COMPACT HERMITIAN SYMMETRIC SPACES, COADJOINT ORBITS, AND THE DYNAMICAL STABILITY OF THE RICCI FLOW

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ABSTRACT. Using a stability criterion due to Kröncke, we show, providing \( n \neq 2k \), the Kähler–Einstein metric on the Grassmannian \( Gr_k(\mathbb{C}^n) \) of complex \( k \)-planes in an \( n \)-dimensional complex vector space is dynamically unstable as a fixed point of the Ricci flow. This generalises the recent results of Kröncke and Knopf–Sesum on the instability of the Fubini–Study metric on \( \mathbb{C}P^n \) for \( n > 1 \). The key to the proof is using the description of Grassmannians as certain coadjoint orbits of \( SU(n) \). We are also able to prove that Kröncke’s method will not work on any of the other compact, irreducible, Hermitian symmetric spaces.

1. Introduction

In 2013 Kröncke proved the surprising result that the Fubini–Study Kähler–Einstein metric on \( \mathbb{C}P^n \), \( n > 1 \), is unstable as a fixed point of the Ricci flow [21]. More precisely he showed that there are certain conformal (and hence non-Kähler) deformations of the Fubini–Study metric from which the Ricci flow never returns. This is in stark contrast to the behaviour of the Kähler–Ricci flow where Tian and Zhu [30] have shown that Kähler–Einstein metrics are essentially global attractors within their Kähler class. In [20] Knopf and Sesum give an independent verification of Kröncke’s result.

The behaviour of Ricci flow on manifolds admitting Kähler metrics is a topic of current interest (for example [14], [15], [18], and [23]). What Kröncke’s result suggests is that behaviour of the Ricci flow near the space of Kähler metrics is more complicated than was initially believed. If a Fano manifold \( M \) with Hodge number \( h^{1,1}(M) > 1 \) admits a Kähler–Einstein metric then it can be destabilised by a harmonic perturbation within the Kähler cone. This method can be used to show many known examples of Kähler–Einstein metrics are unstable. However, as the complex dimension of the Fano manifold grows, there are numerous examples of Kähler–Einstein manifolds with \( h^{1,1}(M) = 1 \). One such class of Kähler–Einstein manifolds are the compact, irreducible, Hermitian symmetric spaces. These manifolds were completely classified by E. Cartan into six types; there are four infinite families and two exceptional spaces. Each of these spaces admits a Kähler–Einstein metric unique up to automorphisms of the complex structure; this metric is the symmetric metric on each manifold. We will henceforth not distinguish between the symmetric space and its Kähler–Einstein metric.

The main result that we prove in this paper is a generalisation of the \( \mathbb{C}P^n \) result to almost all the members of one of the infinite families of Hermitian symmetric space, the Grassmannians of complex \( k \)-planes in an \( n \)-dimensional complex vector space which we denote \( Gr_k(\mathbb{C}^n) \).

**Theorem A.** If \( k, n \in \mathbb{N} \) with \( 1 < k < n \) and \( n \neq 2k \), then \( Gr_k(\mathbb{C}^n) \) is dynamically unstable as a fixed point of the Ricci flow.

We cannot say anything about the stability of the spaces \( Gr_k(\mathbb{C}^{2k}) \) apart from the case when \( k = 1 \) as then \( Gr_1(\mathbb{C}^2) \cong \mathbb{C}P^1 \cong S^2 \). In this case, the Kähler–Einstein metric is the
round metric and this is known to be dynamically stable by a result of Hamilton [16] (Chow later proved that the round metric on $S^2$ is a global attractor for the normalised Ricci flow starting at any initial metric [7]).

The stability criterion proved by Kröncke is very simple to state (c.f. Theorem 2.6): if an Einstein metric with Einstein constant $\frac{1}{\tau} > 0$ admits an eigenfunction of the Laplacian, $f$ say, with eigenvalue $-\frac{1}{\tau}$ and the integral over the manifold of $f^3$ does not vanish, then the metric is dynamically unstable. It is a classical result of Matsushima [22] that, on a Fano Kähler–Einstein manifold, there is a bijection between Killing fields and the eigenspace corresponding to $-\frac{1}{\tau}$. Hence Kähler–Einstein manifolds with large symmetry groups are ideal candidates on which to attempt to use Kröncke’s result to investigate stability.

The methods used in [20] and [21] to show a destabilising eigenfunction exists on $\mathbb{C}P^n$ use the generalised Hopf fibration to lift the problem to finding certain $U(1)$-invariant functions on the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. This paper takes a totally different approach by viewing the Grassmannians as adjoint orbits of $SU(n)$ and using techniques coming from symplectic geometry (such as the Duistermaat–Heckman formula) to construct eigenfunctions and make calculations of the relevant integrals. The benefit of this viewpoint is that it is immediate how to generalise to other manifolds that occur as coadjoint orbits of Lie groups.

We investigate the stability criterion on compact irreducible, Hermitian, symmetric spaces other than Grassmannians (we give an explicit list of these spaces in Section 2). By an application of the Chevalley–Shepherd–Todd theorem we are able to prove the following theorem which effectively means Kröncke’s method will not apply to any other Hermitian symmetric space.

**Theorem B.** Let $(M, g)$ be a compact, irreducible, Hermitian symmetric space other than a Grassmannian $Gr_k(\mathbb{C}^n)$ with $n \neq 2k$ and let $g$ be the canonical Kähler–Einstein metric normalised to have Einstein constant $\frac{1}{2\tau}$. Then any $(-\frac{1}{\tau})$-eigenfunction of the Laplacian, $f$, satisfies

$$\int_M f^3 \, dV_g = 0.$$ 

Theorem B shows that new techniques will be needed to investigate the dynamical stability of most of the compact, irreducible, Hermitian, symmetric spaces. Given the result in Theorem A it might still be reasonable to expect the Kähler–Einstein metrics to be unstable as fixed points of the Ricci flow but as critical points of the $\nu$ functional they have a very high degree of degeneracy.

2. Background

2.1. **Stability.** Einstein metrics $g$ satisfying $\text{Ric}(g) = \frac{1}{2\tau}g$ for $\tau \in \mathbb{R}$, evolve via homothetic scaling under the Ricci flow

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g).$$

It is therefore useful to view Einstein metrics as fixed points of the Ricci flow up to a normalisation of the volume of the metric by homothetic scaling. A natural question is whether a given Einstein metric is stable as a fixed point in the sense that the Ricci flow starting at a small perturbation of the metric will return to the original Einstein metric. Perelman [26] introduced a functional $\nu(g)$ which is stationary at shrinking gradient Ricci solitons (every Einstein metric with $\tau > 0$ is such a soliton) and which is otherwise strictly increasing along
the Ricci flow. This allows the stability of an Einstein metric to be investigated by calculation of the second variation of the \( \nu \) functional along potentially destabilising directions. If the entropy increases along a particular direction then the corresponding perturbation of the Einstein metric will never return under the flow. This process was first carried out for Einstein metrics by Cao, Hamilton, and Ilmanen [4] and later generalised by Cao and Zhu [6].

