Next-to-Leading Order QCD Corrections to $\Delta F = 2$ effective Hamiltonians

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Abstract

The most general QCD next-to-leading anomalous-dimension matrix of all four-fermion dimension-six $\Delta F = 2$ operators is computed. The results of this calculation can be used in many phenomenological applications, among which the most important are those related to theoretical predictions of $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixing in several extensions of the Standard Model (supersymmetry, left-right symmetric models, multi-Higgs models, etc.), to estimates the $B^0_s - \bar{B}^0_s$ width difference, and to the calculation of the $O(1/m_b^2)$ corrections for inclusive $b$-hadron decay rates.
1 Introduction

Theoretical predictions of several measurable quantities, which are relevant in $K$-, $D$- and $B$-meson phenomenology, depend crucially on the matrix elements of some $\Delta F = 2$ four-fermion operators. Examples are given by FCNC effects in SUSY extensions of the Standard Model \[1\]-\[2\] (or other models such as left-right symmetric \[3\] or multi-Higgs models), by the $B^0_s - \bar{B}^0_s$ width difference \[4\], and by the $O(1/m_b^3)$ corrections in inclusive $b$-hadron decay rates (which actually depend on the matrix elements of several four-fermion $\Delta F = 0$ operators \[5\]). In all these cases, the relevant operators have the form

$$Q = C^{\alpha\beta\rho\sigma} (\bar{b}_\alpha \Gamma q_\beta)(\bar{b}_\rho \Gamma q_\sigma),$$

where $\Gamma$ is a generic Dirac matrix acting on (implicit) spinor indices; $\alpha - \sigma$ are colour indices and $C^{\alpha\beta\rho\sigma}$ is either $\delta^{\alpha\beta}\delta^{\rho\sigma}$ or $\delta^{\alpha\sigma}\delta^{\rho\beta}$ (for the $1/m_b^3$ corrections to the inclusive decay rates the flavour structure has the form $(\bar{b}q)(\bar{q}b)$).

All the operators discussed in this paper appear in some “effective” theory, obtained by using the Operator Product Expansion (OPE). As a consequence, in all cases, three steps are necessary for obtaining physical amplitudes from their matrix elements:

i) matching of the original theory to the effective one at some large energy scale;

ii) renormalization-group evolution from the large energy scale to a low scale suitable for the calculation of the hadronic matrix elements (typically 1–5 GeV);

iii) non-perturbative calculation of the hadronic matrix elements.

In this paper we present a calculation of the two-loop anomalous dimension matrix relevant for $\Delta F = 2$ transition amplitudes. This matrix can be used for the Next-to-Leading Order (NLO) renormalization-group evolution of the Wilson coefficient functions of the effective theory from the large to the small energy scale, step ii). The anomalous dimension matrix includes leading and sub-leading corrections of order $\alpha_s$ and $\alpha_s^2$. The calculation has been performed in naive dimensional regularization (NDR) and we give re-

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1 All the formulae of this paper refer to the $\Delta B = 2$ case. Their extension to generic $\Delta F = 2$ transitions is straightforward.
sults for different renormalization schemes. We have also verified that the results obtained in different schemes are compatible. We give many details on the calculation itself, on the definition of the renormalized operators, on the relation between different renormalization schemes (and on the rôle of the corresponding counterterms), on the gauge invariance of the final results etc. We also present a list containing the contribution of all the Feynman diagrams to the anomalous dimension matrix. The list may be useful to check our results and for further applications. In this paper we have preferred to give the results for the one- and two-loop anomalous-dimension matrix with as many details as possible and postpone the phenomenological applications of the results given here to further publications. In section 5, the reader who is not interested in the theoretical and technical aspects of the calculations can find the final results for the anomalous dimension matrix of the operator basis defined in eq. (13) of subsec. 2.3.

Besides presenting the results for the $\Delta B = 2$ ($\Delta F = 2$) operators discussed here, we take the opportunity to clarify several issues related to the regularization and renormalization dependence of the operators and of the corresponding Wilson coefficients. In particular, we discuss in detail the problems related to the precise definition of the so-called "\MS" schemes", which, for composite operators, are not uniquely defined, even for a given regularization [6]–[13]. We also examine the equivalence, and differences, of the most popular renormalization schemes, and the subtleties related to Regularization-Independent renormalization schemes (RI) [14].

The paper is organized as follows. In sec. 2 we introduce the operators relevant for physical applications and define the operator basis for which the anomalous dimension matrix will be given; a general discussion on the Wilson coefficients, their scheme-dependence and renormalization-group evolution will be presented in sec. 3; the strategy for the calculation of the anomalous dimension matrix in the \MS and RI schemes, to be defined below, is given in sec. 4; the final results, together with the one-loop matrices necessary to change renormalization scheme, are also given for some relevant cases in sec. 5.
2 Four-fermion operators

We start this section by introducing the operators that we have in mind in view of future applications; we then illustrate the chiral and Fierz properties of the relevant operators which are used to derive the general form of the mixing matrix.

2.1 Operators relevant for physical applications

In this subsection, we present a list of operators which enter the calculation of the physical quantities mentioned in the introduction:

1) FCNC in SUSY extensions of the Standard Model:

   For $K$- and $B$-meson transitions, these effects have been recently analyzed in detail in a series of papers, see for example refs. [1]-[2]. The relevant operators which enter the effective Hamiltonian are

   \[ Q_1 = \bar{b}^\alpha \gamma_\mu (1 - \gamma_5) q^\alpha \bar{b}^\beta \gamma_\mu (1 - \gamma_5) q^\beta , \]
   \[ Q_2 = \bar{b}^\alpha (1 - \gamma_5) q^\alpha \bar{b}^\beta (1 - \gamma_5) q^\beta , \]
   \[ Q_3 = \bar{b}^\alpha (1 - \gamma_5) q^\beta \bar{b}^\beta (1 - \gamma_5) q^\alpha , \]
   \[ Q_4 = \bar{b}^\alpha (1 - \gamma_5) q^\alpha \bar{b}^\beta (1 + \gamma_5) q^\beta , \]
   \[ Q_5 = \bar{b}^\alpha (1 - \gamma_5) q^\beta \bar{b}^\beta (1 + \gamma_5) q^\alpha , \]

   together with the operators $\tilde{Q}_{1,2,3}$ which can be obtained from the operators $Q_{1,2,3}$ by the exchange $(1 - \gamma_5) \leftrightarrow (1 + \gamma_5)$.

2) The $B_s$–$\bar{B}_s$ width difference $\Delta \Gamma_{B_s}$:

   At lowest order in $1/m_b$, by using the OPE, the width difference $\Delta \Gamma_{B_s}$ can be written in terms of two $\Delta B = 2$ operators [4]

   \[ Q = \bar{b} \gamma_\mu (1 - \gamma_5) s \bar{b} \gamma_\mu (1 - \gamma_5) s , \]
   \[ Q_S = \bar{b} (1 - \gamma_5) s \bar{b} (1 - \gamma_5) s . \]

   where, since the fermion bilinears are colour singlets $(\bar{b} \gamma_\mu (1 - \gamma_5) s = \bar{b} \gamma_\mu (1 - \gamma_5) s^\alpha)$, the colour indices have not been shown explicitly.

   \footnote{Here and in the following we adopt the same notation as in the original papers.}
3) Heavy-hadrons lifetimes ($\tau_{B}$, $\tau_{B_s}$, $\tau_{\Lambda_b}$):

In this case, the $1/m_b^3$ corrections to the lifetime, due to Pauli interference and $W$-exchange, can be written in terms of four operators [5]

\[
\begin{align*}
O_{V-A}^q &= \bar{b}\gamma_{\mu}(1 - \gamma_5)q \, \bar{q}\gamma^{\mu}(1 - \gamma_5)b, \\
O_{S-P}^q &= \bar{b}(1 - \gamma_5)q \, \bar{q}(1 + \gamma_5)b, \\
T_{V-A}^q &= \bar{b}t^A\gamma_{\mu}(1 - \gamma_5)q \, \bar{q}t^A\gamma^{\mu}(1 - \gamma_5)b, \\
T_{S-P}^q &= \bar{b}t^A(1 - \gamma_5)q \, \bar{q}t^A(1 + \gamma_5)b.
\end{align*}
\]

where an implicit sum over colour indices is understood. The operators above are $\Delta B = 0$ operators. They contribute to the decay rates of the $B$-mesons (and $\Lambda_b$s) not only through the so-called “eight” diagrams, but also through tadpole diagrams, in which the light- or heavy-quark fields are contracted in a loop. These “non-spectator” diagrams mix the operators of the basis (4) with the lower dimension operators $\bar{b} \bar{b}$, $\bar{b}D^2b$, $b\sigma_{\mu\nu}G^{\mu\nu}b$, $\bar{q}q$, etc. The mixing matrix is, however, triangular. Thus, it is possible to compute separately, at the NLO, the $4 \times 4$ sub-matrix related to the mixing of the operators appearing in (4) among themselves. For this sub-matrix, the Feynman diagrams entering the calculation are the same as those relevant for the $\Delta B = 2$ operators.

