Unification in models with replicated gauge groups

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Abstract

We examine unification of gauge couplings in four dimensional renormalizable gauge theories inspired by the latticized (deconstructed) SM or MSSM in five dimensions. The models are based on replicated gauge groups, spontaneously broken to the diagonal subgroup. The analysis is performed at one-loop level, with the contribution from the heavy vector bosons included, and compared with the analogous results in the SM or MSSM. Unification at or above the diagonal breaking scale is discussed. We find that in the considered class of extensions of the SM(MSSM) unification is possible for a wide range of unification scales and with the similar accuracy as in the SM(MSSM). Unification above the diagonal breaking scale is particularly attractive: it is a consequence of the SM(MSSM) unification, but with the unification scale depending on the number of replications of the gauge group.
1 Introduction

In recent papers it has been demonstrated \cite{1,2} that there exists an interesting class of renormalizable four dimensional gauge theories that predict the same infrared physics as five-dimensional gauge theories. The gauge group of the four-dimensional theory is a product of $N$ copies of $SU(k)$ groups and is spontaneously broken at some scale $v$ to the diagonal subgroup by a set of link-Higgs fields in the bifundamental representations $(k_n, \bar{k}_{n+1})$ of the $SU(k)_n$ group. In addition to the massless gauge bosons of the unbroken $SU(k)$, the theory predicts the existence of massive gauge bosons transforming in the adjoint representations of the diagonal $SU(k)$. In the limit of large $N$ their mass spectrum and interactions are identical to those of the Kaluza Klein (KK) modes of the compactified 5d $SU(k)$ gauge theory (after proper mapping of the parameters of the two theories; in particular the scale $v$ of the symmetry breaking has to be of the same magnitude as the UV cut-off of the 5d theory).

Thus, the four-dimensional theory at distances larger than $1/v$ predicts the phenomena that are usually attributed to extra dimension (the appearance of massive KK modes, deviations from the Coulomb law). The obvious advantage of the four-dimensional set-up is its renormalizability and, moreover, the possibility to discuss physics also at distances shorter than $1/v$ where the full gauge symmetry is restored. These properties make this class of 4d theories interesting in its own, not just as an UV completion of 5d non-renormalizable theories. Many possibilities for constructing four dimensional models inspired by extra dimensional gauge theories open up, and a number of examples have already been presented \cite{8,9}.

In this paper we examine the running of gauge couplings in a model based on $N$-fold replication of $G_{SM} \equiv SU(3) \times SU(2) \times U(1)$ group of the Standard Model or its supersymmetric extension. The theory is parametrized by $N$ sets of gauge couplings $\alpha_1^{(n)}$, $\alpha_2^{(n)}$, $\alpha_3^{(n)}$ ($n = 1, \ldots N$). At some scale $v$ the full gauge symmetry $[G_{SM}]^N$ is broken to the diagonal subgroup, which we identify with the SM or MSSM group with the gauge couplings $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$. Above the diagonal breaking scale it is natural to discuss the evolution of the couplings $\alpha_i^{(n)}$ of each single factor of $[G_{SM}]^N$, as the full gauge symmetry is restored. Moreover, we can treat all the fields as massless and the running of the gauge couplings proceeds as in the $\overline{MS}$ scheme. Below the scale $v$ one could attempt to construct the effective theories at each stage of decoupling a single massive gauge boson but this is problematic, since the full gauge group is broken to the diagonal subgroup in one step. Instead, we can work with the full spontaneously broken $[G_{SM}]^N$ gauge theory and study the evolution of the couplings $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ that have the clear physical sense. At the scale $v$ the couplings $\tilde{\alpha}_i(v)$ and $\alpha_i^{(n)}(v)$ are connected to each other by the tree-level relation. Beyond the tree-level, at scales lower than $v$ the gauge couplings of the unbroken $SU(3) \times SU(2)$ subgroup can be defined as the full trilinear massless gauge boson effective vertices \cite{14,15}. For the unbroken $U(1)$ factor we define the gauge coupling as the effective vertex with the massless $U(1)$ gauge boson and two fermions. This allows us to include in the running the threshold effects which are sizeable each time the RGE scale is comparable to the gauge bosons masses.

In this paper we examine the evolution of the gauge couplings at one-loop level, including all contributions of the massive gauge and Higgs sectors. It is well known that in the SM and MSSM the two-loop effects are important for precision considerations and one may expect the same to be true in the discussed class of models. Working at one-loop we cannot conclude
about the absolute precision of unification but we can compare the one-loop unification in the replicated models with the one-loop unification in the SM or MSSM. This is the goal of the present paper and we hope, that our conclusions related to such a comparison hold also at the two-loop level.

One can study the gauge coupling unification at the diagonal breaking scale \( v \) or above it and these two cases are qualitatively different. The former option is similar to the power-law unification in five dimensions considered in [4, 5, 3], but in our case the calculation is performed in the renormalizable set-up and can be done rigorously. Unification of the diagonal gauge couplings \( \tilde{\alpha}(v) = \tilde{\alpha} \) means, of course, unification of all the gauge couplings of the \([G_{SM}]^N\), \( \alpha_i^{(n)}(v) = \alpha(v) \). With such a unification scenario, one can think about the theory above the scale \( v \) as e.g. \([SU(5)]^N\) or string theory. Our conclusions with regard to the unification at the diagonal breaking scale are similar to [4]. First of all, unification is possible for a wide range of \( v < 10^{16} \text{GeV} \), but the number of replications required for unification at a given value of \( v \) is almost uniquely fixed. In the five-dimensional language it means a correlation between the cut-off scale and the compactification radius \( R \). Moreover, in order to lower the unification scale sizeably we need a large number of replications. In the presence of \( N \) gauge bosons the appropriate loop expansion parameter is \( N\tilde{\alpha} \equiv \alpha \) rather than \( \tilde{\alpha} \) itself. We find that if we try to lower the unification scale below \( \sim 10^{10} \text{GeV} \), then \( \alpha \) becomes larger than 1 and the perturbative analysis cannot be trusted.

Our second conclusion concerns about the comparison with one-loop unification in the SM or MSSM, which, as is well known, require threshold corrections at the unification scale of order \( O(10\%) \) or \( O(1\%) \), respectively. We find that the similar conclusion holds in the replicated SM, but in the replicated MSSM, to lower the unification scale one needs somewhat larger threshold corrections.

