A strain-damage coupled model and its application to near crack tip localization analysis

V A Kovalev\textsuperscript{1}, Y N Radayev\textsuperscript{2}

\textsuperscript{1} Department of Applied Mathematics, Moscow City Government University of Management, 28 Sretenka Str., Moscow, 107045, Russia
\textsuperscript{2} Department of Continuum Mechanics, Samara State University, 1 Akad. Pavlov Str., Samara, 443011, Russia
E-mail: kovalev@migm.ru
E-mail: radayev@ssu.samara.ru

Abstract. Three-dimensional equations of a strain-damage coupled model for ductile deformation processes in solids are presented in an attempt to find approaches for study of the anisotropic damage effect on the perfect plastic flow. Damage state is represented by a symmetric damage tensor. A modified Tresca yielding condition, associated flow and damage rules are used to formulate the strain-damage coupled constitutive equations, thus allowing to take account of that effect. The principal axes of stresses are chosen as a local frame for representing the static, kinematic and constitutive equations of the coupled model. The closed system of equations represented in the local principal frame needed for formulation of the strain-damage coupled model is obtained. This system is shown belong to the hyperbolic type. Theory of characteristic lines of the strain-damage coupled equations is developed. A numerical solution of the plane strain problem of the strain-damage localization near a mode I crack tip is obtained. It shows the marked effect of damage induced anisotropy on the localization of plastic strains near a crack tip and allows to estimate an extension of the damaged zone ahead of the crack tip. Gradual distortion of the slip line geometry is also determined thus demonstrating how the anisotropic damage affects plastic flow.

1. Introduction
A formulation of continuum damage model requires before all a description and approximation of discontinuous and often nondeterministic distributions of microcracks, generated in the course of the brittle deformation process, internal structure of the metals, strained at high temperature, which are greatly affected by the nucleation of holes and their growth within the grains or on the grain boundaries (all these phenomena are usually referred to as the creep damage), and slip lines systems (for a ductile damaging process in a perfectly plastic solid).

A relevant aspect of damage state analysis is the damage induced anisotropy. Anisotropic damage state is to be primarily described by a directional distribution of damage measured in some way. Damage tensors can be then extracted in a regular way from the directional damage distribution.

An approach to representation of three-dimensional anisotropic damage state has been developed in [1], [2]. A general thermodynamical model of three-dimensional anisotropic damage state based on the canonical formalism has been discussed in [3], [4]. The canonical damage state
variables, defined therein, provide a very simple representation of the thermodynamic damage state potentials. Their derivation needs averaging the local directional damage, so that the importance of averaged directional damage for the formulation of continuum damage model can be seen.

The aim of this paper is to give a reminder of the mathematical description of an anisotropic damage state, discuss requisite mathematical relations for the averaged directional damage and estimate its variation limits for an arbitrary type of damage induced anisotropy, obtain modified yielding criteria, associated flow and damage rules in order to formulate strain-damage coupled constitutive equations which are necessary for study of the anisotropic damage effect on the near crack tip perfectly plastic flow, kinematic and constitutive equations of the coupled model.

After the Introduction, the paper includes (Section 2) a short review of damage state description by the symmetric second rank damage tensor, as proposed in [2] with the objective of establishing a reference framework of notations and terminology. The definition of the second rank damage tensor is reestablished in order to elucidate its origin from the classical Finger strain tensor.

In Section 3 average values of the directional damage are considered. We define and then compute them (those considerations will be restricted by the second order approximations of directional damage distribution).

In Section 4 we give kinematic and constitutive equations of the coupled model. The closed system of equations represented in the local principal frame needed for formulation of the strain-damage coupled model is obtained assuming a nontrivial modification of incremental equations of the perfect plasticity. This system is shown belong to the hyperbolic type. The hyperbolicity enables us to develop theory of characteristic slip lines of the strain-damage coupled equations. Particularly this holds in the case of plane strain thus allowing to generalize slip lines theory known from plane strain perfect plasticity to coupled states.

In Section 5 a numerical solution of the plane strain problem of the strain-damage localization near a mode I crack tip is obtained showing the marked effect of damage induced anisotropy on the localization of plastic strains near a crack tip and allows to estimate an extension of the damaged zone ahead of the crack tip. Gradual distortion of the slip line geometry is also determined thus demonstrating how the anisotropic damage affects plastic flow.

