Chiral Symmetry

Gerhard Ecker

Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, A-1090 Wien, Austria

Abstract. Broken chiral symmetry has become the basis for a unified treatment of hadronic interactions at low energies. After reviewing mechanisms for spontaneous chiral symmetry breaking, I outline the construction of the low-energy effective field theory of the Standard Model called chiral perturbation theory. The loop expansion and the renormalization procedure for this nonrenormalizable quantum field theory are developed. Evidence for the standard scenario with a large quark condensate is presented, in particular from high-statistics lattice calculations of the meson mass spectrum. Elastic pion-pion scattering is discussed as an example of a complete calculation to $O(p^6)$ in the low-energy expansion. The meson-baryon system is the subject of the last lecture. After a short summary of heavy baryon chiral perturbation theory, a recent analysis of pion-nucleon scattering to $O(p^3)$ is reviewed. Finally, I describe some very recent progress in the chiral approach to the nucleon-nucleon interaction.

1 The Standard Model at Low Energies

My first Schladming Winter School took place exactly 30 years ago. Recalling the program of the 1968 School (Urban 1968), many of the topics discussed at the time are still with us today. In particular, chiral symmetry was very well represented in 1968, with lectures by S. Glashow, F. Gursey and H. Leutwyler. In those pre-QCD days, chiral Lagrangians were already investigated in much detail but the prevailing understanding was that due to their nonrenormalizability such Lagrangians could not be taken seriously beyond tree level. The advent of renormalizable gauge theories at about the same time seemed to close the chapter on chiral Lagrangians.

More than ten years later, after an influential paper of Weinberg (1979) and especially through the systematic analysis of Gasser and Leutwyler (1984, 1985), effective chiral Lagrangians were taken up again when it was realized that in spite of their nonrenormalizability they formed the basis of a consistent quantum field theory. Although QCD was already well established by that time the chiral approach was shown to provide a systematic low-energy approximation to the Standard Model in a regime where QCD perturbation theory was obviously not applicable.

Over the years, different approaches have been pursued to investigate the Standard Model in the low-energy domain. Most of them fall into the following three classes:

i. QCD-inspired models

There is a large variety of such models with more or less inspiration from
QCD. Most prominent among them are different versions of the Nambu-Jona-Lasinio model (Nambu and Jona-Lasinio 1961; Bijnens 1996 and references therein) and chiral quark models (Manohar and Georgi 1984; Bijnens et al. 1993). Those models have provided a lot of insight into low-energy dynamics but in the end it is difficult if not impossible to disentangle the model dependent results from genuine QCD predictions.

ii. Lattice QCD

iii. Chiral perturbation theory (CHPT)

The underlying theory with quarks and gluons is replaced by an effective field theory at the hadronic level. Since confinement makes a perturbative matching impossible, the traditional approach (Weinberg 1979; Gasser and Leutwyler 1984, 1985; Leutwyler 1994) relies only on the symmetries of QCD to construct the effective field theory. The main ingredient of this construction is the spontaneously (and explicitly) broken chiral symmetry of QCD.

The purpose of these lectures is to introduce chiral symmetry as a leit-motiv for low-energy hadron physics. The first lecture starts with a review of spontaneous chiral symmetry breaking. In particular, I discuss a recent classification of possible scenarios of chiral symmetry breaking by Stern (1998) and a connection between the quark condensate and the \( V, A \) spectral functions in the large-\( N_c \) limit (Knecht and de Rafael 1997). The ingredients for constructing the effective chiral Lagrangian of the Standard Model are put together. This Lagrangian can be organized in two different ways depending on the chiral counting of quark masses: standard vs. generalized CHPT. To emphasize the importance of renormalizing a nonrenormalizable quantum field theory like CHPT, the loop expansion and the renormalization procedure for the mesonic sector are described in some detail. After a brief review of quark mass ratios from CHPT, I discuss the evidence from lattice QCD in favour of a large quark condensate. The observed linearity of the meson masses squared as functions of the quark masses is consistent with the standard chiral expansion to \( O(p^4) \). Moreover, it excludes small values of the quark condensate favoured by generalized CHPT. Elastic pion–pion scattering is considered as an example of a complete calculation to \( O(p^6) \) in the low-energy expansion. Comparison with forthcoming experimental data will allow for precision tests of QCD in the confinement regime. Once again, the quark condensate enters in a crucial way. In the meson–baryon sector, the general procedure of heavy baryon CHPT is explained for calculating relativistic amplitudes from frame dependent amplitudes. As an application, I review the analysis of Mojžiš (1998) for elastic \( \pi N \) scattering to \( O(p^3) \). Finally, some promising new developments in the chiral treatment of the nucleon–nucleon interaction are discussed.
1.1 Broken Chiral Symmetry

The starting point is an idealized world where $N_f = 2$ or 3 of the quarks are massless ($u, d$ and possibly $s$). In this chiral limit, the QCD Lagrangian

$$\mathcal{L}_{\text{QCD}}^D = -\bar{q}i\gamma^\mu \left( \partial_\mu + ig_s \frac{\lambda^\alpha}{2} G^\alpha_\mu \right) q - \frac{1}{4} G^\alpha_\mu G^{\alpha\mu} + \mathcal{L}_{\text{heavy quarks}}$$

At the effective hadronic level, the quark number symmetry $U(1)_V$ is realized as baryon number. The axial $U(1)_A$ is not a symmetry at the quantum level due to the Abelian anomaly ('t Hooft 1976; Callan et al. 1976; Crewther 1977) that leads for instance to $\mathcal{M}_\eta \neq 0$ even in the chiral limit.

A classical symmetry can be realized in quantum field theory in two different ways depending on how the vacuum responds to a symmetry transformation. With a charge $Q = \int d^3x J_\mu(x)$ associated to the Noether current $J^\mu(x)$ of an internal symmetry and for a translation invariant vacuum state $|0\rangle$, the two realizations are distinguished by the

| Goldstone alternative |
|------------------------|
| $Q|0\rangle = 0$       | $||Q|0\rangle|| = \infty$ |
| Wigner-Weyl            | Nambu-Goldstone          |
| linear representation  | nonlinear realization    |
| degenerate multiplets  | massless Goldstone bosons|
| exact symmetry         | spontaneously broken symmetry |

There is compelling evidence both from phenomenology and from theory that the chiral group $G$ is indeed spontaneously broken:

i. Absence of parity doublets in the hadron spectrum.
ii. The $N_f^2 - 1$ pseudoscalar mesons are by far the lightest hadrons.
iii. The vector and axial-vector spectral functions are quite different as shown in Fig. 1.

iv. The anomaly matching conditions (‘t Hooft 1980; Frishman et al. 1981; Coleman and Grossman 1982) together with confinement require the spontaneous breaking of $G$ for $N_f \geq 3$.

v. In vector-like gauge theories like QCD (with the vacuum angle $\theta_{QCD} = 0$), vector symmetries like the diagonal subgroup of $G$, $SU(N_f)_v$, remain unbroken (Vafa and Witten 1984).

vi. There is by now overwhelming evidence from lattice gauge theories (see below) for a nonvanishing quark condensate.

All these arguments together suggest very strongly that the chiral symmetry $G$ is spontaneously broken to the vectorial subgroup $SU(N_f)_v$ (isospin for $N_f = 2$, flavour $SU(3)$ for $N_f = 3$):

$$G \rightarrow H = SU(N_f)_v.$$ (2)

To investigate the underlying mechanism further, let me recall one of the standard proofs of the Goldstone theorem (Goldstone 1961): starting with the charge operator in a finite volume $V$, $Q^V = \int_V d^3x J^0(x)$, one assumes the existence of a (local) operator $A$ such that

$$\lim_{V \rightarrow \infty} \langle 0 | [Q^V(x^0), A] | 0 \rangle \neq 0,$$ (3)

which is of course only possible if

$$Q | 0 \rangle \neq 0.$$ (4)
Then the Goldstone theorem tells us that there exists a massless state \(|G\rangle\) with
\[
\langle 0|J^0(0)|G\rangle\langle G|A|0\rangle \neq 0 .
\] (5)

The left-hand side of Eq. (3) is called an order parameter of the spontaneous symmetry breaking. The relation (5) contains two nonvanishing matrix elements. The first one involves only the symmetry current and it is therefore independent of the specific order parameter:
\[
\langle 0|J^0(0)|G\rangle \neq 0
\] (6)
is a necessary and sufficient condition for spontaneous breaking. The second matrix element in (5), on the other hand, does depend on the order parameter considered. Together with (6), its nonvanishing is sufficient but of course not necessary for the Nambu–Goldstone mechanism.

In QCD, the charges in question are the axial charges
\[
eq Q_A - Q_L \quad (i = 1, \ldots, N_f^2 - 1).
\] (7)
Which is (are) the order parameter(s) of spontaneous chiral symmetry breaking in QCD? From the discussion above, we infer that the operator \(A\) in (3) must be a colour–singlet, pseudoscalar quark–gluon operator. The unique choice for a local operator in QCD with lowest operator dimension three is
\[
A_i = \bar{q} \gamma_5 \lambda_i q
\] (8)
with
\[
[Q'_A, A_j] = -\frac{1}{2} \bar{q}\{\lambda_i, \lambda_j\} q .
\] (9)
If the vacuum is invariant under \(SU(N_f)_V\),
\[
\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle
\] (10)
Thus, a nonvanishing quark condensate
\[
\langle 0|\bar{q}q|0\rangle \neq 0
\] (11)
is sufficient for spontaneous chiral symmetry breaking. As already emphasized, (11) is certainly not a necessary condition. Increasing the operator dimension, the next candidate is the so-called mixed condensate of dimension five,
\[
\langle 0|\bar{q}\sigma_{\mu\nu}\lambda_0 qG^{a\mu\nu}|0\rangle \neq 0 ,
\] (12)
and there are many more possibilities for operator dimensions \(\geq 6\). All order parameters are in principle equally good for triggering the Goldstone mechanism. As we will see later on, the quark condensate enjoys nevertheless a special status. Although the following statement will have to be made more precise, we are going

\[\text{1 Here, the } \lambda_i \text{ are the generators of } SU(N_f)_V \text{ in the fundamental representation.}\]
to investigate whether the quark condensate is the dominant order parameter of spontaneous chiral symmetry breaking in QCD.

To analyse the possible scenarios, it is useful to consider QCD in a Euclidean box of finite volume $V = L^4$. The Lagrangian for a massive quark in a given gluonic background is

$$\mathcal{L} = \bar{q}(\mathcal{D} + m)q$$

(13)

with hermitian $i\mathcal{D}$. In a finite volume, the Dirac operator has a discrete spectrum:

$$i\mathcal{D}u_n = \lambda_n u_n$$

(14)

with real eigenvalues $\lambda_n$ and orthonormal spinorial eigenfunctions $u_n$. Spontaneous chiral symmetry breaking is related to the infrared structure of this spectrum in the limit $V \to \infty$ (Banks and Casher 1980; Vafa and Witten 1984; Leutwyler and Smilga 1992; ...; Stern 1998).

The main reason for working in Euclidean space is the following. Because of

$$i\mathcal{D}u_n = \lambda_n u_n \rightarrow i\mathcal{D}\gamma_5 u_n = -\lambda_n \gamma_5 u_n ,$$

(15)

the nonzero eigenvalues come in pairs $\pm \lambda_n$. Therefore, the fermion determinant in a given gluon background is real and positive (for $\theta_{QCD} = 0$):

$$\det(\mathcal{D} + m) = m^\nu \prod_{\lambda_n \neq 0} (m - i\lambda_n) = m^\nu \prod_{\lambda_n > 0} (m^2 + \lambda_n^2) > 0 ,$$

(16)

where $\nu$ is the multiplicity of the zero modes. The fermion integration yields a real, positive measure for the gluonic functional integral. Thus, many statements for correlation functions in a given gluon background will survive the functional average over the gluon fields.

The quark two–point function for coinciding arguments can be written as (the subscript $G$ denotes the gluon background)

$$\langle \bar{q}(x)q(x)\rangle_G = - \sum_n \frac{u_n^\dagger(x)u_n(x)}{m - i\lambda_n}$$

(17)

implying

$$\frac{1}{V} \int d^4x \langle \bar{q}(x)q(x)\rangle_G = -\frac{1}{V} \sum_n \frac{1}{m - i\lambda_n} = -\frac{2m}{V} \sum_{\lambda_n > 0} \frac{1}{m^2 + \lambda_n^2} .$$

(18)

This relation demonstrates that the chiral and the infinite-volume limits do not commute. Taking the chiral limit $m \to 0$ for fixed volume yields $\langle \bar{q}q\rangle_G = 0$, in accordance with the fact that there is no spontaneous symmetry breaking in a finite volume. The limit of interest is therefore first $V \to \infty$ for fixed $m$ and then $m \to 0$.

