Positivity and Fermionic Dense Matter

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Euclidean dense matter generically suffers from the fermion sign problem. However, we argue that the sign problem is absent if one considers only low-energy degrees of freedom. Specifically, the low energy effective theory of dense QCD has positive Euclidean path integral measure, which allows one to establish rigorous inequalities showing that the color-flavor locked (CFL) phase is the true vacuum of three flavor, massless QCD. We then describe a method for simulating dense QCD on the lattice. We also discuss applications to electronic systems in condensed matter, such as generalized Hubbard models.

Quark matter is described by a partition function,

\[ Z(\mu) = \int dA \det (M) e^{-S(A)}, \]

(1)

where \( \mu \) is the chemical potential for the quark number and \( M = \gamma^\mu E_D^\mu + \mu \gamma^4_E \) is the Euclidean Dirac operator. Since the Dirac operator \( M \) consists of both anti-Hermitian and Hermitian operators, it has in general complex eigenvalues. Furthermore, it is not related to its Hermitian conjugate by any similarity transformations.

The determinant of \( M \) is therefore complex for generic gauge fields, which has thus far made lattice simulations very difficult \cite{1243}. However, it is shown here that the complexity of the measure of fermionic dense matter can be ascribed to modes far from the Fermi surface, which are irrelevant to dynamics at sufficiently high density in most cases, including quark matter \cite{145}. For modes near the Fermi surface, there is a discrete symmetry, relating particles and holes, which pairs the eigenvalues of the Dirac operator to make its determinant real and non-negative.

As a simple example, let us consider fermionic matter in 1+1 dimensions, where non-relativistic fermions interact with a gauge field \( A \). The action is in general given as

\[ S = \int d\tau dx \, \psi^\dagger [(-\partial_\tau + i\phi + \epsilon_F) - \epsilon(-i\partial_x + A)] \psi, \]

(3)

where \( \epsilon(p) \approx p^2/(2m) + \cdots \) is the energy as a function of momentum. Low energy modes have momentum near the Fermi points and have energy, measured from the Fermi points,

\[ E(p \pm p_F) \approx \pm v_F p, \quad v_F = \frac{\partial E}{\partial p}_{p_F}. \]

(4)

If the gauge fields have small amplitude and are slowly varying relative to scale \( p_F \), the fast modes are decoupled from low energy physics. The low energy effective theory involving quasi particles and gauge fields has a positive, semi-definite determinant.

To construct the low energy effective theory of the fermionic system, we rewrite the fermion fields as

\[ \psi(x, \tau) = \psi_L(x, \tau)e^{+ipFx} + \psi_R(x, \tau)e^{-ipFx}, \]

(5)

where \( \psi_{L,R} \) describes the small fluctuations of quasiparticles near the Fermi points. Using \( e^{\pm ipFx}E(-i\partial_x + A)e^{\mp ipFx}\bar{\psi}(x) \approx \pm v_F(-i\partial_x + A)\bar{\psi}(x) \), we obtain

\[ S_{\text{eff}} = \int d\tau \, dx \left[ \psi_L^\dagger(-\partial_\tau + i\phi + i\partial_x - A)\psi_L + \psi_R^\dagger(-\partial_\tau + i\phi - i\partial_x + A)\psi_R \right]. \]

(6)
Introducing the Euclidean (1+1) gamma matrices \( \gamma_{0,1,2} \) and \( \psi_{L,R} = \frac{1}{2} (1 \pm \gamma_2) \psi \), we obtain a positive action:

\[
S_{\text{eff}} = \int d\tau dx \, \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu) \psi \equiv \int d\tau dx \, \bar{\psi} \mathcal{D} \psi.
\]

Since \( \mathcal{D} = \gamma_2 \mathcal{D}^\gamma \gamma_2 \), the determinant of \( \mathcal{D} \) is positive, semi-definite.

In this example, we see that low energy modes near the Fermi surface can be integrated (in favor of a determinant), leading to an effective theory without any sign problem whatsoever as long as they couple to slowly varying background fields. QCD at high baryon density falls into this category, since the coupling constant is small at high energy due to asymptotic freedom, and the corresponding fluctuations in the gauge fields at high density are small.

A low energy effective theory of QCD at high density, called as High Density Effective Theory (HDET), has been derived by one of us [6]. (Renormalization group analysis of the Fermi surface effective theory appears in [7].) Consider a quark in one of the patches that cover the Fermi surface only once. Its momentum can be decomposed as

\[
p_\mu = \mu v_\mu + l_\mu , \quad |l_\mu| < \Lambda , \quad \Lambda_\perp (\ll \mu ),
\]

where \( \Lambda \) and \( \Lambda_\perp \) are the sizes of patches perpendicular and parallel to the Fermi surface respectively, much smaller than the chemical potential but larger than the scale of interest. The normalization is enforced by a condition,

\[
\sum_{\text{patches}} \int_{\Lambda_\perp} d^2 l_\perp = 4\pi \rho_F^2.
\]

