A linear isoperimetric inequality for the punctured Euclidean plane

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Abstract

It follows from a general theorem of Bonk and Eremenko that contractible closed curves on the Euclidean $\mathbb{R}^2 \setminus \mathbb{Z}^2$ satisfy a linear isoperimetric inequality. In the present note we give an alternative proof of this fact. Our approach is based on a non-standard combinatorial isoperimetric inequality for the group $\langle a, b, c | aba^{-1}b^{-1}c \rangle$. We show that the combinatorial area of every element in the normal closure of $aba^{-1}b^{-1}c$ does not exceed the half of its length in generators $a^{\pm 1}, b^{\pm 1}$ (here $c^{\pm 1}$ is ignored). The proof uses a refinement of the small cancellation theory. In addition, we present an application of the isoperimetric inequality on $\mathbb{R}^2 \setminus \mathbb{Z}^2$ to Hamiltonian dynamics. Combining it with methods of symplectic topology we show that every non-identical Hamiltonian diffeomorphism of the 2-torus has at least linear asymptotic growth of the differential.

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1 Introduction and results

For a smooth contractible curve \( \alpha : S^1 \to \mathbb{R}^2 \setminus \mathbb{Z}^2 \) define its area by

\[
\text{Area}(\alpha) = \inf \int_{D^2} |f^* \omega|,
\]

where the infimum is taken over all maps \( f : D^2 \to \mathbb{R}^2 \setminus \mathbb{Z}^2 \) with \( f|_{S^1} = \alpha \), and \( \omega \) stands for the Euclidean area form. A recent result due to Bonk and Eremenko [BE] which is proved for a much more general class of surfaces yields that \( \text{Area}(\alpha) \leq \mu \text{Length}(\alpha) \) for some universal constant \( \mu \). The proof in [BE] uses the theory of Gromov hyperbolic spaces. In this note we give an alternative proof of this result and present an application to Hamiltonian dynamics.

**Theorem 1.1 (see Bonk-Eremenko [BE]).**

\[
\text{Area}(\alpha) \leq (1 + \sqrt{2}) \text{Length}(\alpha)
\]

for every smooth contractible curve \( \alpha : S^1 \to \mathbb{R}^2 \setminus \mathbb{Z}^2 \).

Theorem 1.1 has the following counterpart in combinatorial group theory. Consider the free group \( \mathbb{F}_3 \) with 3 generators \( a, b, c \) and the word \( r = aba^{-1}b^{-1}c \). Let \( N \subset \mathbb{F} \) be the normal closure of \( r \). For an element \( w \in N \) define its **combinatorial area** \( A(w) = \inf d \), where the infimum is taken over all decompositions of the form

\[
w = \prod_{i=1}^{d} g_i r^{\pm 1} g_i^{-1}, \quad g_i \in \mathbb{F}_3.
\]

Define the \((a, b)\) -length \( l(w) \) as the number of occurrences of symbols \( a^{\pm 1}, b^{\pm 1} \) in a cyclic reduction of \( w \).

**Theorem 1.2.** \( A(w) \leq \frac{1}{2} l(w) \) for every \( w \in N \).

The link between these two inequalities is discussed in §2 where we deduce Theorem 1.1 from Theorem 1.2. Theorem 1.2 is proved in §3.

Some remarks are in order. Inequality 1.2 is sharp (see 3.3 below) while the isoperimetric constant provided by Theorem 1.1 most probably can be
improved. It would be interesting to compare the isoperimetric constant from [1.1] with the one obtained in [BE].

For an embedded curve $\alpha$ one has $\text{Area}(\alpha) \leq \text{Length}(\alpha)$. This follows from the classical theorem of Jarnik in number theory (see [Hua], p. 123).

Contractibility of $\alpha$ cannot be replaced by a homological assumption. Indeed, let $\lambda$ be a primitive of $\omega$ on $\mathbb{R}^2$, that is $\omega = d\lambda$. It is easy to construct a curve $\alpha : S^1 \rightarrow \mathbb{R}^2 \setminus \mathbb{Z}^2$ of arbitrarily large length which is homologous to zero (but non-contractible!) so that

$$\int_{\alpha} \lambda \geq \left(\text{Length}(\alpha)\right)^{1+\delta}$$

for some positive constant $\delta$.

