AXIAL ANOMALY LOW ENERGY TESTS
AND INSTANTON VACUUM MODELS

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ABSTRACT

A low energy theorem concerning a matrix element of the QCD axial anomaly is tested against different instanton models. In the chiral limit the theorem is fulfilled by the Diakonov&Petrov model, whereas it is violated by single-instanton approximations. Beyond the chiral limit the theorem, and two relations established for other matrix elements of the QCD axial anomaly, result to be violated also by the Diakonov&Petrov model.

1. Introduction

In quantum field theory the symmetry of the classical lagrangian may be destroyed by quantum correction. In gauge theories the most important examples of this kind are the axial anomalies in ElectroWeak theory and in QCD. The axial anomaly arises from noninvariance of the fermionic measure against axial transformations in the path integrals of the theory (see also higher-loop corrections). In Euclidean QCD+QED the axial anomaly reads

\[ \partial_\mu j_5^\mu = i N_f g^2 \frac{G_\tilde{G}}{16\pi^2} + i N_c e^2 \frac{Q_f^2 \tilde{F} \tilde{F} + 2 \sum m_f \psi_f^\dagger \gamma_5 \psi_f}{8\pi^2}, \]  

where \( j_5^\mu = \sum_f \psi_f^\dagger \gamma_\mu \gamma_5 \psi_f \) (\( f = u, d, s \)) is the quark singlet axial current, \( \psi_f \) the quark field, \( N_f \) the number of the flavors, \( g \) the QCD coupling constant, \( 2G_\tilde{G} = \epsilon^{\mu\nu\lambda\sigma} G_{\mu\nu}^a G_{\lambda\sigma}^a \) the gluon field strength operator, \( N_c \) the number of colours, \( e \) the QED coupling constant, \( Q_f \) the electric fractional charges of the quarks, \( \tilde{F}_{\mu\nu} \) the photon field strength operator. We have explicitly included the contributions of the current masses of light quarks \( m_f \). It was pointed out that this equation leads to a nontrivial low-energy theorem (LET) and to useful relations.

In the present paper we consider the matrix element of eq. (1) \((i)\) between vacuum and two-photons states and \((ii)\) between vacuum and meson states.

Analysis of case \((i)\), and nonvanishing of the \( \eta' \) meson mass even in chiral limit due to axial anomaly, leads to the conclusion that in the kinematical configuration of zero virtualities of each line in the form factor the matrix element of the divergence of the singlet axial current vanishes, implying the following LET:

\[ \langle 0 | N_f g^2 \frac{G_\tilde{G}}{16\pi^2} | 2\gamma \rangle = N_c \frac{e^2}{4\pi^2} \sum Q_f^2 \tilde{F}^{(2)} \tilde{F}^{(3)} + 2 \sum m_f \langle 0 | \psi_f^\dagger \gamma_5 \psi_f | 2\gamma \rangle, \]  

\[ \text{(2)} \]

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where \( F^{(i)}_{\mu\nu} = \epsilon_{i,\mu} q_{i,\nu} - \epsilon_{i,\nu} q_{i,\mu} \) and \( q_i, \epsilon_i (i = 2, 3) \) are respectively the momenta and polarization vectors of photons.

Eq. (2) gives the exact answer for the matrix element of the gluonic operator. Of course gluons can couple to photons only by virtual quarks. In perturbation theory the left side of the eq. (2) is at least \( O(g^4 e^2) \), whereas the right side of this equation does not contain any strong coupling at all. It is evident that the \( LET \) can be fulfilled only out of the framework of perturbation theory. The factor \( g^2 \) at the left side must be completely cancelled by some nonperturbative contribution, such as the one, of order \( g^{-2} \), provided by instantons. The computation of the left side of eq. (2) amounts to calculating the instanton contribution in the Euclidean space to the three-point function

\[
\tau_{\mu\nu}(x_1, x_2, x_3) = \langle 0| T[g^2 \tilde{G}G(x_1) j_{\mu}^{em}(x_2) j_{\nu}^{em}(x_3)]|0\rangle ,
\]

\((j_{\mu}^{em} \text{ being the electromagnetic current})\) which contains needed information on the matrix element \( \langle 0| \tilde{G}G|2\gamma \rangle \) and also useful physical information concerning \( \eta \to 2\gamma \) decay, \( 2\gamma \)-transitions in heavy quarkonia etc.

