Higgs Dark Matter from a Warped Extra-Dimension
– the truncated-inert-doublet model

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ABSTRACT: We construct a 5D $\mathbb{Z}_2$-symmetric model with three D3-branes: two IR ones with negative tension located at ends of an extra-dimensional interval and one UV-brane with positive tension placed in the middle of the interval. Within this setup we investigate the low-energy effective theory for the bulk SM bosonic sector. The $\mathbb{Z}_2$-even zero-modes correspond to known standard degrees of freedom, whereas the $\mathbb{Z}_2$-odd zero modes might serve as dark sector. We discuss two scenarios for spontaneous breaking of the gauge symmetry, one based on expansion of the bulk Higgs field around extra-dimensional vev with non-trivial profile and the second in which the symmetry breaking is triggered by a vev of Kaluza-Klein modes of the bulk Higgs field. It is shown that they lead to the same low-energy effective theory. The effective low-energy scalar sector contains a scalar which mimics the Standard Model (SM) Higgs boson and a second stable scalar particle (dark-Higgs) that is a dark matter candidate; the latter is a component of the zero-mode of $\mathbb{Z}_2$-odd Higgs doublet. The model that results from the $\mathbb{Z}_2$-symmetric background geometry resembles the Inert Two Higgs Doublet Model. The effective theory turns out to have an extra residual $SU(2) \times U(1)$ global symmetry that is reminiscent of an underlying 5D gauge transformation for odd degrees of freedom. At tree level the SM Higgs and the dark-Higgs have the same mass; however, when leading radiative corrections are taken into account the dark-Higgs turns out to be heavier than the SM Higgs. Implications for dark matter are discussed; it is found that the dark-Higgs can provide only a small fraction of the observed dark matter abundance.

KEYWORDS: Warped Extra Dimensions, Dark matter, Bulk Higgs

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1 Introduction

The seminal work of Randall and Sundrum (RS) [1] provides an elegant solution to the hierarchy problem. Their proposal involves an extra-dimension with a non-trivial warp factor due to the assumed anti-de Sitter (AdS) geometry along the extra-dimension. In their model an AdS geometry on an $S_1/Z_2$ orbifold is considered which is equivalent to a line-element $0 \leq y \leq L$, where $y$ is the coordinate of the fifth-dimension and $L = \pi r_c$, with $r_c$ being the radius of the circle in the fifth-dimension. Moreover, their model involves two D3-branes localized on the fixed points of the orbifold, a “UV-brane” at $y = 0$ and an “IR-brane” at $y = L$ (our nomenclature will become clear below), see Fig. 1. The solution for the RS geometry is [1, 2],

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \quad \text{with} \quad A(y) = -k|y|, \quad (1.1)$$

where $k$ is the inverse of the AdS radius. In the original RS1 model [1] it was assumed that the Standard Model (SM) is localized on the IR-brane, whereas gravity is localized on the
UV-brane and propagates through the bulk to the IR-brane. They famously showed that if the 5D fundamental theory involves only one mass scale $M_*$ – the Planck mass in 5D – then, due to the presence of non-trivial warping along the extra-dimension, the effective mass scale on the IR-brane is rescaled to $m_{KK} \sim ke^{-kL} \sim O(\text{TeV})$ and hence ameliorates the hierarchy problem for mild values of $kL \sim O(35)$.

Soon after the RS proposal, many important improvements to the model were considered. First, a stabilization mechanism for the RS1 setup was proposed by Goldberger and Wise [3]; it employs a real scalar field in the bulk of AdS geometry with localized potentials on both of the branes, see also [4]. A second interesting observation, which could potentially solve the fermion mass hierarchy problem within the SM, was made by many groups [5–9]. The core idea of these works was to allow all the SM fields to propogate in the RS1 bulk, except the Higgs field which was kept localized on the IR-brane. In this way, the zero-modes of these bulk fields correspond to the SM fields and the overlap of y-dependent profiles of fermionic fields with the Higgs field could generate the required fermion mass hierarchy. To suppress the EW precision observables, the symmetry of the gauge group was enhanced by introducing custodial symmetry in Ref. [10]. The common lore, in the RS1 model and its extensions, was to keep the Higgs field localized on the IR-brane in order to solve the hierarchy problem. The first attempt to consider the Higgs field in the bulk of RS1 was made by Luty and Okui [11]. They employed AdS/CFT duality\(^1\) to argue that a bulk Higgs scenario can address the hierarchy problem by making the Higgs mass operator marginal in the dual CFT.

A study of electroweak symmetry breaking (EWSB) within the Bulk Higgs scenario was first performed in the RS1 setup by Davoudiasl et al. [14]; they showed that the zero-mode of the bulk Higgs is tachyonic and hence could lead to a vacuum expectation value (vev) at the TeV scale. Recently there have been many studies where a bulk Higgs scenario is considered from different perspectives — see for example: a study with custodial symmetry in the Higgs sector[15]; models with a soft wall setup [16]; bulk Higgs mediated FCNC’s [17]; suppression of EW precision observables by modifying the warped metric near the IR-brane [18–20]; and, a bulk Higgs as the modulus stabilization field (Higgs–radion unification) [21]. Different phenomenological aspects after the Higgs discovery were explored in [22–28]. These phenomenological studies show that the RS1 model with bulk SM fields and its descendants with modified geometry (RS-like warped geometries in general) are consistent with the current experimental bounds and EW precision data.

A separate category of generalization of the RS models was based on the assumption that the singular branes are replaced with thick branes which are smooth field configurations of the bulk scalar field, see e.g. [4] and [29, 30].

As we discussed above, RS-like warped geometries, being consistent with the experimental data, offer an attractive solution to many of the fundamental puzzles of the SM, mostly through geometric means. In the same spirit, one can ask if RS-like warped extra-dimensions can shed some light on another outstanding puzzle of SM, the lack of a candidate for dark matter (DM) which constitutes 83% of the observed matter in the universe [31].

\(^1\)For the phenomenological applications of AdS/CFT with RS1 geometry, see for example [12, 13].
appears that unlike (flat) universal extra-dimensions (UED), where the KK-modes of the bulk fields can be even and odd under KK-parity (implying that the lowest KK-odd particle (LKP) could be a natural dark matter candidate [32, 33]), RS1-like models (involving two branes and warped bulk) are unable to offer an analogue of KK-parity. The reason lies in the fact that the RS1 geometry is just a single slice of AdS space and, since warped, cannot be symmetric around any point along the extra-dimension and hence does not allow a KK-parity. As a result it cannot accommodate a realistic dark matter candidate. To cure this problem in the warped geometries, usually extra discrete symmetries are introduced such that the SM fields are even while the DM is odd under such discrete symmetries in order to make it stable [34–37]. Another way to mend this problem in warped geometries is to introduce an additional hidden sector with some local gauge symmetries such that only DM is charged under the hidden sector gauge symmetries and it couples to the SM very weakly [38, 39], (see also [40]).

An alternative to introducing additional symmetries, is to extend the RS1-like warped geometry in such a way that the whole geometric setup becomes symmetric around a fixed point in the bulk. Two \(\mathbb{Z}_2\) symmetric warped configurations are possible. In the first, two identical AdS patches are symmetrically glued together at a UV fixed point, while in the second two identical AdS pathes are symmetrically glued together at an IR fixed point. The geometric configuration when the two AdS copies are glued together at the UV fixed point will be referred as “IR-UV-IR geometry”, whereas the geometry corresponding to the setup when two AdS copies are glued at the IR fixed point is called “UV-IR-UV geometry”. We will only consider the IR-UV-IR geometric setup — it is straight forward to extend our analysis to the UV-IR-UV geometries. (A common pathology associated with this latter type of geometry is the appearance of ghosts.) We consider an interval \(y \in [-L, L]\) in the extra-dimension, where on each end of the interval \(y = \pm L\) there is a D3 brane with negative tension (in Sec. 2.1 it will be clear why we need negative tension branes) and at the center of the interval, \(y = 0\), we place a positive tension brane where we assume that gravity is localized. We call the boundary branes “IR-branes” and the brane at \(y = 0\) we term the “UV-brane”. The IR-UV-IR geometry and a pictorial description of such a geometric setup is shown in Fig. 2. Since the brane tensions of the two IR-branes are the same, this geometry is \(\mathbb{Z}_2\) symmetric. We are aware of only two earlier attempts to construct a similar setup. The first [41] treated the lowest odd KK gauge mode as the DM candidate. The second employed a kink-like UV thick brane [42] and the corresponding dark-matter was the first odd KK-radion [43].

In this work, we place all the SM fields, including the Higgs doublet, in the bulk of the IR-UV-IR geometry. The geometric \(\mathbb{Z}_2\) parity (\(y \rightarrow -y\) symmetry) leads to “warped KK-parity”, i.e. there are towers of even and odd KK-modes corresponding to each bulk field. We focus on EWSB induced by the bulk Higgs doublet and low energy aspects of the 4D effective theory for the even and odd zero-modes assuming the KK-mass scale is high enough \(\sim \mathcal{O}(\text{few})\) TeV. In the effective theory the even and odd Higgs doublets mimic a two-Higgs-doublet model (2HDM) scenario – the truncated inert-doublet model – with the odd doublet similar to the inert doublet but without corresponding pseudoscalar and charged scalars. All the parameters of this truncated 2HDM are determined by the
fundamental 5D parameters of the theory and the choice of boundary conditions for the fields at \( \pm L \). (Note that the boundary or “jump” conditions at \( y = 0 \) follow from the bulk equations of motion in the case of even modes, whereas odd modes are required to be zero by symmetry.) There are many possible alternative choices for the b.c. at \( \pm L \). We allow the \( y \)-derivative of a field to have an arbitrary value at \( \pm L \) as opposed to requiring that the field value itself be zero, i.e. we employ Neumann or mixed b.c. rather than Dirichlet b.c. at \( \pm L \). Only the former yields a non-trivial theory allowing spontaneous symmetry breaking (SSB), whereas the latter leads to an explicit symmetry breaking scenario in which there are no Goldstone modes and the gauge bosons do not acquire mass. With these choices, the symmetric setup yields an odd Higgs zero-mode that is a natural candidate for dark matter. We compute the one-loop quadratic (in cutoff) corrections to the two scalar zero modes within the effective theory and discuss their mass splitting. The dark matter candidate is a WIMP — we calculate its relic abundance in the cold dark matter paradigm.

The paper is composed as follows. In Sec. 2, we setup the IR-UV-IR geometric configuration and provide background solutions. We also discuss the manifestation of KK-parity due the \( \mathbb{Z}_2 \) geometric setup. An Abelian Higgs mechanism, with a complex scalar field and a gauge field, is studied in our background geometry in Sec. 3. In the Abelian case we lay down the foundation for SSB due to bulk Higgs, which is later useful for the case of EWSB in the SM. Two apparently different approaches are considered to study SSB in the Abelian case: \( i \) SSB by vacuum expectation values of the KK modes; and, \( ii \) SSB via a vacuum expectation value of the 5D Higgs field. Low energy (zero-mode) 4D effective theories are obtained within the two approaches and we find that the effective theories are identical up to corrections of order \( \mathcal{O}(m_0^2/m_{KK}^2) \), where \( m_0 \) and \( m_{KK} \) are the zero-mode mass and KK-mass scale, respectively. Section 4 contains the main part of our work. There, we focus on EWSB for the SM gauge sector due to the bulk Higgs doublet in our \( \mathbb{Z}_2 \) symmetric geometry and obtain a low-energy 4D effective theory containing all the SM fields plus a real scalar – a dark matter candidate – which is odd under the discrete \( \mathbb{Z}_2 \) symmetry. In the subsequent two subsections of Sec. 4, we consider the quantum corrections to the scalar masses below the KK-scale \( \sim \mathcal{O}(\text{few}) \) TeV and explore the possible implications of the dark-matter candidate by calculating its relic abundance. We summarize and give our conclusions in Sec. 5. For a warmup, Appendix A discusses SSB of a discrete symmetry with a real scalar in the bulk of our geometric setup.
2 A $Z_2$ symmetric warped extra-dimension and KK-parity

In this section we provide the background solution for the $Z_2$ symmetric background (IR-UV-IR) geometry and show how KK-parity is manifested within this geometric setup.

2.1 The IR-UV-IR model

We consider the IR-UV-IR warped geometry compactified on an interval, $-L \leq y \leq L$, where a D3-brane with positive tension is located at the UV fixed point, $y = 0$, and two negative tension D3-branes are located at the IR fixed points, $y = \pm L$. The 5D gravity action $S_G$ for such a geometry can be written as

$$S_G = \int d^5x \sqrt{-g} \left\{ \frac{R}{2} - \Lambda_B - \lambda_{UV} \delta(y) - \lambda_{IR} \delta(y + L) - \lambda_{IR} \delta(y - L) \right\} + S_{GH}, \quad (2.1)$$

where $R$ is the Ricci scalar, $\Lambda_B$ is the bulk cosmological constant and $\lambda_{UV}(\lambda_{IR})$ are the brane tensions at the UV(IR) fixed points. Since our geometry is compact with boundaries, the action contains the Gibbons-Hawking boundary term,

$$S_{GH} = -\int_{\partial M} d^4x \sqrt{-\hat{g}} \mathcal{K}, \quad (2.2)$$

where $\mathcal{K}$ is the intrinsic curvature of the surface of the boundary manifold $\partial M$, given by

$$\mathcal{K} = -\hat{g}^{\mu\nu} \nabla_\mu n_\nu = \hat{g}^{\mu\nu} \Gamma^M_{\mu\nu} n_M, \quad (2.3)$$

with $n_M$ being the unit normal vector to the surface of the boundary manifold $\partial M$ and $\hat{g}_{\mu\nu}$ is the induced boundary metric. For the 5D manifold with 4D Poincaré invariance ($n^5 = 1$ and $n^\mu = 0$), the intrinsic curvature reduces to

$$\mathcal{K} = -\frac{1}{2} \hat{g}^{\mu\nu} \partial_5 \hat{g}_{\mu\nu}. \quad (2.4)$$

The solution of the Einstein equations resulting from the above action is the RS metric (1.1), where the AdS curvature $k$ is related to $\Lambda_B$ by

$$\Lambda_B = 6k^2. \quad (2.5)$$

Since the above setup is compactified on an interval $y \in [-L, L]$, rather than on a circle as in RS1, one needs to be careful and show that the solution (1.1) is compatible with the boundaries and that the effective 4D cosmological constant is zero, see also [44]. We will see below that we need a fine tuning between the 5D cosmological constant $\Lambda_B$ and the brane tensions $\lambda_{UV,IR}$ in order to get zero 4D cosmological constant. One can calculate the effective 4D cosmological constant $\Lambda_4$ from the action (2.1) by integrating out the extra-dimension,

$$\Lambda_4 = -\int_{-L}^{L} dy \sqrt{-g} \left\{ \frac{R}{2} - \Lambda_B - \lambda_{UV} \delta(y) - \lambda_{IR} \left[ \delta(y + L) + \delta(y - L) \right] \right\} + \sqrt{-\hat{g}} \mathcal{K} \bigg|_{-L}^{L}, \quad (2.6)$$

We use the metric signature $(-, +, +, +, +)$ and the unit system such that the 5D Planck mass $M_5 = 1$.

The Gibbons-Hawking boundary term is needed in order to cancel the variation of the Ricci tensor at the boundaries so that the RS metric (1.1) is indeed a solution of the Einstein equations of motion.
where \( R = -20A'^2 - 8A'' \) and \( \Lambda_B = -6A'^2 \) corresponding to the solution (1.1). Using \( A(y) = -k|y| \) we find,

\[
\Lambda_4 = 2e^{-4kL} (\lambda_{IR} + 3k) + (\lambda_{UV} - 6k),
\]

which can only be zero if

\[
\lambda_{UV} = 6k \quad \text{and} \quad \lambda_{IR} = -3k.
\]

This result explicitly shows that one needs a positive tension brane at \( y = 0 \) and two negative tension branes at \( y = \pm L \) in order to obtain zero 4D cosmological constant. This is the usual fine tuning which appears in brane world scenarios [1, 3, 4]. Hence we have a 5D geometry with AdS solution (1.1) with negative bulk cosmological constant and a positive tension brane in the middle and two equal negative tension branes at the end of the interval, see Fig. 2.

### 2.2 KK-Parity

The IR-UV-IR geometry is \( \mathbb{Z}_2 \)-symmetric and we will consider this symmetry to be exact for our 5D theory. If the 5D theory has this \( \mathbb{Z}_2 \)-parity (symmetry) then the Schrödinger-like potential for all the fields is symmetric, resulting in even (symmetric) and odd (antisymmetric) eigenmodes under this parity. Thus, a general field \( \Phi(x, y) \) can be KK decomposed,

\[
\Phi(x, y) = \sum_n \phi_n(x)f_n(y),
\]

and, due to the \( \mathbb{Z}_2 \) geometry, the wave functions \( f_n(y) \) are either even or odd. As a result, we can write

\[
\Phi(x, y) = \Phi^+(x, y) + \Phi^-(x, y),
\]

where

\[
\Phi^+(x, y) = \sum_n \phi_n^+(x)f_n^+(y) \frac{y+y}{y-y} \Phi^+(x, y),
\]

\[
\Phi^-(x, y) = \sum_n \phi_n^-(x)f_n^-(y) \frac{y+y}{y-y} -\Phi^-(x, y).
\]

Due to the symmetry, a single odd KK-mode cannot couple to two even KK-modes, which will ensure that the lowest odd KK-mode will be stable. This is the geometric manifestation of KK-parity in a warped extra-dimension.

Furthermore, as the geometry is \( \mathbb{Z}_2 \) symmetric in \( y \in [-L, L] \), the continuity conditions for odd and even modes at \( y = 0 \) strongly impact the physics scenario. Our choice will be that the odd (even) modes satisfy Dirichlet (Neumann or mixed) boundary conditions (b.c.) at \( y = 0 \), respectively. As for the odd modes, continuity implies that they must be zero at \( y = 0 \), but we could also have demanded the Neumann conditions that their \( y \) derivative be zero at \( y = 0 \). We choose not to impose this additional b.c. in this work. As regards the even modes, one cannot choose Dirichlet b.c. at \( y = 0 \) because of the presence of the UV-brane and associated “jump” conditions following from the equations of motion.
3 SSB in the IR-UV-IR model: the Abelian Higgs mechanism

In this section we will discuss the mechanism of spontaneous symmetry breaking (SSB) for an Abelian case with the Higgs field (a complex scalar) in the IR-UV-IR geometry of Sec. 2.1. The metric is given by Eq. (1.1), we will neglect the back reaction of the bulk fields on the geometry. We will borrow most of our results from Appendix A, and focus here on gauge-symmetric aspects of the model. We start by specifying the 5D Abelian action,

\[ S_{\text{Ab}} = -\int d^5x \sqrt{-g} \left\{ \frac{1}{4} F_{MN} F^{MN} + |D_M H|^2 + \mu^2 B H^* H \\
+ V_{IR}(H) \delta(y + L) + V_{UV}(H) \delta(y) + V_{IR}(H) \delta(y - L) \right\}, \]

(3.1)

where \( D_M = \partial_M - ig_5 A_M \) with the 5D \( U(1) \) coupling constant \( g_5 \) and \( F_{MN} = \partial_M A_N - \partial_N A_M \). We require that the bulk potential and the UV-brane potential have only quadratic terms whereas the IR-brane potential is allowed to have a quartic term:

\[ V_{UV}(H) = \frac{m_{UV}^2}{k} H^* H, \quad V_{IR}(H) = -\frac{m_{IR}^2}{k} H^* H + \frac{\lambda_{IR}}{k} (H^* H)^2. \]

(3.2)

In this way EWSB is mainly triggered by the IR-brane. Above, \( H \) is a complex scalar field and the parametrization is such that \( m_{UV} \) and \( m_{IR} \) have mass dimensions while \( \lambda_{IR} \) is dimensionless. The gauge transformations can be written as

\[ H(x, y) \rightarrow H'(x, y) = e^{iA(x,y)} H(x, y), \]

(3.3)

\[ A_M(x, y) \rightarrow A'_M(x, y) = A_M(x, y) + \frac{1}{g_5} \partial_M \Lambda(x, y), \]

(3.4)

where \( \Lambda(x, y) \) is the gauge parameter.