A linear isoperimetric inequality is obvious for the complement of the "thick" lattice. Let $B(\epsilon)$ be the Euclidean $\epsilon$-neighbourhood of $\mathbb{Z}^2$ in $\mathbb{R}^2$. One can easily show that $\text{Area}(\alpha) \leq C(\epsilon)\text{Length}(\alpha)$ for every contractible curve $\alpha : S^1 \rightarrow \mathbb{R}^2 \setminus B(\epsilon)$. Indeed, the Euclidean metric on $\mathbb{R}^2 \setminus B(\epsilon)$ is equivalent to the hyperbolic one which as it is known satisfies the linear isoperimetric inequality. The problem is that the Lipschitz constant, and hence the isoperimetric constant obtained by this argument, goes to infinity as $\epsilon \rightarrow 0$. Bruce Kleiner pointed out that one can go round this difficulty using the theory of Gromov hyperbolic spaces. He outlined an argument which shows that a linear isoperimetric inequality should be valid for much more general spaces of non-positive curvature. Bonk and Eremenko [BE] established a linear isoperimetric inequality for a fairly general class of open (not necessarily complete) surfaces of non-positive curvature.

The inequality $A(w) \leq |w|$ where $|w|$ stands for the usual word length of $w \in N$ in generators $a, b, c$ is an immediate consequence of the small cancellation theory [LS]. It turns out that in our situation some arguments of this theory can be refined. This enables us to replace the word length by the $(a, b)$-length and to improve the isoperimetric constant.

Finally let us discuss an application of isoperimetric inequality [1.1] to Hamiltonian dynamics. Consider an area-preserving diffeomorphism $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form

$$\psi(x_1, x_2) = (x_1 + \psi_1(x_1, x_2), x_2 + \psi_2(x_1, x_2)),$$

where the functions $\psi_i$, $i = 1, 2$ are 1-periodic in both variables and satisfy

$$\int_0^1 \int_0^1 \psi_i(x_1, x_2)dx_1dx_2 = 0. \quad (1)$$
Diffeomorphisms of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ corresponding to such $\psi$’s are called *Hamiltonian*. They play a fundamental role both in symplectic topology and classical mechanics (see [MS], [HZ], [P1]). Put

$$||d\psi|| = \max |d_\xi \psi(\xi)|,$$

where the maximum is taken over all unit tangent vectors $\xi \in T_x \mathbb{R}^2$. The growth type of the sequence $||d\psi^n||$ is a basic dynamical invariant of $\psi$ (cf. [DG]). Up to a multiplicative constant it is an invariant of the conjugacy class of $\psi$ in the group of all $\mathbb{Z}^2$-periodic diffeomorphisms.

**Theorem 1.3.** Suppose that $\psi \neq id$. Then

$$||d\psi^n|| \geq \kappa n$$

for some $\kappa = \kappa(\psi) > 0$.

This result is sharp. For instance $||d\psi^n||$ grows linearly for $\psi(x_1, x_2) = (x_1, x_2 + \sin x_1)$. On the other hand one has $||d\phi^n|| \equiv 1$ when $\phi$ is a parallel translation $x \to x + \text{const}$ of the plane. Here $\phi$ is still area-preserving and $\mathbb{Z}^2$-periodic, but non-Hamiltonian – condition (1) is violated. We refer to [P2] for an extensive discussion on the growth of differential of symplectic maps including more sophisticated non-Hamiltonian examples and counterexamples on $\mathbb{T}^2$, sharp results for surfaces of higher genus and generalizations to Hamiltonian diffeomorphisms of arbitrary aspherical symplectic manifolds. Theorem 1.3 is proved in §4 below.

