Principle of Minimal Energy
in
Relativistic Schrödinger Theory

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Abstract

The Hamilton-Lagrange action principle for Relativistic Schrödinger Theory (RST) is converted to a variational principle (with constraints) for the stationary bound states. The groundstate energy is the minimally possible value of the corresponding energy functional and the relativistic energy eigenvalue equations do appear as the corresponding variational equations. The matter part of these eigenvalue equations is a relativistic generalization of the well-known Ritz prin-
ciple in non-relativistic quantum mechanics which however disregards
the dynamical character of the particle interactions. If the latter are
included in the proposed principle of minimal energy for the bound
states, one obtains a closed dynamical system for both matter and
gauge fields. The new variational principle enables the development
of variational techniques for solving approximately the energy eigen-
value equations. As a demonstration, the positronium groundstate is
treated in great detail. Here a simple exponential trial function is suffi-
cient in order to reproduce the (exact) result of conventional quantum
mechanics where the relativistic and spin effects are neglected.

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I Introduction and Survey of Results

The existence of an action principle is generally believed to be a very attractive feature of any (quantum) field theory, especially concerning the modern gauge field theories. The reason is that such an action principle does not only admit the deduction of the dynamical equations of the theory but it provides also many additional advantages, e.g. deduction of the conservation laws due to the symmetries of the theory (Noether theorem), canonical and path integral quantization, coupling to other fields, etc. (see, e.g., ref.s [1, 2]). In view of such a considerable relevance of the variational principles, it does not appear as a surprise that there is an extended literature concerning the mathematical structure of the variational principles themselves, e.g. ref.s [3, 4].

One important aspect of these variational principles refers to the possibility of using them for the construction of certain approximation techniques if exact solutions of the dynamical equations cannot easily be found which will be mostly the case. The present paper is also concerned with just this aspect of the variational principles, namely in the context of the energy eigenvalue problems emerging within the framework of Relativistic Schrödinger Theory (RST), a recently established theory of relativistic quantum matter [5, 6, 7]. More precisely, the intention of the paper aims at the construction of a principle of minimal energy for the bound RST field configurations so that the groundstate carries the minimally possible value of the total field energy $E_T$; and furthermore the excited states represent the stationary points ($\delta E_T = 0$) of this energy functional $E_T$. It should be rather evident that the existence of such a minimal-energy principle is of invaluable practical usefulness for
the calculation of the (relativistic) energy levels of the bound systems (i.e. atoms and molecules). The point here is that one is not necessarily forced to look for the exact solutions of the eigenvalue equations but instead one can resort to the invention of appropriate trial configurations of the RST fields with all those symmetries (and other properties) being owned by the unknown exact solution, too. The principle of minimal energy can then be used in order to find the trial configurations with minimal (or stationary) energy which mostly is equivalent to a purely algebraic problem, namely the determination of the ansatz parameters in the chosen trial functions.

For the sake of a simple demonstration and comparison with exactly solvable examples of the conventional theory, we select the positronium ground-state as a typical two-body problem. Restricting ourselves here to the non-relativistic situation, one can easily show that the choice of a simple trial function reproduces exactly the positronium groundstate energy as it is predicted by the conventional quantum mechanics.

These results are elaborated through the following sequence of intermediate steps:

1. **RST Eigenvalue Problem**

As the point of departure for the construction of the desired variational principle, one reconsiders the emergence of the typical energy eigenvalue problem within the framework of RST. **Sect. II** presents a brief sketch of the general theory for two oppositely charged particles such as, e.g., hydrogen atom or positronium. The important point here is that the subsequently defined energy functional $E_T$ can be based upon the generally accepted hy-
hypothesis of field energy which is concentrated in any relativistic field configuration with energy-momentum density \( T_{\mu\nu} \), see equation (II.53) below. This object \( T_{\mu\nu} \) may be deduced from the corresponding RST Lagrangean \( \mathcal{L}_{\text{RST}} \) (II.57) in the usual way, i.e. via the standard Noether theorem. Indeed, the existence of a Hamilton-Lagrange action principle (II.56a)-(II.56b) for RST is essential for the subsequent construction of the desired minimal-energy principle for the bound states since this action principle provides the possibility of introducing the Poisson identities which then work as constraints for the variational procedure (see the discussion of this point in the preceding paper [6]).

Next, the stationary bound states are introduced in Sect. III and lead immediately to the mass eigenvalue equations (III.15) and (III.17) for the time-independent Dirac spinors \( \psi_a(\vec{r}) \) of both particles \((a = 1, 2)\). The interesting point with these mass eigenvalue equations refers to the fact that they can not only be deduced from the original Hamilton-Lagrange action principle (II.56a)-(II.56b) by means of the stationary ansatz (III.1a)-(III.1b) for both Dirac spinors, cf. (III.14) and (III.16), but these mass eigenvalue equations can also be deduced from an appropriately constructed mass functional \( \tilde{M}_TC^2 \) (III.24). It is true, this mass functional can be understood to represent the immediate relativistic generalization of the well-known Hartree-Ritz variational principle (III.33a)-(III.33b) for factorized two-particle wave functions \( \Phi(\vec{r}_1, \vec{r}_2) \) (III.36), but nevertheless \( \tilde{M}_TC^2 \) cannot be accepted to represent our wanted energy functional \( \tilde{E}_T \) because this mass functional \( \tilde{M}_TC^2 \) suffers from the same deficiencies as the conventional Ritz principle; namely, in the latter approach the interaction between both particles is simply taken as
the rigid Coulomb interaction (see the conventional Hamiltonian $\hat{H}_S$ (III.34)) whereas it is well-known that the gauge field, as the mediator of the relativistic interactions, must of course be treated as a dynamical object obeying its own field equations (see the Maxwell equations (II.24)).

And furthermore, there is a second deficiency inherent in the Hartree-Ritz approach which refers to the conventional dogma that wave functions should always be unique and non-singular. However, the RST treatment of positronium in the preceding paper [6] has shown that non-unique wave functions of the type (III.60) must be admitted. Indeed, their use yields then a more realistic prediction of the positronium groundstate energy as compared to the Hartree-Ritz-Schrödinger approach, see ref. [6]. The conclusion from this is that in RST one must both admit a more general type of wave function and treat the particle interaction as a proper dynamical object. Otherwise one cannot expect to achieve a well-working principle of minimal energy which takes adequate account of both the matter and gauge fields!

2. Exotic Quantum States

The treatment of the positronium groundstate in the preceding paper [6] demonstrates that the minimal value of the RST energy functional $E_T$ cannot be reached by admitting exclusively these non-singular and unique wave functions as they are usually required by the Ritz-Hartree-Schrödinger approach in conventional quantum mechanics. For instance, the requirement of physical equivalence of both positronium constituents (i.e. positron and electron) entails that any of the two fermions has vanishing spin component along the z-direction (equation (III.61)) which is quite unusual for fermionic
particles. As a consequence of this unusual behavior of the fermions, several other taboos of conventional quantum theory become broken, too:

(i) The wave functions are singular at the origin \((r = 0)\) and along the whole z-axis from the very beginning, see equations \((\text{III.62a})-(\text{III.62b})\) below; but nevertheless these singularities do not spoil the normalization conditions in the relativistic sense \((\text{II.44})\).

(ii) The wave functions become non-unique, e.g. in the sense of equation \((\text{III.60})\); but the observable physical densities (of charge, current, energy-momentum etc.) generated by these ambiguous wave functions are still unique and physically well-behaved.

(iii) The interaction potentials, generated by the exotic states, are singular at the origin but less singular as the standard Coulomb potential so that their field energy is kept finite and can thus enter the wanted energy functional \(E_T\) without causing infinities, see equations \((\text{III.76})-(\text{III.77})\) below.

(iv) The magnetic moment carried by the bound matter fields amounts only to half a Bohr magneton \(\mu_B\), see the asymptotic form of the magnetic potential in equation \((\text{IV.48})\) below.

3. Positronium Groundstate

As a concrete demonstration, all this theoretical structure is evoked in order to calculate the positronium groundstate energy \(E_0\). First, in the absence of an exact solution to the corresponding RST eigenvalue problem (consisting of the coupled set of relativistic mass eigenvalue equations
One resorts to a self-suggesting variational technique based upon the constructed energy functional $\tilde{E}_T$ (IV.10) which, however, is applied in this paper only in its non-relativistic approximation $\tilde{E}_T^{(0)}$ (IV.26). This means that one has to guess a trial function as realistic as possible (see the simple exponential wave amplitude $\tilde{R}(r)$ (V.5)) and substitutes this into the energy functional $\tilde{E}_T^{(0)}$ (IV.26). This energy functional is additively composed of the kinetic energies $E_{\text{kin}}(a)$ of both particles (IV.20a)-(IV.20b) plus their electrostatic interaction energy $\tilde{E}_R^{(e)}$ (IV.22a); the magnetic interaction energy $\tilde{E}_R^{(m)}$ (IV.22b) is first neglected and treated afterwards as a small perturbation of the electric effects.

The energy functional $\tilde{E}_T^{(0)}$ (IV.26) contains also two constraints which have to be respected for the deduction of the mass eigenvalue and Poisson equations as the variational equations due to that functional (i.e. $\delta \tilde{E}_T^{(0)} = 0$). The first constraint refers to the wave function normalization (as shown, e.g., by equation (IV.47)) and is automatically satisfied by our ansatz (V.9a). However, the second constraint refers to the electric Poisson identities, such as (III.58), and requires a more subtle argument: if one wishes to have the Poisson constraints also automatically satisfied by the trial functions one first has to solve the corresponding Poisson equations; i.e. equation (III.72) for the present situation. But if all constraints are thus satisfied automatically by our trial function $\tilde{R}(r)$, one substitutes this into the energy functional $\tilde{E}_T^{(0)}$ (IV.26) and obtains an ordinary function $\tilde{E}_T^{(0)}(r_*)$ of the ansatz parameter $r_*$, i.e. equation (V.7) which according to the principle of minimal energy adopts the groundstate energy $E_0$ (V.8) as its minimal value. This just coincides
with the corresponding prediction (V.1) of conventional quantum mechanics. However, observe here that this groundstate energy $E_0$ (V.1) owns the status of exactness within the framework of the conventional theory, whereas in RST it appears as an approximation (even if all the relativistic effects including magnetism are disregarded) since our trial function $\tilde{R}(r)$ (V.5) is surely not the exact solution of the non-relativistic RST eigenvalue problem in the electrostatic approximation.

Finally, the magnetic interaction energy $\hat{E}_R^{(m)}$ (IV.56) is estimated in the lowest-order of approximation, equation (V.16) below. It turns out that the RST prediction for the hyperfine splitting of the positronium ground-states $^1S_0$ and $^3S_1$ amounts to only $1.47 \cdot 10^{-4}[eV]$, whereas the experimental value is $8.41 \cdot 10^{-4}[eV]$ [8]. Thus this lowest-order RST prediction shows that for the hyperfine splitting one needs a better trial function.
II Two-Fermion Systems in RST

In order to introduce the relevant notation, a brief sketch of the general two-particle theory is presented first so that the characteristic dynamical structure becomes obvious: matter dynamics, Hamiltonian dynamics, gauge field dynamics, *action principle*, and the associated conservation laws (for a more detailed presentation of the RST dynamics, see the preceding papers [5]-[7]). It is true, the existence of an action principle is common to almost all of the successful field theories, but a pleasant feature of the present RST dynamics refers to the fact that its action principle can be converted to a *principle of minimal energy* for the bound systems. This will subsequently be exploited in order to compute approximately the positronium groundstate energy.

A. Matter Dynamics

The central equation of motion for matter is the *Relativistic Schrödinger Equation* (RSE)

\[ i\hbar c D_\mu \Psi = \mathcal{H}_\mu \Psi , \]  

(II.1)

or if matter is to be described by an intensity matrix \( I \) in place of a pure state \( \Psi \), one applies the *Relativistic von Neumann Equation* (RNE)

\[ D_\mu I = \frac{i}{\hbar c} (I \mathcal{H}_\mu - \mathcal{H}_\mu I) . \]  

(II.2)

In the present paper, we will exclusively deal with pure two-particle states \( \Psi \) which in RST are always the direct (Whitney) sum of the one-particle states \( \psi_a \ (a = 1, 2) \), i.e.

\[ \Psi(x) = \psi_1(x) \oplus \psi_2(x) . \]  

(II.3)
Here the one-particle states $\psi_a(x)$ are four-component Dirac spinor fields so that the two-particle wave function $\Psi(x)$ may be understood as a section of a complex vector bundle over space-time as the base space with typical fibre $\mathbb{C}^8$.

Both particles are interacting with each other via the principle of minimal coupling, i.e. the gauge-covariant derivative in the RSE (II.1) is defined by means of the gauge potential $A_\mu$ (bundle connection) in the usual way as

$$D_\mu \Psi = \partial_\mu \Psi + A_\mu \Psi ,$$

or, resp., in component form

$$D_\mu \Psi = (D_\mu \psi_1) \oplus (D_\mu \psi_2) .$$

Here the gauge-covariant derivatives of the one-particle states $\psi_a(x)$ are given by

$$D_\mu \psi_1 = \partial_\mu \psi_1 - iA_2^1 \psi_1 - iB_\mu \psi_2$$

$$D_\mu \psi_2 = \partial_\mu \psi_2 - iA_1^1 \psi_2 - iB_\mu \psi_1 ,$$

provided the bundle connection $A_\mu$ takes its values in the four-dimensional Lie algebra $U(2)$ of the unitary group $U(2)$ (structure group) and is decomposed with respect to a suitable basis of generators $\{\tau_a, \chi, \bar{\chi}\}$ as follows:

$$A_\mu = \sum_{a=1}^2 A^a_\mu \tau_a + B_\mu \chi - \bar{B}_\mu \bar{\chi} .$$

The (real-valued) electromagnetic potentials $A_\mu^a$ do mediate the electromagnetic interactions between both particles; and similarly the (complex-valued) exchange potentials $B_\mu$ do mediate the exchange interactions which thus are
treated in RST as real forces on the same footing as their electromagnetic counterparts. However the exchange forces (due to $B_\mu$) can be active exclusively among identical particles and must vanish ($B_\mu \equiv 0$) for non-identical particles (see refs.s [5, 7]). Since we restrict ourselves in the present paper to a system of two oppositely charged particles with different or identical masses $M_p$ and $M_e$, resp., the exchange forces must therefore be zero and consequently the covariant derivatives (II.6a)-(II.6b) simplify to

\begin{align}
D_\mu \psi_1 &= \partial_\mu \psi_1 - iA^2_\mu \psi_1 \quad \text{(II.8a)} \\
D_\mu \psi_2 &= \partial_\mu \psi_2 - iA^1_\mu \psi_2. \quad \text{(II.8b)}
\end{align}
B. Hamiltonian Dynamics

The Hamiltonian $\mathcal{H}_\mu$, occurring in the RSE (II.1) or in the RNE (II.2), takes its values in the general linear algebra $GL(2, \mathbb{C})$ and is itself a dynamical object which is to be determined from its field equations, i.e. the integrability condition

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{i}{\hbar c} [\mathcal{H}_\mu, \mathcal{H}_\nu] = i\hbar c F_{\mu\nu}$$  \hspace{1cm} (II.9)\)

and the conservation equation

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \mathcal{H}_\mu = -i\hbar c \left[ \left( \frac{\mathcal{M} \hbar c}{\hbar} \right)^2 + \Sigma_{\mu\nu} F_{\mu\nu} \right].$$  \hspace{1cm} (II.10)\)

The integrability condition (II.9) contains the curvature $F_{\mu\nu}$ of the bundle connection $A_\mu$ (II.7)

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]$$  \hspace{1cm} (II.11)\)

and guarantees the validity of the bundle identities

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu] \Psi = F_{\mu\nu} \Psi$$  \hspace{1cm} (II.12a)\)

$$[\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu] I = [F_{\mu\nu}, I].$$  \hspace{1cm} (II.12b)\)

The conservation equation (II.10) contains the mass operator $\mathcal{M}$ and the Spin(1,3) generators $\Sigma_{\mu\nu}$

$$\Sigma_{\mu\nu} = \frac{1}{4} [\Gamma_\mu, \Gamma_\nu]$$  \hspace{1cm} (II.13)\)

which both are assumed to be covariantly constant

$$\mathcal{D}_\mu \mathcal{M} \equiv 0$$  \hspace{1cm} (II.14a)\)

$$\mathcal{D}_\mu \Sigma_{\lambda\nu} \equiv 0 .$$  \hspace{1cm} (II.14b)\)
The latter constancy condition (II.14b) may be traced back to the covariant constancy of the total velocity operator $\Pi_\mu$

$$\mathcal{D}_\lambda \Pi_\mu \equiv 0 , \quad (\text{II.15})$$

where $\Pi_\mu$ are the direct sum of the one-particle Dirac matrices $\gamma_\mu$

$$\Pi_\mu = (-\gamma_\mu) \oplus \gamma_\mu \quad (\text{II.16})$$

and therefore can be taken as the generators of the required eight-dimensional representation of the Clifford algebra $\mathbb{C}(1,3)$, i.e.

$$\Pi_\mu \Pi_\nu + \Pi_\nu \Pi_\mu = 2 g_{\mu\nu} \cdot 1_{(8)} . \quad (\text{II.17})$$

Observe here that the arrangement of the plus and minus signs in the direct sum (II.16) displays the opposition of both particle charges (positive charge of the first particle and negative charge of the second particle, by convention).

The conservation equation (II.10) is needed for the deduction of the conservation laws from the RST dynamics (see below) and admits an equivalent algebraic formulation:

$$\Pi^\mu \mathcal{H}_\mu = \mathcal{H}_\mu \Pi^\mu = \mathcal{M} c^2 . \quad (\text{II.18})$$

This can be used in order to eliminate the Hamiltonian $\mathcal{H}_\mu$ by recasting the RSE (II.1) into the two-particle Dirac Equation (DE)

$$i \hbar \Pi^\mu D_\mu \Psi = \mathcal{M} c \Psi . \quad (\text{II.19})$$

In component form, this equation reads

$$i \hbar \gamma^\mu D_\mu \psi_1 = - M_p c \psi_1 \quad (\text{II.20a})$$

$$i \hbar \gamma^\mu D_\mu \psi_2 = M_e c \psi_2 , \quad (\text{II.20b})$$
provided the (covariantly constant) mass operator $\mathcal{M}$ is written as

$$\mathcal{M} = i \sum_{a=1}^{2} M^a \tau_a$$  \hspace{1cm} (II.21)

with

$$M^1 \equiv M_e \hspace{1cm} (II.22a)$$

$$M^2 \equiv M_p \hspace{1cm} (II.22b)$$

where $M_p$ and $M_e$ are denoting the rest mass of the positively and negatively charged particle, resp. For the case of pure states, one can eliminate the Hamiltonian $H_\mu$ also by differentiating once more the RSE (II.1) and substituting therein the derivative of $H_\mu$ from the original conservation equation (II.10) which yields a second-order equation of the Klein-Gordon type (KGE):

$$D^\mu D_\mu \Psi + \left( \frac{M_c}{\hbar} \right)^2 \Psi = -\Sigma^{\mu\nu} F_{\mu\nu} \Psi \hspace{1cm} (II.23)$$

However, subsequently we will prefer to deal with the first order equation (II.19).
C. Gauge Field Dynamics

In order to close the RST dynamics, one finally has to specify some field equation for the bundle connection $A_\mu$. Our choice is the non-Abelian Maxwell equation

$$D^\mu F_{\mu\nu} = -4\pi i\alpha_s J_\nu$$

(II.24)

where the current operator $J_\mu$ may be thought to decompose with respect to the structure algebra basis $\{\tau_\alpha, \alpha = 1 \ldots 4\} = \{\tau_a, \chi, \bar{\chi}\}$ as follows:

$$J_\mu = ij^\alpha_\mu \tau_\alpha = i \left(j^1_\mu \tau_1 + j^2_\mu \tau_2 + g_\mu \chi - g^*_\mu \bar{\chi}\right).$$

(II.25)

Here, the Maxwell currents $j^a_\mu (a = 1, 2)$ generate the electromagnetic potentials $A^a_\mu$ (II.7) which is seen by explicitly writing down the electromagnetic part of the general Maxwell equations (II.24) in component form ($a = 1, 2$)

$$D^\mu F^a_{\mu\nu} = 4\pi\alpha_s j^a_\nu.$$  

(II.26)

The exchange currents $j^3_\mu \doteq g_\mu$ and $j^4_\mu \doteq -g^*_\mu$ do generate the exchange potentials $B_\mu$ and $\bar{B}_\mu$ (II.7); but since we are dealing here exclusively with non-identical particles the exchange potentials $B_\mu, \bar{B}_\mu$ must be put to zero so that the Maxwell equations (II.26) become Abelian:

$$\nabla^\mu F^1_{\mu\nu} = 4\pi\alpha_s j^1_\nu.$$  

(II.27a)

$$\nabla^\mu F^2_{\mu\nu} = 4\pi\alpha_s j^2_\nu.$$  

(II.27b)

The formal reason for this is that the bundle curvature $A_\mu$ (II.7) and its curvature $F_{\mu\nu}$ (II.11)

$$F_{\mu\nu} = \sum_{a=1}^2 F^a_{\mu\nu} \tau_a + G_{\mu\nu} \chi - G^*_{\mu\nu} \bar{\chi}$$

(II.28)
become projected onto the Abelian subalgebra $U(1) \oplus U(1)$ when the exchange fields $B_\mu, G_{\mu\nu}$ are put to zero.

