Abstract: A hybrid nonlinear differentiator being fit for rapid convergence is presented, which is based on singular perturbation technique and consists of linear and nonlinear parts. This differentiator is simple and can converge rapidly at all times, and no chattering phenomena happen. The theoretical results are confirmed by computer simulations and an experiment.

Keywords: Differentiator, singular perturbation, rapid convergence, chattering phenomenon

1. Introduction

Differentiation of signals is an old and well-known problem [1, 2, 3] and has attracted more attention [4, 5, 6] in recent years. Rapidly and accurately obtaining the velocities and accelerations of tracked targets is crucial for several kinds of systems with correct and timely performances, such as the missile-interception systems in defence systems [7] and underwater vehicle systems [8]. Therefore, the convergence rates and computation time of differentiators are very significant. In [19], we presented a finite-time-convergent differentiator that is based on singular perturbation technique. The merits of this differentiator exist in three aspects: rapidly finite-time convergence compared with other typical differentiators; no chattering phenomenon; and besides the derivatives of the derivable signals, the generalized derivatives of some classes of signals can be obtained. However, the differentiators in [19] are complicated and difficult to be used in practice. Moreover, the computation time is still large.

In this paper, we expand our work in [19]. The designed differentiator consists of linear and nonlinear parts. The structure of the designed differentiator is simple and the computation time is short comparing with the differentiator in [19]. This paper is organized in the following format. In section 2, rapid convergence is given. In section 3, our previous main results are overviewed. In section 4, a rapid-convergent hybrid differentiator is obtained by comparing the presented three types of differentiators based on singular perturbation technique. In section 5, reduction of peaking phenomenon is given. In section 6, the simulations are given. In section 7, an experiment is shown, and our conclusions are made in section 8.

2. Rapid convergence

In order to design practical differentiators and make their structure simple, we give lemmas 1-3 in the following.

Lemma 1. System

\[
\begin{align*}
\frac{dz_1(t)}{d\tau} &= z_2(t) \\
\frac{dz_2(t)}{d\tau} &= -a_{10}z_1(t)-a_{20}z_2(t)
\end{align*}
\]

is exponentially convergent with respect to the origin \((0, 0)\). Where \(a_{10}>0\) and \(a_{20}>0\) are constants.

Proof. If the Lyapunov function candidate is selected as \(V(z_1(t), z_2(t))=(a_{10}z_1(t)^2+z_2(t)^2)/2\), then the result of the convergence can be obtained. \(\Box\)

The process of convergent velocity for the system (1) is shown in figure 1.
From figure 1, the convergent velocities of the states are rapid when they are far away from the equilibrium point. However, the convergent velocities are slow when they approach the equilibrium point. It is required to improve the process of convergent velocities when the states approach the equilibrium point.

In order to demonstrate the finite-time convergence, we introduce the following Definitions 1-2, Assumptions 1-3, Theorem 3.1 from the corresponding references.

**Definition 1** [20]. “Generalized derivative” denotes that the left and right derivatives of a point in a function trajectory both exist, and they may be not equal to each other.

**Definition 2** [18]. Consider a time-invariant system in the form of

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \]

where \( f : D \to \mathbb{R}^n \) is continuous on open neighborhood \( D \subset \mathbb{R}^n \) of the origin. The origin is said to be a finite-time-stable equilibrium of the above system if there exists an open neighborhood \( N \subset D \) of the origin and a function \( T_f : N \setminus \{0\} \to (0, \infty) \), called the settling-time function, such that the following statements hold:

1) **Finite-time-convergence:** For every \( x \in N \setminus \{0\} \), \( \psi^x \) is the flow starting from \( x \) and defined on \([0, T_f(x))\), \( \psi^x(t) \in N \setminus \{0\} \) for all \( t \in [0, T_f(x)) \), and \( \lim_{t \to T_f(x)} \psi^x(t) = 0 \).

