CYCLICITY OF THE LEFT REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP

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Abstract. We combine harmonic analysis and operator algebraic techniques to give a concise argument that the left regular representation of a locally compact group is cyclic if and only if the group is first countable, a result first proved by Greenleaf and Moskowitz.

Let $G$ be a locally compact group and let $\lambda$ and $\rho$ denote the (unitarily equivalent) left and right regular representations of $G$ on $L^2(G)$, respectively. The group von Neumann algebra $VN(G)$ is the von Neumann algebra generated in $B(L^2(G))$ by $\lambda(G)$. It is well known that the commutant of $VN(G)$ is the von Neumann algebra generated by $\rho(G)$. In [2], an operator algebraic argument viewing $VN(G)$ as arising from a left Hilbert algebra, in combination with a reduction argument using the structure theory of locally compact groups, is used to show that $\lambda$ is cyclic when the group $G$ is first countable. The converse, left open in [2], was established later by the same authors in [3].

The purpose of this note is to give a new and more economical proof of this equivalence. Moreover, we show that these conditions are equivalent to $\sigma$-finiteness of $VN(G)$, the latter condition arising naturally from our techniques. In the commutative case, it is well known that $\sigma$-finiteness of $L^\infty(G)$ characterizes $\sigma$-compactness of $G$, and it is our hope that further development of the techniques we employ will yield natural characterizations of $\sigma$-finiteness of a general locally compact quantum group. An alternative proof of the characterization we establish, exploiting the structure theory of locally compact groups, is given in [5].

Recall that the support of a normal state $\omega$ on a von Neumann algebra $M$ is the minimal projection $S_\omega$ in $M$ for which $\langle \omega, S_\omega \rangle = 1$ and that $\omega$ is faithful if $S_\omega = I$, the identity in $M$, equivalently if $\omega$ takes strictly positive values on strictly positive operators. We record some elementary facts about these concepts. For a vector $\xi$ in a Hilbert space, let $\omega_\xi$ denote the vector functional implemented by $\xi$. The notation $\langle X \rangle$ denotes the norm closed linear span of $X$.

Lemma 0.1. Let $H$ be a Hilbert space, let $M$ a von Neumann algebra in $B(H)$, and let $\xi, \eta \in H$ be unit vectors. The following hold.

1. The projection $S_{\omega_\xi}$ has range $\langle M'\xi \rangle$.
2. $\langle \omega_\xi, S_{\omega_\eta} \rangle = 0$ if and only if $\xi$ is orthogonal to $\langle M'\eta \rangle$.
3. A projection $P$ in $M$ satisfies $P\xi = \xi$ if and only if $S_{\omega_\xi} \leq P$.
4. A normal state $\omega$ on $M$ is faithful if and only if $\langle \omega, U \rangle = \langle \omega, I \rangle$ implies $U = I$, for any unitary $U$ in $M$.

Motivated by the following simple observation, we choose to characterize cyclicity of the right regular representation.

Lemma 0.2. Let $G$ be a locally compact group. A vector $\xi \in L^2(G)$ is cyclic for $\rho$ if and only if $\omega_\xi$ is faithful on $VN(G)$.
Proof. For \( \xi \in L^2(G) \) we have \( \langle \rho(G) \xi \rangle = \langle VN(G)' \xi \rangle \) since span\( \rho(G) \) is strong operator topology dense in \( VN(G)' \). Consequently, the vector \( \xi \) is cyclic for \( \rho \) if and only if \( \langle VN(G)' \xi \rangle = L^2(G) \). As \( S_{\omega} \) has range \( VN(G)' \xi \), the latter assertion is exactly that \( S_{\omega} = I \).

Let \( A(G) \) denote the Fourier algebra of a locally compact group \( G \), which is the predual of \( VN(G) \), and for \( T \in VN(G) \) and \( u \in A(G) \) define \( T^*u \in A(G) \) by
\[
\langle T^*u, S \rangle = \langle u, \tilde{T}S \rangle \quad (S \in VN(G)),
\]
where \( \tilde{T} \) is the image of \( T \) under the adjoint of the check map \( u \mapsto \check{u} \) on \( A(G) \) (here, \( \check{u}(s) = u(s^{-1}) \)). See [1] p.213]. Proposition 3.17 of [1] shows that for \( u \in A(G) \cap L^2(G) \) we have \( T^*u = Tu \), where the right hand side is evaluation of the operator \( T \) at the vector \( u \) in \( L^2(G) \). This fact is needed in the following lemma, which is key to establishing the main result.

