A WEIGHTED CENTRAL LIMIT THEOREM FOR $\log |\zeta(1/2 + it)|$

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Abstract. Under the Riemann Hypothesis, we show that as $t$ varies in $T \leq t \leq 2T$, the distribution of $\log |\zeta(1/2 + it)|$ with respect to the measure $|\zeta(1/2 + it)|^2 dt$ is approximately normal with mean $\log \log T$ and variance $\frac{1}{2} \log \log T$.

1. Introduction and statement of the main results

In the understanding of the value distribution of the Riemann zeta function on the critical line, the milestone is due to Selberg [12], who proved a central limit theorem for $\log |\zeta(1/2 + it)|$, showing that

\begin{equation}
\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log |\zeta(1/2 + it)|}{\sqrt{\frac{1}{2} \log \log T}} \geq V \right\} \sim \int_V^\infty e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}
\end{equation}

for any fixed $V$, as $T$ goes to infinity. Analogous statements hold also in more generality, for example in the case of the imaginary part of $\log \zeta(1/2 + it)$ or other $L$–functions (see e.g. [3]). In 2015 Radziwill and Soundararajan [11] gave a new and simple proof of Selberg’s central limit theorem.

In this context one may ask about the uniformity in $V$ of (1.1), investigating the large values of zeta. Classically it was known that (1.1) holds for $V = V(T) \ll (\log \log T)^{1/2-\varepsilon}$, $\varepsilon > 0$ (see [14]). More recently Radziwill [10] introduced a new method that extended (1.1) to the large deviation range $V \ll (\log \log T)^{1/10-\varepsilon}$. Furthermore he conjectured that the largest range of uniformity for (1.1) is $V = o(\sqrt{\log \log T})$.

Moreover under the Riemann Hypothesis, Soundararajan [13] obtained upper bounds for the measure of the set of large values, in the case $V \gg \sqrt{\log \log T}$. The speculation that gives rise to Soundararajan’s work is that an upper bound like

\begin{equation}
\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log |\zeta(1/2 + it)|}{\sqrt{\frac{1}{2} \log \log T}} \geq V \right\} \ll \frac{1}{V} \exp \left(- \frac{V^2}{2}\right)
\end{equation}

also holds for $V \gg \sqrt{\log \log T}$. Even though Soundararajan does not prove such a precise upper bound, he gets a quasi optimal one under the Riemann Hypothesis, which is enough to derive conditional quasi optimal upper bounds for the moments of the Riemann zeta function. Then the problem of proving (1.2) is still open. For example in the case $V = \sqrt{2 \log \log T}$, Soundararajan only proved that the left hand side of (1.2) is essentially $\ll (\log T)^{-1+o(1)}$ while the conjectural sharp upper bound should be $\ll (\log T \sqrt{\log \log T})^{-1}$ (see [6], [7] and [8] for further discussions).
In this setting, with the aim of studying the large values of the Riemann zeta function, one can investigate the distribution of \( \log |\zeta(1/2 + it)| \) with respect to a different measure. For instance one can “tilt” the measure and study the distribution of \( \log |\zeta(1/2 + it)| \) with respect to the weighted measure

\[
|\zeta|^2 dt := |\zeta(1/2 + it)|^2 dt.
\]

This change of measure means that in integrals which represent probabilities (or moments) we are giving more importance to the contribution of those \( t \) such that \( |\zeta(1/2 + it)| \) is large. For this reason, understanding the distribution of \( \log |\zeta(1/2 + it)| \) with respect to the weighted measure \( |\zeta|^2 dt \) might be of help in the understanding of the large values of \( \zeta \).

**Theorem 1.** Under the Riemann Hypothesis, as \( t \) varies in \( T \leq t \leq 2T \), the distribution of \( \log |\zeta(1/2 + it)| \) is asymptotically Gaussian with mean \( \log \log T \) and variance \( \frac{1}{2} \log \log T \), with respect to the weighted measure \( |\zeta|^2 dt \).

We note that this result is a manifestation of Girsanov’s theorem, which states that if we take a Gaussian random variable and tilt it against an exponential of itself, the resulting random process is again Gaussian with mean and variance related to the original one in a specific way. Theorem 1 shows the same phenomenon for the Riemann zeta function, reinforcing our expectation that \( \log |\zeta(1/2 + it)| \) behaves like a Gaussian in many respects.

We now describe the general strategy to prove Theorem 1. Even though the Euler product formula only holds in the half-plane of convergence, for many purposes the Riemann zeta function behaves like an Euler product also on the critical line (see Principle 1.3 in [8]), thus \( \log |\zeta(1/2 + it)| \) behaves like a Dirichlet polynomial. Roughly speaking we know that for a suitable \( x = x(T) \) we have

\[
\log |\zeta(1/2 + it)| \approx \Re \sum_{p \leq x} \frac{1}{p^{1/2+it}} + \text{(contribution from zeros)}
\]

(see [6], [7] for further and more precise details) and in several applications the contribution from the zeros can be controlled (two important examples are [13] and [6]). This approximation also holds in our setting, as shown by the following proposition:

**Proposition 1.** Let \( T \) be a large parameter. Denote \( P(t) = \sum_{p \leq x} p^{-1/2-it} \), where \( x = T^{\epsilon/k}, \epsilon := (\log \log T)^{-1}, k \) a positive integer. Under the Riemann Hypothesis, there exists a constant \( C > 0 \) such that we have uniformly in \( k \):

\[
\frac{1}{T \log T} \int_T^{2T} \left| \log |\zeta(1/2 + it)| - \Re P(t) \right|^2 |\zeta|^2 dt = O \left( (Ck)^{4k} (\log \log T)^{2k+1/2} \right).
\]

We remark that this is the only point where we rely on the assumption of the Riemann Hypothesis. In fact in order to estimate the contribution of the zeros that appears in (1.3), we need to bound the sum over the non-trivial zeros \( \sum_{0 < \rho \leq T} |\zeta(\rho + i\alpha)|^2 \) with \( |\alpha| \leq 1 \), which is known to be \( \ll T(\log T)^2 \) only conditionally on the Riemann Hypothesis (see [4]).
Thanks to Proposition 1, at this point it suffices to show that the distribution of $\Re P(t)$ is approximately Gaussian with respect to the measure $|\zeta|^2dt$. This is achieved by the method of moments using the following result.

**Proposition 2.** Let $P(t) = \sum_{p \leq x} p^{-1/2-it}$, $x := T^{\varepsilon/k}$, $\varepsilon := (\log \log \log T)^{-1}$. Denote $\mathcal{L} = \sum_{p \leq x} \frac{1}{p}$. Then, for every fixed $k$ integer

$$
\frac{1}{T \log T} \int_T^{2T} (\Re P(t) - \mathcal{L})^k |\zeta|^2 dt = \begin{cases} 
(\frac{k}{2})^k(k-1)!! + O_k(\mathcal{L}^{(k-1)/2}) & \text{if } k \text{ is even} \\
O_k(\mathcal{L}^{(k-1)/2}) & \text{if } k \text{ is odd}.
\end{cases}
$$

Note that by definition of $x$ we know that $\log \log x = \log \log T - \log k + \log \varepsilon$, then for a fixed $k$ we have $\log \log x = \log \log T + O(\log x)$, where $\log x$ denotes the fourth iterated natural logarithm. Hence by Mertens’ theorem $\mathcal{L} = \log \log x + O(1) = \log \log T + O(\log T)$. As a consequence the right hand side in Proposition 2 matches with the moments of a normal with respect to $\mathcal{L}$, whose notations become easier under the Riemann Hypothesis, we have (see [14], equation (5.15)). Putting together the two propositions one has that the moments of $\log \zeta(1/2 + it)$ with respect to the measure $|\zeta|^2 dt$ are asymptotic to the moments of a Gaussian random variable of mean $\log \log T$ and variance $\frac{1}{2} \log \log T$, then the theorem is proved.