**Theorem 2.1** (Cao–Hamilton–Ilmanen [4]). Let \((M, g)\) be an Einstein metric with Einstein constant \( \frac{1}{\tau} > 0 \). Let \( h \in s^2(TM^*) \). Then

\[
\frac{d^2}{ds^2} \nu(g + sh)|_{s=0} = -\tau \frac{\operatorname{Vol}(M, g)}{\operatorname{Vol}(M, g)} \int_M \langle N(h), h \rangle dV_g,
\]

where

\[
N(h) = \frac{1}{2} \Delta h + \operatorname{Rm}(h, \cdot) + \operatorname{div}^* \operatorname{div}(h) + \frac{1}{2} \nabla^2 v_h - \frac{g}{2n\tau \operatorname{Vol}(M, g)} \int_M \operatorname{tr}(h) dV_g, \tag{2.1}
\]

and \( v_h \) is the unique solution to

\[
\Delta v_h + \frac{v_h}{2\tau} = \operatorname{div}(\operatorname{div}(h)).
\]

The diffeomorphism and scale invariance of \( \nu(g) \) means that to check linear stability one only needs to consider perturbations \( h \in s^2(TM^*) \) satisfying

\[
\operatorname{div}(h) = 0 \quad \text{and} \quad \langle h, g \rangle_{L^2} = \int_M \operatorname{tr}(h) dV_g = 0.
\]

In this case, the stability operator \( N \) in Equation (2.1) reduces to

\[
N(h) = \frac{1}{2} \Delta h + \operatorname{Rm}(h, \cdot) = \frac{1}{2} (\Delta_L + \frac{1}{\tau}) h,
\]

where \( \Delta_L \) is the Lichnerowicz Laplacian

\[
\Delta_L h := \Delta h + 2\operatorname{Rm}(h, \cdot) - \operatorname{Ric} \cdot h - h \cdot \operatorname{Ric}.
\]

We thus have the following definitions:

**Definition 2.2** (Linear stability of Einstein metrics). Let \((M, g)\) be a compact Einstein manifold satisfying \( \operatorname{Ric}(g) = \frac{1}{2\tau} g \) and let \(-\kappa\) be the largest eigenvalue of the Lichnerowicz Laplacian restricted to the space of divergence-free, \( g \)-orthogonal tensors.

1. If \( \kappa > \frac{1}{\tau} \), \( g \) is called *linearly stable*.
2. If \( \kappa = \frac{1}{\tau} \), \( g \) is called *neutrally linearly stable*.
3. If \( \kappa < \frac{1}{\tau} \), \( g \) is called *linearly unstable*.

**Definition 2.3** (Dynamical stability of Einstein metrics). Let \((M^n, g_E)\) be a compact Einstein manifold. The metric \( g_E \) is said to be *dynamically stable* for the Ricci flow if for any \( m \geq 3 \) and any \( C^m \)-neighbourhood \( U \) of \( g_E \) in the space of sections \( \Gamma(s^2(TM^*)) \), there exists a \( C^{m+2} \) neighbourhood of \( g_E \), \( V \subset U \), such that:

1. for any \( g_0 \in V \), the volume normalised Ricci flow

\[
\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g) + \frac{2}{n} \left( \int_M \operatorname{scal}(g) dV_g \right) g,
\]

with \( g(0) = g_0 \) exists for all time,
2. the metrics \( g(t) \) converge modulo diffeomorphism to an Einstein metric in \( U \).

3
If the metric $g_E$ is not dynamically stable then it is said to be dynamically unstable.

If a metric is linearly unstable, then Sesum [28] showed that it is dynamically unstable. Similarly (though much more technically difficult to prove), under a mild assumption on the metric, Sesum proved that linear stability implies dynamical stability. In the case that the metric is neutrally linearly stable, nothing can be inferred about the dynamical stability.

In general, it is very difficult to analyse the spectrum of the Lichnerowicz Laplacian for an arbitrary Einstein metric. If the metric has some extra structure then more can be said. In the case the Einstein metric is Kähler–Einstein then there is the following topological condition (originally stated in [4] and proved for the more general class of Kähler–Ricci solitons in [13])

**Proposition 2.4** (Cao–Hamilton–Ilmanen). Let $(M, J, g)$ be a Kähler–Einstein metric. If the Hodge number $h^{1,1}(M) > 1$ then $g$ is linearly unstable.

This proposition can be seen as generalising the fact that any product of Einstein metrics with fixed Einstein constant $\frac{1}{2\pi}$ is unstable under the Ricci flow. The product of any two Kähler–Einstein metrics always has $h^{1,1}(M) > 1$.

In [5], Cao and He made a complete study of the stability of the simply-connected, compact, irreducible, symmetric spaces. The spaces where the metric is Kähler–Einstein can be written in the form $M = G/H$ where $G$ is a connected compact simple Lie group and $H$ is the isotropy subgroup. We note that the identity component of the isometry group $\text{Iso}_0 = G$ and so the Lie algebra of Killing fields $\mathfrak{k} = \text{Lie}(\text{Iso}_0)$ is isomorphic to the Lie algebra $\mathfrak{g}$. All the manifolds in the following theorem have $h^{1,1}(M) = 1$.

**Theorem 2.5** (Cao–He, c.f. Theorem 4.3 in [5]). The linear stability of the irreducible compact Hermitian symmetric spaces $M = G/H$ is as follows:

1. $M$ is linearly unstable if:
   - $M$ is the space of compatible complex structures on $\mathbb{H}^n$, $M = \text{Sp}(n)/U(n)$, for $n > 1$.

2. $M$ is neutrally linearly stable if:
   - $M$ is a complex Grassmannian $\text{Gr}_k(\mathbb{C}^n)$, $M = \text{SU}(n)/S(U(k) \times U(n-k))$, where $n > 2$ and $0 < k < n$,
   - $M$ is a complex hyperquadric $Q_n$, $M = \text{SO}(n+2)/(\text{SO}(n) \times \text{SO}(2))$, where $n \geq 4$,
   - $M$ is a space of orthogonal almost complex structures on $\mathbb{R}^{2n}$, $M = \text{SO}(2n)/U(n)$, for $n > 2$,
   - $M$ is one of the exceptional spaces, $M = E_6/(\text{SO}(10) \times \text{SO}(2))$, or $M = E_7/(E_6 \times \text{SO}(2))$. 
(3) If $M$ is the sphere $S^2 \cong Gr_1(C^2) \cong \frac{SO(4)}{U(2)}$ then $M$ is dynamically stable and so linearly stable.

Missing from this list (as it is not irreducible) is the hyperquadric $Q_2 = \frac{SO(4)}{SO(2) \times SO(2)} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.

It is unstable as it is a product. The hyperquadric $Q_3 \cong \frac{Sp(2)}{U(2)} \times SO(2)$ has $h^{1,1} = 1$ but is nevertheless linearly unstable by a result of Gasqui and Goldschmidt [11].

What the Cao–He result shows is that most of the Hermitian symmetric spaces are neutrally linearly stable. In particular, the complex projective spaces $\mathbb{CP}^n = Gr_1(C^{n+1})$ with $n > 1$ are all neutrally linearly stable.

On any Einstein manifold $(M, g)$ with Einstein constant $\frac{1}{2\tau}$, if there is an eigenfunction $f$ satisfying $\Delta f = -\frac{1}{\tau} f$ then we define the tensor

$$h_f := (\Delta f)g - \text{Hess}(f) + \frac{f}{2\tau} g.$$ 

It can be shown ([4] , [5]) that $h_f$ is divergence free, $L^2$-orthogonal to $g$ and satisfies

$$\Delta_L h_f = -\frac{1}{\tau} h_f.$$ 

In 2013 Kröncke proved the following stability criterion by computing the third variation of the $\nu$ functional.

