Growth Histories in Bimetric Massive Gravity

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Abstract

We perform cosmological perturbation theory in Hassan-Rosen bimetric gravity for general homogeneous and isotropic backgrounds. In the de Sitter approximation, we obtain decoupled sets of massless and massive scalar gravitational fluctuations. Matter perturbations then evolve like in Einstein gravity. We perturb the future de Sitter regime by the ratio of matter to dark energy, producing quasi-de Sitter space. In this more general setting the massive and massless fluctuations mix. We argue that in the quasi-de Sitter regime, the growth of structure in bimetric gravity differs from that of Einstein gravity.
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1 Introduction

The cosmological constant problem [1, 2] is one of the most vexing problems in physics. The main problem is why the vacuum energy densities of quantum field theory seem to contribute to observable gravitational physics so much less than simple estimates would indicate. Presumably, the problem would be resolved in a theory of quantum gravity. String theory contains some quantum gravity, but it is notoriously difficult to address quantum problems in dynamical gravity with the present formulation of string theory. With the observation of the accelerated expansion of the universe in 1998, usually attributed to a dominant dark energy component such as the cosmological constant (or quintessence, which has similar naturalness problems), the issue has been driven to a sharp point. As reviewed in [2], very few of the many proposed solutions to the problem stand a remote chance of success.

Modified gravity is one of the well-known proposed “solutions” that fares particularly poorly in the evaluation of e.g. [2], because modified gravity theories are typically only deep-infrared modifications of gravity (where “deep-infrared” means very low energies, some tiny fraction of an electron volt), which ultimately seems insufficient to solve the problem of quantum field theory contributions from all known fields, including for example around the electron mass of 511 keV. There are suggestions how modified gravity could effectively limit how energy gravitates (a “filter”) at a wider variety of energy scales, as in the proposed “degravitation” mechanism [3], and the earlier discussions of screening mechanisms summarized in [4]. These are intriguing but incomplete.
suggestions, in that it is not yet clear if any of these mechanisms are actually realized in any underlying theory in which the range of applicability of these mechanisms could be reliably evaluated.

Massive gravity, a theory of gravity where the graviton has a mass (which is typically constrained by observations to be extremely small, perhaps $10^{-33}$ eV, see e.g. [5] for a list of references) is at face value a relatively minor and again deep-infrared modification of gravity. The study of massive gravity was initiated in 1939 by Fierz and Pauli [7], but these theories suffered from ghost instabilities at the nonlinear level. In 2010 progress was made when a particular set of nonlinear ghost-free interactions was found by de Rham, Gabadadze and Tolley (dRGT) in a series of papers [17, 18, 19, 20, 21, 22, 23, 24], following seminal earlier work in [38, 39]. The dRGT formulation of nonlinear massive gravity requires a fixed auxiliary two-tensor $f_{\mu\nu}$ with no dynamics of its own. Apart from aesthetic concerns about this, for our purposes it is a deficiency of this theory that it seems to have no homogeneous and isotropic cosmological solutions [18, 30].

Last year, Hassan and Rosen [14, 15, 16] gave dynamics to the tensor $f_{\mu\nu}$ in a bimetric theory, with the nonlinear interactions between $g_{\mu\nu}$ and $f_{\mu\nu}$ imported from the de Rham-Gabadadze-Tolley massive gravity theories. (In fact, the need for two metrics to realize massive gravity covariantly was already appreciated in 1976 [13], but there was no theory without nonlinear instabilities.) The Hassan-Rosen bimetric theory has cosmological solutions, as explored in [8, 9, 11, 10]. In the aforementioned cosmological solutions of Hassan-Rosen bimetric gravity, both $g_{\mu\nu}$ and $f_{\mu\nu}$ have equations of motion, and the background solutions we consider are of general FLRW form for both backgrounds.

Now, bimetric gravity is a more far-reaching modification of gravity than massive gravity, in that the new gravitationally coupled tensor field $f_{\mu\nu}$ has dynamics of its own. (In the formulation we will be using here, it does not couple directly to matter, so “bimetric” is a little bit of a misnomer, “gravity coupled to matter and a symmetric two-tensor” would have been more accurate.) Of course, since the field content and interactions of bimetric theory are different from Einstein gravity, this theory may have different quantum properties at any given scale and not exclusively in the deep infrared. One way to try to understand the quantum properties of this theory would be to try to embed the theory in string theory, but currently it is not known how to do this. On the good side, the Hassan-Rosen bimetric theory (respectively de Rham-Gabadadze-Tolley massive gravity theory) has the kind of rigid structure that one would think could possibly descend from an underlying theory, like string theory. Symmetries constrain the interaction terms to the form $V(f^{-1}g)$, and their relative coefficients are constrained, and it is now understood how to construct these theories in various dimensions and including higher-derivative corrections [29]. It would be somewhat surprising, and a shame, if this structure existed for no reason at all.

To be clear, there is so far no clear indication that even embedding Hassan-Rosen bimetric theory in string theory would particularly help with the cosmological constant problem, but at least the problems could be addressed in a theory that is apparently nonlinearly consistent and also fundamentally different from Einstein gravity already at the level of the low-energy effective action.

On the other hand, it may be easier to rule these kinds of theories out observationally (and
classically) than to properly understand their quantization, so here we pursue strategies to achieve classical observational tests. In this paper, we

- derive the linearized gravitational scalar fluctuation equations for FLRW backgrounds (section 4)
- find convenient gauge invariant variables (section 4)
- solve the equations in the special case of a de Sitter background (section 6) — this is not completely new, see [25, 26]
- develop the quasi-de Sitter (qdS) approximation in bimetric gravity (section 7)
- find solutions of the qdS fluctuation equations, both analytical and numerical (section 7)
- in general, construct some necessary framework for the analysis of growth of structure in Hassan-Rosen bimetric gravity.

Detailed observational and phenomenological analyses are left for the future.

We also mention that there has also been recent related work on multi-metric theory, the natural generalization of this framework to coupling multiple spin-two fields nonlinearly [37, 35, 36]. Finally, there is also progress on related theories in three dimensions [33]. In fact, the “new massive gravity” theory in three dimensions [32], which generated some excitement in the last few years, is a scaling limit of the Hassan-Rosen bimetric theory [34, 29].

2 Hassan-Rosen bimetric massive gravity

The bimetric massive gravity theory found by Hassan and Rosen [15] is given by the action

$$S_{HR} = -\frac{M^2}{2} \int d^4x \sqrt{-g} R(g) - \frac{\mathcal{M}^2}{2} \int d^4x \sqrt{-f} R(f)$$

$$+ m^2 M^2 \sum_{n=0}^{4} \beta_n e_n \left( \sqrt{g^{-1}f} \right) + \int d^4x \sqrt{-g} \mathcal{L}_m (g, \Phi) .$$

and represents a natural generalization of the de Rham-Gabadadze-Tolley massive gravity theory [22] to a theory with two dynamical metrics, as discussed in the introduction. Here $\beta_n$ are free parameters, which in general are the coefficients in the “deformed determinant” of [12]. The interaction terms $e_n (X)$ are elementary symmetric polynomials of the eigenvalues of the matrix $X$, which explicitly are given by

$$e_0 (X) = 1, \quad e_1 (X) = \text{Tr} X, \quad e_2 (X) = \frac{1}{2} \left( (\text{Tr} X)^2 - \text{Tr} X^2 \right) ,$$
\[ e_3(X) = \frac{1}{6} \left( (\text{Tr } X)^3 - 3 \text{Tr } X \text{Tr } X^2 + 2 \text{Tr } X^3 \right), \quad e_4(X) = \det(X) \] (3)

We have chosen to only couple \( g_{\mu\nu} \) to matter, and not \( f_{\mu\nu} \), as in the original papers. We note that this is not the only possible choice and it would be interesting to explore other options.

The equations of motion are given by varying the action with respect to \( g_{\mu\nu} \) and \( f_{\mu\nu} \):

\[
R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R + \frac{m^2}{2} \sum_{n=0}^{3} (-1)^n \beta_n \left[ g_{\mu\lambda} Y_{(n)\nu}^\lambda \left( \sqrt{g^{-1} f} \right) + g_{\nu\lambda} Y_{(n)\mu}^\lambda \left( \sqrt{g^{-1} f} \right) \right] = \frac{1}{M^2_f} T_{\mu\nu},
\] (4)

\[
\bar{R}_{\mu\nu} = -\frac{1}{2} f_{\mu\nu} \bar{R} + \frac{m^2}{2 M^2_f} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \left[ f_{\mu\lambda} Y_{(n)\nu}^\lambda \left( \sqrt{f^{-1} g} \right) + f_{\nu\lambda} Y_{(n)\mu}^\lambda \left( \sqrt{f^{-1} g} \right) \right] = 0,
\] (5)

where \( \bar{R}_{\mu\nu} \) and \( \bar{R} \) are the Ricci tensor and Ricci scalar due to \( f_{\mu\nu} \), and

\[ M^2_* = \frac{M^2_f}{M^2_g}. \] (6)

The matrices \( Y_{(n)\mu}^\lambda (X) \) are given by

\[
Y_{(0)}(X) = 1, \quad Y_{(1)}(X) = X - 1 \text{Tr } X,
\]

\[
Y_{(2)}(X) = X^2 - X \text{Tr } X + \frac{1}{2} 1 \left( (\text{Tr } X)^2 - \text{Tr } X^2 \right),
\]

\[
Y_{(3)}(X) = X^3 - X^2 \text{Tr } X + \frac{1}{2} X \left( (\text{Tr } X)^2 - \text{Tr } X^2 \right) - \frac{1}{6} 1 \left( (\text{Tr } X)^3 - 3 \text{Tr } X \text{Tr } X^2 + 2 \text{Tr } X^3 \right),
\] (7)

where \( 1 \) is the identity matrix. Imposing that \( T_{\mu\nu} \) is covariantly conserved, then from eq. (4), the Bianchi constraint gives

\[
\nabla^\mu \sum_{n=0}^{3} (-1)^n \beta_n \left[ g_{\mu\lambda} Y_{(n)\nu}^\lambda \left( \sqrt{g^{-1} f} \right) + g_{\nu\lambda} Y_{(n)\mu}^\lambda \left( \sqrt{g^{-1} f} \right) \right] = 0.
\] (8)

It can be shown that the corresponding Bianchi constraint given from (5) is equivalent with (8).

