Interior point methods are not worse than Simplex *

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Abstract

Whereas interior point methods provide polynomial-time linear programming algorithms, the running time bounds depend on bit-complexity or condition measures that can be unbounded in the problem dimension. This is in contrast with the simplex method that always admits an exponential bound. We introduce a new polynomial-time path-following interior point method where the number of iterations also admits a combinatorial upper bound $O(2^n n^{1.5} \log n)$ for an $n$-variable linear program in standard form. This complements previous work by Allamigeon, Benchimol, Gaubert, and Joswig (SIAGA 2018) that exhibited a family of instances where any path-following method must take exponentially many iterations.

The number of iterations of our algorithm is at most $O(n^{1.5} \log n)$ times the number of segments of any piecewise linear curve in the wide neighborhood of the central path. In particular, it matches the number of iterations of any path following interior point method up to this polynomial factor. The overall exponential upper bound derives from studying the ‘max central path’, a piecewise-linear curve with the number of pieces bounded by the total length of $2n$ shadow vertex simplex paths.

From the existence of a line segment in the wide neighborhood we derive strong implications on the structure of the corresponding segment of the central path. Our algorithm is able to detect this structure from the local geometry at the current iterate, and constructs a step direction that descends along this segment. The bound $O(n^{1.5} \log n)$ that applies for arbitrarily long line segments is derived from a combinatorial progress measure.

Our algorithm falls into the family of layered least squares interior point methods introduced by Vavasis and Ye (Math. Prog. 1996). In contrast to previous layered least squares methods that partition the kernel of the constraint matrix into coordinate subspaces, our method creates layers based on a general subspace providing more flexibility. Our result also implies the same bound on the number of iterations of the trust region interior point method by Lan, Monteiro, and Tsuchiya (SIOPT 2009).

1 Introduction

In this paper, we explore connections between interior point methods and the simplex method, the two most commonly used classes of algorithms for linear programming. We consider linear programming (LP) in the following primal-dual form:

\[
\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

\[
\begin{align*}
\max & \quad \langle b, y \rangle \\
\text{s.t.} & \quad A^\top y + s = c \\
& \quad s \geq 0
\end{align*}
\]

*(LP)*

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where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, and $\text{rk}(A) = m$. We let

$$
P = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}, \quad D = \{ s \in \mathbb{R}^n : \exists y \ A^\top y + s = c, \ s \geq 0 \}
$$

denote the primal and dual feasible regions. Our focus is on LP algorithms that find exact primal and dual optimal solutions.

The simplex method traverses a path formed by vertices and edges of $P$ according to a certain pivot rule. Albeit efficient in practice, there is no polynomial-time variant known, and there are exponential worst case examples for several pivot rules. The first such construction was given by Klee and Minty [21] for Dantzig’s pivot rule.

Breakthrough developments in the seventies and eighties led to the first polynomial-time algorithms for linear programming: the ellipsoid method by Khachiyan [20], and interior point methods introduced by Karmarkar [19]. The running time of these algorithms is $\text{poly}(n, L)$, where $L$ denotes the encoding-length $L$ of the rational input $(A, b, c)$ of (LP).

While the simplex method may be exponential, it is never worse: for any non-cycling pivot rule, the number of pivot steps can be bounded by the number of bases, at most $\binom{n}{m} < 2^n$. Whereas the bound $\text{poly}(n, L)$ is typically much better, the encoding length $L$ may be arbitrarily large. To the extent of our knowledge, no variant of the ellipsoid or interior point methods have been shown to admit a bound $f(n)$ on the number of iterations for any function $f : \mathbb{N} \to \mathbb{N}$ prior to our work.

Even though LPs with exponential encoding length do not frequently appear in practice, there are examples when the binary encoding is exponential yet one could efficiently implement arithmetic operations using a different encoding, see Megiddo [27]. The net present value problem in project scheduling is a particular example of a natural optimization problem that can be reformulated as an LP of exponential encoding length, see Grinold [16]. From a theoretical perspective, finding an interior point method with an absolute bound $f(n)$ on the number of iterations connects to the fundamental open question on finding a strongly polynomial algorithm to solve linear programming. Besides being polynomial time, such an algorithm must achieve a number of arithmetic operations in $\text{poly}(m, n)$. This question takes its roots in the development of the simplex method, and appears in Smale’s list of open problems for the 21st century [35].

**Interior point methods and the central path** Whereas the simplex method moves on the boundary of the feasible region $P$, interior point methods (IPM) reach an optimal solution by iterating through the strict interior of $P$. Path-following interior point methods are driven to an optimal point by following a smooth trajectory called the central path. In the most standard setting [34], the latter is defined as the parametric curve $\mu \in (0, \infty) \mapsto z(\mu) := (x(\mu), s(\mu))$, where $x(\mu)$ and $(y(\mu), s(\mu))$ are the unique solutions to the system

$$
A x(\mu) = b, \quad x(\mu) > 0 \\
A^\top y(\mu) + s(\mu) = c, \quad s(\mu) > 0 \\
x(\mu), s(\mu)_i = \mu \quad \text{for all } i \in [n].
$$

(1)

This system arises from the optimality conditions of convex problems obtained by penalizing the original linear programs with the logarithmic barrier, i.e., respectively adding terms of the form $-\mu \sum_{i=1}^n \log x_i$ and $\mu \sum_{i=1}^n \log s_i$ to the objective functions of the primal and dual (LP). The weight of the penalty is given by the parameter $\mu > 0$. When $\mu \searrow 0$, the central path $z(\mu)$ converges to a pair of optimal solutions $(x^*, s^*)$ of (LP), which can be easily deduced from the fact that the duality gap of $z(\mu)$ is given by $\langle c, x(\mu) \rangle - \langle b, y(\mu) \rangle = \langle x(\mu), s(\mu) \rangle = \mu$. Accordingly, we define the quantity $\mathcal{P}(z) := (x, s) / n$ for any feasible point $z = (x, s) \in P \times D$, which we refer to as the normalized duality gap of $z$.

Interior point methods iteratively compute approximations of the points on the central path associated with successive values of $\mu$ that decrease geometrically; at most $O(\sqrt{n \log(\mu/\mu')})$ iterations are needed to decrease the normalized duality gap from $\mu$ to $\mu'$. The iterations follow an improvement direction, e.g., a Newton step, while remaining in a certain neighborhood of the central path, and can be implemented in polynomial time. The classical analysis yields a running time $O(n^{3.5} L)$ for solving (LP) for a rational input $(A, b, c)$ of total encoding length $L$. There have been significant improvements in recent years both for general LP as well as for special classes, see Section 1.3.

A running time bound dependent on $L$ requires a rational input; in contrast, the simplex method can be implemented in $2^n \text{poly}(n)$ even in the real model of computation. Whereas standard IPMs use bit-complexity arguments to terminate, they have also been extended to the real model of computation, e.g., by Vavasis and Ye [44]. The running time of such algorithms is parametrized by condition numbers that capture geometric properties of the input. In a remarkable paper, Vavasis and Ye [45] introduced a
layered least squares (LLS) interior point method that runs in $O(n^{3.5} \log(\chi_A + n))$ iterations, where $\chi_A$ is the Dikin–Stuart–Todd condition number associated with the kernel of $A$ (but independent of $b$ and $c$). As a consequence, they also derive a structural characterization of the central path: there are at most $\binom{n}{2}$ ‘short and curved’ segments, possibly separated by ‘long and straight’ segments. The LLS directions are refined Newton steps that can traverse the latter segments.

Lan, Monteiro and Tsuchiya [22] gave a scaling invariant trust region IPM taking $O(n^{3.5} \log(\chi_A + n))$ iterations. Here, $\chi_A$ is the minimum value of $\chi$ that can be achieved by any column rescaling. However, computing the step directions in this algorithm has a weakly polynomial dependence on the right hand side. In recent work, [10] gave a scaling invariant LLS algorithm with iteration bound $O(n^{5.5} \log(n) \log(\chi_A + n))$, where the step directions can be computed in strongly polynomial time, by solving linear systems. We discuss the literature on such IPM methods in more detail in Section 1.3.

Lower bounds on interior point methods LLS methods provide strongly polynomial IPMs whenever $\chi_A \in 2^{\text{poly}(n)}$; this is always the case if the encoding-length of $A$ is polynomially bounded. One may wonder if some variant of IPM could be strongly polynomial for all LPs. A negative answer to this question was given in recent work by Allamigeon, Benchimol, Gaubert, and Joswig: they used tropical geometry to build pathological linear programs on which the number of iterations of IPM has to be exponential (in $m, n$) [1, 2]. Their construction shows that, when the entries of $A, b, c$ and $d$ are of very different orders of magnitude, the central path can be significantly deformed to the boundary of the feasible set. Allamigeon, Gaubert and Vandame later extended this result to the broad class of path-following IPMs using any self-concordant barrier function [3]. They exhibited a counterexample where the feasible set is an $n$-dimensional combinatorial cube and the shape of the central path is analogous to the simplexes on pathological instances of LP for the simplex method, akin to the Klee–Minty cube [21].

The shadow vertex simplex rule We introduce a new IPM method in this paper whose analysis can be related to the shadow vertex simplex rule. Originally dubbed ‘parametric simplex’ by Gass and Saaty [14], this is one of the most extensively analyzed simplex rules from a theoretical perspective. The shadow vertex rule was used in Borgwardt’s average case analysis [5] and in Spielman and Teng’s smoothed analysis [38]. The interested reader may refer to the recent survey for a detailed exposition [9].

Given a pointed polyhedron $P \subseteq \mathbb{R}^n$ and two objectives $c^{(1)}, c^{(2)} \in \mathbb{R}^n$, the shadow vertex rule consists in iterating over the vertices of $P$ successively maximizing the objectives $(1 - \lambda)c^{(1)} + \lambda c^{(2)}$ as $\lambda$ goes from 0 to 1. Under non-degeneracy assumptions, the vertices of the path correspond to those vertices of the two-dimensional projection $\{(c^{(1)}(x), c^{(2)}(x)) : x \in P\}$ that maximize some open interval of objectives $(1 - \lambda)c^1 + \lambda c^2, \lambda \in [0, 1]$ (where $c^1$ and $c^2$ stand for the unit vectors of $\mathbb{R}^2$). We denote by $S\mathcal{P}(c^{(1)}, c^{(2)})$ the number of vertices of the projection of the simplex path in this two-dimensional projection; this corresponds to the number of non-degenerate pivots.

1.1 Contributions

The purpose of this work is to establish a natural connection between the complexity of IPM and that of the simplex method, and deduce combinatorial bounds on the number of iterations. To this end, we introduce an interior point method called IPM with subspace LLS (see Algorithm 1), whose number of iterations is, up to a factor $O(n^{1.5} \log n)$, bounded by the number of pieces of any piecewise linear curve contained a wide neighborhood of the central path. This neighborhood is defined as

$$\mathcal{N}^{-\infty}(\theta) := \left\{ z = (x, s) \in P \times \mathcal{D} : \frac{xs}{\|s\|_\infty} \geq (1 - \theta) \mathbf{1} \right\} \quad (0 < \theta < 1) \quad (2)$$

where $xs \in \mathbb{R}^n$ denotes the Hadamard-product and $\mathbf{1} \in \mathbb{R}^n$ is the $n$-dimensional all-one vector. Our algorithm will however navigate through the narrower $\ell_2$-neighborhood of the central path:

$$\mathcal{N}(\beta) := \left\{ z = (x, s) \in P \times \mathcal{D} : \frac{xs}{\|s\|_\infty} \geq (1 - \theta) \mathbf{1} \right\} \quad (0 < \beta < 1/4). \quad (3)$$

**Theorem 1.1.** Assume that there exists $\Gamma : (0, \mu_0) \rightarrow \mathcal{N}^{-\infty}(\theta), \theta \in (0, 1)$, a piecewise linear curve satisfying $\mathbf{1}(\Gamma(\mu)) = \mu, \forall \mu \in (0, \mu_0)$. Starting from any point $x^0 \in \mathcal{N}(\beta)$ such that $\mathbf{1}(x^0) \leq \mu_0$, the algorithm IPM with subspace LLS finds an optimal solution of (LP) in $O(n^{1.5} \log(\frac{1}{1-\beta}) T)$, where $T$ is the number of linear segments in $\Gamma$.
At a high level, our strategy is to show that any ‘somewhat straight’ segment of the central path, corresponding to a single straight segment in the wide neighborhood $N^{-\infty}(\theta)$, can be decomposed into at most $n$ short segments of length $\text{poly}(n/(1-\theta))$ (as measured by the ratio of the start and end parameter), where consecutive short segments are possibly separated by ‘long and straight’ segments. To traverse the long and straight segments we develop a novel subspace LLS step, which generalizes prior LLS steps from coordinate subspaces to general ones. Before describing this in more details, we present two applications of Theorem 1.1.

An exponential upper bound on the number of iterations The first application relies on a piecewise linear curve that we call the max central path, and that is related with $2n$ simplex paths. It is defined as the parametric curve $g \mapsto z^m(g) := (x^m(g), s^m(g)) \in \mathbb{R}^{2n}$, where $x^m_i(g)$ and $s^m_i(g)$ are the optimal values of the following parametric LP, respectively:

$$\begin{align*}
\max x_i & \\
Ax &= b, \ x \geq 0 \\
(c, x) &\leq v^* + g,
\end{align*}$$

$$\begin{align*}
\max s_i & \\
A^T y + s &= c, \ s \geq 0 \\
(b, y) &\geq v^* - g,
\end{align*}$$

where we denote by $v^*$ the optimal value of (LP). As we show in Section 5, the maps $x^m_i(g)$ and $s^m_i(g)$ are piecewise linear, and the number of pieces can be related to the complexity of the simplex method with the shadow vertex rule.

Recall that $(x^*, s^*)$ is the optimal solution of (LP) at the central path limit point, and $S_P(-s^*, e_i)$ and $S_D(-x^*, e_i)$ denote the number of nondegenerate pivots in the primal and dual shadow vertex paths for the indicated objective functions. We let $V_P$ and $V_D$ denote the number of vertices of the primal and dual feasible polytopes, respectively.

In Lemma 5.4, we show that the number of pieces in the max-central path is bounded by the sum of the quantities $S_P(-s^*, e_i)$ and $S_D(-x^*, e_i)$ for $i \in [n]$. Whereas the max-central path does not necessarily lie in the primal-dual feasible set $P \times D$, we prove that the corresponding shadow-vertex paths induce a piecewise linear curve in the wide neighborhood $N^{-\infty}(\theta)$, for $\theta = 1 - \frac{1}{2n}$, with the same number of pieces; see Theorem 5.6 and an illustration in Figure 2. As a consequence, we obtain the following bound:

**Theorem 1.2.** From any point $z^0 \in N(\beta)$, the algorithm IPM with subspace LLS finds an optimal solution of (LP) in a number of iterations bounded by

$$O\left(n^{1.5} \log n \min \left\{ \sum_{i=1}^{n} S_P(-s^*, e_i) + S_D(-x^*, e_i), V_P + V_D \right\} \right).$$
This in particular implies an $O(2^n n^{1.5} \log n)$ iteration bound for IPM with subspace LLS. Theorem 1.2 thus complements the results of [1, 2] by giving a singly exponential upper bound. We note that the max central path also plays an important if implicit role in the papers [1, 2, 3], as it can be directly related to the tropical central path by the log-limit, see discussion in Section 1.3.

Theorem 1.2 assumes that a feasible starting point $z^0 \in \mathcal{N}(\beta)$ is given. This assumption can be removed e.g. by using the standard homogeneous self-dual embedding [46, Section 5.3.1]. Then, the bounds in the theorem will refer to the shadow vertex paths and the number of vertices in the self-dual program.

Matching the complexity of any path following method The second implication of Theorem 1.1 provides polynomial-time bounds in case the bit-complexity or a condition number such as $\chi_\lambda^2$ is bounded. We show that—apart from a factor $O(n^{1.5} \log (\chi_\lambda^2/\theta))$—the number of iterations of IPM with subspace LLS is at most that of any IPM that stays in the wide neighborhood.

Indeed, any IPM induces a piecewise linear curve formed by the line segments between the successive iterates. Already the wide neighborhood $\mathcal{N}^{-\infty}(1/2)$ is known to contain this piecewise linear curve for a large class of IPM based on the logarithmic barrier; we refer to [1, Section 2] for a detailed discussion. We note that our algorithm matches—up to a polynomial factor—even any IPM that only stays in the extremely wide neighbourhood $\mathcal{N}^{-\infty}(1 - 1/2^{\text{poly}(n)})$.

**Theorem 1.3.** Suppose that an IPM reduces the duality gap from $\mu_0$ to $\mu_1 \geq 0$ in $T$ iterations staying throughout in the wide neighborhood $\mathcal{N}^{-\infty}(\theta)$ for some $0 < \theta < 1$. Then, from any point $z^0 \in \mathcal{N}(\beta)$ satisfying $\overline{p}(z^0) \leq \mu_0$, the algorithm IPM with subspace LLS finds a solution $z^1$ with $\overline{p}(z^1) \leq \mu_1$ in at most $O(n^{1.5} \log (\chi_\lambda^2 + n))$, iterating bound for $\mu_0$, the algorithm IPM with subspace LLS finds a solution $z^1$ with $\overline{p}(z^1) \leq \mu_1$ in at most $O(n^{1.5} \log (\chi_\lambda^2 + n)), by adapting the analysis of the LLS methods [31, 45].

The step directions used by our algorithm are feasible solutions to (6) for a suitable parameter. This implies that the steps of the Trust Region algorithm are always at least as long as the steps in our algorithm; as a consequence, the iteration bounds of our algorithm are also applicable to the Trust Region algorithm. Whereas any individual step of our algorithm could be arbitrarily worse than the one using the trust region step, Theorem 1.3 implies that overall we may only take $O(n^{1.5} \log n)$ more iterations. We emphasize that [22] only provides the $\chi_\lambda^2$ dependent iteration bound, and we do not see any obvious ways to obtain any $f(n)$ bound on their algorithm, other than comparing it to IPM with subspace LLS.

A further advantage of our algorithm is that the iterations can be implemented in strongly polynomial time, using simple projection steps. The description of this paper requires a subspace $V$ that can be obtained as a singular value subspace from a singular value decomposition. However, it suffices to compute rough approximations on the singular values, see Section 4.1.1.

### 1.2 Techniques

We now explain the key ideas behind the proof of Theorem 1.1.

#### 1.2.1 Polarization of the Central Path

The first idea underlying to the proof is the following: every linear segment in the wide neighborhood gives rise to a polarized segment of the central path. A segment of the central path $\text{CP}[\mu_1, \mu_0] := \{z(\mu) : \mu \in [\mu_1, \mu_0]\}, 0 \leq \mu_0 < \mu_1$, is polarized if it admits a partition $B \cup N = [n]$ such that the primal variables in $B$ are essentially fixed and those in $N$ are scaling down linearly with the parameter $\mu$ (vice versa for the dual variables). More precisely, $\forall \mu \in [\mu_1, \mu_0], we require$

\[ \gamma x_i(\mu_0) \leq x_i(\mu) \leq nx_i(\mu_0), \forall i \in B, \]

\[ \frac{\mu}{n\mu_0} x_i(\mu_0) \leq x_i(\mu) \leq \frac{\mu}{\gamma \mu_0} x_i(\mu_0), \forall i \in N, \]

(5)
where $\gamma \in [0,1]$ is a polarization parameter (see Definition 3.1 and Corollary 3.3). By definition of the central path, the same relation holds for dual variables $s(\mu)$, $\mu \in [\mu_1, \mu_0]$, with the roles of $N$ and $B$ swapped. We note that the upper bound on $x_i(\mu)$ for $i \in B$ and the lower bound on $x_i(\mu)$ for $i \in N$ hold for any point of the central path by the near-monotonicity property (Lemma 2.4); the important parts of the definition are the other two bounds.

For simplicity of notation, let us restrict to line segments between two points on the central path. To relate polarization to the wide neighborhood, we show that if the line segment $[z(\mu_1), z(\mu_0)]$ between central path points is contained in the wide neighborhood $\mathcal{N}^{-\infty}(\theta)$, then the corresponding segment of the central path is polarized with $\gamma = \frac{1-\theta^\gamma}{\text{im}(\theta)}$ with respect to some partition $B \cup N = [n]$ (see Lemma 3.5 for the general statement).

