Thermodynamics of Internal Correlations

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Abstract

Previous research has consistently affirmed that Maxwell’s demon must adhere to the second law of thermodynamics. Yet, the unresolved question remains whether the profitability and indispensability of information, which we routinely take for granted, are based on constraints stemming from physical laws. This paper reports a novel generalization of the second law of thermodynamics, answering that when internal correlations, i.e., correlations between subsystems of resource, are intended to be exploited, information is indispensable to extract free energy. Furthermore, the internal correlations, which can grow linearly with the number of subsystems in the resource, allow for control with information that yields significant gains, dwarfing the negligible operational costs in the thermodynamic limit. Thus, the generalized second law presented herein can be interpreted as a fundamental physical principle that ensures the benefit and inevitability of information processing in thermodynamics.

I. INTRODUCTION

In the realm of theoretical physics, the concept of Maxwell’s demon has long served as a catalyst for exploring the boundaries of the second law of thermodynamics. The prominent studies of Szilard, Brillouin, and Bennett demonstrated that the operational cost, specifically, the free energy consumed to maintain the memory available [1–8], is the key to solving the paradox of Maxwell’s demon [9–13]. Recently, advances have been made in the powerful theories of nonequilibrium statistical mechanics, such as the fluctuation theorem [14–20]. This progress has motivated researchers to construct modern models of Maxwell’s demon, including the feedback control [21–41] and information reservoir models [42–55]. These models extend the second law of thermodynamics by factoring in the correlation between an agent and its target. They collectively affirm that the gain (free energy or work) gleaned from information cannot exceed the associated operational costs, thereby upholding the second law’s tenets.

Notwithstanding these advancements, a critical gap persists in our understanding: the recognition of information’s indispensability and profitability, often taken as given in daily life, lacks a solid grounding in physical principles. Our research endeavors to bridge this gap by proposing a novel generalization of the second law that integrates ‘internal correlations’ — the correlations within subsystems. The internal correlations are ubiquitous in the real
world. Indeed, many organisms on Earth exploit internal correlations for survival, acquiring necessary information through sensory inputs. Understanding these internal correlations is crucial for appreciating the significance of information processing in ecosystems.

Our generalization implies that information is essential for harnessing internal correlations. Here, we interpret indispensability of information in the sense that an agent must establish correlations with the target for gain extraction. In alignment with the perspective in [31], information indispensability equates to the necessity of feedback control. Since diminishing internal correlations leads to increased entropy, these correlations represent a free energy resource that can be tapped into by their release. Existing interpretations of the second law, including its variants in prior literature, do not preclude the extraction of work irrespective of feedback control. In contrast, our generalization asserts that the feedback control is indispensable to extract gains from the internal correlations.

Moreover, our generalization explicitly states that when an agent employs feedback control using a resource encompassing internal correlations, the attainable free energy exceeds the operational cost. In such scenarios, the agent can acquire substantially more information than what is stored in memory, allowing the extraction of gains that surpass operational costs. This phenomenon is distinctly articulated in our generalized second law.

II. SETTING

This research explores the dynamics of subsystems in a classical system interfacing with a thermal reservoir at temperature $T$. We partition the entire system into $N$ distinct subsystems. Let $X_k$ and $X'_k$ symbolize the initial and final states of the $k$-th subsystem, respectively. Likewise, $X_{\text{tot}}$ and $X'_{\text{tot}}$ denote the initial and final states of the whole system. To depict subsystems within a specific segment, a colon notation is employed, for example, $X_{j:k} = (X_j, X_{j+1}, \ldots, X_k)$ and $X'_{j:k} = (X'_j, X'_{j+1}, \ldots, X'_k)$. Consequently, $X_{\text{tot}} = X_{1:N}$ and $X'_{\text{tot}} = X'_{1:N}$. We establish the following convention using this colon notation:

$$m \geq 1 \Rightarrow X_{(n+m):n} = \emptyset. \quad (1)$$

For describing dependencies in the evolution of subsystems, we introduce a notation where $A \perp B$ indicates that two random variables $A$ and $B$ are independent. Define $\mathcal{X}_j$ as the set of subsystems impacting $X'_j$: $\mathcal{X}_j := \{X_k \mid X_k \not\perp X'_j, X_k \in X_{\text{tot}} \setminus X_j\}$. Assuming that final
states are solely determined by initial states, we have:

\[ X'_j \perp X'_k \]  \hspace{1cm} (2)

if \( j \neq k \).

III. MAIN RESULT

A. Notation

The conditional Shannon entropy of \( X_j \) conditioning on \( X_k \) is denoted by \( S(X_j \mid X_k) \):

\[ S(X_j \mid X_k) := \langle -\ln P(X_j \mid X_k) \rangle , \]  \hspace{1cm} (3)

where \( \langle \cdot \rangle \) denotes the average over a joint distribution. Similarly, the conditional mutual information is indicated as \( I(X_j; X_k \mid X_i) \):

\[ I(X_j; X_k \mid X_i) := \left\langle \ln \left[ \frac{P(X_j, X_k \mid X_i)}{P(X_j \mid X_i) P(X_k \mid X_i)} \right] \right\rangle . \]  \hspace{1cm} (4)

As defined in Eq. (11), the mutual information is generalized for more than two variables. To represent the increase in these quantities through evolution, \( \Delta \) is set before \( S \) or \( I \), and \( \star \) is used as a superscript to specify subsystems causing this increase, as follows:

\[ \Delta I(A^*; B^* \mid C^*) := I(A'; B' \mid C') - I(A; B \mid C) , \]  \hspace{1cm} (5)

\[ \Delta I(A^*; B^* \mid C) := I(A'; B' \mid C) - I(A; B \mid C) , \]  \hspace{1cm} (6)

\[ \Delta I(A^*; B \mid C) := I(A'; B \mid C) - I(A; B \mid C) , \]  \hspace{1cm} (7)

\[ \Delta I(A; B \mid C^*) := I(A; B \mid C') - I(A; B \mid C) . \]  \hspace{1cm} (8)

If all variables contribute to the increase, we may eliminate \( \star \). For instance,

\[ \Delta I(A; B \mid C) = \Delta I(A^*; B^* \mid C^*) . \]  \hspace{1cm} (9)

We assume a local detailed balance for each subsystem, which holds in a broad class of nonequilibrium dynamics including Langevin dynamics [40]:

\[ \frac{Q_j}{T} = \left\langle \ln \left( \frac{P(X'_j \mid X_j, \hat{X}_j)}{P^B(X'_j \mid X'_j, \hat{X}_j)} \right) \right\rangle , \]  \hspace{1cm} (10)
where $P^B$ denotes the probability distribution of the backward paths. As presented in Appendix A, this assumption results in the entropy bound for each subsystem:

$$\Delta S \left( X_j^* \mid \hat{X}_j \right) + \frac{Q_j}{T} \geq 0,$$

where $Q_j$ denotes the heat transfer from subsystem $j$.

Let us introduce two components that constitute the entropy lower bound when internal correlations are considered. As can be immediately obtained from Eq. (F17), the total entropy production splits into the individual entropy production of each subsystem and the reduction in the correlations between subsystems:

$$\Delta S (X_{\text{tot}}) = \sum_{j=1}^{N} \Delta S (X_j) - \sum_{j=2}^{N} \Delta I (X_j \mid X_{1:j-1}).$$

Let $\Delta I_{\text{tot}}$ be the second term on the right-hand side:

$$\Delta I_{\text{tot}} := \Delta I_{2:N}$$

where

$$\Delta I_k := \Delta I (X_k \mid X_{1:k-1}).$$

$$\Delta I_{2:n} := \sum_{k=2}^{n} \Delta I_k$$

In Appendix B, we show that $\Delta I_{\text{tot}}$ represents the total increase in the internal correlations within the entire system.

Another component is the increase of the mutual information between subsystems and other subsystems that influence them:

$$\gamma := \sum_{i=1}^{N} \gamma_i,$$

where

$$\gamma_i := I \left( X_i^* ; \hat{X}_i \right) = I \left( X_i^* ; X_{\text{tot}} \setminus X_i \setminus \hat{X}_i \right).$$
B. Derivation of the Lower Bound on Entropy Production under Internal Correlations

Upon summing over $j = 1$ to $N$ (or, as inferred from Appendix C and the application of Jensen’s inequality), the subsequent relation is derived.

$$\Delta S_{\text{tot}} \geq \gamma - \Delta I_{\text{tot}}$$  \hspace{1cm} (19)

Here, we define

$$\Delta S_{\text{tot}} := \Delta S(X_{\text{tot}}) + \frac{Q_{\text{tot}}}{T} \hspace{1cm} (20)$$

At first glance, this inequality appears as an extension of the second law of thermodynamics, yet the right-hand side contains negative components, which could provide a looser lower bound compared to the second law.

To extend this inequality to encompass the second law of thermodynamics, it is imperative to eliminate the negative contributions. Specifically, the right-hand side overestimates the utilizable internal correlations by double counting. For instance, in a two-component system utilizing correlations between 1 and 2, $\gamma$ assumes that both subsystems can act as agents to harness the internal correlations. In reality, only one of the subsystems, either 1 or 2, utilizes the internal correlations as an agent. This overestimation manifests as independent negative terms on the right-hand side. Below, we present an extension of the second law of thermodynamics where this double counting is rectified, and the utility of information is more interpretable.

C. Entropy Bounds in the Case of Complete Dependencies

To derive an extension of the second law of thermodynamics, let us first consider how $\bar{\gamma}_{\text{tot}} - \gamma$ can be reformulated in the case of complete dependencies, i.e., when for any subsystem $i$, $\hat{i} = \omega_i$.

For the sake of simplicity, the notation for the Shannon information quantities is abbreviated as follows from this section onwards.

$$i \mid j = S(X_i \mid X_j) \hspace{1cm} (21)$$

$$i ; j \mid k = I(X_i ; X_j \mid X_k) \hspace{1cm} (22)$$
As per [56], the Shannon information quantities $S$ or $I$ can be considered as measures in a measure space, with their set operations corresponding to: ’,’ as $\cap$, ’ ’ as $\setminus$, and ’,’ as $\cup$. Therefore, under these correspondences, one can freely perform operations equivalent to set operations using the above notation. For example, the distributive law holds as follows.

\[(i; j + k; l) ; m = i; j ; m + k; l ; m \] (23)  
\[(i; j + k; l) | m = i; j | m + k; l | m \] (24)

Additionally, the following convention is established.

\[Y; Z | \emptyset = Y; Z \] (25)

Moreover, with $i \in 1 : N, I \subset 1 : N$, we define the symbols as follows.

\[\omega^n_i := (1 : n) \setminus i \] (26)  
\[\omega_i := \omega_i^N \] (27)  
\[\mu(j) := 1 : (j - 1) \] (28)  
\[\mu := \mu(i) \] (29)  
\[\nu(j) := (j + 1) : N \] (30)  
\[\nu := \nu(i) \] (31)  
\[\hat{i} := \{j \mid X_j \in \hat{X}_i\} \] (32)  
\[\hat{I} := \bigcup_{i \in I} \hat{i} \] (33)

In cases where the dependency relationship is complete, the equality $\hat{i} = \omega_i$ holds. Consequently, from the definition presented in Eq. (18), $\gamma$ can be expressed as follows:

\[\gamma = \sum_{i=1}^{N} i^* ; \omega_i \] (34)
Let us introduce symbols to represent this quantity.

\[ \bar{\gamma}_i := i^* ; \omega_i \]  
\[ \bar{\gamma}_{1:i} := \sum_{j=1}^i \bar{\gamma}_j \]  
\[ \bar{\gamma}_{1:N} \]  
\[ \bar{\gamma}_{1:A} := \sum_{i \in A} i^* ; A \setminus i \]  
\[ \bar{\gamma}^j_{1:i} := j^* ; \omega^j \]  
\[ \bar{\gamma}^i_{1:i} := i^* ; \omega_i \]  

(Note that \( \bar{\gamma}_{1:tot} = \bar{\gamma}_i \))

The quantity \( \bar{\gamma}_{tot} \) is equal to \( \gamma \) in the case of the complete dependency. Furthermore, let us define \( a_i \) as

\[ a_i := \bar{\gamma}^i_{1:i} - \Delta I_{1:i} \] (2 \( \leq i \)).

