A Resource Framework for Quantum Shannon Theory

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Abstract

Quantum Shannon theory is loosely defined as a collection of coding theorems, such as classical and quantum source compression, noisy channel coding theorems, entanglement distillation, etc., which characterize asymptotic properties of quantum and classical channels and states. In this paper we advocate a unified approach to an important class of problems in quantum Shannon theory, consisting of those that are bipartite, unidirectional and memoryless.

We formalize two principles that have long been tacitly understood. First, we describe how the Church of the larger Hilbert space allows us to move flexibly between states, channels, ensembles and their purifications. Second, we introduce finite and asymptotic (quantum) information processing resources as the basic objects of quantum Shannon theory and recast the protocols used in direct coding theorems as inequalities between resources. We develop the rules of a resource calculus which allows us to manipulate and combine resource inequalities. This framework simplifies many coding theorem proofs and provides structural insights into the logical dependencies among coding theorems.

We review the above-mentioned basic coding results and show how a subset of them can be unified into a family of related resource inequalities. Finally, we use this family to find optimal trade-off curves for all protocols involving one noisy quantum resource and two noiseless ones.

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1 Introduction

Hitherto quantum and classical information theory have been developed using a “first principles” approach. Each new coding theorem requires importing or re-deriving the basic tools from a previous communication scenario and then applying them in a new, usually more sophisticated, way. This may be compared to computer programming directly in assembly language as opposed to using a high-level programming language like C++. In this work we advocate an alternative to the first principles approach, stemming from the view that all quantum and classical coding theorems are quantitative statements regarding inter-conversions between non-local information processing resources [20]. As an example, consider the scenario in which the sender Alice and receiver Bob have the predefined goal of perfect transmission of a classical message, but have at their disposal only “imperfect” resources such as a noisy channel. This is Shannon’s channel coding problem [52]: allowing the parties arbitrary local operations, they can perform encoding and decoding of the message to effectively reduce the noise level of the given channel. Their performance is measured by two parameters: the error probability and the number of bits in the message, and they want to minimize the former while maximizing the latter. In Shannon theory, we are particularly interested in the memoryless case in which the message is long and the channel is a number of independent realizations of the same noisy channel $N^{A\to B}$. The efficiency of the code is then measured by the rate $R$, the ratio of the number of bits in a message to the number of channel uses. We are specifically concerned with the asymptotic regime of arbitrarily long messages and vanishing error probability. Note that not only the given channel, but also the goal of the parties, noiseless communication, is a resource: the channel which transmits one bit perfectly. The latter resource we call a cbit (“classical bit”) and denote by the symbol $[c \to c]$. Thus coding can be described more generally as the conversion of one resource into another, i.e., simulation of the target resource by using the given resource together with local processing. We express such an asymptotically faithful conversion as a resource inequality (RI)

$$\langle N^{A\to B} \rangle \geq R[c \to c].$$

The left hand side we call the input resource (or consumed resource) and the right hand side the output resource (or created resource). In the asymptotic setting, $R$ can be any real number, and the supremum of $R$ is the capacity of the channel.

Obviously, there exist other useful or desirable resources, such as perfect correlation in the form of a uniformly random bit (abbreviated rbit) known to both parties, denoted by $[cc]$, or more generally some noisy correlation. In quantum information theory, we have further resources: noisy quantum channels and quantum correlations (a.k.a. entanglement) between the parties. Again of particular interest are the noiseless unit resources: $[q \to q]$ is an ideal quantum bit channel (qubit for short), and $[qq]$ is a unit of maximal entanglement, a two-qubit singlet state (ebit).

To illustrate our goals, it is instructive to look at the conversions permitted by the unit resources $[c \to c]$, $[q \to q]$ and $[qq]$, where resource inequalities are finite and exact. The following inequalities always refer to a specific integral number of available resources of a given type, and the protocol introduces no error. For example, it is always possible to use a qubit to send one classical bit, $[q \to q] \geq [c \to c]$, and to distribute one ebit using a qubit channel, $[q \to q] \geq [qq]$. The latter is referred to as entanglement distribution. More inequalities are obtained by combining resources. Super-dense coding (SD) [3] is a coding protocol to send two classical bits using one qubit and one ebit:

$$[q \to q] + [qq] \geq 2[c \to c].$$

(1)

Teleportation (TP) [4] is expressed as

$$2[c \to c] + [qq] \geq [q \to q].$$

(2)

In [4] the following argument was used that the ratio of 1 : 2 between $[q \to q]$ and $[c \to c]$ in these protocols is optimal. Assume, with $R > 1$, $[q \to q] \geq \infty [qq] \geq 2R[c \to c]$; then chaining
this with (TP) gives \( [q \to q] + \infty [qq] \geq R[q \to q] \). Hence by iteration \( [q \to q] + \infty [qq] \geq R^k [q \to q] \geq R^k [c \to c] \) for arbitrary \( k \), which can make \( R^k \) arbitrarily large, and this is easily disproved. Analogously, \( 2[c \to c] + \infty [qq] \geq R[q \to q] \), with \( R > 1 \), gives, when chained with SD, \( 2[c \to c] + \infty [qq] \geq 2R[c \to c] \), which also easily leads to a contradiction. In a similar way, the optimality of the one ebit involved in both SD and TP can be seen.

While the above demonstration looks as if we did nothing but introduce a fancy notation for things understood perfectly otherwise, in this paper we want to make the case for a systematic theory of resource inequalities. We will present a framework general enough to include all unidirectional two-player setups, specifically designed for the asymptotic memoryless regime. There are three main issues there: first, a suitably flexible definition of a protocol, i.e., a way of combining resources (and with it a mathematically precise notion of a resource inequality); second, a justification of the composition (chaining) of resource inequalities; and third, general tools to produce new protocols (and hence resource inequalities) from existing ones.

The benefit of such a theory should be clear. While it does not mean that we get coding theorems “for free”, we do get many protocols by canonical modifications from others, which saves effort and provides structural insights into the logical dependencies among coding theorems. As the above example shows, we also can relate (and sometimes actually prove) the converses, i.e. the statements of optimality, using the resource calculus.

From here, the paper is structured as follows.

**Section 2** (p. 4) covers the preliminaries and describes several complementary formalisms for quantum mechanics, which serve diverse purposes in the study of quantum information processing. Here also some basic facts are collected.

**Section 3** (p. 10) sets up the basic communication scenario we will be interested in. It contains definitions and basic properties of so-called finite resources, and how they can be used in protocols. Building upon these we define asymptotic resources and inequalities between them, in such a way as to ensure natural composability properties.

**Section 4** (p. 29) contains a number of general and useful resource inequalities.

**Section 5** (p. 34) compiles most of the hitherto discovered coding theorems, rewritten as resource inequalities.

**Section 6** (p. 41): Armed with these we give rigorous proofs of a family of resource inequalities from [16], as well as of two general rules for “making protocols coherent”.

**Section 7** (p. 47): Here we discover the sense in which this family of resource inequalities is optimal by exhibiting an entropic characterization of five new resource trade-offs.

**Section 8** (p. 57) concludes the paper with some remarks on open problems and possible future work.
2 Preliminaries

This section is intended to introduce notation and ways of speaking about quantum mechanical information scenarios. We also state several key lemmas needed for the technical proofs. Most of the facts and the spirit of this section can be found in [35]; a presentation slightly more on the algebraic side is [61], appendix A.

2.1 Variations on the formalism of quantum mechanics

We start by reviewing several equivalent formulations of quantum mechanics and discussing their relevance for the study of quantum information processing. As we shall be using several of them in different contexts, it is useful to present them in a systematic way. The main two observations are, first, that a classical random variable can be identified with a quantum system equipped with a preferred basis, and second, that a quantum Hilbert space can always be extended to render all states pure (via a reference system) and all operations unitary (via an environment system) on the larger Hilbert space.

Both have been part of the quantum information processing folklore for at least a decade (the second of course goes back much farther: the GNS construction, Naimark’s and Stinespring’s theorems, see [35]), and roughly correspond to the “Church of the Larger Hilbert Space” viewpoint.

Based on this hierarchy of embeddings C(lassical) ⇒ Q(uantum) ⇒ P(ure), in the above sense, we shall see how the basic “CQ” formalism of quantum mechanics gets modified to (embedded into) CP, QQ, QP, PQ and PP formalisms. (The second letter refers to the way quantum information is presented; the first, how knowledge about this information is presented.) We stress that from an operational perspective they are all equivalent — they are just of variable expressive convenience in different situations.

Throughout the paper we shall use labels such as A (similarly, B, C, etc.) to denote not only a particular quantum system but also the corresponding Hilbert space (and to some degree even the set of bounded linear operators on that Hilbert space). When talking about tensor products of spaces, we will habitually omit the tensor sign, so $A \otimes B = AB$, etc. Labels such as $X, Y, \text{ etc.}$ will be used for classical random variables. For simplicity, all spaces and ranges of variables will be assumed to be finite.

The CQ formalism. This formalism is the most commonly used one in the literature, as it captures most of the operational features of a Copenhagen style quantum mechanics in the Schrödinger picture. The postulates of quantum mechanics can be classified into static and dynamic ones. The static postulates define the static entities of the theory, while the dynamic postulates describe the physically allowed evolution of the static entities.

The most general static entity is an ensemble of quantum states $(p_x, \rho_x)_{x \in X}$. The probability distribution $(p_x)_{x \in X}$ is defined on some set $X$ and is associated with the random variable $X$. The $\rho_x$ are density operators (positive Hermitian operators of unit trace) on the Hilbert space of a quantum system $A$. The state of the quantum system $A$ is thus correlated with the classical index random variable $X$. We refer to $XA$ as a hybrid classical-quantum system, and the ensemble $(p_x, \rho_x)_{x \in X}$ is the “state” of $XA$. We will occasionally refer to a classical-quantum system as a “$\{cq\}$ entity”. Special cases of $\{cq\}$ entities are $\{c\}$ entities (“classical systems”, i.e. random variables) and $\{q\}$ entities (quantum systems).

The most general dynamic entity would be a map between two $\{cq\}$ entities. Let us highlight only a few special cases:

A map between two $\{c\}$ entities is a stochastic map, or a $\{c \rightarrow c\}$ entity. It is defined by a conditional probability distribution $Q(y|x)$, where $x \in X$ and $y \in Y$. 
The most general map from a \{c\} entity to a \{q\} entity is a state preparation map or a \{c \to q\} entity. It is defined by a quantum alphabet \((\rho_x)_{x \in X}\) and maps the classical index \(x\) to the quantum state \(\rho_x\).

Next we have a \{q \to c\} entity, a quantum measurement, defined by a positive operator-valued measure (POVM) \((\Lambda_x)_{x \in X}\), where \(\Lambda_x\) are positive operators satisfying \(\sum_x \Lambda_x = I\), with the identity operator \(I\) on the underlying Hilbert space. The action of the POVM \((\Lambda_x)_{x \in X}\) on some quantum system \(\rho\) results in the random variable defined by the probability distribution \((\text{Tr} \, \rho \Lambda_x)_{x \in X}\) on \(X\). POVMs will throughout the paper be denoted by greek capitals.

A \{q \to q\} entity is a quantum operation, a completely positive and trace preserving (CPTP) map \(N : A \to B\), described (non-uniquely) by its Kraus representation: a set of operators \(\{N_x\}_{x \in X}\), \(\sum_x N_x^\dagger N_x = I_B\), whose action is given by

\[
N(\rho) = \sum_x N_x \rho N_x^\dagger.
\]

(In this paper, \(\dagger\) indicates the adjoint, while \(*\) is reserved for the complex conjugate.) A CP map is defined as above, but with the weaker restriction \(\sum_x A_x^\dagger A_x \leq I_B\), and by itself is unphysical (or rather, it includes a postselection of the system). Throughout, we will denote CP and CPTP maps by calligraphic letters: \(L, M, N, P, \ldots\). A special CPTP map is the identity on a system \(A\), \(\text{id}^A : A \to A\), with \(\text{id}^A(\rho) = \rho\). More generally, for an isometry \(U : A \to B\), we denote — for once deviating from the notation scheme outlined here — the corresponding CPTP map by the same letter: \(U(\rho) = U \rho U^\dagger\).

A \{q \to cq\} entity is an instrument \(\mathbb{P}\), described by an ordered set of CP maps \((\mathcal{P}_x)_{x}\) that add up to a CPTP map. \(\mathbb{P}\) maps a quantum state \(\rho\) to the ensemble \((p_x, \mathcal{P}_x(\rho)/p_x)_{x}\), with \(p_x = \text{Tr} \mathcal{P}_x(\rho)\). A special case of an instrument is one in which \(\mathcal{P}_x = p_x N_x\), and the \(N_x\) are CPTP; it is equivalent to an ensemble of CPTP maps, \((p_x, N_x)_{x \in X}\). Instruments will be denoted by blackboard style capitals: \(L, M, N, P, \ldots\).

A \{cq \to q\} entity is given by an ordered set of CPTP maps \((N_x)_{x}\), and maps the ensemble \((p_x, \rho_x)_{x \in X}\) to \(\sum_x p_x N_x(\rho_x)\).

In quantum information theory the CQ formalism is used for proving direct coding theorems of a part classical – part quantum nature, such as the HSW theorem [54, 57]. In addition, it is most suitable for computational purposes.

For two states, we write \(\varphi^{RA} \supset \rho^A\) to mean that the state \(\rho^A\) is a restriction of \(\varphi^{RA}\), namely \(\rho^A = \text{Tr}_R \varphi^{RA}\). The subsystem \(R\) is possibly null (which we write \(R = \emptyset\)), i.e., a 1-dimensional Hilbert space. Conversely, \(\varphi^{RA}\) is called an extension of \(\rho^A\). Furthermore, if \(\varphi^{RA} = |\varphi\rangle \langle \varphi|^{RA}\) is pure it is called a purification of \(\rho^R\). The purification is unique up to a local isometry on \(R\); this is an elementary consequence of the singular value decomposition, or Schmidt decomposition. These notions carry over to dynamic entities as well. For two quantum operations \(A : A \to BE\) and \(B : A \to B\) we write \(A \supset B\) if \(B = \text{Tr}_E \circ A\). If \(A\) is an isometry, it is called an isometric extension or Stinespring dilation [58] of \(B\), and is unique up to an isometry on \(E\).

Observe that we can safely represent noiseless quantum evolution by isometries between systems (whereas quantum mechanics demands unitarity). This is because our systems are all finite, and we can embed the isometries into unitaries on larger systems. Thus we lose no generality but gain flexibility.

**The CP formalism.** In order to define the CP formalism, it is necessary to review an alternative representation of the CQ formalism that involves fewer primitives. For instance,

- \(\{q\}\). A quantum state \(\rho^A\) is referred to by its purification \(|\phi\rangle^{AR}\).
- \(\{cq\}, \{c \to q\}\). The ensemble \((p_x, \rho_x^A)_{x}\) [resp. quantum alphabet \((\rho_x^A)_{x}\)] is similarly seen as the set of restrictions of a pure state ensemble \((p_x, |\phi_x\rangle^{AR})_{x}\) [resp. quantum alphabet \((|\phi_x\rangle^{AR})_{x}\)].
\begin{itemize}
  \item \{q \rightarrow q\}. A CPTP map $N : A \rightarrow B$ is referred to by its isometric extension $U_N : A \rightarrow BE$.
  \item \{q \rightarrow c\}. A POVM $(\Lambda_x)_x$ on the system $A$ is equivalent to some isometry $U_M : A \rightarrow AE_X$, followed by a von Neumann measurement of the system $E_X$ in basis $\{|x\rangle_{Ex}\}$, and discarding $A$.
  \item \{q \rightarrow cq\}. An instrument $P$ is equivalent to some isometry $U_\psi : A \rightarrow BEE_X$, followed by a von Neumann measurement of the system $E_X$ in basis $\{|x\rangle_{Ex}\}$, and discarding $E$.
  \item \{cq \rightarrow q\} The collection of CPTP maps $(N_x)_x$ is identified with the collection of isometric extensions $(U_{N_x})_x$.
\end{itemize}

In this alternative representation of the CQ formalism all the quantum static entities are thus seen as restrictions of pure states; all quantum dynamic entities are combinations of performing isometries, von Neumann measurements, and discarding auxiliary subsystems. The CP formalism is characterized by never discarding (tracing out) the auxiliary subsystems (reference systems, environments, ancillas); they are kept in the description of our system. As for the auxiliary subsystems that get (von-Neumann-) measured, without loss of generality they may be discarded: the leftover state of such a subsystem may be set to a standard state $|0\rangle$ (and hence decoupled from the rest of the system) by a local unitary conditional upon the measurement outcome. The CP formalism is mainly used in quantum information theory for proving direct coding theorems of a quantum nature, such as the quantum channel coding theorem (see e.g. [15]).

The QP formalism. The QP formalism differs from CP in that the classical random variables, i.e. classical systems, are embedded into quantum systems, thus enabling a unified treatment of the two.

\begin{itemize}
  \item \{c\}. The classical random variable $X$ is identified with a dummy quantum system $X$ equipped with preferred basis $\{|x\rangle^X\}$, in the state $\sigma^X = \sum_x p_x |x\rangle^X \langle x|^X$. The main difference between random variables and quantum systems is that random variables exist without reference to a particular physical implementation, or a particular system “containing” it. In the QP formalism this is reflected in the fact that the state $\sigma^X$ remains intact under the “copying” operation $\sqcup : X \rightarrow XX'$, with Kraus representation $\{|x\rangle^X|x\rangle^{X'} \langle x|^X\}$. In this way, instances of the same random variable may be contained in different physical systems.
  \item \{c \rightarrow c\}. The stochastic map $Q_{y|x}$ becomes the operation $\overline{N} : X' \rightarrow Y$ with Kraus representation $\{\sqrt{Q_{y|x}} |y\rangle^Y \langle x|^X\}_{x \in X, y \in Y}$. Since the operation $\overline{N}$ remains intact under copying the input, we can define the classical extension of $\overline{N}$ by the map $C_{\overline{N}} : X' \rightarrow YX$, 
  \[ C_{\overline{N}} = \overline{N} \circ \sqcup^{X' \rightarrow XX} \]
  The operation $C_{\overline{N}}$ thus implements $\overline{N}$ while storing a copy of the input in the system $X$.
  \item \{cq\}. An ensemble $(p_x, |\phi_x\rangle^{AR})_x$ is represented by a quantum state $\sigma^{XAR} = \sum_x p_x |x\rangle^X \langle x|^X \otimes |\phi_x\rangle^{AR}$.
  \item \{c \rightarrow q\}. A state preparation map $(|\phi_x\rangle^{AR})_x$ is given by the isometry $\sum_x |\phi_x\rangle^{AR} |x\rangle^X \langle x|^X$, followed by tracing out $X$.
  \item \{cq \rightarrow q\}. The collection of isometries $(U_x)_x$ is represented by the controlled isometry $\sum_x |x\rangle^X \langle x|^X \otimes U_x$. 
\end{itemize}
• \{q \to c\}, \{q \to cq\}. POVMs and instruments are treated as in the CP picture, except that the final von Neumann measurement is replaced by a completely dephasing operation \( \mathbb{1}_X : E_X \to X \), defined by the Kraus representation \( \{ |x \rangle^X \langle x |^E_X \} \).

The QP formalism is mainly used in quantum information theory for proving converse theorems.

Other formalisms. The QQ formalism is obtained from the QP formalism by tracing out the auxiliary systems, and is also convenient for proving converse theorems. In this formalism the primitives are general quantum states (static) and quantum operations (dynamic).

The PP formalism involves further “purifying” the classical systems in the QP formalism; it is distinguished by its remarkably simple structure: all of quantum information processing is described in terms of isometries on pure states. There is also a PQ formalism, for which we don’t see much use; one may also conceive of hybrid formalisms, such as QQ/QP, in which some but not all auxiliary systems are traced out. One should remain flexible. We will indicate which formalism is used in a given section.

2.2 Quantities, norms, inequalities, and miscellaneous notation

For a state \( \rho^{RA} \) and quantum operation \( \mathcal{N} : A \to B \) we often abuse notation, identifying

\[
\mathcal{N}(\rho) := (\text{id}^R \otimes \mathcal{N})\rho^{RA}.
\]

With each state \( \rho^B \), associate a quantum operation \( \mathcal{A}^\rho : A \to AB \) that appends the state to the input:

\[
\mathcal{A}^\rho(\sigma^A) = \sigma^A \otimes \rho^B.
\]

The state \( \rho \) and the operation \( \mathcal{A}^\rho \) are clearly equivalent in an operational sense.

Given some state, say \( \rho^{XAB} \), one may define the usual entropic quantities with respect to it. Recall the definition of the von Neumann entropy

\[
H(A) = H(A)_\rho = H(\rho^A) = -\text{Tr}(\rho^A \log \rho^A),
\]

where \( \rho^A = \text{Tr}_B \rho^{XAB} \). Further define the conditional entropy \( I_2 \)

\[
H(A|B) = H(A|B)_\rho = H(AB) - H(B),
\]

the quantum mutual information \( I_2 \)

\[
I(A; B) = I(A; B)_\rho = H(A) + H(B) - H(AB),
\]

the coherent information \( [49, 50] \)

\[
I(A\rangle B) = -H(A|B) = H(B) - H(AB),
\]

and the conditional mutual information

\[
I(A; B|X) = H(A|X) + H(B|X) - H(AB|X) = H(A\!X) + H(B\!X) - H(AB\!X) - H(X).
\]

Note that the conditional mutual information is always non-negative, thanks to strong subadditivity \( [45] \).

It should be noted that conditioning on classical variables (systems) amounts to averaging. For instance, for a state of the form

\[
\sigma^{XA} = \sum_x p_x |x\rangle^X \langle x |^A \otimes \rho^A_x,
\]
\[ H(A|X)_\sigma = \sum_x p_x H(A)_{\rho_x}. \]

We shall freely make use of standard identities for these entropic quantities, which are formally identical to their classical predecessors (see [14], Ch. 2). One such identity is the so-called chain rule for mutual information,

\[ I(A; BC) = I(A; B|C) + I(A; C), \]

and using it we can derive an identity will later be useful:

\[ I(X; AB) = H(A) + I(A)_{BX} - I(A; B) + I(X; B). \] (3)

We shall usually work in situations where the underlying state is unambiguous, but as shown above, we can emphasize the state by putting it in the subscript.