The modes near the Fermi surface are given as

\[
\psi_+ (\bar{v}_F, x) = \frac{1 + \bar{\alpha} \cdot \bar{v}_F}{2} e^{-i\mu \bar{v}_F \cdot \bar{x}} \psi(x),
\]

where \( \bar{\alpha} = \gamma_0 \gamma \) and \( \bar{v}_F \) is the Fermi velocity of the modes. HEDT of quark matter is then described by

\[
\mathcal{L}_{\text{HDET}} = \bar{\psi_+} i\gamma_\mu \partial_\mu \psi_+ - \frac{1}{2\mu} \bar{\psi_+} \gamma^0 (\mathcal{D}_\perp) \psi_+ + \cdots ,
\]

where \( \gamma_\mu = (\gamma^0, \bar{v}_F \bar{v}_F \cdot \vec{\gamma}) = \gamma^\mu - \gamma_\perp^\mu \). We see that the leading term has a positive determinant, since

\[
M_{\text{eff}} = \gamma_\mu^E \cdot D(A) = \gamma_5 M_{\text{eff}}^t \gamma_5.
\]

In order to implement this HDET on lattice, it is convenient to introduce an operator formalism, where the velocity is realized as an operator,

\[
\bar{v} = \frac{-i}{\sqrt{-\nabla^2}} \frac{\partial}{\partial x} .
\]

Then the quasi quarks near the Fermi surface become

\[
\psi = \exp (+i\mu x \cdot v \frac{1 + \alpha \cdot v}{2}) \psi.
\]

Now, neglecting the higher order terms, the Lagrangian becomes with \( X = \exp (i\mu x \cdot v \frac{1 + \alpha \cdot v}{2}) \),

\[
\mathcal{L}_{\text{HDET}} = \bar{\psi_+} \gamma^\mu (\partial_\mu + iA_\mu) \psi_+ ,
\]

where \( A_\mu^+ = X^\dagger A^\mu X \) denotes soft gluons whose momentum \( |p_\mu| < \mu \). Since \( v \cdot \partial + \gamma = \partial \cdot \gamma \), we get \( \gamma^\mu \partial^\mu = \gamma^\gamma \partial^\gamma \), which shows that the operator formalism automatically covers modes near the full Fermi surface.

Integrating out the fast modes, modes far from the Fermi surface and hard gluons, the QCD partition function becomes

\[
Z(\mu) = \int dA_+ \det (M_{\text{eff}}) e^{-S_{\text{eff}}(A_+)} ,
\]

where

\[
S_{\text{eff}} = \int d\tau d^2 x \left( \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{M^2}{16\pi} A_{\perp,\perp}^a A_{\perp,\perp}^a \right) + \cdots ,
\]

and \( A_\perp = A - A_\parallel \), the Debye mass \( M = \sqrt{N_f/(2\pi^2)} g_\mu \). At high density the higher order terms \( \sim \Lambda/\mu \) are negligible and the effective action becomes positive, semi-definite. Therefore, though it has non-local operators, HDET in the operator formalism is free from the sign problem and can be used to simulate the Fermi surface physics like superconductivity. Furthermore, being exactly positive at asymptotic density, HDET allows to establish rigorous inequalities relating bound state masses and forbidding the breaking
of vector symmetries, except baryon number, in dense QCD \[5\].

With the help of the previous two examples, we propose a new way of simulating dense QCD, which evades the sign problem. Integrating out quarks far from the Fermi surface, which are suppressed by \(1/\mu\) at high density, we can expand the determinant of Dirac operator at finite density,

\[
\det (M) = \text{[real, positive]} \left[ 1 + \mathcal{O} \left( \frac{\mathcal{F}}{\mu^2} \right) \right]. \tag{18}
\]

As long as the gauge fields are slowly varying, compared to the chemical potential \(\mu\), the sign problem can be evaded. As a solution to the sign problem, we propose to use two lattices with different spacings, a finer lattice with a lattice spacing \(a_{\text{det}} \sim \mu^{-1}\) for fermions and a coarser lattice with a lattice spacing \(a_{\text{gauge}} \ll \mu^{-1}\) for gauge fields and then compute the determinant on such lattices.

The determinant is a function of plaquettes \(\{U_{x\mu}\}\) which are obtained by interpolation from the plaquettes on the coarser lattice with spacing \(a_{\text{gauge}}\). To get the link variables for the finer lattice, we interpolate the link variables \(U_{x\mu} \in SU(3)\) (see Fig. 1). Connect any two points \(g_1, g_2\) on the group manifold as

\[
g(t) = g_1 + t(g_2 - g_1) , \quad 0 \leq t \leq 1. \tag{19}
\]

Figure 1. Simulation with two lattices with different lattice spacings

For importance sampling in the lattice simulation, one can use the leading part of the determinant, [real, positive]. This proposal provides a nontrivial check on analytic results at asymptotic density and can be used to extrapolate to intermediate density. Furthermore, it can be applied to condensed matter systems like High-\(T_c\) superconductors, which in general suffers from a sign problem.

Acknowledgement

The work of D.K.H. is supported by KOSEF grant number R01-1999-000-00017-0 and also by Pusan National University Research Grant, 1999 – 2003. The work of S.H. was supported in part under DOE contract DE-FG06-85ER40224 and by the NSF through through the USA-Korea Cooperative Science Program, 9982164.

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