Existence, uniqueness, and stability of periodic solutions of an equation of Duffing type

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Abstract. We consider a second-order equation of Duffing type. Bounds for the derivative of the restoring force are given which ensure the existence and uniqueness of a periodic solution. Furthermore, the unique periodic solution is asymptotically stable with sharp rate of exponential decay. In particular, for a restoring term independent of the variable t, a necessary and sufficient condition is obtained which guarantees the existence and uniqueness of a periodic solution that is stable.

§1. Introduction. This paper is devoted to the existence, uniqueness and stability of periodic solutions of the Duffing-type equation

\[ x'' + cx' + g(t, x) = h(t), \]  

(1.1)

where \( g(t, x) \) is a \( T \)-periodic function in \( t \) and \( h(t) \) is a \( T \)-periodic function. The existence and multiplicity of periodic solutions of (1.1) or more general types of nonlinear second-order differential equations have been investigated extensively by many authors since C. Fabry, J. Mawhin and M.N. Nkashama initiated the study of the Ambrosetti–Prodi problem with periodic boundary condition [7]. However, the stability of periodic solutions is less extensively studied. In [18] R. Ortega studied (1.1) from a stability point of view and obtained an Ambrosetti–Prodi-type theorem under an assumption of convex nonlinearity. A.C. Lazer and P.J. McKenna established stability results by converting the equation (1.1) to a fixed-point problem [14]. Recently, more complete results concerning the stability of periodic solutions of (1.1) were obtained by J.M. Alonso and R. Ortega [1,2]. Under the condition that the derivative of the restoring force is independent of \( t \) and positive, they found sharp bounds that guarantee global asymptotic stability. In [1], optimal bounds for stability are obtained. But the above results do not cover our Theorem 1, since in the theorem the derivative of the restoring force may be negative.

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for some $t$. From their results, the key idea is to impose a condition on $g(x, t)$ that can rule out the existence of additional periodic solutions that are subharmonic of order 2. The aim of this paper is to give conditions for existence, uniqueness, and local asymptotic stability. The novelty of our result is not that the friction constant and bounds on the derivative of the restoring force control the stability of the periodic solutions, but that the friction constant alone determines the rate of decay of the other solutions of (1.1) which are near to the unique periodic solution. More precisely, we will show that every solution to (1.1) that is near the unique periodic solution decays uniformly at the same exponential rate of $c/2$. Our method is based on linearization combined with the Floquet theory. The main results are the following.

**Theorem 1.** Assume that $g(x, t) \in C^1(\mathbb{R} \times \mathbb{R})$, and that $g$ is $T$-periodic in $t$, such that

1. $\sup_{x \in \mathbb{R}} g'(t, x) = \alpha(t) < \frac{\pi^2}{T^2} + \frac{c^2}{4}$,
2. there is a $T$-periodic function $\beta(t) \in C_T$ such that $\int_0^T \beta(t) \, dt > 0$, and $g'(t, x) \geq \beta(t)$ for all $x \in \mathbb{R}$.

Then the differential equation (1.1) has a unique $T$-periodic solution, which is asymptotically stable.

Here, we say that the periodic solution $x_0$ of (1.1) is locally asymptotically stable if there exist constants $C > 0$ and $\alpha > 0$ such that if $x$ is another solution with $\|x(0) - x_0(0)\| + \|x'(0) - x'_0(0)\| = d$ sufficiently small, then $\|x(t) - x_0(t)\| + \|x'(t) - x'_0(t)\| < Cde^{-\alpha t}$. The coefficient $\alpha$ in the exponent of this upper bound is called the rate of decay of $x_0$.

**Theorem 2.** Assume that $g(t, x) \in C^1(\mathbb{R} \times \mathbb{R})$, such that for all $x \in \mathbb{R}$, the derivative of $g$ with respect to $x$ is subject to the bounds

$$\frac{n^2 \pi^2}{T^2} + \frac{c^2}{4} \leq g'(t, x) \leq \frac{(n + 1)^2 \pi^2}{T^2} + \frac{c^2}{4}$$

for some $n \geq 1$. Then equation (1.1) has a unique $T$-periodic solution, which is stable with the rate of decay $c/2$.

**Theorem 3.** Assume that $g(t, x) = g(x) \in C(\mathbb{R})$.

If

$$\frac{n^2 \pi^2}{T^2} + \frac{c^2}{4} \leq \frac{g(x) - g(y)}{x - y} \leq \frac{(n + 1)^2 \pi^2}{T^2} + \frac{c^2}{4}$$

then equation (1.1) has a unique $T$-periodic solution. Moreover, if $h \in C_T^1$ such that the set of critical points $C = \{ t \in [0, T] : h'(t) = 0 \}$ of $h$ is Lebesgue-null, then the periodic solution is stable with rate of decay of $c/2$.

**Remark.** The bounds in Theorem 2 are optimal for the rate of decay $c/2$. In the end of this section, we shall give an example to show that the Floquet multipliers associated with (1.1) may be a pair distinct real numbers, as soon as the derivative of the restoring force goes a little bit across the bounds given in Theorem 2.

The following notations will be used throughout the rest of the paper.

1. $L^p_T$: $T$-periodic function $x \in L^p[0, T]$ with $\|x\|_p$ for $1 \leq p \leq \infty$;
2. $C^k_T$: $T$-periodic function $x \in C^k[0,T]$, $k \geq 0$, with $C^k$-norm; 
3. $\alpha(t) \gg \beta(t)$: if $\alpha(t) \geq \beta(t)$ on $[0,T]$ and $\alpha(t) > \beta(t)$ on some subset of positive measure.

