Magnetic flux expulsion in a superconducting wire

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An electric current generates a magnetic field, and magnetic fields cannot exist in the interior of type I superconductors. As a consequence of these two facts, electric currents can only flow near the surface of a type I superconducting wire so that the self-field vanishes in the interior. Here we examine how an electric current flowing through the entire cross-section of a normal conducting wire becomes a surface current when it enters a superconducting portion of the wire. This geometry provides insight into the dynamics of magnetic flux expulsion that is not apparent in the Meissner effect involving expulsion of an externally applied magnetic field. It provides clear evidence that the motion of magnetic field lines in superconductors is intimately tied to the motion of charge carriers, as occurs in classical plasmas (Alfvén’s theorem) and as proposed in the theory of hole superconductivity [1], in contradiction with the conventional London-BCS theory of superconductivity.

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I. INTRODUCTION

Within the conventional theory of superconductivity [2], an externally applied magnetic field is expelled from the interior of a metal becoming superconducting (Meissner effect) without any associated radial motion of charge carriers. This is at the very least surprising. Good electrical conductors resist changes of magnetic flux through their interior, through the generation of eddy currents that generate magnetic fields opposing flux changes (Lenz’s law). Perfect conductors should make it impossible for magnetic field lines to cut through them. So how do superconductors expel magnetic fields?

The conventional theory of superconductivity does not provide an answer to this question. Classical plasmas do. In a perfectly conducting plasma, magnetic field lines can only move if electric charges move together with the field lines [3–5]. Magnetic field lines are frozen into the plasma. That is known as ‘Alfvén’s theorem’ [6]. It is natural to assume that the same physics is at play in superconductors.

In recent work we have proposed that Alfvén’s theorem explains the Meissner effect [7], namely that the expulsion of magnetic field from the interior of a metal becoming superconducting results from outward motion of a conducting fluid that carries the magnetic field lines with it, as in a classical plasma. Furthermore we have argued that without outward motion of charge carriers there cannot be a Meissner effect [8]. In contrast, within the conventional theory of superconductivity [2] the magnetic field is expelled without any outward motion of charge carriers. The conventional theory does not offer a dynamical explanation for how this happens.

In this paper we consider a wire geometry and type I superconductors only. Figure 1 shows schematically the behavior of current streamlines and magnetic field lines of a superconducting cylindrical wire inserted between normal metal leads in steady state. These observable quantities result from solution of London’s equation and Ampère’s law [9]. Fig. 1 shows that when normal current carriers enter the superconducting region they acquire outward radial velocity and flow towards the surface of the wire. The region close to the normal-superconductor boundary where there is radial motion of charge is of order of the London penetration depth, $\lambda_L$. Beyond that region, current carriers flow parallel to the surface of the cylindrical wire within a London penetration depth of its surface and no current flows in the interior. We assume the current is smaller that the critical current

Concurrently, magnetic field lines that exist throughout the interior in the normal conductor move to the surface in the superconducting region following the current streamlines, and are always confined within a London penetration depth of the boundaries of the superconducting region, as the figure shows. If we imagine traveling with a charge carrier along a streamline with a magnetic field line next to us, we will see that the magnetic field line moves with us as we enter the superconducting region and we move towards the surface, as shown in Figure 1. Thus, the motion of magnetic field lines follows the motion of charge carriers, as we proposed is also true, but less evident, in the Meissner effect [7, 8].

Our explanation of the Meissner effect follows from the prediction of the theory of hole superconductivity [1] that when a metal enters the superconducting state electrons forming Cooper pairs expand their orbits giving rise to outward motion of negative charge [10]. This process is driven by lowering of quantum kinetic energy [11]. Concurrently, to preserve both charge and mass neutral-
ity in the interior, normal state hole carriers also move outward [12]. This outward motion of a charge-neutral mass-neutral fluid carries the magnetic field lines with it, as would happen in a classical plasma [7].

In this paper we argue that the known behavior of a superconducting wire carrying a current provides further evidence for our proposed explanation of the Meissner effect.

II. SUPERCONDUCTING WIRE VERSUS PERFECTLY CONDUCTING WIRE

It is generally assumed that superconductors and perfect conductors behave identically in processes that do not involve a change in temperature. Here we point out that this is not so.

As is well known, if we cool a normal metal into the superconducting state in the presence of an external magnetic field, it behaves very differently than if we cool a normal metal into a perfectly conducting state in the presence of an external magnetic field. In the former case the magnetic field is expelled through the development of a Meissner surface current, whereas in the latter case the magnetic field remains frozen in the interior and no surface current flows.

Instead, if we consider a superconductor and a perfect conductor below their critical temperature initially without magnetic fields, they behave identically when we apply an external magnetic field: both develop the same surface current to prevent the magnetic field from penetrating, with the current flowing in a layer of thickness \( \lambda_L \), the London penetration depth, that is a function of carrier density and effective mass. The dynamics of the process by which this state is established is fully accounted for by Maxwell’s equations, in particular Faraday’s law, and Newton’s laws.

