Comparing theories: the dynamics of changing vocabulary.
A case-study in relativity theory.

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Abstract

There are several first-order logic (FOL) axiomatizations of special relativity theory in the literature, all looking different but claiming to axiomatize the same physical theory. In this paper, we elaborate a comparison, in the framework of mathematical logic, between these FOL theories for special relativity. For this comparison, we use a version of mathematical definability theory in which new entities can also be defined besides new relations over already available entities. In particular, we build an interpretation (in Tarski’s sense) of the reference-frame oriented theory \textsf{SpecRel} of \cite{4} into the observationally oriented Signalling theory of James Ax \cite{6}. This interpretation provides \textsf{SpecRel} with an operational/experimental semantics. Then we make precise, “quantitative” comparisons between these two theories via using the notion of definitional equivalence. This is an application of logic to the philosophy of science and physics in the spirit of van Benthem’s \cite{8,9}.

1 Introduction

This paper is about an application of logic to the methodology of science in the spirit of van Benthem’s \cite{8,9}.

There are several axiomatizations of special relativity theory available in the literature, all looking different but claiming to axiomatize the same physical theory. Such are, among many others, the ones in Andréka et al. \cite{4}, Ax \cite{6}, Goldblatt \cite{17}, Schelb \cite{32}, Schutz \cite{33}, Suppes \cite{38}. These papers talk about very different kinds of objects: \cite{4} talks about reference frames, \cite{6} talks about particles and signals, \cite{17} looks like a purely geometrical theory about orthogonality, the central notion of \cite{38} is the so-called Minkowski-metric, etc. While, as usual, one gets a better picture of this area via a variety of different “eyeglasses”, the following questions arise. What are the connections between these theories? Do they all talk about the same thing? If they do, do they capture it to the same extent, or is one axiomatization more detailed or accurate than some of the others? In this paper we want to show how, in the framework of mathematical logic, a concrete, tangible comparison/connection can be elaborated between these theories for special relativity and what we can gain from such an investigation.

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For this comparison, we have to use a form of logical definability theory in which totally new kinds of entities can be defined as opposed to traditional definability theory where only new relations can be defined on already available entities. The existing methods of definability theory had to be modified and refined for the purposes of the present situation. Thus, definability theory, too, profits from such an application. In the present paper, we elaborate in detail one piece of such a comparison: we construct an interpretation of the relativity theory in [4] talking about reference frames into the theory in [6] which talks about particles emitting and absorbing signals. Then we construct an inverse interpretation for showing which versions of the two theories are definitionally equivalent. Since this is a case-study showing applicability of the proposed method for comparing/connecting theories, we tried to give all the detail needed. This is why some sections of the paper may look somewhat technical.

An insight of last century mathematical logic is that it is important to fix the vocabulary of a first-order logic theory and stay inside the so obtained language while in a specific theory (see, e.g., [40]). The symbols in the vocabulary are the concepts that are not analyzed further in the given theory, they are called thus basic (or primitive) concepts. But this is not a forever frozen state: we may decide to analyze further the basic concepts of this vocabulary and we can do this in the form of building an interpretation (in the sense of mathematical logic) into another language the vocabulary of which consists of new basic concepts, and the interpretation gives us the information of how the “old basic concepts” are built up from the “new basic concepts” as refined ones. The interpretation we construct in this paper thus refines the basic concept of a reference frame in terms of just sending and receiving signals. To refine the basic concepts of this Signalling theory, we can interpret it to, say, in a theory of electromagnetism, or in a quantum-mechanical theory.

Such an interpretation may also be regarded as defining a so-called operational semantics for the basic concepts of the first theory. Starting with the Vienna Circle, several authors suggest that a physical theory is a more complex object than just a set of first-order logic (FOL for short) formulas. A physical theory, they propose, is a FOL theory together with instructions for how to interpret the basic symbols (or vocabulary) of this theory “in the real physical world”. (Following Carnap [12], this is often called a “(partially) interpreted theory”.) We want to show in section 6 that such an “operational semantics” can be taken to be an interpretation in the sense of mathematical logic.

Returning to our concrete example, an operational semantics should say something about how we obtain or set up (in the real world) the reference frames for special relativity theory. Usually, rigid meter-rods and standard clocks are used for this purpose (e.g., [44]). However, as [34] points out, we cannot use these rigid meter-rods in astronomy or cosmology. The interpretation we give in detail in this paper results also in an operational/experimental/observational definition for setting up a reference frame by just relying on sending and receiving light-signals. This method can be used, in principle, in the above mentioned astronomical scale.

Summing up, the first language in an interpretation has the theoretical concepts while the basic concepts of the second language are the observational ones (for the observational-theoretical duality see, e.g., [8, 13]). We can look at the same interpretation “from the other direction”: In our example, we may imagine someone living in a space-time, exploring his surroundings by sending and receiving signals, and during this process, he devises so-called theoretical concepts which make thinking more efficient. In particular, he may devise the concept of a reference frame, and even the concept of quantities forming a field, as mental constructs having concrete definitions in terms of observations. The tools of mathematical logic, and more closely those of definability theory.

\footnote{Other names for vocabulary are signature and set of nonlogical constants}
(interpretations are among them) can be used for modeling this emergence of theoretical concepts.

A further aim of the present approach of comparing theories is shifting the emphasis from working inside a single huge theory to working in a modularized hierarchy of smaller theories connected in many ways. Usually this approach is called theory-hierarchy. We note that this is not so much a hierarchy as rather a category of theories, technically the category of all FOL-theories as objects with interpretations as morphisms of the category. This direction of replacing a huge theory with a category of small theories is present in many parts of science. In foundational thinking, [13] emphasizes this. In computer science, it is present in the form of structured programming. “Putting theories together” of Burstall and Goguen [11] refers to the act of computing/generating colimits of certain diagrams in this category. Even in such practical areas as using a huge medical data-base the need of modularizing arises: it is necessary to “break up” the given data-base and generate many smaller ones according to the query at hand [21, 22]. The interpretation going from special relativity as formalized in [4] into the more observational Signalling theory of [6] we build in the present paper is but one morphism of this huge dynamic category of FOL theories.

The content of this paper can also be viewed as preparing the ground for an application of algebraic logic to relativity theory, as follows. The cylindric algebra of a theory is an abstract representation of the structure of concepts expressible/definable in that theory and a homomorphism between two cylindric algebras corresponds to an interpretation between the corresponding theories. Hence the category of all FOL theories is basically the same as the category of cylindric algebras as objects and homomorphisms as morphisms. There are, for example, well known and understood methods for how to compute colimits in this category of algebras.

There are still many questions and phenomena to be understood in this area of application of logic. For example, what are the desirable or good properties of an interpretation for being informative about the theories in one or other respect? Consider for a second the definability/interpretability picture between scientific theories (in FOL) in two versions: (1) in the framework of traditional definability theory, and (2) in the new, extended theory of definability used in the present paper. What are the characteristic differences? We think it is useful to keep this picture/issue in mind.

In section 2 we briefly recall the relativity theory SpecRel from [4], in section 3 we recall Signalling theory SigTh from [6] and we try to give a basic feeling for it by sketching the proof of completeness theorem in [6]. Section 4 is an important part of the present paper, it contains an algorithm for how to set up a reference frame in Signalling theory, this is an “operational semantics” for setting up reference frames of [4]. At the end of the section we outline how the same method could be used for space-times other than special relativistic, e.g., for the Schwarzschild space-time of a black hole. This algorithm is at the heart of the interpretation elaborated in section 6. Section 5 recalls the features of the more refined definability theory that are needed for defining the interpretation of SpecRel into SigTh. Section 7 rounds up the picture between SpecRel and SigTh by interpreting SigTh in a slightly reinforced version of SpecRel and then giving more information about connections between various concrete theories of special relativity. We end the paper with a Conclusion.

2 Special Relativity

In this section we give a list of basic concepts and axioms of the FOL theory SpecRel in [3, 4, 5, 23, 30].
The basic notions not analyzed further in SpecRel are “observers” having reference frames in which they represent the world-lines of bodies (or test particles), of which signals (light-particles, or photons) are special ones. The world-line of a body represents its motion, it is a function that describes the location of the body at each instant. For representing “time” and “location”, observers use quantities, quantities are endowed with addition and multiplication in order to be able to express whether a motion is “uniform” or not. To make life simpler, we treat also observers as special bodies. (Another, equivalent, option would be to treat them as entities of different “kind”, or of different “sort”, than bodies and quantities.) The reference frame or world-view of an observer \( o \) gives the information which bodies \( b \) are present at time \( t \) at location \( x, y, z \); thus \( W(o, b, t, x, y, z) \) expresses that body \( b \) is present at \( t \) in \( ⟨x, y, z⟩ \), according to observer \( o \). We treat quantities as entities of different nature, of different kind, than bodies.

According to the above, the vocabulary of the language of SpecRel is the following: we have two sorts, bodies \( B \) and quantities \( Q \), we have two unary relations \( \text{Obs}, \text{Ph} \) of sort \( B \), we have two binary functions \( +, \star \) of sort \( Q \), and we have a six-place relation \( W \) the first 2 places of which are of sort \( B \) and the rest of sort \( Q \).

Next, we list the five axioms of SpecRel. Concrete formulas and more intuition can be found in, e.g., \([3, 4, 5, 23, 36]\).

\[ \text{AxPh} \quad \text{The world-lines of photons are exactly the straight lines of slope 1, in each reference frame.} \]

\[ \text{AxEv} \quad \text{All observers coordinatize the same physical reality (i.e., the same set of events).} \]

\[ \text{AxSelf} \quad \text{The “owner” of a reference frame sits tight (stays put) at the origin.} \]

\[ \text{AxFd} \quad \text{The quantities form a Euclidean field w.r.t. the operations } +, \star, \text{ this means that } Q, +, \star \text{ form an ordered field in which each positive quantity has a square root.} \]

\[ \text{AxSym} \quad \text{All observers use the same units of measurement: if two events are simultaneous for observers } o, o', \text{ then the spatial distance between them is the same according to } o, o'. \]

\[ \text{SpecRel}_0 \ := \ \{ \text{AxPh, AxEv, AxSelf, AxFd} \} \quad \text{and} \]
\[ \text{SpecRel} \ := \ \{ \text{AxPh, AxEv, AxSelf, AxFd, AxSym} \}. \]

SpecRel may seem to be a rather weak axiom system. However, this is not so. All the well-known theorems/predictions of the (kinematics of) special relativity can be proved even from \( \text{SpecRel}_0 \). Below is a sample of theorems that can be proved from \( \text{SpecRel}_0 \) (for proofs, further theorems provable from SpecRel, and for extensions see the references given earlier as well as \([24, 37]\)):

• Each observer moves uniformly and slower than light in any other observer’s world-view (i.e., the world-line of an observer is a straight line with slope less than 1).

