A classical $q$-hypergeometric approach to the $A_{2}^{(2)}$ standard modules

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November 26, 2018

Dedicated to Krishna Alladi on the occasion of his sixtieth birthday

Abstract

This is a written expansion of the talk delivered by the author at the International Conference on Number Theory in Honor of Krishna Alladi for his 60th Birthday, held at the University of Florida, March 17–21, 2016.

Here we derive Bailey pairs that give rise to Rogers–Ramanujan type identities which are the principally specialized character of the $A_{2}^{(2)}$ standard module $(\ell-2i+2)\Lambda_{0}+(i-1)\Lambda_{1}$ for any level $\ell$, and $i=1,2$.

1 Notation and Motivation

1.1 $q$-series notation and classical results

Let $q$ denote a formal variable. The standard notation for the infinite rising $q$-factorial is

$$(a;q)_{\infty} := \prod_{j=0}^{\infty}(1-aq^{j}).$$

In order to allow for positive and negative values of $n$, we define the finite rising $q$-factorial as

$$(a;q)_{n} := \frac{(a;q)_{\infty}}{(aq^{n};q)_{\infty}}.$$

We will also use the abbreviations $(q)_{n}$ and $(q)_{\infty}$ for $(q;q)_{n}$ and $(q;q)_{\infty}$ respectively. Additionally,

$$(a_{1},a_{2},\ldots,a_{r};q)_{n} := (a_{1};q)_{n}(a_{2};q)_{n} \cdots (a_{r};q)_{n}$$

and

$$(a_{1},a_{2},\ldots,a_{r};q)_{\infty} := (a_{1};q)_{\infty}(a_{2};q)_{\infty} \cdots (a_{r};q)_{\infty}.$$

The bilateral basic hypergeometric series is given by

$$_{r+1}\psi_{1}\left[\begin{array}{c}a_{1},a_{2},\ldots,a_{r}\vspace{1mm}\
\frac{b_{1},b_{2},\ldots,b_{r}}{q,\bar{q}}\end{array};q,z\right] := \sum_{r \in \mathbb{Z}} \frac{(a_{1};q)_{r}(a_{2};q)_{r} \cdots (a_{r};q)_{r}}{(b_{1};q)_{r}(b_{2};q)_{r} \cdots (b_{r};q)_{r}} z^{r}.$$
The \( q \)-binomial coefficient is
\[
\binom{n}{m}_q := \begin{cases} 
\frac{(q)_m}{(q)_n} & \text{if } 0 \leq m \leq n \\
0 & \text{otherwise}
\end{cases}.
\]

We will require the following classical results. For our purposes, \( z = \pm q^r \) for some \( r \in \frac{1}{2} \mathbb{Z} \).

**Triple product identity (Jacobi).** \[\text{[16] p. 15, Eq. (1.6.1)].}\]
\[
\sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n^2} = (q/z, zq, q^2; q^2)_\infty.
\]

**Quintuple product identity (Fricke \[15\]), cf. [16] p. 147, ex. 5.6].**
\[
(-qz^3, -q^2z^{-3}, q, q^3)_\infty - z (-qx^{-3}, -q^2z^3, q^3; q^3)_\infty
= (q/z, z, q; q)_\infty (q/z^2, qz^3, q^3)_\infty.
\]

1.2 Certain affine Kac–Moody Lie algebras and their connection to \( q \)-series

Let \( g \) denote the affine Kac–Moody Lie algebra \( A_1^{(1)} \) or \( A_2^{(2)} \) and let \( h_0, h_1 \) denote the usual basis of a maximal toral subalgebra \( T \) of \( g \). Let \( d \) denote the degree derivation of \( g \) and let \( T := T \oplus Cd \). For all dominant integral \( \lambda \in \check{T}^* \), there is a unique irreducible, integrable, highest weight module \( L(\lambda) \), assuming (without loss of generality) that \( \lambda(d) = 0 \). Also, \( \lambda = s_0 \Lambda_0 + s_1 \Lambda_1 \) where \( \Lambda_0 \) and \( \Lambda_1 \) are the fundamental weights, given by \( \Lambda_i(h_j) = \delta_{ij} \) and \( \Lambda_i(d) = 0 \); \( s_0 \) and \( s_1 \) are nonnegative integers. For \( A_1^{(1)} \), the canonical central element is \( c = h_0 + h_1 \), and for \( A_2^{(2)} \), the canonical central element is \( c = h_0 + 2h_1 \). The level \( \lambda(c) \) of \( L(\lambda) \) is
\[
\lambda(c) = \begin{cases} 
\lambda_0 + \lambda_1 & \text{if } g = A_1^{(1)} \\
\lambda_0 + 2\lambda_1 & \text{if } g = A_2^{(2)}
\end{cases},
\]
(cf. \[15, 20\]). For brevity, it is common to refer to \( L(\lambda) = L(s_0 \Lambda_0 + s_1 \Lambda_1) \) as the \( (s_0, s_1) \)-module.

