Groups with few $p'$-character degrees in the principal block

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Abstract
Let $p \geq 5$ be a prime and let $G$ be a finite group. We prove that $G$ is $p$-solvable of $p$-length at most 2 if there are at most two distinct $p'$-character degrees in the principal $p$-block of $G$. This generalizes a theorem of Isaacs–Smith as well as a recent result of three of the present authors.

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1 Introduction
Let $G$ be a finite group. If all non-linear irreducible characters of $G$ have degree divisible by a prime $p$, then $G$ has a normal $p$-complement by a theorem of Thompson [Tho70, Theorem 1] (see also [Isa06, Corollary 12.2]). Moreover, Berkovich [Ber95, Proposition 9 and the subsequent remark] has shown that $G$ is solvable in this situation. This result was extended in Kazarin–Berkovich [KB99] to the case where $G$ has at most one non-linear character of $p'$-degree. In a recent paper [GRS], three of the present authors proved more generally that $G$ is solvable of $p$-length at most 2 whenever $p \geq 5$ and $|\{\chi(1) : \chi \in \text{Irr}_{p'}(G)\}| \leq 2$ where $\text{Irr}_{p'}(G)$ is the set of irreducible characters of $G$ of $p'$-degree. This has solved Problem 1 in [KB99, p. 588] and Problem 5.3 in [Nav16].

In the present paper we generalize our theorem to blocks. This is motivated by a result of Isaacs and Smith [IS76, Corollary 3] who showed that $G$ has a normal $p$-complement if and only if all non-linear characters in the principal $p$-block of $G$ have degree divisible by $p$. The following is our main theorem.

Theorem A. Let $B_0$ be the principal block of a finite group $G$ with respect to a prime $p \geq 5$. Suppose that $|\{\chi(1) : \chi \in \text{Irr}_{p'}(B_0)\}| \leq 2$. Then $G/O_p(G)$ is solvable and $O_{p'}p'(G) = 1$. In particular, $G$ is $p$-solvable.

As usual we define $O_{p'}(G) := O_p(O_p(G))$ and so on. It is easy to construct groups of $p$-length 2 satisfying the hypothesis of Theorem A (e.g. $G = (C_5 \rtimes C_{11}) \rtimes C_5$ with $p = 5$). In contrast to the main theorem of [GRS] we cannot conclude further that $G$ is solvable since every $p'$-group satisfies the assumption of Theorem A. Furthermore, the examples given in [GRS] show that Theorem A does not extend to $p \in \{2, 3\}$. We also like to mention a conjecture by Malle and Navarro [MNT1], which generalizes the result of Isaacs and Smith to arbitrary

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blocks. More precisely, they conjectured that a \( p \)-block \( B \) of \( G \) is nilpotent if and only if all height 0 characters in \( B \) have the same degree. We do not know if our main result admits a version for arbitrary blocks.

The proof of Theorem A relies on the classification of finite simple groups. In the next section we reduce Theorem A to a statement about simple groups (Proposition 2.1 below), which is proved case-by-case in the following two sections. We care to remark that in the case of alternating groups, Proposition 2.1 is deduced as a consequence of a more general statement giving a lower bound for the number of (extendable) \( p' \)-character degrees in any block of maximal defect. This is Proposition 3.5 below, which we believe is of independent interest.

2 Reduction to simple groups

The following proposition about simple groups will be proven in the next two sections.

Proposition 2.1. Let \( S \) be a finite non-abelian simple group of order divisible by a prime \( p \geq 5 \).

(i) If \( S \neq P\Omega^+_2(q) \), then there exist \( \alpha, \beta \in \text{Irr}(S) \) with the following properties:

- \( \alpha \neq 1 = \beta \),
- \( \alpha(1) \) and \( \beta(1) \) are not divisible by \( p \),
- for every \( S \leq T \leq \text{Aut}(S) \), \( \alpha \) extends to a character in the principal block of \( T \),
- \( \beta \) lies in the principal block of \( S \) and is \( P \)-invariant for some Sylow \( p \)-subgroup \( P \) of \( \text{Aut}(S) \),
- \( \beta(1) \nmid \alpha(1) \).

(ii) If \( S = P\Omega^+_2(q) \), then there exist \( \alpha, \beta \in \text{Irr}(S) \) with the following properties:

- \( \alpha \neq 1 \neq \beta \),
- \( \alpha(1) \) and \( \beta(1) \) are not divisible by \( p \),
- \( \alpha(1) > 2\beta(1) \),
- for every \( S \leq T \leq \text{Aut}(S) \) there exist \( \hat{\alpha}, \hat{\beta} \in \text{Irr}(\text{Aut}(T)) \) in the principal block such that \( \hat{\alpha}_S \in \{ \alpha, 2\alpha \} \) and \( \hat{\beta}_S \in \{ \beta, 2\beta \} \).

We make use of the following results.

Lemma 2.2 (Murai [Mur94 Lemma 4.3]). Let \( N \trianglelefteq G \) be finite groups with principal \( p \)-blocks \( B_N \) and \( B_G \) respectively. Suppose that \( \psi \in \text{Irr}_{p'}(B_N) \) is invariant under a Sylow \( p \)-subgroup of \( G \). Then there exists a character \( \chi \in \text{Irr}_{p'}(B_G) \) lying over \( \psi \).

Lemma 2.3. Let \( \chi, \psi \in \text{Irr}(B_0) \) where \( B_0 \) is the principal \( p \)-block of \( G \). Suppose that \( p \nmid \chi(1) \) and \( \chi \psi \in \text{Irr}(G) \). Then \( \chi \psi \in \text{Irr}(B_0) \).

Proof. Clearly, \( \overline{\psi} \in \text{Irr}(B_0) \). Hence by [Nav98 Corollary 3.25], we have

\[ [\chi \psi, 1]^0 = [\chi, \overline{\psi}]^0 \neq 0. \]

The claim follows from [Nav98 Theorem 3.19].

Now we are in a position to reduce Theorem A to simple groups.

Theorem 2.4. If Proposition 2.1 holds, then Theorem A holds.
Proof. Let $p$, $G$ and $B_0$ be as in Theorem A. Suppose first that $G$ is $p$-solvable. Let $N := \Omega_p(G)$. Then, by [Nav98, Theorem 10.20], $\text{Irr}(B_0) = \text{Irr}(G/N)$. It follows from [GRS, Theorem A] that $G/N$ is solvable and $O^{p^\prime p'}(G/N) = 1$. In particular, $O^{p^\prime p'}(G)/N$ is a $p'$-group. Since $N$ is a $p'$-group, this implies $O^{p^\prime p'}(G) = 1$.

Thus, it suffices to show that $G$ is $p$-solvable. Let $N$ be a minimal normal subgroup of $G$. Since the principal block of $G/N$ lies in $B_0$, we may assume that $G/N$ is $p$-solvable by induction on $|G|$. If $N$ is a $p'$-group or a $p'$-group, then we are done. Therefore, by way of contradiction, we assume that

$$N = S_1 \times \ldots \times S_t$$

with isomorphic non-abelian simple groups $S := S_1 \cong \ldots \cong S_t$ of order divisible by $p$. Since $N$ is the unique minimal normal subgroup, $C_G(N) = 1$. Moreover, $G$ permutes $S_1, \ldots, S_t$ transitively by conjugation.

Case 1: $S \neq P\Omega^+_2(q)$.

