Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras

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Abstract

We extend the classical characterization of a finite-dimensional Lie algebra \( g \) in terms of its Maurer-Cartan algebra—the familiar differential graded algebra of alternating forms on \( g \) with values in the ground field, endowed with the standard Lie algebra cohomology operator—to sh Lie-Rinehart algebras. To this end, we first develop a characterization of sh Lie-Rinehart algebras in terms of differential graded cocommutative coalgebras and Lie algebra twisting cochains that extends the nowadays standard characterization of an ordinary sh Lie algebra (equivalently: Linfty algebra) in terms of its associated generalized Cartan-Chevalley-Eilenberg coalgebra. Our approach avoids any higher brackets but reproduces these brackets in a conceptual manner. The new technical tool we develop is a notion of filtered multi derivation chain algebra, somewhat more general than the standard notion of a multicomplex endowed with a compatible algebra structure. The crucial observation, just as for ordinary Lie-Rinehart algebras, is this: For a general sh Lie-Rinehart algebra, the generalized Cartan-Chevalley-Eilenberg operator on the corresponding graded algebra involves two operators, one coming from the sh Lie algebra structure and the other from the generalized action on the corresponding algebra; the sum of the operators is defined on the algebra while the operators are individually defined only on a larger ambient algebra. We illustrate the structure with quasi Lie-Rinehart algebras.

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1 Introduction

A finite-dimensional Lie algebra $\mathfrak{g}$ can be characterized in terms of its Maurer-Cartan algebra, that is, the algebra of alternating forms on $\mathfrak{g}$ with the (cartan-)Chevalley-Eilenberg differential. The same is true of a Lie-Rinehart algebra $(A,L)$ when $L$ is finitely generated and projective as an $A$-module. A Lie-Rinehart algebra $(A,L)$ is a pair that consists of a commutative algebra $A$ and a Lie algebra $L$ together with an $A$-module structure on $L$ and an $L$-action on $A$ by derivations such that two obvious axioms are satisfied; these axioms are modeled on the standard example $(A,L) = (C^\infty(M),\text{Vect}(M))$ that consists of the smooth functions $C^\infty(M)$ and smooth vector fields $\text{Vect}(M)$ on a smooth manifold $M$. Given a Lie-Rinehart algebra $(A,L)$, the CCE operator on the CCE algebra $\text{Alt}(L,A)$ involves two derivations $\partial [\cdot,\cdot]$ and $\partial^t$, the first coming from the Lie bracket and the second from the $L$-action on $A$, and the sum $d = \partial [\cdot,\cdot] + \partial^t$, at first defined on $\text{Alt}(L,A)$, passes to a derivation on the $A$-valued $A$-multilinear forms $\text{Alt}_A(L,A)$ and turns this algebra into differential graded $R$-algebra, even though the individual derivations $\partial [\cdot,\cdot]$ and $\partial^t$ do not necessarily descend, and we refer to the resulting differential graded $R$-algebra $(\text{Alt}_A(L,A),d)$ as the Maurer-Cartan algebra associated to the data. When $L$ is finitely generated and projective as an $A$-module, this Maurer-Cartan algebra characterizes the Lie-Rinehart algebra. In the situation of the standard example $(A,L) = (C^\infty(M),\text{Vect}(M))$, the Maurer-Cartan algebra is the de Rham algebra of the underlying smooth manifold.

In this paper we generalize the Maurer-Cartan characterization to sh Lie-Rinehart algebras. The idea of an sh Lie algebra or, equivalently, $L_\infty$ algebra, has a history [Hue11, Hue10]; we only mention that the $A_\infty$-algebra concept, prior to the $L_\infty$-algebra concept, was introduced by J. Stasheff in the 1960-s, cf. [Hue11, Hue10]. Sh Lie-Rinehart algebras were introduced in [Kje01] (part of a thesis supervised by J. Stasheff). In [Hue05] we introduced quasi Lie-Rinehart algebras as a higher homotopies generalization of ordinary Lie-Rinehart algebras. Quasi Lie-Rinehart algebras actually arise in mathematical nature in the theory of foliations [Hue05]. In this paper we develop a Maurer-Cartan type characterization of sh Lie-Rinehart algebras. This recovers quasi Lie-Rinehart algebras as a special case of sh
Lie-Rinehart algebras. The new technical tool we introduce for that purpose is a notion of multi derivation chain algebra, more flexible than the traditional concept of a multicomplex endowed with a compatible algebra structure (as we hope to demonstrate in this paper) and also somewhat more general, cf. Remarks 4.1 and 4.6 below. In Theorems 4.7 and 4.11 below, we show how sh Lie-Rinehart algebras can be characterized in terms of the newly developed notion of multi derivation chain algebra. Suffice it to mention here that a multi derivation chain algebra is a graded commutative algebra $A$ together with a filtration $A^0 \supseteq A^1 \ldots$ and a family of degree $-1$ derivations $\{D_j\}_{j\geq 0}$ such that $D_j$ lowers filtration by $j$ and such that $\sum_{j\geq 0} D_j$ is a differential. See Section 4 for details. The crucial observation now is this: An sh Lie-Rinehart algebra $(A, L)$ leads to a multi derivation chain algebra $(A, D_0, D_1, D_2, \ldots)$ of graded symmetric $A$-multilinear forms but, beware, the differential $\sum_{j\geq 0} D_j$ is only linear over the ground ring, such that, for $j \geq 1$, each operator $D_j$ has the form $D_j = \partial_j^{[\cdot, \cdot]} + \partial_j^t$ and such that, just as in the case of ordinary Lie-Rinehart algebras, while $\partial_j^{[\cdot, \cdot]}$ and $\partial_j^t$ are not individually defined on $A$ (only on a larger ambient graded algebra), their sum $D_j$ is defined on $A$. Under a suitable additional assumption ($A$-reflexivity of $L$), these multi derivation chain algebra structures then characterize sh Lie-Rinehart algebra structures on $(A, L)$. A salient feature is that an sh Lie structure of $L$ lives on the cofree differential graded cocommutative coalgebra $S^c[sL]$ on $sL$ over the ground ring whereas the Maurer-Cartan algebra characterization of an sh Lie-Rinehart structure is phrased in terms of an algebra of graded symmetric $A$-multilinear forms on $sL$, viewed as a graded $A$-module.

To make the results more easily accessible, in Section 2 we explain first the special case of ordinary Lie-Rinehart algebras in a language tailored to the general situation. In Section 3 we spell out some technical tools that are indispensable thereafter. Here we borrow from the theory of homological perturbations, cf. e.g., [Hue10] and [Hue11]. We spell out the main results related with sh Lie-Rinehart algebras in Section 5. Theorem 5.15 says that, given the relevant data, under the hypothesis spelled out there, these data constitute an sh Lie-Rinehart algebra if and only if they induce a multi derivation chain algebra structure on the corresponding object. Theorem 5.16 says that, under the stronger hypothesis of this theorem, every multi derivation chain algebra structure on of the kind under discussion arises from a unique sh Lie-Rinehart algebra structure. See also Remark 5.17 below.

On the technical side we note here that we avoid any “bracket yoga”. In $L_\infty$-technology, it is nowadays common to use a bracket zoo which necessarily comes with complications related with signs etc. Our approach in terms of differential graded cocommutative coalgebras and Lie algebra twisting cochains avoids spelling out explicitly any of the corresponding brackets and takes care of any of the complications by itself, once the Eilenberg-Koszul sign convention has been implemented.

The terminology ‘Maurer-Cartan algebra’ goes back at least to [VE89]; among many other things, van Est noticed that the idea of a Maurer-Cartan algebra was used by E. Cartan already in 1936 to characterize the structure of Lie groups and Lie algebras.

A construction aimed at characterizing sh Lie-Rinehart algebras in terms of Maurer-Cartan algebras has been developed in [Vit12]. The approach in that paper tames the corresponding bracket zoo. In [Hue04a] and [Hue05] we used the terminology “multialgebra” for what we now refer to as a multi derivation chain algebra. We hope this avoids confusion with a well established notion of multialgebra in the literature that has a meaning very different from that of multi derivation chain algebra, cf. e.g., [Gra62].

We are much indebted to J. Stasheff, for many discussions on the topic, for having enthui-
siastically insisted that the relationship between the various notions discussed in this paper be conclusively clarified, and for a number of valuable comments on a draft of the manuscript.

2 Ordinary Lie-Rinehart algebras

For ease of exposition, we explain first the case of ordinary Lie-Rinehart algebras, in language and notation tailored to our purposes. We hope this will provide a road map for the reader so that he can more easily digest the material in later sections.

Under suitable circumstances, a Lie-Rinehart algebra can be characterized in terms of its Maurer-Cartan algebra. We will give a precise statement as Theorem 2.7 below. To prepare for it, let \( R \) be a commutative ring with 1; henceforth the unadorned tensor product refers to the tensor product over \( R \). Let \( A \) be a commutative \( R \)-algebra; then the commutator bracket turns the \( A \)-module \( \text{Der}(A/R) \) of derivations of \( A \) into an \( R \)-Lie algebra (beware: this is no longer true when \( A \) is non-commutative). Further, let \( L \) be an \( A \)-module, \([\cdot, \cdot]: L \times L \to L\) a skew-symmetric pairing, and

$$\partial: L \to \text{Der}(A/R)$$

an \( R \)-linear map. Given \( \alpha \in L \) and \( a \in A \) we write \( \alpha(a) = (\partial(\alpha))(a) \). The pair \((A, L)\) is said to constitute a Lie-Rinehart algebra when the pieces of structure satisfy two obvious axioms modeled on the pair \((A, \text{Der}(A/R))\), cf. \cite{Hue90}. These axioms read

\begin{align*}
(a \alpha)(b) &= a (\alpha(b)), \quad \alpha \in L, \ a, b \in A, \\
[\alpha, a \beta] &= a [\alpha, \beta] + \alpha(\alpha) \beta, \quad \alpha, \beta \in L, \ a \in A.
\end{align*}

Given only the pieces of structure \( A, L, [\cdot, \cdot], (2.1) \), consider the \( R \)-algebra \( \text{Alt}(L, A) \) of \( A \)-valued \( R \)-multilinear alternating forms on \( L \), and define two \( R \)-linear derivations \( \partial^f \) and \( \partial^{[\cdot, \cdot]} \) on \( \text{Alt}(L, A) \) by the familiar expressions

\begin{align*}
(-1)^{n-1}(\partial^f)(\alpha_1, \ldots, \alpha_n) &= \sum_{i=1}^n (-1)^{(i-1)}\alpha_i(f(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_n)) \\
(-1)^{n-1}(\partial^{[\cdot, \cdot]})(\alpha_1, \ldots, \alpha_n) &= \sum_{1 \leq j < k \leq n} (-1)^{(j+k)}f([\alpha_j, \alpha_k], \alpha_1, \ldots, \hat{\alpha_j}, \ldots, \hat{\alpha_k}, \ldots, \alpha_n).
\end{align*}

The sign \((-1)^{n-1} = (-1)^{|f|}\) is, perhaps, not entirely classical. According to the Eilenberg-Koszul convention, it is the correct sign, cf. \( (2.1) \) and \( (4.4) \) below. We will justify the notation \( \partial^f \) shortly; suffice it to note for the moment that, when \([\cdot, \cdot]\) is a Lie bracket and \( (2.1) \) an \( L \)-action on \( A \) by derivations, the operator \( \partial^f \) arises from a Lie algebra twisting cochain \( t \) that recovers the \( L \)-action on \( A \). Let

$$d = \partial^f + \partial^{[\cdot, \cdot]}.$$  

The reader will notice right away the following.

**Proposition 2.1.** When \([\cdot, \cdot]\) is a Lie bracket and \( (2.1) \) a morphism of \( R \)-Lie algebras, the derivation \( d \) is a differential, in fact, the classical C(artan-)C(hevalley-)E(ilenberg) operator, and the differential graded algebra \((\text{Alt}(L, A), d)\) computes the CCE cohomology of \( L \) with coefficients in \( A \).

Using terminology that goes back at least to \cite{VE89}, we refer to a differential graded algebra of the kind \((\text{Alt}(L, A), d)\) as a Maurer-Cartan algebra.

Concerning Lie-Rinehart algebras, a crucial observation is now the following.
Proposition 2.2. When \((A, L)\) is a Lie-Rinehart algebra, the derivation \(d\) descends to an \(R\)-linear differential on \(\text{Alt}_A(L, A)\), even though this is not true of the individual operators \(\partial^t\) and \(\partial^{[\cdot, 
abla]}\) unless \(A = R\) and \(\partial^t\) is trivial.

This observation has a long history; see e.g., [Hue04b] for a survey. Under such circumstances, we refer to the differential graded \(R\)-algebra \((\text{Alt}_A(L, A), d)\) as the Maurer-Cartan algebra associated to the Lie-Rinehart algebra and to \(d\) as its CCE operator.

We spell out the proof since it clearly shows how the Lie-Rinehart axioms imply that the derivation \(d\) descends to an \(R\)-linear differential on \(\text{Alt}_A(L, A)\), even though this is not necessarily true of the individual operators \(\partial^t\) and \(\partial^{[\cdot, 
abla]}\).

Proof of Proposition 2.2. We explain the crucial observation; this will help the reader understand the general case in Theorem 5.1 below: Let \(u \in A\) and \(\varphi \in \text{Alt}_A^0(L, A) = \text{Hom}_A(L, A)\). Our aim is to show that \(du\) is \(A\)-linear and that \(d\varphi\) is \(A\)-bilinear. Given \(\alpha, \beta \in L\) and \(a \in A\), exploiting the \(A\)-linearity of \(\varphi\) and the two Lie-Rinehart axioms (2.2) and (2.3), we find

\[
du(a\alpha) = \partial^t u(a\alpha) = (a\alpha)(u) = a((\alpha)(u)) = adu,
\]

\[
d\varphi(\alpha, a\beta) = \partial^t \varphi(\alpha, a\beta) + \partial^{[\cdot, 
abla]} \varphi(\alpha, a\beta)
\]

\[
\partial^t \varphi(\alpha, a\beta) = a\partial^t \varphi(\alpha, \beta) + \alpha(a)\varphi(\beta)
\]

\[
\partial^{[\cdot, 
abla]} \varphi(\alpha, a\beta) = a\partial^{[\cdot, 
abla]} \varphi(\alpha, \beta) - \alpha(a)\varphi(\beta)
\]

\[
d\varphi(\alpha, a\beta) = a \left( \partial^t \varphi(\alpha, \beta) + \partial^{[\cdot, 
abla]} \varphi(\alpha, \beta) \right) = ad\varphi(\alpha, \beta).
\]

A similar reasoning establishes the \(A\)-multilinearity in an arbitrary degree.

Remark 2.3. There is no need to confirm the \(A\)-multilinearity in (upper) degrees \(> 1\): When \(L\) is finitely generated as an \(A\)-module, as an \(R\)-algebra, the graded \(A\)-algebra \(\text{Alt}_A(L, A)\) is generated by its elements in degree 0 and (upper) degree 1. When \(L\) is not finitely generated as an \(A\)-module, in a suitable topology, the graded \(A\)-subalgebra of \(\text{Alt}_A(L, A)\) generated by its elements in degree 0 and (upper) degree 1 is dense in \(\text{Alt}_A(L, A)\).