Additionally \([20]\), there is an infinite product \( F_0 \) associated with \( g \), sometimes called the “fudge factor,” which needs to be divided out of the the principally specialized character \( \chi(L(\lambda)) = \chi(s_0 \Lambda_0 + s_1 \Lambda_1) \), in order to obtain the quantities of interest here. For \( g = A_1^{(1)} \), the fudge factor is given by
\[
F_0 = \begin{cases} 
(q; q^2)_{\infty}^{-1} & \text{if } g = A_1^{(1)} \\
[q; q^2]_{\infty} (q^3; q^3)_{\infty}^{-1} & \text{if } g = A_2^{(2)}
\end{cases}.
\]

Also, \( g \) has a certain infinite-dimensional Heisenberg subalgebra known as the principal Heisenberg vacuum subalgebra \( s \) (consult \[21\] for the construction of \( A_1^{(1)} \) and \[19\] for that of \( A_2^{(2)} \)). As demonstrated in \[22\], the principal character \( \chi(\Omega(s_0 \Lambda_0 + s_1 \Lambda_1)) \), where \( \Omega(\lambda) \) is the vacuum space for \( s \) in \( L(\lambda) \), is
\[
\chi(\Omega(s_0 \Lambda_0 + s_1 \Lambda_1)) = \frac{\chi(L(s_0 \Lambda_0 + s_1 \Lambda_1))}{F_0}.
\]
where \( \chi(L(\lambda)) \) is the principally specialized character of \( L(\lambda) \).

By [24] applied to (1.3) in the case of \( A_1^{(1)} \), the standard modules of odd level correspond to Andrews’ analytic generalization of the Rogers–Ramanujan identities [3], known as the “Andrews–Gordon identity,” and the partition theoretic generalization of the Rogers–Ramanujan identities due to B. Gordon [17]. Bressoud’s even modulus counterpart to the Andrews–Gordon identity [11, p. 15, Eq. (3.4)] and its partition theoretic counterpart [10, p. 64, Theorem, \( j = 0 \) case]; was explained vertex-operator theoretically in [23] and [24] to correspond to the standard modules of even level in \( A_1^{(1)} \).

The combined Andrews–Gordon–Bressoud identity (for both even and odd moduli) and its correspondence to the level \( \ell \) standard modules of \( A_1^{(1)} \) can be stated compactly as

\[
\chi(\Omega((\ell + 1 - i)\Lambda_0 + (i - 1)\Lambda_1)) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq 0} \prod_{j=1}^{\ell} (q)_{n_j - n_{j+1}} (-q)_{n_{k-1}} (q)_{n_{k-1}} = (q; q)_{\infty}^{(q)_{\infty}^{\ell+2}}. \tag{1.4}
\]

where \( k := k(\ell) = 1 + [\ell/2], 1 \leq i \leq k \), and

\[
[[P]] := \begin{cases} 
1 & \text{if } P \text{ is true}, \\
0 & \text{if } P \text{ is false}.
\end{cases}
\]

A pair of sequences \((\alpha_n(a,q), \beta_n(a,q))\) form a Bailey pair with respect to \( a \) if

\[
\beta_n(a,q) = \sum_{s=0}^{n} \frac{\alpha_n(a,q)}{(q)_{n-s}(aq; q)_{n+s}}.
\]

[5, p. 25–26]; cf. [9, pp. 2, 5].

It is well known that identities of Rogers–Ramanujan type may be derived by the insertion of Bailey pairs into limiting cases of Bailey’s lemma [5, p. 25, Thm. 3.3; p. 27, Eq. (3.33)] such as

\[
\sum_{n=0}^{\infty} a^n q^n \beta_n(a,q) = \frac{1}{(aq; q)_{\infty}} \sum_{n=0}^{\infty} a^n q^n \alpha_n(a,q), \tag{1.5}
\]

and setting \( a \) equal to a power of \( q \).

An efficient method for deriving (1.4) for odd \( \ell \) is via the Bailey lattice [2], which is an extension of the Bailey chain ([4, cf. 5, §3.5, pp. 27ff]) built upon the “unit Bailey pair”

\[
\beta_n(1,q) = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{if } n > 0
\end{cases}, \quad \alpha_n(1,q) = \begin{cases} 
1 & \text{if } n = 0 \\
(-1)^n q^{n(n-1)/2}(1 + q^n) & \text{if } n > 0.
\end{cases}
\]

Similarly, for even \( \ell \), (1.4) follows from a Bailey lattice built upon the Bailey pair.
\[ \beta_n(1, q) = \frac{1}{(q^2; q^2)_n}, \]
\[ \alpha_n(1, q) = \begin{cases} 
1 & \text{if } n = 0 \\
(-1)^n2q^n & \text{if } n > 0.
\end{cases} \]

Thus the standard modules of \( A_1^{(1)} \) correspond to two interlaced instances of the Bailey lattice.

In contrast, the standard modules of \( A_2^{(2)} \) are not as well understood, and a uniform \( q \)-series and partition correspondence analogous to what is known for \( A_1^{(1)} \) has to date remained beyond our reach.

As with \( A_1^{(1)} \), there are \( 1 + \lfloor \frac{\ell}{2} \rfloor \) inequivalent level \( \ell \) standard modules associated with the Lie algebra \( A_2^{(2)} \), but the principal characters for the level \( \ell \) standard modules are given by instances of the quintuple product identity (1.2) (rather than the triple product identity) divided by \( (q)_\infty \):

\[ \chi((\ell - 2i + 2)\Lambda_0 + (i - 1)\Lambda_1)) = \frac{(q^\ell, q^{\ell+3-i}, q^{\ell+3}; q^\ell)_{\infty}(q^{\ell+3-2i}; q^\ell)_{\infty}(q^{\ell+2i+3}; q^\ell)_{\infty}}{(q)_\infty}, \quad (1.6) \]

where \( 1 \leq i \leq 1 + \lfloor \frac{\ell}{2} \rfloor \); see [20].

