Initial value problem for the linearized mean field Kramers equation with long-range interactions

Pierre-Henri Chavanis

1Laboratoire de Physique Théorique (IRSAMC), CNRS and UPS,
Université de Toulouse, F-31062 Toulouse, France

We solve the initial value problem for the linearized mean field Kramers equation describing Brownian particles with long-range interactions in the $N \to +\infty$ limit. We show that the dielectric function can be expressed in terms of incomplete Gamma functions. The dielectric functions associated with the linearized Vlasov equation and with the linearized mean field Smoluchowski equation are recovered as special cases corresponding to the no friction limit or to the strong friction limit respectively. Although the stability of the Maxwell-Boltzmann distribution is independent on the friction parameter, the evolution of the perturbation depends on it in a non-trivial manner. For illustration, we apply our results to self-gravitating systems, plasmas, and to the attractive and repulsive BMF models.

I. INTRODUCTION

The statistical mechanics of systems with long-range interactions is currently a topic of active research [1–4]. In most papers devoted to this subject, one assumes that the system is isolated. This corresponds to the microcanonical ensemble in which the energy is conserved. This is the correct description of plasmas, stellar systems, and two-dimensional vortices [5–11]. This is also the correct description of the Hamiltonian mean field (HMF) model [12] which is a toy model of systems with long-range interactions consisting in $N$ particles of unit mass moving on a circle and interacting via a cosine potential. In the collisionless regime, valid for $N \to +\infty$ in a proper thermodynamic limit, Hamiltonian systems with long-range interactions are described by the Vlasov equation. The dynamical stability of a spatially homogeneous steady state of the Vlasov equation has been studied by Landau [13] in a seminal paper by solving an initial value problem (previous treatments by Vlasov [14, 15] and others were not rigorous and led to mathematical difficulties). For the Coulombian potential, Landau showed that the density perturbation exhibits a phenomenon of collisionless damping.1 For the gravitational potential, the density perturbation either decays or grows depending on whether the wavelength of the perturbation is smaller or larger than the Jeans length [17, 18].

In many situations of physical interest, the system is not isolated from the surrounding and it is important to take into account its interaction with the external medium. This interaction usually results in some effects of forcing and dissipation. In the simplest situation, the one that we shall consider here, the forcing and the dissipation satisfy a detailed balance condition so that formally the system can be thought to be in contact with a thermal bath fixing its temperature $T$. In that case, the proper statistical ensemble is the canonical ensemble. We stress that the thermostat is played by a system of another nature (physically different from the system under consideration) which usually has short-range interactions2. We shall consider a system of Brownian particles in interaction for which the deterministic Hamiltonian equations are replaced by stochastic Langevin equations [19–22]. In addition to the long-range interaction, the particles experience a friction force and a stochastic force (noise). If we assume a detailed balance condition, the diffusion coefficient $D$ and the friction coefficient $\xi$ satisfy the Einstein relation $D = \xi k_B T/m$ where $T$ is the temperature of the bath. The self-gravitating Brownian gas has been studied in a series of papers by Chavanis and Sire (see, e.g., [22] and references therein) in the strong friction limit $\xi \to +\infty$ in which the motion of the particles is overdamped. Some interesting analogies with the chemotaxis of bacterial populations, the so-called Keller-Segel model [24], have been developed in these papers. The gravitational collapse of the self-gravitating Brownian gas also presents striking analogies with the Bose-Einstein condensation (in particular, a Dirac peak is formed in the post-collapse regime) [25]. Another example of Brownian systems with long-range interactions is the Brownian mean field (BMF) model [26, 27] which can be viewed as the canonical counterpart of the HMF model. In the collisionless regime, valid for $N \to +\infty$ in a proper thermodynamic limit, Brownian systems with long-range interactions are described by the mean field Kramers equation. In this paper, we solve the initial value problem for the linearized mean field Kramers equation around the spatially homogeneous Maxwell-Boltzmann distribution. We obtain the exact solution of this problem and determine the corresponding dielectric function. We show that it can be expressed

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1 Recently, Mouhot and Villani [16] have obtained an important theorem concerning the nonlinear Landau damping.

