Decisions in elections—transitive or intransitive quantum preferences

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Abstract
Our preferences depend on the circumstances in which we reveal them. We will introduce a dependence which allows us to illustrate the relation between the possibility of winning of a particular candidate in an election and the type of preference. It is generally observed that if voters start to clearly prefer one of the candidates, the significance of intransitive preferences in the quantum model decreases. This dynamic change cannot be observed in the case of the classical model.

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1. Introduction

A relation $\succ$ existing between any three elements of a certain set (e.g. $A$, $B$, and $C$) is called transitive if $A \succ C$ resulting from the fact that $A \succ B$ and $B \succ C$. If this condition is not fulfilled, then the relation will be called intransitive (not transitive).

The axiom concerning the intransitiveness of preference relations is one of the basic assumptions of the choice theory. It is often identified with the rationality of the decisions taken. One of the main arguments put forward by many experts, proving the irrationality of preferences which violate transitivity, is ‘money pump’ (see [1]).

Despite the fact that intransitivity appears to be contrary to our intuition, life provides many examples of intransitive orders. Rivalry between species may be intransitive. For example, in the case of fungi, *Phallus impudicus* replaced *Megacollybia platyphylla*, *M. platyphylla* replaced *Psathyrella hydrophilum*, but *P. hydrophilum* replaced *P. impudicus* [2]. Similarly, the experiment can be performed on bees, which make intransitive choices between flowers [3]. The best known and socially significant example of intransitivity is Condorcet’s voting paradox. Consideration of this paradox led Arrow (in the 20th century) to prove the theorem that there is no procedure of a successful choice that would meet a democratic assumption [4]. Interesting examples of intransitivity are provided by probability models (Efron’s dice [5], Penney’s game [6]).
Intransitive strategies are sometimes a consequence of negligent, rash decisions. They also appear when we are not able to conduct an accurate valuation of the assessed goods. If we lack information indispensable in a conscious decision-making process, we are indecisive. Often the criteria based upon which we decide about a choice refer to a sphere of feelings which are difficult to be compared. Ulam mentioned it in his autobiography [7]. He tried to evaluate the taste of fruits and claimed that this relation is intransitive (plums could be better than nuts, which were better than apples, but apples were better than plums). The division into transitive and intransitive strategies (preferences) does not necessarily have to be equal to that into rationality and irrationality. It may, however, reflect the process of decision making under conditions of uncertainty or the lack of decisiveness of the chooser. Intransitive strategies are still a mysterious feature of the human thinking process and knowing the reasons behind the presence of intransitivity may be useful to better understand thinking mechanisms and for the research concerning modelling of artificial intelligence. This concept is worth analysing also within the context of tools which are provided by the quantum game theory which has recently been developed intensively [8–15], where the formalism of Hilbert’s space is applied for constructing decision-making algorithms. This non-classical approach may lead to interesting and qualitatively new effects which cannot be achieved by means of traditional probabilistic models.

In this paper, we will once again take a look at a very simple game analysed in [16] and a different way of modelling this game known from the previous analyses [1, 17, 18]. We pay attention to an interesting relationship (occurring in the quantum model) between intransitive strategies and the increase of the player’s certainty, who starts to be in favour of one of the available alternatives. We will show that if voters start to clearly prefer one of the candidates, the significance of intransitive preferences in the quantum model decreases. This seems to be a fairly natural and intuitive characteristic, especially if intransitive preferences are interpreted as an expression of uncertainty in the decision-making process. Analysis of a well-known example (Condorcet paradox) will allow us to obtain such a result.

Suppose we have three candidates, A, B and C. Let us assume that each voter’s preference is independently selected from the following (transitive) preferences:

1. \( A \succ B \succ C \)
2. \( B \succ C \succ A \)
3. \( C \succ A \succ B \),