Let $\alpha, \beta \in \text{Irr}(S)$ as in [Proposition 2.1]. We may regard $\alpha$ as a character of $SC_G(S)$, since $SC_G(S)/C_G(S) \cong S/\mathbb{Z}(S) = S$. As such it extends to a character $\hat{\alpha}$ in the principal block of $N_C(S)$, because $N_C(S)/C_G(S) \leq \text{Aut}(S)$. Let $M := N_{C(S)} \cap \ldots \cap N_{C(S)} \leq G$. Since the principal block of $N_C(S)$ covers the principal block $B_M$ of $M$, the restriction $\hat{\alpha}_M$ lies in $B_M$. Now by [Nav18, Corollary 10.5], the tensor product $\psi := \hat{\alpha}^{S_G}$ is an irreducible character of $G$ with $p'$-degree $\psi(1) = \alpha(1)^t$. Let $x_1, \ldots, x_t \in G$ be representatives for the right cosets of $N_C(S)$ in $G$ such that $S_1^x = S_i$. Then for $g \in M$ we obtain

$$\psi(g) = \prod_{i=1}^t \hat{\alpha}^{x_i}(g)$$

from [Nav18, Lemma 10.4]. In particular, $\psi_N = \alpha^{x_1} \times \ldots \times \alpha^{x_t} \in \text{Irr}(N)$ and therefore $\psi_M \in \text{Irr}(M)$ as well. Since $\hat{\alpha}_M$ lies in $B_M$, so does $\hat{\alpha}_M^g$. Hence, by Lemma 2.3 also $\psi_M = \hat{\alpha}_M^g \times \hat{\alpha}_M^g$ lies in $B_M$.

Let $Q$ be a Sylow $p$-subgroup of $M$. Then $Q \cap S_i$ is a Sylow $p$-subgroup of $S_i$. It follows that $C_G(Q) \subseteq C_G(Q \cap S_i) \subseteq N_{C(S)}$ for $i = 1, \ldots, t$ and therefore $C_G(Q) \subseteq M$. Hence, the Brauer correspondent $B_M^Q$ is defined (see [Nav98, Theorem 4.14]) and equals $B_0$ by Brauer’s third main theorem. Every block $B$ of $G$ covering $B_M$ has a defect group containing $Q$ by [Nav98, Theorem 9.26]. Hence by [Nav18, Lemma 9.20], $B$ is regular with respect to $N$ and therefore $B = B_0$ by [Nav98, Theorem 9.19]. Thus, $B_0$ is the only block of $G$ covering $B_M$. This implies $\psi \in \text{Irr}(B_0)$. Since the trivial character in $B_0$ has degree $1$, $d := \psi(1) = \alpha(1)^t$ is the unique non-trivial $p'$-character degree in $B_0$ by hypothesis.

Now we work with $\beta$. Let $P$ be a Sylow $p$-subgroup of $G$ such that $\beta$ is invariant under $N_P(S)$. Without loss of generality, let $\{S_1, \ldots, S_t\}$ be a $P$-orbit. Let $y_i \in P$ such that $S_i^y = S_i$ for $i = 1, \ldots, t$. Then $\beta^y := \beta^{y_i}$ lies in the principal block of $S_i$. By Lemma 2.3, $\beta^1 \times \ldots \times \beta^t$ lies in the principal block of $N$. Moreover, if $\beta^y \in \text{Irr}(S_j)$ for some $y \in P$, then $y_jy_j^y \in N_P(S)$. Since $\beta$ is $N_P(S)$-invariant, it follows that $\beta^y = \beta^y \beta^y \beta^y = \beta^{y_i} = \beta^y$. This shows that $\{\beta_1, \ldots, \beta_t\}$ is $P$-orbit and $\beta^1 \times \ldots \times \beta^t$ is $P$-invariant. If $r < t$, then we consider $\beta_{r+1} := \beta^{x_{r+1}} \in \text{Irr}(S_{r+1})$. By Sylow’s theorem, we can assume after conjugation inside $N_{C(S_{r+1})}$ that $\beta_{r+1}$ is $N_P(S_{r+1})$-invariant. Now we can form the $P$-orbit of $\beta_{r+1}$ to obtain another $P$-invariant character $\beta_{r+1} \times \ldots \times \beta_r \in \text{Irr}(N)$ in the principal block of $N$. We repeat this with every $P$-orbit and eventually get a $PN$-invariant character $\tau := \beta_1 \times \ldots \times \beta_t \in \text{Irr}(N)$ in the principal block of $N$. Since $o(\tau) = 1$ and $\text{gcd}(\tau(1), |P/N|) = 1$, $\tau$ extends to $PN$ (see [Isa06, Corollary 8.16]). By Lemma 2.2, there exists some $\chi \in \text{Irr}_p(B_0)$ such that $\tau$ is a constituent of $\chi_N$. Since $1 \neq \beta(1)^t = \tau(1) | \chi(1)$, it follows that $\chi(1) = d = \psi(1)$. But then $\beta(1)^t \psi(1) = \alpha(1)^t$ and $\beta(1) | \alpha(1)$, a contradiction to the choice of $\alpha$ and $\beta$.

Case 2: $S = P\Omega^+_2(q)$.

Let $\alpha, \beta \in \text{Irr}(S)$ and $\hat{\alpha}, \hat{\beta} \in \text{Irr}(N_C(S))$ as in [Proposition 2.1]. Since the principal block of $N_C(S)$ covers $B_M$, $\hat{\alpha}^g$ is the sum of at most two irreducible characters in $B_M$. If $\alpha_1 \in \text{Irr}(B_M)$ is one of those summands, then $\alpha_1^{x_1} \ldots \alpha_1^{x_t}$ restricts to $\alpha^{x_1} \times \ldots \times \alpha^{x_t} \in \text{Irr}(N)$ [1]. Hence, by Lemma 2.3, $\alpha_1^{x_1} \ldots \alpha_1^{x_t}$ lies in $B_M$. As in Case 1 we see that $(\hat{\alpha}^{S_G})_M$ is a sum of irreducible characters in $B_M$. Moreover, $(\hat{\alpha}^{S_G})_N = d(\alpha^{x_1} \times \ldots \times \alpha^{x_t})$ where

1Miquel Martínez pointed out that this is not always the case. A workaround (due to G. Navarro) will appear in a forthcoming paper of Martínez.
\( d \in \{1, 2^i\} \). Since \( B_0 \) is the only block of \( G \) covering \( B_M \), all irreducible constituents of \( \hat{\alpha}^{\otimes G} \) lie in \( B_0 \). We may choose such a constituent \( \chi \) of \( p' \)-degree. Then \( \chi N = e(\alpha^{x_1} \times \ldots \times \alpha^{x_k}) \) for some integer \( e \leq d \leq 2^i \). Similarly, we choose a constituent \( \psi \) of \( \beta^{\otimes G} \) with \( p' \)-degree. Then by Proposition 2.1 we derive the contradiction
\[
\alpha(1)^t > 2^t \beta(1)^t \geq \psi(1) = \chi(1) \geq \alpha(1)^t.
\]

\[ \square \]

## 3 Alternating groups

This section is devoted to proving Proposition 2.1 for the alternating groups \( S = A_n \) where \( n \geq 5 \). It is well-known that \( \text{Aut}(S) \cong S_n \) is the symmetric group unless \( n = 6 \).