2.2 Chiral and Fierz properties of the operators

The operators considered in 1)-2) can be expressed in terms of linear combinations of independent operators, defined by their colour-Dirac structure, belonging to some basis. In case 3), for the sub-matrix considered here, the same colour-Dirac structure (with obvious replacement of the flavour indices) can also be used. The choice of the basis of reference is, however, arbitrary, and different equivalent possibilities exist. We first present one possible choice, which we find particularly convenient to discuss the chiral properties of the operators:

\[
\begin{align*}
Q_{VL\,VL} &= \bar{\psi}_1^{\alpha}\gamma_{\mu}(1 - \gamma_5)\psi_2^{\alpha} \, \bar{\psi}_3^{\beta}\gamma_{\mu}(1 - \gamma_5)\psi_4^{\beta} \\
Q_{VL\,VR} &= \bar{\psi}_1^{\alpha}\gamma_{\mu}(1 - \gamma_5)\psi_2^{\alpha} \, \bar{\psi}_3^{\beta}\gamma_{\mu}(1 + \gamma_5)\psi_4^{\beta} \\
Q_{RL} &= \bar{\psi}_1^{\alpha}(1 + \gamma_5)\psi_2^{\alpha} \, \bar{\psi}_3^{\beta}(1 - \gamma_5)\psi_4^{\beta}
\end{align*}
\]
\[ Q_{LL} = \bar{\psi}^\alpha_1 (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 (1 - \gamma_5) \psi^\beta_4 \]
\[ Q_{TLTL} = \bar{\psi}^\alpha_1 \sigma_{\mu\nu} (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 \sigma_{\mu\nu} (1 - \gamma_5) \psi^\beta_4 \]
\[ \tilde{Q}_{VLVL} = \bar{\psi}^\alpha_1 \gamma_\mu (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 \gamma_\mu (1 - \gamma_5) \psi^\beta_4 \]
\[ \tilde{Q}_{VLVR} = \bar{\psi}^\alpha_1 \gamma_\mu (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 \gamma_\mu (1 + \gamma_5) \psi^\beta_4 \]
\[ \tilde{Q}_{RL} = \bar{\psi}^\alpha_1 (1 + \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 (1 - \gamma_5) \psi^\beta_4 \]
\[ \tilde{Q}_{LL} = \bar{\psi}^\alpha_1 (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 (1 - \gamma_5) \psi^\beta_4 \]
\[ \tilde{Q}_{TLTL} = \bar{\psi}^\alpha_1 \sigma_{\mu\nu} (1 - \gamma_5) \psi^\beta_2 \bar{\psi}^\beta_3 \sigma_{\mu\nu} (1 - \gamma_5) \psi^\beta_4 \]

where \( \sigma_{\mu\nu} \equiv 1/2[\gamma_\mu, \gamma_\nu] \). In (5), the flavours \( \psi_1 - \psi_4 \) are all different and the operators belong to irreducible representations of the chiral group. To these 10 operators, we have to add those which can be obtained by exchanging left- with right-handed fields. Since, however, strong interactions cannot change chirality, the second set of operators does not mix with the operators defined in (5) and, because parity is conserved, the Anomalous Dimension Matrix (ADM) is the same in the two cases. Thus, in the following, we will only consider the operators of eq. (5).

The previous considerations hold only if one uses a renormalization prescription which preserves chirality (in this respect parity is never a serious issue). It often happens, e.g., in dimensional schemes such as the 't Hooft-Veltman \( \overline{\text{MS}} \) one (HV), that the renormalization procedure violates either chirality, or (and) other symmetries that are manifest at the tree level, for example the Fierz transformation properties. In order to simplify the presentation of the results, we will use in the following a renormalization scheme which preserves all the relevant symmetries (chirality and Fierz). With such a choice, it is then sufficient to consider the basis (5). This renormalization scheme has been recently called the Regularization Independent (RI) scheme \[14\] (MOM in the early literature) to emphasize that the renormalization conditions are independent of the regularization, although they depend on the external states used in the renormalization procedure and on the gauge. The RI scheme offers also a great computational advantage in the calculation of the counterterms which contribute at the two-loop level: as demonstrated in subsec. 4.2, in this scheme it is not necessary to iden-
tify and subtract separately the counterterms relative to the mixing with the “Effervescent Operators” (EOs) which appear in dimensional regularization \([6]–[13]\). The relation between the operators renormalized in some RI scheme and, for instance, those of the standard \(\overline{MS}\) schemes can then be easily found with a simple one-loop calculation. Finally, the RI scheme allows to use, without any further perturbative calculation, the matrix elements of the operators computed in lattice simulations and renormalized non-perturbatively \([15]–[17]\).

Chiral symmetry, and Fierz rearrangement, have further consequences, since they forbid the mixing between some of the operators appearing in eq. (5). For this reason the ADM, \(\hat{\gamma}\), is a block-matrix which only allows mixing between sub-sets of the possible operators. In the convenient representation in which the operators appearing in (5) are components of row vectors

I) \(\vec{Q}_I \equiv (Q_{V_LV_L}, \tilde{Q}_{V_LV_L})\),

II) \(\vec{Q}_{II} \equiv (Q_{V_LV_R}, \tilde{Q}_{V_LV_R})\),

III) \(\vec{Q}_{III} \equiv (Q_{RL}, \tilde{Q}_{RL})\),

IV) \(\vec{Q}_{IV} \equiv (Q_{LL}, \tilde{Q}_{LL}, Q_{T_LT_L}, \tilde{Q}_{T_LT_L})\),

Fierz rearrangement imposes the following restrictions on the form of the mixing matrix:

- For set I) the structure is given by

\[
\hat{\gamma}_I \equiv \begin{pmatrix}
A_I & B_I \\
B_I & A_I
\end{pmatrix};
\]

(6)

- If the mixing-matrix for II) is given by

\[
\hat{\gamma}_{II} \equiv \begin{pmatrix}
A_{II} & B_{II} \\
C_{II} & D_{II}
\end{pmatrix},
\]

(7)

then we have

\[
\hat{\gamma}_{III} \equiv \begin{pmatrix}
D_{II} & C_{II} \\
B_{II} & A_{II}
\end{pmatrix}.
\]

(8)
In order to discuss Fierz rearrangement for the sector IV), we introduce
the Fierz transformation matrix for the sub-basis \( \vec{Q}_{IV} \)
\[
\mathcal{F} = \begin{pmatrix}
0 & -\frac{1}{2} & 0 & \frac{1}{8} \\
-\frac{1}{2} & 0 & \frac{1}{8} & 0 \\
0 & 6 & 0 & \frac{1}{2} \\
6 & 0 & \frac{1}{2} & 0
\end{pmatrix}.
\] (9)

The anomalous dimension matrix must satisfy the relation
\[
\hat{\gamma}_{IV} = \mathcal{F} \hat{\gamma}_{IV} \mathcal{F}
\] (10)

The mixing matrix has a simple form in the Dirac-Fierz basis
\[
\begin{align*}
\vec{Q}_F &\equiv (Q^F_1, Q^F_2, \tilde{Q}^F_1, \tilde{Q}^F_2), \\
Q^F_1 &= Q_{LL} + \frac{1}{4} \hat{Q}_{T_LT_L}, \\
Q^F_2 &= Q_{LL} - \frac{1}{12} \hat{Q}_{T_LT_L}, \\
\tilde{Q}^F_1 &= \tilde{Q}_{LL} + \frac{1}{4} \hat{Q}_{T_LT_L}, \\
\tilde{Q}^F_2 &= \tilde{Q}_{LL} - \frac{1}{12} \hat{Q}_{T_LT_L}.
\end{align*}
\] (11)

In this basis, the ADM can be written as
\[
\hat{\gamma}_{IV} \equiv \begin{pmatrix}
A_{IV} & B_{IV} & C_{IV} & D_{IV} \\
E_{IV} & F_{IV} & G_{IV} & H_{IV} \\
C_{IV} & -D_{IV} & A_{IV} & -B_{IV} \\
-G_{IV} & H_{IV} & -E_{IV} & F_{IV}
\end{pmatrix}
\] (12)

In summary, we have seen that the ADM for all the operators appearing
in (5) can be expressed in terms of 14 quantities, i.e. \( A_I, B_I, A_{II}, \ldots, H_{IV} \).

### 2.3 The Fierz basis

In the case in which \( \psi_1 = \psi_3 \) (or \( \psi_2 = \psi_4 \)), not all the operators in (5) are
independent. In order to take into account the simplifications occurring in
In this particular case, it is more convenient to give the results in the (Fierz) basis

\[ Q^\pm_1 = \frac{1}{2} \left( \bar{\psi}^a_1 \gamma_\mu (1 - \gamma_5) \psi^a_2 \bar{\psi}^\beta_3 \gamma_\mu (1 - \gamma_5) \psi^\beta_4 \pm (\psi_2 \leftrightarrow \psi_4) \right) \]

\[ Q^\pm_2 = \frac{1}{2} \left( \bar{\psi}^a_1 \gamma_\mu (1 - \gamma_5) \psi^a_2 \bar{\psi}^\beta_3 \gamma_\mu (1 + \gamma_5) \psi^\beta_4 \pm (\psi_2 \leftrightarrow \psi_4) \right) \]

\[ Q^\pm_3 = \frac{1}{2} \left( \bar{\psi}^a_1 (1 + \gamma_5) \psi^a_2 \bar{\psi}^\beta_3 (1 - \gamma_5) \psi^\beta_4 \pm (\psi_2 \leftrightarrow \psi_4) \right) \]

\[ Q^\pm_4 = \frac{1}{2} \left( \bar{\psi}^a_1 (1 - \gamma_5) \psi^a_2 \bar{\psi}^\beta_3 (1 - \gamma_5) \psi^\beta_4 \pm (\psi_2 \leftrightarrow \psi_4) \right) \]

\[ Q^\pm_5 = \frac{1}{2} \left( \bar{\psi}^a_1 \sigma_{\mu\nu} (1 - \gamma_5) \psi^a_2 \bar{\psi}^\beta_3 \sigma_{\mu\nu} (1 - \gamma_5) \psi^\beta_4 \pm (\psi_2 \leftrightarrow \psi_4) \right) . \]

In this case, the operators do not belong, in general, to irreducible representations of the chiral group, e.g. both right- and left-handed \( \psi_2 \) fields appear in \( Q^3_3 \). The \( \Delta B = 2 \) operators are obtained from the \( Q^i_+ \)s, by taking \( \psi_1 = \psi_3 = b \) and \( \psi_2 = \psi_4 = q \) (with this choice of flavours the \( Q^i_- \) vanish). In the basis (13), the ADM has the form

\[ \hat{\gamma}^\pm \equiv \begin{pmatrix} A^\pm & 0 & 0 & 0 & 0 \\ 0 & B & \pm C & 0 & 0 \\ 0 & \pm D & E & 0 & 0 \\ 0 & 0 & 0 & F^\pm & G^\pm \\ 0 & 0 & 0 & H^\pm & I^\pm \end{pmatrix}, \]

and there is no mixing between the \( Q^+_i \) and the \( Q^-_i \) operators.

