Ramsey numbers of uniform loose paths and cycles

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Abstract

Recently, determining the Ramsey numbers of loose paths and cycles in uniform hypergraphs has received considerable attention. It has been shown that the 2-color Ramsey number of a \(k\)-uniform loose cycle \(C_k^n\), \(R(C_k^n, C_k^n)\), is asymptotically \(\frac{1}{2}(2k-1)n\). Here we conjecture that for any \(n \geq m \geq 3\) and \(k \geq 3\),

\[ R(P_k^n, P_m^k) = R(C_k^n, C_m^k) + 1 = (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor. \]

Recently the case \(k = 3\) is proved by the authors. In this paper, first we show that this conjecture is true for \(k = 3\) with a much shorter proof. Then, we show that for fixed \(m \geq 3\) and \(k \geq 4\) the conjecture is equivalent to (only) the last equality for any \(2m \geq n \geq m \geq 3\). Consequently, the proof for \(m = 3\) follows.

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1 Introduction

For given \(k\)-uniform hypergraphs \(G\) and \(H\), the \textit{Ramsey number} \(R(G, H)\) is defined to be the smallest integer \(N\) so that in every red-blue coloring of the edges of the complete \(k\)-uniform hypergraph \(K_N^k\) there is a red copy of \(G\) or a blue copy of \(H\). There are various definitions for paths and cycles in hypergraphs. The case we focus on here is called \textit{loose}. A \textit{\(k\)-uniform loose cycle} \(C_k^n\) (shortly, a \textit{cycle of length} \(n\)) is a hypergraph with vertex set \(\{v_1, v_2, \ldots, v_{n(k-1)}\}\) and the set of \(n\) edges \(e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)\), \(i = 1, 2, \ldots, n\). Here, we use mod \(n(k-1)\) arithmetic and adding a number \(t\) to a set \(H = \{v_1, v_2, \ldots, v_k\}\) means a shift, i.e. the set obtained by adding \(t\) to subscripts of each element of \(H\). Similarly, a \textit{\(k\)-uniform loose path} \(P_k^n\) (shortly, a \textit{path of length} \(n\)) is a hypergraph with vertex set \(\{v_1, v_2, \ldots, v_{n(k-1)+1}\}\) and the set of \(n\) edges \(e_i = \{v_1, v_2, \ldots, v_k\} + (i-1)(k-1)\), \(i = 1, 2, \ldots, n\). For an edge \(e_i = \{v_{i-1}(k-1)+1, v_{i-1}(k-1)+2, \ldots, v_{i(k-1)+1}\}\) of a given loose path (also a given loose cycle) \(K\), the first vertex, \(v_{i-1}(k-1)+1\), and the last vertex, \(v_{i(k-1)+1}\), are denoted by \(f_{K,e_i}\) and \(l_{K,e_i}\), respectively.

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The problem of determining or estimating Ramsey numbers is one of the most important problems in combinatorics which has been of interest to many investigators. In contrast to the graph case, there are only a few results on the Ramsey numbers of hypergraphs. Recently, this topic has received considerable attention. The investigation of the Ramsey numbers of hypergraph loose paths and cycles was initiated by Haxell et al. (see [3]). Indeed, they determined the asymptotic value of the Ramsey number of 3-uniform loose cycles. This result was extended by Gyárfás, Sárközy and Szemerédi [2, Theorem 2] to k-uniform loose cycles as follows.

Theorem 1.1. [2] For all \( \eta > 0 \) there exists \( n_0 = n_0(\eta) \) such that for every \( n > n_0 \), every 2-coloring of \( K^k_N \) with \( N = (1 + \eta)\frac{1}{2}(2k - 1)n \) contains a monochromatic copy of \( C^k_n \).

Some interesting results were obtained on the exact values of the Ramsey numbers of loose paths and cycles. Gyárfás and Raeisi [1] determined the values of the Ramsey numbers of two \( k \)-uniform loose triangles and two \( k \)-uniform quadrangles. In [4], the authors proved that for every \( n \geq \lceil \frac{5m}{4} \rceil \), \( R(P^3_n, P^3_m) = 2n + \lceil \frac{m+1}{2} \rceil \). Recently, the Ramsey numbers of 3-uniform loose paths and loose cycles are completely determined; see [5].

These results motivate us to pose the following conjecture:

Conjecture 1. Let \( k \geq 3 \) be an integer number. For any \( n \geq m \geq 3 \),

\[
R(P^k_n, P^k_m) = R(P^k_n, C^k_m) = R(C^k_n, C^k_m) + 1 = (k - 1)n + \left\lceil \frac{m+1}{2} \right\rceil.
\]

In the next section, we provide a proof of Conjecture 1 when \( k = 3 \) (a much shorter proof to that of [3]). For this purpose, first we show that Conjecture 1 is equivalent to the following conjecture (see Theorem 2.2).

Conjecture 2. Let \( k \geq 3 \) be an integer number. For every \( n \geq m \geq 3 \),

\[
R(C^k_n, C^k_m) = (k - 1)n + \left\lceil \frac{m-1}{2} \right\rceil.
\]

Then we will prove Conjecture 2 for \( k = 3 \). In Section 3, we will demonstrate that for fixed \( m \geq 3 \) and \( k \geq 4 \) Conjecture 1 is equivalent to (only) the last equality for any \( 2m \geq n \geq m \geq 3 \). More precisely, we will show that for fixed \( m \geq 3 \) and \( k \geq 4 \), Conjecture 2 is true for each \( n \geq m \) if and only if it is true for each \( 2m \geq n \geq m \geq 3 \). So using Theorem 2.2 we are done. Subsequently, in the last section, we conclude that Conjecture 1 holds for \( m = 3 \) and every \( n \geq 3 \).

The following lemma [1, Lemma 1] shows that the values of the Ramsey numbers in Conjecture 1 are lower bounds for the claimed Ramsey numbers.

Lemma 1.2. [1] For every \( n \geq m \geq 2 \) and \( k \geq 3 \), \((k - 1)n + \left\lceil \frac{m+1}{2} \right\rceil \) is a lower bound for both \( R(P^k_n, P^k_m) \) and \( R(P^k_n, C^k_m) \). Moreover, \( R(C^k_n, C^k_m) \geq (k - 1)n + \left\lceil \frac{m-1}{2} \right\rceil \).

Therefore, in this paper in order to determine the Ramsey numbers, it suffices to verify that the known lower bounds are also upper bounds.

Throughout the paper, for a 2-edge colored hypergraph \( H \) we denote by \( H_{\text{red}} \) and \( H_{\text{blue}} \) the induced hypergraphs on red edges and blue edges, respectively. Also, the number of vertices and edges of \( H \) are denoted by \( ||H|| \) and \( |H| \).
2 3-uniform loose paths and cycles

In this section, we present a proof of Conjecture 1 when \( k = 3 \) (an alternative proof to that of [5]). First, we show that this conjecture is equivalent to Conjecture 2. For this purpose, we sketch how the last equality of (1) for given \( n \geq m \geq 2 \), leads to determine the values of \( R(P_n^k, C_m^k) \) and \( R(P_n^k, P_{m-1}^k) \) and also \( R(P_n^k, P_n^k) \) when \( n = m \).