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2. Proof of Proposition 1

First of all, we recall an important tool which allows us to compute the moments of a sufficiently short Dirichlet polynomial with respect to $|\zeta|^2 dt$ (see [1] and [2]).

**Lemma 2.1.** Let $A(s) = \sum_{n \leq T^s} a(n)n^{-s}$ and $B(s) = \sum_{m \leq T^s} b(m)m^{-s}$ be Dirichlet polynomials with $a(n) \ll n^\varepsilon$, $b(m) \ll m^\varepsilon$ for every $\varepsilon > 0$ and $\theta + 2\sigma \leq 1/2$. Then, denoting $c := 2\gamma + 4 - 2\log 2\pi - 1$, we have:

$$
\int_T^{2T} A(1/2+it)|B(1/2+it)|^2 \zeta^2 dt = T \sum_{n, m} \frac{(a \ast b)(n)b(m)}{[n, m]} \left( \log \left( \frac{T(n, m)^2}{nm} \right) + c \right) + o(T).
$$

Our proof of Proposition 1 is a modification of Theorem 5.1 in [14]. We recall that $P(t) = \sum_{p \leq x} p^{-1/2-it}$ and $x = T^{\varepsilon/k}$ with $\varepsilon = (\log \log \log T)^{-1}$. Following Tsang’s strategy, whose notations become easier under the Riemann Hypothesis, we have (see [14], equation (5.15)):

$$
\log \zeta(1/2 + it) - P(t) = S_1 + S_2 + S_3 + O(R) - L(t)
$$

with

$$
S_1 := \sum_{p \leq x} (p^{-1/2-4/\log x} - p^{-1/2})p^{-it}, \quad S_2 := \sum_{p' \leq x} \frac{p^{-r(1/2+4/\log x+it)}}{r}, \quad S_3 := \sum_{x < p \leq x^3} \frac{\Lambda(n)}{\log n} n^{-1/2-4/\log x-it},
$$

$$
R := \frac{5}{\log x} \left( \sum_{n \leq x^3} \frac{\Lambda(n)}{n^{1/2+4/\log x+it}} + \log T \right)
$$
\[
L(t) := \sum_{\rho} \int_{1/2}^{1/2+4/\log x} \left( \frac{1}{2} + \frac{4}{\log x} - u \right) \frac{1}{u + it - \rho/2 + \frac{1}{\log x} - \rho} du,
\]
where the sum in the definition of \( L(t) \) is over all the non-trivial zeros of \( \zeta \). Hence

\[
(2.2) \quad \log |\zeta(1/2 + it)| - \Re(P(t)) = \Re(S_1) + \Re(S_2) + \Re(S_3) + O(R) - \Re(L(t))
\]

so what remains to do is studying the \( 2k \)-moments of all these objects with respect to the weighted measure \( |\zeta|^2 dt \); to this aim we rely on Lemma 2.1.

Let’s start with the first one:

\[
\frac{1}{T \log T} \int_{T}^{2T} |S_1|^{2k} |\zeta|^2 dt = \frac{1}{T \log T} \int_{T}^{2T} \left| \left( \sum_{p \leq x} \frac{p^{-4/\log x}}{p^{1/2+it} - 1} \right)^k \right| |\zeta|^2 dt
\]

\[
= \frac{1}{T \log T} \int_{T}^{2T} \sum_{n \leq x} \frac{1}{n^{1/2+it}} \sum_{p_1 \cdots p_k = n} \prod_{i=1}^{k} (p_i^{-4/\log x} - 1)^2 |\zeta|^2 dt
\]

\[
\ll \sum_{m,n \leq x} \frac{(m,n)}{mn} \sum_{p_1 \cdots p_k = n} \prod_{i=1}^{k} (p_i^{-4/\log x} - 1)^2 |\zeta|^2 dt
\]

To make the GCD on the numerator explicit, we rewrite the primes \( p_1, \ldots, p_k \) highlighting the multiplicity of these primes:

\[
\{p_1, \ldots, p_k\} = \{p_1', \ldots, p_k'\}
\]

where the \( p_i' \)'s are distinct and we denote \( c_i \geq 1 \) the multiplicity of \( p_i' \) in this set, so \( c_1 + \cdots + c_i = k \). Now we do the same for the \( q_i \)'s and we put in evidence if any \( q_i \) already appears among the \( p_i' \)'s:

\[
\{q_1, \ldots, q_k\} = \{p_1', \ldots, p_k'\} \cup \{q_1', \ldots, q_m'\}
\]

where the \( p_i' \)'s and \( q_i' \)'s are all distinct and we denote \( e_i \geq 0 \) and \( d_i \geq 1 \) the multiplicities of \( p_i' \) and \( q_i' \) respectively. Then we have \( e_1 + \cdots + e_i + d_1 + \cdots + d_m = k \). In the following we drop the symbol \( ' \), just denoting the new primes with \( p_i, q_i \). With these notations, the previous sum is

\[
\ll (k!)^2 \sum_{i \leq k} \sum_{m \leq k} \prod_{i=1}^{l} \left( \sum_{p_i} \frac{|p_i^{-4/\log x} - 1|^{c_i+e_i}}{p_i^{\max(e_i,c_i)}} \right) \prod_{i=1}^{m} \left( \sum_{q_i} \frac{|q_i^{-4/\log x} - 1|^{d_i}}{q_i^{\max(e_i,d_i)}} \right)
\]

and if we ignore the equation for \( c_i, e_i, d_i \) we get

\[
(2.3) \quad \ll (k!)^2 \sum_{i \leq k} \prod_{i=1}^{l} \left( \sum_{c_i \geq 1} \sum_{p_i \leq x} \frac{|p_i^{-4/\log x} - 1|^{c_i+e_i}}{p_i^{\max(e_i,c_i)}} \right) \prod_{i=1}^{m} \left( \sum_{d_i \geq 1} \sum_{q_i \leq x} \frac{|q_i^{-4/\log x} - 1|^{d_i}}{q_i^{\max(e_i,d_i)}} \right).
\]
Now we remark that only in the case $c_i = 1$ and $c_i \leq 1$ the sum over $p_i$ in the first parentheses gives an unbounded contribution. Indeed the remaining cases give

$$
\sum_{c_i \geq 1, e_i \geq 0: \max(c_i, e_i) \geq 2} \left| p_i^{4/\log x} - 1 \right| c_i + e_i / p_i^{\max(c_i, e_i)} \ll \sum_{c_i \geq 1, e_i \geq 0: \max(c_i, e_i) \geq 2} \sum_{p_i \leq x} \frac{1}{p_i^{\max(c_i, e_i) - 3/2}} \frac{1}{p_i^{3/2}} \ll \sum_{c_i \geq 1, e_i \geq 0: \max(c_i, e_i) \geq 2} \sum_{p_i \leq x} \frac{1}{2^{\max(c_i, e_i)}} \ll \sum_{c_i \geq 0} \frac{1}{2^{(c_i + e_i)/2}} \ll 1.
$$

