Conormal varieties on the cominuscule Grassmannian-II

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Abstract
Let $X_w$ be a Schubert subvariety of a cominuscule Grassmannian $X$, and let $\mu : T^*X \to \mathcal{N}$ be the Springer map from the cotangent bundle of $X$ to the nilpotent cone $\mathcal{N}$. In this paper, we construct a resolution of singularities for the conormal variety $T^*_X X_w$ of $X_w$ in $X$. Further, for $X$ the usual or symplectic Grassmannian, we compute a system of equations defining $T^*_X X_w$ as a subvariety of the cotangent bundle $T^*X$ set-theoretically. This also yields a system of defining equations for the corresponding orbital varieties $\mu(T^*_X X_w)$. Inspired by the system of defining equations, we conjecture a type-independent equality, namely $T^*_X X_w = \pi^{-1}(X_w) \cap \mu^{-1}(\mu(T^*_X X_w))$. The set-theoretic version of this conjecture follows from this work and previous work for any cominuscule Grassmannian of type A, B, or C.

We work over an algebraically closed field $k$ of good characteristic (for a definition, see [4]). Let $G$ be a connected algebraic group whose Lie algebra $\mathfrak{g}$ is simple.

For $P$ a conjugacy class of parabolic subgroups of $G$, we denote by $X^P$ the variety of parabolic subgroups of $G$ whose conjugacy class is $P$. For a parabolic group $P$, let $u_P$ denote the Lie algebra of the unipotent radical of $P$. The cotangent bundle $T^*X^P$ of $X^P$ is then given by

$$T^*X^P = \{(P, x) \in X^P \times \mathcal{N} | x \in u_P\},$$

where $\mathcal{N}$ is the variety of nilpotent elements in $\mathfrak{g}$. The map $\mu : T^*X^P \to \mathcal{N}$, given by $\mu(P, x) = x$, is the celebrated Springer map.

Let $B$ be the conjugacy class of Borel subgroups of $G$. The Steinberg variety,

$$Z^P = \{(B, P, x) \in X^B \times X^P \times \mathcal{N} | x \in u_B \cap u_P\},$$

is reducible. Each irreducible component $Z^P_w$ of $Z^P$ is the conormal variety of a $G$-orbit closure (under the diagonal action) in $X^B \times X^P$.

In [11], Lakshmibai and Singh constructed a resolution of singularities for certain $Z^P_w$, namely where $X^P$ is cominuscule (see Sect. 1.10), and the opposite Schubert variety $X^{P,w}$ is smooth. One also obtains a system of equations for these $Z^P_w$ via the standard monomial theory for Kac-Moody groups as developed by Littelmann [9].

In this paper, we extend the aforementioned results of [11], constructing a resolution of singularities for each irreducible component $Z^P_w \subset Z^P$ of the Steinberg variety of a
cominuscule Grassmannian $X^P$. In types A and C, we also provide a system of defining 
equations for each component $Z^P_w$, viewed as a subvariety of $X^B \times X^P \times N$. This also yields 
a system of defining equations for certain orbital varieties, which we discuss later in this 
section.

Before getting into the details, let us first present the irreducible components of $Z^P$ from 
an alternate point of view. We fix a Borel subgroup $B$ in $G$, and a standard parabolic subgroup 
$P$ corresponding to omitting a cominuscule simple root $\gamma$, see Sect. 1.10. Let $X$ be the variety of ‘parabolic subgroups conjugate to $P$’. We have an isomorphism $X \cong G/P$, and further, a $G$-equivariant isomorphism,

$$G \times^B X \cong X^B \times X,$$

given by $(g, P) \mapsto (g B g^{-1}, g P g^{-1})$.

A $B$-orbit $C_w \subset X$ is called a Schubert cell, and its closure $X_w$ is called a Schubert 
variety. The Schubert cell, being $B$-homogeneous, hence smooth, admits a conormal bundle $T^*_X C_w$; its closure in $T^* X$ is precisely the conormal variety $T^*_X X_w$, see Sect. 1.1 and Lemma 1.2.

Consider the map $Z^P_w \to X^B$, given by $(B, P, x) \mapsto B$. We identify $Z^P_w$ as a fibre bundle 
over $X^B$ via this map, with the fibre over the point $B \in X^B$ being precisely the conormal 
variety $T^*_X X_w$. In particular, we have an isomorphism

$$G \times^B T^*_X X_w \cong Z^P_w.$$

From this viewpoint, it is clear that the geometry of $T^*_X X_w$ is closely related to the geometry of $Z^P_w$.

**Remark** A similar (and essentially equivalent) statement can be found in [5, Proposition 3.3.4]; the proof there is different, leveraging the fact that the Springer map $\mu : T^* X \to N$ can be identified with the moment map arising from the $G$-symplectic structure on $T^* X$.

We now present our main results. Let $X_w$ be a Schubert subvariety of a cominuscule Grassmannian $X$. In Sect. 2, we present a variety $\tilde{Z}_w$, which is a vector bundle over a Bott-Samelson variety resolving $X_w$, along with a proper birational $B$-equivariant map, $\theta_w : \tilde{Z}_w \to T^*_X X_w$.

**Theorem A** The map $\theta_w : \tilde{Z}_w \to T^*_X X_w$ is a $B$-equivariant resolution of singularities.

Since the map $\theta_w$ is $B$-equivariant, it also yields a resolution of singularities,

$$G \times^B \theta_w : G \times^B \tilde{Z}_w \to Z^P_w,$$

of the Steinberg component $Z^P_w$.

Next, we study the system of defining equations for the conormal variety $T^*_X X_w$ inside $T^* X$. For $i \geq 1$, let $E(i)$ denote a vector space with basis $e_1, \ldots, e_i$. We work simultaneously in two cases: either $G = SL_n$, or $G = Sp_n$ with the further assumption that $n = 2d$. In the first case, let $X$ be the usual Grassmannian,

$$Gr(d, n) = \left\{ V \subset E(n) \mid \dim V = d \right\}.$$

In the second case, let $\omega$ be the non-degenerate skew-symmetric bilinear form on $E(n)$, given by

$$\omega(e_i, e_j) = \begin{cases} 
\delta_{i+j, n+1} & \text{if } i \leq d, \\
-\delta_{i+j, n+1} & \text{if } i > d,
\end{cases}$$

where $\delta_{i,j}$ is the Kronecker delta.
and let $X$ be the symplectic Grassmannian,

$$\text{SGr}(2d) = \left\{ V \subset E(2d) \mid V = V^\perp \right\}.$$ 

In either case, the cotangent bundle of $X$ has the description

$$T^*X = \left\{ (V, x) \in X \times \mathcal{N} \mid \text{Im} \, x \subset V \subset \ker x \right\},$$

where, recall that $\mathcal{N}$ denotes the corresponding nilpotent cone.

Let $B$ be the Borel subgroup which is the stabilizer of the flag $(E(i))_i$ in $G$. In Theorem B, we provide a system of defining equations for $T^*_X X_w$ in $T^*X$.

**Theorem B** Consider $(V, x) \in T^*X$. Then $(V, x) \in T^*_X X_w$ if and only if $V \in X_w$, and further, for all $1 \leq j < i \leq l + 1$, we have

$$\dim(xE(t_i)/E(t_j)) \leq \begin{cases} r_i - 1 - r_j, \\ c_i - c_{j+1}. \end{cases}$$

The numbers $r_i, c_i, t_i$ are defined in terms of $w$, see Sect. 5.1.

A system of defining equations for $Z^P_w$ in $X^B \times X \times \mathcal{N}$ follows as a consequence. We simply replace the subspaces $E(t_i)$ with the subspace $E'(t_i)$, where $(E'(i))_i$ is the flag fixed by $B'$, the Borel subgroup at the first coordinate in $X^B \times X \times \mathcal{N}$.

Theorem B does not hold for the orthogonal Grassmannian. The key difference between $C_n$ and $D_n$ is the following: Consider the embedding of the Weyl group $W$ into $S_{2n}$. Then, for $C_n$, the Bruhat order on $W$ is identical to the order induced by restricting the (type A) Bruhat order on $S_{2n}$. This is not true for $D_n$.

In Eq. (7.1), we interpret Theorem B in a type-independent manner,

$$T^*_X X_w = \mu^{-1}(\mu(T^*_X X_w)) \cap \pi^{-1}(X_w).$$

Here $\mu : T^*X \to \mathcal{N}$ is the Springer map, and $\pi : T^*X \to X$ is the structure map defining the cotangent bundle.

We conjecture that Eq. (7.1) holds for any Schubert variety $X_w$ in any cominuscule Grassmannian $X$. The containment $\subset$ holds trivially. Besides Theorem B, further evidence in support of this conjecture is provided by Proposition 7.2, which states that Eq. (7.1) holds set-theoretically if $X_w$ is smooth, and by Proposition 7.3, which states that Eq. (7.1) holds scheme-theoretically if the opposite Schubert variety $X_{w^0}$ is smooth. Combining these results, we see that Eq. (7.1) holds set-theoretically for any cominuscule Grassmannian in types A, B, and C.

Finally, let us discuss orbital varieties, and their relationship with the conormal varieties of Schubert varieties. Consider a $G$-orbit $\mathcal{N}_\lambda^G \subset \mathcal{N}$. The irreducible components of the closure $\overline{\mathcal{N}_\lambda^G \cap u_B}$ are called orbital varieties. The reader might consult [6] for a general survey.

**Caution** Some authors define an orbital variety to be an irreducible component of $\mathcal{N}_\lambda^G \cap u_B$, where $\mathcal{N}_\lambda^G$ is $G$-orbit in $\mathcal{N}$. What we call an orbital variety here is an orbital variety closure in their language.

The key fact relating orbital varieties with conormal varieties is the following: Given a conjugacy class $P$, and a Schubert variety $X_w^P$, the image of the conormal variety $T^*_X X^P$ under the Springer map $\mu$ is an orbital variety. Conversely, every orbital variety is of the form $\mu(T^*_X X_w^B)$ for some Schubert variety $X_w^B \subset X^B$. Combined with Theorem B, this yields equations for the corresponding orbital varieties.
Theorem C  Let G, B, P, X, w, and μ be as in Theorem B. Then

\[
\mu(T^*_X X_w) = \left\{ x \in u_B \mid x^2 = 0, \dim(xE(u)/E(v)) \leq \begin{cases} r_i - r_j, & \forall 1 \leq i < j \leq l \\ c_i - c_{j+1} & \end{cases} \right\}.
\]

It is in general an open problem to give a combinatorial description of the inclusion order on orbital varieties. For varieties of matrices x satisfying x^2 = 0, this problem was solved by Melnikov [12] in type A, and in types B and C by Melnikov and Barnea [3]. In Corollary 6.6, we show how their results can be recovered as a simple consequence of Theorem C.

