Lower Bound on Derivatives of Costa’s Differential Entropy

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Abstract
Several conjectures concern the lower bound for the differential entropy $H(X_t)$ of an $n$-dimensional random vector $X_t$ introduced by Costa. Cheng and Geng conjectured that $H(X_t)$ is completely monotone, that is, $C_1(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq 0$. McKean conjectured that Gaussian $X_Gt$ achieves the minimum of $(-1)^{m+1}(d^m/d^mt)H(X_t)$ under certain conditions, that is, $C_2(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq (-1)^{m+1}(d^m/d^mt)H(X_Gt)$. McKean’s conjecture was only considered in the univariate case before: $C_2(1, 1)$ and $C_2(2, 1)$ were proved by McKean and $C_2(i, 1), i = 3, 4, 5$ were proved by Zhang-Anantharam-Geng under the log-concave condition. In this paper, we prove $C_2(1, n)$, $C_2(2, n)$ and observe that McKean’s conjecture might not be true for $n > 1$ and $m > 2$. We further propose a weaker version $C_3(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq (-1)^{m+1} \frac{1}{2}(d^m/d^mt)H(X_Gt)$ and prove $C_3(3, 2), C_3(3, 3), C_3(3, 4), C_3(4, 2)$ under the log-concave condition. A systematical procedure to prove $C_l(m, n)$ is proposed based on semidefinite programming and the results mentioned above are proved using this procedure.

Keyword. Costa’s differential entropy, Mckean’s conjecture, log-concavity, Gaussian optimality, lower bound of differential entropy.

1 Introduction

Shannon’s entropy power inequality (EPI) is one of the most important information inequalities [1], which has many proofs, generalizations, and applications [2 3 4 5 6 8 9 10 11]. In particular, Costa presented a stronger version of the EPI in his seminal paper [12].

Let $X$ be an $n$-dimensional random vector with probability density $p(x)$. For $t > 0$, define $X_t \triangleq X + Z_t$, where $Z_t \sim N_0(0, tI)$ is an independent standard Gaussian random vector with covariance matrix $t \times I$. The probability density of $X_t$ is

$$
p_t(x_t) = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} p(x) \exp \left(-\frac{\|x_t - x\|^2}{2t}\right) dx_t.
$$

Costa’s differential entropy is defined to be the differential entropy of $X_t$:

$$
H(X_t) = -\int_{\mathbb{R}^n} p_t(x_t) \log p_t(x_t) dx_t.
$$

Costa [12] proved that the entropy power of $X_t$, given by $N(X_t) = \frac{1}{2\pi e} e^{(2/n)H(X_t)}$ is a concave function in $t$. More precisely, Costa proved $(d/dt)N(X_t) \geq 0$ and $(d^2/d^2t)N(X_t) \leq 0$. 

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Due to its importance, several new proofs and generalizations for Costa’s EPI were given. Dembo [14] gave a simple proof for Costa’s EPI via the Fisher information inequality. Villani [13] proved Costa’s EPI with advanced theories. Toscani [16] proved that $(d^3/d^3t)N(X_t) \geq 0$ if $p_t$ is log-concave. Cheng and Geng proposed a conjecture [19]:

**Conjecture 1.** $H(X_t)$ is completely monotone in $t$, that is,

$$C_1(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq 0.$$  

Costa’s EPI implies $C_1(1, n)$ and $C_1(2, n)$ [12]. Cheng-Geng proved $C_1(3, 1)$ and $C_1(4, 1)$ [19]. In [20], the multivariate case of Conjecture 1 was considered and $C_1(3, 2), C_1(3, 3), C_1(3, 4)$ were proved.

Let $X_G$ be an $n$-dimensional Gaussian random vector and $X_{Gt} \triangleq X_G + Z_t$ the Gaussian $X_t$. McKean [18] proved that $X_{Gt}$ achieves the minimum of $(d/dt)H(X_t)$ and $-(d^2/d^2t)H(X_t)$ subject to $\text{Var}(X_t) = \sigma^2 + t$, and conjectured the general case, that is

**Conjecture 2.** The following inequality holds subject to $\text{Var}(X_t) = \sigma^2 + t$,

$$C_2(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq (-1)^{m+1}(d^m/d^mt)H(X_{Gt})$$  

McKean proved $C_2(1,1)$ and $C_2(2,1)$ [18]. Zhang-Anantharam-Geng [17] proved $C_2(3,1), C_2(4,1)$ and $C_2(5,1)$ if the probability density function of $X_t$ is log-concave. The work [17] [18] were limited to the univariate case. In this paper, we consider the multivariate case of Conjecture 2 and will prove $C_2(1,n)$ and $C_2(2,n)$, which give the exact lower bounds for $(-1)^{m+1}(d^m/d^mt)H(X_t)$ for $m = 1, 2$. We also notice that in the multivariate case, Conjecture 2 might not be true for $m > 2$ even under the log-concave condition, which motivates us to propose the following weaker conjecture.

**Conjecture 3.** The following inequality holds subject to $\text{Var}(X_t) = \sigma^2 + t$,

$$C_3(m, n) : (-1)^{m+1}(d^m/d^mt)H(X_t) \geq (-1)^{m+1}(d^m/d^mt)H(X_{Gt}).$$  

The three conjectures give different lower bounds for the derivatives of $(-1)^{m+1}H(X_t)$. Also, Conjecture 2 implies Conjecture 3 and Conjecture 3 implies Conjecture 1, since $H(X_{Gt}) \geq 0$ [17].

In this paper, we propose a systematic and effective procedure to prove $C_l(m, n)$, which consists of three main ingredients. First, a systematic method is proposed to compute constraints $R_i, i = 1, \ldots, N_1$ satisfied by $p_t(x_t)$ and its derivatives. The condition that $p_t$ is log-concave can also be reduced to a set of constraints $R_j, j = 1, \ldots, N_2$. Second, proof for $C_l(m, n)$ is reduced to the following problem

$$\exists p_t \in \mathbb{R} \text{ and } Q_j \text{ s.t. } (E - \sum_{i=1}^{N_1} p_t R_i - \sum_{j=1}^{N_2} Q_j R_j = S)$$  

where $Q_j$ is a polynomial in $p_t$ and its derivatives such that $Q_j \geq 0$ and $S$ is a sum of squares (SOS). Third, problem (6) can be solved with the semidefinite programming (SDP) [22] [23]. There exists no guarantee that the procedure will generate a proof, but when succeeds, it gives an exact and strict proof for $C_s(m, n)$.

Using the procedure proposed in this paper, we first prove $C_2(1, n), C_2(2, n)$. Then we prove $C_3(3, 2), C_3(3, 3), C_3(3, 4)$ and $C_3(4, 2)$ under the condition that $p_t$ is log-concave. $C_2(3, 2), C_2(3, 3), C_2(3, 4)$, and $C_2(4, 2)$ cannot be proved with the above procedure even if $p_t$ is log-concave, which motivates us to propose Conjecture 3.

In Table II, we give the data for computing the SOS representation (6) using the Matlab software package in Appendix A, where $\text{Vars}$ is the number of variables, $N_1$ and $N_2$ are the numbers of
To be the set of all derivatives of \( p \). In this section, we give a general procedure to prove \( C_2(1,n) \). In Section 3, we prove \( C_2(2,n) \) using the proof procedure. In Section 4, we prove \( C_3(3,2), C_3(3,3), \) and \( C_3(3,4) \) under the log-concave condition. In Section 5, we prove \( C_3(4,2) \) under the log-concave condition. In Section 6, conclusions are presented.

### 2 Proof Procedure

In this section, we give a general procedure to prove \( C_s(m,n) \) for specific values of \( l,m,n \).

#### 2.1 Notations

Let \([n]_0 = \{0,1,\ldots,n\}\) and \([n] = \{1,\ldots,n\}\), and \(x_t = [x_{1,t}, \ldots, x_{n,t}]\). To simplify the notations, we use \( p_t \) to denote \( p_t(x_t) \) in the rest of the paper. Denote

\[
P_n = \left\{ \frac{\partial^h p_t}{\partial^{h_1} x_{1,t} \cdots \partial^{h_n} x_{n,t}} : h = \sum_{i=1}^{n} h_i, h_i \in \mathbb{N} \right\}
\]

to be the set of all derivatives of \( p_t \) with respect to the differential operators \( \frac{\partial}{\partial x_{i,t}} \), \( i = 1, \ldots, n \) and \( \mathbb{R}[P_n] \) to be the set of polynomials in \( P_n \) with coefficients in \( \mathbb{R} \). For \( v \in P_n \), let \( \text{ord}(v) \) be the order of \( v \). For a monomial \( \prod_{i=1}^{r} v_i^{d_i} \) with \( v_i \in P_n \), its degree, order, and total order are defined to be \( \sum_{i=1}^{r} d_i \), \( \max_{i=1}^{r} \text{ord}(v_i) \), and \( \sum_{i=1}^{r} d_i \cdot \text{ord}(v_i) \), respectively.

A polynomial in \( \mathbb{R}[P_n] \) is called a \( k \)-th-order differentially homogenous polynomial or simply a \( k \)-th-order differential form, if all its monomials have degree \( k \) and total order \( k \). Let \( M_{k,n} \) be the set of all monomials which have degree \( k \) and total order \( k \). Then the set of \( k \)-th-order differential forms is an \( \mathbb{R} \)-linear vector space generated by \( M_{k,n} \), which is denoted as \( \text{Span}_\mathbb{R}(M_{k,n}) \).

We will use Gaussian elimination in \( \text{Span}_\mathbb{R}(M_{k,n}) \) by treating the monomials as variables. We always use the lexicographic order for the monomials to be defined below unless mentioned otherwise. Consider two distinct derivatives \( v_1 = \prod_{i=1}^{r} \frac{\partial^{h_i} p_t}{\partial^{h_{1,i}} x_{1,t} \cdots \partial^{h_{n,i}} x_{n,t}} \) and \( v_2 = \prod_{i=1}^{r} \frac{\partial^{h_i} p_t}{\partial^{h_{1,i}} x_{1,t} \cdots \partial^{h_{n,i}} x_{n,t}} \). We say \( v_1 > v_2 \) if \( h_l > s_l \) and \( h_j = s_j \) for \( j = l + 1, \ldots, n \). Consider two distinct monomials \( m_1 = \prod_{i=1}^{r} v_i^{d_i} \) and

| Vars | \( C_2(3,1) \) | \( C_3(3,2) \) | \( C_3(3,3) \) | \( C_3(3,4) \) | \( C_3(4,2) \) | \( C_2(2,n) \) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( N_1 \) | 6 | 14 | 38 | 38 | 33 | 6 |
| \( N_2 \) | 0 | 0 | 6 | 6 | 3 | 0 |
| Time | 0.18 | 0.53 | 9.00 | 9.02 | 4.49 | 0.32 |
| Proof | Yes | Yes | Yes | Yes | Yes | Yes |

Table 1: Data in computing the SOS with SDP constraints in (6). Time is the running time in seconds collected on a desktop PC with a 3.40GHz CPU and 16G memory, and Proof means whether a proof is given.

