The unique global solvability of the nonhomogeneous incompressible asymmetric fluids with vacuum

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Abstract The present paper deals with the nonhomogeneous incompressible asymmetric fluids equations in dimension $d = 2, 3$. The aim is to prove the unique global solvability of the system with only bounded nonnegative initial density and $H^1$ initial velocities. We first construct the global existence of the solution with large data in 2-D. Next, we establish the existence of local in time solution for arbitrary large data and global in time for some smallness conditions in 3-D. Finally, the uniqueness of the solution is proved under quite soft assumptions about its regularity through a Lagrangian approach. In particular, the initial vacuum is allowed.

Key words: Nonhomogeneous asymmetric fluids; Vacuum; Lagrangian coordinates; The unique global solvability

1 Introduction and Main Results

In the present paper, we consider the following $d$-dimensional (for $d = 2, 3$) nonhomogeneous incompressible asymmetric fluids equations in an open bounded set $\Omega$:

\[
\begin{align*}
\rho(u_t + u \cdot \nabla u) - (\mu + \chi)\Delta u + \nabla P &= 2\chi \text{curl}\omega & \text{in} & \mathbb{R}^+ \times \Omega, \\
\text{div} u &= 0 & \text{in} & \mathbb{R}^+ \times \Omega, \\
\rho(\omega_t + u \cdot \nabla \omega) - \gamma \Delta \omega - \kappa \nabla \text{div}\omega + 4\chi \omega &= 2\chi \text{curl}u & \text{in} & \mathbb{R}^+ \times \Omega, \\
\rho_t + u \cdot \nabla \rho &= 0 & \text{in} & \mathbb{R}^+ \times \Omega, \\
(\rho, u, \omega)|_{t=0} &= (\rho_0, u_0, \omega_0) & \text{in} & \Omega,
\end{align*}
\]

where $u$ is the fluid velocity, $\omega$ is the field of microrotation representing the angular velocity of the rotation of the particles of the fluid, $P$ is the scalar pressure of the flow, while $\rho_0$, $u_0$ and $\omega_0$ are the given initial density, initial velocity and initial angular velocity, respectively, with $\text{div} u_0 = 0$. $\mu$ is the Newtonian kinematic viscosity, $\kappa$ is the angular viscosity, $\chi$ is the micro-rotation viscosity.

For the derivation of system (1.1) and a discussion on their physical meaning, see [9]. Concerning

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applications, the micropolar fluid model has been used, for example, in lubrication theory [18,28], as well as in modeling blood flow in thin vessels [2]. We shall assume that the fluid domain $\Omega$ is either the torus $\mathbb{T}^d$ or a $C^2$ simply connected bounded domain of $\mathbb{R}^d$. For simplicity, we take $\nu = \mu + \chi$.

System (1.1) includes several important models as special cases. When $\rho = \text{const}$, system (1.1) becomes the incompressible micropolar fluid provided that $P$ is an unknown pressure function, which was first introduced in 1965 by Eringen to model micropolar fluids (see Eringen [17], Sections 1 and 6). The micropolar fluid model that will be considered in this article is a generalization of the Navier-Stokes equations, which takes into account the microstructure of the fluid by which we mean the geometry and microrotation of particles. As experiments show the solutions of this model represent flows of many fluids (like, e.g. blood, see Reference [27]) more precisely than solutions of the Navier-Stokes equations. Due to its physical significance and mathematical relevance, there have a lot of works studying on the viscous or inviscid 3-D(2-D) system (see e.g. [7,15,16,31]). When $\omega = 0$, system (1.1) reduces the nonhomogeneous incompressible Navier-Stokes equations, which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [24] for the detailed derivation of this system. Kazhikov [30] first proved the global existence of weak solutions for the 3-D nonhomogeneous incompressible Navier-Stokes equations with $\inf \rho_0 > 0$. Later on, Simon [29] removed the lower bound assumption on $\rho_0$, and Lions [24] proved that $\rho$ is a renormalized solution of the mass equation. However, the uniqueness and smoothness of weak solutions to the nonhomogeneous incompressible Navier-Stokes equations, even for the 2-D case, is still an open problem. Local existence (but without uniqueness) of strong solutions to the nonhomogeneous incompressible Navier-Stokes equations was first established by Antontsev and Kazhikov [1], under the assumptions that the initial density is bounded and away from zero and the initial velocity has $H^1$ regularity. In the bounded domain $\Omega$, Ladyzhenskaya and Solonnikov [22] first constructed global strong solutions in 2-D, and unique local in time maximal strong solution for arbitrary data and global in time for small data in 3-D where $\rho_0 \in C^1(\Omega)$ is bounded away from zero. In general when $\rho_0 \in L^\infty(\mathbb{R}^d)$ with a positive lower bound and $u_0 \in H^2(\mathbb{R}^d)$, Danchin and Mucha [12] proved that the nonhomogeneous incompressible Navier-Stokes equations have a unique local strong solution. Furthermore, with the initial density fluctuation being sufficiently small, they also obtained the global well-posedness. Paicu et al. [26] proved the global existence and uniqueness of the solution to $d$-dimensional (for $d = 2,3$) nonhomogeneous incompressible Navier-Stokes equations with initial density being bounded from above and below by some positive constants, and with initial velocity $u_0 \in H^s(\mathbb{R}^2)(s > 0)$ in 2-D, or $u_0 \in H^1(\mathbb{R}^3)$ satisfying some smallness conditions in 3-D. If it is not assumed that the density
is bounded away from zero, then the analysis of nonhomogeneous incompressible Navier-Stokes equations gets wilder, since the initial density is allowed to have a vacuum. Choe-Kim [8] first proved the local existence and uniqueness of strong solution with initial data \((\rho_0, u_0)\) satisfying 
\[
0 \leq \rho_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), u_0 \in H^2(\mathbb{R}^3) \cap H^1_{0,\sigma}(\mathbb{R}^3)
\]
and the compatibility conditions. Recently, in a bounded domain \(\Omega\), removed the compatibility condition, Li [23] proved the local existence and uniqueness of strong solutions to the 3-D nonhomogeneous Navier-Stokes equations for any initial data \(0 \leq \rho_0 \in L^\infty(\Omega) \cap W^{1,\gamma}(\Omega), u_0 \in H^1_{0,\sigma}(\Omega)\), with \(\gamma > 1\), and if \(\gamma \geq 2\), then the strong solution is unique. More recently, Danchin and Mucha [13] studied the existence and uniqueness issue for the multi-dimensional nonhomogeneous incompressible Navier-Stokes equations supplemented with \(H^1(\Omega)\) initial velocity and only bounded nonnegative density. Specifically, they established the global existence of the solution for general data in 2-D, and the global in time if the velocity satisfies a suitable scaling-invariant smallness condition in 3-D.

Concerning the model considered in this paper, let us recall that, for the 3-D case, under certain assumptions, Lukaszewicz [25] established the existence of weak solutions for a short time by using linearization and an almost fixed point theorem. Braz e Silva and collaborators in [4,5] constructed the existence of global in time weak solutions of system (1.1) when the initial density is not necessarily strictly positive. Local existence of strong solutions to system (1.1) was constructed by Lukaszewicz [25] when the initial density is strictly separated from zero. Using a spectral semi-Galerkin method, when the initial density is bounded and away from zero, Boldrini et al. [6] proved the unique local solvability of strong solution and some global existence results for small data. In 2020, Braz e Silva et al. [3] proved the global existence and uniqueness of solution to the 3-D system (1.1) with the initial velocities \((u_0, \omega_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\) and with initial density being bounded from above and below by some positive constants. Obviously, the result does not allow the presence of initial vacuum. In addition, the corresponding result of the model in 2-D is also unknown.

The main goal of this paper is to study the global well-posedness for \(d\)-dimensional \((d = 2, 3)\) system (1.1) when the initial vacuum is allowed. Now, let us explain some of the main difficulties and the strategies to overcome them in the process. First, since the density is only bounded, it seems impossible to prove the uniqueness of the solution in the Eulerian coordinates as in [8,23]. Indeed, let \((\rho_1, u_1, \omega_1)\) and \((\rho_2, u_2, \omega_2)\) be two different solutions of system (1.1). Then \(\delta \rho = \rho_1 - \rho_2\) satisfies
\[
\partial_t \delta \rho + u_1 \cdot \nabla \delta \rho = -(u_1 - u_2) \cdot \nabla \rho_2.
\]
Without extra assumptions about the regularity of these solutions, the term \((u_1 - u_2) \cdot \nabla \rho_2\) cannot be handled by the energy method because the usual technique to prove uniqueness via Gronwall’s inequality cannot be applied here. And the uniqueness result of Germain [19] cannot be applied
here either, which requires the density function satisfying $\nabla \rho \in L^\infty(0,T; L^n(\mathbb{R}^n))$. To overcome this difficulty, the proof of uniqueness of the solution will use the Lagrangian approach, which is motivated by \[10, 13, 20\]. Second, when the vacuum appears, that is, the initial density is not bounded from below by some positive constant, system (1.1) degenerates in vacuum regions and the terms $\rho v_t, \rho \omega_t$ in the equations are likely to vanish in some parts of the fluid domain, which brings some difficulties for our analysis. Meanwhile, Lemma 9 in [3] which plays a key role in the proof of the uniqueness does not apply to the presence of initial vacuum since it requires positive lower bound of the density, which makes us have to choose another suitable method different from [3] to address the problem. Under the new framework, the estimates of nonlinear terms including curl $u$, curl $\omega$ and $\nabla \text{div} \omega$ in Lagrangian coordinates bring some difficulties. To overcome them, we introduce some useful analysis tools, for example, the Lagrangian vorticity and the Piola identity. Third, when the density is rough and the vacuum is taken into consideration, propagate enough regularity for the velocity is the main difficulty. In most evolutionary fluid mechanics models, the uniqueness issue is closely connected to the Lipschitz control of the flow of the velocity field. In order to achieve the $L^1(0,T; W^{1,\infty})$ estimate of the velocity, the main tools are to perform the time-weighted estimates to the system (1.1) in the spirit of [23] and the shifts of integrability from the time variable to the space variable in [13]. At last, much more complicated nonlinear terms and the coupling effects of system (1.1) will also bring some troubles in our proof.

For simplification, we first define the following some constants:

$$M \overset{\text{def}}{=} \int_\Omega \rho_0 dx,$$

(1.2)

$$C_0 \overset{\text{def}}{=} \|\sqrt{\rho_0} u_0\|_2^2 + \|\sqrt{\rho_0} \omega_0\|_2^2,$$

(1.3)

$$J_0 \overset{\text{def}}{=} \nu \|\nabla u_0\|_2^2 + \gamma \|\nabla \omega_0\|_2^2 + \kappa \|\text{div} \omega_0\|_2^2 + 4 \chi \|\omega_0\|_2^2,$$

(1.4)

and

$$K_0 \overset{\text{def}}{=} \mu \|\nabla u_0\|_2^2 + \gamma \|\nabla \omega_0\|_2^2 + \kappa \|\text{div} \omega_0\|_2^2 + \chi \|\text{curl} u_0 - 2 \omega_0\|_2^2.$$  

(1.5)

Now our main results in this paper can be listed as follows. Let us start with the 2-D case.

**Theorem 1.1.** Let $\Omega$ be a $C^2$ bounded subset of $\mathbb{R}^2$, or the torus $T^2$. Suppose that the initial data $(\rho_0, u_0, \omega_0) \in L^\infty(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$ satisfy for some constant $\rho^* > 0$,

$$0 \leq \rho_0 \leq \rho^*, \quad \text{div} u_0 = 0 \quad \text{and} \quad M > 0.$$  

(1.6)

Then system (1.1) has a unique globally defined solution $(\rho, u, \omega, \nabla P)$ satisfying

$$\rho \in L^\infty(\mathbb{R}^+; L^\infty(\Omega)), \quad u, \omega \in L^\infty(\mathbb{R}^+; H^1_0(\Omega)),$$

$$\sqrt{\rho} u_t, \nabla^2 u, \nabla^2 \omega, \nabla P \in L^2(\mathbb{R}^+; L^2(\Omega))$$
and also, for all $1 \leq r < 2$ and $1 \leq m < \infty$,
\[
\nabla(\sqrt{t}P), \nabla^2(\sqrt{t}u), \nabla^2(\sqrt{t}\omega) \in L^\infty(0, T; L^r(\Omega)) \cap L^2(0, T; L^m(\Omega))^2 \quad \text{for all} \quad T > 0.
\]
Furthermore, we have $\sqrt{t}u, \sqrt{t}\omega \in C([\mathbb{R}^+; L^2(\Omega)]$, $\rho \in C([\mathbb{R}^+; L^p(\Omega)])$ for all $p < \infty$, and $u, \omega \in H^\eta(0, T; L^p(\Omega))$ for all $\eta < \frac{1}{2}$ and $T > 0$.

In the 3-D case we have the following result.

**Theorem 1.2.** Let $\Omega$ be a $C^2$ bounded subset of $\mathbb{R}^3$, or the torus $\mathbb{T}^3$. There exists a constant $\varepsilon_0 > 0$ such that for the initial data $(\rho_0, u_0, \omega_0) \in L^\infty(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$ satisfying (1.6) and
\[
(\rho^*)^{\frac{1}{2}}C_0K_0 \leq \varepsilon_0, \tag{1.7}
\]
Then system (1.1) has a unique globally defined solution $(\rho, u, \omega, \nabla P)$ satisfying
\[
\rho \in L^\infty(\mathbb{R}^+; L^\infty(\Omega)), \quad u, \omega \in L^\infty(\mathbb{R}^+; H^1_0(\Omega)), \quad \sqrt{t}u, \sqrt{t}u, \sqrt{t}\omega, \nabla P \in L^2(\mathbb{R}^+; L^2(\Omega))
\]
and
\[
\nabla(\sqrt{t}P), \nabla^2(\sqrt{t}u), \nabla^2(\sqrt{t}\omega) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^6(\Omega))^2 \quad \text{for all} \quad T > 0.
\]
Furthermore, we have $\sqrt{t}u, \sqrt{t}\omega \in C([\mathbb{R}^+; L^2(\Omega)]$, $\rho \in C([\mathbb{R}^+; L^p(\Omega)])$ for all $p < \infty$, and $u, \omega \in H^\eta(0, T; L^p(\Omega))$ for all $\eta < \frac{1}{2}$ and $T > 0$.

**Remark 1.3.** Compared with [3], our results allow the persistence of the vacuum and show the global well-posedness of solution with large data in 2-D.

**Remark 1.4.** Here, we should point out that the fluid domain $\Omega$ is either the torus $\mathbb{T}^d$ or a $C^2$ simply connected bounded domain of $\mathbb{R}^d (d = 2, 3)$ since we need to use Poincaré’s inequality. The case $\mathbb{R}^d$ will be considered in our future work.