We treat the second parentheses analogously, so that we get a bound for (2.3), which is

$$
\ll (k!)^2 \sum_{l, m \leq k} \left( \sum_{p \leq x} \frac{|p^{4/\log x} - 1|}{p} + \sum_{p \leq x} \frac{|p^{4/\log x} - 1|^2}{p} + O(1) \right)^{l + m}.
$$

In order to bound the first sum we use that $e^{-z} = 1 + O(z)$ for $z \ll 1$, so if $p \leq x^\delta$ (with $\delta \ll 1$) we have

$$
|p^{4/\log x} - 1| = |e^{4\log p/\log x} - 1| \ll \frac{4 \log p}{\log x} \ll \delta
$$

and as a consequence

$$
\sum_{p \leq x^\delta} \frac{|p^{4/\log x} - 1|}{p} \ll \delta \sum_{p \leq x^\delta} \frac{1}{p} \ll \delta (|\log \delta| + \log \log x).
$$

On the other hand, if $x^\delta < p \leq x$ than trivially $|p^{4/\log x} - 1| \leq 2$ hence

$$
\sum_{x^\delta < p \leq x} \frac{|p^{4/\log x} - 1|}{p} \ll \sum_{x^\delta < p \leq x} \frac{1}{p} \ll |\log \delta| + O(1).
$$

Therefore

$$
\sum_{p \leq x} \frac{|p^{4/\log x} - 1|}{p} \ll \delta (|\log \delta| + \log \log x) + |\log \delta| + O(1) \ll \log \log \log T
$$

by selecting $\delta = \frac{\log \log \log x}{\log \log x}$. The second sum is $\ll \log \log \log T$ too, being $|A - 1|^2 \leq |A - 1|$ for $0 < A < 1$. Putting all together, the sum we are considering is:

$$
\ll (k!)^2 \sum_{l, m \leq k} (3 \log \log \log T)^{l + m} \ll (k!)^2 C^{2k} (\log \log \log T)^{2k}
$$

with $C$ a sufficiently large positive constant. In conclusion, uniformly in $k$ we have

$$
\frac{1}{T \log T} \int_T^{2T} (S_1)^{2k} |\zeta|^2 dt \ll (Ck)^{2k} (\log \log \log T)^{2k}.
$$

Now we focus on $S_2$. Using again Lemma 2.1 we have:

$$
\frac{1}{T \log T} \int_T^{2T} |S_2|^{2k} |\zeta|^2 dt \ll \sum_{p_1, \ldots, p_k \leq x} \sum_{q_1, \ldots, q_k \leq x} \sum_{s_1, \ldots, s_k \geq 2} \frac{(p_1 \cdots p_k (q_1 \cdots q_k)}{p_1^{r_1} \cdots p_k^{r_k} q_1^{s_1} \cdots q_k^{s_k}}.
$$
We use the same decomposition of \( \{p_1, \ldots, p_k\} \) and \( \{q_1, \ldots, q_k\} \) as before getting
\[
\ll (k!)^2 \sum_{m,l \leq k} \sum_{p_1 \leq x_{a_1, a_2} \geq 0, p_1 \leq x_{a_1, a_2} \leq x} \frac{1}{p_1^{\max(a_1, b_1)} \cdots p_l^{\max(a_l, b_l)} q_1^{f_1} \cdots q_m^{f_m}}
\]
\[
= (k!)^2 \prod_{m,l \leq k} \left( \sum_{p_i \leq x} \frac{1}{p_i^{\max(a_i, b_i)}} \right) \prod_{i=1}^{m} \left( \sum_{q_i \leq x} \frac{1}{q_i^{f_i}} \right) \ll (k!)^2 \sum_{m,l \leq k} C^{n+l} \ll (Ck)^{2k}.
\]

Let us investigate \( S_3 \) using the same approach. We have
\[
\frac{1}{T \log T} \int_T^{2T} |S_3|^{2k} |\zeta|^2 dt \ll \sum_{x < p_1^{r_1} \cdots p_k^{r_k} \leq x_3} \sum_{x < q_1^{s_1} \cdots q_k^{s_k} \leq x_3} \frac{p_1^{r_1} \cdots p_k^{r_k} q_1^{s_1} \cdots q_k^{s_k}}{p_1^{r_1} \cdots p_k^{r_k} q_1^{s_1} \cdots q_k^{s_k}}.
\]
We begin studying the case when all the exponents \( r_i, s_i \) are equal to 1. We can implement the same technique as before, getting:
\[
\ll (k!)^2 \sum_{m,l \leq k} \sum_{c_1, \ldots, c_{m, l} \geq 1} \sum_{d_1, \ldots, d_{m, l} \geq 0} \frac{1}{p_1^{\max(c_1, c_1)} \cdots p_l^{\max(c_l, c_l)} q_1^{d_1} \cdots q_m^{d_m}}
\]
\[
= (k!)^2 \prod_{l, m \leq k} \left( \sum_{c_1, \ldots, c_{m, l} \geq 0} \frac{1}{p_i^{\max(c_i, c_i)}} \right) \prod_{i=1}^{m} \left( \sum_{d_1, \ldots, d_{m, l} \geq 0} \frac{1}{q_i^{d_i}} \right)
\]
\[
= (k!)^2 \prod_{l, m \leq k} \left( 2 \sum_{x < p_1^{r_1} \leq x_3} \frac{1}{p_i} + O(1) \right) \prod_{i=1}^{m} \left( \sum_{x < q_i^{s_i} \leq x_3} \frac{1}{q_i} + O(1) \right)
\]
\[
\ll (k!)^2 \sum_{l, m \leq k} (2 \log 3 + C)^l \left( \log 3 + C \right)^m \ll (Ck)^{2k}.
\]
The contribution of the case where some exponents are larger than 1 in the right hand side of (2.4) is still \( \ll (Ck)^{2k} \), by a combination of the previous computation and the argument we used in order to bound \( S_2 \).

Now we analyze the error term, which is
\[
R \ll \frac{1}{\log x} \sum_{n \leq x^3} \Lambda(n)n^{-4/\log x} + \frac{k}{\varepsilon}.
\]
Hence
\[
\frac{1}{T \log T} \int_T^{2T} \left| R \right|^{2k} |\zeta|^2 dt \ll 1 \left( \frac{1}{\log x} \right)^{2k} \frac{1}{T \log T} \int_T^{2T} \left| \sum_{n \leq x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right|^{2k} \left| \zeta \right|^2 dt + \frac{k^{2k}}{\varepsilon^{2k}}
\]
\[
\ll \frac{1}{(\log x)^{2k}} \frac{1}{T \log T} \int_T^{2T} \left| \sum_{n \leq x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right|^{2k} \left| \zeta \right|^2 dt + (\log \log T)^{2k} k^{2k}.
\]
We now study the first term, with the aim of proving
\[
(2.5) \quad \frac{1}{T \log T} \int_T^{2T} \left| \sum_{n \leq x^2} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right|^{2k} |\zeta|^2 dt \ll (CK)^{2k} (\log x)^{2k}.
\]

Using our usual approach we get:
\[
(2.6) \quad \frac{1}{T \log T} \int_T^{2T} \left| \sum_{n \leq x^2} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right|^{2k} |\zeta|^2 dt \ll \sum_{p_1, \ldots, p_k, q_1, \ldots, q_k} \log p_1 \cdots \log p_k \log q_1 \cdots \log q_k.
\]