### D. Conservation Laws

The right choice of the gauge field dynamics is not a trivial thing because it must be compatible with the already fixed matter dynamics (for both the pure states and the mixtures). However, this desired compatibility of our choice can be verified in the general case by the following arguments: First, the generally valid bundle identity

$$D^\mu D^\nu F_{\mu\nu} \equiv 0,$$

(II.29)

when applied to the Maxwell equations (II.24), yields the following source equation for the current operator $J_\mu$

$$D^\mu J_\mu \equiv 0,$$

(II.30)

or in component form

$$D^\mu j^\alpha_\mu \equiv 0.$$  

(II.31)

This means that the two-particle Maxwell currents $j^\alpha_\mu (\alpha = 1, \ldots 4)$ must be constructed in terms of the two-particle wave function $\Psi$ in such a way that the covariant source equations (II.31) do actually hold just as a consequence of the RST dynamics!

This compatibility requirement can be satisfied by first constructing the RST currents $j_{\alpha\mu}$ through

$$j_{\alpha\mu} = \bar{\Psi} v_{\alpha\mu} \Psi,$$

(II.32)
with the velocity operators $v_{\alpha \mu}$ being defined through the following anticommutators

$$v_{\alpha \mu} = \frac{i}{2} \{ \tau_\alpha, \Gamma_\mu \} .$$  \hspace{1cm} (II.33)

Indeed, one can easily show that these RST currents $j_{\alpha \mu}$ (II.32) do obey the source equations

$$D_\mu j_{\alpha \mu} \equiv 0 ,$$  \hspace{1cm} (II.34)

provided the wave function $\Psi$ (or intensity matrix $\mathcal{I}$, resp.) does satisfy the RSE (II.1) (or the RNE (II.2), resp.). However, observe here that the RST currents $j_{\alpha \mu}$ (II.32) cannot a priori identified with the Maxwell currents $j^\alpha_\mu$ (II.25) generating the gauge potentials $A^\alpha_\mu$ via the Maxwell equations (II.26). Consequently, there must be established some link between the Maxwell currents $j^\alpha_\mu$ and RST currents $j_{\alpha \mu}$ in such a way that both source equations (II.31) and (II.34) are simultaneously valid! This requirement can be satisfied by conceiving $j^\alpha_\mu$ and $j_{\alpha \mu}$ as contra- and covariant versions of one and the same object; namely by introducing a covariantly constant fibre metric $K_{\alpha \beta}$ for the associated Lie algebra bundle

$$D_\lambda K_{\alpha \beta} \equiv 0 ,$$  \hspace{1cm} (II.35)

and then putting

$$j^\alpha_\mu = K^{\alpha \beta} j_{\beta \mu} ,$$

$$j_{\alpha \mu} = K_{\alpha \beta} j^\beta_\mu .$$  \hspace{1cm} (II.36a)

$$K_{\alpha \beta} = C_1 \, \text{tr} \tau_\alpha \cdot \text{tr} \tau_\beta + C_2 \, \text{tr} (\tau_\alpha \cdot \tau_\beta) ,$$  \hspace{1cm} (II.37)

Actually, such a compatibility tensor $K_{\alpha \beta}$ can be found:
where \( C_1 \) and \( C_2 \) are constants which have to be chosen in such a way that the following constraint for the currents holds:

\[
\sum_{a=1}^{2} j^a_{\mu} = - \sum_{a=1}^{2} j_{a\mu} \equiv - j_{\mu} .
\]  

(II.38)

Here the total current \( j_{\mu} \) of the two-particle system appears as the sum of the Maxwell (or RST) currents and acts as the source of the total electromagnetic field \( F_{\mu\nu} \):

\[
F_{\mu\nu} \equiv F_{1\mu\nu} + F_{2\mu\nu} ,
\]

(II.39)
i.e. one easily deduces from the Abelian Maxwell equations (II.27a)-(II.27b) the total Maxwell equation

\[
\nabla^\mu F_{\mu\nu} = -4\pi\alpha_s j_\nu .
\]

(II.40)

Moreover, an immediate consequence of this Maxwell equation is the continuity equation for the total current \( j_{\mu} \):

\[
\nabla^\mu j_{\mu} \equiv 0 .
\]

(II.41)

Therefore the total charge \( z \) may be defined through

\[
z = \int_{(S)} j_{\mu} dS^\mu ,
\]

(II.42)

which is independent of the chosen hypersurface \( (S) \) but must of course be zero because we are dealing with opposite charges. This may be realized more clearly by expressing the RST currents \( j_{a\mu} \) through the Dirac currents \( k_{a\mu} \)

\[
j_{1\mu} = k_{2\mu} \equiv \bar{\psi}_2 \gamma_{\mu} \psi_2
\]

(II.43a)

\[
j_{2\mu} = -k_{1\mu} \equiv -\bar{\psi}_1 \gamma_{\mu} \psi_1 .
\]

(II.43b)
Thus, since anyone of the two particles is assumed to carry just one charge unit, one will apply the following normalization of the wave functions for the stationary bound states \((a = 1, 2)\)

\[
\int d^3 \vec{r} \,(^a k_0 (\vec{r})) = 1 ,
\] (II.44)

where the hypersurface \((S)\) in (II.42) is taken as a time slice \((t = \text{const.})\) of space-time; and the stationary form of the Dirac currents \(k_{a\mu}\) (II.43a)-(II.43b) is of course

\[
k_{a\mu}(x) = \left(\,(^a k_0 (\vec{r}) ; -\vec{k}_a(\vec{r})\right) .
\] (II.45)

Obviously the total charge \(z\) (II.42) becomes actually zero, namely by simply observing the sum requirement (II.38) and applying the normalization conditions (II.44) together with the relationship (II.43a)-(II.43b) between the RST and Dirac currents.

It is very instructive to consider the local charge conservation (II.41) also from an other viewpoint: The total current \(j_\mu\) may be defined alternatively through

\[
j_\mu = \bar{\Psi} \Gamma_\mu \Psi .
\] (II.46)

Carrying here out the differentiation process (II.41) and using the Dirac equation (II.19) together with the covariant constancy of the total velocity operator \(\Gamma_\mu\) (II.15) actually yields just the total charge conservation law (II.41).

A similar procedure does apply also to the local energy-momentum conservation

\[
\nabla^\mu (T) T^\nu_{\mu} \equiv 0 ,
\] (II.47)

where \((T) T^\nu_{\mu}\) is the total energy-momentum density of the field configuration
and is composed of a matter part \((D)T_{\mu\nu}\) and a gauge field part \((G)T_{\mu\nu}\)

\[
(T)T_{\mu\nu} = (D)T_{\mu\nu} + (G)T_{\mu\nu} .
\] (II.48)

The interesting point here is that the individual sources of the partial densities \((D)T_{\mu\nu}\) and \((G)T_{\mu\nu}\) turn out as the Lorentz forces which are mutually annihilating:

\[
\nabla^\mu (D)T_{\mu\nu} = -\nabla^\mu (G)T_{\mu\nu} = \hbar c F_\alpha^{\mu\nu} j_\alpha
\] (II.49)

so that the local law (II.47) can be true. However the crucial condition for this pleasant result is, that the RST dynamics (i.e. matter and gauge field dynamics) is chosen as described above and that the partial densities are defined as follows:

\(\text{(G)} T_{\mu\nu} = \frac{\hbar c}{4\pi\alpha_s} K_{\alpha\beta} \left( F_\alpha^{\mu\lambda} F^{\beta\lambda}_{\nu} - \frac{1}{4} g_{\mu\nu} F_\sigma^{\alpha\beta} F^{\beta\sigma\lambda} \right)\)  \hspace{1cm} (II.50a)

\(\text{(D)} T_{\mu\nu} = \bar{\Psi} T_{\mu\nu} \Psi\), \hspace{1cm} (II.50b)

with the energy-momentum operator \(T_{\mu\nu}\) being given in terms of the Hamiltonian \(H_\mu\) and total velocity operator \(\Gamma_\mu\) as

\[
T_{\mu\nu} = \frac{1}{4} \left( \Gamma_\mu H_\nu + \bar{H}_\nu \Gamma_\mu + \Gamma_\nu H_\mu + \bar{H}_\mu \Gamma_\nu \right) .
\] (II.51)

If matter can be described by a pure state \(\Psi\) (in place of a mixture) so that the matter density is given in terms of \(\Psi\) by (II.50b), then the Hamiltonian \(H_\mu\) can again be eliminated from the matter density \((D)T_{\mu\nu}\) by means of the DE (II.19) which yields

\[\text{(D)} T_{\mu\nu} = \frac{i\hbar c}{4} \left[ \bar{\Psi} \Gamma_\mu (D_\nu \Psi) - (D_\nu \bar{\Psi}) \Gamma_\mu \Psi + \bar{\Psi} \Gamma_\nu (D_\mu \Psi) - (D_\mu \bar{\Psi}) \Gamma_\nu \Psi \right] .
\] (II.52)
Clearly, the energy-momentum density $(^{(T)}T_{\mu\nu})$ is the crucial object for testing the practical usefulness of the theory, because the corresponding energy content $E_T$ of the field configuration is given by the spatial integral of the time component $(^{(T)}T_{00}(\vec{r}))$, i.e.

$$E_T = \int d^3\vec{r} (^{(T)}T_{00}(\vec{r}))$$  \hspace{1cm} (II.53)

Since the density $(^{(T)}T_{\mu\nu})$ appears as the sum of a matter and gauge field part, the same must hold also for the total energy $E_T$ (II.53)

$$E_T = E_D + E_G$$  \hspace{1cm} (II.54)

with the individual contributions being defined in a self-evident way as

$$E_D = \int d^3\vec{r} (^{(D)}T_{00}(\vec{r}))$$  \hspace{1cm} (II.55a)

$$E_G = \int d^3\vec{r} (^{(G)}T_{00}(\vec{r}))$$  \hspace{1cm} (II.55b)

Subsequently we will clarify the question whether for the groundstate of the stationary two-particle systems the energy functional (II.53) adopts its minimally possible value ($\Rightarrow$ principle of minimal energy).

\textbf{E. Action Principle}

The conservation laws for charge (II.41) and energy-momentum (II.47) can be directly deduced from the general RST dynamics, but a more elegant method is provided by the Noether theorem \cite{9}. For the latter method one needs an action principle

$$\delta W_{\text{RST}} = 0$$  \hspace{1cm} (II.56a)

$$W_{\text{RST}} = \int d^4x L_{\text{RST}}[\Psi, A_\mu]$$  \hspace{1cm} (II.56b)
from which both the matter dynamics (II.19) and the gauge field dynamics (II.24) may be deduced by the usual variational methods. The corresponding RST Lagrangean $\mathcal{L}_{\text{RST}}$ has been specified as a sum of the matter part $\mathcal{L}_D$ and gauge field part $\mathcal{L}_G$ \[ \mathcal{L}_{\text{RST}}[\Psi, A_{\mu}] = \mathcal{L}_D[\Psi] + \mathcal{L}_G[A_{\mu}] , \] (II.57)

where the matter part is given by

$$
\mathcal{L}_D[\Psi] = \frac{i\hbar c}{2} \left[ \bar{\Psi} \gamma^\mu (D_\mu \Psi) - (D_\mu \bar{\Psi}) \gamma^\mu \Psi \right] - \bar{\Psi} \mathcal{M} c^2 \Psi
$$

(ii.58)

and the gauge field part by

$$
\mathcal{L}_G[A_{\mu}] = \frac{\hbar c}{16 \pi \alpha_s} K_{\alpha\beta} F_{\mu\nu}^\alpha F_{\mu\nu}^\beta
$$

(ii.59)

Concerning the latter part (ii.59), observe here that the bundle curvature $F_{\mu\nu}$ (II.11) takes its values in the subalgebra $\mathcal{U}(1) \oplus \mathcal{U}(1)$ because we are dealing with non-identical particles; and thus the gauge field Lagrangean becomes reduced to

$$
\mathcal{L}_G[A_{\mu}] = \frac{\hbar c}{16 \pi \alpha_s} \sum_{a,b=1}^2 K_{ab} F_{\mu\nu}^a F_{\mu\nu}^b
$$

(ii.60)

If the self-interactions are neglected, the fibre submetric $K_{ab}$ is of a very simple shape [5]

$$
\{K_{ab}\} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
$$

(ii.61)

Thus, from the formal point of view, $\mathcal{L}_G[A_{\mu}]$ describes the interaction of the two gauge field modes $F_{\mu\nu}^a (a, b = 1, 2)$, i.e.

$$
\mathcal{L}_G[A_{\mu}] = \frac{\hbar c}{4 \pi \alpha_s} \left( \vec{E}_1 \cdot \vec{E}_2 - \vec{H}_1 \cdot \vec{H}_2 \right)
$$

(ii.62)

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provided the four-tensors $F^a_{\mu\nu}$ are splitted into their space and time components $\vec{E}_a, \vec{H}_a$ as usual

$$\vec{E}_a = \{(a) E^j\} \div \{F^a_{0j}\} \quad \text{(II.63a)}$$

$$\vec{H}_a = \{(a) H^j\} \div \left\{ \frac{1}{2} \varepsilon^{ijk} F^a_{k\ell} \right\} \quad \text{(II.63b)}$$

But once the Lagrangean has been specified, it is an easy exercise to deduce both the RST matter dynamics (II.19) and the gauge field dynamics (II.24) from the action principle (II.56a)-(II.56b) as the corresponding Euler-Lagrange equations. Furthermore, the considered conservation laws of charge (II.30) and energy-momentum (II.47) are just those which are predicted by the Noether formalism, see ref. [9].
III Stationary Bound Systems

In order to make the proposed exercise with the Euler-Lagrange equations somewhat more instructive and fruitful, one may immediately pass over to the stationary systems which are defined through the usual product ansatz for the wave functions

$$\psi_1(\vec{r}, t) = \exp\left(-i\frac{M_1 c^2}{\hbar} t\right) \cdot \psi_1(\vec{r}) \quad (\text{III.1a})$$

$$\psi_2(\vec{r}, t) = \exp\left(-i\frac{M_2 c^2}{\hbar} t\right) \cdot \psi_2(\vec{r}), \quad (\text{III.1b})$$

whereas the electromagnetic potentials become time-independent:

$$A_{\alpha \mu} = \left\{ (^{(a)}A_0(\vec{r}); -\vec{A}_a(r) \right\} \quad (\text{III.2})$$

This time-independence obviously does then apply also for the Dirac currents $$k_{\alpha \mu} \quad (\text{II.43a})-(\text{II.43b})$$, see equation (\text{II.45}).

A. Mass Eigenvalue Equations

The mass eigenvalues $$M_a \ (a = 1, 2)$$, occurring in the stationary ansatz (\text{III.1a})-(\text{III.1b}), must be determined through solving the stationary form of the matter dynamics (\text{II.20a})-(\text{II.20b}). This stationary form (“mass eigenvalue equations”) may be obtained either by direct substitution of the stationary ansatz (\text{III.1a})-(\text{III.1b}) into the coupled Dirac equations (\text{II.20a})-(\text{II.20b}), or by substitution of that ansatz into the matter Lagrangean $$\mathcal{L}_D[\Psi] \quad (\text{II.58})$$ and then carrying out the variational procedure with respect to the spatial
parts $\psi_a(\vec{r})$ of the wave functions. Here it is easy to see that the matter Lagrangean $\mathcal{L}_D[\Psi]$ splits up into a sum of the single-particle contributions $\mathcal{L}_D(a)$, i.e.

$$\mathcal{L}_D[\Psi] = \mathcal{L}_D(1) + \mathcal{L}_D(2), \quad (\text{III.3})$$

with the individual contributions being given by

$$\mathcal{L}_D(1) = -i\hbar c \bar{\psi}_1 \gamma^\mu (D_\mu \psi_1) - M_p c^2 \bar{\psi}_1 \psi_1 \quad (\text{III.4a})$$

$$\mathcal{L}_D(2) = i\hbar c \bar{\psi}_2 \gamma^\mu (D_\mu \psi_2) - M_e c^2 \bar{\psi}_2 \psi_2. \quad (\text{III.4b})$$

However, observe here that these matter contributions to the Lagrangean $\mathcal{L}_{\text{RST}}$ do nevertheless contain the gauge fields which invade the matter Lagrangean via the gauge-covariant derivatives $D_\mu \psi_a$ (II.8a)-(II.8b). Therefore the matter contributions themselves split up into the proper kinetic parts $\mathcal{L}_D^{(\text{kin})}(a)$ and the electric plus magnetic interaction contributions in the following way:

$$\mathcal{L}_D(a) = \mathcal{L}_D^{(\text{kin})}(a) + \mathcal{L}_D^{(e)}(a) + \mathcal{L}_D^{(m)}(a) + \mathcal{L}_D^{(M)}(a), \quad (\text{III.5})$$

i.e. for the first (positively charged) particle

$$\mathcal{L}_D^{(\text{kin})}(1) = -i\hbar c \bar{\psi}_1(\vec{r}) \gamma^\mu \vec{\nabla} \psi_1(\vec{r}) \quad (\text{III.6a})$$

$$\mathcal{L}_D^{(e)}(1) = -\hbar c \bar{A}_0(\vec{r}) \cdot \vec{k}_0(\vec{r}) \quad (\text{III.6b})$$

$$\mathcal{L}_D^{(m)}(1) = \hbar c \vec{A}_2(\vec{r}) \cdot \vec{k}_1(\vec{r}) \quad (\text{III.6c})$$

$$\mathcal{L}_D^{(M)}(1) = -M_1 c^2 \cdot \vec{k}_0(\vec{r}) - M_p c^2 \cdot \bar{\psi}_1(\vec{r}) \psi_1(\vec{r}) \quad (\text{III.6d})$$
and similarly for the second (negatively charged) particle

\[ \mathcal{L}_{D}^{(\text{kin})}(2) = i\hbar c \bar{\psi}_2(\vec{r}) \vec{\nabla} \psi_2(\vec{r}) \]

\[ \mathcal{L}_{D}^{(e)}(2) = \hbar c (1) A_0(\vec{r}) \cdot (2) k_0(\vec{r}) \]

\[ \mathcal{L}_{D}^{(m)}(2) = -\hbar c \vec{A}_1(\vec{r}) \cdot \vec{k}_2(\vec{r}) \]

\[ \mathcal{L}_{D}^{(M)}(2) = M_2 c^2 \cdot (2) k_0(\vec{r}) - M_e c^2 \cdot \bar{\psi}_2(\vec{r}) \psi_2(\vec{r}) \]

(III.7a)

(III.7b)

(III.7c)

(III.7d)

From this reason, the space part (\( \tilde{W}_D \), say)

\[ \tilde{W}_D(a) \overset{\dagger}{\propto} \int d^3 \vec{r} \mathcal{L}_D(a) \]

(III.8)

of the matter contribution to the action integral \( W_{\text{RST}} \) (II.56b) is not only built up by the kinetic and rest mass terms but contains also the electric (e) and magnetic (m) mass equivalents \( M_{I/I}^{(e,m)} c^2 \) of the gauge field energy, i.e. for the first particle

\[ \tilde{W}_D(1) = -Z_1^2 \cdot M_p c^2 - M_1 c^2 \int d^3 \vec{r} (1) k_0(\vec{r}) + \int d^3 \vec{r} \mathcal{L}_{D}^{(\text{kin})}(1) - M_{I}^{(e)} c^2 - M_{I}^{(m)} c^2 \]

(III.9)

and similarly for the second particle

\[ \tilde{W}_D(2) = -Z_2^2 \cdot M_e c^2 + M_2 c^2 \int d^3 \vec{r} (2) k_0(\vec{r}) + \int d^3 \vec{r} \mathcal{L}_{D}^{(\text{kin})}(2) - M_{II}^{(e)} c^2 - M_{II}^{(m)} c^2 \]

(III.10)

Here the mass renormalization factors \( Z_2^2 \) are defined through \( (a = 1, 2) \)

\[ Z_2^2(a) = \int d^3 \vec{r} \bar{\psi}_a(\vec{r}) \psi_a(\vec{r}) \]

(III.11)