The correspondence between the operators of the basis (13) and the operators which are relevant for the physical applications listed in subsec. 2.1 is the following

1) \( Q_1 \rightarrow Q^+_1, \; Q_2 \rightarrow Q^+_4, \; Q_3 \rightarrow -\frac{1}{2} (Q^+_4 - \frac{1}{4} Q^+_5) \),

\[ Q_4 \rightarrow Q^+_3, \; Q_5 \rightarrow -\frac{1}{2} Q^+_2 ; \]

2) \( Q \rightarrow Q^+_1, \; Q_S \rightarrow Q^+_4 ; \)
3) \[ O_{V-A}^q \rightarrow Q_1^+ + Q_1^- , \quad Q_{S-P}^q \rightarrow Q_3^+ + Q_3^- , \]
\[ T_{V-A}^q \rightarrow \frac{1}{2} \left( 1 - \frac{1}{N_c} \right) Q_1^+ - \frac{1}{2} \left( 1 + \frac{1}{N_c} \right) Q_1^- \]
\[ T_{S-P}^q \rightarrow - \frac{1}{2N_c} (Q_3^+ + Q_3^-) - \frac{1}{4} (Q_2^+ - Q_2^-) . \]

3 The Wilson coefficients

The general method for the calculation of the Wilson coefficients, and a detailed discussion on their renormalization-scheme dependence, can be found in the literature [14]–[13]. In this section, we only summarize the main formulae which are necessary to present our results. We also take the opportunity to clarify some important subtleties about the renormalization-scheme dependence.

3.1 Effective theories and Wilson coefficients

In all cases of interest, the matrix elements of the effective Hamiltonian can be written as
\[ \langle F | H_{\text{eff}} | I \rangle = \sum_i \langle F | Q_i(\mu) | I \rangle C_i(\mu) , \]
where the \( Q_i(\mu) \)s are the relevant operators renormalized at the scale \( \mu \) and the \( C_i(\mu) \)s are the corresponding Wilson coefficients. We represent the operators as row vectors \( \vec{Q} \), as in subsec. 2.2, and the coefficients, \( \vec{C}(\mu) \), as column ones. The vectors \( \vec{C}(\mu) \) are expressed in terms of their counter-part, computed at a large scale \( M \), through the renormalization-group evolution matrix \( \hat{W}[\mu, M] \)
\[ \vec{C}(\mu) = \hat{W}[\mu, M] \vec{C}(M) . \]

The initial conditions for the evolution equations, \( \vec{C}(M) \), are obtained by matching the full theory, which includes propagating heavy-vector bosons (\( W \) and \( Z^0 \)), the top quark, SUSY particles, etc., to the effective theory where the \( W \), \( Z^0 \), the top quark and all the heavy particles have been removed simultaneously. In general, \( \vec{C}(M) \) depend on the definition of the operators in a given renormalization scheme. The coefficients \( \vec{C}(\mu) \) obey the
renormalization-group equations:

\[ \left[ -\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \beta_\lambda(\alpha_s) \lambda \frac{\partial}{\partial \lambda} - \frac{\gamma^T(\alpha_s)}{2} \right] \vec{C}(t, \alpha_s(t), \lambda(t)) = 0 , \]  

(20)

where \( t = \ln(M^2/\mu^2) \). The term proportional to \( \beta_\lambda \), the \( \beta \)-function of the gauge parameter \( \lambda(t) \) (for covariant gauges), takes into account the gauge dependence of the Wilson coefficients in gauge-dependent renormalization schemes, such as the RI scheme \[4, 14\]. This term, the rôle of which will be discussed extensively in sec. 4, is absent in standard MS schemes, independently of the regularization which is adopted (NDR, HV or DRED for example) \[7\]–\[13\]. The factor of 2 in eq. (20) normalizes the anomalous dimension matrix as in refs. \[14\]–\[13\]. To simplify the discussion, we only consider the case where there is no crossing of a quark threshold when going from \( M \) to \( \mu \). The relevant formulae for the general case can be found in refs. \[11\]–\[13\].

At the next-to-leading order, we can write

\[ \hat{W}[\mu, M] = \hat{M}[\mu] \hat{U}[\mu, M] \hat{M}^{-1}[M] , \]  

(21)

where \( \hat{U} \) is the leading-order evolution matrix

\[ \hat{U}[\mu, M] = \left[ \frac{\alpha_s(M)}{\alpha_s(\mu)} \right]^{\gamma(0)T/2\beta_0} , \]  

(22)

and the NLO matrix is given by

\[ \hat{M}[\mu] = \hat{1} + \frac{\alpha_s(\mu)}{4\pi} \hat{J}[\lambda(\mu)] . \]  

(23)

By substituting the expression of the \( \vec{C}(\mu) \) given in eq. (19) in the renormalization-group equations (21), and using \( \hat{W}[\mu, M] \) written as in eqs. (21)–(23), we find that the matrix \( J \) satisfies the equation

\[ \hat{J} + \frac{\beta_0}{\beta_0} \lambda \frac{\partial}{\partial \lambda} \left[ \hat{J}, \frac{\gamma(0)T}{2\beta_0} \right] = \frac{\beta_1}{2\beta_0^2} \hat{J} \left[ \frac{\gamma(0)T}{2\beta_0} \right] - \frac{\hat{\gamma}(1)T}{2\beta_0} . \]  

(24)

\[ ^4 \text{In the following, we will denote by } \lambda = 1 \text{ the Feynman gauge and } \lambda = 0 \text{ the Landau gauge.} \]
In eqs. (22) and (24), $\beta_0$, $\beta_1$ and $\beta_0^\lambda$ are the first coefficients of the $\beta$-functions of $\alpha_s$ and of $\lambda$, respectively; $\hat{\gamma}^{(0)}$ and $\hat{\gamma}^{(1)}$ are the LO and NLO anomalous dimension matrices to be defined in sec. 4. $\hat{U}$ is determined by the LO anomalous dimension matrix $\hat{\gamma}^{(0)}$ and is therefore regularization and renormalization-scheme independent; at this order, $\lambda \partial \hat{J} / \partial \lambda$ is also regularization (but not renormalization) scheme independent; the two-loop anomalous dimension matrix $\hat{\gamma}^{(1)}$, and consequently $\hat{J}$ and $\hat{W}[\mu, M]$, are, instead, renormalization-scheme dependent.

### 3.2 Coefficient functions and scheme dependence

In this subsection, we recall some basic aspects of the calculation of the Wilson coefficients and discuss in detail the issues of the regularization and renormalization dependence of the coefficients and of the corresponding operators. We believe that this discussion may be useful to clarify some misunderstandings that can be found in the literature.

In order to compute the Wilson coefficients at a large energy scale $\mu \sim M$, we should consider the full set of current-current, box and penguin diagrams in the full theory, i.e. with propagating heavy particles, including the $O(\alpha_s)$ corrections. To date, for the $\Delta F = 2$ transitions, this part of the calculation has been carried out only in the Standard Model and 2HDM cases [8]–[10].

In the full theory, the direct calculation of the current-current, box and penguin diagrams, including $O(\alpha_s)$ corrections has the form

$$\langle H_{eff} \rangle \sim \langle \vec{Q}^{(0)T} \rangle \cdot \left[ \vec{T}^{(0)} + \frac{\alpha_s}{4\pi} \vec{T}^{(1)} \right] = \langle \vec{Q}^T(\mu) \rangle \cdot \vec{C}(\mu) , \tag{25}$$

where $\langle \vec{Q}^{(0)T} \rangle$ are the tree-level matrix elements and the vector $\vec{T}^{(1)}$ depends on the external quark (and gluon) states chosen for the calculation. By inserting the renormalized operators of the effective Hamiltonian, we then compute, at order $\alpha_s$, the one-loop diagrams between the same external states as in the full theory, using the same regularization. In this case we obtain [11]

$$\langle \vec{Q}(\mu) \rangle = \left( 1 + \frac{\alpha_s}{4\pi} \hat{r} \right) \langle \vec{Q}^{(0)} \rangle . \tag{26}$$

5 The most convenient method to define the matrix elements is by projectors on the tree-level colour-Dirac structures of the operators belonging to the four-dimensional basis [10].
The coefficients \( \vec{C}(\mu) \) are obtained by comparing eq. (25) with eq. (26); if we (formally) choose the renormalization scale \( \mu = M \), all the logarithms related to anomalous dimensions of the operators disappear and

\[
\vec{C}(M) = \vec{T}^{(0)} + \frac{\alpha_s}{4\pi} \left( \vec{T}^{(1)} - \hat{r}^T \vec{T}^{(0)} \right).
\]  

(27)

\( \vec{T}^{(1)} \) and \( \hat{r}^T \) depend on the external states. However, their difference depends only on the renormalization scheme, but not on the external states. For this reason, the dependence on the external momenta \( \sim \ln(-p^2) \) of \( \hat{r} \) in \( \vec{C}(M) \) and \( \vec{T}^{(1)} \) cancels out, for details see refs. [11]–[13]. In the following \( \hat{r} \) and \( \vec{T}^{(1)} \) denote only non-logarithmic terms.

