NEW ENTROPY CONDITIONS FOR SCALAR CONSERVATION
LAWS WITH DISCONTINUOUS FLUX

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Abstract. We propose new Kruzhkov type entropy conditions for one dimensional scalar conservation law with a discontinuous flux. We prove existence and uniqueness of the entropy admissible weak solution to the corresponding Cauchy problem merely under assumptions on the flux which provide the maximum principle. In particular, we allow multiple flux crossings and we do not need any kind of genuine nonlinearity conditions.

In the current contribution, we consider the following problem

\[
\begin{aligned}
\partial_t u + \partial_x \left( H(x)f(u) + H(-x)g(u) \right) &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u|_{t=0} &= u_0(x) \in L^\infty(\mathbb{R}), \\
x &\in \mathbb{R}
\end{aligned}
\]

where \( u \) is the scalar unknown function; \( u_0 \) is a function such that \( a \leq u_0 \leq b \), \( a, b \in \mathbb{R} \); \( H \) is the Heaviside function; and \( f, g \in C^1(\mathbb{R}) \) are such that \( f(a) = f(b) = g(a) = g(b) = 0 \).

Problems such as (1) are non-trivial generalization of scalar conservation law with smooth flux, and they describe different physical phenomena (flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, gas flow in a variable duct...). Therefore, beginning with eighties (probably from \([35]\)), problems of type (1) are under intensive investigations.

As usual in conservation laws, the Cauchy problem under consideration in general does not possess classical solution, and it can have several weak solutions. Since it is not possible to directly generalize standard theory of entropy admissible solutions \([23]\), in order to choose a proper weak solution to (1) many admissibility conditions were proposed. We mention minimal jump condition \([17]\), minimal variation condition and \( \Gamma \) condition \([10, 11]\), entropy conditions \([19, 1]\), vanishing capillary pressure limit \([18]\), admissibility conditions via adapted entropies \([6, 8]\) or via conditions at the interface \([2, 3, 12]\).

But, in every of the mentioned approaches, in order to prove existence or uniqueness of a weak solution to the considered problem, some structural hypothesis on the flux (such as convexity or genuine nonlinearity) or on the form of the solution (see \([2, 3]\)) were assumed.

Recently, in \([26]\), we have proved existence and uniqueness in the multidimensional situation. Still, due to certain technical obstacles, admissible solutions selected in that paper are rather special.

Here, we propose admissibility conditions which involve much less restrictions than in previous works on the subject (excluding \([26]\) where there are no restrictions), and we still can make many different stable semigroups depending on the physical situation under considerations.

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Since one can find excellent overviews on the subject in many papers [5, 3, 7, 8, 12, 28] which are easily available via internet (e.g. www.math.ntnu.no/conservation), in this introduction, we shall restrict our attention on papers [19], [21], and [28] which are in the closest connection to our contribution. Later, in Section 2, we shall comment how our admissibility conditions can be considered as a generalization of the entropy solution of type \((A,B)\) given in [8] (see Definition 1.1 in the current paper).

In [19], degenerate parabolic equation with discontinuous flux is considered:

\[
\begin{aligned}
&\partial_t u + \partial_x \left( H(x) f(u) + H(-x) g(u) \right) = \partial_{xx} A(u), \quad (t,x) \in (0,T) \times \mathbb{R} \\
&u|_{t=0} = u_0(x) \in BV(\mathbb{R}) \cap L^1(\mathbb{R}), \quad x \in \mathbb{R},
\end{aligned}
\]

where \(A\) is non-decreasing with \(A(0) = 0\). Assuming that \(A \equiv 0\) we obtain the problem of type \((1)\). In order to obtain uniqueness of a weak solution to the problem, the Kruzhkov type entropy admissibility condition [23] is used:

**Definition 0.1.** [19] Let \(u\) be a weak solution to problem \((1)\).

We say that \(u\) is an entropy admissible weak solution to \((1)\) if the following entropy condition is satisfied for every fixed \(\xi \in \mathbb{R}\):

\[
\partial_t |u - \xi| + \partial_x \left\{ \text{sgn}(u - \xi) \left[ H(x) (f(u) - f(\xi)) + H(-x) (g(u) - g(\xi)) \right] \right\}
- |f(\xi) - g(\xi)| \delta(x) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}).
\]

Still, merely such entropy condition was insufficient to prove stability of the admissible weak solution to the considered problem. Two more things were necessary.

First, one needs the following technical assumption:

**Crossing condition:** For any states \(u, v\) the following crossing condition must hold:

\[f(u) - g(u) < 0 < f(v) - g(v) \Rightarrow u < v.\]

Geometrically, the crossing condition requires that either the graph of \(f\) and \(g\) do not cross, or the graph \(g\) lies above the graph of \(f\) to the left of the crossing point (see Figure 1). The functions \(f\) and \(g\) appearing in \((1)\) do not necessarily satisfy the crossing conditions, but it is possible to transform them so that the crossing condition is satisfied (see Figure 2 and Figure 4).

Next, in [21] existence of strong traces at the interface \(x = 0\) was necessary. We provide appropriate definition.

**Definition 0.2.** Let \(W : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}\) be a function that belongs to \(L^\infty(\mathbb{R} \times \mathbb{R}^+)\).

By the right and left traces of \(W(\cdot,t)\) at the point \(x = 0\) we understand functions \(t \mapsto W(0^\pm , t) \in L^\infty_{loc}(\mathbb{R}^+)\) that satisfy for a.e. \(t \in \mathbb{R}^+\):

\[
\text{esslim}_{x \downarrow 0} |W(t,x) - W(t,0^+)| = 0, \quad \text{esslim}_{x \uparrow 0} |W(t,x) - W(t,0^-)| = 0.
\]

Assuming the crossing condition and the existence of traces, we have the following theorem:

**Theorem 0.3.** [19] Assume that weak solutions \(u\) and \(v\) to \((1)\) with the initial conditions \(u_0\) and \(v_0\), respectively, satisfy entropy admissibility conditions from Definition 0.1 and admit left and right strong traces at the interface \(x = 0\).

Then for any \(T, R > 0\) there exist constants \(C, \bar{R} > 0\) such that:

\[
\int_0^T \int_{-R}^R |v(t,x) - u(t,x)| \, dx \, dt \leq CT \int_{-\bar{R}}^\bar{R} |v_0(x) - u_0(x)| \, dx.
\]
Remark 1. It is important to notice that Theorem 0.3 remains to hold if in (1), instead of $\partial_t u$, we put $\partial_t(\alpha(u)H(x) + \beta(u)H(-x))$, for some strictly increasing bijections $\alpha : [a,b] \to [a',b']$ and $\beta : [a,b] \to [a'',b'']$, $a',a'',b',b'' \in \mathbb{R}$. Indeed, since we did not put a function depending on $t \in \mathbb{R}^+$ under the derivative $\partial_t$, and since $\alpha$ and $\beta$ are increasing bijections (we can extract all the information on $u$ knowing only $\beta(u)$ or $\alpha(u)$), we can safely use results from [21] on the equation $\partial_t(\alpha(u)H(x) + \beta(u)H(-x)) + \partial_x(f(u)H(x) + g(u)H(-x)) = 0$.

Figure 1. Functions $f$ (normal line) and $g$ (dashed line) satisfying the crossing condition.

First, we shall explain how to force the crossing condition and existence of traces. We shall use the idea from [28]. In [28], the following problem was considered

$$\begin{align*}
\partial_t u + \partial_x f(\alpha(x,u)) &= 0, \\
u|_{t=0} &= u_0(x),
\end{align*}$$

where $\alpha$ is a function discontinuous in $x \in \mathbb{R}$ and strictly increasing with respect to $u$. Then, we can write:

$$v = \alpha(x,u) \Rightarrow u = \beta(x,v).$$

Problem (3) becomes

$$\begin{align*}
\partial_t \beta(x,v) + \partial_x f(v) &= 0, \\
v|_{t=0} &= \alpha(x,u_0).
\end{align*}$$

Thus, the discontinuity in $x$ is removed out of the derivative in $x$, and we can apply standard vanishing viscosity approach:

$$\begin{align*}
\partial_t \beta(x,v_z) + \partial_x f(v_z) &= \varepsilon \partial_{xx} v_z, \\
v_z|_{t=0} &= \alpha(x,u_0),
\end{align*}$$

to obtain the sequence $(v_z)$ strongly converging in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$ to a unique Kruzhkov admissible weak solution $v$ of (4) which immediately gives uniqueness of appropriate weak solution to (3).

It is important to notice that the existence and uniqueness are actually obtained thanks to the appropriate choice of the viscosity term. Such choice enables the author to control the flux corresponding to (3).

Using this observation, we shall propose new admissibility conditions which will enable us to control the flux corresponding to (1) in an extent which will provide uniqueness in a rather general situation. Informally speaking, we shall consider the following vanishing viscosity regularization to (1):

$$\begin{align*}
\begin{cases}
\partial_t u + \partial_x (H(x)f(u) + H(-x)g(u)) = \varepsilon \partial_{xx} (\tilde{\alpha}(u)H(x) + \tilde{\beta}(u)H(-x)), \\
u|_{t=0} = u_0(x),
\end{cases}
\end{align*}$$

\[(\alpha, \beta)\text{-ENTROPY CONDITIONS} 3\]
where \( \hat{\alpha} : [a, b] \to [a', b'] \) and \( \hat{\beta} : [a, b] \to [a'', b''] \) are smooth strictly increasing bijections.

