Instability and control of a periodically-driven Bose-Einstein condensate

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(Dated: September 18, 2008)

We investigate the dynamics of a Bose-Einstein condensate held in an optical lattice under the influence of a strong periodic driving potential. Studying the mean-field version of the Bose-Hubbard model reveals that the condensate becomes highly unstable when the effective intersite tunneling becomes negative. We further show how controlling the sign of the tunneling can be used as a powerful tool to manage the dispersion of an atomic wavepacket, and thus to create a pulsed atomic soliton laser.

PACS numbers: 03.75.Lm, 03.75.Kk, 03.65.Xp

Introduction – The spectacular experimental progress in confining Bose-Einstein condensates in optical lattice potentials has provided a powerful tool for investigating many-body quantum dynamics. Such optical potentials are extremely clean and controllable, and together with their long decoherence times, this allows the observation of many coherent lattice phenomena which are extremely challenging to study in other solid-state systems. One such effect is “coherent destruction of tunneling” (CDT) [1], observed very recently in atomic systems [2, 3], in which a periodic driving field acts to renormalise the tunneling between lattice sites. This control over the dynamics of the condensate is achieved without altering any of the parameters of the optical lattice, and has natural applications to quantum information, since it preserves the system’s coherence. However, it is crucial to know the stability of the condensate during its time evolution, particularly the presence of dynamical instability, in which deviations from a steady state grow exponentially with time. The case of a static potential was analyzed in Ref. [1], and studied experimentally in Refs. [2, 3], and it was found that dynamical instability occurs above a certain critical quasimomentum. Later work [4] examined the case of a uniformly accelerated lattice, and found the non-intuitive result that dynamical instability was enhanced in the limit of low acceleration.

In this paper we build upon this approach to analyze the richer and more complex case where the optical potential is periodically rocked. To achieve this we carry out the stability analysis about the Floquet states [1] of the system, which are the appropriate generalization of energy eigenstates to the case of a time-periodic Hamiltonian. An important point for experiment is to minimize or avoid instabilities, and so we first find the critical interaction strength at which dynamical instability occurs. We then connect this with the behavior of the effective tunneling \( J_{\text{eff}} \), and show how manipulating \( J_{\text{eff}} \) via CDT can be used to control the dynamics of the condensate, providing control over matter-wave dispersion [5] and thus allowing the creation of bright solitons.

Method – A system of cold bosons held in an optical lattice can be described very accurately [6] by the Bose-Hubbard Hamiltonian

\[
H_{\text{BH}} = -J \sum_{(m,n)} (a_m^\dagger a_n + H.c.) + \frac{U}{2} \sum_m n_m (n_m - 1),
\]

where \( a_m^\dagger / a_m \) are the boson creation/annihilation operators, and \( n_m = a_m^\dagger a_m \) is the number operator. The properties of the system are governed by the hopping parameter \( J \), and the Hubbard interaction \( U \) which describes the potential energy between two bosons occupying the same lattice site. An extremely valuable means of studying and controlling such systems is to accelerate the lattice by varying the phase-difference between the two laser beams forming the standing wave potential. In the rest frame of the lattice this acceleration manifests itself as an inertial force which effectively “tilts” the potential. If instead of a uniform acceleration the lattice is periodically accelerated and decelerated, it is possible to produce a potential that oscillates periodically in time, \( H_I = K \cos \omega t \sum_m n_m \), where \( K \) and \( \omega \) parametrize its amplitude and frequency respectively.

In order to study the stability of the driven condensate we will first pass to a mean-field description of this model, analogous to the Gross-Pitaevskii equation, and then linearize about the ground-state to obtain the Bogoliubov equations for the condensate excitations. We first write the Heisenberg equations of motion for the boson operators \( a_n \), and then take the classical field approximation and treat them simply as c-numbers \( \alpha_n \). It is then straightforward to show that the classical amplitudes obey the equation of motion

\[
i \frac{\partial \alpha_n}{\partial t} = -J(\alpha_{n+1} + \alpha_{n-1}) + g|\alpha_n|^2 + K \cos \omega t \, n \alpha_n
\]

where for convenience we have scaled the interaction as \( g = U/N \). Note that we also take \( \hbar = 1 \), and will measure all energies in units of \( J \).