2) **Lyapunov stability:** For every open neighborhood \( U_\varepsilon \) of zero, there exists an open subset \( U_\delta \) of \( N \) containing zero such that, for every \( x \in U_\delta \setminus \{0\} \), \( \psi^x(t) \in U_\varepsilon \) for all \( t \in [0, T_f(x)) \).

The origin is said to be a globally finite-time-stable equilibrium if it is a finite-time-stable equilibrium with \( D = N = \mathbb{R}^n \). Then the system is said to be finite-time-convergent with respect to the origin.

**Assumption 1.** Suppose the origin is a finite-time-stable equilibrium of [18, Theorem 4.3] of

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= f(z_1, z_2, \ldots, z_n)
\end{align*}
\]

And the settling-time function \( T_f \) is continuous at zero, where \( f(\cdot) \) is continuous and \( f(0) \). Let \( N \) be as in Definition 2 and let \( \theta \in (0,1) \). Then there exists a continuous scalar function \( V \) such that 1) \( V \) is positive definite.
and 2) \( \dot{V} \) is real valued and continuous on \( N \) and there exists \( c > 0 \) such that
\[
\dot{V} + cV^q \leq 0 \quad \text{for any } \theta \in (0,1)
\] (4)

**Assumption 2.** There exists a Lipschitz Lyapunov function \( V \) satisfying (4) with Lipschitz constant \( M \).

**Assumption 3.** For (3), there exist \( \rho_i \in (0,1], i = 0, 1, \ldots, n-1 \), and a nonnegative constant \( \overline{a} \) such that
\[
|f(z_1, z_2, \ldots, z_n) - f(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)| \leq \overline{a} \sum_{i=1}^{n} |z_i - \bar{z}_i|^{\rho_i-1}
\] (5)
where \( z_i, \bar{z}_i \in \mathbb{R}, i = 1, \ldots, n \).

**Assumption 4.** \( v(t) \) is a continuous and piecewise \( n \)-order derivable signal with the following properties. The derivatives of \( v(t) \) up to order \( n-2 \) exist on the whole time domain and \( v(t) \) is not \( (n-1) \)-order differentiable at some instants \( t_j, j = 1, \ldots, k \). However, the \( (n-1) \)-order left derivative \( v_{\tau}^{(n-1)}(t_j) \) and right derivative \( v_{\tau}^{(n-1)}(t_j) \) exist [20], respectively, and \( v_{\tau}^{(n-1)}(t_j) \neq v_{\tau}^{(n-1)}(t_j) \), \( j = 1, \ldots, k \), may be satisfied.

**Remark 1.** There are a number of nonlinear functions actually satisfying this assumption. For example, one such function is \( x^\rho \) since
\[
|x^\rho - \bar{x}^\rho| \leq 2^{1-\rho} |x - \bar{x}|^\rho, \quad \rho_i \in (0,1].
\] Moreover, there are smooth functions also satisfying this property. In fact, it is easy to verify that
\[
|\sin x - \sin \bar{x}| \leq 2|x - \bar{x}|^\rho \quad \text{for any } \rho_i \in (0,1].
\]

**Denotations:**
\[
\text{sgn}(y)^\alpha = |y|^\alpha \text{sgn}(y), \quad \alpha > 0.
\]
It is obvious that \( \text{sgn}(y)^\alpha = y^\alpha \) only if \( \alpha = q/p \), where \( p, q \) are positive odd numbers. Moreover,
\[
\frac{d}{dy} y^\alpha = (\alpha + 1) y^{\alpha+1}, \quad \frac{d}{dy} \text{sgn}(y)^\alpha = (\alpha + 1) y^{\alpha+\frac{q}{p}}
\] (6)

**Theorem 3.1 in [15].** Let the differential equation
\[
z(\tau) = f(z_1(\tau), \ldots, z_n(\tau))
\] (7)
be locally homogeneous of degree \( q < 0 \) with respect to dilation \( (r_1, \ldots, r_n) \) and let the equilibrium point \( z = 0 \) of (7) be globally uniformly asymptotically stable. Then the differential equation (7) is globally uniformly finite time stable.

