Local Relaxation and Collective Stochastic Dynamics

H. Neal Bertram and Xiaobin Wang

Center for Magnetic Recording Research, University of California,
San Diego, 9500 Gilman Drive, La Jolla, Ca 92093-0401

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Abstract

Damping and thermal fluctuations have been introduced to collective normal modes of a magnetic system in recent modeling of dynamic thermal magnetization processes. The connection between this collective stochastic dynamics and physical local relaxation processes is investigated here. A system of two coupled magnetic grains embedded in two separate oscillating thermal baths is analyzed with no a priori assumptions except that of a Markovian process. It is shown explicitly that by eliminating the oscillating thermal bath variables, collective stochastic dynamics occurs in the normal modes of the magnetic system. The grain interactions cause local relaxation to be felt by the collective system and the dynamic damping to reflect the system symmetry. This form of stochastic dynamics is in contrast to a common phenomenological approach where a thermal field is added independently to the dynamic equations of each discretized cell or interacting grain. The dependence of this collective stochastic dynamics on the coupling strength of the magnetic grains and the relative local damping is discussed.
I. INTRODUCTION

A new model has recently been developed [1], [2], [3] to study thermal noise and dynamic thermal reversal in interacting magnetic systems. In this approach, damping and thermal fluctuations are introduced to the independent normal modes of the magnetic system, corresponding to the analogy of temperature defined by independent particles in an ideal gas. The damping term in the dynamic equations differ from that of Landau-Lifshitz [4] and, for even a single domain particle or film, reflects the asymmetry of the magnetic energy [3]. Generalization of the LLG equations to reflect the magnetic symmetry has been discussed in general [5] with specific analysis for the conduction electron relaxation process [6]. Collective normal mode processes have also been examined through analysis of relaxation to the complete spin-wave spectrum in thin films [7]. Here we derive the stochastic differential equations (SDE) specifically for the case of local damping in a system of interacting grains.

Historically, stochastic differential equations have been developed by simply adding a thermal fluctuation field to the Landau-Lifshitz-Gilbert equations [8]. This approach has been widely utilized to analyze the role of thermal fluctuations, for example, in non-uniformly magnetized materials, such as a thin film, by discretizing the film and solving the coupled LLG equations for each cell with a statistically independent random field added to each cell [9], [10]. One argument in favor of this individual particle approach is that for physically localized relaxation processes damping and thermal fluctuations can be conveniently introduced to individual particles as an effective field.

However, from a collective normal mode point of view, even physically localized relaxation processes should give collective stochastic dynamics in the normal modes. Application of this approach to analyze thermal noise in a thin film appears to give better agreement with experiment than the LLG-Brown approach [11]. Here, collective stochastic dynamics are explicitly derived through a system-reservoir interaction model. We consider two coupled magnetic grains embedded in two separate thermal baths, focusing on small amplitude oscillations near equilibrium. No a priori assumption is made concerning the form of the
dynamic damping. This configuration provides a simplified picture for localized relaxation processes. We expand the analysis in [3] for a single grain utilizing the method of [13] to add a generalized thermal bath to the magnetization dynamics. The technique is to explicitly eliminate the oscillating bath variables to obtain a closed stochastic equation for the magnetic system. Under a Markovian approximation it can be shown that the magnetic system obeys collective stochastic dynamics in the form of damped harmonic oscillators in the normal modes. Thus, the original conjecture is verified that damping and additive thermal fluctuations should be added to the normal modes of a magnetic system, even if the physical relaxation processes are local.

Section II introduces our model configuration. Section III obtains the dynamic equations for the magnetic system by explicitly eliminating the bath variables. In Section IV Markovian and rotational wave approximations are utilized to obtain the collective stochastic dynamics for the independent normal modes of a magnetic system. Section V discusses the dependence of this collective stochastic dynamics on the magnetic interaction and a comparison to the individual particle picture is given.

II. TWO INTERACTING GRAINS EMBEDDED IN DIFFERENT LOCAL THERMAL BATHS

We consider two interacting nonidentical cubic magnetic grains of diameter $D$ embedded in two different localized thermal baths, as shown in Fig. 1. Initially, neglecting the coupling to the thermal baths, the energy for the host magnetic grains, normalized by $M_s^2 V$, is:

$$E_m = M_s^2 V \left[ \frac{1}{2} h_{k1} (1 - m_1^2) + \frac{1}{2} h_{k2} (1 - m_2^2) - h_{ex} \vec{m}_1 \cdot \vec{m}_2 \right]$$

$$+ (\vec{m}_1 \cdot \vec{m}_2 - 3m_{1z}m_{2z})$$

(1)

where $\vec{m}_1 = \vec{M}_1/M_s$ and $\vec{m}_2 = \vec{M}_2/M_s$ are normalized magnetizations of the two grains. The normalized anisotropy fields are $h_{k1} = 2K_1/M_s^2$ and $h_{k2} = 2K_2/M_s^2$. The normalized exchange field, assuming an interaction strength $A$ (acting uniformly between grain centers),
is $h_{ex} = 2A/D^2M_s^2$ (although here normalization is by $M_s^2$ rather than by $K$ typical in micromagnetic simulations of recording media [14]). Normalization by $M_s^2V$ yields the magnetostatic term (using a dipole approximation) without any multiplicative constant.