For given external states, and for a given gauge, the matrix \( \hat{r} \) completely specifies the renormalization scheme (\( \overline{\text{MS}}, \text{RI} \), etc.). In this respect, all the renormalization schemes, including the \( \overline{\text{MS}} \) ones, are regularization independent. The \( \overline{\text{MS}} \) schemes simply amount to some specific choice of \( \hat{r} \). This is also demonstrated by the following observation: even when the regularization is specified, for example the NDR one, the so-called “\( \overline{\text{MS}} \) scheme” is not unique. The renormalized operators, and consequently \( \hat{r} \), depend in general on the basis chosen in the regularized theory to implement the minimal subtraction procedure, the projectors, i.e. the definition of the EOs, etc. [11]–[13]. Thus, in order to define completely the “\( \overline{\text{MS}} \) scheme”, we should specify all the variables (regularized basis, EOs, etc.) entering the calculation. In practice, this is equivalent to fix \( \hat{r} \), i.e. the renormalization prescription. Summarizing, the regularization dependence must always be understood as a “renormalization-scheme dependence”. In all the renormalization schemes (both \( \overline{\text{MS}} \) and RI), the same information is contained in \( \hat{r} \), once that the external states and the gauge are specified: we will then use eq. (26) to define the renormalized operators. The explicit expressions of the matrix \( \hat{r} \) in the different schemes (and the corresponding states and gauge) can be found in sec. 4.

In subsection 4.2, it will be shown that the combination

\[
\hat{G} = \hat{\gamma}^{(1)} - \left[ \hat{r}, \hat{\gamma}^{(0)} \right] - 2\beta\hat{r} - 2\beta_0\lambda \frac{\partial \hat{r}}{\partial \lambda}
\]

(28)

is renormalization-scheme independent. It can be easily shown that a consequence of eq. (28) is the independence of the combination

\[
\hat{J}_{RI} = \hat{J} + \hat{r}^T
\]

(29)
of the renormalization scheme (but not of the external states and, in general, of the gauge on which \( \hat{r} \) is computed), see also refs. [14]–[13].

The independence of \( \hat{J}_{RI} \) (and of \( \hat{G} \)) of the renormalization scheme means the following. As mentioned above, for given external states, and in a given gauge, we compute the matrix element of the renormalized operators. These operators are renormalized in some scheme, for example one of the possible \( \overline{\text{MS}} \) schemes. If we change scheme, \( \hat{r} \) and \( \hat{J} \) will accordingly change, whilst \( \hat{J}_{RI} \) will remain the same. Using eqs. (24) and (28), we find indeed

\[
\hat{J}_{RI} + \frac{\beta_0}{\beta_0} \frac{\partial \hat{J}_{RI}}{\partial \lambda} - \left[ \hat{J}_{RI}, \frac{\hat{\gamma}^{(0)T}}{2\beta_0} \right] = \frac{\beta_1}{2\beta_0^2} \frac{\gamma^{(0)T}}{2\beta_0} - \frac{\hat{G}^T}{2\beta_0}. \tag{30}
\]

The scheme independence of \( \hat{G} \) and \( \hat{\gamma}^{(0)} \) guarantees the independence of the solution of eq. (30). In turn, this implies the renormalization-scheme independence of the matrix \( \hat{J}_{RI} \). Note also that, since \( \hat{J}_{\text{MS}} \) is gauge independent, \( \partial \hat{J}_{RI}/\partial \lambda = \partial \hat{r}^T_{\text{MS}}/\partial \lambda \).

The renormalization-invariant properties discussed above can be used to introduce schemes which respect all the symmetries of the tree-level theory. Using eqs. (21)–(23) and (27), we introduce a new set of Wilson coefficients \( \vec{C}'(M) \)

\[
\vec{C}(\mu) = \hat{M}[^{\mu} \hat{U}[\mu,M] \hat{N}^{-1}[M] \vec{C}'(M), \tag{31}
\]

where \( \hat{M}[\mu] \) has been defined in eq. (23) and

\[
\hat{N}[M] = \hat{1} + \frac{\alpha_s(M)}{4\pi} \left( \hat{J} + \hat{r}^T \right), \quad \vec{C}'(M) = \vec{T}^{(0)} + \frac{\alpha_s}{4\pi} \vec{T}^{(1)}. \tag{32}
\]

In the above equations we have neglected higher order terms in \( \alpha_s \). By a suitable change of the renormalization scheme, corresponding to

\[
\vec{V}^T(\mu) = \vec{Q}^T(\mu) \left( 1 - \frac{\alpha_s(\mu)}{4\pi} \hat{r}^T \right), \tag{33}
\]

and \( \hat{M}[\mu] \rightarrow \hat{N}[\mu] \) one gets

\[
\vec{C}'(\mu) = \hat{N}[\mu] \hat{U}[\mu,M] \hat{N}^{-1}[M] \vec{C}'(M). \tag{34}
\]

Equation (34) has the following interpretation: it corresponds to the general expression (21), with the matrices \( \hat{M}[\mu] \) and \( \hat{M}[M] \) given in terms of \( \hat{J}_{RI} \),
which satisfies eq. (30). In this renormalization scheme, \( \hat{G} = \gamma_R^{(1)} \), since \( \hat{r} = \hat{r}_{RI} = 0 \). Clearly, the result is independent of the renormalization scheme of the original operators \( \vec{Q}(\mu) \). The scheme dependence is implicitly contained in the states and the gauge on which \( \hat{r} \) is computed. Thus, in the following, we will call FRI (LRI) the scheme with matrix elements computed in the Feynman (Landau) gauge.

A remark is in order at this point. In refs. [18]–[20], for \( \Delta B = 1 \) transitions, the authors used the so-called regularization-scheme-independent coefficients (corresponding to our \( C'(\mu) \)) introduced in ref. [11] and computed in ref. [18]. In this reference, the preference for using this particular renormalization scheme was justified with the argument that also the operator matrix elements computed with factorization (and used in [18]) are scheme independent. It was therefore argued that the regularization-independent coefficients are more suited to obtain the physical amplitudes. This argument is clearly illusory; the coefficients, though regularization independent, depend on the external states and on the gauge at which the renormalization conditions have been imposed. There is no way to match the external quark and gluon states, used in the perturbative calculation of the Wilson coefficients, to the hadronic states on which the operator matrix elements are computed (not to speak about gauge invariance). A similar argument applies to the scheme used in ref. [21], where the authors try to get rid of the \( \mu \) dependence of the non-leptonic amplitudes computed with factorization. In any case, in the absence of a consistent calculation in which both the coefficients and the matrix elements of the operators are computed with the same renormalization, a preferred scheme does not exist.

4 Anomalous dimensions at one and two loops

In this section we recall the procedure for the calculation of the anomalous dimension matrix \( \hat{\gamma} \) in dimensional regularization. The section is divided in two parts: in the first part, we introduce the general formulae which define the anomalous dimension matrix, in the second we give the practical recipe

\footnote{Indeed the renormalization scheme of ref. [11] has never been completely specified, because the external states, on which the renormalization conditions were imposed, have not been given explicitly.}
to compute it at the NLO.

### 4.1 General definitions and scheme dependence

The ADM for the operators appearing in the effective theory is given by the matrix

\[
\hat{\gamma} = 2 \hat{Z}^{-1} \mu^2 \frac{d}{d\mu^2} \hat{Z},
\]

(35)

where \( \hat{Z} \equiv \hat{Z}(\alpha_s, \lambda) \), defined by the relation

\[
\hat{Q} = \hat{Z}^{-1} \hat{Q}^B,
\]

(36)

gives the renormalized operators in terms of the bare ones. \( \lambda \) is the renormalized gauge parameter on which, in general, \( \hat{Z} \) may depend, see for example ref. [6]. Note that in refs. [7]–[13] the dependence on the gauge parameter was ignored because, in \( \text{MS} \) schemes, \( \hat{Z} \) is gauge independent. In this work, since we will compare anomalous-dimension matrices between schemes in which \( \hat{Z} \) can be gauge dependent, such as the RI scheme, this dependence has to be taken explicitly into account.