Denote by \( \alpha \) and \( \beta \) the inverse functions of the functions \( \hat{\alpha} \) and \( \hat{\beta} \), respectively. Introducing the change of the unknown function:

\[
v = \hat{\alpha}(u)H(x) + \hat{\beta}(u)H(-x) \Rightarrow u = \alpha(v)H(x) + \beta(v)H(-x),
\]

and denoting \( f_\alpha = f \circ \alpha \) and \( g_\beta = g \circ \beta \), we have from (6):

\[
\left\{ \begin{array}{l}
\partial_t(\alpha(v)H(x) + \beta(v)H(-x)) + \partial_x(H(x)f_\alpha(v) + H(-x)g_\beta(v)) = \varepsilon \partial_x v,
\end{array} \right.
\]

\[
|v|_{t=0} = \hat{\alpha}(u_0)H(x) + \hat{\beta}(u_0)H(-x).
\]

(7)

So, instead of dealing with the flux \( H(x)f(u) + H(-x)g(u) \), we deal with the new flux \( H(x)f_\alpha(v) + H(-x)g_\beta(v) \). As we shall see later, by choosing appropriate functions \( \alpha \) and \( \beta \) we can always make the new flux to satisfy "the crossing condition" at least in the range of the solution (see Figure 2 and Figure 4 as important special cases). Now, we can introduce the definition of admissibility that we shall use.

**Definition 0.4.** Let \( u \) be a weak solution to problem (1). Let \( \hat{\alpha} : [a, b] \to [a', b'] \) and \( \hat{\beta} : [a, b] \to [a'', b''] \) be smooth strictly increasing bijections. Denote by \( \sigma \) and \( \mu \) the inverse functions to \( \hat{\alpha} \) and \( \hat{\beta} \), respectively.

We say that \( u \) is an \((\alpha, \beta)\)-entropy admissible solution to (1) if

(D.1) \( u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) and \( u(t, x) \in [a, b] \) for almost every \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^d \);

(D.2) the function \( v = \hat{\alpha}(u)H(x) + \hat{\beta}(u)H(-x) \) satisfies the following entropy condition for every fixed \( \xi \in \mathbb{R}^d \):

\[
\partial_t \left\{ \text{sgn}(v - \xi)[H(x)(\alpha(v) - \alpha(\xi)) + H(-x)(\beta(v) - \beta(\xi))] \right\}
\]

\[
+ \partial_x \left\{ \text{sgn}(v - \xi)[H(x)(f_\alpha(v) - f_\alpha(\xi)) + H(-x)(g_\beta(v) - g_\beta(\xi))] \right\}
\]

\[
- |f_\alpha(\xi) - g_\beta(\xi)| \delta(x) \leq 0,
\]

where, as before, \( f_\alpha = f \circ \alpha \) and \( g_\beta = g \circ \beta \).

From the previous analysis, appealing on [19], we conclude that we need only existence of traces to obtain the uniqueness. The question of existence of traces is rather serious in itself [24, 27, 36], but it was shown in [27] that they exist practically in all relevant situations. In order to formulate a necessary theorem, we need the notion of the quasi-solution.

**Definition 0.5.** We say that the function \( u \in L^\infty(\mathbb{R}^d) \) is a quasi-solution to the scalar conservation law

\[
\text{div}_x F(u) = 0, \quad x \in \mathbb{R}^d,
\]

where \( F = (F_1, \ldots, F_d) \in C(\mathbb{R}^d, \mathbb{R}) \) if it satisfies for every \( \xi \in \mathbb{R}^d \):

\[
\text{div}_x \text{sgn}(u - \xi)(F(u) - F(\xi)) = \gamma_k \text{ in } \mathcal{D}'(\mathbb{R}^d),
\]

where \( \gamma_k \) is a locally bounded Borel measure.

Next theorem can be found in [27]. We adapt it to our situation.

**Theorem 0.6.** [27] Let \( h, f \in C(\mathbb{R}) \).

Suppose that the function \( u \) is a quasi-solution to

\[
\partial_t h(u) + \partial_x f(u) = 0,
\]

where the vector \((h, f)\) is such that the mappings \( \lambda \mapsto h(\lambda) \) and \( \lambda \mapsto f(\lambda) \) are not constant on any non-degenerate interval.
Then, the function $u$ admits right and left strong traces at $x = 0$.

Now, the situation with traces is clear and we need to cope with the existence of a solution admissible in the sense of Definition 0.4.

In the case of a scalar conservation law with a smooth flux, the proof of existence is based on the BV-estimates for a sequence of solutions to the corresponding Cauchy problem regularized with the vanishing viscosity. Such estimates are not available if the flux is discontinuous. Therefore, we need to apply more subtle arguments involving singular mapping [35], local variation bounds [9], compensated compactness [20, 21, 22, 33], difference schemes [3, 19, 22] or $H$-measures [15, 16, 30, 34].

In general, using e.g. the compensated compactness, it is possible to prove that the sequence $(u_\varepsilon)$ of solutions to (6) weakly converges to a weak solution $u$ of (1). However, it is not possible to state that the weak solution satisfies wanted admissibility conditions. In order to be sure that $u$ is admissible, in principle, we need to prove that the corresponding sequence $(u_\varepsilon)$ strongly converges strongly in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ to $u$ (still, not necessarily; see [29]) which, at least in the framework of the compensated compactness (or the $H$-measures whose consequences we are going to use), can be proved only by assuming the genuine nonlinearity condition given by the following definition.

**Definition 0.7.** Let $h : \mathbb{R}^2 \to \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$.

We say that the vector $(h(x, \lambda), H(x)f(\lambda) + H(-x)g(\lambda))$ is genuinely nonlinear if for almost every $x \in \mathbb{R}$ and every $(\xi_0, \xi_1) \in S^1$, $S^1 \subset \mathbb{R}^2$ is two dimensional sphere, the mapping

$$(a, b) \ni \lambda \mapsto \xi_1 h(x, \lambda) + \xi_1 (H(x)f(\lambda) + H(-x)g(\lambda)),$$

is different from a constant on any non-degenerate interval $(a, \beta) \subset (a, b)$.

The latter condition provides the following theorem to hold.

**Theorem 0.8.** [30] Assume that the vector $(h(x, u), H(x)f(u) + H(-x)g(u))$, $(x, u) \in \mathbb{R} \times \mathbb{R}$, is genuinely nonlinear in the sense of Definition 0.7.
Then, the following statement holds:
Each family \((v_\varepsilon(t, x)) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}), \ a \leq v_\varepsilon \leq b, \varepsilon > 0, \) such that for every \(c \in \mathbb{R}\), the quantity
\[
\partial_t(H(v_\varepsilon - c)(h(x, v_\varepsilon) - h(x, c))) + \partial_x(H(v_\varepsilon - c)((H(x)(f(v_\varepsilon) - f(c)) + H(-x)(g(v_\varepsilon) - g(c))))
\]
is precompact in \(W^{-1,2}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}),\) contains a subsequence convergent in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}).\)

So, our last obstacle is the genuine nonlinearity condition. In order to overcome it we shall use an idea from [21] which is further developed in [4]. In [21, 4], existence of solution to a Cauchy problem of type (1) is proved. Roughly speaking, the key point of the proof is based on a lemma stating that if in (1) we assume \(u_0 \in BV(\mathbb{R}),\) then, for the sequence \((u_\varepsilon)\) of solutions to (6), it holds \(\|\partial_t u_\varepsilon\|_{L^1(\mathbb{R})} \leq \text{const}\) for every fixed \(t, \varepsilon \in \mathbb{R}^+\). This actually means that for any function \(h(x, \lambda), x, \lambda \in \mathbb{R},\) which is Lipshitz continuous in \(\lambda,\) it holds \(\|\partial_t h(x, u_\varepsilon)\|_{L^1(\mathbb{R})} \leq \text{const}\) for every fixed \(t, \varepsilon \in \mathbb{R}^+\).

Next, it is not difficult to prove that it holds for the sequence \((u_\varepsilon)\) of solutions to (6)
\[
\partial_t(H(u_\varepsilon - c)(u_\varepsilon - c)) + \partial_x(H(u_\varepsilon - c)(H(x)(f(u_\varepsilon) - f(c)) + H(-x)(g(u_\varepsilon) - g(c))))
\]
is precompact in \(W^{-1,2}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\). However, since \(\|\partial_t u_\varepsilon\|\) is the sequence bounded in the space of Radon measures, we also have:
\[
\partial_t(H(u_\varepsilon - c)(H(x)(h_R(u_\varepsilon) - h_R(c)) + H(-x)(h_L(u_\varepsilon) - h_L(c))))
\]
is precompact in \(W^{-1,2}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)\) if \(h_L, h_R \in \text{Lip}(\mathbb{R})\) (Lipschitz continuous functions). Furthermore, if we choose \(h_L\) and \(h_R\) so that the vector \((H(x)h_R(u) + H(-x)h_L(u), H(x)f(u) + H(-x)g(u))\) is genuinely nonlinear, we can apply Theorem 0.8 to conclude about strong \(L^1_{\text{loc}}\) precompactness of the family \((u_\varepsilon)\. It is clear that a \(L^1_{\text{loc}}\) limit along a subsequence of the family \((u_\varepsilon)\) will represent wanted admissible weak solution to (1). Furthermore, according to Theorem 0.6, we infer about the existence of traces at the interface \(x = 0\) for the previously constructed weak solution which immediately gives uniqueness. Of course, it is not always possible to choose \(h_R\) and \(h_L\) so that we have both, the genuine nonlinearity and the crossing conditions fulfilled. Still, as we shall see, using truncation functions \(s_{l,k}(u) = \max\{l, \min\{k, u\}\}, \ l < k, l, k \in \mathbb{R}\), (first used in [27] for this kind of problems; see also [15]), we are able to localize and thus deal with the segments where the genuine nonlinearity is unobtainable.

