Non-linear resonance in nearly geodesic motion in low-mass X-ray binaries

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(Received 2002 December 22; accepted 2003 January 15)

Abstract

We have explored the ideas that parametric resonance affects nearly geodesic motion around a black hole or a neutron star, and that it may be relevant to the high frequency (twin) quasi-periodic oscillations occurring in some low-mass X-ray binaries. We have assumed the particles or fluid elements of an accretion disc to be subject to an isotropic perturbation of a hypothetical but rather general form. We find that the parametric resonance is indeed excited close to the radius where epicyclic frequencies of radial and meridional oscillations are in a 2 : 3 ratio. The location and frequencies of the highest amplitude excitation vary with the strength of the perturbation. These results agree with actual frequency ratios of twin kHz QPOs that have been reported in some black hole candidates, and they may be consistent also with correlation of the twin peaks in Sco X-1.

Key words: Accretion – General relativity – X-rays: binaries – X-rays: individual (Sco X-1, J1550-564) – QPOs

1. Introduction

Pairs of high frequencies, known as kilohertz quasi-periodic oscillations (kHz QPOs), have been detected in the X-ray emission of low-mass X-ray binaries (LMXBs) for more than 20 neutron stars and several black holes. It has been suggested that a non-linear resonance within an accretion disc in a general-relativistic space-time metric plays a role in the excitation of the two oscillations (Abramowicz & Kluźniak 2001, Kluźniak & Abramowicz 2001).$^1$

It is now recognized that the frequencies in the two peaks in the power spectrum of X-ray variability in black holes are in rational ratios, $\omega_1/\omega_2 = m/n$, with $m/n = 2/3$ for two sources, and $m/n = 3/5$ in a third source (Abramowicz & Kluźniak 2001; Kluźniak & Abramowicz 2002; Remillard et al. 2002, Abramowicz et al. 2002a). These black hole QPOs are thought to have stable frequencies, and until the discovery of the puzzling rational ratios they have been thought to correspond to a trapped g-mode or c-mode of disc oscillation in the Kerr metric

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$^3$ Non-linear resonance has been favored as an explanation of QPOs also by Titarchuk (2002) in a rather different (“transition layer”) model. In another context, a tidally forced, Newtonian, non-linear vertical resonance in accretion discs has been discussed for close binary systems (Lubow 1981; Goodman 1993).
...geodesic, nearly circular, and nearly planar. In this paper we investigate a simple mathematical model of such motions. To this aim, we discuss the properties of parametric resonance in the Paczyński-Wiita (1980) model of the Schwarzschild metric.

2. Nearly-geodesic motion

Let us consider a fluid in which the flow lines are only slightly non-circular, and slightly off the $\theta = \pi/2$ symmetry plane. In spherical coordinates,

$$r(t) = r_0 + \delta r(t), \quad \theta(t) = \frac{3}{2} \pi + \delta \theta(t), \quad \phi(t) = \Omega t.$$  

With accuracy to the third order in $\delta r \ll r_0$ and $\delta \theta \ll \pi/2$, equations of fluid motion read:

$$\ddot{\delta \theta} + \omega_\theta^2 \left[ 1 + \frac{(\omega_\theta^2)' \delta r + \frac{1}{2} (\omega_\theta^2)'' \delta r^2}{\omega_\theta^2} \right] \delta \theta$$

$$+ \frac{2}{r} \left( 1 - \frac{\delta r}{r} \right) \delta \theta \delta r + \frac{1}{6r^2} \left( \frac{\partial^2 U}{\partial \theta^2} \right)_t \delta \theta^3 = f_\theta,$$  

$$\ddot{\delta r} + \omega_r^2 \delta r + \frac{1}{2} (\omega_r^2)' \delta r^2 + \frac{1}{2} (\omega_r^2)'' \delta r^3$$

$$- r \left( \frac{\delta \theta}{r} \right)^2 + \delta r \left( \frac{\delta \theta}{r} \right)^2 = f_r,$$  

$$\ddot{\ell} = r^2 \sin^2 \theta \, f_\theta.$$  

(4)

