THE MODIFIED $q$-EULER NUMBERS AND POLYNOMIALS

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ABSTRACT. In the recent paper (see [6]) we defined a set of numbers inductively by

$$E_{0,q} = 1, \quad q(qE + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $E^n$ by $E_{n,q}$. These numbers $E_{k,q}$ are called “the $q$-Euler numbers” which are reduced to $E_k$ when $q = 1$. In this paper we construct the modified $q$-Euler numbers $E_{k,q}$

$$E_{0,q} = \frac{[2]_q}{2}, \quad (qE + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $E^i$ by $E_{i,q}$. Finally we give some interesting identities related to these $q$-Euler numbers $E_{i,q}$.

§1. Introduction

Let $p$ be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-1/p - 1}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. The ordinary Euler numbers are defined by the generating function as follows:

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf. } [6].$$

From this equation, we derive the following relation:

$$E_0 = 1, \quad (E + 1)^n + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$
where we use the technique method notation by replacing $E^n$ by $E_{n,q}$ ($n \geq 0$), symbolically. In the recent (see [6,8]), we defined “the $q$-Euler numbers” as

$$E_{0,q} = 1, \quad q (qE + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $E^n$ by $E_{n,q}$. These numbers are reduced to $E_k$ when $q = 1$. From (1), we also derive

$$E_{n,q} = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{l+1}}, \quad \text{(see [6])},$$

where $\binom{n}{l} = \frac{n(n-1)\ldots(n-l+1)}{l!}$. The $q$-extension of $n \in \mathbb{N}$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1},$$

and

$$[n]_{-q} = \frac{1 - (-q)^n}{1 + q} = 1 - q + q^2 - \cdots + (-q)^{n-1}, \quad \text{cf. [4,5,7,9]}.$$

In [1,2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \quad (q\xi + 1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad \text{(2)}$$

with the usual convention of replacing $\xi^i$ by $\xi_i$. These numbers are $q$-extension of ordinary Bernoulli numbers $B_k$, but they do not remain finite when $q = 1$. So, he modified the Eq.(2) as follows:

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases} \quad \text{(3)}$$

These numbers $\beta_k = \beta_k(q)$ are called “the $q$-Bernoulli numbers”, which are reduced to $B_k$ when $q = 1$, see [1,2]. Some properties of $\beta_k$ were investigated by many authors (see [1,2,3,4,9]). In [3,10], the definition of modified $q$-Bernoulli numbers $B_{k,q}$ are introduced by

$$B_{0,q} = \frac{q - 1}{\log q}, \quad (qB + 1)^k - B_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad \text{(3)}$$

with the usual convention of replacing $B^i$ by $B_{i,q}$. For a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z},$$

$$X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp, (a,p) = 1} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \}, \quad \text{cf. [3,4,5,6,7,8]},$$

$$2$$
where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

We say that $f$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ have a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, an invariant $p$-adic $q$-integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{see [3, 4].}$$

From this we can derive

$$qI_q(f_1) = I_q(f) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0), \quad \text{see [3],} \quad \text{(5)}$$

where $f_1(x) = f(x + 1)$, $f'(0) = \frac{df(0)}{dx}$.

In the sense of fermionic, let us define

$$qI_{-q}(f_1) = \lim_{q \to -q} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x), \quad \text{see [6].} \quad \text{(6)}$$

Thus, we have the following integral relation:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where $f_1(x) = f(x + 1)$. Let $I_{-1}(f) = \lim_{q \to 1} I_{-q}(f)$. Then we see that

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{see [6,8].}$$

In the present paper we give a new construction of $q$-Euler numbers which can be uniquely determined by

$$E_{0,q} = [2]_q \frac{1}{2}, \quad (qE + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $E^n$ by $E_{n,q}$. These $q$-Euler numbers are corresponding to $q$-Bernoulli numbers $B_{k,q}$. Finally we shall consider $q$-zeta function which interpolates $E_{k,q}$ at negative integers. As an application of these numbers $E_{k,q}$, we will investigate some interesting alternating sums of powers of consecutive $q$-integers.

