THE QUANTUM SUPER YANGIAN AND 
CASIMIR OPERATORS OF $U_q(gl(M|N))$

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The $\mathbb{Z}_2$ graded Yangian $Y_q(gl(M|N))$ associated with the Perk - Schultz $R$ matrix is introduced. Its structural properties, the central algebra in particular, are studied. A $\mathbb{Z}_2$ graded associative algebra epimorphism $Y_q(gl(M|N)) \rightarrow U_q(gl(M|N))$ is obtained in explicit form. Images of central elements of the quantum super Yangian under this epimorphism yield the Casimir operators of the quantum supergroup $U_q(gl(M|N))$ constructed in an earlier publication.

1 INTRODUCTION

Quantum supergroups[1] are a particularly interesting class of deformations of the universal enveloping algebras of the basic classical Lie superalgebras. Their origin may be traced back to the Perk – Schultz solution[2] of the Yang – Baxter equation, and the work of Bazhanov and shadrikov[3] on the $R$ – matrices associated to the vector representation of $osp(M|2N)$ (For a quantum supergroup interpretation of these $R$ – matrices, see [4]). However, systematic studies of these algebraic structures only began very recently, largely motivated by applications to soluble lattice models in statistical mechanics and knot theory. Now the understanding of the so – called type I quantum supergroups is rather complete, but many important problems remain open for the type II quantum supergroups.

Another kind of algebraic structure closely related to the quantum supergroups is quantum super Yangians, which are generalizations of Drinfeld’s Yangian algebras[5], and also arise naturally from the algebraic description of soluble lattice models in statistical mechanics. The classical super Yangians have been studied by Nazarov[6], who in particular explored the hidden super Yangian symmetry of the classical Lie
superalgebras. In this letter we will study the structural features of the quantum super Yangian associated with the Perk – Schultz $R$ matrix, and also to investigate the interrelationship between this quantum super Yangian and the quantum supergroup $U_q(gl(M|N))$, thus to better our understanding of the structures of the latter from a Yangian point of view. We wish to point out that Yangians provide a new framework for studying the ordinary Lie algebras, within which many of the structural and representation theoretical problems can be treated in a unified manner. For example, the recent Yangian interpretations of the Gelfand - Zetlin bases for irreducible modules and the central algebra of $gl(M)$ are fascinating developments in the theory of Lie algebras (for a review see [8]).

We will define the quantum super Yangian $Y_q(gl(M|N))$ associated with the Lie superalgebra $gl(M|N)$ employing the Faddeev – Reshetikhin – Takhtajian approach to quantization of algebras, starting from the Perk – Schultz solution of the Yang – Baxter equation. The Hopf algebraic structure of the quantum super Yangian will be elucidated, and a set of central elements will be constructed explicitly. We will also demonstrate that there exists a homomorphism of $\mathbb{Z}_2$ graded associative algebras from $Y_q(gl(M|N))$ to $U_q(gl(M|N))$, which is surjective, and explicitly realized by the universal $L$ – operator constructed in [9]. Images of the $Y_q(gl(M|N))$ central elements under this epimorphism are Casimir operators of [10], which turn out to be those constructed in [9] using the $q$ – (super)trace technique developed in earlier publications [10] [1].

As the ordinary quantum group $U_q(gl(M))$ is nothing else but $U_q(gl(M|N))$ at $N = 0$, $Y_q(gl(M|N))$ at $N = 0$ reduces to the quantum Yangian studied in [12]. Our results on central elements also yield, in this case, a Yangian interpretation of the $U_q(gl(M))$ Casimir operators of [10], which will be shown to generate the entire center of $U_q(gl(M))$.

2 VECTOR REPRESENTATIONS OF $U_q(gl(M|N))$

We will largely follow the notation of [9], but adopt a different convention for the co-multiplication etc.. To avoid confusion, we present the definition of the quantum supergroup $U_q(gl(M|N))$ here. We will also briefly explain the $R$ - matrices associated with its vector and dual vector modules, which are of crucial importance for studying the quantum super Yangian $Y_q(gl(M|N))$.

