Particle dynamics on hyperboloid and unitary representation of $SO(1, N)$ group

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Abstract

We analyze particle dynamics on $N$ dimensional one-sheet hyperboloid embedded in $N + 1$ dimensional Minkowski space. The dynamical integrals constructed by $SO^\uparrow(1, N)$ symmetry of spacetime are used for the gauge-invariant Hamiltonian reduction. The physical phase-space parametrizes the set of all classical trajectories on the hyperboloid. In quantum case the operator ordering problem for the symmetry generators is solved by transformation to asymptotic variables. Canonical quantization leads to unitary irreducible representation of $SO^\uparrow(1, N)$ group on Hilbert space $L^2(S^{N-1})$.

1 Introduction

In recent papers [1-3] we investigated classical and quantum dynamics of a relativistic particle on 2-dimensional Lorentzian manifold with constant curvature. Locally, such manifolds (with the same curvature) are isometric, they are described by the Liouville equation and have the symmetry associated with $sl(2, R)$ algebra. However, the global properties of these manifolds can be different.

For the classical description we have used the following scheme: Reparametrization invariant Lagrangian of the system has a symmetry of spacetime. The dynamical integrals constructed by this symmetry define the
physical phase-space $\Gamma_{ph}$, which is specified by the admissible values of dynamical integrals on the constraint surface. The physical phase-space fixes the Casimir number of $sl(2, R)$ algebra and gives $C = m^2 r^2$, where $m$ is particle mass and $r$ is radius of hyperboloid. Independent physical variables are used to construct the symplectic form on $\Gamma_{ph}$ [4]. Locally, the physical-phase space and spacetime have the same symmetry, which is expressed by different structures (symplectic and metric, respectively). The physical phase-space parametrizes the set of all classical trajectories in spacetime. In this way we find the correlation between global properties of spacetime and physical phase-space.

The symplectic structure on $\Gamma_{ph}$ is used for the quantization of the reduced system. For the operators of the dynamical integrals one has ambiguity connected with operator ordering. It is shown that this problem can be solved in case $\Gamma_{ph}$ has global $SO_\uparrow(1, 2)$ symmetry.

The 2d one-sheet hyperboloid embedded in 3d Minkowski space is an example of a constant curvature manifold with global $SO_\uparrow(1, 2)$ symmetry. In this case the physical phase space has the same global symmetry. The corresponding quantum theory describes unitary irreducible representation of $SO_\uparrow(1, 2)$ group.

The present paper is a generalization of these results for particle dynamics on $N$ dimensional hyperboloid, embedded in $N + 1$ dimensional Minkowski space, for arbitrary $N$. Such manifold has constant curvature [5] and global $SO_\uparrow(1, N)$ symmetry, which plays the same role in this curved spacetime as the Poincare group in flat Minkowski space.

In classical case we follow the scheme just described for the case $N = 2$. To have a covariant description we use the coordinates of $N + 1$ dimensional Minkowski space, which leads to the additional constraints.

In quantum theory we have the problem of operator ordering for the symmetry generators. Generalizing the method of [1], we find unitary irreducible representation of the symmetry group. As far as we know it is a new representation of $SO_\uparrow(1, N)$ group.

2 Classical dynamics on hyperboloid

Let $x^\mu$ ($\mu = 0, 1, ..., N$) be the coordinates on $N + 1$ dimensional Minkowski space with the metric tensor $\eta_{\mu\nu} = diag(+, -, ..., -)$. A one-sheet hyper-
boloid $H_N$ is defined by

\[ x^2 + r^2 = 0 \quad (x^2 := \eta_{\mu\nu}x^\mu x^\nu), \]

where $r > 0$ is a fixed parameter. The induced metric on $H_N$ has the Lorentzian signature. The topology of $H_N$ is $R^1 \times S^{N-1}$ and $H_N$ can be considered as a model of ‘universe’ with a compact $(S^{N-1})$ space.