§2. A note on $q$-Bernoulli and Euler numbers associated with $p$-adic $q$-integrals on $\mathbb{Z}_p$

In [4], it was known that

$$\int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_q(x) = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{n!}, \quad \text{(7)}$$
where $\beta_n$ are Carlitz’s $q$-Bernoulli numbers. By (5) and (7), we see that

$$(q - 1) + t = qI_q(e^{(1+q[x]_q)t}) - I_q(e^{[x]_q t}), \quad \text{cf. [3]}. $$

Thus, we have

$$\beta_0 = 1, \quad q(q\beta + 1)^n - \beta_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

**Lemma 1.** For $n \in \mathbb{N}$, we have

$$q^n I_q(f_n) = I_q(f) + (q - 1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} q^l f'(l),$$

where $f_n(x) = f(x + n)$.

**Proof.** By Eq.(5) and induction, Lemma 1 can be easily proved.

It was known that

$$\int_{\mathbb{Z}_p} [x + y]^n d\mu_q(x) = \beta_n(x), \quad \text{see [3,4], (9)},$$

where $\beta_n(x)$ are Carlitz’s $q$-Bernoulli polynomials. Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$. Then, by (8), we have

$$q^n \int_{\mathbb{Z}_p} [x + n]^k d\mu_q(x) = \int_{\mathbb{Z}_p} [x]^k d\mu_q(x) + (q - 1) \sum_{l=0}^{n-1} q^l [l]^k_q + k \sum_{l=0}^{n-1} q^{2l}[l]^k_q. \quad (10)$$

By (7), (8) and (10), we obtain the following:

**Proposition 2.** For $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$, we have

$$q^n \beta_k(n) - \beta_k = (q - 1) \sum_{l=0}^{n-1} q^l [l]^k_q + k \sum_{l=0}^{n-1} q^{2l}[l]^k_q. \quad (11)$$

**Remark.** If we take $n = 1$ in Proposition 2, then we have

$$q(q\beta + 1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases}$$

In [10], it was also known that the modified $q$-Bernoulli numbers and polynomials can be represented by $p$-adic $q$-integral as follows:

$$\int_{\mathbb{Z}_p} q^{-x}[x]^n d\mu_q(x) = B_{n,q}, \quad \text{and} \quad \int_{\mathbb{Z}_p} q^{-y}[y + x]^n d\mu_{-q}(y) = B_{n,q}(x). \quad (12)$$

From the definition of $p$-adic $q$-integral, we easily derive

$$I_q(q^{-x}f_1) = I_q(q^{-x}f) + \frac{q - 1}{\log q} f'(0). \quad (12)$$

By (12), we obtain the following lemma:
Lemma 3. For \( n \in \mathbb{N} \), we have

\[
I_q(q^{-x} f_n) = I_q(q^{-x} f) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} f'(l),
\]

where \( f_n(x) = f(x + n) \). That is,

\[
\int_{\mathbb{Z}_p} q^{-x} f(x + n) d\mu_q(x) = \int_{\mathbb{Z}_p} q^{-x} f(x) d\mu_q(x) + \frac{q - 1}{\log q} \sum_{l=0}^{n-1} f'(l).
\]

From (11) and (13), we note that

\[
B_{k,q}(n) - B_{k,q} = k \sum_{l=0}^{n-1} q^l \left[ l \right]_q^{k-1}, \quad \text{cf. [5,7]},
\]

where \( n \in \mathbb{N}, \ k \in \mathbb{Z}_+. \) By the definition of \( I_{-q}(f) \), we show that

\[
qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).
\]

From (15) and induction, we derive the following integral equation:

\[
q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l),
\]

where \( n \in \mathbb{N}, \ f_n(x) = f(x + n). \) When \( n \) is an odd positive integer, we have

\[
q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l).
\]

If \( n \) is an even natural number, then we see that

\[
q^n I_{-q}(f_n) - I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l f(l).
\]

By (17) and (18), we obtain the following lemma:

Lemma 4. Let \( n \) be an odd positive integer. Then

\[
[2]_q \sum_{l=0}^{n-1} q^l \left[ l \right]_q^m = q^n E_{m,q}(n) + E_{m,q}.
\]
If \( n (=\text{even}) \in \mathbb{N} \), then we have

\[
q^n E_{m,q}(n) - E_{m,q} = [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l [l]_q^m.
\]

Let us consider the modified \( q \)-Euler numbers and polynomials. For any non-negative integer \( n \), the modified \( q \)-Euler numbers \( E_{n,q} \) are defined by

\[
E_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]^n d\mu_{-q}(x) = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l}.
\]

By using \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \), we can also consider the modified \( q \)-Euler polynomials as follows:

\[
E_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-t} [x+t]^n d\mu_{-q}(t) = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^x t^l}{1+q^t}.
\]

From (6) and (15), we derive the following \( p \)-adic \( q \)-integral relation:

\[
I_{-q}(q^{-x} f_1) + I_{-q}(q^{-x} f) = [2]_q f(0).
\]

Thus, we obtain the following proposition:

**Proposition 5.** For \( n \in \mathbb{N} \), we have

\[
I_{-q}(q^{-x} f_n) + (-1)^{n-1} I_{-q}(q^{-x} f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l).
\]

If we take \( f(x) = e^{[x]q^t} \) in Eq.(19), then we have

\[
[2]_q = \int_{\mathbb{Z}_p} q^{-x} e^{[x+1]q^t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} q^{-x} e^{[x]q^t} d\mu_{-q}(x)
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left( \frac{n}{l} \right) q^l E_{l,q} + E_{l,q} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} ((qE + 1)^n + E_{n,q}) \frac{t^n}{n!},
\]

with the usual convention of replacing \( E^n \) by \( E_{n,q} \).

Therefore we obtain the following theorem:
Theorem 6. Let \( n \in \mathbb{Z}_+ \). Then

\[
(qE + 1)^n + E_{n,q} = \begin{cases} 
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n \neq 0,
\end{cases}
\]

with the usual convention of replacing \( E^i \) by \( E_{i,q} \).

Note that \( \lim_{q \to 1} E_{n,q} = E_n \), where \( E_n \) are ordinary Euler numbers. From (19) and (20), we can derive the following theorem:

Theorem 7. Let \( k (=\text{even}) \in \mathbb{N} \), and let \( n \in \mathbb{Z}_+ \). Then we have

\[
E_{n,q} - E_{n,q}(k) = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]^n_q.
\]

If \( k (=\text{odd}) \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \), then we see that

\[
E_{n,q} + E_{n,q}(k) = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]^n_q.
\]

Let \( \chi \) be the Dirichlet’s character with conductor \( d (=\text{odd}) \in \mathbb{N} \). Then we consider the modified generalized \( q \)-Euler numbers attached to \( \chi \) as follows:

\[
E_{n,\chi,q} = \int_X [x]_q^n q^{-x} \chi(x) \, d\mu_{-q}(x).
\]

From this definition, we derive

\[
E_{n,\chi,q} = \int_X \chi(x)[x]_q^n q^{-x} \, d\mu_{-q}(x)
\]

\[
= [d]_q^n [2]_q [2]_q^{-d} \sum_{a=0}^{d-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \frac{a}{d} + x \, q^{-d} \, d\mu_{-q^a}(x)
\]

\[
= [d]_q^n [2]_q [2]_q^{-d} \sum_{a=0}^{d-1} \chi(a)(-1)^a E_{n,q^a}(\frac{a}{d}).
\]

§ 3. \( q \)-zeta function associated with \( q \)-Euler numbers and polynomials

In this section we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). Let \( F_q(t, x) \) be the generating function of \( E_{k,q}(x) \) as follows:

\[
F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]
Then, we show that

\[ F_q(t, x) = \sum_{m=0}^{\infty} \left( \frac{[2]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-1)^l \frac{q^lx}{1+q^l} \right) \frac{t^m}{m!} \]

\[ = [2]_q \sum_{m=0}^{\infty} \left( \frac{1}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-1)^l q^lx \sum_{k=0}^{\infty} (-1)^k q^{kl} \right) \frac{t^m}{m!} \]

\[ = [2]_q \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} ([k + x]_q \frac{t^m}{m!}) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]q} t. \]

