Thermodynamics of a charged relativistic ideal Boltzmann gas

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Abstract

This paper presents a toy model of a charged relativistic classical gas in flat spacetime of an arbitrary number of dimensions equipped with some “pair production” mechanism. Working with the microcanonical ensemble, the charge is taken to be conserved in contrast with the total number of “particles” plus “antiparticles” which varies with temperature $T$. Thermodynamics of the classical gas is detailed studied in the nonrelativistic ($Mc^2 \gg kT$) as well as in the ultrarelativistic ($Mc^2 \ll kT$) regimes. It is shown that, although we are dealing with classical distributions, at the ultrarelativistic regime the behavior of the gas is Planckian. We also compare the thermodynamics of the toy model with that of quantum gases.

Keywords: charged classical gas; charged quantum gas; blackbody radiation.

1 Introduction

In 1928, Paul Dirac solved the negative energy problem in the Klein-Gordon equation by assuming that there was an object resembling a particle but with an opposite electric charge \textsuperscript{1}. His theory would be experimentally confirmed in 1932 by Carl Anderson with the discovery of the electron’s antiparticle, the positron \textsuperscript{2}. Since then, many experimental and theoretical discoveries have shown that at extreme energies (e.g., either inside modern accelerators or in the early universe) pair production must be taken into account.

When one tries to extend classical statistical mechanics of an ideal gas with $N$ particles of mass $M$, in a cavity of volume $V$, to the realm of relativity the prescription seems to be simply to consider that the particles have relativistic energies. Such a prescription (see, e.g., ref. \textsuperscript{3}) results in the familiar equation of state: \textsuperscript{4}

\begin{equation}
PV = NkT,
\end{equation}

and either in the nonrelativistic internal energy:

\begin{equation}
U = NMc^2 + \frac{3}{2}NkT, \quad Mc^2 \gg kT,
\end{equation}

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\textsuperscript{2}Fundamental constants have the usual meaning.
or in the ultrarelatistic one:

\[ U = 3NkT, \quad Mc^2 \ll kT, \]  

(3)

with both expressions for \( U \) holding in leading order. In deriving eqs. (1) to (3) it is assumed that \( N \) is conserved, which is an oversimplification as far as eq. (3) is concerned. And the reason is simple. For instance, if particles are electrons, as temperature grows eventually positrons will appear resulting that \( N \) (the number of electrons plus the number of positrons) increases while the net charge \( Q \) (the number of electrons minus the number of positrons) remains the same according to what quantum field theory at finite temperature prescribes [4].

It is worth remarking that even quantum statistical mechanics with \( N \) conserved would lead to eq. (3) for high enough \( T \) [3]. The issue was resolved in 1981 by Haber and Weldon [6] who considered an ideal Bose gas with conserved net charge \( Q \) (the number of particles minus the number of antiparticles). Analogous considerations can be applied to an ideal Fermi gas [4, 7], resulting that for both quantum gases eq. (3) is replaced by the following Planckian internal energy,

\[ U = \sigma V T^4, \quad Mc^2 \ll kT, \]  

(4)

where the value of the “Stefan-Boltzmann” constant \( \sigma \) depends on the nature of the gas [3]. It should be added that, corresponding to the ultrarelativistic regime in eq. (4), now \( P = \sigma T^4/3 \).

After going through the literature, one can see that a toy model which has been overlooked is that of a charged relativistic ideal classical gas equipped with some mechanism of “pair production”, which could be of any nature. It would be interesting to compare the associated thermodynamics with that of charged quantum gases, identifying limitations of the toy model. And this is the aim of the present paper whose organization is as follows. For the sake of generalization, we will work in \( D \)-dimensional Minkowski spacetime.