Furthermore the electric mass equivalents of the interaction energy appear as

\[ M_{I}^{(e)} c^2 = \hbar c \int d^3 \vec{r} (2) A_0(\vec{r}) \cdot (1) k_0(\vec{r}) \]

(III.12a)

\[ M_{II}^{(e)} c^2 = -\hbar c \int d^3 \vec{r} (1) A_0(\vec{r}) \cdot (2) k_0(\vec{r}) \]

(III.12b)
and analogously for the magnetic mass equivalents

\[ M^{(m)}_1 c^2 = -\hbar c \int d^3\vec{r} \vec{A}_2(\vec{r}) \cdot \vec{k}_1(\vec{r}) \]  \hfill (III.13a)

\[ M^{(m)}_2 c^2 = \hbar c \int d^3\vec{r} \vec{A}_1(\vec{r}) \cdot \vec{k}_2(\vec{r}) \]  \hfill (III.13b)

But with this explicit structure of the matter Lagrangean \( L_D[\Psi] \) (III.3) it is easy to see that the variation of the action integral \( W_{\text{RST}} \) (II.56a)-(II.56b) with respect to the first wave function \( \psi_1 \)

\[ \delta (1) W_{\text{RST}} = \delta (1) W_D(1) = 0 \]  \hfill (III.14)

yields just the first mass eigenvalue equation

\[ i\vec{\gamma} \cdot \vec{\nabla} \psi_1(\vec{r}) + (2) A_0(\vec{r}) \gamma^0 \psi_1(\vec{r}) - \vec{A}_2(\vec{r}) \cdot \vec{\gamma} \psi_1(\vec{r}) = - \left( \frac{M_1 c}{\hbar} \gamma_0 + \frac{M_p c}{\hbar} \right) \cdot \psi_1(\vec{r}) \]  \hfill (III.15)

and similarly the variation of \( W_{\text{RST}} \) with respect to the second wave function \( \psi_2 \)

\[ \delta (2) W_{\text{RST}} = \delta (2) W_D(2) = 0 \]  \hfill (III.16)

yields the second mass eigenvalue equation

\[ i\vec{\gamma} \cdot \vec{\nabla} \psi_2(\vec{r}) + (1) A_0(\vec{r}) \gamma^0 \psi_2(\vec{r}) - \vec{A}_1(\vec{r}) \cdot \vec{\gamma} \psi_2(\vec{r}) = - \left( \frac{M_2 c}{\hbar} \gamma_0 - \frac{M_e c}{\hbar} \right) \cdot \psi_2(\vec{r}) \]  \hfill (III.17)

\section*{B. Mass Functional}

An interesting property of the mass eigenvalue equations (III.15) and (III.17) refers to the fact that they are linear with respect to the wave functions \( \psi_a \). This implies that one can multiply any solution \( \psi_a(\vec{r}) \) by some
constant and then obtains a further solution, which necessarily can not modify the value of the matter functionals $\tilde{W}_D(a)$ upon the whole class of such solutions. On the other hand, the matter functionals $\tilde{W}_D(a)$ (III.8) are bilinear with respect to the wave functions and their (pseudo-) Hermitian conjugates $\bar{\psi}_a$; and from this one concludes that $\tilde{W}_D(a)$ must take the value zero upon the solutions of the mass eigenvalue equations ($a = 1, 2$):

$$\tilde{W}_D(a) = 0 .$$  \hspace{1cm} (III.18)

This is the reason why one is not forced to impose some normalization condition upon the wave functions $\psi_a(\vec{r})$ when one deduces the mass eigenvalue equations as the Euler-Lagrange equations due to the RST variational principle (II.56a)-(II.56b).

However, it is just this property (III.18) of the matter functionals $\tilde{W}_D(a)$ which enables one to resolve these equations for the mass eigenvalues $M_a$, where it is convenient to apply the normalization conditions (II.44) for the wave functions $\psi_a(\vec{r})$; and these mass functionals ($M_a c^2$, say) emerge then in the following form:

$$-M_{[1]} c^2 = \mathcal{Z}_{(1)}^2 M_p c^2 + 2T_{\text{kin}(1)} + M_{[1]}^{(e)} c^2 + M_{[1]}^{(m)} c^2 \quad (\text{III.19a})$$

$$M_{[2]} c^2 = \mathcal{Z}_{(2)}^2 M_e c^2 + 2T_{\text{kin}(2)} + M_{[2]}^{(e)} c^2 + M_{[2]}^{(m)} c^2 . \quad (\text{III.19b})$$

Here the mass renormalization factors $\mathcal{Z}_{(a)}^2$ have already been specified by equation (III.11) and the mass equivalents of the interaction energies by (III.12a)-(III.13b). The emergence of a pre-factor of two in front of the
kinetic energies $T_{\text{kin}}(a)$, being given by

$$T_{\text{kin}(1)} = \frac{i}{2} \hbar c \int d^3\vec{r} \bar{\psi}_1 \vec{\gamma} \cdot \vec{\nabla} \psi_1(\vec{r}) \quad (\text{III.20a})$$

$$T_{\text{kin}(2)} = -\frac{i}{2} \hbar c \int d^3\vec{r} \bar{\psi}_2 \vec{\gamma} \cdot \vec{\nabla} \psi_2(\vec{r}) , \quad (\text{III.20b})$$

is a relativistic effect and is compensated by the mass renormalization factors $Z_2^{(a)}$ (see the discussion of this effect in ref.s [6, 10]). But clearly, in place of resolving the equations (III.18) for the mass eigenvalues $M_a$ in order to obtain the mass functionals $M_{[a]}c^2$ (III.19a)-(III.19b), one could equally well multiply through the original mass eigenvalue equations (III.15) and (III.17) by $\bar{\psi}_1$ and $\bar{\psi}_2$ and integrating over whole three-space in order to arrive again at the same mass functionals $M_{[a]}c^2$ (III.19a)-(III.19b).

The relevance of the mass functionals originates now from the fact that they are stationary upon the solutions of the mass eigenvalue equations (III.15) and (III.17), albeit with regard of the constraints of wave function normalization (II.44). In order to take account of these constraints, one introduces the Lagrangean multipliers $\lambda_D(a)$ and considers the modified mass functionals $\bar{M}_{[a]}c^2$

$$-\bar{M}_{[1]}c^2 = -M_{[1]}c^2 + \lambda_D(1) \cdot N_D(1) \quad (\text{III.21a})$$

$$\bar{M}_{[2]}c^2 = M_{[2]}c^2 + \lambda_D(2) \cdot N_D(2) , \quad (\text{III.21b})$$

with the constraints $N_D(a)$ being given by the wave function normalization ($a = 1, 2$), i.e.

$$N_D(a) \doteq \int d^3\vec{r} \bar{\psi}_a \gamma^0 \psi_a(\vec{r}) - 1 = 0 . \quad (\text{III.22})$$

Indeed, carrying through here the variational procedure and comparing the emerging variational equations to the former mass eigenvalue equations (III.15)
and (III.17) just yields the identity of both sets of equations, provided one fixes the Lagrangean multipliers $\lambda_D(a)$ in terms of the mass eigenvalues $M_a$ as follows:

$$\lambda_{D(1)} = M_1 c^2 \quad \text{(III.23a)}$$
$$\lambda_{D(2)} = -M_2 c^2 \quad \text{(III.23b)}$$

Thus the Lagrangean multipliers just turn out to be identical to the mass eigenvalues (up to sign). Clearly such a result strongly remembers one of the well-known Ritz variational principle of conventional quantum mechanics which has frequently been applied in the early days of atomic physics [11] and in the meantime has been advanced to a standard exercise for any student [12].

Indeed, anyone of the mass functionals $M[a] c^2$ (III.19a)-(III.19b) appears to be of a very plausible form, namely as the sum of three kinds of energy: rest mass energy, kinetic energy and interaction energy of the electric (e) and magnetic (m) type. Therefore it is very tempting to think that the total energy of the bound two-particle system could be identified with the sum ($\tilde{M}_T c^2$) of both mass eigenvalues

$$\tilde{M}_T c^2 = -\tilde{M}_{[1]} c^2 + \tilde{M}_{[2]} c^2 \quad \text{(III.24)}$$

especially because the corresponding variational equations just coincide with the mass eigenvalue equations (III.15) and (III.17). However, such a supposition is incorrect; and a closer inspection of how the Ritz variational principle emerges as the non-relativistic limit of the present mass functional approach will reveal the origin of the deficiencies connected with the non-relativistic Ritz method. Furthermore this analysis provides one with a valuable hint
on how to overcome the shortcomings of those non-relativistic variational methods and to construct the relativistic energy functional $\tilde{E}_T$.

**C. Ritz Variational Principle**

In order to deduce the non-relativistic limit form of both the mass functionals $M_{[\phi]}c^2$ (III.19a)-(III.19b) and of their variational equations (III.15) and (III.17) one conceives the Dirac four-spinors $\psi_a(\vec{r})$ as a direct sum of two-component Pauli spinors $(^a)\varphi_{\pm}(\vec{r})$ $(a = 1, 2)$, i.e. one puts

$$\psi_a(\vec{r}) = (^a)\varphi_+(\vec{r}) \oplus (^a)\varphi_-(\vec{r}),$$

(III.25)

and then one deduces the corresponding eigenvalue equations for these Pauli spinors from the original mass eigenvalue equations (III.15) and (III.17). This yields for the first particle $(a = 1)$

$$i\vec{\sigma} \cdot \nabla^{(1)} \varphi_+(\vec{r}) + (^0)A_0(\vec{r}) \cdot (^0)\varphi_-(\vec{r}) - \vec{A}_2 \cdot \vec{\sigma}^{(1)} \varphi_+(\vec{r}) = \frac{M_p - M_1}{\hbar} c \cdot (^0)\varphi_-(\vec{r}),$$

(III.26a)

$$i\vec{\sigma} \cdot \nabla^{(1)} \varphi_-(\vec{r}) + (^0)A_0(\vec{r}) \cdot (^0)\varphi_+(\vec{r}) - \vec{A}_2 \cdot \vec{\sigma}^{(1)} \varphi_-(\vec{r}) = -\frac{M_p + M_1}{\hbar} c \cdot (^0)\varphi_+(\vec{r}),$$

(III.26b)

and similarly for the second particle $(a = 2)$

$$i\vec{\sigma} \cdot \nabla^{(2)} \varphi_+(\vec{r}) + (^1)A_0(\vec{r}) \cdot (^1)\varphi_-(\vec{r}) - \vec{A}_1 \cdot \vec{\sigma}^{(2)} \varphi_+(\vec{r}) = -\frac{M_e + M_2}{\hbar} c \cdot (^1)\varphi_-(\vec{r}),$$

(III.27a)

$$i\vec{\sigma} \cdot \nabla^{(2)} \varphi_-(\vec{r}) + (^1)A_0(\vec{r}) \cdot (^1)\varphi_+(\vec{r}) - \vec{A}_1 \cdot \vec{\sigma}^{(2)} \varphi_-(\vec{r}) = \frac{M_e - M_2}{\hbar} c \cdot (^1)\varphi_+(\vec{r}).$$

(III.27b)
Next one approximately resolves the eigenvalue equations (III.26a) and (III.27a) for the “small” components \( \varphi_{-}(\vec{r}) \)

\[
(1) \varphi_{-}(\vec{r}) \simeq \frac{i\hbar}{2M_p c} \vec{\sigma} \cdot \vec{\nabla} \varphi_{+}(\vec{r}) \quad \text{(III.28a)}
\]

\[
(2) \varphi_{-}(\vec{r}) \simeq -\frac{i\hbar}{2M_e c} \vec{\sigma} \cdot \vec{\nabla} \varphi_{+}(\vec{r}) \quad \text{(III.28b)}
\]

and substitutes this into the eigenvalue equations (III.26b) and (III.27b) for the “small” components \( \varphi_{-}(\vec{r}) \) in order to obtain the well-known Pauli equations for the “large” components:

\[
-\frac{\hbar^2}{2M_p} \Delta \varphi_{+}(\vec{r}) + \hbar c A_0(\vec{r}) \cdot \varphi_{+}(\vec{r}) = E_{S(1)} \cdot \varphi_{+}(\vec{r}) \quad \text{(III.29a)}
\]

\[
-\frac{\hbar^2}{2M_e} \Delta \varphi_{+}(\vec{r}) - \hbar c A_0(\vec{r}) \cdot \varphi_{+}(\vec{r}) = E_{S(2)} \cdot \varphi_{+}(\vec{r}) \quad \text{(III.29b)}
\]

Here the Pauli-Schrödinger energy eigenvalues \( E_{S(a)} \) are introduced through

\[
E_{S(1)} = -\left( M_p c^2 + M_1 c^2 \right) \quad \text{(III.30a)}
\]

\[
E_{S(2)} = M_2 c^2 - M_e c^2 \quad \text{(III.30b)}
\]

which again demonstrates that the mass eigenvalue of the first particle is negative \( (M_1 < 0) \), in contrast to the second eigenvalue \( (M_2 > 0) \). Furthermore the magnetic interaction terms \( \sim \vec{A}_a \) are omitted since the corresponding magnetic interaction energy is mostly much smaller than its electric counterpart described by \( (a)A_0(\vec{r}) \). As a consequence of this omission, the Pauli spinors can be assumed to occupy a fixed direction in spinor space, e.g.

\[
(1) \varphi_{+}(\vec{r}) = \varphi_0(\vec{r}) \cdot | \uparrow > 
\]

so that for the scalar wave functions \( \varphi_0(\vec{r}) \) there emerge the conventional
Schrödinger equations from the Pauli system (III.29a)-(III.29b):

\[
-\frac{\hbar^2}{2M_p} \Delta \varphi_1(\vec{r}) + \hbar c^{(2)} A_0(\vec{r}) \cdot \varphi_1(\vec{r}) = E_{S(1)} \cdot \varphi_1(\vec{r}) \quad \text{(III.32a)}
\]

\[
-\frac{\hbar^2}{2M_e} \Delta \varphi_2(\vec{r}) - \hbar c^{(1)} A_0(\vec{r}) \cdot \varphi_2(\vec{r}) = E_{S(2)} \cdot \varphi_2(\vec{r}) \quad . \quad \text{(III.32b)}
\]

Observe here that either of the two particles feels the attractive force due to the other one because the first potential \(^{(1)}A_0(\vec{r})\) (due to the positively charged particle) is positive and the second potential \(^{(2)}A_0(\vec{r})\) is negative! (See below for the discussion of the corresponding Poisson equations (III.47a)-(III.47d)).

Now the interesting point with this non-relativistic approximation (III.32a)-(III.32b) to the properly relativistic eigenvalue equations (III.15) and (III.17) refers to the fact that the non-relativistic system may be also deduced from a variational principle; i.e. the well-known Ritz principle [13]

\[
\delta W_S = 0 \quad \text{(III.33a)}
\]

\[
W_S = \int\int d^3\vec{r}_1 \ d^3\vec{r}_2 \ \Phi(\vec{r}_1, \vec{r}_2) \hat{H}_S \Phi(\vec{r}_1, \vec{r}_2) , \quad \text{(III.33b)}
\]

where the conventional Hamiltonian \(\hat{H}_S\) is given by

\[
\hat{H}_S = \frac{\vec{p}_1^2}{2M_p} + \frac{\vec{p}_2^2}{2M_e} - \frac{e^2}{||\vec{r}_1 - \vec{r}_2||} . \quad \text{(III.34)}
\]

Indeed, carrying out the variational procedure (III.33a), with the constraint of wave normalization, lets emerge the conventional Schrödinger equation as the corresponding variational equation

\[
\hat{H}_S \Phi(\vec{r}_1, \vec{r}_2) = E_S \cdot \Phi(\vec{r}_1, \vec{r}_2) . \quad \text{(III.35)}
\]

It is generally believed that the associated conventional eigenvalue \(E_S\) is the “true” (albeit non-relativistic) energy of such a two-particle system being
specified by the Schrödinger Hamiltonian (III.34); and consequently all other predictions can at most appear to be approximations to that true value! For instance, one may try (for the groundstate) the following product ansatz (Hartree approximation)

$$\Phi(\vec{r}_1, \vec{r}_2) = \varphi_1(\vec{r}_1) \cdot \varphi_2(\vec{r}_2)$$  \hfill (III.36)

and use this for carrying out the variational procedure (III.33a)-(III.33b) which then yields the following one-particle eigenvalue equations:

$$-\frac{\hbar^2}{2M_p} \Delta \varphi_1(\vec{r}) + \hbar c (2) V_{HS}(\vec{r}) \cdot \varphi_1(\vec{r}) = -\lambda_{S(1)} \cdot \varphi_1(\vec{r}) \quad \text{(III.37a)}$$

$$-\frac{\hbar^2}{2M_e} \Delta \varphi_2(\vec{r}) - \hbar c (1) V_{HS}(\vec{r}) \cdot \varphi_2(\vec{r}) = -\lambda_{S(2)} \cdot \varphi_2(\vec{r}). \quad \text{(III.37b)}$$

Here the normalization conditions for the conventional wave functions $\varphi_a(\vec{r})$

$$\int d^3\vec{r} \varphi_a^*(\vec{r}) \varphi_a(\vec{r}) - 1 = 0$$  \hfill (III.38)

are respected by application of the method of Lagrangean multipliers; and the Hartree-Schrödinger potentials $^{(a)}V_{HS}(\vec{r})$ are given in terms of the one-particle wave functions $\varphi_a(\vec{r})$ through

$$^{(1)}V_{HS}(\vec{r}) = \alpha_s \int d^3\vec{r}_1 \frac{\varphi_1^*(\vec{r}_1) \cdot \varphi_1(\vec{r}_1)}{||\vec{r} - \vec{r}_1||}$$  \hfill (III.39a)

$$^{(2)}V_{HS}(\vec{r}) = -\alpha_s \int d^3\vec{r}_2 \frac{\varphi_2^*(\vec{r}_2) \cdot \varphi_2(\vec{r}_2)}{||\vec{r} - \vec{r}_2||}. \quad \text{(III.39b)}$$

Clearly, the Lagrangean multipliers $\lambda_S(a)$ in the Hartree-Schrödinger eigenvalue equations (III.37a)-(III.37b) are to be identified again with the energy eigenvalues $E_S(a) \ (a = 1, 2)$

$$\lambda_S(a) = -E_S(a), \quad \text{(III.40)}$$
cf. the relativistic version (III.23a)-(III.23b); and thus the Hartree-Schrödinger system (III.37a)-(III.37b) is identical to the non-relativistic RST limit (III.32a)-(III.32b), provided one can show (see below) that the electric RST potentials \( A_0(\vec{r}) \) are identical to the Hartree-Schrödinger potentials \( V_{\text{HS}}(\vec{r}) \) (III.39a)-(III.39b). This important identification of the Hartree-Schrödinger (or more generally: Hartree-Fock) approach with the non-relativistic limit of the RST eigenvalue equations suggests that the RST variational method

\[
\delta \tilde{M}_T = 0
\]  
(III.41)

(with the mass functional \( \tilde{M}_T \) given by equations (III.21a)-(III.24)) may be considered as a viable relativistic generalization of the Hartree-Ritz variational principle (or Hartree-Fock approach, resp). But since the latter approach is in general used as merely an approximation to the conventional Schrödinger theory, being based upon the “exact” non-relativistic eigenvalue equation (III.35), it may seem that RST is afflicted with all those deficiencies of the Hartree-Fock approach, albeit on a relativistic level (for a critical discussion of the limits of the HF approach see ref. [14]). However this conclusion is not valid because RST is capable of overcoming the main deficiencies of both the Ritz principle and the HF approach (apart from their non-relativistic character). These deficiencies are the following:

i) the Ritz principle (III.33a)-(III.34) relies upon the instantaneous Coulomb interaction of the particles and thus violates the true spirit of relativity

ii) the Hartree-Fock approach relies upon unique wave functions \( \varphi_a(\vec{r}) \) (III.36), whereas not the wave functions themselves (as unobservable
objects) need be unique but rather the observable quantities (i.e. the physical densities in RST).