Based on the theory of finite-time convergence, we give two simple finite-time-convergent systems in the following.

**Lemma 2.** System
\[
\frac{d(z_1(\tau))}{d\tau} = z_2(\tau)
\]
\[
\frac{d(z_2(\tau))}{d\tau} = -a_{11} \text{sgn}(z_1(\tau))^{\frac{q}{p}} - a_{21} \text{sgn}(z_2(\tau))^{\alpha}
\] (8)
is finite time convergent with respect to the origin \((0, 0)\), i.e.,
\[
z_1(\tau) \equiv 0 \quad \text{and} \quad z_2(\tau) \equiv 0 \quad \text{for} \quad \tau \geq \tau_s
\] (9)
where \( 0 < \alpha < 1, \ a_{11} \) and \( a_{21} \) are positive constants, \( \tau_s \) is time constant.
Proof. Let the Lyapunov function candidate be:

\[ V(z_1(\tau), z_2(\tau)) = \frac{a_{11}(2 - \alpha)}{2} z_1(\tau)^{\frac{2}{2 - \alpha}} + \frac{1}{2} z_2^2(\tau) \] (10)

Therefore, we have

\[
\frac{d(V(z_1(\tau), z_2(\tau)))}{d\tau} = a_{11}\text{sig}(z_1(\tau))^{\alpha} z_2(\tau) + z_2(\tau)^2 \left( -a_{11}\text{sig}(z_1(\tau))^{\alpha} - a_{21}\text{sig}(z_2(\tau))^2 \right)
\]

\[ = -a_{21}|z_2(\tau)|^{\alpha} < 0 \] (11)

Note that the uniqueness of the solution of the system can be obtained from [16]. Then based on LaSalle invariant theory, the system (8) is asymptotically stable. Therefore, based on the negative homogeneity [15, Theorem 3.1], we have that

\[ e^\tau z_2(\tau) = e^{\alpha \tau + k} z_2(\tau) \]

\[ -a_{11}\text{sig}(e^\tau z_1(\tau))^{\alpha} - a_{21}\text{sig}(e^\tau z_2(\tau))^2 = e^{\alpha \tau + k}\left( -a_{11}\text{sig}(z_1(\tau))^{\alpha} - a_{21}\text{sig}(z_2(\tau))^2 \right) \] (12)

Therefore we have that

\[ r_2 = r_1 + k, r_1 = \frac{\alpha}{2 - \alpha} = r_2 + k \] (13)

i.e.,

\[ r_1 = k \frac{2 - \alpha}{\alpha - 1}, r_2 = k \frac{1}{\alpha - 1} \] (13)

Therefore, system (8) is finite time convergent with respect to the origin (0, 0), i.e.,

\[ z_1(\tau) \equiv 0 \text{ and } z_2(\tau) \equiv 0 \text{ for } \tau \geq \tau_s \] (14)

Remark 2. From [17], system

\[ \frac{d(z_1(\tau))}{d\tau} = z_2(\tau) \] (15)

\[ \frac{d(z_2(\tau))}{d\tau} = -\text{sig}(z_1(\tau))^{\alpha} - \text{sig}(z_2(\tau))^2 \]

is finite time convergent with respect to the origin (0, 0), i.e.,

\[ z_1(\tau) \equiv 0 \text{ and } z_2(\tau) \equiv 0 \text{ for } \tau \geq \tau_s \] (16)

where 0 < \alpha_2 < 1, and 1 > \alpha_1 > \alpha_2/(2 - \alpha_2).

The process of convergent velocity for the system (8) or (15) is shown by solid line in figure 2.

From figure 2, although the convergent velocities of the states are rapid when they approach the equilibrium point, the convergent velocities of the states are slow when they are far away from the equilibrium point. Therefore, it is required to improve the convergent velocities whether the states are near to the equilibrium point or not, i.e., rapid convergence should be guaranteed at all times.