2 Bailey pairs for \( A_2^{(2)} \)

Let \((\alpha_n^{(\ell, 1)}, \beta_n^{(\ell, 1)})\) denote the Bailey pair which, upon insertion into (1.5) with \( a = 1 \), gives the principally specialized character of the \( A_2^{(2)} \) standard module \((\ell - 2i + 2)\Lambda_0 + (i - 1)\Lambda_1\).

2.1 Bailey pairs for \( \chi((\ell)\Lambda_0) \)

\[ \alpha_n = \alpha_n^{(\ell, 1)}(1, q) = \begin{cases} 
1 & \text{if } n = 0 \\
q^{\frac{2}{3}((\ell-3)r^2 - \frac{1}{2}(\ell-3)r) + \frac{2}{3}((\ell-3)r^2 + \frac{1}{2}(\ell-3)r) - \frac{2}{3}(\ell-3)r^2 - \frac{1}{3}(\ell-3)r} & \text{if } n = 3r + 1 \\
q^{\frac{2}{3}((\ell-3)r^2 + \frac{1}{2}(\ell-3)r) - \frac{2}{3}(\ell-3)r^2 - \frac{1}{3}(\ell-3)r} & \text{if } n = 3r - 1
\end{cases} \]

\[ \beta_n^{(\ell, 1)}(1, q) = \sum_{s=0}^{n} \frac{\alpha_s^{(\ell, 1)}(1, q)}{(q)_{n-s}(q)_{n+s}} = \frac{\alpha_0}{(q)_n^2} + \sum_{r \geq 1} \frac{\alpha_{3r}}{(q)_{n-3r}(q)_{n+3r}} + \sum_{r \geq 0} \frac{\alpha_{3r+1}}{(q)_{n-3r-1}(q)_{n+3r+1}} + \sum_{r \geq 1} \frac{\alpha_{3r-1}}{(q)_{n-3r+1}(q)_{n+3r-1}} = \frac{1}{(q)_n^2} + \sum_{r \geq 1} \frac{q^{\frac{2}{3}((\ell-3)r^2 + \frac{1}{2}(\ell-3)r) + \frac{2}{3}((\ell-3)r^2 - \frac{1}{2}(\ell-3)r)}}{(q)_{n-3r}(q)_{n+3r}}. \]
For each $\ell = 1, 2, \ldots$, the series expression in (2.1) is a limiting case of a very-well-poised bilateral basic hypergeometric series.

For example, we have

$$
\frac{q^{-n}(q)_{n}\psi(q)_{n+1}}{1-q} B^{(\ell, 1)}(1, q)
$$

$$
= \left\{ \begin{array}{l}
\lim_{\psi \to 0} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, e, e; q^{3}, q^{3n+6} ; e^{3} \right] \quad \text{if } \ell = 3 \\
\lim_{\psi \to 0} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, e, -q^{2} ; q^{q^{3n+4}} ; -q^{3n+4} ; e^{2} \right] \quad \text{if } \ell = 4 \\
\lim_{\psi \to 0} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, -q^{2} ; q^{3}, -q^{3n+2} ; q^{3n+2} ; e \right] \quad \text{if } \ell = 5 \\
\lim_{\psi \to \infty} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, -q^{2} ; q^{3}, -q^{3n} ; q^{3n+2} ; e \right] \quad \text{if } \ell = 6 \\
\lim_{\psi \to \infty} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, -q^{2} ; q^{q^{3n+6}} ; q^{3n+2} ; e^{2} \right] \quad \text{if } \ell = 7 \\
\lim_{\psi \to \infty} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, e, e ; q^{q^{3n+6}} ; q^{3n+2} ; e^{2} \right] \quad \text{if } \ell = 8 \\
\lim_{\psi \to \infty} \phi_{\mu} \left[ q^{2}, -q^{2}, q^{-n}, q^{1-n}, q^{2-n}, e, e, e ; q^{q^{3n+6}} ; q^{3n+2} ; e^{2} \right] \quad \text{if } \ell = 9
\end{array} \right.
$$

Observe that the easiest cases are $\ell = 5, 6, 7$, as these are instances of Bailey’s summable bilateral very-well-poised $\phi_{\mu}$ [S, Eq. (4.7)]; cf. [10] p. 357, Eq. (II.33)]. Indeed, Slater evaluated the cases $\ell = 5$ and 7 [28] p. 464, Eqs. (3.4) and (3.3) resp., while McLaughlin and Sills evaluated the case $\ell = 6$ [25] p. 772, Table 3.1, line (P2)].
We have
\[
\beta_n^{(5,1)}(1, q) = \frac{q^{n^2}}{(q)_{2n}}, \quad (2.2)
\]
\[
\beta_n^{(6,1)}(1, q) = \frac{q^n(-1; q^3)_n}{(-1; q)_n(q)_{2n}}, \quad (2.3)
\]
\[
\beta_n^{(7,1)}(1, q) = \frac{q^n}{(q)_{2n}}. \quad (2.4)
\]