The fact that \((k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor \) is a lower bound for \( R(P_n^k, C_m^k) \) follows from Lemma 1.2. To see that it is the upper bound, assume that \( k^{(k-1)n+\left\lfloor \frac{m+1}{2} \right\rfloor} \) is 2-edge colored red and blue. Since

\[ R(C_n^k, C_m^k) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor < (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor, \]

we have a red copy of \( C_n^k \) or a blue copy of \( C_m^k \). If there is a blue copy of \( C_m^k \), we are done.

Otherwise, the existence of a red copy of \( C_n^k \) implies that there is a red copy of \( P_n^k \) by [1, Lemma 2]. Now we show that

\[ R(P_n^k, P_{m-1}^k) = (k-1)n + \left\lfloor \frac{m}{2} \right\rfloor. \]

To see this, assume that \( k^{(k-1)n+\left\lfloor \frac{m}{2} \right\rfloor} \) is 2-edge colored red and blue. Again, we have a red copy of \( C_n^k \) or a blue copy of \( C_m^k \). If the first case holds, by [1, Lemma 2], we do not have anything to prove. Otherwise, a blue copy of \( C_m^k \) contains a blue copy of \( P_{m-1}^k \). This observation and Lemma 1.2 complete the proof.

Now let \( n = m \). By applying Lemma 2 of [1] the existence of a monochromatic \( C_n^k \) in a 2-edge colored \( k^{(k-1)n+\left\lfloor \frac{n+1}{2} \right\rfloor} \) implies that there is a monochromatic \( P_n^k \). So using Lemma 1.2 we have

\[ R(P_n^k, P_n^k) = (k-1)n + \left\lfloor \frac{n+1}{2} \right\rfloor. \]

In fact, we have the following theorem.

**Theorem 2.1.** Let \( n \geq m \geq 2 \) be given integers and \( R(C_n^k, C_m^k) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor \). Then \( R(P_n^k, C_m^k) = (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor \) and \( R(P_n^k, P_{m-1}^k) = (k-1)n + \left\lfloor \frac{m}{2} \right\rfloor \). Moreover, for \( n = m \) we have \( R(P_n^k, P_n^k) = (k-1)n + \left\lfloor \frac{n+1}{2} \right\rfloor \).

Using Theorem 2.1 we have the following result.

**Theorem 2.2.** Two Conjectures 1 and 2 are equivalent.

In the rest of this section, we will demonstrate that Conjecture 2 is true for \( k = 3 \). For this purpose, we need some definitions.

Let \( \mathcal{H} \) be a 2-edge colored complete 3-uniform hypergraph, \( \mathcal{P} \) be a loose path in \( \mathcal{H} \) and \( W \) be a set of vertices with \( W \cap V(\mathcal{P}) = \emptyset \). By a \( \pi_3 \)-configuration, we mean a copy of \( P_2^3 \) with edges \( \{x, a_1, a_2\} \) and \( \{a_2, a_3, y\} \) so that \( \{x, y\} \subseteq W \) and \( S = \{a_j : 1 \leq j \leq 3\} \subseteq \{e_{i-1} \setminus \{f_{\mathcal{P}, e_{i-1}}\}\} \cup e_i \cup e_{i+1} \) is a set of unordered vertices of three consecutive edges of \( \mathcal{P} \) with \( |S \cap \{e_{i-1} \setminus \{f_{\mathcal{P}, e_{i-1}}\}\}| \leq 1 \). The vertices \( x \) and \( y \) are called the end vertices of this configuration. A \( \pi_3 \)-configuration, \( S \subseteq \{e_{i-1} \setminus \{f_{\mathcal{P}, e_{i-1}}\}\} \cup e_i \cup e_{i+1}, \) is good if at least one of the vertices of \( e_{i+1} \setminus e_i \) is not in \( S \). We say that a monochromatic path \( \mathcal{P} = e_1e_2 \ldots e_n \)
is maximal with respect to $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ (in brief, maximal w.r.t. $W$) if there is no $W' \subseteq W$ so that for some $1 \leq r \leq n$ and $1 \leq i \leq n - r + 1$,
\[ \mathcal{P}' = e_1e_2 \cdots e_i e_i' e_{i+1}' \cdots e_{i+r} e_{i+r} \cdots e_n, \]
is a monochromatic path with $n + 1$ edges and the following properties:

(i) $V(\mathcal{P}') = V(\mathcal{P}) \cup W'$,

(ii) if $i = 1$, then $f_{\mathcal{P}'_{e_i'}} = f_{\mathcal{P}_{e_i}}$,

(iii) if $i + r - 1 = n$, then $l_{\mathcal{P}'_{e_i'}} = l_{\mathcal{P}_{e_n}}$.

Clearly, if $\mathcal{P}$ is maximal w.r.t. $W$, then it is maximal w.r.t. every $W' \subseteq W$ and also every loose path $\mathcal{P}'$ which is a sub-hypergraph of $\mathcal{P}$ is again maximal w.r.t. $W$.

**Lemma 2.3.** Let $\mathcal{H} = K_4$ be 2-edge colored red and blue and let $\mathcal{P} = e_1e_2 \cdots e_n \subseteq \mathcal{H}_{\text{red}}$ be a maximal path w.r.t. $W$, where $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ and $|W| \geq 3$. Set $A_1 = \{f_{\mathcal{P}_{e_1}}\}$ and $A_i = e_{i-1} \setminus \{f_{\mathcal{P}_{e_{i-1}}}\}$ for $i > 1$. Then for every two consecutive edges $e_i$ and $e_{i+1}$ of $\mathcal{P}$ and for each $u \in A_i$ there is a good $\varpi_S$-configuration, say $C$, in $\mathcal{H}_{\text{blue}}$ with end vertices in $W$ and
\[ S \subseteq ((e_i \setminus \{f_{\mathcal{P}_{e_i}}\}) \cup \{u\}) \cup (e_{i+1} \setminus \{v\}), \]
for some $v \in A_{i+2}$ such that each vertex of $W$, with the exception of at most one vertex, can be considered as an end vertex of $C$.