The procedure is inspired by the work [12, 15, 17, 19], and uses basic ideas introduced therein. In particular, our approach can be basically considered as a generalization of [17] from the univariate case to the multivariate case and as a generalization of [20] by adding the log-concave constraints. Also, the log-concave constraints considered in this paper are more general than those in [17].

The rest of this paper is organized as follows. In Section 2, we give the proof procedure and prove \( C_2(1,n) \). In Section 3, we prove \( C_2(2,n) \) using the proof procedure. In Section 4, we prove \( C_3(3,2), C_3(3,3), \) and \( C_3(3,4) \) under the log-concave condition. In Section 5, we prove \( C_3(4,2) \) under the log-concave condition. In Section 6, conclusions are presented.
\[ m_2 = \prod_{i=1}^{r} v_i^{e_i}, \] where \( v_i \in \mathcal{P}_n \) and \( v_i < v_j \) for \( i < j \). We define \( m_1 > m_2 \) if \( d_t > e_t \), and \( d_i = e_i \) for \( i = l + 1, \ldots, r \).

From (1), \( p_t : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) is a function in \( x_t \) and \( t \). So each polynomial \( f \in \mathbb{R}[\mathcal{P}_n] \) is also a function in \( x_t \) and \( t \), \( \tilde{f}(t) = \int_{\mathbb{R}^n} f(t) dx_t \) is a function in \( t \), and the expectation of \( f \) with respect to \( x_t \), \( \mathbb{E}[f] \equiv \int_{\mathbb{R}^n} p_t f(t) dx_t \) is also a function in \( t \). By \( f \geq 0 \), \( \tilde{f} \geq 0 \), and \( \mathbb{E}[f] \geq 0 \), we mean \( f(x_t, t) \geq 0 \), \( \tilde{f}(t) \geq 0 \), and \( \mathbb{E}[f](t) \geq 0 \) for all \( x_t \in \mathbb{R}^n \) and \( t > 0 \).

### 2.2 The proof procedure

In this section, we give the procedure to prove \( C_s(m, n) \), which consists of four steps.

In step 1, we reduce the proof of \( C_s(m, n) \) into the proof of an integral inequality, as shown by the following lemma whose proof will be given in section 2.3

**Lemma 2.1.** Proof of \( C_s(m, n), s = 1, 2, 3 \) can be reduced to show

\[
\int_{\mathbb{R}^n} E_{s,m,n}^n p_t^m dx_t \geq 0
\]  

(7)

where \( E_{s,m,n} = \sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} E_{s,m,n,a_m}, \ a_m = (a_1, \ldots, a_m) \), \( E_{s,m,n,a_m} \) is a 2\( m \)th-order differential form in \( \mathbb{R}[\mathcal{P}_{m,n}] \), and

\[
P_{m,n} = \left\{ \frac{\partial^h p_t}{\partial^{h_1} x_{a_1,t} \cdots \partial^{h_m} x_{a_m,t}} : h \in [2m - 1]; a_i \in [n], i \in [m] \right\}.
\]

(8)

In step 2, we compute the constraints which are relations satisfied by the probability density \( p_t \) of \( X_t \). In this paper, we consider two types of constraints: integral constraints and log-concave constraints which will be given in Lemmas 2.3 and 2.4 respectively. Since \( E_{s,m,n} \) in (7) is a 2\( m \)th-order differential form, we need only the constraints which are 2\( m \)th-order differential forms.

**Definition 2.2.** An \( m \)th-order integral constraint is a 2\( m \)th-order differential form \( R \) in \( \mathbb{R}[\mathcal{P}_n] \) such that \( \int_{\mathbb{R}^n} \frac{R}{p_t^{m-1}} dx_t = 0 \).

**Lemma 2.3** ([20]). There is a systematical method to compute the \( m \)th-order integral constraints \( \mathcal{C}_{m,n} = \{ R_i, i = 1, \ldots, N_1 \} \).

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called log-concave if \( \log f \) is a concave function. In this paper, by the log-concave condition, we mean that the density function \( p_t \) is log-concave.

**Definition 2.4.** An \( m \)th-order log-concave constraint is a 2\( m \)th-order differential form \( \mathcal{R} \) in \( \mathbb{R}[\mathcal{P}_n] \) such that \( \mathcal{R} \geq 0 \) under the log-concave condition.

The following lemma computes the log-concave constraints, whose proof is given in section 2.4

**Lemma 2.5.** Let \( H(p_t) \in \mathbb{R}[\mathcal{P}_n]^{n \times n} \) be the Hessian matrix of \( p_t \), \( \nabla p_t = (\frac{\partial p_t}{\partial x_{1,t}}, \ldots, \frac{\partial p_t}{\partial x_{n,t}}) \),

\[
L(p_t) \equiv p_t H(p_t) - \nabla^T p_t \nabla p_t,
\]

(9)

and \( \triangle_{k,l}, l = 1, \ldots, L_k \) the \( k \)th-order principle minors of \( L(p_t) \). Then the \( m \)th-order log-concave constraints are

\[
\mathcal{C}_{m,n} = \left\{ \prod_{i=1}^{s} (-1)^{k_i} \triangle_{k_i,l_i} T_{k_1, \ldots, k_s} : \sum_{i=1}^{s} k_i \leq m \right\}
\]

(10)
where \( T_{k_1,\ldots,k_s} \in \text{Span}_R(\mathcal{M}_{2(m-2)\sum_{i=1}^{k_i} k_i},n) \) and \( T_{k_1,\ldots,k_s} \geq 0 \). For convenience, denote these constraints as
\[
\mathbb{C}_{m,n} = \{ P_j Q_j, j = 1, \ldots, N_2 \},
\]
where \( P_j \) represents \( \prod_{i=1}^{k_i} (-1)^{k_i} \triangle_{k_i,l_i} \) and \( Q_j \) is the corresponding \( T_{k_1,\ldots,k_s} \).

In step 3, we give a procedure to write \( E_{s,m,n} \) as an SOS under the constraints, detail of which will be given in section 2.5.

Procedure 2.6. For \( E_{s,m,n} \) in Lemma 2.1, \( \mathbb{C}_{m,n} = \{ R_i, i = 1, \ldots, N_1 \} \) in Lemma 2.3 and \( \mathbb{C}_{m,n} = \{ P_j Q_j, j = 1, \ldots, N_2 \} \) in Lemma 2.5, the procedure computes \( e_i \in \mathbb{R} \) and \( Q_j \in \text{Span}_R(\mathcal{M}_{2(m-\deg P_j),n}) \) such that
\[
E_{s,m,n} - \sum_{i=1}^{N_1} e_i R_i - \sum_{j=1}^{N_2} P_j Q_j = S \text{ and } Q_j \geq 0, \quad j = 1, \ldots, N_2
\]
where \( S \) is an SOS. The procedure is not complete in the sense that it may fail to find \( e_i \) and \( Q_j \).

To summarize the proof procedure, we have

**Theorem 2.7.** If Procedure 2.6 finds (12) and (13) for certain \( s, m, n \), then \( C_s(m,n) \) is true.

**Proof.** By Lemma 2.1, we have the following proof for \( C_s(m,n) \):
\[
\int_{\mathbb{R}^n} E_{s,m,n} \frac{dt}{p_t^{2m-1}} dx_t = \int_{\mathbb{R}^n} \sum_{i=1}^{N_1} e_i R_i + \sum_{j=1}^{N_2} P_j Q_j + S \frac{dt}{p_t^{2m-1}} dx_t \geq \int_{\mathbb{R}^n} S \frac{dt}{p_t^{2m-1}} dx_t \geq 0.
\]
Equality S1 is true, because \( R_i \) is an integral constraint by Lemma 2.3. By Lemma 2.5 and (13), \( P_j Q_j \geq 0 \) is true under the log-concave condition, so inequality S2 is true under the log-concave condition. If the log-concave condition is not needed, we may set \( Q_j = 0 \) for all \( j \). Finally, inequality S3 is true, because \( S \geq 0 \) is an SOS. \( \square \)

### 2.3 Proof of Lemma 2.1

Costa [12] proved the following basic properties for \( p_t \) and \( H(X_t) \)
\[
\frac{dp_t}{dt} = \frac{1}{2} \nabla^2 p_t, \quad \frac{dH(X_t)}{dt} = -\frac{1}{2} \mathcal{E}[\nabla^2 \log p_t] = \frac{1}{2} \int_{\mathbb{R}^n} \frac{||\nabla p_t||^2}{p_t} dX_t,
\]
where \( \nabla p_t = (\frac{\partial p_t}{\partial x_{1,t}}, \ldots, \frac{\partial p_t}{\partial x_{n,t}}), \) \( \nabla^2 p_t = \sum_{i=1}^{n} \frac{\partial^2 p_t}{\partial x_{i,t}^2} \) and \( \mathcal{E}[\nabla^2 \log p_t] \) is the expectation of \( \nabla^2 \log p_t \). Equation (15) shows that \( p_t \) satisfies the heat equation.

For \( s = 1 \), Lemma 2.1 was proved in [20].
Lemma 2.8 \( [20] \). For \( m \in \mathbb{N}_{m > 1} \), we have
\[
(-1)^{m+1} \frac{d^m}{dt^m} H(X_t) = \int_{\mathbb{R}^n} \frac{E_{1,m,n}}{p_t^{2m-1}} \, dx_t,
\]
where \( E_{1,m,n} = p_t^{2m-1} \left\{ (-1)^{m+1} \frac{1}{2} \frac{d^{m-1}}{dt^{m-1}} \left( \|\nabla p_t\|_2 \right)^2 \right\} = \sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} E_{1,m,n,a_m} \) is a 2mth-order differential form in \( \mathbb{R}[P_{m,n}] \).