Finally, by performing the constant rescaling

\[
f_{\mu\nu} \rightarrow \frac{M^2_g}{M^2_f} f_{\mu\nu}, \quad \beta_n \rightarrow \left( \frac{M_f}{M_g} \right)^n \beta_n,
\] (9)

we set \( M^2_* \) to unity. In other words, \( M^2_* \) was a redundancy that we do not consider a separate free parameter.
3 Review of background equations

Cosmological solutions of the Hassan-Rosen bimetric theory were studied in [8, 9, 11, 10]. We have a separate metric ansatz for each of $g_{\mu\nu}$ and $f_{\mu\nu}$ (specialized to the case of flat spatial sections):

$$ds_g^2 = -dt^2 + a^2(t) \, dx^2,$$

$$ds_f^2 = -X^2(t) \, dt^2 + Y^2(t) \, dx^2.$$  \tag{10}

The Bianchi constraint given in eq. (8) gives

$$3m^2 \frac{a}{\dot{a}} \left( \beta_1 + 2 \frac{Y}{a} \beta_2 + \frac{Y^2}{a^2} \beta_3 \right) \left( \dot{Y} - \dot{\alpha}X \right) = 0,$$  \tag{12}

where overdots denote differentiation with respect to $t$. There are two options for solving the Bianchi identity, which we refer to as Case A and Case B. In Case A, which we will not use,

$$\beta_1 + 2 \frac{Y}{a} \beta_2 + \frac{Y^2}{a^2} \beta_3 = 0,$$  \tag{13}

which gives a cosmological solution that is degenerate with GR, for which the fluctuation equations reduces to identical equations to those of GR (as noted in [11]). Thus, we focus on Case B, which is

$$X = \frac{\dot{Y}}{\dot{a}}.$$  \tag{14}

The Friedmann equations derived from eq. (4) and eq. (5), together with the Bianchi identity, are

$$-3 \left( \frac{\dot{a}}{a} \right)^2 + m^2 \left( \beta_0 + 3 \beta_1 \frac{Y}{a} + 3 \beta_2 \frac{Y^2}{a^2} + \beta_3 \frac{Y^3}{a^3} \right) = \frac{1}{M_g^2} T_0^0,$$  \tag{15}

$$-3 \left( \frac{\dot{a}}{Y} \right)^2 + m^2 \left( \beta_1 + 3 \beta_3 \frac{a}{Y} + 3 \beta_2 \frac{a^2}{Y^2} + \beta_3 \frac{a^3}{Y^3} \right) = 0.$$  \tag{16}

The acceleration equations can be shown to follow from the two Friedmann equations when using the Bianchi constraint.

In this paper we will only consider the simplest class of solutions, corresponding to $\beta_1 = \beta_3 = 0$, as discussed in [9]. Combining the two Friedmann equations then gives

$$H^2 = \frac{\beta_4}{\beta_4 - 3 \beta_2} \frac{\rho}{3 M_g^2} + m^2 \frac{\beta_0 \beta_4 - 9 \beta_2^2}{3 (\beta_4 - 3 \beta_2)},$$  \tag{17}

$$\frac{Y^2}{a^2} = \frac{\rho}{m^2 M_g^2 (\beta_4 - 3 \beta_2)} + \frac{\beta_0 - 3 \beta_2}{\beta_4 - 3 \beta_2},$$  \tag{18}

where $H = \dot{a}/a$ and $\rho = -T_0^0$ corresponds to the pressureless matter density.
We now define four effective parameters, to be used in our fluctuation analysis, according to

\[ H_{\text{dS}}^2 \equiv m^2 \frac{\beta_0 \beta_4 - 9 \beta_2^2}{3 (\beta_4 - 3 \beta_2)} , \quad M_P^2 \equiv m^2 \frac{\beta_4 - 3 \beta_2}{\beta_4} \]

\[ M^2 \equiv 2m^2 \beta_2 (1 + c^2), \quad c^2 \equiv \frac{\beta_0 - 3 \beta_2}{\beta_4 - 3 \beta_2} . \]  

(19)

The importance of these particular combinations of parameters in the action (2) is as follows. First observe in the action that \( \beta_0 \) can be thought of as setting the usual g cosmological constant, that \( \beta_4 \) likewise can be thought of as setting the f cosmological constant, but that the effective “observable” cosmological constant that actually appears in (17) is a combination of \( \beta_0, \beta_2 \) and \( \beta_4 \). In the \( \rho \to 0 \) limit of (17), there is a de Sitter solution, and its Hubble constant \( H_{\text{dS}}^2 \) is then related to the effective cosmological constant induced by the interaction potential, that is in turned fixed by the \( \beta_n \) parameters. If we also consider \( \rho \to 0 \) in (18), we see that also \( f_{\mu \nu} \) will have a de Sitter solution with possibly different overall normalization, and the parameter \( c \) is the proportionality constant between \( g_{\mu \nu} \) and \( f_{\mu \nu} \) in the de Sitter spacetime. Further, \( M^2 \) is the mass of the spin-2 helicity modes when linearizing \( g_{\mu \nu} \) and \( f_{\mu \nu} \) around such proportional background metrics (see appendix E). Finally, \( M_P^2 \) is the effective gravitational coupling constant for \( \rho \) in the cosmological framework (note that this will not be the coupling constant for the fluctuations, nor does it necessarily describe the coupling in local solutions).

In terms of these parameters, the \( H \) equation can be written

\[ H^2 = \frac{\rho}{3 M_P^2} + H_{\text{dS}}^2 . \]  

(20)

As usual, the continuity equation for equation of state \( p = w \rho \) with constant equation of state parameter \( w \) reads

\[ \frac{d \ln \rho}{d \ln a} = -3(1 + w) \]  

(21)

with solution \( \rho = \rho_0 a^{-3(1+w)} \). Pressureless matter \( (w = 0) \) evolves as \( a^{-3} \), and normalizing the scale factor at the present time to \( a_0 = 1 \), we can rewrite eqs. (20) and (18) as

\[ \left( \frac{H}{H_0} \right)^2 = \frac{1}{a^3} \left[ 1 - \left( \frac{H_{\text{dS}}}{H_0} \right)^2 \right] + \left( \frac{H_{\text{dS}}}{H_0} \right)^2 \]  

(22)

\[ \frac{Y^2}{a^2} = c^2 \frac{2 (1 + c^2) H^2 - M^2}{2 (1 + c^2) H_{\text{dS}}^2 - M^2} . \]  

(23)

where we also used the relation

\[ M_P^2 = M_g^2 (1 + c^2) \frac{M^2 - 2 H_{\text{dS}}^2}{M^2 - 2 (1 + c^2) H_{\text{dS}}^2} . \]  

(24)

Equations (22) and (23) are our final forms of the background equations, expressed entirely in terms of the parameters (19).
In terms of background cosmology, since the form (22) is equivalent to ΛCDM, the only relevant parameter is 
\( H^2_{\text{dS}}/H^2_0 \), which can be constrained by observational data to be close to 0.7, by relating it to the usual ratios to critical densities:

\[
\Omega_\Lambda = \frac{H^2_{\text{dS}}}{H^2_0}, \quad \Omega_m = 1 - \frac{H^2_{\text{dS}}}{H^2_0}.
\] (25)

It is therefore only the specific combination of \( \beta_n \) given by the definition of \( H_{\text{dS}} \) that is constrained by the expansion history of the universe, leaving \( M^2_P, M^2 \) and \( c^2 \) as unconstrained parameters, possibly to be constrained by structure formation data, but there are some further restrictions, as we shall see.

Finally, a comment on the range of these parameters, in particular of the mass parameter \( M \). Since \( H^2 \geq H^2_{\text{dS}} \) always (see e.g. fig. 3), we note that if \( 2(1 + c^2) H^2_{\text{dS}} < M^2 \) and at some time it happens that \( 2(1 + c^2) H^2(t) \geq M^2 \) (which can occur in the early universe), then from (23) the \( f \) scale factor \( Y \) will be imaginary, which we consider unphysical. Therefore, we demand that \( M^2 \leq 2(1 + c^2) H^2_{\text{dS}} \). But then, we see that \( M^2_\beta \) will be negative if also \( M^2 > 2H^2_{\text{dS}} \). Negative values of \( M^2_\beta \) would be unphysical, since the matter density would then need to be negative in order to have expanding background solutions originating in a hot and dense state (as demanded by observations of the cosmic microwave background).

To summarize, if we demand that \( Y \) should be real and \( M^2_\beta \) positive, we require

\[
M^2 < 2H^2_{\text{dS}}.
\] (26)

In these bimetric models, we thus need to violate the Higuchi bound \( M^2 > 2H^2_{\text{dS}} \) already at the level of the background. We will comment more on this later, and see also fig. 2.

\section{Perturbations}

We will first consider a general background for \( g_{\mu\nu} \) and \( f_{\mu\nu} \). In this paper we will only consider scalar gravitational perturbations, except for a brief review of tensor perturbations in appendix E. For the scalar perturbations, following Weinberg \[43\] we make the ansatz

\[
\begin{align*}
\text{for } g: & \quad ds^2_g = -(1 + E_g) dt^2 + 2a\partial_i F_g dx^i dt + a^2 \left( (1 + A_g) \delta_{ij} + \partial_i \partial_j B_g \right) dx^i dx^j, \\
\text{for } f: & \quad ds^2_f = -X^2 (1 + E_f) dt^2 + 2XY \partial_i F_f dx^i dt + Y^2 \left( (1 + A_f) \delta_{ij} + \partial_i \partial_j B_f \right) dx^i dx^j.
\end{align*}
\] (27) (28)

so our set of eight (non-gauge-invariant) scalar gravitational fluctuations is \( \{ E_g, F_g, A_g, B_g \} \) and \( \{ E_f, F_f, A_f, B_f \} \). Note that at this point, the ansatz is completely symmetric between the \( g \) and \( f \) metrics, as far as the perturbations are concerned.

\footnote{It might be interesting to explore this branch of solutions, for example by picking a sufficiently large \( M \) that moves this region to the very early universe where the current model is in any case not applicable. We will not consider such models in this paper.}
4.1 Gauge invariant variables

We form gauge invariant combinations of the perturbations in (27). As usual there is no uniqueness in the choice of gauge invariant variables (any combination of gauge invariant variables is gauge invariant) but we find the following variables convenient:

\[
\begin{align*}
\Psi_g &= -\frac{1}{2} A_g + \frac{H}{2} \left[a^2 B_g - 2a F_g \right] \\
\Phi_g &= \frac{1}{2} E_g - \frac{1}{2} \left[a^2 B_g - 2a F_g \right] \\
\mathcal{B} &= \frac{1}{2} (B_f - B_g)
\end{align*}
\]

\[
\begin{align*}
\Psi_f &= -\frac{1}{2} A_f + \frac{K}{2X} \left[\frac{Y^2}{X} B_f - 2Y F_f \right] \\
\Phi_f &= \frac{1}{2} E_f - \frac{1}{2X} \left[\frac{Y^2}{X} B_f - 2Y F_f \right] \\
\mathcal{F} &= F_f - \frac{a X}{Y} F_g + \frac{1}{2} \left[\frac{X a^2}{Y} B_g - \frac{Y}{X} \dot{B}_f \right]
\end{align*}
\]

with the definitions

\[
H \equiv \dot{a}/a, \quad K \equiv \dot{Y}/Y, \quad \quad \quad (30)
\]

so \(K\) is the Hubble function for the \(f\) metric. The ansatz (29) is roughly speaking “as symmetric as possible” between \(g\) and \(f\), but complete symmetry is unattainable as the backgrounds are generically different.