We describe the dynamics of a relativistic particle on $H_N$ by the action

\[ S = \int d\tau [ -m\sqrt{\dot{x}^2} + \lambda (x^2 + r^2) ], \]

where $\tau$ is an evolution parameter, $\lambda$ plays a role of Lagrange multiplier, $m$ is particle mass and $\dot{x}^2 := \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ ($\dot{x}^\mu := dx^\mu/d\tau$). The coordinate $x^0$ is associated with time and we assume that $\dot{x}^0 > 0$.

The action (2) is invariant under the Lorentz transformations

\[ x^\mu \to \Lambda^\mu_\nu x^\nu, \quad \Lambda^\mu_\nu \in SO(1,N) \]

and the corresponding dynamical integrals constructed by the Noether theorem have the form

\[ M_{\mu\nu} = p_{\mu}x_{\nu} - p_{\nu}x_{\mu}, \]

where $p_{\mu}$ are the canonical momenta. The variables $(x^\mu, p_{\mu})$ define $2(N+1)$ dimensional extended phase-space $\Gamma_e$.

In the Hamiltonian formulation the Dirac procedure [6] leads to the following constraints

\[ \Phi_1 = x^2 + r^2 = 0, \quad \Phi_2 = p^2 - m^2 = 0, \quad \Phi_2 = px = 0, \]

where $p^2 := \eta^{\mu\nu}p_{\mu}p_{\nu}$, $px := p_{\mu}x^\mu$. These constraints fix the Casimir number

\[ C := \frac{1}{2}M_{\mu\nu}M^{\mu\nu} = m^2 r^2. \]

The constraint surface $\Gamma_c$, given by (5), is $2N-1$ dimensional submanifold of $\Gamma_e$. By Eq.(4) we have the map $F$ from $\Gamma_c$ to the space of dynamical integrals $\Gamma_d$. The physical phase-space $\Gamma_{ph}$ is a submanifold of $\Gamma_d$ defined by $\Gamma_{ph} = F(\Gamma_c)$. To specify $\Gamma_{ph}$ we introduce the variables

\[ \xi_n := \frac{J_n}{J}, \quad \eta_n := \frac{J_nM_{mn}}{J}, \quad (m,n = 1, ..., N), \]

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where $M_{mn}$ are generators of space rotations in $N + 1$ dimensional Minkowski space, $J_n := M_{n0}$ are generators for boosts and $J := \sqrt{\mathcal{J}_k \mathcal{J}_k}$.

Due to (7) we have

$$\xi_k \xi_k = 1 \quad \text{and} \quad \xi_k \eta_k = 0. \quad (8)$$

Therefore, Eq.(7) gives a map from $\Gamma_d$ to $T S^{N-1}$. The map (7) is invertible on $\Gamma_{ph}$, since from (4) – (8) we obtain

$$M_{mn} = \xi_m \eta_n - \xi_n \eta_m, \quad J_n = \xi_n \sqrt{\kappa^2 + \eta^2}, \quad (9)$$

where $\kappa := mr$ and $\eta := \sqrt{\eta^k \eta_k}$.

Thus, we conclude that the manifold $T S^{N-1}$, given by (8), represents the physical phase-space $\Gamma_{ph}$.

To find the symplectic structure on $\Gamma_{ph}$ we use the method based on calculation of a reduced 1-form $p_\mu dx^\mu$ in physical variables (see [4], [7]). For this purpose we parametrize the constraint surface (5) by the physical variables $\xi_n, \eta_n$ and the parameter $s = - \tanh^{-1}(m x_0/ \rho_0)$. From (4), (5) and (7) we get

$$p_0 = \frac{1}{r} \sqrt{\kappa^2 + \eta^2} \cosh s, \quad x_0 = \frac{1}{m} \sqrt{\kappa^2 + \eta^2} \sinh s, \quad (10)$$

$$p_n = - \frac{\eta_n}{r} \cosh s - m \xi_n \sinh s, \quad x_n = \frac{\eta_n}{m} \sinh s + r \xi_n \cosh s$$

(we take into account that $p_0 < 0$, since $x^0 > 0$).