To prove Lemma 2.1 for \( s = 2, 3 \), we need to compute \( (d^m/dt^m) H(X_{Gt}) \). Let \( X_G \sim N_n(\mu, \sigma^2 I) \) be an \( n \)-dimensional Gaussian random vector and \( X_{Gt} \triangleq X_G + Z_t \), where \( Z_t \sim N_n(0,tI) \) is introduced in Section 1. Then \( X_{Gt} \sim N_n(\mu_t, (\sigma^2 + t)I) \) and the probability density of \( X_{Gt} \) is
\[
\tilde{p}_t = \frac{1}{(2\pi)^{n/2} \sigma^2 + t} \exp(-\frac{1}{2(\sigma^2 + t)} \|x_t - \mu\|^2).
\]

Lemma 2.9. Let \( T = \nabla^2 \log p_t \) and \( T_G = \nabla^2 \log \tilde{p}_t \). Then under the log-concave condition, we have
\[
\mathbb{E}[(-T)^m] \geq \mathbb{E}(-T)^m \geq \mathbb{E}(-T_G)^m = (-1)^{m+1} \frac{2m-1}{(m-1)!} \frac{d^m}{dt^m} H(X_{Gt}).
\]

Proof. We claim \( T \leq 0 \) under the log-concave condition, which implies inequality (a). From (16),
\[
T = \frac{p_t \nabla^2 p_t - \|\nabla p_t\|^2}{p_t^2} = \frac{1}{p_t^n} \sum_{a=1}^{n} \left( p_t \frac{\partial^2 p_t}{\partial^2 x_{a,t}} - \left( \frac{\partial p_t}{\partial x_{a,t}} \right)^2 \right).
\]
By Lemma 2.5 under the log-concave condition \( \frac{\partial p_t}{\partial x_{a,t}} = 0 \) for \( a = 1, \ldots, n \), so \( T \leq 0 \) and the claim is proved.

To prove inequality (b), we need the concept of Fisher information \([7]\): \( J(X_t) \triangleq \mathbb{E} \left( \frac{\|\nabla p_t\|^2}{p_t} \right) \). By simple computation, we have
\[
J_G = \nabla^2 \log \tilde{p}_t = -\frac{n}{\sigma^2 + t},
\]
\[
\mathbb{E}(-T) = -\mathbb{E}(\nabla^2 \log p_t) = \int \frac{\|\nabla p_t(x_t)\|^2}{p_t(x_t)} \, dx_t = J(X_t).
\]
From (6), we have \( J(X_t) \geq J(X_{Gt}) \). Then \( \mathbb{E}(-T) = J(X_t) \geq J(X_{Gt}) \leq \mathbb{E}(-T_G) > 0 \), and hence inequality (b).

For equation (c), we first have \( H(X_{Gt}) = \frac{n}{2} + \frac{1}{2} \log(2\pi) + \frac{n}{2} \log(\sigma^2 + t) \) and then equation (c):
\[
(-1)^{m+1} \frac{d^m}{dt^m} H(X_{Gt}) = \frac{n(n-1)!}{2(\sigma^2 + t)^{n/2}} \mathbb{E}(-T_G)^m.
\]

Lemma 2.10. For \( m \in \mathbb{N}_{m > 1} \), we have
\[
\mathbb{E}[(-T)^m] = \int_{\mathbb{R}} \frac{E_{0,m,n}}{p_t^{2m-1}} \, dx_t
\]
where \( E_{0,m,n} = \sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} E_{0,m,n,\alpha_m} \), \( \alpha_m = (a_1, \ldots, a_m) \), and \( E_{0,m,n,\alpha_m} \) is a 2mth-order differential form in \( \mathbb{R}[P_{m,n}] \).

Proof. From (19), we have \( \mathbb{E}[(-T)^m] = \int \frac{(\|\nabla p_t\|^2 - p_t \nabla^2 p_t)^m}{p_t^{2m-1}} \, dx_t \), so \( E_{0,m,n} = (\|\nabla p_t\|^2 - p_t \nabla^2 p_t)^m = \sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} E_{0,m,n,\alpha_m} \), where \( E_{0,m,n,\alpha_m} \) is a 2mth-order differentially form in \( \mathbb{R}[P_{m,n}] \), since \( \text{ord}((\|\nabla p_t\|^2 - p_t \nabla^2 p_t) = 2 \) and \( m > 1 \).
We can now prove Lemma 2.1 for \( s = 2, 3 \). Let
\[
E_{2,m,n} = E_{1,m,n} - \frac{(m-1)!}{2^{m-1}} E_{0,m,n} \\
E_{3,m,n} = E_{1,m,n} - \frac{(m-1)!}{2^{m-1}} E_{0,m,n}
\] (23)
where \( E_{1,m,n} \) and \( E_{0,m,n} \) are from Lemmas 2.8 and 2.10. By Lemma 2.9, \( C_s(m, n) \) is true if \( \int_\mathbb{R} \frac{E_{2,m,n}}{p_t} \, dx_t \geq 0 \) for \( l = 2, 3 \).

As a consequence of Lemma 2.9, we can prove \( C_2(1, n) \), that is

**Theorem 2.11.** Subject to \( \text{Var}(X_t) = (\sigma^2 + t) \times I \), \((1)^{n+1}\frac{d}{dt} H(X_t)\) achieves the minimum when \( X_t \) is Gaussian with variance \((\sigma^2 + t) \times I\) for \( t > 0 \) and \( n \geq 1 \).

**Proof.** By \( (18) \), \( \mathbb{E}(T) \geq \mathbb{E}(-T_G) \). By \( (16) \) and \( (21) \), \((d/dt) H(X_t) = \frac{1}{2} \int \|\nabla_{p_t(x_t)}\|^2 \, dx_t = \frac{1}{2} \mathbb{E}(T) \geq \frac{1}{2} \mathbb{E}(-T_G) = (d/dt) H(X_{Gt}).\) The theorem is proved. \( \square \)

### 2.4 Proof of Lemma 2.5

In this section, we prove Lemma 2.5 which computes the \( m \)-th order log-concave constraints.

A symmetric matrix \( M \in \mathbb{R}^{n \times n} \) is called negative semidefinite and is denoted as \( M \leq 0 \), if all its eigenvalues are nonpositive. From [22], \( p_t \) is log-concave if and only if for all \( x \in \mathbb{R}^n \) and \( t > 0 \), \( L(p_t) \) in \( [9] \) is negative semidefinite. By the knowledge of linear algebra, \( L(p_t) \leq 0 \) if and only if
\[
(-1)^k \triangle_{k,l} \geq 0 \quad \text{for} \quad 1 \leq k \leq n, \quad 1 \leq l \leq \binom{n}{k}
\] (24)
where \( \triangle_{k,l} \) is a \( k \)-order principle minors of \( L(p_t) \). Note that elements of \( L(p_t) \) are quadratic differential forms in \( \mathbb{R}[P_n] \). Then \( (-1)^k \triangle_{k,l} \) is a \( k \)-th order log-concave constraint. As a consequence, \( \prod_{i=1}^n (-1)^k \triangle_{k_i, l_i} Q_{k_1, k_2, \ldots, k_n} \) is an \( m \)-th order log-concave constraint, if \( Q_{k_1, k_2, \ldots, k_n} \in \text{Span}_\mathbb{R}(M_{2m-2} \Sigma_{i=1}^n k_i, n) \) and \( Q_{k_1, k_2, \ldots, k_n} \geq 0 \). This proves Lemma 2.5.

As an illustrative example, assume that \( m = 2, \ n = 2 \). From [9],
\[
L(p_t) = \begin{bmatrix}
  p_t \frac{\partial^2 p_t}{\partial x_{1,t}^2} - (\frac{\partial p_t}{\partial x_{1,t}})^2 & p_t \frac{\partial^2 p_t}{\partial x_{1,t} \partial x_{2,t}} - (\frac{\partial p_t}{\partial x_{1,t}})(\frac{\partial p_t}{\partial x_{2,t}}) \\
  p_t \frac{\partial^2 p_t}{\partial x_{2,t} \partial x_{1,t}} - (\frac{\partial p_t}{\partial x_{2,t}})^2
\end{bmatrix}.
\]

From \( (24) \), \( \triangle_{1,1} = p_t \frac{\partial^2 p_t}{\partial x_{1,t}^2} - (\frac{\partial p_t}{\partial x_{1,t}})^2, \ \triangle_{1,2} = p_t \frac{\partial^2 p_t}{\partial x_{2,t}^2} - (\frac{\partial p_t}{\partial x_{2,t}})^2, \ \triangle_{2,1} = |L(p_t)|. \) From Lemma 2.5, the second order log-concave constraints are
\[
R_{1,1} = -\triangle_{1,1} Q_{1,1}, \text{ where } Q_{1,1} = q_{1,1,1}(\frac{\partial p_t}{\partial x_{1,t}})^2 + q_{1,1,2}(\frac{\partial p_t}{\partial x_{1,t}})(\frac{\partial p_t}{\partial x_{2,t}}) + q_{1,1,3}(\frac{\partial p_t}{\partial x_{2,t}})^2 \text{ and } Q_{1,1} \geq 0, \\
R_{1,2} = -\triangle_{1,2} Q_{1,2}, \text{ where } Q_{1,2} = q_{1,2,1}(\frac{\partial p_t}{\partial x_{1,t}})^2 + q_{1,2,2}(\frac{\partial p_t}{\partial x_{1,t}})(\frac{\partial p_t}{\partial x_{2,t}}) + q_{1,2,3}(\frac{\partial p_t}{\partial x_{2,t}})^2 \text{ and } Q_{1,2} \geq 0, \\
R_{2,1} = \triangle_{2,1}, \ R_3 = \triangle_{1,1} \triangle_{1,2}
\]
where \( Q_{1,1}, Q_{1,2} \in \text{Span}_\mathbb{R}(M_{2,2}) \) and \( M_{2,2} = \{ p_t \frac{\partial^2 p_t}{\partial x_{1,t}^2}, p_t \frac{\partial^2 p_t}{\partial x_{2,t}^2}, (\frac{\partial p_t}{\partial x_{1,t}})^2, (\frac{\partial p_t}{\partial x_{2,t}})^2 \}. \) The monomials \( p_t \frac{\partial^2 p_t}{\partial x_{1,t}^2} \) and \( p_t \frac{\partial^2 p_t}{\partial x_{2,t}^2} \) do not appear in \( Q_{1,1} \) and \( Q_{1,2} \) due to the condition \( Q_{1,1} \geq 0 \) and \( Q_{1,2} \geq 0 \).
2.5 Procedure 2.6

In this section, we present Procedure 2.6 which is a modification of the proof procedure given in [20].

Procedure 2.12. Input: $E_{s,m,n}$; $R_i, i = 1, \ldots, N_1$ are 2mth-order differential forms; $P_j$ is a 2kth-order differential form for $j = 1, \ldots, N_2$.

Output: $e_i \in \mathbb{R}$ and $Q_j \in \text{Span}_\mathbb{R}(M_{2(m-k_j),n})$ such that (12) and (13) are true; or fail meaning that such $e_i$ and $Q_j$ are not found.

S1. Treat the monomials in $M_{m,n}$ as new variables $m_l, l = 1, \ldots, N_{m,n}$, which are all the monomials in $\mathbb{R}[P_n]$ with degree $m$ and total order $m$. We call $m_l m_s$ a quadratic monomial.