The magnetostatic and exchange interactions are assumed to be small compared to the anisotropy energies $h_{k1}, h_{k2} \gg 1 \gg h_{ex}$. With this assumption the equilibrium state is (Fig.1):

$$\vec{m}_1 = e_z, \quad \vec{m}_2 = e_z$$ (2)

For small excitations around equilibrium ($e_z$), we only need consider second order variations so that:

$$m_{1z} = 1 - \frac{1}{2}(m_{1x}^2 + m_{1y}^2)$$

$$m_{1z}m_{2z} = 1 - \frac{1}{2}(m_{1x}^2 + m_{1y}^2 + m_{2x}^2 + m_{2y}^2)$$ (3)

The magnetic energy of the host grains, to second order, is:

$$E_m = M_s^2V[-\frac{1}{2}(-h_{k1} + 1 - h_{ex})(m_{1x}^2 + m_{1y}^2) - \frac{1}{2}(-h_{k2} + 1 - h_{ex})(m_{2x}^2 + m_{2y}^2)$$

$$+ (1 - h_{ex})m_{1y}m_{2y} - (2 + h_{ex})m_{1x}m_{2x}]$$ (4)

For each grain ($j$), we transform the two magnetization components orthogonal to the equilibrium direction ($m_{jx}, m_{jy}$) into (linearized) rotating magnetization components ($a_j^*, a_j$) (e.g. [12]):

$$a_j = \frac{m_{jx} + im_{jy}}{\sqrt{2}}, \quad a_j^* = \frac{m_{jx} - im_{jy}}{\sqrt{2}}$$ (5)

In these coordinates, again in the lowest order quadratic variation, the magnetic energy (4) can be written as:

$$E_m = M_s^2V[-(-h_{k1} + 1 - h_{ex})a_1a_1^* - (-h_{k2} + 1 - h_{ex})a_2a_2^*$$

$$+ 1/2(1 - h_{ex})(a_1a_2^* + a_2^*a_1 - a_1a_2 - a_1^*a_2^*)$$

$$- 1/2(2 + h_{ex})(a_1a_2 + a_2^*a_1^* + a_1a_2^* + a_2a_1^*)]$$ (6)
In order to consider localized relaxation processes the host magnetic grains are bilinearly coupled to two separate oscillating thermal baths \[13\]. This is a simplified model for local relaxation processes of the interacting magnetization system. A physical example would be relaxation by localized high moment Rare Earth impurities (e.g. see \[15\]). The total energy including magnetic energy, thermal bath energy and interaction energy is:

\[
E_t = E_m + E_b + E_I
\]  (7)

\[
E_m = E_m(a_1, a_1^*, a_2, a_2^*)
\]

\[
E_b = \frac{M_s V}{\gamma} \sum_k \omega_{1k} b_{1k} b_{1k}^* + \frac{M_s V}{\gamma} \sum_k \omega_{2k} b_{2k} b_{2k}^*
\]

\[
E_I = \frac{M_s V}{\gamma} \sum_k g_{1k} (b_{1k} a_1^* + b_{1k}^* a_1) + \frac{M_s V}{\gamma} \sum_k g_{2k} (b_{2k} a_2^* + b_{2k}^* a_2)
\]

where \(b_{jk}, b_{jk}^*\) are the oscillating thermal bath variables. \(g_{1k}, g_{2k}\) represent the coupling strength of the magnetic systems and the thermal baths. Note that the bath terms are in the form of independent harmonic oscillators. Using the transformation \(5\), the interaction term \(E_I\) can be viewed as simply a Zeeman coupling with a thermal field. Here following \[13\], we assume that thermal bath is in equilibrium and the magnetic coupling is only a small perturbation to thermal baths.

### III. MODEL DYNAMIC EQUATIONS

The standard procedure is to obtain a closed dynamic equation in the magnetization variables by explicitly eliminating the thermal bath variables \[13\]. The Hamiltonian equations for the magnetic system are:

\[
\frac{da_j}{dt} = -i \frac{\gamma}{M_s V} \frac{\partial E}{\partial a_j^*}
\]

\[
\frac{da_j^*}{dt} = i \frac{\gamma}{M_s V} \frac{\partial E}{\partial a_j}
\]

which can be written in matrix form as:
\[
\frac{d}{dt} \begin{pmatrix}
    a_1 \\
    a_1^* \\
    a_2 \\
    a_2^*
\end{pmatrix} = iG \begin{pmatrix}
    a_1 \\
    a_1^* \\
    a_2 \\
    a_2^*
\end{pmatrix} + i \begin{pmatrix}
    -\sum_k g_{1k} b_{1k} \\
    \sum_k g_{1k}^* b_{1k}^* \\
    -\sum_k g_{2k} b_{2k} \\
    \sum_k g_{2k}^* b_{2k}^*
\end{pmatrix} \tag{9}
\]

where:

\[
G = \begin{pmatrix}
    (-h_{k1} + 1 - h_{ex}) & 0 & (1 + 2h_{ex})/2 & 3/2 \\
    0 & (-h_{k1} + 1 - h_{ex}) & -3/2 & -(1 + 2* h_{ex})/2 \\
    (1 + 2h_{ex})/2 & 3/2 & (-h_{k2} + 1 - h_{ex}) & 0 \\
    -3/2 & -(1 + 2* h_{ex})/2 & 0 & -(-h_{k2} + 1 - h_{ex})
\end{pmatrix} \gamma M_s \tag{10}
\]

is the matrix for gyromagnetic precession.

In order to obtain a closed equation for magnetic system variables \((a_1, a_1^*, a_2, a_2^*)\), we need to formally represent \((b_{1k}, b_{1k}^*, b_{2k}, b_{2k}^*)\) in terms of \((a_1, a_1^*, a_2, a_2^*)\) in (9). We expect the closed equation for \((a_1, a_1^*, a_2, a_2^*)\) to be a stochastic differential equation under a Markov approximation. However, before starting to represent \((b_{1k}, b_{1k}^*, b_{2k}, b_{2k}^*)\) in terms of \((a_1, a_1^*, a_2, a_2^*)\), we notice that (9) has mixed times because of nondiagonal terms in the matrix (10). Thus, due to intergranular interactions, the distinct magnetic system time scales are not well presented in the rotating magnetization components \((a_1, a_1^*, a_2, a_2^*)\). In the Markov approximation for stochastic dynamics, distinct system time scales must be separated from thermal bath time scales. So we first need to obtain explicitly system time scales [16]. This is done by normal mode analysis (e.g. [12]).