We will finish this section §1 by showing that the bounds given in Theorem 2 are optimal for rate of decay $c/2$.

Consider the linear equation

$$x'' + cx' + q(t)x = 0,$$

where

$$q(t) = \begin{cases} \frac{2}{\pi} + (w + \epsilon)^2 & \text{for } 0 \leq t < \pi, \\ \frac{2}{\pi} + (w - \epsilon)^2 & \text{for } \pi \leq t \leq \pi. \end{cases}$$

By the transformation $y(t) = e^{-ct/2}x(t)$, the damping term can be eliminated and equation (1.2) reduced to the more familiar form of the Hill equation,

$$y'' + q(t)y = 0.$$

Evidently, the nontrivial solution $x(t)$ of (1.2) has the rate of decay $c/2$ if and only if $y(t)$ is a bounded nontrivial solution of (1.3).

The fundamental solutions of the equation $y'' + w^2y = 0$ are $x_1 = \cos wt$, $x_2 = \sin wt/w$. Let $T = 2\pi$, the monodromy matrix associated with (1.3) is $A = A_2 \cdot A_1$, where $w_1 = (w + \epsilon)^2$ and $w_2 = (w - \epsilon)^2$, and $A_i$ is defined by

$$A_i = \begin{pmatrix} \cos \pi w_i & \sin \pi w_i / w_i \\ -w_i \sin \pi w_i & \cos \pi w_i \end{pmatrix}$$

and the discriminant function is given by

$$\text{tr } A = 2 \cos \pi w_1 \cdot \cos \pi w_2 - \left( \frac{w_1}{w_2} + \frac{w_2}{w_1} \right) \sin \pi w_1 \sin \pi w_2.$$ 

The boundary of the zone of stability of (1.3) is determined by $|\text{tr } A(\epsilon)| = 2$. According to [4] on page 120 (or by a straightforward computation) it can be expressed asymptotically by

$$w = k \pm \frac{\epsilon^2}{k^2} + o(\epsilon^2) \quad \text{or} \quad w = k + \frac{1}{2} \pm \frac{\epsilon}{\pi(k + \frac{1}{2})} + o(\epsilon).$$

If we choose $(w, \epsilon)$ in the parametric resonance region, see Figure (101) in [4], such that $(w, \epsilon)$ is near $(n/2, 0)$, where $n$ is an integer, then one of the Floquet multipliers of (1.3) is greater than one and another is less than one. In this case, the trivial solution of (1.2) is still asymptotically stable, but the rate of decay is different from $c/2$. Thus if the derivative of restoring force crosses a little bit over the bounds given in Theorem 2, the conclusion of Theorem 2 no longer holds. Hence the bounds are optimal for rate of decay $c/2$.

This paper is organized as follows. In §2 we recall some basic results about topological methods and prove a few lemmas that are crucial for the proofs of the main results. The proofs of Theorems 1–3 are given in §3.
2. Linear periodic problems. In this section we shall recall some basic results about topological methods. Consider the periodic boundary value problem

\[ x' = f(t, x), \quad (2.1) \]
\[ x(0) = x(T), \quad (2.2) \]

where \( f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function and \( T \)-periodic in \( t \). In order to use a homotopic method to compute the degree, we assume that \( h : [0, T] \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n \) is a continuous function such that

\[ h(t, x, 1) = f(t, x), \]
\[ h(t, x, 0) = g(x), \]

where \( g(x) \) is continuous. The following continuation theorem is due to J. Mawhin [17].

**Lemma 2.1.** Let \( \Omega \subset C_T \) be an open bounded set such that the following conditions are satisfied.

1. There is no \( x \in \partial \Omega \) such that \( x' = h(t, x, \lambda) \quad \forall \lambda \in [0, 1) \).
2. \( \deg(g, \Omega \cap \mathbb{R}^n, 0) \neq 0 \).

Then \((2.1) - (2.2)\) has at least one solution.

Let us consider the Liénard equation

\[ x'' + f(x)x' + g(t, x) = h(t), \quad (2.3) \]

where \( h(t) \in C_T \). Evidently the periodic solution of \((2.3)\) is equivalent to the planar system

\[ x' = y - F(x), \]
\[ y' = h(t) - g(t, x), \quad (2.4) \]

where \( F(x) \) is a primitive of \( f(x) \). A natural choice for the homotopy in applying Lemma 2.1 is to take

\[ h(t, x, y, \lambda) = \{ y - F(x), \lambda h(t) + (1 - \lambda)\bar{h} - [\lambda g(t, x) + (1 - \lambda)\bar{g}(x)] \}, \]

where \( \bar{g}(x) = \frac{1}{T} \int_0^T g(t, x) \, dt \) is the average of \( g(t, x) \). Since \( h(t, x, y, 0) = (y - F(x), \bar{h} - \bar{g}(x)) = G(x) \), then the condition (2) in Lemma 2.1 reduces to

\[ \deg(G, \Omega \cap \mathbb{R}^2, 0) \neq 0. \quad (2.5) \]

Next we consider the system \((2.1)\) for \( n = 2 \). We denote by \( x(t, x_0) \) the initial-value solution of \((2.1)\) and introduce the Poincaré map \( P : x_0 \rightarrow x(T, x_0) \). It is well known that \( x(t, x_0) \) is a \( T \)-periodic solution of system \((2.1)\) if and only if \( x_0 \) is a fixed point of \( P \). If \( x \) is an isolated \( T \)-periodic solution of \((2.1)\), then \( x_0 \) is an isolated fixed point of \( P \).
Definition. A $T$-periodic solution $x$ of (2.1) will be called a nondegenerate $T$-periodic solution if the linearized equation

$$y' = f_x(t, x)y$$

(2.6)
does not admit a nontrivial $T$-periodic solution.