The quantum supergroup $U_q(gl(M|N))$ is a $\mathbb{Z}_2$-graded unital algebra generated by $E^a_{\pm 1}, E^a_a, a, a \pm 1 = 1, 2, ..., M + N$, subject to the following relations

\[
[E^a_a, E^b_b] = 0, \quad [E^a_a, E^b_{b\pm 1}] = (\delta^b_a - \delta^b_{b\pm 1})E^b_{b\pm 1}, \quad \forall a, b
\]

\[
[E^a_{a+1}, E^b_{b+1}] = (-1)^{|a|} \delta_{ab} \frac{q^h - q^{-h}}{q - q^{-1}}, \quad (E^m_{m+1})^2 = (E^m_m)^2 = 0,
\]

\[
E^a_{a+1}E^b_{b+1} = E^b_{b+1}E^a_{a+1}, \quad E^a_a E^b_{b+1} = E^b_{b+1} E^a_a, \quad |a - b| \geq 2,
\]

\[
(E^a_{a+1})^2 E^{a\pm 1}_{a\pm 1} - (q + q^{-1}) E^a_{a+1} E^{a\pm 1}_{a\pm 1} E^a_{a+1} + E^a_{a\pm 1+1} (E^a_{a+1})^2 = 0, \quad a \neq m,
\]

\[
(E^a_{a+1})^2 E^{a\pm 1}_{a\pm 1} - (q + q^{-1}) E^a_{a+1} E^{a\pm 1}_{a\pm 1} E^a_{a+1} + E^a_{a\pm 1+1} (E^a_{a+1})^2 = 0, \quad a \neq m,
\]

\[
[E^m_{m+1}, E^m_{m+1}] = [E^m_{m-1}, E^m_{m-1}] = 0,
\]

(1)
where

\[ h_a = (-1)^{[a]} E_a^a - (-1)^{[a+1]} E_a^{a+1}, \]

with \([a] = 0\) for \(a \leq m\) and \([a] = 1\) for \(a > m\); \([\cdot, \cdot]\) represents the standard graded commutator; and \(E_{M+1}^{M-1}, E_{M+2}^{M-2}\) are the \(a = M - 1, b = M + 1\) case of the following elements

\[
\begin{align*}
E^a_b &= E^a_c E^b_c - q^{-[a]} E^a_b E^b_c, \\
E^a_a &= E^a_c E^c_a - q^{-[a]} E^a_b E^b_c, \quad a < c < b.
\end{align*}
\]

Throughout the paper, we assume that \(q \in \mathbb{C}\) is not a root of unity.

The co-multiplication \(\Delta : U_q(gl(M|N)) \rightarrow U_q(gl(M|N)) \otimes U_q(gl(M|N))\) is taken to be

\[
\Delta(E^a_{a+1}) = E^a_{a+1} \otimes 1 + q^{ha} \otimes E^a_{a+1},
\]
\[
\Delta(E^{a+1}_a) = E^{a+1}_a \otimes q^{-ha} + 1 \otimes E^{a+1}_a,
\]
\[
\Delta(E^a_a) = E^a_a \otimes 1 + 1 \otimes E^a_a;
\]

and the co - unit \(\epsilon : U_q(gl(M|N)) \rightarrow \mathbb{C}\) is defined by

\[
\epsilon(E^a_{a+1}) = \epsilon(E^a_a) = 0,
\]
\[
\epsilon(1) = 1.
\]

Then the corresponding antipode \(S : U_q(gl(M|N)) \rightarrow U_q(gl(M|N))\) is given by

\[
\begin{align*}
S(E^a_{a+1}) &= -q^{-ha} E^a_{a+1}, \\
S(E^{a+1}_a) &= -E^{a+1}_a q^{ha}, \\
S(E^a_a) &= -E^a_a;
\end{align*}
\]

which is a \(\mathbb{Z}_2\)-graded algebra anti-automorphism, i.e., for homogeneous elements \(x, y \in U_q(gl(M|N))\), we have \(S(xy) = (-1)^{[x][y]} S(y)S(x)\), and generalizing to inhomogeneous elements through linearity. We will denote the opposite co-multiplication by \(\Delta'\).

