Conformally covariant differential operators for the diagonal action of $O(p, q)$ on real quadrics

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Abstract

Let $X = G/P$ be a real projective quadric, where $G = O(p, q)$ and $P$ is a parabolic subgroup of $G$. Let $(\pi_{\lambda, \epsilon}, \mathcal{H}_{\lambda, \epsilon})_{(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}}$ be the family of (smooth) representations of $G$ induced from the characters of $P$. For $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$, a differential operator $D^{reg}_{(\lambda, \epsilon), (\mu, \eta)}$ on $X \times X$, acting $G$-covariantly from $\mathcal{H}_{\lambda, \epsilon} \otimes \mathcal{H}_{\mu, \eta}$ into $\mathcal{H}_{\lambda+1, -\epsilon} \otimes \mathcal{H}_{\mu+1, -\eta}$ is constructed.

Introduction

Let $S = S^n$ be the sphere of dimension $n$, equipped with its standard Riemannian structure. The group $G = O(1, n + 1)$ acts conformally on $S$. For $\lambda \in \mathbb{C}$, let

$$\mathcal{H}_\lambda = \{ f(x)(dx)^{\frac{n}{\lambda}} \}, \quad f \in C^\infty(S)$$

be the space of smooth $\frac{n}{\lambda}$-densities. The space $C^\infty(S)$ correspond to $\lambda = 0$, whereas the space of measures on $S$ having a smooth density with respect to the Lebesgue measure $dx$ on $S$ corresponds to $\lambda = n$. The natural action of $G$ on $\mathcal{H}_\lambda$ induces a (smooth) representation $\pi_\lambda$ of $G$ on $\mathcal{H}_\lambda$. The family $(\pi_\lambda)_{\lambda \in \mathbb{C}}$ is known in semisimple harmonic analysis as the scalar principal series of representations of $G$.

Now let $G$ act diagonally on $S \times S$. The tensor product $\pi_\lambda \otimes \pi_\mu$ of two representations of the principal series has a natural realization on a space $\mathcal{H}_{\lambda, \mu}$ of sections of a certain line bundle over $S \times S$. In [1], R. Beckmann and the present author constructed a family of differential operators on $S \times S$, depending on two complex parameters $(\lambda, \mu)$, which are covariant with respect to $(\pi_\lambda \otimes \pi_\mu, \pi_{\lambda+1} \otimes \pi_{\mu+1})$. The construction of these operators uses the heavy machinery of Knapp-Stein intertwining operators (see [6] for a general presentation). Whereas the covariance property of the operators is intrinsic to their definition, the fact that they are differential operators is
much more involved. The problem is transferred (by using a stereographic projection) to the non-compact picture or flat model $\mathbb{R}^n \times \mathbb{R}^n$, and is solved through a long computation, using the Fourier transform on $\mathbb{R}^n$. See also [3] Section 11 for a slightly different presentation of these results. This procedure was generalized recently to the geometric framework of completion of simple real Jordan algebras (see [2]).

The present paper gives a more elementary construction of these operators in the geometric setting of the real quadrics. The philosophy behind the present construction is based on the following observation. Let $X$ be a real quadric, and let $G$ be its group of conformal transformations. Then $G$ has an open dense orbit in its diagonal action on $X \times X$ which is a reductive symmetric space (see Proposition 3.1 for a more explicit statement). This rich underlying geometric structure explains that it is easy to construct $G$-covariant differential operators on this open orbit. The next question is to study whether such a differential operator can be smoothly extended to $X \times X$.

The real quadric $X$ is realized as the projective variety associated to the isotropic cone $\Xi$ of the ambient space $(V, Q)$, where $Q$ is a quadratic form on a real vector $V$. The group of conformal transformations of $X$ is $O(Q)$, acting projectively on $X$. To construct covariant differential operators on $X$ (or on $X \times X$), it is wise to start with a homogenous $G$-invariant differential operator on $V$ (or $V \times V$) and try to induce a differential operator on $\Xi$ (or $\Xi \times \Xi$). This is possible only if the operator on $V$ is “tangential along $\Xi$” (or along $\Xi \times \Xi$). The corresponding verification is obtained through computations in the Weyl algebra (= algebra of differential operators with polynomial coefficients) of $V$.