To simplify the analysis we use periodic boundary conditions. In the limit of large lattice sizes, however, the choice of boundary conditions does not affect the underlying physics, and we will later use Dirichlet boundary conditions to simulate the time-evolution of the condensate. In the absence of interactions, the eigenstates of the
system will simply be plane waves \( \alpha_n = \exp \{ i n p \} \) where \( p \) is the angular momentum. With this in mind, we take as a trial solution \( \alpha_n = \exp \{ i (n \phi + \theta) \} \), where \( \phi \) and \( \theta \) are functions to be determined. Substituting this solution in Eq\[2\], we have

\[
\begin{align*}
\dot{\phi}(t) &= p - \frac{K}{\omega} \sin \omega t \quad (3) \\
\dot{\theta}(t) &= 2J(\cos p S(t) - \sin p C(t)) - gt \quad (4)
\end{align*}
\]

where the functions \( S(t)/C(t) \) are defined in terms of Bessel functions as

\[
C(t) = \sum_{m=-\infty}^{\infty} \frac{\cos m\omega t - 1}{m\omega} J_m(K/\omega), \quad (5)
\]

\[
S(t) = \sum_{m=-\infty}^{\infty} \frac{\sin m\omega t}{m\omega} J_m(K/\omega). \quad (6)
\]

Since the Hamiltonian of the system is periodic in time, the Floquet theorem dictates that the solutions of the time-dependent Schrödinger equation can be written in the form \( \exp \{ i \epsilon t \} u(t) \), where \( \epsilon \) is termed the quasienergy, and \( u(t) \) is a \( T \)-periodic function called the Floquet state. To obtain the quasienergies, we thus simply have to extract the terms from the solution which are not \( T \)-periodic, giving the result

\[
\epsilon(p) = 2 \cos p J_0(K/\omega) + g. \quad (7)
\]

In the absence of the driving the quasienergies thus form a normal single-particle bandstructure, the interaction \( g \) acting merely to shift the entire spectrum. The driving then acts to renormalise the width of the spectrum by the Bessel function \( J_0 \), as was previously observed in a theoretical analysis [10] for semiconductor superlattice systems. In Fig[1], we show numerical results for the quasienergies of an 8-site system, obtained directly from the time-evolution of the system, which beautifully corroborate the expected behavior. In particular, when the Bessel function becomes zero, the spectrum collapses to a point and the system will manifest CDT.

In order to analyze the dynamical stability of the ground state \( (p = 0) \), we now introduce a perturbation \( \alpha_0(t) = \alpha_0^0(t) (1 + u(t) \exp \{ i q \nu \} + v^* (t) \exp \{ -i q \nu \} ) \), where \( \alpha_0^0(t) \) is the unperturbed solution, and \( q \) and \( \omega \) are the momentum and energy of the excitation. We then linearize Eq\[2\] about this solution to obtain the Bogoliubov de Gennes equations for \( u(t) \) and \( v(t) \)

\[
i \frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \mathcal{L}(q,t) \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad (8)
\]

where the elements of the matrix \( \mathcal{L}(q,t) \) are given by

\[
\begin{align*}
\mathcal{L}_{11}(q,t) &= 4J \sin(q/2) \sin(q/2 - K/\omega \sin \omega t) + g, \\
\mathcal{L}_{12}(q,t) &= g = -\mathcal{L}_{21}(q,t), \\
\mathcal{L}_{22}(q,t) &= -4J \sin(q/2) \sin(q/2 + K/\omega \sin \omega t) - g.
\end{align*}
\]

We can note that, similarly to the Hamiltonian, the operator \( \mathcal{L}(q,t) \) is \( T \)-periodic. Consequently we can also apply the Floquet theorem to describe the time-evolution of the quasi-particle excitation \( (u,v) \). To find the corresponding Floquet states, we numerically solve Eq\[8\] over one period of driving, using the \( 2 \times 2 \) identity matrix as the initial state. The result of this procedure is the single-period propagator \( U \). The eigenstates of \( U \) are then the Floquet states, while its eigenvalues are related to the excitation quasienergies via \( \lambda_\pm = \exp \{ -i\omega t \} \).