Unification above the diagonal breaking scale \( v \) is a new interesting possibility in the discussed class of four-dimensional theories, with no counterpart in the five-dimensional theories [6]. In this case unification of the gauge couplings \( \alpha^{(n)}_{i}(M_{\text{GUT}}) = \alpha^{(n)}(M_{\text{GUT}}) \) does not necessarily imply \( \alpha^{(n)}(M_{\text{GUT}}) = \alpha(M_{\text{GUT}}) \). We consider the models such that the contribution to the beta functions of any two couplings \( \alpha^{(n)}_{i}(Q) \) and \( \alpha^{(m)}_{i}(Q) \) can differ by a contribution of complete representations of \( SU(5) \), e.g. matter fields. Since we are not interested in the values of the unified couplings, for our purpose the indices \( n \) will often be in the following. The unification can take place at \( M_{\text{GUT}} \gg v \) well below \( 10^{16} \text{GeV} \) and has several interesting aspects. The unification itself remains the prediction of the theory to the same extent as it is in the SM or MSSM, provided the beta functions for the couplings \( \alpha_{i} \) above \( v \) are the same as in SM(MSSM), up to full \( SU(5) \) multiplets. Furthermore, the unification takes place for a wide range of \( N \), with \( M_{\text{GUT}} \approx 10^{16/N} v^{(N-1)/N} \) and in this scenario the unification scale can be lowered sizeably even for small \( N \). Therefore, contrary to the previous case of unification at the diagonal breaking scale, \( v \) and \( N \) remain independent parameters of the theory. Finally, at least at one-loop, the GUT scale threshold corrections needed for unification remain of the same order of magnitude as in the SM or MSSM, respectively. Thus, the models with \( N \)-fold replication of the MSSM predict unification in the same sense as does the MSSM, but with the

\footnote{The other logical possibility - unification below the scale \( v \) - is very similar to the usual logarithmic running, since the heavy gauge bosons become operative not far below that scale.}
unification scale depending on \( N \).

## 2 The Model

In this section we define our set-up in which we calculate the running of gauge couplings. Our guideline is the assumption that the success of the MSSM unification is not accidental. Therefore we construct our model in such a way that we replicate the gauge and Higgs structures, which are crucial for successful unification in the MSSM. We do not replicate the matter fields which affect only the value of the gauge couplings at the unification point and not the unification itself. \footnote{Replicating matter fields would make the unified coupling \( \alpha \) bigger and perturbativity would be violated for rather small \( N \).}

The SM or MSSM gauge group is replicated \( N \)-times. We also add \( N - 1 \) bifundamental link-Higgs fields \( \Phi^{(n)} \) which fill out the \((5, \bar{5})\) representations of \( SU(5) \). Under the \( n \)-th and \((n + 1)\)-th \( SU(3) \times SU(2) \times U(1) \) these link fields split into \( \Phi_{33} \) in \((3, 1/\sqrt{15}, \bar{3}, -1/\sqrt{15})\), \( \Phi_{22} \) in \((2, -2/\sqrt{15}, \bar{2}, 2/\sqrt{15})\), \( \Phi_{32} \) in \((3, 1/\sqrt{15}, \bar{2}, 2/\sqrt{15})\) and \( \Phi_{23} \) in \((2, -2/\sqrt{15}, \bar{3}, -1/\sqrt{15})\). We choose to work with the model in which the first and the \( N \)-th \( SU(3) \times SU(2) \times U(1) \) group factors are not linked by the Higgs field.

Further, we assume that \( \Phi_{33}^{(n)} \) and \( \Phi_{22}^{(n)} \) link-Higgs fields acquire common vacuum expectation values \( \langle \Phi^{(n)} \rangle = v \). In the non-supersymmetric case this can be obtained by choosing the scalar potential for these Higgs fields of the form:

\[
V = \sum_n \text{Tr} \left[ (\Phi^{(n)} \dagger \Phi^{(n)}) - v^2 \right] \tag{1}
\]

In the supersymmetric case we can construct the model along the lines of \footnote{Replicating matter fields would make the unified coupling \( \alpha \) bigger and perturbativity would be violated for rather small \( N \).}. The D-term potential for the link Higgs fields is:

\[
V_D = \frac{1}{2} g^2 \sum_{n=1}^{N} \left( \text{Tr}[\Phi^{(n)} \dagger T^a \Phi^{(n)} - \Phi^{(n-1)} T^a \Phi^{(n-1)} \dagger] \right)^2 \tag{2}
\]

where \( \Phi^{(0)} = \Phi^{(N)} = 0 \) and \( T^a \) are the generators of the fundamental representation of the SM gauge group. This potential has a set of minima of the form \( \Phi^{(n)} = v_n I \) where \( v_n \) are its flat directions. Note that, with the link-Higgs fields alone, no renormalizable superpotential can be constructed. Therefore, the flat directions of the potential have to be lifted by adding soft susy breaking scalar mass terms. Without further explanation we assume that all the VEVs are non-zero and equal: \( v_n = v \). Expanding the link-Higgs field around the minimum, \( \Phi^{(n)} = v I + \phi^{(n)} \), we find that under the diagonal gauge group each of the link-Higgs fields split into the following three representations: two (real) adjoint \( h_n^a = \text{Tr}[T^a(\phi^{(n)} + (\phi^{(n)})\dagger)] \), \( G_n^a = i\text{Tr}[T^a(\phi^{(n)} - (\phi^{(n)})\dagger)] \) and the singlet \( h_n = \text{Tr}[\phi^{(n)}] \). The adjoints \( G_n^a \) are Goldstone bosons and become longitudinal components of the massive gauge bosons. The mass matrix of the physical scalars \( h_n^a \) is identical to the one of the gauge bosons and to the mass matrices of the Dirac fermions \( \Psi_n^a \) constructed from the superpartners of the gauge bosons \( x_n^a \) and the adjoint fermions coming from the superpartner of the link-Higgs fields \( \sqrt{2}\text{Tr}[T^a \Psi^{(n)}] \). Thus, at
each gauge boson mass level we have a full $\mathcal{N} = 2$ Yang-Mills supermultiplet. Of course, at one-loop the singlets do not affect the running of the gauge couplings and are irrelevant for our discussion.