One should read this last statement as saying that segment of central path is ‘approximately linear’ if and only if it is polarized (in fact, a segment is 1-polarized if and only if it is linear, see Lemma 3.7). The link between polarization and linearity is surprisingly elementary; it follows from the analysis of the evolution of variables on a segment. We note that the scaling direction $(\Delta x, \Delta s)$ itself is long, and that we are given the starting point $\mu = \text{im}(\theta)$ for the general statement).

The link between polarization and linearity is surprisingly elementary; it follows from the analysis of the evolution of variables on a segment. We note that the scaling direction $(\Delta x, \Delta s)$ itself is long, and that we are given the starting point $\mu = \text{im}(\theta)$ for the general statement).

Let $CP_{\mu_1 \mu_0}$, $0 \leq \mu_1 < \mu_0$, be a $\gamma$-polarized segment with partition $B \cup N = [n]$. For simplicity of presentation, let us assume that given any iterate $(x, s)$ in the narrow neighborhood $\mathcal{N}(1/6)$ used in our algorithm, we can jump to the exact central path point $z(\mu) \in CP = \mathcal{N}(0)$ with $\mu = \text{im}(x,s)$ for free. Let us further assume that the algorithm knows the partition $B,N$ (we discuss how to effectively compute it at the end) and that we are given the starting point $z(0):= z(\mu_0)$.

Our abstract algorithm will thus compute iterates $z^{(0)}, z^{(1)}, \ldots$ on the central path $CP$ with $\text{im}(z^{(0)}) > \text{im}(z^{(1)}) > \ldots$. To move from $z^{(t)}$ to $z^{(t+1)}$, we first compute a movement direction

$$D_{z^{(t)}} = (\Delta x^{(t)}, \Delta s^{(t)}) \in \ker(A) \times \text{im}(A^\top) := W \times W^\perp,$$

together with a step-length $\alpha^{(t)} \in [0,1]$, chosen such that $z^{(t)} + \alpha \Delta z^{(t)} \in \mathcal{N}(1/6)$, $0 \leq \alpha \leq \alpha^{(t)}$. Lastly, we jump for free to $z^{(t+1)} \in CP$ satisfying $\text{im}(z^{(t+1)}) = \text{im}(z^{(t)}) + \alpha^{(t)} \Delta z^{(t)}$.

Given this setup, our goal is to compute movement directions, such that after $k = O(n^{1.5} \log(n/\gamma))$ iterations, we have $\text{im}(z^{(k)}) \leq \mu_1$, i.e., that we have crossed the segment. We would like to emphasize that our algorithm will in fact compute the movement direction $\Delta z^{(t)}$ using only local information at $z^{(t)}$, without any explicit knowledge of the polarized segment.

A natural movement direction is affine scaling used in predictor-corrector methods, see Section 2.2. This direction guarantees $1 - \Omega(1/\sqrt{n})$ decrease in normalized gap per step. Hence, if $\mu_0/\mu_1 \leq \text{poly}(n, 1/\gamma)$, then simply using $\sqrt{n}\text{im}(\mu_0/\mu_1)$ affine scaling iterations is sufficient for our purposes.

Thus, we may assume that $\mu_0/\mu_1 \gg \text{poly}(n, 1/\gamma)$. In this case, we show that computing an affine scaling direction $(\Delta x, \Delta s)$ at the current iterate $(z^{(t)}, s^{(t)})$ reveals the correct partition $B \cup N = [n]$ of
the current polarized segment. This is because the standard affine scaling step itself exhibits a polarized behaviour: we can simply select \(B\) as the set of coordinates \(i\) where \(|\Delta x_i^t/x_i^{(t)}| < |\Delta s_i^t/s_i^{(t)}|\), i.e., the relative primal movement is smaller than the relative dual movement (see Definition 4.4).

**Trust Region Programs and Subspace LLS** The trust region programs by Lan, Monteiro, Tsuchiya [22] provide a good starting point for defining our movement direction \(\Delta z^{(t)} = (\Delta x^{(t)}, \Delta s^{(t)}) \in W \times W^\perp\) from an iterate \(z^{(t)} = (x^{(t)}, s^{(t)}) \in CP[\mu_1, \mu_0]\) and a given a partition \([n] = B \cup N:\)

\[
\begin{align*}
\min_{\Delta x \in W} & \left\{||\Delta x_N + x_N^t|| / x_N^t : ||\Delta x_B / x_B^t|| \leq \theta\right\} \\
\min_{\Delta s \in W^\perp} & \left\{||\Delta s_B + s_B^t|| / s_B^t : ||\Delta s_N / s_N^t|| \leq \theta\right\}
\end{align*}
\]

where \(\theta = 1/100\) is sufficient for our purposes. We use the notation \(\Delta x/x^{(t)} := (\Delta x_1/x_1^{(t)}, \ldots, \Delta x_n/x_n^{(t)})\) and similarly for \(\Delta s/s^{(t)}\). The norms \(\|x/x^{(t)}\|\) and \(\|s/s^{(t)}\|\) are the so-called primal and dual *local norms* at \(x^{(t)}\) and \(s^{(t)}\). By definition, the optimal primal trust region direction \(\Delta x^*\) achieves a maximal multiplicative decrease on the coordinates in \(N\) while ‘barely moving’ the coordinates in \(B\) as measured in the local norm. The optimal dual direction \(\Delta s^*\) achieves the same on the dual side with the role of \(N\) and \(B\) swapped.

Note that these directions mesh very well with polarization of the segment \(CP[\mu_1, \mu_0]\). In particular, they reflect the idea that the coordinates of \(x(\mu)\) in \(N\) should be linearly scaling down while those in \(B\) are staying mostly fixed, and vice versa for \(s(\mu)\). As shown in [22], moving in any direction \(\Delta z^{(t)} = (\Delta x^{(t)}, \Delta s^{(t)})\) corresponding to feasible solutions to (6), the normalized gap can be reduced as

\[
\frac{\pi(z^{(t+1)})}{\pi(z^{(t)})} \leq ||x_N^t + \Delta x_N^t|| / x_N^t + ||s_B^t + \Delta s_B^t|| / s_B^t.
\]

That is, we can achieve a drop that corresponds to the sum of primal and dual objective values.

In many ways, the trust region direction can be seen as the ‘optimal’ movement direction. However, [22] solves the quadratic convex programs in (6) in weakly polynomial time with dependence on the vectors \(b\) and \(c\) in (LP). It is not known whether a strongly polynomial algorithm (with dependence only on \(n\)) exists. Further, the analysis in [22] relies on combinatorial progress measures adapted from the LLS analyses. It remains unclear how to analyze the convergence of the trust region steps by only using only the fact that they are maximally long.

Instead of optimally solving (6), we introduce what we call *subspace LLS steps* that yield ‘good enough’ approximate solutions for our purposes. We restrict the set of primal and dual directions to come from carefully selected subspaces \(V^{(t)} \subseteq W\) and \(U^{(t)} \subseteq W^\perp\) satisfying:

\[
\begin{align*}
||\Delta x_B / x_B^t|| & \leq \tau ||\Delta x_N / x_N^t||, \quad \forall \Delta x \in V^{(t)} \\
||\Delta s_N / s_N^t|| & \leq \tau ||\Delta s_B / s_B^t||, \quad \forall \Delta s \in U^{(t)}
\end{align*}
\]

where we set \(\tau = \theta/(16\sqrt{n})\). We call any such subspaces \(U^{(t)}, V^{(t)}\) *cheap lift subspaces*. Note that every such solution automatically satisfies the constraints in program (6). Hence, the optimal solutions can be computed by solving systems of linear equations that correspond to minimum-norm points in the local norms.

In terms of the choice of subspaces, there is quite a lot of flexibility. A canonical choice, which we use for simplicity in the analysis, consists of choosing a space spanned by the singular vectors of a certain ‘lifting map’ whose corresponding singular values are at most \(\tau\). While singular values (and their corresponding spaces) are not computable in strongly polynomial time, one can indeed compute sufficiently good approximations for our purposes (see Section 4.1.1).

**Analyzing Subspace LLS** At each iteration, our algorithm computes the affine scaling steps and the subspace LLS steps as above, and uses the one that enables more progress by choosing the largest possible step-length. We use the subspaces \(U^{(t)}, V^{(t)}\) to compute the subspace LLS direction \(\Delta z^{(t)}\) as above.

Let us now explain the key idea in showing that subspace LLS steps can reach the end of the current \(\gamma\)-polarized segment \(CP[\mu_1, \mu_0]\) in \(k = O(n^{1.5} \ln(n/\gamma))\) iterations. Let \(k = \Omega(\sqrt{n} \ln(n/\gamma))\). Given any iterate \(z^{(t)} \in CP[\mu_1, \mu_0]\), if \(\pi(z^{(t+k)}) > \mu_1\) i.e., we have not reached the end of the segment — then we show that both \(\dim(U^{(t+k)}) > \dim(U^{(t)})\) and \(\dim(V^{(t+k)}) > \dim(V^{(t)})\). The overall bound follows since this can occur at most \(n\) times.
To get this result, we analyze the evolution of what we call the ‘empirical gradient’ at \( z^{(t)} \), which we define to be \( \Delta z^{(t)} := z(\mu_t) - z^{(t)} \), i.e., the difference between the current iterate and the end of the segment. A crucial observation is that if \( \Delta z^{(t)} \) were a feasible solution to (6), then following this direction would get to within a poly\((n^2/\gamma)\) factor for the end of the segment in one step (though we do not know how to compute it). Furthermore, the empirical gradient is never far from being feasible, in particular, it is feasible if the bound of \( \beta \) is replaced by \( O(n) \). We show the following dichotomy: Given an iteration \( z^{(t)} \), either the empirical gradient \( \Delta z^{(t)} \) is mostly “aligned” with the LLS subspaces \( U^{(t)} \times V^{(t)} \), in which case we get close to the end of the segment in one step, or we can extract from \( \Delta z^{(t)} \) an additional “cheap lift” dimension in the next \( O(\sqrt{n} \ln(n^2/\gamma)) \) iterations. In the latter case, we use the polarization property to analyze the evolution of the singular values of lifting maps.

This concludes our overview of the proof of Theorem 1.1. For more details, see Section 4.

### 1.3 Related Work

Interior points methods have been a tremendously active and fruitful research area since the seminal works of Karmarkar [19] and Nesterov [34] in the 80’s. Remarkable advances have been made both in speed as well as applicability of IPMs. We first briefly review works that—unlike the present paper—aim for \( \varepsilon \)-approximate solutions. A key ingredient have been using different, self-concordant barrier functions. Like the logarithmic barrier, every such function gives rise to a notion of central path. In the general setting, the iteration complexity to get an \( \varepsilon \)-approximation of the optimal value is bounded by \( O((\log \varepsilon)^{-1}) \), where \( \vartheta \) is a complexity parameter specific to the barrier function. General bounds on self-concordant barriers were given by Nesterov and Nemirovski [33], improved recently by Lee and Yue [25]. Specific barrier functions include Vaidya’s volumetric barrier [39], the entropic barrier by Bubeck and Eldan [6], and the weighted log-barrier by Lee and Sidford [23, 24].

Recent improvements make use of efficient data structures to amortize the cost of the iterative updates, and work with approximate computations, see Cohen, Lee and Song [8], van den Brand [40], and van den Brand, Lee, Sidford, and Song [43]. For special classes of LP such as network flow and matching problems, even faster algorithms have been obtained using, among other techniques, fast Laplacian solvers [37], see e.g. [4, 11, 13, 26, 42, 41], culminating in the very recent near-linear time minimum-cost flow algorithm [7].

Layered least squares IPMs, initiated by Vavasis and Ye [15] find exact optimal solutions and their running time bound is independent of \( b \) and \( c \). Improved LLS algorithms were given by Megiddo, Mizuno, and Tsuchiya [28] and Monteiro and Tsuchiya [31, 32]. As discussed previously, scaling invariant algorithms with a \( \chi_A \) dependence are the Trust Region algorithm by Lan, Monteiro, and Tsuchiya [22], and the LLS algorithm [10] that relies on approximating circuit imbalances.

There is an interesting connection between IPMs and differential geometry. Sonnevend, Stoer, and Zhao [36] introduced a primal-dual curvature concept for the central path, and related the curvature integral to the iteration complexity of IPMs. Monteiro and Tsuchiya [30] showed that a curvature integral is bounded by \( O(n^{3.5} \log(\chi_A^* + n)) \). This has been extended to SDP and symmetric cone programming [18], and also studied in the context of information geometry [17].

Relating the central path with a simplex path has been mainly used to build LP with pathological properties. On top of the construction of [3] that we already discussed, Deza, Nematollahi and Terlaky [12] built a Klee–Minty cube with exponentially many redundant inequalities where the central path is distorted into the neighborhood of the simplex path that visits the \( 2^n \) vertices.

The max central path studied in this paper is related to the tropical central path in [1, 2, 3]. The latter arises when studying parametric families of LP where the input \((A, b, c)\) depend on a parameter \( t > 1 \). The tropical central path is defined as the log-limit, i.e., the limit as \( t \to \infty \) of the image under the map \( z \mapsto \log_t z = \frac{\log z}{\log t} \) of the central path of these LP. In [1, 2, 3], it was shown that the tropical central path corresponds to the greatest point (entrywise) of the log-limit of the feasible sets of (4). This turns to be precisely the log-limit of the max central path.

### 1.4 Organization of the Paper

In Section 2, we recall some basic properties related to the central path and its neighborhoods (Section 2.1). We also discuss the affine scaling steps used in predictor-correct methods (Section 2.2). We finally introduce the lifting maps used in the subspace LLS step (Section 2.3). Section 3 deals with the polarized segments of the central path and their connection with linear segments in the wide neighborhood.
Section 4 provides the complexity analysis of the algorithm IPM WITH SUBSPACE LLS for traversing polarized segments. Section 5 studies the max central path. We give a direct proof of the polarization of the central path along the max central path (Section 5.2), and an alternative proof via a piecewise linear curve in the wide neighborhood induced by the max central path (Section 5.3). Omitted proofs are deferred to the Appendix.

2 Preliminaries

**Notation** We let $\mathbb{R}_{+}^n$ denote the set of positive reals, and $\mathbb{R}_{+}$ the set of nonnegative reals. For $n \in \mathbb{N}$, we let $[n] = \{1, 2, \ldots, n\}$. Let $e^i \in \mathbb{R}^n$ denote the $i$th unit vector, and $1 \in \mathbb{R}^n$ the all 1s vector. For two points $x, y \in \mathbb{R}^n$, we let $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ denote the line-segment connecting $x$ and $y$.

We let $\ker(M)$ denote the kernel of the matrix $M \in \mathbb{R}^{m \times n}$. For $S \subseteq [n]$, $M_S \in \mathbb{R}^{m \times |S|}$ refers to the submatrix formed by the columns in $S$. For $i \in [n]$, we set $M_{\leq i} = M_{\{j \in [n] : j \leq i\}}$ and $M_{\geq i} = M_{\{j \in [n] : j \geq i\}}$. We define $\sigma_i(M)$ to be its $i$-th smallest singular value. The norm $\|\cdot\|$ refers to the usual Euclidean norm. The inner product of the two vectors is denoted as $\langle x, y \rangle = x^T y$.

For a vector $x \in \mathbb{R}^n$, we let $\text{diag}(x) \in \mathbb{R}^{n \times n}$ denote the diagonal matrix with $x$ on the diagonal. For $x, y \in \mathbb{R}^n$, we use the notation $xy \in \mathbb{R}^n$ for the Hadamard product $xy = (x_i y_i)_{i \in [n]}$. For a vector $x \in \mathbb{R}^n$ and a subspace $W \subseteq \mathbb{R}^n$, we use the notation $xW = \{xw : w \in W\}$. For $p \in \mathbb{Q}$, we also use the notation $x^p$ to denote the vector $(x_i^p)_{i \in [n]}$. Similarly, for $x, y \in \mathbb{R}^n$, we let $x/y$ denote the vector $(x_i/y_i)_{i \in [n]}$.

For a subspace $W \subseteq \mathbb{R}^n$, we define $\Pi_W : \mathbb{R}^n \to W$ to be the orthogonal projection onto $W$. We define $W^\perp := \{x \in \mathbb{R}^n : \langle x, y \rangle = 0, \forall y \in W\}$ as the orthogonal complement of $W$. For a non-empty index set $I \subseteq [n]$, we let $\pi_I : \mathbb{R}^n \to \mathbb{R}^I$ to denote the coordinate projection onto $I$, that is $\pi_I(x) = x_I$, and for a subset $S \subseteq \mathbb{R}^n$, $\pi_I(S) = \{x_I : x \in S\}$. We also define $\mathbb{R}^I_+ := \{x \in \mathbb{R}^n : x_i > 0, i \in [n] \setminus I\}$ to be the set of vectors in $\mathbb{R}^n$ with support contained in $I$.

We will often use the following identity:

**Proposition 2.1.** For $\emptyset \neq I \subseteq [n]$ and a linear subspace $W \subseteq \mathbb{R}^n$, $\pi_I(W)^\perp = \pi_I(W^\perp \cap \mathbb{R}^I_+)$ holds.

We use the real model of computation, allowing basic arithmetic operations $+, -, \times, \/, \text{comparisons}$, and square root computations. Exact square root computations could be omitted by using approximate square roots; we assume exact computations for simplicity.

**Subspace formulation** It will be more convenient for our algorithm and analysis to represent (LP) in an equivalent subspace language. Throughout the paper, we let $W = \ker(A) \subseteq \mathbb{R}^n$. Using this notation, (LP) can be written in the form

$$\begin{align*}
\min & \quad \langle c, x \rangle \\
\text{s.t.} & \quad x \in W + d, \quad x \geq 0, \\
\max & \quad \langle d, c - s \rangle \\
\text{s.t.} & \quad s \in W^\perp + e, \quad s \geq 0,
\end{align*}$$

where $d \in \mathbb{R}^n$ is any solution $Ad = b$ and $W^\perp = \text{im}(A^\top)$ is the orthogonal complement of $W$. A natural choice of $d$ is the minimum norm solution, namely, $d = \arg \min \{\|x\| : Ax = b\}$.

Note that $s \in W^\perp + e$ is equivalent to $\exists y \in \mathbb{R}^m$ such that $A^\top y + c = s$. Hence, the original variable $y$ is implicit. The feasible regions can be written as

$$\mathcal{P} = \{x \in \mathbb{R}^n : x \in W + d, x \geq 0\}, \quad \mathcal{D} = \{s \in \mathbb{R}^n : s \in W^\perp + e, s \geq 0\}.$$  

2.1 Preliminaries on Interior-Point Methods

In this section, we recall standard properties of the central path and IPM that will be required for our algorithm. To ensure that the central path is well-defined, we assume that $\mathcal{P}$ and $\mathcal{D}$ admit strictly feasible solutions, i.e., the sets

$$\mathcal{P}_{++} := \{x \in \mathcal{P} : x > 0\}, \quad \mathcal{D}_{++} := \{s \in \mathcal{D} : s > 0\}$$

are both nonempty. We use $z^{(p)}(\mu) = (x^{(p)}(\mu), s^{(p)}(\mu))$ to denote the central path point at $\mu$ rather than $z(\mu) = (x(\mu), s(\mu))$ used in the Introduction.

Given $z = (x, s) \in \mathcal{P} \times \mathcal{D}$, we recall that the normalized duality gap is defined as $\overline{\pi}(z) = \frac{\langle x, s \rangle}{\mu}$. The following identity is useful in comparing duality gaps:
Proposition 2.2. Given \( x, x' \in W + d, s, s' \in W^\perp + c \), we have that
\[
\langle x, s \rangle + \langle x', s' \rangle = \langle x, s' \rangle + \langle x', s \rangle.
\]
In particular, if \( \langle x', s' \rangle = 0 \), then
\[
\langle x, s \rangle = \langle x, s' \rangle + \langle x', s \rangle.
\]

The next proposition shows that the normalized duality gap is a linear function for convex combinations of points.

Proposition 2.3 (Linearity duality gap). Given \( x^{(1)}, \ldots, x^{(k)} \in W + d, s^{(1)}, \ldots, s^{(k)} \in W^\perp + c \) forming the sequence \( z^{(1)} = (x^{(1)}, s^{(1)}), \ldots, z^{(k)} = (x^{(k)}, s^{(k)}) \) and \( \lambda \in \mathbb{R}^k \) such that \( \sum_{i=1}^k \lambda_i = 1 \), we have that
\[
\mathfrak{p} \left( \sum_{i=1}^k \lambda_i z^{(i)} \right) = \sum_{i=1}^k \lambda_i \mathfrak{p}(z^{(i)}).
\]

A key property of the central path is ‘near monotonicity’, formulated in the following lemma, see [45, Lemma 16].