2 Indeed, it is not possible to define the notion of thermostat for a purely long-range system (i.e. to divide the system into a subsystem + a reservoir) since the energy is non-additive [4].
in terms of incomplete Gamma functions. The zeros of the dielectric function determine the complex pulsations of the density perturbations depending on the temperature $T$, the wavenumber $k$, and the friction coefficient $\xi$. We show that the stability of the spatially homogeneous Maxwell-Boltzmann distribution is independent on the friction coefficient. By contrast, the complex pulsations that determine the evolution of the perturbation depend on it in a non-trivial manner. For $\xi \to 0$ (no friction limit) we recover the results of the Vlasov equation and for $\xi \to +\infty$ (strong friction limit) we recover the results of the Smoluchowski equation. These results are illustrated for self-gravitating systems, plasmas, and for the attractive and repulsive BMF models.

II. BROWNIAN PARTICLES IN INTERACTION: INERTIAL MODEL

A. The Langevin equations

We consider a system of $N$ Brownian particles in interaction. The dynamics of these particles is governed by the coupled stochastic Langevin equations

\[
\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \\
\frac{d\mathbf{v}_i}{dt} = -\frac{1}{m}\nabla_i U(\mathbf{r}_1, ..., \mathbf{r}_N) - \xi \mathbf{v}_i + \sqrt{2D} \mathbf{R}_i(t). \tag{1}
\]

The particles interact through the potential $U(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{i<j} m^2 u(|\mathbf{r}_i - \mathbf{r}_j|)$. The Hamiltonian is $H = \sum_{i=1}^{N} m \mathbf{v}_i^2/2 + U(\mathbf{r}_1, ..., \mathbf{r}_N)$. $\mathbf{R}_i(t)$ is a Gaussian white noise satisfying $\langle \mathbf{R}_i(t) \rangle = 0$ and $\langle \mathbf{R}_i^a(t) \mathbf{R}_i^b(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t-t')$ where $i = 1, ..., N$ label the particles and $\alpha = 1, ..., d$ the coordinates of space. $D$ and $\xi$ are respectively the diffusion and friction coefficients. The former measures the strength of the noise, whereas the latter quantifies the dissipation to the external environment. We assume that these two effects have the same physical origin, like when the system interacts with a heat bath. In particular, we suppose that the temperature $T$ of the bath satisfies the Einstein relation

\[
D = \frac{\xi k_B T}{m}. \tag{2}
\]

The temperature measures the strength of the stochastic force for a given friction coefficient. For $\xi = D = 0$, we recover the Hamiltonian equations of particles in interaction which conserve the energy $E = H$.

B. The $N$-body Kramers equation

The evolution of the $N$-body distribution function is governed by the Fokker-Planck equation \[20\]:

\[
\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \left( \mathbf{v}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} + \mathbf{F}_i \frac{\partial P_N}{\partial \mathbf{v}_i} \right) = \sum_{i=1}^{N} \frac{\partial}{\partial \mathbf{v}_i} \left( D \frac{\partial P_N}{\partial \mathbf{v}_i} + \xi P_N \mathbf{v}_i \right), \tag{3}
\]

where $\mathbf{F}_i = -\frac{1}{m} \nabla_i U$ is the force per unit mass acting on particle $i$. This is the so-called $N$-body Kramers equation. In the absence of forcing and dissipation ($\xi = D = 0$), it reduces to the Liouville equation. The $N$-body Kramers equation satisfies an $H$-theorem for the free energy

\[
F[P_N] = E[P_N] - TS[P_N], \tag{4}
\]

where $E[P_N] = \int P_N H \, d\mathbf{r}_1 d\mathbf{v}_1 ... d\mathbf{r}_N d\mathbf{v}_N$ is the energy and $S[P_N] = -k_B \int P_N \ln P_N \, d\mathbf{r}_1 d\mathbf{v}_1 ... d\mathbf{r}_N d\mathbf{v}_N$ is the entropy. A simple calculation gives

\[
\dot{F} = -\sum_{i=1}^{N} \int \frac{\xi m}{P_N} \left( \frac{k_B T}{m} \frac{\partial P_N}{\partial \mathbf{v}_i} + P_N \mathbf{v}_i \right)^2 d\mathbf{r}_1 d\mathbf{v}_1 ... d\mathbf{r}_N d\mathbf{v}_N. \tag{5}
\]

Therefore, $\dot{F} \leq 0$ and $\dot{F} = 0$ if, and only, if $P_N$ is the canonical distribution defined by Eq. \[4\] below. Because of the $H$-theorem, the system converges towards the canonical distribution for $t \to +\infty$. 

