Existence and asymptotic behavior of least energy sign-changing solutions for Schrödinger-Poisson systems with doubly critical exponents

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Abstract: In this paper, we are concerned with the following Schrödinger-Poisson system with critical nonlinearity and critical nonlocal term due to the Hardy-Littlewood-Sobolev inequality

\[
\begin{align*}
-\Delta u + u + \lambda \phi |u|^3 u &= |u|^4 u + |u|^{q-2} u, & x \in \mathbb{R}^3, \\
-\Delta \phi &= |u|^5, & x \in \mathbb{R}^3,
\end{align*}
\]

where \( \lambda \in \mathbb{R} \) is a parameter and \( q \in (2, 6) \). If \( \lambda \geq (\frac{2+2}{q-2})^2 \) and \( q \in (2, 6) \), the above system has no nontrivial solution. If \( \lambda \in (\lambda^*, 0) \) for some \( \lambda^* < 0 \), we obtain a least energy radial sign-changing solution \( u_\lambda \) to the above system. Furthermore, we consider \( \lambda \) as a parameter and analyze the asymptotic behavior of \( u_\lambda \) as \( \lambda \to 0^- \).

Keywords: Schrödinger-Poisson system; Doubly critical exponents; Least energy; Sign-changing solutions

Mathematics Subject Classification: 35J50; 35J47; 47J30

1 Introduction and main results

In this article, we are interested in the existence, nonexistence and asymptotic behavior of least energy sign-changing solutions for the following Schrödinger-Poisson system

\[
\begin{align*}
-\Delta u + u + \lambda \phi |u|^3 u &= |u|^4 u + |u|^{q-2} u, & x \in \mathbb{R}^3, \\
-\Delta \phi &= |u|^5, & x \in \mathbb{R}^3,
\end{align*}
\]

where \( \lambda \in \mathbb{R} \) is a parameter and \( q \in (2, 6) \).

When it comes to the following more general cases of Schrödinger-Poisson system

\[
\begin{align*}
-\Delta u + V(x)u + \lambda \phi(x)|u|^3 u &= f(u), & x \in \mathbb{R}^3, \\
-\Delta \phi &= |u|^5, & x \in \mathbb{R}^3,
\end{align*}
\]

which has been studied extensively, where \( \lambda \in \mathbb{R} \) is a parameter and \( s \in [\frac{6}{5}, 5] \), the potential function \( V(x) \) is continuous and the nonlinearity \( f \) satisfies some suitable assumptions. Notice

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that the numbers $\frac{5}{3}$ and $5$ are called the lower and the upper critical exponents due to the Hardy-Littlewood-Sobolev inequality (see Proposition 2.2 below), respectively. In the past decades, much attention has been attracted to system (1.1) with $s \in \left[\frac{5}{3}, 5\right)$ (i.e., the subcritical nonlocal case), see for example, [3, 18].

Especially, for the case of $s \equiv 2$, system (1.1) turns to the following system

$$
\begin{cases}
-\Delta u + V(x)u + \lambda \phi(x)u = f(u), & x \in \mathbb{R}^3, \\
-\Delta \phi = |u|^2, & x \in \mathbb{R}^3.
\end{cases}
$$  \tag{1.2}

System (1.2) was firstly proposed by Benci and Fortunato [4] in 1998 to describe the interaction between a charged particle and the electrostatic field in quantum mechanics. For more detailed physical background of system (1.2), we refer the readers to [1, 5] and the reference therein. Due to its physical significance, which has been investigated by many researchers for the existence and nonexistence of positive solutions, multiple solutions, least energy solutions, radial solutions, sign-changing solutions and so on, see for example, [10–12, 25, 26, 30, 31, 33–36]. In [10, 25], the existence of positive solutions was investigated for the subcritical nonlinearity $f$. The existence of least energy solutions for system (1.2) has been considered in [27, 33]. Moreover, the existence of least energy sign-changing solutions for system (1.2) with subcritical nonlinearity was investigated in [26, 31], and the existence of least energy sign-changing solutions for system (1.2) with critical nonlinearity has been considered in [11, 12, 30, 34–36].

However, all the papers mentioned above investigated the subcritical nonlocal term. To the best of our knowledge, Schrödinger-Poisson system with critical nonlocal term (i.e., $s \equiv 5$) has only been studied in [2, 13, 16, 17, 21]. Azzollini and d’Avenia [2] firstly studied the Schrödinger-Poisson system with critical nonlocal term as follows:

$$
\begin{cases}
-\Delta u = \lambda u + q \phi |u|^3 u, & x \in B_R, \\
-\Delta \phi = q |u|^5, & x \in B_R, \\
u = \phi = 0, & \text{on } \partial B_R,
\end{cases}
$$

where $B_R(0) \subset \mathbb{R}^3$ is a ball centered in $0$ with radius $R$. The authors considered the existence and nonexistence of nontrivial solutions involving the range of $\lambda$. After this, many researchers are devoted to investigating the Schrödinger-Poisson system with critical nonlocal term. As we know, compared with the Schrödinger-Poisson system with subcritical nonlocal term, there are few results to investigate the Schrödinger-Poisson system with critical nonlocal term.

In [21], by concentration-compactness principle, Liu proved the existence of positive solutions for the following system

$$
\begin{cases}
-\Delta u + V(x)u - K(x) \phi(x) |u|^3 u = f(u), & x \in \mathbb{R}^3, \\
-\Delta \phi = K(x) |u|^5, & x \in \mathbb{R}^3,
\end{cases}
$$  \tag{1.3}

where $V, K$ and $f$ are asymptotically periodic functions with respect to $x$. Li, Li and Shi [16] investigated the existence of positive radial symmetric solutions for system (1.3) with $V(x) \equiv b$ (a positive constant) and $K(x) \equiv 1$, by using variational methods without usual compactness
conditions. Later, in [17], they also paid attention to the existence, nonexistence and multiplicity of positive radial solutions for the following system

\[
\begin{cases}
-\Delta u + u + \lambda |u|^3 u = \mu |u|^{p-1} u, & x \in \mathbb{R}^3, \\
-\Delta \phi = |u|^5, & x \in \mathbb{R}^3,
\end{cases}
\]

where \( \mu \geq 0 \) and \( \lambda \in \mathbb{R} \) are parameters. Recently, in [13], He considered a fractional Schrödinger-Poisson system with critical nonlocal term, and the system studied there was as follows:

\[
\begin{cases}
(-\Delta)^s u + V(x)u - \phi(x)u^{2^*_s-3} u = |u|^{2^*_s-2} u + f(u), & x \in \mathbb{R}^3, \\
(-\Delta)^s \phi = |u|^{2^*_s-1}, & x \in \mathbb{R}^3,
\end{cases}
\]

(1.4)

where \( s \in (0, 1) \) and \( V(x) \) is a coercive potential, the author proved that system (1.4) possesses at least one positive solutions by employing mountain pass theorem.

Motivated by the above works, especially the results obtained in [13, 16, 17, 21], in this paper, we investigate the existence of least energy sign-changing solutions for system \((SP)\) with \( \lambda \in \mathbb{R} \), which involves critical nonlinearity and critical nonlocal term. As far as we know, in the case of \( \lambda > 0 \), it seems that there is no result about the existence of nontrivial solutions for system \((SP)\). Here we consider it. In what follows, we state the nonexistence result of nontrivial solutions for system \((SP)\).

**Theorem 1.1.** For \( \lambda \geq (\frac{q+2}{8})^2 \) and \( q \in (2, 6) \), then system \((SP)\) has no nontrivial solution.

**Remark 1.2.** As has been pointed out above, Theorem 1.1 is the first attempt to investigate the nonexistence of nontrivial solutions for system \((SP)\) with critical nonlinearity and critical nonlocal term. We will give the proof of Theorem 1.1 with the help of Young’s inequality.

**Remark 1.3.** The existence or nonexistence results for nontrivial solutions of system \((SP)\) are completely different from the system with subcritical nonlocal term. In [35], authors obtained a least energy sign-changing solution for the following system

\[
\begin{cases}
-\Delta u + u + \phi u = |u|^q u + |u|^{q-2} u, & x \in \mathbb{R}^3, \\
-\Delta \phi = |u|^2, & x \in \mathbb{R}^3,
\end{cases}
\]

where \( q \in (5, 6) \). However, Theorem 1.1 shows that system \((SP)\) has no nontrivial solution for \( \lambda \geq (\frac{q+2}{8})^2 \) and \( q \in (2, 6) \). Thereby, this indicates the difference between the Schrödinger-Poisson system with a critical nonlocal term and that with a subcritical nonlocal term.

Now, we turn to investigate the existence of sign-changing solutions for system \((SP)\). Inspired by the papers mentioned above, there is no result to study the existence of sign-changing solutions for system \((SP)\) which involves critical nonlinearity and a critical nonlocal term, a natural question is whether system \((SP)\) possesses a least energy sign-changing solution or not? In the following theorem, we will give an affirmative answer to this question. The Schrödinger-Poisson system with critical nonlinearity and critical nonlocal term is much more difficult to obtain the existence of sign-changing solutions. The first difficulty is the lack of compactness. The second
difficulty is the competition between the critical nonlocal term and the critical nonlinearity. Now, we are in a position to state the existence result of least energy sign-changing solutions for system (SP). In the following results, we will work in the space $H^1_r(\mathbb{R}^3)$ which contains all radial symmetric functions of standard Hilbert space $H^1(\mathbb{R}^3)$.

In the following, we consider the case of $\lambda < 0$, which is a quite crucial result of this paper.

**Theorem 1.4.** For $q \in (5, 6)$, there exists $\lambda^* < 0$ such that for all $\lambda \in (\lambda^*, 0)$, then system (SP) possesses at least one least energy radial sign-changing solution $u_\lambda$.

**Remark 1.5.** Theorem 1.4 is the first existence result about least energy sign-changing solutions for system (SP) which involves critical nonlinearity and critical nonlocal term. Here we would like to emphasize that the existence of sign-changing solutions for Schrödinger-Poisson system with subcritical nonlocal term has been considered in many papers for the case $\lambda > 0$ not the case $\lambda < 0$. It is worthwhile noticing that the existence of sign-changing solutions for system (SP) with critical nonlocal term has not been investigated in either $\lambda > 0$ or $\lambda < 0$ case. Here we consider the both cases.

**Remark 1.6.** Compared with [11, 12, 30, 34–36], where authors investigated a class of Schrödinger-Poisson system with critical nonlinearity and subcritical nonlocal term, we investigate the existence of least energy sign-changing solutions for Schrödinger-Poisson system with doubly critical exponents. Instead of the single critical exponent, the presence of doubly critical exponents make it more difficult to recover the compactness condition and estimate energy.

Now, we give the main ideas for the proof of Theorem 1.4. Since our main purpose is to investigate the existence of sign-changing solutions for system (SP) involving critical nonlinearity, the first thing we need to do is to construct a minimizing Palais-Smale sequence ((PS) sequence for short) on the sign-changing Nehari manifold, we borrow some ideas from [9] to obtain it, see Lemma 4.5 below. The second thing is to prove the (PS)$_{m_\lambda}$ condition with the help of the upper bound of the least energy $m_\lambda$ on the sign-changing Nehari manifold $\mathcal{M}_\lambda$. It is worthwhile noticing that to estimate the least energy $m_\lambda$ on the sign-changing Nehari manifold $\mathcal{M}_\lambda$ is a key point in the proof of Theorem 1.4, we established some more subtle estimates to obtain it.

Note that when $\lambda \equiv 0$, system (SP) reduces to the following equation
\[-\Delta u + u = |u|^4u + |u|^{q-2}u, \quad x \in \mathbb{R}^3. \quad (1.5)\]

Then according to Theorem 1.4, we obtain the following result.

**Corollary 1.7.** For $q \in (5, 6)$, then problem (1.5) possesses at least one least energy radial sign-changing solution with exactly two nodal domains, and its energy is larger than twice that of least energy radial solutions.

**Remark 1.8.** In view of Theorem 1.1, Theorem 1.4 and Corollary 1.7, it is still an open question to study the existence of nontrivial solutions for the case $\lambda \in (0, (\frac{2q+2}{8})^2)$, if nontrivial solutions exist, whether least energy sign-changing solutions exist. Hence, it is worth exploring a new technique to study the existence of nontrivial solutions of system (SP) for the case $\lambda \in (0, (\frac{2q+2}{8})^2)$. 
We further study the asymptotic behavior of the least energy radial sign-changing solution $u_{\lambda}$ obtained in Theorem 1.4 as $\lambda \to 0^{-}$, which indicates the relationship between $\lambda < 0$ and $\lambda = 0$ in system $(SP)$.

**Theorem 1.9.** Let $u_{\lambda_n}$ be a least energy radial sign-changing solution of system $(SP)$ with $\lambda = \lambda_n$ obtained in Theorem 1.4, then for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0^{-}$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $u_{\lambda_n}$ converges to $u_0$ weakly in $H^1_0(\mathbb{R}^3)$ as $n \to \infty$, where $u_0$ is a least energy radial sign-changing solution of problem $(1.5)$ which has precisely two nodal domains.

**Remark 1.10.** To our knowledge, this paper is the first attempt to prove the existence and asymptotic behavior of least energy radial sign-changing solutions for system $(SP)$.

Moreover, for proving the asymptotic behavior of least energy sign-changing solutions of system $(SP)$ like Theorem 1.9, we refer the interested readers to [11, 30]. However, the methods of analyzing the asymptotic behavior used in [11, 30] heavily depend on the following inequality:

$$0 < \rho < \|u_{\lambda_n}^\pm\|^2 \leq \int_{\mathbb{R}^3} |u_{\lambda_n}^\pm|^6 \, dx + \mu \int_{\mathbb{R}^3} f(u_{\lambda_n}^\pm)u_{\lambda_n}^\pm \, dx \leq 2\mu \int_{\mathbb{R}^3} f(u_{\lambda_n}^\pm)u_{\lambda_n}^\pm \, dx,$$

for $\mu$ large enough, which is used to prove $u_0^\pm \neq 0$. Therefore, they only analyzed the asymptotic behavior of least energy sign-changing solutions for Schrödinger-Poisson system which does not involve a critical nonlinearity, which indicates that the method of analyzing the asymptotic behavior in [11, 30] is not applicable to our paper. Since problem $(1.5)$ involves a critical nonlinearity, the main difficulty we encounter in the proof of Theorem 1.9 is to obtain $u_0^\pm \neq 0$. Here we will use a method of proof by contradiction and some technical analysis to overcome it.

This paper is constructed as follows. In Section 2, we present the variational framework. In Section 3, we prove Theorem 1.1 for the case $\lambda \geq (\frac{q+2}{8})^2$. In Section 4, we complete the proof of Theorem 1.4 for the case $\lambda < 0$. Section 5 is interested in proving Theorem 1.7 for the case $\lambda = 0$. In Section 6, we analyze the asymptotic behavior of the least energy radial sign-changing solution $u_{\lambda}$ obtained in Theorem 1.4 as $\lambda \to 0^{-}$, and prove Theorem 1.9.

## 2 Preliminaries

In this section, we provide some notations, work space stuff and present some useful propositions which are crucial for proving our main results. We firstly present some necessary notations which will be used throughout this paper.

- “$\rightharpoonup$” (“$\to$”) denotes the weak (strong) convergence.
- is the norm in the usual Lebesgue space $L^p(\mathbb{R}^3)$ for $p \in [1, +\infty)$. $L^p(\mathbb{R}^3)$ is the usual Lebesgue space with the norm

$$|u|_p = \left( \int_{\mathbb{R}^3} |u|^p \, dx \right)^{\frac{1}{p}} \text{ for all } p \in [1, \infty), \quad \text{and } |u|_\infty = \text{ess sup}_{x \in \mathbb{R}^3} |u(x)|.$$
• Denote $D^{1,2}(\mathbb{R}^3) := \{ u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \}$ equipped with the norm
  \[ \|u\|_{D^{1,2}(\mathbb{R}^3)} := \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \]
• Let $H^1(\mathbb{R}^3) := \{ u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \}$ be the Hilbert space endowed with the inner product and norm
  \[ \langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx, \quad \|u\| = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}. \]
• $C_0^\infty(\mathbb{R}^3)$ contains infinitely times differentiable functions with compact support in $\mathbb{R}^3$.
• $o(1)$ denotes a quantity which goes to 0 as $n \to \infty$.
• $O(\varepsilon)$ denotes a bounded quantity as $\varepsilon \to 0$.
• $C, C_i \ (i \in \mathbb{N}^+)$ denote various positive constants.