To finish this introduction, let us mention an application of these operators, which is not developed in this article. By restriction to the diagonal, they provide covariant bi-differential operators from $X \times X$ to $X$. As it is possible to compose (appropriate) covariant differential operators on $X \times X$, the restriction process also yields higher order covariant bi-differential operators. These bi-differential operators are generalizations of the classical Rankin-Cohen brackets (see [1] Theorem 3.4 or [2] Section 8). A similar approach for Juhl’s conformally covariant differential operators from $S^n$ to $S^{n-1}$ was proposed in [4].
1 The real quadric and a series of representations of $O(p, q)$

Let $V$ be a real vector space of dimension $n = p + q$ where $p, q$ are natural integers such that $p, q \geq 1, p + q \geq 3$, and let $Q$ be a quadratic form on $V$ of signature $(p, q)$. Choose a basis $e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n$ such that the quadratic form $Q$ is given by

$$Q(v) = Q(x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2.$$ 

The corresponding symmetric bilinear form will be denoted also by $Q$, namely for $v = (x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_n)$ and $w = (y_1, \ldots, y_p, y_{p+1}, \ldots, y_n)$

$$Q(v, w) = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_ny_n.$$ 

For $v \in V, v \neq 0$, let $[v] = \mathbb{R}^* v$ be its corresponding element in the projective space $\mathbb{P}(V)$.

Consider the proper isotropic cone $\Xi = \{ v \in V, v \neq 0, Q(v) = 0 \}$. For $v \neq 0$, the differential $dQ(v) = 2Q(v, \cdot)$ is $\neq 0$ and hence $Q = 0$ is a regular equation of $\Xi$ near any point of $\Xi$. The projective quotient $X = \Xi/\mathbb{R}^*$ is a real quadric.

The group $G = O(Q) \simeq O(p, q)$ preserves $\Xi$. As the action of $G$ commutes with the dilations, the group $G$ acts naturally on $X$. As a consequence of Witt theorem, this action is transitive.

An open subset $O$ (resp. $\Omega$) of $V \setminus \{0\}$ (resp. $\Xi$) is said to be conical if $O$ (resp. $\Omega$) is stable by all dilations $v \mapsto rv, r \in \mathbb{R}^*$.

For $\lambda \in \mathbb{C}, \epsilon \in \{\pm\}$ and for $r \in \mathbb{R}^*$, let

$$r^{\lambda, \epsilon} = \begin{cases} 
|r|^{\lambda} & \text{if } \epsilon = + \\
\text{sgn}(r)|r|^{\lambda} & \text{if } \epsilon = - 
\end{cases}.$$ 

Let $O$ be a conical open subset of $V \setminus \{0\}$, and let $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$. Set

$$\mathcal{F}_{\lambda, \epsilon}(O) = \{ F \in C^\infty(O), F(rv) = r^{-\lambda \epsilon}F(v) \quad \forall r \in \mathbb{R}^*, v \in O \}.$$ 

Similarly, for $\Omega$ a conical open subset of $\Xi$, let

$$\mathcal{H}_{\lambda, \epsilon}(\Omega) = \{ F \in C^\infty(\Omega), F(rv) = r^{-\lambda \epsilon}F(v), \quad \forall r \in \mathbb{R}^*, v \in \Omega \},$$

and simply let $\mathcal{H}_{\lambda, \epsilon} = \mathcal{H}_{\lambda, \epsilon}(\Xi)$, equipped with its natural Fréchet topology.
For $g \in G$, and $F \in \mathcal{H}_{\lambda,\epsilon}$, let
\[ \pi_{\lambda,\epsilon}(g) F = F \circ g^{-1}. \]
Then $\pi_{\lambda,\epsilon}(g) F$ belongs to $\mathcal{H}_{\lambda,\epsilon}$ and this defines a (smooth) representation $\pi_{\lambda,\epsilon}$ of $G$ on $\mathcal{H}_{\lambda,\epsilon}$.

Homogenous functions on $\Xi$ are interpreted as sections of a corresponding line bundle on $X$, and conversely, differential operators for these line bundles over $X$ are viewed as differential operators acting on homogenous functions on $\Xi$. These identifications are tacitly used in the sequel.