The next section is devoted to deriving equilibrium particle and antiparticle distributions by means of the microcanonical ensemble [8]. Two Lagrange multipliers arise corresponding to the charge \( Q \) and energy \( E \). In Section 3 chemical potential \( \mu \), temperature \( T \) and pressure \( P \) are obtained from the equilibrium entropy as a function of \( U \equiv E, V \) and \( Q \), where \( V \) is the \((D - 1)\)-dimensional volume of the cavity containing an ideal gas with particles and antiparticles of mass \( M \). Thermodynamic quantities are expressed in terms of modified Bessel functions of the second kind, \( K_\nu(Mc^2/kT) \), with \( \nu \) determined by \( D \). In Sections 3.1 and 3.2 respectively, thorough investigations of thermodynamics at the nonrelativistic (\( Mc^2 \gg kT \)) and ultrarelativistic (\( Mc^2 \ll kT \)) regimes are performed by using the leading order behaviors of \( K_\nu(z) \) as \( z \) is big and small. In Section 4 we close with a comparison between the thermodynamics of our toy model with that of realistic quantum gases where rather curious features are spotted. For convenience, a short appendix is included containing a few properties of \( K_\nu(z) \).

## 2 Most probable distributions

We begin with a relativistic classical ideal gas with charge

\[ Q = N_+ - N_-, \]  

(5)

It is assumed that \( T \) is high enough, i.e., much bigger than the Fermi temperature (for fermions) or bigger than the critical temperature (for bosons).

\[ \sigma = \pi^2 k^4/15(hc)^3 \text{ or } \sigma = 7\pi^2 k^4/60(hc)^3 \]  

for bosons and fermions, respectively.
where $N_+$ is the number of particles and $N_-$ is the number of antiparticles. If $n_{(+)}(i)$ and $n_{(-)}(i)$ are occupation numbers of the $i$th cell of the phase space with $K$ cells, then

$$N_+ = \sum_{i=1}^{K} n_{(+)}(i), \quad N_- = \sum_{i=1}^{K} n_{(-)}(i). \quad (6)$$

To each set of occupation numbers \{($n_{(+)}(i)$, $n_{(-)}(i)$)\} corresponds to a certain number of states $\Omega$, and the total number of states is given by

$$\Gamma = \sum_{\{(n_{(+)}(i), n_{(-)}(i))\}} \Omega \left(\{(n_{(+)}(i), n_{(-)}(i))\}\right). \quad (7)$$

Considering the number of ways one can arrange distinguishable objects into $K$ cells, it follows that

$$\Omega \propto \prod_{i=1}^{K} \frac{1}{n_{(+)}(i)!n_{(-)}(i)!}, \quad (8)$$

where the “correct Boltzmann counting” has been used.

The distributions $\bar{n}_{(+)}(i)$ and $\bar{n}_{(-)}(i)$ for which $\ln \Omega$ is maximum will now be determined. Two constraints will be taken into account along with their Lagrangian multipliers $\alpha$ and $\beta$. One is the net charge in eq. (5) [see also eq. (6)], and the other is the gas total energy:

$$E = \sum_{i=1}^{K} \left[ n_{(+)}(i) + n_{(-)}(i) \right] \epsilon_i, \quad \epsilon_i = c\sqrt{M^2c^2 + p_i^2}. \quad (9)$$

Thus, by noting eqs. (5), (6), (8) and (9), the solution of

$$\delta \left[ \ln \Omega + \alpha \sum_{i=1}^{K} (n_{(+)}(i) - n_{(-)}(i)) - \beta \sum_{i=1}^{K} (n_{(+)}(i) + n_{(-)}(i)) \epsilon_i \right]_{(n_{(+)}(i), n_{(-)}(i)) = (\bar{n}_{(+)}(i), \bar{n}_{(-)}(i))} = 0$$

is given by:

$$\bar{n}_{(+)}(i) = e^\alpha e^{-\beta \epsilon_i}, \quad \bar{n}_{(-)}(i) = e^{-\alpha} e^{-\beta \epsilon_i}, \quad (10)$$

where Stirling’s approximation,

$$\ln n! = n \ln n - n + \cdots, \quad (11)$$

has been considered as usual.