As one can see from the toy model discussed in Appendix A, the fields in the IR-UV-IR setup have even and odd bulk wave functions implied by the geometric KK-parity. Hence, in our Abelian model, it is convenient to decompose the generic Higgs and the gauge field into fields of definite parity as follows

\[ H(x, y) = H^{(+)}(x, y) + H^{(-)}(x, y), \quad A_M(x, y) = A^{(+)}_M(x, y) + A^{(-)}_M(x, y), \]

(3.5)

where \( \pm \) denotes the even and odd states. The gauge transformations for the even and odd parity modes are,

\[ A^{(\pm)}_\mu(x, y) \rightarrow A^{(\pm)}_\mu(x, y) = A^{(\pm)}_\mu(x, y) + \frac{1}{g_5} \partial_\mu \Lambda^{(\pm)}(x, y), \]

(3.6)

\[ A^{(\pm)}_5(x, y) \rightarrow A^{(\pm)}_5(x, y) = A^{(\pm)}_5(x, y) + \frac{1}{g_5} \partial_5 \Lambda^{(\pm)}(x, y), \]

(3.7)

\[ \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\Lambda^{(+)}_1}e^{i\Lambda^{(-)}_1}H^{(+)} \\ e^{i\Lambda^{(-)}_1}e^{i\Lambda^{(+)}_1}H^{(-)} \end{pmatrix}, \]

(3.8)

\( \Lambda^{(+)1} \) has mass dimension \(-1/2\).
where $\mathbb{1}$ is a $2 \times 2$ unit matrix, whereas $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix.

With this decomposition the above action can be written as

$$ S_{Ab} = - \int d^5 x \sqrt{-g} \left\{ \frac{1}{4} F_{\mu\nu}^{(+)} F_{\mu\nu}^{(+)} + \frac{1}{2} F_{\mu\sigma}^{(-)} F_{\mu\sigma}^{(-)} + D_M H^{(+)*} D_M H^{(+)} + \mu_B H^{(+)*} H^{(+)} \\
+ \frac{1}{4} F_{\mu\nu}^{(-)} F_{\mu\nu}^{(-)} + \frac{1}{2} F_{\mu\sigma}^{(-)} F_{\mu\sigma}^{(-)} + D_M H^{(-)*} D_M H^{(-)} + \mu_B H^{(-)*} H^{(-)} \\
+ V_{IR}(H^{(\pm)}) \delta(y + L) + V_{UV}(H^{(+)}) \delta(y) + V_{IR}(H^{(\pm)}) \delta(y - L) \right\} , \quad (3.9) $$

where,

$$ F_{\mu\nu}^{(\pm)} \equiv \partial_\mu A_\nu^{(\pm)} - \partial_\nu A_\mu^{(\pm)}, \quad F_{\mu\sigma}^{(\pm)} \equiv \partial_\mu A_\sigma^{(\pm)} - \partial_\sigma A_\mu^{(\pm)}. \quad (3.10) $$

The brane localized potentials for the Higgs field, $V_{UV}(H)$ and $V_{IR}(H)$, can be written in terms of even and odd parity modes as

$$ V_{UV}(H^{(+)}) = \frac{m_{UV}^2}{k} |H^{(+)}|^2, \quad (3.11) $$

$$ V_{IR}(H^{(\pm)}) = - \frac{m_{IR}^2}{k} |H^{(+)}|^2 - \frac{m_{IR}^2}{k} |H^{(-)}|^2 + \frac{\lambda_{IR}}{k^2} |H^{(+)}|^4 + \frac{\lambda_{IR}}{k^2} |H^{(-)}|^4 \\
+ \frac{4 \lambda_{IR}}{k^2} |H^{(+)|^2}|H^{(-)|^2} + \frac{\lambda_{IR}}{k^2} \left( (H^{(+)|^2} H^{(-)|^2} + h.c. \right). \quad (3.12) $$

In the above, we have not written $H^{(-)}$ terms in $V_{UV}$ since $H^{(-)}(0) = 0$. Moreover, we have not written terms in the above action, including the potentials, which are explicitly odd as they will not contribute after integration over the $y$-coordinate. One can easily check that the above brane potentials are invariant under the gauge transformations defined above. Also note that $F_{\mu\nu}^{(\pm)}$ and $F_{\mu\sigma}^{(\pm)}$ are gauge invariant under the gauge transformations (3.6) and (3.7). In the even/odd basis, the covariant derivatives $D_\mu$ and $D_5$ following from $D_M = \partial_M - ig_5 A_M$, take the form

$$ D_\mu \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} = \begin{bmatrix} \partial_\mu - ig_5 \left( A_\mu^{(+) A_\mu^{(-)}} \right) \\ \partial_\mu - ig_5 \left( A_\mu^{(-) A_\mu^{(+)}} \right) \end{bmatrix} \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \quad (3.13) $$

$$ D_5 \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} = \begin{bmatrix} \partial_5 - ig_5 \left( A_5^{(+) A_5^{(-)}} \right) \\ \partial_5 - ig_5 \left( A_5^{(-) A_5^{(+)}} \right) \end{bmatrix} \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}. \quad (3.14) $$

Under the gauge transformations the covariant derivative transforms as

$$ D_M \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \rightarrow D'_M \begin{pmatrix} H'^{(+)} \\ H'^{(-)} \end{pmatrix} = e^{i\Lambda^{(+)}} \mathbb{1}_2 e^{i\Lambda^{(-)} \tau_1} D_M \begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix}, \quad (3.15) $$

i.e. it transforms the same way as complex scalar field transforms (3.8). It is important to note that the above action is manifestly gauge invariant under the gauge group $U(1)' \times U(1)$.

The next two subsections are devoted to two possible strategies for implementing spontaneous gauge symmetry breaking for this Abelian $U(1)$ symmetric case. We are going to describe and compare: (i) SSB by vacuum expectation values of the KK modes and (ii)
SSB by a $y$-dependent vacuum expectation value of the 5D Higgs field. Readers can either follow and continue, or, as we would advise, one may consider warming up within a toy model of a real scalar field and spontaneous symmetry breaking as discussed in Appendix A and then return to the following subsections. In Appendix A we also consider the above two possible approaches to SSB.

### 3.1 SSB by vacuum expectation values of KK modes

In this case we will choose the 5D axial gauge, $A_5^{(\pm)} = 0$. This gauge is realized by choosing the gauge parameter such that,

$$\Lambda^{(\pm)}(x, y) = -g_5 \int dy A_5^{(\mp)}(x, y) + \tilde{\Lambda}^{(\pm)}(x),$$

where $\tilde{\Lambda}^{(\pm)}(x)$ is the integration constant (residual gauge freedom) and only depends on $x^5$. Note that the $\tilde{\Lambda}^{(-)}(x)$, being an odd function of $y$, must vanish. Consequently, we are left with only one 4D gauge function, $\tilde{\Lambda}^{(+)}(x)$.

In this gauge, the Abelian action reduces to,

$$S_{Ab} = -\int d^5x \sqrt{-g} \left( \frac{1}{4} F_{\mu\nu}^{(+)} F_{\mu\nu}^{(+)} + \frac{1}{2} \partial_{\mu} A^{(+)}_5 \partial^{\mu} A^{(+)}_5 + D_M H^{(+)*} D_M H^{(+)} + \mu_B^2 |H^{(+)}|^2 ight.$$

$$+ \frac{1}{4} F_{\mu\nu}^{(-)} F_{\mu\nu}^{(-)} + \frac{1}{2} \partial_{\mu} A^{(-)}_5 \partial^{\mu} A^{(-)}_5 + D_M H^{(-)*} D_M H^{(-)} + \mu_B^2 |H^{(-)}|^2$$

$$+ V_{IR}(H^{(+)}_5 \delta(y + L) + V_{UV}(H^{(+)}_5 \delta(y) + V_{IR}(H^{(\pm)}_5 \delta(y - L) \right) \right),$$

where the brane potentials are given by Eqs. (3.11) and (3.12). It is convenient to parametrize the complex scalar field $H^{(\pm)}(x, y)$ in the following form,

$$\begin{pmatrix} H^{(+)} \\ H^{(-)} \end{pmatrix} \equiv e^{i\pi \tau_3(n^+ \mathbf{1} + n^- \tau_1)} \begin{pmatrix} \Phi^{(+)} \\ \Phi^{(-)} \end{pmatrix},$$

where $\Phi^{(\pm)}(x, y)$ and $\pi^{(\pm)}(x, y)$ are real scalar fields. We KK-decompose the scalar fields $\Phi^{(\pm)}(x, y)$, $\pi^{(\pm)}(x, y)$ and the gauge fields $A_5^{(\pm)}(x, y)$ as

$$\Phi^{(\pm)}(x, y) = \sum_n \Phi_n^{(\pm)}(x) f_n^{(\pm)}(y),$$

$$\pi^{(\pm)}(x, y) = \sum_n \pi_n^{(\pm)}(x) a_n^{(\pm)}(y),$$

$$A_5^{(\pm)}(x, y) = \sum_n A_{n5}^{(\pm)}(x) a_n^{(\pm)}(y),$$

where the wave-functions $f_n^{(\pm)}(y)$ satisfy Eq. (A.8) from Appendix A.1. (We borrow the results for the wave-functions $f_n^{(\pm)}(y)$ from Appendix A.1.) We choose that gauge wave-functions $a_n^{(\pm)}(y)$ to satisfy

$$-\partial_y \left( e^{2A(y)} \partial_y a_n^{(\pm)}(y) \right) = m_n^2 a_n^{(\pm)}(y),$$

where $m_n^2$ is a function of $n$.
as consistent with the equations of motion following from Eq. (3.17). The $a_n^{(\pm)}$ satisfy the following orthonormality conditions,

$$\int_{-L}^{+L} dy a_m^{(\pm)}(y)a_n^{(\pm)}(y) = \delta_{mn}. \quad (3.23)$$

It is worth commenting here that the gauge field $A_\mu^{(\pm)}(x, y)$ and the scalar field $\pi^{(\pm)}(x, y)$ share the same $y$-dependent KK-eigen bases $a_n^{(\pm)}(y)$, as is necessitated (see below) by the Higgs mechanism. The KK-modes satisfy $\Box^{(4)} A_{n\mu}^{(\pm)}(x) = m_n^2 A_{n\mu}^{(\pm)}(x)$. The boundary (jump) conditions for $a_n^{(\pm)}(y)$ at $y = 0$ and $y = \pm L$ are,

$$\partial_5 a_n^{(+)}(y) \bigg|_{0+} = 0, \quad a_n^{(-)}(y) \bigg|_{0+} = 0, \quad \partial_5 a_n^{(\pm)}(y) \bigg|_{\pm L^+} = 0. \quad (3.24)$$

We choose the Neumann b.c. for $a_n^{(\pm)}(y)$ at $y = 0, \pm L$ in order to ensure that we get non-zero even zero-mode gauge profiles. With regard to the odd modes, we have chosen the Neumann b.c. of $\partial_5 a_n^{(-)}(\pm L) = 0$ — the other choice of $a_n^{(-)}(\pm L) = 0$ would lead to a trivial theory with $a_n^{(-)}(y) = 0$ everywhere.

With the above KK-decomposition we can write the effective 4D action for the Abelian case, with the Higgs brane localized potentials, as

$$S_{Ab} = -\int d^4x \left\{ \frac{1}{4} F^{\mu\nu}_{n(+) - F^{\mu\nu}_{n(-)}} + \frac{1}{2} m_n^2 A_{n(+) - A_{n(-)}} + \frac{1}{4} F^{\mu\nu}_{n(+) - F^{\mu\nu}_{n(-)}} + \frac{1}{2} m_n^2 A_{n(+) - A_{n(-)}} \right\}$$

$$+ \partial_\mu \Phi_n^{(+)} \partial_\mu \Phi_n^{(+)} + m_2^{n(+) - \Phi_n^{(+)}} + \partial_\mu \Phi_n^{(-)} \partial_\mu \Phi_n^{(-)} + m_2^{n(-) - \Phi_n^{(-)}}$$

$$+ \left( g_{klnm}^{(+)} \Phi_{k}^{(+)} \Phi_{l}^{(+)} + g_{klnm}^{(-)} \Phi_{k}^{(-)} \Phi_{l}^{(-)} \right) \left( A_{k\mu}^{(+)} - \partial_\mu \pi_{k}^{(+)} \right) \left( A_{l\mu}^{(+)} - \partial_\mu \pi_{l}^{(+)} \right)$$

$$+ \left( g_{klnm}^{(-)} \Phi_{k}^{(+)} \Phi_{l}^{(-)} + g_{klnm}^{(-)} \Phi_{k}^{(-)} \Phi_{l}^{(+)} \right) \left( A_{k\mu}^{(-)} - \partial_\mu \pi_{k}^{(-)} \right) \left( A_{l\mu}^{(-)} - \partial_\mu \pi_{l}^{(-)} \right)$$

$$+ 4g_{klnm}^{(+)} \Phi_{k}^{(+)} \Phi_{l}^{(+)} \Phi_{m}^{(+)} \Phi_{n}^{(+)} + \lambda_{klnm}^{(-)} \Phi_{k}^{(-)} \Phi_{l}^{(-)} \Phi_{m}^{(-)} \Phi_{n}^{(-)}$$

$$+ \lambda_{klnm}^{(+)} \Phi_{k}^{(+)} \Phi_{l}^{(+)} \Phi_{m}^{(+)} \Phi_{n}^{(+)} \right\}, \quad (3.25)$$

where the indices in this action are raised and lowered by Minkowski metric and the coupling constants are given as

$$\lambda_{klnm}^{(\pm)} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} \left| f_{k}^{(\pm)} f_{l}^{(\pm)} f_{m}^{(\pm)} f_{n}^{(\pm)} \right|_{L}, \quad \lambda_{klnm} = e^{4A(L)} \frac{\lambda_{IR}}{k^2} \left| f_{k}^{(+)} f_{l}^{(+)} f_{m}^{(-)} f_{n}^{(-)} \right|_{L}, \quad (3.26)$$

$$g_{klnm}^{(\pm)} = g_5^2 \int_{-L}^{L} dy e^{2A(y)} a_k^{(\pm)} a_l^{(\pm)} f_m^{(\pm)} f_n^{(\pm)}, \quad (3.27)$$

$$g_{klnm}^{(+)} = g_5^2 \int_{-L}^{L} dy e^{2A(y)} a_k^{(+)} a_l^{(+)} f_m^{(-)} f_n^{(-)}, \quad (3.28)$$

$$g_{klnm}^{(-)} = g_5^2 \int_{-L}^{L} dy e^{2A(y)} a_k^{(-)} a_l^{(-)} f_m^{(+)} f_n^{(+)}, \quad (3.29)$$

where the superscripts $\pm$ on the coupling constants are just for notational purposes and do not refer to the parity.
The above action is valid for all KK-modes. Assuming that the KK-scale is high enough, i.e. \( m_{KK} \sim \mathcal{O}(\text{few}) \text{ TeV} \), we can employ the effective theory where only the lowest modes (zero-modes with masses much below \( m_{KK} \)) are considered. Equation (3.22) along with the b.c. (3.24) imply that the odd zero-mode wave-function of the gauge boson is zero, i.e. \( a_0^{(-)} = 0 \). As a result, in the effective theory the odd zero-mode gauge boson \( A_0^{(-)} \) and odd parity Goldstone mode \( \pi_0^{(-)} \) are not present. In contrast, the even zero-mode wave-function for the gauge boson has a constant profile in the bulk, i.e. \( a_0^{(+)} = 1/\sqrt{2L} \), implying that the couplings of the even zero-mode gauge boson \( g_{00}^{\pm} \) and \( g_{00}^{00} \) are equal (see Eq. (3.27)), which, in turn, implies \( g_{00}^{(+)0} = g_{00}^{(+)0} = g_4 \delta_{mn} \), with \( g_4 \equiv g_5/\sqrt{2L} \). The forms of the scalar zero-mode wave functions \( f_0^{(+)0} (y) \) are given by Eq. (A.25). We can now write down the low-energy effective action for the zero-modes:

\[
S_{Ab}^{\text{eff}} = -\int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^{(+)0} F_{0(+)\mu\nu}^{(+)0} + \partial^\mu \Phi_0^{(+)0} \partial^\nu \Phi_0^{(+)0} - \mu^2 \Phi_0^{(+)0}^2 + \partial^\mu \Phi_0^{(-)0} \partial^\nu \Phi_0^{(-)0} - \mu^2 \Phi_0^{(-)0}^2 \\
+ \left( g_{0000}^{(+)0} \Phi_0^{(+)0} g_{0000}^{(-)0} \Phi_0^{(-)0} \right) \left( A_0^{(+)0} - \partial_\mu \pi_0^{(+)0} \right)^2 \\
+ \lambda_0 (\Phi_0^{(+)0})^4 + \lambda_0 (\Phi_0^{(-)0})^4 + 6 \lambda_0 (\Phi_0^{(+)0})^2 (\Phi_0^{(-)0})^2 \right),
\]

(3.30)

where the couplings can be read from Eqs. (3.26) and (3.27) and the mass parameter \( \mu \) is defined as \( \mu^2 \equiv (1 + \beta) m_{KK}^2 \delta_{IR} \), with the parameters defined as

\[
\delta_{IR} \equiv \frac{m_{KK}^2}{k^2} - 2(2 + \beta), \quad m_{KK} \equiv ke^{-kL} \quad \text{and} \quad \beta \equiv \sqrt{4 + \mu_B^2/k^2}.
\]

(3.31)

By using the results from Appendix A, we get the following couplings in terms of the parameters of the fundamental theory:

\[
\lambda_0 (\Phi_0^{(+)0})^2 = \lambda_0 \equiv \lambda_0 \lambda_0^2, \quad \lambda_0 (\Phi_0^{(-)0})^2 = \lambda_0 \equiv \frac{g_5}{\sqrt{2L}}.
\]

(3.32) and

(3.33)

Our effective theory could also be parameterized by redefining \( A_0^{(+)0} (x) \equiv A_0 (x) \), \( \pi_0^{(+)0} (x) \equiv \pi (x) \) and

\[
H_1(x) \equiv e^{i g_4 \pi (x)} \Phi_0^{(+)0} (x), \quad H_2(x) \equiv e^{i g_4 \pi (x)} \Phi_0^{(-)0} (x),
\]

(3.34)
in which case the above effective action can be written in a nice gauge covariant form as

\[
S_{Ab}^{\text{eff}} = -\int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^\mu F_{\mu\nu}^\nu + D_\mu H_1^* D^\mu H_1 + D_\mu H_2^* D^\mu H_2 + V(H_1, H_2) \right\},
\]

(3.35)

where the covariant derivative is defined as

\[
D_\mu \equiv \partial_\mu - ig_4 A_\mu,
\]

(3.36)

and the scalar potential can be written as

\[
V(H_1, H_2) = -\mu^2 |H_1|^2 - \mu^2 |H_2|^2 + \lambda |H_1|^4 + \lambda |H_2|^4 + 6 \lambda |H_1|^2 |H_2|^2.
\]
It is important to note that, after choosing the gauge \( A_5(x, y) = 0 \), we are left with a residual gauge freedom with a single purely 4D gauge parameter \( \tilde{\Lambda}^{(+)}(x) \) such that the above Lagrangian is invariant under the \( U(1) \) gauge transformation,

\[
A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{g_4} \partial_\mu \tilde{\Lambda}^{(+)}(x),
\]

\[
H_1(x) \rightarrow H'_1(x) = e^{ig_4 \tilde{\Lambda}^{(+)}} H_1(x), \quad H_2(x) \rightarrow H'_2(x) = e^{ig_4 \tilde{\Lambda}^{(+)}} H_2(x).
\]

Thus, besides \( \mathbb{Z}_2' \times \mathbb{Z}_2 \) symmetry, the above potential is invariant under \( U(1)' \times U(1) \). One \( U(1) \) has been gauged while the other is a remnant of the global unbroken symmetry associated with the odd gauge transformation \( (\Lambda^{(-)}) \) defined in Eqs. (3.7)-(3.8).