The zero modes will not be relevant in the infinite–volume limit.
For $V \to \infty$, the eigenvalues $\lambda_n$ become dense and we must replace the sum over eigenvalues by an integral over a density $\rho(\lambda)$:

$$\frac{1}{V} \sum_n V \to \infty \int d\lambda \rho(\lambda).$$

Averaging the relation (18) over gluon fields and taking the infinite-volume limit, one gets

$$\langle 0|\bar{q}q|0 \rangle = -\int_{-\infty}^{\infty} \frac{d\lambda \rho(\lambda)}{m - i\lambda} = -2m \int_0^\infty \frac{d\lambda \rho(\lambda)}{m^2 + \lambda^2}. \quad (19)$$

In the chiral limit, we obtain the relation of Banks and Casher (1980):

$$\lim_{m \to 0} \langle 0|\bar{q}q|0 \rangle = -\pi \rho(0). \quad (20)$$

For free fields, $\rho(\lambda) \sim \lambda^3$ near $\lambda = 0$. Thus, the eigenvalues must accumulate near zero to produce a nonvanishing quark condensate. Although the Banks–Casher relation does not tell us which gauge field configurations could be responsible for $\rho(0) \neq 0$, many suggestions are on the market (instantons, monopoles, ...).

This is a good place to recall the gist of the Vafa–Witten argument for the conservation of vector symmetries (Vafa and Witten 1984):

$$\lim_{m \to m_d} \frac{d\lambda \rho(\lambda) (m_u - m_d)}{m_u - i\lambda (m_d - i\lambda)} = (m_u - m_d) \int_{-\infty}^{\infty} \frac{d\lambda \rho(\lambda)}{(m_u - i\lambda)(m_d - i\lambda)}.$$ \quad (21)

Unlike in the chiral limit, the integrand in (21) does not become singular in the equal-mass limit and the vacuum remains $SU(N_f)_{\text{v}}$ invariant.

The previous discussion concentrated on one specific order parameter for spontaneous chiral symmetry breaking, the quark condensate. Stern (1998) has recently performed a similar analysis for a quantity that is directly related to the Goldstone matrix element (6). Consider the correlation function

$$\Pi^{\mu \nu}_{LR}(q) \delta_{ij} = 4i \int d^4 x e^{i q x} \langle 0|TL_i^\mu(x)R_j^\nu(0)|0 \rangle \quad (22)$$

$$L_i^\mu = \frac{\bar{q}L \gamma^\mu \lambda_i}{2} q_L, \quad R_i^\mu = \frac{\bar{q}R \gamma^\mu \lambda_i}{2} q_R.$$ \quad (23)

In the chiral limit, the correlator vanishes for any $q$ unless the vacuum is asymmetric. In particular, one finds in the chiral limit

$$\lim_{m_q \to 0} \Pi^{\mu \nu}_{LR}(0) = -F^2 g^{\mu \nu}. \quad (23)$$
where the constant $F$ (the pion decay constant in the chiral limit) characterizes the Goldstone matrix element (6):

$$
(0|q\gamma^\mu\gamma_5q|\varphi_j(p)) = i\delta_{ij}F[1 + O(m_q)]p^\mu e^{-ipx}.
$$

(24)

Thus, $\Pi^{\mu\nu}_{LR}(0) \neq 0$ is a necessary and sufficient condition for spontaneous chiral symmetry breaking.

Introducing the average (over all gluon configurations) number of states $N(\varepsilon, L)$ with $|\lambda| \leq \varepsilon$, Stern (1998) defines a mean eigenvalue density $\hat{\rho}$ in finite volume as

$$
\hat{\rho}(\varepsilon, L) = \frac{N(\varepsilon, L)}{2\varepsilon V}.
$$

(25)

Of course,

$$
\rho(0) = \lim_{\varepsilon \to 0} \lim_{L \to \infty} \hat{\rho}(\varepsilon, L)
$$

(26)

with the previously introduced density $\rho$. With similar techniques as before (again in Euclidean space), Stern (1998) has derived a relation for the decay constant $F$:

$$
F^2 = \pi^2 \lim_{\varepsilon \to 0} \lim_{L \to \infty} L^4 J(\varepsilon, L)\hat{\rho}(\varepsilon, L)^2
$$

(27)

in terms of an average transition probability between states with $|\lambda| \leq \varepsilon$:

$$
J(\varepsilon, L) = \frac{1}{N(\varepsilon, L)^2} \langle \langle \sum_{kl} J_{kl} \rangle \rangle, \quad J_{kl} = \frac{1}{4} \sum_\mu \left| \int d^4x u_k^\dagger(x)\gamma_\mu u_l(x) \right|^2
$$

(28)

where $\langle \langle ... \rangle \rangle$ denotes an average over gluon configurations. The formula (27) closely resembles the Greenwood–Kubo formula for electric conductivity (see Stern 1998).

As already emphasized, the eigenvalues $\lambda_n$ must accumulate near zero to trigger spontaneous chiral symmetry breaking. A crucial parameter is the critical exponent $\kappa$ defined as (Stern 1998)

$$
\langle \langle \lambda_n \rangle \rangle \sim L^{-\kappa}
$$

(29)

for $\lambda_n$ near zero and $L \to \infty$. Up to higher powers in $\varepsilon$, the average number of states and the mean eigenvalue density depend on $\kappa$ as

$$
N(\varepsilon, L) = \left( \frac{2\varepsilon}{\mu} \right)^\kappa (\mu L)^4 + \ldots
$$

(30)

$$
\hat{\rho}(\varepsilon, L) = \left( \frac{2\varepsilon}{\mu} \right)^\kappa \frac{1}{\mu^3} + \ldots
$$

(31)

in terms of some energy scale $\mu$. As is obvious from the definition (29) and from the expressions (30),(31), the eigenvalues with maximal $\kappa$ are the relevant ones.
The completeness sum rule $\sum_i J_{ki} = 1$ for the transition probabilities yields an upper bound for $F^2$ (Stern 1998):

$$F^2 \leq \pi^2 \mu^2 \lim_{\epsilon \to 0} \left( \frac{2\epsilon}{\mu} \right)^{\frac{4}{\kappa} - 2}.$$ (32)

Therefore, while $\kappa = 1$ for free fields, spontaneous chiral symmetry breaking requires $\kappa \geq 2$. With the same notation, we also have

$$\langle 0 | \bar{q} q | 0 \rangle = -\pi \mu^3 \lim_{\epsilon \to 0} \left( \frac{2\epsilon}{\mu} \right)^{\frac{4}{\kappa} - 1}$$ leading to $\kappa = 4$ for a nonvanishing quark condensate (Leutwyler and Smilga 1992). On rather general grounds, the critical index is bounded by

$$1 \leq \kappa \leq 4.$$ (34)

Stern (1998) has argued that the existence of an effective chiral Lagrangian analytic in the quark masses suggests that the exponent $4/\kappa$ is actually an integer$^3$. In this case, only $\kappa = 1$ or $\kappa = 2, 4$ would be allowed, the latter two cases being compatible with spontaneous chiral symmetry breaking.

There are then two preferred scenarios for spontaneous chiral symmetry breaking (Stern 1998):

i. $\kappa = 2$:

The density of states near $\epsilon = 0$ is too small to generate a nonvanishing quark condensate, but the high "quark mobility" $J$ induces $F \neq 0$.

ii. $\kappa = 4$:

Here, the density of states is sufficiently large for $\rho(0) \neq 0$. This option is strongly supported by lattice data (see below) favouring a nonvanishing quark condensate. With hindsight, the scenario most likely realized in nature is at least consistent with the previous analyticity hypothesis.

Are there other indications for a large quark condensate? Knecht and de Rafael (1997) have recently found an interesting relation between chiral order parameters and the vector and axial-vector spectral functions in the limit of large $N_c$. They consider again the correlation function (22). In the chiral limit, it can be expressed in terms of a single scalar function $\Pi_{LR}(Q^2)$:

$$\Pi_{LR}(q) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2), \quad Q^2 = -q^2.$$ (35)

$^3$There are explicit counterexamples to this analyticity assumption in less than four dimensions (L. Alvarez-Gaumé, H. Grosse and J. Stern, private communications).
Because it is a (nonlocal) order parameter, $\Pi_{LR}(Q^2)$ vanishes in all orders of QCD perturbation theory for a symmetric vacuum. The asymptotic behaviour for large and small $Q^2$ ($Q^2 \gtrsim 0$) is

$$\Pi_{LR}(Q^2) = -\frac{4\pi}{Q^6} [\alpha_s + O(\alpha_s^2)] \langle \bar{u}u \rangle^2 + O\left( \frac{1}{Q^8} \right)$$  \hspace{1cm} (36)$$

$$-Q^2 \Pi_{LR}(Q^2) = F^2 + O(Q^2) .$$  \hspace{1cm} (37)

For the large-$Q^2$ behaviour (36) (Shifman et al. 1979), $N_c \to \infty$ has already been assumed to factorize the four-quark condensate into the square of the (two-)quark condensate. In the same limit, the correlation function $\Pi_{LR}(Q^2)$ is determined by an infinite number of stable vector and axial-vector states:

$$-Q^2 \Pi_{LR}(Q^2) = F^2 + \sum_A \frac{F_A^2 Q^2}{M_A^2 + Q^2} - \sum_V \frac{F_V^2 Q^2}{M_V^2 + Q^2} ,$$  \hspace{1cm} (38)

where $M_I, F_I (I = V, A)$ are the masses and the coupling strengths of the spin-1 mesons to the respective currents. Comparison with the asymptotic behaviour (36) yields the two Weinberg sum rules (Weinberg 1967)

$$\sum_V F_V^2 - \sum_A F_A^2 = F^2$$  \hspace{1cm} (39)$$

$$\sum_V F_V^2 M_V^2 - \sum_A F_A^2 M_A^2 = 0$$  \hspace{1cm} (40)

and allows (38) to be rewritten as

$$-Q^2 \Pi_{LR}(Q^2) = \sum_A \frac{F_A^2 M_A^4}{Q^2(M_A^2 + Q^2)} - \sum_V \frac{F_V^2 M_V^4}{Q^2(M_V^2 + Q^2)} .$$  \hspace{1cm} (41)

This expression can now be matched once more to the asymptotic behaviour (36). Referring to Knecht and de Rafael (1997) for a general discussion, I concentrate here on the simplest possibility assuming that the $V, A$ spectral functions can be described by single resonance states plus a continuum. The experimental situation for the $I = 1$ channel shown in Fig. 1 is clearly not very far from this simplest case. In addition to the inequality $M_V < M_A$ following from the Weinberg sum rules (39), (40), the matching condition requires

$$4\pi [\alpha_s + O(\alpha_s^2)] \langle \bar{u}u \rangle^2 = F^2 M_V^2 M_A^2$$  \hspace{1cm} (42)$$

or approximately

$$4\pi \alpha_s \langle \bar{u}u \rangle^2 \simeq F^2 M_V^2 M_A^2 .$$  \hspace{1cm} (43)$$

From the last relation, Knecht and de Rafael (1997) extract a quark condensate

$$\langle \bar{u}u \rangle (\nu = 1 \text{ GeV}) \simeq -(303 \text{ MeV})^3$$  \hspace{1cm} (44)
with $\nu$ the QCD renormalization scale in the $\overline{\text{MS}}$ scheme. In view of the assumptions made, especially the large-$N_c$ limit, this value is quite compatible with

$$\langle \bar{u} u \rangle (\nu = 1 \text{ GeV}) = - [(229 \pm 9) \text{ MeV}]^3$$

from a recent compilation of sum rule estimates (Dosch and Narison 1998).

The conclusion is that the $V, A$ spectrum is fully consistent with both sum rule and lattice estimates for the quark condensate. We come back to this issue in the discussion of light quark masses.

### 1.2 Effective Field Theory

The pseudoscalar mesons are not only the lightest hadrons but they also have a special status as (pseudo-) Goldstone bosons. In the chiral limit, the interactions of Goldstone bosons vanish as their energies tend to zero. In other words, the interactions of Goldstone bosons become arbitrarily weak for decreasing energy no matter how strong the underlying interaction is. This is the basis for a systematic low-energy expansion with an effective chiral Lagrangian that is organized in a derivative expansion.

There is a standard procedure for implementing a symmetry transformation on Goldstone fields (Coleman et al. 1969; Callan et al. 1969). Geometrically, the Goldstone fields $\phi = [\pi, K, \eta]$ can be viewed as coordinates of the coset space $G/H$. They are assembled in a matrix field $u(\phi) \in G/H$, the basic building block of chiral Lagrangians. Different forms of this matrix field (e.g., the exponential representation) correspond to different parametrizations of coset space. Since the chiral Lagrangian is generically nonrenormalizable, there is no distinguished choice of field variables as for renormalizable quantum field theories.

An element $g$ of the symmetry group $G$ induces in a natural way a transformation of $u(\phi)$ by left translation:

$$u(\phi) g \in G \mapsto g u(\phi) = u(\phi') h(g, \phi).$$

The so-called compensator field $h(g, \phi)$ is an element of the conserved subgroup $H$ and it accounts for the fact that a coset element is only defined up to an $H$ transformation. For $g \in H$, the symmetry is realized in the usual linear way (Wigner–Weyl) and $h(g)$ does not depend on the Goldstone fields $\phi$. On the other hand, for $g \in G$ corresponding to a spontaneously broken symmetry ($g \not\in H$), the symmetry is realized nonlinearly (Nambu–Goldstone) and $h(g, \phi)$ does depend on $\phi$.