Assume that \( o, o' \) are moving relative to each other.

• Events that are separated in \( o \)’s world-view in a direction of \( o' \)’s motion and simultaneous according to \( o \), are not simultaneous according to \( o' \).

• Events that are simultaneous according to both \( o \) and \( o' \) are exactly the ones that are separated orthogonally to the direction of motion of \( o' \).
• Assume that \(o\) and \(o'\) use the same units of measurement, i.e., the spatial distance between events that are simultaneous to both of them is the same according to them. Then a-synchronicity, time-dilation and length-contraction between \(o\) and \(o'\) are exactly according to the known formula of special relativity, see e.g., [3, p.633].

• The world-view transformations between observers in \(\text{SPECREL}_0\) are exactly the bijections that preserve Minkowski-equidistance; these bijections are the so-called Poincaré-transformations composed with dilations and field-automorphisms.

• The world-view transformations between observers in \(\text{SPECREL}\) are exactly the bijections that preserve Minkowski-distance; these bijections are the so-called Poincaré-transformations.

In \(\text{SPECREL}\), a reference frame is a basic (or primitive) notion, just an “out-of-the-blue” assigning space-and-time coordinates to events, which all together have to satisfy some regularities (our axioms). The theory does not address the question of how an observer sets up his reference frame. As already outlined in the introduction, according to some authors, a physical theory (a theory about our physical reality), should say something about the meaning (in the “real” physical world) of the basic concepts, if not otherwise, then in natural language one could amend the theory with a set of so-called operational rules about how the basic concepts (the reference frames in our case) are set up (experimentally). Here usually meter-rods and wrist-watches, or standard clocks, are used, see e.g., Taylor and Wheeler [44, Fig.s 9,135], L. E. Szabo [34]. In section 4 we give a more ambitious algorithm for setting up coordinate systems.

3 James Ax’s Signalling theory

The intention of Ax’s theory is to give an axiom system for special relativity so that its basic symbols and axioms are designed to be observational. The players of this theory are experimenters that can “communicate with each other” by sending signals to each other. Together, as a team the experimenters can “map” (or explore) space-time, without having rigid meter rods or clocks. A definition of an introduced (or defined) term in this first-order logic theory can be viewed as an experiment designed to establish whether the defined term holds or not. The basic terms of space and time are defined this way. (Indeed, in this theory one can define “rigid meter rods” and “clocks” from signalling experiments.) The results of the experiments we make can be built into axioms then (which are designed to be observational-oriented), and they can tell us in what kind of space-time we live in. Euclidean? Special relativistic? Hyperbolic space with relativistic time? Newtonian? General relativistic? Etc. All this amounts to an implementation of Leibnizian relational notion of space and time. We return to this subject in more detail in the next section.

We begin to describe Ax’s theory which we call Signalling theory \(\text{SIGTH}\). In the vocabulary of \(\text{SIGTH}\) we have two sorts, \(\text{Par}\) for “particles” (or experimenters, or agents) and \(\text{Sig}\) for “signals” (or light-signals); and we have two binary relations \(T, R\) between particles and signals. The intended meanings of \(aT\sigma\) and \(aR\sigma\) are “\(a\) transmits (or emits, or sends out) \(\sigma\)”, and “\(a\) receives (or absorbs) \(\sigma\)”, respectively. Ax [6] uses an impersonal terminology of particle physics, particles emit and absorb signals. We are more attracted to a terminology of communication between active experimenters. These experimenters (players) of \(\text{SIGTH}\) are somewhat analogous to the observers of \(\text{SPECREL}\). In this paper when talking about Ax’s Signalling theory, we will use the terms experimenter and particle interchangeably.
The “standard” (or intended) model we have in mind is the following: Let us fix a Euclidean field $F$. Then $\text{Par}$ is the set of all straight lines in $F^4$ with slope less than 1, and $\text{Sig}$ is the set of all directed finite segments (including the segments of length zero) of straight lines with slope 1. A particle $a$ transmits a signal $\sigma$ iff the beginning point of $\sigma$ lies on $a$, and $a$ receiving the signal $\sigma$ means that the endpoint of $\sigma$ lies on $a$. Let us denote this structure by $\mathcal{M}(F)$.

The main result of [6] is a finite set $\Sigma$ of axioms, our $\text{SigTh}$, which characterizes the class of standard models, i.e., the models of $\Sigma$ are exactly the standard models $\mathcal{M}(F)$ over some Euclidean field $F$ (Thm.1 in [6]). $\text{SigTh}$ consists of three groups of axioms, altogether it has 23 elements. Instead of listing these 23 axioms, in this paper we will use Ax’s completeness theorem, since that implies that a formula $\psi$ is provable from $\text{SigTh}$ iff $\psi$ is true in all the standard models.

The question immediately arises: what does this theory $\text{SigTh}$ have to do with special relativity? Do not we lose much expressive power by using such meager resources? Where do a-variety? Do not we lose much expressive power by using such meager resources? Where do a-

To give a feeling for $\text{SigTh}$ and the expressive power of its language, we briefly outline the proof of Ax’s completeness theorem. Let’s begin by making a little elbow-room for working. We will need to express things such as “two signals are received by an experimenter at the same time”, and “signal $\sigma$ was received by an experimenter just when he transmitted signal $\gamma$”. Since we have no notion of time in our language, we have to express these notions just by using the basic concepts of transmitting and receiving signals. Here comes how we can do this. The (open) formula $\phi := \forall a \ aT\sigma \rightarrow aT\gamma$ is true in a standard model just when the beginning points of the segments $\sigma, \gamma$ coincide, we say that $\phi$ expresses this fact. Similarly, $\forall a \ aR\sigma \rightarrow aT\gamma$ expresses that the endpoint of $\sigma$ coincides with the beginning point of $\gamma$, etc. Now can we express that two particles/experimenters meet? Well, they meet if there is a signal that both of them transmit. From now on we will use similar statements without translating them to the language of $\text{SigTh}$.

To begin outlining the idea of Ax’s completeness proof, let $\mathcal{M}$ be any model of $\text{SigTh}$, and let $e$ be any experimenter in this model. We will construct an isomorphism between $\mathcal{M}$ and a standard model $\mathcal{M}(F)$ which takes $e$ to the time-axis in $\mathcal{M}(F)$. From now on, in this section $e$ denotes this fixed experimenter.

We define the set $\text{Space}$ of “places” or “locations” for experimenter $e$ to consist of those particles which are motionless w.r.t. $e$. For expressing that two particles are motionless w.r.t. each other, any formula expressing this in the standard models will do. Ax uses the following formula: $e'$ is motionless w.r.t. $e$ exactly when $e$ and $e'$ do not meet (if they are not equal) and there are two other particles $d, c$ which meet them and each other in 5 distinct events. Ax then expresses the betweenness relation $Bw$ for such places as well as the equidistance relation $Ed$ with suitable formulas. Having all this, the first group of axioms in $\text{SigTh}$ states Tarski’s axioms for axiomatizing Euclidean geometry over the Euclidean fields (see [41]). From Tarski’s theorem then Ax gets a Euclidean field $F$ and an isomorphism between $(F^3, Bw, Ed)$ and $(\text{Space}, Bw, Ed)$.

Having $\text{Space}$ for our experimenter $e$, what is “time” for it? What are the things that we mark

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2In the formulas, the scope of a quantifier is till the end of the formula if not indicated otherwise. Lower case Roman and Greek letters denote variables of sorts $\text{Par}$ and $\text{Sig}$, respectively. In place of conjunction $\land$ we will simply write a comma.
with time? The events. And what are the events? In the present vocabulary we take them to be “particle b emits/receives a signal σ”, more precisely we take the equivalence classes of them described when we made the elbow-room for this proof (e.g., particle b may send out signal σ in the same event when it sends out another signal γ or when it receives γ). Then our experimenter e’s time will be the events that happened to e. For simplicity, we will represent events with special signals, as explained below.

In the standard models, there are special signals that are received by everyone who transmitted them, we call these signals events, we will denote them by variants of ε:

$$\text{Ev}(\varepsilon) :\iff \forall a \ a T \varepsilon \rightarrow a R \varepsilon.$$ 

In the standard models, events are the light-like segments of zero length, so they correspond to elements of $F^4$. (These zero-length signals may look counter-intuitive to some readers. It is just handy and not important that we use or have these at hand, everything works with a slight modification if we omit these short signals from the standard models.) We say that event $\varepsilon$ happened to experimenter e, or in other words, experimenter $e$ participated in event $\varepsilon$, if $e$ transmitted (and then also received) $\varepsilon$. The events that happened to $e$ will constitute $e$’s world-line.

We can then express simultaneity of events by using that the speeds of light-signals are the same (see Figure 1). Ax then states an axiom to the effect that signals make a one-to-one correspondence between $e$’s world-line and the simultaneous events on any given line in Space. This makes $e$’s world-line isomorphic to $F$, we take this to be the time-axis. From now on it is more or less straightforward what we have to include to SigTh in order to make $M$ isomorphic to $M(F)$. E.g., we can state that for any event $\varepsilon$ there is a simultaneous event $\varepsilon'$ on the world-line of $e$, and there is a particle $e'$ that participates in $\varepsilon'$ and is motionless w.r.t. $e$. This concludes the proof-idea.

4 An algorithm for setting up coordinate systems

The purpose of this section is twofold. Firstly, in section 6 we want to give an interpretation of SpecRel into SigTh, and for this we need concrete formulas representing the proof-idea given in the previous section. For example, Ax used Tarski’s theorem for getting the Euclidean field $F$, but we will need to exhibit concrete formulas defining this $F$. Secondly, we want to make the previous proof-idea into an algorithm for setting up coordinate systems (i.e., reference frames) with the use of just light-signals and freely moving particles. This could also be viewed as providing operational semantics to the basic notion of a coordinate system of SpecRel. What we give in this section will not be an algorithm in the strict sense, it will be more like a recipe for how to design experiments/measurements for assigning coordinates to events. These experiments will also be suitable for finding out/confirming that we live in a special relativistic space-time (if we do). For this reason, we will try to make the formulas “executable” when possible. There will be plenty of room for improving on this aspect, the reader is invited to design more practical experiments.

Assume that we are given a model $M$ of SigTh, and $e$ is an experimenter in this model. Just as in the previous section, this experimenter $e$ is fixed throughout this section. We are going to give $e$ a recipe for defining a field $F$ of quantities and for assigning four quantities to each event. Such an assignment is called a coordinate system (or reference frame). These coordinate systems will satisfy the axioms of SpecRel0.