**Notations.** We assume $C$ be a positive generic constant throughout this paper that may vary at different places. By $\nabla$ we denote the gradient with respect to space variables, and by $\partial_t u$ or $u_t$, the time derivative of function $u$. By $\| \cdot \|_p$, we mean $p$-power Lebesgue norms over $\Omega$; we denote by $H^s$ and $W^s_p$ the Sobolev (Slobodeckij for $s$ not an integer) space, and put $H^s = W^s_2$. Finally, as a great part of our analysis will concern $H^1$ regularity and will work indistinctly in a bounded domain or in the torus, we shall adopt slightly abusively the notation $H^1_0(\Omega)$ to designate the set of $H^1(\Omega)$ functions that vanish at the boundary if $\Omega$ is a bounded domain, or general $H^1(\mathbb{T}^d)$ functions if $\Omega = \mathbb{T}^d$.

The rest of the paper unfolds as follows. In the next section, we will prove the global existence of the solution for system (1.1) and some of the time-weighted estimates on time derivatives in 2-D. In Section 3, we will deal with the 3-D case. At last, section 4 is devoted to the proof of the uniqueness of the solution to system (1.1).
2 Existence of solution and weighted energy method in 2-D

2.1 Existence of solution in 2-D

The proof is based on a priori bounds for suitable smoothed out approximate solutions with no vacuum, then to pass to the limit. Let $j_\delta$ be the standard Friedrich’s mollifier and define

\[ u^\delta_0 = j_\delta * u_0, \quad \omega^\delta_0 = j_\delta * \omega_0, \]

and

\[ \rho^\delta_0 = j_\delta * \rho_0, \quad \delta \leq \rho^\delta_0 \leq \rho^*. \]

In what follows, we shall only derive a priori uniform energy estimates for the approximate sequences $(\rho^\delta, u^\delta, \omega^\delta)$. Then the existence part of Theorem 1.1 essentially follows from the a priori estimates and a standard compactness argument. We omit the superscript $\delta$ to keep the notation simple.

From $\rho_t = -u \cdot \nabla \rho$, we can easily get

\[ \|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}, \]

and with (1.6), we have

\[ 0 \leq \rho(t) \leq \rho^*, \quad (t, x) \in [0, \infty) \times T^2. \] (2.1)

Taking the $L^2$-scalar product of the first equation of system (1.1) with $u$ and the third equation with $\omega$ respectively and the combining together, using $\rho_t = -u \cdot \nabla \rho$ and integrating it over $[0, t]$, we obtain

\[ \|\sqrt{\rho} u\|_2^2 + \|\sqrt{\rho} \omega\|_2^2 + \int_0^t \left( \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 \right) d\tau \leq CC_0, \] (2.2)

where $C$ is a universal positive constant depending the parameters $\nu, \gamma, \kappa$ and $\chi$.

**Proposition 2.1.** Under the assumptions of Theorem 1.1. Let $(\rho, u, \omega)$ be a smooth enough solution to system (1.1) satisfying (2.1) and $T > 0$. There exists a constant $C_1$ depending only on $M, \|\rho_0\|_2, \|\sqrt{\rho_0} u_0\|_2$ and $\rho^*$ so that for all $t \in [0, T)$, we have

\[ \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2 + \|\nabla P\|_2^2 \right) d\tau \leq \left( e + CJ_0 \right) \exp(CC_1 C_0). \] (2.3)

Furthermore, for all $p \in [1, \infty)$ and $t \in [0, T)$, we have

\[ \|u\|_p + \|\omega\|_p \leq C \frac{C_0}{M} + C_p \left( 1 + \frac{\|M - \rho\|_2}{M} \right) \left( \|\nabla u\|_2 + \|\nabla \omega\|_2 \right). \] (2.4)
Proof. Taking the $L^2$-scalar product of the first equation of system (1.1) with $u_t$ and the third equation with $\omega_t$ respectively, we obtain from $\rho_t = -u \cdot \nabla \rho$ that
\[ \int_{T_2} \rho |u_t|^2 dx + \frac{\nu}{2} \int_{T_2} |\nabla u|^2 dx = 2\chi \int_{T_2} \text{curl} \omega \cdot u_t dx - \int_{T_2} (\rho u \cdot \nabla) \cdot u_t dx, \]
and
\[ \int_{T_2} \rho |\omega_t|^2 dx + \frac{\gamma}{2} \int_{T_2} |\nabla \omega|^2 dx + \frac{\kappa}{2} \int_{T_2} |\text{div} \omega|^2 dx + 2\chi \int_{T_2} |\omega|^2 dx = 2\chi \int_{T_2} \text{curl} \omega \cdot \omega_t dx - \int_{T_2} (\rho u \cdot \nabla) \cdot \omega_t dx. \]
Adding the two identities and using Hölder’s and Young’s inequalities yield that
\[ \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \frac{1}{2} \frac{d}{dt} \left( \nu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + 4\chi \|\omega\|_2^2 \right) \]
\[ = -\int_{T_2} (\rho u \cdot \nabla) \cdot u_t dx - \int_{T_2} (\rho u \cdot \nabla) \cdot \omega_t dx \]
\[ \leq \frac{1}{2} \int_{T_2} \rho |u_t|^2 dx + \frac{1}{2} \int_{T_2} \rho |\omega_t|^2 dx + \frac{1}{2} \int_{T_2} \rho |u \cdot \nabla u|^2 dx + \frac{1}{2} \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx, \]
which implies that
\[ \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \frac{d}{dt} \left( \nu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + 4\chi \|\omega\|_2^2 \right) \leq \int_{T_2} \rho |u \cdot \nabla u|^2 dx + \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx. \]
Due to
\[ 4\chi \|\text{curl} u, \omega\| \leq \chi \|\nabla u\|_2^2 + 4\chi \|\omega\|_2^2, \]
then there exist two positive constants $c_1$ and $c_2$ such that
\[ c_1 \alpha_1 \leq \nu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + 4\chi \|\omega\|_2^2 + 4\chi \|\text{curl} u, \omega\| \leq c_2 \alpha_1 \]
with $\alpha_1 \overset{\text{def}}{=} \nu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + 4\chi \|\omega\|_2^2$. This along with (2.5) ensures that
\[ \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \frac{d}{dt} \left( \nu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + 4\chi \|\omega\|_2^2 \right) \leq C \left( \int_{T_2} \rho |u \cdot \nabla u|^2 dx + \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx \right). \]
It follows from integrating with respect to time over $[0, t]$ on the both sides of (2.6) that
\[ \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 \right) dt \]
\[ \leq C J_0 + C \left( \int_0^t \int_{T_2} \rho |u \cdot \nabla u|^2 dx dt + \int_0^t \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx dt \right), \]
where $J_0$ is given by (1.4).
In order to bound the second derivatives of \((u, \omega)\), let us take the \(L^2\)-scalar product of the first equation of system (1.1) with \(-\Delta u\) and the third equation with \(-\Delta \omega\) respectively. Adding the resulting equations, we easily arrive at

\[
\nu\|\nabla^2 u\|^2 + \gamma\|\nabla^2 \omega\|^2 + \kappa\|\nabla \text{div} \omega\|^2 + 4\chi\|\nabla \omega\|^2
= 4\chi(\text{curl} \omega, -\Delta u) + (\rho u_t, \Delta u) + (\rho \omega_t, \Delta \omega)
+ (\rho u \cdot \nabla u, \Delta u) + (\rho u \cdot \nabla \omega, \Delta \omega),
\]

where we have used the fact \((\text{curl} \omega, -\Delta u) = (\text{curl} \omega, -\Delta \omega)\).

Hence, using Young’s inequality yields that

\[
\|\nabla^2 u\|^2 + \|\nabla^2 \omega\|^2 \leq C\left(\|\sqrt{\rho} u_t\|^2 + \|\sqrt{\rho} \omega_t\|^2 + \int_{\mathbb{T}^2} \rho |u| \cdot \nabla u|^2 dx + \int_{\mathbb{T}^2} \rho |u| \cdot \nabla \omega|^2 dx\right).
\]  

(2.8)

Next, we deal with the gradient of the pressure. Taking the divergence of the linear momentum equation gives

\[
\Delta P = -\Delta (\rho u_t + \rho u \cdot \nabla u),
\]

and consequently the pressure \(P\) may be recovered by

\[
\nabla P = -\nabla \Delta^{-1} \text{div} (\rho u_t + \rho u \cdot \nabla u).
\]

By the bounded of Riesz’ operator, we have

\[
\|\nabla P\|^2 \leq C\left(\|\sqrt{\rho} u_t\|^2 + \int_{\mathbb{T}^2} \rho |u| \cdot \nabla u|^2 dx\right).
\]  

(2.9)

We finally conclude from (2.7), (2.8) and (2.9), that

\[
\|\nabla u\|^2 + \|\nabla \omega\|^2 + \int_0^t \left(\|\sqrt{\rho} u_t\|^2 + \|\sqrt{\rho} \omega_t\|^2 + \|\nabla^2 u\|^2 + \|\nabla^2 \omega\|^2 + \|\nabla P\|^2\right) d\tau
\leq C J_0 + C \left(\int_0^t \int_{\mathbb{T}^2} \rho |u| \cdot \nabla u|^2 dx d\tau + \int_0^t \int_{\mathbb{T}^2} \rho |u| \cdot \nabla \omega|^2 dx d\tau\right).
\]  

(2.10)

Here and in what follows, we bound the last term on the right-hand side of the inequality above. Using Hölder’s inequality and 2-D Ladyzhenskaya’s inequality \(21\): \(\|v\|_4^2 \leq C\|v\|_2\|\nabla v\|_2\), we deduce that

\[
\int_{\mathbb{T}^2} \rho |u| \cdot \nabla u|^2 dx \leq \|\sqrt{\rho} u\|^2 \|\sqrt{\rho} \nabla u\|^2 \leq C\sqrt{\rho} \sqrt{\rho} \|u\|^2 \|\nabla u\|_2, \quad \text{(2.11)}
\]

and

\[
\int_{\mathbb{T}^2} \rho |u| \cdot \nabla \omega|^2 dx \leq \|\sqrt{\rho} u\|^2 \|\sqrt{\rho} \nabla \omega\|^2 \leq C\sqrt{\rho} \sqrt{\rho} \|u\|^2 \|\nabla \omega\|_2 \|\nabla^2 \omega\|_2. \quad \text{(2.12)}
\]

For the term \(\|\sqrt{\rho} u\|^2\), we have

\[
\|\sqrt{\rho} u\|^2 \leq C\|\sqrt{\rho} u\|^2 \|\nabla u\|^2 \log \left(e + \frac{\|\rho_0 - M\|^2}{M^2} + \frac{\rho^* \|\nabla u\|^2}{\|\sqrt{\rho} u\|^2}\right),
\]
where we have used the following improvement of Ladyzhenskaya’s inequality that has been pointed out by B. Desjardins in [14]:

\[ \| \sqrt{\rho} v \|_2 \leq C \| \sqrt{\rho} v \|_2 \| \nabla v \|_2 \log^2 \left( e + \frac{\| \rho - M \|_2^2}{M^2} + \rho^* \| \nabla v \|_2^2 \right), \]

where for all \( v \in H^1(\mathbb{T}^2) \) and \( \rho \in L^\infty(\mathbb{T}^2) \) with \( 0 \leq \rho \leq \rho^* \).

Observing that the function \( z \to z \log(e + \frac{1}{z}) \) is increasing, and reverting to (2.11) and (2.12), we end up with

\[ \int_{T_2} \rho |u| \cdot |\nabla u|^2 \, dx \leq \varepsilon \| \nabla^2 u \|_2^2 + C_{\rho^*} \| \sqrt{\rho} u \|_2^2 \| \nabla u \|_2^2 \]

\[ \leq \varepsilon \| \nabla^2 u \|_2^2 + C_{\rho^*} C_0 \| \nabla u \|_2^2 \log \left( e + \frac{\| \rho_0 - M \|_2^2}{M^2} + \frac{\rho^* \| \nabla u \|_2^2}{C_0} \right), \]  

(2.13)

and

\[ \int_{T_2} \rho |u| \cdot |\nabla \omega|^2 \, dx \leq \varepsilon \| \nabla^2 \omega \|_2^2 + C_{\rho^*} \| \sqrt{\rho} u \|_2^2 \| \nabla \omega \|_2^2 \]

\[ \leq \varepsilon \| \nabla^2 \omega \|_2^2 + C_{\rho^*} C_0 \| \nabla u \|_2^2 \| \nabla \omega \|_2^2 \log \left( e + \frac{\| \rho_0 - M \|_2^2}{M^2} + \frac{\rho^* \| \nabla u \|_2^2}{C_0} \right), \]  

(2.14)

where \( \varepsilon \) is arbitrary small positive constant.

Then combining (2.13) and (2.14) with (2.10) yields that

\[ \| \nabla u \|_2^2 + \| \nabla \omega \|_2^2 + \int_0^t \left( \| \sqrt{\rho} u \|_2^2 + \| \sqrt{\rho} \omega \|_2^2 + \| \nabla^2 u \|_2^2 + \| \nabla^2 \omega \|_2^2 + \| \nabla P \|_2^2 \right) \, d\tau \]

\[ \leq C_{J_0} + C \varepsilon \left( \int_0^t \log \left( e + \frac{\| \rho_0 \|_2^2}{M^2} \right) \log \left( e + \frac{\| \rho_0 \|_2^2}{M^2} \right) \, d\tau \right) \]

(2.15)

with \( C_1 \) depending only on \( \rho^*, C_0, M \) and \( \| \rho_0 \|_2 \).

Denoting \( f(t) := C_1 \| \nabla u \|_2^2 \) and

\[ X(t) := \| \nabla u \|_2^2 + \| \nabla \omega \|_2^2 + \int_0^t \left( \| \sqrt{\rho} u \|_2^2 + \| \sqrt{\rho} \omega \|_2^2 + \| \nabla^2 u \|_2^2 + \| \nabla^2 \omega \|_2^2 + \| \nabla P \|_2^2 \right) \, d\tau. \]

It follows from the inequality (2.15) that

\[ X(t) \leq C_{J_0} + C \int_0^t f(\tau) X(\tau) \log \left( e + X(\tau) \right) \, d\tau. \]

(2.16)

Setting \( g(t) = \int_0^t f(\tau) X(\tau) \log \left( e + X(\tau) \right) \, d\tau \), from (2.16), we obtain \( e + X(t) \leq e + C_{J_0} + C g(t) \).