with probabilities \( w_1, w_2, \) and \( w_3 \), respectively. First, we assume that \( w_1 = w_2 = w_3 = 1/3 \). Collective preference is intransitive: with probability \( 2/3 \) \( A \succ B \) (from (1) and (3)), with probability \( 2/3 \) \( B \succ C \) (from (1) and (2)) and with probability \( 2/3 \) \( C \succ A \) (from (2) and (3)). The significance of the collective intransitive preference will reduce with the reduction of probabilities \( w_2 \) and \( w_3 \) (reduction of probability that \( C \succ A \)) and the increase of probability \( w_1 \) (i.e. voters start to clearly prefer candidate A). In the Condorcet paradox, pure strategies, i.e. transitive preferences, are mixed. This model therefore favours transitive preferences, and thus cannot be treated as an effective tool for solving the conflict between transitive and intransitive preferences with all related implications. This paper refers to a model devoid of this shortcoming and takes both intransitive and transitive preferences equally into account. Illustrating the problem, we will relate our interpretation to an election interpretation of the game proposed in [1] (see the protocol for quantum voting in [19]). This will allow us to present the problem in a real-life context. We will analyse the influence of the increase of support (understood as a probability of winning the election) for one of the candidates for the significance of intransitive preferences.
Owing to the quantum approach, the paper naturally singles out the class of relevant intransitive strategies which has not been subjected to a prior analysis.

**Definition 1.** The intransitive strategy will be called the relevant strategy if there is no transitive strategy of the same consequences with the same assumption.

Relevant intransitive strategies may occur only in the quantum game model and the decrease in their significance is proportional to the increase in the player’s determination to make a given choice. In everyday life, we often tend to be perfectly sure about our decisions. On the other hand, however, we are often equally insecure about them (we are not absolutely convinced that we have made a good choice). It is extremely difficult to construct a mathematical model describing such relations, one which would fully render their complexity. Therefore, it must be emphasized that the conclusions of this dissertation relate only to a simple model of behaviour, and it is difficult to say whether they would be suitable in the case of more complex models. In this paper, we refer to the well-known game model. A simple modification of its parameters allows for an interesting characteristic to be observed concerning an increase in the player’s determination (which is defined as the probability of deciding on one of the possible options). The mathematical rules employed in this paper have been introduced (partially) in [1]. Significant terms are explained in the following section in order to facilitate reading and make the paper complete.

### 2. The model

Let us assume that three candidates (0, 1 and 2) take part in the election. We do not refer to any concrete election procedure here; however, we assume that the elections ensue in two stages. In the first stage there is an elimination of one of the candidates in order to make a final choice in the second stage (of one of the two remaining candidates). We can observe such a construction of an election in many countries (mainly European) where two candidates with the highest social support enter the second stage (the so-called second round of the elections). In this type of elections, the candidate’s chances also depend on the strength of the candidate with whom he will have to compete in the second round. It may happen that the candidate with the largest support in the first round would lose in the second round (the choice depends on the context; we often vote ‘against’ and not ‘for’).

Let $q_i$ denote the probability that candidate $i$ will not enter the second round. We denote by $P(C_i|B_j)$ the probability of the choice of candidate $i$ in the second round when the decision concerns a pair of candidates in which candidate $j$ is not present. The probability of the victory of candidate $i$ is denoted by $\omega_k$:

$$\omega_k = \sum_{j \neq k} P(C_k|B_j) q_j, \quad j, k = 0, 1, 2. \quad (1)$$

Any six conditional probabilities ($P(C_1|B_0)$, $P(C_2|B_0)$, $P(C_0|B_1)$, $P(C_2|B_1)$, $P(C_0|B_2)$, and $P(C_1|B_2)$) that for a fixed triples $(q_0, q_1, q_2)$ and $(\omega_0, \omega_1, \omega_2)$ fulfill (1) will be called a voter’s optimal strategy (‘the collective voter’—the electorate).

After elementary modification, we introduce the following relation:

$$q_0 = \frac{1}{d} (-P(C_2|B_1) \omega_0 + P(C_2|B_1) P(C_0|B_2) + (1 - P(C_2|B_1) - P(C_0|B_2)) \omega_2),$$

$$q_1 = \frac{1}{d} (-P(C_0|B_2) \omega_1 + P(C_0|B_2) P(C_1|B_0) + (1 - P(C_0|B_2) - P(C_1|B_0)) \omega_0),$$