A location for $e$ was defined as a particle that is motionless w.r.t. $e$. In the previous section we recalled a formula, from [6], expressing whether $e'$ is motionless w.r.t. $e$ (in symbols, $e'||e$).
However, the algorithm suggested by that formula is not very convenient since it involves deciding whether \( e \) meets \( e' \) or not, and for this \( e \) has to know all the events that happened and will happen to him. This is not very practical as an experiment, since \( e \) may need to “wait” for an infinity of time before he could know the result. Using the Affine Desargues Property (ADP for short, see, e.g., [17, p.20]) one can design a more realistic experiment which decides \( e'|e \) “in a finite time”, we are going to describe it now. We note that in the standard models \( \mathfrak{M}(F) \) the ADP is true, because it is true in the affine space \( F^4 \), for any field \( F \).

For a while, it will be easier to think in 4-dimensional space-time than tracing motion in 3-dimensional space. Geometrically, \( e' \) is motionless w.r.t. \( e \) iff the world-line of \( e' \) is parallel to that of \( e \). The conclusion of the ADP is that two lines are parallel, but in the hypothesis part parallelism of two other sets of lines are used. We are lucky: we have light-signals and their speeds are the same in both directions, thus we can use parallelism of world-lines of two sets of light-signals in the hypothesis part of the ADP. The experiment is depicted in geometrical form in the left-hand part of Figure 1. Here is the “non-geometrical” description of the experiment: Assume \( e \) wants to decide whether \( e' \) is motionless w.r.t. him or not. He asks a brother (another experimenter) to throw towards him three “test” particles (“balls”) \( b_1, b_2, b_3 \) at once (in one event \( \varepsilon \)), \( b_1 \) faster than \( b_2 \) and \( b_3 \) faster than \( b_2 \) in such a way that when \( b_1 \) meets \( e \), the latter sends out a signal towards \( b_2 \) that \( b_2 \) reflects back and the reflected signal reaches \( e \) just when \( b_3 \) reaches \( e \). (The brother and \( e \) have to experiment a little while till finding the right velocities for such three particles.) After checking that \( b_1, b_2, b_3 \) have the desired property, \( e \) asks \( e' \) to do the same: when \( b_1 \) reaches \( a' \), he should send a signal towards \( b_2 \) that reflects this signal back towards \( e' \). If the reflected signal reaches \( e' \) just when \( b_3 \) reaches \( e' \), then \( e' \) is motionless w.r.t. \( e \); otherwise \( e' \) is not motionless w.r.t. \( e \). It is best to imagine this experiment to take place in outer space, far from heavy heavenly objects so that gravity and friction do not bend the world-lines of the “balls”\(^3\). From now on, we will use “locations” and “places” as being particles/experimenters motionless w.r.t. our fixed experimenter \( e \).

Two events \( \varepsilon, \varepsilon' \) are defined to be simultaneous w.r.t. \( e \) iff there is a place \( \varepsilon' \) such that from \( \varepsilon' \) two signals can be sent at the same event towards the locations of \( \varepsilon \) and \( \varepsilon' \) respectively such that if these signals are sent back from \( \varepsilon \) and \( \varepsilon' \) right away, they will arrive back to \( \varepsilon' \) at the same event, see middle of Figure 1. Formally: \( \varepsilon \equiv e \varepsilon' : \Leftrightarrow \exists e'[e, \sigma_1, \ldots, \sigma_4, e_1, e_2 EV(\varepsilon_1), EV(\varepsilon_2), (e_1, \sigma_1, \varepsilon), (e_1, \sigma_2, \varepsilon'), (\varepsilon, \sigma_3, \varepsilon_2), (\varepsilon', \sigma_4, \varepsilon_2), T(\varepsilon_1), T(\varepsilon_2), \varepsilon_1, \varepsilon_2) \). where \((e, \sigma, \gamma)\) means that \( e, \gamma \) are the events of sending and receiving \( \sigma \), respectively, formally: \((e, \sigma, \gamma) : \Leftrightarrow (Beg(\sigma, \varepsilon), End(\sigma, \gamma))\) where \( Beg(\sigma, \varepsilon) \) expresses that \( \sigma, \varepsilon \) are sent out at the same event, formally: \( Beg(\sigma, \varepsilon) : \Leftrightarrow \forall v b T(\sigma) \rightarrow b T(\varepsilon) \) and a similar definition for \( End \). (Note that if we want a more experiment-friendly formula for \( Beg \), then we can use the following: \( Beg(\sigma, \varepsilon) \Leftrightarrow (\exists b, c b \neq c, b T(\sigma), b T(\varepsilon), c T(\varepsilon), e T(\varepsilon)) \). We even can provide instructions for where to look for such a place \( \varepsilon' \): it can be chosen to be the midpoint of the line-segment connecting the locations of \( \varepsilon \) and \( \varepsilon' \). (We can use this experiment for setting two clocks at the places of \( \varepsilon, \varepsilon' \) which “tick simultaneously”.)

We get an ordering on all the events from the fact that we send a signal earlier than receiving it, namely \( \varepsilon \) is earlier than \( \varepsilon' \) iff we can send a signal at \( \varepsilon \) to an event from where it bounces back to \( \varepsilon' \) (\( \varepsilon \prec \varepsilon' \) : \( \Leftrightarrow \exists e''(\varepsilon, \sigma_1, \sigma_2 (e, \sigma_1, e''), (\varepsilon'', \sigma_2, e'')) \)). For example, \( \varepsilon_1 \) is earlier than \( \varepsilon_2 \) in the middle part of Figure 1. We note that, while two events being simultaneous or not depends on which experimenter makes the experiment deciding simultaneity, one event being earlier than another

\(^3\) Or, if we are content with more approximate measurements, we can imagine all this happening on a big lake covered with smooth ice (but then we have to take space to be 2-dimensional).
find out whether that we have addition, we do not stop before having a space-trajectories of signals are (3-dimensional) straight lines in the standard models: happens earlier than ε and arrive back to ε, and let ε₁, ..., ε₄ exist as parameter. For defining \( \text{Ev}(e), \text{Ev}(e') \), (ε₁, σ₁, ε), (ε₂, σ₂, ε₂), (ε', σ₄, ε₄), ε'Tₑ, ε'Tₑ', and \( \text{Edt}_e(ε₁, ε₂, ε₃, ε₄), ε₃ < ε₁ \implies \text{Edt}_e(ε₃, ε₄, ε₁, ε₂), ε₃ < ε₁ \). Note that \( \text{Edt}_e(ε₁, ..., ε₄) \) implies that ε₁ happens earlier than ε₂ iff ε₃ happens earlier than ε₄. By using time-equidistance, we can define \text{addition} by selecting a “zero” time o ∈ Timeₑ as parameter, namely \( \tau = τ₁ + τ₂ \implies + (τ, τ₁, τ₂, a, o) \implies \text{Edt}_e(o, τ₁, τ₂, τ) \). Now that we have addition, we do not stop before having \text{multiplication}. For this we have to choose a unit time \( \iota \in \text{Time}_ₑ \), distinct from o and happening later than o, as another parameter. For defining multiplication, we will need the \text{collinearity} relation on locations, we will get this by noticing that the space-trajectories of signals are (3-dimensional) straight lines in the standard models: Col(a₁, a₂, a₃) iff exist signals σ₁, σ₂, σ₃ and events ε₁, ε₂, ε₃ such that \( (ε₁, σ₁, ε₁), (ε₂, σ₂, ε₂), (ε₃, σ₃, ε₃) \) and \( a₁ Tₑ₁, a₂ Tₑ₂, a₃ Tₑ₃ \), for some permutation \( i, j, k \) of 1, 2, 3.

We define \( τ₁ \star τ₂ \) for the case when \( τ₁ \) happened later than \( \iota \), and \( τ₂ \) happened later than \( o \). See the left-hand part of Figure 2. (The other cases are similar, we leave them out.) Here is how we find out whether \( τ \) is \( τ₁ \star τ₂ \): we find two places \( b₁ \) and \( b₂ \) collinear with \( e \) and we find a particle \( p \) such that if \( b₁ \) and \( b₂ \) send towards \( e \), simultaneously, at time zero, a light-signal and \( p \), and another light-signal and another particle \( q \) “with the same speed” as \( p \), then these four arrive (to \( e \)) at times \( \iota, τ₁, τ₂, τ \), respectively. Formally: \( \tau = τ₁ \star τ₂ \iff * (τ, τ₁, τ₂, e, o, i) \iff \exists b₁ \| e, b₂ \| e', τ', σ₁, σ₂, p, q \{ \text{Ev}(e'), \text{Ev}(τ₂'), \text{Col}(e, b₁, b₂), o \equiv e, o' \equiv e', b₁ Tₑ', b₂ Tₑ₂', (τ', σ₁, e), (τ₂, σ₂, τ₂), p Tₑ', p T₁, q Tₑ₂'.

Figure 1: On the left: Experiment for checking whether \( e' \) is motionless w.r.t. \( e \). In the middle: Experiment to make sure that \( ε, e' \) are simultaneous w.r.t. \( e \). On the right: Time-equidistance of events \( ε₁, ..., ε₄ \).
are a special case of Hilbert’s coordinatization procedure, see e.g., \[17, pp.23-28\] or \[23, pp.296-308\].

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Figure 2: On the left: Experiment for computing \( \tau_1 \ast \tau_2 \). In the middle: distance between locations \( e, b \). On the right: Spatial coordinates of location \( b \). In the picture, \( \gamma_x = \delta(e, p_x) \). In this part points represent (3-dimensional) locations, while in the previous pictures points represent (4-dimensional) events.

The reader will have noticed that the above definitions of addition and multiplication on \( \text{Time}_e \) are a special case of Hilbert’s coordinatization procedure, see e.g., \[17, pp.23-28\] or \[23, pp.296-308\]. By the above, we have a structure \( F(e, o, i) = \langle \text{Time}_e, +, \ast \rangle \) which is isomorphic to our field \( F \) in the intended models \( \mathfrak{M}(F) \). We define the above structure to be the field of quantities of our fixed experimenter \( e \). We define the time-coordinate of an arbitrary event \( \varepsilon \) as an element of this field, namely the unique event on \( e \)’s world-line which is simultaneous with it (simultaneous according to \( e \)). Next, we define three coordinates, three elements of this field, for each location \( b \). From now on, let \( \text{Space}_e \) denote the set of locations for \( e \).

We begin by defining a geometric structure on \( \text{Space}_e \), namely we will define distance of locations, parallelism and orthogonality of (3-dimensional) spatial lines.