4.2 Equations of motion: general background

Using the gauge invariant variables in the previous section, the equations of motion for the scalar perturbations in the \(g\) sector become

\[
- \frac{1}{a^2} \nabla^2 \Psi_g + 3H \left( H \Phi_g + \dot{\Psi}_g \right) + \frac{m^2 Y P}{2a^3} \left[ 3 \left( -\Psi_f + \Psi_g - \frac{Y K}{X} \mathcal{F} \right) + \nabla^2 \mathcal{B} \right] = \frac{1}{2M_g^2} \delta T_0^0
\]

\[
- \partial_i \left( \dot{\Psi}_g + H \Phi_g \right) + \frac{m^2 Y X P}{2a (a X + Y)} \partial_i \left( \mathcal{F} + \frac{Y}{X} \dot{\mathcal{B}} \right) = \frac{\delta T_i^0}{2M_g^2}
\]

\[
\dot{\Psi}_g + H \Phi_g + 3H \left( H \Phi_g + \dot{\Psi}_g \right) + 2H \Phi_g + \frac{1}{2a^2} \left( \partial_j^2 + \partial_k^2 \right) (\Phi_g - \Psi_g) + \\
+ \frac{m^2}{2a^2} \left\{ P \left[ X (\Phi_f - \Phi_g) - \left( Y \mathcal{F} \right) \right] + Y Q \left[ 2 \left( -\Psi_f + \Psi_g - \frac{Y K}{X} \mathcal{F} \right) + (\partial_j^2 + \partial_k^2) \mathcal{B} \right] \right\} = \frac{1}{2M_g^2} \delta T_i^i
\]

\[
- \frac{1}{2a^2} \partial_j \partial_k (\Phi_g - \Psi_g) - \frac{m^2 Y Q}{2a^2} \partial_i \partial_j \mathcal{B} = \frac{1}{2M_g^2} \delta T_j^i
\]

where \(j, k\) are not equal to \(i\). In the \(f\)-sector we have

\[
- \frac{1}{Y^2} \nabla^2 \Psi_f + \frac{1}{X^2} K \left( K \Phi_f + \dot{\Psi}_f \right) - \frac{m^2 a}{2Y^3} P \left[ 3 \left( -\Psi_f + \Psi_g - \frac{Y H}{X} \mathcal{F} \right) + \nabla^2 \mathcal{B} \right] = 0
\]
\[-\frac{1}{X^2} \partial_i \left( \dot{\Psi}_f + K \Phi_f \right) - \frac{m^2 P}{2X^2(aX + Y)} \partial_i \left( \mathcal{F} + \frac{a^2 X}{Y} \mathcal{B} \right) = 0 \quad (36)\]

\[\frac{1}{X^2} \ddot{\Psi}_f - \frac{X}{X^3} \dot{\Psi}_f + \frac{1}{X^2} K \Phi_f + 3 \frac{K}{X^2} \left( K \Phi_f + \dot{\Psi}_f \right) + \frac{2}{X} \left( \frac{K}{X} \Phi_f + \frac{1}{2Y^2} \left( \partial_j^2 + \partial_k^2 \right) (\Phi_f - \Psi_f) \right) + (37)\]

\[-m^2 \frac{2}{2XY^2} \left\{ P \left[ \Phi_f - \Phi_g - \left( \frac{Y}{X} \mathcal{F} \right)^* \right] + aQ \left[ 2 \left( -\Psi_f + \Psi_g - \frac{YH}{X} \mathcal{F} \right) + (\partial_j^2 + \partial_k^2) \mathcal{B} \right] \right\} = 0\]

\[-\frac{1}{2Y^2} \partial_i \partial_j (\Phi_f - \Psi_f) + \frac{m^2 aQ}{2XY^2} \partial_i \partial_j \mathcal{B} = 0 \quad (38)\]

with the definitions

\[P \equiv (\beta_1 a^2 + 2\beta_2 aY + \beta_3 Y^2), \quad Q \equiv [a\beta_1 + \beta_2 (aX + Y) + \beta_3 XY]. \quad (39)\]

At this point we recall that although the separate Einstein-Hilbert actions for $g$ and $f$ in eq. (1) are of course invariant under separate diffeomorphisms of the two metrics, the mass terms are only invariant under diagonal diffeomorphisms that preserve $g^{-1}f$. Thus, the $\Psi_g$, $\Phi_g$ and $\Phi_f$, $\Psi_f$ can only appear as differences in the mass terms, and we see this manifestly in the equations.

### 4.3 Massless limit

If we were to turn off the interaction potential in the action, we might expect to find two decoupled sets of fluctuations. In fact we observe that if $\beta_1 = \beta_2 = \beta_3 = 0$ then the two combinations $P$ and $Q = 0$ both vanish, so $\mathcal{F}$ and $\mathcal{B}$ drop out of the equations entirely, and the $g$ fluctuations and $f$ fluctuations constitute decoupled sectors.

This is a simple observation purely in terms of the fluctuation equations. However, whether the fluctuations truly represent decoupled physics is a subtle issue. For example, the condition (14) from the Bianchi identity relates the $g$ and $f$ background solutions for arbitrarily small $M$, but there is a priori no reason to impose this condition in the strictly massless theory. (In the language of that section, one can revert to Case A, in which case (14) need not be imposed.) But if the condition imposed on the background differs between $M \to 0$ and $M = 0$, there is a potential “cosmological vDVZ discontinuity” [40, 41], i.e. the $M \to 0$ and $M = 0$ theories could potentially be different no matter how small $M$ is taken in the limit. Of course, there could still be a Vainshtein mechanism [42] that resolves the discontinuity in the nonlinear regime, but this would not be evident in our linear approximation. In figure [5] below, we see some hint of a discontinuity, but it is somewhat subtle here as we have several parameters to play with. We will not resolve the issue of the existence of a smooth limit here, but see also the recent interesting discussions by [28, 29].

Now we consider more special backgrounds, first a two-component fluid solution that we will refer to as the “exact solution”, then de Sitter and then quasi-de Sitter.
5 Exact solution

It is well-known that in the approximation of a two-component fluid of pressureless matter (dust) with equation of state \( p = 0 \) \( (w = 0) \) and cosmological constant with equation of state \( p = -\rho \) \( (w = -1) \), the combination of equations (22) and (21) admits the exact solution

\[
a(t) = c_1 \sinh \left( \frac{3}{2} H_{\text{dS}} t \right)^{2/3}
\]

where the constants \( c_1 \) and \( H_{\text{dS}} \) are

\[
c_1 = \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^{1/3} = \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{1/3}, \quad H_{\text{dS}} = \sqrt{\Omega_\Lambda H_0}.
\]

We can write \( \dot{a} = Ha \) in (15) to express \( Y(t) \) in terms of \( a(t) \):

\[
Y(t) = \frac{\sqrt{3}}{\sqrt{\beta_4 m}} \sqrt{H^2(t) - \beta_2 m^2} \cdot a(t).
\]

and then using the effective parameters (19) we obtain for the scale factor of the \( f \) metric:

\[
Y = \sqrt{\frac{2 (1 + c^2) H^2_{\text{dS}} \coth^2 \left( \frac{3}{2} H_{\text{dS}} t \right) - M^2}{2 (1 + c^2) H^2_{\text{dS}} - M^2}} \cdot c \cdot c_1 \sinh \left( \frac{3}{2} H_{\text{dS}} t \right)^{2/3}
\]

Ideally one would now simply use these background scale factors in the fluctuation equations and solve them numerically, which would lead to a model for growth of structure in bimetric theory at any time \( t \). Unfortunately, we have not been able to complete this program, and instead we will focus on special cases and simplifying approximations.

The simplest special case is that \( H = \text{constant} \) as in pure de Sitter space, then (43) tells us that \( Y(t) \propto a(t) \), with the constant of proportionality given by

\[
c \equiv \frac{Y_{\text{dS}}(t)}{a_{\text{dS}}(t)}.
\]

We will in general not limit ourselves to pure de Sitter space, but it provides a useful starting point. Any departure of the \( g_{\mu\nu} \) metric from pure de Sitter space breaks the proportionality between the \( g_{\mu\nu} \) scale factor and the \( f_{\mu\nu} \) scale factor.

6 Pure de Sitter space

Matter dilutes away as the universe expands, and the universe approaches a de Sitter (pure dark energy) solution in the future. To provide some feeling for the numbers, if the evolution would proceed according to GR, then it will take around 10 Gyr after present for the exact FLRW scale factor \( a(t) \) to agree with the de Sitter scale factor \( a_{\text{dS}}(t) \) to within 1% accuracy.
For large time, the exact solutions in the previous sections reduce to the approximate solutions
\( a(t) \to a_{\text{dS}}(t), \ Y(t) \to Y_{\text{dS}}(t) \) where

\[
a_{\text{dS}}(t) = c_2 \exp(H_{\text{dS}} t) \quad (45)
\]

\[
Y_{\text{dS}}(t) = c \cdot c_2 \exp(H_{\text{dS}} t) \quad (46)
\]

where

\[
c_2 = \left( \frac{1 - \Omega_\Lambda}{4\Omega_\Lambda} \right)^{1/3} \quad (47)
\]

and \( c \) is the proportionality constant from (43). Although these are of course exact de Sitter solutions in their own right, it is useful to consider them as limits of the exact solution for normalization purposes. In particular, since there is no Big Bang in pure de Sitter, there would have been no way to normalize the scale factor.

### 6.1 Gauge invariant variables in dS

The general gauge invariant variables of (29) have the following dS limits:

\[
\Psi_g = -\frac{1}{2} A_g + \frac{H_{\text{dS}}}{2} \left[a^2 \dot{B}_g - 2a F_g \right] \quad \Psi_f = -\frac{1}{2} A_f + \frac{H_{\text{dS}}}{2} \left[a^2 \dot{B}_f - 2a F_f \right]
\]

\[
\Phi_g = \frac{1}{2} E_g - \frac{1}{2} \left[a^2 \dot{B}_g - 2a F_g \right] \quad \Phi_f = \frac{1}{2} E_f - \frac{1}{2} \left[a^2 \dot{B}_f - 2a F_f \right]
\]

\[
\mathcal{B} = \frac{1}{2} (B_f - B_g) \quad \mathcal{F} = F_f - F_g + \frac{a}{2} \left[\dot{B}_g - \dot{B}_f \right]
\]

Defining the linear combinations of fields\(^2\)

\[
\Phi_+ = \Phi_g + c^2 \Phi_f \quad \Phi_- = \Phi_g - \Phi_f \quad (49)
\]

\[
\Psi_+ = \Psi_g + c^2 \Psi_f \quad \Psi_- = \Psi_g - \Psi_f \quad (50)
\]

we are able to separate the scalar gravitational fluctuation equations (31)-(38) into a system of massless equations for \( \Phi_+, \Psi_+ \) and massive equations for \( \Phi_-, \Psi_- \). The \( \mathcal{F} \) and \( \mathcal{B} \) fields appear only in the massive field equations. The matter perturbations will appear in both sectors.