Lemma 2.4. For the central path points at \( 0 \leq \mu' \leq \mu \), we have
\[
\left\| \frac{x^{cP}(\mu')}{x^{cP}(\mu)} + \frac{s^{cP}(\mu')}{s^{cP}(\mu)} \right\|_\infty \leq n.
\]

We will also require following lemma which was implicit in the proof of [45, Lemma 16].

Lemma 2.5. For the central path points at \( 0 \leq \mu' \leq \mu \), we have
\[
\left\| \frac{x^{cP}(\mu')}{x^{cP}(\mu)} + \frac{s^{cP}(\mu')}{s^{cP}(\mu)} \right\|_1 \leq 2n.
\]

Proof. There is nothing to prove if \( \mu = \mu' = 0 \); let us assume \( \mu > 0 \), and denote \( (x, s) = (x^{cP}(\mu), s^{cP}(\mu)) \), \( (x', s') = (x^{cP}(\mu'), s^{cP}(\mu')) \). Using Proposition 2.2,
\[
\|x'/x + s'/s\|_1 = \mu^{-1}(\langle s, x' \rangle + \langle x, s' \rangle) = \mu^{-1}(\langle x, s \rangle + \langle x', s' \rangle)
\]
\[
= \mu^{-1}(n\mu + n\mu' + 0) = n(1 + \mu^{-1}\mu') \leq 2n.
\]

Central path neighbourhoods   The neighbourhoods \( \mathcal{N}(\beta) \) and \( \mathcal{N}^{-\infty}(\theta) \) introduced in (3) and (2) comprise the points \( z = (x, s) \in P \times D \) such that the centrality error, i.e., the norm of the vector \( \langle x, s \rangle - 1 \), is bounded. They use the \( \ell_2 \)-norm and the \( \ell_\infty \)-seminorm \( \|u\|_\infty := \max_{1 \leq i \leq n} \max(0, -u_i) \), respectively.

The following proposition gives a bound on the distance between a point \( z \in \mathcal{N}(\beta) \) in the \( \beta \)-neighbourhood and the corresponding central path point with the same normalized duality gap \( z(\mu) \) for \( \mu = \mathfrak{p}(z) \). See e.g. [15, Lemma 5.4], [31, Proposition 2.1].

Proposition 2.6. Let \( z = (x, s) \in \mathcal{N}(\beta) \) for \( \beta \in (0, 1/4] \) and \( \mu = \mathfrak{p}(z) \), and consider the central path point \( z^{\infty P}(\mu) = (x^{cP}(\mu), s^{cP}(\mu)) \). For each \( i \in [n] \),
\[
\frac{x_i}{1 + 2\beta} \leq \frac{1 - 2\beta}{1 - \beta} x_i \leq x_i^{cP}(\mu) \leq \frac{x_i}{1 - \beta}, \quad \text{and} \quad \frac{s_i}{1 + 2\beta} \leq \frac{1 - 2\beta}{1 - \beta} s_i \leq s_i^{cP}(\mu) \leq \frac{s_i}{1 - \beta}.
\]

We will often use the following proposition which is immediate from the definition of \( \mathcal{N}(\beta) \).

Proposition 2.7. Let \( z = (x, s) \in \mathcal{N}(\beta) \) for \( \beta \in (0, 1/4] \), and \( \mu = \mathfrak{p}(z) \). Then for each \( i \in [n] \)
\[
(1 - \beta)\mu \leq s_i x_i \leq (1 + \beta)\mu.
\]

We will need the following lemma regarding the near-optimality of the choice \( \mathfrak{p}(z) \) as \( (x, s)/n \) for a point \( z = (x, s) \) with respect to minimizing centrality error.
Lemma 2.8 ([32, Lemma 4.4]). For \( \beta \in (0, 1/4] \), let \( z = (x, s) \in \mathcal{P}_{++} \times \mathcal{D}_{++} \) and \( \mu' > 0 \) satisfy \( \|xs/\mu' - 1\|_2 \leq \beta \). Then,

\[
(1 - \beta/\sqrt{n})\mu' \leq \overline{p}(z) \leq (1 + \beta/\sqrt{n})\mu \quad \text{and} \quad z \in \mathcal{N}(\beta/(1 - \beta)).
\]

The next lemma relates a point in the wide neighbourhood to the corresponding central path point.

Lemma 2.9. Let \( z' = (x', s') \in \mathcal{N}^{-\infty}(\theta), \theta \in [0, 1) \). Then for \( \mu = \overline{p}(z) \), and the corresponding central path point \( z^{\text{CP}}(\mu) = (x^{\text{CP}}(\mu), s^{\text{CP}}(\mu)) \), we have that

1. \( \frac{1}{2\beta}x' \leq x^{\text{CP}}(\mu) \leq \frac{2n}{1 - \beta}x' \).
2. \( \frac{1}{2\beta}s' \leq s^{\text{CP}}(\mu) \leq \frac{2n}{1 - \beta}s' \).

Proof. We only prove (1), as the proof of (2) is symmetric. Let \( (x, s) = (x^{\text{CP}}(\mu), s^{\text{CP}}(\mu)) \). Using Proposition 2.2, for \( i \in [n] \) we have that

\[
\frac{x_i'}{x_i} = \frac{x_is_i'}{\mu} \leq \frac{1}{\mu}(\langle x', s \rangle + \langle x, s' \rangle) = \frac{1}{\mu}(\langle x', s' \rangle + \langle x, s \rangle) = 2n.
\]

This proves the first inequality of (1); note that this part does not use \( z \in \mathcal{N}^{-\infty}(\theta) \), but only that \( z \in \mathcal{P}_{++} \times \mathcal{D}_{++} \). For the second inequality, \( z \in \mathcal{N}^{-\infty}(\theta) \) by definition implies

\[
\frac{x_i}{x_i'} = \frac{x_is_i}{\mu(1 - \theta)} \leq \frac{1}{\mu(1 - \theta)}(\langle x', s' \rangle + \langle x, s \rangle) = \frac{2n}{1 - \theta},
\]

as needed. \( \square \)

2.2 Predictor-Corrector Methods

Given \( z = (x, s) \in \mathcal{P}_{++} \times \mathcal{D}_{++} \), the search directions commonly used in interior-point methods are obtained as the solution \((\Delta x, \Delta s)\) to the following linear system for some \( \nu \in [0, 1] \).

\[
\begin{align*}
\Delta x & \in W, \\
\Delta s & \in W^\perp, \\
s\Delta x + x\Delta s & = \nu\mu 1 - xs
\end{align*}
\]

Predictor-corrector methods, such as the Mizuno–Todd–Ye Predictor-Corrector (MTY P-C) algorithm [29], alternate between two types of steps. In predictor steps, we use \( \nu = 0 \). This direction is also called the affine scaling direction, and will be denoted as \( \Delta z^a = (\Delta x^a, \Delta s^a) \) throughout. In corrector steps, we use \( \nu = 1 \). This gives the centrality direction, denoted as \( \Delta z^c = (\Delta x^c, \Delta s^c) \).

In the predictor steps, we make progress along the central path. Given the search direction on the current iterate \( z = (x, s) \in \mathcal{N}(\beta) \), the step-length is chosen maximal such that we remain in \( \mathcal{N}(2\beta) \), i.e.

\[
\alpha^a := \sup\{\alpha \in [0, 1] : \forall \alpha' \in [0, \alpha] : z + \alpha'\Delta z^a \in \mathcal{N}(2\beta)\}.
\]

Thus, we obtain a point \( z^+ = z + \alpha^a\Delta w^a \in \mathcal{N}(2\beta) \). The corrector step finds a next iterate \( z^c = z^+ + \Delta z^c \), where \( \Delta z^c \) is the centrality direction computed at \( z^+ \). The next proposition summarizes well-known properties, see e.g. [46, Section 4.5.1].

Proposition 2.10. Let \( z = (x, s) \in \mathcal{N}(\beta) \) for \( \beta \in (0, 1/4] \).

(i) For the affine scaling step, we have \( \overline{p}(z^+) = (1 - \alpha^a)\overline{p}(z) \).

(ii) The affine scaling step-length is

\[
\alpha^a \geq \max\left\{ \frac{\beta}{\sqrt{n}} 1 - \frac{\|\Delta x^a\Delta s^a\|}{\overline{p}(z)} \right\}.
\]

(iii) For \( z^+ \in \mathcal{N}(2\beta) \), and \( z^c = z^+ + \Delta w^c \), we have \( \overline{p}(z^c) = \overline{p}(z^+) \) and \( z^c \in \mathcal{N}(\beta) \).

(iv) After a sequence of \( O(\sqrt{n}) \) predictor and corrector steps, we obtain an iterate \( z' = (x', s') \in \mathcal{N}(\beta) \) such that \( \overline{p}(z') \leq \overline{p}(z)/2^k \).
Minimum-norm viewpoint We introduce some useful notation for the algorithm, and derive the minimum-norm interpretation of the affine scaling steps. For \( z = (x, s) \in P_{++} \times D_{++} \), we let
\[
\xi (z) = x^{1/2}s^{1/2} / \rho(z)^{1/2} \in \mathbb{R}^n, \quad \hat{x} = x\xi^{-1}(z) = x^{1/2}s^{-1/2}/\rho(z)^{1/2} \in \mathbb{R}^n, \quad \hat{s} = s\xi^{-1}(z) = s^{1/2}x^{-1/2}/\rho(z)^{1/2} \in \mathbb{R}^n.
\]
(14)

If clear from the context, we simply use \( \xi \). If \( z = (x, s) \) falls on the central path, that is, \( x,s = \rho(z)1 \), then \( \xi (z) = 1, \hat{x} = x \) and \( \hat{s} = s \). The variables \( \hat{x} \) and \( \hat{s} \) represent natural adjustments for points off the central path. The following is a simple corollary of Proposition 2.7.

Proposition 2.11. For \( z = (x, s) \in \mathcal{N}(\beta) \) for \( \beta \in (0, 1/4] \), we have \( \| \xi \| = \sqrt{n} \). Moreover,
\[
\sqrt{1 - \beta} \leq \xi \leq \sqrt{1 + \beta}, \quad \sqrt{1 - \beta} x \leq \hat{x} \leq \sqrt{1 + \beta} x \quad \text{and} \quad \sqrt{1 - \beta} s \leq \hat{s} \leq \sqrt{1 + \beta} s.
\]

We will frequently use the rescaled subspaces \( \hat{x}^{-1}W \) and \( \hat{s}^{-1}W^\perp \) that correspond to using the local geometry at the point \( z = (x, s) \). Throughout, we will refer to \( \|w/\hat{x}\| \) and \( \|w/\hat{s}\| \) as the primal and dual local norms of the vector \( w \in \mathbb{R}^n \) at the point \( z = (x, s) \in P_{++} \times D_{++} \). The following statement is immediate from the definitions using \( \hat{x}s = \rho(z)1 \).

Proposition 2.12. The subspaces \( \hat{x}^{-1}W \) and \( \hat{s}^{-1}W^\perp \) are orthogonal.

Equation (13) for the predictor step \( (\nu = 0) \) with update direction \( (\Delta x^h, \Delta s^h) \) can be written as
\[
x^{-1}\Delta x^h + x^{-1}\Delta s^h = -1,
\]
or equivalently,
\[
\hat{x}^{-1}\Delta x^h + \hat{s}^{-1}\Delta s^h = -\xi,
\]
which serves the purpose that now \( \hat{x}^{-1}\Delta x^h \in \hat{x}^{-1}W \) and \( \hat{s}^{-1}\Delta s^h \in \hat{s}^{-1}W^\perp \) are orthogonal vectors (Proposition 2.12). Thus, \( \hat{x}^{-1}\Delta x^h \) and \( \hat{s}^{-1}\Delta s^h \) give an orthogonal decomposition of \( -\xi \). This leads to the following formulas:
\[
\Delta x^h = -\hat{x} \Pi_{\hat{x}^{-1}W}(\xi), \quad 
\Delta s^h = -\hat{s} \Pi_{\hat{s}^{-1}W^\perp}(\xi).
\]
(17)

Equivalently, we can see \( \Delta x^h = (\Delta x^h, \Delta s^h) \) as the optimal solutions of the following minimum-norm problems:
\[
\Delta x^h = \arg\min_{\Delta x \in W} \|\hat{x}^{-1}(x + \Delta x)\|, \\
\Delta s^h = \arg\min_{\Delta s \in W^\perp} \|\hat{s}^{-1}(s + \Delta s)\|.
\]
(18)

A further equivalent way to express these movement directions is by projections in the rescaled subspaces \( \hat{x}^{-1}W \) and \( \hat{s}^{-1}W^\perp \); this viewpoint will be used in Section 4.
\[
\Delta x^h = \hat{x} \arg\min_{\delta \in \hat{x}^{-1}W} \|\xi + \delta\|, \\
\Delta s^h = \hat{s} \arg\min_{\delta \in \hat{s}^{-1}W^\perp} \|\xi + \delta\|.
\]
(19)

The equivalence of the two forms follows by noting that \( \hat{x}^{-1}x \equiv \hat{s}^{-1}s = \xi \).

Step-length estimates We will also need good estimates on the size on predictor steps beyond affine scaling. Our main estimate in this regard is given below.

Lemma 2.13 (Step-length estimate for general directions). Let \( z = (x, s) \in \mathcal{N}(\beta) \), \( \beta \in (0, 1/6] \). Consider directions \( \Delta x \in W, \Delta s \in W^\perp \) that satisfy \( \|\Delta x \Delta s\| \leq \beta \mu/4 \). Let
\[
\gamma = \frac{\|(x + \Delta x)(s + \Delta s)\|}{\mu}.
\]
Then \( (x + \alpha \Delta x, s + \alpha \Delta s) \in \mathcal{N}(2\beta) \) and \( \bar{\rho}(x + \alpha \Delta x, s + \alpha \Delta s) \leq (1 + \frac{3}{2}\beta/\sqrt{n})(1 - \alpha)\mu \), for all \( 0 \leq \alpha \leq 1 - \frac{\gamma}{\mu} \).
Proof. Let \( z_\alpha := (x + \alpha \Delta x, s + \alpha \Delta s) \). We first bound the centrality error using the estimate \((1 - \alpha) \nu\) for \( \mu(z_\alpha) \) as follows:

\[
\left\| \frac{(x + \alpha \Delta x)(s + \alpha \Delta s)}{(1 - \alpha) \nu} - 1 \right\| = \left\| \frac{(1 - \alpha)x + \alpha(x + \Delta x)(s + \Delta s) - \alpha(1 - \alpha)\Delta x \Delta s}{(1 - \alpha) \nu} - 1 \right\|
\]

\[
\leq \left\| \frac{x}{\nu} - 1 \right\| + \frac{\alpha}{1 - \alpha} \left\| \frac{(x + \Delta x)(s + \Delta s)}{\nu} \right\| + \alpha \left\| \frac{\Delta x \Delta s}{\nu} \right\|
\]

\[
\leq \beta + \frac{\alpha}{1 - \alpha} \gamma + \frac{\alpha \beta}{4} \leq \frac{3}{2} \beta,
\]

where the last inequality follows since \( \frac{\alpha}{1 - \alpha} \gamma \leq \beta/4 \) for \( 0 \leq \alpha \leq 1 - 4 \gamma/\beta \).

Given (20), by Lemma 2.8, we get that \( \mu(z_\alpha) \leq (1 + \frac{\beta}{\sqrt{n}})(1 - \alpha) \nu\) and \( z_\alpha \in N\left( \frac{1}{2} \beta \right) \subseteq N(2 \beta) \), for \( \beta \in (0, 1/6] \).

2.3 Lifting Maps

Our algorithm in Section 4 uses a layered least squares step that consists of solving a minimum-norm point in a smaller subspace first, and then extending it to the entire space. A crucial operation in both computing the layered step as well as in identifying the appropriate subspaces is the lifting map defined next.

Definition 2.14. Given a partition \( I \cup J = [n] \) and a subspace \( W \subseteq \mathbb{R}^n \), we define the lifting map \( L^W_I : \mathbb{R}^I \rightarrow W \subseteq \mathbb{R}^n \) as follows:

\[
L^W_I(x) := \arg \min \{ \| w \| : w \in W, w_I = \Pi_{\pi_I(W)}(x) \}. \quad (21)
\]

We further define \( \ell^W_I : \mathbb{R}^I \rightarrow \mathbb{R}^J \) by

\[
\ell^W_I(x) := (L^W_I(x))_J = \pi_J(L^W_I(x)) . \quad (22)
\]

Note that if \( x \in \pi_I(W) \), \( w = L^W_I(x) \) is the minimum-norm point in \( W \) with \( w_I = x \). The following lists the key properties of the lifting map.

Lemma 2.15. For a partition \( I \cup J = [n] \) and a linear subspace \( W \), \( L^W_I : \mathbb{R}^I \rightarrow W \) and \( \ell^W_I : \mathbb{R}^I \rightarrow \mathbb{R}^J \) are linear maps. Moreover, for \( x \in \mathbb{R}^I \), \( w = L^W_I(x) \) is the unique solution to the following linear system:

\[
w \in W ,
\]

\[
w_I \in \pi_I(W)^\perp + x ,
\]

\[
w_J \in \pi_J(W^\perp) . \quad (23)
\]

We now give the fundamental duality relation between lifting maps. For this purpose, we define \( \ell^{W*}_I : \mathbb{R}^J \rightarrow \mathbb{R}^I \) to denote the adjoint of \( \ell^W_I \), namely, the map satisfying \( \langle \ell^{W*}_I(y), x \rangle = \langle y, \ell_I(x) \rangle \), \( \forall y \in \mathbb{R}^J, x \in \mathbb{R}^I \). If we are expressing \( \ell^W_I \) as matrix \( M \in \mathbb{R}^{J \times I} \), then \( \ell^{W*}_I \) is represented by \( M^\top \). That is, if \( \ell^W_I(x) = Mx \), then \( \ell^{W*}_I(y) = M^\top y \).

Lemma 2.16. For a partition \( I \cup J = [n] \) and a linear subspace \( W \), we have that \( \ell^W_I = \ell^{W*}_I \). In particular, the non-zero singular values of \( \ell^W_I \) and \( \ell^{W*}_I \) are the same.

3 Polarization of the Central Path

We now introduce the notion of polarized segments of the central path. For \( 0 \leq \mu_1 \leq \mu_0 \), the central path segment between these values is denoted by

\[
\text{CP}[\mu_1, \mu_0] := \{ z^{cp}(\mu) : \mu_1 \leq \mu \leq \mu_0 \} . \quad (24)
\]

Definition 3.1 (Polarization). For \( \gamma \in [0, 1] \) and \( \mu_0 \geq \mu_1 \geq 0 \), we say that the segment \( \text{CP}[\mu_1, \mu_0] \) is \( \gamma \)-polarized if there exists a partition \( B \cup N = [n] \) such that for all \( \mu \in [\mu_1, \mu_0] \):

\[
x^{cp}_i(\mu) \geq \gamma x^{cp}_i(\mu_0), \quad \forall i \in B ,
\]

\[
s^{cp}_i(\mu) \geq \gamma s^{cp}_i(\mu_0), \quad \forall i \in N .
\]
Remark 3.2. As stated, the notion of polarization requires an inequality to hold for all \( \mu \in [\mu_1, \mu_0] \). At the cost of losing a factor \( n \) however, it is in fact sufficient to check the polarization condition only at \( \mu = \mu_1 \). This follows by the near-monotonicity of the central path (Lemma 2.4):

\[
\frac{x^{\text{CP}}_i(\mu)}{x^{\text{CP}}_i(\mu_0)} = \frac{x^{\text{CP}}_i(\mu_1)}{x^{\text{CP}}_i(\mu_0)} = \frac{1}{n} \cdot \frac{x^{\text{CP}}_i(\mu_1)}{x^{\text{CP}}_i(\mu_0)}, \quad \forall i \in [n],
\]

The same is true for \( s(\mu) \) by a symmetric argument.

As a direct consequence of the definition together with near-monotonicity, we deduce the following crucial corollary:

Corollary 3.3. Let \( \text{CP}[^{\mu_1, \mu_0}] \), \( 0 \leq \mu_1 \leq \mu_0 \), be \( \gamma \)-polarized with respect to the partition \( B \cup N = [n] \). Then, for all \( \mu \in [\mu_1, \mu_0] \), the following holds:

1. \( \gamma x_i(\mu_0) \leq x_i(\mu) \leq n x_i(\mu_0), \quad i \in B \).
2. \( \gamma s_i(\mu_0) \leq s_i(\mu) \leq n s_i(\mu_0), \quad i \in N \).
3. \( \frac{\mu_i}{\gamma x_i(\mu_0)} \leq x_i(\mu) \leq \frac{\mu_i}{\gamma s_i(\mu_0)}, \quad i \in N \).
4. \( \frac{\mu_i}{\gamma x_i(\mu_0)} \leq s_i(\mu) \leq \frac{\mu_i}{\gamma s_i(\mu_0)}, \quad i \in B \).