Given \( n \in \mathbb{N} \) we let \( \mathcal{P}(n) \) be the set of partitions of \( n \). Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}(n) \). Adopting the notation of [Ols93, Chapter 1] we let \( f(\lambda) = k \) denote the number of parts of \( \lambda \), and \( \Upsilon(\lambda) \) be the Young diagram of \( \lambda \).

Given a node \( (i, j) \in \Upsilon(\lambda) \) we denote by \( h_{ij}(\lambda) \) the length of the hook corresponding to \( (i, j) \). If \( q \in \mathbb{N} \) then the \( q \)-core \( C_q(\lambda) \) of \( \lambda \) is the partition obtained from \( \lambda \) by successively removing all hooks of length \( q \) (usually called \( q \)-hooks). We denote by \( \mathcal{H}^q(\lambda) \) the subset of nodes of \( \Upsilon(\lambda) \) having associated hook-length divisible by \( q \). A partition \( \gamma \) is called a \( q \)-core if \( \mathcal{H}^q(\lambda) = \emptyset \).

The set \( \text{Irr}(S_n) \) is naturally in bijection with \( \mathcal{P}(n) \). Given \( \lambda \in \mathcal{P}(n) \) we let \( \chi^\lambda \) be the corresponding irreducible character of \( S_n \). Let \( p \) be a prime and \( \lambda, \mu \in \mathcal{P}(n) \). By [JK81, 6.1.21] we know that \( \chi^\lambda \) and \( \chi^\mu \) lie in the same \( p \)-block of \( S_n \) if and only if \( C_p(\lambda) = C_p(\mu) \). If \( \gamma \) is a \( p \)-core partition then we denote by \( B(\mathfrak{S}_n, \gamma) \) the corresponding \( p \)-block of \( S_n \). We use the notation \( \lambda \vdash_p n \) to say that \( \chi^\lambda \) has degree coprime to \( p \).

The following result follows from [Mac71] and it will be extremely useful for our purposes.

**Lemma 3.1.** Let \( p \) be a prime and let \( n \) be a natural number with \( p \)-adic expansion \( n = \sum_{j=0}^k a_j p^j \). Let \( \lambda \) be a partition of \( n \). Then \( \lambda \vdash_p n \) if and only if \( |\mathcal{H}^p(\lambda)| = a_k \) and \( C_p(\lambda) \vdash_p n - a_k p^k \).

A straightforward consequence of Lemma 3.1 is that \( \text{Irr}(B(\mathfrak{S}_n, \gamma)) \neq \emptyset \) if and only if \( |\gamma| < p \).

For \( \lambda \in \mathcal{P}(n) \), we denote by \( \lambda' \) its conjugate partition. From [JK81, 2.5.7] we know that \( \psi^\lambda := (\chi^\lambda)_{\mathfrak{S}_n} \) is irreducible if and only if \( \lambda \neq \lambda' \). In this case \( \chi^\lambda \) and \( \chi^{\lambda'} \) are all the extensions of \( \psi^\lambda \) to \( S_n \). Let \( \lambda, \mu \) be non-self-conjugate partitions of \( n \). Then \( \psi^\lambda \) and \( \psi^\mu \) lie in the same \( p \)-block of \( S_n \) if and only if \( C_p(\lambda) \in \{ C_p(\mu), C_p(\mu') \} \).

It follows that also \( p \)-blocks of \( S_n \) can be labeled by \( p \)-core partitions, by keeping in mind that conjugated \( p \)-cores label the same \( p \)-block. We denote by \( B(n; \gamma) \) the \( p \)-block of \( \mathfrak{S}_n \) labeled by \( \gamma \).

In order to show that Proposition 2.1 holds for alternating groups, we introduce the following conventions.

**Notation 1.** Let \( B \) be a \( p \)-block of \( \mathfrak{S}_n \). We let \( \text{cd}_{\mathfrak{S}_n}(B) \) be the set of degrees of irreducible characters in \( B \) of degree coprime to \( p \) that extend to an irreducible character of \( S_n \). Moreover, when \( S \) is a subset of \( \mathcal{P}(n) \) we let \( \text{cd}(S) = \{ \chi^\lambda(1) \mid \lambda \in S \} \).

Observe that if \( B \) is the principal \( p \)-block of \( \mathfrak{S}_n \) and \( \psi^\lambda \) lies in \( B \) and extends to \( S_n \), then one of the two extensions of \( \psi^\lambda \) lies in the principal \( p \)-block of \( S_n \). This is explained in [Ols90]. Even if in this article we are mainly interested in studying the principal block, in Proposition 3.5 below we are going to compute an explicit lower bound for \( |\text{cd}_{\mathfrak{S}_n}(B(n; \gamma))| \), for any \( p \)-core \( \gamma \) such that \( |\gamma| < p \).

Given \( \gamma = (\gamma_1, \ldots, \gamma_t) \vdash n \) and natural numbers \( x \) and \( y \), we denote by \( \gamma \ast (x, y) \) the partition of \( n + x + y \) defined by
\[
\gamma \ast (x, y) = (\gamma_1 + x, \gamma_2, \ldots, \gamma_t, 1^y).
\]

We start by proving a technical lemma that will be useful later in this section.

**Lemma 3.2.** Let \( p \) be a prime, let \( m, n, w \in \mathbb{N} \) be such that \( m < p \) and \( n = m + pw \). Let \( \gamma \vdash m \) and let \( a \in \mathbb{N} \) be such that \( |\gamma| + 1 \leq a \leq w \). Setting \( \lambda = \gamma \ast (ap, (w - a)p) \) and \( \mu = \gamma \ast ((a - 1)p, (w - a + 1)p) \), we have that \( \chi^\lambda(1) < \chi^\mu(1) \).
Proof. For \( \nu \vdash n \) we let \( \pi(\nu) := \prod h_{ij}(\nu) \) be the product of the hook-lengths in \( \nu \). From the hook length formula \([\text{JKST}]\, 2.3.21\) it follows that \( \chi^\nu(1) \pi(\nu) = n! \). We let \( h^i = h_{ii}(\gamma) \) and \( h_{ij} = h_{ij}(\gamma) \) for all \( i \in \{1, \ldots, \gamma_1\} \) and \( j \in \{1, \ldots, \ell(\gamma)\} \). It follows that

\[
\pi(\lambda) = (ap)! \cdot ((w-a)p)! \prod_{i=2}^{\gamma_1} (h^i + ap) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w-a)p) \cdot \hat{\gamma} \cdot (h_{11}(\gamma) + pw),
\]

where \( \hat{\gamma} \) is the product of the hook lengths \( h_{ij}(\gamma) \) for all \( i, j \geq 2 \). Similarly,

\[
\pi(\mu) = ((a-1)p)! \cdot ((w-a+1)p)! \prod_{i=2}^{\gamma_1} (h^i + (a-1)p) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w-a+1)p) \cdot \hat{\gamma} \cdot (h_{11}(\gamma) + pw).
\]

It follows that \( \pi(\lambda)/\pi(\mu) = A \cdot B \cdot C \), where

\[
A = \prod_{i=1}^p \frac{(a-1)p+i}{(w-a)p+i}, \quad B = \prod_{i=2}^{\gamma_1} \frac{h^i + ap}{h^i + (a-1)p}, \quad \text{and} \quad C = \prod_{i=2}^{\ell(\gamma)} \frac{h_i + (w-a)p}{h_i + (w-a+1)p}.
\]