The vector module over \(U_q(gl(M|N))\) is a \(\mathbb{Z}_2\)-graded vector space \(V\) of dimension \(M+N\), for which we choose a basis \(\{v^a | a = 1, 2, ..., M+N\}\), that is homogeneous with \(v^a\) being even if \([a] = 0\) and odd if \([a] = 1\). The action of \(U_q(gl(M|N))\) is now defined by \(E^b_c v^c = \delta^c_b v^a\), for \(b = a - 1, a, a + 1\). We denote the associated vector representation of \(U_q(gl(M|N))\) by \(\pi\). Then in this basis \(\pi(E^b_b) = e^b_b\), where \(e^b_b \in End(V)\) are the standard matrix units.

We denote the dual vector module over \(U_q(gl(M|N))\) by \(V^*\), and the corresponding representation by \(\pi^*\). A useful basis for \(V^*\) is \(\{v^*_a | a = 1, 2, ..., M+N\}\) defined by \(v^*_a(v^b) = \delta^b_a\). The action of \(U_q(gl(M|N))\) is defined by \(E^b_c v^*_c(v^d) = (-1)^{([a]+[b])[c]} v^*_c(S(E^b_c)v^d)\). Let \(\triangleleft\) denote the graded transposition on matrices defined by \((e^a_b)^\triangleleft = (-1)^{([a]+[b])[a]} e^a_b\). Then we have \(\pi^*(E^a_{a\pm1}) = -q^{1} e^a_{a\pm1}^\triangleleft, \pi^*(E^a_a) = -e^a_a\).

The Perk - Schultz solution of the Yang - Baxter equation is associated with the tensor product module \(V \otimes V\) over \(U_q(gl(M|N))\), and is given by

\[
R(x) = q \sum_a e^a_a \otimes e^a_a (-1)^{[a]} - x^{-1} q^{-1} \sum_a e^a_a \otimes e^a_a (-1)^{[a]} + (q - q^{-1})(x^{-1} \sum_{a<b} e^b_a \otimes e^b_a (-1)^{[b]} + \sum_{a>b} e^b_a \otimes e^b_a (-1)^{[b]}, \quad x \in \mathbb{C}.
\]
For later use we also give the solution $R(x)^*$ of the Yang-Baxter equation associated with the $U_q(gl(M|N))$ module $V^* \otimes V$, which can be expressed in terms of $R(x)$ through

$$R^*(x) = (R(x)^{-1})^{+1}(q - x^{-1}q^{-1})(q^{-1} - x^{-1}q),$$

where $^{+1}$ denotes the graded transposition over the first space, and the numerical factor is introduced for convenience. Explicitly,

$$R^*(x) = x^{-1} \left[ q \sum_a e_a^a \otimes e_{1a}^{(-1)} + (q - q^{-1}) \sum_{a < b} e_a^b \otimes e_a^b (1)^{[a][b] + [a] + [b]} \right]$$

$$- \left[ q - \sum_a e_a^a \otimes e_{1a}^{(-1)} - (q - q^{-1}) \sum_{a > b} e_a^b \otimes e_a^b (1)^{[a][b] + [a] + [b]} \right].$$

$R^*$ satisfies the following equations

$$R_{12}^*(x)R_{13}^*(xy)R_{23}(y) = R_{23}(y)R_{13}^*(xy)R_{12}^*(x),$$

$$R^*(x)(\pi^* \otimes \pi)\Delta(u) = (\pi^* \otimes \pi)\Delta'(u)R^*(x), \quad \forall u \in U_q(gl(M|N)). \quad (3)$$

