Heuristic Description of Perpendicular Transport

A. Shalchi
Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
E-mail: andreasm4@yahoo.com

Abstract. The problem of the transport of energetic particles across a mean magnetic field is known since more than 50 years. Previous attempts to describe perpendicular transport theoretically were either based on complicated non-linear theories or computationally expensive simulations. In either case it remained unclear how particles really experience perpendicular transport. In this paper I will present a heuristic approach to solve this problem. Simple arguments will lead to several formulas for the perpendicular diffusion coefficient. These formulas include well-known cases such as compound sub-diffusion and the field line random walk limit but also newer cases such as the collisionless Rechester and Rosenbluth limit. Furthermore, analytical theories such as NLGC and UNLT theories contain a correction factor $a^2$ which is often assumed to be $1/3$. The heuristic approach discussed in this article explains this value as well.

1. Introduction
The theoretical exploration of the interaction between electrically charged energetic particles and magnetized plasmas plays a significant role in a variety of physical systems ranging from fusion devices, over the solar wind, to the interstellar media of our own and external galaxies. In such scenarios energetic particles experience scattering due to interactions with mean magnetic fields $\vec{B}_0$ and turbulent magnetic fields $\delta \vec{B}$. Early analytical work was based on perturbation theory also known as quasi-linear theory (see Jokipii (1966)). However, in general this approach fails (see, e.g., Shalchi (2009) for a review of the problems in diffusion theory). Heuristic arguments as well as systematic theories have been developed focusing on electron heat transport in fusion plasmas where collisions play a significant role (see, e.g., Rechester & Rosenbluth (1978), Kadomtsev & Pogutse (1979), and Krommes et al. (1983)). However, in space plasmas such as the solar wind collisions are insignificant. Therefore, it was concluded that the aforementioned approaches are not applicable in such scenarios. The assumption of exponential field line separation (see again Rechester & Rosenbluth (1978)) was also questioned (see Matthaeus et al. (2003)). In the context of astrophysical plasmas, however, one still finds perpendicular diffusion in most cases as shown via test-particle simulations (see, e.g., Giacalone & Jokipii (1999) and Qin et al. (2002)). Until recently it remained unclear what the mechanisms behind this type of transport are.

Over the past few decades systematic non-linear theories have been developed (see Shalchi (2020) for a review). A major step forward had been achieved due to the derivation of the so-called non-linear guiding center (NLGC) theory of Matthaeus et al. (2003), the unified non-linear transport (UNLT) theory of Shalchi (2010), as well as its time-dependent generalization (see Shalchi (2017) and Lasuik & Shalchi (2017)). The non-linear integral equation of diffusive
UNLT theory contains asymptotic solutions which were derived in Shalchi (2015a). The validity of such solutions depends on the parallel mean free path of the particle $\lambda_\parallel$ as well as the Kubo number (see Kubo (1963)) defined via

$$K = \frac{\ell_\parallel}{\ell_\perp} \frac{\delta B_x}{B_0}. \quad (1)$$

Here we have used the parallel and perpendicular bendover scales $\ell_\parallel$ and $\ell_\perp$ of the turbulence, the $x$-component of the turbulent magnetic field $\delta B_x$, and the mean magnetic field $B_0$. Time-dependent UNLT theory is no longer based on a diffusion approximation and can, for instance, describe compound sub-diffusion in slab turbulence perfectly. Furthermore, the theory explains why diffusion is restored and this is entirely due to transverse complexity becoming important. However, there are at least two remaining problems in the theory of perpendicular diffusion. First, there is a discrepancy between theory and simulations in the large Kubo number regime which was previously balanced out by incorporating the factor $a^2$ (see Section 2 of this paper) and by setting $a^2 = 1/3$ (see Matthaeus et al. (2003)). Furthermore, the question remains what the physics behind collisionless perpendicular diffusion is. Therefore, a heuristic approach was developed in Shalchi (2019a). This approach provides an answer to both questions.

The reminder of this paper is organized as follows. In Sect. 2 we very briefly review systematic analytical theories derived before for describing perpendicular transport. The heuristic approach is based on three rules which are discussed in Sect. 3. The following Sect. 4 discusses the perpendicular diffusion coefficient based on those rules. Sect. 5 provides a comparison between simulations, systematic theories, and the new heuristic approach. Sect. 6 summarizes and concludes.

2. Systematic Analytical Theories for Perpendicular Transport

Before employing heuristic arguments we briefly review existing theories for perpendicular transport. A major step forward was the development of the so-called non-linear guiding center (NLGC) theory proposed by Matthaeus et al. (2003). After employing a set of approximations the latter authors deduced the following non-linear integral equation for the perpendicular diffusion coefficient

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \int d^3k \frac{P_{xx}(\hat{k})}{\kappa_\parallel k_\parallel^2 + \kappa_\perp k_\perp^2 + v/\lambda_\parallel}. \quad (2)$$

The solution of this equation depends on the components of the spectral tensor $P_{nm}$ describing the magnetic fluctuations, the parallel mean free path $\lambda_\parallel = 3\kappa_\parallel/v$, the particle speed $v$, and the mean magnetic field $B_0$.