Let $M(t)$ be the fundamental matrix of (2.6) and $\mu_1$ and $\mu_2$ be the eigenvalues of the matrix $M(T)$. Then $x(t, x_0)$ is asymptotically stable if and only if $|\mu_i| < 1$ ($i = 1, 2$). Otherwise, if there is an eigenvalue of $M(T)$ with modulus greater than one, then $x(t, x_0)$ is unstable.

Consider the homogeneous periodic equation

$$L_\alpha x = x'' + cx' + \alpha(t)x = 0,$$

(2.7)

where $c$ is constant and $\alpha(t) \in L_T$. From now on, without further mention, we always suppose that the frictional constant $c > 0$.

The following lemma is due to R. Ortega [18].

**Lemma 2.2.** Assume that

$$\alpha(t) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}.$$  

(2.8)

Then equation (2.7) does not admit negative Floquet multipliers.

The next lemmas are crucial to the argument for stability.

**Lemma 2.3.** Assume that $\alpha(t) \in L_T$ satisfying (2.8) such that $\alpha(t) > 0$.

Then the moduli of the Floquet multipliers associated with equation (2.7) are less than one. In other words, the trivial $T$-periodic solution is asymptotically stable.

**Proof.** We consider the following two cases.

Case 1. If the multipliers are a pair of conjugate numbers, the conclusion of Lemma 2.3 follows immediately from the Jacobi–Liouville formula.

Case 2. If the Floquet multipliers are real numbers, then Lemma 2.2 rules out negative multipliers, so it is sufficient to show that the moduli of the positive multipliers are less than one. If $x(t)$ vanishes at some $t_0$, then it must vanish at $T + t_0$. Thus $x(t)$ solves the boundary condition problem of the following equation:

$$(e^{ct}x')' + (\alpha(t)e^{ct})x = 0, \quad x(t_0) = x(T + t_0) = 0.$$  

(2.9)

Next, we consider the B.V.P.

$$(e^{ct}x')' + \lambda e^{ct}x = 0, \quad x(t_0) = x(t_0 + T) = 0.$$  

(2.10)

It is easy to verify that the $n$-th eigenvalue of (2.10) is

$$\lambda_n = \frac{\pi^2n^2}{T^2} + \frac{c^2}{4}$$

and

$$y(t) = e^{-\frac{c}{2}(t-t_0)}\sin\frac{\pi(t-t_0)}{T}$$
is the first eigenfunction corresponding to the first eigenvalue

\[ \lambda_1 = \frac{\pi^2}{T^2} + \frac{c^2}{4} \]

We compare \( y(t) \) with the solution \( x(t) \) of the B.V.P. (2.9).

Since \( \alpha(t) \ll \lambda_1 \), it follows from the Sturm comparison theorem that \( y(t) \) has a zero in \((t_0, T + t_0)\), a contradiction. Thus \( x(t) \) does not change sign. We may assume that \( x(t) > 0 \), since \( x'(T) = \rho x'(0) \) and \( \frac{x'(T)}{x(T)} = \frac{x'(0)}{x(0)} \). Dividing (2.7) by \( x(t) \) and integrating by parts gives that

\[
\int_0^T \frac{x'(t)^2}{x(t)^2} \, dt + c \ln \rho + \int_0^T \alpha(t) \, dt = 0.
\]

The hypothesis of the lemma implies that \( \ln \rho < 0 \). Hence \( 0 < \rho < 1 \).

**Lemma 2.4.** Assume that there is an integer \( n \geq 1 \) such that

\[
\frac{n^2 \pi^2}{T^2} + \frac{c^2}{4} \ll \alpha(t) \ll \frac{(n + 1)^2 \pi^2}{T^2} + \frac{c^2}{4}.
\]

Then (2.7) does not admit real Floquet multipliers.

**Proof.** If the conclusion does not hold, then there is a real Floquet multiplier \( \rho \) and a nontrivial solution \( x(t) \) such that \( x(t + T) = \rho x(t) \). Since \( \alpha(t) \gg \lambda_1 \), it follows from Sturm’s separation theorem that \( x(t) \) has a zero \( t_0 \in [0, T] \). Thus \( \rho x(t_0) = x(T + t_0) = 0 \), i.e., \( x(t) \) is a solution of the B.V.P. (2.9). The assumption of the lemma implies that \( \lambda_n \ll \alpha(t) \ll \lambda_{n+1} \). Therefore \( x(t) \equiv 0 \), a contradiction.

