On the convexity of relativistic ideal magnetohydrodynamics

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Abstract

We analyze the influence of the magnetic field in the convexity properties of the relativistic magnetohydrodynamics system of equations. To this purpose we use the approach of Lax, based on the analysis of the linearly degenerate/genuinely nonlinear nature of the characteristic fields. Degenerate and non-degenerate states are discussed separately and the non-relativistic, unmagnetized limits are properly recovered. The characteristic fields corresponding to the material and Alfvén waves are linearly degenerate and, then, not affected by the convexity issue. The analysis of the characteristic fields associated with the magnetosonic waves reveals, however, a dependence of the convexity condition on the magnetic field. The result is expressed in the form of a generalized fundamental derivative written as the sum of two terms. The first one is the generalized fundamental derivative in the case of purely hydrodynamical (relativistic) flow. The second one contains the effects of the magnetic field. The analysis of this term shows that it is always positive, leading to the remarkable result that the presence of a magnetic field in the...
fluid reduces the domain of thermodynamical states for which the equation of state is non-convex.

Keywords: convexity, special relativity, magnetohydrodynamics

1. Introduction

There are many astrophysical scenarios governed by relativistic magnetohydrodynamical processes as, for example, the production of relativistic jets emanating from active galactic nuclei, the structure and dynamics of pulsar wind nebulae, the mechanisms triggering the explosion in core-collapse supernovae, or the production of gamma ray bursts. These scenarios are nowadays the subject of intensive research by means of numerical simulations thanks to recent advances in numerical relativistic magnetohydrodynamics (RMHD) that exploit the fact that the RMHD equations obeying a causal equation of state (EOS) form a hyperbolic system of conservation laws [1].

Matter at densities higher than nuclear matter density can undergo first-order phase transitions to various phases of matter, such as pion condensates [2], hyperonic matter [3] or deconfined quark matter [4, 5]. Several authors [6–8] have studied, from different points of view, the influence that those exotic states of matter at extreme high densities have on, e.g., the dynamics of stellar core collapse supernovae, the evolution of proto-neutron stars, or the collapse to black hole.

The classical Van der Waals (VdW) EOS is a well known example of EOS displaying a first-order phase transition. Fluids having a thermodynamics governed by a VdW-like EOS exhibit, outside the region of the phase transition, non-classical gasdynamic behaviors in a range of thermodynamic conditions characterized by the negative value of the so-called fundamental derivative,

\[ G := -\frac{1}{2} V \left( \frac{\partial p}{\partial V} \right)_s \]  

\( p \) being the pressure, \( V := 1/\rho \) the specific volume (\( \rho \) is the rest-mass density) and \( s \) the specific entropy. The fundamental derivative measures the convexity of the isentropes in the \( p - V \) plane and if \( G > 0 \) then the isentropes in the \( p - V \) plane are convex, leading to expansive rarefaction waves (and compressive shocks) [12]. In a VdW-like EOS, or in general in a non-convex EOS, rarefaction waves can change to compressive and shock waves to expansive depending on the specific thermodynamical state of the system. These non-classical phenomena have been observed experimentally and their study is, currently, of interest in many engineering applications [13, 14].

Besides this thermodynamical interpretation of convexity, there is an equivalent definition due to Lax [15] that connects with the mathematical properties of the hyperbolic system. According to Lax’s approach, a hyperbolic system of conservation laws\(^8\) is convex if all its characteristic fields are either genuinely nonlinear or linearly degenerate. A characteristic field

\(^8\) The books by LeVeque [16] and Toro [17] are recommendable references for those readers interested in the basic theory of hyperbolic systems of conservation laws. The monograph of [18] on finite-volume methods for hyperbolic problems pays special attention to non-convex flux functions (see their sections 13.8.4—definitions of genuine nonlinearity and linear degeneracy, and their relationship with convexity, and 16.1—devoted entirely to the study of scalar conservation laws with non-convex flux functions).
\( \lambda \) is said to be genuinely nonlinear or linearly degenerate if, respectively

\[
P = \nabla_\mu \lambda \cdot r \neq 0, \quad (2)
\]

\[
P = \nabla_\mu \lambda \cdot r = 0, \quad (3)
\]

for all \( \mathbf{u} \), where \( \nabla_\mu \lambda \) is the gradient of \( \lambda(\mathbf{u}) \) in the space of conserved variables, \( \mathbf{r} \) is the corresponding eigenvector, and the dot stands for the inner product in the space of physical states.

In a non-convex system, non-convexity is associated with those states \( \mathbf{u} \) for which the factor \( P \) corresponding to a genuinely nonlinear field, equation (2), is zero and changes sign in a neighborhood of \( \mathbf{u} \).

A virtue of Lax’s approach is that it can be applied to other hyperbolic systems in which the convex or non-convex character of the dynamics is governed by other ingredients beyond the EOS. Among these systems are those of relativistic hydrodynamics (RHD) and classical magnetohydrodynamics (MHD). In these two cases, the convexity of the system has been characterized with the sign of a generalized fundamental derivative that includes an extra term depending of the local speed of sound (in the case of RHD [19]) and the magnetic field (in the case of MHD [20]).

In this work we use the approach of Lax to characterize, from a theoretical point of view, the effects of magnetic fields in the convexity properties of the RMHD system of equations as a previous step to explore its possible impact in the dynamical evolution of different astrophysical scenarios. The result is presented in the form of an extended fundamental derivative whose sign determines the convex/non-convex character of the RMHD system at a given state. Our result recovers the proper non-relativistic and unmagnetized limits.