We define the distance of any two locations. Let \( b \in \text{Space}_e \) be arbitrary. We define the distance of \( b \) from our fixed \( e \) as the event when a signal sent from \( b \) at time zero arrives to \( e \), see the middle part of Figure 2. This definition corresponds to a convention that we measure spatial distances in light-years (if we measure time in years). Having this, we get the distance between any two locations \( b_1, b_2 \) by measuring the distance between their parallel translated versions so that \( b_1 \) gets to \( e \), i.e., \( \delta(e, b) = \varepsilon \iff \exists \varepsilon', \sigma[\text{Ev}(\varepsilon'), \varepsilon' \equiv e, b \in \text{Time}_e], \) and \( \delta(b_1, b_2) = \varepsilon : \iff \varepsilon = \delta(e, b), pa(b_1, b_2, e, b), pa(b_1, e, b_2, b) \) means that the spatial lines defined by \( b_1, b_2 \) and \( b_3, b_4 \) are parallel, we easily can express this by using the collinearity relation \( \text{Col} \) between locations as defined earlier in this section.

We also need the orthogonality relation which is definable from the equidistance of pairs of locations. We define orthogonality of two intersecting lines only. We call the lines going through \( a, b \) and \( a, c \) orthogonal, if \( a \neq b, a \neq c \) and there is a \( b' \neq b \) on the spatial line going through \( a, b \) such that the distances between \( a, b' \) and \( a, b \) equal, and also those between \( c, b' \) and \( c, b \) equal (\( \text{Orth}(a, b, a, c) \iff \exists b'[\text{Col}(b', a, b), \delta(a, b') = \delta(a, b), \delta(c, b') = \delta(c, b)] \)). By now we defined a structure \( \langle \text{Space}_a, \text{Col}, \text{pa}, \text{Orth} \rangle \) and we defined distance \( \delta : \text{Space}_a \longrightarrow \text{Time}_e \).

Setting up a coordinate system needs three more parameters, the three space-axes. Let \( a_x, a_y, a_z \in \text{Space}_e \) be such that \( e, a_x, e, a_y \) and \( e, a_z \) are pairwise orthogonal. We have everything for defining the usual spatial coordinates of the place \( b \). See the right-hand part of Figure 2. The spatial coordinates of a location \( b \) are defined the usual way by “projecting” \( b \) to the three coordinate axes,
along lines parallel with some of the axes, and measuring the distance of the projected points from the origin (our experimenter $e$ in our case). See the formula $\text{cor}$ below.

We can now round up the definition of the coordinate system our experimenter $e$ is setting up. We already defined the time-coordinate of an event $\varepsilon$, and we define the space-coordinates of $\varepsilon$ to be the spatial coordinates just defined for the “location of $\varepsilon$”, the latter being the unique particle participating in $\varepsilon$ and motionless w.r.t. our experimenter $e$. The formula $\text{cor}(\varepsilon, \tau, \gamma_x, \gamma_y, \gamma_z, e, o, \iota, o, a_x, a_y, a_z)$ defined below expresses that the coordinates of the event $\varepsilon$ are $\tau, \gamma_x, \gamma_y, \gamma_z$ in the coordinate system specified by $e, o, \iota, a_x, a_y, a_z$.

$$\text{cor}(\varepsilon, \tau, \gamma_x, \gamma_y, \gamma_z, e, o, \iota, a_x, a_y, a_z) \iff \varepsilon \equiv e, \exists b, p_x, p_y, p_z \in \text{Space}_e[b, \Gamma, e, \tau, x, y, z].$$

$$\text{cor}(\varepsilon, e, o, \iota, a_x, a_y, a_z) = (\tau, \gamma_x, \gamma_y, \gamma_z) \iff \text{cor}(\varepsilon, \tau, \gamma_x, \gamma_y, \gamma_z, e, o, \iota, a_x, a_y, a_z).$$

By the above, we have defined coordinate systems to each particle $e \in \text{Par}$. Such a coordinate system is defined by six parameters: $e, o, \iota, a_x, a_y, a_z$. Before going on, we show that the relativistic (or, in other words, Minkowski-) distance between events can be defined in these coordinate systems. We call two events $\varepsilon, \varepsilon'$ time-like separated iff there is a particle participating in both. For simplicity, we will define relativistic distance between time-like separated events only. See Figure 3.

![Figure 3: Relativistic distance $\xi = \mu_{\varepsilon, o}(\varepsilon) = \mu_{\varepsilon, o}(\varepsilon_1, \varepsilon_2)$ between events $\varepsilon_1, \varepsilon_2$.](image)

The relativistic distance we are going to define will depend on experimenter $e$ and on the chosen zero $o$ of its coordinate system. Let first $\varepsilon \succ o$ be any event time-like separated from $o$. Then $\mu_{\varepsilon, o}(o, \varepsilon) = \xi$ iff there is an event $\varepsilon'$ which is simultaneous with $o$ both according to $e$ and according to the unique observer participating in $o, \varepsilon$, and there are signals from $\varepsilon'$ to $\varepsilon$ and from $\varepsilon'$ to $\xi$, respectively. It can be checked that in any standard model $\mathcal{M}(F)$, if $o, \varepsilon, \xi$ are in the above described configuration, then the “standard” Minkowski-distances between $o, \varepsilon$ and $o, \xi$ are the same. Conversely, if these two distances agree then there exists an event $\varepsilon'$ as in Figure 3. Let now $\varepsilon_1, \varepsilon_2$ be any two time-like separated events, $\varepsilon_1 \prec \varepsilon_2$. Then the relativistic distance between $\varepsilon_1, \varepsilon_2$
is the same as that between the “parallel translations” $o, \varepsilon$ of these, where the “parallel translation” happens according to Figure 3 (where for $\varepsilon''$ it is important only that it is connected to both $o$ and to $\varepsilon_1$ with a light-signal, e.g., it is not important that $o < \varepsilon''$). If $\varepsilon_1 > \varepsilon_2$ then we define $\mu_{e,o}(\varepsilon_1, \varepsilon_2) = -\mu_{e,o}(\varepsilon_2, \varepsilon_1)$. We note that, while this relativistic distance strongly depends on the parameters $e, o$, the relativistic equidistance relation we get from this does not depend on $e, o$ any more. So, let us define relativistic equidistance, or 4-equidistance, as

$$Edr(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \iff \mu_{e,o}(\varepsilon_1, \varepsilon_2) = \mu_{e,o}(\varepsilon_3, \varepsilon_4), \quad \text{for any } e \in \text{Par} \text{ and event } o \text{ on } e's \text{ world-line.}$$

Having defined the desired coordinate systems in $\text{SigTh}$, we conclude this section with some remarks on what this method can give us, what it can be used for.

We asked earlier, in section 3, where the paradigmatic effects — a-synchronicity, time-dilation, length-contraction — of special relativity theory came into the picture in Signalling theory. One answer is the following. We defined natural coordinate systems to the particles. (These coordinate systems correspond to the observers in $\text{SpecRel}$, this correspondence will be made explicit in section 6.) Now, the coordinate-transformations between these are so that the three paradigmatic effects of special relativity (mentioned in section 2) hold in a version where we can recalibrate the units of measurement.

This section contains definitions only, definitions (with some parameters) in the language of $\text{SigTh}$ that in the standard models define coordinate systems for the particles/experimenters. We can get an axiom system characterizing the standard models (thus doing the job of $\text{SigTh}$) via using these definitions. Namely, we can state as axioms that the coordinate systems defined for the experimenters have all the good properties we want (e.g., the beginning and end-points of light-signals are exactly those of the ordered segments of slope 1). This alternative axiom system would be more complicated and less natural than $\text{SigTh}$ of [6], however, it would be the result of a clear-cut method that can be used in many other situations, as indicated below.

We can use the method of this section for exploring space-times other than the special relativistic one, and for using signals of various different nature, too. We mention some examples briefly, we think that elaborating these examples would be worthwhile.

We can use the method of setting up a coordinate system as described in this section, for example, for a particle moving faster-than-light (FTL) in a special relativistic space-time. So, let us take as standard models the standard models $\mathfrak{M}(F)$ modified so that the particles are the lines with slope more than 1 (and not the ones with slope less than 1). If we apply our method to these modified models, then the FTL experimenter $e$ will find that its space $\text{Space}_e$ is a 3-dimensional Minkowski-space $MS := \langle F^3, Bw, Edr \rangle$, and not a Euclidean space $\langle F^3, Bw, Ed \rangle$. He can reach by signals directly, and check whether they are motionless w.r.t. him, only those places/brothers that are time-like separated from him in terms of $MS$, but he can get indirect information about the rest of places by communicating with these primarily reachable brothers. By working through the details, we can get an axiom system $\text{SigTh}^{\text{ftl}}$ axiomatizing the signalling models of FTL experimenters that would be quite analogous to Ax's $\text{SigTh}$. The main difference would be that the first group of axioms for 3-dimensional Euclidean space would be replaced by an analogous axiom system for 3-dimensional Minkowski space. For this we can use the one devised by Goldblatt in [17, Appendix A]. For a slightly different approach for including FTL observers in this setting see [20].

However, communicating with directed signals (as in $\text{SigTh}$) between FTL experimenters is rather restricted if we want to take the experiments to be executable (FTL experimenters can get information this way only about the part of their space $MS$ which is in their “past” in terms of $MS$.
as a Minkowski-space). We can change the nature of signals to be undirected (but otherwise letting their speed to be 1), imagining that if two events are connected with a signal, then the information this signal carries appears at both events “at once”. This is connected somehow to time-travel, a subject strongly connected to FTL motion. The method given in the present section is suitable for exploring space-time with undirected signals, too.

The method given in this section can also be used for giving meaning to two-dimensional time. Time being 2-dimensional could simply mean that the events happening with the experimenters can be best described by, say, the structure \((\mathbb{F}^2, \prec)\). For example, one could assume that our experimenter lives in a world characterized by the 2+2-dimensional Minkowski-metric \(\sqrt{t_1^2 + t_2^2 - x^2 - y^2}\) and then apply our method to see what kind of coordinate system he would set up for himself, and in general, what kind of responses he would get to his experiments.

Finally, we can imagine using signals of infinite velocity, this way we can explore the Newtonian space-time characterized by absolute time. Or, we can use bent signals of general relativity. For example, we can explore the outer part of the Schwarzshild black hole (the space-time outside the event horizon) with the same method. We would take as experimenters a team of densely placed suspended observers (spaceships in outer space using their drives to maintain their desired positions), constantly checking positions by communicating with photons (as light-signals), and using freely-falling spaceships (or astronauts) as messengers.

5 Defining new entities, interpretations

Our aim is to clarify the connections between \textsc{SpecRel} and \textsc{SigTh}. Not only the vocabularies of these two theories are disjoint, even on the intuitive level they speak of different kinds of things. We can see that somehow photons and observers of \textsc{SpecRel} correspond to signals and particles of \textsc{SigTh}, but what correspond to quantities in \textsc{SigTh}? Quantities of \textsc{SpecRel} do not seem to enter the picture in \textsc{SigTh}. Yet, in section 4 we defined something that intuitively could correspond to quantities in \textsc{SigTh}. In this section we recall some tools from mathematical definability theory by which we can make explicit the way quantities arise in \textsc{SigTh}.