The symmetries of \( U \), combined with the normalisation condition obeyed by the quasiparticle excitation, \(|u|^2 - |v|^2 = 1\), allow the characteristic equation to be written in the particularly simple form

\[
\lambda_\pm = \text{Re} \left[ U_{11} \right] \pm \sqrt{\text{Re} [U_{11}]^2 - 1}. \quad (9)
\]

The condition of dynamical stability can now be recast in terms of the eigenvalues of \( \mathcal{L}(q,t) \), in analogy with the use of Lyapunov exponents in classical mechanics. The solution is stable against an excitation with momentum \( q \) if there is no quasienergy with a positive imaginary component, or equivalently, if \( \text{Re} [U_{11}] \leq 1 \). This suggests a simple scheme to map out the stability zones of the driven system. For a given choice of \( K \) and \( \omega \) we select a value of \( g \) and scan over the range of \( q \). If \( \text{Re} [U_{11}] \leq 1 \) for all values of \( q \) we can declare that the system is stable for these parameters, and that to induce instability we need to increase \( g \) to a higher value. In this way a standard bisection scheme can be used to locate the instability boundary, \( g_c \).

**Results** – We show the results of this procedure for two driving frequencies in Fig[1]. We can first observe that for \( K/\omega = 0 \) it is possible to directly diagonalise \( \mathcal{L}(q) \) to obtain the result \( \epsilon_\pm = \pm 2\sqrt{2} \sin(q/2) \sqrt{2J^2 \sin^2(q/2) + Jg} \). As expected, this duplicates the familiar result for the Bogoliubov excitations of an undriven, stationary condensate [11]. It is also clear from this expression that dynamical instability will not occur if the interaction is repulsive, since the eigenfrequencies will not become complex unless the product \( (J,g) < 0 \). Accordingly, as \( K/\omega \to 0 \) we can see from Fig[1] that the value of \( g_c \) diverges. For larger \( K/\omega \) the value of \( g_c \) then rapidly drops, passing through a broad local minimum before again diverging as \( K/\omega \) approaches 2.4048 – the first zero of \( J_0 \). This corresponds to the onset of CDT; as the effective tunneling is reduced, the dynamics of the condensate is suppressed, and stability is regained.

Passing through the zero of \( J_0 \), we can see that dynamical stability is then abruptly lost. In this region the condensate becomes dynamically unstable for any positive value of the interaction. We can obtain some insight into this effect from Eq\[7\] by defining an effective tunneling, \( J_{\text{eff}} = J_0 g/K/\omega \). When \( K/\omega \) is increased from 2.4048, the Bessel function changes sign and \( J_{\text{eff}} \)
becomes negative. The physical significance of this sign-change has been observed previously in experiment [2], where it caused the momentum distribution function to be discretely shifted by $\pi$. This occurs because the tunneling, as well as being renormalized in amplitude, acquires a phase-factor of $\exp[i\pi]$ [12]. Accordingly, if we view the driving field as acting simply to renormalise the tunneling, the product ($J_{\text{eff}} g$) now becomes negative and dynamical instability can indeed occur. A similar effect would occur if instead $g$ were made negative (for example, by using a Feshbach resonance), for which the condensate would become attractive and thus unstable toward collapse. We can note that in Ref. [2] the condensate was close to non-interacting, and thus avoided this instability. When $J_{\text{eff}}$ becomes zero at $K/\omega = 5.52$ we can see that $g_c$ again diverges due to CDT. The same pattern of behavior then repeats. Fig.1 also shows that $g_c$ scales quite accurately as $g_c \sim \omega^2$. Surprisingly, the zone of stability thus becomes wider at high driving frequencies, although the acceleration of the lattice is much larger. A similar feature was seen in the analysis of an accelerated condensate [7], which found an increasing propensity to dynamical instability in the limit of low acceleration.

![Quasienergy spectrum of the driven mean-field Bose-Hubbard model, for an 8-site system with $\omega = 16$.](image)

FIG. 1: (a) Quasienergy spectrum of the driven mean-field Bose-Hubbard model, for an 8-site system with $\omega = 16$. The width of the spectrum is modulated by the Bessel function $J_0(K/\omega)$ and displaced by the interaction energy $g = 0.5$, in full agreement with the analytical solution Eq.7. (b) Plot of the critical interaction, $g_c$, at which the system becomes dynamically unstable, for driving frequencies $\omega = 4$ and 8. This quantity diverges at $K/\omega = 0$ and at the Bessel function zeros (vertical dashed lines), and is zero when $J_{\text{eff}}$ is negative, for which dynamical instability occurs for any positive value of $g$.