$$q_2 = \frac{1}{d} (-P(C_1|B_0) \omega_2 + P(C_1|B_0) P(C_2|B_1) + (1 - P(C_1|B_0) - P(C_2|B_1)) \omega_1). \quad (2)$$
The above relation defines the mapping $A : D_3 \rightarrow T_2$ of a three-dimensional cube ($D_3$) into a triangle ($T_2$) of two-dimensional simplex ($q_0 + q_1 + q_2 = 1$ and $q_i \geq 0$), where $d$ is the determinant of the matrix of parameters $P(C_j|B_k)$ (see [16]). The barycentric coordinates of a point of this triangle are interpreted as probabilities $q_0, q_1$ and $q_2$ (see figure 1). For further deliberations (in the quantum case), we will assume the construction of conditional probabilities $P(C_k|B_j)$ proposed in [17]. It is based on the replacement of a cube $D_3$ with a sphere $S_2$ with the use of a probabilistic interpretation of vector coordinates in a two-dimensional Hilbert space $H_2$ and on the concept of the so-called conjugate bases which have already played an essential role e.g. in quantum cryptography [20] and quantum market games [21]. The coordinates of the same strategy $|z⟩ \in H_2$ read (measured) in various bases define voters’ preferences towards a candidate pair represented by the base vectors. The squares of their moduli, after normalization, measure the conditional probability of voters which make decision in choosing a particular candidate, when the choice is related to the suggested candidate pair. Ultimately, the probabilities $P(C_k|B_j)$ which are of interest to us have the following form [17]:

$$
P(C_0|B_2) = \frac{1 - x_3}{2}, \quad P(C_1|B_2) = \frac{1 + x_3}{2},$$

$$
P(C_0|B_1) = \frac{1 + x_1}{2}, \quad P(C_2|B_1) = \frac{1 - x_1}{2},$$

$$
P(C_1|B_0) = \frac{1 + x_2}{2}, \quad P(C_2|B_0) = \frac{1 - x_2}{2},$$

(3)

where $(x_1, x_2, x_3) \in S_2$.

A combination of the above projection and (2) results in the projection $A_q : S_2 \rightarrow T_2$ of a two-dimensional sphere $S_2$ into a triangle $T_2$.

In the case of random selections, we may talk about an order relation:

candidate 0 < candidate 1,

when from pair (0, 1) we are willing to choose candidate 1 ($P(C_0|B_2) < P(C_1|B_2)$). We deal with an intransitive choice (strategy) if one of the following conditions is fulfilled:
Figure 2. A simplex area for which there exist optimal strategies in the quantum model (all, intransitive, transitive) in the case $\omega_0 = \omega_1 = \omega_2 = 1/3$.

- $P(C_2|B_1) < \frac{1}{2}$, $P(C_1|B_0) < \frac{1}{2}$, $P(C_0|B_2) < \frac{1}{2}$.
- $P(C_2|B_1) > \frac{1}{2}$, $P(C_1|B_0) > \frac{1}{2}$, $P(C_0|B_2) > \frac{1}{2}$.

3. Decrease in importance of intransitive strategies

In this section, we will look at the geometrical interpretation of the above-discussed problem which will allow us to track the changes of the significance of intransitive strategies if the chance of winning of one of the candidates increases (in relation to the remaining two). It corresponds to the increase of decisiveness of the electorate—the collective voter—that is to some extent the decrease of uncertainty in decision making. We present a range of representation $\mathcal{A}_q$ for 10 000 randomly chosen points with the Fubini–Study measure on a sphere $S_2$.

Figure 2 presents the areas of probability $q_m$ for which optimal strategies of different types exist (with the assumption that $\omega_0 = \omega_1 = \omega_2 = 1/3$ [17]). Candidates have equal chances of winning (none of them have the advantage over the others), which means that the electorate cannot make a decisive choice (i.e. they are not in favour of a concrete candidate), and give equal support to each of them. It is an exceptional situation which is at the same time extremely interesting from the theoretical point of view, since it illustrates the uncertainty while making decisions when we are not able to conduct the valuation of the assessed goods (in our case the candidates) [1]. This lack of decisiveness may be a consequence of very similar (practically indiscernible) or incomparable features according to which we make an assessment of the available alternatives (similar to the case of the above-mentioned assessment of fruit tastes which S M Ulam tried to conduct). Let us note that in this case the intransitive strategies (associated exactly with the uncertainty while making decisions) are of great significance. It is clearly observable in the figure presenting transitive strategies. In this case, the dotted area (probabilities $q_m$ for which there is an optimal transitive strategy) does not cover the whole area corresponding to optimal strategies of a random type. The non-dotted area in the central part of the figure corresponds to the probabilities $q_m$ for which there is only an optimal intransitive strategy (the relevant intransitive strategy). It is an area of overlapping of two intransitive orders. Let us see how the size of this area is influenced by an increase in support for one of the candidates, i.e. the appearance of an election leader. In this case, we increase the probability $\omega_i$ in relation to the remaining two. Figures 3 and 4 present such a situation (in the classical and quantum case) assuming that the probability of winning of candidate 2 ($\omega_2$) increases and the chances of the other candidates are similar ($\omega_0 = \omega_1$). By symmetry, the result is
equivalent whichever candidate is chosen to be the leading candidate. We illustrated only the transitive strategies, since in this case the change of importance of intransitive strategies is visible. Figure 3 presents areas which are of interest to us for $\omega_2 = 0.42$, $\omega_2 = 0.52$ and $\omega_2 = 0.54$, respectively.