To evaluate \(\beta_n^{(\ell,1)}(1, q)\) for levels \(\ell = 3, 4, 8, 9\), we can use the following identity \[16\] p. 147, exercise 5.11], analogous to Bailey’s \(\psi_6\) sum: for nonnegative integer \(n\),
\[
s\psi_8 \left[ \frac{q^{\sqrt{n}} - q^{\sqrt{n}}c, d, e, f, aq^{-n}, q^{-n}, q^{n}, q^{n+1}, aq^{n+1}; q, q^2q^{2n+2}; cdef}{\sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, q, q; q, q} \right] = (aq, a, aq, aq, aq; q)_n \psi_4 \left[ e, f, \frac{aq^{n+1}}{aq^n}, q^{-n}; q, q \right]. \quad (2.5)
\]

Note further that (2.5) is a bilateral analog of Watson’s \(q\)-analog of Whipple’s theorem \[52\] (cf. \[16\] p. 360, Eq. (III.17))

Notice that for level \(\ell = 1\) and 2, \(q^{-n}(q)_n(q^2; q)\psi_6^{(\ell,1)}(1, q)\) is a limiting case of a \(10\psi_1\), while for \(\ell > 6\), it is a limiting case of a \(\psi_t\) with \(t = \ell - 1 + (1 + (-1)^t)/2\).

More precisely, if \(\ell\) is even and \(\ell \geq 6\),
\[
q^{-n}(q)_n(q^2; q)\psi_6^{(\ell,1)}(1, q) = \lim_{c \to \infty} \ell^\psi c \left[ q^{\frac{\ell}{2}}, q^{\frac{\ell}{2}}, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, \ldots, q^{-n}, q^{n+1}, q^{n+2}, q^{n+3}, q^{n+4}, q^{n+5}, q^{\ell-6}; e, e, e, \ldots, e, q^2, q^4, q^6, \ldots, q^{3n+2\ell-12}; e^{\ell-6} \right]
\]

while if \(\ell\) is odd and \(\ell > 6\),
\[
q^{-n}(q)_n(q^2; q)\psi_6^{(\ell,1)}(1, q) = \lim_{c \to \infty} \psi_1(-1) \left[ q^{\frac{\ell}{2}}, q^{\frac{\ell}{2}}, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, \ldots, q^{-n}, q^{n+1}, q^{n+2}, q^{n+3}, q^{n+4}, q^{n+5}, q^{\ell-6}; e, e, e, \ldots, e, q^2, q^4, q^6, \ldots, q^{3n+2\ell-12}; e^{\ell-6} \right]
\]

Then, to obtain the series and product expressions for \(\chi(\Omega(\ell \Lambda_0))\), one inserts the Bailey pair \((\alpha_n^{(\ell,1)}(1, q), \beta_n^{(\ell,1)}(1, q))\) into \[16\] with \(a = 1\), and upon applying (1.1) and (1.2), we find that
\[
\sum_{m=0}^{\infty} q^{m+2} \left( q^{-6m+1} \beta_{3m}^{(\ell,1)}(1, q) + \beta_{3m+1}^{(\ell,1)}(1, q) + q^{6m+1} \beta_{3m+1}^{(\ell,1)}(1, q) \right)
\]

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And thus in (2.6), we have a uniform series-product identity for the principally specialized character of the \((\ell, 0)\) standard module of \(A^{(2)}_2\) for any \(\ell\).

To express the \(\beta_\ell^{(\ell, 1)}\) as a multisum for arbitrary \(\ell\), one may employ the Andrews–Baxter–Forrester bilateral very-well poised \(q\)-hypergeometric summation formula \([5, p. 83, Eq. (8.56)]\); cf. \([7, Appendix B, pp. 261–265]\).

The level 3 case will be considered in detail in the next section.

### 2.2 Bailey pairs for \(\chi(\Omega((\ell - 2)\Lambda_0 + \Lambda_1))\)

\[
\alpha_n = \alpha_n^{(\ell, 2)}(1, q) = \begin{cases} 
1 & \text{if } n = 0 \\
q^{\frac{1}{2}(\ell-3)r^2 - \frac{1}{2}(9-\ell)r} + q^{\frac{1}{2}(\ell-3)r^2 + \frac{1}{2}(9-\ell)r} & \text{if } n = 3r > 0 \\
q^{\frac{1}{2}(\ell-3)r^2 + \frac{1}{2}(\ell-3)r + 1} & \text{if } n = 3r + 1 \\
q^{\frac{1}{2}(\ell-3)r^2 - \frac{1}{2}(\ell-3)r + 1} & \text{if } n = 3r - 1 
\end{cases}.
\]

The calculation of \(\beta_n^{(\ell, 2)}\) parallels that of \(\beta_n^{(\ell, 1)}\). The details of the \(\ell = 7\) case are given by Slater \([30, p. 464]\).

\[
\beta_n^{(\ell, 2)}(1, q) = \sum_{s=0}^{n} \alpha_n^{(\ell, 2)}(1, q) = \frac{1}{(q)_n} \sum_{r\in\mathbb{Z}} (1 - q^{2r+1})(q^{-n}; q)_{3r} (-1)^r q^{\frac{1}{2}(\ell-6)r^2 + \frac{1}{2}(\ell-6)r}. \tag{2.7}
\]