We measure the distance between two quantum states \( \rho^A \) and \( \sigma^A \) by the trace norm,

\[ \| \rho^A - \sigma^A \|_1, \]

where \( \| \omega \|_1 = \text{Tr} \sqrt{\omega^\dagger \omega} \). An important property of the trace distance is its monotonicity under quantum operations \( \mathcal{N} \):

\[ \| \mathcal{N}(\rho^A) - \mathcal{N}(\sigma^A) \|_1 \leq \| \rho^A - \sigma^A \|_1. \]

The trace distance is operationally connected to the distinguishability of the states. If \( \rho \) and \( \sigma \) have uniform prior, by Helstrom’s theorem [32] the maximum probability of correct identification of the state by a POVM is \( \frac{1}{2} + \frac{1}{4}\| \rho - \sigma \|_1 \).

The following lemma is a trivial application of Fannes’ inequality [24].

**Lemma 2.1** For the quantity \( I(\cdot|\cdot) \) defined on a system \( AB \) of total dimension \( d \), if \( \| \rho^{AB} - \sigma^{AB} \|_1 \leq \epsilon \) then

\[ |I(A|B)_\rho - I(A|B)_\sigma| \leq \eta(\epsilon) + K\epsilon \log d, \]

where \( \lim_{\epsilon \to 0} \eta(\epsilon) = 0 \) and \( K \) is some constant. The same holds for \( I(A; B) \) and other entropic quantities. \( \square \)

Define a distance measure between two quantum operations \( \mathcal{M}, \mathcal{N} : A_1A_2 \to B \) with respect to some state \( \omega^{A_1} \) by

\[ \| \mathcal{M} - \mathcal{N} \|_{\omega^{A_1}} := \max_{\xi^{R^{A_1}A_2}} \| (\text{id}^R \otimes \mathcal{M})\xi^{R^{A_1}A_2} - (\text{id}^R \otimes \mathcal{N})\xi^{R^{A_1}A_2} \|_1. \] (4)

The maximization may, w.l.o.g., be performed over pure states \( \xi^{R^{A_1}A_2} \). This is due to the monotonicity of trace distance under the partial trace map. Important extremes are when \( A_1 \) or \( A_2 \) are null. The first case measures absolute closeness between the two operations (and in fact, \( \| \cdot \|_\emptyset \) is the dual of the cb-norm, see [43]), while the second measures how similar they are relative to a particular input state. (4) is written more succinctly as

\[ \| \mathcal{M} - \mathcal{N} \|_{\omega} := \max_{\xi^{R^{A_1}A_2}} \| (\mathcal{M} - \mathcal{N})\xi \|_1. \]

We say that \( \mathcal{M} \) and \( \mathcal{N} \) are \( \epsilon \)-close with respect to \( \omega \) if

\[ \| \mathcal{M} - \mathcal{N} \|_{\omega} \leq \epsilon. \]

Note that \( \| \cdot \|_\omega \) is a norm only if \( \omega \) has full rank; otherwise, different operations can be at distance 0. If \( \rho \) and \( \sigma \) are \( \epsilon \)-close then so are \( \mathcal{A}^\rho \) and \( \mathcal{A}^\sigma \) (with respect to \( \emptyset \), hence every state).
Define the fidelity of two density operators with respect to each other as

\[ F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1^2 = \left( \text{Tr} \sqrt{\sqrt{\rho} \sqrt{\sigma}} \right)^2. \]

For two pure states |\phi\rangle, |\psi\rangle this amounts to

\[ F(\phi, \psi) = |\langle \phi | \psi \rangle|^2. \]

We shall need the following relation between fidelity and the trace distance \[ 2^{\frac{1}{4}} \]

\[ 1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}, \]

(5)

the second inequality becoming an equality for pure states. Uhlmann’s theorem \[ \text{[59, 40]} \]
states that, for any fixed purification |\phi\rangle of \sigma,

\[ F(\rho, \sigma) = \max_{|\psi\rangle \geq \rho} F(\psi, \phi). \]

As the fidelity is only defined between two states living on the same space, we are, of course, implicitly maximizing over extensions \psi that live on the same space as \phi.

**Lemma 2.2** If \|\rho - \sigma\|_1 \leq \epsilon and \sigma' \supseteq \sigma, then there exists some \rho' \supseteq \rho for which \|\rho' - \sigma'\|_1 \leq 2\sqrt{\epsilon}.

**Proof** Fix a purification |\phi\rangle|\phi\rangle^{ABC} \supseteq |\sigma'\rangle^{AB} \supseteq |\sigma\rangle^A.

By Uhlmann’s theorem, there exists some |\psi\rangle|\psi\rangle^{ABC} \supseteq |\rho\rangle^A such that

\[ F(\psi, \phi) = F(\rho, \sigma) \geq 1 - 2\epsilon, \]

using also \[ B \]

Define \rho^{ABC} = \text{Tr}_C |\psi\rangle|\psi\rangle^{ABC}.

By the monotonicity of trace distance under the partial trace map and \[ M \]

we have

\[ \|\rho' - \sigma'\|_1 \leq \|\psi - \phi\|_1 \leq 2\sqrt{\epsilon}, \]

as advertised. \( \square \)

**Corollary 2.3** Given an orthonormal basis \{|x\rangle\}, let \|\sum_x p_x|x\rangle - \sum_x q_x|x\rangle\|_1 \leq \epsilon. Define

\[ |\psi\rangle = \sum_x \sqrt{p_x}|x\rangle \]

and \[ |\phi\rangle = \sum_x \sqrt{q_x}|x\rangle. \]

Then

\[ \|\psi - \phi\|_1 \leq 2\sqrt{\epsilon}. \]

(6)

**Lemma 2.4** The following statements hold for density operators \omega^A, \omega'^{AA'} , \pi^A, \rho^A, \Omega^A, and quantum operations \mathcal{M}',\mathcal{N}' : AA'B \to C, \mathcal{M}, \mathcal{N} : AB \to C, \mathcal{K}, \mathcal{L} : AB' \to C', and \mathcal{M}_i, \mathcal{N}_i : A_iB_i \to A_{i+1}C_{i+1}.

1. If \omega' \supseteq \omega then \|\mathcal{M}' - \mathcal{N}'\|_{\omega'} \leq \|\mathcal{M}' - \mathcal{N}'\|_{\omega}.
2. \|\mathcal{M} - \mathcal{N}\|_{\omega} \leq \|\mathcal{M} - \mathcal{N}\|_{\sigma} + 4\sqrt{\|\omega - \sigma\|_1}.
3. \|\mathcal{M} \otimes \mathcal{K} - \mathcal{N} \otimes \mathcal{L}\|_{\omega \otimes \rho} \leq \|\mathcal{M} - \mathcal{N}\|_{\omega} + \|\mathcal{K} - \mathcal{L}\|_{\rho}.
4. \|\mathcal{M}_k \circ \cdots \circ \mathcal{M}_1 - \mathcal{N}_k \circ \cdots \circ \mathcal{N}_1\|_\Omega \leq \sum_i \|\mathcal{M}_i - \mathcal{N}_i\|_\Omega.\]

**Proof** Straightforward. \( \square \)

Finally, \[ \{n\} \]

denotes the set \{1, \ldots, n\} and if we have systems \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n, we use the shorthand

\[ \mathcal{A}^n = \mathcal{A}_1 \cdots \mathcal{A}_n. \]
3 Information processing resources

In this section, the notion of a information processing resource will be rigorously introduced. Unless stated otherwise, we shall be using the QQ formalism (and occasionally the QP formalism) in order to treat classical and quantum entities in a unified way.

3.1 The distant labs paradigm

The communication scenarios we will be interested in involve two or more separated parties. Each party is either active or passive. Active parties are allowed to perform arbitrary local operations in their lab for free, while passive ones are not allowed to perform any operations at all. Non-local operations (a.k.a. channels) and states connecting the parties are the principal objects of our theory. They are valuable resources and are carefully accounted for. In this paper, we consider the following parties:

- **Alice (A)** Alice is an active party, usually in the role of the sender.
- **Bob (B)** Bob is an active party, usually in the role of the receiver. In this paper we consider only problems involving communication from Alice to Bob. This means we work with channels from Alice to Bob (i.e. of the form $N : A' \rightarrow B$) and arbitrary states $\rho_{AB}$ shared by Alice and Bob. More generally we have feedback channels $N : A' \rightarrow AB$ with outputs on both sides.
- **Eve (E)** In the CP and QP formalisms, we purify noisy channels and states by giving a share to the environment. Thus, we replace $N : A' \rightarrow B$ with the isometry $U_N : A' \rightarrow BE$ and replace $\rho_{AB}$ with $\psi_{ABE}$. \(^1\) We consider a series of operations equivalent when they differ only by a unitary rotation of the environment.
- **Reference (R)** Suppose Alice wants to send an ensemble of states $\{p_i, |\alpha_i|^A\}$ to Bob with average density matrix $\rho^A = \sum_i p_i |\alpha_i|^A$. We would like to give a lower bound on the average fidelity of this transmission in terms only of $\rho$. Such a bound can be accomplished (in the CP/QP formalisms) by extending $\rho^A$ to a pure state $|\phi|^AR \supseteq \rho^A$ and finding the fidelity of the resulting state with the original state when $A$ is sent through the channel and $R$ is left untouched \([1]\). Here the reference system $R$ is introduced to guarantee that transmitting system $A$ preserves its entanglement with an arbitrary external system. Like the environment, $R$ is always inaccessible and its properties are not changed by local unitary rotations. Indeed the only freedom in choosing $|\phi|^AR$ is given by a local unitary rotation on $R$. Both the Reference and Eve are passive.
- **Source (S)** In most coding problems Alice can choose how she encodes the message, but cannot choose the message that she wants to communicate to Bob; it can be thought of as externally given. Taking this a step further, we can identify the source of the message as another protagonist (S), who begins a communication protocol by telling Alice which message to send to Bob. Alice’s communication task becomes to redirect the channel originating at the Source to Bob (Fig. 1). Introducing $S$ is useful in cases when the Source does more than simply send a state to Alice. For example, in distributed compression, the Source distributes a bipartite state to Alice and Bob. The Source is a passive party as it is not allowed to code.

To each party corresponds a class of quantum or classical systems which they control or have access to at different times. The systems corresponding to Alice are labeled by $A$ (for example, $A'$,

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\(^1\)In our paper, we think of Eve as a passive environment, but other work, for example on private communication \([15, 2]\), treats Eve as an active participant who is trying to maximize her information. In these settings, we introduce private environments for Alice and Bob $E_A$ and $E_B$, so that they can perform noisy operations locally without leaking information to Eve.
Figure 1: A channel (a) between Alice and Bob may be used in a source coding problem (b) to convert a channel from the Source to Alice, into a channel from the Source to Bob.

A1, XA, etc.), while Bob’s systems are labeled by B. When two classical systems, such as XA and XB, have the same principal label it means that they are instances of the same random variable. In our example, XA is Alice’s copy and XB is Bob’s copy of the random variable X.

We turn to some important examples of quantum states and operations. Let A, B, A’, XA and XB be d-dimensional systems with respective distinguished bases \{|x⟩^A\}, \{|x⟩^B\}, etc. The standard maximally entangled state on AB is given by

$$|Φ_d⟩^{AB} = \frac{1}{\sqrt{d}} \sum_{x=1}^{d} |x⟩^A |x⟩^B.$$  

The decohered, “classical”, version of this state is

$$Φ_{X_A X_B}^{X_A X_B} = \frac{1}{d} \sum_{x=1}^{d} |x⟩⟨x|_{X_A} \otimes |x⟩⟨x|_{X_B},$$  

which may be viewed as two maximally correlated random variables taking values on the set \([d] := \{1, \ldots, d\}\). The local restrictions of either of these states is the maximally mixed state \(τ_d := \frac{1}{d} I_d\). (We write \(τ\) to remind us that it is also known as the tracial state.) Define the identity quantum operation \(id_d : A’ \rightarrow B\) by the isometry \(\sum_x |x⟩^B ⟨x|^{A’}\) (Note that this requires fixed bases of \(A’\) and \(B\)!). It represents a perfect quantum channel between the systems \(A’\) and \(B\). Its classical counterpart is the completely dephasing channel \(id_d : X_A’ \rightarrow X_B\), given in the Kraus representation by \(\{ |x⟩^B ⟨x|^{X_A’}\}_{x \in [d]}\). It corresponds to a perfect classical channel because it perfectly transmits density operators diagonal in the preferred basis, i.e. random variables. The channel \(Δ_d : X_A’ \rightarrow X_A X_B\) with Kraus representation \(\{ |x⟩^X_B ⟨x|^{X_A}⟨x|^{X_A’}\}_{x \in [d]}\) is a variation on \(i_d\) in which Alice first makes a (classical) copy of the data before sending it through the classical channel. The two channels are essentially interchangeable. Of considerable interest is the so-called coherent channel \(Δ_d : A’ \rightarrow AB\), given by the isometry \(\sum_x |x⟩^A |x⟩^B ⟨x|^{A’}\) which is a coherent version of the noiseless classical channel with feedback, \(Δ_d\). Here and in the following, “coherent” is meant to say that the operation preserves coherent quantum superpositions.

The maximally entangled state \(|Φ_d⟩^{AB}\) and perfect quantum channel \(id_d : A’ \rightarrow B\) are locally basis covariant: \((U ⊗ U^*)|Φ_d⟩^{AB} = |Φ_d⟩^{AB}\) and \(U^† \circ id_d \circ U = id_d\) for any unitary \(U\). On the other hand, \(Φ_d, i_d, Δ_d\) and \(Δ_d\) are all locally basis-dependent.

### 3.2 Finite resources

In this subsection we introduce finite or non-asymptotic resources. The central theme, which will carry over to the asymptotic setting, is that of comparing two resources. We introduce the notion of a protocol in which resource 1 is consumed in order to simulate resource 2. We then consider resource 1 to be at least as strong as resource 2 (for any asymptotic communication task).
Definition 3.1 (Static and dynamic resources) A finite static resource is a quantum state $\rho^{AB}$ shared between Alice and Bob. Let $\mathcal{N}: A' \rightarrow AB$ be a quantum operation which takes states living on Alice’s system $A'$ to a system $AB$ shared by Alice and Bob. The test state $\omega^{A^{\text{rel}}}$ lives on a subsystem $A^{\text{rel}}$ of $A = A^{\text{abs}} A^{\text{rel}}$. A finite dynamic resource is the ordered pair $(\mathcal{N}: \omega)$. A static resource $\rho$ is a special kind of dynamic resource because of its equivalence to appending maps.

States and channels can be used to perform information processing tasks of interest. Hence the name “resource”. The operation $\mathcal{N}$ comes with a test state because it “expects” an extension of $\omega$ as input. This will be formalized in Definition 3.6. If $A^{\text{rel}} = \emptyset$, we identify $(\mathcal{N}: \omega)$ with the proper dynamic resource $\mathcal{N}$. This is the usual notion of a quantum channel which can be used without restriction. Note that $\mathcal{A}^o$ is always a proper dynamic resource, as it has no inputs. The dynamic resource $(\mathcal{N}: \omega)$ is called relative if $A^{\text{abs}} = \emptyset$.

Definition 3.2 ( Protected resources) Let $\mathcal{N}: S \rightarrow AB$ be a quantum operation which takes states living on the Source system $S$ to a system $AB$ shared by Alice and Bob. The source state $\omega^S$ lives on the system $S$. A finite protected resource is the ordered pair $(\mathcal{N}: \omega)$.

A protected resource differs from a relative dynamic resource only in that it originates at the Source. An example of a “source coding” problem is Schumacher compression. There Alice expects a particular state from the Source, channeled through $\mathcal{N}$. Information coming from the Source is supposed to be preserved (albeit redirected — see Fig. 1), and restrictions exist on the allowed operations. Hence the adjective “protected”. This is formalized in Definition 3.6.

We now unify the concepts of protected and unprotected (static and dynamic) resources.

Definition 3.3 (Generalized resources) Let $\mathcal{N}: A'S \rightarrow AB$ be a quantum operation which takes states living on the joint Alice-Source system $A'S$ to a system $AB$ shared by Alice and Bob. Define $\omega^{A'^{\text{rel}}}$ and $\rho^S$ as above. A finite generalized resource is the ordered pair $(\mathcal{N}: [\omega \otimes \rho])$.

In the next couple of paragraphs when we speak of resources we mean finite generalized resources. We will often omit the system labels and absorb $\rho$ into $\omega$.

A resource $(\mathcal{N}: \omega)$ is called pure if $\mathcal{N}$ is an isometry. It is called classical if $\mathcal{N}$ is a $\{c \rightarrow c\}$ entity.

Define a distance measure between two dynamic resources $(\mathcal{N}: \omega)$ and $(\mathcal{N}': \omega')$ with the same test state as

$$|| (\mathcal{N}' : \omega) - (\mathcal{N} : \omega) || := || \mathcal{N}' - \mathcal{N} ||_{\omega}.$$ (If they have different test states then the distance is undefined.) Define the tensor product of resources as

$$(\mathcal{N}_1 : \omega_1) \otimes (\mathcal{N}_2 : \omega_2) := (\mathcal{N}_1 \otimes \mathcal{N}_2 : \omega_1 \otimes \omega_2).$$

Definition 3.4 ( Reduction) We are given two resources $(\mathcal{N} : \omega)$ and $(\mathcal{N}': \omega')$. We say that $(\mathcal{N} : \omega)$ reduces to $(\mathcal{N}': \omega')$ and write

$$(\mathcal{N} : \omega) \overset{\ast}{\rightarrow} (\mathcal{N}': \omega')$$

if there exist encoding and decoding channels $\mathcal{E}$ and $\mathcal{D}$ such that $\omega = \mathcal{E}(\omega')$ and $\mathcal{N}'(\rho) = \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\rho)$.

The reduction has an operational significance. One can simulate $(\mathcal{N}': \omega')$ using the resource $(\mathcal{N} : \omega)$ by means of feeding some dummy input $\omega''$ along with the “genuine” input $\omega'$. By definition $\omega''$ is of the form $\omega^{A'^{\text{rel}}} \otimes \rho^S$, and may thus be locally prepared. The canonical example of a reduction is that from $(\mathcal{N} : \omega)^{\otimes 2}$ to $(\mathcal{N} : \omega)$. This natural reduction would cease to hold had we allowed source/test states to be generally correlated states between spatially separated parties.
Resources as defined above are atomic primitives. If you have a resource $N$, you are allowed to apply the operation $N$ only once. This is why we speak of consuming resources. If you have a resource $N_1 \otimes N_2$, you have to apply the two channels in parallel. You would not be able to use the output of $N_1$ as an input to $N_2$. We extend our original definition in order to allow for such sequential use of resources.

**Definition 3.6 (Depth-$\ell$ resources)** A finite depth-$\ell$ resource is an unordered collection of “component” resources

$$(N : \omega)^\ell := ((N_1 : \omega_1), \ldots, (N_\ell : \omega_\ell)).$$

What we previously called resources are now identified as depth-$1$ resources. To avoid notational confusion, for $\ell$ copies of the same resource, $((N : \omega), \ldots, (N : \omega))$, we reserve the notation $(N : \omega)^\ell$.

The definition of the distance measure naturally extends to the case of two depth-$\ell$ resources:

$$\|(N' : \omega)^\ell - (N : \omega)^\ell\| := \min_{\pi \in S_\ell} \sum_{j=1}^{\ell} \| (N'_j : \omega_j) - (N_{\pi(j)} : \omega_{\pi(j)}) \|.$$ 

Here $S_\ell$ is the set of permutations on $\{1, \ldots, \ell\}$ objects; we need to minimize over it to reflect the fact that we are free to use depth-$\ell$ resources in an arbitrary order.

To combine resources there is no good definition of a tensor product (which operations should we take the products of?), but we can take tensor powers of a resource:

$$(N : \omega)^\ell \otimes k := ((N_1 : \omega_1)^\otimes k, \ldots, (N_\ell : \omega_\ell)^\otimes k).$$

The way we combine a depth-$\ell$ and a depth-$\ell'$ resource is by concatenation. From $(N : \omega)^\ell$ and $(N' : \omega')^\ell'$ we obtain

$$(N_1 : \omega_1), \ldots, (N_\ell : \omega_\ell), (N'_1 : \omega'_1), \ldots, (N'_{\ell'} : \omega'_{\ell'})).$$

For resources with depth $> 1$, $(N : \omega)^\ell = ((N_1 : \omega_1), \ldots, (N_\ell : \omega_\ell))$ and $(N' : \omega')^\ell = ((N'_1 : \omega'_1), \ldots, (N'_{\ell'} : \omega'_{\ell'}))$, we say that $(N : \omega) \geq (N' : \omega')$ if there exists an injective function $f : [\ell'] \rightarrow [\ell]$ such that for all $i \in [\ell']$, $(N_{f(i)} : \omega_{f(i)}) \geq (N'_i : \omega'_i)$. In other words, for each $(N'_i : \omega'_i)$ there is a unique $(N_j : \omega_j)$ that reduces to $(N'_i : \omega'_i)$. Note that this implies $\ell \geq \ell'$.

Now we are in a position to define a protocol as a general way of simulating or creating a depth-$1$ resource while consuming a depth-$\ell$ resource. At the same time we introduce the notions of approximation that will be essential for the treatment of asymptotic resources in Section 3.3.

**Definition 3.6 (Protocol)** A depth-$\ell$ protocol $P$ is a map taking a depth-$\ell$ resource to a depth-1 resource. Define the depth-$\ell$ resource $(N : \omega \otimes \rho)^\ell$ by the operations $N_i : A_i[\omega_i \rightarrow A_i B_i]$, $A'_i = A'_{i}^{\text{rel}} A'_{i}^{\text{abs}}$ and test/source states $\omega_i^{\text{rel}} \otimes \rho_{i}^{\text{rel}}$, $i = 1, \ldots, \ell$. Then $P((N : \omega \otimes \rho)^\ell)$ is the finite depth-$\ell$ resource $P : \Omega^{\text{rel}} \otimes \Theta^S$, where $\Theta^S$ is a restriction of $\bigotimes_i \rho_i^{S_i}$ to a subsystem $S$ of $S'$; the quantum map $P : A' S \rightarrow AB$, $A' = A'^{\text{rel}} A'^{\text{abs}}$, is the following composition of operations:

1. select a permutation $\pi$ of the integers $\{1, \ldots, \ell\}$;
2. perform local operations $E_0 : A' \rightarrow A'_0 A'^{\text{aux}}$;

We use diverse notation to emphasize the role of the systems in question. The primed systems, such as $A'_i$, are channel inputs. Test systems like $A'^{\text{rel}}_i$ are always subsystems of the corresponding channel input $A'_i$. In case of operations originating at the Source, the test system is the full input system $S_i$. The systems with no superscript, such as $B_i$, are channel outputs. Furthermore, there are auxiliary systems, such as $A'^{\text{aux}}$. Of course many of these systems can be null (i.e. one-dimensional).
Definition 3.8 (Tensor product of protocols)
Given protocols $P$ and $Q$, the simulation of the flattened version of any resource. Definition 3.6 does not account for the simulation of resources of arbitrary depth. It does allow the arbitrary permutation of the resources $\pi$ so that depth-$\ell$ resources do not have to be used in a fixed order. Denote by $P_\pi$ the composition of all operations through step 3(a). Define $P_\pi$ to be $P_\pi$ followed by a restriction onto $A^0_\text{rel}$. The protocol $P$ is called $\eta$-valid on the input finite resource $(N : \omega \otimes \rho)^\ell$ if the conditions
\[
\|P_\pi(\xi) - \omega_{\pi(i)}^{A^0_\text{rel}}\|_1 \leq \eta
\]
are met for all $i$ and for all extensions $\xi$ of $\Omega^{A^0_\text{rel}} \otimes \Theta^{S}$. Whenever the input resource is clear from the context, we will just say that the protocol is $\eta$-valid.