As illustrated in the toy model Appendix A.1, we choose the vacuum such that the even parity Higgs \( H_1 \) acquires a vev, whereas the odd parity Higgs \( H_2 \) does not. That choice of vacuum implies values of \( v_1 \) and \( v_2 \) given by,

\[
v_1^2 = \frac{\mu^2}{\lambda}, \quad v_2 = 0.
\]

Now let us consider the fluctuations of the above fields around our choice of the vacuum,

\[
H_1(x) = \frac{1}{\sqrt{2}} \left( v_1 + h \right) e^{ig_4 \pi(x)}, \quad H_2(x) = \frac{1}{\sqrt{2}} \chi e^{ig_4 \pi(x)}.
\]

We rewrite our effective action (3.35) only up to the quadratic order in fluctuations as

\[
S^{(2)}_{Ab} = - \int d^4 x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g_4^2 v_1^2}{2} \left( A_\mu - \partial_\mu \pi \right)^2 + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} m_h^2 h^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m_\chi^2 \chi^2 \right\}.
\]

The mixing between \( A_\mu \) and \( \pi \) in the above action can be removed by an appropriate 4D gauge choice. Here we will choose the 4D unitary gauge such that \( \pi = 0 \) and the gauge field acquires mass. The remaining scalars are \( h \) and \( \chi \) with masses

\[
m_h^2 = m_\chi^2 = 2\mu^2.
\]

Hence, the full effective Abelian action can be written in the 4D unitary gauge as

\[
S^{eff}_{Ab} = - \int d^4 x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} m_h^2 h^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m_\chi^2 \chi^2 \\
+ \frac{\mu}{\sqrt{\lambda}} h A_\mu A^\mu + \frac{1}{2} g_4^2 \left( h^2 + \chi^2 \right) A_\mu A^\mu \right\},
\]

where

\[
m_A^2 = g_4^2 v_1^2 = g_4^2 \frac{\mu^2}{\lambda}.
\]

To summarize, the zero-mode effective theory for the Abelian case has two real scalars with equal mass and a massive gauge boson. Also, the above action is invariant under the \( \mathbb{Z}_2 \) symmetry \( h \rightarrow h \) and \( \chi \rightarrow -\chi \).
3.2 SSB by a vacuum expectation value of the 5D Higgs field

In this subsection, we write the complex scalar fields $H^{(±)}$ as

$$
\begin{pmatrix}
H^{(+)}(x, y) \\
H^{(-)}(x, y)
\end{pmatrix} = \frac{1}{\sqrt{2}} e^{i g_5 (π^{(±)}(x, y))} F^{(±)(x, y)} \begin{pmatrix}
v(y) + h^{(+)}(x, y) \\
h^{(-)}(x, y)
\end{pmatrix}.
\tag{3.45}
$$

As mentioned above, the vev $v(y)$ is only associated with the even Higgs field $H^{(+)}$. The fluctuations $h^{(+)}(x, y)$ and $π^{(±)}(x, y)$ are even, whereas the fluctuations $h^{(-)}(x, y)$ and $π^{(±)}(x, y)$ are odd under the $Z_2$ geometric parity.

We can write the action Eq. (3.9) up to quadratic order in fields as

$$
S^{(2)}_{AB} = - \int d^5 x \left\{ \frac{1}{4} F^{(μν)}_{μν} F^{(μν)}_{μν} + \frac{1}{2} e^{2 A(y)} \left( (\partial_μ A_5^{(μ)})^2 + (\partial_μ A_5^{(μ)})^2 + g_5^2 v^2 A_μ^{(μ)} A_5^{(μ)} \right) \\
+ \frac{1}{2} e^{2 A(y)} \left( (\partial_μ v + \partial_μ h^{(μ)})^2 + g_5^2 v^2 \left( A_5^{(μ)} - \partial_μ π^{(μ)} \right)^2 + μ_B(v + h^{(μ)})^2 \right) \\
+ \frac{1}{4} F^{(μν)}_{μν} F^{(μν)}_{μν} + \frac{1}{2} e^{2 A(y)} \left( (\partial_μ A_5^{(μ)})^2 + (\partial_μ A_5^{(μ)})^2 + g_5^2 v^2 A_μ^{(μ)} A_5^{(μ)} \right) \\
+ \frac{1}{2} e^{2 A(y)} \left( (\partial_μ h^{(μ)})^2 + g_5^2 v^2 \left( A_5^{(μ)} - \partial_μ π^{(μ)} \right)^2 + μ_B h^{(μ)} \right) \right\},
\tag{3.46}
$$

where the indices are raised and lowered by the Minkowski metric. The bulk equation of motion for the background Higgs vev corresponding to the above action is

$$
\left( - \frac{1}{2} \partial_ν \left( e^{4 A(y)} \partial_ν \right) + \frac{1}{2} μ_B^2 e^{4 A(y)} \right) v(y) = 0,
\tag{3.47}
$$

and the bulk equations of motion for all the fluctuations are

$$
\square^{(4)} A_μ^{(±)} + \partial_ν \left( M^2_A \partial_ν A_μ^{(±)} \right) - M^2_A A_μ^{(±)} - \partial_ν \left( e^{2 A(y)} A_ν^{(±)} \right) - \partial_ν \left( e^{2 A(y)} A_ν^{(±)} \right) = 0,
\tag{3.48}
$$

$$
\square^{(4)} A_5^{(±)} - \partial_ν \left( e^{2 A(y)} A_ν^{(±)} \right) - M^2_A \left( A_5^{(±)} - \partial_ν π^{(±)} \right) = 0,
\tag{3.49}
$$

$$
\square^{(4)} π^{(±)} - \partial_ν A_ν^{(±)} - M^2_A \left( e^{2 A(y)} A_5^{(±)} - \partial_ν π^{(±)} \right) = 0,
\tag{3.50}
$$

where $M^2_A \equiv g_5^2 v^2(y) e^{2 A(y)}$. The jump conditions at the UV-brane following from the equations of motion above are:

$$
\left. \left( \partial_5 - \frac{∂V_{UV} (v)}{∂v} \right) v(y) \right|_{0^+} = 0, \quad \left. \left( \partial_5 - \frac{∂^2 V_{UV} (v)}{∂v^2} \right) h^{(±)}(x, y) \right|_{0^+} = 0,
\tag{3.52}
$$

whereas the odd fields must vanish at $y = 0$. In addition, we choose the boundary conditions at $±L$ following the logic of our earlier discussion (see Eq. (3.24) and Appendix A.2):

$$
\left. \left( ± \partial_5 + \frac{∂V_{IR} (v)}{∂v} \right) v(y) \right|_{±L^±} = 0, \quad \left. \left( ± \partial_5 + \frac{∂^2 V_{IR} (v)}{∂v^2} \right) h^{(±)}(x, y) \right|_{±L^±} = 0,
\tag{3.53}
$$
\[ \partial_\mu A_5^{(\pm)}(x, y) - \partial_5 A_\mu^{(\mp)}(x, y) \bigg|_{\pm L^\mp} = 0, \quad A_5^{(\mp)}(x, y) - \partial_5 \pi^{(\pm)}(x, y) \bigg|_{\pm L^\mp} = 0, \quad (3.54) \]

where \( L^\pm \equiv L \pm \epsilon \) for \( \epsilon \rightarrow 0 \).

In the above action there are mixing terms of \( \partial_\mu A^{(\pm)} \) with the scalars \( \pi^{(\pm)} \) and \( A_5^{(\pm)} \), which can be canceled by adding the following gauge fixing Lagrangian to the above action,

\[ S_{GF} = -\int d^5x \left\{ \frac{1}{2\xi} \left[ \partial_\mu A^{\mu(+)} - \xi \left( M_A^2 \pi^{(+)} - \partial_5 \left( e^{2A(y)} A_5^{(-)} \right) \right) \right]^2 + \frac{1}{2\xi} \left[ \partial_\mu A^{\mu(-)} - \xi \left( M_A^2 \pi^{(-)} - \partial_5 \left( e^{2A(y)} A_5^{(+)} \right) \right) \right]^2 \right\}. \quad (3.55) \]

One can identify the Goldstone modes from the above two Eqs. (3.46) and (3.55):

\[ \Pi^{(+)}(x, y) \equiv M_A^2 \pi^{(+)} - \partial_5 \left( e^{2A(y)} A_5^{(-)} \right), \quad (3.56) \]
\[ \Pi^{(-)}(x, y) \equiv M_A^2 \pi^{(-)} - \partial_5 \left( e^{2A(y)} A_5^{(+)} \right), \quad (3.57) \]

along with the two pseudoscalars \( A_5^{(\pm)}(x, y) \) given as

\[ A_5^{(+)}(x, y) \equiv A_5^{(+)} - \partial_5 \pi^{(-)}, \quad (3.58) \]
\[ A_5^{(-)}(x, y) \equiv A_5^{(-)} - \partial_5 \pi^{(+)}. \quad (3.59) \]

The resulting four pseudoscalars above along with the two \( h^{(\pm)} \) scalar fields agrees with the naive counting before SSB of three even-parity scalars \( (h^{(+)}(x, y), \pi^{(+)>(x, y)} \) and \( A_5^{(+)>(x, y)} \)) and three odd-parity scalars \( (h^{(-)}(x, y), \pi^{(-)}(x, y) \) and \( A_5^{(-)}(x, y) \)). It is seen from the Eq. (3.46) quadratic action that both the even and odd gauge bosons \( A_\mu^{(\pm)}(x, y) \) acquire mass from the Higgs mechanism, whereby the two Goldstone bosons are eaten up by these gauge bosons.

In order to obtain an effective 4D Lagrangian we need to integrate the above quadratic Lagrangian over the \( y \)-coordinate. The first step to achieve this is to decompose all the fields in KK-modes. We will use the following decomposition,

\[ A_\mu^{(\pm)}(x, y) = \sum_n A_n^{(\pm)}(x)\tilde{a}_n^{(\pm)}(y), \quad (3.60) \]
\[ \Pi^{(\pm)}(x, y) = \sum_n \Pi_n^{(\pm)}(x)\tilde{a}_n^{(\pm)}(y)\tilde{m}_n^{(\pm)}(y), \quad A_5^{(\pm)}(x, y) = \sum_n A_5^{(\pm)}(x)\eta_n^{(\pm)}(y), \quad (3.61) \]
\[ h^{(+)}(x, y) = \sum_n h_n(x)\tilde{f}_n^{(+)}(y), \quad h^{(-)}(x, y) = \sum_n \chi_n(x)\tilde{f}_n^{(-)}(y) \quad (3.62) \]

where \( \tilde{a}_n^{(\pm)}(y), \eta_n^{(\pm)}(y) \) and \( \tilde{f}_n^{(\pm)}(y) \) are the 5D profiles for the vector fields (the same for the Goldstone fields), the pseudoscalars and the Higgs bosons, respectively. The Higgs profiles \( \tilde{f}_n^{(\pm)}(y) \) are exactly the same as in Appendix A.2 since they follow the same e.o.m and b.c.; thus, we use the same forms here. The e.o.m. for the wave-functions \( \tilde{a}_n^{(\pm)}(y) \) and \( \eta_n^{(\pm)}(y) \) are

\[ -\partial_5 \left( e^{2A(y)} \partial_5 \tilde{a}_n^{(\pm)}(y) \right) + M_A^2 \tilde{a}_n^{(\pm)}(y) = \tilde{m}_n^{(\pm)}(y) \tilde{a}_n^{(\pm)}(y), \quad (3.63) \]
\[ -\partial_5 \left( M_A^2 \partial_5 (M_A^2 e^{2A(y)} \eta_n^{(\pm)}(y)) \right) + M_A^2 \eta_n^{(\pm)}(y) = m_{A_n}^{(\pm)} \eta_n^{(\pm)}(y), \tag{3.64} \]

where \( m_{A_n}^{(\pm)} \) and \( m_{A_n}^{(\mp)} \) are the KK-masses for \( A_{5\mu}^{(\pm)}(x) \) and \( \phi_n^{(\pm)}(x) \). The normalization conditions for the \( \tilde{a}_n(y) \) and \( \eta_n(y) \) wave-functions are

\[ \int_{-L}^{+L} dy \tilde{a}_n^{(+)}(y) \tilde{a}_n^{(-)}(y) = \delta_{nn}, \quad \int_{-L}^{+L} dy \frac{M_A^2 e^{2A(y)}}{m_{A_n}^{(\pm)} m_{A_n}^{(\mp)}} \eta_n^{(\pm)}(y) \eta_n^{(\mp)}(y) = \delta_{nn}. \tag{3.65} \]

Following the general strategy mentioned in Sec. 2.2, we choose the \( y = 0 \) b.c. for the even wave functions as Neumann b.c., whereas all the odd-mode wave functions satisfy Dirichlet b.c. at \( y = 0 \):

\[ \partial_5 \tilde{a}_n^{(+)}(y) \bigg|_0 = 0, \quad \tilde{a}_n^{(-)}(y) \bigg|_0 = 0, \quad \partial_5 \eta_n^{(+)}(y) \bigg|_0 = 0 \quad \eta_n^{(-)}(y) \bigg|_0 = 0. \tag{3.66} \]

The b.c. for wave-functions \( \tilde{a}_n^{(\pm)} \) and \( \eta_n^{(\pm)} \) at \( y = \pm L \) follow from Eq. (3.54),

\[ \partial_5 \tilde{a}_n^{(\pm)}(y) \bigg|_{\pm L} = 0, \quad \eta_n^{(\pm)}(y) \bigg|_{\pm L} = 0. \tag{3.67} \]

One can also easily find the KK-decomposition of the fluctuation fields \( A_{5\mu}^{(\pm)}(x, y) \) and \( \pi^{(\pm)}(x, y) \) in terms of Goldstone bosons \( \Pi^{(\pm)} \) and the physical scalars \( A_{5\mu}^{(\pm)} \) from Eqs. (3.56)-(3.59):

\[ A_{5\mu}^{(\pm)}(x,y) = \sum_n \left( \frac{\Pi_n^{(\pm)}(x)}{m_{A_n}^{(\pm)}} \partial_5 \tilde{a}_n^{(\pm)}(y) - \frac{M_A^2}{(m_{A_n}^{(\pm)})^2} A_{5\mu}^{(\pm)}(x) \eta_n^{(\pm)}(y) \right), \tag{3.68} \]

\[ \pi^{(\pm)}(x,y) = \sum_n \left( \frac{\Pi_n^{(\pm)}(x)}{m_{A_n}^{(\pm)}} \tilde{a}_n^{(\pm)}(y) - \frac{M_A^2}{(m_{A_n}^{(\pm)})^2} \partial_3 \left( \frac{M_A^2 e^{2A(y)}}{m_{A_n}^{(\pm)} m_{A_n}^{(\mp)}} \eta_n^{(\mp)}(y) \right) A_{5\mu}^{(\mp)}(x) \right), \tag{3.69} \]

By using the above KK-decomposition, one gets the 4D effective action

\[ S_{Ab}^{(2)} = - \int d^5 x \left\{ \frac{1}{2} A_{5\mu}^{(\pm)} \left[-\eta^{\mu\nu}\square^{(4)} + \left(1 - \frac{1}{\xi} \right) \partial_5 \partial^\nu + m_{A_n}^{(\pm)} \eta_n^{(\pm)} \right] A_{5\mu}^{(\pm)} \right. \]

\[ + \frac{1}{2} A_{5\mu}^{(-)} \left[-\eta^{\mu\nu}\square^{(4)} + \left(1 - \frac{1}{\xi} \right) \partial_5 \partial^\nu + m_{A_n}^{(-)} \eta_n^{(-)} \right] A_{5\mu}^{(-)} \right. \]

\[ + \frac{1}{2} \left( \partial_{\mu} \Pi_n^{(\pm)} \right)^2 + \frac{1}{2} \xi m_{A_n}^{(\pm)} (\Pi_n^{(\pm)})^2 + \frac{1}{2} \left( \partial_{\mu} \Pi_n^{(-)} \right)^2 + \frac{1}{2} \xi m_{A_n}^{(-)} (\Pi_n^{(-)})^2 \]

\[ + \frac{1}{2} \left( \partial_{\mu} A_{5\mu}^{(\pm)} \right)^2 + \frac{1}{2} m_{A_n}^{(\pm)} (A_{5\mu}^{(\pm)})^2 + \frac{1}{2} \left( \partial_{\mu} A_{5\mu}^{(-)} \right)^2 + \frac{1}{2} m_{A_n}^{(-)} (A_{5\mu}^{(-)})^2 \]

\[ + \frac{1}{2} \left( \partial_{\mu} h_n \right)^2 + \frac{1}{2} m_n^2 (h_n)^2 + \frac{1}{2} \left( \partial_{\mu} \chi_n \right)^2 + \frac{1}{2} m_n^2 (\chi_n)^2 \right\}. \tag{3.70} \]

The above quadratic action is valid for all KK-modes. Below, we consider the low-energy effective theory obtained by assuming that the KK-mass scale is high enough that we can integrate out all the heavier KK modes and keep only the zero-modes of the theory. From here on, we choose the unitary gauge such that \( \xi \to \infty \) which implies \( \Pi_n^{(\pm)}(x) \to 0 \). Moreover, with our choice of boundary conditions for \( a_0^{(-)}(y) \) and \( \eta_0^{(\pm)}(y) \) in Eqs. (3.66) and
in Fig. 3, \( \mu \) where finite corrections which are suppressed by \( O \) the leading order the vev profile and zero-mode profiles are the same. However, there are

We can now write down the effective theory for the zero-modes in the unitary gauge:

there will be no zero-modes (3.67) one can see that the corresponding wave-functions for zero-modes are vanishing, i.e.

this latter value being that required (see main text) to solve the hierarchy problem.) The right plot is the same as the left but focused near the origin.