Notice that

\[
q^n \beta_n^{(\ell, 2)}(1, q) = \beta_n^{(\ell, 1)}(1, q). \tag{2.8}
\]

And so it follows that the series and product expressions for \(\chi(\Omega((\ell - 2)\Lambda_0 + \Lambda_1))\) are

\[
\sum_{m=0}^{\infty} q^{mn^2} \left( q^{-6m+1} \beta_n^{(\ell, 2)}(1, q) + q^{6m+1} \beta_n^{(\ell, 2)}(1, q) \right)
= \frac{(q^2, q^{\ell+1}, q^{\ell+3}; q^{\ell+6})_\infty (q^{\ell-1}, q^{\ell+7}; q^{2\ell+6})}{(q)_\infty}. \tag{2.9}
\]

### 3 Level 3

Let us consider the case \(\ell = 3\) in detail. This level is of particular interest as it was the study of the the level 3 standard modules of \(A^{(2)}_2\) that led S. Capparelli to discover two new Rogers–Ramanujan type partition identities \([1, 6, 13, 14]\). See \([29, \S 3]\) for some historical notes.

From the \((3, 0)\)-module, Capparelli conjectured (and later proved \([14]\), although the first proof was due to Andrews \([4]\)) the following partition
identity. A partition $\lambda$ of an integer $n$ is a finite weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers that sum to $n$; each $\lambda_i$ is called a part of the partition $\lambda$.

**Theorem 1** (Capparelli’s first partition identity). Let $c_1(n)$ denote the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ wherein

- $\lambda_i \neq 1$ for $i = 1, 2, \ldots, l$,
- $\lambda_i - \lambda_{i+1} \geq 2$, for $i = 1, 2, \ldots, l - 1$,
- $\lambda_i - \lambda_{i+1} = 2$ only if $\lambda_i \equiv 1 \pmod{3}$,
- $\lambda_i - \lambda_{i+1} = 3$ only if $\lambda_i \equiv 0 \pmod{3}$.

Let $c_2(n)$ denote the number of partitions of $n$ into distinct parts $\neq \pm 1 \pmod{6}$. Let $c_3(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 2, \pm 3 \pmod{12}$. Then $c_1(n) = c_2(n) = c_3(n)$ for all $n$.

From the $(1,1)$-module, Capparelli obtained the companion identity:

**Theorem 2** (Capparelli’s second partition identity). Let $d_1(n)$ denote the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of $n$ wherein

- $\lambda_i \neq 2$ for $i = 1, 2, \ldots, l$,
- $\lambda_i - \lambda_{i+1} \geq 2$, for $i = 1, 2, \ldots, l - 1$,
- $\lambda_i - \lambda_{i+1} = 2$ only if $\lambda_i \equiv 1 \pmod{3}$,
- $\lambda_i - \lambda_{i+1} = 3$ only if $\lambda_i \equiv 0 \pmod{3}$.

Let $d_2(n)$ denote the number of partitions of $n$ into distinct parts $\neq \pm 2 \pmod{6}$.

Then $d_1(n) = d_2(n)$ for all $n$.

### 3.1 $\chi(\Omega(3\Lambda_0))$

In order to use (2.35), we need to consider three cases, $n = 3m$, $3m + 1$, and $3m - 1$.

In (2.35), replace $q$ by $q^3$; then set $a = q$, $n = m$, $f = q^{2-3m}$, and $c = d = e$, to obtain

$$\beta_{3m}^{(3,1)}(1,q) = \frac{q^n(1-q)}{(q)_{3m}(q)_{3m+1}} \lim_{s \to 0^+} \left[ \frac{q^{-7/2}, q^{-7/2}, e, e, e, q^{2-3m}, q^{2-3m}, q^{1-3m}, q^{-3m}}{q^{1/2}, q^{1/2}, q^{1/2}, q^{1/2}, q^{1/2}, q^{1/2}, q^{1/2}, q^{1/2}} \right]$$

$$\sum_{r=-m}^m \frac{(-1)^{m+r}}{(q^2; q^3)_{2m}(q^3; q^3)_{m+r}(q; q^3)_{m-r}(q^3; q^3)_{m-r}}$$

$$\sum_{r=0}^m (-1)^{m+r} q^{2r^2 + 2r} (q^3; q^3)^{m-r}(q^3; q^3)_{m-r} (q^3; q^3)_{3r} \quad (3.1)$$

In (2.35), replace $q$ by $q^3$; then set $a = q$, $n = m$, $f = q^{1-3m}$, and $c = d = e$, to obtain
and with a bit of elementary algebra, documented in [27], one can find that

\[ \beta_{3m+1}(1, q) = \frac{q^n(1 - q)}{(q^n)(q^m+1)} \lim_{s \to 0} \left[ q^{-7/2}, -q^{-7/2}, e, e, e, q^{-1-3m}, q^{1-3m}, q^{-3m}, q^{-3m}, q^{1/2}, -q^{1/2}, \frac{4}{e}, \frac{4}{e}, \frac{4}{e}, q^{3m+5}, q^{3m+5}, q^{3m+4}; q^3, q^3 \right] \]

\[ = \frac{q^{3m+1}(1 - q)}{(q^n)(q^m+1)} \lim_{s \to 0} \left( \frac{q^n, q^m, q^{3m+5}, q^{-3m}}{q^{3m+5}, q^{3m+4}} \right) \]

\[ = \sum_{r = -m}^{m} \frac{(-1)^{m+r} q^{3r^2 + 2r + 1} (q^2; q^4)_r}{(q^2; q^4)_m (q^2; q^4)_m (q^2; q^4)_m} \quad (3.2) \]