It is important to observe that $V^* \otimes V$ can be decomposed into the direct sum of two irreducible $U_q(gl(M|N))$ modules, $V^* \otimes V = A \oplus \{|0\}$, with $\{|0\}$ being trivial, and $A$ a quantum deformation of the adjoint module of $gl(M|N)$. The tensor product $V \otimes V^*$ also has these properties, namely, $V \otimes V^* = \tilde{A} \oplus \{|0\}$. Here the modules are defined with respect to the co-multiplication $\Delta$. We introduce the projection operators

$$P[A](V^* \otimes V) = \tilde{A}, \quad \tilde{P}[A](V \otimes V^*) = A,$$

$$P[0](V^* \otimes V) = \{|0\}, \quad \tilde{P}[0](V \otimes V^*) = \{|0\},$$

$$P[A] + P[0] = P, \quad \tilde{P}[A] + \tilde{P}[0] = \tilde{P},$$

where $P : V^* \otimes V \rightarrow V \otimes V^*$ and $\tilde{P} : V \otimes V^* \rightarrow V^* \otimes V$ are the graded permutation operators respectively defined by $P(v_a^b \otimes v^b) = (-1)^{[a][b]}v^b \otimes v_a^b$, and $P(v^b \otimes v_a^a) = (-1)^{[a][b]}v_a^a \otimes v^b$. Needless to say, these projection operators all commute with $\Delta(u), \forall u U_q(gl(M|N))$, e.g., $(\pi \otimes \pi^*)\Delta(u)P[A] = P[A](\pi^* \otimes \pi)\Delta(u)$, and also satisfy the relations $P[0]P[A] = P[A]P[0] = 0$, $P[0]P[0] + P[A]P[A] = id_{V \otimes V^*}$, and similar relations obtained by exchanging the orders of the two sets of operators. These relations define the projectors uniquely.

In terms of the projectors and the permutation operators, $R$ and $R^*$ can be expressed as

$$PR^*(x) = (1 - x^{-1})P[A] + (q^{N-M} - x^{-1}q^{M-N})P[0],$$

$$(R^*(x))^{-1} \tilde{P} = \frac{\tilde{P}[A]}{1 - x^{-1}} + \frac{\tilde{P}[0]}{q^{N-M} - x^{-1}q^{M-N}}. \quad (4)$$

For the purpose of this letter, we will only need explicit matrix forms of the projection operators $\tilde{P}[0]$ and $P[0]$, which we spell out below

$$P[0] = \frac{1}{SD_q(V)} \sum_{a,b} (-1)^{[b][a] + [b]} e_b^a \otimes e_b^a,$$

$$\tilde{P}[0] = \frac{1}{SD_q(V)} \sum_{a,b} (-1)^{[a][a] + [b]} \pi(q^{h_2}) e_b^a \pi(q^{-h_2}) \otimes e_b^a,$$
where \( h_{2p} \) is the linear sum of \( h_a \)’s such that

\[
[h_{2p}, E_{a+1}^a] = 2(-1)^{|a|}(1 - \delta_{am})E_{a+1}^a, \quad [h_{2p}, E_{a}^{a+1}] = -2(-1)^{|a|}(1 - \delta_{am})E_{a}^{a+1},
\]

and \( SD_q(V) \) is the \( q \)-superdimension of \( V \).

