A CAUCHY-DAVENPORT THEOREM FOR SEMIGROUPS

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Abstract. We generalize the Davenport transform and use it to prove that, for a (possibly non-commutative) cancellative semigroup $A = (A, +)$ and non-empty subsets $X, Y$ of $A$ such that the subsemigroup generated by $Y$ is commutative, we have $|X + Y| \geq \min(\omega(Y), |X| + |Y| - 1)$, where

$$\omega(Y) := \sup_{y_0 \in Y \cap A} \inf_{y \not\in Y \setminus \{y_0\}} |\langle y - y_0 \rangle|.$$ 

This carries over the Cauchy-Davenport theorem to the broader setting of semigroups, and it implies, in particular, an extension of I. Chowla’s and S.S. Pillai’s theorems for cyclic groups and a notable strengthening of another generalization of the same Cauchy-Davenport theorem to commutative groups, where $\omega(Y)$ in the above is replaced by the minimal order of the non-trivial subgroups of $A$.

1. Introduction

The present paper deals with the structure theory of semigroups. We refer to [B2, B1, Chapter I, Sections 1-2, 4, and 6], and [H0, Chapter 1] for all necessary prerequisites as well as for notation and terminology used but not defined here.

Semigroups are a natural framework for developing large parts of theories traditionally presented in much less general contexts. Not only this can suggest new directions of research and shed light on questions primarily focused on groups, but it also makes methods and results otherwise restricted to “richer settings” applicable to significantly larger classes of problems.

Here, a semigroup is a pair $A = (A, +)$ consisting of a (possibly empty) set $A$, referred to as the carrier of $A$, and an associative binary operation $+$ on $A$ (unless otherwise specified, all semigroups considered below are written additively, but are not necessarily commutative).

Given subsets $X, Y$ of $A$, we define as usual the sumset, relative to $A$, of the pair $(X, Y)$ as the set $X + Y := \{x + y : x \in X, y \in Y\}$, which is written as $x + Y$.
(respectively, \(X + y\)) if \(X = \{x\}\) (respectively, \(Y = \{y\}\)). Furthermore, we extend the notion of difference set from groups to semigroups by

\[
X - Y := \{z \in A : (z + Y) \cap X \neq \emptyset\}, \quad -X + Y := \{z \in A : (X + z) \cap Y \neq \emptyset\}.
\]

Expressions of the form \(Z_1 + \cdots + Z_n\) or \(\sum_{i=1}^{n} Z_i\), involving one or more summands, as well as \(-x + Y\) and \(X - y\) for \(x, y \in A\) are defined in a similar way; in particular, we use \(nZ\) for \(Z_1 + \cdots + Z_n\) if \(Z_1 = \cdots = Z_n =: Z\).

We say that \(A\) is unital, or a monoid, if there exists \(0 \in A\) such that \(z + 0 = 0 + z = z\) for all \(z\); when this is the case, \(0\) is unique and called the identity of \(A\). Then, we let \(A^\times\) be the set of units of \(A\), in such a way that \(A^\times := \emptyset\) if \(A\) is not a monoid; this is simply denoted as \(A^\times\) if there is no likelihood of confusion. If \(A\) is unital with identity 0, a unit of \(A\) is now an element \(z\) for which there exists \(z^\dagger\), provably unique and called the inverse of \(z\) in \(A\), such that \(z + z^\dagger = z^\dagger + z = 0\). Moreover, for \(Z \subseteq A\) we write \(\langle Z \rangle_A\) for the smallest subsemigroup of \(A\) containing \(Z\), and given \(z \in A\) we use \(\langle z \rangle_A\) for \(\{\langle z \rangle\}_A\) and \(\text{ord}_A(z)\) for the order of \(z\) in \(A\), that is \(\text{ord}_A(z) := |\langle z \rangle_A|\), so generalizing the notion of order for the elements of a group.

Here and later, the subscript ‘\(A\)’ may be omitted if \(A\) is clear from the context. Finally, we say that \(A\) is cancellative if for \(x, y, z \in A\) it holds \(z + x = z + y\) or \(x + z = y + z\) only if \(x = y\); we notice that any group is a cancellative monoid.

Sumsets in (mostly commutative) groups have been intensively investigated for several years (see [Ru] for a recent survey), and interesting results have been also obtained in the case of commutative cancellative monoids [G] (in A. Geroldinger’s work these are simply termed monoids). The present paper aims to extend aspects of the theory to the more general setting of possibly non-commutative semigroups.

Historically, the first significant achievement in the field is probably the Cauchy-Davenport theorem, originally established by A.-L. Cauchy [C] in 1813, and independently rediscovered by H. Davenport [D1, D2] more than a century later:

**Theorem 1** (Cauchy-Davenport theorem). Let \((A, +)\) be a group of prime order \(p\) and \(X, Y\) non-empty subsets of \(A\). Then, \(|X + Y| \geq \min(p, |X| + |Y| - 1)\).

The result has been the subject of numerous papers, and received many different proofs, each favoring alternative points of view and eventually leading to progress on analogous questions. In fact, the main contribution here is an extension of Theorem 1 to cancellative semigroups (this is stated in Section 2).

The Cauchy-Davenport theorem applies especially to the additive group of the integers modulo a prime. Extensions to composite moduli have been given by several authors, and notably by I. Chowla [Ch] and S.S. Pillai [P]. These results, reported below for the sake of exposition and used by Chowla and Pillai in relation to Waring’s problem, are further strengthened, in Section 2 by Corollary [E] which can be viewed as a common generalization of both of them, and whose proof is sensibly shorter than each of the proofs appearing in [Ch] and [P] (not to mention that it comes as a by-product of a deeper result). Here and later, for \(m \in \mathbb{N}^+\)
we write, as usual, $\mathbb{Z}/m\mathbb{Z}$ for the integers modulo $m$, endowed with their usual additive and multiplicative structure.