The white fusiform slit (in the quantum case of figure 3) in the central part of the triangles corresponds to the area in which two intransitive orders are overlapping (only the relevant intransitive strategy exists, i.e. there is no transitive strategy of identical effect of activity). It means that for a certain distribution of probability $q_m$ of the occurrence of concrete pairs in the second round of elections, the voters achieve an established distribution of support $(\omega_0, \omega_1, \omega_2)$ only owing to the intransitive strategy.

We can see that with the increase of $\omega_2$ (i.e. the chances of winning of candidate 2), the white slit in the quantum model becomes smaller (i.e. the importance of intransitive strategies decreases). It will disappear for $\omega_2 \approx 0.55$. An increase in support for one of the candidates caused the decrease in importance of intransitive strategies (the decrease in the area in which these strategies are relevant). Let us recall that intransitive strategies are associated with an uncertainty and lack of decisiveness. Perhaps by reducing uncertainty we may limit the importance of intransitive strategies.
4. Remarks

In the above-discussed model, when the support of a candidate reaches about 55% (i.e. it exceeds a half which guarantees winning), the intransitive strategies become irrelevant (they still exist but we can always select a transitive strategy of a similar result). It is interesting, since such a level of support means that the voters have made a choice indicating a concrete winner (the candidate who obtained over 50% of votes). This decisiveness of voters is accompanied by the decrease in the importance of intransitive preferences. Perhaps a certain flaw of the model is the fact that it does not ensue at 51% already (i.e. the minimum support which is necessary for winning). On the other hand, maybe this minimum advantage does not mean that voters have not decided for a particular candidate. In this case, even a minimum decrease in support may change the result of election.

The decrease in importance of intransitive orders which accompanies the growth of support for one of the candidates is an interesting property of the quantum game model. This dynamic change cannot be observed in the case of the classical model (see figure 4) in which for each intransitive strategy we may select a transitive strategy of the same effect. In the case of the classical models, the relevant strategies (intransitive strategies) cannot occur, since the use of mixed strategies in the description which is necessary in classical models will successfully fill the gaps situated in the surroundings of areas occupied by transitive strategies. Perhaps exactly in the context of the quantum model, the division into transitive and intransitive strategies will allow us to characterize the decisiveness (certainty) or the lack of it (uncertainty) in the process of decision making. The elementary model presented here constitutes an important example of opportunities provided by non-classical ways of decision-making process description. Extending it with mixed strategies does not remove the effect of the occurrence of relevant intransitive preferences [17]. It is also worth emphasizing that the classical model (although intuitively clear and intelligible) has, however, certain flaws which may raise certain reservations. The strategies (conditional probabilities) creating a three-dimensional cube do not have an equal mathematical and information status. In the apex of the dice, we have deterministic pure strategies, whereas the remaining ones are mixed ones. Strategies may provide different pieces of information (which can be measured with the Boltzmann/Shannon entropy). The quantum model of strategies (pure states) is free from such flaws—all strategies have equal informative values (the zero entropy); hence, treating them in an equal way (i.e. measuring by means of the Fubini–Study measure) is natural and does not raise any controversy similar to those of the constant measure (Laplace’s principle of insufficient reason [22]) in the classical model.

5. Quantization model for the space with dimension higher than 2

In the quantization model, we received conditional probabilities which were of interest to us on the basis of measurements of the same vector of complex Hilbert spaces in different complementary bases. We built our model in a two-dimensional Hilbert space \( H_2 \). The maximal set of complementary bases in this space have three elements which are exactly the same as the number of possible pairs of candidates in the considered election game model. The above-mentioned method (see section 2) for obtaining conditional probabilities by measurements in different complementary bases of space \( H_2 \) can also be moved to spaces of dimension \( d = 2^m \), where \( m \in \mathbb{N} \) and \( m > 1 \) (m-qubit system).