1 Preliminaries on Schubert varieties and conormal varieties

In this section, we recall some standard results about conormal varieties, Schubert varieties, and cominuscule Grassmannians.

Let k be an algebraically closed field of good characteristic, g a simple Lie algebra over k, and G a connected algebraic group for which g = Lie(G). We fix a maximal torus T in G, and a Borel subgroup B containing T.

Let Δ be the root system of g with respect to t = Lie(T), and let S and Δ^+ be the set of simple roots and positive roots respectively, corresponding to the choice of Borel subalgebra b = Lie(B). For α ∈ Δ, we will write g_α for the corresponding root space.

1.1 Conormal varieties

Let X be a smooth variety, Y a smooth (not necessarily closed) subvariety of X, and TX and TY the corresponding tangent bundles. The conormal bundle of Y in X is a vector bundle, T^*X → Y, whose fibre at a point p ∈ Y is the annihilator of the tangent subspace T_pY in T^*_pX, i.e.,

\[
(T^*_X Y)_p = \left\{ x \in T^*_pX \mid x(v) = 0, \forall v \in T_pY \right\}.
\]

Now suppose Y is a closed, but not necessarily smooth, subvariety of X, and let Y^{sm} be the smooth locus of Y. The closure (in T^*X) of the conormal bundle T^*_X Y^{sm} is called the conormal variety T^*_X Y of Y in X. Restricting the natural projection T^*X → X to the conormal variety induces a structure map, T^*_X Y → Y. The conormal variety T^*_X Y is not, in general, a vector bundle over Y.

Lemma 1.2 Suppose Y^o is a dense open subset of Y^{sm}. The closure of T^*_X Y^o in T^*X is precisely the conormal variety T^*_X Y.

Proof Since Y^o is open in Y^{sm}, we have T_p Y^o = T_p Y^{sm}, and hence (T^*_X Y^o)_p = (T^*_X Y^{sm})_p, for all p ∈ Y^o. It follows that T^*_X Y^o, being the pull-back of the vector bundle T^*_X Y^{sm} along the open inclusion Y^o ↪ Y^{sm}, is dense in T^*_X Y^{sm}. Consequently, we have T^*_X Y^o = T^*_X Y^{sm} = T^*_X Y.

1.3 Standard parabolic subgroups

A subgroup Q ⊂ G is called parabolic if the quotient X^Q = G/Q is proper. We will say that Q is a standard parabolic subgroup if B ⊂ Q.
Let \( \{ s_\alpha \mid \alpha \in S \} \) be the set of simple reflections in the Weyl group \( W = N_G(T)/T \); here \( N_G(T) \) is the normalizer of \( T \) in \( G \). For any subset \( R \subset S \), we have a subgroup \( W_R \subset W \), given by \( W_R = \{ s_\alpha \mid \alpha \in R \} \). The subgroup \( BW_R B \subset G \), given by,

\[
BW_R B = \left\{ b_1 w b_2 \mid b_1, b_2 \in B, w \in W_R \right\},
\]

is a standard parabolic subgroup; further, the map \( R \leftrightarrow BW_R B \) is a bijective correspondence from subsets of \( S \) to the standard parabolic subgroups of \( G \).

### 1.4 Schubert varieties

Let \( Q \) be a standard parabolic subgroup of \( G \), corresponding to some subset \( S_Q \subset S \). A \( B \)-orbit \( C_Q^w \subset X_Q^Q \) is called a Schubert cell. The pull-back of \( C_Q^w \) along the quotient map \( G \to G/Q = X_Q^Q \) is

\[
BwQ = \{ bwq \mid b \in B, q \in Q \}.
\]

The closure \( X_Q^Q \) of the Schubert cell \( C_Q^w \) is called a Schubert variety. The Schubert varieties \( X_Q^Q \subset X_Q^Q \) are indexed by \( w \in W_Q \), where

\[
W_Q \overset{\text{def}}{=} \{ w \in W \mid w(\alpha) > 0, \forall \alpha \in S_Q \}.
\]  

(1.5)

The set \( W_Q \) is called the set of minimal representatives of \( W \) with respect to \( Q \). It is clear from Eq. (1.5) that we have \( W_P \subset W_Q \) whenever \( Q \subset P \).

### 1.6 Bott–Samelson varieties

Let \( w = (s_1, \ldots, s_r) \) be a minimal word for \( w \), i.e., the \( s_i \) are simple reflections such that \( w = s_1 \ldots s_r \), and further, there is no sub-sequence of \( w \) whose product is \( w \).

Let \( P_i \) be the standard parabolic subgroup \( B s_i B \). The Bott-Samelson variety,

\[
\widetilde{X}_w^Q \overset{\text{def}}{=} P_1 x^B \cdots x^B P_r / B,
\]

provides a resolution of singularities of \( X_Q^Q \) via the map \( \rho_w^Q : \widetilde{X}_w^Q \to X_Q^Q \), given by,

\[
(p_1, \ldots, p_r) \mapsto p_1 \cdots p_r (\mod Q).
\]

Let \( P_i^0 \) denote the open set \( Bs_i B \subset P_i \). The map \( \rho_w^Q \) induces an isomorphism,

\[
(\widetilde{X}_w^Q)^0 = P_1^0 x^B \cdots x^B P_r^0 / B \overset{\sim}{\to} C_w^Q.
\]

### 1.7 The cotangent bundle \( T^*X_Q^Q \)

The cotangent bundle \( \pi : T^*X_Q^Q \to X_Q^Q \) is the vector bundle whose fibre \( T^*_pX_Q^Q \) at any point \( p \in X_Q^Q \) is precisely the cotangent space of \( X_Q^Q \) at \( p \). We call \( \pi \) the structure map defining the cotangent bundle.

Recall that the characteristic of \( k \) is a good prime. We have (cf. [1, Ch. 5]),

\[
T^*X_Q^Q = G \times^Q u_Q = (G \times u_Q)/Q,
\]  

(1.8)

where the quotient is with respect to the action \( q \cdot (g, x) = (gq, Ad(q^{-1}x)) \).
Table 1 Dynkin diagrams with cominuscule simple roots marked in black

| Diagram | Diagram |
|---------|---------|
| $A_n$   | $B_n$   |
| $C_n$   | $D_n$   |
| $E_6$   | $E_7$   |

1.9 The Springer map

Let $\mathcal{N}$ be the nilpotent cone of $g$, i.e.,

$$\mathcal{N} = \{ x \in g \mid ad(x) \text{ is nilpotent} \}.$$  

The Springer map $\mu^Q : T^*X^Q \to \mathcal{N}$, given by,

$$\mu^Q(g, x) = Ad(g)x,$$

is a proper map. The product map,

$$(\pi, \mu^Q) : T^*X^Q \to X^Q \times \mathcal{N}, \quad (g, x) \mapsto (g, \mu^Q(x)),$$

is a closed immersion, see [5] for details.

1.10 Cominuscule Grassmannians

A simple root $\gamma \in S$ is called cominuscule if the coefficient of $\gamma$ in any positive root is either 0 or 1, i.e.,

$$\alpha \in \Delta^+ \implies 2\gamma \nless \alpha.$$  

The cominuscule roots for various Dynkin diagrams are labelled in Table 1. The Dynkin diagrams not present in Table 1 do not admit any cominuscule roots.

Following Sect. 1.3, let $P$ be the standard parabolic subgroup corresponding to the subset $S\backslash\{\gamma\}$, where $\gamma$ is some cominuscule root. The variety $G/P$ is called a cominuscule Grassmannian.

Example 1.12 Let $E(n)$ be an $n$–dimensional vector space. The variety of $d$–dimensional subspaces of $E(n)$ is called the usual Grassmannian variety,

$$Gr(d, n) \overset{\text{def}}{=} \{ V \subset E(n) \mid \dim V = d \}.$$  

It is a cominuscule Grassmannian corresponding to the group $G = SL_n$, and the cominuscule root $\alpha_d \in A_{n-1}$, see Table 1.

Example 1.13 Let $\omega$ be a symplectic form on an $n$-dimensional vector space $E(n)$, where $n = 2d$ is necessarily even. The variety of Lagrangian subspaces in $E(n)$,

$$SGr(n) \overset{\text{def}}{=} \left\{ V \subset E(n) \mid V = V^\perp \right\},$$
is called the symplectic Grassmannian. It is a cominuscule Grassmannian corresponding to the group \( G = Sp_n \), and the cominuscule root \( \alpha_d \in C_d \), see Table 1.

2 A resolution of singularities of the conormal variety

Let \( G, B, T, \Delta, \Delta^+ \), and \( S \) be as in the previous section. We fix a cominuscule root \( \gamma \in S \). Let \( P \) be the standard parabolic subgroup corresponding to \( S \setminus \{ \gamma \} \), and let \( u \) be the Lie algebra of the unipotent radical of \( P \). We will denote the variety \( X^P = G/p \) as simply \( X \), and the Schubert varieties \( X^P_w \) as simply \( X_w \).

In this section, we study the conormal variety \( \pi_w : T^*_X X_w \to X_w \) of a Schubert variety \( X_w \) in \( X \). Following Lemma 2.1, the closure of the conormal bundle \( T^*_X C_w \) is precisely the conormal variety \( T^*_X X_w \). We describe the structure of \( T^*_X C_w \) in Lemma 2.2, which we then use to construct a resolution of singularities of \( T^*_X X_w \) in Theorem A.

**Lemma 2.1** For any \( w \in W^P \), the subspace \( u_w \) defined \( = u \cap Ad(w^{-1})u_B \) is \( B \)-stable.

**Proof** The subspaces \( u \) and \( Ad(w^{-1})u_B \) are \( T \)-stable, and so their intersection \( u_w \) is also \( T \)-stable. Further, since \( Ad(w^{-1})g_\alpha = g_{w^{-1}(\alpha)} \), we have,

\[
u_w = \bigoplus_{\alpha \leq \gamma} g_\alpha \cap \bigoplus_{\alpha \in \Delta^+} g_{w^{-1}(\alpha)} = \bigoplus_{\alpha \in R} g_\alpha,
\]

where \( R = \{ \alpha \in \Delta \mid \alpha \geq \gamma, w(\alpha) > 0 \} \). Since \( B \) is generated by the torus \( T \) and the root subgroups \( U_\alpha, \alpha \in S \), it suffices to show that \( u_w \) is \( U_\alpha \)-stable for all \( \alpha \in S \). This follows from the claim,

\[
\alpha \in R, \beta \in S, \alpha + \beta \in \Delta \implies \alpha + \beta \in R,
\]

which we now prove. We first consider the case \( \beta = \gamma \). For any \( \alpha \in R \), we have \( \gamma \leq \alpha \), hence \( 2\gamma \leq \alpha + \beta \). Now, since \( \gamma \) is cominuscule, we have \( \alpha + \beta \notin \Delta \).

