BOUNDEDNESS OF MULTILINEAR PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN THE HÖRMANDER CLASS $S_{0,0}$

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Abstract. The multilinear pseudo-differential operators with symbols in the multilinear Hörmander class $S_{0,0}$ are considered. A complete identification of the cases where those operators define bounded operators between local Hardy spaces is given. Some results for the boundedness between Wiener amalgam spaces are also given. These are extensions and improvements of the results known in the bilinear case.

1. Introduction

1.1. Introduction and main result. Throughout this paper, we fix positive integers $n$ and $N$. We consider $N$-linear pseudo-differential operators acting on functions on $\mathbb{R}^n$.

If $\sigma = \sigma(x,\xi_1,\ldots,\xi_N)$, $(x,\xi_1,\ldots,\xi_N) \in (\mathbb{R}^n)^{N+1}$, is a measurable function on $(\mathbb{R}^n)^{N+1}$ satisfying the estimate $|\sigma(x,\xi_1,\ldots,\xi_N)| \leq c(1 + |\xi_1| + \cdots + |\xi_N|)^L$ with some $c$ and $L \in \mathbb{R}$, then the $N$-linear pseudo-differential operator $T_{\sigma}$ is defined by

$$T_{\sigma}(f_1,\ldots,f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix \cdot (\xi_1 + \cdots + \xi_N)} \sigma(x,\xi_1,\ldots,\xi_N) \prod_{j=1}^N \hat{f}_j(\xi_j) \, d\xi_1 \cdots d\xi_N,$$

where $x \in \mathbb{R}^n$, $f_1,\ldots,f_N \in S(\mathbb{R}^n)$, and $\hat{f}_j$ denotes the Fourier transform. The function $\sigma$ is called the symbol of the operator $T_{\sigma}$. If $\sigma(x,\xi_1,\ldots,\xi_N)$ does not depend on the variable $x$, then the function $\sigma = \sigma(\xi_1,\ldots,\xi_N)$ is called the multiplier and $T_{\sigma}$ is called the $N$-linear Fourier multiplier operator.

We shall be interested in the boundedness of the $N$-linear pseudo-differential operators. For the boundedness of $T_{\sigma}$, we use the following terminology. Let $X_1,\ldots,X_N$, and $Y$ be function spaces on $\mathbb{R}^n$ equipped with quasinorms $\| \cdot \|_{X_j}$ and $\| \cdot \|_Y$, respectively. If there exists a constant $C$ for which the inequality

$$\|T_{\sigma}(f_1,\ldots,f_N)\|_Y \leq C \prod_{j=1}^N \|f_j\|_{X_j}$$

holds for all $f_j \in S \cap X_j$, then, with a slight abuse of terminology, we say that $T_{\sigma}$ is bounded from $X_1 \times \cdots \times X_N$ to $Y$ and write $T_{\sigma} : X_1 \times \cdots \times X_N \to Y$. The smallest constant $C$ of (1.1) is denoted by $\|T_{\sigma}\|_{X_1 \times \cdots \times X_N \to Y}$. If $A$ is a class of symbols, we denote by $\text{Op}(A)$ the class of all operators $T_{\sigma}$ corresponding to $\sigma \in A$. If $T_{\sigma} : X_1 \times \cdots \times X_N \to Y$ for all $\sigma \in A$, then we write $\text{Op}(A) \subset B(X_1 \times \cdots \times X_N \to Y)$.

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Notice that $T_\sigma$ is originally defined for $f_j \in S(\mathbb{R}^n)$. If $T_\sigma : X_1 \times \cdots \times X_N \to Y$ in the sense given above, then, in many cases, we can employ some limiting argument to extend the definition of $T_\sigma$ to general $f_j \in X_j$ and prove that the inequality (1.1) holds for all $f_j \in X_j$.

In this paper, we shall consider the boundedness of $T_\sigma$ from $h^{p_1} \times \cdots \times h^{p_N}$ to $h^p$, where $h^r$ denotes the local Hardy space of Goldberg [15]. We shall also consider function spaces $L^p$ (the Lebesgue space), $H^p$ (the Hardy space), $BMO$, $bmo$ (the local BMO space), and the Wiener amalgam spaces $W_s^{p,q}$. The definitions of these function spaces will be given in Subsection 1.3 and Section 2.

It is known that if $p, p_1, \ldots, p_N \in (0, \infty]$ and for every $\sigma \in C^\infty_c((\mathbb{R}^n)^N)$ the Fourier multiplier operator $T_\sigma$ is bounded from $h^{p_1} \times \cdots \times h^{p_N}$ to $h^p$ then $1/p = 1/p_1 + \cdots + 1/p_N$; see [17] Chapter 7, Proposition 7.3.7, where the assertion is given for $L^p$ spaces but the proof can be modified to cover the case of $h^p$ spaces. Thus we shall consider the boundedness $T_\sigma : h^{p_1} \times \cdots \times h^{p_N} \to h^p$ only for $p, p_1, \ldots, p_N \in (0, \infty]$ that satisfy $1/p \leq 1/p_1 + \cdots + 1/p_N$.

We shall consider the class of symbols defined as follows.

**Definition 1.1.** For $m \in \mathbb{R}$, the class $S_{0,0}^m(\mathbb{R}^n, N)$ is defined to be the set of all $C^\infty$ functions $\sigma = \sigma(x, \xi_1, \ldots, \xi_N)$ on $(\mathbb{R}^n)^{N+1}$ that satisfy the estimate

$$\left| \partial_x^\alpha \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_N}^{\beta_N} \sigma(x, \xi_1, \ldots, \xi_N) \right| \leq C_{\alpha,\beta_1,\ldots,\beta_N}(1 + |\xi_1| + \cdots + |\xi_N|)^m$$

for all multi-indices $\alpha, \beta_1, \ldots, \beta_N \in \mathbb{N}^n_0 = \{0, 1, 2, \ldots\}^n$.

The class $S_{0,0}^m(\mathbb{R}^n, N)$ is an $N$-linear version of the well-known Hörmander class considered in the theory of linear pseudo-differential operators. The class $S_{0,0}^m(\mathbb{R}^n, N)$ for $N = 2$ was considered by Bényi–Torres [3, 4] and investigated by Bényi–Maldonado–Naibo–Torres [2] and Bényi–Bernicot–Maldonado–Naibo–Torres [1]. See these papers for the basic properties of the class $S_{0,0}^m(\mathbb{R}^n, 2)$ including symbolic calculus, duality, and interpolation.

We shall consider this problem: for $N \geq 2$ and for given $p, p_1, \ldots, p_N \in (0, \infty]$, identify those $m \in \mathbb{R}$ such that $\text{Op}(S_{0,0}^m(\mathbb{R}^n, N)) \subset B(h^{p_1} \times \cdots \times h^{p_N} \to h^p)$. The origin of this problem may go back to Bényi–Torres [4], where the authors proved that for any $1 \leq p, p_2, p < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$ there exists a multiplier in $S_{0,0}^0(\mathbb{R}^n, 2)$ for which the corresponding bilinear Fourier multiplier operator is not bounded from $L^{p_1} \times L^{p_2}$ to $L^p$. Thus in order to have the relation $\text{Op}(S_{0,0}^m(\mathbb{R}^n, 2)) \subset B(L^{p_1} \times L^{p_2} \to L^p)$ the number $m$ must be negative.

In the case $N = 2$, after the works of Michalowski–Rule–Staubach [25] and Bényi–Bernicot–Maldonado–Naibo–Torres [1], the following theorem was proved by the second and the third named authors of the present paper.

**Theorem A** ([28]). Let $0 < p, p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $m \in \mathbb{R}$. Then the boundedness

$$\text{Op}(S_{0,0}^m(\mathbb{R}^n, 2)) \subset B(h^{p_1} \times h^{p_2} \to h^p),$$

where if $p_1, p_2$, or $p$ is equal to $\infty$ then the corresponding $h^{p_1}$, $h^{p_2}$, or $h^p$ should be replaced by $bmo$, holds if and only if

$$m \leq -n \left( \max \left\{ \frac{1}{2}, \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{p_2}, \frac{1}{p_2} - \frac{1}{p_1} - \frac{1}{2} \right\} \right).$$

The purpose of the present paper is to generalize this theorem to the case $N \geq 2$ and remove the restriction $1/p = 1/p_1 + 1/p_2$. Our result also gives an improvement of Theorem A in the case $p = \infty$; we prove that $\text{Op}(S_{0,0}^m(\mathbb{R}^n, 2)) \subset B(bmo \times bmo \to L^\infty)$, whereas Theorem A gives the weaker result $\text{Op}(S_{0,0}^{-n}(\mathbb{R}^n, 2)) \subset B(bmo \times bmo \to bmo)$. 
The main result of the present paper reads as follows.

**Theorem 1.2.** Let \( N \geq 2, 0 < p, p_1, \ldots, p_N \leq \infty, 1/p \leq 1/p_1 + \cdots + 1/p_N, \) and \( m \in \mathbb{R}. \) Then the boundedness

\[
\text{Op}\left(S_{0,0}^m(\mathbb{R}^n, N)\right) \subset B(h^{p_1} \times \cdots \times h^{p_N} \to h^p)
\]

holds if and only if

\[
m \leq \min\left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max\left\{ \frac{n}{p_j}, \frac{n}{2} \right\}.
\]

If (1.4) is satisfied and if some of the \( p_j \)’s are equal to \( \infty, \) then (1.3) holds with the corresponding \( h^{p_j} \) replaced by \( \text{bmo}. \)

Observe that if \( N = 2 \) and \( 1/p = 1/p_1 + 1/p_2 \) then (1.4) coincides with (1.2).

