Current production in ring condensates with a weak link

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We consider attractive and repulsive condensates in a ring trap stirred by a weak link, and analyze the spectrum of solitonic trains dragged by the link, by means of analytical expressions for the wave functions, energies and currents. The precise evolution of current production and destruction in terms of defect formation in the ring and in terms of stirring is studied. We find that any excited state can be coupled to the ground state through two proposed methods: either by adiabatically tuning the link’s strength and velocity through precise cycles which avoid the critical velocities and thus unstable regions, or by having the link still while setting an auxiliary potential and imprinting a nonlinear phase as the potential is turned off. We also analyze hysteresis cycles through the spectrum of energies and currents.

I. INTRODUCTION

Condensates in ring geometries present a wealth of superfluid and nonlinear effects and yield a potential for the development of atomtronic devices [1]. Production and decay of supercurrents, supersonic flow, hysteresis cycles, and the ability to sustain solitonic solutions have been widely studied theoretically and experimentally [2–11]. The production and control of supercurrents is a crucial step towards future quantum devices. For instance, the atomic analog of the superconducting quantum interference device, which was realized experimentally in [12], is based on the stirring of a weak link across a condensate trapped in a ring geometry. This process has been extensively investigated and it has been shown capable to produce superposition of current states [13, 14].

One dimensional rings offer the opportunity to analyze the spectrum much more precisely, and to tackle new effects which are in general masked by higher dimensional dynamics, such as non-vortex-antivortex phase slips [10]. Experimentally, the production of currents can be induced by a rotating weak link. The link produces a low density region and can be rotated to stir the condensate and produce superfluid currents [15, 18]. Alternatively, a phase can be directly imprinted on the condensate [19, 20]. In the latter case, however, Bose-Josephson Junction (BJJ) oscillations are found in the case of a large enough defect or a small enough nonlinearity [21].

With the aim to better understand the behavior of currents on ring condensates, various works have analyzed these systems in the mean field limit and at zero temperature. Current dynamics have been studied through either a rotational drive [22], through the interaction between symmetry breaking potentials and rotation such as in lattice rings [23, 25], or through rotating defects [26–28]. Solutions of the Gross-Pitaevskii equation (GPE) for a 1D ring, in the free case and with various sets of potentials, have been established by analyzing its spectrum either numerically and/or through the use of Jacobi elliptic functions [29, 34]. The spectrum for a moving link and repulsive interactions was analyzed thoroughly in [35].

Studying how current states are coupled to either the ground state or dark solitonic states, which are found to trigger phase slips, has proven essential to understand production and decay of currents, and how to build more robust states.

In this article we complement previous studies by focusing on determining and describing the spectrum and critical velocities for both attractive and repulsive stirred condensates. The use of analytical solutions releases us from the limitation to study the ground state, and also allows us to explore the current dynamics of stirred excited states. The main features of the spectrum are laid out in Sec. II and details are given in Appendix A. In Sec. III, we analyze how currents depend on the link’s velocity and strength, through regular stirring (Sec. IIIA) and through a set of adiabatic cycles which are able to couple any stationary current to the ground state (Sec. IIIB). We also connect the energy and current spectra to a set of hysteresis cycles in Sec. IIIC. In Sec. IIID, we present an alternative method for stable current production in rings with weak links. This protocol does not involve the movement of the link, but setting an auxiliary potential and phase imprinting a nonlinear slope, so that no BJJ oscillations are found. We conclude in Sec. IV.

II. EXACT SPECTRUM OF A STIRRED BEC

Within the mean field limit, and at zero temperature, the condensate wave function on a ring $\psi(\theta, t)$ is determined by the Gross-Pitaevskii equation (GPE),

$$i\hbar \partial_t \psi(\theta, t) = -\frac{\hbar^2}{2MR} \partial^2_{\theta} \psi(\theta, t) + g|\psi(\theta, t)|^2 \psi(\theta, t) + V(\theta, t)\psi(\theta, t),$$