**Proof.** Let $\mathcal{P} = e_1e_2 \cdots e_n \subseteq \mathcal{H}_{\text{red}}$ be a maximal path w.r.t. $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$, where
\[ e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}, \quad i = 1, 2, \ldots, n. \]
Assume that $W = \{x_1, \ldots, x_t\} \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$. Consider the edges $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and $e_{i+1} = \{v_{2i+1}, v_{2i+2}, v_{2i+3}\}$. If the edge $\{u, v_{2i}, x\}$ (resp. the edge $\{v_{2i+2}, v_{2i+3}, x\}$) is red for some $x \in W$, then the maximality of $\mathcal{P}$ w.r.t. $W$ implies that for arbitrary vertices $x' \neq x'' \in W \setminus \{x\}$ the edges $\{x', v_{2i+1}, v_{2i}\}$ and $\{v_{2i+2}, v_{2i+3}, x''\}$ (resp. $\{x', v_{2i+1}, v_{2i+2}\}$ and $\{v_{2i+2}, v_{2i}, x''\}$) are blue and there is a good $\varpi_S$-configuration $C = \{x', v_{2i+1}, v_{2i}\} \{v_{2i+2}, v_{2i+3}, x''\}$ (resp. $C = \{x', v_{2i+1}, v_{2i+2}\} \{v_{2i+2}, v_{2i}, x''\}$) with
\[ S = \{v_{2i}, v_{2i+1}, v_{2i+2}\} \subseteq ((e_i \setminus \{v_{2i-1}\}) \cup \{u\}) \cup (e_{i+1} \setminus \{v_{2i+3}\}). \]
So we may assume that for each vertex $x \in W$ both edges $\{u, v_{2i}, x\}$ and $\{v_{2i+2}, v_{2i+3}, x\}$ are blue. If there is a vertex $y \in W$ such that at least one of the edges $f_1 = \{u, v_{2i+1}, y\}$, $f_2 = \{v_{2i}, v_{2i+1}, y\}$, $f_3 = \{u, v_{2i+2}, y\}$ or $f_4 = \{v_{2i}, v_{2i+2}, y\}$, say $f_i$, is blue, then there is a good $\varpi_S$-configuration $C = \{u, v_{2i}, x\} f_i$, where $x \neq y$, with
\[ S = \{u, v_{2i}\} \cup (f \setminus \{y\}) \subseteq ((e_i \setminus \{v_{2i-1}\}) \cup \{u\}) \cup (e_{i+1} \setminus \{v_{2i+3}\}). \]
Otherwise, we may assume that for every $y \in W$ the edges $f_1, f_2, f_3$ and $f_4$ are red. Therefore, maximality of $\mathcal{P}$ w.r.t. $W$ implies that for every $y' \in W$ the edge $\{v_{2i}, v_{2i+3}, y'\}$ is blue (otherwise, replacing $e_i e_{i+1}$ by $f_3 f_2 \{v_{2i}, v_{2i+3}, y'\}$, where $y \neq y'$, in $\mathcal{P}$ yields a red path $\mathcal{P}'$ greater than $\mathcal{P}$; this is a contradiction). Thus, for every $a \neq b \in W$, $C = \{u, a, v_{2i}\} \{v_{2i}, v_{2i+3}, b\}$ is a good $\varpi_S$-configuration with the desired properties, so that
\[ S = \{u, v_{2i}, v_{2i+3}\} \subseteq ((e_i \setminus \{v_{2i-1}\}) \cup \{u\}) \cup (e_{i+1} \setminus \{v_{2i+2}\}). \]
Corollary 2.4. Let $\mathcal{H} = K^3_t$ be 2-edge colored red and blue and $\mathcal{P} = e_1e_2\ldots e_n$ with $n \geq 2$ be a maximal red path w.r.t. $W$, where $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ and $|W| \geq 3$. Then for some $r \geq 0$ and $W' \subseteq W$ there is a blue path $Q = f_{1}f_{2}\ldots f_{q}$ between $W'$ and $\mathcal{P}' = e_{1}e_{2}\ldots e_{n-r}$ so that $W' = \{f_{Q,f_{i}}\cup \{Q,f_{j}\} | 1 \leq i \leq q/2 \}$ and $V(Q) \setminus W' \subseteq \bar{\mathcal{P}}$. Moreover, $Q$ does not contain at least one of the vertices of $e_{n-r} \setminus e_{n-r-1}$ as a vertex, $\|Q\| = q = 2(|W'| - 1) = n - r$ and either $x = |W \setminus W'| \in \{0, 1\}$ or $x \geq 2$ and $0 \leq r \leq 1$.

Proof. Let $\mathcal{P} = e_1e_2\ldots e_n$ be a maximal red path w.r.t. $W \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ where

$$e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}, \quad i = 1, 2, \ldots, n.$$

Step 1: Set $\mathcal{P}_1 = \mathcal{P}$, $W_1 = W$ and $\mathcal{P}'_1 = e_1e_2$. Since $\mathcal{P}$ is maximal w.r.t. $W_1$, using Lemma 2.3 there is a good $\mathcal{P}_1$-configuration, say $Q_1$, with end vertices $x_1$ and $y_1$ in $W_1$ so that $S \subseteq \mathcal{P}'_1$ and $Q_1$ does not contain at least one of the vertices of $e_2 \setminus e_1$, say $u_1$. Set $W_2 = W$ and $\mathcal{P}_2 = e_3e_4\ldots e_n$. If $|W_2| \leq 3$ or $\|\mathcal{P}_2\| \leq 1$, then $Q = Q_1$ is a blue path between $W_2' = W_2 \cap V(Q_1)$ and $\bar{\mathcal{P}} = e_1e_2$ with the desired properties. Otherwise, go to Step 2.

Step $k$ ($k \geq 2$): Clearly $|W_k| > 3$ and $\|\mathcal{P}_k\| > 1$. Set $\mathcal{P}_{k+1}' = (e_{2k-1} \setminus \{f_{\mathcal{P}_k,e_{2k-1}}\}) \cup \{u_{k-1}\}e_{2k}$. Since $\mathcal{P}$ is maximal w.r.t. $W_k$, using Lemma 2.3 there is a good $\mathcal{P}_{k}'$-configuration, say $Q_k$, with end vertices in $W_k$ so that $S \subseteq \mathcal{P}_{k}'$ and $Q_k$ does not contain at least one of the vertices of $e_{2k} \setminus e_{2k-1}$, say $u_k$. Since each vertex of $W_k$, with the exception at most one, can be considered as an end vertex of $Q_k$, we may assume that $\bigcup_{i=1}^{k} Q_i$ is a blue path with end vertices $x_k$ and $y_k$ in $W_k$. Set $\mathcal{P}_{k+1} = e_{2k+1}e_{2k+2}\ldots e_n$ and $W_{k+1} = (W \setminus \bigcup_{i=1}^{k} V(Q_i)) \cup \{x_k, y_k\}$. If $|W_{k+1}| \leq 3$ or $\|\mathcal{P}_{k+1}\| \leq 1$, then $Q = \bigcup_{i=1}^{k} Q_i$ is a blue path between $W' = W \cap \bigcup_{i=1}^{k} V(Q_i)$ and $\bar{\mathcal{P}} = e_1e_2\ldots e_{2k}$ with the desired properties. Otherwise, go to Step $k+1$.

Let $t \geq 2$ be the minimum integer for which we have $|W_t| \leq 3$ or $\|\mathcal{P}_t\| \leq 1$. Let $W' = W \cap \bigcup_{i=1}^{t-1} V(Q_i)$. Clearly $|W \setminus W'| = 0, 1$ or $|W \setminus W'| \geq 2$ and $0 \leq \|\mathcal{P}_t\| \leq 1$. So $Q = \bigcup_{i=1}^{t-1} Q_i$ is a blue path between $\bar{\mathcal{P}} = e_1e_2\ldots e_{n-r}$ and $W'$ with the desired properties, where $r = \|\mathcal{P}_t\|$. Note that, we have $|W'| = t$ and $n - r = 2(t - 1)$. 

Lemma 2.5. Let $n \geq m \geq 3$, $(n, m) \neq (3,3), (4, 3), (4, 4)$ and $\mathcal{H} = K^3_{2n+\lfloor \frac{m-1}{2} \rfloor}$ be 2-edge colored red and blue. Assume that there is no copy of $C^3_n$ in $\mathcal{H}_{\text{red}}$ and $\mathcal{C} = C^3_{n-1}$ is a loose cycle in $\mathcal{H}_{\text{red}}$. Then there is a copy of $C^3_m$ in $\mathcal{H}_{\text{blue}}$.

Proof. Let $C = e_1e_2\ldots e_{n-1}$ be a copy of $C^3_{n-1}$ in $\mathcal{H}_{\text{red}}$ with edges $e_i = \{v_1, v_2, v_3\} + 2(i-1)$ (mod $2(n-1)$), $i = 1, \ldots, n-1$ and $W = V(\mathcal{H}) \setminus V(C)$. We consider the following cases.