### 2 The covariant differential operator $\widetilde{\Box}$

Let $q \in \mathbb{C}[V]$ be a polynomial on $V$. There is a unique constant coefficients differential operator, denoted by $q \left( \frac{\partial}{\partial x} \right)$ such that for any $y \in V$
\[ q \left( \frac{\partial}{\partial x} \right) e^{Q(x,y)} = q(y) e^{Q(x,y)}. \]

The operator $q \left( \frac{\partial}{\partial x} \right)$ is $G$-invariant (i.e. commutes with the action of $G$) if and only if $q$ is a $G$-invariant polynomial on $V$. Choosing $q = Q$, this yields the d’Alembertian operator $\Box = Q \left( \frac{\partial}{\partial x} \right)$, which in the coordinates $(x_1, x_2, \ldots, x_n)$ reads
\[ \Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_n^2}. \]

The Weyl algebra is the algebra of differential operators on $V$ having polynomial coefficients. For $q \in \mathbb{C}[V]$, the multiplication operator by $q$ is simply denoted by $q$ or $q(x)$, depending on the context. The composition of operators in the Weyl algebra is usually denoted by $\circ$. However when multiplication by a polynomial is performed after a constant coefficient differential operators, the symbol $\circ$ may be omitted.

The construction of covariant differential operators on the quadric $X$ is well-known (see e.g. [5]) and is recalled here, as it is used and serves as a model for the more elaborate constructions to come.

**Lemma 2.1.**
\[ \Box \circ Q = 2n + 4E + Q \Box, \]
where $E$ is the Euler operator given by $E = E \left( x, \frac{\partial}{\partial x} \right) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$.

Proof. Straightforward computation. □

Let $F \in \mathcal{H}_{\lambda, \epsilon}$. It is possible to extend $F$ to a function $\overline{F} \in \mathcal{F}_{\lambda, \epsilon}(\mathcal{O})$ for $\mathcal{O}$ a conical neighborhood of $\Xi$ in $V \setminus \{0\}$. The restriction of $\Box \overline{F}$ to $\Xi$ belongs to $\mathcal{H}_{\lambda + 2, \epsilon}$. However, the extension is not unique and the restriction of $\Box \overline{F}$ to $\Xi$ usually depends on the extension.

Proposition 2.1. Let $F \in \mathcal{H}_{\frac{n}{2} - 2, \epsilon}$. Let $\overline{F}$ an extension of $F$ to a conical neighborhood of $\Xi$ as above. Then the restriction of $\Box \overline{F}$ to $\Xi$ only depends on the values of $F$ on $\Xi$.

Proof. It is enough to show that if $\overline{F}$ vanishes on $\Xi$, then $\Box \overline{F}$ vanishes on $\Xi$. But such a function can be written as $\overline{F} = QG$, where $G$ is defined in a conical neighborhood $\mathcal{O}$ of $\Xi$ and satisfies $G(rx) = r^{-\frac{n}{2} - \epsilon}G(x)$ for all $r \in \mathbb{R}^*$ and $x \in \mathcal{O}$. But differentiating this relation at $r = 1$ yields $EG = -\frac{n}{2}G$. Hence by Lemma 2.1 $\Box(QG) = Q\Box G$, which implies for $x \in \Xi$

$$\Box F(x) = Q(x) \Box G(x) = 0.$$ □

Proposition 2.1 defines a differential operator on $X$

$$\Box : F \mapsto \overline{F} \mapsto \Box \overline{F} \mapsto \Box \overline{F}|_{\Xi},$$

mapping $\mathcal{H}_{\frac{n}{2} - 2, \epsilon}$ into $\mathcal{H}_{\frac{n}{2}, \epsilon}$. Moreover, as $\Box$ commutes with the natural action of $G$ on functions, the operator $\Box$ intertwines $\pi_{\frac{n}{2} - 2, \epsilon}$ and $\pi_{\frac{n}{2}, \epsilon}$.