We have so far used an ideal flat optical lattice potential. In experiment, however, an additional harmonic trap potential is usually present which can substantially modify the dynamics of the system, and introduce new effects. To investigate this, we now apply an additional quadratic potential $V = kr_1^2$, where $r_1$ is measured from the center of the system. We initialize the system in the ground state of the the mean-field Hamiltonian (2) in the presence of the trap, and then displace the trapping potential by a distance of 25 lattice spacings, thereby exciting the condensate into motion.

In Fig.2a, we show the evolution of the condensate in the absence of the periodic driving. The condensate makes a periodic oscillation of constant amplitude, very similar to the center of mass motion observed in experiment [5]. The period of oscillation is governed by the intersite tunneling, or equivalently, by the condensate’s effective mass ($m^* \propto J_{\text{eff}}^{-1}$), and thus allows these quantities to be measured directly. In Fig.2b the system is subjected to a driving with $K/\omega = 2$. Clearly the oscillation period has increased, corresponding to the expected reduction of the tunneling by $J_0$, which can alternatively be interpreted as an enhancement of the effective mass. Increasing $K/\omega$ further to the first zero of $J_0$ produces CDT, and so the system remains frozen in its initial state (Fig.2c). In this case the effective mass has become infinite.

A further increase of $K/\omega$ means that the Bessel function changes sign, and thus $J_{\text{eff}}$ becomes negative. We have seen already that this sign-change has a significant effect on the dynamical stability of the ground-state, and as we show in Fig.2d it has an equally dramatic effect on the dynamics of the condensate. Instead of oscillating, the condensate now rapidly accelerates away from the center of the trap. The reason for this becomes evident when we examine the terms of the Hamiltonian. Making $J_{\text{eff}}$ negative is clearly equivalent to time-reversed evolution with a positive tunneling, but with reversed signs for the trapping potential and non-linearity $g$. Thus when $J_{\text{eff}}$ changes sign, the condensate behaves as an attractive condensate in an inverted potential, and so is quickly expelled from the center of the trap.

Controlling the parameters of the periodic driving field thus allows us to tune the effective mass of the condensate to be positive, negative, or infinite. A particularly important application of this control is the production of solitons. Solitons remain stable by balancing dispersion with the interparticle interaction, and thus bright soliton solutions of the Gross-Pitaevskii equation demand either using an attractive interaction, or the use of complicated staggered phase-imprinting techniques. Making the effective mass negative, however, would allow bright solitons to be created in repulsive condensates, without having to reverse the sign of the interaction or requiring phase imprinting.

To investigate this possibility, we revisit a proposal made by Carr and Brand in Ref. [12] for manipulating a trapped condensate to produce a train of solitons. The
interaction must be set to an appropriately small value. This is particularly important for the determination of the condensate into a train of solitonic pulses, as we have seen previously, both of these processes can be accomplished simultaneously by tuning the sign of \( J_{\text{eff}} \). As we have seen in Ref. [13], and the initial state has almost completely converted into a soliton train.

We have also shown how manipulating \( J_{\text{eff}} \) in this way may also be used as a novel tool to control the dynamics of a condensate, in a complementary way to the well-known method of controlling the interaction via Feshbach resonances. This both extends the possibility of manipulating the condensate to systems which do not possess convenient resonances, and provides a new means to investigate the interplay between nonlinearity and dispersion, notably the production of solitons.

The author was supported by a Ramón y Cajal Fellowship.

Conclusions – In summary, we have shown how an oscillating driving potential can be used to renormalise the effective tunneling, \( J_{\text{eff}} \), of a Bose-Einstein condensate. This parameter crucially determines when the condensate is dynamically unstable, and when \( J_{\text{eff}} \) is negative the condensate becomes unstable for any positive value of interaction. This is particularly important for the design of experiments, since in such parameter ranges the interaction must be set to an appropriately small value.