The right-hand side of Eq. (19) coincides with \( a_N \) in the case of the complete dependency relationships. Hence, the aim of this section, as initially stated, is to reformulate \( a_N \) into a form that is more readily interpretable. The quantities defined below will appear in the result.

\[ b(A, B) := A^* ; B + A ; B^* - A^* ; B^* \]  
\[ b_i := b(i, \mu(i)) \]  
\[ b^A_i := b(i, \mu(i) \cap A) \]

1. Recurrence relation for \( a_n \)

To rewrite \( a_n \) in an interpretable manner, we first provide the recurrence relation it satisfies. For this purpose, we present the following lemma.

**Lemma 1.** It holds that:

\[ \bar{\gamma}^i_{1:i} - \bar{\gamma}^{i-1}_{1:i-1} = i^* ; \mu + i ; \mu^* - a_{i-1} ; i \]  

**Proof.** From the definition of \( a_n \) given in Eq. (11), it is sufficient to show the following relation:

\[ \bar{\gamma}^i_{1:i} - \bar{\gamma}^{i-1}_{1:i-1} = i^* ; \mu + i ; \mu^* - (\bar{\gamma}^{i-1}_{1:i-1} - \Delta I_{1:i-1}) ; i \]
To demonstrate this, we decompose $\gamma_{1;:i}$. By definition,

$$\gamma_{1;:i} = \sum_{k=1}^{i-1} k^* ; \omega_k^i + i^* ; \mu$$

(47)

Applying Eq. (F3) to $i$ in $\omega_k^i$ yields:

$$\sum_{k=1}^{i-1} k^* ; \omega_k^i = \sum_{k=1}^{i-1} (k^* ; i + k^* ; \omega_k^{i-1} - k^* ; i ; \omega_k^{i-1}) .$$

(48)

Using Eq. (F17), we have:

$$\sum_{k=1}^{i-1} k^* ; i = i ; \mu^* + \sum_{k=1}^{i-1} k^* ; \mu(k)^* ; i \defeq i ; \mu^* + \Delta I_{1;:i-1} ; i$$

(49)

(50)

By the definition,

$$\sum_{k=1}^{i-1} k^* ; \omega_k^{i-1} = \gamma_{1;:i-1}$$

(51)

$$\sum_{k=1}^{i-1} k^* ; i ; \omega_k^{i-1} = \gamma_{1;:i-1} ; i$$

(52)

Inserting Eqs. (50)-(52) into Eq. (48) yields Eq. (45).

Let us now provide a recurrence relation for $a_n$.

**Proposition 1.** The sequence $a_n$ satisfies the following recurrence relation:

$$a_2 = b_2$$

(53)

$$a_n - a_{n-1} = b_n - a_{n-1} ; n$$

(54)

Proof. The identity $a_2 = b_2$ is trivial from the definition. By the same token, from the definition,

$$a_n - a_{n-1} = \gamma_{1:n}^n - \gamma_{1:n-1}^{n-1} - (\Delta I_{1:n} - \Delta I_{1:n-1})$$

(55)

$$= \gamma_{1:n}^n - \gamma_{1:n-1}^{n-1} - \Delta I_n$$

(56)

Substituting Eq. (F3) from Lemma into the above,

$$a_n - a_{n-1} = n^* ; \mu(n) + n ; \mu(n)^* - \Delta I_n$$

$$= b_n - a_{n-1} ; n$$

(57)

(58)

This completes the proof. □

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Let us solve the recurrence relation for $a_n$ as given in Proposition. For this purpose, let us introduce the following Lemma. Note that one has $m + 1 : m = \emptyset$ and $b_i \mid \emptyset = b_i$.

**Lemma 2.** Assuming $a_2 = b_2$, and for $3 \leq m \leq n - 1$ the following holds true:

$$a_m = \sum_{i=2}^{m} b_i \mid (i + 1 : m) \quad (59)$$

Then, for $n \geq 4$ the subsequent expression is valid:

$$\sum_{j=3}^{n-1} a_{j-1} \mid j = \sum_{i=2}^{n-2} b_i \mid (i + 1 : n - 1) \quad (60)$$

**Proof.** Let $\ell$ be an arbitrary integer such that $2 \leq \ell \leq n - 2$. Under the assumption of Eq. (59), the left and right sides of Eq. (60) represent a linear sum with respect to $b_\ell$ under the product denoted by a semicolon. Hence, it suffices to extract the terms containing $b_\ell$ from both sides and show their equality. Let us denote the term extracted from $\star$ containing $b_\ell$ by $\|\star\|_\ell$. From assumption Eq. (59), for $\ell \leq j - 1$ and $j \geq 3$,

$$\|a_{j-1}\|_\ell = b_\ell \mid (\ell + 1 : j - 1) \quad (61)$$

Thus, for $n \geq 4$:

$$\|\text{L.H.S of (60)}\|_\ell = \sum_{j=3}^{n-1} b_\ell \mid (\ell + 1 : j - 1) \quad (62)$$

Clearly, for $n \geq 4$:

$$\|\text{R.H.S of (60)}\|_\ell = b_\ell \mid (\ell + 1 : n - 1) \quad (63)$$

We shall demonstrate the equality of the right-hand sides of Eq. (62) and Eq. (63) by induction on $n$. For $n = 4$, both are simply $b_2 \mid 3$, which is evidently equal. Assuming the right-hand sides of Eq. (62) and Eq. (63) are equal for $n = m - 1$, under this assumption, for $n = m$:

$$\text{R.H.S of (62)} = \sum_{j=3}^{m-2} b_\ell \mid (\ell : j - 1) + b_\ell \mid (m - 1) \mid (\ell + 1 : m - 2) \quad (64)$$

$$= b_\ell \mid (\ell + 1 : m - 2) + b_\ell \mid (m - 1) \mid (\ell + 1 : m - 2) \quad (65)$$

(by induction hypothesis)

$$= b_\ell \mid (\ell + 1 : m - 1) \quad (66)$$

$$= \text{R.H.S of (63)} \quad (67)$$
Thus, by induction, we have shown that the right-hand sides of Eq. (62) and Eq. (63) are equal for \( n \geq 4 \). This implies \( \| \text{L.H.S of (60)} \|_{\ell} = \| \text{R.H.S of (60)} \|_{\ell} \). The validity of Eq. (60) has been demonstrated.

Using the above Lemma, let us now determine the solution for \( a_n \).

**Proposition 2.** Given that \( a_n \) follows the recurrence relations Eq. (53) and Eq. (54), \( a_n \) can be expressed using \( b_i \) as follows:

\[
a_n = \sum_{i=2}^{n} b_i \mid (i + 1 : N) \quad (68)
\]

Proof. Taking into account Eqs. (1) and (25), it suffices to prove the following:

\[
a_n = \sum_{i=2}^{n} b_i - \sum_{i=2}^{n-1} b_i ; (i + 1 : n) \quad (69)
\]

We proceed by induction on \( n \). For the base case \( n = 2 \), Eq. (69) simplifies to \( a_2 = b_2 \). Hence, for \( n = 2 \), Eq. (69) is valid by virtue of Eq. (53).

Assume Eq. (69) holds for \( n = m - 1 \). Under this assumption, we aim to demonstrate its validity for \( n = m \). Substituting the expression for \( n = m - 1 \) from Eq. (69) and for \( n = m \) from Eq. (54) into the right-hand side of \( a_m = a_{m-1} - (a_m - a_{m-1}) \), we obtain

\[
a_m = a_{m-1} - (a_m - a_{m-1}) \quad (70)
\]

\[
= \sum_{i=2}^{m-1} b_i - \sum_{i=2}^{m-2} b_i ; (i + 1 : m - 1) + b_m - a_{m-1} ; m. \quad (71)
\]

The first and third terms on the right-hand side of Eq. (71) can be rewritten as:

\[
\sum_{i=2}^{m-1} b_i + b_m = \sum_{i=2}^{m} b_i. \quad (72)
\]

Utilizing Eq. (69) for \( n = m - 1 \), the second and fourth terms on the right-hand side of
Eq. (71) can be reformulated as follows:

\[
\sum_{i=2}^{m-2} b_i; (i + 1 : m - 1) + a_{m-1} ; m
\]

\[
= \sum_{i=2}^{m-2} b_i; (i + 1 : m - 1) + \sum_{i=2}^{m-1} b_i; m - \sum_{i=2}^{m-2} b_i; (i + 1 : m - 1) ; m
\]  

(73)

\[
= \sum_{i=2}^{m-1} b_i; (i + 1 : m - 1, m)
\]  

(74)

\[
= \sum_{i=2}^{m-1} b_i; (i + 1 : m).
\]  

(75)

From Eqs. (71), (72), and (75), it follows that Eq. (69) also holds for \( n = m \). This concludes the proof by mathematical induction.

Based on the aforementioned results, we can immediately derive the following equation:

\[
\bar{\gamma}_{\text{tot}} - \Delta I_{\text{tot}} = \sum_{i=2}^{N} b_i | \nu.
\]  

(76)

Therefore, when every subsystem pair is interdependent, the subsequent inequality is valid:

\[
\Delta S_{\text{tot}} \geq \sum_{i=2}^{N} b_i | \nu.
\]  

(77)

As previously discussed, it is important to note that the right-hand side of this expression contains negative components, which could potentially violate the second law of thermodynamics, \( \Delta S_{\text{tot}} \geq 0 \). In the following section, we address the removal of these negative contributions and extend our analysis to general dependencies.

D. The Second Law of Thermodynamics Considering Internal Correlations

Thus far, we have simplified \( \bar{\gamma}_{\text{tot}} - \Delta I_{\text{tot}} \) to obtain Eq. (77). Denoting the right-hand side of Eq. (17) as \( \mathfrak{B} \) and its \( i \)-th term as \( \mathfrak{B}_i \), we express them as follows:

\[
\mathfrak{B}_i := b_i | \nu, \quad \text{ (78)}
\]

\[
\mathfrak{B} := \sum_{i=2}^{N} \mathfrak{B}_i. \quad \text{ (79)}
\]

Regarding the right-hand side of Eq. (19), we have:

\[
\gamma - \Delta I_{\text{tot}} = \bar{\gamma}_{\text{tot}} - \Delta I_{\text{tot}} - (\bar{\gamma}_{\text{tot}} - \gamma)
\]

\[
= \mathfrak{B} - (\bar{\gamma}_{\text{tot}} - \gamma)
\]  

(80)
Thus, the difference $\mathfrak{B} - (\bar{\gamma}_{\text{tot}} - \gamma)$, after excluding its negative components, represents the lower bound for entropy increase when considering internal correlations under a general dependency relationship. We will determine this in this section.