We will now develop an alternate characterization of Lie-Rinehart algebras, to be given as Theorem 2.5 below. This characterization will pave the way for developing a notion of sh Lie-Rinehart algebra in Section 5 below. To this end, let \(sL\) be the suspension of \(L\), that is, \(sL\) is the \(A\)-module \(L\) regraded up by one, and consider the (graded) exterior \(R\)-algebra \(\Lambda[sL]\); to avoid misunderstanding or confusion, we note that we take \(\Lambda[sL]\) to be the graded symmetric \(R\)-algebra on \(sL\). The familiar shuffle diagonal turns \(\Lambda[sL]\) into an \(R\)-bialgebra, in particular, into an \(R\)-coalgebra, and the skew-symmetric bracket \([\cdot, 
abla]\) on \(L\) (not assumed to satisfy the Jacobi identity) determines and is determined by a degree \(-1\) coderivation \(\partial^{[\cdot, 
abla]}\) on \(\Lambda[sL]\). This coderivation induces the derivation \(\partial^{[\cdot, 
abla]}\) on \(\text{Alt}(L, A) \cong \text{Hom}(\Lambda[sL], A)\) given by (2.5); at this stage the sign in (2.5) is the correct one. Moreover, \(\text{Alt}(L, \text{Der}(A|R)) \cong \text{Hom}(\Lambda[sL], \text{Der}(A|R))\) acquires a graded Lie algebra structure as well as a graded \(\text{Alt}(L, A)\)-module structure, the action of \(\text{Der}(A|R)\) on \(A\) extends to an action

\[
([\cdot, 
abla]: \text{Alt}(L, \text{Der}(A|R)) \times \text{Alt}(L, A) \longrightarrow \text{Alt}(L, A))
\]
by derivations and, with the structure of mutual interaction, the pair

\[(\text{Alt}(L, A), \text{Alt}(L, \text{Der}(A|R))) \cong (\text{Hom}(\Lambda[sL], A), \text{Hom}(\Lambda[sL], \text{Der}(A|R)))\]  
\hspace{1cm} (2.8)

constitutes a graded Lie-Rinehart algebra. This graded Lie-Rinehart algebra is a special case of (3.20) below. The \(R\)-linear map \(\vartheta: L \rightarrow \text{Der}(A|R),\) cf. (2.1), determines (and is determined by) the degree \(-1\) morphism

\[t = \vartheta \circ s^{-1}: sL \longrightarrow \text{Der}(A|R),\]  
\hspace{1cm} (2.9)

the composite of the desuspension \(s^{-1}\) with \(\vartheta,\) and \(\partial^f = [t, \cdot]\) is the degree \(-1\) derivation on \(\text{Hom}(\Lambda[sL], A) \cong \text{Alt}(L, A)\) given by (2.11). The various operators are now related by the identity

\[\left( [\partial^f \cdot \cdot, \partial^f] + \partial^f \partial^f \right) \varphi = [t \partial^f \cdot \cdot] + t \wedge t, \varphi \in \text{Alt}(L, A).\]  
\hspace{1cm} (2.10)

Notice that the composite

\[\Lambda[sL] \xrightarrow{\partial^f \cdot \cdot} \Lambda[sL] \xrightarrow{t} \text{Der}(A|R)\]

is a homogeneous degree \(-2\) member of \(\text{Hom}(\Lambda[sL], \text{Der}(A|R))\) in an obvious manner, and here and below we use the familiar notation \(t \wedge t = \frac{1}{2}[t, t].\) Later in the paper, we will generalize the identity (2.10) to (4.21).

Let \(C\) be a coaugmented differential graded cocommutative coalgebra and \(g\) a differential graded Lie algebra. Then the cup bracket \([\cdot, \cdot]\) induced by the diagonal of \(C\) and the bracket \([\cdot, \cdot]\) of \(g\) (where the notation \([\cdot, \cdot]\) is abused) turns \(\text{Hom}(C, g)\) into a differential graded Lie algebra, the differential \(D\) on \(\text{Hom}(C, g)\) being the ordinary Hom-differential; a \textit{Lie algebra twisting cochain} is a homogeneous morphism \(t: C \rightarrow g\) of \(R\)-modules of degree \(-1\) whose composite with the coaugmentation \(\eta: R \rightarrow C\) is zero and which, with the notation \(t \wedge t = \frac{1}{2}[t, t],\) satisfies the identity

\[Dt + t \wedge t = 0.\]  
\hspace{1cm} (2.11)

See e.g., [HS02], [Moo71], [Qui69]. The sign here is the same as that in [Qui69]; it differs from that in [HS02]. The present sign convention simplifies our formulas.

The following is immediate; we spell it out for ease of exposition.

\textbf{Proposition 2.4.} (i) \textit{The bracket \([\cdot, \cdot]\) on \(L\) is a Lie bracket, i.e., satisfies the Jacobi identity if and only if} \(\partial^f \cdot \cdot \cdot \partial^f \cdot \cdot = 0.\)

(ii) \textit{Suppose that the bracket \([\cdot, \cdot]\) on \(L\) is a Lie bracket and let} \(t\) \textit{be the degree \(-1\) morphism of \(R\)-modules given by (2.9). Then the following are equivalent:}

- \textit{The morphism} \(t\) \textit{is a Lie algebra twisting cochain} \((\Lambda[sL], \partial^f \cdot \cdot) \rightarrow \text{Der}(A|R));\)
- \textit{the degree \(-1\) morphism} \(t\) \textit{satisfies the identity} \(t \partial^f \cdot \cdot + t \wedge t = 0;\)
- \textit{the degree zero morphism} \(\vartheta,\) \textit{cf. (2.1), is a morphism of} \(R\)-\textit{Lie algebras.}\]

The following observation characterizes a Lie-Rinehart algebra structure on \((A, L)\) entirely in terms of the corresponding coderivation on \(\Lambda[sL]\) and the corresponding degree \(-1\) morphism \(\Lambda[sL] \rightarrow \text{Der}(A|R)).\)
Theorem 2.5. Given the data \((A, L, \cdot, \cdot, \vartheta)\), as before, let \(t = \vartheta \circ s^{-1} : \Lambda[sL] \to \text{Der}(A|R)\), cf. (2.9). The following are equivalent.

(i) The data \((A, L, \cdot, \cdot, \vartheta)\) constitute a Lie-Rinehart algebra.

(ii) The coderivation \(\partial_{\cdot, \cdot}\) on \(\Lambda[sL]\) is a differential, i.e., \(\partial_{\cdot, \cdot} \circ \partial_{\cdot, \cdot} = 0\), the degree \(-1\) morphism \(t\) is \(A\)-linear, and \(\partial_{\cdot, \cdot}\) and \(t\) are related by the following identities:

\[
\begin{align*}
t \partial_{\cdot, \cdot} + t \wedge t &= 0, \\
\partial_{\cdot, \cdot}(\alpha_1, a\alpha_2) &= (t(\alpha_1)(a))a_2 + a\partial_{\cdot, \cdot} \partial_{\cdot, \cdot} \alpha_2, \quad \alpha_1, \alpha_2 \in sL, \ a \in A.
\end{align*}
\]

Proof. This is straightforward and left to the reader. \(\square\)

Corollary 2.6. Suppose that the \(A\)-module \(L\) has the property that the canonical map

\[
L \longrightarrow \text{Hom}_A(\text{Hom}_A(L, A), A)
\]

from \(L\) to its double \(A\)-dual \(\text{Hom}_A(\text{Hom}_A(L, A), A)\) is an injection of \(A\)-modules. Then the pair \((A, L)\) constitutes a Lie-Rinehart algebra if and only if the derivation \(d = \vartheta + \partial_{\cdot, \cdot}\) on \(\Lambda[sL]\) passes to an \(R\)-linear differential on \(\text{Alt}(L, A)\), necessarily a derivation.

The proof is straightforward, cf. e.g., [Hue05] (Lemma 2.2.11). In Theorem 4.7 below, we will generalize the sufficiency claim. To prepare for this generalization, we will now give a technically more involved proof of Corollary 2.6, which extends to the more general situation of Theorem 4.7, see also Remark 4.10 below.

Proof of Corollary 2.6. Proposition 2.2 shows that the condition is necessary. To show that it is sufficient, suppose that the derivation \(d = \vartheta + \partial_{\cdot, \cdot}\) on \(\Lambda[sL]\) passes to an \(R\)-linear differential on \(\text{Alt}(L, A)\).

Let \(a \in A\). Since \(\partial_{\cdot, \cdot} \circ d a = 0\),

\[
0 = dd a = (\partial_{\cdot, \cdot} + \partial') (\partial_{\cdot, \cdot} + \partial') a
= \left(\partial_{\cdot, \cdot}, \partial' + \partial' \partial'\right) a
= \left[t \partial_{\cdot, \cdot} + t \wedge t, a\right].
\]

Since \(a\) is arbitrary, we conclude

\[
t \partial_{\cdot, \cdot} + t \wedge t = 0,
\]

whence \(\vartheta : L \to \text{Der}(A|R)\), cf. (2.1), is compatible with the brackets. Consequently, on \(\text{Alt}(L, A)\) (beware: not on \(\text{Alt}_A(L, A)\), since this would not even make sense on \(\text{Alt}_A(L, A)\))

\[
[\partial_{\cdot, \cdot}, \partial'] + \partial' \partial' = 0.
\]

Next, let \(\varphi \in \text{Hom}_A(sL, A)\), and view \(\varphi\) as a member of \(\text{Hom}(sL, A)\). Then

\[
0 = dd \varphi = (\partial_{\cdot, \cdot} + \partial') (\partial_{\cdot, \cdot} + \partial') \varphi
= \left(\partial_{\cdot, \cdot} + \partial' + \partial' \partial'\right) \varphi
= \partial_{\cdot, \cdot} \partial_{\cdot, \cdot} \varphi + \left[t \partial_{\cdot, \cdot} + t \wedge t, \varphi\right]
= \partial_{\cdot, \cdot} \partial_{\cdot, \cdot} \varphi.
\]
Let \( x \in \Lambda_3[sL] \). Then
\[
0 = (dd\varphi)(x) = (\partial^{[\cdot,\cdot]}\partial^{[\cdot,\cdot]}\varphi)(x) = \varphi(\partial^{[\cdot,\cdot]}\partial^{[\cdot,\cdot]}(x)) \in A.
\]
Since \( \varphi \) is arbitrary, and since \((2.13)\) is injective, we conclude that \( \partial^{[\cdot,\cdot]}\partial^{[\cdot,\cdot]}(x) = 0 \) \( \in sL \).
Since \( x \) is arbitrary, we see that \( \partial^{[\cdot,\cdot]}\partial^{[\cdot,\cdot]} = 0: \Lambda_3[sL] \rightarrow \Lambda_1[sL] = sL \), whence the bracket \([\cdot,\cdot]\) on \( L \) satisfies the Jacobi identity.

Let \( a, b \in A \) and \( \alpha \in sL \). Since \( \partial^{[\cdot,\cdot]}(a) = 0 \), the hypothesis of the corollary implies that
\[
b(t(a))(b) = b(\partial^t a)(b) = (\partial^t a)(b) = (t(ba))(a)
\]
whence, since \( a \) is arbitrary, \( t \) is \( A \)-linear or, in other words, the data satisfy the axiom \((2.2)\).

Finally, let \( a \in A \), \( \alpha, \beta \in sL \), and let \( \varphi \in \text{Hom}_A(sL, A) \). Then
\[
(\partial^t \varphi)(\alpha, a\alpha_2) = a(\partial^t \varphi)(\alpha, \alpha_2) - \varphi(((t\alpha_1)(a))\alpha_2).
\]
Indeed,
\[
(\partial^t \varphi)(\alpha, a\alpha_2) = [t, \varphi](\alpha, a\alpha_2)
= a[t, \varphi](\alpha, \alpha_2) - ((t\alpha_1)(a))\varphi(\alpha_2)
= a(\partial^t \varphi)(\alpha, \alpha_2) - \varphi(((t\alpha_1)(a))\alpha_2).
\]
Moreover,
\[
\partial^{[\cdot,\cdot]}\varphi(\alpha, a\alpha_2) = \varphi(\partial^{[\cdot,\cdot]}(\alpha, a\alpha_2)).
\]
Since \( \partial^t + \partial^{[\cdot,\cdot]} \) passes to \( \text{Alt}_A(L, A) \), we conclude
\[
(\partial^t + \partial^{[\cdot,\cdot]})(\varphi(\alpha, a\alpha_2) = a(\partial^t + \partial^{[\cdot,\cdot]})(\varphi(\alpha, \alpha_2)
= a(\partial^t \varphi)(\alpha, \alpha_2) - \varphi(((t\alpha_1)(a))\alpha_2) + \varphi(\partial^{[\cdot,\cdot]}(\alpha, a\alpha_2))
= a(\partial^t \varphi)(\alpha, \alpha_2) + a\varphi(\partial^{[\cdot,\cdot]}(\alpha, \alpha_2))
\]
Since \( \varphi \) is arbitrary, the hypothesis of the corollary implies the identity \((2.13)\), viz.
\[
\partial^{[\cdot,\cdot]}(\alpha, a\alpha_2) = ((t\alpha_1)(a))\alpha_2 + a\partial^{[\cdot,\cdot]}(\alpha, \alpha_2).
\]

Theorem \(2.5\) implies that the data \((A, L, [\cdot,\cdot], \theta)\) constitute a Lie-Rinehart algebra. In particular, the identity \((2.13)\) implies that the data satisfy the axiom \((2.3)\).

**Theorem 2.7.** Suppose that the canonical \( A \)-module morphism \((2.14)\) from \( L \) to its double \( A \)-dual \( \text{Hom}_A(\text{Hom}_A(L, A), A) \) is an isomorphism of \( A \)-modules. Let \( d \) be an \( R \)-linear derivation on \( \text{Alt}_A(L, A) \) that turns the graded \( A \)-algebra \( \text{Alt}_A(L, A) \) into a differential graded \( R \)-algebra. Then \((\text{Alt}_A(L, A), d)\) is the Maurer-Cartan algebra associated to a (necessarily unique) Lie-Rinehart structure on \((A, L)\).

For example, the hypothesis of Theorem \(2.7\) as well as that of Corollary \(2.6\) is satisfied when \( L \) is a finitely generated projective \( A \)-module.

See [Hue15] (Lemma 2.2.15) for a traditional proof. We now sketch a proof in the language and terminology of the proof of Corollary \(2.6\) above. An extension of this reasoning yields the proof of a more general result, Theorem \(4.11\) below.
Proof of Theorem 2.7

Let \( t: sL \rightarrow \text{Der}(A|R) \subseteq \text{End}(A, A) \) be the adjoint of the composite
\[
d: A \rightarrow \text{Hom}_A(sL, A) \subseteq \text{Hom}(sL, A)
\]
of the derivation \( d \) with the injection into \( \text{Hom}(sL, A) \) as displayed. Then the derivation
\[
\partial^f: \text{Hom}_A(\Lambda[sL], A) \rightarrow \text{Hom}(\Lambda[sL], A)
\]
is defined, and hence the \( R \)-module morphism \( \partial \), cf. (2.1).