For the special case of chiral symmetry $G = SU(N_f)_L \times SU(N_f)_R$, parity relates left- and right-chiral transformations. With a standard choice of coset representatives, the general transformation (46) takes the special form

$$u(\phi') = g_R u(\phi) h(g, \phi)^{-1} = h(g, \phi) u(\phi) g_L^{-1}$$

$$g = (g_L, g_R) \in G.$$
For practical purposes, one never needs to know the explicit form of $h(g, \varphi)$, but only the transformation property (47). In the mesonic sector, it is often more convenient to work with the square of $u(\varphi)$. Because of (47), the matrix field $U(\varphi) = u(\varphi)^2$ has a simpler linear transformation behaviour:

$$U(\varphi) \rightarrow g_R U(\varphi) g_L^{-1}.$$  

(48)

It is therefore frequently used as basic building block for mesonic chiral Lagrangians.

When non-Goldstone degrees of freedom like baryons or meson resonances are included in the effective Lagrangians, the nonlinear picture with $u(\varphi)$ and $h(g, \varphi)$ is more appropriate. If a generic hadron field $\Psi$ (with $M_\Psi \neq 0$ in the chiral limit) transforms under $H$ as

$$\Psi \xrightarrow{\Psi} h_\Psi (h) \Psi$$  

(49)

according to a given representation $h_\Psi$ of $H$, the compensator field in this representation furnishes immediately a realization of all of $G$:

$$\Psi \xrightarrow{\Psi} h_\Psi (g, \varphi) \Psi.$$  

(50)

This transformation is not only nonlinear in $\varphi$ but also space–time dependent requiring the introduction of a chirally covariant derivative. We will come back to this case in the last lecture on baryons and mesons.

Before embarking on the construction of an effective field theory for QCD, we pause for a moment to realize that there is in fact no chiral symmetry in nature. In addition to the spontaneous breaking discussed so far, chiral symmetry is broken explicitly both by nonvanishing quark masses and by the electroweak interactions of hadrons.

The main assumption of CHPT is that it makes sense to expand around the chiral limit. In full generality, chiral Lagrangians are therefore constructed by means of a two-fold expansion in both

- derivatives (~ momenta) and
- quark masses:

$$\mathcal{L}_{\text{eff}} = \sum_{i,j} \mathcal{L}_{ij}, \quad \mathcal{L}_{ij} = O(p^i m^j_\varphi).$$  

(51)

The two expansions become related by expressing the pseudoscalar meson masses in terms of the quark masses $m_\varphi$. If the quark condensate is nonvanishing in the chiral limit, the squares of the meson masses start out linear in $m_\varphi$ (see below). The constant of proportionality is a quantity $B$ with

$$B = -\frac{\langle \bar{u} u \rangle}{F^2}.$$  

(52)
in the chiral limit. Assuming the linear terms to provide the dominant contributions to the meson masses corresponds to a scale (the product $Bm_q$ is scale invariant)

$$B(\nu = 1 \text{ GeV}) \simeq 1.4 \text{ GeV}.$$  

(53)

This standard scenario of CHPT (Weinberg 1979; Gasser and Leutwyler 1984, 1985; Leutwyler 1994) is compatible with a large quark condensate as given for instance in (45). The standard chiral counting

$$m_q = O(M^2) = O(p^2)$$  

(54)

reduces the two-fold expansion (51) to

$$L_{\text{eff}} = \sum_n L_n, \quad L_n = \sum_{i+2j=n} L_{ij}. \quad (55)$$

For mesons, the chiral expansion proceeds in steps of two ($n = 2, 4, 6, \ldots$) because the index $i$ is even.

Despite the evidence in favour of the standard scenario, the alternative of a much smaller or even vanishing quark condensate (e.g., for $\kappa = 2$ in the previous classification of chiral symmetry breaking) is actively being pursued (Fuchs et al. 1991; Stern et al. 1993; Knecht et al. 1993, 1995; Stern 1997 and references therein). This option is characterized by

$$B(\nu = 1 \text{ GeV}) \sim O(F\pi)$$  

(56)

with the pion decay constant $F\pi = 92.4 \text{ MeV}$. The so-called generalized CHPT amounts to a reordering of the effective chiral Lagrangian (55) on the basis of a modified chiral counting with $m_q = O(p)$. We will come back to generalized CHPT in several instances, in particular during the discussion of quark masses, but for most of these lectures I will stay with the mainstream of standard CHPT.

Both conceptually and for practical purposes, the best way to keep track of the explicit breaking is through the introduction of external matrix fields (Gasser and Leutwyler 1984, 1985) $v_\mu, a_\mu, s, p$. The QCD Lagrangian (1) with $N_f$ massless quarks is extended to

$$\mathcal{L} = \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma^\mu (v_\mu + a_\mu \gamma_5) q - \bar{q} (s - i p \gamma_5) q$$  

(57)

to include electroweak interactions of quarks with external gauge fields $v_\mu, a_\mu$ and to allow for nonzero quark masses by setting the scalar matrix field $s(x)$ equal to the diagonal quark mass matrix. The big advantage is that one can perform all calculations with a (locally) $G$ invariant effective Lagrangian in a manifestly chiral invariant manner. Only at the very end, one inserts the appropriate external fields to extract the Green functions of quark currents or matrix elements of interest. The explicit breaking of chiral symmetry is automatically taken care of by this spurion technique. In addition, electromagnetic gauge invariance is manifest.
Table 1. The effective chiral Lagrangian of the Standard Model

| $L_{\text{chiral dimension}}$ ( # of LECs) | loop order |
|------------------------------------------|------------|
| $L_2(2) + L_4^{\text{odd}}(0) + L_2^{\Delta S=1}(2) + L_6^{(1)}$ | $L = 0$ |
| $+ L_7^{\pi N}(1) + L_8^{\tau N}(7) + \ldots$ |            |
| $+ L_4^{\text{even}}(10) + L_6^{\text{odd}}(32) + L_4^{\Delta S=1}(22, \text{octet}) + L_7^{(14)}$ | $L = 1$ |
| $+ L_8^{\pi N}(23) + L_4^{\tau N}(?) + \ldots$ |            |
| $+ L_8^{\text{even}}(112 \text{ for } SU(N_f)) + \ldots$ | $L = 2$ |

Although this procedure produces all Green functions for electromagnetic and weak currents, the method must be extended in order to include virtual photons (electromagnetic corrections) or virtual $W$ bosons (nonleptonic weak interactions). The present status of the effective chiral Lagrangian of the Standard Model is summarized in Table 1. The purely mesonic Lagrangian is denoted as $L_2 + L_4 + L_6$ and will be discussed at length in the following lecture. Even (odd) refers to terms in the meson Lagrangian without (with) an $\varepsilon$ tensor. The pion-nucleon Lagrangian $\sum_n L_n^{\pi N}$ will be the subject of the last lecture. The chiral Lagrangians for virtual photons (superscript $\gamma$) and for nonleptonic weak interactions (superscript $\Delta S = 1$) will not be treated in these lectures. The numbers in brackets denote the number of independent coupling constants or low-energy constants (LECs) for the given Lagrangian. They apply in general for $N_f = 3$ except for the $\pi N$ Lagrangian ($N_f = 2$) and for the mesonic Lagrangian of $O(p^6)$ (general $N_f$). The different Lagrangians are grouped together according to the chiral order that corresponds to the indicated loop order. The underlined parts denote completely renormalized Lagrangians.

A striking feature of Table 1 is the rapidly growing number of LECs with increasing chiral order. Those constants describe the influence of all states that are not represented by explicit fields in the effective chiral Lagrangians. Although the general strategy of CHPT has been to fix those constants from experiment and then make predictions for other observables there is obviously a natural limit for such a program. This is the inescapable consequence of a nonrenormalizable effective Lagrangian that is constructed solely on the basis of symmetry consid-
erations. Nevertheless, I will try to convince you that even with 112 coupling constants one can make reliable predictions for low-energy observables.

2 Chiral Perturbation Theory with Mesons

The effective chiral Lagrangian for the strong interactions of mesons is constructed in terms of the basic building blocks $U(\varphi)$ and the external fields $v_\mu$, $a_\mu$, $s$ and $p$. With the standard chiral counting described previously, the chiral Lagrangian starts at $O(p^2)$ with

$$\mathcal{L}_2 = \frac{F^2}{4} \left( D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \right)$$

$$\chi = 2B(s + ip) \quad D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)$$

where $\langle \ldots \rangle$ stands for the $N_f$-dimensional trace. We have already encountered both LECs of $O(p^2)$. They are related to the pion decay constant and to the quark condensate:

$$F_\pi = F[1 + O(m_q)] = 92.4 \text{ MeV}$$

$$\langle 0|\bar{u}u|0\rangle = -F^2B[1 + O(m_q)]$$

Expanding the Lagrangian (58) to second order in the meson fields and setting the external scalar field equal to the quark mass matrix, one can immediately read off the pseudoscalar meson masses to leading order in $m_q$, e.g.,

$$M_{\pi^+}^2 = (m_u + m_d)B$$

As expected, for $B \neq 0$ the squares of the meson masses are linear in the quark masses to leading order. The full set of equations ($N_f = 3$) for the masses of the pseudoscalar octet gives rise to several well-known relations:

$$F_\pi^2 M_{\pi^+}^2 = -(m_u + m_d)\langle 0|\bar{u}u|0\rangle$$ (Gell-Mann et al. 1968)

$$\frac{M_{\pi^+}^2}{m_u + m_d} = \frac{M_{K^+}^2}{m_s + m_u} = \frac{M_{K^0}^2}{m_s + m_d}$$ (Weinberg 1977)

$$3M_{\eta^8}^2 = 4M_K^2 - M_{\pi}^2$$ (Gell-Mann 1957; Okubo 1962)

Having determined the two LECs of $O(p^2)$, we may now calculate from the Lagrangian (58) any Green function or S-matrix amplitude without free parameters. The resulting tree-level amplitudes are the leading expressions in the low-energy expansion of the Standard Model. They are given in terms of $F_\pi$ and meson masses and they correspond to the current algebra amplitudes of the sixties if we adopt the standard chiral counting.

The situation becomes more involved once we go to next-to-leading order, $O(p^4)$. Before presenting the general procedure, we observe that no matter how many higher-order Lagrangians we include, tree amplitudes will always be real. On the other hand, unitarity and analyticity require complex amplitudes in
A good example is elastic pion–pion scattering where the partial-wave amplitudes $t_l^f(s)$ satisfy the unitarity constraint

$$\Im t_l^f(s) \geq (1 - \frac{4M^2}{s})^{1/2} |t_l^f(s)|^2 .$$

(64)

Since $t_l^f(s)$ starts out at $O(p^2)$ (for $l < 2$), the partial-wave amplitudes are complex from $O(p^4)$ on.

This example illustrates the general requirement that a systematic low-energy expansion entails a loop expansion. Since loop amplitudes are in general divergent, regularization and renormalization are essential ingredients of CHPT. Any regularization is in principle equally acceptable, but dimensional regularization is the most popular method for well-known reasons.

Although the need for regularization is beyond debate, the situation is more subtle concerning renormalization. Here are two recurrent questions in this connection:

- Why bother renormalizing a quantum field theory that is after all based on a nonrenormalizable Lagrangian?
- Why not use a “physical” cutoff instead?

The answer to both questions is that we are interested in predictions of the Standard Model itself rather than of some cutoff version no matter how “physical” that cutoff may be. Renormalization ensures that the final results are independent of the chosen regularization method. As we will now discuss in some detail, renormalization amounts to absorbing the divergences in the LECs of higher-order chiral Lagrangians. The renormalized LECs are then measurable, although in general scale dependent quantities. In any physical amplitude, this scale dependence always cancels the scale dependence of loop amplitudes.

2.1 Loop Expansion and Renormalization

This part of the lectures is on a more technical level than the rest. Its purpose is to demonstrate that we are taking the quantum field theory aspects of chiral Lagrangians seriously.

The strong interactions of mesons are described by the generating functional of Green functions (of quark currents)

$$e^{iZ[j]} = \langle 0 \text{ out} | 0 \text{ in} \rangle = \int [d\varphi] e^{iS_{\text{eff}}[\varphi, j]}$$

(65)

where

$$j \sim v, a, s, p$$

denotes collectively the external fields.
The chiral expansion of the action

\[ S_{\text{eff}}[\varphi,j] = S_2[\varphi,j] + S_4[\varphi,j] + S_6[\varphi,j] + \ldots \] (66)

\[ S_n[\varphi,j] = \int d^4x \mathcal{L}_n(x) \]

is accompanied by a corresponding expansion of the generating functional:

\[ Z[j] = Z_2[j] + Z_4[j] + Z_6[j] + \ldots \] (67)

Functional integration of the quantum fluctuations around the classical solution gives rise to the loop expansion. The classical solution is defined as

\[ \varphi_{\text{cl}}[j] = 0 \] (68)

and it can be constructed iteratively as a functional of the external fields \( j \). Note that we define \( \varphi_{\text{cl}}[j] \) through the lowest-order Lagrangian \( \mathcal{L}_2(\varphi,j) \) at any order in the chiral expansion. In this case, \( \varphi_{\text{cl}}[j] \) carries precisely the tree structure of \( O(p^2) \) allowing for a straightforward chiral counting. This would not be true any more if we had included higher-order chiral Lagrangians in the definition of the classical solution.