**Remark 1.3.** (1) In the case \( N = 1, \) the operator \( T_m \) is a linear pseudo-differential operator and the class \( S_{0,0}^m(\mathbb{R}^n, 1) \) is the usual Hörmander class. For this case, the following assertion holds. Let \( 0 < p_1 \leq p \leq \infty \) and \( m \in \mathbb{R}. \) Then the boundedness

\[
\text{Op}\left(S_{0,0}^m(\mathbb{R}^n, 1)\right) \subset B(h^{p_1} \to h^p)
\]

with \( h^{p_1} (h^p, \text{resp.}) \) replaced by \( \text{bmo} \) when \( p = \infty \) (\( p_1 = \infty, \text{resp.} \)) holds if and only if

\[
m \leq \min\left\{ \frac{n}{p}, \frac{n}{2} \right\} - \max\left\{ \frac{n}{p_1}, \frac{n}{2} \right\}.
\]

In fact, this result is already known. The ‘if’ part for \( p_1 = p \) is given by Calderón–Vaillancourt [5] (the case \( p_1 = p_2 = 2 \)), by Fefferman [9] and Coifman–Meyer [6] (the case \( 1 < p_1 = p < \infty \)), and by Miyachi [27] and Päivärinta–Somersalo [29] (the case \( 0 < p_1 = p \leq 1 \)). The ‘if’ part for the case \( p_1 < p \) can be deduced from the case \( p_1 = p \) with the aid of the mapping properties of the fractional integration operator and symbolic calculus in \( S_{0,0}^m(\mathbb{R}^n, 1) \). The ‘only if’ part for the case \( p_1 = p \) can be found, e.g., in [26, Section 5]. It should be remarked that the method of the present paper can be applied, with only slight modification, to the case \( N = 1 \) to prove the above assertion.

(2) In the case \( N = 1, \) the target space \( h^\infty \) has to be replaced by \( \text{bmo} \), in particular, the boundedness \( \text{Op}(S_{0,0}^{m/2}(\mathbb{R}^n, 1)) \subset B(L^\infty \to L^\infty) \) does not hold (see [27, Section 5, p. 151]). However, if \( N \geq 2 \) then we have the boundedness with the target space \( h^\infty = L^\infty \). This is related to the fact that Lemma 2.5 to be given in Section 2 holds only for \( N \geq 2 \).

### 1.2. Refined version of the main theorem.

In fact, we shall give a slightly refined version of Theorem 1.2. To give the refined version, we use the following.

**Definition 1.4.** For \( m = (m_1, \ldots, m_N) \in \mathbb{R}^N, \) the class \( S_{0,0}^m(\mathbb{R}^n, N) \) is defined to be the set of all \( C^\infty \) functions \( \sigma = \sigma(x, \xi_1, \ldots, \xi_N) \) on \((\mathbb{R}^n)^{N+1}\) that satisfy the estimate

\[
\left| \partial_x^{\alpha} \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_N}^{\beta_N} \sigma(x, \xi_1, \ldots, \xi_N) \right| \leq C_{\alpha, \beta_1, \ldots, \beta_N} \prod_{j=1}^N (1 + |\xi_j|)^{m_j}
\]

for all \( \alpha, \beta_1, \ldots, \beta_N \in \mathbb{N}_0^n. \) For \( m \in \mathbb{R}, \) we write \( S_{0,0}^m(\mathbb{R}^n, N)^\times \) to denote the set of all \( \sigma \in S_{0,0}^m(\mathbb{R}^n, N) \) that do not depend on the variable \( x. \) We consider \( \sigma \in S_{0,0}^m(\mathbb{R}^n, N)^\times \) as a function defined on \((\mathbb{R}^n)^N.\)
Notice that if \( m_1, \ldots, m_N \leq 0 \) and if \( m_1 + \cdots + m_N = m \) then \( S^0_{0,0}(\mathbb{R}^n, N) \subset S^m_{0,0}(\mathbb{R}^n, N) \). Thus the assertion (1) of the following theorem is a refined version of the ‘if’ part of Theorem 1.2 for the case \( p < \infty \). The assertion (2) of this theorem is essentially a restatement of the ‘if’ part of Theorem 1.2 for the case \( p = \infty \).

**Theorem 1.5.** Let \( N \geq 2 \), \( 0 < p, p_1, \ldots, p_N \leq \infty \), and \( 1/p \leq 1/p_1 + \cdots + 1/p_N \).

1. If \( p < \infty \) and if \( m = (m_1, \ldots, m_N) \in \mathbb{R}^N \) satisfies
   \[
   -\max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\} < m_j < \frac{n}{2} - \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}, \quad j = 1, \ldots, N,
   \]
   then \( \text{Op} \left( S^m_{0,0}(\mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \cdots \times h^{p_N} \to h^p). \)

2. If \( m = -\sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}, \) then \( \text{Op} \left( S^m_{0,0}(\mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \cdots \times h^{p_N} \to L^\infty). \)

In the above assertions, if some of the \( p_j \)'s are equal to \( \infty \), then the conclusions hold with the corresponding \( h^{p_j} \) replaced by \( bmo \).

The next theorem covers all \( N \geq 1 \). This theorem is a slightly refined version of the ‘only if’ part of Theorem 1.2 since \( H^p \subset h^p \) and \( L^\infty \subset bmo \subset BMO \).

**Theorem 1.6.** Let \( N \geq 1 \), \( 0 < p, p_1, \ldots, p_N \leq \infty \), \( 1/p \leq 1/p_1 + \cdots + 1/p_N \), and \( m \in \mathbb{R} \). If
\[
\text{Op}(S^m_{0,0}(\mathbb{R}^n, N)) \subset B(H^{p_1} \times \cdots \times H^{p_N} \to L^p),
\]
with \( L^p \) is replaced by \( BMO \) when \( p = \infty \), then (1.4) holds.

One of the main ideas of this paper is to use Wiener amalgam spaces. We first prove the estimate of pseudo-differential operators in Wiener amalgam spaces and then derive the estimate in \( h^p \) spaces by using embedding relations between Wiener amalgam spaces and \( h^p \).

Similar method is also used in the papers [22] and [23] by the same authors.

### 1.3. Function spaces, etc.

We give the definitions and properties of some function spaces and give some notations.

\( L^p(\mathbb{R}^n) \), \( 0 < p \leq \infty \), denotes the usual Lebesgue space on \( \mathbb{R}^n \). \( L^{p,q}(\mathbb{R}^n) \), \( 0 < p < \infty \), \( 0 < q \leq \infty \), denotes the Lorentz space; see, e.g., [16] Chapter 1, Section 1.4. In particular, the space \( L^{p,\infty}(\mathbb{R}^n) \), \( 0 < p < \infty \), consists of all measurable functions \( f \) on \( \mathbb{R}^n \) such that
\[
\|f\|_{L^{p,\infty}} = \sup_{0 < \lambda < \infty} \lambda |\{x \in \mathbb{R}^n | |f(x)| > \lambda\}|^{1/p} < \infty,
\]
where \( |E| \) denotes the Lebesgue measure of \( E \subset \mathbb{R}^n \). It holds that \( L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 0 < p < \infty \).

For \( 0 < q < \infty \), \( \ell^q(\mathbb{Z}^n) \) denotes the class of all complex sequences \( a = \{a_k\}_{k \in \mathbb{Z}^n} \) such that
\[
\|a\|_{\ell^q(\mathbb{Z}^n)} = \left( \sum_{k \in \mathbb{Z}^n} |a_k|^q \right)^{1/q} < \infty.
\]

Let \( \phi \in S(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} \phi(x) \, dx \neq 0 \) and let \( \phi_t(x) = t^{-n} \phi(x/t) \) for \( t > 0 \). The space \( H^p = H^p(\mathbb{R}^n) \), \( 0 < p \leq \infty \), consists of all \( f \in S'(\mathbb{R}^n) \) such that \( \|f\|_{H^p} = \|\sup_{0 < t < \infty} |\phi_t * f|\|_{L^p} < \infty \). The space \( h^p = h^p(\mathbb{R}^n) \), \( 0 < p \leq \infty \), consists of all \( f \in S'(\mathbb{R}^n) \) such that \( \|f\|_{h^p} = \|\sup_{0 < t < 1} |\phi_t * f|\|_{L^p} < \infty \). It is known that \( H^p \) and \( h^p \) do not depend on the choice of the function \( \phi \) up to the equivalence of quasinorm.
Obviously $H^p \subset h^p$. If $1 < p \leq \infty$, then $H^p = h^p = L^p$ with equivalent norms. If $0 < p \leq 1$, then the inequality $\|f\|_{L^p} \lesssim \|f\|_{h^p}$ holds for all $f \in h^p$ which are defined by locally integrable functions on $\mathbb{R}^n$.