Once again we start with the case where all the exponents are equal to 1 and we rewrite the sum in the usual way
\[
(k!)^2 \sum_{l,m \leq k} \sum_{c_1, \ldots, c_l \geq 1} \sum_{c_1+\cdots+c_l=k} \sum_{e_1, \ldots, e_l \geq 0} \prod_{i=1}^l \left( \sum_{p_i} \frac{\log p_i}{p_i^{1+e_i}} \right) \prod_{i=1}^m \left( \sum_{q_i} \frac{\log q_i}{q_i} \right) = (k!)^2 \sum_{l,m \leq k} \sum_{0 \leq e_1, \ldots, e_l \leq 1} \prod_{i=1}^l \left( \log x + O(1) \right)^{1+e_i} \prod_{i=1}^m \left( \log x + O(1) \right)
\]
and this is \(\ll (\log x)^{2k} (CK)^{2k}\). As before, if some exponents among the \(r_i, s_j\) in (2.6) are larger than 1, then the contribution of this case in (2.6) is still \(\ll (\log x)^{2k} (CK)^{2k}\), by a combination of the previous computation and the technique we used to study \(S_2\). This proves (2.5) and as a consequence we get
\[
(2.7) \quad \frac{1}{T \log T} \int_T^{2T} \left( \frac{1}{\log x} \left( \sum_{n \leq x^2} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right) + \log T \right)^{2k} |\zeta|^2 dt \ll (\log \log \log T)^{2k} (CK)^{2k}.
\]

What remains to investigate is the contribution of \(L(t)\). Following Tsang ([14], equation (5.21)) we have:
\[
(2.8) \quad \Re L(t) \ll L_1(t) + L_2(t)
\]
where denoting with $\rho = \frac{1}{2} + i \gamma$ the non-trivial zeros of $\zeta$

$$L_1(t) := \sum_\rho \left( \frac{4}{\log x} \right)^2 \frac{1}{\left| \frac{4}{\log x} + i(t-\gamma) \right|^2} \int_{\frac{1}{2}}^{1/2+4/\log x} \frac{|u - \frac{1}{2}|}{(u - \frac{1}{2})^2 + (t-\gamma)^2} du$$

$$L_2(t) := \left( \frac{4}{\log x} \right)^2 \sum_\rho \frac{1}{\left| \frac{4}{\log x} + i(t-\gamma) \right|^2}$$

so we need to study the weighted moments of $L_1(t)$ and $L_2(t)$. The latter is not difficult; indeed Selberg proved that (see [14], equation (5.20))

$$\sum_\rho \frac{1}{\left| \frac{4}{\log x} + i(t-\gamma) \right|^2} \ll \log x$$

hence in view of (2.7) we know that the $2k$-th moment of $L_2(t)$ is $\ll (\log \log \log T)^{2k} (Ck)^{2k}$.

To deal with $L_1(t)$, we denote $\eta_t := \min_\rho |t - \gamma|$ and $\log^+ t := \max(\log t, 0)$. From Tsang’s computation ([14], p.93) we know that:

$$L_1(t) \ll \frac{1}{\log x} \left( \left| \sum_{n\leq x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right| + \log T \right)$$

$$+ \frac{1}{\log x} \log^+ \left( \frac{1}{\eta_t \log x} \right) \left( \left| \sum_{n\leq x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right| + \log T \right)$$

and the first term here is not a problem for the same reason as before. As a last step we study the $2k$-th moment of the second term. Applying the Cauchy-Schwarz inequality:

$$\frac{1}{(\log x)^{2k}} \int_T^{2T} \left( \log^+ \frac{1}{\eta_t \log x} \right)^{2k} \left( \left| \sum_{n\leq x^3} \frac{\Lambda(n)n^{-4/\log x}}{n^{1/2+it}} \right| + \log T \right)^{2k} |\zeta|^2 dt$$

$$\ll \sqrt{T \log T} (\log \log T)^{2k} (Ck)^{2k} \sqrt{\int_T^{2T} \left( \log^+ \frac{1}{\eta_t \log x} \right)^{4k} |\zeta|^2 dt}.$$ 

The proposition follows if we bound the remaining integral. Here the Riemann Hypothesis plays a central role, in the form of a result due to Gonek, which is a consequence of the Landau-Gonek formula (see [5], p93, Theorem 2):

**Lemma 2.2.** (Gonek) Assume RH, let $T$ be a large parameter, $|\alpha| \leq \frac{\log T}{2\pi}$, then

$$\sum_{0 < \gamma \leq T} \left| \zeta \left( \frac{1}{2} + i \left( \gamma + \frac{2\pi \alpha}{\log T} \right) \right) \right|^2 = \left( 1 - \left( \frac{\sin \pi \alpha}{\pi \alpha} \right)^2 \right) \frac{T}{2\pi} (\log T)^2 + O(T(\log T)^{7/4}).$$
For us the uniform upper bound \( \ll T(\log T)^2 \) for \(|\alpha| \leq \frac{\log T}{2\pi \log x}\) will be sufficient. Using this result we get:

\[
\int_T^{2T} \left( \log^{+} \frac{1}{\eta \log x} \right)^{4k} |\xi|^2 dt \\
\leq \sum_{T^{-\log x} \leq \gamma \leq 2T + \frac{1}{\log x}} \int_0^{1/\log x} \left( \log^{+} \frac{1}{w \log x} \right)^{4k} |\zeta(1/2 + i(w + \gamma))|^2 \, dw \\
= \sum_{T^{-\log x} \leq \gamma \leq 2T + \frac{1}{\log x}} \int_0^1 \left( \log^{+} \frac{1}{t} \right)^{4k} |\zeta(1/2 + i(\gamma + \frac{t}{\log x}))|^2 \, dt \\
= \frac{1}{\log x} \int_0^1 (\log t)^{4k} \sum_{T^{-\log x} \leq \gamma \leq 2T + \frac{1}{\log x}} |\zeta(1/2 + i(\gamma + \frac{t}{\log x}))|^2 \, dt \\
= \frac{T(\log T)^2}{\log x} \int_0^1 (\log t)^{4k} \, dt = \frac{T \log T \log (\log x)}{\log x} \\
\ll T \log T \int_T^{2T} |\Re L(t)|^{2k} \, dt \\
\ll (Ck)^4 k^{1/2} \log ^{2k+1/2} \log T \log \log T \log \log \log T \log \log \log \log T.
\]

since \( \int_0^1 (\log t)^{4k} \, dt = \int_0^\infty e^{-t} t^{4k} \, dt = \Gamma(4k+1) = (4k)! \ll (4k)^{4k} \). Putting this into (2.10) one has that also the 2k—th moment of \( L_1(t) \) is bounded by \((Ck)^4 k^{1/2} \log ^{2k+1/2} \log T \log \log T \log \log \log T \log \log \log \log T\).

Then the contribution of the zeros is under control, being

\[
\frac{1}{T \log T} \int_T^{2T} |\Re L(t)|^{2k} \, dt \\
\ll (Ck)^4 k^{1/2} \log ^{2k+1/2} \log T \log \log T \log \log \log T \log \log \log \log T.
\]

and the proposition follows.