Let $S$ be the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, that is,
\[ S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^3} |u|^6 dx \right)^{\frac{1}{3}}}. \tag{2.1} \]
By [32, Theorem 1.42], $S$ is attained by the following function
\[ \xi_{\varepsilon}(x) := \frac{\varepsilon^{\frac{2}{3}}}{(\varepsilon + |x|^2)^{\frac{1}{3}}}, \tag{2.2} \]
for any $\varepsilon > 0$ and $x \in \mathbb{R}^3$, which satisfies that $\|\xi_{\varepsilon}\|_{D^{1,2}(\mathbb{R}^3)}^2 = \|\xi_{\varepsilon}\|_6^6 = S^2$. Moreover, Let us define a subspace of $H^1(\mathbb{R}^3)$ which contains all radial symmetric functions as
\[ H_1^r(\mathbb{R}^3) := \{ u \in H^1(\mathbb{R}^3) : u(x) = u(|x|) \}. \]

Next, we would like to show the following embedding proposition which will be used frequently.

**Proposition 2.1** (See [29, Theorem B]). *The embedding $H_1^r(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for $r \in [2,6]$. Moreover, the embedding $H_1^r(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is compact for $r \in [2,6]$. *

The well known Hardy-Littlewood-Sobolev inequality presented as follows which plays a crucial role in constructing the variational framework for system $(SP)$.

**Proposition 2.2** (Hardy-Littlewood-Sobolev inequality, see [19, 20]). *Let $r, t > 1$ and $0 < \alpha < 3$ with $\frac{1}{r} + \frac{1}{t} + \frac{\alpha}{2} = 2$, $u \in L^r(\mathbb{R}^3)$ and $v \in L^t(\mathbb{R}^3)$. Then, there exists a sharp constant $C(r, t, \alpha) > 0$ (independent of $u$ and $v$) such that
  \[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)v(y)}{|x-y|^\alpha} dx dy \leq C(r, t, \alpha) |u|_r |v|_t. \tag{2.3} \]
In particular, if $r = t = \frac{6}{6-\alpha}$, then
  \[ C(r, t, \alpha) = C(\alpha) = \frac{\Gamma((3-\alpha)/2)^{\pi/2}}{\Gamma((3-\alpha)/2) \Gamma(3/2)} \left( \frac{\Gamma(3)}{\Gamma(3/2)} \right)^{(3-\alpha)/3}, \]
  where $\Gamma$ is the gamma function.*
and there is equality in (2.3) if and only if \( u \equiv (\text{const.})v \) and
\[
v(x) = A \left( 1 + \lambda^2 |x - z|^2 \right)^{-\frac{6-n}{2}}
\]
for some given constants \( A \in \mathbb{C}, \lambda \in \mathbb{R} \setminus \{0\} \) and for some point \( z \in \mathbb{R}^3 \).

Motivated by Proposition 2.2, for any \( u \in H^1_r(\mathbb{R}^3) \), we know that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^s |u(y)|^s}{|x - y|} \, dx \, dy
\]
is well-defined if for \( u \in L^{st}(\mathbb{R}^3) \) satisfying
\[
\frac{2}{t} + \frac{1}{3} = 2,
\]
i.e., \( t = \frac{5}{3} \). Then, the Sobolev continuous embedding \( H^1_2(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3) \) for \( p \in [2, 6] \) shows that
\[
2 \leq st \leq 6 \Rightarrow \frac{5}{3} \leq s \leq 5.
\]
Thus, (2.4) is well-defined if \( s \in [\frac{5}{3}, 5] \). The numbers \( \frac{5}{3} \) and 5 are called the lower and the upper critical exponents with respect to the Hardy-Littlewood-Sobolev inequality, respectively.

For a given \( u \in H^1(\mathbb{R}^3) \), Lax-Milgram theorem implies that there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that \( -\Delta \phi = |u|^5 \) in a weak sense. Moreover, according to [20, Theorem 6.21], we see that
\[
\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x - y|} \, dy > 0,
\]
and
\[
\int_{\mathbb{R}^3} \phi_u |u|^5 \, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^5 |u(y)|^5}{|x - y|} \, dx \, dy.
\]
It derives from the above equality and Fubini theorem that
\[
\int_{\mathbb{R}^3} \phi_u^+ |u^+|^5 \, dx = \int_{\mathbb{R}^3} \phi_u^- |u^-|^5 \, dx > 0.
\]

Next, we list some properties of \( \phi_u \), which can directly deduce from [16, Lemma 2.1].

**Proposition 2.3.** For any \( u \in H^1_r(\mathbb{R}^3) \), \(-\Delta \phi = |u|^5 \) possesses a unique weak solution \( \phi_u \geq 0 \) in \( D^{1,2}(\mathbb{R}^3) \). Moreover, there hold

1. \( \| \phi_u \|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u |u|^5 \, dx \);
2. \( \phi_{su} = s^5 \phi_u \), for any \( s > 0 \);
3. \( \| \phi_u \|_{D^{1,2}(\mathbb{R}^3)} \leq C_1 \| u \|^5 \), \( \int_{\mathbb{R}^3} \phi_u |u|^5 \, dx \leq C_2 \| u \|^5 \) for some \( C_1, C_2 > 0 \) (independent of \( u \));
4. if \( u \) is a radial function, and \( \phi_u \) is also radial;
5. if \( u \rightharpoonup u \) in \( H^1_1(\mathbb{R}^3) \) and \( u_n \rightharpoonup u \) a.e. in \( \mathbb{R}^3 \), then \( \phi_{u_n} \rightharpoonup \phi_u \) in \( D^{1,2}(\mathbb{R}^3) \) and
\[
\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 \, dx - \int_{\mathbb{R}^3} \phi_{u-u} |u_n - u|^5 \, dx = \int_{\mathbb{R}^3} \phi_u |u|^5 \, dx + o(1).
\]
Substituting (2.5) into system (SP), we rewrite system (SP) into the following equation
\[-\Delta u + u + \lambda \phi_u |u|^3 u = |u|^4 u + |u|^{q-2} u, \quad x \in \mathbb{R}^3.\]

Next, we define the energy functional \( I_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R} \) corresponding to system (SP) by
\[ I_\lambda(u) = \frac{1}{2} \|u\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q dx. \]

Clearly, \( I_\lambda \) is well-defined in \( H^1(\mathbb{R}^3) \) and of class \( C^1 \) for any \( \lambda \in \mathbb{R} \), and for any \( u, v \in H^1(\mathbb{R}^3) \),
\[ \langle I'_\lambda(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^3 uv dx - \int_{\mathbb{R}^3} |u|^4 uv dx - \int_{\mathbb{R}^3} |u|^{q-2} uv dx. \quad (2.6) \]

Thereby, we can directly use the variational methods to investigate the nontrivial weak solutions of system (SP) by studying the critical points of the functional \( I_\lambda \) in \( H^1(\mathbb{R}^3) \).

In what follows, we recall some definitions of critical points for the functional \( I_\lambda \).

**Definition 2.4.** (1) If \( u \in H^1(\mathbb{R}^3) \) such that \( \langle I'_\lambda(u), v \rangle = 0 \) for any \( v \in H^1(\mathbb{R}^3) \), then we say that \( u \) is a weak solution of system (SP).

(2) A nontrivial solution \( u \) of (SP) is called a ground state (or least energy) solution if
\[ I_\lambda(u) = c_\lambda := \inf_{v \in \mathcal{N}_\lambda} I_\lambda(v), \quad (2.7) \]
where
\[ \mathcal{N}_\lambda := \{ u \in H^1_\tau(\mathbb{R}^3) \setminus \{0\} : \langle I'_\lambda(u), u \rangle = 0 \}. \]

(3) If \( u \in H^1(\mathbb{R}^3) \) is a weak solution of system (SP) with \( u^\pm \neq 0 \), then \( u \) is a sign-changing solution of system (SP), where \( u^+ := \max\{u, 0\} \) and \( u^- := \min\{u, 0\} \).

(4) A sign-changing solution \( u \) of (SP) is called a least energy sign-changing solution if
\[ I_\lambda(u) = m_\lambda := \inf_{v \in \mathcal{M}_\lambda} I_\lambda(v), \quad (2.8) \]
where
\[ \mathcal{M}_\lambda := \{ u \in H^1_\tau(\mathbb{R}^3) : u^\pm \neq 0, \langle I'_\lambda(u), u^+ \rangle = \langle I'_\lambda(u), u^- \rangle = 0 \}. \]

**3 Nonexistence of nontrivial solutions for the case** \( \lambda \geq (\frac{q+2}{8})^2 \)**

In this section, we are ready to prove the nonexistence of nontrivial solutions for system (SP) for the case of \( \lambda \geq (\frac{q+2}{8})^2 \), and prove Theorem 1.1.

**Proof of Theorem 1.1.** By using \( q \in (2, 6) \) and Young’s inequality, we see that
\[ \int_{\mathbb{R}^3} |u|^q dx \leq \frac{6 - q}{4} \int_{\mathbb{R}^3} |u|^2 dx + \frac{q - 2}{4} \int_{\mathbb{R}^3} |u|^6 dx. \quad (3.1) \]

By \(-\Delta \phi_u = |u|^5\), it holds that
\[ \int_{\mathbb{R}^3} |u|^6 dx = \int_{\mathbb{R}^3} -\Delta \phi_u |u| dx \leq \frac{1}{2\tau^2} \int_{\mathbb{R}^3} |
abla \phi_u|^2 dx + \frac{\tau^2}{2} \int_{\mathbb{R}^3} |
abla |u||^2 dx \]

By the Hölder inequality, we have
\[ \int_{\mathbb{R}^3} |u|^4 u dx \leq \frac{1}{2\tau} \int_{\mathbb{R}^3} |
abla \phi_u|^2 dx + \frac{\tau}{2} \int_{\mathbb{R}^3} |
abla |u||^2 dx. \]

Therefore, by (2.5) and (2.6), we obtain
\[ \int_{\mathbb{R}^3} |u|^4 u dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^3 u dx \leq 2\tau \int_{\mathbb{R}^3} |
abla \phi_u|^2 dx + \tau \int_{\mathbb{R}^3} |
abla |u||^2 dx. \]

By using (3.1), we can show that
\[ \int_{\mathbb{R}^3} |u|^4 u dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^3 u dx \leq \frac{6 - q}{4} \int_{\mathbb{R}^3} |u|^2 dx + \frac{q - 2}{4} \int_{\mathbb{R}^3} |u|^6 dx. \]

This completes the proof of the theorem.
3.2 has no nontrivial solution if \( \lambda \leq \left( \frac{2+2}{8} \right)^2 \), taking \( \tau^2 = \lambda - \frac{1}{2} \), then

\[
\int_{\mathbb{R}^3} \phi_u |u|^5 \, dx \geq 2\lambda^{-\frac{1}{2}} \int_{\mathbb{R}^3} |u|^6 \, dx - \lambda^{-1} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.
\]

(3.2)

If \((u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a solution of system (SP), it follows that

\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^5 \, dx - \int_{\mathbb{R}^3} |u|^6 \, dx - \int_{\mathbb{R}^3} |u|^q \, dx = 0.
\]

Then, it obtains from \( \lambda \geq \left( \frac{2+2}{8} \right)^2 \), (3.1) and (3.2) that

\[
0 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^5 \, dx - \int_{\mathbb{R}^3} |u|^6 \, dx - \int_{\mathbb{R}^3} |u|^q \, dx
\]

\[
\geq \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \lambda \left( 2\lambda^{-\frac{1}{2}} \int_{\mathbb{R}^3} |u|^6 \, dx - \lambda^{-1} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)
\]

\[
- \int_{\mathbb{R}^3} |u|^6 \, dx - \frac{6-q}{4} \int_{\mathbb{R}^3} |u|^2 \, dx - \frac{q-2}{4} \int_{\mathbb{R}^3} |u|^6 \, dx
\]

\[
= \frac{q-2}{4} \int_{\mathbb{R}^3} u^2 \, dx + \left( 2\lambda^{\frac{1}{2}} - \frac{q+2}{4} \right) \int_{\mathbb{R}^3} |u|^6 \, dx
\]

\[
\geq \frac{q-2}{4} \int_{\mathbb{R}^3} u^2 \, dx.
\]

Since \( q \in (2, 6) \), then \( u \equiv 0 \). Thus, system (SP) has no nontrivial solution if \( \lambda \geq \left( \frac{2+2}{8} \right)^2 \). \( \square \)

**Remark 3.1.** From the above nonexistence result, we know that system (SP) can only possess nontrivial solutions when \( \lambda < \left( \frac{2+2}{8} \right)^2 \) and \( q \in (2, 6) \).

## 4 Least energy radial sign-changing solutions for the case \( \lambda < 0 \)

In this section, we are devoted to investigating the existence of least energy radial sign-changing solutions for system (SP) with \( \lambda < 0 \), and prove Theorem 1.4. This section will be divided into four subsections: proving some preliminary lemmas which are crucial for proving Theorem 1.4, constructing a sign-changing (PS)\(_{m_\lambda}\) sequence for the functional \( \mathcal{I}_\lambda \), estimating the least energy \( m_\lambda \) on the sign-changing Nehari manifold \( \mathcal{M}_\lambda \) and proving the (PS)\(_{m_\lambda}\) condition, respectively.

### 4.1 Some preliminary lemmas

In this subsection, we will prove some preliminary results which play an important role in the proof of Theorem 1.4. First we show that the sign-changing Nehari manifold \( \mathcal{M}_\lambda \) is non-empty.

**Lemma 4.1.** Assume that \( \lambda < 0 \) holds. Let \( u \in H^1_r(\mathbb{R}^3) \) with \( u^+ \neq 0 \), then there exists a unique pair \((s_u, t_u) \in (0, \infty) \times (0, \infty) \) such that \( s_u u^+ + t_u u^- \in \mathcal{M}_\lambda \). Moreover,

\[
\mathcal{I}_\lambda (s_u u^+ + t_u u^-) = \max_{s,t \geq 0} \mathcal{I}_\lambda (s u^+ + t u^-).
\]
Proof. Firstly, we prove the existence of \((s_u, t_u)\). Define the function \(\Phi : [0, \infty) \times [0, \infty) \to \mathbb{R}\) by

\[
\Phi(s, t) := \mathcal{I}_\lambda(su^+ + tu^-) = \frac{s^2}{2} \|u^+\|^2 + \frac{t^2}{2} \|u^-\|^2 + \frac{\lambda s^{10}}{10} \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^5 dx + \frac{\lambda t^{10}}{10} \int_{\mathbb{R}^3} \phi_{u^-} |u^-|^5 dx
\]

\[
+ \frac{\lambda s^5 t^5}{10} \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^5 dx + \frac{\lambda s^5 t^5}{10} \int_{\mathbb{R}^3} \phi_{u^-} |u^-|^5 dx - \frac{s^6}{6} \int_{\mathbb{R}^3} |u^+|^6 dx
\]

\[
- \frac{t^6}{6} \int_{\mathbb{R}^3} |u^-|^6 dx - \frac{s^9}{q} \int_{\mathbb{R}^3} |u^+|^9 dx - \frac{t^9}{q} \int_{\mathbb{R}^3} |u^-|^9 dx,
\]

where \(u^\pm \neq 0\). Through a simple calculation, we obtain that \(su^+ + tu^- \in \mathcal{M}_\lambda\) is equivalent to \((\frac{\partial \Phi(s,t)}{\partial s}, \frac{\partial \Phi(s,t)}{\partial t}) = (0,0)\) with \(s,t > 0\).

Secondly, we prove the uniqueness of the pair \((s_u, t_u)\) obtained above. By the definition of \(\Phi\),

\[
\Phi(s, t) \leq \frac{s^2}{2} \|u^+\|^2 + \frac{t^2}{2} \|u^-\|^2 - \frac{s^6}{6} \int_{\mathbb{R}^3} |u^+|^6 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u^-|^6 dx,
\]

from which we obtain that

\[
\lim_{s^2 + t^2 \to \infty} \Phi(s, t) \leq \lim_{s^2 + t^2 \to \infty} \left( \frac{s^2}{2} \|u^+\|^2 + \frac{t^2}{2} \|u^-\|^2 - \frac{s^6}{6} \int_{\mathbb{R}^3} |u^+|^6 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u^-|^6 dx \right) = -\infty.
\]

Thus, \(\Phi(s, t)\) has at least one global maximum point on \([0, \infty) \times [0, \infty)\). On the other hand, since the function \(\Phi\) is strictly concave, then any critical point is a maximum point and there exists at most one maximum point. Thereby, there exists a unique maximum point for the function \(\Phi\) on \([0, \infty) \times [0, \infty)\).