C. The canonical distribution

When the system is in contact with a thermal bath, the relevant statistical ensemble is the canonical ensemble. The statistical equilibrium state is described by the canonical distribution

\[ P_N(r_1, v_1, ..., r_N, v_N) = \frac{1}{Z(\beta)} e^{-\beta H(r_1, v_1, ..., r_N, v_N)}, \]

(6)

where \( \beta = 1/(k_B T) \) is the inverse temperature and \( Z(\beta) = \int e^{-\beta H} \prod_i dr_i dv_i \) is the partition function determined by the normalization condition \( \int P_N \prod_i dr_i dv_i = 1 \). The canonical distribution (6) is the steady state of the N-body Kramers equation (3) provided that the Einstein relation (2) is satisfied.

We define the free energy by

\[ F = -T S + k_B T \left( \sum_i \ln \left( \frac{Z_N}{Z_N^i} \right) \right), \]

(13)

where \( Z_N^i = \int e^{-\beta H} \prod_i dr_i dv_i \) is the partition function determined by \( N \) variables we find that the evolution of the distribution function \( f(r, v, t) = NmP_1(r, v, t) \) is governed by the mean field Kramers equation (8):

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left( D \frac{\partial f}{\partial v} + \xi f v \right), \]

(8)

where

\[ \Phi(r, t) = \int u(r - r')\rho(r', t) \, dr, \]

(9)

is the mean potential and \( \rho(r, t) = \int f(r, v, t) \, dv \) is the density. For \( \xi = D = 0 \), Eq. (8) reduces to the Vlasov equation which describes the collisionless evolution of a Hamiltonian system with long-range interactions. Using the Einstein relation (2), the mean field Kramers equation (8) may be rewritten as

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left( \xi \left( \frac{k_B T}{m} \frac{\partial f}{\partial v} + f v \right) \right). \]

(10)

The mean field Kramers equation satisfies an \( H \)-theorem for the free energy

\[ F[f] = E[f] - TS[f] = \frac{1}{2} \int f v^2 \, dr \, dv + \frac{1}{2} \int \rho \Phi \, dr + k_B T \int \frac{f}{m} \ln \left( \frac{f}{N m} \right) \, dr \, dv. \]

(11)

Its expression can be obtained from Eq. (1) by using the mean field approximation. In terms of the free energy, the mean field Kramers equation may be written as a gradient flow

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ \xi f \frac{\partial f}{\partial v} \left( \frac{\delta F}{\delta f} \right) \right]. \]

(12)

A simple calculation gives

\[ \dot{F} = -\int \xi f \left( \frac{\partial f}{\partial v} \left( \frac{\delta F}{\delta f} \right) \right)^2 \, dr \, dv = -\int \frac{\xi}{f} \left( \frac{k_B T}{m} \frac{\partial f}{\partial v} + f v \right)^2 \, dr \, dv. \]

(13)
Therefore, $\dot{F} \leq 0$ and $\dot{F} = 0$ if, and only if, $f$ is the mean field Maxwell-Boltzmann distribution

$$f(r, v) = A e^{-\beta m [\frac{v^2}{2} + \Phi(r)]},$$

(14)

with the temperature of the bath $T$. Because of the $H$-theorem, the system converges, for $t \to +\infty$, towards a mean-field Maxwell-Boltzmann distribution that is a (local) minimum of free energy at fixed mass. If several minima exist at the same temperature, the selection depends on a notion of basin of attraction. The relaxation time is $t_B \sim 1/\xi$, independent of $N$.

III. BROWNIAN PARTICLES IN INTERACTION: OVERDAMPED MODEL

A. The Langevin equations

In the strong friction limit $\xi \to +\infty$, the inertia of the particles can be neglected. This corresponds to the overdamped model. The stochastic Langevin equations (1) reduce to

$$\frac{d\mathbf{r}_i}{dt} = -\mu \nabla U(\mathbf{r}_1, ..., \mathbf{r}_N) + \sqrt{2D_*} \mathbf{R}_i(t),$$

(15)

where $\mu = 1/(\xi m)$ is the mobility and $D_* = D/\xi^2$ is the diffusion coefficient in physical space. The Einstein relation (2) may be rewritten as

$$D_* = \frac{k_B T}{\xi m} = \mu k_B T.$$

(16)

The temperature measures the strength of the stochastic force (for a given mobility).