Lemma 3. System

\[ \frac{d(z_1(\tau))}{d\tau} = z_2(\tau) \] (17)

\[ \frac{d(z_2(\tau))}{d\tau} = -a_{10}z_1(\tau) - a_{11}\text{sig}(z_1(\tau))^{\alpha} - a_{20}z_2(\tau) - a_{21}\text{sig}(z_2(\tau))^2 \]

is finite time convergent with respect to the origin (0, 0), i.e.,
\[ z_1(\tau) \equiv 0 \text{ and } z_2(\tau) \equiv 0 \text{ for } \tau \geq \tau_s \] (18)

where \( 0 < \alpha < 1, a_{10}, a_{20}, a_{11} \) and \( a_{21} \) are positive constants, \( \tau_s \) is time constant.

**Proof.** We select the Lyapunov function candidate as:

\[ V(z_1(\tau), z_2(\tau)) = a_{11}(2-\alpha)z_1^2(\tau) + \frac{1}{2}(a_{10}z_1^2 + a_{20}z_2^2(\tau)) \] (19)

Therefore, we have

\[
\frac{dV(z_1(\tau), z_2(\tau))}{d\tau} = a_{11}\text{sig}(z_1(\tau))\alpha^\alpha z_2(\tau) + z_2(\tau)\left(-a_{10}z_1(\tau) - a_{11}\text{sig}(z_1(\tau))\alpha^\alpha - a_{20}z_2(\tau) - a_{21}\text{sig}(z_2(\tau))\alpha^\alpha\right)
\]

\[ + a_{10}z_1(\tau)z_2(\tau) \]

\[ = -a_{21}|z_2(\tau)|^{1+\alpha} - a_{20}z_2^2(\tau) < 0 \] (20)

Note that the unique of the solution of the system can be obtained from [16]. Then based on LaSalle invariant theory, the system (17) is asymptotically stable.

From [15], \(-a_{10}z_1(\tau) - a_{20}z_2(\tau)\) is locally uniformly bounded, whereas the right-hand side of the nominal model (8) is continuous and globally homogeneous of degree \( q = k < 0 \) with respect to dilation \( r = (r_1, r_2) \). Hence, the condition \( k + r_2 \leq 0 \), required by Theorem 3.2 in [15], is satisfied, and Theorem 3.2 in [15] is applicable to the globally equiuniformly asymptotically stable (17). By applying Theorem 3.2 in [15], (17) is thus globally equiuniformly finite time stable. The proof of Lemma is completed.

**Remark 3.** From [17] and [15], system

\[
\frac{d(z_1(\tau))}{d\tau} = z_2(\tau) \]

\[
\frac{d(z_2(\tau))}{d\tau} = -a_{10}z_1(\tau) - a_{11}\text{sig}(z_1(\tau))\alpha^\alpha - a_{20}z_2(\tau) - a_{21}\text{sig}(z_2(\tau))\alpha^\alpha \]

is finite time convergent with respect to the origin \((0,0)\), i.e.,

\[ z_1(\tau) \equiv 0 \text{ and } z_2(\tau) \equiv 0 \text{ for } \tau \geq \tau_s \] (22)

where \( 0 < \alpha < 1, \) and \( 1 > \alpha > \alpha_j/(2-\alpha_j) \), \( a_{10}, a_{20}, a_{11} \) and \( a_{21} \) are positive constants, \( \tau_s \) is time constant.

The process of convergent velocity for the system (17) or (21) is shown by solid line in figure 3. From figure 3, we can see that the convergent velocities keep rapid whether the states approach to the equilibrium point or not, that is to say, the rapid-convergent velocities can be guaranteed at all times.