**Theorem 2.6** (Kröncke, Theorem 1.7 in [21]). Let $(M, g)$ be an Einstein metric with Einstein constant $\frac{1}{2\tau}$ and let $f$ be an eigenfunction of the Laplacian with eigenvalue $-\frac{1}{\tau}$. If the integral

$$\int_M f^3 dV_g \neq 0,$$

then $g$ is dynamically unstable as a fixed point of the Ricci flow and is destabilised by the tensor $h_f$.

Kröncke then constructed a eigenfunction satisfying the condition (2.2) for the spaces $\mathbb{CP}^n$ with $n > 1$ and proved:

**Corollary 2.7.** The Fubini–Study metrics on $\mathbb{CP}^n$, $n > 1$ are dynamically unstable as fixed points of the Ricci flow.

This result was somewhat unexpected as a long-standing conjecture in the field had included $\mathbb{CP}^2$ on the list of stable, four-dimensional geometries for the Ricci flow. Theorem [A] can be seen as a generalisation of the $\mathbb{CP}^n$ results of Kröncke and Knopf–Sesum; however, as mentioned in the introduction, our construction of eigenfunctions and method of evaluating the integral is totally different to the methods used in [20] and [21].

**2.2. Geometry of coadjoint orbits.** In this section $G$ is a compact, semisimple Lie group. Hence the Killing form $\langle \cdot, \cdot \rangle$ is non-degenerate and so the adjoint representation of $G$ is Euclidean. We can also use the Killing form (or any $Ad_G$-invariant inner product) to identify $g$ and $g^*$. The coadjoint action of $G$ on $g^*$ is defined by

$$\text{Ad}_g^*(\xi)(X) := \xi(\text{Ad}_g^{-1}(X))$$
for \( g \in G, \xi \in g^* \) and \( X \in g \). If \( \xi(\cdot) = \langle \cdot, x \rangle \) then \( \text{Ad}_g^*(\xi)(\cdot) = \langle \cdot, \text{Ad}_g(x) \rangle \) and we have a straightforward identification of coadjoint and adjoint orbits via the Killing form.

For \( \xi \in g \) we consider the orbit \( O_\xi \) of \( \xi \) under the adjoint action of \( G \). Denote by \( H \) stabiliser of \( \xi \) and let \( h \subset g \) be its Lie algebra. Then \( g = h \oplus m \) where \( m \) can be identified with the tangent space to \( O_\xi \) at \( \xi \). The subalgebra \( h \) is the kernel of the map \( \text{ad}(\xi) : g \rightarrow g \) and thus \( m \) is the image.

We let \( T \) be a maximal torus of \( G \) and take \( t = \text{Lie}(T) \) to be its Lie algebra. The Weyl group \( W = N_G(T)/T \) where \( N_G(T) \) is the normaliser of \( T \) in \( G \). A classical theorem (see for example Bott [3]) yields:

1. \( O_\xi \cap t \neq \emptyset \),
2. \( O_\xi \cap t \) is a \( W \)-orbit.

This means we lose nothing by taking the element representing the orbit \( \xi \in t \).

The orbits have the structure of a complex manifold. Decomposing the complexified Lie algebra \( g \otimes \mathbb{C} \) we get

\[
g \otimes \mathbb{C} = t_\mathbb{C} \oplus \bigoplus_{\alpha : \langle \alpha, \xi \rangle = 0} R_\alpha \oplus A \oplus \bar{A}
\]

where \( \alpha \in t_\mathbb{C} \) are the roots of \( G \), \( R_\alpha \) is the root space of \( \alpha \), and \( A \) is the span of the root spaces satisfying

\[
[\xi, r_\alpha] = i\langle \alpha, \xi \rangle r_\alpha, \text{ with } \langle \alpha, \xi \rangle > 0 \text{ for all } r_\alpha \in R_\alpha.
\]

One can identify \( m_\mathbb{C} \cong A \oplus \bar{A} \) and show that \( A \) and \( t_\mathbb{C} \oplus A \) are Lie subalgebras of \( g \otimes \mathbb{C} \). By defining \( m^{(1,0)} = A \) we get a \( G \)-invariant complex structure on \( O_\xi \).

The Kirillov–Kostant– Souriau symplectic form is defined as

\[
\omega_\xi(x, y) = -\langle \xi, [x, y] \rangle,
\]

for \( x, y \in m \). This is extended over \( O_\xi \) using the adjoint action. This form is compatible with the complex structure and gives the orbit the structure of a Kähler manifold. In the case when the center of \( H \) has dimension 1 (which will be the cases that we wish to consider in this article), the induced metric is Kähler–Einstein (c.f.[2] Proposition 8.85).

2.3. Properties of the eigenfunctions. We will now show how to construct eigenfunctions for the Laplacian of the Kähler–Einstein metric on \( O_\xi \). We begin by recalling a theorem of Matsushima [22] which says that for any Fano Kähler–Einstein manifold \((M, g, J)\) there is an isomorphism between the \((-1/\tau)\)-eigenspace, \( E_{(-1/\tau)} \), and the Lie algebra of Killing vector fields \( \mathfrak{k} \) given by

\[
\phi \rightarrow -J\nabla \phi,
\]

where \( J \) is the complex structure.

All compact Riemannian symmetric spaces can be constructed in the form \( M = \text{Iso}/\text{Iso}_p \) where \( \text{Iso} \) is the isometry group of \((M, g)\) and \( \text{Iso}_p \) is the isotropy group of isometries fixing a point. For the spaces \( G/H \) in Theorem 2.5 we have

\[
g \cong \text{Lie}(\text{Iso}) \cong \mathfrak{k}.
\]

This map can be realised by the assignment

\[
\eta \rightarrow \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(t\eta)}(Z) = [\eta, Z],
\]
Another isomorphism between the Lie algebra $\mathfrak{g}$ and $E_{-(1/\tau)}$ is given by the following map $\varphi : \mathfrak{g} \to E_{-(1/\tau)}$

$$\varphi(\eta) = f_\eta(Z) := \langle \eta, Z \rangle - \frac{1}{\text{Vol}(O_\xi)} \int_{O_\xi} \langle \eta, p \rangle \omega^n(p),$$

where $\eta \in \mathfrak{g}$. The following lemma is standard but we include a short proof as it does not appear in the literature in this form.

**Lemma 2.8.** If the Kähler–Einstein metric on $O_\xi$ has Einstein constant $\frac{1}{2\tau}$, then the functions $f_\eta$ defined in Equation (2.3), satisfy

$$\Delta f_\eta = -\frac{1}{\tau} f_\eta.$$

**Proof.** Let $X \in \mathfrak{m}$ and consider

$$F(t) = f_\eta(\text{Ad}_{\exp(tX)} \xi) = \langle \eta, \text{Ad}_{\exp(tX)} \xi \rangle = \langle \text{Ad}_{\exp(-tX)} \eta, \xi \rangle.$$

Taking derivatives we see

$$\frac{dF}{dt} \bigg|_{t=0} = -\langle \xi, [X, \eta] \rangle = -\omega(X, \eta) = g(X, J\eta).$$

Hence $\nabla f_\eta = J\eta$ (here we identify $\eta$ with the Killing field it generates on $O_\xi$). As $\eta$ is a holomorphic Killing field we invoke Matsushima’s theorem which says the map

$$\phi \to \nabla \phi,$$

is an isomorphism between the eigenspace $E_{-1/\tau}$ and $J\mathfrak{k}$ where $\mathfrak{k}$ is the space of Killing fields. Hence, as $f_\eta$ has mean value zero by construction, we see $f_\eta$ is an eigenfunction of the Laplacian with eigenfunction $-\frac{1}{\tau}$.