In the parametrization (10) the canonical 1-form $\Theta = p_\mu dx^\mu$ reads

$$\Theta := p_\mu dx^\mu |_{\Phi_1 = 0 = \Phi_2 = \Phi_3} = \eta_k d\xi_k - \kappa ds. \quad (11)$$

The unit vector $\xi_n$ can be parametrized by $N - 1$ independent coordinates $\bar{\theta}^a$ ($a = 1, ..., N - 1$) and for the orthogonal vector $\eta_n$ one can use $N - 1$ additional parameters $\bar{\rho}_b$, uniquely defined by

$$\eta_n = \partial_a \xi_n(\bar{\theta}) g^{ab}(\bar{\theta}) \bar{\rho}_b, \quad (12)$$

where $g^{ab}(\bar{\theta})$ is the inverse to the induced metric tensor on the unit sphere $\xi_k \xi_k = 1$.

Neglecting the exact form $\kappa ds$ in (11) and using the independent coordinates $(\bar{\theta}^a, \bar{\rho}_a)$, we obtain the canonical 1-form on $\Gamma_{ph}$

$$\Theta = \bar{\rho}_a d\bar{\theta}^a. \quad (13)$$
Due to the isomorphism between $T S^{N-1}$ and $T^* S^{N-1}$ (realized by the induced metric on the unit sphere $\xi_k \xi_k = 1$), the variables $\xi_n$ and $\eta_n$ can be considered as functions on $T^* S^{N-1}$. Thus, Eq.(9) defines all dynamical integrals (4) as functions on $T^* S^{N-1}$, i.e. $M_{\mu \nu} = M_{\mu \nu}(\hat{\rho}, \hat{\theta})$.

Note that the gauge fixing $x^0 - \tau = 0$ in the initial action (2) gives time-dependent non-singular Lagrangian with the configuration space $S^{N-1}$. In the Hamiltonian formulation it leads to the phase-space $T^* S^{N-1}$, which coincides with our gauge invariant reduction procedure.

Excluding the variable sinh $s$, from (10) we find

$$x_n = \frac{x_0}{\sqrt{\kappa^2 + \eta^2}} \eta_n + \frac{r \sqrt{\kappa^2 + \eta^2 + m^2 x_0^2}}{\sqrt{\kappa^2 + \eta^2}} \xi_n,$$

which describes the particle trajectories on $H_N$. A given point $(\xi_n, \eta_n)$ of $\Gamma_{ph}$ defines the particle trajectory uniquely. Thus, $\Gamma_{ph}$ can be associated with the space of trajectories as well. According to (14) the ‘space vector’ $x_n$ lays on the plane defined by two orthogonal vectors $\xi_n$ and $\eta_n$. At the ‘moment’ $x_0 = t$ the vector $x_n$ is on the circle of radius $\sqrt{x_n x_n} = \sqrt{r^2 + t^2}$. The rotation angle $\alpha(t)$ of the vector $x_n$ from $x_0 = 0$ to $x_0 = t$ is given by

$$\alpha(t) = \arcsin \frac{\eta t}{\sqrt{\kappa^2 + \eta^2} \sqrt{t^2 + r^2}}.$$

Since $\alpha(t) < \pi/2$, one can speculate that geodesics of two signals sent in different directions never cross. Thus, an ‘observer’ can never see multiple images of a ‘cosmic object’. On the other hand, comparing the ‘distance’ between two nearby geodesics the observer can detect expansion (for $x_0 > 0$) of the ‘universe’.

### 3 Quantization

The canonical quantization of our reduced system implies realization of commutation relations

$$[\hat{\rho}_a, \hat{\theta}^b] = -i \hbar \delta_a^b \tag{16}$$

and construction of $\hat{M}_{\mu \nu}$ operators by classical expressions

$$\hat{M}_{\mu \nu} := M_{\mu \nu}(\hat{\rho}, \hat{\theta}) \tag{17}$$
using definite prescription for the ordering of $\hat{\rho}_a$ and $\hat{\theta}^a$ operators in (17). This prescription should lead to a self-adjoint representation of $\mathfrak{so}(1, N)$ algebra. In the case the symmetry generators $M_{\mu\nu}$ are linear in momenta $\tilde{\rho}_a$ the ordering problem can be solved by a simple symmetrization procedure [1] (see below).