With a mass-independent regularization method like dimensional regularization, it is straightforward to compute the degree of homogeneity of a generic Feynman amplitude as a function of external momenta and meson masses. This number is called the chiral dimension \( D \) of the amplitude and it characterizes the order of the low-energy expansion. For a connected amplitude with \( L \) loops and with \( N_n \) vertices of \( O(p^n) \) \( (n = 2,4,6,\ldots) \), it is given by (Weinberg 1979)

\[ D = 2L + 2 + \sum_n (n - 2)N_n , \quad n = 4,6,\ldots \] (69)

For a given amplitude, the chiral dimension obviously increases with \( L \). In order to reproduce the (fixed) physical dimension of the amplitude, each loop produces a factor \( 1/F^2 \). Together with the geometric loop factor \( (4\pi)^{-2} \), the loop expansion suggests

\[ 4\pi F_\pi = 1.2 \text{ GeV} \] (70)

as natural scale of the chiral expansion (Manohar and Georgi 1984). Restricting the domain of applicability of CHPT to momenta \( |p| \lesssim O(M_K) \), the natural expansion parameter of chiral amplitudes is therefore expected to be of the order

\[ \frac{M_K^2}{16\pi^2 F_\pi^2} = 0.18 . \] (71)

As we will see soon, these terms often appear multiplied with chiral logarithms. Substantial higher-order corrections in the chiral expansion are therefore to be expected for chiral \( SU(3) \). On the other hand, for \( N_f = 2 \) and for momenta \( |p| \lesssim O(M_\pi) \) the chiral expansion is expected to converge considerably faster.
The formula (69) implies that $D = 2$ is only possible for $L = 0$: the tree-level amplitudes from the Lagrangian $\mathcal{L}_2$ are then polynomials of degree 2 in the external momenta and masses. The corresponding generating functional is given by the classical action:

$$Z_2[j] = \int d^4x \mathcal{L}_2(\varphi_c[j], j).$$

(72)

Already at next-to-leading order, the amplitudes are not just polynomials of degree $D = 4$, but they are by definition of the chiral dimension always homogeneous functions of degree $D$ in external momenta and masses. For $D = 4$, we have two types of contributions: either $L = 0$ with $N_4 = 1$, i.e., exactly one vertex of $O(p^4)$, or $L = 1$ and only vertices of $O(p^2)$ (which, as formula (69) demonstrates, do not modify the chiral dimension). Explicitly, the complete generating functional of $O(p^4)$ consists of

$$L = 0 \quad Z_{\text{WZW}}[\varphi_c[j], v, a] \quad \text{chiral action of } O(p^4)$$

$$L = 1 \quad Z^{(L=1)}_4[j] \quad \text{one-loop functional}$$

In addition to the Wess–Zumino–Witten functional $Z_{\text{WZW}}$ (Wess and Zumino 1971; Witten 1983) accounting for the chiral anomaly, the $L = 0$ part involves the general chiral Lagrangian $\mathcal{L}_4$ with 10 LECs (Gasser and Leutwyler 1985):

$$\mathcal{L}_4 = \sum_i L_i \left( D_\mu U U^\dagger D_\mu U \right)^2 + \sum D_\mu U U^\dagger D_\nu U + \sum L_i \mathcal{L}_i$$

where $F^\mu_\nu$, $F^\mu_\nu$ are field strength tensors associated with the external gauge fields. This is the most general Lorentz invariant Lagrangian of $O(p^4)$ with (local) chiral symmetry, parity and charge conjugation.

The one-loop functional can be written in closed form as

$$Z^{(L=1)}_{\text{WZW}}[j] = \frac{i}{2} \ln \det D_2 = \frac{i}{2} \text{Tr} \ln D_2$$

(74)

in terms of the determinant of a differential operator associated with the Lagrangian $\mathcal{L}_2$. In accordance with general theorems of renormalization theory (e.g., Collins 1984), its divergent part takes the form of a local action with all the symmetries of $\mathcal{L}_2$ and thus of QCD. Since the chiral dimension of this divergence action is 4, it must be of the form (73) with divergent coefficients:

$$\mathcal{L}_{\text{div}}^{(L=1)} = -\Lambda(\mu) \sum_i P_i$$

where $P_i$ is the $i$-th LEC.

(75)

$$\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left[ \ln 4\pi + 1 + \Gamma'(1) \right] \right\}$$

Here, $\Gamma'(1)$ is the derivative of the Gamma function at 1.
Table 2. Phenomenological values of the renormalized LECs $L_i^r(M_\rho)$, taken from Bijnens et al. (1995), and $\beta$ functions $\Gamma_i$ for these coupling constants.

| $i$ | $L_i^r(M_\rho) \times 10^3$ | source | $\Gamma_i$ |
|-----|--------------------------|--------|-----------|
| 1   | $0.4 \pm 0.3$            | $K_{ee}, \pi\pi \to \pi\pi$ | 3/32      |
| 2   | $1.35 \pm 0.3$           | $K_{ee}, \pi\pi \to \pi\pi$ | 3/16      |
| 3   | $-3.5 \pm 1.1$           | $K_{ee}, \pi\pi \to \pi\pi$ | 0         |
| 4   | $-0.3 \pm 0.5$           | Zweig rule | 1/8       |
| 5   | $1.4 \pm 0.5$            | $F_K : F_\pi$ | 3/8       |
| 6   | $-0.2 \pm 0.3$           | Zweig rule | 11/144    |
| 7   | $-0.4 \pm 0.2$           | Gell-Mann-Okubo, $L_5, L_8$ | 0         |
| 8   | $0.9 \pm 0.3$            | $M_K^0 - M_{K^+}, L_5,$ | 5/48      |
|     |                          | $(2m_s - m_u - m_d) : (m_d - m_u)$ |           |
| 9   | $6.9 \pm 0.7$            | $r^2 \gamma_\nu$ | 1/4       |
| 10  | $-5.5 \pm 0.7$           | $\pi \to \nu\gamma$ | $-1/4$    |

with the conventions of Gasser and Leutwyler (1985) for $\overline{MS}$. The coefficients $\Gamma_i$ are listed in Table 2.

Renormalization to $O(p^4)$ proceeds by decomposing

$$ L_i = L_i^r(\mu) + \Gamma_i A(\mu) \quad (76) $$

such that

$$ Z_4 - Z_{\text{WZW}} = Z_4^{(L=1)} + \int d^4 x \mathcal{L}_4(L_i) = Z_4^{(L=1)}(\mu) + \int d^4 x \mathcal{L}_4(L_i^r(\mu)) \quad (77) $$

is finite and independent of the arbitrary scale $\mu$. The generating functional and therefore the amplitudes depend on scale dependent LECs that obey the renormalization group equations

$$ L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{(4\pi)^2} \ln \frac{\mu_1}{\mu_2}. \quad (78) $$

The current values of these constants come mainly from phenomenology to $O(p^4)$ and are listed in Table 2.

Many recent investigations in CHPT have included effects of $O(p^6)$ (see below for a discussion of elastic $\pi\pi$ scattering). The following contributions are also shown pictorially in Fig. 2:

$$ D = 6 : \quad L = 0, \quad N_6 = 1 $$
$$ L = 0, \quad N_4 = 2 $$
$$ L = 1, \quad N_4 = 1 $$
$$ L = 2 $$
Fig. 2. Skeleton diagrams of $O(p^6)$. Normal vertices are from $\mathcal{L}_2$, crossed circles denote vertices from $\mathcal{L}_4$ and the square in diagram f stands for a vertex from $\mathcal{L}_6$. The propagators and vertices carry the full tree structure associated with the lowest-order Lagrangian $\mathcal{L}_2$.

Unlike for the one-loop functional (74), no simple closed form for the two-loop functional $Z^{(L=2)}_2[j]$ is known. General theorems of renormalization theory guarantee that

- the sum of the irreducible loop diagrams a, b, d in Fig. 2 is free of subdivergences, and that
- the sum of the one-particle-reducible diagrams c, e, g is finite and scale independent (at least for the form of $\mathcal{L}_4$ given in (73)).

As a consequence, $Z_{6,\text{div}}^{(L=2)}$ is again a local action with all the symmetries of $\mathcal{L}_2$ and the corresponding divergence Lagrangian is of the general form $\mathcal{L}_6$ with divergent coefficients. For general $N_f$, this Lagrangian has 115 terms (Bijnens et al. 1998b), 112 measurable LECs and three contact terms. For $N_f = 3$, this Lagrangian was first written down by Fearing and Scherer (1996) but some of their terms are redundant.

How does renormalization at $O(p^6)$ work in practice? To simplify the discussion, we consider chiral $SU(2)$ with a single mass scale $M$ (the pion mass at lowest order). The LECs in chiral $SU(2)$ and their associated $\beta$ functions are usually denoted $\lambda_i, \gamma_i$ (Gasser and Leutwyler 1984). Since the divergences occur only in polynomials in the external momenta and masses, we consider a generic dimensionless coefficient $Q$ of such a polynomial, e.g., $m_6, f_6$ in the chiral expansions of the pion mass and decay constant, respectively (Bürgi 1996; Bijnens
et al. 1997):

\[ M_{\tau}^2 = M^2 \left\{ 1 + m_4 \frac{M^2}{F^2} + m_6 \frac{M^4}{F^4} + O(F^{-6}) \right\} \]  

(79)

\[ F_{\pi} = F \left\{ 1 + f_4 \frac{M^2}{F^2} + f_6 \frac{M^4}{F^4} + O(F^{-6}) \right\} . \]  

(80)

Working from now on in \( d \) dimensions, we obtain from the (irreducible) diagrams a,b,d and f

\[ Q = Q_{\text{loop}} + Q_{\text{tree}} \]  

(81)

with

\[ Q_{\text{loop}}(d) = \frac{J(0)^2 x(d)}{\text{diagrams a,b}} + J(0) \sum_i y_i(d) \]  

(82)

\[ J(0) = \frac{1}{i} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - M^2)^2} . \]

The coefficients \( x(d), y_i(d) \) are expanded to \( O(\omega^2) \) in \( \omega = \frac{1}{2}(d-4) \):

\[ x(d) = x_0 + x_1 \omega + x_2 \omega^2 + O(\omega^3) \]  

(83)

\[ y_i(d) = y_{i0} + y_{i1} \omega + y_{i2} \omega^2 + O(\omega^3) . \]

Likewise, for \( J(0) \) and the (unrenormalized) \( l_i \) we perform a Laurent expansion in \( \omega \):

\[ J(0) = \frac{M^{2\omega}}{(4\pi)^2} \frac{\Gamma(-\omega)}{2\omega} = \frac{(c\mu)^{2\omega}}{(4\pi)^2} \left( \frac{M}{\mu} \right)^{2\omega} \frac{\Gamma(-\omega)}{(4\pi)^\omega} \]  

(84)

\[ = \frac{(c\mu)^{2\omega}}{(4\pi)^2} \left[ -\frac{1}{\omega} + b(M/\mu) + a(M/\mu)\omega + O(\omega^2) \right] \]

\[ l_i = \frac{(c\mu)^{2\omega}}{(4\pi)^2} \left[ \frac{\gamma_i}{2\omega} + \beta_i(\mu) + \alpha_i(\mu)\omega + O(\omega^2) \right] . \]  

(85)

In the \( \overline{MS} \) scheme with

\[ 2\ln c = -1 - \ln 4\pi - \Gamma'(1) \]  

(86)

one gets

\[ b(M/\mu) = -2 \ln \frac{M}{\mu} - 1 \]  

(87)

\[ \beta_i(\mu) = (4\pi)^2 l_i^r(\mu) \]

where the \( l_i^r(\mu) \) are the standard renormalized LECs of Gasser and Leutwyler (1984).

An important consistency check is due to the absence of nonlocal divergences of the type

\[ \frac{\ln M/\mu}{\omega} \]
implying (Weinberg 1979)

\[ 4x_0 = \sum_i \gamma_i y_{i0} \]  

(88)

For SU(Nf), there are 115 such relations between two-loop and one-loop quantities due to the 115 independent monomials in the chiral Lagrangian of O(p6). We have recently verified these conditions by explicit calculation (Bijnens et al. 1998b).