Now let us consider the current-carrying wire shown in Fig. 1. If initially the middle section is in the normal state, current will flow throughout its cross section giving rise to magnetic field lines throughout the interior just like in the normal metal leads. This magnetic field is generated by the current itself; it is not an external magnetic field. If we now cool and the middle section is generated by the current itself, it is not an external giving rise to magnetic field lines throughout the interior mal state, current will flow throughout its cross section in Fig. 1. If initially the middle section is in the normal’s law, and Newton’s laws.

counted for by Maxwell’s equations, in particular Faraday’s law, that is a function of an external magnetic field. In the former case the magnetic field is expelled through the development of a Meissner surface current, whereas in the latter case the magnetic field remains frozen in the interior and no surface current flows.

In the case of the superconductor, current is cooled into a perfectly conducting state, the same current will continue to flow throughout the interior of the perfect conductor, as shown in the figure.

In the next sections we discuss the quantitative solution of these equations. Here we consider how they apply to a perfect conductor.

Recall that London’s equation (2) can be derived by starting from the superconductor in the absence of a magnetic field and calculating the response of the system to electromagnetic fields assuming it is a perfect conductor [9]. The current is given by

\[
\vec{J}_s(\vec{r}, t) = n_s e \vec{v}_s(\vec{r}, t)
\]

with \( \vec{v}_s \) the carrier velocity and \( n_s \) the carrier density. The equation of motion assuming only electromagnetic forces is

\[
\frac{d\vec{v}_s}{dt} = \frac{e}{m_e} (\vec{E} + \frac{\vec{v}_s}{c} \times \vec{B})
\]

and yields

\[
\frac{\partial \vec{v}_s}{\partial t} + \nabla (\frac{v_s^2}{2}) - \frac{e}{m_e} \vec{E} = \vec{v}_s \times (\nabla \times \vec{B}) + \frac{e}{m_e c} \vec{B}
\]

Defining the generalized vorticity as

\[
\vec{\omega}(\vec{r}, t) = \nabla \times \vec{v}_s(\vec{r}, t) + \frac{e}{m_e c} \vec{B}(\vec{r}, t)
\]

the equation of motion for \( \vec{\omega} \) is [9, 13]

\[
\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v}_s \times \vec{\omega})
\]

so if initially \( \vec{\omega}(\vec{r}, t = 0) = 0 \), it cannot change with time, neither in the superconductor nor in the perfect conductor. So the condition

\[
\vec{\omega}(\vec{r}, t) = \nabla \times \vec{v}_s(\vec{r}, t) + \frac{e}{m_e c} \vec{B}(\vec{r}, t) = 0
\]
or equivalently

\[
\nabla \times \mathbf{j} + \frac{c}{4\pi\lambda_L^2} \mathbf{B} = 0
\]

(8b)

remains valid at all times if it is valid initially, as will happen if initially \( \mathbf{j} = \mathbf{B} = 0 \). When we apply an external magnetic field or an external voltage to a superconductor or to a perfect conductor, if a current develops it has to satisfy Eq. (8).

For a superconductor that can happen when we apply a voltage across the wire, and the result is the steady state depicted in Fig. 1. However, a perfect conductor, unlike a superconductor, cannot be in the steady state depicted in Fig. 1. The streamlines in Fig. 1 indicate that carriers acquire an acceleration in the radial direction when they enter the superconducting region, but there is no electromagnetic force to provide that acceleration. Nor can the perfect conductor develop the uniform current density shown in Fig. 2, because it requires the interior magnetic field to change in the interior of the perfect conductor from its initial value 0 to a finite value, which cannot happen according to Faraday’s law. Indeed, the state shown in Fig. 2 does not satisfy \( \mathbf{v} = 0 \) (Eq. (8)) since \( \nabla \times \mathbf{v} = 0 \) and \( \mathbf{B} \neq 0 \) in the interior.

Instead, what will happen when we apply a voltage across a perfect conductor is that a charge polarization will develop to account for the absence of voltage drop across the perfect conductor.

We consider first a planar instead of a cylindrical geometry, as shown in Fig. 4, since the mathematics is simpler (trigonometric instead of Bessel functions). We assume that current flows in the \( z \) direction in the normal region, and the superconducting region is \(-b \leq z \leq b\). In the perpendicular \((\hat{y})\) direction, the current is confined to the region \(-a \leq y \leq a\). In the normal region we assume uniform resistivity and hence uniform current density \( \mathbf{j} = J \hat{z} \). In the superconducting region the current is given by

\[
\mathbf{j} = J_y(y,z)\hat{y} + J_z(y,z)\hat{z}.
\]

(9)

The magnetic field points in the \( x \) direction,

\[
\mathbf{B} = B_x(y,z)\hat{x}.
\]

(10)

From Ampere’s law Eq. (1),

\[
\frac{\partial B_x}{\partial y} = \frac{4\pi}{c} J_y
\]

(11a)

\[
\frac{\partial B_x}{\partial z} = -\frac{4\pi}{c} J_z
\]

(11b)

hence in the normal region the magnetic field is given by

\[
B_x(y,z \leq b) = -\frac{4\pi}{c} J_y.
\]

(12)

In the superconducting region, no normal current flows in steady state because it is shorted by the supercurrent. The supercurrent is determined by the London equation (2), which is

\[
\frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z} = -\frac{c}{4\pi\lambda_L^2} B_x
\]

