SUPERPOSITION OF A STATIC PERFECT FLUID AND A
RADIAL ELECTRIC FIELD*

Mauricio Cataldo†
Departamento de Física, Facultad de Ciencias, Universidad del Bío-Bío, Concepción, Chile

Patricio Salgado‡
Departamento de Física, Facultad de Ciencias, Universidad de Concepción, Concepción, Chile

Abstract

We obtain a two-parameter set of solutions, which represents a spherically symmetric space-time with a superposition of a neutral fluid and an electric field. The electromagnetic four-potential of this Einstein-Maxwell space-time is taken in the form $A = q r^n dt$, when $n \neq 0$ and $A = q \ln r dt$, when $n=0$ (where $q$ and $n$ are arbitrary constants). Some particular solutions obtainable from the general solution are presented.

I. INTRODUCTION

In this paper we formulate an approach that gives us a procedure to write down in an easy way the Einstein-Maxwell equations and solve them for a metric with certain symmetries. This procedure is formulated on the basis of a generalization of the approach to generation of Einstein-Maxwell fields proposed by J. Horský and N.V. Mitskievitch [1] and later developed in [2–7], where Einstein-Maxwell exact solutions are generated from a seed isometric

*gr-qc/9605040
†e-mail: mcataldo@zeus.dci.ubiobio.cl
‡e-mail: psalgado@halcon.dpi.udec.cl
gravitational field. In particular, in [1] is formulated a conjecture about the connection between isometries of vacuum space-time and the existence of corresponding space-times with electromagnetic fields for which the electromagnetic four-potential is proportional to a Killing vector of the vacuum seed metric. Some electrovacuum solutions are obtained in [2, 7]. The new method of generation of electrovacuum solutions was later generalized to the case of non-vacuum seed solutions. Some superpositions of electrically neutral stiff matter with an electric field are obtained in [3, 4]. In this case the four-potential must be proportional to a Killing vector which is orthogonal to the four-velocity of the neutral fluid in the seed gravitational field. Self-consistent fields with a charged perfect fluid are generated in [5, 6]. We will show that one can leave out the seed solution and consider only the chosen form of the self-consistent line element and four-potential, from which we obtain all the required parameters for the electromagnetic field.

On the other hand, we apply our approach to the construction of a superposition of a radial electric field and a static neutral perfect fluid, based on the idea of the re-interpretation of the exterior electrovacuum solution of Kottler-Reissner-Nordström [8, 6] as a superposition of a coulombian type electric field and a static perfect fluid with \( \mu = -p = \Lambda/\chi \), where \( \Lambda \) is the cosmological constant, \( \mu \) and \( p \) are the energy density and pressure of the fluid, respectively. In Sec. II we discuss the main ideas of the construction of the above mentioned superposition for the space-time with the most general static spherically symmetric line element. In Sec. III we obtain our two-parameter set of metrics for the electromagnetic four-potential determined by (9). Finally in Sec. IV some particular solutions attainable from the general self-consistent field are considered.

II. AN APPROACH FOR THE CONSTRUCTION OF SELF-CONSISTENT SPHERICALLY SYMMETRIC SOLUTIONS

The metric for an arbitrary static, spherically symmetric space-time can be taken in the form
\[
\begin{align*}
\frac{ds^2}{d^2} &= e^{2\alpha(r)} dt^2 - e^{2\beta(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2).
\end{align*}
\]

To write Einstein’s equations we will use the tetrad formalism and Cartan structure equations [9]. A convenient orthonormal basis for the metric in (1) is

\[
\begin{align*}
\theta^{(0)} &= e^{\alpha} dt, \quad \theta^{(1)} = e^{\beta} dr, \quad \theta^{(2)} = r \, d\theta, \quad \theta^{(3)} = r \sin \theta \, d\varphi.
\end{align*}
\]