Finally, it is sufficient to verify that the maximum point cannot be achieved on the boundary of \([0, \infty) \times [0, \infty)\). Suppose by contradiction that \((\bar{s}, 0)\) is a maximum point of \(\Phi\) with \(\bar{s} \geq 0\). If \(t > 0\) small enough, it holds that

\[
\frac{\partial \Phi(\bar{s}, t)}{\partial t} = t \|u\|^2 + \lambda t^9 \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^5 dx + \frac{\lambda \bar{s}^5 t^4}{2} \int_{\mathbb{R}^3} \phi_{u^-} |u^-|^5 dx + \frac{\lambda \bar{s}^5 t^4}{2} \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^5 dx
\]

\[
- t^5 \int_{\mathbb{R}^3} |u^-|^6 dx - t^{q-1} \int_{\mathbb{R}^3} |u^-|^q dx > 0,
\]

that is, \(\Phi(\bar{s}, t)\) is an increasing function with respect to \(t\) if \(t > 0\) small enough, which yields to a contradiction. Similarly, \(\Phi(s, t)\) cannot achieve its global maximum on \((0, \bar{t})\). So, the proof is completed.

\[\square\]

Let \((s_u, t_u) \in (0, \infty) \times (0, \infty)\) be the unique pair obtained by Lemma 4.1. In the following, we will study some further properties of \((s_u, t_u)\).

**Lemma 4.2.** For any \(u \in H^1_r(\mathbb{R}^3)\) with \(u^\pm \neq 0\), there hold

(1) the functionals \(s, t\) are continuous in \(H^1_r(\mathbb{R}^3)\);
(2) if \(u_n^+ \to 0\) in \(H^1_r(\mathbb{R}^3)\) and \(u^-_n \to u^- \neq 0\) in \(H^1_r(\mathbb{R}^3)\) as \(n \to \infty\), one obtains that \(s_{u_n} \to \infty\); if \(u^-_n \to 0\) in \(H^1_r(\mathbb{R}^3)\) and \(u^+_n \to u^+ \neq 0\) in \(H^1_r(\mathbb{R}^3)\) as \(n \to \infty\), one has that \(t_{u_n} \to \infty\);
(3) if \(\{u_n\} \subset \mathcal{M}_\lambda\), \(\lim_{n \to \infty} \mathcal{I}_\lambda(u_n) = m_\lambda\), then \(m_\lambda > 0\), \(\Lambda_2 \leq \|u_n\|^2 \leq \Lambda_1\) for some \(\Lambda_1, \Lambda_2 > 0\).
Proof. (1) Take a sequence \( \{u_n\} \subset H_r^1(\mathbb{R}^3) \) such that \( u_n \to u \) in \( H_r^1(\mathbb{R}^3) \), then we get \( u_n^+ \to u^+ \) in \( H_r^1(\mathbb{R}^3) \). Using Lemma 4.1, there exist \((s_{u_n}, t_{u_n}) \), \((s_u, t_u) \) such that \( s_{u_n} u_n^+ + t_{u_n} u_n^- \in \mathcal{M}_\lambda \), \( s_u u^+ + t_u u^- \in \mathcal{M}_\lambda \). By the definition of \( \mathcal{M}_\lambda \) and \( \lambda < 0 \), we see that

\[
\begin{align*}
    s_{u_n}^2 \|u_n^+\|^2 &\geq 2s_{u_n} \|u_n^+\|^2 + s_{u_n}^{10} \int_{\mathbb{R}^3} \phi_{u_n^+}|u_n^+|^5 \, dx + \lambda s_{u_n}^5 t_{u_n}^5 \int_{\mathbb{R}^3} \phi_{u_n^-}|u_n^-|^5 \, dx \\
    &= s_{u_n}^6 \int_{\mathbb{R}^3} |u_n^+|^6 \, dx + s_{u_n}^6 \int_{\mathbb{R}^3} |u_n^-|^6 \, dx,
\end{align*}
\]
\begin{equation}
(4.1)
\end{equation}

\[
\begin{align*}
    t_{u_n}^2 \|u_n^-\|^2 &\geq 2t_{u_n} \|u_n^-\|^2 + \lambda t_{u_n}^{10} \int_{\mathbb{R}^3} \phi_{u_n^-}|u_n^-|^5 \, dx + \lambda s_{u_n}^5 t_{u_n}^5 \int_{\mathbb{R}^3} \phi_{u_n^+}|u_n^+|^5 \, dx \\
    &= t_{u_n}^6 \int_{\mathbb{R}^3} |u_n^-|^6 \, dx + t_{u_n}^6 \int_{\mathbb{R}^3} |u_n^+|^6 \, dx.
\end{align*}
\]
\begin{equation}
(4.2)
\end{equation}

We claim that \( \{s_{u_n}\} \) and \( \{t_{u_n}\} \) are bounded in \( \mathbb{R}^+ \). It yields from (4.1) that

\[
s_{u_n}^5 \int_{\mathbb{R}^3} |u_n^+|^6 \, dx \leq s_{u_n}^2 \|u_n^+\|^2,
\]

which implies that \( \{s_{u_n}\} \) is bounded in \( \mathbb{R}^+ \). Analogously, \( \{t_{u_n}\} \) is bounded in \( \mathbb{R}^+ \). Therefore, up to a subsequence if necessary, still denoted by \( \{s_{u_n}\} \) and \( \{t_{u_n}\} \), there exists a pair of nonnegative numbers \( (s_0, t_0) \) such that

\[
\lim_{n \to \infty} s_{u_n} = s_0 \quad \text{and} \quad \lim_{n \to \infty} t_{u_n} = t_0.
\]

Passing to the limit in (4.1) and (4.2), we deduce that

\[
\begin{align*}
    s_0^2 \|u^+\|^2 &+ \lambda s_0^{10} \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 \, dx + \lambda s_0^5 t_0^5 \int_{\mathbb{R}^3} \phi_{u^-}|u^-|^5 \, dx = s_0^6 \int_{\mathbb{R}^3} |u^+|^6 \, dx + s_0^6 \int_{\mathbb{R}^3} |u^-|^6 \, dx, \\
    t_0^2 \|u^-\|^2 &+ \lambda t_0^{10} \int_{\mathbb{R}^3} \phi_{u^-}|u^-|^5 \, dx + \lambda s_0^5 t_0^5 \int_{\mathbb{R}^3} \phi_{u^+}|u^+|^5 \, dx = t_0^6 \int_{\mathbb{R}^3} |u^-|^6 \, dx + t_0^6 \int_{\mathbb{R}^3} |u^+|^6 \, dx,
\end{align*}
\]

which indicates that \( s_0 u^+ + t_0 u^- \in \mathcal{M}_\lambda \). According to the uniqueness of \( (s_u, t_u) \), we conclude that \( s_u = s_0 \) and \( t_u = t_0 \).

(2) Here we only need to show that \( u_n^+ \to 0 \) in \( H_r^1(\mathbb{R}^3) \) and \( u_n^- \to u^- \neq 0 \) in \( H_r^1(\mathbb{R}^3) \). Analogously, \( t_n \to 0 \) in \( H_r^1(\mathbb{R}^3) \) and \( u_n^+ \to u^+ \neq 0 \) in \( H_r^1(\mathbb{R}^3) \). In fact, by contradiction, if \( u_n^+ \to 0 \) in \( H_r^1(\mathbb{R}^3) \) and \( u_n^- \to u^- \neq 0 \) in \( H_r^1(\mathbb{R}^3) \), there exists a constant \( M > 0 \) such that \( s_{u_n} \leq M \). By the Sobolev inequality, Proposition 2.3(3) and \( q \in (2, 6) \), it gives that

\[
\begin{align*}
    s_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 \, dx &\leq C_1 \|u_n^+\|^6 = o(\|u_n^+\|^2), \\
    s_{u_n}^q \int_{\mathbb{R}^3} |u_n^+|^q \, dx &\leq C_2 \|u_n^+\|^q = o(\|u_n^+\|^2), \\
    s_{u_n}^8 \int_{\mathbb{R}^3} \phi_{u_n^+}|u_n^+|^5 \, dx &\leq C_3 \|u_n^+\|^10 = o(\|u_n^+\|^2).
\end{align*}
\]
\begin{equation}
(4.3)
\end{equation}
\begin{equation}
(4.4)
\end{equation}
\begin{equation}
(4.5)
\end{equation}

By (2.6) and \( \lambda < 0 \), it holds that

\[
0 = \frac{\langle \mathcal{I}_\lambda(u_n^+ + t_n u_n^-), t_n u_n^+ \rangle}{t_n^2} \leq \|u_n^-\|^2 - t_n^4 \int_{\mathbb{R}^3} |u_n^-|^6 \, dx,
\]

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which implies that $\{t_{n}\}$ is bounded in $\mathbb{R}^+$. Then, by $u^{-}_n \to u^{-} \neq 0$ in $H^1_r(\mathbb{R}^3)$, it yields that

$$s^8_{un} \int_{\mathbb{R}^3} \phi_{un} |u^+_n|^{5}dx \leq C_4 |u^+_n|^{5}|u^-_n|^{5} = o(|u^+_n|^2).$$

(4.6)

From (4.3)–(4.6), we obtain that

$$0 = \langle I'_\lambda(s_{un}^+, t_{un}^+), s_{un}^+, t_{un}^+ \rangle = |u^+_n|^2 + \lambda s^8_{un} \int_{\mathbb{R}^3} \phi_{un} |u^+_n|^{5}dx + \lambda s^3_{un} t^5_{un} \int_{\mathbb{R}^3} \phi_{un} |u^-_n|^{5}dx$$

$$- s^4_{un} \int_{\mathbb{R}^3} |u^+_n|^{6}dx - s^6_{un} \int_{\mathbb{R}^3} |u^-_n|^{9}dx$$

$$\geq |u^+_n|^2 - o(|u^+_n|^2) > 0,$$

which is a contradiction. Thus, if $u^+_n \to 0$ in $H^1_r(\mathbb{R}^3)$ and $u^-_n \to u^- \neq 0$ in $H^1_r(\mathbb{R}^3)$, $s_{un} \to \infty$.

(3) One hand, for any $\{u_n\} \subset \mathcal{M}_\lambda \subset \bar{\mathcal{N}}_\lambda$, it holds that $\langle I'_\lambda(u_n) , u_n \rangle = 0$, and thus

$$m_\lambda + o(1) = I_\lambda(u_n) = I_\lambda(u_n) - \frac{1}{q} \langle I'_\lambda(u_n) , u_n \rangle$$

$$= \left( \frac{1}{2} - \frac{1}{q} \right) |u_n|^2 + \left( \frac{\lambda}{10} - \frac{\lambda}{q} \right) \int_{\mathbb{R}^3} \phi_{un} |u_n|^{5}dx + \left( \frac{1}{6} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n|^{6}dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{q} \right) |u_n|^2 \geq \left( \frac{1}{2} - \frac{1}{q} \right) |u^+_n|^2 > 0,$$

which means that $m_\lambda > 0$ and there exists $\Lambda_1 > 0$ such that $|u^+_n| \leq \Lambda_1$. On the other hand, by $\{u_n\} \subset \mathcal{M}_\lambda$ and the Sobolev inequality, one derives from $|u^{-}_n| \leq \Lambda_1$ that

$$0 < |u^+_n|^2 \leq \int_{\mathbb{R}^3} |u^+_n|^{6}dx + \int_{\mathbb{R}^3} |u^+_n|^{9}dx - \lambda \int_{\mathbb{R}^3} \phi_{un} |u^+_n|^{5}dx - \lambda \int_{\mathbb{R}^3} \phi_{un} |u^-_n|^{5}dx$$

$$\leq C_1 |u^+_n|^6 + C_2 |u^+_n|^9 + C_3 |u^+_n|^{10} + C_4 |u^-_n|^5 |u^+_n|^5$$

$$\leq C_1 |u^+_n|^6 + C_2 |u^+_n|^9 + C_3 |u^+_n|^{10} + C_5 |u^+_n|^5.$$

Thus, there exists $\Lambda_2 > 0$ such that $|u^+_n| \geq \Lambda_2$. Similarly, we also infer that $|u^-_n| \geq \Lambda_2$. Hence, $\Lambda_2 \leq |u^+_n| \leq \Lambda_1$ for some $\Lambda_1, \Lambda_2 > 0$. This completes the proof.

Inspired by [8], the following results hold. Since the proof is standard, we omit it here.

**Lemma 4.3.** Suppose that $\lambda < 0$ and $q \in (2,6)$ hold. It holds that

(1) for any $u \in H^1_r(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $\bar{s}_n > 0$ such that $\bar{s}_n u \in \mathcal{N}_\lambda$. Moreover,

$$I_\lambda(\bar{s}_n u) = \max_{s \geq 0} I_\lambda(s u);$$

(2) system $(SP)$ possesses a positive least energy solution $v_0 \in \mathcal{N}_\lambda$ such that $I_\lambda(v_0) = c_\lambda$.

**Remark 4.4.** Recall that $v_0$ is a positive least energy solution of system $(SP)$. We can directly deduce from [15, Theorem 1.11] that $v_0 \in L^\infty(\mathbb{R}^3)$ and $v \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3)$ for some $0 < \alpha < 1$. The boundedness and regularity of $v_0$ play a crucial role in the proof of Theorem 1.4, see Lemma 4.8 below.
4.2 Constructing the sign-changing \((PS)_{m\lambda}\) sequence

Due to the presence of a critical nonlinearity in system \((SP)\), our first task is to construct a sign-changing \((PS)_{m\lambda}\) sequence. Inspired by the spirit of [9], we present some definitions firstly. Let the functional \(l_\lambda(u, v)\) defined on \(H^1_\sigma(\mathbb{R}^3)\) by

\[
l_\lambda(u, v) := \begin{cases} 
\frac{\int_{\mathbb{R}^3} |u|^6 dx + \int_{\mathbb{R}^3} |v|^9 dx - \lambda \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \lambda \int_{\mathbb{R}^3} \phi_v |v|^5 dx}{\|u\|^2}, & \text{if } u \neq 0; \\
0, & \text{if } u = 0.
\end{cases}
\]

Obviously, \(l_\lambda(u, v) > 0\) if \(\lambda < 0\) and \(u \neq 0\). \(u \in \mathcal{M}_\lambda\) if and only if \(l_\lambda(u^+, u^-) = l_\lambda(u^-, u^+) = 1\). Next we define

\[
U_\lambda := \left\{ u \in H^1_\sigma(\mathbb{R}^3) : \frac{1}{2} < l_\lambda(u^+, u^-) < \frac{3}{2}, \frac{1}{2} < l_\lambda(u^-, u^+) < \frac{3}{2} \right\}.
\]

Let \(P\) be the cone of nonnegative functions in \(H^1_\sigma(\mathbb{R}^3)\) and \(Q \in [0, 1] \times [0, 1]\). \(\Sigma\) denotes the set which contains continuous maps \(\sigma\) satisfying the following two conditions:

(I) \(\sigma(s, 0) = 0, \sigma(0, t) \in P\) and \(\sigma(1, t) \in -P\);

(II) \((\mathcal{I}_\lambda \circ \sigma)(s, 1) \leq 0, \frac{\int_{\mathbb{R}^3} |\sigma(s, 1)|^6 dx + \int_{\mathbb{R}^3} |\sigma(s, 1)|^9 dx - \lambda \int_{\mathbb{R}^3} \phi_{\sigma(s, 1)} |\sigma(s, 1)|^5 dx}{\|\sigma(s, 1)\|^2} \geq 2\),

where \(s, t \in [0, 1]\). For any \(u \in H^1_\sigma(\mathbb{R}^3)\) with \(u^\pm \neq 0\), taking \(\sigma(s, t) = kt(1-s)u^+ + kstu^-\), where \(k > 0, s, t \in [0, 1]\). Notice that \(\sigma(s, t) \in \Sigma\) for \(k > 0\) large enough, which indicates that \(\Sigma \neq \emptyset\).

**Lemma 4.5.** There exists a sequence \(\{u_n\} \subset U_\lambda\) satisfying \(\mathcal{I}_\lambda(u_n) \to m\lambda\) and \(\mathcal{I}_\lambda(u_n) \to 0\).

**Proof.** We will divide the proof into the following three claims.

**Claim 1:** \(\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u) = \inf_{u \in \mathcal{M}_\lambda} \mathcal{I}_\lambda(u) = m\lambda\).