The diagonal subgroup, to which the full gauge group $[G_{\text{SM}}]^N$ is broken by the VEVs of the link-Higgs fields, is identified with the gauge group of the SM (MSSM). The SM (MSSM) gauge couplings $\tilde{\alpha}_i$ are given in terms of the couplings $\alpha_i^{(n)}$ of $SU(3)_n \times SU(2)_n \times U(1)_n$ by the formula

$$\frac{1}{\tilde{\alpha}_i(v)} = \sum_{n=1}^{N} \frac{1}{\alpha_i^{(n)}(v)}. \quad (3)$$

For the sake of simplicity we assume that the gauge couplings at the scale $v$ are the same for all the $N$ group factors (which is the case in the models of deconstructed dimensions of $[1, 2]$) so that the formula for the low energy gauge couplings takes the form:

$$\frac{1}{\tilde{\alpha}_i(v)} = \frac{N}{\alpha_i(v)} \quad (4)$$

The gauge bosons corresponding to the diagonal $SU(3) \times SU(2) \times U(1)$ subgroup remain massless (down to the scale of electroweak symmetry breaking). The remaining gauge boson acquire masses equal to:

$$(M_n^{(i)})^2 = 16\pi\alpha_i(v)v^2\sin^2\left(\frac{n\pi}{2N}\right) \quad (5)$$

In the supersymmetric case the same masses are acquired by the corresponding Dirac gaugino and the scalar field.

We also replicate $N$-times the electroweak Higgs doublet(s), i.e. we assume the existence of $N$ scalars $H_n$ transforming as $(2)_{1/2}$ of the $n$-th $SU(2)$ group. Next, we introduce trilinear couplings to the link-Higgs fields of the form:

$$\mathcal{L} \sim m_n(H_n^\dagger \phi^{(n)} H_{n+1}) + \text{H.c.} \quad (6)$$

where $m_n$ are dimensionful couplings. The expectation value of the link-Higgs field produces a mass matrix for the electroweak Higgs. We choose the parameters $m_n$ so that the tower of Higgs boson masses is the same as that of $SU(2)$ gauge bosons masses:

$$(M_n)^2 = 16\pi\alpha_2(v)v^2\sin^2\left(\frac{n\pi}{2N}\right) \quad (7)$$

In the supersymmetric case we need $2N$ chiral supermultiplets $H_n^{(k)}$ with the weak hypercharges $Y = \pm 1/2$. The mass tower similar as in the non-supersymmetric case can be obtained by choosing the superpotential of the form: $W \sim \sum_n (\epsilon H_n^{(2)} \Phi^{(n)}_{12} H_{n+1}^{(1)} + \mu_n H_{n+1}^{(1)} H_n^{(2)}).$ Again, by tuning the parameters of the superpotential the tower of electroweak Higgs mass eigenstates with masses given by (7) can be constructed.

The matter fields, which are assumed to be those of the SM or MSSM, are not replicated and transform under one, say the first, of the $G_{\text{SM}}$ groups $\mathbf{3}$. The field $\phi$ in representation $r$
and charge $g$ of $G_{1\text{SM}}$, under the diagonal group transforms in the same representation with charge $\tilde{g} = \frac{g}{\sqrt{N}}$. In addition, $\phi$ couples to the whole tower of massive gauge bosons, and the charge is given by $\tilde{g} = \sqrt{2} \cos \left( \frac{n\pi}{2N} \right) \frac{g}{\sqrt{N}}$.

3 Unification at the diagonal breaking scale

In order to define what we understand by ‘successful unification’ let us first recall the one-loop renormalization group equations in the SM and MSSM. At one-loop the gauge couplings $\tilde{\alpha}_i$ of the three group factors of $G_{\text{SM}}$ run according to the equations:

$$\frac{1}{\tilde{\alpha}_i(Q)} = \frac{1}{\alpha_i(M_Z)} - \frac{b^{(i)}_0}{2\pi} \ln \left( \frac{Q}{M_Z} \right) + \delta_i$$

(8)

Here, $1/\tilde{\alpha}_i(M_Z) = (58.98 \pm 0.04, 29.57 \pm 0.03, 8.40 \pm 0.14)$ \cite{17} are the experimental values of the gauge couplings at the $Z^0$-pole and $b^{(i)}_0$ are the one-loop coefficients of the relevant beta-functions. In the limit in which all (s)particles are massless, they read $b_0 = \left( \frac{1}{10} + \frac{4}{3} N_g, -\frac{43}{6} + \frac{4}{3} N_g; -11 + \frac{4}{3} N_g \right)$ in the SM and $b_0 = \left( \frac{3}{5} + 2 N_g; -5 + 2 N_g; -9 + 2 N_g \right)$ in the MSSM, where $N_g$ is the number of generations. Threshold corrections (e.g. from heavy GUT gauge bosons \cite{11}) are represented by the parameters $\delta_i$.

In the bottom-up approach one can speak about the gauge coupling unification if in some range of scales $Q$ the couplings defined by eq. (8) with, in general, $Q$-dependent $\delta_i(Q)$ can take a common value $\alpha_i(Q) = \alpha_{\text{GUT}}$ for reasonably small values of $\delta_i(Q)$ (compared to $\alpha_{\text{GUT}}^{-1}$)\footnote{Whether there exists a unified model able to provide such values of $\delta_i(Q)$’s is a different question.}

The condition for the unification can be succinctly written as

$$\epsilon_{ijk} \left( \frac{1}{\tilde{\alpha}_i(M_Z)} + \delta_i \right) (b^{(j)}_0 - b^{(k)}_0) = 0$$

(9)

Putting in the experimental values for $\alpha_i(M_Z)$ and the beta-functions we get:

$$-41.1 + 3.8\delta_1 - 11.1\delta_2 + 7.3\delta_3 = 0 \quad \text{(SM)}$$
$$-0.9 + 4\delta_1 - 9.6\delta_2 + 5.6\delta_3 = 0 \quad \text{(MSSM)}$$

(10)

We see, that to achieve the gauge coupling unification at one-loop level we need the threshold corrections $\delta_i$ to be of order $10\% \alpha_{\text{GUT}}^{-1}$ in the SM, while in the MSSM we need only $\delta_i \sim 1\% \alpha_{\text{GUT}}^{-1}$. The latter number suggests that the gauge coupling unification in the MSSM can be very successful. Indeed, in precision calculations \cite{10,12} non-negligible two-loop corrections are approximately cancelled out by superpartner threshold corrections for superpartner masses $O(1 \text{ TeV})$ \cite{12}.