As pointed out in Matthaeus et al. (2003) and Bieber et al. (2004), the theory can successfully reproduce some of the performed test-particle simulations as well as solar wind observations. However, after more detailed work it was concluded that the theory has two major weaknesses:

(i) For pure slab turbulence, the theory provides a finite result for the perpendicular diffusion coefficient (see, e.g., Shalchi et al. (2004)). However, it is well-known that for this type of turbulence perpendicular transport should be sub-diffusive (see, e.g., Qin et al. (2002)). Furthermore, it became clear that NLGC theory does not agree with simulations if three-dimensional turbulence with small and intermediate Kubo numbers is considered (see, e.g., Shalchi & Hussein (2014)).

(ii) For slab/2D turbulence and three-dimensional turbulence with large Kubo numbers, the theory works well but only if the correction factor $a^2$ is incorporated (see, e.g., Eq. (2) of this paper) and if one sets $a^2 = 1/3$. However, it remained unclear why one needs this value and what the physical meaning of this parameter is.
Because of these problems the unified non-linear transport (UNLT) theory was developed in Shalchi (2010) based on the cosmic ray Fokker-Planck equation. Within diffusive UNLT theory the perpendicular diffusion coefficient is now given by

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \int d^3 k \frac{P_{xx}(\vec{k})}{F(k_\parallel, k_\perp) + (4/3)\kappa_\perp k_\perp^2 + v/\lambda_\parallel}$$  \hspace{1cm} (3)$$

instead of Eq. (2). In Eq. (3) we have used the function $F(k_\parallel, k_\perp) = (v k_\parallel)^2/(3\kappa_\perp k_\perp^2)$. Asymptotic solutions of Eq. (3) and the importance of the Kubo number defined via Eq. (1) have been discussed in Shalchi (2015a). Eq. (3) shows good agreement with most test-particle simulations in particular with those performed for three-dimensional turbulence with small and intermediate Kubo numbers (see, e.g., Heusen & Shalchi (2017)). Furthermore, Eq. (3) contains quasi-linear theory as well as the non-linear theory of field line random walk (FLRW) developed by Matthaeus et al. (1995).

In order to obtain a time-dependent theory for perpendicular transport, time-dependent UNLT theory has been derived in Shalchi (2017) as well as Lasuk & Shalchi (2017). This theory describes perpendicular transport based on mean square displacements $\langle (\Delta x)^2 \rangle$ rather than diffusion coefficients $\kappa_\perp$. Therefore, the theory considered here does not rely on a diffusion approximation. As demonstrated in the aforementioned papers, one can derive the following integro-differential equation for the mean square displacement across the mean magnetic field

$$\frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = \frac{2a^2}{B_0^2} \int d^3 k P_{xx}(\vec{k}, t) \xi(k_\parallel, t) e^{-\frac{1}{2} \langle (\Delta x)^2 \rangle k_\perp^2}. \hspace{1cm} (4)$$

As shown in Shalchi et al. (2011), the parallel correlation function used in Eq. (4) is given by

$$\xi(k_\parallel, t) = \frac{v^2}{3} \frac{1}{\omega_+ - \omega_-} \left[ \omega_+ e^{\omega_+ t} - \omega_- e^{\omega_- t} \right] \hspace{1cm} (5)$$

where

$$\omega_\pm = -\frac{v}{2\lambda_\parallel} \pm \sqrt{\left( \frac{v}{2\lambda_\parallel} \right)^2 - \frac{1}{3} v^2 k_\parallel^2}. \hspace{1cm} (6)$$

Differential equation (4) can be evaluated numerically for any given turbulence model including dynamical turbulence. In some cases analytical solutions can be found as well (see Shalchi (2018) and Shalchi (2020)). Eq. (3) can be derived from Eq. (4) by employing a diffusion approximation of the form $\langle (\Delta x)^2 \rangle = 2\kappa_\perp t$ and integrating over time. Furthermore, the theory explains why diffusion is restored and this is entirely due to transverse complexity becoming important. Due to the exponential factor in Eq. (4), this means that diffusion is obtained as soon as $\langle (\Delta x)^2 \rangle \geq 2\ell_\perp^2$ is satisfied.

However, diffusive and time-dependent UNLT theories still require the parameter $a^2$ in some cases and one still needs to set $a^2 = 1/3$ in order to achieve agreement with simulations. Furthermore, the question remained what the physics behind collisionless perpendicular diffusion is. The heuristic approach developed in Shalchi (2019a) resolves these problems and is explained in the following three sections.

3. The Three Rules of Perpendicular Diffusion

We now formulate rules allowing us to derive formulas for the perpendicular diffusion coefficient without employing systematic theories. Those rules are:\footnote{Note, these three rules were taken word-by-word from Shalchi (2019a).}
(i) Perpendicular transport is only controlled by three effects, namely parallel transport, the random walk of magnetic field lines, as well as transverse complexity. The last of these three effects leads to the particles getting scattered away from the original magnetic field lines they were tied to.