We briefly recall the tools that we will use in the next section for making connections between theories for special relativity in a very precise sense. We elaborated these tools in [1, 23] for the specific purpose of establishing a strong connection between two versions of special relativity theory, the so-called observer-independent geometrical and the reference-frame oriented ones. We only recall the syntactic form to be used in specifying a concrete interpretation together with some background intuition. We elaborated a more extensive definability theory for this kind of connecting theories that we do not recall here. We will say some words about it at the end of this section. For simplicity, we will treat function symbols as special relation symbols.

In “traditional”, one-sorted definability theory, an interpretation of a theory \(\textsc{Th}'\) in language \(\mathcal{L}'\) into a theory \(\textsc{Th}\) in another language \(\mathcal{L}\) is the following. For each \(n\)-place relation symbol \(R\) of \(\mathcal{L}'\) we assign a formula \(\varphi_R\) of \(\mathcal{L}\) with at most \(n\) free variables. (We think of \(\varphi_R\) as the “definition of \(R\)” in \(\mathcal{L}\).) This then defines a natural translation function \(\text{tr} : \mathcal{L}' \rightarrow \mathcal{L}\) by replacing each atomic formula \(R(v_1, \ldots, v_n)\) with \(\varphi_R(v_1, \ldots, v_n)\). This is an interpretation of \(\mathcal{L}'\) into \(\mathcal{L}\). This interpretation is an interpretation of \(\textsc{Th}'\) into \(\textsc{Th}\) iff \(\textsc{Th}\) proves the translated theory \(\textsc{Th}'\), i.e.,

\[
\text{Th} \models \text{tr}(\psi) \text{ whenever } \textsc{Th}' \models \psi, \text{ for all } \psi \in \mathcal{L}'.
\]

On the semantic side, an interpretation of \(\textsc{Th}'\) into \(\textsc{Th}\) “constructs” a model of \(\textsc{Th}'\) inside each model.
of Th. Namely, it associates a model \( \text{tr}(\mathcal{M}) \) of \( \mathcal{L}' \) to each model \( \mathcal{M} \) of \( \mathcal{L} \) in such a way that the universe of \( \text{tr}(\mathcal{M}) \) is the same as that of \( \mathcal{M} \), and for each assignment \( k \) of the variables into this universe we have

\[
\star \star \text{tr}(\mathcal{M}) \models \psi[k] \text{ if and only if } \mathcal{M} \models \text{tr}(\psi)[k], \text{ for each formula } \psi \text{ in } \mathcal{L}'.
\]

In the new, “non-traditional” or “generalized” definability theory we will use a notion of interpretation that does the same thing, except that the universe of \( \text{tr}(\mathcal{M}) \) will not necessarily be a subset of the universe of \( \mathcal{M} \), therefore its definition and the property analogous to (\( \star \star \)) above will be more involved. We will define new entities as elements of new “sorts”. Using many-sorted FOL is not an essential feature of this generalized definability theory, just it is convenient in many cases, as it is in our present task.

We illustrate the idea of defining new sorts with a simple example. The language of affine planes in, e.g., [17] is two-sorted, we have two sorts Points, Lines and we have a binary relation between them, the relation \( I \) of incidence (or membership) between a point and a line. Another language in use for the same is one-sorted, see, e.g., [42], we have one sort Points and we have a three-place relation \( \text{Col} \) of “collinearity” between three points. Everyone can connect the two ways of thinking about affine planes immediately: a line is the set of all points collinear with given two distinct points. Thus a line \( \ell \) is a subset of the old universe, given two distinct points \( p, q \) the line \( \ell \) going through them is defined by

\[
\ell(p, q) := \{x : \text{Col}(x, p, q)\}.
\]

But the new sort Lines stands for the set of all these subsets! We can specify one line with the open formula \( \text{Col}(x, p, q) \) with one free variable \( x \), but how can we define the set of all lines? Well, we will define the set of the parameters \( p, q \) specifying the individual lines: we identify the set of all lines with the set of pairs of distinct points. Thus the formula defining the new sort Lines will have two free variables \( p, q \) and it will state \( p \neq q \). We are almost there, except that different pairs of distinct points may specify the same line, and we have to take this into account when talking about equality of lines, i.e., when interpreting the equality symbol on the sort Lines. We can do this again with a formula using 4 free variables \( p, q, p', q' \) stating when the lines specified by \( p, q \) and \( p', q' \) coincide. In our case this formula can be taken to be \( \text{Col}(p', p, q) \land \text{Col}(q', p, q) \).

So far we have defined the universe of the new sort Lines and the equality relation of this new sort by two formulas in the “old” language, i.e., in the language talking about Points and Col. Having defined a universe means that we have variables ranging over this universe (and we can quantify over them). In other words, we have to introduce variables \( \text{Var} \langle \text{Lines} \rangle \) of sort Lines. Then, in order to be able to use the definition of the new sort Lines, we need to connect \( \text{Var} \langle \text{Lines} \rangle \) to variables used in the definition for Lines, i.e., to \( \text{Var} \langle \text{Points} \rangle \). We can state this connection by matching a variable \( \ell \) of sort Lines with variables denoting its “defining parameters”, e.g., we can state that \( \ell_p, \ell_q \) denote parameters that define \( \ell \). After this we can define the incidence relation, too: \( I(x, \ell) : \iff \text{Col}(x, \ell_p, \ell_q) \) , where \( x \) is a variable of sort Points and \( \ell \) is a variable of sort Lines.

Summing up: defining the new sort Lines goes by defining the variables \( \text{Var} \langle \text{Lines} \rangle \) of the new sort and matching them to the variables of the old sort \( \text{Var} \langle \text{Points} \rangle \) occurring in the defining formula of the sort Lines, defining the equality on the sort Lines, and defining the non-logical symbol of incidence \( I \) which involves the sort Lines. Thus we can interpret the 2-sorted language of affine planes into the one-sorted one by the following data:

\[
\text{var} : \ell \mapsto \langle \ell_p, \ell_q \rangle \text{ for } \ell \in \text{Var} \langle \text{Lines} \rangle,
\]
The above data then define a translation function $\text{tr}$ from the 2-sorted language of affine planes to their one-sorted language as follows:

\[
\text{tr}(\exists \ell \psi) := \exists \ell_p, \ell_q \quad \ell_p \neq \ell_q, \text{tr}(\psi),
\]
\[
\text{tr}(\ell = h) := \text{Col}(\ell_p, h_p, h_q), \text{Col}(\ell_q, h_p, h_q),
\]
\[
\text{tr}(I(x, \ell)) := \text{Col}(x, \ell_p, \ell_q),
\]

the rest of the definition of $\text{tr}$ is more or less straightforward.

The new feature in this translation function, over the traditional one, is that we translate the quantifiers according to the defining formula and variable-matching of the new sort and we translate equality on the new sort, too. Throughout, we will use the above variable matching $\text{var} : \ell \mapsto \langle \ell_p, \ell_q \rangle$ without recalling it.

This translation is not only recursive and structural, it is also meaning preserving in the sense analogous to $(\star \star)$. In more detail: let $\mathcal{M} = \langle P, \text{Col} \rangle$ be a model of the one-sorted language. We will construct its “translation”, a model $\text{tr}(\mathcal{M})$ of the two-sorted language. Let

\[
U := \{\langle x, y \rangle \in P \times P : x \neq y\}, \text{ and let } E \subseteq U \times U \text{ be defined by}
\]
\[
E := \{\langle u, v \rangle \in U \times U : \text{Col}(u_1, v_1, v_2), \text{Col}(u_2, v_1, v_2)\}.
\]

Assume that $E$ is an equivalence relation on $U$, then define

\[
\text{tr}(\mathcal{M}) := \langle P, L, I \rangle \text{ where}
\]
\[
L := U/E \quad \text{and}
\]
\[
I := \{\langle x, u \rangle \in P \times L : \text{Col}(x, v_1, v_2) \text{ for some } v \in u/E\}.
\]

Let $\text{Var}_P, \text{Var}_L$ denote the sets of variables in the 2-sorted language of the affine planes and let

\[
\text{Var}_P := \text{Var}_P \cup (\text{Var}_L \times \{1\}) \cup (\text{Var}_L \times \{2\})
\]

be the variables of the one-sorted language. Now, let $k : \text{Var}_P \cup \text{Var}_L \rightarrow \text{tr}(\mathcal{M})$ be any evaluation of the variables of the 2-sorted language, and let $\text{tr}(k) : \text{Var}_P \rightarrow \mathcal{M}$ be an evaluation of the variables of the one-sorted language such that $\text{tr}(k)(x) = k(x)$ if $x \in \text{Var}_P$, and if $\ell \in \text{Var}_L$ then $(\text{tr}(k)(\ell, 1), \text{tr}(k)(\ell, 2))$ is an arbitrary element of $k(\ell)$. Then the following is true for each formula $\psi$ of the 2-sorted language:

\[\text{\texttt{(\star \star') } } \text{tr}(\mathcal{M}) \models \psi[k] \text{ if and only if } \mathcal{M} \models \text{tr}(\psi)[\text{tr}(k)].\]

The above $(\star \star')$ expresses that the translation function preserves meaning when we talk about the 2-sorted model constructed inside the one-sorted model.

Now, such a translation $\text{tr}$ is an interpretation from $\text{Th}'$ into $\text{Th}$ iff, just as before,

\[\text{\texttt{(\star') } } \text{Th} \models \text{tr}(\psi) \text{ whenever } \text{Th}' \models \psi, \text{ for all } \psi \in \mathcal{L}'.\]
Definitional equivalence of theories $\mathbf{Th}',\mathbf{Th}$ in different languages $\mathcal{L}',\mathcal{L}$ is a strong connection between them, much stronger than mutual interpretability requiring that the two interpretations be inverses of each other, up to isomorphism. (Cf. Ex.4.3.46, p.266.)