The nondiagonal matrix (10) can be diagonalized into the following form:
Thus, the system gyromagnetic motion alone without a thermal bath or equivalently a relaxation mechanism can be written as:

\[
\frac{d}{dt} \begin{pmatrix}
    c_1 \\
    c_1^* \\
    c_2 \\
    c_2^*
\end{pmatrix} = \begin{pmatrix}
    -i\omega_1 & 0 & 0 & 0 \\
    0 & i\omega_1 & 0 & 0 \\
    0 & 0 & -i\omega_2 & 0 \\
    0 & 0 & 0 & i\omega_2
\end{pmatrix} \begin{pmatrix}
    c_1 \\
    c_1^* \\
    c_2 \\
    c_2^*
\end{pmatrix}
\] (12)

where \(c_1, c_1^*, c_2, c_2^*\) are the system normal modes and distinct system times scales can be determined as \(1/\omega_1\) and \(1/\omega_2\). \(\omega_1\) and \(\omega_2\) are the magnetic system resonant frequencies.

Now we represent total Hamiltonian (4) using normal mode coordinate:

\[
E = \frac{M_s V}{\gamma} [\omega_1 c_1 c_1^* + \omega_2 c_2 c_2^*] + \frac{M_s V}{\gamma} \sum_k \omega_{1k} b_{1k} b_{1k}^* + \frac{M_s V}{\gamma} \sum_k \omega_{2k} b_{2k} b_{2k}^* + \frac{M_s V}{\gamma} \sum \left( c_1 (g_{1k} v_{11} b_{1k} + g_{1k} u_{11} b_{1k}^* + g_{2k} v_{21} b_{2k} + g_{2k} u_{21} b_{2k}^*) \\
    c_1^* (g_{1k} u_{11} b_{1k} + g_{1k} v_{11} b_{1k}^* + g_{2k} u_{21} b_{2k} + g_{2k} v_{21} b_{2k}^*) \\
    c_2 (g_{1k} v_{12} b_{1k} + g_{1k} u_{12} b_{1k}^* + g_{2k} v_{22} b_{2k} + g_{2k} u_{22} b_{2k}^*) \\
    c_2^* (g_{1k} u_{12} b_{1k} + g_{1k} v_{12} b_{1k}^* + g_{2k} u_{22} b_{2k} + g_{2k} v_{22} b_{2k}^*) \right)
\] (13)

It should be pointed out here that diagonalizing the matrix \(G\) in (11) together with the requirement that the normal mode energy in form of \(E = \frac{MV}{\gamma} [\omega_1 c_1 c_1^* + \omega_2 c_2 c_2^*]\) (13) determines the transformation matrix \(U\) uniquely. The dynamic equations for the magnetic system can be obtained from the Hamiltonian:
\[
\frac{dc_i}{dt} = -i \frac{\gamma}{M_s V} \frac{\partial E}{\partial c_i^*}
\]

so that:

\[
\frac{dc_1}{dt} = -i \omega_1 c_1 - i \sum_k \left( g_{1k} u_{11} b_{1k} + g_{1k} v_{11} b_{1k}^* + g_{2k} u_{21} b_{2k} + g_{2k} v_{21} b_{2k}^* \right)
\]

\[
\frac{dc_2}{dt} = -i \omega_2 c_2 - i \sum_k \left( g_{1k} u_{12} b_{1k} + g_{1k} v_{12} b_{1k}^* + g_{2k} u_{22} b_{2k} + g_{2k} v_{22} b_{2k}^* \right)
\]

The dynamic equations for bath variables are:

\[
\frac{db_{ik}}{dt} = -i \frac{\gamma}{M_s V} \frac{\partial E}{\partial b_{ik}^*}
\]

that is:

\[
\frac{db_{1k}}{dt} = -i \omega_1 b_{1k} - i \left( g_{1k} u_{11} c_1 + g_{1k} v_{11} c_1^* + g_{1k} u_{12} c_2 + g_{1k} v_{12} c_2^* \right)
\]

\[
\frac{db_{2k}}{dt} = -i \omega_2 b_{2k} - i \left( g_{2k} u_{21} c_1 + g_{2k} v_{21} c_1^* + g_{2k} u_{22} c_2 + g_{2k} v_{22} c_2^* \right)
\]

Equation (17) can be formally solved as:

\[
b_{1k}(t) = b_{1k}(0) e^{-i \omega_1 t} - i g_{1k} \int_0^t dt' (u_{11} c_1 + v_{11} c_1^* + u_{12} c_2 + v_{12} c_2^*) e^{-i \omega_1 (t-t')} \]

\[
b_{2k}(t) = b_{2k}(0) e^{-i \omega_2 t} - i g_{2k} \int_0^t dt' (u_{21} c_1 + v_{21} c_1^* + u_{22} c_2 + v_{22} c_2^*) e^{-i \omega_2 (t-t')} \]

In order to simplify the calculation, in the following we assume the two thermal baths are identical except for a weighting factor in coupling strength to the magnetic system:

\[
g_{2k} = \beta g_{1k}, \ \omega_{1k} = \omega_{2k}
\]