## 2 Reduction to an algebraic problem

Let $G_0$ be the unit grid on the plane with integer vertices. Fix $\epsilon \in (0; 0.1)$ and write $B(\epsilon)$ for the $\epsilon$-neighbourhood of $\mathbb{Z}^2$ in $\mathbb{R}^2$. Consider the 1-complex

$$G_\epsilon = (G_0 \setminus B(\epsilon)) \cup \partial B(\epsilon)$$

(see figure 1). A 1-cycle on $G_\epsilon$ is called *geometrically irreducible* if it has no pair of consecutive edges which coincide but have opposite orientations. We use the convention that a trivial 1-cycle (a vertex) is geometrically irreducible. Clearly, every closed curve on $G_\epsilon$ is free homotopic to a geometrically
irreducible cycle, and every geometrically irreducible cycle minimizes the Euclidean length among closed curves on $G_{\epsilon}$ in its free homotopy class. This follows from the fact that the universal cover of $G_{\epsilon}$ is a tree.

For a homotopy $\theta : S^1 \times [0; 1] \rightarrow \mathbb{R}^2$ define its area as the integral of $|\theta^{*}\omega|$ over $S^1 \times [0; 1]$.

**Lemma 2.1.** Every smooth curve $\alpha : S^1 \rightarrow \mathbb{R}^2 \setminus B(\epsilon)$ can be homotoped in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ to a geometrically irreducible cycle $\gamma : S^1 \rightarrow G_{\epsilon}$ with $\text{Length}(\gamma) \leq 2\text{Length}(\alpha)$ by a homotopy of area $\leq \sqrt{2}\text{Area}(\alpha)$.

The proof is given at the end of this section. Let us now look more attentively at curves on $G_{\epsilon}$. First consider the complex $G_{\epsilon}/\mathbb{Z}^2$. Its fundamental group equals $\mathbb{F}_3$. Let $a, b$ and $c$ be generators of this fundamental group represented by the following paths on $G_{\epsilon}$ (see figure 1): $a = P_1E_1MP_2, b = P_1LE_2P_3$ and $c = P_1LME_1P_1$. In Euclidean coordinates we have $P_1 = (0, -\epsilon), P_2 = (0, 1 - \epsilon), P_3 = (1, 1 - \epsilon), E_1 = (-\epsilon, 0), E_2 = (1 - \epsilon, 0), L = (\epsilon, 0)$ and $M = (0, \epsilon)$. Points denoted by the same letter with different indices have the same projections to $G_{\epsilon}/\mathbb{Z}^2$. The fundamental group described above corresponds to the base point $P_1 \mod \mathbb{Z}^2$.

Consider the homomorphism of $\mathbb{F}_3 = \langle a, b, c \rangle$ to $\mathbb{Z}^2$ which sends $a$ and $b$ to $(0, 1)$ and $(1, 0)$ respectively and vanishes on $c$. Every word $w \in \mathbb{F}_3$
which lies in its kernel represents a closed curve on \( G \) denoted \( \gamma'(w) \). By Seifert-van Kampen theorem this curve is contractible on \( \mathbb{R}^2 \setminus B(\epsilon) \) if and only if \( w \) belongs to the normal closure \( N \) of the element \( r = aba^{-1}b^{-1}c \). Indeed, \( \mathbb{R}^2 \setminus B(\epsilon) \) is a cover of the surface which is obtained from \( G/\mathbb{Z}^2 \) by gluing a disc along the cycle \( r \). Write \( \gamma(w) \) for the geometrically irreducible cycle homotopic to \( \gamma'(w) \).

**Lemma 2.2.** Let \( w \in N \) be a cyclically reduced word. Then

\[
l(w) \leq (1 - 2\epsilon)^{-1} \text{Length}(\gamma(w)).
\]

**Proof:** Note that consecutive edges of \( \gamma'(w) \) which coincide but have opposite orientation can occur only on circular parts of curves corresponding to letters \( a^{\pm 1}, b^{\pm 1} \) in \( w \). The length of the segment of the grid \( G_0 \) corresponding to each such letter is \( 1 - 2\epsilon \). Hence \( \text{Length}(\gamma(w)) \geq (1 - 2\epsilon)l(w) \). □