The stress-energy tensor for a perfect fluid, which fills the space-time, for our signature is defined by

\[
T_{(\mu)(\nu)} \theta^{(\mu)} \otimes \theta^{(\nu)} = \left[ (\mu + \nu)U_{(\mu)}U_{(\nu)} - pg_{(\mu)(\nu)} \right] \times \theta^{(\mu)} \otimes \theta^{(\nu)},
\]

where \( \mu \) and \( p \) are the mass-energy density and the pressure of the fluid, respectively. \( U_{(\mu)} \) is its timelike 4-velocity. If we take the four velocity \( U = \theta^{(0)} \), then (3) becomes

\[
T_{(\mu)(\nu)}^{P.F.} = \mu \theta^{(0)} \theta^{(0)} + p \left( \theta^{(1)} \theta^{(1)} + \theta^{(2)} \theta^{(2)} + \theta^{(3)} \theta^{(3)} \right).
\]

Now, to construct Einstein-Maxwell fields, we must consider Maxwell’s equations and stress-energy tensor of the electromagnetic field. This, with respect to Eq. (2), is defined by

\[
T_{(\mu)(\nu)} \theta^{(\mu)} \otimes \theta^{(\nu)} = -\frac{1}{4\pi} \left[ F_{(\mu)(\gamma)} F_{(\nu)}^{(\gamma)} - \frac{1}{4} g_{(\mu)(\nu)} F^{(\gamma)(\delta)} F_{(\gamma)(\delta)} \right] \theta^{(\mu)} \otimes \theta^{(\nu)}.
\]

To get its components, of course we must compute \( F_{(\mu)(\nu)} \). The general form for a Maxwell tensor which shares the static and spherical symmetries of the space-time is, [10],

\[
F_{(\mu)(\nu)} = 2B(r) \delta_{(\mu)}^{(\nu)} + 2C(r) \delta_{(\mu)}^{\theta} \delta_{(\nu)}^{\varphi}.
\]

However, our goal in this paper is to obtain superpositions of an electromagnetic field and a neutral perfect fluid. To do this, we must consider the source-free Maxwell’s equations for the self-consistent problem. This means that one can set \( C=0 \) in Eq. (3) since all the cases with an electric field without sources can be reformulated to the corresponding cases with a magnetic field (dual to the initial electric one) or mixtures of both fields (duality rotation). The electromagnetic stress-energy tensor is the same in all these cases.
In general, in the coordinate basis, the Maxwell tensor can be expressed in terms of a four-potential covector $A$ in the following way:

$$dA = F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$  

(7)

For the considered Maxwell tensor (with $C=0$) one can take the 4-potential in the form

$$A = q f(r) dt,$$  

(8)

where $f(r)$ is an arbitrary function of the $r$ coordinate, and $q$ is a constant coefficient which is introduced for further use in the self-consistent solution (for switching off the electric field).

Lastly, we must note that with the help of Eq. (8) the source-free equations are replaced by an algebraic condition which considerably simplifies the solution of the self-consistent equations.

III. SPHERICALLY SYMMETRIC SPACE-TIME WITH A SUPERPOSITION OF NEUTRAL FLUID AND ELECTRIC FIELD

According to the expression (8), we choose the electromagnetic potential in the form

$$A = \begin{cases} \frac{q}{n} r^n dt & \text{if } n \neq 0 \\ q \ln r dt & \text{if } n = 0 \end{cases}$$  

(9)

where $n$ is an arbitrary constant. Then

$$dA = qr^{(n-1)} dr \wedge dt,$$  

(10)

and the Maxwell tensor $F_{\mu\nu}$, in the coordinate basis, has the form

$$F_{\mu\nu} = 2q r^{(n-1)} \delta_{[\mu}^r \delta_{\nu]}^t,$$  

(11)

or in the basis (2)

$$F_{(\mu)(\nu)} = qr^{(n-1)} e^{-\alpha-\beta} \delta_{[\mu}^1 \delta_{\nu]}^0.$$  