On the one hand, for any \(u \in \mathcal{M}_\lambda\), there exists \(\sigma(s, t) = kt(1-s)u^+ + kstu^- \in \Sigma\) for \(k > 0\) large enough. Thus, it follows from Lemma 4.1 that

\[
\mathcal{I}_\lambda(u) = \max_{s, t \geq 0} \mathcal{I}_\lambda(su^+ + tu^-) \geq \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u) \geq \inf_{\sigma \in \Sigma} \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u),
\]

which indicates that

\[
\inf_{u \in \mathcal{M}_\lambda} \mathcal{I}_\lambda(u) \geq \inf_{\sigma \in \Sigma} \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u). \tag{4.8}
\]

On the other hand, for each \(\sigma \in \Sigma\) and \(t \in [0, 1]\), we get that \(\sigma(0, t) \in P\) and \(\sigma(1, t) \in -P\), thus

\[
\begin{align*}
l_\lambda(\sigma^+(0, t), \sigma^-(0, t)) - l_\lambda(\sigma^-(0, t), \sigma^+(0, t)) &= l_\lambda(\sigma^+(0, t), \sigma^-(0, t)) \
l_\lambda(\sigma^+(1, t), \sigma^-(1, t)) - l_\lambda(\sigma^-(1, t), \sigma^+(1, t)) &= -l_\lambda(\sigma^-(1, t), \sigma^+(1, t)) \leq 0. \tag{4.9}
\end{align*}
\]

Meanwhile, from the definition of \(\Sigma\), for any \(\sigma \in \Sigma\) and \(s \in [0, 1]\), we deduce that

\[
l_\lambda(\sigma^+(s, 1), \sigma^-(s, 1)) + l_\lambda(\sigma^-(s, 1), \sigma^+(s, 1)) \geq \frac{\int_{\mathbb{R}^3} |\sigma(s, 1)|^6 dx + \int_{\mathbb{R}^3} |\sigma(s, 1)|^9 dx - \lambda \int_{\mathbb{R}^3} \phi_{\sigma(s, 1)} |\sigma(s, 1)|^5 dx}{\|\sigma(s, 1)\|^2} \geq 2,
\]

with the help of the elementary inequality \(\frac{a}{c} + \frac{b}{d} \geq \frac{b+c}{a+c}\) for any \(a, b, c, d > 0\). Thus,

\[
l_\lambda(\sigma^+(s, 1), \sigma^-(s, 1)) + l_\lambda(\sigma^-(s, 1), \sigma^+(s, 1)) - 2 \geq 0, \tag{4.11}
\]

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Hence, combining (4.12) with (4.9)–(4.12), there exists \((s_\sigma, t_\sigma) \in Q\) such that

\[
l_\lambda(\sigma^+(s, 0), \sigma^-(s, 0)) + l_\lambda(-\sigma^-(s, 0), \sigma^+(s, 0)) - 2 = -2 < 0.
\]  

Combining Miranda’s theorem \([23]\) with (4.9)–(4.12), there exists \((s_\sigma, t_\sigma) \in Q\) such that

\[
0 = l_\lambda(\sigma^+(s_\sigma, t_\sigma), \sigma^-(s_\sigma, t_\sigma)) - l_\lambda(\sigma^-(s_\sigma, t_\sigma), \sigma^+(s_\sigma, t_\sigma)) = l_\lambda(\sigma^+(s_\sigma, t_\sigma), \sigma^-(s_\sigma, t_\sigma)) + l_\lambda(\sigma^-(s_\sigma, t_\sigma), \sigma^+(s_\sigma, t_\sigma)) - 2,
\]

which directly gives that

\[
l_\lambda(\sigma^+(s_\sigma, t_\sigma), \sigma^-(s_\sigma, t_\sigma)) = l_\lambda(-\sigma^-(s_\sigma, t_\sigma), \sigma^+(s_\sigma, t_\sigma)) = 1.
\]

This indicates that for any \(\sigma \in \Sigma\), there exists \(u_\sigma = (s_\sigma, t_\sigma) \in \sigma(Q) \cap M_\lambda\), which yields that

\[
\sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u) \geq \inf_{u \in M_\lambda} \mathcal{I}_\lambda(u),
\]

that is,

\[
\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u) \geq \inf_{u \in M_\lambda} \mathcal{I}_\lambda(u).
\]  

Hence, combining (4.8) with (4.13), we conclude that

\[
\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(Q)} \mathcal{I}_\lambda(u) = \inf_{u \in M_\lambda} \mathcal{I}_\lambda(u) = m_\lambda.
\]

**Claim 2:** There exists a \((PS)_{m_\lambda}\) sequence \(\{u_n\} \subset H^1_\epsilon(\mathbb{R}^3)\) for the functional \(\mathcal{I}_\lambda\).

Consider a minimizing sequence \(\{w_n\} \subset M_\lambda\) and \(\sigma_n(s, t) = k t (1 - s) w_n^+ + k t s w_n^- \in \Sigma\), then

\[
\lim_{n \to \infty} \max_{w \in \sigma_n(Q)} \mathcal{I}_\lambda(w) = \lim_{n \to \infty} \mathcal{I}_\lambda(w_n).
\]

In view of a variant form of the classical deformation lemma \([24]\) due to Hofer \([14]\), we assert that there exists \(\{u_n\} \subset H^1_\epsilon(\mathbb{R}^3)\) such that, as \(n \to \infty\),

\[
\mathcal{I}_\lambda(u_n) \to m_\lambda, \quad \mathcal{I}'_\lambda(u_n) \to 0 \quad \text{and} \quad \text{dist}(u_n, \sigma_n(Q)) \to 0.
\]  

Suppose by contradiction, there exists \(\delta > 0\) such that \(\sigma_n(Q) \cap V_\delta = \emptyset\) for \(n\) large enough, where

\[
V_\delta = \{u \in H^1_\epsilon(\mathbb{R}^3) : \exists v \in H^1_\epsilon(\mathbb{R}^3), \ \text{s.t.} \ ||v - u|| \leq \delta, \ ||\mathcal{I}'_\lambda(v)|| \leq \delta, \ ||\mathcal{I}_\lambda(v) - m_\lambda|| \leq \delta\}.
\]

Inspired by \([14, \text{Lemma } 1]\), there exists a continuous map \(\eta : [0, 1] \times H^1_\epsilon(\mathbb{R}^3) \to H^1_\epsilon(\mathbb{R}^3)\) such that for some \(\epsilon \in (0, \frac{m_\lambda}{2})\) and all \(t \in [0, 1]\),

(a) \(\eta(0, u) = u, \ \eta(t, -u) = -\eta(t, u)\);
(b) \(\eta(t, u) = u\) for any \(u \in \mathcal{I}_\lambda^\epsilon - \{0\} \cup (H^1_\epsilon(\mathbb{R}^3) \setminus \mathcal{I}_\lambda^\epsilon)\);
(c) \(\eta(1, \mathcal{I}_\lambda^\epsilon - \{0\}) \subset \mathcal{I}_\lambda^\epsilon - \{0\}\);
(d) \(\eta(1, \mathcal{I}_\lambda^\epsilon - \{0\} \setminus P) \subset \mathcal{I}_\lambda^\epsilon - \{0\} \setminus P\), where \(\mathcal{I}_\lambda^k := \{u \in H^1_\epsilon(\mathbb{R}^3) : \mathcal{I}_\lambda(u) \leq k\}\).

Since \(\lim_{n \to \infty} \max_{w \in \sigma_n(Q)} \mathcal{I}_\lambda(w) = \lim_{n \to \infty} \mathcal{I}_\lambda(w_n) = m_\lambda\), choosing \(n\) large enough such that

\[
\sigma_n(Q) \subset \mathcal{I}_\lambda^\epsilon - \{0\} \quad \text{and} \quad \sigma_n(Q) \cap V_\delta = \emptyset.
\]  

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Denote $\tilde{\sigma}(s,t) := \eta(1, \sigma(n(s,t)))$ for all $(s,t) \in Q$. We declare that $\tilde{\sigma} \in \Sigma$, it derives from (4.15) and property (c) of $\eta$ that $\tilde{\sigma}(Q) \subset \mathcal{I}^{mn+\frac{\nu}{2}}_{\lambda}$, which leads to a contradiction with

$$m_{\lambda} = \inf_{\sigma \in \Sigma} \sup_{w \in \sigma(Q)} \mathcal{I}_{\lambda}(w) \leq \max_{w \in \tilde{\sigma}_{n}(Q)} \mathcal{I}_{\lambda}(w) \leq m_{\lambda} - \frac{\epsilon}{2}.$$ 

Actually, property (b) of $\eta$ and $\sigma_{n} \in \Sigma$ imply that $\tilde{\sigma}_{n}(s,0) = \eta(1, \sigma_{n}(s,0)) = \eta(1,0) = 0$. One hand, it follows from $\sigma_{n}(0,t) \in P$, (4.15) and property (d) of $\eta$ that $\tilde{\sigma}_{n}(0,t) \in P$. On the other hand, thanks to $\sigma_{n}(1,t) \in -P$ and (4.15), we deduce that $-\sigma_{n}(1,t) \in \mathcal{I}^{mn+\frac{\nu}{2}}_{\lambda} \setminus \mathcal{V}_{\delta}$, which implies that $\tilde{\sigma}_{n}(1,t) = \eta(1, \sigma_{n}(1,t)) = \eta(1, -\sigma_{n}(1,t)) \in -P$ with the help of properties (a) and (d) of $\eta$. Then, $\tilde{\sigma}_{n}$ satisfies property (I). Moreover, using the fact that $(\mathcal{I}_{\lambda} \circ \sigma_{n})(s,1) \leq 0$ and property (b) of $\eta$, we obtain that $\tilde{\sigma}_{n}(s,1) = \eta(1, \sigma_{n}(s,1)) = \sigma_{n}(s,1)$, then $\tilde{\sigma}_{n}$ satisfies property (II). Hence, due to the continuity of $\eta$ and $\sigma_{n}$, we derive that $\tilde{\sigma}_{n} \in \Sigma$.

**Claim 3:** The sequence $\{u_{n}\}$ obtained in Claim 2 satisfies $\{u_{n}\} \subset U_{\lambda}$ for $n$ large enough.

Since $\mathcal{I}_{\lambda}(u_{n}) \to 0$, we obtain that $\langle \mathcal{I}_{\lambda}(u_{n}), u_{n} \rangle = o(1)$. Hence, to complete the proof of Claim 3, it suffices to check that $u_{n}^{\pm} \neq 0$, which means that $l_{\lambda}(u_{n}^{+}, u_{n}^{-}) \to 1$, $l_{\lambda}(u_{n}^{-}, u_{n}^{+}) \to 1$, and then $\{u_{n}\} \subset U_{\lambda}$ for $n$ large enough. From (4.14), there exists a sequence $\{\nu_{n}\}$ such that

$$\nu_{n} = s_{n}w_{n}^{+} + t_{n}w_{n}^{-} \in \sigma_{n}(Q), \quad \|\nu_{n} - u_{n}\| \to 0.$$  (4.16)

Thus, to prove $u_{n}^{\pm} \neq 0$ is equivalent to prove $s_{n}w_{n}^{+} \neq 0$ and $t_{n}w_{n}^{-} \neq 0$ for $n$ large enough. By $\{w_{n}\} \subset M_{\lambda}$ and Lemma 4.2(3), we just need to check that $s_{n} \to 0$ and $t_{n} \to 0$ for $n$ large enough. Suppose by contradiction that $s_{n} \to 0$, by the continuity of $\mathcal{I}_{\lambda}$ and (4.16), we see that

$$0 < m_{\lambda} = \lim_{n \to \infty} \mathcal{I}_{\lambda}(\nu_{n}) = \lim_{n \to \infty} \mathcal{I}_{\lambda}(s_{n}w_{n}^{+} + t_{n}w_{n}^{-}) = \lim_{n \to \infty} \mathcal{I}_{\lambda}(t_{n}w_{n}^{-}).$$

It derives from Lemma 4.2(3) that $\Lambda_{2} \leq \|w_{n}^{-}\| \leq \Lambda_{1}$, which indicates that $t_{n} \to 0$ and $\{t_{n}\}$ is bounded. Thereby, by $\lambda < 0$, $q \in (2,6)$, $\Lambda_{2} \leq \|w_{n}^{+}\| \leq \Lambda_{1}$ and Lemma 4.1, one concludes that $s_{n}$ enough,

$$m_{\lambda} = \lim_{n \to \infty} \mathcal{I}_{\lambda}(w_{n}) = \lim_{n \to \infty} \max_{s,t \geq 0} \mathcal{I}_{\lambda}(sw_{n}^{+} + tw_{n}^{-})$$

$$\geq \lim_{n \to \infty} \max_{s \geq 0} \mathcal{I}_{\lambda}(sw_{n}^{+} + t_{n}w_{n}^{-})$$

$$\geq \lim_{n \to \infty} \max_{s \geq 0} \left( \frac{s^{2}}{2}\|w_{n}^{+}\|^{2} + \frac{\lambda s^{10}}{10} \int_{\mathbb{R}^{3}} \phi_{w_{n}^{+}}\|w_{n}^{+}\|^{5}dx + \frac{\lambda s^{10}}{5} \int_{\mathbb{R}^{3}} \phi_{w_{n}^{-}}\|w_{n}^{-}\|^{5}dx ight.$$  

$$- \frac{s^{6}}{6} \int_{\mathbb{R}^{3}} |w_{n}^{+}|^{6}dx - \frac{s^{5}}{q} \int_{\mathbb{R}^{3}} |w_{n}^{-}|^{6}dx \left. + \lim_{n \to \infty} \mathcal{I}_{\lambda}(t_{n}w_{n}^{-}) \right)$$

$$\geq \lim_{n \to \infty} \max_{s \geq 0} \left[ \frac{s^{2}}{2}\|w_{n}^{-}\|^{2} - C_{1} \left( s^{10}\|w_{n}^{+}\|^{10} + s^{5}\|w_{n}^{+}\|^{5} + s^{6}\|w_{n}^{+}\|^{6} + s^{6}\|w_{n}^{-}\|^{6} + s^{9}\|w_{n}^{-}\|^{9} \right) \right] + m_{\lambda}$$

$$\geq \max_{s \geq 0} \left[ C_{2}s^{2} - C_{3} \left( s^{10} + s^{5} + s^{6} + s^{9} \right) \right] + m_{\lambda}$$

$$> m_{\lambda},$$

which leads to a contradiction. Then $\{u_{n}\} \subset U_{\lambda}$ for $n$ large enough. \qed
4.3 Estimating the least energy $m_\lambda$

In this subsection, we are devoted to estimating the least energy $m_\lambda$ on the sign-changing Nehari manifold $\mathcal{M}_\lambda$. For this purpose, let us denote $u_\varepsilon := \varphi \circ \xi_\varepsilon$, where $\xi_\varepsilon$ is defined by (2.2) and $\varphi$ is a cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi|_{B_{r_0}(0)} \equiv 1$ and $\text{supp}(\varphi) \subset B_{2r_0}(0)$ for some $r_0 > 0$.