Case 1. For some edge $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$, $1 \leq i \leq n - 1$, there is a vertex $z \in W$ such that at least one of the edges $\{z, v_{2i}, v_{2i+1}\}$ or $\{v_{2i-1}, v_{2i}, z\}$ is red.

Assume that the edge $g = \{z, v_{2i}, v_{2i+1}\}$ is red. Set $\mathcal{P} = e_{i+1}e_{i+2}\ldots e_{n-1}e_1e_2\ldots e_{i-2}e_{i-1}$ and $W_0 = W \setminus \{z\}$ (If the edge $\{v_{2i-1}, v_{2i}, z\}$ is red, consider $\mathcal{P} = e_{i-1}e_{i-2}\ldots e_2e_1e_n\ldots e_{i+2}e_{i+1}$ and do the following process to get a blue copy of $C^3_m$).

First let $m \leq 4$. Therefore, $|W_0| = 2$. Assume that $W_0 = \{u, v\}$. We show that $\mathcal{H}_{\text{blue}}$ contains $C^3_m$ for each $m \in \{3,4\}$. Since $n \geq 5$ and there is no red copy of $C^3_n$, the
edges $f_1 = \{u, v_{2i-2}, v_{2i}\}$, $f_2 = \{v_{2i}, v, v_{2i-1}\}$, $f_3 = \{v_{2i-1}, z, u\}$ are blue (if the edge $f_j$ for $1 \leq j \leq 3$ is red, then $f_j e_i \ldots e_i \ldots e_i$ is a red copy of $C_3$, a contradiction). Thereby $f_1 f_2 f_3$ is a blue copy of $C_3$. Moreover, $P' = e_{i-3} e_{i-2}$ (we use mod $(n - 1)$ arithmetic) is maximal w.r.t. $W = W_0 \cup \{z\}$. Using Lemma 2.3 there is a good $\varpi$-configuration, say $C$, in $\mathcal{H}_{\text{blue}}$ with end vertices in $W$ so that $S \subseteq e_{i-3} \cup e_{i-2}$. Without loss of generality assume that $u$ is an end vertex of $C$ in $W$. Again, since there is no red copy of $C_3$, $C\{v, z, v_{2i-1}\}\{v_{2i-1}, v_{2i}, u\}$ is a blue copy of $C_3$.

Now let $m \geq 5$. Clearly $|W_0| = \frac{m-1}{2} + 1 \geq 3$. Since there is no red copy of $C_3$, $P$ is a maximal path w.r.t. $W_0$. Now, using Corollary 2.4 there is a blue path of length $\ell'$ between $P$, the path obtained from $P$ by deleting the last $r$ edges, and $W' \subseteq W_0$ with the properties mentioned in Corollary 2.4. Let $Q$ be such a blue path so that $\ell'$ is maximum. Since $|P| = n - 2$, we have $\ell' = 2(|W'| - 1) = n - 2 - r$. Let $x$ and $y$ be the end vertices of $Q$ in $W'$ and $T = W_0 \setminus W'$. If $|T| \geq 2$, then $r \geq 3$, a contradiction to Corollary 2.4. Therefore, we have $|T| \leq 1$. First let $T = \emptyset$. Clearly $\ell' = 2\left(\frac{m-1}{2}\right)$. If $m$ is even, then $\ell' = m - 2$. Assume that $w$ is a vertex of $e_{i-1} \setminus e_{i-2}$ so that $w \notin V(Q)$ (the existence of $w$ is guaranteed by Corollary 2.4). Since there is no red copy of $C_3$, the edges $g_1 = \{w, z, x\}$ and $g_2 = \{y, v_{2i}, w\}$ are blue (if the edge $g_j$ for $1 \leq j \leq 2$ is red, then $g_j e_i \ldots e_i \ldots e_i$ is a red copy of $C_3$, a contradiction). Thus $g_1Qg_2$ is a blue copy of $C_3$. When $m$ is odd, we have $\ell' = m - 1$. In this case, remove the last two edges of $Q$ to get a blue path $Q'$ of length $m - 3$ so that $v_{2i-2} \notin Q'$. Now we may assume that vertices $x$ and $y' \neq y$ of $W'$ are the end vertices of $Q'$. Again, since there is no red copy of $C_3$,

$$Q'\{y', v_{2i-2}, v_{2i}\}\{v_{2i}, y, v_{2i-1}\}\{v_{2i-1}, z, x\}$$

is a copy of $C_3$ in $\mathcal{H}_{\text{blue}}$.

Now let $T = \{u\}$. Clearly $\ell' = 2\left(\frac{m-1}{2}\right) - 2$. If $m$ is odd, then $\ell' = m - 3$ and $r \geq 1$. Thereby,

$$Q\{y, v_{2i-2}, v_{2i}\}\{v_{2i}, u, v_{2i-1}\}\{v_{2i-1}, z, x\}$$

is a blue copy of $C_3$. Now suppose that $m$ is even. So $\ell' = m - 4$ and $r \geq 2$. Let $w$ be a vertex of $e_{i-3} \setminus e_{i-4}$ so that $w \notin V(Q)$. Using Lemma 2.3 there is a good $\varpi$-configuration, say $C$, in $\mathcal{H}_{\text{blue}}$ with end vertices in $W = \{x, y, u, z\}$ so that $S \subseteq (e_{i-2} \setminus \{f_{p,e_{i-2}}\} \cup \{w\}) \cup e_{i-1}$. Since $Q$ is maximum with the properties mentioned in Corollary 2.4 we can assume that $y$ and $z$ are end vertices of $C$ in $W$. Clearly $QC\{z, u, w'\}\{w', v_{2i}, x\}$ is a blue copy of $C_3$, where $w' \in e_{i-1} \setminus (V(C) \cup \{f_{p,e_{i-1}}\})$.

**Case 2.** For every edge $e_i = \{v_{2i-1}, v_{2i}, v_{2i+1}\}$, $1 \leq i \leq n - 1$, and every vertex $z \in W$ the edges $\{v_{2i-1}, v_{2i}, z\}$ and $\{v_{2i}, v_{2i+1}, z\}$ are blue.

Assume that $W = \{x_1, x_2, \ldots, x_{\frac{m-1}{2} + 2}\}$. For $1 \leq i \leq m - 1$, set

$$f_i = \begin{cases} \{x_{\frac{i+1}{2}}, v_{\frac{i+1}{2}+1}, v_{\frac{i+1}{2}+1}\} & \text{if } i \text{ is odd,} \\ \{v_{\frac{i}{2}}, v_{\frac{i}{2}+1}, x_{\frac{i}{2}+1}\} & \text{if } i \text{ is even.} \end{cases}$$

Let $Q = f_1 f_2 \ldots f_{m-1}$. So for $m$ even, $Q\{x_1, v_{\frac{m}{2}}, v_{\frac{m}{2}+1}\}$ and for $m$ odd, $Q\{x_{\frac{m+1}{2}}, v_1, v_2\}$ is a blue copy of $C_3$. \hfill \Box
Before we give the main result, we need the following lemmas.

Lemma 2.6. [1] For every \( k \geq 3 \),
(a) \( R(P_{3k}^k, P_{3k}^k) = R(C_{3k}^k, P_{3k}^k) = R(C_{3k}^k, C_{3k}^k) + 1 = 3k - 1 \),
(b) \( R(P_{nk}^k, P_{nk}^k) = R(C_{nk}^k, P_{nk}^k) = R(C_{nk}^k, C_{nk}^k) + 1 = 4k - 2 \).