Then,

\[ \frac{d}{dt} g(t) = f(t) X(t) \log \left( e + X(t) \right) \]

\[ \leq f(t) \left( e + X(t) \right) \log \left( e + X(t) \right) \]

\[ \leq f(t) \left( e + C_{J_0} + C g(t) \right) \log \left( e + C_{J_0} + C g(t) \right), \]

from which we get, for all \( t \geq 0 \),

\[ e + C_{J_0} + C g(t) \leq \left( e + C_{J_0} \right)^{\exp \left( C \int_0^t f(\tau) \, d\tau \right)}. \]
Hence,
\[
\|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho}u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2 + \|\nabla P\|_2^2 \right) d\tau
\]
\[
\leq \left( e + CJ_0 \right) \exp \left( C_1 C_0 \right).
\]

In order to prove (2.4), denoting by \( \bar{u}(t), \bar{\omega}(t) \) the average of \( u(t), \omega(t) \) on \( T^2 \), for all \( p \in [1, \infty) \), it then follows from Sobolev embedding that
\[
\|u(t)\|_p + \|\omega(t)\|_p \leq \|\bar{u}(t)\| + \|\bar{w}(t)\| + \|u(t) - \bar{u}(t)\|_p + \|\omega(t) - \bar{w}(t)\|_p
\]
\[
\leq \|\bar{u}(t)\| + \|\bar{w}(t)\| + C_p \left( \|\nabla u(t)\|_2 + \|\nabla \omega(t)\|_2 \right).
\]  

On the other hand, applying Poincaré’s inequality yields that
\[
M \left( \|\bar{u}(t)\| + \|\bar{\omega}(t)\| \right) = \int_{T^2} \rho u dx + \int_{T^2} \rho \omega dx + \int_{T^2} (M - \rho)(u - \bar{u}) dx
\]
\[
+ \int_{T^2} (M - \rho)(\omega - \bar{\omega}) dx
\]
\[
\leq CC_0 + \|M - \rho\|_2 \left( \|\nabla u\|_2 + \|\nabla \omega\|_2 \right).
\]

Putting that latter inequality into (2.17) yields (2.4).

### 2.2 Weighted energy method in 2-D.

In order to obtain the shift integrability from time to space variables, our next aim is to exploit some bounds, for example, \( (\sqrt{\rho u_t}, \sqrt{\rho \omega_t}) \) in \( L^\infty([0, T]; L^2) \) and \( (\sqrt{T \nabla u_t}, \sqrt{T \nabla \omega_t}) \) in \( L^2([0, T]; L^2) \) respectively, in terms of the data.

**Lemma 2.2.** Assume \( d = 2 \) and that the solution is smooth enough of system (1.1) with no vacuum. Then for all \( t \geq 0 \), we have
\[
\|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \int_0^t \tau \|\nabla u_t\|_2^2 d\tau + \int_0^t \tau \|\nabla \omega_t\|_2^2 \leq \exp \left( \int_0^t h_1(\tau) d\tau \right) - 1
\]  

with \( h_1 \in L^1_{\text{loc}}(\mathbb{R}^+) \) depending only on \( \rho^*, \|\sqrt{\rho_0} u_0\|_2, \|\sqrt{\rho_0} \omega_0\|_2 \) and \( K_0 \).

**Proof.** Differentiating (1.1) and (1.1) with respect to \( t \), respectively, we have
\[
\rho u_{tt} + \rho u_t + \rho u_t \cdot \nabla u + \rho u_t \cdot \nabla u_t + \nu \Delta u_t + \nabla P = 2\chi \text{curl}\omega_t,
\]
and
\[
\rho \omega_{tt} + \rho \omega_t + \rho u \cdot \nabla \omega + \rho u_t \cdot \nabla \omega + \rho u \cdot \nabla \omega_t - \gamma \Delta \omega_t - \kappa \nabla \text{div}\omega_t + 4\chi \omega_t = 2\chi \text{curl}\omega_t.
\]

Then, multiplying by \( \sqrt{\tau} \) the above two inequalities respectively yields that
\[
\rho(\sqrt{\tau} u_t) + \frac{1}{2\sqrt{\tau}} \rho u_t + \sqrt{\tau} \rho u_t + \sqrt{\tau} \rho u_t \cdot \nabla u + \sqrt{\tau} \rho u_t \cdot \nabla u_t + \sqrt{\tau} \rho u \cdot \nabla u_t - \nu \Delta (\sqrt{\tau} u_t) + \nabla (\sqrt{\tau} P)
\]
\[
= 2\chi \text{curl}(\sqrt{\tau} \omega_t),
\]  

(2.19)
and

\[ \rho(\sqrt{t}\omega_l)_t - \frac{1}{2\sqrt{t}}\rho\omega_l + \sqrt{t}\rho u_t \cdot \nabla \omega + \sqrt{t}\rho u_t \cdot \nabla \omega + \sqrt{t}\rho u_t \cdot \omega - \gamma\Delta(\sqrt{t}\omega_l) \]
\[ - \kappa \nabla \text{div}(\sqrt{t}\omega_l) + 4\chi(\sqrt{t}\omega_l) = 2\chi \text{curl}(\sqrt{t}u_t). \]  

(2.20)

Taking the $L^2$ scalar product of (2.19) with $\sqrt{t}u_t$ and (2.20) with $\sqrt{t}\omega_l$ respectively, we get

\[ \frac{1}{2} \frac{d}{dt} \int_{T^2} \rho|u_t|^2 dx + \nu \int_{T^2} t|\nabla u_t|^2 dx \]
\[ \leq \frac{1}{2} \int_{T^2} \rho|u_t|^2 dx - \frac{1}{2} \int_{T^2} t\rho_t|u_t|^2 dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla u) \cdot (\sqrt{t}u_t)dx - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla \omega) \cdot (\sqrt{t}\omega_l)dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \omega) \cdot (\sqrt{t}u_t)dx + 2\chi \int_{T^2} \text{curl}(\sqrt{t}\omega_l) \cdot (\sqrt{t}u_t)dx, \]  

(2.21)

and

\[ \frac{1}{2} \frac{d}{dt} \int_{T^2} \rho|\omega_l|^2 dx + \gamma \int_{T^2} t|\nabla \omega_l|^2 dx + 2\chi \int_{T^2} \text{curl}(\sqrt{t}u_t) \cdot (\sqrt{t}\omega_l)dx \]
\[ \leq \frac{1}{2} \int_{T^2} \rho|\omega_l|^2 dx - \frac{1}{2} \int_{T^2} t\rho_t|\omega_l|^2 dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla \omega) \cdot (\sqrt{t}\omega_l)dx - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla \omega) \cdot (\sqrt{t}\omega_l)dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \omega) \cdot (\sqrt{t}u_t)dx + 2\chi \int_{T^2} \text{curl}(\sqrt{t}u_t) \cdot (\sqrt{t}\omega_l)dx. \]  

(2.22)

Note that

\[ 2\chi \int_{T^2} \text{curl}(\sqrt{t}\omega_l) \cdot (\sqrt{t}u_t)dx + 2\chi \int_{T^2} \text{curl}(\sqrt{t}u_t) \cdot (\sqrt{t}\omega_l)dx \]
\[ = 4\chi \int_{T^2} \text{curl}(\sqrt{t}u_t) \cdot (\sqrt{t}\omega_l)dx \]
\[ \leq 4\chi \|\sqrt{t}\nabla u_t\|_2 \|\sqrt{t}\omega_l\|_2 \]
\[ \leq \chi \|\sqrt{t}\nabla u_t\|_2^2 + 4\chi \|\sqrt{t}\omega_l\|_2^2, \]

which together with (2.21) and (2.22) implies that

\[ \frac{d}{dt} \left( \|\sqrt{t}u_t\|_2^2 + \|\sqrt{t}\omega_l\|_2^2 \right) dx + \int_{T^2} t|\nabla u_t|^2 dx + \int_{T^2} t|\nabla \omega_l|^2 dx \]
\[ \leq \int_{T^2} (\rho|u_t|^2 + \rho|\omega_l|^2) dx + \int_{T^2} (t|\rho u_t|^2 + t|\rho_t|\omega_l|^2) dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla u) \cdot (\sqrt{t}u_t) \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \nabla \omega) \cdot (\sqrt{t}\omega_l) dx \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \omega) \cdot (\sqrt{t}u_t) \]
\[ - \int_{T^2} (\sqrt{t}\rho u_t \cdot \omega) \cdot (\sqrt{t}\omega_l) dx \]
\[ = \sum_{i=1}^{5} I_i. \]  

(2.23)
In what follows, we estimate term by term above. For $I_2$, thanks to $\rho_t = -u \cdot \nabla \rho$, we have

$$I_2 \leq C \left| \int_{T_2} t \mathrm{div}(\rho u)|u_t|^2 \, dx + \int_{T_2} t \mathrm{div}(\rho u)\omega_t \, dx \right|$$

$$\leq C \int_{T_2} t|\rho||\nabla u_t||u_t| \, dx + \int_{T_2} t|\rho||\nabla \omega_t||\omega_t| \, dx$$

$$\leq C \left( \int_{T_2} \rho t|u_t|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{T_2} t|\rho|\omega_t^2 \, dx \right)^{\frac{1}{2}}$$

$$+ C \left( \int_{T_2} \rho t|\omega_t|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{T_2} t|\rho u_t|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq C\|\sqrt{\rho} u_t\|_2\|u\|_\infty\|\sqrt{\rho} \nabla u_t\|_2 + C\|\sqrt{\rho} \omega_t\|_2\|u\|_\infty\|\sqrt{\rho} \nabla \omega_t\|_2$$

$$\leq \varepsilon \left( \|\sqrt{\rho} \nabla u_t\|_2^2 + \|\sqrt{\rho} \nabla \omega_t\|_2^2 \right) + C\|u\|_\infty^2 \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 \right).$$

For $I_3$, according to $\rho_t = -u \cdot \nabla \rho$ and then performing an integration by parts, we get

$$I_3 \leq \left| - \int_{T_2} (\sqrt{\rho} \rho u \cdot \nabla u) \cdot (\sqrt{\rho} u_t) \, dx - \int_{T_2} (\sqrt{\rho} \rho u \cdot \nabla \omega) \cdot (\sqrt{\rho} \omega_t) \, dx \right|$$

$$\leq \left| - \int_{T_2} t \rho u \cdot \nabla \left( (u \cdot \nabla u) \cdot u_t \right) \, dx - \int_{T_2} t \rho u \cdot \nabla \left( (u \cdot \nabla \omega) \cdot \omega_t \right) \, dx \right|$$

$$\leq \int_{T_2} t|\rho|\left( \|\nabla u_t\|_2^2 |u_t| + |u||\nabla^2 u||u_t| + |u||\nabla u||\nabla u_t| \right)$$

$$+ |\nabla u||\nabla \omega||\omega_t| + |u||\nabla^2 \omega||\omega_t| + |u||\nabla \omega||\nabla \omega_t| \right) \, dx$$

$$\leq \sum_{i=1}^{6} I_{3i}.$$
For $I_4$ and $I_5$, using Hölder’s and Young’s inequalities, we deduce that

$$I_4 \leq \left\| \nabla u \right\|_2 \left\| \sqrt{\rho t^\varepsilon} u \varepsilon \right\|_2^2 + \left\| \nabla \omega \right\|_2 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4$$

$$\leq (\rho^*)^{3/4} \left\| \nabla u \right\|_2 \left\| \sqrt{\rho t^\varepsilon} u \varepsilon \right\|_2^2 + (\rho^*)^{3/4} \left\| \nabla \omega \right\|_2 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4$$

$$\leq C(\rho^*)^{3/4} \left\| \nabla u \right\|_2 \left\| \sqrt{\rho t^\varepsilon} u \varepsilon \right\|_2^2 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4 + C(\rho^*)^{3/4} \left\| \nabla \omega \right\|_2 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4 \left\| \sqrt{\rho t^\varepsilon} \omega \varepsilon \right\|_4$$

$$\leq \varepsilon \left( \| \nabla \sqrt{\rho t} u \|_2^2 + \| \nabla \sqrt{\rho t} \omega \|_2^2 \right) + C_{T, \rho^*} \left( \| \nabla u \|_2^2 + \| \nabla \omega \|_2^2 \right) \left( \| \nabla \sqrt{\rho t^\varepsilon} u \|_2^2 + \| \nabla \sqrt{\rho t^\varepsilon} \omega \|_2^2 \right),$$

(2.26)

and

$$I_5 \leq \left| - \int_{T^2} (\sqrt{\rho t} \nabla \cdot \nabla u ) \cdot (\sqrt{\rho t} \nabla u ) dx - \int_{T^2} (\sqrt{\rho t} \nabla \cdot \nabla \omega ) \cdot (\sqrt{\rho t} \nabla \omega ) dx \right|$$

$$\leq \varepsilon \left( \| \nabla \sqrt{\rho t} u \|_2^2 + \| \nabla \sqrt{\rho t} \omega \|_2^2 \right) + C \rho^* \| u \|_{2, \infty}^4 \left( \| \nabla \sqrt{\rho t^\varepsilon} u \|_2^2 + \| \nabla \sqrt{\rho t^\varepsilon} \omega \|_2^2 \right).$$

(2.27)

Therefore, for some constant $C_{T, \rho^*}$ depending only on $\rho^*$ and $T$, from (2.23), we conclude that

$$\frac{d}{dt} \left( \| \sqrt{\rho t} u \|_2^2 + \| \sqrt{\rho t} \omega \|_2^2 \right) + \| \nabla \sqrt{\rho t} u \|_2^2 + \| \nabla \sqrt{\rho t} \omega \|_2^2$$

$$\leq C \left( (1 + \rho^*) \| u \|_{2, \infty}^2 + \| u \|_{4, \infty}^4 \right) \left( \| \nabla \sqrt{\rho t} u \|_2^2 + \| \nabla \sqrt{\rho t} \omega \|_2^2 \right) + C_{T, \rho^*} \left( \| \nabla u \|_4^4 + \| \nabla \omega \|_4^4 + \| \nabla^2 u \|_2^2 \right.$$  

$$+ \| \nabla^2 \omega \|_2^2 + \| u \|_{4, \infty}^4 \| \nabla u \|_2^2 + \| \nabla \omega \|_2^2 \right) + (1 + \| u \|_2^2 + \| \nabla \omega \|_2^2) \left( \| \nabla \sqrt{\rho t^\varepsilon} u \|_2^2 + \| \nabla \sqrt{\rho t^\varepsilon} \omega \|_2^2 \right) \right).$$

Set

$$h_1(t) = C \left( (1 + \rho^*) \| u \|_{2, \infty}^2 + \| u \|_{4, \infty}^4 \right) + C_{T, \rho^*} \left( \| \nabla u \|_4^4 + \| \nabla \omega \|_4^4 + \| \nabla^2 u \|_2^2 + \| \nabla^2 \omega \|_2^2 \right.$$  

$$+ \| u \|_{4, \infty}^4 \| \nabla u \|_2^2 + \| \nabla \omega \|_2^2 \right) + (1 + \| u \|_2^2 + \| \nabla \omega \|_2^2) \left( \| \nabla \sqrt{\rho t^\varepsilon} u \|_2^2 + \| \nabla \sqrt{\rho t^\varepsilon} \omega \|_2^2 \right),$$

then, $h_1(t) \in L^1_{\text{loc}}(\mathbb{R}^+)$ depending only on $\rho^*, \| \sqrt{\rho_0} u_0 \|_2, \| \sqrt{\rho_0} \omega_0 \|_2$ and $K_0$. Indeed, from (2.2), (2.3) and the 2-D Gagliardo-Nirenberg interpolation inequality $\| u \|_{4, \infty}^4 \leq \| u \|_2^2 \| \nabla^2 u \|_2^2$, we get $(u, \omega) \in L^2((0, T; L^\infty))$, and naturally $(u, \omega) \in L^2((0, T; L^\infty))$. Similarly, we also get $(\nabla u, \nabla \omega) \in L^2((0, T; L^4), (\nabla^2 u, \nabla^2 \omega) \in L^2(\mathbb{R}^+; L^2)$.