With the threshold corrections of the right order of magnitude, the unification scale can be estimated from the equation:

$$\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} - \frac{1}{2\pi} (b^{(1)}_0 - b^{(2)}_0) \ln \left( \frac{M_{\text{GUT}}}{M_Z} \right) + (\delta_1 - \delta_2) = 0$$

(11)
For the sake of concreteness, in this paper we shall always assume that \( \delta_1 = \delta_2 = 0 \) and that all threshold corrections are accounted for by \( \delta_3 \) (thus, the unification point is assumed to be where \( \alpha_1 \) and \( \alpha_2 \) intersect). Putting in the experimental numbers and the beta-function coefficients we get \( M_{\text{GUT}} \approx 1 \times 10^{13} \text{ GeV} \) in the SM and \( M_{\text{GUT}} \approx 2 \times 10^{16} \text{ GeV} \) in the MSSM.

We now turn to models with replicated gauge groups. In this section we investigate the possibility that unification occurs at the scale at which the \([SU(3) \times SU(2) \times U(1)]^N\) group is broken down to its diagonal subgroup by the Higgs boson VEV’s. In practice, it is convenient to choose for this scale the mass of the heaviest gauge boson \( M_{\text{GUT}} \equiv M_{N-1} = 2g_v \sin((N-1)\pi/2N) \). The massive gauge bosons corresponding to broken symmetries transform in the adjoint representation of the unbroken group and thus contribute to the running of the SM (MSSM) coupling constants. Working at one-loop level, we include them in the step-function approximation, in which only the particles lighter than the actual scale contribute to the coefficients of the beta-functions. For the gauge couplings defined as full vertices, the step-function decoupling procedure is justified in Appendix A. The one-loop renormalization group equations for scales \( Q \) such that \( M_{n(i)} < Q < M_{n+1} \) are

\[
\frac{1}{\hat{\alpha}_i(Q)} = \frac{1}{\hat{\alpha}_i(M_Z)} - \frac{b_0(i)}{2\pi} \ln \left( \frac{Q}{M_Z} \right) - \frac{\bar{b}(i)}{2\pi} \ln \left( \frac{Q^n}{M_1^{(i)} \cdots M_n^{(i)}} \right) + \delta_i \tag{12}
\]

The coefficients of the beta-functions for the massive gauge sector are (see Appendix A) \( \bar{b} = (\frac{1}{10}, -\frac{41}{6}, -\frac{21}{2}) \) in the SM and \( \bar{b} = (\frac{3}{5}, -3, -6) \) in the MSSM. Assuming, that with the threshold corrections \( \delta_i \) included, all \( \alpha_i \) take a common value at the scale \( M_{\text{GUT}} \) we find that the KK mass levels are the same for each gauge group: \( M_{n(1)} = M_{n(2)} = M_{n(3)} \). Therefore it is useful to define:

\[
F_N \equiv \ln \left( \frac{M_{\text{GUT}}^{N-1}}{M_1 \cdots M_{N-1}} \right) = \ln \left[ \frac{2^{N-1}}{\sqrt{N}} \sin^{N-1} \left( \frac{(N-1)\pi}{2N} \right) \right] \tag{13}
\]

We see that eqs. (8), after substituting \( \delta_i \rightarrow \delta_i - \frac{\bar{b}(i)}{2\pi} F_N \), become identical to eqs. (12). In the present case (which is qualitatively the same as the case of extra dimensions [4, 5]) we have a new source of threshold corrections, with the function \( F_N \) uniquely determined by the number of replications of the SM(MSSM) gauge group. The condition for unification is now:

\[
\epsilon_{ijk} \left( \frac{1}{\hat{\alpha}_i(M_Z)} + \delta_i - \frac{\bar{b}(i)}{2\pi} F_N \right) \left( b_0^{(j)} - b_0^{(k)} \right) = 0 \tag{14}
\]

Putting in the experimental numbers and the coefficients of the beta-functions we obtain:

\[
\begin{align*}
-41.1 + 3.8\delta_1 - 11.1\delta_2 + 7.3\delta_3 + 0.01 F_N & = 0 \quad (\text{SM}) \\
-0.9 + 4\delta_1 - 9.6\delta_2 + 5.6\delta_3 + 0.38 F_N & = 0 \quad (\text{MSSM}) \tag{15}
\end{align*}
\]

Let us investigate the impact of \( F_N \) on the unification. With \( \delta_1 = \delta_2 = 0 \), the unification scale can be determined from the equation:

\[
\frac{1}{\hat{\alpha}_1(M_Z)} - \frac{1}{\hat{\alpha}_2(M_Z)} - \frac{1}{2\pi} (b_0^{(1)} - b_0^{(2)}) \ln \left( \frac{M_{\text{GUT}}}{M_Z} \right) - \frac{1}{2\pi} (\bar{b}(1) - \bar{b}(2)) F_N = 0 \tag{16}
\]
The results in the supersymmetric case are summarized in the Table 1 in which the required threshold correction $\delta_3$ is also given. It is clearly seen that in order to lower the unification scale significantly one needs large $N$. We observe also that even for the unification scale as low as 10 TeV the required threshold corrections are of order 6%. However, for unification scales below $\sim 10^{10}$ GeV, the effective loop expansion parameter $\alpha \equiv N \tilde{\alpha}$ becomes larger than 1 which disfavours this possibility.

Table 1: Unification at the scale $v$ in the step-function approximation in the supersymmetric case.

| $M_{\text{GUT}}$ (GeV) | $v$ (GeV) | $N$ | $F$ | $\tilde{\alpha}_{\text{GUT}}$ | $\delta_3/\tilde{\alpha}_{\text{GUT}}$ | $\alpha$ |
|------------------------|-----------|-----|-----|------------------------------|---------------------------------|--------|
| $1.2 \times 10^9$      | $1.4 \times 10^4$ | 62  | 40.2| 47.6                         | -5.4%                           | 1.30   |
| $1.1 \times 10^6$      | $1.3 \times 10^5$ | 57  | 36.8| 45.6                         | -5.1%                           | 1.25   |
| $1.3 \times 10^8$      | $1.8 \times 10^7$ | 46  | 29.3| 41.3                         | -4.4%                           | 1.11   |
| $1.1 \times 10^{10}$   | $1.5 \times 10^9$ | 36  | 22.4| 37.3                         | -3.7%                           | 0.96   |
| $1.3 \times 10^{12}$   | $2.1 \times 10^{11}$ | 25  | 15.0| 33.0                         | -2.6%                           | 0.76   |
| $1.2 \times 10^{15}$   | $3.0 \times 10^{14}$ | 9   | 4.32| 26.8                         | -0.5%                           | 0.34   |

In the non-supersymmetric case the threshold corrections are little sensitive to the size of $F_N$ because of the small coefficient in front of $F_N$ in eq. (15). Thus, we can lower the unification scale down to $\sim 10$ TeV and still maintain the threshold corrections of the order of 10% $\alpha_{\text{GUT}}^{-1}$. The comment on large $N$ also applies in this case.