Substituting (18) into (15), we obtain:
\[
\frac{dc_1}{dt} = -i\omega_1 c_1 - i \sum_k g_{1k} [u_{11} b_{1k}(0) + \beta u_{21} b_{2k}(0)] e^{-i\omega_k t}
\]

\[
-i \sum_k g_{1k} [v_{11} b_{1k}^*(0) + \beta v_{21} b_{2k}^*(0)] e^{i\omega_k t}
\]

\[
- \sum_k g_{1k}^2 \int_0^t dt' e^{-i\omega_k (t-t')} (u_{11}^2 c_1 + u_{11} v_{11} c_1^* + u_{11} u_{12} c_2 + u_{11} v_{12} c_2^*)
\]

\[
- \sum_k \beta^2 g_{1k}^2 \int_0^t dt' e^{-i\omega_k (t-t')} (u_{21}^2 c_1 + u_{21} v_{21} c_1^* + u_{21} u_{22} c_2 + u_{21} v_{22} c_2^*)
\]

\[
+ \sum_k g_{1k}^2 \int_0^t dt' e^{i\omega_k (t-t')} (v_{11}^2 c_1 + v_{11} u_{11} c_1^* + v_{11} v_{12} c_2 + v_{11} u_{12} c_2^*)
\]

\[
+ \sum_k \beta^2 g_{1k}^2 \int_0^t dt' e^{i\omega_k (t-t')} (v_{21}^2 c_1 + v_{21} u_{21} c_1^* + v_{21} v_{22} c_2 + v_{21} u_{22} c_2^*)
\]

and similarly for \(c_2\).

**IV. STOCHASTIC DYNAMIC EQUATIONS**

We assume that the thermal bath is in thermal equilibrium with a continuum density of states:

\[
< b_{ik}(0) >= < b_{ik}^*(0) >= 0
\]

\[
< b_{ik}(0) b_{ik'}^*(0) >= \delta_{kk'} n_k
\]

where \(i = 1, 2\), \(n_k\) is the energy of the \(k\)th component of the oscillating thermal bath, proportional to temperature if the thermal bath is in equilibrium. The summation over discrete energy levels can be approximated by a continuous integration:

\[
\sum_k g_{1k}^2 \rightarrow \int d\omega_k D(\omega_k) g^2(\omega_k)
\]

and where occupation number in (21) is directly proportional to the temperature: \(n_k \sim k_B T\).

We need to single out the magnetic system time scales from the coupled total system time scales. This is done by the following transformation [13]:
Substituting (22) and (23) into (20), we obtain:

\[ \tilde{c}_1(t) = c_1(t)e^{i\omega_1 t}, \quad \tilde{c}_2(t) = c_2(t)e^{i\omega_2 t} \]  

(23)

Substituting (22) and (23) into (20), we obtain:

\[
\frac{d\tilde{c}_1}{dt} = f_1(t) - \int d\omega_k D(\omega_k)g^2(\omega_k) \int_0^t dt' 
\begin{bmatrix}
(u_{11}^2 + \beta^2 u_{21}^2)\tilde{c}_1(t')e^{-i(\omega_k - \omega_1)(t-t')} & (u_{11}v_{11} + \beta^2 u_{21}v_{21})\tilde{c}_1(t')e^{-i(\omega_k + \omega_1)(t-t')}e^{2i\omega_1 t} \\
(u_{11}u_{12} + \beta^2 u_{21}u_{22})\tilde{c}_2(t')e^{-i(\omega_k - \omega_1)(t-t')} & (u_{11}v_{12} + \beta^2 u_{21}v_{22})\tilde{c}_2(t')e^{-i(\omega_k + \omega_1)(t-t')}e^{2i\omega_1 t} \\
-(v_{11}^2 + \beta^2 v_{21}^2)\tilde{c}_1(t')e^{i(\omega_k - \omega_1)(t-t')} & -(v_{11}v_{12} + \beta^2 v_{21}v_{22})\tilde{c}_1(t')e^{i(\omega_k + \omega_1)(t-t')}e^{2i\omega_1 t} \\
-(v_{11}v_{12} + \beta^2 v_{21}v_{22})\tilde{c}_2(t')e^{i(\omega_k + \omega_2)(t-t')} & -(v_{11}v_{12} + \beta^2 v_{21}v_{22})\tilde{c}_2(t')e^{i(\omega_k + \omega_2)(t-t')}e^{2i\omega_2 t}
\end{bmatrix}
\]

where

\[
f_1(t) = -i \sum_k g_{1k}[u_{11}b_{1k}(0) + \beta u_{21}b_{2k}(0)]e^{-i(\omega_k - \omega_1)t}
\]

(25)

\[
-f_1(t) f_1^*(t') = D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)\delta(t-t')
\]

(26)

A similar expression for the second mode can be obtained.

In the Markovian approximation [13], [16], we neglect memory for the slow variable \((\tilde{c}_1(t') \to \tilde{c}_1(t))\). For long times \(t >> 1/(\omega_1 - \omega_k)\) the upper integral limit may be put to infinity and using:

\[
\int_0^\infty dx e^{i\omega x} = \pi \delta(\omega) + iP.V.\left(\frac{1}{\omega}\right)
\]

(27)

only the following terms remain:
Because $e^{2i\omega_1 t}$ is a fast oscillating term for $t >> 1/\omega_1$, the $-(v_{11}u_{11} + \beta^2 v_{21}v_{22}) \tilde{c}_1(t)e^{i(\omega_k - \omega_1)(t-t')}e^{2i\omega_1 t}$ term can be included into the fluctuating term. This is the rotational wave approximation [16]. If the gyromagnetic system is not degenerate ($\omega_1 \neq \omega_2$) and the gyromagnetic rotating frequency gap is sufficiently large $t >> 1/(\omega_1 - \omega_2)$, $e^{i(\omega_1 - \omega_2)t}$ is also a fast oscillating term and the $(u_{11}u_{12} + \beta^2 u_{21}u_{22}) \tilde{c}_2(t)e^{-i(\omega_k - \omega_1)(t-t')}e^{i(\omega_1 - \omega_2)t}$ term also enters into the fluctuating terms. As discussed in section V, for a wide range of parameters, even for identical grains, nondegenerate gyromagnetic rotation is guaranteed.