The paper is organized as follows.
In Section 1, we solve (1) under additional assumptions on the flux. We find the section important since it sheds (another) light on paper [8] where the crossing condition is bypassed by using so called adapted entropies (see [6]). We show that admissibility conditions that we introduced in Definition 0.4 can be considered as a generalization of the approach from [8], which is actually an explanation how adapted entropies enabled avoiding (or maybe better to say forced) the crossing conditions.

In Section 2, by passing to the measure valued solution concept [13], we show existence and uniqueness in the general situation.
1. New Entropy Admissibility Conditions

The basic purpose of the section is to explain connection between our \((\alpha, \beta)\)-entropy solutions and the entropy solutions of type \((A, B)\) used in [8]. Furthermore, we find that this section represents a good introduction into the general situation considered in Section 3.

We shall consider here \((1)\) under the additional assumptions that the mappings

\[ \lambda \mapsto f(\lambda), \quad \lambda \mapsto g(\lambda) \]

are nonconstant and strictly positive on any subinterval of the interval \((a, b)\) (notice that this assumption is weaker than the appropriate assumption \([8, (1.2)]\) which demands a genuine nonlinearity of \(f\) and \(g\)).

To proceed, let us briefly recall the concept from \([8]\). First, we need the function \(c^{AB}\) (see \([8, (11)]\)):

\[ c^{AB}(x) = \begin{cases} A, & x \leq 0 \\ B, & > 0 \end{cases} \]

In \([8]\), the function \(c^{AB}\) is used to form the function \(u \mapsto |u - c^{AB}(x)|\) which is an example of what is in \([6]\) called an adapted entropy. Still, in \([6]\), the existence of infinitely many adapted entropies was necessary to prove uniqueness (see also \([28]\)) while in \([8]\) only the entropy \(u \mapsto |u - c^{AB}(x)|\) was sufficient (together with the classical Kruzhkov entropies out of the interface). The function \(c^{AB}\) is called a connection if it represents a weak solution to \((1)\), i.e., if \(f(B) = g(A)\) (see Remark 2 for a more precise explanation). We remark that the notion of the connection originated from \([2]\). The following admissibility conditions were used in \([8]\):

**Definition 1.1.** \([8, \text{Definition } 3.1.]\) (Entropy solution of type \((A, B)\)). A measurable function \(u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\), representing a weak solution to \((1)\) is an entropy solution of type \((A, B)\) if it satisfies the following conditions:

1. \((D.1)\) \(u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\); \(u(t, x) \in [a, b]\) for a.e. \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\).
2. \((D.2)\) For any test function \(0 \leq \varphi \in \mathcal{D}([0, T] \times \mathbb{R})\), \(T > 0\), which vanishes for \(x \geq 0\), and any \(\xi \in \mathbb{R}\), the following holds:

\[ \int_0^T \int_{\mathbb{R}} (\langle u - \xi \rangle \varphi_t + \text{sgn}(u - \xi)(f(u) - f(\xi))\varphi_x) \, dx \, dt + \int_{\mathbb{R}} |u_0 - \xi| \varphi(0, x) \, dx \geq 0, \]

and for any test function \(0 \leq \varphi \in \mathcal{D}([0, T] \times \mathbb{R})\), \(T > 0\), which vanishes for \(x \leq 0\)

\[ \int_0^T \int_{\mathbb{R}} (\langle u - \xi \rangle \varphi_t + \text{sgn}(u - \xi)(g(u) - g(\xi))\varphi_x) \, dx \, dt + \int_{\mathbb{R}} |u_0 - \xi| \varphi(0, x) \, dx \geq 0, \]

3. \((D.3)\) The following Kruzhkov-type entropy inequality holds for any test function \(0 \leq \varphi \in \mathcal{D}([0, T] \times \mathbb{R})\), \(T > 0\),

\[ \int_0^T \int_{\mathbb{R}} \left( |u - c^{AB}(x)| \varphi_t + \text{sgn}(u - c^{AB}(x))(H(u)(f(u) - f(A)) + H(-x)(g(u) - g(B)))\varphi_x \right) \, dx \, dt + \int_{\mathbb{R}} |u_0 - c^{AB}(x)| \varphi(0, x) \, dx \geq 0. \]

In the next theorem, we state that the \((\alpha, \beta)\)-entropy admissible solution from Definition 0.4 is, under certain conditions, at the same time an entropy solution of
type \((A, B)\) from Definition 1.1. In Remark 2 after the theorem, we shall explain why such conditions are always fulfilled in the case of the flux given in [8].

**Theorem 1.2.** Assume that the function \(u\) is an \((\alpha, \beta)\)-entropy admissible solution to (1) in the sense of Definition 0.4 where \(\alpha\) and \(\beta\) satisfy:

- \(\alpha, \beta : [a, b] \to [a, b]\);
- there exists \(c \in (a, b)\) such that \(\alpha(c) = B\) and \(\beta(c) = A\) where \(f(B) = g(A)\);
- the functions \(f \circ \alpha\) and \(g \circ \beta\) satisfy the crossing conditions.

Then, the \((\alpha, \beta)\)-entropy admissible solution to (1) is at the same time the entropy solution of type \((A, B)\).

**Proof.** First, notice that, according to the choice of \(\alpha\) and \(\beta\), the function \(e^{AB}\) will represent an \((\alpha, \beta)\)-entropy admissible solution to (1) in the sense of Definition 0.4. Taking another \((\alpha, \beta)\)-entropy admissible solution to (1), say \(u = \alpha(v)H(x) + \beta(v)H(-x)\), and applying the procedure from [19] leading to [19, (2.34)] (keep in mind that \(f_\alpha\) and \(g_\beta\) satisfy the crossing conditions), we reach to the following (well known) relation:

\[
\begin{aligned}
&\partial_t \text{sgn}(v - c) \left( (\alpha(v) - \alpha(c)) + H(-x)(\beta(v) - \beta(c)) \right) \\
&+ \partial_x \text{sgn}(v - c) (H(x)(f_\alpha(v) - f_\alpha(c)) + H(-x)(g_\beta(v) - g_\beta(c))) \leq 0.
\end{aligned}
\]  

(11)

Since \(\alpha\) and \(\beta\) as well as their inverses \(\tilde{\alpha}\) and \(\tilde{\beta}\) are increasing bijections, it holds

\[
\text{sgn}(v - c) = \text{sgn} \left( (\tilde{\alpha}(u) - \tilde{\alpha}(B))H(x) + (\tilde{\beta}(u) - \tilde{\beta}(A))H(x) \right) = \text{sgn}(u - c^{AB}).
\]

From here, we see that (11) is actually condition (D.3) from Definition 1.1 meaning that the \((\alpha, \beta)\)-entropy admissible solution \(u\) is, at the same time, an entropy solution of type \((A, B)\) (conditions (D.1.) and (D.2.) from Definition 1.1 are easily checked).

**Remark 2.** The notion of connection used in [8] relied on the case when the functions \(f\) and \(g\) forming the flux in (1) were such that they admit unique local maxima points \(u^*_f \in (a, b)\) and \(u^*_g \in (a, b)\), respectively. Then, the pair \((A, B)\) is called a connection if

\[
f(B) = g(A) \quad \text{with} \quad u^*_g \leq A \leq B \leq u^*_f.
\]

(12)

In this case, we can always find functions \(\alpha\) and \(\beta\) such that conditions of Theorem 1.2 are satisfied.

Indeed, assume that \(u^*_f < u^*_g\) (other two situations \(u^*_f > u^*_g\) and \(u^*_f = u^*_g\) can be resolved similarly). Denote by \(\tilde{\alpha}\) and \(\tilde{\beta}\) inverse functions to the functions \(\alpha\) and \(\beta\), respectively. Choose \(\tilde{\alpha}\) and \(\tilde{\beta}\) on the intervals \([u^*_f, b]\) and \([a, u^*_g]\) to be linear and such that \(\tilde{\alpha}(u^*_f) > c > \tilde{\beta}(u^*_g)\) (see Figure 2; the situation plotted there is more general but completely analogical with the one we are considering at the moment).

To extend the function \(\tilde{\alpha}\) in the interval \([a, u^*_f]\), we will construct its inverse \(\alpha\) in the interval \([a, c]\). Take an arbitrary decreasing function \(\tilde{\alpha}\) connecting the points \((a, 0)\) and \((c, f_\alpha(c))\) such that \(\tilde{\alpha} \leq g_\beta\) on \([a, c]\). This is always possible since \(g_\beta > 0\) on \((a, c)\); for instance, we can take \(\tilde{\alpha}\) to be the convex hull of \(g_\beta\) on \([a, c]\). Then, put \(\alpha = f^{-1} \circ \tilde{\alpha}\) i.e. \(\tilde{\alpha} = \alpha^{-1}\) on \([a, u^*_f]\) (this is permitted since \(f^{-1}\) is monotonic on \((a, c)\)). We choose \(\tilde{\beta}\) on \([u^*_g, b]\) in the completely same manner (see Figure 2 for further clarification). It is clear that \(\alpha = \tilde{\alpha}^{-1}\) and \(\beta = \tilde{\beta}^{-1}\) chosen in such a way satisfy conditions of Theorem 1.2.
Actually, from the latter discussion, we can conclude that the conditions given in Theorem 1.2 are a generalization of the notion of connection. More precisely, we can say that a pair \((A, B)\) is a connection if there exist functions \(\alpha\) and \(\beta\) satisfying conditions of Theorem 1.2. As we shall see in Theorem 1.3, such conditions provide existence and uniqueness of the \((\alpha, \beta)\)-entropy admissible solution to (1). In particular, the function \(BH(x) + AH(-x)\) will be the \((\alpha, \beta)\)-entropy admissible shock.