The paper is organized as follows. In section 2, the equations of RMHD are introduced as a hyperbolic system of conservation laws. The transformation between primitive and conserved variables are explicitly written. In section 3 the characteristic structure of the RMHD equations is discussed and the analysis of convexity in non-degenerate states presented. In section 4 the analysis of convexity is extended to degenerate states. The non-relativistic, unmagnetized limits are recovered in section 5. Section 6 includes a short summary and presents the conclusions. Finally, there is an appendix that displays the Jacobian matrices of the RMHD system in quasi-linear form, necessary for the characteristic analysis of section 3.

2. The equations of ideal RMHD

Let \( J^\mu \), \( T^{\mu \nu} \) and \( *F^{\mu \nu} \) be the components of the rest-mass current density, the energy–momentum tensor and the Maxwell tensor of an ideal (infinite conductivity) magneto-fluid, respectively

\[
J^\mu = \rho u^\mu, \quad (4)
\]

\[
T^{\mu \nu} = \rho h^a u^\mu u^\nu + g^{\mu \nu} p^a - b^\mu b^\nu, \quad (5)
\]

\[
*F^{\mu \nu} = u^\nu b^\mu - u^\mu b^\nu, \quad (6)
\]

where \( \rho \) is the proper rest-mass density, \( h^a = 1 + e + p/\rho + b^2/\rho \) is the specific enthalpy including the contribution from the magnetic field (\( b^2 \) stands for \( b^a b_a \)), \( e \) is the specific

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9 Throughout this paper, Greek indices will run from 0 to 3, while Roman run from 1 to 3, or, respectively, from \( t \) to \( z \) and from \( x \) to \( z \) in Cartesian coordinates.
internal energy, $p$ is the thermal pressure, $p^* = p + b^2/2$ is the total pressure, and $g^{\mu\nu}$ is the metric of the space–time where the fluid evolves. Throughout the paper we use units in which the speed of light is $c = 1$ and the $(4\pi)^{1/2}$ factor is absorbed in the definition of the magnetic field. The four-vectors representing the fluid velocity, $u^\mu$, and the magnetic field measured in the comoving frame (CF), $b^\mu$, satisfy the conditions $u^\mu u_\mu = -1$ and $u^\mu b_\mu = 0$.

The equations of ideal RMHD correspond to the conservation of rest-mass and energy–momentum, and the Maxwell equations. In a flat space–time and Cartesian coordinates, these equations read:

\begin{align}
J^\mu_{,\mu} &= 0, \\
T^\mu_{,\mu} &= 0, \\
\ast F^\mu\nu_{,\mu} &= 0,
\end{align}

where subscript $(,\mu)$ denotes partial derivative with respect to the corresponding coordinate, $(t, x, y, z)$, and the standard Einstein sum convention is assumed.

The above system can be written as a system of conservation laws as follows

\begin{equation}
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = 0,
\end{equation}

where $\mathbf{V} = (\rho, v^i, e, B^j)^T$ is the set of primitive variables. The state vector (the set of conserved variables) $\mathbf{U}$ and the fluxes, $\mathbf{F}^i$, are, respectively:

\begin{equation}
\mathbf{U} = \begin{pmatrix} D \\ S^i \\ \tau \\ B^j \end{pmatrix},
\end{equation}

\begin{equation}
\mathbf{F}^i = \begin{pmatrix} Dv^i \\ S^i v^j + p^* \delta^i_{j} - b^j B^i / W \\ \tau v^i + p^* v^j - b^0 B^i / W \\ v^k B^k - v^i B^i \end{pmatrix}.
\end{equation}

In the preceding equations, $D$, $S^i$ and $\tau$ stand, respectively, for the rest-mass density, the momentum density of the magnetized fluid in the $j$-direction, and its total energy density, all of them measured in the laboratory (i.e., Eulerian) frame:

\begin{align}
D &= \rho W, \\
S^i &= \rho h^* W^2 v^i - b^0 b^i, \\
\tau &= \rho h^* W^2 - p^* - \left( b^0 \right)^2 - D.
\end{align}

The components of the fluid velocity trivector, $v^i$, as measured in the laboratory frame, are related with the components of the fluid four-velocity according to the following expression:

\begin{equation}
u^\mu = W (1, v^i), \text{ where } W \text{ is the flow Lorentz factor, } W^2 = 1/(1 - \mathbf{v}_\lambda \mathbf{v}^\lambda).
\end{equation}

The components of the magnetic field four-vector in the CF and the three vector components $B^j$ measured in the laboratory frame satisfy the relations:
Finally, the square of the modulus of the magnetic field can be written as
\[ b^2 = \frac{B_k B^k}{W^2} + (\nabla B)^2. \]  
(18)

The preceding system must be complemented with the time component of equation (9), that becomes the usual divergence constraint
\[ \frac{\partial B^i}{\partial t} = 0. \]  
(19)

An EOS \( p = p(\rho, e) \) closes the system. Accordingly, the (relativistic) sound speed
\[ a_s = \sqrt{\frac{\partial p}{\partial \rho}} \]  
and \( \kappa = \frac{\partial p}{\partial e} \), satisfies
\[ \chi \kappa = \rho \hspace{1cm} (e = \rho(1 + \epsilon)), \]  
where \( \chi \) and \( \kappa \) are displayed in the appendix.