In dimensional regularization, using eq. (35), one gets

\[
\hat{\gamma} = 2 \hat{Z}^{-1} \left[ (-\epsilon \alpha_s + \beta(\alpha_s)) \frac{\partial}{\partial \alpha_s} \hat{Z} + \lambda \beta_\lambda(\alpha_s) \frac{\partial}{\partial \lambda} \hat{Z} \right],
\]

(37)

where \( \epsilon = (4 - D)/2 \). \( \beta(\alpha_s) \) and \( \beta_\lambda(\alpha_s) \) are the \( \beta \) functions which govern the evolution of the effective coupling constant and renormalized gauge parameter \( \lambda \), respectively

\[
\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s), \quad \mu^2 \frac{d\lambda}{d\mu^2} = \lambda \beta_\lambda(\alpha_s),
\]

(38)

with

\[
\beta(\alpha_s) = -\beta_0 \frac{\alpha_s^2}{4\pi} - \beta_1 \frac{\alpha_s^3}{(4\pi)^2} + O(\alpha_s^4), \quad \beta_\lambda(\alpha_s) = -\frac{\alpha_s}{4\pi} \beta_\lambda^0 + O(\alpha_s^2).
\]

(39)

\( \beta_0, \beta_1 \) and \( \beta_\lambda^0 \) are given by

\[
\beta_0 = \frac{(11N - 2n_f)}{3}, \quad \beta_1 = \frac{34}{3} N^2 - \frac{10}{3} N n_f - \frac{(N^2 - 1)}{N} n_f,
\]

\[
\beta_\lambda^0 = -\frac{N}{2} \left( \frac{13}{3} - \lambda \right) + \frac{2}{3} n_f,
\]

(40)
where $n_f$ is the number of active flavours. The strong coupling constant $\alpha_s$ and the gauge parameter $\lambda$ are renormalized in the $\overline{\text{MS}}$ scheme. This does not imply that also the operators must be computed in $\overline{\text{MS}}$, because the definition of the composite operators is an independent step that has nothing to do with the procedure which renormalizes the parameters of the strong-interaction Lagrangian.

From eq. (37), by writing $\hat{\gamma}$ and $\hat{Z}$ as series in the strong coupling constant

$$\hat{\gamma} = \frac{\alpha_s}{4\pi} \hat{\gamma}^{(0)} + \frac{\alpha_s^2}{(4\pi)^2} \hat{\gamma}^{(1)} + \cdots, \quad (41)$$

and

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi} \hat{Z}^{(1)} + \frac{\alpha_s^2}{(4\pi)^2} \hat{Z}^{(2)} + \cdots, \quad (42)$$

we derive the following relations

$$\hat{\gamma}^{(0)} = -2\epsilon \hat{Z}^{(1)} \quad (43)$$

and

$$\hat{\gamma}^{(1)} = -4\epsilon \hat{Z}^{(2)} - 2\beta_0 \hat{Z}^{(1)} + 2\epsilon \hat{Z}^{(1)} \hat{Z}^{(1)} - 2\eta_0 \hat{Z}^{(1)} - 2\beta_0^0 \lambda \frac{\partial \hat{Z}^{(1)}}{\partial \lambda}. \quad (44)$$

We can expand $\hat{Z}^{(i)}$ in eqs. (43) and (44) in inverse powers of $\epsilon$

$$\hat{Z}^{(i)} = \sum_{j=0}^{i} \left( \frac{1}{\epsilon} \right)^j \hat{Z}^{(i-j)}. \quad (45)$$

The requirement that anomalous dimension is finite as $\epsilon \to 0$ implies a relation between the one- and two-loop coefficients of $\hat{Z}$ (note that in all the regularizations $\hat{Z}^{(1)}_1$ is gauge invariant for gauge invariant operators)

$$4\hat{Z}^{(2)}_2 + 2\beta_0 \hat{Z}^{(1)}_2 - 2\hat{Z}^{(1)}_1 \hat{Z}^{(1)}_1 = 0, \quad (46)$$

which can be used as a check of the calculations. In addition, from the eqs. (43) and (44) we obtain

$$\hat{\gamma}^{(0)} = -2\hat{Z}^{(1)} \quad (47)$$
and

\[ \hat{\gamma}^{(1)} = -4\hat{Z}_1^{(2)} - 2\beta_0\hat{Z}_0^{(1)} + 2(\hat{Z}_1^{(1)}\hat{Z}_0^{(1)} + \hat{Z}_0^{(1)}\hat{Z}_1^{(1)}) - 2\beta_0\lambda \frac{\partial Z_0^{(1)}}{\partial \lambda}. \]  

Thus, it is sufficient to compute the pole and finite part of \( \hat{Z}^{(1)} \) and the single pole of \( \hat{Z}^{(2)} \), together with \( \beta_0 \) and \( \beta_0^0 \), in order to obtain the two-loop anomalous dimension. Note that the last term in eq. (48) is absent in refs. \([7]-[13]\). Eq. (48) tells us how to derive \( \hat{\gamma}^{(1)} \). In dimensional regularizations, such as HV, NDR or DRED, the calculation is complicated by the presence of the so-called “effervescent” operators, which appear in the intermediate steps of the calculation \([6, 7]\). The EOs are independent operators which are present in D dimensions but disappear in the physical basis of the 4-dimensional operators. Because of the presence of the EOs, the products of the matrices \( \hat{Z}_j^{(1)} \) in eq. (48) have to be done by summing indices over the full set of operators, including the EOs. Only at the end of the calculation we can restrict the set of operators to those of the physical 4-dimensional basis. As explained below, the identification of the EOs, and of the corresponding mixing matrix, can be completely avoided in RI schemes.

### 4.2 Extraction of the the one- and two-loop anomalous dimension matrix

We now derive the general expression of the coefficients \( Z_j^{(i)} \)'s in an arbitrary renormalization scheme, as obtained by using dimensional regularization. The derivation is general and, with trivial modifications, holds also with other regularizations, such as for example the lattice one \([14]\). Let us consider the matrix elements of generic bare operators, denoted as \( \hat{Q}_B \), computed in a covariant gauge, between assigned quark and gluon external states. We define the matrix elements of \( \hat{Q}_B \) as the 1PI bare Green functions \( \Gamma_{Q_B} \) multiplied by the renormalization constants of the external fields

\[ \langle \hat{Q}_B \rangle = Z_\psi^{-2} \Gamma_{Q_B}, \]

where \( Z_\psi \) will be defined below. By calling \( \alpha_0 \) the dimensionless bare coupling constant, for \( p^2 = -\mu^2 \), where \( p^2 \) denotes generically the squared momentum.
of the external states, we have

\[ \langle \vec{Q}_B \rangle = \left[ 1 + \frac{\alpha_0}{4\pi} \left( \hat{A}_0 + \frac{\hat{A}_1}{\epsilon} \right) + \left( \frac{\alpha_0}{4\pi} \right)^2 \left( \hat{B}_0 + \frac{\hat{B}_1}{\epsilon} + \frac{\hat{B}_2}{\epsilon^2} \right) \right] \langle \vec{Q}^{(0)} \rangle. \]  

(50)

\( \langle \vec{Q}^{(0)} \rangle \) are the tree-level matrix elements of all the operators of the regularized theory, including the EOs. At one loop, for gauge-invariant operators, the matrix \( \hat{A}_1 \) is gauge and regularization independent, whilst \( \hat{A}_0 \) can be written in the form

\[ \hat{A}_0(\lambda_0) = \hat{A}_0(0) + \lambda_0 \frac{\partial \hat{A}_0}{\partial \lambda_0} \]  

(51)

where \( \lambda_0 \) is the bare gauge-parameter. Note that also \( \partial \hat{A}_0 / \partial \lambda_0 \) is regularization independent.

In eq. (50), we substitute the bare parameters \( \alpha_0 \) and \( \lambda_0 \) with their renormalized counter-parts, according to

\[ \lambda_0 = \lambda \left( 1 - \frac{\alpha_s}{4\pi} \frac{\beta_0}{\epsilon} + \ldots \right), \quad \alpha_0 = \alpha_s \left( 1 - \frac{\alpha_s}{4\pi} \frac{\beta_0}{\epsilon} + \ldots \right). \]  

(52)

At the NLO, and taking into account that \( \hat{A}_1 \) is gauge invariant, we can ignore all other terms which relate the bare and the renormalized coupling constant and gauge parameter.

For a given, generic renormalization scheme, we can write the following relation between matrix elements

\[ \langle Q_R \rangle = \hat{Z}^{-1} \langle Q_B \rangle = \left( 1 + \frac{\alpha_s}{4\pi} \hat{r} \right) \langle Q^{(0)} \rangle, \]  

(53)

where the matrix \( \hat{r} \) defines the renormalization scheme. With a little algebra, a comparison of eqs. (51), (52) and (53) gives the mixing matrix \( \hat{Z} \) in terms

When working in the \( \overline{\text{MS}} \) scheme, it is convenient to express the poles in terms of \( 1/\bar{\epsilon} = 1/\epsilon - \gamma_E + \ln(4\pi) \), where \( \gamma_E \) is the Euler gamma. The formulae below are valid also in the \( \overline{\text{MS}} \) scheme if one interprets \( 1/\epsilon \) as \( 1/\bar{\epsilon} \).
$\hat{Z}_0^{(1)} = \hat{A}_0 - \hat{r}$, $\hat{Z}_1^{(1)} = \hat{A}_1$, $\hat{Z}_1^{(2)} = \hat{B}_1 - \hat{A}_1 \hat{r} - \beta_0 \hat{A}_0 - \beta_0^0 \lambda \partial \hat{r} \over \partial \lambda$, $\hat{Z}_2^{(2)} = \hat{B}_2 - \beta_0 \hat{A}_1$.