with $\theta \in [0, 2\pi)$, $g$ the reduced 1D coupling, and $V(\theta)$ an external potential. From here onwards we work in natural units, where the ring’s radius $R$, the atomic mass $M$, and $h$ are $R = M = h = 1$, and in the frame of reference comoving with the link. In this frame of reference, where
the link is modeled by a static Dirac delta, the stationary wave function $\phi(\theta)$ and chemical potential $\mu$ are fully determined by

\begin{align}
-\frac{1}{2} \phi''(\theta) + g |\phi(\theta)|^2 \phi(\theta) &= \mu \phi(\theta), \\
\phi(0) - e^{i2\pi \Omega} \phi(2\pi) &= 0, \\
\phi'(0) - e^{i2\pi \Omega} \phi'(2\pi) &= 2\alpha \phi(0),
\end{align}

where $\alpha$ and $\Omega$ are the link’s strength and velocity, and where the wave function is normalized to \( \int_0^{2\pi} d\theta |\phi(\theta)|^2 = 1 \). This framework allows us to use analytical expressions for the wave functions, chemical potentials, and currents. In particular, the chemical potential is given by

\[ \mu(k,m) = \frac{1}{4\pi} (3g + 2k^2(m - 2) + 3k\eta), \]

with $k \geq 0$ and $m \in [0,1]$ the frequency and elliptic modulus of the Jacobi solution, and $\eta = E[JA(k(\theta - \theta_0), m)] + E[JA(k\theta_0, m)]$. $\theta_0$ is a shift depending on $k$ and $m$ such that the wave is continuous at $\theta = 0, 2\pi$. $E$ is the elliptic integral of the second kind, and $JA$ the Jacobi amplitude. Equations (2)-(5) then provide $\alpha(k,m)$, $\Omega(k,m)$, and $\mu(k,m)$, from which we compute $\mu(\alpha, \Omega)$. More details are given in Appendix A and in [33].

The full energy spectrum of dragged solitons in the link’s frame of reference is plotted in Fig. 1 for reduced 1D couplings $g = -1, 0, 1$ and delta strengths $\alpha = 0, \frac{1}{2}, \frac{1}{3}$. The general solution has the form of a moving gray solitonic train, which may become a dark solitonic train at $\Omega = n, \frac{n}{2}$, with $n$ integer (black dots in the Fig. 1), or plane waves in the absence of a link, at $\alpha = 0$ (parabolas in Fig. 1(a), (d) and (g)). This spectrum is characterized, for either attractive or repulsive condensates, by a set of concatenated swallowtail diagrams, each forming an energy band corresponding to solutions with a different number of solitons, and indicated by different colors in Fig 1. These levels only cross for $\alpha = 0$. In this case, the parabolas correspond to plane waves, and represent the energies of vortex states from the point of view of an observer moving at $\Omega$. The lines crossing among these parabolas correspond to gray solitons freely moving at $\Omega$.

The spectrum of dragged solutions, at $\alpha \neq 0$, and its connection to the free ones, at $\alpha = 0$, which move in the absence of a link, is qualitatively different for attractive and repulsive condensates. Firstly, for $g < 0$, swallowtails point upward, while for $g > 0$ they point downward. This implies that a given stirring protocol produces solutions with a different number of solitons in repulsive and attractive condensates. For example, if a link is set on the first vortex state, for any $g \geq 0$ (connecting the middle blue lines in Fig. 1(a) with the ones in (b)), the condensate presents a density dip at the position of the link as well as a gray soliton in the opposite site. In contrast, for $g \lesssim 0$, the solution with current $J \approx 1$ at $\alpha > 0$ (orange line in Fig. 1(h)) shows two gray solitons and a density dip at the position of the link. In both cases, $g > 0$ and $g < 0$, the two and three dips created by setting the delta on the first vortex state, become deeper as the delta strength increases, leading to two and three fully formed dark solitons, respectively, as $g \to \infty$ (see Fig. 6 in Appendix A). Secondly, the band formed by the ground states at different $\Omega$ for $g < 0$, contains no swallowtail structure and forms a continuous set of solutions without critical velocities. This means that, in attractive condensates, solutions with one gray soliton and with largely different currents can be coupled among them through a simple adiabatic variation of the link’s velocity $\Omega$.