Two theories $\mathbf{Th}'$ and $\mathbf{Th}$ are said to be \textit{definitionally equivalent} if they have a common definitional extension. Here, two theories are said to be the same if they prove the same formulas. But what is a definitional extension? In the one-sorted case, definitional extension of $\mathbf{Th}$ is $\mathbf{Th} \cup \Delta$ where $\Delta$ is a union of definitions of the form $\Delta(R) := \{ R(v_1,\ldots,v_n) \leftrightarrow \varphi_R(v_1,\ldots,v_n) \}$ with $\varphi_R$ as above (see, e.g., [19] pp.60-61). For telling what definitional extension is in the many-sorted case, we return to our previous example of defining the sort Lines. Let us write $\varepsilon(p,q,p',q')$ for $p \neq q$ and $\text{Col}(p',p,q),\text{Col}(q',p,q)$ respectively, for the formulas defining the “domain” and the “equality” on the new sort Lines. The explicit definition of the sort Lines will also involve a new relation $\pi$ fixing the connection of the new sort to the old ones. Now, $\Delta(\text{Lines}, \pi)$ is defined to be the set of the following sentences

\[
\exists p,q (\pi(p,q,\ell),\pi(p,q,\ell')) \leftrightarrow \ell = \ell',
\]

\[
\exists \ell (\pi(p,q,\ell),\pi(p',q',\ell)) \leftrightarrow \varepsilon(p,q,p',q'),
\]

\[
\exists \ell (\pi(p,q,\ell)) \leftrightarrow \delta(p,q).
\]

We note that the intuitive meaning of $\pi(p,q,\ell)$ is that “$p,q$ are distinct points lying on $\ell$”, or, “$p,q$ code, or represent, line $\ell$”. So far it was the variable matching that played this role and, intuitively, $\pi(p,q,\ell)$ is an explicit way of saying $\varepsilon(p',q',\ell_p,\ell_q)$.

After having defined the new sort Lines, the definition $\Delta(I)$ of the incidence relation is the same as in the one-sorted case:

\[
I(p,\ell) \leftrightarrow \exists p',q' (\pi(p',q',\ell),\text{Col}(p',q')).
\]

Now, $\mathbf{Th} \cup \Delta(\text{Lines}, \pi) \cup \Delta(I)$ is a definitional extension of $\mathbf{Th}$, where $\mathbf{Th}$ is the “one-sorted” theory of affine planes. A \textit{definitional extension} of any theory $\mathbf{Th}$ is $\mathbf{Th} \cup \Delta$ where $\Delta$ is a union of definitions of the above form. Instead of describing the above in more detail, we refer to [1]. [23] sec.4.3, [24] sec.6.3] where many examples can also be found.

The notion of definitional equivalence is important for our purposes, and we believe that it is an important one in understanding how we form our concepts. We try to illustrate this with an example. We will see that the theory $\text{EFd}$ of Euclidean fields and the theory $\text{SigTh}$ of special relativity are mutually interpretable into each other. However, they are not definitionally equivalent\footnote{Similar observations apply to a slight variant $\text{SpecRel}_0 + \text{Compl}$ of $\text{SpecRel}$ in place of $\text{SigTh}$ (cf. Thm.7.1 in section 7). This can be extended to the Newtonian theory in [2] sec.4.1, p.423.} Namely, $\text{SigTh}$ and $\text{EFd}$ cannot have a common definitional extension because of the following two reasons. (i) $\text{SigTh}$ has to be an “information-losing” reduct of any definitional extension of $\text{EFd}$, and (ii) any theory is an “information-preserving” reduct of any of its own definitional extensions. We note that (ii) holds because the very idea of “definitional extension” is an extension based on “information” contained in the unextended theory; thus by forgetting this extra structure we lose nothing, we can recover it from the unextended theory. We explain (i): In a definitional extension of $\text{EFd}$ of which $\text{SigTh}$ is a reduct, we will define the new sort $\text{Par}$ of experimenters together with a projection function $\pi_P$ which ties the behavior of $\text{Par}$ to $\text{EFd}$. Such a projection function will single out the experimenter whose world-line is the time-axis, in other words, we can single out
“the” motionless experimenter. Absolute motion! However, the essence of relativity theory is that motion is relative. This is formalized in the so-called Special Principle of Relativity, which states that all the experimenters are equivalent, we cannot tell which one is motionless and which one moves. Indeed, any experimenter can be taken to any other experimenter by an automorphism, in any model of SigTh. Thus, when making the reduct of a definitional extension of EFd in order to obtain SigTh, we have to forget \( \pi_P \), otherwise we do not get the right concept of experimenter. This is “information-loss” since we cannot recover \( \pi_P \) from SigTh. This shows that “forgetting” is an important part in forming the concept of experimenter in this case. “Less is more” in this case. Definitional equivalence keeps track of these kinds of “forgetting”, while mutual interpretability may not do this.

We conclude this section with a few words about interpretations. We already wrote about the philosophical importance of interpretations between theories in the introduction. Here we write about more technical aspects. An interpretation \( \text{tr} \) from theory \( \text{Th}' \) to \( \text{Th} \) is a connection between them, and this connection imports some properties of one theory to the other. For example, if \( \text{Th} \) is consistent, then \( \text{Th}' \) is also consistent. If \( \text{tr} \) is faithful and \( \text{Th}' \) is undecidable, then \( \text{Th} \) is also undecidable, and if \( \text{Th}' \) and \( \text{Th} \) are mutually interpretable in each other, then an axiom system for \( \text{Th} \) can be imported to \( \text{Th}' \) via any two mutual interpretations. Definitional equivalence induces a strong duality between \( \text{Th}' \) and \( \text{Th} \). For these kinds of application of interpretations see, e.g., \[14, 13, 28, 23\]. The present paper intends to show the usefulness of interpretations in physical theories, e.g., defining operational semantics for a physical theory. We note that definability theory is quite extensively used in geometry, see, e.g., \[17, \text{App.B}, 28, 29, 30, 42\].

Versions of the general interpretability we use in this paper appeared in various different forms as early as in 1969, see \[31, 35, 27, 10, 25, 26, 19\]. Almost all of these works use a syntactic device similar to ours, let’s call it explicit definitions, but they all elaborate on different semantical aspects of this general definability. For example, \[10\] characterizes when a functor of a given form is the semantical part of an interpretation. \[27, 25\] recast model theory in a categorical form, where both the syntactical and semantical parts of an interpretation are functors between pretoposes, and it is proved that both functors are equivalences when one is. This theorem is called a conceptual completeness theorem. For the model theoretical forms, meaning and impacts of this completeness theorem we refer to \[18\]. (We refer specifically to \[18\] section 6, item (3) for connections with the notion of general interpretability.) In \[11, 23\], it is shown that our form of explicit definitions outlined in this section is not ad hoc in the sense that any sensible definition can be brought to this form. Namely, a notion of implicit definability suggests itself as a necessary condition for these new entities to be called “defined”, see, e.g., \[23, \text{sec.4.3.1}\] and \[19\]. An analogue of the Beth definability theorem (\[23, 4.3.48\]) states that if a sort of new elements is implicitly definable, then it is explicitly definable, too. We note that the powerset of the universe of an infinite model is not implicitly definable in the sense of \[23, \text{sec.4.3.1}\], while, say, the set of two-element subsets of it is implicitly (and thus also explicitly) definable.

We hope that the content of this section is enough to give us a guiding intuition for what comes in the rest of this paper.

---

5The easiest way of making this precise is that there are fields with no automorphisms at all, e.g., the field of real numbers, and this means that the structure \((\text{Par}, \pi_P, \text{EFd})\) will have no automorphism, either.
6 Reducing SpecRel to Signalling theory: an interpretation

In this section we define in detail an interpretation of SPECREL in SigTh. We have to define (over SigTh) the new sorts $Q$ and $B$, and the new operations and relations $+,*$, Obs, Ph, W that involve these new sorts.

We begin with defining the new sort $Q$. In section 4 we already defined a field $\mathbf{F}(e,o,\iota)$, that will provide the definition to our new sort $Q$ and to $+,*$. However, that definition had three parameters $e,o,\iota$ (the particle who was setting up his coordinate system, the “beginning of the era”, and duration of one year). Up to isomorphism, we get the same field no matter how we choose these 3 parameters, but their universes strongly depend on the parameters (namely, the universe of $\mathbf{F}(e,o,\iota)$ is the set of events on $e$’s world-line). Which one should we take as the set of elements of sort $Q$? The answer is: take neither one, take all of them! Intuitively, this means that we take the disjoint union of all the fields belonging to the different parameters, and then we define an equivalence relation on this set that relates the isomorphic images of the same element.

For this, in our explicit definition of the quantity sort we need a uniform formula that defines the isomorphisms between the fields $\mathbf{F}(e,o,\iota)$. One such formula is given in [23, p.305]. Here we give a simpler formula defining the isomorphisms between the various incarnations of our field. We can give this simpler formula because relativistic equidistance is available for us, while [23] used only the betweenness relation.

We are going to define the isomorphisms sought for between the fields $\mathbf{F}(e,o,\iota)$. See Figure 4. Let $e,o,\iota,e',o',\iota'$ be suitable parameters for defining the fields (as in section 4). The isomorphism between them will take $o$ to $o'$, $\iota$ to $\iota'$ and it will take an arbitrary $\xi$ on the world-line of $e$ to $\xi' := \xi''/\iota''$ where $\xi''$, $\iota''$ are events on $e'$’s world-line such that $\text{Edr}(\xi,o,\xi'',o')$ and $\text{Edr}(\iota,o,\iota'',o')$, further $/$ denotes the division operation of the field belonging to $e'$, $o'$, $\iota'$. Let $\varphi_{\text{iso}}(\xi,\xi',e,o,\iota,e',o',\iota')$ denote the formula expressing the above. We denote the isomorphism as $\varphi_{\text{iso}}(e,o,\iota,e',o',\iota')$, and we denote the unique $\xi$ with the property $\varphi_{\text{iso}}(\xi,\xi',e,o,\iota,e',o',\iota')$ as $\xi = \varphi_{\text{iso}}(e,o,\iota,e',o',\iota')$.

![Figure 4: The isomorphism $\varphi_{\text{iso}}(e,o,\iota,e',o',\iota')$ between $\mathbf{F}(e,o,\iota)$ and $\mathbf{F}(e',o',\iota')$.](image)

Let $\mathbf{Fp}(e,o,\iota)$ express that $e,o,\iota$ are appropriate parameters for a field $\mathbf{F}(e,o,\iota)$, let $U$ be the disjoint union of the universes of all the fields $\mathbf{F}(e,o,\iota)$, and let $E$ denote the binary relation relating isomorphic elements, i.e.,

$$\mathbf{Fp}(e,o,\iota) \iff \text{Ev}(o), \text{Ev}(\iota), o \neq \iota, o < \iota, e \mathbin{E} o, e \mathbin{E} \iota,$$
\[ U := \{ \langle \xi, e, o, i \rangle : \mathcal{F}_p(e, o, i), \xi \in \mathcal{F}(e, o, i) \} \]

\[ E := \{ \langle \langle \xi, e, o, i \rangle, (\xi', e', o', i') \rangle : \varphi_{\text{iso}}(\xi, e, o, i, e', o', i') \} \].

It can be shown that \(E\) is an equivalence relation on \(U\), in each standard model of \(\text{SigTh}\). Our quantity sort will be \(U/E\).