We remark that we always regard empty products as equal to 1. We observe that \( B \geq 1 \). Since \( a-1 \geq w-a+1 \) by hypothesis, it is clear that \( A > 1 \). Hence, if \( \ell(\gamma) > 1 \) then \( C = 1 \) and clearly \( A \cdot B \cdot C > 1 \). Suppose that \( \ell(\gamma) \geq 2 \). Then observe that \( p > |\gamma| > h_2 > h_3 > \cdots > h_{\ell(\gamma)} \geq 1 \). Hence for all \( i \in \{2, \ldots, \ell(\gamma)\} \) we have that \( \frac{(a-1)p+h_i}{(w-a)p+h_i} \) is one of the factors appearing in \( A \). Moreover

\[
\frac{(a-1)p+h_i}{(w-a)p+h_i} \geq \frac{h_i + (w-a)p}{h_i + (w-a+1)p} \geq 1,
\]

since \( a-1 \geq w-a+1 \). We conclude that \( A \cdot B \cdot C \geq A \cdot C > 1 \) and therefore that \( \chi^\lambda(1) < \chi^\mu(1) \). \( \square \)

**Definition 3.3.** Let \( p \) be a prime and \( n = wp + m \), for some \( m < p \). Let \( \gamma \) be a \( p \)-core partition of \( m \). We let \( H(n; \gamma) \) be the subset of \( \mathcal{P}(n) \) defined by

\[
H(n; \gamma) = \{ \lambda \vdash wp + m \mid C_p(\lambda) = \gamma \}.
\]

We also set \( \Omega(n; \gamma) = \{ \lambda \vdash H(n; \gamma) \mid \lambda_1 > (\lambda')_1 \} \).

**Lemma 3.4.** Let \( n = \sum_{i=0}^k a_i p^i \) be the \( p \)-adic expansion of \( n \), with \( a_k \neq 0 \). If \( \gamma \vdash a_0 \), then

\[
|\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)| \geq \left(\frac{a_k + 1}{2}\right) \cdot \prod_{i=1}^{k-1} (a_i + 1).
\]

**Proof.** Let \( \lambda = \gamma \ast (x, n-a_0-x) \), for some \( 0 \leq x \leq n-a_0 \). Let \( x = \sum_{i=0}^k b_ip^i \) be the \( p \)-adic expansion of \( x \). By definition of \( H(n; \gamma) \) we have that \( \lambda \in H(n; \gamma) \) if and only if \( \lambda \vdash wp + m \) and \( C_p(\lambda) = \gamma \). In turn, this is equivalent to have that \( p \) divides \( x \) (and \( n-a_0-x \)) so that \( C_p(\lambda) = \gamma \) and by [Lemma 3.3] to have that \( b_0 = 0 \) and \( 0 \leq b_i \leq a_i \) for all \( i \geq 1 \). It follows that \( |H(n; \gamma)| = \prod_{i=1}^{k} (a_i + 1) \). Moreover, if \( b_k \geq \lceil a_k/2 \rceil + 1 \), then certainly \( \lambda_1 > (\lambda')_1 \) and therefore \( \lambda \in \Omega(n; \gamma) \). It follows that

\[
|\Omega(n; \gamma)| \geq \left(\frac{a_k + 1}{2}\right) \cdot \prod_{i=1}^{k-1} (a_i + 1).
\]

We conclude by observing that [Lemma 3.2] implies that given \( \lambda, \mu \in \Omega(n; \gamma) \) we have that \( \chi^\lambda(1) \neq \chi^\mu(1) \) and hence that \( |\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)| \). \( \square \)

Given \( \lambda \in \Omega(n; \gamma) \) we have that \( \chi^\lambda \) lies in \( B(\mathfrak{S}_n; \gamma) \) and that \( (\chi^\lambda)_{\mathfrak{A}_n} \) is irreducible and lies in \( B(n; \gamma) \). As explained in Notation 1 above, \( \text{cd}_{\gamma}^{\mathfrak{A}_n}(B(n; \gamma)) \) denotes the set of degrees of irreducible characters of \( B(n; \gamma) \) of degree coprime to \( p \) that extend to \( B(\mathfrak{S}_n; \gamma) \).

In the following proposition we are able to establish a lower bound for the number of extendable \( p' \)-character degrees lying in any given \( p \)-block of \( \mathfrak{A}_n \). We believe this statement of independent interest from the topic of this article.
Proposition 3.5. Let \( n = \sum_{i=0}^{k} a_i p^i \) be the \( p \)-adic expansion of \( n \), with \( a_k \neq 0 \). Let \( \gamma \vdash a_0 \), then

\[
|cd^\text{ext}_p(B(n;\gamma))| \geq \left( \frac{q_k+1}{2} \right) \cdot \prod_{i=1}^{k-1} (a_i + 1).
\]

Proof. By definition, for every partition \( \lambda \in \Omega(n;\gamma) \) we have that \((\chi^\lambda)_\gamma\) is a \( p' \)-degree character that lies in \( B(n;\gamma) \) and extends to \( \chi^\lambda \) in \( B(\mathfrak{S}_n;\gamma) \). The statement now follows from Lemma 3.4.

Proposition 3.6. Let \( n \geq 5 \) be a natural number and \( p > 3 \) be a prime. Then Proposition 2.1 holds for \( \mathfrak{A}_n \). In particular if \( n \geq 7 \) then \( |cd^\text{ext}_p(B_0(\mathfrak{A}_n))| \geq 3 \).

Proof. Direct verification proves that Proposition 2.1 holds for \( \mathfrak{A}_5 \) and \( \mathfrak{A}_6 \). Suppose that \( n \geq 7 \) and that \( n = a_0 + \sum_{i=1}^{k} a_i p^i \) is the \( p \)-adic expansion of \( n \), with \( a_i \neq 0 \) for all \( i \geq 1 \) and with \( n_1 < n_2 < \cdots < n_k \). Since \( p \) is odd, for \( P \in \text{Syl}_p(\mathfrak{S}_n) \) we have that \( P \leq \mathfrak{A}_n \) and hence that all irreducible characters in \( B_0(\mathfrak{A}_n) \) are \( P \)-invariant. Thus we just need to show that \( |cd^\text{ext}_p(B_0(\mathfrak{A}_n))| \geq 3 \). From Proposition 3.5 we deduce that \( |cd^\text{ext}_p(B_0(\mathfrak{A}_n))| \geq 3 \), whenever \( k \geq 3 \). Suppose that \( k \leq 2 \). If \( a_0 \leq 1 \) then \( \text{Irr}_p(B_0(\mathfrak{A}_n)) = \text{Irr}_p(\mathfrak{A}_n) \) and the statement follows from [GRS] Proposition 3.5. Hence we can assume that \( a_0 \geq 2 \) and consider \( \lambda, \mu \in \mathcal{P}(n) \) to be defined as follows.

\[
\lambda = (a_0, 1^{n-a_0}), \quad \text{and} \quad \mu = (a_0, 2, 1^{n-a_0-2}).
\]

It is clear that both \((\chi^\lambda)_\mathfrak{A}, \mu \) lie in the principal \( p \)-block of \( \mathfrak{A}_n \) and extend to the principal \( p \)-block of \( \mathfrak{S}_n \), to \( \chi^\lambda \) and \( \chi^\mu \) respectively. Moreover \( \lambda \) and \( \mu \) label characters of degree coprime to \( p \) by Lemma 3.1 using the hook-length formula we verify that \( 1 = \chi(n)(1) < \chi^\lambda(1) < \chi^\mu(1) \). The proof is complete.