The space $BMO = BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{BMO} = \sup_R \frac{1}{|R|} \int_R |f(x) - f_R| \, dx < \infty,$$

where $f_R = |R|^{-1} \int_R f(x) \, dx$ and $R$ ranges over all the cubes in $\mathbb{R}^n$. The space $bmo = bmo(\mathbb{R}^n)$ consists of all locally integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{bmo} = \sup_{|R| \leq 1} \frac{1}{|R|} \int_R |f(x) - f_R| \, dx + \sup_{|R| \geq 1} \frac{1}{|R|} \int_R |f(x)| \, dx < \infty,$$

where $R$ denotes cubes in $\mathbb{R}^n$.

The embedding $L^\infty \subset bmo \subset BMO$ holds.

The definition of Wiener amalgam spaces together with their embedding properties will be given in Section 2.

The following notations are used throughout the paper.

For two nonnegative functions $A(x)$ and $B(x)$ defined on a set $X$, we write $A(x) \lesssim B(x)$ for $x \in X$ to mean that there exists a positive constant $C$ such that $A(x) \leq CB(x)$ for all $x \in X$. We often omit to mention the set $X$ when it is obviously recognized. Also $A(x) \approx B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

$C_0^\infty(\mathbb{R}^d)$ denotes the set of all the $C^\infty$ functions on $\mathbb{R}^d$ that have compact supports.

The symbols $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the Schwartz class of rapidly decreasing smooth functions and the space of tempered distributions on $\mathbb{R}^d$, respectively. The Fourier transform and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx,$$

$$\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(\xi) \, d\xi,$$

respectively. For $m \in \mathcal{S}'(\mathbb{R}^d)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$m(D) f = \mathcal{F}^{-1} [m \cdot \mathcal{F} f].$$

For $1 \leq p \leq \infty$, the conjugate index $p'$ is defined by $1/p + 1/p' = 1$.

For $x \in \mathbb{R}^d$, we write $(x) = (1 + |x|^2)^{1/2}$.

The contents of the rest of the paper is as follows. In Section 2 we recall some embedding relations between Wiener amalgam spaces, $h^p$, and $bmo$ and prove some inequalities that are used in the proof of Theorem 1.5. In Sections 3 and 4, we prove Theorems 1.5 and 1.6 respectively. Notice that the main theorem, Theorem 1.2 directly follows from Theorems 1.5 and 1.6.

2. Preliminaries for the proof of Theorem 1.5

We begin with the definition of the Wiener amalgam spaces.
Definition 2.1. Let $\kappa \in C^\infty_0(\mathbb{R}^n)$ be a function such that

\begin{equation}
\sum_{k \in \mathbb{Z}^n} \kappa(\xi - k) \geq 1, \quad \xi \in \mathbb{R}^n.
\end{equation}

Then for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, the Wiener amalgam space $W_{s}^{p,q}$ is defined to be the set of all $f \in S'(\mathbb{R}^n)$ such that the quasinorm

$$
\|f\|_{W_{s}^{p,q}} = \left\| |(k)^s \kappa(D - k)f(x)|_{L_s^q(\mathbb{Z}^n)} \right\|_{L_p^s(\mathbb{R}^n)}
$$

is finite. If $s = 0$, we write $W_{0}^{p,q} = W_{0}^{p,q}$.

The basic definition of the Wiener amalgam space is due to Feichtinger [10] and Triebel [30]. The usual definition of the Wiener amalgam space is given by a function $\kappa$ that satisfies $\sum_{k \in \mathbb{Z}^n} \kappa(\cdot - k) \equiv 1$. However, a bit more general $\kappa$ satisfying (2.1) will be convenient in our argument. The space $W_{s}^{p,q}$ does not depend on the choice of the function $\kappa$ up to the equivalence of quasinorm. The space $W_{s}^{p,q}$ is a quasi-Banach space (Banach space if $1 \leq p, q \leq \infty$) and $S \subset W_{s}^{p,q} \subset S'$. If $0 < p, q < \infty$, then $S$ is dense in $W_{s}^{p,q}$.

Several quasinorms equivalent to $\|f\|_{W_{s}^{p,q}}$ are known. Firstly, $\|f\|_{W_{s}^{p,q}}$ is equivalent to its continuous version:

\begin{equation}
\|f\|_{W_{s}^{p,q}} \approx \left\| |(y)^s \kappa(D - y)f(x)|_{L_s^q(\mathbb{R}^n)} \right\|_{L_p^s(\mathbb{R}^n)}.
\end{equation}

If we write $\varphi = F^{-1}\kappa$, then $\kappa(D - y)f(x) = \langle \varphi(x - z)e^{-iyz}, f(z) \rangle e^{iyx}$ and hence (2.2) for $s = 0$ implies

\begin{equation}
\|f\|_{W_{s}^{p,q}} \approx \left\| \langle \varphi(x - z)e^{-iyz}, f(z) \rangle |_{L_s^q(\mathbb{R}^n)} \right\|_{L_p^s(\mathbb{R}^n)}
\end{equation}

\begin{equation}
= \left\| \langle \kappa(-y + \xi)e^{i\xi x}, \hat{f}(\xi) \rangle |_{L_s^q(\mathbb{R}^n)} \right\|_{L_p^s(\mathbb{R}^n)}.
\end{equation}

The quasinorm of the right hand side of (2.3) is sometimes adopted for the definition of $W_{s}^{p,q}$. The quasinorm (2.4) is the quasinorm of $\hat{f}$ in the modulation space. If $q = 2$, then (2.3) combined with Plancherel’s theorem yields

$$
\|f\|_{W_{s}^{p,2}} \approx \left\| \langle \varphi(x - z)f(z) \rangle |_{L_s^2(\mathbb{R}^n)} \right\|_{L_p^s(\mathbb{R}^n)},
$$

the right hand side of which is the quasinorm of $f$ in the $L^2$-based amalgam space. For these facts, see [10, 11, 30, 18, 13, 24, 33, 19]. For modulation space, see [11, 13, 18, 24, 33]. For amalgam spaces, see [12] and [21].

The following embedding relations are known.

Lemma 2.2. The following embeddings hold:

\begin{equation}
W_{s}^{p_1,q_1} \hookrightarrow W_{s}^{p_2,q_2}, \quad 0 < p_1 \leq p_2 \leq \infty, \quad 0 < q_1 \leq q_2 \leq \infty;
\end{equation}

\begin{equation}
L^p \hookrightarrow W_{s}^{p,p'}, \quad 1 \leq p \leq 2;
\end{equation}

\begin{equation}
L^p \hookrightarrow W_{s}^{p,2}, \quad 2 \leq p \leq \infty;
\end{equation}

\begin{equation}
h^p \hookrightarrow W_{s}^{p,2}_{n(1/2-1/p)}, \quad 0 < p \leq 2;
\end{equation}

\begin{equation}
bmo \hookrightarrow W_{s}^{\infty,2}.
\end{equation}
Proposition 3.4], where the case of extension of functions on $Z$ functions

$$W^{p,2} \hookrightarrow h^p, \quad 0 < p \leq 2;$$

$$W^{p,p'} \hookrightarrow L^p, \quad 2 \leq p \leq \infty. \tag{2.11}$$

The embeddings (2.6), (2.7), and (2.11) are given in [8 Theorems 1.1 and 1.2]. The embeddings (2.8) and (2.10) are given in [20 Theorem 1.2]. For (2.5) and (2.9), see [8, 19, 20, 23].

Lemma 2.3 (29 Lemma 2.2]). For each $M \in \mathbb{N}$, there exist functions $\{\chi_\ell\}_{\ell \in \mathbb{Z}^n} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\text{supp } \chi_\ell \subset [-1,1]^n, \quad \sup \|\mathcal{F}^{-1} \chi_\ell\|_{L^1} \lesssim 1,$$

$$\sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^{-2M} \chi_\ell(\zeta) \langle \zeta \rangle^{2M} = 1 \quad \text{for all } \zeta \in \mathbb{R}^n.$$

Here the implicit constant in $\lesssim$ depends only on $n$ and $M$.

Lemma 2.4. Let $N \geq 2, 1 < r < \infty$, and let $a_1, \ldots, a_N$ be real numbers satisfying $-n/2 < a_j < 0$ and $\sum_{j=1}^N a_j = n/r - Nn/2$. Then the following inequality holds for all nonnegative functions $A_1, \ldots, A_N$ on $\mathbb{Z}^n$:

$$\left\| \sum_{\nu_1, \ldots, \nu_N \in \mathbb{Z}^n, \nu_1 + \cdots + \nu_N = \mu} \prod_{j=1}^N (1 + |\nu_j|^r)^{a_j} A_j(\nu_j) \right\| \lesssim \prod_{j=1}^N \|A_j\|_{\ell^r(\mathbb{Z}^n)}.$$