Next, we consider \( \beta \in S \setminus \{ \gamma \} \). In this case, since \( w \in W^P \), it follows from Eq. (1.5) that \( w(\beta) > 0 \). Now, for any \( \alpha \in R \), we have \( w(\alpha) > 0 \), hence

\[
w(\alpha + \beta) = w(\alpha) + w(\beta) > 0.
\]

It follows from the definition of \( R \) that if \( \alpha + \beta \in \Delta \), then \( \alpha + \beta \in R \).

**Lemma 2.2** The conormal bundle \( T^*_X C_w \to C_w \) is isomorphic to the vector bundle \( BwB \times^B u_w \to C_w \), given by \( (bw, x) \mapsto bw(mod P) \).

**Proof** Let \( pr : X^B_B \to X_w \) be restriction of the quotient map \( G/B \to G/p \) to \( X^B_w \). Since \( w \in W^P \), the map \( pr \) restricts to an isomorphism of Schubert cells \( C^B_w \cong C_w \), see [8]. The claim now follows from the observation (cf. [11, Sect. 4.3]) that for any \((bw, x) \in T^*_X \), we have \((bw, x) \in T^*_X C_w \) if and only if \( x \in u_w \).

2.3 The subgroup \( Q \)

As a consequence of Lemma 2.1, we see that \( Stab_G(u_w) \) is a standard parabolic subgroup. Let \( Q \) be any standard parabolic subgroup contained in \( P \) that stabilizes \( u_w \), i.e.,

\[
Q \subset Stab_G(u_w) \cap P.
\]
We define a vector bundle \( \pi^Q_w : Z^Q_w \to X^Q_w \), where,
\[
Z^Q_w = Bw^Q \times^Q u_w, \quad \pi^Q_w(g, x) = g \pmod Q.
\]

Following Sect. 1.4, we see that \( w \in W^Q \), and the \( B \)-orbit closure \( Bw^Q/Q \) is precisely the Schubert variety indexed \( X^Q_w \).

Let \( w = (s_1, \ldots, s_r) \) be a minimal word for \( w \). Recall the Bott-Samelson variety \( \widetilde{X}_w \) from Sect. 1.6. We lift \( Z^Q_w \) to a vector bundle \( \pi_w : \widetilde{Z}_w \to \widetilde{X}_w \), given by
\[
\widetilde{Z}_w = P_1 \times^B \cdots \times^B P_r \times^B u_w,
\]
and \( \pi_w(p, x) = p \pmod B \), for \( p \in P_1 \times^B \cdots \times^B P_r \).

**Proposition 2.5** Let \( \tau \) be the quotient map \( G \times^Q u \to G \times^B u \). Viewing \( Z^Q_w \) as a subvariety of \( G \times^Q u \), we have \( \tau(Z^Q_w) \subset T^*_X X_w \). Let \( \tau_w : Z^Q_w \to T^*_X X_w \) denote the induced map. We have a commutative diagram,

\[
\begin{array}{ccc}
\widetilde{Z}_w & \xrightarrow{\theta^Q_w} & Z^Q_w \xrightarrow{\tau_w} T^*_X X_w \\
\downarrow{\pi_w} & & \downarrow{\pi^Q_w} \downarrow{\pi_w} \\
\widetilde{X}_w & \xrightarrow{\rho^Q_w} & X^Q_w \xrightarrow{pr} X_w
\end{array}
\]

Here \( pr : X^Q_w \to X_w \) is the restriction of the quotient map \( G/Q \to G/P \) to \( X^Q_w \), and \( \theta^Q_w : \widetilde{Z}_w \to Z^Q_w \) is the map given by \( \theta^Q_w(p_1, \ldots, p_r, x) = (p_1 \ldots p_r, x) \).

**Proof** Let \( (Z^Q_w)^o \) be the restriction of the vector bundle \( \pi^Q_w : Z^Q_w \to X^Q_w \) to the Schubert cell \( C^Q_w \). The quotient map \( G/B \to G/Q \) induces an isomorphism \( C^B_w \sim C^Q_w \) of Schubert cells. Consequently, the quotient map,
\[
Z^o_w = BwB \times^B u_w \to Bw^Q \times^Q u_w, \quad (bw, x) \mapsto (bw, x), \quad (2.6)
\]
is an isomorphism. Observe that this map is the inverse of \( \tau((Z^Q_w)^o) \), and so,
\[
\tau((Z^Q_w)^o) = T^*_X C_w \subset T^*_X X_w.
\]

Now, since \( T^*_X X_w \) is a closed subvariety, it follows that \( \tau(Z^Q_w) \subset T^*_X X_w \).

Finally, the commutativity of the diagram is a simple verification based on the formulae defining the various maps. \( \square \)

Before we prove Theorem A, let us recall some standard results about proper maps, which the reader can find, for example, in [7, Ch. 2].

**Proposition 2.7** The following properties are true:

1. Closed immersions are separated and proper.
2. The composition of proper maps is proper.
3. If \( g : X \to Y \) is a proper map, then \( g \times \text{id}_Z : X \times Z \to Y \times Z \) is proper.
4. Let \( f : Y \hookrightarrow Z \) be a closed immersion. A map \( g : X \to Y \) is proper if and only if \( f \circ g \) is proper.

**Theorem A** The maps \( \theta^Q_w \) and \( \tau_w \) are proper and birational, and the composite map \( \theta^Q_w \overset{\text{def}}{=} \tau_w \circ \theta^Q_w \) is a \( B \)-equivariant resolution of singularities \( \theta^Q_w : \widetilde{Z}_w \to T^*_X X_w \). The map \( \theta^Q_w \) is independent of the choice of \( Q \).
The birationality of $\tau_w$ is a consequence of Eq. (2.6). Recall from Sect. 1.6 that $\rho^Q_w$ induces an isomorphism $(\check{X}_w^Q)^{\circ} \sim \check{C}_w^Q$. Consequently, $\theta^Q_w$ induces an isomorphism $(\check{Z}_w^Q)^{\circ} \sim (Z_w^Q)^{\circ}$. It follows that $\theta^Q_w$ is birational.

Consider now the commutative diagram

\begin{equation}
\begin{array}{ccc}
\check{Z}_w^Q & \xrightarrow{\theta^Q_w} & Z_w^Q \\
f & & g \\
\check{X}_w^Q \times N & \xrightarrow{\rho^Q_w \times \id_N} & X_w^Q \times N \\
& & h \\
\end{array}
\end{equation}

where $f$, $g$, $h$ are the closed immersions given by

$$f(p_1, \ldots, p_r, x) = (\pi_w(p_1, \ldots, p_r, x), \Ad(p_1 \cdots p_r)x),$$

$$g(a, x) = (\pi_w^Q(a, x), \Ad(a)x),$$

$$h(a, x) = (\pi_w(a, x), \Ad(a)x).$$

Observe that the map,

$$(\pr \times \id_N) \circ (\rho^Q_w \times \id) = \rho_w \times \id_N,$$

is independent of the choice of $Q$, and therefore, the map $\theta_w = \tau_w \circ \theta^Q_w$ is also independent of the choice of $Q$.

Next, the maps $\rho^Q_w$ and $\pr$ are proper; hence $\rho^Q_w \times \id_N$ and $\check{X}_w^Q \times N$ are proper. Consequently, $\theta^Q_w$ and $\tau_w$ are proper.

Finally, observe that $\check{Z}_w^Q$, being a vector bundle over the smooth variety $\check{X}_w^Q$, is itself a smooth variety. Therefore, the map $\theta_w$ is a resolution of singularities.

2.9 Relationship with the Nash blowup of $X_w$

Let $Z \hookrightarrow M$ be a closed subvariety of a smooth variety $M$, let $N(Z) \to Z$ be the Nash blowup of $Z$ in $M$, and let $T^*_N Z$ be the conormal variety of $Z$ in $M$. A result of Sabbah [14, Sect. 1.2] states that the strict transform of $T^*_N Z$ along $N(Z) \to Z$ is a vector bundle over $N(Z)$. This relates our construction of the vector bundle $Z_w^Q \to X_w^Q$ to a recent result of Richmond, Slofstra and Woo [13, Theorem 2.1] identifying the map $X_w^Q \to X_w$ as a Nash blowup for an appropriate choice of $Q$.

3 The type A Grassmannian

The simple linear group $SL_n$ is a semisimple group corresponding to the Dynkin diagram $A_{n-1}$; every simple root of $A_{n-1}$ is cominuscule, see Table 1. For every integer $d < n$, the space $Gr(d, n)$ of $d$-dimensional subspaces of an $n$-dimensional vector space, is a cominuscule Grassmannian, corresponding to the root $\alpha_d$ in $A_{n-1}$. In this section, we recall the classical theory of Schubert varieties in type A. The primary reference for this section is [10].
3.1 The root system of $SL_n$

Let $E(n)$ be an $n$–dimensional vector space with a privileged basis $\{e_1, \ldots, e_n\}$. The group $G = SL_n$ acts on $E(n)$ by left multiplication with respect to this basis. The Lie algebra $\mathfrak{g}$ of $G$ is precisely the set of traceless $n \times n$ matrices, i.e.,

$$\mathfrak{g} = \{ x \in \text{Mat}_n(k) \mid \text{trace}(x) = 0 \} .$$

Let $t$ be the set of diagonal matrices in $\mathfrak{g}$. For $1 \leq i \leq n$, let $\epsilon_i \in t^*$ be the linear functional given by

$$\langle \epsilon_i, \sum_{j=1}^n a_j E_{j,j} \rangle = a_i,$$

where $E_{j,j}$ is the diagonal matrix with entry 1 in the $j^{th}$ position and zero elsewhere.

We fix $B$ to be the set of upper triangular matrices in $G$. The Lie algebra $b$ of $B$ is then precisely the set of upper triangular matrices in $\mathfrak{g}$. The root system $\Delta$ of $\mathfrak{g}$ with respect to $(b, t)$ is precisely,

$$\Delta = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n \},$$

with the simple root $\alpha_i \in S = A_{n-1}$ being given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$. The root $\epsilon_i - \epsilon_j$ is positive if and only if $i < j$.

We denote by $E_{i,j}$ the elementary $n \times n$ matrix with 1 in the $(i, j)$ position and 0 elsewhere; and by $[E_{i,j}]$ the one-dimensional subspace of $\mathfrak{g}$ spanned by $E_{i,j}$. Then $[E_{i,j}]$ is precisely the root space corresponding to the root $\epsilon_i - \epsilon_j$.

3.2 Partial flag varieties

Let $q = (q_0, \ldots, q_r)$ be an integer-valued sequence satisfying $0 = q_0 \leq q_1 \leq \cdots \leq q_r = n$.