Once the eigenvalues and eigenvectors are known, we can analyze the convexity of the system studying the expression
\[ \lambda = \xi \alpha \cdot U \]  
(see the Introduction). Finally, we can take advantage of the fact that, since \( \mathcal{A}^0 \) is non-singular, then \( \mathcal{P}^0 \) is non-singular, and perform the analysis of convexity in terms of \( \mathcal{P}^0 \).

The eigenvalues \( \lambda_a \) are the solutions of the following polynomial expression for \( \lambda \)
\[ \lambda \left( \mathcal{E}a^2 - B^2 \right) \left( b^2 + \rho a \right) a^2 G - W_a^2 \rho a^2 - a^2 G B^2 = 0, \]  
(20)
where \( \mathcal{E} = \rho b + b^2, W_a^2 = 1 - a^2 \) and quantities \( a, G \) and \( B \) were defined in [1], \( a = \phi U a, G = \phi \partial_a, B = \phi b^a \), being, in our case, \( \phi = (\frac{\partial}{\partial \kappa}, \frac{\partial}{\partial \xi}) \) the normal to the wavefront propagating with speed \( \lambda \) in the spatial direction given by the unit vector \( \xi \).

As it is well known, the system of (R)MHD is not-strictly hyperbolic [21]. This means that in some cases, two or more eigenvalues can be equal leading to well studied cases of degeneracy (see [1, 22], for the relativistic case). In Type I degeneracy, the magnetic field is normal to the propagation direction of the wavefront (i.e., \( \xi \partial_k B^k = 0 \)). In Type II degeneracy,
ζ \mathbf{B}^k \neq 0$, but the eigenvalues associated with, at least, one Alfvén wave and one magneto-sonic wave are degenerate. Leaving aside the particular cases associated with both degeneracy types, that will be discussed later, the following list compiles the roots of the characteristic equation (20), $\lambda_\alpha = \lambda_\alpha^*$, the right eigenvectors, and their corresponding scalar products in the non-degenerate, general case.

(i) $\lambda = \lambda_{\text{null}} := 0$. In this case, $P_{\text{null}}^*$ is trivially zero. This eigenvalue is spurious and is associated with the fact that although the RMHD system (10) consists of eight conservation equations, only seven components of the fluxes are non-trivial. Due to the antisymmetric character of the induction equation, the flux of $\zeta B_k$ in the $\zeta$-direction is identically zero.

(ii) $\lambda = \lambda_0 := \zeta v^k$ is the eigenvalue associated with the material waves. The corresponding eigenvector is $\mathbf{r}_0^\alpha = (-\kappa, 0', \chi, 0')^T$, where $\kappa$ and $\chi$ are thermodynamical derivatives defined at the end of the previous Section, and $0' = 0$ ($i = 1, 2, 3$). The scalar product is $P_0^* = 0$ and, consequently, the characteristic field defined by $\lambda_0$ is linearly degenerate.

(iii) $\lambda = \lambda_{\text{A}}$ are the roots of the second-order polynomial in $\lambda$, $A := EA^2 - B^2$. They define the Alfvén waves. Since $\zeta \mathbf{B}^k \neq 0$, then $\alpha \neq 0$ and the corresponding eigenvectors are

$$
\mathbf{r}_a^\alpha = \left( 0, r_2^a, 0, r_4^a \right)^T, \quad (22)
$$

where $r_2^a = a_1 B^k + a_3 v^l + a_5 \zeta^s$, $r_4^a = Wa^{-1}(r_2^a \zeta B^k - B^s \zeta r_2^a)$. The coefficients $a_p (p = 1, 2, 3)$ are such that $\nu_k r_2^a = 1$, $\zeta k r_4^a = -W a$, and $B_k r_4^a = -\nu_k B^k W^2$. The scalar products are

$$
P_{a*}^\alpha = \frac{\partial \lambda_a}{\partial \nu^l} r_2^a + \frac{\partial \lambda_a}{\partial B^k} r_4^a \propto \left( \zeta k r_4^a + W a \left( \nu_k r_2^a \right) \right) = 0, \quad (23)
$$

in agreement with the linearly degenerate character of the Alfvén waves.

(iv) The four eigenvalues $\lambda_f, \lambda_s, \lambda_{\text{A}}, \lambda_{\text{as}}$ are the roots of the fourth-order polynomial in $\lambda$, $\mathcal{N}_4 := \left( b^2 + \rho ha^2 \right) a^2 G - W_s^{-2} \rho ha^4 - a_s^2 GB^2$, (24)

associated with the fast and slow magnetosonic wavespeeds, respectively. Since $\zeta \mathbf{B}^k \neq 0$, then $\alpha \neq 0$ and the corresponding eigenvectors are

$$
\mathbf{r}_m^s = \left( \eta_1, \eta_2, \eta_3, \eta_4 \right)^T, \quad (m = f, s),
$$

10 The expressions of the eigenvectors have been obtained after tedious algebraic manipulations. They can be verified by direct substituting in the eigenvalue equation, $(\zeta \mathbf{A} - \lambda^* \mathbf{A}) \mathbf{r}_a^\alpha = 0$.