Likewise, notice that the composite
\[
d: \text{Hom}_A(sL, A) \rightarrow \text{Hom}_A(\Lambda_2[sL], A) \subseteq \text{Hom}(\Lambda_2[sL], A) \cong \text{Alt}^2(L, A)
\]
is defined, and let
\[
\bar{\partial}[\cdot, \cdot] = d - \partial^f: \text{Hom}_A(sL, A) \rightarrow \text{Hom}(\Lambda_2[sL], A).
\]

Consider the pairing
\[
L \otimes L \otimes \text{Hom}_A(L, A) \rightarrow A
\]
\[
\alpha \otimes \beta \otimes \varphi \mapsto \pm(\overline{\partial}[\cdot, \cdot](\varphi \circ s^{-1}))(s\alpha, s\beta).
\]
Since the map (2.14) from \( L \) to its double \( A \)-dual is an \( A \)-module isomorphism, this pairing induces a skew symmetric \( R \)-linear bracket \( [\cdot, \cdot] \) on \( L \), and hence a coderivation
\[
\partial[\cdot, \cdot]: \Lambda[sL] \rightarrow \Lambda[sL].
\]

By construction, the pieces of structure \((A, L), \vartheta, \) cf. (2.1), and \([\cdot, \cdot]\) satisfy the hypotheses of Corollary 2.6. Hence the pair \((A, L)\), endowed with \( \vartheta \) and the bracket \([\cdot, \cdot]\), constitutes a Lie-Rinehart algebra. Still by construction, the differential graded \( R \)-algebra \((\text{Alt}_A(L, A), d)\) is the Maurer-Cartan algebra associated to that Lie-Rinehart algebra.

Remark 2.8. Corollary 2.6 has the following consequence: If the derivation \( d = \partial^f + \bar{\partial}[\cdot, \cdot] \) on \( \text{Alt}(L, A) \) restricts to an \( R \)-linear differential on \( \text{Alt}_A(L, A) \subseteq \text{Alt}(L, A) \), it is necessarily a differential on all of \( \text{Alt}(L, A) \).

Remark 2.9. Let \( g \) be an ordinary Lie algebra. The identity (2.11) characterizing a Lie algebra twisting fixes the operator \( \partial[\cdot, \cdot] \) on \( \Lambda[g] \). Indeed, the desuspension \( t = s^{-1}: sg \rightarrow g \) is the universal Lie algebra twisting cochain for \( g \). Let \( x, y \in g \). Since \(Dt = t\partial[\cdot, \cdot] \) and since
\[
(t \wedge t)(sx, sy) = \frac{1}{2}[\cdot, \cdot] \circ t \otimes t(sx, sy) = [y, x]
\]
we find
\[
t\partial[\cdot, \cdot](sx, sy) = -t \wedge t(sx, sy) = [x, y]
\]
\[
\partial[\cdot, \cdot](sx, sy) = s[x, y].
\]
Remark 2.10. Let $M$ be a smooth manifold. In the standard formalism the de Rham differential $d$ is given by the formulas

$$df(X) = X(f)$$
$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha[X, Y]$$

etc. Here $f$ is a smooth function on $M$, $X$ and $Y$ are smooth vector fields, and $\alpha$ is a smooth 1-form. While in degree 1, the sign of (2.24) is the same as that of the corresponding operator $\partial_t (=\partial_t + \partial_{[\cdot, \cdot]})$, cf. (2.4), in degree 1, the sign of (2.25) is opposite to that of $\partial_t + \partial_{[\cdot, \cdot]}$. The sign in (2.25) arises by abstraction from the naive evaluation of a 2-form of the kind $df dh$ on a pair of vector fields by means of the product formula

$$\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y)$$

involving two 1-forms. However when we systematically exploit the Eilenberg-Koszul convention, the product formula takes the form

$$\alpha \wedge \beta(X, Y) = -\alpha(X)\beta(Y) + \beta(X)\alpha(Y)$$

and accordingly, the resulting sign coincides with that of the operator $\partial_t + \partial_{[\cdot, \cdot]}$.

3 Some technicalities

We work over a commutative ground ring $R$ that contains the rational numbers as a subring. Graded objects are graded over the integers. We understand a differential as an operator that lowers degree by 1, and we then indicate the degree by subscripts if need be. At times it is convenient to switch notationally to superscripts; here our convention is $A^q = A_{-q}$, so that the differential takes the form $d: A^q \to A^{q+1}$. Henceforth ‘graded’ means externally graded, cf. e.g., [Mac67] (p. 175 ff.), that is, we work only with homogeneous constituents.

3.1 Hom-differential

Given two chain complexes $C_1$ and $C_2$, we write the Hom-differential on $\text{Hom}(C_1, C_2)$ as $\mathcal{D}$; this differential turns $\text{Hom}(C_1, C_2)$ into a chain complex. We recall that, given a homogeneous member $f$ of $\text{Hom}(C_1, C_2)$, the value $\mathcal{D}(f)$ is given by

$$\mathcal{D}(f) = d \circ f + (-1)^{|f|+1} f \circ d.$$  

(3.1)

3.2 Hopf algebra of forms and beyond

Later in the paper we will develop an an sh Lie-Rinehart concept that involves sh Lie-algebras. An sh Lie algebra structure is characterized in terms of a cofree differential graded cocommutative coalgebra. Under our circumstances, the ground ring is assumed to contain the rational numbers as a subring, and to develop the sh Lie-Rinehart concept later in the paper we must express things in terms of the coalgebra that underlies the symmetric Hopf algebra. We now explain the requisite technical details.

Let $V$ be a graded $R$-module and let $S[V]$ denote the graded symmetric $R$-algebra on $V$. The diagonal map of $V$ induces a graded cocommutative diagonal map on $S[V]$, indeed, the
familiar shuffle diagonal map, that turns $S[V]$ into a graded bialgebra. Let $T_c[V]$ denote the graded tensor coalgebra on $V$ in the category of $R$ modules, and let $S^c[V] \subseteq T_c[V]$ be the largest graded cocommutative subcoalgebra of $T^c[V]$ containing $V$, cf. [Moo71] (p. 338) where this construction is taken over a field of characteristic different from 2. The universal property of $T_c[V]$ entails the existence of a unique extension of the identity of $V$ to a morphism $S[V] \to T^c[V]$ of graded coalgebras and, since $S[V]$ is cocommutative, the values of that morphism lie in $S^c[V]$, that is, the identity of $V$ induces a canonical morphism

$$S[V] \to S^c[V]$$

of graded cocommutative coalgebras. Since $R$ is assumed to contain the rational numbers as a subring, (3.2) is an isomorphism. Indeed, in degree $n$, the composite $V \otimes V \to S^c[V] \to S^n[V]$ of the multiplication map mult with multiplication by $\frac{1}{n!}$, restricted to $S^c[V] \subseteq V \otimes V$, yields the inverse of (3.2). The coalgebra $S^c[V]$ is the cofree graded cocommutative coalgebra on $V$ in the category of $R$-modules. The addition of $V$ induces a graded commutative multiplication that turns $S^c[V]$ into a graded bialgebra, and (3.2) is an isomorphism of graded bialgebras.

For both $S[V]$ and $S^c[V]$, multiplication on $V$ by $-1$ induces an antipode turning each of $S[V]$ and $S^c[V]$ into a graded Hopf algebra over $R$.

The dual of (3.2) has the form

$$\text{Hom}(S^c[V], R) \to \text{Hom}(S[V], R),$$

(3.3)

and $\text{Hom}(S[V], R)$ is traditionally interpreted as an algebra of multilinear graded symmetric forms (alternating when $V$ is concentrated in odd degrees and symmetric in the usual ungraded sense when $V$ is concentrated in even degrees). Thus the algebra structure of the algebra $\text{Hom}(S[V], R)$ of multilinear graded symmetric forms is induced by the shuffle diagonal on $S[V]$.

Remark 3.1. To the browsing reader, the distinction between $S[V]$ and $S^c[V]$ might appear pedantic, so here is one hint at the difference between the two: An $R$-linear map $q: S^c_2[V] \to R$ on the degree 2 constituent of $S^c[V]$ is a (graded) quadratic form, and the composite with the canonical map $V \otimes V \to S^c_2[V] \to S^c[V]$ is the associated (graded) polar form.

Remark 3.2. The graded coalgebra $S^c[V]$ is the graded coalgebra that underlies the divided power Hopf algebra $\Gamma[V]$ on $V$, cf. [Car55], [EML54] (§§17 and 18). On the other hand, the assignment to $x \in V$ of $\gamma_k(x) = \frac{1}{n!} x^k$ ($k \geq 0$) turns $S[V]$ into a divided power Hopf algebra. In terms of these divided powers, the diagonal map $\Delta$ of $S[V]$ is given by the identity

$$\Delta(\gamma_n(x)) = \sum_{j+k=n} \gamma_j(x) \otimes \gamma_k(x), \quad n \geq 1, \quad x \in V,$$

Now (3.2) is an isomorphism of divided power Hopf algebras.

In terms of the notation $V^* = \text{Hom}(V, R)$, when $V$ is of finite type (finitely generated in each degree), (3.3) can be written as

$$S[V^*] \to S^c[V^*]$$

(3.4)

and is formally exactly of the same kind as (3.2), with $V^*$ substituted for $V$. 

11
3.3 Perturbations

Let \((C, d)\) be a chain complex. Recall that a perturbation of \(d\) is an operator \(\partial\) on \(C\) such that \(d + \partial\) is a differential. When \((C, d)\) is a differential graded coalgebra with counit \(\varepsilon : C \to R\), a perturbation \(\partial\) of \(d\) that is also a coderivation is a coalgebra perturbation. Likewise when \((A, d)\) is a differential graded algebra, a perturbation \(\partial\) of \(d\) that is also a derivation is an algebra perturbation.

Let \(C\) be a coaugmented differential graded coalgebra, the coaugmentation being written as \(\eta : R \to C\), write the resulting coaugmentation filtration as

\[ R = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_j \subseteq \ldots, \tag{3.5} \]

and suppose that \(C\) is cocomplete, that is, \(C = \cup C_j\). It then makes sense to require that a coalgebra perturbation lowers filtration. We warn the reader that, to avoid an orgy of notation, here the subscripts refer to the filtration degree and not to the ordinary degree. Henceforth, whenever we use subscripts of this kind, it will be clear from the context whether filtration degree or ordinary degree is intended.

Let \(\partial_{\ldots, j}\) be a coalgebra perturbation that lowers filtration. Suppose that \(\partial_{\ldots, j}\) can be written in the form

\[ \partial_{\ldots, j} = \partial^1_{\ldots, j} + \ldots + \partial^j_{\ldots, j} + \ldots \tag{3.6} \]

such that \(\partial^j_{\ldots, j}\), when non-zero, lowers filtration by \(j\) and not by \(j + 1\) \((j \geq 1)\). We will then say that \(\partial_{\ldots, j}\) is a filtered coalgebra perturbation. The bracket subscript is intended as a mnemonic that, for our purposes, later in the paper, such a perturbation characterizes an sh Lie algebra structure. In particular, given a filtered coalgebra perturbation \(\partial_{\ldots, j}\), the counit \(\varepsilon : C \to R\) being a morphism of chain complexes, for \(j \geq 1\), the constituent \(\partial^j_{\ldots, j}\) vanishes on \(C_j\) whence, since \(C\) is cocomplete, the infinite sum \((3.6)\) converges naïvely in the sense that, applied to a specific element of \(C\), only finitely many terms are non-zero.

Given a filtration decreasing coderivation of degree \(-1\) of the kind \((3.6)\) such that \(\partial^j_{\ldots, j}\), when non-zero, lowers filtration by \(j\) and not by \(j + 1\) \((j \geq 1)\), save that \(\partial^j_{\ldots, j}\) is not required to be a perturbation of the coalgebra differential \(d^0\) of \(C\), we refer to a filtered degree \(-1\) filtration decreasing coderivation. The wording of Theorem \(4.7\) involves a filtered degree \(-1\) filtration lowering coderivation; Theorem \(4.7\) is a crucial step for Theorems \(4.11, 5.11, 5.15\) and \(5.16\). For our purposes, a filtered degree \(-1\) filtration lowering coderivation generalizes a skew-symmetric bracket which is not assumed to be a Lie bracket (i.e., does not necessarily satisfy the Jacobi identity). For later reference, we spell out the following.

**Proposition 3.3.** A filtered degree \(-1\) filtration lowering coderivation \(\partial_{\ldots, j}\) is a filtered coalgebra perturbation of \(d^0\) if and only if, for \(j \geq 1\),

\[ d^0 \partial^j_{\ldots, j} + \partial^j_{\ldots, j} d^0 + \sum_{k=1}^{j-1} \partial^k_{\ldots, j} \partial^{j-k}_{\ldots, j} = 0. \tag{3.7} \]

For the benefit of the reader we note that \((3.7)\) reads

\[ d^0 \partial^1_{\ldots, j} + \partial^1_{\ldots, j} d^0 = 0 \tag{3.8} \]

\[ d^0 \partial^2_{\ldots, j} + \partial^2_{\ldots, j} d^0 + \partial^1_{\ldots, j} \partial^1_{\ldots, j} = 0 \tag{3.9} \]

\[ d^0 \partial^3_{\ldots, j} + \partial^3_{\ldots, j} d^0 + \partial^1_{\ldots, j} \partial^2_{\ldots, j} + \partial^2_{\ldots, j} \partial^1_{\ldots, j} = 0 \tag{3.10} \]

etc.
3.4 Filtered Lie algebra twisting cochains

Let $C$ be a cocomplete coaugmented differential graded cocommutative coalgebra and $g$ a differential graded Lie algebra. Let
\[ t = t_1 + t_2 + \ldots : C \rightarrow g \]  
be a Lie algebra twisting cochain such that, for $j \geq 1$, the constituent $t_j$, if non-zero, is zero on the constituent $C_{j-1}$ but not on the constituent $C_j$ of the coaugmentation filtration of $C$, cf. (3.5). Since $C$ is cocomplete, the infinite sum (3.11) converges naively in the sense that, applied to a specific element of $C$, only finitely many terms are non-zero. We will then say that $t$ is a filtered Lie algebra twisting cochain.

**Remark 3.4.** Write the differential graded Lie algebra $\text{Hom}(C, g)$ as $L$, write $L^0 = L$ and, for $j \geq 0$, let
\[ L^{j+1} = \ker(\text{Hom}(C, g) \rightarrow \text{Hom}(C_j, g)). \]
The coaugmentation filtration of $C$ induces the descending filtration
\[ L = L^0 \supseteq L^1 \supseteq \ldots \supseteq \ldots \]  
of differential graded Lie algebras. The Lie algebra twisting cochain (3.11) being filtered means that, for $j \geq 1$, the constituent $t_j$ lies in $L^j$ but not in $L^{j+1}$.

We will refer to a homogeneous morphism of degree $-1$ of the kind (3.11) that is not necessarily a Lie algebra twisting cochain as a filtered degree $-1$ morphism. We need this terminology to be able to phrase Theorem 4.7, a crucial step for Theorems 4.11, 5.11, 5.15, and 5.16.

For later reference, we spell out the following.