(3.67) one can see that the corresponding wave-functions for zero-modes are vanishing, i.e. there will be no zero-modes \( A^{(-)}_0(x) \) and \( A^{(±)}_0(x) \) in our effective theory. The \( y \)-dependent vev and the zero-mode profiles for even and odd Higgs are given by (see Appendix A.2),

\[
\begin{align*}
\tilde{f}^{(±)}(|y|) & \approx \sqrt{k(1 + \beta)} e^{k L e^{(2 + \beta) k(|y| - L)}}, \\
\tilde{f}^{(−)}(y) & = c(y) \tilde{f}^{(−)}(|y|),
\end{align*}
\]

where \( \mu^2 \equiv (1 + \beta) m^2_{KK} \delta_{IR} \) and \( \lambda \equiv \lambda_{IR}(1 + \beta)^2 \). It is important to comment here that at the leading order the vev profile and zero-mode profiles are the same. However, there are finite corrections which are suppressed by \( O \left( m^2_\mu/m^2_{KK} \right) \) as given below and also depicted in Fig. 3,

\[
\frac{\tilde{f}^{(±)}(|y|)}{f_v(|y|)} = 1 + \frac{m^2_h}{m^2_{KK}} \left( 1 - \frac{e^{2k(|y| - L)}}{4(1 + \beta)} \right) + O \left( \frac{m^2_\mu}{m^2_{KK}} \right).
\]

We can now write down the effective theory for the zero-modes in the unitary gauge:

\[
S_{eff} = - \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \tilde{m}^2_A A_\mu A^\mu + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} \tilde{m}^2_A h^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} \tilde{m}^2_\chi \chi^2 \\
+ \frac{1}{4} \lambda h^4 + \frac{1}{4} \lambda \chi^4 + \frac{3}{2} \lambda h^2 \chi^2 + \lambda v_4 h \left( h^2 + 3 \chi^2 \right) \\
+ \lambda v_4 h A_\mu A^\mu + \frac{1}{2} g^2 \left( h^2 + \chi^2 \right) A_\mu A^\mu \right\},
\]

where we have denoted \( A^{(+)}_0(x) \equiv A_\mu(x) \) and we have suppressed the zero-mode subscript ‘0’ for all modes. Adopting the results of Appendix A.2, one can easily find the masses of the zero-mode scalars at leading order:

\[
\tilde{m}_h^2 \approx \hat{m}_h^2 \simeq 2 \mu^2,
\]

where \( \mu \) was defined above. For the mass of the gauge boson \( m_A \) one finds at the leading order

\[
\tilde{m}_A^2 \simeq \frac{1}{2 L} \int_{-L}^L dy M_A^2 = \tilde{g}_4^2 v_4^2, \quad \text{where} \quad \tilde{g}_4 \equiv g_5/\sqrt{2L}.
\]

Figure 3: The left graph shows the profile of the vev \( f_v(y) \) and the zero-mode profiles \( \tilde{f}^{(±)}_0(y) \) as functions of \( y \) with all the fundamental parameters of order one and \( kL = 5 \). (We use a mild value of \( kL \) to show the small differences near the IR-brane which will be hard to see if we use \( kL \approx 35 \), this latter value being that required (see main text) to solve the hierarchy problem.) The right plot is the same as the left but focused near the origin.
In order to facilitate comparison between the two approaches, we collect information concerning all the low-energy degrees of freedom for both pictures in Table 1. Comparing the effective theories obtained within EWSB induced by the Higgs KK-mode vev and by a 5D-Higgs vev in (3.43) and (3.74) one finds that both approaches give exactly the same zero-mode effective theory up to $\mathcal{O}(m_h^2/m_{KK}^2) \sim 10^{-3}$ corrections. We have checked that the scalar masses are exactly same to all orders in the expansion parameter $m_h^2/m_{KK}^2$. In contrast, the gauge boson masses and the couplings can have subleading differences of order $\mathcal{O}(m_h^2/m_{KK}^2)$. Note that we have neglected all the effects due to the non-zero KK-modes, such effects being suppressed by their masses, i.e. $\mathcal{O}(m_n^2/m_{KK}^2)$. Hence we conclude that the two approaches to EWSB discussed above give the same low-energy (zero-mode) effective theory aside from small deviations of order $\mathcal{O}(m_h^2/m_{KK}^2)$. To make this comparison more transparent we summarize the parameters of both effective theories in terms of the fundamental parameters of the 5D theory in Table 2. The observed agreement is a non-trivial verification of the results obtained here.

| EWSB by KK mode vev | EWSB by 5D Higgs vev |
|---------------------|----------------------|
| **5D fields**       | **KK-modes**         | **n = 0** | **n ≠ 0** | **KK-modes** | **n = 0** | **n ≠ 0** |
| Re$H^+(0)$          | $\phi_n^{(+)}(x)$    | ✓         | ✓         | $h_n(x)$     | ✓         | ✓         |
| Re$H^-(0)$          | $\phi_n^{(-)}(x)$    | ✓         | ✓         | $\chi_n(x)$  | ✓         | ✓         |
| Im$H^+(0)$          | $\pi_n^{(+)}(x)$     | ✓         | ✓         | $\Pi_n^{(+)}(x)$ | ✓         | ✓         |
| Im$H^-(0)$          | $\pi_n^{(-)}(x)$     | ✓         | ✓         | $\Pi_n^{(-)}(x)$ | ✓         | ✓         |
| $A_5^{(+)}$         | $A_n^{(1)}(x)$       | ✓         | ✓         | $A_n^{(1)}(x)$ | ✓         | ✓         |
| $A_5^{(-)}$         | $A_n^{(n)}(x)$       | ✓         | ✓         | $A_n^{(n)}(x)$ | ✓         | ✓         |
| $A_\mu^{(+)}$       | $A_n^{(\mu)}(x)$     | ✓         | ✓         | $A_n^{(\mu)}(x)$ | ✓         | ✓         |
| $A_\mu^{(-)}$       | $A_n^{(\mu)}(x)$     | ✓         | ✓         | $A_n^{(\mu)}(x)$ | ✓         | ✓         |

**Table 1:** Comparison of dynamical d.o.f. between KK-mode-vev and 5D-Higgs-vev EWSB. The b.c. (boundary condition) and g.c. (gauge choice) show why a given mode is not present in the corresponding effective theory. Note that $\Pi_n^{(\pm)}$ is a mixture of $\pi_n^{(\pm)}$ and $A_n^{(\pm)}$, see (3.56)-(3.57).

| EWSB by KK mode vev | EWSB by 5D Higgs vev | Comment |
|---------------------|----------------------|---------|
| $f_0(y) \simeq \sqrt{k(1 + \beta)}e^{KL}e^{(2+\beta)k(|y|L)}$ | $f_0(y) \simeq \sqrt{k(1 + \beta)}e^{KL}e^{(2+\beta)k(|y|L)}$ | same |
| $v_1^2 = \frac{\mu^2}{\chi}(1 - \mathcal{O}(\frac{m_h^2}{m_{KK}^2}))$ | $v_1^2 = \frac{\mu^2}{\chi}$ and $f_v(y) = \tilde{f}_0(y)$ | $\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$ |
| $m_h^2 = m_{KK}^2 = 2\mu^2 \left(1 - \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | $\tilde{m}_h^2 = \tilde{m}_{KK}^2 = 2\mu^2 \left(1 - \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | same |
| $a_0(y) = \frac{1}{\sqrt{2L}}$ | $\tilde{a}_0(y) = \frac{1}{\sqrt{2L}} \left(1 + \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | $\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$ |
| $g_4 = \frac{g_5}{\sqrt{2L}}$ | $\tilde{g}_4 = \frac{g_5}{\sqrt{2L}} \left(1 - \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | $\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$ |
| $m_A^2 = \frac{g_5^2}{2L} \frac{\mu^2}{\chi} \left(1 - \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | $\tilde{m}_A^2 = \frac{g_5^2}{2L} \frac{\mu^2}{\chi} \left(1 - \mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)\right)$ | $\mathcal{O}\left(\frac{m_h^2}{m_{KK}^2}\right)$ |

**Table 2:** Comparison of the effective parameters in terms of the fundamental parameters of the 5D theory, where $\mu^2 \equiv (1 + \beta)m_{KK}^2 \delta_{IR}$ and $\lambda = (1 + \beta)^2 \lambda_{IR}$. Here we have explicitly shown corrections of order the expansion parameter $m_h^2/m_{KK}^2 \sim 10^{-3}$ and have neglected any differences of order $\mathcal{O}(e^{-2kkL})$; the latter are extremely small for $\beta > 0$ and $kL \sim 35$. 

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4 SM EWSB by bulk Higgs doublet – the truncated-inert-doublet model

In this section we consider all the SM fields in the bulk and study the phenomenological implications of our symmetric geometry.

The 5D action for the electroweak sector of the SM can be written as

\[ S = - \int d^5x \sqrt{-g} \left\{ \frac{1}{4} F_{MN}^a F^{aMN} + \frac{1}{4} B_{MN} B^{MN} + |D_M H|^2 + \mu^2 |H|^2 \\
+ V_{IR}(H) \delta(y + L) + V_{UV}(H) \delta(y) + V_{IR}(H) \delta(y - L) \right\}, \]  

(4.1)

where \( F_{MN}^a \) and \( B_{MN} \) are the 5D field strength tensors for \( SU(2) \) and \( U(1)_Y \), respectively with \( a \) being the number generators of \( SU(2) \). Above, \( H \) is the \( SU(2) \) doublet and its brane potentials are

\[ V_{UV}(H) = \frac{m^2_{UV}}{k} |H|^2, \quad V_{IR}(H) = -\frac{m^2_{IR}}{k} |H|^2 + \frac{\lambda_{IR}}{k^2} |H|^4. \]  

(4.2)

In our approach, we do not put the Higgs quartic terms in the bulk or on the UV-brane since we want EWSB to take place near the IR-brane. The covariant derivative \( D_M \) is defined as follows:

\[ D_M = \partial_M - i \frac{g_5}{2} \tau^a A_M^a - i \frac{g_5'}{2} B_M, \]  

(4.3)

where \( \tau^a \) are Pauli matrices and \( g_5(g_5') \) is the coupling constant for the \( A_M^a(B_M) \) fields.

It is instructive to make the usual redefinition of the gauge fields,

\[ W_M^\pm = \frac{1}{\sqrt{2}} \left( A_M^1 \mp i A_M^2 \right), \]  

(4.4)

\[ Z_M = \frac{1}{\sqrt{g_5^2 + g_5'^2}} \left( g_5 A_M^3 - g_5' B_M \right), \]  

(4.5)

\[ A_M = \frac{1}{\sqrt{g_5^2 + g_5'^2}} \left( g_5 A_M^3 + g_5 B_M \right). \]  

(4.6)

Analogous to the 4D procedure, we define the 5D Weinberg angle \( \theta \) as follows:

\[ \cos \theta = \frac{g_5}{\sqrt{g_5^2 + g_5'^2}}, \quad \sin \theta = \frac{g_5'}{\sqrt{g_5^2 + g_5'^2}}, \]  

(4.7)

The 5D gauge fields corresponding to the gauge group \( SU(2) \times U(1)_Y \) are then

\[ A_M(x, y) \equiv \begin{pmatrix} \sin \theta A_M + \frac{\cos^2 \theta - \sin^2 \theta}{2 \cos \theta} Z_M \ \\
\frac{1}{\sqrt{2}} W_M \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} W_M^+ \\
- \frac{1}{\sqrt{2}} W_M^- \end{pmatrix}. \]  

(4.8)

The gauge transformations for the Higgs doublet \( H(x, y) \) and gauge matrix \( A_M \) under the gauge group \( SU(2) \times U(1)_Y \) can be written as

\[ H(x, y) \rightarrow H'(x, y) = U(x, y) H(x, y), \]  

(4.9)

\[ A_M(x, y) \rightarrow A_M'(x, y) = U(x, y) A_M(x, y) U^{-1}(x, y) - i \frac{g_5}{g_5} (\partial_M U(x, y)) U^{-1}(x, y), \]  

(4.10)
where \( U(x, y) \) is the unitary matrix corresponding to the fundamental representation of \( SU(2) \times U(1)_Y \) gauge transformations.

We will choose the 5D axial gauge analogous to the Abelian case by taking \( \mathcal{A}_5(x, y) = 0 \). Note that we can always find \( U(x, y) \) such that the axial gauge is manifest, i.e. \( \mathcal{A}_5(x, y) = 0 \). We employ an axial gauge choice for the non-Abelian case of the form

\[
U(x, y) = \hat{U}(x)\mathcal{P}e^{-ig_5\int y^5 d\mathcal{A}_5(x,y')},
\]

where \( \hat{U}(x) \) is the residual 4D gauge transformation and \( \mathcal{P} \) denotes path-ordering of the exponential. Another key point for the later discussion is that this 4D residual gauge transformation \( \hat{U}(x) \) is independent of \( y \) and thus automatically even under the geometric parity.

As we have demonstrated in Sec. 3, due to the symmetric geometry the background fields in the IR-UV-IR setup separate into even and odd bulk wave functions. Hence, it is straightforward to generalize the results obtained in Sec. 3 for the Abelian model to the electroweak sector of the SM. Let us start by decomposing the Higgs doublet and gauge fields into components of definite parity as follows:

\[
H(x, y) = H^+(x, y) + H^-(x, y), \quad V_M(x, y) = V_M^+(x, y) + V_M^-(x, y),
\]

where \( V_M \equiv (A_M, W^+_M, Z_M) \). We can write the action (4.1) up to quadratic level in the \( \mathcal{A}_5(x, y) = 0 \) gauge as

\[
S^{(2)} = -\int d^5x\sqrt{-g}\left\{ \frac{1}{2} W_\mu^+ W_-\mu^+ + \partial_\mu W_\mu^+ \partial^\mu W^-_\mu + \frac{1}{4} Z_\mu^+(Z^-_\mu) + \frac{1}{2} \partial_\mu Z^+(\partial^\mu Z^-) + \frac{1}{2} \partial_\mu Z^-(\partial^\mu Z^+)
\]

\[+ \frac{1}{4} F_\mu^+ F_-\mu^+ + \frac{1}{2} \partial_\mu A_\mu^+(\partial^\mu A^-_\mu) + \frac{1}{4} F_\mu^- F_-\mu^- + \frac{1}{2} \partial_\mu A_\mu^-(\partial^\mu A^-_\mu)
\]

\[+ \mathcal{D}_M H^+(\mathcal{D}^M H^+) + \mu^2_B |H^+|^2 + \mathcal{D}_M H^-(\mathcal{D}^M H^-) + \mu^2_B |H^-|^2
\]

\[+ \frac{m^2_V}{k} \delta(y) - \frac{m^2_R}{k} \left( |H^+|^2 + |H^-|^2 \right) \left[ \delta(y + L) + \delta(y - L) \right] \right\},
\]

where we have adopted the following definitions:

\[
\tilde{V}_\mu^{(\pm)} = \partial_\mu \tilde{V}_\nu^{(\pm)} - \partial_\nu \tilde{V}_\mu^{(\pm)}, \quad F_\mu^{(\pm)} = \partial_\mu A_\nu^{(\pm)} - \partial_\nu A_\mu^{(\pm)},
\]

\[
\mathcal{D}_\mu \left( \begin{array}{c} H^+ \\ H^- \end{array} \right) \equiv -\partial_\mu - ig_5 \left( A^-_\mu A^+_\mu \right) \left( \begin{array}{c} H^+ \\ H^- \end{array} \right),
\]

\[
\mathcal{D}_5 \left( \begin{array}{c} H^+ \\ H^- \end{array} \right) \equiv -\partial_5 - ig_5 \left( A^-_5 A^+_5 \right) \left( \begin{array}{c} H^+ \\ H^- \end{array} \right),
\]

where \( \tilde{V}_\mu \equiv (W^\pm_\mu, Z_\mu) \) and \( A^{(\pm)}_\mu \) was defined in (4.8).

It is convenient to write the Higgs doublets in the following form:

\[
\left( \begin{array}{c} H^+ \\ H^- \end{array} \right) = e^{ig_5 (\Pi^+(1) + \Pi^-(1))} \left( \begin{array}{c} \mathcal{H}^+ \\ \mathcal{H}^- \end{array} \right),
\]

where \( \Pi^{(1)} = (\Pi^+(1), \Pi^-(1)) \).

\[\text{(4.17)}\]
where $\mathcal{H}$ and $\Pi$ are defined as (the parity indices are suppressed)

$$\mathcal{H}(x, y) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x, y) \end{pmatrix}, \quad (4.18)$$

$$\Pi(x, y) \equiv \begin{pmatrix} \cos^2 \theta - \sin^2 \theta \pi Z \\ \frac{1}{\sqrt{2}} \sin \theta \pi W \\ - \frac{1}{2 \cos \theta} \pi Z \end{pmatrix}. \quad (4.19)$$

We KK-decompose the Higgs doublets $H^{(\pm)}(x, y)$ and the gauge fields $V^{(\pm)}_{\mu}(x, y)$ as

$$\mathcal{H}^{(\pm)}(x, y) = \sum_n \mathcal{H}^{(\pm)}_n(x) f^{(\pm)}_n(y), \quad (4.20)$$

$$\pi^{(\pm)}_{\nu}(x, y) = \sum_n \pi^{(\pm)}_{\nu n}(x) a^{(\pm)}_{\nu n}(y), \quad (4.21)$$

$$V^{(\pm)}_{\mu}(x, y) = \sum_n V^{(\pm)}_{\mu n}(x) a^{(\pm)}_{\nu n}(y), \quad (4.22)$$

where the wave-functions $f^{(\pm)}_n(y)$ and $a^{(\pm)}_{\nu n}(y)$ satisfy

$$-\partial_5 (e^{A(y)} \partial_5 f^{(\pm)}_n(y)) + \mu_5^2 e^{A(y)} f^{(\pm)}_n(y) = m^2_{\nu n} e^{2A(y)} f^{(\pm)}_n(y), \quad (4.23)$$

$$-\partial_5 (e^{2A(y)} \partial_5 a^{(\pm)}_{\nu n}(y)) = m^2_{\nu n} a^{(\pm)}_{\nu n}(y), \quad (4.24)$$

and, for our background geometry, $A(y) = -k|y|$. The $y$-dependent wave functions $f^{(\pm)}_n(y)$ and $a^{(\pm)}_{\nu n}(y)$ satisfy the following orthonormality conditions:

$$\int_{-L}^{+L} dy e^{2A(y)} f^{(\pm)}_n(y) f^{(\pm)}_m(y) = \delta_{mn}, \quad \int_{-L}^{+L} dy a^{(\pm)}_{\nu n}(y) a^{(\pm)}_{\nu m}(y) = \delta_{mn}. \quad (4.25)$$

The even modes are subject to jump conditions at $y = 0$ while the odd modes are constrained by continuity at $y = 0$, resulting in the following boundary conditions:

$$\left. \begin{pmatrix} \partial_5 - \frac{m^2_{\nu n}}{k} \\ \partial_5 a^{(\pm)}_{\nu n} \end{pmatrix} f^{(\pm)}_n(y) \right|_{y=0} = 0, \quad \left. f^{(\pm)}_n(y) \right|_{y=0} = 0, \quad (4.26)$$

$$\left. \partial_5 a^{(\pm)}_{\nu n} \right|_{y=0} = 0, \quad a^{(\pm)}_{\nu n} \left|_{y=0} \right. = 0. \quad (4.27)$$

The b.c. at $y = \pm L$ are:

$$\left. \begin{pmatrix} \pm \partial_5 - \frac{m^2_{\nu n}}{k} \\ \partial_5 a^{(\pm)}_{\nu n} \end{pmatrix} f^{(\pm)}_n(y) \right|_{y=L} = 0, \quad \left. \partial_5 a^{(\pm)}_{\nu n} \right|_{y=L} = 0. \quad (4.28)$$

As pointed out in the Abelian case, the choices of b.c. for $a^{(\pm)}_{\nu n}(y)$ at $y = 0, \pm L$ are motivated by the requirement that the even zero-mode profiles for gauge boson be non-zero.