(12)
The contravariant density components of (11) are

$$\sqrt{-g} F^{\mu\nu} = q r^{(n-1)} \sin \theta \ e^{-\alpha-\beta} \delta^{[\mu}_{\nu}]$$,

where $$\sqrt{-g} = r^2 e^{\alpha \beta} \sin \theta$$. It is clear that the source-free Maxwell’s equations

$$(\sqrt{-g} F^{\mu\nu})_{,\nu} = 0$$

are satisfied if

$$r^{n+1} = Ae^{\alpha+\beta}$$,

where $$A$$ is a constant of integration. Without any loss of generality, by re-scaling the $$r$$ coordinate, we can set $$A=1$$. In the particular case of an electric field in the space-time (I), we have locally that $$F = -E_r dt \wedge dr$$ and we obtain from Eq. (10) that the radial electric field has the form

$$E_r = q r^{n-1}$$.

Now let us write Einstein’s equations. By using Cartan exterior forms calculus one can obtain from the tetrad (2) the next non-trivial components of the curvature tensor:

$$R^{(0)}_{(1)(1)(0)} = e^{-2\beta} (\alpha'' + \alpha'' - \alpha' \beta')$$,

$$R^{(0)}_{(2)(2)(0)} = R^{(0)}_{(3)(3)(0)} = \frac{\alpha'}{r} e^{-2\beta}$$,

$$R^{(1)}_{(2)(1)(2)} = R^{(1)}_{(3)(1)(3)} = \frac{\beta'}{r} e^{-2\beta}$$,

$$R^{(2)}_{(3)(2)(3)} = \frac{1}{r^2} (1 - e^{-2\beta})$$;

where the differentiation with respect to $$r$$ is denoted by $$'$$.

The stress-energy tensor of the electromagnetic field can be obtained using the definition (5). The electromagnetic field invariant

$$F_{(\mu)(\nu)} F^{(\mu)(\nu)} = -2q^2 r^{2(n-1)} e^{-2\alpha-2\beta}$$

is negative. This means that we have an electric-type field according to Eq. (15). Then
\[ T^{E,F}_{(\mu)(\nu)} \theta^{(\mu)} \otimes \theta^{(\nu)} = \frac{q^2 r^{2(n-1)} e^{-2\alpha - 2\beta}}{8\pi} \times [\theta^{(0)} \otimes \theta^{(0)} - \theta^{(1)} \otimes \theta^{(1)} + \theta^{(2)} \otimes \theta^{(2)} + \theta^{(3)} \theta^{(3)}]. \] (20)

Thus, from equations (14), (16) – (19) and (20) we obtain the non-trivial components of Einstein’s equations

\[ \frac{1}{r^2} - 2 \frac{\beta'}{r} - \frac{e^{2\beta}}{r^2} = - \frac{\chi q^2 r^{2(n-1)} e^{-2\alpha}}{8\pi} - \chi \mu e^{2\beta}, \] (21)

\[ \frac{e^{2\beta}}{r^2} - \frac{1}{r^2} - 2 \frac{\alpha'}{r} = - \frac{\chi q^2 r^{2(n-1)} e^{-2\alpha}}{8\pi} - \chi \mu e^{2\beta}, \] (22)

\[ \alpha' \beta' - \alpha'^2 - \alpha'' - \alpha' + \beta' = - \frac{\chi q^2 r^{2(n-1)} e^{-2\alpha}}{8\pi} - \chi \mu e^{2\beta}. \] (23)

This implies that the self-consistent system consists of equations (14), (21), (22) and (23).

By subtracting equations (22) and (23) and taking into account Eq. (14) (with \( A = 1 \)), we obtain

\[ (e^{2\alpha})'' - (n + 1) \frac{(e^{2\alpha})'}{r} - 2(n + 2) \frac{e^{2\alpha}}{r^2} = \frac{\chi q^2}{2\pi} r^{2(n-1)} - 2r^{2n}, \] (24)

which implies

\[ e^{2\alpha} = \frac{Q^2 r^{2n}}{n^2 - 3n - 2} + \frac{r^{2(n+1)}}{2 - n^2} + Ar^{I_1(n)} + Br^{I_2(n)}, \] (25)

where

\[ I_1(n) = \frac{1}{2} \left( n + 2 + \sqrt{(n + 2)(n + 10)} \right), \] (26)

\[ I_2(n) = \frac{1}{2} \left( n + 2 - \sqrt{(n + 2)(n + 10)} \right). \] (27)