(ii) We assume that the bendover scales $\ell_{\parallel}$ and $\ell_{\perp}$, the integral scales $L_{\parallel}$ and $L_{\perp}$, the ultra-scale $L_U$, as well as the Kolmogorov scale $L_K$ are finite and non-zero. Furthermore, the parallel motion is assumed to be ballistic at early times and thereafter turns into a diffusive motion described by the parallel diffusion coefficient $\kappa_{\parallel}$. The FLRW is initially ballistic and becomes diffusive for larger distances. In this case it is described by the field line diffusion coefficient $\kappa_{\text{FL}}$ which depends on some of the aforementioned scales.

(iii) In order to obtain normal diffusion, the particles need to leave the original magnetic field lines they followed. This happens as soon as transverse complexity becomes significant corresponding to

$$\left\langle (\Delta x)^2 \right\rangle \geq 2\ell_{\perp}^2.$$  \hspace{1cm} (7)

What the perpendicular diffusion coefficient is depends solely on the state of parallel and field line transport at the time particles start to satisfy condition (7).

4. The Perpendicular Diffusion Coefficient

In the following we construct the perpendicular diffusion coefficient $\kappa_{\perp}$ based on the three rules formulated above. We shall derive eight cases which are summarized in Table 1. As demonstrated, there are four different routes to perpendicular diffusion. Those are listed in Table 2.

4.1. The Field Line Random Walk Limit

First we assume that the random walk of magnetic field lines is diffusive in the scenario of interest

$$\left\langle (\Delta x)^2 \right\rangle = 2\kappa_{\text{FL}} |z|.$$  \hspace{1cm} (8)

If we assume that there are no collisions and no pitch-angle scattering, we can set $z = v\mu t$ where we used the pitch-angle cosine $\mu$. Combining this with Eq. (8) and averaging over $\mu$ yields

$$\left\langle (\Delta x)^2 \right\rangle = v\kappa_{\text{FL}} t$$

and, thus,

$$\kappa_{\perp} = \frac{v}{2} \kappa_{\text{FL}}$$  \hspace{1cm} (9)

corresponding to the FLRW limit. This limit is stable because if condition (7) is met, it does not alter the transport. This case is highly relevant in the limit of long parallel mean free paths corresponding to high particle energies (see Figs. 1 and 2 of the current paper).

4.2. Compound Sub-diffusion

If there is strong pitch-angle scattering the parallel motion is diffusive meaning that

$$\left\langle (\Delta z)^2 \right\rangle = 2\kappa_{\parallel} t.$$  \hspace{1cm} (10)

Assuming that field lines are diffusive and particles follow field lines, we can combine Eqs. (8) and (10) to find

$$\left\langle (\Delta x)^2 \right\rangle \approx 2\kappa_{\text{FL}} \sqrt{2\kappa_{\parallel} t}.$$  \hspace{1cm} (11)

$^2$ It is assumed here that $\ell_{\perp}$ is the scale at which transverse complexity becomes significant. In principle this could be a different scale such as the integral scale $L_{\perp}$. To use the bendover scale, however, is motivated by time-dependent UNLT theory (see Sect. 2).
Table 1. The eight cases of perpendicular transport. The perpendicular diffusion coefficient depends on parallel transport, the field line random walk (FLRW), and the transverse complexity (TC) of the turbulence. The results based on the heuristic approach are compared with limits contained in diffusive and time-dependent UNLT theories. As indicated by this table, the heuristic approach agrees with UNLT theories for most cases except the last two. In these cases UNLT theories agree with the heuristic approach only for small and intermediate Kubo numbers but not for large Kubo numbers.

| Case | Parallel T. | FLRW | TC | Perpendicular T. | Diffusion Coeff. | UNLT |
|------|-------------|------|----|------------------|------------------|------|
| 1    | Ballistic   | Ballistic | No | Ballistic        | $d_\perp(t) = \frac{v^2 \delta B^2}{3} \ell$ | Yes |
| 2    | Ballistic   | Ballistic | Yes| Double-ballistic | $\kappa_\perp = \sqrt{\frac{2}{3}} v \ell_\perp \delta B \ell$ | Yes |
| 3    | Ballistic   | Diffusive | No | FLRW Limit       | $\kappa_\perp = \frac{v}{2} \kappa_{FL}$ | Yes |
| 4    | Ballistic   | Diffusive | Yes| FLRW Limit       | $\kappa_\perp = \frac{v}{2} \kappa_{FL}$ | Yes |
| 5    | Diffusive   | Ballistic | No | Fluid Limit      | $\kappa_\perp = \frac{\delta B^2}{B_0^2} \kappa_\parallel$ | Yes |
| 6    | Diffusive   | Ballistic | Yes| Fluid Limit      | $\kappa_\perp = \frac{\delta B^2}{B_0^2} \kappa_\parallel$ | Yes |
| 7    | Diffusive   | Diffusive | No | Compound Sub-diff.| $d_\perp(t) = \kappa_{FL} \sqrt{\frac{\kappa_\parallel}{\kappa_\perp}}$ | Yes / No |
| 8    | Diffusive   | Diffusive | Yes| CLRR Limit       | $\kappa_\perp = \left(\frac{\kappa_{FL}}{\ell_\perp}\right)^2 \kappa_\parallel$ | Yes / No |