In the following, we will explicitly demonstrate that the treatment of the interaction fields as proper dynamical variables together with the use of non-unique wave functions may actually lead to predictions which can compete with those of the conventional quantum theory. Indeed, this improvement of the standard Ritz-Hartree-Fock approaches will then result in the desired principle of minimal energy for the stationary bound states.
D. Poisson Equations

Surely, if the particle interaction is to be considered as a dynamical object, then it must obey some equation of motion and cannot be specified by the rigid Coulomb interaction. However in RST, the existence of such a (relativistic) equation of motion for the interaction fields is a matter of course, since the RST variational principle (II.56a)-(II.56b) includes the gauge field $A_\mu$ a priori. It is merely necessary to carry out the variational procedure (II.56a) with respect to the bundle connection $A_\mu$ which then yields the general Maxwell equations (II.24), or their Abelian specialization (II.27a)-(II.27b), resp. Since for the present Abelian situation (for non-identical particles) the field strengths $F^a_{\mu\nu}$ degenerate to the simple curls of the potential $A^a_\mu$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

(III.42)

(see ref. [7] for the case of identical particles) the Abelian Maxwell equations (II.27a)-(II.27b) yield just the well-known wave equations of classical electrodynamics ($a = 1, 2$):

$$\partial^\mu \partial_\mu A^a_\nu = 4\pi \alpha_s j^a_\nu ,$$

(III.43)

provided one applies the usual Lorentz gauge condition

$$\partial^\mu A^a_\mu \equiv 0 .$$

(III.44)

For the present stationary situation (III.2), this splits up into the (electric) time component ($a = 1, 2$)

$$\Delta^{(a)} A_0(\vec{r}) = -4\pi \alpha_s (a) j_0(\vec{r})$$

(III.45)

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and (magnetic) space component

\[ \Delta \vec{A}_a(\vec{r}) = -4\pi\alpha_s \vec{j}_a(\vec{r}) . \]  

(III.46)

Or, if the Maxwell currents \( j^a_\mu = \{ (a) j_0(\vec{r}), -j_a(\vec{r}) \} \) are written in terms of the stationary Dirac currents \( k_{a\mu} \) one has the Poisson equations

\[ \Delta^{(1)} A_0(\vec{r}) = -4\pi\alpha_s^{(1)} k_0(\vec{r}) \]  

(III.47a)

\[ \Delta^{(2)} A_0(\vec{r}) = 4\pi\alpha_s^{(2)} k_0(\vec{r}) \]  

(III.47b)

\[ \Delta \vec{A}_1(\vec{r}) = -4\pi\alpha_s \vec{k}_1(\vec{r}) \]  

(III.47c)

\[ \Delta \vec{A}_2(\vec{r}) = 4\pi\alpha_s \vec{k}_2(\vec{r}) . \]  

(III.47d)

Observe here that the fibre submetric \( K_{ab} \) has just the right form in order to get the Poisson equations adapted to the positive and negative charges carried by the particles! The standard solutions of the Poisson equations are adopted to be of the usual form

\[ (1) A_0(\vec{r}) = \alpha_s \int d^3\vec{r}' \frac{(1) k_0(\vec{r}')}{||\vec{r} - \vec{r}'||} \]  

(III.48a)

\[ (2) A_0(\vec{r}) = -\alpha_s \int d^3\vec{r}' \frac{(2) k_0(\vec{r}')}{||\vec{r} - \vec{r}'||} \]  

(III.48b)

\[ A_1(\vec{r}) = \alpha_s \int d^3\vec{r}' \frac{\vec{k}_1(\vec{r}')}{||\vec{r} - \vec{r}'||} \]  

(III.48c)

\[ A_2(\vec{r}) = -\alpha_s \int d^3\vec{r}' \frac{\vec{k}_2(\vec{r}')}{||\vec{r} - \vec{r}'||} . \]  

(III.48d)

The non-relativistic limits of these formally exact solutions are obtained by substituting herein the non-relativistic approximations for the Dirac densities.
of charge \( (a)k_0(\vec{r}) \) and current \( \vec{k}_a(\vec{r}) \) \([7]\):

\[
(a)k_0(\vec{r}) = \tilde{\psi}_a(\vec{r})\gamma_0\psi_a(\vec{r}) = (a)\varphi_{+}^\dagger(\vec{r}) \cdot (a)\varphi_+(\vec{r}) + (a)\varphi_{-}^\dagger(\vec{r}) \cdot (a)\varphi_-(\vec{r}) \quad (III.49a)
\]

\[
\vec{k}_a(\vec{r}) = \tilde{\psi}_a(\vec{r}) \cdot \vec{\gamma} \cdot \psi_a(\vec{r}) = (a)\varphi_+(\vec{r}) \cdot \vec{\sigma} \cdot (a)\varphi_-(\vec{r}) + (a)\varphi_{-}^\dagger(\vec{r}) \cdot \vec{\sigma} \cdot (a)\varphi_+(\vec{r})
\quad (III.49b)
\]

Observe here that for the non-relativistic limit the “small” Pauli components \((a)\varphi_-(\vec{r})\) can be neglected against its “large” counterparts \((a)\varphi_+(\vec{r})\) only for the charge density \((a)k_0(\vec{r})\) \((III.49a)\), but not for the current densities \((III.49b)\) as the sources of the magnetic fields! This is the reason why we omitted the magnetic terms when deducing the Hartree-Schrödinger eigenvalue equations \((III.32a)\)-(\(III.32b)\) for the “large” components from their properly relativistic RST form \((III.15)\) and \((III.17)\). But inserting now the approximate form \((III.49a)\) of the charge densities \((a)k_0(\vec{r})\) into the formal solutions \((a)A_0(\vec{r})\) \((III.48a)\)-(\(III.48b)\) of the Poisson equations yields

\[
(1)A_0(\vec{r}) \approx \alpha_s \int d^3\vec{r}' \frac{\varphi_1(\vec{r}') \cdot \varphi_1(\vec{r}')}{||\vec{r} - \vec{r}'||}
\quad (III.50a)
\]

\[
(2)A_0(\vec{r}) \approx -\alpha_s \int d^3\vec{r}' \frac{\varphi_2(\vec{r}') \cdot \varphi_2(\vec{r}')}{||\vec{r} - \vec{r}'||},
\quad (III.50b)
\]

and this is just the result which is needed in order to identify the non-relativistic RST limit \((III.32a)\)-(\(III.32b)\) with the conventional Hartree-Schrödinger eigenvalue system \((III.37a)\)-(\(III.37b)\).

This is the way in which RST cures the first one (i) of the Ritz-Hartree-Fock deficiencies mentioned above, namely by adopting the Poisson equations (or more generally: the Maxwell equations) for the determination of the gauge potentials and thus accepting the action of the gauge forces as a truly
The dynamical process. However, observe in this context that the Poisson equations emerge here as the variational equations due to the Hamilton-Lagrange action principle (II.56a)-(II.56b), not due to the variation of the RST mass functional $\tilde{M}_T c^2$ (III.24)! Therefore it is not yet possible to identify this mass functional with the wanted energy functional as the ultimate goal of the present investigation. This goal will be readily achieved in the next section but can be prepared here by first regarding an important property of the time-independent gauge part $\hat{W}_G[A_\mu]$

$$\hat{W}_G[A_\mu] = \int d^3\vec{r} \mathcal{L}_G[A_\mu]$$ (III.51)

of the original RST action principle (II.56b). Indeed, substituting here the Lagrangean density $\mathcal{L}_G$ in the form (II.62) with the electromagnetic three-vector fields $\vec{E}_a(\vec{r})$ and $\vec{H}_a(\vec{r})$ being written in terms of the corresponding potentials $(^a)A_0(\vec{r}), \vec{A}_a(\vec{r})$ as

$$\vec{E}_a(\vec{r}) = -\vec{\nabla} (^a)A_0(\vec{r})$$ (III.52a)

$$\vec{H}_a(\vec{r}) = \vec{\nabla} \times \vec{A}_a(\vec{r})$$ (III.52b)

lets appear that time-independent gauge part $\hat{W}_G[A_\mu]$ in the following form:

$$\hat{W}_G[A_\mu] = \frac{\hbar c}{4\pi\alpha_s} \int d^3\vec{r} \left\{ \vec{\nabla} (^1)A_0(\vec{r}) \cdot \vec{\nabla} (^2)A_0(\vec{r}) - \left( \vec{\nabla} \times \vec{A}_1(\vec{r}) \right) \cdot \left( \vec{\nabla} \times \vec{A}_2(\vec{r}) \right) \right\}$$ (III.53)

Furthermore, by resorting to the gauge field contributions $\mathcal{L}_{D}^{(e,m)}$ (III.6b), (III.6c) and (III.7b), (III.7c) due to the matter Lagrangean $\mathcal{L}_D$, one has the corresponding electric (e) and magnetic (m) action constituents as

$$\hat{W}_D^{(e)} = -\hbar c \int d^3\vec{r} \left\{ (^2)A_0(\vec{r}) \cdot (^1)k_0(\vec{r}) - (^1)A_0(\vec{r}) \cdot (^2)k_0(\vec{r}) \right\}$$ (III.54a)

$$\hat{W}_D^{(m)} = \hbar c \int d^3\vec{r} \left\{ \vec{A}_2(\vec{r}) \cdot \vec{k}_1(\vec{r}) - \vec{A}_1(\vec{r}) \cdot \vec{k}_2(\vec{r}) \right\}.$$ (III.54b)
Thus the former Poisson equations (III.47a)-(III.47d) are actually recovered from here by variation of the partial sum of action integrals $W_G + W_D + W_D$ with respect to the static gauge potentials.

Now the interesting point with this variational procedure for the gauge fields is that it leads us to global identities which subsequently will be needed as constraints for the principle of minimal energy, i.e. the Poisson identities. These global relations between the gauge fields and their sources emerge from the Hamiltonian-Lagrange action principle (II.56a)-(II.56b) by considering the scaling variations for the potentials, e.g. for the first electrostatic potential

$$(1)A_0(\vec{r}) \rightarrow (1)A'_0(\vec{r}) = C_* \cdot (1)A_0(\vec{r}) \ ,$$

with the scaling factor $C_*$ being a constant over three-space. Similar arguments do hold also for the other potentials $(2)A_0(\vec{r}), \vec{A}_a(\vec{r})$. By inserting this special variation (III.55) into the RST action integral, its relevant parts become

$$\circ \ W^{(e)}_G + \circ \ W^{(e)}_D \Rightarrow C_* \left[ \frac{\hbar c}{4\pi\alpha_s} \int d^3\vec{r} \left( \vec{\nabla} \cdot (1)A_0(\vec{r}) \cdot \vec{\nabla} \cdot (2)A_0(\vec{r}) + 4\pi\alpha_s (1)A_0(\vec{r}) \cdot (2)k_0(\vec{r}) \right) \right] \ .$$

Since the Hamiltonian-Lagrange action principle demands stationarity of the action integral $W_{RST}$ with respect to the choice of $C_*:

$$\left. \frac{dW_{RST}(C_*)}{dC_*} \right|_{C_*=1} = 0 \ ,$$

one concludes from equation (III.56) that the following integral relation must hold:

$$N^{(e)}_G(1) \div \int d^3\vec{r} \left[ \left( \vec{\nabla} \cdot (1)A_0(\vec{r}) \right) \cdot \left( \vec{\nabla} \cdot (2)A_0(\vec{r}) \right) + 4\pi\alpha_s (1)A_0(\vec{r}) \cdot (2)k_0(\vec{r}) \right] \equiv 0 \ ,$$

(III.58)
and analogously for the other gauge potentials

\[ N^{(e)}_G(2) \triangleq \int d^3\vec{r} \left[ (\tilde{\nabla} A_0^{(1)}(\vec{r})) \cdot (\tilde{\nabla} A_0^{(2)}(\vec{r})) - 4\pi\alpha_s A_0^{(2)}(\vec{r}) \cdot \vec{k}_0(\vec{r}) \right] \equiv 0 \]

(III.59a)

\[ N^{(m)}_G(1) \triangleq \int d^3\vec{r} \left[ (\nabla \times A_1)(\vec{r}) \cdot (\nabla \times A_2)(\vec{r}) + 4\pi\alpha_s A_1(\vec{r}) \cdot \vec{k}_2(\vec{r}) \right] \equiv 0 \]

(III.59b)

\[ N^{(m)}_G(2) \triangleq \int d^3\vec{r} \left[ (\nabla \times A_1)(\vec{r}) \cdot (\nabla \times A_2)(\vec{r}) - 4\pi\alpha_s A_2(\vec{r}) \cdot \vec{k}_1(\vec{r}) \right] \equiv 0 . \]

(III.59c)

Clearly, these Poisson identities may be obtained also directly from the Poisson equations (III.47a)-(III.47d) by multiplying through with the appropriate potentials and integrating by parts. But their deduction from the RST action principle does better elucidate their meaning for the variational procedure: obviously, when looking (by trial and error) for those gauge potentials which yield stationarity of the action integral \( W_{RST} \) (II.56b), one can restrict oneself to those potentials which obey the Poisson identities. It is just with reference to this meaning of restrictive conditions that the Poisson identities will readily be used in order to set up the RST principle of minimal energy!

**E. Double-Valued Wave Functions**

After the first deficiency (i) of the Ritz-Hartree-Schrödinger approach is now eliminated, one can turn to the next critical point, namely the conventional assumption (ii) that the wave functions must always be unique. Indeed we will relax now this presumption and will (as a counterexample) admit double-valued wave functions of the type

\[ \Psi(r, \vartheta, \phi + 2\pi) = -\Psi(r, \vartheta, \phi) \]

(III.60)
where \( \{r, \vartheta, \phi\} \) are the usual spherical polar coordinates. We will readily see that such a more general class of wave functions can generate unconventional gauge potentials, namely via the solutions of the Poisson equations (III.48a)-(III.48d) or their non-relativistic approximations resp; and the corresponding unusual form of interaction force may then yield energy levels which are closer to the conventional Schrödinger predictions than it is possible for the Dirac-Fock approach [15]. But clearly, the admitted non-uniqueness (III.60) of the wave functions must not imply the non-uniqueness of the physical densities, e.g. of four-current \( k_{a\mu} \) (II.45) or of energy-momentum density \( ^{(D)}T_{\mu\nu} \) (II.50b), etc. Indeed, it is easy to see that those physical densities are bilinear constructions of \( \Psi \) and \( \bar{\Psi} \) and therefore remain invariant against the change (III.60) of the wave function!

For a concrete exemplification of those double-valued wave functions, one may resort to two basis systems \( \{\omega_0^{(+)}, \omega_0^{(-)}\} \) and \( \{\omega_1^{(+)}, \omega_1^{(-)}\} \) of the two-dimensional Pauli spinor space which are eigenvectors (with zero eigenvalue) of the total angular momentum \( \hat{J}_z = \hat{L}_z + \hat{S}_z \):

\[
\hat{J}_z \omega_0^{(+)} = \hat{J}_z \omega_0^{(-)} = \hat{J}_z \omega_1^{(+)} = \hat{J}_z \omega_1^{(-)} = 0 ,
\]

see ref. [6] for the details. Since these basis spinors themselves are already double-valued (i.e. \( \omega_0^{(+)}(r, \vartheta, \phi + 2\pi) = -\omega_0^{(+)}(r, \vartheta, \phi) \); etc.), one can decompose the Pauli spinors \( ^{(a)}\varphi_{\pm}(\vec{r}) \) (III.25) with respect to these double-valued basis systems as follows \( (a = 1, 2) \)

\[
^{(a)}\varphi_+(\vec{r}) = (r \sin \vartheta)^{-\frac{1}{2}} \left[ ^{(a)}\bar{R}_+(r, \vartheta) \cdot \omega_0^{(+)} + ^{(a)}\bar{S}_+(r, \vartheta) \cdot \omega_0^{(-)} \right] \quad \text{(III.62a)}
\]

\[
^{(a)}\varphi_-(\vec{r}) = -i(r \sin \vartheta)^{-\frac{1}{2}} \left[ ^{(a)}\bar{R}_-(r, \vartheta) \cdot \omega_1^{(+)} + ^{(a)}\bar{S}_-(r, \vartheta) \cdot \omega_1^{(-)} \right] \quad \text{(III.62b)}
\]

\[
^{(a)}\varphi_{\pm}(r, \vartheta, \phi + 2\pi) = -^{(a)}\varphi_{\pm}(r, \vartheta, \phi) ,
\]

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and then both these Pauli spinors and the corresponding Dirac spinors \(\psi_a(\vec{r})\) are double-valued in the sense of equation (III.60), provided the wave amplitude \((a)\tilde{R}_\pm(r, \vartheta)\) and \((a)\tilde{S}_\pm(r, \vartheta)\) are \textit{single-valued}! Moreover, the latter property of uniqueness is transferred also to the Dirac densities \(k_{a\mu}(\vec{r}) = \{(a)k_0(\vec{r}), -\vec{k}_a(\vec{r})\}\) (II.45) since these appear as bilinear constructions of the Pauli spinors \((a)\phi_\pm(\vec{r})\), i.e.

\[
(a)k_0(\vec{r}) = (a)\varphi_+(\vec{r}) \cdot (a)\varphi_+(\vec{r}) + (a)\varphi_-(\vec{r}) \cdot (a)\varphi_-(\vec{r}) = \frac{(a)\tilde{R}_+^2 + (a)\tilde{S}_+^2 + (a)\tilde{R}_-^2 + (a)\tilde{S}_-^2}{4\pi r \sin \vartheta} \tag{III.63a}
\]

\[
\vec{k}_a(\vec{r}) = (a)\varphi_+(\vec{r}) \cdot \vec{\sigma} \cdot (a)\varphi_-(\vec{r}) + (a)\varphi_-(\vec{r}) \cdot \vec{\sigma} \cdot (a)\varphi_+(\vec{r}) \equiv (a)k_\varphi(\vec{r}) \cdot \vec{e}_\varphi , \tag{III.63b}
\]

with the azimuthal component \((a)k_\varphi\) of the Dirac currents \(\vec{k}_a(\vec{r})\) being given by

\[
(a)k_\varphi = \frac{\sin \vartheta \left((a)\tilde{R}_+ \cdot (a)\tilde{R}_- - (a)\tilde{S}_+ \cdot (a)\tilde{S}_-\right) - \cos \vartheta \left((a)\tilde{S}_+ \cdot (a)\tilde{R}_- + (a)\tilde{R}_+ \cdot (a)\tilde{S}_-\right)}{2\pi r \sin \vartheta} . \tag{III.64}
\]

Observe here, that through the choice of \textit{real-valued} wave amplitudes \((a)\tilde{R}_\pm, (a)\tilde{S}_\pm\), the radial \((a)k_r\) and longitudinal \((a)k_\varphi\) components of the Dirac currents \(\vec{k}_a(\vec{r})\) do vanish (i.e. \((a)k_r = (a)k_\varphi \equiv 0\)), so that these three-currents \(\vec{k}_a(\vec{r})\) encircle the axis of the spherical polar coordinates \((r, \vartheta, \varphi)\). Naturally, this symmetry of the three-currents may then be transferred also to the vector potentials \(\vec{A}_a(\vec{r})\) which thus appear in the following form

\[
\vec{A}_a(\vec{r}) = (a)A_\varphi(r, \vartheta) \cdot \vec{e}_\varphi , \tag{III.65}
\]

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from which the magnetic fields \( \vec{H}_a(\vec{r}) \) can be computed by means of the usual curl relation (III.52b).

Summarizing the properties of the wave functions to be used for the RST description of bound states, one first has to mention their double-valuedness (III.60) and moreover we will assume that the wave amplitudes \((a)\tilde{R}_\pm, (a)\tilde{S}_\pm\) are unique and non-singular (real-valued) functions over space time. But observe here that, through this second assumption, the Pauli spinors \((a)\varphi_\pm(\vec{r})\) (III.62a)-(III.62b) and therefore also the original Dirac spinors \(\psi_a(\vec{r})\) (III.25) become both singular and double-valued (\(\sim\) “exotic states”). It should appear as a matter of course that such exotic states will imply further unconventional elements of the theory, e.g. the form of the gauge potentials. Notice, however, that the observable objects of the theory (i.e. the densities of charge, current, energy-momentum etc.) are well-defined and unique objects over space-time, the singularities of which (if present at all) do not induce any pathological element into the theory. Therefore the wave amplitudes \((a)\tilde{R}_\pm, (a)\tilde{S}_\pm\) (III.62a)-(III.62b) as the unique and (mostly) regular constituents of the non-unique and singular wave functions \(\psi_a(\vec{r})\) (III.25) will appear as the solutions of a well-defined eigenvalue problem. The corresponding eigenvalue equations are to be deduced from the original eigenvalue equations (III.26a)-(III.27b) for the double-valued Pauli spinors \((a)\varphi_\pm(\vec{r})\) and
appear then in the following form, e.g. for the first particle \((a = 1)\) [7]

\[
\frac{\partial(1)\tilde{R}_+}{\partial r} + \frac{1}{r} \frac{\partial(1)\tilde{S}_+}{\partial \vartheta} + (2)A_0 \cdot (1)\tilde{R}_- - (2)A_\varphi \left( \sin \vartheta \cdot (1)\tilde{R}_+ - \cos \vartheta \cdot (1)\tilde{S}_+ \right) = \frac{M_p - M_1}{\hbar} c \cdot (1)\tilde{R}_-
\]

(III.66a)

\[
\frac{\partial(1)\tilde{S}_+}{\partial r} - \frac{1}{r} \frac{\partial(1)\tilde{R}_+}{\partial \vartheta} + (2)A_0 \cdot (1)\tilde{S}_- + (2)A_\varphi \left( \sin \vartheta \cdot (1)\tilde{S}_+ + \cos \vartheta \cdot (1)\tilde{R}_+ \right) = \frac{M_p - M_1}{\hbar} c \cdot (1)\tilde{S}_-
\]

(III.66b)

\[
\frac{1}{r} \frac{\partial(1)\tilde{R}_-}{\partial r} - \frac{1}{r} \frac{\partial(1)\tilde{S}_-}{\partial \vartheta} - (2)A_0 \cdot (1)\tilde{R}_+ - (2)A_\varphi \left( \sin \vartheta \cdot (1)\tilde{R}_- - \cos \vartheta \cdot (1)\tilde{S}_- \right) = \frac{M_p + M_1}{\hbar} c \cdot (1)\tilde{R}_+
\]

(III.66c)

\[
\frac{1}{r} \frac{\partial(1)\tilde{S}_-}{\partial r} + \frac{1}{r} \frac{\partial(1)\tilde{R}_-}{\partial \vartheta} - (2)A_0 \cdot (1)\tilde{S}_+ - (2)A_\varphi \left( \sin \vartheta \cdot (1)\tilde{S}_- + \cos \vartheta \cdot (1)\tilde{R}_- \right) = \frac{M_p + M_1}{\hbar} c \cdot (1)\tilde{S}_+
\]

(III.66d)

An analogous set of four eigenvalue equations does apply to the second particle \((a = 2)\) which, however, needs not explicitly be reproduced here because it can be obtained simply by means of the particle permutation symmetry, see ref. [7].