**Lemma 0.3.** Let \( G \) be a locally compact group. Every nonzero projection in \( VN(G) \) has a nonzero continuous function in its range.

**Proof.** Let \( P \in VN(G) \) be a nonzero projection and choose a unit vector \( \xi \) in its range. Since positive functions span \( C_c(G) \), which is in turn dense in \( L^2(G) \), we may find a positive \( f \in C_c(G) \) of norm one in \( L^2(G) \) and not orthogonal to \( \langle \rho(G) \xi \rangle \), so that \( \langle \rho(f), S_{\omega} \xi \rangle \neq 0 \) by Lemma 0.1.

The function \( \omega_f \) in \( A(G) \) is positive definite and pointwise positive, so that \( \omega_f = \omega_f \), and is in \( A(G) \cap L^2(G) \) because \( f \) has compact support, whence
\[
S_{\omega} (\omega_f)(e) = \langle S_{\omega} \omega_f \rangle (e) = \langle S_{\omega} \omega_f, \lambda(e) \rangle = \langle \omega_f, S_{\omega} \xi \rangle = \langle \omega_f, S_{\omega} \xi \rangle \neq 0.
\]
Thus \( S_{\omega} (\omega_f) = S_{\omega} \omega_f \) is nonzero and in \( A(G) \), hence continuous, and is in the range of \( P \) because \( S_{\omega} \leq P \), by Lemma 0.1.

**Theorem 0.4.** Let \( G \) be a locally compact group. The following are equivalent:

1. \( G \) is first countable.
2. \( VN(G) \) is \( \sigma \)-finite.
3. The left (equivalently, right) regular representation is cyclic.

**Proof.** Suppose (1) holds. Let \( (U_n)_{n=1}^{\infty} \) be a countable neighborhood base at the identity in \( G \) and define \( \omega_n = |U_n^{-1} \omega|_{U_n} \). We show that the normal state \( \omega = \sum_{n=1}^{\infty} 2^{-n} \omega_n \) is faithful. Let \( T \) be a positive operator in \( VN(G) \) with \( \langle \omega, T \rangle = 0 \) and let \( P \) be the range projection of \( T \), so that \( \langle \omega, P \rangle = 0 \) (see, e.g., [1] Remark 7.2.5). Given any vector \( \eta \) in the range of \( T \) we have \( S_{\omega} \leq P \) and thus \( 0 \leq \langle \omega_n, S_{\omega_n} \rangle \leq \langle \omega_n, P \rangle \leq \langle \omega, P \rangle = 0 \), implying that \( \eta \) is orthogonal to \( \langle \rho(G) \chi_{U_n} \rangle \) for each \( n \geq 1 \). If \( \eta \) is continuous, then
\[
\eta(s) = \lim_n |U_n s|^{-1} \int_{U_n} \eta \lim_n |U_n s|^{-1} \langle \eta, \rho(s^{-1}) \chi_{U_n} \rangle \Delta(s)^{\frac{1}{2}} = 0
\]
for every \( s \in G \). Thus \( P = 0 \) by Lemma 0.3, hence \( T = 0 \) and \( \omega \) is faithful, so (2) holds.

Normal states on \( VN(G) \), being positive definite functions in \( A(G) \), are vector states, so that statements (2) and (3) are equivalent by Lemma 0.2.

We provide the argument of [6] establishing that (2) implies (1). Suppose (2) holds and let \( \omega \) be a faithful normal state on \( VN(G) \). Fix a compact neighborhood \( K \) of the identity in \( G \) and let \( V \) be any open neighborhood of the identity contained in \( K \). We show that the sets \( U_n = \{ s \in K : \omega(s) - 1 < \frac{1}{n} \} \) form a neighborhood base at the identity, for which it suffices to establish that \( U_n \) is contained in \( V \) for some \( n \geq 1 \). For any \( s \in G \) with \( \omega(s) = 1 \), Lemma 0.1 entails that \( s = e \), since \( \omega(s) = \langle \omega, \lambda(s) \rangle \). Compactness of \( K \setminus V \) then implies that \( \epsilon = \inf \{ \omega(s) - 1 : s \in K \setminus V \} \) is strictly positive. Choosing \( N \geq 1 \) with \( \frac{1}{N} < \epsilon \), if \( s \in U_N \), then that \( s \in K \) and \( |\omega(s) - 1| < \epsilon \) together imply that \( s \in V \). Thus \( U_N \subset V \), as required. □


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