Before associating $\alpha$ and $\beta$ in eq. (10) with thermodynamic parameters, let us covert summations over $i$ into integrations. This can be achieved by taking an arbitrary small volume in phase space, denoted by $h^{D-1}$, where the constant $h$ has dimensions of distance $\times$ momentum. For example, corresponding to the most probable distributions in eq. (10), eq. (6) yields

$$N_+ = e^\alpha \frac{V}{\Lambda^{D-1}}, \quad N_- = e^{-\alpha} \frac{V}{\Lambda^{D-1}}, \quad (12)$$

That is, original $\Omega$ has been divided by $N_+! N_-!$. 

\[\text{3}\]
with
\[ \Lambda^{-(D-1)} = \frac{1}{\hbar^{D-1}} \int e^{-\beta c \sqrt{M^2 c^2 + p^2}} d^{D-1}p. \] (13)

Integrations in eq. (13) run from \(-\infty\) to \(\infty\) over \(D - 1\) components of the Euclidean vector \(p\) whose squared norm is \(p^2\). We will call \(\Lambda\) “thermal wavelength” though it is clearly an abuse of language once we are dealing with a classical gas. Shortly a closed formula for \(\Lambda\) will be determined. Noting eq. (12) one sees that \(Q\) in eq. (5) and
\[ N = N_+ + N_- \] (14)
can also be written in terms of \(\Lambda\), i.e.,
\[ Q = 2 \sinh(\alpha) \frac{V}{\Lambda^{D-1}}, \quad N = 2 \cosh(\alpha) \frac{V}{\Lambda^{D-1}}, \] (15)
which gives rise a neat relation, namely:
\[ N = \sqrt{Q^2 + \frac{4V^2}{\Lambda^{2(D-1)}}}. \] (16)

The total energy in eq. (9) has the following expression in terms of \(\Lambda\), more precisely in terms of its derivative:
\[ E = -2 \cosh(\alpha) V \frac{\partial}{\partial \beta} \Lambda^{-(D-1)}, \] (17)
as can be quickly checked by looking at eqs. (10) and (13).

Considering eqs. (12), (14) and the densities
\[ n_+ = \frac{N_+}{V}, \quad n_- = \frac{N_-}{V}, \quad n = \frac{N}{V}, \] (18)
it follows relative densities:
\[ \frac{n_+}{n} = \frac{1}{e^{-2\alpha} + 1}, \quad \frac{n_-}{n} = \frac{1}{e^{2\alpha} + 1}, \] (19)
to which we will return later in the text. Notice that \(n_+/n + n_-/n = 1\) as should be.

We turn now to determining a closed form for the “thermal wavelength” in eq. (13) in terms of modified Bessel functions of the second kind (see Appendix A). Spherical polar coordinates (see, e.g., ref. [8]), allow us to write,
\[ \Lambda^{-(D-1)} = \frac{2\pi(D-1)/2}{\hbar^{D-1} \Gamma \left(\frac{D-1}{2}\right)} \int_0^\infty p^{D-2} e^{-\beta c \sqrt{M^2 c^2 + p^2}} dp. \] (20)

Setting \(p = Mc \sinh(\omega)\), eq. (20) yields
\[ \Lambda^{-(D-1)} = \frac{2\pi(D-1)/2}{\hbar^{D-1} \Gamma \left(\frac{D-1}{2}\right)} (Mc)^{D-1} \int_0^\infty \cosh(\omega) \sinh^{D-2}(\omega) e^{-\beta Mc^2 \cosh(\omega)} d\omega. \]
Thus, eq. (47) leads to:

$$\Lambda^{-(D-1)} = \frac{2^{D/2} \pi^{(D-2)/2}}{h^{D-1}} \beta^{(2-D)/2} M^{D/2} c K_{D/2} (\beta Mc^2).$$  

(21)

By looking at the expression for $N$ in eq. (15), eq. (21) can be readily used in eq. (17), resulting in

$$E = N \beta^{-1} \left[ D - 1 + \beta Mc^2 \frac{K_{D-1}(\beta Mc^2)}{K_{D/2}(\beta Mc^2)} \right],$$  

(22)

where the second identity in eq. (46) has been considered.