2. The symmetric second rank damage tensor of continuum damage mechanics

An adequate description of anisotropic damage state requires an appropriate directional damage variable $\varsigma(n)$, where $n$ is a unit three-dimensional vector often referred to as director. In the actual damaged state the value of $\varsigma$ associated with the direction pointed by $n$ is the damage measured in appropriate terms. Since the early classical work of L M Kachanov [5], the variables representing the damage are traditionally interpreted in terms of effective load-carrying area reduction. In a perfectly plastic solid the effective area reduction is caused by micronecking. Corresponding slip line mechanism and kinematic are shown by Fig. 1.

In the current damaged state, due to distributed microcracks, cavities or slip lines mechanism, the load carrying area $dA^*(n)$ of a plane element normal to $n$ is less than its observed geometrical area $dA(n)$. Thus, the definition of damage variable $\varsigma(n)$ appears as

$$\varsigma(n) = \frac{dA^*(n)}{dA(n)}.$$  

The directional damage variable $\varsigma(n)$ is of primary importance for the all following consideration. As $\varsigma$ is decreasing when the damage is growing, this variable may be more correctly called as the directional integrity.
Figure 1. Effective net area reduction in a perfectly plastic solid due to slip line mechanism (plane strain state, the Onat–Prager (1955) scheme).

As it was elucidated in our previous discussion [2], the symmetric second rank damage tensor $D$ determines the reduction of the effective area of the plane element normal to the director $n$ according to the relation

$$\varsigma = \sqrt{\text{tr}[(I - D)^2n \otimes n]}.$$  \hspace{1cm} (2)

Symmetry of the damage tensor provides a clear mechanical interpretation for the damage principal directions and values (the latter is called as principal damages).

In view of symmetry, the damage tensor can be represented in the spectral form

$$D = \sum_{\alpha = 1}^{3} D^{(\alpha)} d^{(\alpha)} \otimes d^{(\alpha)},$$  \hspace{1cm} (3)

wherein $d_{(1)}$, $d_{(2)}$, $d_{(3)}$ are vectors of the orthonormal eigenbasis, and $D_{(1)}$, $D_{(2)}$, $D_{(3)}$ — the damage tensor eigenvalues (principal damages).

Substitution of the spectral decomposition (3) into equation (2) gives

$$\varsigma = \sqrt{(1 - D_{(1)})^2n_{(1)}^2 + (1 - D_{(2)})^2n_{(2)}^2 + (1 - D_{(3)})^2n_{(3)}^2},$$  \hspace{1cm} (4)

where $n_{(i)}$ are the components of the unit vector $n$ with respect to the damage eigenbasis $d_{(1)}$, $d_{(2)}$, $d_{(3)}$.

For a plane element, orthogonal to the principal axis of damage labelled by $\gamma$, we obtain from the latter equation the following formula

$$D_{(\gamma)} = \frac{dA_{(\gamma)} - dA_{(\gamma)}^*}{dA_{(\gamma)}} \quad \text{(no sum on } \gamma; \ \gamma = 1, 2, 3),$$  \hspace{1cm} (5)
that accords to the classical Kachanov notion [5] of a damage variable.

If one renumber the principal damages in the following order
\[ D(3) \leq D(2) \leq D(1), \]  
then for an arbitrary orientation \( \mathbf{n} \) the following double sided estimation is derived:
\[ D(3) \leq 1 - \varsigma \leq D(1). \]  
The principal damages can be represented in terms of the principal damage stretches \( L_D^{(\alpha)} \) (see discussion in [3]) as follows
\[ 1 - D(\gamma) = L_D^{(1)} L_D^{(2)} L_D^{(3)} L_D^{(\gamma)}, \]  
or vice versa
\[ L_D^{(1)} = \sqrt{\frac{(1 - D(2))(1 - D(3))}{(1 - D(1))}}, \quad L_D^{(2)} = \sqrt{\frac{(1 - D(1))(1 - D(3))}{(1 - D(2))}}, \quad L_D^{(3)} = \sqrt{\frac{(1 - D(1))(1 - D(2))}{(1 - D(3))}}. \]  

3. Directional average of the second order damage state approximation
Consider the mean of the directional damage variable \( \varsigma \) as represented by the second order approximation (4). We will follow [6] and shall use the notations \( \theta \) and \( \varphi \) for the spherical angles. Let \( C(j) = (1 - D(j))^2 \) \((j = 1, 2, 3)\) and a pair of triangle brackets \( <> \) denotes the averaging over the sphere of unit directions.