The matrix elements of eq. (3) between vacuum and meson states lead to the relations

\[
\langle 0|N_f \frac{g^2}{16\pi^2} \tilde{G}G|\eta \rangle = 2im_s \langle 0|\psi_s^3 \gamma_5 \psi_s|\eta \rangle ,
\]

\[
\langle 0|N_f \frac{g^2}{16\pi^2} \tilde{G}G|\pi^0 \rangle = i(m_u - m_d) \langle 0|\psi^3 \gamma_5 \psi|\pi_0 \rangle ,
\]

which provide, together with \( LET \), more stringent tests for instanton models.

Instantons - whose presence in QCD is a well established fact, at least at a phenomenological level and in numerical simulations of QCD vacuum - constitute the main input to our calculations, especially in connection with quark propagation. To our present knowledge the instanton structure of QCD vacuum is concentrated in an average size \( \rho \) and in an average interinstanton distance \( R \), such that

\[
\rho = 1/3 \text{fm}, \quad R = 1 \text{fm}.
\]

Therefore the packing parameter \((\rho/R)^4 = 0.012\) is small, legitimating independent averaging over positions and orientations of the instantons; moreover in some cases it is even possible to use the quark propagator in the single-instanton field.

In a previous work \( LET \) was used as a test for the Diakonov-Petrov (DP) ansatz of the low-energy QCD effective action in the chiral limit. The ansatz, based on an interpolation formula for the quark propagator in the field of a single instanton, does satisfy \( LET \) to \( \sim 17\% \) accuracy.

One of the aims of our present work is to formulate more stringent tests of the DP model, by taking explicitly into account the contribution of the current quark masses \( m_f \). We also test the validity of different instanton vacuum models by calculating \( \tau_{\mu\nu} \) in the range of small virtualities and testing it against \( LET \).

In section 2 we calculate \( \tau_{\mu\nu} \) in single-instanton approximation (SIA). In section 3 we rederive the DP effective action starting from results of Lee & Bardeen (LB) and accounting for current quark masses \( m_f \). In section 4 we calculate \( \tau_{\mu\nu} \) by this
effective action and check it by LET. In section 5 we calculate in the same framework both sides of relations (8) and (9) and compare each left side with its respective right side.

2. Single-instanton approximation

A lot of papers have been devoted to instanton calculus to single-instanton approximation (see for example the calculations of the correlator of two vector currents in [16,17,18]). In this approximation the calculation of $\tau_{\mu\nu}$ amounts to computing integrals like

$$\int d^4z dU \langle (g^2G\tilde{G})(x_1)j_{\mu}^em(x_2)j_{\nu}^em(x_3) \rangle = \sum_f Q_f^2 \int d^4z (G\tilde{G})(x_1)Tr(\gamma_{\mu}S_{\pm}(x_2,x_3)\gamma_{\nu}S_{\pm}(x_3,x_2)),$$

where $z_{\pm}$ and $U_{\pm}$ are the position and orientation of the (anti-)instanton, assuming instanton integration sizes to be peaked at $\rho$, eq. (6). Here

$$(g^2G\tilde{G})(x)_{\pm} = \pm f(x - z) = \pm \frac{192 \rho^4}{(\rho^2 + (x - z)^2)^4}$$

and $S_{\pm} = (i\hat{D}_{\pm} + im)^{-1}$ is the full quark propagator in presence of a single (anti)instanton.