(13)

with \( \lambda_L \) the London penetration depth, together with Eq. (11). From Eqs. (11) and (13) the supercurrent satisfies

\[
\nabla^2 J_y = \frac{1}{\lambda_L^2} J_y
\]

(14a)
\[ \nabla^2 J_z = \frac{1}{\lambda_L^2} J_z \] (14b)

as well as the continuity equation \( \nabla \cdot \vec{J} = 0 \), i.e.

\[ \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0. \] (15)

A solution of Eq. (14a) which is odd in \( y \) as required by symmetry and satisfies the boundary conditions \( J_y(a, z) = J_y(-a, z) = 0 \) is

\[ J_y(y, z) = \sin\left(\frac{\pi \ell}{a} y\right) e^{\pm \sqrt{\frac{\pi^2}{a^2} + \left(\frac{a^2}{z^2}\right)^2}} \] (16)

with \( \ell \) an integer. Using the symmetry condition \( J_y(y, z) = -J_y(y, -z) \), the general solution is

\[ J_y(y, z) = J \sum_{\ell=1}^{\infty} A_\ell \sin\left(\frac{\pi \ell}{a} y\right) \sinh\left(\frac{z}{a_\ell}\right) \] (17a)

with

\[ a_\ell = \frac{1}{\sqrt{\frac{\pi^2}{a^2} + \left(\frac{a^2}{z^2}\right)^2}} \] (17b)

and the coefficients \( A_\ell \) determined by the boundary conditions. From the continuity Eq. (15), an equation for \( \partial J_z/\partial z \) is obtained, and using the boundary conditions \( J_z(y, -b) = J_z(y, b) = J \) we find

\[ J_z(y, z) = J \frac{a}{\lambda_L \sinh\left(\frac{a \ell}{a}\right)} - J \pi \sum_{\ell=1}^{\infty} \frac{a}{a_\ell} A_\ell \cos\left(\frac{\pi \ell}{a} y\right) \cosh\left(\frac{z}{a_\ell}\right) \] (18a)

and

\[ A_\ell = \frac{2(-1)^\ell}{\pi \ell} \frac{a}{a \cosh(b/a_\ell)}. \] (18b)

The magnetic field in the superconducting region is determined by the London Eq. (13)

\[ B_x(y, z) = -\frac{4\pi J_\ell}{c} \left( \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z} \right) \] (19)

so that

\[ B_x(y, z) = -\frac{4\pi J_\ell}{c} \frac{\sinh\left(\frac{\pi \ell}{a}\right)}{\sinh\left(\frac{a \ell}{a}\right)} \] (20)

\[ -\sum_{\ell=1}^{\infty} \frac{a_\ell}{a_\ell^2} A_\ell \sin\left(\frac{\pi \ell}{a} y\right) \cosh\left(\frac{z}{a_\ell}\right). \]

which properly satisfies the boundary condition \( B(y, \pm b) = -(4\pi/c) J_y \), which is the magnetic field in the normal region.

For a wire with \( b >> a \) and for \( z \) far from the boundaries \( z = \pm b \), the currents and field reduce to

\[ J_y(y, z) = 0 \] (21a)

\[ B_x(y, z) = -\frac{4\pi J_\ell}{c} \frac{\sinh\left(\frac{\pi \ell}{a}\right)}{\sinh\left(\frac{a \ell}{a}\right)} \] (21b)

and

\[ B_x(y, z) = -\frac{4\pi J_\ell}{c} \frac{\sinh\left(\frac{\pi \ell}{a}\right)}{\sinh\left(\frac{a \ell}{a}\right)} \] (21c)

and for \( \lambda_L << a \) they further simplify to (for \( y > 0 \))

\[ J_z(y, z) = J \frac{a}{\lambda_L \sinh\left(\frac{\pi \ell}{a}\right)} \] (22a)

\[ B_x(y, z) = -\frac{4\pi J_\ell}{c} e^{(y-a)/\lambda_L}. \] (22b)

IV. STREAMLINES

The streamlines for the charge motion, denoted by \( y(z) \), satisfy

\[ \frac{dy(z)}{dz} = \frac{J_y(y, z)}{J_z(y, z)} \] (23)

so that in the superconducting region

\[ y(z) = y_0 + \int_{-b}^{z} \frac{dz'}{J_z(y(z'), z')} \frac{J_y(y(z'), z')}{J_z(y(z'), z')} \] (24)

with \( y_0 = y(z = -b) \). In the normal region \( J_y = 0 \) so the streamlines are parallel to the \( z \)-axis. Figure 5 shows one example. It can be seen that the slope of the streamline changes discontinuously at the normal-superconductor boundary.

In the superconducting region electrons move along the streamlines with velocity given by

\[ \vec{v}(y, z) = \frac{1}{en_s} \vec{J}(y, z) \] (25)
with $e < 0$ the electron charge. In the normal region, charge carriers move in the $z$ direction with velocity $\vec{v}_n$ that is independent of $y$. The normal carrier velocity is

$$\vec{v}_n = \frac{1}{en} J \hat{z}$$

(26)

where $n$ is the normal carrier density.