Also, remark that conditions (12) can be naturally generalized by assuming that

\[
f(B) = g(A) \quad \text{with} \quad A \in (u^*_\alpha, b), \; B \in (a, u^*_\beta),
\]

where \(u^*_\alpha\) and \(u^*_\beta\) are the rear right local maximum of the function \(g\) and the rear left local maximum of the function \(f\), respectively. Repeating the procedure from the beginning of the remark, we can find the function \(\alpha\) and \(\beta\) such that conditions of Theorem 1.2 are satisfied (see Figure 2).

Finally, notice that if \(A = u^*_g\) and \(B = u^*_\beta\), we cannot state that the functions \(\alpha\) and \(\beta\) satisfying conditions of Theorem 1.2 exist (for instance, if the functions \(f\) and \(g\) have several local maxima, and all of them have the same values).

The following theorem is the main theorem of the section:

**Theorem 1.3.** There exists a pair of function \((\alpha, \beta)\) from Definition 0.4 such that there exists a unique \((\alpha, \beta)\)-entropy admissible solution to (1).

For such \(\alpha\) and \(\beta\) any two \((\alpha, \beta)\)-entropy admissible solutions \(u\) and \(v\) to (1) satisfy (2).

Before we prove the theorem, we shall need several auxiliary statements and explanations.

In order to construct an \((\alpha, \beta)\)-entropy admissible solution to (1), we use a non-standard vanishing viscosity approximation with regularized flux. First, introduce the following change of the unknown function \(u\):

\[
u(t, x) = \tilde{\alpha}(v(t, x))H(x) + \tilde{\beta}(v(t, x))H(-x),
\]

for increasing functions \(\tilde{\alpha}, \tilde{\beta} : [a, b] \to [a, b]\). Denote by \(\alpha = \tilde{\alpha}^{-1}\) and \(\beta = \tilde{\beta}^{-1}\). Equation (1) becomes:

\[
\partial_t (H(x)\alpha(v) + H(-x)\beta(v)) + \partial_x (H(x)f_\alpha(v) + H(-x)g_\beta(v)) = 0.
\]

Then, take the following regularization of the Heaviside function \(H\), \(H_\epsilon(x) = \int_{-\infty}^{x/\epsilon} \omega(z)dz\), where \(\omega\) is a smooth even compactly supported function with total mass one. Let \(\chi_\epsilon\) be a smooth function equal to one in the interval \((-1/\epsilon, 1/\epsilon)\) and zero out of the interval \((-2/\epsilon, 2/\epsilon)\). Consider the following regularized problem:

\[
\partial_t (H_\epsilon(x)\alpha(v_\epsilon) + H_\epsilon(-x)\beta(v_\epsilon)) \\
+ \partial_x (H_\epsilon(x)f_\alpha(v_\epsilon) + H_\epsilon(-x)g_\beta(v_\epsilon)) = \epsilon \partial_{xx}v_\epsilon \\
v_\epsilon \big|_{\epsilon=0} = (\tilde{\alpha}(u_0)H(x) + \tilde{\beta}(u_0)H(-x)) \star \frac{1}{\epsilon} \omega(\cdot/\epsilon)\chi_\epsilon(x)
\]

Obviously, for every fixed \(\epsilon > 0\) quasilinear parabolic Cauchy problem (15) will have a unique smooth solution \(v_\epsilon\).

Since \(\alpha\) and \(\beta\) are strictly increasing functions which map interval \([a, b]\) into itself, slightly modifying the methodology from [21], we obtain the following three lemmas.
Lemma 1.4. [21, Lemma 4.1] \([L^\infty\text{-bound}]\) There exists constant \(c_0 > 0\) such that for all \(t \in (0, T)\),
\[
\|v_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq c_0.
\]
More precisely,
\[
a \leq v_\varepsilon \leq b.
\]

Lemma 1.5. [21, Lemma 4.2] \([\text{Lipshitz regularity in time}]\) Assume that the initial function \(u_0\) from (1) has bounded variation. Then, there exists constant \(c_1\), independent of \(\varepsilon\), such that for all \(t > 0\),
\[
\int_\mathbb{R} |\partial_t v_\varepsilon(\cdot, t)| \, dx \leq c_1.
\]

Lemma 1.6. [21, Lemma 4.3] \([\text{Entropy dissipation bound}]\) There exists a constant \(c_2\) independent from \(\varepsilon\) such that for all \(t > 0\),
\[
\varepsilon \int_\mathbb{R} (\partial_x v_\varepsilon(t, x))^2 \, dx \leq c_2,
\]

To proceed, we need Murat’s lemma:

Lemma 1.7. [14] Assume that the family \((Q_\varepsilon)\) is bounded in \(L^p(\Omega), \Omega \subset \mathbb{R}^d, p > 2\).
Then,
\[
(\text{div} \, Q_\varepsilon)_\varepsilon \in W^{-1,2}_{c,\text{loc}} \quad \text{if} \quad \text{div} \, Q_\varepsilon = p_\varepsilon + q_\varepsilon,
\]
with \((q_\varepsilon)_\varepsilon \in W^{-1,2}_{c,\text{loc}}(\Omega)\) and \((p_\varepsilon)_\varepsilon \in \mathcal{M}_{b,\text{loc}}(\Omega)\).

Now, we can prove a crucial lemma for obtaining the existence of the \((\alpha, \beta)\)-entropy admissible solution to (1).

Lemma 1.8. Denote for a fixed \(\xi \in \mathbb{R}\):
\[
q(x, \lambda) = H(\lambda - \xi) \left( H(x)(f_\alpha(\lambda) - f_\alpha(\xi)) + H(-x)(g_\beta(\lambda) - g_\beta(\xi)) \right),
\]
\[
\bar{q}(x, \lambda) = H(\lambda - \xi) \left( H(x)(f_\alpha^2(\lambda) - f_\alpha^2(\xi)) + H(-x)(g_\beta^2(\lambda) - g_\beta^2(\xi)) \right), \quad (16)
\]
\[
q_{\alpha, \beta}(x, \lambda) = H(\lambda - \xi) \left( H(x)(\alpha(\lambda) - \alpha(\xi)) + H(-x)(\beta(\lambda) - \beta(\xi)) \right).
\]
If the initial function \(u_0\) from (1) has bounded variation then the family
\[
\partial_t \bar{q}(x, v_\varepsilon) + \partial_x q(x, v_\varepsilon), \quad \varepsilon > 0,
\]
is precompact in \(W^{-1,2}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})\).

Proof:
Denote \(n'(\lambda) = H(\lambda - \xi)\). Define the entropy flux which corresponds to (15):
\[
q^e(x, \lambda) = H(\lambda - \xi) \left( H_e(x)(f_\alpha(\lambda) - f_\alpha(\xi)) + H_e(-x)(g_\beta(\lambda) - g_\beta(\xi)) \right),
\]
\[
q_{\alpha, \beta}^e(x, \lambda) = H(\lambda - \xi) \left( H_e(x)(\alpha(\lambda) - \alpha(\xi)) + H_e(-x)(\beta(\lambda) - \beta(\xi)) \right).
\]
Denote $\delta_{\varepsilon}(x) = H_\varepsilon'(x)$, $i = 1, 2$. After multiplying (15) by $\eta'(v_\varepsilon)$, we obtain in the sense of distributions:

$$
\partial_t q_{\alpha, \beta}^\varepsilon(x, v_\varepsilon) + \partial_x q^\varepsilon(x, v_\varepsilon)
= (\delta_{\varepsilon}(x)f_\alpha(\xi) - \delta_{\varepsilon}(x)g_\beta(\xi) + \varepsilon(\partial_x(v_{xx} \eta'(v_\varepsilon))) - (v_{xx})^2 \eta''(v_\varepsilon)
\leq \delta_{\varepsilon}(x)(f_\alpha(\xi) - g_\beta(\xi)) + \varepsilon(\partial_x(v_{xx} \eta'(v_\varepsilon))).
$$

From here, according to the Schwartz lemma for non-negative distributions, we conclude that there exists a positive Radon measure $\mu^\varepsilon(t, x)$ such that:

$$
\partial_t q_{\alpha, \beta}(x, v_\varepsilon) + \partial_x q(x, v_\varepsilon)
= \delta_{\varepsilon}(f_\alpha(\xi) - g_\alpha(\xi)) + \varepsilon(\partial_x(v_{xx} \eta'(v_\varepsilon))) - \mu^\varepsilon(t, x).
$$

Rewrite expression (19) in the form:

$$
\partial_t \tilde{q}(x, v_\varepsilon) + \partial_x q(x, v_\varepsilon)
= \partial_t (\tilde{q}(x, v_\varepsilon) - q_{\alpha, \beta}^\varepsilon(x, v_\varepsilon)) + \partial_x (q^\varepsilon(x, v_\varepsilon) - q(x, v_\varepsilon))
+ \delta_{\varepsilon}(f_\alpha(\xi) - g_\beta(\xi)) + \varepsilon(\partial_x(v_{xx} \eta'(v_\varepsilon))) - \mu^\varepsilon(t, x).
$$

Since, clearly, $q^\varepsilon(x, v_\varepsilon) - q(x, v_\varepsilon) \to 0$ as $\varepsilon \to 0$ pointwisely, we derive the statement of the lemma from the Lebesgue dominated convergence theorem, Lemmas 1.4-1.6, and Lemma 1.7. For details please consult [4, Theorem 2.6.] □

From Lemma 1.8 and Theorem 0.8, it is easy to prove that for any choice of the functions $\alpha$ and $\beta$ from Definition 0.4 there exists an $(\alpha, \beta)$-entropy admissible solution to (1) provided $u_0 \in BV(\mathbb{R})$:

**Theorem 1.9.** Assume that $u_0 \in BV(\mathbb{R})$, and that $f$ and $g$ satisfy (10), where $u_0$, $f$ and $g$ are given in (1). For any bijections $\alpha, \beta : [a, b] \to [a, b]$ from Definition 0.4 there exists an $(\alpha, \beta)$-entropy admissible weak solution to (1).