It is worth mentioning here that the choice of writing the Higgs doublets $H^{(\pm)}$ in the form of Eq. (4.17) and using the KK decomposition for the pseudoscalars $\pi^{(\pm)}_{\nu}$ as given in Eq. (4.21) are both motivated by model-building considerations discussed below. The other possibility is to choose different KK bases and b.c. for the pseudoscalars $\pi^{(\pm)}_{\nu}$ such
that after SSB these pseudoscalars become Nambu-Goldstone bosons (NGB). The even zero-mode gauge bosons would then acquire masses by eating up the even-parity NGB, whereas the odd-parity NGB would remain in the spectrum (the odd zero-mode gauge boson fields being zero, see below). An effective potential for the odd-parity NGB would be generated through their interactions with gauge bosons, hence making them pseudo-NGB. We don’t follow this approach here but it is an interesting possibility in which the neutral odd pseudo-NGB would be a composite dark Higgs in the dual CFT description.\footnote{At the final stages of the present work, Ref.\ [45] appeared where the authors considered composite dark sectors. A similar construction can be naturally realized as a CFT dual to the model considered here.}

We assume that the KK-scale is high enough, i.e. \( m_{KK} \sim O(\text{few}) \) TeV, that we can consider an effective theory where only the lowest modes (zero-modes with masses much below \( m_{KK} \)) are kept. It is important to note that the odd zero-mode wave functions obey \( a_{V_0}^{(-)}(y) = 0 \), as can be easily seen from Eq. (4.24) along with the b.c. (4.27) and (4.28). As a consequence of \( a_{V_0}^{(-)}(y) = 0 \), the odd zero-mode gauge fields \( V_{0\mu}^{(-)}(x) \) and the odd Goldstone modes \( \pi_{V_0}^{(-)}(x) \) will not be present in the effective 4D theory. Moreover, the even zero-mode gauge profile is constant, i.e. \( a_{V_0}^{(+)}(y) = 1/\sqrt{2L} \). Using the results from Sec. 3, we can determine the values of the couplings and mass parameters in the effective 4D theory in terms of the parameters of the fundamental 5D theory. The result is that we can write down the effective 4D action for the zero-modes as

\[
S^{(2)}_{\text{eff}} = - \int d^4x \left\{ \frac{1}{4} F^{\mu\nu}_{(0)(+)} F_{(0)(+)}^{\mu\nu} + \frac{1}{4} Z^{(0)(+)} Z_{(0)(+)}^{\mu\nu} + \frac{1}{2} W^{(0)(+)}_{\mu\nu} W_{(0)(+)}^{-\mu\nu} + \partial_{\mu} \mathcal{H}_0^{(+)} \partial^\mu \mathcal{H}_0^{(+)} \\
+ i g_4 \mathcal{H}_0^{(+)} \mathcal{M}_\mu \partial^\mu \mathcal{H}_0^{(+)} + g_4^2 \mathcal{H}_0^{(+)} \mathcal{M}_\mu^\dagger \mathcal{M}_\mu^\dagger \mathcal{H}_0^{(+)} + g_4^2 \mathcal{H}_0^{(+)} \mathcal{M}_\mu \mathcal{M}_\mu^\dagger \mathcal{H}_0^{(+)} \right\},
\]

(4.29)

where \( \mathcal{M}_\mu \) is defined as

\[
\mathcal{M}_\mu \equiv U^\dagger \mathcal{A}_0^{(+)} U + \frac{i}{g_4} \mathcal{U}^{\dagger} \partial_\mu \mathcal{U},
\]

(4.30)

with \( \mathcal{U} \equiv e^{ig_4 \mathcal{R}^{(+)}_0} \) and \( g_4 \equiv g_5/\sqrt{2L} \). In the above action \( \mathcal{H}_0^{(\pm)} \) are real doublets defined in Eq. (4.18), implying that \( \mathcal{H}_0^{(\pm)\dagger} = \mathcal{H}_0^{(\pm)\dagger} \), whereas \( \mathcal{A}_0^{(+)} \) and \( \mathcal{P}_0^{(+)\dagger} \) are defined as (below we suppress the parity indices and zero-mode index):

\[
\mathcal{A}_\mu(x) \equiv \begin{pmatrix} \sin \theta A_\mu & + \cos^2 \theta - \sin^2 \theta Z_\mu \frac{1}{\sqrt{2}} W^+_\mu \\ \frac{1}{\sqrt{2}} W^-_\mu & - \frac{1}{\sqrt{2}} \cos \theta Z_\mu \end{pmatrix},
\]

(4.31)

\[
\mathcal{P}(x) \equiv \begin{pmatrix} \cos^2 \theta - \frac{\sin^2 \theta}{2} Z & \frac{1}{\sqrt{2}} \pi^- Z \\ \frac{1}{\sqrt{2}} \pi^+ Z & - \frac{1}{\sqrt{2}} \cos \theta Z \end{pmatrix}.
\]

(4.32)

It is important to comment here that the above action is manifestly gauge invariant under the following \( SU(2) \times U(1)_Y \) gauge transformation,

\[
\mathcal{A}_\mu^{(+)} \rightarrow \mathcal{U} \mathcal{A}_\mu^{(+)} \mathcal{U}^{\dagger} - \frac{i}{g_4} (\partial_\mu \mathcal{U}) \mathcal{U}^{\dagger}, \quad \mathcal{U} \rightarrow \mathcal{U} e^{ig_4 \mathcal{P}^{(+)\dagger}}.
\]

(4.33)
whereas the $\mathcal{H}_0^{(\pm)}$ are gauge invariant under the 4D residual gauge transformation $\hat{U}$. Equation (4.29) is a non-Abelian analog of the Abelian zero-mode action given by (3.30).

We introduce a convenient notion for our effective theory by redefining $V_{0\mu}^{(\pm)}(x) \equiv V_\mu(x)$, $\pi_{0\mu}^{(+)}(x) \equiv \pi(0, x)$, $\hat{H}_0^{(\pm)}(x) \equiv \hat{H}(x)$ and

$$H_1(x) \equiv e^{i\theta_1(\bar{0})}H_0^{(\pm)}(x), \quad H_2(x) \equiv e^{i\theta_2(\bar{0})}H_0^{(-)}(x). \quad (4.34)$$

Now the above effective action, including the scalar interaction terms, can be written in a nice gauge covariant form as\(^6\)

$$S_{\text{eff}} = -\int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \tilde{Z}_{\mu\nu} \tilde{Z}^{\mu\nu} + \frac{1}{2} W_{\mu\nu} W^{-\mu\nu} + (D_\mu H_1) \dagger D^\mu H_1 + (D_\mu H_2) \dagger D^\mu H_2 + V(H_1, H_2) \right\}, \quad (4.35)$$

where the scalar potential can be written as

$$V(H_1, H_2) = -\mu^2 |H_1|^2 - \mu^2 |H_2|^2 + \lambda |H_1|^4 + \lambda |H_2|^4 + 6\lambda |H_1|^2 |H_2|^2. \quad (4.36)$$

The covariant derivative $D$ is defined as

$$D_\mu = \partial_\mu - ig_4 \hat{A}_\mu^{(+)} \quad (4.37)$$

where $\hat{A}_\mu^{(+)}$ is defined in Eq. (4.31). In the above scalar potential the mass parameter $\mu$ is defined as (see Sec. A.1 and Appendix A.1),

$$\mu^2 \equiv -m_0^{2(\pm)} = (1 + \beta) m_{KK}^2 \delta_{IR},$$

where $\delta_{IR}$, $m_{KK}$ and $\beta$ are defined in Eq. (3.31).

Concerning the symmetries of the above potential, one can notice that $V(H_1, H_2)$ is invariant under $[SU(2) \times U(1)_Y]' \times [SU(2) \times U(1)_Y]$, where one of the blocks has been gauged while the other one survived as a global symmetry. The zero-modes of the four odd vector bosons ($W^{(-\pm)}_\mu, Z^{(-\pm)}_\mu$ and $A^{(-\pm)}_\mu$) and the three would-be-Goldstone bosons $\Pi_0^{(-)}$ have been removed by appropriate b.c., implying that the corresponding gauge symmetry has been broken explicitly. What remains is the truncated inert doublet model, that contains $H_{1,2}$, and the corresponding residual $SU(2) \times U(1)_Y$ global symmetry of the action. Symmetry under the above mentioned $U(1)' \times U(1)$ implies in particular that $V(H_1, H_2)$ is also invariant under various $\mathbb{Z}_2$’s, for example $H_1 \rightarrow -H_1, H_2 \rightarrow -H_2$ and $H_1 \rightarrow \pm H_2$.

As explained in the Abelian case, we choose the vacuum such that the even parity Higgs field $H_1$ acquires a vev, whereas the odd parity Higgs field $H_2$ does not, i.e.

$$v_1^2 \equiv v^2 = \frac{\mu^2}{\lambda}, \quad v_2 = 0. \quad (4.38)$$

Let us now consider fluctuations around the vacuum of our choice

$$H_1(x) = \frac{1}{\sqrt{2}} e^{i\theta_1(\bar{0})} \begin{pmatrix} 0 \\ v + h \end{pmatrix}, \quad H_2(x) = \frac{1}{\sqrt{2}} e^{i\theta_2(\bar{0})} \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad (4.39)$$

\(^6\)Note that the action of Eq. (4.35) is a non-Abelian version of the Abelian zero-mode action (3.35).
where $\hat{\Pi}$ (defined in Eq. (4.32)) contains the pseudoscalar Goldstone bosons $\pi_{W^\pm, Z}$. We choose the unitary gauge in which $\pi_{W^\pm, Z}$ are gauged away, that is they are eaten up by the massive gauge bosons $W_\mu^\pm$ and $Z_\mu$. Hence in the unitary gauge our effective action up to the quadratic order in fluctuations is

$$S_{\text{eff}}^{(2)} = -\int d^4 x \left\{ \frac{1}{2} W_\mu^+ W_-^{\mu \nu} + \frac{1}{4} Z_{\mu \nu} Z^{\mu \nu} + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + m_W^2 W_\mu^+ W_-^{\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^{\mu} + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} m_h^2 h^2 + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m_\chi^2 \chi^2 \right\},$$

(4.40)

where the masses are,

$$m_h^2 = m_\chi^2 = 2 \mu^2, \quad m_W^2 = \frac{1}{4} g_4^2 \mu^2, \quad m_Z^2 = \frac{1}{4} \left( g_4^2 + g_4'^2 \right) \mu^2 = \frac{m_W^2}{\cos^2 \theta_W}. \quad (4.41)$$

It is worth noticing here that the Higgs mass $m_h$ and the dark scalar mass $m_\chi$ are degenerate at the tree level. However, as we demonstrate below, this degeneracy is lifted by the quantum corrections predicted by the effective theory below the KK-mass scale. The interaction terms for effective theory can be written as

$$S_{\text{int}} = -\int d^4 x \left\{ \lambda v h^3 + \frac{3}{4} \lambda h^4 + \frac{3}{4} \lambda \chi^2 + 3 \lambda v h \chi^2 + \frac{3}{2} \lambda h^2 \chi^2 + \frac{g_4^2}{2} v W_\mu^+ W_-^{\mu} h + \frac{g_4'^2}{4} W_\mu^+ W_-^{\mu} (h^2 + \chi^2) + \frac{1}{4} (g_4^2 + g_4'^2) v h Z_\mu Z^{\mu} + \frac{1}{8} (g_4^2 + g_4'^2) Z_\mu Z^{\mu} (h^2 + \chi^2) \right\},$$

(4.42)

where we have omitted terms involving gauge interactions alone as they are irrelevant to our discussion below.

### 4.1 Quantum corrections to scalar masses

In this subsection we will study the quantum corrections to the tree-level scalar masses of the Higgs boson $h$ and the dark matter candidate $\chi$.

Before proceeding further, we want to point out here that in this work we have not studied fermions in our geometric setup since our focus is on the bosonic sector of the SM and EWSB. For the sake of self-consistency, we mention here three possibilities for fermion localization and their implications in our geometric setup:

1. In this first scenario, one takes the heavy (top) quarks to be localized towards the IR-brane, while the light quarks and leptons are localized towards the UV-brane. Through this geometric localization one can address the fermion mass hierarchy problem. In this scenario the even and odd zero-modes corresponding to the heavy quarks will be almost degenerate in our symmetric geometry, whereas the odd zero-modes corresponding to the light quarks could be much heavier than their corresponding even zero-modes \cite{41, 42}.

2. In the second scenario, all the fermions have flat zero-mode profiles. This can be achieved by the choice of appropriate bulk mass parameters for the fermions. As a consequence of flat profiles the odd fermion zero-modes have to disappear and the even zero-modes will correspond to the SM fermions (in this case the fermion mass hierarchy problem is reintroduced).
3. In the third scenario all the fermions are localized towards UV-brane. In this case the masses of all odd zero-modes of the fermions could be heavier than their corresponding even zero-modes.

In this study we implicitly limit ourselves to the last two cases in order that the dark Higgs be the lightest odd particle and all the other odd zero-modes are either not present in our low-energy effective theory or they are much heavier that the dark Higgs, which will therefore be the only relevant dark matter candidate. For either of the choices 2. or 3. above, the top Yukawa coupling $y_t$ in the low-energy effective theory will be the same as in the SM and the top-quark loop correction to the SM Higgs boson mass will be exactly as in the SM up to the KK cutoff. In case 2., the $n \neq 0$ fermion KK-modes are all much heavier than the KK cutoff, $m_{KK}$, and will not significantly influence the radiative corrections to the SM Higgs mass. We leave the study of the complete fermionic sector associated with our geometric setup for future studies.

The quantum corrections to the Higgs boson ($h$) mass and the dark-Higgs ($\chi$) mass within our effective theory below the KK-scale are quite essential for breaking the mass degeneracy of Eq. (4.41). For instance, at the 1-loop level of the perturbative expansion, the main contributions (quadratically divergent) to the masses of the SM Higgs and the dark-Higgs come from the exchanges of the top quark ($t$), massive gauge bosons ($W, Z$), Higgs boson ($h$) and the dark-Higgs ($\chi$) in the loop, see Fig. 4.\footnote{Another scalar which could be potentially present in our effective theory is the radion, which is responsible for the stabilization of the set-up. The stabilization mechanism is beyond the scope of the present work, as here we assume a rigid geometrical background with no fluctuations of the 5D metric. However, we want to comment here that if the radion were present in our effective theory, because of its bosonic nature it would likely reduce the fine-tuning much in the manner that the $\chi$ does.} It is instructive to write the general 1-loop effective scalar potential $V_{\text{eff}}(H_1, H_2)$ for our effective theory, described in the previous section, as \footnote{Note that in this section we are considering the Higgs doublets $H_{1,2}$ in the unitary gauge, such that $H_1(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 + h \end{pmatrix}$ and $H_2(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 + \chi \end{pmatrix}$, where at tree level our choice was $v_1 = v$ and $v_2 = 0.$}

$$V_{\text{eff}}(H_1, H_2) = V_0(H_1, H_2) + V_1(H_1, H_2),$$

where $V_0(H_1, H_2)$ is the tree level scalar potential given by Eq. (4.36) and $V_1(H_1, H_2)$ is the 1-loop effective potential, given by (see for example Refs. [46–48])

$$V_1(H_1, H_2) = \frac{\Lambda^2}{32\pi^2} \left[ 3 \left( g_1^2 + \frac{1}{2} \left( g_4^2 + g_4'^2 \right) + 8\lambda \right) \left( |H_1|^2 + |H_2|^2 \right) - 12y_t^2 |H_1|^2 \right] + \cdots,$$
where $y_t$ is the top Yukawa coupling, related to top mass through $m_t^2 = y_t^2 v^2/2$. We use the momentum cut-off regularization. Also it is important to comment here that $H_2$ is odd under the geometric $\mathbb{Z}_2$ parity, implying that it does not couple to the even zero-mode fermions. Moreover, we consider only the quadratically divergent part of the effective scalar potential and the ellipses in the above equation represent the terms which are not quadratically divergent.

The minimization of the effective potential $V_{\text{eff}}(H_1, H_2)$, i.e.

$$\frac{\partial V_{\text{eff}}}{\partial H_i} \bigg|_{H_i = \langle H_i \rangle} = 0,$$

where $\langle H_i \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v_i \end{array} \right)$, $i = 1, 2$ (4.45)

gives the following set of conditions for the global minimum,

$$\lambda v_1^2 = \mu^2 - \delta\mu^2 - 3\lambda v_2^2, \quad \text{or} \quad v_1 = 0, \quad (4.46)$$

and

$$\lambda v_2^2 = \mu^2 - \delta\mu^2 + 3\Lambda^2 \frac{g_t^2}{8\pi^2} - 3\lambda v_1^2, \quad \text{or} \quad v_2 = 0, \quad (4.47)$$

where $\delta\mu^2$ is given by

$$\delta\mu^2 = \frac{3\Lambda^2}{32\pi^2} \left[ g_t^2 + \frac{1}{2}(g_t^2 + g_t'^2) + 8\lambda - 4y_t^2 \right].$$

(4.48)

Of the four possible global minima of Eqs. (4.46) and (4.47), we will choose the vacuum such that $H_1$ acquires the vev, whereas $H_2$ does not:

$$v_1 = v, \quad v_2 = 0,$$

where $v \simeq 246$ GeV is the vacuum expectation value of the SM Higgs doublet. With this choice of vacuum, the 1-loop corrected masses for the fluctuations around the vacuum are

$$m_h^2 = \frac{\partial^2 V_{\text{eff}}(H_1, H_2)}{\partial H_1^2} \bigg|_{H_1 = v, H_2 = 0} = (-\mu^2 + \delta\mu^2) + 3\lambda v^2 = 2\lambda v^2, \quad (4.49)$$

$$m_\chi^2 = \frac{\partial^2 V_{\text{eff}}(H_1, H_2)}{\partial H_2^2} \bigg|_{H_1 = v, H_2 = 0} = (-\mu^2 + \delta\mu^2) + 3\lambda v^2 + \frac{3\Lambda^2}{8\pi^2} y_t^2,$n

$$= 2\lambda v^2 + \frac{3\Lambda^2}{4\pi^2 v^2} m_t^2. \quad (4.50)$$

To get $m_h = 125$ GeV, equivalent to $v \simeq 246$ GeV, we need to fine-tune the parameters of the theory. To quantify the level of fine-tuning, we employ the Barbieri–Giudice fine-tuning measure $\Delta_{m_h}$ [47–49]:

$$\Delta_{m_h} \equiv \left| \frac{\delta\mu^2}{\mu^2} \right| = \left| \frac{\delta m_h^2}{m_h^2} \right|. \quad (4.51)$$

We plot the fine-tuning measure $\Delta_{m_h}$ as a function of the effective cutoff scale $\Lambda$ in Fig. 5. If one allows fine-tuning of about 10%, i.e. $\Delta_{m_h} = 10$, then the effective cutoff scale is $\Lambda \simeq 2$ TeV. The most stringent bounds on the KK-scale $m_{KK}$ in RS1 geometry with a bulk Higgs come from EWPT by fitting the $S$, $T$ and $U$ parameters [25]. The lower bound
Figure 5: The left plot gives the value of the fine-tuning measure $\Delta_m$ for a Higgs mass of 125 GeV as a function of the cutoff $\Lambda$. The right plot shows the dependence of $m_\chi$ on $\Lambda$ for $m_h = 125$ GeV. In our model $\Lambda = m_{KK}$. The vertical gray line indicates the current lower bound on the KK mass scale coming from EWPT as computed in our model for $\beta = 0$, $m_{KK} \gtrsim 2.5$ TeV.

on the KK mass scale in our model (AdS geometry, i.e. $A(y) = -k|y|$) is $m_{KK} \gtrsim 2.5$ TeV for $\beta = 0$ and $m_{KK} \gtrsim 4.3$ TeV for $\beta = 10$ at 95% C.L. [25]. This implies a tension between fine-tuning (naturalness) and the lower bound on the KK mass scale $m_{KK}$. The region within the gray lines in Fig. 5 shows the current bounds on the KK mass scale for our geometry and the associated fine-tuning. It is important to comment here that a slight modification to the AdS geometry (for example, models with soft wall or thick branes) leads to a considerable relaxation of the above mentioned lower bound on KK mass scale [18–20].

For instance, a mild modification to the AdS metric in the vicinity of the IR-brane can relax the KK mass scale to $m_{KK} \gtrsim 1$ TeV [18–20, 28]. Needless to say, the generalization of our model to modified AdS geometries with soft walls or thick branes is possible.

The 1-loop quantum corrected dark matter squared mass $m_\chi^2$ is:

$$m_\chi^2 = m_h^2 + \frac{3}{4\pi^2 v^2} \Lambda^2 m_t^2.$$  \hspace{1cm} (4.52)

Hence, $m_\chi$ is raised linearly with the cut-off scale $\Lambda$. This is illustrated in Fig. 5. An interesting aspect of our model is that dark matter is predicted to be heavier than the SM Higgs boson. A natural value of the cutoff coincides with the mass of the first KK excitations, which are experimentally limited [50] to lie above a few TeV (depending on model details and KK mode sought). Requiring that the fine-tuning measure $\Delta_m$ be less than 100 implies that $m_{KK}$ should be below about 6 TeV. Meanwhile, the strongest version of the EWPT bound requires $m_{KK} \gtrsim 2.5$ TeV, corresponding to $m_\chi \sim 500$ GeV, for which $\Delta_m$ is a very modest $\sim 18$. In short, our model is most consistent for $500$ GeV $< m_\chi < 1200$ GeV.