### 3. Overview of our previous results of finite-time-convergent differentiator:

**Theorem 1** [19]. If

\[
\tilde{z}_1 = \tilde{z}_2 \\
\vdots \\
\tilde{z}_{n-1} = \tilde{z}_n \\
\tilde{z}_n = f(\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_n) \] (23)

is satisfied with Assumptions 1-3, signal \( v(t) \) is satisfied with Assumption 4. Then for system
\[
\frac{dx_1}{dt} = x_2 \\
\vdots \\
\frac{dx_{n-1}}{dt} = x_n \\
\varepsilon^n \frac{dx_n}{dt} = f(x_1 - v(t), \varepsilon x_2, \cdots, \varepsilon^{n-1} x_n)
\]

(24)

there exist \( \gamma > 0 \) (where \( \rho \gamma > n \)) and \( \Gamma > 0 \) such that

\[
x_j - v^{(i-1)}(t) = O(\varepsilon^{\rho \gamma - i+1})
\]

(25)

for \( t_j > t \geq t_{j-1} + \varepsilon \Gamma (\Xi(t), \xi(j-1)) \), \( j = 1, \cdots, k + 1 \), (let \( t_0 = 0 \) and \( t_{k+1} = \infty \)) \( i = 1, \cdots, n \) with

\[
x_n(t_j) - v^{(n-1)}(t_j) = O(\varepsilon^{\rho \gamma - n+1})
\]

(26)

where \( \varepsilon > 0 \) is the perturbation parameter and \( O(\varepsilon^{\rho \gamma - n+1}) \) denotes the approximation of \( \varepsilon^{\rho \gamma - n+1} \) order [13] between \( x_i \) and \( v^{(i)}(t) \); and \( \gamma = (1-\theta)/\theta \), \( \theta \epsilon (0, \min(\rho/(\rho+n), 1/2)) \), \( n \geq 2 \). \( e_i = x_i - v^{(i)}, j = 1, \cdots, n \). \( e = [e_1 \cdots e_n]^T \), \( e_i(t_j) = [e_i(t_{j-1}) \cdots e_n(t_{j-1})] \), \( e_n(t_{j-1}) = x_n - v^{(n-1)}(t_{j-1}) \), and \( \Xi(e) = \text{diag}(e, \cdots, e^{n-1}) \).

The practical form of differentiator in [19] is:

\[
\dot{x}_i(t) = (x_i(t)
\varepsilon^2 \dot{x}_2(t) = u
u = -sat_{x_i} \left( \text{sign}(\phi_a(x_1 - v(t), \varepsilon x_2), \phi_a(x_1 - v(t), \varepsilon x_2)^{2/(2-\alpha)} \right) - sat_{x_i} \left( \text{sign}(x_2, \varepsilon x_2)^{2/(2-\alpha)} \right)
\]

where

\[
\phi_a(x_1 - v(t), \varepsilon x_2) = x_1 - v(t) + \frac{\text{sign}(x_2, \varepsilon x_2)^{2-\alpha}}{2-\alpha}, \quad \text{sat}_{x_i}(x) = \begin{cases} x, & |x| < \varepsilon b \\ \varepsilon b \text{sign}(x), & |x| \geq \varepsilon b \end{cases}
\]

Its structure is too complicated. In the following, we will design more simple differentiators based on Theorem 1 in [19], and the designed hybrid differentiator can keep more rapidly convergent all times.

4. Design of differentiators

We design three types of differentiator, and give Theorems 1-3 in the following.

1) Linear differentiator

**Theorem 1.** The linear differentiator is designed as:

\[
\dot{x}_1 = x_2 \\
\varepsilon^2 \dot{x}_2 = -a_{10} (x_1 - v(t)) - a_{20} \varepsilon x_2 \\
y = x_2
\]

(27)

for a signal \( v(t) \) (where \( v(t) \) is a continuous and piecewise second-order derivable signal, we have that

\[
x_2(t) - \dot{v}(t) = O(\varepsilon) \quad \text{for} \ t \to \infty
\]

(28)
where $a_{10}, a_{20} > 0$, $\epsilon > 0$ is perturbation parameter, $O(\epsilon)$ denotes the approximation of $\epsilon$ order $[13]$ between $x_2$ and $\dot{v}(t)$.