Table 2. The four routes to perpendicular diffusion. In the first three cases perpendicular transport starts as ballistic motion which then turns into a diffusive motion. The perpendicular diffusion coefficient is then given by the corresponding case listed in Table 1. In the fourth case the ballistic motion is followed by a sub-diffusive regime and then, at later times, diffusion is restored.

| Route | Final State              | Diffusion Coefficient                     |
|-------|--------------------------|-------------------------------------------|
| 1 → 2 | Double-ballistic diffusion| $\kappa_\perp = \sqrt{\frac{2}{3}} v \ell_\perp \delta B \ell$ |
| 1 → 3 → 4 | FLRW Limit          | $\kappa_\perp = \frac{v}{2} \kappa_{FL}$ |
| 1 → 5 → 6 | Fluid Limit          | $\kappa_\perp = \frac{\delta B^2}{B_0^2} \kappa_\parallel$ |
| 1 → 7 → 8 | CLRR Limit           | $\kappa_\perp = \left(\frac{\kappa_{FL}}{\ell_\perp}\right)^2 \kappa_\parallel$ |
The running perpendicular diffusion coefficient is then

\[ d_\perp(t) = \frac{d}{2} \frac{d\langle(\Delta x)^2\rangle}{dt} \approx \kappa_{FL} \frac{\kappa_\parallel}{2t} \]  

(12)
corresponding to sub-diffusive transport. However, diffusion will be restored as soon as condition (7) is satisfied as discussed in the next paragraph. For slab turbulence, on the other hand, this condition is never satisfied due to \( \ell_\perp = \infty \) and thus, we find compound sub-diffusion as the final state of perpendicular transport. Please note that a much more comprehensive discussion of compound sub-diffusion can be found in Webb et al. (2006).

4.3. The Collisionless Rechester & Rosenbluth Regime

We now assume that diffusion is restored as soon as the particles scatter away from the original field lines they were tied to. This happens as soon as condition (7) is satisfied. We also assume that this happens after the particles travel the distance \( L_K \) in the parallel direction leading to

\[ \frac{\kappa_\perp}{\kappa_\parallel} = \frac{\langle(\Delta x)^2\rangle}{\langle(\Delta z)^2\rangle} = \frac{\ell^2_\perp}{L^2_K}. \]  

(13)

In order to eliminate \( L_K \) we use the field line diffusion coefficient

\[ \kappa_{FL} = \frac{\langle(\Delta x)^2\rangle}{2|z|} = \frac{\ell^2_\perp}{L_K}. \]  

(14)

Combining Eqs. (13) and (14) yields

\[ \kappa_\perp \approx \left( \frac{\kappa_{FL}}{\ell_\perp} \right)^2 \kappa_\parallel. \]  

(15)

Alternatively, we can use Eq. (14) in order to replace the scale \( \ell_\perp \) in Eq. (13) so that we derive

\[ \kappa_\perp \approx \frac{\kappa_{FL} \kappa_\parallel}{L_K}. \]  

(16)
in agreement with equation (8) of Rechester & Rosenbluth (1978) as well as equation (4) of Krommes et al. (1983). The quantity \( L_K \) is either called the Kolmogorov-Lyapunov length or just the Kolmogorov length (see, e.g., Krommes et al. (1983)). However, here \( L_K \) is not an exponentiation length as in Rechester & Rosenbluth (1978) but a characteristic distance along the mean field at which transverse complexity becomes significant. The scale \( L_K \) also occurs in field line separation theory (see Shalchi (2019b)). Furthermore, Eqs. (15) and (16) were obtained without assuming collisions and, thus, we call this result the collisionless Rechester & Rosenbluth (CLRR) limit.

One can also obtain this by using a slightly different derivation. We assume that we find compound sub-diffusion until the particles satisfy condition (7) which happens at the diffusion time \( t_d \) so that Eqs. (11) and (12) become

\[ 2\ell^2_\perp = 2\kappa_{FL} \sqrt{2\kappa_\parallel t_d} \]  

(17)
as well as

\[ \kappa_\perp = \kappa_{FL} \sqrt{\frac{\kappa_\parallel}{2t_d}}. \]  

(18)
Combining the latter two equations in order to eliminate $t_d$ yields again Eq. (15).