### 6.2 Perturbations in dS

The equations of motion reduce as follows. In the massless sector:

\[
-\frac{1}{a^2} \nabla^2 \Psi_+ + 3H_{\text{dS}} \left(H \Phi_+ + \dot{\Psi}_+ \right) = \frac{\delta T^0_0}{2M_g^2} \quad (51)
\]

\(^2\)Had we not set \( M_\star = 1 \) by rescaling, it would also enter in these combinations.
\[ -\partial_i \left( \dot{\Psi}_+ + H_{\text{dS}} \Phi_+ \right) = \frac{\delta T^0_{ij}}{2M_g^2} \]  \(52\)

\[ \ddot{\Psi}_+ + H \dot{\Phi}_+ + 3H_{\text{dS}} \dot{\Psi}_+ + 3H_{\text{dS}}^2 \Phi_+ + \frac{1}{2a^2} \left( \partial_j^2 + \partial_k^2 \right) (\Phi_+ - \Psi_+) = \frac{\delta T^i_{ij}}{2M_g^2} \]  \(53\)

\[ -\frac{1}{2a^2} \partial^i \partial_j (\Phi_+ - \Psi_+) = \frac{\delta T^i_{ij}}{2M_g^2}. \]  \(54\)

We immediately note that these equations are of exactly the same form as the analogous equations for perturbations in Einstein gravity. In the massive sector:

\[ -\frac{1}{a^2} \nabla^2 \Psi_- + 3H_{\text{dS}} \left( H \Phi_- + \dot{\Psi}_- \right) + \frac{m^2 P}{2a^2} \left( 1 + \frac{c^2}{c} \right) \left( 3\Psi_- - 3aH_{\text{dS}} F + \nabla^2 B \right) = \frac{\delta T^0_{ij}}{2M_g^2} \]  \(55\)

\[ -\partial_i \left( \dot{\Psi}_- + H_{\text{dS}} \Phi_- \right) + \frac{m^2 P}{4a} \left( 1 + \frac{c^2}{c} \right) \partial_i \left( F + aB \right) = \frac{\delta T^0_{ij}}{2M_g^2} \]  \(56\)

\[ \ddot{\Psi}_- + H_{\text{dS}} \left( 3\dot{\Psi}_- + \dot{\Phi}_- \right) + 3H_{\text{dS}}^2 \Phi_- + \frac{1}{2a^2} \left( \partial_j^2 + \partial_k^2 \right) (\Phi_- - \Psi_-) + \]  

\[ + \frac{m^2 P}{2a^2} \left( 1 + \frac{c^2}{c} \right) \left\{ [-\Phi_- - (aF)^*] + [2\Psi_- - 2H_{\text{dS}} aF + (\partial_j^2 + \partial_k^2) B] \right\} = \frac{\delta T^i_{ij}}{2M_g^2} \]  \(57\)

\[ -\frac{1}{2a^2} \partial^i \partial_j (\Phi_- - \Psi_-) - \frac{m^2 P}{2a^2} \left( 1 + \frac{c^2}{c} \right) \partial_i \partial_j B = \frac{\delta T^i_{ij}}{2M_g^2} \]  \(58\)

where \( P \) and \( Q \) are defined in \([39]\). The earlier statement that only differences of the \( \Phi \) and \( \Psi \) fields can appear in the mass terms now translates into the statement that mass terms only appear for the \( \Phi_- \) and \( \Psi_- \) fields. Thus it must be that the \( \Phi_+ \) and \( \Psi_+ \) fields are massless, which is clear above. This was also observed in \([25]\).

### 6.3 dS solutions: Massless sector

We will consider only pressureless matter (dust), both for background and for fluctuations. This means that \( \delta T^i_{ij} = 0 \). However we will only literally use this condition in this section, since we will have sources generating effective pressure and anisotropic stress in the next section. Also, we will spatially Fourier transform the perturbations, i.e. write Fourier modes with spatial wave number \( k = |k| \). For practical purposes these wave numbers will correspond to wavelengths below the horizon scale, so \( k > H_{\text{dS}} \).
With this understanding, the massless ("plus") gravitational potentials in dS are

\[ \Psi_{+,dS} = \Phi_{+,dS} = C_1 e^{-H_{dS}t} + C_2 e^{-3H_{dS}t} \]  

(59)

where as before \( H_{dS} = \sqrt{\Omega_\Lambda} H_0 \). The matter perturbations are given from the 00 and 0i component of the equations of motion:

\[ \frac{\delta \rho_{dS}}{M_g^2} = -12 H_{dS}^2 C_2 e^{-3H_{dS}t} - \frac{2}{c_s^2} k^2 (C_1 e^{-3H_{dS}t} + C_2 e^{-5H_{dS}t}) \]  

(60)

\[ \frac{\delta u_{dS}}{M_g^2} = 4 C_2 H_{dS} e^{-3H_{dS}t}. \]  

(61)

Again these are identical with the corresponding GR solutions. As in GR, there are two integration constants \( C_1 \) and \( C_2 \), and the remaining fields are determined from these.

### 6.4 dS solutions: Massive sector

The massive ("minus") sector can be manipulated to yield a single wave equation from which the other fields are determined. Solving for \( \mathcal{F} \) and \( \mathcal{B} \) gives

\[ \mathcal{F} = \frac{2 M_g^{-2} \delta u + 4 H \Phi_- + 4 \Psi_-}{a M^2} - a \dot{B} \]  

(62)

\[ \mathcal{B} = \frac{\Psi_- - \Phi_- - M_g^{-2} \chi a^2}{a^2 M^2} \]  

(63)

where the anisotropic stress \( \chi \) is defined through \( \delta T^i_j = \partial_i \partial_j \chi \). Subtracting the 00 equation from the ii equation, and defining

\[ \Xi \equiv \Phi_- + \Psi_- \]  

(64)

gives the following second-order equation for the massive fluctuation \( \Xi \):

\[ \ddot{\Xi} - H_{dS} \dot{\Xi} + \frac{\nabla^2 \Xi}{a^2} + \left( M^2 - 2 H_{dS}^2 \right) \Xi = J \]  

(65)

with source

\[ J = \frac{1}{M_g^2} \left( -\delta p - \delta \rho + 2 H_{dS} \delta u - 2 \delta \dot{u} + \frac{1}{3} \nabla^2 \chi - H_{dS} a^2 \chi - a^2 \ddot{\chi} \right), \]  

(66)

where \( \delta p = (1/3) \delta T^k_k \) and \( \delta \rho = -\delta T^0_0 \). Note that generically, the massive field \( \Xi \) is excited by matter sources because the definition of \( \Xi \) in [64] contains \( g \) fluctuations that do couple to matter. Because of this mixing we cannot think of \( \Xi \) as purely "new physics", and in fact it contains a piece that is present also in GR, as we will see more explicitly below.

We note that if we define a new scalar field

\[ \Pi = \frac{\Xi}{a^2} \]  

(67)
its equation of motion is a (sourced) Klein-Gordon equation

\[(\square - M^2) \Pi = -J.\]  

(68)

where the covariant box operator is

\[
\square = -\frac{\partial}{\partial t^2} - 3H_{dS} \frac{\partial}{\partial t} + \nabla^2. \tag{69}
\]

The identification of the scalar wave equation in massive gravity scenario was previously considered in [44], following older work like [46].

For solving the massive scalar wave equation, it will be convenient to introduce a new “time” variable \(x\) that goes to zero as \(t \to \infty\)

\[
x = \frac{k}{a_{dS} H_{dS}} = \frac{k}{c_2 H_{dS}} e^{-H_{dS} t}. \tag{70}
\]

with \(c_2\) from (45), and we introduce a rescaled field

\[
y(x) = \sqrt{x} \Xi(x), \tag{71}
\]

i.e. \(y(t) = \sqrt{k/(c_2 H_{dS})} e^{-H_{dS} t/2} \Xi(t)\), then \(y(x)\) precisely satisfies the inhomogeneous Bessel equation:

\[
y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = J(x) \tag{72}
\]

where the \(\nu\) parameter (the order of the Bessel function) is

\[
\nu^2 = \frac{9}{4} - \frac{M^2}{H_{dS}^2}. \tag{73}
\]

Here \(M\) is given in (19) and the source \(J(x)\) is found from the massless sector, i.e. from (60) and (61). Setting \(\chi = 0\) and \(\delta p = 0\) it simplifies to

\[
J(x) = \frac{2c_2 C_1 H_{dS} x^{3/2}}{k} + \frac{20c_3 C_2 H_{dS}^3 x^{3/2}}{k^3} + \frac{2c_3 C_2 H_{dS}^3 x^{7/2}}{k^3}. \tag{74}
\]

We now proceed to write down the solutions of (72) for \(y(x)\) and use them to recover \(\Pi\) of (68). We will only consider \(M^2 < (9/4)H_{dS}^2\) here, such that \(\nu\) in (73) is real; the solution for \(\nu\) imaginary is discussed somewhat further in the appendix. We give representative plots of the corresponding \(J_\nu\) Bessel functions in fig. 1.

### 6.5 Solutions of massive wave equations in dS: real case

As we argue in the appendix, the solution of (72) is

\[
y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x) + y_p(x) \tag{75}
\]
where the particular solution $y_p$ is obtained from (74) and (114) as

$$y_p(x) = \frac{2c_5}{k} C_1 H d s_{5/2,\nu}(x) + \frac{20c_5^3 C_2 H^3 d s_{5/2,\nu}(x)}{k^3} + \frac{2c_5^3 C_2 H^3 d s_{5/2,\nu}(x)}{k^3}. \tag{76}$$

The $s_{\mu,\nu}(x)$ are special functions that are solutions to the inhomogeneous Bessel equation with power-like source, also called Lommel functions, and are defined in appendix A. By the asymptotics given in that appendix, this $y_p$ vanishes as $x^{7/2}$ for $x \to 0$ (late times). If we now fix $c_Y = 0$ in (75), the homogeneous solution $J_{\nu}(x)$ is proportional to $x^\nu$ for $x \to 0$. The parameter $\nu$ depends on the mass parameter $M$ and is given in (73). We see that since $\nu < 7/2$ (compare fig. 2), the homogeneous solution will be leading (i.e., its leading power of $x$ will be lower) compared to the particular solution, at late times. Of course, we can also turn off the homogeneous solution at will by setting $c_J = 0$, but the particular solution always remains in the massive scalar $\Pi$.

In fig. 1, some of the solutions of the homogeneous equation blow up at late time. Moreover, this is just the $J_{\nu}$ solution; the $Y_{\nu}$ solution blows up at late times for all $\nu$ in our range. This is not necessarily a problem for the theory, though it is a problem for our approximation; all it means here is that if these modes would be excited, linear perturbation theory breaks down in the (possibly distant) future. Even if nonlinear fluctuations around this background did produce some instability in the future, we do not believe this is relevant to our phenomenological objectives in this paper, simply because the far future is more of an auxiliary device here than a region of interest.