V. THE N-S BOUNDARY

At the N-S boundary, the current streamlines acquire discontinuously motion in the $y$-direction, as seen in Fig. 5. For $\lambda_L << a, b$ the current in the $y$ direction at the phase boundary takes the simple form:

$$J_y(y, -b) = \frac{J}{\lambda_L} y.$$  

(27)

This is easily seen from Eqs. (17) and (18b), using the Fourier expansion for $y$

$$y = -\frac{2a}{\pi} \sum \frac{(-1)^f}{f} \sin \left( \frac{\pi fy}{a} \right).$$

(28)

From Eq. (27), the speed of the superconducting carriers in the $y$ direction is then

$$v_y(y, -b) = \frac{1}{en_s} \frac{J}{\lambda_L} y = n_s \frac{y}{\lambda_L} v_n$$

(29)

which is much larger than the drift velocity of carriers in the normal region $v_n$ since $y >> \lambda_L$ except very near the center.

Eq. (29) implies that as carriers enter the superconducting region they suddenly acquire a very large impulse in direction parallel to the phase boundary. Presumably this occurs over a very short time scale, which implies that an enormous force in the $y$ direction is acting on the charge carriers as they enter the superconducting region. The conventional theory of superconductivity provides no insight into the physical origin of this force. It does not explain why the process of Cooper pair formation would give rise to such a force.

To understand the physical origin of this force, we note that we can replace the current $J$ in Eq. (29) in terms of the magnetic field at the phase boundary Eq. (12) and obtain

$$v_y(y, -b) = -\frac{e}{m_e} \lambda_L B_x(y, -b)$$

(30)

where we have used that [2]

$$\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{m_e c^2}.$$ 

(31)

Eq. (30) indicates that it is the magnetic field $B_x$ that imparts the impulse to the carriers in the $y$ direction as they become superconducting: the impulse is zero if $B_x = 0$ and it is directly proportional to the local value of $B_x$ for a given $y$, with the same proportionality constant independent of $y$. And it points in direction perpendicular to $\vec{B}$. So we ask: how can a magnetic field impart momentum to electric charges in an amount that is proportional to its magnitude at that point in space and in a direction perpendicular to its direction?

The answer is, of course, the magnetic Lorentz force [14]. That is the only way that magnetic fields exert forces on charges according to the laws of physics. The magnetic Lorentz force on a charge $e$ is

$$\vec{F}_L = -\frac{e}{c} \vec{v} \times \vec{B}$$

(32)

whether we are talking about the macroscopic or the microscopic realm, and whether we are talking about normal metals or superconductors. This has been known since 1895 [15], and the equivalent Ampere force law since 1822 [16].

Therefore, to understand this physics, we simply have to look at Eq. (32). The velocity in Eq. (32) is the velocity of the charge $e$ upon which the magnetic Lorentz force acts. In order for the charge to get an impulse in the positive $y$ direction in the region $y > 0$ where $B$ points in the $+\hat{z}$ direction, $\vec{v}$ in Eq. (32) has to point in the $-\hat{z}$ direction. Similarly to get the required impulse in the direction $-y$ for carriers in the region $y < 0$ entering the superconducting region, $\vec{v}$ has to also point in the $-\hat{z}$ direction since the direction of $B$ is reversed in that region.

The carriers flowing from the normal into the superconducting region acquire the velocity $v_y$ instantly as they cross the phase boundary. Let us assume that the instant they cross the phase boundary $z = -b$ they recoil backward (in the $-\hat{z}$ direction) a distance $\Delta s$ in a very short time interval $\Delta t$, so in Eq. (32) $v = \Delta s / \Delta t$. In order to acquire the speed in the $y$ direction given by Eq. (30) under the action of the Lorentz force Eq. (32) it is necessary that $\Delta s = \lambda_L$, so that $\vec{F}_L \Delta t = (e/c) \Delta s \vec{B} = m_e v_y$. This is shown schematically in Fig. 6.

FIG. 6: In order for carriers to acquire the speed Eq. (30) in the $y$ or $-y$ direction as they enter the superconducting region, they have to undergo a sudden motion in the $-\hat{z}$ direction a distance $\lambda_L$. The magnetic field that provides the impulse in the $y$ direction points into (out of) the paper in the region $y > 0 / y < 0$ as indicated by the crosses and circles. This process presumably occurs in a region of thickness $\lambda_L$ in the $z$ direction around the phase boundary, indicated in grey.
Then, when charges leave the condensate at \( z = +b \), they have to acquire a sudden impulse in the same direction and of the same magnitude as when they entered, to cancel the momentum in the \( y \) direction that they acquired as they approached the phase boundary \( z = +b \) (see streamlines in Fig. 5 near \( z = b \)). Figure 7 shows the corresponding process for carriers leaving the superconducting region. Again this would result if they move backward (in the \(-y \) direction) a distance \( \Delta s = \lambda_L \).

The theory of hole superconductivity explains why this happens. When normal carriers pair and join the condensate their orbits expand from a microscopic radius to radius \( 2\lambda_L \), as we discussed extensively elsewhere \([10–12]\) and is shown schematically in Fig. 8. In the presence of a magnetic field, they acquire angular velocity that gives rise to the tangential velocity given by Eq. (30). Conversely, when pairs leave the condensate their orbits contract and their tangential velocity goes to zero.

