# Bethe Ansatz solutions for two qudits

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**Abstract.** We use the famous Weyl recipe to introduce Bethe Ansatz (BA) solutions for the two-magnon sector in the case of the Heisenberg magnet consisting of a small number \(N\) of nodes. In accordance with the Weyl recipe, the obvious symmetry group of the model is the translation group \(C_N\), whereas the group of all automorphisms \(\text{Aut}C_N\) constitutes the hidden symmetry, and, in fact, manifests important symmetry properties of the system. With the use of the action of the \(\text{Aut}C_N\) on the set of eigenvalues of BA, we propose the way of drawing energy bands for the problem being discussed.

1. **Introduction**

Considerations concerning the one-dimensional XXX Heisenberg model of a 1/2-spin magnet [1, 2, 3] are often related to solutions obtained by using the Bethe Ansatz method [4, 5, 6, 7]. Owing to the implementation of this famous substitution one can evaluate eigenvalues and eigenfunctions of the problem being discussed for relatively small number of nodes \(N\) and spin deviations \(r\). However, this method is not sufficient if we want to have an insight into the shape of energy bands, and so in our paper we propose the way which is helpful in drawing such bands. We focus our attention on the two-magnon sector \((r = 2)\) for the magnet consisting of \(N = 5\) nodes.

Let’s consider the set

\[
\tilde{N} = \{j = 1, 2, \ldots, N\}
\]

of nodes of the magnet. We can think of it as a computer memory in the language of quantum computation. Furthermore, the set of spin projections on every node

\[
\tilde{2} = \{+,-\}
\]

means a single bit. Then the set

\[
\tilde{2}^{\tilde{N}} = \{f : \tilde{N} \to \tilde{2}\}
\]

constitutes an orthonormal basis in the Hilbert space \( \mathcal{H} = \sum_{r=0}^{N} \oplus \mathcal{H}^{(r)} \), and describes the register of a classical computer with memory \(N\). The single-node space – a qubit – can be introduced as \(\text{lcc}\tilde{2}\), whereas \(h = \text{lcc}\tilde{N} = \mathcal{H}^{(1)}\) describes the single-magnon space – a qudit. In the case of two qudits we have \(h \otimes h \neq \mathcal{H}^{(2)}\), what means that physical states can not be considered as the tensor product of two single-magnon spaces, since decomposition

\[
h \otimes h = \mathcal{H}_{\text{hc}} \oplus \mathcal{H}_{\text{ph}} \oplus \mathcal{H}_{\text{a}}
\]

contains hard-core, physical and antisymmetric part, respectively, where in fact \(\mathcal{H}_{\text{ph}} = \mathcal{H}^{(2)}\).
2. The Weyl recipe

One of the tools which is helpful in investigating a symmetry of the Heisenberg magnet is so-called the Weyl recipe [8, 9]. In general, it is a useful way in considering some algebraic structure connected with a physical system that manifests some symmetry described by a group $G$. According to this method, $G$ is a group of the obvious symmetry of a physical system, while it is interesting to take into account a group $\text{Aut} G$ of all automorphisms of $G$, called a group of the hidden symmetry of a set, as a group $\text{Aut} G$ contains important properties of the symmetry of a set being investigated. In the case of the Heisenberg ring the obvious symmetry is $G = C_N$, and it constitutes a translational symmetry, so that, according to the Weyl recipe, it reflects in the hidden symmetry $\text{Aut} C_N$.

The group $\text{Aut} C_N$ is formed by elements $\tau$, for which the image coincides with the group $C_N$, e.g.

$$\text{Aut} C_N = \left\{ \tau : C_N \rightarrow C_N \mid \tau(j_1)\tau(j_2) = \tau(j_1j_2), \, j_1, j_2 \in C_N \right\}. \quad (5)$$

Hence, the group of automorphisms is formed by elements $\tau_r$ satisfying relation

$$\tau_r = \begin{pmatrix} j \\ r \cdot j \mod N \end{pmatrix}, \quad j \in \tilde{N}, \quad \text{if} \quad \gcd(r, N) = 1. \quad (6)$$

$\text{Aut} C_N$ acts on the Brillouin zone $B$ and transfers one of quasimomentum values $k \in B$ into another one $k' \in B$:

$$P : \text{Aut} C_N \times B \rightarrow B. \quad (7)$$

Orbits of the action $P$

$$P = \begin{pmatrix} k \\ r \cdot k \mod N \end{pmatrix}, \quad k \in B, \quad \text{if} \quad \tau_r \in \text{Aut} C_N \quad (8)$$

on the Brillouin zone $B$

$$B = \bigcup_{\kappa \in K(N)} B^{(\kappa)} \quad (9)$$

are generalized stars of the form

$$B^{(\kappa)} = \{ k \in B \mid \gcd(k, N) = \kappa \}. \quad (10)$$

With a concept of generalized star one can relate the structure of a characteristic polynomial of the $\hat{C}_N$ operator. The action of such an operator on any regular orbit of the group $C_N$ has the form:

$$\hat{C}_N | t, j \rangle = | t, (j + 1) \mod N \rangle. \quad (11)$$

Then characteristic polynomial of the $\hat{C}_N$ operator can be introduced as

$$w(x) = x^N - 1 = \prod_{\kappa \in K(N)} w_N^{(\kappa)}(x), \quad (12)$$

where polynomials $w_N^{(\kappa)}(x)$ are indecomposable over the field $\mathbb{Q}$ of rational numbers, and their decomposition into linear factors are related with generalized stars $B^{(\kappa)}$ via the formula

$$w_N^{(\kappa)}(x) = \prod_{k \in B^{(\kappa)}} (x - \omega^k), \quad (13)$$
so that, roots of the polynomial \( w_N^\kappa(x) \) constitute the set

\[
\text{RootOf}(w_N^\kappa(x)) = \{ \omega^k \mid k \in B(\kappa) \},
\]

(14)

with \( \omega = \exp(2\pi i/N) \) being the first primary \( N \)-th root of 1. In the case of the group Aut \( C_N \) of automorphisms, it turns out that it is helpful in determination of energy bands in the energy spectrum.

It is worth to mention, that, in general, the Hamiltonian commutates with \( \hat{C}_N \) operator

\[
[\hat{H}, \hat{C}_N] = 0,
\]

(15)

and so the total quasimomentum \( k \) is a motion constant, while in general it does not commute with automorphisms \( \tau_r \in \text{Aut} \, C_N \):

\[
[\hat{H}, \tau_r] \neq 0.
\]

(16)

Automorphisms \( \tau \) transfer quantum states one into another with different energies and quasimomenta. In next section we present our considerations for the case of Heisenberg magnet consisted of \( N = 5 \) nodes and \( r = 2 \) spin deviations.

3. An example: \( N = 5, r = 2 \)

For the case of \( N = 5 \) nodes system, characteristic polynomial has the form:

\[
w(x) = x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1),
\]

(17)

where \( w' \equiv w_N^1(x) = x^4 + x^3 + x^2 + x + 1 \) is the polynomial, indecomposable over the field \( \mathbb{Q} \). Then we can introduce \( \Delta \)

\[
\Delta = \{ \omega \in \mathbb{C} \mid w(\omega) = 0 \} = \{ \omega, \omega^2, \omega^3, \omega^4, \omega^5 \equiv 1 \}
\]

(18)

being the set of roots of polynomial \( w \). Table 1 contains the action of elements of the group Aut \( C_5 \) on elements of the group \( C_5 \), the action of elements of the group Aut \( C_5 \) on the set of roots \( \Delta \) of polynomial \( w \) and the table of multiplication group for the magnet consisted of \( N = 5 \), respectively.

**Table 1.** From the left: table of the action of elements of the group Aut \( C_5 \) on elements of the group \( C_5 \), table of the action of elements of the group Aut \( C_5 \) on the set of roots of polynomial \( w \) and multiplication table for the case \( N = 5 \) according to the rule \( \tau_r \tau_r' = \tau_{rr'} \mod N \).

| \( \tau \) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| \( \tau_1 \) | 1 | 2 | 3 | 4 | 5 |
| \( \tau_2 \) | 2 | 4 | 1 | 3 | 5 |
| \( \tau_3 \) | 3 | 1 | 4 | 2 | 5 |
| \( \tau_4 \) | 4 | 3 | 2 | 1 | 5 |

| \( \tau \) | \( \omega^5 \equiv 1 \) | \( \omega \) | \( \omega^2 \) | \( \omega^3 \) | \( \omega^4 \) |
|---|---|---|---|---|---|
| \( \tau_1 \) | 1 | \( \omega \) | \( \omega^2 \) | \( \omega^3 \) | \( \omega^4 \) |
| \( \tau_2 \) | 2 | \( \omega^2 \) | \( \omega^4 \) | \( \omega \) | \( \omega^3 \) |
| \( \tau_3 \) | 3 | \( \omega^3 \) | \( \omega \) | \( \omega^4 \) | \( \omega^2 \) |
| \( \tau_4 \) | 4 | \( \omega^4 \) | \( \omega^3 \) | \( \omega^2 \) | \( \omega \) |

Under the action of the translation group \( C_N \) in the case of \( N = 5, r = 2 \) the Hilbert space \( \mathcal{H}^{(2)} \) decomposes into orbits \( (t, j) \) in following way:
two-magnon states can be constructed according to the formula:

\[
H^{(2)} = iC_C \begin{cases} 
  t_1 = (1, 4) & t_2 = (2, 3) \\
  |j_1j_2 \rangle & |j_1j_2 \rangle \\
  12 & 13 \\
  23 & 24 \\
  34 & 35 \\
  45 & 14 \\
  15 & 25 
\end{cases}.
\]