The following iterated integral
\[ <> = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sqrt{(C(1)\sin^2\theta\cos^2\varphi + C(2)\sin^2\theta\sin^2\varphi + C(3)\cos^2\theta\sin\theta)} \ d\theta d\varphi, \]  
by making use of the result [7] (formula 2.597.2) becomes
\[ <> = \frac{2}{\pi} \int_0^1 \mathbf{E}(k) \sqrt{(C(3) - C(2))\tau^2 + C(2)} \ d\tau, \quad k = \sqrt{\frac{(1 - \tau^2)(C(2) - C(1))}{(C(3) - C(2))\tau^2 + C(2)}}, \]  
where \( \mathbf{E}(k) \) is the complete elliptic integral of the second kind, \( k \) is the modulus.

The number of the independent parameters in (11) can be reduced by introducing ratios of the eigenvalues \( C(1), C(2), C(3) \):
\[ \frac{\pi}{2\sqrt{C}} <> = I(p_1, p_2) \quad (p_1 = C(1)/C(2), \ p_2 = C(2)/C(3), \ C = C(3)), \]  
where
\[ I(p_1, p_2) = \int_0^1 \sqrt{(1 - p_2)\tau^2 + p_2} \mathbf{E}(\sqrt{p_2(1 - \tau^2)(1 - p_1)((1 - p_2)\tau^2 + p_2)^{-1}}) \ d\tau. \]
The following expressions for the variables $p_1$, $p_2$, $C$ in terms of the principal damage stretches (see equations (9)) are valid:

\[
\sqrt{p_1} = \frac{L_D^{(2)}}{L_D^{(1)}}, \quad \sqrt{p_2} = \frac{L_D^{(3)}}{L_D^{(2)}}, \quad \sqrt{C} = L_D^{(1)}L_D^{(2)}. \tag{13}
\]

The mean of the integral $I = I(p_1, p_2)$ is found as (numerical result)

\[
<I> = \int_0^1 \int_0^1 I(p_1, p_2)dp_1 dp_2 = 1.157790... .
\]

The average value (over the unit square) of the ratio $<\langle>\sqrt{C}$ can be numerically obtained as

\[
\frac{\langle\langle\rangle\rangle}{\sqrt{C}} = \frac{2}{\pi} <I> = 0.737072... .
\]

One more useful representation for $<\langle>\sqrt{C}$ is

\[
\frac{\pi <\langle>}{2\sqrt{C}} = q^{-1}(1 + p^2 q^2)J(p, q),
\]

where the improper integral

\[
J(p, q) = \int_0^q \frac{kE(k)dk}{\sqrt{q^2 - k^2 (1 + p^2 k^2)^2}}
\]

depends on the parameters

\[
p = \frac{L_D^{(2)} L_D^{(3)} - L_D^{(1)} L_D^{(2)} 2 L_D^{(3)} 2}{L_D^{(2)} L_D^{(3)} - L_D^{(1)} L_D^{(3)} 2}, \quad q = \sqrt{1 - \frac{L_D^{(2)} 2}{L_D^{(1)} 2}}.
\]

A more symmetric formula for the ratio $<\langle>\sqrt{C}$ must be remarked

\[
\frac{\pi <\langle>}{2\sqrt{C}} = (1 + p^2 q^2)J^*(p^*, q^*), \tag{14}
\]

wherein

\[
J^*(p^*, q^*) = \int_0^q \frac{k^*E(k^* k^*)dk^*}{\sqrt{1 - k^* 2 (1 + p^{2*} k^* 2)^2}}: \quad p^* = \sqrt{\frac{L_D^{(2)} 2}{L_D^{(3)} 2}}, \quad q^* = 1 - \frac{L_D^{(2)} 2}{L_D^{(1)} 2}.
\]

This latter integral $J^*$ for a comparatively weak damage anisotropy (i.e. $q^* \to 0$) after asymptotic evaluations is obtained as

\[
J^*(p^*, q^*) = \frac{\pi}{2p^{*4}} (1 + \frac{1}{4} q^{*2} p^{*-2}) \left\{ \frac{1}{4} (1 + p^{*-2})^{-3/2} \ln \frac{1 + p^{*-2} + 1}{\sqrt{1 + p^{*-2} - 1}} + \frac{1}{2p^{*-2}(1 + p^{*-2})} \right\} - \frac{1}{8p^{*4}} q^{*2} \frac{1}{\sqrt{1 + p^{*-2} - 1}} \ln \frac{1 + p^{*-2} + 1}{\sqrt{1 + p^{*-2} - 1}} + O(q^{*4}).
\]