A protocol is thus defined to be the most general way one can use the available resources to generate a new one. Each use of a resource $(N_i : \omega_i \otimes \rho_i)$ is preceded by Alice’s encoding layer (the operations $E_i$) which prepares an appropriate input based on feedback from the preceding layer and memory exemplified in the auxiliary system $A^\text{aux}_i$. The $\eta$-validity condition ensures that each operation $N_i$ acts on a extension of a state close to $\omega_i \otimes \rho_i$. The Source has a passive role and is not allowed to freely shape her input states like Alice can. Thus we require that the source state $\Theta^S$ for the created resource is a restriction of the source state $\otimes \rho_i^{\text{rel}_i}$ for the consumed resources. In contrast, the test systems are virtual, and change from $A^0_\text{rel}_1 \ldots A^0_\text{rel}_\ell$ to $A^0_\text{rel}$.

The protocol $P$ is completely characterized by the ordered $\ell + 3$-tuple $(\pi, E_0, \ldots, E_{\ell+1}, D)$. Thus we may write
\[
P = (\pi, E_0, \ldots, E_{\ell+1}, D).
\]

The notion of a reduction from Definition 3.4 provides a simple example of a 0-valid protocol. If $(N : \omega) \supseteq (N' : \omega')$ then there exists a protocol $R$ such that $R[(N : \omega)] = (N' : \omega')$. Another important example is given below.

Definition 3.7 (Standard protocol) Define the standard protocol $S$, which is a 0-valid elementary protocol on a depth-$\ell$ finite resource $(N : \omega)^\ell$, by
\[
S[(N : \omega)^\ell] = \bigotimes_{i=1}^{\ell} (N_i : \omega_i).
\]

This protocol takes a collection of resources and “flattens” them into a depth-1 tensor product. The standard protocol will play a major role in the asymptotic theory of resources in Section 3.3. Definition 3.4 does not account for the simulation of resources of arbitrary depth. It does allow the simulation of the flattened version of any resource.

We can define a tensor product of two protocols by their parallel execution.

Definition 3.8 (Tensor product of protocols) Given protocols $P_1 = (\pi_1, E_{1,0}, \ldots, E_{1,\ell_1+1}, D_1)$ and $P_2 = (\pi_2, E_{2,0}, \ldots, E_{2,\ell_2+1}, D_2)$ acting on two separate systems, define $P_1 \otimes P_2$ by
\[
P_1 \otimes P_2 = ((\pi_1, \pi_2), E_{1,0}, \ldots, E_{1,\ell_1+1}, E_{2,0}, \ldots, E_{2,\ell_2+1}, D_1 \otimes D_2).
\]

\[3\text{recall, } B^\ell = B_1 \ldots B_\ell.
\]

\[4\text{The simplest situation, which is seen in Fig. 3, is when the input states for the consumed and created resource are identical.}\]
Corollary 3.9 If $P_1$ and $P_2$ are $\eta$-valid on $(N_1 : \omega_1)^{\ell_1}$ and $(N_2 : \omega_2)^{\ell_2}$, respectively, then $P_1 \otimes P_2$ is $\eta$-valid on $((N_1 : \omega_1)^{\ell_1}, (N_2 : \omega_2)^{\ell_2})$.

The following three lemmas are straightforward exercises in applying the definitions and are given without proof.

Lemma 3.10 If $(N_1 : \omega_1)^{\ell_1} \geq (N_2 : \omega_2)^{\ell_2}$ then

$$S[(N_1 : \omega_1)^{\ell_1}] \geq S[(N_2 : \omega_2)^{\ell_2}].$$

Lemma 3.11 If $P$ is an $\eta$-valid protocol for which

$$\|P[(N_2 : \omega_2)^{\ell_2}] - S[(N_3 : \omega_3)^{\ell_3}]\|_1 \leq \epsilon$$

and

$$(N_1 : \omega_1)^{\ell_1} \geq (N_2 : \omega_2)^{\ell_2}$$

$$(N_3 : \omega_3)^{\ell_3} \geq (N_4 : \omega_4)^{\ell_4}$$

then there is an $\eta$-valid protocol $P'$ such that

$$\|P'[(N_1 : \omega_1)^{\ell_1}] - S[(N_4 : \omega_4)^{\ell_4}]\|_1 \leq \epsilon.$$

Lemma 3.12 If, for $i = 1, 2$,

$$\|P_i[(N_i : \omega_i)^{\ell_i}] - S[(N_i' : \omega_i')^{\ell_i'}]\|_1 \leq \epsilon_i,$$

then

$$\|P_1 \otimes P_2)[((N_1 : \omega_1)^{\ell_1}, (N_2 : \omega_2)^{\ell_2})] - S[((N_1' : \omega_1')^{\ell_1'}, (N_2' : \omega_2')^{\ell_2'})]\|_1 \leq \epsilon_1 + \epsilon_2.$$
Lemma 3.14 (Continuity) If some protocol $P$ is $\eta$-valid on $[(N : \omega)\ell]$ and
\[ \| (N : \omega)^\ell - (N' : \omega)^\ell \|_1 \leq \epsilon, \]
then it is $(\epsilon + \eta + 4\ell\sqrt{\eta})$-valid on $(N' : \omega)^\ell$ and
\[ \| P[(N : \omega)^\ell] - P[(N' : \omega)^\ell] \| \leq \epsilon + 4\ell\sqrt{\eta}. \]

Proof Let $(P : \Omega) = P[(N : \omega)^\ell]$ and $(P' : \Omega) = P[(N' : \omega)^\ell]$. By definition Lemma 3.6, $P$ is of the form
\[ P = E_{\ell+1} \circ N_\ell \circ E_{\ell} \circ \cdots \circ N_1 \circ E_1 \circ E_0, \]
and similarly for $P'$. The $\eta$-validity condition reads, for all $i$ and for all extensions $\xi$ of $\Omega$,
\[ \| \hat{P}_i(\xi) - \omega_i \|_1 \leq \eta. \]
By part 4 of Lemma 2.4,
\[ \| P - P' \|_{\xi} \leq \sum_i \| N'_i - N_i \|_{P,(\xi)}. \]
By part 1 of Lemma 2.4
\[ \| N'_i - N_i \|_{P,(\xi)} \leq \| N'_i - N_i \|_{P,(\xi)}. \]
By part 2 of Lemma 2.4 and $\eta$-validity
\[ \| N'_i - N_i \|_{P,(\xi)} \leq \| N'_i - N_i \|_{\omega_i} + 4\sqrt{\eta} \]
Hence
\[ \| P - P' \|_{\xi} \leq \epsilon + 4\ell\sqrt{\eta}, \]
which is one of the statements of the lemma. To estimate the validity of $P$ on $[(N' : \omega)^\ell]$, note that one obtains in the same way as above, for all $i$,
\[ \| \hat{P}_i - \hat{P}'_i \|_{\xi} \leq \epsilon + 4\ell\sqrt{\eta}. \]
Combining this with the $\eta$-validity condition via the triangle inequality finally gives
\[ \| \hat{P}'_i(\xi) - \omega_i \|_1 \leq \epsilon + \eta + 4\ell\sqrt{\eta}, \]
concluding the proof. \qed

Recall that we can only simulate a depth-$\ell$ resource flattened by the standard protocol. The following lemma states that the standard protocol is in a sense sufficient to generate any other, under some i.i.d.-like assumptions. Thus working with depth-1 resources is not overly restrictive.

Lemma 3.15 (Sliding) If for some depth-$\ell$ finite resource $(N : \omega)^\ell = ((N_1 : \omega_1), \ldots, (N_\ell : \omega_\ell))$ and quantum operation $C$,
\[ \| (C : \bigotimes_i \omega_i) - S[(N : \omega)^\ell] \|_1 \leq \epsilon, \] (7)
then for any integer $m \geq 1$ and for any $\eta$-valid protocol $P$ on $(N' : \omega)^\ell$, there exists an $(m + \ell - 1)(\epsilon + 4\sqrt{\eta})$-valid protocol $K$ on $(C : \bigotimes_i \omega_i)^{\times (m + \ell - 1)}$, such that
\[ \| K[(C : \bigotimes_i \omega_i)^{\times (m + \ell - 1)}] - (P[(N' : \omega)^\ell])^{\otimes m} \| \leq (m + \ell - 1)(\epsilon + 4\sqrt{\eta}). \]
Figure 2: The sliding protocol $K$.

**Proof** Let $(P : \Omega) = P[(N : \omega)^i]$ and $(M, \otimes, \omega_i) = S[(N : \omega)^i]$. Let $P = (\pi, \mathcal{E}_1, \ldots, \mathcal{E}_{\ell+1}, D)$, absorbing $\mathcal{E}_0$ into $\mathcal{E}_1$ and w.l.o.g. assuming that $\pi$ is the identity permutation $id$. We start by defining the sliding protocol $K$ for which we show that

$$K[(S[(N : \omega)^i]) \times (m + \ell - 1)] = (P[(N : \omega)^i]) \otimes^m.$$  

(8)

In other words, the protocol $K$ effects the map $P = D \circ N_\ell \cdots N_1 \circ \mathcal{E}_1$ on each of the $m$ realizations $\Omega^{(1)}, \ldots, \Omega^{(m)}$ of the test/source state $\Omega$. In the $i$th round of the protocol, $i = 1, \ldots, m + \ell - 1$, a realization $M_i$ of the map $M$ must be applied to the input $\otimes_j \omega_j$. The structure of the protocol is shown in Fig. 2 which is perhaps more useful than the formal description below. We proceed to decribe the elements of $K = (id, F_0, F_1, \ldots, F_{m + \ell}, D')$. Let $\xi_i^{A_{\text{rel}}}$ be some dummy locally prepared extension of $\omega_i^{A_{\text{rel}}}$. $F_0$ consists of Alice appending a number of such states, yielding

$$\xi_{\ell} \otimes \cdots \otimes \xi_2 \otimes \Omega^{(1)} \otimes \cdots \otimes \Omega^{(m)} \otimes \xi_1 \otimes \cdots \otimes \xi_1.$$

For $1 \leq i \leq \ell + 1$,

$$F_i = I \otimes^{\ell - 1} \otimes \mathcal{E}_i \otimes \cdots \otimes \mathcal{E}_1 \otimes I \otimes^m \otimes^{\ell - 1}.$$

For $\ell + 2 \leq i \leq m$,

$$F_i = I \otimes^{\ell - 2} \otimes \mathcal{E}_{\ell + 1} \otimes \cdots \otimes \mathcal{E}_1 \otimes I \otimes^m \otimes^{\ell - 1}.$$

For $m + 1 \leq i \leq m + \ell$,

$$F_i = I \otimes^{\ell - 2} \otimes \mathcal{E}_{\ell + 1} \otimes \cdots \otimes \mathcal{E}_{-m + 1} \otimes I \otimes^{\ell - 1}.$$

For $1 \leq i \leq m + \ell - 1$,

$$M_i = I \otimes^{i - 1} \otimes N_\ell \otimes \cdots \otimes N_1 \otimes I \otimes^{m + \ell - i - 1}.$$

Finally Bob performs $D$ on the middle $m$ systems and traces out the first and last $\ell - 1$; in other words,

$$D' = Tr_i \otimes^{\ell - 1} \otimes D \otimes^m \otimes Tr_i \otimes^{\ell - 1}.$$  

The claim follows by inspection. Observe that if $P$ is $\eta$-valid then $K$ is $\ell \eta$-valid.

Condition (7) implies that

$$\|(C : \bigotimes_i \omega_i)^\times(m + \ell - 1) - (S[(N : \omega)^i])^\times(m + \ell - 1)\|_1 \leq (m + \ell - 1)\epsilon.$$  

(9)
Relative resources are only guaranteed to work properly when applied to the corresponding test state. Here we show that using shared randomness, some of the standard relative resources can be “absolutized,” removing the restriction to a particular input state.

Lemma 3.16 For an operation $\mathcal{N} : A' \to AB$ which is either the perfect quantum channel $\text{id}_d$, the coherent channel $\Delta_d$ or the perfect classical channel $\text{id}_d$, there exists a 0-valid protocol $\mathbf{P}$ such that

$$\mathbf{P}[\Phi^{X_A X_B}, (\mathcal{N} : \tau^{A'})] = \mathcal{N} \otimes \Phi^{X_A X_B},$$

where $\dim X_A = (\dim A')^2$, and $\tau^{A'}$ is the maximally mixed state on $A'$.

Proof Consider first the case where $\mathcal{N}$ is either $\text{id}_d$ or the coherent channel $\Delta_d$. The main observation is that there exists a set of unitary operations $\{U_x\}_{x \in [d^2]}$ (the generalized Pauli, or discrete Weyl, operators) living on a $d$-dimensional Hilbert space such that

(i) For any state $\rho$

$$d^{-2} \sum_x U_x \rho U_x^\dagger = \tau_d,$$

(ii) There exists a set of unitary operators $\{V_x\}_{x \in [d^2]}$ such that

$$\Delta_d \circ U_x = [V_x \otimes U_x] \circ \Delta_d.$$

Let Alice and Bob share the common randomness state

$$\Phi^{X_A X_B} = d^{-2} \sum_{x=1}^{d^2} |x\rangle \langle x|^{X_A} \otimes |x\rangle \langle x|^{X_B},$$

where $d := \dim A'$. Consider an arbitrary input state $|\phi\rangle^{RA'}$, possibly entangled between Alice and a reference system $R$. Alice performs the conditional unitary $\sum_x |x\rangle \langle x|^{X_A} \otimes U_x^{A'}$, yielding a state whose restriction to $A'$ is precisely $\tau^{A'}$. She then applies the operation $\mathcal{N}$ (this is 0-valid!), which gives the state

$$d^{-2} \sum_{x=1}^{d^2} |x\rangle \langle x|^{X_A} \otimes |x\rangle \langle x|^{X_B} \otimes (\mathcal{N} \circ U_x^{A'}) |\phi\rangle^{RA'}.$$

In the case of the $\text{id}_d$ channel, Bob simply applies the conditional unitary $\sum_x |x\rangle \langle x|^{X_B} \otimes (U_x^{-1})^{B'}$. In the case of the $\Delta_d$ channel Alice must also perform

$$\sum_x |x\rangle \langle x|^{X_A} \otimes (V_x^{-1})^{A'}.$$

In either case, the final state is

$$\Phi^{X_A X_B} \otimes \mathcal{N}(|\phi\rangle^{RA'}),$$

as advertised.

The case where $\mathcal{N}$ is the perfect classical channel $\text{id}_d$ is a classical analogue of the above. The observation here is that there exists a set of $d$ unitaries $\{U_x\}_{x \in [d]}$ (all the cyclic shifts of the basis vectors), such that

(i) $\text{(10)}$ holds for any state $\rho$ diagonal in the preferred basis.

(ii) $U_x \circ \text{id}_d = \text{id}_d \circ U_x \circ \text{id}_d.$
Alice first applies a local $\operatorname{id}_d$ on the $A'$ system (making the state of $A'$ input diagonal) before proceeding as above. This concludes the proof.

In the above lemma, the final output of $\mathcal{N}$ is uncorrelated with the shared randomness that is used. In the QQ formalism, this is immediate from the tensor product form of (12). Thus we say that the shared randomness is (incoherently) decoupled from the rest of the protocol.

If we move to the QP formalism, so $\mathcal{N}$ replaced by $U_{\mathcal{N}} : A \rightarrow BE$, this decoupling need not hold any more. When $\mathcal{N} = \operatorname{id}_d$, the common randomness will remain coupled to the $E$ system for a particular input state $\sigma_R^{RA'}$. In a cryptographic setting this means that Eve has acquired information about the key $\mathcal{F}_{X_A X_B}$. When $\mathcal{N}$ is an isometry such as $\Delta_d$ or $\operatorname{id}_d$ then the shared randomness is decoupled even from the environment. This stronger form of decoupling is called coherent decoupling. Below we extend these notions of decoupling to arbitrary classical resources.\(^5\)

**Definition 3.17 (Incoherent decoupling of input resources)** Consider some $\{c \rightarrow c\}$ entity $\mathcal{N}_1 : X'_1 \rightarrow Y_1$ with classical extension $\mathcal{C}_{\mathcal{N}_1} : X'_1 \rightarrow Y_1 X_1$. This induces a modification of the depth-$\ell$ resource $(\mathcal{N} : \omega)^\ell$,

$$(\mathcal{N} : \omega)^\ell = ((\mathcal{C}_{\mathcal{N}_1} : \omega_1), (\mathcal{N}_2 : \omega_2), \ldots, (\mathcal{N}_\ell : \omega_\ell)).$$

For some protocol $\mathcal{P}$, define

$$(\mathcal{P} : \Omega) = \mathcal{P}[(\mathcal{N} : \omega)^\ell].$$

Assume that for all extensions $\xi$ of $\Omega$

$$\|\sigma^{X_1 Q} - \sigma^{X_1} \otimes \sigma^Q\|_1 \leq \epsilon,$$

where $\sigma^{X_1,Q} = \mathcal{P}(\xi)$. Then we say that the classical resource $(\mathcal{N}_1 : \omega_1)$ is $\epsilon$-incoherently decoupled (or just $\epsilon$-decoupled) with respect to the protocol $\mathcal{P}$ on $(\mathcal{N} : \omega)^\ell$.

**Definition 3.18 (Coherent decoupling of input resources)** Consider the setting of the previous definition. Now adopt a QP view in which all operations except for the classical $\mathcal{N}_1$ are isometrically extended. Thus $(\mathcal{N} : \omega)^\ell$ is replaced by

$$(\mathcal{N}'' : \omega)^\ell = ((\mathcal{C}_{\mathcal{N}_1} : \omega_1), (\mathcal{U}_{\mathcal{N}_2} : \omega_2), \ldots, (\mathcal{U}_{\mathcal{N}_\ell} : \omega_\ell))$$

and $\mathcal{P} = (\pi, \mathcal{E}_0, \ldots, \mathcal{D})$ is replaced by $\mathcal{P}' = (\pi, \mathcal{U}_{\mathcal{E}_0}, \ldots, \mathcal{U}_{\mathcal{D}})$. Let

$$(\mathcal{P}' : \Omega) = \mathcal{P}'[(\mathcal{N}'' : \omega)^\ell].$$

Assume that for all extensions $\xi$ of $\Omega$

$$\|\sigma^{X_1 QE} - \sigma^{X_1} \otimes \sigma^{QE}\|_1 \leq \epsilon,$$

where $\sigma^{X_1,QE} = \mathcal{P}'(\xi)$. Then we say that the classical resource $(\mathcal{N}_1 : \omega_1)$ is $\epsilon$-coherently decoupled with respect to the protocol $\mathcal{P}$ on $(\mathcal{N} : \omega)^\ell$.

The above definitions naturally extend to the case where $(\mathcal{N}_1 : \omega_1)$ is replaced by a resource of arbitrary depth. In this case each component resource must be $\epsilon$-decoupled.

**Definition 3.19 (Coherent decoupling of output resources)** Let $\mathcal{P}$ be a protocol mapping $(\mathcal{N} : \omega)^\ell$ to $(\mathcal{P}_1 : \Omega_1) \otimes (\mathcal{P}_2 : \Omega_2)$, where $\mathcal{P}_1 : X'_1 \rightarrow Y_1$ is a $\{c \rightarrow c\}$ entity with classical extension

\(^5\)The notion of an “oblivious” protocol for remotely preparing quantum states is similar to coherent decoupling, but applies instead to quantum messages [14].
$C_P : X'_1 \to Y_1X_1$. Let $(N' : \omega)^\ell$ be the modification of $(N : \omega)^\ell$ in which all operations are isometrically extended. Replace $P = (\pi, \xi_0, \ldots, \mathcal{D})$ by $P' = (\pi, \Sigma^{X'_1 \to X_1}, \xi_0, \ldots, \mathcal{D})$. Let

$$
(P' : \Omega) = P'[\{(N' : \omega)^\ell\}].
$$

Assume that for all extensions $\xi$ of $\Omega$

$$
\|\sigma^{X_1Y_1}QE - \sigma^{X_1Y_1} \otimes \sigma^{QE}\|_1 \leq \epsilon,
$$

(15)

where $\sigma^{X_1Y_1}QE = P'(\xi)$. Then we say that the that the output classical resource $(P_1 : \Omega_1)$ is $\epsilon$–coherently decoupled with respect to the protocol $P$ on $(N : \omega)^\ell$.

One simple example of decoupling is when a protocol involves several pure resources (i.e. isometries) and one noiseless classical resource. In this case, decoupling the classical resource is rather easy, since pure resources don’t involve the environment. However, it is possible that the classical communication is correlated with the ancilla system $Q$ that Alice and Bob are left with. If $Q$ is merely discarded, then the cbits will be incoherently decoupled. To prove that coherent decoupling is in fact possible, one has to carefully account for the ancillas produced by the classical communication. This was performed in [27], which proved that classical messages sent through isometric channels can always be coherently decoupled.

In this paper, we will instead focus on examples of decoupled classical communication obtained through noisy channel coding.

### 3.3 Asymptotic resources

**Definition 3.20 (Asymptotic resources)** An asymptotic resource $\alpha$ is defined by a sequence of finite depth-$\ell$ resources $(\alpha_n)_{n=1}^\infty$, where $\alpha_n = (N_n : \omega_n)^\ell := ((N_{n,1} : \omega_{n,1}), (N_{n,2} : \omega_{n,2}), \ldots, (N_{n,\ell} : \omega_{n,\ell}))$, such that

1. for all sufficiently large $n$

$$
\alpha_n \geq \alpha_{n-1};
$$

(16)

2. for any $\delta > 0$, any integer $k$ and all sufficiently large $n$,

$$
\alpha_{n(1+\delta)} \geq (\alpha_{n/k}) \otimes_k \geq \alpha_{n(1-\delta)}.
$$

(17)

Denote the set of asymptotic resources by $\mathcal{R}$.