Here the time derivative is denoted by a dot, the radial derivative by a prime, and $f_i$ are components of a small force of non-gravitational origin (pressure, viscous, magnetic, or other). The epicyclic eigenfrequencies $\omega_\theta$ and $\omega_r$ are defined, in terms of the effective potential $U = \Phi(r, \theta) + \ell^2/(2r^2 \sin^2 \theta)$ and the specific angular momentum $\ell = \dot{\delta} r^2 \sin^2 \theta$, by

$$\omega_\theta^2 = \frac{1}{r^2} \left( \frac{\partial^2 U}{\partial \theta^2} \right)_t, \quad \omega_r^2 = \left( \frac{\partial^2 U}{\partial r^2} \right)_t.$$  

(5)

Derivation of these equations assumes equatorial plane symmetry for the gravitational potential $\Phi$. Here we adopt a spherically symmetric potential

$$\Phi(r) = -\frac{GM}{r - r_c}, \quad r_c = \frac{2GM}{c^2}.$$  

(6)

From eqs. (5) and (6) it follows

$$\omega_r^2 = \frac{GM}{(r - r_c)^3} \left( 1 - \frac{r_{ms}}{r} \right) < \omega_\theta^2 = \frac{GM}{r(r - r_c)^2},$$  

(7)

where $r_{ms} = 3r_c$ is the marginally stable orbit.

3. Strong gravity’s 2 : 3 resonance

A parametric resonance instability occurs near $\omega_r = 2\omega_\theta/n$ for $n = 1, 2, 3, \ldots$, in an oscillator that obeys the Mathieu type equation of motion (Landau & Lifshitz 1976),

$$\ddot{\delta \theta} + \omega_\theta^2 \left[ 1 + h_1 \cos(\omega_r t) \right] \delta \theta + \lambda \delta \dot{\theta} = 0.$$  

(8)

When the coupling is weak ($h_1$ small), the strongest instability occurs for the lowest value of $n$ possible (see Figure 1).

With this textbook mathematics in mind, let us consider a simplified version of eqs. (2)–(4), with $f_i = 0$, and some of the higher order terms neglected. With accuracy up to linear terms in (3) one obtains, obviously, a solution $\delta r(t) \propto \cos \omega_r t$. Substituting this solution in (2) brings this equation close to the standard form (8). We have solved these equations for $n \omega_r = 2 \omega_\theta$, with $n$ a positive real parameter. The instability regions for a lower-order version of equations (2)–(3) resemble the well-known tongues of instability of the Mathieu equation. Since $\omega_r < \omega_\theta$ in general relativity, the lowest value for which the resonance can occur is $n = 3$, and the first two tongues of instability in Fig. 1 will be absent; i.e., the ratio of the two eigenfrequencies in (8) is 2 : 3 for the strongest resonance (Kluźniak & Abramowicz 2002).

However, the behaviour of the resonance can only be properly studied if the third-order terms in (2) and the second-order terms in (3) are retained. These terms provide the non-linearity necessary to saturate the amplitude at a finite value and the damping which affects the frequency of oscillations. Using the standard analytic method of successive approximations (Landau & Lifshitz 1976), we verified that, as expected, no parametric resonance occurs for strictly geodesic motion, i.e., for $f_i = 0$ in eqs. (2)–(4). We also checked numerically that there is no resonance in the geodesic case. However, the resonance

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See Paczyński & Wiita (1980). This form is known to be a convenient model for the external gravitational field of a non-rotating black hole or a neutron star; it captures the essential feature of Einstein’s strong gravity relevant here, i.e., the fact that $\omega_r(r) < \omega_\theta(r)$, and $\omega_r = 0$ at the marginally stable orbit.
does occur (as anticipated in Kluźniak & Abramowicz 2002) for even very slightly non-geodesic motion, when the higher order terms are influenced by non-geodesic effects of a rather general form and right amplitude.