3 The operators $D_{(\lambda, \epsilon), (\mu, \eta)}$ on $(X \times X)^\times$.

Let

$$(V \times V)^\times = \{ (x, y) \in V \times V, \quad Q(x, y) \neq 0 \}.$$

Clearly, $(V \times V)^\times$ is a conical dense open subset of $V \times V$, which is invariant under the diagonal action of $G$ on $V \times V$. Similarly, let

$$(\Xi \times \Xi)^\times = \{ (x, y) \in \Xi \times \Xi, \quad Q(x, y) \neq 0 \}.$$

Consider the corresponding projective situation, i.e. let

$$(X \times X)^\times = (\Xi \times \Xi)^\times / (\mathbb{R}^* \times \mathbb{R}^*).$$
Then $H G \in \text{subspace } (\mathbb{R}^2, \text{diagonal action of } G)$ isomorphic to loosing any generality, it is possible to assume that $Q$ be the transformation which is $+1$ on $Q$ and the restriction of $Q$ to ($X \times X$) such that $Q(x', y') = 1$. Recall that $Q(x) = Q(x') = 0$ and $Q(y) = Q(y') = 0$.

By Witt theorem, there exists an isometry $g$ of $(V, Q)$ such that $g(x) = x', g(y) = y'$, thus proving the first part of the proposition.

Next, let $(x, y) \in (V \times V)^\times$. The restriction of $Q$ to the 2-subspace $\mathbb{R}^x \oplus \mathbb{R}^y$ is of signature $(1, -1)$. Hence

$$V = (\mathbb{R}^x \oplus \mathbb{R}^y) \oplus (\mathbb{R}^x \oplus \mathbb{R}^y)^\perp,$$

and the restriction of $Q$ to $\mathbb{R}^x \oplus \mathbb{R}^y$ is of signature $(p - 1, q - 1)$. Let $\sigma$ be the transformation which is $+1$ on $\mathbb{R}^x \oplus \mathbb{R}^y$ and $-1$ on $(\mathbb{R}^x \oplus \mathbb{R}^y)^\perp$. If $g \in G$ stabilizes both $[x]$ and $[y]$, then $g$ stabilizes $\mathbb{R}^x \oplus \mathbb{R}^y$ and its orthogonal subspace $(\mathbb{R}^x \oplus \mathbb{R}^y)^\perp$, so that $\sigma \circ g \circ \sigma = g$. Let

$$H = \{g \in G, \sigma \circ g \circ \sigma \}.$$

Then $H \cong O(1, -1) \times O(p - 1, q - 1)$ is a symmetric reductive subgroup of $G$, and the stabilizer $G^{[x][y]}$ of $([x], [y])$ in $G$ is the subgroup of $H$ of index 2, isomorphic to $\mathbb{R}^* \times O(p - 1, q - 1)$.

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$. For $\mathcal{O}$ a conical open set of $(V \times V)^\times$, let

$$\mathcal{F}_{(\lambda, \epsilon)(\mu, \eta)}(\mathcal{O}) = \{F \in C^\infty(\mathcal{O}), \quad F(rx, sy) = r^{-\lambda \epsilon} s^{-\mu \eta} F(x, y)\}$$

for all $(x, y) \in \mathcal{O}$ and $r, s \in \mathbb{R}^*$. Similarly, for $\Omega$ a conical open subset of $(\Xi \times \Xi)^\times$ let $\mathcal{H}_{(\lambda, \epsilon)(\mu, \eta)}(\Omega)$ be the space of all functions $F \in C^\infty(\Omega)$ such that

$$F(rx, sy) = r^{-\lambda \epsilon} s^{-\mu \eta} F(x, y), \quad \text{for all } (x, y) \in \Omega \text{ and } r, s \in \mathbb{R}^*.$$ 

The space corresponding to $\Omega = (\Xi \times \Xi)^\times$ is denoted by $\mathcal{H}_{(\lambda, \epsilon)(\mu, \eta)}^\times$. The diagonal action of $G$ on $\Xi \times \Xi$ induces a representation of $G$ on $\mathcal{H}_{(\lambda, \epsilon)(\mu, \eta)}^\times$. 