**F. Unconventional Potentials**

The interesting point with these double-valued wave functions is now that they do generate a rather unusual form of the gauge potentials \(^{(a)}A_0(\vec{r})\) and \(\vec{A}_a(\vec{r})\) by means of the recipe (III.48a)-(III.48d). In order to see this more clearly, one substitutes the charge and current densities \(^{(1)}k_0(\vec{r})\) (III.63a)
and \( \tilde{k}_a(\vec{r}) \) into those formal solutions of the Poisson equations which yields explicitly for the electric potentials in terms of the unique wave amplitudes

\[
(1) A_0(r, \vartheta) = \frac{\alpha_s}{4\pi} \int \frac{d^3\vec{r}'}{r' \sin \vartheta'} \left( \frac{1}{||\vec{r} - \vec{r}'||} \right) \left( (1) \tilde{R}_+^2(r', \vartheta') + (1) \tilde{S}_+^2(r', \vartheta') + (1) \tilde{R}_-^2(r', \vartheta') + (1) \tilde{S}_-^2(r', \vartheta') \right)
\]

(III.67a)

\[
(2) A_0(r, \vartheta) = -\frac{\alpha_s}{4\pi} \int \frac{d^3\vec{r}'}{r' \sin \vartheta'} \left( \frac{1}{||\vec{r} - \vec{r}'||} \right) \left( (2) \tilde{R}_+^2(r', \vartheta') + (2) \tilde{S}_+^2(r', \vartheta') + (2) \tilde{R}_-^2(r', \vartheta') + (2) \tilde{S}_-^2(r', \vartheta') \right)
\]

(III.67b)

and similarly for the magnetic potentials \((a) A_\varphi(r, \vartheta)\).

In order to estimate qualitatively the new feature of these potentials due to the non-singular wave amplitudes \( (a) \tilde{R}_\pm, (a) \tilde{S}_\pm \), it may be sufficient for the moment to adopt a simple model \( (b) k_0(\vec{r}) \) for a non-spherically symmetric and singular charge distribution being normalized to unity, cf. (II.44), i.e. we put

\[
(b) k_0(r) \overset{\text{def}}{=} \frac{(b) \tilde{k}_0(r)}{4\pi r \sin \vartheta} = (4\pi r \sin \vartheta)^{-1} \cdot \frac{8}{\pi r_\ast^2} \exp \left( -2 \frac{r}{r_\ast} \right)
\]

(III.68)

with the regular and spherically symmetric charge distribution \( (b) \tilde{k}_0(r) \) being normalized as follows:

\[
\int_0^\infty dr \ r \ (b) \tilde{k}_0(r) = \frac{2}{\pi}
\]

(III.69)

The corresponding electric potential \((p) A_0(\vec{r})\)

\[
(p) A_0(\vec{r}) = \alpha_s \int \frac{d^3\vec{r}'}{4\pi r' \sin \vartheta'} \left( \frac{(b) \tilde{k}_0(\vec{r}')}{||\vec{r} - \vec{r}'||} \right)
\]

(III.70)
will then be found to be also non-spherically symmetric, but it can be shown [6] that the binding ability of this potential is supplied mainly by its spherically symmetric part $(p)A_0(r)$, say). The latter part may be defined by suitable expansion of the denominator $||\vec{r} - \vec{r}'||$ in the integral (III.70), see ref. [6] for this method; or otherwise one may substitute the anisotropic density $(b)k_0(r)$ (III.68) into the RST action principle (II.56a)-(II.56b) and may then determine the desired isotropic part $(p)A_0(r)$ of $(b)A_0(r)$ (III.70) via the solution of the corresponding variational (i.e. Poisson) equation. Resorting here to the second method it suffices to consider merely the electrostatic part $W^{(e)}_{\text{RST}}$ of the two-particle action integral (II.56b) with $(1)A_0 = -(2)A_0 \div (p)A_0$ and $(1)\tilde{k}_0 = (2)\tilde{k}_0 \div (b)\tilde{k}_0$ which yields

$$W^{(e)}_{\text{RST}} = \int d^3\vec{r} \left( 2L_D^{(e)} + L_G^{(e)} \right)$$

$$= \int d^3\vec{r} \left( 2\hbar c \cdot (p)A_0(r) \cdot (b)k_0(r) - \frac{\hbar c}{4\pi\alpha_s} ||\nabla (p)A_0(r)||^2 \right)$$

$$= \hbar c \int dr \left[ \pi (p)A_0(r) \cdot (b)\tilde{k}_0(r) - \frac{r}{\alpha_s^2} \left( \frac{d[p]A_0(r)}{dr} \right)^2 \right].$$

Thus the electrostatic variational equation ($\delta W_{\text{RST}} = 0$) emerges as a spherically symmetric Poisson equation:

$$\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) [p]A_0(r) = -\frac{\pi}{2} \frac{\alpha_s}{r} (b)\tilde{k}_0(r) - \frac{r}{\alpha_s^2} \left( \frac{d[p]A_0(r)}{dr} \right)^2.$$  (III.72)

Finally, substituting here the assumed charge density $(b)\tilde{k}_0(r)$ (III.68) yields for the electric potential $(p)A_0(r)$

$$(p)A_0(r) = \frac{\alpha_s}{r} \left( 1 - \exp \left[ -\frac{2r}{r_s} \right] \right).$$  (III.73)

This interaction potential, being typical for the exotic quantum states, has some peculiar properties: First, it approaches the Coulomb potential ($\sim \frac{\alpha_s}{r}$).
at spatial infinity \((r \to \infty)\) as expected since it is generated by just one electric charge unit (i.e. elementary charge). Second, the potential remains finite at the origin \((r = 0)\)

\[
[p]A_0(0) = \frac{2\alpha_s}{r_*} \quad \text{(III.74)}
\]
as well as the corresponding electric field strength \(\vec{E}_p\) \(\text{(III.52a)}\)

\[
[p]E_r \bigg|_{r=0} = -\frac{d[p]A_0(r)}{dr} \bigg|_{r=0} = -\frac{2\alpha_s}{r_*^2}. \quad \text{(III.75)}
\]

Moreover, if the length parameter \(r_*\) tends to zero \((r_* \to 0)\), both the potential \(\text{(III.74)}\) and its field strength \(\text{(III.75)}\) approach infinity which says that the asymptotic Coulomb form fills then the whole three-space. This is clear because, in this limit \((r_* \to 0)\), the charge distribution \((b)\tilde{k}_0(r)\) becomes pointlike (see fig. 1 of ref. [6] for a sketch of the unconventional potentials).

However from the physical point of view, the most interesting feature of those potentials \((a)A_0(\vec{r})\) due to the exotic quantum states surely refers to the fact that they carry a finite energy content, in contrast to the Coulomb potential. Indeed, the electrostatic interaction energy \(\hat{E}^{(e)}_R\) of both charges is given by \([6]\)

\[
\hat{E}^{(e)}_R = -\frac{\hbar c}{4\pi\alpha_s} \int d^3\vec{r} \left\| \vec{\nabla} [p]A_0(\vec{r}) \right\|^2, \quad \text{(III.76)}
\]

and when the spherically symmetric approximation \([p]A_0(r)\) \(\text{(III.73)}\) is substituted herein, one finds the following result

\[
\hat{E}^{(e)}_R = -\frac{e^2}{r_*}. \quad \text{(III.77)}
\]

Incidentally, this is just the interaction energy of two point charges separated by the distance \(r_*\) which plays the role of a length parameter for our model charge distribution \((b)\tilde{k}_0(r)\) \(\text{(III.68)}\). This charge distribution becomes
pointlike when the length parameter $r_*$ tends to zero and, clearly, for this limit the interaction energy $\hat{E}_R^{(e)}$ (III.77) of both extended charge distributions becomes infinite, just as is the case with two point charges of vanishing separation ($r_* \to 0$).

Of course, the interaction energy $\hat{E}_R^{(e)}$ of the two particles is only a fraction of their total energy $E_T$ which must contain also the kinetic form of the particle energy. Indeed, this latter form of energy deserves a closer inspection, too; and this can be performed most adequately by setting up now the total energy functional $E_T$ through adequately exploiting the intrinsic RST logic.
IV Energy Functional

Besides the use of exotic quantum states and their unconventional potentials, it is necessary to introduce a further new element into the theory in order to deal successfully with the energy spectra of the bound systems: This refers to the construction of a suitable energy functional $E_T$, which equips the RST field configurations with an energy content being then immediately accessible to spectroscopic test. Recall here the fact that, though the variational approach (III.41) due to the total mass functional $\tilde{M}_T c^2$ (III.24) can be viewed as the relativistic generalization of the Ritz variational principle (III.33a)-(III.33b), this relativistic approach nevertheless fails to establish the gauge field equations and exclusively reproduces the eigenvalue equations for the matter fields; see the critical comments (i) and (ii) mentioned above. Therefore it suggests itself to restart from the original notion of field energy $E_T$ (II.53)-(II.55b) and to convert this to the wanted energy functional $\tilde{E}_T$.

A. Relativistic Construction

First, observe that the underlying energy-momentum densities $(D) T_{\mu\nu}$ and $(G) T_{\mu\nu}$ are already specified by equations (II.50a)-(II.52); and if one substitutes therein the stationary form of the matter and gauge fields (III.1a)-(III.2), one finds the individual energy contributions $E_D$ and $E_G$ (II.55a)-
appearing in the following form [7]:

\[ E_D = - \left( M_1 c^2 + M_1^{(e)} c^2 \right) + \left( M_2 c^2 - M_II^{(e)} c^2 \right) = M_T c^2 - \left( M_II^{(e)} c^2 + M_I^{(e)} c^2 \right) \]

\[ E_G \rightarrow \hat{E}_R = \frac{\hbar c}{4\pi \alpha_s} \int d^3 \vec{r} \left[ \vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r}) + \vec{H}_1(\vec{r}) \cdot \vec{H}_2(\vec{r}) \right] = \hat{E}_R^{(e)} + \hat{E}_R^{(m)}. \]

Here, the mass equivalents \( M_I^{(e)} c^2 \) and \( M_{II}^{(e)} c^2 \) of the electrostatic interaction energy have already been defined previously through equations (III.12a)-(III.12b). Furthermore, the gauge field energy \( E_G \) consists exclusively of the energy content \( \hat{E}_R \) due to the real gauge field modes \( A^a_\mu \), since the complex field modes \( B_\mu \) must be put to zero together with their energy content \( \hat{E}_C \) (see the discussion below equation (II.26)).

Clearly it is very tempting now to consider the total energy functional \( E_T \) (II.55b), with \( E_D \) and \( E_G \) being specified by the present equations (IV.1a)-(IV.1b), as the wanted object of our interest. The corresponding variational procedure (\( \delta E_T = 0 \)) must then be complemented by the former constraints of wave function normalization (III.22), which had to be applied already in connection with the mass functional approach (III.41). By this arrangement, one would be led to the following first proposal \( \tilde{E}_T^{(l)} \) for the desired energy functional:

\[ \tilde{E}_T^{(l)} = E_T + \lambda_{D(1)} \cdot N_{D(1)} + \lambda_{D(2)} \cdot N_{D(2)} \]

\[ = M_T c^2 + \left( \hat{E}_R^{(e)} - M_I^{(e)} c^2 - M_{II}^{(e)} c^2 \right) + \hat{E}_R^{(m)}. \]
equivalents $M_1^{(e)}$ and $M_{II}^{(e)}$ does spoil the partial success already obtained with the mass functional $\tilde{M}_T c^2$ (III.24) from which the matter eigenvalue equations can actually be deduced. The reason is that those mass equivalents (III.12a)-(III.13b) do also contain the wave functions $\psi_a(\vec{r})$, namely via the densities $(^a)k_\mu(\vec{r})$ (III.49a)-(III.49b). Therefore we have to eliminate again these redundant mass equivalents from our first proposal (IV.2), which can be achieved by expressing them in terms of the electromagnetic gauge field energy $\hat{E}_R^{(e)}$ and $\hat{E}_R^{(m)}$ (IV.1b) as follows:

$$\hat{E}_R^{(e)} = M_1^{(e)} c^2 = M_{II}^{(e)} c^2$$
$$\hat{E}_R^{(m)} = -M_1^{(m)} c^2 = -M_{II}^{(m)} c^2 ,$$

and these relations represent nothing else than the Poisson identities (III.58)-(III.59c). Thus using this electric coincidence (IV.3a) in order to eliminate the electric mass equivalents from the previous proposal $\tilde{E}_T^{(i)}$ (IV.2), one arrives at the next proposal $\tilde{E}_T^{(ii)}$:

$$\tilde{E}_T^{(ii)} = \tilde{M}_T c^2 - \hat{E}_R^{(e)} + \hat{E}_R^{(m)} .$$

This second proposal displays now some pleasant features and therefore must be expected to come close to the wanted final result: First, $\tilde{E}_T^{(ii)}$ contains the matter fields $\psi_a(\vec{r})$ only in form of the mass functional $\tilde{M}_T c^2$ and therefore the variational equations of the functional $\tilde{E}_T^{(ii)}$ must correctly reproduce the mass eigenvalue equations (III.15) and (III.17)! Second, returning for the moment to the matter energy $E_D$ (IV.1a) and substituting there the mass eigenvalues $M_a c^2$ (III.19a)-(III.19b) lets the matter energy $E_D$ appear
essentially as a sum of single-particle contributions

\[ E_D = \sum_{a=1}^{2} E_{D(a)} \]  \hspace{1cm} (IV.5)

with the individual contributions \( E_{D(a)} \) being given by

\[ E_{D(1)} = -\left( M_1 c^2 + M_1^{(e)} \right) = Z_{(1)}^2 \cdot M_p c^2 + 2T_{\text{kin}(1)} + M_1^{(m)} c^2 \]  \hspace{1cm} (IV.6a)

\[ E_{D(2)} = \left( M_2 c^2 - M_2^{(e)} \right) = Z_{(2)}^2 \cdot M_e c^2 + 2T_{\text{kin}(2)} + M_2^{(m)} c^2 . \]  \hspace{1cm} (IV.6b)

This physically plausible result says that the matter energy \( E_{D(a)} \) of either particle \( (a = 1, 2) \) consists of rest mass energy (first terms) plus kinetic energy (second terms) plus magnetic interaction energy (third terms), while for these single-particle energies \( E_{D(a)} \) there appears no electric interaction energy. The emergence of the magnetic kind of interaction energy seems to be somewhat unreasonable; but this is to be understood as the field theoretic counterpart of the minimal substitution \((\hat{p} \to \hat{p} - \frac{e}{c} \hat{A})\) for the conventional energy \( (\hat{H} = \frac{\hat{p}^2}{2m}) \) of a point particle moving in a magnetic field \( \hat{H} = \nabla \times \hat{A} \).

The third interesting point with that second proposal \( \hat{E}_T^{(||)} \) (IV.4) is now that by use of the explicit form (III.19a)-(III.19b) of the mass functionals together with the electric and magnetic Poisson identities, cf. (IV.3a)-(IV.3b), this proposal can be rewritten as the sum of the individual rest mass and kinetic energies plus the gauge field energy of the electric (e) and magnetic (m) type:

\[ E_T^{(||)} = \left( Z_{(1)}^2 \cdot M_p c^2 + Z_{(2)}^2 \cdot M_e c^2 \right) + 2 \left( T_{\text{kin}(1)} + T_{\text{kin}(2)} \right) + \left( \hat{E}_R^{(e)} - \hat{E}_R^{(m)} \right) . \]  \hspace{1cm} (IV.7)

Here, the validity of the normalization conditions (III.22) has tacitly been assumed and therefore they do not appear explicitly in the present
result for $E_{T}^{(||)}$. Observe however that the physically reasonable form (IV.7) of the energy functional is a consequence of the fact that its preliminary form $\tilde{E}_{T}^{(||)}$ (IV.4) contains the gauge field energies of electric and magnetic type with different signs. This important fact is the reason why the double counting of the electric term (IV.3a) in the sum $\tilde{M}_{T}c^{2}$ (III.24) of mass eigenvalues $M_{[a]}c^{2}$ (III.19a)-(III.19b) becomes compensated (see the discussion of this effect in refs. [5, 10]); and then the electric field energy $\hat{E}_{R}^{(e)}$ appears only once in the third proposal $E_{T}^{(||)}$ (IV.7). For the magnetic field energy $\hat{E}_{R}^{(m)}$ there occurs an analogous effect since its (negative) double-counting in the sum $\tilde{M}_{T}c^{2}$ (IV.4) is weakened so that the magnetic field energy $\hat{E}_{R}^{(m)}$ appears now in the third proposal (IV.7) with the opposite sign relative to its electric counterpart $\hat{E}_{R}^{(e)}$! This circumstance however does not influence the lowest-order approximation of the atomic energy levels because these are dominated by the electric interactions. Nevertheless for the higher-order approximations, the negative sign of the magnetic term will leave its imprint upon the predictions and therefore must receive confirmation or rejection by the observational data (see below).

The final step for the construction of the wanted energy functional must now be based upon the somewhat amazing circumstance that the third proposal $E_{T}^{(||)}$ (IV.7) would numerically produce the same energy upon an exact solution of the RST eigenvalue problem as does the original functional $E_{T}$ (II.53)-(II.55b), too. The reason is that the transcription of $E_{T}$ to $E_{T}^{(||)}$ relies exclusively upon the use of the Poisson identities which, however, do automatically hold for any exact solution of the RST eigenvalue problem! Nevertheless, this third form $E_{T}^{(||)}$ (IV.7) cannot be used for the
deduction of the mass eigenvalue and Poisson equations as the corresponding variational equations since this functional (IV.7) contains no coupling at all between the matter fields $\psi_a(\vec{r})$ and the gauge fields $^{(a)}A_0(\vec{r}), \vec{A}_a(\vec{r})$. Indeed, the coupling of matter and gauge fields has been eliminated on the way from the original $E_T$ (II.53) to the present $E_T^{(||)}$ (IV.7) via the Poisson identities, albeit under simultaneous preservation of the numerical value of the energy functional.