In order to evaluate this further, we consider two sub-cases, namely small and large values of the Kubo number, respectively. For small Kubo numbers the field line diffusion coefficient is given by the quasi-linear limit (see for instance Shalchi (2020) for a detailed derivation of this formula)

$$\kappa_{FL} = L_{\parallel} \frac{\delta B_x^2}{B_0^2}$$

where we have used the parallel integral scale defined via

$$\delta B_x^2 L_{\parallel} = \pi \int d^3 k P_{xx}(\vec{k}) \delta \left( k_{||} \right).$$

Using Eq. (19) in Eq. (15) yields

$$\kappa_{\perp} \approx \left( \frac{L_{\parallel}}{\ell_{\perp}} \right)^2 \frac{\delta B_x^4}{B_0^4} \kappa_{||} \propto \frac{\ell_{\parallel}^2}{\ell_{\perp}^2} \frac{\delta B_x^4}{B_0^4} \kappa_{||},$$

(21)

where we have assumed that the parallel integral scale is directly proportional to the parallel bendover scale $\ell_{\parallel}$. Eq. (21) in agreement with the scaling obtained from diffusive UNLT theory in Shalchi (2015a). Furthermore, we derive from Eq. (14)

$$L_K \propto \frac{\ell_{\perp}^2 B_0^2}{\delta B_x^2}.$$  

(22)

Alternatively, this can be written as

$$\frac{L_K}{\ell_{\parallel}} \propto \frac{\ell_{\parallel}^2 B_0^2}{\ell_{\perp}^2 \delta B_x^2} = K^{-2}.$$  

(23)

Since we consider the limit of small Kubo numbers, the distance we need to travel in order to experience transverse complexity is long.

For large Kubo numbers, on the other hand, we have (see again Shalchi (2020) for details)

$$\kappa_{FL} = L_U \frac{\delta B_x}{B_0}$$

with the ultra-scale $L_U$ defined via

$$L_U^2 \delta B_x^2 = \int d^3 k P_{xx}(\vec{k}) k_{\perp}^{-2}.$$  

(25)

Eq. (24) is either called the non-linear or Bohmian limit of FLRW and is similar compared to the field line diffusion coefficient obtained by Kadomtsev & Pogutse (1979). Therewith, Eq. (15) becomes

$$\kappa_{\perp} \approx \left( \frac{L_U}{\ell_{\perp}} \right)^2 \frac{\delta B_x^2}{B_0^2} \kappa_{||},$$

(26)

and the Kolmogorov scale is

$$L_K = \frac{\ell_{\perp}^2 B_0}{L_U \delta B_x^2}.$$  

(27)

For most turbulence spectra we expect $L_U \propto \ell_{\perp}$ (see, e.g., Sect. 5.1 of this paper) and, therefore,

$$L_K \propto \ell_{\perp} B_0 \delta B_x.$$  

(28)
as well as
\[
\frac{L_L}{\ell_\parallel} \propto \frac{\ell_\perp B_0}{\ell_\parallel \delta B_x} = K^{-1}.
\]  
(29)

Since we now consider the case of large Kubo numbers, the distance we need to travel to experience transverse complexity is now short.

4.4. The Fluid Limit
Let us now assume that parallel transport is diffusive but magnetic field lines are still ballistic when the particles start to satisfy condition (7). From the field line equation \(dx = dz\delta B_x/B_0\) we can easily derive
\[
\langle (\Delta x)^2 \rangle = \langle (\Delta z)^2 \rangle \frac{\delta B_x^2}{B_0^2} = 2\kappa_\parallel \delta B_x^2/B_0^2
\]  
(30)

and, thus, the perpendicular particle diffusion coefficient is
\[
\kappa_\perp = \frac{\delta B_x^2}{B_0^2} \kappa_\parallel
\]  
(31)

which Krommes et al. (1983) called the fluid limit.

4.5. The Initial Free-Streaming Regime
The simplest case is obtained for the early times when parallel and field line transport are ballistic. For ballistic parallel transport, Eq. (30) turns into
\[
\langle (\Delta x)^2 \rangle = \langle (\Delta z)^2 \rangle \frac{\delta B_x^2}{B_0^2} = \left(\frac{v\mu t}{3}\right)^2 \frac{\delta B_x^2}{B_0^2}
\]  
(32)

Averaging over \(\mu\) yields
\[
\langle (\Delta x)^2 \rangle = \frac{v^2}{3} \frac{\delta B_x^2}{B_0^2} \ell^2
\]  
(33)

so that the running perpendicular diffusion coefficient becomes
\[
d_\perp(t) = \frac{v^2}{3} \frac{\delta B_x^2}{B_0^2} t
\]  
(34)

corresponding to ballistic perpendicular transport. However, this is not a stable regime since we only find this type of transport before condition (7) is met.

4.6. Double-ballistic Diffusion
We now consider a scenario where the transport is still ballistic when the particles start to satisfy condition (7). Therefore, we use Eqs. (33) and (34) to derive
\[
2\ell_\perp^2 = \frac{v^2}{3} \frac{\delta B_x^2}{B_0^2} \ell_\perp^2
\]  
(35)

as well as
\[
\kappa_\perp = \frac{v^2}{3} \frac{\delta B_x^2}{B_0^2} \ell_d.
\]  
(36)

Combining the latter two equations leads to
\[
\kappa_\perp = \sqrt{\frac{2\ell}{\frac{3}{3} v\ell_\perp} \frac{\delta B_x}{B_0}}.
\]  
(37)

A similar result can be derived from Eq. (4) by assuming a ballistic perpendicular motion (see Shalchi (2020) for more details).
4.7. Time-scale Arguments
In order to determine which case is valid for which scenario, one needs to explore at which time a certain process takes place. In the parallel direction particles need to travel a parallel mean free path in order to get diffusive and, thus,

$$t_\parallel = \frac{\lambda_\parallel}{v} = \frac{3\kappa_\parallel}{v^2}. \quad (38)$$