The following stochastic differential equation for $\tilde{c}_1(t)$ is obtained:

$$\frac{d\tilde{c}_1}{dt} = \tilde{F}_1(t) - D(\omega_1)g^2(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2) \tilde{c}_1(t)$$

where the damping term is defined as:

$$\eta_1 = D(\omega_1)g^2(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)$$

Weak interactions between the system and the thermal reservoir is usually assumed for a Markovian approximation and this corresponds to $\eta_1 << 1$. The thermal fluctuation term(29) is:

$$\tilde{F}_1(t) = -i \sum_k \{[u_{11}b_{1k}(0) + \beta u_{21}b_{2k}(0)]e^{-i(\omega_k - \omega_1)t}$$

$$+ [v_{11}b_{1k}^*(0) + \beta v_{21}b_{2k}^*(0)]e^{i(\omega_k + \omega_1)t}\}$$

$$- \int d\omega_k D(\omega_k)g^2(\omega_k) \int_0^\infty dt(u_{11}u_{12} + \beta^2 u_{21}u_{22}) \tilde{c}_2(t)e^{-i(\omega_k - \omega_1)(t-t')}e^{i(\omega_1 - \omega_2)t}$$

$$+ \int d\omega_k D(\omega_k)g^2(\omega_k) \int_0^\infty dt'(v_{11}u_{11} + \beta^2 v_{21}v_{22}) \tilde{c}_1^*(t)e^{i(\omega_k - \omega_1)(t-t')}e^{2i\omega_1 t}$$
Notice that the last two terms are from the fast oscillating terms and have the magnitude proportional to $\eta_1$. The thermal fluctuation has zero mean and, to leading order (neglecting $\eta_2^2$), the variance is:

$$<\tilde{F}_1(t)\tilde{F}^*_1(t')>=D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)\delta(t-t')$$  \hspace{1cm} (33)

A similar equation for $\tilde{c}_2(t)$ can be obtained based upon the same conditions:

$$\frac{d\tilde{c}_2}{dt} = \tilde{F}_2(t) - \eta_2\tilde{c}_1(t)$$ \hspace{1cm} (34)

To summarize, the thermal fluctuation in the two modes have to leading order the following properties:

$$<\tilde{F}_1(t)> = 0, <\tilde{F}_2(t)> = 0$$ \hspace{1cm} (35)

$$<\tilde{F}_1(t)\tilde{F}^*_1(t')>=D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)\delta(t-t')$$

$$<\tilde{F}_2(t)\tilde{F}^*_2(t')>=D(\omega_2)g^2(\omega_2)n(\omega_2)(\beta^2 u_{22}^2 + u_{12}^2)\delta(t-t')$$

$$<\tilde{F}_1(t)\tilde{F}^*_2(t')>=0$$

The last condition in (35) gives uncorrelated thermal fluctuations in the two normal modes. It should be pointed out that in principle the fast oscillating terms in the thermal noise could result in correlation between $\tilde{F}_1(t)$ and $\tilde{F}_2(t)$. However, those terms are of order $\eta^2$ and can be neglected for a consistent Markovian approximation with weak interactions between the magnetic system and thermal reservoir.

Transforming (29) and (34) into $c_1(t)$ and $c_2(t)$ coordinates, we obtain the following stochastic equations for the collective normal modes of the interacting magnetic system:
\[ \frac{dc_1}{dt} + \eta_1 c_1 = -i\omega_1 c_1 + f_1(t) \]  
\[ \frac{dc_2}{dt} + \eta_2 c_2 = -i\omega_2 c_2 + f_2(t) \]  
\[ \eta_1 = D(\omega_1)g^2(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2) \]  
\[ \eta_2 = D(\omega_2)g^2(\omega_2)(\beta^2 u_{22}^2 + u_{12}^2) \]  
\[ < f_1(t) >= 0, < f_2(t) >= 0 \]  
\[ < f_1(t)f_1^*(t') >= D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)\delta(t-t') \]  
\[ < f_2(t)f_2^*(t') >= D(\omega_2)g^2(\omega_2)n(\omega_2)(\beta^2 u_{22}^2 + u_{12}^2)\delta(t-t') \]  
\[ < f_1(t)f_2^*(t') >= 0 \]

Thus, this analysis without any a priori assumptions has given stochastic differential equations for the two independent normal modes in the form of independent damped harmonic oscillators driven by uncorrelated thermal fluctuations. Note that the damping terms \( \eta_1 (\omega_1) \) and \( \eta_2 (\omega_2) \) are directly proportional to thermal variance terms \( < f_1(t)f_1^*(t') >= D_{c1} \) and \( < f_2(t)f_2^*(t') >= D_{c2} \) respectively. The ratio of fluctuation magnitudes in two modes are:

\[ \frac{D_{c1}}{D_{c2}} = \frac{D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2 u_{21}^2)}{D(\omega_2)g^2(\omega_2)n(\omega_2)(\beta^2 u_{22}^2 + u_{12}^2)} \]  

For a magnetic system that satisfies a canonical distribution around equilibrium, the magnitude of the damping coefficients \( \eta_1 \) and \( \eta_2 \) are related to the thermal fluctuations terms \( D_{c1} \) and \( D_{c2} \) through the fluctuation-dissipation condition [13]:

\[ \eta_1 = \frac{D_{c1}M_sV}{\gamma k_B T} \omega_1 = \alpha q_1 \omega_1 \]  
\[ \eta_2 = \frac{D_{c2}M_sV}{\gamma k_B T} \omega_2 = \alpha q_2 \omega_2 \]

where \( \eta_1 \) and \( \eta_2 \) are damping rate.