11 For the scalar products $P_{a*}^\alpha$ and $P_{a*}^\alpha$, the partial derivatives of the corresponding eigenvalues with respect to the primitive variables, $V$, have been computed by implicit derivation of the characteristic equations for $\lambda_{a*}$ and $\lambda_{a*}$, respectively, i.e., $A = 0$ and $\mathcal{N}_4 = 0$ (see below).
The scalar products are
\[ p_{mz}^* = \left( \frac{\partial \lambda_m}{\partial v^i} \right) n + \left( \frac{\partial \lambda_m}{\partial c} \right) r_3 + \left( \frac{\partial \lambda_m}{\partial B^2} \right) r_4, \]
\[ = \frac{W^2 a^4 G^2}{2a_s^2} p^*_1 p^*_5, \tag{27} \]
where \( d \), the derivative of \( \mathcal{N} \) with respect to \( \lambda = \lambda_m, \pm \ (m = f, s) \), \( \mathcal{N}(\lambda_m, \pm) \), is
\[ d = a_s^2 G^2 B \left( \xi_B^2 B^2 - \left( G - \lambda_m aW^{-1} \right) \rho hW_s^{-2} a^4 \right), \tag{28} \]
and
\[ p^*_1 = b^2 G - \rho ha^2, \tag{29} \]
\[ p^*_2 = \rho \left\{ \frac{a_s^2}{2a_s^2} \left( + \left( \frac{\partial a_s^2}{\partial c} \right) W^2 \left( B^2 \right)^2 - E \right) \right. \]
\[ - b^2 \left( 3 - a_s^2 \right) - 2\rho h a^2 + \frac{a_s^2 (5 - 3a_s^2) B^2}{a^2} \right\}. \tag{30} \]

It is interesting to note that \( d \) can only be zero in degenerate states, since it is only in these states where both \( \mathcal{N}_d(\lambda) = 0 \) and \( \mathcal{N}_s(\lambda) = 0 \) are satisfied simultaneously. Let us now discuss the conditions under which the remaining factors in equation (27) can become zero. Quantity \( a \) is non-zero as far as \( \xi \neq 0 \). On the other hand, it can be proven by simple algebraic manipulation of equations \( \mathcal{N}_d(\lambda) = 0 \) and \( \mathcal{N}_s(\lambda) = 0 \) that \( p^*_1 = 0 \) if and only if the corresponding magnetosonic eigenvalue is also an Alfvén eigenvalue (i.e., Type II degeneracy). Since we are avoiding degenerate states, and \( G \) is always non-zero, we shall concentrate on the changes of sign of \( p^*_2 \), in order to analyze the possible loss of convexity associated with the magnetosonic waves. Since in the case of zero magnetic field, the purely relativistic result has to be recovered, we shall try now to rewrite expression (30) in terms of the relativistic fundamental derivative
\[ \mathcal{G} = 1 + \frac{\rho}{2a_s^2} \left. \frac{\partial a_s^2}{\partial \rho} \right|_v - a_s^2 \]
\[ \mathcal{G}_M, \] derived in [19]. The sought expression is
\[ p^*_2 = -2a_s^2 W^2 E (1 - R) \mathcal{G}_M, \tag{32} \]
with \( \mathcal{G}_M \), the fundamental derivative for relativistic, magnetized fluids, being
\[
\hat{G}_M := \hat{G} + F, \tag{33}
\]

where
\[
F := \frac{3}{2} W_{w}^{-4} \left( \frac{c_s^2}{a_s^2} - R \right), \tag{34}
\]

In the previous expressions, \( R := \frac{\rho_s^2}{c_a^2} \), and \( c_s^2 := \frac{\rho_s^2}{c_a^2} \) stands for the square of the Alfvén velocity. Moreover, in deriving expression (32) from (30) we have used the following relation among thermodynamical derivatives
\[
\frac{\partial}{\partial \rho} \mathbb{P} = \frac{\partial}{\partial \rho} \mathbb{P} + \frac{\partial}{\partial c} \mathbb{P}.
\]

It is important to note that \( R = 1 \) if and only if the eigenvalue corresponds to an Alfvén wavespeed (i.e., it satisfies equation \( \lambda(\lambda) = 0 \)). Since we are not considering degeneracies, we conclude that \( R \neq 1 \) for magnetosonic waves and, consequently, (1) the denominator in the second term of \( \hat{G}_M \) is well defined, and (2) \( \mathbb{P}_2 = 0 \) if and only if \( \hat{G}_M = 0 \).

The price to pay for using primitive (or conserved) variables in our analysis of convexity is the loss of covariance and a dependence of the fundamental derivative \( \hat{G}_M \) on kinematics through quantity \( R \). For fast and slow magnetosonic fields, let us carry out the analysis of the magnetic correction to the purely hydrodynamic (relativistic) fundamental derivative (equation (34)) in the CF \( \mu^\mu = \delta_{\mu}^\mu \), which we will name \( F_{\text{CF},m} \) \((m = f, s)\) henceforth. A simple algebraic calculation leads to
\[
F_{\text{CF},m} = \frac{3}{2} W_{w}^{-2} \left( \frac{c_m^2 - a_m^2}{c_m^2 - c_a^2} \right), \tag{35}
\]

where \( c_m^2 \) are the solutions of the quadratic equation in \( \lambda^2, \mathbb{N}_{\lambda,\text{CF}}(\lambda) = 0 \), namely
\[
c_m^2 = \frac{1}{2} \left( \left( \omega^2 + a_s^2 c_A^2 \right) \pm \left( \left( \omega^2 + a_s^2 c_A^2 \right)^2 - 4a_s^2 c_A^2 \right)^{1/2} \right). \tag{36}
\]

with \( c_A^2 = \frac{(\delta \delta^\mu\mu)^2}{\epsilon} \) and \( W_{w}^{-2} := 1 - \omega^2, a_s^2 = a_s^2 + c_s^2 - a_s^2 c_s^2 \).