**Proof of Theorem 1.1:** Let \( \alpha \) be a contractible curve on \( \mathbb{R}^2 \setminus \mathbb{Z}^2 \). Fix \( \epsilon > 0 \) so small that \( \alpha \) lies in \( \mathbb{R}^2 \setminus B(\epsilon) \). Lemma 2.1 provides a homotopy of \( \alpha \) to a geometrically irreducible cycle \( \gamma \) on \( G(\epsilon) \) with \( \text{Length}(\gamma) \leq 2\text{Length}(\alpha) \). Let \( w \in N \) be a cyclically reduced word so that \( \gamma = \gamma(w) \). Clearly, \( \text{Area}(\gamma) \leq (1 - \pi\epsilon^2)A(w) \). By Theorem 1.2 \( A(w) \leq l(w)/2 \). Combining this with Lemma 2.2 we get that

\[
\text{Area}(\gamma) \leq (1 - \pi\epsilon^2)(1 - 2\epsilon)^{-1} \text{Length}(\alpha).
\]

Recall now that the area of the homotopy between \( \alpha \) and \( \gamma \) does not exceed \( \sqrt{2} \text{Length}(\alpha) \). Hence, taking \( \epsilon \to 0 \) we get

\[
\text{Area}(\alpha) \leq (1 + \sqrt{2}) \text{Length}(\alpha)
\]

as required. □

**Proof of Lemma 2.1:** Given a curve \( \alpha : S^1 \to \mathbb{R}^2 \setminus B(\epsilon) \), perturb it so that it will be in general position with respect to \( G_\epsilon \). This means that \( G_\epsilon \) does not contain self-intersection points of \( \alpha \) and \( \alpha \) intersects \( G_\epsilon \) transversally. Assume first that \( \alpha \cap G_\epsilon \neq \emptyset \). Then there exists a finite partition \( [0; 1] = \bigcup_{i=1}^n I_i \) such that \( \alpha|_{I_i} \) lies entirely in a fundamental domain of \( \mathbb{R}^2 \setminus B(\epsilon) \). Fix a segment \( I = [t_i; t_{i+1}] \) of the partition, and put \( X = \alpha(t_i), Y = \alpha(t_{i+1}) \) and \( f = \alpha|_I \). Denote by \( Q \) the fundamental domain containing \( f \). There are 3 possibilities.
**Case 1.** Points $X$ and $Y$ lie on different and non-opposite linear edges of $Q$ (see figure 2a).

**Case 2.** Points $X$ and $Y$ lie on opposite linear edges of $Q$ (see figure 2b).

**Case 3.** Points $X$ and $Y$ lie on the same linear edge of $Q$ (see figure 2c).

Let us outline the proof of the Lemma. In each case, we are going to homotope (with fixed end points) the curve $f$ to a curve $g$ which is a part of $\partial Q$ between $X$ and $Y$. We will see that $\text{Length}(g) \leq 2\text{Length}(f)$ and the area of the homotopy does not exceed $\sqrt{2}\text{Length}(f)$. Clearly, this will enable us to homotope $\alpha$ to a curve $\gamma$ on $G(\epsilon)$ which satisfies the inequalities of the Lemma. Shortening $\gamma$ if necessary, we achieve that $\gamma$ is geometrically irreducible. We treat each of these three cases separately.

![Figure 2](image_url)

**Case 1.** Choose $g$ so that it contains a circular arc whose endpoints lie on the linear segments passing through $X$ and $Y$ (see figure 2a). We work in polar coordinates $(\rho, \phi)$ with the origin at $O$. Let $(\rho(t), \phi(t)), t \in [0; \pi/2]$ be a (re)parameterization of $f$. Consider an arbitrarily small perturbation of $g$ of the form $h = (u(t), t), t \in [0; \pi/2]$. Then $u(\phi(t)) \leq \rho(t)$ for all $t$. Consider the homotopy

$$\theta(s, t) = ((1-s)\rho(t) + su(\phi(t)), \phi(t)), s \in [0; 1]$$

between $f$ and the curve $h'(t) = (u(\phi(t)), \phi(t))$. In polar coordinates the area form $\omega$ is given by $\rho d\rho \wedge d\phi$ so

$$|\theta^* \omega| = |((1-s)\rho(t)+su(\phi(t)))u(\phi(t)) - \rho(t)||\dot{\phi}(t)| ds \wedge dt \leq \rho(t)\sqrt{2}|\dot{\phi}(t)| ds \wedge dt.$$
Hence the area of homotopy $\theta$ can be estimated as follows:

$$\int |\theta^* \omega| \leq \sqrt{2} \int_0^{\pi/2} \rho |\dot{\phi}| dt \leq \sqrt{2} \text{Length}(f).$$

Note that $h$ and $h'$ are homotopic with fixed end points in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by a homotopy of area 0. Combining it with $\theta$, we get the desired homotopy.