For $0 \leq i \leq n$, we denote by $E(i)$, the subspace of $E(n)$ with basis $e_1, \ldots, e_i$, and by $E(q)$, the partial flag $E(q_0) \subset \cdots \subset E(q_r)$. Let $Q$ be the parabolic subgroup of $SL_n$ corresponding to the subset $S_Q = \{ \alpha_j \mid j \neq q_i, 1 \leq i \leq r \}$. The variety $X^Q = G/Q$ is precisely the variety of partial flags of shape $q$.

$$X^Q = \{ F(q_0) \subset \cdots \subset F(q_r) \mid \dim F(q_i) = q_i \} .$$

For brevity, we will denote a partial flag $F(q_0) \subset \cdots \subset F(q_r)$ of shape $q$ by $F(q)$.

As a particular example, let $P$ be the standard parabolic subgroup corresponding to the subset $S \setminus \{ \alpha_d \}$. Then $X^d := G/P$ is precisely,

$$X^d = Gr(d, n) = \{ 0 \subset V \subset E(n) \mid \dim V = d \} .$$

3.3 The Weyl group

The Weyl group of $G$ is isomorphic to $S_n$, the symmetric group on $n$ elements. The action of $W$ on $\Delta$ is given by the formula,

$$w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)} .$$

In particular, $w(\epsilon_i - \epsilon_j) > 0$ if and only if $w(i) < w(j)$.
The set of minimal representatives with respect to $Q$ is given by,

$$S_n^Q = \left\{ w \in S_n \mid w(q_i + 1) < w(q_i + 2) < \cdots < w(q_i + 1), \forall 0 \leq i \leq r \right\}.$$  

(3.4)

### 3.5 Schubert varieties

For $w \in S_n$, let $m_w(i, j)$ be the number of non-zero entries in the top left $i \times j$ sub-matrix of the permutation matrix $\sum E_{w(k), k}$, i.e.,

$$m_w(i, j) \overset{\text{def}}{=} \# \{w(1), \ldots, w(j)\} \cap \{1, \ldots, i\} = \#\{(k, w(k)) \mid k \leq j, w(k) \leq i\}.$$  

(3.6)

The Schubert cells, $C^Q_w \subset X^Q$ and $C^d_w \subset X$, are given by,

$$C^Q_w = \left\{ F(q) \in X^Q \mid \dim(F(q_i) \cap E(j)) = m_w(j, q_i), 1 \leq j \leq n, 1 \leq i \leq r \right\},$$

$$C^d_w = \left\{ V \in X^d \mid \dim(V \cap E(j)) = m_w(j, d), 1 \leq j \leq n \right\},$$

while the Schubert varieties, $X^Q_w \subset X^Q$ and $X^d_w \subset X^d$, are given by

$$X^Q_w = \left\{ F(q) \in X^Q \mid \dim(F(q_i) \cap E(j)) \geq m_w(j, q_i), 1 \leq j \leq n, 1 \leq i \leq r \right\},$$

$$X^d_w = \left\{ V \in X \mid \dim(V \cap E(j)) \geq m_w(j, d), 1 \leq j \leq n \right\}.$$  

(3.7)

In particular, we have $F(q) \in X^Q_w$, if and only if $F(q_i) \in X^Q_{d_i}$ for all $i$.

### 3.8 The projection map

Suppose $d = q_i$ for some $i$, and consider the projection map $pr_d : X^Q \to X^d$, given by $F(q) \mapsto F(d)$.

For any $w \in S_n$, we have $pr_d(X^Q_w) = X^d_w$. Further, if $w$ satisfies

$$w(1) > \cdots > w(d),$$

$$w(d + 1) > \cdots > w(n),$$

then $pr_d^{-1}(X^d_w) = X^Q_w$, see [10]. Any $w \in S_n$ satisfying Eq. (3.9) is called a maximal representative with respect to $P$.

The following lemmata are easy consequences of standard results on Schubert varieties. They are used repeatedly in the proofs of Propositions 5.14, 5.17 and 6.1.

**Lemma 3.10** Suppose we have integers $0 \leq k \leq d \leq n$, a permutation $w \in S_n$, and a $k$-dimensional subspace $U \subset E(n)$. If $U \in X^k_w$, then

$$\dim(U \cap E(i)) \geq m_w(i, d) - (d - k) \quad \forall 1 \leq i \leq n.$$  

Conversely, suppose the above inequalities hold, and further, $w(1) > \cdots > w(d)$. Then $U \in X^k_w$.

**Proof** Observe that for $1 \leq i \leq n$, we have,

$$m_w(i, k) \geq \max\{0, m_w(i, d) - (d - k)\}.$$  

with equality holding for all $i$ if $w(1) > \cdots > w(d)$. \hfill $\square$
Lemma 3.11 Consider integers $0 \leq k \leq d \leq n$, and a permutation $w \in S_n$, satisfying $w(k + 1) > \cdots > w(n)$. Given $U \in X^d_w$, we have $U \in X^d_w$, if and only if,

$$\dim(U \cap E(i)) \geq m_w(i, k) \quad \forall 1 \leq i \leq n.$$  

**Proof** Consider first $U \in X^d_w$. It follows from Eqs. (3.6) and (3.7) that

$$\dim(U \cap E(i)) \geq m_w(i, d) \geq m_w(i, k) \quad \forall 1 \leq i \leq n.$$  

We now prove that any $U \in X^d_w$ satisfying the inequalities of the lemma belongs to $X^d_w$. Since $k \leq d = \dim U$, there exists $1 \leq j \leq n$ such that $\dim(U \cap E(j)) = k$. Let $V = U \cap E(j)$. It is clear that $V \in X^k_w$. Now, since the statement of the lemma only involves $w$ via the integers $m_w(i, k)$ and $m_w(i, d)$, we may assume without loss of generality that $w(1) > \cdots > w(k)$.

Let $Q$ be the parabolic group corresponding to the sequence $q = (k, d)$. Then, we have $\text{pr}_k(V \subset U) = V$ and $\text{pr}_d(V \subset U) = U$. It follows from Sect. 3.8 that,

$$\text{pr}_k^{-1}(X^k_w) = X^Q_w \implies \text{pr}_d(\text{pr}_k^{-1}(X^k_w)) = X^d_w.$$  

Consequently, we obtain $U \in X^d_w$. $\square$

**Proposition 3.12** Consider integers $0 \leq k \leq d \leq m \leq n$, and a permutation $w \in S_n$. Given subspaces $U \subset V \subset E(n)$ satisfying $\dim U = k$, and

$$\dim(U \cap E(i)) \geq m_w(i, d) - (d - k) \quad \forall 1 \leq i \leq n,$$

$$\dim(V \cap E(i)) \geq m_w(i, m) \quad \forall 1 \leq i \leq n,$$

there exists $U' \in X^d_w$ satisfying $U \subset U' \subset V$.

**Proof** Set $l = \dim V$. Observe that

$$l = \dim(V \cap E(n)) \geq m_w(n, m) = m.$$  

Let $Q'$ be the parabolic group corresponding to the sequence $(k, l)$, and $Q$ the parabolic group corresponding to the sequence $(k, d, l)$. We have a projection map $\text{pr} : X^Q \to X^{Q'}$, given by $F(k, d, l) \mapsto F(k, l)$.

Since the statement of the proposition only involves $w$ via the integers $m_w(i, d)$ and $m_w(i, m)$, we may replace $w$ by any permutation $v$ satisfying

$$\{v(1), \ldots, v(d)\} = \{w(1), \ldots, w(d)\},$$

$$\{v(d + 1), \ldots, v(m)\} = \{w(d + 1), \ldots, w(m)\},$$

$$\{v(m + 1), \ldots, v(n)\} = \{w(m + 1), \ldots, w(n)\},$$  

without changing the statement. In particular, we may assume that

$$w(1) > w(2) > \cdots > w(d),$$

$$w(d + 1) > \cdots > w(m),$$

$$w(m + 1) > \cdots > w(n).$$  

(3.13)

Using Lemmas 3.10 and 3.11, we deduce that $(U \subset V) \in X^{Q'}_w$. Further, it follows from Eq. (3.13) that the projection map $X^Q_w \to X^{Q'}_w$ is surjective, see [10]. In particular, there exists a partial flag $F(k, d, l) \in X^{Q'}_w$, for which $F(k) = U$ and $F(l) = V$. This partial flag yields the required subspace $F(d)$, satisfying $U \subset F(d) \subset V$, and $F(d) \in X^d_w$. $\square$
4 The symplectic Grassmannian

Let \( n = 2d \). The symplectic group \( Sp_n \) is a semisimple group corresponding to the Dynkin diagram \( C_d \); it admits a cominuscule root \( \alpha_d \) (see Table 1). The corresponding cominuscule Grassmannian is the so-called symplectic Grassmannian, \( SGr(n) \). In this section, we recall some facts about \( Sp_n \), its root system, and the Schubert subvarieties of \( SGr(n) \). We follow closely the presentation in [10].

4.1 The Bilinear form

Let \( E(n) \) be an \( n \)–dimensional vector space with a privileged basis \( \{e_1, \ldots, e_n\} \). For \( 1 \leq i \leq n \), we define,

\[
\tilde{i} \overset{\text{def}}{=} n + 1 - i.
\]

Consider the non-degenerate skew-symplectic bilinear form \( \omega \) on \( E(n) \) given by,

\[
\omega(e_i, e_j) = \begin{cases} 
\delta_{i,j} & \text{if } i \leq d, \\
-\delta_{i,j} & \text{if } i > d.
\end{cases}
\]

For \( V \) a subspace of \( E(n) \), we define the orthogonal subspace,

\[
V^\perp = \{ u \in E(n) \mid \omega(u, v) = 0, \ \forall v \in V \}.
\]

A simple calculation yields \( E(i)^\perp = E(n - i) \), for \( 1 \leq i \leq n \). Further, as a consequence of the non-degeneracy of \( \omega \), we have the formulae,

\[
\dim V + \dim V^\perp = n, \quad U^\perp \cap V^\perp = (U + V)^\perp, \quad (V^\perp)^\perp = V,
\]

for any subspaces \( U, V \subset E(n) \).

4.3 The symplectic group \( Sp_n \)

Let \( G = \text{Stab}_{SL_n}(\omega) \), i.e.,

\[
G = \{ g \in SL_n \mid \omega(gu, gv) = \omega(u, v), \ \forall u, v \in E(n) \}.
\]

The group \( G \) is the symplectic group \( Sp_n \). It is a semisimple group with Dynkin diagram \( C_d \). Its Lie algebra \( g \) is given by

\[
g = \{ x \in sl_n \mid \omega(xu, v) + \omega(u, xv) = 0, \ \forall u, v \in E(n) \}.
\]

Let \( T' \) (resp. \( t' \)) be the set of diagonal matrices, and \( B' \) (resp. \( b' \)) the set of upper triangular matrices in \( SL_n \) (resp. \( sl_n \)). The subgroup \( T = T' \cap G \) is a maximal torus in \( G \), and the subgroup \( B = B' \cap G \) is a Borel subgroup of \( G \).