Given two resources $\alpha = (\alpha_n)_{n=1}^\infty$ and $\beta = (\beta_n)_{n=1}^\infty$, if $\alpha_n \geq \beta_n$ for all sufficiently large $n$, then we write $\alpha \geq \beta$.

Unless otherwise stated, we shall henceforth abbreviate “asymptotic resource” to “resource”.

**Definition 3.21 (I.i.d. resources)** We call a resource $\alpha$ independent and identically distributed (i.i.d.) if $\alpha_n = (N : \omega)^n$ for some depth-1 finite resource $(N : \omega)$. We use the shorthand notation $\alpha = \langle N : \omega \rangle$.

We shall use the following notation for unit asymptotic resources:

- **ebit $[q q]$ := $\langle \Phi_2 \rangle$**
- **rbit $[c c]$ := $\langle \Phi_2 \rangle$**
- **qubit $[q \to q]$ := $\langle \text{id}_2 \rangle$**
- **cbit $[c \to c]$ := $\langle \text{id}_2 \rangle$**
However, there is no formal reason that they cannot be used interchangeably.

We also can define versions of the dynamic resources with respect to the standard “reference” state $\rho_0^A = I^A_2/2$: a qubit in the maximally mixed state. These are denoted as follows:

- cobit $[q \to qg] := \langle \Delta_2 \rangle$

In this paper, we tend to use symbols for asymptotic resource inequalities (e.g. “$(\mathcal{N}) \geq R[c \to c]$”) and words for finite protocols (e.g. “$\mathcal{N} \otimes n$ can be used to send $n(R - \delta_n)$ qubits with error $\leq \epsilon_n$”).

We need to verify that $R$ is closed under multiplication. Before we do so, it will be convenient to introduce some notation. Let $n$ be an integer and $z_1, \ldots, z_a$ be positive real numbers. By $[z_1, z_2, \ldots, z_a; n]$ we denote the set of numbers of the form $[z_{\pi(1)} [z_{\pi(2)} [z_{\pi(3)} [\ldots [z_{\pi(a)} n] \ldots]]]$, where $\pi$ is some permutation of $\{1, \ldots, a\}$. There can be an arbitrary number of $[$ brackets as long as they all contain $n$. For instance, $[z[w] n]$ and $[w[z] n]$ satisfy this requirement, while $[[z w] n]$ does not. It can be shown that for all $\delta > 0$ and all $n \geq N$, where $N = N(z_1, \ldots, z_a, \delta)$,

$$b - \delta \leq b_0 \leq b_8$$

(18)

holds for all $b_\nu \in [z_1, z_2, \ldots, z_a; (1 + \nu) n]$.

Define $\beta := z \alpha$, so that $\beta_n = \alpha_{[zn]}$. Condition 1 of Definition 3.20 is trivially verified for $\beta$. For $\delta > 0$, all $k$ and all sufficiently large $n$

$$\alpha_{[zn(1+\delta)^3]} \geq \alpha_{[zn(1+\delta)n/(1+\delta)]} \geq (\alpha_{[zn(1+\delta)n/k]})^\otimes k \geq (\alpha_{[zn/k]})^\otimes k.$$  

The first and third inequality follow from (18) and (16), and the second from (17). Thus we get $\beta_{[zn(1+\delta)^3]} \geq (\beta_{[n/k]})^\otimes k$. Analogously it can be shown that $(\beta_{[n/k]})^\otimes k \geq \beta_{[n(1-\delta)^3]}$. Thus $\beta$ satisfies condition 2 of Definition 3.20.

Our next goal is to define what it means to simulate one (asymptotic) resource by another. This is the central definition of the paper.
Definition 3.24 (Asymptotic resource inequality) A resource inequality \( \alpha \geq \beta \) holds between two resources \( \alpha = (\alpha_n)_n \) and \( \beta = (\beta_n)_n \) if for any \( \delta > 0 \) there exists an integer \( k \) such that for any \( \epsilon > 0 \) there exists an integer \( N \) such that for all \( n \geq N \) there exists an \( \epsilon \)-valid protocol \( P^{(n)} \) on \( (\alpha_{n/k})^\times k \) for which

\[
\| P^{(n)}[\alpha_{n/k}]^\times k - S[\beta_{(1-\delta)n}] \| \leq \epsilon.
\]

\( \alpha \) is called the input or consumed resource, \( \beta \) is called the output or created resource, \( n \) is the blocklength, \( \delta \) is the inefficiency and \( \epsilon \) (which bounds both the validity and the error) is called the accuracy or error.

At first glance it may seem that we are demanding rather little from asymptotic resource inequalities: we allow the depth of the input resource to grow arbitrarily, while requiring only a depth-1 output. This definition is nevertheless strong enough to allow the sort of protocol manipulations we would like. We show this in Theorem 3.29 using tools like the sliding lemma.

Definition 3.24 is slightly inadequate for source coding. There the data stream coming from the Source needs to be redirected in its entirety. In contrast, our definition allows a fraction \( \delta \) of the Source-supplied data to get lost. Alice and Bob can fix this problem by replacing this perishable data by fake data. Section 3.4 is dedicated to this issue.

Corollary 3.25 If \( \alpha^* \geq \beta \) then \( \alpha \geq \beta \).

Resources that consist entirely of states and one-way channels never require protocols with depth \( > 1 \). This fact will later be useful in proving converses, i.e. statements about which resource inequalities are impossible.

Lemma 3.26 (Flattening) Suppose \( \alpha \geq \beta \) and \( \alpha \) is a one-way resource, meaning that it consists entirely of static resources \( (A^\rho) \) and dynamic resources which leave nothing on Alice’s side (e.g. \( N_{A^1\rightarrow BE} \)). Then for any \( \epsilon, \delta > 0 \) for sufficiently large \( n \) there is an \( \epsilon \)-valid protocol \( P^{(n)} \) on \( \alpha_n \) such that

\[
\| P^{(n)}[\alpha_n] - S[\beta_{(1-\delta)n}] \| \leq \epsilon.
\]

Proof To prove the lemma, it will suffice to convert a protocol on \( (\alpha_{n/k})^\times k \) to a protocol on \( (\alpha_{n/k})^\otimes k \). The lemma then follows from \( \alpha_{n(1+\delta)} \geq (\alpha_{n/k})^\otimes k \) and a suitable redefinition of \( n \) and \( \delta \).

Since \( \alpha \) is a one-way resource, any protocol that uses it can be assumed to be of the following form:

1. First the Source applies all of its protected maps;
2. Alice applies all of the appending maps;
3. Alice applies all of her encoding operations;
4. Alice applies all of the dynamic resources;
5. Bob performs his decoding operation.

The one-way nature of the protocol means that Alice can apply the dynamic resources last: they have no outputs on her side, so none of her other operations can depend on them. The protected and appending maps can be pushed to the beginning because they require no inputs from Alice. Thus \( (\alpha_{n/k})^\times k \) can be simulated using \( (\alpha_{n/k})^\otimes k \), completing the proof. \( \square \)
Definition 3.27 (Asymptotic decoupling of input resources) Let the inequality \( \alpha + \beta \geq \gamma \) hold, with a classical resource \( \beta \). Referring to Definition 3.24, if \( (\beta_{[n/k]})^{\times k} \) is \( \epsilon \)-(coherently) decoupled with respect to \( P^{(n)} \) for each \( \epsilon > 0 \) and all sufficiently large \( n \), then we say that \( \beta \) is (coherently) decoupled in the resource inequality.

Definition 3.28 (Asymptotic decoupling of output resources) Let the resource inequality \( \alpha \geq \beta + \gamma \) hold with \( \beta \) a classical resource. Referring to Definition 3.24, if \( \beta_n \) is \( \epsilon \)-coherently decoupled with respect to \( P^{(n)} \) for each \( \epsilon > 0 \) and all sufficiently large \( n \), then we say that \( \beta \) is coherently decoupled in the resource inequality.

The central purpose of our resource formalism is contained in the following “composability” theorem, which states that resource inequalities can be combined via concatenation and addition. In other words, the origin of a resource (like cbits) doesn’t matter; whether they were obtained via a quantum channel or a carrier pigeon, they can be used equally well in any protocol that takes cbits as an input. A well-known example of composability in classical information theory is Shannon’s joint source-channel coding theorem which states that a channel with capacity \( \geq C \) can transmit any source with entropy rate \( \leq C \); the coding theorem is proved trivially by composing noiseless source coding and noisy channel coding.

**Theorem 3.29 (Composability)** For resources in \( \mathcal{R} \):

1. if \( \alpha \geq \beta \) and \( \beta \geq \gamma \) then \( \alpha \geq \gamma \)
2. if \( \alpha \geq \beta \) and \( \gamma \geq \varepsilon \) then \( \alpha + \gamma \geq \beta + \varepsilon \)
3. if \( \alpha \geq \beta \) then \( z\alpha \geq z\beta \)

**Proof**

1. Since \( \alpha \geq \beta \) and \( \beta \geq \gamma \), according to Definition 3.24, \( \forall \delta > 0, \exists k, k', \forall \varepsilon > 0, \exists N, \forall n \geq N \)

\[
\|P_1[\alpha_{[n(1-\delta)^2/(mk')]}^{\times k'}] - S[\beta_{[n(1-\delta)^2/(mk')]}^{\times (1-\delta)}]\| \leq \varepsilon,
\]

(19)

\[
\|P_2[\beta_{[n(1-\delta)^4/m]}^{\times k'}] - S[\gamma_{[n(1-\delta)^4/m]}^{\times (1-\delta)}]\| \leq \varepsilon,
\]

(20)

with \( m \geq k'\ell/\delta \), where \( \ell \) is the depth of \( \beta \), and where \( P_1 \) and \( P_2 \) are both \( \epsilon \)-valid protocols. For sufficiently large \( n \)

\[
(\gamma_{[n(1-\delta)^4/m]}^{\times (1-\delta)})^{\otimes m} \geq (\gamma_{[n(1-\delta)^4/m]}^{\times m})^{\otimes m} \geq (\gamma_{[n(1-\delta)^4/m]}^{\times m})^{\otimes m} \geq (\gamma_{[n(1-\delta)^4]}^{\times m})^{\otimes m}.
\]

The first and third reductions follow from (18) and (16), and the second from (17). Together they imply the existence of a \( 0 \)-valid protocol \( R_{\gamma} \) such that

\[
R_{\gamma}[\gamma_{[n(1-\delta)^4/m]}^{\otimes m}] = S[\gamma_{[n(1-\delta)^4]}^{\otimes m}].
\]

Applying \( R_{\gamma} \) and lemmas 3.12 and 3.11 to (20):

\[
\|R_{\gamma} \circ P_2^{\otimes m}[(\beta_{[n(1-\delta)^4/m]}^{\times k'})^{\times k'} - S[\gamma_{[n(1-\delta)^4]}^{\otimes m}]]_1 \leq m\varepsilon.
\]

(21)
Define $K = \lfloor mk'(1 + \delta) \rfloor$. Then $|n/K| \geq \lfloor n(1 - \delta)/(mk') \rfloor$, which, combined with (18) and (16), gives (for sufficiently large $n$)

$$\alpha_{n/K} \geq \alpha_{\lfloor n(1 - \delta)/(mk') \rfloor} \geq \alpha_{\lfloor n(1 - \delta)^2/(mk')/k \rfloor}.$$  

Equations (18) and (16) also imply

$$\beta_{\lfloor n(1 - \delta)^2/(mk') \rfloor} \geq \beta_{\lfloor n(1 - \delta)^2/m \rfloor/k'}.$$  

Applying lemmas 3.12 and 3.11 to (19), there exists an $\epsilon$-valid protocol $\tilde{P}_1^{\otimes k'}$ such that

$$\|\ell - S[\beta_{\lfloor n(1 - \delta)^2/m \rfloor/k'}] \times k']\| \leq k'\epsilon.$$  

where $\ell = \tilde{P}_1^{\otimes k'}[(\alpha_{n/K})^{kk'}]$. By the Sliding Lemma 3.15 and (22), there exists some $\epsilon'$-valid protocol $K$ such that

$$\|K^{[\lfloor m + k'\ell - 1 \rfloor]} - P_2^{\otimes m}[\beta_{\lfloor n(1 - \delta)^2/m \rfloor/k'}]^{kk'}\| \leq \epsilon',$$

where

$$\epsilon' = (m + k'\ell - 1)(k'\epsilon + 4\sqrt{k'\ell}) + k'\ell.$$  

Combining with (21) and invoking Lemma 3.11 the protocol $K \circ (\tilde{P}_1^{\otimes k'})^{kk'(m + k'\ell - 1)}$ (which is $(\epsilon + 2\sqrt{\epsilon'})$-valid) obeys

$$\|K \circ (\tilde{P}_1^{\otimes k'})^{kk'(m + k'\ell - 1)}[\alpha_{n/K}]^{kk'} - S[\gamma_{\lfloor n(1 - \delta) \rfloor}]\| \leq \epsilon' + m\epsilon.$$  

Since $K \geq kk'(m + k'\ell - 1)$,

$$(\alpha_{n/K})^{kk'} \geq (\alpha_{n/K})^{kk'(m + k'\ell - 1)}.$$  

Finally, by Lemma 3.11 there exists a $(\epsilon + 2\sqrt{\epsilon})$-valid protocol $P_3$ such that

$$\|P_3[\alpha_{n/K}]^{K} - S[\gamma_{\lfloor n(1 - \delta) \rfloor}]\| \leq \epsilon' + m\epsilon.$$  

Fixing $\delta$, which controls the inefficiency, since $k, k'$ and $m$ are functions of $\delta$, the accuracy can be made arbitrarily small for a suitable choice of $\epsilon$. Therefore $\alpha \geq \gamma$.

2. Since $\alpha \geq \beta$ and $\gamma \geq \epsilon$, according to Definition 3.24 $\forall \delta > 0, \exists k, k', \forall \epsilon > 0, \exists N, \forall n \geq N$

$$\|P_1[\alpha_{n/k'}]^{\otimes k} - S[\beta_{\lfloor n/k' \rfloor}]^{(1-\delta)}\| \leq \epsilon,$$  

$$\|P_2[\gamma_{n/k'}]^{\otimes k'} - S[\epsilon_{\lfloor n/k \rfloor}]^{(1-\delta)}\| \leq \epsilon.$$  

For sufficiently large $n$

$$(\beta_{\lfloor n/k' \rfloor}^{(1-\delta)})^{\otimes k'} \geq (\beta_{\lfloor n(1-\delta)^2/k' \rfloor}^{(1-\delta)})^{\otimes k'} \geq (\beta_{\lfloor n(1-\delta)^2 \rfloor}^{(1-\delta)}) \geq \beta_{\lfloor n(1-\delta)^2 \rfloor}.$$  

(25)
The first and third reductions follow from (18) and (16), and the second from (17). Thus
there exists a 0-valid reduction $R_1$ mapping the LHS of (25) to the flattened version of
the RHS. Combining $R_1$ with (28) via Lemmas 3.12 and 3.11 and (16) gives

$$\|R_1 \circ P_1 \otimes k'[\alpha_{[n/(1-k^2)]}] \times k' - S[\beta_{[n(1-\delta)^*]}]\|_1 \leq k'\epsilon.$$  

Similarly there exists a reduction $R_2$ such that

$$\|R_2 \circ P_2 \otimes k'\alpha_{[n/(1-k^2)]} \times k' - S[\beta_{[n(1-\delta)^*]}]\|_1 \leq k\epsilon.$$  

Again invoking Lemma 3.12 the $\epsilon$-valid $P = P_1 \otimes P_2 \otimes k$ satisfies

$$\|P_3[(\alpha + \gamma)_{[n/(1-k^2)]}] \times k' - S[\beta + \epsilon_{[n(1-\delta)^*]}]\|_1 \leq (k + k')\epsilon.\tag{26}$$

Hence $\alpha + \gamma \geq \beta + \epsilon$.

3. Immediate from the definitions.

It is worth noting that our definitions of resources and resource inequalities were carefully
chosen with the above theorem in mind; as a result the proof exposes most of the important features
of our definitions. (It is a useful exercise to try changing aspects of our definitions to see where the
above proof breaks down.)

**Definition 3.30 (Equivalent resources)** Define an equivalence between resources $\alpha \equiv \beta$ iff $\alpha \geq \beta$ and $\beta \geq \alpha$.

**Example 3.31** It is easy to see that $R[qq] = (\Phi_D) \equiv (D'_n = [2^{nR}]$.

**Lemma 3.32** For resources in $\alpha, \beta \in R$ and $z, w \geq 0$:

1. $(zw)\alpha \equiv z(w\alpha)\equiv z\alpha + z\beta$
2. $z(\alpha + \beta) = z\alpha + z\beta$
3. $(z + w)\alpha \equiv z\alpha + w\alpha$

**Proof** 1. It suffices to show that $\alpha_{[zn]} \geq \alpha_{[zn]}$ and $\alpha_{[zn]} \geq \alpha_{[zn]}$. These
follow from (18) and (19).

2. Immediate from the definitions.

3. Consider first the $\geq$ direction. From the first two parts of this lemma it suffices to prove the
statement when $z = w = 1$. Define $\beta = z\alpha + w\alpha$. Fix $\delta < 0$. Let $k = [zm]$ and $k' = m - [zm]$, 
where $m$ is chosen such that $k, k' \geq 1/\delta$. Clearly, $(1 - \delta)k/m \leq z \leq (1 + \delta)k/m$ and $(1 - \delta)k'/m \leq
w \leq (1 + \delta)k'/m$. Hence, for sufficiently large $n$,

$$\begin{align*}
(\alpha_{[n/m]}^k \geq \alpha_{[m/n(1-\delta)]} \geq \alpha_{[m/n(1-\delta)]} \geq \alpha_{[n(1-\delta)^3]} \geq \alpha_{[n(1-\delta)^3]}. 
\end{align*}$$

The first inequality follows from (17), and the last two from (18) and (19). Similarly it can be
shown that $(\alpha_{[n/m]}^k \geq \alpha_{[w_m]}^k)$. Since

$$S[\alpha_{[n/m]}^k] = S[(\alpha_{[n/m]}^k, (\alpha_{[n/m]}^k)^{k'}],$$
by Lemma 3.11 there exists a 0-valid protocol \( P \) such that
\[
P[(\alpha \lfloor n/m \rfloor) \times m] = S[\beta \lfloor n(1-\delta^3) \rfloor].
\]
Hence \( \alpha \geq \beta \).

To prove the \( \leq \) direction we observe that
\[
\alpha [z \lfloor n(1+\delta^3) \rfloor] \otimes \alpha [w \lfloor n(1+\delta^3) \rfloor] \geq \alpha [\lfloor n/m \rfloor] \otimes k \otimes (\alpha [\lfloor n/m \rfloor])^{k'}
\]
\[
= (\alpha [\lfloor n/m \rfloor])^m
\]
\[
\geq \alpha [\lfloor n(1-\delta^3) \rfloor].
\]

Since
\[
S[\beta \lfloor n(1+\delta^3) \rfloor] = S[\alpha [z \lfloor n(1+\delta^3) \rfloor] \otimes \alpha [w \lfloor n(1+\delta^3) \rfloor]],
\]
by Lemma 3.11 there exists a 0-valid protocol \( P \) such that
\[
P[\beta \lfloor n(1+\delta^3) \rfloor] = S[\alpha \lfloor n(1-\delta^3) \rfloor].
\]
Hence \( \beta \geq \alpha \).

\[\blacksquare\]

**Definition 3.33 (Equivalence classes of resources)** Denote by \( \tilde{\alpha} \) the equivalence class of \( \alpha \), i.e. the set of all \( \alpha' \) such that \( \alpha' \equiv \alpha \). Define \( \tilde{R} \) to be the set of equivalence classes of resources in \( R \). Define the relation \( \geq \) on \( \tilde{R} \) by \( \tilde{\alpha} \geq \tilde{\beta} \) if \( \alpha' \geq \beta' \) for all \( \alpha' \in \tilde{\alpha} \) and \( \beta' \in \tilde{\beta} \). Define the operation \( + \) on \( \tilde{R} \) such that \( \tilde{\alpha} + \tilde{\beta} \) is the union of \( \alpha' + \beta' \) over all \( \alpha' \in \tilde{\alpha} \) and \( \beta' \in \tilde{\beta} \). Define the operation \( \cdot \) on \( \tilde{R} \) such that \( z \tilde{\alpha} \) is the union of \( z\alpha' \) over all \( \alpha' \in \tilde{\alpha} \).

**Lemma 3.34** For resources in \( R \):

1. \( \tilde{\alpha} \geq \tilde{\beta} \) iff \( \alpha \geq \beta \)
2. \( \tilde{\alpha} + \tilde{\beta} = \tilde{\alpha'} + \tilde{\beta'} \)
3. \( z\tilde{\alpha} = z\tilde{\alpha'} \)

**Proof** Regarding the first item: it suffices to show the “if” direction. Indeed, for any \( \alpha' \in \tilde{\alpha} \) and \( \beta' \in \tilde{\beta} \)
\[
\alpha' \geq \alpha \geq \beta \geq \beta',
\]
by Theorem 3.29. Regarding the second item: it suffices to show that if \( \alpha' \equiv \alpha \), \( \beta' \equiv \beta \) then \( \alpha' + \beta' \equiv \alpha + \beta \). This follows from Theorem 3.29. Similarly, for the third item it suffices to show that if \( \alpha' \equiv \alpha \) then \( z\alpha' \equiv z\alpha \), which is true by Theorem 3.29.

We now state a number of additional properties of \( \tilde{R} \), each of which can be easily verified.

**Proposition 3.35** The relation \( \geq \) forms a partial order on the set \( \tilde{R} \):

1. \( \tilde{\alpha} \geq \tilde{\alpha} \) (reflexivity)
2. if \( \tilde{\alpha} \geq \tilde{\beta} \) and \( \tilde{\beta} \geq \tilde{\gamma} \) then \( \tilde{\alpha} \geq \tilde{\gamma} \) (transitivity)
3. if \( \tilde{\alpha} \geq \tilde{\beta} \) and \( \tilde{\beta} \geq \tilde{\alpha} \) then \( \tilde{\alpha} = \tilde{\beta} \) (antisymmetry)

\[\blacksquare\]

**Proposition 3.36** The following properties hold for the set \( \tilde{R} \) with respect to + and multiplication by positive real numbers.