In general, these swallowtail shapes imply a set of critical velocities, or bands of solutions with limiting velocities, which are determined by the tips of the tails (triangles) in each diagram. In the case of downward swallowtails, they also imply hysteresis [36]. These velocities are computed through Eqs. (2)-(4). In the limit $\alpha \to 0$ and for $g \neq 0$, where plane wave and gray solitonic solutions merge, these velocities have a simple analytical form given by

\[ \Omega_l = \pm \tilde{\Omega}_n \pm l, \]

with $\tilde{\Omega}_n = \sqrt{\frac{g}{2\pi} + \frac{n^2}{4}}$ and with two integers $n > 0$ and
l ≥ 0. As the barrier strength increases, the critical velocities Ωc monotonically decrease. In the limit \( \alpha \to \infty \), where gray solitons become fully formed dark solitons with their corresponding phase jump and zero valued density dip, Ωc converge to a value of \( \pm \frac{\sqrt{2}}{4} \pm l \).

Beyond these critical velocities, solutions for the corresponding band are not stationary anymore, and the condensate is not able to sustain solitons comoving with the link. This does not happen in the linear case, \( g = 0 \), where the tails shrink and vanish and the above ranges of velocities become zero, or for the ground state of attractive condensates, where there are no swallowtail structures in the first place. In principle, in both of these cases, the stirring link velocity can be increased indefinitely, such that solitonic waves are always dragged by the link.

The spectrum plotted in Fig. 1 is then essential to qualitatively understand how to avoid critical velocities and how the different excited states can be coupled among them and to the ground state by tuning the link strength \( \alpha \) and velocity \( \Omega \). It also provides a basis to study hysteresis cycles. We illustrate a sample of such stirring protocols in the following sections.

### III. CURRENT PRODUCTION

The current, \( J = -\frac{i}{2} \int d\theta (\phi^* \phi' - \phi \phi'^*) \), for a link modeled by a Dirac delta, is given by

\[
J = \pm 2\pi \gamma + 2\pi n, \tag{7}
\]

with \( n \) an integer, and

\[
\gamma = \frac{1}{g(2\pi)^{3/2}} \sqrt{g + k \eta \sqrt{g - 2\pi k^2 + k \eta}} \times \sqrt{g - 2\pi k^2(1 - m) + k \eta}. \tag{8}
\]

Together with \( \alpha(k, m) \) and \( \Omega(k, m) \), the current can be found in terms of \( \alpha \) and \( \Omega \) by scanning the well defined parameter space given by the frequency \( k \geq 0 \) and elliptic modulus \( m \in [0, 1] \), as done with the chemical potential. The analytical results shown in the following plots are corroborated by simulations of the time-dependent GPE in the lab frame, where a peaked Gaussian potential explicitly moves around the ring. Solutions from both methods are found to overlap for Gaussian amplitude widths \( \sigma = 2\pi/200 \).

#### A. Adiabatic, regular stirring

Figure 2 shows the current evolution for three different cases, \( g = -1, 0, 1 \), as a link is set on the ground and first excited states and then stirred by adiabatically increasing the velocity from \( \Omega = 0 \) to \( \Omega \approx 0.5 \).

For \( g < 0 \), a link can be set in the ground state and its velocity increased indefinitely. The current can be steadily increased, and the solutions alternate between a shallow gray soliton moving at \( \Omega = n \) and a dark soliton moving at \( \Omega = n + \frac{1}{2} \). See the red lines in Fig. 1 (h) and (i) for the energy oscillations between these two solutions, and the red line in Fig. 2 (a), for the current evolution in the path \( \Omega = 0 \to 0.5 \). This current is produced more abruptly for attractive interactions closer to zero. For weak interactions, \( g \lesssim 0 \), the first excited state consists in two dark solitons, with \( J = 0 \). When this state is stirred, a current \( J \approx 1 \) in the stirring direction is produced at very small velocities, and then is kept roughly constant up to the critical velocity, \( \Omega_c \geq 0.5 \) (see blue line in Fig. 2 (a)). Setting a link in the first vortex state, produces a density with three gray solitons, and its energy corresponds to the crossing orange lines in the center of Fig. 1 (h). If the link is moved in the same direction of the vortex current, it very soon encounters a critical velocity, \( \Omega \in (0, 1 - \tilde{\Omega}_2) \), beyond which solutions comoving with the link do not exist. If it is stirred in the opposite direction, the current remains constant up to \( \Omega \lesssim 0.5 \), at which point it decreases abruptly to zero, and then a current in the direction of the link is produced. See the orange line in panel (a) of Fig. 2.