**Theorem 2** (Chowla’s theorem). Let $m$ be an integer $\geq 1$. If $X, Y$ are non-empty subsets of $\mathbb{Z}/m\mathbb{Z}$ such that $0 \in Y$ and $\gcd(m, y) = 1$ for each $y \in Y \setminus \{0\}$, then $|X + Y| \geq \min(m, |X| + |Y| - 1)$.

**Theorem 3** (Pillai’s theorem). Given an integer $m \geq 1$, pick non-empty subsets $X, Y$ of $\mathbb{Z}/m\mathbb{Z}$. Let $\delta$ be the maximum of $\gcd(m, y - y_0)$ for distinct $y, y_0 \in Y$ if $|Y| \geq 2$, and set $\delta := 1$ otherwise. Then, $|X + Y| \geq \min(\delta^{-1}m, |X| + |Y| - 1)$.

A partial account of further results in the same spirit can be found in [N, Section 2.3], along with an entire chapter dedicated to Kneser’s theorem [N, Chapter 4], which, among the other things, implies Theorem 2 and then also Theorem 1; see [N, Section 4.6, Exercises 5 and 6]. Generalizations of the Cauchy-Davenport theorem of a somewhat different flavor have been furnished, still in recent years, by several authors.

For, assume for the rest of the paper that $A = (A, +)$ is a fixed, arbitrary semigroup (unless differently specified), and let $0$ be the identity of the unitization, $A^{(1)}$, of $A$: If $A$ is not unital, $A^{(1)}$ is the pair $(A \cup \{A\}, +)$, where $+$ is, by an abuse of notation, the unique extension of $+$ to a binary operation on $A \cup \{A\}$ for which $A$ serves as an identity (note that $A \not\in A$, so loosely speaking we are just adjoining a distinguished element to $A$ and extending the structure of $A$ in such a way that the outcome is a monoid whose identity is the adjoined element); otherwise $A^{(1)} := A$ (cf. [Ho, p. 2]). We denote by $p(A)$ the minimum of $\text{ord}_{A^{(1)}}(z)$ as $z$ ranges in the carrier of $A^{(1)}$ and $z \neq 0$, with the convention that $p(A) := |N|$ if $A^{(1)} = \{0\}$, namely $A^{(1)}$ is trivial. Then we have:

**Theorem 4** (folklore). If $A$ is a commutative group and $X, Y$ are non-empty subsets of $A$, then $|X + Y| \geq \min(p(A), |X| + |Y| - 1)$.

Theorem 4 is another (straightforward) consequence of Kneser’s theorem. While it applies to both finite and infinite commutative groups, an analogous result holds true for all groups:

**Theorem 5** (Hamidoune-Károlyi theorem). If $A$ is a group and $X, Y$ are non-empty subsets of $A$, then $|X + Y| \geq \min(p(A), |X| + |Y| - 1)$.

This was first proved by Károlyi in the case of finite groups, relying on the structure theory of group extensions, by reduction to finite solvable groups in the light of the Feit-Thompson theorem, and then by Hamidoune in the general case, based on the isoperimetric method; see [K] for details.

A further result from the literature that is significant in relation to the subject matter is due to J.H.B. Kemperman [Ke], and reads as follows:
Theorem 6 (Kemperman’s inequality for torsion-free groups). Let $\mathbb{A}$ be a group, and let $X, Y$ be non-empty subsets of $A$. Suppose that every non-zero element of $A$ has order $\geq |X| + |Y| - 1$ in $\mathbb{A}$. Then, $|X + Y| \geq |X| + |Y| - 1$.

In fact, \[Ke\] is focused on cancellative semigroups (there simply called semigroups), and it is precisely in this framework that Kemperman establishes a series of results, mostly related to the number of different representations of an element in a sumset, eventually leading to Theorem 6, a weak version of which will be proved in Section 5 as a corollary of our main theorem (namely, Corollary 13).

For the rest, Hamidoune and coauthors, see [CHS, Theorem 3], have proved a Cauchy-Davenport theorem for acyclic monoids (these are termed acyclic semigroups in [CHS], but they are, in fact, monoids in our terminology), and it would be interesting to find a common pattern among their result and the ones in the present paper; unluckily, the author has no clue on this for the moment (in particular, note that acyclic semigroups in [CHS] are not cancellative semigroups).

Organization. In Section 2 we define the Cauchy-Davenport constant of a pair of sets in a semigroup and state our main results. In Section 3 we establish a few basic lemmas. Section 4 is devoted to generalized Davenport transforms and their fundamental properties. We demonstrate the central theorem of the paper (namely, Theorem 8) in Section 5 and give a couple of applications in Section 6.

2. The statement of the main results

With all the above in mind, we can now proceed to the heart of the paper.