For convenience, let us define the abbreviation

\[ \sigma(m, r) := \frac{(-1)^{m} q^{3r^2 + 2r + 1} (q^2; q^4)_r}{(q^2; q^4)_m (q^2; q^4)_m (q^2; q^4)_m} \quad (3.3) \]

so that we have immediately

\[ \beta_{3m}^{(3,1)}(1, q) = \sum_{r = 0}^{2m} \sigma(m, r), \]

and with a bit of elementary algebra,

\[ \beta_{3m+1}^{(3,1)}(1, q) = \sum_{r = 0}^{2m} \frac{\sigma(m, r)}{1 - q^{3m+2}} \left( \frac{1}{1 - q^{3m+1}} - 1 \right), \]

for \( m \geq 0. \)

The author could not find a direct substitution into \( (2.5) \), analogous to the \( n = 3m \) and \( n = 3m + 1 \) cases, which yields the \( n = 3m - 1 \) case. So we resort to an alternate method to obtain the \( n = 3m - 1 \) case.

From the Paule–Riese qZeil Mathematica package available for download at http://www.risc.jku.at/research/combinat/software/qZeil/index.php and documented in [27], one can find that \( \beta_n^{(3,1)}(1, q) \) satisfies the recurrence

\[ \beta_n = \frac{-q^2 + q^{2n} + q^{2n+1}}{q^2(1 - q^{2n})(1 - q^{2n-1})^2} \beta_{n-1} - \frac{1}{(1 - q^{2n})(1 - q^{2n+1})} \beta_{n-2} \quad (3.4) \]

as certified by the rational function

\[ \frac{q^{-n-6r-2} (q^{3r} - q^{n}) (q^{3r+1} - q^{n}) (q^{3r+2} - q^{n})}{(q^n - 1) (q^n + 1) (q^{2n} - q) (q^{3r+1} - 1)} \]

Setting \( n = 3m + 1 \) in \( (3.4) \) and rearranging, we see how to express \( \beta_{3m-1} \) in terms of the two known expressions \( \beta_{3m} \) and \( \beta_{3m+1} \):

\[ \beta_{3m-1} = -(1 - q^{6m+2})(1 - q^{6m+1}) \beta_{3m+1} - (1 - q^{6m} - q^{6m+1}) \beta_{3m+1} \quad (3.5) \]

\[ = -(1 - q^{6m+2})(1 - q^{6m+1}) \sum_{r = 0}^{2m} \frac{\sigma(m, r)}{1 - q^{6m+2}} \left( \frac{1}{1 - q^{3r+1}} - 1 \right) \]
\[ + (1 - q^{6m+1}) \beta_{3m} - (1 - q^{6m} - q^{6m+1}) \beta_{3m} \]
\[ = -(1 - q^{6m+1}) \sum_{r=0}^{2m} \frac{\sigma(m, r)}{1 - q^{3r+1}} + q^{6m} \beta_{3m} \]
\[ = \sum_{r=0}^{2m} \sigma(m, r) \left( q^{6m} - \frac{1 - q^{6m+1}}{1 - q^{3r+1}} \right), \quad (3.6) \]

for \( m \geq 1 \).

Inserting \( \left( \alpha_n^{(3,1)}(1, q), \beta_n^{(3,1)}(1, q) \right) \) into (1.5) with \( a = 1 \), and applying (1.1) and (1.2), we find that
\[
\sum_{m=0}^{\infty} q^{9m^2} \left( q^{-6m+1} \beta_{3m-1}^{(3,1)}(1, q) + \beta_{3m}^{(3,1)}(1, q) + q^{6m+1} \beta_{3m+1}^{(3,1)}(1, q) \right) \\
= \sum_{m=0}^{\infty} \sum_{r=0}^{2m} q^{9m^2} \sigma(m, r) \left( q^{1-6m} \left( q^{6m} - \frac{1 - q^{6m+1}}{1 - q^{3r+1}} \right) + 1 \right. \\
+ \left. \frac{q^{6m+1}}{1 - q^{9m+2}} \left( 1 - \frac{1}{1 - q^{3r+1}} \right) \right) \\
= \frac{1}{(q, q^3, q^6, q^{10}; q^{12})_{\infty}} - \frac{q^2}{(q^2, q^5, q^9, q^{10}; q^{12})_{\infty}} \\
= (-q^2; q^2)_{\infty} (-q^3; q^6)_{\infty}. \quad (3.7) \]

The series expansion of \((q^2, q^3, q^6, q^{10}; q^{12})_{\infty}^{-1}\) in (3.7) is quite different from others that have appeared in the literature, due to Alladi, Andrews, and Gordon [28] pp. 648–649, Lemma 2(b)], (cf. [25] p. 399, Eq. (1.3)), the author [28] p. 399, Eq. (1.4) and Eq. (1.5)], and Bringmann and Mahlburg [12].