S2. Write monomials in $C_{m,n} = \{R_i, i = 1, \ldots, N_1\}$ as quadratic monomials if possible. Doing Gaussian elimination to $C_{m,n}$ by treating the monomials as variables and according to a monomial order such that a quadratic monomial is less than a non-quadratic monomial, we obtain

$$\tilde{C}_{m,n} = C_{m,n,1} \cup C_{m,n,2},$$

where $C_{m,n,1}$ is the set of quadratic forms in $m_i$, $C_{m,n,2}$ is the set of non-quadratic forms, and $\text{Span}_\mathbb{R}(C_{m,n}) = \text{Span}_\mathbb{R}(\tilde{C}_{m,n})$.

S3. There may exist relations among the variables $m_i$, which are called intrinsic constraints. For instance, for $m_1 = p_1^2 \left( \frac{\partial p_1}{\partial x_1} \right)^2$, $m_2 = p_1 \left( \frac{\partial p_1}{\partial x_1} \right)^2 \frac{\partial^2 p_1}{\partial x_1^2}$, and $m_3 = (\frac{\partial p_1}{\partial x_1})^4$ in $M_{1,n}$, an intrinsic constraint is $m_1 m_3 - m_2^2 = 0$. Add the intrinsic constraints which are quadratic forms in $m_i$ to $\tilde{C}_{m,n,1}$ to obtain

$$\tilde{C}_{m,n,1} = \{\tilde{R}_i, i = 1, \ldots, N_3\}.$$

S4. Let $M_{2(m-k_j),n} = \{m_{j,k}, k = 1, \ldots, V_j\}$ and $Q_j = \sum_{k=1}^{V_j} q_{j,k} m_{j,k}$, where $q_{j,k}$ are variables to be found later. Let $R_j$ be obtained from $P_j Q_j$ by writing monomials in $P_j Q_j$ as quadratic monomials and eliminating the non-quadratic monomials with $C_{m,n,2}$, such that $R_j - P_j Q_j \in \text{Span}_\mathbb{R}(C_{m,n})$ and $R_j = \sum_{l=1}^{V_j} q_{j,l} h_{j,l}$, where $h_{j,l} \in \mathbb{R}[m_i]$ is a quadratic form. Delete those $R_j$ which are not quadratic forms in $m_i$ and still denote these constraints as $R_j, j = 1, \ldots, N_2$.

S5. Let $\tilde{E}_{s,m,n}$ be obtained from $E_{s,m,n}$ by eliminating the non-quadratic monomials using $C_{m,n,2}$ such that $E_{s,m,n} - \tilde{E}_{s,m,n} \in \text{Span}_\mathbb{R}(C_{m,n,2}) \subset \text{Span}_\mathbb{R}(C_{m,n})$.

S6. Since $\tilde{E}_{s,m,n}$, $\tilde{R}_i, i = 1, \ldots, N_3$ and $R_j, j = 1, \ldots, N_2$ are quadratic forms in $m_i$, we can use the Matlab program given in Appendix A to compute $p_i, q_{j,s} \in \mathbb{R}$ such that

$$\tilde{E}_{s,m,n} - \sum_{i=1}^{N_3} p_i \tilde{R}_i - \sum_{j=1}^{N_2} R_j = S, \quad (25)$$

$$R_j = \sum_{l=1}^{V_j} q_{j,l} h_{j,l}, j = 1, \ldots, N_2$$

$$Q_j = \sum_{l=1}^{V_j} q_{j,l} m_{j,l} \geq 0, j = 1, \ldots, N_2 \quad (26)$$

where $S = \sum_{i=1}^{N_{m,n}} c_i (\sum_{j=1}^{N_{m,n}} e_{ij} m_j)^2$ is an SOS, $c_i, e_{ij} \in \mathbb{R}$ and $c_i \geq 0$. If (25) and (26) cannot be found, return fail.

S7. Since $R_i$, $E_{s,m,n} - \tilde{E}_{s,m,n}$, $R_j - P_j Q_j$ are all in $\text{Span}_\mathbb{R}(C_{m,n})$, equations (12) and (13) can be obtained from (25) and (26), respectively.

Remark 2.13. Let $R$ be an intrinsic constraint. Then $R$ becomes zero, when replacing $m_i$ by its corresponding monomial in $M_{m,n}$. So $\text{Span}_\mathbb{R}(\tilde{C}_{m,n,1}) = \text{Span}_\mathbb{R}(C_{m,n,1}) \subset \text{Span}_\mathbb{R}(\tilde{C}_{m,n})$ in $\mathbb{R}[P_n]$, that is, we do not need to include the intrinsic constraints in (25). But these intrinsic constraints are needed when using the Matlab program in Appendix A.
2.6 An illustrative example

As an illustrative example, we prove $C_2(3, 1)$ under the log-concave condition using the proof procedure given in section 2.2. Since $n = 1$, denote $x_t = x_{1,t}$, $f := f_0 := p_t, f_n := \frac{\partial^2 p_t}{\partial x_{1,t}^2}, n \in \mathbb{N}_{\geq 0}$.

In step 1, By Lemma 2.1 and (7), we have

$$\frac{d^2 (H(X_t)) - \frac{\partial}{\partial t} \frac{1}{2} \left( \frac{f_1^2}{f} \right) \right)}{\int f_1 dx_t} = \int \frac{E_{2,3,1}}{f^5} dx_t$$

where $E_{2,3,1} = \frac{1}{2} \left( f_1^2 f_2 - f_3 f_1 f_3 f_2 + \frac{1}{2} f_1^2 f_1 f_5 + \frac{1}{2} f_1 f_1 f_5 - \frac{3}{2} f_2 f_2 f_2 f_2 - 3 f_1 f_1 f_2 - 2 f_1 f_2 + f_3 f_3 f_3 + 3 f_1 f_2 f_2 - f_1^6$ is a 6th-order differential form.

In step 2, we compute the constraints with Lemmas 2.3 and 2.5. With Lemma 2.3 we find 6 third order constraints 29: $C_{3,1} = \{R_i, i = 1, \ldots, 6\}$:

$$R_1 = 5 f f_1 f_2 - 4 f_1^3, \quad R_2 = f f_1 f_2 f_3 + f_3^3 - 2 f_1 f_2 f_2^2,$$

$$R_3 = f_1^2 f_2^2 + 2 f_1 m_2 - 2 m_2^2, \quad R_4 = f_1^2 f_2 + 2 f_1 f_2 f_3 - 2 f_1 f_2^2,$$

$$R_5 = f_1^2 f_2 f_4 + m_1^2 - m_2^2, \quad R_6 = f_1^2 f_2 f_4 + 2 f_1 m_2 + 6 m_2^2 - \frac{24}{5} m_3^2.$$  

With Lemma 2.5, we have one third order order log-concave constraint: $C_{3,1} = \{P_i Q_i\}$, where $P_i = f f_2 - f_1^2$, $Q_i \in \text{Span}_R(M_{3,1})$, and $Q_1 \geq 0$.

In step 3, we use Procedure 2.12 to compute the SOS representation 12 and 13 with input $E_{2,3,1}, C_{3,1} = \{R_i, i = 1, \ldots, 6\}, P_1 = f f_2 - f_1^2$.

**S1.** The new variables are $M_{3,1} = \{m_1 = f^2 f_3, m_2 = f f_1 f_2, m_3 = f^3\}$, which are listed from high to low in the lexicographical monomial order.

**S2.** Writing monomials in $C_{3,1}$ as quadratic monomials in $m_i$ if possible and doing Gaussian elimination to $C_{3,1}$, we have

$$C_{3,1,1} = \{\tilde{R}_1 = 5 m_2 m_3 - 4 m_3^2, \quad \tilde{R}_2 = m_1 m_2 + 3 m_2^2 - \frac{4}{5} m_3^2\},$$

$$C_{3,1,2} = \{\tilde{R}_1 = f_1^2 f_3 + 2 m_1 m_2 - 2 m_2^2, \quad \tilde{R}_2 = f_1^2 f_1 f_5 - 3 m_1 m_2 + 6 m_2^2 - \frac{24}{5} m_3^2\}.$$  

**S3.** There exist no intrinsic constraints and thus $\tilde{C}_{3,1,1} = \{\tilde{R}_1, \tilde{R}_2\}$ and $N_3 = 2$.

**S4.** $M_{4,1} = \{f^3 f_1 f_2, f^3 f_1 f_3, f^3 f_2 f_3, f^3 f_2, f^2 f_2 f_1\}$. Then $Q_1 = q_1 t f_1 f_2 + q_1 t f_2 f_3 + q_1 t f_1^3$. Monomials $f^3 f_1 f_2, f^3 f_1 f_3$ do not appear in $Q_1$ due to $Q_1 \geq 0$. Writing monomials in $P_i Q_i$ as quadratic monomials if possible and using $C_{3,1,2}$ to eliminate non-quadratic monomials, we obtain $R_1 = P_i Q_i - \left(\frac{8}{5} q_1 t \tilde{R}_1 - \tilde{q}_1 t \tilde{R}_1 - \tilde{q}_1 t \tilde{R}_1\right) = q_1 (2 m_1 m_2 - m_2^2) + q_1 (\frac{2}{5} m_2^2 - m_2^2) + \frac{3}{5} m_3^2$.

**S5.** Writing $E_{2,3,1}$ as a quadratic form in $m_i$, we have

$$\tilde{E}_{2,3,1} = E_{2,3,1} - \frac{3}{5} \tilde{R}_1 - \frac{1}{5} \tilde{R}_2 + \frac{2}{5} \tilde{R}_4 = \frac{1}{5} m_3^3 - 3 m_1 m_2 - \frac{4}{5} m_2^2 + 2 m_3^2.$$  

**S6.** Since $\tilde{E}_{3,1}, \tilde{R}_1, \tilde{R}_2, \tilde{R}_4$ are quadratic forms in $m_i$, we can use the Matlab program in Appendix A to obtain the following SOS representation

$$\tilde{E}_{2,3,1} = \sum_{i=1}^{3} p_i \tilde{R}_1 + \sum_{i=1}^{3} c_i \left(\sum_{j=1}^{3} m_i m_j\right)^2, \quad P_1 \geq 0,$$

where $p_1 = \frac{4}{5}, p_2 = -2, c_1 = \frac{1}{2}, c_2 = -3, c_3 = 2, q_1 = q_2 = q_3 = c_2 = c_3 = 0$.

**S7.** Since $q_1 = q_2 = q_3 = 0$, the log-concave constraint $R_1$ is not needed and we obtain

$$E_{2,3,1} = \frac{3}{4} R_1 + R_2 + \frac{1}{4} R_3 + \frac{1}{8} R_4 - \frac{7}{4} R_5 - \frac{1}{4} R_6 + \sum_{i=1}^{3} c_i \left(\sum_{j=1}^{3} m_i m_j\right)^2$$

From Theorem 2.7, a proof for $C_2(3, 1)$ is given based on the above SOS representation.
3 Proof of $C_2(2, n)$

In this section, we prove $C_2(2, n)$ using the procedure given in section [22] that is,

**Theorem 3.1.** Subject to $Var(X_t) = (\sigma^2 + t) \times I$, Gaussian $X_t$ with variance $(\sigma^2 + t) \times I$ achieves the minimum of $(-1)^{n+1} \frac{d^2}{dt^2} H(X_t)$ for $t > 0$ and $n \geq 1$.