$\hat{\gamma}^{(1)}$ is then readily obtained by substituting the relations (54) in eq. (48). To this purpose, we express the regularization and renormalization-independent combination (obviously $\hat{\gamma}^{(0)} = -2 \hat{A}_1$)

$$\hat{G} = \hat{\gamma}^{(1)} - \left[ \hat{r}, \hat{\gamma}^{(0)} \right] - 2 \beta_0 \hat{r} - 2 \beta_0^0 \lambda \partial \hat{r} \over \partial \lambda$$

in terms of the matrices $\hat{A}_0, \ldots \hat{B}_2$,

$$\hat{G} = -4 \left[ \hat{B}_1 - \frac{1}{2} \left( \hat{A}_1 \hat{A}_0 + \hat{A}_0 \hat{A}_1 \right) - \frac{1}{2} \beta_0 \hat{A}_0 - \frac{1}{2} \beta_0^0 \lambda \partial \hat{A}_0 \over \partial \lambda \right]. \quad (56)$$

Equation (56) demonstrates that $\hat{G}$ is renormalization-scheme independent since the r.h.s. does not depend on $\hat{r}$. $\hat{G}$ is also regularization independent as the following, simple argument demonstrates. Let us compute the matrix elements of the renormalized operators using two different regularizations, but the same external quark and gluon states and in the same gauge. Irrespectively of the regularization used in the calculations, we can define the operators in the same renormalization scheme in terms of the mixing matrix $\hat{r}$ in eq. (53). Since the renormalized operators are the same, they obey the same renormalization-group equations. Thus, not only $\hat{r}$, but also $\hat{\gamma}^{(1)}$ is the same in the two cases. This demonstrates that the l.h.s. of eq. (56) is also regularization independent.

Note that the last term of eq. (55) is absent in refs. [7]–[13]. This is because in all $\overline{\text{MS}}$ schemes the derivative $\partial \hat{r} / \partial \lambda$ is regularization invariant, i.e. it is the same for two different $\overline{\text{MS}}$ regularizations. Thus, in these schemes, the difference between the two-loop ADMs is given by:

$$\Delta \hat{\gamma}^{(1)} = \left[ \Delta \hat{r}, \hat{\gamma}^{(0)} \right] + 2 \beta_0 \Delta \hat{r}$$

On the other hand, the combination $\hat{G}$ depends on the external states used in the calculation, and on the gauge, because $\hat{r}$ depends on these variables.
The RI scheme is defined, for given external states and at a fixed gauge, by the condition \( \hat{r} = 0 \). Thus, in this scheme, \( \hat{G} \) coincides with the two-loop anomalous dimension.

Equation (56) provides also a practical method to compute the ADM in terms of the one-loop matrices \( \hat{A}_1 \) and \( \hat{A}_0 \) and of the two-loop pole term \( \hat{B}_1 \). In refs. [7] and [13], it was demonstrated that the equation which allows to compute \( \hat{\gamma}^{(1)} \) in terms of the one- and two-loop renormalization matrices is valid diagram-by-diagram. In ref. [13] it was also shown that the relations between the anomalous dimensions in different regularizations/renormalizations can also be established on a diagram-by-diagram basis. Using these observations, and eq. (56), we give the recipe to obtain in a very simple way the anomalous dimension matrix in the RI scheme, \( \hat{\gamma}^{(1)}_{RI} \):

i) Choose the set of external states, and the gauge, which define the RI scheme of interest.

ii) Compute a given two-loop diagram where the bare operator is inserted.

iii) Subtract to it the result obtained by substituting, to any internal subdiagram, one half of the amplitude of the corresponding one-loop diagram computed at \( p^2 = -\mu^2 \):

\[
\frac{1}{2} \frac{\alpha_s}{4\pi} \left( \hat{A}_0 + \frac{\hat{A}_1}{\epsilon} \right) \langle \vec{Q}^{(0)} \rangle ,
\]

i.e. one half of the contribution of the subdiagram to \( \hat{A}_0 \) and \( \hat{A}_1 \).

iv) When the internal subdiagram contributes to the renormalization of \( \alpha_s \) and of the gauge parameter \( \lambda \), apply iii) by inserting, in the two-loop diagram, only the divergent part of one-loop diagram (always with a factor \( 1/2 \)). This rule corresponds to the choice of renormalized \( \alpha_s \) and \( \lambda \) in the MS scheme.

v) The coefficient of the single pole obtained from steps i) - iv) is the contribution of the given two-loop diagram to the combination (56).

With this procedure, we do not need to isolate the EOs from the operators of the 4-dimensional basis. The reason is two-fold. On the one hand, in the
RI scheme, the counterterms corresponding to the EO$s$ and to the operators of the 4-dimensional basis (i.e. the combination $\hat{A}_1\hat{A}_0 + \hat{A}_0\hat{A}_1$ of eq. (56)) are both subtracted with the same factor $1/2$. As shown in eq. (61) below, in the $\overline{\text{MS}}$ scheme, instead, different factors, namely $1$ and $1/2$, enter the subtraction for the 4-dimensional and effervescent counterterms. Moreover, the subtracted diagrams, as obtained from steps i) to v), only contain simple poles or finite terms, while the double poles completely cancel out. Thus, the projection on the physical 4-dimensional basis cannot give rise to further single-pole terms due to the EO$s$.

For completeness, we now give the definition of the quark wave-function renormalization which we used in the RI scheme. We introduce the two-point Green function, computed in the same gauge as the RI scheme at hand,

$$\Gamma_\psi(p^2) = \frac{i}{48}\text{tr}\left(\gamma^\mu \frac{\partial S(p)}{\partial p^\mu}\right),$$

(59)

where the trace is taken over colour and spin indices. The renormalization condition for the quark fields is given by

$$Z^{-1}_\psi(\mu^2)\Gamma_\psi(p^2 = \mu^2) = 1.$$  

(60)

Note that, in the calculation of the four-point Green functions, in any given scheme, different choices of the wave-function renormalization correspond to different choices of the quark external states. Thus, they also imply, in practice, different definitions of the renormalized operators. Obviously, all these choices are equivalent in principle, and they do not affect the calculation of the physical quantities at the order we are working. In the RI scheme, however, the specific choice of eq. (60) has the advantage that the vector and axial-vector currents, renormalized according to the same rules used for the four-fermion operators, satisfy automatically the relevant Ward identities. This is true for all the regularizations used in the intermediate steps and thus provides a useful check of the calculations. The validity of the Ward identities among renormalized quantities is not a-priori guaranteed, and it does not occur, for instance, in the HV- or DRED-$\overline{\text{MS}}$ schemes, because of the chiral symmetry breaking induced by the regularization. In the latter cases, the finite one-loop coefficient, entering the forward matrix element of the axial-vector current, does not vanish. These finite corrections are
compensated, in the evolution equation of the Wilson coefficients, by a term appearing in the two-loop current anomalous dimension.

In order to obtain the anomalous dimension in the $\overline{\text{MS}}$ scheme, one can proceed in two ways. The first was explained in refs. [7]–[13] and the details will not be given here. It is based on the relation

$$\hat{\gamma}^{(1)}_{\overline{\text{MS}}} = -4 \left[ \hat{B}_1 - \left( \hat{A}_1 \right) \left( \hat{A}_0 \right) - \frac{1}{2} \left( \hat{A}_1 \right) \left( \hat{A}_0 \right) - \beta_0 \hat{A}_0 - \beta_0^\lambda \frac{\partial \hat{A}_0}{\partial \lambda} \right]. \quad (61)$$

which can be derived from eqs. (55) and (56) by using $\hat{r}_{\overline{\text{MS}}} = \hat{A}_0$. In eq. (61), we denoted as $\hat{A}_i$ the matrix elements restricted to the operators of the four-dimensional basis, and as $\tilde{A}_i$ those connecting the operators of the four-dimensional basis with the effervescent ones. $\hat{\gamma}^{(1)}_{\overline{\text{MS}}}$ is obviously the restricted matrix. The second method to obtain $\hat{\gamma}^{(1)}_{\overline{\text{MS}}}$ is by using the relation

$$\hat{\gamma}^{(1)}_{\text{RI}} = \hat{\gamma}^{(1)}_{\overline{\text{MS}}} - 2 \left( \hat{A}_1 \hat{A}_0 - \hat{A}_0 \hat{A}_1 \right) - 2 \beta_0 \hat{A}_0 - 2 \beta_0^\lambda \frac{\partial \hat{A}_0}{\partial \lambda} \quad (62)$$

which follows from eqs. (56) and (61) and it is also valid diagram by diagram. Note that the change of scheme in eq. (62) is equivalent to the substitution

$$\hat{J}_{\overline{\text{MS}}} = \hat{J}_{\text{RI}} - \hat{r}_{\overline{\text{MS}}}^T \quad (63)$$

discussed in eq. (29) of subsec. 3.2.

### 4.3 Checks of the calculation

In this subsection, we illustrate several checks that have been made in order to verify the correctness of our calculations:

- in the NDR-$\overline{\text{MS}}$ renormalization scheme, the ADM of the operators $Q_1^\pm$, $Q_2^\pm$ and $Q_3^\pm$ can be extracted from the results of refs. [11]–[13]. They agree with the results presented in this paper;

- we have computed the anomalous dimension matrix both in $\overline{\text{MS}}$ and in RI and verified that the result satisfy eq. (55), i.e. that we get exactly the same $\hat{G}$ in the two schemes;
by computing the two-loop diagrams both in the $\overline{\text{MS}}$ and in the FRI schemes, we verified that the relation in eq. (62) holds, as we mentioned above, diagram by diagram [13].