Definition 7. For a subset $Z$ of $A$, we let

\begin{equation}
\omega_A(Z) := \sup_{z_0 \in Z \cap A^\times} \inf_{z \in Z \setminus \{z_0\}} \text{ord}(z - z_0).
\end{equation}

Then, given $X, Y \subseteq A$ we define $\Omega_A(X, Y) := 0$ if either of $X$ or $Y$ is empty; $\Omega_A(X, Y) := \max(|X|, |Y|)$ if $X \times Y \neq \emptyset$ and either $X$ or $Y$ is infinite, and

\[\Omega_A(X, Y) := \min(\omega_A(X, Y), |X| + |Y| - 1)\]

otherwise, where $\omega_A(X, Y) := \max(\omega_A(X), \omega_A(Y))$. We refer to $\Omega_A(X, Y)$ as the Cauchy-Davenport constant of $(X, Y)$ relative to $A$ (again, the subscript ‘$A$’ may be omitted from the notation if there is no danger of ambiguity).

Here and later, we assume that the supremum of the empty set is 0, while its infimum is $|N|$, so any pair of subsets of $A$ has a well-defined Cauchy-Davenport constant (relative to $A$). In particular, $\omega(Z)$ is zero for $Z \subseteq A$ if $Z \cap A^\times = \emptyset$. However, this is not the case, for instance, when $Z \neq \emptyset$ and $A$ is a group, which is the “base” for the following non-trivial bound:
Theorem 8. Suppose $\mathcal{A}$ is cancellative and let $X, Y$ be subsets of $A$ such that $\langle Y \rangle$ is commutative. Then, $|X + Y| \geq \min(\omega(Y), |X| + |Y| - 1)$ if both of $X$ and $Y$ are finite and non-empty, and $|X + Y| \geq \Omega(X, Y)$ otherwise.

Theorem 8 represents the central contribution of the paper. Not only it extends the Cauchy-Davenport theorem to the broader and more abstract setting of semigroups (see Section 6), but it also provides a strengthening and a generalization of Theorem 4 due to the following lemma.

Lemma 9. If $Z$ is a subset of $A$ such that $Z \cap A^\times \neq \emptyset$, then $\omega(Z) \geq p(\mathcal{A})$.

Proof. Pick $z_0 \in Z \cap A^\times$ using that $Z \cap A^\times \neq \emptyset$. If $Z$ is a singleton, the assertion is trivial since then $\inf_{z \in Z \setminus \{z_0\}} \text{ord}(z - z_0) = |\mathbb{N}|$. In the other case, taking $z \in Z \setminus \{z_0\}$ gives $\text{ord}(z - z_0) \geq p(\mathcal{A})$ by the definition of $p(\mathcal{A})$.

Lemma 9 applies, on the level of groups, to any non-empty subset (see Corollary 12 below), and the stated inequality is strict in significant cases: For a concrete example, pick $k, q \in \mathbb{N}^+$ and set $m := qk$ and $X := \{(1+ik) \mod m : i = 1, \ldots, q\}$. Then observe that $|2X| = \Omega_{\mathbb{Z}/m\mathbb{Z}}(X, X) = q$, while $p(\mathbb{Z}/m\mathbb{Z})$ is the smallest prime, say $p$, dividing $m$, to the effect that $p(\mathbb{Z}/m\mathbb{Z})$ is “much” smaller than $\Omega_{\mathbb{Z}/m\mathbb{Z}}(X, X)$ if $p$ is “much” smaller than $q$.

Theorem 8 can be “symmetrized” and further strengthened in the case where each summand generates a commutative subsemigroup, leading to the following corollaries, whose proofs are straightforward in the light of Definition 7.

Corollary 10. Assume $\mathcal{A}$ is cancellative and let $X, Y$ be subsets of $A$ such that $\langle X \rangle$ is commutative. Then, $|X + Y| \geq \min(\omega(X), |X| + |Y| - 1)$ if both of $X$ and $Y$ are finite and non-empty, and $|X + Y| \geq \Omega(X, Y)$ otherwise.

Corollary 11. If $\mathcal{A}$ is cancellative and $X, Y$ are subsets of $A$ such that both of $\langle X \rangle$ and $\langle Y \rangle$ are commutative, then $|X + Y| \geq \Omega(X, Y)$.

Moreover, the result specializes to groups as follows:

Corollary 12. If $\mathcal{A}$ is a group and $X, Y$ are non-empty subsets of $A$ such that $\langle Y \rangle$ is commutative. Then, $|X + Y| \geq \min(\omega(Y), |X| + |Y| - 1)$, where

$$\omega(Y) = \sup_{y_0 \in Y} \inf_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0),$$

and indeed $\omega(Y) = \max_{y_0 \in Y} \inf_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0)$ if $Y$ is finite.

Proof. Immediate by Theorem 8 for on the one hand $\mathcal{A}$ being a group implies $Y = Y \cap A^\times$, and on the other, a supremum over a finite set is a maximum.

The next corollary is now a partial generalization of Theorem 8 to cancellative semigroups: its proof is straightforward by Corollary 11 and Lemma 9. Here, we say that $\mathcal{A}$ is torsion-free if $p(\mathcal{A})$ is infinite (in fact, this is an abstraction of the analogous definition for groups).
Corollary 13. If $A$ is cancellative and $X, Y$ are non-empty subsets of $A$ such that every element of $A \setminus \{0\}$ has order $\geq |X| + |Y| - 1$ in $A$ (this is especially the case when $A$ is torsion-free) and either of $\langle X \rangle$ or $\langle Y \rangle$ is abelian, then $|X + Y| \geq |X| + |Y| - 1$.

Theorem 8 is proved in Section 5. The argument is inspired by the transformation proof originally used for Theorem 1 by Davenport in [D1]. This leads us to the definition of what we call a generalized Davenport transform. The author is not aware of an earlier use of the same technique in the literature, all the more in relation to semigroups. With few exceptions, remarkably including [HR] and A.G. Vosper’s original proof of his famous theorem on critical pairs [V], even the “classical” Davenport transform has not been greatly considered by practitioners in the area, especially in comparison with similar “technology” such as the Dyson transform [N, p. 42].