B. The $N$-body Smoluchowski equation

The evolution of the $N$-body distribution function $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is governed by the $N$-body Fokker-Planck equation (20):

$$\frac{\partial P_N}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial \mathbf{r}_i} \left[ D_* \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu P_N \frac{\partial U(\mathbf{r}_1, ..., \mathbf{r}_N)}{\partial \mathbf{r}_i} \right].$$

(17)

This is the so-called $N$-body Smoluchowski equation. It can be derived directly from the stochastic equations (15). Alternatively, it can be obtained from the $N$-body Kramers equation (3) in the strong friction limit $\xi \to +\infty$ (30). In that limit, using the Einstein relation (2), we find that

$$P_N(\mathbf{r}_1, \mathbf{v}_1, ..., \mathbf{r}_N, \mathbf{v}_N, t) = \left( \frac{\beta m}{2\pi} \right)^{dN/2} P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t) e^{-\beta m \sum_{i=1}^{N} \frac{v_i^2}{2}} + O(\xi^{-1}),$$

(18)

where the evolution of $P_N(\mathbf{r}_1, ..., \mathbf{r}_N, t)$ is governed by Eq. (17). The $N$-body Smoluchowski equation satisfies an H-theorem for the free energy

$$F[P_N] = \int P_N U d\mathbf{r}_1...d\mathbf{r}_N + k_B T \int P_N \ln P_N d\mathbf{r}_1...d\mathbf{r}_N - \frac{d}{2} N k_B T \ln \left( \frac{2\pi k_B T}{m} \right).$$

(19)

The expression (19) can be obtained from the free energy (4) by using Eq. (18). A simple calculation gives

$$\dot{F} = -\sum_{i=1}^{N} \int \frac{m}{\xi P_N} \left( \frac{k_B T}{m} \frac{\partial P_N}{\partial \mathbf{r}_i} + \frac{1}{m} P_N \frac{\partial U}{\partial \mathbf{r}_i} \right)^2 d\mathbf{r}_1...d\mathbf{r}_N.$$ 

(20)

Therefore, $\dot{F} \leq 0$ and $\dot{F} = 0$ if, and only if, $P_N$ is the canonical distribution in physical space defined by Eq. (21) below. Because of the $H$-theorem, the system converges towards the canonical distribution (21) for $t \to +\infty$. 
C. The canonical distribution

The statistical equilibrium state in configuration space is described by the canonical distribution

\[ P_N(r_1, ..., r_N) = \frac{1}{Z_{\text{conf}}(\beta)} e^{-\beta U(r_1, ..., r_N)}, \]  

(21)

where \( Z_{\text{conf}}(\beta) = \int e^{-\beta U} \prod_i d r_i \) is the configurational partition function determined by the normalization condition \( \int P_N dr_1 ... dr_N = 1 \). The canonical distribution \( \text{(21)} \) is the steady state of the \( N \)-body Smoluchowski equation \( \text{(17)} \) provided that the Einstein relation \( \text{(16)} \) is satisfied. It can also be obtained from Eq. \( \text{(6)} \) by integrating over the density \( \rho \).

The expression \( \text{(26)} \) can be obtained from Eq. \( \text{(19)} \) by using the mean field approximation \( \text{(22)} \). It can also be obtained from Eq. \( \text{(11)} \) by using Eq. \( \text{(24)} \). In terms of the free energy, the mean field Smoluchowski equation may be written as a gradient flow

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \right], \]  

(23)

where \( \Phi(r, t) \) is given by Eq. \( \text{(9)} \). The mean field Smoluchowski equation \( \text{(23)} \) can also be obtained from the mean field Kramers equation \( \text{(10)} \) in the strong friction limit \( \xi \rightarrow +\infty \). In that limit, the distribution function is close to the Maxwellian

\[ f(r, v, t) = \left( \frac{\beta m}{2\pi} \right)^{d/2} \rho(r, t) e^{-\beta m \frac{v^2}{2}} + O(\xi^{-1}), \]  

(24)

with the temperature of the bath, and the evolution of the density is governed by Eq. \( \text{(25)} \). The mean field Smoluchowski equation \( \text{(23)} \) may be written in the form of an integro-differential equation as

\[ \xi \frac{\partial \rho}{\partial t} = \frac{k_B T}{m} \Delta \rho + \nabla \cdot \left[ \rho \nabla \int u(r - r') \rho(r', t) \, dr' \right]. \]  