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1. \((zw)\tilde{\alpha} = z(w\tilde{\alpha})\)

2. \((z + w)\tilde{\alpha} = z\tilde{\alpha} + w\tilde{\alpha}\)

3. \(z(\tilde{\alpha} + \tilde{\beta}) = z\tilde{\alpha} + z\tilde{\beta}\)

4. \(1\tilde{\alpha} = \tilde{\alpha}\)

\[\square\]

**Proposition 3.37** For equivalence classes in \(\tilde{\mathcal{R}}\):

1. if \(\tilde{\alpha}_1 \geq \tilde{\alpha}_2\) and \(\tilde{\beta}_1 \geq \tilde{\beta}_2\) then \(\tilde{\alpha}_1 + \tilde{\beta}_1 \geq \tilde{\alpha}_2 + \tilde{\beta}_2\)

2. if \(\tilde{\alpha} \geq \tilde{\beta}\) then \(z\tilde{\alpha} \geq z\tilde{\beta}\)

\[\square\]

Lemma [3.34] has essentially allowed us to replace resources with their equivalence classes and \(\equiv\) with \(=\). Henceforth we shall equate the two, and drop the ~ superscript.

### 3.4 Source coding and improper resource inequalities

In this subsection we will introduce *improper* resource inequalities as a means for overcoming the slight inadequacy of Definition [3.24]. In this definition consumed resources correspond to block length \(n\) (or rather \(k\) blocks of length \(\lfloor n/k \rfloor\)), while created resources correspond to block length \(\lfloor (1-\delta)n \rfloor\). In source coding we insist that created and consumed resources are of the same blocklength. We will indicate this requirement with a superscript \(s\) (for “source coding”) above the resource > sign. Noting that there is no advantage in breaking up a protected resource \(\gamma = (\gamma_n)_n\) into a resource of depth > 1, we extend Definition [3.24] as follows.

**Definition 3.38 (Improper RI)** An improper resource inequality

\[
\alpha + \gamma^s \geq \beta + \gamma'
\]

holds for general resources \(\alpha = (\alpha_n)_n\) and \(\beta = (\beta_n)_n\) and protected resources \(\gamma = (\gamma_n)_n\) and \(\gamma' = (\gamma'_n)_n\), if for any \(\delta > 0\) there exists an integer \(k\) such that for any \(\epsilon > 0\) there exists \(N\) such that for all \(n \geq N\) there exists an \(\epsilon\)-valid protocol \(P^{(n)}(\alpha^k_{(n/k)}, \gamma(1-\delta/2)n] - S[\beta(1-\delta)n], \gamma'_(1-\delta/2)n] \|_1 \leq \epsilon.\)

While the unprotected resources \(\alpha\) and \(\beta\) appear as in Definition [3.24], the protocol consumes slightly less of the protected resource \(\gamma\) and creates slightly more of its “partner” protected resource \(\gamma'\).

A simple example of a source coding resource inequality is the one illustrated in figure II. A channel between Alice and Bob may be used in a source coding problem to convert the channel from the Source to Alice into a channel from the Source to Bob.

\[
\langle \text{id}^{A'\rightarrow B} : \rho^{A'} \rangle + \langle \text{id}^{S\rightarrow A} : \rho^S \rangle \geq \langle \text{id}^{S\rightarrow B} : \rho^S \rangle.
\]

In contrast, the proper RI (from Definition [3.24])

\[
\langle \text{id}^{A'\rightarrow B} : \rho^{A'} \rangle + \langle \text{id}^{S\rightarrow A} : \rho^S \rangle \geq \langle \text{id}^{S\rightarrow B} : \rho^S \rangle.
\]

allows a fraction \(\delta\) of the Source-supplied data to get lost.
The problem with Definition 3.38 is that composition of protocols via the sliding lemma will always introduce a small inefficiency $\delta$. Thus improper resource inequalities cannot be composed. In general we will have to switch back and forth between proper and improper resource inequalities. To prove an improper resource inequality we typically prove its proper version first, and then convert it to the improper version. Rules for doing this appear in the next section as Lemmas 4.10 and 4.11.
4 General resource inequalities

In this section, we present several resource inequalities and theorems that will be useful for manipulating and combining other resource inequalities.

Lemma 4.1 The following resource inequalities hold:
1. \( \langle N^{A' \rightarrow AB} \rangle \geq \langle N^{A' \rightarrow AB} : \omega_A' \rangle \)
2. \( \langle N^S \rightarrow AB : \rho^S \rangle \geq \langle N^S \rightarrow AB(\rho^S) \rangle \),
3. \( \langle N^S \rightarrow A'B' : \rho^S \rangle + \langle M^{A' \rightarrow AB} : Tr_{B'}[N^S \rightarrow A'B'(\rho^S)] \rangle \geq \langle N^S \rightarrow A'B' \circ M^{A' \rightarrow AB} : \rho^A \rangle \),
4. \( \langle \rho^{A'B'} \rangle + \langle N^{A' \rightarrow AB} : Tr_A[M^{A' \rightarrow AB} \circ N^S \rightarrow A'B'(\rho^S)] \rangle \geq \langle N^S \rightarrow A'B' \circ M^{A' \rightarrow AB} : \rho^A \rangle \),
5. If \( (N_1 : \omega_1) \geq (N_2 : \omega_2) \) then \( (N_1 : \omega_1) \geq (N_2 : \omega_2) \).

Proof Immediate from definitions.

Lemma 4.2 (Closure) Given \( z_0 > 0 \) and \( \alpha, \beta \in \mathbb{R} \), if \( z\alpha \geq \beta \) for every \( z > z_0 \) then \( z_0 \alpha \geq \beta \).

Proof The statement is equivalent to \( z_0 \alpha \geq (1 - \delta) \beta, \ \forall \delta > 0 \), which by Definition 3.24 implies the statement for \( \delta = 0 \).

The case of \( z_0 = 0 \) is special and corresponds to the use of a sublinear amount of a resource.

Definition 4.3 (Sublinear \( o \) terms) We write \( \alpha + o \gamma \geq \beta \)
if for every \( z > 0 \)
\( \alpha + z \gamma \geq \beta \).

At the other extreme we might consider a setting in which we are allowed an arbitrary rate of some resource.

Definition 4.4 (\( \infty \) terms) We write \( \alpha + \infty \gamma \geq \beta \)
if there exists an \( z \) for which \( \alpha + z \gamma \geq \beta \).

Note that “\( \infty \gamma \)” does not actually mean that our protocols may use an arbitrary amount of the resource \( \gamma \); more precisely, they may, in the asymptotic limit, use an arbitrary but finite rate.

Let us focus on sublinear terms. In general we cannot neglect sublinear resources. In entanglement dilution, for instance, they are both necessary [28, 31] and sufficient [47]. This situation only occurs when the sublinear resources cannot be generated from the other resources being consumed in the protocol.
Lemma 4.5 (Removal of o terms) For $\alpha, \beta, \gamma \in \mathbb{R}$, if
\[ \alpha + o\gamma \geq \beta \]
\[ z\alpha \geq \gamma \]
for some real $z > 0$, then
\[ \alpha \geq \beta. \]

Proof For any $z' > 0$
\[ (1 + z'z) \alpha \geq \alpha + z'\gamma \geq \beta. \]
The lemma follows by the Closure Lemma (4.2).

One place that sublinear resources often appear is as catalysts, meaning they are used to enable a protocol without themselves being consumed. Repeating the protocol many times reduces the cost of the catalyst to sublinear:

Lemma 4.6 (Cancellation) For $\alpha, \beta, \gamma \in \mathbb{R}$, if
\[ \alpha + \gamma \geq \beta + \gamma, \quad \text{then} \quad \alpha + o\gamma \geq \beta. \]

Proof Combine $N$ copies of the inequality (using part 1 of Theorem 3.29) to obtain
\[ \gamma + N\alpha \geq \gamma + N\beta. \]
Divide by $N$:
\[ N^{-1}\gamma + \alpha \geq N^{-1}\gamma + \beta \geq \beta. \]
As $N^{-1}$ is arbitrarily small, the result follows.

This cancellation result motivates us to extend the set $\mathbb{R}$ of all resources into the negative domain: we will in the future also call expressions $\alpha - \beta$ “resources”. The rules of arithmetic will be clear, including the one implicit in the above Lemma, $\alpha - \alpha = 0\alpha$. We only need to define what the inequality sign means. Also that is straightforward, by declaring, for $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$,
\[ \alpha - \beta \geq \alpha' - \beta' \quad \iff \quad \alpha + \beta' + o\beta \geq \alpha' + \beta. \] (29)
Allowing negative terms is mostly for notational convenience, but it often also helps to concisely state a resource inequality.

Often we will find it useful to use shared randomness as a catalyst. The condition for this to be possible is that the randomness be incoherently decoupled.

Lemma 4.7 (Recycling common randomness) If $\alpha$ and $\beta$ are resources for which
\[ \alpha + z[cc] \geq \beta, \]
and the $[cc]$ is incoherently decoupled in the above RI, then
\[ \alpha + o[cc] \geq \beta. \]

Proof Since $[cc]$ is asymptotically independent of the $\beta$ resource, by definitions 3.27 and 3.27 it follows that
\[ \alpha + z[cc] \geq \beta + z[cc]. \]
An application of the Cancellation Lemma 4.6 yields the desired result.

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Corollary 4.8 If $\alpha \geq [c c]$ and $\beta$ is pure then
\[
\alpha + z [c c] \geq \beta
\]
can always be derandomized to
\[
\alpha \geq \beta.
\]

Proof It suffices to notice that for a pure output resource $\beta$, equation (13) is automatically satisfied. ☐

The following theorem tells us that in proving channel coding theorems one only needs to consider the case where the input state is maximally mixed. A similar result was shown in [1], though with quite different techniques and formalism.

Theorem 4.9 (Absolutization) The following resource inequalities hold:

1. $[q \rightarrow q : \tau] = [q \rightarrow q]$
2. $[q \rightarrow qq : \tau] = [q \rightarrow qq]$
3. $[c \rightarrow c : \tau] = [c \rightarrow c]$

Proof The lemma is a direct consequence of Lemma 3.16. By part 1 of Lemma 4.1 it suffices to show the $\geq$ direction. We shall only prove item 1.; the proofs of 2. and 3. are identical. By Lemma 3.16 we know that
\[
[q \rightarrow q : \tau] + 2 [c c] \geq [q \rightarrow q] + 2 [c c].
\]

By the cancellation lemma,
\[
[q \rightarrow q : \tau] + o [c c] \geq [q \rightarrow q].
\]

Since
\[
[q \rightarrow q : \tau] \geq [c c],
\]
by Lemma 3.16 the $o$ term can be dropped, and we are done. ☐

In section 3.4 we showed how to write source coding problems as improper resource inequalities. We need to be able to move between proper and improper resource inequalities in order to take advantage of composability properties of proper resources inequalities.

Lemma 4.10 (Faking the Source) If for some resources $\alpha$ and $\beta$
\[
\alpha + \langle N^{S \rightarrow AB} : \rho^S \rangle \geq \beta
\]
and $\beta$ does not refer to the Source system S then the protected resource $\langle N^{S \rightarrow AB} : \rho^S \rangle$ may be “faked” by Alice and Bob alone:
\[
\alpha + \langle N^{S \rightarrow AB} (\rho^S) \rangle \geq \beta.
\]

Proof Obvious. ☐

Lemma 4.11 (Improper and proper resource inequalities) Let $\langle N^{S \rightarrow AB} : \omega^S \rangle$ and $\langle M^{S \rightarrow AB} : \omega^S \rangle$ be two i.i.d. protected resources, and $\alpha$ and $\beta$ be arbitrary resources in $\mathcal{R}$.

1. If
\[
\alpha + \langle M : \omega \rangle \geq \beta + \langle N : \omega \rangle
\]
then
\[
\alpha + \langle M : \omega \rangle \geq \beta + \langle N : \omega \rangle.
\]
(2) Conversely, if \( \mathcal{M}(\sigma) \) holds then
\[
\alpha + (\mathcal{M} : \omega) + o(\mathcal{M}(\omega)) \geq \beta + (\mathcal{N} : \omega)
\]

**Proof** Item (1) is immediate from definitions 3.24 and 3.38. Item (2) needs also the following observation (cf. Lemma 4.10): if
\[
P[(\mathcal{N}_1 \rightarrow AB : \omega_1^S_1), (\mathcal{N}_2 \rightarrow AB : \omega_2^S_2)] = (\mathcal{M}_2 \rightarrow AB : \omega_2^S_2)
\]
then
\[
P[(\mathcal{N}_1(\omega_1^S_1) \rightarrow AB, (\mathcal{N}_2 \rightarrow AB : \omega_2^S_2)] = (\mathcal{M}_2 \rightarrow AB : \omega_2^S_2).
\]
In other words, sources originating at \( S_2 \) don’t care if we can “fake” data coming from an independent source \( S_1 \). \( \square \)

Finally, we note how convex combinations of static resources can be thought of as states conditioned on classical variables.

**Theorem 4.12** Consider the static i.i.d. resource \( \alpha = \langle \sigma \rangle \), where
\[
\sigma^{AXA \otimes XB} = \sum_x p_x |x\rangle\langle x|^X_A \otimes |x\rangle\langle x|^X_B \otimes \rho_x^{AB}.
\]
In other words, Alice and Bob share a bipartite state chosen from an ensemble and both parties have the classical information identifying the state. Denote \( \alpha_x = (\rho_x) \). Then
\[
\alpha \geq \sum_x p_x \alpha_x.
\]

**Proof** We will show that for all \( \epsilon, \delta > 0 \) and sufficiently large \( n \), \( \sigma^{\otimes n} \) can be transformed into a state \( \epsilon \)-close to \( \omega_{|n(1-\delta)|} \), where
\[
\omega_n = \bigotimes_x \rho_x^{\otimes [p_x n]}.
\]
Recall the notion of the typical set \( T_{p,\delta}^n \). For any \( x^n \in T_{p,\delta}^n \),
\[
|n_x - p_x n| \leq \delta n,
\]
where \( n_x \) is the number of occurrences of the symbol \( x \) in \( x^n \). In addition, \( \rho^{\otimes n}(T_{p,\delta}^n) \geq 1 - \epsilon \) for any \( \epsilon, \delta > 0 \) and sufficiently large \( n \). Then
\[
\left\| \sigma^{\otimes n} - \sum_{x^n \in T_{p,\delta}^n} \rho^{\otimes n}(x^n) |x^n\rangle\langle x^n|^X_A \otimes |x^n\rangle\langle x^n|^X_B \otimes \rho_x^{AB} \right\|_1 \leq \epsilon.
\]
For any \( x^n \in T_{p,\delta}^n \) there is, clearly, a unitary \( U_{x^n}^A \otimes U_{x^n}^B \), that maps \( \rho_{x^n} \) to \( \omega_{[|n(1-\delta)|]} \otimes \hat{\rho}_{x^n} \) exactly for some state \( \hat{\rho}_{x^n} \). Performing
\[
\left( \sum_{x^n} |x^n\rangle\langle x^n|^X_A \otimes U_{x^n}^A \right) \otimes \left( \sum_{x^n} |x^n\rangle\langle x^n|^X_B \otimes U_{x^n}^B \right)
\]
and tracing out subsystems thus brings \( \sigma^{\otimes n} \) \( \epsilon \)-close to \( \omega_{[|n(1-\delta)|]} \). Hence the claim. \( \square \)

In fact, the above result could be strengthened to the equality
\[
\alpha = \sum_x p_x \alpha_x + H(X_A)_{\sigma} [c|c], \tag{32}
\]
but we will not need this fact, so omit the proof. However, we will show how a similar statement to Theorem 4.12 can be made about source coding.

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Theorem 4.13 Consider a source state of the form
\[ \rho^{X_S S} = \sum_x p_x |x \rangle \langle x |^{X_S} \otimes \omega^S_x. \]

Then
\[ \sum_x p_x (\text{id}^{A' \rightarrow B} : \omega^A_x) + (\Delta^{X_S \rightarrow X_A X_B} \otimes \text{id}^{S \rightarrow A} : \rho^{X_S S}) \geq (\Delta^{X_S \rightarrow X_A X_B} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S}). \]

Proof The proof is very similar to that of the previous theorem and is hence omitted. \(\square\)

Corollary 4.14 In the setting of the above theorem, let \(\overline{N}^{X_S \rightarrow Y_A}\) be a \(\{c \rightarrow c\}\) entity and let
\[ \overline{N}^{X_S \rightarrow Y_A} (\rho^{X_S S}) = \sum_y q_y |y \rangle \langle y |^{Y_A} \otimes \sigma^S_y. \]

Define \(\overline{N}^{X_S \rightarrow Y_A Y_B} = \Delta^{Y_A \rightarrow Y_A Y_B} \circ \overline{N}^{X_S \rightarrow Y_A}. \) Then
\[ \sum_y q_y (\text{id}^{A' \rightarrow B} : \sigma^A_x) + (\overline{N}^{X_S \rightarrow Y_A Y_B} \otimes \text{id}^{S \rightarrow A} : \rho^{X_S S}) \geq (\overline{N}^{X_S \rightarrow Y_A Y_B} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S}). \]

\(\square\)
5  Known coding theorems and converses expressed as resource inequalities

There have been a number of quantum and classical coding theorems discovered to date, typically along with so-called converse theorems which prove that the coding theorems cannot be improved upon. The theory of resource inequalities has been developed to provide an underlying unifying principle. This direction was initially suggested in [20].

We shall state theorems such as Schumacher compression, the classical reverse Shannon theorem, the instrument compression theorem, the classical-quantum Slepian-Wolf theorem, the HSW theorem, and common randomness concentration as resource inequalities. Then we will show how some of these can be used as building blocks, yielding transparent and concise proofs of other important results.

We shall work within the QQ formalism.

**Schumacher compression.** The quantum source compression theorem was proven by Schumacher in [41, 48]. Given a quantum state $\rho^{A'}$, define $\sigma^B := \text{id}_{A'} \rightarrow B(\rho^{A'})$. Then the following RI holds:

$$ (H(B)_{\sigma} + \delta)[q \rightarrow q] \geq \langle \text{id}_{A'} \rightarrow B : \rho^{A'} \rangle, \quad (33) $$

if and only if $\delta \geq 0$.

Note that this formulation simultaneously expresses both the coding theorem and the converse theorem.

The Source version of this theorem states that

$$ (H(B)_{\sigma} + \delta)[q \rightarrow q] + \langle \text{id}^{S \rightarrow A} : \rho^{S} \rangle \overset{s}{\geq} \langle \text{id}^{S \rightarrow B} : \rho^{S} \rangle, \quad (34) $$

if and only if $\delta \geq 0$.

**Entanglement concentration.** The problem of entanglement concentration was solved in [3], and is, in a certain sense, a static counterpart to Schumacher’s compression theorem. Entanglement concentration can be thought of as a coding theorem which says that given a pure bipartite quantum state $|\phi^{AB}\rangle$ the following RI holds:

$$ \langle \phi^{AB} \rangle \geq H(B)_{\phi}[q q]. $$

The reverse direction is known as *entanglement dilution* [3], and thanks to Lo and Popescu [47], it is known that

$$ H(B)_{\phi}[q q] + o[c \rightarrow c] \geq \langle \phi^{AB} \rangle. $$

Were it not for the $o[c \rightarrow c]$ term, we would have the equality $\langle \phi^{AB} \rangle = H(B)_{\phi}[q q]$. However, if it turns out that the $o[c \rightarrow c]$ term cannot be avoided [28, 31]. This means that the strongest equality we can state has a sublinear amount of classical communication on both sides:

$$ H(B)_{\phi}[q q] + o[c \rightarrow c] = \langle \phi^{AB} \rangle + o[c \rightarrow c]. \quad (35) $$

Note how (35) states the converse in a form that is in some ways stronger than (33), since it implies the transformation is not only optimal, but also asymptotically reversible. We can also state a converse when unlimited classical communication is allowed:

$$ \langle \phi^{AB} \rangle + \infty[c \rightarrow c] \geq (H(B)_{\phi} - \delta)[q q] $$

if $\delta \geq 0$; and similarly for entanglement dilution.
**Shannon compression.** Shannon’s classical compression theorem was proven in \[52\]. Given a classical state \(\rho_{XA}\) and defining
\[
\sigma_{XB} = \text{id}_{X\rightarrow XA}(\rho_{XA}),
\]
Shannon’s theorem says that
\[
(H(X_B)_{\sigma} + \delta)[c \rightarrow c] \geq \frac{1}{c} \sum_x p_x \langle x|X_A \otimes X_B \rangle, \tag{36}
\]
if and only if \(\delta \geq 0\). The Source version of this theorem reads
\[
(H(X_B)_{\sigma} + \delta)[c \rightarrow c] + (H_{X} \circ X_{\sigma})^s \geq \frac{1}{c} \sum_x p_x \langle x|X_A \otimes X_B \rangle, \tag{37}
\]
if and only if \(\delta \geq 0\).