V. DISCUSSION

We have shown here (36) that local physical relaxation mechanisms give collective stochastic dynamics in the system normal modes for interacting magnetic grains or con-
tinuum discretization cells. For a given system the specific damping terms can be evaluated using (38). These results can be generalized for any system of $N$ interacting magnetic sub-units. The general operative approach is to first diagonalize the $(2N \times 2N)$ matrix of the gyromagnetic precession (near equilibrium) without any damping terms to find the collective modes and their eigenvalues or resonance frequencies. Following that, using expanded forms of (38), damping and thermal fields are added to each of the collective mode dynamics. The total energy in each mode is $k_B T$ and the fluctuation-dissipation theorem relates the variance of the thermal term to the damping. As will be shown in the specific examples below, there is no contradiction between local relaxation mechanisms and collective stochastic dynamics. The interactions between the elements allow damping in one cell to be felt by all the others. The interactions also cause the dynamics to reflect the overall sample geometry or anisotropy.

Here we evaluate some examples for this two-grain system. The most general case of the dependence of the collective stochastic dynamics on interaction strength can be obtained from (11). We begin with the results for a system of identical grains with no exchange ($h_{ex} = 0$). The mode resonance frequencies are:

$$\frac{\omega_1}{\gamma H_{K1}} = \sqrt{1 - \frac{3 M_s}{H_{K1}}}$$

$$\frac{\omega_2}{\gamma H_{K1}} = \sqrt{1 - \frac{M_s}{H_{K1}} - 2 \left( \frac{M_s}{H_{K1}} \right)^2}$$

In Fig.2a these normalized frequencies are potted versus interaction strength $M_s/H_{K1}$. With weak or vanishing magnetostatic interactions the frequencies are simply that of independent grains with $\omega_1 = \gamma H_{K1}$ and $\omega_2 = \gamma H_{K2} = H_{K1}$. With increasing interaction both resonance frequencies decrease. The modes in zero field are asymmetric coherent rotation (mode 1) and asymmetric fanning (mode 2) [17]. The lowest frequency mode is almost coherent, because there is less magnetostatic energy to rotate into the particle axis direction ($x$: Fig.1). As the magnetostatic interaction is increased the energy barrier (or quadratic curvature) decreases and at $M_s/H_{K1} \approx 0.33$ the frequency vanishes and the equilibrium direction
moves from the “z” direction to be along the line joining the particles (the “x” direction). For the higher energy mode the frequency $\omega_2$ also decreases with interaction strength and eventually the energy barrier will vanish ($\omega_2 \rightarrow 0$), but at a higher $M_s/H_{K1} = 0.5$ due to the larger magnetostatic energy of that mode.

In Fig.2b normalized resonance frequencies are plotted versus exchange $h_{ex}$ for the case $M_s/H_{K1} = 0.09$ and $H_{K2} = 1.95H_{K1}$. At $h_{ex} = 0$ the resonance frequencies are almost the ratio of the $H_K$ values. The quasi-coherent mode frequency ($\omega_1$) hardly varies with exchange, as expected, since coherent motion does not involve exchange energy. The incoherent mode frequency $\omega_2$ increases rapidly with exchange because of the increased exchange energy of that non-uniform mode. We emphasize that the stochastic dynamic modeling in this paper is for $\omega_2 \neq \omega_1$ (and sufficiently different). However, the plots shown here indicate that for almost all cases of interest, this condition holds, even for identical particles with finite coupling.

We now explore the variation of the mode damping parameters with magnetostatic interaction for $h_{ex} = 0$, as in Fig.2a and with slightly differing anisotropy ($H_{K2} = 1.001H_{K1}$). As can be shown using (11), for $M_s/H_{K1} = 0$, we have $u_{11} = u_{22} = 1$, $v_{11} = v_{22} = 0$, $u_{21} = v_{21} = 0$, $u_{12} = v_{12} = 0$. The normal modes are just those of the individual particles: $(c_1 = a_1, c_2 = a_2)$ and the damping terms (36) are just the local damping for the individual grains:

$$\eta_1 = D(\omega_1)g^2(\omega_1)$$

$$\eta_2 = \beta^2 D(\omega_2)g^2(\omega_2)$$

In Fig.3a we plot normalized damping terms $\eta_1/D(\omega_1)g^2(\omega_1)n(\omega_1), \eta_2/D(\omega_2)g^2(\omega_2)n(\omega_2)$ versus magnetostatic interaction $M_s/H_{K1}$ for the case of $\beta = 1$. For $M_s/H_{K1} = 0$, the damping terms are just those of the independent grains. As seen in (38) the damping terms are proportional to the mode frequencies and that variation is seen in Fig.3a. It is noteworthy that for identical damping mechanisms and a finite grain interaction, the collective modes are not identical.
The case for \( \beta = 0 \), corresponding to relaxation only in one cell, is shown in Fig. 3b. In this example with no interactions \((M_s/H_{K1} = 0 \text{ and } h_{ex} = 0)\), \( \eta_2 = 0 \) corresponds to no thermal bath coupling for grain 2. For a very slight non-uniformity \( (H_{K2} = 1.001H_{K1}) \), as \( M_s/H_{K1} \) increases, the normal modes become a combination of the individual particles and for \((H_{K2} - H_{K1})/H_{K1} = 0.001\), this occurs when the magnetostatic field surpasses the difference of the anisotropy fields \((M_s/H_{K1} \approx 0.001)\). For \( M_s/H_{K1} > 0.001 \) the normalized damping terms become almost equal as the coupling dominates \((\eta_1 \approx \eta_2 \approx 0.5)\) and then decrease following the frequency dependence in (38).

In Fig. 4 the damping terms are plotted versus exchange for a fixed magnetostatic interaction \( M_s/H_{K1} = 0.09 \) and \( H_{K2} = 1.95H_{K1} \), as in Fig. 2b. In this case \( \beta^2 = 0.25 \), corresponding to a weaker relaxation in grain 2 compared to grain 1. The initial values for \( h_{ex} = 0 \) reflect the frequency dependence of the damping, between the values as seen in Figs. 3a, 3b. With increasing exchange the damping of mode 1 hardly varies and the damping of mode 2 increases rapidly, following the results of Fig. 2b and the accompanying discussion.