For the repulsive case, stirring the ground state with increasing velocity leaves the current practically constant, and a critical velocity is found at \( \Omega \gtrsim 0.5, \Omega \in (0.5, \tilde{\Omega}_1) \), as shown in Fig. 2 (c) (red line). This velocity marks the tip of the lowest and right swallowtail in the top plots of Fig. 1 and is connected to the solu-
tion with one dark soliton through a set of unstable solutions [35]. The first excited state corresponds to the first vortex state and for \( \alpha \geq 0 \) contains two gray solitons. Its energy is represented by the crossing blue lines in Fig. 1 (b). In the limit \( \alpha \rightarrow 0 \), it turns into a plane wave with one unit of angular momentum. If this vortex is stirred, the current remains roughly constant for small velocities, and abruptly decreases to \( J = 0.5 \) at \( \Omega \lesssim 0.5 \) as plotted in the blue line in Fig. 2 (c). Finally, stirring the state with two dark solitons, abruptly produces a current in the direction opposite to the link’s velocity, until a critical velocity is found at \( \Omega > 0.5 \), see the orange line in Fig. 2 (c).

The linear case, \( g = 0 \), in panel (b) of Fig. 2 presents similar current dynamics, except that initial states are all static \( (J = 0) \), and no critical velocities are encountered in any stirring. In this case, link velocities can be increased and decreased indefinitely.

In the beginning of the stirring process, when the link is set and at zero velocity, the ground and dark solitonic states have zero current. For the first vortex states, with \( J = \pm 1 \) in the absence of the potential, the initial currents decrease with the link’s strength if \( g \neq 0 \), as plotted in the bottom panels of Fig. 2.

The adiabatic paths described above can also be understood in reverse, that is, in terms of a decreasing stirring velocity. Moreover, we note that, due to rotational symmetry, the evolution of the currents of these paths is also valid for the same states and stirrings but with velocities \( \Omega \rightarrow \pm \Omega + n \) and currents \( J \rightarrow \pm J + n \), with \( n \) an integer.

B. Adiabatic, excitation stirring

The stirring procedures of Fig. 2 consisting in a steady increase of the link’s velocity up to \( \Omega \simeq 0.5 \), allow us to produce currents \( |\Delta J| \lesssim 1 \). Passed these velocities, critical velocities are encountered, and the condensate cannot be adiabatically excited anymore. However, there are cases where critical velocities are not a limitation. In particular, the ground state of attractive condensates, any state for the linear case, and in general dark solitonic states in the limit \( \alpha \rightarrow \infty \), where the tails in the energy spectrum shrink and vanish. These states can be continuously excited to states with larger currents by constantly increasing the velocity of the link.

We follow similar procedures in attractive and repulsive condensates that couple the ground state to excited states so that \( |\Delta J| \geq 1 \). For repulsive condensates, a link is set in the ground state while rotating at a velocity \( \Omega_i \). Initial velocities \( \Omega_i \in (\Omega_n, \Omega_{n+1}) \) access the \( n^{th} \) excited state, i.e. the one with \( n + 1 \) solitons. For example, in Fig. 1 (a), the energy of the ground state from the point of view of an observer moving at \( \Omega \in (\tilde{\Omega}_1, \tilde{\Omega}_2) \), corresponds to the blue line in the right branch of the center parabola. This state, at \( \alpha = 0 \), is coupled to the one with two dragged gray solitons at \( \alpha > 0 \), represented by the blue line in Fig. 1 (b) in the same range of velocities. The velocity is then decreased down to the other side of the swallowtail diagram, and then the weak link is turned off. Two examples of such cycles, that excite the repulsive condensate to \( J = 1 \) (red) and \( J = 2 \) (blue) for attractive \( (g = -1 \), solid lines) and repulsive condensates \( (g = 1 \), dashed lines). All cycles involve setting and unsetting a weak link of \( \alpha = 0.3 \), as plotted in (a). For repulsive condensates it is necessary to start at a finite velocity and then decrease it, while for attractive interactions, the velocity is steadily increased (plot (b)). Plot (c) shows the evolution of the current for these two types of cycles and for \( J = 1, 2 \).