3 THE QUANTUM SUPER YANGIAN AND ITS CENTRAL ALGEBRA

With the above preparations, we can now define \( Y_q(gl(M|N)) \), the quantum super Yangian associated with \( gl(M|N) \), and investigate its properties. \( Y_q(gl(M|N)) \) is a \( \mathbb{Z}_2 \) graded algebra generated by \( \{T_a^b[k] \mid 0 < k \in \mathbb{Z}, \ a, b = 1, 2, ..., M + N \} \), with the following quadratic relations

\[
R_{12}(x)L_1(xy)L_2(y) = L_2(y)L_1(xy)R_{12}(x), \quad (5)
\]

where

\[
L(x) = \sum_{a,b}(-1)^{|b|}e_b^a \otimes T_a^b(x), \quad T_a^b(x) = (-1)^{|b|}\delta_a^b + \sum_{0 < k \in \mathbb{Z}_+} x^{-k}T_a^b[k].
\]

The element \( T_a^b[k] \) is assumed to be even if \(|a| + |b| \equiv 0\,(mod\,2)\), and odd otherwise.

This algebra admits co-algebra structures compatible with the associative multiplication defined by equation (5). Explicitly, there exist a co-unit \( \sigma : Y_q(gl(M|N)) \to \mathbb{C} \), \( T_a^b[k] \mapsto \delta_{0,k}\delta_a^b(-1)^{|a|} \), and a co-multiplication \( \hat{\Delta} : Y_q(gl(M|N)) \to Y_q(gl(M|N)) \otimes Y_q(gl(M|N)) \), \( L(x) \mapsto L(x) \otimes L(x) \). An antipode \( \gamma : Y_q(gl(M|N)) \to Y_q(gl(M|N)) \), \( L(x) \mapsto L^{-1}(x) \) can also be introduced, thus turning \( Y_q(gl(M|N)) \) into a \( \mathbb{Z}_2 \) graded Hopf algebra. Note that matrix elements of \( \gamma(L(x)) = 1 \otimes 1 + \sum_{0 < n \in \mathbb{Z}_+}(1 \otimes 1 - L(x))^n \) are in \( Y_q(gl(M|N)) \), and well defined on \( \mathbb{C}[|x^{-1}|] \) under a \( p \)-adic type topology.

For convenience, we write

\[
\gamma(L(x)) = \sum_{a,b}(-1)^{|b|}e_b^a \otimes \bar{T}_a^b(x), \quad \bar{T}_a^b(x) = \sum_{n \in \mathbb{Z}_+} x^{-1}\bar{T}_a^b[n], \quad \bar{T}_a^b[0] = (-1)^{|b|}\delta_a^b,
\]

and define

\[
L^*(x) := (\gamma(L(x)))^+ = \sum_{a,b}(-1)^{|a|+|a|+|b|}e_a^b \otimes \bar{T}_a^b(x).
\]

Then in terms of \( L^*(x) \), the defining relations for \( Y_q(gl(M|N)) \) can be rewritten as

\[
R_{12}^*(x)L_1^*(xy)L_2(y) = L_2(y)L_1^*(xy)R_{12}^*(x), \quad (6)
\]

Pre and post multiplying (3) by \( (R_{12}^*(x))^{-1} \) results in an equation, which has simple poles at \( x = 1 \) and \( x = q^{2(M-N)} \), as can be easily seen by recalling the explicit form of \( (R^*(x))^{-1} \) given in (3). Extracting the residue of this equation at \( x = q^{2(M-N)} \), we arrive at

\[
\bar{P}_{12}[0]P_{12}L_2(y)L_1^*(yq^{2(M-N)})\bar{P}_{12} = L_1^*(yq^{2(M-N)})L_2(y)\bar{P}_{12}[0], \quad (7)
\]
with, in explicit form, reads

\[ \sum_{a,r,s} \pi(q^{h_{2\nu}}) e_a^a \otimes e_s^a \otimes \sum_b (\pi(q^{-h_{2\nu}}))_b^b T_b^r(y) \tilde{T}_{\bar{s}}^t(y q^{2(M-N)})(-1)^{[b]} \zeta \]
\[ = \sum_{b,r,s} e_b^a \pi(q^{-h_{2\nu}}) \otimes e_b^a \otimes \sum_a (\pi(q^{h_{2\nu}}))_a^a \tilde{T}^r_a(y q^{2(M-N)}) T_{\bar{s}}^a(y)(-1)^{[a]} + \eta, \]

with

\[ \zeta \equiv [r] + [s] + [r][s] + [a][a + [r]](mod\ 2), \]
\[ \eta \equiv [r] + [s][r] + [b](mod\ 2). \]