4 Sporadic groups and groups of Lie type

Proposition 4.1. Proposition 2.1 holds for all sporadic simple groups \( S \) and the Tits group \( ^2F_4(2)' \).

Proof. Recall that \( |\text{Aut}(S) : S| \leq 2 \). Hence, we may take a \( p' \)-character \( \alpha \) in the principal block of \( \text{Aut}(S) \) such that \( \alpha \vdash \alpha_\mathfrak{G} \neq 1 \) is irreducible. For \( \beta \) we can choose any non-trivial \( p' \)-character in the principal block of \( S \). Now it can be checked with GAP [GAP] that there are choices such that \( \beta(1) \neq \alpha(1) \).

Now we consider simple groups \( S \) of Lie type, by which we mean groups of the form \( G/\mathbb{Z}(G) \), where \( G = G^F \) is the set of fixed points of a simple simply connected algebraic group under a Steinberg morphism \( F \). In the case where \( \mathbb{Z}(G) \) is trivial, we define \( G = G \), and otherwise we let \( G = G \) be a regular embedding, as in [CEI] Section 15], so that \( \mathbb{Z}(G) \) is connected, \( [\tilde{G}, G] = [G, G] \), and \( G \) is normal in \( \tilde{G} := G^F \). We write \( \tilde{S} \) for the group \( \tilde{G}/\mathbb{Z}(\tilde{G}) \), so \( \text{Aut}(\tilde{S}) \) may be viewed as generated by \( \tilde{S} \) and graph and field automorphisms.

Recall that the set \( \text{Irr}(\tilde{G}) \) can be partitioned into so-called Lusztig series \( \mathcal{E}(\tilde{G}, s) \), where \( s \) is a semisimple element of the dual group \( \tilde{G}^* \), up to conjugacy. Each series \( \mathcal{E}(\tilde{G}, s) \) has a unique character of degree \( \tilde{G}^* : C_{\tilde{G}, s} \), where \( F_q \) is the field over which \( G \) is defined, called a semisimple character. Further, the characters in the series \( \mathcal{E}(\tilde{G}, 1) \) are called unipotent characters, and for a prime \( p \), any \( p \)-block containing a unipotent character is called a unipotent block.

When \( G \) is type \( A_{n-1} \) (that is, in the case of lineal and unitary groups), we will use the notation \( PSL_n(q) \) to denote \( PSL_n(q) \) for \( \epsilon = 1 \) and \( PSU_n(q) \) for \( \epsilon = -1 \), and similar for \( GL_n(q) \) and \( SL_n(q) \). Similarly, \( A_{n-1}(q) \) will denote the untwisted case \( A_{n-1}(q) \) when \( \epsilon = 1 \) and the twisted case \( 2A_{n-1}(q) \) when \( \epsilon = -1 \). We also remark that the group \( P^{2A_{n-1}}(q) \) corresponds to \( D_n(q) \) and \( P^{2A_{n-1}}(q) \) corresponds to \( 2D_n(q) \).

The following result settles Proposition 2.1 for most simple groups in defining characteristic.
Proposition 4.2. Let $S$ be a simple group of Lie type defined over $F_q$, where $q$ is a power of $p > 3$ not in the following list: $\PSL_2(q)$, $\PSL_3(q)$, or $\PSp_4(q)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \Irr(\hat{B}_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:

- If $S \not\cong \Omega^+_q(3)$, then for every $S \leq T \leq \Aut(S)$, each of $\chi_1$ and $\chi_2$ extend to a character in the principal $p$-block of $T$.

- If $S = \Omega^+_q(3)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \Aut(S)$, for $i = 1, 2$, there exist $\tilde{\chi}_i$ in the principal $p$-block of $T$ such that $\tilde{\chi}_i|_S \in \{\chi_1, 2\chi_1\}$.

Proof. In the proof of [GRS] Proposition 4.3, it is shown that there exist two characters $\chi_1, \chi_2 \in \Irr(\hat{S})$ that restrict irreducibly to $S$, extend to characters of $\Aut(S)$, have different degrees, and are obtained from characters of $\hat{G}$ trivial on $Z(\hat{G})$. Now, since $\Irr(\hat{G}) = \Irr(\hat{B}_0(\hat{G}))$ (using, for example, [CE04 6.18, 6.14, 6.15]) and using [CE04] Lemma 17.2, we see that in fact these characters are members of the principal block of $\hat{S}$, and their restrictions are members of the principal block of $S$.

Now, let $S \leq T \leq \Aut(S)$. Then for $i = 1, 2$, $\chi_i|_{T \cap \hat{S}}$ is in the principal block of $T \cap \hat{S}$, since $B_0(\hat{S})$ covers a unique block of $T \cap \hat{S}$. Note that by [Nav98] Theorem 9.4, there must be a character of $B_0(T)$ lying above $\chi_i|_{T \cap \hat{S}}$. If $S \not\cong \Omega^+_q(3)$, we have $\Aut(S) \cap S$ is abelian, and hence every character of $T$ lying above $\chi_i|_{T \cap \hat{S}}$ is an extension, completing the proof in this case.

If $S = \Omega^+_q(3)$, then $\Aut(S) \cap \hat{S}$ is of the form $\varphi \times C$, where $C$ is cyclic. Then the character $\tilde{\chi}_i$ in $B_0(T)$ lying above $\chi_i|_{T \cap \hat{S}}$ must be such that $\tilde{\chi}_i|_S \in \{\chi_1, 2\chi_1\}$, as desired. Switching the roles of the semisimple elements $s_1$ and $s_2$ constructed in [GRS] Proposition 4.3, we further see that the characters have been constructed to satisfy $\chi_1(1) > 2\chi_2(1)$, since the centralizers of $s_1$ and $s_2$ have types $A_1 \times T_1$ and $A_1^+ \times T_2$ with $T_1$ and $T_2$ appropriate tori, and $2|C_{G^T}(s_i)|_{p'} < |C_{G^T}(s_j)|_{p'}$. □

The following handles the exceptional cases left by Proposition 4.2.

Proposition 4.3. Let $S$ be one of $\PSL_2(q)$, $\PSL_3(q)$, or $\PSp_4(q)$, where $q$ is a power of a prime $p > 3$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \Irr(\hat{B}_0(S))$ such that $\chi_2(1) \neq \chi_1(1)$; $\chi_2$ is invariant under a Sylow $p$-subgroup of $\Aut(S)$; and for every $S \leq T \leq \Aut(S)$, $\chi_1$ extends to a character in the principal $p$-block of $T$.

Proof. In this case, characters $\chi_1$ and $\chi_2$ are constructed in the proof of [GRS] Lemma 4.4] that satisfy all of the needed properties, except possibly the property that for every $S \leq T \leq \Aut(S)$, $\chi_1$ extends to a character in the principal block of $T$. However, $\chi_1$ is again constructed from a character of $\hat{G}$ trivial on $Z(\hat{G})$ that restricts irreducibly to $G$. Hence since again $\Aut(S) \cap \hat{S}$ is abelian, the proof is complete arguing as in the second paragraph of Proposition 4.2. □

For the remainder of the section, we consider the case of non-defining characteristic. That is, we assume $p > 3$ is a prime and $S$ is a simple group of Lie type defined in characteristic different than $p$.