According to (9) and (12) the generators of space rotations $M_{mn}$ are linear in momenta $\tilde{\rho}_a$, whereas the boosts $J_n$ are not. In [1] we have examined the problem of linearization (in momenta) of symmetry generators for the case $N = 2$. We have found transformation to new canonical variables which linearize all three generators $M_{01}, M_{02}$ and $M_{12}$. In what follows we show that the linearization procedure can be generalized for arbitrary $N$ in a covariant form and we use it for the construction of $\hat{M}_{\mu\nu}$ operators.

Let us consider transformation to the new physical variables $u_n$ and $v_n$

$$u_n := \frac{\kappa}{\sqrt{\kappa^2 + \eta^2}} \xi_n + \frac{1}{\sqrt{\kappa^2 + \eta^2}} \eta_n, \quad v_n := -\frac{\eta^2}{\sqrt{\kappa^2 + \eta^2}} \xi_n + \frac{\kappa}{\sqrt{\kappa^2 + \eta^2}} \eta_n. \quad (18)$$

Transformation (18) is a rotation in $(\xi_n, \eta_n)$ plane by the angle

$$\beta = \arcsin \frac{\eta}{\sqrt{\kappa^2 + \eta^2}}. \quad (19)$$

Comparing (19) with (15), we see that Eq.(18) describes transformation to the asymptotic variables at $t \to +\infty$.

Making use of (8) and (9) we obtain

$$u_k u_k = 1, \quad u_k v_k = 0, \quad (20)$$

$$v^2 := v_k v_k = \eta^2, \quad \eta_k d\xi_k = u_k dv_k - \frac{\kappa dv^2}{2(\kappa^2 + v^2)}, \quad (21)$$

and

$$M_{mn} = u_m v_n - u_n v_m, \quad J_n = \kappa u_n - v_n. \quad (22)$$

The variables $u_n$ satisfy the equations

$$\{M_{\mu\nu}, u_n\}, u_m = 0 \quad \text{and} \quad \{u_n, u_m\} = 0$$

(for any $\mu, \nu, n, m$), which corresponds to the choice of polarization in the method of geometric quantization. In [1] these conditions were used to find the variables linearizing the dynamical integrals for the case $N = 2$. 

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Due to (20), the variables \((u_n, v_n)\) define the manifold \(T^*S^{N-1}\) in the same way as \((\xi_n, \eta_n)\). We parametrize the unit sphere \(u_k u_k = 1\) by the coordinates \(\theta^a\) (without tilde) and again introduce the coordinates \(\rho_a\) (without tilde) defined by (see (12))

\[
v_n = \partial_a u_n(\theta) g^{ab}(\theta) \rho_b,
\]

where now \(g^{ab}(\theta)\) is the inverse to

\[
g_{ab}(\theta) := \partial_a u_k(\theta) \partial_b u_k(\theta).
\]

In the new coordinates \((\rho_a, \theta^a)\) we have again the canonical symplectic form \(\sigma := d\Theta = d\rho_a \wedge d\theta^a\) (see (13) and (21)).

It is natural to choose \(L^2(S^{N-1})\) as a Hilbert space, with the scalar product

\[
\langle \Psi_2 | \Psi_1 \rangle := \int d\theta \sqrt{g(\theta)} \overline{\Psi_2^*(\theta)} \Psi_1(\theta),
\]

where \(g(\theta) := det g_{ab}(\theta)\) and \(d\theta \sqrt{g(\theta)}\) is the invariant measure on \(S^{N-1}\).

The Hermitian operators \(\hat{\rho}_a\), which satisfy (16), have the form

\[
\hat{\rho}_a = -i\hbar \partial_a - i\hbar \frac{\partial_a g}{4g}.
\]

All generators \(M_{\mu\nu}\) are now linear in \(\rho_a\) coordinates

\[
M_{\mu\nu}(\rho, \theta) = \rho_a A^a_{\mu\nu}(\theta) + B_{\mu\nu}(\theta),
\]

where \(B_{\mu\nu}(\theta)\) are functions and \(A^a_{\mu\nu}(\theta)\) are the components of vector fields on \(S^{N-1}\). For the corresponding operators (17) we apply the following symmetrization prescription

\[
\rho_a A^a_{\mu\nu}(\theta) \rightarrow \frac{1}{2} [\hat{\rho}_a A^a_{\mu\nu}(\theta) + A^a_{\mu\nu}(\theta) \hat{\rho}_a].
\]

The prescription (28) provides Hermiticity of \(\hat{M}_{\mu\nu}\) operators and preserves the classical commutation relations of the dynamical integrals [1].