Therefore it finally becomes necessary to restore that lost coupling of matter and gauge fields, again under preservation of the numerical value of the energy functional. Naturally one expects that such a restoration of the desired coupling must be performed with regard again of the Poisson identities which thus have to take over the role of constraints for the variational procedure (see also ref. [6]). In this sense, one resorts to the method of Lagrangean multipliers with respect to both the wave function normalizations (III.22) and the Poisson identities (III.58)-(III.59c); and thus one complements the third proposal $E_T^{(||)}$ (IV.7) to the final result $\tilde{E}_T$ in the following way:

$$\tilde{E}_T = E_T^{(||)} + \sum_{a=1}^{2} \left( \lambda_{D(a)} \cdot N_{D(a)} + \lambda_{G(e)(a)} \cdot N_{G(e)(a)} + \lambda_{G(m)(a)} \cdot N_{G(m)(a)} \right). \quad \text{(IV.8)}$$

Here it is now a standard exercise to convince oneself of the fact that the variational equations due to this functional $\tilde{E}_T$ actually are just the mass eigenvalue equations (III.15) and (III.17) together with the electric and magnetic Poisson equations (III.47a)-(III.47d), provided the Lagrangean matter multipliers $\lambda_{D(a)}$ are given in terms of the mass eigenvalues $M_a$ as shown by equations (III.23a)-(III.23b) and furthermore the gauge field multipli-
ers $\lambda_{G(a)}^{(e,m)}$ are specified as follows ($a = 1, 2$)

$$\lambda_{G(a)}^{(e)} = -\frac{\hbar c}{4\pi\alpha_s}$$  \hspace{1cm} (IV.9a)

$$\lambda_{G(a)}^{(m)} = \frac{\hbar c}{4\pi\alpha_s}.$$  \hspace{1cm} (IV.9b)

Thus collecting all the partial results, the ultimate form of the wanted energy functional $\tilde{E}_T$ is the following:

$$\tilde{E}_T = Z_1^2 \cdot M_p c^2 + Z_2^2 \cdot M_e c^2 + 2 \left( T_{\text{kin}(1)} + T_{\text{kin}(2)} \right) + \left( \hat{E}_R^{(e)} - \hat{E}_R^{(m)} \right)$$

$$+ M_1 c^2 \cdot N_{D(1)} - M_2 c^2 \cdot N_{D(2)} + \frac{\hbar c}{4\pi\alpha_s} \sum_{a=1}^{2} \left( N_{G(a)}^{(m)} - N_{G(a)}^{(e)} \right).$$  \hspace{1cm} (IV.10)

The practical usefulness of this ultimate energy functional $\tilde{E}_T$ (IV.10) refers mainly to those situations where the RST eigenvalue problem cannot be solved exactly so that one is forced to look for approximate solutions (which will be mostly the case). But fortunately, a convenient approximation method is now at hand in form of the energy functional $\tilde{E}_T$, so that one can test certain trial functions for the Dirac spinors $\psi_a(\vec{r})$ and for the gauge potentials $^{(a)}A_0(\vec{r}), \vec{A}_a(\vec{r})$. These trial functions will depend upon some parameters ($b_k$) so that, after substitution of the trial functions into the energy functional $\tilde{E}_T$ (IV.10), the latter becomes an ordinary function of the ansatz parameters $b_k$: $\tilde{E}_T = \tilde{E}_T(b_k)$. Finally, looking for the minimally possible value of that function $\tilde{E}_T(b_k)$ yields a more or less good approximation for the wanted energy eigenvalue of the RST eigenvalue problem. For an example of this type see ref. [6]. However a further improvement of this general approximation procedure may be achieved by not trying some independent functions for the gauge potentials $^{(a)}A_0(\vec{r})$ and $\vec{A}_a(\vec{r})$ but by trying merely
for the (normalized) wave functions $\psi_a(\vec{r})$ and then calculating (exactly) the associated gauge potentials from their Poisson equations (III.47a)-(III.47d), preferably in form of the solutions (III.48a)-(III.48d). Clearly, through such a procedure the Poisson identities (III.58)-(III.59c) will be satisfied exactly, though the associated solution of the RST eigenvalue problem is an approximation. But the advantage is here that all the constraints (second line on the right of equation (IV.10)) can be omitted and one can concentrate upon the physical terms (first line) which effectively is $E^{(\text{II})}_T$ (IV.7). Thus it will be sufficient to look for the minimum of the corresponding function $E^{(\text{II})}_T(b_k)$. Subsequently we will exemplify this procedure by means of the positronium groundstate.

As a preparation of this groundstate treatment, it is very instructive and convenient to specify the functional $E^{(\text{II})}_T$ in terms of the wave amplitudes $(a)^{\tilde{R}_\pm}$ and $(a)^{\tilde{S}_\pm}$ (III.62a)-(III.62b). First, the mass renormalization factors $Z^2_{(a)}$ (III.11) are found to be of the following form

$$Z^2_{(a)} = \frac{1}{2} \int d^2\vec{r} \left( (a)^{\tilde{R}^2_+} + (a)^{\tilde{S}^2_+} - (a)^{\tilde{R}^2_-} - (a)^{\tilde{S}^2_-} \right),$$

(IV.11)

where the unique wave amplitudes $(a)^{\tilde{R}_\pm}$, $(a)^{\tilde{S}_\pm}$ are assumed (for the sake of simplicity) to depend only upon the radial ($r$) and longitudinal ($\vartheta$) variables: $(a)^{\tilde{R}_\pm}(r, \vartheta)$; $(a)^{\tilde{S}_\pm}(r, \vartheta)$; and the remaining two-dimensional volume element $d^2\vec{r}$ is then given in terms of these variables as

$$d^2\vec{r} = rdrd\vartheta.$$  

(IV.12)

Next, the kinetic energies $T_{\text{kin}(a)}$ (III.20a)-(III.20b) of both particles ($a = 1, 2$) are found to split up into the radial ($T_r$) and longitudinal ($T_\vartheta$)
part, i.e.

\[ T_{\text{kin}(a)} = T_{r(a)} + T_{\vartheta(a)} \]  

(IV.13)

with the radial part being given by

\[ T_{r(a)} = (-1)^{a-1} \frac{\hbar c}{4} \int d^2 \vec{r} \left( \frac{(a)\tilde{R}_-}{r} \cdot \frac{\partial (a)\tilde{R}_+}{\partial r} - \frac{(a)\tilde{R}_+}{r} \cdot \frac{\partial (r(a)\tilde{R}_-)}{\partial r} \right. \]

\[ + \frac{(a)\tilde{S}_-}{r} \cdot \frac{\partial (a)\tilde{S}_+}{\partial r} - \frac{(a)\tilde{S}_+}{r} \cdot \frac{\partial (r(a)\tilde{S}_-)}{\partial r} \left. \right) \]  

(IV.14)

and analogously the longitudinal part by

\[ T_{\vartheta(a)} = (-1)^{a-1} \frac{\hbar c}{4} \int \frac{d^2 \vec{r}}{r} \left( \frac{(a)\tilde{S}_-}{r} \cdot \frac{\partial (a)\tilde{S}_+}{\partial \vartheta} - \frac{(a)\tilde{S}_+}{r} \cdot \frac{\partial (r(a)\tilde{S}_-)}{\partial \vartheta} \right. \]

\[ + \frac{(a)\tilde{R}_-}{r} \cdot \frac{\partial (a)\tilde{R}_+}{\partial \vartheta} - \frac{(a)\tilde{R}_+}{r} \cdot \frac{\partial (r(a)\tilde{R}_-)}{\partial \vartheta} \left. \right) \]  

(IV.15)

Furthermore, the electric and magnetic field energies \( \tilde{E}_R^{(e)} \) and \( \tilde{E}_R^{(m)} \) (IV.11b) read in terms of the gauge potentials \((a)A_0(\vec{r})\) and \((a)A_\varphi(\vec{r})\) (III.65)

\[ \tilde{E}_R^{(e)} = \frac{\hbar c}{2\alpha_s} \int d^2 \vec{r} r \sin \vartheta \left( \frac{\partial (1)A_0(\vec{r})}{\partial r} \cdot \frac{\partial (2)A_0(\vec{r})}{\partial r} + \frac{1}{r^2} \frac{\partial (1)A_0(\vec{r})}{\partial \vartheta} \cdot \frac{\partial (2)A_0(\vec{r})}{\partial \vartheta} \right) \]  

(IV.16a)

\[ \tilde{E}_R^{(m)} = \frac{\hbar c}{2\alpha_s} \int dr d\vartheta \sin \vartheta \left( \frac{\partial (r(1)A_\varphi)}{\partial r} \cdot \frac{\partial (r(2)A_\varphi)}{\partial r} \right. \]

\[ \left. + \frac{1}{\sin^2 \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta (1)A_\varphi \right) \cdot \frac{\partial}{\partial \vartheta} \left( \sin \vartheta (2)A_\varphi \right) \right) \]  

(IV.16b)

Finally, both kinds of constraints, i.e. the normalization conditions (II.44) and the Poisson identities (III.58)-(III.59c), must also be rewritten in terms of the wave amplitudes; but it is not necessary to reproduce this here because for the subsequent treatment of the positronium groundstate we will use trial
functions satisfying a priori all those constraints and therefore we can rely directly upon the truncated functional $E^{(||)}_T$ (IV.7) without loss of accuracy.

Now in order to support the confidence in the established functional $E_T$ (IV.10), one can look for both the mass eigenvalue and Poisson equations in terms of the wave amplitudes $(a)\tilde{R}_\pm, (a)\tilde{S}_\pm$ by carrying out the variational procedure ($\delta E_T = 0$) just with respect to these wave amplitudes and gauge potentials. Clearly, one will then actually recover the former mass eigenvalue equations (III.66a)-(III.66d) together with the Poisson equations (III.47a)-(III.47d). For their magnetic part (III.47c)-(III.47d) one may resort for the moment to the special case of circular flow around the z-axis, cf. (III.65); and in this special case the magnetic Poisson equations for the azimuthal component $(a)A_\phi$ read

\[
\Delta^{(1)} A_\phi - \frac{(1) A_\phi}{r^2 \sin^2 \vartheta} = -4\pi \alpha_s^{(1)} k_\phi \tag{IV.17a}
\]

\[
\Delta^{(2)} A_\phi - \frac{(2) A_\phi}{r^2 \sin^2 \vartheta} = 4\pi \alpha_s^{(2)} k_\phi , \tag{IV.17b}
\]

with the circular current components $(a)k_\phi$ being given by equation (III.64).

**B. Non-Relativistic Approximation**

For a first practical test of the present construction of an RST energy functional $E_T$ (IV.10), it may be sufficient to restrict oneself to the non-relativistic approximation. Clearly, if such an approximation would fail to meet with the well-known results of ordinary non-relativistic quantum mechanics, one would not try to further elaborate the corresponding relativistic situation. Fortunately, the subsequent demonstration by means of the positronium groundstate points just into the other direction: The *conventional* groundstate energy can be *exactly* reproduced by an appropriate trial
function for the non-relativistic limit of the RST functional \( \tilde{E}_T \) [IV.10], or \( E_T^{(||)} \) [IV.7], resp. In order to find the desired non-relativistic limit of the functional \( E_T^{(||)} \), it is merely necessary to look for the non-relativistic forms of its constituents, i.e. rest mass and kinetic energy and the field energy of the electric \( \tilde{E}_R^{(e)} \) and magnetic kind \( \tilde{E}_R^{(m)} \).

Naturally, the non-relativistic situation becomes even further simplified if one restricts oneself to the spherically symmetric approximation by neglecting the magnetic interactions. As a matter of course, the electric fields \( \tilde{E}_a(\vec{r}) \) can easily be visualized to be spherically symmetric (\( \sim \) hedgehog configuration) in contrast to the magnetic fields which mostly obey a dipole (or higher) symmetry. Therefore it is favorable to start with the spherically symmetric configurations of the purely electric type.

Turning here first to the mass eigenvalue equations (III.66a)-(III.66d), one usually assumes that the “negative” Pauli components \( (a)\tilde{R}_- \), \( (a)\tilde{S}_- \) are much smaller than their “positive” counterparts \( (a)\tilde{R}_+ \) and \( (a)\tilde{S}_+ \), so that the non-relativistic form of the mass eigenvalue equations is obtained by simply eliminating those negative components \( (a)\tilde{R}_- \), \( (a)\tilde{S}_- \). The residual eigenvalue equations for the positive components of the first particle \( (a = 1) \) do appear then in the following form:

\[
-\frac{\hbar^2}{2M_p} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot \frac{\partial (1)\tilde{R}_+}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (1)\tilde{R}_+}{\partial \vartheta^2} \right] + \hbar c (2) A_0(\vec{r}) \cdot (1)\tilde{R}_+ = E_{S(1)} \cdot (1)\tilde{R}_+ \tag{IV.18a}
\]

\[
-\frac{\hbar^2}{2M_p} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot \frac{\partial (1)\tilde{S}_+}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (1)\tilde{S}_+}{\partial \vartheta^2} \right] + \hbar c (2) A_0(\vec{r}) \cdot (1)\tilde{S}_+ = E_{S(1)} \cdot (1)\tilde{S}_+ . \tag{IV.18b}
\]
Here, the magnetic interactions are neglected \(^{(a)}A_\phi \to 0\) together with the relativistic effects because both phenomena are mostly of the same (small) order of magnitude. Moreover, the (conventional) non-relativistic Schrödinger eigenvalues \(E_{S(a)}\) are defined as in equations [III.30a]-[III.30b]. The case of the second particle \((a = 2)\) is not written down because it can easily be supplied by means of the particle permutation symmetry \((1 \leftrightarrow 2)\), see ref. [7]. However the important points with the non-relativistic eigenvalue equations [IV.18a]-[IV.18b] refer now to the facts that (i) they are not of the usual Schrödinger form [III.32a]-[III.32b] and (ii) the spin-up \((\sim \hat{R}_+)\) and spin-down \((\sim \hat{S}_+)\) configurations are decoupled. The latter circumstance admits us to conceive either of the two single-particle spins to point definitely into the positive or negative \(z\)-direction and their combination to the para- and ortho-states of the two-particle system will then intuitively be evident.

Naturally, one expects that these non-relativistic eigenvalue equations, such as [IV.18a]-[IV.18b], should emerge as the variational equations due to the non-relativistic approximation \((\tilde{E}_T^{(0)}, \text{say})\) of the original RST energy functional \(\tilde{E}_T\) [IV.10]. Indeed, one is easily convinced that this supposition is true; namely the elimination of the negative Pauli components \((a)\tilde{R}_-, (a)\tilde{S}_-\) from the relativistic kinetic energies \(T_{\text{kin}(a)}\) [IV.13]-[IV.15] yields [6]

\[
(Z_1^2 - 1)M_p c^2 + 2T_{\text{kin}(1)} \Rightarrow E_{\text{kin}(1)} + E_{W(1)} \quad (IV.19a)
\]

\[
(Z_2^2 - 1)M_e c^2 + 2T_{\text{kin}(2)} \Rightarrow E_{\text{kin}(2)} + E_{W(2)} \quad (IV.19b)
\]
with the non-relativistic kinetic energies $E_{\text{kin}(a)}$ being given by

$$E_{\text{kin}(1)} = \frac{\hbar^2}{4M_p} \int d^2\mathbf{r} \left[ \left( \frac{\partial (1) \tilde{R}_+}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial (1) \tilde{R}_+}{\partial \theta} \right)^2 \right] \quad (\text{IV.20a})$$

$$E_{\text{kin}(2)} = \frac{\hbar^2}{4M_e} \int d^2\mathbf{r} \left[ \left( \frac{\partial (2) \tilde{S}_+}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial (2) \tilde{S}_+}{\partial \theta} \right)^2 \right], \quad (\text{IV.20b})$$

and the “winding energies” $E_{W(a)}$ being given by

$$E_{W(1)} = \frac{\hbar^2}{4M_p} \int d^2\mathbf{r} \left[ \frac{\partial (1) \tilde{R}_+}{\partial r} \cdot \frac{\partial (1) \tilde{S}_+}{\partial \theta} - \frac{\partial (1) \tilde{S}_+}{\partial r} \cdot \frac{\partial (1) \tilde{R}_+}{\partial \theta} \right] \quad (\text{IV.21a})$$

$$E_{W(2)} = \frac{\hbar^2}{4M_e} \int d^2\mathbf{r} \left[ \frac{\partial (2) \tilde{R}_+}{\partial r} \cdot \frac{\partial (2) \tilde{S}_+}{\partial \theta} - \frac{\partial (2) \tilde{S}_+}{\partial r} \cdot \frac{\partial (2) \tilde{R}_+}{\partial \theta} \right]. \quad (\text{IV.21b})$$

Observe here that, for the kinetic energies $E_{\text{kin}(a)}$, we made use of the non-relativistic decoupling of the spin-up and spin-down components and thus adopted the first spin (IV.18a) pointing in the positive z-direction ($\sim (1) \tilde{R}_+$) and the second spin (IV.18b) in the negative z-direction ($\sim (2) \tilde{S}_+$). Clearly, the other combinations of the spin directions \{(1) $\tilde{R}_+$, (2) $\tilde{S}_+$\}, \{(1) $\tilde{S}_+$, (2) $\tilde{R}_+$\}, \{(1) $\tilde{R}_+$, (2) $\tilde{S}_+$\} are equally well possible, see below for the para- and ortho-configurations. Fortunately, through this choice of definite spin directions for any particle, the winding energies (IV.21a)-(IV.21b) become zero so that one can restrict oneself to the kinetic energies (IV.20a)-(IV.20b) alone. Of course the rest mass energies need not be taken into account for a non-relativistic treatment and therefore have been omitted, cf (IV.19a)-(IV.19b).

Next, the non-relativistic form of the electric and magnetic gauge field energies $\hat{E}^{(e)}_R$ and $\hat{E}^{(m)}_R$ remains the same as in the relativistic case, i.e. in
terms of the static gauge potentials \(^{(a)}A_0(\vec{r})\) and \(\vec{A}_a(\vec{r})\) (cf. (III.52a)-(III.52b):

\[
\hat{E}^{(e)}_R = \frac{\hbar c}{4\pi\alpha_s} \int d^3\vec{r} \left( \vec{\nabla}^{(1)}A_0(\vec{r}) \right) \cdot \left( \vec{\nabla}^{(2)}A_0(\vec{r}) \right) \quad \text{(IV.22a)}
\]

\[
\hat{E}^{(m)}_R = \frac{\hbar c}{4\pi\alpha_s} \int d^3\vec{r} \left( \vec{\nabla} \times \vec{A}_1 \right) \cdot \left( \vec{\nabla} \times \vec{A}_2 \right). \quad \text{(IV.22b)}
\]

But for the explicit calculation of the non-relativistic potentials \(^{(a)}A_0(\vec{r})\) and \(^{(a)}A_\phi(\vec{r})\) from the Poisson equations (III.47a)-(III.47b) and (III.47c)-(III.47d) one will use the corresponding non-relativistic approximations for the charge and current densities, cf. (III.63a) and (III.64)

\[
(1)^{k_0}(\vec{r}) \overset{\text{def}}{=} \frac{(1)^{k_0}(r, \vartheta)}{4\pi r \sin \vartheta} \Rightarrow \frac{(1)^{\tilde{R}_+^2}(r, \vartheta)}{4\pi r \sin \vartheta} \quad \text{(IV.23a)}
\]

\[
(2)^{k_0}(\vec{r}) \overset{\text{def}}{=} \frac{(2)^{k_0}(r, \vartheta)}{4\pi r \sin \vartheta} \Rightarrow \frac{(2)^{\tilde{S}_+^2}(r, \vartheta)}{4\pi r \sin \vartheta} \quad \text{(IV.23b)}
\]

\[
(1)^{k_\phi}(\vec{r}) \overset{\text{def}}{=} \frac{(1)^{k_\phi}(r, \vartheta)}{2\pi r \sin \vartheta} \Rightarrow \frac{(1)^{\tilde{R}_+} \cdot (1)^{\tilde{R}_-}}{2\pi r} \quad \text{(IV.23c)}
\]

\[
(2)^{k_\phi}(\vec{r}) \overset{\text{def}}{=} \frac{(2)^{k_\phi}(r, \vartheta)}{2\pi r \sin \vartheta} \Rightarrow -\frac{(2)^{\tilde{S}_+} \cdot (2)^{\tilde{S}_-}}{2\pi r}. \quad \text{(IV.23d)}
\]

Observe here again that, in the contrast to the charge densities \(^{(a)}k_0(\vec{r})\), the current densities \(^{(a)}k_\phi(\vec{r})\) are built up by both the positive \(^{(a)}\tilde{R}_+, (a)\tilde{S}_+\) and negative \(^{(a)}\tilde{R}_-, (a)\tilde{S}_-\) wave amplitudes while, properly speaking, the negative amplitudes \(^{(a)}\tilde{R}_-, (a)\tilde{S}_-\) should be neglected against their positive counterparts \(^{(1)}\tilde{R}_+, (1)\tilde{S}_+\) for the non-relativistic limit. This demonstrates that it appears somewhat inconsequent to retain the magnetic (i.e. spin-spin) interactions for the non-relativistic approximation because their order of magnitude may be the same as that of the other dominant relativistic effects. Nevertheless we will not drop the magnetic effects for our non-relativistic
approach because one can still deal with the effect of ortho-para splitting of
the energy levels from a more qualitative viewpoint.