Starting from the expression of $S_{\pm}$ given in ref.10, we obtain

$$\tau_{\mu\nu}(x_1,x_2,x_3) = N \frac{192 N_c \rho^6}{V 3 \pi^4} \sum_f Q_f^2 \int d^4z h^4(x_1 - z) \frac{h(x_2 - z)h(x_3 - z)}{(x_2 - x_3)^2}$$

$$\times \left[ \frac{h(x_2 - z) + h(x_3 - z)}{(x_2 - x_3)^2} + h(x_2 - z)h(x_3 - z) \right] \epsilon_{\mu\nu\alpha\beta} (x_2 - z)_{\alpha}(x_3 - z)_{\beta},$$

where $h(x) = (\rho^2 + x^2)^{-1}$. Then the correlator, that is the Fourier transform $\hat{\tau}_{\mu\nu}$ of the three-point function $\tau_{\mu\nu}$, to be tested against LET, results to be

$$\hat{\tau}_{\mu\nu}(q_1,q_2,q_3) = N \frac{N_c \rho^2}{V 3 \pi^4} \sum_f Q_f^2 (2\pi)^4 \delta(\sum_i q_i) \hat{f}(q_1^2)$$

$$\times 4! \int_0^1 \int_0^1 \int_0^1 db da (a(1 - a) + ab(1 - b))^{-3/2} \epsilon_{\mu\nu\alpha\beta}(\frac{\partial^2}{\partial p_{2,\alpha}\partial p_{3,\beta}}) J(P),$$

as can be shown by applying the Feynman integration technique. We have set

$$\hat{f}(q_1^2) = \int d^4x \frac{192 \rho^4}{(\rho^2 + x_1^2)^4}, \exp(iq_1x_1),$$

$$J(P) = \int d^8Y [Y^2 + r^2]^{-5} \exp iP Y,$$
having introduced the 8-dim vectors \( P = (p_2, p_3), Y = (y_2, y_3) \), with
\[
p_2 = \frac{q_2 + bq_3}{(1 - a + ab(1 - b))^{1/2}}, \quad p_3 = \frac{q_3}{a^{1/2}}, \quad \rho^2 = \rho^2(1 - ab).
\]
The latter integral (11) can be easily calculated and is reduced to the MacDonald function, i.e.,
\[
J(P) = \frac{\pi^4}{4!} P_{rK_1}(Pr), \quad P = (P^2)^{1/2}.
\]
(12)

In the limit of small momenta
\[
K_1(Pr) = \frac{1}{Pr} + \frac{Pr}{2}[\ln \frac{Pr}{2} + C - 1/2] + ..., \tag{13}
\]
where \( C = 0.577215... \) is the Euler constant.

Eqs. (10), (12), (13) imply that \( \hat{\tau}_{\mu\nu}(q_1, q_2, q_3) \) is divergent for \( q_i^2 \to 0 \), that is, the model badly violates \( LET \). Nor does agree with \( LET \) the improved single-instanton approximation, where the zero-mode contribution to the propagator has been modified by replacing \( m \to m^* \), where \( m^* \) is the effective mass of the quark accounting for the influence of the surrounding instantons. We conclude that single-instanton approximation badly violates \( LET \), therefore it is not suitable for calculating three-point functions at small momenta. By the way we note that two-point functions of vector currents are not so sensitive to large distance effects. The violation of \( LET \) - and in particular the divergence in massless quark approximation - is related to the slow decrease of \( \tau_{\mu\nu} \) at large distances. Indeed we have neglected rescattering effects of quarks by other instantons, which, during quark propagation, lead to the formation of the constituent quark, producing a suitable effective mass and providing needed exponential decrease with distance. Such effects can be described by an effective action.