Similar arguments as [7, Lemma 1.1], we have the following estimates

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx = S^3 + O(\varepsilon^\frac{4}{3}), \quad \left( \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx \right)^{\frac{1}{3}} = S^\frac{1}{2} + O(\varepsilon^\frac{2}{3})$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon|^r \, dx = \begin{cases} O(\varepsilon^\frac{r}{2}), & r \in [2, 3); \\
O(\varepsilon^\frac{r}{2} |\ln \varepsilon|), & r = 3; \\
O(\varepsilon^{\frac{r-2}{r}}), & r \in (3, 6). \end{cases}$$

Lemma 4.6. Let $A, B, C > 0$ and define $g : [0, \infty) \to \mathbb{R}$ by

$$g(t) = \frac{A}{2} t^2 + \frac{\lambda B}{10} t^{10} - \frac{C}{6} t^6,$$

where $\lambda < 0$ is a constant. Then

$$\max_{t \geq 0} g(t) = \left( \frac{C - \sqrt{C^2 - 4 \lambda AB}}{2 \lambda B} \right)^\frac{1}{2} \frac{12 \lambda AB - C^2 + C \sqrt{C^2 - 4 \lambda AB}}{30 \lambda B}.$$

Proof. For $t \geq 0$, we have $g'(t) = t \left( A + \lambda B t^8 - C t^4 \right)$. Let $h(t) = A + \lambda B t^8 - C t^4$, we arrive at

$$t^4 = \frac{C - \sqrt{C^2 - 4 \lambda AB}}{2 \lambda B}.$$

Substituting it into $g(t)$, we obtain the result, and the proof is completed. \qed

Lemma 4.7. Let $f(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx + \frac{\lambda t^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 \, dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx$, for $t \geq 0$ and $\lambda < 0$. Then we obtain that as $\varepsilon \to 0$,

$$\max_{t \geq 0} f(t) \leq \left( 1 - \frac{\sqrt{1 - 4 \lambda}}{2} \right)^\frac{1}{2} \frac{12 \lambda - 1 + \sqrt{1 - 4 \lambda}}{30 \lambda} S^\frac{2}{3} + O(\varepsilon^\frac{1}{2}).$$

Proof. From $-\Delta \phi_{u_\varepsilon} = |u_\varepsilon|^5$, it holds that

$$\int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx = \int_{\mathbb{R}^3} -\Delta \phi_{u_\varepsilon} |u_\varepsilon|^6 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_{u_\varepsilon}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u_\varepsilon||^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx.$$

For $\varepsilon > 0$ small enough, it derives from the above inequality and (4.17) that

$$\int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 \, dx \geq 2 \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx - \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx = S^\frac{3}{2} + O(\varepsilon^\frac{1}{2}),$$

which together with Lemma 4.6 and (4.17) yields that

$$f(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \, dx + \frac{\lambda t^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 \, dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx$$
\[
\leq \frac{t^2}{2} \left(S_2^\lambda + O(\varepsilon^{\frac{2}{7}})\right) + \frac{\lambda t^{10}}{10} \left(S_2^\lambda + O(\varepsilon^{\frac{1}{7}})\right) - \frac{t^6}{6} \left(S_3^\lambda + O(\varepsilon^{\frac{1}{7}})\right)
\]

for \(\varepsilon > 0\) small enough and \(\lambda < 0\). So, the proof is completed. \(\square\)

For the simplicity of notation, let us denote \(c_\lambda^* := \left(\frac{1-\frac{1-4\lambda}{2\lambda}}{2\lambda}\right)^2 \frac{12\lambda - 1 + \sqrt{1 - 4\lambda}}{30\lambda} S_2^\lambda\). Next, we will estimate the upper bound of the least energy \(m_\lambda\) on the sign-changing Nehari manifold \(\mathcal{M}_\lambda\) by using a test function, which is a key point in this paper.

**Lemma 4.8.** Assume that \(\lambda < 0\) and \(q \in (5, 6)\) hold. Then \(m_\lambda < c_\lambda + c_\lambda^*\), where \(m_\lambda\) and \(c_\lambda\) defined by (2.8) and (2.7) respectively.

**Proof.** The main idea of this lemma is to look for an element in \(\mathcal{M}_\lambda\) such that the energy value of this element is strictly less than \(c_\lambda + c_\lambda^*\). We break the proof into two parts.

Firstly, we assert that there exist \(s_\varepsilon, t_\varepsilon > 0\) such that \(s_\varepsilon v_0 - t_\varepsilon u_\varepsilon \in \mathcal{M}_\lambda\), where \(v_0\) is a positive least energy solution of system (\(\mathcal{S}\)) obtained by Lemma 4.3. From Remark 4.4, we obtain that \(v_0 \in L^\infty(\mathbb{R}^3)\). Let us define \(\psi(r) := \frac{1}{r} v_0 - u_\varepsilon\) with \(r > 0\), define \(r_1 = \sup\{r \in \mathbb{R}^+ : \psi^-(r) \neq 0\}\) and \(r_2 = \inf\{r \in \mathbb{R}^+ : \psi^-(r) \neq 0\}\). Because of the positivity and regularity of \(v_0\), it is easy to verify that \(r_1 = \infty\) and \(0 < r_2 < r_1\). As \(r \to r_2^+\), this immediately implies that \(\psi^-(r) \to 0\) and \(\psi^+(r) \to \frac{1}{r_2} v_0 - u_\varepsilon \neq 0\). Then, we deduce from Lemma 4.2(2) that \(t(\psi(r)) \to \infty\). Similar to the proof of Lemma 4.2(1), we see that \(\{s(\psi(r))\}\) is bounded in \(\mathbb{R}^+\), and thus as \(r \to r_2^+\),

\[s(\psi(r)) - t(\psi(r)) \to -\infty.\]

As \(r \to r_1 = \infty\), \(\psi^+(r) \to 0\), it follows from Lemma 4.2(2) and the proof of Lemma 4.2(1) that \(s(\psi(r)) \to \infty\) and \(\{t(\psi(r))\}\) is bounded in \(\mathbb{R}^+\), and thus

\[s(\psi(r)) - t(\psi(r)) \to \infty.\]

Then, Lemma 4.2(1) implies that there exists \(r_2 \in (r_2, r_1)\) such that \(s(\psi(r_2)) = t(\psi(r_2))\). Let us denote \(s_\varepsilon = \frac{1}{r_2} s(\psi(r_2))\) and \(t_\varepsilon = t(\psi(r_2))\), it is easy to obtain that

\[s(\psi(r_2)) \psi(r_2) = s(\psi(r_2)) \psi^+(r_2) + t(\psi(r_2)) \psi^-(r_2) = s_\varepsilon v_0 - t_\varepsilon u_\varepsilon \in \mathcal{M}_\lambda.\]

Moreover, it follows from Lemma 4.1 that \(\mathcal{I}_\lambda(s_\varepsilon v_0 - t_\varepsilon u_\varepsilon) = \max_{s,t \geq 0} \mathcal{I}_\lambda(sv_0 - tu_\varepsilon)\).

Secondly, we show that \(m_\lambda < c_\lambda + c_\lambda^*\). It is easy to check that \(\mathcal{I}_\lambda(s_\varepsilon v_0 - t_\varepsilon u_\varepsilon) < 0\) if \(s_\varepsilon\) or \(t_\varepsilon\) large enough. Additionally, the continuity of \(\mathcal{I}_\lambda\) with respect to \(t\) implies that \(\mathcal{I}_\lambda(sv_0 - tu_\varepsilon) < c_\lambda + c_\lambda^*\) if \(t\) small enough. Thus, it suffices to consider the case that \(s_\varepsilon, t_\varepsilon\) contained in a bounded interval. Through a simple calculation, we can obtain that

\[m_\lambda \leq \mathcal{I}_\lambda(s_\varepsilon v_0 - t_\varepsilon u_\varepsilon) = \mathcal{I}_\lambda(s_\varepsilon v_0) + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6,\]

where

\[\Pi_1 = \frac{1}{2} \|t_\varepsilon u_\varepsilon\|_{D^{1,2}(\mathbb{R}^3)}^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{t_\varepsilon u_\varepsilon} |t_\varepsilon u_\varepsilon|^5 dx - \frac{1}{6} \int_{\mathbb{R}^3} |t_\varepsilon u_\varepsilon|^6 dx,\]

and...
\[
\Pi_2 = \frac{1}{2} \| s_\varepsilon v_0 - t_\varepsilon u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{1}{2} \| s_\varepsilon v_0 \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{1}{2} \| t_\varepsilon u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)},
\]
\[
\Pi_3 = \frac{1}{2} \int_{\mathbb{R}^3} (|s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^2 - |s_\varepsilon v_0|^2) \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |t_\varepsilon u_\varepsilon|^q \, dx,
\]
\[
\Pi_4 = \frac{1}{q} \int_{\mathbb{R}^3} (|s_\varepsilon v_0|^q + |t_\varepsilon u_\varepsilon|^q - |s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^q) \, dx,
\]
\[
\Pi_5 = \frac{1}{6} \int_{\mathbb{R}^3} (|s_\varepsilon v_0|^6 + |t_\varepsilon u_\varepsilon|^6 - |s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^6) \, dx,
\]
\[
\Pi_6 = \frac{\lambda}{10} \int_{\mathbb{R}^3} (\phi_{s_\varepsilon v_0 - t_\varepsilon u_\varepsilon} |s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^5 - \phi_{s_\varepsilon v_0} |s_\varepsilon v_0|^5 - \phi_{t_\varepsilon u_\varepsilon} |t_\varepsilon u_\varepsilon|^5) \, dx.
\]

By Lemma 4.7, it is easy to prove that as \( \varepsilon \to 0 \),
\[
\Pi_1 \leq \max_{t \geq 0} \left( \frac{t^2}{2} \| u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} + \frac{\lambda t^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 \, dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 \, dx \right) = c_\lambda^* + O(\varepsilon^{\frac{1}{2}}). \tag{4.20}
\]
Furthermore, since \( |a - b|^2 \leq |a|^2 + |b|^2 \) for all \( a, b \in \mathbb{R}^+ \), by a simple calculation, we arrive at
\[
\Pi_2 = \frac{1}{2} \| s_\varepsilon v_0 - t_\varepsilon u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{1}{2} \| s_\varepsilon v_0 \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{1}{2} \| t_\varepsilon u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} \leq 0, \tag{4.21}
\]
and by (4.18), it follows from \( q \in (5, 6) \) that
\[
\Pi_3 \leq \frac{1}{2} \int_{\mathbb{R}^3} |t_\varepsilon u_\varepsilon|^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |t_\varepsilon u_\varepsilon|^q \, dx = C_1 |u_\varepsilon|^2 - C_2 |u_\varepsilon|^q = C_1 \varepsilon^{\frac{2}{r}} - C_2 \varepsilon^{\frac{6-q}{4}}. \tag{4.22}
\]
To proceed further, we need the following inequality: \( |a - b|^r - a^r - b^r \geq -C(a^{r-1}b + ab^{r-1}) \) for all \( a, b \geq 0 \) and \( r \geq 1 \) (see [28, Calculus Lemma]). This together with \( q \in (5, 6) \), \( v_0 \in L^{\infty}(\mathbb{R}^3) \), H"older inequality and the boundedness of \( s_\varepsilon, t_\varepsilon \), it holds that
\[
\Pi_4 \leq C_1 \int_{\mathbb{R}^3} \left( |s_\varepsilon v_0|^{q-1} |t_\varepsilon u_\varepsilon| + |s_\varepsilon v_0| |t_\varepsilon u_\varepsilon|^{q-1} \right) \, dx \\
\phantom{\Pi_4} \leq C_2 |v_0|^{q-1} \int_{|x| \leq 2r_0} |u_\varepsilon| \, dx + C_3 |v_0| \int_{\mathbb{R}^3} |u_\varepsilon|^{q-1} \, dx \\
\phantom{\Pi_4} \leq C_4 \left( \int_{|x| \leq 2r_0} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} + C_5 \varepsilon^{\frac{q-7}{4}} \\
\phantom{\Pi_4} \leq C_6 \varepsilon^{\frac{1}{4}} + C_5 \varepsilon^{\frac{7-q}{4}} \leq C_7 \varepsilon^{\frac{1}{4}}, \tag{4.23}
\]
and
\[
\Pi_5 = C_1 \int_{\mathbb{R}^3} \left( |s_\varepsilon v_0|^5 |t_\varepsilon u_\varepsilon| + |s_\varepsilon v_0| |t_\varepsilon u_\varepsilon|^5 \right) \, dx \\
\phantom{\Pi_5} \leq C_2 |v_0| \int_{|x| \leq 2r_0} |u_\varepsilon| \, dx + C_3 |v_0| \int_{\mathbb{R}^3} |u_\varepsilon|^5 \, dx \\
\phantom{\Pi_5} \leq C_4 \left( \int_{|x| \leq 2r_0} |u_\varepsilon|^2 \, dx \right)^{\frac{5}{2}} + C_5 \varepsilon^{\frac{7}{4}} \\
\phantom{\Pi_5} \leq C_6 \varepsilon^{\frac{1}{4}}. \tag{4.24}
\]
In view of this, it suffices to prove \( \Pi_6 \). Recall that \( |a - b|^r - a^r - b^r \geq -C(a^{r-1}b + ab^{r-1}) \) for all \( a, b \geq 0 \) and \( r \geq 1 \), it holds that
\[
\int_{\mathbb{R}^3} \phi_{s_\varepsilon v_0 - t_\varepsilon u_\varepsilon} |s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^5 \, dx
\]
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\[
\begin{align*}
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left[(s_\varepsilon v_0 - t_\varepsilon u_\varepsilon(x))(y)\right]^5}{|x-y|} (s_\varepsilon v_0 - t_\varepsilon u_\varepsilon(x))^5 \, dx \, dy \\
& \geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|s_\varepsilon v_0(x)|^5 |s_\varepsilon v_0(x) - t_\varepsilon u_\varepsilon(x)|^5}{|x-y|} \, dx \, dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|t_\varepsilon u_\varepsilon(y)|^5 |s_\varepsilon v_0(x) - t_\varepsilon u_\varepsilon(x)|^5}{|x-y|} \, dx \, dy \\
& \quad - C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left(|s_\varepsilon v_0(y)|^4 |t_\varepsilon u_\varepsilon(y)| + |s_\varepsilon v_0(y)||t_\varepsilon u_\varepsilon(y)||t_\varepsilon u_\varepsilon(x)|^4\right) |s_\varepsilon v_0(x) - t_\varepsilon u_\varepsilon(x)|^5}{|x-y|} \, dx \, dy \\
& \geq \int_{\mathbb{R}^3} \phi_{s_\varepsilon v_0}|s_\varepsilon v_0|^5\, dx + \int_{\mathbb{R}^3} \phi_{t_\varepsilon u_\varepsilon}|t_\varepsilon u_\varepsilon|^5\, dx \\
& \quad - C_2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|s_\varepsilon v_0(x)|^5 \left(|s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy \\
& \quad - C_2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|t_\varepsilon u_\varepsilon(x)|^5 \left(|s_\varepsilon v_0(x)|^4 |t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy \\
& \quad - C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left(|s_\varepsilon v_0(y)|^4 |t_\varepsilon u_\varepsilon(y)| + |s_\varepsilon v_0(y)||t_\varepsilon u_\varepsilon(y)||t_\varepsilon u_\varepsilon(x)|^4\right) |s_\varepsilon v_0(x) - t_\varepsilon u_\varepsilon(x)|^5}{|x-y|} \, dx \, dy. \quad (4.25)
\end{align*}
\]

With the help of the following inequality \(|a - b|^r \leq 2^{r-1}(|a| + |b|)^r\), (4.25) turns into

\[
\int_{\mathbb{R}^3} \phi_{s_\varepsilon v_0-t_\varepsilon u_\varepsilon}|s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^5\, dx \\
\geq \int_{\mathbb{R}^3} \phi_{s_\varepsilon v_0}|s_\varepsilon v_0|^5\, dx + \int_{\mathbb{R}^3} \phi_{t_\varepsilon u_\varepsilon}|t_\varepsilon u_\varepsilon|^5\, dx \\
- C_3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|s_\varepsilon v_0(y)|^5 \left(|s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy \\
- C_4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|t_\varepsilon u_\varepsilon(x)|^5 \left(|s_\varepsilon v_0(x)|^4 |t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy. \quad (4.26)
\]

According to (4.17), (4.18), (4.26), Hardy-Littlewood-Sobolev inequality (see Proposition 2.2) and the boundedness of \(s_\varepsilon, t_\varepsilon\), we conclude that as \(\varepsilon \to 0\),

\[
\Pi_6 = \frac{\lambda}{10} \int_{\mathbb{R}^3} \left(\phi_{s_\varepsilon v_0-t_\varepsilon u_\varepsilon}|s_\varepsilon v_0 - t_\varepsilon u_\varepsilon|^5 - \phi_{s_\varepsilon v_0}|s_\varepsilon v_0|^5 - \phi_{t_\varepsilon u_\varepsilon}|t_\varepsilon u_\varepsilon|^5\right) \, dx \\
+ C_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|s_\varepsilon v_0(y)|^5 \left(|s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy \\
+ C_2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|t_\varepsilon u_\varepsilon(x)|^5 \left(|s_\varepsilon v_0(x)|^4 |t_\varepsilon u_\varepsilon(x)| + |s_\varepsilon v_0(x)||t_\varepsilon u_\varepsilon(x)||t_\varepsilon u_\varepsilon(x)|^4\right)}{|x-y|} \, dx \, dy \\
\leq C_3 \left(|v_0|_{L^5}^5|u_\varepsilon|_{L^6}^5 + |v_0|_{L^5}^5|v_0|_{L^6}^4\right) + C_4 \left(|u_\varepsilon|_{L^5}^5|v_0^3 u_\varepsilon|_{L^6}^3 + |u_\varepsilon|_{L^5}^5|v_0 u_\varepsilon|_{L^6}^5\right) \\
\leq C_5 |u_\varepsilon|_{L^6}^3 + C_6 |u_\varepsilon|_{L^6}^4 \leq C_7 \left(\int_{|x| \leq 2 r_0} |u_\varepsilon|_{L^6}^4 \, dx\right)^{\frac{1}{4}} + C_8 \varepsilon^{\frac{1}{4}} \\
\leq C_9 \varepsilon^{\frac{1}{4}}. \quad (4.27)
\]
So, substituting (4.20)–(4.24) and (4.27) into (4.19), by Lemma 4.3 and \( q \in (5, 6) \), we obtain
\[
m_\lambda \leq \mathcal{I}_\lambda (s_\varepsilon v_0 - t_\varepsilon u_\varepsilon) \leq \mathcal{I}_\lambda (v_0) + c_\lambda^* + C_1 \varepsilon^{\frac{2}{q}} - C_2 \varepsilon^{\frac{6-q}{2}} < c_\lambda + c_\lambda^*,
\]
(4.28)
as \( \varepsilon \to 0 \). This completes the proof. \( \square \)

4.4 The (PS)\(_{m_\lambda}\) condition

In what follows, we will show that the functional \( \mathcal{I}_\lambda \) satisfies the (PS)\(_{m_\lambda}\) condition.