An element $X \in \mathfrak{g}$ is said to be regular if its centraliser in $\mathfrak{g}$ is of smallest possible dimension. The regular elements of $\mathfrak{t}$ we will denote by $\mathfrak{t}_{\text{reg}}$. The following lemma will be vital in our analysis.

**Lemma 2.9** (Bott [3]). Let $D \in \mathfrak{t}$ be regular. Then the function $f_D$ defined by Equation (2.3) satisfy:

1. The critical points of $f_D$ are non-degenerate and of even index.
2. The critical points arise as the orbit of $\xi \in \mathfrak{t}$ under the action of Weyl group $W$.

We remark that $f_D$ is a Hamiltonian function for the action on $O_\xi$ generated by $D$. If $\Lambda \subset \mathfrak{t}$ is the weight lattice, then choosing $D \in \Lambda \otimes \mathbb{Z} \mathbb{Q}$ will generate an $S^1$-action. We denote the set of elements in $\mathfrak{t}$ that generate closed orbits by $\mathfrak{t}_{\text{c}}$. The set $\mathfrak{t}_{\text{reg}} \cap \mathfrak{t}_{\text{c}}$ is dense in $\mathfrak{t}$ (with respect to the Euclidean topology). It turns out that the eigenfunctions $f_D$ generated by $D \in \mathfrak{t}_{\text{reg}} \cap \mathfrak{t}_{\text{c}}$ are the only ones that one needs to check the stability condition (2.2) on.

**Proposition 2.10.** Let $G/H = O_\xi$ be one of the symmetric spaces in Theorem 2.5 and let $T$ be a maximal torus in $G$ with Lie algebra $\mathfrak{t}$. Suppose that there exists $f \in E_{-(1/\tau)}$ such that

$$\int_{O_\xi} f^3 \omega^n \neq 0,$$

then there exists $D \in \mathfrak{t}_{\text{reg}} \cap \mathfrak{t}_{\text{c}}$ such that

$$\int_{O_\xi} f_D^3 \omega^n \neq 0.$$
Proof. As the map \( \varphi : g \to E_{-1/\tau} \) given by \( \varphi(X) = f_X \) is an isomorphism, there must exist \( \eta \in g \) such that \( f = \varphi \). As remarked in Section 2.2, given a fixed maximal torus \( T \) with Lie algebra \( t \), any element of \( g \) is in the adjoint orbit of some element in \( t \). If \( D = \text{Ad}_g(\eta) \in t \) for some \( g \in G \) then, by the Ad-invariance of the inner product,

\[
\varphi(\eta)(Z) = \langle \eta, Z \rangle = \langle \text{Ad}_g^{-1}(D), Z \rangle = \langle D, \text{Ad}_g(Z) \rangle = f_D(\text{Ad}_g(Z)).
\]

In other words, \( f_\eta = \text{Ad}_g^* f_D \). Note as \( \text{Ad}_g : O_\xi \to O_\xi \) is an orientation-preserving diffeomorphism

\[
\int_{O_\xi} f_\eta^3 \omega^n = \int_{O_\xi} (\text{Ad}_g^* f_D)^3 \omega^n = \int_{O_\xi} (\text{Ad}_g^* f_D)^3 (\text{Ad}_g^* \omega)^n = \int_{O_\xi} f_D^3 \omega^n,
\]

where the penultimate equality holds because \( \text{Ad}_g^* \omega = \omega \) as \( \text{Ad}_g \) is an isometry. Finally we see we can take \( D \in t_{\text{reg}} \cap t_c \) as this set is dense in \( t \).

\[\Box\]

2.4. Computing integrals. We will need to be able to compute integrals of powers of \( f \) over the orbit. This is achieved by the famous Duistermaat–Heckman formula [9] (see [24] for the form we are using). On a symplectic manifold \((M, \omega)\) with a Hamiltonian circle action that has an associated Hamiltonian function \( \varphi \) with non-degenerate critical points, the Duistermaat–Heckman formula is

\[
\int_M e^{-t\varphi(p)} \omega^n(p) = \frac{n!}{t^n} \sum_{q \text{ critical}} e^{-t\varphi(q)} \overline{\omega}(q),
\]

where the \( \overline{\omega}(q) \) is the product of the weights of the circle action that is induced on the tangent space at each fixed point \( q \).

The holomorphic tangent space at a critical point \( q \) is identified with the span of the root spaces \( R_\alpha \) satisfying \( \langle q, \alpha \rangle > 0 \), we will denote this set \( P(q) \). The derivative of \( r_\alpha \to \text{ad}_{\exp(tD)} r_\alpha \) at \( t = 0 \) is \( \langle \alpha, D \rangle r_\alpha \). Hence the weight at each fixed point \( q \in t \) is given by

\[
\overline{\omega}(q) = \prod_{\alpha \in P(q)} \langle \alpha, D \rangle.
\]

Proposition 2.11. Let \( D \in t_{\text{reg}} \cap t_c \) and let \( f_D(Z) = \langle Z, D \rangle \). Then

\[
\int_{O_\xi} e^{-tf(p)} \omega^n(p) = \frac{n!}{t^n} \sum_{w \in W/\text{stab}(\xi)} e^{-tf(w, \xi)} \prod_{\alpha \in P(w, \xi)} \langle \alpha, D \rangle,
\]

where \( n \) is the complex dimension of the orbit \( O_\xi \).

Proof. We simply apply the Duistermaat–Heckman formula to the function \( f_D \) which is the Hamiltonian for the circle action generated by \( D \in t_{\text{reg}} \cap t_c \). As the element \( D \) is regular, Lemma 2.9 says the critical points are non-degenerate precisely the orbit of \( \xi \) under the action of the Weyl group \( W \). The value of the weights follows from the previous discussion. \[\Box\]

Formulae similar to (2.4) occur in the theory of the orbit method developed by Kirillov [19]. In this theory integrals over certain coadjoint orbits correspond to the characters of the representation corresponding to the orbit. Similar expressions also appear elsewhere in the literature, for example in the papers of Berline and Vergne [1], Paradan [25], and Rossman [27].
3. The proof of Theorem B

The proof of Theorem B is based on the classification of certain invariant polynomial algebras. If we let $G$ be one of the compact simple Lie groups appearing as the larger group in Theorem 2.5, then we denote by $\mathbb{R}[\mathfrak{g}]^G$ the graded algebra of $G$-invariant polynomials functions on $\mathfrak{g}$. If $T$ is a maximal torus of $G$ with Lie algebra $t$ and Weyl group $W$, then the Chevalley restriction theorem (see for example Section 23 in [17]) yields an isomorphism $r : \mathbb{R}[\mathfrak{g}]^G \to \mathbb{R}[t]^W$ where $r(P)$ is simply restriction of a polynomial $P \in \mathbb{R}[\mathfrak{g}]^G$ to $t$. By the Chevalley–Shephard–Todd theorem and the Shephard–Todd classification of complex reflection groups, see [8] and [29], the algebra $\mathbb{R}[t]^W$ is a polynomial algebra, with generators of well defined degrees. We list them in the following table with the groups in brackets having Lie algebra with the corresponding root system:

| Root system | Degree of generators in $\mathbb{R}[t]^W$ |
|-------------|---------------------------------|
| $A_n (SU(n+1))$ | $2, 3, \ldots, n+1$ |
| $B_n (SO(2n+1))$ | $2, 4, \ldots, 2n$ |
| $C_n (Sp(n))$ | $2, 4, \ldots, 2n$ |
| $D_n (SO(2n))$ | $2, 4, \ldots, 2(n-1); n$ |
| $E_6$ | $2, 5, 6, 8, 9, 12$ |
| $E_7$ | $2, 6, 8, 10, 12, 14, 18$ |

The following lemma shows that the stability integral (2.2) can be thought of as an element of $\mathbb{R}[\mathfrak{g}]^G$.