Obviously, if the solution is smooth with density bounded away from zero, then we have

$$\lim_{t \to 0^+} \int_{T^2} \rho t (|u|_t^2 + |\omega|_t^2) dx = 0.$$  

Thus, integrating with respect to time from 0 to $t$ for the following inequality

$$\frac{d}{dt} \left( \| \sqrt{\rho t} u \|_2^2 + \| \sqrt{\rho t} \omega \|_2^2 \right) + \int_0^t \tau \| \nabla u \|_2^2 d\tau + \int_0^t \tau \| \nabla \omega \|_2^2 d\tau$$

$$\leq h_1(t) \left( 1 + \| \sqrt{\rho t} u \|_2^2 + \| \sqrt{\rho t} \omega \|_2^2 \right),$$

we conclude that (2.18) holds for $t \geq 0$.

As in the process starting from time $t_0$, we also have the following lemma.

**Lemma 2.3.** Assume $d = 2$ and that the solution is smooth with no vacuum. Then for all $t_0, T \geq 0$, we have

$$\sup_{t_0 \leq t \leq t_0 + T} \int_{T^2} \rho (t - t_0) (|u|_t^2 + |\omega|_t^2) dx + \int_{t_0}^{t_0 + T} \int_{T^2} (t - t_0) \left( |\nabla u|_t^2 + |\nabla \omega|_t^2 \right) dx dt \leq c(T)$$

(2.28)

with $c(T)$ going to zero as $T \to 0.$
Furthermore, denoting by \( \overline{u_t} \) the average of \( u_t \), we have
\[
\int_{\mathbb{T}^2} \rho u_t \, dx = M(\overline{u_t}) + \int_{\mathbb{T}^2} \rho (u_t - \overline{u_t}) \, dx.
\]
Thus,
\[
M(\overline{|u_t|}) \leq \|\rho\|_2 \|\nabla u_t\|_2 + M^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_2.
\]
Similarly,
\[
M(\overline{|\omega_t|}) \leq \|\rho\|_2 \|\nabla \omega_t\|_2 + M^{\frac{1}{2}} \|\sqrt{\rho} \omega_t\|_2.
\]
Adding the above two inequalities yields that
\[
M\left(\overline{|u_t|} + \overline{|\omega_t|}\right) \leq \|\rho\|_2 \left(\|\nabla u_t\|_2 + \|\nabla \omega_t\|_2\right) + M^{\frac{1}{2}} \left(\|\sqrt{\rho} u_t\|_2 + \|\sqrt{\rho} \omega_t\|_2\right).
\]
Since \( \|\rho\|_2 \) and \( M \) are time independent, we get by Sobolev embedding,
\[
\|u_t\|_p + \|\omega_t\|_p \leq \|u_t - \overline{u_t}\|_p + \|\overline{u_t}\| + \|\omega_t - \overline{\omega_t}\|_p + \|\overline{\omega_t}\| \leq \left(C_p + \frac{\|\rho\|_2}{M}\right) \left(\|\nabla u_t\|_2 + \|\nabla \omega_t\|_2\right) + \frac{1}{M^{1/2}} \left(\|\sqrt{\rho} u_t\|_2 + \|\sqrt{\rho} \omega_t\|_2\right),
\]
which implies that, for all \( p < \infty \),
\[
\|\sqrt{t} u_t\|_{L^2(0,T;L^p)} + \|\sqrt{t} \omega_t\|_{L^2(0,T;L^p)} \leq \left(C_p + \frac{\|\rho\|_2}{M}\right) \left(\|\sqrt{t} \nabla u_t\|_{L^2(0,T;L^2)} + \|\sqrt{t} \nabla \omega_t\|_{L^2(0,T;L^2)}\right) + \frac{1}{M^{1/2}} \left(\|\sqrt{t} \rho u_t\|_{L^2(0,T;L^2)} + \|\sqrt{t} \rho \omega_t\|_{L^2(0,T;L^2)}\right).
\]
According to (2.18), we deduce that for \( p < \infty \),
\[
\|\left(\sqrt{t} u_t, \sqrt{t} \omega_t\right)\|_{L^2(0,T;L^p)} \leq c(T) \quad \text{with} \quad c(T) \to 0 \quad \text{for} \quad T \to 0.
\]
In order to obtain some strong sense convergence of the approximate sequences \((u^\delta, \omega^\delta)\), as in [13], we also need the following control on the regularity of \( u, \omega \) with respect to the time variable.

**Lemma 2.4.** Let \( p \in [1, \infty] \) and \( u, \omega \) satisfy \( u, \omega \in L^2(0,T;L^p) \) and \( \sqrt{t} u_t, \sqrt{t} \omega_t \in L^2(0,T;L^2) \). Then \( u, \omega \in H^{1/2-\alpha}(0,T;L^p) \) for all \( \alpha \in (0,1/2) \) and
\[
\|u, \omega\|_{H^{\frac{1}{2}-\alpha}}^2 \leq \|u, \omega\|_{L^2(0,T;L^p)}^2 + C_{\alpha,T} \|\sqrt{t} u_t, \sqrt{t} \omega_t\|_{L^2(0,T;L^2)}^2,
\]
with \( C_{\alpha,T} \) depending only on \( \alpha \) and on \( T \).

### 3 Existence of solution and weighted energy method in 3-D

#### 3.1 Existence of solution in 3-D

Similar to 2-D, here, we only present the *a priori* estimates for smooth enough solutions \((\rho, u, \omega)\) of system (1.1) in what follows.
Proposition 3.1. Under the assumptions of Theorem 1.2. Let \((\rho, u, \omega)\) be a smooth enough solution of system (1.1) satisfying (2.1). There exist a universal positive constant \(C\) and \(T > 0\) such that

\[ T \leq \frac{C}{(\rho^*)^3 C_0 K_0^3}. \tag{3.1} \]

Then, for all \(t \in [0, T]\), we have

\[ \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2 + \|\nabla P\|_2^2 \right) d\tau \leq C K_0. \tag{3.2} \]

Furthermore, if (1.7) is satisfied then (3.2) holds true for all \(t \in [0, \infty)\). At last, inequality (2.4) holds true for all \(p \in [1, 6]\).

**Proof.** Taking the \(L^2\)-scalar product of the first equation of system (1.1) with \(u_t\) and the third equation with \(\omega_t\) respectively, we get by \(\rho_t = -u \cdot \nabla \rho\) that

\[ \int_{T_2} \rho |u_t|^2 dx + \frac{\nu d}{2 dt} \int_{T_2} |\nabla u|^2 dx = 2\chi \int_{T_2} \text{curl} \cdot u_t dx - \int_{T_2} (\rho u \cdot \nabla u) \cdot u_t dx, \]

and

\[ \int_{T_2} \rho |\omega_t|^2 dx + \frac{\gamma d}{2 dt} \int_{T_2} |\nabla \omega|^2 dx + \frac{\kappa d}{2 dt} \int_{T_2} |\text{div} \omega|^2 dx + 2\chi \int_{T_2} |\omega|^2 dx \]

\[ = 2\chi \int_{T_2} curl u \cdot \omega_t dx - \int_{T_2} (\rho u \cdot \nabla \omega) \cdot \omega_t dx. \]

Adding the two identities above and using Hölder’s and Young’s inequalities and the fact \(\text{curl}(\text{curl} u) = -\Delta u\) (for \(\text{div} u = 0\)) yield that

\[ \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \frac{d}{dt} \left( \mu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + \chi \|\text{curl} u - 2\omega\|_2^2 \right) \]

\[ = -\int_{T_2} (\rho u \cdot \nabla u) \cdot u_t dx - \int_{T_2} (\rho u \cdot \nabla \omega) \cdot \omega_t dx \]

\[ \leq \frac{1}{2} \int_{T_2} \rho |u_t|^2 dx + \frac{1}{2} \int_{T_2} \rho |\omega_t|^2 dx + \frac{1}{2} \int_{T_2} \rho |u \cdot \nabla u|^2 dx + \frac{1}{2} \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx, \]

which gives that

\[ \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \frac{d}{dt} \left( \mu \|\nabla u\|_2^2 + \gamma \|\nabla \omega\|_2^2 + \kappa \|\text{div} \omega\|_2^2 + \chi \|\text{curl} u - 2\omega\|_2^2 \right) \]

\[ \leq \int_{T_2} \rho |u \cdot \nabla u|^2 dx + \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx. \tag{3.3} \]

Furthermore, we also get from integrating with respect to time over \([0, t]\) on the both sides of (3.3) that

\[ \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 \right) d\tau \]

\[ \leq CK_0 + C \left( \int_0^t \int_{T_2} \rho |u \cdot \nabla u|^2 dx d\tau + \int_0^t \int_{T_2} \rho |u \cdot \nabla \omega|^2 dx d\tau \right). \tag{3.4} \]
where $K_0$ is given by (1.5).

From (3.4), (2.8) and (2.9), we finally conclude that

$$
\|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2 + \|\nabla P\|_2^2 \right) d\tau 
\leq CK_0 + C \left( \int_0^t \int_{\mathbb{T}^3} |u| \cdot |\nabla u|^2 dx d\tau + \int_0^t \int_{\mathbb{T}^3} |u| \cdot |\nabla \omega|^2 dx d\tau \right).
$$

(3.5)

In what follows, we will bound the last term on the right-hand side of the inequality above. To this end, it follows from Hölder’s and Young’s inequalities and Sobolev embedding $\dot{H}^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$ that

$$
\int_{\mathbb{T}^3} \rho |u \cdot \nabla u|^2 dx \leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} |u|^2 \|\nabla u\|_2^2
\leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} |u| \|\nabla u\|_6^2 \|\nabla \omega\|_6^2
\leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} \|\nabla u\|_2^2 \|\nabla \omega\|_2^2 \|\nabla \omega\|_2^2
\leq \varepsilon \|\nabla \omega\|_2^2 + C(\rho^*)^3 \|\nabla u\|_2^2 \|\nabla \omega\|_2^2,
$$

and

$$
\int_{\mathbb{T}^3} \rho |u \cdot \nabla \omega|^2 dx \leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} |u|^2 \|\nabla \omega\|_2^2
\leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} \|\nabla u\|_2^2 \|\nabla \omega\|_2^2 \|\nabla \omega\|_2^2
\leq (\rho^*)^\frac{1}{3} \left( \rho^* \right)^\frac{2}{3} \|\nabla \omega\|_2^2 \|\nabla \omega\|_2^2 \|\nabla \omega\|_2^2
\leq \varepsilon \|\nabla \omega\|_2^2 + C(\rho^*)^3 \|\nabla u\|_2^2 \|\nabla \omega\|_2^2 \|\nabla \omega\|_2^2,
$$

from which, together with (2.2) and (3.5) implies that

$$
\|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 + \int_0^t \left( \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} \omega_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2 + \|\nabla P\|_2^2 \right) d\tau
\leq CK_0 + C \int_0^t (\rho^*)^3 \|\nabla u\|_2^2 \|\nabla \omega\|_2^2 \left( \|\nabla u\|_2^2 + \|\nabla \omega\|_2^2 \right) d\tau.
$$

(3.6)

That is

$$
X(t) \leq CK_0 + C \int_0^t f_1(\tau) X^3(\tau) d\tau
$$

(3.7)

with $f_1(t) = (\rho^*)^3 \|\nabla u\|_2^2 \|\nabla \omega\|_2^2$.

Setting $g_1(t) = \int_0^t f_1(\tau) X^3(\tau) d\tau$, from (3.7), we get $X(t) \leq CK_0 + C g_1(t)$. Thus,

$$
\frac{d}{dt} g_1(t) = f_1(t) X^3(t)
\leq f_1(t) \left( CK_0 + C g_1(t) \right)^3.
$$

Hence, whenever $T$ satisfies $2CK_0^2 \int_0^T f_1(\tau) d\tau \leq \frac{1}{4}$, we obtain

$$
\left( CK_0 + C g_1(t) \right)^2 \leq \frac{CK_0^2}{1 - 2CK_0^2 \int_0^T f_1(\tau) d\tau} \text{ for } t \in [0, T],
$$
which ensures that

\[ X^2(t) \leq \frac{CK_0^2}{1 - 2CK_0^2 \int_0^t f_1(\tau) d\tau}. \] (3.8)

Thus, we conclude that (3.2) holds for \( t \in [0, T] \). Furthermore, according (2.2), we have

\[ \int_0^T f_1(t) dt \leq (\rho^*)^3 C_0^2 T \sup_{t \in [0, T]} X(t), \]

which implies that (3.1) holds.

On the other hand, if \( \varepsilon_0 \) is small enough in (1.7), from (2.2) and (3.8), for \( t \in [0, \infty) \), we also have

\[ 2CK_0^2 \int_0^t f_1(\tau) d\tau \leq \frac{1}{2}, \]

which yields that

\[ X(t) \leq CK_0 \quad \text{for} \quad t \in [0, \infty). \]

The proof of the last part of the theorem is similar to 2-D case. The only difference is that Sobolev embedding \( H^1(T^3) \hookrightarrow L^p(T^3) \) holds true only for \( p \leq 6 \). Here, we omit it.

### 3.2 Weighted energy method in 3-D

In 3-D case, our aim is also to obtain bounds \((\sqrt{\rho^*} u_t, \sqrt{\rho^*} \omega_t)\) in \( L^\infty([0, T]; L^2) \) and \((\sqrt{\dot{t}} \nabla u_t, \sqrt{\dot{t}} \nabla \omega_t)\) in \( L^2([0, T]; L^2) \) respectively.

**Lemma 3.2.** Assume \( d = 3 \) and that the solution is smooth enough of system (1.1) with no vacuum. Then for all \( t \geq 0 \), we have

\[ \|\sqrt{\rho^*} u_t\|_2^2 + \|\sqrt{\rho^*} \omega_t\|_2^2 + \int_0^t \tau \|\nabla u_t\|_2^2 d\tau + \int_0^t \tau \|\nabla \omega_t\|_2^2 d\tau \leq \exp \left( \int_0^t h_2(\tau) d\tau \right) - 1 \] (3.9)

with \( h_2 \in L^1_{\text{loc}}(\mathbb{R}^+) \) depending only on \( \rho^*, \|\sqrt{\rho^*} u_0\|_2, \|\sqrt{\rho^*} \omega_0\|_2 \) and \( K_0 \).