Proof. The first inequalities in (1) and (2) are the definition of \( \gamma \)-polarization and the second inequalities are from Lemma 2.4. (3) and (4) are equivalent to (1) and (2) using the central path relations \( x(\mu_0)s(\mu_0) = \mu_01 \) and \( x(\mu)s(\mu) = \mu1 \). \( \square \)

Section 4 introduces the algorithm \textit{IPM with subspace LLS} that can traverse \( \gamma \)-polarized segments in \( O(n^{1.5} \log(n/\gamma)) \) iterations. Theorem 1.1 follows by combining this algorithm with the following decomposition result that is the main result of this section.

Theorem 3.4. Let \( \Gamma : (\mu_1, \mu_0) \rightarrow \mathcal{N}^{-\infty}(\theta), \quad \theta \in (0,1), \quad 0 \leq \mu_1 \leq \mu_0 \leq \infty \), be a piecewise linear curve satisfying \( \mathcal{P}(\Gamma(\mu)) = \mu, \quad \forall \mu \in (\mu_1, \mu_0) \) consisting of \( T \) linear segments. Then, \( \text{CP}[\mu_1, \mu_0] \) can be decomposed into \( T \) segments that are \( (1-\theta)^2 \)-polarized.

Theorem 1.2 showing the combinatorial bound on the number of iterations of algorithm \textit{IPM with subspace LLS} follows from the existence of a piecewise linear curve traversing the wide neighborhood with at most \( O(2^n) \) pieces. This will be shown in Section 5, where we extract such a path from the max central path. We will also give a direct proof that the central path can be decomposed into \( O(2^n) \) polarized segments, by showing that linear segments of the max-central path correspond to polarized segments of the central path. Theorem 3.4 is a direct consequence of the following key lemma.

Lemma 3.5. For \( \theta \in (0,1) \), let \( [z^{(0)}, z^{(1)}] \subseteq \mathcal{N}^{-\infty}(\theta), \quad \mathcal{P}(z^{(0)}) > \mathcal{P}(z^{(1)}) \). Then, \( \text{CP}[\mathcal{P}(z^{(1)}), \mathcal{P}(z^{(0)})] \) is \( (1-\theta)^2 \)-polarized.

The proof requires the following simple technical lemma that allows us to relate approximatecentrality along lines to polarization.

Lemma 3.6. For any \( u, v > 0 \),

\[
\min_{\alpha \in [0,1]} \frac{(1-\alpha + \alpha u)(1-\alpha + \alpha v)}{1-\alpha + \alpha uv} = \min \left\{ 1, \left( \frac{\sqrt{u} + \sqrt{v}}{1 + \sqrt{uv}} \right)^2 \right\} \leq 2(u + v). \tag{25}
\]

Proof. To show the equality, let \( \mu := uv \). Note that

\[
\min_{\alpha \in [0,1]} \frac{(1-\alpha + \alpha u)(1-\alpha + \alpha v)}{1-\alpha + \alpha uv} = \min_{\alpha \in [0,1]} \frac{(1-\alpha)^2 + \alpha^2 \mu + \alpha(1-\alpha)(u+v)}{1-\alpha + \alpha \mu} = 1 + \min_{\alpha \in [0,1]} \left( u + v - (1 + \mu) \right) \frac{\alpha (1-\alpha)}{1-\alpha + \alpha \mu}. \tag{26}
\]

Case I: \( \left( \frac{\sqrt{u} + \sqrt{v}}{1 + \sqrt{uv}} \right)^2 \geq 1 \). In this case, we need to show that the minimum of the expression is 1. It is easy to see that the condition equivalent to \( u + v \geq 1 + uv = 1 + \mu \). Thus, the minimum value of (26) is clearly 1, attained at \( \alpha \in [0,1] \).
Case II: \((\frac{\sqrt{u} + \sqrt{v}}{1 + \sqrt{uv}})^2 < 1\), or equivalently, \(u + v < 1 + \mu\). In this case, the minimizer of (26) corresponds to the maximizer of \(\frac{\alpha(1-\alpha)}{u+\alpha \mu}\). This function takes value 0 at \(\alpha \in \{0,1\}\) and is strictly positive for \(0 < \alpha < 1\). Furthermore, the unique critical point in the interval [0,1] occurs at \(\alpha^* = \frac{1}{1+\sqrt{\mu}}\), which is thus the maximizer. The minimum value of (26) is therefore

\[
1 + (u + v - (1 + \mu)) \frac{\alpha^*(1 - \alpha^*)}{1 - \alpha^* + \alpha^* \mu} = \frac{u + v + 2\sqrt{\mu}}{(1 + \sqrt{\mu})^2},
\]
as required. The inequality in the statement follows easily as

\[
\left(\frac{\sqrt{u} + \sqrt{v}}{1 + \sqrt{uv}}\right)^2 \leq (\sqrt{u} + \sqrt{v})^2 = 2(u + v) - (\sqrt{u} - \sqrt{v})^2.
\]

\[\square\]

**Proof of Lemma 3.5.** For \(\alpha \in [0,1]\), let \(z^{(\alpha)} := (x^{(\alpha)}, s^{(\alpha)}) := (1 - \alpha)z^{(0)} + \alpha z^{(1)}\). By Proposition 2.3, we first note that the normalized gap function \(\overline{\gamma}(z)\) is in fact linear on \([z^{(0)}, z^{(1)}]\). That is,

\[
\mu_\alpha = \overline{\gamma}(z^{(\alpha)}) = (1 - \alpha)\overline{\gamma}(z^{(0)}) + \alpha \overline{\gamma}(z^{(1)}).
\]

For any \(i \in [n]\), \(z^{(\alpha)} \in \mathcal{N}^{-\infty}(\theta)\) implies

\[
\frac{x^{(\alpha)}_i s^{(\alpha)}_i}{(1 - \alpha)x^{(0)}_i s^{(0)}_i + \alpha x^{(1)}_i s^{(1)}_i} \geq \frac{(1 - \theta)\mu_\alpha}{n - (1 - \alpha)\overline{\gamma}(z^{(0)}) + \alpha \overline{\gamma}(z^{(1)})} \geq \frac{1 - \theta}{n}.
\]

From Lemma 3.6 for \(u = x^{(1)}_i / x^{(0)}_i\), \(v = s^{(1)}_i / s^{(0)}_i\) we get that

\[
1 - \theta \leq \frac{2}{n} \left(\frac{x^{(1)}_i}{x^{(0)}_i} + \frac{s^{(1)}_i}{s^{(0)}_i}\right).
\]

Let

\[
B := \left\{ i \in N : \frac{x^{(1)}_i}{x^{(0)}_i} \geq \frac{s^{(1)}_i}{s^{(0)}_i} \right\}, \quad N := [n] \setminus B.
\]

Then, \(x^{(1)}_i / x^{(0)}_i \geq \frac{1 - \theta}{n}\), for all \(i \in B\), and \(s^{(1)}_i / s^{(0)}_i \geq \frac{1 - \theta}{4n}\), for all \(i \in N\).

For any \(\alpha \in [0,1]\) and \(i \in B\),

\[
\frac{x^{(\alpha)}_i}{x^{(0)}_i} = (1 - \alpha) + \frac{x^{(1)}_i}{x^{(0)}_i} \geq \min \left\{ 1, \frac{1 - \theta}{4n} \right\} = \frac{1 - \theta}{4n}.
\]

Similarly, for \(i \in N\), \(s^{(\alpha)}_i / s^{(0)}_i \geq \frac{1 - \theta}{4n}\).

For the central path point \(z^{cp} (\mu_\alpha) = (x^{cp}(\mu_\alpha), s^{cp}(\mu_\alpha))\) at \(\mu_\alpha\), the bounds in Lemma 2.9 relating points in a neighborhood with central path points give

\[
\frac{x^{cp}(\mu_\alpha)}{x^{(0)}_i} \geq \frac{x^{(\alpha)}_i / (2n)}{2^{cp} x^{(0)}_i} \geq \frac{(1 - \theta)^2}{16 n^3}, \quad \forall i \in B.
\]

By a similar argument, we also have \(s^{cp}(\mu_\alpha) / s^{(0)}_i \geq \frac{(1 - \theta)^2}{16 n^3} \), \(\forall i \in N\). Thus, \(CP[\mu_1, \mu_0] \) is \(\frac{(1 - \theta)^2}{16 n^3}\) polarized.

To conclude this section, we show that linearity of the central path can be equivalently restated in terms of polarization. While not needed in the sequel, we believe it to be of independent interest. The proof can be found in the Appendix.

**Lemma 3.7.** For \(\mu_0 > \mu_1 \geq 0\), let \(CP[\mu_1, \mu_0] \) be \(\gamma\)-polarized. Then \(\gamma \in [0,1]\). Furthermore, \(\gamma = 1\) if and only if \(CP[\mu_1, \mu_0] \) is linear.
4 The Subspace Layered Least Squares IPM

In this section, we introduce the algorithm IPM with subspace LLS (Algorithm 1) and prove the following result.

**Theorem 4.1** (Proof on 26). Let CP[μ₁, μ₀] be γ-polarized. Then, given an iterate \( z \in \mathcal{N}(β) \) with parameter \( \Pi(z) \in (μ₁, μ₀) \), the algorithm IPM with subspace LLS (Algorithm 1) takes \( O\left( n^{1.5} \log(n/γ) \right) \) many iterations to find \( z' \in \mathcal{N}(β) \) such that \( \Pi(z') ≤ μ₁ \).

Recall from Theorem 3.4 that the existence of a line segment in the wide neighborhood of the central path implies corresponding central path segment. Hence, Theorem 1.1 directly follows from Theorem 3.4 and Theorem 4.1. Recall from the discussion after the statement of Theorem 3.4 in Section 3 how Theorem 1.2 and Theorem 1.3 can also be derived. We stress that IPM with subspace LLS does not have any information about the polarization of CP[μ₁, μ₀], but only defines the steps using local information.

The subspace LLS direction. We now introduce a new update direction, called the subspace layered least squares update direction. At a given point \( z \in \mathcal{N}(β) \), this step direction is specified by a partition \( B \cup N = [n] \) and two subspaces \( V \subseteq π_N(\hat{x}^{-1}W) \) and \( U \subseteq π_B(\hat{s}^{-1}W^\perp) \).

Recall the notation \( δ \) for the local error scaling and the thereby adjusted versions \( \hat{v} \) of \( x \) and \( s \) defined in (14). For a given partition \( B \cup N = [n] \) such that \( B, N ≠ \emptyset \), recall the lifting maps introduced in Section 2.3. We use the following shorthand notation

\[ \ell_{z,N} := \ell_N^{-1}W \quad \text{and} \quad \ell_{z,B} := \ell_B^{-1}W^\perp. \]

Thus, the linear map \( \ell_{z,N} : R^N → R^B \) computes a minimum-norm lift in the rescaling \( \hat{x}^{-1}W \) corresponding to the local geometry at the point \( z \). Recall from Lemma 2.16 that the linear map \( \ell_{z,B} : R^B → R^N \) is the adjoint of \( \ell_{z,N} \), thus, they are represented by matrices that are transposed of each other.

Also recall the affine scaling direction \((\Delta x^a, Δs^a)\); the most convenient formulation in the context of the next definition is (19).

**Definition 4.2** (Subspace LLS direction). Let \( z \in \mathcal{N}(β), \) and let \( \xi, \hat{x}, \hat{s} \) be defined as in (14). Assume we are given a partition \( B \cup N = [n] \) and two subspaces \( V \subseteq π_N(\hat{x}^{-1}W) \) and \( U \subseteq π_B(\hat{s}^{-1}W^\perp) \). The Subspace LLS update direction \((Δx^f, Δs^f)\) ∈ \( W × W^\perp \) at \( z \) with respect to \((B, N,U,V)\) is defined as follows. If \( B, N ≠ \emptyset \), then

\[
\begin{align*}
δ^V &:= \min_{δ ∈ V} \|ξ_N + δ\|, \\
δ^U &:= \min_{δ ∈ U} \|ξ_B + δ\|, \\
δ^x &:= (\ell_{z,N}(δ^V), δ^U), \\
δ^s &:= (δ^U, \ell_{z,B}(δ^U)), \\
Δx^f &:= \hat{x}δ^x, \\
Δs^f &:= \hat{s}δ^s.
\end{align*}
\]

If \( N = \emptyset \), then we let \((Δx^f, Δs^f) = (0, Δs^a)\) and if \( B = \emptyset \), then we let \((Δx^f, Δs^f) = (Δx^a, 0)\).

The formula defining \( δ^V \) is similar to the definition (19) of the affine scaling direction. However, when defining \( δ^V \), we restrict ourselves to norm minimization in \( π_N(\hat{x}^{-1}W) \), and within that, we require \( δ^V ∈ V \). This step is then extended to the coordinates in \( B \) using a minimum norm lift in the rescaled subspace \( \hat{x}^{-1}W \).

Note that we can equivalently write \( δ^V = −Π_V(ξ_N) \) and \( δ^U = −Π_U(ξ_B) \). An equivalent definition of \( Δx^f \), using the primal local norm in the original space \( W \) — similarly to (18) — is the following:

\[
\begin{align*}
Δx^f &:= \min \left\{ \|\hat{x}^{-1}(x_N + Δv)\| : Δv ∈ \hat{x}V \right\}, \\
Δx^f &:= \min \left\{ \|\hat{x}^{-1}Δx\| : Δx_N = Δx^f, Δx ∈ W \right\}.
\end{align*}
\] (27)

Analogous formulas can be given for \( Δs^f \).

For the subspace LLS direction as in Definition 4.2, let us define the residuals

\[
\begin{align*}
δ^V &:= ξ + δ^x = \frac{x + Δx^f}{\hat{x}}, \\
δ^U &:= ξ + δ^s = \frac{s + Δs^f}{\hat{s}}.
\end{align*}
\] (28)
Cheap lifts and singular values To argue for the usefulness of the above defined step direction, and to select suitable subspaces \( V \) and \( U \) for a given partition \((B, N)\), we recall the discussion of the trust region step from the Introduction (Section 1). As long as the step primal and dual directions \((\Delta x, \Delta s)\) are feasible to the systems (6) for a suitably small threshold \( \tau \), we are guaranteed to make progress as measured by the primal and dual objective values as in (7).

Simply selecting \( V = \pi_N(\hat{x}^{-1}W) \) and \( U = \pi_B(\hat{s}^{-1}W^+) \) would attain the smallest possible objective values; however, the constraints bounding the local norms of \( \Delta x_B^t \) and \( \Delta s_N^t \) in (6) could be arbitrarily violated. Therefore, we select the lifts \( \Delta x_B^t = \hat{x}N\ell_{z,N}(\delta^V) \) and \( \Delta s_N^t = \delta_N\ell_{z,B}^t(\delta^U) \) will have small local norms; that is, \( ||\ell_{z,N}(\delta^V)|| \) and \( ||\ell_{z,B}^t(\delta^U)|| \) are suitably bounded.

Using that \( ||\delta^V|| \leq ||\xi_N|| \) as \( \delta^V \) is the projection of \( \xi_N \), and Proposition 2.11, we get

\[
||\delta^V|| \leq ||\xi_N|| \leq \sqrt{n}.
\]

Hence, if we can guarantee that the lifting map satisfies \( ||\ell_{z,N}(\delta)|| \leq \tau||\delta|| \) for all \( \delta \in V \) for a suitably small threshold \( \tau \), we can guarantee \( \Delta x^t \) to be feasible to the primal trust region program (6); analogous arguments can be made for the dual direction. Naturally, we would like to select the largest subspaces \( V \) and \( U \) with this property.

Let us define the threshold

\[
\tau := \frac{\beta}{16\sqrt{n}}.
\]

Our goal is to ensure that

\[
||\ell_{z,N}(\delta)|| \leq \tau||\delta||, \quad \forall \delta \in V \quad \text{and} \quad ||\ell_{z,B}^t(\delta)|| \leq \tau||\delta||, \quad \forall \delta \in U. \tag{30}
\]

This holds if \( V \) is the subspace spanned by the singular vectors of the map \( \ell_{z,N} \) corresponding to the singular values \( \leq \tau \) in absolute value; analogously for \( \ell_{z,B}^t \) and \( U \). In the following definition, we use a full Singular Value Decomposition (SVD) of the matrix \( M \in \mathbb{R}^{[B] \times [N]} \) representing the map \( \ell_{z,N} \). The SVD gives \( M = U\Sigma V^T \) where \( U \in \mathbb{R}^{[B] \times [B]} \) and \( V \in \mathbb{R}^{[N] \times [N]} \) are orthogonal matrices, and \( \Sigma \in \mathbb{R}^{[B] \times [N]} \) is a rectangular diagonal matrix also including the zero singular values.

**Definition 4.3** (Cheap Subspaces). For a partition \( B \cup N = [n] \) with \( B, N \neq \emptyset \) and \( z \in N(\beta) \), consider an SVD decomposition

\[
\ell_{z,N} = U\Sigma V^T, \quad \ell_{z,B}^t = V\Sigma^t U^T.
\]

Depending on a threshold parameter \( t \in \mathbb{R}_{>0} \), we set

\[
V_{z,N}(t) := \text{im}(V_S) \cap \pi_N(\hat{x}^{-1}W) \quad \text{for} \quad S := \{i : ||\Sigma_i|| \leq t\},
\]

\[
U_{z,B}(t) := \text{im}(U_T) \cap \pi_B(\hat{s}^{-1}W^+) \quad \text{for} \quad T := \{i : ||\Sigma_i|| \leq t\}.
\]

Further, we let \( \overline{V}_{z,N}(t) \) be the orthogonal complement of \( V_{z,N}(t) \) in \( \pi_N(\hat{x}^{-1}W) \), that is, \( V_{z,N}(t) \oplus \overline{V}_{z,N}(t) = \pi_N(\hat{x}^{-1}W) \) and \( V_{z,N}(t) \perp \overline{V}_{z,N}(t) \). Analogously we define \( \overline{U}_{z,B}(t) \) such that \( U_{z,B}(t) \oplus \overline{U}_{z,B}(t) = \pi_B(\hat{s}^{-1}W^+) \) and \( U_{z,B}(t) \perp \overline{U}_{z,B}(t) \).

Note that \( ||\Sigma_i|| \) and \( ||\Sigma_i|| \) are the absolute values of the singular values corresponding to the \( i \)-th columns of \( V \) and \( U \), respectively. It is clear from the definition that \( V = V_{z,N} \) and \( U = U_{z,B} \) satisfy (30).

Definition 4.3 gives a natural way of defining \( V = V_{z,N}(\tau) \) and \( U = U_{z,B}(\tau) \) for our algorithm. However, this requires an exact SVD decomposition of the matrix \( M \) representing the lifting map \( \ell_{z,N} \). It is important to note that our arithmetic model does not allow for computing an exact decomposition. Since we would like to implement each iteration of the algorithm in strongly polynomial time, we cannot use a numerical approximation for the SVD decomposition, as such approximations would depend on the norm of the matrix. However, a weaker property suffices for our analysis. Namely, we need that the subspace \( V \subseteq \pi_N(\hat{x}^{-1}W) \) has singular values at most \( \tau \)—and thus satisfies (30)—and further it includes the subspace of right singular vectors corresponding to the singular values \( \leq \tau/n^c \) for some constant \( c > 0 \); analogously for \( U \). A strongly polynomial subroutine **APPROX-SVD** finding such a subspace is given in Section 4.1.1.
The associated partition Definition 4.2 and Definition 4.3 are applicable for any partition \( B \cup N = [n] \), \( B, N \neq \emptyset \) and \( z \in \mathcal{N}(\beta) \). Our algorithm chooses a natural partition derived from the relative step lengths in the affine scaling step:

Definition 4.4 (Associated partition). For \( z = (x, s) \in \mathcal{N}(\beta) \), let \((\Delta x^a, \Delta s^a)\) be the affine scaling step as in (17). Let us define the associated partition \( \tilde{B}_z \cup \tilde{N}_z = [n] \) as

\[
\tilde{B}_z := \left\{ i : \left| \frac{\Delta x^a_i}{x_i} \right| < \left| \frac{\Delta s^a_i}{s_i} \right| \right\}; \quad \tilde{N}_z := [n] \setminus \tilde{B}_z.
\]

Further, let \( \tilde{\ell}_z := \ell_{z, \tilde{N}_z} \) and \( \tilde{\ell}_z^\perp := \ell_{z, \tilde{B}_z}^\perp \) denote the lifting maps corresponding to this partition, assuming \( \tilde{B}_z, \tilde{N}_z \neq \emptyset \).