Proof. (The following argument is a slight modification of the one given in [23, Proof of Proposition 3.4], where the case $r = 2$ was treated.) By duality and by an appropriate extension of functions on $\mathbb{Z}^n$ to functions on $\mathbb{R}^n$, it is sufficient to prove the inequality

$$\int_{(\mathbb{R}^n)^N} A_0(x_1 + \cdots + x_N) \prod_{j=1}^N (x_j)^{a_j} A_j(x_j) \, dx_1 \cdots dx_N \lesssim \|A_0\|_{L^r(\mathbb{R}^n)} \prod_{j=1}^N \|A_j\|_{L^2(\mathbb{R}^n)}$$

for all nonnegative functions $A_0, A_1, \ldots, A_N$ on $\mathbb{R}^n$. In [23, Proof of Proposition 3.4], it is proved that the following inequality holds for all nonnegative functions $A_0, A_1, \ldots, A_N, f_1, \ldots, f_N$ on $\mathbb{R}^n$:

$$\int_{(\mathbb{R}^n)^N} A_0(x_1 + \cdots + x_N) \prod_{j=1}^N f_j(x_j) A_j(x_j) \, dx_1 \cdots dx_N$$

$$\lesssim \|A_0\|_{L^{r_0}(\mathbb{R}^n)} \prod_{j=1}^N \|f_j\|_{L^{r_j}(\mathbb{R}^n)} \|A_j\|_{L^{Q_j}(\mathbb{R}^n)}$$

for

$$\frac{1}{Q_0} + \sum_{j=1}^N \left( \frac{1}{p_j} + \frac{1}{q_j} \right) = N,$$

$$0 < 1/q_0, 1/p_j, 1/q_j < 1,$$

$$0 < 1/p_j + 1/q_j < 1, \quad j = 1, \ldots, N,$$
Lemma 2.5. Let 
\[ s_0, s_j, t_j \in [1, \infty], \quad j = 1, \ldots, N, \quad \text{and} \quad \frac{1}{s_0} + \sum_{j=1}^{N} \left( \frac{1}{s_j} + \frac{1}{t_j} \right) = 1. \]

We take 
\[ q_0 = r, \quad q_1 = \cdots = q_N = 2, \]
\[ s_0 = s_1 = \cdots = s_N = t_1 = \cdots = t_{N-2} = \infty, \quad t_{N-1} = t_N = 2. \]

Then the above result combined with the embedding \( L^r \hookrightarrow L^{r, \infty}, \ r < \infty, \) implies 
\[
\int_{(\mathbb{R}^n)^N} A_0(x_1 + \cdots + x_N) \prod_{j=1}^{N} f_j(x_j) A_j(x_j) \, dx_1 \cdots dx_N \lesssim \|A_0\|_{L^r} \prod_{j=1}^{N} \|f_j\|_{L^{p_j, \infty}} \|A_j\|_{L^2}
\]
for 
\[
\sum_{j=1}^{N} \frac{1}{p_j} = \frac{N}{2} - \frac{1}{r}, \quad 0 < \frac{1}{r} < 1, \quad 0 < \frac{1}{p_j} < \frac{1}{2} \quad (j = 1, \ldots, N).
\]
The desired inequality now follows if we take \( p_j = -n/a_j \) and \( f_j(x_j) = \langle x_j \rangle^{a_j} \). \qed

**Lemma 2.5.** Let \( N \geq 2 \). Then the following inequality holds for all nonnegative functions \( A_1, \ldots, A_N \) on \( \mathbb{Z}^n \):

\[
(2.12) \quad \sum_{\nu_1, \ldots, \nu_N \in \mathbb{Z}^n} (1 + |\nu_1| + \cdots + |\nu_N|)^{-Nn/2} \prod_{j=1}^{N} A_j(\nu_j) \lesssim \prod_{j=1}^{N} \|A_j\|_{\ell^2(\mathbb{Z}^n)}.
\]

**Proof.** If \( N = 2 \) and \( n = 1 \), then the inequality reads as 
\[
\sum_{\nu_1, \nu_2 \in \mathbb{Z}} (1 + |\nu_1| + |\nu_2|)^{-1} A_1(\nu_1) A_2(\nu_2) \lesssim \|A_1\|_{\ell^2(\mathbb{Z})} \|A_2\|_{\ell^2(\mathbb{Z})}.
\]
This inequality is known by the name of Hilbert’s inequality and is derived from the \( L^2 \)-boundedness of the Hilbert transform; see, e.g., [14, Chapter 11, Theorem 11.4.1]. If \( N = 2 \) and \( n \geq 2 \), then, since 
\[
(1 + |\nu_1| + |\nu_2|)^{-n} \leq \prod_{k=1}^{n} (1 + |\nu_{1,k}| + |\nu_{2,k}|)^{-1}, \quad \nu_j = (\nu_{j,1}, \ldots, \nu_{j,n}),
\]
the inequality can be proved by repeated application of Hilbert’s inequality.

We shall prove the inequality for \( N \geq 3 \). Applying the Cauchy–Schwarz inequality to the sum \( \sum_{\nu_1 \in \mathbb{Z}^n} \), we obtain 
(\text{the left hand side of } (2.12))
\[
\leq \sum_{\nu_2, \ldots, \nu_N \in \mathbb{Z}^n} \|A_1\|_{\ell^2}^2 \left( \sum_{\nu_1 \in \mathbb{Z}^n} (1 + |\nu_1| + \cdots + |\nu_N|)^{-Nn} \right)^{1/2} \prod_{j=2}^{N} A_j(\nu_j)
\]
\[
\approx \sum_{\nu_2, \ldots, \nu_N \in \mathbb{Z}^n} \|A_1\|_{\ell^2} (1 + |\nu_2| + \cdots + |\nu_N|)^{-(N-1)n/2} \prod_{j=2}^{N} A_j(\nu_j).
\]
Repeating this process \( N - 2 \) times, we obtain 
(\text{the left hand side of } (2.12))
\[
\leq \prod_{j=1}^{N-2} \|A_j\|^{2} \sum_{\nu_{N-1},\nu_N \in \mathbb{Z}^n} (1 + |\nu_{N-1}| + |\nu_N|)^{-n} A_{N-1}(\nu_{N-1})A_N(\nu_N) =: (\ast).
\]

Finally using the case \( N = 2 \) of (2.12), we obtain \( (\ast) \leq \prod_{j=1}^{N} \|A_j\|^{2} \). \( \square \)

3. PROOF OF THEOREM 1.5

We use the following notation: \( \xi = (\xi_1, \ldots, \xi_N) \in (\mathbb{R}^n)^N, \nu = (\nu_1, \ldots, \nu_N) \in (\mathbb{Z}^n)^N, \) and \( k = (k_1, \ldots, k_N) \in (\mathbb{Z}^n)^N \). We also write \( d\xi = d\xi_1 \cdots d\xi_N \) and \( \xi \cdot k = \xi_1 \cdot k_1 + \cdots + \xi_N \cdot k_N \).

Proof of Theorem 1.5. We first give some argument which can be applied to general multilinear pseudo-differential operators and prove some result concerning the estimate of the operators in Wiener amalgam spaces.

For the moment, we assume \( \sigma \in S^m(\mathbb{R}^n, N) \) with \( m \in \mathbb{R}^N \) or \( \sigma \in S^m(\mathbb{R}^n, N) \) with \( m \in \mathbb{R} \).

We shall follow the method of utilizing Fourier series expansion, which may go back at least to Coifman and Meyer [6, 7]. We take a function \( \varphi \) such that

\[
\varphi \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } \varphi \subset [-1,1]^n, \quad \sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu) = 1 \quad (\xi \in \mathbb{R}^n)
\]

and decompose the symbol \( \sigma \) as

\[
\sigma(x, \xi) = \sum_{\nu \in (\mathbb{Z}^n)^N} \sigma_\nu(x, \xi),
\]

\[
\sigma_\nu(x, \xi) = \sigma(x, \xi) \prod_{j=1}^{N} \varphi(\xi_j - \nu_j).
\]

If we define

\[
S_\nu(x, \xi) = \sum_{\mu \in \mathbb{Z}^n} \sigma_\nu(x, \xi - 2\pi \mu),
\]

then \( S_\nu(x, \xi) \) is a \( 2\pi \mathbb{Z}^N \)-periodic function with respect to the variable \( \xi \) and \( S_\nu(x, \xi) = \sigma_\nu(x, \xi) \) for \( \xi \in \nu + [-\pi, \pi]^N \). Hence we can take a function \( \tilde{\varphi} \) such that

\[
\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n), \quad 0 \leq \tilde{\varphi} \leq 1, \quad \text{supp } \tilde{\varphi} \subset [-\pi, \pi]^n, \quad \tilde{\varphi} = 1 \text{ on } [-1,1]^n,
\]

and

\[
\sigma_\nu(x, \xi) = S_\nu(x, \xi) \prod_{j=1}^{N} \tilde{\varphi}(\xi_j - \nu_j).
\]

Expanding \( S_\nu(x, \xi) \) into the Fourier series with respect to \( \xi \), we obtain

\[
\sigma_\nu(x, \xi) = \sum_{\mathbf{k} \in (\mathbb{Z}^n)^N} e^{i \xi \cdot \mathbf{k}} P_{\nu, \mathbf{k}}(x) \prod_{j=1}^{N} \tilde{\varphi}(\xi_j - \nu_j),
\]

\[
P_{\nu, \mathbf{k}}(x) = \frac{1}{(2\pi)^{2n}} \int_{\nu + [-\pi, \pi]^N} e^{-i \mathbf{k} \cdot \eta} \sigma_\nu(x, \eta) \, d\eta.
\]

Integration by parts gives

\[
P_{\nu, \mathbf{k}}(x) = (\mathbf{k})^{-2L} Q_{\nu, \mathbf{k}}(x),
\]
We further decompose $Q_{\nu,k}$ by using the partition of unity of Lemma 2.3 as

$$Q_{\nu,k}(x) = \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^{-2M} Q_{\nu,k,\ell}(x),$$

$$Q_{\nu,k,\ell}(x) = \mathcal{F}^{-1} \left[ \chi_{\ell}(\zeta) \langle \zeta \rangle^{2M} \widehat{Q}_{\nu,k}(\zeta) \right](x) = (\mathcal{F}^{-1}\chi_{\ell}(x)) \ast (I - \Delta_x)^M Q_{\nu,k}(x).$$