3.2 \( \chi(\Omega(\Lambda_0 + \Lambda_1)) \)

In light of (2.3), it is trivial to obtain \( \beta_n^{(3,2)}(1, q) \) from \( \beta_n^{(3,1)}(1, q) \), and upon inserting \( \left( \alpha_n^{(3,2)}(1, q), \beta_n^{(3,2)}(1, q) \right) \) into (1.5) with \( a = 1 \), and applying (1.1) and (1.2), we find that
\[
\sum_{m=0}^{\infty} \sum_{r=0}^{2m} q^{9m^2-3m} \sigma(m, r) \left( q^{2-6m} \left( q^{6m} - \frac{1 - q^{6m+1}}{1 - q^{3r+1}} \right) + 1 \right. \\
+ \left. \frac{q^{6m}}{1 - q^{9m+2}} \left( 1 - \frac{1}{1 - q^{3r+1}} \right) \right) \\
= \frac{1}{(q^2, q^4, q^6; q^{12})_{\infty}} - \frac{q^2}{(q^2, q^4, q^6, q^{10}; q^{12})_{\infty}} \\
= (-q^2; q^2)_{\infty} (-q^3; q^6)_{\infty}. \quad (3.8) \]

3.3 Nandi’s recent work on level 4

It should be noted that recently D. Nandi, in his Ph.D. thesis [25] conjectured the partition identities corresponding to the three inequivalent level 4 standard modules \((4, 0), (2, 1)\) and \((0, 2)\). These identities, while still
in the spirit of the Rogers–Ramanujan and Capparelli identities, involve difference conditions that are much more complicated than anything that has been considered previously in the theory of partitions. It is no wonder that after Capparelli’s discoveries for level 3, it took a quarter century to successfully perform the analogous feat for level 4.

4 Bailey pairs for levels 3 through 9 summarized

4.1 Level 3

\[
\beta^{(3,1)}_{3m} (1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2 - \frac{3}{2}m + 3mr + 2r^2 - \frac{1}{2}r} (q^2; q^3)^{2m}}{(q)_{2m} (q^2; q^3)_{m+r}} \left[ \frac{2m}{m+r} \right] q^3 \\
= \sum_{r=0}^{2m} \frac{(-1)^r q^{2r^2 - \frac{1}{2}r} (q^2; q^3)^{2m-r}(q^2; q^3)_{2m-r}}{(q^3)^{2m} (q^2; q^3)_{m+r}} 
\]

\[
\beta^{(3,1)}_{3m+1} (1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2 + 3mr + 2r^2 + \frac{1}{2}r + 1} (q^2; q^3)^{2m}}{(q)_{6m+1} (q^2; q^3)_{m+r}} \left[ \frac{2m}{m+r} \right] q^3 \\
= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2 - 3mr + 3r^2} (q^2; q^3)^{2m-r}(q^2; q^3)_{2m-r}}{(q^3)^{2m+1} (q^2; q^3)_{m+r+1}} 
\]

\[
\beta^{(3,1)}_{3m+1} (1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2 + 3mr + 2r^2 + \frac{1}{2}r + 1} (q^2; q^3)^{2m}}{(q)_{6m+1} (q^2; q^3)_{m+r+1}} \left[ \frac{2m}{m+r} \right] q^3 \\
= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2 - 3mr + 3r^2 + 3r + 1} (q^2; q^3)^{2m-r}(q^2; q^3)_{2m-r}}{(q^3)^{2m+1} (q^2; q^3)_{m+r+1}} 
\]

\[
q^{3m} \beta^{(3,1)}_{3m+1} = -(1-q^{6m+1})(1-q^{6m+2})(1-q^{6m}) \beta^{(3,1)}_{3m+1} - (1-q^{6m}-q^{6m+1}) \beta^{(3,1)}_{3m+1} 
\]

4.2 Level 4

\[
\beta^{(4,1)}_{3m} (1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2 + 3mr + 2r^2 - \frac{1}{2}r} (q^2; q^3)^{2m}}{(q)_{6m} (q^2; q^3)_{m+r}} \left[ \frac{2m}{m+r} \right] q^3 \\
= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2 - 3mr + 3r^2} (q^2; q^3)^{2m-r}(q^2; q^3)_{2m-r}}{(q^3)^{2m} (q^2; q^3)_{m+r}} 
\]

\[
\beta^{(4,1)}_{3m+1} (1,q) = \sum_{r=-m}^{m} \frac{(-1)^{m+r} q^{3m^2 + 3mr + 2r^2 + \frac{1}{2}r + 1} (q^2; q^3)^{2m}}{(q)_{6m+1} (q^2; q^3)_{m+r+1}} \left[ \frac{2m}{m+r} \right] q^3 \\
= \sum_{r=0}^{2m} \frac{(-1)^r q^{3m^2 - 3mr + 3r^2 + 3r + 1} (q^2; q^3)^{2m-r}(q^2; q^3)_{2m-r}}{(q^3)^{2m+1} (q^2; q^3)_{m+r+1}} 
\]

\[
q^{3m} \beta^{(4,1)}_{3m+1} = -(1-q^{6m+1})(1-q^{6m+2})(1+q^{6m}) \beta^{(4,1)}_{3m+1} - q^{3m} \beta^{(4,1)}_{3m+1} 
\]
4.3 Level 5
\[
\beta_{3m}^{(5,1)}(1, q) = \frac{q^{3m}}{(q)_{6m}}, \\
\beta_{3m+1}^{(5,1)}(1, q) = \frac{q^{3m+6m+1}}{(q)_{6m+2}},
\]
and
\[
q^{6m-1}\beta_{3m-1} = (1 - q^{6m})(1 - q^{6m-1})\beta_{3m},
\]
which, together, simplifies to Eq. (2.2).