With the summation convention for \( i \) implied, the complete loop contribution

\[ Q_{\text{loop}} = \frac{\mu^4}{(4\pi)^4} \left\{ -\frac{x_0}{\omega^2} + \frac{\left[x_1 - \beta_i(\mu)y_{i0} - \frac{1}{2}\gamma_i y_{i1}\right]}{\omega} \right\} + x_0 b(M/\mu)^2 + \left[-2x_1 + \beta_i(\mu)y_{i0} + \frac{1}{2}\gamma_i y_{i1}\right] b(M/\mu) + x_2 - \beta_i(\mu)y_{i1} - \frac{1}{2}\gamma_i y_{i2} - \alpha_i(\mu)y_{i0} + O(\omega) \}

is renormalized by the tree-level contribution from \( \mathcal{L}_6 \):

\[ Q_{\text{tree}}(d) = z(d) \]

\[ = \frac{\mu^4}{(4\pi)^4} \left\{ \frac{x_0}{\omega^2} - \frac{\left[x_1 - \beta_i(\mu)y_{i0} - \frac{1}{2}\gamma_i y_{i1}\right]}{\omega} + (4\pi)^4 z'(\mu) + O(\omega) \right\} \]

(90)

where \( z \) is the appropriate combination of (unrenormalized) LECs of O(p6). The total contribution from diagrams a,b,d,f is now finite and scale independent:

\[ Q = \lim_{d \to 4} [Q_{\text{loop}}(d) + Q_{\text{tree}}(d)] \]

\[ = \frac{1}{(4\pi)^4} \left\{ x_0 \left[1 + 2 \ln \frac{M}{\mu}\right]^2 + \left[2x_1 - \frac{1}{2}\gamma_i y_{i1} - (4\pi)^2 \bar{l}_i(\mu)y_{i0}\right] \left(1 + 2 \ln \frac{M}{\mu}\right) + x_2 - \frac{1}{2}\gamma_i y_{i2} - (4\pi)^2 \bar{l}_i(\mu)y_{i1} + (4\pi)^4 z'(\mu) \right\} \]

in terms of a redefined \(^4\) combination \( z'(\mu) \) of LECs,

\[ z'(\mu) = z'(\mu) - \frac{\alpha_i(\mu)y_{i0}}{(4\pi)^4} \]

(92)

that obeys the renormalization group equation

\[ \frac{d z'(\mu)}{d\mu} = \frac{2}{(4\pi)^4} \left[2x_1 - (4\pi)^2 \bar{l}_i(\mu)y_{i0} - \gamma_i y_{i1}\right]. \]

\(^4\) This process independent (Bijnens et al. 1998b) redefinition absorbs the redundant expansion coefficients \( \alpha_i(\mu) \).
Table 3. Complete calculations to $O(p^6)$ in standard CHPT.

| Process                                      | Reference               |
|----------------------------------------------|-------------------------|
| $\gamma\gamma \to \pi^0\pi^0$               | Bellucci et al. (1994)  |
| $\gamma\gamma \to \pi^+\pi^-$               | Bürgi (1996)            |
| $\pi \to \ell\nu\gamma$                     | Bijnens and Talavera (1997) |
| $\pi\pi \to \pi\pi$                         | Bijnens et al. (1996, 1997) |
| $\pi$ form factors                           | Bijnens et al. (1998a)  |
| $VV, AA$ form factors                        | Golowich and Kambor (1995, 1997) |

Remarks:

i. Weinberg's relation (88) implies that the coefficient of the leading chiral log $\ln^2 M/\mu$ can be extracted from a one–loop calculation (cf. Kazakov 1988).

ii. There are in general additional finite contributions (including chiral logs) from the reducible diagrams c, e.g. of Fig. 2.

In Table 3, I list the complete two–loop calculations that have been performed up to now. The first five entries are for chiral $SU(2)$, the last two for $N_f = 3$.

2.2 Light Quark Masses

In the framework of standard CHPT, the (current) quark masses $m_q$ always appear in the combination $m_q B$ in chiral amplitudes. Without additional information on $B$ through the quark condensate [cf. Eq. (59)], one can only extract ratios of quark masses from CHPT amplitudes.

The lowest–order mass formulas (62) together with Dashen's theorem on the lowest–order electromagnetic contributions to the meson masses (Dashen 1969) lead to the ratios (Weinberg 1977)

$$
\frac{m_u}{m_d} = 0.55, \quad \frac{m_s}{m_d} = 20.1.
$$

(94)

Generalized CHPT, on the other hand, does not fix these ratios even at lowest order but only yields bounds (Fuchs et al. 1990), e.g.,

$$
6 \leq r := \frac{m_s}{m} \leq r_2 := \frac{2M_K^2}{M_Z^2} - 1 \simeq 26
$$

(95)

with $2\bar{m} := m_u + m_d$. The ratios (94) receive higher–order corrections. The most important ones are corrections of $O(p^4) = O(m_q^2)$ and $O(e^2 m_s)$. Gasser
and Leutwyler (1985) found that to \( O(p^4) \) the ratios

\[
\frac{M_K^2 - M_s^2}{M_L^2} = \frac{m_s + \bar{m}}{m_u + m_d} [1 + \Delta_M + O(m_s^2)]
\]  
(96)

\[
\frac{(M_{K^0}^2 - M_{K^+}^2)_{QCD}}{M_K^2 - M_s^2} = \frac{m_d - m_u}{m_s - \bar{m}} [1 + \Delta_M + O(m_s^2)]
\]  
(97)

depend on the same correction \( \Delta_M \) of \( O(m_s) \). The ratio of these two ratios is therefore independent of \( \Delta_M \) and it determines the quantity

\[
Q^2 := \frac{m_s^2 - \bar{m}^2}{m_d^2 - m_u^2}.
\]  
(98)

Without higher-order electromagnetic corrections for the meson masses,

\[ Q = Q_D = 24.2 \],

but those corrections reduce \( Q \) by up to 10% \( (\text{Donoghue et al. 1993; Bijnens 1993; Duncan et al. 1996; Kambor and Leutwyler 1996; Leutwyler 1996a; Baur and Urech 1996; Bijnens and Prades 1997; Moussallam 1997}) \). Plotting \( m_s/m_d \) versus \( m_u/m_d \) leads to an ellipse \( \text{(Leutwyler 1990)} \). In Fig. 3, the relevant quadrant of the ellipse is shown for \( Q = 24 \) (upper curve) and \( Q = 21.5 \) (lower curve).

Kaplan and Manohar (1986) pointed out that due to an accidental symmetry of \( \mathcal{L}_2 + \mathcal{L}_4 \) the separate mass ratios \( m_u/m_d \) and \( m_s/m_d \) cannot be calculated to \( O(p^4) \) from S-matrix elements or \( V, A \) Green functions only. Some additional input is needed like resonance saturation (for (pseudo-)scalar Green functions), large-\( N_c \) expansion, baryon mass splittings, etc. Some of those constraints are also shown in Fig. 3. A careful analysis of all available information on the mass ratios was performed by Leutwyler (1996b, 1996c), with the main conclusion that the quark mass ratios change rather little from \( O(p^2) \) to \( O(p^4) \). In Table 4, I compare the so-called current algebra mass ratios of \( O(p^2) \) with the ratios including \( O(p^4) \) corrections, taken from Leutwyler (1996b, 1996c). The errors are Leutwyler’s estimates of the theoretical uncertainties as of 1996. Although theoretical errors are always open to debate, the overall stability of the quark mass ratios is evident.

Let me now turn to the absolute values of the light quark masses. Until recently, the results from QCD sum rules \( (\text{de Rafael 1998 and references therein}) \) tended to be systematically higher than the quark masses from lattice QCD. Some lattice determinations were actually in conflict with rigorous lower bounds on the quark masses \( (\text{Lellouch et al. 1997}) \). Recent progress in lattice QCD \( (\text{e.g., Lüscher 1997}) \) has led to a general increase of the (quenched) lattice values. Table 5 contains the most recent determinations of both \( \bar{m} \) and \( m_s \) that I am aware of. Judging only on the basis of the entries in Table 5, sum rule and lattice values for the quark masses now seem to be compatible with each other. The values are given at the \( \overline{\text{MS}} \) scale \( \nu = 2 \text{ GeV} \) as is customary in lattice QCD.
Fig. 3. First quadrant of Leutwyler's ellipse for $Q = 24$ (upper curve) and $Q = 21.5$ (lower curve). The dotted lines correspond to $\Theta_{\eta'} = -15^\circ$ (upper line) and $-25^\circ$ (lower line) for the $\eta - \eta'$ mixing angle. The bounds defined by the two dashed lines come from baryon mass splittings, $\rho - \omega$ mixing and $\Gamma(\psi' \rightarrow \psi \pi^0)/\Gamma(\psi' \rightarrow \psi \eta)$ (Leutwyler 1996b, 1996c) for the ratio $R = (m_s - \bar{m})/(m_d - m_u)$ ($35 \leq R \leq 50$).

Table 4. Quark mass ratios at $O(p^2)$ (Weinberg 1977) and to $O(p^4)$ (Leutwyler 1996b, 1996c).

|          | $m_u/m_d$ | $m_s/m_d$ | $m_s/\bar{m}$ |
|----------|-----------|-----------|---------------|
| $O(p^2)$ | 0.55      | 20.1      | 25.9          |
| $O(p^4)$ | 0.55 ± 0.04 | 18.9 ± 0.8 | 24.4 ± 1.5 |

Except for chiral logs, the squares of the meson masses are polynomials in $m_q$. It is remarkable if not puzzling that many years of lattice studies have not seen any indications for terms higher than linear in the quark masses. An impressive example from the high-statistics spectrum calculation of the CP-PACS Collaboration (Aoki et al. 1998) is shown in Fig. 4. The ratio $M^2/(m_1 + m_2)$ appears to be flat over the whole range of quark masses accessible in the simulations. The different values of $\beta$ stand for different lattice spacings but for each $\beta$ the ratio is constant to better than 5%. Since lattice calculations have found evidence for nonlinear quark mass corrections to baryon masses (e.g., Aoki
Table 5. Light quark masses in MeV at the $\overline{MS}$ scale $\nu = 2$ GeV. The most recent values from QCD sum rules and (quenched) lattice calculations are listed.

| $\bar{m}$ | $m_s$ |
|-----------|--------|
| 4.9 ± 0.9 | 125 ± 25 |
| Prades (1998) | Jamin (1998) |
| 5.7 ± 0.1 ± 0.8 | 130 ± 2 ± 18 |
| Giménez et al. (1998) |

Fig. 4. $2M^2/(m_1 + m_2)$ as a function of $(m_1 + m_2)/2$ (from Aoki et al. 1998).

et al. 1998), it is difficult to blame this conspicuous linearity\(^5\) between $M^2$ and $m_q$ on the limitations of present-day lattice methods only.

In order to see whether the lattice findings are consistent with CHPT, I take the $O(p^4)$ result (Gasser and Leutwyler 1985) for $M_K^2$ and vary $m_1 = \bar{m}$, $m_2 = m_s$. Since the actual quark masses on the lattice are still substantially bigger than $\bar{m}$, the $SU(2)$ result for $M_K^2$ cannot be used for this comparison.

\(^5\) Quenching effects are estimated to be $\sim 5\%$ at the lightest $m_q$ presently available on the lattice (Sharpe 1997; Golterman 1997).
Writing $M^2$ instead of $M^2_K$ for general $m_1, m_2$, one finds

$$M^2 = (m_1 + m_2)B \left\{ 1 + \frac{(m_1 + 2m_2)B}{72\pi^2F^2} \ln \frac{2(m_1 + 2m_2)B}{3\mu^2} \right. + \frac{8(m_1 + m_2)B}{F^2}(2L_8^r(\mu) - L_6^r(\mu)) + \frac{16(2m_1 + m_2)B}{F^2}(2L_6^r(\mu) - L_4^r(\mu)) \right\}$$

with the scale dependent LECs given in Table 2. As can easily be checked with the help of Eq. (78), $M^2$ in (99) is independent of the arbitrary scale $\mu$ as it should be.

Since the $L_i$ are by definition independent of quark masses, it is legitimate to use the values in Table 2 also when varying $m_1, m_2$. Let me first consider the standard scenario with $B(\nu = 1 \text{ GeV}) = 1.4 \text{ GeV}^6$ together with the mean values of the $L_i^r(M_\rho)$ in Table 2. In Fig. 5, $M^2$ is plotted as a function of the average quark mass $(m_1 + m_2)/2$ for two extreme cases: $m_1 = m_2$ or $m_1 = 0$. The second case with a massless quark can of course not be implemented on the lattice. As the figure demonstrates, there is little deviation from linearity at least up to $M \simeq 600 \text{ MeV}$ although this deviation is in general bigger than suggested by Fig. 4 (for the range of LECs in Table 2).