Denote the the right hand side of (7) by $RHS_{(7)}$. We have

\[ P_{12}[0]RHS_{(7)}P_{12}[0] = P_{12}[0] \otimes \text{str}_\pi[\pi(q^{h_{2\nu}}) L^{-1}(y q^{2(M-N)}) L(y)]. \]

Define

\[ C(x) = \text{str}_\pi[\pi(q^{h_{2\nu}}) L^{-1}(x q^{2(M-N)}) L(x)] \in Y_q(gl(M|N)). \]  

We claim that

**Lemma 1** $C(x)$ can be expanded into

\[ C(x) = \sum_{n \in \mathbb{Z}_+} x^{-1} C_n, \]

with $C_n \in Y_q(gl(M|N)), n = 0, 1, ..., \text{belonging to the center of } Y_q(gl(M|N)), \text{namely,}$

\[ [C_n, T_b^a[k]] = 0, \ \forall a, b, k, n. \]  

**Proof:** Let us examine the following equation

\[ L_1^*[xyz] L_2^*[y z] L_3^*[z] (R_{12}^*[x])^{-1}(R_{13}^*[x])^{-1} (R_{23}^*[y])^{-1} \]
\[ = (R_{23}^*[y])^{-1}(R_{13}^*[x])^{-1} (R_{12}^*[x])^{-1} L_3^*[z] L_2^*[y z] L_1^*[x y z], \]  

which, thought appears rather complicated, in fact is a direct consequence of the the Yang - Baxter equation obeyed by $R$ and $R^*$ and the defining relation (8) for the quantum super Yangian $Y_q(gl(M|N))$. Equation (10) has various simple poles. One of them is located at $x = q^{2(M-N)}$, the residue of which yields the following equation

\[ L_1^*[y z q^{2(M-N)}] L_2^*[y z] L_3^*[z] \tilde{P}_{12}[0] P_{12}(R_{13}^*[y q^{2(M-N)}])^{-1} (R_{23}^*[y])^{-1} \tilde{P}_{12} \]
\[ = (R_{23}^*[y])^{-1}(R_{13}^*[y q^{2(M-N)}])^{-1} \tilde{P}_{12}[0] P_{12}[0] L_3^*[z] L_2^*[y z] L_1^*[y z q^{2(M-N)}] \tilde{P}_{12}. \]  

To simplify this equation, we consider the factor

\[ K_{123}(y, z) := \tilde{P}_{12}[0] P_{12}(R_{13}^*[y q^{2(M-N)}])^{-1} (R_{23}^*[y])^{-1} \]
\[ = (R_{23}^*[y])^{-1}(R_{13}^*[y q^{2(M-N)}])^{-1} \tilde{P}_{12}[0] P_{12}, \]

which appears on both sides of (11). An important property of $K_{123}(y, z)$ is that

\[ K_{123}(y, z)(\pi^* \otimes \pi \otimes \pi) \Delta^{(2)}(u) = (\pi^* \otimes \pi \otimes \pi) \Delta^{(2)}(u) K_{123}(y, z), \]
\[ \forall u \in U_q(gl(M|N)), \]  