1. **Partition of Eq. (81)**

According to Yeung’s theory, the Shannon entropy can be equated to calculations in set theory (measure theory). In the context of Yeung’s theory, we denote by $\mathcal{Y}(Z)$ the set conjugate to a random variable $Z$. When $\mathcal{Y}(A) \cap \mathcal{Y}(B) = \emptyset$, in this manuscript, we refer to $A$ and $B$ as exclusive components. The act of decomposing into exclusive components is referred to as partitioning, analogous to set theory.

**Lemma 3.** $A; Z$ and $S(Z \mid A)$ are exclusive.

**Proof.** $[\mathcal{Y}(A) \cap \mathcal{Y}(Z)] \cap [\mathcal{Y}(Z) \setminus \mathcal{Y}(A)] = \emptyset$. The proof is thus complete. □

The negative terms that must be excluded are the independent negative components that appear when $\mathfrak{B} - (\bar{\gamma}_{\text{tot}} - \gamma)$ is partitioned. To determine this, we perform partitioning of $\Delta I_i$. To represent the exclusive components of $\Delta I_i$, we define the following quantities: for...
any \( u, v, w \in (1 : N) \cup 2^{1:N} \),

\[
\mathcal{T}_{u,v,w} := (u, w)' ; u ; v - (u, v)' ; u ; v = u ; v \mid (u, w)',
\]

(82)

\[
\mathcal{C}_{u,v} := -u^* ; u ; v - \mathcal{T}_{u,v,v},
\]

(83)

\[
\mathcal{C}_u := \mathcal{C}_{u,\mu(u)},
\]

(84)

\[
\mathcal{B}_{u,v} := -v^* ; u ; v - \mathcal{T}_{u,v,v},
\]

(85)

\[
\mathcal{M}_{u,v}^L := u ; v^* \mid u^* , v,
\]

(86)

\[
\mathcal{M}_{u,v}^R := u^* ; v \mid u, v^*,
\]

(87)

\[
\mathcal{M}_{u,v} := \mathcal{M}_{u,v}^L + \mathcal{M}_{u,v}^R.
\]

(88)

As depicted in Fig. 1, the decomposition of \( \gamma_{\text{tot}} - \gamma \) can be expressed using \( \mathcal{T}_{i,\mu,\mu}, \mathcal{C}_i, \mathcal{B}_{i,j} \). Additionally, the decomposition of \( \mathfrak{B} \) also involves \( \mathcal{M}_{i,\mu} \). These components can be interpreted as follows:

- \( \mathcal{T}_{i,\mu,\mu} \): The reduction in the correlation between subsystem \( i \) and \( \mu \) in the initial state that arises from changes of either \( i \) or \( \mu \).

- \( \mathcal{C}_i \): The reduction in the correlation between \( i \) and \( \mu \) in the initial state that arises solely from the change of \( i \).

- \( \mathcal{B}_{i,j} \): The reduction in the correlation between \( i \) and \( j \) in the initial state that arises solely from the change of \( j \).

- \( \mathcal{M}_{i,\mu} \): The correlation between \( \mu \) and the changes in \( i \) that is newly formed and lost due to the change from \( \mu \) to \( \mu' \).

Furthermore, when \( j \in \mu \), \( \mathcal{T}_{i,j,\mu} = \mathcal{T}_{i,\mu,\mu} ; j \), which signifies that \( \mathcal{T}_{i,j,\mu} \) is the portion of \( \mathcal{T}_{i,\mu,\mu} \) that is specific to \( j \).

Using the quantities introduced earlier, let us rewrite \( \mathfrak{B} - (\gamma_{\text{tot}} - \gamma) \). Now,

\[
i^* ; \omega_i - i^* ; \hat{i}^\ast \hat{\omega}_i = i^* ; \omega_i - i^* ; \hat{(i, \hat{i})} ; \hat{\omega}_i
\]

(89)

\[
= i^* ; \omega_i - i^* ; \omega_i ; \hat{i}
\]

(90)

\[
\equiv i^* ; \omega_i \mid \hat{i},
\]

(91)
thus, \( \tilde{\gamma}_{\text{tot}} - \gamma \) can be reformulated as:

\[
\tilde{\gamma}_{\text{tot}} - \gamma \overset{\text{def}}{=} \sum_{i=1}^{N} \left[ i^*; \hat{i} - i^*; \omega_i \right]
\]

(92)

\[
= \sum_{i=1}^{N} i^*; \omega_i | \hat{i}
\]

(93)

Herein,

\[
i^*; \omega_i | \hat{i} \overset{\text{F2}}{=} i^*; \mu | \hat{i}, \nu + i^*; \nu | \hat{i}
\]

(94)

\[
i^*; \mu | \hat{i}, \nu + \sum_{j=i+1}^{N} i^*; j | \hat{i}, \nu(j)
\]

(95)

Therefore,

\[
\tilde{\gamma}_{\text{tot}} - \gamma = \sum_{i=2}^{N} \left[ i^*; \mu | \hat{i}, \nu + \sum_{j=i+1}^{N} i^*; j | \hat{i}, \nu(j) \right]
\]

(96)

From Eq. (G13), the first term of Eq. (96) can be rewritten as:

\[
i^*; \mu | \hat{i}, \nu = -\left[ \mathcal{T}_{i,\mu,\mu} | \hat{i} + \mathcal{C}_i | \hat{i} \right] | \nu
\]

(97)

Herein,

\[
\mathcal{T}^2 := \mathcal{T}_{i,\mu,\mu} | \hat{i}
\]

is defined. Upon applying Eq. (G13) to the second term of Eq. (96), one obtains

\[
\sum_{i=2}^{N} \sum_{j=i+1}^{N} i^*; j | \hat{i}, \nu(j) = -\sum_{i=2}^{N} \sum_{j=i+1}^{N} \left[ \mathcal{T}_{i,j,j} | \hat{i}, \nu(j) + \mathcal{C}_{i,j} | \hat{i}, \nu(j) \right]
\]

(99)

In this context, the first term of Eq. (99) can be reformulated as

\[
\sum_{i=2}^{N} \sum_{j=i+1}^{N} \mathcal{T}_{i,j,j} | \hat{i}, \nu(j) = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \mathcal{T}_{i,j,j} | \hat{j}, \nu.
\]

(100)

This is due to setting \( f(i, j) = \mathcal{T}_{i,j,j} | \hat{i}, \nu \) in Eq. (G1), which leads to

\[
\sum_{i=2}^{N} \sum_{j=i+1}^{N} \mathcal{T}_{i,j,j} | \hat{i}, \nu(j) = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \mathcal{T}_{j,i,i} | \hat{j}, \nu
\]

(101)

\[
= \sum_{i=2}^{N} \sum_{j=1}^{i-1} \mathcal{T}_{i,j,j} | \hat{j}, \nu.
\]

(102)
To demonstrate the second equality, we utilize the following relation:

\[
T_{j,i,i} \overset{\text{def}}{=} i ; j | (i, j)' = j ; i | (j, i)' = T_{i,j,j}.
\] (103)

Regarding Eq. (102), we introduce the following notation:

\[
T^3 := \sum_{j=1}^{i-1} T_{i,j,j} | \hat{j}
\] (104)

Subsequently, the second term of Eq. (99) becomes

\[
\sum_{i=2}^{N} \sum_{j=i+1}^{N} C_{i,j} | \tilde{i}, \nu(j) = \sum_{i=2}^{N} \sum_{j=1}^{i-1} C_{i,i} | \hat{j}, \nu \overset{(G1)}{=} \sum_{i=2}^{N} \sum_{j=1}^{i-1} B_{i,j} | \hat{j}, \nu.
\] (105)

Hence,

\[
-(\gamma_{\text{tot}} - \gamma) = \sum_{i=2}^{N} \left\{ T_{i,\mu,\mu} | \tilde{i} + C_{i} | \tilde{i} + \sum_{j=1}^{i-1} T_{i,j,j} | \hat{j} + \sum_{j=1}^{i-1} B_{i,j} | \hat{j} \right\} | \nu.
\] (106)

Additionally, based on Eq. (G25), the expression for \( \mathfrak{B} \) can be reformulated as follows.

\[
\mathfrak{B} = \sum_{i=2}^{N} (-T_{i,\mu,\mu} + M_{i}) | \nu
\] (107)

The right-hand side is then defined by

\[
T^1 := T_{i,\mu,\mu}
\] (108)

From the foregoing observations, we can now rewrite \( \mathfrak{B} - (\gamma_{\text{tot}} - \gamma) \) as follows:

\[
\mathfrak{B} - (\gamma_{\text{tot}} - \gamma) = \sum_{i=2}^{N} \left[ -T_{i,\mu,\mu} + T_{i,\mu,\mu} | \tilde{i} + \sum_{j=1}^{i-1} T_{i,j,j} | \hat{j} + C_{i} | \tilde{i} + \sum_{j=1}^{i-1} B_{i,j} | \hat{j} + M_{i} \right] | \nu
\] (109)

\[
= \sum_{i=2}^{N} \left[ -T_{1} + T_{2} + T_{3} + C_{i} | \tilde{i} + \sum_{j=1}^{i-1} B_{i,j} | \hat{j} + M_{i} \right] | \nu.
\] (110)
2. The Second Law with Internal Correlations in General Dependencies

We shall now transform $\mathfrak{B} - (\tau_{\text{tot}} - \gamma)$ into a form where the independent negative contributions have been eliminated and the interpretation has been simplified. Let us denote the independent negative terms as $\tau^-$. From Corollary 11, it follows that all terms constituting Eqs. (108) and (107) are mutually exclusive for different $i$. Thus, if we can identify independent negative terms for a certain $i$, then the sum over that $i$ represents $\tau^-$. 

Among $\tau^1, \tau^2, \tau^3, C_i, B_i, M_i$, the components $M_i, C_i, B_i$ are exclusive with respect to the other components and are all positive. Consequently, $\tau^-$ is contained within $-\tau^1 + \tau^2 + \tau^3$. In the following, we will identify $\tau^-$ by partitioning $-\tau^1 + \tau^2 + \tau^3$ into exclusive components. Firstly, we rewrite $\tau^3$ as follows:

$$\tau^3 \equiv \sum_{j=1}^{i-1} \left( \tau_{i,j,\mu} | \hat{j} + \tau_{i,j,j} ; \mu' | \hat{j} \right) \tag{112}$$

$$= \sum_{j=1}^{i-1} \left( \tau_{i,j,\mu} | \hat{j}, \mu(j) + \tau_{i,j,j} ; \mu(j) | \hat{j} + \tau_{i,j,j} ; \mu' | \hat{j} \right) \tag{113}.$$

For simplicity, we denote each term in the above equation as follows:

$$\tau^{3,1} := \sum_{j=1}^{i-1} \tau_{i,j,\mu} | \hat{j}, \mu(j), \tag{114}$$

$$\tau^{3,2} := \sum_{j=1}^{i-1} \tau_{i,j,j} ; \mu(j) | \hat{j}, \tag{115}$$

$$\tau^{3,3} := \sum_{j=1}^{i-1} \tau_{i,j,j} ; \mu' | \hat{j}. \tag{116}$$
Taking into account Eq. (F17), the decomposition of \( \sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{j} \) into \( T^{3,1} \) and \( T^{3,2} \) corresponds to the decomposition of the total entropy and internal correlations from the sum of individual entropies. Therefore, \( T^{3,2} \) is encompassed by \( T^{3,1} \). Moreover, \( T^{3,3} \) is exclusive of \( T^1 \) with respect to \( \mu' \). Hence, the independent negative term \( T^- \) satisfies \( \mathcal{Y}(T^-) = \mathcal{Y}(T^1 - T^2) \setminus \mathcal{Y}(T^{3,1}) \). Thus, by partitioning \( T^1 - T^2 \) into terms that are exclusive of \( T^{3,1} \) and those that are included within \( T^{3,1} \), the first term of this partition will be \( T^- \).