**Lemma 2.5.** Under the condition of Lemma 2.4, the rate of decay of any nontrivial solution of (2.9) is \( c/2 \).

**Proof.** Consider the corresponding system

\[ X'(t) = A(t)X(t) \tag{2.11} \]

where the column vector function \( X(t) = (x(t), x'(t))^T \) and \( A(t) \) is the matrix function

\[
A(t) = \begin{pmatrix} 0 & 1 \\ -p(t) & -c \end{pmatrix}.
\]

Let \( M(t) \) be a fundamental matrix solution of (2.11). It is well known that \( M(t) \) has the form

\[ M(t) = P(t)e^{Bt} \tag{2.12} \]

where \( P(t) \) and \( B \) are \( 2 \times 2 \) matrices, \( P(t) = P(t + T) \) and \( B \) is constant. Let \( \rho_1 = e^{T \lambda_1} \) and \( \rho_2 = e^{T \lambda_2} \) be the Floquet multipliers, so that \( \lambda_1 \) and \( \lambda_2 \) are the Floquet exponents associated with \( \rho_1 \) and \( \rho_2 \). Let \( x_1 \) and \( x_2 \) be the eigenvectors of the matrix \( e^{TB} \). It follows from Lemma 2.2 that \( \rho_1 \) and \( \rho_2 \) are a complex conjugate pair. Thus the eigenvectors that are associated with different eigenvalues are linearly independent. Therefore \( y_i = \rho_i(t)e^{\lambda_i t} \) \((i = 1, 2)\) form the fundamental solutions of equation (2.11). On the other hand, by applying the Jacobi–Liouville formula we have

\[
|\rho_1|^2 = \rho_1 \rho_2 = e^{-\int_0^T c \, dt} = e^{-cT}
\]
and
\[
\text{Re} \lambda_1 = \text{Re} \lambda_2 = \frac{1}{2} \text{Re}(\lambda_1 + \lambda_2) = \frac{1}{2} \ln(\rho_1 \rho_2) = \frac{c}{2}.
\]
Since every solution is a linear combination of \(y_1(t)\) and \(y_2(t)\), \(p_i(t)\) is \(T\)-periodic, hence it is bounded. Therefore every nonzero solution of the equation (2.11) decays at the same exponential rate of \(c/2\).

§3. Proof of main results. Now we prove our main results.

Proof of Theorem 2. We divide the proof into three steps.

Step 1. Existence. Without loss of generality, we may assume that \(g(0, t) = 0\), for otherwise we can subtract \(g(0, t)\) from both sides of equation (1.1). Consider the parametrized equation
\[
x'' + cx' + \tau g(t, x) + (1 - \tau)a x = \tau h(t) \tag{3.1}
\]
for some \(a \in (\lambda_n, \lambda_{n+1})\). We claim that there is an \(R > 0\) such that equation (3.1) has no solution on \(\partial B_R\) for all \(\tau \in [0, 1]\). If there is not such an \(R\), let \(x_n\) be a sequence such that \(\|x_n\| \to \infty\) and \(\tau_n \in [0, 1]\), and denote by \(z_n\) the ratio \(x_n/\|x_n\|\).

Dividing (3.1) by \(\|x_n\|\), then multiplying by \(\varphi(t) \in C^2_T\) and integrating by parts, we have that
\[
\int_0^T \frac{z_n \varphi'' - c \varphi' + [\tau_n g(t, x_n) + (1 - \tau_n)a] \varphi}{\|x_n\|} dt = \tau_n \int_0^T \frac{\varphi h_n}{\|x_n\|} dt. \tag{3.2}
\]
The condition of Theorem 2 implies that \([\tau_n g(t, x_n) + (1 - \tau_n)a] x_n/\|x_n\|\) is bounded. It is pre-compact in the weak* topology in \(L^1([0, T])\). Thus there is a subsequence such that \(g(t, x_n)/x_n \to \beta(t)\) and \(\tau_n \to \tau\). Taking the limit in the equation (3.2), one obtains that
\[
\int_0^T \{z \varphi'' - cz \varphi' + w(t, \varphi)\} dt = 0 \tag{3.3}
\]
where \(w(t) = \tau \beta(t) + (1 - \tau)a\). Since \(w(t)\) satisfies the condition of Lemma 2.4 and \(z(t)\) is a \(T\)-periodic solution, it follows from Lemma 2.4 that \(z(t) \equiv 0\), which contradicts \(\|z(t)\| = 1\). Next, by applying the homotopic invariance property, we have that
\[
\deg(\phi_0, B_R, 0) = \deg(\phi_1, B_R, 0) = 1.
\]
This completes the proof of existence.

Step 2. Uniqueness. Let \(x_1\) and \(x_2\) be two distinct \(T\)-periodic solutions of the equation (1.1), and \(x = x_1 - x_2\). Then \(x\) satisfies the equation
\[
x'' + cx' + p(t)x = 0, \tag{3.4}
\]
where \(p(t) = [g(t, x_1(t)) - g(t, x_1(t))]/[x_1 - x_2]\). Since \(x(t)\) is a nontrivial \(T\)-periodic solution of (3.4), it will imply that \(x\) is an eigenfunction associated with a Floquet multiplier equal to one. Again, Lemma 2.4 rules out this possibility. Therefore \(x \equiv 0\), a contradiction.
Step 3. Stability. Let \( x(t) \) be the unique \( T \)-periodic solution obtained by Step 2. Consider the linearized equation

\[
v'' + cv' + g'(t, x(t))v = 0.
\]

The condition of Theorem 2 implies that \( p(t) = g'(t, x(t)) \), verifying the hypotheses of Lemma 2.3, which gives the stability results about the linearized equation.

To show that every solution of the nonlinear equation (1.1) locally decays to the unique \( T \)-periodic solution with rate of decay \( c/2 \), we need the following \( C^1 \) version of the Hartman-Grobman theorem [12].