Recall that we are in the process of defining \(\text{SpecRel}_0\) over \(\text{SigTh}\).

We are ready to define the quantity sort \(Q\) explicitly, by using the tools we introduced in the previous section. If \(q\) is a variable of the (new) sort \(Q\), then \(q_\xi, q_e, q_o, q_i\) denote the corresponding variables of the (old) sorts \(\text{Sig}\) and \(\text{Par}\). We can think of this variable matching as \(q\) denotes an equivalence block of \(E\) (i.e., an element of \(U/E\)), and \(\langle q_\xi, q_e, q_o, q_i \rangle\) denotes an arbitrary (unknown) element in the equivalence block \(q\). Intuitively, \(q\) denotes an “abstract” quantity, and \(\varphi_{\text{iso}}(q_\xi, q_e, q_o, q_i)\) is the corresponding “concrete” quantity in the field \(\mathcal{F}(e, o, i)\). Let us denote this last thing as

\[ \text{rep}(q, e, o, i) := \varphi_{\text{iso}}(q_\xi, e, o, i, q_e, q_o, q_i) \].

This situation is somewhat analogous to the concept of a manifold in general relativity theory, the elements of the manifold are the “observer-independent” entities, and the charts/observers associate concrete values to these. Below comes the definition of the sort \(Q\):

\[ \text{var} : q \mapsto \langle q_\xi, q_e, q_o, q_i \rangle \quad \text{for } q \in \text{Var}_Q \].

\[ \text{Q}(q) :\Leftrightarrow q_e T q_\xi, Ev(q_\xi), F_p(q_e, q_o, q_i), q = q' :\Leftrightarrow \varphi_{\text{iso}}(q_\xi, q'_\xi, q_e, q_o, q_i, q'_e, q'_o, q'_i) \].

Note that this definition of the sort \(Q\) is analogous to the one given for the new sort \(\text{Lines}\) in the example of affine planes in the previous section.

We get the definitions for \(+, \star\) from writing up the definitions given in section 4 as follows. Recall the formula \(+ (\tau, \tau_1, \tau_2, e, o)\) from section 4.

Now, here is the definition of addition of sort \(Q\):

\[ + (q, q_1, q_2) := + (q_\xi, \varphi_{\text{iso}}(q_1 \xi, q_e, q_o, q_1 e, q_1 o, q_1 i), \varphi_{\text{iso}}(q_2 \xi, q_e, q_o, q_2 e, q_2 o, q_2 i), q_e, q_o) \].

The formula defining multiplication of sort \(Q\) is obtained analogously.

The rest of this section (interpreting \(\text{SpecRel}_0\) in \(\text{SigTh}\)) will be relatively straightforward.

We turn to defining the sort \(B\). We will define the sort \(B\) of bodies as the union of observers and photons. So first we define the entities that we will call photons. A photon will be defined just as a signal \(\sigma\) that is not an event. The world-line of this photon will be defined as the set of all events that lie on the 4-dimensional line defined by the beginning and end points of \(\sigma\). This way, many photons will share the same world-line, just as in the case of affine planes many pairs of distinct points define the same line, and we will define two photons to be equal if they share the same world-line. An observer will be defined to be a coordinate system. We recall from section 4 that six parameters are required for defining a coordinate system, namely the experimenter \(e\), a “zero” \(o\) and a time-unit \(i\), and three locations \(a_x, a_y, a_z\) specifying the space coordinate axes. These parameters have to satisfy the conditions below, which we will denote by \(\text{Op}\) (\(\text{Op}\) refers to “observer parameters”):

19
Two observers will be defined equal if they assign the same coordinates to all events.

We are ready to formalize these definitions by using the tools we introduced in section \[\text{Var}_B\] Let \(\text{Var}_B\) denote the set of variables of sort \(B\). If \(b\) is a variable of sort \(B\), then \(b_\sigma, b_e, b_0, b_t, b_x, b_y, b_z\) will denote the corresponding variables of “old” sorts. Intuitively, this body will be \(b_\sigma\) if this is a “real”, non-degenerate signal (i.e., if \(b_\sigma\) is not an event), and if \(b_\sigma\) is “degenerate” (i.e., if it is an event), then the body \(b\) will be the observer \((b_e, b_0, b_t, b_x, b_y, b_z)\). We are ready to define the new sort \(B\) together with the unary formulas \(\text{Ph}(b)\) and \(\text{Obs}(b)\):

\[
\text{var} : b \mapsto (b_\sigma, b_e, b_0, b_t, b_x, b_y, b_z) \quad \text{for } b \in \text{Var}_B.
\]

\[
\text{Ph}(b) : \iff \neg \text{Ev}(b_\sigma),
\]

\[
\text{Obs}(b) : \iff \text{Ev}(b_\sigma), \text{Op}(b_e, b_0, b_t, b_x, b_y, b_z),
\]

\[
B(b) : \iff \text{Ph}(b) \lor \text{Obs}(b),
\]

We are going now to define the equality relation on this new sort \(B\). For stating equality of photons, first we express that three events are on one light-like line \((\lambda(\varepsilon_1, \varepsilon_2, \varepsilon_3))\), then we express that an event is on the world-line of a signal \((wl(\varepsilon, \sigma))\).

\[
\lambda(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \iff \wedge \{ \exists \sigma[(\varepsilon_i, \sigma, \varepsilon_j) \lor (\varepsilon_j, \sigma, \varepsilon_i)] : i, j \in \{1, 2, 3\}\},
\]

\[
wl(\varepsilon, \sigma) : \iff \exists \varepsilon_1, \varepsilon_2 \lambda(\varepsilon_1, \varepsilon_2), \text{Beg}(\sigma, \varepsilon_1), \text{End}(\sigma, \varepsilon_2).
\]

Recall from section \[\text{Var}_B\] that the formula \(\text{cor}(\varepsilon, e, o, t, a_x, a_y, a_z) = (\tau, \gamma_x, \gamma_y, \gamma_z)\) expresses that the coordinates of the event \(\varepsilon\) are \(\tau, \gamma_x, \gamma_y, \gamma_z\), in the coordinate system specified by \(e, o, t, a_x, a_y, a_z\).

\[
b = b' : \iff
\]

\[
(\neg \text{Ev}(b_\sigma), \neg \text{Ev}(b'_\sigma), \forall \varepsilon \text{wl}(\varepsilon, b_\sigma) \leftrightarrow \text{wl}(\varepsilon, b'_\sigma)) \lor
\]

\[
(\text{Ev}(b_\sigma), \text{Ev}(b'_\sigma), \forall \varepsilon \text{cor}(\varepsilon, b_e, b_0, b_t, b_x, b_y, b_z) = \text{cor}(\varepsilon, b'_e, b'_0, b'_t, b'_x, b'_y, b'_z)).
\]

It remains to define the world-view relation \(W\). The intuitive meaning of the formula \(W(m, b, t, x, y, z)\) will be that \(m\) is an observer, and the event at place \(t, x, y, z\) in \(m\)'s coordinate system is on the world-line of \(b\). Let \(m, b\) be variables of sort \(B\) and let \(t, x, y, z\) be variables of sort \(Q\). Assume that \(m\) is an observer, i.e., \(\text{Ev}(m_\sigma)\). Let us denote the concrete value of an abstract quantity \(q\) in \(m\)'s coordinate system by

\[
m(q) := \text{rep}(q, m_e, m_o, m_t).
\]

We can now define \(W\) as follows:

\[
W(m, b, t, x, y, z) : \iff \exists \varepsilon \text{cor}(\varepsilon, m(t), m(x), m(y), m(z), m_e, m_o, m_t, m_x, m_y, m_z),
\]

\[
(\neg \text{Ev}(b_\sigma) \rightarrow \text{wl}(\varepsilon, b_\sigma)), (\text{Ev}(b_\sigma) \rightarrow b_e \varepsilon), \text{Ev}(m_\sigma).
\]

By the above, we gave definitions for all the sort and relation symbols of the language of \(\text{SPECREL}\) in the language of \(\text{SIGTH}\). This defines a translation function \(\text{tr}\) between the two languages. Let \(\equiv_Q\) and \(\equiv_B\) stand for the equality relations between terms of sort \(Q\) and \(B\), respectively. In the next theorem we state, without proof, that we indeed obtained an interpretation.
Theorem 6.1  \( \text{tr} \) as given in this section is an interpretation of \( \text{SpecRel}_0 \) into \( \text{SigTh} \), that is, the following are true:

\[
\begin{align*}
\text{SigTh} \models & \quad \text{“} =_Q \text{ and } =_B \text{ are equivalence relations”,} \\
\text{SigTh} \models & \quad \text{“the formulas defining } +, \ast, \text{Ph, Obs, W are invariant under } =_Q, =_B \text{”,} \\
\text{SigTh} \models & \quad \text{tr}(\psi) \quad \text{for all } \psi \in \text{SpecRel}_0.
\end{align*}
\]

Having defined the desired interpretation of \( \text{SpecRel}_0 \) into \( \text{SigTh} \), in the next section we extend this interpretation to a definitional equivalence between a slightly stronger version of \( \text{SpecRel}_0 \) and \( \text{SigTh} \).

7  Definitional equivalence between SpecRel and Signalling theory

In this section we investigate interpretability and definitional equivalence between some of the FOL theories formalizing special relativity. We show that a slightly reinforced version of \( \text{SpecRel}_0 \) is definitionally equivalent to \( \text{SigTh} \). We mean interpretability and definitional equivalence in the sense of the generalized definability theory of [1, 2, 23] outlined in section 5.

The interpreted theory \( \text{tr}(\text{SpecRel}_0) \) is stronger than the original one in the sense that there are sentences \( \psi \) in the language of \( \text{SpecRel}_0 \) such that \( \text{SigTh} \models \text{tr}(\psi) \) while \( \text{SpecRel}_0 \not\models \psi \). Such a sentence is, e.g., “all lines of slope less than 1 are world-lines of observers”. We can express exactly how much more is true in the translated models by amending \( \text{SpecRel}_0 \) with some existence, extensionality, and time-orientation axioms (see below) and showing that the so obtained theory is definitionally equivalent with \( \text{SigTh} \). This is what we are going to do now.

The formulas describing the “difference” between \( \text{SpecRel}_0 \) and \( \text{SigTh} \) are as follows. Formulas expressing that we have all kinds of possible observers (from each point, in each direction, for each velocity less than the speed of light there is an observer moving in that direction with that speed, each observer can re-coordinatize its coordinate-system with any space-isometry, each observer can set the unit of its clock arbitrarily), and otherwise we are as economic as possible (at most one photon through any two distinct events, only one observer with the same coordinate-system, only photons and observers as bodies, only one time-orientation for each observer).