As a comparison of the results of this analysis with the conventional LLG-Brown approach, following [3], we compare the spectra of thermal white noise excitation of this system. We take the case of damping only in mode 1 \((\beta = 0)\). The analysis is in Appendix I. For \( M_s/H_{K1} = 0.25, H_{K2} = H_{K1} \), and \( h_{ex} = 0.5 \), following Fig. 4 for the collective analysis, we use \( \eta_2/\eta_1 = 1.5 \). For the LLG-Brown equations we use \( \alpha_1 = 0.1, \alpha_2 = 0 \). Both curves have been normalized by matching the first resonance peak and linewidth (by adjusting \( \eta_2/\alpha_1 \)). Both curves exhibit the same resonance frequencies for the two modes; the resonance frequencies are (to first order) independent of damping. The spectral shapes differ, however. As in [3], the low frequency PSD of the LLG approach is 3-4 dB above that of the collective result (greater than 3 dB because of the proximity of the two resonance frequencies). The second mode peak is lower and broader for the collective model.

In this work the problem of introducing damping and thermal fluctuations for an interacting magnetic system is addressed using a physical model of system-reservoir interactions. No a priori assumption is made concerning the form of the dynamic damping. This ap-
The approach is quite different from the LLG-Brown approach [8], where a dynamic damping is assumed in the LLG format for each individual grain or discretization cell. In the LLG-Brown approach, the underlying physical processes of damping and thermal fluctuations are not explicitly considered. There have been two papers [18], [19] that have attempted to justify the LLG-Brown model for interacting magnetic units. In [18] it is argued that the LLG-Brown approach gives dynamic equations that can be cast in the form of generalized Newtonian dynamics (specifically in the form of an RLC circuit of coupled oscillators). However, this argument is only inferential and is not derived using any basic physics model. The result is equivalent to assuming the application of LLG-Brown to individual grains or discretized cells of a continuous medium, as is also the essence of [19]. These analyses do not derive a stochastic differential equation with damping and thermal fluctuating terms from system-reservoir interactions.

The arguments in [18], [19] for independent thermal fluctuations for independent particles in an interacting magnetic system is, in fact, a thermodynamic consistent condition required for localized LLG damping for individual magnetic units. However, in this paper, we have explicitly constructed a system-reservoir model without any \textit{a priori} assumption of the damping format or thermal fluctuations. The explicit elimination of thermal reservoir variables in the model gives stochastic dynamic equations of a damped harmonic oscillator driven by thermal noise only in the collective normal modes. As we show in Fig.5, different thermal noise power spectra are obtained. The analysis presented here verifies that damping and additive fluctuations must be added to the normal modes of an interacting magnetic system, even if the physical relaxation processes are local.

\textbf{VI. CONCLUSION}

A fundamental analysis of the stochastic dynamic equations for local coupling to a thermal bath has been performed for a system of two interacting grains. Here no \textit{a priori} assumptions about the form of the dynamic damping term have been assumed. The results
are in the form of damped harmonic oscillators driven by thermal fields in the collective modes of the system, a result that previously was derived rigorously for a single grain with anisotropic energy variations about equilibrium. The LLG-Brown formalism in which a thermal field is added to each grain or discretization cell of a continuous medium is shown not to apply. Noise power spectra are evaluated and it is shown that the two resonance frequencies are broadened by the collective damping terms. Localized relaxation is felt by the system collective stochastic dynamics due to the intergranular interactions. The form of the dynamic damping term, also due to the interactions, reflects the overall system symmetry. For approximately identical grains these damping constants are about equal even if the physical damping occurs in only one of the grains. Noise power spectra are shown which give significant differences for the two models, thus providing guidance to future experimental analysis.

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VII. APPENDIX I

The stochastic differential equations (36) for two independent normal modes can be written as:

\[
\frac{dc_1}{dt} + \eta_1 c_1 = -i\omega_1 c_1 + f_1(t) \\
\frac{dc_2}{dt} + \eta_2 c_2 = -i\omega_2 c_2 + f_2(t) \\
< f_1(t)f_1^*(t') > = 2D_{c1} \\
< f_2(t)f_2^*(t') > = 2D_{c2}
\]

where the ratio of fluctuation magnitudes is:

\[
\frac{D_{c1}}{D_{c2}} = \frac{D(\omega_1)g^2(\omega_1)n(\omega_1)(u_{11}^2 + \beta^2u_{21}^2)}{D(\omega_2)g^2(\omega_2)n(\omega_2)(\beta^2u_{22}^2 + u_{12}^2)}
\]
For magnetic systems that satisfy a canonical distribution around equilibrium, the magnitude of damping coefficients $\eta_1$ and $\eta_2$ are related to the thermal fluctuations $D_{c1}$ and $D_{c2}$ through the fluctuation-dissipation condition:

$$
\eta_1 = \frac{D_{c1}M_sV}{\gamma k_B T} \omega_1 = \alpha Q_1 \omega_1 \quad (43)
$$

$$
\eta_2 = \frac{D_{c2}M_sV}{\gamma k_B T} \omega_2 = \alpha Q_2 \omega_2
$$

The spectra of the normal modes can be calculated based upon (41). Here the calculation is done in a non-dimensional format and the frequency is normalized by $\gamma M_s$: $\tilde{\omega} = \omega / \gamma M_s$.