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Consider the differential operator on \((V \times V)^\times\) given by

\[ E_{\lambda, \mu} = \text{sgn}(Q(x, y)) \times |Q(x, y)|^{\frac{n}{2} - \mu - 1} \circ \Box \circ |Q(x, y)|^{-\lambda + \mu} \circ \Box \circ |Q(x, y)|^{-\frac{n}{2} + 2 + \lambda}. \]

The operator \(E_{\lambda, \mu}\) is well defined on \(C^\infty((V \times V)^\times)\), and commutes with the diagonal action of \(G\) on \((V \times V)^\times\). Let \(F \in \mathcal{H}^\times_{(\lambda, \epsilon), (\mu, \eta)}\). Extend it to a function \(\mathcal{F} \in \mathcal{F}_{(\lambda, \epsilon), (\mu, \eta)}(\mathcal{O})\) where \(\mathcal{O}\) is conical neighborhood of \((\Xi \times \Xi)^\times\) in \((V \times V)^\times\). For \(y\) fixed, the function

\[ G_y : x \mapsto |Q(x, y)|^{-\frac{n}{2} + 2 + \lambda} \mathcal{F}(x, y) \]

is defined and smooth on a conical neighborhood of \(\Xi_y = \{ x \in \Xi, Q(x, y) \neq 0 \}\) and homogeneous of degree \(-\frac{n}{2} + 2\). Hence, by (a localized version of) Proposition 2.1, the restriction to \(\Xi_y\) of \(\Box \left( \frac{\partial}{\partial x} \right) G_y\) depends only on the values of \(\mathcal{F}\) on \(\Xi_y\).

Similarly, for \(x\) fixed the function

\[ H_x : y \mapsto |Q(x, y)|^{-\lambda + \mu} \Box \left( \frac{\partial}{\partial x} \right) \left( |Q(x, y)|^{-\frac{n}{2} + 2 + \lambda} \mathcal{F}(x, y) \right) \]

is defined and smooth on a conical neighborhood of \(\Xi_x\) and homogeneous of degree \(-\frac{n}{2} + 2\). Hence, by the same argument as above, the restriction to \(\Xi_x\) of the function \(\Box \left( \frac{\partial}{\partial y} \right) H_x\) depends only on the values of \(\mathcal{F}\) on \(\Xi_x\).

These observations and some elementary verifications about the homogeneity and the action of \(G\) yields the following proposition.

**Proposition 3.2.** The operator \(E_{\lambda, \mu}\) induces a differential operator

\[ D_{(\lambda, \epsilon), (\mu, \eta)} : \mathcal{H}^\times_{(\lambda, \epsilon), (\mu, \eta)} \longrightarrow \mathcal{H}^\times_{(\lambda + 1, -\epsilon), (\mu + 1, -\eta)}. \]

The induced operator commutes with the natural actions of \(G\) on each of the function spaces involved.

To have a better understanding of the behavior of the operator \(E_{\lambda, \mu}\) near the singular set where \(Q(x, y) = 0\), a more explicit expression of the operator \(E_{\lambda, \mu}\) is needed.
Proposition 3.3. The following identity holds on \((V \times V)^\times\)

\[
E_{\lambda, \mu} = 
\left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \lambda\right) \left(-\frac{n}{2} + \mu\right) \left(-\frac{n}{2} + \mu - 1\right) Q(x, y)^{-3} Q(x) Q(y) 
\] (I)

\[
+ 2 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \lambda\right) Q(x, y)^{-2} Q(y) \circ \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j}\right) 
\] (II)

\[
+ \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \lambda\right) (2n - 4 + 4\mu) \ Q(x, y)^{-1} 
\] (III)

\[
+ 4 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \lambda\right) Q(x, y)^{-1} Q(y) \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j}\right) 
\] (IV)

\[
+ \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \lambda\right) Q(x, y)^{-1} Q(y) Q \left(\frac{\partial}{\partial y}\right)
\] (V)

\[
+ 2 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \mu\right) \left(-\frac{n}{2} + \mu\right) Q(x) Q(x, y)^{-2} \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j}\right) 
\] (VI)

\[
+ 2 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \mu\right) Q(x, y)^{-1} \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right)
\] (VII)

\[
+ 2 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-\frac{n}{2} + 1 + \mu\right) Q(x, y)^{-1} \left(\sum_{j=1}^{n} \sum_{k=1}^{n} x_j y_k \frac{\partial^2}{\partial x_k \partial y_j}\right)
\] (VIII)