**Lemma 4.9.** Assume that there exists \( \lambda^* < 0 \) such that for all \( \lambda \in (\lambda^*, 0) \), \( \{u_n\} \subset U_\lambda \) satisfying
\[
\mathcal{I}_\lambda (u_n) \to m_\lambda \in (0, c_\lambda + c_\lambda^*), \quad \mathcal{I}'_\lambda (u_n) \to 0
\]
contains a convergent subsequence.

**Proof.** It obtains from \( \mathcal{I}_\lambda (u_n) \to m_\lambda \), \( \mathcal{I}'_\lambda (u_n) \to 0 \) and \( q \in (2, 6) \) that
\[
m_\lambda + 1 + \|u_n\| \geq \mathcal{I}_\lambda (u_n) - \frac{1}{q} \langle \mathcal{I}'_\lambda (u_n), u_n \rangle
\]
\[
= \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2 + \left( \frac{\lambda}{10} - \frac{\lambda}{q} \right) \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n - \phi_u |u|^3 u \) \, dx
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{q} \right) \|u_n\|^2,
\]
which indicates that \( \{u_n\} \) is bounded in \( H^1_\lambda (\mathbb{R}^3) \). Then, up to a subsequence if necessary, still denoted by \( \{u_n\} \), we assume that there exists \( u \in H^1_\lambda (\mathbb{R}^3) \) such that for any \( r \in [2, 6) \),
\[
u_n \rightharpoonup u \text{ in } H^1_\lambda (\mathbb{R}^3), \quad u_n \to u \text{ in } L^r (\mathbb{R}^3), \quad u_n (x) \to u(x) \text{ a.e. in } \mathbb{R}^3.
\]
(4.29)

Firstly, we prove that \( \mathcal{I}'_\lambda (u) = 0 \). It suffices to verify that \( \langle \mathcal{I}'_\lambda (u), \psi \rangle = 0 \) for all \( \psi \in C_0^\infty (\mathbb{R}^3) \). Observe that
\[
\langle \mathcal{I}'_\lambda (u_n) - \mathcal{I}'_\lambda (u), \psi \rangle = \langle u_n - u, \psi \rangle + \lambda \int_{\mathbb{R}^3} (\phi_{u_n} |u_n|^3 u_n - \phi_u |u|^3 u) \psi \, dx
\]
\[
- \int_{\mathbb{R}^3} (|u_n|^4 u_n - |u|^4 u) \psi \, dx - \int_{\mathbb{R}^3} (|u_n|^{q-2} u_n - |u|^{q-2} u) \psi \, dx.
\]
(4.30)

In view of \( u_n \rightharpoonup u \) in \( H^1_\lambda (\mathbb{R}^3) \), then \( \langle u_n - u, \psi \rangle \to 0 \). By Hölder’s inequality and \( |a^m - b^m| \leq L \max \{a^{m-1}, b^{m-1}\} |a - b| \) for \( a, b \geq 0, m \geq 1 \) and some \( L > 0 \), there hold
\[
\left| \int_{\mathbb{R}^3} (|u_n|^4 u_n - |u|^4 u) \psi \, dx \right| \leq \int_{\mathbb{R}^3} |u_n|^4 |u_n - u| |\psi| \, dx + \int_{\mathbb{R}^3} |u_n|^4 - |u|^4 |u\psi| \, dx
\]
\[
\leq |\psi|_\infty |u_n|^4 \left( \int_{\text{supp } \psi} |u_n - u|^5 \, dx \right)^{\frac{1}{5}}
\]
\[
+ C|\psi|_\infty |u|^5 \left( \int_{\text{supp } \psi} |u_n|^3 + |u|^3 \right) \left( \int_{\text{supp } \psi} |u_n - u|^5 \, dx \right)^{\frac{1}{5}} \to 0,
\]
(4.31)
as \( n \to \infty \). Similarly,
\[
\left| \int_{\mathbb{R}^3} (|u_n|^{q-2} u_n - |u|^{q-2} u) \psi \, dx \right| \to 0, \quad \text{as } n \to \infty.
\]
(4.32)
For this, it remains to prove that

\[
\int_{\mathbb{R}^3} (\phi_{u_n}|u_n|^3u_n - \phi_u|u|^3u) \psi dx \to 0, \quad \text{as } n \to \infty. \tag{4.33}
\]

In deed, by Proposition 2.3(5), we get \( \phi_{u_n} \to \phi_u \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \) and so \( \phi_{u_n} \to \phi_u \) in \( L^6(\mathbb{R}^3) \). Then

\[
\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) |u|^3u \psi dx \to 0, \quad \text{as } n \to \infty. \tag{4.34}
\]

Since \( u_n \to u \) a.e. in \( \mathbb{R}^3 \) and

\[
\int_{\mathbb{R}^3} |\phi_{u_n} (|u_n|^3u_n - |u|^3u)|^\frac{6}{5} dx = \int_{\mathbb{R}^3} |\phi_{u_n} (|u + \theta(u_n - u)|^\frac{6}{5} |u_n - u|^{\frac{6}{5}} dx
\]

\[
\leq C_1 |\phi_{u_n}|_6^6 |u_n|_6^6 + |u|_6^6 |u_n - u|_6^6,
\]

\[
\leq C_2,
\]

where \( 0 < \theta < 1 \), we see that \( \phi_{u_n} (|u_n|^3u_n - |u|^3u) \to 0 \) in \( L^\frac{6}{5}(\mathbb{R}^3) \) and thus

\[
\int_{\mathbb{R}^3} \phi_{u_n} (|u_n|^3u_n - |u|^3u) \psi dx \to 0, \quad \text{as } n \to \infty,
\]

this together with (4.34), we conclude that (4.33) holds. Substituting (4.31)–(4.33) into (4.30), using the fact that \( u_n \to u \) in \( H^1(\mathbb{R}^3) \), it holds that

\[
\langle \mathcal{I}_\lambda'(u), \psi \rangle = \lim_{n \to \infty} \langle \mathcal{I}_\lambda'(u_n), \psi \rangle = 0,
\]

for any \( \psi \in C_0^\infty(\mathbb{R}^3) \), which means that \( \mathcal{I}_\lambda'(u) = 0 \).

Secondly, we will show that \( u \neq 0 \). Suppose by contradiction that \( u \equiv 0 \), that is, \( u^+ \equiv 0 \) and \( u^- \equiv 0 \). By applying (2.1), (4.29), \( \mathcal{I}_\lambda'(u_n) \to 0 \) and the Hardy-Littlewood-Sobolev inequality (see Proposition 2.2), we infer that

\[
\lim_{n \to \infty} \|u_n^+\|^2 = -\lambda \lim_{n \to \infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n^+|^5 dx + \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^6 dx + \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^9 dx
\]

\[
\leq -\lambda C_1 \lim_{n \to \infty} |u_n^+|_{10}^{10} - \lambda C_1 \lim_{n \to \infty} |u_n^-|_{10}^{10} |u_n^+|_{10}^{5} + \lambda |u_n^+|_{10}^{5} \leq -\lambda C_1 S^{-5} \lim_{n \to \infty} \|u_n^+\|^5 + S^{-3} \lim_{n \to \infty} \|u_n^+\|^6.
\]

It gets from Lemma 4.2(3) that \( A_2 \leq \lim_{n \to \infty} \|u_n^+\| \leq A_1 \). Thus, the above inequality turns into

\[
0 < \lim_{n \to \infty} \|u_n^+\|^2 \leq -\lambda C_2 S^{-5} \lim_{n \to \infty} \|u_n^+\|^{10} + S^{-3} \lim_{n \to \infty} \|u_n^+\|^6.
\]

Let \( 0 < t := \lim_{n \to \infty} \|u_n^+\|^2 \), then \( 0 < t \leq -\lambda C_2 S^{-5} t^5 + S^{-3} t^3 \). A simple calculation yields that

\[
t^2 \geq t_s^2 := \frac{-S^{-3} + \sqrt{S^{-6} - 4\lambda C_2 S^{-5}}}{-2\lambda C_2 S^{-5}}
\]

that is,

\[
\lim_{n \to \infty} \|u_n^+\|^2 \geq t_s = \left( \frac{-S^{-3} + \sqrt{S^{-6} - 4\lambda C_2 S^{-5}}}{-2\lambda C_2 S^{-5}} \right)^{\frac{1}{2}}. \quad \text{(4.35)}
\]
Similarly,
\[
\lim_{n \to \infty} \|u_n\|^2 \geq t_* = \left( \frac{-S^{-3} + \sqrt{S^{-6} - 4\lambda C_2 S^{-5}}}{-2\lambda C_2 S^{-5}} \right)^{\frac{2}{3}}. \tag{4.36}
\]

Then, we derive from \(\lambda < 0\), (4.29), (4.35) and (4.36) that

\[
m_\lambda = \lim_{n \to \infty} \mathcal{I}_\lambda(u_n)
= \frac{1}{2} \lim_{n \to \infty} \|u_n\|^2 + \frac{\lambda}{10} \lim_{n \to \infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^6 dx - \frac{1}{q} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^q dx
= \frac{1}{2} \lim_{n \to \infty} \|u_n\|^2 + \frac{\lambda}{10} \lim_{n \to \infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^6 dx

\]
\[
\geq \frac{1}{2} \lim_{n \to \infty} \|u_n\|^2 + \frac{\lambda}{10} \lim_{n \to \infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx - \frac{1}{6} \lim_{n \to \infty} \|u_n\|^2
\]
\[
\geq \frac{1}{3} \lim_{n \to \infty} \|u_n\|^2 = \frac{1}{3} \lim_{n \to \infty} \|u_n^+\|^2 + \frac{1}{3} \lim_{n \to \infty} \|u_n^-\|^2
\]
\[
\geq \frac{2}{3} t_* \geq \frac{2}{3} \left( \frac{-S^{-3} + \sqrt{S^{-6} - 4\lambda C_2 S^{-5}}}{-2\lambda C_2 S^{-5}} \right)^{\frac{2}{3}}. \tag{4.37}
\]

As \(\lambda \to 0^-\), we see that

\[
\lim_{\lambda \to 0^-} \frac{-S^{-3} + \sqrt{S^{-6} - 4\lambda C_2 S^{-5}}}{-2\lambda C_2 S^{-5}} = \lim_{\lambda \to 0^-} \frac{-2\lambda C_2 S^{-5} \left( S^{-6} - 4\lambda C_2 S^{-5} \right)^{-\frac{1}{2}}}{-2\lambda C_2 S^{-5}}
= \lim_{\lambda \to 0^-} \left( S^{-6} - 4\lambda C_2 S^{-5} \right)^{-\frac{1}{2}}
= S^3.
\]

Therefore, as \(\lambda \to 0^-\), we infer from (4.37) and the above inequality that

\[
c_0 + c_0^* > m_0 \geq \frac{2}{3} S^\frac{3}{2},
\]

where \(c_0^* = \frac{1}{3} S^\frac{3}{2}\) and \(c_0 < \frac{1}{3} S^\frac{3}{2}\) (obtained in Remark 5.3), which leads to a contradiction. Hence, there exists \(\lambda^* < 0\) such that for all \(\lambda \in (\lambda^*, 0)\), we infer that \(u^+ \neq 0\) and \(u^- \neq 0\), that is, \(u \neq 0\).

Which together with \(\mathcal{I}_\lambda(u) = 0\), it holds that \(u \in \mathcal{N}_\lambda\) and \(\mathcal{I}_\lambda(u) \geq c_\lambda\).

Lastly, we will prove that \(u_n \to u\) in \(H^1_r(\mathbb{R}^3)\). Denote \(v_n := u_n - u\), it yields from Brézis-Lieb lemma (see [6, Theorem 1]), Proposition 2.3(5) and the compactness embedding of \(H^1_r(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)\) for \(r \in [2, 6)\) that

\[
m_\lambda = \mathcal{I}_\lambda(u_n) + o(1)
= \frac{1}{2} \|u_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx - \frac{1}{q} \int_{\mathbb{R}^3} |u_n|^q dx + o(1)
= \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 dx
\]

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- \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx + o(1)
= I_{\lambda}(u) + \frac{1}{2} \|v_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1), \quad (4.38)
and
0 = \langle I'_{\lambda}(u), u_n \rangle + o(1)
= \|u\|^2 + \|v_n\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_n} |u|^5 \, dx + \lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx
- \int_{\mathbb{R}^3} |u|^6 \, dx - \int_{\mathbb{R}^3} |v_n|^6 \, dx - \int_{\mathbb{R}^3} |u|^q \, dx + o(1)
= \langle I_{\lambda}(u), u \rangle + \|v_n\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1)
= \|v_n\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1). \quad (4.39)

If \( v_n \to 0 \) in \( H^1_0(\mathbb{R}^3) \), the proof of Lemma 4.9 is completed. So we suppose by contradiction that 
\( v_n \to 0 \) and \( v_n \rightharpoonup 0 \) in \( H^1_0(\mathbb{R}^3) \). Then by (4.39), we may assume that, for \( n \) large enough,
\[
\|v_n\|^2 \to l, \quad \lambda \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx \to a, \quad \int_{\mathbb{R}^3} |v_n|^6 \, dx \to b. \quad (4.40)
\]
Notice that for any \( \tau > 0 \), by \( -\Delta \phi_{v_n} = |v_n|^5 \), it holds that
\[
\int_{\mathbb{R}^3} |v_n|^6 \, dx = \int_{\mathbb{R}^3} -\Delta \phi_{v_n} |v_n| \, dx \leq \frac{1}{2\tau^2} \int_{\mathbb{R}^3} |\nabla \phi_{v_n}|^2 \, dx + \frac{\tau^2}{2} \int_{\mathbb{R}^3} |\nabla |v_n||^2 \, dx
\leq \frac{1}{2\tau^2} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx + \frac{\tau^2}{2} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx.
\]
Thus, passing to the limit as \( n \to \infty \), it holds that
\[
a + t = b \leq \frac{a}{2\lambda \tau^2} + \frac{\tau^2}{2} l. \quad (4.41)
\]
With the help of \( \lambda < 0 \), choosing
\[
\tau^2 = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} > 0,
\]
which together with (4.41) implies that \( a \leq -\frac{2\lambda + 1 - \sqrt{1 - 4\lambda}}{2\lambda} l \). It follows from (4.38)-(4.40) that
\[
\frac{1}{2} \|v_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1) = l - \frac{a}{15} \geq \frac{12\lambda - 1 + \sqrt{1 - 4\lambda}}{30\lambda} l. \quad (4.42)
\]
On the other hand, it follows from (2.1) that
\[
\int_{\mathbb{R}^3} |v_n|^6 \, dx \leq S^{-3} \|v_n\|_{D^{1,2}(\mathbb{R}^3)}^6 \leq S^{-3} \|v_n\|^6.
\quad (4.43)
\]
So, it gives that
\[
b \leq S^{-3} l^3. \quad (4.44)
\]
By (2.1) and Hölder inequality, we obtain that
\[
\int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx \leq \left( \int_{\mathbb{R}^3} |\phi_{v_n}|^6 \, dx \right)^{\frac{5}{6}} \left( \int_{\mathbb{R}^3} |v_n|^6 \, dx \right)^{\frac{1}{6}}
\leq S^{-\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla \phi_{v_n}|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |v_n|^6 \, dx \right)^{\frac{5}{6}}
\leq S^{-\frac{1}{2}} \left( \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |v_n|^6 \, dx \right)^{\frac{5}{6}},
\]
which directly yields from (4.43) that
\[
\int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx \leq S^{-1} \left( \int_{\mathbb{R}^3} |v_n|^6 \, dx \right)^{\frac{5}{6}} \leq S^{-6} \|v_n\|^{10}.
\]
Then, it gives that
\[
a \geq \lambda S^{-6} l^5. \tag{4.45}
\]
Thus, it follows from (4.39), (4.40), (4.44) and (4.45) that
\[
l = b - a \leq S^{-3} l^3 - \lambda S^{-6} l^5.
\]
Thus, \(l = 0\) or \(l^2 \geq \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} S^3\). If \(l^2 \geq \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} S^3\), combining with (4.42), it yields that
\[
\frac{1}{2} \|v_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1) \geq \frac{12\lambda - 1 + \sqrt{1 - 4\lambda}}{30\lambda}
\geq \frac{12\lambda - 1 + \sqrt{1 - 4\lambda}}{30\lambda} \left( \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} \right)^{\frac{1}{2}} S^{3/2}
= c^*_\lambda.
\]
Hence,
\[
m_\lambda = I_\lambda(u_n) + o(1) = I_\lambda(u) + \left( \frac{1}{2} \|v_n\|^2 + \frac{\lambda}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx \right) + o(1) \geq c_\lambda + c^*_\lambda,
\]
which leads to a contradiction with our assumption \(m_\lambda \in (0, c_\lambda + c^*_\lambda)\). \(\square\)