Thus $\sigma_{\nu}$ is written as

$$\sigma_{\nu}(x, \xi) = \sum_{k \in (\mathbb{Z}^n)^N} \sum_{\ell \in \mathbb{Z}^n} \langle k \rangle^{-2L} \langle \ell \rangle^{-2M} Q_{\nu,k,\ell}(x) e^{i\xi \cdot k} \prod_{j=1}^N \varphi(\xi_j - \nu_j).$$

By writing

$$F^j_{\nu_j,k_j}(x) = \varphi(D - \nu_j) f_j(x + k_j),$$

we have

$$T_{\sigma_{\nu}}(f_1, \ldots, f_N)(x) = \sum_{k \in (\mathbb{Z}^n)^N} \sum_{\ell \in \mathbb{Z}^n} \langle k \rangle^{-2L} \langle \ell \rangle^{-2M} Q_{\nu,k,\ell}(x) \prod_{j=1}^N F^j_{\nu_j,k_j}(x).$$

Thus, since $\sigma = \sum_{\nu} \sigma_{\nu}$,

$$T_{\sigma}(f_1, \ldots, f_N)(x) = \sum_{k \in (\mathbb{Z}^n)^N} \sum_{\ell \in \mathbb{Z}^n} \langle k \rangle^{-2L} \langle \ell \rangle^{-2M} \sum_{\nu \in (\mathbb{Z}^n)^N} Q_{\nu,k,\ell}(x) \prod_{j=1}^N F^j_{\nu_j,k_j}(x).$$

The functions $Q_{\nu,k,\ell}(x)$ have the following estimate. If $\sigma \in S_{0,0}^m(\mathbb{R}^n, N)$ with $m \in \mathbb{R}^N$, then the symbol $\sigma_{\nu}(x, \xi)$ satisfies

$$\left| \partial_x^{\alpha} \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_N}^{\beta_N} \sigma_{\nu}(x, \xi) \right| \lesssim \prod_{j=1}^N (1 + |\nu_j|)^{m_j}$$

and hence $Q_{\nu,k}(x)$ satisfies

$$|(I - \Delta_x)^M Q_{\nu,k}(x)| \lesssim \prod_{j=1}^N (1 + |\nu_j|)^{m_j},$$

which combined with $\|\mathcal{F}^{-1}\chi_{\ell}\|_{L^1} \lesssim 1$ (Lemma 2.3) implies

$$|Q_{\nu,k,\ell}(x)| \lesssim \prod_{j=1}^N (1 + |\nu_j|)^{m_j}. \tag{3.3}$$
If \( \sigma \in S_{0,0}^m(\mathbb{R}^n, N) \) with \( m \in \mathbb{R} \), then
\[
\left| \partial_x^\alpha \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_N}^{\beta_N} \sigma_{\nu}(x, \xi) \right| \lesssim (1 + |\nu_1| + \cdots + |\nu_N|)^m
\]
and hence, by the same reason as above,
\[
(3.4) \quad |Q_{\nu, k, \ell}(x)| \lesssim (1 + |\nu_1| + \cdots + |\nu_N|)^m.
\]
Notice that the implicit constants in (3.3) and (3.4) do not depend on \( k \) and \( \ell \).

Let us consider the estimate of \( W^{s,t} \)-quasinorm of \( T_\sigma(f_1, \ldots, f_N) \) for \( s, t \in (0, \infty) \). If \( s, t \in (0, \infty) \) are given, taking the numbers \( L \) and \( M \) sufficiently large and using (3.2), we obtain
\[
\| T_\sigma(f_1, \ldots, f_N) \|_{W^{s,t}} \lesssim \sup_{k, \ell} \left\| \sum_{\nu} Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j \right\|_{W^{s,t}}.
\]
Notice that
\[
(3.5) \quad \text{supp } \mathcal{F}Q_{\nu, k, \ell} \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \ell| \lesssim 1 \},
\]
\[
(3.6) \quad \text{supp } \mathcal{F}F_{\nu_j, k_j}^j \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \nu_j| \lesssim 1 \}
\]
and recall that the function \( \kappa \) used in the definition of \( W^{s,t} \)-quasinorm has compact support. Hence we have
\[
\left\| \sum_{\nu} Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j \right\|_{W^{s,t}}
= \left\| \sum_{\nu: |\nu_1| + \cdots + |\nu_N| + \ell < \mu} \kappa(D - \mu) \left[ Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j \right] \right\|_{L^s(\mathbb{R}^n)}
\lesssim \sum_{\tau: |\tau| \leq 1} \left\| \kappa(D - \mu + \tau - \ell) \left[ \sum_{\nu: |\nu_1| + \cdots + |\nu_N| = \mu} Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j \right] \right\|_{L^s(\mathbb{R}^n)}
=: (\ast).
\]
We write
\[
h_\mu = \sum_{\nu: |\nu_1| + \cdots + |\nu_N| = \mu} Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j.
\]
Then \( \text{supp } \hat{h}_\mu \subset \{ \zeta \mid |\zeta - \mu - \ell| \lesssim 1 \} \) by (3.5) and (3.6). Recall that \( \text{supp } \kappa \) is also compact. Hence by Nikol’skij’s inequality (see, e.g., [31 Section 1.3.2, Remark 1]), we have
\[
|\kappa(D - \mu + \tau - \ell) h_\mu(x)| \leq \left\| (\mathcal{F}^{-1} \kappa)(y) h_\mu(x - y) \right\|_{L^1_y}
\lesssim \left\| (\mathcal{F}^{-1} \kappa)(y) h_\mu(x - y) \right\|_{L^{\min(1,s,t)}_y}.
\]
(3.7)
The implicit constant in (3.7) can be taken depending only on \( s, t, n \), and the diameters of \( \text{supp } \hat{h}_\mu \) and \( \text{supp } \kappa \); in particular, the inequality (3.7) holds uniformly with respect to \( \mu, \tau, \)
and $\ell$. By (3.7) and Minkowski’s inequality, we have

$$(*) \lesssim \left\| \left\| (\mathcal{F}^{-1} \kappa)(y) h_\mu(x - y) \right\|_{L^{\min(1,s,t)}_\mu} \right\|_{L^s(\mathbb{R}^n)} \lesssim \left\| h_\mu \right\|_{\ell^p_\mu(\mathbb{Z}^n)} \right\|_{L^s(\mathbb{R}^n)}.$$ 

To sum up, we see that the inequality

$$(3.8) \quad \| T_\sigma(f_1, \ldots, f_N) \|_{W^{s,t}} \lesssim \sup_{k, \ell} \left\| \sum_{\nu, \mu_1 + \cdots + \nu_N = \mu} Q_{\nu, k, \ell} \prod_{j=1}^N F_{\nu_j, k_j}^j \right\|_{L^p_\mu(\mathbb{Z}^n)}$$

holds for each $s, t \in (0, \infty]$.

Now we shall proceed to the estimate of the operators considered in Theorem 1.5. We introduce some notation. If $p_1, \ldots, p_N \in (0, \infty]$ and function spaces $X_1, \ldots, X_N, Y$ on $\mathbb{R}^n$ are given, then we define $p_0$ by $1/p_0 = 1/p_1 + \cdots + 1/p_N$, define the sets $J$ and $J^c$ by

$$J = \{ j \in \{1, \ldots, N \} \mid 2 \leq p_j \leq \infty \}, \quad J^c = \{ j \in \{1, \ldots, N \} \mid 0 < p_j < 2 \},$$

and write the boundedness $T_\sigma : X_1 \times \cdots \times X_N \to Y$ as

$$T_\sigma : \prod_{j \in J} X_j \times \prod_{j \in J^c} X_j \to Y.$$ 

We devide the rest of the arguments into three parts.

**Part I: Proof of Theorem 1.5 (1) for the case $0 < p \leq 2$.**

Assume $m \in \mathbb{R}^N$ satisfies the conditions of Theorem 1.5 (1) with $0 < p \leq 2$ and assume $\sigma \in S^m_{0,0}(\mathbb{R}^n, N)$. We shall prove the estimate

$$(3.9) \quad T_\sigma : \prod_{j \in J} W^{p_j,2} \times \prod_{j \in J^c} W^{p_j,2}_{n(1/2 - 1/p_j)} \to W^{p_0,2}.$$ 

If this is proved, then combining it with the embeddings

$$(3.10) \quad h^{p_j} \hookrightarrow W^{p_j,2} \quad \text{for} \quad j \in J,$$

$$(3.11) \quad h^{p_j} \hookrightarrow W^{p_j,2}_{n(1/2 - 1/p_j)} \quad \text{for} \quad j \in J^c,$$

$$W^{p_0,2} \hookrightarrow W^{p,2} \hookrightarrow h^p \quad \text{for} \quad p_0 \leq p \leq 2$$

(see Lemma 2.2), we obtain the desired boundedness $T_\sigma : h^{p_1} \times \cdots \times h^{p_N} \to h^p$. Since the embedding $bmo \hookrightarrow W^{\infty,2}$ also holds, if $p_j = \infty$ then we can replace $h^{p_j}$ by $bmo$.