### 3 Thermodynamics

Entropy is defined by $S = k \ln \Gamma$, with $\Gamma$ given in eq. (7). Since $\pi_{(+)}$ and $\pi_{(-)}$ contribute overwhelmingly, it follows that

$$S = k \ln \Omega \left( \{(\pi_{(+)}), (\pi_{(-)})\} \right).$$  

(23)

Now, neglecting an unimportant constant, by using eqs. (8) to (11) and also eqs. (5), (6) and (14), we can show that $S$ in eq. (23) can be recast as

$$S = k \beta E - k \alpha Q + k N.$$  

(24)

The Lagrangian multipliers $\alpha$ and $\beta$ can be associated with thermodynamic quantities by taking into account that they are functions of $E$, $V$ and $Q$. Thus, recalling that $\Lambda = \Lambda(\beta)$, and noticing eqs. (15), (17) and (24), the usual definitions

$$\beta = \frac{1}{kT}, \quad \alpha = \frac{\mu}{kT}.$$  

(25)

Likewise,

$$P = T \left( \frac{\partial S}{\partial V} \right)_{E,Q}$$

yields eq. (1); but now with the crucial difference that $N$ is not conserved [cf. eq. (16)]. Putting these findings together, eq. (24) becomes

$$U = TS - PV + \mu Q,$$  

(26)

where $E$ has been identified as the internal energy $U$, as usual.

An expression for the chemical potential in eq. (25) can be obtained by noting eq. (15). One sees that

$$\mu = kT \arcsinh \left( \Lambda^{D-1} \frac{Q}{2V} \right).$$  

(27)
It is worth remarking that \( \mu \) can be recast as

\[
\mu = kT \ln \left( \Lambda^{D-1} n_+ \right),
\]

(28)

\[
= -kT \ln \left( \Lambda^{D-1} n_- \right),
\]

(29)

where eqs. (5), (14), (16) and (18) have been used.

It follows immediately from eqs. (25) and (27) that \( \mu \) vanishes if and only if \( \alpha = 0 \) and the gas is neutral (i.e., \( Q = 0 \)). Then the relative densities in eq. (19) are both equal to 1/2. In other words when \( \mu \) vanishes the numbers of particles and antiparticles are the same for all temperatures:

\[
n_+ = n_- = \frac{n}{2} = \frac{1}{\Lambda^{D-1}}, \quad \mu = 0.
\]

(30)

In deriving eq. (30) we have noted eqs. (16) and (18). Now we go into a more detailed study of thermodynamics by looking at asymptotic regimes of large and small masses. Unless stated otherwise, only contributions due to the leading order terms in eq. (18) will appear in the formulae below.

### 3.1 Nonrelativistic regime: \( Mc^2 \gg kT \)

By considering eqs. (21), (25) and the second expression in eq. (48), one ends up with

\[
\Lambda^{D-1} = \lambda^{D-1} e^{\frac{Mc^2}{kT}}, \quad \lambda = \sqrt{\frac{h^2}{2\pi M kT}}.
\]

(31)

Correspondingly, eqs. (28) and (29) yield

\[
\mu = Mc^2 + kT \ln \left( \Lambda^{D-1} n_+ \right),
\]

(32)

\[
= -Mc^2 - kT \ln \left( \Lambda^{D-1} n_- \right),
\]

(33)

respectively. Either eqs. (32) and (33) or eq. (31) quickly show that for a neutral gas (i.e., \( \mu = 0 \)) the densities in eq. (30) and \( P \) in eq. (1) are exponential small.

Let us look more carefully at the densities when \( Q \neq 0 \). Then, eq. (16) leads to

\[
n = \frac{|Q|}{V} \sqrt{1 + \frac{4V^2}{Q^2 \Lambda^{2(D-1)}}}.
\]

(34)

For the nonrelativistic regime, we use eq. (31) in eq. (34), resulting

\[
n = \frac{|Q|}{V}
\]

(35)

which, up to an exponential small correction that has been omitted, does not depend on \( T \). Moreover, it follows also that, when \( Q > 0 \), \( n_+ \) essentially equals \( n \) in eq. (35), whereas \( n_- \) is exponential.