\[
+ 2 \left(-\frac{n}{2} + 2 + \lambda\right) \left(-2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j \partial y_j} + \sum_{k=1}^{n} y_k \frac{\partial}{\partial x_k}\right) \circ Q \left(\frac{\partial}{\partial y}\right)
\] (IX)

\[
+ \left(-\frac{n}{2} + 2 + \mu\right) \left(-\frac{n}{2} + 1 + \mu\right) Q(x, y)^{-1} Q(x) \left(\frac{\partial}{\partial x}\right)
\] (X)

\[
+ 2 \left(-\frac{n}{2} + 2 + \mu\right) \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j}\right) \circ Q \left(\frac{\partial}{\partial x}\right)
\] (XI)

\[
+ Q(x, y) Q \left(\frac{\partial}{\partial y}\right) Q \left(\frac{\partial}{\partial x}\right)
\] (XII).

**Proof.** Computations are first made on \(\{(x, y) \in V \times V, Q(x, y) > 0\}\) and it will be indicated at the end how to handle the situation when \(Q(x, y) < 0\). With this extra assumption, it is possible, for \(\rho\) any complex number, to
replace $|Q(x, y)|^\rho$ by simply $Q(x, y)^\rho$. An intermediate calculation yields

$$Q(x, y)^{-\lambda+\mu} \circ Q \left( \frac{\partial}{\partial x} \right) \circ Q(x, y)^{\frac{n}{2}+2+\lambda} =$$

$$(-\frac{n}{2} + 2 + \lambda)(\frac{n}{2} + 1 + \lambda)Q(x, y)^{-\frac{n}{2}+\mu} \circ Q(y)$$

$$+ 2(-\frac{n}{2} + 2 + \lambda)Q(x, y)^{-\frac{n}{2}+1+\mu} \circ \left( \sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)$$

$$+ Q(x, y)^{-\frac{n}{2}+2+\mu} \circ Q \left( \frac{\partial}{\partial x} \right).$$

After a long but straightforward computation, the formula of Proposition 3.3 is obtained.

To finish the proof, it is enough to justify that the same formula is valid on the domain where $Q(x, y) < 0$. To see this, let $Q' = -Q$ be the opposite quadratic form, and let $E_{\lambda, \mu}'$ be the differential operator obtained from $Q'$ by the same procedure as for obtaining $E_{\lambda, \mu}$ from $Q$. If $Q(x, y) < 0$, $Q'(x, y) > 0$, so that the previous computation can be used to evaluate $E_{\lambda, \mu}'$ using $Q'$ instead of $Q$. Now each term (from (I) to (XII) corresponding to the explicit expression of $E_{\lambda, \mu}'$, can be rewritten using $Q = -Q'$. But each of the twelve terms labeled from (I) to (XII) is easily seen to be changed to its opposite when changing $Q'$ to $Q = -Q'$. The conclusion follows as it is easily seen directly from their definition that $E_{\lambda, \mu}' = -E_{\lambda, \mu}$.

4 The operators $D^{reg}_{(\lambda, \epsilon)(\mu, \eta)}$ on $X \times X$

For reasons to be explicited later, the term labeled (VIII) has to be written differently. Let

$$E_{(VIII)} = Q(x, y)^{-1} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j y_k \frac{\partial^2}{\partial x_k \partial y_j},$$

and

$$E \left( x, \frac{\partial}{\partial y} \right) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial y_j}, \quad E \left( y, \frac{\partial}{\partial x} \right) = \sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j}.$$
Lemma 4.1.
\[
\left[ E_{(VIII)} , Q(x) \right] = 2E \left( x, \frac{\partial}{\partial y} \right) \quad (1)
\]
\[
\left[ Q \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) , Q(x) \right] = 2E \left( x, \frac{\partial}{\partial y} \right) \quad . \quad (2)
\]

Consider the differential operator
\[
F = E_{(VIII)} - Q \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)
\]

Proposition 4.1.
\[
F \circ Q(x) = Q(x) \circ F , \quad F \circ Q(y) = Q(y) \circ F . \quad (3)
\]

Proof. Use Lemma 4.1, the second half of the statement being obtained by exchanging \( x \) and \( y \). \( \square \)

Proposition 4.2. Let \( f \) be a smooth function on \((\Xi \times \Xi)^\times\). Let \( \overline{f} \) be a smooth extension of \( f \) to a conical neighborhood \( \mathcal{O} \) of \((\Xi \times \Xi)^\times \) in \((V \times V)^\times\). The restriction of \( F(\overline{f}) \) to \((\Xi \times \Xi)^\times \) depends only on \( f \) and not of the particular extension used.