Given the large number of the heavy vector boson thresholds it is interesting to go beyond the step-function approximation and to investigate, by means of the numerical integration, the running with the full mass-dependent beta-functions (A.3). Although these mass effects are formally of higher order, one can check they are not enhanced by the accumulation of thresholds. The results of the numerical integration are presented in Fig. 1 (solid lines) and compared with those obtained in the step-function approximation. As could be expected, the difference between the two results increases with $N$ but we see, that the full mass effects are indeed of the same order of magnitude as the two-loop effects in the MSSM. Typical evolution of (inverse) gauge couplings in models with replicated gauge groups is shown in Fig. 2a and b for $v = 10^5$ GeV, $N = 60$ and $v = 10^{10}$ GeV, $N = 32$, respectively.

Concluding this section, in models with $N$-fold replication of the SM gauge and Higgs structures unification at the scale of symmetry breaking to the diagonal subgroup requires as large threshold corrections as the unification in the SM itself, $O(10\%)$. The threshold corrections are minimized only for strongly correlated values of $v$ and $N$. The latter remains true also for the replication of the MSSM. In this case the minimal threshold corrections required at one-loop level and in the step-function approximation remain small and for $M_{\text{GUT}} = 10^{10}$ GeV one needs $\delta_3 \sim 4\%$ for a successful unification. The impact of the full mass dependent beta-functions is non-negligible, but remains of the order of the known two-loop corrections in the MSSM. Thus, threshold effects of the massive gauge bosons should be considered on equal footing with two-loop corrections and other subdominant effects (e.g. e.g. corrections to the initial values...
Figure 1: $M_{\text{GUT}}$ and $\delta_3/\alpha_{\text{GUT}}^{-1}$ (in %) as functions of $N$ for replicated MSSM. Solid lines show the results of the numerical integration of the full mass-dependent beta-functions. Dashed lines correspond to the step-function approximation.
Figure 2: Evolution of (inverse) gauge couplings in replicated MSSM for $v = 10^5$ GeV, $N = 60$ (panel a) and $v = 10^{10}$ GeV, $N = 32$ (panel b). Solid lines show the results of the integration of the mass-dependent beta-functions. Dashed lines show the results obtained in the step-function approximation.
of the couplings in the mass dependent renormalization scheme) in a more precise study of unification. The unification scale below $10^{10}$ TeV requires very large $N$, so that the effective loop expansion parameter $N\tilde{\alpha}$ becomes larger than one.

4 Unification above the scale of symmetry breaking

If, for a fixed $v$, the number of replications $N$ is smaller than the value required for the unification at the scale $v$, the gauge couplings $\tilde{\alpha}_i$ will still differ at that scale. However, above the scale $v$, the full symmetry group $[G_{SM}]^N$ is restored. In our model, the gauge couplings $\alpha_i^{(a)}$ of all $N$ SM group factors have a common value $\alpha_i$ at the scale of the heaviest gauge boson $M_{N-1}^{(i)}$ and satisfy the relation:

$$\alpha_i(M_{N-1}^{(i)}) = N\tilde{\alpha}_i(M_{N-1}^{(i)})$$

in which $\tilde{\alpha}_i(M_{N-1}^{(i)})$ are determined by the running of gauge couplings below the diagonal breaking scale. Of course, now $\alpha_i(M_{N-1}^{(i)}) \neq \alpha_j(M_{N-1}^{(j)})$ for $i \neq j$. Recall that in the step-decoupling approximation:

$$\frac{1}{\tilde{\alpha}_i(M_{N-1}^{(i)})} = \frac{1}{\tilde{\alpha}_i(M_Z)} - \frac{b_0^{(i)}}{2\pi} \ln \left( \frac{M_{N-1}^{(i)}}{M_Z} \right) - \frac{\tilde{b}^{(i)}}{2\pi} F_N$$

Above the scale $M_{N-1}^{(i)}$ each of the gauge couplings $\alpha^{(i)}$ runs as in ordinary (super-)Yang-Mills theory coupled to matter. Treating all gauge, matter and Higgs fields (including the link-Higgs fields) as massless above that scale we have:

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i(M_{N-1}^{(i)})} - \frac{\tilde{b}^{(i)}}{2\pi} \ln \left( \frac{Q}{M_{N-1}^{(i)}} \right) + \delta_i \quad (Q > M_{N-1}^{(i)})$$

The beta-function coefficients of all but the first gauge group $G_{SM}$ are $\tilde{b}_0 = (53/30, -11/2, -28/3)$ in the non-supersymmetric case and $\tilde{b}_0 = (28/5, 0, -4)$ in the supersymmetric case and include the contribution from the bifundamental Higgs fields. Combining eqs. (17), (18) and (19) we get:

$$\frac{1}{\alpha_i(Q)} = \frac{1}{N} \left( \frac{1}{\alpha_i(M_Z)} - N\tilde{b}^{(i)}_0 \frac{b_0^{(i)}}{2\pi} \ln \left( \frac{Q}{M_{N-1}^{(i)}} \right) - \frac{b_0^{(i)}}{2\pi} \ln \left( \frac{Q}{M_Z} \right) - \frac{\tilde{b}^{(i)}}{2\pi} F_N + N\delta_i \right)$$

To obtain a transparent formula for the unification scale we need to assume that the heaviest gauge bosons corresponding to all the gauge factors are approximately equal $M_{N-1}^{(1)} = M_{N-1}^{(2)} = M_{N-1}^{(3)} = M_{N-1} = M_G$ (in numerical calculations, whose results are shown in the Fig. 3, they usually differ by a factor of $2 - 5$). Then assuming $\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT}) \equiv \alpha$ we get the equations:

5The coefficient of the beta-function of the first $G_{SM}$ factor gets, in addition, the contribution from the (MS)SM matter fields.
\[
\frac{1}{\hat{\alpha}_1(M_Z)} - \frac{1}{\hat{\alpha}_2(M_Z)} = -\frac{1}{2\pi}(b_0^{(1)} - b_0^{(2)}) \ln \left( \frac{M_U^N}{M_{N-1} M_Z} \right) - \frac{1}{2\pi}(\tilde{b}^{(1)} - \tilde{b}^{(2)}) F_N + N(\delta_1 - \delta_2) = 0
\]

\[
\frac{1}{\hat{\alpha}_2(M_Z)} - \frac{1}{\hat{\alpha}_3(M_Z)} = -\frac{1}{2\pi}(b_0^{(2)} - b_0^{(3)}) \ln \left( \frac{M_U^N}{M_{N-1} M_Z} \right) - \frac{1}{2\pi}(\tilde{b}^{(2)} - \tilde{b}^{(3)}) F_N + N(\delta_2 - \delta_3) = 0 \tag{21}
\]