If a damage state is axially symmetric (when for the case $q^* = 0$), the above equation provides the following exact formula for integral $J^*$ in equation (14):

\[
J^*(p^*, 0) = \frac{\pi}{2p^{*4}} \left\{ \frac{1}{4} (1 + p^{*-2})^{-3/2} \ln \frac{1 + p^{*-2} + 1}{\sqrt{1 + p^{*-2} - 1}} + \frac{1}{2p^{*-2}(1 + p^{*-2})} \right\}. \tag{15}
\]
4. A strain-damage coupled model

The equations required for mathematical modeling damaging processes in a perfectly plastic solid we formulate in isostatic co-ordinate net.

We start from the spectral decomposition of the stress tensor $\sigma$

$$\sigma = \sigma_1 l \otimes l + \sigma_2 m \otimes m + \sigma_3 n \otimes n,$$

where $\sigma_1$, $\sigma_2$, $\sigma_3$ — principal stresses, $l$, $m$, $n$ — orthonormal system of stress eigenvectors. Then the static equilibrium equation

$$\nabla \cdot \sigma = 0$$

(17)

by introducing

$$d_1 = l \cdot \nabla, \quad d_2 = m \cdot \nabla, \quad d_3 = n \cdot \nabla$$

can be represented in the isostatic co-ordinate net as follows

$$\begin{cases}
    d_1 \sigma_1 + \kappa_{23} (\sigma_1 - \sigma_2) + \kappa_{32} (\sigma_1 - \sigma_3) = 0, \\
    d_2 \sigma_2 + \kappa_{31} (\sigma_2 - \sigma_3) + \kappa_{13} (\sigma_2 - \sigma_1) = 0, \\
    d_3 \sigma_3 + \kappa_{12} (\sigma_3 - \sigma_1) + \kappa_{21} (\sigma_3 - \sigma_2) = 0,
\end{cases}$$

(18)

where $\kappa_{ij}$ are the principal curvatures of the co-ordinate surfaces.

The compatibility equation for strain increments is

$$dS = \nabla \times d\varepsilon \times \nabla = 0.$$

The incompatibility tensor $dS$ can be given by its physical components in the isostatic co-ordinate net. Here we present the minimally necessary set of them (others can be obtained by the usual cyclic permutations)

$$dS_{11\rangle} = -d_2 d_3 d\varepsilon_3 - d_3 d_2 d\varepsilon_{22\rangle} + (\kappa_{21}^2 - \kappa_{31}^2) (d\varepsilon_3 - d\varepsilon_{22\rangle}) +
+ d_3 (\kappa_{21} (d\varepsilon_3 - d\varepsilon_{22\rangle})) - d_2 (\kappa_{31} (d\varepsilon_3 - d\varepsilon_{22\rangle})) -
- \kappa_{23}\kappa_{32} (d\varepsilon_{22\rangle} + d\varepsilon_3 - 2d\varepsilon_{11\rangle}) - \kappa_{31} d_3 d\varepsilon_3 -
- \kappa_{21} d_3 d\varepsilon_{22\rangle} - \kappa_{32} d_1 d\varepsilon_{22\rangle} - \kappa_{23} d_1 d\varepsilon_3 +
+ 2\kappa_{32} d_2 d\varepsilon_{12\rangle} + (2\kappa_{32}\kappa_{13} + \kappa_{23}\kappa_{31} + \kappa_{32}\kappa_{31} + (d_2 \kappa_{32})) d\varepsilon_{12\rangle},$$

$$dS_{12\rangle} = d_2 d_1 d\varepsilon_3 + d_2 [\kappa_{32} (d\varepsilon_3 - d\varepsilon_{11\rangle})] + \kappa_{31} d_1 (d\varepsilon_3 - d\varepsilon_{22\rangle}) -
- \kappa_{23} d_2 d\varepsilon_3 + \kappa_{31} (d\varepsilon_3 - d\varepsilon_{11\rangle}) (\kappa_{32} - \kappa_{23}) +
+ d_3 d_3 d\varepsilon_{12\rangle} + (\kappa_{21} + \kappa_{12}) d_3 d\varepsilon_{12\rangle} +
+ (-\kappa_{31}^2 + \kappa_{32}\kappa_{23} + 2\kappa_{13}\kappa_{31} + 2\kappa_{21}\kappa_{12} - \kappa_{21}^2 - d_2 \kappa_{33} + d_3 \kappa_{12}) d\varepsilon_{12\rangle}.$$  

We proceed our considerations to the notion of effective stress. For a simple case of uniaxial tension of damaging specimen the effective stress $\sigma^*$, the actual stress $\sigma$ and the damage $D$ are involved in the equation

$$\sigma^* = \frac{\sigma}{1 - D}.$$  

(21)

As $\sigma^* > \sigma$ in view of $0 < D < 1$, the latter equation in the simplest form takes account of the stress magnifying effect in a progressively damaging solid.