The log-concave conditions are not needed, so we may set $Q_j = 0$ and compute $e_i \in \mathbb{R}$ such that $E_{2,2,n} - \sum_{i=1}^{N_1} e_i R_i = S$ in [12].

### 3.1 Compute $E_{2,2,n}$

In step 1, we compute $E_{2,2,n}$ with [23]:

\[
-\frac{d^2 H(X_t)}{d^2 t} - \frac{1}{2n} E(\|\nabla p_t\|^2 - p_t \nabla^2 p_t)^2 = \int E_{2,2,n} p_t^2 dx_t
\]

where

\[
E_{2,2,n} = -\frac{d}{dt}(\|\nabla p_t\|^2) - \frac{1}{2n} E(\|\nabla p_t\|^2 - p_t \nabla^2 p_t)^2
\]

\[
= \frac{1}{2n} \left( \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} + \frac{1}{3} \frac{\partial (\frac{\partial p_t}{\partial x_{a,t}})^2}{\partial x_{b,t}} \right)
\]

\[
T_{1,a,b} = \frac{1}{2n} \left( \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} + \frac{1}{3} \frac{\partial (\frac{\partial p_t}{\partial x_{a,t}})^2}{\partial x_{b,t}} \right)
\]

\[
T_{2,a,b} = \left( (\frac{\partial p_t}{\partial x_{a,t}})^2 - p_t \frac{\partial^2 p_t}{\partial x_{a,t}} \right) \left( (\frac{\partial p_t}{\partial x_{b,t}})^2 - p_t \frac{\partial^2 p_t}{\partial x_{b,t}} \right)
\]

### 3.2 The second order constraints

In step 2, we compute the second order integral constraints. Due to the summation structure of $E_{2,2,n}$ in [23], we introduce the following notations

\[
V_{a,b} = \left\{ \frac{\partial^h p_t}{\partial x_{a,t} \partial x_{b,t}} : h = h_1 + h_2 \in [3]_0 \right\}
\]

where $a, b$ are variables taking values in $[n]$. Then $P_{2,n} = \bigcup_{a=1}^{n} \bigcup_{b=1}^{n} V_{a,b}.

The second order integral constraints are [20]:

\[
C_{2,n} = \{ R_{i,a,b}^{(2)}, R_{j}^{(0)} : i = 1, \ldots, 17; j = 1, 2; a, b \in [n] \},
\]

where $R_{i,a,b}^{(2)}$ can be found in [20], $R_{j}^{(0)} = \sum_{i=1}^{n} \sum_{b=1}^{n} R_{i,a,b}^{(0)}, i = 1, 2$, and

\[
R_{i,a,b}^{(0)} = p_t \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} \frac{\partial p_t}{\partial x_{a,t}} + \frac{\partial^2 p_t}{\partial x_{a,t}} \left[ p_t \frac{\partial^2 p_t}{\partial x_{b,t}} - p_t \left( \frac{\partial p_t}{\partial x_{b,t}} \right)^2 \right],
\]

\[
R_{2,a,b}^{(0)} = p_t \frac{\partial^2 p_t}{\partial x_{a,t} \partial x_{b,t}} \frac{\partial p_t}{\partial x_{a,t}} + 2 \frac{\partial p_t}{\partial x_{a,t}} \left[ p_t \frac{\partial^2 p_t}{\partial x_{b,t}} \frac{\partial p_t}{\partial x_{b,t}} - p_t \frac{\partial p_t}{\partial x_{b,t}} \right]^2.
\]

### 3.3 Prove $C_2(2, n)$

In step 3, we use Procedure [2.12] to prove $C_2(2, n)$ with $E_{2,2,n}$ and $C_{2,n}$ in [31] as input. It suffices to write

\[
E_{2,2,n} - \sum_{R \in C_{2,n}} c_R R = S \geq 0
\]
where $c_R \in \mathbb{R}$ and $S$ is an SOS. From (33), a proof for $C_2(2, n)$ can be given based on Theorem 2.7 Since $C_2(2, 1)$ was proved in [18, 17], we will consider $C_2(2, n)$, $n \geq 2$. The general case cannot be proved directly with Procedure 2.12 due to the existence of the parameter $n$. We will reduce the general case to a “finite” problem which can be solved with Procedure 2.12.

From (28) and (31), to prove (33), it suffices to solve Problem I. There exist $c_1, c_2 \in \mathbb{R}$ and an SOS $S$ such that

$$E_{2, n} = \sum_{a=1}^{n} \sum_{b=1}^{n} (T_{1,a,b} - \frac{1}{2n} T_{2,a,b} + c_1 R_{(b,a)}^{(0)} + c_2 R_{(a,b)}^{(0)})$$

under the constraints $R_{(i,a,b)}^{(2)}, i = 1, \ldots, 17$ given in (31).

Motivated by symmetric functions, for any function $f(a, b)$, we have

$$\sum_{a,b=1}^{n} f(a, b) = \sum_{1 \leq a < b} \left\{ \frac{1}{n-1} [f(a, a) + f(b, b)] + [f(a, b) + f(b, a)] \right\}. \quad (34)$$

By (31), we have

$$L_{2,n} = \sum_{a=1}^{n} \sum_{b=1}^{n} (T_{1,a,b} - \frac{1}{2n} T_{2,a,b} + c_1 R_{(b,a)}^{(0)} + c_2 R_{(a,b)}^{(0)})$$

$$= \sum_{a,b} \sum_{a < b} \left\{ \frac{1}{n} T_{2,a,b} - \frac{1}{2n} T_{2,a,a} + T_{1,a,b} - \frac{1}{n-1} T_{2,a,b} + c_1 (R_{(b,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)}) \right\}$$

$$= \sum_{a < b} \left\{ \frac{1}{n-1} (T_{1,a,a} + T_{1,b,b}) - \frac{1}{2n} (T_{2,a,a} + T_{2,b,b}) + \frac{1}{n-1} c_1 (R_{(b,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)}) \right\}$$

$$+ \sum_{a,b} \left\{ \frac{1}{n-1} [(T_{2,a,a} + T_{2,b,b}) - (T_{2,a,b} + T_{2,b,a}) + c_1 (R_{(b,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)}) \right\}$$

$$+ \frac{1}{2n} \left\{ [(T_{2,a,a} + T_{2,b,b}) - (T_{2,a,b} + T_{2,b,a})] + [(T_{1,a,b} + T_{1,b,a}) + c_1 (R_{(b,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)}) \right\}$$

$$= \sum_{a < b} \left\{ \frac{1}{n-1} L_{1,a,b} + \frac{1}{2n} L_{2,a,b} + L_{3,a,b} \right\},$$

where

$$L_{1,a,b} = (T_{1,a,a} + T_{1,b,b}) - \frac{1}{2} (T_{2,a,a} + T_{2,b,b}) + c_1 (R_{(1,a,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)})$$

$$L_{2,a,b} = (T_{2,a,a} + T_{2,b,b}) - (T_{2,a,b} + T_{2,b,a}),$$

$$L_{3,a,b} = (T_{1,a,b} + T_{1,b,a}) + c_1 (R_{(b,a)}^{(0)} + R_{(1,b,b)}^{(0)}) + c_2 (R_{(a,b)}^{(0)} + R_{(1,b,b)}^{(0)}).$$

To prove Problem I, it suffices to prove

Problem II. There exist $c_1, c_2 \in \mathbb{R}$ and SOSs $S_1, S_2, S_3$ such that $L_{1,a,b} = S_1, L_{2,a,b} = S_2, L_{3,a,b} = S_2$ under the constraints $R_{(i,a,b)}^{(2)}, i = 1, \ldots, 17$.

In Problem II, the subscripts $a$ and $b$ are fixed and we can prove Problem II with Procedure 2.12 with $L_{1,a,b}, L_{2,a,b}, L_{3,a,b}$ and $R_{(i,a,b)}^{(2)}, i = 1, \ldots, 17$ as input.

Step S1. The new variables are all the monomials in $\mathbb{R}[\mathbb{V}_{a,b}]$ with degree 2 and total order 2 ($\mathbb{V}_{a,b}$ is defined in (30)):

$$m_1 = \frac{\partial p(x)}{\partial x_{a,t}}^2, m_2 = \left( \frac{\partial p(x)}{\partial x_{b,t}} \right)^2, m_3 = \frac{\partial^2 p(x)}{\partial x_{a,t} \partial x_{b,t}},$$

$$m_4 = p(x) \frac{\partial^2 p(x)}{\partial x_{a,t} \partial x_{b,t}}, m_5 = p(x) \frac{\partial^2 p(x)}{\partial^2 x_{a,t}}, m_6 = p(x) \frac{\partial^2 p(x)}{\partial^2 x_{b,t}}.$$
Step S2. We obtain $C_{2,n,1} = \{ \tilde{R}_i, i = 1, \ldots, 7 \}$ and $C_{2,n,2} = \{ \tilde{R}_i, i = 1, \ldots, 10 \}$ using Gaussian elimination, where

\[
\begin{align*}
\tilde{R}_1 &= m_1 m_6 - 2 m_2^2 + 2 m_3 m_4, & \tilde{R}_2 &= -2 m_2 m_3 + m_2 m_4 + 2 m_3 m_6, \\
\tilde{R}_3 &= -2 m_2^2 + 3 m_3 m_6, & \tilde{R}_4 &= -2 m_1 m_3 + m_1 m_4 + 2 m_3 m_5, \\
\tilde{R}_5 &= m_2 m_5 - 2 m_3^2 + 2 m_3 m_4, & \tilde{R}_6 &= -2 m_2 m_3 + 3 m_3 m_4, \\
\tilde{R}_7 &= -2 m_2^2 + 3 m_3 m_5. \\
\end{align*}
\]

\[
\begin{align*}
\tilde{R}_1 &= p_1^2 \left( \frac{\partial p_1}{\partial x_{1,t}} \right)^2 - m_2 m_6 + m_6^2, & \tilde{R}_2 &= p_1^2 \left( \frac{\partial p_1}{\partial x_{1,t}} \right)^2 - m_1 m_5 + m_5^2, \\
\tilde{R}_3 &= p_1^2 \left( \frac{\partial p_1}{\partial x_{1,t}} \right)^2 - m_3 m_4 + m_4^2, & \tilde{R}_4 &= p_1^2 \left( \frac{\partial p_1}{\partial x_{1,t}} \right)^2 - m_3 m_4 + m_4^2. \\
\end{align*}
\]

$\tilde{R}_k, k = 5, \ldots, 10$ are not given, because they are not used in the proof.