**Proposition 3.5.** Let $(C, d^0)$ be a coaugmented differential graded cocommutative coalgebra, and let $\partial_{\{\cdot, \cdot\}}$ be a filtered coalgebra perturbation of $d^0$. A filtered degree $-1$ morphism
\[ t = t_1 + t_2 + \ldots : C \rightarrow g \]
is a Lie algebra twisting cochain
\[ t : (C, d^0 + \partial_{\{\cdot, \cdot\}}) \rightarrow g \]
if and only if, for $j \geq 1$,
\[ d_0 t^j + t^j d^0 + \sum_{k=1}^{j-1} t^k \partial_{\{\cdot, \cdot\}}^{j-k} + \sum_{k=1}^{j-1} t^k \wedge t^{j-k} = 0. \]  
(3.13)

Explicitly, the identities (3.13) take the form
\[ d_0 t^1 + t^1 d^0 = 0 \]  
(3.14)
\[ d_0 t^2 + t^2 d^0 + t^1 \partial_{\{\cdot, \cdot\}}^{1} + t^1 \wedge t^1 = 0 \]  
(3.15)
\[ d_0 t^3 + t^3 d^0 + t^1 \partial_{\{\cdot, \cdot\}}^{2} + t^2 \partial_{\{\cdot, \cdot\}}^{1} + [t^1, t^2] = 0 \]  
(3.16)
\[ d_0 t^4 + t^4 d^0 + t^1 \partial_{\{\cdot, \cdot\}}^{3} + t^2 \partial_{\{\cdot, \cdot\}}^{2} + t^3 \partial_{\{\cdot, \cdot\}}^{1} + [t^1, t^3] + t^2 \wedge t^2 = 0, \]  
(3.17)
etc. For clarity, we note that, as for the term $\sum_{k=1}^{j-1} t^k \wedge t^{j-k}$ in (3.13), this identity is a concise version of the two identities

$$d_0 t^{2j} + t^{2j} d^0 + \sum_{k=1}^{2j-1} t^k \partial_{t^j} [\cdot, \cdot] + \sum_{k=1}^{j-1} [t^k, t^{2k-1}] + t^j \wedge t^j = 0 \quad (j \geq 1)$$  \hspace{1cm} (3.18)$$

$$d_0 t^{2j+1} + t^{2j+1} d^0 + \sum_{k=1}^{2j} t^k \partial_{t^{j+1}} [\cdot, \cdot] + \sum_{k=1}^{j} [t^k, t^{2k-1}] = 0 \quad (j \geq 0)$$  \hspace{1cm} (3.19)$$

### 3.5 Lie-Rinehart structures associated to Hom$(C, A)$

Let $C$ be differential graded cocommutative coalgebra and $A$ a differential graded commutative algebra. The Hom-differential $D$ and the cup product turn Hom$(C, A)$ into a differential graded commutative algebra. Likewise Hom$(C, \text{Der}(A|R))$ acquires a differential graded $R$-Lie algebra structure and a differential graded Hom$(C, A)$-module structure. Furthermore, with this structure of mutual interaction, the pair

$$(A, \mathcal{L}) = \text{Hom}(C, A), \text{Hom}(C, \text{Der}(A|R))) \quad (3.20)$$

is a differential graded Lie-Rinehart algebra.

Let $t \in \text{Hom}(C, \text{Der}(A|R))$ be homogeneous of degree $-1$, at first not necessarily a Lie algebra twisting cochain $C \rightarrow \text{Der}(A|R)$. The morphism $t$ determines a derivation $\partial^t: \text{Hom}(C, A) \longrightarrow \text{Hom}(C, A)$. \hspace{1cm} (3.21)

Indeed, consider the universal differential graded algebra $U_A[\text{Der}(A|R)]$ associated to the differential graded Lie-Rinehart algebra $(A, \text{Der}(A|R))$, and let $[\cdot, \cdot]$ denote the ordinary commutator bracket of $U_A[\text{Der}(A|R)]$. In terms of this bracket, we write the action of $\text{Der}(A|R)$ on $A$ as the bracket operation

$$\text{Der}(A|R) \otimes A \longrightarrow A, \quad (\delta, a) \mapsto [\delta, a].$$  \hspace{1cm} (3.22)$$

This bracket, in turn, induces a bracket pairing

$$[\cdot, \cdot]: \text{Hom}(C, \text{Der}(A|R)) \otimes \text{Hom}(C, A) \longrightarrow \text{Hom}(C, A),$$  \hspace{1cm} (3.23)$$

and the operator $\partial^t$ is given by the expression

$$\partial^t(\alpha) = [t, \alpha], \quad \alpha \in \text{Hom}(C, A).$$  \hspace{1cm} (3.24)$$

We will now suppose that $C$ is coaugmented.

**Proposition 3.6.** The degree $-1$ morphism $t: C \rightarrow \text{Der}(A|R)$ of the underlying graded $R$-modules is a Lie algebra twisting cochain if and only if the derivation $\partial^t$ is an algebra perturbation of the Hom-differential $D$ on Hom$(C, A)$, that is, if and only if $D + \partial^t$ is a(n algebra) differential on Hom$(C, A)$.

**Proof.** This is a consequence of the identity

$$(D\partial^t + \partial^t D + \partial^t \partial^t)(\alpha) = [Dt + t \wedge t, \alpha], \quad \alpha \in A = \text{Hom}(C, A).$$
4 Multi derivation Maurer-Cartan algebras

Let \((\mathcal{A}, D_0)\) be a differential graded algebra, endowed with a filtration
\[
\mathcal{A} = \mathcal{A}^0 \supseteq \mathcal{A}^1 \supseteq \cdots
\]
that is compatible with the differential \(D_0\). We warn the reader that, to avoid an orgy of notation, here the superscripts refer to the filtration degree and not to the ordinary degree (written in superscripts). Henceforth, whenever we use superscripts of this kind, it will be clear from the context whether filtration degree or ordinary degree is intended. Let \(D = \sum_{j \geq 1} D_j\) be an algebra perturbation of \(D_0\) such that, for \(j \geq 1\), the derivation \(D_j\), when non-zero, lowers filtration by \(j\) but not by \(j + 1\) \((j \geq 1)\) in the following sense: for any \(\ell \geq 0\), the derivation \(D_j\), restricted to \(\mathcal{A}^\ell\), has the form
\[
D_j: \mathcal{A}^\ell \to \mathcal{A}^{\ell+j}
\]
but does not factor through \(\mathcal{A}^{\ell+j+1}\) as a composite of the kind \(\mathcal{A}^\ell \to \mathcal{A}^{\ell+j+1} \subseteq \mathcal{A}^{\ell+j}\). Here we implicitly assume that \(\sum_{j \geq 1} D_j\) converges, either naively in the sense that, given a homogeneous member \(\alpha\) of \(\mathcal{A}\), only finitely many values \(D_j(\alpha)\) are non-zero or, more generally, in this sense: the filtration is complete, that is, the canonical map \(\mathcal{A} \to \lim(\mathcal{A}/\mathcal{A}^j)\) is an isomorphism, and \(\sum_{j \geq 1} D_j\) converges. We will then say that
\[
(\mathcal{A}, D_0, D_1, \ldots)
\]
is a multi derivation chain algebra.

Remark 4.1. Given a filtered algebra of the kind \((4.1)\), suppose that \(\mathcal{A}\) admits a bigrading \(\{\mathcal{A}^{p,q}\}_{p,q}\) with filtration degree \(p \geq 0\) and complementary degree \(q\); here the meaning of the term ‘complementary degree’ is that \(p + q\) recovers the total degree. Then a special kind of multi derivation chain algebra structure on \(\mathcal{A}\) is one of the kind where the operators \(D_j\) \((j \geq 0)\) take the familiar form
\[
D_j: \mathcal{A}^{p,q} \to \mathcal{A}^{p+j,q-j+1}.
\]
We will refer to this kind of multi derivation chain algebra structure as a bigraded multi derivation chain algebra structure.

Remark 4.2. Given a filtered algebra of the kind \((4.1)\), let \(E_0(\mathcal{A})\) denote the associated bigraded algebra, with bigrading
\[
E_0(\mathcal{A})^{p,q} = E_0(\mathcal{A})^p_{-q} = (\mathcal{A}^p/\mathcal{A}^{p+1})_{-(p+q)},
\]
filtration degree \(p \geq 0\) and complementary degree \(q\). A multi derivation chain algebra structure \(D_0, D_1, \ldots\) on \(\mathcal{A}\) induces a bigraded multi derivation chain algebra structure \(\hat{D}_0, \hat{D}_1, \ldots\) on \(E_0(\mathcal{A})\); we will refer to this bigraded multi derivation chain algebra structure as the associated bigraded multi derivation chain algebra structure. A bigraded multi derivation chain algebra is isomorphic to its associated bigraded multi derivation chain algebra via the obvious map from the former to the latter.

Let \(C\) be a cocomplete coaugmented differential graded cocommutative coalgebra and \(A\) a differential graded commutative algebra. We write the differential of \(C\) as \(d^0\). Furthermore, let
\( \partial \cdot \cdot \cdot \) be a filtered degree \(-1\) filtration lowering coderivation on \( C \) and \( t: C \to \text{Der}(A|R) \) a filtered degree \(-1\) morphism. We use the notation in \((3.6)\) for the constituents \( \partial^j \cdot \cdot \cdot \) and that in \((3.11)\) for the constituents \( t_j \) of \( t \) \((j \geq 1)\). Consider the differential graded algebra \( \text{Hom}(C, A) \), endowed with the Hom-differential \( D_0 \), as well as the graded Lie algebra \( \text{Hom}(C, \text{Der}(A|R)) \), endowed with the Hom-differential \( D_0 \), where the notation is slightly abused. For \( j \geq 1 \), the operator \( \partial^j \), cf. \((3.24)\), is a derivation of \( \text{Hom}(C, A) \), and the coderivation \( \partial^j \cdot \cdot \cdot \) of \( C \) induces a derivation \( \partial^j \cdot \cdot \cdot \) of \( \text{Hom}(C, A) \); explicitly, given \( \varphi \in \text{Hom}(C, A) \) homogeneous,

\[
\partial^j \cdot \cdot \cdot (\varphi) = (-1)^{|\varphi|+1} \varphi \circ \partial^j \cdot \cdot \cdot .
\]

Thus the derivations

\[
\partial \cdot \cdot \cdot = \sum \partial^j \cdot \cdot \cdot \quad \quad \quad \quad (4.5)
\]

\[
\partial^j = \sum \partial^j \quad \quad \quad \quad (4.6)
\]

\[
\mathcal{D} \cdot \cdot \cdot = D_0 + \partial^0 \cdot \cdot \cdot \quad \quad \quad (4.7)
\]

\[
\mathcal{D}_j = \partial^j + \partial^j \cdot \cdot \cdot \quad (j \geq 1)
\]

\[
\mathcal{D} = \mathcal{D} \cdot \cdot \cdot + \partial^j = D_0 + \partial^j \cdot \cdot \cdot + \partial^j = D_0 + \sum \mathcal{D}_j \quad (4.9)
\]

of \( \text{Hom}(C, A) \) are defined. With a slight abuse of notation, we denote the corresponding operator on the graded Lie algebra \( \text{Hom}(C, \text{Der}(A|R)) \) by

\[
\mathcal{D} \cdot \cdot \cdot = D_0 + \partial^0 \cdot \cdot \cdot : \text{Hom}(C, \text{Der}(A|R)) \to \text{Hom}(C, \text{Der}(A|R)) \quad (4.10)
\]

as well. For ease of exposition, we recollect the following (obvious) statements in the next proposition.

**Proposition 4.3.**

1. The filtered degree \(-1\) filtration lowering coderivation \( \partial \cdot \cdot \cdot \) is a perturbation of the coalgebra differential \( d^0 \) on \( C \) if and only if

\[
[d^0, \partial \cdot \cdot \cdot] + \partial \cdot \cdot \cdot \partial \cdot \cdot \cdot = 0. \quad (4.11)
\]

2. Suppose that \( \partial \cdot \cdot \cdot \) is a (coalgebra) perturbation of the differential \( d^0 \) on \( C \). Then \( t \) is a Lie algebra twisting cochain of the kind \((C, d^0 + \partial \cdot \cdot \cdot) \to \text{Der}(A|R) \) if and only if

\[
\mathcal{D} \cdot \cdot \cdot t + t \wedge t = 0 \in \text{Hom}(C, \text{Der}(A|R)). \quad (4.12)
\]

3. The derivation \( \partial^0 \cdot \cdot \cdot \) is an algebra perturbation of the algebra differential \( D_0 \) on \( \text{Hom}(C, A) \) if and only if

\[
[D_0, \partial^0 \cdot \cdot \cdot] + \partial^0 \cdot \cdot \cdot \partial^0 \cdot \cdot \cdot = 0. \quad (4.13)
\]

4. The identity \((4.11)\) implies the identity \((4.13)\). Thus when \( d^0 + \partial \cdot \cdot \cdot \) is a (coalgebra) differential on \( C \), the derivation \( \mathcal{D} \cdot \cdot \cdot = D_0 + \partial^0 \cdot \cdot \cdot \) is an algebra differential on \( \text{Hom}(C, A) \).
5. The system of derivations \( \{D_j\}_{j \geq 0} \) turns \( \text{Hom}(C, A) \) into a multi derivation chain algebra, that is, the derivation \( D \), cf. (4.3), is an algebra differential on \( \text{Hom}(C, A) \), if and only if

\[
[D_0, \partial^{[\cdot \cdot \cdot]}] + [D_0, \partial'] + [\partial^{[\cdot \cdot \cdot]}, \partial'] + \partial^{[\cdot \cdot \cdot]} \partial' + \partial' \partial' = 0. \tag{4.14}
\]

6. The identities (4.11) and (4.12) together imply the identity (4.14). Thus when \( \partial^{[\cdot \cdot \cdot]} \) is a (coalgebra) perturbation on \( C \) and \( t: (C, \delta^0 + \partial^{[\cdot \cdot \cdot]}) \to \text{Der}(A|R) \) a Lie algebra twisting cochain, the derivation \( D \), cf. (4.9), is an algebra differential on \( \text{Hom}(C, A) \).

Let \( V \) be a differential graded \( A \)-module. The \( A \)-module structure being from the left, \( V \) also acquires an obvious differential graded right \( A \)-module structure—this involves the Eilenberg-Koszul convention—, and the graded tensor powers \( V^{\otimes_A n} \) \((n \geq 1)\) are defined. Accordingly, let \( S_A[V] \) be the graded symmetric \( A \)-algebra on \( V \), and endow it with the differential graded \( A \)-module structure it acquires in an obvious manner. The differential graded symmetric \( R \)-algebra \( S[V] \) is defined in the standard way, and the canonical map \( S[V] \to S_A[V] \) is a morphism of \( R \)-algebras. As noted before, the diagonal map \( V \to V \oplus V \) induces the standard shuffle diagonal on \( S[V] \) and, likewise, a shuffle diagonal on \( S_A[V] \) in such a way that \( S[V] \to S_A[V] \) is compatible with the diagonals but, beware, \( S[V] \to S_A[V] \) is not in a naive manner a morphism of coalgebras.