4.4 The Weyl group

Let \( s_1, \ldots, s_d \) denote the simple reflections in Weyl group \( W \) of \( G \), and let \( r_1, \ldots, r_{n-1} \) denote the simple reflections of \( S_n \). We have an embedding \( W \hookrightarrow S_n \), given by,

\[
s_i \mapsto \begin{cases} 
rr_{n-i} & \text{for } 1 \leq i < d, \\
r_d & \text{for } i = d.
\end{cases}
\]
Via this embedding, we have,
\[ W = \left\{ w \in S_n \mid w(i) = w(j), \ 1 \leq i \leq d \right\}. \]

The Bruhat order on \( S_n \) induces a partial order on \( W \). This induced order is precisely the Bruhat order on \( W \).

### 4.5 The root system of \( Sp_n \)

Recall from Sect. 3.1, the linear functionals \( \epsilon_1, \ldots, \epsilon_n \) on \( t' \). By abuse of notation, we also denote by \( \epsilon_i \), the restriction \( \epsilon_i | t \). Following [10], we present the root system of \( G \) with respect to \( (B, T) \). The simple root \( \alpha_i \in S = C_d \) is given by
\[
\alpha_i = \begin{cases} 
\epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i < d, \\
2\epsilon_d & \text{for } i = d.
\end{cases}
\]

The set of roots \( \Delta \), and the set of positive roots \( \Delta^+ \), are given by
\[
\Delta = \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq d \} \cup \{ \pm 2\epsilon_i \mid 1 \leq i \leq d \},
\]
\[
\Delta^+ = \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq d \} \cup \{ 2\epsilon_i \mid 1 \leq i \leq d \}.
\]

The corresponding root spaces are given by \( g_{2\epsilon_j} = [E_{i,j}], g_{-2\epsilon_j} = [E'_{i,j}] \),
\[ g_{\epsilon_i+\epsilon_j} = [E_{i,j} + E_{j,i}], \quad g_{-\epsilon_i-\epsilon_j} = [E'_{i,j} + E'_{j,i}], \quad g_{\epsilon_i-\epsilon_j} = [E_{i,j} - E_{j,i}]. \]

### 4.6 The subspace \( u_w \)

Let \( v \) be the Lie algebra of the unipotent radical of \( P' \), and \( u \) the Lie algebra of the unipotent radical of \( P \). We have
\[
v = \bigoplus_{i \leq j < d} [E_{i,j}], \quad u = \bigoplus_{1 \leq i < j \leq d} g_{\epsilon_i+\epsilon_j} = \bigoplus_{1 \leq i < j \leq d} [E_{i,j} + E_{j,i}].
\]

In particular, we have \( u = v \cap g \). Recall the subspace \( u_w \) from Lemma 2.1. Since \( g \) is stable under the action of \( Ad(w^{-1}) \), we have,
\[
u_w = u \cap Ad(w^{-1})b = (v \cap g) \cap Ad(w^{-1})(b' \cap g) = v \cap Ad(w^{-1})b' \cap g. \quad (4.7)
\]

### 4.8 Standard parabolic subgroups

Let \( q = (q_0, \ldots, q_r) \) be any integer-valued sequence satisfying \( 0 = q_0 \leq q_1 \leq \cdots \leq q_r = n \), and further, \( q_i + q_{r-i} = n \) for \( 1 \leq i \leq r \). Suppose \( Q' \) is the standard parabolic subgroup of \( SL_n \) corresponding to the subset,
\[
\{ \alpha_j \in A_{n-1} \mid j \neq q_i, \ 1 \leq i \leq r - 1 \}.
\]

Then \( Q = Q' \cap G \) is the parabolic subgroup of \( G \) corresponding to the subset,
\[
\{ \alpha_j \in S \mid j \neq q_i, \ 1 \leq i \leq \lceil r/2 \rceil \}.
\]

The variety \( X^Q = G/Q \) is precisely the variety of isotropic flags of shape \( q \), i.e.,
\[
X^Q = \left\{ F(q) \in SL_n/Q' \mid F(q_i) \perp F(q_{r-i}) \right\}.
\]
As a particular example, let $P'$ be the standard parabolic subgroup of $SL_n$ corresponding to the subset $A_{n-1}\setminus\{e_d\}$, and let $P = P' \cap G$. Then $P$ is the standard parabolic corresponding to $S\setminus\{e_d\}$, and further,

$$X = G/P = \left\{ V \subseteq E(n) \mid V = V^\perp \right\}.$$ 

Observe that the condition $V = V^\perp$ ensures that $\dim V = d$, see Eq. (4.2).

### 4.9 Schubert varieties

Consider an element $w \in W^Q$. By viewing $w$ as an element of $S_n$, we define the numbers $m_w(i, k)$ precisely as in Eq. (3.6). The Schubert cell $C_w^Q$ is given by,

$$C_w^Q = \left\{ F(q) \in X^Q \mid \dim (F(q_i) \cap E(j)) = m_w(j, q_i), \ 1 \leq i \leq r, \ 1 \leq j \leq n \right\},$$

and the Schubert variety $X_w^Q$ is given by,

$$X_w^Q = \left\{ F(q) \in X^Q \mid \dim (F(q_i) \cap E(j)) \geq m_w(j, q_i), \ 1 \leq i \leq r, \ 1 \leq j \leq n \right\} \ (4.10)$$

In particular, any Schubert subvariety of $X^Q$ can be identified as the intersection of a Schubert subvariety of $SL_n/Q'$ with the subvariety $Sp_n/Q \subset SL_n/Q'$. In fact, this identification is scheme-theoretic, see [10, Sect. 6.1.1.2].

### 4.11 Numerical redundancy

Viewing $w$ as an element of $S_n$ (see Sect. 4.4), we define the integers $t_i, t_i', r_i, c_i$ exactly as in Sect. 5.1 and Eq. (5.2). Observe that since $w(i) = \overline{w(i)}$ for all $1 \leq i \leq n$, the permutation matrix of $w$ is symmetric across the anti-diagonal. Consequently, for any $0 \leq i \leq l$, we have,

$$r_i + c_{l-i} = d, \quad t_i + t_{l-i} = n. \quad (4.12)$$

In particular, we have $E(t_i) = E(t_{l-i})$.

The conditions defining the Schubert variety $X_w^Q \subset X^Q$, described in Eq. (4.10), are not minimal. We describe this redundancy in the next lemma.

**Lemma 4.13** Consider $F(q) \in X^Q$. Then $F(q) \in X_w^Q$ if and only if

$$\dim (F(q_i) \cap E(j)) \geq m_w(j, q_i), \quad 1 \leq i \leq l, \ 1 \leq j \leq n.$$

**Proof** Since the permutation matrix of $w$ is symmetric across the anti-diagonal, the number of non-zero entries in the top left $i \times j$ corner of $w$ equals the number of entries in the bottom right $i \times j$ corner. Further, since each row and column of this matrix has precisely one non-zero entry, we have,

$$m_w(i, j) = \# \{(k, w(k)) \mid k > n - j, \ w(k) > n - i\}$$

$$= \# \{(k, w(k)) \mid k \leq n - j\} \cup \{(k, w(k)) \mid w(k) \leq n - i\}$$

$$= n - ((n - j) + (n - i) - m_w(n - i, n - j)).$$

Hence, for $1 \leq i, j \leq n$, we have the formula,

$$n - (i + j - m_w(i, j)) = m_w(n - i, n - j).$$
Consider some $F(q) \in X^Q$ satisfying the inequalities of the lemma. Given $1 \leq i \leq l$ and $1 \leq j \leq n$, we have,
\[
\dim(F(q_i) \cap E(j)) \geq m_w(j, q_i) \\
\implies \dim(F(q_i) + E(j)) \leq q_i + j - m_w(j, q_i) \\
\implies \dim((F(q_i) + E(j))^\perp) \geq n - (q_i + j - m_w(j, q_i)) \\
\implies \dim(F(q_i)^\perp \cap E(n - j)) \geq m_w(n - j, n - q_i).
\]
The final inequality follows from the penultimate as a consequence of Eq. (4.2). We see that $F(\underline{q})$ satisfies Eq. (4.10), and hence obtain $F(\underline{q}) \in X_w^Q$. \hfill $\square$

5 Defining equations for the conormal variety in types A and C

Fix integers $d < n$. We work simultaneously in two cases: either $G = SL_n$, or $G = Sp_n$ with the further assumption that $n = 2d$. Let $B$ be the subgroup of upper triangular matrices in $G$, and $P$ the standard parabolic subgroup of $G$ corresponding to the subset $\delta \setminus \{a_d\}$ of simple roots. As discussed in Sects. 3 and 4, the variety $X = G/P$ is either the usual Grassmannian $Gr(d, n)$ or the symplectic Grassmannian $SGr(2n)$.

Consider a Schubert variety $X_w \subset X$ corresponding to some $w \in W^P$. In Sect. 5.10, we fix a particular standard parabolic subgroup $Q$ satisfying Eq. (2.4). We use this choice to prove Theorem B, which states that a point $p \in T^*X$ is in $T^*_X X_w$ if and only if $\pi(p) \in X_w$ and $\mu(p)$ satisfies Eq. (5.21). Here $\pi : T^*X \to X$ is the structure map, and $\mu$ the Springer map.

Recall the commutative diagram from Proposition 2.5. We show in Proposition 5.13 that for any point in $Z_w^Q$, its image under $\mu \circ \tau_w$ satisfies the equations in Theorem B. Conversely, we show in Propositions 5.14 and 5.17 that any point in $T^*X$ lying over $X_w$, which further satisfies Eq. (5.21), belongs to $\tau_w(Z_w^Q) = T^*_X X_w$.