Lemma 2.7. Let \( k \geq 3 \) be an integer number. Then
\[ R(C_{4k}^k, C_{4k}^k) = 4k - 3. \]

Proof. Suppose indirectly that the edges of \( K_{4k-3}^k \) can be colored with no red copy of \( C_{3k}^k \) and no blue copy of \( C_{3k}^k \). By Theorem 2.6 we must have a blue copy of \( C_{4k}^k \). Let \( C = e_1e_2e_3e_4 \) be a blue copy of \( C_{4k}^k \) with edges \( e_i = \{v_1, \ldots, v_k\} + (k-1)(i-1)(\text{mod} \ 4(k-1)) \). Since there is no blue copy of \( C_{3k}^k \),
\[(e_2 \cup \{v_{4k-4}\} \setminus \{v_k\})(e_4 \cup \{v_k\} \setminus \{v_1\})(e_1 \cup \{v_{3k-3}\} \setminus \{v_1\})(e_3 \cup \{v_1\} \setminus \{v_{3k-2}\})\]
is a red copy of \( C_{4k}^k \). This contradiction completes the proof. \[\square\]

Theorem 2.8. Conjecture [1] is true for \( k = 3 \).

Proof. We give a proof by induction on \( m+n \). The cases \( n = m \leq 4 \) follow from Lemma 2.6. By Lemma 2.7 we may assume that \( n \geq 5 \). Suppose to the contrary that the edges of \( H = K_{2n+\lfloor \frac{m-1}{2} \rfloor}^3 \) can be colored red and blue with no red copy of \( C_m^3 \) and no blue copy of \( C_m^3 \). For \( n = m \) by the induction hypothesis,
\[ R(C_{n-1}^3, C_{n-1}^3) = 2(n-1) + \left\lfloor \frac{n-2}{2} \right\rfloor < 2n + \left\lfloor \frac{n-1}{2} \right\rfloor. \]
So we may assume that there is a red copy of \( C_{n-1}^3 \) in \( H \). Using Lemma 2.6 we have a blue copy of \( C_n^3 \); a contradiction. For \( n > m \), we have \( n-1 \geq m \) and since
\[ R(C_{n-1}^3, C_m^3) = 2(n-1) + \left\lfloor \frac{m-1}{2} \right\rfloor < 2n + \left\lfloor \frac{m-1}{2} \right\rfloor, \]
we may assume that \( C_{n-1}^3 \subseteq H_{\text{red}} \). Using Lemma 2.5 we have a blue copy of \( C_m^3 \); a contradiction.

Using Theorems 2.2 and 2.8 we have the following.

Theorem 2.9. Conjecture [1] is true for \( k = 3 \).

3 An equivalent variation of Conjecture [1] for \( k \geq 4 \)

In this section, we show that for fixed \( m \geq 3 \) and \( k \geq 4 \) Conjecture [1] is equivalent to (only) the last equality for any \( 2m \geq n \geq m \geq 3 \). Indeed, we demonstrate that for fixed \( m \geq 3 \) and \( k \geq 4 \), Conjecture [2] holds for every \( n \geq m \) if and only if it holds for every \( 2m \geq n \geq m \). Then, using Theorem 2.2, the desired result is achieved. We will use the following lemma later as follows: We will exhibit two disjoint red copies of \( C_{\lceil n_0/2 \rceil}^k \) and \( C_{\lceil n_0/2 \rceil}^k \) and this will be enough to ensure the existence of a red copy of \( C_{n_0}^k \) or a blue copy of \( C_m^k \) for \( n_0 \geq 2m + 1 \).
Lemma 3.1. Let $n, m, t$ and $k$ be positive integer numbers so that $n \geq m \geq 3$, $k \geq 4$ and $t \geq (k - 1)(n + m)$. Let $\mathcal{H} = K^k_t$ be 2-edge colored red and blue and $C_1 = C^k_n$ and $C_2 = C^k_m$ be two disjoint cycles in $\mathcal{H}_{\text{red}}$. There is a red copy of $C^k_{n+m}$ or a blue copy of $C^k_k$ for every $3 \leq \ell \leq m$.

Proof. Let $C_1 = e_1e_2 \ldots e_n$ and $C_2 = f_1f_2 \ldots f_m$ be copies of $C^k_n$ and $C^k_m$ in $\mathcal{H}_{\text{red}}$ with edges
\[
e_i = \{v_1, v_2, \ldots, v_k\} + (i - 1)(k - 1)(\text{mod } n(k - 1))
\]
and
\[
f_i = \{u_1, u_2, \ldots, u_k\} + (i - 1)(k - 1)(\text{mod } m(k - 1)).
\]
Without loss of generality assume that there is no copy of $C^k_{n+m}$ in $\mathcal{H}_{\text{red}}$. Let
\[
g_1 = (e_1 \setminus \{v_{k-1}, v_k\}) \cup \{u_{k-1}, u_k\}
\]
and
\[
h_1 = (f_1 \setminus \{u_{k-1}, u_k\}) \cup \{v_{k-1}, v_k\}.
\]
At least one of the edges $g_1$ or $h_1$, say $s_1$, is blue (otherwise, $h_1e_2e_3 \ldots e_ne_1f_2f_3 \ldots f_m$ is a red copy of $C^k_{n+m}$, a contradiction to our assumption). Assume that $x_2 \in s_1 \cap e_1$ and $y_2 \in s_1 \cap f_1$ are vertices with maximum indices. Now set $P_1 = s_1$,
\[
g_2 = (e_2 \setminus \{v_k, v_{2k-2}, v_{2k-1}\}) \cup \{x_2, u_{2k-2}, u_{2k-1}\}
\]
and
\[
h_2 = (f_2 \setminus \{u_k, u_{2k-2}, u_{2k-1}\}) \cup \{y_2, v_{2k-2}, v_{2k-1}\}.
\]
Similarly, at least one of the edges $g_2$ or $h_2$, say $s_2$, is blue (otherwise, $h_2e_3e_4 \ldots e_ne_1g_1f_2f_3f_4 \ldots f_mf_1$ is a red copy of $C^k_{n+m}$, a contradiction). Set $P_2 = P_1s_2$ and continue this process $\ell - 2$ times. Note that, with this approach, in the $i$-th step, $2 \leq i \leq \ell - 2$, we have
\[
g_i = (e_i \setminus \{v_{(i-1)(k-1)+1}, v_{i(k-1)}, v_{i(k-1)+1}\}) \cup \{x_i, u_{i(k-1)}, u_{i(k-1)+1}\}
\]
and
\[
h_i = (f_i \setminus \{u_{(i-1)(k-1)+1}, u_{i(k-1)}, u_{i(k-1)+1}\}) \cup \{y_i, v_{i(k-1)}, v_{i(k-1)+1}\},
\]
where $x_i \in s_{i-1} \cap e_{i-1}$ and $y_i \in s_{i-1} \cap f_{i-1}$ are vertices with maximum indices. Since there is no red copy of $C^k_{n+m}$, at least one of the edges $g_i$ or $h_i$, say $s_i$, is blue. So clearly $P_i = P_{i-1}s_i$ is a blue path of length $i$. Now let
\[
g_{\ell-1} = (e_n \setminus \{v_{(n-1)(k-1)+1}, v_{(n-1)(k-1)+2}, v_1\}) \cup \{x_1, u_{(n-1)(k-1)}, u_{(n-1)(k-1)+1}\}
\]
and
\[
h_{\ell-1} = (f_{\ell-1} \setminus \{u_{(\ell-2)(k-1)+1}, u_{(\ell-1)(k-1)}, u_{(\ell-1)(k-1)+1}\}) \cup \{y_{\ell-1}, v_{(n-1)(k-1)+1}, v_{(n-1)(k-1)+2}\},
\]
and
so that $x_1 \in s_1 \cap e_1$ is a vertex with minimum index and $y_{\ell-1} \in s_{\ell-2} \cap f_{\ell-2}$ is a vertex with maximum index. Again, at least one of the edges $g_{\ell-1}$ or $h_{\ell-1}$ is blue (otherwise, $g_{\ell-1}e_1e_2 \ldots e_{n-1}h_{\ell-1}f_{\ell-2} \ldots f_mf_1 \ldots f_1$ is a red copy of $C^k_{n+m}$, a contradiction). If the edge $g_{\ell-1}$ is blue, then set