**Lemma 3.1.** Let \( U \) be an open neighborhood of 0, and \( f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a \( C^1 \) function such that \( f'_x(0): \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a contraction mapping. Then \( f \) \( C^1 \) is conjugate equivalent to \( f'_x(0) \).

**Remark.** The proof of the theorem depends on a \( C^1 \) version of the Hartman-Grobman theorem. More detailed results concerning smooth equivalence can be found in [6], [19], [21], [22]. In general, a \( C^1 \) hyperbolic map is only topologically conjugate equivalent to the linear part in a small neighborhood of the fixed point, thus Lemma 3.1 no longer holds for \( C^1 \) maps near the hyperbolic fixed point without assuming that \( f \) is contracting.

Now we will complete the proof of Theorem 2.

Consider the planar system associated with equation (1.1),

\[
\begin{align*}
    x' &= y - cx, \\
    y' &= h(t) - g(t, x).
\end{align*}
\]

(3.5)

Let \( X_0(t) = (x_0(t), y_0(t)) \) be the unique \( T \)-periodic solution determined by the initial condition \( X_0(0) = (x_0, y_0) \). Then \( X_0 \) corresponds to the unique fixed point of the Poincaré mapping \( PX = U(T, X) \), where \( U(t, X) \) is the initial-value solution of (3.5) with \( U(0, X) = X \). Let \( M(t) \) be the fundamental matrix solution of the linearization

\[
X' = A(t)X
\]

(3.6)

of (3.5), where

\[
A(t) = \begin{pmatrix} -c & 1 \\ -p(t) & 0 \end{pmatrix}.
\]

By the differentiability of \( X(t) \) with respect to the initial value, the Poincaré mapping can be expressed in terms of the initial value \( X \) by the following formula:

\[
PX - X_0 = M(T)(X - X_0) + o(X - X_0).
\]

(3.7)

Referring to Lemma 2.4, \( M(T) \) has a pair of conjugate eigenvalues \( \lambda, \bar{\lambda} \) with \( |\lambda| = e^{-cT/2} \). Thus \( P(X) \) is a contracting mapping. According to Lemma 3.1, there is a \( C^1 \) diffeomorphism \( \varphi \) which is near enough to the identity that \( PX - X_0 \) is conjugate equivalent to \( M(T) \). There is an invertible constant matrix \( C \) such that

\[
C^{-1}M(T)C = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = D(\lambda),
\]

where \( D(\lambda) \) is a diagonal matrix.
and we may suppose that
\[ \frac{1}{2} |X - X_0| < |\varphi(X) - \varphi(X_0)| < 2 |X - X_0| \] (3.8)
for \( X - X_0 \) small, since \( \varphi \) is near the identity. Therefore, the Liapunov exponent is given by
\[
\mu_x = \lim_{n \to \infty} \frac{1}{nT} \ln |P^n X - X_0|
= \lim_{n \to \infty} \frac{1}{nT} \ln |\varphi \circ M(T)^n \circ \varphi^{-1}(X) - \varphi \circ M(T)^n \circ \varphi^{-1}(X_0)|
= \lim_{n \to \infty} \frac{1}{nT} \ln |D(\lambda)^n \circ C^{-1}[\varphi^{-1}(X) - \varphi^{-1}(X_0)]| = -\frac{c}{2}.
\]
The third equality follows from (3.8) and the fact that \( C \) is invertible.

Remark. The above result shows that the Liapunov exponent is invariant under a \( C^1 \) conjugate transformation. From the proof, the conclusion is still true if \( \varphi \) is a Lipschitz mapping.

Hence, the rate of decay of the solution to the unique \( T \)-periodic solution is \( c/2 \), independently of the initial value \( X_0 \). Theorem 1 can be proved by exactly the same method.

Proof of Theorem 3. In order to prove Theorem 3, let us consider the following Liénard equation:
\[ x'' + f(x)x' + g(x) = h(t). \] (3.9)
The following lemma seems to be well known; however, we have been unable to find a published reference. Some similar results concerning sufficient conditions for existence were given in [9], [10]. We give a proof here, for completeness.

Lemma 3.2. Let \( f, g \in C(\mathbb{R}) \) such that \( |f(x)| > k \) for some \( k > 0 \) and \( g(x) \) is increasing. Then (3.9) has a \( T \)-periodic solution if and only if \( \overline{h} \in g(\mathbb{R}) \).

Proof. The trivial necessary condition for existence can be obtained by integrating (1.1) and applying the mean value theorem, and the condition turns out to be sufficient. In fact, we consider the parametrized equation
\[ x'' + f(x)x' + g(x) = \lambda h(t) + (1 - \lambda)\overline{h} = h_\lambda(t), \] (3.10)
where \( \lambda \in [0, T] \).