3. Proof of Proposition 2

3.1. Sketch of the proof. In order to prove Proposition 2, we need to perform a precise asymptotic analysis for the moments of \( \Re P(t) \). First of all, since the polynomial is short \((n \leq x = T^{e/k} = T^{o(l/k)})\) one can easily compute its mean and variance by standard applications of Lemma 2.1 Indeed for any \( r, s \) integers one has

\[
\int_T^{2T} P(t)\overline{P(t)}^s |\zeta|^2 dt \\
= T \sum_{p_1 \cdots p_r \leq x} \frac{(p_1 \cdots p_r, q_1 \cdots q_s)}{p_1 \cdots p_r q_1 \cdots q_s} \left( \log \left( \frac{T(p_1 \cdots p_r, q_1 \cdots q_s)^2}{p_1 \cdots p_r q_1 \cdots q_s} \right) + c \right) + o(T)
\]

then, since \( 2\Re P(t) = P(t) + \overline{P(t)} \), the mean of \( \Re P(t) \) is

\[
\frac{1}{T \log T} \int_T^{2T} \Re P(t) |\zeta|^2 dt = \frac{1}{T \log T} \sum_{p \leq x} \frac{\log T - \log p + c}{p} + o\left( \frac{1}{\log T} \right) \\
= \mathcal{L} - \frac{c}{k} + O\left( \frac{\log \log T}{\log T} \right) = \mathcal{L} + o(1).
\]
Similarly
\[ \frac{1}{T \log T} \int_{T}^{2T} (\Re P(t))^2 |\zeta|^2 dt = \frac{1}{T \log T} \int_{T}^{2T} \left( \frac{1}{4} P(t)^2 + \frac{1}{2} P(t) \overline{P(t)} + \frac{1}{4} \overline{P(t)}^2 \right) |\zeta|^2 dt \]
\[ = \frac{1}{2 \log T} \sum_{p_1 p_2 \leq x} \log T - \log(p_1 p_2) + c + \sum_{p, q \leq x} (p, q) \log \left( \frac{T(p, q)^2}{pq} \right) + O \left( \frac{1}{\log T} \right) \]
\[ = \frac{1}{2 \log T} \left( 2L^2 \log T - 4L \log x + L \log T + O(\log T) \right) = L^2 + \frac{L}{2} - \frac{2\varepsilon L}{k} + O(1). \]

Hence the variance is
\[ \frac{1}{T \log T} \int_{T}^{2T} (\Re P(t) - L)^2 |\zeta|^2 dt \sim \frac{L}{2}. \]

To prove Proposition 2, we now have to compute the \( k \)-th moment of \( \Re P(t) - L \) with respect to \( |\zeta|^2 dt \), for every \( k \) integer. Here we give a simplified sketch of the proof, leaving the rigorous one for the following section. First of all, since
\[ \log \left( \frac{T(p_1 \cdots p_r q_1 \cdots q_s)^2}{p_1 \cdots p_r q_1 \cdots q_s} \right) + c = \log T + \log \left( \frac{(p_1 \cdots p_r q_1 \cdots q_s)^2}{p_1 \cdots p_r q_1 \cdots q_s} \right) + c \]
then expanding out the \( k \)-th power and using (3.1) one has
\[ \frac{1}{T \log T} \int_{T}^{2T} (\Re P(t) - L)^k |\zeta|^2 dt \]
\[ = \sum_{j + h = k} \binom{k}{h} (-1)^j L^j 2^{-h} \sum_{r + s = h} \binom{h}{r} \sum_{\substack{p_1, \ldots, p_r \leq x \\ q_1, \ldots, q_s \leq x \\ p_1, \ldots, p_r, q_1, \ldots, q_s}} \frac{(p_1 \cdots p_r q_1 \cdots q_s)}{p_1 \cdots p_r q_1 \cdots q_s} + \cdots \]
where the dots come from the contributions of the second and third terms in (3.2), which we are going to ignore in the following. Indeed the contribution of the constant \( c \) is clearly analogous but smaller than the one coming from \( \log T \). Even though the second term in (3.2) is not negligible compared to the first one, its contribution in the right hand side of (3.3) can be computed in a similar way to the contribution of the first one, with the important difference that in this case the main term will cancel out. Thus we ignore it as well for now, focusing on the first term.

Let’s suppose now that the primes \( p_1, \ldots, p_r \) are distinct and the primes \( q_1, \ldots, q_s \) are distinct as well. In order to compute explicitly the GCD, we fix an integer \( m \), which is smaller than both \( r \) and \( s \), and we suppose that \( m \) repetitions occur among the \( p_i \) and the \( q_j \). Because of the previous assumptions, it can happen in \( \binom{r}{m} \binom{s}{m} m! \) ways (selecting \( m \) primes among the \( p_i \) and \( m \) primes among the \( q_j \), then permuting the two blocks multiplying by \( m! \)), hence
\[ \sum_{\substack{p_1, \ldots, p_r \leq x \\ q_1, \ldots, q_s \leq x \\ p_1, \ldots, p_r, q_1, \ldots, q_s \text{ distinct}}} \frac{(p_1 \cdots p_r q_1 \cdots q_s)}{p_1 \cdots p_r q_1 \cdots q_s} = \sum_{m \leq \min(r, s)} \binom{r}{m} \binom{s}{m} m! \sum_{\substack{p_1, \ldots, p_r + s - m \leq x \\ q_1, \ldots, q_s \leq x \\ p_1, \ldots, p_r + s - m \text{ distinct}}} \frac{1}{p_1 \cdots p_r + s - m}. \]

We now drop the condition in the inner sum that the primes are distinct. As we will show in the following section, all these assumptions about distinct primes do not affect the asymptotic of the moment we are interested in. Indeed the errors coming from all
these extra assumptions will all cancel out and give a contribution which is negligible with respect to the main term. With this assumption the previous sum becomes
\[
\sum_{m \leq \min(r,s)} \frac{r!}{m! (r-m)!} \frac{s!}{(s-m)!} \mathcal{L}^{r+s-m}.
\]
Putting this into (3.3), recalling that \(r!/(r-m)! = \partial^m_X [X^r]_{X=1}\), for \(k\) even we get:
\[
\frac{1}{T \log T} \int_T^{2T} (|\Re P(t) - \mathcal{L}|^k)^2 dt
\]
\[
= \sum_{j+h=k} \left( \frac{k}{h} \right) (-1)^j \mathcal{L}^{j^2-h} \sum_{r+s=h} \left( \frac{h}{r} \right) \sum_{m \leq \min(r,s)} \frac{r!}{m! (r-m)!} \frac{s!}{(s-m)!} \mathcal{L}^{r+s-m} + \cdots
\]
\[
= \sum_{m \leq \frac{k}{2}} \frac{\mathcal{L}^{k-m}}{m!} \left( \frac{k!}{(k-2m)!} \left( \frac{X+Y}{2} - 1 \right)^{k-2m} 2^{-2m} \right)_{X=Y=1} + \cdots
\]
\[
= \sum_{m \leq \frac{k}{2}} \frac{\mathcal{L}^{k-m}}{2^{2m} m! (k-2m)!} \sum_{r+s=k} \frac{k!}{2^k (k/2)!} \mathcal{L}^{k/2} + \cdots = \left( \frac{\mathcal{L}}{2} \right)^{k/2} (k-1)!! + \cdots
\]
since \(k! = 2^k (k/2)! (k-1)!!\) for any even \(k\). Otherwise if \(k\) is odd, then the main term vanishes, being \(m \leq (k-1)/2\).