4. Non-geodesic coupling

In general it is not possible to specify the form of the very small force \( f_i \) exactly, because the nature of non-geodesic forces in accretion discs is not yet very accurately known. Here, we only explore the basic mathematical form of the solutions, not the physical origin of the \( f_i \) force. We assume ad hoc that the possible non-geodesic effects have the form of a non-linear isotropic coupling between the two components of geodesic deviation:

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\begin{align*}
  f_\theta &= +\alpha_\theta \left[ \frac{1}{2} (\omega_0^2)^2 \delta r^2 + \frac{1}{6r^2} \left( \frac{\partial^2 U}{\partial \theta^2} \right) \delta \theta^2 \right] \delta \theta, \\
  f_r &= -\alpha_r \left[ 1 - \frac{\delta r}{r} \right] r \left( \delta \theta \right)^2.
\end{align*}
\]

Here, \( \alpha_r \) and \( \alpha_\theta \) measure the strength of non-geodesic forces in comparison with the geodesic terms of corresponding form and order. We first assumed \( \alpha_r = \alpha_\theta = \alpha \) for the sake of simplicity.\(^3\)

After choosing a particular value of \( \alpha \), we solved eqs. (2)–(3) numerically by using the Runge-Kutta adaptive step-size routine. The integration was performed over a wide range of starting radii, \( 6 \leq r_0 (c^2/GM) \leq 15 \). For the initial conditions, we took \( \delta \theta(0) = 0.5, \delta \theta(0) = 0.01, \delta r(0) = 0.5, \) and \( \delta r(0) = 0.01 \). We then repeated the same procedure for another value of \( \alpha \), and in this way we constructed a sequence of solutions that is parameterized by \( \alpha \) and \( r_0 \).

One can locate radii \( r_p(\alpha) \) for which \( |\delta \theta(t; r_p; \alpha)| \) grows in time up to a maximum amplitude. The saturation amplitude, \( A \), is shown in Fig. 2 (right panel) as a function of radius. Clearly, the largest amplitudes correspond to parametric resonance. Close to those radii the test-particle epicyclic frequencies are in rational ratios \( n : m \); \( r_p = r_{2:3}, r_{4:2}, \) and \( r_{5:2} \). Note that the growth rate of the amplitudes \( |\delta \theta(t)|, |\delta r(t)| \) decreases with decreasing \( \alpha \). The 2 : 3 resonance is the strongest one, as expected.

The exact slope of the 2 : 3 resonance trace \( r_p(\alpha) \), starting from the point \( (r_0, \alpha) = (9.2, 1) \), depends on the choice of coupling parameters in eqs. (9)–(10), so the above assumption of identical values for \( \alpha_\theta = \alpha_r \) is not universal. For example, assuming \( \alpha_r = \kappa \alpha_\theta \) and setting \( \kappa = 0.2 \), one finds that the end of the trace is moved to the point \( (9.0, 0.95) \). It is thus shifted in the \((r, \alpha)\)-plane with respect to the case \( \kappa = 1 \), for which the trace ends at the point \( (8.8, 0.95) \), as shown in the left panel of Fig. 2. Also the initial conditions for \( \delta \theta \) and \( \delta r \) have an influence on the trace slope. Therefore, the corresponding frequencies that are present in the solutions for \( \delta r(t), \delta \theta(t) \) depend on the details of the solution. Nevertheless, the ratio of the frequencies stays at 2 : 3 at the maximum amplitude. This fact is in agreement with the intuition about the properties of eqs. (2)–(3), and we checked it by Fourier-analyzing their solution.

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\(^3\) Perturbations considered here may be consistent with the formation of non-axially symmetric coherent structures in accretion discs ("planets", "vortices", "magnetic flux tubes" or "spiral waves"), as proposed by several authors: Abramowicz et al. (1992); Goodman, Narayan, & Goldreich (1987); Adams & Watkins (1995); Bracco et al. (1998); Karas (1999); Kato (2002, 2003).

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Fig. 2. Left: Location of the first three resonances is shown for a particular choice of initial amplitude, \( \delta \theta(0) = 0.5 \), by encoding the growth rate of \( |\delta \theta| \) from eq. (2) with different levels of shading in the \((r, \alpha)\)-plane (radius versus strength of perturbation). Note that \( \alpha \) decreases along the \( y \)-axis. For \( \alpha = 1.00 \), the resonant orbits are located at radii \( r_{m:n} \) (indicated by vertical lines) where \( \omega_\theta \) and \( \omega_\theta \) are in rational ratios, \( m : n \). In particular, the trace of the 2 : 3 resonance \((n = 3)\) starts from \( r_0 = 9.2GM/c^2 \) in the Paczyński-Wiita potential. The frequencies and radius of resonant orbits vary when \( \alpha \) is changed. Right: The maximum amplitude \( A \) of \( \delta \theta(t) \) for a fixed value of \( \alpha = 0.98 \) as a function of the radius of the unperturbed circular orbit.
In this paper we assumed the gravitational potential of eq. (6) for the sake of simplicity. When a more accurate, full formulation in general relativity is developed together with a physical mechanism responsible for the non-geodesic coupling term (9)–(10), one will be able to constrain the absolute value of the mass of the accreting body.