For example, one can picture a scenario where the current model in Hassan-Rosen bimetric theory is replaced by another effective theory at late times, where the growing solution is matched to a decaying solution in the new theory. It would be interesting to learn that this is not possible and actually the presence of these modes rule out the theory for some values of $M$, but at the moment this is not clear to us. See also the conclusions for more comments on this.
We summarize in figure 2 why $\nu > 1/2$ seems the only reasonable choice in the current model.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Summary of some interesting values of the mass parameter $M$ and the corresponding $\nu$ values from (73). We indicate when only one homogeneous mode goes to infinity at late time, or when both modes do, and when $\nu$ becomes imaginary (which is not a problem in itself, but see fig 1) and when $Y$ and $M_P^2$ become negative (which are big problems for the background in the current model).}
\end{figure}

6.6 Parameter values

We now have four integration constants that need to be specified: First, the constants $C_1$ and $C_2$ of (59) for $\Psi_{+,dS}$, which we can think of as initial conditions (ICs) for $\Psi_{+,dS}$ and $\dot{\Psi}_{+,dS}$, i.e. the gravitational potential perturbation of the massless sector. Or equivalently, we can think of them as ICs on the values (but not the derivatives) of $\delta \rho_{dS}$ and $\delta u_{dS}$. Second, we have the the constants $c_J$ and $c_Y$ from the homogeneous solution for $\Xi_{dS}$, which we can think of as ICs on $\Xi_{dS}$ and $\dot{\Xi}_{dS}$, i.e. the “wave-like” field in the massive sector. Thus in general we have four parameters $\{C_1, C_2, c_J, c_Y\}$.

For convenience we give the mass $M^2$ and the comoving wavenumber $k^2$ in terms of the Hubble parameter value $H_{dS} = \sqrt{\Omega_M H_0}$ that is asymptotically approached in the future, i.e. not in terms of the Hubble parameter value $H_0$ of today.

6.7 Conclusions in dS: nothing new for cosmological geometrical probes

Because the perturbation equations for the “+” subscript fields, eqs. (51)-(54), are identical to the GR equations, and the “−” subscript fields only appear in the massive equations (55)-(58), and never appear in the GR-like equations at this order, the matter perturbations and hence the linear growth of structure will be identical to that seen in general relativity. It is important for this conclusion that the matter perturbations are completely determined by the 00 and $\theta \theta$ components of Einstein’s equation, which are of course constraints. In general in massive gravity this may not be the case; see the conclusions for related comments.

We also note that the argument in the previous paragraph leaves much to be desired from a phenomenological standpoint. For example, as light rays propagate along geodesics of the $g$ metric,
presumably one cannot form linear combinations of fluctuations in the electromagnetic field as we have done here that look identical to the GR equations, even in pure de Sitter, so one could expect that bending of light and similar local experiments will be grossly affected even where the cosmological matter perturbations are not. However, we emphasize that local solutions are not yet well understood, and we follow the practice of [9] of lumping these unknowns into $M_P$, that is never directly relevant in global equations.

7 Quasi-de-Sitter space

We have seen that bimetric cosmological gravitational perturbations in de Sitter space are identical to those of GR. For this and other reasons, it is of interest to consider a universe that represents a small deviation from the de Sitter background, and then consider cosmological perturbation theory in this slightly generalized background. This kind of approach works well in inflation, but it is not often used for late-time cosmology where the presence of matter complicates things. We will see some of these complications, and why the quasi-de-Sitter approach is still useful for our purposes. For some background on quasi-de-Sitter, it is useful to consult a review of inflation, e.g. [6].

7.1 Background

We define quasi-de-Sitter space as near-exponential expansion of the scale factor

$$\tilde{\epsilon} \equiv -\frac{\dot{H}}{H^2} > 0$$

where as usual $H(t)$ is defined as $H = \dot{a}/a$, and the square in the denominator makes $\tilde{\epsilon}$ dimensionless. For pure de Sitter space, $H$ is strictly constant so $\tilde{\epsilon} = 0$ in pure dS. (The reason for the tilde will become apparent shortly.) We remind the reader that also in inflation $\epsilon$ is often first defined in terms of the geometry, just like in (77). The inflationary slow-roll parameters in inflation are then given in terms of some scalar field potential, for which we have no direct analog here.

It is a simple matter to compute $-\dot{H}/H^2$ from the Friedmann equation (22) to obtain

$$\tilde{\epsilon} \approx \frac{3}{2} \frac{\Omega_m}{\Omega_m a^3}$$

where we have used the zeroth-order relation $H(t) \approx H_{dS}$. We see that $\tilde{\epsilon}$ encodes the fraction of matter in a universe dominated by dark energy. Clearly, this should be a small parameter in the future, and as we approach matter domination, the quasi de Sitter approximation breaks down in the past. $^3$

$^3$In fact, the expansion in e.g. Ch. 8 of Weinberg [43] using the correction factor $C(x)$ is the inverse expansion of this, there it is the ratio of dark energy to matter, which breaks down around present and more severely in the future.
We split the Robertson-Walker scale factors of the two metrics into products of de Sitter scale factors and correction factors:

\[ a(t) = q_a(t)a_{dS}(t), \quad q_a(t \to \infty) = 1 \]  

\[ Y(t) = q_Y(t)Y_{dS}(t), \quad q_Y(t \to \infty) = 1 \]

so that the functions \( q_a(t) \) and \( q_Y(t) \) capture the “quasi-ness” of the expansion. Let us consider the exact solution (40), (45) for the two-component fluid, then we can extract the quasi-ness for \( a(t) \) as

\[ q_a(t) = \frac{a(t)}{a_{dS}(t)} = 2^{2/3} \frac{\sinh \left( \frac{2}{3} H_{dS} t \right)^{2/3}}{\exp H_{dS} t} = \left( 1 - \frac{1}{6} \epsilon(t) \right)^{2/3} \]  

exactly, where now we define

\[ \epsilon(t) \equiv 6e^{-3H_{dS} t} \]  

which agrees with (78) to lowest order (hence the tilde in (78)). For some numbers to keep in mind, \( \epsilon \approx 0.01 \) at \( H_0 t \approx 2.5 \), \( \epsilon \approx 0.1 \) at \( H_0 t \approx 1.6 \) and \( \epsilon \approx 0.5 \) at \( H_0 t \approx 1 \). We will prefer to stay at \( \epsilon < 1 \), which limits us to \( H_0 t > 0.7 \) as a matter of principle. (In actual examples, we will find greater limitations than this.) For small \( \epsilon \), we can expand the quasi-ness of \( a(t) \) in \( \epsilon \):

\[ q_a(t) = 1 - \frac{1}{9} \epsilon(t) - \frac{1}{324} \epsilon(t)^2 + \ldots \]  

We can then easily compute the (square of the) Hubble function:

\[ H^2 \equiv \frac{\dot{a}^2}{a^2} = H_{dS}^2 \left( 1 + \frac{2}{3} \epsilon + \frac{2}{9} \epsilon^2 \right) \]  

leading to

\[ -\frac{\dot{H}}{H^2} = \epsilon(t) + \ldots . \]  

Note that because we defined \( \epsilon(t) \) as (82), this is not exactly \( \tilde{\epsilon}(t) \) of (77). This distinction is one of convenience and merely amounts to a rearrangement of higher-order perturbation theory in the “true” quasi-de Sitter parameter \( \tilde{\epsilon} \).

For the \( f \) metric one can argue similarly. From (43) and with the dS solution in (45),

\[ Y(t) = Y_{dS} \left( 1 - \frac{c_Y}{9} \epsilon - \frac{c_{Y,2}}{324} \epsilon^2 + \ldots \right) \]  

where

\[ c_Y = \frac{M^2 + 4(1 + c^2)H_{dS}^2}{M^2 - 2(1 + c^2)H_{dS}^2} \]  

\[ c_{Y,2} = \frac{M^4 + 44(1 + c^2)M^2 H_{dS}^4 - 20(1 + c^2)^2 H_{dS}^4}{(M^2 - 2(1 + c^2)H_{dS}^2)^2} . \]

This qdS expansion of the scale factor \( Y \) captures the loss of proportionality between \( Y(t) \) and \( a(t) \) as we leave the pure dS regime and enter the quasi-de Sitter regime. (Note that the constant \( c_Y \) is never unity.) We summarize the results for qdS expansion coefficients for the various derived background quantities in appendix C at linear order, which is all we will use explicitly in this paper.
Figure 3: Background solutions for the Hubble functions $H = \dot{a}/a$ and $K = \dot{Y}/Y$, with the latter for $c = 6$ and $c = 1/6$, cf. (44). (Here the $c = 1/6$ curves are included for illustration only, we never use values for $c$ this low). Linear (in $\epsilon$) qdS in black, quadratic qdS in dashed red, exact solution in dotted blue, and $H_0 t \sim 1$ is roughly present. We see that for times $H_0 t \gtrsim 0.7$, qdS remains a good approximation for the $g$ background, and also for the $f$ background for $c = 6$.

7.2 Perturbations in qdS

We write the general perturbation equations in section 4.2 as $D_{mn} \phi^n = J_m$ for a collection of fields $\phi^m$ enumerated by $m = 1, \ldots, 8$ and a differential operator $D_{mn}$ and sources $J_m$. We organize the expansion as follows:

$$D_{mn} = D^0_{mn} + \epsilon D^\epsilon_{mn}, \quad \phi^m = \phi^m_0 + \epsilon \phi^m_\epsilon, \quad J_m = J^0_m + \epsilon J^\epsilon_m,$$

from which we write the first order equation as

$$D^0_{mn}(\epsilon \phi^m_\epsilon) = \epsilon J^\epsilon_m - \epsilon D^\epsilon_{mn} \phi^m_0 \quad (90)$$

$$= \epsilon (J^\epsilon_m + \tilde{J}^\epsilon_m) \quad (91)$$

using the zeroth order de Sitter equations $D^0_{mn} \phi^m_0 = J^0_m$, and neglecting quadratic order in $\epsilon$. As expected, the zeroth order (pure de Sitter) fields act as additional sources $\tilde{J}^\epsilon_m$ for the first order quasi-de Sitter fields. It is useful that from this vantage point, the differential operator on the left-hand side of (91) is the unperturbed de Sitter differential operator $D^0_{mn}$.

Note that (91) contains terms with the time derivative $\dot{\epsilon}$ of our perturbation parameter. In principle by using (82) we can express these entirely in terms of $\epsilon$, and the latter would then drop out of the equation. In practice, it is convenient to work with the unperturbed dS differential
operator also in qdS, so we will not substitute in for \( \dot{\epsilon} \) and instead simply write

\[
\phi^m = \phi_{\text{dS}}^m + \phi_{\text{qdS}}^m .
\]  

(92)

The price to pay for this convenience is that viewed as expansions in \( x \sim e^{-H_{\text{dS}} t} \), it is not guaranteed that every term in \( \phi_{\text{qdS}}^m \) is suppressed compared to every term in \( \phi_{\text{dS}}^m \), and in fact it will generically not be the case, but the overall series in \( x \) does display relative suppression. We summarize this fact in table [I]. For more explicit comments on this issue in the simpler setting of ordinary GR, we refer the reader to appendix [D].