Let us partition \( T^1 - T^2 \) accordingly:

\[
T^1 + T^2 = -\sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{i} \quad \quad \quad (117)
\]

\[
= -\sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{i} | \mu(j) \quad \quad \quad (118)
\]

\[
= -\sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{i} | \mu(j) - \sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{j}, \mu(j). \quad \quad \quad (119)
\]

Since the right-hand side is mutually exclusive with respect to \( \hat{j} \), this indeed forms a partition of \( T^1 - T^2 \). Furthermore, the first term on the right-hand side is exclusive of \( T^{3,1} \) with respect to \( \hat{j} \), and the second term is equivalent to \( T^{3,1} | \hat{i} \), which is included within \( T^{3,1} \). Consequently, the decomposition in the above equation satisfies the conditions of the partition described earlier. Hence, we derive the following:

\[
T^- = -\sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{i}, \hat{j} | \mu(j). \quad \quad \quad (120)
\]

We denote the second term by the following symbol:

\[
\Delta = -\sum_{j=1}^{i-1} T_{i,j,\mu} | \hat{i} | \hat{j}, \mu(j) \quad \quad \quad (121)
\]

This component corresponds to \( \Delta \) in Fig. 2.

In light of the above results, we obtain the following expression:

\[
\mathcal{B}_i - (\hat{\tau}_{\text{tot}} - \gamma) = \sum_i \left[ T^{3,1} + T^{3,2} + T^{3,3} - \Delta - T^- + C_i | \hat{i} + \sum_{j=1}^{i-1} B_{i,j} | \hat{j} + M_i \right] \bigg| \nu. \quad (122)
\]

\( \Delta \) is included within \( T^{3,1} \), and

\[
T^{3,1} - \Delta = \sum_{j=1}^{i-1} \left[ T_{i,j,\mu} | \mu(j), \hat{j} - T_{i,j,\mu} | \hat{i} | \mu(j), \hat{j} \right] \quad \quad \quad (123)
\]

\[
= \sum_{j=1}^{i-1} T_{i,j,\mu} | \mu(j), \hat{i}, \hat{j} \quad \quad \quad (124)
\]
FIG. 3. The gray area represents the usable internal correlations. The large circle corresponds to the reduction in the internal correlations, while the small circle corresponds to the components on the right-hand side of Eq. (127). The gray area arises entirely through feedback control, illustrating the essentiality of information-based control to utilize the reduction in internal correlations.

holds true. This corresponds to the component denoted by $\bullet$ in Fig. 2. We represent this as follows:

$$\bar{T}^{3,1} = \sum_{j=1}^{i-1} T_{i,j,\mu} \mid \mu(j), \hat{i}, \hat{j}. \quad (125)$$

Therefore, we have

$$\Delta S_{\text{tot}} \geq \sum_{i=2}^{N} \left[ \bar{T}^{3,1} + T^{3,2} + B_{i,j} \mid \hat{j} + C_i \mid \hat{i} + M_i \right] \nu \quad (126)$$

$$= \sum_{i=2}^{N} \left\{ \sum_{j=1}^{i-1} \left[ T_{i,j,\mu} \mid \mu(j), \hat{i}, \hat{j} + (T_{i,j,\mu} ; \mu(j) + T_{i,j,j} ; \mu') \mid \hat{j} \right] \right.$$  
$$+ C_i \mid \hat{i} + \sum_{j=1}^{i-1} B_{i,j} \mid \hat{j} + M_i \right\} \nu. \quad (127)$$

This is our main result, the Second Law considering internal correlations. The interpretation of this result will be discussed in the following section.

E. The Indispensability of Information

Utilizing the result from the previous section, Eq. (127), let us express the internal correlations available as a resource in the form of a sum of interpretable components. From Eq. (77), we have

$$-\Delta I_{\text{tot}} = \mathcal{B} - \tau_{\text{tot}} \quad (128)$$
Let us consider Eq. (110) in the case of $\gamma = 0$. This leads to the right-hand side equating to $\mathcal{B} - \tau_{\text{tot}}$, thereby aligning with $-\Delta I_{\text{tot}}$. Furthermore, the pure decrement component on the right-hand side, as can be inferred from the derivation process, corresponds to the condition where $\hat{i} = \emptyset, \hat{j} = \emptyset$. Therefore, the pure internal correlation decrement component of $-\Delta I_{\text{tot}}$, that is

$$\Delta I_{\text{tot}} := - \sum_{i=2}^{N} \Delta I_{\hat{i}, \mu},$$

may be expressed as follows:

$$\Delta I_{\text{tot}} = \sum_{i=2}^{N} \left[ \sum_{j=1}^{i-1} T_{i,j,j} + C_{i} + \sum_{j=1}^{i-1} B_{i,j} \right] \nu. \quad (130)$$

Hence, if we represent $\mathcal{I}$ as the internal correlations that can be potentially transformed into free energy, it can be articulated in the following manner:

$$\mathcal{I} = \text{the } i; j \text{ components of } (130) - (127) \quad (131)$$

$$\mathcal{I} = \sum_{i=2}^{N} \left\{ C_{i}; \hat{i} + \sum_{j=1}^{i-1} B_{i,j}; \hat{j} \right. \right. \left. \left. \left. + \sum_{j=1}^{i-1} \left[ T_{i,j,j} - T_{i,j,\mu} \mid \mu(j); \hat{i}, \hat{j} - (T_{i,j,\mu}; \mu(j) + T_{i,j,j}; \mu') \right] \right\} \nu. \quad (132)$$

Herein,

$$\sum_{j=1}^{i-1} T_{i,j,j} = \sum_{j=1}^{i-1} [T_{i,j,\mu} + T_{i,j,j}; \mu'] \quad (133)$$

$$= \sum_{j=1}^{i-1} [T_{i,j,\mu}; \mu(j) + T_{i,j,j}; \mu'] \quad (134)$$

Consequently, The fundamental advantage of utilizing information is as follows.

$$\mathcal{I} = \sum_{i=2}^{N} \left\{ \sum_{j=1}^{i-1} \left[ T_{i,j,\mu}; (\hat{i}, \hat{j}) \mid \mu(j) + (T_{i,j,\mu}; \mu(j) + T_{i,j,j}; \mu') \right] \right. \right. \left. \left. \right. \right. \left. \left. \left. \left. C_{i}; \hat{i} + \sum_{j=1}^{i-1} B_{i,j}; \hat{j} \right\} \nu. \quad (135)$$

This result is depicted in the Information Diagram in Fig. 3. The first area from the left indicates that 'Only the information known to the subsystem capable of controlling $j$ can be utilized for $T_{i,j,\mu} \mu(j)$', the second region denotes 'Only the information known to
FIG. 4. Schematic model of work extraction from internal correlations of a disposable resource. The first subsystem is the agent’s memory, which possesses a storage capacity of 1 bit. The other subsystems consist of the gas molecules enclosed in the container. Initially, all molecules are located on the same side, and the agent’s memory has a correlation with their positions, symbolized by the squiggly line $\sim$. The agent operates the barrier in accordance with its memory’s status, which in turn unlocks the initial mutual correlation across the subsystems, thus yielding positive work for the agent who regards the container as a one-time resource rather than a cycle engine.

IV. EXAMPLES

A. Case Study 1

To exemplify the crucial role and the positive outcomes of the feedback control, we delve into the model depicted in Fig. 4. Within this model, the first subsystem serves as the controller of the barrier to facilitate work extraction. The target of this control encapsulates $N - 1$ ideal gas molecules, which are contained in the container as subsystem 2 : $N$. The entire system is in contact with a reservoir at temperature $T$.

Initially, the barrier bisects the container, with molecules probabilistically distributed in the same division with equal probability of 0.5. Hence, the target subsystems manifest mutual correlations: $I (X_j ; X_k) = \ln 2$ for $j, k \geq 2$. Our discussion is confined to scenarios in which the final state corresponds to the maximum entropy state, allowing molecules to traverse the entire container.
The conceptualization of the model draws inspiration from the Szilard engine \cite{Szilard}, which functions as a cycle engine. In contrast, our model’s container does not function cyclically. Rather, the agent utilizes it as a single-use resource. In the pre-encounter phase, the gas within the container is pre-filled on one side, with the agent lacking knowledge of the specific side where the molecules are located. Following the extraction of work, the agent proceeds to discard the container without restoring it to its original state. Owing to the inherent correlations among the particles, the agent is able to determine the initial collective position of all molecules by observing just a single molecule.

In this narrative, we juxtapose feedback control (FBC) with non-feedback control (NFC). In FBC, the agent is privy to the initial confinement of the molecules: \( I(X_1; X_j) = \ln 2 \) for \( 1 < j \leq N \). Conversely, in NFC, the agent lacks prior knowledge of the molecular states.

We now turn our attention to the right-hand side of Eq. (127) in this case study. To evaluate this expression, it is necessary to determine the interdependencies among subsystems. We assume that the state of the agent’s memory remains unchanged during the barrier’s manipulation. Within the container, the ideal gas particles is subject to the agent’s influence via barrier movement in both FBC and NFC. The inter-system dependencies hence can be articulated as:

\[
\hat{X}_k = \begin{cases} 
\emptyset & (k = 1) \\
\{X_1\} & (k = 2, 3, \ldots, N). 
\end{cases}
\] (136)

As there is no emergence of new correlations, \( \mathcal{M}_i = 0 \) in both FBC and NFC.

In the context of FBC, the initial correlation amongst any two subsystems is \( \ln 2 \). Consequently, all factors except for

\[
\mathcal{C}_i; \hat{\tau} = \begin{cases} 
0 & (i = 1) \\
\ln 2 & (i \geq 2),
\end{cases}
\] (137)

are equal to zero in Eqs. (127) and (135). As a result, the entropy production can attain a value of zero:

\[
\Delta S(X_{\text{tot}}) + \frac{Q_{\text{tot}}}{T} \geq 0
\] (138)

and the gain attributed to feedback control is

\[
\mathcal{I} = (N - 1) \ln 2.
\] (139)
FIG. 5. The model examined in Section IV B exemplifies the occurrence of the loss of new correlations. Initially, the molecule is enclosed to either side of the container, and the 1-bit memory is in a reset state. Subsequent to the measurement linking the memory with the initial molecular position, the container’s barrier is dismantled. As a consequence, the memory does not exhibit correlation with the final molecular state, despite being correlated with the initial state.

Given the minimal operational cost is $T \ln 2$, the utmost net gain is $T(N-2) \ln 2$. This indicates the full potential of the free energy $T(N-1) \ln 2$ to be harnessed as work. The feedback controller’s operational cost $T \ln 2$ is significantly less than the work extractable in the thermodynamic limit $N \to \infty$.