4.2 Dark matter relic abundance

In this subsection we calculate the dark matter relic abundance. The diagrams contributing to dark matter annihilation are shown in Fig. 6. The squared amplitudes $|\mathcal{M}|^2$ correspond-
Figure 6: Dark matter annihilation diagrams.

ing to the contribution of each final state to dark matter annihilation are:

\[ |M(\chi \chi \rightarrow \tilde{V} \tilde{V})|^2 = \frac{4m_{\tilde{V}}^4}{S_{\tilde{V}}v^4} \left(1 + \frac{3m_h^2}{s - m_h^2}\right)^2 \left[2 + \left(1 - \frac{s}{2m_{\tilde{V}}^2}\right)^2\right], \]  

(4.53)

\[ |M(\chi \chi \rightarrow f \bar{f})|^2 = 18N_c \frac{m_f^2m_h^4}{v^4} \frac{s - 4m_f^2}{(s - m_h^2)^2}, \]  

(4.54)

\[ |M(\chi \chi \rightarrow hh)|^2 = \frac{9m_h^4}{2v^4} \left[1 + 3m_h^2 \left(\frac{1}{s - m_h^2} + \frac{1}{t - m_h^2} + \frac{1}{u - m_h^2}\right)\right]^2, \]  

(4.55)

where \(\tilde{V} = W, Z\) and \(S_W = 1\) and \(S_Z = 2\) are the symmetry factors accounting for the identical particles in the final state; \(N_c\) refers to the number of “color” degrees of freedom for the given fermion and \(s, t, u\) are the Mandelstam variables. Here, we ignore the loop-induced \(\gamma \gamma\) and \(Z \gamma\) final states, which are strongly suppressed. Note that the first term in the parenthesis in eq. (4.53) and the first term in the square bracket in eq. (4.55) arise from the \(\chi \chi \tilde{V} \tilde{V}\) and the \(\chi \chi hh\) contact interactions, respectively. The former channel is present in our model since \(\chi\) is a component of the (truncated) odd \(SU(2)\) doublet.

In the left panel of Fig. 7 we plot the annihilation cross-section for the contributing channels as a function of \(m_{\chi}\). (Note that the parameter \(\Lambda\) would only enter if we performed this calculation at the one-loop level.) As seen from the plot, the total cross section is dominated by \(WW\) and \(ZZ\) final states. The main contributions for these final states are those generated by the contact interactions \(\chi \chi \tilde{V} \tilde{V}\). In fact, in our model, the \(\tilde{V} \tilde{V}\) final states are additionally enhanced by a constructive interference of the contact \(\chi \tilde{V} \tilde{V}\) interaction with the s-channel Higgs-exchange diagram. In addition, for low \(m_{\chi}\), there is a comparable contribution from \(\chi \chi\) annihilation into \(hh\). (The dip at \(m_{\chi} \sim 210\) GeV is caused by cancellation between the contact interaction and \(s, t, u\)-channel diagrams.) Fermionic final states are always irrelevant; even \(\chi \chi \rightarrow tt\) production is very small in comparison to \(\chi \chi \rightarrow \tilde{V} \tilde{V}\). Then, adopting the standard s-wave cold dark matter approximation [51], we find the present \(\chi\) abundance \(\Omega_{\chi}h^2\) shown in the right panel of Fig. 7. We observe that \(\Omega_{\chi}h^2 < 10^{-4}\) once the EWPT bound of \(m_{\chi} \gtrsim 500\) GeV is imposed. Clearly, some other dark matter component is needed within this model to satisfy the Planck measurement, \(\Omega_{\chi}h^2 \sim 0.1\) [31].

\[ |M(\chi \chi \rightarrow h) | = \frac{9m_h^4}{2v^4} \left[1 + 3m_h^2 \left(\frac{1}{s - m_h^2} + \frac{1}{t - m_h^2} + \frac{1}{u - m_h^2}\right)\right]^2, \]  

(4.55)
Figure 7: The above graphs show the annihilation cross-section $\sigma_0$ for different final states (left) and the $\chi$ abundance $\Omega_\chi h^2 \times 10^4$ (right) as a function of dark matter mass $m_\chi$.

5 Summary

In this paper, we constructed a model with $\mathbb{Z}_2$ geometric symmetry such that two identical AdS patches are glued together at $y = 0$, where $y$ is the coordinate of the fifth dimension. We considered three D3-branes, one at $y = 0$ referred to as the UV-brane where gravity is assumed to be localized and two branes at $y = \pm L$ referred to as IR-branes. The motivation of this work is twofold: (i) to analyze the situation where EWSB is due to the bulk Higgs in this $\mathbb{Z}_2$ symmetric geometry; and (ii) to discuss the lowest odd KK-mode as a dark matter candidate. Concerning EWSB, we discussed in detail many important aspects of SSB due to a bulk Higgs. We first considered the Abelian gauge group and then generalized to the SM gauge group. In the Abelian case, we followed two apparently different approaches to SSB due to a bulk Higgs. We first considered the Abelian gauge group and then generalized to the SM gauge group. In the Abelian case, we followed two apparently different approaches to SSB due to a bulk Higgs field acquiring a vacuum expectation value. In one approach, symmetry breaking is triggered by a vev of the KK zero-mode of the bulk Higgs field. The second approach is based on the expansion of the bulk Higgs field around an extra-dimensional vev with non-trivial $y$ profile. The comparison between the two Abelian scenarios is summarized in Tables 1 and 2. The (zero-mode) effective theories obtained from the two approaches are identical and the most intriguing feature of the Abelian Higgs mechanism is that the even and odd Higgs zero-modes have degenerate mass at tree-level — a feature that is also present in the SM case.

To achieve SSB, the choice of boundary conditions for the fields at $\pm L$ is critical. In both the above two approaches to the Abelian case, we allowed the $y$-derivative of a field to have an arbitrary value at $\pm L$ as opposed to requiring that the field value itself be zero, i.e. we employed Neumann or mixed b.c. rather than Dirichlet b.c. at $\pm L$. The latter choice would have led to an explicit symmetry breaking scenario in which there are no Goldstone modes and the gauge bosons do not acquire mass. (Note that the boundary or “jump” conditions at $y = 0$ follow from the bulk equations of motion in the case of even modes, whereas odd modes are required to be zero by symmetry.)

Following this introductory material, we considered EWSB assuming that the SM gauge group is present in the bulk of our $\mathbb{Z}_2$ symmetric 5D warped model. The zero-mode effective theory appropriate at scales below the KK scale, $m_{KK}$, was obtained. For appropriate Higgs field potentials in the bulk and localized at the UV and IR-branes, SM-
like EWSB is obtained when only the IR-branes have a quartic potential term. In contrast, quadratic mass-squared terms are allowed both on the branes and in the bulk. Of course, to achieve spontaneous EWSB, we employed the same boundary conditions as delineated above for the Abelian model. The resulting model has the following features.

- Due to warped KK-parity all fields develop even and odd towers of KK-modes in the 4D effective theory.
- Assuming that the KK-scale is high enough ($m_{KK} \sim \mathcal{O}(\text{few}) \text{ TeV}$), we derive the low energy effective theory which includes only zero-modes of the theory.
- In the effective theory, the symmetry of the model is $[SU(2) \times U(1)_Y]' \times [SU(2) \times U(1)_Y]$, where the unprimed symmetry group is gauged while the other stays as a global symmetry. The zero-mode odd gauge fields and the corresponding Goldstone modes from the odd Higgs doublet are eliminated due to the b.c.
- In the low energy effective theory, we are left with all the SM fields plus a dark-Higgs – the odd zero-mode Higgs. This dark-Higgs and the SM Higgs (the even zero-mode) are degenerate at tree level.
- In order to get the SM Higgs mass of 125 GeV, we need to fine-tune the 5D fundamental parameters of theory to about $1\%-5\%$, where the upper bound is determined by the lower bound on the KK scale coming from EWPT requirements.
- We computed the one-loop quantum corrections to the tree-level masses of the SM Higgs and the dark Higgs assuming that the cutoff scale of our effective theory is the KK-scale, $m_{KK}$. One finds that the dark-Higgs mass is necessarily larger than the SM Higgs mass, the difference being quadratically dependent on $m_{KK}$.
- Requiring that the fine-tuning measure $\Delta_m_h$ be less than 100 implies that $m_{KK}$ should be below about 6 TeV. Meanwhile, the strongest version of the EWPT bound requires $m_{KK} \gtrsim 2.5 \text{ TeV}$, corresponding to $m_{\chi} \sim 500 \text{ GeV}$, for which $\Delta_m_h$ is a very modest $\sim 18$. In short, our model is most consistent for 500 GeV $< m_{\chi} < 1200 \text{ GeV}$.
- We calculate the relic abundance of the dark-Higgs in the cold dark matter approximation. For $m_{\chi}$ in the above preferred range, $\Omega_{\chi} h^2 < 10^{-4}$ as compared to the current experimental value of $\sim 0.1$. To obtain a more consistent dark matter density, one needs to either assume another DM particle or perform a more rigorous analysis of our model by considering the even and odd higher KK-modes in the effective theory.

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A SSB in the IR-UV-IR model: the real scalar case

In this Appendix, we consider a real scalar in the geometric setup layout in Sec. 2.1 in order to understand some of the general properties of fields in this geometry. The action for this toy model with a real scalar field \( \Phi(x,y) \) is:

\[
S_{\text{toy}} = - \int d^5x \sqrt{-g} \left\{ \frac{1}{2} g^{MN} \nabla_M \Phi \nabla_N \Phi + \frac{1}{2} \mu_B^2 \Phi^2 
+ V_{IR}(\Phi) \delta(y + L) + V_{UV}(\Phi) \delta(y) + V_{IR}(\Phi) \delta(y - L) \right\}, \tag{A.1}
\]

where \( \mu_B \) is bulk mass parameter and,

\[
V_{UV}(\Phi) = \frac{m_{UV}^2}{2k} \Phi^2, \quad V_{IR}(\Phi) = -\frac{m_{IR}^2}{2k} \Phi^2 + \frac{\lambda_{IR}}{4k^2} \Phi^4, \tag{A.2}
\]

are the scalar field potentials localized on the UV and IR-branes, respectively. The background metric for the IR-UV-IR geometric setup is given by Eq. (1.1). It is important to note that the above action has a discrete \( \mathbb{Z}_2 \) symmetry, i.e. \( \Phi(x,y) \rightarrow -\Phi(x,y) \). In the following two sub-Appendices we consider two different strategies for spontaneous symmetry breaking (SSB) of the discrete symmetry: \( (i) \) SSB by vacuum expectation values of KK modes, and \( (ii) \) SSB by a vacuum expectation value of the 5D scalar field. Later we will compare the effective theories obtained within the two approaches.

A.1 SSB by vacuum expectation values of KK modes

In this approach, we KK-decompose the scalar field \( \Phi(x,y) \) of the above action (A.1) as

\[
\Phi(x,y) = \sum_n \Phi_n(x) f_n(y). \tag{A.3}
\]

The wave-functions \( f_n(y) \) satisfy the following e.o.m and orthonormality condition in the bulk,

\[
-\partial_5 \left( e^{A(y)} \partial_5 f_n(y) \right) + m_{UV}^2 e^{A(y)} f_n(y) + e^{A(y)} m_{IR}^2 f_n(y) \delta(y) = m_n^2 e^{2A(y)} f_n(y), \quad \int_{-L}^{+L} dy e^{2A(y)} f_m(y) f_n(y) = \delta_{mn}. \tag{A.4, A.5}
\]

In the presence of Dirac delta function \( \delta(y) \) (UV-brane) at \( y = 0 \) there is a discontinuity (jump) in the first derivatives of \( f_n(y) \). The corresponding jump condition at \( y = 0 \) and the boundary conditions at \( y = \pm L \) are:

\[
\left( \partial_5 - \frac{m_{UV}^2}{2k} \right) f_n(y) \bigg|_{y=0^+} = 0, \quad \left( \pm \partial_5 - \frac{m_{IR}^2}{2k} \right) f_n(y) \bigg|_{y=\pm L} = 0, \tag{A.6}
\]
where $0^\pm \equiv 0 + \epsilon$ and $L^\pm \equiv L \pm \epsilon$ with $\epsilon \to 0$. Equation (A.4) together with the jump-boundary conditions (A.6) defines the basis for the KK decomposition. It is easy to see that the choice of (A.4) implied by the orthogonality relations (A.5) eliminates non-diagonal bulk mass terms (i.e. quadratic in $f_n(y)$) in the effective 4D action. The first condition in (A.6) is dictated by integrating (A.4) around $y = 0$ (jump across the UV-brane), while the second one is imposed in order to eliminate non-diagonal terms $\propto (A.6)$ is dictated by integrating (A.4) around $y = 0$ (jump across the UV-brane), while the second one is imposed in order to eliminate non-diagonal terms $\propto (A.6)$. The strategy will be often employed hereafter and in the main text to get rid of the non-diagonal quadratic KK terms.

Since our the background geometry is $\mathbb{Z}_2$ symmetric under extra-dimension $y$ therefore the solutions of the Eq. (A.4) will have even and odd solutions w.r.t. $y$. We denote the even and odd wave-functions as $f_n^{(+)}(y)$ and $f_n^{(-)}(y)$, respectively. It is instructive to write the KK-decomposition (A.51) as

$$
\Phi(x, y) = \Phi^{(+)}(x, y) + \Phi^{(-)}(x, y),
$$

$$
= \sum_n \Phi_n^{(+)}(x)f_n^{(+)}(|y|) + \epsilon(y)\sum_n \Phi_n^{(-)}(x)f_n^{(-)}(|y|),
$$

(7.7)

where $\epsilon(y)$ is $\pm 1$ for $y \geq 0$. We rewrite Eq. (A.4) as

$$
- \partial_5 (e^{4A(y)}\partial_5 f_n^{(\pm)}(y)) + \mu^2 e^{4A(y)}f_n^{(\pm)}(y) + e^{4A(y)}\frac{m_{UV}^2}{k} f_n^{(\pm)}(y)\delta(y) = m_n^2 e^{2A(y)} f_n^{(\pm)}(y).
$$

(8)

The jump and boundary conditions for the even and odd profiles follow from Eq. (A.6),

$$
\left(\partial_5 - \frac{m_{UV}^2}{2k}\right) f_n^{(+)}(y)|_{0^+} = 0, \quad f_n^{(-)}(y)|_{0^+} = 0.
$$

(9)

$$
\left(\pm \partial_5 - \frac{m_{IR}^2}{k}\right) f_n^{(\pm)}(y)|_{\pm L^\mp} = 0.
$$

(10)

For our geometric setup, we find the following general solutions of Eq. (A.8):

$$
f_n^{(+)}(y) = \frac{e^{2|y|}}{N_n^{(+)}} \left[ J_\beta \left(m_n^{(+)} e^{k|y|}/k\right) + b_n^{(+)} Y_\beta \left(m_n^{(+)} e^{k|y|}/k\right) \right],
$$

(11)

$$
f_n^{(-)}(y) = \epsilon(y) \frac{e^{2|y|}}{N_n^{(-)}} \left[ J_\beta \left(m_n^{(-)} e^{k|y|}/k\right) + b_n^{(-)} Y_\beta \left(m_n^{(-)} e^{k|y|}/k\right) \right],
$$

(12)

where $J_\beta$ and $Y_\beta$ are the Bessel functions with weight $\beta \equiv \sqrt{4 + \mu_{IR}^2/k^2}$, which parameterizes the bulk mass. Above $N_n^{(+)}$ and $b_n^{(\pm)}$ are the two integration constants for each parity mode. Where $N_n^{(\pm)}$ are fixed by the orthogonality condition (A.5). The coefficients $b_n^{(\pm)}$ and $b_n^{(\pm)}$ are implied by the jump conditions at $y = 0$ for even and odd wave functions:

$$
b_n^{(+)} = -\frac{k^2 \delta_{UV} J_\beta \left(m_n^{(+)}/k\right) + km_n^{(+)} J_\beta J_{\beta+1} \left(m_n^{(+)}/k\right)}{k^2 \delta_{UV} Y_\beta \left(m_n^{(+)}/k\right) + km_n^{(+)} Y_\beta Y_{\beta+1} \left(m_n^{(+)}/k\right)},
$$

(13)

$$
b_n^{(-)} = \frac{J_\beta \left(m_n^{(-)}/k\right)}{Y_\beta \left(m_n^{(-)}/k\right)},
$$

(14)

\footnote{The other choice $f_n(y)|_{\pm L} = 0$ eliminates all the IR-brane interactions, which is not interesting case.}
where $\delta_{UV}$ is the UV-brane dimensionless parameter defined as

$$\delta_{UV} \equiv \frac{m_{UV}^2}{k^2} - 2(2 + \beta). \quad (A.14)$$

Now the boundary condition at $y = \pm L$ implies the following equation whose roots determine the mass spectrum of the KK-modes,

$$\frac{m_n^{(\pm)}}{m_{KK}} \left( J_{\beta + 1} \left( \frac{e^{kLm_n^{(\pm)}}}{k} \right) + b_n^{(\pm)} \left( \frac{e^{kLm_n^{(\pm)}}}{k} \right) \right) = -\frac{1}{2} \delta_{IR} \left( J_\beta \left( \frac{e^{kLm_0^{(\pm)}}}{k} \right) + b_0^{(\pm)} \left( \frac{e^{kLm_0^{(\pm)}}}{k} \right) \right), \quad (A.15)$$

where $m_{KK}$ and the IR-brane dimensionless parameter $\delta_{IR}$ are defined in Eq. (3.31). We solve the above equation for the KK mass eigenvalues in the approximation $kL \gg 1$ and $m_n^{(\pm)} \ll k$, such that one can set $b_n^{(\pm)} \approx 0$. With this simplification we get the following zero-mode mass $m_0^{(\pm)}$ eigenvalue equation:

$$\frac{m_0^{(\pm)}}{m_{KK}} \frac{J_{\beta + 1} \left( \frac{m_0^{(\pm)}}{m_{KK}} \right)}{J_\beta \left( \frac{m_0^{(\pm)}}{m_{KK}} \right)} \approx -\frac{1}{2} \delta_{IR}. \quad (A.16)$$

Expanding around $m_0^{(\pm)} \sim 0$, we find the following mass for the zero-mode,

$$m_0^{2(\pm)} \approx -(1 + \beta)m_{KK}^2 \delta_{IR} \left[ 1 - \frac{2 \delta_{IR}}{2 + \beta} + \frac{2 \delta_{IR}^2}{(2 + \beta)^2(3 + \beta)} + O(\delta_{IR}^3) \right]. \quad (A.17)$$

Note that the above zero-mode mass must be negative in order to trigger the spontaneous symmetry breaking. Therefore we assume at this point that $\delta_{IR} > 0$. The above solution implies that for the values of $\delta_{IR} \sim O(1)$ the zero-mode mass is of the order of KK mass scale $m_{KK}$. In order to have a light zero-mode mass (order of the electroweak scale), we need to fine-tune $\delta_{IR}$ such that $|m_0^{(\pm)}| \ll m_{KK}$ which implies that $\delta_{IR} \sim 10^{-3}$ for $m_{KK} \sim O(\text{few})$ TeV (required by the recent data from EWPT, see Sec. 4.1). For the non-zero KK mode masses (excited KK states for $n \neq 0$) we assume that $m_n^{(\pm)} \geq m_{KK}$ so that we can treat $\delta_{IR} \approx 0$. Hence the non-zero KK-mode masses $m_n^{(\pm)}$ read:

$$m_n^{(\pm)} \approx \left( n + \frac{\beta}{2} - \frac{3}{4} \right) \pi m_{KK}, \quad (A.18)$$

which implies that the masses of even and odd excited KK-modes is degenerate in the approximations considered here.