Step S3. There exists one intrinsic constraint: $\tilde{R}_8 = m_1 m_2 - m_3^2$ and $N_4 = 8$.

We do not need Step S4, since there exist no log-concave constraints.

Step S5. Eliminating the non-quadratic monomials in $L_{1,a,b}, L_{2,a,b}$, and $L_{3,a,b}$ using $C_{2,n,2}$, and doing further reduction by $C_{2,n,1}$, we have

\[
\begin{align*}
\hat{L}_{1,a,b} &= L_{1,a,b} + \left( \frac{1}{4} - c_1 \right) \tilde{R}_1 + \left( \frac{1}{2} - c_1 \right) \tilde{R}_2 - \left( \frac{1}{4} + c_2 \right) \tilde{R}_3 - \left( \frac{1}{2} + c_2 \right) \tilde{R}_7 = 0, \\
\hat{L}_{2,a,b} &= L_{2,a,b} - 2 \tilde{R}_1 + \frac{1}{2} \tilde{R}_3 - 2 \tilde{R}_5 + \frac{1}{2} \tilde{R}_7 \\
&= -\frac{1}{2} m_1 m_5 - \frac{1}{2} m_3 m_6 + 6 m_2^2 - 8 m_3 m_4 + m_3^2 - 2 m_5 m_6 + m_6^2, \\
\hat{L}_{3,a,b} &= L_{3,a,b} + \left( \frac{1}{4} - c_1 \right) \tilde{R}_3 + \left( \frac{1}{2} - c_1 \right) \tilde{R}_4 + \left( c_1 - c_2 - \frac{1}{4} \right) \tilde{R}_5 \\
&= m_3^2 - 2 m_3 m_4 + m_4^2 + c_1 \left( -4 m_3^2 + 6 m_3 m_4 - 2 m_4^2 + 2 m_5 m_6 \right)
\end{align*}
\]

which are quadratic forms in $m_i$.

Step S6. Using the Matlab program in Appendix A, we obtain the following SOS representation

\[
\begin{align*}
\hat{L}_{1,a,b} &= 0, & \hat{L}_{2,a,b} = k \sum_{k=1}^{8} p_k \tilde{R}_k + (m_1 - m_2 - m_3 + m_6)^2, & \hat{L}_{3,a,b} &= (m_3 - m_4)^2, \\
\end{align*}
\]

where $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$, $p_3 = 2$, $p_5 = -2$, $p_7 = -2$, $c_1 = c_2 = p_4 = p_5 = p_8 = 0$. So, Problem II is solved and thus $C_2(2, n)$ is proved.

## 4 Proof of $C_3(3, n)$ for $n = 2, 3, 4$ under the log-concave condition

We use the procedure in section 2.2 to prove $C_3(3, n)$ for $n = 2, 3, 4$ under the log-concave condition.

### 4.1 Compute $E_{3,3,n}$

In step 1, we compute $E_{3,3,n}$ in (17) and (23):

\[
\frac{1}{2} \int_{\mathbb{R}^n} \left( \frac{\|\nabla p_t\|^2}{p_t} \right) - \frac{1}{n^3} E \left( \frac{\|\nabla p_t\|^2 - p_t \nabla^2 p_t}{p_t^4} \right)^3 \int_{\mathbb{R}^n} E_{3,3,n} \, dx_t
\]

where $E_{3,3,n} = \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} E_{3,a,b,c}$ and

\[
E_{3,a,b,c} = \begin{align*}
&\frac{p_4}{4} \frac{\partial p_1}{\partial x_{a,t}} \frac{\partial p_1}{\partial x_{b,t}} \frac{\partial p_1}{\partial x_{c,t}} - \frac{p_5}{4} \frac{\partial p_1}{\partial x_{a,t}} \frac{\partial p_2}{\partial x_{b,t}} \frac{\partial p_2}{\partial x_{c,t}} + \frac{p_6}{4} \frac{\partial p_1}{\partial x_{a,t}} \frac{\partial p_3}{\partial x_{b,t}} \frac{\partial p_3}{\partial x_{c,t}} \\
&\quad - \frac{p_7}{4} \frac{\partial p_1}{\partial x_{a,t}} \frac{\partial p_4}{\partial x_{b,t}} \frac{\partial p_4}{\partial x_{c,t}} + \frac{p_8}{4} \frac{\partial p_1}{\partial x_{a,t}} \frac{\partial p_5}{\partial x_{b,t}} \frac{\partial p_5}{\partial x_{c,t}} - \frac{p_9}{8} \left( \frac{\partial p_1}{\partial x_{a,t}} \right)^2 \frac{\partial^2 p_1}{\partial^2 x_{b,t}} \frac{\partial^2 p_1}{\partial^2 x_{c,t}} \\
&\quad - \frac{p_10}{n^3} \left( \frac{\partial p_1}{\partial x_{a,t}} \right)^2 - p_1 \left( \frac{\partial^2 p_1}{\partial x_{b,t}} \right)^2 - p_2 \left( \frac{\partial^2 p_1}{\partial x_{c,t}} \right)^2 - p_3 \left( \frac{\partial^2 p_1}{\partial x_{b,t} \partial x_{c,t}} \right)^2 - p_4 \left( \frac{\partial^2 p_1}{\partial x_{a,t} \partial x_{b,t} \partial x_{c,t}} \right)^2.
\end{align*}
\]
4.2 Compute the third order constraints

In step 2, we obtain the third order constraints. Similar to \(30\), we introduce the notation

\[
\mathcal{V}_{a,b,c} = \{ \frac{\partial^hp}{\partial^{h_1}x_{a,t} \partial^{h_2}x_{b,t} \partial^{h_3}x_{c,t}} : h = h_1 + h_2 + h_3 \in \{0, 1, \ldots, 5\}\} 
\]

where \(a, b, c\) are variables taking values in \([n]\). Then \(\mathcal{P}_{3,n} = \bigcup_{a=1}^{n} \bigcup_{b=1}^{n} \bigcup_{c=1}^{n} \mathcal{V}_{a,b,c}\).

The third order integral constraints are [20]:

\[
\mathcal{C}_{3,n} = \{ R_{i,a,b,c}^{(3)} : i = 1, \ldots, 955; a, b, c \in [n] \},
\]

where \(R_{i,a,b,c}^{(3)}\) can be found in [20]. Note that we do not use all the third order constraints in [20].

From Lemma 2.3 we can compute the third order log-concave constraints:

\[
\mathcal{C}_{3,2} = \{ R_1 = -\Delta_{1,1}Q_1, R_2 = -\Delta_{1,2}Q_2, R_3 = \Delta_{2,1}Q_3 \},
\]

where \(Q_1, Q_2 \in \text{Span}_k(M_{4,4})\) and \(Q_3 \in \text{Span}_k(M_{4,2})\). Note that \(\mathcal{C}_{3,2}\) does not contain all the log-concave constraints in Lemma 2.3. The constraints \(\mathcal{C}_{3,2}\) are enough for our purpose in this paper.

For \(n > 2\), we give certain log-concave constraints in a special form, which are needed in the proof procedure in section 4.3. Let \(\nabla_1 p_i = (\frac{\partial p_i}{\partial x_{a,t}}, \frac{\partial p_i}{\partial x_{b,t}}, \frac{\partial p_i}{\partial x_{c,t}})\), \(L_1(p_i) \triangleq p_i H_1(p_i) = \nabla_1^T p_i \nabla_1 p_i\), where

\[
H_1(p_i) = \begin{bmatrix}
\frac{\partial^2 p_i}{\partial x_{a,t}^2} & \frac{\partial^2 p_i}{\partial x_{a,t} \partial x_{b,t}} & \frac{\partial^2 p_i}{\partial x_{a,t} \partial x_{c,t}} \\
\frac{\partial^2 p_i}{\partial x_{b,t} \partial x_{c,t}} & \frac{\partial^2 p_i}{\partial x_{b,t}^2} & \frac{\partial^2 p_i}{\partial x_{b,t} \partial x_{c,t}} \\
\frac{\partial^2 p_i}{\partial x_{c,t}^2} & \frac{\partial^2 p_i}{\partial x_{b,t} \partial x_{c,t}} & \frac{\partial^2 p_i}{\partial x_{c,t}^2}
\end{bmatrix}
\]

and \(\Delta_{i,l}, l = 1, \ldots, L_k\) the kth-order principle minors of \(L_1(p_i)\). Let \(\mathcal{M}_k^*\) be the set of all monomials in \(\mathcal{V}_{a,b,c}\) (defined in [37]) which have degree \(k\) and total order \(k\). We have

\[
\mathcal{C}_{3,n} = \{ -\Delta_{1,1}Q_{1,1}, -\Delta_{1,2}Q_{1,2}, -\Delta_{1,3}Q_{1,3}, \Delta_{2,1}Q_{2,1}, \Delta_{2,3}Q_{2,3} - \Delta_{3,1}Q_{3,1} \}
\]

where \(Q_{1,1} \in \text{Span}_k(\mathcal{M}_k'), Q_{2,3} \in \text{Span}_k(\mathcal{M}_k'), \) and \(Q_{3,1} \in \mathbb{R}\).

4.3 Proof of \(C_{5}(3, 2)\)

The proof follows Procedure 2.12 with \(E_{3,3,3}\) given in [35] and the constraints in [35] and [39] as input.

In Step S1, the new variables are \(M_{4,2}\) and are listed in the lexicographical monomial order:

\[
m_1 = p_1^2 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{2,t} \partial x_{3,t}}, m_2 = p_1^2 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{2,t} \partial x_{4,t}}, m_3 = p_1^2 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{3,t} \partial x_{4,t}}, m_4 = p_1^2 \frac{\partial^3 p}{\partial x_{2,t} \partial x_{3,t} \partial x_{4,t}}
\]

\[
m_5 = p_1^4 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{2,t} \partial x_{4,t}}, m_6 = p_1^4 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{3,t} \partial x_{4,t}}, m_7 = p_1^4 \frac{\partial^3 p}{\partial x_{2,t} \partial x_{3,t} \partial x_{4,t}},
\]

\[
m_8 = p_1^4 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{2,t}^2 \partial x_{3,t}}, m_9 = p_1^4 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{2,t} \partial x_{4,t}^2}, m_{10} = p_1^4 \frac{\partial^3 p}{\partial x_{1,t} \partial x_{3,t} \partial x_{4,t}^2}, m_{11} = p_1^4 \frac{\partial^3 p}{\partial x_{2,t} \partial x_{3,t} \partial x_{4,t}^2}, m_{12} = \left( \frac{\partial p}{\partial x_{3,t}} \right)^3, m_{13} = \left( \frac{\partial p}{\partial x_{4,t}} \right)^3, m_{14} = \left( \frac{\partial p}{\partial x_{1,t}} \right)^3.
\]

In Step S2, the constraints are \(\mathcal{C}_{3,2} = \{ R_{j,a,b,c}^{(3)} : j = 1, \ldots, 955; a, b, c \in \{2\} \}.\) Removing the repeated ones, we have \(N_1 = 135\). We obtain \(\mathcal{C}_{3,2,1}\) and \(\mathcal{C}_{3,2,2}\) which contain 48 and 52 constraints, respectively.