Taking into account that, for non-degenerate states\(^{12}\), \( a_s^2, c_s^2 \in (c_s^2, c_f^2) \), we have that \( F_{\text{CF},m} > 0 \) \((m = f, s)\). Now, the transformation of \( R \) as a scalar ensures that \( E_m > 0 \) \((m = f, s)\) in any reference frame, with important consequences for the influence of the magnetic field on the convexity of the system.

### 4. Analysis of convexity in degenerate states

#### 4.1. Type I degeneracy

This degeneracy appears in states in which \( \zeta \ll B^2 = 0 \). Now, the roots of the characteristic equation (20), the right eigenvectors, and the corresponding scalar products have the following properties:

(i) \( \lambda = \lambda_{\text{null}} := 0 \). It is again the spurious eigenvalue analyzed in the previous section associated with the null flux component. \( \mathbb{P}_{\text{null}} \) is trivially zero.

\(^{12}\) In the CF it can be easily proven that \( \mathbb{N}_{\lambda,\text{CF}}(\zeta) < 0 \) and \( \mathbb{N}_{\lambda,\text{CF}}(\zeta) < 0 \), implying that both \( a_s^2 \) and \( c_s^2 \) are between the roots of \( \mathbb{N}_{\lambda,\text{CF}}(\lambda) = 0 \), namely \( c_s^2, c_f^2 \).
iii) The eigenvalue \( \lambda = \lambda_0 := \zeta_k v^k \) has multiplicity 5. The corresponding eigenvectors are of the form \( \mathbf{r}_0 = (r_1, a_1 B^i + a_2 \zeta_k^i, r_2, a_3 B^i + a_4 \zeta_k^i)^T \), where \( \zeta_k^i \) is an arbitrary vector orthogonal to \( \zeta^i \) and \( B^i \), and \( r_1, r_2, \) and \( a_p \) (\( p = 1, 2, 3, 4 \)) are functions of the primitive variables. Since only the derivative \( \partial \lambda/\partial v^k \) \( (= \zeta_k) \) is different from zero, the scalar product is

\[
P_0^s = \zeta \left( a_1 B^i + a_2 \zeta_k^i \right) = 0. \tag{37}
\]

Hence, the characteristic fields defined by \( \lambda_0 \) are linearly degenerate.

(iii) \( \lambda_{f, \pm} \) are the solutions of the quadratic equation in \( \lambda \)

\[
\begin{bmatrix}
   b^2 + \rho a_1^2 - a_2^2 \left( v_k B^k \right)^2 \\
   G - W_2^{-2} \rho a_1^2
\end{bmatrix} = 0,
\tag{38}
\]

and are associated with the fast magnetosonic wavespeeds. The explicit expression of these eigenvalues when \( \zeta_k = (1, 0, 0) \) can be found in [23]. The corresponding eigenvectors can be obtained from those of the fast magnetosonic eigenvalues in the general case (see equation (25)) making \( \zeta_k B^k \) \( = 0 \), i.e., \( B = a(v_k B^k) \). The scalar products are

\[
P_{fs}^s = \frac{W_2^2 G^2}{2 \rho h} P_1^s P_2^s, \tag{39}
\]

where

\[
P_1^s = \frac{E - \left( v_k B^k \right)^2}{1 - \zeta_k v^k} \tag{40}
\]

\[
P_2^s = \left( \rho \frac{\partial a_2^2}{\partial \rho} + \frac{p}{\rho} \frac{\partial a_2^2}{\partial \rho} \right) W_2^2 \left( \left( v_k B^k \right)^2 - \rho \right) - b^2 \left( 3 - a_2^2 \right) - 2 \rho a_1^2 + \frac{5 - 3 a_2^2}{a_1^2} \left( v_k B^k \right)^2. \tag{41}
\]

From equation (18), \( b^2 - (v_k B^k)^2 \geq 0 \) and then \( P_1^s \) is always positive. Hence the possible changes of sign of \( P_{fs}^s \) coincide with those of \( P_2^s \). Let us note that the expression for \( P_2^s \) coincides with that of the general case (equation (30)) making \( B = a(v_k B^k) \). Then, proceeding in exactly the same way as in the general case we conclude that the fundamental derivative for relativistic, magnetized fluids for Type I degenerate states is

\[
\tilde{G}_{M, \ deg \ I} = \tilde{G} + \frac{3}{2} W_2^{-4} \left( \frac{\zeta_k}{a_2^2} - 1 - R_{\text{deg} \ I} \right), \tag{42}
\]

where now, \( R_{\text{deg} \ I} = \frac{\zeta_k}{a_2^2} \). As discussed in the non-degenerate case, \( R_{\text{deg} \ I} \neq 1 \), and the corresponding factor is \( F_{\text{deg} \ I} > 0 \).

The special case when \( v_k B^k = 0 \) is obtained by making \( R_{\text{deg} \ I} = 0 \) in the previous expression. The same result for this case is obtained through a purely hydrodynamical

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\(^{13}\) As in the non-degenerate case, for the scalar products \( P_{fs}^s \), the partial derivatives of the corresponding eigenvalues with respect to the primitive variables, \( \mathbf{V} \), have been computed by implicit derivation of the characteristic equation (38).
approach (see appendix in [24]) by building up a thermodynamically consistent EOS incorporating the effects of the magnetic field.