In order to demonstrate that the near-linearity is specific for standard CHPT, we now lower the value of $B$ as suggested by the proponents of generalized CHPT. Remember that $B = O(F^r)$ is considered to be a reasonable value in

\[ ^6 \text{Note that the quark masses in Fig. 4 correspond to } \nu = 2 \text{ GeV, however.} \]
that scenario. To show the dramatic changes required by a small $B$, I choose an intermediate value $B = 0.3$ GeV. Of course, in order to obtain the observed meson masses, at least some of the LECs have to be scaled up. Leaving the signs of the LECs unchanged, Eq. (99) requires to scale up $L_5$ to obtain realistic meson masses for a similar range of quark masses as before. But this is precisely the suggestion of generalized CHPT that the LECs associated with mass terms in $\mathcal{L}_4$ may have been underestimated (Stern 1997) by standard CHPT. For the following plot, I therefore take $L_5(M_p) = 20 \cdot 10^{-3}$. The two cases considered before ($m_1 = m_2$ or $m_1 = 0$) are now practically indistinguishable and they lead to a strong deviation from linearity as exhibited in the first graph of Fig. 6. The second graph can be compared with the lattice results in Fig. 4. Please make sure to compare the scales of the ordinates: whereas the lattice ratios vary by at most 5%, this ratio would now have to change by more than a factor of four (!) over the same range of quark masses.

The conclusion of this exercise is straightforward: lattice QCD is incompatible with a small quark condensate. Unless lattice simulations for the meson mass spectrum are completely unreliable, the observed linearity of $M^2$ in the quark masses favours standard CHPT and excludes values of $B$ substantially smaller than the standard value.

2.3 Pion–Pion Scattering

There are several good reasons for studying elastic pion–pion scattering:
i. The elastic scattering of the lightest hadrons is a fundamental process for testing CHPT: the only particles involved are SU(2) pseudo-Goldstone bosons. One may rightfully expect good convergence of the low-energy expansion near threshold.

ii. The behaviour of the scattering amplitude near threshold is sensitive to the mechanism of spontaneous chiral symmetry breaking (Stern et al. 1993), or more precisely, to the size of the quark condensate.

iii. After a long period without much experimental activity, there are now good prospects for significant improvements in the near future. K_{e4} experiments to extract pion–pion phase shifts due to the final–state interactions of the pions are already in the analysis stage at Brookhaven (Lowe 1997) or will start this year at the \( \Phi \) factory DA\( \Phi \)NE in Frascati (Baillargeon and Franzini 1995; Lee-Franzini 1997). In addition, the ambitious DIRAC experiment (Adeva et al. 1994; Schacher 1997) is being set up at CERN to measure a combination of \( S \)-wave scattering lengths through a study of \( \pi^+ \pi^- \) bound states.

In the isospin limit \( m_u = m_d \), the scattering amplitude is determined by one scalar function \( A(s, t, u) \) of the Mandelstam variables. In terms of this function, one can construct amplitudes with definite isospin \( (I = 0, 1, 2) \) in the \( s \)-channel. A partial-wave expansion gives rise to partial-wave amplitudes \( t_I^f(s) \) that are described by real phase shifts \( \delta_I^f(s) \) in the elastic region \( 4M_\pi^2 \leq s \leq 16M_\pi^2 \) in the usual way:

\[
t_I^f(s) = \left(1 - \frac{4M_\pi^2}{s}\right)^{-1/2} \exp i\delta_I^f(s) \sin \delta_I^f(s).
\] (100)

The behaviour of the partial waves near threshold is of the form

\[
\Re t_I^f(s) = q^2 \{a_I^f + q^2b_I^f + O(q^4)\},
\] (101)

with \( q \) the center-of-mass momentum. The quantities \( a_I^f \) and \( b_I^f \) are referred to as scattering lengths and slope parameters, respectively.

The low-energy expansion for \( \pi\pi \) scattering has been carried through to \( O(p^6) \) where two-loop diagrams must be included. Before describing the more recent work, let me recall the results at lower orders.

\[ O(p^2) \ (L = 0) \]

As discussed previously in this lecture, only tree diagrams from the lowest–order Lagrangian \( L_2 \) contribute at \( O(p^2) \). The scattering amplitude was first written down by Weinberg (1966):

\[
A_2(s, t, u) = \frac{s - M_\pi^2}{F_\pi^2}.
\] (102)

At the same order in the standard scheme, the quark mass ratios are fixed in terms of meson mass ratios, e.g., \( r = r_2 \) in the notation of Eq. (95).
In generalized CHPT, some of the terms in \( \mathcal{L}_4 \) in the standard counting appear already at lowest order. Because there are now more free parameters, the relation \( r = r_2 \) is replaced by the bounds (95). The \( \pi\pi \) scattering amplitude of lowest order in generalized CHPT is (Stern et al. 1993)

\[
A_2(s, t, u) = s - \frac{4}{3} \frac{M_\pi^2}{F_\pi^2} + \alpha \frac{M_\pi^2}{3 F_\pi^2}
\]

(103)

\[
\alpha = 1 + \frac{\delta (r_2 - r)}{r_2 - 1}, \quad \alpha \geq 1.
\]

The amplitude is correlated with the quark mass ratio \( r \). Especially the \( S \)-wave is very sensitive to \( \alpha \): the standard value of \( \alpha_0 = 0.16 \) for \( \alpha = 1 \) \( (r = r_2) \) moves to \( \alpha_0 = 0.26 \) for a typical value of \( \alpha \approx 2 \) \( (r \approx 10) \) in the generalized scenario. As announced before, the \( S \)-wave amplitude is indeed a sensitive measure of the quark mass ratios and thus of the quark condensate. To settle the issue, the lowest-order amplitude is of course not sufficient.

\[
O(p^4) \quad (L \leq 1)
\]

To next-to-leading order, the scattering amplitude was calculated by Gasser and Leutwyler (1983):

\[
F_\pi^4 A_4(s, t, u) = c_1 M_\pi^4 + c_2 M_\pi^2 s + c_3 s^2 + c_4 (t - u)^2 + F_1(s) + G_1(s, t) + G_1(s, u)
\]

(104)

\( F_1, G_1 \) are standard one-loop functions and the constants \( c_i \) are linear combinations of the LECs \( l_7^\mu(\mu) \) and of the chiral log \( \ln(M_\pi^2/\mu^2) \). It turns out that many observables are dominated by the chiral logs. This applies for instance to the \( I = 0 \) \( S \)-wave scattering length that increases from 0.16 to 0.20. This relatively big increase of 25% makes it necessary to go still one step further in the chiral expansion.

\[
O(p^6) \quad (L \leq 2)
\]

Two different approaches have been used. In the dispersive treatment (Knecht et al. 1995), \( A(s, t, u) \) was calculated explicitly up to a crossing symmetric subtraction polynomial

\[
[b_1 M_\pi^4 + b_2 M_\pi^2 s + b_3 s^2 + b_4 (t - u)^2]/F_\pi^4 + [b_5 s^3 + b_6 s(t - u)^2]/F_\pi^6
\]

(105)

with six dimensionless subtraction constants \( b_i \). Including experimental information from \( \pi\pi \) scattering at higher energies, Knecht et al. (1996) evaluated four of those constants \( (b_3, \ldots, b_6) \) from sum rules. The amplitude is given in a form compatible with generalized CHPT.
The field theoretic calculation involving Feynman diagrams with $L = 0, 1, 2$ was performed in the standard scheme (Bijnens et al. 1996, 1997). Of course, the diagrammatic calculation reproduces the analytically nontrivial part of the dispersive approach. To arrive at the final renormalized amplitude, one needs in addition the following quantities to $O(p^6)$: the pion wave function renormalization constant (Bürgi 1996), the pion mass (Bürgi 1996) and the pion decay constant (Bijnens et al. 1996, 1997). Moreover, in the field theoretic approach the previous subtraction constants are obtained as functions

$$b_i(M_\pi/F_\pi, M_\pi/\mu, l_i^\tau(\mu), k_i^\tau(\mu)),$$

where the $k_i^\tau$ are six combinations of LECs of the $SU(2)$ Lagrangian of $O(p^6)$.

Compared to the dispersive approach, the diagrammatic method offers the following advantages:

i. The full infrared structure is exhibited to $O(p^6)$. In particular, the $b_i$ contain chiral logs of the form $(\ln M_\pi/\mu)^n$ ($n \leq 2$) that are known to be numerically important, especially for the infrared-dominated parameters $b_1$ and $b_2$.

ii. The explicit dependence on LECs makes phenomenological determinations of these constants and comparison with other processes possible. This is especially relevant for determining $l_1^\tau, l_2^\tau$ to $O(p^6)$ accuracy (Colangelo et al. 1998).

iii. The fully known dependence on the pion mass allows one to evaluate the amplitude even at unphysical values of the quark mass (remember that we assume $m_u = m_d$). One possible application is to confront the CHPT amplitude with lattice calculations of pion-pion scattering (Colangelo 1997).

In the standard picture, the $\pi\pi$ amplitude depends on four LECs of $O(p^4)$ and on six combinations of $O(p^6)$ couplings. The latter have been estimated with meson resonance exchange that is known to account for the dominant features of the $O(p^4)$ constants (Ecker et al. 1989). It turns out (Bijnens et al. 1997) that the inherent uncertainties of this approximation induce small (somewhat bigger) uncertainties for the low (higher) partial waves. The main reason is that the higher partial waves are more sensitive to the short-distance structure.

However, as the chiral counting suggests, the LECs of $O(p^4)$ are much more important. Eventually, the $\pi\pi$ amplitude of $O(p^6)$ will lead to a more precise determination of some of those constants (Colangelo et al. 1998) than presently available. For the time being, one can investigate the sensitivity of the amplitude to the $l_i^\tau$. In Table 6, some of the threshold parameters are listed for three sets of the $l_i^\tau$ (Bijnens et al. 1997; Ecker 1997): set I is mainly based on phenomenology to $O(p^4)$ (Gasser and Leutwyler 1984; Bijnens et al. 1994), for set II the $\pi\pi$ $D$-wave scattering lengths to $O(p^6)$ are used as input to fix $l_1^\tau, l_2^\tau$, whereas for set III resonance saturation is assumed for the $l_i^\tau$ renormalized at $\mu = M_\eta$. Although some of the entries in Table 6 are quite sensitive to the choice of the $l_i^\tau$, two points are worth emphasizing:
- The $S$-wave threshold parameters are very stable, especially the $I = 0$ scattering length, whereas the higher partial waves are more sensitive to the choice of LECs of $O(p^4)$ (and also of $O(p^6)$).
- The resonance dominance prediction (set III) is in perfect agreement with the data although the agreement becomes less impressive for $\mu > M_\eta$.

Table 6. Threshold parameters in units of $M_{\pi^+}$ for three sets of LECs $l_i^+$ (Bijnens et al. 1997; Ecker 1997). The values of $O(p^4)$ correspond to set I. The experimental values are from Dumbrajs et al. (1983).

|               | $O(p^2)$ | $O(p^4)$ | $O(p^6)$ | $O(p^8)$ | $O(p^{10})$ | experiment |
|---------------|----------|----------|----------|----------|-------------|------------|
| $a_0^0$       | 0.16     | 0.20     | 0.217    | 0.206    | 0.209       | 0.26 ± 0.05|
| $b_0^0$       | 0.18     | 0.25     | 0.275    | 0.249    | 0.261       | 0.25 ± 0.03|
| $2a_0^0 - 5a_0^2$ | 0.55   | 0.61     | 0.641    | 0.634    | 0.626       | 0.66 ± 0.05|
| $-10b_0^0$    | 0.91     | 0.73     | 0.72     | 0.80     | 0.75        | 0.82 ± 0.08|
| $10a_1^0$     | 0.30     | 0.37     | 0.40     | 0.38     | 0.37        | 0.38 ± 0.02|
| $10^2a_2^0$   | 0        | 0.18     | 0.27     | input    | 0.19        | 0.17 ± 0.03|

In Fig. 7, the phase shift difference $\delta_0^0 - \delta_1^1$ is plotted as function of the center-of-mass energy and compared with the available low-energy data. The two-loop phase shifts describe the $K_{\pi^4}$ data (Rosselet et al. 1977) very well for both sets I and II, with a small preference for set I. The curve for set III is not shown in the figure, it lies between those of sets I and II.

To conclude this part on $\pi\pi$ scattering, let me stress the main features:

- The low-energy expansion converges reasonably well. The main uncertainties are not due to the corrections of $O(p^6)$, but they are related to the LECs of $O(p^4)$. This will in turn make a better determination of those constants possible (Colangelo et al. 1998).
- Many observables, especially the $S$-wave threshold parameters, are infrared dominated by the chiral logs. This is the reason why the $I = 0 S$-wave scattering length is rather insensitive to the LECs of $O(p^4)$. From the calculations in standard CHPT, a value

$$a_0^0 = 0.21 \div 0.22$$ (107)
Chiral Symmetry

is well established. This will be a crucial test for the standard framework once the data become more precise. On the basis of available experimental information, there is at present no indication against the standard scenario of chiral symmetry breaking with a large quark condensate.

- Altogether, there is good agreement with the present low-energy data as both Table 6 and Fig. 7 demonstrate.

- Isospin violation and electromagnetic corrections have to be included. First results are already available (Meißner et al. 1997; Knecht and Urech 1997).

3 Baryons and Mesons

A lot of effort has been spent on the meson-baryon system in CHPT (e.g., Bernard et al. 1995; Walcher 1998). Nevertheless, the accuracy achieved is not comparable to the meson sector. Here are some of the reasons.