**Common randomness concentration.** This is the classical analogue of entanglement concentration, and a static counterpart to Shannon’s compression theorem. It states that, if Alice and Bob have a copy of the same random variable \(X\), embodied in the classical bipartite state
\[
\rho_{XAXB} = \sum_x p_x |x\rangle \langle x| \otimes |x\rangle \langle x|,
\]
then
\[
\langle \rho_{XAXB} \rangle \geq H(X_B)_{\rho}[c c]. \tag{38}
\]
Incidentally, common randomness dilution can do without the \(o[c \rightarrow c]\) term:
\[
H(X_B)_{\rho}[c c] \geq \langle \rho_{XAXB} \rangle.
\]

Thus we obtain a simple resource equality:
\[
H(X_B)_{\rho}[c c] = \langle \rho_{XAXB} \rangle.
\]

**Classical reverse Shannon theorem (CRST).** This theorem was proven in \[8, 62\], and it generalizes Shannon’s compression theorem to compress probability distributions of classical states instead of pure classical states. Given a classical channel \(\mathcal{N} : X_{A'} \rightarrow Y_B\) and a classical state \(\rho_{X_{A'}}\), the CRST states that
\[
I(X_A; Y_B)_{\sigma}[c \rightarrow c] + H(X_A|Y_B)_\sigma[c c] \geq \langle \mathcal{N} : \rho_{X_{A'}} \rangle, \tag{39}
\]
where
\[
\sigma_{XAXY} = C_{\mathcal{N}^X_{X_{A'} \rightarrow Y_{AX}}}(\rho_{X_{A'}}).
\]

We can also express this in the Source formalism,
\[
I(X_A; Y_B)_{\sigma}[c \rightarrow c] + H(X_A|Y_B)_\sigma[c c] + (H_{X} \circ X_{\sigma})^s \geq \langle \mathcal{N}^X_{X_{A'} \rightarrow X_{A'}} : \rho_{X_{A'}} \rangle.
\]

Moreover, given a modified classical channel \(\mathcal{N}' : X_{A'} \rightarrow Y_AY_B\) which also provides Alice with a copy of the channel output,
\[
\mathcal{N}' = \mathcal{N}^{Y_B \rightarrow Y_{A}Y_B} \circ \mathcal{N},
\]
the following stronger RI also holds:
\[
I(X_A; Y_B)_{\sigma}[c \rightarrow c] + H(X_A|Y_B)_\sigma[c c] \geq \langle \mathcal{N}' : \rho_{X_{A'}} \rangle, \tag{40}
\]
In fact, this latter RI can be reversed to obtain the equality
\[ I(X_A; Y_B)_\sigma[c \rightarrow c] + H(X_A|Y_B)_\sigma[c,c] = \langle N \colon \rho^{X_A'} \rangle. \] (41)

However, in the case without feedback, the best we can do is a tradeoff curve between cbits and rbits, with \( N \) representing the case of unlimited randomness consumption. The full tradeoff will be given by an RI of the following form
\[ a[c \rightarrow c] + b[c,c] \geq \langle N \colon \rho^{X_A'} \rangle \]
where \((a, b)\) range over some convex set \( CR(N) \). It can be shown that \((a, b) \in CR(N)\) iff there exist channels \( \overline{N}_1 : X_{A'} \rightarrow W_C, \overline{N}_2 : W_C \rightarrow Y_B \) such that \( N = \overline{N}_2 \circ \overline{N}_1 \) and \( a \geq I(X_A; W_C)_\omega, b \geq I(X_AY_B; W_C)_\omega \), where
\[
\omega_{X_AW_CY_B} := \overline{C}_{\overline{N}_2} W_C \rightarrow Y_B W_C \circ \overline{C}_{\overline{N}_1} X_{A'} \rightarrow W_C X_A (\rho^{X_A'}). 
\]

**Classical compression with quantum side information.** This problem was solved in [19, 61], and is a generalization of Shannon’s classical compression theorem in which Bob has quantum side information about the source. Suppose Alice and Bob are given an ensemble
\[
\rho^{X_AB} = \sum_x p_x |x\rangle \langle x|^{X_A} \otimes \rho^B_x,
\]
and Alice wants to communicate \( X_A \) to Bob, which would give them the state
\[
\sigma^{X_B} := \overline{id}^{X_A \rightarrow X_B} (\rho^{X_A}).
\]
To formalize this situation, we use the Source as one of the protagonists in the protocol, so that the coding theorem redirects a channel from the Source to Alice and Bob \( \langle id^{X_S \rightarrow X_A} \otimes id^{S \rightarrow B} : \rho^{X_S} \rangle \) to a channel from the Source entirely to Bob. The coding theorem is then
\[
(H(X_B|B)_\sigma + \delta)[c \rightarrow c] + \langle id^{X_S \rightarrow X_A} \otimes id^{S \rightarrow B} : \rho^{X_S} \rangle \geq \langle id^{X_S \rightarrow X_B} \otimes id^{S \rightarrow B} : \rho^{X_S} \rangle, \] (42)
which holds iff \( \delta > 0 \).

Of course, with no extra resource cost Alice could keep a copy of \( X_A \).

**Instrument compression theorem.** This theorem was proven in [63], and is a generalization of the CRST. Given a remote instrument \( T : A' \rightarrow AX_B \), and a quantum state \( \rho^{A'} \), the following RI holds:
\[ I(R; X_B)_\sigma[c \rightarrow c] + H(X_B|R)_\sigma[c,c] \geq \langle T : \rho^{A'} \rangle, \] (43)
where
\[
\sigma^{RX_XB} = T(\psi^{RA'})
\]
and \( |\psi\rangle \psi^{RA'} \supseteq \rho^{A'} \). Moreover, given a modified remote instrument which also provides Alice with a copy of the instrument output,
\[ T' = \sum^{X_B \rightarrow X_A X_B} T \circ \sigma, \]
the RI still holds:
\[ I(R; X_B)_\sigma[c \rightarrow c] + H(X_B|R)_\sigma[c,c] \geq \langle T' : \rho^{A'} \rangle. \] (44)
Only this latter RI is known to be optimal (up to a trivial substitution of \([c \rightarrow c]\) for \([c,c]\)); indeed
\[ a[c \rightarrow c] + b[c,c] \geq \langle T' : \rho^{A'} \rangle, \] (45)
iff \( a \geq I(R; X_B)_\sigma \) and \( a + b \geq H(X_B)_\sigma \).

By contrast, only the communication rate of \( X_B \) is known to be optimal; examples are known in which less randomness is necessary.
Teleportation and super-dense coding. Teleportation \[4\] and super-dense coding \[9\] are finite protocols, and we have discussed them already in the introduction. In a somewhat weaker form they may be written as resource inequalities. Teleportation (TP):

\[
2 [c \rightarrow c] + [qq] \geq [q \rightarrow q].
\] (46)

Super-dense coding (SD):

\[
[q \rightarrow q] + [qq] \geq 2 [c \rightarrow c].
\] (47)

Finally, entanglement distribution:

\[
[q \rightarrow q] \geq [qq].
\] (48)

All of these protocols are optimal (we neglect the precise statements), but composing them with each other (e.g. trying to reverse teleportation by using super-dense coding) is wasteful. By replacing classical communication with coherent classical communication (below), the protocols become reversible.

Coherent classical communication identity. In \[26\] two more resource inequalities involving unit resources were discovered. Coherent versions of teleportation and super-dense coding, respectively:

\[
[q \rightarrow q] + [qq] \geq 2 [q \rightarrow qq],
\]

\[
2 [q \rightarrow qq] + [qq] \geq [q \rightarrow q] + 2 [qq].
\]

The \([qq]\) term on the left hand side of the second inequality may be canceled completely by Lemma 4.6, Lemma 4.5 and the fact that \([q \rightarrow qq] \geq [qq]\). This brings us to the coherent communication identity

\[
[q \rightarrow qq] = \frac{1}{2} (q \rightarrow q) + [qq],
\] (49)

which will turn out to be an important tool for constructing new protocols.

Holevo-Schumacher-Westmoreland (HSW) theorem. The direct part of this theorem was proven in \[34, 51\] and the converse in \[33\]. Together they say that given a quantum channel \(N : A' \rightarrow B\), for any ensemble

\[
\rho_{X A A'} = \sum_x p_x |x\rangle \langle x| \otimes \rho_{x A'}
\]

the following RI holds:

\[
\langle N : \rho_{A'} \rangle \geq (I(X) ; Y_B)_{\sigma} - \delta [c \rightarrow c],
\] (50)

iff \(\delta \geq 0\), where

\[
\sigma_{X A B} = N_{A' \rightarrow B}(\rho_{X A A'}).
\]

Shannon’s noisy channel coding theorem. This theorem was proven in \[52\] and today can be understood as a special case of the HSW theorem. One version of the theorem says that given a classical channel \(\mathcal{N} : X_{A'} \rightarrow Y_B\) and any classical state \(\rho_{X A'}\) the following RI holds:

\[
\langle \mathcal{N} \rangle \geq (I(X_A ; Y_B)_{\sigma} - \delta)[c \rightarrow c],
\] (51)

iff \(\delta \geq 0\) and where

\[
\sigma_{X A Y_B} := \mathcal{C}_{X_{A'} \rightarrow Y_B X_A}(\rho_{X A'}).\]

If we optimize over all input states, then we find that

\[
\langle \mathcal{N} \rangle \geq C[c \rightarrow c]
\] (53)

iff there exists an input \(\rho_{X A'}\) such that \(C \leq I(X_A ; Y_B)_{\sigma}\), with \(\sigma\) given by \[52\].
Entanglement-assisted capacity (EAC) theorem. This theorem was proven in \(|8, 36, 39\). The direct coding part of the theorem says that, given a quantum channel \(N : A' \to B\), for any quantum state \(\rho^{A'}\) the following RI holds:

\[
\langle N : \rho^{A'} \rangle + H(R|q) \geq I(R; B|c \to c)
\]

(54)

where

\[
\sigma^{RB} = N(\psi^{RA'})
\]

for an arbitrary \(\psi\) satisfying \(|\psi\rangle \langle \psi|^{RA'} \supseteq \rho^{A'}\).

The only converse proven in \([8, 36]\) was for the case of infinite entanglement: they found that \(\langle N \rangle + \infty[q] \geq C[c \to c]\) iff \(C \leq I(R; B)\) for some appropriate \(\sigma\). Ref. \([56]\) gave a full solution to the tradeoff problem for entanglement-assisted classical communication which we will present an alternate converse for in Section 7.

Quantum capacity (LSD) theorem. This theorem was conjectured in \([49, 50]\), a heuristic (but not universally accepted) proof given by Lloyd \([46]\) and finally proven by Shor \([54]\) and with an independent method by Devetak \([15]\). The direct coding part of the theorem says that, given a quantum channel \(N : A' \to B\), for any quantum state \(\rho^{A'}\) the following RI holds:

\[
\langle N \rangle \geq I(R; B|q \to q)
\]

(55)

where

\[
\sigma^{RB} = N(\psi^{RA'})
\]

for any \(\psi^{RA'}\) satisfying \(|\psi\rangle \langle \psi|^{RA'} \supseteq \rho^{A'}\).

Noisy super-dense (NSD) coding theorem. This theorem was proven in \([37]\). The direct coding part of the theorem says that, given a bipartite quantum state \(\rho^{AB}\), the following RI holds:

\[
\langle \rho^{AB} \rangle + H(A)_{\rho} [q \to q] \geq I(A; B|c \to c).
\]

(56)

A converse was proven in \([37]\) only for the case when an infinite amount of \(\langle \rho^{AB} \rangle\) is supplied, but we will return to this problem and provide a full trade-off curve in Section 7.

Entanglement distillation. The direct coding theorem for one-way entanglement distillation is embodied in the hashing inequality, proved in \([22, 21]\): given a bipartite quantum state \(\rho^{AB}\),

\[
\langle \rho^{AB} \rangle + I(A; E)_{\psi} [c \to c] \geq I(A; B)_{\psi} [q q],
\]

(57)

where \(|\psi\rangle \langle \psi|^{ABE} \supseteq \rho^{AB}\).

Again, the converse was previously only known for the case when an unlimited amount of classical communication was available \([19, 50, 22, 24]\). In Section 7 we will give an expression for the full trade-off curve.

State merging. The state merging RI was proved in \([38]\)

\[
\langle U^{S \to AB} : \rho^{S} \rangle + I(A; E)_{\psi} [c \to c] + H(A|B)_{\psi} [q q] \geq \langle id^{S \to B} : \rho^{S} \rangle,
\]

(58)

where \(U^{S \to AB}\) is an isometry, \(\rho^{AB} = U^{S \to AB}(\rho^{S})\) and \(\psi^{ABE}\) is defined as above. It holds irrespectively of the sign of \(H(A|B)\). It implies entanglement distillation via Lemmas \([4, 11]\) and \([4, 10]\). Conversely, the protocol \([22]\) implementing \([17]\) may be easily modified (replacing Eve with the reference system) to give \([58]\) for \(H(A|B)_{\psi} < 0\).
Lemma 14.11 says that proper and improper resource inequalities are equivalent up to $o$ terms. In this vein, we may equivalently write \[56\] as

$$I(A;E)_\psi[c \rightarrow c] + H(A|B)_\psi[q \rightarrow q] \geq \langle \id^{S \rightarrow B} : \rho^S \rangle - \langle U^{S \rightarrow AB} : \rho^S \rangle,$$

reflecting the fact that the redirection of protected resources (in this case from Alice to Bob) is the information processing task Alice and Bob are trying to accomplish. Taking this a step further, one may be inclined to disregard the Source altogether and define

$$\langle \id^{A \rightarrow B'} : \rho^{AB} \rangle := \langle U^{S \rightarrow AB} : \rho^S \rangle - \langle \id^{S \rightarrow B} : \rho^S \rangle,$$

in analogy to the Source-free version of Schumacher compression \[83\] (strictly speaking, our current formalism does not permit this). Curiously, $\langle \id^{A \rightarrow B'} : \rho^{AB} \rangle$ on the right hand side of a RI can be an asset or liability, depending on whether $H(A|B)_\psi$ is negative or positive.

**Noisy teleportation.** This RI was discovered in \[16\]. Given a bipartite quantum state $\rho^{AB}$,

$$\langle \rho^{AB} \rangle + I(A;B)_\rho[c \rightarrow c] \geq I(A|B)_\rho[q \rightarrow q].$$

Indeed, letting $|\psi\rangle^ABE \geq \rho^{AB}$,

$$\langle \rho^{AB} \rangle + I(A;B)_\psi[c \rightarrow c] = \langle \rho^{AB} \rangle + I(A;E)_\psi[c \rightarrow c] + 2I(A|B)_\psi[q \rightarrow q] \geq I(A|B)_\psi[q \rightarrow q].$$

The first inequality follows from \[67\] and the second from teleportation.

**Quantum compression with classical side information** Suppose Alice is given the ensemble

$$\rho^{X_A} = \sum_x p_x |x\rangle \langle x| \otimes \rho_x,$$

and she wants Bob to end up with the quantum part $A$ \[29\]. The resources at their disposal are $[c \rightarrow c]$ and $[q \rightarrow q]$. As in the classical compression with quantum side information problem above, we first give $\rho^{X_A}$ to the Source (and rename it $\rho^{X_A}S$). For any classical channel $\mathcal{N} : X_S \rightarrow Y_B$, the following RI holds \[29\]:

$$\langle \id^{X_{S \rightarrow X_A} \otimes \id^{S \rightarrow A}} : \rho^{X_A}S \rangle + H(B|Y_B)_\sigma[q \rightarrow q] + I(X_A;Y_B)_\sigma[c \rightarrow c] \geq \langle \id^{S \rightarrow B} : \rho^S \rangle,$$

where

$$\sigma^{X_AY_B} = (\mathcal{N}^{X_S \rightarrow Y_BX_A} \otimes \id^{S \rightarrow B})_\rho^{X_A}S.$$

Conversely, if $a[q \rightarrow q] + b[c \rightarrow c]$ is $\geq$ to the right hand side of \[60\] then there exists a classical channel $\mathcal{N} : X_A \rightarrow Y_B$ with corresponding state $\sigma$ such that $a \geq H(B|Y_B)_\sigma$ and $b \geq I(X_A;Y_B)_\sigma$.

We shall now show how the proof from \[29\] may be written very succinctly in terms of the resource calculus. Define $\mathcal{N}' = \mathcal{N}^{X_{A'} \rightarrow Y_AY_B} \circ \mathcal{N}$. By the Classical Reverse Shannon Theorem \[30\]

$$I(X_A;Y_B)_\sigma[c \rightarrow c] + H(X_A|Y_B)_\sigma[c | c] \geq \langle \mathcal{N}'^{X_{A'} \rightarrow Y_AY_B} : \rho^{X_{A'}} \rangle.$$

Combining with part 3 of Lemma 14.1 gives

$$\langle \id^{X_{S \rightarrow X_A} \otimes \id^{S \rightarrow A}} : \rho^{X_A}S \rangle + I(X_A;Y_B)_\sigma[c \rightarrow c] + H(X_A|Y_B)_\sigma[c | c] \geq \langle \mathcal{N}^{X_S \rightarrow Y_AY_B} \otimes \id^{S \rightarrow A} : \rho^{X_A}S \rangle.$$
On the other hand, combining Schumacher compression \[(33)\] with Corollary \[(4.14)\] gives

\[
H(B|Y_B)_{\sigma[q \rightarrow q]} + \langle N_{X_S \rightarrow X_A Y_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle \geq \langle N_{X_S \rightarrow X_A Y_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle.
\]

Adding the two equations gives

\[
\langle \text{id}^{X_S \rightarrow X_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle + H(B|Y_B)_{\sigma[q \rightarrow q]} + I(X_A;Y_B)_{\sigma[c \rightarrow c]} + H(X_A|Y_B)_{\sigma[c|c]} \geq \langle \text{id}^{S \rightarrow B} : \rho^S \rangle.
\]

The last line is by part 4 of Lemma \[(4.1)\] Derandomizing via Corollary \[(4.8)\] gives

\[
H(B|Y_B)_{\sigma[q \rightarrow q]} + I(X_A;Y_B)_{\sigma[c \rightarrow c]} + \langle \text{id}^{X_S \rightarrow X_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle \geq \langle \text{id}^{S \rightarrow B} : \rho^S \rangle.
\]

Invoking Lemma \[(4.11)\] and Lemma \[(4.5)\] yields the desired result \[(60)\].

**Common randomness distillation.** This theorem was originally proven in \[(20)\]. Given an ensemble

\[
\rho^{X_A B} = \sum_x p_x |x\rangle \langle x| \otimes \rho^B_x,
\]

the following RI holds:

\[
\langle \rho^{X_A B} \rangle + H(X_A|B)_{\rho[c \rightarrow c]} \geq H(X_A)_{\rho[c|c]}.
\]

Our formalism makes transparent the intimate relation between \[(63)\] and the problem of classical compression with quantum side information \[(42)\].

\[
\langle \text{id}^{X_S \rightarrow X_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle + H(X_A|B)_{\rho[c \rightarrow c]} \geq \langle \text{id}^{X_S \rightarrow X_A X_B} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle \geq \langle \text{id}^{X_S \rightarrow X_A X_B} : \rho^{X_S} \rangle \geq \langle \text{id}^{X_S \rightarrow X_A X_B} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle \geq H(X_A | c|c). \quad (64)
\]

The first inequality is by \[(42)\] and Lemma \[(4.11)\], the second and third are by parts 5 and 2, respectively, of Lemma \[(4.1)\]. The last inequality is common randomness concentration \[(38)\]. By Lemma \[(4.10)\] \[
\langle \text{id}^{X_S \rightarrow X_A} \otimes \text{id}^{S \rightarrow B} : \rho^{X_S S} \rangle \]

can be replaced by

\[
\langle \rho^{X_A B} \rangle = \langle \text{id}^{X_S \rightarrow X_A} \otimes \text{id}^{S \rightarrow B} (\rho^{X_S S}) \rangle,
\]

proving \[(63)\].
6 A family of quantum protocols.

6.1 The family tree.

A large class of problems in quantum Shannon theory involves transforming a noisy resource, such as a channel or bipartite state, into a noiseless one (such as c bits, ebits or qubits), perhaps by consuming some other noiseless resource. In the prequel to this paper [16] we gave a unified treatment of four such protocols that were already known together with three new such protocols. This section and the next one are devoted to a detailed treatment of these results. This is now possible because of the rigorous theory of resource inequalities developed above. All of the RIs presented in this section involve a single noisy resource. The “static” members of the family involve a noisy bipartite state \( \rho_{AB} \), while the “dynamic” members involve a general quantum channel \( N : A' \to B \). In the former case one may define a class of purifications \( |\psi\rangle\langle\psi|_{ABE} \supseteq \rho_{AB} \).

In the latter case one may define a class of pure states \( |\psi\rangle_{RBE} \), which corresponds to the outcome of sending half of some \( |\phi\rangle_{RA'} \) through the channel’s isometric extension \( U_N : A' \to BE, U_N \supseteq N \).

Recall the identities, for a tripartite pure state \( |\psi\rangle_{ABE} \),

\[
\frac{1}{2} I(A; B)_{\psi} + \frac{1}{2} I(A; E)_{\psi} = H(\psi),
\]

\[
\frac{1}{2} I(A; B)_{\psi} - \frac{1}{2} I(A; E)_{\psi} = I(A \rangle B)_{\psi}.
\]

Henceforth, all entropic quantities will be defined with respect to \( |\psi\rangle_{RBE} \) or \( |\psi\rangle_{ABE} \), depending on the context, so we shall drop the \( \psi \) subscript.

The two “parent” resource inequalities were introduced in [16]. The “mother” RI reads

\[
\langle \rho \rangle + \frac{1}{2} I(A; B)_{\psi} + \frac{1}{2} I(A; E)_{\psi} \geq \frac{1}{2} I(A; B)_{\psi} + \frac{1}{2} I(A; E)_{\psi} \geq \frac{1}{2} I(A; B)_{\psi} + \frac{1}{2} I(A; E)_{\psi}.
\]

By the cancellation lemma,

\[
\langle \rho \rangle + \frac{1}{2} I(A; E)_{\psi} |c \to c| + \frac{1}{2} I(A; E)_{\psi} |q q| \geq \langle \rho \rangle + \frac{1}{2} I(A; E)_{\psi} |q q| \geq \frac{1}{2} I(A; E)_{\psi} |q q| \geq 0.
\]

This is slightly weaker than (67) itself. Further combining with teleportation gives a variation on noisy teleportation (68):

\[
\langle \rho \rangle + I(A; B)_{\psi} |c \to c| + o[|q q|] \geq I(A \rangle B)_{\psi} |q q|.
\]
The third child is noisy super-dense coding \((56)\), obtained by combining the mother with super-dense coding:

\[
H(A) [q \rightarrow q] + \langle \rho \rangle = \frac{1}{2} I(A; B) [q \rightarrow q] + \frac{1}{2} I(A; E) [q \rightarrow q] + \langle \rho \rangle \\
\geq \frac{1}{2} I(A; B) [q \rightarrow q] + \frac{1}{2} I(A; B) [q q] \\
\geq I(A; B) [c \rightarrow c].
\]

The father happens to have only two children (that we know of). One of them is the entanglement-assisted classical capacity RI \((54)\), obtained by combining the father with super-dense coding:

\[
H(R) [q q] + \langle N \rangle = \frac{1}{2} I(R; B) [q q] + \frac{1}{2} I(R; E) [q q] + \langle N \rangle \\
\geq \frac{1}{2} I(R; B) [q q] + \frac{1}{2} I(R; B) [q \rightarrow q] \\
\geq I(R; B) [c \rightarrow c].
\]

The second is a variation on the quantum channel capacity result \((55)\). It is obtained by combining the father with entanglement distribution:

\[
\frac{1}{2} I(R; E) [q q] + \langle N \rangle \geq \frac{1}{2} I(R; B) [q \rightarrow q] \\
= \frac{1}{2} I(R; E) [q q] + \frac{1}{2} I(R; E) [q \rightarrow q] \\
= \frac{1}{2} I(R; E) [q q] + \frac{1}{2} I(R) [q q].
\]

Hence, by the cancellation lemma

\[
\langle N \rangle + o[q q] \geq I(R) [q \rightarrow q].
\] (69)

In the following subsection we give a rigorous proof of the parent RIs using so-called coherification rules.

### 6.2 Constructing the parent protocols using coherification rules.

Having demonstrated the power of the parent resource inequalities, we now address the question of constructing protocols implementing them. The lessons learned in \([15, 22, 21]\) regarding making protocols coherent and the observations of \([26]\) (in particular the coherent communication identity \((49)\)), lead us to two general rules regarding making classical communication coherent. When coherently-decoupled cbits are in the input to a protocol, Rule I (“input”) says that replacing them with cobits not only performs the protocol, but also has the side effect of generating entanglement. Rule O (“output”) is simpler; it says that if a protocol outputs coherently-decoupled cbits, then it can be modified to instead output cobits. Using these rules, we can give simple proofs of the parent protocols by making coherent previously known protocols.