In this sense one has to renounce on the inclusion of the magnetic ef-
fects for the purpose of deducing the non-relativistic eigenvalue equations
(IV.18a)-(IV.18b) from the desired non-relativistic version $\tilde{E}_T^{(0)}$ of the or-
3
iginal functional $\tilde{E}_T$ (IV.10). Consequently one drops also the magnetic con-
straints $N_{G(m)}$ (III.59b)-(III.59c) from the latter functional and retains only
the electric constraints $N_{G(e)}$ (III.58)-(III.59a) which do then appear in the
following form:

$$N_{G(e)}(1) \Rightarrow N_{G(0)}(1) = \int d^3 \vec{r} \left[ \left( \vec{\nabla}(1)A_0(\vec{r}) \right) \cdot \left( \vec{\nabla}(2)A_0(\vec{r}) \right) + \alpha_s \frac{(1)A_0(\vec{r}) \cdot (2)\tilde{S}^2_+}{r \sin \vartheta} \right]$$

(IV.24a)

$$N_{G(e)}(2) \Rightarrow N_{G(0)}(2) = \int d^3 \vec{r} \left[ \left( \vec{\nabla}(1)A_0(\vec{r}) \right) \cdot \left( \vec{\nabla}(2)A_0(\vec{r}) \right) - \alpha_s \frac{(2)A_0(\vec{r}) \cdot (1)\tilde{R}^2_+}{r \sin \vartheta} \right] ,$$

(IV.24b)

where the non-relativistic approximations (IV.23a)-(IV.23b) of the charge
densities $(^0)k_0(\vec{r})$ have already been respected. Clearly, the latter approx-
imations for the charge densities must also be used for the constraints of
wave function normalization (III.22) which then appear in their following
non-relativistic forms $N_{D(0)}$:

$$N_{D(1)} \Rightarrow N_{D(0)}(1) = \frac{1}{2} \int d^2 \vec{r} (1)\tilde{R}^2_+(r, \vartheta) - 1 = 0$$

(IV.25a)

$$N_{D(2)} \Rightarrow N_{D(0)}(2) = \frac{1}{2} \int d^2 \vec{r} (2)\tilde{S}^2_+(r, \vartheta) - 1 = 0 .$$

(IV.25b)

Finally, collecting all the non-relativistic approximations and applying
again the method of Lagrangean multipliers lets appear the wanted non-
relativistic approximation $\tilde{E}_T^{(0)}$ of the original functional $\tilde{E}_T$ (IV.10) in the following form:

$$\tilde{E}_T^{(0)} = E_{\text{kin}(1)} + E_{\text{kin}(2)} + \tilde{E}_R^{(e)}$$

$$+ \sum_{a=1}^{2} \lambda_S(a) \cdot N_D^{(0)}(a) - \frac{\hbar c}{4\pi\alpha_s} \sum_{a=1}^{2} N_G^{(0)}(a).$$

(IV.26)

Here it is again a nice consistency check to convince oneself of the fact that the usual variational procedure ($\delta \tilde{E}_T^{(0)} = 0$) actually does reproduce the claimed non-relativistic forms (IV.18a)-(IV.18b) and (III.47a)-(III.47b) of the mass eigenvalue and Poisson equations. The non-relativistic multipliers $\lambda_{S(a)}$ turn out as the conventional Schrödinger energie eigenvalues

$$\lambda_{S(a)} = -E_{S(a)},$$

(IV.27)

which compares to the analogous result (III.40) of the Ritz-Hardy-Schrödinger approach. Clearly according to our present choice of the negative z-direction for the second particle spin, the second eigenvalue equation (IV.18b) for the first particle must be replaced for the present situation by

$$-\frac{\hbar^2}{2Me} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \cdot \frac{\partial (2) \tilde{S}_+}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (2) \tilde{S}_+}{\partial \vartheta^2} \right] - \hbar c \cdot (1) A_0 \cdot (2) \tilde{S}_+ = -\lambda_{S(2)} \cdot (2) \tilde{S}_+$$

(IV.28)

for the second particle. Furthermore, the non-relativistic approximations of the electric Poisson equations do now appear as the following variational equations:

$$\Delta^{(1)} A_0 = -\alpha_s \frac{\tilde{R}_+^2}{r \sin \vartheta}$$

(IV.29a)

$$\Delta^{(2)} A_0 = \alpha_s \frac{\tilde{S}_+^2}{r \sin \vartheta}.$$
The important point here is that, despite the many similarities between the non-relativistic limit of RST and the conventional Ritz-Hartree-Schrödinger approach, there are also characteristic differences of both approaches which are in favour of RST. This will readily be demonstrated by considering a numerical example. The main difference refers to the gauge potentials, e.g. those of the electric type (III.48a)-(III.48b), which by means of the non-relativistic approximations (IV.23a)-(IV.23b) for the charge densities \( k_0(\vec{r}) \) appear as

\[
(1) A_0(\vec{r}) = \frac{\alpha_s}{4\pi} \int \frac{d^3\vec{r}'}{r'\sin\vartheta'} \cdot \frac{(1)\tilde{R}_2^2(r', \vartheta')}||r - \vec{r}'||
\]

\[
(2) A_0(\vec{r}) = -\frac{\alpha_s}{4\pi} \int \frac{d^3\vec{r}'}{r'\sin\vartheta'} \cdot \frac{(2)\tilde{S}_2^2(r', \vartheta')}||r - \vec{r}'||
\]

(IV.30a) (IV.30b)

Obviously, these gauge potentials due to the exotic states must be more singular as the Hartree potentials (III.50a)-(III.50b) which are due to the non-singular Hartree wave functions \( \varphi_a(\vec{r}) \) (III.31), see the example (III.73).

C. Magnetic Interactions

In the contrast to the electric fields \( \vec{E}_a(\vec{r}) \) (III.52a), the magnetic fields \( \vec{H}_a(\vec{r}) \) (III.52b) cannot obey the SO(3) symmetry because they have dipole character rather than monopole character like their electric counterparts. Therefore it will become necessary to apply more complicated approximation techniques; but fortunately it is not necessary to explicitly solve the magnetic Poisson equations (III.47c)-(III.47d) for the three-vector potentials \( \vec{A}_a(\vec{r}) \), e.g. in form of the special solutions (III.48c)-(III.48d). Rather it is sufficient to determine the magnetic fields \( \vec{H}_a(\vec{r}) \) \( (a = 1, 2) \) directly from the Abelian Maxwell equations

\[
\vec{\nabla} \times \vec{H}_a = 4\pi\alpha_s j_a
\]

(IV.31)
which is the three-vector form of the relativistic versions (II.27a)-(II.27b).

Nevertheless, one has to insist on the existence of the corresponding vector potentials \( \vec{A}_a(\vec{r}) \) (III.52b), namely in order that the magnetic Poisson identities (III.59b)-(III.59c) can be satisfied and thus the corresponding magnetic constraints in the energy functional \( \tilde{E}_T \) (IV.10) can be dropped. Indeed, in the latter case one can restrict oneself to the physical terms of the energy \( \tilde{E}_T \) (i.e. the first line on the right-hand side of (IV.10)), where the magnetic interaction energy is then simply given by \( \tilde{E}_R^{(m)} \) (IV.11) in terms of the magnetic fields \( \vec{H}_a \) themselves. But clearly if there is no difficulty with the determination of the vector potentials \( A_a(\vec{r}) \) (III.52b) directly from their Poisson equations, one may calculate the magnetic interaction energy \( \tilde{E}_R^{(m)} \) also in terms of these vector potentials \( \vec{A}_a(\vec{r}) \) as shown by equation (IV.22b).

Following here the first path (i.e. determination of the magnetic fields \( \vec{H}_a(\vec{r}) \) directly from the Maxwell equations (IV.31) with omission of the vector potentials \( \vec{A}_a(\vec{r}) \)), one additionally has to impose the conditions of vanishing sources

\[
\vec{\nabla} \cdot \vec{H}_a(\vec{r}) = 0 , \tag{IV.32}
\]

in order to ensure the existence of the vector potentials \( \vec{A}_a(\vec{r}) \). Since the Maxwell currents \( \vec{j}_a(\vec{r}) \) are connected to the Dirac currents \( \vec{k}_a(\vec{r}) \) by equations (II.43a)-(II.43b), the Abelian Maxwell equations (IV.31) read in the component form of the spherical polar coordinates \( (a = 1,2) \)

\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} (r \cdot (a) H_\phi) - \frac{\partial (a) H_r}{\partial \theta} \right] = 4\pi \alpha_s (a) k_\phi , \tag{IV.33}
\]

where the azimuthal components \( (a) k_\phi \) of the Dirac currents \( \vec{k}_a(\vec{r}) \) are specified by equations (IV.23c)-(IV.23d). However for the present non-relativistic
limit, the “negative” Pauli wave amplitudes \( (a)\tilde{R}_- \), \( (a)\tilde{S}_- \) must be approximately traced back to their “positive” counterparts \( (a)\tilde{R}_+ \), \( (a)\tilde{S}_+ \) through [6, 7]

\[
(1)\tilde{R}_+ \simeq \frac{\hbar}{2M_p c} \left( \frac{\partial(1)\tilde{R}_+}{\partial r} + \frac{1}{r} \frac{\partial(1)\tilde{S}_+}{\partial \vartheta} \right) \Rightarrow \frac{\hbar}{2M_p c} \frac{\partial(1)\tilde{R}_+}{\partial r}
\]

(IV.34a)

\[
(1)\tilde{S}_+ \simeq \frac{\hbar}{2M_p c} \left( \frac{\partial(1)\tilde{S}_+}{\partial r} - \frac{1}{r} \frac{\partial(1)\tilde{R}_+}{\partial \vartheta} \right) \Rightarrow \frac{\hbar}{2M_p c} \frac{\partial(1)\tilde{S}_+}{\partial r}
\]

(IV.34b)

and analogously for the second particle (1 \( \rightarrow \) 2; \( M_p \rightarrow M_e \)). This approximate procedure recasts the non-relativistic current components \( (a)k_\phi \) (IV.23c)-(IV.23d) to the following form:

\[
(1)k_\phi \simeq \frac{\hbar}{2M_p} \cdot \frac{(1)\tilde{R}_+ \cdot \frac{\partial(1)\tilde{R}_+}{\partial r}}{2\pi r}
\]

(IV.35a)

\[
(2)k_\phi \simeq -\frac{\hbar}{2M_e} \cdot \frac{(2)\tilde{S}_+ \cdot \frac{\partial(2)\tilde{S}_+}{\partial r}}{2\pi r}
\]

(IV.35b)

Evidently, the currents \( \vec{k}_a(\vec{r}) \) become singular at the origin (\( r \rightarrow 0 \)) unless the wave amplitudes \( (1)\tilde{R}_+ \), \( (1)\tilde{S}_+ \) or their derivatives do vanish for \( r \rightarrow 0 \).

Of course, these singular currents will imply a corresponding singular behavior of the magnetic components \( (a)H_r \) and \( (a)H_\vartheta \) as solutions of the Maxwell equations (IV.33). In order to elaborate this singular behavior in some more detail, one splits off the short-range magnetic field \( \tilde{H}_a'(\vec{r}) \) from its far-range counterpart which can always be taken as a gradient field, i.e. we put

\[
\tilde{H}_a(\vec{r}) = \tilde{H}_a'(r) + \vec{\nabla} \eta(a)
\]

(IV.36)

The inclusion of such a gradient field is necessary in order to satisfy the divergence relation (IV.32) which yields for the magnetic potentials \( \eta(a) \)

\[
\Delta \eta(a) = - \left( \vec{\nabla} \cdot \tilde{H}_a' \right)
\]

(IV.37)
Thus the magnetic Maxwell equations (IV.31) do fix only the curl of the short-range fields \( \vec{H}_a'(\vec{r}) \) and leave the determination of the magnetic potentials \( \eta(a) \) to the Poisson equation (IV.37).

Observe also that the superposition of a magnetic potential \( \eta(a) \) to the original magnetic field \( \vec{H}_a'(r) \) (IV.36), so that the resulting field \( \vec{H}_a(\vec{r}) \) gets vanishing divergence (IV.32), does not only ensure the existence of a vector potential \( \vec{A}_a(\vec{r}) \) for \( \vec{H}_a(=\vec{\nabla} \times \vec{A}_a) \) but additionally implies a further effect which refers to the magnetic field energy \( \hat{E}_R^{(m)} \) (IV.1b). Indeed the latter object reads by use of the superposition (IV.36)

\[
\hat{E}_R^{(m)} = \frac{\hbar c}{4\pi \alpha_s} \int d^3 \vec{r} \vec{H}'_1(\vec{r}) \cdot \vec{H}'_2(\vec{r})
\]

\[
= \frac{\hbar c}{4\pi \alpha_s} \int d^3 \vec{r} \left( \vec{H}'_1 \cdot \vec{H}'_2 + \vec{H}'_1 \cdot \vec{\nabla} \eta(2) + \vec{H}'_2 \cdot \vec{\nabla} \eta(1) + \vec{\nabla} \eta(1) \cdot \vec{\nabla} \eta(2) \right).
\]

(IV.38)

Here it is easy to see that for given fields \( \vec{H}_a'(r) \) this functional of the magnetic potentials \( \eta(a) \) is stationary just upon the solutions of the Poisson equations (IV.37)! Thus the introduction of the magnetic potentials \( \eta(a) \) does not only guarantee the existence of the vector potentials \( \vec{A}_a(\vec{r}) \) but it additionally lets the magnetic energy functional \( \hat{E}_R^{(m)} \) appear stationary (independent of the stationarity of the total functional \( \hat{E}_T \)).

The ansatz for the components of the short-range fields \( \vec{H}_a' \) is now \( (a = 1, 2) \):

\[
(a)H'_r(r, \vartheta) = (a)H'_\varphi(r, \vartheta) \equiv 0 \quad \text{(IV.39a)}
\]

\[
(a)H'_\varphi(r, \vartheta) = h_a(r) \quad \text{(IV.39b)}
\]

i.e. the integral lines of the short-range fields \( \vec{H}_a' = h_a(r)\vec{e}_\varphi \) are circles
\[ r = \text{const.} \text{ in the two-planes } \phi = \text{const.} \text{ with the center located at the origin } r = 0. \text{ Clearly, such a field must necessarily be singular along the z-axis but this singularity does not contribute to the magnetic field energy } \hat{E}_{R}^{(m)} . \]

The magnetic ansatz functions \( h_a(r) \) are linked to the azimuthal components \( (a)k_\phi(r) \) \[\text{[(IV.35a)-[IV.35b]]} \] via the Maxwell equations \[\text{[(IV.33)]} \], which do apply also to the short-range components \( (a)H'_r, (a)H'_\phi \), yielding
\[
\begin{align*}
\frac{1}{r} \frac{d}{dr} (r h_1) &= 4\pi \alpha_s (1)^2 \, k_\phi \\
\frac{1}{r} \frac{d}{dr} (r h_2) &= -4\pi \alpha_s (2)^2 \, k_\phi ,
\end{align*}
\]

But since the current components \( (a)k_\phi \) are just of that specific form \[\text{[(IV.35a)-[IV.35b]]} \], the solutions of the present differential equations \[\text{[(IV.40a)-[IV.40b]]} \] are easily found as
\[
\begin{align*}
h_1(r) &= \frac{\alpha_s \hbar}{2 M_p c} \cdot (1)\hat{R}_+(r)^2 \\
h_2(r) &= \frac{\alpha_s \hbar}{2 M_e c} \cdot (2)\hat{S}_+(r)^2 ,
\end{align*}
\]
i.e. the short-range fields \( \vec{H}'_a(r) = \{ (a)H'_r, (a)H'_\phi \} \) \[\text{[(IV.39a)-[IV.39b]]} \] can be directly traced back to the non-relativistic wave amplitudes \( \{ (1)\hat{R}_+, (2)\hat{S}_+ \} \).

Unfortunately, the determination of the long-range magnetic potentials \( \eta_A \) from their Poisson equations \[\text{[(IV.37)]} \] is technically somewhat more complicated. It is true, the source of the short-range fields \( \vec{H}'_a(\vec{r}) \) is relatively simple
\[
\vec{\nabla} \cdot \vec{H}'_a(r) = \cot \frac{\vartheta}{r} \cdot h_a(r) .
\]
Thus the corresponding standard solutions for the magnetic potentials \( \eta_A \) are given by
\[
\eta_A \equiv (a)\eta(r, \vartheta) = \frac{1}{4\pi} \int \frac{d^3 \vec{r}' \cot \vartheta' \cdot h_a(r')}{||\vec{r}' - \vec{r}||} ,
\]
\[72\]
where the radial functions \( h_a(r) \) are specified by equations (IV.41a)-(IV.41b). However it seems here that the integral cannot be calculated in terms of analytic functions, not even for the simple exponential trial form (III.68) for the wave amplitudes \( \tilde{R}_+^2, \tilde{S}_+^2 \). Therefore one will be forced to apply more or less effective approximation methods. For the present purpose one expands the denominator in the integral (IV.43) as follows

\[
\frac{1}{||\vec{r} - \vec{r}'||} = \frac{1}{\sqrt{r^2 + r'^2}} \cdot \left[ 1 + \frac{\vec{r} \cdot \vec{r}'}{r^2 + r'^2} + \frac{3}{2} \left( \frac{\vec{r} \cdot \vec{r}'}{r^2 + r'^2} \right)^2 + \ldots \right]. \tag{IV.44}
\]

Here the first (i.e. monopole) term does not contribute to the magnetic potential \( \eta_a \) (IV.43) so that we may be satisfied in the lowest order with the dipole approximation (second term). Thus the magnetic potential \( \eta_a \) becomes then in this dipole approximation

\[
\eta_a \Rightarrow (D)\eta_a(r, \vartheta) = \frac{\pi}{4} r \cos \vartheta \int_0^{\infty} dr' r'^2 \frac{h_a(r')}{\sqrt{r^2 + r'^2}} . \tag{IV.45}
\]

The dipole character of this result becomes evident from its asymptotic behavior \( r \to \infty \):

\[
(D)\eta_a(r \to \infty, \vartheta) = \frac{\pi}{4} \cdot \cos \vartheta \cdot \frac{\int_0^{\infty} dr' r'^2 h_a(r')}{r^2} . \tag{IV.46}
\]

If the preceding results (IV.41a)-(IV.41b) are used here, with observation of the non-relativistic normalization conditions (III.69) reading explicitly, e.g., for the second particle

\[
\int_0^{\infty} dr \ r \ \tilde{S}_+^2(r) = \frac{2}{\pi} , \tag{IV.47}
\]

then the second magnetic potential (IV.43) appears in the asymptotic region \( r \to \infty \) as

\[
(D)\eta_2(r \to \infty, \vartheta) = \frac{e}{\hbar c} \cdot \frac{\mu_B}{2} \cdot \frac{\cos \vartheta}{r^2} . \tag{IV.48}
\]
Apart from the dimensional factor \( \frac{\hbar}{\alpha e} \), which is due to our use of geometric units for the potentials and field strengths (see equation (II.11)), the present result (IV.48) for the asymptotic magnetic potential is the usual one for a magnetic dipole which however carries only half of a Bohr magneton \( \mu_B \left( \frac{\hbar}{2M_e c} \right) \). Clearly, this is a further unconventional feature of the exotic wave functions \( \psi_a(\vec{r}) \); namely besides their singular character for \( r \to 0 \) (III.62a)-(III.62b), their doubled-valuedness (III.60), and their integer spin eigenvalue (III.61).

It must be stressed, however, that these exotic states do not induce any pathological feature into the theory, neither with respect to the electric field nor for its magnetic counterpart. As a brief demonstration one may inspect the magnetic fields in the vicinity of the origin \( (r \to 0) \). First, the scalar magnetic potential \( \eta(2) \) is rewritten as

\[
^{(D)}\eta_2(r, \vartheta) = \frac{e\mu_B}{2\hbar c} \cdot \frac{\cos \vartheta}{r^2} g_2(r) ,
\]

with the dipole screening factor \( g_2(r) \) being given by

\[
g_2(r) = \frac{\pi M_e c}{\alpha_e \hbar} \int_0^\infty dr' \frac{r^2 \cdot h_2(r')}{\sqrt{r^2 + r'^2}} r' \to 1 ,
\]

so that the asymptotic dipole behavior (IV.48) is immediately manifest. (For the first particle, \( a = 1 \), the same arguments do hold with merely the electron mass \( M_e \) being replaced by the mass \( M_p \) of the positive particle). But the crucial point with the scalar magnetic potentials \( ^{(D)}\eta_2(r, \vartheta) \) (IV.45) is now that they do not diverge at the origin. One is easily convinced of this assertion by tentatively substituting for the magnetic ansatz function \( h_2(r) \) its non-relativistic form (IV.41b) with the wave amplitude \( ^{(2)}\tilde{S} \) being deduced from
the charge density \((b)\tilde{k}_0(r)\) \((\text{III.68})\) as

\[
(2)\tilde{S}_+(r) = \sqrt{\frac{8}{\pi r^2}} \cdot \exp \left(-\frac{r}{r_*}\right).
\]

This then yields for the dipole screening factor \(g_2(r)\) \((\text{IV.50})\)

\[
g_2(r) = \left(\frac{2}{r_*}\right)^2 r^3 \cdot \int_0^\infty dr' \frac{r' \exp \left(-\frac{2r'}{r_*}\right)}{\sqrt{r'^2 + r^2}},
\]

which by substitution of the integration variable \(r'\)

\[
\rho \equiv \frac{r'}{r}
\]

adopts the following form

\[
g_2(r) = \left(\frac{2}{r_*}\right)^2 r^2 \int_0^\infty d\rho \frac{\rho \cdot \exp \left(-\frac{2\rho}{r_*}\right)}{\sqrt{1 + \rho^2}},
\]

However in this form, it is easy to see that in the vicinity of the origin \((r \to 0)\) the screening factor \(g_2(r)\) looks as follows

\[
g_2(r) \to \left(\frac{2}{r_*}\right)^2 \cdot r^2
\]

and thus yields a finite value of the magnetic dipole potential \((D)\eta_2(r, \theta)\) \((\text{IV.49})\) around the origin. This compares to the analogous behavior of the electric potential \((a)A_0(r)\), see the discussion below equation \((\text{III.73})\).