In the NFC scenario, the agent and the molecules are not inter-correlated. Thus, every factor excluding

$$C_i | \hat{\mathcal{X}} = \begin{cases} 0 & (i = 1) \\ \ln 2 & (i \geq 2), \end{cases}$$

is zero in Eqs. (127) and (135). Hence, entropy production remains non-zero:

$$\Delta S(X_{\text{tot}}) + \frac{Q_{\text{tot}}}{T} \geq (N - 1) \ln 2 \quad (141)$$

and the gain occurred by using the information is nullified:

$$I = 0. \quad (142)$$

The occurrence of positive gain is precluded by Eq. (141), despite the fact that the container experiences a loss of free energy equal to $T(N-1) \ln 2$. This finding corroborates our assertion that an agent devoid of feedback control cannot derive positive work from the internal correlation.

B. Case Study 2

To inspect the phenomenon of the loss of new correlations, we analyze a bipartite system composed of subsystems 1 and 2. Subsystem 1 consists of single molecule within a container
and subsystem 2 is the 1-bit memory of the agent. Initially, subsystem 1 is probabilistically located within either side of the container with equal probability of 0.5 and subsystem 2 is in the reset state, as illustrated in Fig. 5. Subsequent to subsystem 2’s measurement that determines the initial location of subsystem 1, the container’s partition is eliminated. Thus, subsystem 2 retains a correlation with the initial but not the terminal state of subsystem 1. The interdependency in this context is depicted as $\hat{X}_1 = \emptyset$ and $\hat{X}_2 = \{X_1\}$. Every factor except for

$$M_i = \ln 2$$

does not occur in Eq. (127). Hence, we conclude $\Delta S(X_{tot}) + \frac{Q_{tot}}{T} \geq \ln 2$. The maximal work extraction is zero, even though there is an entropy production of $T \ln 2$ at the memory. As demonstrated in the preceding analysis, the emergence of redundancy in the recorded measurement outcomes precipitates the loss of new correlations. This redundancy is a consequence of the evolution of the measured subsystems, which in turn diminishes their potential for work extraction.

V. CONCLUSION

In this study, we have introduced an extended formulation of the second law of thermodynamics that takes into account the role of internal correlations. This refined framework suggests the essentiality and net benefit of information. Such an extension of the second law could serve as a fundamental principle in analyzing systems where information processing is pivotal.

The implications of this extended second law may provide insights into the mechanisms by which biological entities maintain homeostasis through the utilization of expendable resources, such as nutrients, highlighting the significance of information processing in the perpetuation of life. This extended law also holds potential as a cornerstone in the development of high-efficiency engines that capitalize on the consideration of internal correlations to minimize losses.

There is still a need for future investigations to shed light on information processing within more intricate frameworks and its underlying principles. Exploring information processing over successive time intervals may be crucial for understanding anticipatory processes like
forecasting and learning. Furthermore, probing into systems composed of competing entities could lead to the discovery of new forms of generalized second laws that underscore the intrinsic value of information processing in the realm of game theory. Additionally, given that our current findings indicate that the local detailed balance condition is tied to the essentiality and net advantage of information, further exploration into the roots of this condition could enhance our comprehension of the fundamental aspects of information processing.

Appendix A: Entropy bound for a subsystem

In this section, we provide a derivation of Eq. (11) based on the premise formulated in Eq. (10). Let us denote by $P^\dagger$ a conjugate probability distribution of $P$. We determine $P^\dagger$ to satisfy the following relations:

$$P^\dagger(\hat{X}_j) = P(\hat{X}_j), \quad (A1)$$
$$P^\dagger(X'_j | \hat{X}_j) = P(X'_j | \hat{X}_j), \quad (A2)$$
$$P^\dagger(X_j | X'_j, \hat{X}_j) = P^B(X_j | X'_j, \hat{X}_j). \quad (A3)$$

Let $\bar{X}_j := X_{tot} \setminus X_j \setminus \hat{X}_j$ and $\bar{X}_j' := X'_{tot} \setminus X'_j \setminus \hat{X}_j$. Using the chain rule, we obtain

$$\frac{P^\dagger(X'_j, X_j, \hat{X}_j)}{P(X_j, X'_j, \hat{X}_j)} = \frac{P^\dagger(X'_j, \hat{X}_j) P^\dagger(X_j | X'_j, \hat{X}_j)}{P(X_j, \hat{X}_j) P(X'_j | X_j, \hat{X}_j)} \quad (A4)$$

and

$$P(X_{tot}, X'_{tot}) = P(\bar{X}_j, \bar{X}_j', \hat{X}_j' | X_j, X'_j, \hat{X}_j) P(X_j, X'_j, \hat{X}_j). \quad (A5)$$

Thus, we infer

$$\left\langle \frac{P^\dagger(X'_j, X_j, \hat{X}_j)}{P(X_j, X'_j, \hat{X}_j)} \right\rangle = \int P(X_{tot}, X'_{tot}) \frac{P^\dagger(X'_j, X_j, \hat{X}_j)}{P(X_j, X'_j, \hat{X}_j)} dX_{tot} dX'_{tot} \quad (A6)$$

$$= \int P(\bar{X}_j, \bar{X}_j', \hat{X}_j' | X_j, X'_j, \hat{X}_j) d\bar{X}_j d\bar{X}_j' d\hat{X}_j$$

$$\times \int P(X_j, X'_j, \hat{X}_j) \frac{P^\dagger(X'_j, X_j, \hat{X}_j)}{P(X_j, X'_j, \hat{X}_j)} dX_j dX'_j d\hat{X}_j \quad (A7)$$

$$= 1. \quad (A8)$$
Moreover, we can infer

\[
\left\langle \frac{P^\dagger(X_j, X_j, \hat{X}_j)}{P(X_j, X_j', \hat{X}_j)} \right\rangle = \left\langle \frac{P^\dagger(X_j', \hat{X}_j) P^\dagger(X_j | X_j', \hat{X}_j)}{P(X_j, X_j') P(X_j | X_j, \hat{X}_j)} \right\rangle = \left\langle \exp \left\{ \ln P(X_j' | \hat{X}_j) - \ln P(X_j | \hat{X}_j) + \ln \frac{P(X_j' | X_j, \hat{X}_j)}{P^\dagger(X_j | X_j', \hat{X}_j)} \right\} \right\rangle
\]

(A9)

By combining Eqs. (C8), (C9), (A8), and (A11),

\[
\left\langle \exp \left\{ \ln P(X_j' | \hat{X}_j) - \ln P(X_j | \hat{X}_j) + \ln \frac{P(X_j' | X_j, \hat{X}_j)}{P^\dagger(X_j | X_j', \hat{X}_j)} \right\} \right\rangle = 1.
\]

(A12)

Based on the premises expressed in Eq. (10), the application of Jensen’s inequality to Eq. (A12) produces inequality (11).

Appendix B: Total increase in internal correlation

Here, we establish that $\Delta I_{\text{tot}}$ represents the increase in the internal correlation of the entire system by mathematical induction with respect to $N$. If $N = 1$, then $\Delta I_{\text{tot}}$ coincides with the increase in the internal correlation of the entire system, because both quantities are equal to zero. Let us proceed to the induction step of the proof. Suppose that $\sum_{j=1}^{k-1} \Delta I_j$ coincides with the internal correlation increase in subsystems $1 : k - 1$. When subsystem $k$ is appended to subsystem $1 : k - 1$, the increase in the internal correlation grows by $\Delta I_k$. Thus, $\sum_{j=1}^k \Delta I_j$ is equal to the total internal correlation increase in subsystems $1 : k$. This completes the induction step.

Appendix C: Fluctuation Theorem for Correlated Subsystems

In this section, we demonstrate the following relation:

\[
\left\langle \exp \left[ \Delta s_{\text{tot}} + \frac{q_{\text{tot}}}{T} - d \right] \right\rangle = 1.
\]

(C1)
Herein, we denote stochastic quantities such as the Shannon information and the dissipated heat by lowercase letters. For instance,

\[ s_{\text{tot}} = -\ln P(X_{\text{tot}}), \quad (C2) \]

\[ i(X_k; X_{1:k-1}) = \ln \frac{P(X_k, X_{1:k-1})}{P(X_k)P(X_{1:k-1})}, \quad (C3) \]

\[ q_{\text{tot}} = T \ln \frac{P^B(X_j | X'_j, \hat{X}_j)}{P(X'_j | X_j, \hat{X}_j)}, \quad (C4) \]

\[ \Delta i_{\text{tot}} = \sum_k \Delta i(X_k; X_{1:k-1}). \quad (C5) \]

Furthermore, we define

\[ d := -\Delta i_{\text{tot}} + \sum_k i(X'_k; \hat{X}_k). \quad (C6) \]

Under this definition, the ensemble average of \( d \) coincides with the right-hand side of Eq. (19):

\[ \langle d \rangle = \gamma - \Delta I_{\text{tot}}. \]

We specify \( P^\dagger \) to fulfill the subsequent relations:

\[ P^\dagger(\hat{X}_j) = P(\hat{X}_j), \quad (C7) \]

\[ P^\dagger(X'_j | \hat{X}_j) = P(X'_j | \hat{X}_j), \quad (C8) \]

\[ P^\dagger(X_j | X'_j, \hat{X}_j) = P^B(X_j | X'_j, \hat{X}_j). \quad (C9) \]

We shall rewrite

\[ \prod_{k=1}^N \frac{P^\dagger(X'_k, X_k, \hat{X}_k)}{P(X_k, X'_k, \hat{X}_k)} \]

in two different manners. Firstly,

\[ \frac{P(X_{1:j-1}, X'_{1:j-1})P(X_j, X'_j, \hat{X}_j)}{P^\dagger(X'_{1:j-1}, X_{1:j-1})P^\dagger(X'_j, X_j, \hat{X}_j)} \]

\[ = \frac{P(X_{1:j-1}, X'_{1:j-1})P(X_j, X'_j | \hat{X}_j)P(\hat{X}_j)}{P^\dagger(X'_{1:j-1}, X_{1:j-1})P^\dagger(X'_j, X_j | \hat{X}_j)P^\dagger(\hat{X}_j)}. \quad (C11) \]

Due to the independence,

\[ P(X_j, X'_j | \hat{X}_j) = P(X_j, X'_j | X_{1:j-1}, X'_{1:j-1}), \quad (C12) \]

\[ P^\dagger(X'_j, X_j | \hat{X}_j) = P^\dagger(X'_j, X_j | X_{1:j-1}, X'_{1:j-1}). \quad (C13) \]
Thus, by the chain rule,

\[
C_{11} = \frac{P(X_{1:j-1}, X'_{1:j-1})P(X_j, X'_j | X_{1:j-1}, X'_{1:j-1})}{P^\dagger(X_{1:j-1}, X'_{1:j-1})P^\dagger(X'_j, X_j | X_{1:j-1}, X'_{1:j-1})} = \frac{P(X_1, X'_1)}{P^\dagger(X_1, X'_1)}. \tag{C14}
\]

By similar transformations,

\[
\frac{P(X_1, X'_1)P(X_2, X'_2 | \tilde{X}_2, \tilde{X}'_2)}{P^\dagger(X'_1, X_1)P(X'_2, X_2 | \tilde{X}'_2, \tilde{X}_2)} = \frac{P(X_1, X'_1, X_2, X'_2)}{P^\dagger(X'_1, X_1, X'_2, X_2)}. \tag{C16}
\]