Figure 9 shows the resulting state. At the boundary \( z = -b \), this extra velocity acquired by the carriers is in the \(+y \) direction for \( y > 0 \) and in the \(-y \) direction for \( y < 0 \), since the magnetic field points in opposite directions. In the interior, the velocity of neighboring orbits point in opposite directions and cancel out. At the boundary \( z = +b \), the orbits shrink again and the carriers in the region \( y > 0 \) lose their velocity pointing in the \(-y \) direction, which corresponds to suddenly acquiring momentum in the \(+y \) direction as they leave the superconducting region, in accordance with the streamlines shown in Fig. 5.

In summary, the same physics that explains how electrons spontaneously acquire the speed of the Meissner current when a system is cooled into the superconducting state \([10]\), and also explains why electrons slow down when a rotating normal metal becomes superconducting \([17]\), explains how streamlines acquire and lose their velocity perpendicular to the normal current flow as carriers enter and leave the superconductor. The situation discussed here shows the underlying physics more clearly than in the cases of the Meissner effect and the rotating superconductor, because the value of the magnetic field changes with position.

We conclude from this analysis, or simply from consid-

**VI. FORCE ACTING ON CARRIERS**

The profile of streamlines is determined by London’s equation, Ampere’s law, and the boundary conditions. It is interesting to ask: what are the forces acting on carriers that make them flow along the streamlines?

We assume the superfluid charge carriers are negatively charged electrons of carrier density \( n_s \). Their equation of motion assuming only electric and magnetic forces is given by Eq. (5). However for further generality we will assume that there could be another ‘quantum force’ \( \vec{F}_q \) acting on electrons that derives from a potential, i.e. \( \nabla \times \vec{F}_q = 0 \). Including that force and using the London
condition Eq. (8a), Eq. (5) yields
\[ \frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla} \left( \frac{v^2_s}{2} \right) = e \frac{\vec{E}}{m_e} + \frac{1}{m_e} \vec{F}_q \equiv \frac{1}{m_e} \vec{F}_0 \] (33)
where \( \vec{F}_0 \) is the sum of electric and quantum forces. In terms of the supercurrent Eq. (3), neglecting possible small variations of \( n_s \) with position,
\[ \frac{\partial \vec{J}_s}{\partial t} + \frac{1}{2n_e e} \vec{\nabla} J^2_s = \frac{n_s e}{m_e} \vec{F}_0 \] (34)
and under stationary conditions
\[ \frac{1}{2n_s e} \vec{\nabla} J^2_s = \frac{n_s e}{m_e} \vec{F}_0. \] (35)

Therefore, this equation determines the non-magnetic forces acting on the charge carriers in the superconductor in terms of the supercurrent \( J_s \). Using Eq. (35), we can rewrite Eq. (4) (generalized to include the quantum force) in terms of the current as
\[ \frac{d\vec{J}_s}{dt} = \frac{1}{n_s e} \left[ \frac{1}{2} \vec{\nabla} J^2_s + \frac{e}{4\pi \lambda_L^2} \vec{J}_s \times \vec{B} \right]. \] (36)

The second term on the right side of Eq. (36) is the magnetic Lorentz force on the carriers, the first term is the sum of electric and quantum forces \( \vec{F}_0 \) which we will call generalized force.

Finally, we can rewrite the total derivative on the left side of Eq. (36) in terms of the derivative with respect to \( z \), using that
\[ dz = \frac{J_z}{n_s e} dt \] (37)
and Eq. (36) yields
\[ \frac{d\vec{J}_s}{dz}_z = \frac{1}{2} \vec{\nabla} J^2_s + \frac{e}{4\pi \lambda_L^2} \vec{J}_s \times \vec{B} \] (38)
where the derivative on the left side of Eq. (38) follows the streamlines, i.e.
\[ \frac{dJ_z}{dz} = \lim_{dz \to 0} \frac{J_z(y + dy, z + dz) - J_z(y, z)}{dz} \] (39a)
with
\[ dy = \frac{J_y}{J_z} dz. \] (39b)

Figure 10 shows the direction and magnitude (in arbitrary units) of the total force on carriers along a typical streamline in the region \( y > 0, z < 0 \). This force determines the time evolution of the carriers after the initial kick received by carriers when they enter the superconducting region, discussed in Sect. V. Note that the \( y \) component of the net force is negative, as required so that the total momentum transfer in the \( -y \) direction along the trajectory cancels the \( y \) momentum acquired as the electrons enter and leave the superconducting region discussed in Sect. V.

The net force in Fig. 10 is the sum of generalized force \( \vec{F}_0 \) and magnetic Lorentz force \( \vec{F}_B \), that are shown in Fig. 11. The scale of the forces here is reduced by a factor of 3 with respect to Fig. 10. This means that the net force in Fig. 10 results from a net near cancellation of magnetic and generalized force in nearly opposite directions.

More generally the pattern of this generalized force is shown in Fig. 12. It can be seen that it pushes carriers out of the superconducting region towards the nearest boundary. It is associated with the current pattern and becomes very small in the interior where the current is small.
FIG. 12: Spatial distribution of the generalized force $F_0$ for
the case $\lambda_L = 0.2$.

The physical origin of these non-magnetic forces is not clear. If part or all of $F_0$ is electric, it implies that there is some charge redistribution in the interior of superconductors carrying a current. The potential that gives rise to the force $F_0$ (Eq. (33)) is called the Bernoulli potential. Various explanations for its origin within the conventional theory of superconductivity are discussed in ref. [18].