It would be useful to know when the linear (in \( \epsilon \)) qdS approximation is valid to some some prescribed accuracy for the perturbations. However, we currently do not have perturbations in an exact solution to compare to in bimetric theory, so it is hard to be absolutely precise about this (and if we did have an exact solution, the question would be rather pointless). As a first check, we compute the relative errors of the linear and quadratic qdS approximations versus the exact solution in cosmological perturbation theory in pure Einstein gravity in the aforementioned appendix [D]. As a second check, we have performed some preliminary analyses of the quadratic (in \( \epsilon \)) qdS approximation also in bimetric theory, in particular how the bimetric perturbations in the quadratic qdS approximation differ from the linear qdS approximation (typically if second order perturbation theory produces significant changes, perturbation theory has broken down). To be clear, for the purposes of this paper we only use the quadratic qdS approximation for auxiliary checks and we do not display it in plots. We find numerically that the linear approximation in \( \epsilon \) is good to about 10 to 30% for the bimetric perturbations (depending on the field) for \( H_0 t \gtrsim 1.3 - 1.5 \). This will be indicated by shading the region below this in the plots.

From the good accuracy of the qdS background in section 3 one could have hoped that the linear qdS perturbations would have extended further back than \( H_0 t \gtrsim 1.3 - 1.5 \), since the future is of no direct use for phenomenology, but there was of course no guarantee that this would be the case. On the good side, since going to quadratic order seems to give some improvement in our preliminary analyses, we believe that the qdS approximation at higher orders will be useful also for phenomenological purposes, and not only as indirect checks of numerical solutions of the perturbation equations in the exact background.

### 7.3 Parameters

We use wavenumbers \( k = 10H_{\text{dS}} \), \( k = (5/2)H_{\text{dS}} \) and \( k = (1/2)H_{\text{dS}} \) as representative cases, the latter only for internal checking of the analytics, as we will describe later. For \( k = (5/2)H_{\text{dS}} \) we will impose the following values on the matter perturbations at \( H_0 t' = 1.5 \). (One would have liked to do this at present \( t = t_0 \), but the qdS approximation needs to be valid in the region where we set initial conditions.) We obtain the values from GR (see appendix [D]):

\[
\frac{\rho(t')}{M_P^2} = 8.93 \cdot 10^{-5}, \quad \frac{u(t')}{M_P^2} = 0.72 \cdot 10^{-5}
\]  

(93)
For $k = (1/2)H_{\text{dS}}$, we have instead
\[
\frac{\rho(t')}{M_P^2} = 2.43 \cdot 10^{-5}, \quad \frac{u(t')}{M_P^2} = 0.72 \cdot 10^{-5}.
\] (94)

We view these as having roughly 10% accuracy. It is nontrivial to extract the values directly from data, but doing so would be useful in a more phenomenological analysis, rather than comparing directly to Einstein gravity, since we are of course modifying gravity.

### 7.4 Analytical series solution

The qdS equations are more complicated than the dS equations, but as the differential operator on the left-hand side of (91) is the unperturbed de Sitter differential operator, the general strategy for solving the differential equations is the same. In particular we again find a massive inhomogeneous Bessel equation for $\Xi_{\text{qdS}}$, just with more complicated sources. We will not turn on this homogeneous solution, since if we did, it should have been included at dS order (if this is unclear, it may help to consult our analogous comments in GR in appendix D). We introduce $y = x^{1/2}$, cf. (70). We compute series expansions of the right hand sides of all the perturbation equations to see which powers of $y$ actually occur, and arrive at the following series ansatz:

\[
\begin{align*}
\Phi_+ &= \Phi_{(7)} y^7 + \Phi_{(8)} y^8 + \Phi_{(11)} y^{11} + \Phi_{(12)} y^{12}, \\
\Psi_+ &= \Psi_{(7)} y^7 + \Psi_{(8)} y^8 + \Psi_{(11)} y^{11} + \Psi_{(12)} y^{12}, \\
\delta \rho &= \delta \rho_{(7)} y^7 + \delta \rho_{(8)} y^8 + \delta \rho_{(11)} y^{11} + \delta \rho_{(12)} y^{12}, \\
\delta u &= \delta u_{(7)} y^7 + \delta u_{(8)} y^8 + \delta u_{(11)} y^{11} + \delta u_{(12)} y^{12}, \\
\Xi &= \Xi_{(7)} y^7 + \Xi_{(8)} y^8 + \Xi_{(11)} y^{11} + \Xi_{(12)} y^{12}, \\
\Phi_- &= \Phi_{-(7)} y^7 + \Phi_{-(8)} y^8 + \Phi_{-(11)} y^{11} + \Phi_{-(12)} y^{12}.
\end{align*}
\] (95), (96), (97), (98), (99), (100)

(The $F$ and $B$ fields are determined from these, as before.) The explicit expressions for the coefficients obtained in this way are not terribly illuminating so we do not present them in full, but to give an idea of what they look like for our fixed parameter values (in particular $\nu = 1$), we find coefficients of the rather manageable form

\[
\begin{align*}
\Phi_{(7)} &= -\frac{6c_2^2H_{\text{dS}}^3}{5k^4} \cdot \frac{48M^4 - 176M^2H_{\text{dS}}^2 + 205H_{\text{dS}}^4}{(M^2 - 74H_{\text{dS}}^2)(M_{\text{dS}}^2 - 2H_{\text{dS}}^2)} \cdot C_J = \frac{128c_2^2H_{\text{dS}}^3}{97} \cdot C_J, \\
\Phi_{(8)} &= -\frac{8c_2^2H_{\text{dS}}^4}{3k^4} \cdot \frac{M^2 + 34H_{\text{dS}}^2}{(M^2 - 74H_{\text{dS}}^2)(M_{\text{dS}}^2 - 2H_{\text{dS}}^2)} \cdot C_1 = -\frac{376c_2^2H_{\text{dS}}^4}{291} \cdot C_1
\end{align*}
\] (101), (102)

and so on. For our parameters, $2(1 + c^2) = 74$, so both denominators are $(M^2 - 2(1 + c^2)H_{\text{dS}}^2)(M^2 - 2H_{\text{dS}}^2)$, i.e. the expansion breaks down not only for early times but also for two of the distinguished mass parameter values in figure 2. The breakdown point $M^2 = 2(1 + c^2)H_{\text{dS}}^2$ could have been anticipated from the background expansion (87).

For $k = (1/2)H_{\text{dS}}$, we find that the present time $t = t_0$ corresponds to $x = 0.47$ (see (70)), and for $k = (5/2)H_{\text{dS}}$ we find $x = 2.37$. So for the larger $k$ values, the more phenomenologically
Table 1: Leading expansion powers in the analytical solution. The notation means that the leading term at late time is $x^\Delta$.}

| field | $\Delta_{dS}$ | $\Delta_{qdS}$ | field | $\Delta_{dS}$ | $\Delta_{qdS}$ |
|-------|----------------|----------------|-------|----------------|----------------|
| $\Psi_+$ | 1 | $7/2$ | $\Xi$ | 3 | 4 |
| $\Phi_+$ | 1 | $7/2$ | $\Phi_-$ | 3 | 4 |
| $\delta \rho$ | 3 | $7/2$ | $B$ | 5 | 5 |
| $\delta u$ | 3 | $7/2$ | $F$ | 4 | 4 |

interesting ones, we expect this analytic version of the qdS expansion to break down, even when a numerical qdS approach would still be valid. (However, the analytics may be somewhat better than this, since many terms are actually expansions in $x/k$, which is independent of $k$.) Therefore we focus on $k = (1/2)H_{dS}$ when we use the analytical series solution. In all, the lessons we learn from the analytical solution is that there are certain degenerate special parameters, and we can quantify when a given approximation breaks down at least for small wavenumber $k$. None of this will be evident in the following, since we have already identified useful parameter values and we will not bother to show plots comparing the analytical and numerical results, we will just state here that they agree to the extent we expect them to. Perhaps the most important use is as cross-check with the numerics for low $k$. We now turn to the numerics.

### 7.5 Numerical solution: general

Naively we would expect that the energy density would receive a slight positive correction since going away from pure de Sitter expansion means that there is less expansion and the friction term due to the expansion is thus smaller. But in bimetric gravity the situation is more involved since both the massless and massive background sector will contribute to the correction. For example, the initial conditions set for the massive wave (i.e. the two integration constants for the homogeneous solution) will affect the correction to the energy density.

One way to fix parameters would be to use $\rho$, $u$ and $\dot{\rho}$ at present to fix ICs $C_1$, $C_2$ and $c_J$ (we always set $c_Y = 0$). Another way is to fix $c_J = 0$ and fix $C_1$, $C_2$ from $\rho$, $u$ at present, which is what we will do here (with the exception of figure 7). For the values (94) for $\delta \rho$ and $\delta u$, and $k = (5/2)H_{dS}$, we obtain

$$C_1 = -2.19 \cdot 10^{-5}, \quad C_2 = 3.33 \cdot 10^{-5}, \quad c_J = 0.$$  \quad (103)

where we have normalized the gravitational potential as in GR (see appendix D).

### 7.6 Numerical solution: gravitational potential

We first show two plots of the gravitational potential in figure 4. One generally expect that for larger
Figure 4: Numerical qdS plots for the gravitational potential $\Psi_+$ for $k = (5/2)H_{dS}$ (left panel) and $k = 10H_{dS}$ (right panel). and $M^2/H_{dS}^2 = \{1/50, 4/5, 5/4, 3/2\}$. The shaded area is our estimate for when the qdS approximation breaks down.

wavenumber $k$, the natural time variable $x$ in (70) is larger, so if we were to series expand the Bessel (and Lommel) functions, we would need to keep more terms. In other words, for large wavenumber $k$, the solutions “explore” the Bessel functions more, and the oscillations there can carry over to the gravitational potential.

### 7.7 Numerical solution: density contrast

We form the density contrast

$$\delta = \frac{\delta \rho - 3H\delta u}{\rho}$$

where $\rho$ is the background matter density. This is used to compute the growth factor.

We observe that although the $\Psi_+$ field depends on $M$, the density contrast $\delta$ seems to depend on $M$ much less. This is partially because of the way we fix boundary conditions, which is imposed directly on $\delta$ and therefore only indirectly on $\Psi_+$. Nevertheless, although the $\delta$ we see here does not differ appreciably from GR around $H_0t \gtrsim 1.3$, it does differ in the future, so we would expect that if we go away some time from the point at which we give the ICs (here $H_0t = 1.5$), there would in fact be some controllable difference, which is where the phenomenology of matter perturbations could begin.

It is of great importance whether there is a Vainshtein-like mechanism here. If there is a finite gap between the GR solution and the bimetric solution for any value of $M$, one might be tempted to conclude that there is in fact such a mechanism at work. However, there are also other parameters, for example $c_J$, that can be turned on to try to mimic GR at zeroth order. See figure 7.
Figure 5: Numerical qdS plots of the density contrast $\delta$ for various wavenumbers $k$. Left panel: small $k$ values, $k/H_{dS} = \{5/2, 3, 7/2, 4\}$. Right panel: intermediate $k$ values, $k/H_{dS} = \{19/2, 10, 21/2, 11\}$. Each curve is plotted for two $M$ values, $M^2/H_{dS}^2 = \{1/50, 3/2\}$, but the curves for the two $M$ values are nearly coincident in some cases. The shaded area is our estimate for when the qdS approximation breaks down.