(12)
where $\Delta^{(2)} = (\Delta \otimes id)\Delta$, and $\Delta'^{(2)} = (\Delta' \otimes id)\Delta'$. Since

$$P[0]P(\pi^* \otimes \pi)\Delta(u) = (\pi^* \otimes \pi)\Delta'(u)P[0]P \quad = \epsilon(u), \quad \forall u \in U_q(gl(M|N)),$$

we readily see that

$$[K_{123}(y, z), 1 \otimes 1 \otimes \pi(u)] = 0, \quad \forall u \in U_q(gl(M|N)),$$

and Schur’s Lemma forces

$$K_{123}(y, z) = K_{12}(y, z) \otimes 1,$$

where $K_{12}(y, z) \in End(V^* \otimes V)$. Using equation (12) again, we can deduce that

$$K_{12}(y, z) = f(y, z)\overline{P}_{12}[0]P_{12},$$

where $f(y, z) \neq 0$ is a complex function of $y$ and $z$. Inserting $K_{123}(y, z)$ back into (11) leads to

$$L_1^*(yq^{2(\nu-N)})L_2(y)\overline{P}_{12}[0]L_3(z) = L_3(z)P_{12}[0]P_{12}L_2(y)L_1^*(yq^{2(\nu-N)})P_{12},$$

which, when sandwiched between two $P_{12}[0]$’s, gives rise to

$$[T^\alpha_b(z), C(y)] = 0.$$

Now a few remarks are in order. Set $x = q^\theta$. In the $q \to 1$ limit, $R(x)/(q - q^{-1})$ reduces to the rational solution $S(\theta) = 1 - P/\theta$ of the Yang - Baxter equation. Using $S(\theta)$, Nazarov\[6] defined the classical super Yangian associated with $gl(M|N)$, and studied its central algebra. Our results presented here reduce to his in the classical limit.

In studying the central algebra of the ordinary Yangian associated with the general linear algebra $gl(M)$, a prominent role is played by the quantum determinant, which is obtained by applying to the $M$ - fold tensor product of the Yangian generating matrix(with appropriate choices for the spectral parameters) the projection operator mapping the $M$ - th rank tensor product of the $gl(M)$ vector module with itself to the totally antisymmetric component, i.e., the trivial module\[8]. However, no generalization of this construction to superalgebras seems to be possible, primarily due to the reason that no tensor product of the $gl(M|N)$ vector module with itself contains a trivial module as an irreducible component. Although in our construction of the $Y_q(gl(M|N))$ central elements we made essential use of a trivial $U_q(gl(M|N))$ module too, but it was manufactured by taking the tensor product of the vector module with its dual.

Since the ordinary general linear Lie algebra $gl(M)$ is recovered from $gl(M|N)$ by setting $N$ to zero, our treatment of $Y_q(gl(M|N))$ covers the ordinary quantum Yangian\[12] as a special case. It is clear that the algebra defined by equation (5) at $N = 0$ is identical to the quantum Yangian of \[12]\, , but the set of invariants $C_n$ are very different from those obtained from quantum determinants.
We now examine the super Yangian structure of $gl(M|N)$, then apply the results obtained to study the center of this quantum supergroup. Let us define

$$l(+) = q\sum_a e_a^{\alpha} \otimes E_a^{\beta}(1) \{1 \otimes 1 + (q - q^{-1}) \sum_{a<b} e_a^b \otimes E_a^b(-1)^{[a]}\},$$
$$l(-) = \{1 \otimes 1 - (q - q^{-1}) \sum_{a<b} e_a^b \otimes E_a^b(-1)^{[b]}\}q^{-\sum_a e_a^{\alpha} \otimes E_a^{\alpha}(-1)^{[a]}},$$
$$l(x) = \{l(+) - x^{-1}l(-)\} / (1 - x^{-1}). \quad (14)$$

Let $T : \text{End}(V) \otimes U_q(gl(M|N)) \to U_q(gl(M|N)) \otimes \text{End}(V)$ be the twisting map defined by $T(e \otimes u) = (-1)^{[u]} u \otimes e$. Then $(1 - x^{-1})T(l(x))$ coincides with the universal $L$ operator of $[4]$. From the results of that paper, we can deduce that $l(x)$ satisfies the following relations

$$l(x)(\pi \otimes id)\Delta(u) = (\pi \otimes id)\Delta'(u)l(x), \quad \forall u \in U_q(gl(M|N)) \quad (15)$$
$$R_{12}(x)l_1(xy)l_2(y) = l_2(y)l_1(xy)R_{12}(x). \quad (16)$$