Following Sect. 1.9, we implicitly identify, throughout this section, the cotangent bundle $T^*X$ with its image under the closed embedding $(\pi, \mu) : T^*X \hookrightarrow X \times \mathcal{N}$, 
\[
T^*X = \{(V, x) \in X \times \mathcal{N} \mid x E(n) \subset V, \ xV = 0\}.
\]

5.1 The numbers $r_i, c_i$

For integers $a, b$, let $(a, b]$ denote the sequence, 
\[
{a + 1, a + 2, \ldots, b}.
\]
We fix $w \in W^P$, which we view as an element of $S_n$, using Sect. 4.4 if necessary. Following Eq. (3.4), we have, 
\[
w(1) < w(2) < \cdots < w(d), \quad w(d + 1) < \cdots < w(n).
\]
Consequently, $w$ is completely determined by the sequence $w(1), \ldots, w(d)$, which we now write as the following concatenation of contiguous sequences, 
\[
(t'_1, t_1], (t'_2, t_2], \ldots, (t'_l, t_l].
\]
Here the $t_i, t'_i$ are certain integers satisfying, 
\[
0 \leq t'_1 < t_1 < t'_2 < \cdots < t_{l-1} < t'_l < t_l \leq n,
\]

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and $\sum (t_i - t_i') = d$. For convenience, we set $t_0 = 0$ and $t_{l+1}' = n$. The sequence $w(d + 1), \ldots, w(n)$ is precisely,

$$(t_0, t_1', (t_1, t_2'), \ldots, (t_l, \ldots, t_{l+1}')).$$

Consider the partial sums $r_0, \ldots, r_l$, and $c_0, \ldots, c_{l+1}$, given by,

$$r_i \overset{\text{def}}{=} \sum_{1 \leq j \leq i} (t_j - t_j'), \quad c_i \overset{\text{def}}{=} \sum_{1 \leq j \leq i} (t_j' - t_j).$$

(5.2)

For $1 \leq i \leq l$, we have $t_i = r_i + c_i$. Further, we have $r_l = d, c_{l+1} = n - d$, and

$$m_w(t_i, r_j) = \min\{r_i, r_j\} = r_{\min(i,j)},$$

$$m_w(t_i, d + c_j) = r_i + \min\{c_i, c_j\} = r_i + c_{\min(i,j)},$$

(5.3)

for all $1 \leq i, j \leq l$.

**Example 5.4** We illustrate the combinatorics of Sect. 5.1 with an example. Let $n = 15$ and $d = 7$, and consider the permutation $w \in S^d_n$, given by the one line notation,

$$w = (2, 3, 7, 8, 9, 11, 15, 1, 4, 5, 6, 10, 12, 13, 14).$$

Written as a concatenation of contiguous sequences, we have,

$$(w(1), \ldots, w(7)) = (2, 3, 7, 8, 9, 11, 15)$$

$$= (1, 3), (6, 9), (10, 11), (14, 15),$$

$$(w(9), \ldots, w(15)) = (1, 4, 5, 6, 10, 12, 13, 14)$$

$$= (0, 1), (3, 6), (9, 10), (11, 14), (15, 15).$$

Observe that $(15, 15)$ is the empty sequence, which we have listed only to match the notation in Sect. 5.1. Following said notation, we have $l = 4$,

$$(t_1', \ldots, t_4') = (1, 6, 10, 14), \quad (t_1, \ldots, t_4) = (3, 9, 11, 15),$$

$$(r_0, \ldots, r_4) = (0, 2, 5, 6, 7), \quad (c_0, \ldots, c_5) = (0, 1, 4, 5, 8, 8).$$

Observe that the consecutive differences

$$(r_1 - r_0, r_2 - r_1, r_3 - r_2, r_4 - r_3) = (2, 5, 6, 7),$$

$$(c_1 - c_0, c_2 - c_1, c_3 - c_2, c_4 - c_3, c_5 - c_4) = (1, 3, 1, 3, 0)$$

are precisely the sizes of the contiguous sub-sequences of the sequence $w(1), \ldots, w(d)$ and $w(d + 1), \ldots, w(n)$ respectively.

### 5.5 The sequence $q$

Let the integers $m_w(i, j), r_i$, and $c_i$ be as in Eqs. (3.6) and (5.2) respectively. We define integers $q_0, q_1, \ldots, q_{2l+1}$ by

$$q_i = \begin{cases} r_i & \text{for } 0 \leq i \leq l, \\ d + c_{i-l} & \text{for } l < i \leq 2l + 1. \end{cases}$$

For example, if $w$ is as in Example 5.4, then $(q_0, \ldots, q_9) = (0, 2, 5, 6, 7, 8, 11, 12, 15, 15)$. 

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Proposition 5.6 Suppose $G = SL_n$, and let $Q$ be the standard parabolic subgroup associated to the sequence $q = (q_0, \ldots, q_{2l+1})$. Then $Q$ satisfies Eq. (2.4), i.e., $Q \subset \text{Stab}_G(u_w) \cap P$.

Proof Observe that $q_i = d$; hence, we have $Q \subset P$. It remains to show that $Q$ stabilizes $u_w$. Recall the set $R$ and the subspace $u_w$ from Lemma 2.1. It follows from Sect. 5.1 that

$$R = \{ \epsilon_i - \epsilon_j \mid \exists k \text{ such that } i \leq q_k, j > q_{k+1} \},$$

$$u_w = \{ x \in g \mid xE(q_{l+i}) \subset E(q_{l-1}), \forall 1 \leq i \leq l + 1 \}. \quad (5.7)$$

Now, since $Q$ stabilizes the flag $E(q)$, it also stabilizes $u_w$.

Proposition 5.8 Suppose $G = Sp_n$, and consider the sequence $q = (q_0, \ldots, q_{2l})$. Then $q$ corresponds to a standard parabolic subgroup $Q$ of $G$, in the sense of Sect. 4.8. Further, $Q$ satisfies Eq. (2.4), i.e., $Q \subset \text{Stab}_G(u_w) \cap P$.

Proof First observe that Eq. (4.12) yields $q_i + q_{2l-i} = n$ for all $1 \leq i \leq 2l$, and hence the sequence $q = (q_0, \ldots, q_{2l})$ does indeed correspond to a parabolic subgroup of $G$ in the sense of Sect. 4.8.

It follows also from Eq. (4.12) that $c_{l+1} = c_l = d$, hence $q_{2l+1} = q_{2l} = n$. Therefore, the standard parabolic subgroup $Q' \subset SL_n$, associated to $(q_0, \ldots, q_{2l+1})$, is the same as the standard parabolic subgroup of $SL_n$, associated to $(q_0, \ldots, q_{2l})$.

Next, it follows from Eqs. (4.7) and (5.7) that

$$u_w = \{ x \in g \mid xE(q_{l+i}) \subset E(q_{l-1}), \forall 1 \leq i \leq l + 1 \}. \quad (5.9)$$

Now, since $Q'$ stabilizes $u_w$, and since $Q = Q' \cap G$, we have $Q \subset \text{Stab}_G(u_w)$. Finally, since $q_l = d$, we have $Q \subset P$, hence $Q \subset \text{Stab}_G(u_w) \cap P$.

5.10 Combinatorial description of $X_w^Q$

For $G = SL_n$, let $q$ and $Q$ be as in Proposition 5.6. For $G = Sp_n$, let $q$ and $Q$ be as in Proposition 5.8. Following Sect. 1.4, we see that $w$ indexes a Schubert variety $X_w^Q \subset X^Q$. We see from Eqs. (3.7), (5.3) and (4.10) that $F(q) \in X_w^Q$ if and only if

$$\dim(F(q_i) \cap E(t_j)) \geq \min\{r_i, r_j\} = r_{\min(i,j)} \quad \forall 1 \leq i, j \leq l,$$

$$\dim(F(q_{l+i}) \cap E(t_j)) \geq r_j + \min\{c_i, c_j\} = r_j + c_{\min(i,j)} \quad \forall 1 \leq i, j \leq l.$$

Observe that setting $i = j$ yields $F(q_i) \subset E(t_i) \subset F(q_{l+i})$.

5.11 The variety $Z_w^Q$

Recall the variety $Z_w^Q$ from Sect. 2.3, and the descriptions of $u_w$ from Eqs. (5.7) and (5.9). Using the closed embedding $f$ from (2.8), we obtain,

$$Z_w^Q = \{(F(q), x) \in X_w^Q \times \mathcal{N} \mid xF(q_{l+i}) \subset F(q_{l-1}), \forall 1 \leq i \leq l + 1 \}. \quad (5.12)$$

Theorem B states that given $(V, x) \in T^*X$, we have $(V, x) \in T^*_X \mathcal{X}_w$, if and only if $V \in X_w$, and $x$ satisfies Eq. (5.21). The purpose of the following proposition is to show that Eq. (5.21) is necessary, i.e., if $(V, x) \in T^*_X \mathcal{X}_w$, then $x$ satisfies Eq. (5.21).
Proposition 5.13 For any point \((F(q), x) \in Z_w^Q\), we have, for \(1 \leq j < i \leq l\),
\[
\dim(xE(t_i)/E(t_j)) \leq \left\{ \begin{array}{ll}
 r_{i-1} - r_j, \\
 c_i - c_{j+1}.
\end{array} \right.
\]

Proof Consider \((F(q), x) \in Z_w^Q\), and integers \(1 \leq j < i \leq l\). We see from Sect. 5.10 that \(E(t_i) \subset F(q_{i+1})\), and from Eq. (5.12) that \(xF(q_{i+1}) \subset F(q_{i-1})\). Consequently, we have \(xE(t_i) \subset F(q_{i-1})\), and hence,
\[
\dim(xE(t_i)/E(t_j)) \leq \dim(F(q_{i-1})/E(t_j)) = \dim(F(q_{i-1}) \cap E(t_j)) \\
\leq r_{i-1} - r_j,
\]
where the final inequality follows from Sect. 5.10. Next, we see from Sect. 5.10 and Eq. (5.12) that \(xF(q_{j+l+1}) \subset F(q_j) \subset E(t_j)\). In particular, \(F(q_{j+l+1})\) is contained in the kernel of the map,
\[
F(q_{i+l}) \to E(n)/E(t_j), \quad v \mapsto xv(mod(E(t_j))).
\]

Since the image of this map is precisely \(xF(q_{i+l})/E(t_j)\), we have,
\[
\dim(xF(q_{i+l})/E(t_j)) \leq \dim(F(q_{i+l}) - \dim(F(q_{j+l+1})
\]
\[
= q_{i+l} - q_{j+l+1} = c_i - c_{j+1}.
\]
Finally, since \(E(t_i) \subset F(q_{i+1})\), we deduce that \(\dim(xE(t_i)/E(t_j)) \leq c_i - c_{j+1}\). \(\square\)

The following two propositions lay the groundwork required to prove that Eq. (5.21) is sufficient to determine whether a point is in the conormal variety.

Proposition 5.14 Consider \((V, x) \in X_w \times N\) satisfying \(\text{Im } x \subset V \subset ker x\), and
\[
\dim(xE(n)/E(t_j)) \leq r_{i-1} - r_j, \quad 0 \leq j < i \leq l + 1. \tag{5.15}
\]

Then, there exists a sequence of subspaces \(V_0 \subset \cdots \subset V_l = V\), satisfying,
\[
dim V_i = q_i, \\
xE(t_i) \subset V_i \subset E(t_i), \\
\dim(V_i \cap E(t_j)) \geq \min\{r_i, r_j\} = m_w(t_j, q_i), \tag{5.16}
\]
for all \(1 \leq i, j \leq l\).