VII. ALFVEN’S THEOREM

Figures 1 and 4 show qualitatively that magnetic field lines are carried along with the streamlines, as determined by Alfvén’s theorem. Let us examine this question quantitatively. The convective time derivative of the magnetic field, following the motion of the streamlines, is given by

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v}_s \cdot \nabla) \vec{B}. \tag{40}$$

We have

$$(\vec{J}_s \cdot \nabla) \vec{B} = J_z \frac{\partial B_z}{\partial z} + J_y \frac{\partial B_x}{\partial y} = 0 \tag{41}$$

which follows immediately from Ampere’s law Eq. (11). Therefore,

$$\frac{d\vec{B}}{dt} = 0 \tag{42}$$

for the stationary flow depicted in Fig. 4. This means that the value of the magnetic field does not change along a streamline, it stays constant at its normal state value:

$$B(y(z), z) = -\frac{4\pi}{c} y(z = -b) J \tag{43}$$

for a given streamline $y(z)$, which indicates that the carriers carry the magnetic field with them.

A more general condition for Alfvén’s theorem to hold follows from the identity [3]

$$\frac{d}{dt} \int_{S_m} \vec{B} \cdot d\vec{S} = \int_{S_m} \left[ \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{S} \tag{44}$$

for any surface $S_m$ moving with the fluid that is moving with velocity $\vec{u}(\vec{r})$. For a perfect conductor the integrand is zero, and in particular for stationary flow,

$$\nabla \times (\vec{J}_s \times \vec{B}) = 0. \tag{45}$$

We have

$$\nabla \times (\vec{J}_s \times \vec{B}) = -\left( \frac{\partial}{\partial y} (J_y B_x) + \frac{\partial}{\partial z} (J_z B_x) \right) = 0 \tag{46}$$

from Ampere’s law and the continuity equation. Therefore,

$$\frac{d}{dt} \int_{S_m} \vec{B} \cdot d\vec{S} = \int_{S_m} \left[ \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{S} = 0 \tag{47}$$

for any arbitrary surface $S_m$ that moves together with the fluid. This means that magnetic field lines are frozen into the fluid and move together with the fluid as required by Alfvén’s theorem [3–5].

VIII. TEMPERATURE DEPENDENCE

If the current flowing in the normal state is very small, upon cooling the system will enter the superconducting state at a temperature close to the critical temperature $T_c$, and the London penetration depth will be a significant fraction of the sample’s dimensions.

Consider for example a cylindrical wire of radius 1 mm, carrying a current $I = 1\mu A$. This corresponds to a current density

$$J = 0.32 \times 10^{-4} \frac{A}{cm^2} = 0.95 \times 10^5 \frac{statA}{cm^2}. \tag{48}$$

Let us assume the maximum magnetic field at the boundary of the sample is the critical field $H_c(T)$ at temperature $T$:

$$H_c(T) = \frac{4\pi}{c} J a = 4 \times 10^{-6} G \tag{49}$$

Assuming the relations for the two-fluid model

$$H_c(T) = H_c(0) \left[ 1 - \left( \frac{T}{T_c} \right)^2 \right] \tag{50a}$$

$$\frac{1}{\lambda_L(T)^2} = \frac{1}{\lambda_L(0)^2} \left[ 1 - \left( \frac{T}{T_c} \right)^4 \right] \tag{50b}$$
yields close to $T_c$

$$\frac{1}{\lambda_L(T)^2} = \frac{2}{\lambda_L(0)^2} \frac{H_c(T)}{H_c(0)}$$

hence

$$\lambda_L(T) = \frac{\lambda_L(0) H_c(T)}{\sqrt{2} H_c(0)}^{1/2}$$

For $H_c(0) = 1000G$, $\lambda_L(0) = 500\AA$ Eq. (52) yields

$$\lambda_L(T) = 0.56mm = 0.56a.$$ (53)

So with those parameters, the system will enter the superconducting state at a temperature close to $T_c$, with a London penetration depth that is of order half of the system half-width. Upon cooling further, the London penetration depth will rapidly decrease.

Figure 13 shows the evolution of streamlines as the temperature is lowered under those conditions. The magnetic field, not shown in Fig. 13, follows the behavior of the streamlines, as shown quantitatively in Sect. VI and qualitatively in Fig. 4. It moves out together with the streamlines.

Figure 13 shows that as the system is cooled and enters deeper into the superconducting state, charge carriers carrying the current along the streamlines move towards the surface, and carry the magnetic field lines out with them. This clearly illustrates that Alfvén’s theorem governs the behavior of charges and magnetic fields in a superconducting wire.