Attractive states with \( g \lesssim 0 \) allow for two possible excitation processes. Firstly, and analogously to the repulsive case, a link can be set at a finite velocity \( \Omega_i \). Initial rotations \( \Omega_i \in (\tilde{\Omega}_n, \tilde{\Omega}_{n+1}) \) produce solutions corresponding to the \( n^{th} \) excited state. In this case, the velocity must be increased up to the following swallowtail diagram, such that critical velocities are avoided, see Fig. 1. Secondly, and perhaps more naturally, we can stir by setting a link at zero velocity, and then speed up the stirring. This process takes advantage that no critical velocities are encountered when stirring the ground state of attractive condensates. Unsetting the link at \( \Omega = n \) will then produce vortex states with \( n \) quanta of angular momentum. The corresponding cycles to produce states with \( J = 1 \) and \( J = 2 \), are shown in Fig. 3 (dashed lines). These cycles include an intermediate dark solitonic solution (with \( n + 1 \) dark solitons), and the current increases more abruptly in the middle points of the paths.

We have focused our attention to \( |g| \) small enough such that for attractive condensates the ground state is always coupled through the stirring strength and velocity to all other states. This is not the case at \( g \ll 0 \) (see Appendix A), where new types of solutions appear. In this case, to adiabatically excite the condensate from the ground state, one must use the cycles where the link is produced while moving at a finite velocity.
C. Hysteresis cycles

Hysteresis due to a rotating weak link in a Bose gas was first experimentally observed in [6]. These cycles are understood in terms of swallowtail diagrams [35], and their widths were numerically computed in [27]. Here we discuss how for a delta type link the widths and heights of the hysteresis cycles, $\Delta \Omega$ and $\Delta J$, can be computed using the exact spectrum presented in this work.

Fig. 1 proves useful to illustrate the main features of hysteresis. On the one hand, it shows that only repulsive condensates present downward swallowtail structures, and therefore the associated hysteresis cycles only exist for $g > 0$. This is because when the critical velocity is reached at the tip of an upward swallowtail, the state with lower energy to which the condensate decays belongs to a lower set of concatenated swallowtails. This effectively impedes to excite the condensate back to the upper swallowtail through any adiabatic variation of $\Omega$, and therefore to close the cycle. On the other hand, hysteresis cycles on repulsive condensates are not characteristic of a stirring of the ground state, where the condensate undergoes a transition $\Delta J \simeq 1$. Stirring of excited states also present hysteresis, each excited state implying a different width $\Delta \Omega$ and height $\Delta J$, features not discussed in previous works. Moreover, these cycles can be analyzed in terms of the nonlinearity and link’s strength. In general, the range of velocities limited by $\Omega_c$, and the associated widths of the swallowtails and hysteresis cycles, $\Delta \Omega$, become smaller as larger transitions $\Delta J$ are considered, as $g$ decreases, or as the link’s magnitude $\alpha$ becomes stronger.

In Fig. 2 we present two hysteresis cycles, one corresponding to stirring the ground state, where $\Delta J \simeq 1$, and another associated to setting a link in a vortex state, and then stirring, effectively coupling $J \simeq -1$. Stirring of excited states both present hysteresis, each excited state implying a different width $\Delta \Omega$ and height $\Delta J$, features not discussed in previous works. Moreover, these cycles can be analyzed in terms of the nonlinearity and link’s strength. In general, the range of velocities limited by $\Omega_c$, and the associated widths of the swallowtails and hysteresis cycles, $\Delta \Omega$, become smaller as larger transitions $\Delta J$ are considered, as $g$ decreases, or as the link’s magnitude $\alpha$ becomes stronger.