In the following we focus on the case of short $\lambda_\parallel$. For small Kubo numbers the field lines become diffusive for $|z| \approx \ell_\parallel$. For diffusive parallel transport the corresponding time is then given by

$$t_{FL} \approx \frac{\ell_\parallel^2}{2\kappa_\parallel}. \quad (39)$$

Then, on the other hand, if we assume that condition (7) is satisfied while the field lines are still ballistic, we have

$$2\ell_\perp^2 = 2\kappa_\parallel t_{Fluid} \frac{\delta B_x^2}{B_0^2} \quad (40)$$

giving us the time

$$t_{Fluid} = \frac{\ell_\perp^2}{\kappa_\parallel} \frac{B_0^2}{\delta B_x^2}. \quad (41)$$

Therefore, for $t_\parallel < t_{Fluid} < t_{FL}$ the final state is the fluid limit because then we find that parallel transport becomes diffusive first and then we meet condition (7). If, on the other hand, $t_\parallel < t_{FL} < t_{Fluid}$ the field lines become diffusive before condition (7) is met. This means that we find compound sub-diffusion first. At even later time condition (7) is eventually met and diffusion is restored. The corresponding diffusion coefficient is then the CLRR limit. Using the formulas for the times discussed above, this means that we find CLRR diffusion for

$$\lambda_\parallel^2 < t_\parallel^2 < \ell_\parallel L_K \quad (42)$$

where the Kolmogorov length $L_K$ is given by Eq. (23). Thus for $\lambda_\parallel \ll \ell_\parallel$ we either find the fluid limit or CLRR diffusion. If additionally $\ell_\parallel \gg L_K$ we find the fluid limit but for $\ell_\parallel \ll L_K$ we get CLRR diffusion. It follows from Eq. (23) that $L_K / \ell_\parallel \approx K^{-2} \gg 1$ meaning that for small Kubo numbers we should always find CLRR diffusion.

For large Kubo numbers, on the other hand, it is more difficult to estimate the time $t_{FL}$ since it is unclear for which distance the field lines become diffusive. For pure two-dimensional turbulence corresponding to $K = \infty$, the field lines should become diffusive for $\langle (\Delta x)^2 \rangle \approx 2\ell_\perp^2$ corresponding to the transverse complexity condition. Therefore, the field lines should become diffusive for $z \approx L_K$. If we assume that until this condition is satisfied, the field lines are ballistic, we can estimate

$$2\ell_\perp^2 \approx z^2 \frac{\delta B_x^2}{B_0^2} \quad (43)$$

leading to

$$z \approx \sqrt{2\ell_\perp} \frac{B_0}{\delta B_x}. \quad (44)$$

If we assume that parallel transport is already diffusive when this condition is met, we find

$$t_{FL} \approx \frac{\ell_\perp^2}{\kappa_\parallel} \frac{B_0^2}{\delta B_x^2}. \quad (45)$$
We conclude that this time is identical compared to the time given by Eq. (41). Therefore, for large Kubo number turbulence, the time needed in order to obtain the fluid limit and CLRR diffusion are very similar. Therefore, the obtained diffusion coefficient can be somewhere between the fluid limit and the CLRR limit. For most applications in space and astrophysics, on the other hand, one deals with intermediate Kubo numbers and, thus, the perpendicular diffusion coefficient should be given by the CLRR limit. This statement is, of course, only true for short parallel mean free path. For long parallel mean free path, corresponding the high particle energies, one should obtain the FLRW limit.

4.8. A Composite Formula

A problem of the heuristic approach is that the obtained formulas are only valid in asymptotic limits. Since the two most important cases are CLRR and FLRW limits, we propose for the perpendicular mean free path defined via \( \lambda_\perp = 3 \kappa_\perp / v \), the formula (see Shalchi (2019a) and Shalchi (2020) for more details)

\[
\frac{\lambda_\perp}{\ell_\perp} = \frac{9}{16} \left[ \frac{1 + \frac{8 \kappa_{FL} \lambda_\parallel}{3 \ell_\perp^2}}{1} \right]^{2}.
\] (46)

Eq. (46) was chosen so that for \( \lambda_\parallel \to 0 \) we obtain Eq. (15) and for \( \lambda_\parallel \to \infty \) we get Eq. (9). Note that Eq. (46) does not contain the fluid limit given by Eq. (31) and, thus, it has some limitations.

4.9. Further Comments

The results obtained here are sometimes not comparable to previous results. First of all there are cases such as slab or two-dimensional (2D) turbulence. In the former case condition (7) is never satisfied leading to compound sub-diffusion as the final state. In the 2D case parallel transport is not diffusive (see, e.g., Arendt & Shalchi (2018)) violating the second rule. In some work (see, e.g., Matthaeus et al. (2003) and Shalchi et al. (2004)) a flat spectrum at large scales was used for the 2D modes. For this type of spectrum the ultra-scale is not finite also violating the second rule.