The effective stress concept is easily generalized to the three-dimensional case by using as previously the isostatic co-ordinate representation

$$\sigma^*_j = \frac{\sigma_j}{1 - D_j} \quad \text{(no sum on } j) .$$  

(22)
The generalized effective stress notion is then applied to formulation of the yielding condition that describes those zones in a solid which are progressively damaging in the course of plastic flow. A general form of the yielding condition for an isotropic perfectly plastic solid
\[ f(\sigma_1, \sigma_2, \sigma_3) = 0 \]
changes into a new form of the principal effective stresses \( \sigma_j^* \)
\[ f(\sigma_1^*, \sigma_2^*, \sigma_3^*) = 0, \quad (23) \]
which enters the framework of our mathematical model of a damage state in perfectly plastic body. Thus we obtain, for example, a formulation of the Tresca yielding condition taking account of anisotropic damage:
\[ \sigma_1 - \sigma_2 - 2k = 0 \quad (24) \]
for the facet of the Tresca prism, and
\[ \sigma_2 - \sigma_3 - 2k = 0, \quad (25) \]
for the edge.

The damage accumulation law is given by linear equations for principal damage and strains increments as follows
\[ dD_j = K_j \text{sgn} (d\epsilon_P^j) d\epsilon_P^j \quad (\text{no sum on } j), \quad (26) \]
where \( K_j \) are material constants.

For given yielding condition (23) the associated flow rule in the principal axes of stress (and also damage) tensor is
\[ d\epsilon_P^j = (d\Lambda) \frac{\partial f}{\partial \sigma_j^*} \quad (27) \]
with an undetermined increment \( d\Lambda \).

The associated flow rule can be formulated in terms of the principal effective stresses, which leads to
\[ d\epsilon_P^j = (d\Lambda) \frac{1}{1 - D_j} \left[ (\beta - 1) \frac{\partial f}{\partial \sigma_j^*} \right] (\text{no sum on } j). \]

In the following discussion we shall consider plane strain coupled states. In this case the flow rule is represented by
\[ d\epsilon_P^1 = \frac{d\Lambda}{1 - D_1}, \quad d\epsilon_P^2 = -\frac{d\Lambda}{1 - D_2}, \quad d\epsilon_P^3 = 0. \]

In view of (26) assuming \( dD_3 = 0 \) we have
\[ d\Lambda = \frac{(1 - D_1)^2 [(\beta - 1) d\sigma_2 - d\sigma_1]}{K_1 \text{sgn} (d\epsilon_P^1) \sigma_1 + (\beta - 1)^2 K_2 \text{sgn} (d\epsilon_P^2) \sigma_2}, \quad (28) \]
where
\[ \beta - 1 = \frac{1 - D_1}{1 - D_2}. \]

Hence, for plane strain states the flow rule is obtained as
\[ \begin{align*}
\frac{d\epsilon_P^1}{F} &= (\beta - 1) d\sigma_2 - d\sigma_1, \\
\frac{d\epsilon_P^2}{F} &= - (\beta - 1)^2 d\sigma_2 + (\beta - 1) d\sigma_1, \\
\frac{d\epsilon_P^3}{F} &= 0.
\end{align*} \quad (29) \]
with the following notation

\[ F = \frac{1 - D_1}{K_1 \text{sgn} (d\varepsilon^P_1) \sigma_1 + (\beta - 1)^3 K_2 \text{sgn} (d\varepsilon^P_2) \sigma_2}. \]

Strain incompatibility equation reads (\( \kappa_1, \kappa_2 \) — the stress principal lines curvatures)

\[
d_2 d_2 d\varepsilon_1 - (\beta - 1) d_1 d_1 d\varepsilon_1 - ((1 + \beta) \kappa_2 + 2d_1 \beta) d_1 d\varepsilon_1 +
+ (1 + \beta) \kappa_1 d_2 d\varepsilon_1 + (d_1 d_1 \beta + \kappa_2 d_1 \beta) -
- \kappa_1 d_2 \beta - \beta d_1 \kappa_2 + \beta d_2 \kappa_1 - \beta \kappa_2^2 (30)
\]