Remark 14. A couple of things are worth mentioning before proceeding. While every commutative cancellative semigroup embeds as a subsemigroup into a group (as it follows from the standard construction of the group of fractions of a commutative monoid; see [B1, Chapter I, Section 2.4]), nothing similar is true in the non-commutative case. This is linked to a well-known question in the theory of semigroups, first answered by A.I. Mal’cev in [M], and serves as a fundamental motivation for the present paper, in that it shows that the study of sumsets in cancellative semigroups cannot be systematically reduced, in the absence of commutativity, to the case of groups (at the very least, not in any obvious way).

On the other hand, it is true that every cancellative semigroup can be embedded into a cancellative monoid (through the unitization process mentioned in the comments preceding the statement of Theorem 4 in Section 1), to the effect that, for the specific purposes of the manuscript, we could have assumed in most of our statements that the “ambient” is a monoid rather than a semigroup, but we did differently for the assumption is not really necessary. We will see, however, that certain parts take a simpler form when an identity is made available somehow, as in the case of lemmas in Section 3 or in the proof of Theorem 8.

We provide two applications of Theorem 8 in Section 6 (others will be investigated in future work): The first is a generalization of Theorem 2, the second is an improvement on a previous result by Ø.J. Rødseth [R, Section 6] relying on Hall’s “marriage theorem”. As for the former (which is stated below), we will use the following specific notation: Given $m \in \mathbb{N}^+$ and a non-empty $Z \subseteq \mathbb{Z}/m\mathbb{Z}$, we let

$$\delta_Z := \min_{z_0 \in Z} \max_{z \in \mathbb{Z} \setminus \{z_0\}} \gcd(m, z - z_0)$$

if $|Z| \geq 2$, and $\delta_Z := 1$ otherwise. Based on this, the next result holds:
**Corollary 15.** For an integer \( m \geq 1 \) let \( X \) and \( Y \) be non-empty subsets of \( \mathbb{Z}/m\mathbb{Z} \) and define \( \delta := \min(\delta_X, \delta_Y) \). Then, \(|X + Y| \geq \min(\delta^{-1}m, |X| + |Y| - 1)\). More in particular, \(|X + Y| \geq \min(m, |X| + |Y| - 1)\) if there exists \( y_0 \in Y \) such that \( m \) is prime with \( y - y_0 \) for each \( y \in Y \setminus \{y_0\} \) (or dually with \( X \) in place of \( Y \)).

In fact, Corollary 15 contains Chowla’s theorem (Theorem 2) as a special case: With the same notation as above, it is enough to assume that the identity of \( \mathbb{Z}/m\mathbb{Z} \) belongs to \( Y \), and \( \gcd(m, y) = 1 \) for each non-zero \( y \in Y \). Furthermore, it is clear from 2 that the result is a strengthening of Pillai’s theorem (Theorem 3).

Many questions arise. Most notably: Is it possible to further extend Corollary 11 in such a way to get rid of the assumption that summands generate commutative subsemigroups? This sounds particularly significant, for a positive answer would provide a comprehensive generalization of about all the extensions of the Cauchy-Davenport theorem reviewed in Section 1 and remarkably of Theorems 5 and 9.

### 3. Preliminaries

This section collects basic results used later to introduce the generalized Davenport transforms and prove Theorem 8. Some proofs are direct and standard (and thus omitted without further explanation), but we have no reference to anything similar in the context of semigroups, so we include them here for completeness.

**Lemma 16.** Pick \( n \in \mathbb{N}^+ \) and subsets \( X_1, Y_1, \ldots, X_n, Y_n \) of \( A \) such that \( X_i \subseteq Y_i \) for each \( i \). Then, \( \sum_{i=1}^{n} X_i \subseteq \sum_{i=1}^{n} Y_i \) and \(|\sum_{i=1}^{n} X_i| \leq |\sum_{i=1}^{n} Y_i|\).

**Lemma 17.** Assume \( A \) is cancellative and pick an integer \( n \geq 2 \) and non-empty \( X_1, \ldots, X_n \subseteq A \). Then, \(|\sum_{i=2}^{n} X_i| \leq |\sum_{i=1}^{n-1} X_i| \) and \(|\sum_{i=1}^{n-1} X_i| \leq |\sum_{i=1}^{n} X_i|\).

For the next lemma, whose proof is straightforward by a routine induction, we assume that \( 0 \cdot \kappa := \kappa \cdot 0 := 0 \) for every cardinal \( \kappa \).

**Lemma 18.** For \( n \in \mathbb{N}^+ \) and \( X_1, \ldots, X_n \subseteq A \) it holds \(|\sum_{i=1}^{n} X_i| \leq \prod_{i=1}^{n} |X_i|\).

Let \( X, Y \subseteq A \). No matter if \( A \) is cancellative, nothing similar to Lemmas 17 and 18 applies, in general, to the difference set \( X - Y \), in the sense that this can be infinite even if both of \( X \) and \( Y \) are finite. On another hand, we get by symmetry and Lemma 17 that, in the presence of cancellativity, the cardinality of the sumset \( X + Y \) is preserved under translation, namely \(|z + X + Y| = |X + Y + z| = |X + Y|\) for every \( z \in A \). This is a point in common with the case of groups, save for the fact that we cannot profit from it, at least in general, to “normalize” either of \( X \) or \( Y \) in such a way as to contain some distinguished element of \( A \).