Making use of (26) and (28) we obtain

\[
\hat{M}_{\mu\nu} = -i\hbar [A_{\mu\nu} + B_{\mu\nu}(\theta)] - \frac{i\hbar}{2} \nabla (A_{\mu\nu}),
\]
where $A_{\mu\nu}$ is a vector field on $S^{N-1}$

$$A_{\mu\nu} := A^a_{\mu\nu}(\theta) \partial_a,$$  \hspace{1cm} (30)

and $\nabla_a$ is the operator of covariant derivative on the Riemannian manifold [8]. Thus, the operators $\hat{M}_{\mu\nu}$ do not depend on the choice of $\theta^a$ coordinates.

To calculate the term $\nabla(A_{\mu\nu})$ we introduce the vector fields $Y_n$ ($n = 1, ..., N$) defined as solutions of the equations

$$G(Y_n, \cdot) = du_n(\cdot),$$  \hspace{1cm} (32)

where $G := g_{ab}d\theta^a d\theta^b$ is the symmetric 2-form of the induced metric on $S^{N-1}$. Due to nondegeneracy of $G$, Eq.(32) has unique solution for each $n$ given by

$$Y_n = Y^a_n(\theta) \partial_a,$$  \hspace{1cm} (33)

The vector fields $Y_n$ and the functions $u_n$ satisfy the following relations (see Appendix)

$$Y_n(u_m) = \delta_{mn} - u_m u_n,$$  \hspace{1cm} (34)

$$\nabla(Y_n) = -(N - 1)u_n.$$  \hspace{1cm} (35)

According to (22), (23) and (27) the vector fields $A_{\mu\nu}$ associated with the space rotations and the boosts are $A_{mn} = u_m Y_n - u_n Y_m$ and $A_{n0} = -Y_n$ respectively; the functions $B_{\mu\nu}$ read: $B_{mn} = 0$ and $B_{n0} = \kappa u_n$. Due to (31), (34) and (35) we have

$$\nabla(u_m Y_n) = \delta_{mn} - N u_m u_n,$$  \hspace{1cm} (36)

which leads to $\nabla(A_{mn}) = 0$. As a result we obtain

$$\dot{M}_{mn} = -i\hbar(u_m Y_n - u_n Y_m), \quad \dot{J}_n = \left(\kappa - \frac{i\hbar}{2} (N - 1)\right) u_n + i\hbar Y_n.$$  \hspace{1cm} (36)

The operators (36) realize the commutation relations of $so(1,N)$ algebra, which follows from (34) and (A.6).

Taking into account that $u_n Y_n = 0$, we find the quantum Casimir number corresponding to (6)

$$\hat{C} = \frac{1}{2}\hat{M}_{\mu\nu} \hat{M}^{\mu\nu} = \kappa^2 + \frac{\hbar^2(N - 1)^2}{4}.$$  \hspace{1cm} (37)
Let us specify the representation of $SO_{\uparrow}(1, N)$ group corresponding to (36). For any $\Lambda^\mu_\nu \in SO_{\uparrow}(1, N)$ one has the decomposition $\Lambda^\mu_\nu = R^\mu_\beta B^\beta_\nu$, where $R^\mu_\nu$ is the space rotation

$$R^0_0 = 1, \quad R^0_n = 0 = R^n_0, \quad R^m_n = \Lambda^m_n - \frac{\Lambda^m_0 \Lambda^0_n}{1 + \Lambda^0_0}$$

and $B^\mu_\nu$ is the boost

$$B^0_0 = \Lambda^0_0, \quad B^0_n = \Lambda^0_n = B^n_0, \quad B^m_n = \delta^m_n + \frac{\Lambda^0_m \Lambda^0_n}{1 + \Lambda^0_0}. \quad (39)$$