We have used the fact that \(\tilde{b}_0^{(i)} - \tilde{b}_0^{(j)} = b_0^{(i)} - b_0^{(j)}\) because the bifundamental Higgs fields form complete representations of SU(5). These equations have the same form as in the case of unification at the scale \(v\) but now \[8\]:

\[
M_{\text{GUT}} \rightarrow M_{\text{GUT}}^{N} \over M_{N-1} \tag{22}
\]

We also have \(\delta_k \rightarrow N\delta_k\). This means that threshold corrections needed for successful unification, in absolute numbers, may be much smaller. But we always express the threshold corrections in the units of \(\alpha^{-1}\) at the GUT scale which is also approximately \(N\)-times smaller than in the case of unification at the scale \(v\).

In Fig. [3] we present the results of numerical calculations in the step-decoupling approximation (left panels) and using full mass dependent beta functions (right panels). As expected from the approximate eq. (21), for a chosen value of \(v\), the gauge couplings do unify for any number \(N\) of replications. No correlation between \(v\) and \(N\) is needed and, in this sense, unification of the gauge couplings is the prediction of the model to the same extent as in the MSSM itself, but with \(N\) dependent \(M_{\text{GUT}}\). A small number of replications is sufficient for reaching unification at low scale (for low \(v\); of course \(v\) is constrained from below \(v \gtrsim 1\) TeV by experimental limits on massive vector bosons). The precision of one-loop unification is as satisfactory as in the MSSM. The effects of the full mass dependent beta-functions are very small for not too large value of \(N\), say \(N < 10\). Moreover, for such small \(N\), \(\alpha_{\text{GUT}}\) stays well within the perturbative region. For large \(N\) the couplings unify just above the diagonal breaking scale and the unification scale and the threshold corrections are similar as in the case discussed in the previous section.

5 Conclusions

We have discussed the evolution of the gauge couplings in renormalizable gauge theories based on \(N\)-fold replication of the SM(MSSM) gauge groups, spontaneously broken at some scale \(v\) to the diagonal subgroup identified with the SM(MSSM) group. A systematic one-loop calculation has been performed below and above the scale \(v\).

Unification at the diagonal breaking scale \(v\) is possible down to \(v \sim 10^5\) GeV; for any value of \(v\) there exists a narrow range of \(N\), that gives unification at that scale. This result has the obvious counterpart in the power-law unification in five dimensions [4, 5].

Unification above the diagonal breaking scale is a new feature of the considered class of models [3], with no counterpart in extra dimensions. It is particularly attractive as it does not require any correlation between \(v\) and \(N\). It is a prediction of the model to the same extent as
Figure 3: $M_{\text{GUT}}$, $\alpha_{\text{GUT}}$ and the GUT scale threshold corrections $\delta_3/\alpha_{\text{GUT}}^{-1}$ as functions of $N$ for the unification above the diagonal breaking scale in the replicated MSSM. Left (right) panels show the results in the step-function approximation (with the mass-dependent beta functions). Various lines correspond to $v = 10^4$ (solid), $10^6$ (dashed), $10^8$ (dotted), $10^{10}$ GeV (dash-dotted).
unification in the MSSM and is a consequence of the latter. However, in this case the unification scale is $M_{\text{GUT}} \approx v^{(N-1)/N} 10^{16/N}$, and can be very low even for small $N$.

The accuracy of one-loop unification in the considered class of models, particularly unification above the diagonal breaking scale and with not too large number of replications, is as satisfactory as in the MSSM. Of course, low unification scale raises such problems as the proton decay, but they do not rule out the idea of low scale unification [4, 16].

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**Appendix A**

In this appendix we specify our method of computing the RGE running of the gauge couplings and give the formulae for the full mass-dependent beta-functions which automatically take into account threshold effects due to the massive gauge bosons. For the unbroken non-abelian group factors it is convenient to define the scale dependent gauge couplings as the scalar factor of the full trilinear massless vector boson vertex.

$$V(p_1, p_2) f^{abc} g_{\mu\nu\rho} + \text{other Lorentz structures}$$

$$g_{\mu\nu\rho} = g_{\mu\nu}(p_1 - p_2)_{\rho} + g_{\nu\rho}(p_2 - p_3)_{\mu} + g_{\rho\mu}(p_3 - p_1)_{\nu}$$

**Figure 4**: 1-PI three-gluon vertex.

At one-loop the beta-functions are given by the expression:

$$\beta(q^2) = q^2 \frac{\partial}{\partial q^2} \left( 2V(q^2) - 3g\Pi(q^2) \right)$$  \hspace{1cm} (A.1)

In eq. (A.1) $V(q^2)$ is the sum of one-loop contributions to the vertex defined in fig. 4 evaluated at euclidean external momenta $p_1^2 = p_2^2 = p_3^2 = -q^2 < 0$, and $\Pi(q^2)$ is given by the sum of the one-loop vacuum polarization diagrams contributing to the vector boson self-energy $-i(g_{\mu\nu}p^2 - p_{\mu}p_{\nu})\Pi(p^2)$, also evaluated at $p^2 = -q^2 < 0$. 