**Lemma 19.** Let \( X \) and \( Y \) be subsets of \( A \). The following are equivalent:

(i) \( X + 2Y \subseteq X + Y \).
(ii) $X + nY \subseteq X + Y$ for all $n \in \mathbb{N}^+$.
(iii) $X + \langle Y \rangle = X + Y$.

**Proof.** Points (ii) and (iii) are clearly equivalent, as $X + \langle Y \rangle = \bigcup_{n=1}^{\infty} (X + nY)$, and (i) is obviously implied by (ii). Thus, we are left to prove that (ii) follows from (i), which is immediate (by induction) using that, if $X + nY \subseteq X + Y$ for some $n \in \mathbb{N}^+$, then $X + (n+1)Y = (X + nY) + Y \subseteq (X + Y) + Y = X + 2Y \subseteq X + Y$. □

The above result is as elementary as central in the plan of the paper, for the properties of the generalized Davenport transform used later, in Section 5, in the proof of Theorem 8 are strongly dependent on it.

On another hand, the following lemma shows that, in reference to Theorem 8, there is no loss of generality in assuming that the ambient semigroup is unital, for any semigroup embeds as a subsemigroup into its unitization.

**Lemma 20.** Let $(B, \star)$ be a semigroup, $\varphi$ an injective function from $A$ to $B$ such that $\varphi(z_1 + z_2) = \varphi(z_1) \star \varphi(z_2)$ for all $z_1, z_2 \in A$, and $X_1, \ldots, X_n \subseteq A$ ($n \in \mathbb{N}^+$). Then, $|X_1 + \cdots + X_n| = |\varphi(X_1) \star \cdots \star \varphi(X_n)|$.

We close the section with a few properties of units. Here and later, given $X \subseteq A$ we use $C_A(X)$ for the centralizer of $X$ in $A$, namely the set of all $z \in A$ such that $z + x = x + z$ for every $x \in X$.

**Lemma 21.** Let $A$ be a monoid, $X$ a subset of $A$, and $z$ a unit of $A$ with inverse $\bar{z}$. Then the following conditions hold:

(i) $X - z = X + \bar{z}$, $-z + X = \bar{z} + X$ and $|-z + X| = |X - z| = |X|$.
(ii) If $z \in C_A(X)$ then $\bar{z} \in C_A(X)$; in addition to this, $\langle X - z \rangle$ and $\langle -z + X \rangle$ are commutative if $\langle X \rangle$ is commutative.

**Proof.** (i) By symmetry, it suffices to prove that $X - z = X + \bar{z}$ and $|X - z| = |X|$. As for the first identity, it holds $w \in X - z$ if and only if there exists $x \in X$ such that $w + z = x$, which in turn is equivalent to $x + \bar{z} = (w + z) + \bar{z} = w$, namely $w \in X + \bar{z}$. In order to conclude, it is then sufficient to observe that the function $A \to A : \xi \mapsto \xi + \bar{z}$ is a bijection.

(ii) Pick $z \in C_A(X)$ and $x \in X$. It is then seen that $x + \bar{z} = \bar{z} + x$ if and only if $x = (x + \bar{z}) + z = \bar{z} + x + z$, and this is certainly verified as our standing assumptions imply $\bar{z} + x + z = \bar{z} + z + x = x$. It follows that $\bar{z} \in C_A(X)$.

Suppose now that $\langle X \rangle$ is a commutative semigroup and let $v, w \in \langle X - z \rangle$. By point (i) above, there exist $k, \ell \in \mathbb{N}^+$ and $x_1, \ldots, x_k, y_1, \ldots, y_\ell \in X$ such that $v = \sum_{i=1}^{k} (x_i + \bar{z})$ and $w = \sum_{i=1}^{\ell} (y_i + \bar{z})$, to the effect that $v + w = w + v$ by induction on $k + \ell$ and the observation that for all $u_1, u_2 \in X$ it holds

$$(u_1 + \bar{z}) + (u_2 + \bar{z}) = u_1 + u_2 + 2\bar{z} = u_2 + u_1 + 2\bar{z} = (u_2 + \bar{z}) + (u_1 + \bar{z}),$$

where we use that $\bar{z} \in C_A(X)$, as proved before, and $\langle X \rangle$ is commutative. Hence, $\langle X - z \rangle$ is commutative too, which completes the proof by symmetry. □
Remark 22. There is a subtleness in Definition 7 which we have “overlooked” so far, but should be noticed. For, suppose that $A$ is a monoid and pick $x, y \in A$. In principle, $x - y$ and $-y + x$ are not elements of $A$: In fact, they are (difference) sets, and no other meaningful interpretation is possible a priori. However, if $y$ is a unit of $A$ and $\tilde{y}$ is the inverse of $y$, then $x - y = \{x + \tilde{y}\}$ and $-y + x = \{\tilde{y} + x\}$ by point (i) of Lemma 21, and we are allowed to identify $x - y$ with $x + \tilde{y}$ and $-y + x$ with $\tilde{y} + x$, which will turn to be useful in various places.

4. The Davenport transform revisited

As mentioned in Section 2, Davenport’s proof [D1, Statement A] of Theorem 1 is a transformation proof. Assuming that $A$ is a commutative group, the idea is to map a pair $(X, Y)$ of non-empty subsets of $A$ to a new pair $(X, Y')$, which is smaller than $(X, Y)$ in an appropriate sense, and specifically such that $|Y'| < |Y|$, $|X + Y'| + |Y| \leq |X + Y| + |Y'|$. We then refer to $(X, Y')$ as a Davenport transform of $(X, Y)$; see, for instance, [HR]. For this to be possible, the classical approach requires that $X + 2Y \not\subseteq X + Y$ and $0 \in Y$, to the effect that $|Y| \geq 2$.