We now highlight the main difference from the classical case [11]. There one easily sees that \(\int P(t)^r \overline{P(t)}^s |\zeta|^2 dt\) is non negligible only if \(r\) equals \(s\). Therefore just the diagonal term \(r = s = k/2\) contributes to the main term of the \(k\)-th moment of \(\Re P(t)\). On the other hand this is no longer true in the weighted case, since all the integrals \(\int P(t)^r \overline{P(t)}^s |\zeta|^2 dt\) give a contribution of order \(T \log T \mathcal{L}^{-k}\). The main point is that in the classical case the mean of \(\Re P(t)\) is 0, while with respect to the weighted measure \(|\zeta|^2 dt\) the mean is \(\sim \mathcal{L}\). Thus, even though in the weighted case the size of the \(k\)-th moment of \(\Re P(t)\) is \(\mathcal{L}^k\), the \(k\)-th moment of \(\Re P(t) - \mathcal{L}\) has order \(\mathcal{L}^{k/2}\). Showing this cancellation from \(k\) to \(k/2\) is the bulk of the proof.

3.2. Proof of Proposition 2. We now prove the result, following the line of the previous computation. Expanding out the \(k\)-th power and using \(2\Re P(t) = P(t) + \overline{P(t)}\), one finds
\[
(3.4) \int_T^{2T} (|\Re P(t) - \mathcal{L}|^k |\zeta|^2 dt = \sum_{j+h=k} \left( \frac{k}{h} \right) (-1)^j \mathcal{L}^{j^2-h} \sum_{r+s=h} \left( \frac{h}{r} \right) \int_T^{2T} P(t)^r \overline{P(t)}^s |\zeta|^2 dt
\]
and the inner integral equals
\[
T \sum_{p_1, \ldots, p_r \leq x} \sum_{q_1, \ldots, q_s \leq x} \left( \log \left( \frac{T(p_1 \cdots p_r, q_1 \cdots q_s)^2}{p_1 \cdots p_r q_1 \cdots q_s} \right) \right) + o(T)
\]
in view of (3.1). Since \(\log t = \partial_w [t^w]_{w=0}\), one gets
\[
(3.5) \int_T^{2T} P(t)^r \overline{P(t)}^s |\zeta|^2 dt = T (\log T + c) f_x(0) + T \partial_w [f_x(w)]_{w=0} + o(T)
\]
where

\[ f_x(w) = \sum_{p_1, \ldots, p_r \leq x \atop q_1, \ldots, q_s \leq x} \frac{(p_1 \cdots p_r, q_1 \cdots q_s)^{2w+1}}{(p_1 \cdots p_r q_1 \cdots q_s)^{w+1}} \]

In order to be able to compute explicitly the GCD, we put in evidence the possible repetitions among the primes, re-writing the \( p_i \) and the \( q_i \) as follows. First we put in evidence the repetitions among the primes \( p_i \), writing

\[ p_1, \ldots, p_r \rightarrow p_1, \ldots, p_{r-v_1}, p_{v_1}^{\alpha_1}, \ldots, p_{u_1}^{\alpha_u} \]

where \( p_1, \ldots, p_{r-v_1}, p_{v_1}', \ldots, p_{u_1}' \) are all distinct, \( \alpha_1 + \cdots + \alpha_u = v_1 \), \( \alpha_i \geq 2 \) for every \( i \). With this change of variable we need a normalization \( \frac{r!}{(r-v_1)!} c_{\alpha} \) where \( c_{\alpha} \) is a positive coefficient smaller than 1, which does not depend on \( r \) but just on the configuration \( \alpha_1, \ldots, \alpha_u \). Notice that if \( v_1 = 0 \), then \( c_{\alpha} = 1 \). Now we highlight the multiplicities of the primes \( q_j \) and we put in evidence those ones that already appear among the \( p_i' \). Then we write

\[ q_1, \ldots, q_s \rightarrow q_1, \ldots, q_{s-v_2}, p_1', \ldots, p_{v_2}' q_1^{\beta_1}, \ldots, q_{v_2}'^{\beta_{v_2}} p_{v_1}^{\gamma_1}, \ldots, p_{u_1}^{\gamma_u} \]

with \( q_i \) distinct, \( q_i' \) distinct, \( q_i' \neq p_j' \) for every \( i, j \), \( q_i \neq q_j' \) for every \( i, j \) and \( \beta_1 + \cdots + \beta_{v_2} + \gamma_1 + \cdots + \gamma_{u_1} + a_2 = v_2 \), \( \beta_i \geq 2 \), \( \gamma_i \neq 1 \) for every \( i \). Also in this case the change of variable brings into play a normalization \( \frac{(s-v_2)!}{(s-v_2-1)!} c_{\beta, \gamma} \) where once again \( c_{\beta, \gamma} \) only depends on the configuration \( \beta_1, \ldots, \beta_{v_2}, \gamma_1, \ldots, \gamma_{u_1} \) and it is equal to 1 when \( u_2 = 0 \) and \( \gamma_i = 0 \) for every \( i \). Then we have

\[
f_x(w) = \sum_{p_{v_1} \leq a_2 \leq s} \left( \sum_{p_{v_2} \leq a_2 \leq s} \sum_{\beta_1 + \cdots + \beta_{v_2} + \gamma_1 + \cdots + \gamma_{u_1} = v_2} \sum_{\beta_i \geq 2, \gamma_i \neq 1} \frac{c_{\alpha}}{c_{\beta, \gamma}} \right) \frac{r!}{(r-v_1)!} \frac{s!}{(s-v_2)!} \sum_{p_i q_j' \text{ distinct and } \neq p_i'} \frac{(p_1^{\alpha_1} \cdots p_{u_1}'^{\alpha_u} p_1' \cdots p_{v_1}' \gamma_1 \cdots p_{u_1}^{\gamma_u})^{2w+1}}{(p_1 \cdots p_{r-v_1} q_1 \cdots q_{s-v_2-a_2})^{2w+1}} \times \frac{(p_1 \cdots p_{r-v_1} q_1 \cdots q_{s-v_2-a_2})^{2w+1}}{(p_1 \cdots p_{r-v_1} q_1 \cdots q_{s-v_2-a_2})^{w+1}}
\]

For the sake of brevity let’s denote \( p' \) and \( q' \) the product of \( p_i' \) and \( q_i' \) respectively with their exponents (for instance \( p'_{u_1} := p_1^{\alpha_1} \cdots p_{u_1}'^{\alpha_u} \)). To be able to compute the GCD between \( p \) and \( q^{\beta_j} \) in the inner sum, we put in evidence the repetitions among the \( p_i \) and the \( q'_j \). Let’s say we have \( a_1 \) primes among the \( p_i \) which coincide with some \( q'_j \). Then, denoting
\( r' := r - v_1 - a_1 \) and \( s' := s - v_2 - a_2 \), we get

\[
\begin{align*}
f_x(w) &= \sum_{v_1, u_1 \leq r} \sum_{v_2, u_2 \leq s} \sum_{a_1, a_2} c(\alpha, \beta, \gamma, \delta) \frac{r!}{(r-v_1-a_1)!} \frac{s!}{(s-v_2-a_2)!} \\
&\quad \sum_{p'_i, q'_j \text{ distinct}} \frac{(p'^{\alpha}_i, p'^{\beta}_i)^{2w+1}(q'^{\gamma}_j)^w}{(p'^{\alpha}_i p'^{\beta}_i)^{\alpha \beta \gamma} \cdot \gamma \\gamma)} \sum_{p'_i, q'_j \text{ distinct and } \neq p'_i, q'_j} \frac{(p_1 \cdots p_{r'}, q_1 \cdots q_{s'})^{2w+1}}{(p_1 \cdots p_r q_1 \cdots q_{s'})^{w+1}}
\end{align*}
\]