**Lemma 3.1.** Let $I^k : \mathfrak{g} \to \mathbb{R}$ be defined by

$$I^k(\eta) = \int_{O_{\xi}} f^k(p) \omega^n(p).$$

Then $I^k$ is a (possibly trivial) $Ad_G$-invariant, homogenous, degree-$k$ polynomial.

**Proof.** The $Ad_G$-invariance comes from the $Ad_G$-invariance of the Kähler form $\omega$. The proof that the function is a homogenous, degree-$k$ polynomial is straightforward if one picks a basis of $\mathfrak{g}$ and then calculates in coordinates. □

We now give the proof of Theorem B for all the cases other than the Grassmannians $Gr_k(\mathbb{C}^{2k})$. The fact that the integral (2.2) vanishes on these spaces is a consequence of the calculation in section 5 specifically Equation (5.2).

**Proof.** We note by Proposition 2.10 we might as well assume that a destabilising eigenfunction is of the form $f_D$ for $D \in t_{reg} \cap t_c$. If the symmetric space is not a Grassmanian then it is of the form $G/H$ with $G$ being one of the groups $B_n, D_n, E_6$ or $E_7$. Lemma 3.1 shows that $I^3$ is a degree 3, homogenous $G$-invariant polynomial and so by the Chevalley restriction theorem yields a degree 3 element of $\mathbb{R}[t]^W$. However, the above table shows this function must vanish unless $G = D_3 = SO(6) = A_3$. One can show that $SO(6)/U(3) \cong \mathbb{C}P^3$ and $Q_4 \cong Gr(2, 4)$ and so the stability of these spaces follows from the type $A_n$ consideration. □

We also note that this method of proof also shows that the functions defined by $\langle \eta, Z \rangle$ for $\eta \in \mathfrak{g}$ and $Z \in O_{\xi}$ are automatically eigenfunctions; there is no need for the projection to the set of functions with mean value 0. This follows as there are no non-zero homogeneous, degree 1 polynomials that are invariant under the Weyl groups of the compact connected Lie groups we are considering in this article.
4. Combinatorial properties of certain determinants

In order to prove Theorem A we first need to collect some results on the power series of certain matrices that will appear after the manipulation of the righthand side of Equation (2.4). For \( m_1, m_2, \ldots, m_n \in \mathbb{R} \) and \( 0 < k \leq \lfloor n/2 \rfloor \) we consider the matrix valued function \( M : \mathbb{R} \to \text{Mat}_{\mathbb{R}}^{n \times n} \) given by

\[
M(t) = \begin{pmatrix}
  e^{-m_1 t} & e^{-m_2 t} & \cdots & e^{-m_n t} \\
  e^{-m_1 t} m_1 & e^{-m_2 t} m_2 & \cdots & e^{-m_n t} m_n \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{-m_1 t} m_1^{k-1} & e^{-m_2 t} m_2^{k-1} & \cdots & e^{-m_n t} m_n^{k-1} \\
  1 & 1 & \cdots & 1 \\
  m_1 & m_2 & \cdots & m_n \\
  \vdots & \vdots & \ddots & \vdots \\
  m_1^{n-k-1} & m_2^{n-k-1} & \cdots & m_n^{n-k-1}
\end{pmatrix}.
\] (4.1)

We will be interested in computing the derivatives of \( \det(M(t)) \) when \( t = 0 \); it is therefore useful to think of \( \det(M(t)) \) as the sum of various products of \( k \) functions. To compute the \( p^{th} \) derivative we can use the formula

\[
\frac{d^p}{dt^p} \det(M(t)) = \sum_{d_1+d_2+\cdots+d_k=p} \left( \frac{p!}{d_1!d_2!\cdots d_k!} \right) \det(M(R_1^{(d_1)}, R_2^{(d_2)}, \ldots, R_k^{(d_k)})),
\]

where \( d_i \in \mathbb{N} \cup \{ 0 \} \) and \( M(R_1^{(d_1)}, R_2^{(d_2)}, \ldots, R_k^{(d_k)}) \) is the matrix formed by taking \( d_i \) derivatives of the \( i^{th} \) row.

It is clear that, evaluating at \( t = 0 \), this formula is going to require the calculation of determinants of matrices of the form

\[
A = \begin{pmatrix}
  m_1^{e_1} & m_2^{e_1} & \cdots & m_n^{e_1} \\
  m_1^{e_2} & m_2^{e_2} & \cdots & m_n^{e_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_1^{e_k} & m_2^{e_k} & \cdots & m_n^{e_k} \\
  m_1^{n-k-1} & m_2^{n-k-1} & \cdots & m_n^{n-k-1} \\
  m_1^{n-k-2} & m_2^{n-k-2} & \cdots & m_n^{n-k-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_1 & m_2 & \cdots & m_n \\
  1 & 1 & \cdots & 1
\end{pmatrix},
\]

for exponents \( e_1, e_2, \ldots, e_k \in \mathbb{N} \). In fact, a matrix of the form \( A \) that gives a non-zero determinant can be written (after possibly reordering rows) in the form

\[
A_{ij} = m_j^{\lambda_i + n - i},
\] (4.2)

for some vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \) with \( \lambda_i \geq \lambda_{i+1} \). Such determinants are all multiples of the Vandermonde determinant \( V \) given by

\[
V = |m_j^{n-i}| = \prod_{1 \leq i < j \leq n} (m_i - m_j),
\]

which is the determinant of the \( \lambda = (0, 0, \ldots, 0) \) case of Equation (4.2). This leads to the definition of the Schur polynomials.
**Definition 4.1** (Schur polynomial). Given \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n \) with \( \lambda_i \geq \lambda_{i+1} \) the Schur polynomial \( S_\lambda \) is given by

\[
S_\lambda(m_1, m_2, \ldots, m_n) = \frac{|m_j^{\lambda_i+n-i}|}{V}.
\]

The Schur polynomial \( S_\lambda \) is a homogeneous, \( \text{Sym}_n \)-invariant multinomial of degree \( \sum_{i=1}^n \lambda_i \). It is straightforward to write a short list of these in degrees 0 to 3:

\[
\begin{align*}
S_{(0,0,\ldots,0)} &= 1, \\
S_{(1,0,\ldots,0)} &= m_1 + m_2 + \cdots + m_n, \\
S_{(2,0,\ldots,0)} &= \sum_{i=1}^{i=n} m_i^2 + \sum_{1 \leq i < j \leq n} m_im_j, \\
S_{(1,1,0,\ldots,0)} &= \sum_{i=1}^{i=n} m_im_j, \\
S_{(3,0,0,\ldots,0)} &= \sum_{i=1}^{i=n} m_i^3 + \sum_{1 \leq i < j \leq n} m_im_j^2 + \sum_{1 \leq i < j < l \leq n} m_im_jm_l, \\
S_{(2,1,0,\ldots,0)} &= \sum_{1 \leq i < j \leq n} m_i^2m_j + 2\sum_{1 \leq i < j < l \leq n} m_im_jm_l, \\
S_{(1,1,1,0,\ldots,0)} &= \sum_{1 \leq i < j < l \leq n} m_im_jm_l.
\end{align*}
\]