As expected, many difficulties arise when attempting to adapt the same approach to semigroups, all the more if these are non-commutative. Even the possibility of embedding a semigroup into a monoid does not resolve anything, since the fundamental problem is that, contrary to the case of groups, cardinality is not preserved “under subtraction”. Namely, if $A$ is an arbitrary monoid with identity $0$ (as intended for the rest of the section, unless differently stated), $X$ is a subset of $A$, and $z$ is an element of $A$, then $|X|$, $|X - z|$ and $| - z + X|$ can be greatly different from each other, even supposing that $A$ is cancellative; cf. point (i) of Lemma 21. Thus, unless $A$ is a group in disguise or, more generally, embeds as a submonoid into a group, we are not allowed to assume, for instance, that $0 \in Y$ by picking an arbitrary element $y_0 \in Y$ and replacing $(X, Y)$ with the “shifted” pair $(X + y_0, -y_0 + Y)$; cf. the comments following Lemma 18.

In fact, the primary goal of this section is to show that, in spite of these issues, Davenport’s original ideas can be extended and used for a proof of Theorem 8.

To start with, let $X$ and $Y$ be subsets of $A$ such that $mX + 2Y \not\subseteq X + Y$ for some positive integer $m$. For the sake of brevity, define

$$Z := (mX + 2Y) \setminus (X + Y).$$

Our assumptions give $Z \neq \emptyset$. So fix $z \in Z$, and take $x_z \in (m - 1)X$ and $y_z \in Y$ for which $z \in x_z + X + Y + y_z$, where $0X := \{0\}$. Finally, set

$$\hat{Y}_z := \{y \in Y : z \in x_z + X + Y + y\}, \quad Y_z := Y \setminus \hat{Y}_z.$$

We refer to $(X, Y_z)$ as a generalized Davenport transform of $(X, Y)$ (relative to $z$), and based on this notation we have the next proposition:
Proposition 23. If \( Y \neq \emptyset \), then the triple \((X, Y, \tilde{Y})\) satisfies the following:

(i) \( Y \) and \( \tilde{Y} \) are non-empty disjoint proper subsets of \( Y \), and \( \tilde{Y} = Y \setminus Y \).
(ii) If \( A \) is cancellative, then \((x + X + Y) \cup (z - \tilde{Y}) \subseteq x + X + Y\).
(iii) \((x + X + Y) \cap (z - \tilde{Y}) = \emptyset\) if \( Y \) is commutative.
(iv) If \( A \) is cancellative, then \(|z - \tilde{Y}| \geq |\tilde{Y}|\).
(v) \(|X + Y| + |Y| \geq |X + Y| + |Y|\) if \( A \) is cancellative and \( Y \) commutative.

Proof. [1] \( Y \) and \( \tilde{Y} \) are non-empty because \( \tilde{y} \in Y \) by construction. Also, [3] gives \( Y \subseteq Y \) and \( Y \cap \tilde{Y} = \emptyset \), so that \( Y \setminus Y = Y \setminus (Y \setminus \tilde{Y}) = \tilde{Y} \) and \( Y \subseteq Y \).

(ii) Since \( Y \subseteq Y \) by point (i) above, \( x + X + Y \subseteq x + X + Y \) by Lemma 17. On the other hand, if \( w \in z - \tilde{Y} \) then there exists \( y \in \tilde{Y} \) such that \( z = w + y \).

But \( y \in \tilde{Y} \) implies by (3) that \( z = \tilde{w} + y \) for some \( \tilde{w} \in x + X + Y \), whence \( w = \tilde{w} \) by right cancellativity, namely \( w \in x + X + Y \).

(iii) Assume the contrary and let \( w \in (x + X + Y) \cap (z - \tilde{Y}) \). There then exist \( x \in X \), \( y_1 \in Y \) and \( y_2 \in \tilde{Y} \) such that \( w = x + y + y_1 \) and \( z = w + y_2 \). Using that \( Y \) is commutative, it follows that \( z = x + y + y_1 + y_2 = x + y + y_2 + y_1 \), which in turn implies \( y_1 \in \tilde{Y} \) by (3), since \( Y \), \( \tilde{Y} \subseteq Y \) by point (i). This is, however, absurd as \( Y \cap \tilde{Y} = \emptyset \), by the same point (3).

(iv) We have from (3) that for each \( y \in Y \) there exists \( w \in x + X + Y \) such that \( z = w + y \), and hence \( w \in z - \tilde{Y} \). On the other hand, since \( A \) is left cancellative, it cannot happen that \( w + y_1 = w + y_2 \) for some \( w \in A \) and distinct \( y_1, y_2 \in \tilde{Y} \). Thus, \( \tilde{Y} \) embeds as a set into \( z - \tilde{Y} \), with the result that \(|z - \tilde{Y}| \geq |\tilde{Y}|\).