$$g_{\ell} = (e_{\ell-1} \setminus \{v(\ell-2)(k-1)+1, v(\ell-1)(k-1), v(\ell-1)(k-1)+1\}) \cup \{x_{\ell-1}, u(\ell-2)(k-1)+k-2, u(\ell-1)(k-1)+1\}$$

and

$$h_{\ell} = (f_{\ell-1} \setminus \{u(\ell-2)(k-1)+1, u(\ell-2)(k-1)+k-2, u(\ell-1)(k-1)+1\}) \cup \{y_{\ell-1}, v(\ell-1)(k-1), v(\ell-1)(k-1)+1\},$$

where $x_{\ell-1} \in s_{\ell-2} \cap e_{\ell-2}$ and $y_{\ell-1} \in s_{\ell-2} \cap f_{\ell-2}$ are vertices with maximum indices. Clearly, at least one of the edges $g_{\ell}$ or $h_{\ell}$ is blue (otherwise, $h_{\ell}e_\ell e_{\ell+1} \ldots e_n e_1 \ldots e_{\ell-2}g_{\ell} f_{\ell-1} \ldots f_{m} f_1 \ldots f_{\ell-2}$ is a red copy of $C^k_{n+m}$). If the edge $g_{\ell}$ is blue, then $P_{\ell-2}g_{\ell-1}$ is a blue copy of $C^k_{n}$, Otherwise, $P_{\ell-2}h_{\ell}s_{\ell-1}$ is a blue copy of $C^k_{n}$. Now we may assume that the edge $g_{\ell-1}$ is red and so the edge $h_{\ell-1}$ is blue. In this case set

$$g'_{\ell} = (e_n \setminus \{v(n-1)(k-1)+2, v_1\}) \cup \{u_{m}(k-1), y_1\}$$

and

$$h'_{\ell} = (f_m \setminus \{u_{m}(k-1), u_1\}) \cup \{v(n-1)(k-1)+2, x_1\},$$

where $x_1 \in s_1 \cap e_1$ and $y_1 \in s_1 \cap f_1$ are vertices with minimum indices. Since at least one of the edges $g'_{\ell}$ and $h'_{\ell}$, say $s_{\ell}$, is blue, then $P_{\ell-2}s_{\ell-1}s_{\ell}$ is a blue copy of $C^k_{n}$, which completes the proof.

\[\square\]

**Theorem 3.2.** Let $m \geq 3$ be a fixed integer, $k \geq 4$ and $A_m = \{n \geq m : R(C^k_n, C^k_m) = (k-1)n + \lfloor \frac{m-1}{2} \rfloor \}$. If $\lfloor m, 2m \rfloor \subseteq A_m$, then $A_m = [m, \infty]$.

**Proof.** Suppose to the contrary that $A_m \neq [m, \infty]$. Let $n_0$ be the minimum element of $[m, \infty] - A_m$. Note that $n_0 \geq 2m+1$. Let $H = K_{(k-1)n_0 + \lfloor \frac{m-1}{2} \rfloor}$ be 2-edge colored red and blue with no red copy of $C^k_{n_0}$ and no blue copy of $C^k_{m}$. We consider the following cases:

**Case 1.** $n_0$ is even.

Clearly $\frac{m+n_0}{2} > m$. Since $n_0$ is the minimum integer in $[m, \infty] - A_m$, then

$$R(C^k_{\frac{m+n_0}{2}}, C^k_{m}) = (k-1)\frac{n_0}{2} + \lfloor \frac{m-1}{2} \rfloor < (k-1)n_0 + \lfloor \frac{m-1}{2} \rfloor.$$ 

Therefore, there is a copy of $C^k_{\frac{m+n_0}{2}}$ in $H_{\text{red}}$, say $C_1$. Remove $C_1$ from $H$ to get a hypergraph $H'$. Since the reminder hypergraph $H'$ has equal to $R(C^k_{\frac{m+n_0}{2}}, C^k_{n_0})$ vertices, there is red copy of $C^k_{\frac{m+n_0}{2}}$, say $C_2$, disjoint from $C_1$. Using Lemma 3.3.1 we have a red copy of $C^k_{n_0}$ or a blue copy of $C^k_{m}$, a contradiction to our assumption.

**Case 2.** $n_0$ is odd.

By an argument similar to case 1, we may assume that there are disjoint copies of $C^k_{\frac{m+n_0-1}{2}}$ and $C^k_{\frac{m+n_0+1}{2}}$ in $H_{\text{red}}$. So using Lemma 3.3.1 we have a red copy of $C^k_{n_0}$ or a blue copy of $C^k_{m}$. This contradiction completes the proof. 

\[\square\]
Remark 3.3. Note that Theorem 5.2 shows that for fixed $m \geq 3$ and $k \geq 4$, Conjecture 2 holds for each $n \geq m$ if and only if it holds for each $2m \geq n \geq m$. This show, using Theorem 2.2 that for fixed $m \geq 3$ and $k \geq 4$, Conjecture 1 is equivalent to (only) the last equality for any $2m \geq n \geq m \geq 3$. We will use this result in the next section, to demonstrate that Conjecture 1 is true for $m = 3$.

4 Cycle-triangle Ramsey number in uniform hypergraphs

In this section, we show that Conjecture 1 holds for any $n \geq m = 3$. By Remark 3.3, it only suffices to prove that Conjecture 2 is true for $m = 3$ and $3 \leq n \leq 6$. First, we establish some essential lemmas.

Lemma 4.1. Let $H = K_{i(k-1)+1}^k$, $i = 5, 6$, be 2-edge colored red and blue with no blue copy of $C_3^k$. If there is a blue copy of $C_4^k$, then there is a copy of $C_4^k$ in $H_{red}$.