13
The vector boson self energy gets the following contribution:

| Field Type             | Contribution                                                                 |
|------------------------|------------------------------------------------------------------------------|
| Massless gauge boson   | $\Pi = g^2 \frac{C_2(G)}{(4\pi)^2} \left( \frac{5}{2} B_0(m, m) - \frac{1}{9} \right)$ |
| Massive gauge boson    | $\Pi = g^2 \frac{C_2(G)}{(4\pi)^2} \left( \frac{3}{2} B_0(m, m) - 2\left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) \right)$ |
| Real scalar            | $\Pi = g^2 \frac{C_2(r)}{(4\pi)^2} \left( -\frac{1}{2} B_0(m, m) - \frac{2}{3} \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) + \frac{1}{9} \right)$ |
| Weyl fermion           | $\Pi = g^2 \frac{C_2(r)}{(4\pi)^2} \left( -\frac{2}{3} B_0(m, m) + \frac{4}{3} \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) - \frac{4}{9} \right)$ |

The vertex correction $V(q^2)$ gets the following contributions:

| Field Type             | Contribution                                                                 |
|------------------------|------------------------------------------------------------------------------|
| Massless gauge boson   | $V = g^3 \frac{C_2(G)}{(4\pi)^2} \left( \frac{2}{3} B_0(m, m) - \frac{1}{36} q^2 C_0(m, m, m) - \frac{31}{12} \right)$ |
| Massive gauge boson    | $V = g^3 \frac{C_2(G)}{(4\pi)^2} \left( \frac{1}{2} B_0(m, m) - 2m^2 C_0(m, m, m) + \frac{1}{12} q^2 C_0(m, m, m) - \frac{33}{12} \right)$ |
| Real scalar            | $V = g^3 \frac{C_2(r)}{(4\pi)^2} \left( -\frac{1}{6} B_0(m, m) - \frac{1}{3} m^2 C_0(m, m, m) + \frac{1}{9} q^2 C_0(m, m, m) - \frac{1}{6} \right)$ |
| Weyl fermion           | $V = g^3 \frac{C_2(r)}{(4\pi)^2} \left( -\frac{2}{3} B_0(m, m) + \frac{2}{3} m^2 C_0(m, m, m) + \frac{8}{9} q^2 C_0(m, m, m) - \frac{2}{3} \right)$ |

Using the above formulae we can easily calculate the contribution of the various fields to the mass dependent beta-function. $A_\mu^{(0)}, A_\mu^{(a)}, \phi, \psi$ denote massless gauge boson, massive gauge boson, real scalar field and Weyl fermion, respectively.
\[ A_{\mu}^{(0)} \quad \beta = g^3 C_2(G) 4^2 \frac{\partial}{\partial q^2} \left( -\frac{11}{3} B_0(m, m) \right) \]

\[ A_{\mu}^{(n)} \quad \beta = g^3 C_2(G) 4^2 \frac{\partial}{\partial q^2} \left( -\frac{7}{2} B_0(m, m) + 6 \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) + \left( -4m^2 + \frac{1}{6} g^2 \right) C_0(m, m, m) \right) \]

\[ \phi \quad \beta = g^3 C_2(r) 4^2 \frac{\partial}{\partial q^2} \left( + \frac{1}{6} B_0(m, m) + 2 \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) + \left( -\frac{2}{3} m^2 + \frac{2}{3} g^2 \right) C_0(m, m, m) \right) \]

\[ \psi \quad \beta = g^3 C_2(r) 4^2 \frac{\partial}{\partial q^2} \left( + \frac{2}{3} B_0(m, m) - 4 \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, m) + \frac{m^2}{q^2} \right) + \left( \frac{4}{3} m^2 + \frac{8}{9} g^2 \right) C_0(m, m, m) \right) \]

\( C_2(r) \) is the quadratic Casimir in the representation \( r \), by \( G \) we denote the adjoint representation. Recall that \( C_2(G) = N \) for the \( SU(N) \) group and \( C_2(N) = \frac{1}{2} \) for the fundamental representation of \( SU(N) \). The functions \( A, B_0, C_0 \) are defined as in [13]

\[
A(m) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
B_0(p_1^2, m_1, m_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m_1^2][(k + p_1)^2 - m_2^2]} \\
C_0(p_1, p_2, m_1, m_2, m_3) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m_1^2][(k + p_1)^2 - m_2^2][(k + p_1 + p_2)^2 - m_3^2]}
\]

In the formulae for the beta functions the above functions are evaluated at \( p_1^2 = p_2^2 = -2p_1p_2 = -q^2 \).

It is worthwhile to show that far above and far below the mass threshold our beta functions reduce to their \( MS \) scheme counterparts. For \( -q^2 \gg m^2 \) we need the asymptotic expansions:

\[
A(m) = m^2(-\eta - 1 + \ln(m^2)), \quad B_0(m, m) = -\eta + \ln(p^2) + \mathcal{O}\left(\frac{m^2}{q^2}\right), \quad q^2 C_0(m, m, m) \sim \text{const} + \mathcal{O}\left(\frac{m^2}{q^2}\right)
\]

(where \( \eta = \frac{2}{4d} + \ln(4\pi) - \gamma_E \)). Thus, up to \( \mathcal{O}\left(\frac{m^2}{q^2}\right) \) corrections, far above threshold the beta function is given by the coefficients multiplying the \( B_0(m, m) \) function. The same coefficient multiplies the \( \frac{2}{4d} \) singularity which yields the beta function in the \( MS \) scheme. For \( -q^2 \ll m^2 \) we need \( B_0(m, m) = -\eta - 1 + \ln(m^2) + \mathcal{O}\left(\frac{m^2}{q^2}\right) \) and \( C_0 \sim \mathcal{O}\left(\frac{m^2}{q^2}\right) \). It is straightforward to see that all terms of order \( \mathcal{O}\left(\frac{m^2}{q^2}\right) \) and \( \mathcal{O}(1) \) cancel, thus far below threshold the beta function is vanishing up to \( \mathcal{O}\left(\frac{m^2}{q^2}\right) \) corrections.

For the \( U(1) \) factor we of course do not have the trilinear gauge boson couplings. In this case it is convenient to define the scale dependent gauge coupling as the scalar factor of the full vertex with two fermions (e.g. electrons).

The one-loop beta function is given by:

\[
\beta(q^2) = q^2 \frac{\partial}{\partial q^2} \left( 2V(q^2) - 2g\Sigma(q^2) - g\Pi(q^2) \right)
\]  

In eq. (A.2) \( V(q^2) \) is the sum of one-loop contributions to the vertex defined in fig. 1; \( \Pi(q^2) \) is the the vector boson self-energy and \( \Sigma(q^2) \) is the fermion self-energy; all diagrams being
evaluated at euclidean external momenta $p_1^2 = p_2^2 = p_3^2 = -q^2 < 0$. $Y$ is the SU(5) normalized hypercharge of the fermion.