To prove (3.9), we use (3.8) with $s = p_0, t = 2$, and use (3.3). Then we obtain

$$\| T_\sigma(f_1, \ldots, f_N) \|_{W^{p_0,2}} \lesssim \sup_k \left\| \sum_{\nu, \mu_1 + \cdots + \nu_N = \mu} \prod_{j=1}^N (1 + |\nu_j|)^{m_j} \left\| F_{\nu_j, k_j}^j \right\|_{L^p_\mu(\mathbb{Z}^n)} \right\|_{L^p_\mu(\mathbb{Z}^n)}$$

$$= \sup_k \left\| \sum_{\nu, \mu_1 + \cdots + \nu_N = \mu} \prod_{j \in J} (\nu_j)^{m_j} \prod_{j \in J^c} (\nu_j)^{m_j - n(1/2 - 1/p_j)} \prod_{j \in J} F_{\nu_j, k_j}^j \prod_{j \in J^c} \tilde{F}_{\nu_j, k_j}^j \right\|_{L^p_\mu(\mathbb{Z}^n)}$$

$$=: (\dagger),$$
where \( \tilde{F}^j_{\nu_j, k_j} = (\nu_j)^{n(1/2 - 1/p_j)} F^j_{\nu_j, k_j} \). Now using Lemma 2.4 with \( r = 2 \) and using Hölder’s inequality with the indices \( 1/p_0 = 1/p_1 + \cdots + 1/p_N \), we obtain

\[
(\dagger) \lesssim \sup_k \left\| \prod_{j \in J} \left\| F^j_{\nu_j, k_j} \right\|_{\ell_j} \left\| \tilde{F}^j_{\nu_j, k_j} \right\|_{L^{p_j}} \right\|_{L^{p_0}} \lesssim \sup_k \prod_{j \in J} \left\| \left\| F^j_{\nu_j, k_j} \right\|_{\ell_j} \left\| \tilde{F}^j_{\nu_j, k_j} \right\|_{L^{p_j}} \right\|_{L^{p_j}}.
\]

Finally, the choice of \( \tilde{\varphi} \) implies

\[
\left\| \left\| F^j_{\nu_j, k_j} \right\|_{\ell_j} \right\|_{L^{p_j}} = \left\| \tilde{\varphi}(D - \nu_j) f_j(x + k_j) \right\|_{L^{p_j}} \lesssim \| f_j \|_{W^{p_j, 2}},
\]

\[
\left\| \left\| \tilde{F}^j_{\nu_j, k_j} \right\|_{\ell_j} \right\|_{L^{p_j}} = \left\| \langle \nu_j \rangle^{n(1/2 - 1/p_j)} \tilde{\varphi}(D - \nu_j) f_j(x + k_j) \right\|_{L^{p_j}} \approx \| f_j \|_{W^{p_j, 2}},
\]

and thus the estimate (3.9) follows.

**Part II: Proof of Theorem 1.5 (1) for the case \( 2 < p < \infty \).**

Assume \( m \in \mathbb{R}^N \) satisfies the conditions of Theorem 1.5 (1) with \( 2 < p < \infty \) and assume \( \sigma \in S^m_{0,0}(\mathbb{R}^n, N) \). In this case, we shall prove the estimate

\[
(3.12) \quad T_\sigma : \prod_{j \in J} W^{p_j, 2} \times \prod_{j \in J^c} W^{p_j, 2}_{n(1/2 - 1/p_j)} \to W^{p_0, p'}.
\]

Combining this with the embeddings (3.10), (3.11), and

\[
W^{p_0, p'} \hookrightarrow W^{p, p'} \hookrightarrow L^p = h^p, \quad p_0 \leq p, \quad 2 < p < \infty
\]

(see Lemma 2.2), we obtain \( T_\sigma : h^{p_1} \times \cdots \times h^{p_N} \to h^p \). If \( p_j = \infty \), then we can replace \( h^{p_j} \) by \( bmo \).

To prove (3.12), we use (3.8) with \( s = p_0, \ t = p' \), and also use (3.3). This yields

\[
\| T_\sigma (f_1, \ldots, f_N) \|_{W^{p_0, p'}} \lesssim \sup_k \left\| \sum_{\nu_1, \ldots, \nu_N = \mu} \prod_{j \in J} \nu_j^{m_j} \prod_{j \in J^c} \nu_j^{m_j - n(1/2 - 1/p_j)} \right\| \left\| \prod_{j \in J} F^j_{\nu_j, k_j} \right\|_{L^{p_j}} \left\| \prod_{j \in J^c} \tilde{F}^j_{\nu_j, k_j} \right\|_{L^{p_j}} \left\| \right\|_{L^{p_0}} =: (\dagger'),
\]

where \( \tilde{F}^j_{\nu_j, k_j} \) is the same as in Part I. Using Lemma 2.4 with \( r = p \) and using Hölder’s inequality with the indices \( 1/p_0 = 1/p_1 + \cdots + 1/p_N \), we obtain

\[
(\dagger') \lesssim \sup_k \left\| \prod_{j \in J} \left\| F^j_{\nu_j, k_j} \right\|_{\ell_j} \left\| \tilde{F}^j_{\nu_j, k_j} \right\|_{L^{p_j}} \right\|_{L^{p_0}} \lesssim \sup_k \prod_{j \in J} \left\| \left\| F^j_{\nu_j, k_j} \right\|_{\ell_j} \left\| \tilde{F}^j_{\nu_j, k_j} \right\|_{L^{p_j}} \right\|_{L^{p_j}}.
\]
Thus (3.12) is proved.

**Part III: Proof of Theorem 1.5 (2).**

Assume \( p = \infty, \ m \in \mathbb{R} \) is given as in Theorem 1.5 (2), and assume \( \sigma \in S_{0,0}^{-\infty}(\mathbb{R}^n, N) \). We shall prove the estimate

\[
T_{\sigma} : \prod_{j \in J} W^{p_j,2} \times \prod_{j \in J^c} W^{p_j,2}_{n(1/2 - 1/p_j)} \to W^{p_0,1}.
\]

(3.13)

Combining this estimate with the embeddings (3.10), (3.11), and

\[
W^{p_0,1} \hookrightarrow W^{\infty,1} \hookrightarrow L^{\infty}
\]

(see Lemma 2.2), we obtain the desired boundedness \( T_{\sigma} : h^{p_1} \times \cdots \times h^{p_N} \to L^{\infty} \). If \( p_j = \infty \), then we can replace \( h^{p_j} \) by \( \text{bmo} \).

By (3.4), we have

\[
|Q_{\nu_k}(x)| \lesssim (1 + |\nu_k| + \cdots + |\nu_N|)^m
\]

Observe that \( m = -Nn/2 + \sum_{j \in J^c} (n/2 - n/p_j) \) in the present case. Hence the above inequality implies

\[
|Q_{\nu_k}(x)| \lesssim (1 + |\nu_k| + \cdots + |\nu_N|)^{Nn/2} \prod_{j \in J^c} (\nu_j)^{n(1/2 - 1/p_j)}.
\]

(3.14)

We use (3.8) with \( s = p_0, t = 1 \), and use (3.14) to obtain

\[
\|T_{\sigma}(f_1, \ldots, f_N)\|_{W^{p_0,1}} \lesssim \sup_k \left\| \sum_{\nu \in \Sigma} (1 + |\nu_k| + \cdots + |\nu_N|)^{-Nn/2} \prod_{j \in J} F_{\nu,j,k_j} \prod_{j \in J^c} \tilde{F}_{\nu,j,k_j} \right\|_{L^{p_0}} =: (\dagger \dagger \dagger),
\]

where \( \tilde{F}_{\nu,j,k_j} \) is the same as in Part I. Now by Lemma 2.3 and Hölder’s inequality, we obtain

(\dagger \dagger \dagger) \lesssim \sup_k \left\| \prod_{j \in J} F_{\nu,j,k_j} \right\|_{\ell^1_{\nu_k}} \prod_{j \in J^c} \left\| \tilde{F}_{\nu,j,k_j} \right\|_{\ell^1_{\nu_j}} L^{p_0}

\leq \sup_k \prod_{j \in J} \left\| F_{\nu,j,k_j} \right\|_{L^{p_j}} \prod_{j \in J^c} \left\| \tilde{F}_{\nu,j,k_j} \right\|_{L^{p_j}}

\approx \prod_{j \in J} \left\| f_j \right\|_{W^{p_j,2}} \prod_{j \in J^c} \left\| f_j \right\|_{W^{p_j,2}_{n(1/2 - 1/p_j)}}.

Thus (3.13) is proved. This completes the proof of Theorem 1.5. \( \square \)
4. Proof of Theorem 1.6

The key fact we shall use here is that for each \( p_j \in (1, \infty) \) there exists a function \( f_j \in L^{p_j}(\mathbb{R}^n) \) whose Fourier transform satisfies

\[
|\hat{f}_j(\xi)| \approx \begin{cases} 
|\xi|^{n/p_j - n - \epsilon} & \text{(when } 1 < p_j < 2) \\
|\xi|^{-n/2 - \epsilon} & \text{(when } 2 \leq p_j < \infty) 
\end{cases}
\]

for \( \ell \in \mathbb{Z}^n \setminus \{0\} \) and \( |\xi - \ell| \ll 1 \). If \( 1 < p_j \leq 2 \), the function \( f_j \) can be easily found by setting \( f_j(x) \approx |x|^{-n/p_j + \epsilon} \) for \( |x| \) small. If \( 2 < p_j < \infty \), the existence of \( f_j \) is a classical fact but is not so elementary. Here to give a precise argument, we use the following result due to Wainger [32].