The boost (39) is characterized by the transformation parameter $\sigma$ and the unit vector $\vec{\zeta} := (\zeta_1, ..., \zeta_N)$ defined by

$$\cosh \sigma = \Lambda^0_0, \quad \zeta_n \sinh \sigma = \Lambda^0_n \quad (\sigma > 0). \quad (40)$$

One can prove (see Appendix) that the action of the boost operator

$$\hat{B}(\sigma, \vec{\zeta}) := \exp \left( -\frac{i}{\hbar} \sigma \zeta_n \hat{J}_n \right)$$

on the wave function $\Psi \in L^2(S^{N-1})$ has the form

$$\hat{B}(\sigma, \vec{\zeta}) \Psi(\vec{w}) = [\cosh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle \sinh \sigma]^{-z} \Psi \left( \vec{F}(\vec{w}; \sigma, \vec{\zeta}) \right), \quad (42)$$

where

$$\vec{w} := (w_1, ..., w_N) \in S^{N-1} \quad (u_n(\vec{w}) = w_n), \quad \langle \vec{w} \cdot \vec{\zeta} \rangle := w_n \zeta_n, \quad z := \frac{N-1}{2} + \frac{i \kappa}{\hbar},$$

and $\vec{F}(\vec{w}; \sigma, \vec{\zeta})$ is the flow generated by the vector field $Y_{\vec{\zeta}} := \zeta_n Y_n$. This flow is given by (see Appendix)

$$\vec{F}(\vec{w}; \sigma, \vec{\zeta}) := \frac{\vec{w} + [\sinh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle (\cosh \sigma - 1)] \vec{\zeta}}{\cosh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle \sinh \sigma}. \quad (43)$$

Since the operators $\hat{M}_{mn}$ are vector fields, the action of the rotation operator on the wave function $\Psi(w_n)$ transforms only its argument

$$w_n \rightarrow w_m R^m_n. \quad (44)$$

Making use of (38)-(44) we find the transformation corresponding to $\Lambda^\mu_\nu$

$$\Psi(w_n) \rightarrow [\Lambda^0_0 + \Lambda^k_0 w_k]^{-z} \Psi \left( \frac{\Lambda^m_n w_m + \Lambda^0_0}{\Lambda^0_0 + \Lambda^0_l w_l} \right), \quad (45)$$

which defines the unitary irreducible representation of $SO_{\uparrow}(1, N)$ group. For the case $N = 2$ it is equivalent to the representation of $SL(2, R)$ group [9].
4 Remarks

It should be noted that the prescription (28) for the operator ordering is not unique. One can show that Hermiticity of the symmetry generators could be achieved by the more general prescription

\[ M_{\mu\nu} \rightarrow \hat{M}_{\mu\nu} := \hat{M}_{\mu\nu} + c\nabla(A_{\mu\nu}), \]

where \( \hat{M}_{\mu\nu} \) is given by (29) and \( c \) is any real number. This freedom and related details of the representation (45) will be discussed elsewhere.

In the case \( N = 4 \), the induced metric tensor on the hyperboloid satisfies the Einstein equation

\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \]

with the cosmological term \( \Lambda = 3/r^2 \). Therefore, it is de Sitter’s type metric and the corresponding spacetime can be considered as a ‘toy model’ of the Universe.

As it was mentioned in Introduction (discussing the case \( N = 2 \)), there exists the relationship between global properties (topology, symmetry) of spacetime and physical phase-space. The corresponding quantum system should take into account this relationship. As we show the correlation among spacetime, physical phase-space and the corresponding quantum system can be generalized to the case of \( N \) dimensional hyperboloid. Due to this correlation one can consider the inverse problem: finding spacetime(s), which leads to a given physical phase-space and corresponding quantum system. Such a problem could be interesting in the context of cosmology.

Massive scalar particle is the simplest physical object. As we have shown, its dynamics (both classical and quantum) can be described completely, as in the case of the flat Minkowski space. It would be interesting to investigate the dynamics of other systems like spinning particle, (super)string, etc.