5. Frequency correlation in Sco X-1

As noted in Section 1, the frequency ratio of the QPOs in some black hole sources is accurately $2:3$. Specifically, in GRO J1655–40 the frequency ratio is that of 300 Hz and 450 Hz, while in XTE J1550–564 it is that of 184 Hz and 276 Hz (Abramowicz & Kluzniak 2001, Remillard et al. 2002). The properties of parametric resonance, discussed above, provide a natural explanation for this ratio.

Can the same mechanism be responsible for the observed frequencies of the kHz QPOs in neutron stars? In neutron star sources the two kHz QPOs vary considerably in frequency, and their ratio is not always $2:3$. A case in point is Sco X-1, whose two kHz frequencies are linearly correlated (Fig. 3), but not directly proportional to each other (van der Klis et al. 1997). However, the distribution of points along the line of correlation is not uniform, and the resulting distribution of frequency ratios has a prominent peak at about $2:3$ value, strongly suggesting the presence of a non-linear resonance (Abramowicz et al. 2002b).

We note that QPOs are not coherent oscillations, and that the typical integration time (minutes) greatly exceeds the coherence time of the signal (fraction of a second). We find that the slope of Sco X-1 frequency correlation can be reproduced if we assume that the signal observed at a given time corresponds to the frequencies originating at a particular radius, for example the radius $r_\nu(\alpha)$ discussed in the previous section. If this is not strictly equal to the radius where the maximum amplitude is attained, the ratio of the frequencies departs from $2:3$. Specifically, taking an arbitrary initial amplitude $\delta \theta(0)$ we constructed Fourier power spectra of $\delta \theta(t; r_\nu, \alpha)$, and $\delta r(t; r_\nu, \alpha)$. These power spectra peak at frequencies $\omega_\nu^*(\alpha)$ and $\omega_r^*(\alpha)$, which we interpret here as the observed pair of oscillations.

Figure 3 shows the correlation between $\omega_r^*$ and $\omega_\nu^*$ along the $\alpha$-trace from Fig. 2. We assumed $\kappa = 1$ as before. The actual value of $\delta \theta(0)$ has been selected to match the slope of Sco X-1 data (van der Klis 1997). Because $\alpha$ measures the deviation from geodesicity, its value is likely to be a function of the accretion rate in the disc, $\alpha(M)$. This notion agrees with the reported correlation between the frequencies and the position of the source along the $Z$ curve in the color-color diagram. Fig. 3 illustrates that the simple scheme that we introduced in the present paper may, indeed, explain the slope of $\omega_r^*(\omega_\nu^*)$ relation.

6. Conclusions

We have expanded, through the third order, relativistic equations of test motion about a circular geodesic, and have shown that the lowest order expansion corresponds to Mathieu’s equation with non-standard damping. As expected, the solutions of these (unperturbed) equations do not show any resonant phenomenon—they simply describe geodesic motion.

When a sufficiently large non-geodesic perturbation is imposed, the deviations grow rapidly at certain resonant radii and the motion departs from circular geodesics. This is caused by parametric resonance between the meridional and radial epicyclic motions, and the strongest resonance occurs when the two dominant frequencies are near ratio $2:3$.

We thank Michiel van der Klis for providing the Sco X-1 data and for a very helpful discussion. Most of this work was done during MAA’s and VK’s visit to the supercomputer centre UKAFF at Leicester University, and MAA’s, WK’s and WHL’s visits to IAP (Paris) and SISSA (Trieste). WK held a post rouge at CESR, funded by CNRS. VK acknowledges support from GACR 205/03/0902 and GAUK 188/2001, and WHL from CONACYT (36632E). The Center for Particle Physics is supported by the Czech Ministry of Education Project LN00A006.
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