where \( c(\alpha, \beta, \gamma, \delta) \) is a bounded coefficient which does not depend on \( r \) and \( s \) and it is equal to 1 when \( u_i = v_i = a_i = 0 \) for \( i = 1, 2 \). Note that the sum over \( p'_i \) and \( q'_j \) is bounded when \( w \) is close to 0, since both \( \beta \) and \( \max(\alpha_i, \gamma_i + 1) \) are \( \geq 2 \). Lastly we want to put in evidence the repetitions among the \( p_i \) and the \( q_j \), in order compute explicitly the last greatest common divisor \( (p_1 \cdots p_{r'}, q_1 \cdots q_{s'}) \) in the inner sum. If \( m \) repetitions occur, for any \( m \leq \min(r', s') \), we finally have \( r' + s' - m \) distinct primes and the coefficient of normalization is \( (\binom{r'}{m}) (\binom{s'}{m}) m! \). Therefore

\[
(3.6)
\]

\[
\begin{align*}
f_x(w) &= \sum_{v_1, u_1 \leq r} \sum_{v_2, u_2 \leq s} \sum_{a_1, a_2} c(\alpha, \beta, \gamma, \delta) \frac{r!}{(r-v_1-a_1)!} \frac{s!}{(s-v_2-a_2)!} \sum_{p'_i, q'_j \text{ distinct}} \frac{(p'^{\alpha}_i, p'^{\beta}_i)^{2w+1}(q'^{\gamma}_j)^w}{(p'^{\alpha}_i p'^{\beta}_i)^{\alpha \beta \gamma} \cdot \gamma \\gamma)} \sum_{p'_i, q'_j \text{ distinct and } \neq p'_i, q'_j} \frac{1}{(p_1 \cdots p_{r'+s'-2m} q_1 \cdots q_{s'} q_1 \cdots q_m)^{w+1}}
\end{align*}
\]

After computing the GCD, we now remove the extra conditions in the inner sum, which force the primes \( p_i \) and \( q_j \) to be all distinct and \( \neq p'_i, q'_j \). We get rid of the condition that forces the primes to be all distinct by using basic combinatorics and we remove the last condition \( p_1, \ldots, p_{r'+s'-2m}, q_1, \ldots, q_m \neq p'_i, q'_j \), splitting the inner sums as

\[
\sum_{p \leq x} \frac{1}{p^s} = \sum_{p \leq x} \frac{1}{p^s} - \sum_{i=1}^{u_1} \frac{1}{p^s} - \sum_{i=1}^{u_2} \frac{1}{q^s}
\]

and expanding out the powers by Newton’s binomial formula. Hence we have (denote \( h' := r' + s' \))
$f_x(w) = \sum_{v_1, u_1 \leq r} \sum_{v_2, u_2 \leq s} \sum_{\substack{\alpha, \beta, \gamma \text{ distinct}}} \frac{(p^\alpha, p^\beta, p^\gamma)^{2w + 1}(q^\gamma)^w}{(p^\alpha p^\beta p^\gamma)^{w + 1}} \sum_{m \leq \min(r', s')} \frac{r!}{(r' - m)!} \frac{s!}{(s' - m)!} \sum_{t_1 \leq h' - 2m, t_2 \leq m, t_3 \leq t_1 + t_2} \prod_{i=1}^{t_3} (\#R_i - 1)!(-1)^{\#R_i - 1} \left( \sum_{p \neq p', q'} \frac{1}{p^i} \right) c(\alpha, \beta, \gamma, \alpha, t)

\sum_{i \leq l_2 \leq m - t_2 - l_2} \left( \sum_{p \leq x} \frac{1}{p^{1 + w}} \right)^{h' - 2m - t_1 - l_1} \sum_{i = 1}^{l_1}(h' - 2m - t_1 - l_1)! \left( \sum_{p \leq x} \frac{1}{p^{1 + w}} \right)^{l_1} \theta_{m - t_2 - l_2}

where $c(\alpha, \beta, \gamma, \alpha, t)$ is a bounded coefficient not depending on $r, s, m$, which is equal to 1 if the parameters $v_i, u_i, t_i$ are all equal to 0 and $Part$ denotes the set of partitions of the set of the exponents of primes appearing in the inner sum in (3.6).

We are ready to plug the formula we got for $f_x(w)$ into the formula for the $k$–th moment of $\Re P(t) - \mathcal{L}$. Putting (3.4) and (3.5) together one has

$$\int_T^{2T} (\Re P(t) - \mathcal{L})^k |\zeta|^2 dt$$

(3.7) $$= (T \log T + c) \left[ \sum_{j \neq h = k} \binom{k}{h} (-1)^j \mathcal{L}^{j - h} \sum_{r + s = h} \binom{h}{r} f_x(w) \right]_{w=0}$$

$$+ \partial_w \left[ T \sum_{j \neq h = k} \binom{k}{h} (-1)^j \mathcal{L}^{j - h} \sum_{r + s = h} \binom{h}{r} f_x(w) \right]_{w=0} + o(T)$$

Now we exchange the order of summation, bringing the sum over $j, h$ inside in order to appreciate the cancellation. By the explicit expression we got for $f_x(w)$ we have

$$\sum_{j \neq h = k} \binom{k}{h} (-1)^j \mathcal{L}^{j - h} \sum_{r + s = h} \binom{h}{r} f_x(w)$$

$$= \sum_{m \leq \frac{t}{2}} \sum_{\substack{p_1, p_2, p_3, q_1, q_2 \text{ distinct}}} F_{\alpha, \beta, \gamma \alpha, \beta, \gamma, \alpha, \beta, \gamma, \alpha, \beta, \gamma} \frac{1}{(m - t_2 - l_2)!} \mathcal{L}^{m - t_2 - l_2}$$

(3.8) $$\sum_{j \neq h = k} \binom{k}{h} (-1)^j \mathcal{L}^{j - h} \frac{(h' - 2m)!}{(h' - 2m - t_1 - l_1)!} \left( \sum_{p \leq x} \frac{1}{p^{1 + w}} \right)^{h' - 2m - t_1 - l_1}$$

$$\sum_{r + s = h} \binom{h}{r} \frac{r!}{(r - v_1 - a_1 - m)!} \frac{s!}{(s - v_2 - a_2 - m)!}$$
where we denote \( k' := k - v_1 - v_2 - a_1 - a_2 \) and

\[
F_{\omega, \alpha, \beta, \gamma, \eta, \xi, \psi} (p', q'; w) := \frac{(p'^{\omega} p'^{\psi} z)^{2w+1}(q')^{w}}{(p u^\alpha q u^\beta z^{2})^{w+1} c(\omega, \beta, \gamma, \eta, \xi, \psi)} \left( \sum_{p \neq p', q'} \frac{1}{p'} \right)
\]

\[
\prod_{i=1}^{l_1} \left( \prod_{i=1}^{w_1} \frac{1}{p^{1+w}} - \sum_{i=1}^{w_2} \frac{1}{q^{1+w}} \right) l_1 \left( \sum_{i=1}^{w_1} \frac{1}{p} - \sum_{i=1}^{w_2} \frac{1}{q} \right)^2.
\]