Below, we give formal statements of rules I and O, deferring their proofs till the end of the section. We shall be working in the CP picture.

**Theorem 6.1 (Rule I)** If for resources \(\alpha, \beta \in \mathcal{R}\)

\[
\alpha + R[c \rightarrow c : \tau] \geq \beta
\]

and the classical resource \(R[c \rightarrow c : \tau]\) is coherently decoupled then

\[
\frac{\alpha}{2} [q \rightarrow q] \geq \frac{\beta}{2} [q q].
\]
There is also an incoherent version of Rule I which is easy to prove (cf. Lemma 4.7):

**Proposition 6.2 (Incoherent Rule I)** If for resources $\alpha, \beta \in \mathcal{R}$

\[
\alpha + R[c \rightarrow c : \tau] \geq \beta
\]

and the classical resource $R[c \rightarrow c : \tau]$ is incoherently decoupled then

\[
\alpha + R[c \rightarrow c : \tau] \geq \beta + R[c \rightarrow c].
\]

**Theorem 6.3 (Rule O)** If for resources $\alpha, \beta \in \mathcal{R}$

\[
\alpha \geq \beta + R[c \rightarrow c]
\]

and the classical resource $R[c \rightarrow c]$ is coherently decoupled then

\[
\alpha \geq \beta + R[2^Q] + R[2^{Q1}] + R[2^{Q2}].
\]

**Corollary 6.4** The mother inequality is obtained from the hashing inequality by applying rule I. It can be readily checked that the classical message in [22, 21]'s protocol is coherently decoupled and is uniformly random (so the protocol is 0-valid).

**Corollary 6.5** The father inequality follows from the EAC inequality by applying rule O. In [39] it was shown explicitly that the conditions of rule O hold for the protocol implementing the EAC inequality exhibited therein. These conditions also hold for the original protocol of [8].

**Corollary 6.6** The mother inequality also follows from the NSD inequality by applying rule O. The proof is almost the same as for the previous corollary. It is easy to see that the conditions of rule O hold for the protocol from [37].

We now give the proofs of rules I and O.

**Proof (of rule I)** In what follows we shall fix $\epsilon$ and consider a sufficiently large blocklength $n$ so that the protocol $P_n$ is $\epsilon$-valid, $\epsilon$-decoupled and accurate to within $\epsilon$. Whenever the resource inequality features $[c \rightarrow c]$ in the input this means that Alice performs a von Neumann measurement on some subsystem $A_1^{i}$ of dimension $D_i$ such that $\sum_i D_i = \lfloor n(R + \delta) \rfloor$. The outcome of this measurement is sent to Bob who at the end of the protocol performs an isometry depending on the received information. Before Alice’s von Neumann measurement, the joint state of $A_1^{i}$ and the remaining quantum system $Q$ is

\[
\sum_x \sqrt{p_x} |x\rangle_{A_1^{i}} |\phi_x\rangle^Q,
\]

where

\[
\left\| \sum_x p_x |x\rangle_{A_1^{i}} - \tau_{D_1} \right\|_1 \leq \epsilon,
\]

and $\tau_D$ is the $D$ dimensional maximally mixed state. At the end of the protocol Bob performs some isometry $U_x$ on $Q$, leaving it $\epsilon$-decoupled from $x$:

\[
\left\| \sum_x p_x |x\rangle_{A_1^{i}} \otimes \theta_x^{Q2} - \sum_x p_x |x\rangle_{A_1^{i}} \otimes \overline{\theta}^{Q2} \right\|_1 \leq \epsilon.
\]

| If the protocol has depth $> 1$, then in the $i$th round a measurement is performed on some $A_1^{i}$ of dimension $D_i$ such that $\sum_i D_i = \lfloor n(R + \delta) \rfloor$. In the analysis below we simply refer to $D$.

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where $|\theta_x\rangle = U_x|\phi_x\rangle$ and $\bar{\theta} = \sum x p_x \theta_x$. Combining (70) and (71) gives

$$\left\| \sum_x p_x |x\rangle \langle x| A_1 \otimes \theta_x^Q - \tau^{A_1} \otimes \bar{\theta}^Q \right\|_1 \leq 2\epsilon. \tag{72}$$

If Alice refrains from the measurement and instead sends $A_1$ through a coherent channel, the resulting state is

$$\sum_x \sqrt{p_x} |x\rangle \langle x| A_1 \otimes |\phi_x\rangle Q.$$  

Bob now performs the controlled unitary $\sum x |x\rangle \langle x| B_1 \otimes U_{B_1}^x$, giving rise to

$$|\Upsilon\rangle^{A_1B_1Q} = \sum_x \sqrt{p_x} |x\rangle \langle x| A_1 \otimes |\theta_x\rangle Q.$$  

(71) may be written as

$$\| \Upsilon^{A_1Q} - \tau^{A_1} \otimes \Upsilon^Q \|_1 \leq 2\epsilon.$$

Invoking Lemma 2.2 there exists an isometry $V$: $B_1 \rightarrow B_2B_3$ on Bob’s side taking $\Upsilon$ to $\Upsilon'$ such that

$$\| \Upsilon'^{A_1B_2B_3Q} - \Phi_{A_1B_2} \otimes \xi^{B_3Q} \|_1 \leq 2\sqrt{2}\epsilon,$$

for some purification $\xi^{B_3Q}$ of $\Upsilon^Q$. Tracing out subsystems gives

$$\| \Upsilon'^{A_1B_2} - \Phi_{A_1B_2} \| \leq 2\sqrt{2}\epsilon.$$  

Thus, the total effect of replacing $[c \rightarrow c^\epsilon : \tau]$ by $[q \rightarrow qq : \tau]$ is the generation of a state close to $\Phi_D$. This mapping preserves the $\epsilon$-validity of the original protocol (with respect to the inputs of $\alpha$) since all of Alice’s reduced density operators are the same. It also preserves the $\epsilon$-accuracy of the protocol concerning the $\beta$ resource, as the final state of $Q$ is the same. We have thus shown

$$\alpha + R[q \rightarrow qq] \geq \beta + R[q q].$$

Equation (49) and lemmas 4.5 and 4.6 give the desired result

$$\alpha + \frac{R}{2} |q \rightarrow q| \geq \beta + \frac{R}{2} |q q|.$$  

Proof (of rule O) Again we fix $\epsilon$ and consider a sufficiently large blocklength $n$ so that the protocol $P_n$ is $\epsilon$-valid, $\epsilon$-decoupled and accurate to within $\epsilon$. Now the roles of Alice and Bob are somewhat interchanged. Assume that the message $x$ being sent is uniformly distributed over a set of size $D$, $\log D = \lfloor n(R + \delta) \rfloor$. Alice performs a unitary operation depending on $x$. At the end of the protocol Bob performs a von Neumann measurement on some subsystem $B_1$ of dimension $D$, yielding outcome $x'$ with some probability $p_{x'|x}$. By the $\epsilon$-accuracy of the protocol

$$\left\| \frac{1}{D} \sum_{x,x'} p_{x'|x} |x\rangle \langle x| \otimes |x'\rangle \langle x'| - \frac{1}{D} \sum_x |x\rangle \langle x| \otimes |x\rangle \langle x| \right\| \leq \epsilon. \tag{73}$$

Before Bob’s measurement, the state of $B_1$ and the remaining quantum system $Q$ conditioned on Alice’s message being $x$ is

$$\sum_{x'} \sqrt{p_{x'|x}} |x\rangle^{B_1} \otimes |\phi_{x'}\rangle^Q.$$
Based on the outcome $x'$ of his measurement, Bob performs some unitary $U_{x'}$ on $Q$, yielding the state

$$\ket{\gamma_x}^{B_1Q} = \sum_{x'} \sqrt{p_{x|x'} x'}^{B_1Q},$$

where $|x', x\rangle^{B_1Q} = |x'\rangle^{B_1|\theta_{x'x}\rangle^Q}$ and $|\theta_{x'x}\rangle = U_{x'}|\phi_{x'x}\rangle$. The decoupling condition says that the state

$$\sigma^{A_1B_1Q} = \frac{1}{D} \sum_{x} |x\rangle^A A_1 \otimes \gamma_x^{B_1Q}$$

satisfies

$$\|\sigma^{A_1B_1Q} - \sigma^{A_1B_1} \otimes \sigma^{Q}\|_1 \leq \epsilon.$$ (74)

The above protocol may be modified to implement coherent communication in lieu of ordinary classical communication. Given a subsystem $A_1$ in the state $|x\rangle^{A_1}$, Alice encodes via controlled unitary operations, eventually yielding

$$|x\rangle^{A_1} \sum_{x'} \sqrt{p_{x|x'} x'}^{B_1Q} |\phi_{x'x}\rangle^Q.$$

Bob refrains from measuring $B_1$ and instead performs the controlled unitary $\sum_{x'} |x'\rangle^{B_1} \otimes U_{x'}^{Q}$, giving rise to $|x\rangle^{A_1} \ket{\gamma_x}^{B_1Q}$. Applying the protocol on the purification $|\Phi_D\rangle^{RA_1}$ yields

$$|\gamma_x^{RA_1B_1Q} := \frac{1}{\sqrt{D}} \sum_{x} |x\rangle^R \otimes |x\rangle^{A_1} \otimes |x', x\rangle^{B_1Q}.$$

By (74) may be rewritten as

$$\left\| \frac{1}{D} \sum_{x,x'} p_{x|x'} |x\rangle^{RA_1B_1Q} - \frac{1}{D} \sum_{x} |x\rangle^R \otimes |x\rangle^{A_1} \otimes |x', x\rangle^{B_1Q} \right\| \leq \epsilon.$$

From this and Corollary 2.3, we get

$$\|\gamma^{RA_1B_1Q} - \Gamma^{RA_1B_1Q}\|_1 \leq 2\sqrt{\epsilon},$$

where

$$\Gamma^{RA_1B_1Q} := \frac{1}{\sqrt{D}} \sum_{x} |x\rangle^R \otimes |x\rangle^{A_1} \otimes |x', x\rangle^{B_1Q}.$$

Since $\Gamma^{RB_1} = \Phi_D^{RB_1}$,

$$\|\gamma^{RB_1} - \Phi_D^{RB_1}\|_1 \leq 2\sqrt{\epsilon}.$$

By (74)

$$\|\gamma^{RB_1Q} - \gamma^{RB_1} \otimes \gamma^{Q}\|_1 \leq \epsilon.$$

Combining the two gives

$$\|\gamma^{RB_1Q} - \Phi_D^{RB_1} \otimes \gamma^{Q}\|_1 \leq \epsilon + 2\sqrt{\epsilon}.$$ 

Define the GHZ state

$$|\Phi_{GHZ}\rangle^{RA_1B_1} = \frac{1}{\sqrt{D}} \sum_{x} |x\rangle^R \otimes |x\rangle^{A_1} \otimes |x\rangle^{B_1},$$

so that

$$\Phi_D^{RA_1B_1} = \Delta^{A_1 \rightarrow A_1 B_1} (\Phi_D^{RA_1}).$$

Invoking Lemma 2.2, there exists an isometry $V : A_1 \rightarrow A_2 A_3$ on Alice’s side taking $\gamma$ to $\gamma'$ such that

$$\|\gamma'^{RB_1A_2A_3Q} - \Phi_{GHZ}^{RB_1A_2} \otimes \xi^{A_3Q}\|_1 \leq 2\sqrt{\epsilon + 2\sqrt{\epsilon}},$$

45
for some purification $\xi^{A_3 Q}$ of $\Upsilon^Q$. Tracing out subsystems gives

$$\|\Upsilon^{RB_1 A_2} - \Phi_{GHZ}^{RB_1 A_2}\|_1 \leq 2\sqrt{\epsilon}.\]

Thus we have successfully replaced $[c \to c]$ by $[q \to qq]$. This mapping preserves the $\epsilon$-validity of the original protocol (with respect to the inputs of $\alpha$) since all of Alice’s reduced density operators are the same. It also preserves the $\epsilon$-accuracy of the protocol concerning the $\beta$ resource, as the final state of $Q$ is the same. We have thus shown that

$$\alpha \geq \beta + R[q \to qq : \tau].$$

Using Theorem 4.9 and (49) gives the desired result

$$\alpha \geq \beta + \frac{R}{2} [q q] + \frac{R}{2} [q \to q].$$

$\square$
7 Two dimensional trade-offs for the family

It is natural to ask about the optimality of our family of resource inequalities. In this section we show that they indeed give rise to optimal two dimensional capacity regions, the boundaries of which are referred to as trade-off curves. To each family member corresponds a theorem identifying the operationally defined capacity region $C(\rho^{AB})$ (or $C(N)$) with a formula $\tilde{C}(\rho^{AB})$ ($\tilde{C}(N)$) given in terms of entropic quantities evaluated on states associated with the given noisy resource $\rho^{AB}$ ($N$).

Each such theorem consists of two parts: the direct coding theorem which establishes $\tilde{C} \subseteq C$ and the converse which establishes $C \subseteq \tilde{C}$.

7.1 Grandmother protocol

To prove the trade-offs involving static resources, we will first need to extend the mother protocol to a “grandmother” RI by combining it with instrument compression.

**Theorem 7.1** (Grandmother) Given a static resource $\rho^{AB}$, for any remote instrument $T : A \to A' X_B$, the following RI holds

$$\frac{1}{2} I(A'; E|X_B)_\sigma [q \to q] + I(X_B; B|E)_\sigma [c \to c] + \langle \rho^{AB} \rangle \geq \frac{1}{2} I(A'; B|X_B)_\sigma [q q].$$

(75)

In the above, the state $\sigma^{X_B A' B E E'}$ is defined by

$$\sigma^{X_B A' B E E'} = T^{A \to A'} X_B (\psi^{ABE}),$$

where $|\psi\rangle |\psi\rangle^{ABE} \geq \rho^{AB}$ and $T : A \to A' X_B$ is a QP extension of $T$.

**Proof** By the instrument compression RI, for any $T : A \to A' X_B$

$$\langle \rho^{AB} \rangle + I(X_B; B|E)_\sigma [c \to c] + H(X|B)_\sigma [c] \geq \langle \rho^{AB} \rangle + \langle \tilde{\Delta}^{X_B \to X_A X_B} \circ T : \rho^A \rangle \geq \langle \tilde{\Delta}^{X_B \to X_A X_B} (\sigma^{X_B A}) \rangle.$$

On the other hand, by Theorem 4.12 and the mother inequality, for any $T : A \to A' X_B$

$$\langle \tilde{\Delta}^{X_B \to X_A X_B} (\sigma^{X_B A'}) \rangle + \frac{1}{2} I(A'; E|X_B)_\sigma [q \to q] \geq \frac{1}{2} I(A'; B|X_B)_\sigma [q q].$$

The grandmother RI is obtained by adding the above RIs, followed by a derandomization via Corollary 4.8.

**Corollary 7.2** In the above theorem, one may consider the special case where $T : A \to A' X_B$ corresponds to some ensemble of operations $(p_x, \mathcal{E}_x)$, $\mathcal{E}_x : A \to A'$, via the identification

$$T : \rho^A \mapsto \sum_x p_x |x\rangle |x\rangle^{X_B} \otimes \mathcal{E}_x (\rho^A).$$

Then the $[c \to c]$ term from (75) vanishes identically.

7.2 Trade-off for noisy super-dense coding

Now that we are comfortable with the various formalisms, the formulae will reflect the QP formalism, whereas the language will be more in the CQ spirit.

Given a bipartite state $\rho^{AB}$, the noisy super-dense coding capacity region $C_{NSD}(\rho^{AB})$ is the two-dimensional region in the $(Q, R)$ plane with $Q \geq 0$ and $R \geq 0$ satisfying the RI

$$\langle \rho^{AB} \rangle + Q [q \to q] \geq R [c \to c].$$

(76)
Theorem 7.3 The capacity region $C_{NSD}(\rho^{AB})$ is given by

$$C_{NSD}(\rho^{AB}) = C_{NSD}^\infty(\rho^{AB}) := \bigcup_{n=1}^\infty \frac{1}{n} C_{NSD}^<(\rho^{AB} \otimes^n),$$

where the $\overline{\cdot}$ means the closure of a set $S$ and $C_{NSD}^<(\rho^{AB})$ is the set of all $R \geq 0, Q \geq 0$ such that

$$R \leq Q + \max_\sigma \{ I(A'|BX)_\sigma : H(A'|X) \leq Q \}.$$ 

In the above, $\sigma$ is of the form

$$\sigma^{XA'B} = \sum_x p_x |x\rangle \langle x|_X \otimes E_x^{A \rightarrow A'}(\rho^{AB}). \tag{77}$$

for some ensemble of operations $(p_x, E_x), E_x : A \rightarrow A'$.

Proof We first prove the converse. Fix $n, R, Q, \delta, \epsilon$, and use the Flattening Lemma 3.26 so that we can assume that $k = 1$. The resources available are

- The state $(\rho^{AB})^\otimes^n$ shared between Alice and Bob. Let it be contained in the system $A^nB^n$, of total dimension $d^n$, which we shall call $AB$ for short.
- A perfect quantum channel $\text{id} : A' \rightarrow A'$, $\text{dim} A' = 2^nQ$, from Alice to Bob (after which $A'$ belongs to Bob despite the notation!).

The resource to be simulated is the perfect classical channel of size $D = 2^{n(R-\delta)}$ on any source, in particular on the random variable $X$ corresponding to the uniform distribution $\tau_D$.

In the protocol (see Fig. 3), Alice performs a $\{cq \rightarrow q\}$ encoding $(E_x : A \rightarrow A')_x$, depending on the source random variable, and then sends the $A'$ system through the perfect quantum channel. After time $t$ Bob performs a POVM $\Lambda : A'B \rightarrow X'$, on the system $A'B$, yielding the random variable $X'$. The protocol ends at time $t_f$. Unless otherwise stated, the entropic quantities below refer to the state of the system at time $t$.

Since at time $t_f$ the state of the system $XX'$ is supposed to be $\epsilon$-close to $\Phi_D$, Lemma 2.4 implies

$$I(X;X')_{t_f} \geq n(R - \delta) - \eta(\epsilon) - K\epsilon n R.$$
By the Holevo bound \[33\],
\[ I(X; X')_{t_f} \leq I(X; A'B). \]
Recall from \[3\] the identity
\[ I(X; A'B) = H(A') + I(A'BX) - I(A'; B) + I(X; B). \]
Since \( I(A'; B) \geq 0 \), and in our protocol \( I(X; B) = 0 \), this becomes
\[ I(X; A'B) \leq H(A') + I(A'BX). \]
Observing that
\[ nQ \geq H(A') \geq H(A'|X), \]
these all add up to
\[ R \leq Q + \frac{1}{n} I(A'BX) + \delta + KRe \frac{\eta(\epsilon)}{n}. \]
As these are true for any \( \epsilon, \delta > 0 \) and sufficiently large \( n \), the converse holds.
Regarding the direct coding theorem, it suffices to demonstrate the RI
\[ \langle \rho^{AB} \rangle + H(A'|X)_\sigma [q \rightarrow q] \geq I(A'; B|X)_\sigma [c \rightarrow c]. \]
This, in turn, follows from linearly combining Corollary \[7.2\] with super-dense coding \[47\] much in the same way the noisy super-dense coding RI \[50\] follows from the mother \[66\].

### 7.3 Trade-off for quantum communication assisted entanglement distillation

Given a bipartite state \( \rho^{AB} \), the quantum communication assisted entanglement distillation capacity region (or “mother” capacity region for short) \( C_M(\rho^{AB}) \) is the set of \((Q, E)\) with \( Q \geq 0 \) and \( E \geq 0 \) satisfying the RI
\[ \langle \rho^{AB} \rangle + Q [q \rightarrow q] \geq E [q q]. \]
(This RI is trivially false for \( Q < 0 \) and trivially true for \( Q \geq 0 \) and \( E \geq 0 \).)

**Theorem 7.4** The capacity region \( C_M(\rho^{AB}) \) is given by
\[ C_M(\rho^{AB}) = \widetilde{C}_M(\rho^{AB}) := \bigcup_{n=1}^{\infty} \frac{1}{n} \widetilde{C}_M^{(1)}((\rho^{AB}) \otimes^n), \]
where \( \widetilde{C}_M^{(1)}(\rho^{AB}) \) is the set of all \( Q \geq 0, E \geq 0 \) such that
\[ E \leq Q + \max_{\sigma} \left\{ I(A'BX)_\sigma : \frac{1}{2} I(A'; EE'|X)_\sigma \leq Q \right\}. \]

In the above, \( \sigma \) is the QP version of \[74\], namely
\[ \sigma^{X'A'BEE'} = \sum_x p_x |x\rangle |x\rangle X \otimes U_{x}^{A \rightarrow A'E'} (\psi^{ABE}). \]
for some ensemble of isometries \((p_x, U_x)\), \( U_x : A \rightarrow A'E' \), and purification \( |\psi\rangle |\psi\rangle^{ABE} \geq \rho^{AB} \).
Proof We first prove the converse, which in this case follows from the converse for the noisy super-dense coding trade-off. The main observation is that super-dense coding \((47)\) induces an invertible linear map \(f\) between the \((Q,E)\) and \((Q,R)\) planes corresponding to the mother capacity region and that of noisy super-dense coding, respectively, defined by

\[
f : (Q,E) \mapsto (Q + E, 2E).
\]

By adding superdense coding (i.e. \(E[qq] + E[q \to q] \geq 2E[c \to c]\)) to the mother \((78)\), we find \(f(C_M) \subseteq C_{NSD}\).

On the other hand, by inspecting the definitions of \(\tilde{C}_{NSD}\) and \(\tilde{C}_M\), we can verify \(\tilde{C}_{NSD} = f(\tilde{C}_M)\).

The converse for the noisy super-dense coding trade-off is written as \(C_{NSD} \subseteq \tilde{C}_{NSD}\). As \(f\) is a bijection, putting everything together we have

\[
C_M \subseteq f^{-1}(C_{NSD}) \subseteq f^{-1}(\tilde{C}_{NSD}) = \tilde{C}_M,
\]

which is the converse for the mother trade-off.