Multiplying (3.10) by \( x' \) and integrating, we have that
\[
k \int_0^T (x')^2 \, dt \leq \left| \int_0^T x' h_\lambda(t) \, dt \right|
\]
and the Hölder inequality gives that
\[
\|x'\|_2 \leq \frac{1}{k} \|h\|_2. \tag{3.11}
\]
On the other hand, integrating (3.10) shows that there is a \( \tau \in [0, T] \) such that \( g(x(\tau)) = \overline{h} \). Since \( g(x) \) is increasing, hence \( x(\tau) = g^{-1}(\overline{h}) = c \) is unique, independently of \( \lambda \), and then a \( C^0 \) bound can be obtained from the following formula:
\[
|x(t)| = |x(\tau) + \int_\tau^t x'(s) \, ds| \leq |c| + \frac{1}{k} \|h\|_2. \tag{3.12}
\]
It is easy now to obtain a \( C^2 \) bound for equation (3.10).
Let $r_1$ and $r_2$ be sufficiently large, and set

$$\Omega = \{ (x,y) \mid |x| \leq r_1, |y| \leq r_2 \}.$$ 

It follows from the estimates obtained above that the equivalent planar system defined in Section 2,

$$x' = h(t, x, y, \lambda),$$

has no solutions on $\partial \Omega$ for $\lambda \in [0,1]$, and the computation of the degree for $r_1$ and $r_2$ large enough is given by

$$\text{deg}(G, \Omega, 0) = \text{deg}(g(x) - \overline{h}, (-r_1, r_2), 0) = 1.$$ (3.13)

By applying Lemma 2.1, we have that the equation (3.9) has at least one $T$-periodic solution.

The necessary and sufficient conditions for existence of a periodic solution can be obtained by Lemma 3.2. In order to obtain the stability result of Theorem 3 the following lemma concerning the regularity of solutions of the initial-value problem of (1.1) is needed. Roughly speaking, under Lipschitz nonlinearity, the solution of the initial-value problem of (1.1) is still smooth provided that the force term $h$ does not oscillate too violently.

**Lemma 3.3.** Let $u(t, \xi, \eta)$ be the solution of the initial-value problem

$$\begin{cases} 
  u'' + cu' + g(u) = h(t), \\
  u(0) = \xi, \quad u'(0) = \eta. 
\end{cases}$$ (3.14)

Assume that $g$ is a Lipschitz function, and that $h \in C^1_T$ such that the set of critical points $C = \{ t \in [0,T] : h'(t) = 0 \}$ of $h$ is Lebesgue-null. Then, for $t \in [0,t^*]$, the partial derivatives of $u$ and $u'$ with respect to $\xi, \eta$ exist and are continuous. Moreover, if

$$X(t) = \left( \begin{array}{cc} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\
 \frac{\partial u'}{\partial \xi} & \frac{\partial u'}{\partial \eta} \end{array} \right),$$

then

$$X'(t) = A(t)X(t), \quad X(0) = \text{Id},$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\
 -g'(u(t)) & -c \end{pmatrix}.$$ 

**Proof.** The lemma says under Lipschitz nonlinearity the solution of (3.14) is still smooth provided that the force term $h(t)$ does not oscillate violently.

We were inspired by Lazer and McKenna's result concerning the regularity of solutions of (1.1) for the case $g(u) = au^+ - bu^-$ in [15], for the general case of $g(u)$ a Lipschitz function; the proof is convoluted and requires delicate analysis. We shall divide our proof into three steps, since it involves a great deal of real analysis.

Step 1.
Claim. Let \( u(t) \) be a solution of (3.14). Let \( B \subset u([0, \bar{t}]) \) be null. Then under the assumptions of Lemma 3.3, the pre-image \( A = u^{-1}(B) \) is null in \([0, \bar{t}]\).

The following result concerning measure theory is needed.

1. Let \( f \) be differentiable on \([a, b]\), and let \( A \) be a measurable subset of \([a, b]\). If \( m(f(A)) = 0 \), then \( f'(x) = 0 \) a.e. \( x \in A \).

Setting
\[
E_n = \left\{ x \in A : \frac{|f(y) - f(x)|}{|y - x|} > \frac{1}{n}, \text{whenever } |y - x| < \frac{1}{n} \right\},
\]
it is easy to check that
\[
E = \{ x \in A : |f'(x)| > 0 \} = \bigcup_{n=1}^{\infty} E_n.
\]
In order to show that \( E \) is null, it is sufficient to show that \( E_n \) is null for any \( n \).

Now, for \( n \) fixed, by the additive property of the measure, it is enough to show that for any small interval \( I \) with length \(|I| < \frac{1}{n}\), the measure \( m(I \cap E_n) = 0 \). Setting \( B = I \cap E_n \), since \( f(B) \subset f(A) \), by assumption \( m(f(B)) = 0 \). By the definition of a null set, there is a sequence of intervals such that
\[
\bigcup_{k=1}^{\infty} I_k \supset f(B), \quad \sum_{k=1}^{\infty} I_k < \epsilon.
\]
Setting \( B_k = f^{-1}(I_k) \cap B \), then \( B_k \subset E_n \cap I \) and \( \bigcup_{k=1}^{\infty} B_k = B \). Noting that \( f(B_k) \subset I_k \), we have that
\[
m^*(B) \leq \sum_{k=1}^{\infty} m^*(B_k) \leq \sum_{k=1}^{\infty} \text{diam}(B_k)
\]
\[
\leq \sum_{k=1}^{\infty} n \text{diam}(f(B_k)) \leq n \sum_{k=1}^{\infty} m(I_k) \leq n\epsilon.
\]
Since \( \epsilon \) is arbitrary, it follows that \( m(I \cap E_n) = 0 \). Hence, \( m(E) = 0 \): namely, for almost all \( x \in A \), we have \( f'(x) = 0 \). The proposition can be obtained also by directly applying the one-dimensional area formula. According to Theorem 3.2.5 in [8] on page 244 or in [16] on page 106, for a given mapping, the area of its image and its derivative are related by the following formula:

2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a Lipschitz mapping, and for any measurable set \( A \), let \( N(f \mid A, y) \) be the cardinal number of \( f^{-1}(y) \cap A \). Then \( N(f \mid A, y) \in L^1(\mathbb{R}) \), and
\[
\int_A |f'(x)| \, dx = \int_{\mathbb{R}} N(f \mid A, y) \, dy.
\]
(3.15)

The cardinal number \( N(f \mid A, y) \neq 0 \) if and only if \( y \in f(A) \), so (3.15) combined with this assumption gives that
\[
\int_A |f'(x)| \, dx = \int_{f(A)} N(f \mid A, y) \, dy = 0.
\]
Thus, for almost all $x \in A$, we have $f'(x) = 0$.