Note that the function \( F_{\omega, \alpha, \beta, \gamma, \eta, \xi, \psi} (p', q'; w) \) makes the sum over \( p_i, q_j \) in (3.8) converge. Moreover notice that in the case trivial case \( v_i = u_i = a_i = t_i = l_i = 0 \) for every \( i \) then we have that \( F_{\omega, \alpha, \beta, \gamma, \eta, \xi, \psi} (p', q'; w) = 1 \). Now we recall that the three quotients involving \( r! \), \( s! \) and \( h! \) can be expressed in terms of derivatives (for instance \( r!/(r-v_1-a_1-m)! = \partial_X^{v_1+a_1+m}[X^r]_{X=1} \) then (3.3) becomes

\[
\sum_{m \leq \frac{k}{2}} \sum_{p \neq p', q' \text{ distinct}} F_{\omega, \alpha, \beta, \gamma, \eta, \xi, \psi} (p', q'; w) \frac{1}{(m - t_2 - l_2)!} L^{m-t_2-l_2}
\]

\[
\partial_X^{v_1+a_1+m} \partial_Y^{v_2+a_2+m} \partial_Z^{t_1+l_1} \left[ \left( \sum_{p \leq x} \frac{1}{p^{1+w}} \right)_{-v_1-v_2-a_1-a_2-2m-t_1-l_1} Z_{-v_1-v_2-a_1-a_2-2m} \right.
\]

\[
\left. \sum_{j+h=k} \binom{k}{h} (-1)^j L^j 2^{-h} Z^h \left( \sum_{p \leq x} \frac{1}{p^{1+w}} \right)^h (X + Y)^h \right]_{X=Y=Z=1}.
\]

Carrying out the computation straightforwardly, denoting \( y = v_1 + v_2 + a_1 + a_2 + t_1 + l_1 \), it yields

\[
\sum_{j+h=k} \binom{k}{h} (-1)^j L^j 2^{-h} \sum_{r+s=h} \binom{h}{r} f_x(w)
\]

\[
= \sum_{m \leq \frac{k}{2}} \sum_{p \neq p', q' \text{ distinct}} F_{\omega, \alpha, \beta, \gamma, \eta, \xi, \psi} (p', q'; w) \frac{L^{m-t_2-l_2}}{(m - t_2 - l_2)!}
\]

\[
k(k-1) \cdots (k-y-2m+1) \frac{2y-t_1-l_1+2m+1}{2^{y-t_1-l_1+2m+1}} \left( \sum_{p \leq x} \frac{1}{p^{1+w}} - L \right)^{k-y-2m}.
\]

Now, recalling (3.7), we have to study the right hand side of (3.9) and its derivative at \( w = 0 \). As we will see soon, only the former contributes to the main term of the \( k \)-th moment we are considering.

By definition of \( L := \sum_{p \leq x} \frac{1}{p!} \) if \( w = 0 \) then the expression in the parentheses on the right hand side of (3.9) vanishes. This forces its exponent to be zero, otherwise all the
contribution vanishes. Hence we get

\[
(3.10) \quad \left[ \sum_{j+h=k} \binom{k}{h} (-1)^j \mathcal{L}^{j-2h} \sum_{r+s=h} \binom{h}{r} f_x(w) \right]_{w=0}
\]

\[
= \sum_{m \leq \frac{k}{2}} \sum_{p'_i, q'_j \text{ distinct}} F_{u, u', a, \beta, \gamma, \delta, \rho, \varphi}(p'_i, q'_j; w) \frac{\mathcal{L}^{m-t_2-l_2}}{(m-t_2-l_2)!} \cdot \frac{k(k-1) \cdots (k-y-2m+1)}{2^{y-t_1-l_1+2m}} \cdot \mathbb{1}_{2m=k-y}
\]

The main term is given by the largest \( m \) possible, i.e. \( m = \frac{k}{2} \) if \( k \) is even. Since \( 2m = k-y \), then \( y = 0 \) hence all the parameters that individuate the configuration vanish. Therefore

\[
(3.11) \quad \left[ \sum_{j+h=k} \binom{k}{h} (-1)^j \mathcal{L}^{j-2h} \sum_{r+s=h} \binom{h}{r} f_x(w) \right]_{w=0} = \frac{k!}{2^k(k/2)!} \mathcal{L}^{k/2} + O_k(\mathcal{L}^{k/2-1})
\]

which matches with the \( k \)-th moment of a Gaussian by basic properties of the double factorial, since \( k! = 2^{k/2}(k/2)! (k-1)! \) for any even \( k \). Note that the error term in (3.11) is given by the term \( m = k/2 - 1 \) hence it is \( O_k(\mathcal{L}^{k/2-1}) \). Of course if \( k \) is odd one can immediately see that the right hand side of (3.10) is \( O_k(\mathcal{L}^{(k-1)/2}) \).

Let’s now analyze the derivative

\[
\partial_w \left[ \sum_{j+h=k} \binom{k}{h} (-1)^j \mathcal{L}^{j-2h} \sum_{r+s=h} \binom{h}{r} f_x(w) \right]_{w=0}
\]

\[
= \partial_w \sum_{m \leq \frac{k}{2}} \sum_{p'_i, q'_j \text{ distinct}} F_{u, u', a, \beta, \gamma, \delta, \rho, \varphi}(p'_i, q'_j; w) \frac{1}{(m-t_2-l_2)!} \mathcal{L}^{m-t_2-l_2} \cdot \frac{k(k-1) \cdots (k-y-2m+1)}{2^{y-t_1-l_1+2m}} \cdot \left( \left( \sum_{p \leq x} \frac{1}{p^{1+w}} \right) - \mathcal{L} \right)_{w=0}^{k-y-2m}.
\]

Recall that this term will be multiplied by a factor \( T \) in (3.7), while the other one by \( T \log T \). When we compute the derivative using Leibniz’s rule, the term where the derivative of \( F \) appears is trivially \( O_k(\mathcal{L}^k / \log T) \), which is negligible. Indeed the sum over \( p'_i, q'_j \) is still bounded because the exponents of the variables are larger that 2 and computing derivatives just \( \log p_i \) or \( \log q_j \) come out. We finally have to deal with the derivative of the inner term. Since

\[
\partial_w \left[ \sum_{p \leq x} \frac{1}{p^{1+w}} \right]_{w=0} = - \sum_{p \leq x} \frac{\log p}{p} \ll \log x = \frac{\varepsilon}{k} \log T
\]
we get that the contribution coming from derivative of $p^{-1-w}$ in (3.12) is
\[
\ll \sum_{m \leq k} \sum_{\substack{p_1 \leq x, x_1 \leq m+1 \atop \text{distinct}}} \frac{1}{(m - l_2 - l_2)!} L^{m - l_2 - l_2} \frac{k(k-1) \cdots (k-y-2m+1)}{2^{y-l_1-l_1+2m}} \partial_w \left[ \left( \sum_{p \leq x} \frac{1}{p^{1+w} - L} \right)^{k-y-2m} \right]_{w=0}
\]
which is $O_k(\varepsilon \log TL^{(k-1)/2})$ by the same argument as before. Hence
\[
(3.13) \quad \partial_w \left[ \sum_{j+h=k} \binom{k}{h} (-1)^j L^{j/2-h} \sum_{r+s=k} \binom{h}{r} f_\varepsilon(w) \right]_{w=0} = O_k(\varepsilon \log TL^{(k-1)/2})
\]
Putting both (3.11) and (3.13) into (3.7) the proof is complete.

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