(v) Since \( A \) is cancellative and \( X \neq \emptyset \) (otherwise \( Z = \emptyset \)), we have \(|X + Y| \geq \max(|X|, |Y|)\) by symmetry and Lemma 17. This implies the claim if \( Y \) is infinite, since then either \(|X + Y| > |Y|\), and hence

\[ |X + Y| + |Y| = |X| = |X + Y| + |Y|, \]

or instead \(|X + Y| = |Y|\), and accordingly

\[ |X + Y| + |Y| = |Y| = |X + Y| + |Y|. \]

We are using here the axiom of choice, which is assumed in the background as part of our foundations, to say that \( |X + Y| = \max(|X|, |Y|) \) if \( X \) and \( Y \) are both infinite. So we are left with the case when \( Y \) is finite, for which the inclusion-exclusion principle, points (ii)-(iv) and Lemma 17 give, by symmetry, that

\[ |X + Y| = |x + X + Y| \geq |x + X + Y| + |z - \tilde{Y}| = |X + Y| + |z - \tilde{Y}| \geq |X + Y| + |\tilde{Y}|. \]

But \( \tilde{Y} = Y \setminus Y \) and \( Y \subseteq Y \) by point (i) above, so in the end we get \(|X + Y| \geq |X + Y| + |Y| - |Y|\), and the proof is complete. ■
5. The Proof of the Main Theorem

Lemma 23 is used here to establish the main contribution of the paper.

Proof of Theorem 8. Since every semigroup embeds as a subsemigroup into its unitization, and the unitization of a cancellative semigroup is cancellative in its own right, Lemma 20 and Definition 7 imply that there is no loss of generality in assuming, as we do, that $A$ is unital.

Thus, suppose by contradiction that the theorem is false. There then exists a pair $(X, Y)$ of subsets of $A$ for which $|X + Y| < \min(|Y|, |X| + |Y| - 1)$, whence

$$2 \leq |X|, |Y| < |\mathbb{N}|.$$  

In fact, if either of $X$ or $Y$ is a singleton or infinite then $|X + Y| = \max(|X|, |Y|)$, and Definition 7 gives $|X + Y| = \Omega(|X, Y|)$, contradicting the standing assumptions. It follows from (11) and (11) that

$$|X + Y| < \sup_{y \in Y \cap A^x} \inf_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0), \quad |X + Y| \leq |X| + |Y| - 2.$$  

Again without loss of generality, we also assume that $|X| + |Y|$ is minimal over the pairs of subsets of $A$ for which (11) and (11) are presumed to hold.

Now, since $|X + Y|$ is finite, thanks to (11) and Lemma 18, we get by (11) and the same equation (11) that there exists $y_0 \in Y \cap A^x$ such that

$$|X + Y| < \inf_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0) = \min_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0).$$  

So letting $0$ be the identity of $A$ and taking $W_0 := Y - y_0$ imply

$$|X + W_0| < \min_{w \in W_0 \setminus \{0\}} \text{ord}(w), \quad |X + W_0| \leq |X| + |W_0| - 2$$

in view of (11) and (11). In fact, on the one hand $|Y - y_0| = |Y|$ and $|X + Y - y_0| = |X + Y|$ by point (11) of Lemma 21, and on the other hand, $y \in Y \setminus \{y_0\}$ only if $y - y_0 \in (Y - y_0) \setminus \{0\}$, as well as $w \in (Y - y_0) \setminus \{0\}$ only if $w + y_0 \in Y \setminus \{y_0\}$ (see also Remark 22). We claim that

$$Z := (X + 2W_0) \setminus (X + W_0) \neq \emptyset.$$  

For, suppose the contrary. Then, $X + W_0 = X + \langle W_0 \rangle$ by Lemma 19, so that

$$|X + W_0| = |X + \langle W_0 \rangle| \geq |\langle W_0 \rangle| \geq \max_{w \in W_0} \text{ord}(w) \geq \min_{w \in W_0 \setminus \{0\}} \text{ord}(w),$$

which contradicts the standing assumptions. Hence, $Z$ is non-empty.
where we use, in particular, Lemma 17 for the first inequality and the fact that
\[ |W_0| \geq 2 \] for the last one. But this contradicts (7), so (8) is proved.

Pick \( z \in Z \) and let \( (X, W'_0) \) be a generalized Davenport transform of \( (X, W_0) \) relative to \( z \). Since \( \langle Y \rangle \) is a commutative subsemigroup of \( A \) (by hypothesis), the same is true for \( \langle W_0 \rangle \), by point \( \langle \mathrm{ii} \rangle \) of Lemma 21. Moreover, \( 0 \in W_0 \), and thus
\[ 0 \in W'_0 \neq \emptyset, \quad W'_0 \subseteq W_0, \]
when taking into account Remark 24 and point \( \langle \mathrm{i} \rangle \) of Proposition 23. As a consequence, point \( \langle \mathrm{v} \rangle \) of the same Proposition 23 yields, together with (7), that
\[ |X + W'_0| + |W_0| \leq |X + W'_0| + |W'_0| \leq |X| + |W_0| - 2 + |W'_0|, \]
which means, since \( |W_0| = |Y - \bar{y}_0| = |Y| < |N| \) by (4) and the above, that
\[ |X + W'_0| \leq |X| + |W'_0| - 2. \]
It follows from (9) that \( 1 \leq |W'_0| < |W_0| \), and in fact \( |W'_0| \geq 2 \), as otherwise we would have \( |X| = |X + W'_0| \leq |X| - 1 \) by (10), in contradiction with the fact that \( |X| < |N| \) by (4). To summarize, we have found that
\[ 2 \leq |W'_0| < |W_0| < |N|. \]
Furthermore, (10) and (9) entail that
\[ |V_0 + W'_0| \leq |V_0 + W_0| < \min_{w \in W'_0 \setminus \{0\}} \text{ord}(w), \]
where we use that \( \min(C_1) \leq \min(C_2) \) if \( C_1 \) and \( C_2 \) are sets of cardinal numbers with \( C_2 \subseteq C_1 \). Thus, since \( 0 \in W'_0 \cap A^\times \), we get by (12) that
\[ |X + W'_0| < \sup_{w_0 \in W'_0 \cap A^\times} \min_{w \in W'_0 \setminus \{w_0\}} \text{ord}(w), \]
which is however in contradiction, due to (11), (10) and (11), with the minimality of \( |X| + |Y| \), for \( |W'_0| < |W_0| = |Y| \), and hence \( |X| + |W'_0| < |X| + |Y| \).