**Proof.** Taking Laplace transformation of (27), we have

$$sX_1(s) = X_2(s)$$

$$\epsilon^2 sX_2(s) = -a_{10} \left( X_1(s) - V(s) \right) - a_{20} \epsilon X_2(s)$$

Therefore,

$$\epsilon^2 sX_2(s) = -a_{10} \left( \frac{X_2(s)}{s} - V(s) \right) - a_{20} \epsilon X_2(s)$$

Then, we have

$$sV(s) = 1 + \frac{\epsilon^2 s^2 + a_{20} \epsilon s}{a_{10}}$$

i.e.,

$$\frac{X_2(s)}{V(s)} = \frac{s}{1 + \frac{\epsilon^2 s^2 + a_{20} \epsilon s}{a_{10}}}$$

It is clear that

$$\lim_{\epsilon \to 0} \frac{X_2(s)}{V(s)} = \lim_{\epsilon \to 0} \frac{s}{1 + \frac{\epsilon^2 s^2 + a_{20} \epsilon s}{a_{10}}} = s$$

It means that $x_2(t)$ approximates the derivative $\dot{v}(t)$. This ends the proof of Theorem 1. $\Box$

**Remark 3.** In fact, the differentiator (27) is equivalent to the one presented in [4]. Select

$$x_1 = w_1 - \alpha_2 w_2 / a_1, x_2 = w_2$$

We have

$$\dot{w}_1 = w_2 - \frac{\alpha_2 (w_1 - \dot{v}(t))/\epsilon}{\alpha_1}$$

$$\dot{w}_2 = -a_1 (w_1 - \dot{v}(t))/\epsilon^2$$

$$y = w_2$$

From (32), we see that the analog differentiation is achieved over a limited frequency range $1/\epsilon$. State $x_2(t)$ of differentiator (32) is the output of a concatenated ideal 1-order differentiator and a low-pass filter of order 2. By increasing the differentiation order 2, noise is more attenuated, but bandwidth becomes smaller than the usual one. The controllable canonical form (27) seems to be so interesting when the differentiator is used in closed-loop configurations. In addition, we see that $\dot{v}(t)$ just appears in the last equation, so a great amount of eventual additive noise shall be eliminated because of the presence of the successive 2 integrators.

System (27) is the boundary layer of system (1), and the convergent velocities of the state variables are slow in the nonlinear region of the system dynamics, the lagging phenomenon is inevitable. This is confirmed by figure 1.

In order to remove the slow convergence in the nonlinear region of the system dynamics, we present an algorithm of nonlinear differentiator in the following.
2) Nonlinear differentiator

**Theorem 2.** The nonlinear differentiator is designed as:

\[
\dot{x}_1 = x_2
\]

\[
\varepsilon^2 \dot{x}_2 = -a_{11}\text{sig}(x_1 - v(t))^{\alpha} - a_{21}\text{sig}(\alpha x_2)^{\alpha}
\]

\[
y = x_2
\]

(36)

For a continuous and piecewise two-order derivative signal \(v(t)\), there exists \(\gamma > 0\) (where \(\rho\gamma > 2\) and \(\rho = \min[\alpha,\alpha/(2-\alpha)] = \alpha/(2-\alpha)\), such that

\[
x_j - v^{(i-1)}(t) = O(e^{\rho\gamma t})
\]

(37)

for \(t \geq \varepsilon T(\Xi e(0)), i = 1\), and \(t_j > t \geq t_{j-1} + \varepsilon T(\Xi e_j(t_{j-1})), j = 1,\ldots,k+1\), \(i = 2\), respectively, with

\[
x_2(t_j) - v_j(t_j) = O(e^{\rho\gamma t_j}), j = 1,\ldots,k\text{, where } \alpha \in (0,1).
\]

In fact, we know that (8) is finite-time-convergent with respect to the origin with a finite time \(T_f\). From Theorem 1 in [19], we have the result (37) for (36).