3. A low energy QCD effective action

It is natural to choose the singular gauge for the instantons in describing many instanton effects in the propagation of the quarks. In the case of a small packing parameter it is possible to do the following ansatz for the background instanton field:
\[
A_{\mu}(x) = \sum_{+}^{N_+} A_{+,\mu}(x; \xi_+) + \sum_{-}^{N_-} A_{-,\mu}(x; \xi_-), \quad (\xi_\pm = (z_\pm, U_\pm, \rho_\pm)), \tag{14}
\]
where, \( z_i, U_i \) and \( \rho_i \) the position, orientation and size of the \( i \)-th instanton. The canonical partition function of the \( N_+ \)instantons and \( N_- \) antiinstantons can be schematically written as
\[
Z_{N_+,N_-} = \int det_N \exp(-V_g) \prod_{i}^{N_+N_-} d^4z_idU_idn(\rho_i), \tag{15}
\]
where \( V_g \) is the instanton-(anti)instanton interaction potential generated by the gluon field action and \( det_N \) is a quark determinant in the instanton field. The main assumption of the instanton model is that \( V_g \) is repulsive at small distances between instanton
and antiinstanton. This should provide the stabilization of the instanton sizes and of the interinstanton distances, as discussed in the introduction. We mainly deal with \( \det_N \), which describes the influence of light quarks.

Lee & Bardeen (LB) calculated the quark propagator in a more sophisticated approximation than SIA. In particular they found that

\[
\det_N = \det B, \quad B_{ij} = i m \delta_{ij} + a_{ij},
\]

(16)

where \( a_{ij} \) is the overlapping matrix element of the quark zero-modes \( \Phi_{\pm,0} \) generated by instantons. This matrix element is nonzero only between instantons and antiinstantons (and vice versa) due to the chiral factor in \( \Phi_{\pm,0} \), i.e.,

\[
a_{-+} = - \langle \Phi_{-,0} | i \hat{\partial} | \Phi_{+,0} \rangle.
\]

(17)

The overlapping of the quark zero-modes causes quark jumping from one instanton to another one during propagation.

It is clear from (16) that for \( N_+ \neq N_- \) \( \det_N \sim m^{|N_+ - N_-|} \), so the fluctuations of \( |N_+ - N_-| \) are strongly suppressed due to presence of light quarks. Therefore we assume \( N_+ = N_- = \bar{N}/2 \).

Following the procedure suggested in ref. [19], we get the fermionization of LB’s result, i.e.,

\[
\det_N = \int D\psi D\psi^\dagger \exp(\int d^4x \sum_f \psi_f^\dagger i \hat{\partial} \psi_f)
\]

\[
\times \prod_{f}^{N_+} (\prod_f (im_f + V_+[\psi_f^\dagger, \psi_f])) \prod_{f}^{N_-} (im_f + V_-[\psi_f^\dagger, \psi_f]),
\]

(18)

where

\[
V_\pm[\psi_f^\dagger, \psi_f] = \int d^4x (\psi_f^\dagger(x) i \hat{\partial} \Phi_{\pm,0}(x; \xi_\pm)) \int d^4y (\Phi_{\pm,0}^\dagger(y; \xi_\pm) i \hat{\partial} \psi_f(y)).
\]

(19)

Eq. (18) coincides with the ansatz for the fixed \( N \) partition function postulated by DP, except for the sign in front of \( V_\pm \). Keeping in mind the low density of the instanton media, which allows independent averaging over positions and orientations of the instantons, eq. (18) leads to the partition function

\[
Z_N = \int D\psi D\psi^\dagger \exp(\int d^4x \psi^\dagger i \hat{\partial} \psi) W_+^{N_+} W_-^{N_-},
\]

(20)

where

\[
W_\pm = \int d^4\xi_\pm \prod_f (V_\pm[\psi_f^\dagger, \psi_f] + im_f) = (-i)^N_f \left( \frac{4\pi^2 \rho^2}{N_c} \right)^N_f \int \frac{d^4z}{V} \det(iJ_\pm(z) - \frac{m}{4\pi^2 \rho^2})
\]

(21)

and

\[
J_\pm(z)_{fg} = \int \frac{d^4k d^4l}{(2\pi)^8} \exp(-i(k - l)z) F(k^2) F(l^2) \psi_f^\dagger(k) \frac{1}{2}(1 \pm \gamma_5) \psi_g(l).
\]

(22)
The form factor $F$ is related to the zero-mode wave function in momentum space $\Phi_\pm(k; \xi_\pm)$ and is equal to

$$F(k^2) = -t \frac{d}{dt} [I_0(t) K_0(t) - I_1(t) K_1(t)], \quad t = \frac{1}{2} \sqrt{k^2 \bar{\rho}}.$$