4.2. Type II degeneracy

Now, $\zeta B^k \neq 0$ and, at least, one eigenvalue associated with an Alfvén wave and an eigenvalue associated with a magnetosonic wave are degenerated. Three cases are distinguished. In cases 1 ($c_A > a_s$) and 2 ($c_A < a_s$) one fast or slow magnetosonic eigenvalue, respectively, and an Alfvén eigenvalue are degenerated. In these cases, as discussed in the previous Section, the quantity $P^1_I$ defined in equation (29) is zero for the degenerate eigenvalues and, hence, the corresponding characteristic fields are linearly degenerate. When $c_a = a_s$ (case 3), an Alfvén eigenvalue is degenerated with a pair (slow and fast) of magnetosonic eigenvalues. Now, quantity $d$ defined in equation (28) is also 0, and we have an indetermination in $P^1_m$ (equation (27)). In this case, we have checked that the dot product of the magnetosonic eigenvectors associated with the degenerated fields and the gradient of the Alfvén eigenvalue is zero, which means that the degenerate characteristic field is again linearly degenerate.

5. Purely hydrodynamical and classical limits

The purely (relativistic) hydrodynamical limit can be obtained as a particular case of the Type I degeneracy, in which besides having $\zeta = a_s B^k = 0$ and $v_B = 0$, we make $b^2 = 0$. Hence, from equation (33), and making $R = 0$ and $c_a = 0$, we have $\tilde{G}_{M, b^2=0} = \tilde{G}$. We now discuss the classical (magnetized) limits for both degenerate and non-degenerate states. These limits are obtained by expanding all the quantities in the definition of the fundamental derivative in powers of $c$ ($c$ is the speed of light) and keeping the leading term. On one hand, the relativistic (non-magnetized) fundamental derivative is $\tilde{G} = G + \mathcal{O}(1/c^2)$, where $G$ is the classical (non-magnetized) counterpart [19]. On the other hand, $R = (\zeta_B B^k)^2/(\rho a_{m, cl} c^2) + \mathcal{O}(1/c^2)$, where $a_{m, cl}$ ($m = f, s$) is $a_{m, cl} = \frac{1}{\sqrt{2}}$

$$\left(\frac{a_{s, cl}^2 + B^2/\rho \pm \sqrt{(a_{s, cl}^2 + B^2/\rho)^2 - 4a_{s, cl}^2 (\zeta_B B^k)^2/\rho}}{1 - (\zeta_B B^k)^2/\rho} \right)^{1/2},$$

and $a_{s, cl}$ stands for the classical definition of the sound speed. Hence, we get from equation (33)

$$\tilde{G}_{M, cl} := G + \frac{3}{2} \left( \frac{c_{s, cl}^2/\rho a_{s, cl} B^k}{1 - (\zeta_B B^k)^2/\rho} \right).$$

In the previous expression, $c_{s, cl}$ stand for the classical definition of the Alfvén speed, $\sqrt{B^2/\rho}$.

It can be shown that, taking $\zeta_B = (1, 0, 0)$, the resulting expression of $\tilde{G}_{M, cl}$ is proportional to the nonlinearity factor for the nonlinear fields of the (classical) MHD system obtained in [20] (see their equation (17)).

For Type I degenerate states, since $R = \mathcal{O}(1/c^2)$,

$$\tilde{G}_{M, deg. I, cl} = G + \frac{3}{2} \left( \frac{c_{s, cl}^2}{a_{s, cl} B^k} \right),$$

proportional to the corresponding result obtained in [20] (see their table 1).

Finally for Type II degenerate states, the eigenvalues that are degenerated lead to characteristic fields which are linearly degenerate, whereas the (hypothetical) non-degenerate
magnetosonic field (subcases 1 and 2) is genuinely nonlinear and its properties in relation with convexity are governed by the fundamental derivative in equation (43), with \( c_{m,cl} = c_{s,cl} \) (subcase 1), and \( c_{m,cl} = c_{f,cl} \) (subcase 2).

6. Summary and conclusions

In this paper we have analyzed the influence of the magnetic field in the convexity properties of the RMHD equations. To this purpose we have used the approach of Lax, based on the analysis of the linearly degenerate/genuinely nonlinear nature of the characteristic fields. Degenerate and non-degenerate states have been discussed separately and the non-relativistic, unmagnetized limits are properly recovered. The characteristic fields corresponding to the material and Alfvén waves are linearly degenerate and, then, not affected by the convexity issue. The analysis of the characteristic fields associated with the magnetosonic waves reveals, however, a dependence of the convexity condition on the magnetic field.

The result is expressed in the form of a generalized fundamental derivative, equation (33), written as the sum of two terms. The first one is the generalized fundamental derivative in the case of purely hydrodynamical (relativistic) flow already obtained in [19]. The second one contains the effects of the magnetic field. The analysis of this term in the CF (extendable to any other reference system given the scalar nature of the term) shows that it is always positive leading to the remarkable result that the presence of a magnetic field in the fluid reduces the domain of thermodynamical states for which the EOS is non-convex, as it happens in the non-relativistic MHD limit [20].