The Cauchy formulae for displacements increments \( du_{<j>} \) are

\[
d\varepsilon^P_1 = \kappa_1 du_{<2>} + d_1 du_{<1>},
\]

\[
d\varepsilon^P_2 = \kappa_2 du_{<1>} + d_2 du_{<2>}.
\]

Thus we obtain the hyperbolic system of incremental equations. The characteristic lines of the system are determined by the following ordinary differential equation

\[
\frac{ds_1}{ds_2} = \pm \sqrt{\frac{1 - D_1}{1 - D_2}} (32)
\]

where \( ds_j \) is an elementary length of the characteristic line. The latter equation is of crucial importance for numerical analysis of plane strain problems formulated within the framework of the developed strain-damage coupled model.

5. Numerical analysis of strain-damage localization near a mode I crack tip

Localization of plastic strains and anisotropic damage near a linear mode I semi-infinite crack tip in a perfectly plastic solid under plane strains conditions is of particular interest as a sample application of the developed strain-damage coupled model to study of these localization phenomena. Perfectly plastic analysis of this problem (see [8] for details) gives the near crack tip stress distribution in the form

\[
\begin{cases}
\frac{\sigma_{rr}}{2k} = \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2} \cos 2\varphi, & 0 \leq \varphi \leq \pi/4, \\
\frac{\sigma_{\varphi\varphi}}{2k} = \frac{1}{2} + \frac{\pi}{2} + \frac{1}{2} \cos 2\varphi, & \frac{\pi}{2} \leq \varphi \leq \pi/2, \\
\frac{\sigma_{r\varphi}}{2k} = \frac{1}{2}, & 0 \leq \varphi \leq \pi.
\end{cases}
\]

Here \( k \) is the shear yield stress. This stress distribution is employed as initial.

Slip lines geometry and perfectly plastic strains localization zone are shown by Fig. 2. Very small localization zone of this kind is used as an initial approximation.

Numerical analysis of the strain-damage localization is based on the plane strain coupled equations given in the previous section of the paper taking account of the strong hyperbolicity
of equations of the coupled model. The values of the dimensionless constants are: $K_2 = 1.5$ and comparatively small value of the constant $K_1 = 0.05$. The standard finite difference technique has been applied to the coupled differential equations. The Courant necessary condition is verified by the aid of equation (32). Results of the numerical analysis show a marked effect of anisotropic damage on linear dimensions of near crack tip localization zone. If $h$ denotes the linear horizontal dimension of the near crack localization zone before the crack tip (see Fig. 3) then the following data are obtained: $0.731h$ for linear dimension of the localization zone on the crack surface; $0.43h$ for vertical dimension of the ”triangle” localization zone ahead of the crack tip; $0.861h$ maximal vertical dimension of the strain-damage localization zone. An important characteristics of the strain-damage localization is the plastic strains concentration coefficient. Numerical estimation gives the value 1.214 for this coefficient.

References
[1] Murakami S 1988 Mechanical modeling of material damage J. Appl. Mech. 55 No. 2 280
[2] Murakami S and Radayev Y N 1996 Mathematical model of three-dimensional anisotropic damage state Izvestiya Rossiijskoi Akademii Nauk. Mech. of Solids 4 93 (in Russian)
[3] Radayev Y N 1996 Thermodynamical model of anisotropic damage growth. Part I. Canonical damage state variables of continuum damage mechanics and thermodynamical functions of three-dimensional anisotropic damage state J. Non-Equilib. Thermodyn. 21 129
[4] Radayev Y N 1996 Thermodynamical model of anisotropic damage growth. Part II. Canonical damage growth rate equations and theory of damage invariants J. Non-Equilib. Thermodyn. 21 197
[5] Kachanov L M 1958 On the creep rupture time Izvestiya Akademii Nauk SSSR. Otd. Tekh. Nauk 8 26 (in Russian)
[6] Radayev Y N 2004 On directional average of the local anisotropic damage Int. J. Fracture 128 293
[7] Gradshteyn I S and Ryzhik I M 1965 Table of Integrals, Series, and Products translation from the Russian ed. by A Jeffrey (New York, London: Academic Press)
[8] Kachanov L M 1969 Foundations of the Theory of Plasticity (Moscow: Nauka) (in Russian)