Sometimes the FLRW limit as given by Eq. (9) together with Eq. (24) is called the Kadomtsev & Pogutse limit. However, there is another limit which is sometimes also called the Kadomtsev & Pogutse limit (see, e.g., table 1 of Krommes et al. (1983)) which is

\[
\kappa_\perp = \kappa_{FL} \frac{\sqrt{\kappa_\parallel} \chi_\perp}{\ell_\perp}.
\] (47)

where \( \chi_\perp \) is the perpendicular diffusion coefficient due to collisions. However, in astrophysical scenarios there are no collisions and perpendicular diffusion is entirely caused by pitch-angle scattering and transverse complexity. Thus we set \( \chi_\perp = \kappa_\perp \) so that the aforementioned equation turns into \( \kappa_\perp = \kappa_{FL}^{2} \kappa_\parallel / \ell_\perp^{2} \) in perfect agreement with Eq. (15). Therefore, we conclude that in the collisionless case the second Kadomtsev & Pogutse limit and the Rechester & Rosenbluth limit are the same.

In order to determine the form of the field line diffusion coefficient, we have used the Kubo number. However, in some turbulence models (see, e.g., Goldreich & Sridhar (1995)) there is only one scale and, thus, the Kubo number becomes \( K = \delta B_z / B_0 \) often called the Alfvénic Mach number. The arguments presented above are still valid.

5. Comparison Between Theory and Simulations

As indicated by Table 1, the heuristic approach discussed in the current paper agrees with UNLT theories in most cases. The latter theory, on the other hand, was positively tested for a variety
of turbulence models (see, for instance, Shalchi & Hussein (2014) and Heusen & Shalchi (2017)). However, for large Kubo numbers and short parallel mean free paths there is only agreement between theory and simulations if one sets $a^2 = 1/3$. In the following we consider some examples in order to explore this matter in more detail.

5.1. Slab/2D Turbulence
As a first example we employ a two-component turbulence model where we approximate turbulence by using a superposition of slab and two-dimensional modes. In the case of 2D turbulence we have for the spectral tensor

$$P_{xx}(\vec{k}) = g^{2D}(k_{\perp}) \delta(k_{\parallel}) \frac{k^2_{\parallel}}{k^2_{\perp}}$$  \hspace{1cm} (48)

and for the spectrum we employ the form proposed by Shalchi & Weinhorst (2009), namely

$$g^{2D}(k_{\perp}) = \frac{2D(s, q)}{\pi} \delta B^2_{2D} \ell_{\perp} \frac{(k_{\perp} \ell_{\perp})^q}{[1 + (k_{\perp} \ell_{\perp})^2]^{(s+q)/2}}$$  \hspace{1cm} (49)

where we used the bendover scale $\ell_{\perp}$ denoting the turnover from the energy range to the inertial range. Furthermore, we used the inertial range spectral index $s$ as well as the energy range spectral index $q$. The normalization function is given by

$$D(s, q) = \frac{\Gamma[(s + q)/2]}{2\Gamma[(s - 1)/2] \Gamma[(q + 1)/2]}$$  \hspace{1cm} (50)

where we have used gamma functions. The ultra-scale is defined via Eq. (25). For 2D turbulence defined via Eq. (48), and the spectrum given by Eq. (49) this yields

$$L_U = \sqrt{\frac{s - 1}{q - 1} \ell_{\perp}}$$  \hspace{1cm} (51)

requiring $q > 1$ for the energy range spectral index and $1 < s < 2$ for the inertial range spectral index. With the parameter $a^2$ included, non-linear theories provide in the limit of short parallel mean free paths and 2D turbulence (see, e.g., Shalchi et al. (2004) and Zank et al. (2004))

$$\kappa_{\perp} = a^2 \frac{\delta B^2_{2D}}{B^2_0} \kappa_{\parallel}.$$  \hspace{1cm} (52)

According to the heuristic approach we expect CLRR diffusion in the considered parameter regime. Comparing Eqs. (52) and (26) yields $a = L_U/\ell_{\perp}$ and using Eq. (51) for the ultra-scale gives us

$$a^2 = \frac{s - 1}{q - 1}.$$  \hspace{1cm} (53)

Previously it was often assumed that $s = 5/3$ and $q = 3$ (see for instance Arendt & Shalchi (2018)) leading to $a^2 = 1/3$. Although it was already stated in Matthaeus et al. (2003) that $a^2 = 1/3$ is needed to achieve agreement between theory and simulations, the heuristic approach discussed here provides for the first time an explanation of this value. It has to be noted that this result was obtained for a specific form of the spectrum. Alternative spectra and the associated turbulence scales have been discussed in Matthaeus et al. (2007). For some of those spectra one obtains an ultra-scale larger than the bendover scale. In such cases, however, one would expect that the diffusion coefficient is close to the fluid limit and, thus, $a^2 \approx 1$. 

Figure 1. Results for a spectral tensor based on the critical balance condition of Goldreich & Sridhar (1995). For $\delta B/B_0 = 1$ the field line diffusion coefficient is in this case $\kappa_{FL} = 0.38\ell$. Shown are the simulations (dots) of Sun & Jokipii (2011), the result of diffusive UNLT theory for $a^2 = 1$ (solid line), the CLRR limit (dashed line) as given by Eq. (15), the FLRW limit (dotted line) as given by Eq. (9), and the composite formula (grey line) as given by Eq. (46). Reprinted with permission from The American Astronomical Society - Shalchi (2019a).