\( A \) and \( B \) are constants of integration and \( Q^2 = \chi q^2 / 4\pi \). From equations (24), (27) it follows that the possible range for values of \( n \) is \( n \leq -10 \) or \( n \geq -2 \) (for nonvanishing constants \( A \) and \( B \)). On the other hand, equation (14) yields

\[ e^{2\beta} = \frac{r^{2(n+1)}}{n^2 - 3n - 2} + \frac{r^{2(n+1)}}{2 - n^2} + Ar^{I_1(n)} + Br^{I_2(n)}. \] (28)
Now substituting the functions $\alpha$ and $\beta$ in equations (21) and (22), and performing a few algebraic manipulations, we find that for the energy density of the fluid,

$$\chi_\mu(r) = \frac{Q^2(n + 1)(4 - n)}{2(n^2 - 3n - 2)r^4} + \frac{n^2 - 1}{(2 - n^2)r^2} + r^{-2(n+2)}$$  \hspace{2cm} (29)$$

$$\times \left\{ A(2n + 1 - I_1)r^{I_1(n)} + B(2n + 1 - I_2)r^{I_2(n)} \right\} ,$$

and for its pressure,

$$\chi_p(r) = \frac{Q^2n(n + 1)}{2(n^2 - 3n - 2)r^4} + \frac{(n + 1)^2}{(2 - n^2)r^2} + r^{-2(n+2)} \times \left\{ A(I_1 + 1)r^{I_1(n)} + B(I_2 + 1)r^{I_2(n)} \right\} .$$  \hspace{2cm} (30)$$

Thus we arrive at the space-time metric

$$ds^2 = e^{2\alpha}dt^2 - r^{2(n+1)}e^{-2\alpha}dr^2 - r^2(\theta^2 + \sin^2\theta d\phi^2) ,$$  \hspace{2cm} (31)$$

where $e^{2\alpha}$ is given by Eq. (25).

**IV. LIMITING CASES AND THE WEYL TENSOR FOR THE FOUND SPACE-TIME**

First we will consider the case when the electric field is switched off. This means that $q = 0$ in our solution. Thus we obtain an analytical solution which depends on an arbitrary parameter $n$ for the case where the only source of the gravitational field is a perfect fluid, and the geometry is static and possesses spherical symmetry. When $n = -1$ we have the Kottler solution [11], and when $A = 0$ the Schwarzschild geometry [12]. Now, if also $A = B = 0$, the solution becomes

$$ds^2 = r^{2(n+1)}dt^2 - (2 - n^2)dr^2 - r^2(\theta^2 + \sin^2\theta d\phi^2) ,$$  \hspace{2cm} (32)$$

with
\[ \chi \mu = \frac{1 - n^2}{(2 - n^2)r^2} \]  
(33)

and

\[ \chi p = \frac{(n + 1)^2}{(2 - n^2)r^2}. \]  
(34)

If the fluid obeys a \( \gamma \)-law equation of state, i.e., its \( \mu \) and \( p \) are related by an equation of the form

\[ p = (\gamma - 1)\mu, \]  
(35)

where \( \gamma \) is a constant (which, for physical reasons [13] satisfies the inequality \( 1 \leq \gamma \leq 2 \)), the constant \( \gamma \) may be expressed as

\[ \gamma = \frac{2}{1 - n}. \]  
(36)

Therefore, the possible range of values of \( n \) (when \( A = B = Q = 0 \)) is \(-1 \leq n \leq 0\). In the present case, when \( n = -1/2 \) we get exactly the Klein metric for a static spherically symmetric distribution of incoherent radiation [6,14], and when \( n = 0 \) a space-time filled with stiff matter.

Returning to the case where the electric field is present, we let \( n = -1 \), so that our solution represents the well known exterior electrovacuum of charged Kottler spacetime [3]. Moreover, if \( A = 0 \) we obtain the Reissner-Nordström solution [15].