The affine scaling step is the canonical candidate for an improving direction. Namely, for each \( i \in \tilde{B}_z \) the variable \( s_i \) decreases at a faster rate than \( x_i \), and vice versa for \( i \in \tilde{N}_z \). As shown in the analysis (Lemma 4.11), for a sufficiently long polarized segment, \((\tilde{B}_z, \tilde{N}_z)\) reveals the polarizing partition.

Description of the algorithm We are ready to describe the predictor-corrector algorithm IPM with subspace LLS, shown in Algorithm 1. We are given a starting point \((x^0, s^0) \in \mathcal{N}(\beta)\). In each iteration, we compute the affine scaling direction \((\Delta x^a, \Delta s^a)\) and identify the associated partition \((\tilde{B}, \tilde{N})\). Using this partition, we approximate the cheap subspaces \( U \) and \( V \) using APPROX-SVD (Algorithm 2). We then compute the subspace LLS direction \((\Delta x', \Delta s')\) for \((\tilde{B}, \tilde{N}, V, U)\). For both directions, compute the feasible step-lengths according to the bounds in Proposition 2.10 and Lemma 2.13. We use the better of these two possible steps, and obtain the next iterate after a corrector step.

**Algorithm 1: IPM with subspace LLS**

**Input**: Instance of (LP) and initial \((x^0, s^0) \in \mathcal{N}(\beta), \beta \in (0, 1/6)\).

**Output**: Optimal solution \((x^*, s^*)\) to (LP).

1. \( x \leftarrow x^0, s \leftarrow s^0 \);
2. while \((x, s) > 0\) do
3. Compute affine scaling direction \((\Delta x^a, \Delta s^a)\);
4. Set \( \alpha^a \) for \((\Delta x^a, \Delta s^a)\) according to Proposition 2.10(ii);
5. \((\tilde{x}^a, \tilde{s}^a) \leftarrow (x + \alpha^a \Delta x^a, s + \alpha^a \Delta s^a)\);
6. \( \tilde{B} \leftarrow \left\{ i : \left| \frac{\Delta x^a_i}{x_i} \right| < \left| \frac{\Delta s^a_i}{s_i} \right| \right\} \), \( \tilde{N} \leftarrow [n] \setminus \tilde{B} \);
7. \( V \leftarrow \text{APPROX-SVD}(\tilde{\ell}_{(x,s),\tilde{N}})\);
8. \( i \leftarrow \max\{ j : \|v_{\ell_{(x,s),\tilde{N}}}(W)\|_2 \leq \frac{\beta}{n} \};\)
9. \( V \leftarrow \text{im}(V_{\leq i}) \cap \pi_{\tilde{B}}(\tilde{x}^{-1}W)\);
10. \( U \leftarrow \text{APPROX-SVD}(\tilde{\ell}_{(x,s),\tilde{B}}^\perp)\);
11. \( i \leftarrow \max\{ j : \|u_{\ell_{(x,s),\tilde{B}}^\perp}(U)\|_2 \leq \frac{\beta}{n} \};\)
12. \( U \leftarrow \text{im}(U_{\leq i}) \cap \pi_{\tilde{N}}(s^{-1}W^\perp)\);
13. Find subspace LLS direction \((\Delta x', \Delta s')\) according to Definition 4.2 for \( V \) and \( U \);
14. Set \( \alpha^e \) for \((\Delta x', \Delta s')\) according to Lemma 2.13;
15. \((\tilde{x}', \tilde{s}') \leftarrow (x + \alpha^e \Delta x', s + \alpha^e \Delta s')\);
16. if \( \pi(\tilde{x}', \tilde{s}') \leq \pi(\tilde{x}^a, \tilde{s}^a) \) then
17. \( (x, s) \leftarrow (\tilde{x}', \tilde{s}')\)
18. else
19. \( (x, s) \leftarrow (\tilde{x}', \tilde{s}')\)
20. \((\Delta x^e, \Delta s^e) = \text{CORRECTOR}(x, s);\)
21. \( x \leftarrow x + \Delta x^e, s \leftarrow s + \Delta s^e;\)
22. return \((x, s)\).

The main potential in the analysis is the \( \sigma(z) \), the non-decreasing vector of non-zero singular values of \( \ell_z \) and \( \ell_z^\perp \), which are the same by Lemma 2.16.
In particular, we have
\[ \sigma(z)_i = \sigma_{i+\dim(\ker(\ell_z))}(\ell_z) = \sigma_{i+\dim(\ker(\ell_w))}(\ell_w^+). \] (31)

for all \( i \in [\dim(\ker(\ell_z))]. \)

Throughout this section, we will analyze the behavior of **IPM WITH SUBSPACE LLS** on a fixed \( \gamma \)-polarized segment \( CP[\mu_1, \mu_0] \) with respect to the polarizing partition \( B \cup N = [n] \). This partition will be fixed throughout, and for \( z \in \mathcal{N}(\beta) \) with \( \overline{p}(z) \in [\mu_1, \mu_0] \), we use the shorthands
\[ \ell_z := \ell_{z,N} = \ell_N^{-1}W, \quad \ell_z^+ := \ell_{z,B} = \ell_B^{-1}W^+. \]

### 4.1 Algorithmic Tools

#### 4.1.1 Polynomial approximation of singular values

To compute the subspace LLS direction, we need to identify the linear spaces \( U \) and \( V \) that are obtained from an exact SVD decomposition of the matrix \( M \) representing the lifting map \( \hat{\ell}_z \). It is important to note that our arithmetic model does not allow for computing an exact decomposition. Since our goal is to be able to implement each iteration of the algorithm in strongly polynomial time, we cannot use a numerical approximation for the SVD decomposition, as such approximations would depend on the norm of the matrix. However, the analysis of the main algorithm (Algorithm 1) is robust, and it suffices to identify a subspace \( V \leq \tau^{-1}W \) such that the lifting map from \( V \) has singular values at most \( \tau \) and includes the subspace of right singular vectors corresponding to the singular values \( \leq \tau/n^\gamma \) for some threshold \( \tau \) and \( c > 0 \). In the following we provide such an approximation for \( c = 1/2 \).

**Algorithm 2: APPROX-SVD**

**Input**: A matrix \( M \in \mathbb{R}^{n \times n} \).

**Output**: A matrix \( V \in \mathbb{R}^{n \times n} \).

1. \( V \leftarrow I_n; \)
2. for \( i = 1, \ldots, n \) do
3. \( V \leftarrow \text{ORTHOGONALIZE}(I_n, V); \)
4. \( V_{\geq i} \leftarrow \text{ORTHOGONALIZE}(M, V_{\geq i}); \)
5. \( c \leftarrow \arg \min_{1 \leq j \leq n} \|MV_j\|; \)
6. \( V \leftarrow V \) with column \( i \) swapped with column \( c; \)
7. \( V^{(i)} \leftarrow V; \) // Only needed for the analysis
8. return \( V \);
9. 
10. **procedure** Orthogonalize(\( N, V \)):

    **Input**: Matrices \( N \in \mathbb{R}^{m \times n} \) and \( V \in \mathbb{R}^{n \times k} \).

    **Output**: A matrix \( \hat{V} \in \mathbb{R}^{n \times k} \) with \( \text{im}(V_{\leq i}) = \text{im}(\hat{V}_{\leq i}) \) for all \( i \in [k] \) and \( N \)-orthogonal columns, i.e., \( \hat{V}^T N^T N \hat{V} \) is diagonal. // If \( m = 0 \) return \( V \).

We denote the set of \( i \)-dimensional linear subspaces of \( \mathbb{R}^n \) by \( S(i) \). Recall the min-max principle for singular values:
\[ \sigma_i(M) = \min_{S \in S(i)} \max_{u \in S \setminus \{0\}} \frac{\|Mu\|}{\|u\|}, \]
\[ = \max_{S \in S(n-i+1)} \min_{u \in S \setminus \{0\}} \frac{\|Mu\|}{\|u\|}. \] (32)

The next statement gives an approximate version based on the approximate representatives for \( i \)-dimensional linear subspaces of \( \mathbb{R}^n \).

**Lemma 4.5.** Algorithm 2 returns a matrix \( V \) with orthogonal columns such that
\[ \frac{\sigma_i(M)}{n} \leq \frac{1}{n} \max_{v \in \text{im}(V_{\leq i} \setminus \{0\})} \frac{\|Mv\|}{\|v\|} \leq \frac{\|MV_i\|}{\|V_i\|} \leq \sqrt{n} \min_{v \in \text{im}(V_{\leq i} \setminus \{0\})} \frac{\|Mv\|}{\|v\|} \leq \sqrt{n} \sigma_i(M), \] (33)

for all \( i \in [n] \).
Proof. Note that the first and last inequality follow from (32). We prove the second and the third inequality. Note that by choice of \( j \) in line 5 we have for all \( i \leq j \leq n \) that
\[
\frac{\|MV_i^j\|}{\|V_i^j\|} \leq \frac{\|MV_j^i\|}{\|V_j^i\|}.
\] (34)

Therefore, by \( M \)-orthogonality of \( V_{\geq i}^j \) we have
\[
\frac{\|MV_{\geq i}^j\|}{\|V_{\geq i}^j\|} = \frac{\sqrt{\sum_{j \geq i} u_j^2} \|MV_j^i\|^2}{\|V_{\geq i}^j\|} \geq \frac{\|MV_j^i\| \sqrt{\sum_{j \geq i} u_j^2} \|V_{\geq i}^j\|^2}{\|V_j^i\| \left( \sum_{j \geq i} u_j \|V_j^i\| \right)} \geq \frac{1}{\sqrt{n}} \frac{\|MV_j^i\|}{\|V_j^i\|}.
\] (35)

Note that \( \text{im}(V_{\geq i}^j) = \text{im}(V_j^i) \) for and \( V_j^i = \text{V}_i \) for all \( i \in [n] \).

Therefore,
\[
\min_{v \in \text{im}(V_{\geq i}^j)} \frac{\|Mv\|}{\|v\|} = \min_{v \in \text{im}(V_j^i)} \frac{\|Mv\|}{\|v\|} \overset{(35)}{=} \frac{1}{\sqrt{n}} \frac{\|MV_j^i\|}{\|V_j^i\|},
\] (36)

which proves the third inequality of the lemma.

It remains to prove the second inequality of the lemma. Note that by the \( I_n \)-orthogonality of the columns of \( V \) we have for all \( i \in [n] \) that
\[
\max_{v \in \text{im}(V_{\geq i}^j) \setminus \{0\}} \frac{\|Mv\|}{\|v\|} = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\|MV_{\leq i}^j u\|}{\|V_{\leq i}^j u\|} \overset{\Delta-ineq.}{\leq} \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{j \leq i} \|MV_j^i \| u_j}{\sqrt{\sum_{j \leq i} \|V_j^i \|^2 u_j^2}} \overset{AM-QM}{\leq} \frac{1}{\sqrt{n}} \max_{u \in \mathbb{R}^n \setminus \{0\}} \sqrt{\frac{\sum_{j \leq i} \|MV_j^i \| u_j}{\sum_{j \leq i} \|V_j^i \| u_j}}.
\] (37)

Further, we have for all \( j \leq i \) that \( V_j \in \text{im}(V_{\geq j}^j) = \text{im}(V_{\geq j}^i) \). Therefore,
\[
\frac{\|MV_j^i\|}{\|V_j^i\|} \geq \min_{v \in \text{im}(V_{\geq j}^i)} \frac{\|Mv\|}{\|v\|} \overset{(36)}{=} \frac{1}{\sqrt{n}} \frac{\|MV_j^i\|}{\|V_j^i\|}.
\] (38)

Combining (38) with (37) gives the second inequality of the lemma. \( \square \)

4.1.1 Stability of singular values on polarized segments

We now present two statements that describe the stability and evolution of singular values of the map \( \ell_z \) on polarized segments of the central path.

Lemma 4.6 (Stability of singular values for multiplicative perturbation). Let \( y \in \mathbb{R}_{>0}^* \), \( W \subseteq \mathbb{R}^n \) a subspace and let \( \ell_N^{y^{-1}W} : \mathbb{R}^N \to \pi_B(y^{-1}W) \) and \( \tilde{\ell}_N^{W} : \mathbb{R}^N \to \pi_B(W) \) be defined according to Definition 2.14.

Let \( \sigma_1 \leq \cdots \leq \sigma_N \) and \( \tilde{\sigma}_1 \leq \cdots \leq \tilde{\sigma}_N \) their respective singular values. Then, for all \( k \in [N] \), we have
\[
\frac{1}{\|y_B\|_{\infty} \|y_N\|_{\infty}} \sigma_k \leq \tilde{\sigma}_k \leq \|y_B\|_{\infty} \|y_N^{-1}\|_{\infty} \sigma_k.
\] (39)

Proof. We denote \( \ell := \ell_N^{y^{-1}W} \) and \( \tilde{\ell} := \tilde{\ell}_N^{W} \).

We prove the second inequality; the first inequality follows by replacing \( y \) with \( y^{-1} \) and \( \sigma \) with \( \tilde{\sigma} \).

Note that for any \( x \in \pi_N(y^{-1}W) \), we have \( y_N x = (yL_N^{y^{-1}W}(x))_N \in \pi_N(W) \) and \( (yL_N^{y^{-1}W}(x))_B = y_B \ell(x) \in \pi_B(W) \). By definition of \( \tilde{\ell} \), we deduce that
\[
\|\tilde{\ell}(y_N x)\| \leq \|y_B \ell(x)\| \leq \|y_B\|_{\infty} \|\ell(x)\|.
\] (40)
Consider the singular value decomposition $\ell = \mathbf{U} \Sigma \mathbf{V}^\top$ such that the diagonal of $\Sigma$ is non-decreasing. For $k \in [N]$, let $V^{(k)} = \text{im} \left((\mathbf{V}^\top)_{\leq k}\right) \subseteq \pi_N(y^{-1}W)$ be the column span of the first $k$ columns of $\mathbf{V}^\top$. This is the subspace corresponding to the smallest $k$ singular values of $\ell$. Let further $V^{(k)} := y_N V^{(k)}$ be the rescaled subspace in $\pi_N(W)$. Now, for any $\tilde{v} \in V^{(k)}$, we have for $v := y_N^{-1} \tilde{v} \in V^{(k)}$ that $\|\ell(v)\| \leq \sigma_k \|v\|$ and so
\[
\|\tilde{\ell}(\tilde{v})\| = \|\tilde{\ell}(y_N v)\| \leq \|y_B\| \|\ell(v)\| \leq \|y_B\| \|\sigma_k\| \|v\| \leq \sigma_k \|y_B\| \|y_N^{-1}\| \infty \|\tilde{v}\|.
\] (41)

Hence, by (32) we get
\[
\tilde{\sigma}_k \leq \max_{v \in V^{(k)}} \|\tilde{\ell}(v)\| \|\tilde{v}\| \leq \|y_B\| \|y_N^{-1}\| \infty \|\sigma_k\|.
\]

**Lemma 4.7** (Stability of singular values on polarized segments). Let $CP[\mu_1, \mu_0]$ be a $\gamma$-polarized segment of the central path with partition $B \cup N = [n]$. Let $z, \tilde{z} \in N(\beta)$ for $\beta \in (0, 1/6)$, such that $\mu := \pi(z)$ and $\tilde{\mu} := \pi(\tilde{z})$ satisfy $\mu_0 \geq \mu \geq \mu_1$. Then we have:
\[
\frac{\gamma^2}{4n^2} \frac{\tilde{\mu}}{\mu} \sigma(z) \leq \sigma(\tilde{z}) \leq \frac{\mu_1}{\gamma^2} \frac{\mu_0}{\tilde{\mu}} \sigma(z).
\] (42)

**Proof.** Let $z := (x, s)$ and $\tilde{z} := (\tilde{x}, \tilde{s})$. We denote $\xi(z) = x^{1/2}s^{1/2}\mu^{-1/2}$ and $\xi(\tilde{z}) = \tilde{x}^{1/2}\tilde{s}^{1/2}\tilde{\mu}^{-1/2}$ by $\xi$ and $\xi$, respectively. We want to apply Lemma 4.6 with $y = x\xi^{-1}\tilde{x}\xi^{-1}$. By Proposition 2.11 we have
\[
\sqrt{1 - \beta} \leq \xi, \xi^{-1} \leq \sqrt{1 + \beta}.
\] (43)

With Proposition 2.6 and Corollary 3.3 we get
\[
\|y_B\| = \|x_B x_B^\top x_B^\top x_B\| \leq \frac{(1 + 2\beta)(1 + \beta)^{1/2}}{(1 - \beta)^{3/2}} \left\|\frac{x(\mu)}{x(\mu)}\right\| \leq \frac{2n}{\gamma}
\]
(44)
and
\[
\|y_N^{-1}\| = \|x_N^{-1} x_N^\top x_N^\top x_N\| \leq \frac{(1 + 2\beta)(1 + \beta)^{1/2}}{(1 - \beta)^{3/2}} \left\|\frac{x(\tilde{\mu})}{x(\tilde{\mu})}\right\| \leq \frac{2n\mu}{\tilde{\mu}}
\]
(45)
using $\frac{(1 + 2\beta)(1 + \beta)^{1/2}}{(1 - \beta)^{3/2}} \leq 2$ for $\beta \in (0, 1/6)$ and $\left\|\frac{x(\mu)}{x(\mu)}\right\| = \frac{x(\mu)}{x(\mu)}\tilde{\mu}$ for the maximizing index $i$ (as well as the analogous terms of the other inequalities).

Plugging these estimates into Lemma 4.6 yields the result. □

### 4.1.3 Simple properties of the subspace LLS step

Here and in the rest of the analysis, $\| \cdot \|$ denotes the usual $\ell_2 \to \ell_2$ operator norm. By the choice of $V$ and $U$ in the algorithm, and using the second inequality of Lemma 4.5 we obtain the following bound.

**Lemma 4.8.** $\|\ell_z\| \|\ell_z^\perp\| \leq \tau$.

A simple and useful bound on $\delta^V$ and $\delta^N$ is the following.

**Lemma 4.9.** We have
\[
\|\delta^V\|, \|\delta^U\| \leq \sqrt{n},
\]
and
\[
\|\delta^V\| \leq \tau \sqrt{n}, \quad \text{and} \quad \|\delta^N\| \leq \tau \sqrt{n}.
\]

**Proof.** By definition, $\delta^V$ is the projection of some coordinates of $\xi$, and therefore $\|\delta^V\| \leq \|\xi\| = \sqrt{n}$ according to Proposition 2.11. Further, $\|\delta^V\| = \|\ell_z(\delta^V)\|$. Thus, Lemma 4.8 yields $\|\delta^V\| \leq \tau \|\delta^V\| \leq \tau \sqrt{n}$. The analogous arguments give the bounds $\|\delta^U\|$ and $\|\delta^N\|$. □
4.1.4 The empirical gradient and its projections

For a point \( z = (x, s) \in \mathcal{N}(\beta) \) such that \( \mathcal{P}(z) \in [\mu_1, \mu_0] \), the empirical gradient is the direction that points directly to the endpoint of the polarized segment \( z^\text{P}(\mu_1) \), that is,

\[
\Delta x^\text{emp} := x^\text{cp}(\mu_1) - x, \quad \text{and} \quad \Delta s^\text{emp} := s^\text{cp}(\mu_1) - s.
\]

This would be a desirable direction to directly shoot down along the polarized segment, however, it requires explicit knowledge of the point \( z^\text{P}(\mu_1) \). Nevertheless, a key idea of the analysis is to measure the discrepancy between the subspace LLS direction and the empirical gradient. We will analyze the quantities

\[
\psi_V := ||\Pi_V(\xi_N - \hat{x}_N^{-1}x^\text{cp}(\mu_1)N)|| \quad \text{and} \quad \psi_U := ||\Pi_U(\xi_B - \hat{s}_B^{-1}s^\text{cp}(\mu_1)B)||.
\]

These are the projections of a rescaling of the empirical gradient onto the respective ‘non-cheap’ subspaces. These quantities are crucial in proving Theorem 4.14. Lemma 4.11 shows that if both \( \psi_V \) and \( \psi_U \) are small, then the partition associated to the affine scaling direction matches the polarizing partition. Furthermore, if both are small then we make significant progress on the central path (Theorem 4.13). Intuitively, in this case ‘most of’ \( \Delta x^\text{emp} / \hat{x}_N \) is in \( V \) and ‘most of’ \( \Delta s^\text{emp} / \hat{s}_B \) is in \( U \). Since the subspace LLS step is optimized subject to the constraints \( \Delta x_N / \hat{x}_N \in V \), \( \Delta s_N / \hat{s}_N \in U \), we can move in this direction nearly as far as with the empirical gradient.