D. Auxiliary potential and phase imprinting

The adiabatic paths presented so far involve the movement of the link or a stirring. An alternative procedure to excite the condensate is to imprint a phase through an electromagnetic field [20]. Phase imprinting provides a fast way to excite the condensate, but when the ring contains a static defect, which might happen naturally in experiment, the state obtained is not stationary, and one encounters BJJ oscillations [21], or other nonlinear effects if $g$ is large enough. These oscillations can be understood intuitively through simplified hydrodynamics considerations. If a linear phase is imprinted, all the atoms throughout the annular trap acquire the same momentum, which implies a smaller current at the low density region created by the defect. The condensate thus accumulates at the side of the defect, slows down, and bounces to the other side of the barrier. This effect can be partially reduced by imprinting a nonlinear phase such that a larger kick is provided to the condensate at the low density region, producing thus a current which is roughly stationary, i.e., $J(\theta) = \rho(\theta) \beta'(\theta) \simeq constant$. This idea can be quantitatively analyzed since the excited stationary state with current $J \simeq 1$ and a delta defect is well known in terms of Jacobi elliptic functions. One can in principle imprint the phase of these states on the ground state, but the unperturbed density would still differ from the densities of stationary ones, which, apart from the dip produced by the link, contain other gray solitons. Therefore, the final states would not be stable. To solve this,
and taking advantage of adiabatic processes, we design a protocol to produce the density of the desired excited state through an auxiliary potential [37, 38], and leaving the link fixed. Once this condensate’s density is obtained, the phase is imprinted and the auxiliary potential is turned off.

More explicitly, if the final stationary solution we want to obtain, in presence of the delta potential, is $\psi_s(\theta) = r_s(\theta)e^{i\beta_s(\theta)}$, we first set an auxiliary potential $V_{aux}$ such that the ground state is $\psi_g = r_s(\theta)$, and therefore satisfies

$$\mu r_s = -\frac{1}{2}r''_s + gr'_s + V_{aux} r_s.$$ (9)

On the other hand, the final excited state we want to build is determined by

$$\mu r_s = -\frac{1}{2}[r''_s - r_s \beta'^2_s + i(r_s \beta''_s + 2r'_s \beta'_s)] + gr'^2_s.$$ (10)

The imaginary part is zero as long as $\beta'_s = \frac{\gamma}{r'_s}$, with $\gamma$ being a constant representing the current. Subtracting both equations, and neglecting the constant $\mu - \mu$, we find

$$V_{aux}(\theta) = \frac{1}{2} \beta'_s(\theta)^2 = \frac{\gamma^2}{2r_s(\theta)^4}.$$ (11)

The protocol then consists in gradually turning on $V_{aux}(\theta)$, for example by increasing its overall factor from zero to one, and then turn if off while phase imprinting $\beta_s(\theta)$. The stationary state $r_s(\theta)e^{i\beta_s(\theta)}$ with current $J \simeq n > 0$ is thus accessed. How precisely these final currents differ from that of plane waves, $J = n$, depends on the link’s strength, as shown in plots (d) and (f) in Fig. [2] for the first vortex state ($n = 1$).

This protocol is reproduced through simulations in the time-dependent GPE, as shown in Fig. [3] obtaining a steady current $J \simeq 1$. This is in contrast with regular phase imprinting, which for large enough defects, in our tested case $\alpha = 0.3$, produces BJJ oscillations, as also shown in the figure. Apart from the protocol described above, we study the evolution of the current when only the auxiliary potential is used —only the density of the stationary current with a delta is imitated, and a regular slope $2\pi x$ is imprinted—, and when only the non-linear phase imprinting is used on the ground state with a delta. We observe that in all cases, both the auxiliary potential and the nonlinear phase imprinting, serve independently to produce more self-trapping in the final state.

IV. CONCLUSIONS

We have described the spectrum of solitons moving at constant velocity in a ring condensate, either freely, or being dragged by a weak link. Their energies, densities, and currents have been thoroughly analyzed in terms of the link’s strength $\alpha$ and velocity $\Omega$, and found that all states are coupled at $\alpha = 0$.