Applying mathematical induction using Eq. (C16) and Eq. (C15), we obtain

\[
C_{10} = \frac{P^\dagger(X'_{\text{tot}}, X_{\text{tot}})}{P(X_{\text{tot}}, X'_{\text{tot}})}. \tag{C17}
\]

Consequently,

\[
\langle C_{10} \rangle = 1 \tag{C18}
\]

On the other hand, by the chain rule,

\[
\frac{P^\dagger(X'_k, X_k, \tilde{X}_k)}{P(X_k, X'_k, \tilde{X}_k)} = \frac{P^\dagger(X'_k | \tilde{X}_k)P^\dagger(X_k | X'_k, \tilde{X}_k)P^\dagger(\tilde{X}_k)}{P(X_k | \tilde{X}_k)P(X'_k | X_k, \tilde{X}_k)P(\tilde{X}_k)}. \tag{C19}
\]

Therefore,

\[
C_{10} = \exp \left[ \Delta s(X'_k | \tilde{X}_k) + \frac{q_k}{T} \right]. \tag{C20}
\]

Thus,

\[
\prod_k \langle C_{10} \rangle = \exp \left\{ \sum_k \left[ \Delta s(X'_k | \tilde{X}_k) + \frac{q_k}{T} \right] \right\}
= \exp \left\{ \sum_k \left[ \Delta s(X_k) - i(X'_k; \tilde{X}_k) + \frac{q_k}{T} \right] \right\}
= \exp \left\{ \left[ \Delta s(X_{\text{tot}}) + \Delta i_{\text{tot}} - \sum_k i(X'_k; \tilde{X}_k) + \frac{q_{\text{tot}}}{T} \right] \right\}
= \exp \left[ \Delta s(X_{\text{tot}}) + \frac{q_{\text{tot}}}{T} - d \right]. \tag{C21}
\]

Hence, Eq. (C1) is validated.
Appendix D: Derivation of the generalized second law

Let us derive inequality (19) by summing inequality (11) for all subsystems. Let $\sigma_j$ be the left-hand side of inequality (11), which results in $\sigma_j \geq 0$. We can rewrite $\sigma_j$ with individual entropy production and $\gamma_j$ as follows:

$$\sigma_j \equiv \Delta S(X_j) - \Delta I(X_j; \hat{X}_j) + \frac{Q_j}{T} \quad \text{(D1)}$$

$$\equiv \Delta S(X_j) - \gamma_j + \frac{Q_j}{T}. \quad \text{(D2)}$$

By Eq. (F17) and definition (13),

$$\sum_{j=1}^{N} \Delta S(X_j) = \Delta S(X_{\text{tot}}) + \Delta I_{\text{tot}}. \quad \text{(D3)}$$

Since $\gamma = \sum_{j=1}^{N} \gamma_j$ and $Q_{\text{tot}} = \sum_{j=1}^{N} Q_j$,

$$\sum_{j=1}^{N} \sigma_j = \Delta S(X_{\text{tot}}) + \Delta I_{\text{tot}} - \gamma + \frac{Q_{\text{tot}}}{T}. \quad \text{(D4)}$$

By the positivity of $\sigma_j$, Eq. (D4) is always positive. Based on the definition of $D$ expressed in Eq. (13), the positivity of Eq. (D4) coincides with inequality (19).

Appendix E: Correlation in independent evolution

In this section, we confirm that no new correlation occurs between subsystems that evolve independently of each other. We begin by describing a formal definition of the independence of random variables:

**Definition 1** (independence [57]). For random variables $A, B,$ and $Z$, $A$ is independent of $B$ conditioning on $Z$, denoted by $A \perp B \mid Z$, if

$$P(A, B, Z)P(Z) = P(A, Z)P(B, Z) \quad \text{(E1)}$$

for all $A, B,$ and $Z$.

If $P(Z) \neq 0$, then by dividing Eq. (E1) by $P(Z)$ twice, we can obtain

$$P(A, B \mid Z) = P(A \mid Z)P(B \mid Z). \quad \text{(E2)}$$
Therefore, if $A \perp B \mid Z$ and $P(B \mid Z) \neq 0$, then
\[
P(A \mid Z) = \frac{P(A, B \mid Z)}{P(B \mid Z)} = P(A \mid B, Z).
\] (E3)

The following shows the constraints to obtain new correlations.

**Proposition 3.** Let $A, A', B, B'$, and $Z$ be random variables, and consider the evolution from $(A, B)$ to $(A', B')$. Then,
\[
I(A'; B' \mid A, B, Z) = 0. 
\] (E4)

Furthermore, if $A$ evolves independently from $B$ conditioned on $Z$, i.e., $A' \perp B \mid Z$, then
\[
I(A'; B \mid A, Z) = 0, 
\] (E5)
\[
\Delta I(A^*; B) = -I(A; B \mid A'). 
\] (E6)

**Proof.** **Proof of Eq. (E4).** By the chain rule,
\[
P(A', B' \mid A, B, Z)
= P(A' \mid B', A, B, Z)P(B' \mid A, B, Z).
\] (E7)

Because we have assumed that $A' \perp B'$, as expressed in Eq. (2), Eq. (E3) leads to
\[
P(A' \mid B', A, B, Z) = P(A' \mid A, B, Z). 
\] (E8)

By substituting Eq. (E8) into Eq. (E7),
\[
P(A', B' \mid A, B, Z)
= P(A' \mid A, B, Z)P(B' \mid A, B, Z).
\] (E9)

Therefore,
\[
I(A'; B' \mid A, B, Z)
def\left\langle \ln \frac{P(A', B' \mid A, B, Z)}{P(A' \mid A, B, Z)P(B' \mid A, B, Z)} \right\rangle
\equiv 0. 
\] (E10) (E11)

This completes the proof.
Proof of Eq. (E5). Using the chain rule and Eq. (E3),

\[
P(A', B \mid A, Z) = P(A' \mid A, Z)P(B \mid A', A, Z) \tag{E12}
\]

\[
= P(A' \mid A, Z)P(B \mid A, Z). \tag{E13}
\]

Therefore,

\[
I(A', B \mid A, Z) \overset{\text{def.}}{=} \langle \ln \frac{P(A', B \mid A, Z)}{P(A' \mid A, Z)P(B \mid A, Z)} \rangle \tag{E13}
\]

\[
= 0. \tag{E14}
\]

This completes the proof.

Proof of Eq. (E6). We can rewrite \(I(A' ; B)\) as follows:

\[
I(A' ; B) \overset{\text{F12}}{=} I(A' ; A; B) + I(A' ; B \mid A) \tag{E15}
\]

\[
= I(A ; B; A'). \tag{E16}
\]

Further, based on Eq. (F12),

\[
I(A; B; A') - I(A; B) = -I(A; B \mid A') \tag{E17}
\]

Upon combining these observations, we can obtain Eq. (E6). \(\square\)

Appendix F: Shannon Information Measures

In this appendix, we compile the expressions related to Shannon information that are utilized in our analysis. Let us consider random variables \(A, B, C, Z\) and a sequence of random variables \(A_{1:n} = \{A_1, A_2, \ldots, A_n\}\). The following identities are well-established in the literature [58]:

\[
S(A \mid Z) = S(A \mid B, Z) + I(A; B \mid Z), \tag{F1}
\]

\[
I(A; (B, C) \mid Z) = I(A; B \mid Z) + I(A; C \mid B, Z), \tag{F2}
\]

\[
= I(A; B \mid Z) + I(A; C \mid Z) - I(A; B; C \mid Z). \tag{F3}
\]

Given the above, we can deduce:

\[
A; (B, C); Z \overset{\text{F3}}{=} A; B; Z + A; C; Z - A; B; C; Z, \tag{F4}
\]
which leads to the subsequent relationship:

\[ A; B = 0 \Rightarrow A; (B, C); Z = A; C; Z. \]  \hspace{1cm} (F5)

Additionally, we have:

\[ Z; A = Z; B \Rightarrow Z; A; Y = Z; B; Y \]  \hspace{1cm} (F6)

as demonstrated by the following derivation under the condition \( Z; A = Z; B \):

\[
\begin{align*}
Z; A; Y &= Z; A - Z; A | Y \\
    &= Z; B - Z; B | Y \\
    &= Z; B; Y.
\end{align*}
\]  \hspace{1cm} (F7) \hspace{1cm} (F8) \hspace{1cm} (F9)

By setting \( Z; B = 0 \), we obtain:

\[ Z; A = 0 \Rightarrow Z; A; B = 0. \]  \hspace{1cm} (F10)

The interaction information, also referred to as the multivariate mutual information, is defined recursively by extending Eq. (F1) [59, 60]:

\[
I(A_1; A_2; \ldots; A_n | Z) := I(A_1; A_2; \ldots; A_{n-1} | Z) - I(A_1; A_2; \ldots; A_{n-1} | A_n, Z). \]  \hspace{1cm} (F11)

For the case where \( n = 3, A_1 = A, A_2 = B, \) and \( A_3 = C \), we derive:

\[ I(A; B | Z) = I(A; B; C | Z) + I(A; B | C, Z). \]  \hspace{1cm} (F12)

Thus, if \( I(A; B; C | Z) > 0 \), it follows that:

\[ I(A; B | Z) > 0. \]  \hspace{1cm} (F13)

The subsequent proposition provides an insight into the partitioning of the total entropy production:

**Proposition 4.** For random variables \( A_{1:n}, B, \) and \( Z \),

\[
I(A_{1:n}; B | Z) = \sum_{j=1}^{n} I(A_j; B | Z) - \sum_{j=2}^{n} I(A_j; A_{1:j-1}; B | Z). \]  \hspace{1cm} (F14)
Proof. By iteratively applying Eq. (F2), we arrive at:

\[
I(A_1; B \mid Z)
= I(A_1; B \mid Z) + \sum_{j=2}^{n} I(A_j; B \mid A_{1:j-1}, Z). \tag{F15}
\]

Utilizing Eq. (F12), we have:

\[
I(A_j; B \mid A_{1:j-1}, Z)
= I(A_j; B \mid Z) - I(A_j; A_{1:j-1}; B \mid Z). \tag{F16}
\]

Substituting Eq. (F16) into Eq. (F15) yields Eq. (F14).

In the case of \( B = \emptyset \), Eq. (F14) signifies the decomposition of the total entropy into the entropy of each random variable and the mutual information among all random variables:

\[
S(A_1; \ldots; A_n \mid Z)
= \sum_{j=1}^{n} S(A_j \mid Z) - \sum_{j=2}^{n} I(A_j; A_{1:j-1} \mid Z). \tag{F17}
\]

Furthermore, we have the following relationship:

\[
I(A_j; B \mid Z) - I(A_j; A_{1:j-1}; B \mid Z) = I(A_j; B \mid A_{1:j-1}, Z). \tag{F18}
\]

Thus, we can express:

\[
I(A_1; B \mid Z)
= \sum_{j=1}^{n} I(A_j; B \mid A_{1:j-1}, Z). \tag{F19}
\]

Here, let \( A_{1:0} = \emptyset \). By reversing the order of indices, we obtain:

\[
I(A_1; B \mid Z)
= \sum_{j=1}^{n} I(A_j; B \mid A_{j+1:n}, Z). \tag{F20}
\]

Here, let \( A_{n+1:n} = \emptyset \).