(23)

4. Calculation of the correlator by the $DP$ effective action

In quasiclassical (saddle point) approximation any gluon operator derives its main contribution from instanton background. In the following the operator $g^2 G \tilde{G}(x)$ will be considered. Owing to the low density of the instanton medium, it is possible to neglect the overlap of the fields of different instantons. In that case, the matrix element of $g^2 G \tilde{G}(x)$ with any other quark operator $Q$ is

$$\langle g^2 G \tilde{G}(x) Q \rangle_N = Z_N^{-1} \int D\psi D\psi^\dagger \exp(\int d^4 x \bar{\psi} i \hat{\partial} \psi)
\times \left( N_+ \left( W_{G \tilde{G}+}(x) Q \right) W_{N+}^{N+} W_{N-}^{N-} + N_- \left( W_{G \tilde{G}-}(x) Q \right) W_{N+}^{N+} W_{N-}^{N-} \right),$$

where

$$W_{G \tilde{G} \pm} = \pm \left( \frac{4\pi^2 \rho^2}{N_c} \right)^{N_f} \int \frac{d^4 z}{V} f(x - z) \det(J_\pm(z) + i \frac{m N_c}{4\pi^2 \rho^2}).$$

(25)

It is useful to introduce the external fields $\kappa(x)$, coupled to $g^2 G \tilde{G}$, and $a$, such that $\hat{D} = \hat{\partial} - ie Q_f \hat{\alpha}$. Starting from (25), we find that the partition function $\hat{Z}[\kappa, a]$ describing mesons in presence of such external fields is

$$\hat{Z}[\kappa, a] = \int D\Phi_+ D\Phi_- \exp(-W[\Phi_+, \Phi_-]),$$

(26)

where

$$W[\Phi_+, \Phi_-] = \int d^4 x (w_a + w_b - w_c),$$

$$w_a = (N_f - 1) \frac{N}{2V} \prod_f M_f^{-1} \det(\Phi_+)^{(N_f - 1)^{-1}} + (+ \rightarrow -),$$

$$w_b = \frac{N_c}{4\pi^2 \rho^2} Tr(m (\Phi_+ + \Phi_-)),\quad \text{(27)}$$

$$w_c = \sum_f Tr \ln \frac{i \hat{\partial} + i F^2 (\Phi_+ A_+ + \Phi_- A_-)}{i \hat{\partial} + i m_f},$$

$$A_\pm = \left[ (1 \pm (\kappa f))^N \right] \frac{1}{2} (1 \pm \gamma_5).$$
The saddle point of the integral (26) is located at $(\Phi_{\pm})_{fg} = M_f \delta_{fg}$, a self-consistency equation for the effective quark mass, i.e.,

$$\int \frac{d^4 k}{(2\pi)^4} \frac{M_f^2 F^4(k^2)}{M_f^4 F^4(k^2) + k^2} = N + \frac{m_f M_f V N_c}{2\pi^2 \rho^2},$$  \hspace{1cm} (28)$$

being imposed. Eq. (28) describes also the shift of the effective mass of the quark $M_f$ due to current mass $m_f$. Expanding (28) with respect to $m_f$, we have

$$M_f = M_0 + \gamma m_f,$$

where

$$\gamma^{-1} = \rho^2 \int_0^\infty dk^2 \frac{k^4 F^4(k^2)}{(M_0^2 F^4(k^2) + k^2)^2}.$$  \hspace{1cm} (29)$$

Such integrals converge due to the form factor $F(k^2)$. Assuming for the instanton model parameters $\rho$ and $R$ the values (6) - which correspond to $M_0 = 340$ MeV - we find

$$\gamma = 2.4.$$

(30)

The quark condensate is then given by

$$-V^{-1} Z^{-1}_N \frac{dZ_N}{dm} |_{\kappa=0} = - \frac{N_c M_0}{2\pi^2 \rho^2} = -(265 \ MeV)^3 \sim N_c \frac{1}{\rho R^2}.$$  \hspace{1cm} (31)$$