In order to keep track of signs we introduce the following function \( \sigma : \mathbb{Z} \rightarrow \{0, 1\} \) defined by

\[
\sigma(N) = \begin{cases} 
0 & \text{if } N \equiv 0, 1 \pmod{4}, \\
1 & \text{if } N \equiv 2, 3 \pmod{4}.
\end{cases}
\]

**Lemma 4.2.** Let \( k, n, \) and \( M(t) \) be as in Equation (4.1) and let \( V \) be the Vandemonde determinant. Then the power series expansion about 0 of \( \det(M(t)) \) begins

\[
\det(M(t)) = \frac{\varepsilon_{k,n}V}{(k(n-k))!} t^{k(n-k)} + \ldots ,
\]

where

\[
\varepsilon_{k,n} = (-1)^{k(n-k)+\sigma(k)+\sigma(n-k)},
\]

and

\[
c_0 = (k(n-k))! \prod_{i=1}^{i=k} \frac{(i-1)!}{(n-k+i-1)!}.
\]

**Proof.** In order to get a non-zero determinant when \( t = 0 \), we need the first \( k \) rows to yield the powers \( m_j^{n-k}, m_j^{n-k+1}, \ldots, m_j^{n-1} \). Given that there are existing powers \( m_i^0, m_i^1, \ldots, m_i^{k-1} \) we require

\[
\sum_{j=n-k}^{j=n-1} j - \sum_{j=k-1}^{j=k-1} j = \frac{k(2n - k - 1)}{2} - \frac{k(k - 1)}{2} = k(n-k),
\]

derivatives. The \( (k(n-k))^{th} \) derivative of \( \det(M(t)) \) evaluated at \( t = 0 \) is then some multiple of the Vandemonde determinant \( V \).
For $1 \leq i \leq k$ let $l_i = (n - k) + k - i = n - i$. For each $\sigma \in \text{Sym}_k$, let the number of derivatives of the $\sigma(i)^{th}$ line be given by

$$(l_i - \sigma(i) + 1).$$

(Hence the power of $m$ contributed by the $\sigma(i)^{th}$ row is $l_i$). Using the multinomial version of the Leibniz rule and weighting the resulting matrix by the sign of the permutation $\sigma$ (needed to reorder the rows so the powers of $m_i$ run from 1 to $n - 1$) we obtain the $(k(n - k))^\text{th}$ derivative of $\det(M(t))$ at $t = 0$ is given by

$$
\varepsilon_{k,n} V \left( \sum_{\sigma \in \text{Sym}_k} \text{sgn}(\sigma) \frac{[k(n - k)]!}{(l_1 - \sigma(1) + 1)! (l_2 - \sigma(2) + 1)! \ldots (l_k - \sigma(k) + 1)!} \right).
$$

The result now follows from the discussion in [10] where the quantity inside the brackets is shown to compute the number of standard Young tableau of row structure

$$(n - k, n - k, \ldots, n - k) \in \mathbb{N}^k.$$ 

The formula for $c_0$ can be computed by the famous Hook Length formula (see Section 4.1 in [10]). This gives the result.

The formula for $c_0$ (up to factors of $\pi$) recovers the volume of $Gr_k(\mathbb{C}^n)$ as first computed by Schubert [12]. We will see that $c_0$ is essentially the first term in the expansion of the Duistermaat–Heckman integral (2.4) which indeed should be the volume of the manifold computed with respect to the symplectic form.

In order to compute the stability integral (2.2), we will require the coefficient of $t^{k(n - k) + 3}$ in the power series expansion about 0 of $\det(M(t))$. After factoring out $V$, this coefficient will be a combination of the Schur polynomials $S_{(3,0,\ldots,0)}$, $S_{(2,1,0,\ldots,0)}$, and $S_{(1,1,1,0,\ldots,0)}$. The coefficient of each polynomial can be computed in terms constant $c_0$ given by Equation (4.3).

**Lemma 4.3.** Let $c_3 = \frac{c_3}{(k(n - k) + 3)!}$ be the coefficient of $t^{k(n - k) + 3}$ in the power series expansion about 0 of $\det(M(t))$ and let $\varepsilon_{k,n}$ be as in Lemma 4.2. Then

$$c_3 = \varepsilon_{k,n} V (c_{(3,0,0)} S_{(3,0,\ldots,0)} + c_{(2,1,0)} S_{(2,1,0,\ldots,0)} + c_{(1,1,1)} S_{(1,1,1,0,\ldots,0)}),$$

where

$$
\begin{align*}
\varepsilon_{(3,0,0)} &= \frac{[k(n - k) + 1][k(n - k) + 2][k(n - k) + 3]}{6} \frac{k(k + 1)(k + 2)}{n(n + 1)(n + 2)} c_0, \quad (4.4) \\
\varepsilon_{(2,1,0)} &= \frac{[k(n - k) + 1][k(n - k) + 2][k(n - k) + 3]}{3} \frac{(k - 1)k(k + 1)}{(n - 1)n(n + 1)} c_0, \quad (4.5) \\
\varepsilon_{(1,1,1)} &= \frac{[k(n - k) + 1][k(n - k) + 2][k(n - k) + 3]}{6} \frac{(k - 2)(k - 1)k}{(n - 2)(n - 1)n} c_0. \quad (4.6)
\end{align*}
$$

**Proof.** Let $\chi_1 = (3,0,\ldots,0)$, $\chi_2 = (2,1,0,\ldots,0)$, and $\chi_3 = (1,1,1,0,\ldots,0) \in \mathbb{Z}^k$. For $i = 1, 2$ or 3, let $\nu = (n - k, n - k, \ldots, n - k) + \chi_i$. Furthermore, for $j = 1, 2, \ldots, k$ let

$$l_j = (v_j + k - j).$$

Let row $\sigma(j)$ have $(l_j - \sigma(j) + 1)$ derivatives so, as in the proof of Lemma 4.3, the power of the ms in row $\sigma(j)$ is $l_j$. For a fixed element $\sigma \in \text{Sym}_k$, computing the derivative of $\det(M(t))$ at 0 and after dividing through by $V$, the distribution of the powers shows we get (abusing notation slightly) the Schur polynomial $S_{\chi_i}$.