Further,

$$\text{Length}(f) \geq \sqrt{OX^2 + OY^2} \geq (OX + OY)/\sqrt{2} \geq (OX + OY - 2\epsilon + \pi\epsilon/2)/\sqrt{2} = \text{Length}(g)/\sqrt{2}.$$ 

**Case 2.** Choose $g$ so that $\text{Length}(g) \leq 2$. We work in Euclidean coordinates $(p,q)$ with the origin at $O$. After a (re)parameterization we write $f(t) = (p(t),q(t)), t \in [0;1]$. Perturb $g$ to a curve $h(t) = (u(t),t), t \in [0;1]$. Consider the homotopy

$$\theta(s,t) = ((1 - s)p(t) + su(q(t)),q(t)), s \in [0;1]$$

between $f$ and the curve $h'(t) = (u(q(t)),q(t))$. Since $\omega = dp \wedge dq$ we have

$$|\theta^* \omega| = |u(q(t)) - p(t)| |\dot{q}(t)| ds \wedge dt \leq |\dot{q}(t)| ds \wedge dt,$$

so the area of $\theta$ does not exceed

$$\int_0^1 |\dot{q}(t)| dt \leq \text{Length}(f).$$

Note that $h$ and $h'$ are homotopic with fixed end points in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by a homotopy of area 0. Combining it with $\theta$, we get the desired homotopy.

Further, $\text{Length}(f) \geq 1 \geq \text{Length}(g)/2$.

**Case 3.** Choose $g = XY$. In Euclidean coordinates $(p,q)$ with the origin at $O$ one has $g(t) = (0,t)$ and $f(t) = (p(t),q(t)), t \in [g_0;g_1]$. Here $X = (0,g_0)$ and $Y = (0,q_1)$. One computes that the area of the homotopy $(sp(t),q(t)), s \in [0;1]$ does not exceed $\text{Length}(f)$. One can combine it with the obvious homotopy with fixed end points in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ of area 0 to get the desired homotopy between $f$ and $g$. Further, $\text{Length}(f) \geq \text{Length}(g)$.

This finishes off the proof of the Lemma in the case when $\alpha \cap G_\epsilon \neq \emptyset$. Suppose now that this intersection is empty. Then $\alpha$ is entirely contained in some fundamental domain $Q$ of $\mathbb{R}^2 \setminus B(\epsilon)$. Arguing exactly as in Step 3 above we get the desired homotopy. This completes the proof. $\Box$. 

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3 Combinatorial isoperimetric inequality

Let $w \in F_3$ be a cyclically reduced word. A simple conjugate of $w$ is a word of the form $\beta \alpha$ where $\alpha$ is an initial sub-word of $w$ and $w = \alpha \beta$. Denote by $R$ the set of 10 words consisting of simple conjugates of $r^\pm 1$, that is $R = \{r_1^\pm, \ldots, r_5^\pm\}$, where $r_1 = r = aba^{-1}b^{-1}c$, $r_2 = ba^{-1}b^{-1}ca$, $r_3 = a^{-1}b^{-1}cab$, $r_4 = b^{-1}caba^{-1}$ and $r_5 = caba^{-1}b^{-1}$. A piece is an initial sub-word of some $r \in R$.

**Theorem 3.1.** Every cyclically reduced word $w \in N$ contains a piece of length 4.

Note that the small cancellation theory guarantees existence of such a piece of length 3 only.