By studying how the different stationary states are connected through an adiabatic variation of $\alpha$ and $\Omega$, we have laid out three different methods to modify the state of the condensate in a controlled manner. The first, involves a purely adiabatic variation of the link’s strength and velocity. Setting a link and then stirring by increasing its velocity, allows to excite the ground state of attractive condensates, but in all other cases critical velocities are encountered, and current variations are limited to $|\Delta J| \lesssim 1$. To access excited states, the link must be set while rotating at a finite velocity. Secondly, we have considered processes in which the link moves but its strength is left fixed. In this case, the link’s velocity surpasses the
critical one and the condensate is assumed to decay to the immediate lower state. For repulsive condensates, these paths, consisting in both an adiabatic excitation part and a non-adiabatic decay, can be closed by moving the weak link in both directions, and effectively producing hysteresis cycles. Here, we have shown that these hysteresis cycles can also be produced in excited states, although they are limited by different critical velocities, and that hysteresis cycles cannot exist for attractive condensates. Finally, we have made use of an auxiliary potential to adiabatically modify the ground state density, and to then imprint a nonlinear phase while the potential is turned off. The auxiliary potential and phase are precisely designed such that the state produced is an excited but stationary state, and no BJJ oscillations are found.

This work illustrates, from an analytical point of view, the physical mechanisms involved in the production of currents in weakly interacting Bose gases in a ring trap. It also provides a theoretical description which allows for further exploration of the system, including ground states as well as excited states.

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Appendix A: Spectrum

The spectrum of normalized solutions is more easily analyzed in the delta comoving frame, which is determined by Eqs. (2)-(4). The solutions of these equations are parametrized through a density \( \rho \) and a phase \( \phi(\theta) = \sqrt{\rho(\theta)}e^{i\beta(\theta)} \), and are given in analytical form in terms of Jacobi elliptic functions \( F \), \( \rho(\theta) = A + B F^2(k(\theta - \theta_0), m) \), and where \( \rho(\theta) \beta'(\theta) = \gamma \) is constant. In our case, we set the Jacobi function \( F = dn. \) \( A, B \) and \( \theta_0 \) depend on the frequency \( k \geq 0 \) and elliptic modulus \( m \in [0, 1] \). \( \rho(\theta) \) in general oscillates around a finite value, has no zeros, and is smooth except for the derivative jump due to the delta at \( \theta = 0 \). In the limits \( A \to 0 \) and \( B \to 0 \), these solutions become dark solitons and plane waves, respectively. In the following we illustrate the main features of the spectrum according to Fig. 1.

In the linear and free case, \( g = 0 \) and \( \alpha = 0 \), the solutions are plane waves or vortex states, with the chemical potential quantized by periodic conditions and given by \( \mu = \frac{2\pi}{\Omega} \). Each parabola in Fig. 1 (d) represents the energy of each of these states, \( \mu = \frac{2\pi}{\Omega}(\Omega + n)^2 \), as measured by an observer moving around the ring at constant velocity \( \Omega \). In the lab frame each parabola represents the same solution with energy \( \frac{\mu^2}{2} \).

When atomic interactions are finite, \( g \neq 0 \), and no link is present, \( \alpha = 0 \), the condensate is governed by the GPE with periodic boundary conditions, which also has as solutions plane waves. Their energy includes the same kinetic term as in the linear case, but also a potential term which shifts the energy parabolas upward and downward for the repulsive and attractive cases, \( \mu = \frac{2\pi}{\Omega} + \frac{1}{2}(\Omega + n)^2 \). Moreover, there are new sets of solutions, consisting in gray solitons moving at constant velocity \( \Omega \). Their energies as a function of \( \Omega \), in the frame of reference of the moving solitons, are shown in Fig. 1 (a) and (g) as the curves crossing between the plane wave parabolas. The middle points of these crossing lines, marked as black dots, correspond to dark solitons. Even number of dark solitons move at velocities \( \Omega = n \), while odd number of dark solitons travel at \( \Omega = n + \frac{1}{2} \), with \( n \) an integer. These waves are always non-moving with respect to the condensate. As the gray solitons become shallower, their velocities depart from \( \Omega = n, n + \frac{1}{2} \), until the densities become completely flat and these solutions merge with the plane waves. These sets of solutions can be understood as energy bands in the lab frame. Each band consists in solutions with a fixed number of solitons with velocities ranging from \( m \) to \( \Omega = m \pm n \). A more rigorous analysis of the first band of gray solitonic solutions can be found in [39].