\[\text{arcsinh}(x) = \ln \left( x + \sqrt{x^2 + 1} \right).\]
small. Now, when $Q < 0$, this time is $n_-$ that essentially equals $n$ in eq. (35) with $n_+$ exponential small. These remarks result from [cf. eqs. (5) and (14)]

$$N_+ = \frac{N + Q}{2}, \quad N_- = \frac{N - Q}{2},$$

(36)

and eq. (18). Thus, in the norelativistic regime, one has roughly either a gas of particles ($Q > 0$) or a gas of antiparticles ($Q < 0$), with relative densities $n_+ / n = 1$ and $n_- / n = 1$, respectively. Although we should not expect that the present classical toy model is realistic when $T$ is close to absolute zero, it is worth pointing out that as $T \to 0$ eqs. (32) and (33) lead to $\mu = M c^2$ or $\mu = - M c^2$, corresponding to $Q > 0$ and $Q < 0$, respectively. A word should be added regarding eqs. (1) and (16) at the nonrelativistic regime where eq. (35) holds. The equation of state is essentially given by

$$P V = |Q| k T.$$

(37)

By taking into account also the subleading correction in the second expansion in eq. (18), eq. (22) yields the following nonrelativistic internal energy and heat capacity at constant volume:

$$U = |Q| M c^2 + \frac{D - 1}{2} |Q| k T, \quad C_V = \frac{D - 1}{2} |Q| k.$$

(38)

To complete, noticing eqs. (32) or (33), eq. (35) and the text just after it, we use eqs. (37) and (38) in eq. (26) to obtain

$$S = k |Q| \ln \left( \frac{V}{|Q| \lambda^{D-1}} \right) + \frac{D + 1}{2} k |Q|,$$

(39)

which is the Sackur-Tetrode equation of our toy model.

### 3.2 Ultrarelativistic regime: $M c^2 \ll k T$

Let us turn now to the ultrarelativistic regime. By considering again eqs. (21), (25) and now the first expression in eq. (18), it results in

$$\Lambda = \frac{1}{2 \pi} \left[ \pi^{D/2} \Gamma(D/2) \right]^{1/2} \frac{h c}{k T},$$

(40)

instead of eq. (31). Now, by inserting eq. (40) into eq. (27), we arrive at the leading contribution

$$\mu = \frac{Q}{V} \frac{(h c)^{D-1}}{2^D \pi^{D/2-1} \Gamma(D/2)(k T)^{D-2}};$$

(41)

which goes to zero as $T$ progressively increases. Correspondingly, the densities are essentially given by eq. (30), i.e., $n_+ = n_- = n/2$, with

$$n = 2^D \pi^{(D-2)/2} \Gamma(D/2) \left( \frac{k T}{h c} \right)^{D-1};$$

(42)

---

7As part of our classical toy model we assume that the gas is very diluted, i.e., $|V/Q|$ is taken to be arbitrary large.
and where eq. (40) has been used. Alternatively one could have arrived at eq. (42) by considering eq. (16). Note that mass and charge appear only in small corrections to the "Planckian" density in eq. (42) which holds exactly when $M$ and $Q$ vanish. It is worth remarking that $\mu = QkT/N$ up to corrections.

Considering eq. (22) and taking the first expression in eq. (48) into account leaves us with

$$U = (D - 1)NkT,$$

which for $D = 4$ resembles eq. (3) but, again, with the important difference that now $N$ depends on $T$. By using further eq. (12), it results as leading contribution a "Planckian" internal energy, namely

$$U = \frac{2^{D} \pi^{(D-2)/2}}{(hc)^{D-1}} (D - 1)VT (D/2)(kT)^{D}. \quad (43)$$

Now we can use eqs. (1), (26), and (41) to (43), to derive the following “blackbody” identities

$$PV = \frac{U}{D - 1}, \quad S = \frac{D}{D - 1} \frac{U}{T}, \quad C_V = (D - 1)S, \quad (44)$$

which are satisfied by leading contributions.

4 Discussion

This paper addressed the thermodynamics of an ideal gas with charge $Q$ that is conserved whereas the total number of particles $N$ varies with $T$. It is a classical toy model in $D$-dimensional Minkowski spacetime that assumes some mechanism of pair production. By working with the microcanonical ensemble we have shown that at the nonrelativistic regime $N$ hardly depend on $T$ and equals $|Q|$, up to exponential small corrections. The corresponding thermodynamics is essentially that of an ordinary ideal gas where the Sackur-Tetrode equation holds [cf. eqs. (37), (38) and (39)]. In sharp contrast, at the ultrarelativistic regime, up to negligible corrections, thermodynamics is that of a “Planck gas” [cf. eqs. (43) and (44)] where entropy $S = kDN \propto T^{D-1}$.