Proof. It is enough to prove that if \( \overline{f} \) vanishes on \((\Xi \times \Xi)^\times\), then \( F(\overline{f}) \) vanishes on \((\Xi \times \Xi)^\times\). Now such a function \( \overline{f} \) can be written (in a conical neighborhood \( \mathcal{O} \) of a given ray \( \mathbb{R}^\times(x_0, y_0) \subset (\Xi \times \Xi)^\times \)) as
\[
\overline{f}(x, y) = Q(x)g(x, y) + Q(y)h(x, y) .
\]
where \( g \) and \( h \) are smooth functions on \( \mathcal{O} \). From Lemma 4.1 follows
\[
[ F , Q(x) ] = [ F , Q(y) ] = 0
\]
and hence
\[
F(\overline{f}) = Q(x)F(g) + Q(y)F(h) .
\]
When \((x, y)\) belongs to \((\Xi \times \Xi)^\times\), then \( Q(x) = Q(y) = 0 \) and the proposition follows. \( \square \)

Consider now the following decomposition
\[
E_{\lambda,\mu} = \text{E}^{\text{sing}}_{\lambda,\mu} + \text{E}^{\text{reg}}_{\lambda,\mu}
\]
where
\[
\text{E}^{\text{sing}}_{\lambda,\mu} = (I) + \cdots + (V II) + 2(-\frac{n}{2} + 1 + \lambda)(-\frac{n}{2} + \mu)F + (X)
\]
\[
\text{E}^{\text{reg}}_{\lambda,\mu} = 2(-\frac{n}{2} + 1 + \lambda)(-\frac{n}{2} + \mu) Q \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) + (IX) + (XI) + (XII)
\]
Proposition 4.3. Let $\mathcal{O}$ be a conical neighborhood of $(\Xi \times \Xi)^\times$, and let $f \in \mathcal{F}(\lambda, \epsilon, \mu, \eta)(\mathcal{O})$. Then the restriction of $E_{\lambda, \mu}^{\text{reg}} f$ to $(\Xi \times \Xi)^{\text{reg}}$ only depends on the restriction of $f$ to $(\Xi \times \Xi)^{\text{reg}}$.

Proof. From Proposition 3.2, it is equivalent to prove the similar statement for $E_{\lambda, \mu}^{\text{sing}}$. As already argued above, it is sufficient to prove that if $f$ vanishes on $(\Xi \times \Xi)^\times$, then $E_{\lambda, \mu}^{\text{sing}} f$ vanishes on $(\Xi \times \Xi)^\times$. In the expression of $E_{\lambda, \mu}^{\text{sing}} f$, terms corresponding to factors (I), (II), (V), (VI), (X) vanish on $(\Xi \times \Xi)^{\text{reg}}$ as they contain either a factor $Q(x)$ or $Q(y)$. Terms (III), (IV) and (VII) when evaluated on $f$ vanish on $(\Xi \times \Xi)^{\text{reg}}$. This is trivially true for term (III). For terms (IV) and (VII), use the homogeneity condition of $f$ to justify the statement. The extra term in the definition of $E_{\lambda, \mu}^{\text{sing}}$ is proportional to $F$. But $F(f)$ vanishes on $(\Xi \times \Xi)^\times$ as a consequence of Proposition 4.2. This achieves the proof.

Theorem 4.1. The operator $E_{\lambda, \mu}^{\text{reg}}$ induces a map from $\mathcal{H}(\lambda, \epsilon, \mu, \eta)$ into $\mathcal{H}(\lambda + 1, -\epsilon, \mu + 1, -\eta)$. Viewed as an operator on sections of line bundles over $\Xi \times \Xi$, it is a differential operator $D_{\lambda, \mu}^{\text{reg}}$ which intertwines the representations $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and $\pi_{\lambda + 1, -\epsilon} \otimes \pi_{\mu + 1, -\eta}$.