**Proof of Theorem 3.1:** We prove $A(w) \leq l(w)/2$ by the induction in the word length $|w|$ of $w \in N$. If $|w| = 5$ then $w \in R$ (one can see this directly on the Cayley complex, see figure 3 below). Hence $A(w) = 1$ while $l(w) = 4$ so the inequality holds. Take $w \in N$ and assume that our inequality is already proved for all words of length $< |w|$. By Theorem 3.1 $w$ contains a piece, say $u$, of length 4. The piece $u$ is the initial sub-word of some unique $s \in R$. Define $u_s$ by $s = uu_s^{-1}$. Consider a new word $w' \in N$ which is obtained from $w$ by replacing $u$ with $u_s$ and cyclic reduction. Clearly, one has that $A(w) \leq A(w') + 1$, $l(w') \leq l(w) - 2$ and $|w'| \leq |w| - 3$. By the inductive assumption, $A(w') \leq l(w')/2$. Combining the inequalities we get $A(w) \leq l(w)/2$ as required. □

It remains to prove Theorem 3.1. Denote by $K$ the Cayley complex of the group $F_2 = F_3/N$ associated to the presentation $F_2 = <a, b, c|aba^{-1}b^{-1}c>$. It is easy to see that $K$ is a 2-dimensional simply connected PL-manifold with boundary (in fact $K$ is homeomorphic to the universal cover of $\mathbb{T}^2 \setminus D^2$). Every vertex, say $O$ of $K$ lies on the boundary of $K$. The union of all faces which have $O$ as a vertex is presented on figure 3.

**Proof of Theorem 3.1 :** Let $w$ be a cyclically reduced word representing an element from $N$. Consider the cycle on $K$ which corresponds to $w$ and starts at 1. This cycle has an embedded subcycle, say $\gamma$, which corresponds to some sub-word $w'$ of $w$ (see figure 4). Let $U \subset K$ be a subcomplex (which is topologically a disc) bounded by $\gamma$. Write $U = \Delta_1 \cup \ldots \cup \Delta_F$, where $\Delta_i$ are faces of $K$. 

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Lemma 3.2. Either $U$ consists of exactly 1 face, or there exist at least 2 distinct faces $\Delta_i, \Delta_j$ such that each of them intersects $\partial U$ along 4 edges.

Assume the Lemma. If $U$ is a face then $w' \in R$ so $w$ has a piece of length 4. Otherwise, let $\Delta$ be a face whose intersection with $\partial U$ contains 4 edges as provided by the Lemma. Since $\gamma$ is embedded we conclude that these 4 edges must be consecutive edges for $\gamma$. Thus $\gamma$ contains two subsegments with disjoint interiors corresponding to pieces of length 4. At most one of them contains the point $P$ (see figure 4) as an interior point. Hence the second segment supplies us with a piece of length 4 for $w'$, and hence for $w$. This completes the proof of Theorem \[3.1\] modulo Lemma 3.2. □

Proof of Lemma 3.2: Let $V, E$ and $F$ be the number of vertices, edges and faces of the subcomplex $U$. Denote by $E'$ the number of boundary
edges. Then $V = E'$ since every vertex of $U$ lies on the boundary of $K$ and hence on the boundary of $U$. Since every face has precisely 5 edges one has $5F = 2E - E'$. Substituting this into the Euler formula $V - E + F = 1$ we get that $E' = 3F + 2$. But the total number of faces equals $F$, so either there is a face with 5 boundary edges (which means $F = 1$) or there are 2 faces with at least 4 boundary edges. This completes the proof. □

3.3 Example. Consider a sequence of elements $u_k \in \mathbb{N}$, $k \in \mathbb{N}$ defined as

$$u_k = (b^{-1}c)^k ab^{k-1}.$$ 

We claim that the ratio $A(u_k)/l(u_k)$ goes to $1/2$ as $k \to \infty$, and therefore the isoperimetric constant in Theorem 1.2 cannot be improved.

First of let us check that $A(u_k) \leq k$. We use induction in $k$. The word $u_1$ is contained in $R$, so its combinatorial area equals 1. Further, $u_kr = (b^{-1}c)^k ab^{k+1}a^{-1}b^{-1}c$ and hence is conjugate to $u_{k+1}$. Therefore, $A(u_{k+1}) \leq A(u_k) + 1 \leq k + 1$, as required.

Further, consider a homomorphism $\phi : F_3 \to \mathbb{R}$ such that $\phi(a) = \phi(b) = 0$ and $\phi(c) = 1$. Since $\phi(r) = 1$ we have that $|\phi(w)| \leq A(w)$ for every $w \in \mathbb{N}$. Note that $\phi(u_k) = k$. Hence $A(u_k) \geq k$.