In this case, the maximal set of complementary bases have exactly \( d + 1 \) elements. Construction of the model could proceed as follows. We choose the space \( H_d \). Let us denote
one of the complementary bases of this space by \([|0\rangle_{m,j}, |1\rangle_{m,j}, \ldots, |d\rangle_{m,j}]\). Let the measurement of the vector \(|\psi\rangle \in H_d\),

\[
|\psi\rangle = \alpha_0|0\rangle_{m,j} + \alpha_1|1\rangle_{m,j} + \cdots + \alpha_d|d\rangle_{m,j},
\]
in this base defines the preferences of voters to the pair \((m, j)\) as follows:

\[
P(C_m|B_{m,j}) = |\alpha_0|^2 + |\alpha_1|^2 + \cdots + |\alpha_{m-1}|^2, \quad P(C_j|B_{m,j}) = |\alpha_{m+1-1}|^2 + |\alpha_{m+1-2}|^2 + \cdots + |\alpha_{m-2}|^2.
\]

The sum of the squares of moduli for the first half of coordinates of the \(|\psi\rangle\) vector defines the probability of choosing the candidate with the number \(m\) of the \((m, j)\) pair, and similarly, the other coordinates define the probability of choosing candidate \(j\) from this pair. Analogically, we define the preference of voters in relation to the other pairs of candidates, each of which shall be identified with another complementary base of the \(H_d\) space.

Illustrating the above construction, we apply it (to our three-candidate model) in the space \(H_d\) (a system of two qubits). The maximum sets of complementary bases of this space consist of five elements \(B_t\) (for \(t = 0, 1, 2, 3, 4\))

\[
B_0 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\},
B_1 = \{\frac{1}{2}(1, 1, 1), \frac{1}{2}(1, 1, -1), \frac{1}{2}(1, -1, 1), \frac{1}{2}(1, -1, -1)\},
B_2 = \{\frac{1}{2}(1, -1, -i, -i), \frac{1}{2}(1, -1, i, i), \frac{1}{2}(1, 1, i, -i), \frac{1}{2}(1, 1, -i, i)\},
B_3 = \{\frac{1}{2}(1, -i, -i, -i), \frac{1}{2}(1, -i, i, i), \frac{1}{2}(1, i, i, -i), \frac{1}{2}(1, i, -i, i)\},
B_4 = \{\frac{1}{2}(1, -i, -i, i), \frac{1}{2}(1, -i, i, i), \frac{1}{2}(1, i, i, i), \frac{1}{2}(1, i, -i, i)\}.
\]

Let us denote by \(|j\rangle_{B_t}\) the \(j\)th element of basis \(B_t\). For the construction of conditional probabilities, let us choose for example bases \(B_0, B_1\) and \(B_2\), corresponding to pairs \((0, 1)\), \((0, 2)\) and \((1, 2)\), respectively. Let us write the vector \(|\psi\rangle \in H_d\) in three selected bases:

\[
|\psi\rangle := a |0\rangle_{B_0} + b |1\rangle_{B_0} + c |2\rangle_{B_0} + d |3\rangle_{B_0} = \frac{1}{2}(a + b + c + d)|0\rangle_{B_1} + \frac{1}{2}(a + b - c - d)|1\rangle_{B_1} + \frac{1}{2}(a - b + c - d)|2\rangle_{B_1} + \frac{1}{2}(a - b - c + d)|3\rangle_{B_1} = \frac{1}{2}(a + b + c + d)|0\rangle_{B_2} + \frac{1}{2}(a + b - c - d)|1\rangle_{B_2} + \frac{1}{2}(a - b + c - d)|2\rangle_{B_2} + \frac{1}{2}(a - b - c + d)|3\rangle_{B_2}.
\]

where \(a, b, c, d \in \mathbb{C}\). According to the interpretation given above, we obtain three independent conditional probabilities:

\[
P(C_0|B_2) = |a|^2 + |b|^2,
\]

\[
P(C_0|B_1) = \frac{1}{2}|a + b + c + d|^2 + \frac{1}{2}|a + b - c - d|^2,
\]

\[
P(C_1|B_0) = \frac{1}{2}|a - b + c + d|^2 + \frac{1}{2}|a - b - c + d|^2.
\]