Note that equation (13) is nothing else but the defining relation of the quantum super Yangian. Thus the universal $L$ operator yields a $\mathbb{Z}_2$ graded algebra homomorphism $\Lambda : Y_q(gl(M|N)) \to U_q(gl(M|N))$,

$$\Lambda(L(x)) = l(x).$$

This map is surjective, as the $E_a^\alpha$’s generate $U_q(gl(M|N))$ (in fact they form a quantum analogue of the Cartan – Weyl basis). Therefore, the images of the $Y_q(gl(M|N))$ central elements $C_n$ lie in the center of $U_q(gl(M|N))$. Set

$$x = q^\theta,$$
$$\Gamma = \frac{(l(-))^{-1}l(+) - 1}{q - q^{-1}},$$

we have

$$\Lambda(C(q^\theta)) = str_{\pi} \left[ \pi(q^{h_2\theta}) \left( \Gamma - \frac{q^{-\theta - 2(M-N)} - 1}{q - q^{-1}} \right)^{-1} \left( \Gamma - \frac{q^{-\theta} - 1}{q - q^{-1}} \right) \right].$$

From this equation we clearly see that $\Lambda(C_n)$’s can be expressed as linear combinations of the following $U_q(gl(M|N))$ Casimir operators

$$I_n = str_{\pi} \left[ \pi(q^{h_2\theta}) \Gamma^n \right], \quad n = 0, 1, 2, \ldots,$$

and vice versa. Note that the $I_n$’s are precisely the $U_q(gl(M|N))$ Casimir operators constructed in $[4]$ using the $q$ - supertrace approach, which bears no similarity to the Yangian construction presented here. As pointed out in $[4]$, this set of invariants reduces to the $gl(M|N)$ Casimirs obtained by Jarvis and Green, and Scheunert $[13]$, which were known to generate the center of $gl(M|N)$. Therefore, there is a good chance that the $I_n$’s generate the entire center of $U_q(gl(M|N))$.

In the special case $N = 0$, $Y_q(gl(M|N))$ reduces to the quantum Yangian associated with the ordinary general linear algebra $gl(M)$, and $I_n$’s become the $U_q(gl(M))$ Casimir operators of $[14]$. In this case, our arguments can be sharpened, leading to
Lemma 2 When $N = 0$, the $I_k$, $k = 0, 1, \ldots$, generate the entire center of $U_q(gl(M))$ at generic $q$.

A detailed proof of the Lemma is out of the scope of this letter, but we can sketch the main arguments involved. Set $q = \exp(t)$. The quantum group $U_q(gl(M))$ is an associative algebra over the polynomial ring $C[[t]]$ completed with respect to the $t$-adic topology; it is also a deformation of the universal enveloping algebra $U(gl(M))$ of $gl(M)$ in the sense of Gerstenhaber. Since the second Hochschild cohomology group of $U(gl(M))$ with coefficients in itself is trivial, it follows that all associative algebraic deformations of $U(gl(M))$ are trivial. This in particular guarantees the existence of an algebra isomorphism $F_t : U_q(gl(M)) \rightarrow U(gl(M))$ with $F_t = id + tf^{(1)} + t^2f^{(2)} + \ldots$. Therefore, as shown by Drinfeld, the centers of these algebras are canonically isomorphic. We have pointed out that $I_n(mod\ t)$ generate the center of $U(gl(M))$, thus so do $F_t(I_n)$, $n = 0, 1, 2, \ldots$. It then follows that $\{I_n \mid n = 0, 1, 2, \ldots\}$ is a complete set of generators for the central algebra of $U_q(gl(M))$. Following the same line of reasoning we can show that the quantum Casimir operators constructed in [10] generate the centers for all quantum groups. A detailed proof of this statement will be reported in a separate publication.
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