**Proof:** First, notice that the vector $(\tilde{q}(x, \lambda), q(x, \lambda))$ from (16) is genuinely nonlinear. Indeed, for $x > 0$ the vector reduces to $(f_\alpha^2(\lambda), f_\alpha(\lambda))$ and this is obviously genuinely nonlinear vector according to (10). Similarly, we conclude about the genuine nonlinearity for $x < 0$.

Now, from Theorem 0.8 and Lemma 1.8, we conclude that the family $(v_\varepsilon)$ of solutions to (15) is strongly precompact in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Denote by $v$ the $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ limit along a subsequence of the family $(v_\varepsilon)$. Clearly, $u = \alpha(v)H(x) + \beta(v)H(-x)$ will represent the $(\alpha, \beta)$-entropy admissible solution to (1). □

Now, we can prove the main theorem of the section.

**Proof of Theorem 1.3:** We need to find the functions $\alpha$ and $\beta$ so that the functions $f_\alpha$ and $g_\beta$ satisfy the crossing conditions. As explained in Remark (2), we choose the points $A, B \in (a, b)$ satisfying (13), and construct the functions $\alpha$ and $\beta$ so that for appropriate $c \in (a, b)$ it holds $\alpha(c) = B, \beta(c) = A,$ and $f_\alpha \geq g_\beta$ on $[c, b],$ and $f_\alpha \leq g_\beta$ on $[a, c]$ which is nothing else but the crossing condition for $f_\alpha$ and $g_\beta$.

Next, assume that $u_0 \in BV(\mathbb{R})$ and denote by $u$ the $(\alpha, \beta)$-entropy admissible solution to (1) (it is given by Theorem 1.9). Notice that from the construction (it is enough to let $\varepsilon \to 0$ in (20)) and Lemmas 1.4-1.6, it follows that the function $v = \tilde{\alpha}(u)H(x) + \tilde{\beta}(u)H(-x)$ is, at the same time, a quasi-solution to the equation:

$$
\partial_t \left( H(x)f_\alpha^2(v) + H(-x)g_\beta^2(v) \right) + \partial_x \left( H(x)f_\alpha(v) + H(-x)g_\beta(v) \right) = 0,
$$

□
Since the vector \((H(x)f_\alpha^2(\lambda) + H(-x)g_\beta^2(\lambda), H(x)f_\alpha(\lambda) + H(-x)g_\beta(\lambda))\) is genuinely nonlinear (see (10)), according to Theorem 0.6, the function \(v\) admits strong traces at the interface \(x = 0\).

Similarly, from the construction again and according to the choice of the function \(\alpha\) and \(\beta\), we see that \(v\) is an entropy admissible solution in the sense of Definition 0.1 to the Cauchy problem

\[
\begin{align*}
\partial_t (\alpha(v)H(x) + \beta(v)H(-x)) + \partial_x (H(x)f_\alpha(v) + H(-x)g_\alpha(v)) &= 0, \\
v|_{t=0} &= \alpha(u_0)H(x) + \beta(u_0)H(-x),
\end{align*}
\]

(22)

where \(f_\alpha\) and \(g_\beta\) satisfy the crossing condition.

According to Theorem 0.3, we conclude that \(v\) is a unique entropy admissible solution to (22) in the sense of Definition 0.1 implying that \(u = \alpha(v)H(x) + \beta(v)H(-x)\) is a unique \((\alpha, \beta)\)-entropy admissible solution to (1).

Now, assume that \(u_0 \notin BV(\mathbb{R})\). Approximate the function \(u_0\) by a sequence \((u_{\delta}) \in BV(\mathbb{R})\) so that

\[u_0 - u_{\delta} \to 0 \text{ as } \delta \to 0\]

strongly in \(L^1_{loc}(\mathbb{R})\). Then, we find a unique \((\alpha, \beta)\)-entropy admissible solution \(u_{\delta}\) to (1) where \(u_{\delta}|_{t=0} = u_0\) (given \(\alpha\) and \(\beta\) for which we have uniqueness i.e. such that \(f_\alpha\) and \(g_\beta\) satisfy the crossing conditions). According to Theorem 1.3, the family \((u_{\delta})\) satisfy the following stability relation:

\[
\int_0^T \int_{-R}^R |u_{\delta_1} - u_{\delta_2}|dxdt \leq CT \int_{-R}^R |u_{\delta_1} - u_{\delta_2}|dx,
\]

where \(R\) and \(T\) are arbitrary positive constants, and \(C, \tilde{R}\) are constants depending on \(R\), the functions \(f, g, \alpha\) and \(\beta\). Since the right-hand side of the latter expression is uniformly small with respect to \(\delta_1\) and \(\delta_2\), from the Cauchy criterion we conclude that there exists \(u \in L^1_{loc}\) such that \(u_{\delta} \to u\) strongly in \(L^1_{loc}(\mathbb{R}^d)\). Clearly, the function \(u\) will represent an \((\alpha, \beta)\)-entropy admissible solution to (1).

Since, according to (10) and Theorem 0.6, the function \(u\) admits strong traces at \(x = 0\), we conclude that it must be a unique \((\alpha, \beta)\)-entropy admissible solution to (1). □

2. General case

At the beginning, notice that there are many examples of fluxes from (1) when we cannot apply the procedure from the previous section (see Figure 3). Therefore, in this section, we shall demonstrate how to apply the \((\alpha, \beta)\)-entropy admissibility concept on (1) in a general case. More precisely, we shall only assume that \(f, g \in C^1(\mathbb{R})\) are such that \(f(a) = f(b) = g(a) = g(b) = 0\), and, for simplicity, that there exists a finite number of intervals \((a_{r_i}, a_{r_{i+1}}), j = 1, \ldots, k_r, (b_i, b_{i+1}), i = 1, \ldots, k_1, k_1, k_r \in \mathbb{N}\), such that the mappings

\[
\lambda \mapsto g(\lambda) \text{ and } \lambda \mapsto f(\lambda) \text{ are constant on the intervals}
\]

\((b_i, b_{i+1}), i = 1, \ldots, k_1\), and \((a_{r_j}, a_{r_{j+1}}), j = 1, \ldots, k_r\), respectively.

(23)

For a convenience, assume that \([a, b] = \bigcup_{i=1}^{n_1}[a_i, a_{i+1}]\) and \([a, b] = \bigcup_{i=1}^{n_2}[b_i, b_{i+1}]\), where \(n_1, n_2 \in \mathbb{N}\), and \(a_{1} = b_1 = a\), and \(a_{n_1} = b_{n_2} = b\).

We shall need the notion of Young measures (we will be highly selective and, for an application of Young measures in conservation laws, address a reader on famous paper [13]).
Figure 3. Functions $f$ (normal line) and $g$ (dashed line) do not satisfy the crossing condition and there exist no increasing bijections $\alpha, \beta : [a, b] \to [a, b]$ such that $f \circ \alpha$ and $g \circ \beta$ satisfy the crossing conditions.

Theorem 2.1. \cite{31} Assume that the sequence $(u_{\varepsilon k})$ is uniformly bounded in $L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, $p \geq 1$. Then, there exists a subsequence (not relabeled) $(u_{\varepsilon k})$ and a family of probability measures $\nu_{t,x} \in \mathcal{M}(\mathbb{R})$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$

such that the limit

$$\bar{g}(t, x) := \lim_{k \to \infty} g(u_{\varepsilon k}(t, x))$$

exists in the distributional sense for all $g \in C(\mathbb{R})$. The limit is represented by the expectation value

$$\bar{g}(t, x) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(\lambda) d\nu_{t,x}(\lambda),$$

for almost all points $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. We refer to such a family of measures $\nu = (\nu_{t,x})_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}}$ as the Young measure associated to the sequence $(u_{\varepsilon k})_{k \in \mathbb{N}}$.

Furthermore,

$$u_{\varepsilon k} \rightharpoonup u \text{ in } L^r_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d), 1 \leq r < p$$

if and only if

$$\nu_{t,x}(\lambda) = \delta(\lambda - u(t, x)) \text{ a.e. } (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

where $\delta$ is the Dirac distribution.

Introduce the truncation operator $s_{l,k}(u) = \max\{l, \min\{k, u\}\}, l < k, l, k \in \mathbb{R}$. The following important lemma holds.

Lemma 2.2. Denote by $(v_\varepsilon)$ family of solutions to (15) where $u_0 \in BV(\mathbb{R}; [a, b])$ and $f$ and $g$ satisfy (23). Assume that the mapping $\lambda \mapsto f(\lambda)$ is not constant on any subinterval of an interval $(l, k)$. Then, the sequence $(H(x)s_{l,k}(v_\varepsilon))$ is strongly precompact in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$.

Similarly, if the mapping $\lambda \mapsto g(\lambda)$ is not constant on any subinterval of an interval $(l, k)$. Then, the sequence $(H(-x)s_{l,k}(v_\varepsilon))$ is strongly precompact in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$.