We are now in a position to finish proving the claim.

Suppose on the contrary that $m(u^{-1}(B)) > 0$. It follows from above argument that $u'(t) = 0$ a.e. $t \in A = u^{-1}(B)$. We may assume that $A$ consists of accumulation points only, since the isolated points of $A$ are countable, hence it is null. By Rolle’s theorem, any accumulation point of zeros of $u'(t)$ must be a zero of $u''(t)$. Thus, $u''(t) = 0$ a.e. $t \in A$, so the Duffing’s equation (3.14) reduces to

$$g(u(t)) = h(t), \quad u'(t) = 0 \quad \text{a.e.} \ t \in A. \quad (3.16)$$

The above equation gives that $h'(t) = 0$ a.e. $t \in A$, because

$$\frac{|g[u(t + s)] - g[u(t)]|}{|s|} \leq L\frac{|u(t + s) - u(t)|}{|s|},$$

where $L$ is the Lip constant of $g$. This contradicts our assumption that the set of critical points of $h$ is null because of $m(A) > 0$, which completes our claim.

Step 2. The existence of partial derivatives.

Let $L$ be the Lipschitz constant of $g$ and let $\{\xi_n\}_1^\infty$ be a sequence of numbers such that $\xi_n \to \xi_0$ as $n \to \infty$. Let

$$u_n = u(t, \xi_n, \eta), \quad u_0 = u(t, \xi_0, \eta)$$

and let

$$\Psi_n = (u_n - u_0)/(\xi_n - \xi_0).$$

Since $u''_n + cu'_n + g(u_n) = h(t)$ for $n \geq 0$, we have that

$$|\Psi_n(t)'| \leq c|\Psi_n(t)| + L|\Psi_n(t)|. \quad (3.17)$$

Moreover, we have that $\Psi_n(0) = 1$, $\Psi_n'(0) = 0$, so for $n \geq 1$ and $t \geq 0$,

$$\Psi_n(t) = 1 + \int_0^t \Psi_n(s) ds, \quad \Psi_n'(t) = 1 + \int_0^t \Psi_n'(s) ds. \quad (3.18)$$

Let $M = 1 + c + L$; then (3.17) gives that

$$|\Psi_n(t)| + |\Psi_n'(t)| \leq 1 + M \int_0^t |\Psi_n(s)| + |\Psi_n'(s)| ds.$$

It follows from Gronwall’s inequality that

$$|\Psi_n(t)| + |\Psi_n'(t)| \leq \exp(Mt). \quad (3.19)$$

From (3.17) and (3.19) it follows that the sequences $\{\Psi_n(t)\}$ and $\{\Psi_n'(t)\}$ are equicontinuous and uniformly bounded on $[0, \ell]$, so there exists a subsequence [still denoted by $\{\Psi_n(t)\}$] and a $C^1$ $z(t)$ such that $\Psi_n(t) \to z$ and $\Psi_n'(t) \to z'$ as $n \to \infty$ uniformly on $[0, \ell]$. Moreover, $\{\Psi_n(t)\}$ satisfies the following equation:

$$\Psi_n(t)'' + c\Psi_n(t)' + \frac{[g(u_n(t)) - g(u_0(t))]}{\xi_n - \xi_0} = 0. \quad (3.20)$$
Since $g$ is a Lipschitz function, it follows that $g'(u)$ exists for almost every $u$. Let $E$ be a set such that the derivative of $g$ does not exist; then $m(E) = 0$. Since $\Psi_n(t) \to z$, thus, the limits of the last term in (3.20) exist for $u_0(t) \in u_0([0, t]) \setminus E$. Let $A = u_0^{-1}(E)$ be a pre-image of $E$. It follows from step 1 that $A$ is null, namely, for almost all $t \in [0, t]$ the limit

$$
\lim_{n \to \infty} \frac{g(u_n(t)) - g(u_0(t))}{\xi_n - \xi_0} = g'(u_0(t))z(t),
$$

and the Lipschitz condition gives that the above sequence is bounded. Taking the limit in (3.18), it follows from the Lebesgue bounded dominated convergence theorem that

$$
z(t) = 1 + \int_0^t z(s) \, ds, \quad z'(t) = -\int_0^t c z'(s) + g'(u_0(s))z(s) \, ds.
$$

Thus, $z'(t)$ is precisely the function satisfying

$$
z'' + cz' + g'(u_0(t))z = 0, \quad z(0) = 1, \quad z'(0) = 0.
$$

Since this determines $z$ uniquely, the original sequences $\{\Psi_n(t)\}$ and $\{\Psi'_n(t)\}$ must converge to $z$ and $z'$ respectively, which gives the existence of $\partial u/\partial \xi$ and $\partial u'/\partial \xi$. Similarly, we can show that $\partial u/\partial \eta$ and $\partial u'/\partial \eta$ exist.