But once it is guaranteed that the vector potentials \(\vec{A}_a(\vec{r})\) do really exist, one can use this fact in order to recast the magnetic interaction energy \(\tilde{E}_R^{(m)}\) \((\text{IV.1b})\) in a new form which exclusively is based upon the Dirac currents \(\vec{k}_a(\vec{r})\):

\[
\tilde{E}_R^{(m)} = -e^2 \int \int d^3\vec{r}_1 d^3\vec{r}_2 \frac{\vec{k}_1(\vec{r}_1) \cdot \vec{k}_2(\vec{r}_2)}{||\vec{r}_1 - \vec{r}_2||}.
\]
Indeed in order to arrive at this result, one merely has to substitute the vector potentials \( \vec{A}_a(\vec{r}) \) in the magnetic mass equivalents \( M_1^{(m)} c^2 \) or \( M_H^{(m)} c^2 \) of the Poisson identities (IV.3b) by the formal solution (III.48c)-(III.48d). Furthermore, both Dirac currents \( \vec{k}_a(\vec{r}) \) are of the azimuthal form (III.63b) with \( (a) k_\phi \) being specified by equations (IV.23c)-(IV.23d), which in their non-relativistic form appear as shown by equations (IV.35a)-(IV.35b). Thus the magnetic energy \( \hat{E}_R^{(m)} \) does finally emerge in the following form for identical rest masses \( (M_e = M_p \div M) \)

\[
\hat{E}_R^{(m)} = \left( \frac{e \hbar}{8 \pi M c} \right)^2 \int \int \frac{d^3 r_1}{r_1} \cdot \frac{d^3 r_2}{r_2} \frac{\frac{d}{dr_1} (1) \tilde{R}_+ (r_1)^2 \cdot \frac{d}{dr_2} (2) \tilde{S}_+ (r_2)^2}{||\vec{r}_1 - \vec{r}_2||} .
\]

(IV.57)

Recalling here the fact that for the groundstate both particles must be in the same quantum state (apart from the spin direction), one puts

\[
(1) \tilde{R}_+ (r) = (2) \tilde{S}_+ (r) \Rightarrow \tilde{R}(r) = \sqrt{\frac{8}{\pi r_s^2}} \cdot \exp \left[ -\frac{r}{r_s} \right] , \quad (IV.58a)
\]

and thus the magnetic energy (IV.57) becomes

\[
\hat{E}_R^{(m)} = \left( \frac{2 e \hbar}{\pi^2 M c r_s^2} \right)^2 \int \int \frac{d^3 r_1}{r_1} \cdot \frac{d^3 r_2}{r_2} \frac{\exp \left[ -\frac{2}{r_s} (r_1 + r_2) \right]}{||\vec{r}_1 - \vec{r}_2||} \]

= \left( \frac{2 \alpha_s}{\pi} \right)^2 \left( \frac{a_B}{r_s} \right)^2 \frac{e^2}{r_s} .
\]

(IV.59)

Here it is reasonable to assume that the optimal value of the variational parameter \( r_s \) will be of the order of magnitude of the Bohr radius \( a_B \); and this implies that the magnetic interaction energy \( \hat{E}_R^{(m)} \) is smaller than its electric counterpart \( \hat{E}_R^{(e)} \) (III.77) by the factor \( \alpha_s^2 \lesssim 10^{-4} \). This is also the
order of magnitude of the other relativistic effects; and therefore the ground-
state energy difference of ortho- and para-positronium cannot be expected
to be properly predicted by the present purely magnetic result (IV.59) (see
below).
V  Positronium Groundstate

For the situation where both particles masses are identical ($M_e = M_p = M$), it is reasonable to assume that both the first (positively charged) particle ($a = 1$) and the second (negatively charged) particle ($a = 2$) do always occupy physically equivalent states. According to this assumption, the positronium energy spectrum is expected to be essentially a one-particle spectrum which is in perfect agreement with the observational data [8]. The conventional classification of the positronium energy levels relies on the composition law for angular momenta so that the groundstate appears as the doublet $1^1S_0$ and $1^3S_1$ corresponding to whether the total spin $S$ is zero ($S = s_1 - s_2 = 0$; para-positronium) or is unity ($S = s_1 + s_2 = 1$; ortho-positronium), see e.g. ref. [16]. However in RST as a fluid-dynamical theory, it is more adequate to base the classification upon the relative orientation of both magnetic fields $\vec{H}_a(\vec{r})$ rather than upon the angular momentum composition law which is adequate for the conventional tensor product of Hilbert spaces but not for the present Whitney sum of single-particle bundles.

But in any case, the inclusion of the magnetic (i.e. spin-spin) interactions is an additional complication; and it is therefore convenient to first simplify the problem by neglecting the magnetic interactions completely and considering the residual problem alone (\textsuperscript{\textasciitilde} “electrostatic approximation”). In the non-relativistic conventional theory, this truncated problem is then described by the two-particle Hamiltonian $\hat{H}_S$ \textsuperscript{[11.34]} and can be solved exactly by introducing the relative and center-of-mass coordinates \textsuperscript{[17]}. The
corresponding conventional groundstate energy \( E_{0|\text{con}} \) is then easily found as

\[
E_{0|\text{con}} = \frac{1}{4} \frac{e^2}{a_B} = -\frac{1}{4} \alpha_s^2 \cdot M c^2 \simeq -6.80 \text{ [eV] (V.1)}
\]

where \( a_B (= \hbar^2/Me^2) \) is the Bohr radius and \( \alpha_s (= e^2/\hbar c) \) is the fine structure constant.

Indeed, this result (V.1) is nothing else than the conventional hydrogen groundstate energy due to a fixed nucleus, with merely the electron mass \( M \) being replaced by the reduced mass \( M/2 \) due to the comoving positron. Naturally, one will demand from any new theory of quantum matter that it should reproduce this standard result (V.1) in its lowest order of approximation; and afterwards one may proceed to compare the higher-order predictions of the various theoretical approaches. Therefore we will now first clarify the way in which the standard result (V.1) emerges in RST, and afterwards one can turn to the magnetic effects as small corrections of the electrostatic results.

Amazingly enough, we will recover just the standard result (V.1) as an approximate (i.e. variational) solution within the RST framework, namely by resorting to the RST principle of minimal energy \( \delta \tilde{E}_T^{(0)} = 0 \) (IV.26).

\[ A. \text{ Electrostatic Approximation} \]

Reasonably, the electric properties of both particles may be adopted to be approximately independent of the different magnetic arrangements. Thus the Dirac densities \( ^{(a)}k_0 (\vec{r}) \) can be assumed to be the same for both particles:

\[
^{(1)}k_0 (\vec{r}) \equiv ^{(2)}k_0 (\vec{r}) \equiv ^{(b)}k_0 (\vec{r}) . \quad \text{(V.2)}
\]

Furthermore, since these charge densities generate the electric potentials
\( A_0(\vec{r}) \) according to the Poisson equations (III.47a)-(III.47b), both potentials can differ at most in sign, i.e.

\[
(1)A_0(\vec{r}) \equiv \mp (2)A_0(\vec{r}) \overset{\text{(p)}}{=} (p)A_0(\vec{r}) ,
\]

with the common potential \((p)A_0(\vec{r})\) obeying the Poisson equation

\[
\Delta (p)A_0(\vec{r}) = -4\pi\alpha_s(\vec{r}).
\]

If we resort here to the non-relativistic approximations (IV.23a)-(IV.23b) and tentatively put for the non-relativistic wave amplitudes, cf. (III.68),

\[
(2)\tilde{S}+ \equiv (1)\tilde{R}_+ \Rightarrow \tilde{R}(r) = \sqrt{(b)\tilde{k}_0(r)} \overset{\text{(b)}}{=} \sqrt{\frac{8}{\pi r^2}} \cdot \exp \left(-\frac{r}{r_*}\right)
\]

then we just recover the former model potential \([b]A_0(r)\) (III.73) as the common potential \((V.3)\) for both particles. And correspondingly, their electrostatic interaction energy \(\hat{E}^{(e)}_R\) (IV.22a) is then just given by equation (III.77).

Here it is important to remark that, due to the non-relativistic trial function \(\tilde{R}(r)\) (V.5), the potential \([b]A_0(r)\) is the exact solution of the non-relativistic Poisson equation (III.72) and therefore the Poisson constraints (III.58)-(III.59a) are exactly satisfied in their non-relativistic form (IV.24a)-(IV.24b). Furthermore, it is easy to see that the non-relativistic trial function \(\tilde{R}(r)\) (V.5) actually obeys the normalization conditions (IV.25a)-(IV.25b). Thus both constraints for the (second line of the) non-relativistic energy functional \(\hat{E}^{(0)}_T\) (IV.26) are automatically satisfied by our spherically symmetric trial wave amplitude \(\tilde{R}(r)\) (V.5); and therefore one is concerned solely with the physical contributions (first line) to the energy functional.

However, since the field energy \(\hat{E}^{(e)}_R\) (as the interaction energy of both particles) is already specified by equation (III.77), one is left with the determi-
nation of the non-relativistic kinetic energies $E_{\text{kin}(a)}$ (IV.20a)-(IV.20b). Observing here the spherical symmetry of our trial function (V.5) together with the fact that both kinetic energies must be identical (i.e. $E_{\text{kin}(1)} = E_{\text{kin}(2)}$), one arrives at the total kinetic energy $E_{\text{kin}}$ as

$$E_{\text{kin}} \equiv E_{\text{kin}(1)} + E_{\text{kin}(2)} = \frac{\hbar^2}{2M} \int d^2 \vec{r} \left( \frac{d \tilde{R}(r)}{dr} \right)^2 = \frac{\hbar^2}{Mr^2_*}.$$ (V.6)

Consequently, the value of the non-relativistic functional $	ilde{E}_T^{(0)}$ upon our spherically symmetric trial function $	ilde{R}(r)$ (V.5) becomes the following ordinary function $	ilde{E}_T^{(0)}(r_*)$ of the ansatz parameter $r_*$

$$\tilde{E}_T^{(0)}(r_*) = E_{\text{kin}}(r_*) + \tilde{E}_R^{(e)}(r_*) = \frac{\hbar^2}{Mr^2_*} - \frac{e^2}{r_*}.$$ (V.7)

According to the established principle of minimal energy, the positronium groundstate energy $E_0$ in the spherically symmetric approximation is given by the minimal value of this function $\tilde{E}_T^{(0)}(r_*)$ (V.7), i.e.

$$E_0 = \tilde{E}_T^{(0)} \bigg|_{\text{min}} = -\frac{e^2}{4a_B},$$ (V.8)

and this minimum occurs for the value $r_* \Rightarrow 2a_B$ of the ansatz parameter $r_*$. Thus the non-relativistic approximation of the RST principle of minimal energy yields just the conventional Schrödinger value (V.1) for the positronium groundstate! This, however, is surely an amazing result in a two-fold way:

(i) Despite the very different mathematical structure of both approaches (Whitney sum vs. tensor product) the present RST prediction (V.8) coincides exactly with the conventional Schrödinger prediction (V.1).

(ii) However, in contrast to the conventional prediction, which is adopted to be exact within the standard framework of quantum mechanics, the
corresponding RST prediction (V.8) is based upon the choice of an appropriate trial function, cf. (V.5), and therefore is an approximate result within the RST framework. Thus the interesting question arises how close the exact (but non-relativistic) RST prediction would come to the exact conventional prediction (V.1)?

Surely this is a difficult question because its answer would require to find the exact solution of the non-relativistic RST eigenvalue problem which consists of the (non-relativistic) eigenvalue equations (IV.18a)-(IV.18b) and the coupled (non-relativistic) Poisson equations (IV.29a)-(IV.29b).

B. Hyperfine Splitting

The effect of level splitting by the magnetic (i.e. spin-spin) interactions is experimentally well established and is found to amount to 0.0008...[eV] for the positronium groundstate [8]. It should be clear that such a small energy difference between the triplet (3S1) and singlet (1S0) state falls into the order of magnitude of the relativistic effects which such compete with the magnetic interaction effects. Therefore it seems very unlikely that the total energy difference due to the hyperfine splitting of the groundstate should be caused by the magnetic effects alone, but nevertheless it may be interesting to estimate their relative contribution to the hyperfine splitting of the groundstate within the present framework of RST.

For this purpose, one first has to demonstrate the specific way in which this level dichotomy does emerge in RST. Such an effect, however, is rather obvious since in the non-relativistic limit one can alternatively put the spin-
down component \((2)\bar{S}_+ (\text{III.62a})\) to zero and retain only the spin-up component \((2)\bar{R}_+\) which itself must then coincide with the first wave amplitude \((1)\bar{R}_+\); i.e. in place of the former arrangement \((\text{IV.58a})-(\text{IV.58b})\) one puts now for the non-vanishing wave amplitudes

\[
(2)\bar{R}_+ \equiv (1)\bar{R}_+ \equiv \bar{R}(r) = \sqrt{\frac{8}{\pi r^*}} \exp\left(-\frac{r}{r_*}\right) \tag{V.9a}
\]

\[
(1)\bar{S}_+ = (2)\bar{S}_+ = 0. \tag{V.9b}
\]

This yields the parallelity of both Dirac currents \(\vec{k}_a(\vec{r})(= (a)k_\phi \vec{e}_\phi)\),

\[
(a)k_\phi = \frac{\hbar}{2Mc} \cdot \frac{\bar{R}(r) \cdot \frac{\partial}{\partial r} \bar{R}(r)}{2\pi r} \quad (a = 1, 2), \tag{V.10}
\]

in contrast to the antiparallelity of the former case \((\text{IV.35a})-(\text{IV.35b})\). According to the relationships \((\text{II.36a})-(\text{II.36b})\) and \((\text{II.43a})-(\text{II.43b})\) between the Dirac currents \(k_{a\mu}\) and Maxwell currents \(j_{a\mu}\), the Maxwell equations \((\text{II.27a})-(\text{II.27b})\) in three-vector notation \((\text{IV.31})\) say that the parallelity of the Dirac currents \(\vec{k}_a\) imply the antiparallelity of the magnetic fields \(\vec{H}_a\) and vice versa. These arrangements of the RST fields suggest the following magnetic classification of the positronium states [6]

\begin{align*}
\text{ortho-positronium} & \quad \text{\(1S_0\)} \\
\vec{k}_1 \equiv -\vec{k}_2 \equiv \vec{k}_p &= (p)k_\phi \vec{e}_\phi \tag{V.11a} \\
\vec{j}_1 \equiv \vec{k}_1 = \vec{k}_p; \vec{j}_2 \equiv -\vec{k}_2 = \vec{k}_p &= \tag{V.11b} \\
\vec{A}_1 \equiv \vec{A}_2 \equiv \vec{A}_b &= (b)A_\phi \vec{e}_\phi \tag{V.11c} \\
\vec{H}_1 \equiv \vec{H}_2 \equiv \vec{H}_b &= (b)H_r \vec{e}_r + (b)\vec{H}_\theta \vec{e}_\theta, \tag{V.11d}
\end{align*}

and analogously

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Thus the present RST ortho-positronium corresponds to the conventional singlet states \( (1S_0) \) and RST para-positronium to the triplet states \( (3S_1) \). It is true, this RST classification of the positronium states is based upon the \((non-relativistic)\) decoupling of the spin-up and spin-down configurations \([6]\) but is assumed to hold also for the relativistic case where the spin-up and spin-down amplitudes \((^a)\tilde{R}_\pm, (^a)\tilde{S}_\pm\) remain coupled so that spherically symmetric configurations are not possible, see the eigenvalue equations \((III.66a)-(III.66d)\).

The present magnetic dichotomy of the positronium states lends itself now to a very simple calculation of the groundstate hyperfine splitting. Namely, for a lowest-order estimate one may resort to the two trial configurations of the parallel magnetic-fields \((IV.58a)-(IV.58b)\) \((\sim ortho-positronium)\) or of antiparallel fields \((V.9a)-(V.9b)\) \((\sim para-positronium)\). The corresponding magnetic field energy \(\hat{E}^{(m)}_R\) \((IV.1b)\) differs then only in sign:

\[
\hat{E}^{(m)}_R|_{ortho} = \frac{\hbar c}{4\pi \alpha_s} \int d^3 \vec{r} \left| \left| \vec{H}_b \right| \right|^2 \equiv -\hat{E}^{(m)}_R|_{para} \quad \text{ (V.13a)}
\]

\[
\hat{E}^{(m)}_R|_{para} = -\frac{\hbar c}{4\pi \alpha_s} \int d^3 \vec{r} \left| \left| \vec{H}_p \right| \right|^2 \equiv -\hat{E}^{(m)}_R|_{ortho} , \quad \text{ (V.13b)}
\]

provided the magnetic field \(\vec{H}_{b/p}\) is computed approximately by means of the trial functions for the amplitude combinations \(\{^{(1)}\tilde{R}_+, (2)\tilde{S}_+\}\) and \(\{^{(1)}\tilde{R}_+, (2)\tilde{R}_+\}\), as demonstrated in subsection IV.C \((Magnetic \ Interactions)\). Thus referring to the magnetic interaction energy \(\hat{E}^{(m)}_R\) \((IV.59)\) of ortho-positronium, one
ends up with the following total energy $\tilde{E}_T^{(0)}$:

$$\tilde{E}_T^{(0)}(r_*) = \frac{\hbar^2}{Mr_*^2} - \frac{e^2}{r_*} \mp \left( \frac{2\alpha_s a_B}{\pi} \right)^2 \cdot \frac{e^2}{r_*^3},$$  \tag{V.14}

which of course is the magnetic generalization of the simpler purely electric case \(\text{(V.7)}\). (The upper/lower sign refers to the ortho/para-configurations \(\text{(V.11a)-(V.12d)}\)).

The minimal value of the total energy $\tilde{E}_T^{(0)}(r_*)$ occurs now at the slightly shifted position

$$r_{\min} = a_B \left[ 1 + \sqrt{1 \mp 3 \left( \frac{2\alpha_s}{\pi} \right)^2} \right] \simeq 2a_B \left[ 1 \mp 3 \left( \frac{\alpha_s}{\pi} \right)^2 \right], \quad \tag{V.15}$$

and the corresponding minimal value of the energy becomes now in the order of $\alpha_s^2$

$$\tilde{E}_{T,\min}^{(0)} = -\frac{e^2}{4a_B} \left[ 1 \pm 2 \left( \frac{\alpha_s}{\pi} \right)^2 \right]. \quad \tag{V.16}$$

Therefore the hyperfine splitting $\Delta\tilde{E}_T^{(0)}$ is predicted by the present estimate as

$$\Delta\tilde{E}_T^{(0)} \equiv \tilde{E}_T^{(0)}|_{\text{ortho}} - \tilde{E}_T^{(0)}|_{\text{para}} \equiv -2\tilde{E}_R^{(m)} = -\left( \frac{2\alpha_s}{\pi} \right)^2 \cdot \frac{e^2}{4a_B}. \quad \tag{V.17}$$

This is much smaller than the electrostatic binding energy of $6,800 [eV]$, cf. \(\text{(V.8)}\), namely

$$\Delta\tilde{E}_T^{(0)} = -(2,15 \cdot 10^{-5}) \cdot 6,800 [eV] = -1,46 \cdot 10^{-4} [eV]. \quad \tag{V.18}$$

It is true, this is qualitatively in agreement with the experimental fact that the binding energy of the ortho-system is greater than that of the para-system \[8\]; but the experimental value of the hyperfine splitting is $-8,41 \cdot$
$10^{-4} [eV]$ which is six times larger than the present RST prediction (V.18). Such a discrepancy may be understood in the sense that for the positronium hyperfine splitting in the order of $\alpha_s^2$ it is necessary to use some trial function $\tilde{R}(r)$ which is closer to the exact solution than the simple exponential function (V.9a). In any case, it seems worthwhile to look for the exact solutions of both the relativistic and non-relativistic RST eigenvalue problem in order to test its theoretical accuracy in comparison to the experimental situation and the other theoretical approaches.
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