**Proof.** Compared with the proof of lemma 2.3 for the 2-D case, we here only show some different parts for \( I_2-I_5 \) in what follows. For \( I_3 \), we also have (2.25). The method of processing \( I_{31} \) and \( I_{34} \) in 2-D is not applicable to 3-D. However, combining Hölder’s inequality and Sobolev embedding \( \dot{H}^1(T^3) \hookrightarrow L^p(T^3) \) for some constant \( C_{T, \rho^*} \) depending only on \( T \) and \( \rho^* \), we have

\[
I_{31} \leq \sqrt{\rho^* T} \|\sqrt{\rho^*} u_t\|_4 \|u_6\|_6 \|\nabla u\|_{24/7}^2
\leq \sqrt{\rho^* T} \|\sqrt{\rho^*} u_t\|_6^{1/4} \|\sqrt{\rho^*} u_t\|_6^{3/4} \|u_6\|_6 \|\nabla u\|_{24/7}^2
\leq \varepsilon \|\nabla \sqrt{\rho^*} u_t\|_2 + C_{T, \rho^*} \|\sqrt{\rho^*} u_t\|_2 \|\nabla u\|_{24/7} \|\nabla u\|_2^{8/5}.
\]

Due to

\[ \|\nabla u\|_{24/7}^{16/5} \leq C \|\nabla u\|_2^{6/5} \|\nabla^2 u\|_2^2, \]

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thus,

\[
I_{31} \leq \varepsilon \|\nabla \sqrt{t}u_t\|_2^2 + C_{T,\rho^*} \|\sqrt{t}u_t\|_2^{14/5} \|\nabla u\|_2^{14/5} \|\nabla u\|_2^2 \\
\leq \varepsilon \|\nabla \sqrt{t}u_t\|_2^2 + C_{T,\rho^*} \left(\|\sqrt{t}u_t\|_2^2 + \|\nabla u\|_2^{7/2}\right) \|\nabla^2 u\|_2^2.
\]

Similarly,

\[
I_{34} \leq \sqrt{\rho^* T} \|\sqrt{\rho^* \omega_t}\|_4 \|u\|_6 \|\nabla u\|_{24/7} \|\nabla \omega\|_{24/7} \\
\leq \sqrt{\rho^* T} \|\sqrt{\rho^* \omega_t}\|_2^{1/4} \|\sqrt{\rho^* \omega_t}\|_6^{3/4} \|u\|_6 \|\nabla u\|_{24/7} \|\nabla \omega\|_{24/7} \\
\leq \varepsilon \|\nabla \sqrt{t}\omega_t\|_2^2 + C_{T,\rho^*} \sqrt{\rho^* \omega_t} \|\nabla \omega_t\|_2^{11/5} \|\nabla^2 \omega\|_2^5 \|\nabla^2 \omega\|_2^2 \\
\leq \varepsilon \|\nabla \sqrt{t}\omega_t\|_2^2 + C_{T,\rho^*} \left(\|\sqrt{\rho^* \omega_t}\|_2^2 + \|\nabla \omega\|_{22/5} \|\nabla \omega\|_2^2 \right) \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right).
\]

Other items are treated exactly the same in 3-D as in 2-D.

Therefore, we get from (2.23) for some constant $C_{T,\rho^*}$ depending only on $\rho^*$ and $T$,

\[
\frac{d}{dt} \left(\|\sqrt{\rho^* t}u_t\|_2^2 + \|\sqrt{\rho^* \omega_t}\|_2^2\right) + \|\nabla \sqrt{t}u_t\|_2^2 + \|\nabla \sqrt{t}\omega_t\|_2^2 \\
\leq C \left(1 + \rho^*\right) \|u\|_\infty^2 + \|u\|_\infty^4 \left(\|\sqrt{\rho^* t}u_t\|_2^2 + \|\sqrt{\rho^* \omega_t}\|_2^2\right) \\
+ C_{T,\rho^*} \left(\|\sqrt{\rho^* t}u_t\|_2^2 + \|\sqrt{\rho^* \omega_t}\|_2^2\right) \left(\|\nabla^2 u\|_2^2 + \|\nabla \omega\|_2^2\right) + \|u\|_\infty^4 \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right) \\
+ \|u\|_\infty^4 \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right) \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right).
\]

Set

\[
h_2(t) = C \left(1 + \rho^*\right) \|u\|_\infty^2 + \|u\|_\infty^4 + C_{T,\rho^*} \left(\|u\|_\infty^4 \left(\|\nabla^2 u\|_2^2 + \|\nabla \omega\|_2^2\right) \\
+ \|u\|_\infty^4 \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right) \left(\|\nabla^2 u\|_2^2 + \|\nabla^2 \omega\|_2^2\right)\right),
\]

thus, $h_2(t) \in L^1_{\text{loc}}(\mathbb{R}^+)$ depending only on $\rho^*$, $\|\sqrt{\rho^* u_0}\|_2$, $\|\sqrt{\rho^* \omega_0}\|_2$ and $K_0$. In fact, from (2.2), (3.2) and the 3-D Gagliardo-Nirenberg interpolation inequality $\|v\|_\infty^4 \leq \|\nabla v\|_2^2 \|\nabla^2 v\|_2^2$, we get $(u, \omega) \in L^3(\mathbb{R}^+; L^\infty)$. Similar to the case of 2-D, we can also get that $(u, \omega) \in L^2(0, T; L^\infty)$, 

$(\nabla u, \nabla \omega) \in L^4(0, T; L^3)$ and $(\nabla^2 u, \nabla^2 \omega) \in L^2(\mathbb{R}^+; L^2)$.

Thus, from

\[
\frac{d}{dt} \left(\|\sqrt{t}u_t\|_2^2 + \|\sqrt{t}u_t\|_2^2\right) + \int_0^t \|\nabla u_t\|_2^2 d\tau + \int_0^t \|\nabla \omega_t\|_2^2 d\tau \\
\leq h_2(t) \left(1 + \sqrt{\rho t} u_t^2 + \sqrt{\rho^* \omega_t}\right),
\]

and

\[
\lim_{t \to 0^+} \int_{\mathbb{T}^2} pt(u_t^2 + \omega_t^2) dx = 0,
\]

we conclude that (3.9) holds for $t \geq 0$.

In a similar way, we know that Lemma 2.3 still hold for 3-D, and (2.29) holds for all $p < 6$ in 3-D.
4 The proof of uniqueness

The purpose of this section is to present the proof to the uniqueness part of both Theorems 1.1 and 1.2.

4.1 More regularity of the solutions

In order to prove the uniqueness parts of theorems 1.1 and 1.2, we need more information on the regularity of the solution to system (1.1) obtained in previous section. Our first goal is to achieve the bound \((\nabla u, \nabla \omega)\) in \(L^1(0,T;L^\infty)\) in terms of the data and of \(T\) by performing the shift of integrability method. This is given by the following two lemmas in 2-D and 3-D respectively.

**Lemma 4.1.** Assume \(d = 2\), then \(\forall T > 0, p \in [2, \infty]\) and \(\epsilon\) small enough, we have

\[
\|\nabla^2 \sqrt{t}u, \nabla^2 \sqrt{t}\omega, \sqrt{t}\omega\|_{L^p(0,T;L^{p^*}-\epsilon)} + \|\nabla \sqrt{t}P\|_{L^p(0,T;L^{p^*}-\epsilon)} \leq C_{0,T},
\]

where \(p^* \triangleq \frac{2p}{p-2}\) and \(C_{0,T}\) depends only on \(\rho^*, \|\sqrt{t_0}u_0\|_2, \|\sqrt{t_0}\omega_0\|_2, K_0, p, \epsilon\).

Furthermore, \(\forall s \in [1, 2)\), there exists \(\theta > 0\) such that

\[
\int_0^T \left(\|\nabla u\|^s_\infty + \|\nabla \omega\|^s_\infty\right) dt \leq C_{0,T} T^\theta.
\]

**Proof.** From (1.1) and (1.13), we have

\[
\begin{aligned}
- \nu \Delta \sqrt{t}u + \nabla \sqrt{t}P &= 2\chi \text{curl} (\sqrt{t}\omega) - \rho \sqrt{t}(u_t + u \cdot \nabla u) \\
\text{div} \sqrt{t}u &= 0 \\
- \gamma \Delta \sqrt{t}\omega - \kappa \text{div} (\sqrt{t}\omega) + 4\chi \sqrt{t}\omega &= 2\chi \text{curl} (\sqrt{t}u) - \rho \sqrt{t} (\omega_t + u \cdot \nabla \omega)
\end{aligned}
\]

in \((0, T) \times \mathbb{T}^2\).

Using (2.1) and (2.18) yields that

\((\rho \sqrt{t}u_t, \rho \sqrt{t}\omega_t) \in L^\infty(0,T;L^2)\).

According to (2.29), we have

\((\rho \sqrt{t}u_t, \rho \sqrt{t}\omega_t) \in L^2(0,T;L^q)\) for \(q < \infty\).

Therefore, we get by interpolation inequality,

\[
\|\rho \sqrt{t}u_t\|_{L^p(0,T;L^r)} \leq \|\rho \sqrt{t}u_t\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{p}} \|\rho \sqrt{t}u_t\|_{L^2(0,T;L^q)}^{\frac{2}{p}},
\]

\[
\|\rho \sqrt{t}\omega_t\|_{L^p(0,T;L^r)} \leq \|\rho \sqrt{t}\omega_t\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{p}} \|\rho \sqrt{t}\omega_t\|_{L^2(0,T;L^q)}^{\frac{2}{p}}
\]

with \(\frac{1}{r} = \frac{p-2}{2p} + \frac{2}{pq}, 2 \leq r < p^*\).

Thus,

\[
\|\rho \sqrt{t}u_t, \rho \sqrt{t}\omega_t\|_{L^p(0,T;L^r)} \leq C_{0,T} \quad \text{for} \quad p \in [2, \infty], \quad r \in [2, p^*). \quad (4.4)
\]
Similarly, it is known from (2.3) that $\langle \nabla u, \nabla \omega \rangle$ is bounded at $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$. By interpolation inequality, we get for $\frac{1}{r} = \frac{m-2}{2p} + \frac{2}{pq}$, $2 \leq r < p^*$

$$
\|\nabla u\|_{L^p(0,T;L^2)} \leq \|\nabla u\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{r}} \|\nabla u\|_{L^2(0,T;L^p)}^{\frac{2}{r}} \\
\leq \|\nabla u\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{r}} \|\nabla u\|_{L^2(0,T;H^1)}^{\frac{2}{r}},
$$

and then

$$
\|\nabla \omega\|_{L^p(0,T;L^2)} \leq \|\nabla \omega\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{r}} \|\nabla \omega\|_{L^2(0,T;L^p)}^{\frac{2}{r}} \\
\leq \|\nabla \omega\|_{L^\infty(0,T;L^2)}^{1-\frac{2}{r}} \|\nabla \omega\|_{L^2(0,T;H^1)}^{\frac{2}{r}},
$$

which implies that

$$
\|\nabla u, \nabla \omega\|_{L^p(0,T;L^2)} \leq C_{0,T} \quad \text{for} \quad p \geq 2, \quad r < p^*,
$$

and then

$$
\|\text{curl}(\sqrt{t}u), \text{curl}(\sqrt{t}\omega)\|_{L^p(0,T;L^2)} \leq C_{0,T} \quad \text{for} \quad p \geq 2, \quad r < p^*.
$$

As obvious, since $u, \omega$ (and thus $\sqrt{t}u, \sqrt{t}\omega$) is bounded in all spaces $L^q(0, T; L^r)$ (except $q = r = \infty$), we conclude that

$$
\|\sqrt{t}u \cdot \nabla u, \sqrt{t}u \cdot \nabla \omega\|_{L^p(0,T;L^2)} \leq C_{0,T} \quad \text{for} \quad p \in [2, \infty], \quad r \in [2, p^*).
$$

Applying the maximal regularity estimate for the Stokes equations and the standard estimate for elliptic equations for (4.3) yields that

$$
\|\nabla^2 \sqrt{t}u, \nabla^2 \sqrt{t}\omega, \nabla \sqrt{t}P\|_{L^p(0,T;L^2)} \leq C_{0,T} \quad \text{for} \quad p \in [2, \infty], \quad r \in [2, p^*).
$$

Fix $p \in [2, \infty)$ so that $ps < 2(p-s)$ and $1 \leq s < 2$, which means that $(\int_0^T t^{-\frac{ps}{2p-2s}} dt)^{\frac{1}{s}} \leq C_{0,T}$.

Taking $r \in (2, p^*)$ such that the embedding $W^s_1 \hookrightarrow L^\infty$, we obtain from (4.8)

$$
\left( \int_0^T \|\nabla u\|^s_I dt \right)^{\frac{1}{s}} + \left( \int_0^T \|\nabla \omega\|^s_I dt \right)^{\frac{1}{s}} \\
\leq C \left( \int_0^T (t^{-1/2}\|\sqrt{t}u\|_{W^s_1})^s dt \right)^{\frac{1}{s}} + C \left( \int_0^T (t^{-1/2}\|\sqrt{t}\omega\|_{W^s_1})^s dt \right)^{\frac{1}{s}} \\
\leq C \left( \int_0^T t^{-\frac{ps}{2p-2s}} dt \right)^{\frac{1}{s}} \left( \|\nabla \sqrt{t}u\|_{L^p(0,T;W^s_1)} + \|\nabla \sqrt{t}\omega\|_{L^p(0,T;W^s_1)} \right) \\
\leq C_{0,T} T^{\frac{2p-2s-ps}{2p}},
$$

which yields (4.2).

**Lemma 4.2.** Assume $d = 3$, then for all $T > 0$, $p \in [2, \infty]$, we have

$$
\|\nabla^2 \sqrt{t}u, \nabla^2 \sqrt{t}\omega, \nabla \sqrt{t}P\|_{L^p(0,T;L^2)} \leq C_{0,T} \quad \text{for} \quad 2 \leq r \leq \frac{6p}{3p-4},
$$

where $C_{0,T}$ depends only on $\rho^*, \|\sqrt{\rho}u_0\|_2, \|\sqrt{\rho}\omega_0\|_2, K_0, p$.