In general, for the two-magnon sector \( r = 2 \) Fourier operator expanded on the space of two-magnon states can be constructed according to the formula:

\[
|Bt, k\rangle = \frac{1}{\sqrt{N}} \sum_j \omega^{-kj} |t, j\rangle \equiv \frac{1}{\sqrt{N}} \sum_j \omega^{(N-1)kj} |t, j\rangle,
\]

so that, for \( N = 5 \) it has the form:

\[
\hat{F} = \begin{pmatrix}
  5^{\frac{1}{2}} & 0 & 5^{\frac{1}{2}} \\
  0 & 5^{\frac{1}{2}} & 0 \\
  5^{-\frac{1}{2}} & 0 & 5^{-\frac{1}{2}} \\
\end{pmatrix}
\]

Hamiltonian matrix in the basis of orbits for the case being discussed has then the form:

\[
\hat{H} = \begin{pmatrix}
  t_1 = (1, 4), j & t_2 = (2, 3), j \\
  1 & 2 & 3 & 4 & 5 \\
  1 & 0 & 0 & 0 & 0 \\
  2 & 0 & 0 & 0 & 0 \\
  3 & 0 & 0 & -2 & 0 \\
  4 & 0 & 0 & 0 & -2 \\
  5 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

while in the basis of wavelets [10] Hamiltonian matrix appears in quasidiagonal form with respect to the total quasimomentum \( k \) as a result of the action \( \hat{F}^\dagger \hat{H} \hat{F} \).
Finally, determination of energy bands is related to the action of elements \( \tau \) on energy values

\[
\begin{pmatrix}
\omega \\
\omega^2 \\
\omega^3
\end{pmatrix}
\]

where

\[
E = \begin{pmatrix}
\omega \\
\omega^2 \\
\omega^3
\end{pmatrix}
\]

An example:

The action of the element \( \tau_3 \) on the energy value \( E_{r',k} = E_{2,-1} \equiv E_{2,1} = -3 + 2\omega^2 + 2\omega^3 \):

\[
[\tau_3(\omega)]E_{2,-1} = -3 + 2(\omega^2)^3 + 2(\omega^3)^3 = -3 + 2\omega^6 + 2\omega^9 = -3 + 2\omega + 2\omega^4 = -3 + 2\omega + 2(-1 - \omega - \omega^2 - \omega^3) = -5 - 2\omega^2 - 2\omega^3 = E_{2,2} \equiv E_{2,-2},
\]

where \( \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0 \). Results of the action of an each element of the group \( \text{Aut} C_5 \) on energy values \( E_{r',k} \) are introduced in Table 3 and in Figure 1.

| \( r' \) | \( E_{r',k} \) |
|---|---|
| 0 | \( E_{0,0} = 0 \) |
| 1 | \( E_{1,\pm1} = -3 - \omega^2 - \omega^3 \) |
| | \( E_{1,\pm2} = -2 + \omega^2 + \omega^3 \) |
| 2 | \( E_{2,0} = -4 \) |
| | \( E_{2,1} = -3 + 2\omega^2 + 2\omega^3 \) |
| | \( E_{2,2} = -5 - 2\omega^2 - 2\omega^3 \) |
Table 3. The action of elements of the group Aut $C_5$ on eigenvalues spectrum of $r' = 1$ and $r' = 2$ sector for $N = 5$. Table contains only positive $k$ values but it is also true for negative ones because of degeneracy $E_{1,1} \equiv E_{1,-1}$, $E_{1,2} \equiv E_{1,-2}$, $E_{2,1} \equiv E_{2,-1}$, $E_{2,2} \equiv E_{2,-2}$.

| $E_{r',k}$ | $r' = 1$ | $r' = 2$ |
|------------|-----------|-----------|
| $\tau_1$  | $E_{1,1}$ | $E_{1,2}$ | $E_{2,1}$ | $E_{2,2}$ |
| $\tau_2$  | $E_{1,2}$ | $E_{1,1}$ | $E_{2,2}$ | $E_{2,1}$ |
| $\tau_3$  | $E_{1,2}$ | $E_{1,1}$ | $E_{2,2}$ | $E_{2,1}$ |
| $\tau_4$  | $E_{1,1}$ | $E_{1,2}$ | $E_{2,1}$ | $E_{2,2}$ |

Figure 1. Energy spectrum for the case $N = 5$ in terms of $\omega$, with energy bands for sectors $r' = 1$ and $r' = 2$.

4. Discussion
The Weyl recipe method applied in this paper for investigation of Heisenberg magnet enables us to link eigenvalues expressed in terms of roots, which results in drawing energy bands. For the case $N = 5$, $r = 2$ bands can be separated immediately because of small number of states. However, together with the growth of $N$ one can observe an increase of the number of energy levels for a given value of $k \in B$ so that the way of connecting neighbour points of the Brillouin zone into bands becomes more and more complicated task. Then the action of the group Aut $C_N$ makes up an important advice, and we believe that with additional considerations it can be realized for a greater number of nodes $N$.

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