In Step S3, there exist 15 intrinsic constraints:

\[
m_5 m_8 = m_6 m_7, m_5 m_{10} = m_6 m_9, m_5 m_{12} = m_6 m_{11}, m_5 m_{13} = m_6 m_{12}, m_5 m_{14} = m_6 m_{13},
\]

\[
m_7 m_{10} = m_7 m_9, m_7 m_{12} = m_7 m_{11}, m_7 m_{13} = m_7 m_{12}, m_7 m_{14} = m_7 m_{13}, m_7 m_{12} = m_7 m_{11},
\]

\[
m_8 m_{13} = m_8 m_{12}, m_9 m_{14} = m_9 m_{13}, m_{11} m_{13} = m_{12} m_{14}, m_{12} m_{14} = m_{13}.
\]
Thus, $\hat{C}_{3,2,1}$ contains 63 constraints and $N_3 = 63$.

In Step $S4$, we obtain $\hat{C}(3, 2)$ which contains 3 quadratic form constraints.

In Step $S5$, eliminating the non-quadratic monomials in $E_{3,3,2}$ using $C_{3,2,2}$ to obtain a quadratic form in $m_i$ and then simplifying the quadratic form using $C_{3,2,1}$, we have

$$
\hat{E}_{3,3,2} = -\frac{147}{8}m_{14}^2 + \frac{31}{40}m_{14}^2 - \frac{3}{5}m_{14}m_{10} + \frac{15}{4}m_3^2 - \frac{25}{8}m_4^2 - \frac{31}{15}m_9m_{11} + \frac{207}{8}m_9m_{13} - \frac{3}{2}m_7^2 + \frac{1}{2}m_1^2
$$

$$
-\frac{7}{2}m_1m_5 + \frac{7}{4}m_{11}^2 + \frac{7}{2}m_2^2 + \frac{7}{4}m_4m_6 - \frac{5}{4}m_4m_7 + \frac{9}{8}m_3^2 - \frac{7}{4}m_7^2 - \frac{9}{4}m_4m_{10}.
$$

In Step $S6$, using the Matlab program in Appendix A with $\hat{E}_{3,3,2}$, $\hat{C}_{3,2,1}$ and $\hat{C}_{3,2}$ as input, we find an SOS representation for $\hat{E}_{3,3,2}$. Thus, $C_{3}(3,2)$ is proved under the log-concave condition. The Maple program to prove $C_{3}(3,2)$ can be found in [https://github.com/cmyuanmmrc/codeforepi/](https://github.com/cmyuanmmrc/codeforepi/).

**Remark 4.1.** We fail to prove $C_{3}(3,2)$ even under the log-concave condition similar to the above procedure. Specifically, we cannot find an SOS representation for $\hat{E}_{2,3,2}$ in Step $S6$. Since the SDP algorithm is not complete for problem [25], we cannot say that an SOS representation does not exist for $\hat{E}_{2,3,2}$. The Maple program for $C_{2}(3,2)$ can be found in [https://github.com/cmyuanmmrc/codeforepi/](https://github.com/cmyuanmmrc/codeforepi/).

### 4.4 Proof of $C_{3}(3,3)$ and $C_{3}(3,4)$

In this subsection, we want to prove $C_{3}(3,3), C_{3}(3,4)$.

Motivated by symmetric functions, for any function $f(a,b,c)$, we have

$$
\sum_{a,b,c=1}^{n} f(a,b,c) = \sum_{1 \leq a < b < c}^{n} \frac{2}{(n-1)(n-2)} [f(a,a,a) + f(b,b,b) + f(c,c,c)] + \frac{1}{n-2} \sum_{1 \leq a < b < c}^{n} f(a,a,b) + f(a,b,a)
$$

$$
+ f(b,a,a) + f(a,c,a) + f(c,a,a) + f(b,b,a) + f(a,a,b) + f(h,b,c) + f(c,b,c) + f(h,c,c) + f(c,c,b) + f(c,b,c) + f(c,c,c) + f(c,b,c) + f(c,c,c)
$$

$$
+ f(a,a,b) + f(a,c,b) + f(b,c,a) + f(c,a,b) + f(c,a,b) + f(c,b,a).
$$

(41)

From [20] and [11], we obtain

$$
E_{3,3,n} = \sum_{a,b=1}^{n} \sum_{1 \leq a < b < c}^{n} E_{3,a,b,c} = \sum_{1 \leq a < b < c}^{n} J_{3,3,n},
$$

where

$$
J_{3,3,n} = \frac{2}{(n-1)(n-2)} [E_{3,a,a,a} + E_{3,b,b,b} + E_{3,c,c,c}] + \frac{1}{n-2} [E_{3,a,a,b} + E_{3,a,b,a} + E_{3,b,a,a} + E_{3,a,a,c}
$$

$$
+ E_{3,a,c,a} + E_{3,c,a,a} + E_{3,b,b,a} + E_{3,b,b,b} + E_{3,b,b,c} + E_{3,b,c,b} + E_{3,c,b,b} + E_{3,c,b,c} + E_{3,c,c,c} + E_{3,b,c,c} + E_{3,b,c,b}]
$$

(42)

Thus, if we prove $J_{3,3,n} \geq 0$, then $E_{3,3,n} \geq 0$. It is clear that $J_{3,3,n}$ contains much smaller terms than $E_{3,3,n}$.

In $J_{3,3,n}$ given in [12], and the constraints in [38] and [10], we may consider $\frac{\partial}{\partial x_{a,b}}$, $\frac{\partial}{\partial x_{b,c}}$, and $\frac{\partial}{\partial x_{c,t}}$ as the differential operators without giving concrete values to $a, b, c$.

First, we prove of $C_{3}(3,3)$ using Procedure [12] with $J_{3,3,3}$ given in [12] and the constraints in [38] and [10] as the input.

In Step $S1$, the new variables are $\mathcal{M}_3 = \{m_i, i = 1, \ldots, 38\}$.

In Step $S2$, the constraints are: $C_{3,n} = \{R_{r,a,b,c}^{(3)} : i = 1, \ldots, 955\}$, $N_1 = 955$. We obtain $C_{3,n,1}$ and $C_{3,n,2}$, which contain 350 and 328 constraints, respectively.

In Step $S3$, there exist 189 intrinsic constraints. In total, $\hat{C}_{3,n,1}$ contains 539 constraints. Using $\mathbb{R}$-Gaussian elimination in $\text{Span}_R(\hat{C}_{3,n,1})$ shows that 512 of these 539 constraints are linearly independent, so $N_3 = 512$. 


In Step S4, we obtain \( \hat{C}_{3,n} \) from \( C_{3,n} \) which contains 6 constraints.

In Step S5, eliminating the non-quadratic monomials in \( J_{3,3,3} \) using \( C_{3,n,2} \) and then simplify the expression using \( C_{3,n,1} \), we have

\[
\hat{J}_{3,3,3} = \frac{31}{5} m_2^3 + \frac{29}{2} m_2^2 + \frac{88}{3} m_2 m_1^2 - \frac{20}{3} m_2 m_3 - \frac{17}{2} m_2 m_4 + \frac{88}{3} m_2 m_{34} + \frac{202}{9} m_2 m_{32} - \frac{145}{4} m_2 - \frac{29}{4} m_2^4 + \frac{17}{2} m_3^2 + \frac{88}{3} m_3^2 m_4 + \frac{202}{9} m_3^2 m_{32} - \frac{145}{4} m_3^2 - \frac{29}{4} m_3^4 + \frac{17}{2} m_4^2 + \frac{88}{3} m_4^2 m_3 - \frac{20}{3} m_4 m_2 - \frac{17}{2} m_4 m_1 + \frac{88}{3} m_4 m_{34} - \frac{202}{9} m_4 m_{32} + \frac{145}{4} m_4^2 - \frac{29}{4} m_4^4.
\]

In Step S6, using the Matlab program in Appendix A with \( \hat{J}_{3,3,3}, \hat{C}_{3,n,1} \) and \( \hat{C}_{3,n} \) as input, we find an SOS representation for \( \hat{J}_{3,3,3} \). Thus, \( C_{3}(3,3) \) is proved. The Maple program to prove \( C_{3}(3,3) \) can be found in https://github.com/cmyuanmmrc/codeforepi/.

To prove \( C_{3}(3,4) \), we just need to replace the input from \( J_{3,3,3} \) to \( J_{3,3,4} \) in the Step S5 in the above procedure. In the same way, \( C_{3}(3,4) \) can be proved. The Maple program to prove \( C_{3}(3,4) \) can be found in https://github.com/cmyuanmmrc/codeforepi/.

### 5 Proof of \( C_{3}(4,2) \)

We use the procedure in section 2.2 to prove \( C_{3}(4,2) \) under the log-concave condition.

In step 1, we compute \( E_{3,4,n} \) in (4) and (47):

\[
\frac{1}{2} d^3 \left( \| \nabla p_t \|^2 - p_t \| \frac{\nabla^2 p_t}{p_t} \| \right)^4 \int_{R^n} E_{3,4,n} \frac{dX_i}{p_t^4}
\]

where \( E_{3,4,n} = \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{c=1}^{n} \sum_{d=1}^{n} E_{4,a,b,c,d} \). For brevity, we omit the concrete expression of \( E_{4,a,b,c,d} \).

In step 2, based on Lemma 2.3, we obtain 589 fourth order constraints

\[
C_{4,2} = \{ R_{i,1,2}^{(2)} : i = 1, \ldots, 589 \} \subseteq R[P_2] \quad \text{and} \quad N_1 = 589.
\]

By Lemma 2.5, we obtain three 4th-order log-concave constraints:

\[
C_{4,2} = \{ -\Delta_{1,1} Q_{1,1}, -\Delta_{1,2} Q_{1,2}, \Delta_{2,1} Q_{2,1} \}
\]

where \( Q_{1,1}, Q_{1,2} \in \text{Span}_{R}(M_{6,2}) \) and \( Q_{2,1} \in \text{Span}_{R}(M_{4,2}) \).

In step 3, we use Procedure 2.6 to compute the SOS representation (2) and (18) with \( E_{3,4,n}, C_{4,2}, \) and \( \hat{C}_{3} \) as input.