We conclude that $A(u_k) = k$, while $l(u_k) = 2k + 1$. The claim follows.

4 An application to dynamics

In this section we prove Theorem 1.3 by combining a method developed in [P2] with isoperimetric inequality 1.1

The famous Arnold conjecture proved in [CZ] states that $\psi$ has at least two fixed points, say $x$ and $y$, with distinct projections to $\mathbb{T}^2$. Join them by
any curve $\gamma$ and consider a loop $\alpha : S^1 \to \mathbb{R}^2$ formed by $\gamma$ and its image under $\psi$ taken with the opposite orientation. Extend $\alpha$ to any map $f : D^2 \to \mathbb{R}^2$. Its symplectic area

$$\delta(x, y, \psi) = \int_{D^2} f^* \omega$$

does not depend on the particular choice of $\gamma$ and $f$ and is called the action difference of $x$ and $y$. Moreover, this quantity behaves nicely under iterations:

$$\delta(x, y, \psi^n) = n\delta(x, y, \psi)$$  \hspace{1cm} (2)

for all $n \in \mathbb{N}$. Modern developments in symplectic topology led to the following refinement of the Conley-Zehnder result cited above: $\psi$ has a pair of fixed points with positive action difference (see [Sch]).

So take fixed points $x, y$ with $\delta(x, y, \psi) > 0$ and a map $f$ as above. There exists a vector $v \in \mathbb{Z}^2$ and a sufficiently large positive integer $m$ so that the lattice $L = x + v + m\mathbb{Z}^2$ is disjoint from the image of $f$. Then $\psi(\gamma)$ and $\gamma$ are homotopic with fixed end points in $\mathbb{R}^2 \setminus L$. Since $L$ consists of fixed points of $\psi$ we see that $\psi^k(\gamma)$ is homotopic to $\psi^{k-1}(\gamma)$ with fixed end points for all $k \in \mathbb{N}$. Thus the loop $\alpha_n$ formed by $\gamma$ and $\psi^n(\gamma)$ is contractible in $\mathbb{R}^2 \setminus L$. Therefore Theorem 1.1 yields

$$\text{Area}(\alpha_n) \leq \mu \text{Length}(\alpha_n),$$  \hspace{1cm} (3)

where the constant $\mu$ depends only on the lattice $L$ and not on $n$.

Note now that the symplectic area of any loop does not exceed its Euclidean area, so

$$\delta(x, y, \psi^n) \leq \text{Area}(\alpha_n).$$

Combining this with inequalities (2) and (3) we get that

$$\text{Length}(\psi^n(\gamma)) = \text{Length}(\alpha_n) - \text{Length}(\gamma) 
\geq \mu^{-1} \text{Area}(\alpha_n) - \text{Length}(\gamma) \geq \mu^{-1} \delta(x, y, \psi^n) - \text{Length}(\gamma) 
= n\mu^{-1} \delta(x, y, \psi) - \text{Length}(\gamma).$$

On the other hand

$$\text{Length}(\psi^n(\gamma)) \leq ||d\psi^n||\text{Length}(\gamma).$$

Therefore $||d\psi^n|| \geq \kappa n$ with

$$\kappa = (1 + \text{Length}(\gamma))^{-1} \mu^{-1} \delta(x, y, \psi).$$
This completes the proof of Theorem 1.3.

**Remark 3.1.** Exactly the same method enables us to prove that $||d\psi^n|| \geq \text{const} \, n$ for any non-Hamiltonian $\mathbb{Z}^2$-periodic area-preserving diffeomorphism $\psi$ which has a fixed point. If $x$ is such a fixed point, then one can choose $v \in \mathbb{Z}^2$ so that $\delta(x, x + v, \psi) > 0$. Existence of $v$ follows from the theory of Flux homomorphism, see [MS],[P1]. The rest of the argument goes through without changes. Combining this with Theorem 1.3 we see that $||d\psi^n|| \geq \text{const} \, n$ for every area-preserving diffeomorphism $\psi \neq \text{id}$ of $\mathbb{T}^2$ which is isotopic to identity and has a fixed point.

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