Proof Since \(V \in X_w\), it follows from Sect. 5.10 that \(V_i = V\) satisfies Eq. (5.16). We construct the subspaces \(V_i\) inductively. In particular, given subspaces \(V_i, \ldots, V_l\) satisfying Eq. (5.16), we construct \(V_{i-1}\).

Applying Eq. (5.15) with \(j = i - 1\), we have \(xE(t_i) \subset E(t_{i-1})\). Further, Eq. (5.16) yields \(xE(t_i) \subset xE(t_{i+1}) \subset V_i\). Hence, we have,
\[
xE(t_i) \subset V_i \cap E(t_{i-1}).
\]

Set \(U_1 = xE(t_i)\), and \(U_2 = V_i \cap E(t_{i-1})\). Applying Eq. (5.15) with \(j = 0\), we see that \(\dim U_1 \leq r_{i-1}\). Let \(k = r_{i-1} - \dim U_1\).

Observe that \(U_1 \cap E(t_j)\) is the kernel of the quotient map \(U_1 \to U_1/E(t_j)\). Now, since \(\dim(xE(t_i)/E(t_j)) \leq r_{i-1} - r_j\), we have, for \(1 \leq j \leq l\),
\[
\dim(U_1 \cap E(t_j)) = \dim U_1 - \dim(U_1/E(t_j))
\]
\[ \geq (r_{i-1} - k) - (r_{i-1} - r_j) = r_j - k \]
\[ \geq \min\{r_{i-1}, r_j\} - k \]
\[ = m_w(t_j, q_{i-1}) - k. \]

On the other hand, observe that,
\[
U_2 \cap E(t_j) = \begin{cases} 
V_i \cap E(t_{i-1}) & \text{if } i \leq j, \\
V_i \cap E(t_j) & \text{if } i > j,
\end{cases}
\]
\[ \implies \dim(U_2 \cap E(t_j)) \geq \begin{cases} 
r_{i-1} & \text{if } i \leq j, \\
r_j & \text{if } i > j,
\end{cases}
\]
\[ = \min\{r_{i-1}, r_j\} = m_w(t_j, q_{i-1}). \]

It now follows from Proposition 3.12 that there exists a subspace \( V_{i-1} \) satisfying \( xE(t_i) \subset V_{i-1} \subset U_2 \subset E(t_{i-1}) \), and Eq. (5.16).

\[ \square \]

**Proposition 5.17** Consider \((V, x) \in X_w \times \mathcal{N}\) satisfying \( \text{Im } x \subset V \subset \ker x, \) and
\[
\dim(xE(t_i)/E(t_j)) \leq c_i - c_{i+1}, \quad \forall 0 \leq j < i \leq l + 1. \tag{5.18}
\]
Then, there exists a sequence of subspaces \( V = V_l \subset V_{l+1} \subset \cdots \subset V_{2l+1} \), satisfying,
\[
\dim V_{l+i} = q_{l+i}, \quad V_{l+i} \subset \ker x + E(t_i),
\]
\[
\dim(V_{l+i} \cap E(t_j)) \geq r_j + \min\{c_i, c_j\} = m_w(t_j, q_{l+i}). \tag{5.19}
\]
for all \( 1 \leq i, j \leq l. \)

**Proof** Since \( V \in X_w \), it follows from Sect. 5.10 that \( V_l = V \) satisfies Eq. (5.19). We construct the subspaces \( V_{l+i} \) inductively. In particular, given subspaces \( V_l, \ldots, V_{l+i-1} \) satisfying Eq. (5.19), we construct \( V_{l+i}. \)

We see from Eq. (5.19) that \( V_{l+i-1} \subset \ker x + E(t_i). \) Set \( U = \ker x + E(t_i). \) We first prove that,
\[
\dim(U \cap E(t_j)) \geq r_j + \min\{c_i, c_j\} = m_w(t_j, q_{l+i}) \quad \forall 1 \leq j \leq l + 1. \tag{5.20}
\]
For \( j \leq i \), we have \( E(t_j) \subset U \), hence \( U \cap E(t_j) = E(t_j) \). It follows that,
\[ \dim(U \cap E(t_j)) = t_j = r_j + c_j = r_j + \min\{c_i, c_j\}. \]
For \( j > i \), consider the map,
\[ E(t_j) \to xE(t_i)/E(t_{i-1}), \quad v \mapsto xv \text{ (mod } E(t_{i-1})) \text{.} \]
Since \( xE(t_i) \subset E(t_{i-1}) \), the subspace \( U \cap E(t_j) \) is contained in the kernel of this map. Further, Eq. (5.18) states that \( \dim(xE(t_i)/E(t_j)) \leq c_j - c_{i+1} \text{, hence} \)
\[ \dim(U \cap E(t_j)) \geq t_j - (c_j - c_i) \]
\[ = r_j + c_i = r_j + \min\{c_i, c_j\}. \]
This finishes the proof of Eq. (5.20). It now follows from Eq. (5.20), Lemma (3.10) and Proposition 3.12 that there exists a subspace \( V_{l+i} \) satisfying \( V_{l+i-1} \), and further satisfying Eq. (5.19).

\[ \square \]
Theorem B Consider \((V, x) \in T^*X\). Then \((V, x) \in T^*_X X_w\) if and only if \(V \in X_w\), and further, for all \(1 \leq j < i \leq l + 1\), we have,

\[
\dim(xE(t_i)/E(t_j)) \leq \begin{cases} r_i - r_j, \\ c_i - c_{j+1}. \end{cases} \tag{5.21}
\]

Proof Recall the map \(\tau_w : Z^Q \rightarrow T_X^* X_w\) from Proposition 2.5, given by,

\[
\tau_w(F(q), x) = (F(d), x).
\]

The map \(\tau_w\) is proper and birational (see Theorem A), hence surjective. It follows that \((V, x) \in T^*_X X_w\) if and only if there exists \(F(q) \in X^Q_w\) such that \(F(d) = V\), and \((F(q), x) \in Z^Q_w\).

Consider \((V, x) \in T^*_X X_w\). It follows from Theorem A that \(V \in X_w\), and from Proposition 5.13 that Eq. (5.21) holds. Conversely, consider \((V, x) \in T^*_X X_w\) satisfying \(V \in X_w\) and Eq. (5.21). We will construct \(F(q) \in X^Q_w\) such that \((F(q), x) \in Z^Q_w\) and \(\tau_w(F(q), x) = (V, x)\).

Using Proposition 5.14, we construct subspaces \(V_0, \ldots, V_l = V\) satisfying Eq. (5.16). Similarly, we use Proposition 5.17 to construct subspaces \(V_{l+1}, \ldots, V_{2l+1}\) satisfying Eq. (5.19).

Suppose first that \(G = SL_n\). We set,

\[
F(q) = V_0 \subset V_1 \subset \cdots \subset V_{2l+1}.
\]

Observe that \(F(d) = V_l = V\). It follows from Eqs. (5.16) and (5.19) that \(F(q) \in X^Q_w\). Further, for \(1 \leq i \leq l + 1\), we have,

\[
F(q_{l+i}) \subset \ker x + E(t_i) \implies xF(q_{l+i}) \subset xE(t_i) \subset F(q_{l-1}).
\]

This is precisely the condition for \((F(q), x)\) to belong to \(Z^Q_w\).

Suppose next that \(G = Sp_n\). Let \(\overline{F(q)}\) be the partial flag given by,

\[
F(q_i) = \begin{cases} V_i & \text{for } i \leq l, \\ V_{2l-i} & \text{for } l < i \leq 2l. \end{cases}
\]

In particular, we have \(F(d) = V_l = V\). It follows from Lemma 4.13 and Proposition 5.14 that \(F(q) \in X^Q_w\). It remains to show that \((F(q), x) \in Z^Q_w\).

For \(0 \leq i \leq l\), we have \(xE(t_{i+1}) \subset V_i\), hence \(\omega(xE(t_{i+1}), V^\perp_i) = 0\). It follows from Sect. 4.3 that \(\omega(E(t_{i+1}), x(V^\perp_i)) = 0\), hence,

\[
xF(q_{2l-i}) = x(V^\perp_i) \subset E(t_{i+1})^\perp = E(t_{l-i-1}).
\]

The final equality is a consequence of Eq. (4.12). Substituting \(i \mapsto l - i\) yields \(xF(q_{l+i}) \subset E(t_{l-i})\) for all \(0 \leq i \leq l\). It follows that \((F(q), x) \in Z^Q_w\), hence \((F(d), x) \in T^*_X X_w\).

\[\square\]

6 Orbital varieties

Let \(G, B\) be as in the previous sections. Consider a \(G\)-orbit closure \(\mathcal{N}_\lambda \subset \mathcal{N}\). The irreducible components of \(\mathcal{N}_\lambda \cap U_q\) are called orbital varieties. Orbital varieties are closely related to the conormal varieties of Schubert varieties.
**Proposition 6.1** (cf. [15]) Given a standard parabolic subgroup $Q$, and a Schubert variety $X_Q^Q \subset X^Q$, the image of the conormal variety $T^*_X X_Q^Q$ under the Springer map $\mu^Q : T^* X^Q \to N$ is an orbital variety.

For more details on the relationship between conormal varieties and orbital varieties, the reader may consult [6,15]. Providing a combinatorial description of the inclusion order on orbital varieties, and providing the defining equations for an orbital variety viewed as a subvariety of $u_B$, are both open problems in general. For certain orbital varieties in types A, B, C (those corresponding to the nilpotent orbits satisfying $x^2 = 0$), these problems were solved in [3,12].

As in Sect. 5, we work simultaneously in two cases: either $G = SL_n$, or $G = Sp_n$ with the further assumption that $n = 2d$. Let $P$ be the standard parabolic group corresponding to $S\{\alpha_d\}$. We derive, in Theorem C, a system of defining equations for orbital varieties of the form $\mu(T^*_X X_w)$. This is an easy consequence of Theorem B, and recovers some of the results of [3,12].