Proposition 4.4. Let $p > 3$ be a prime and let $S$ be a simple group of Lie type defined over $F_q$, where $q$ is a power of a prime different than $p$ and $S$ is not in the following list: $\PSL_2(q)$, $\PSL_3(q)$ with $p \mid (q + e)$, $2B_2(2^{2e+1})$ with $p \mid (2^{2e+1} - 1)$, or $G_2(3^{2e+1})$ with $p \mid (3^{2e+1} - 1)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \Irr(\hat{B}_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:

- If $S \not\cong \Omega^+_q(3)$, then for every $S \leq T \leq \Aut(S)$, each of $\chi_1$ and $\chi_2$ extend to a character in the principal $p$-block of $T$.

- If $S = \Omega^+_q(3)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \Aut(S)$, for $i = 1, 2$, there exist $\tilde{\chi}_i$ in the principal $p$-block of $T$ such that $\tilde{\chi}_i|_S \in \{\chi_1, 2\chi_1\}$. 
Proof. We adapt our proof of [GRS] Proposition 4.5, ensuring that we may choose unipotent characters of $p'$-degree satisfying the principal block conditions required here. That is, we will exhibit unipotent characters of $\tilde{G}$ with different degree (and in the case of $P\Pi L_2^d(q)$, satisfying $\chi_1(1) > 2\chi_2(1)$) that are contained in $\text{Irr}_p(B_0(\tilde{G}))$, which as unipotent characters must be trivial on $Z(\tilde{G})$ and restrict irreducibly to $G$. Then the restriction lies in $B_0(G)$, since $B_0(\tilde{G})$ covers a unique block of $G$, and by [CF84] Lemma 17.2, the resulting characters of $\bar{S}$ and $S = G/Z(\tilde{G})$ also lie in the principal blocks. By [Ma08] Theorems 2.4 and 2.5, every unipotent character extends to its inertia group in $\text{Aut}(S)$, and except for some specifically stated exceptions, the inertia group is all of $\text{Aut}(S)$. Then arguing as in Proposition 4.2, the required properties will hold for all $S \leq T \leq \text{Aut}(S)$.

To see that the unipotent characters exhibited are indeed of $p'$-degree, it will often be useful to recall that $q^x - 1 = \prod_{\Phi_m} \Phi_m$, and note that $p \mid \Phi_m$ if and only if $m = dp^i$ for some non-negative integer $i$, where $\Phi_m$ denotes the $m$-th cyclotomic polynomial in $q$ and $d$ is the order of $q$ modulo $p$. Further, $p^2 \equiv \Phi_m$ only if $m = d$. (This is [Ma07] Lemma 5.2.)

First, we consider groups of exceptional type. If $S$ is one of $2G_2(3(2^{a+1})$ or $2B_2(2^{2a+1})$ but not one of the exceptions of the statement, then the unipotent characters mentioned in the proof of [GRS] Proposition 4.5 work here, since by [H90] Proposition 3.2, respectively [B79] Section 2, there is a unique unipotent block of maximal defect. If $S$ is $2F_4(2^{2a+1})$, then by [Mal00] Bemerkung 1, there is again a unique unipotent block of maximal defect unless $p \mid (2^{2a+1} - 1)$, in which case the principal block contains the Steinberg character and two more unipotent characters of $p'$-degree. Hence we are also done in this case. If $S = 3D_4(q)$, then there is either a unique unipotent block of maximal defect or the principal block contains the Steinberg character and one other unipotent character of $p'$-degree, using [DMS] Propositions 5.6 and 5.8, so we are similarly finished in this case.

Now let $S$ be one of $G_2(q)$, $F_4(q)$, $E_6(q)$, $2E_6(q)$, $E_7(q)$, or $E_8(q)$. Let $d$ be the order of $q$ modulo $p$. Using [E00] Theorem A1, we have the unipotent blocks of $\tilde{G}$ are indexed by conjugacy classes of pairs $(L, \lambda)$ for $L$ a $d$-split Levi subgroup and $\lambda$ a $d$-cuspidal unipotent character. In particular, the characters in the $d$-Harish-Chandra series indexed by such an $(L, \lambda)$ all lie in the same block of $\tilde{G}$. Further, [Ma07] Corollary 6.6 then yields that if a unipotent character in the series indexed by $(L, \lambda)$ has $p'$-degree, then $L$ is the centralizer of a Sylow $d$-torus. Now, using this and [BMM93] Theorem 5.1, we see that either such an $L$ is a maximal torus (yielding a unique block containing unipotent characters of $p'$-degree, and hence we are done using [GRS] Proposition 4.5) or we may use the decompositions in [BMM93] Table 2 to see there are at least two non-trivial unipotent characters in the principal block with different degrees relative prime to $p$. (For an example of the argument in the latter situation, consider $E_6(q)$ in the case $d = 7$. Then Line 58 of [BMM93] Table 2 shows that the trivial character and the unipotent characters $\phi_{8,91}$ and $\phi_{400,7}$ in the notation of [Ca85] Section 13.9, which have degree $q^{91} \Phi_3 \Phi_8 \Phi_{12} \Phi_{20} \Phi_{24}$ and $\frac{1}{2} g^{6} f_{1}^{2} f_{2}^{2} f_{4}^{2} f_{6}^{2} f_{8}^{2} f_{10}^{2} f_{14}^{2} f_{15}^{2} f_{18} f_{20} f_{24} f_{30}$, respectively, lie in the same $d$-Harish-Chandra series, and hence the same block. Since $p \mid \Phi_7$ and $p \neq 2$, we see these two non-trivial character degrees are $p'$ and distinct.)

We are left to consider the classical groups, in which case the unipotent characters of $\tilde{G}$ are parametrized by certain partitions or symbols. By a symbol of rank $n$, we mean a pair of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_\mu)$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_a$, $\mu_1 < \mu_2 < \cdots < \mu_b$, $\lambda_1$ and $\mu_1$ are not both 0, and $n = \sum_{i} \lambda_i + \sum_{j} \mu_j - \left\lfloor \frac{a+b-1}{2} \right\rfloor$. The defect of a symbol is $|b-a|$. Given an integer $e$, an $e$-hook is a pair of non-negative integers $(x, y)$ with $y-x = e$, $x \not\in \lambda$ (resp. $\mu$), and $y \in \lambda$ (resp. $\mu$). The $e$-core of a symbol is obtained by successively removing $e$-hooks, which means replacing $y$ by $x$ in $\lambda$ (resp. $\mu$) and then replacing the result with an equivalent symbol satisfying that $\lambda_1$ and $\mu_1$ are not both 0. An $e$-cohook is defined similarly, except that $x \not\in \lambda$ and $y \in \mu$ (or $x \not\in \mu$ and $y \in \lambda$), and the $e$-cocore is obtained by removing $e$-cohooks, which means removing $y$ from $\mu$ and adding $x$ to $\lambda$ (resp. removing $y$ from $\lambda$ and adding $x$ to $\mu$), and again replacing the result with an equivalent symbol satisfying that $\lambda_1$ and $\mu_1$ are not both 0.