7.8 Numerical solution: massive wave

The massive wave $\Pi/\Xi$ is plotted in Fig. 6. Again, the “bumps” are in the region where the approximation has already broken down, and so cannot be trusted. Still, also here we learn something about the massive wave around $H_0 t \sim 1.5$, and we see that it does depend on $M$, as one would expect.

One can use the existence of the homogenous mode to see if one can recreate GR. We show some simple attempts in this direction in figure 7.

8 Towards $\Lambda$CDM

We see that it is in principle possible to obtain good accuracy with the quasi-de Sitter approximation, but what suffices depends on the detailed application. In particular, with our current understanding it seems that it would be beneficial to automate an arbitrary-order qdS approach in symbolic manipulation software, especially if one wants to go to relevent eras such as $z \sim 1$. This is certainly possible, but outside the scope of this work. Even better would be if one could solve the fluctuation equations in the exact solution. At the moment we do not know if this is feasible in practice.
Figure 6: Numerical qdS plots for the massive scalar gravitational fluctuation $\Pi = \Xi/a^2$, for wavenumber $k = (5/2)H_{dS}$ (left panel) and $k = 10H_{dS}$ (right panel), and for mass parameter $M^2/H_{dS}^2 = \{1/50, 4/5, 5/4, 3/2\}$.

Figure 7: Numerical qdS plots for the density contrast $\delta$, for the homogeneous mode in $\Xi$ turned on, i.e. $c_J \neq 0$, unlike in the other plots in this section, where $c_J = 0$. 
9 Conclusions and outlook

We have computed the scalar fluctuation equations in de Sitter space and quasi de Sitter space, and we have found some analytical and some numerical solutions. There is much left to do as regards phenomenology.

It would also be very interesting to perform the analogous Hamiltonian analysis to analyze linear and nonlinear stability of these fluctuations. In GR, we can solve the gravitational perturbation separately, which then completely determine the matter perturbations. It is not clear that this strategy should automatically work here, as some of the constraints may become dynamical. But at least in quasi-de Sitter, there seems to be no real issue with this. To completely understand this issue, we would also need to perform a Hamiltonian analysis, which is beyond the scope of this work.

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A The inhomogenous Bessel equation

A.1 General

Let us begin by recalling some elementary facts about general inhomogenous 2nd order ODEs to set notation:

\[ D[y] = y'' + p(x)y' + q(x)y = g(x) . \] (105)

for some polynomials \( p(x), q(x) \). Let \( y_1 \) and \( y_2 \) be fundamental (linearly independent and normalized) solutions to the homogenous ODE \( D[y] = 0 \). We set for the general solution

\[ y_g(x) = u_1(x)y_1(x) + u_2(x)y_2(x) , \] (106)

for two unknown coefficient functions \( u_1, u_2 \). Using variation of parameters we find solutions for \( u_1(x) \) and \( u_2(x)\):

\[ u_1(x) = - \int dx \frac{y_2(x)g(x)}{W(x)} , \quad u_2(x) = \int dx \frac{y_1(x)g(x)}{W(x)} . \] (107)

where the Wronskian \( W \) is the usual determinant

\[ W[y_1, y_2] = y_1y_2' - y_2y_1' . \] (108)
A.2 Bessel

For Bessel functions, the fundamental solutions are $y_1(x) = J_\nu(x)$ and $y_2(x) = Y_\nu(x)$ and the Wronskian is quite simple:

$$W[J_\nu,Y_\nu] = J_{\nu+1}Y_\nu - J_\nu Y_{\nu+1} = \frac{2}{\pi x}$$

so the coefficient functions become

$$u_1(x) = -\frac{\pi}{2} \int dx Y_\nu(x) g(x) \cdot x, \quad u_2(x) = \frac{\pi}{2} \int dx J_\nu(x) g(x) \cdot x.$$  

where again $g(x)$ is the right-hand side of the inhomogenous equation, and the general (and generic) solution is simply

$$y_g = u_1(x) y_1(x) + u_2(x) y_2(x).$$

A simple way to specify ICs is to fix $\int_{x_0}^x \text{ in (112) and write}$

$$y = y_h + y_g,$$

with the usual two free integration constants in the homogeneous piece $y_h$, and no free constants in $y_g$. Then (112) above represents the \textit{particular} solution. If we specialize to a power-like right-hand side, $g(x) = x^\mu$, and fix $x_0 = 0$, then

$$y_{\text{particular}}(x) = \left(-\frac{\pi}{2} \int_0^x dx Y_\nu(x) x^{\mu+1}\right) J_\nu(x) + \left(\frac{\pi}{2} \int_0^x dx J_\nu(x) x^{\mu+1}\right) Y_\nu(x)$$

where $s_{\mu,\nu}(x)$ is a Lommel function. The series expansion of this Lommel function at $x = 0$ is

$$s_{\mu,\nu}(x) = \frac{x^{\mu+1}}{(\mu - \nu + 1) (\mu + \nu + 1)} + O(x^{\mu+3})$$

i.e. the order of vanishing at $x = 0$ is independent of $\nu$, which is not evident from the integral representation (114).

B Massless limit

In the limit $M \to 0 \ (\beta_2 \to 0)$ we find

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi}} \frac{x \sin x - x \cos x}{x^{3/2}}, \quad Y_{3/2}(x) = -\sqrt{\frac{2}{\pi}} \frac{x \sin x + \cos x}{x^{3/2}}$$

Using this, we find from (114) that the associated Lommel functions reduce to simple powers:

$$s_{3/2,3/2}(x) = \frac{x^2 + 2}{x^{3/2}}, \quad s_{5/2,3/2}(x) = x^{3/2}.$$
C Replacement rules

The range of $t$ is such that the dimensionless $\epsilon(t)$ is considered small. In the linear $\epsilon$ expansion, we have for the scale factors

$$a(t) = a_{dS}(1 - a_\epsilon \epsilon) \quad (120)$$

$$Y(t) = Y_{dS}(1 - a_\epsilon c_Y \epsilon) \quad (121)$$

with

$$a_\epsilon = \frac{1}{9} \quad (122)$$

$$c_Y = \frac{M^2 + 4(1 + c^2)H_{dS}^2}{M^2 - 2(1 + c^2)H_{dS}^2} \quad (123)$$

and for the derived quantities

$$H(t) \equiv \frac{\dot{a}}{a} = H_{dS}(1 + H_\epsilon \epsilon(t)) \quad (124)$$

$$K(t) \equiv \frac{\dot{Y}}{Y} = H_{dS}(1 + K_\epsilon \epsilon(t)) \quad (125)$$

$$X(t) \equiv \frac{\dot{Y}}{a} = c(1 + X_\epsilon \epsilon(t)) \quad (126)$$

where the coefficients are constants

$$H_\epsilon = 3a_\epsilon Y_\epsilon \quad (127)$$

$$K_\epsilon = cY/3 \quad (128)$$

$$X_\epsilon = 2a_\epsilon (Y_\epsilon - 1) \quad (129)$$

These simple expressions are sufficient to eliminate all derivatives on the background.

D Quasi-de-Sitter expansion in Einstein gravity

In this appendix we apply the qdS expansion to scalar perturbations in Einstein gravity. We compare the results of first and second order qdS expansions to the exact solution for the two-component fluid (with only dust and dark energy, which should be a good approximation to physical cosmology in this time interval).

The well known GR scalar perturbation equations for dust with the conventions used in this paper can be written as:

$$-\frac{1}{a^2} \nabla^2 \Psi + 3H \left( H \Phi + \dot{\Psi} \right) = -\frac{\delta \rho}{2M_P^2} \quad (130)$$
\[- \partial_i \left( \dot{\Psi} + H\Phi \right) = \frac{\partial_i \delta u}{2M_P^2} \]  \hspace{1cm} (131)
\[\dot{\Psi} + H\dot{\Phi} + 3H \left( H\Phi + \dot{\Psi} \right) + 2\dot{H}\Phi = 0\]
\[- \frac{1}{2a^2} \partial^i \partial_j (\Phi - \Psi) = 0\]

Recall from eq. (83) that the scale factor expanded to second order can be written as:
\[a(t) = a_{\text{as}}(t) \left( 1 - \frac{1}{9} \epsilon(t) - \frac{1}{324} \epsilon(t)^2 \right)\]

Using this expansion, the equations of motion for the perturbations can easily be recast in the general form
\[D_{ij} \phi^j = \left( D_{ij}^0 + D_{ij}^\varepsilon + D_{ij}^{\varepsilon^2} \right) \left( \phi_0^j + \phi_\varepsilon^j + \phi_\varepsilon^{\varepsilon^2} \right) = J_0^i + J_\varepsilon^i + J_\varepsilon^{\varepsilon^2}\]

which we split order by order into three sets of equations:
\[D_{ij}^0 \phi_0^j = J_0^i\]
\[D_{ij}^0 \phi_\varepsilon^j = J_\varepsilon^i - D_{ij}^\varepsilon \phi_0^j\]
\[D_{ij}^0 \phi_{\varepsilon^2}^j = J_\varepsilon^{\varepsilon^2} - D_{ij}^\varepsilon \phi_\varepsilon^j - D_{ij}^{\varepsilon^2} \phi_0^j\]

It is possible to analytically solve these equations, obtaining for the gravitational potential:
\[\Psi_0 = C_1 e^{-H_{\text{as}} t} + C_2 e^{-3H_{\text{as}} t}\]
\[\Psi_\varepsilon = \frac{8}{15} \left( 5C_1 e^{2H_{\text{as}} t} + 3C_2 \right) e^{-6H_{\text{as}} t}\]
\[\Psi_{\varepsilon^2} = \frac{7}{90} \left( 50C_1 e^{2H_{\text{as}} t} + 27C_2 \right) e^{-9H_{\text{as}} t}\]

where we have set the integration constants of the 1st and 2nd order equations to zero. In fact, the solutions of the homogenous versions of these equations are effectively of zeroth order, so to be consistent with the qdS expansion we should turn them off. With this understanding, the three expressions above are nicely separated in order. In the language of the main text, we have \(\Delta \Psi_0 = 1\), \(\Delta \Psi_1 = 4\), \(\Delta \Psi_2 = 7\) (which is a compact way of stating that the leading terms for large \(t\) are \(e^{-H_{\text{as}} t}\), \(e^{-4H_{\text{as}} t}\) and \(e^{-7H_{\text{as}} t}\), respectively). Here the suppressed terms in each fluctuation are less suppressed than the next order, as one would expect. This will be different for the matter perturbations below.