This leads to \((a = x_0 + ix_1, b = x_2 + ix_3, c = x_4 + ix_5, d = x_6 + ix_7)\)

\[
P(C_0|B_2) = x_0^2 + x_1^2 + x_2^2 + x_3^2,
\]

\[
P(C_0|B_1) = \frac{1}{2}(x_0 + x_2 + x_4 + x_6)^2 + \frac{1}{2}(x_1 + x_3 + x_5 + x_7)^2
\]

\[
+ \frac{1}{2}(x_0 + x_2 - x_4 - x_6)^2 + \frac{1}{2}(x_1 + x_3 - x_5 - x_7)^2,
\]

\[
P(C_1|B_0) = \frac{1}{2}(x_0 - x_2 - x_4 - x_6)^2 + \frac{1}{2}(x_1 - x_3 + x_5 + x_7)^2
\]

\[
+ \frac{1}{2}(x_0 - x_2 + x_4 + x_6)^2 + \frac{1}{2}(x_1 - x_3 - x_5 - x_7)^2,
\]

where \((x_0, \ldots, x_7) \in S_7\).

The above equations define the projection \(S_7 \rightarrow D_3\) of a seven-dimensional sphere \(S_7\) into a three-dimensional cube \(D_3\). The combination of this projection with \(D_1 \rightarrow T_2\). The combination of this projection with \(D_1 \rightarrow T_2\).
results in the projection \( S_7 \rightarrow T_2 \) of a seven-dimensional sphere \( S_7 \) into a triangle \( T_2 \), which allows us to compare the applicability scope of various types of optimal strategies.

Figure 5 presents the areas (for 10000 randomly chosen points with respect to the constant probability distribution on \( S_7 \)) of probability \( q_m \) for which optimal strategies of different types exist (with the assumption \( \omega_0 = \omega_1 = \omega_2 = 1/3 \)).

Let us note that optimal strategies of any type (figure 5–all) are achievable within the frequency \( q_m \) area of the surface similar to the one from the quantization model on one qubit. A similar correspondence can be observed in the case of optimal intransitive strategies. The comparison of the areas corresponding to optimal transitive strategies reveals an essential difference. In the model presented here, this area overlaps with the area of all optimal strategies. There are no relevant intransitive strategies which were a specific property of a model based on one qubit. Hence, we do not observe any decline in the importance of intransitive strategies with the increasing support for one of the candidates. This property remains the characteristic for the model based on one qubit. It is a consequence of a special, linear form of conditional probabilities obtained in this model, each of which is dependent on one of the coordinates of the sphere \( S_2 \).

References

[1] Makowski M 2009 Phys. Lett. A 373 2125
[2] Boddy L 2000 FEMS Microbiol. Ecol. 31 185
[3] Shafir S 1994 Anim. Behav. 48 55
[4] Arrow K J 1951 Social Choice and Individual Values (New York: Yale University Press)
[5] Gardner M 1970 Sci. Am. 223 110
[6] Penney W 1969 J. Recreat. Math. 2 241
[7] Ulam S M 1976 Adventures of a Mathematician (New York: Scribner)
[8] Eisert J, Wilkens M and Lewenstein M 1999 Phys. Rev. Lett. 83 3077
[9] Meyer D 1999 Phys. Rev. Lett. 8 1052
[10] Flitney A P and Abbott D 2002 Fluct. Noise Lett. 2 R175
[11] Guo H, Zhang J and Koehler G J 2008 Decis. Support Syst. 46 318
[12] Piotrowski E W and Sladkowski J 2003 Int. J. Theor. Phys. 42 1089
[13] Miakisz K, Piotrowski E W and Sladkowski J 2006 Theor. Comput. Sci. 358 15
[14] Piotrowski E W and Sladkowski J 2004 Mathematical Physics Research at the Cutting Edge ed C V Benton (New York: Nova Science) pp 247–68
[15] Vaidman L 1999 Found. Phys. 29 615
[16] Piotrowski E W and Makowski M 2005 Fluct. Noise Lett. 5 L85
[17] Makowski M and Piotrowski E W 2006 Phys. Lett. A 355 250
[18] Makowski M and Piotrowski E W 2011 J. Phys. A: Math. Theor. 44 075301
[19] Vaccaro J A, Spring J and Chefles A 2007 Phys. Rev. A 75 012333
[20] Wiesner S 1983 SIGACT News 15 78
[21] Pakula I, Piotrowski E W and Sładkowski J 2007 Physica A 385 397
[22] Dupont P 1977/78 Rend. Sem. Mat. Univ. Politec. Torino 36 125