This quantity can be also calculated by formula (i2) $-i \langle \psi^\dagger \psi >_{Euclid} = i Tr S$, yielding

$$-i \langle \psi^\dagger \psi >_{Euclid} = -4 N_c \int \frac{d^4 k}{(2\pi)^4} \frac{M_0 F^2(k^2)}{M_0^2 F^4(k^2) + k^2} = -(255 \ MeV)^3 \sim N_c \frac{1}{\rho R^2}.$$  \hspace{1cm} (32)$$

Although coming from different formulas, predictions (31) and (32) have the same parametric dependence and agree with the phenomenological value, i.e.,

$$-i \langle \psi^\dagger \psi >_{Euclid} = -(240 - 250 \ MeV)^3.$$

The three-point function can be derived from the functional relation

$$\tau_{\mu\nu}(x_1, x_2, x_3) = \frac{\delta \hat{W}[\kappa, a]}{\delta \kappa(x_1) \delta a_\mu(x_2) \delta a_\nu(x_3)} |_{\kappa, a=0},$$  \hspace{1cm} (33)$$

where

$$\hat{W}[\kappa, a] = \sum_f Tr ln \frac{i\hat{D} + i M_f F^2 (A_+ + A_-)}{i\hat{D} + i m_f}.$$  \hspace{1cm} (34)$$

It is clear from eq. (34) that the vertex factors in the diagram of the process are $e Q_f \gamma_\mu$ and $i M_f F^2 N_f^{-1} \gamma_5$. Taking the Fourier transform of (35), we get

$$\hat{\tau}_{\mu\nu}(q_1, q_2, q_3) = \hat{f}(q_1^2) N_c e^2 \sum_f Q_f^2 8 M_f^2 e^{\mu\nu\lambda\sigma} q_{2\lambda} q_{3\sigma} \Gamma_f(q_1^2, q_2^2, q_3^2).$$  \hspace{1cm} (35)$$
where $\Gamma_f(q_1^2, q_2^2, q_3^2)$, the form factor coming from the diagram of the process considered, may be calculated analytically if we approximate the form factor $F$ by 1. As a result, the left side of eq. (2) reduces to

$$ (N_f \frac{g^2}{16\pi^2}(\frac{4e^2 N_c}{g^2 N_f} \sum_f Q_f^2)) F^{(2)} \tilde{F}^{(3)}, $$

which coincides with the first term at the right side of eq. (2). If we take into account the form factor $F$ in (35), and give the model parameters the values (3), in the chiral limit we find a variation of $\sim 17\%$. Beyond the chiral limit the left side of (2) receives the contribution

$$ -\frac{N_c e^2}{2\pi^2} \sum_f Q_f^2 0.2 \gamma \rho m_f \epsilon^{\mu\nu\lambda\sigma} q_2 \lambda q_3 \sigma. $$

This quantity has to be compared with the explicit contribution of the current quark masses to the right side of (2), which, too, may be calculated by formulas (34) and (33) substituting $iM_f F^2 N_f^{-1}$ by $2im_f$ in the $\gamma_5$ vertex. Approximating again $F \sim 1$, we obtain

$$ 2i \sum_f m_f \langle 0 | \psi_f^+ \gamma_5 \psi_f | 2 \gamma \rangle = \frac{N_c e^2}{2\pi^2} \sum_f Q_f^2 \frac{m_f}{M_f} \epsilon^{\mu\nu\lambda\sigma} q_2 \lambda q_3 \sigma. $$

The ratio of (37) to (38) at the parameter values (3) results to be

$$ -0.2 \gamma \rho M_0 = -0.28 $$

and not 1, as demanded by LET. We stress the neat contradiction with the theorem, not only in magnitude but also in the sign. So the $DP$ model (eq. (20)) fails to reproduce $LET$ beyond chiral limit.

5. Other refined tests for instanton models

The matrix elements of the anomaly (11) between vacuum and $\eta$-meson or $\pi^0$ states lead to more stringent tests of the $DP$ model.