5 The anomalous dimension matrix

In this section, we give the results for the anomalous dimension matrix in the Feynman-gauge RI scheme, which will be defined precisely in subsec. 5.1, and the matrices necessary to pass from FRI to a) NDR-$\overline{\text{MS}}$, as defined in refs. [7]–[13]; b) the Landau-gauge RI scheme, which is the most suitable for the calculation of the matrix elements on the lattice, using operators renormalized non-perturbatively [15]–[17]. The FRI scheme is presented only because it is the simplest for doing the perturbative calculations. In practical cases, we expect that the $\overline{\text{MS}}$ and LRI schemes will be used for phenomenological applications.

5.1 The anomalous dimension matrix at LO and at the NLO in FRI

In this subsection, we present the results of the leading order ADM and of the next-to-leading order ADM in the FRI scheme.

The results in FRI have been obtained by computing the one- and two-loop Feynman diagrams shown in figs. 1-2 in the Feynman gauge. At one loop, we have taken the external quark momenta as indicated in fig. 1; in the two-loop case, when the external subdiagrams are $D_1$, $D_2$ or $D_3$, the external momenta have been chosen as the corresponding ones in fig. 1. The field renormalization constant is computed in FRI according to eq. (60).

Although the results in RI only depend on the external momenta and the gauge, but not on the regularization, we specify that we did the calculation using NDR. The choice of the external momenta may appear rather strange, since it is different for the different one-loop diagrams. It is, however, particularly convenient for the perturbative calculation. In the $\overline{\text{MS}}$ scheme, the results for the ADM are not affected, since the independence of the external states is valid diagram-by-diagram. In tables 1 and 2 we give the complete...
list of the single poles necessary to compute the one- and two-loop anomalous dimension matrix.

In order to present results for the ADM, we expand the coefficients of \( \hat{\gamma}_{FRI} \), written as in eq. (14), in powers of \( \alpha_s \)

\[
A^\pm = \frac{\alpha_s}{4\pi} A^\pm_1 + \frac{\alpha_s^2}{(4\pi)^2} A^\pm_2 + \ldots
\]

\[
B = \frac{\alpha_s}{4\pi} B_1 + \frac{\alpha_s^2}{(4\pi)^2} B_2 + \ldots
\]

\[
I^\pm = \frac{\alpha_s}{4\pi} I^\pm_1 + \frac{\alpha_s^2}{(4\pi)^2} I^\pm_2 + \ldots
\]

and give the expression for these quantities.

At one-loop the ADM is independent of renormalization scheme, external

\[
I^\pm = \frac{\alpha_s}{4\pi} I^\pm_1 + \frac{\alpha_s^2}{(4\pi)^2} I^\pm_2 + \ldots
\]
Table 1: Single pole contributions of the one- and two-loop diagrams to the ADM in the FRI scheme. In this scheme the double poles are absent. The label $i \to j$ denotes the indices of the mixing-matrix. Thus $4 \to 5$ corresponds to $[\hat{Z}_1^{(2)}]_{45}$, i.e. the mixing of the bare operator $Q_4$ with $Q_5$, without colour factors. The first column refers to the diagram labels defined in fig. 1 and 3, the second column to the diagram multiplicity. For the last three diagrams, we indicate separately the term proportional to $N_c$ or $n_f$ coming from the gluon vacuum-polarization of the internal gluon line, see fig. 3.
Figure 2: One- and two-loop diagrams for the quark propagator.

Table 2: Single pole contributions to the ADM of the one- and two-loop self-energy diagrams in the FRI scheme. For $S_2$ the contributions proportional to $N_c$ (3rd column) and $n_f$ (4th column) are shown separately.
Figure 3: Two-loop Feynman diagrams.
states and gauge. The results in this case are the following

\[
\begin{align*}
A^+_1 &= 6 - \frac{6}{N_c} \\
B_1 &= \frac{6}{N_c} \\
D_1 &= 0 \\
F^+_1 &= 6 - 6N_c + \frac{6}{N_c} \\
G^+_1 &= \frac{1}{2} - \frac{1}{N_c} \\
H^+_1 &= -24 - \frac{48}{N_c} \\
I^+_1 &= 6 + 2N_c - \frac{2}{N_c}
\end{align*}
\]

\[
\begin{align*}
A^-_1 &= -6 - \frac{6}{N_c} \\
C_1 &= 12 \\
E_1 &= -6N_c + \frac{6}{N_c} \\
F^-_1 &= -6 - 6N_c + \frac{6}{N_c} \\
G^-_1 &= -\frac{1}{2} - \frac{1}{4N_c} \\
H^-_1 &= 24 - \frac{48}{N_c} \\
I^-_1 &= -6 + 2N_c - \frac{2}{N_c}.
\end{align*}
\]

(65)

Our results agree with those obtained in ref. [2].

For the two-loop ADM we obtained (with \(\beta_\lambda^0\) computed from eq. (40) for \(\lambda = 1\))

\[
\begin{align*}
A^\pm_2 &= -\frac{209}{3} - \frac{57}{2N_c^2} \pm \frac{39}{N_c} \pm \frac{355N_c}{6} + \frac{32n_f}{3} + \frac{32n_f}{3N_c} \pm 3\beta_\lambda^0 \left(1 \mp \frac{1}{N_c}\right) \\
B_2 &= \frac{355}{6} + \frac{15}{2N_c^2} - \frac{32n_f}{3N_c} + \frac{3\beta_\lambda^0}{N_c} \\
C_2 &= -\frac{6}{N_c} + \frac{418N_c}{3} - \frac{64n_f}{3} + 6\beta_\lambda^0 \\
D_2 &= \frac{9}{N_c} - \frac{9n_c}{4} \\
E_2 &= \frac{481}{6} + \frac{15}{2N_c^2} - \frac{445N_c^2}{6} - \frac{32n_f}{3N_c} + \frac{32n_f}{3} - 3\beta_\lambda^0 \left(1 - \frac{N_c}{N_c}\right) \\
F^\pm_2 &= -\frac{209}{3} - \frac{272N_c}{3} \pm \frac{272}{3} - \frac{445N_c^2}{6} + \frac{32n_f}{3} - \frac{32n_f}{3N_c} + \frac{32n_f}{3N_c} \\
&\quad \pm 3\beta_\lambda^0 \left(1 \mp \frac{1}{N_c} \mp N_c\right) \\
G^\pm_2 &= -\frac{263}{18} - \frac{2}{N_c^2} \pm \frac{4}{N_c} \pm \frac{59N_c}{9} \mp \frac{8n_f}{9} + \frac{16n_f}{9N_c} \pm \frac{1}{4} \beta_\lambda^0 \left(1 \mp \frac{2}{N_c}\right)
\end{align*}
\]

(66)
For $\gamma_{\text{FRI}}^\pm$, we have shown explicitly those terms, proportional to $\beta_0^0$, which cancel $\lambda \partial \hat{J}_{\text{FRI}} / \partial \lambda$ in eq. (30). From the one- and two-loop matrix elements of $\hat{\gamma}_{\text{FRI}}^\pm$, by solving eq. (30), one can easily compute $\hat{J}_{\text{FRI}}$. By writing $\hat{J}_{\text{FRI}}$ as

$$
\hat{J}_{\text{FRI}} \equiv \begin{pmatrix}
J_{11}^- & 0 & 0 & 0 & 0 \\
0 & J_{22} & \pm J_{23} & 0 & 0 \\
0 & \pm J_{32} & J_{33} & 0 & 0 \\
0 & 0 & 0 & J_{44}^+ & J_{44}^- \\
0 & 0 & 0 & J_{45}^+ & J_{45}^- 
\end{pmatrix},
$$

(67)

we obtain

$$
J_{11}^+ = \frac{-23931 - 2862 n_f + 128 n_f^2}{6 (33 - 2 n_f)^2}
$$

$$
J_{11}^- = \frac{28089 - 3114 n_f + 128 n_f^2}{3 (33 - 2 n_f)^2}
$$

$$
J_{22} = \frac{-1437345 - 221058 n_f + 13488 n_f^2 - 256 n_f^3}{24 (33 - 2 n_f)^2 (30 - n_f)}
$$

$$
J_{23} = \frac{45}{16 (30 - n_f)}
$$

$$
J_{32} = \frac{-4347675 + 2468583 n_f - 294786 n_f^2 + 14928 n_f^3 - 256 n_f^4}{4 (30 - n_f) (3 - n_f) (33 - 2 n_f)^2}
$$

$$
J_{33} = \frac{-15575085 + 2142036 n_f - 115572 n_f^2 + 2048 n_f^3}{24 (33 - 2 n_f)^2 (30 - n_f)}
$$

$$
J_{44}^+ = \frac{4176675 - 5048688 n_f + 669548 n_f^2 - 36624 n_f^3 + 640 n_f^4}{3 (33 - 2 n_f)^2 (125 - 132 n_f + 4 n_f^2)}
$$

$$
J_{44}^- = \frac{1208435 - 14286828 n_f + 1744892 n_f^2 - 86016 n_f^3 + 1408 n_f^4}{3 (33 - 2 n_f)^2 (125 - 132 n_f + 4 n_f^2)}
$$

(68)
$J_{45}^+ = \frac{20 \left( -277425 - 767424 n_f + 118876 n_f^2 - 7056 n_f^3 + 128 n_f^4 \right)}{3 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

$J_{45}^- = \frac{4 \left( 791235 + 1102188 n_f - 152932 n_f^2 + 7776 n_f^3 - 128 n_f^4 \right)}{3 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