Therefore, we obtain the following:

**Theorem 8.** Let \( F_q(t, x) \) be the generating function of \( E_{k,q}(x) \). Then we have

\[ F_q(t, x) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]q} t = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \]

From Theorem 8, we note that

\[ E_{k,q}(x) = \frac{d^k}{dt^k} F_q(t, x)|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n [n + x]_q^k. \]

**Corollary 9.** For \( k \in \mathbb{Z}_+ \), we have

\[ E_{k,q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n [n + x]_q^k. \]

**Definition 10.** For \( s \in \mathbb{C} \), we define \( q \)-zeta function as follows:

\[ \zeta_q(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^k}{[n + x]_q^s}. \]

Note that \( \zeta_q(-n, x) = E_{n,q}(x) \), for \( n \in \mathbb{N} \cup \{0\} \). Let \( d (=\text{odd}) \) be a positive integer. From the generating function of \( E_{n,q}(x) \), we derive

\[ E_{n,q}(x) = [d]_q^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{d-1} (-1)^a E_{n,q^d}(\frac{x + a}{d}). \]

Therefore we obtain the following:
Theorem 11. For \(d (= \text{odd}) \in \mathbb{N}, n \in \mathbb{Z}^+\), we have

\[
E_{n,q}(x) = [d]_q^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{d-1} (-1)^a E_{n,q^d}(\frac{x + a}{d}).
\]

Let \(\chi\) be the Dirichlet’s character with conductor \(d (= \text{odd}) \in \mathbb{N}\) and let \(F_{\chi,q}(t)\) be the generating function of \(E_{n,\chi,q}\) as follows:

\[
F_{\chi,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,q} t^n n!
n!
\]

Then we see that

\[
F_{\chi,q}(t) = \sum_{n=0}^{\infty} \left( [d]_q^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{d-1} \chi(a) (-1)^a E_{n,q^d}(\frac{a}{d}) \right) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{d-1} \chi(a) (-1)^a \sum_{n=0}^{\infty} E_{n,q^d}(\frac{a}{d}) \frac{[d]_q^a t^n}{n!}
\]

\[
= [2]_q \sum_{a=0}^{d-1} \chi(a) (-1)^a \sum_{k=0}^{\infty} (-1)^k e^{[kd+a]_q t}
\]

\[
= [2]_q \sum_{n=0}^{\infty} \chi(n) (-1)^n e^{[n]_q t}
\]

Therefore, we obtain the following theorem:

Theorem 12. Let \(F_{\chi,q}(t)\) be the generating function of \(E_{n,\chi,q}\). Then we have

\[
F_{\chi,q}(t) = [2]_q \sum_{n=0}^{\infty} \chi(n) (-1)^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}.
\]

From Theorem 12, we derive

\[
E_{k,\chi,q} = \frac{d^k}{dt^k} F_{\chi,q}(t)|_{t=0} = [2]_q \sum_{n=1}^{\infty} \chi(n) (-1)^n [n]_q^k.
\]

Corollary 13. For \(k \in \mathbb{Z}^+\), we have

\[
E_{k,\chi,q} = [2]_q \sum_{n=1}^{\infty} \chi(n) (-1)^n [n]_q^k.
\]

For \(s \in \mathbb{C}\), we define a \(l\)-series which interpolates the modified generalized \(q\)-Euler numbers attached to \(\chi\) at a negative integer as follows:

\[
l_q(s, \chi) = [2]_q \sum_{n=1}^{\infty} \frac{\chi(n) (-1)^n}{[n]_q^s}.
\]

From Corollary 13, we easily derive

\[
l_q(-n, \chi) = E_{n,\chi,q}, \quad n \in \mathbb{Z}^+.
\]
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