**IX. CYLINDRICAL GEOMETRY**

For completeness we now give results for a cylindrical wire. We consider a cylinder of radius $a$ and length $2b$ in the region $-b \leq z \leq b$. The current density is uniform in the normal region, so the magnetic field in the normal region is given by

$$\vec{B}(r, z) = \frac{2\pi r}{c} J \hat{\theta} \equiv B_\theta \hat{\theta}.$$ (54)

In the superconducting region, the current components are given by [19]

$$J_r = \frac{J_0}{\lambda_L^2} \sum_{\ell=1}^{\infty} \frac{a_\ell}{\xi_\ell} J_\ell(\frac{\xi_\ell r}{a}) \frac{\sinh \frac{\xi_\ell a}{a}}{\cosh \frac{\xi_\ell a}{a}}$$

and the magnetic field by

$$B_\theta = \frac{4\pi}{c} \frac{J a i}{2} \frac{J_1(\xi r / \lambda_L)}{J_0(\xi a / \lambda_L)} + \frac{J_0}{\lambda_L^2} \sum_{\ell=1}^{\infty} \frac{a_\ell^2}{a} J_\ell(\frac{\xi_\ell r}{a}) \frac{\cosh \frac{\xi_\ell a}{a}}{\sinh \frac{\xi_\ell a}{a}}$$

with

$$a_\ell = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{\xi_\ell^2}{a^2}}}$$

where $J_0$ and $J_1$ are Bessel functions of zero and first order and $\xi_\ell$’s are the zeros of $J_1$, given in Appendix A.

**FIG. 13: Evolution of streamlines as the temperature is lowered and $\lambda_L$ decreases in a planar superconducting region $-b \leq z \leq b$, $-a \leq y \leq a$, for $a = 1$, $b = 2$, for various values of $\lambda_L$**

**FIG. 14: Streamlines for charge motion in the z direction in a cylindrical superconducting region $-b \leq z \leq b$, $r \leq a$, for $a = 1$, $b = 2$, $\lambda_L = 0.2$**

Figure 14 shows streamlines for a cylindrical wire of radius $a = 1$ and London penetration depth $\lambda_L = 0.2$. They look qualitatively similar to the planar case, Fig. 5, except that the outward motion of streamlines is considerably less than in the planar case for given $\lambda_L$. This follows simply from the fact that the magnetic field here Eq. (54) is half as large as for the planar case Eq. (12) for
the same distance to the central axis. Another important difference with the planar case is that here the magnetic field is not constant along streamlines. The material time derivative of the magnetic field is given by

\[
\frac{dB_\theta}{dt} = -\frac{1}{n_e e} \frac{J_e B_\theta}{r}
\]

(59)
or as a function of \(z\)

\[
\frac{dB_\theta}{dz} = \frac{J_r B_\theta}{J_z r}.
\]

(60)

Fig. 15 shows the value of the magnetic field along the streamlines shown in Fig. 14. It decreases, which means that magnetic field is being expelled even faster that expected from the motion of the charge carriers, unlike the situation in the planar geometry where the magnetic field is constant along the streamlines.

![Image of magnetic field values along streamlines](image)

**FIG. 15:** Magnetic field values along the streamlines shown in the upper left quadrant of Fig. 14.

Furthermore the condition Eq. (45) doesn’t hold, since we have

\[
\nabla \times (\vec{J}_s \times \vec{B}) = \frac{2B_0 J_x}{r} \hat{\theta}
\]

(61)

hence

\[
\frac{d}{dt} \int_{S_m} \vec{B} \cdot d\vec{S} = -\frac{2}{n_e e} \int_{S_m} \frac{B_0 J_x}{r} \hat{\theta} \cdot d\vec{S}
\]

(62)

The extra factor of 2 compared to Eq. (59) is because the surface \(S_m\) shrinks as it moves together with the streamlines, as a consequence the flux through it decreases even faster.

In conclusion, we find that in a cylindrical wire the magnetic field lines move out even faster than the streamlines. Still, the motion of magnetic field lines is closely associated with the motion of charges.

We can also show analytically that when the wire carrying a current is cooled from the normal into the superconducting state there is a radial outflow of charge carriers. For a point far from the boundaries with the normal leads the velocity of carriers is given by

\[
\vec{v}_c(\vec{r}) = v_z(r) \hat{z} + v_r(r) \hat{r}.
\]

(63)

Initially when the system is in the normal state, \(v_r = 0\) and the current is uniform so that \(v_z\) is independent of \(r\), hence \(\nabla \times \vec{v}_c = 0\). The generalized vorticity Eq. (6) is given by

\[
\vec{w}(r, t = 0) = \frac{e}{n_e c} B_\theta(r, t = 0) \hat{\theta}
\]

(64)

with \(B_\theta\) given by Eq. (54). Hence \(\vec{w}(r, t = 0) = w_\theta(r, t = 0) \hat{\theta} \neq 0\). When the system is in the superconducting state, \(w_\theta(r) = 0\) according to Eq. 8. The equation of motion for \(\vec{w}\) is, from Eq. (7)

\[
\frac{\partial w_\theta(r, t)}{\partial t} = -\frac{\partial}{\partial r}[v_r(r, t) w_\theta(r, t)].
\]

(65)

It shows that \(\vec{w}\) cannot evolve from its initial nonzero value Eq. (64) to zero unless \(v_r \neq 0\), which means that there is necessarily radial motion of charge carriers during the process.

**X. DISCUSSION**

It is generally stated that the difference between perfect conductors and superconductors is that the superconductor can only be in one single state for given external conditions, independent of history, while a perfect conductor can reach a variety of different states dependent of history, including the one that the superconductor adopts. We pointed out here that this is not so in the case of a wire. Instead, a perfectly conducting wire, no matter what the history, can never adopt the unique state that the superconducting wire carrying a current adopts. For this reason, analyzing the superconducting wire scenario can yield new insight beyond analyzing the situation where an external magnetic field is applied, in which case the superconducting and perfectly conducting bodies can reach the same state. Here we have shown that the wire scenario provides further evidence in support of the physics that we have proposed explains the Meissner effect [12].