The direct coding theorem follows immediately from Corollary 7.2.

\[\square\]

7.4 Trade-off for noisy teleportation

Given a bipartite state \(\rho^{AB}\), the noisy super-dense coding capacity region \(C_{NTP}(\rho^{AB})\) is a two-dimensional region in the \((R,Q)\) plane with \(R \geq 0\) and \(Q \geq 0\) satisfying the RI

\[
\langle \rho^{AB} \rangle + R[c \to c] \geq Q[q \to q].
\]

Theorem 7.5 The capacity region \(C_{NTP}(\rho^{AB})\) is given by

\[
C_{NTP}(\rho^{AB}) = \tilde{C}_{NTP}(\rho^{AB}) := \bigcup_{n=1}^{\infty} \frac{1}{n} \tilde{C}_{NTP}((\rho^{AB})^\otimes n),
\]

where \(\tilde{C}_{NTP}^{(1)}(\rho^{AB})\) is the set of all \(R \geq 0, Q \geq 0\) such that

\[
Q \leq \max_{\sigma} \left\{ I(A';BX)_{\sigma} : I(A';B|X)_{\sigma} + I(X;BE)_{\sigma} \leq R \right\}.
\]

In the above, \(\sigma\) is of the form

\[
\sigma^{X'BE} = T(\psi^{ABE}),
\]

for some instrument \(T : A \to A'X\) and purification \(|\psi\rangle \langle \psi|^{ABE} \geq \rho^{AB}\).

Proof We first prove the converse. Fix \(n, Q, R, \delta, \epsilon,\) and use the Flattening Lemma so we can assume that the depth is one. The resources available are

- The state \((\rho^{AB})^\otimes n\) shared between Alice and Bob. Let it be contained in the system \(A^nB^n\), which we shall call \(AB\) for short.
- A perfect classical channel of size \(2^{nR}\).
Figure 4: A general protocol for noisy teleportation.

The resource to be simulated is the perfect quantum channel \( \text{id}_D : A_1 \rightarrow B_1 \), \( D = \dim A_1 = 2^n(Q - \delta) \), from Alice to Bob, on any source, in particular on the maximally entangled state \( \Phi^{A'A_1} \).

In the protocol (see Fig. 4), Alice performs a POVM \( \Lambda : AA_1 \rightarrow X \) on the system \( AA_1 \), and sends the outcome random variable \( X \) through the classical channel. After time \( t \) Bob performs a \( \{cq \rightarrow q\} \) decoding quantum operation \( D : XB \rightarrow B_1 \). The protocol ends at time \( t_f \). Unless otherwise stated, the entropic quantities below refer to the time \( t_f \).

Our first observation is that performing the POVM \( \Lambda \) induces an instrument \( T : A \rightarrow A'X \), so that the state of the system \( XA'BE \) at time \( t_f \) is indeed of the form of (85).

Since at time \( t_f \) the state of the system \( A'B_1 \) is supposed to be \( \epsilon \)-close to \( \Phi_D \), Lemma 2.1 implies
\[
I(A'\rangle B_1)_{t_f} \geq n(Q - \delta) - \eta(\epsilon) - K\epsilon nQ.
\]

By the data processing inequality,
\[
I(A'\rangle B_1)_{t_f} \leq I(A'\rangle BX).
\]

Thus
\[
Q \leq \frac{1}{n} I(A'\rangle BX) + \delta + KQ\epsilon + \frac{\eta(\epsilon)}{n}.
\]  

(86)

To bound \( R \), start with the identity
\[
I(X; A'BE) = H(A') + I(A'\rangle BEX) - I(A'; BE) + I(X; BE).
\]

Since \( I(A'; BE) = 0 \), \( H(A') \geq H(A'\rangle X) \) and \( I(A'\rangle BEX) \geq I(A'\rangle BX) \), this becomes
\[
I(X; A'BE) \geq I(A'; B|X) + I(X; BE).
\]

Combining this with
\[
nR \geq H(X) \geq I(X; A'BE)
\]

gives the desired
\[
R \geq \frac{1}{n} [I(A'; B|X) + I(X; BE)].
\]  

(87)

As \( \text{S3} \) and \( \text{S4} \) are true for any \( \epsilon, \delta > 0 \) and sufficiently large \( n \), the converse holds.

\footnote{Indeed, first a pure ancilla \( A'A_1 \) was appended, then another pure ancilla \( X \) was appended, the system \( AA'A_1X \) was rotated to \( A'E'X \), and finally \( X \) was measured and \( E' \) was traced out.}
Regarding the direct coding theorem, it suffices to demonstrate the RI
\[
\langle \rho^{AB} \rangle + (I(A'; B|X) + I(X; BE)_{\sigma})_{c \rightarrow c} \geq I(A'|BX)_{\sigma} [q \rightarrow q].
\] (88)

Linearly combining the grandmother RI ((75)) with teleportation ((46)), much in the same way the variation on the noisy teleportation RI ((68)) was obtained from the mother ((65)), we have
\[
\langle \rho^{AB} \rangle + (I(A'; B|X) + I(X; BE)_{\sigma})_{c \rightarrow c} + o[q.q] \geq I(A'|BX)_{\sigma} [q \rightarrow q].
\]

Equation (88) follows by invoking Lemma 4.5 and (57). □

7.5 Trade-off for classical communication assisted entanglement distillation

Given a bipartite state \(\rho^{AB}\), the classical communication assisted entanglement distillation capacity region (or “entanglement distillation” capacity region for short) \(C_{\text{ED}}(\rho^{AB})\) is the two-dimensional region in the \((R, E)\) plane with \(R \geq 0\) and \(E \geq 0\) satisfying the RI
\[
\langle \rho^{AB} \rangle + R [c \rightarrow q] \geq E[q.q].
\] (89)

Theorem 7.6 The capacity region \(C_{\text{ED}}(\rho^{AB})\) is given by
\[
C_{\text{ED}}(\rho^{AB}) = \tilde{C}_{\text{ED}}(\rho^{AB}) := \bigcup_{n=1}^{\infty} \frac{1}{n} \tilde{C}_{\text{ED}}^{(1)}((\rho^{AB})^{\otimes n}),
\]
where \(\tilde{C}_{\text{ED}}^{(1)}(\rho^{AB})\) is the set of all \(R \geq 0, E \geq 0\) such that
\[
E \leq \max_{\sigma} \{I(A'|BX)_{\sigma} : I(A'|EE'|X)_{\sigma} + I(X; BE)_{\sigma} \leq R\},
\] (90)

In the above, \(\sigma\) is the fully QP version of (56), namely
\[
\sigma^{XA'E'E'} = T'(\psi^{ABE}),
\] (91)
for some instrument \(T : A \rightarrow A'E'E'X\) with pure quantum output and purification \(\langle \psi\rangle \langle \psi\rangle^{ABE} \supseteq \rho^{AB}\).

Proof We first prove the converse, which in this case follows from the converse for the noisy teleportation trade-off. The argument very much parallels that of the converse for the mother trade-off. The main observation is that teleportation ((46)) induces an invertible linear map \(g\) between the \((R, E)\) and \((R, Q)\) planes corresponding to the entanglement distillation capacity region and that of noisy teleportation, respectively, defined by
\[
g : (R, E) \mapsto (R + 2E, E).
\]

By applying TP to (58), we find
\[
g(C_{\text{ED}}) \subseteq C_{\text{NTP}}.
\] (92)

On the other hand, from the definitions of \(\tilde{C}_{\text{ED}}\) and \(\tilde{C}_{\text{NTP}}\) ((60) and (64)), we have
\[
\tilde{C}_{\text{ED}} = g(\tilde{C}_{\text{NTP}}).
\] (93)

The converse for the noisy teleportation trade-off is written as \(C_{\text{NTP}} \subseteq \tilde{C}_{\text{NTP}}\). As \(g\) is a bijection, putting everything together we have
\[
C_{\text{ED}} \subseteq g^{-1}(C_{\text{NTP}}) \subseteq g^{-1}(\tilde{C}_{\text{NTP}}) = \tilde{C}_{\text{ED}},
\]
which is the converse for the entanglement distillation trade-off.

Regarding the direct coding theorem, it suffices to demonstrate the RI

\[ \langle \rho^{AB} \rangle + (I(A'; EE'|X)_{\sigma} + I(X; BE)_{\sigma})_{[c \rightarrow c]} \geq I(A'BX)_{\sigma} [q q]. \] (94)

Linearly combining the grandmother RI ((75)) with teleportation (46), much in the same way the variation on the hashing RI ((67)) was obtained from the mother ((65)), we have

\[ \langle \rho^{AB} \rangle + (I(A'; EE'|X)_{\sigma} + I(X; BE)_{\sigma})_{[c \rightarrow c]} + o[q q] \geq I(A'BX)_{\sigma} [q \rightarrow q]. \]

(94) follows by invoking Lemma 4.5 and (57). \( \Box \)

7.6 Trade-off for entanglement assisted quantum communication

Given a noisy quantum channel \( N : A' \rightarrow B \), the entanglement assisted quantum communication capacity region (or “father” capacity region for short) \( C_F(N) \) is the region of \((E, Q)\) plane with \( E \geq 0 \) and \( Q \geq 0 \) satisfying the RI

\[ \langle N \rangle + E[q q] \geq Q[q \rightarrow q]. \] (95)

**Theorem 7.7** The capacity region \( C_F(N) \) is given by

\[ C_F(N) = \overline{C}_F(N) := \bigcup_{n=1}^{\infty} \frac{1}{n} \overline{C}_F^{(1)}(N^\otimes n), \]

where \( \overline{C}_F^{(1)}(N) \) is the set of all \( E \geq 0, Q \geq 0 \) such that

\[ Q \leq E + I(A)_{B\sigma} \]

\[ Q \leq \frac{1}{2} I(A; B)_{\sigma}. \]

In the above, \( \sigma \) is of the form

\[ \sigma_{ABE} = U_N \circ E(\phi^{AA''}), \]

for some pure input state \( |\phi^{AA''}\rangle \), encoding operation \( E : A'' \rightarrow A' \), and where \( U_N : A' \rightarrow BE \) is an isometric extension of \( N \).

This tradeoff region includes two well-known limit points. When \( E = 0 \), the quantum capacity of \( N \) is \( I(A)_{B} \), and for \( E > 0 \), entanglement distribution \( ([q \rightarrow q] \geq [q q]) \) means it should still be bounded by \( I(A)_{B} + E \). On the other hand, when given unlimited entanglement, the classical capacity is \( I(A; B) \), and thus the quantum capacity is never greater than \( \frac{1}{2} I(A; B) \) no matter how much entanglement is available. These bounds meet when \( E = \frac{1}{2} I(A; E) \) and \( Q = \frac{1}{2} I(A; E) \), the point corresponding to the father protocol. Thus, the goal of our proof is to show that the father protocol is optimal.

**Proof** We first prove the converse. Fix \( n, E, Q, \delta, \epsilon \), and use the Flattening Lemma to reduce the depth to one. The resources available are

- The channel \( N^\otimes n : A'^n \rightarrow B^n \) from Alice to Bob. We shall shorten \( A'^n \) to \( A' \) and \( B^n \) to \( B \).
- The maximally entangled state \( \Phi^{T_A T_B} \), \( \dim T_A = \dim T_B = 2^{nE} \), shared between Alice and Bob.
The resource to be simulated is the perfect quantum channel $id_D : A_1 \rightarrow B_1$, $D = \dim A_1 = 2^{n(Q - \delta)}$, from Alice to Bob, on any source, in particular on the maximally entangled state $\Phi^{RA_1}$.

In the protocol (see Fig. 5), Alice performs a general encoding map $E : A_1 T_A \rightarrow A' E'$ and sends the system $A'$ through the noisy channel $\mathcal{N} : A' \rightarrow B$. After time $t$ Bob performs a decoding operation $\mathcal{D} : B T_B \rightarrow B_1$. The protocol ends at time $t_f$. Unless otherwise stated, the entropic quantities below refer to the time $t_f$.

Define $A := RT_B$ and $A'' := A_1 T_A$. Since at time $t_f$ the state of the system $RB_1$ is supposed to be $\epsilon$-close to $\Phi^D$, Lemma 2.1 implies

$$I(R)B_1)_{t_f} \geq n(Q - \delta) - \eta(\epsilon) - K\epsilon nQ.$$ 

By the data processing inequality,

$$I(R)B_1)_{t_f} \leq I(RBT_B).$$

Together with the inequality

$$I(R)BT_B) \leq I(RT_B)B) + H(T_B),$$

since $E = H(T_B)$, the above implies

$$Q \leq E + \frac{1}{n}I(A)B + \delta + K\epsilon + \frac{\eta(\epsilon)}{n}.$$ 

Combining this with

$$H(A) = H(R) + H(T_B) = nQ + nE.$$ 

gives

$$Q \leq \frac{1}{2n}I(A;B) + \delta/2 + K\epsilon/2 + \frac{\eta(\epsilon)}{2n}.$$ 

As these are true for any $\epsilon, \delta > 0$ and sufficiently large $n$, the converse holds.

Regarding the direct coding theorem, it follows directly form the father RI

$$\langle N \rangle + \frac{1}{2}I(A;E)_{\sigma} [q \rightarrow q] \geq \frac{1}{2}I(A;B)_{\sigma} [q \rightarrow q].$$
Figure 6: A general protocol for entanglement assisted classical communication.

7.7 Trade-off for entanglement assisted classical communication

The result of this subsection was first proved by Shor in [56]. Here we state it for completeness, and give an independent proof of the converse. An alternative proof of the direct coding theorem was sketched in [18] and is pursued in [17] to unify this result with the father trade-off.

Given a noisy quantum channel $\mathcal{N}: A' \rightarrow B$, the entanglement assisted classical communication capacity region (or “entanglement assisted” capacity region for short) $C_{EA}(\mathcal{N})$ is the set of all points $(E, R)$ with $E \geq 0$ and $R \geq 0$ satisfying the RI

$$\langle N \rangle + E[q,q] \geq R[c \rightarrow c].$$

(96)

**Theorem 7.8** The capacity region $C_{EA}(\mathcal{N})$ is given by

$$C_{EA}(\mathcal{N}) = \bar{C}_{EA}(\mathcal{N}) := \bigcup_{n=1}^{\infty} \frac{1}{n} \bar{C}^{(1)}_{EA}(\mathcal{N}^\otimes n),$$

where $\bar{C}^{(1)}_{EA}(\mathcal{N})$ is the set of all $E \geq 0$, $R \geq 0$ such that

$$R \leq \max_{\sigma} \{ I(AX;B)_{\sigma} : E \geq H(A|X)_{\sigma} \}.$$

(97)

In the above, $\sigma$ is of the form

$$\sigma^{XAB} = \sum_{x} p_{x|x}^{X} \otimes \mathcal{N}(\phi_{x}^{AA'}),$$

(98)

for some pure input ensemble $(p_{x}, |\phi_{x}^{AA'}\rangle_{x})$.

**Proof** We first prove the converse. Fix $n, E, Q, \delta, \epsilon$, and again use the flattening lemma to reduce depth to one. The resources available are

- The channel $\mathcal{N}^\otimes n: A'^{n} \rightarrow B^{n}$ from Alice to Bob. We shall shorten $A'^{n}$ to $A'$ and $B^{n}$ to $B$.
- The maximally entangled state $\Phi_{T A T B}^{T A', T B}$, $\dim T_A = \dim T_B = 2^n E$, shared between Alice and Bob.

The resource to be simulated is the perfect classical channel of size $D = 2^{n(R-\delta)}$ on any source, in particular on the random variable $X$ corresponding to the uniform distribution $\tau_{D}$.

In the protocol (see Fig. 6). Alice performs a $\{cq \rightarrow q\}$ encoding $(\mathcal{E}_{x}: T_A \rightarrow A')_{x}$, depending on the source random variable, and then sends the $T_A$ system through the noisy channel $\mathcal{N}: A' \rightarrow BE$. 

55
After time $t$ Bob performs a POVM $\Lambda : T_B B \to X'$, on the system $T_B B$, yielding the random variable $X'$. The protocol ends at time $t_f$. Unless otherwise stated, the entropic quantities below refer to the state of the system at time $t$.

Since at time $t_f$ the state of the system $XX'$ is supposed to be $\epsilon$-close to $\Phi D$, Lemma 2.1 implies

$$I(X; X')_{t_f} \geq n(R - \delta) - \eta(\epsilon) - K\epsilon nR.$$ 

By the Holevo bound

$$I(X; X')_{t_f} \leq I(X; T_B B).$$

Using the chain rule twice, we find

$$I(X; T_B B) = I(X; B | T_B) + I(X; T_B) = I(X T_B; B) + I(X; T_B) - I(T_B; B)$$

Since $I(T_B; B) \geq 0$ and in this protocol $I(X; T_B) = 0$, this becomes

$$I(X; T_B B) \geq I(X T_B; B).$$

These all add up to

$$R \leq \frac{1}{n} I(X T_B; B) + \delta + K\epsilon + \frac{\eta(\epsilon)}{n},$$

while on the other hand,

$$nE \geq H(T_B | X).$$

As these are true for any $\epsilon, \delta > 0$ and sufficiently large $n$, we have thus shown a variation on the converse with the state $\sigma$ from 48 replaced by $\tilde{\sigma}$,

$$\tilde{\sigma}^{XABE'} = \sum_x p_x |x\rangle X \otimes N \circ U_x^{A'' \to A'E'}(\phi^{AA''}),$$

defining $A := T_B$ and letting $U_x : T_A \to A'E'$ be the isometric extension of $E_x$.

However, this is a weaker result than we would like; the converse we have proved allows arbitrary noisy encodings and we would like to show that isometric encodings are optimal, or equivalently that the $E'$ register is unnecessary. We will accomplish this, following Shor 55, by using a standard trick of measuring $E'$ and showing that the protocol can only improve. If we apply the dephaser map $\text{id} : E' \to Y$ to $\tilde{\sigma}^{ABE'}$, we obtain a state of the form

$$\sigma^{XYAB} = \sum_{xy} p_{xy} |x\rangle X \otimes |y\rangle Y \otimes N(\psi_{xy}^{AA'}).$$

The converse now follows from

$$I(B; AX)_{\tilde{\sigma}} \leq I(B; AXY)_{\sigma}$$

$$H(A|X)_{\tilde{\sigma}} \geq H(A|XY)_{\sigma}.$$ 

\[\square\]
We have shown how to set up a systematic theory of quantum information resources. We restricted attention to communication scenarios with two active protagonists connected by unidirectional channels with passive feedback. After mastering the formal foundations, this theory allows for fairly flexible play with existing protocols, and derivation of new ones. The main tools for the latter turned out to be derandomization and coherification. Then we went on to prove trade-off converses for a family of protocols. Again the general resource calculus came in handy to save work, and to organize the converse proofs.

The primary limitation is that our approach is most successful when considering one-way communication and when dealing with only one noisy resource at a time. These, and other limitations, suggest a number of ways in which we might imagine revising the notion of an asymptotic resource given in Definition 3.21. For example, if we were to explore unitary and/or bidirectional resources more carefully, then we would need to reexamine our treatments of depth and of relative resources. Recall that we (1) always simulate the depth-1 version of the output resource, (2) are allowed to use a depth-$k$ version of the input resource where $k$ depends only on the target inefficiency and not the target error. These features were chosen rather delicately in order to guarantee the convergence of the error and inefficiency in the Composability Theorem 3.29, which in turn gets most of its depth blow-up from the double-blocking of the Sliding Lemma 3.15. However, it is possible that a different model of resources would allow protocols which deal with depth differently. This won’t make a difference for one-way resources due to the Flattening Lemma 3.26, but there is evidence that depth is an important issue in bidirectional communication [42]; on the other hand, it is unknown how quickly depth needs to scale with $n$.

Relative resources are another challenge for studying bidirectional communication. As we discussed in Section 3.2, if $\rho^{AB}$ cannot be bilocally prepared then $\langle N : \rho^{AB} \rangle$ fails to satisfy (10) and is thus not a valid resource. The problem is that being able to simulate $n$ uses of a channel on $n$ copies of a correlated or entangled state is not necessarily stronger than the ability to simulate $n-1$ uses of the channel on $n-1$ copies of the state. The fact that many bidirectional problems in classical information theory [53] remain unsolved is an indication that the quantum versions of these problems will be difficult. On the other hand, it is possible that special cases, such as unitary gates or Hamiltonians, will offer simplifications not possible in the classical case [6, 13].

Another challenge to our definition of a resource comes from unconventional “pseudo-resources” that resemble resources in many ways but fail to satisfy the quasi-i.i.d. requirement (17). For example, the ability to remotely prepare an arbitrary non qubit state cannot be simulated by the ability to remotely prepare $k$ states of $n(1+\delta)/k$ qubits each. There are many fascinating open questions surrounding this “single-shot” version of remote state preparation (RSP); for example, is the RSP capacity of a channel ever greater than its quantum capacity?8 The case of a noiseless channel was treated in [7]. Another example comes from the “embezzling states” of [30]. The $n$-qubit embezzling state can be prepared from $n$ cbits and $n$ ebits (which are also necessary [31]) and can be used as a resource for entanglement dilution and for simulating noisy quantum channels on non-i.i.d. inputs [52]; however, it also cannot be prepared from $k$ copies of the $n(1+\delta)/k$-qubit embezzling state. These pseudo-resources are definitely useful and interesting, but it is unclear how they should fit into our resource formalism.

Other extensions of the theory will probably require less modification. For example, it will not a priori be hard to extend the theory to multi-user scenarios. Resources and capacities can even be defined in non-cooperative situations pervasive in cryptography (see e.g. [63]), which will mostly require a more careful enumeration of different cases. We can also consider privacy to be a resource. Our definitions of decoupled classical communication are a step in this direction. Also there are expressions for the private capacity of quantum channels [15] and states [22], and there are cryptographic versions of our Composability Theorem [2, 60].

8Thanks to Debbie Leung for suggesting this question.
Our expressions for trade-off curves also should be seen more as first steps rather than final answers. For one thing, we would ultimately like to have formulae for the capacity that can be efficiently computed, which will probably require replacing our current regularized expressions with single-letter ones. This is related to the additivity conjectures, which are equivalent for some channel capacities \(^{57}\), but are false for others \(^{23}\).

A more reasonable first goal is to strengthen some of the converse theorems, so that they do not require maximizing over as many different quantum operations. As inspiration, note that \(^{1}\) showed that isometric encodings suffice to achieve the optimal rate of quantum communication through a quantum channel. However, the analogous result for entanglement-assisted quantum communication is not known. Specifically, in Fig. 5 we suspect that the \(E'\) register (used to discard some of the inputs) is only necessary when Alice and Bob share more entanglement than the protocol can use. Similarly, it seems plausible to assume that the optimal form of protocols for noisy teleportation (Fig. 4) is to perform a general CPTP preprocessing operation on the shared entanglement, followed by a unitary interaction between the quantum data and Alice’s part of the entangled state. These are only two of the more obvious examples and there ought to be many possible ways of improving our formulae.

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