We notice that eqs. (37) and (38) are not affected by taking $h \to 0$ (i.e., by considering the small volume $h^{D-1}$ in the phase space arbitrary small), whereas such a limit causes divergences in eqs. (43) and (44). Clearly, this is related to the intrinsic quantum nature of an ultrarelativistic gas. Namely, if $\Lambda$ in eq. (40) is considered as a genuine thermal wavelength, by noting eq. (42) one sees quickly that $\Lambda$ is comparable to the intermolecular distance $n^{1/(1-D)}$.

It should be remarked that to derive the “Planckian” internal energy in eq. (43) we did not take into account if we were dealing with bosons or fermions, since our toy model is classical. Nevertheless, neglecting this fact results that if we redo the calculations for, say, a Bose gas we will eventually notice a missing factor $\zeta(D)$ in eq. (43), which comes with the use of Bose distributions for particles and antiparticles. In spite of limitations such as this one, it is rather curious that a classical toy model contains the thermodynamics of an ordinary ideal gas [see eqs. (37), (38) and (39)] and that of blackbody radiation [see eqs. (43) and (44)] as well.

Before closing, we wish to address further a couple of issues that are relevant from the physical point of view. Although our toy model is classical, in a sense it takes into account the Heisenberg uncertainty principle by assuming an arbitrary small but non-vanishing volume $h^{D-1}$ in the phase space. Quantum statistical mechanics has another important ingredient that is missing in the toy model. Namely, quantum statistics. Nevertheless, it seems fair to say that the toy model will described reasonably well any relativistic gas with some kind of “pair production” mechanism,
as long as temperatures involved are not that low as to be comparable with the quantum gases characteristic temperatures.

There is a point we would like to stress. The term “charge” in this paper should be understood in a broad sense. It means the difference of two species particle numbers regardless of the phenomenon responsible for keeping “charge” conserved as thermodynamic equilibrium is established. Such a “pair production” mechanism could be in QED or chemistry, and its details are irrelevant to the final outcome which is the stable thermodynamic equilibrium of the whole system.

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A Some properties of $K_\nu(z)$

An integral representation of the modified Bessel function of the second kind, $K_\nu(z)$, is \cite{9,10,11}:

$$K_\nu(z) = (z/2)^\nu \frac{\Gamma(1/2)}{\Gamma(\nu + 1/2)} \int_0^\infty \sinh^{2\nu}(\omega) e^{-z \cosh(\omega)} d\omega$$  \hspace{1cm} (45)

with $Re(\nu) > -1/2$ and $Re(z) > 0$. $K_\nu(z)$ satisfies identities such as

$$z K_{\nu-1}(z) - z K_{\nu+1}(z) = -2\nu K_\nu(z)$$

and

$$z K_{\nu+1}(z) = \nu K_\nu(z) - z \frac{d}{dz} K_\nu(z), \quad z K_{\nu-1}(z) = -\nu K_\nu(z) - z \frac{d}{dz} K_\nu(z).$$  \hspace{1cm} (46)

By taking the derivative of eq. (45) and using the first identity in eq. (46), one obtains the integral representation used in the text, i.e.,

$$K_{\nu+1}(z) = (z/2)^\nu \frac{\Gamma(1/2)}{\Gamma(\nu + 1/2)} \int_0^\infty \cosh(\omega) \sinh^{2\nu}(\omega) e^{-z \cosh(\omega)} d\omega.$$  \hspace{1cm} (47)

We also have the following leading behaviors for small and large values of $z$:

$$K_\nu(z \to 0) = 2^{\nu-1} \Gamma(\nu) z^{-\nu} + \cdots, \quad K_\nu(z \to \infty) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{4\nu^2 - 1}{8z} + \cdots \right],$$  \hspace{1cm} (48)

with $\nu > 0$ in the first expression.

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