Furthermore, for $s \in [1, \frac{3}{2})$, then for some $\theta > 0$, we have

$$
\int_0^T \left( \|\nabla u\|^s_I + \|\nabla \omega\|^s_I \right) dt \leq C_{0,T} T^\theta.
$$
Proof. From (2.1) and (3.9), we get
\[ \rho \sqrt{t} u_t, \rho \sqrt{t} \omega_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \]
and from \( \dot{H}^1(T^3) \hookrightarrow L^6(T^3) \), we have
\[ \rho \sqrt{t} u_t, \rho \sqrt{t} \omega_t \in L^\infty(0, T; L^2) \cap L^2(0, T; L^q) \quad \text{with} \quad q \leq 6. \]
It then follows from interpolation inequality that
\[ \| \rho \sqrt{t} u_t \|_{L^p(0, T; L^r)} \leq \| \rho \sqrt{t} u_t \|_{L^\infty(0, T; L^2)}^{1-\frac{2}{p}} \| \rho \sqrt{t} u_t \|_{L^2(0, T; L^q)}^{\frac{2}{p}}, \]
and
\[ \| \rho \sqrt{t} \omega_t \|_{L^p(0, T; L^r)} \leq \| \rho \sqrt{t} \omega_t \|_{L^\infty(0, T; L^2)}^{1-\frac{2}{p}} \| \rho \sqrt{t} \omega_t \|_{L^2(0, T; L^q)}^{\frac{2}{p}}, \]
with \( \frac{1}{r} = \frac{p-2}{2p} + \frac{2}{pq} \). Here, when \( q \) takes 6, then \( r \) may take the maximum value of \( \frac{6p}{3p-4} \).
Thus, we readily get
\[ \| \rho \sqrt{t} u_t, \rho \sqrt{t} \omega_t \|_{L^p(0, T; L^r)} \leq C_{0, T} \quad \text{for} \quad p \in [2, \infty], \quad r \in [2, \frac{6p}{3p-4}], \quad (4.11) \]
According to (3.2), we have \( \nabla u, \nabla \omega \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \), and from \( \dot{H}^1(T^3) \hookrightarrow L^6(T^3) \), we get \( \nabla u, \nabla \omega \in L^\infty(0, T; L^2) \cap L^2(0, T; L^q) \) with \( q \leq 6 \). By interpolation inequality, we obtain
\[ \| \nabla u \|_{L^p(0, T; L^r)} \leq \| \nabla u \|_{L^\infty(0, T; L^2)}^{1-\frac{2}{p}} \| \nabla u \|_{L^2(0, T; L^q)}^{\frac{2}{p}}, \]
and
\[ \| \nabla \omega \|_{L^p(0, T; L^r)} \leq \| \nabla \omega \|_{L^\infty(0, T; L^2)}^{1-\frac{2}{p}} \| \nabla \omega \|_{L^2(0, T; L^q)}^{\frac{2}{p}}, \]
with \( \frac{1}{r} = \frac{p-2}{2p} + \frac{2}{pq} \).
Then
\[ \| \nabla u, \nabla \omega \|_{L^p(0, T; L^r)} \leq C_{0, T} \quad \text{for} \quad p \in [2, \infty], \quad r \in [2, \frac{6p}{3p-4}], \]
which means that
\[ \nabla u, \nabla \omega \in L^4(0, T; L^3), \quad (4.12) \]
and
\[ \| \text{curl}(\sqrt{t} u), \text{curl}(\sqrt{t} \omega) \|_{L^p(0, T; L^r)} \leq C_{0, T}, \quad \text{for all} \quad p \in [2, \infty], \quad r \in [2, \frac{6p}{3p-4}], \]
On the other hand, using Gagliardo-Nirenberg interpolation inequality \( \| v \|_{L^2}^4 \leq C \| \nabla v \|_{L^2}^2 \| \nabla^2 v \|_{L^2}^2 \)
leads to
\[ \| u \|_{L^4(0, T; L^\infty)} \leq \| \nabla u \|_{L^\infty(0, T; L^2)}^{\frac{1}{4}} \| \nabla^2 u \|_{L^2(0, T; L^2)}^{\frac{3}{4}}, \]
and
\[ \| \omega \|_{L^4(0, T; L^\infty)} \leq \| \nabla \omega \|_{L^\infty(0, T; L^2)}^{\frac{1}{4}} \| \nabla^2 \omega \|_{L^2(0, T; L^2)}^{\frac{3}{4}}. \]
Thanks to (2.1) and (3.2), we conclude that
\[
\sqrt{t\rho u}, \sqrt{t\rho \omega} \in L^4(0, T; L^\infty). \tag{4.13}
\]

Using Hölder’s inequality, and combining with (4.12) and (4.13), we get
\[
\sqrt{t\rho u} \cdot \nabla u, \sqrt{t\rho u} \cdot \nabla \omega \in L^2(0, T; L^3).
\]

Similarly,
\[
\sqrt{t\rho u}, \sqrt{t\rho \omega} \in L^\infty(0, T; L^6) \quad \text{and} \quad \nabla u, \nabla \omega \in L^\infty(0, T; L^2),
\]
which implies that
\[
\sqrt{t\rho u} \cdot \nabla u, \sqrt{t\rho u} \cdot \nabla \omega \in L^\infty(0, T; L^{3/2}).
\]

It follows from interpolating inequality and Hölder’s inequality that
\[
\|\sqrt{t\rho u} \cdot \nabla u\|_{L^p(0, T; L^r)} \leq \|\sqrt{t\rho u} \cdot \nabla u\|_{L^2(0, T; L^3)} \|\sqrt{t\rho u} \cdot \nabla u\|_{L^\infty(0, T; L^{3/2})}^{1 - \frac{2}{p}}.
\]
\[
\|\sqrt{t\rho u} \cdot \nabla \omega\|_{L^p(0, T; L^r)} \leq \|\sqrt{t\rho u} \cdot \nabla \omega\|_{L^2(0, T; L^3)} \|\sqrt{t\rho u} \cdot \nabla \omega\|_{L^\infty(0, T; L^{3/2})}^{1 - \frac{2}{p}},
\]
with \(\frac{2}{p} + \frac{3}{r} = 2, p \geq 2\). Using the maximal regularity estimate for the Stokes equations and the standard estimate for elliptic equations for (4.3) yields that
\[
\|\nabla^2 \sqrt{t}u, \nabla^2 \sqrt{t} \omega, \sqrt{t} \omega\|_{L^p(0, T; L^r)} + \|\nabla \sqrt{t}P\|_{L^p(0, T; L^r)} \leq C_{0,T} \quad \text{for} \quad p \geq 2 \quad \text{and} \quad \frac{2}{p} + \frac{3}{r} = 2.
\tag{4.14}
\]

Furthermore, using the bound for \((\rho u, \rho \omega)\) in \(L^\infty(0, T; L^6)\) and the embedding \(W^1_r(T^3) \hookrightarrow L^q(T^3)\) with \(\frac{3}{q} = \frac{3}{r} - 1\) if \(1 \leq r < 3\) (which implies that \((\nabla \sqrt{t}u, \nabla \sqrt{t} \omega)\) is bounded in \(L^p(0, T; L^q)\) with \(\frac{2}{q} + \frac{3}{r} = 2\)), we get (4.14) for the full range of indices. Fix \(p \in (2, 4)\) such that \(ps < 2p - 2s\) and take \(r = \frac{6p}{2p - 4}\). Thanks to \(W^1_r \hookrightarrow L^\infty\) (because \(r > 3\) for \(2 < p < 4\)), we have
\[
\left(\int_0^T \|\nabla u\|_{L^s}^s dt\right)^\frac{1}{s} + \left(\int_0^T \|\nabla \omega\|_{L^s}^s dt\right)^\frac{1}{s} \leq C\left(\int_0^T \|\sqrt{t} \nabla u\|_{W^1_r}^p \frac{dt}{\sqrt{t}}\right)^\frac{1}{p} + C\left(\int_0^T \|\sqrt{t} \nabla \omega\|_{W^1_r}^p \frac{dt}{\sqrt{t}}\right)^\frac{1}{p} \leq C\left(\int_0^T t^{\frac{p - 2s}{2p - 4s}} dt\right)^\frac{1}{p - 1} \left(\|\nabla \sqrt{t}u\|_{L^p(0, T; W^1_r)} + \|\nabla \sqrt{t} \omega\|_{L^p(0, T; L^r)}\right) \leq C_{0,T} T^{\frac{2p - 2s - ps}{2p - 4s}},
\]
which concludes that (4.10) holds.

### 4.2 Lagrangian formulation

As in [12, 14, 26], we shall prove the uniqueness part of both Theorems 1.1 and 1.2 using the Lagrangian formulation of system (1.1). First, we introduce the flow \(X : \mathbb{R}^+ \times \mathbb{T}^d \rightarrow \mathbb{T}^d\) of \(u\) by
\[
\partial_t X(t, y) = u(t, X(t, y)), \quad X(0, y) = y.
\]

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Note that
\[ X(t, y) = y + \int_0^t u(\tau, X(\tau, y)) d\tau, \]
and
\[ \nabla_y X(t, y) = 1d + \int_0^t \nabla_y u(\tau, X(\tau, y)) d\tau. \]
In Lagrangian coordinates \((t, y)\), a solution \((\rho, u, \omega, P)\) to system (1.1) recasts in \((\bar{\rho}, \bar{u}, \bar{\omega}, \bar{P})\) with
\[
\begin{align*}
\bar{\rho}(t, y) &= \rho(t, X(t, y)), \quad \bar{u}(t, y) = u(t, X(t, y)), \\
\bar{\omega}(t, y) &= \omega(t, X(t, y)), \quad \bar{P}(t, y) = P(t, X(t, y)),
\end{align*}
\]
and the triplet \((\bar{\rho}, \bar{u}, \bar{\omega}, \bar{P})\) thus satisfies
\[
\begin{align*}
\begin{cases}
\bar{\rho} u_t - \nu \Delta_u \bar{u} + \nabla_u \bar{P} = 2\chi \text{curl}_u \bar{\omega} & \text{in } (0, T) \times T^d, \\
\text{div}_u \bar{u} = 0 & \text{in } (0, T) \times T^d, \\
\bar{\rho} \bar{\omega}_t - \gamma \Delta_u \bar{\omega} - \kappa \nabla_u \text{div}_u \bar{\omega} + 4\chi \bar{\omega} = 2\chi \text{curl}_u \bar{\omega} & \text{in } (0, T) \times T^d, \\
\bar{\rho} t &= 0 & \text{in } (0, T) \times T^d, \\
\bar{\rho}(0, y) = \rho_0(y), \bar{u}(y, 0) = u_0(y), \bar{\omega}(y, 0) = \omega_0(y) & \text{in } T^d,
\end{cases}
\end{align*}
\]
where operators \(\nabla_u, \Delta_u, \text{div}_u, \text{curl}_u\) and \(\text{div}_u\) correspond to the original operators \(\nabla, \Delta, \text{div}, \text{curl}\) and \(\text{div}\), respectively, after performing the change to the Lagrangian coordinates.

As pointed out in [12,14,26], in our regularity framework, that latter system (4.16) is equivalent to system (1.1). Thanks to (4.2) and (4.10), we can take the time \(T\) to be small enough so that
\[
\int_0^T \| \nabla u \|_{\infty} d\tau \leq \frac{1}{2}. \tag{4.17}
\]
Set
\[
A = (\nabla X)^{-1} \text{(inverse of deformation tensor)},
\]
\[
J = \det \nabla X \text{(Jacobian determinant)},
\]
\[
a = JA \text{(transpose of cofactor matrix)}.
\]
Thus, in the \((t, y)\)-coordinates, operators \(\nabla, \text{div}, \text{curl} \text{ and } \Delta\) translate into
\[
\nabla_u := \nabla_y, \quad \text{div}_u := \text{div}_y(A\cdot), \quad \text{curl}_u := \nabla_u \wedge \cdot, \quad \text{and} \quad \Delta_u := \text{div}_u \nabla_u. \tag{4.18}
\]
Moreover, given some matrix \(N\), we define the divergence operator (acting on vector fields \(v\)) by the formulation
\[
\text{div}_u N v = \text{div}_y (N \cdot v) \overset{\text{def}}{=} T N : \nabla v, \tag{4.19}
\]
where \(N : B = \sum_{i,j} N_{ij} B_{ji}\) for \(N = (N_{ij})_{1 \leq i,j \leq d}\) and \(B = (B_{ij})_{1 \leq i,j \leq d}\) two \(d \times d\) matrices.

Of course, if the condition (4.17) is fulfilled then we have
\[
A = \left( Id + (\nabla_y X - Id) \right)^{-1} = \sum_{k=0}^{+\infty} (-1)^k \left( \int_0^t \nabla_y \bar{u}(\tau, \cdot) d\tau \right)^k. \tag{4.20}
\]

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which yields that
\[
\delta A = \left( \int_0^t \nabla \delta u \, dt \right) \cdot \left( \sum_{k \geq 1} \sum_{0 \leq j < k} C_j^1 C_2^{k-1-j} \right) \quad \text{with} \quad C_i(t) = \int_0^t \nabla \bar{u}^i \, dt,
\]
(4.21)
where \( \delta A \overset{\text{def}}{=} A_2 - A_1 \) and \( \delta u \overset{\text{def}}{=} \bar{u}^2 - \bar{u}^1 \).

We also make use of the following permutation symbol
\[
\varepsilon_{ijk} = \begin{cases} 
1, & \text{even permutation of 1,2,3}, \\
-1, & \text{odd permutation of 1,2,3}, \\
0, & \text{otherwise}, 
\end{cases}
\]
and the basic identity regarding the \( i \)th component of the curl of a vector field \( u \)
\[
\left( \text{curl} u \right)_i = \varepsilon_{ijk} \bar{u}^k_j.
\]
The chain rule shows that
\[
\left( \text{curl} u(X) \right)_i = \left( \text{curl} \bar{u} \right)_i := \varepsilon_{ijk} A^j_s \bar{u}^k_s.
\]
(4.22)
Here, we also present the following Piola identity, that is, the columns of every cofactor matrix
are divergence-free and satisfy
\[
a^j_{ik} = 0.
\]
(4.23)
Here, it is pointed out that we use the notation \( F_{ik} \) to denote \( \frac{\partial F}{\partial x^k} \), the \( k \)th-partial derivative of \( F \) for \( k = 1, 2, 3 \), and omit Einstein’s summation convention in (4.22) and (4.23).

### 4.3 The proof of the uniqueness

Let \((\rho^1, u^1, \omega^1, P^1)\) and \((\rho^2, u^2, \omega^2, P^2)\) be two solutions of system (1.1) fulfilling the properties of Theorems 1.1 and 1.2, with the same initial data, and denote by \((\bar{\rho}^1, \bar{u}^1, \bar{\omega}^1, \bar{P}^1)\) and \((\bar{\rho}^2, \bar{u}^2, \bar{\omega}^2, \bar{P}^2)\) in Lagrangian coordinates. Of course, we have \( \bar{\rho}^1 = \rho^2 = \rho_0 \), which explains the choice of our approach here. In what follows, we shall use repeatedly the fact that for \( i = 1, 2 \), we have
\[
\begin{align*}
&\frac{1}{t^2} \nabla \bar{u}^i \in L^2(0, T; L^\infty), \quad \frac{1}{t^{1/3}} \nabla \omega^i \in L^2(0, T; L^\infty), \quad \frac{1}{t^2} \nabla \bar{P}^i \in L^2(0, T; L^3), \quad \frac{1}{t^2} \bar{u}^i \in L^{4/3}(0, T; L^6), \\
&\nabla \bar{u}^i \in L^1(0, T; L^\infty) \cap L^2(0, T; L^6) \cap L^4(0, T; L^3), \quad \bar{u}^i \in L^4(0, T; L^\infty).
\end{align*}
\]
(4.24)
It should be noted that the former four items in (4.24) is less than or equal to \( c(T) \), where \( c(T) \) designates a nonnegative continuous increasing function of \( T \), with \( c(0) = 0 \) and \( c(T) \to 0 \) when \( T \to 0 \). For example, in 3-D, using Gagliardo-Nirenberg interpolation inequality, (2.3) and Lemma
4.2, we have
\[ ||t^{2}V \delta y||^{2}_{L^{2}(0,T;L^{2})} = \int_{0}^{T} t^{2}A \nabla u^{i}||^{2}_{\infty}dt \]
\[ \leq C \int_{0}^{T} t||\nabla u^{i}||^{2}_{L^{2}}||\nabla^{2} u^{i}||^{2}_{L^{2}}dt \]
\[ \leq C \sup_{t\in[0,T]}||\nabla u^{i}||^{2}_{L^{2}} \int_{0}^{T} t||\nabla^{2} u^{i}||^{2}_{L^{2}}dt \]
\[ \leq CT^{2}||\nabla \nabla^{2} u^{i}||^{2}_{L^{2}(0,T;L^{6})} \]
\[ \leq c(T). \]

Denoting \( \delta \omega \equiv \bar{\omega}^{2} - \bar{\omega}^{1} \), and \( \delta P \equiv \bar{P}^{2} - \bar{P}^{1} \), we get
\[
\begin{aligned}
\rho \partial_{t}\delta u - \nu \Delta_{u^{i}} \delta u + \nabla_{u^{i}} \delta P - 2\chi \text{curl} \nabla_{u^{i}} \delta \omega &= \delta f_{1}, \\
\text{div}_{u^{i}} \delta u &= (\text{div}_{u^{i}} - \text{div}_{u^{2}}) \bar{u}^{2}, \\
\rho \partial_{t} \delta \omega - \gamma \Delta_{u^{2}} \delta \omega - \kappa \nabla_{u^{i}} \text{div}_{u_{i}} \delta \omega + 4\chi \delta \omega - 2\chi \text{curl} \nabla_{u^{i}} \delta u &= \delta f_{2}, \\
(\delta u, \delta \omega)|_{t=0} &= (0,0),
\end{aligned}
\]
with \( \delta f_{1} \equiv \nu(\Delta_{u^{2}} - \Delta_{u^{1}}) \bar{u}^{2} - (\nabla_{u^{2}} - \nabla_{u^{1}}) \bar{P}^{2} + 2\chi(\text{curl}_{u^{2}} - \text{curl}_{u^{1}}) \bar{\omega}^{2} \),
\( \delta f_{2} \equiv \gamma(\Delta_{u^{2}} - \Delta_{u^{1}}) \bar{\omega}^{2} + \kappa(\nabla_{u^{2}} \text{div}_{u^{2}} - \nabla_{u^{1}} \text{div}_{u^{1}}) \bar{\omega}^{2} + 2\chi(\text{curl}_{u^{2}} - \text{curl}_{u^{1}}) \bar{u}^{2} \).