**Lemma 4.1** ([32 Theorem 10]). If \( 1 \leq p \leq \infty \), \( 0 < a < 1 \), and \( b > \frac{a}{2} + (1-a)(\frac{a}{2} - \frac{a}{p}) \), then there exists a function \( g_{a,b} \in L^p([-\pi, \pi]^n) \) whose Fourier coefficients are given by

\[
\frac{1}{(2\pi)^n} \int_{[-\pi, \pi]^n} g_{a,b}(x)e^{-i\ell \cdot x} \, dx = \begin{cases} 
|\xi|^{-b}e^{2\pi i \ell \cdot a} & \text{for } \ell \in \mathbb{Z}^n \setminus \{0\}, \\
0 & \text{for } \ell = 0.
\end{cases}
\]

Using this lemma, we shall prove the lemma below, which contains the essential part of Theorem 1.6.

**Lemma 4.2.** Suppose \( N \geq 1 \), \( 1 < p_1, \ldots, p_N < \infty \), \( 0 < p < \infty \), and \( m \in \mathbb{R} \) satisfy

\[
\text{Op}(S^m_{0,0}(\mathbb{R}^n, N)^\times) \subset B(L^{p_1} \times \cdots \times L^{p_N} \to L^p).
\]

Then (1.4) holds.

**Proof.** We set \( J = \{ j \in \{1, \ldots, N\} \mid 2 \leq p_j < \infty \} \) and \( J^c = \{ j \in \{1, \ldots, N\} \mid 1 < p_j < 2 \} \). The proof will be divided into three cases.

**Case I:** \( 0 < p \leq 2 \).

From (1.1), the closed graph theorem implies that there exists a positive integer \( M \) and a constant \( C \) such that

\[
\|T_\sigma\|_{L^{p_1} \times \cdots \times L^{p_N} \to L^p} \leq C \max_{|\beta_1|, \ldots, |\beta_N| \leq M} \| (1 + |\xi|)^{-m} \partial^\beta_\xi \sigma(\xi) \|_{L^\infty(\mathbb{R}^{Nn})}
\]

for all \( \sigma \in S^m_{0,0}(\mathbb{R}^{n}, N)^\times \).

Let \( \varphi, \tilde{\varphi} \in S(\mathbb{R}^n) \) be functions such that

\[
\text{supp } \varphi \subset [-1/4, 1/4]^n, \quad |\mathcal{F}^{-1}\varphi(x)| \geq 1 \text{ on } [-\pi, \pi]^n,
\]

\[
\text{supp } \tilde{\varphi} \subset [-1/2, 1/2]^n, \quad \tilde{\varphi} = 1 \text{ on } [-1/4, 1/4]^n.
\]

Let \( \{c_k\}_{k \in (\mathbb{Z}^n)^N} \) be a sequence of complex numbers satisfying \( \sup_{k} |c_k| \leq 1 \) and let \( D \) be a finite subset of \( (\mathbb{Z}^n)^N \). We consider the multiplier

\[
\sigma(\xi) = \sum_{k \in D} c_k (1 + |k|)^m \prod_{j=1}^N \varphi(\xi_j - k_j).
\]

As far as \( \sup_k |c_k| \leq 1 \), the multiplier \( \sigma \) satisfies

\[
|\partial^\beta_\xi \sigma(\xi)| \leq c(1 + |\xi|)^m
\]
with a constant $c$ independent of $\{c_k\}$ and $D$, and hence (4.2) implies
\begin{equation}
\|T_\sigma\|_{L^{p_1} \times \cdots \times L^{p_N} \to L^p} \lesssim 1
\end{equation}
with the implicit constant independent of $\{c_k\}$ and $D$.

Let $g_{a,b}$ be the function given in Lemma 4.1. We extend $g_{a,b}$ to the $2\pi \mathbb{Z}^n$-periodic function on $\mathbb{R}^n$ and denote it by the same symbol $g_{a,b}$. Using this $g_{a,b}$, we define the functions $f_{a_j,b_j}$, $j = 1, \ldots, N$, on $\mathbb{R}^n$ as follows. We take $\epsilon > 0$ and $0 < a_j < 1$, and define
\begin{equation}
b_j = \frac{n}{2} + (1 - a_j) \left( \frac{n}{2} - \frac{n}{p_j} \right) + \epsilon
\end{equation}
and
\begin{equation}
f_{a_j,b_j}(x) = g_{a_j,b_j}(x)F^{-1}_j(\tilde{\varphi}(x)).
\end{equation}
From Lemma 4.1 it follows that
\begin{equation}
f_{a_j,b_j} \in L^{p_j}(\mathbb{R}^n)
\end{equation}
and its Fourier transform is given by
\begin{equation}
\hat{f}_{a_j,b_j}(\xi_j) = \sum_{\ell_j \in \mathbb{Z}^n \setminus \{0\}} |\ell_j|^{-b_j} e^{2\pi i |\ell_j|^{a_j}} \tilde{\varphi}(\xi_j - \ell_j).
\end{equation}
From our choice of $\varphi$ and $\tilde{\varphi}$, we have
\begin{align*}
T_\sigma(f_{a_1,b_1}, \cdots, f_{a_N,b_N})(x) &= \sum_{k \in D} c_k (1 + |k|)^m \prod_{j=1}^N |k_j|^{-b_j} e^{2\pi i |k_j|^{a_j}} F^{-1}_{-}\varphi(\cdot - k_j)(x) \\
&= \left\{ F^{-1}_j(\varphi(x)) \right\}^N \sum_{k \in D} c_k (1 + |k|)^m \prod_{j=1}^N |k_j|^{-b_j} e^{2\pi i |k_j|^{a_j}} e^{ik_j \cdot x}.
\end{align*}

Let $\{r_k(\omega)\}_{k \in \mathbb{Z}^n}$ be mutually independent random variables on a probability space $(\Omega, P)$ such that $P\{r_k = 1\} = P\{r_k = -1\} = 1/2$. We consider the following $\{c_k\}$:
\begin{equation}
c_k = r_{k_1 + \cdots + k_N}(\omega) \prod_{j=1}^N e^{-2\pi i |k_j|^{a_j}}.
\end{equation}
Then we have
\begin{align*}
T_\sigma(f_{a_1,b_1}, \cdots, f_{a_N,b_N})(x) &= \left\{ F^{-1}_j(\varphi(x)) \right\}^N \sum_{k \in D} r_{k_1 + \cdots + k_N}(\omega) (1 + |k|)^m \prod_{j=1}^N |k_j|^{-b_j} e^{i k_j \cdot x} \\
&= \left\{ F^{-1}_j(\varphi(x)) \right\}^N \sum_{k \in \mathbb{Z}^n} r_k(\omega) e^{i x \cdot k} d_k
\end{align*}
with
\begin{equation}
d_k = \sum_{k \in D, k_1 + \cdots + k_N = k} (1 + |k|)^m \prod_{j=1}^N |k_j|^{-b_j}.
\end{equation}
Since $D$ is a finite set, $d_k = 0$ except for a finite number of $k \in \mathbb{Z}^n$. From (4.5) and from our choice of $\varphi$, it follows that

$$
\| T_\sigma(f_{a_1, b_1}, \cdots, f_{a_N, b_N}) \|_{L^p(\mathbb{R}^n)} = \left\{ \mathcal{F}^{-1} \varphi(x) \right\}^N \sum_{k \in \mathbb{Z}^n} r_k(\omega) e^{ix \cdot k} d_k \right\|_{L^p_{\omega}(\mathbb{R}^n)} \geq \left\| \sum_{k \in \mathbb{Z}^n} r_k(\omega) e^{ix \cdot k} d_k \right\|_{L^p_{\omega}([-\pi, \pi]^n)}.
$$

(4.6)

Now combining (4.3), (4.4), and (4.6), we obtain

$$
\left\| \sum_{k \in \mathbb{Z}^n} r_k(\omega) e^{ix \cdot k} d_k \right\|_{L^p_{\omega}([-\pi, \pi]^n)} \lesssim 1,
$$

where the constant in $\lesssim$ does not depend on $\omega \in \Omega$ and $D$. Raising to the power $p$ and averaging over $\omega \in \Omega$, we obtain

$$
\int_{[-\pi, \pi]^n} \int_{\Omega} \left\| \sum_{k \in \mathbb{Z}^n} r_k(\omega) e^{ix \cdot k} d_k \right\|_{L^p_{\omega}(\mathbb{R}^n)}^p dP(\omega) dx \lesssim 1.
$$

By Khintchine’s inequality, the above inequality is equivalent to

$$
\left( \sum_{k \in \mathbb{Z}^n} |d_k|^2 \right)^{1/2} \lesssim 1.
$$

(4.7)

Notice that the implicit constant in (4.7) does not depend on the finite set $D \subset (\mathbb{Z}^n)^N$.