When \( n = -2 \) the static equation can be written as \( \mu + 3p = 0 \), and when \( n = -\frac{1}{2} \) we obtain a generalization of the Klein metric where a superposition of an electric field and a perfect fluid is present. Another generalization of the Klein metric has been obtained in [3], where the static perfect fluid is charged.

On the other hand, the only curvature singularity of the metric (31) is located at \( r = 0 \) and the horizons at \( e^{2\alpha} = 0 \), forming purely coordinate singularities.

Now, let us consider a special case of the found metric setting \( A = B = 0 \). Then
\[ ds^2 = \left( \frac{Q^2 r^{2n}}{n^2 - 3n - 2} + \frac{r^{2(n+1)}}{2 - n^2} \right) dt^2 - r^{2(n+1)} \left( \frac{Q^2 r^{2n}}{n^2 - 3n - 2} + \frac{r^{2(n+1)}}{2 - n^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \]  

(37)

\[ r^{2n} \left( \frac{Q^2 r^{2n}}{n^2 - 3n - 2} + \frac{r^{2(n+1)}}{2 - n^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \]

\[ \chi_{\mu} = \frac{Q^2 (n + 1)(4 - n)}{2(n^2 - 3n - 2)r^4} + \frac{1 - n^2}{(2 - n^2)r^2}, \]  

(38)

\[ \chi_{\rho} = \frac{Q^2 n(n + 1)}{(n^2 - 3n - 2)r^4} + \frac{(n + 1)^2}{(2 - n^2)r^2}, \]  

(39)

In this case we can characterize the behaviour of the fluid as follows. At

\[ r_1 = \sqrt{\frac{Q^2 (n^2 - 2)(n - 1)}{(n^2 - 3n - 2)(2n + 1)}} \]

as an incoherent radiation \((\mu = 3\rho)\), at

\[ r_2 = \sqrt{\frac{Q^2 (2 - n^2)(2 - n)}{2n(n^2 - 3n - 2)}} \]

as a stiff matter \((\mu = \rho)\), at

\[ r_3 = \sqrt{\frac{Q^2 n(n^2 - 2)}{2(n^2 - 3n - 2)(n + 1)}} \]

as a dust, and at

\[ r_4 = \sqrt{\frac{Q^2 (n^2 - 2)}{n^2 - 3n - 2}} \]

as an electrified de Sitter space-time \((\mu = -\rho)\). The latter spherical surface is the horizon for the resulting static spacetime \((37)\). For their existence, it must be true that all of the values of the spherical surfaces have to be positive for a determined value of \(n\). Thus, for example, the horizon exists only if \(\frac{3 - \sqrt{17}}{2} < n < \sqrt{2}\) or \(n > \frac{3 + \sqrt{17}}{2}\) or \(n < -\sqrt{2}\).

Lastly, we find that the Weyl coefficients \(\psi_\alpha\) for the general solution \((31)\), with respect to the complex null tetrad
\[ \Theta^{(0)} = \sqrt{\frac{1}{2}} (\theta^{(0)} - \theta^{(1)}), \]
\[ \Theta^{(1)} = \sqrt{\frac{1}{2}} (\theta^{(0)} + \theta^{(1)}), \]
\[ \Theta^{(2)} = \Theta^3 = \sqrt{\frac{1}{2}} (\theta^{(0)} + \theta^{(1)}), \]

are

\[ \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0 \]

\[ \psi_2 = -\frac{Q^2}{6r^4} + \frac{(n^2 - 1)}{3(n^2 - 2)r^2} - \]
\[ \frac{A}{12} I_1(n) \{ I_1(n) + 3n + 4 \} r^{(I_1(n) - 2n - 4)} - \]
\[ \frac{B}{12} I_2(n) \{ I_2(n) + 3n + 4 \} r^{(I_2(n) - 2n - 4)}. \]

We see that in general, as is well known for any static spherically symmetric space-time, the solution (31) belongs to Petrov type D, degenerating locally to the conformally flat case (type 0) when \( \psi_2 = 0 \). For the special case (37), type 0 takes place for the spherical surface

\[ r = \sqrt{\frac{Q^2(n^2 - 2)}{2(n^2 - 1)}} \]

and asymptotically when \( r \to \infty \).