5.2. Goldreich & Sridhar Turbulence
Fig. 1 compares diffusive UNLT theory and the heuristic approach with simulations performed by Sun & Jokipii (2011). The latter authors employed a spectral tensor based on the critical balance condition of Goldreich & Sridhar (1995). In this particular case diffusive UNLT theory agrees very well with the simulations and the parameter $a^2$ is not required. The heuristic approach agrees with the simulations as well but is less accurate compared to systematic transport theory. This is what one would naturally expect since systematic theories should be more accurate than simple heuristic discussions. Nevertheless, our present understanding of perpendicular diffusion is confirmed via the comparison visualized in Fig. 1. Furthermore, we can easily see that FLRW and CLRR limits have to be understood as asymptotic limits.

5.3. Noisy Reduced MHD Turbulence
Another example is presented in Fig. 2 where we show a comparison with simulations performed for a spectral tensor based on the noisy reduced MHD (NRMHD) model of Ruffolo & Matthaeus (2013). In this particular case UNLT theory only agrees well with simulations if the correction factor $a^2$ is used and if we set $a^2 = 1/3$. This can be understood as the last remaining problem in systematic theories for perpendicular transport. The heuristic approach, however, agrees almost perfectly with the simulations. Very clearly we can see the turnover from CLRR diffusion to the FLRW limit.

6. Summary and Conclusion
The heuristic arguments discussed in the current paper cannot substitute systematic theories due to the lack of accuracy in the general case. Furthermore, there a certain limitations of the heuristic approach. For high energy particles, for instance, the Larmor radius can
Figure 2. Results for a spectral tensor based on the NRMHD model of Ruffolo & Matthaeus (2013). For $\delta B/B_0 = 1$ the field line diffusion coefficient is in this case $\kappa_{FL} = 0.23\ell_\perp$. Shown are the simulations (dots) of Shalchi & Hussein (2014), the results of diffusive UNLT theory for $a^2 = 1/3$ and $a^2 = 1$ (solid lines), the CLRR limit (dashed line) as given by Eq. (15), the FLRW limit (dotted line) as given by Eq. (9), and the composite formula (grey line) as given by Eq. (46). Reprinted with permission from The American Astronomical Society - Shalchi (2019a).

exceed the perpendicular bendover scale. In this case finite Larmor radius effects can become important (see, e.g., Shalchi (2015b) and Shalchi (2016)). Those are neglected in the heuristic approach discussed in the present paper. In the low energy regime, on the other hand, dynamical turbulence effects (see, e.g., Bieber et al. (1994)) become important and can no longer be neglected. This also includes wave propagation effects as, for instance, explained comprehensively in Schlickeiser (2002).

As discussed in this paper and in more detail in the review of Shalchi (2020), time-dependent UNLT theory works very well in most cases. There is good agreement between simulations, heuristic considerations, and systematic theories. However, there is one exception and that is perpendicular transport for the case of short parallel mean free paths, corresponding to strong pitch-angle scattering, and large Kubo numbers. Strictly speaking, UNLT theories provide the fluid limit in this regime whereas the heuristic approach provides a CLRR type of transport. This failure of the theory can be balanced out via the correction parameter $a^2$. Ideally, however, this should not be required. Therefore, one has to achieve a further improvement of the theory.

The non-linear guiding center (NLGC) theory is based on the approximation

$$\langle v_z(t)v_z(0)e^{i\vec{k}\cdot\vec{x}} \rangle \approx \langle v_z(t)v_z(0) \rangle \langle e^{i\vec{k}\parallel z} \rangle \langle e^{i\vec{k}\perp \cdot \vec{x}} \rangle$$

whereas UNLT theories are based on the approximation (see Shalchi (2017))

$$\langle v_z(t)v_z(0)e^{i\vec{k}\cdot\vec{x}} \rangle \approx \langle v_z(t)v_z(0) e^{ik_\parallel z} \rangle \langle e^{i\vec{k}\perp \cdot \vec{x}} \rangle.$$  

In the limit of small Kubo numbers the latter approximation becomes exact explaining why UNLT theories work so well in this case. However, for large Kubo number turbulence, approximations (54) and (55) are equal explaining why NLGC and UNLT theories provide very
similar results for large Kubo number turbulence. However, this also means that both theories are inaccurate in these cases and require the correction factor $a^2$. In the general case there can be a strong correlation between velocities and the perpendicular position of the particle. It was already shown in Shalchi (2005) that the assumption of uncorrelated positions and velocities omits compound sub-diffusion. Since CLRR diffusion comes after the sub-diffusive regime, theories based on approximation (54) are incomplete and do not provide sub-diffusion and CLRR diffusion for large Kubo numbers.

The next step in the theoretical description of perpendicular transport is, therefore, the formulation of a theory which does not rely on this approximation regardless of what the Kubo number is.

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