Proof. Notice that from Lemma 1.8, it follows that for the family of functions $v_\varepsilon$ and any $k, l \in \mathbb{R}$, the families

$$\partial_t \bar{q}(x, H(x)s_{l,k}(v_\varepsilon)) + \partial_x q(x, H(x)s_{l,k}(v_\varepsilon)) \text{ and}$$

$$\partial_t \bar{q}(x, H(-x)s_{l,k}(v_\varepsilon)) + \partial_x q(x, H(-x)s_{l,k}(v_\varepsilon)).$$

(24)
where the functions \( \tilde{q}, q \) given by 16, are strongly precompact in \( W_{loc}^{-1,2}(\mathbb{R}^+ \times \mathbb{R}) \).

Indeed, notice that
\[
q(x, H(x)s_{l,k}(v_\varepsilon)) = H(x)q(x, s_{l,k}(v_\varepsilon)) - H(-\xi)H(-x)(g_\beta(0) - g_\beta(\xi)) \\
\tilde{q}(x, H(x)s_{l,k}(v_\varepsilon)) = H(x)\tilde{q}(x, s_{l,k}(v_\varepsilon)) - H(-\xi)H(-x)(g_\beta^2(0) - g_\beta^2(\xi)).
\]

(25)

Since \( \partial_t q(x, s_{l,k}(v_\varepsilon)) + \partial_x q(x, s_{l,k}(v_\varepsilon)) \) is strongly precompact in \( W_{loc}^{-1,2}(\mathbb{R}^+ \times \mathbb{R}) \) if \( \partial_t \tilde{q}(x, v_\varepsilon) + \partial_x \tilde{q}(x, v_\varepsilon) \) is (see [30, Theorem 6]), we conclude from (25) that (24) holds.

Furthermore, notice that if the mapping \( \lambda \mapsto f(\lambda) \) is not constant on any subinterval of an interval \((k, l)\) then the vector \((\tilde{q}(x, \lambda), q(x, \lambda))\) from (16) is genuinely nonlinear on the interval \((l, k)\) and \( x > 0 \). Indeed, for \( x > 0 \) the vector reduces to \((f^2_\alpha(\lambda), f_\alpha(\lambda))\) and this is obviously genuinely nonlinear vector since, due to the assumptions of the lemma, for any \( \xi_0, \xi_1 \in \mathbb{R} \), it holds \( \xi_0 f_\alpha^2(\lambda) \neq \xi_1 f(\lambda) \) for a.e. \( \lambda \in (k, l) \).

Now, from Theorem 0.8 and Lemma 1.8, we conclude that the family \((H(x)s_{k,l}(v_\varepsilon))\) is strongly precompact in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \).

In the completely same way, we conclude that the family \((H(-x)s_{k,l}(v_\varepsilon))\) is strongly precompact in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) if the mapping \( \lambda \mapsto g(\lambda) \) is different from a constant on every subinterval of the interval \((k, l)\).

Next lemma deals with precompactness properties of the family \((f(v_\varepsilon)H(x) + g(v_\varepsilon)H(-x))\).

**Lemma 2.3.** Assume that the flux functions \( f \) and \( g \) from (1) satisfy (23). Denote by \((v_\varepsilon)\) family of solutions to (15) with \( u_0 \in BV(\mathbb{R}; [a, b]) \). Then, there exists a function \( v \in L^\infty(\mathbb{R}) \) such that
\[
f(v_\varepsilon)H(x) + g(v_\varepsilon)H(-x) \to f(v)H(x) + g(v)H(-x)
\]
strongly in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \). Moreover, the function \( v \) admits left and right traces at the interface \( x = 0 \).

**Proof.** Denote
\[
\tilde{v}_\varepsilon(t, x) = \begin{cases} 
v_\varepsilon(t, x), & v_\varepsilon(t, x) \notin \cup_{i=1}^k [a_i, a_{i+1}), \quad x > 0 \\
v_\varepsilon(t, x), & v_\varepsilon(t, x) \notin \cup_{i=1}^k [b_i, b_{i+1}), \quad x \leq 0, \\
ar_{r_j}, & v_\varepsilon(t, x) \in [a_{r_j}, a_{r_j+1}), \quad x > 0, \\
b_{r_j}, & v_\varepsilon(t, x) \in [b_{r_j}, b_{r_j+1}), \quad x \leq 0.
\end{cases}
\]

(27)

Notice that \( f(v_\varepsilon)H(x) + g(v_\varepsilon)H(-x) = f(\tilde{v}_\varepsilon)H(x) + g(\tilde{v}_\varepsilon)H(x) \) according to assumptions (23). Then, notice that
\[
\tilde{v}_\varepsilon = H(x) \left( \sum_{i=1}^{n_1} s_{a_i, a_{i+1}}(\tilde{v}_\varepsilon) - \sum_{i=2}^{n_1} a_i \right) + H(-x) \left( \sum_{i=1}^{n_2} s_{b_i, b_{i+1}}(\tilde{v}_\varepsilon) - \sum_{i=2}^{n_2} b_i \right).
\]

(28)

According to Lemma 2.2 and the definition of the function \( \tilde{v}_\varepsilon \), it is easy to see that \((\tilde{v}_\varepsilon)\) is strongly precompact in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) (since this property has each of the summands on the right-hand side of (28)). Denote an accumulation point of the family \((\tilde{v}_\varepsilon)\) by \( v \). Clearly, the function \( v \) satisfies (26).

In order to prove that the function \( v \) admits traces at the interface, denote by \( H(x)s_{a_i, a_{i+1}}, \; i = 1, \ldots, n_r \), and \( H(-x)s_{b_i, b_{i+1}}, \; i = 1, \ldots, n_l \), strong \( L^1_{loc} \)-limits along subsequences of the families \((s_{a_i, a_{i+1}}(\tilde{v}_\varepsilon)), \; i = 1, \ldots, n_r \), and \((s_{b_i, b_{i+1}}(\tilde{v}_\varepsilon)), \; i = 1, \ldots, n_l \),
Assume that the functions \( \nu \) and \( \phi \) function satisfying (4). By applying the standard procedure (see proof of Lemma 2.3), it is not difficult to see that every \( v \) satisfies for every \( \xi \in \mathbb{R}:
\begin{align*}
\partial_t (\text{sign}(v_i - \xi)) ( (\alpha(v_i) - \alpha(\xi)) H(x) + (\beta(v_i) - \beta(\xi)) H(-x) ) \\
+ \partial_x ((f_\alpha(v_i) - f_\alpha(\xi)) H(x) + (g_\beta(v_i) - g_\beta(\xi)) H(-x)) \leq O_D(\varepsilon),
\end{align*}

where \( O_D(\varepsilon) \) is a family of distributions tending to zero in the sense of distributions as \( \varepsilon \to 0 \). Letting \( \varepsilon \to 0 \) in (30) and taking Lemma 2.3 and Theorem 2.1 into account, we obtain in \( D'(\mathbb{R}^+ \times \mathbb{R}) \):
\begin{align*}
\partial_t \int_{\mathbb{R}} \text{sign}(\lambda - \xi) ( (\alpha(\lambda) - \alpha(\xi)) H(x) + (\beta(\lambda) - \beta(\xi)) H(-x) ) d\nu_{t,x}(\lambda) \\
+ \partial_x ((f_\alpha(v) - f_\alpha(\xi)) H(x) + (g_\beta(v) - g_\beta(\xi)) H(-x)) \leq 0,
\end{align*}

where \( \nu_{t,x} \) is a Young measure corresponding to the sequence \( (v_i) \), and \( v \) is the function satisfying (26). The Young measure \( \nu_{t,x} \) and the function \( v \) (admitting strong traces at \( x = 0 \)), we shall call an \((\alpha, \beta)\)-entropy admissible measure valued solution to (1).

Denote by \( \sigma_{t,x} \) a Young measure and by \( w \) a function representing an \((\alpha, \beta)\)-entropy admissible measure valued solution to (1) corresponding to initial data \( v_0 \in BV(\mathbb{R}; [a, b]) \).

Using the classical arguments by DiPerna \cite{DiPerna1985}, we conclude that for any test function \( \varphi \in C^0_c(\mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})) \) it holds (keep in mind that \( \alpha \) and \( \beta \) are strictly increasing functions):
\begin{align*}
\int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^2} |\alpha(\lambda) - \alpha(\xi)| H(x) + |\beta(\lambda) - \beta(\eta)| H(-x) \partial_t \varphi d\nu_{t,x}(\lambda) d\sigma_{t,x}(\eta) dx dt \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}} ( (f_\alpha(v) - f_\alpha(w)) H(x) + (g_\beta(v) - g_\beta(w)) H(-x) ) \partial_x \varphi dx dt \geq 0.
\end{align*}