These additional axioms, except the one about setting the clocks, are denoted as \( \text{AxThEx} \), \( \text{AxCoord} \), \( \text{AxExtOb} \), \( \text{AxExtPh} \), \( \text{AxNobody} \), \( \text{Ax}^\uparrow \) in [3 sec. 2.5]. Let \( \text{AxClock} \) formulate that each observer can set the unit of its clock arbitrarily (in the spirit of the above axioms). Let \( \text{COMPL} \) denote the set of these axioms and let \( \text{SpecRel}_0^+ \) denote the theory \( \text{SpecRel}_0 \) amended with these formulas:

\[
\begin{align*}
\text{COMPL} & \ := \ \{ \text{AxThEx, AxCoord, AxClock, AxExtOb, AxExtPh, AxNobody, Ax}^\uparrow \}, \\
\text{SpecRel}_0^+ & \ := \ \text{SpecRel}_0 + \text{COMPL}.
\end{align*}
\]

To state definitional equivalence between \( \text{SpecRel}_0^+ \) and \( \text{SigTh} \), we now define an interpretation \( \text{Tr} \) of \( \text{SigTh} \) into \( \text{SpecRel}_0^+ \). We have to define the universes \( \text{Par}, \text{Sig} \) of particles and signals and the relations \( T, R \) of transmitting and receiving, inside \( \text{SpecRel}_0^+ \). Intuitively, particles are defined to be observers, with two particles being equal if their world-lines coincide:

\[ \text{var} : a \mapsto a_b \quad \text{for } a \in \text{VarPar}, \text{where } a_b \in \text{VarB}. \]
Par(a) :⇔ \( \text{Obs}(a_b) \),
\[
a = a' :⇔ \forall t, x, y, z \ W(a, a', t, x, y, z) \leftrightarrow x = y = z = 0.
\]

Signals are defined to be photons with two events on their world-lines representing the beginning and end-points of the signal. We represent the two events with observers meeting the photon. The following formulae express in \( \text{SpecRel}_0 \) that “in b’s world-view, p meets a at time \( t' \)”, and “a, p, e meet in one event”, respectively:

\[
\text{Meet}(b, p, a, t) :⇔ \exists x, y, z \ W(b, p, t, x, y, z), W(b, a, t, x, y, z),
\]
\[
\text{meet}(a, p, e) :⇔ \exists b, t \ \text{Meet}(b, a, p, t), \text{Meet}(b, a, e, t).
\]

Now we are ready to interpret signals in \( \text{SpecRel}_0 \):

\[
\text{var} : \sigma \mapsto \langle \sigma_b, \sigma_p, \sigma_e \rangle \quad \text{for } \sigma \in \text{Var}_{\text{Sig}}, \text{ where } \sigma_b, \sigma_p, \sigma_e \in \text{Var}_B.
\]

\[
\text{Sig}(\sigma) :⇔ \text{Ph}(\sigma_p), \text{Obs}(\sigma_b), \text{Obs}(\sigma_e), \exists t \leq t' \ \text{Meet}(\sigma_b, \sigma_p, \sigma_b, t), \text{Meet}(\sigma_b, \sigma_p, \sigma_e, t').
\]

\[
\sigma = \sigma' :⇔ \text{meet}(\sigma_b, \sigma'_b, \sigma_p), \text{meet}(\sigma_c, \sigma'_c, \sigma_p), \neg \text{meet}(\sigma_b, \sigma_p, \sigma_e) \rightarrow \sigma_p = \sigma'_p.
\]

Finally,

\[
a \text{Tr} \sigma :⇔ \text{meet}(a_b, \sigma_b, \sigma_p),
\]
\[
a \text{R} \sigma :⇔ \text{meet}(a_b, \sigma_e, \sigma_p).
\]

The above define a translation function \( \text{Tr} \) as indicated in section 5. We state in the next theorem, without proof, that this \( \text{Tr} \) interprets \( \text{SigTh} \) in \( \text{SpecRel}_{0+} \), and moreover, together with the interpretation \( \text{tr} \) defined in the previous section it forms a definitional equivalence between \( \text{SpecRel}_{0+} \) and \( \text{SigTh} \). This is the main theorem of this paper.

**Theorem 7.1** \( \text{SigTh} \) is definitionally equivalent to \( \text{SpecRel}_0 + \text{Compl} \), the pair \( \text{tr}, \text{Tr} \) of interpretations forms a definitional equivalence between them.

We can read the above theorem as saying that what the theory \( \text{SigTh} \) tells about special relativity is exactly what the theory \( \text{SpecRel}_0 + \text{Compl} \) says. Since no axiom in \( \text{Compl} \) follows from \( \text{SpecRel}_0 \), we can conclude that \( \text{SigTh} \) tells more than \( \text{SpecRel}_0 \), the amount of “more” is exactly \( \text{Compl} \). However, we did not include the axioms of \( \text{Compl} \) into \( \text{SpecRel} \), because we do not need them in proving the main predictions of relativity theory; we feel that they do not belong to the core of the physical theory. Moreover, of the axioms of \( \text{Compl} \), we consider only \( \text{Ax}\uparrow \) as having a physically (or even philosophically) relevant content, namely it says that “time is oriented”.

On the other hand, we will see that \( \text{SpecRel} \) has a content that \( \text{SigTh} \) does not say about special relativity theory. This is the axiom \( \text{Ax}\uparrow \) of \( \text{SpecRel} \). So, what is the connection between \( \text{Ax}\uparrow \) and \( \text{SigTh} \)? Below we answer this question.

The interpretation \( \text{tr} \) we defined in the previous section does not interpret \( \text{SpecRel} \) in \( \text{SigTh} \), because \( \text{tr}(\text{Ax}\uparrow) \) does not follow from \( \text{SigTh} \) (i.e., it is not true in the standard models \( \mathcal{M}(F) \))
The reason for this is the following. \(AxSym\) states that any two observers use the same units of measurement. We can express in the language of \(SigTh\) that “two observers use the same units of measurement”, and this defines an equivalence relation on the set of all observers. For \(AxSym\) to be true, we should select any one of the blocks of this equivalence relation (since \(AxSym\) states that any two observers use the same units of measurements). But which one should we select? The question might sound familiar. In the previous analogous case (that concerned the various incarnations \(F(e, o, i)\) of the field \(F\)) we took all the classes “up to isomorphism”. However, in the present case there are no definable bijections between the blocks of this equivalence relation.

We can get around this problem by adding to the models \(M(F)\) of \(SigTh\) a “unit of measurement”. We can do this, e.g., the following way. We add a new basic two-place relation symbol \(Tu\) (short for “Time unit”) of sort \(Sig\) to the language of Signalling theory. In each standard model \(M(F)\) we interpret \(Tu\) as the set of pairs of events with Minkowski-distance 1. (We note that these relations are not definable in \(M(F)\) in the language of Signalling theory.) Let us denote the so expanded standard models by \(M(F)\). This technique is elaborated in, e.g., [23, 19].

We included \(AxSym\) into \(SpecRel\) as a tool for convenience, it seems to carry no philosophical or physical importance. \(AxSym\) is only a simplifying assumption.

Concerning some of the other theories for special relativity, we mention that \(SigTh\) is definitionally equivalent to Goldblatt’s theory for special relativity in [17] Appendix A] amended with time-orientation. I.e., the two theories are almost the same, the only difference is that \(SigTh\) assumes time-orientation while Goldblatt’s theory does not. The proof of this last statement can be put together from the definitions and ideas in sections 4, 6. Also, (a slight variant of) our \(SpecRel\) is definitionally equivalent to (a slight variant of) Suppes’s axiomatization of special relativity in [38, 39].

---

\(\text{COMPL}^- := \{AxThEx, AxCoord, AxExtOb, AxExtPh, AxNobody, Ax\uparrow\}\),

To our minds, the following theorem clarifies the connection between \(AxSym\) and \(SigTh\). It says that the content of \(AxSym\) is to set the time-unit: the difference between \(SigTh^+\) and \(SigTh\) is that in \(SigTh^+\) we can express Minkowski-distance, while in \(SigTh\) we have only Minkowski equidistance.

**Theorem 7.2** The following (i),(ii) hold:

(i) \(SigTh^+\) is definitionally equivalent to \(SpecRel + \text{COMPL}^-\).

(ii) \(SigTh^+\) is not definitionally equivalent to \(SigTh\).

The proof of part (i) of the above theorem goes by extending the interpretation \(Tr\) to \(SigTh^+\), this amounts to defining the new relation \(Tu\) in \(SpecRel\); and also making some (minor) changes in the definition of \(tr\). The proof of part (ii) of the above theorem goes by showing that the automorphism groups of members of \(SigTh\) and \(SigTh^+\) differ from each other, this technique is elaborated in, e.g., [23, 19].

We included \(AxSym\) into \(SpecRel\) as a tool for convenience, it seems to carry no philosophical or physical importance. \(AxSym\) is only a simplifying assumption.

Concerning some of the other theories for special relativity, we mention that \(SigTh\) is definitionally equivalent to Goldblatt’s theory for special relativity in [17] Appendix A] amended with time-orientation. I.e., the two theories are almost the same, the only difference is that \(SigTh\) assumes time-orientation while Goldblatt’s theory does not. The proof of this last statement can be put together from the definitions and ideas in sections 4, 6. Also, (a slight variant of) our \(SpecRel\) is definitionally equivalent to (a slight variant of) Suppes’s axiomatization of special relativity in [38, 39].

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\(^6\) We note that \(SpecRel\) can be interpreted in \(SigTh\) in the way that we interpret \(SpecRel\) in the field \(Q, +, \times\).
8 Conclusion

We intended to show in this paper some results the methods of mathematical logic can provide for other branches of science, in particular, for physics and the methodology of science. Using the tools of definability theory of first-order logic, we compared in detail two rather different axiom systems for special relativity theory. One of these, SpecRel of [5], is coordinate-system-, or reference frame-oriented, while the other, SigTh of [6], uses meager resources and talks about particles emitting and absorbing signals. The two theories use disjoint languages and talk about different kinds of entities. Yet, a precise comparison was made possible by using mathematical logic, and we obtained the following: SigTh can express and states everything that SpecRel does, except for the relativistic (Minkowski) distance between events (implied by AxSym in SpecRel), while in addition it states time-orientation for space-time together with some auxiliary simplifying axioms (Compl). Informally,

\[
\text{SigTh} = \text{SpecRel} - \text{relativistic distance} + \text{time-orientation} + \text{auxiliaries},
\]

and a little more formally

\[
\text{SigTh} + \text{AxSym} = \text{SpecRel} + \text{Compl}^{-}.
\]

A byproduct of these investigations is a concrete operational semantics for special relativity theory. We believe that interpreting one theory in another is a flexible methodology for connecting physical theories with each other as well as with the “physical reality”.

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