Using the multinomial version of the Leibniz rule where we sum only over the distribution
of derivatives that yield powers of \( m \) running \( 1, 2, \ldots, n + 2 \) and weighting the resulting matrix by the sign of the permutation \( \sigma \) we obtain that this part of the \( (k(n - k) + 3)^{rd} \) derivative of \( \det(\mathcal{M}(t))/V \) at \( t = 0 \) (and hence the coefficient of \( S_{\chi_i} \)) is given by

\[
\varepsilon_{k,n} V \left( \sum_{\sigma \in \text{Sym}_k} \text{sgn}(\sigma) \frac{[k(n-k)+3]!}{(l_1 - \sigma(1)+1)!(l_2 - \sigma(2)+1)! \cdots (l_k - \sigma(k)+1)!} \right).
\]

Again, the discussion in [10] shows that the quantity inside the brackets computes the number of standard Young tableau of row structure \((n-k, n-k, \ldots, n-k) + \chi_i \). The result follows from the Hook Length formula. \( \square \)

5. Proof of Theorem \( \text{A} \)

The Lie algebra of \( SU(n) \), \( \mathfrak{su}(n) \), is identified with trace-free \( n \times n \) skew-Hermitian matrices. The rank of \( SU(n) \) is \( n-1 \) with a maximal torus \( T \) being given by diagonal matrices. Hence the Lie algebra of \( T \), \( t \), can be identified with

\[
\text{Diag}(\sqrt{-1} \mu_1, \sqrt{-1} \mu_2, \ldots, \sqrt{-1} \mu_n) \text{ with } \sum_i \mu_i = 0.
\]

The roots can identified with \( e_j - e_l \) for \( j \neq l \) where \( e_j \) is the diagonal matrix with the entry \( \sqrt{-1} \) in the \( j^{th} \) coefficient and we are using the inner product \( \langle X, Y \rangle = \text{tr}(X^*Y) \) to identify \( \mathfrak{su}(n) \) and \( \mathfrak{su}^*(n) \). The Weyl group \( W \cong \text{Sym}_n \) acts on \( T \) by permuting the elements of the diagonal and \( W \) acts on \( t \) permuting the \( \sqrt{-1} \mu_i \).

The adjoint orbits we consider can be represented by an element \( \xi \in t \). We let

\[
\xi = \text{Diag}(\sqrt{-1} \mu_1, \ldots, \sqrt{-1} \mu_{k}, \sqrt{-1} \mu_{k}, \ldots, \sqrt{-1} \mu_n)
\]

where \( \mu_1 > 0 \) and \( k \mu_1 + (n-k) \mu_2 = 0 \). In fact, it will be useful to fix

\[
\mu_1 = \frac{n-k}{n} \quad \text{and} \quad \mu_2 = -\frac{k}{n},
\]

so that \( \mu_1 - \mu_2 = 1 \). We get an identification of \( \mathcal{O}_\xi \) with \( \text{Gr}_k(\mathbb{C}^n) \) by considering the \( k \)-plane generated by the span of the \( \sqrt{-1} \mu_1 \) eigenspace at each point in the orbit.

The vector \( D \in t \) given by \( D = \text{Diag}(2\pi \sqrt{-1} m_1, 2\pi \sqrt{-1} m_2, \ldots, 2\pi \sqrt{-1} m_n) \), where \( m_j \in \mathbb{Z} \), \( m_j \neq m_l \) for \( j \neq l \), and \( \sum_j m_j = 0 \), generates a circle action on \( \mathcal{O}_\xi \). Recall the function \( f_D : \mathcal{O}_\xi \to \mathbb{R} \) defined by Equation (2.3). By Lemma 2.8 \( f_D \) is an eigenfunction of the Laplacian and by Lemma 2.9 the fixed points of the circle action (or equivalently the critical points of \( f_D \)) are the orbit of \( \xi \) under the Weyl group \( \text{Sym}_n \) which are the vectors consisting of to the \( n \)-plane of possible placements of the \( \mu_1 \)s. If we index a fixed point of the circle action by the \( k \)-element set \( J \subset \{1, 2, \ldots, n\} \) corresponding to this placement and denoting such a fixed point \( q_J \), then the value of the function \( f_D \) at this point is

\[
f_D(q_J) = 2\pi \left( \mu_1 \sum_{j \in J} m_j + \mu_2 \sum_{j \in J^c} m_j \right).
\]

The set of roots \( \alpha \in \Lambda_R \) such that \( \langle \alpha, q_J \rangle > 0 \) are \( e_{j_1} - e_{j_2} \) where \( j_1 \in J \) and \( j_2 \in J^c \). Hence the weight of the induced action on the holomorphic tangent space at \( q_J \) is

\[
\varpi(q_J) = \prod_{j \in J, l \in J^c} (m_j - m_l).
\]
The Duistermaat–Heckman formula yields
\[ \int_{O_k} e^{-tf_D(p)} \omega^{n-k}(p) \, dp = \frac{[k(n-k)]!}{V_k((n-k))} \sum_{J \subset \{1,2,\ldots,n\} \colon |J|=k} e^{-\left(\mu_1 \sum_{j \in J} m_j + \mu_2 \sum_{j \in J^c} m_j \right)2\pi t} \prod_{j \in J} (m_j - m_l) \prod_{j \in J^c} (m_j - m_l). \]

We manipulate this expression by pulling out the Vandermonde factor
\[ V = \prod_{1 \leq j < l \leq n} (m_j - m_l), \]
yielding
\[ \frac{[k(n-k)]!}{V_k((n-k))} \sum_{J \subset \{1,2,\ldots,n\} \colon |J|=k} \varepsilon_J e^{-\left(\mu_1 \sum_{j \in J} m_j + \mu_2 \sum_{j \in J^c} m_j \right)2\pi t} \prod_{j \in J} (m_j - m_l) \prod_{j < l \in J^c} (m_j - m_l), \]
where \(\varepsilon_J\) is the sign of the permutation sending \(1, \ldots, k\) to the sequence \(j_1 < j_2 < \cdots < j_k \in J\) and \(k+1, k+2, \ldots, n\) to the elements of \(J^c\).

The sum is the (Laplace) expansion of the determinant of the following matrix
\[ M(t) = \begin{pmatrix} e^{-\mu_1 m_1 t} & e^{-\mu_1 m_2 t} & \cdots & e^{-\mu_1 m_n t} \\ e^{-\mu_1 m_1 t} m_1 & e^{-\mu_1 m_2 t} m_2 & \cdots & e^{-\mu_1 m_n t} m_n \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\mu_2 m_1 t} m_1^{k-1} & e^{-\mu_2 m_2 t} m_2^{k-1} & \cdots & e^{-\mu_2 m_n t} m_n^{k-1} \\ e^{-\mu_2 m_1 t} m_1 & e^{-\mu_2 m_2 t} m_2 & \cdots & e^{-\mu_2 m_n t} m_n \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\mu_2 m_1 t} m_1^{n-k-1} & e^{-\mu_2 m_2 t} m_2^{n-k-1} & \cdots & e^{-\mu_2 m_n t} m_n^{n-k-1} \end{pmatrix}. \]

Hence, we have
\[ \int_{O_k} e^{-tf_D(p)} \omega^{k(n-k)}(p) \, dp = \frac{[k(n-k)]!}{V_k((n-k))} \det(M(2\pi t)) = \varepsilon_{k,n} \frac{[k(n-k)]!}{V_k((n-k))} \det(M(2\pi t)), \quad (5.1) \]
where \(M(t)\) is the matrix \((4.1)\) and \(\varepsilon_{k,n}\) is the signed quantity from Lemma 4.2. The final equality in Equation (5.1) follows as \(\sum_j m_j = 0\) and \(\mu_1 - \mu_2 = 1\).