On the other hand, if one of \( \psi_V \) and \( \psi_U \) is relatively big, then the fact that the rescaled empirical gradient has lifting costs that are bounded by \( n \) shows that one of the non-cheap subspaces \( V \) or \( U \) contains a direction that is not too expensive.

4.1.5 Partition identification

This section is concerned about the difference between the partition associated to the affine scaling direction and the polarizing partition. We will show that if the residuals \( \hat{e}_B^U \) and \( \hat{e}_N^V \) are small (recall their definition in (28)), then both partitions coincide (Lemma 4.10). This in turn can be used to show that these two partitions coincide, if the norm of the projections of the empirical gradient \( \psi_V \) and \( \psi_U \) are small (Lemma 4.11).

**Lemma 4.10 (Identify partition).** Let \( z = (x, s) \in \mathcal{N}(\beta), \beta \in (0, 1/2], t \in [0, \tau] \) and define \( V := V_{z,B}(t) \) and \( U := U_{z,B}(t) \). If \( \|\hat{e}_B^U\| + \|\hat{e}_N^V\| \leq 1/16 \), then the associated partition (see Def. 4.4) \( \hat{B} = \{ i \in [n] : |\Delta x_i^a / x_i| < |\Delta s_i^a / s_i| \} \), \( \hat{N} = [n] \setminus \hat{B} \) agrees with the polarizing partition, that is \( B = \hat{B}, N = \hat{N} \).

**Proof.** Since \( \hat{x}^{-1} \Delta x^a, \hat{s}^{-1} \Delta s^a \in \hat{x}^{-1}W^a \) and \( \hat{s}^{-1} \Delta s^a, \hat{s}^{-1} \Delta s^a \in \hat{s}^{-1}W^a \) are in orthogonal spaces, we have that

\[
\|
\delta^s - \hat{x}^{-1} \Delta x^a\| + \|
\delta^s - \hat{s}^{-1} \Delta s^a\| = \|
\delta^s + \delta^s - (\hat{x}^{-1} \Delta x^a + \hat{s}^{-1} \Delta s^a)\| = (16) \|
\delta^s + \delta^s + \xi\|.
\]

By the triangle inequality

\[
\|
\delta^s + \delta^s + \xi\| \leq (\|
\delta^s + \xi\| + \|
\delta^s + \xi\|) + (\|
\delta^s + \xi\| + \|
\delta^s + \xi\|) = \|
\hat{e}_B^U\| + \|
\hat{e}_N^V\| + \|
\delta^s + \delta^s\| \leq \frac{1}{16} + 2\sqrt{n}\tau = \frac{1}{16} + \frac{1}{8} \beta \leq \frac{1}{8},
\]

where we used Lemma 4.9 to bound \( \|
\delta^s_B\| \) and \( \|
\delta^s_N\| \). Recall from Proposition 2.11 that \( |\hat{x}_i / x_i|, |\hat{s}_i / s_i| \leq \sqrt{1 + \beta} < 2 \) for each \( i \in [n] \). Therefore, for \( i \in N \), we have

\[
\left| \frac{\Delta s_i^a}{s_i} \right| \leq 2 \left| \frac{\Delta s_i^a}{s_i} \right| \leq 2 \left( |\delta^s_i| + \frac{1}{8} \right) \leq \frac{1}{4} + 2\sqrt{n}\tau < \frac{1}{2}, \tag{46}
\]

and for \( i \in B \), we have

\[
\left| \frac{\Delta x_i^a}{x_i} \right| \leq 2 \left| \frac{\Delta x_i^a}{x_i} \right| \leq 2 \left( |\delta^s_i| + \frac{1}{8} \right) \leq \frac{1}{4} + 2\sqrt{n}\tau < \frac{1}{2}. \tag{47}
\]

Since \( \Delta x^a / x + \Delta s^a / s = -1 \), we must have max \( \{ |\Delta x_i^a / x_i|, |\Delta s_i^a / s_i| \} \geq 1/2, \forall i \in [n] \). Combining with (46) and (47), we have that \( |\Delta x_i^a / x_i| < |\Delta s_i^a / s_i|, \forall i \in \hat{B} \), and \( |\Delta x_i^a / x_i| > |\Delta s_i^a / s_i|, \forall i \in \hat{N} \), as needed. \( \square \)
Lemma 4.11 (Partition identification via empirical gradient). Let \( t \in [0, \tau] \) and define \( V := V_{z,N}(t) \) and \( U := U_{z,B}(t) \). We have
\[
\|\eta_B^U\| + \|\eta_N^V\| \leq \psi^\nabla + \psi^\nabla + 4n^{1.5}\gamma^{-1}\frac{\mu_1}{\mu}.
\]
If
\[
\psi^\nabla + \psi^\nabla + 4n^{1.5}\gamma^{-1}\frac{\mu_1}{\mu} \leq \frac{1}{16},
\]
then the associated partition \((\hat{B}, \hat{N})\) coincides with the polarizing partition \((B, N)\), that is \( B = \hat{B}, N = \hat{N}\).

Proof. The second part is immediate from the first part and Lemma 4.10. We have \( \xi_N = \eta_N^V - \eta_N^V \). By definition, \( \delta^V = -\Pi_V^\nabla(\xi_N) \). Since \( V \) and \( \nabla \) are orthogonal complements, it follows that \( \psi_N^V = \Pi_V^\nabla(\xi_N) \).
Hence,
\[
\psi^\nabla \equiv \|\Pi_V^\nabla(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N)\| = \|\eta_N^V - \Pi_V^\nabla(\hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N)\|
\]
\[
\geq \|\eta_N^V\| - \|\hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N\| \geq \|\eta_N^V\| - 2n^{1.5}\gamma^{-1}\frac{\mu_1}{\mu}.
\]
Analogously \( \xi_B = \eta_B^U = -\delta^U \) and so
\[
\psi^\nabla \geq \|\eta_B^U\| - 2n^{1.5}\gamma^{-1}\frac{\mu_1}{\mu}.
\]

4.1.6 Large singular values vs empirical gradient
The following lemma helps us to upper bound either the smallest singular value or \( \psi \) on either the primal or dual side.

Lemma 4.12. For the polarizing partition \( B \cup N = [n] \) let \( Y, Y \subseteq \mathbb{R}^N \) be any subspaces such that \( Y \perp \nabla\), \( Y \perp \nabla = \pi_N(\hat{x}_N^{-1}W) \) and analogously let \( Z, Z \subseteq \mathbb{R}^B \) such that \( Z \perp \nabla\), \( Z \perp \nabla = \pi_B(\hat{x}_N^{-1}W^*) \). Let \( \sigma^\nabla \) be the smallest singular value of \( \ell_{z|\nabla} \) and analogously let \( \sigma^\nabla \) be the smallest singular value of \( \ell_{z|\nabla} \).
Then,
\[
\psi^\nabla \sigma^\nabla \leq 4n + (2\mu_1\gamma^{-1} + 1) n^{1.5} \|\ell_{z|\nabla}\|,
\]
\[
\psi^\nabla \sigma^\nabla \leq 4n + (2\mu_1\gamma^{-1} + 1) n^{1.5} \|\ell_{z|\nabla}\|.
\]

Proof. We only prove the first statement, since the second statement can be shown analogously. From Lemma 2.5, we have
\[
\|\ell_z(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N)\| \leq \|\xi_B - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)B\| \leq \|\xi_B\| + \|\hat{x}_N^{-1}x^{\text{cp}}(\mu_1)B\|
\]
\[
\leq \sqrt{n} + (1 + \beta) \|x^{\text{cp}}(\mu_1)B\| \leq 4n.
\]
Since \( \sigma^\nabla \) is the smallest singular value of \( \ell_{z|\nabla} \), we can bound
\[
\psi^\nabla \sigma^\nabla \leq \|\ell_z(\Pi_{\nabla}(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N))\|
\]
\[
= \|\ell_z(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N) - \ell_z(\Pi_{\nabla}(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N))\|
\]
\[
\leq \|\ell_z(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N)\| + \|\ell_z(\Pi_{\nabla}(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N))\|
\]
\[
\leq 4n + \|\ell_z\| \|\Pi_{\nabla}(\xi_N - \hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N)\|
\]
\[
\leq 4n + \|\ell_z\| \|\Pi_{\nabla}(\xi_N)\| + \|\Pi_{\nabla}(\hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N))\|
\]
\[
\leq 4n + \|\ell_z\| \|\Pi_{\nabla}(\xi_N)\|. \tag{51}
\]
Where the last inequality used \( \|\Pi_{\nabla}(\xi_N)\| \leq \|\xi_N\| \leq \|\xi\| = \sqrt{n} \). From Proposition 2.11 and Corollary 3.3 we have that
\[
\|\hat{x}_N^{-1}x^{\text{cp}}(\mu_1)N\| \leq \sqrt{1 + \beta} \|x^{\text{cp}}(\mu_1)N\| \leq (1 + \beta) \|x^{\text{cp}}(\mu_1)N\| \leq 2n^{1.5}\mu_1\gamma^{-1}.
\]
The claim follows by substituting this in (51). 

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4.2 Analysis

The main result of this section is Theorem 4.14, which shows that in case that the residuals are small enough, then within a single step we either come within polynomial factors of the end of the polarized segment, or the smallest singular value of the expensive subspace becomes polynomial. Theorem 4.1 is then a simple consequence and will eventually be proven on Page 26.

We introduce the special index $\zeta(z) \in [n]$ as

$$\zeta(z) := 1 + \max \left\{ i : \sigma(i) \leq \frac{\tau}{n^{1.5}} \right\}. \quad (53)$$

Then $\zeta(z) - 1$ corresponds to the number of non-zero singular values of $\ell_z$ (and $\ell_z^2$) that are not larger than $\frac{\tau}{n^{1.5}}$.

**Theorem 4.13.** Let $\text{CP}[\mu_1, \mu_0]$ be $\gamma$-polarized and let $z = (x, s) \in N(\beta)$ be an iterate with parameter $\Pi(z) \in (\mu_1, \mu_0)$, and let $z^+ = (x^+, s^+)$ be the next iterate of IPM with subspace LLS run with subspaces $V$ and $U$. Let $t \in [0, \tau]$ and let $Y := V_{z,N}(t)$ and $Z := U_{z,B}(t)$. If

$$\psi^T + 4n^{1.5}\gamma^{-1}\mu_1 - \frac{1}{16}, \quad (54)$$

then

$$\Pi(z^+) = O \left( n^{4.5}\gamma^{-1} \left( \mu_1 + \sigma(z)^{-1}\mu \right) \right).$$

**Proof.** By Lemma 4.11, $\|\delta_N^Z\| + \|\delta_N^Y\| \leq 1/16$, and the associated partition agrees with the polarizing partition: $\bar{B} = B$, $\bar{N} = N$.

From Proposition 2.11 we can bound

$$\|\delta_N^Y\| = \|\xi_N\| \leq \|\delta_N^Y\|_\infty + \|\xi\|_\infty \leq \frac{1}{16} + \sqrt{1 + \beta} \leq 2,$$

$$(55)$$

$$\|\delta_N^B\| = \|\xi_B - \xi_N\|_\infty \leq \|\delta_N^B\|_\infty + \|\xi\|_\infty \leq \frac{1}{16} + \sqrt{1 + \beta} \leq 2.$$

Recall from Lemma 4.9 that

$$\|\delta_N^B\| \leq \tau \sqrt{n}, \quad \text{and} \quad \|\delta_N^Y\| \leq \tau \sqrt{n}.$$

Consequently,

$$\left\| \frac{\Delta x^d \Delta s^d}{\mu} \right\|^2 = \left\| \delta_N^B \delta_N^Y \right\|^2 + \left\| \delta_N^B \delta_N^Y \right\|^2 \leq \left\| \delta_N^B \left\| \delta_N^B \right\|^2 + \left\| \delta_N^Y \right\|^2 \left\| \delta_N^Y \right\|^2 \leq 4 \left( \|\delta_N^B\|^2 + \|\delta_N^Y\|^2 \right) \leq 8n \tau^2. \quad (56)$$

Note that this bound gives us for $(\Delta x^d, \Delta s^d)$ the assumptions of Lemma 2.13 as $\sqrt{8n \tau} \leq \beta/4$.

Further, note that

$$\left\| \frac{(x + \Delta x^d)(s + \Delta s^d)}{\mu} \right\|^2 = \left\| \frac{(x_B + \Delta x_B^d)(s_B + \Delta s_B^d)}{\mu} \right\|^2 + \left\| \frac{(x_N + \Delta x_N^d)(s_N + \Delta s_N^d)}{\mu} \right\|^2$$

$$= \left\| \delta_N^B \delta_N^Y \right\|^2 + \left\| \delta_N^B \delta_N^Y \right\|^2 \leq \left\| \delta_N^B \right\|^2 + \left\| \delta_N^Y \right\|^2 \leq 4 \left( \|\delta_N^B\|^2 + \|\delta_N^Y\|^2 \right) \leq 4 \left( \|\delta_N^B\|^2 + \|\delta_N^Y\|^2 \right),$$

where the penultimate inequality used that

$$\|\delta_N^B\| \leq \|\xi_B\|_\infty + \|\delta_N^B\|_\infty \leq \sqrt{1 + \beta} + \sqrt{n \tau} \leq 2, \quad (58)$$

due to Proposition 2.11 and Lemma 4.9. Analogously, $\|\delta_N^Y\| \leq 2$. Together with Lemma 4.11, we get

$$\left\| \frac{(x + \Delta x^d)(s + \Delta s^d)}{\mu} \right\| \leq 2 \psi^T + 2 \psi^T + 8n^{1.5}\gamma^{-1}\mu_1 - \frac{1}{16} \mu.$$
Lemma 2.13 guarantees a step-length
\[
\alpha = 1 - \frac{4}{\beta} \frac{2}{(1 - \alpha)\mu} \frac{\| (x + \Delta x^\ell)(s + \Delta s^\ell) \|}{\mu} \tag{59}
\]
such that \( z^+ := (x + \alpha \Delta x^\ell, s + \alpha \Delta s^\ell) \in N'(\beta) \) and
\[
\overline{\mu}(z^+) \leq \left( 1 + \frac{3}{2} \frac{\beta}{\sqrt{n}} \right) (1 - \alpha)\mu \leq \frac{8\mu}{\beta} \frac{\| (x + \Delta x^\ell)(s + \Delta s^\ell) \|}{\mu} \leq \frac{16}{\beta} \left( 4\sqrt{n}\mu_1 \gamma^{-1} + \psi^\mu + \psi^\mu \right).
\tag{60}
\]

Using that \((\tilde{B}_z, \tilde{N}_z) = (B, N)\) and therefore \( \tilde{\ell}_z = \ell_z, \tilde{\ell}_z^\perp = \ell_z^\perp \), Lemma 4.8 implies \( \| \ell_z \|, \| \ell_z \| \leq \tau \). Let \( p := \dim(V), r := \dim(U) \) and \( q := \dim(\im(\ell_z)) = \dim(\im(\ell_z^\perp)) \). Note that by Lemma 4.5 we have
\[
\sigma(z)_{q-p+1} \geq \sigma^\nu \geq \frac{1}{\sqrt{n}} \frac{\| \ell_z(V_{|N|-p+1}) \|}{\| V_{|N|-p+1} \|} > \frac{1}{\sqrt{n} \tau} \frac{\tau}{n^{1.5}},
\tag{61}
\]
where the third inequality follows by definition of \( V \). This in particular implies that \( q - p + 1 \geq \zeta(z) \) and so \( \sigma(z)_{q-p+1} \geq \sigma(z)_{\zeta(z)} \). Further, we have by Lemma 4.5 that
\[
\sigma^\nu \geq \frac{1}{n^{1.5}} \sigma(z)_{q-p+1}.
\tag{62}
\]
Analogously, we have that
\[
\sigma(z)_{r-p+1} \geq \sigma(z)_{\zeta(z)} \quad \text{and} \quad \sigma^\nu \geq \frac{1}{n^{1.5}} \sigma(z)_{r-p+1}.
\tag{63}
\]
Together with Lemma 4.12 with \( Y := V, Z := U \), we have
\[
\psi^\nu \sigma(z)_{\zeta(z)} \leq \psi^\nu \sigma(z)_{q-p+1} \leq n^{1.5} \psi^\nu \sigma^\nu \leq n^{1.5} (4n + (2\mu_1^{-1} \gamma^{-1} + 1) n^{1.5} \tau) \leq n^{1.5} (4n + (2\gamma^{-1} + 1) n^{3} \sigma(z)_{\zeta(z)}),
\tag{64}
\]
where the last inequality follows by definition of the index \( \zeta(z) \). Analogously we get on the dual side
\[
\psi^\nu \sigma(z)_{\zeta(z)} \leq n^{1.5} (4n + (2\gamma^{-1} + 1) n^{3} \sigma(z)_{\zeta(z)}).
\tag{65}
\]
Plugging these into (60), we get
\[
\overline{\mu}(z^+) \leq \frac{256}{\beta \gamma} \left( 4\sqrt{n}\mu_1 + n^{4.5} \mu \sigma(z)_{\zeta(z)}^{-1} \right),
\tag{66}
\]
which proves the theorem.

With Theorem 4.13 at hand we can show Theorem 4.14, which shows that if the residuals are not small, then this is easy to handle and we can nonetheless make a similar conclusion to the one of Theorem 4.13. Theorem 4.14 shows that we have two alternatives after a single step in the algorithm.

Let \( \mu = \overline{\mu}(z) \) throughout.

**Theorem 4.14.** Let \( \text{CP}[^{\mu_1, \mu_0}] \) be \( \gamma \)-polarized with polarizing partition \( B \sqcup N = [n] \). Then, given an iterate \( z = (x, s) \in N'(\beta) \) with parameter \( \overline{\mu}(z) \in (\mu_1, \mu_0) \) in IPM with subspace LLS (Algorithm 1), the next iterate \( z^+ = (x^+, s^+) \in N'(\beta) \) satisfies one of the following properties:

(i) \( \sigma(z^+)_{\zeta(z)} = O(n^{6.5} \beta^{-1} \gamma^{-2}) \),

(ii) \( \overline{\mu}(z^+) \leq O(n^{4.5} \beta^{-3} \gamma^{-2} \mu_1) \).

**Proof of Theorem 4.14.** Algorithm 1 runs the subspace LLS with partition \( \tilde{B} \sqcup \tilde{N} = [n] \). Let \( V := V_{z,N}(\tau/n) \) and \( U = U_{z,B}(\tau/n) \) (defined in Definition 4.3). We distinguish two cases based on whether the terms \( \psi^\nu \) and \( \psi^\nu \) are small.

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Case I. $\psi^T + \psi^F + 4n^{1.5} \gamma^{-1} \frac{m}{\mu} \geq \frac{1}{16}$. 

Then, by Lemma 4.12 with $Y := V_{z,N}(\tau/n)$ and $Z := U_{z,B}(\tau/n)$, we have that

$$2 \cdot (4n + (2\mu_1 \mu^{-1} \gamma^{-1} + 1) n^{1.5} (\tau/n)) \sigma(z_{(z)}^{-1} \mu + 4n^{1.5} \gamma^{-1} \frac{\mu_1}{\mu}) \geq \frac{1}{16}. \quad (67)$$

In particular, for the next iterate $z^+$, we have $\overline{\rho}(z^+) \leq \mu = O(n^{1.5} \gamma^{-1} (\sigma_{(z)}^{-1} \mu + \mu_1))$.

Case II. $\psi^T + \psi^F + 4n^{1.5} \gamma^{-1} \frac{m}{\mu} \leq \frac{1}{16}$. 

By Theorem 4.13 we have that $\overline{\rho}(z^+) = O \left( n^{4.5} \beta^{-1} \gamma^{-1} \left( \mu_1 + \sigma_{(z)}^{-1} \mu \right) \right)$.