Now, we follow \cite{DiPerna1985}. Take the function
\begin{align*}
\mu_h(x) &= \begin{cases} 
\frac{1}{h}(x + h), & x \in [-2h, -h] \\
1, & x \in [-h, h] \\
\frac{1}{h}(2h - x), & |x| > 2h
\end{cases}
\end{align*}
and for an arbitrary \( \psi \in C^1_0(\mathbb{R}^+ \times \mathbb{R}) \), put \( \varphi = (1 - \mu_h)\psi \) in (32). We obtain:

\[
\begin{multline}
\int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^2} \left( |(\alpha(\lambda) - \alpha(\xi)|H(x) + |\beta(\lambda) - \beta(\eta)|H(-x) \right) \partial_t \psi d\nu_{t,x}(\lambda) d\sigma_{t,x}(\eta) dx dt \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}} \left( (\alpha(v) - \alpha(w))H(x) + (g_\beta(v) - g_\beta(w))H(-x) \right) \partial_x \psi dx dt \geq -J(h) + O(h),
\end{multline}

where \( J(h) = \int_{\mathbb{R}^+ \times \mathbb{R}} (\alpha(\lambda) - \alpha(\xi)|H(x) + |\beta(\lambda) - \beta(\eta)|H(-x) \right) \mu_h dx dt \), while \( O(h) \) is the standard Landau symbol. Since \( v \) and \( w \) admit strong traces at \( x = 0 \), and since \( \alpha \) and \( \beta \) satisfy the crossing conditions, as in [19, Theorem 2.1], we conclude that \( \lim_{h \to 0} J(h) \geq 0 \). From here, after letting \( h \to 0 \) in (33), we conclude:

\[
\begin{multline}
\int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^2} \left( |(\alpha(\lambda) - \alpha(\xi)|H(x) + |\beta(\lambda) - \beta(\eta)|H(-x) \right) \partial_t \psi d\nu_{t,x}(\lambda) d\sigma_{t,x}(\eta) dx dt \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}} \left( (\alpha(v) - \alpha(w))H(x) + (g_\beta(v) - g_\beta(w))H(-x) \right) \partial_x \psi dx dt \geq 0,
\end{multline}

and from here, using well known procedure [23], we conclude that for any \( T, R > 0 \) and appropriate \( C, \mathcal{R} \) depending on \( R \), the functions \( f, g, \alpha \) and \( \beta \):

\[
\int_0^T \int_{-R}^R (|\alpha(\lambda) - \alpha(\xi)|H(x) + |\beta(\lambda) - \beta(\eta)|H(-x) \right) d\nu_{t,x}(\lambda) d\sigma_{t,x}(\eta) dx dt \leq CT \int_{-R}^R |u_0 - v_0| dx.
\]

Taking \( u_0 = v_0 \), we see from (34) that for almost every \( (t, x) \in [0, T] \times \mathbb{R} \) the Young measures \( \nu_{t,x} \) and \( \sigma_{t,x} \) are the same and they are supported at the same point (since \( \alpha \) and \( \beta \) are increasing functions). This actually means that \( \sigma_{t,x}(\lambda) = \nu_{t,x}(\lambda = \delta(\lambda - u(t,x)) \) for a function \( u \), where \( \delta \) is the Dirac \( \delta \) function. From Theorem 2.1, we conclude that \( v \to u \) strongly in \( L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}) \) along a subsequence. The function \( u \) will obviously represent the \( (\alpha, \beta) \)-entropy admissible solution to (1).

In order to prove that \( u \) is a unique \( (\alpha, \beta) \)-entropy admissible solution to (1), we basically need to repeat the procedure from the first part of the proof.

Accordingly, take two \( (\alpha, \beta) \)-entropy admissible solutions \( u \) and \( v \) to (1) corresponding to initial data \( u_0 \) and \( v_0 \), respectively. By using the same argumentation as before, we reach to the relation analogous to (33):

\[
\begin{multline}
\int_{\mathbb{R}^+ \times \mathbb{R}} \left( |(\alpha(u) - \alpha(v)|H(x) + |\beta(u) - \beta(v)|H(-x) \right) \partial_t \psi dx dt \\
+ \int_{\mathbb{R}^+ \times \mathbb{R}} \left( (\alpha(u) - \alpha(v))H(x) + (g_\beta(u) - g_\beta(v))H(-x) \right) \partial_x \psi dx dt \geq -\tilde{J}(h) + O(h),
\end{multline}

where \( \tilde{J}(h) = \int_{\mathbb{R}^+ \times \mathbb{R}} (\alpha(u) - \alpha(v))H(x) + (g_\beta(u) - g_\beta(v))H(-x) \right) \mu_h(x) \psi dx dt \).
Introduce the functions
\[
\tilde{u}(t, x) = \begin{cases} 
    u(t, x), & u(t, x) \notin \bigcup_{i=1}^{k_1} [a_i, a_{i+1}), \quad x > 0 \\
    u(t, x), & u(t, x) \notin \bigcup_{i=1}^{k_2} [b_i, b_{i+1}], \quad x \leq 0,
\end{cases}
\]
\[
\tilde{v}(t, x) = \begin{cases} 
    v(t, x), & v(t, x) \notin \bigcup_{i=1}^{k_1} [a_i, a_{i+1}), \quad x > 0 \\
    v(t, x), & v(t, x) \notin \bigcup_{i=1}^{k_2} [b_i, b_{i+1}], \quad x \leq 0,
\end{cases}
\]

Using the same arguments as in Lemma 2.3, we conclude that the functions \(\tilde{u}\) and \(\tilde{v}\) have strong traces at the interface \(x = 0\). Moreover, \(f(u)H(x) + g(u)H(-x) = f(\tilde{u})H(x) + g(\tilde{u})H(-x)\) and \(f(v)H(x) + g(v)H(-x) = f(\tilde{v})H(x) + g(\tilde{v})H(-x)\).

Having this in mind, we conclude
\[
\lim_{h \to 0} J(h) = - \int_{0}^{T} \left( (f_0(\tilde{\alpha}^+) - f_0(\tilde{\alpha}^-))H(x) + (g_0(\tilde{\alpha}^+) - g_0(\tilde{\alpha}^-))H(-x) \right) \psi(t, 0) dt,
\]
where \(\tilde{\alpha}^+\) and \(\tilde{\alpha}^-\) are right and left traces of the function \(\tilde{u}\), while \(\tilde{\nu}^+\) and \(\tilde{\nu}^-\) are right and left traces of the function \(\tilde{v}\). Now, relying on [19, Theorem 2.1] again, we conclude that \(\lim_{h \to 0} J(h) \leq 0\). From here, letting \(h \to 0\) in (35), we obtain:
\[
\int_{\mathbb{R}^+ \times \mathbb{R}} (|\alpha(u) - \alpha(v)|H(x) + |\beta(u) - \beta(v)|H(-x)) \partial_t \psi dx dt + \int_{\mathbb{R}^+ \times \mathbb{R}} (f_0(u) - f_0(v))H(x) + (g_0(u) - g_0(v))H(-x) \partial_x \psi dx dt \geq 0,
\]
and from here, as usual,
\[
\int_{0}^{T} \int_{-R}^{R} (|\alpha(u) - \alpha(v)|H(x) + |\beta(u) - \beta(v)|H(-x)) dx dt \leq CT \int_{-R}^{R} \text{sign}(u - v)((\alpha(u_0) - \alpha(v_0))H(x) + (\beta(u_0) - \beta(v_0))H(-x)) dx dt.
\]

Since \(\alpha\) and \(\beta\) are increasing functions on the range of \(u\) and \(v\), from the above we immediately obtain the \(L^1_{\text{loc}}\) stability of the \((\alpha, \beta)\)-entropy admissible solutions to (1).

Now, as in the last part of the proof of Theorem 1.3, we consider the case \(u_0 \notin BV(\mathbb{R})\). We recall briefly the arguments providing the statement of the theorem in this case. First, we take a sequence \((u_{0\varepsilon})\) of the functions of bounded variation such that \(u_{0\varepsilon} \to u_0\) in \(L^1_{\text{loc}}(\mathbb{R})\). Then, we take the sequence \((u_\varepsilon)\) of \((\alpha, \beta)\)-entropy admissible solutions to (1) with \(u_0 = u_{0\varepsilon}\). The sequence \((u_\varepsilon)\) satisfy:
\[
\int_{0}^{T} \int_{-R}^{R} |u_{\varepsilon_1} - u_{\varepsilon_2}| dx dt \leq CT \int_{-R}^{R} |u_{0\varepsilon_1} - u_{0\varepsilon_2}| dx,
\]
where \(R\) and \(T\) are arbitrary positive constants, and \(C, \tilde{R}\) are constants depending on \(R\), the functions \(f, g, \alpha\) and \(\beta\). This readily implies that the sequence \((u_\varepsilon)\) is convergent in \(L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})\). Its limit is clearly an \((\alpha, \beta)\)-entropy admissible solution to (1). Uniqueness of such \((\alpha, \beta)\)-entropy admissible solution is proved in the completely same way as when \(u_0 \in BV(\mathbb{R}; [a, b])\).
A simple corollary of Theorem 2.4 is the maximum principle for an \((\alpha, \beta)\)-entropy admissible solution to (1).

**Corollary 1.** Assume that \(u\) and \(v\) are two \((\alpha, \beta)\)-admissible weak solutions to (1) corresponding to the initial data \(u_0 \in L^1(\mathbb{H}; [a,b])\) and \(v_0 \in L^1(\mathbb{H}; [a,b])\) such that \(u_0(x) \leq v_0(x)\) for a.e. \(x \in \mathbb{H}\). Furthermore, assume that \(f_\alpha\) and \(g_\beta\) satisfy the crossing conditions. Then, it holds

\[ u(t, x) \leq v(t, x) \quad \text{a.e.} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \]

**Proof.** It is enough to notice that, since \(|u|^+ = \frac{|u| + u}{2}\), i.e. \(\text{sign}_+(u) = (|u|^+') = \frac{|\text{sign}(u)|}{2}\), relation (36) holds if we replace there \(\text{sign}\) by \(\text{sign}_+\). From that relation, the standard arguments provide

\[
\int_0^T \int_{-R}^R |u(t, x) - v(t, x)|^+ dxdt \leq CT \int_{-R}^R |u_0(x) - v_0(x)|^+ dx.
\]

From here, the statement of the corollary immediately follows. \(\square\)

Now, we shall prove that we can always find \(\alpha\) and \(\beta\) so that there exists a unique \((\alpha, \beta)\)-entropy admissible solutions to (1).

**Figure 4.** Functions \(f\) (normal line) and \(g\) (dashed line) on the left plot do not satisfy the crossing condition. On the other hand, for appropriate \(k_L > k_R\), the functions \(f(\cdot + k_R)\) and \(g(\cdot + k_L)\) on the right plot satisfy the crossing conditions.