We will now apply the previous discussion to \( C = S[V] \) and \( A \), as well as, suitably adjusted, to \( S_A[V] \) and \( A \). The graded commutative algebra \( \text{Hom}(C, A) \) (endowed with the cup multiplication) is that of graded symmetric \( A \)-valued \( R \)-multi linear forms on \( V \), the differentials on \( V \) and \( A \) induce the algebra differential written before as \( D_0 \), and we continue to use this notation. Likewise, \( \text{Hom}_A(S_A[V], A) \) is the graded \( A \)-module of graded symmetric \( A \)-valued graded \( A \)-multi linear forms on \( V \). A little thought reveals that, since \( S_A[V] \) is a differential graded \( A \)-module, with a slight abuse of the notation \( D_0 \), this operator passes to \( \text{Hom}_A(S_A[V], A) \) and, with respect to the ordinary multiplication of forms (induced by the shuffle diagonal on \( S_A[V] \) and the multiplication of \( A \)), turns \( \text{Hom}_A(S_A[V], A) \) into a differential graded commutative \( R \)-algebra. We write this differential graded algebra as \( (\text{Sym}_A(V, A), D_0) \).

Recall the coaugmentation filtration \( R = C_0 \subseteq C_1 \subseteq \ldots \) of the coaugmented differential graded cocommutative \( R \)-coalgebra \( C = S[V] \) and, for \( j \geq 0 \), let

\[
\text{Hom}(S[V], A)^{j+1} = \ker(\text{Hom}(C, A) \to \text{Hom}(C_j, A)).
\]

Relative to the differential \( D_0 \),

\[
\text{Hom}(S[V], A) = \text{Hom}(S[V], A)^0 \supseteq \text{Hom}(S[V], A)^1 \supseteq \ldots \supseteq \ldots \tag{4.15}
\]

is a descending filtration of differential graded algebras and, in an obvious manner, still relative to the differential \( D_0 \), this filtration induces as well a filtration

\[
\text{Sym}_A(V, A) = \text{Sym}_A(V, A)^0 \supseteq \text{Sym}_A(V, A)^1 \supseteq \ldots \supseteq \text{Sym}_A(V, A)^j \supseteq \ldots \tag{4.16}
\]

of differential graded \( R \)-algebras.

As noted earlier, since the ground ring \( R \) contains the rational numbers as a subring, as a Hopf algebra, the differential graded symmetric algebra \( S[V] \) is actually canonically isomorphic to the cofree differential graded cocommutative coalgebra \( S^0[V] \) on \( V \); in particular, the coaugmentation filtration coincides with the tensor power filtration.

As before, let \( \partial^{[\cdot \cdot \cdot]} \) be a filtered degree \(-1\) filtration lowering coderivation on \( C = S[V] \) and \( t: C = S[V] \to \text{Der}(A|R) \) a filtered degree \(-1\) morphism. The following is immediate.
Proposition 4.4. Suppose that, for \( j \geq 1 \), each derivation \( D_j \) of \( \text{Hom}(S[V], A) \) (though not necessarily the individual constituents \( \partial^j \) and \( \partial_j^{[\cdot, \cdot]} \) of \( D_j = \partial^j + \partial_j^{[\cdot, \cdot]} \)) passes to a derivation of \( \text{Sym}_A(V, A) = \text{Hom}_A(S_A[V], A) \), necessarily lowering the filtration (4.16) by \( j \) (with a slight abuse of the notation \( D_j \)). When \( D \) is an algebra differential—equivalently, when \( \sum_{j \geq 1} D_j \), cf. (4.9), is an algebra perturbation of the algebra differential \( D_0 \) on \( \text{Sym}_A(V, A) \)—the data

\[
(\text{Sym}_A(V, A), D_0, D_1, D_2, \ldots)
\]  

constitute a multi derivation chain algebra.

Corollary 4.5. Suppose that \( \partial_{[\cdot, \cdot]} \) is a (coalgebra) perturbation on \( C = S[V] \) and \( t \) a filtered Lie algebra twisting cochain \((C, d^0 + \partial_{[\cdot, \cdot]}) \to \text{Der}(A|R)\). Suppose that, furthermore, for \( j \geq 1 \), each derivation \( D_j \) of \( \text{Hom}(S[V], A) \) (though not necessarily the individual constituents \( \partial^j \) and \( \partial_j^{[\cdot, \cdot]} \) of \( D_j = \partial^j + \partial_j^{[\cdot, \cdot]} \)) passes to a derivation of \( \text{Sym}_A(V, A) = \text{Hom}_A(S_A[V], A) \). Then (4.17) is a multi derivation chain algebra.

Remark 4.6. The graded commutative algebras \( \text{Hom}(S[V], A) \) and \( \text{Hom}_A(S_A[V], A) \) are bi-graded in an obvious manner: Write the grading of \( A \) in superscripts, so that \( A_{-q} = A^q \) and so that the differential of \( A \) takes the form \( d_0 : A^q \to A^{q+1} \). A homogeneous member of \( \text{Hom}(S[V], A) \) of the kind

\[
f : V_{j_1} \times \ldots \times V_{j_p} \to A^{p+q-j_1-j_2-\ldots-j_p}
\]

has filtration degree \( p \) and complementary degree \( q \), cf. Remark 4.1, and the resulting bigrading of \( \text{Hom}(S[V], A) \) passes to \( \text{Hom}_A(S_A[V], A) \).

We will now explore the question to what extent the converse of the statement of Corollary 4.5 holds. Here is a generalization of the sufficiency claim of Corollary 2.6 above.

Theorem 4.7. Given the filtered degree \(-1\) filtration lowering coderivation \( \partial_{[\cdot, \cdot]} \) and the filtered degree \(-1\) morphism \( t \), suppose that, for \( j \geq 1 \), each derivation \( D_j \) of \( \text{Hom}(S[V], A) \) (though not necessarily the individual constituents \( \partial^j \) and \( \partial_j^{[\cdot, \cdot]} \) of \( D_j = \partial^j + \partial_j^{[\cdot, \cdot]} \)) passes to a derivation of \( \text{Sym}_A(V, A) = \text{Hom}_A(S_A[V], A) \) and that (4.17) is a multi derivation chain algebra. Suppose, furthermore, that the canonical morphism

\[
V \longrightarrow \text{Hom}_A(V, \text{Hom}_A(V, A), A)
\]  

(4.18)

of graded \( A \)-modules (from \( V \) into its double \( A \)-dual) is injective. Then \( \partial_{[\cdot, \cdot]} \) is a coalgebra perturbation of the coalgebra differential \( d^0 \) of \( C = S[V] \), and \( t \) is a Lie algebra twisting cochain \((C, d^0 + \partial_{[\cdot, \cdot]}) \to \text{Der}(A|R)\).

Remark 4.8. In Theorem 4.7 we do not assume that the multi derivation chain algebra (4.17) comes from a multi derivation chain algebra structure on the ambient algebra \( \text{Sym}(V, A) = \text{Hom}(S[V], A) \).

However, Theorem 4.7 has the following curious consequence, cf. Remark 2.8 above:

Corollary 4.9. Under the circumstances of Theorem 4.7, the data

\[
(\text{Sym}(V, A), D_0, D_1, D_2, \ldots) = (\text{Hom}(S[V], A), D_0, D_1, D_2, \ldots)
\]  

(4.19)

necessarily constitute a multi derivation chain algebra as well.
Proof of Theorem 4.7. We must confirm the identities (3.7) and (3.13).

We note first that, on Hom(S[V], A),

\[
\sum_{k=0}^{j} D_k D_{j-k} = \left( [D_0, \partial^j] + \sum_{k=1}^{j-1} [\partial_k^{[\cdot \cdot \cdot]}, \partial^{j-k}] + \sum_{k=1}^{j-1} \partial^k \partial^{j-k} \right) + [D_0, \partial_j^{[\cdot \cdot \cdot]}] + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial_{j-k}^{[\cdot \cdot \cdot]}. \tag{4.20}
\]

Let \( \ell \geq 0, j \geq 1 \), and let \( a \in A \). Exploiting the bracket pairing (3.23), we find

\[
\left( [D_0, \partial^j] + \sum_{k=1}^{j-1} [\partial_k^{[\cdot \cdot \cdot]}, \partial^{j-k}] + \sum_{k=1}^{j-1} \partial^k \partial^{j-k} \right) a = \left[ d_0 \ell^j + \ell^j d^0 + \sum_{k=1}^{j-1} \ell^k \partial^{j-k} + \sum_{k=1}^{j-1} \ell^k \partial^{j-k}, a \right]. \tag{4.21}
\]

Since \( a \) and \( j \) are arbitrary, and since the second constituent

\[
[D_0, \partial_j^{[\cdot \cdot \cdot]}] + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial_{j-k}^{[\cdot \cdot \cdot]}
\]
on the right-hand side of (4.20), evaluated at \( a \in A \) is zero, we conclude that the identities (3.13) hold.

To establish the identities (3.7), let again \( j \geq 1 \). We note first that, \( j \) having been fixed, the identity (3.13), in turn, implies that the identity

\[
[D_0, \partial^j] + \sum_{k=1}^{j-1} [\partial_k^{[\cdot \cdot \cdot]}, \partial^{j-k}] + \sum_{k=1}^{j-1} \partial^k \partial^{j-k} = 0 \tag{4.22}
\]

holds on Hom(S[V], A) (not just on Hom\(_A\)(S[V], A), in fact it would not even make sense on Hom\(_A\)(S[V], A)).

Let \( x \in S^{j+1}[V] \) homogeneous (beware, \( x \) is not taken to be a member of \( S^{j+1}_A[V] \)) and \( \varphi \in \text{Hom}_A(V, S^{j+1}_A[V]) \) homogeneous. For the moment, view \( \varphi \) as a member of Hom\(_A\)(V, S\(_A^{j+1}\)(V)).

By construction

\[
\left( [D_0, \partial_j^{[\cdot \cdot \cdot]}] + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial_{j-k}^{[\cdot \cdot \cdot]} \right) \varphi(x) = \pm \varphi \left( d^0 \partial_j^{[\cdot \cdot \cdot]} + \partial_j^{[\cdot \cdot \cdot]} d^0 + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial^{j-k} \right) (x).
\]

However, in view of (4.20) and (4.22),

\[
\left( [D_0, \partial_j^{[\cdot \cdot \cdot]}] + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial_{j-k}^{[\cdot \cdot \cdot]} \right) \varphi = \left( \sum_{k=0}^{j} D_k D_{j-k} \right) \varphi. \tag{4.23}
\]

Now

\[
\left( d^0 \partial_j^{[\cdot \cdot \cdot]} + \partial_j^{[\cdot \cdot \cdot]} d^0 + \sum_{k=1}^{j-1} \partial_k^{[\cdot \cdot \cdot]} \partial^{j-k} \right) (x) \in V.
\]
By assumption, \( \sum_{k=0}^{j} D_k D_{j-k} \) is zero on \( \text{Sym}_A(V, A) \) and \( \varphi \) was taken in \( \text{Sym}_A(V, A) \) whence
\[
\varphi \left( \left( d^0 \partial^j_{[\cdots, \cdots]} + \partial^j_{[\cdots, \cdots]} d^0 + \sum_{k=1}^{j-1} \partial^k_{[\cdots, \cdots]} \partial^{j-k}_{[\cdots, \cdots]} \right)(x) \right) = 0. \quad (4.24)
\]
Since the canonical morphism \( (4.18) \) of graded \( A \)-modules is supposed to be injective and since \( \varphi \) is arbitrary, we conclude
\[
\left( d^0 \partial^j_{[\cdots, \cdots]} + \partial^j_{[\cdots, \cdots]} d^0 + \sum_{k=1}^{j-1} \partial^k_{[\cdots, \cdots]} \partial^{j-k}_{[\cdots, \cdots]} \right)(x) = 0. \quad (4.25)
\]
Since \( x \in S^{j+1}[V] \) is arbitrary, we find
\[
d^0 \partial^j_{[\cdots, \cdots]} + \partial^j_{[\cdots, \cdots]} d^0 + \sum_{k=1}^{j-1} \partial^k_{[\cdots, \cdots]} \partial^{j-k}_{[\cdots, \cdots]} = 0, \quad (4.26)
\]
that is, the identity \( (3.17) \) holds as well. \( \square \)

**Remark 4.10.** It is instructive to illustrate the reasoning in the above proof in low degrees:

For \( j = 1 \), the identity \( (4.20) \) reads
\[
[D_0 D_1 + D_1 D_0 = [D_0, \partial^1] + [D_0, \partial^1_{[\cdots, \cdots]}],
\]
the hypothesis \( 0 = D_0 D_1 + D_1 D_0 \) comes down to
\[
0 = [D_0, \partial^1] + [D_0, \partial^1_{[\cdots, \cdots]}]: \text{Hom}_A(S^1_A[V], A) \rightarrow \text{Hom}_A(S^2_A[V], A), \quad (4.27)
\]
and the identity \( (4.21) \) amounts to
\[
[D_0, \partial^1]a = [d_0 t_1 + t_1 d^0, a], \quad a \in A.
\]
Since \( a \in A \) is arbitrary, we conclude that \( t_1 \) satisfies the identity \( (3.14) \), viz. \( d_0 t_1 + t_1 d^0 = 0 \), whence \( t_1 : V \rightarrow A \) is a morphism of chain complexes. Consequently the derivation \( [D_0, \partial^1] \)
is zero on all of \( \text{Hom}(S[V], A) \). Since the sum \( [D_0, \partial^1] + [D_0, \partial^1_{[\cdots, \cdots]}] \) is zero, in view of
\[
\left( [D_0, \partial^1] + [D_0, \partial^1_{[\cdots, \cdots]}] \right) \varphi(x) = [d_0 t_1 + t_1 d^0, \varphi](x) \pm \varphi((\partial^1_{[\cdots, \cdots]} d^0 + d^0 \partial^1_{[\cdots, \cdots]})(x)),
\]
for any \( \varphi \in \text{Hom}_A(V, A) \) and any \( x \in S^2[V] \), we deduce the identity \( (3.8) \), viz.
\[
d^0 \partial^1_{[\cdots, \cdots]} + \partial^1_{[\cdots, \cdots]} d^0 = 0.
\]
Consequently the derivation \( [D_0, \partial^1_{[\cdots, \cdots]}] \) is zero on all of \( \text{Hom}(S[V], A) \).

For \( j = 2 \), the hypothesis \( D_0 D_2 + D_2 D_0 + D_1 D_1 = 0 \) implies
\[
[D_0, \partial^2] + [D_0, \partial^2_{[\cdots, \cdots]}] + [\partial^1, \partial^1_{[\cdots, \cdots]}] + \partial^1 \partial^1 + \partial^1_{[\cdots, \cdots]} \partial^1_{[\cdots, \cdots]} = 0 \quad (4.28)
\]
Given \( a \in A \), exploiting the identity \( (4.20) \), we find that
\[
\left( [D_0, \partial^2] + \partial^1_{[\cdots, \cdots]} \partial^1 + \partial^1 \partial^1 \right)(a) = [d_0 t_2 + t_2 d^0 + t_1 \partial^1_{[\cdots, \cdots]} + t_1 \wedge t_1, a]
\]
is zero. Since \( a \in A \) is arbitrary, we conclude that \( d_0 t_2 + t_2 d^0 + t_1 \partial^1_{[\cdot, \cdot]} + t_1 \wedge t_1 \) is zero, that is, \( t_1 \) and \( t_2 \) satisfy the identity \([3.15]\). Consequently \( [D_0, \partial^2_{[\cdot, \cdot]}] + \partial^2_{[\cdot, \cdot]} \partial^1_{[\cdot, \cdot]} + \partial^2_{[\cdot, \cdot]} \partial^1_{[\cdot, \cdot]} = 0 \), whence \( [D_0, \partial^2_{[\cdot, \cdot]}] + \partial^2_{[\cdot, \cdot]} \partial^1_{[\cdot, \cdot]} = 0 \), and hence \([3.9]\) holds, viz.

\[
d^0 \partial^2_{[\cdot, \cdot]} + \partial^2_{[\cdot, \cdot]} d^0 + \partial^1_{[\cdot, \cdot]} \partial^1_{[\cdot, \cdot]} = 0.
\]

The next result extends Theorem 2.7 above.