**Theorem C** Let $G, B, P, X, w$, and $\mu$ be as in Theorem B. Then,

$$\mu(T^*_X X_w) = \left\{ x \in u_B \mid x^2 = 0, \dim \left( \frac{\langle x \rangle}{\langle E(t_l) \rangle} \right) \leq \begin{array}{l} r_{i-1} - r_j, \\
 c_i - c_{j+1}, \end{array} \quad \forall \ 1 \leq i < j \leq l \right\}.$$  

**Proof** Consider $x \in u_B$ satisfying $x^2 = 0$, and

$$\dim \left( \frac{\langle x \rangle}{\langle E(t_l) \rangle} \right) \leq \begin{array}{l} r_{i-1} - r_j, \\
 c_i - c_{j+1}, \end{array} \quad \forall \ 1 \leq j < i \leq l. \quad (6.2)$$

Substituting $j = 0$ in Eq. (6.2), we obtain,

$$\dim(\langle x \rangle) = \dim(\langle E(t_l) \rangle) \leq c_i - c_1 \quad \Rightarrow \quad \dim(\ker x \cap \langle E(t_l) \rangle) = \dim(\ker(\langle x \rangle))$$

$$\geq t_l - (c_i - c_1) = r_l + c_1 \geq r_l.$$

Let $k = d - \dim(\langle x \rangle)$. Substituting $i = l$ in Eq. (6.2) yields,

$$\dim(\langle x \rangle) \leq r_{l-1} - r_l \leq d \quad \Rightarrow \quad \dim(\langle x \rangle + \langle E(t_l) \rangle) \leq (d - r_l) + t_l$$

$$\Rightarrow \quad \dim(\langle x \rangle \cap \langle E(t_l) \rangle) = \dim(\langle x \rangle) + \dim(\langle E(t_l) \rangle) - \dim(\langle x \rangle + \langle E(t_l) \rangle) \geq (d - k) + t_l - (d - r_l + t_l) = r_l - k.$$  

Observe that since $x^2 = 0$, we have $\langle x \rangle \subset \ker x$. It now follows from Proposition 3.12 and Sect. 5.10 that there exists $V \in X_w$ such that,

$$\langle x \rangle \subset V \subset \ker x,$$

i.e., $(V, x) \in T^*_X X_w$. Consequently, we have $x \in \mu(T^*_X X_w)$. \qed

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6.3 Orbital varieties in type A

When $G = SL_n$, the $G$-orbits in $N$ are indexed by the partitions of $n$. For $\lambda$ a partition of $n$, the irreducible components of $\Lambda^*_\lambda \cap u_B$ are indexed by the standard Young tableaux of shape $\lambda$. For $T$ a standard Young tableau, we denote the corresponding orbital variety by $O_T$.

In this case, the relationship between conormal varieties of Schubert varieties, and orbital varieties, as described in Proposition 6.1, provides a geometric realization of the Robinson–Schensted correspondence.

**Proposition 6.4** (cf. [16]) Suppose $T$ is the left Robinson–Schensted tableau of some $w \in S_n$. Then $O_T = \mu^B (T^*_X X^B_w)$.

**Proposition 6.5** Let $T$ be a standard Young tableau with exactly two columns. Then there exists a standard parabolic subgroup $P \subset G$, and a Schubert variety $X_w$ in $X = G/P$, such that $O_T = \mu (T^*_X X_w)$.

**Proof** Let $k$ be the number of boxes in the first column of $T$, and let $P$ be the standard parabolic subgroup of $G$ corresponding to $A_{n-1} \setminus \{\alpha_k\}$. The longest element $w_p$ of $W_P$ is given by

$$ w_p(i) = \begin{cases} k + 1 - i & \text{for } i \leq k, \\ n + 1 - k & \text{for } i > k. \end{cases} $$

Let $a_1, \ldots, a_k$ be the entries in the first column of $T$, written in increasing order, i.e., top to bottom; and $b_1, \ldots, b_{n-k}$ the entries in the second column, also written in increasing order. We consider the element $w \in S_n$ given by,

$$ w(i) = \begin{cases} a_i & \text{for } i \leq k, \\ b_{i-k} & \text{for } i > k. \end{cases} $$

Let $v = w w_p$. Since $w \in S_n^P$, the Schubert variety $X^B_v$ is a fibre bundle over $X_w$ with fibre $B/P$, and we have a Cartesian diagram,

$$ \begin{array}{ccc} X^B_v & \hookrightarrow & X^B \\
pr & & \downarrow \\
X^P_w & \hookrightarrow & X^P. \end{array} $$

The map $X^B \to X^P$ is precisely the quotient map $G/B \to G/P$. Consequently, $T^*_X X^B_v$ is a $B/P$-fibre bundle over $T^*_X X^B_w$, with the map $T^*_X X^B_v \to T^*_X X^B_w$ being simply the restriction (to $T^*_X X^B_v \subset T^*_X X \subset X^B \times N$) of the map,

$$ \text{pr} \times \text{id}_N : X^B \times N \to X \times N. $$

This yields us $\mu^B (T^*_X X^B_w) = \mu (T^*_X X_w)$. Finally, we verify that $T$ is the left RSK-tableau of $v$, thus obtaining $O_T = \mu^B (T^*_X X^B_w) = \mu (T^*_X X_w)$.

**Corollary 6.6** Let $T$ be a two-column standard Young tableau. Consider integers $0 \leq j < i \leq n$, and the skew-tableau $\setminus \{1, \ldots, j, i+1, \ldots, n\}$. Let $T^j_i$ denote the tableau obtained from this skew-tableau via ‘jeu de taquin’. Then,

$$ O_T = \left\{ x \in N \mid J(x^j_i) \leq T^j_i \right\}. $$
where $x_j^i$ is the square sub-matrix of $x$ with corners $(t_j + 1, t_j + 1)$ and $(t_i, t_i)$, $J(x_j^i)$ denotes the Jordan type of $x_j^i$, and $\leq$ denotes the inclusion order on the set of $G$-orbits $\mathcal{N}_\lambda$.

**Proof** This statement is proved in [12]. We explain here how it also follows as a consequence of Theorem C and Sect. 6.3.

Since $x^2 = 0$, we have $(x_j^i)^2 = 0$ for all $i, j$. Consequently, the inequality $J(x_j^i) \leq T_j^i$ is equivalent to the inequality $rk(x_j^i) \leq f_j^i$, where $f_j^i$ is the number of boxes in the second column of $T_j^i$. On the other hand, it follows from Theorem C and Sect. 6.3 that

$$\mathcal{O}_T = \left\{ x \in \mathcal{N} \mid rk(x_j^i) \leq g_j^i \right\},$$

for certain integers $g_j^i$. It is a simple exercise to verify that the integers $f_j^i$ and $g_j^i$ defined here are equal. $\square$

### 7 A type independent conjecture

In this section, we assume that $X$ is a cominuscule Grassmannian corresponding to some Dynkin diagram. We conjecture, for any Schubert variety $X_w \subset X$, the following equality,

$$T_X^* X_w = \mu^{-1}(\mu(T_X^* X_w)) \cap \pi^{-1}(X_w). \quad (7.1)$$

The question is well-posed in both set-theoretic and scheme-theoretic settings.

Suppose $X_w \subset X$ is a smooth Schubert subvariety. We prove in Proposition 7.2 that Eq. (7.1) holds set-theoretically in this case.

Next, let $w_0$ denote the longest element in the Weyl group $W$. We show in Proposition 7.3 that if $X_w \subset X$ is a Schubert variety such that the opposite Schubert variety $X_{w_0 w}$ is smooth, then Eq. (7.1) holds scheme-theoretically. This is a straightforward corollary to [11, Theorem 1.1].

When $X$ is the usual Grassmannian or the symplectic Grassmannian, the set-theoretic version is a consequence of Theorems B and C. In type B, the only cominuscule Grassmannian is the one corresponding to $G = SO_{2n+1}$, and the cominuscule root $\alpha_1$. In this case, one easily verifies that for each $w \in W^P$, either $X_w$ is smooth, or $X_{w_0 w}$ is smooth, hence settling the set-theoretic version of our conjecture for all cominuscule Grassmannians in types A, B, and C.

One would like to know in which of these cases Eq. (7.1) holds scheme-theoretically, and also whether Eq. (7.1) holds for types D and E. If it does, can we find a uniform, type independent proof?

**Proposition 7.2** Suppose $X_w$ is smooth. Then the conormal variety $T_X^* X_w$ satisfies Eq. (7.1) set-theoretically.

**Proof** A Schubert variety $X_w$ in a cominuscule Grassmannian $X$ is smooth if and only if $X_w$ is homogeneous for some standard parabolic subgroup $L$, see [2].

Suppose $X_w$ is homogeneous for some standard parabolic subgroup $L$; let $S_L$ be the corresponding subset of $S$, and $w_L$ the longest word of $W$ supported on $S_L$. Then $w$ is the minimal representative of $w_L$ in $W^P$. Further, the subspace $u_w \subset u$ from Lemma 2.1 is precisely,

$$u_w = \bigoplus_{\alpha \geq \gamma, \text{ Supp}(\alpha) \notin S_L} \mathfrak{g}_\alpha.$$
In particular, $u_w$ is $L$-stable.

The quotient map $G/B \to G/P$ induces an isomorphism $L/B \sim T^*_X X_w$, and the conormal variety $T^*_X X_w \to X_w$ is simply the vector bundle $L \times_B u_w \to L/B$. Consequently, we have,

$$\mu(T^*_X X_w) = \{ Ad(l_0)x_0 \mid l_0 \in L, \ x_0 \in u_w \}. \quad (7.2)$$

Now, consider some $(l, x) \in G \times P u$, satisfying $\pi(l) \in X_w$ and $\mu(l, x) \in \mu(T^*_X X_w)$. We may assume, without loss of generality, that $l \in L$. As a consequence of Eq. (7.2), there exist $l_0 \in L$, and $x_0 \in u_w$, such that,

$$\mu(l, x) = Ad(l)x = Ad(l_0)x_0$$

$$\implies x = Ad(l^{-1}l_0)x_0.$$

Now, since $u_w$ is $L$-stable, we have $x \in u_w$, hence $(l, x) \in T^*_X X_w$.

\textbf{Proposition 7.3} Suppose the opposite Schubert variety $X_{w_0 w}$ is smooth for some $w \in W_P$. Then $T^*_X X_w$ satisfies Eq. (7.1) scheme-theoretically.

\textbf{Proof} Let $D_0$ denote the Dynkin diagram of $G$, and let $D$ be the corresponding extended Dynkin diagram. The loop group $LG = G(k[t, t^{-1}])$ is an affine Kac-Moody group corresponding to the extended Dynkin diagram $D$. Let $G_0, G_d, \text{ and } P$ be parabolic subgroups of $LG$ corresponding to the subsets $D \backslash \{a_0\}, D \backslash \{a_d\}, \text{ and } D \backslash \{a_0, a_d\}$ respectively.

Following [11], there exists an embedding $\phi : T^*_X X_w \to LG/P$ such that $\phi(T^*_X X_w)$ is an open subset of some Schubert subvariety of $LG/P$. Further, we can identify the structure map $\pi$ and the Springer map $\mu$ as the restriction to $\phi(T^*_X X_w)$ of the quotient maps $\pi_d : LG/P \to LG/G_d$ and $\pi_0 : LG/P \to LG/G_0$ respectively.

Now, for any Schubert variety $Y \subset LG/P$, we have the scheme-theoretic equality,

$$Y = \pi_0^{-1}(\pi_0(Y)) \cap \pi_d^{-1}(\pi_d(Y)).$$

From this, we deduce that Eq. (7.1) holds for $T^*_X X_w$ scheme-theoretically.

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