In either case, we have

$$\overline{\rho}(z^+) = O \left( n^{4.5} \beta^{-1} \gamma^{-1} \left( \mu_1 + \sigma_{(z)}^{-1} \mu \right) \right). \quad (68)$$

If $\mu_1 \geq \sigma_{(z)}^{-1} \mu$, then $\overline{\rho}(z^+) = O(n^{4.5} \beta^{-1} \gamma^{-1} \mu_1)$, so we are in case (ii). Otherwise, $\overline{\rho}(z^+) = O(n^{4.5} \beta^{-1} \gamma^{-1} \sigma_{(z)}^{-1} \mu)$. By Lemma 4.7 we have that $\sigma(z^+) \leq 4n^2 \gamma^{-2} \overline{\rho}(z^+) \mu^{-1} \sigma(z)$. In particular,

$$\sigma(z^+) \leq 4n^2 \gamma^{-2} \overline{\rho}(z^+) \mu^{-1} \sigma(z) = O(n^{6.5} \beta^{-1} \gamma^{-2}). \quad (69)$$

Hence, we are in case (i). This proves the theorem. \hfill \square

From Theorem 4.14 we can now derive the proof of the main theorem as follows.

**Proof of Theorem 4.1.** Consider any two consecutive iterates $z = (x, s)$ and $z^+ = (x^+, s^+)$ in IPM WITH SUBSPACE LLS. In case (ii) of Theorem 4.14, the end of the polarized segment will be reached within $O(\beta^{-1} \sqrt{n} \log(n/\gamma))$ iterations, since the affine scaling step itself leads to an $1 - \Omega(\beta/\sqrt{n})$ decrease in the normalized gap, and we always choose the better of the two steps.

In case (i) of Theorem 4.14, we have $\sigma(z^+) \leq O(n^{5.5} \beta^{-1} \gamma^{-1})$. Using Lemma 4.7, after $O(\beta^{-1} \sqrt{n} \log(n/\gamma))$ additional iterations, any subsequent iterate $z^{++} = (x^{++}, s^{++})$ satisfies $\sigma(z^{++}) \leq \sigma(z) < \tau/n^{1.5}$. By definition of $\zeta(\cdot)$ in (53), we have $\zeta(z^{++}) > \zeta(z)$ for all these iterates. Clearly, such an event may occur at most $n$ times. \hfill \square

5 The Max Central Path

In this section, we deal with the properties of the max central path that we introduced in Section 1. Given $g \geq 0$, we denote by

$$P_g := \{ x \in \mathbb{R}^n : Ax = b, x \geq 0, \langle c, x \rangle \leq v^* + g \},$$

$$D_g := \{ s \in \mathbb{R}^m : 3y A^T y + s = c, s \geq 0, \langle b, y \rangle \geq v^* - g \}$$

the feasible sets of the linear programs in (4). They correspond to the sets of the primal and dual feasible points $(x, s) \in P \times D$ with objective value within $g$ from the optimum $v^*$, respectively.

We recall that the duality gap of any pair $(x, (y, s))$ of primal-dual feasible points of (LP) fulfills $\langle c, x \rangle - \langle b, y \rangle = \langle x, s \rangle$. In particular, we have $\langle x, s^* \rangle = \langle c, x \rangle - v^*$. Thus, the two sets $P_g$ and $D_g$ are equivalently given by

$$P_g = \{ x \in P : \langle x, s^* \rangle \leq g \}, \quad D_g = \{ s \in D : \langle x^*, s \rangle \leq g \}.$$

These expressions are in fact independent of the choice of optimal solutions $(x^*, s^*)$. The following claim is immediate by our assumption that $P_{++}$ and $D_{++}$ are non-empty.

**Proposition 5.1.** For all $g \geq 0$, the sets $P_g$ and $D_g$ are bounded.

We denote by $\text{MCP} = \{ z^m(g) : g \geq 0 \}$ the whole max central path. The max central path point $z^m(g) = (x^m(g), s^m(g))$ is the entry-wise maximum of the set $P_g \times D_g$.

While the points of the max central path are not feasible in general, the following theorem shows that the max central path shares important similarities with the central path:

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**Theorem 5.2** (Centrality of the max central path). For all $g ≥ 0$, we have that

\[ g ≤ x^m(g)s^m(g) ≤ 2g \quad \forall i \in [n]. \]

**Proof.** We first prove the upper bound. For $i \in [n]$, let $x^{(i)} \in \arg \max \{ x_i : x \in \mathcal{P}_g \}$ and $s^{(i)} \in \arg \max \{ s_i : s \in \mathcal{D}_g \}$. Note that $x^{(i)}, s^{(i)}$ exist by Proposition 5.1. Then,

\[ x^m_i(g)s^m_i(g) = x^{(i)}_i s^{(i)}_i ≤ \langle x^{(i)}, s^{(i)} \rangle = \langle x^{(i)}, s^* \rangle + \langle x^*, s^{(i)} \rangle ≤ 2g, \]

where the last equality follows from Proposition 2.2. We now prove the lower bound. We assume $g > 0$, since the statement is trivial otherwise.

Note that the dual program of $\max \{ x_i : x \in W + d, x ≥ 0, \langle x, s^* \rangle ≤ g \}$ can be expressed as

\[ \min \{ \alpha g + \langle u, x^* \rangle : \alpha s^* + u ≥ e^i, u \in W^\perp, \alpha ≥ 0 \}, \]

using that $\langle u, x^* \rangle = \langle u, d \rangle$ since $d - x^* \in W, u \in W^\perp$. Similarly, the dual program of $\max \{ s_i : s \in W^\perp + c, s ≥ 0, \langle s, x^* \rangle ≤ g \}$ can be expressed as

\[ \min \{ \beta g + \langle v, s^* \rangle : \beta x^* + v ≥ e^i, v \in W, \beta ≥ 0 \}. \]

Let us pick optimal $(\alpha, u)$ and $(\beta, v)$ to these two programs. The product of the objective values is thus equal to $x^m_i(g)s^m_i(g)$; the proof is complete by showing a lower bound $g$.

We first claim that

\[ \langle u, x^* \rangle ≥ 0 \quad \text{and} \quad \langle v, s^* \rangle ≥ 0. \tag{70} \]

By symmetry, it suffices to prove the first claim. For a contradiction, assume $\langle u, x^* \rangle < 0$. Then, there exists an index $j \in [n]$ such that $x^*_j > 0$ and $u_j < 0$. By complementarity, $s^*_j = 0$. This contradicts $\alpha s^*_j + u_j ≥ c^j_i$.

Next, note that the constraints in the two programs imply

\[ 1 = \langle e^i, e^i \rangle ≤ (\alpha s^* + u, \beta x^* + v) = \alpha \langle v, s^* \rangle + \beta \langle u, x^* \rangle. \tag{71} \]

Now, the product of the objective values can be written as

\[ x^m_i(g)s^m_i(g) = (\alpha g + \langle u, x^* \rangle)(\beta g + \langle v, s^* \rangle) = \alpha \beta g^2 + g (\alpha \langle v, s^* \rangle + \beta \langle u, x^* \rangle) + \langle u, x^* \rangle \cdot \langle v, s^* \rangle ≥ g, \]

using inequalities (70) and (71). This concludes the proof.\hfill \Box

Given the above, we have the following straightforward relation between the max central path and central path.

**Lemma 5.3.** For $\mu > 0$, we have that

\[ z^m(n\mu) ≥ z^{cp}(\mu) ≥ \frac{z^m(n\mu)}{2n}. \]

**Proof.** Recall that $z^{cp}(\mu) = (x^{cp}(\mu), s^{cp}(\mu))$ with $\langle x^{cp}(\mu), s^{cp}(\mu) \rangle = (x^{cp}(\mu), s^*) + \langle x^*, s^{cp}(\mu) \rangle = n\mu$ using Proposition 2.2. Therefore, $x^{cp}(\mu) \in \mathcal{P}_{n\mu}$ and $s^{cp}(\mu) \in \mathcal{D}_{n\mu}$. By definition of the max central path, $z^{cp}(\mu) = (x^{cp}(\mu), s^{cp}(\mu)) ≤ (x^m(n\mu), s^m(n\mu)) = z^m(n\mu)$. For the second inequality, note that

\[ x^{cp}(\mu) = \frac{\mu}{s^{cp}(\mu)} ≥ \frac{\mu}{s^m(n\mu)} = \frac{\mu}{x^m(n\mu)s^m(n\mu)}x^m(n\mu) ≥ 5.2 \frac{\mu}{2n\mu}x^m(n\mu) = \frac{x^m(n\mu)}{2n}. \]

By a symmetric argument, $s^{cp}(\mu) ≥ s^m(n\mu)/2n$.\hfill \Box
5.1 The Shadow Vertex Simplex Rule

Given a pointed polyhedron $P \subseteq \mathbb{R}^n$ and two objectives $c^{(1)}, c^{(2)} \in \mathbb{R}^n$ such that $\max_{x \in P} \langle c^{(1)}, x \rangle < \infty$ and $\max_{x \in P} \langle c^{(2)}, x \rangle < \infty$, we recall that the shadow vertex simplex rule consists in pivoting over vertices of $P$ maximizing the objectives $(1-\lambda)c_1 + \lambda c_2$ as $\lambda$ goes from 0 to 1. More formally, a sequence of vertices $v^{(1)}, \ldots, v^{(k)} \in P$ is a $(c^{(1)}, c^{(2)})$-shadow vertex path on $P$ if

- $[v^{(i)}, v^{(i+1)}]$ is an edge of $P$, $\forall i \in [k-1]$,
- $\langle c^{(2)}, v^{(i)} \rangle < \langle c^{(2)}, v^{(i+1)} \rangle$, $\forall i \in [k-1]$, and
- there exists $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{k-1} < \lambda_k = 1$ such that $\forall i \in [k]$, $\langle v^{(i)}, (1-\alpha)c^{(1)} + \alpha c^{(2)} \rangle = \max_{x \in P} \langle x, (1-\alpha)c^{(1)} + \alpha c^{(2)} \rangle$ for all $\alpha \in [\lambda_{i-1}, \lambda_i]$.

To analyze shadow vertex paths further, we define the two-dimensional projection

$$P[c^{(1)}, c^{(2)}] := \{ \{ \langle c^{(1)}, x \rangle, \langle c^{(2)}, x \rangle \} : x \in P \} = (c^{(1)}, c^{(2)})^T : P.$$ 

Under non-degeneracy assumptions (which are easily satisfied by infinitesimally perturbing the constraints), there is a unique shadow vertex path with respect to $c^{(1)}$ and $c^{(2)}$. Non-degeneracy implies that $\lambda_0 < \lambda_1 < \cdots < \lambda_k$ above, and thus the maximizing objective moves strictly closer to $c^{(2)}$ after each simplex step. In this case, the vertices of the shadow vertex simplex path project under the map $x \mapsto (\langle c^{(1)}, x \rangle, \langle c^{(2)}, x \rangle)$ precisely to the subset of the vertices of the 2-dimensional projection $P[c^{(1)}, c^{(2)}]$ maximizing some open interval of objectives $(1-\lambda)c_1 + \lambda c_2$, $\lambda \in [0, 1]$, where $c_1$ and $c_2$ stand for the unit vectors of $\mathbb{R}^2$. As $P[c^{(1)}, c^{(2)}]$ is the shadow (projection) of $P$ onto the plane spanned by $(c^{(1)}, c^{(2)})$, this justifies the name ‘shadow vertex simplex rule’.

In the general setting, the vertices of $P[c^{(1)}, c^{(2)}]$ maximizing an open interval of objectives in $(1-\lambda)c_1 + \lambda c_2$, $\lambda \in [0, 1]$ are precisely the projections of vertices $v^{(i)}, i \in [k]$, on the shadow path such that $\lambda_{i-1} < \lambda_i$. The degenerate vertices $v^{(i)}, i \in [k]$, such that $\lambda_{i-1} = \lambda_i$, will in fact in general project into the interior of edges of $P[c^{(1)}, c^{(2)}]$.

We define $S_P(c^{(1)}, c^{(2)})$ as the number of vertices of $P[c^{(1)}, c^{(2)}]$ maximizing an open interval of objectives in $(1-\lambda)c_1 + \lambda c_2$, $\lambda \in [0, 1]$. By the preceding observations, we have that $S_P(c^{(1)}, c^{(2)})$ is a lower bound on the number of vertices of any $(c^{(1)}, c^{(2)})$-shadow vertex path.

In the above, we restricted both starting and ending objectives $c^{(1)}, c^{(2)}$ to have finite objective value on $P$. It will be useful in the sequel to extend to the case where $c^{(2)}$ might be unbounded. In this case, we define the shadow vertex path as above, with the only modification being that we let $\lambda_k := \max \{ \lambda \in [0, 1] : \max_{x \in P} \langle x, (1-\lambda)c^{(1)} + \lambda c^{(2)} \rangle < \infty \}$, that is, the simplex path stops just before reaching an unbounded ray for $c^{(2)}$. In this setting, note that $S_P(c^{(1)}, c^{(2)})$ is still well-defined and continues to be a lower bound on the number of vertices of any $(c^{(1)}, c^{(2)})$ shadow vertex path.

The following lemma now gives the main relation between shadow vertex paths and the number of linear segments of the max central path. Precisely, a segment

$$\text{MCP}[g_1, g_0] := \{ z^m(g) : g_0 \geq g \geq g_1 \}$$

is linear if $z^m((1-\alpha)g_0 + \alpha g_1) = (1-\alpha)z^m(g_0) + \alpha z^m(g_1), \forall \alpha \in [0, 1]$.

**Lemma 5.4 (Piecewise Linearity of MCP).**

(i) $\forall i \in [n], g \mapsto x^m_i(g)$ is piecewise linear non-decreasing with $S_P(-s^*, e^{(i)})$ pieces.

(ii) $\forall i \in [n], g \mapsto s^m_i(g)$ is piecewise linear non-decreasing with $S_D(-x^*, e^{(i)})$ pieces.

(iii) $g \mapsto z^m(g)$ is piecewise linear and entry-wise non-decreasing with at most

$$\min \left\{ \sum_{i=1}^n S_P(-s^*, e^{i}) + S_D(-x^*, e^{i}), \mathcal{V}_P + \mathcal{V}_D \right\}$$

pieces, where $\mathcal{V}_P$ and $\mathcal{V}_D$ denote the number of vertices of $P$ and $D$, respectively.
Proof. Proof of (i) For \( i \in [n] \), let \( Q_i = P[\mu^*, e^i] \). We note that \( x^m_i(g) = \max\{v_2 : (v_1, v_2) \in Q_i, v_1 \leq g\} \).
In particular, the map \( x^m_i \) is a non-decreasing concave function of \( g \). Again by definition, \( S_P(-s^*, e^i) \) equals the number of vertices of \( Q_i \) maximizing an open interval of objectives in \( O := \{ -(1 - \lambda)e^1 + \lambda e^2 : \lambda \in [0, 1] \} \subset \mathbb{R}^2 \).

Define \( \bar{u}_i(g) := \sup\{v_2 : (g, v_2) \in Q_i\} \), which is defined to equal \( -\infty \) if \( \{(g, v_2) \in Q_i\} = \emptyset \). By Proposition 5.1, note that \( \bar{u}_i(g) < \infty \) for all \( g \geq 0 \). By convexity of \( Q_i \), \( \bar{u}_i \) is concave function on \( \mathbb{R}_+ \). Let \( h = \sup\{\bar{u}_i(g) : g \geq 0\} \).

Assume \( h = \infty \). By concavity, \( \bar{u}_i \) must be a strictly increasing function on \( \mathbb{R}_+ \). In particular, \( \bar{u}_i(g) = x^m_i(g) \). This gives, we see that the linear pieces of \( x^m_i(g) \) are in one to one correspondence with the edges of \( Q_i \) on the upper convex hull whose projection onto the \( e_1 \)-axis have positive length (i.e., excluding the potential edge \( \{(0, v_2) \in Q_i\} \)). Since \( Q_i \subseteq \mathbb{R}_+^2 \), every such edge can be uniquely associated with its left endpoint (which is always a vertex of \( Q_i \)). It is now easy to check geometrically that the set of such endpoints exactly corresponds to the set of vertices that are maximizers of the objectives in an open interval of \( O \).

Assume \( h < \infty \). Let \( g_h := \min\{g \geq 0 : x^m_i(g) = h\} \). It is direct to see that \( x^m_i(g) = \bar{u}_i(g) \) if \( g \leq g_h \) and that \( x^m_i(g) = h \) for \( g \geq g_h \). Furthermore, \( x^m_i(g) \) is strictly increasing on \( [0, g_h] \). From this, it is easy to see geometrically that the number of linear pieces of \( x^m_i \) is one plus the number of edges of \( Q_i \) on the upper convex hull lying in the band \( \{ (v_1, v_2) : 0 \leq v_1 \leq g \} \), where the extra linear segment corresponds to constant segment between \( g_h \) and \( \infty \). As in the previous case, these linear segments can be uniquely identified with their left endpoints, which correspond to vertices of \( Q_i \). Furthermore, it is easy to check that these correspond to vertices of \( Q_i \) maximizing an open interval of objectives in \( O \).

Proof of (ii). Same as (i), swapping the role of \( D \) and \( x^* \) for \( P \) and \( s^* \).

Proof of (iii). Let \( 0 = g_1^p < g_2^p < \cdots < g_k^p \), denote the parameters corresponding to breakpoints of linear segments of \( x^m_i \), and similarly let \( 0 = g_1^d < g_2^d < \cdots < g_k^d \) correspond to breakpoints for \( s^m_i \). Finally, let \( 0 = g_1 \leq g_2 < \cdots < g_T \) denote an ordering of the merged sequence of breakpoints (suppressing duplicates) of each \( x^m_i, s^m_i, \forall i \in [n] \). Since \( x^m_i, s^m_i, \forall i \in [n] \), is linear on each interval \( [g_i, g_{i+1}] \), \( i \in [T - 1] \), and on the interval \( [g_T, \infty) \), we get that \( z^m_i \) is also linear on these intervals. Furthermore, they are exactly the breakpoints of the linear segments of \( z^m_i \), since \( z^m_i \) is linear on an interval iff \( x^m_i, s^m_i, i \in [n] \), are linear on the interval. By associating each linear segment with its left endpoint, we see that the number of linear segments of \( m \) is \( T \leq \sum_{i=1}^n k^i_p + k^i_d = \sum_{i=1}^n S_P(-s^*, e^i) + S_D(-x^*, e^i) \). Furthermore, note that for each \( g_j, j \in [T] \), there exists \( i \in [n] \) such that either \( (g_j, x^m_i(g_j)) \) is a vertex of \( P[s^*, e^i] \) or \( (g_j, s^m_i(g_j)) \) is a vertex of \( P[x^*, e^i] \). In particular, there exists either a vertex \( x_j \) of \( P \) such that \( (s^*, x_j) = g_j \) or a vertex \( y_j \) of \( D \) such that \( (x^*, y_j) = g_j \). This association between breakpoints and vertices is clearly injective (since the \( g_j \) are all distinct). Therefore, we also get the bound \( T \leq V_P + V_D \) as needed.

From the above discussion, note that Lemma 5.4 implies that the number of linear pieces of the max-central path is at most the number of vertices on \( 2n \) shadow vertex paths on \( P \) and \( D \).

5.2 Direct Proof of Polarization along Max Central Path Segments

In this subsection, we give a simple proof that the central path can be decomposed into \( T \) polarized segments, where \( T \) is the number of linear segments of the max central path. The proof avoids using the wide neighborhood and instead directly compares the central path with the max central path. By virtue of being more direct, it also achieves a better polarization parameter. Since the polarization parameter appears under a logarithm, this improvement does not change the asymptotics of our algorithm.

Lemma 5.5. Let \( \text{MCP}[\mu_1, \mu_0] \) be a linear segment of the max-central path with \( \mu_0 \geq \mu_1 \geq 0 \). Then \( \text{CP}[\mu_1/n, \mu_0/n] \) is \( 1/(16n) \)-polarized.