**Theorem 4.11.** Suppose that the canonical morphism \([4.18]\) of graded \( A \)-modules is an isomorphism. Let \( \{D_j\}_{j \geq 1} \) be a family of derivations of \( \Sym_A(V, A) \) such that \( D_j \) lowers the filtration \([4.10]\) by \( j \). Suppose, furthermore, that

\[
(Sym_A(V, A), D_0, D_1, D_2, \ldots) \tag{4.29}
\]

is a multi derivation chain algebra. Then the family \( \{D_j\}_{j \geq 1} \) arises from a (necessarily unique) filtered (coalgebra) perturbation \( \partial_{[\cdot, \cdot]} \) of the coalgebra differential \( d^0 \) on \( C = S[V] \) and a filtered Lie algebra twisting cochain \( t: (C, d^0 + \partial_{[\cdot, \cdot]}) \to \text{Der}(A|R) \).

**Proof.** For \( j \geq 1 \), let

\[
t_j: S^j[V] \to \text{Der}(A|R) \subseteq \text{End}(A, A) \tag{4.30}
\]

be the adjoint of the composite

\[
D_j: A \to \text{Hom}_A(S_A^j[V], A) \subseteq \text{Hom}(S^j_A[V], A)
\]

of the derivation \( D_j \) with the injection into \( \text{Hom}(S^j_A[V], A) \). This yields a filtered degree \(-1\) morphism \( t: S[V] \to \text{Der}(A|R) \). By construction, the derivations

\[
\partial^j: \text{Hom}_A(S[V], A) \to \text{Hom}(S[V], A) \quad (j \geq 1) \tag{4.31}
\]

are then defined.

Likewise, let \( j \geq 1 \), notice that the composite

\[
D_j: \text{Hom}_A(V, A) \to \text{Hom}_A(S_A^{j+1}[V], A) \subseteq \text{Hom}(S^{j+1}_A[V], A) \tag{4.32}
\]

is defined, and let

\[
\bar{\partial}^j_{[\cdot, \cdot]} = D_j - \partial^j: \text{Hom}_A(V, A) \to \text{Hom}(S^{j+1}_A[V], A). \tag{4.33}
\]

Consider the pairing

\[
V^\otimes(j+1) \otimes \text{Hom}_A(V, A) \to A
\]

\[
\alpha_1 \otimes \ldots \otimes \alpha_{j+1} \otimes \varphi \mapsto \bar{\partial}^j_{[\cdot, \cdot]} \varphi(\alpha_1, \ldots, \alpha_{j+1}).
\]

Since the map from \( V \) to its double \( A \)-dual is an \( A \)-module isomorphism, this pairing induces an operation

\[
\bar{\partial}^j_{[\cdot, \cdot]}: S^{j+1}_A[V] \to V,
\]

and we extend \( \bar{\partial}^j_{[\cdot, \cdot]} \) to a coderivation

\[
\partial^j_{[\cdot, \cdot]}: S[V] \to S[V].
\]
This yields a filtered degree $-1$ filtration lowering coderivation $\partial_{[\cdot, \cdot, \cdot]} = \sum_{j \geq 0} \partial_{[\cdot, \cdot, \cdot]}^j$.

By construction, the hypotheses of Theorem 4.7 are satisfied. In view of that theorem, $\partial_{[\cdot, \cdot, \cdot]}$ is a filtered coalgebra perturbation of the coalgebra differential $d^0$ of $C = S[V]$, and $t$ is a filtered Lie algebra twisting cochain $(C, d^0 + \partial_{[\cdot, \cdot, \cdot]}) \to \text{Der}(A[R])$. By construction, the family $\{D_j\}_{j \geq 1}$ of derivations arises from these data as asserted.

Under the circumstances of Theorem 4.7 when each derivation $D_j$ is $A$-multilinear in such a way that (4.17) is a multi derivation chain algebra, we refer to the multi derivation chain algebra as the multi derivation Maurer-Cartan algebra associated to the data. Theorem 4.11 says that, when (4.18) is an isomorphism, any multi derivation chain algebra structure on $\text{Sym}_A(V, A)$ is the multi derivation Maurer-Cartan algebra associated to a filtered coalgebra perturbation $\partial_{[\cdot, \cdot, \cdot]}$ on $S[V]$ and a filtered Lie algebra twisting cochain $t: (S[V], d^0 + \partial_{[\cdot, \cdot, \cdot]}) \to \text{Der}(A[R])$ having the property that each derivation of the kind $D_j$, cf. (1.18), is $A$-multilinear.

5 sh Lie-Rinehart algebras

5.1 sh Lie algebras

Let $\mathfrak{g}$ be an $R$-chain complex. An sh-Lie algebra or, equivalently, $L_\infty$-algebra structure on $\mathfrak{g}$ is a coaugmentation filtration lowering coalgebra perturbation $\partial_{[\cdot, \cdot, \cdot]}: S^c[\mathfrak{g}] \to S^c[\mathfrak{g}]$ of the coalgebra differential $d^0$ on $S^c[\mathfrak{g}]$. We will then refer to the pair $(\mathfrak{g}, \partial_{[\cdot, \cdot, \cdot]})$ as an sh Lie algebra.

In the literature, it is common to write the structure in terms of higher order brackets. While we do not use the bracket formalism in the paper, for the benefit of the reader, we now explain how the higher order brackets arise: For $n \geq 2$, consider the graded symmetrization map

$$(\mathfrak{g})_s^{\otimes n} \xrightarrow{\text{sym}} S_n^c[\mathfrak{g}], \quad a_1 \otimes \ldots \otimes a_n \mapsto \frac{1}{n!} \sum \pm a_{\sigma 1} \otimes \ldots \otimes a_{\sigma n},$$

and use the bracket notation

$$[\cdot, \ldots, \cdot]: \mathfrak{g}^{\otimes n} \xrightarrow{s^{\otimes n}} (\mathfrak{g})^{\otimes n} \xrightarrow{\text{sym}} S_n^c[\mathfrak{g}] \xrightarrow{\partial^0_{[\cdot, \cdot, \cdot]}^{n-1}} S_1^c[\mathfrak{g}] \xrightarrow{\tau_{\mathfrak{g}}} \mathfrak{g} \quad (5.1)$$

for the depicted $\mathfrak{g}$-valued operation of $n$-variables ranging over $\mathfrak{g}$; by construction, the operation $[\cdot, \ldots, \cdot]$ has homogeneous degree $n - 2$ and is graded skew symmetric.

Let $(\mathfrak{g}, \partial_{[\cdot, \cdot, \cdot]})$ be an $L_\infty$-algebra and suppose that $\mathfrak{g}$ is concentrated in degrees $\leq 0$; we then write $\mathfrak{g}^j = \mathfrak{g}_{-j}$ ($j \geq 0$). The $L_\infty$-structure is given by a system of bracket operations

$$[\cdot, \ldots, \cdot]_n: \mathfrak{g}^{j_1} \times \ldots \times \mathfrak{g}^{j_n} \to \mathfrak{g}^{j_1 + \ldots + j_n - n + 2} \quad (5.2)$$

Thus

$$[\cdot, \cdot]_2: \mathfrak{g}^{j_1} \times \mathfrak{g}^{j_2} \to \mathfrak{g}^{j_1 + j_2} \quad (5.3)$$

$$[\cdot, \ldots, \cdot]: \mathfrak{g}^{j_1} \times \mathfrak{g}^{j_2} \times \mathfrak{g}^{j_3} \to \mathfrak{g}^{j_1 + j_2 + j_3 - 1} \quad (5.4)$$

$$[\cdot, \cdot, \cdot]: \mathfrak{g}^{1} \times \mathfrak{g}^{0} \times \mathfrak{g}^{0} \to \mathfrak{g}^{0} \quad (5.5)$$

etc.
5.2 sh Lie algebra action by derivations

Let $A$ be a differential graded commutative $R$-algebra. Given an sh Lie algebra $(g, \partial)$, we define an sh-action of $(g, \partial)$ on $A$ by derivations to be a Lie algebra twisting cochain

$$t : (S^c[sg], d^0 + \partial_{[-,-]}) \longrightarrow \text{Der}(A|R). \quad (5.6)$$

Let $t : S^c[sg] \to \text{Der}(A|R)$ and $t' : S^c[sg'] \to \text{Der}(A'|R)$ be two sh actions by derivations and $\varphi : A \to A'$ a morphism of differential graded algebras.

**Definition 5.1.** Given a morphism

$$\Phi : (S^c[sg], d^0 + \partial_{[-,-]}) \longrightarrow (S^c[sg'], d^0 + \partial'_{[-,-]})$$

of differential graded coalgebras, the pair

$$((\varphi, \Phi)) : (A, g, \partial_{[-,-]}, t) \longrightarrow (A', g', \partial'_{[-,-]}, t') \quad (5.7)$$

is a morphism of sh actions by derivations when the adjoints $t^\flat : S^c[sg] \otimes A \to A$ and $(t')^\flat : S^c[sg'] \otimes A' \to A'$ of $t$ and $t'$, respectively, make the diagram

$$S^c[sg] \otimes A \quad \Phi \otimes \varphi \quad \varphi \quad \longrightarrow \quad A'$$

$$S^c[sg'] \otimes A' \quad (t')^\flat \quad \longrightarrow \quad A'$$

commutative.

For reasons of variance, a general morphism of sh actions does not induce a morphism between the associated Maurer-Cartan algebras, see Remark [5.4] below. In ordinary Lie algebra cohomology theory, one takes care of the variance problem by means of comorphisms. To extend the comorphism concept to the present situation, define

$$t^\flat : S^c[sg] \longrightarrow \text{Der}(A, A') \quad (5.9)$$

by $t^\flat(x) = \varphi \circ (t(x))$, $x \in S[sg]$ and

$$\varphi^\flat : \text{Der}(A'|R) \longrightarrow \text{Der}(A, A') \quad (5.10)$$

by $\varphi^\flat(\delta) = \delta \circ \varphi$.

**Definition 5.2.** Given a morphism

$$\Phi : (S^c[sg'], d^0 + \partial'_{[-,-]}) \longrightarrow (S^c[sg], d^0 + \partial_{[-,-]})$$

of differential graded coalgebras, the pair

$$((\varphi, \Phi)) : (A', g', \partial'_{[-,-]}, t') \longrightarrow (A, g, \partial_{[-,-]}, t) \quad (5.11)$$

is a comorphism of sh actions by derivations when the diagram

$$S^c[sg'] \\ \varphi^\flat \quad \longrightarrow \quad S^c[sg]$$

$$\varphi'' \quad \longrightarrow \quad \text{Der}(A'|R) \quad \Phi \quad \longrightarrow \quad \text{Der}(A, A')$$

is commutative.
Notice that, between the two definitions (5.1) and (5.2), there is a difference of variance. Notice also when \( g \) and \( g' \) are ordinary Lie algebras and \( t \) and \( t' \) come from ordinary Lie algebra actions by derivations, in terms of the bracket notation \([\cdot, \cdot]: g \times A \to A\) and \([\cdot, \cdot]: g' \times A' \to A'\) for these actions, the commutativity of (5.12) comes down to the familiar identity

\[
\varphi[\Phi(x), a] = [x, \varphi(a)], \quad x \in g', \; a \in A. \tag{5.13}
\]

This identity says that \( \varphi \) is a morphism of \( g' \)-modules when \( g' \) acts on \( A \) through \( \Phi: g' \to g \). The following proposition generalizes a classical observation in ordinary Lie algebra cohomology theory.

**Proposition 5.3.** A comorphism \((\varphi, \Phi): (A', g', \partial'_{[\cdot, \cdot], t'}) \to (A, g, \partial_{[\cdot, \cdot], t})\) of sh actions by derivations induces a morphism

\[
(\varphi_*, \Phi^*): (\text{Hom}(S^c[s\mathfrak{g}], A), D_0 + \partial[\cdot, \cdot, \cdot] + \partial') \to (\text{Hom}(S^c[s\mathfrak{g}], A'), D_0 + (\partial[\cdot, \cdot, \cdot]') + \partial'') \tag{5.14}
\]

of differential graded \( R \)-algebras.

**Remark 5.4.** A comorphism \((\varphi, \Phi): (A', g', \partial'_{[\cdot, \cdot], t'}) \to (A, g, \partial_{[\cdot, \cdot], t})\) of sh actions by derivations having \( A = A' \) and \( \varphi = \text{Id} \) is simply a morphism

\[
(\text{Id}, \Phi): (A, g', \partial'_{[\cdot, \cdot], t'}) \to (A, g, \partial_{[\cdot, \cdot], t})
\]

of sh actions by derivations. However, for reasons of variance, a general morphism of sh actions by derivations cannot induce a morphism of the kind (5.14).

An sh Lie algebra action may be written in terms of bracket operations in the following manner; again we do not need these brackets but spell them out for the benefit of the reader.

Let \( t \) be an sh-action of the kind (5.6) of \((g, \partial_{[\cdot, \cdot]})\) on \( A \) by derivations. For \( n \geq 1 \), consider the composite

\[
g^\otimes n \xrightarrow{s^\otimes n} (s\mathfrak{g})^\otimes n \xrightarrow{\text{sym}} S^c_n[s\mathfrak{g}] \xrightarrow{t_n} \text{Der}(A|R), \tag{5.15}
\]

and write its adjoint as an \( A \)-valued operation

\[
\{\cdot, \ldots, \cdot\}: g^\otimes n \otimes A \to A \tag{5.16}
\]

having \( n \) arguments from \( g \) and one argument from \( A \). Given homogeneous \( x_1, \ldots, x_n \in g \), the operation

\[
\{x_1, \ldots, x_n\cdot\}: A \to A
\]

is a homogeneous derivation of \( A \) of degree \(|x_1| + \ldots + |x_n| + n - 1\).

### 5.3 sh Lie-Rinehart structure

In terms of the formalism so far developed, in the spirit of [Kje01] (Def. 4.9 p. 157), we will now propose a definition of an sh Lie-Rinehart algebra.

Let \( A \) be a differential graded commutative \( R \)-algebra and \((L, \partial_{[\cdot, \cdot], t})\) an sh Lie algebra over \( R \). In view of the isomorphism (3.2), we will henceforth identify \( S^c[sL] \) with \( S[sL] \), endowed with the graded shuffle diagonal. Let \( t: S[sL] \to \text{Der}(A|R) \) be an sh-action of \( L \) on \( A \) by derivations, and suppose that \( L \) carries a differential graded \( A \)-module structure.