Now we can write the 4D effective action for the toy model (A.1) by using the above KK-decomposition and integrating over the extra-dimension $y$, as:

$$S_{\text{toy}} = -\int d^4x \left\{ \frac{1}{2} \partial_\mu \Phi_n^{(+)} \partial^\mu \Phi_n^{(+)} + \frac{1}{2} \partial_\mu \Phi_n^{(-)} \partial^\mu \Phi_n^{(-)} + \frac{1}{2} m_n^{2(+)\Phi_n^{2(+)}} + \frac{1}{2} m_n^{2(-)\Phi_n^{2(-)}} \right\}$$

\footnote{For the zero-mode mass this approximation does not hold, as $m_0^{(\pm)}/m_{KK}$ is of same order as $\delta_{IR} \sim 10^{-3}$.}
\[
\begin{align*}
&+ \frac{\lambda_{klmn}^{(0)}}{4} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(+)} \Phi_n^{(+)} + \frac{\lambda_{klmn}^{(-)}}{4} \Phi_k^{(-)} \Phi_l^{(-)} \Phi_m^{(-)} \Phi_n^{(-)} + \frac{3}{2} \lambda_{klmn} \Phi_k^{(+)} \Phi_l^{(+)} \Phi_m^{(-)} \Phi_n^{(-)} \Bigg) ,
\end{align*}
\]

(A.19)

where \(\lambda_{klmn}^{(\pm)}\) and \(\lambda_{klmn}\) are the quartic couplings given by,

\[
\lambda_{klmn}^{(\pm)} = e^{4A(L)} \frac{\lambda_{IR} f_0^{(\pm)} f_0^{(\pm)}}{k^2 f_k^{(\pm)} f_0^{(\pm)} f_0^{(\pm)}} \Bigg|_{L}, \quad \lambda_{klmn} = e^{4A(L)} \frac{\lambda_{IR} f_0^{(\pm)} f_0^{(\pm)} f_0^{(\pm)}}{k^2 f_k^{(\pm)} f_0^{(\pm)} f_0^{(\pm)}} \Bigg|_{L} .
\]

(A.20)

The above action (A.19) is symmetric under \(Z_2 \times Z_2\) under which \(\Phi_n^{(+)} \rightarrow -\Phi_n^{(+)}\) and \(\Phi_n^{(-)} \rightarrow -\Phi_n^{(-)}\), respectively. Note that it has been taken into account that the integration of the IR-brane delta functions will provide a factor of 1/2 instead of 1, as our geometry is an interval, assuming that there is nothing outside the \([-L, +L]\) interval. The above effective action is valid for all the KK-modes. In order to get the low-energy effective action we limit ourself to zero-modes only. The low energy (zero-mode) effective action for our toy model is:

\[
S_{eff}^{toy} = -\int d^4 x \left\{ \frac{1}{2} \partial_{\mu} \Phi_0^{(+)} \partial^{\mu} \Phi_0^{(+)} + \frac{1}{2} \partial_{\mu} \Phi_0^{(-)} \partial^{\mu} \Phi_0^{(-)} - \frac{1}{2} \mu^2 \Phi_0^{2(+)} - \frac{1}{2} \mu^2 \Phi_0^{2(-)} 
+ \frac{\lambda_{0000}^{(0)}}{4} \Phi_0^{4(\pm)} + \frac{\lambda_{0000}^{(-)}}{4} \Phi_0^{4(\pm)} + \frac{3}{2} \lambda_{0000} \Phi_0^{2(\pm)} \Phi_0^{2(\pm)} \right\},
\]

(A.21)

where \(\lambda_{0000}^{(\pm)}\) and \(\lambda_{0000}\) are given by Eq. (A.20), whereas \(\mu\) is the zero-mode mass parameter defined as

\[
-m_0^{(\pm)} \approx \mu^2 \equiv (1 + \beta) m_{KK}^2 \delta_{IR},
\]

(A.22)

In order to get more insight of the above results we determined the wave-function for the zero-modes \(f_0^{(\pm)}(y)\).\footnote{One can also get the solutions for the zero-mode wave functions \(f_0^{(\pm)}(y)\) by solving the Eq. (A.4) for \(m_0^{(\pm)} = 0\), instead of following our approach.} In the above mentioned approximation, \(kL \gg 1\) and \(m_0^{(\pm)} \ll k\) such that \(b_0^{(\pm)} \approx 0\), the wave-functions for zero-modes \(f_0^{(\pm)}(y)\) have the following simple form,

\[
f_0^{(\pm)}(|y|) \approx e^{2k|y|} J_\beta \left( \frac{m_0^{(\pm)}}{k} e^{k|y|} \right),
\]

(A.23)

where \(f_0^{(-)}(y) = e(y) f_0^{(-)}(|y|)\). The normalization \(N_0^{(\pm)}\) can be fixed by Eq. (A.5):

\[
N_0^{2(\pm)} = 2 \int_0^L dy e^{2k y} J_\beta \left( \frac{m_0^{(\pm)}}{k} e^{k y} \right)^2 \approx e^{2kL} \frac{m_0^{(\pm)}}{k} e^{kL} \left( \frac{m_0^{(\pm)}}{k} e^{kL} \right)^2.
\]

(A.24)

We get the zero-mode wave-functions \(f_0^{(\pm)}(y)\), after expanding the Bessel functions around the zero arguments \((m_0^{(\pm)} \approx 0)\), as

\[
f_0^{(\pm)}(|y|) \approx \sqrt{k(1 + \beta)} e^{kL} e^{(2 + \beta)k|y| - L}.
\]

(A.25)
Hence for $f_0^\pm(\pm L) \simeq \sqrt{k(1+\beta)} e^{kL}$, the quartic couplings are:

$$\lambda_{0000}^\pm = \lambda_{0000} \simeq \lambda, \quad \text{where} \quad \lambda \equiv \lambda_{IR}(1 + \beta)^2.$$  \hspace{1cm} (A.26)

After calculating all the parameters of the effective action (A.21) in terms of the fundamental 5D parameters, we can proceed further to find the vevs of the scalar fields $\Phi^\pm$ (from hereon in the Appendix we drop the subscripts 0 from all the zero-modes). We can write the scalar potentials for even and odd fields as

$$V(\Phi^\pm) = -\frac{1}{2} \mu^2 \Phi^{\pm2} - \frac{1}{2} \mu^2 \Phi^{\mp2} + \frac{\lambda}{4} \Phi^{\pm4} + \frac{\lambda}{4} \Phi^{\mp4} + \frac{3}{2} \lambda \Phi^{\pm2} \Phi^{\mp2}. \hspace{1cm} (A.27)$$

Note that the above scalar potential has $\mathbb{Z}_2' \times \mathbb{Z}_2$ symmetry as pointed out earlier. Below we illustrate that one of the discrete symmetry of $\mathbb{Z}_2' \times \mathbb{Z}_2$ will be spontaneously broken when one of the field from $\Phi^\pm$ will acquire a vev. We find the following conditions for global minima after minimizing the above potential:

$$v^{(\pm)} = \left( \frac{\mu^2}{\lambda} - 3v^{(\mp2)} \right), \quad \text{or} \quad v^{(\pm)} = 0, \hspace{1cm} (A.28)$$

and

$$v^{(\mp)} = \left( \frac{\mu^2}{\lambda} - 3v^{(\pm2)} \right), \quad \text{or} \quad v^{(\mp)} = 0. \hspace{1cm} (A.29)$$

One can easily see from Fig. 8 that the scalar potential $V(\Phi^{(\pm)}, \Phi^{(\mp)})$ has four degenerate global minima at $(\pm v^{(\pm)}, 0)$ and $(0, \pm v^{(\mp)})$. One can choose any of these global vacuum. We select the vacuum where the even mode $\Phi^{(\pm)}$ acquires a vev, whereas $\Phi^{(\mp)}$ has zero vev. In this case the above minimization condition is,

$$v^{(\pm)} = \frac{\mu}{\sqrt{\lambda}}, \quad v^{(\mp)} = 0. \hspace{1cm} (A.30)$$

This choice of the vacuum breaks $\mathbb{Z}_2'$ spontaneously.\textsuperscript{12} Now we perturb the even and odd modes around the vacuum of our choice as:

$$\Phi^{(\pm)}(x) = v^{(\pm)} + \phi(x), \quad \Phi^{(\mp)}(x) = \chi(x). \hspace{1cm} (A.31)$$

\textsuperscript{12}Spontaneous breaking of a discrete symmetry generically implies a proliferation of domain walls or other topological defects that may have dramatic cosmological consequences [52] (universe expanding too homogeneous and too fast). The issue of possible formation of domain walls and their cosmological consequences lies beyond the scope of this work and will not be discussed here. Hereafter we assume that the problem is somehow resolved, possibly through cosmological inflation.
The effective toy action in terms of the even and odd fluctuations can be written as

$$\mathcal{S}_{\text{toy}}^{\text{eff}} = - \int d^4 x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} m^2 \chi^2 + \frac{1}{4} \phi^4 + \frac{1}{4} \chi^4 + \sqrt{\lambda} \mu \phi \left( \phi^2 + 3 \chi^2 \right) + \frac{3}{2} \sqrt{\lambda} \phi^2 \chi^2 \right\},$$

(A.32)

where $m^2 \equiv 2 \mu^2$. It is important to note that both even and odd fluctuations have the same mass $m$ and the same quartic coupling $\lambda$. Moreover, the above action has an unbroken discrete $\mathbb{Z}_2$ symmetry, under which the fields transform as $\phi \rightarrow + \phi$ and $\chi \rightarrow - \chi$.

### A.2 SSB by a vacuum expectation value of the 5D scalar field

In this approach we perturb the 5D scalar field $\Phi(x,y)$ around a $y$-dependent vev $v(y)$ as

$$\Phi(x,y) = v(y) + \phi(x,y),$$

(A.33)

where $v(y)$ is the background solution for the toy action (A.1) and $\phi(x,y)$ is a perturbation around the background vev $v(y)$. The e.o.m. for $v(y)$ and $\phi(x,y)$ read as:

$$\left( - \partial_5 \left( e^{4A(y)} \partial_5 \right) + \mu_B^2 e^{4A(y)} \right) v(y) = -e^{4A(y)} \left( \frac{\partial V_{IR}(v)}{\partial v} \delta(y + L) + \frac{\partial V_{UV}(v)}{\partial v} \delta(y) + \frac{\partial V_{IR}(v)}{\partial v} \delta(y - L) \right),$$

(A.34)

$$\left( -e^{2A(y)} \left( \frac{\partial^2 V_{IR}(v)}{\partial v^2} \right) \phi \right) + \left( e^{4A(y)} \partial_5 + \mu_B^2 e^{4A(y)} \right) \phi(x,y) = -e^{4A(y)} \left( \frac{\partial^2 V_{IR}(v)}{\partial v^2} \phi \delta(y + L) + \frac{\partial V_{UV}(v)}{\partial v^2} \phi \delta(y) + \frac{\partial^2 V_{IR}(v)}{\partial v^2} \phi \delta(y - L) \right).$$

(A.35)

Note that the above e.o.m. for the perturbation $\phi(x,y)$ (A.35) is obtained by Taylor series expansion of the potentials around the background $v(y)$ and it is only up to quadratic order in field $\phi(x,y)$ such that the KK-mass matrix for the field $\phi(x,y)$ is diagonal. The presence of delta functions on the r.h.s. of the above equations suggests that there should be jumps across the branes. The following jump and boundary conditions for $v(y)$ and $\phi(x,y)$ are implied by the general strategy adopted in the main text,

$$\left( \partial_5 - \frac{m_{UV}^2}{2k} \right) v(y) \bigg|_{y=0} = 0, \quad \left( \pm \partial_5 - \frac{m_{IR}^2}{2k} + \frac{\lambda_{IR}}{2k^2} v^2(y) \right) v(y) \bigg|_{y=L^\mp} = 0,$$

\hspace{1cm} (A.36)

$$\left( \partial_5 - \frac{m_{UV}^2}{2k} \right) \phi(x,y) \bigg|_{y=0} = 0, \quad \left( \pm \partial_5 - \frac{m_{IR}^2}{2k} + \frac{3\lambda_{IR}}{2k^2} v^2(y) \right) \phi(x,y) \bigg|_{y=L^\mp} = 0.$$  

(A.37)

Our geometric setup is $\mathbb{Z}_2$ symmetric, therefore the solutions corresponding to the bulk e.o.m. for $v(y)$ Eq. (A.34) are even and odd under $y \rightarrow -y$. Hence, we can look for the solutions possessing a definite parity; the even $v^{(+)}(y)$ and the odd $v^{(-)}(y)$. As discussed in the main text, the jump conditions for the even and odd solutions are different at $y = 0$:

$$\left( \partial_5 - \frac{m_{UV}^2}{2k} \right) v^{(+)}(y) \bigg|_{y=0} = 0, \quad v^{(-)}(y) \bigg|_{y=0} = 0,$$

(A.38)
The b.c. at \( y = \pm L \) for the even and odd solutions are the same:

\[
\left( \pm \partial_5 - \frac{m^2_B}{2k} + \frac{\lambda_{IR}}{2k^2} v^{(\pm)}(y) \right) v^{(\pm)}(y) \bigg|_{\pm L^\mp} = 0. \tag{A.39}
\]

We find the following solutions for the even and odd background vevs:

\[
v^{(+)}(y) = A^{(+)\epsilon}(2+\beta)k|y| + B^{(+)\epsilon}(2-\beta)k|y|, \quad -L \leq y \leq L, \tag{A.40}
\]

\[
v^{(-)}(y) = A^{(-)\epsilon}(y \left( e^{(2+\beta)k|y|} - e^{(2-\beta)k|y|} \right), \quad -L \leq y \leq L, \tag{A.41}
\]

where \( A^{(+)\epsilon} \), \( B^{(+)\epsilon} \) and \( A^{(-)\epsilon} \) are the integration constants. (Note the second integration constant \( B^{(-)\epsilon} \) for the odd solution \( v^{(-)}(y) \) is fixed by the continuity condition at \( y = 0 \)) These integration constants can be fixed by the b.c. \( (A.38) \) and \( (A.39) \). Above \( \beta \) parameterizes the bulk mass as \( \beta \equiv \sqrt{1 + \mu_B^2/k^2} \). For the even background solution, we apply the jump condition \( (A.38) \) at \( y = 0 \) to get one of the integration constant as

\[
B^{(+)\epsilon} = -\frac{m^2_B}{m^2_{UV}} - 2(2+\beta)k^2 \right) A^{(+)\epsilon} = -\frac{\delta_{UV}}{\delta_{UV} + 4\beta} A^{(+)\epsilon}, \tag{A.42}
\]

where \( \delta_{UV} \) is defined in Eq. \( (A.14) \). Since \( \beta > 0 \) therefore the second term in the above solution for \( v^{(+)\epsilon}(y) \) has negligible effect on the shape of the profile near the IR-brane. The constant \( A^{(+)\epsilon}(y) \) is fixed by the boundary condition at \( y = \pm L \):

\[
A^{(+)\epsilon} = \pm \frac{k^3}{\lambda_{IR}} \left( \delta_{IR} - \frac{\delta_{UV}(\delta_{IR} + 4\beta)}{\delta_{UV} + 4\beta} e^{-2\beta k L} \right) e^{-2(2+\beta)kL} \left( 1 - e^{-2\beta k L} \delta_{UV} \right)^{-3/2}, \tag{A.43}
\]

where \( \delta_{IR} \) is defined in Eq. \( (3.31) \). For \( kL \gg 1 \) and \( \beta > 0 \), the second terms in both parentheses above are negligible in the region of interest (near the IR-brane). Hence the background even solution for the scalar field can be written as:

\[
v^{(+)\epsilon}(y) \simeq \pm \frac{k^3 \delta_{IR}}{\lambda_{IR}} e^{(2+\beta)k|y| - L}, \tag{A.44}
\]

From the above solution we conclude that for \( \lambda_{IR} > 0 \) (as required by the positivity of the tree-level potential) one needs \( \delta_{IR} > 0 \), i.e. \( m^2_B/k^2 > 2(2+\beta) \).

Similarly one can determine the boundary condition \( A^{(-)\epsilon}(y) \) by applying the boundary condition at \( y = \pm L \), for \( kL \gg 1 \), we find:

\[
A^{(-)\epsilon}(y) = \pm \frac{k^3}{\lambda_{IR}} \delta_{IR} e^{-(2+\beta)kL}. \tag{A.45}
\]

Hence the solution for odd profile \( v^{(-)\epsilon}(y) \) reads:

\[
v^{(-)\epsilon}(y) = \pm \frac{k^3 \delta_{IR}}{\lambda_{IR}} \epsilon(y) \left( e^{(2+\beta)k|y| - L} - e^{-2\beta k L} e^{(2-\beta)k|y| - L} \right). \tag{A.46}
\]
Again the second term in the above solution is negligible for $\beta > 0$ and $kL \gg 1$, therefore the final form of the odd solution is,

$$v^-(y) \simeq \pm \sqrt{\frac{k^3 \delta_{IR}}{\lambda_{IR}} \epsilon(y) e^{(2+\beta)k(|y|-L)}}. \quad (A.47)$$

We have shown above that the background solutions of 5D scalar field are even or odd parity under extra-dimensional $\mathbb{Z}_2$ parity. Since our geometric setup is symmetric, therefore we also choose the even background solution $v^+(y)$ for the scalar field. With this choice of the background solution, the fluctuations will have a definite parity. It is instructive to rewrite the even background solution (A.44) as

$$v^+(y) \simeq v_4 f_v(y), \quad (A.48)$$

where the constant vev $v_4$ and the $y$-dependent vev profile $f_v(y)$ are:

$$v_4 = \sqrt{\frac{m_{KK}^2 \delta_{IR}}{\lambda_{IR} (1+\beta)}}, \quad f_v(y) = \sqrt{k(1+\beta)} e^{kL} e^{-k(2+\beta)k(|y|-L)}. \quad (A.49)$$

The $y$-dependent vev profile satisfies the orthonormality condition

$$\int_{-L}^{L} dy e^{2A(y)} f_v^2(y) = 1. \quad (A.50)$$

As we will show below, comparing the effective theories obtained by the two different mechanisms of SSB discussed in this appendix, the constant factor of $y$-dependent vev, i.e. $v_4$ has the same value as the vev $v_1$ obtained in the previous section. This is an important result and we will comment on this at the end this section. It is worth mentioning here that the quartic term in the IR-brane potential is crucial for a non-trivial vev profile $v^\pm(y)$. If the quartic term would have been absent in the $V_{IR}$, i.e. $\lambda_{IR} = 0$, then the b.c. (A.38) and (A.39) would have implied $v^\pm(y) = 0$. Even though the quartic term is not in the bulk (only localized at the IR-brane), nevertheless, the b.c. imply the non-zero profile in the bulk.