Proof. Let $f \in \mathcal{H}(\lambda, \epsilon, \mu, \eta)$, extend it, respecting the homogeneities, to a neighborhood of $\Xi \times \Xi$, still denoted by $f$. By Proposition 4.3, the value of $E_{\lambda, \mu}^{\text{reg}} f$ on $(\Xi \times \Xi)^{\text{reg}}$ depends only on the values of $f$ on $(\Xi \times \Xi)^{\text{reg}}$. The operator $E_{\lambda, \mu}^{\text{reg}}$ has polynomial coefficients on $V \times V$. Hence by continuity, the value of $E_{\lambda, \mu}^{\text{reg}} f$ at a point in $(\Xi \times \Xi)$ is well defined and depends only on the values of $f$ on $\Xi \times \Xi$. The invariance property of $E_{\lambda, \mu}^{\text{reg}}$ with respect to the diagonal action of $G$ on $V \times V$ follows immediately from the definition of $E_{\lambda, \mu}^{\text{reg}}$ and the intertwining relation for $D_{\lambda, \mu}^{\text{reg}}$.

Now notice that the term (IX) can be rewritten using

$$\sum_{k=1}^n y_n \frac{\partial}{\partial x_n} = \sum_{k=1}^n (y_k - x_k) \frac{\partial}{\partial x_k} + E \left( x, \frac{\partial}{\partial x} \right).$$

A similar modification is possible for the term (XI).

Define the operator

$$F_{\lambda, \mu} = Q(x, y) \circ Q \left( \frac{\partial}{\partial y} \right) \circ Q \left( \frac{\partial}{\partial x} \right)$$

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\[ +2 \left( -\frac{n}{2} + 1 + \mu \right) \left( \sum_{j=1}^{n} (x_k - y_k) \frac{\partial}{\partial y_k} \right) \circ Q \left( \frac{\partial}{\partial x} \right) \]

\[ +2 \left( -\frac{n}{2} + 1 + \lambda \right) \left( \sum_{j=1}^{n} (y_k - x_k) \frac{\partial}{\partial x_k} \right) \circ Q \left( \frac{\partial}{\partial y} \right) \]

\[ -2\mu \left( -\frac{n}{2} + 1 + \mu \right) Q \left( \frac{\partial}{\partial x} \right) \quad - 2\lambda \left( -\frac{n}{2} + 1 + \lambda \right) Q \left( \frac{\partial}{\partial y} \right) \]

\[ +4 \left( -\frac{n}{2} + 1 + \lambda \right) \left( -\frac{n}{2} + 1 + \mu \right) Q \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \]

**Proposition 4.4.** Let \( f \in \mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)} \). Extend \( f \) to a neighborhood of \( \Xi \times \Xi \), preserving the homogeneities. Then the restriction of \( F_{\lambda, \mu} f \) to \( \Xi \times \Xi \) does not depend on the particular extension of \( f \) and is equal to \( D_{(\lambda, \epsilon), (\mu, \eta)}^{\text{reg}} f \).

**Proof.** Use (4) and the homogeneities of the extension of \( f \) to prove that \( F_{\lambda, \mu} f \) coincides with \( E_{(\lambda, \epsilon), (\mu, \eta)}^{\text{reg}} f \). \( \square \)

**Remark.** The operator \( F_{\lambda, \mu} \) exhibits symmetry with respect to the couples \((x \leftrightarrow y), (\lambda \leftrightarrow \mu)\) which did not exist for the initial operator \( E_{\lambda, \mu} \). In particular, it is possible to start with the operator

\[ \text{sgn}(Q(x, y)) \times \]

\[ |Q(x, y)|^{\frac{1}{2} - \lambda - 1} \circ \Box \left( \frac{\partial}{\partial x} \right) \circ |Q(x, y)|^{\lambda - \mu} \circ \Box \left( \frac{\partial}{\partial y} \right) \circ |Q(x, y)|^{-\frac{n}{2} + 2 + \mu}, \]

which is not equal to \( E_{\lambda, \mu} \). However, the process of regularization produces the same differential operator \( D_{(\lambda, \epsilon), (\mu, \eta)}^{\text{reg}} \) on \( X \times X \).

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