Take \( t = t_1 + t_2 + \ldots \) to be a filtered Lie algebra twisting cochain in the obvious manner, that is, let \( t_j (j \geq 1) \) be the component of \( t \) defined on the \( j \)th graded symmetric power
$S^j[sL]$ of the suspension $sL$ of $L$. Likewise take $\partial_{[\cdot,\cdot]} = \partial^1_{[\cdot,\cdot]} + \partial^2_{[\cdot,\cdot]} + \ldots$ to be a filtered coalgebra perturbation in the obvious way, that is, for $j \geq 1$, let $\partial^j_{[\cdot,\cdot]}$ denote the coderivation determined by the constituent

$$\partial_{[\cdot,\cdot]}: S^{j+1}[sL] \to sL$$

of $\partial_{[\cdot,\cdot]}$. Endow the suspension $sL$ of $L$ with the induced differential graded $A$-module structure.

**Definition 5.5.** The data $(A, L, \partial_{[\cdot,\cdot]}, t)$ constitute an *sh Lie-Rinehart algebra* when $t$ is graded $A$-multilinear and the data satisfy the axiom (5.17) below (for $j \geq 1$):

$$\partial^j_{[\cdot,\cdot]}(\alpha_1, \ldots, \alpha_j, a \alpha_{j+1}) = t_j(\alpha_1, \ldots, \alpha_j)(a)\alpha_{j+1}$$

$$+ (-1)^{(|\alpha_1| + \ldots + |\alpha_j| + 1)a} a \partial^j_{[\cdot,\cdot]}(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})$$

(5.17)

where $\alpha_1, \ldots, \alpha_j, \alpha_{j+1}$ are homogeneous members of $sL$ and $a$ is a homogeneous member of $A$.

**Remark 5.6.** For $j \geq 1$, the condition (5.17) measures the deviation of $\partial^j_{[\cdot,\cdot]}$ from being graded $A$-multilinear.

**Remark 5.7.** In [Kje01] (Def. 4.9 p. 157), the terminology is ‘sh Lie-Rinehart pair’. An ‘sh Lie-Rinehart pair’ is there defined via suitable brackets. In terms of the brackets (5.1) and (5.16), the $A$-multilinearity of $t$ is equivalent to

$$\{ax_1, \ldots, x_n|b\} = (-1)^{|a|} a \{x_1, \ldots, x_n|b\}, \ n \geq 1,$$

(5.18)

and (5.17) is equivalent to

$$[x_1, \ldots, x_n, ax_{n+1}] = \{x_1, \ldots, x_n|a\} x_{n+1}$$

$$+ (-1)^{(|x_1| + \ldots + |x_n| + n+1)|a} a [x_1, \ldots, x_n, x_{n+1}],$$

(5.19)

for $n \geq 1$, where $x_1, \ldots, x_n, x_{n+1}$ are homogeneous members of $L$ and $a, b$ homogeneous members of $A$.

**Remark 5.8.** For $n = 1$, the $A$-multilinearity of $t$ or, equivalently, the condition (5.18), and the axiom (5.17) or, equivalently, (5.19), come down to (2.2) and (2.3), respectively, adjusted to the graded situation.

**Remark 5.9.** Write the differential on $A$ as a unary bracket $\{\cdot\}: A \to A$ and that on $L$ as a unary bracket $[\cdot]: L \to L$. In terms of this notation, (5.19) extends to the identity

$$[ax] = \{a\}x + (-1)^{|a|}a[x]$$

(5.20)

involving the unary brackets. The identity (5.20) precisely says that $L$ is a differential graded $A$-module. A unitary bracket of the kind $[\cdot]: L \to L$ encapsulating a differential occurs, e.g., in [Vor05].

Let $(A, L, \partial_{[\cdot,\cdot]}, t)$ and $(A', L', \partial'_{[\cdot,\cdot]}, t')$ be two sh Lie-Rinehart algebras. Given a morphism $\varphi: A \to A'$ of differential graded algebras, we view $(S_A[sL'], \partial')$ as a differential graded $A$-module via $\varphi$ in the obvious manner.
Definition 5.10. A morphism

\[(\varphi, \Phi): (A, L, \partial_{[\ldots]}, t) \longrightarrow (A', L', \partial'_{[\ldots]}, t')\]

of sh actions is a **morphism of sh Lie-Rinehart algebras** when \(\Phi\) passes to a morphism \(\Phi: (S_A[sL], d^0) \rightarrow (S_A'[sL'], d^0)\) of differential graded \(A\)-modules (where the notation \(\Phi\) is abused).

This extends the notion of an ordinary morphism of Lie-Rinehart algebras, cf. [Hue90].

5.4 **Multi derivation Maurer-Cartan characterization of an sh Lie-Rinehart structure**

Let \((L, d^0)\) be a chain complex, let \(C = S^c[sL]\) and write the induced coalgebra differential on \(S^c[sL]\) as \(d^0\). Using the canonical isomorphism \(S[sL] \rightarrow S^c[sL]\) of differential graded cocommutative coalgebras, cf. [3.2], we interpret

\[
\hom(C, A) = \hom(S^c[sL], A)
\]

as the algebra \(\text{Sym}(sL, A)\) of \(A\)-valued \(R\)-multilinear graded symmetric functions on \(sL\).

We apply the results in the previous section, with \(V = sL\). We maintain the notation \((4.4)-(4.9)\).

**Theorem 5.11.** Let \((A, L, \partial_{[\ldots]}, t)\) be an sh Lie-Rinehart algebra. For \(j \geq 1\), the derivation \(D_j\) of \(\hom(S[sL], A)\) passes to a derivation of \(\text{Sym}_A(sL, A) = \hom_A(S_A[sL], A)\), and the resulting data of the kind \((4.17), \) viz.

\[
(\text{Sym}_A(sL, A), D_0, D_1, D_2, \ldots),
\]

constitute a multi derivation Maurer-Cartan algebra.

**Proof.** Let \(j \geq 1\). Consider the operator \(D_j = \partial^{[\ldots]_j} + \partial_j\) on \(\hom(S[sL], A)\). By construction

\[
D_j = \partial^{[\ldots]_j} : A \rightarrow \hom(S^j[sL], A)
\]

\[
D_j = \partial^{[\ldots]_j} + \partial^{j} : \hom(sL, A) \rightarrow \hom(S^{j+1}[sL], A).
\]

Let \(u, a \in A\) and \(\alpha_1, \ldots, \alpha_j \in sL\) be homogeneous. Since \(t_j\) is \(A\)-multilinear,

\[
\partial^j u(aa_1, \ldots, \alpha_j) = [t_j, u](aa_1, \ldots, \alpha_j)
\]

\[
= (-1)^{|u|(|a|+|\alpha_1|+\ldots+|\alpha_j|)}t_j(aa_1, \ldots, \alpha_j)(u)
\]

\[
= (-1)^{|u|a}(-1)^{|u|(|\alpha_1|+\ldots+|\alpha_j|)}(-1)^{|a|at_j(\alpha_1, \ldots, \alpha_j)}(u)
\]

\[
= (-1)^{|[u]+1}a[t_j, u](\alpha_1, \ldots, \alpha_j)
\]

\[
= (-1)^{([u]+1)a}a(\partial^j u)(\alpha_1, \ldots, \alpha_j).
\]

Likewise, let \(a \in A, \varphi \in \hom(sL, A)\) and \(\alpha_1, \ldots, \alpha_j, \alpha_{j+1} \in sL\) be homogeneous. We claim first that

\[
\partial^{[\ldots]_j}\varphi(\alpha_1, \ldots, \alpha_j, aa_{j+1}) = (-1)^{|[a_1]+\ldots+|\alpha_j|+|a|}[\varphi]^{j+1}t_j(\alpha_1, \ldots, \alpha_j)(a)\varphi(\alpha_{j+1})
\]

\[
+ (-1)^{|[\alpha_1]+\ldots+|\alpha_j|+1+|a|}a_{[\ldots]_j}\varphi(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}).
\]

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Indeed, by (5.17), viz.

\[ \partial^i_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_n, a\alpha_{j+1}) = t_j(\alpha_1, \ldots, \alpha_j)(a)\alpha_{j+1} \\
+ (-1)^{|\alpha_1|+\ldots+|\alpha_j|+|a|}a\partial^i_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

\[ \partial^j_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{|\varphi|+1}\varphi(\partial^j_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})) \]

\[ = (-1)^{|\varphi|+1}\varphi(t_j(\alpha_1, \ldots, \alpha_j)(a)\alpha_{j+1}) \\
+ (-1)^{|a|+|\alpha_1|+|\alpha_j|+|\varphi|+1}a\varphi(\partial^j_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})) \]

\[ = (-1)^{|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}t_j(\alpha_1, \ldots, \alpha_j)(a)\varphi(\alpha_{j+1}) \\
+ (-1)^{|a|+|\alpha_1|+|\alpha_j|+|\varphi|+1}a\varphi(\partial^j_{[\cdot \cdot \cdot]}(\alpha_1, \ldots, \alpha_j, \alpha_{j+1})) \]

whence (5.26).

Next we claim

\[ \partial_t^j(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{|a|+|\alpha_1|+|\alpha_j|+|\varphi|+1}a\partial_t^j(\varphi)(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

\[ + (-1)^{|a|+|\alpha_1|+|\alpha_j|+|\varphi|+1}a\partial_t^j(\varphi)(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

(5.27)

Indeed, for \( j = 1 \)

\[ [t_1, \varphi](\alpha_1, a\alpha_2) = [\cdot \cdot \cdot] \circ (t_1 \otimes \varphi)(\alpha_1 \otimes (a\alpha_2) + (-1)^{|a|}(a\alpha_2 \otimes \alpha_1) \]

\[ = (-1)^{|\alpha_1|+|\varphi|}a[t_1, \varphi](\alpha_1, a\alpha_2) + (-1)^{|\varphi|}t_1(\alpha_1)(a)\varphi(a_2). \]

The same kind of reasoning shows that, for general \( j \geq 1 \),

\[ [t_j, \varphi](\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{|a|+|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}a[t_j, \varphi](\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \\
+ (-1)^{|a|+|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}a[t_j, \varphi](\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

whence (5.27).

Combining (5.26) and (5.27), since the summands involving \( t_j(\alpha_1, \ldots, \alpha_j)(a)\varphi(\alpha_{j+1}) \) cancel out, we find

\[ D_j(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = \partial^i_{[\cdot \cdot \cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) + \partial^j_{[\cdot \cdot \cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) \]

\[ = (-1)^{|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}a\partial^i_{[\cdot \cdot \cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \\
+ (-1)^{|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}a\partial^j_{[\cdot \cdot \cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

\[ = (-1)^{|\alpha_1|+\ldots+|\alpha_j|+|\varphi|+1}aD_j(\varphi)(\alpha_1, \ldots, \alpha_j, \alpha_{j+1}) \]

as asserted. \( \square \)

Under the circumstances of Theorem 5.11, we refer to the multi derivation chain algebra \( (A, L, \partial_{[\cdot \cdot \cdot]}, t) \) as the \textit{multi derivation Maurer-Cartan algebra associated to the sh-Lie-Rinehart algebra} \((A, L, \partial_{[\cdot \cdot \cdot]}, t)\).
Remark 5.12. By construction, relative to the filtration degree and the complementary degree, the multi derivation Maurer-Cartan algebra associated to an sh-Lie-Rinehart algebra is actually a bigraded multi derivation Maurer-Cartan algebra, cf. Remarks 4.1 and 4.6 above.

The multi derivation Maurer-Cartan algebra associated to an sh Lie-Rinehart algebra is natural in a sense we now explain.

Definition 5.13. A comorphism 

$$(\varphi, \Phi): (A', L', \partial'_{[\ldots]}, t') \longrightarrow (A, L, \partial_{[\ldots]}, t)$$

of sh actions is a comorphism of Lie-Rinehart algebras when $\Phi: S[sL'] \rightarrow S[sL]$ passes to a morphism $\hat{\Phi}: S_A'[sL'] \rightarrow A' \otimes_A S_A[sL]$ of differential graded $A'$-modules making the diagram

$$
\begin{array}{ccc}
\Phi: S_A'[sL'] & \xrightarrow{\Phi} & A' \otimes_A S_A[sL] \\
\downarrow & & \downarrow \\
\text{Der}(A'|R) & \xrightarrow{\varphi^\flat} & \text{Der}(A, A')
\end{array}
$$

(5.28)

commutative.

Proposition 5.14 now extends to the following.

Proposition 5.14. A comorphism $$(\varphi, \Phi): (A', L', \partial'_{[\ldots]}, t') \longrightarrow (A, L, \partial_{[\ldots]}, t)$$ of sh Lie-Rinehart algebras induces a morphism

$$(\varphi_* \circ \Phi^*): (\text{Sym}_A(sL, A), D_0, D_1, D_2, \ldots) \longrightarrow (\text{Sym}_{A'}(sL', A'), D_0', D_1', D_2' + \ldots)$$

(5.29)

between the associated multi derivation Maurer-Cartan algebras.

We note that, by construction,

$$\text{Sym}_A(sL, A) \longrightarrow \text{Sym}_{A'}(sL', A')$$

is the composite of

$$\varphi_*: \text{Hom}_A(S_A[sL], A) \longrightarrow \text{Hom}_{A'}(A' \otimes_A S_A[sL], A')$$

with

$$\Phi^* : \text{Hom}_A(A' \otimes_A S_A[sL], A') \longrightarrow \text{Hom}_A(S_{A'}[sL'], A').$$

Proposition 5.14 says that this composite is compatible with the multi derivation Maurer-Cartan algebra structures.

The comorphism concept for sh Lie-Rinehart algebras generalizes that for ordinary Lie-Rinehart algebras, cf. [HM93], where this is explained for Lie algebroids. As already noted in Remark 5.4, for reasons of variance, a morphism

$$(\varphi, \Phi): (A, L, \partial_{[\ldots]}, t) \longrightarrow (A', L', \partial'_{[\ldots]}, t')$$

of sh Lie-Rinehart algebras, as defined in Subsection 5.3 above, does not induce a morphism between the associated multi derivation Maurer-Cartan algebras in an obvious manner except when $A = A'$ and $\varphi$ is the identity—in this case the notions of morphism and comorphism coincide, not even for the special case of a morphism of ordinary Lie-Rinehart algebras (except when $A = A'$ and $\varphi$ is the identity).
Theorem 5.15. Let $A$ be a differential graded commutative algebra and $L$ a differential graded $A$-module having the property that the canonical $A$-module morphism from $L$ to its double $A$-dual is injective. Let $\partial_{[-\cdot,\cdot]}$ be a filtered degree $-1$ filtration lowering coderivation on $S[L]$ and $t$ a filtered degree $-1$ morphism of the kind (5.6), and let $D_0$ denote the algebra differential on $\text{Sym}_A(sL, A)$ induced from the differentials on $L$ and $A$. Then $(A, L, \partial_{[-\cdot,\cdot]}, t)$ is an sh Lie-Rinehart algebra if and only if $(\text{Sym}_A(sL, A), D_0, D_1, D_2, \ldots)$ is a multi derivation chain algebra, necessarily the multi derivation Maurer-Cartan algebra associated to $(A, L, \partial_{[-\cdot,\cdot]}, t)$.

Proof. Theorem 5.11 says that the condition is necessary. Thus suppose that

$$(\text{Sym}_A(sL, A), D_0, D_1, D_2, \ldots)$$

is a multi derivation chain algebra. Theorem 4.7 implies that $(S[sL], \partial_{[-\cdot,\cdot]})$ is a differential graded cocommutative coalgebra, that is, that $(L, \partial_{[-\cdot,\cdot]})$ is an sh Lie algebra, and that $t: S[L] \rightarrow \text{Der}(A|R)$ is a Lie algebra twisting cochain. Thus it remains to show that $t$ is $A$-multilinear and satisfies the axiom (5.17). Formally exactly the same reasoning as that in the proof of Corollary 2.6 above establishes these claims.