Step 3. Continuity of partial derivatives.

In order to show the continuity of the partial derivatives with respect to the initial value, it is sufficient to show that if $\{\xi_n\}_1^\infty$, $\{\eta_n\}_1^\infty$ are sequences such that $\xi_n \to \xi_0$, $\eta_n \to \eta_0$, if $u_n = u(t, \xi_n, \eta_n)$, $u_0 = u(t, \xi_0, \eta_0)$ and if

$$
y''_n + cy'_n + g'(u_0(t))y_n = 0, \quad y_n = 1, \quad y_n'(0) = 0,
$$

then $y_n \to y_0$, $y_n' \to y_0'$ uniformly on $[0, t]$.

First, let us show that $g'(u_n(t)) \to g'(u_0(t))$ in $L^1$.

Since $g'(u) = \lim_{n \to \infty} n[g(u + \frac{1}{n}) - (g(u))]$ is the limit of a continuous function, it follows that $g'(u)$ is a measurable function on $X = u_0([0, t])$. By Lusin’s theorem, [8] on page 76, for every $\delta > 0$ there is a bounded closed set $E_1$ of $X = u_0([0, t])$ with $m(X \setminus E_1) < \delta$ such that $g'(u)$ is continuous on $E_1$. Again, by applying Lusin’s theorem to the subset $(X \setminus E_1)$, there is a closed set $E_2$ of $(X \setminus E_1)$ with $m(X \setminus (E_1 \cup E_2)) < \delta/2$ such that $g'(u)$ is continuous on $E_2$. Evidently, $E_1 \cap E_2 = \emptyset$.

Thus, the distance $d(E_1, E_2) > 0$, which implies that the two sets do not have a limit point in common, so $g'(u)$ is continuous on $E_1 \cup E_2$. By repeating this process, we obtain a sequence of closed subsets $F_k$ of $X$ such that $F_1 = E_1$, $F_2 = E_1 \cup E_2$, $\ldots$, $F_k = \bigcup_{j=1}^k E_j$, with the following properties: 1) $g'(u)$ is continuous on $F_k$; 2) $F_k \subset F_{k+1}$, $m(X \setminus F_k) \to 0$, $k \to \infty$. Next, set

$$
G_k = \{ t \in [0, t] : u_0(t, \xi_0, \eta_0) \in X \setminus F_n \}.
$$

We claim that $m(G_k) \to 0$ as $n \to \infty$. If not, since $G_k \supset G_{k+1}$, there is an $\varepsilon_0 > 0$ such that $m(G_n) > \varepsilon_0$. By the monotone property of the measure, we have that

$$
m \left( \bigcap_{k=1}^\infty G_k \right) = \lim_{k \to \infty} m(G_k) > \varepsilon_0.
$$
But the image of $A = \bigcap_{k=1}^{k=\infty} G_k$ under $u_0$ is contained in $\bigcap_{k=1}^{k=\infty} (X \setminus F_k)$, which is a null subset of $X$. According to step 1, $m(A) = 0$, which leads to a contradiction.

It follows from the theorem concerning continuous dependence on the initial value that $u_n \to u_0$ uniformly on $[0, \bar{t}]$. Now since $m(G_k) \to 0$ as $n \to \infty$, we may choose $k$ large enough that $m(G_k) < \epsilon$. Now we fix $k$, and since $g'$ depends continuously on $F_k$, it follows that $g'(u_n(t)) \to g'(u_0(t))$ uniformly on $G_k^c = [0, \bar{t}] \setminus G_k$, which in turn implies that

$$
\lim_{n \to \infty} \int_0^{\bar{t}} |g'(u_n(t)) - g'(u_0(t))| \, dt = \lim_{n \to \infty} \int_{G_k} |g'(u_n(t)) - g'(u_0(t))| \, dt + \int_{G_k^c} |g'(u_n(t)) - g'(u_0(t))| \, dt \\
\leq 0 + 2L\epsilon.
$$

Therefore, $g'(u_n(t)) \to g'(u_0(t))$ in the sense of $L^1$.

The rest of the proof is similar to that of Lemma 2.2 in [15]. Setting $v_n(t) = y_n(t) - x_n(t)$, then

$$
\begin{align*}
\left\{ \begin{array}{l}
v''_n + cv'_n + g'(u_n(t))v_n = [g'(u_0(t)) - g'(u_n(t))]v'_n, \\
v_n(0) = v'_n(0).
\end{array} \right.
\end{align*}
$$

Denote the $L^1$ norm of $g'(u_n(t)) - g'(u_0(t))$ by $\varepsilon_n$ and the $C^0$ norm of $v_0$ by $k$. Then from (3.21), we have that

$$
|v_n(t)| + |v'_n(t)| \leq \varepsilon_n + M \int_0^{\bar{t}} \|v_n(t)| + |v'_n(t)| \| \, dt,
$$

where $M = 1 + c + L$. The Gronwall inequality implies that

$$
|v_n(t)| + |v'_n(t)| \leq \varepsilon_n k e^{MT}.
$$

This completes the proof that $y_n \to y_0$, $y'_n \to y'_0$ uniformly on $[0, \bar{t}]$.

On the same lines as the proof of Theorem 2, the stability result of Theorem 3 follows from Lemma 3.3.

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