4.4 Level 6
\[
\beta_{3m}^{(6,1)}(1, q) = \frac{q^{3m}(-1; q)_{3m}}{(-1; q^{3})(q)_{6m}}, \\
\beta_{3m+1}^{(6,1)}(1, q) = \frac{q^{3m+1}(-1; q^{3})_{3m+1}}{(q)_{6m+2}(-1; q^{3})_{3m+1}},
\]
and
\[
(q + q^{6m-1} - q^{3m})\beta_{3m-1} = (1 - q^{6m})(1 - q^{6m-1})\beta_{3m},
\]
and thus (2.3) holds.

4.5 Level 7
\[
\beta_{3m}^{(7,1)}(1, q) = \frac{q^{3m}}{(q)_{6m}}, \\
\beta_{3m+1}^{(7,1)}(1, q) = \frac{q^{3m+1}}{(q)_{6m+2}},
\]
and
\[
q\beta_{3m-1} = (1 - q^{6m})(1 - q^{6m-1})\beta_{3m},
\]
and thus (2.3) holds.

4.6 Level 8
\[
\beta_{3m}^{(8,1)}(1, q) = \sum_{r = -m}^{m} \frac{q^{2r^2 + \frac{1}{2}r + 3m + 1}(-q^{3}; q^{3})_{r}(q^{2}; q^{3})_{2m}}{(q)_{6m+1}(-q^{2}; q^{3})_{m}(q^{3}; q^{3})_{m+r}} \begin{bmatrix} 2m \cr m + r \end{bmatrix} q^{3r},
\]
\[
= \sum_{r = 0}^{2m} \frac{q^{3m+2+3m^2 + r - 6mr}}{(-q^{3}; q^{3})_{m-r}(q^{3}; q^{3})_{2m}(-q^{2}; q^{3})_{m}(q^{3}; q^{3})_{m+r}} \begin{bmatrix} 2m \cr m + r \end{bmatrix} q^{3r},
\]
\[
\beta_{3m+1}^{(8,1)}(1, q) = \sum_{r = -m}^{m} \frac{q^{2r^2 + \frac{1}{2}r + 3m + 1}(-q^{3}; q^{3})_{r}(q^{2}; q^{3})_{2m}}{(q)_{6m+1}(-q^{2}; q^{3})_{m}(q^{3}; q^{3})_{m+r+1}} \begin{bmatrix} 2m \cr m + r \end{bmatrix} q^{3r},
\]
\[
= \sum_{r = 0}^{2m} \frac{q^{3m+2+3m^2 + r - 6mr+1}}{(-q^{3}; q^{3})_{m-r}(q^{3}; q^{3})_{2m+1}(-q^{2}; q^{3})_{m}(q^{3}; q^{3})_{m+r+1}} \begin{bmatrix} 2m \cr m + r \end{bmatrix} q^{3r},
\]
\[
q^{2}\beta_{3m-1} = -(1-q^{6m+1})(1-q^{6m+2}) + q(1+q+q^{3m}-q^{6m+1})\beta_{3m}.
\]
4.7 Level 9

\[ \beta_{3m}^{(9,1)}(1, q) = \sum_{r=-m}^{m} \frac{q^{3r^2 + r + 3m}(q^3; q^3)_{2m}}{(q)_{6m}(q^2; q^3)_{m + r}} \left[ \frac{2m}{m + r} \right] q^3 \]

\[ = \sum_{r=0}^{2m} q^{3m^2 + 2m + 3r + 6mr} \beta_{3m+1}^{(9,1)}(1, q) = \sum_{r=0}^{2m} (q; q^3)_{2m}(q^2; q^3)_r(q^3; q^3)_{2m-r}(q^3; q^3)_r, \]

\[ q^3 \beta_{3m-1} = -(1 - q^{6m+1})(1 - q^{6m+2})\beta_{3m+1} + q(1 + q - q^{6m+1})\beta_{3m}. \]

5 Conclusion and Open Questions

It is the hope of the author that the results presented here will help to provide some insight into the structure of \( A_2^{(2)} \) that can be exploited by vertex operator algebraists. Questions of course remain. For instance, as pointed out by Ole Warnaar during the question-and-answer period following my talk at the Alladi conference, it is not at all clear how the series expressions in \( (3.7) \) and \( (3.8) \) enumerate the partition functions \( c_1(n) \) and \( c_2(n) \) respectively. It would be very nice indeed if this connection could be established. Christian Krattenthaler pointed out that it was conceivable that there are other families of Bailey pairs that could give rise to identities with the same product sides. While this is true, the choice of the \( \alpha_n \) employed here is motivated by classical work; in particular the level 5 and 7 identities and the corresponding Bailey pairs coincide with the work of Slater [30, 31]. Further the availability of the Andrews–Baxter–Forrester transformation to express the \( \beta_n \) as a multisum for any level \( \ell \) is an encouraging sign that this may be a fruitful direction to pursue in the effort to better understand \( A_2^{(2)} \) as a whole.

Acknowledgments

The author thanks Jim Lepowsky and Robert Wilson for assistance with the exposition in Section 1.2.

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