**Remark 4.** Based on finite-time convergence of (15) (in Remark 2), we have another simple differentiator as follow:

\[
\dot{x}_1 = x_2
\]

\[
\varepsilon^2 \dot{x}_2 = -\text{sig}(x_1 - v(t))^{\alpha} - a_{21}\text{sig}(\alpha x_2)^{\alpha}
\]

\[
y = x_2
\]

(38)

where \(0 < \alpha_2 < 1\), and \(1 > \alpha_i > \alpha_2/(2-\alpha_2)\).

Systems (36) and (38) are the boundary layers of system (8) and (15), respectively. However, rapid convergence is local from figure 2. For this differentiator, the convergence of the states is still slow although it is rapid when the states approach to the equilibrium point. To make the rapid convergence global, i.e., to make rapid convergence guaranteed at all times, a hybrid form of differentiator is presented in the following.

3) Hybrid differentiator

**Theorem 3.** The hybrid differentiator with linear and nonlinear parts is designed as:

\[
\dot{x}_1 = x_2
\]

\[
\varepsilon^2 \dot{x}_2 = -a_{10}(x_1 - v(t))^{\alpha_1} - a_{20}\text{sig}(\alpha x_2)^{\alpha_2} - a_{21}\text{sig}(\alpha x_2)^{\alpha_2}
\]

\[
y = x_2
\]

(39)

For a continuous and piecewise two-order derivative signal \(v(t)\), there exists \(\gamma > 0\) (where \(\rho\gamma > 2\) and \(\rho = \min[\alpha,\alpha/(2-\alpha)] = \alpha/(2-\alpha)\), such that

\[
x_j - v^{(i-1)}(t) = O(e^{\rho\gamma t})
\]

(40)

for \(t \geq \varepsilon T(\Xi e(0)), i = 1\), and \(t_j > t \geq t_{j-1} + \varepsilon T(\Xi e_j(t_{j-1})), j = 1,\ldots,k+1\), \(i = 2\), respectively, with

\[
x_2(t_j) - v_j(t_j) = O(e^{\rho\gamma t_j}), j = 1,\ldots,k\text{, where } \alpha \in (0,1).
\]
In fact, we know that (17) is finite-time-convergent with respect to the origin with a finite time $T_f$. From Theorem 1 in [19], we have the result (40) for (39).

**Remark 5.** Based on finite-time convergence of (21) (in Remark 3), we have another simple differentiator as follow:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\epsilon^2 \dot{x}_2 &= -a_{10} (x_1 - v(t)) - a_{11} \text{sign}(x_1 - v(t)) - a_{20} \dot{x}_2 - a_{21} \text{sign}(\dot{x}_2) \\
y &= x_2
\end{align*}
\]  

(41)

where $0 < a_2 < 1$, and $1 > a_1 > a_2/(2 - a_2)$, $a_{10}$, $a_{20}$, $a_{21}$ and $a_{21}$ are positive constants.

System (39) and (41) is the boundary layer of system (17) and (21), respectively. We can see that the hybrid differentiator has the rapid convergence at all times from figure 3, and due to the simple construct, the computation time is very short. Moreover, because of the continuity of the system, no chattering phenomena happen. This type of differentiator is adapted to the systems requiring rapid convergence.

5. Reduction of peaking phenomenon

Unfortunately, the smooth variation of the differentiation gain does not remove the peaking phenomenon but it can be reduced by choosing variant $\epsilon$, i.e., the way to reduce the peaking phenomenon is

\[
1/\epsilon = \begin{cases} 
\mu t & \text{if } 0 \leq t \leq t_{\text{max}} \\
\mu_{\text{max}} & \text{otherwise}
\end{cases}
\]

(42)

where $\mu$ and $t_{\text{max}}$ are chosen according to the desired maximum error that depends on the value of $\gamma_{\text{max}} = \mu \cdot t_{\text{max}}$.

6. Simulations

Because (27), (38) and (41) are the boundary layer of systems (1), (15) and (21), respectively, we give the convergences of (1), (8) and (17) firstly, as follows:

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -z_1 - z_2 \\
\dot{z}_1 &= z_2 \text{sign}(z_1) \quad \text{if } 0 \leq t \leq t_{\text{max}} \\
\dot{z}_2 &= -z_1 - z_2 \text{sign}(z_2) \quad \text{otherwise}
\end{align*}
\]

Fig. 4. The convergences of three systems

In the following, we select the functions of sin$\pi t$ and triangular wave, respectively, as the input signal $v(t)$.