We claim for sufficiently small \( T > 0 \),
\[ \int_{0}^{T} \int_{\Omega} \left( |\delta u(t,y)|^{2} + |\delta \omega(t,y)|^{2} + |
\text{curl} \delta u(t,y)|^{2} + |
\text{curl} \delta \omega(t,y)|^{2} \right) dy dt = 0. \]

To prove our claim, we first decompose \( \delta u \) into
\[ \delta u = \varphi + \phi, \]
with \( \varphi \) is the solution given by Lemma 5.1 to the following problem:
\[
\text{div}_{u^{i}} \varphi = (\text{div}_{u^{i}} - \text{div}_{u^{2}}) \bar{u}^{2} = \text{div}(\delta A \bar{u}^{2}).
\]

Then, (5.3) and (4.20) ensure that there exist two universal positive constants \( c \) and \( C \) such that if
\[ ||\nabla \bar{u}^{1}||_{L^{1}(0,T;L^{\infty})} + ||\nabla \bar{u}^{1}||_{L^{2}(0,T;L^{2})} \leq c, \]
and then the following inequalities hold true:
\[ ||\varphi||_{L^{4}(0,T;L^{2})} \leq C ||\delta A \bar{u}^{2}||_{L^{4}(0,T;L^{2})}, \quad ||\nabla \varphi||_{L^{2}(0,T;L^{2})} \leq C ||T \delta A : \nabla \bar{u}^{2}||_{L^{2}(0,T;L^{2})} \]
and \[ ||\varphi||_{L^{4/3}(0,T;L^{3/2})} \leq C ||\delta A \bar{u}^{2}||_{L^{4}(0,T;L^{2})} + C ||(\delta A \bar{u}^{2})||_{L^{4/3}(0,T;L^{3/2})}. \]

Now, let us bound the r.h.s. of (4.29). Regarding \( T \delta A : \nabla \bar{u}^{2} \), it follows from H"{o}lder’s inequality, (4.28) and (4.21) that
\[
\sup_{t \in [0,T]} ||t^{-1/2} \delta A||_{2} \leq C \sup_{t \in [0,T]} ||t^{-1/2} \int_{0}^{t} \nabla \delta u dt||_{2} \leq C ||\nabla \delta u||_{L^{2}(0,T;L^{2})}. \]
According to (4.24) and (4.30), we obtain

\[
\| T \delta A : \nabla \bar{u}^2 \|_{L^2(0,T;L^2)} \leq \sup_{t \in [0,T]} \| t^{-1/2} \delta A \|_2 \| t^{1/2} \nabla \bar{u}^2 \|_{L^2(0,T;L^\infty)} \\
\leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)}.
\]

Similarly,

\[
\| \delta A \bar{u}^2 \|_{L^4(0,T;L^2)} \leq \| t^{-1/2} \delta A \|_{L^\infty(0,T;L^2)} \| t^{1/2} \bar{u}^2 \|_{L^4(0,T;L^\infty)}.
\]

Using (4.24), (4.29) and (4.30) yields that

\[
\| \nabla \varphi \|_{L^2(0,T;L^2)} \leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)},
\]

and

\[
\| \varphi \|_{L^4(0,T;L^2)} \leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)}.
\]

In order to bound \( \varphi_t \), it suffices to derive an appropriate estimate in \( L^{4/3}(0,T;L^{3/2}) \) for

\[
(\delta A \bar{u}^2)_t = \delta A \bar{u}^2_t + (\delta A)_t \bar{u}^2.
\]

Thanks to (4.24) and (4.30), we have

\[
\| (\delta A)_t \bar{u}^2 \|_{L^{4/3}(0,T;L^{3/2})} \leq \| t^{-1/2} \delta A \|_{L^\infty(0,T;L^2)} \| t^{1/2} \nabla \bar{u}^2 \|_{L^{4/3}(0,T;L^6)}
\]

\[
\leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)}.
\]

For the term \( (\delta A)_t \bar{u}^2 \), it follows from Hölder’s inequality that

\[
\| (\delta A)_t \bar{u}^2 \|_{L^{4/3}(0,T;L^{3/2})} \leq \| (\delta A)_t \|_{L^2(0,T;L^6)} \| \bar{u}^2 \|_{L^4(0,T;L^6)}.
\]

Furthermore, differentiating (4.21) with respect to \( t \) and using (4.28) for \( \bar{u}^1 \) and \( \bar{u}^2 \) yield that

\[
\| (\delta A)_t \|_{2} \leq C \left( \| \nabla \delta u \|_{2} + \| t^{-1/2} \int_0^t \nabla \delta u d\tau \|_{2} (\| t^{1/2} \nabla \bar{u}^1 \|_{\infty} + \| t^{1/2} \nabla \bar{u}^2 \|_{\infty}) \right).
\]

This gives

\[
\| (\delta A)_t \|_{L^2(0,T;L^6)} \leq C \| \nabla \delta u \|_{L^2(0,T;L^6)},
\]

which together with (4.24) implies that

\[
\| (\delta A)_t \bar{u}^2 \|_{L^{4/3}(0,T;L^{3/2})} \leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)}.
\]

Thus,

\[
\| \varphi_t \|_{L^{4/3}(0,T;L^{3/2})} \leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)}.
\]

Combining with (4.29), (4.31), (4.32) and (4.33), we have

\[
\| \varphi \|_{L^4(0,T;L^2)} + \| \nabla \varphi \|_{L^2(0,T;L^6)} + \| \varphi_t \|_{L^{4/3}(0,T;L^{3/2})} \leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^6)},
\]

(4.34)
Next, let us restate the equations for \((\delta u, \delta \omega, \delta P)\) as the following system for \((\phi, \delta \omega, \delta P)\):

\[
\begin{aligned}
\rho_0 \partial_t \phi - \nu \Delta u_1 \phi + \nabla u_1 \delta P &= \delta f_1 - \rho_0 \partial_t \varphi + \nu \Delta u_1 \varphi + 2 \chi \text{curl}_u \delta \omega, \\
\text{div}_{u_1} \phi &= 0, \\
\rho_0 \partial_t \delta \omega - \gamma \Delta u_1 \delta \omega - \kappa \nabla u_1 \text{div}_u \delta \omega + 4 \chi \delta \omega &= \delta f_2 + 2 \chi \text{curl}_u \phi + 2 \chi \text{curl}_u \phi.
\end{aligned}
\]

Due to \(\text{div}_{u_1} \phi = 0\), we have

\[
\int_{\mathbb{T}^d} (\nabla u_1 \delta P) \cdot \phi dx = - \int_{\mathbb{T}^d} \text{div}_{u_1} \phi \cdot \delta P dx = 0.
\]

Note that

\[
2 \chi \int_{\mathbb{T}^d} \text{curl}_{u_1} \delta \omega \cdot \phi dx + 2 \chi \int_{\mathbb{T}^d} \text{curl}_{u_1} \phi \cdot \delta \omega dx = 4 \chi \int_{\mathbb{T}^d} \text{curl}_{u_1} \phi \cdot \delta \omega dx \leq 4 \chi \| \nabla_{u_1} \phi \|_2 \| \delta \omega \|_2 \\
\leq \chi \| \nabla_{u_1} \phi \|_2^2 + 4 \chi \| \delta \omega \|_2^2.
\]

Therefore, taking the \(L^2\)-scalar product of the first equation to system (4.35) with \(\phi\) and the third equation with \(\delta \omega\) respectively yields that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \rho_0 (|\phi|^2 + |\delta \omega|^2) dx + \int_{\mathbb{T}^d} \left( \mu |\nabla u_1 \phi|^2 + \gamma |\nabla u_1 \delta \omega|^2 + \kappa |\text{div}_u \delta \omega|^2 \right) dx
\]

\[
= - \int_{\mathbb{T}^d} \rho_0 \partial_t \varphi \cdot \phi dx + \nu \int_{\mathbb{T}^d} \Delta u_1 \varphi \cdot \phi dx + \nu \int_{\mathbb{T}^d} (\Delta u_2 - \Delta u_1) \bar{u}^2 \phi dx
\]

\[
- \int_{\mathbb{T}^d} (\nabla u_2 - \nabla u_1) \bar{P} \cdot \phi dx + 2 \chi \int_{\mathbb{T}^d} (\text{curl}_{u_2} - \text{curl}_{u_1}) \bar{\omega}^2 \cdot \phi dx
\]

\[
+ 2 \chi \int_{\mathbb{T}^d} \text{curl}_{u_1} \varphi \cdot \delta \omega dx + \gamma \int_{\mathbb{T}^d} (\Delta u_2 - \Delta u_1) \bar{\omega}^2 \cdot \delta \omega dx
\]

\[
+ \kappa \int_{\mathbb{T}^d} (\nabla u_2 \text{div}_u - \nabla u_1 \text{div}_u) \bar{\omega}^2 \cdot \delta \omega dx + 2 \chi \int_{\mathbb{T}^d} (\text{curl}_{u_2} - \text{curl}_{u_1}) \bar{u}^2 \cdot \delta \omega dx
\]

\[
\triangleq \sum_{k=1}^{9} I_{I_1}.
\]

Here and in what follows, we estimate term by term above. For \(I_{I_1}\), it follows from Hölder’s inequality that

\[
\int_0^T I_{I_1}(t) dt \leq \| \rho_0 \|_{L^{3/2}(0,T;L^3)} \| \varphi_t \|_{L^{6/3}(0,T;L^{6/2})} \| \rho_0 \|^{1/4}_{L^1(0,T;L^1)}.
\]

Using Hölder’s inequality and the Sobolev embedding \(H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)\) yields that

\[
\| \rho_0 \|^{1/4}_{L^1(0,T;L^1)} \leq \| \sqrt{\rho_0 \phi} \|_{L^6(0,T;L^2)} \| \phi \|_{L^2(0,T;L^6)} \leq C \| \sqrt{\rho_0 \phi} \|_{L^6(0,T;L^2)} \| \phi \|_{L^2(0,T;H^1)}^{1/2},
\]

Employing Poincaré’s inequality in the unit torus \(\mathbb{T}^d\) in [14]: \(|\phi|_{H^1} \leq C(|\sqrt{\rho_0 \phi}|_2 + |\nabla \phi|_2)\) with constant \(C\) depending only on \(\rho_0\), and taking advantage of (4.21) and (4.34), we conclude that

\[
\int_0^T I_{I_1}(t) dt \leq c(T) \left( \| \sqrt{\rho_0 \phi} \|_{L^6(0,T;L^2)} + \| \nabla \phi \|_{L^2(0,T;\mathbb{T}^d)} \right)^{1/2} \| \sqrt{\rho_0 \phi} \|_{L^6(0,T;L^2)} \| \nabla \delta u \|_{L^2(0,T;\mathbb{T}^d)}.
\]

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For $II_2$, it follows from integrating by parts and using (4.34) that
\[
\int_0^T II_2(t)dt \leq \nu \int_0^T \left| \int_{\mathbb{T}^d} \nabla u^1 \varphi \nabla u^1 \phi dx \right| dt
\]
\[
\leq \nu \int_0^T \int_{\mathbb{T}^d} \left| \nabla u^1 \varphi \right| \left| \nabla u^1 \phi \right| dx dt
\]
\[
\leq \frac{\nu}{2} \int_0^T \left\| \nabla u^1 \phi \right\|_2^2 dt + \frac{\nu}{2} \int_0^T \left\| \nabla u^1 \varphi \right\|_2^2 dt
\]
\[
\leq \frac{\nu}{2} \int_0^T \left\| \nabla u^1 \phi \right\|_2^2 dt + c(T) \int_0^T \left\| \nabla u \right\|_2^2 dt.
\]

For $II_3$, using (4.18) and (4.21), we obtain
\[
II_3 \leq C \int_{\mathbb{T}^d} \left| \text{div}((\delta A^T A_2 + A_1^T \delta A) \nabla \bar{u}^2) \cdot \phi dx \right|
\]
\[
\leq C \int_{\mathbb{T}^d} \left| \delta A^T A_2 + A_1^T \delta A \right| \nabla \bar{u}^2 \nabla \phi dx
\]
\[
\leq C \left\| t^{-1/2} \delta A \right\|_2 \left\| t^{1/2} \nabla \bar{u}^2 \right\|_\infty \left\| \nabla \phi \right\|_2,
\]
which together with (4.24) and (4.30) implies that
\[
\int_0^T II_3(t)dt \leq \left\| t^{-1/2} \delta A \right\|_{L^\infty(0,T;L^2)} \left\| t^{1/2} \nabla \bar{u}^2 \right\|_{L^2(0,T;L^\infty)} \left\| \nabla \phi \right\|_{L^2(0,T \times \mathbb{T}^d)}
\]
\[
\leq c(T) \left\| \nabla u \right\|_{L^2(0,T;L^2)} \left\| \nabla \phi \right\|_{L^2(0,T \times \mathbb{T}^d)}.
\]

For $II_4$, using Hölder’s inequality, we obtain
\[
II_4(t) \leq \left| \int_{\mathbb{T}^d} \delta A \nabla \bar{P}^2 \cdot \phi dx \right| \leq C \left| t^{-1/2} \delta A \right|_2 \left| t^{1/2} \nabla \bar{P}^2 \right|_3 \| \phi \|_6.
\]
It then follows from (4.24), (4.30) and Sobolev embedding that
\[
\int_0^T II_4(t)dt \leq \left| t^{-1/2} \delta A \right|_{L^\infty(0,T;L^2)} \left| t^{1/2} \nabla \bar{P}^2 \right|_{L^2(0,T;L^3)} \| \phi \|_{L^2(0,T;H^1)}
\]
\[
\leq c(T) \left\| \nabla u \right\|_{L^2(0,T;L^2)} \left( \| \sqrt{\rho_0} \phi \|_{L^\infty(0,T;L^2)} + \| \nabla \phi \|_{L^2(0,T \times \mathbb{T}^d)} \right).
\]