The energies of the gray solitons increase and decrease with \( g \), and this implies the appearance of new static solutions \( (\Omega = 0) \) as \( |g| \) grows larger [29], in particular of new ground states for \( g < 0 \). In this article we have focused on \( |g| \) small enough so that the ground state for attractive condensates stays coupled to the rest of the spectrum. To illustrate how the spectrum qualitatively depends on \( g \), we consider two particular cases, one for attractive condensates and one for repulsive ones. First, as \( g \) decreases from \( g = 0 \), the lowest of these crossing lines (blue lines in Fig. 1 (g)) move downward, until their left and right limits coincide with the bottom points of the parabolas, at \( \Omega_1 = 0, g = -\frac{\pi}{2} \). For \( g < -\frac{\pi}{2} \), the ground state as a function of \( \Omega \) forms a continuous line uncoupled from the rest of parabolas. In general, new uncoupled states appear at \( g < -\frac{\pi^2}{2} \), with integer \( n > 0 \). Another example is the appearance of a new second excited state as \( g \) grows and the red solitonic line in Fig. 1 (a) crosses the axis \( \Omega = 0 \). More precisely, at \( g = \frac{3\pi}{2} \), such that the left limit coincides with the vertical axis, \( 1 - \Omega_1 = 0 \), a new solution appears at \( \Omega = 0 \) between the first vortex state and the first dark solitonic solution.

As a link is turned on, \( \alpha > 0 \), the spectrum of plots (a), (d), and (g) described above splits into a set of diagrams separated by a gap. For finite \( g \), these diagrams have the shape of downward \( (g > 0) \) and upward \( (g < 0) \) swallowtails. The gap among the swallowtails grows for larger delta strengths and nonlinearities. Each set of concatenated swallowtail diagrams represents a set of solutions with a fixed number of solitons. The densities of the lowest set of swallowtails have only the downward kink created by the delta. Then, each superior set has, apart
from the dip in the density produced by the link, one more gray soliton. The middle points of these diagrams still represent dark solitons, each previous dark solitonic solution at $g = 0$ now split into two. In the solution with higher energy, the dark soliton coinciding with the delta corresponds to a derivative jump, satisfying delta conditions. The solution with lower energy consists in a periodic and smooth wave such that one of the zeros coincides with the position of the link. Solutions of this type trivially satisfy delta conditions for any $\alpha$, since the derivatives and the function at the position of the delta are zero. In Fig. 6 this means the black dots in the red lines, the ones in the blue lines at $\Omega = n$, and the ones at the orange lines at $\Omega = n + \frac{1}{n}$, have the same $\mu$ across the panels in each row. As a sample, the densities of these pairs of dark solitonic trains, and the ground and other excited states, are plotted in Fig. 6 for $\alpha = \frac{1}{4}, g = -1, 1$ and $\Omega = 0, 0.5$.

We herewith have thoroughly described the set of solitonic solutions in correspondence to Fig. 1. To sum up, the solutions for $\alpha = 0$ consist of vortex states with current $J = n$, of $m$ dark solitons moving at $\tilde{\Omega} = \frac{\pi}{2} + n$, and of $m$ gray solitons traveling at $\Omega \in (\frac{\pi}{2} + n - |\tilde{\Omega}_m - \frac{\pi}{2}|, \frac{\pi}{2} + n + |\tilde{\Omega}_m - \frac{\pi}{2}|)$, with integers $n, m$. For a rotating link, the dragged solutions comprise trains with $m$ gray solitons coupling solutions with $m$ and $+1$ dark solitons moving at $\Omega = \frac{\pi}{2} + n$ and $\Omega = \frac{\pi}{2} + n$, respectively, and limited by critical velocities $\Omega_c$.

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