6. A COUPLE OF APPLICATIONS

First, we show how to use Theorem 8 to prove the extension of Chowla’s theorem for composite moduli mentioned in Section 2.

Proof of Corollary 17. The claim is trivial if either of \( X \) or \( Y \) is a singleton. Otherwise, \( \mathbb{Z}/m\mathbb{Z} \) being a commutative finite group and \( \text{ord}(z - z_0) = m/\gcd(m, z - z_0) \) for \( z, z_0 \in \mathbb{Z}/m\mathbb{Z} \) imply \( |X + Y| \geq \min(|Y|, |X| + |Y| - 1) \) by Corollary 12 where
\[ \omega(Y) = \max_{y_0 \in Y} \min_{y \in Y \setminus \{y_0\}} \text{ord}(y - y_0) = m \cdot \max_{y_0 \in Y \setminus \{y_0\}} \min_{y \in \mathbb{Z}/m\mathbb{Z} \setminus \{y_0\}} \frac{1}{\gcd(m, y - y_0)} = \delta^{-1}_ym. \]
Now in an entirely similar way, it is found, in view of Corollary 10 that
\[ |X + Y| \geq \min(\delta^{-1}_Xm, |X| + |Y| - 1). \]
This concludes the proof, considering that \( \delta_Y = 1 \) if there exists \( y_0 \in Y \) such that \( m \) is coprime with \( y - y_0 \) for every \( y \in Y \setminus \{y_0\} \) (and symmetrically with \( X \)).

We now use P. Hall’s theorem about distinct representatives [H] to say something on how to “localize” some elements of a subset.

**Theorem 25** (Hall’s theorem). Let \( S_1, \ldots, S_k \) be sets \((k \in \mathbb{N}^+)\). There then exist \((pairwise)\) distinct elements \( s_1, \ldots, s_k \) such that \( s_i \in S_i \) if and only if for each \( h = 1, \ldots, k \) the union of any \( h \) of \( S_1, \ldots, S_k \) contains at least \( h \) elements.

More precisely, suppose \( A \) is a cancellative semigroup and let \( X, Y \) be non-empty finite subsets of \( A \) such that \( |X + Y| < \omega(Y) \). Clearly, this implies \( Y \cap A^x \neq \emptyset \).

Define \( k := |X| \) and \( \ell := |Y| \), and let \( x_1, \ldots, x_k \) be a numbering of \( X \) and \( y_1, \ldots, y_\ell \) a numbering of \( Y \). Then consider the \( k \)-by-\( \ell \) matrix, say \( \alpha(X,Y) \), whose entry in the \( i \)-th row and \( j \)-th column is \( x_i + y_j \). Any element of \( X + Y \) appears in \( \alpha(X,Y) \), and vice versa any entry of \( \alpha(X,Y) \) is an element of \( X + Y \). Also, Theorem 25 and our hypotheses give \( |X + Y| \geq k + \ell - 1 \). So it is natural to try to gain some information about where in the matrix \( \alpha(X,Y) \) to look for \( k + \ell - 1 \) distinct elements of \( X + Y \). In this respect we have the following proposition, whose proof is quite similar to the one of a weaker result in [R, Section 6], which is, in turn, focused on the less general case of a group of prime order:

**Proposition 26.** Assume that \( \langle Y \rangle \) is commutative and let \( Z \) be any subset of \( X + Y \) of size \( \ell - 1 \), for instance \( Z = x_1 + \{y_1, \ldots, y_{\ell-1}\} \). Then we can choose one element from each row of \( \alpha(X,Y) \) in such a way that \( Z \) and these elements form a subset of \( X + Y \) of size \( k + \ell - 1 \).

**Proof.** For each \( i = 1, \ldots, k \) let \( Z_i := (x_i + Y) \setminus Z \) and note that \( Z_i \) is a subset of the \( i \)-th row of \( \alpha(X,Y) \). Then \( Z_{i_1} \cup \cdots \cup Z_{i_h} = (\{x_{i_1}, \ldots, x_{i_h}\} + Y) \setminus Z \) for any positive integer \( h \leq k \) and all distinct \( i_1, \ldots, i_h \in \{1, \ldots, k\} \), with the result that

\[
|Z_{i_1} \cup \cdots \cup Z_{i_h}| \geq |\{x_{i_1}, \ldots, x_{i_h}\} + Y| - |Z| \geq h + \ell - 1 - (\ell - 1) = h,
\]

thanks to Theorem 25 and the fact that \( |\{x_{i_1}, \ldots, x_{i_h}\} + Y| \leq |X + Y| < \omega(Y) \) by Lemma 16 and the assumption that \( |X + Y| < \omega(Y) \). It follows from Hall’s theorem that we can find \( k \) distinct elements \( z_1, \ldots, z_k \) such that \( z_1 \in Z_1, \ldots, z_k \in Z_k \), and these, together with the \( \ell - 1 \) elements of \( Z \), provide a total amount of \( k + \ell - 1 \) elements in \( X + Y \), since \( Z \cap Z_1 = \cdots = Z \cap Z_k = \emptyset \) (by construction). 

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