Now, we complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** From Lemma 4.5, we know that there exists a sequence \(\{u_n\} \subset U_\lambda\) satisfying \(I_\lambda(u_n) \rightarrow m_\lambda \) and \(I'_\lambda(u_n) \rightarrow 0\) as \(n \rightarrow \infty\). Combining Lemma 4.8 with Lemma 4.9, we obtain that \(\{u_n\}\) contains a convergent subsequence, still denoted by \(\{u_n\}\). Then there exists \(u_\lambda \in H^1_\lambda(\mathbb{R}^3)\) such that \(u_n \rightharpoonup u_\lambda\) in \(H^1_\lambda(\mathbb{R}^3)\) as \(n \rightarrow \infty\), and by the continuity of \(I_\lambda\) and \(I'_\lambda\), we see that \(I_\lambda(u_\lambda) = m_\lambda\) and \(I'_\lambda(u_\lambda) = 0\). Furthermore, from \(\{u_n\} \subset U_\lambda\), we have \(\frac{1}{2} < l_\lambda(u_\lambda^+, u_\lambda^-) < \frac{3}{2}\) and \(\frac{1}{2} < l_\lambda(u_n^+, u_n^-) < \frac{3}{2}\) by (4.7), then
\[
\frac{1}{2} \|u_n^+\|^2 < \int_{\mathbb{R}^3} |u_n^+|^6 \, dx + \int_{\mathbb{R}^3} |u_n^+|^4 \, dx - \lambda \int_{\mathbb{R}^3} \phi_{u_n^+} |u_n^+|^5 \, dx - \lambda \int_{\mathbb{R}^3} \phi_{u_n^-} |u_n^-|^5 \, dx
\leq C_1 \|u_n^+\|^6 + C_2 \|u_n^+\|^4 + C_3 \|u_n^+\|^{10} + C_4 \|u_n^-\|^5 \|u_n^-\|^5.
\]
Similarly, \( \|u_\lambda^-\| \geq \rho > 0 \). Thus, \( u_\lambda \) is a least energy radial sign-changing solution of \((SP)\).  

5 Least energy radial sign-changing solutions for the case \( \lambda = 0 \)

In this section, we are interested in the existence of least energy radial sign-changing solutions for problem \((1.5)\), and prove Corollary 1.7. Before proving Corollary 1.7, we give some definitions firstly. Define the energy functional \( \mathcal{I}_0 \) associating with problem \((1.5)\) by

\[
\mathcal{I}_0(u) := \frac{1}{2} \|u\|^2 - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx.
\]

Meanwhile, let us define

\[
c_0 := \inf_{u \in \mathcal{N}_0} \mathcal{I}_0(u), \quad m_0 := \inf_{u \in \mathcal{M}_0} \mathcal{I}_0(u),
\]

where

\[
\mathcal{N}_0 := \{ u \in H^1_r(\mathbb{R}^3) \setminus \{0\} : \langle \mathcal{I}'_0(u), u \rangle = 0 \},
\]

and

\[
\mathcal{M}_0 := \{ u \in H^1_r(\mathbb{R}^3) : u^\pm \neq 0, \langle \mathcal{I}'_0(u), u^\pm \rangle = 0 \}.
\]

We first show the following lemma, in view of our nonlinearity, which can be directly conclude from [11, Lemma 2.2] with \( \lambda \equiv 0 \). So we omit it here.

**Lemma 5.1.** Let \( v \in H^1_r(\mathbb{R}^3) \) with \( v^\pm \neq 0 \), then there exists a unique pair \((s_v, t_v) \in (0, \infty) \times (0, \infty)\) such that \( s_v v^+ + t_v v^- \in \mathcal{M}_0 \). Moreover,

\[
\mathcal{I}_0(s_v v^+ + t_v v^-) = \max_{s, t \geq 0} \mathcal{I}_0(s v^+ + t v^-).
\]

**Lemma 5.2.** If \( \langle \mathcal{I}'_0(v), v^+ \rangle = 0 \), then \( s_v = t_v = 1 \), where \((s_v, t_v)\) is obtained by Lemma 5.1.

**Proof.** Since \( s_v v^+ + t_v v^- \in \mathcal{M}_0 \), then

\[
s_v^2 \|v^+\|^2 = s_v^6 \int_{\mathbb{R}^3} |v^+|^6 \, dx + s_v^q \int_{\mathbb{R}^3} |v^+|^q \, dx. \tag{5.1}
\]

Since \( \langle \mathcal{I}'_0(v), v^+ \rangle = 0 \), it yields that

\[
\|v^+\|^2 = \int_{\mathbb{R}^3} |v^+|^6 \, dx + \int_{\mathbb{R}^3} |v^+|^q \, dx. \tag{5.2}
\]

Combining (5.1) with (5.2), we obtain that

\[
(1 - s_v^{2-q}) \|v^+\|^2 = (1 - s_v^{6-q}) \int_{\mathbb{R}^3} |v^+|^6 \, dx.
\]

This together with \( q \in (2, 6) \) derives that \( s_v = 1 \). Similarly, \( t_v = 1 \). So, the proof is complete.  

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\textbf{Proof of Corollary 1.7.} Similar arguments as \cite[Theorem 1.1]{35} with \( \phi \equiv 0 \), we obtain that problem \eqref{eq:1.5} has a least energy radial sign-changing solution \( z_0 \).

In what follows, we will prove that the least energy radial sign-changing solution \( z_0 \) has exactly two nodal domains. Suppose by contradiction that \( z_0 \) has at least three nodal domains satisfying
\[
    z_0 = z_1 + z_2 + z_3
\]
with \( z_i \neq 0 \) for \( i = 1, 2, 3 \) and \( z_i \geq 0 \) and \( \text{supp}(z_i) \cap \text{supp}(z_j) = \emptyset \) for \( i \neq j, \, i, j = 1, 2, 3 \) and
\[
    \langle I_0(z_0), z_i \rangle = 0, \quad \text{for } i = 1, 2, 3.
\]

Setting \( w := z_1 + z_2 \), we obtain that \( w^+ = z_1, \, w^- = z_2 \), i.e., \( w^\pm \neq 0 \). By applying Lemma \ref{lem:5.1}, there exists a unique pair \( (s_w, t_w) \in (0, \infty) \times (0, \infty) \) such that
\[
    s_w w^+ + t_w w^- = s_w z_1 + t_w z_2 \in \mathcal{M}_0 \quad \text{and} \quad I_0(s_w z_1 + t_w z_2) \geq m_0.
\]
Using the fact that \( \langle I_0(z_0), z_i \rangle = 0 \) for \( i = 1, 2, 3 \), it follows that \( \langle I_0(w), w^\pm \rangle = 0 \), then \( s_w = t_w = 1 \) by Lemma \ref{lem:5.2}. On the other hand, we deduce that
\[
    0 = \frac{1}{2} \langle I_0(z_0), z_3 \rangle = \frac{1}{2} \|z_3\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} |z_3|^6 dx - \frac{1}{2} \int_{\mathbb{R}^3} |z_3|^3 dx < I_0(z_3).
\]
Then, we see that
\[
    m_0 \leq I_0(s_w z_1 + t_w z_2)
    = I_0(s_w z_1 + t_w z_2) - \frac{1}{q} \langle I_0(s_w z_1 + t_w z_2), s_w z_1 + t_w z_2 \rangle
    = \left( \frac{1}{2} - \frac{1}{q} \right) \|s_w z_1\|^2 + \left( \frac{1}{q} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |s_w z_1|^6 dx
    + \left( \frac{1}{2} - \frac{1}{q} \right) \|t_w z_2\|^2 + \left( \frac{1}{q} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |t_w z_2|^6 dx
    = \left( \frac{1}{2} - \frac{1}{q} \right) \|z_1\|^2 + \left( \frac{1}{q} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |z_1|^6 dx
    + \left( \frac{1}{2} - \frac{1}{q} \right) \|z_2\|^2 + \left( \frac{1}{q} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |z_2|^6 dx
    = I_0(z_1) - \frac{1}{q} I_0(z_1) + I_0(z_2) - \frac{1}{q} I_0(z_2) + I_0(z_3)
    = I_0(z_1) + I_0(z_2) + I_0(z_3)
    = I_0(z_0) = m_0,
\]
which is a contradiction. That is, \( z_3 = 0 \), and \( z_0 \) has exactly two nodal domains.

Lastly, it remains to show that \( m_0 \geq 2c_0 \). Similar arguments as Lemma \ref{lem:5.1}, there exist \( \bar{s}, \bar{t} > 0 \) such that \( \bar{s} z_0^+, \bar{t} z_0^- \in \mathcal{N}_0 \). Then, it follows from Lemma \ref{lem:5.1} that
\[
    m_0 = I_0(z_0) \geq I_0(\bar{s} z_0^+ + \bar{t} z_0^-) = I_0(\bar{s} z_0^+) + I_0(\bar{t} z_0^-) \geq 2c_0.
\]
Thus, the proof of Corollary \ref{cor:1.7} is completed. \( \square \)

\textbf{Remark 5.3.} Similar arguments as \cite[Lemma 3.3]{35} with \( \phi \equiv 0 \), which gives that \( m_0 < c_0 + \frac{1}{3} S_3^2 \). This together with Corollary \ref{cor:1.7}, it holds that \( m_0 \geq 2c_0 \). Thus, \( c_0 < \frac{1}{3} S_3^2 \) and \( m_0 < \frac{2}{3} S_3^2 \).
6 Asymptotic behavior of sign-changing solutions

Now, we consider the asymptotic behavior of \( u_\lambda \) as \( \lambda \to 0^- \), and prove Theorem 1.9. In what follows, we regard \( \lambda < 0 \) as a parameter in system \((\mathcal{SP})\) and show the relationship between the case \( \lambda < 0 \) and \( \lambda = 0 \) in system \((\mathcal{SP})\). We first present the following lemma, which will be used in the proof of Theorem 1.9.

**Lemma 6.1.** Let \( m_\lambda \) be defined by (2.8), then \( 0 < m_\lambda < \frac{2}{3}S^3_2 \).

**Proof.** We first claim that \( m_\lambda < c_\lambda + \frac{4}{3}S^3_2 \). We infer from (4.20) in Lemma 4.8 that

\[
\Pi_1 \leq \max_{t \geq 0} \left( \frac{t^2}{2} \| u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} + \frac{\lambda \varepsilon^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \right)
\leq \max_{t \geq 0} \left( \frac{t^2}{2} \| u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \right)
= \frac{1}{3} \left( \int_{\mathbb{R}^3} \| u_\varepsilon \|^3_{D^{1,2}(\mathbb{R}^3)} \right)^{\frac{2}{3}} = \frac{1}{3} S^\frac{4}{3} + O(\varepsilon^\frac{1}{3}),
\]

provided that \( \lambda < 0 \). The other part of the proof is the same as Lemma 4.8. In view of this, the claim holds. In order to complete the proof of Lemma 6.1, it remains to prove \( c_\lambda < \frac{1}{3}S^\frac{4}{3} \), where \( c_\lambda \) defined by (2.7).

Indeed, it is obvious that \( I_\lambda(\alpha u_\varepsilon) > 0 \) for \( \alpha > 0 \) small, and \( I_\lambda(\alpha u_\varepsilon) \to -\infty \) for \( \alpha \to \infty \), where \( u_\varepsilon \) defined in Subsection 4.3. Hence, there exists \( \alpha_\varepsilon > 0 \) such that \( I_\lambda(\alpha_\varepsilon u_\varepsilon) = \max_{\alpha \geq 0} I_\lambda(\alpha u_\varepsilon) \). It follows from Lemma 4.3 that \( c_\lambda \leq I_\lambda(\alpha_\varepsilon u_\varepsilon) \). Moreover, it is easy to verify that \( \alpha_\varepsilon \) contains in a bounded interval. In view of this, it suffices to show that \( I_\lambda(\alpha_\varepsilon u_\varepsilon) < \frac{1}{3}S^\frac{4}{3} \). We infer from (4.18), \( q \in (5, 6) \) and \( \lambda < 0 \) that

\[
\begin{align*}
I_\lambda(\alpha_\varepsilon u_\varepsilon) &= \frac{1}{2} \| \alpha_\varepsilon u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} + \frac{\lambda \varepsilon^{10}}{10} \int_{\mathbb{R}^3} \phi_{\alpha_\varepsilon u_\varepsilon} |\alpha_\varepsilon u_\varepsilon|^5 dx - \frac{1}{6} \int_{\mathbb{R}^3} |\alpha_\varepsilon u_\varepsilon|^6 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\alpha_\varepsilon u_\varepsilon|^2 dx - \frac{1}{q} \int_{\mathbb{R}^3} |\alpha_\varepsilon u_\varepsilon|^q dx \\
&\leq \max_{\alpha \geq 0} \left( \frac{\alpha^2}{2} \| u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} + \frac{\lambda \varepsilon^{10}}{10} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^5 dx - \frac{\alpha^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \right) + C_1 \varepsilon^\frac{1}{3} - C_2 \varepsilon^\frac{5-\varepsilon}{4} \\
&\leq \max_{\alpha \geq 0} \left( \frac{\alpha^2}{2} \| u_\varepsilon \|^2_{D^{1,2}(\mathbb{R}^3)} - \frac{\alpha^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \right) + C_1 \varepsilon^\frac{1}{3} - C_2 \varepsilon^\frac{5-\varepsilon}{4} \\
&\leq \frac{1}{3} S^\frac{4}{3} + C_1 \varepsilon^\frac{1}{3} - C_2 \varepsilon^\frac{5-\varepsilon}{4} < \frac{1}{3}S^\frac{4}{3},
\end{align*}
\]

as \( \varepsilon \to 0. \) Therefore, \( m_\lambda < \frac{2}{3}S^3_2 \), and we complete the proof of Lemma 6.1. \( \square \)

**Proof of Theorem 1.9.** Recall that \( u_\lambda \in H^1_0(\mathbb{R}^3) \) is a least energy sign-changing solution of system \((\mathcal{SP})\) obtained in Theorem 1.4, this together with Lemma 6.1 implies that \( I_\lambda(u_\lambda) = m_\lambda < \frac{2}{3}S^3_2 \) and \( I'_\lambda(u_\lambda) = 0 \). We split the proof into three claims which can yield to Theorem 1.9 directly.

**Claim 1:** For any sequence \( \{\lambda_n\} \) with \( \lambda_n \to 0^- \) as \( n \to \infty \), \( \{u_{\lambda_n}\} \) is bounded in \( H^1_0(\mathbb{R}^3) \).
For any sequence \{\lambda_n\}, there exists a subsequence of \{u_{\lambda_n}\} such that \(I_{\lambda_n}(u_{\lambda_n}) = m_{\lambda_n} < \frac{2}{3}S^3\) and \(I'_{\lambda_n}(u_{\lambda_n}) = 0\). Then, let \(n \to \infty\), it follows that
\[
\frac{2}{3}S^3 > I_{\lambda_n}(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{q}(I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}) \geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_{\lambda_n}\|^2,
\]
which indicates that \{u_{\lambda_n}\} is bounded in \(H^1_\lambda(\mathbb{R}^3)\).