It is then straightforward to compute energy density and velocity perturbations using (130) and (131). For the energy density we obtain:
\[\frac{\delta \rho_0}{M_P^2} = \left( 12C_2 H_{\text{as}}^2 + \frac{2k^2}{c^2} C_1 \right) e^{-3H_{\text{as}} t} - \frac{2k^2}{c^2} C_2 e^{-5H_{\text{as}} t}\]
\[\frac{\delta \rho_\varepsilon}{M_P^2} = \frac{4}{15} \left( 135H_{\text{as}}^2 C_1 e^{2H_{\text{as}} t} - \frac{30k^2}{c^2} C_1 + 225H_{\text{as}}^2 C_2 - \frac{22k^2}{c^2} C_2 e^{-2H_{\text{as}} t} \right) e^{-6H_{\text{as}} t}\]
\[ \frac{\delta \rho_{e^2}}{M_P^2} = \frac{1}{45} \left( 7560H_{dS}^2 C_1 e^{2H_{dS}t} - \frac{810k^2}{c^2} C_1 + 7452H_{dS}^2 C_2 - \frac{521k^2}{c^2} C_2 e^{-2H_{dS}t} \right) e^{-9H_{dS}t}. \]

We observe that \( \Delta_{\delta \rho_0} = 3, \Delta_{\delta \rho} = 4, \Delta_{\delta \rho_{e^2}} = 7. \) Because the fields begin to mix at first order in the qdS approximations, also fields that are suppressed at zeroth order, as \( \delta \rho \) is, receive a first correction that is relatively big if the other fields are relatively big. In particular, \( \delta \rho_0 \) starts at \( e^{-3H_{dS}t} \) and the linear qdS field has a \( e^{-4H_{dS}t} \) piece. Moreover, there can be terms in the lower order fields, just from solving the equations, that strictly belong to higher orders in the expansion; this is the case for the \( e^{-5H_{dS}t} \) term in \( \delta \rho_0 \). We will typically keep such terms at the order at which they appear, but they cannot be considered reliable for truncation at the given order.

For the velocity perturbation:

\[ \frac{\delta u_0}{M_P^2} = 4C_2H_{dS}e^{-3H_{dS}t} \]

\[ \frac{\delta u_e}{M_P^2} = 12H_{dS} \left( C_1 e^{-4H_{dS}t} + C_2 e^{-6H_{dS}t} \right) \]

\[ \frac{\delta u_{e^2}}{M_P^2} = \frac{4}{5}H_{dS} \left( 40C_1 e^{-7H_{dS}t} + 29C_2 e^{-9H_{dS}t} \right) \]

for which \( \Delta_{\delta u_0} = 3, \Delta_{\delta u_e} = 4 \) and \( \Delta_{\delta u_{e^2}} = 7, \) like for \( \delta \rho. \)

To compare these results with the ΛCDM solutions, in the following called \( \Psi_f, \delta \rho_f, \) and \( \delta u_f, \) we have chosen \( C_1 \) and \( C_2 \) such that \( \Psi_f \) asymptotically matches \( \Psi_0 \) in the future. Notice that this is a slightly different choice of integration constants from that used for bimetric theory in the main text, but it is more suited to this analysis. Typically the two choices give very similar results.

With the initial conditions \( \Psi_f(t_*) = 10^{-5}, \dot{\Psi}_f(t_*) = 0, \) where \( t_* \) is \( H_0 t_* = 0.002 \) (during recombination) we found\(^4\)

\[ C_1 \simeq -2.28 \cdot 10^{-5}, \quad C_2 \simeq 4.89 \cdot 10^{-5}. \]  \hspace{1cm} \text{(132)}

Perhaps the best way to get an intuitive idea about how good our first and second order qdS approximations are is to consider \( \Psi \) plots like those of figure 8. A more precise measure is the relative error:

\[ \left| \frac{\Psi_f - (\Psi_0 + \Psi_e)}{\Psi_f} \right|_{H_0t=1} \simeq 0.015, \quad \left| \frac{\Psi_f - (\Psi_0 + \Psi_e + \Psi_{e^2})}{\Psi_f} \right|_{H_0t=1} \simeq 0.009. \]

so both the first and second order expansions are good to about 1% around present. For our purposes, it is also important to have an idea when the approximations break down. We find that

\[ \left| \frac{\Psi_f - (\Psi_0 + \Psi_e)}{\Psi_f} \right|_{H_0t=0.7} \simeq 0.1, \quad \left| \frac{\Psi_f - (\Psi_0 + \Psi_e + \Psi_{e^2})}{\Psi_f} \right|_{H_0t=0.5} \simeq 0.1 \]

i.e. we are down to 10% accuracy at \( H_0t \sim 0.7 \) and \( H_0t \sim 0.5 \) for the first and second order qdS approximations, respectively.

\(^4\)We have not been careful with the overall factor here, since we do no actual phenomenology in this paper. If the factor changes, all fields would simply be multiplied by the same correction factor, since we are doing linear perturbation theory.
Figure 8: Comparing approximations for the GR gravitational potential $\Psi$. The first order \(qdS\) approximation is black solid, the second order \(qdS\) approximation is red dashed, and the exact $\Lambda$CDM solution is blue dotted.

The range of validity of the various approximations for the matter perturbations might vary with \(k\). We analyzed what happens if \(1/k^2\) lies between the horizon scale and two orders of magnitude below the horizon scale, i.e. when:

\[ H_0^2 dS < k^2 < 100 H_0^2 dS \]

Around present, for all \(k\) in this range, the approximations do not differ more than 10\% from the exact solution, i.e.

\[ 0.1 \lesssim \left| \frac{\delta \rho_f - (\delta \rho_0 + \delta \rho_\epsilon)}{\delta \rho_f} \right|_{H_0 t=1} \lesssim 0.01, \quad 0.007 \lesssim \left| \frac{\delta \rho_f - (\delta \rho_0 + \delta \rho_\epsilon + \delta \rho_{\epsilon^2})}{\delta \rho_f} \right|_{H_0 t=1} \lesssim 0.005. \]

To be more precise, we found a small range of \(k\) in which the density perturbation \(qdS\) expansions are as good as the $\Psi$ expansions or even better, see figure 9 and 10, in particular we find:

\[
\begin{aligned}
3H_0 dS &\lesssim k^2 \lesssim 12H_0 dS \quad \Rightarrow \quad \left| \frac{\delta \rho_f - (\delta \rho_0 + \delta \rho_\epsilon)}{\delta \rho_f} \right|_{H_0 t > 0.7} < 0.1 \\
3H_0 dS &\lesssim k^2 \lesssim 12H_0 dS \quad \Rightarrow \quad \left| \frac{\delta \rho_f - (\delta \rho_0 + \delta \rho_\epsilon + \delta \rho_{\epsilon^2})}{\delta \rho_f} \right|_{H_0 t > 0.5} < 0.1
\end{aligned}
\]

Having computed density and velocity perturbations \(qdS\) expansions, it is simple to compute the corresponding expansions for the comoving density contrast $\delta = (\delta \rho - 3H \delta u)/\rho$:

\[
\delta_1 = \delta_0 + \delta_\epsilon \quad \delta_2 = \delta_0 + \delta_\epsilon + \delta_{\epsilon^2}
\]
Concerning the validity of \( \delta \) approximations we find the same results as for \( \delta \rho \) above. This is simply because the errors in the \( \delta u \) expansion are smaller than the errors in the \( \delta \rho \) expansions for all values of \( k^2 \).

Concerning the growth index we find that in general, the values computed with our 1st and 2nd order qdS approximations are trustable only for \( H_0 t \gtrsim 1.5 \). It is not hard to understand why this happens. We recall the explicit expression for the growth index:

\[
\gamma = \frac{\ln f}{\ln (\Omega_m/a^3)},
\]

with

\[
f = \frac{d \ln \delta}{d \ln a}.
\]

Now, given errors in \( \delta \) and \( \dot{\delta} \), it is straightforward to compute the expected error in \( \gamma \):

\[
\Delta \gamma = \left( \frac{\Delta \delta}{\delta} + \frac{\Delta \dot{\delta}}{\dot{\delta}} \right) \frac{1}{\ln (\Omega_m/a^3)}
\]

In general for \( H_0 t \lesssim 1.5 \) we have \( |\Delta \dot{\delta}| \gg |\Delta \delta| \). This is why the growth index expansions start deviating from the exact function before (in the sense of coming from the future) the other quantities of interest, i.e. the approximation for \( \gamma \) itself is a little worse than that for e.g. \( \Psi \).
E  Bimetric tensor modes in de Sitter

We are not directly interested in tensor models in this paper, but we would like to compare our mass parameters to the mass parameter of the tensor fluctuation. Consider gravitational waves traveling in the z-direction. The conditions for tracelessness and divergence-freeness of the perturbations are solved by the following ansatz

\[
\begin{align*}
\frac{ds_y^2}{g} &= -dt^2 + a(t)^2 \left( dx^2 + 2h_{xy}^g(t,z) dx dy + h_{xx}^g(t,z)(dx^2 - dy^2) \right), \\
\frac{ds_f^2}{g} &= -c^2 dt^2 + c^2 a(t)^2 \left( dx^2 + 2h_{xy}^f(t,z) dx dy + h_{xx}^f(t,z)(dx^2 - dy^2) \right),
\end{align*}
\]

where \( a(t) = c_2 e^{H_{dS} t} \). In terms of the linear combinations

\[
\begin{align*}
h_{xx}^+ &= h_{xx}^g + c^2 h_{xx}^f, & h_{xx}^- &= h_{xx}^g - h_{xx}^f, \\
h_{xy}^+ &= h_{xy}^g + c^2 h_{xy}^f, & h_{xy}^- &= h_{xy}^g - h_{xy}^f,
\end{align*}
\]

the linearized equations of motion become

\[
\begin{align*}
\left( \frac{\partial^2}{\partial t^2} + 3H_{dS} \frac{\partial}{\partial t} - \frac{1}{a^2} \frac{\partial^2}{\partial z^2} + M^2 \right) h_{xx,xy}^- &= 0, \\
\left( \frac{\partial^2}{\partial t^2} + 3H_{dS} \frac{\partial}{\partial t} - \frac{1}{a^2} \frac{\partial^2}{\partial z^2} \right) h_{xx,xy}^+ &= 0,
\end{align*}
\]

where

\[
M^2 = \left( 1 + \frac{1}{c^2} \right) m^2 (c\beta_1 + 2c^2\beta_2 + c^3\beta_3),
\]

Figure 10: Density perturbations for \( k^2 = 5H_{dS} \) and velocity perturbations: the first order qdS approximation is black solid, the second order qdS approximation is red dashed, and the exact \( \Lambda CDM \) solution is blue dotted.
which can be compared to (19), and where we used the background $g$ and $f$ equations to reexpress $\beta_0$ and $\beta_4$ as

$$\beta_0 = 3H^2/m^2 - 3\beta_1 c - 3\beta_2 c^2 - \beta_3 c^3$$  \hspace{1cm} (140)

$$c^4 \beta_4 = 3c^2 H^2/m^2 - \beta_1 c - 3\beta_2 c^2 - 3\beta_3 c^3.$$  \hspace{1cm} (141)

Figure 11: Comoving density contrast $\delta$ and growth index $\gamma$ for $k^2 = 5H_{dS}$: the second order qdS approximation, dashed, the first order, solid, and the $\Lambda$ CDM solution, dotted.

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