Next we deal with the fluctuation field $\phi(x,y)$ by KK-decomposing it as

$$\phi(x,y) = \sum_n \phi_n(x) \tilde{f}_n(y), \quad (A.51)$$

such that the e.o.m. for the wave-function $\tilde{f}_n(y)$ follows from Eq. (A.35),

$$- \partial_5 (e^{4A(y)} \partial_5 \tilde{f}_n(y)) + \mu_{IR}^2 e^{4A(y)} \tilde{f}_n(y) + e^{4A(y)} \frac{m_{UV}^2}{k} \tilde{f}_n(y) \delta(y) = m_n^2 e^{2A(y)} \tilde{f}_n(y), \quad (A.52)$$

where $A(y) = -k|y|$ and the KK-modes satisfy $\Box^{(4)} \phi_n(x) = m_n^2 \phi_n(x)$. Our symmetric $\mathbb{Z}_2$ geometry implies that the solutions of the Eq. (A.52) are even and odd under $y \to -y$. As the wave functions are even $\tilde{f}_n^+(y)$ and odd $\tilde{f}_n^-(y)$ therefore it is instructive to rewrite the KK-decomposition for the fluctuation field $\phi(x,y)$ (A.51) as:

$$\phi(x,y) = \sum_n \phi_n^{(+)}(x) \tilde{f}_n^+(|y|) + \epsilon(y) \sum_n \phi_n^{(-)}(x) \tilde{f}_n^-(|y|), \quad (A.53)$$
where $\epsilon(y)$ is $\pm 1$ for $y \gtrless 0$. The even and odd solutions are subject to different jump conditions at $y = 0$. The jump and boundary conditions for $f_n^{(\pm)}(y)$ follow from Eq. (A.37) as:

$$
\left. \left( \partial_5 - \frac{m_{UV}^2}{2k} \right) f_n^{(+)}(y) \right|_{y=0} = 0, \quad \left. f_n^{(-)}(y) \right|_{y=0} = 0 \tag{A.54}
$$

$$(\pm \partial_5 - \frac{m_{IR}^2}{2k} + 3\lambda_{IR} v^2(y)) \left. f_n^{(\pm)}(y) \right|_{y=\pm L} = 0. \tag{A.55}
$$

Now we write the 4D effective action for the toy model (A.1) by using the above KK-decomposition and integrating over the extra-dimension $y$, as

$$
S_{\text{toy}}^{\text{eff}} = -\int d^4x \left\{ \frac{1}{2} \partial_\mu \phi_n^{(+)} \partial^\mu \phi_n^{(+)} + \frac{1}{2} \partial_\mu \phi_n^{(-)} \partial^\mu \phi_n^{(-)} + \frac{1}{2} \tilde{m}_n^{2(+)} \phi_n^{2(+)} + \frac{1}{2} \tilde{m}_n^{2(-)} \phi_n^{2(-)} + \tilde{\lambda}_{lmn}^+ \phi_l^{(+)} \phi_n^{(+)} \phi_n^{(-)}(+) + \frac{\tilde{\lambda}_{kmn}^+}{4} \phi_k^{(+)} \phi_l^{(+)} \phi_m^{(-)} \phi_n^{(-)}(+) + 3\tilde{\lambda}_{lmn}^- \phi_l^{(-)} \phi_m^{(-)} \phi_n^{(+)} \phi_n^{(+)}(+) \right\}, \tag{A.56}
$$

where,

$$
\tilde{\lambda}_{lmn}^{(+)} = \frac{\lambda_{IR}}{k^2} \epsilon^{+A(y)} f_v(y) \tilde{f}_l^{(+)}(y) \tilde{f}_m^{(\pm)}(y) \tilde{f}_n^{(\pm)}(y) \left. \right|_L, \tag{A.57}
$$

$$
\tilde{\lambda}_{kmn}^{(\pm)} = \frac{\lambda_{IR}}{k^2} \epsilon^{\pm A(y)} \tilde{f}_k^{(\pm)}(y) \tilde{f}_m^{(\pm)}(y) \tilde{f}_n^{(\pm)}(y) \left. \right|_L, \tag{A.58}
$$

$$
\tilde{\lambda}_{kmn}^{(\pm)} = \frac{\lambda_{IR}}{k^2} \epsilon^{\pm A(y)} \tilde{f}_k^{(\pm)}(y) \tilde{f}_m^{(\pm)}(y) \tilde{f}_n^{(\pm)}(y) \left. \right|_L. \tag{A.59}
$$

For the warped (AdS) geometry the general solutions for Eq. (A.52) corresponding to the even and odd $\tilde{f}_n^{(\pm)}(y)$ are the same as in Eqs. (A.11) and (A.12). From the b.c. at $y = \pm L$ we get the following equation whose roots will determine the mass spectrum of the KK-modes,

$$
\frac{\tilde{m}_n^{(\pm)}}{m_{KK}} \left. \left( J_{\beta+1} \left( \frac{e^{kL \tilde{m}_n^{(\pm)}}}{k} \right) + b_n^{(\pm)} Y_{\beta+1} \left( \frac{e^{kL \tilde{m}_n^{(\pm)}}}{k} \right) \right) \right| = \delta_{IR} \left( J_{\beta} \left( \frac{e^{kL \tilde{m}_n^{(\pm)}}}{k} \right) + b_n^{(\pm)} Y_{\beta} \left( \frac{e^{kL \tilde{m}_n^{(\pm)}}}{k} \right) \right). \tag{A.60}
$$

where $m_{KK}$ and $\delta_{IR}$ are defined in Eq. (3.31). In the approximation $kL \gg 1$ and $\tilde{m}_n^{(\pm)} \ll k$, we find $b_n^{(\pm)} \approx 0$. Hence the above mass eigenvalue equation takes the following form for the zero-mode masses $\tilde{m}_0^{(\pm)}$,

$$
\frac{\tilde{m}_0^{(\pm)}}{m_{KK}} \left. \left( J_{\beta+1} \left( \frac{\tilde{m}_0^{(\pm)}}{m_{KK}} \right) \right) \right| = \delta_{1IR}. \tag{A.61}
$$
We expand the above expression around $\tilde{m}_0^{(±)} \sim 0$ to get the following masses for the zero-modes,

$$\tilde{m}_0^{(±)} \simeq 2(1 + \beta) m_{KK}^2 \delta_{IR} \left[ 1 - \frac{\delta_{IR}}{2 + \beta} + \frac{2 \delta_{IR}^2}{(2 + \beta)^2(3 + \beta)} + O(\delta_{IR}^3) \right].$$

(A.62)

As explained in the above sub-Appendix, in order to have the light zero-mode mass $\sim O(100)$ GeV, we need to fine-tune $\delta_{IR} \sim 10^{-3}$. The above result also implies that the odd zero-mode is degenerate in the mass with the even scalar mass. For the non-zero modes (excited KK-modes) we assume that $\tilde{m}_n^{(±)}/m_{KK} \gg \delta_{IR}$, hence the $\tilde{m}_n^{(±)}$ for the non-zero KK-modes ($n \neq 0$) are:

$$\tilde{m}_n^{(±)} \simeq \left( n + \frac{\beta}{2} - \frac{3}{4} \right) \pi m_{KK},$$

(A.63)

which implies that the masses of the even and odd excited KK-modes are degenerate in the approximations considered above.

Assuming the KK-scale $m_{KK}$ is large enough, we can write down the low-energy effective action for the lowest even and odd modes $\phi_0^{(±)}$ from the toy action (A.56) as:

$$S_{\text{eff}}^{\text{toy}} = - \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi_0^{(±)} \partial^\mu \phi_0^{(±)} + \frac{1}{2} \partial_\mu \phi_0^{(-)} \partial^\mu \phi_0^{(-)} + \frac{1}{2} \tilde{m}_0^{(+)} \phi_0^{(+)} + \frac{1}{2} \tilde{m}_0^{(-)} \phi_0^{(-)} + \frac{1}{2} \tilde{m}_0^{+(+)} \phi_0^{(+)} \phi_0^{(+)} + \frac{1}{2} \tilde{m}_0^{(-(-)} \phi_0^{(-)} \phi_0^{(-)} + \frac{1}{2} \tilde{m}_0^{+(+)} \phi_0^{(+)} \phi_0^{(+)} \phi_0^{(+)} \phi_0^{(+)} + \frac{1}{2} \tilde{m}_0^{(-(-)} \phi_0^{(-)} \phi_0^{(-)} \phi_0^{(-)} \phi_0^{(-)} + \frac{1}{2} \tilde{m}_0^{+(+)} \phi_0^{(+)} \phi_0^{(+)} \phi_0^{(+)} \phi_0^{(+)} \right\},$$

(A.64)

where the vev profile and zero-modes at the location of IR-brane, i.e., $\tilde{f}_v(L) \approx f_0^{(±)}(L) \approx \sqrt{k(1 + \beta)}e^{kL}$ (note that zero-mode profile $f_0^{(±)}(y) = f_0^{(±)}(y)$ where $f_0^{(±)}(y)$ is given in Eq. (A.25)). The couplings take the following values from Eqs. (A.57) and (A.59):

$$\tilde{\lambda}_0^{(±)} \simeq \sqrt{\lambda}_0^{(±)} \approx \lambda, \quad \tilde{\lambda}_0^{(±)} \simeq \lambda_0^{(±)} \simeq \lambda,$$

(A.65)

where we defined

$$v_4 = \sqrt{\frac{m_{KK}^2 \delta_{IR}}{\lambda_{IR}(1 + \beta)}} = \frac{\mu}{\sqrt{\lambda}},$$

(A.66)

with $\mu^2 \equiv (1 + \beta)m_{KK}^2 \delta_{IR}$ and $\lambda \equiv \lambda_{IR}(1 + \beta)^2$. Simplifying our notation ($\phi_0^{(+)} = \phi$, $\phi_0^{(-)} = \chi$ and $\tilde{m}_0 = \tilde{m}$), we can write the above effective action (A.64) in the following form:

$$S_{\text{eff}} = - \int d^4x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \lambda \partial^\mu \chi + \frac{1}{2} \tilde{m}^2 \phi^2 + \frac{1}{2} \tilde{m}^2 \chi^2 + \frac{1}{4} \phi^4 + \frac{1}{4} \chi^4 + \frac{3}{2} \lambda \phi^2 \chi^2 \right\},$$

(A.67)

where $\tilde{m}_0$ is the mass of the zero-modes given by Eq. (A.62); the leading term is, $\tilde{m}_0^2 = 2\mu^2$.

Let us conclude this Appendix with the comparison of the effective theories obtained in the previous sub-Appendix (A.32) and here (A.67). It is straight forward to see that...
the low energy (zero-mode) d.o.f. in the both actions are same. Moreover, the effective
theories are identical, for example, all the masses and coupling constants are same in
terms of 5D fundamental parameters of the theory. This is a non-trivial matching and
the core of this matching lies in the fact, as discussed above, that the $y$-dependent vev
$v(y)$ can be written as constat $v_4$ times the normalized $y$-dependent profile $f_0(y)$ such that
$v_4 = v_1 = \frac{\phi}{\sqrt{\Lambda}}$ and $f_0(y) \simeq f_0(y)$, where $v_1$ and $f_0(y)$ are the vev and the zero-mode profile,
respectively, obtained in the previous sub-appendix. Once this matching is achieved then
we can conceive ourselves that all the masses and coupling constants in both low-energy
effective theories are exactly identical. In other words, consistently with our expectations,
the KK-expansion and the expansion around $y$-dependent vev do commute (at least for
the zero modes).

References

[1] L. Randall and R. Sundrum, A Large mass hierarchy from a small extra dimension, 
Phys.Rev.Lett. 83 (1999) 3370–3373, [hep-ph/9905221].

[2] L. Randall and R. Sundrum, An Alternative to compactification, Phys.Rev.Lett. 83 (1999)
4690–4693, [hep-th/9906064].

[3] W. D. Goldberger and M. B. Wise, Modulus stabilization with bulk fields, Phys.Rev.Lett. 83
(1999) 4922–4925, [hep-ph/9907447].

[4] O. DeWolfe, D. Freedman, S. Gubser, and A. Karch, Modeling the fifth-dimension with
scalars and gravity, Phys.Rev. D62 (2000) 046008, [hep-th/9909134].

[5] H. Davoudiasl, J. Hewett, and T. Rizzo, Bulk gauge fields in the Randall-Sundrum model,
Phys.Lett. B473 (2000) 43–49, [hep-ph/9911262].

[6] Y. Grossman and M. Neubert, Neutrino masses and mixings in nonfactorizable geometry,
Phys.Lett. B474 (2000) 361–371, [hep-ph/9912408].

[7] S. Chang, J. Hisano, H. Nakano, N. Okada, and M. Yamaguchi, Bulk standard model in the
Randall-Sundrum background, Phys.Rev. D62 (2000) 084025, [hep-ph/9912498].

[8] T. Gherghetta and A. Pomarol, Bulk fields and supersymmetry in a slice of AdS, Nucl.Phys.
B586 (2000) 141–162, [hep-ph/0003129].

[9] S. J. Huber and Q. Shafi, Fermion masses, mixings and proton decay in a Randall-Sundrum
model, Phys.Lett. B498 (2001) 256–262, [hep-ph/0010195].

[10] K. Agashe, A. Delgado, M. J. May, and R. Sundrum, RS1, custodial isospin and precision
tests, JHEP 0308 (2003) 050, [hep-ph/0308036].

[11] M. A. Luty and T. Okui, Conformal technicolor, JHEP 0609 (2006) 070, [hep-ph/0409274].

[12] N. Arkani-Hamed, M. Porrati, and L. Randall, Holography and phenomenology, JHEP 0108
(2001) 017, [hep-th/0012148].

[13] R. Rattazzi and A. Zaffaroni, Comments on the holographic picture of the Randall-Sundrum
model, JHEP 0104 (2001) 021, [hep-th/0012248].

[14] H. Davoudiasl, B. Lillie, and T. G. Rizzo, Off-the-wall Higgs in the universal
Randall-Sundrum model, JHEP 0608 (2006) 042, [hep-ph/0508279].
[15] G. Cacciapaglia, C. Csaki, G. Marandella, and J. Terning, *The Gaugephobic Higgs*, JHEP **0702** (2007) 036, [hep-ph/0611358].

[16] A. Falkowski and M. Perez-Victoria, *Electroweak Breaking on a Soft Wall*, JHEP **0812** (2008) 107, [arXiv:0806.1737].

[17] A. Azatov, M. Toharia, and L. Zhu, *Higgs Mediated FCNC’s in Warped Extra Dimensions*, Phys.Rev. **D80** (2009) 035016, [arXiv:0906.1990].

[18] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Warped Electroweak Breaking Without Custodial Symmetry*, Phys.Lett. **B697** (2011) 208–214, [arXiv:1011.2205].

[19] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Suppressing Electroweak Precision Observables in 5D Warped Models*, JHEP **1105** (2011) 083, [arXiv:1103.1388].

[20] J. A. Cabrer, G. von Gersdorff, and M. Quiros, *Improving Naturalness in Warped Models with a Heavy Bulk Higgs Boson*, Phys.Rev. **D84** (2011) 035024, [arXiv:1104.3149].

[21] M. Geller, S. Bar-Shalom, and A. Soni, *Higgs-radion unification: radius stabilization by an SU(2) bulk doublet and the 126 GeV scalar*, Phys.Rev. **D89** (2014) 095015, [arXiv:1312.3331].

[22] P. R. Archer, *The Fermion Mass Hierarchy in Models with Warped Extra Dimensions and a Bulk Higgs*, JHEP **1209** (2012) 095, [arXiv:1204.4730].

[23] M. Frank, N. Pourtolami, and M. Toharia, *Higgs Bosons in Warped Space, from the Bulk to the Brane*, Phys.Rev. **D87** (2013), no. 9 096003, [arXiv:1301.7692].

[24] R. Malm, M. Neubert, K. Novotny, and C. Schmell, *5D perspective on Higgs production at the boundary of a warped extra dimension*, JHEP **1401** (2014) 173, [arXiv:1303.5702].

[25] P. R. Archer, M. Carena, A. Carmona, and M. Neubert, *Higgs Production and Decay in Models of a Warped Extra Dimension with a Bulk Higgs*, JHEP **1501** (2015) 060, [arXiv:1408.5406].

[26] B. M. Dillon and S. J. Huber, *Non-Custodial Warped Extra Dimensions at the LHC?*, arXiv:1410.7345.

[27] K. Agashe, A. Azatov, Y. Cui, L. Randall, and M. Son, *Warped Dipole Completed, with a Tower of Higgs Bosons*, arXiv:1412.6468.

[28] A. M. Iyer, K. Sridhar, and S. K. Vempati, *Bulk RS models, Electroweak Precision tests and the 125 GeV Higgs*, arXiv:1502.06206.

[29] A. Ahmed and B. Grzadkowski, *Brane modeling in warped extra-dimension*, JHEP **1301** (2013) 177, [arXiv:1210.6708].

[30] A. Ahmed, L. Dulny, and B. Grzadkowski, *Generalized Randall-Sundrum model with a single thick brane*, Eur.Phys.J. **C74** (2014) 2862, [arXiv:1312.3577].

[31] Planck Collaboration, P. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, arXiv:1502.01589.

[32] G. Servant and T. M. Tait, *Is the lightest Kaluza-Klein particle a viable dark matter candidate?*, Nucl.Phys. **B650** (2003) 391–419, [hep-ph/0206071].

[33] H.-C. Cheng, J. L. Feng, and K. T. Matchev, *Kaluza-Klein dark matter*, Phys.Rev.Lett. **89** (2002) 211301, [hep-ph/0207125].
[34] K. Agashe and G. Servant, Warped unification, proton stability and dark matter, Phys.Rev.Lett. 93 (2004) 231805, [hep-ph/0403143].

[35] G. Panico, E. Ponton, J. Santiago, and M. Serone, Dark Matter and Electroweak Symmetry Breaking in Models with Warped Extra Dimensions, Phys.Rev. D77 (2008) 115012, [arXiv:0801.1645].

[36] E. Ponton and L. Randall, TeV Scale Singlet Dark Matter, JHEP 0904 (2009) 080, [arXiv:0811.1029].

[37] L. Vecchi, Majorana dark matter in warped extra dimensions, Phys.Rev. D90 (2014), no. 2 025017, [arXiv:1310.7862].

[38] T. Gherghetta and B. von Harling, A Warped Model of Dark Matter, JHEP 1004 (2010) 039, [arXiv:1002.2967].

[39] B. von Harling and K. L. McDonald, Secluded Dark Matter Coupled to a Hidden CFT, JHEP 1208 (2012) 048, [arXiv:1012.5298].

[40] A. R. Frey, R. J. Danos, and J. M. Cline, Warped Kaluza-Klein Dark Matter, JHEP 0911 (2009) 102, [arXiv:0908.1387].

[41] K. Agashe, A. Falkowski, I. Low, and G. Servant, KK Parity in Warped Extra Dimension, JHEP 0804 (2008) 027, [arXiv:0712.2455].

[42] A. D. Medina and E. Ponton, Warped Universal Extra Dimensions, JHEP 1106 (2011) 009, [arXiv:1012.5298].

[43] A. D. Medina and E. Ponton, Warped Radion Dark Matter, JHEP 1109 (2011) 016, [arXiv:1104.4124].

[44] Z. Lalak and R. Matyszkiewicz, Boundary terms in brane worlds, JHEP 0111 (2001) 027, [hep-th/0110141].

[45] A. Carmona and M. Chala, Composite Dark Sectors, arXiv:1504.00332.

[46] B. Grzadkowski and J. Wudka, Pragmatic approach to the little hierarchy problem: the case for Dark Matter and neutrino physics, Phys.Rev.Lett. 103 (2009) 091802, [arXiv:0902.0628].

[47] C. F. Kolda and H. Murayama, The Higgs mass and new physics scales in the minimal standard model, JHEP 0007 (2000) 035, [hep-ph/0003170].

[48] J. Casas, J. Espinosa, and I. Hidalgo, Implications for new physics from fine-tuning arguments. 1. Application to SUSY and seesaw cases, JHEP 0411 (2004) 057, [hep-ph/0410298].

[49] R. Barbieri and G. Giudice, Upper Bounds on Supersymmetric Particle Masses, Nucl.Phys. B306 (1988) 63.

[50] Particle Data Group Collaboration, K. Olive et al., Review of Particle Physics, Chin.Phys. C38 (2014) 090001.

[51] E. W. Kolb and M. S. Turner, The Early Universe, Front.Phys. 69 (1990) 1–547.

[52] Y. Zeldovich, I. Y. Kobzarev, and L. Okun, Cosmological Consequences of the Spontaneous Breakdown of Discrete Symmetry, Zh.Eksp.Teor.Fiz. 67 (1974) 3–11.