In view of Claim 1 and Proposition 2.1, there exists a subsequence of \{\lambda_n\} satisfying \(\lambda_n \to 0\) as \(n \to \infty\), still denoted by \{\lambda_n\} and there exists \(u_0 \in H^1_\lambda(\mathbb{R}^3)\) such that, for any \(r \in [2, 6]\),
\[
u_{\lambda_n} \to u_0 \text{ in } H^1_\lambda(\mathbb{R}^3), \quad u_{\lambda_n} \to u_0 \text{ in } L^r(\mathbb{R}^3), \quad u_{\lambda_n} \to u_0 \text{ a.e. in } \mathbb{R}^3.
\]

**Claim 2:** \(u_0\) is a radial sign-changing solution of problem (1.5).

Since \(u_{\lambda_n}\) is a least energy radial sign-changing solution of system \((\mathcal{SP})\) with \(\lambda = \lambda_n\), then
\[
\int_{\mathbb{R}^3} (\nabla u_{\lambda_n} \cdot \nabla v + u_{\lambda_n} v)dx + \lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}} |u_{\lambda_n}|^3 u_{\lambda_n} vdx
= \int_{\mathbb{R}^3} |u_{\lambda_n}|^4 u_{\lambda_n} vdx + \int_{\mathbb{R}^3} |u_{\lambda_n}|^{q-2} u_{\lambda_n} vdx,
\]
for any \(v \in C_0^\infty(\mathbb{R}^3)\). In view of Hardy-Littlewood-Sobolev inequality (see Proposition 2.2) and Hölder inequality, we conclude that
\[
\left|\int_{\mathbb{R}^3} \phi_{u_{\lambda_n}} |u_{\lambda_n}|^3 u_{\lambda_n} vdx\right| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_{\lambda_n}(y)|^5 |u_{\lambda_n}(x)|^3 u_{\lambda_n}(x)v(x)}{|x-y|} dydx
\leq C_1 \left(\int_{\mathbb{R}^3} |u_{\lambda_n}|^6 dx\right)^{\frac{\theta}{6}} \left(\int_{\mathbb{R}^3} |u_{\lambda_n}|^\frac{24}{\theta} |v|^\frac{6}{\theta} dx\right)^{\frac{\theta}{6}}
\leq C_2 \|u_{\lambda_n}\|^5 \|u_{\lambda_n}\|^4 \|v\|
\leq C_3 \|v\|.
\]

Then combining with (4.31), (4.32) and (6.1), as \(n \to \infty\), it deduces from (6.2) that
\[
\int_{\mathbb{R}^3} (\nabla u_0 \cdot \nabla v + u_0 v)dx = \int_{\mathbb{R}^3} |u_0|^4 u_0 vdx + \int_{\mathbb{R}^3} |u_0|^{q-2} u_0 vdx,
\]
for any \(v \in C_0^\infty(\mathbb{R}^3)\). Thereby, \(u_0\) is a weak solution of problem (1.5).

In order to finish the proof of Claim 2, it suffices to prove \(u_0^\pm \neq 0\). Since \(u_{\lambda_n} \in \mathcal{M}_{\lambda_n}\), that is,
\[
\|u_{\lambda_n}^+\|^2 + \lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}^+} |u_{\lambda_n}^+|^5 dx + \lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}^-} |u_{\lambda_n}^-|^5 dx = \int_{\mathbb{R}^3} |u_{\lambda_n}^+|^6 dx + \int_{\mathbb{R}^3} |u_{\lambda_n}^-|^q dx.
\]
For this, we can directly derive from Lemma 4.2(3) that \(\Lambda_2 \leq \|u_{\lambda_n}^\pm\| \leq \Lambda_1\) for some \(\Lambda_1, \Lambda_2 > 0\) (independent of \(\lambda\) and \(n\)). This together with \(\lambda_n \to 0\) as \(n \to \infty\), we infer that
\[
\left|\lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}^+} |u_{\lambda_n}^+|^5 dx\right| \leq -C_1 \lambda_n \|u_{\lambda_n}^+\|^{10} \leq -C_2 \lambda_n \to 0,
\]
and
\[
\left|\lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}^-} |u_{\lambda_n}^-|^5 dx\right| \leq -C_3 \lambda_n \|u_{\lambda_n}^+\|^{10} \|u_{\lambda_n}^-\|^5 \leq -C_4 \lambda_n \to 0,
\]
which implies that
\[ 0 < \Lambda_2^2 \leq \lim_{n \to \infty} \|u_{\lambda_n}^\pm\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^\pm|^6 \, dx + \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx. \]  

If \( \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx \neq 0 \), it follows from (6.3) that \( \|u_0^\pm\|^2 \geq \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx > 0 \), which implies that \( u_0^\pm \neq 0 \). Therefore, the proof of Claim 2 is completed. Otherwise, if \( \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx = 0 \) or \( \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx = 0 \), which leads to the following three cases:

- Case i: \( u_0^+ \equiv 0 \) and \( u_0^- \equiv 0 \);
- Case ii: \( u_0^+ \neq 0 \) and \( u_0^- \equiv 0 \);
- Case iii: \( u_0^+ \equiv 0 \) and \( u_0^- \neq 0 \).

We will show those cases can not happen.

If Case i happens, it follows from (2.1) and (6.4) that
\[ 0 < \Lambda_2^2 \leq \lim_{n \to \infty} \|u_{\lambda_n}^\pm\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^\pm|^6 \, dx \leq S^{-3} \lim_{n \to \infty} \|u_{\lambda_n}^\pm\|^6, \]  
then \( \lim_{n \to \infty} \|u_{\lambda_n}^\pm\|^2 \geq S^{\frac{2}{3}} \). By (6.5) and \( \lambda_n \to 0 \) as \( n \to \infty \), we conclude that
\[ m_0 = \lim_{n \to \infty} m_{\lambda_n} = \lim_{n \to \infty} I_{\lambda_n}(u_{\lambda_n}) = \frac{1}{2} \lim_{n \to \infty} \|u_{\lambda_n}\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}|^6 \, dx \]
\[ = \frac{1}{2} \lim_{n \to \infty} \|u_{\lambda_n}^+\|^2 + \frac{1}{2} \lim_{n \to \infty} \|u_{\lambda_n}^-\|^2 \]
\[ - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^+|^6 \, dx + \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^-|^6 \, dx \]
\[ = \frac{1}{3} \lim_{n \to \infty} \|u_{\lambda_n}^+\|^2 + \frac{1}{3} \lim_{n \to \infty} \|u_{\lambda_n}^-\|^2 \]
\[ \geq \frac{2}{3} S^{\frac{2}{3}}, \]
which leads to a contradiction with \( m_0 < \frac{4}{3} S^{\frac{2}{3}} \) (obtained by Remark 5.3).

If Case ii happens, which leads to
\[ 0 < \Lambda_2^2 \leq \lim_{n \to \infty} \|u_{\lambda_n}^-\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^-|^6 \, dx \leq S^{-3} \lim_{n \to \infty} \|u_{\lambda_n}^-\|^6, \]  
and
\[ 0 < \Lambda_2^2 \leq \lim_{n \to \infty} \|u_{\lambda_n}^+\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^+|^6 \, dx + \int_{\mathbb{R}^3} |u_0^+|^q \, dx. \]

Then (6.6) shows \( \lim_{n \to \infty} \|u_{\lambda_n}^-\|^2 \geq S^{\frac{2}{3}} \). Recall that \( u_0 \) is a weak solution of problem (1.5), i.e.,
\[ \|u_0^\pm\|^2 = \int_{\mathbb{R}^3} |u_0^\pm|^6 \, dx + \int_{\mathbb{R}^3} |u_0^\pm|^q \, dx, \]
which together with Brézis-Lieb lemma, we obtain that (6.7) is equivalent to
\[ \lim_{n \to \infty} \|u_{\lambda_n}^- - u_0^+\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^- - u_0^+|^6 \, dx \leq S^{-3} \lim_{n \to \infty} \|u_{\lambda_n}^- - u_0^+\|^6. \]

Now, if \( \lim_{n \to \infty} \|u_{\lambda_n}^- - u_0^+\|^2 = 0 \), i.e., \( u_{\lambda_n}^- \to u_0^+ \) in \( H^1_1(\mathbb{R}^3) \), by (6.6), there holds
\[ m_0 = \lim_{n \to \infty} m_{\lambda_n} = \lim_{n \to \infty} I_{\lambda_n}(u_{\lambda_n}) = \frac{1}{2} \lim_{n \to \infty} \|u_{\lambda_n}^+\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_{\lambda_n}^+|^6 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u_0^+|^q \, dx \]
\[
\frac{1}{2} \lim_{n \to \infty} \|u_n^-\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^-|^6 \, dx
\]
\[
+ \frac{1}{2} \lim_{n \to \infty} \|u_n^+\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^6 \, dx
\]
\[
= I_0(u_0^+) + \frac{1}{2} \lim_{n \to \infty} \|u_n^-\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^-|^6 \, dx
\]
\[
= I_0(u_0^+) + \frac{1}{3} \lim_{n \to \infty} \|u_n^-\|^2
\]
\[
\geq c_0 + \frac{1}{3} S_{\mathbb{R}^3}^2,
\]
which leads to a contradiction with \( m_0 < c_0 + \frac{1}{3} S_{\mathbb{R}^3}^2 \). Thus, \( \lim_{n \to \infty} \|u_n^+ - u_0^+\|^2 > 0 \), and (6.9) indicates that \( \lim_{n \to \infty} \|u_n^+ - u_0^+\|^2 \geq S_{\mathbb{R}^3}^2 \). By Brézis-Lieb lemma, (6.6), (6.8) and (6.9), then
\[
m_0 = \lim_{n \to \infty} m_{\lambda_n} = \lim_{n \to \infty} I_{\lambda_n}(u_{\lambda_n})
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \|u_n^+\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^6 \, dx - \frac{1}{q} \int_{\mathbb{R}^3} |u_0^+|^q \, dx
\]
\[
+ \frac{1}{2} \lim_{n \to \infty} \|u_n^-\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^-|^6 \, dx
\]
\[
= \frac{1}{2} \lim_{n \to \infty} \|u_n^+\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^6 \, dx - \frac{1}{q} \left( \|u_0^+\|^2 - \int_{\mathbb{R}^3} |u_0^+|^6 \, dx \right) + \frac{1}{3} \lim_{n \to \infty} \|u_n^-\|^2
\]
\[
\geq \frac{1}{2} \lim_{n \to \infty} \|u_n^+ - u_0^+\|^2 - \frac{1}{6} \lim_{n \to \infty} \int_{\mathbb{R}^3} \|u_n^+ - u_0^+\|^6 \, dx + \frac{1}{3} \lim_{n \to \infty} \|u_n^-\|^2
\]
\[
= \frac{1}{3} \lim_{n \to \infty} \|u_n^+ - u_0^+\|^2 + \frac{1}{3} \lim_{n \to \infty} \|u_n^-\|^2
\]
\[
\geq \frac{2}{3} S_{\mathbb{R}^3}^2,
\]
which also leads to a contradiction with \( m_0 < \frac{2}{3} S_{\mathbb{R}^3}^2 \).

Similarly, we can deduce a contradiction if Case iii happens.

Claim 3: Problem (1.5) possesses a least energy radial sign-changing solution \( z_0 \). Moreover, there exists a unique pair \((s_{\lambda_n}, t_{\lambda_n}) \in (0, \infty) \times (0, \infty)\) such that \( s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^- \in \mathcal{M}_{\lambda_n}\) and \((s_{\lambda_n}, t_{\lambda_n}) \to (1, 1)\) as \( n \to \infty \).

Recall from Theorem 1.7 that problem (1.5) has a least energy radial sign-changing solution \( z_0 \), that is, \( I_0(z_0) = m_0 \) and \( I_0'(z_0) = 0 \). Using Lemma 4.1, we get that there exists a unique pair \((s_{\lambda_n}, t_{\lambda_n}) \in (0, \infty) \times (0, \infty)\) such that \( s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^- \in \mathcal{M}_{\lambda_n}\). Next, we aim to show that \((s_{\lambda_n}, t_{\lambda_n}) \to (1, 1)\) as \( n \to \infty \). Since \( s_{\lambda_n} z_0^+ + t_{\lambda_n} z_0^- \in \mathcal{M}_{\lambda_n}\), then
\[
s_{\lambda_n}^2 \|z_0^+\|^2 + \lambda_n s_{\lambda_n}^{10} \int_{\mathbb{R}^3} \phi_{z_0^+}^5 \, dx + \lambda_n s_{\lambda_n}^{5} \int_{\mathbb{R}^3} \phi_{z_0^+} |z_0^+|^5 \, dx
\]
\[
= s_{\lambda_n}^6 \int_{\mathbb{R}^3} |z_0^+|^6 \, dx + s_{\lambda_n}^q \int_{\mathbb{R}^3} |z_0^+|^q \, dx,
\]
\[
t_{\lambda_n}^2 \|z_0^-\|^2 + \lambda_n t_{\lambda_n}^{10} \int_{\mathbb{R}^3} \phi_{z_0^-}^5 \, dx + \lambda_n t_{\lambda_n}^{5} \int_{\mathbb{R}^3} \phi_{z_0^-} |z_0^-|^5 \, dx
\]
\[
= t_{\lambda_n}^6 \int_{\mathbb{R}^3} |z_0^-|^6 \, dx + t_{\lambda_n}^q \int_{\mathbb{R}^3} |z_0^-|^q \, dx.
\]

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This together with $q \in (2, 6)$, we conclude that $\{s_{\lambda_n}\}$ and $\{t_{\lambda_n}\}$ are bounded in $\mathbb{R}^+$. Then, up to a subsequence such that $s_{\lambda_n} \to s_0$ and $t_{\lambda_n} \to t_0$ as $n \to \infty$, there hold
\[
s_0^2 \|z_0^+\|^2 = s_0^6 \int_{\mathbb{R}^3} |z_0^+|^6 dx + s_0^q \int_{\mathbb{R}^3} |z_0^+|^q dx,
\]
\[
t_0^2 \|z_0^-\|^2 = t_0^6 \int_{\mathbb{R}^3} |z_0^-|^6 dx + t_0^q \int_{\mathbb{R}^3} |z_0^-|^q dx.
\]
(6.10)
Recall that $z_0$ is a least energy radial sign-changing solution of problem (1.5), that is,
\[
\|z_0^\pm\|^2 = \int_{\mathbb{R}^3} |z_0^\pm|^6 dx + \int_{\mathbb{R}^3} |z_0^\pm|^q dx.
\]
(6.12)
Then, we can derive from (6.10)–(6.12) that
\[
\left(1 - s_0^{2-q}\right) \|z_0^+\|^2 = \left(1 - s_0^{6-q}\right) \int_{\mathbb{R}^3} |z_0^+|^6 dx,
\]
\[
\left(1 - t_0^{2-q}\right) \|z_0^-\|^2 = \left(1 - t_0^{6-q}\right) \int_{\mathbb{R}^3} |z_0^-|^6 dx.
\]
From $q \in (2, 6)$, we can directly check that $(s_0, t_0) = (1, 1)$, which leads to the Claim 3.

In view of this, it suffices to prove that $u_0$ obtained in Claim 2 is a least energy radial sign-changing solution of problem (1.5). Actually, by Claim 3 and Lemma 4.1, it yields that
\[
\mathcal{I}_0(z_0) \leq \lim_{n \to \infty} \mathcal{I}_{\lambda_n}(u_{\lambda_n}) \leq \lim_{n \to \infty} \mathcal{I}_{\lambda_n}(s_{\lambda_n}z_0^+ + t_{\lambda_n}z_0^-) = \mathcal{I}_0(z_0^+ + z_0^-) = \mathcal{I}_0(z_0).
\]
This indicates that $u_0$ is a least energy radial sign-changing solution of problem (1.5). Similar arguments as proof of Theorem 1.7, we obtain that $u_0$ has precisely two nodal domains. Hence, we complete the proof of Theorem 1.9. \hfill \Box

**Conflict of interest statements**

The authors declare that they have no conflict of interest.

**Data availability statements**

Data sharing not applicable to this article as no new data were created or analyzed in this study.

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