Proof. Suppose that the edges of $H$ are 2-colored with no blue copy of $C_3^k$. Let $C = e_1e_2e_3e_4$ be a blue copy of $C_4^k$ with edges

$$e_i = \{v_1, \ldots, v_k\} + (i - 1)(k - 1)(\text{mod } 4(k - 1)), \quad 1 \leq i \leq 4,$$

and $W = V(H) \setminus V(C)$. For $i = 5$, since there is no blue copy of $C_3^k$, 

$$(e_2 \cup \{v_{4k-4}\} \setminus \{v_k\})(e_4 \cup \{v_k\} \setminus \{v_1\})(e_1 \cup \{v_{3k-3}\} \setminus \{v_1\})$$

$$(\{v_{3k-3}, v_1\} \cup W \setminus \{w_1, w_2\})(e_3 \cup \{v_1, w_1\} \setminus \{v_{3k-3}, v_{3k-2}\}),$$

where $w_1, w_2 \in W$, is a red copy of $C_5^k$. For $i = 6$, partition the vertices of $W$ into three sets $A, B$ and $C$ with $|A| = |B| = k - 2$ and $|C| = 3$. Let $C = \{u_1, u_2, u_3\}$. Since there is no blue copy of $C_3^k$, 

$$(e_2 \cup \{u_1, v_{4k-4}\} \setminus \{v_k, v_{k+1}\})(e_4 \cup \{v_{k+1}\} \setminus \{v_1\})(e_3 \cup \{v_k, u_2\} \setminus \{v_{2k-1}, v_{2k}\})$$

$$(A \cup \{v_k, v_{2k}\})(e_1 \cup \{v_{2k}\} \setminus \{v_k\})(B \cup \{v_1, v_{2k-1}\})$$

is a red copy of $C_6^k$. \qed

Also, we need the following remark of 7 Remark 3. For the sake of completeness, we also represent it’s proof here.

Remark 4.2. If $K_N^k$ with $N \geq k + 1$ is 2-edge colored and both colors are used at least once, then there are two edges of distinct colors intersecting in $k - 1$ vertices.

Proof. Select a red edge $e$ and a blue edge $f$ with maximum intersection and suppose that $m = |e \cap f| < k - 1$. Let $g$ be an edge that contains $e \cap f$ and intersects $e \setminus f$ in $\left\lfloor \frac{k-m}{2} \right\rfloor$ vertices and intersects $f \setminus e$ in $\left\lfloor \frac{k-m}{2} \right\rfloor$ vertices. Now, either $g, e$ or $g, f$ are two edges of distinct colors intersecting in more than $m$ vertices, this contradicts the choice of $e$ and $f$. \qed

The following lemma is a modified version of Lemma 7 in 7. But, for the sake of completeness, we state a proof here.

10
Lemma 4.3. Let $k \geq 3$ and $t \geq 5$. If $\mathcal{H} = K_{t(k-1)+1}^k$ is 2-edge colored red and blue with no red copy of $C_5^k$ and no blue copy of $C_3^k$, then there exist two intersecting pairs of red-blue edges $e_1, e_2$ and $f_1, f_2$ so that $(e_1 \cup e_2) \cap (f_1 \cup f_2) = \emptyset$.

Proof. Note that both colors are used at least once, for otherwise there is either a red copy of $C_5^k$ or a blue copy of $C_3^k$ (see Remark 4.2), a contradiction. As noted in Remark 4.2, we can select $e_1 = \{v_1, v_2, \ldots, v_k\} \in \mathcal{H}_{\text{red}}$ and $e_2 = \{v_2, v_3, \ldots, v_{k+1}\} \in \mathcal{H}_{\text{blue}}$. Let $W = V(\mathcal{H}) \setminus \{v_1, v_2, \ldots, v_{k+1}\}$. If $W$ is not monochromatic, we can find favorable edges $f_1$ and $f_2$ in $W$ based on Remark 4.2. Thus all edges of $W$ are red (otherwise, there is a blue copy of $C_3^k$, a contradiction). Let $W_1$ and $W_2$ be disjoint subsets of $W$ so that $|W_1| = |W_2| = k - 1$. We may assume that $g_1 = \{v_1\} \cup W_1$ is red. If it is not, let $g_2 = \{v_{k+1}\} \cup W_2$. If $g_2$ is red, then $g_2, e_2$ and $W_1 \cup \{w\}, g_1$, where $w \in W \setminus (W_1 \cup W_2)$, are favorable red-blue pairs. If $g_2$ is blue, then $e_1, g_1$ and $W_2 \cup \{w\}, g_2$, where $w \in W \setminus (W_1 \cup W_2)$, are desired. Indeed, for every $W' \subseteq W$ with $|W'| = k - 1$, the edge $\{v_1\} \cup W'$ is red. Note that the edge $f_1 = \{v_k\} \cup W_2$ is blue. Otherwise, the edges $f_1, e_1, g_1$ with the $t - 3$ suitable edges with vertices in $W$ give a red copy of $C_5^k$. Actually, using a similar argument, for every $i, 2 \leq i \leq k$, and every $W' \subseteq W$ with $|W'| = k - 1$, the edge $\{v_{k-i}+1\} \cup W'$ is blue. Therefore, $g_1, \{v_{k-1}\} \cup W_1$ and $\{w\} \cup W_2, f_1$, where $w \in W \setminus (W_1 \cup W_2)$, are two disjoint red-blue pairs that satisfy the mentioned condition.

Corollary 4.4. Let $k \geq 3$, $t \geq 5$ and $\mathcal{H} = K_{t(k-1)+1}^k$ be 2-edge colored red and blue. If there is no red copy of $C_5^k$ and no blue copy of $C_3^k$, then there exist two disjoint intersecting pairs of red-blue edges $e_1, e_2$ and $f_1, f_2$ so that $|e_1 \cap e_2| = |f_1 \cap f_2| = k - 1$.

Proof. Using Lemma 4.3, we can find two intersecting pairs of red-blue edges $e_1, e_2$ and $f_1, f_2$ so that $(e_1 \cup e_2) \cap (f_1 \cup f_2) = \emptyset$. Now, consider the complete hypergraphs on $V(e_1) \cup V(e_2)$, say $G_1$, and $V(f_1) \cup V(f_2)$, say $G_2$. Since $G_1$ and $G_2$ have edges of both colors, applying Remark 4.2 we can find two edges of distinct colors intersecting in $k - 1$ vertices in each of $G_1$ and $G_2$.

Lemma 4.5. Let $k \geq 4$ be an integer number. Then

$$R(C_5^k, C_3^k) = 5k - 4.$$ 

Proof. Suppose to the contrary that the edges of $\mathcal{H} = K_{5k-4}^k$ can be colored red and blue with no red copy of $C_5^k$ and no blue copy of $C_3^k$. Apply Corollary 4.4 for $t = 5$ to find

$$e_1 = \{v_1, v_2, \ldots, v_k\} \in \mathcal{H}_{\text{red}}, e_2 = \{v_2, v_3, \ldots, v_{k+1}\} \in \mathcal{H}_{\text{blue}}$$

and

$$f_1 = \{w_1, w_2, \ldots, w_k\} \in \mathcal{H}_{\text{red}}, f_2 = \{w_2, w_3, \ldots, w_{k+1}\} \in \mathcal{H}_{\text{blue}}$$

so that

$$|e_1 \cap e_2| = |f_1 \cap f_2| = k - 1, (e_1 \cup e_2) \cap (f_1 \cup f_2) = \emptyset.$$ 

Set

$$W = V(\mathcal{H}) \setminus \{v_1, v_2, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\}$$

and partition the vertices of $W$ into three sets $A, B$ and $C$ with $|A| = |B| = |C| = k - 2$ (note that this is possible since $|W| = 3k - 6$). Without loss of generality assume that $T$
and $S$ are two subsets of $W$ so that $T \subseteq A$, $S \subseteq B$ and $|T| + |S| = k - 2$. Set $T' = A \setminus T$, $S' = B \setminus S$, $t \in T$ and $s \in S'$. The proofs of the following claims are similar, so we only give a proof for Claim 1.