We speculate with the possibility that our findings can be relevant in the context of massive stellar core collapse. Depending mostly on the pre-collapse stellar magnetic field and on the gradient of the rotational velocity, dynamically relevant magnetic fields may develop after the core bounce (see, e.g., [25–27]). Should these magnetic fields become as large as the existing numerical models point out, then our results indicate that the loss of convexity would be rather limited, if existing at all. However, it is still a matter of debate what is the actual level of magnetic field saturation due to the action of the magneto rotational instability (MRI; see, e.g., [28, 29]), and hence, whether or not the MRI-amplified magnetic field may have the sufficient strength as to impede the development of non-convex regions in the collapsed core. It is very likely that under the most common conditions (namely, non-rotating or slowly rotating cores), the magnetic field will not play central dynamical role in the post-collapse evolution, though it may set the time scale for supernova explosions (e.g., [30]). In such cases, we foresee that there might exist a range of physical conditions in which a non-convex EOS may render a convexity loss in the post-collapse core that cannot be compensated by the growth of pre-collapse magnetic fields, e.g., in slowly rotating (including non-rotating) massive stellar cores. Addressing this issue by means of numerical simulations is beyond the scope of the present work, and will be considered elsewhere.

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Appendix. Jacobian matrices of the RMHD system in quasi-linear form

Matrices $\mathcal{A}^0$ and $\zeta \, \mathcal{A}^k$ associated with the system (10) in quasilinear form are:

\[
\mathcal{A}^0 = \begin{pmatrix}
W & \rho W^3 v_j & 0 & 0_j \\
(\mathcal{A}^0)^{S}_{\rho} & (\mathcal{A}^0)^{S}_{\nu j} & (\mathcal{A}^0)^{S}_{\epsilon} & (\mathcal{A}^0)^{S}_{B} \\
(\mathcal{A}^0)^{T}_{\rho} & (\mathcal{A}^0)^{T}_{\nu j} & (\mathcal{A}^0)^{T}_{\epsilon} & (\mathcal{A}^0)^{T}_{B} \\
0' & 0_j' & 0' & \delta_j' 
\end{pmatrix},
\]

where

\[
(\mathcal{A}^0)^{S}_{\rho} = (1 + \epsilon + \chi) W^2 \nu, \\
(\mathcal{A}^0)^{S}_{\nu j} = B'B_j + B^2 \delta_j' + h W^2 \left( \delta_j' + 2 W^2 v_j \right), \\
(\mathcal{A}^0)^{S}_{\epsilon} = (\rho + \kappa) W^2 \nu, \\
(\mathcal{A}^0)^{S}_{B} = -\delta_j' v_k B^k - B^2 v_j + 2 \nu B_j, \\
(\mathcal{A}^0)^{T}_{\rho} = (1 + \epsilon) W^2 - W + \chi \left( W^2 - 1 \right), \\
(\mathcal{A}^0)^{T}_{\nu j} = -B_j v_k B^k + v_j \left[ B^2 + \rho W^3 (2hW - 1) \right], \\
(\mathcal{A}^0)^{T}_{\epsilon} = \rho W^2 + \kappa \left( W^2 - 1 \right), \\
(\mathcal{A}^0)^{T}_{B} = -v_j v_k B^k + B_j \left( 2 - 1 / W^2 \right).
\]

\[
\zeta \, \mathcal{A}^k = \begin{pmatrix}
W \zeta_k \nu^k & (\zeta \, \mathcal{A}^k)^D_{\nu j} & 0 & 0_j \\
(\zeta \, \mathcal{A}^k)^{S}_{\rho} & (\zeta \, \mathcal{A}^k)^{S}_{\nu j} & (\zeta \, \mathcal{A}^k)^{S}_{\epsilon} & (\zeta \, \mathcal{A}^k)^{S}_{B} \\
(\zeta \, \mathcal{A}^k)^{T}_{\rho} & (\zeta \, \mathcal{A}^k)^{T}_{\nu j} & (\zeta \, \mathcal{A}^k)^{T}_{\epsilon} & (\zeta \, \mathcal{A}^k)^{T}_{B} \\
0' & B' \zeta_j - \delta_j' \zeta_k B^k & 0' & \delta_j' \zeta_k \nu^k - \nu' \zeta_j 
\end{pmatrix},
\]
where
\[
\begin{aligned}
&\left(\zeta \mathcal{A}^i\right)_D^\rho = \rho W \left( W^2 \nu_j \xi + \zeta_j \right), \\
&\left(\zeta \mathcal{A}^i\right)_D^\rho = (1 + \epsilon + \chi) W^2 \nu_i \xi + \chi \zeta_i, \\
&\left(\zeta \mathcal{A}^i\right)_D^{B^k} = \left( \zeta_j B_j - \delta^i_j \mathcal{A}^i \right) \nu_k B^k + B^2 \left( \delta^i_j \mathcal{A}^i \nu_k - \zeta_j \nu_k + \nu_k \zeta_j \right) \\
&\quad - B^i \left( \zeta_j \nu_k B^k + B_j \mathcal{A}^i - 2\nu_j \mathcal{A}^i + B_j \zeta_k \right) - \nu_i \zeta_j B_k \\
&\quad + \rho h W^2 \left( \delta^i_j \mathcal{A}^i \nu_k + \nu_i \zeta_j + 2W^3 \nu_i \nu_k \zeta_j \right), \\
&\left(\zeta \mathcal{A}^i\right)_D^{\mathcal{E}_k} = \nu_i (\rho + \kappa) W^2 \mathcal{E}_k + \zeta_k \zeta_i, \\
&\left(\zeta \mathcal{A}^i\right)_D^{\mathcal{B}^k} = \zeta_i \nu_j \nu_k B^k - \delta^i_j \nu_k B^k \nu_i - \nu_i \zeta_j \nu_k - B^i \zeta_j \nu_k \\
&\quad - \nu_i \left( \zeta_j \nu_k B^k + \nu_j \mathcal{A}^i + 2\nu_j \mathcal{A}^i \nu_k - 2B_j \zeta_k \right) - W^{-2} \left( \zeta_j \left( hW - 1 \right) + \nu_j \mathcal{A}^i W^2 (2hW - 1) \right), \\
&\left(\zeta \mathcal{A}^i\right)_D^\rho = (1 + \epsilon + \chi) W^2 \mathcal{E}_k - W^2 \mathcal{E}_k \nu_k, \\
&\left(\zeta \mathcal{A}^i\right)_D^{\mathcal{G}_i} = -B_j \zeta_k B^k + B^2 \zeta_j + \rho W \left[ \zeta_j (hW - 1) + \nu_j \mathcal{A}^i W^2 (2hW - 1) \right], \\
&\left(\zeta \mathcal{A}^i\right)_D^{\mathcal{G}_i} = (\rho + \kappa) W^2 \mathcal{E}_k \nu_k, \\
&\left(\zeta \mathcal{A}^i\right)_B^{\mathcal{B}^k} = 2B_j \zeta_k B^k - \nu_j \mathcal{A}^i B^k - \zeta_j \nu_k B^k.
\end{aligned}
\]