- The baryons are not Goldstone particles. Therefore, their interactions are less constrained by chiral symmetry than for pseudoscalar mesons.
Due to the fermionic nature of baryons, there are terms of every positive order in the chiral expansion. In the meson case, only even orders can contribute.

- There are no "soft" baryons because the baryon masses stay finite in the chiral limit. Only baryonic three-momenta may be soft.

- In a manifestly relativistic framework (Gasser et al. 1988), the baryon mass destroys the correspondence between loop and chiral expansion that holds for mesons.

In this lecture, I will only consider chiral SU(2), i.e., pions and nucleons only. Some of the problems mentioned have to do with the presence of the "big" nucleon mass that is in fact comparable to the scale $4\pi F_\pi$ of the chiral expansion. This comparison suggests a simultaneous expansion in

\[
\frac{P}{4\pi F} \quad \text{and} \quad \frac{P}{m}
\]

where $P$ is a small three-momentum and $m$ is the nucleon mass in the chiral limit. On the other hand, there is an essential difference between $F$ and $m$: whereas $F$ appears only in vertices, the nucleon mass enters via the nucleon propagator. To arrive at a simultaneous expansion, one therefore has to shift $m$ from the propagator to the vertices of some effective Lagrangian. That is precisely the procedure of heavy baryon CHPT (Jenkins and Manohar 1991; Bernard et al. 1992), in close analogy to heavy quark effective theory.

### 3.1 Heavy Baryon Chiral Perturbation Theory

The main idea of heavy baryon CHPT is to decompose the nucleon field into "light" and "heavy" components. In fact, the light components will be massless in the chiral limit. The heavy components are then integrated out not unlike other heavy degrees of freedom. This decomposition is necessarily frame dependent but it does achieve the required goal: at the end, we have an effective chiral Lagrangian with only light degrees of freedom where the nucleon mass appears only in inverse powers in higher-order terms of this Lagrangian.

Since the derivation of the effective Lagrangian of heavy baryon CHPT is rather involved, I will exemplify the method only for the trivial case of a free nucleon with Lagrangian

\[
\mathcal{L}_0 = \bar{\Psi}(i\not\partial - m)\Psi .
\]

In terms of a time-like unit four-vector $v$ (velocity), one introduces projectors $P_v^\pm = \frac{1}{2}(1 \pm \not\partial)$. In the rest system with $v = (1,0,0,0)$, for instance, the $P_v^\pm$ project on upper and lower components of the Dirac field in the standard representation of $\gamma$ matrices. With these projectors, one defines (Georgi 1990) velocity-dependent fields $N_v, H_v$:

\[
N_v(x) = \exp[imv \cdot x]P_v^+ \bar{\Psi}(x)
\]

\[
H_v(x) = \exp[imv \cdot x]P_v^- \bar{\Psi}(x).
\]
The Dirac Lagrangian is now rewritten in terms of these fields:

$$\mathcal{L}_0 = (N_v + H_v)e^{imv \cdot x}(i\partial - m)e^{-imv \cdot x}(N_v + H_v)$$

$$= \overline{N_v}(iv \cdot \partial N_v - \overline{H_v}(iv \cdot \partial + 2m)H_v + \text{mixed terms}.$$  

After integrating out the heavy components $H_v$ in the functional integral with the fully relativistic pion-nucleon Lagrangian (Gasser et al. 1988), one arrives indeed at an effective chiral Lagrangian for the field $N_v$ (and pions) only, with a massless propagator

$$\frac{iP^+}{v \cdot k + ie}.$$  

At every order except the leading one, $O(p)$, this Lagrangian consists of two pieces: the first one is the usual chiral Lagrangian of $O(p^2)$ with a priori unknown LECs. The second part comes from the expansion in $1/m$ and it is completely given in terms of LECs of lower than $n$-th order. Since the only nucleon field in this Lagrangian is $N_v$ with a massless propagator, there is a straightforward analogue to chiral power counting in the meson sector given by formula (69). For a connected $L$-loop amplitude with $E_B$ external baryon lines and $N_{n,n_B}$ vertices of chiral dimension $n$ (with $n_B$ baryon lines at the vertex), the analogue of (69) is (Weinberg 1990, 1991)

$$D = 2L + 2 - \frac{E_B}{2} + \sum_{n,n_B} (n - 2 + \frac{n_B}{2}) N_{n,n_B}.$$  

However, as we will discuss later on in connection with nucleon-nucleon scattering, this formula is misleading for $E_B \geq 4$. On the other hand, no problems arise for the case of one incoming and one outgoing nucleon ($E_B = 2$) where

$$D = 2L + 1 + \sum_n [(n - 2)N_{n,0} + (n - 1)N_{n,2}] \geq 2L + 1.$$  

This formula is the basis for a systematic low-energy expansion for single-nucleon processes, i.e., for processes of the type $\pi N \rightarrow \pi \ldots \pi N$, $\gamma N \rightarrow \pi \ldots \pi N$, $l N \rightarrow l \pi \ldots \pi N$ (including nucleon form factors), $\nu l N \rightarrow l \pi \ldots \pi N$. The corresponding effective chiral Lagrangian is completely known to $O(p^3)$ (Bernard et al. 1992; Ecker and Mojžiš 1996; Fettes et al. 1998) including the full renormalization at $O(p^3)$ (Ecker 1994):

$$\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \ldots$$

$$\mathcal{L}_{\pi N}^{(1)} = \overline{N_v}(iv \cdot \nabla + g_A S \cdot u)N_v$$

$$u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) + \text{external gauge fields}, \quad \mathcal{S}^\mu = i\gamma_5 \sigma^{\mu
u} \mathcal{V}_{\nu}/2$$

with a chiral and gauge covariant derivative $\nabla$ and with $g_A$ the axial-vector coupling constant in the chiral limit.

Two remarks are in order at this point.
Table 7. Relations between relativistic covariants and the corresponding quantities in the initial nucleon rest frame \((v = p_{in}/m_N, q = p_{out} - p_{in}, t = q^2)\) with \(\bar{u}(p_{out})\Gamma u(p_{in}) = \bar{u}(p_{out})P_e^+\hat{\Gamma}P_e^+ u(p_{in})\).

| \(\Gamma\)   | \(\hat{\Gamma}\)                                                                 |
|-------------|-----------------------------------------------------------------------------------|
| 1           | 1                                                                                |
| \(\gamma_5\)| \(\frac{q \cdot S}{m_N(1 - t/4m^2_N)}\)                                       |
| \(\gamma^\mu\)| \((1 - t/4m^2_N)^{-1} \left(v^\mu + \frac{q^\mu}{2m_N} + \frac{i}{m_N}\epsilon^{\mu\nu\rho\sigma}q^\nu v^\rho S_\sigma\right)\) |
| \(\gamma^\mu\gamma_5\)| \(2S^\mu - \frac{q \cdot S}{m_N(1 - t/4m^2_N)} v^\mu\)                        |
| \(\sigma^{\mu\nu}\)| \(2\epsilon^{\mu\nu\rho\sigma}v^\rho S_\sigma + \frac{1}{2m_N(1 - t/4m^2_N)}\left\{i(q^\mu v^\nu - q^\nu v^\mu) + 2(v^\mu \epsilon^{\nu\lambda\rho\sigma} - v^\nu \epsilon^{\mu\lambda\rho\sigma})q_\lambda v^\rho S_\sigma\right\}\) |

Since the Lagrangian (114) was derived from a fully relativistic Lagrangian it defines a Lorentz invariant quantum field theory although it depends explicitly on the arbitrary frame vector \(v\) (Ecker and Mojžiš 1996). Reparametrization invariance (Luke and Manohar 1992) is automatically fulfilled.

- The transformation from the original Dirac field \(\Psi\) to the velocity-dependent field \(N_v\) leads to an unconventional wave function renormalization of \(N_v\) that is in general momentum dependent (Ecker and Mojžiš 1997).

Since the theory is Lorentz invariant it must always be possible to express the final amplitudes in a manifestly relativistic form. Of course, this will only be true up to the given order in the chiral expansion one is considering. The general procedure of heavy baryon CHPT for single-nucleon processes can then be summarized as follows.

i. Calculate the heavy baryon amplitudes to a given chiral order with the Lagrangian (114) in a frame defined by the velocity vector \(v\).

ii. Relate those amplitudes to their relativistic counterparts which are independent of \(v\) to the order considered. For the special example of the initial nucleon rest frame with \(v = p_{in}/m_N\), the translation is given in Table 7 (Ecker and Mojžiš 1997).

iii. Apply wave function renormalization for the external nucleons.

As an application of this procedure, I will now discuss elastic pion-nucleon scattering to \(O(p^3)\) in the low-energy expansion. For other applications of CHPT to single-nucleon processes, I refer to the available reviews (Bernard et al. 1995;
3.2 Pion–Nucleon Scattering

Elastic $\pi N$ scattering is maybe the most intensively studied process of hadron physics, with a long history both in theory and experiment (e.g., Höhler 1983). The systematic CHPT approach is however comparatively new (Gasser et al. 1988). I am going to review here the first complete calculation to $O(p^3)$ by Mojžiš (1998). As for $\pi\pi$ scattering, isospin symmetry is assumed.

A comparison with elastic $\pi\pi$ scattering displays the difficulties of the $\pi N$ analysis. Although calculations have been performed to next-to-next-to-leading order for both processes, this is only $O(p^3)$ for $\pi N$ compared to $O(p^6)$ for $\pi\pi$. Of course, this is due to the fact that, unlike for mesons only, every integer order can contribute to the low-energy expansion in the meson–baryon sector. The difference in accuracy also manifests itself in the number of LECs: the numbers are again comparable despite the difference in chiral orders. Finally, while we now know the $\pi\pi$ amplitude to two-loop accuracy the $\pi N$ amplitude is still not completely known even at the one-loop level as long as the $p^4$ amplitude has not been calculated.

The amplitude for pion–nucleon scattering

$$\pi^a(q_1) + N(p_1) \rightarrow \pi^b(q_2) + N(p_2)$$

(115)
can be expressed in terms of four invariant amplitudes $D^\pm$, $B^\pm$:

$$T_{ab} = T^+ \delta_{ab} - T^- i\epsilon_{abc}\tau_c$$

(116)

$$T^\pm = \bar{u}(p_2) \left[ D^\pm(\nu, t) + \frac{i}{2m_N} \sigma^{\mu\nu} q_2\mu q_1\nu B^\pm(\nu, t) \right] u(p_1)$$

with

$$s = (p_1 + q_1)^2 , t = (q_1 - q_2)^2 ,$$

$$u = (p_1 - q_2)^2 , \nu = \frac{s - u}{4m_N} .$$

(117)

With the choice of invariant amplitudes $D^\pm$, $B^\pm$, the low-energy expansion is straightforward: to determine the scattering amplitude to $O(p^n)$, one has to calculate $D^\pm$ to $O(p^n)$ and $B^\pm$ to $O(p^{n-2})$.

In the framework of CHPT, the first systematic calculation of pion–nucleon scattering was performed by Gasser et al. (1988). In heavy baryon CHPT, the pion–nucleon scattering amplitude is not directly obtained in the relativistic form (116) but rather as (Mojžiš 1998)

$$\bar{u}(p_2) P^+ v \left[ \alpha^\pm + i\epsilon^{\mu\nu\rho\sigma} q_1\mu q_2\nu \sigma^\rho S_{\sigma\beta^\pm} \right] P^+ v(u(p_1)) .$$

(118)
The amplitudes $\alpha^\pm$, $\beta^\pm$ depend on the choice of the velocity $v$. A natural and convenient choice is the initial nucleon rest frame with $v = p_1/m_N$. In this frame, the relativistic amplitudes can be read off directly from Table 7:

$$D^\pm = \alpha^\pm + \frac{vt}{4m_N^2} \beta^\pm$$

$$B^\pm = -m_N \left( 1 - \frac{t}{4m_N^2} \right) \beta^\pm$$

(119)

Also the amplitudes $D^\pm$, $B^\pm$ in (119) will depend on the chosen frame. However, as discussed before, they are guaranteed to be Lorentz invariant up to terms of at least $O(p^{n+1})$ if the amplitude (118) has been calculated to $O(p^n)$.

From Eq. (113) one finds that tree-level diagrams with $D = 1, 2, 3$ and one-loop diagrams with $D = 3$ need to be calculated. After proper renormalization, including the nonstandard nucleon wave function renormalization, the final amplitudes depend on the kinematical variables $\nu, t, m_N, M_\pi$, on the lowest-order LECs $F_\pi, g_A$, on four constants of the $p^2$ Lagrangian and on five combinations of LECs of $O(p^3)$.

The invariant amplitudes $D^\pm, B^\pm$ can be projected onto partial-wave amplitudes $f_{\pm\pm}^\pm(s)$. Threshold parameters are defined as in Eq. (101):

$$\text{Re} f_{\pm\pm}^\pm(s) = q^{2i} \left\{ a_{\pm\pm}^\pm + q^2 b_{\pm\pm}^\pm + O(q^4) \right\}.$$

(120)

To confront the chiral amplitude with experiment, Mojžiš (1998) has compared 16 of these threshold parameters with the corresponding values extrapolated from experimental data on the basis of the Karlsruhe-Helsinki phase-shift analysis (Koch and Pietarinen 1980).