**Theorem 2.5.** Denote by \(\chi_{[a,b]}\) the characteristic function of the interval \([a, b]\). For the functions \(\alpha(u) = \alpha_T(u) = u + k_R\) and \(\beta(u) = \beta_T(u) = u + k_L\) such that the functions \(f_\alpha = (f\chi_{[a,b]}) \circ \alpha\) and \(g_\beta = (g\chi_{[a,b]}) \circ \beta\) satisfy the crossing conditions, there exists a unique \((\alpha, \beta)\)-entropy admissible solution to (1).

**Proof.** First, notice that it is always possible to find constants \(k_R\) and \(k_L\) such that the translation functions \(\alpha_T(u) = u + k_R\) and \(\beta_T(u) = u + k_L\) make \(f_\alpha\) and \(g_\beta\) to satisfy the crossing conditions (see Figure 4). Furthermore, the constants \(a\) and \(b\) represent \((\alpha_T, \beta_T)\)-entropy admissible solutions to

\[
\begin{aligned}
\partial_t u + \partial_x \left( H(x)(f\chi_{[a,b]})(u) + H(-x)(g\chi_{[a,b]})(u) \right) &= 0, \\
|u|_{t=0} &= u_0(x) \in L^\infty(\mathbb{R}), \\
x \in \mathbb{R}.
\end{aligned}
\] (37)

Indeed, denoting \(k(x) = \begin{cases} k_L, & x \leq 0 \\
 k_R, & x > 0 \end{cases}\), according to Definition 0.4, we see that we need to check whether the function \(v(t, x) = a - k(x)\) satisfies (8). After substituting it there, we see that we need to check whether (see also [26, Remark 2])

\[
\left| (\text{sgn}(a - k_L - \xi)(g\chi_{[a,b]})(\xi + k_L) - \text{sgn}(a - k_R - \xi)(f\chi_{[a,b]})(\xi + k_L) - |(f\chi_{[a,b]})(\xi + k_R) - (f\chi_{[a,b]})(\xi + k_L)|) \delta(x) \right| \leq 0.
\] (38)
for every $\xi \in \mathbb{R}$. Clearly, if $\xi \in \mathbb{R}$ is such that $\min\{a - k_R - \xi, a - k_L - \xi\} \geq 0$ or $\max\{a - k_R - \xi, a - k_L - \xi\} \leq 0$, then (38) holds with the equality sign. Otherwise, it must hold $a - k_R - \xi \leq 0 \leq a - k_L - \xi$ (see Figure 4). However, if this is a case, then $\xi + k_L \leq a$. This implies $(g\chi_{[a,b]})(\xi + k_L) = 0$ from where (38) easily follows. Similarly, we prove that $u(t, x) \equiv b$ represents an $(\alpha_T, \beta_T)$-entropy admissible solution to (1).

From here, using Corollary 1, we conclude that, for the $\alpha_T$ and $\beta_T$ chosen above (Figure 4), the $(\alpha_T, \beta_T)$-entropy admissible solutions to (1), say $u$, such that $a \leq u_0 \leq b$, must satisfy $a \leq u(t, x) \leq b$ for a.e. $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. This actually means that the $(\alpha_T, \beta_T)$-entropy admissible solution to (37) is, at the same time, $(\alpha_T, \beta_T)$-entropy admissible solution to (1) (since on the range of the solution it holds $f\chi_{[a,b]} \equiv f$ and $g\chi_{[a,b]} \equiv g$). Since $f_{\alpha_T}$ and $g_{\beta_T}$ satisfy the crossing conditions, according to Theorem 2.4, we conclude that the $(\alpha_T, \beta_T)$-entropy admissible solution to (37) is unique making it a unique solution to (1).

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References

[1] (MR2028700) Adimurthi, G. D. Veerappa Gowda, Conservation laws with discontinuous flux, J. Math. (Kyoto University), 43(1) (2003), 27–70.
[2] (MR2195983) Adimurthi, S. Mishra, G. D. Veerappa Gowda, Optimal entropy solutions for conservation laws with discontinuous flux functions, J. of Hyperbolic Differ. Equ., 2 (2005), 783–837.
[3] (MR2291815) Adimurthi, S. Mishra, G. D. Veerappa Gowda, Existence and stability of entropy solutions for a conservation law with discontinuous non-convex fluxes, Netw. Heterog. Media, 2 (2007), 127–157.
[4] (MR2604627) J. Aleksić, D. Mitrovic, On the compactness for two dimensional scalar conservation law with discontinuous flux, Comm. Math. Sciences, 4 (2009), 963–971.
[5] B. Andreianov, K. H. Karlsen, N. H. Risebro, On vanishing viscosity approximation of conservation laws with discontinuous flux, preprint available at www.math.ntnu.no/conservation/2009.
[6] (MR2129374) A. Audusse, B. Perthame, Uniqueness for scalar conservation law via adapted entropies, Proc. Roy. Soc. Edinburgh Sect. A, 135 (2005), 253–265.
[7] (MR2209759) F. Bachmann, J. Vovelle, Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients, Comm. Partial Differential Equations, 31 (2006), 371–395.
[8] (MR2505870) R. Burger, K. H. Karlsen, J. Towers, On Enquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections, SIAM J. Numer. Anal., 3 (2009), 1684–1712.
[9] (MR2396491) R. Burger, A. Garcia, K. H. Karlsen, J. Towers, A family of schemes for kinematic flows with discontinuous flux, J. Engrg. Math., 60 (2008), 387–425.
[10] (MR1356452) S. Diehl, On scalar conservation law with point source and discontinuous flux function modelling continuous sedimentation, SIAM J. Math. Anal., 6 (1995) 1425–1451.
[11] (MR1381652) S. Diehl, A conservation law with point source and discontinuous flux function modelling continuous sedimentation, SIAM J. Appl. Anal., 2 (1996), 388–419.
[12] (MR2512505) S. Diehl, A uniqueness condition for non-linear convection-diffusion equations with discontinuous coefficients, J. Hyperbolic Diff. Eq., 6 (2009), 127–159.
[13] (MR0775191) R. J. DiPerna, Measure-valued solutions to conservation laws, Arch. Rat. Mech. Anal., 88 (1985), 223–270.
[14] (MR1034481) L.C.Evans, Weak convergence methods in nonlinear partial differential equations, AMS, Providence, Rhode Island, No 74, 1990.
(MR2604627) H. Holden, K. Karlsen, D. Mitrovic, Zero diffusion dispersion limits for a scalar conservation law with discontinuous flux function, International Journal of Differential Equations, Volume 2009, Article ID 279818, 33 pages doi:10.1155/2009/279818.

(16) (MR1135919) P. Gerard, Microlocal Defect Measures, Comm. Partial Differential Equations, 11 (1986), 1761–1794.

(17) (MR1109304) T. Gimse, N. H. Risebro, Riemann problems with discontinuous flux function, in Proc. 3rd Int. Conf. Hyperbolic Problems Studentlitteratur, Uppsala (1991), 488–502.

(18) (MR1696184) E. Kaasschieter, Solving the Buckley-Leverett equation with gravity in a heterogeneous porous media, Comput. Geosci., 3 (1999), 23–48.

(19) (MR2024741) K. H. Karlsen, N. H. Risebro, J. Towers, L^1-stability for entropy solutions of nonlinear degenerate parabolic connection-diffusion equations with disc. coeff., Skr.K.Nor.Vid.Selsk, 3 (2003), 1–49.

(20) (MR1938389) K. Karlsen, N. H. Risebro, J. Towers, On a nonlin. degenerate parabolic transport-diff. eq. with a disc. coeff., Electronic J. of Differential Equations, 93 (2002), 23 pp. (electronic)

(21) (MR2334842) K. Karlsen, M. Rascle, and E. Tadmor On the existence and compactness of a two-dimensional resonant system of conservation laws, Communications in Mathematical Sciences 2 (2007), 253–265.

(22) (MR20086124) K. Karlsen, J. Towers, Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space- time dependent flux, Chinese Ann. Math. Ser. B, 3 (2004), 287–318.

(23) (MR2601993) D. Mitrovic, Existence and Stability of a Multidimensional Scalar Conservation Law with Discontinuous flux, Netw. Het. Media, 5 (2010), 163–188.

(24) (MR2374223) E. Yu. Panov, Existence of Strong Traces for Quasi-Solutions of Multidimensional Conservation Laws, J. of Hyperbolic Differential Equations, 4 (2007), 729–770.

(25) (MR2568808) E. Yu. Panov, On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux, J. of Hyperbolic Differential Equations, 3 (2009), 525–548.

(26) (MR2505851) E. Yu. Panov, On Weak Completeness of the Set of Entropy Solutions to a Scalar Conservation Law, SIAM J. Math. Anal., 1 (2009), 26–36.

(27) (MR2599291) E. Yu. Panov, Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux, Arch. Rational Mech. Anal., 195 (2010), 643–673.

(28) (MR1452107) P. Pedregal, Parametrized Measures and Variational Principles, Progress in Nonlinear Partial Differential Equations and Their Applications Vol. 30, Birkhauser, Basel, 1997.

(29) (MR1913839) K. Karlsen, N. H. Risebro, J. Towers, On a nonlin. degenerate parabolic connection-diffusion equations with disc. coeff., Electronic J. of Differential Equations, 93 (2002), 23 pp. (electronic)

(30) (MR1201239) P. L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidim. scalar cons. law and related equations, J. Amer. Math. Soc., 1 (1994), 169–191.

(31) (MR1452107) P. Pedregal, Parametrized Measures and Variational Principles, Progress in Nonlinear Partial Differential Equations and Their Applications Vol. 30, Birkhauser, Basel, 1997.
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