**Claim 1.** If there is $i \in \{2, 3, \ldots, k\}$ so that the edge $e = \{v_{k+1}, w_{k+2-i}\} \cup T \cup S$ is blue, then for every $j \in \{2, 3, \ldots, k\}$, the edge $f = \{w_{k+1}, v_{k+2-j}\} \cup (T \setminus \{t\}) \cup (S \cup \{s\})$, is also blue.

**Proof.** For every $\ell \in \{2, 3, \ldots, k\} \setminus \{i\}$ (resp. $\ell \in \{2, 3, \ldots, k\} \setminus \{j\}$), the edge $h_1 = (T' \cup \{t\}) \cup S' \cup \{w_{k+2-\ell}\}$ (resp. $h_2 = C \cup \{v_{k+2-\ell}, w_{k+2-i}\}$) is red. Otherwise $e_2 h_1 e$ (resp. $e_2 h_2 e$) is a blue copy of $C_3^k$, a contradiction to our assumptions. Now, if the edge $f$ is red, then $f_1 h_1 f_2 e h_2$ is a red copy of $C_5^k$. This contradiction finishes the proof. \qed

**Claim 2.** If there is $i \in \{2, 3, \ldots, k\}$ so that the edge $\{w_{k+1}, v_{k+2-i}\} \cup T \cup S$ is blue, then for every $j \in \{2, 3, \ldots, k\}$, the edge $\{v_{k+1}, w_{k+2-j}\} \cup (T \setminus \{t\}) \cup (S \cup \{s\})$ is also blue.

**Claim 3.** If there are $i, j \in \{2, 3, \ldots, k\}$ so that $\{v_{k+2-i}, w_{k+2-j}\} \cup T \cup S$ is blue, then for every $\ell \in \{2, 3, \ldots, k\} \setminus \{i\}$ and $\ell' \in \{2, 3, \ldots, k\} \setminus \{j\}$, the edge $\{v_{k+2-\ell}, w_{k+2-\ell'}\} \cup (T \setminus \{t\}) \cup (S \cup \{s\})$ is also blue.

One can easily check that the edge $g_1 = A \cup \{v_{k+1}, w_k\}$ is red. Otherwise, apply Claims 1 and 2 alternatively $k - 3$ times to find a blue copy of $C_3^k$ if $k$ is even and apply Claims 1 and 2 alternatively $k - 2$ times to find a blue copy of $C_5^k$ if $k$ is odd. In the first case, $g_1 e_2 (B \cup \{w_{k+1}, v_k, a\} \setminus \{b\})$, where $a \in A$ and $b \in B$, is a blue copy of $C_3^k$ that is a contradiction to our assumptions. In the second case, $g_1 f_2 (B \cup \{w_{k+1}, v_k\}) e_2$ is a blue copy of $C_3^k$. Now, using Lemma 4.5 we can find a red copy of $C_5^k$, a contradiction. Also, a similar discussion can be used to show that the edges $g_2 = C \cup \{w_{k+1}, v_{k-1}, u\} \setminus \{u'\}$, where $u \in A$ and $u' \in C$, and $g_3 = B \cup \{v_k, w_{k-1}\}$ are red. Thereby, $g_1 f_1 g_3 e_1 g_2$ is a copy of $C_5^k$ in $\mathcal{H}_{\text{red}}$. This contradiction completes the proof. \qed

The proof of the following statement is similar to the proof of Lemma 4.5. So we only present the outline of the proof.

**Lemma 4.6.** Let $k \geq 4$ be an integer number. Then

$$R(C_3^k, C_5^k) = 6k - 5.$$ 

**Proof.** Suppose that the edges of $\mathcal{H} = K_{6k-5}^k$ are colored red and blue with no red copy of $C_6^k$ and no blue copy of $C_3^k$. Using Corollary 4.4 select edges $e_1 = \{v_1, v_2, \ldots, v_k\} \in \mathcal{H}_{\text{red}}$, $e_2 = \{v_2, v_3, \ldots, v_{k+1}\} \in \mathcal{H}_{\text{blue}}$ and $f_1 = \{w_1, w_2, \ldots, w_k\} \in \mathcal{H}_{\text{red}}$, $f_2 = \{w_2, w_3, \ldots, w_{k+1}\} \in \mathcal{H}_{\text{blue}}$.

Set $W = V(\mathcal{H}) \setminus \{v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}\}$ and for a vertex $u \in W$ partition the vertices of $W \setminus \{u\}$ into four sets $A, B, C$ and $D$ of size $k - 2$. Suppose that $T$ and $S$ are two subsets of $W$ so that $T \subseteq A$, $S \subseteq B$ and $|T| + |S| = k - 2$. Set $T' = A \setminus T$, $S' = B \setminus S$, $t \in T$ and $s \in S'$. Arguments
similar to the proofs of Claims \([1,2]\) in Lemma 4.5, yield the same claims here (note that, for proofs, more details are required). Consequently, we may assume that the edges 
\[ g_1 = C \cup \{v_{k+1}, w_k\} \] 
and 
\[ g_2 = D \cup \{v_{k-1}, w_{k+1}, u'\} \setminus \{u''\}, \]
where \( u' \in C \) and \( u'' \in D \), are red. Also one can easily see that the following claim holds.

**Claim 4.** If there is \( i \in \{2,3,\ldots,k\} \) so that the edge 
\[ f = \{v_{k+2-i}, u\} \cup T \cup S \]
is blue, then for every \( j \in \{2,3,\ldots,k\} \setminus \{i\} \), the edge 
\[ f' = \{w_{k+1}, v_{k+2-j}\} \cup (T \setminus \{t\}) \cup (S \setminus \{s\}) \]
is also blue.

**Proof.** Using the above arguments we may assume that for every \( 2 \leq \ell \leq k \), the edge 
\[ h_1 = \{v_{k+1}, w_{k+2-\ell}\} \cup C \]
is red. Also, for any \( \ell' \) and \( \ell'' \) with \( 2 \leq \ell', \ell'' \leq k \) and \( \ell' \neq i, j \), the edge 
\[ h_2 = \{v_{k+2-\ell'}, t, w_{k+2-\ell''}\} \cup (D \setminus \{d\}) \] (resp. \( h_3 = \{v_{k+1}, u \cup T' \cup S'\} \)
is red, where \( d \in D \). Otherwise \( e_2fh_2 \) (resp. \( e_2fh_3 \)) is a blue copy of \( C^k_3 \), a contradiction. Now if \( f' \) is red, then \( h_3f'c_1h_2f_1h_1 \) is a red copy of \( C^k_6 \), a contradiction. \( \Box \)

Similarly, we conclude that the edges 
\[ g_3 = A \cup \{u, w_{k-1}\} \] 
and 
\[ g_4 = B \cup \{u, v_k\} \]
are red. Therefore, \( g_1f_1g_3g_4g_1f_2 \) is a red copy of \( C^k_6 \). This contradiction finishes the proof. \( \Box \)

The following theorem is an immediate consequence of Theorems 2.8 and 3.2, and Lemmas 2.6, 2.7, 4.5 and 4.6 (see Remark 3.3).

**Theorem 4.7.** Conjecture [2] holds for \( m = 3 \).

Using Theorems 2.2 and 4.7 we have:

**Theorem 4.8.** Conjecture [1] is true for \( m = 3 \).

**References**

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