**CONCLUSIONS**

From some properties of static, spherically symmetric space-times, it has been shown that one can choose the electromagnetic four-potential in the form (3). This choice allows one to construct the two-parameter set of metrics (31), which are solutions of the Einstein-Maxwell equations. The results obtained in Sec. III lead us to conclude that the obtained self-consistent space-time describes a field, where is a point charge located at the origin \( r = 0 \). It is important to remark that the general solution (31) is valid only outside of the electric source since are considered the sourceless Maxwell equations. In fact
we have a neutral fluid and an electric field given by (15). This leads us to think that the electric field can be interpreted as a superposition of a spherically symmetric coulomb field (external electromagnetic field) and another field caused by “polarization” of the fluid, given by \( E_f = \frac{q}{r^2} (r^{n+1} - 1) \) which vanishes when \( n = -1 \). This is in accordance with the found solution since, if \( n = -1 \), the fluid is “switched off” and we obtain the Kottler-Reissner-Nordström electrovacuum solution. Also, if \( A = 0 \), we obtain the charged black hole (Reissner-Nordström solution).

Finally we can say that the self-consistent solution depends on the function \( f(r) \) which appears in (8) for the electromagnetic potential. Also setting \( q = 0 \) one obtains an exact solution for Einstein’s equations.

\section*{ACKNOWLEDGMENTS}

We would like to thank the referee for his helpful comments. We also want to thank P. Minning and H. Miranda for their help in the redaction of the manuscript.

This work was supported in part by Dirección de Promoción y Desarrollo de la Universidad del Bío-Bío through Grants # 942305-1C, 951105-1 and in part by Dirección de Investigación de la Universidad de Concepción through Grant P.I. # 94.11.09-1.
REFERENCES

[1] J.Horský and N.V.Mitskievitch, *Czech. J. Phys.* **B39**, 957 (1989).

[2] M.Cataldo, J.Horský and N.V.Mitskievitch, Differential Geometry and its Applications. ed. J.Janyska and D.Krupka (Singapore:World Scientific,1990) 297.

[3] M.Cataldo and N.V.Mitskiévic, *J.Math.Phys.* **31**, 2425 (1990).

[4] J.Horský and N.V.Mitskievitch, *Class. Quantum Grav.* **7**, 1523 (1990).

[5] N.V.Mitskiévič and G.Tsalakou, *Class. Quantum Grav.* **8**, 209 (1991).

[6] M.Cataldo and N.V.Mitskiévič, *Class. Quantum Grav.* **9**, 454 (1992).

[7] M.Cataldo, K.K. Kumaraadtya and N.V. Mitskievich, *Gen. Rel. Grav.* **26**, 847 (1994).

[8] D.Kramer, H.Stephani, M.MacCallum, and E. Herlt: *Exact Solutions of Einstein’s Field Equations* (Berlin:DVW, 1980).

[9] W.Israel: *Differential Forms in General Relativity* (Communications of the Dublin Institute for Advanced Studies, 1970).

[10] R.Wald: *General Relativity* (The University of Chicago Press, 1984).

[11] F.Kottler, *Annalen Physik* Vol. **56**, 410 (1918).

[12] K.Schwarzchild, *Sitzungsber. Dtsch. Akad. Wiss. Berlin Kl. Math. Phys. Techn.*, 189 (1916).

[13] C.B. Collins, *J.Math.Phys.* **26**, 2268 (1985).

[14] O.Klein, *Ark.Mat.Astr.Fys.* **A34**, 1 (1947).

[15] H.Reissner, *Annalen Physik* **50**, 106 (1916), G.Nordström, *Proc. Kon. Ned. Akad. Wet* **20**, 1238 (1918).