Tables 1 through 4 describe two unipotent characters for each classical type satisfying the properties described in the first paragraph and not in the list of exceptions of [Ma08] Theorem 2.5. For each type, we include a brief discussion, but we remark that a more complete description of the degrees of such characters and the partitions and symbols can be found in [Ca85] Section 13.8, and a more complete discussion of their distribution into blocks may be found in [ESS92] [FS93]. We will include the details for type $A_{n-1}$ in this respect, and note that the other types have similar arguments.
Types $A_{n-1}$ and $2A_{n-1}$. Here $\tilde{G} = GL_n(q)$. In this case, let $e$ be the order of $eq$ modulo $p$. The unipotent characters are in bijection with partitions of $n$, and two such characters are in the same block if and only if they have the same $e$-core. In particular, the trivial character is given by the partition $(n)$, which has $e$-core $(r)$, where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by $e$. Table 1 lists the desired unipotent characters in this case when $n \geq 4$. Indeed, consider the case $e = 1$. The partitions listed have $e$-core $(r)$, and hence the corresponding characters are in the principal block and it suffices to show that they have $p'$-degree. Since $p \mid q$, we need only consider the part of the degree relatively prime to $q$, which are listed following Section 13.8. If $e = 1$, then since $p > 3$, the character $\chi_1$ in the cases of line 1 or line 2 has $p'$-degree, since $(q^d - 1)/(q - 1)$ is divisible by $p$ in this case and if only if $d$ is divisible by $p$. Hence, for $\chi_1$, we may assume $e \neq 1$. Consider line 3 of Table 1 in this case. Since $me + k$ is not divisible by $e$ for $1 \leq k < e$, we see $(q^{me+k}-1)\cdots(q^n-1)$ is not divisible by $p$. Similarly, if $r + 1 \neq e$, then $(q^{me-r-1})$ is not divisible by $p$. If $r + 1 = e$, then $(q^{me-e-1})/(q^e-1)$ is divisible by $p$ only if $p \mid (m-1)$, so that $(q^{me-e-1})$ has factors of the form $\Phi_{p^i}$, with $i \geq 1$. Hence the character listed in line 3 has $p'$-degree, given the stated conditions, and similar for lines 6 and 7. Line 5 refers to the Steinberg character, which is certainly of $p'$-degree. So, consider the characters in lines 4 and 8, of degree $\prod_{i=1}^{e} \frac{q^{n_{i}-1}}{q-1}$, with $p \mid (m-1)$. If $p$ divides $\prod_{i=1}^{e} \frac{q^{n_{i}-1}}{q-1}$, then $p \mid (q^{n-r-1})/(q^e-1) = (q^{me}-1)/(q^e-1)$, and hence $p \mid m$, a contradiction. The argument is similar in the case $e = -1$.

Finally, if $n = 3$ and $p \nmid (q + e)$, then note that $e = 1$ or $3$, $r < 2$, and the characters listed in Table 1 still satisfy our conditions. (In this case, the two characters are the Steinberg character and the unipotent character of degree $q(q + e)$.)

**Types $B_n$ and $C_n$.** Here the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank $n$ and odd defect. In this case, we let $e$ be the order of $q^2$ modulo $p$. Then two symbols are in the same block if and only if they have the same $e$-core, respectively $e$-co-core, if $p \mid q^2 - 1$, respectively $p \mid q^2 + 1$. The trivial character is represented by the symbol $(\emptyset)$, which has $e$-core and $e$-co-core $(\emptyset)$, where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by $e$. Table 2 lists the desired unipotent characters in this case, as long as $n \geq 3$. For types $D_n$ and $2D_n$. In this case the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank $n$ and defect $0 \pmod{4}$, respectively $2 \pmod{4}$ in case $D_n$, respectively $\tilde{2D}_n$. Again, let $e$ be the order of $q^2$ modulo $p$, and let $n = me + r$ where $0 \leq r < e$ is the remainder when $n$ is divided by $e$. The block distribution is described the same way as for types $B_n$ and $C_n$.

For type $D_n(q)$, the trivial character is represented by the symbol $(\emptyset)$, which has $e$-core $(\emptyset)$ if $e \nmid n$ and $(\emptyset)$ if $e \mid n$. It has $e$-co-core $(\emptyset)$ if $m$ is even and $e \nmid n$; $(\emptyset)$ if $m$ is odd and $e \nmid n$; $(\emptyset)$ if $m$ is even and $e \mid n$; and $(\emptyset)$ if $m$ is odd and $e \mid n$. Table 3 lists the desired unipotent characters as long as $n \geq 5$. In some cases, more than two characters are listed. We remark that if $n = e$, then it must be that $p \mid (q^e - 1)$.

For $D_4(q) = PO_4^+(q)$, note that $1 \leq e \leq 3$ and that $p \mid (q^2 + 1)$ when $e = 2$. Then the Steinberg character of degree $q^{12}$, labeled by $(0~1~2~3)$, may be taken for $\chi_1$. For $\chi_2$, we take the character labeled by $(1~2~3)$, of degree $q(q^2 + 1)^2$ when $e = 1$ or $3$, and $(1~3)$ of degree $\frac{1}{2}q^2(q+1)^3(q^3+1)$ when $e = 2$. In either case, we have $\chi_1(1) > 2\chi_2(1)$.

For type $2D_n(q)$, the trivial character is represented by the symbol $(\emptyset)$, which has $e$-core $(\emptyset)$ when $e \nmid n$ and $(\emptyset)$ if $e \mid n$. The $e$-co-core is $(\emptyset)$ if $e \nmid n$ and $m$ is even, $(\emptyset)$ if $e \nmid n$ and $m$ is odd, $(\emptyset)$ if $e \mid n$ and $m$ is even, and $(\emptyset)$ if $e \mid n$ and $m$ is odd. Table 4 lists the desired unipotent characters in this case.

**Proposition 4.5.** Let $p > 3$ be a prime and let $q$ be a power of a prime different than $p$. Let $S$ be one of $\PSL_2(q)$, $\PSL_3(q)$ with $p \mid (q + e)$, $2B_2(2^{2e+1})$ with $p \mid (2^{2e+1} - 1)$, or $2G_2(3^{2e+1})$ with $p \mid (3^{2e+1} - 1)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \Irr_p(B_2(S))$ such that $\chi_2(1) \nmid \chi_1(1)$; $\chi_2$ is invariant under a Sylow $p$-subgroup of $\Aut(S)$; and for every $S \leq T \leq \Aut(S)$, $\chi_1$ extends to a character in the principal $p$-block of $T$. 

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Table 1: Some unipotent characters in \( \text{Irr}_p(B_0(S)) \) for type \( A_{n-1}(q) \) with \( n \geq 4 \) and \( p \nmid q \)

| Additional condition on \( n = me + r, r < e \) | Partition | \( \chi(1)_{q^r} \) |
|-------------------------------------------------|----------|-------------------|
| \( e = 1 \) and \( p \mid (n - 1) \)          | \((2, n - 2)\) | \( \frac{(q^n - q^r)}{(q^r - 1)} \) |
| \( e = 1 \) and \( p \mid (n - 1) \)          | \((1, n - 1)\) | \( \frac{1}{(q^r - 1)} \) |
| \( 1 \neq e \neq r + 1 \) or \( p \mid (m - 1) \) | \((r + 1, m - 1)\) | \( \frac{(q^{me+e-m-1})(q^{me+e-m-2}) \cdots (q^n - q^r)}{(q^r - 1)(q^{r+e-1})(q^{r+e+1})} \) |
| \( 1 \neq e = r + 1 \) and \( p \mid (m - 1) \) | \((1^{r+1}, m - 1)\) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})} \) |
| \( r < 2 \)                                    | \((1^r)\)  | \( 1 \) |