Reading backwards the reasoning in the proof of Theorem 5.11 yields the details: Indeed, let $j \geq 1$. Let $u, a \in A$ and $\alpha_1, \ldots, \alpha_j \in sL$ be homogeneous. Since $\partial_{[-\cdot,\cdot]}^{[\cdot,\cdot]}(u) = 0$ and since $\partial_{[-\cdot,\cdot]}^{[\cdot,\cdot]}u = [t_j, u]$, the hypothesis implies that

$$t_j(a\alpha_1, \ldots, \alpha_j)(u) = (-1)^{(\lvert a \rvert + \lvert \alpha_1 \rvert + \ldots + \lvert \alpha_j \rvert)\lvert u \rvert}t_j(u)(a\alpha_1, \ldots, \alpha_j)$$

whence, since $u$ is arbitrary, $t_j$ is $A$-multilinear.

Likewise, let $a \in A$, $\varphi \in \text{Hom}(sL, A)$ and $\alpha_1, \ldots, \alpha_j, \alpha_{j+1} \in sL$ be homogeneous. We already know that (5.26) holds, viz.

$$\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{(\lvert \alpha_1 \rvert + \ldots + \lvert \alpha_j \rvert + (\lvert a \rvert + 1)\lvert \varphi \rvert)\lvert a \rvert}a\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}(\varphi)(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})$$

Since the operator $D_j$ passes to $\text{Sym}_A(sL, A)$, we conclude that (5.26) holds, viz.

$$\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}\varphi(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{(\lvert \alpha_1 \rvert + \ldots + \lvert \alpha_j \rvert + (\lvert a \rvert + 1)\lvert \varphi \rvert)\lvert a \rvert}a\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}\varphi(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})$$

Moreover,

$$\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}\varphi(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1}) = (-1)^{(\lvert \varphi \rvert + 1)\lvert a \rvert}\varphi(\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})).$$

Consequently

$$\varphi(\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})) = \varphi(t_j(\alpha_1, \ldots, \alpha_j)(a)\alpha_{j+1})$$

$$+ (-1)^{(\lvert \alpha_1 \rvert + \ldots + \lvert \alpha_j \rvert + (\lvert a \rvert + 1)\lvert \varphi \rvert)\lvert a \rvert}\varphi(a\partial_{[\cdot,\cdot]}^{[\cdot,\cdot]}(\alpha_1, \ldots, \alpha_j, a\alpha_{j+1})).$$

Since $\varphi$ is arbitrary, the hypothesis implies that the identity (5.17) holds. □
Theorems 4.7 and 4.11 imply the following:

**Theorem 5.16.** Let $A$ be a differential graded commutative algebra and $L$ a differential graded $A$-module having the property that the canonical $A$-module morphism from $L$ to its double $A$-dual is an isomorphism, and let $D_0$ denote the algebra differential on $\text{Sym}_A(sL, A)$ induced from the differentials on $L$ and $A$. Sh Lie-Rinehart structures on $(A, L)$ extending the differentials on $A$ and $L$ and multi derivation chain algebra structures on $\text{Sym}_A(sL, A)$ extending the algebra differential $D_0$ are equivalent notions. The equivalence between the two notions is that spelled out explicitly in Theorem 5.15.

**Remark 5.17.** Theorem 5.15 says that, given the data $(A, L, \partial[\cdot, \cdot], t)$, under the hypothesis spelled out there, these data constitute an sh Lie-Rinehart algebra if and only if they induce a multi derivation chain algebra structure on $\text{Sym}_A(sL, A)$. On the other hand, Theorem 5.16 says that, under the stronger hypothesis of this theorem, every multi derivation chain algebra structure on $\text{Sym}_A(sL, A)$ of the kind under discussion arises from a unique sh Lie-Rinehart algebra structure on $(A, L)$.

### 5.5 Quasi Lie-Rinehart algebras

Let $\mathcal{A}$ be a differential graded commutative algebra concentrated in degrees $\leq 0$ and, as before, we then write $\mathcal{A}^j = A_{-j}$ so that $\mathcal{A}^j = 0$ for $j < 0$. Furthermore, let $Q$ be a differential graded $\mathcal{A}$-module whose underlying graded $\mathcal{A}$-module is an induced module of the kind $Q = \mathcal{A} \otimes A\mathcal{Q}$ where $A = \mathcal{A}^0$ and where $Q$ is concentrated in degree zero. Suppose that the canonical $\mathcal{A}$-module morphism from $Q$ to its double $A$-dual is an isomorphism.

In [Hue05] we introduced quasi Lie-Rinehart algebras. We recall that a quasi-Lie-Rinehart algebra structure on $(\mathcal{A}, Q)$ involves the following three items:

— a graded skew-symmetric $R$-bilinear pairing of degree zero

$$\langle \cdot, \cdot \rangle_Q: Q \otimes_R Q \to Q,$$

(5.32)

— an $R$-bilinear pairing of degree zero

$$\langle \cdot, \cdot \rangle: Q \otimes_R \mathcal{A} \to \mathcal{A}, \quad (\xi, \alpha) \mapsto \xi(\alpha), \quad \xi \in Q, \; \alpha \in \mathcal{A},$$

(5.33)

such that, given $\xi \in Q$, the operation $(\xi, \cdot)$ is a homogeneous $R$-linear derivation of $\mathcal{A}$ of degree $|\xi|$;

— an $\mathcal{A}$-trilinear operation of degree $-1$ (beware: in upper degrees)

$$\langle \cdot, \cdot, \cdot \rangle_Q: Q \otimes_A Q \otimes_A \mathcal{A} \to \mathcal{A}$$

(5.34)

which is graded skew-symmetric in the first two variables (i.e. in the $Q$-variables), such that, given $\xi, \theta \in Q$, the operation $\langle \xi, \theta, \cdot \rangle$ is a homogeneous $A$-linear derivation of $\mathcal{A}$ of degree $-1$.

These pieces of structure are subject to a number of constraints, see [Hue05] for details. We do not spell out these constraints here; instead we will now explain directly the Maurer-Cartan algebra characterization of the structure. This will illustrate the technology developed in the present paper. The sign of the Lie-Rinehart operator (generalized CCE operator) in [Hue05] is the negative of the sign of the Lie-Rinehart operator in the present paper.

We note first that

$$\text{Sym}_A(sQ, \mathcal{A}) = \text{Hom}_A(S[sQ], \mathcal{A}) \cong \text{Alt}_A(Q, \mathcal{A}).$$

(5.35)
Now, let $\delta^1_{\cdots} : S[sQ] \to S[sQ]$ denote the coderivation induced by (5.32) and, likewise, $t_1 : S^1[sQ] = sQ \to \text{Der}(A|R)$ the degree $-1$ morphism of $R$-modules given by the composite of the desuspension with the adjoint of (5.33). As before, let $\delta^1_{\cdots}$ and $\delta^{t_1}$ denote the derivations on $\text{Hom}(S[sQ], A)$ induced by $\delta^1_{\cdots}$ and $t_1$, respectively, cf. (4.4) and (3.24). The compatibility conditions among the pieces of structure (5.32) and (5.33) (not spelled out here) imply that the sum $D_1 = \delta^1_{\cdots} + \delta^{t_1}$ passes to a derivation $D_1$ on $\text{Alt}_A(Q, A)$, formally a CCE operator with respect to (5.32) and (5.33). Indeed, given $\xi_1, \ldots, \xi_{p+1} \in Q$:

$$(-1)^{|f|}(D_1 f)(\xi_1, \ldots, \xi_{p+1}) = \sum_{j=1}^{p+1} (-1)^j \xi_j (f(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}))$$

$$+ \sum_{1 \leq j < k \leq p+1} (-1)^{j+k} f([\xi_j, \xi_k]_Q, \xi_1, \ldots, \hat{\xi}_j, \ldots, \hat{\xi}_k, \ldots, \xi_{p+1});$$

where the notation $[\cdot, \cdot]_Q$ refers to the restriction to the degree zero constituent $Q$ (of $Q$) of the bracket (5.32) on $Q$. Next, given $f \in \text{Alt}_A^p(Q, A^q) = \text{Hom}_A(N_A^p[sQ], A^q)$, define the value $D_2(f)(\xi_1, \ldots, \xi_{p+2})$ by

$$(-1)^{|f|}(D_2 f)(\xi_1, \ldots, \xi_{p+2}) = \sum_{1 \leq j < k \leq p+2} (-1)^{j+k} \langle \xi_j, \xi_k; f(\xi_1, \ldots, \hat{\xi}_j, \ldots, \hat{\xi}_k, \ldots, \xi_{p+2}) \rangle_Q$$

(5.36)

where $\xi_1, \ldots, \xi_{p+2} \in Q$.

**Remark 5.18.** The expression in [Hue05] (4.8.2) that corresponds to (5.36) above involves the additional sign factor $(-1)^p$. This sign factor has its origin in the relationship with Lie-Rinehart triples, cf. [Hue05] (4.11) and (4.12). More precisely, in terms of the notation in [Hue05], given $x_1, \ldots, x_{q-1} \in H$, $\alpha \in \text{Alt}^q(H, A) = A^q$, in [Hue05] (4.11.7), the value of the pairing (5.34) is defined by

$$\langle \xi, \eta; \alpha \rangle_Q (x_1, \ldots, x_{q-1}) = \alpha(\delta(\xi, \eta), x_1, \ldots, x_{q-1});$$

however, when we define that value by

$$\langle \xi, \eta; \alpha \rangle_Q (x_1, \ldots, x_{q-1}) = (-1)^{|\alpha|} \alpha(\delta(\xi, \eta), x_1, \ldots, x_{q-1}),$$

(5.37)

the factor $(-1)^p$ in [Hue05] (4.8.2) disappears.

Let $D_0$ denote the algebra differential on $\text{Alt}_A(Q, A)$ induced from the differentials on $A$ and $Q$. By [Hue05] (Theorem 4.10), (5.32), (5.33), and (5.34) turn $(A, Q)$ into a quasi Lie-Rinehart algebra if and only if $(\text{Alt}_A(Q, A), D_0, D_1, D_2)$ is a multi algebra, and any quasi Lie-Rinehart algebra structure arises in this manner. In view of (5.35), Theorem 5.16 shows that the multi algebra $(\text{Alt}_A(Q, A), D_0, D_1, D_2)$, in turn, characterizes a special kind of sh Lie-Rinehart structure on $(A, Q)$. Hence a quasi Lie-Rinehart structure on $(A, Q)$ is equivalent to a special kind of sh Lie-Rinehart structure.

Let $(A, Q, [\cdot, \cdot]_Q, (\cdot, \cdot), \langle \cdot, \cdot, \cdot \rangle)$ be a quasi Lie-Rinehart algebra. With hindsight, it is now interesting to spell out the corresponding sh Lie-Rinehart structure on $(A, Q)$ arising from the quasi Lie-Rinehart algebra structure. This is straightforward: The bracket (5.32) is already defined and, as before, we define $t_1 : S^1[sQ] = sQ \to \text{Der}(A|R)$ to be the degree
−1 morphism of $R$-modules given by the composite of the desuspension with the adjoint of $(5.33)$. Now, noting that a general sh Lie algebra structure on $Q$ is given by a system of brackets of the kind

$$[\cdots, \cdots] : Q^{j_1} \times \cdots \times Q^{j_n} \to Q^{j_1+j_2+\cdots+j_n-n+1}, \quad j_1, \ldots, j_n \geq 0;$$

we see that the 3-variable bracket we are looking for must be of the kind

$$[\cdots, \cdots, \cdots] : Q^{j_1} \times Q^{j_2} \times Q^{j_3} \to Q^{j_1+j_2+j_3-1}, \quad j_1, j_2, j_3 \geq 0;$$

in particular, the constituent of $[\cdots, \cdots, \cdots]$ on the degree zero component $Q \times Q \times Q$ of $Q \times Q \times Q$ is necessarily zero. Define the operation

$$[\cdots, \cdots] : Q \otimes Q \otimes Q \to Q \quad (5.38)$$

by setting

$$[x_1, x_2, ax_3]_3 = (x_1, x_2; a)_Q x_3, \quad x_1, x_2, x_3 \in Q, \quad a \in A. \quad (5.39)$$

Write the operations $(\cdot, \cdot)$ and $(\langle \cdot, \cdot \rangle)_{\mathcal{Q}}$ as

$$\{\cdot, \cdot\} : Q \otimes A \to A \quad \text{and} \quad \{\cdot, \cdot\} : Q \otimes A \otimes A \to A,$$

respectively and, using $(5.18)$, extend $\{\cdot, \cdot\}$ to an operation

$$\{\cdot, \cdot\} : Q \otimes A \otimes A \to A. \quad (5.40)$$

Thereafter, extend $(5.38)$ to

$$[\cdots, \cdots] : Q \otimes_R Q \otimes_R Q \to Q \quad (5.41)$$

in such a way that $(5.18)$ holds, i.e., given homogeneous $x_1, x_2, x_3 \in Q$ and $a \in A$,

$$[x_1, x_2, ax_3]_3 = \{x_1, x_2; a\}_Q x_3 + (-1)^{|x_1|+|x_2|+1} |a| a[x_1, x_2, x_3]_3. \quad (5.42)$$

Then $(A, Q, [\cdot, \cdot]_Q, [\cdot, \cdot, \cdot], \{\cdot, \cdot\}, \{\cdot, \cdot\})$ is the sh Lie-Rinehart algebra we are looking for.

It is also interesting to spell out explicitly how the 3-variable bracket controls the failure of the 2-variable bracket to satisfy the Jacobi identity: By construction, the control of the failure of the 2-variable bracket to satisfy the Jacobi identity is encapsulated in the identity $(3.9)$, viz.

$$d^0 \partial^2_{[\cdots, \cdots]} + \partial^1_{[\cdots, \cdots]} \partial^1_{[\cdots, \cdots]} + \partial^2_{[\cdots, \cdots]} d^0 = 0. \quad (5.43)$$

Given $\xi, \eta, \vartheta \in Q$, for degree reasons, $[\xi, \eta, \vartheta]$ is zero, as noted already, and $(5.43)$ entails

$$\sum_{(\xi, \eta, \vartheta)} cyclic \quad [\xi, \eta]_Q, \vartheta]_Q = \pm ([d_0 \xi, \eta, \vartheta]_3 + [\xi, d_0 \eta, \vartheta]_3 + [\xi, \eta, d_0 \vartheta]_3).$$

In the Lie-Rinehart triple case (a concept not explained here), this identity is equivalent to $\text{Hue05}$ $(1.9.6)$. See also $\text{Hue05}$ $(2.8.5(v))$ and the proof of Theorem 4.10 in $\text{Hue05}$, especially item (v) in this proof.

In $\text{Hue05}$ we have shown that a foliation determines a Lie-Rinehart triple and that a Lie-Rinehart triple determines a quasi Lie-Rinehart algebra. This quasi Lie-Rinehart algebra encapsulates the higher homotopies behind the foliation. In particular, it has the spectral sequence of the foliation as an invariant of the structure.
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