Incidentally, we also note that according to the analysis in this paper, a ‘perfect conductor’ is paradoxically unable to conduct any current unless it became perfectly conducting after the current started flowing. To our knowledge this has not been pointed out before [20].

The analysis of this paper confirms that there is fundamental physics missing in the conventional understanding of superconductivity. The notion that the motion of magnetic field lines in superconductors is tied to the motion of charge carriers is alien to both London theory [9] and to BCS theory [2, 21]. Within BCS theory magnetic field lines move spontaneously out in the Meissner effect with no outward motion of charge carriers. This
in appearance violates Faraday’s law, Newton’s law, and thermodynamic laws, as we pointed out in earlier work \[12, 22, 23\]. The dynamics of this process, and how it is able to circumvent these fundamental laws of physics, has not been addressed by BCS theory in the 64 years since its formulation, when it supposedly explained the Meissner effect \[24\].

In contrast, we have pointed out in this paper that for a superconducting wire carrying a current the motion of magnetic field lines is intimately tied to the motion of charge carriers, within BCS-London theory. Namely, the motion of magnetic field lines follows the motion of charge carriers in the streamlines, both as a function of position in the steady state, and as a function of temperature as the temperature is changed. This has been known for over 70 years \[9\], however its significance has not been appreciated.

The superconducting wire conducting current discussed here reveals key information about the physics of superconductivity. When normal carriers enter the superconducting region they experience a ‘kick’ that changes their direction of flow towards the surface of the wire, as indicated by the discontinuity in the slope of the streamlines at the N-S boundaries shown in the figures. This follows directly from the solution of London’s and Ampère’s equations. This ‘kick’ that transfers momentum to the carriers does not occur for a perfect conductor. It is a quantum effect that occurs when normal carriers form Cooper pairs as they enter the superconducting region and join the condensate. Similarly carriers experience a ‘kick’ when they exit the superconducting region, i.e. transition from Cooper pairs to normal electrons. As discussed in Sect. V, the momentum acquired by the carriers in these processes is in direction orthogonal to the current flow and to the magnetic field and is directly proportional to the local magnitude of the magnetic field. The conclusion that it originates in the magnetic Lorentz force \[14\] is compelling. We have shown in Sect. V how it can be understood by the radial expansion and contraction of the orbit proposed within the theory of hole superconductivity, that also explains the Meissner effect and the behavior of rotating superconductors. Instead, BCS-London theory provides no physical explanation for the phenomena studied theoretically and experimentally by Nikulov and coworkers \[27\].

### Appendix A: Bessel functions

We give here expressions for the Bessel functions used in Sect. IX for the convenience of readers. Series expansions for the Bessel functions are:

\[
J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \tag{A1a}
\]

\[
J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r+1)!} \left(\frac{x}{2}\right)^{2r+1} \tag{A1b}
\]

These expressions are useful for numerical computations for small \(x\) but not for large \(x\). For large \(x\) we use

\[
J_0(x) = \sqrt{\frac{2}{\pi x}} \left(1 - \frac{1}{16x^2} + \frac{53}{512x^4}\right) \cos(x - \frac{\pi}{4} - \frac{1}{8x} + \frac{25}{384x^3}) \tag{A2a}
\]

\[
J_1(x) = \sqrt{\frac{2}{\pi x}} \left(1 + \frac{3}{16x^2} - \frac{99}{512x^4}\right) \cos(x - \frac{3\pi}{4} - \frac{3}{8x} - \frac{21}{128x^3}) \tag{A2b}
\]

Using 10 terms in the series Eq. (A1), the results match those of Eq. (A2) to 8 decimal places for \(x = 2\), so we use Eq. (A1) with \(0 \leq r \leq 10\) for \(x \leq 2\) and Eq. (A2) for \(x \geq 2\). For the zeros of \(J_1\) we find that the formula

\[
x_n = n\pi + \frac{\pi}{4} - \frac{3}{8(n\pi + \pi/4)} \tag{A3}
\]

gives accurate answers (7 digit accuracy) for \(n \geq 5\), for smaller \(n\) we use the tabulated values \(\xi_1 = 3.831705, \xi_2 = 7.015586, \xi_3 = 10.17347, \xi_4 = 13.23269, \xi_5 = 16.47063\).

For Bessel functions of imaginary argument we use

\[
J_0(ix) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \tag{A4a}
\]
\[ J_1(ix) = i \sum_{r=0}^{\infty} \frac{1}{r!(r+1)!} \left( \frac{x^2}{2} \right)^{2r+1} \quad \text{for } x \geq 1.7. \]

with \( r_{\text{max}} = 10 \) for \( x \leq 1.7 \), and

\[ J_\alpha(ix) = i^\alpha \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{4\alpha^2 - 1}{8x} + \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)(4\alpha^2 - 25)}{2!(8x)^2} \right] - \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)(4\alpha^2 - 25)}{3!(8x)^3}. \]

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