\[ \text{Table 2: Some unipotent characters in } \text{Irr}_p(B_0(S)) \text{ for types } B_n(q), C_n(q) \text{ with } n \geq 2, p \nmid q, (n, q) \neq (2, 2^{2a+1}) \]

| Conditions on \( n = me + r, r < e \) | Symbol | \( \chi(1)_{q^r} \) (possibly excluding factors of \( \frac{1}{2} \)) |
|--------------------------------------|--------|-----------------------------------------------------------------|
| \( p \mid (q^r - 1) \)              | \(0 \) \( 1 \) \( m + 1 \) \( m + 1 \) | \( \text{conditions not given} \) |
| \( p \mid (q^r + 1), m \) odd       | \(0 \) \( m + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r + 1), m \) even      | \(r+1 \) \( m + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( e \mid n \)                       | \(1 \) \( 1 \) \( n - 1 \) \( n - 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r - 1), e \mid n, e \neq r + 1 \) or \( p \mid (m - 1) \) | \(0 \) \( m + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r - 1), e \mid n, \) | \(0 \) \( 1 \) \( r + 2 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r + 1), e \mid n, m \) odd | \(1 \) \( r + 2 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r + 1), e \mid n, m \) even | \(0 \) \( 1 \) \( r + 2 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |

\[ \text{Table 3: Some unipotent characters in } \text{Irr}_p(B_0(S)) \text{ for type } D_n(q) \text{ with } n \geq 5, p \nmid q \]

| Conditions on \( n = me + r, r < e \) | Symbol | \( \chi(1)_{q^r} \) (possibly excluding factors of \( \frac{1}{2} \)) |
|--------------------------------------|--------|-----------------------------------------------------------------|
| \( e \mid n \)                       | \(m + e \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r - 1), e \mid n, \) | \(0 \) \( 1 \) \( r + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r - 1), e \mid n, \) | \(0 \) \( 1 \) \( r + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r - 1), e \mid n, \) | \(0 \) \( 1 \) \( r + 1 \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r + 1), e \mid n, m \) odd | \(0 \) \( e \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
| \( p \mid (q^r + 1), e \mid n, m \) even | \(0 \) \( e \) | \( \prod_{i=1}^r \frac{1}{(q^{r+i})(q^{r+i-1})} \) |
Table 4: Some unipotent characters in \(\text{Irr}_{p}(B_0(S))\) for type \(2D_n(q)\) with \(n \geq 4\), \(p \nmid q\)

| Conditions on \(n = me + r\), \(r < e\) | Symbol | \(\chi(1)|_{p}^{r} \) (possibly excluding factors of \(\frac{q}{2}\)) |
|------------------------------------------|--------|---------------------------------------------------|
| \(p \mid (q^e + 1), 1 \neq e \mid n, m \text{ odd} \) | \((0 \ 1 \ n)\) | \(\frac{(q^{2e}+1)-1}{q-1} \) |
| \(p \mid (q^e - 1), p \nmid (n-1)\) | \((1 \ n-1)\) | \(\frac{(q^e-1)}{q-1} \) |
| \(p \mid (q^e + 1), 1 \neq e \mid n, m \text{ even} \) | \((0 \ 1 \ n-e)\) | \(\frac{(q^{2e}+1)-1}{q-1} \) |
| \(p \mid (q^e - 1), 1 \neq e \mid n\) | \((1 \ n-e)\) | \(\frac{(q^e-1)}{q-1} \) |

Proof. First suppose \(S\) is \(PSL_2(q)\) or \(PSL_4(q)\) with \(p \mid (q + c)\). In these cases the order of \(q\) modulo \(p\) is 1 or 2, and there is a unique unipotent block of maximal defect, so \(\chi_1\) may still be taken to be the Steinberg character. Let \(\delta\) be an element of order \(p\) in \(F_{\ell}\). Write \(q = \ell^a\), for some prime \(\ell \neq p\), and write \(a = pb\) with \(p \nmid c\). Then \(p \mid \ell^{2c} - 1\) since the order of \(\ell\) modulo \(p\) divides 2, and hence \(2c\). Then \(\delta\) is either fixed or inverted by \(F_{\ell}\), where \(F_{\ell}\) is the generating field automorphism. In particular, since the semisimple classes of \(\tilde{G}^{\ast} \cong GL_2(q)\), resp. \(GL_4(q)\), are determined by their eigenvalues, this means that a semisimple element \(s\) of \(\tilde{G}^{\ast}\) with eigenvalues \(\{\delta, \delta^{-1}\}\), respectively \(\{\delta, \delta^{-1}, 1\}\) is conjugate to its image under \(F_{\ell}\). Then the corresponding semisimple character of \(G\) is fixed by \(F_{\ell}\), and hence \(H\)-Sylow \(p\)-subgroup of \(\text{Aut}(S)\). Further, \(s\) satisfies (1)-(2) of \([GRS\text{ Section }4.1.1]\), that is, \(s\) is a member of \([\tilde{G}^{\ast}, \tilde{G}^{\ast}] \cong SL_2(q)\), resp. \(SL_4(q)\), and is not conjugate to \(sz\) for any \(z \in Z(\tilde{G}^{\ast})\), since \(|\delta| \geq 5\). Then this character is irreducible on \(G\) and trivial on the center. Further, it has degree \((q - \eta)\), where \(\eta \in \{\pm 1\}\) is such that \(p \mid (q + \eta)\) for \(PSL_2(q)\), and degree \(q^2 - \epsilon\) for \(PSL_4(q)\) with \(p \mid (q + c)\). Since \(s\) is a \(p\)-element, the character lies in a unipotent block, and hence \(B_0(\tilde{G})\), using \([CEF2\text{ Theorem }9.12]\). Then as in the first paragraph of \(\text{Proposition }4.4\), the corresponding character of \(S\) lies in the principal block. It also has non-trivial degree prime to \(q\), which therefore does not divide the degree of the Steinberg character. Hence this character satisfies our conditions.

Now let \(S\) be \(2B_2(q^2)\) with \(q^2 = 2^{a+1}\) and \(p \mid (q^3 - 1)\) and write \(2a + 1 = pb^c\) with \(p \nmid c\). Let \(s\) be such that \(\gamma^s\) has order \(p \mid (2^e - 1)\), where \(\gamma\) has order \(q^2 - 1\). Then using \([BY9\text{ Section }2]\) and arguing as in the case above, we see that a slight modification of the characters used in \([GRS\text{ Lemma }4.8]\) works here: we may take \(\chi_1\) to be the Steinberg character and \(\chi_2\) to be the character \(\chi_5(s)\) in CHEVIE notation.

Finally, let \(S\) be \(2G_2(q^2)\) with \(q^2 = 3^{2a+1}\) and \(p \mid (q^3 - 1)\). Again write \(2a + 1 = pb^c\) with \(p \nmid c\). Using \([H0\text{ Proposition }3.2]\), there is a unique unipotent block of maximal defect, so we may take \(\chi_1\) again to be the Steinberg character. For \(\chi_2\), it follows from \([H0\text{ Proposition }4.1]\) and arguments as above that we may take the character \(\chi_{11}(s)\) in CHEVIE notation, where now \(s\) is such that \(\gamma^s\) has order \(p \mid (3^e - 1)\) and \(\gamma\) has order \(q^2 - 1\).

\(\square\)

\textbf{Proposition 2.1} now follows from \(\text{Propositions }3.6\text{ and }4.1\text{ through }4.5\) completing the proof of \textbf{Theorem A}. 

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