**Proposition 5.** The following relationship holds true:

\[
Z; A \mid B + Z; B \mid A = Z; (A, B) - Z; A; B. \tag{F21}
\]
\textit{Proof.} Considering that 
\[
Z; B \mid A = Z; B - Z; A; B, \tag{F22}
\]
we have 
\[
\text{LHS of (F21)} = Z; A \mid B + Z; B - Z; A; B. \tag{F23}
\]
Now, given that 
\[
Z; B + Z; A \mid B = Z; (A, B), \tag{F24}
\]
the proof is complete. 

\textbf{Proposition 6.} \textit{The following equality is satisfied:}
\[
S (A \mid B) - I (A; C) = S (A \mid B, C) - I (A; B; C) \tag{F25}
\]
\textit{Proof.} From Eq. (F3), we have 
\[
S (A \mid B) = A; C \mid B + S (A \mid B, C), \tag{F26}
\]
\[
I (A; C) = A; C \mid B + I (A; B; C). \tag{F27}
\]
Subtracting the corresponding sides yields the proof. 

\textbf{Proposition 7.} \textit{The following statements are valid:}
\begin{enumerate}
\item \textit{I (Y; Z) and S (Y \mid Z, W) are mutually exclusive.}
\item \textit{S (V \mid W, Y, Z) and I (Y; Z) are mutually exclusive.}
\item \textit{Given j \in A, \{k, l\} \in A^2, S (j \mid A \setminus j) and I (k; l) are mutually exclusive.}
\end{enumerate}
\textit{Proof.} Statements 1 and 2 can be shown through simple set operations. Let us prove statement 3. 
\begin{enumerate}
\item[i)] If j = k: Since l \in A \setminus j, we have 
\[
S (j \mid A \setminus j) = S (k \mid l, A \setminus (k, l)). \tag{F28}
\]
From statement 1, the right-hand side and I (k; l) are mutually exclusive. This proves the case when j = k. 
\item[ii)] The case when j = l can be shown similarly to i). 
\item[iii)] If j \neq k, j \neq l: Since k \in A \setminus j and l \in A \setminus j, statement 2 implies statement 3. This completes the proof. \hfill \Box
\end{enumerate}
Proposition 8. The following holds:

\[
\bar{S}(1 : N) = \sum_{i=1}^{N} \bar{S}(i \mid \omega_i) + \sum_{i=2}^{N} \sum_{j=1}^{i-1} i; j \mid \mu \setminus j. \quad \text{(F29)}
\]

\[
\begin{align*}
&= f(N) \\
&= g(N)
\end{align*}
\]

Proof. We prove this by mathematical induction on \( N \). For \( N = 2 \), Eq. (F29) is easily verified. Assume Eq. (F29) holds for \( N = n - 1 \). Then,

\[
\bar{S}(1 : n) = \bar{S}(n \mid \omega_n^n) + \bar{S}(1 : n - 1). \quad \text{(F30)}
\]

By the induction hypothesis,

\[
\bar{S}(1 : n - 1) = \sum_{i=1}^{n-1} \bar{S}(i \mid \omega_i^{n-1}) + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} i; j \mid \mu \setminus j. \quad \text{(F31)}
\]

Since \( \omega_i^n = \{ \omega_i^{n-1}, n \} \), from Eq. (F3), we have

\[
\sum_{i=1}^{n-1} \bar{S}(i \mid \omega_i^{n-1}) = \sum_{i=1}^{n-1} \bar{S}(i \mid \omega_i^n) + \sum_{i=1}^{n-1} n; i \mid \omega_i^{n-1}. \quad \text{(F32)}
\]

Using the above results, we can rewrite Eq. (F30) as follows:

\[
\bar{S}(1 : n) = \bar{S}(n \mid \omega_n^n) + \sum_{i=1}^{n-1} \bar{S}(i \mid \omega_i^n) + \sum_{i=1}^{n-1} n; i \mid \omega_i^{n-1} + \mathcal{A} + g(n - 1) \quad \text{(F33)}
\]

\[
= f(n) + g(n - 1) + \mathcal{A}. \quad \text{(F34)}
\]

Therefore, to prove Eq. (F29) for \( N = n \), it suffices to show the following:

\[
g(n) = g(n - 1) + \mathcal{A}. \quad \text{(F35)}
\]

Let us demonstrate this. From the definition, we have that

\[
\mathcal{A} = \sum_{j=1}^{n-1} n; j \mid \mu(n) \setminus j \quad \text{(F36)}
\]

By separating the \( n \)-th element of the sum over \( i \) in \( g(n) \), we obtain

\[
\sum_{i=2}^{n} \sum_{j=1}^{i-1} i; j \mid \mu \setminus j = \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} i; j \mid \mu \setminus j + \sum_{j=1}^{n-1} n; j \mid \mu(n) \setminus j. \quad \text{(F37)}
\]

By definition, the left-hand side is \( g(n) \), the first term on the right-hand side is \( g(n - 1) \), and the second term is \( \mathcal{A} \). Hence, Eq. (F35) holds. This completes the proof by mathematical induction. \( \square \)
Proposition 9. The following is true:

\[ A; B = A; (B, C). \quad (F38) \]

Proof. Utilizing Eq. (F3) to decompose the terms \( B \) and \( C \) on the right-hand side, it can be demonstrated that they are equivalent to the left-hand side. \( \square \)

Appendix G: Miscellaneous

Proposition 10. Let \( f : (\mathbb{N}, \mathbb{N}) \to \mathbb{R} \) be a function. For any natural number \( N \geq 2 \), the following relation holds:

\[
\sum_{i=1}^{N} \sum_{j=i+1}^{N} f(i, j) = \sum_{i=1}^{j-1} \sum_{j=2}^{N} f(i, j). \quad (G1)
\]

Proof. The proof is by induction on \( N \).

For the base cases, it is easy to show that the equation holds for \( N = 2 \).

Now, assume that the proposition holds for some \( N = k \), where \( k \geq 2 \). That is, assume that

\[
\sum_{i=1}^{k} \sum_{j=i+1}^{k} f(i, j) = \sum_{i=1}^{j-1} \sum_{j=2}^{k} f(i, j) \quad (G2)
\]

is true. We need to show that it also holds for \( N = k + 1 \).

Expanding the left-hand side (LHS) for \( N = k + 1 \) gives

\[
\sum_{i=1}^{k+1} \sum_{j=i+1}^{k+1} f(i, j) = \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} f(i, j),
\]

since the terms for \( i = k + 1 \) do not contribute any valid summation.

For the right-hand side (RHS), we have

\[
\sum_{i=1}^{j-1} \sum_{j=2}^{k+1} f(i, j) = \sum_{i=1}^{k+1} \sum_{j=i+1}^{j-1} f(i, j).
\]

Utilizing the inductive hypothesis for the terms where \( j \leq k \), we recognize that these terms on both sides of the equation are equal by our assumption. Specifically, the part of the summations where \( j \) ranges from \( i + 1 \) to \( k \) on the LHS, and where \( j \) ranges from \( 2 \) to \( k \) on the RHS, are equivalent by the inductive hypothesis.
Now, we address the additional terms that appear when \( j = k + 1 \). On the LHS, the additional terms are of the form \( f(i, k + 1) \) for \( i = 1, 2, \ldots, k \), which come from the extension of the summation in \( j \) to \( k + 1 \). This results in the inclusion of terms \( f(1, k + 1), f(2, k + 1), \ldots, f(k, k + 1) \) in the LHS.

Similarly, on the RHS, when \( j = k + 1 \), the inner summation \( \sum_{i=1}^{j-1} f(i, j) \) for \( j = k + 1 \) includes the same terms: \( f(1, k + 1), f(2, k + 1), \ldots, f(k, k + 1) \).

Consequently, when these additional terms are added to the respective sides, the LHS and RHS remain equivalent. The LHS, expanded as \( \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} f(i, j) \), and the RHS, expanded as \( \sum_{j=2}^{k+1} \sum_{i=1}^{j-1} f(i, j) \), both include exactly the same terms and thus are equal.

Therefore, the equation holds for \( N = k + 1 \), and by the principle of mathematical induction, it holds for all natural numbers \( N \). This completes the proof. \( \square \)

**Proposition 11.** Let \( X_1, X_2, \ldots, X_n \) be sets. Define \( \mu(i) = \bigcup_{k=1}^{i-1} X_k \) and \( \nu(i) = \bigcup_{k=i+1}^{n} X_k \).

For any \( i, \tilde{i} \in \{1, 2, \ldots, n\} \) with \( i \neq \tilde{i} \), and for any subsets \( J \subset \mu(i) \) and \( \bar{J} \subset \mu(\tilde{i}) \), it holds that

\[
(X_i \cap J \setminus \nu(i)) \cap (X_{\tilde{i}} \cap \bar{J} \setminus \nu(\tilde{i})) = \emptyset.
\]

**Proof.** Consider two cases based on the indices \( i \) and \( \tilde{i} \): \( i < \tilde{i} \) and \( i > \tilde{i} \).

**Case 1** \((i < \tilde{i})\): In this case, \( \nu(i) \) includes \( X_i \). Therefore, any element in \( X_i \) is also in \( \nu(i) \). Let \( x \) be an arbitrary element in \( X_i \cap J \setminus \nu(i) \). This implies \( x \) is in \( X_i \) and \( J \) but not in \( \nu(i) \). Since \( X_i \subseteq \nu(i) \), it follows that \( x \) cannot be in \( X_i \). Consequently, \( x \) cannot be an element of \( X_i \cap \bar{J} \setminus \nu(\tilde{i}) \), thus the intersection is empty.

**Case 2** \((i > \tilde{i})\): Here, \( \nu(\tilde{i}) \) includes \( X_i \). Therefore, any element in \( X_i \) is also in \( \nu(\tilde{i}) \). Let \( x \) be an arbitrary element in \( X_i \cap \bar{J} \setminus \nu(\tilde{i}) \). This means \( x \) is in \( X_i \) and \( \bar{J} \) but not in \( \nu(\tilde{i}) \). Since \( X_i \subseteq \nu(\tilde{i}) \), \( x \) cannot be in \( X_i \). Therefore, \( x \) cannot be an element of \( X_i \cap J \setminus \nu(i) \), resulting in an empty intersection.