The contributions to the $U(1)$ gauge boson self-energy are similar to the non-abelian case, with the group factors $(C_2(G), C_2(r))$ replaced by $(0, Y_r^2)$, respectively. The contributions to the fermion self-energy are:

| Type                      | Contribution                                                                 |
|---------------------------|------------------------------------------------------------------------------|
| Massless gauge boson      | $\Sigma = g^2 \frac{Y^2}{(4\pi)^2} (\frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, 0))$ |
| Massless gaugino          | $\Sigma = g^2 \frac{Y^2}{(4\pi)^2} (\frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, 0))$ |
| Massive gauge boson       | $\Sigma = g^2 \kappa^2 \frac{Y^2}{(4\pi)^2} \left( \frac{A(m)}{q^2} - \frac{m^2}{q^2} B_0(m, 0) \right)$ |
| Massive gaugino           | $\Sigma = g^2 \kappa^2 \frac{Y^2}{(4\pi)^2} \left( \frac{A(m)}{q^2} + \frac{m^2}{q^2} B_0(m, 0) \right)$ |

The factors $\kappa = \sqrt{2\cos[\frac{n\pi}{2N}]}$ arise because the massive gauge bosons at different mass levels have different couplings to the matter fields. $Y_r$ is the SU(5) normalized hypercharge of the (scalar/Weyl fermion) flying in the loop. The contributions to the 1PI vertex corrections are:
Massless gauge boson
\[ \Sigma = g^3 \frac{Y^2}{(4\pi)^2} \left( -B_0(0,0) - 1 + \frac{1}{3} p^2 C_0(0,0,0) \right) \]

Massless gaugino
\[ \Sigma = g^3 \frac{\kappa^2 Y^2}{(4\pi)^2} \left( -B_0(0,0) + 1 + \frac{2}{3} p^2 C_0(0,0,0) \right) \]

Massive gauge boson
\[ \Sigma = g^3 \frac{\kappa^2 Y^2}{(4\pi)^2} \left( -\frac{4}{3} B_0(m,0) + \frac{1}{3} B_0(0,0) - 1 + \frac{1}{3} p^2 C_0(m,0,0) \right) \]

Massive gaugino
\[ \Sigma = g^3 \frac{\kappa^2 Y^2}{(4\pi)^2} \left( -\frac{2}{3} B_0(m,0) - \frac{1}{3} B_0(0,0) + 1 + (\frac{2}{3} p^2 - 2m^2) C_0(m,0,0) \right) \]

Finally, we present the contributions of various fields to the mass dependent beta-functions of the U(1) gauge coupling. \( A^{(0)}_\mu, \chi^{(0)}, A^{(n)}_\mu, \chi^{(n)}, \phi, \psi \) denote massless gauge boson, massless gaugino, massive gauge boson, massive gauginos, real scalar field and Weyl fermion, respectively.

| Field | \( \beta \) |
|-------|-------------|
| \( A^{(0)}_\mu \) | \( \beta = 0 \) |
| \( \chi^{(0)} \) | \( \beta = 0 \) |
| \( A^{(n)}_\mu \) | \( \beta = g^3 Y^2 \kappa^2 \mu^2 \frac{\partial}{\partial \mu^2} \left( -\frac{2}{3} B_0(m,0) + \frac{2}{3} B_0(0,0) + 2 \left( \frac{A(m)}{\mu^2} - \frac{m^2}{\mu^2} B_0(m,0) \right) + \frac{2}{3} q^2 C_0(m,0,0) \right) \) |
| \( \chi^{(n)} \) | \( \beta = g^3 Y^2 \kappa^2 \mu^2 \frac{\partial}{\partial \mu^2} \left( \frac{2}{3} B_0(m,0) - \frac{2}{3} B_0(0,0) - 2 \left( \frac{A(m)}{\mu^2} - \frac{m^2}{\mu^2} B_0(m,0) \right) + \left( \frac{4}{3} q^2 - 4m^2 \right) C_0(m,0,0) \right) \) |
| \( \phi \) | \( \beta = g^3 \frac{Y^2}{(4\pi)^2} \mu^2 \frac{\partial}{\partial \mu^2} \left( +\frac{1}{6} B_0(m,m) + \frac{2}{3} \left( \frac{A(m)}{\mu^2} - \frac{m^2}{\mu^2} B_0(m,m) + \frac{m^2}{\mu^2} \right) \right) \) |
| \( \psi \) | \( \beta = g^3 \frac{Y^2}{(4\pi)^2} \mu^2 \frac{\partial}{\partial \mu^2} \left( +\frac{2}{3} B_0(m,m) - \frac{4}{3} \left( \frac{A(m)}{\mu^2} - \frac{m^2}{\mu^2} B_0(m,m) + \frac{m^2}{\mu^2} \right) \right) \) |

Now we are ready to write the expressions for the full beta-functions in the supersymmetric version of our model. Apart from the (approximately massless) MSSM spectrum with \( N_g \) generations we have a tower of N-1 gauge multiplets of \( \mathcal{N} = 2 \) supersymmetry consisting of a vector boson, two Weyl fermions and a scalar, all in the adjoint represantion of the SM gauge group. The members of this multiplet have masses given by \( M_i^{(0)} = 2g(v) \sin(n\pi/2N) \). There are also \( \tilde{N}_h \) towers of \((N-1) SU(2)\) doublet chiral supermultiplets (a scalar and a Weyl fermion) which for simplicity at every level are assumed to have the same mass as the SU(2) massive supermultiplet.
The beta-functions turn out to be:

\[
\beta^{(k)}(q^2) = g^3 \frac{1}{(4\pi)^2} \left( b_0^{(k)} + q^2 \frac{\partial}{\partial q^2} \sum_{n=1}^{N-1} \tilde{b}^{(k)}(M_n^{(k)}) \right)
\]

\[
b_0^{(3)} = -9 + 2N_g
\]

\[
b_0^{(2)} = -5 + 2N_g
\]

\[
b_0^{(1)} = \frac{3}{10} + 2N_g
\]

\[
\tilde{b}^{(3)}(m) = -6B_0(m,m) - 6m^2C_0(m,m,m) + \frac{13}{2}q^2C_0(m,m,m)
\]

\[
\tilde{b}^{(2)}(m) = (-4 + \frac{1}{2}N_h)B_0(m,m) - 4m^2C_0(m,m,m) + (\frac{13}{2} + \frac{3}{2}N_h)q^2C_0(m,m,m)
\]

\[
\tilde{b}^{(1)}(m) = \frac{3}{10}N_hB_0(m,m) + \kappa^2(2q^2 - 4m^2)C_0(m,0,0)
\]

(A.3)

In the step-decoupling approximation the contribution to the beta coefficients from the massive modes is given by the coefficient multiplying \(B_0\) for \(-q^2 > m^2\) and is vanishing for \(-q^2 < m^2\).

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