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5 Appendix

According to (33) one has

\[ Y_n(u_m) = g^{ab} \partial_a u_n \partial_b u_m, \quad (A.1) \]

which does not depend on the choice of local coordinates \( \theta^a \) on \( S^{N-1} \). We choose \( \theta^a = u_a, \, a = 1, \ldots, N-1 \). The corresponding metric tensor (24) takes the form

\[ g_{ab} = \delta_{ab} + \frac{u_a u_b}{u_2^2}, \quad \text{where} \quad u_2^2 = 1 - \sum_{a=1}^{N-1} u_a^2. \quad (A.2) \]

The inverse to the matrix (A.2) is

\[ g^{ab} = \delta_{ab} - u_a u_b, \quad (A.3) \]

and (A.1) leads to (34), for any \( n, m \in \{1, \ldots, N\} \).

The \( N-1 \) dimensional ‘vector’ \( u_a (a = 1, \ldots, N-1) \) is an eigenvector of the matrix (A.2) with the eigenvalue \( 1/u_2^2 \). Other \( N-2 \) eigenvectors are orthogonal to \( u_a \) and have eigenvalues equal to one. Thus, the determinant \( g \) of the matrix (A.2) reads

\[ g := det g_{ab} = 1/u_2^2. \quad (A.4) \]

By (A.2) and (A.4), for any \( n \in \{1, \ldots, N\} \), we obtain

\[ \nabla(Y_n) := \partial_a (g^{ab} \partial_b u_n) + \frac{\partial g}{2g} g^{ab} \partial_b u_n = -(N-1)u_n. \quad (A.5) \]

The commutator of vector fields \( Y_m \) and \( Y_n \) has the form

\[ Y_m Y_n - Y_n Y_m = u_m Y_n - u_n Y_m. \quad (A.6) \]

The validity of (A.6) results from the fact that the action of both sides of (A.6) on \( u_k \), for any \( k \in \{1, \ldots, N\} \), gives the same result: \( u_m \delta_{nk} - u_n \delta_{mk} \). Since the set of functions \( u_k \) forms (over)complete set on \( S^{N-1} \), one has (A.6).

The flow \( \vec{F} := \vec{F}(\vec{w}; \sigma, \zeta) \) corresponding to the vector field \( Y_\zeta = \zeta_n Y_n \) is defined by (see (34))

\[ \partial_\sigma \vec{F} = \zeta - \langle \vec{F} : \vec{\zeta} \rangle \vec{F}, \quad (A.7) \]
and
\[ \vec{F}(\vec{w}; 0, \vec{\zeta}) = \vec{w}, \quad (A.8) \]

where (A.8) is the initial value for (A.7).

Multiplying (A.7) by the vector \( \vec{\zeta} \), we get
\[ \partial_\sigma \langle \vec{F} \cdot \vec{\zeta} \rangle = 1 - \langle \vec{F} \cdot \vec{\zeta} \rangle^2. \quad (A.9) \]

Solution to (A.8)-(A.9) reads
\[ \langle \vec{F} \cdot \vec{\zeta} \rangle = \sinh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle \cosh \sigma \]
\[ \cosh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle \sinh \sigma \]
\[ (A.10) \]

Substitution of (A.10) into (A.7) gives the linear differential equations for \( \vec{F} \).

Integration of these equations leads to (40).

Taking into account the form of \( \hat{J}_n \) operators, from (36) and (38) we have
\[ \hat{B}(\sigma, \vec{\zeta}) \Psi(\vec{w}) = f(\vec{w}; \sigma, \vec{\zeta}) \Psi(\vec{F}(\vec{w}; \sigma, \vec{\zeta})), \quad (A.11) \]

where \( f \) is some function. Since the right hand side of (A.11) should define one parameter group (with respect to \( \sigma \)) and \( f(\vec{w}; 0, \vec{\zeta}) = 1 \), we find
\[ f(\vec{w}; \sigma, \vec{\zeta}) = [\cosh \sigma + \langle \vec{w} \cdot \vec{\zeta} \rangle \sinh \sigma]^{-2}. \]

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