In both cases, we have shown that the intersection \( (X_i \cap J \setminus \nu(i)) \cap (X_{\tilde{i}} \cap \bar{J} \setminus \nu(\tilde{i})) \) is empty. This concludes that the proposition holds for all \( i, \tilde{i} \) where \( i \neq \tilde{i} \). \( \square \)

The above proposition immediately results in the following corollary.
Corollary 1. For any \( i, \bar{i} \in \{1, 2, \ldots, n\} \) with \( i \neq \bar{i} \), it holds that
\[
\mathcal{Y}(\mathcal{T}_{i, \mu} \mid \nu) \cap \mathcal{Y}(\mathcal{T}_{\bar{i}, \mu} (\bar{i}) \mid \nu(\bar{i})) = \emptyset, \tag{G3}
\]
\[
\mathcal{Y}(\mathcal{T}_{i,j} \mid \nu) \cap \mathcal{Y}(\mathcal{T}_{i,j} \mid \nu(\bar{i})) = \emptyset, \tag{G4}
\]
\[
\mathcal{Y}(\mathcal{C}_i \mid \nu) \cap \mathcal{Y}(\mathcal{C}_i \mid \nu(\bar{i})) = \emptyset, \tag{G5}
\]
\[
\mathcal{Y}(\mathcal{B}_i \mid \nu) \cap \mathcal{Y}(\mathcal{B}_i \mid \nu(\bar{i})) = \emptyset. \tag{G6}
\]

Proposition 12. Let \((\Omega, \Sigma, P)\) be a measure space, where \(\Omega = \{A_1, A_2, \ldots, A_n\}\), and \(\Sigma\) is the power set of \(\Omega\). For \( n \geq 1 \), define \(\mu(n+1) := \{1, 2, \ldots, n\}\) and \(\nu(j, n) := \{j+1, j+2, \ldots, n\}\) for \(1 \leq j < n\). Then the following equation holds:
\[
P(A_{n+1} \cap A_{\mu(n+1)}) = \sum_{j=1}^{n-1} P(A_{n+1} \cap A_j \setminus A_{\nu(j,n)}) + P(A_{n+1} \cap A_n). \tag{G7}
\]

Proof. Consider the measure space \((\Omega, \Sigma, P)\) as defined. We aim to show that
\[
P(A_{n+1} \cap A_{\mu(n+1)}) = \sum_{j=1}^{n-1} P(A_{n+1} \cap A_j \setminus A_{\nu(j,n)}) + P(A_{n+1} \cap A_n).
\]

First, expand the right-hand side (RHS) as follows:
\[
P(A_{n+1} \cap A_n) + \sum_{j=1}^{n-1} P(A_{n+1} \cap A_j \setminus A_{\nu(j,n)}) = P(A_{n+1} \cap A_n) + P(A_{n+1} \cap A_{n-1} \setminus A_n) + \cdots + P(A_{n+1} \cap A_1 \setminus A_{\{2,3,\ldots,n\}}).
\]

Each term \(P(A_{n+1} \cap A_j \setminus A_{\nu(j,n)})\) represents \(A_{n+1}\) intersecting with a unique portion of \(A_j\) not shared with \(A_{j+1}, A_{j+2}, \ldots, A_n\).

Applying the probability rule \(P(A \cap B) + P(A \cap B^c) = P(A)\) for disjoint sets \(A\) and \(B\), these terms can be merged to form:
\[
P(A_{n+1} \cap A_n) + P(A_{n+1} \cap A_{n-1} \setminus A_n) + \cdots + P(A_{n+1} \cap A_1 \setminus A_{\{2,3,\ldots,n\}}) = P(A_{n+1} \cap A_{1,2,\ldots,n}).
\]

The RHS coincides with \(P(A_{n+1} \cap A_{\mu(n+1)})\) by definition. It completes the proof. \(\square\)

Based on the above proposition, we can show the following result.

Proposition 13. Let \((\Omega, \Sigma, P)\) be a measure space, where \(\Omega = \{A_1, A_2, \ldots, A_n\}\), \(\Sigma\) is the power set of \(\Omega\), and \(P\) is a probability measure. Define \(\mu(i) := \{1, 2, \ldots, i-1\}\) and \(\nu(i,n) := \{i+1, i+2, \ldots, n\}\) for \(1 \leq i < n\). Then, for \(n \geq 2\), the following equation holds:
\[
\sum_{i=2}^{n} P(A_i \cap A_{\mu(i)}) = \sum_{j=1}^{n-1} P(A_{\nu(j,n)} \cap A_j).
\]
Proof. To prove the proposition, we use mathematical induction on \(n\).

**Base Case:** For \(n = 2\), the proposition simplifies to \(P(A_2 \cap A_1) = P(A_2 \cap A_1)\), which trivially holds.

**Inductive Step:** Assume the proposition holds for some \(n \geq 2\), i.e.,

\[
\sum_{i=2}^{n} P(A_i \cap A_{\mu(i)}) = \sum_{j=1}^{n-1} P(A_{\nu(j,n)} \cap A_j). \tag{G8}
\]

We need to show it holds for \(n + 1\). Consider the left-hand side (LHS) for \(n + 1\):

\[
\sum_{i=2}^{n+1} P(A_i \cap A_{\mu(i)}) = \sum_{i=2}^{n} P(A_i \cap A_{\mu(i)}) + P(A_{n+1} \cap A_{\mu(n+1)}). \tag{G9}
\]

By the inductive hypothesis and Eq. (G7), Eq. (G8) is equivalent to Eq. (G9).

**Conclusion:** By the principle of mathematical induction, since the base case and the inductive step have been verified, the proposition holds for all \(n \geq 2\).

**Proposition 14.** The following relations hold:

\[
\mathcal{T}_{i,j,\mu} = \mathcal{T}_{i,j,j} \mid \mu' \tag{G10}
\]

\[
\mathcal{T}_{i,j,j} = \mathcal{T}_{i,j,\mu} + \mathcal{T}_{i,j,j} \mid \mu'. \tag{G11}
\]

\[
\mathcal{Y}(\mathcal{T}_{i,j,\mu}) \subset \mathcal{Y}(\mathcal{T}_{i,j,j}) \tag{G12}
\]

**Proof.** These can be readily demonstrated from the definitions.

**Proposition 15.** The following equality is satisfied:

\[
i^\star \mid \hat{i} = - (\mathcal{T}_{i,k,k} + \mathcal{C}_i) \mid \hat{i}. \tag{G13}
\]

**Proof.** From the definitions, we have

\[
\text{R.H.S.} = i^\star \mid \hat{i} \tag{G14}
\]

\[
= (i' \mid i; k - i \mid k) \mid \hat{i}. \tag{G15}
\]

On the other hand,

\[
\text{L.H.S.} = i' \mid \hat{i} - i \mid \hat{i} \tag{G16}
\]

\[
= i' \mid i; k \mid \hat{i} \mid \hat{i} - i \mid k \mid \hat{i}. \tag{G17}
\]

Given that it is conditioned on \(\hat{i}\), we can apply Eq. (E5), yielding

\[
i' \mid i; \hat{i} = 0 \tag{G18}
\]

This demonstrates that the left-hand side and the right-hand side are equal.
Proposition 16. The following is true:

\[ B_i = i'; i; \mu | \mu'. \] \hfill (G19)

Proof. By definition, we have

\[ B_i = -\mu^*; i; \mu - T_{i,\mu,\mu} \] \hfill (G20)
\[ = -\mu^*; i; \mu - i; \mu | (i, \mu)'. \] \hfill (G21)

Now, considering

\[ -\mu^*; i; \mu = i; \mu - i; \mu; \mu' \] \hfill (G22)
\[ = i; \mu | \mu' \] \hfill (G23)
\[ = i; \mu | (i, \mu)' + i; \mu; i' | \mu' \] \hfill (G24)

and substituting this into Eq. (G21), we obtain Eq. (G19).

\[ \square \]

Proposition 17. It is established that:

\[ b_i = -T_{i,\mu,\mu} + M_i. \] \hfill (G25)

Proof. From the definition of \( b_i \), it suffices to show the following:

\[ i^*; \mu + i; \mu^* - i^*; \mu^* = -T_{i,\mu,\mu} + M_i. \] \hfill (G26)

We shall demonstrate that for each component included in \( i; \mu \) and those exclusive to it, the left-hand side and the right-hand side are equivalent. The first term on the right-hand side, \( -T_{i,\mu,\mu} \), is a component contained within \( i; \mu \), while the second term \( M_i \) represents the component that is exclusive to \( i; \mu \). Furthermore, the components of the left-hand side that are included in \( i; \mu \) are as follows:

\[ b_i; i; \mu = i^*; \mu; i + i; \mu^*; \mu - i^*; \mu^*; i; \mu \] \hfill (G27)

From the definitions, we have

\[ i; \mu^*; \mu = -B_i - T_{i,\mu,\mu} \] \hfill (G28)

and

\[ i^*; \mu^*; i; \mu = i'; \mu'; i; \mu - i; \mu \] \hfill (G29)
\[ = i'; i; \mu - i; \mu - i'; i; \mu | \mu' \] \hfill (G30)
\[ = i^*; \mu; i - i'; i; \mu | \mu'. \] \hfill (G31)
Thus,

\[ b_i; i; \mu = -B_i - T_{i,\mu,\mu} + i'; i; \mu | \mu' \]  \hspace{1cm} (G32)

Therefore, the component of \( b_i \) included in \( i; \mu \) is \(-T_{i,\mu,\mu}\).

Next, let us consider the constituents of \( b_i \) that is exclusive of \( i; \mu \). We aim to show the following equation:

\[ b_i \mid (i; j) = \mathcal{M}_{i,\mu}. \]  \hspace{1cm} (G34)

Firstly, by definition, we have

\[ b_i \mid (i; j) = i^\ast; \mu \mid i; j + i; \mu^\ast \mid i; j - i^\ast; \mu^\ast \mid i; j \]  \hspace{1cm} (G35)

and

\[ i^\ast; \mu \mid i; j = i'; \mu \mid i \]  \hspace{1cm} (G36)

\[ i; \mu^\ast \mid i; j = i; \mu' \mid \mu \]  \hspace{1cm} (G37)

\[ i^\ast; \mu^\ast \mid i; j = i'; \mu' \mid i; \mu \]  \hspace{1cm} (G38)

Therefore,

\[ b_i \mid (i; j) = i'; \mu \mid i + i; \mu' \mid \mu - i'; \mu' \mid i; \mu. \]  \hspace{1cm} (G39)

On the other hand,

\[ \mathcal{M}_{i,\mu} \overset{\text{def}}{=} i^\ast; \mu \mid (i, \mu^\ast) + i; \mu^\ast \mid (i^\ast, \mu) \]  \hspace{1cm} (G40)

\[ = i'; \mu \mid (i, \mu') + i; \mu' \mid (i', \mu) \]  \hspace{1cm} (G41)

\[ = i'; \mu \mid i + i; \mu' \mid \mu - \mu; i'; \mu' \mid i - i; i'; \mu' \mid \mu \]  \hspace{1cm} (G42)

Hence, it suffices to demonstrate the following:

\[ i'; \mu' \mid i; \mu = \mu; i' \mid i + i; i'; \mu' \mid \mu. \]  \hspace{1cm} (G43)

Rewriting the left-hand side, we get

\[ i'; \mu' \mid i; \mu = i'; \mu' - i'; \mu' \mid i; \mu \]  \hspace{1cm} (G44)
where

\[ i' \mid \mu' = i' \mid (i, \mu) + i' \mid (i, \mu) \]

\[ \tag{G45} \]

\[ i' \mid \mu' = i' \mid (i, \mu). \]

\[ \tag{G46} \]

Thus, the left-hand side of Eq. (G43) can be rewritten as

\[ i' \mid \mu' \mid i \mid \mu = i' \mid (i, \mu) - i' \mid \mu' \mid i \mid \mu. \]

\[ \tag{G47} \]

The right-hand side of Eq. (G43) can be rewritten as

\[ \mu \mid i' \mid \mu' \mid i + i' \mid \mu' \mid \mu \mid (i, \mu) - i' \mid \mu' \mid i \mid \mu. \]

\[ \tag{G48} \]

Since the right-hand sides of Eqs. (G47) and (G48) match, Eq. (G43) is proven. This completes the proof.

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