We remark again, in order to be able to apply Kröhnke's test we need \(f_D\) to be a genuine eigenfunction of the Laplacian. This means we must normalise \(f_D\) so that it has mean value zero or, equivalently, so that the first derivative of
\[ \int_{O_k} e^{-tf_D(p)} \omega^{k(n-k)}(p) \]
vanishes when \(t = 0\). The \(t^{k(n-k)+1}\) coefficient of the power series expansion of \(M(t)\) is a multiple of \(S_{(1,0,\ldots,0)} = \sum_j m_j\) which vanishes and hence \(f_D\) is an eigenfunction. As mentioned after the proof of Theorem [13] in Section [3] the function \(f_D\) does not really need normalising as there are no homogenous, degree 1, \(\text{Sym}_n\)-invariant polynomials by considering the \(A_n\) part of the Chevalley–Shephard–Todd theorem.

To prove the theorem we need to show that the third derivative of the integral expansion does not vanish. By Lemma 4.3 the third derivative of the expansion at 0 will be a combination of the Schur polynomials \(S_{(3,0,\ldots,0)}\), \(S_{(2,1,0,\ldots,0)}\), and \(S_{(1,1,1,0,\ldots,0)}\). This yields a symmetric, homogenous degree 3 multinomial in the variables \(m_1, m_2, \ldots, m_n\) which we denote \(P_3(m_1, \ldots, m_n)\). The fact that the \(e^{(3,0,0)}\) coefficient in Lemma 4.3 is non-zero
means that the coefficient of the $m_i^3$ in $\mathcal{P}_3$ (which is $c(3,0,0)$) is non-zero. Hence the stability integral (2.2) will not be identically zero if we can show that $\mathcal{P}_3$ is not a multiple of $S_{(1,0,\ldots,0)}$.

Using the notation of Lemma 4.3, suppose that

$$c_3 = S_{(1,0,\ldots,0)}(\alpha S_{(2,0,\ldots,0)} + \beta S_{(1,1,0)})$$

for some $\alpha, \beta \in \mathbb{R}$. Then by equating coefficients of $m_i^3, m_i^2 m_j$, and $m_i m_j m_l$ we obtain

$$c(3,0,0) = \alpha,$$

$$c(3,0,0) + c(2,1,0) = 2\alpha + \beta,$$

$$c(3,0,0) + 2c(2,1,0) + c(1,1,1) = 3(\alpha + \beta).$$

This system is consistent if

$$c(3,0,0) - c(2,1,0) + c(1,1,1) = 0.$$  

Evaluating using Lemma 4.3 this yields

$$k(n-k)(2k-n) = 0.$$  

Hence we see that the third derivative vanishes identically if and only if $n = 2k$. The if part is easy to see in fact as, in the case $2k = n$, $\mu_1 = -\mu_2$ and so the determinant of the matrix $M(t)$ is clearly an even function of $t$ if $k$ is even and an odd function if $k$ is odd.

This finishes the proof of Theorem A and the $Gr_k(\mathbb{C}^{2k})$ case of Theorem B.

**References**

[1] Berline, N., and Vergne, M. Fourier transforms of orbits of the coadjoint representation. In *Representation theory of reductive groups (Park City, Utah, 1982)*, vol. 40 of Progr. Math. Birkhäuser Boston, Boston, MA, 1983, pp. 53–67.

[2] Besse, A. L. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[3] Bott, R. The geometry and representation theory of compact Lie groups. In *Proceedings of the SRC/LMS Research Symposium held in Oxford, June 28–July 15, 1979*, G. L. Luke, Ed., vol. 34 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge-New York, pp. v+341.

[4] Cao, H.-D., Hamilton, R., and Ilmanen, T. Gaussian densities and stability for some Ricci solitons. - (2004), preprint, arXiv:math/0404165 [math.DG].

[5] Cao, H.-D., and He, C. Linear stability of Perelman’s $\nu$-entropy on symmetric spaces of compact type. *J. Reine Angew. Math.* 709 (2015), 229–246.

[6] Cao, H.-D., and Zhu, M. On second variation of Perelman’s Ricci shrinker entropy. *Math. Ann.* 353, 3 (2012), 747–763.

[7] Chow, B. The Ricci flow on the 2-sphere. *J. Differential Geom.* 33, 2 (1991), 325–334.

[8] Coxeter, H. S. M. The product of the generators of a finite group generated by reflections. *Duke Math. J.* 18 (1951), 765–782.

[9] Duistermaat, J., and Heckman, G. On the variation in the cohomology of the symplectic form of the reduced phase space. *Inventiones Mathematicae* 69 (1982), 259–269.

[10] Fulton, W., and Harris, J. *Representation theory*, vol. 129 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[11] Gasqui, J., and Goldschmidt, H. *Radon transforms and the rigidity of the Grassmannians*, vol. 156 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2004.

[12] Griffiths, P., and Harris, J. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[13] Hall, S. J., and Murphy, T. On the linear stability of Kähler-Ricci solitons. *Proc. Amer. Math. Soc.* 139, 9 (2011), 3327–3337.

[14] Hall, S. J., and Murphy, T. Variation of complex structures and the stability of Kähler-Ricci solitons. *Pacific J. Math.* 265, 2 (2013), 441–454.

[15] Hall, S. J., and Murphy, T. On the spectrum of the Page and the Chen-LeBrun-Weber metrics. *Ann. Global Anal. Geom.* 46, 1 (2014), 87–101.
[16] Hamilton, R. S. The Ricci flow on surfaces. In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, vol. 71 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.

[17] Humphreys, J. E. *Introduction to Lie algebras and representation theory*, vol. 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[18] Isenberg, J., Knopf, D., and Sesum, N. Non-Kähler Ricci flow singularities that converge to Kähler-Ricci solitons. preprint, arXiv:1703.02918 [math.DG].

[19] Kirillov, A. A. Merits and demerits of the orbit method. *Bull. Amer. Math. Soc. (N.S.)* 36, 4 (1999), 433–488.

[20] Knopf, D., and Sesum, N. Dynamic instability of $\mathbb{CP}^N$ under Ricci flow. *J. Geom. Anal.* (to appear). arXiv:1709.01005.

[21] Kröncke, K. Stability of Einstein metrics under Ricci flow. *Comm. Anal. Geom.* (to appear). arXiv:1312.2224 [math.DG].

[22] Matsushima, Y. Remarks on Kähler–Einstein manifolds. *Nagoya Math. J.* 46 (1972), 161–173.

[23] Máximo, D. On the blow-up of four-dimensional Ricci flow singularities. *J. Reine Angew. Math.* 692 (2014), 153–171.

[24] McDuff, D., and Salamon, D. *Introduction to symplectic topology*, second ed. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[25] Paradan, P.-E. The Fourier transform of semi-simple coadjoint orbits. *J. Funct. Anal.* 163, 1 (1999), 152–179.

[26] Perelman, G. The entropy formula for the Ricci flow and its geometric applications. - (2002). preprint, arXiv:math/0211159 [math.DG].

[27] Rossmann, W. Kirillov’s character formula for reductive Lie groups. *Invent. Math.* 48, 3 (1978), 207–220.

[28] Sesum, N. Linear and dynamical stability of Ricci-flat metrics. *Duke Mathematical Journal* 133, 1 (2006), 1–26.

[29] Shephard, G. C., and Todd, J. A. Finite unitary reflection groups. *Canadian J. Math.* 6 (1954), 274–304.

[30] Tian, G., and Zhu, X. Convergence of the Kähler-Ricci flow on Fano manifolds. *J. Reine Angew. Math.* 678 (2013), 223–245.

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