In step S1, the new variables are \( M_{4,2} = \{ m_i, i = 1, \ldots, 33 \} \).

In step S2, using Gaussian elimination to \( C_{4,2} = \{ R_{i,1,2}^{(2)} : i = 1, \ldots, 589 \} \), we obtain \( C_{4,2,1} \) and \( C_{4,2,2} \) which contain 266 and 182 constraints, respectively.

In step S3, there exist 182 intrinsic constraints. Thus, \( \hat{C}_{4,2,1} \) contains 448 constraints. Using \( R \)-Gaussian elimination in \( \text{Span}_{R}(\hat{C}_{4,2,1}) \) shows that 417 of these 448 constraints are linearly independent, so \( N_2 = 417 \).

In step S4, we obtain \( \hat{C}(4,2) \) which contain 3 log-concave constraints, so \( N_2 = 3 \).

In step S5, eliminating the non-quadratic monomials in \( E_{3,4,2} \) using \( C_{4,2,2} \) to obtain a quadratic form in
In this paper, two conjectures concerning the lower bounds for the derivatives of $H(X_t)$ are considered. We first consider a conjecture of McKean $C_2(m, n) : (-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_t) \geq (-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_{G(t)})$ in the multivariate case. We propose a general procedure to prove inequalities similar to $C_2(m, n)$. Using the procedure, we prove $C_2(1, n)$ and $C_2(2, n)$. We notice that $C_2(m, n)$ cannot be proved for $m > 2$ and $n > 1$ with the procedure even under the log-concave condition, which motivates us to propose the following weaker conjecture $C_3(m, n) : (-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_t) \geq (-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_{G(t)})$. Using our procedure, we prove $C_3(3, 2), C_3(3, 3), C_3(3, 4)$ and $C_3(4, 2)$ under the log-concave condition.

From $C_2(1, n)$ and $C_2(2, n)$ proved in this paper, the exact lower bounds for $(-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_t)$ are $(-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_{G(t)})$ for $m = 1$ and 2, respectively. The high order cases are widely open and we give a brief summary of the known results below.

First consider the univariate case ($n = 1$). $C_1(3, 1)$ and $C_1(4, 1)$ were true [19] and $C_1(5, 1)$ cannot be proved with the SDP approach [17] [20]. $C_2(3, 1), C_2(4, 1)$, and $C_2(5, 1)$ were true under the log-concave condition [17]. $C_2(6, 1)$ is considered in this paper. The software in Appendix A shows that $E_{2,6,1}$ $\geq 0$ under the log-concave condition. However, due to the accuracy of the SDP solver, we cannot find an explicit SOS representation. So if the SDP software is correct, $C_2(6, 1)$ is proved under the log-concave condition. From these results, a reasonable target is to prove $C_1(m, 1)$ or $C_2(m, 1)$ under the log-concave condition.

For the multivariate case, $C_1(3, 2), C_1(3, 3), C_1(3, 4)$ were true and $C_1(4, 2)$ cannot be proved with the SDP approach [20]. In this paper, $C_3(3, 2), C_3(3, 3), C_3(3, 4)$, and $C_3(4, 2)$ were proved under the log-concave condition, and $C_3(3, 2), C_3(3, 3), C_3(3, 4)$, and $C_3(4, 2)$ cannot be proved with the SDP approach under the log-concave condition. From these results, a guess for the lower bound is $(-1)^{m+1} \left( \frac{d^m}{dt^m} \right) H(X_t) \geq (-1)^{m+1} A(n) \left( \frac{d^m}{dt^m} \right) H(X_{G(t)})$, where $A(n)$ is a function in $n$ such that $0 \leq A(n) \leq 1$.

In order to use the SDP approach to prove more difficult problems such as $C_3(3, n)(n > 4)$ and $C_3(3, n)(n > 4)$ under the log-concave condition, two kinds of improvements are needed. First, it is easy to see that the size of $E_{x}(m, n)$ and the numbers of the constraints increase exponentially as $m$ and $n$ becomes larger. Thus, we need to find certain rules which could be used to simplify the computation. Second, in many cases, such as $C_2(3, 2)$ under the log-concave constraint, the SDP program terminates and gives a negative answer. Since the SDP method is not complete for our problem, we do not know whether an SOS representation exists. We thus need a complete method to solve problem [12]. Another problem is to find more constraints besides those used in this paper in order to increase the power of the approach.

In this paper, when we say $C_3(3, n)$ cannot be proved with the SDP approach, we mean that the software in Appendix A terminates and gives a negative answer for problem [28].
Appendix A. Sum of square of quadratic forms based on SDP

We first restate the problem. Let \( f = \hat{f}_{m,n}, \) \( g_i = \hat{g}_i, i = 1, \ldots, N_3, \) \( R_j = \sum_{l=1}^{V_j} q_{j,l} h_{j,l}, j = 1, \ldots, N_2, \) where \( f, g_i, h_{j,s} \) are quadratic forms in \( \mathbb{R}[\{M_{m,n}\}] \) and \( q_{j,l} \) are variables to be determined. For simplicity, let \( x = M_{m,n} = \{x_1, \ldots, x_u\}, U = N_3, V = N_2. \) \( Q_j = \sum_{l=1}^{V_j} q_{j,l} m_{j,l}, \) where \( q_{j,l} \) are variables to be found and \( m_{j,s} \in M_{2d_j,n} \) are monomials in \( \mathbb{R}[\{P_n\}] \) of degree \( 2d_j \) and total order \( \sum_{i=1}^{N_2} d_i. \) We need to compute \( \pi, q_{j,l} \in \mathbb{R} \) such that

\[
\begin{align*}
\sum_{i=1}^{U} p_i g_i - \sum_{j=1}^{V_j} q_{j,l} h_{j,l} &= S, \\
Q_j &= \sum_{l=1}^{V_j} q_{j,l} m_{j,l} \geq 0, j = 1, \ldots, V
\end{align*}
\]

where \( S = \sum_{i=1}^{u} c_i (\sum_{j=1}^{u} e_{i,j} x_j)^2 \) is an SOS, \( c_i, e_{i,j} \in \mathbb{R} \) and \( c_i \geq 0. \)

We first reduce \( Q_j \) into quadratic forms. Let \( y_j = M_{d_j,n} = \{y_{j,1}, \ldots, y_{j,w_j}\}. \) Write the monomials of \( Q_j \) as quadratic monomials in \( y_{j,s}, \) and still denote results as \( Q_j. \) If \( Q_j \) is not a quadratic form in \( y_{j,s}, \) then just set the coefficients of those non-quadratic monomials to zero. Then, constraint (46) becomes

\[
Q_j = \sum_{l=1}^{V_j} q_{j,l} t_{j,l} \geq 0, j = 1, \ldots, N_2
\]

where each \( t_{j,l} \) is a quadratic monomial in \( y_{j,k}. \)

A polynomial \( f \) in \( \mathbb{R}[x] \) is called **positive semidefinite** and is denoted as \( f \geq 0, \) if \( \forall \hat{x} \in \mathbb{R}^u, f(\hat{x}) \geq 0. \)

**Lemma B.** Let \( f \in \mathbb{Q}[x] \) be a quadratic form. Then \( f \geq 0 \) if and only if

\[
f = \sum_{i=1}^{u} c_i (\sum_{j=1}^{u} e_{i,j} x_j)^2,
\]

where \( c_i, e_{i,j} \in \mathbb{Q}, c_i \geq 0, \) and \( e_{i,j} \neq 0 \) if \( c_i \neq 0, \) for \( i = 1, \ldots, u \) and \( j = i, \ldots, u. \)

Based on Lemma B, problem (45) is equivalent to the following problem.

\[
\begin{align*}
\exists \pi, \quad q_{j,l} \in \mathbb{R}, \text{ s.t.} \\
\sum_{i=1}^{U} p_i g_i - \sum_{j=1}^{V_j} q_{j,l} h_{j,l} &= S, \\
Q_j &= \sum_{l=1}^{V_j} q_{j,l} m_{j,l} \geq 0, j = 1, \ldots, N_2
\end{align*}
\]

Problem (49) can be solved with semidefinite programming (SDP). For details of SDP, please refer to \[22, 23\]. A symmetric matrix \( M \in \mathbb{R}^{n \times n} \) is called **positive semidefinite** and is denoted as \( M \succeq 0, \) if all of its eigenvalues are nonnegative. Rewrite

\[
f(x) = x C x^T, \quad g_i(x) = x A_{i} x^T, \quad h_{j,l}(x) = x A_{j,l} x^T, \quad t_{j,l} = y_{j,l} B_{j,l} y_{j,l}^T, \quad j = 1, \ldots, V_j.
\]

where \( C, A_{j,i} \) are \( u \times u \) real symmetric matrices, \( B_{j,i} \) is \( w_j \times w_j \) real symmetric matrix. Then, problem (49) is equivalent to the following SDP problem:

\[
\begin{align*}
\min_{\pi, q_{j,l} \in \mathbb{R}} & \quad 0 \\
\text{subject to} & \quad C - \sum_{i=1}^{U} p_i A_i - \sum_{j=1}^{V_j} \sum_{l=1}^{V_j} q_{j,l} A_{j,l} \succeq 0, \\
& \quad \sum_{l=1}^{V_j} q_{j,l} B_{j,l} \succeq 0, j = 1, \ldots, V
\end{align*}
\]
The dual of problem (50) is

\[
\begin{align*}
\min_{X \in \mathbb{R}^{n \times u}, X_{j,l} \in \mathbb{R}^{n_j \times n_l}} & \quad \langle X, C \rangle \\
\text{subject to} & \quad \langle X, A_i \rangle = 0, \ i = 1, 2, \ldots, U \\
& \quad \langle X, A_{j,l} \rangle - \langle X_{j,l}, B_{j,l} \rangle = 0, \ j = 1, \ldots, V, s = 1, \ldots, V_j.
\end{align*}
\]  

(51)

where \(X_1\) and \(X_{j,l}\) are symmetric matrices and \(\langle \cdot \rangle\) is the inner product by treating matrices as vectors.

Problem (51) can be solved with the following Matlab program which computes \(P = (p_1, \ldots, p_U, q_1, \ldots, q_{V,V})\) with \(C, A_{j,l}\) and \(B_{j,l}\) as the input. This program uses the CVX package in Matlab \[24\] to solve SDPs.

```matlab
cvx_begin
variable X(u,u) Xjs(w_j,w_j), j=1,...,V, s=1,...,V_j symmetric
dual variable P
minimize(trace(C*X))
subject to [trace(A_1*X),trace(A_i*X), i=1,...,U,
trace(A_js*X-B_js*Xjs), j=1,...,V, s=1,...,V_j,
zeros(r,1): P;
X == semidefinite(n);
X_\{js\} == semidefinite(n_js); j=1,...,V, s=1,...,V_j
cvx_end
```

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