The spectral densities for two modes are:

$$
S_1(\tilde{\omega}) = <c_1^*(\tilde{\omega})c_1(\tilde{\omega})> = \left| \frac{1}{(-j\tilde{\omega} + j\tilde{\omega}_1 - \alpha Q_1 \tilde{\omega}_1)} \right|^2 \frac{2\alpha Q_1 k_B T}{M_s^2 V} \quad (44)
$$

$$
S_2(\tilde{\omega}) = <c_2^*(\tilde{\omega})c_2(\tilde{\omega})> = \left| \frac{1}{(-j\tilde{\omega} + j\tilde{\omega}_2 - \alpha Q_1 \tilde{\omega}_2)} \right|^2 \frac{2\alpha Q_2 k_B T}{M_s^2 V}
$$

The magnetization can be represented by the normal modes as:

$$
\begin{pmatrix}
m_{1x} \\
m_{2x} \\
m_{1y} \\
m_{2y}
\end{pmatrix} = P
\begin{pmatrix}
c_1 \\
c_1^* \\
c_2 \\
c_2^*
\end{pmatrix}
\quad (45)
$$

where:

$$
P = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\
-i/\sqrt{2} & i/\sqrt{2} & 0 & 0 \\
0 & 0 & -i/\sqrt{2} & i/\sqrt{2}
\end{pmatrix}
\quad (46)
$$

The spectral density function for the magnetization is defined as:

$$
S(\omega) = <\bar{m}(\omega)^+ \bar{m}(\omega)>
\quad (47)
$$
where + denotes conjugate transpose and $\overline{m} = [m_{1x}, m_{2x}, m_{1y}, m_{2y}]$. The magnetization spectral density is related to the spectral density of the normal modes through the transformation matrix:

$$TR = U^+ P^+ PU$$  \hspace{1cm} (48)$$

Thus, the magnetization spectral density is:

$$S(\tilde{\omega}) = [TR_{11}S_1(\tilde{\omega}) + TR_{22}S_1(\tilde{\omega}) + TR_{33}S_2(\tilde{\omega}) + TR_{44}S_2(\tilde{\omega})]$$

$$= \left[ \frac{\alpha Q_1 (TR_{11} + TR_{22})}{|(-j\tilde{\omega} + j\tilde{\omega}_1 + \alpha Q_1 \tilde{\omega}_1)|^2} + \frac{\alpha Q_2 (TR_{33} + TR_{44})}{|(-j\tilde{\omega} + j\tilde{\omega}_2 + \alpha Q_2 \tilde{\omega}_1)|^2} \right] \frac{2k_B T}{M_s^2 V}$$  \hspace{1cm} (49)$$

For the LLG model, damping and thermal fluctuations are added to individual particles. For each individual particle:

$$\frac{d\overline{M}}{dt} = -\gamma \overline{M} \times \overline{H} - \alpha \gamma \frac{\overline{M} \times (\overline{M} \times \overline{H})}{M_s}$$  \hspace{1cm} (50)$$

where $\overline{H}$ includes the effective field $-\partial E/\partial \overline{M}$ and the thermal fluctuating fields $\overline{H}^T$. $\overline{H}^T$ satisfies the fluctuation-dissipation condition: $< H_i^T(t) >= 0$ and $< H_i^T(t)H_j^T(t') >= (2\alpha k_B T/\gamma M_s V) \delta(t - t') \delta_{ij}$. The normalized LLG equations to leading order are:

$$\frac{d}{dt} \begin{pmatrix} m_{1x} \\ m_{2x} \\ m_{1y} \\ m_{2y} \end{pmatrix} = \gamma M_s A \begin{pmatrix} m_{1x} \\ m_{2x} \\ m_{1y} \\ m_{2y} \end{pmatrix} + \gamma \begin{pmatrix} H_{1y}^T \\ H_{2y}^T \\ -H_{1x}^T \\ -H_{1y}^T \end{pmatrix}$$  \hspace{1cm} (51)$$

where:

$$A = \begin{pmatrix} \alpha_1(-h_{k1} + 1 - h_{ex}) & \alpha_1(2 + h_{ex}) & -h_{k1} + 1 - h_{ex} & -(1 - h_{ex}) \\ \alpha_2(2 + h_{ex}) & \alpha_2(-h_{k2} + 1 - h_{ex}) & -(1 - h_{ex}) & -h_{k2} + 1 - h_{ex} \\ -(h_{k1} + 1 - h_{ex}) & -(2 + h_{ex}) & \alpha_1(-h_{k1} + 1 - h_{ex}) & -\alpha_1(1 - h_{ex}) \\ -(2 + h_{ex}) & -(h_{k2} + 1 - h_{ex}) & -\alpha_2(1 - h_{ex}) & \alpha_2(-h_{k2} + 1 - h_{ex}) \end{pmatrix}$$  \hspace{1cm} (52)$$
and $\alpha_1$ and $\alpha_2$ are different damping parameters for the two particles. The nonzero correlations between fluctuation terms are:

$$< H_{1y}^T(t)H_{1y}^T(t') > = \frac{2\alpha_2 k_B T}{\gamma M_s V} \delta(t - t')$$
$$< H_{2y}^T(t)H_{2y}^T(t') > = \frac{2\alpha_2 k_B T}{\gamma M_s V} \delta(t - t')$$
$$< H_{1x}^T(t)H_{1x}^T(t') > = \frac{2\alpha_1 k_B T}{\gamma M_s V} \delta(t - t')$$
$$< H_{2x}^T(t)H_{2x}^T(t') > = \frac{2\alpha_1 k_B T}{\gamma M_s V} \delta(t - t')$$

Using (51), (52), we can calculate the nondimensionalized correlation matrix as:

$$Cor(\tilde{\omega}) = \frac{1}{(j\tilde{\omega}I - A^+)} \cdot \frac{1}{(-j\tilde{\omega}I - A)}$$

where $I$ is the unit matrix. The spectral density can be obtained from the correlation matrix and the fluctuation magnitudes (53):

$$S(\tilde{\omega}) = [\alpha_1 Cor_{11} + \alpha_2 Cor_{22} + \alpha_1 Cor_{33} + \alpha_2 Cor_{44}] \frac{2k_B T}{M_s^2 V}$$
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