The partition function (27) describes mesons with matrices $\Phi_{\pm}$, whose usual decomposition is

$$ \Phi_{\pm} = \exp(\pm \frac{i}{2} \phi) M \sigma \exp(\pm \frac{i}{2} \phi), $$

$\phi$ and $\sigma$ being $N_f \times N_f$ matrices. The saddle-point condition demands $\sigma = 1$, $\Phi_{\pm} = 0$. The usual decomposition for the pseudoscalar fields $\phi = \sum_0 \lambda_i \phi_i$ may be used, where $Tr \lambda_i \lambda_j = 2\delta_{ij}$ and $\phi_{8(3)}$ can be identified with the $\eta(\pi^0)$-meson state.

Firstly we consider the matrix element in which the $\eta$-meson is involved. Neglecting the small admixture factor ($\sim 0.1$) with the pure singlet state (21), the matrix element of the divergence (2) between the $\eta$-meson and the vacuum leads to eq. (4), which can be used as a test for instanton models.
As it is clear from previous considerations, the factor $g^2G\tilde{G}$ generates the vertex $iM_fF^2\gamma_5N^{-1}_f$ and the $\eta$-meson gives rise to $iM\lambda_8F^2\gamma_5$. The structure of the mass matrix is

$$M = M_0 + \gamma(m_s\left(\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8\right) + m_u\frac{1 + \tau_3}{2} + m_d\frac{1 - \tau_3}{2}).$$

Then at small $q$

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\eta\rangle = 2\gamma m_s\left(-\frac{1}{\sqrt{3}}tr(\lambda_8)^2\right)4N_c\int \frac{d^4k}{(2\pi)^4}\frac{M_0F^4(k^2)}{M_0^2F^4(k^2) + k^2}. \quad (41)$$

Applying the same procedure that led to eq. (28), we get

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\eta\rangle = 2\gamma m_s\left(-\frac{2}{\sqrt{3}}\right)\frac{N}{VM_0} \sim m_s\frac{N^{1/2}}{\rho R^2}. \quad (42)$$

The right side of (4) is

$$2m_s\langle 0|\psi_s^\dagger\gamma_5\psi_s|\eta\rangle = 2\gamma m_s\left(-\frac{2}{\sqrt{3}}\right)\frac{N}{VM_0} \sim m_s\frac{N^{1/2}}{\rho R^2}. \quad (43)$$

On the other hand eq. (32) yields

$$2m_s\langle 0|\psi_s^\dagger\gamma_5\psi_s|\eta\rangle = 2\gamma m_s\left(-\frac{2}{\sqrt{3}}\right)i <\psi^\dagger\psi> \quad (44)$$

Now let us confront such equations with some consequences of the relation (5), where the $\pi^0$-meson is involved. Repeating for this case the calculations applied to relation (4), the left side of equation (5) yields

$$\langle 0|N_f\frac{g^2}{16\pi^2}G\tilde{G}|\pi^0\rangle = 2\gamma(m_u - m_d)\frac{N}{VM_0}, \quad (45)$$

while the right side results in

$$2i(m_u - m_d)\langle 0|\frac{\psi_s^\dagger\tau_3}{2}\gamma_5\psi|\pi^0\rangle = 2(m_u - m_d)i <\psi^\dagger\psi> \quad (46)$$

The ratio of (42) to (44) equals the ratio of (45) to (46) and at the parameter values (6) yields

$$\frac{N}{VM_0i <\psi^\dagger\psi>} = 0.66 \quad (47)$$

and not 1, as demanded by relations (4) and (5). Again the $DP$ model (eq. (20)) strongly contradicts some consequences of the operator equation (1) beyond the chiral limit.

6. Conclusions

We recall the main results of our treatment. $LET$ and relations (11)-(13) constitute sensitive tests for instanton models. The single-instanton approximation[14][13][18] and
its modifications badly violate LET. On the contrary the DP model, which takes into account the multiinstanton effects in the propagation of the quarks, agrees satisfactorily with this theorem in the chiral limit. However, beyond such a limit, even this model violates LET and the relations (4)-(7).

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