For $II_5$, thanks to (4.18), (4.24) and Hölder’s inequality, we get
\[
\int_0^T II_5(t)dt \leq C \int_0^T \left| \int_{\mathbb{T}^d} \delta A \nabla \cdot \bar{\omega} \cdot \phi dx \right| dt
\]
\[
\leq C \left| t^{-1/2} \delta A \right|_{L^\infty(0,T;L^2)} \left| t^{1/2} \nabla \bar{\omega} \right|_{L^2(0,T;L^4)} \| \phi \|_{L^2(0,T;L^4)}
\]
\[
\leq c(T) \left\| \nabla u \right\|_{L^2(0,T;L^2)} \| \phi \|_{L^2(0,T;H^1)}
\]
\[
\leq c(T) \left\| \nabla u \right\|_{L^2(0,T;L^2)} \left( \| \sqrt{\rho_0} \phi \|_{L^\infty(0,T;L^2)} + \| \nabla \phi \|_{L^2(0,T \times \mathbb{T}^d)} \right).
\]

For $II_6$, we get from (4.34)
\[
\int_0^T II_6(t)dt \leq C \int_0^T \left| \int_{\mathbb{T}^d} \text{curl}_{\mu,1} \delta \omega \cdot \varphi dx \right| dt
\]
\[
\leq C \int_0^T \left\| \nabla \delta \omega \right\|_2 \left\| \varphi \right\|_2 dt
\]
\[
\leq C \left\| \varphi \right\|_{L^2(0,T;L^2)} \left\| \nabla \delta \omega \right\|_{L^2(0,T;L^2)}
\]
\[
\leq C \left\| \varphi \right\|_{L^1(0,T;L^2)} \left\| \nabla \delta \omega \right\|_{L^2(0,T;L^2)}
\]
\[
\leq c(T) \left\| \nabla u \right\|_{L^2(0,T;L^2)} \left\| \nabla \delta \omega \right\|_{L^2(0,T;L^2)}.
\]
For $I_{II_7}$, using (4.18) and $\|A_i\|_\infty < \infty (i = 1, 2)$, we have

$$I_{II_7}(t) \leq \gamma \left| \int_{T^d} \text{div} \left( (T A_2 A_2 - T A_1 A_1) \nabla \omega^2 \right) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} \text{div} \left( (T \delta A A_2 + T A_1 \delta A) \nabla \omega^2 \right) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} [T \delta A A_2 + T A_1 \delta A] \| \nabla \omega^2 \| \| \nabla \delta \omega \| dx \right|$$

$$\leq C \| t^{-\frac{1}{2}} \delta A \|_2 \| t^\frac{1}{2} \nabla \omega^2 \|_\infty \| \nabla \delta \omega \|_2,$$

which along with (4.24) and (4.30) yields that

$$\int_0^T I_{II_7}(t) dt \leq C \int_0^T \| t^{-\frac{1}{2}} \delta A \|_2 \| t^\frac{1}{2} \nabla \omega^2 \|_\infty \| \nabla \delta \omega \|_2 dt$$

$$\leq C \| t^{-\frac{1}{2}} \delta A \|_{L^\infty (0,T;L^2)} \| t^\frac{1}{2} \nabla \omega^2 \|_{L^2(0,T;L^\infty)} \| \nabla \delta \omega \|_{L^2(0,T;L^2)}$$

$$\leq c(T) \| \nabla \delta u \|_{L^2(0,T;L^2)} \| \nabla \delta \omega \|_{L^2(0,T;L^2)}.$$

For $I_{II_8}$, note that $\text{div} u = 0$, then $J = 1$, $a_i = A_i (i = 1, 2)$. From Piola identity (4.23), we get

$$a_{2j,k}^k = a_{1j,k}^k = 0. \quad (4.37)$$

Combining with (4.19), (4.37), Hölder’s inequality and $\|A_i\|_\infty < \infty (i = 1, 2)$, we have

$$I_{II_8}(t) \leq \kappa \left| \int_{T^d} \left( (\nabla u^2 \text{div} u^2 - \nabla u^1 \text{div} u^1) \omega^2 \right) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} (T A_2 \nabla \text{div}(A_2 \omega^2) - T A_1 \nabla \text{div}(A_1 \omega^2)) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} (T a_2 \nabla \text{div}(A_2 \omega^2) - T a_1 \nabla \text{div}(A_1 \omega^2)) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} (T a_{2k} \partial_k (T A_2 : \nabla \omega^2) - T a_{1k} \partial_k (T A_1 : \nabla \omega^2)) \cdot (\delta \omega)^j dx \right|$$

$$\leq \left| \int_{T^d} (a_{2j} \partial_k (T A_2 : \nabla \omega^2) - a_{1j} \partial_k (T A_1 : \nabla \omega^2)) \cdot (\delta \omega)^j dx \right|$$

$$\leq C \left| \int_{T^d} \partial_k \left( a_{2j} (T A_2 : \nabla \omega^2) - a_{1j} (T A_1 : \nabla \omega^2) \right) \cdot (\delta \omega)^j dx \right|$$

$$\leq C \left| \int_{T^d} \text{div} \left( a_2 \cdot (T A_2 : \nabla \omega^2) - a_1 \cdot (T A_1 : \nabla \omega^2) \right) \cdot \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} \left( (a_2 - a_1) \cdot (T A_2 : \nabla \omega^2) + a_1 \cdot (T \delta A : \nabla \omega^2) \right) : \nabla \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} \left( \delta A \cdot (T A_2 : \nabla \omega^2) + A_1 \cdot (T \delta A : \nabla \omega^2) \right) : \nabla \delta \omega dx \right|$$

$$\leq C \left| \int_{T^d} \left( \delta A \cdot (T A_2 : \nabla \omega^2) + A_1 \cdot (T \delta A : \nabla \omega^2) \right) : \nabla \delta \omega dx \right|$$

$$\leq C \left( \| t^{-\frac{1}{2}} \delta A \|_{L^2} \| T A_2 \|_{L^\infty} + \| t^\frac{1}{2} \nabla \omega^2 \|_{L^\infty} \right) \| \nabla \delta \omega \|_{L^2}$$

$$\leq C \| t^{-\frac{1}{2}} \delta A \|_{L^2} \| t^\frac{1}{2} \nabla \omega^2 \|_{L^\infty} \| \nabla \delta \omega \|_{L^2},$$
where $\tau A^j_{ik}$ denotes the $j^{th}$ row and $k^{th}$ column component of the matrix $\tau A_i(i = 1, 2)$. From (4.24) and (4.30), we obtain
\[
\int_0^T I_8(t)dt \leq C\|t^{-\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\|t^{\frac{1}{2}}\nabla \omega^2\|_{L^2(0,T;L^\infty)}\|\nabla \delta \omega\|_{L^2(0,T;L^2)}
\leq c(T)\|\nabla \delta u\|_{L^2(0,T;L^2)}\|\nabla \delta \omega\|_{L^2(0,T;L^2)}.
\]
Finally, for $II_9$, using (4.18), (4.22) and Hölder’s inequality, we have
\[
II_9(t) \leq 2\chi \int_{\mathbb{T}^d} (\text{curl} u^2 - \text{curl} u^1) \bar{u}^2 \cdot \delta \omega dx
\leq 2\chi \int \text{curl} (\text{curl} u^2 - \text{curl} u^1) \delta \omega \cdot \bar{u}^2 dx
\leq C \int_{\mathbb{T}^d} (\varepsilon_{ij}A^k_{ij}(\delta \omega)^k_a - \varepsilon_{ij}A^k_{ij}(\delta \omega)^k_a) \cdot (\bar{u}^2)^i dx
\leq C \int_{\mathbb{T}^d} \varepsilon_{ij}A^k_{ij}(\delta \omega)^k_a \cdot (\bar{u}^2)^i dx
\leq C\|t^{\frac{1}{2}}\delta A\|_{L^2}\|t^{\frac{1}{2}}\bar{u}^2\|_{L^\infty}\|\nabla \delta \omega\|_{L^2},
\]
from which, together with (4.24) and (4.30) yields that
\[
\int_0^T II_9(t)dt \leq C\int_0^T \|t^{\frac{1}{2}}\delta A\|_{L^2}\|t^{\frac{1}{2}}\bar{u}^2\|_{L^\infty}\|\nabla \delta \omega\|_{L^2} dt
\leq C\|t^{\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\|t^{\frac{1}{2}}\bar{u}^2\|_{L^2(0,T;L^\infty)}\|\nabla \delta \omega\|_{L^2(0,T;L^2)}
\leq CT\|t^{\frac{1}{2}}\delta A\|_{L^\infty(0,T;L^2)}\|\bar{u}^2\|_{L^4(0,T;L^\infty)}\|\nabla \delta \omega\|_{L^2(0,T;L^2)}
\leq c(T)\|\nabla \delta u\|_{L^2(0,T;L^2)}\|\nabla \delta \omega\|_{L^2(0,T\times\mathbb{T}^d)}.
\]
So altogether, this gives for all small enough $T > 0$,
\[
\sup_{t \in [0,T]} \left(\sqrt{\rho_0}\phi, \sqrt{\rho_0}\delta \omega\right)^2 + \|\nabla \delta u, \nabla \delta \omega\|^2_{L^2(0,T;L^2)} \leq c(T)\|\nabla \delta u, \nabla \delta \omega\|^2_{L^2(0,T;L^2)}.
\] (4.38)
Combining with (4.34), we conclude that
\[
\|\nabla \delta u, \nabla \delta \omega\|^2_{L^2(0,T;L^2)} \leq c(T)\|\nabla \delta u, \nabla \delta \omega\|^2_{L^2(0,T;L^2)}.
\]
Hence $\nabla \delta u = \nabla \delta \omega \equiv 0$ on $[0, T] \times \mathbb{T}^d$ if $T$ is small enough. Then, plugging that information into (4.38) yields
\[
\left(\sqrt{\rho_0}\phi, \sqrt{\rho_0}\delta \omega\right)^2_{L^\infty(0,T;L^2)} + \|\nabla \phi, \nabla \delta \omega\|^2_{L^2(0,T\times\mathbb{T}^d)} = 0.
\]
Thus, we get $\delta \omega \equiv 0$ on $[0, T] \times \mathbb{T}^d$ if $T$ is small enough. Combining with Lemma 5.3 finally implies that $\phi \equiv 0$ on $[0, T] \times \mathbb{T}^d$, and (4.34) clearly yields $\varphi \equiv 0$. Therefore, for small enough $T > 0$, we finally conclude that
\[
\tilde{u}^1 = \bar{u}^2, \quad \tilde{\omega}^1 = \bar{\omega}^2 \quad \text{on} \quad [0, T] \times \mathbb{T}^d.
\]
Reverting to Eulerian coordinates, we conclude that the two solutions of system (1.1) coincide on $[0, T] \times \mathbb{T}^d$. Then standard connectivity arguments yield uniqueness on the whole $\mathbb{R}^+$. 30
5 Appendix

We here list the useful lemmas and inequalities that have been used several times in the proof of uniqueness.

**Lemma 5.1.** [11, 13] Let $A$ be a matrix valued function on $[0, T] \times \mathbb{T}^d$ satisfying

$$\det A \equiv 1.$$  \hspace{1cm} (5.1)

There exists a constant $c$ depending only on $d$, such that if

$$\|Id - A\|_{L\infty(0,T;L\infty)} + \|A_t\|_{L^2(0,T;L^6)} \leq c,$$  \hspace{1cm} (5.2)

then for all function $g : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ satisfying $g \in L^2(0, T \times \mathbb{T}^d)$ and

$$g = \text{div}\, R \quad \text{with} \quad R \in L^4(0, T; L^2) \quad \text{and} \quad R_t \in L^{4/3}(0, T; L^{3/2}),$$

the equation

$$\text{div}(Aw) = g \quad \text{in} \quad [0, T] \times \mathbb{T}^d$$

admits a solution $w$ in the space

$$X_T := \left\{ v \in L^4(0, T; L^2(\mathbb{T}^d)) , \nabla v \in L^2(0, T; L^2(\mathbb{T}^d)) \text{ and } v_t \in L^{4/3}(0, T; L^{3/2}(\mathbb{T}^d)) \right\}$$

satisfying the following inequalities for some constant $C = C(d)$:

$$\|w\|_{L^4(0,T;L^2)} \leq C\|R\|_{L^4(0,T;L^2)}, \quad \|
abla w\|_{L^2(0,T;L^2)} \leq C\|g\|_{L^2(0,T;L^2)}$$

and

$$\|w_t\|_{L^{4/3}(0,T;L^{3/2})} \leq C\|R\|_{L^4(0,T;L^2)} + C\|R_t\|_{L^{4/3}(0,T;L^{3/2})}. \hspace{1cm} (5.3)$$

In the bounded domain case, the previous lemma can be adapted as follows.

**Lemma 5.2.** [11, 13] Let $\Omega$ be a $C^2$ bounded domain of $\mathbb{R}^d$, and $A$, a matrix valued function on $[0, T] \times \Omega$ satisfying (5.1). If (5.2) is fulfilled then for all function $R : [0, T] \times \Omega \to \mathbb{R}^d$ satisfying

$$\text{div}\, R \in L^2(0, T \times \Omega), \quad R \in L^4(0, T; L^2), \quad R_t \in L^{4/3}(0, T; L^{3/2}) \quad \text{and} \quad R \cdot n \equiv 0 \text{ on } (0, T) \times \partial\Omega,$$

the equation

$$\text{div}(Aw) = \text{div}\, R =: g \quad \text{in} \quad [0, T] \times \Omega$$

admits a solution in the space

$$X_T := \left\{ v \in L^2(0, T; H^1_0(\Omega)) , \quad v \in L^4(0, T; L^2(\Omega)) \text{ and } v_t \in L^{4/3}(0, T; L^{3/2}(\Omega)) \right\},$$

that satisfies Inequalities (5.3).

**Lemma 5.3.** [14] Let $a : (0,1)^d \to \mathbb{R}$ be a nonnegative and nonzero measurable function. Then we have for all $z$ in $H^1(\mathbb{T}^d)$,

$$\|z\|_2 \leq \frac{1}{M} \left| \int_{\mathbb{T}^d} az \, dx \right| + \left(1 + \frac{1}{M}\|M - a\|_2\right)\|\nabla z\|_2 \quad \text{with} \quad M := \int_{\mathbb{T}^d} ax \, dx.$$

Furthermore, in dimension $d = 2$, there exists an absolute constant $C$ so that

$$\|z\|_2 \leq \frac{1}{M} \left| \int_{\mathbb{T}^d} az \, dx \right| + C \log \frac{1}{M} \left( e + \frac{\|M - a\|_2}{M} \right) \|\nabla z\|_2.$$
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