We take the finite set $D \subset (\mathbb{Z}^n)^N$ defined as follows. We take a sufficiently large number $L > 0$ and for sufficiently large $A \in \mathbb{N}$ we set

$$
D = D_A = \left\{ k \in (\mathbb{Z}^n)^N : 2^{A-1} \leq |k_1 + \cdots + k_N| \leq 2^{A+1} \right. \leq 2^{A-L-1} \leq |k_j| \leq 2^{A-L}, \quad j = 1, \cdots, N-1 \right\}.
$$

If $L > 0$ is chosen sufficiently large, then we have

$$
k \in D_A \Rightarrow 2^{A-2} \leq |k_N| \leq 2^{A+2},
$$

and thus for all $k \in \mathbb{Z}^n$ satisfying $2^{A-1} \leq |k| \leq 2^{A+1}$ we have

$$
d_k \approx \sum_{k \in D_A, \atop k_1 + \cdots + k_N = k} 2^{A(m-b_1-\cdots-b_N)} \approx 2^{A(m-b_1-\cdots-b_N)} \times \text{card} \left\{ (k_1, \ldots, k_{N-1}) \in (\mathbb{Z}^n)^{N-1} \mid 2^{A-L-1} \leq |k_j| \leq 2^{A-L}, \ j = 1, \cdots, N-1 \right\} \approx 2^{A(m-b_1-\cdots-b_N)} 2^{An(N-1)}. \n$$

Thus

$$
\left( \sum_{k \in \mathbb{Z}^n} |d_k|^2 \right)^{1/2} \approx 2^{A(m-b_1-\cdots-b_N)} 2^{An(N-1)} 2^{An/2}. \n$$

Hence (4.7) implies

$$
2^{A(m-b_1-\cdots-b_N)} 2^{An(N-1)} 2^{An/2} \lesssim 1.
$$
Since this holds for arbitrarily large $A$, we have
\[ m \leq \sum_{j=1}^{N} b_j - n(N - 1) - \frac{n}{2}. \]
Taking limit as $a_j \to 0$ for $j \in J^c$, $a_j \to 1$ for $j \in J$, and $\epsilon \to 0$, we obtain
\[ m \leq \sum_{j \in J^c} \left( n - \frac{n}{p_j} \right) + \sum_{j \in J} n \frac{n}{2} - n(N - 1) - \frac{n}{2} \]
\[ = \frac{n}{2} - \sum_{j \in J^c} \frac{n}{p_j} - \sum_{j \in J} \frac{n}{2}, \]
which is the inequality (1.4).

Case II: $2 < p < \infty$ and $J \neq \emptyset$.
In order to simplify notation, we assume $1 \in J$, that is $2 \leq p_1 < \infty$. Suppose the multipliers $\sigma = \sigma(\xi_1, \xi_2, \ldots, \xi_N)$ and $\tau = \tau(\xi_1, \xi_2, \ldots, \xi_N)$ are related by
\[ \tau(\eta, \xi_2, \ldots, \xi_N) = \sigma(-\xi_2 - \cdots - \xi_N - \eta, \xi_2, \ldots, \xi_N). \]
Then
\[ \int_{\mathbb{R}^n} T_\sigma(f_1, f_2, \cdots, f_N)(x) g(x) \, dx = \int_{\mathbb{R}^n} T_\tau(g, f_2, \cdots, f_N)(x) f_1(x) \, dx \]
and thus duality implies
\[ T_\sigma : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_N} \to L^p \iff T_\tau : L^{p'} \times L^{p_2} \times \cdots \times L^{p_N} \to L^{p'}. \]
Also it is easy to see that
\[ \sigma \in S_{0,0}^m(\mathbb{R}^n, N)^\times \iff \tau \in S_{0,0}^m(\mathbb{R}^n, N)^\times. \]
Hence (4.1) implies
\[ \text{Op}(S_{0,0}^m(\mathbb{R}^n, N)^\times) \subset B(L^{p'} \times L^{p_2} \times \cdots \times L^{p_N} \to L^{p'}). \]
In the present case, we have $1 < p' < 2$ and $1 < p_1' \leq 2$. Hence by what has been proved in Case I, we have
\[ m \leq \frac{n}{2} - \frac{n}{p'} - \sum_{j \in J \setminus \{1\}} \frac{n}{2} - \sum_{j \in J^c} \frac{n}{p_j} = \frac{n}{p} - \sum_{j \in J} \frac{n}{2} - \sum_{j \in J^c} \frac{n}{p_j}, \]
which is the inequality (4.4).

Case III: $2 < p < \infty$ and $J = \emptyset$.
In this case, $1 < p_j < 2$ for all $j$ and the condition (1.4) reads as
\[ (4.8) \quad m \leq \frac{n}{p} - \sum_{j=1}^{N} \frac{n}{p_j}. \]
Take a function $\Psi \in C_0^\infty((\mathbb{R}^n)^N)$ such that
\[ \text{supp } \Psi \subset \left\{ \xi \in (\mathbb{R}^n)^N \mid 2^{-1/2}N \leq \sum_{j=1}^{N} |\xi_j| \leq 2^{1/2}N \right\}, \]
\[ \Psi(\xi) = 1 \text{ if } 2^{-1/4}N \leq \sum_{j=1}^{N} |\xi_j| \leq 2^{1/4}N. \]
Let $m \in \mathbb{R}$ and consider the multiplier
\[
\sigma(\xi) = \sum_{j=0}^{\infty} 2^{jm} \Psi(2^{-j} \xi), \quad \xi \in (\mathbb{R}^n)^N,
\]
which certainly belongs to the class $S^{m}_{0,0}(\mathbb{R}^n, N)^{\times}$. Take a nonzero function $\psi \in C^\infty_0(\mathbb{R}^n)$ such that
\[
\text{supp } \psi \subset \{ \xi \in \mathbb{R}^n \mid 2^{-1/4} \leq |\xi| \leq 2^{1/4} \}.
\]
Let $f_{j,k} \in \mathcal{S}(\mathbb{R}^n)$ be the functions whose Fourier transforms are given by
\[
\hat{f}_{j,k}(\xi_j) = 2^{kn(1/p_j-1)} \psi(2^{-k} \xi_j), \quad j = 1, \ldots, N, \quad k \in \mathbb{N}.
\]
Then
\[
\|f_{j,k}\|_{L^{p_j}} = \|F^{-1}\psi\|_{L^{p_j}} < \infty.
\]
From the conditions on $\Psi$ and $\psi$, we have
\[
T_{\sigma}(f_{1,k}, \ldots, f_{N,k})(x) = 2^{km} \prod_{j=1}^{N} F^{-1} \left( 2^{kn(1/p_j-1)} \psi(2^{-k} \xi_j) \right)(x)
\]
and hence
\[
\|T_{\sigma}(f_{1,k}, \ldots, f_{N,k})\|_{L^p} = c \cdot 2^{k(m+\sum_{j=1}^{N} n/p_j-n/p)}
\]
with $c = \|(F^{-1}\psi)^N\|_{L^p} \in (0, \infty)$. Thus if (4.11) holds we have $2^{k(m+\sum_{j=1}^{N} n/p_j-n/p)} \lesssim 1$ for all $k \in \mathbb{N}$, which implies (4.8). This completes the proof of Lemma 4.2. \hfill \Box

**Proof of Theorem 1.6** Let $N \geq 1$, $0 < p, p_1, \ldots, p_N \leq \infty$, and $1/p \leq 1/p_1 + \cdots + 1/p_N$. We assume there exists an $\epsilon > 0$ such that
\[
\text{Op} \left( S^{m}_{0,0}(\mathbb{R}^n, N)^{\times} \right) \subset B(H^{p_1} \times \cdots \times H^{p_N} \to L^p),
\]

(4.9)
\[
m = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^{N} \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\} + \epsilon,
\]

(4.10)
with $L^\infty$ replaced by $BMO$ when $p = \infty$, and will derive a contradiction. In [23, Theorem 6.1, Example 1.4] or in our Theorem 1.5 (or by Plancherel’s theorem in the case $N = 1$), it is already proved that
\[
\text{Op} \left( S^{m}_{0,0}(\mathbb{R}^n, N)^{\times} \right) \subset B(L^2 \times \cdots \times L^2 \to L^2),
\]

(4.11)
\[
m = \frac{n}{2} - \frac{Nn}{2}.
\]

From (4.9)-(4.10) and (4.11), by complex interpolation, it follows that
\[
\text{Op} \left( S^{m}_{0,0}(\mathbb{R}^n, N)^{\times} \right) \subset B(L^{\tilde{p}_1} \times \cdots \times L^{\tilde{p}_N} \to L^{\tilde{p}}),
\]

(4.12)
where $\tilde{p}_j, \tilde{p}, \tilde{m}$ are given by
\[
\frac{1}{\tilde{p}_j} = \frac{1 - \theta}{2} + \frac{\theta}{p_j}, \quad j = 1, \ldots, N,
\]
\[
\frac{1}{\tilde{p}} = \frac{1 - \theta}{2} + \frac{\theta}{p},
\]
\[
\tilde{m} = \sum_{j=1}^{N} n/p_j - n/p.
\]

\[\]
Lemma 4.2. Thus Theorem 1.6 is proved.

\[ \tilde{m} = (1 - \theta) \left( \frac{n}{2} - \frac{N}{2} \right) + \theta \left( \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^{N} \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\} + \epsilon \right) \]

\[ = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^{N} \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\} + \theta \epsilon, \]

and \( 0 < \theta < 1 \) is a sufficiently small number. Notice that we have \( 1 < \tilde{p}_j < \infty \) for all \( j \) if \( \theta \) is sufficiently small. Hence (4.12) with the above \( \tilde{p}_j, \tilde{p}, \tilde{m} \) cannot hold as we have shown in Lemma 4.2. Thus Theorem 1.6 is proved. \qed

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