Proof. Let us fix \( i \in [n] \), and let \( u = \frac{x^m_i(\mu_1)}{x^m_i(\mu_0)} \) and \( v = \frac{s^m_i(\mu_1)}{s^m_i(\mu_0)} \). Then for \( \alpha \in [0, 1] \),
\[
(1 - \alpha + \alpha u) (1 - \alpha + \alpha v) = ((1 - \alpha)x^m_i(\mu_0) + \alpha x^m_i(\mu_1))(1 - \alpha)s^m_i(\mu_0) + \alpha s^m_i(\mu_1))
\]
\[
\frac{(1 - \alpha)x^m_i(\mu_0) + \alpha x^m_i(\mu_1)}{(1 - \alpha)s^m_i(\mu_0) + \alpha s^m_i(\mu_1)} \geq \frac{1}{2(1 - \alpha) + \alpha} \quad \text{(by Theorem 5.2)}
\]
Therefore Lemma 3.6 yields \( u + v \geq \frac{1}{4} \). Let \( B := \left\{ i \in [n] : \frac{z^m_i(\mu)}{x^i_1(\mu)} \geq \frac{z^m_i(\mu_0)}{x^i_1(\mu_0)} \right\} \) and \( N := [n] \setminus B \). Thus, \( \frac{z^m_i(\mu)}{x^i_1(\mu)} \geq 1/8, \forall i \in B \), and \( \frac{z^m_i(\mu_0)}{x^i_1(\mu_0)} \geq 1/8, \forall i \in N \).

We now show that \( \text{CP}[\mu_1/n, \mu_0/n] \) is \( \gamma = \frac{1}{16n} \)-polarized with respect to the partition \( B, N \) as above. For \( \mu \in [\mu_1/n, \mu_0/n] \) and \( i \in B \), we have that

\[
\frac{x^i_1(\mu)}{x^i_1(\mu_0)} \geq \frac{2nx^m_i(\mu_0)}{2nx^m_i(\mu)} \geq \frac{1}{16n},
\]

as needed. The inequality \( \frac{x^i_1(\mu)}{x^i_1(\mu_0)} \geq \frac{1}{16n}, \forall i \in N, \mu \in [\mu_0/n, \mu_1/n] \), follows by a symmetric argument. \( \square \)

### 5.3 Converting the Max Central Path into a Wide Neighborhood Path

While the max central path does not correspond to a feasible path inside \( \mathcal{P} \times \mathcal{D} \), we now show that it is in fact close to a piecewise linear path that lives inside the wide neighborhood of the central path having the same number of breakpoints.

![Figure 2: The path \( \bar{\Gamma}^d(g) \) as defined in the proof of Theorem 5.6 for the system max \( -y_1-y_2 \) s.t. \( A^\top y + s = c, s \geq 0 \), where and \( A = \begin{bmatrix} 1 & 2 & 2.5 & -5.5 \\ -5 & -1 & 4 & 2 \end{bmatrix}, c = \begin{bmatrix} 0 & 9 & 27 & 0 \end{bmatrix}^\top \). Dashed lines correspond to level sets at breakpoints.](image)

**Theorem 5.6.** There exists a piecewise linear curve \( \Gamma : \mathbb{R}_+ \rightarrow \mathcal{N}^{-\infty}(1 - \frac{1}{2n}) \) with at most as many linear segments as \( g \mapsto z^m(g) \) satisfying \( \mu(\Gamma(s)) = s, \forall s \geq 0 \).

**Proof.** As in the proof of Lemma 5.4 part (3), let \( 0 = g_1 < \cdots < g_T \) denote the breakpoints for \( g \mapsto z^m(g) \).

From here, pick \( x^{i,j} \in \mathcal{P}_{g_j} \), \( s^{i,j} \in \mathcal{D}_{g_j} \), \( i \in [n], j \in [T] \), such that \( x^{i,j}_1 = x^i_1(g_j), s^{i,j}_1 = s^i_1(g_j) \), and such that \( \langle x^{i,1}, s^1 \rangle \leq \cdots \leq \langle x^{i,T}, s^T \rangle \) and \( \langle x^*, s^{1,1} \rangle \leq \cdots \leq \langle x^*, s^{1,T} \rangle \). Further, for all \( i \in [n] \), choose \( r^{i,p} \) and \( r^{i,d} \) in the recession cone of \( \mathcal{P} \) and \( \mathcal{D} \) respectively, such that \( \langle r^{i,p}, s^1 \rangle \in \{0,1\}, \langle x^*, r^{i,d} \rangle \in \{0,1\} \) and \( \forall t \geq 0, x^{i,p}(g_T + t) = x^{i,T} + tr^{i,p}, s^{i,d}(g_T + t) = s^{i,T} + tr^{i,d} \). Define \( \Gamma^i(p)(g) = g + g_j x^{i,j} + g_j x^{i,j} + g_j x^{i,j+1} \). If \( g_j < g \leq g_{j+1} \), \( j \in [T-1] \) and \( \Gamma^i(p)(g) = x^{i,T} + (g - g_T)r^{i,p} \) if \( g > g_T \). Define the dual counterpart \( \Gamma^i,d \) similarly. By construction, \( \forall i \in [n] \), we have that \( \Gamma^i,p, \Gamma^i,d \) are piecewise linear with breakpoints \( 0 < g_1 < \cdots < g_T, \) and \( \forall g \geq 0, \Gamma^i,p(g) \in \mathcal{P}_g, \Gamma^i,p(g), = x^{i,p}(g), \Gamma^i,d(g) \in \mathcal{D}_g, \Gamma^i,d(g) = s^m_i(g) \).

From here, define

\[
\bar{\Gamma}(g) = (\bar{\Gamma}^p(g), \bar{\Gamma}^d(g)) := \left( \frac{1}{n} \sum_{i=1}^n \Gamma^i,p(g), \frac{1}{n} \sum_{i=1}^n \Gamma^i,d(g) \right).
\]
Clearly $\Gamma(g) \in \mathcal{P}_g \times \mathcal{D}_g, \forall g \geq 0$. Therefore, $\forall i \in [n]$, by Proposition 2.2 we have

$$\overline{p}(\Gamma(g)) = \frac{1}{n} \langle \Gamma^p(g), \Gamma^d(g) \rangle = \frac{1}{n} \left( \langle \Gamma^p(g), s^* \rangle + \langle x^*, \Gamma^d(g) \rangle \right) \leq \frac{2g}{n}.$$  

Furthermore, for $i \in [n]$, we have

$$\Gamma^p(g), \Gamma^d(g) = \left( \frac{1}{n} \sum_{i=1}^{n} \Gamma^{i,p}(g_i) \right) \left( \frac{1}{n} \sum_{i=1}^{n} \Gamma^{i,d}(g_i) \right) \geq \frac{1}{n^2} \sum_{i=1}^{n} \Gamma^{i,p}(g) \Gamma^{i,d}(g) = \frac{g}{n^2} \geq \frac{\overline{p}(\Gamma(g))}{2n}.$$  

In particular, we have that $\Gamma(g) \in \mathcal{N}^{-\infty}(1 - \frac{1}{n})$.

Note that by construction $\Gamma$ has at most $T$ linear segments, where $T$ is the number of linear segments of the max central path. To construct $\Gamma$, we will simply reparametrize $\bar{\Gamma}$ with respect to $\Gamma^1$. Furthermore, for $i \in [n]$, $\Gamma^1_i$ is increasing in $g$. Since $\Gamma^1_i(g) \geq g$ at least one of $x^m_i$ or $s^m_i$ must be unbounded. Assume without loss of generality that $x^m_i$ is unbounded. By concavity of $x^m_i$, $x^m_i$ is strictly increasing. In particular $\Gamma^1_i(g) \in \mathcal{P}_g$ and $\forall g \geq 0$, $\langle x^*, \Gamma^1_i(g) \rangle = x^m_i(g)$ implies that $\langle \Gamma^1_i(g), s^* \rangle = g$, which is clearly increasing in $g$. This proves the lemma. 

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A Missing Proofs in Section 2

Proof of Proposition 2.2. Since $x - x' \in W$ and $s - s' \in W^\perp$, we have that

$$0 = \langle x - x', s - s' \rangle \Leftrightarrow \langle x, s \rangle + \langle x', s' \rangle = \langle x, s' \rangle + \langle x', s \rangle.$$
Proof of Proposition 2.3. Using that \( \sum_{i=1}^{k} \lambda_i = 1 \) and the orthogonality of \( x^{(i)} - d \in W \) and \( s^{(i)} - d \in W^\perp \) for all \( i \in [k] \) we first get
\[
\left\langle \sum_{i=1}^{k} \lambda_i x^{(i)}, \sum_{j=1}^{k} \lambda_j s^{(i)} \right\rangle = \left\langle \sum_{i=1}^{k} \lambda_i (x^{(i)} - d) + d, \sum_{j=1}^{k} \lambda_j (s^{(i)} - c) + c \right\rangle = (c, d) + \sum_{i=1}^{k} \lambda_i \left( \langle x^{(i)} - d, c \rangle + \langle d, s^{(i)} - c \rangle \right) = \sum_{i=1}^{k} \lambda_i \left( \langle d, c \rangle + \langle x^{(i)} - d, c \rangle + \langle d, s^{(i)} - c \rangle \right) = \sum_{i=1}^{k} \lambda_i \left( \langle x^{(i)}, s^{(i)} \rangle \right).
\]
Division by \( n \) yields the respective normalized duality gap.

Proof of Proposition 2.7. By definition of \( \mathcal{N}(\beta) \) we have for all \( i \in [n] \) that \( |\frac{z_i}{\mu} - 1| \leq \frac{\|x\|}{\mu} - 1 \leq \beta \) and so \((1 - \beta)\mu \leq |z_i| \leq (1 + \beta)\mu \).

Proof of Lemma 2.15. Note first that by construction, the solution set of \( L^W(x) \) is non-empty. Furthermore, the minimal norm solution exists and is unique by strict convexity of the square Euclidean norm. Thus, \( w = L^W(x) \) is well-defined.

We now show that \( w = L^W_I(x) \) is a solution to the linear system. Clearly, \( w \in W \) and \( w_I = \pi_{\pi_I(W)}(x) = x - \pi_{\pi_I(W)}(x) \in \pi_I(W)^\perp + x \). It remains to show that \( w_J \in \pi_I(W^\perp) = \pi_I((W \cap \mathbb{R}^n)^\perp) \). Take any \( z \in \mathbb{R}^n \cap W \) with \( \|z\|^2 = 1 \). It suffices to show that \( \langle w, z \rangle = \langle w_J, z_J \rangle = 0 \). Noting that \( w + \beta z \in W \) and \( (w + \beta z)_I = w_I = \pi_{\pi_I(W)}(x) \), by the definition of \( w = L^W_I(x) \), we must have that \( \|w\|^2 + \min_{\beta \in \mathbb{R}} \|w + \beta z\|^2 = \|w\|^2 \). In particular, \( \langle z, w \rangle = 0 \), as needed.

Now take any \( w \in W \), \( w_J \in \pi_J(W^\perp) \), \( w_I \in \pi_I(W)^\perp + x \). We wish to show that \( w = L^W_I(x) \).

Firstly, by the above argument, this system always has a solution, namely \( L^W_I(x) \). Secondly, note that \( z \in W \) implies that \( w_I \in (\pi_I(W)^\perp + x) \cap \pi_I(W) \). By the uniqueness of the orthogonal decomposition, this implies that \( z_I = \pi_{\pi_I(W)}(x) \). Thus, \( z \) is in the solution set of the \( L^W_I(x) \) program. It remains to show that \( z \) has minimum norm. Letting \( w' \in W \setminus \{w\} \) satisfy \( w'_I = \pi_{\pi_I(W)}(x) \), we must show that \( \|w'\|^2 > \|w\|^2 \). Noting that \( z = w' - w \in W \cap \mathbb{R}^n \setminus \{0\} \) and recalling that \( w \in (W \cap \mathbb{R}^n)^\perp \), we have that \( \|w + z\|^2 = \|w\|^2 + 2 \langle z, w \rangle + \|z\|^2 = \|w\|^2 + \|z\|^2 > \|w\|^2 \). Thus, \( w = L^W_I(x) \) as needed.

Given the above, we see that the solution to the linear system always exists and is unique (since \( L^W_I(x) \) is well-defined). Secondly, since \( L^W_I(x) \) is the unique solution to a linear system of equations depending linearly on \( x \), we have that \( L^W_I(x) \) is a linear map.

Proof of Lemma 2.16. To prove the statement, it suffices to show that for all \( x \in \mathbb{R}^I \) that
\[
L^W_I(x) = (\Pi_{\pi_I(W)}(x), \ell^W_J(x)) = (\Pi_{\pi_I(W)}(x), -\ell^W_J^*(x)).
\]
Letting \( z = (\Pi_{\pi_I(W)}(x), -\ell^W_J^*(x)) \), by Lemma 2.15, it suffices to show that
1. \( z \in W \),
2. \( z_I \in \pi_I(W)^\perp + x \),
3. \( z_J \in \pi_J(W^\perp) \).

Property (2) follows directly from \( z_I = \Pi_{\pi_I(W)}(x) = x - \Pi_{\pi_I(W)^\perp}(x) \). To show (1), it suffices to show that \( \langle v, z \rangle = 0 \), \( \forall v \in W^\perp \). For \( v \in W^\perp \), we see that
\[
\langle z, v \rangle = \langle \Pi_{\pi_I(W)}(x), v_I \rangle - \ell^W_J^*(x), v_J \rangle = \langle \Pi_{\pi_I(W)}(x), v_I \rangle - \langle v, \ell^W_J^*(v_J) \rangle = \langle \Pi_{\pi_I(W)}(x), v_I \rangle - \langle v, \ell^W_J^*(v_J) \rangle = \langle v, v_I - \ell^W_J(v_J) \rangle = \langle v^W(x), (v_I - \ell^W_J^*(v_J), 0_J) \rangle = \langle L^W_I(x), v - L^W_J(v_J) \rangle \quad (\text{since } v_J \in \pi_J(W^\perp)) = 0.
\]
where the last equality follows since $L^w(x) \in W$ and $v - L^w_j(v_j) \in W^\perp$.

To show (3), we must show that $(z, w) = 0$, $\forall w \in W \cap \mathbb{R}_j^n$. For $w \in W \cap \mathbb{R}_j^n$, we first claim that $L^w_j(w_j) = 0$. By Lemma 2.15, this follows since 0 a solution to the linear system

$$x_j \in \pi_j(W^\perp) = \pi_j(W \cap \mathbb{R}_j^n) = w_j, \quad x_j \in W^\perp, \quad x_j \in W_j. \quad (72)$$

From the claim, we must have $\ell^w_j(w_j) = (L^w_j(w_j))_I = 0_I$. Therefore,

$$\langle z, w \rangle = - \langle \ell^w_j(w_j), w_j \rangle = - \langle x, \ell^w_j(w_j) \rangle = - \langle x, 0_I \rangle = 0.$$

Thus, $\ell^w_j = -\ell^w_j^{\ast}$ as needed. The equality of singular values follows from the fact that adjoints always have the same non-zero singular values. □

B Missing Proofs in Section 3

Proof of Lemma 3.7. By Proposition 2.2, we have

$$\|x(\mu_1)/x(\mu_0)\|_1 + \|s(\mu_1)/s(\mu_0)\|_1 = \frac{1}{\mu_0} (\langle x(\mu_1), s(\mu_0) \rangle + \langle x(\mu_0), s(\mu_1) \rangle)$$

$$= \frac{1}{\mu_0} (\langle x(\mu_0), s(\mu_0) \rangle + \langle x(\mu_1), s(\mu_1) \rangle) = (1 + \frac{\mu_1}{\mu_0}) n. \quad (73)$$

Let $B \cup N = [n]$ be the polarization partition. Now assume that $\gamma \geq 1$. We will show then that $\gamma = 1$ and that $CP[\mu_1, \mu_0]$ is linear.

Then for $i$, we see that

$$\frac{x_i(\mu_1)}{x_i(\mu_0)} + \frac{s_i(\mu_1)}{s_i(\mu_0)} = \frac{x_i(\mu_1) + \mu_1 x_i(\mu_0)}{\mu_0 x_i(\mu_2)} \geq \min \left( \beta + \frac{1}{\beta} \mu_0 \right) = \gamma + \frac{1}{\gamma} \mu_0. \quad (74)$$

where the last inequality uses $\gamma \geq 1$, $\mu_1/\mu_0 \leq 1$, and that the function $\beta \to \beta + \frac{1}{\beta} \mu_0$ is increasing for $\beta \geq \sqrt{\mu_0}$.

Swapping the role of $x$ and $s$, we also get $\frac{x_i(\mu_1)}{x_i(\mu_0)} + \frac{s_i(\mu_1)}{s_i(\mu_0)} \geq \gamma + \frac{\mu_0}{\mu_1} \gamma$, $\forall i \in N$. Combining with (73), we get that

$$n \left( \gamma + \frac{\mu_0}{\mu_1} \gamma \right) \leq \|x(\mu_1)/x(\mu_0)\|_1 + \|s(\mu_1)/s(\mu_0)\|_1 = n \left( 1 + \frac{\mu_0}{\mu_1} \right).$$

For $\gamma \geq 1$, this inequality can only hold if $\gamma = 1$. If $\gamma = 1$, then all the inequalities in Equation (74) and their analogs for $i \in N$ must hold at equality. In particular, we get that $x_i(\mu_1) = x_i(\mu_0)$, $i \in B$, and $s_i(\mu_1) = s_i(\mu_0)$, $i \in N$.

We now use this to show that $CP[\mu_1, \mu_0]$ is linear. Define

$$\mu_\alpha := (1 - \alpha)\mu_0 + \alpha\mu_1 \quad \text{and} \quad z^{(\alpha)} = (x^{(\alpha)}, s^{(\alpha)}) := (1 - \alpha)z(\mu_0) + \alpha z(\mu_1) \quad \text{for} \quad \alpha \in [0, 1].$$

To prove linearity, it suffices to show that $z(\mu_\alpha) = z_\alpha$. To see this, note that for $i \in B$, we have $x_i^{(\alpha)} = (1 - \alpha)x_i(\mu_0) + \alpha x_i(\mu_1) = (1 - \alpha)x_i(\mu_0) + \alpha x_i(\mu_0) = x_i(\mu_0)$ and $s_i^{(\alpha)} = s_i(\mu_0)$, $i \in N$. That implies

$$x_i^{(\alpha)} s_i^{(\alpha)} = x_i(\mu_0) \left( (1 - \alpha) s_i(\mu_0) + \alpha \frac{\mu_1}{\mu_0} s_i(\mu_0) \right) = x_i(\mu_0) s_i(\mu_0) \left( (1 - \alpha) + \alpha \frac{\mu_1}{\mu_0} \right) = \mu_\alpha.$$
Thus, by Lemma 3.6 applied to \( \frac{x(1)}{x(0)} \) and \( \gamma = 1 \), using that \( \frac{x(1)}{x(0)} \) and \( \frac{s(1)}{s(0)} = \frac{\mu_1}{\mu_0} \), we get that

\[
\frac{x(1)}{x(0)} + \frac{s(1)}{s(0)} \geq \left(1 + \sqrt{\frac{\mu_1}{\mu_0}}\right)^2 - 2\sqrt{\frac{\mu_1}{\mu_0}} = 1 + \frac{\mu_1}{\mu_0}.
\]

Since this holds for all \( i \in [n] \), by the same argument as above, the inequality above must hold at equality for all \( i \in [n] \). In particular, for each \( i \in [n] \), we must have either (1) \( \frac{x(1)}{x(0)} = 1 \) and \( \frac{s(1)}{s(0)} = \frac{\mu_1}{\mu_0} \) or (2) \( \frac{x(1)}{x(0)} = \frac{\mu_1}{\mu_0} \) and \( \frac{s(1)}{s(0)} = 1 \) (note that \( \frac{\mu_1}{\mu_0} < 1 \) implies that these cases are disjoint). Let \( B \subseteq [n] \) denote the indices satisfying case (1) and \( N = [n] \setminus B \) be the indices satisfying case (2). It is now direct to verify \( B, N \) yield a 1-polarized partition for \( CP[\mu_1, \mu_0] \), as needed.

\[\square\]

C Missing Proofs in Section 5

Proof of Proposition 5.1. We restrict to the proof of the boundedness of \( P_g \), since the proof is analogous for \( D_g \). Let \( s^0 \in D_{++} \) be a strictly feasible point of the dual, and \( x \in P_g \). By Proposition 2.2, we have

\[
\langle x, s^* \rangle + \langle x^*, s^0 \rangle = \langle x, s^0 \rangle + \langle x^*, s^0 \rangle.
\]

Since \( \langle x^*, s^0 \rangle = 0 \), we deduce that \( \langle x, s^0 \rangle \leq g + \langle x^*, s^0 \rangle \). As \( s^0 > 0 \), this implies that \( x_i \leq (g + \langle x^*, s^0 \rangle)/s_i^0 \) for all \( i \in [n] \).

\[\square\]