All the quantities appearing in the definition of the matrices are defined in the body of the paper and \(0^i = (0, 0, 0)^t, 0_j = (0, 0, 0)\) and \(0_j^i\) is the null 3 × 3 matrix.

References

[1] Anile M A 1989 Relativistic Fluids and Magneto-Fluids (Cambridge: Cambridge University Press)
[2] Kostyuk A, Gorenstein M I, Stöcker H and Greiner W 2001 Phys. Lett. B 500 273
[3] Gulminelli F, Raduta Ad R, Oertel M and Margueron J 2013 Phys. Rev. C 87 055809
[4] Gorenstein M I, Gaździcki M and Greiner W 2005 Phys. Rev. C 72 024909
[5] Gorenstein M I 2012 Phys. Rev. C 87 055809
[6] Dimmelmeier H, Bejger M, Haensel P and Zdunik J L 2009 Mon. Not. R. Astron. Soc. 396 2269
[7] Bombaci I, Logoteta D, Providencia C and Vidaña I 2012 Second Iberian Nuclear Astrophysics Meeting on Compact Stars J. Phys.: Conf. Ser. 342 012001
[8] Peres B, Oertel M and Novak J 2013 Phys. Rev. D 87 043006
[9] Thompson P A 1971 Phys. Fluids 14 1843
[10] Menikoff R and Plohr B J 1989 Rev. Mod. Phys. 61 75
[11] Guardone A, Zamfirescu C and Colonna P 2010 J. Fluid Mech. 642 127
[12] Rezzolla L and Zanotti O 2013 Relativistic Hydrodynamics (Oxford: Oxford University Press)
[13] Cinnella P and Congedo P M 2007 J. Fluid Mech. 580 179
[14] Cinnella P, Congedo P M, Pediroda V and Parussini L 2011 Phys. Fluids 23 116101
[15] lax P D 1975 Commun. Pure Appl. Math. 10 537
[16] LeVeque R J 1992 Numerical Methods for Conservation Laws (Basel: Birkhäuser)
[17] Toro E F 2009 Riemann Solvers and Numerical Methods for Fluid Dynamics: a Practical Introduction 3rd edn (Berlin: Springer)
[18] LeVeque R J 2002 Finite-Volume Methods for Hyperbolic Problems (Cambridge: Cambridge University Press)
[19] Ibáñez J M, Cordero-Carrión I, Martí J M and Miralles J A 2013 Class. Quantum Grav. 30 057002
[20] Serna S and Marquina A 2014 Phys. Fluids 26 016101
[21] Brio M and Wu C C 1988 J. Comput. Phys. 75 400
[22] Antón L, Miralles J A, Martí J M, Ibáñez J M, Aloy M A and Mimica P 2010 Astrophys. J. Suppl. 188
[23] Leismann T, Antón L, Aloy M A, Müller E, Martí J M, Miralles J A and Ibáñez J M 2005 Astron. Astrophys. 436 503
[24] Romero R, Martí J M, Pons J A, Ibáñez J M and Miralles J A 2005 J. Fluid Mech. 544 323
[25] Akiyama S, Wheeler J C, Meier D L and Lichtenstadt I 2003 Astrophys. J. 584 954
[26] Obergaulinger M, Aloy M A, Dimmelmeier H and Müller E 2006 Astron. Astrophys. 457 209
[27] Sawai H, Yamada S and Suzuki H 2013 Astrophys. J. 770 L19
[28] Obergaulinger M, Cerdá-Durán P, Müller E and Aloy M A 2009 Astron. Astrophys. 498 241
[29] Pessah M E 2010 Astrophys. J. 716 1012
[30] Obergaulinger M, Janka H T and Aloy M A 2014 Mon. Not. R. Astron. Soc. 445 3169