Six of the threshold parameters ($D$ and $F$ waves) turn out to be independent of the low-energy constants of $O(p^2)$ and $O(p^3)$. The results are shown in Table 8 and compared with Koch and Pietarinen (1980).

The main conclusion from Table 8 is a definite improvement seen at $O(p^3)$. Since there are no low-energy constants involved (except, of course, $M_\pi, F_\pi, m_N$ and $g_A$), this is clear evidence for the relevance of loop effects. The numbers shown in Table 8 are based on the calculation of Mojžiš (1998), but essentially the same results were obtained by Bernard et al. (1997).

The altogether nine LECs beyond leading order were then fitted by Mojžiš (1998) to the ten remaining threshold parameters, the $\pi N$ $\sigma$-term and the Goldberger-Treiman discrepancy. Referring to Mojžiš (1998) for the details, let me summarize the main results:

- The fit is quite satisfactory although the fitted value of the $\sigma$-term tends to be larger than the canonical value (Gasser et al. 1991).
- In many cases, the corrections of $O(p^3)$ are sizable and definitely bigger than what naive chiral order-of-magnitude estimates would suggest.
- The fitted values of the four LECs of $O(p^3)$ agree very well with an independent analysis of Bernard et al. (1997). Moreover, those authors have shown
Table 8. Comparison of two D-wave and four F-wave threshold parameters up to the first, second and third order (the two columns differ by higher-order terms) with (extrapolated) experimental values (Koch and Pietarinen 1980). The theoretical values are based on the calculation of Mojžiš (1998). Units are appropriate powers of GeV$^{-1}$.

|       | $O(p)$ | $O(p^2)$ | $O(p^3)$ | HBCHPT $O(p^3)$ | exp.   |
|-------|--------|----------|----------|----------------|--------|
| $a_{2+}^+$ | 0      | -48      | -35      | -36            | $-36 \pm 7$ |
| $a_{2-}^+$ | 0      | 48       | 56       | 56             | $64 \pm 3$  |
| $a_{3+}^+$ | 0      | 0        | 226      | 280            | $440 \pm 140$ |
| $a_{3-}^+$ | 0      | 14       | 26       | 31             | $160 \pm 120$ |
| $a_{1+}^-$ | 0      | 0        | -158     | -210           | $-260 \pm 20$ |
| $a_{1-}^-$ | 0      | -14      | 65       | 57             | $100 \pm 20$  |

that the specific values can be understood on the basis of resonance exchange (baryons and mesons). It seems that the LECs of $O(p^2)$ in the pion–nucleon Lagrangian are under good control, both numerically and conceptually.

- The LECs of $O(p^3)$ are of “natural” magnitude but more work is needed here.

Using the results of Mojžiš (1998), Datta and Pakvasa (1997) have also calculated $\pi N$ phase shifts near threshold. Again, a clear improvement over tree-level calculations can be seen in most cases. As an example, I reproduce their results for the $S_{11}$ phase shift in Fig. 8.

The main conclusions for the present status of elastic $\pi N$ scattering are:

1. The results of the first complete analysis (Mojžiš 1998) to $O(p^3)$ in the low-energy expansion are very encouraging.
2. Effects of $O(p^4)$ (still $L \leq 1$) need to be included to check the stability of the expansion.

3.3 Nucleon–Nucleon Interaction

When Weinberg (1990, 1991) investigated the nucleon–nucleon interaction within the chiral framework, he pointed out an obvious clash between the chiral expansion and the existence of nuclear binding. Unlike for the meson–meson interaction.

$^7$ After the School, a new calculation of Fettes et al. (1998) appeared where both threshold parameters and phase shifts are considered.
that becomes arbitrarily small for small enough momenta (and meson masses),
the perturbative expansion in the NN-system must break down already at low
energies. Therefore, the chiral dimension defined in (112) cannot have the same
interpretation as for mesonic interactions or for single-nucleon processes.

In heavy baryon CHPT, the problem manifests itself through a seeming in-
frared divergence associated with the massless propagator of the "light" field
$N_\nu$. To make the point, we neglect pions for the time being and consider the
lowest-order four-nucleon coupling without derivatives ($n = 0$ and $n_B = 4$ in
the notation of Eq. (112)). The vertex is characterized by the tree diagram in
the first line of Fig. 9. If we now calculate the chiral dimension of the one-loop
diagram (second diagram in the first line of the figure) according to (112) we
find

$$D = 2L + 2 - \frac{E_B}{2} = 2.$$  \hspace{1cm} (121)

However, this result is misleading because the diagram is actually infrared diver-
gent with the propagator (111). Of course, this is an artifact of the approximation
made since nucleons are everything else but massless. The way out is to include
higher-order corrections in the nucleon propagator. The leading correction is

![Graph](image-url)
due to $\mathcal{L}^{(2)}_{\pi N}$ in (114). The kinetic terms to this order are

$$\mathcal{L}_{\text{kin}} = \overline{\eta_v} \left(i \nu \cdot \nabla + \frac{1}{2m}[(\nu \cdot \nabla)^2 - \nabla^2]\right) \eta_v$$

$$= \overline{\eta_v} \left(i \partial_0 + \frac{1}{2m} \partial^2\right) \eta_v \tag{122}$$

where the last expression applies for $\nu = (1,0,0,0)$, which now denotes the center-of-mass system. The corresponding propagator in this frame is

$$\frac{i}{k^0 - \frac{k^2}{2m} + i\varepsilon} \tag{123}$$

Following Kaplan et al. (1998), we now specialize to $NN$ scattering in the $^1S_0$ channel and denote the incoming momenta as

$$p_{1,2} = \left(\frac{E}{2}, \pm p\right), \quad E = \frac{p^2}{m} + \ldots \tag{124}$$

neglecting higher orders in the expression for the cms-energy $E$. Including higher orders in derivatives and quark masses, Kaplan et al. (1998) write the general tree amplitude (in $d$ dimensions) for nucleon–nucleon scattering in the $^1S_0$ channel as

$$A_{\text{tree}} = -\left(\frac{\mu}{2}\right)^{4-d} \sum_{n \geq 0} C_{2n}(\mu) p^{2n} = -\left(\frac{\mu}{2}\right)^{4-d} C(p^2, \mu). \tag{125}$$

The relevance of the subtraction scale $\mu$ will soon become clear. For a general vertex $C_{2n}$ of chiral dimension $2n$, the loop diagram considered before (second diagram in Fig. 9) is easily evaluated (Kaplan et al. 1998) in dimensional regularization:

$$I_n = -i \left(\frac{\mu}{2}\right)^{4-d} \int \frac{d^d k}{(2\pi)^d} k^{2n} \frac{i}{E - k^0 + \frac{k^2}{2m} + i\varepsilon} \frac{i}{E - k^0 - \frac{k^2}{2m} + i\varepsilon}$$

$$= -m(mE)^n (-mE - i\varepsilon)^{\frac{d-2}{2}} \Gamma\left(\frac{3 - d}{2}\right) \left(\frac{\mu}{4\pi}\right)^{4-d} \tag{126}$$

The seeming infrared divergence of before now manifests itself as a divergence for $m \to \infty$. The diagram is actually finite for $d = 4$ and clearly of $O(p^{2n+1})$ invalidating the general formula for the chiral dimension that gave $D = 2$ for $n = 0$.

Kaplan et al. (1998) make the point that the diagram would be divergent in $d = 3$ dimensions with

$$I_n \approx \frac{m(mE)^n \mu}{4\pi (d-3)} \tag{127}$$

near $d = 3$. Although this would not seem to have any great physical significance at first sight, Kaplan et al. (1998) suggest to subtract nevertheless the pole at
$d = 3$ that actually corresponds to a linear ultraviolet divergence in a cutoff regularization. This unconventional subtraction procedure is in line with the observation of other authors (e.g., Lepage 1997; Richardson et al. 1997; Beane et al. 1998) that standard dimensional regularization is not well adapted to the problem at hand.

The one-loop amplitude with the subtraction prescription of Kaplan et al. (1998) is then

$$I_n = -(mE)^n \frac{m}{4\pi} (\mu + ip).$$

Anticipating the following discussion, we now iterate the one-loop diagram and sum the resulting bubble chains to arrive at the final amplitude (Kaplan et al. 1998)

$$A = \frac{-C(p^2, \mu)}{1 + \frac{m}{4\pi} (\mu + ip) C(p^2, \mu)}.$$

This amplitude is related to the phase shift as

$$e^{2i\delta} - 1 = \frac{ipm}{2\pi} A$$

or, with the effective range approximation for $S$-waves in terms of scattering length $a$ and effective range $r_0$,

$$p \cot \delta = ip + \frac{4\pi}{m\lambda} = -\frac{4\pi}{mC(p^2, \mu)} - \mu$$

$$= -\frac{1}{a} + \frac{1}{2} r_0 p^2 + O(p^4).$$

Note that the (traditional) definition of the scattering length used here has the opposite sign compared to (101) for $\pi\pi$ scattering. With the relations (131), the coefficients $C_{2n}$ can be expressed in terms of $a, r_0, \ldots$:

$$C_0(\mu) = \frac{4\pi}{m} \frac{1}{-\mu + 1/a} \quad C_2(\mu) = \frac{2\pi r_0}{m} \left( \frac{1}{-\mu + 1/a} \right)^2.$$

It is known from potential scattering (e.g., Goldberger and Watson 1964) that $r_0$ and the higher-order coefficients in the effective range approximation are bounded by the range of the interaction. This also applies to $NN$ scattering in the $1S_0$ channel: $r_0 \approx 2.7$ fm $\approx 2/M_\pi$. On the other hand, the scattering length is sensitive to states near zero binding energy (e.g., Luke and Manohar 1997) and may be much bigger than the interaction range. Therefore, Kaplan et al. (1998) distinguish two scenarios.

- Normal-size scattering length
  In this case, also the scattering length is governed by the range of the interaction. The simplest choice $\mu = 0$ (minimal subtraction) leads to expansion coefficients $C_{2n}$ in (132) in accordance with chiral dimensional analysis. This corresponds to the usual chiral expansion as in the meson or in the single-nucleon sector.
Large scattering length

In the $^1S_0$ channel of $NN$ scattering, the scattering length is much larger than the interaction range (the situation is similar in the deuteron channel)

$$a = -23.714 \pm 0.013 \text{ fm } \simeq -16/M_\pi.$$  \hspace{1cm} (133)

With the same choice $\mu = 0$ as before, the coefficients $C_{2n}$ are unnaturally large leading to big cancellations between different orders. Kaplan et al. (1998) therefore suggest to use instead $\mu = O(M_\pi)$ which leads to $C_{2n}$ of natural chiral magnitudes.

The choice $\mu = O(M_\pi)$ immediately explains why we have to sum the iterated loop diagrams that led to amplitude $A$ in (129). Let us consider such a bubble chain graph with coefficients $C_{2n}$ at each four-nucleon vertex. From (132) and the obvious generalization to higher-order coefficients, one obtains $C_{2n} = O(p^{-n-1})$. Altogether, this implies a factor $C_{2n}p^{2n} = O(p^{n-1})$ at each vertex. On the other hand, each loop produces a factor of order $mp/4\pi$ as can be seen from Eq. (128). As a consequence, only the chain graphs with $C_0$ at each vertex have to be resummed because all such diagrams are of the same order $p^{-1}$. All other vertices can be treated perturbatively in the usual way.

The chiral expansion of the scattering amplitude (everything still in the $^1S_0$ channel) for $\mu = O(p)$ then takes the form (Kaplan et al. 1998)

$$A = A_{-1} + A_0 + A_1 + \ldots \quad (134)$$

$$A_{-1} = \frac{-C_0}{[1 + \frac{m}{4\pi}(\mu + ip)C_0]} \quad A_0 = \frac{-C_2p^2}{[1 + \frac{m}{4\pi}(\mu + ip)C_0]^2}. \quad (135)$$

This is also shown pictorially in Fig. 9.

So far, pions have been neglected. Inclusion of pions leaves $A_{-1}$ unchanged but modifies $A_0, A_1, \ldots$. Altogether, to next-to-leading order, $O(p^0)$, the amplitude for $NN$ scattering in the $^1S_0$ channel depends on three parameters: $C_0(M_\pi)$, $C_2(M_\pi)$, $D_2(M_\pi)$. Kaplan et al. (1998) fit these three parameters to the $^1S_0$ phase shift and obtain remarkable agreement with the experimental phase shift all the way up to $p = 300$ MeV. They also apply an analogous procedure to the $^3S_1 - ^3D_1$ channels (deuteron).

After many attempts during the past years, a systematic low-energy expansion of nucleon–nucleon scattering seems now under control. This is an important step towards unifying the treatment of hadronic interactions at low energies on the basis of chiral symmetry.

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Fig. 9. Feynman graphs contributing to the leading amplitudes for $^1S_0$ nucleon-nucleon scattering (from Kaplan et al. 1998).

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