$J_{54}^+ = \frac{-898695 + 1066800 n_f - 142204 n_f^2 + 7632 n_f^3 - 128 n_f^4}{36 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

$J_{54}^- = \frac{5 \left( 1208169 - 1422948 n_f + 169780 n_f^2 - 8064 n_f^3 + 128 n_f^4 \right)}{36 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

$J_{55}^+ = \frac{-11915775 + 14548416 n_f - 2050844 n_f^2 + 119952 n_f^3 - 2176 n_f^4}{9 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

$J_{55}^- = \frac{-957555 + 1949172 n_f - 126428 n_f^2 - 2016 n_f^3 + 128 n_f^4}{9 \left( 33 - 2 n_f \right)^2 \left( 125 - 132 n_f + 4 n_f^2 \right)}$

5.2 Relation between FRI and other renormalization schemes

In this subsection, we give the recipe to pass from FRI to other schemes which may be useful for practical applications: LRI, for lattice calculations, and the standard MS NDR scheme. All we need to know is the shift matrix

$$\hat{J}_{LRI} = \hat{J}_{FRI} + \hat{r}_{FRI}^T$$

$$\hat{J}_{MS} = \hat{J}_{FRI} + \hat{r}_{FRI}^T - \hat{r}_{MS}^T.$$

From the knowledge of $\hat{J}$ in a given renormalization scheme, we can immediately obtain the evolution matrix $\hat{W}[\mu, M]$ using eqs. (21) and (23).

As discussed in subsec. 3.2, the renormalization scheme is completely defined by the matrix $\hat{r}$ of eq. (59), computed for given external momenta and gauge. We choose quarks with equal momentum $p$ as external states and the Landau gauge. The field renormalization constant is computed in LRI according to eq. (60), which gives $Z_\psi = 1$. We denote as $\hat{r}_{FRI}$, $\hat{r}_{LRI}$ and $\hat{r}_{MS}$ the three cases considered here: $\hat{r}_{LRI}$ is obviously zero and $\hat{r}_{MS}$ is a $20 \times 20$
matrix because this regularization (and consequently the corresponding MS-renormalization scheme) does not respect chiral and Fierz symmetries. By denoting with $\hat{r}^{++}$ the $5 \times 5$ sub-matrix in the $Q_i^+$ sector and similarly for $\hat{r}^{+-}$, $\hat{r}^{-+}$ and $\hat{r}^{--}$, we get (see also ref. [14] for the operators $Q_{i,1,2,3}^{\pm}$),

\[
\begin{align*}
(\hat{r}^{++}_{FRI})_{11} &= \pm \left( \frac{3}{2} + 12 \ln 2 \right) - \frac{3}{2} \frac{\ln 2}{N_c} - \frac{12 \ln 2}{N_c} \\
(\hat{r}^{++}_{FRI})_{22} &= \frac{1}{2 N_c} - 2 \frac{\ln 2}{N_c} \\
(\hat{r}^{++}_{FRI})_{23} &= \pm (1 - 4 \ln 2) \\
(\hat{r}^{++}_{FRI})_{32} &= \mp \left( \frac{1}{2} + \ln 2 \right) \\
(\hat{r}^{++}_{FRI})_{33} &= \frac{1}{2 N_c} - 2 \frac{\ln 2}{N_c} - 3 \frac{N_c}{2} \\
(\hat{r}^{++}_{FRI})_{44} &= \pm \left( \frac{1}{2} + 4 \ln 2 \right) + 2 \frac{\ln 2}{N_c} - \frac{5 N_c}{2} \\
(\hat{r}^{++}_{FRI})_{45} &= \pm \left( \frac{5}{24} + \frac{2 \ln 2}{3} \right) - \frac{1}{6 N_c} - \frac{5 \ln 2}{6 N_c} \\
(\hat{r}^{++}_{FRI})_{54} &= \mp (2 - 32 \ln 2) - \frac{8}{N_c} - \frac{40 \ln 2}{N_c} \\
(\hat{r}^{++}_{FRI})_{55} &= \pm \left( \frac{7}{6} + \frac{28 \ln 2}{3} \right) - \frac{5}{6 N_c} - \frac{26 \ln 2}{3 N_c} + \frac{N_c}{2}
\end{align*}
\]  

and

\[
\begin{align*}
(\hat{r}^{++}_{MS})_{11} &= \mp (7 - 12 \ln 2) + \frac{7}{N_c} - \frac{12 \ln 2}{N_c} \\
(\hat{r}^{++}_{MS})_{22} &= - \frac{2}{N_c} - 2 \frac{\ln 2}{N_c} \\
(\hat{r}^{++}_{MS})_{23} &= \mp (4 + 4 \ln 2) \\
(\hat{r}^{++}_{MS})_{32} &= \pm (1 - \ln 2) \\
(\hat{r}^{++}_{MS})_{33} &= - \frac{2}{N_c} - 2 \frac{\ln 2}{N_c} + 4 \frac{N_c}{2} \\
(\hat{r}^{++}_{MS})_{44} &= \mp \left( \frac{39}{8} - 4 \ln 2 \right) - \frac{9}{2 N_c} - \frac{2 \ln 2}{N_c} + 4 \frac{N_c}{2} \\
(\hat{r}^{++}_{MS})_{45} &= \mp \left( \frac{31}{96} - \frac{2 \ln 2}{3} \right) + \frac{17}{24 N_c} - \frac{5 \ln 2}{6 N_c} + \frac{N_c}{16}
\end{align*}
\]  

\(31\)
\[
\begin{align*}
(r_{54}^{\pm\pm})_{\overline{\text{MS}}} &= \pm \left( \frac{13}{2} + 32 \ln 2 \right) + \frac{34}{N_c} - \frac{40 \ln 2}{N_c} - 7 N_c \\
(r_{55}^{\pm\pm})_{\overline{\text{MS}}} &= \mp \left( \frac{95}{24} - \frac{28 \ln 2}{3} \right) + \frac{7}{6 N_c} - \frac{26 \ln 2}{3 N_c} \\
(r_{44}^{\pm\pm})_{\overline{\text{MS}}} &= \pm \frac{17}{8} - \frac{1}{2 N_c} \\
(r_{45}^{\pm\pm})_{\overline{\text{MS}}} &= \pm \frac{3}{32} + \frac{3}{8 N_c} - \frac{N_c}{16} \\
(r_{54}^{\pm\pm})_{\overline{\text{MS}}} &= \pm \frac{21}{2} - \frac{22}{N_c} + 7 N_c \\
(r_{55}^{\pm\pm})_{\overline{\text{MS}}} &= \mp \frac{13}{8} + \frac{1}{2 N_c}
\end{align*}
\]

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References

[1] E. Gabrielli, A. Masiero and L. Silvestrini, Phys. Lett. B374 (1996) 80; F. Gabbanini, E. Gabrielli, A. Masiero and L. Silvestrini, Nuc. Phys. B477 (1996) 321 and refs. therein.

[2] J.A. Bagger, K.T. Matchev and R.J. Zhang, preprint JHU-TIPAC-97011, [hep-ph/9707225] and refs. therein.

[3] G. Beall, M. Bander and A. Soni, Phys. Rev. Lett. 48 (1982) 848.

[4] M. Beneke, G. Buchalla and I. Dunietz, Phys. Rev. D54 (1996) 4419 and refs. therein.

[5] M. Neubert and C.T. Sachrajda, Nucl. Phys. B483 (1997) 339 and refs. therein.
[6] G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, Nucl. Phys. B187 (1981) 461.

[7] A.J. Buras, P.H. Weisz, Nucl. Phys. B333 (1990) 66.

[8] A.J. Buras, M. Jamin and P.H. Weisz, Nucl. Phys. B347 (1990) 491.

[9] S. Herrlich and U. Nierste, Nucl. Phys. B419 (1994) 292; Phys. Rev. D52 (1995) 6505; Nucl. Phys. B476 (1996) 27.

[10] J. Urban, F. Krauss, U. Jentschura and G. Soff, HEPPH-9710245, hep-ph/9710245.

[11] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Nucl. Phys. B370 (1992) 69; Addendum, Nucl. Phys. B375 (1992) 501.

[12] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Nucl. Phys. B400 (1993) 37; A.J. Buras, M. Jamin and M.E. Lautenbacher, Nucl. Phys. B400 (1993) 75.

[13] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. B415 (1994) 403.

[14] M. Ciuchini et al., Z. Phys. C68 (1995) 239.

[15] G. Martinelli et al., Nucl. Phys. B445 (1995) 81.

[16] A. Donini et al., Phys. Lett. B360 (1996) 83; M. Crisafulli et al., Phys. Lett. B369 (1996) 325; A. Donini et al., Nucl. Phys. B(Proc. Suppl.)53 (1997) 883; M. Talevi, talk presented at the Workshop “Lattice QCD and Parallel Computers”, Tsukuba, March 1997, hep-lat/9705016.

[17] JLQCD Collaboration, S. Aoki et al., hep-lat/9705033; Nucl. Phys. B(Proc. Suppl.)53 (1997) 349.

[18] N.G. Deshpande and X.-G. He, Phys. Rev. Lett. 74 (1995) 26.

[19] R. Aleksan et al., Phys. Lett. B356 (1995) 95.

[20] P.S. Marroccoli and N. Paver, hep-ph/9702353.

[21] A. Ali and C. Greub, preprint DESY 97-126, hep-ph/9707251.