1) Linear differentiator (differentiator (27)). Parameters: $a_{10}=5$, $a_{11}=2$, $\epsilon=1/300$. 

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\[ \dot{x}_1 = x_2 \]
\[ \frac{1}{300^2} \dot{x}_2 = -5(x_1 - v(t)) - \frac{2}{300} x_2 \]
\[ y = x_2 \]

(5-a) Tracking “sin” wave  
(5-b) The magnified figure of (5-a)  
(5-c) Tracking “Triangular” wave

Fig.5. Linear differentiator

2) Nonlinear differentiator (differentiator (38)). Parameters: \( \varepsilon = 1/300, a_{11} = 5, a_{21} = 2, \alpha = 0.5. \)
\[ \dot{x}_1 = x_2 \]
\[ \frac{1}{300^2} \dot{x}_2 = -5 \text{sgn}(x_1 - v(t))^{0.5} - 2 \text{sgn} \left( \frac{x_2}{300} \right)^{0.5} \]
\[ y = x_2 \]

(6-a) Tracking “sin” wave  
(6-b) The magnified figure of (6-a)  
(6-c) Tracking “Triangular” wave

Fig.6. Nonlinear differentiator

3) Hybrid differentiator (differentiator (41)). Parameters: \( a_{10} = 5, a_{11} = 2, a_{20} = 0.5, a_{21} = 0.5, \varepsilon = 1/300, \alpha_1 = \alpha_2 = 0.5. \)
\[ \dot{x}_1 = x_2 \]
\[ \frac{1}{300^2} \dot{x}_2 = -5(x_1 - v(t)) - 0.5 \text{sgn}(x_1 - v(t))^{0.5} - 2 \frac{x_2}{300} - 0.5 \text{sgn} \left( \frac{x_2}{300} \right)^{0.5} \]
\[ y = x_2 \]

(7-a) Tracking “sin” wave  
(7-b) The magnified figure of (7-a)  
(7-c) Tracking “Triangular” wave

Fig.7. Hybrid differentiator
From the simulations above, for the representative input signal $v(t) = \sin(t)$, we can see that the convergent time $t_c = 0.08s$ in linear differentiator based on singular perturbation technique, computation time: $0.8s$, however, the lagging phenomenon is obvious; the convergent time $t_c = 0.05s$ in nonlinear differentiator based on singular perturbation technique, computation time: $1s$, the lagging phenomenon happen; the convergent time $t_c = 0.008s$ in hybrid differentiator based on singular perturbation technique, computation time: $1.2s$, the effect of tracking has high precision and no chattering phenomenon happen.

7. Experiment for hybrid differentiator

In the following, the rotor acceleration of a motor from the rotor speed and the velocity of triangular wave of convertor will be carried out respectively through a hybrid differentiator. The simulation is given as follow:

The equipment of motor system is shown in figure 8.

![Fig. 8. The equipment of motor system](image)

The rotor speed and triangular wave of convertor are sinusoidal and “triangular” waveforms, respectively, as the input signal $v(t)$, and the hybrid differentiator is designed in DSP6713 (Digital Signal Processor), the differential equation is carried out by the method of 4-order Runge-Kutta (while in the simulations, the differential equation is carried out by Simulink with S-function). For the apparatus in figure 10, through the oscillograph, we can obtain the derivatives in Figure 11.

![Fig. 9. An experiment for hybrid differentiator](image)
8. Conclusions

In this paper, we present an algorithm of hybrid differentiator based on singular perturbation technique and compare it with other approaches of differentiation. The designed hybrid nonlinear differentiator consists of continuous linear and nonlinear parts, and it can keep rapid-convergent velocity at all times. Moreover, the chattering phenomenon can be avoided.

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