(25)

It satisfies an \( H \)-theorem for the free energy

\[ F[\rho] = \frac{1}{2} \int \rho \Phi \, dr + k_B T \int \frac{\rho}{m} \ln \left( \frac{\rho}{N m} \right) \, dr - \frac{d}{2} N k_B T \ln \left( \frac{2\pi k_B T}{m} \right) \]  

(26)

The expression \( \text{(26)} \) can be obtained from Eq. \( \text{(19)} \) by using the mean field approximation \( \text{(22)} \). It can also be obtained from Eq. \( \text{(11)} \) by using Eq. \( \text{(24)} \). In terms of the free energy, the mean field Smoluchowski equation may be written as a gradient flow

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{\rho}{\xi} \nabla \left( \frac{\delta F}{\delta \rho} \right) \right]. \]  

(27)

A simple calculation gives

\[ \dot{F} = - \int \frac{\rho}{\xi} \left( \nabla \left( \frac{\delta F}{\delta \rho} \right) \right)^2 \, dr = - \int \frac{1}{\xi \rho} \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right)^2 \, dr. \]

Therefore, \( \dot{F} \leq 0 \) and \( \dot{F} = 0 \) if, and only if, \( \rho \) is the mean field Boltzmann distribution

\[ \rho(r) = A' e^{-\beta m \Phi(r)}, \]  

(29)
with the temperature of the bath $T$. This distribution can also be obtained from the mean field Maxwell-Boltzmann distribution \(^{14}\) by integrating over the velocity. Because of the $H$-theorem, the system converges, for $t \to +\infty$, towards a mean-field Boltzmann distribution that is a (local) minimum of free energy at fixed mass.\(^3\) If several minima exist at the same temperature, the selection depends on a notion of basin of attraction. The relaxation time is $t_B \sim 1/\xi$, independent of $N$.

The mean field Smoluchowski equation \(^{23}\) may also be written as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right], \quad (30)$$

where $p(r, t)$ is a pressure related to the density by the isothermal equation of state

$$p(r, t) = \rho(r, t) \frac{k_B T}{m}. \quad (31)$$

This equation of state can be obtained from the expression of the local kinetic pressure $p(r, t) = \frac{1}{d} \int f(r, v, t) \left[ v - u(r, t) \right]^2 dv$ where $u(r, t) = (1/\rho) \int v f dv$ is the local velocity, combined with the expression \(^{24}\) of the distribution function valid in the strong friction limit. The steady states of the mean field Smoluchowski equation satisfy the equation

$$\nabla p + \rho \nabla \Phi = 0, \quad (32)$$

which may be interpreted as a condition of hydrostatic equilibrium. A generalization of these results to other barotropic equations of state $p(\rho)$ is developed in \(^{22, 31–34}\). In that case, the free energy is given by

$$F[\rho] = \frac{1}{2} \int \rho \Phi \, dr + \int \rho \int \rho(\rho') \frac{p(\rho')}{\rho'^2} \, dr', \quad (33)$$

up to an additional constant.

**Remark:** at $T = 0$, the free energy reduces to the potential energy $W = (1/2) \int \rho \Phi \, dr$ and the $H$-theorem \(^{28}\) becomes $W = -\int (\rho/\xi) (\nabla \Phi)^2 \, dr \leq 0$. In that case, the system relaxes towards the state of minimum potential energy.

### IV. THE GENERAL SOLUTION OF THE INITIAL VALUE PROBLEM USING GREEN FUNCTIONS

The mean field Kramers equation writes

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \xi \frac{\partial f}{\partial v} \left( \frac{k_B T}{m} \frac{\partial f}{\partial v} + f v \right), \quad (34)$$

$$\Phi(r, t) = \int u(r - r') \rho(r', t) \, dr'. \quad (35)$$

The spatially homogeneous steady state of this equation is the Maxwell-Boltzmann distribution

$$f(v) = \left( \frac{\beta m}{2\pi} \right)^{d/2} \frac{1}{\rho} e^{-\frac{\beta mv^2}{2}}. \quad (36)$$

Considering a small perturbation $\delta f(r, v, t) \ll f(v)$ about this steady state, we obtain the linearized mean field Kramers equation

$$\frac{\partial \delta f}{\partial t} + v \cdot \frac{\partial \delta f}{\partial r} - \nabla \delta \Phi \cdot \frac{\partial \delta f}{\partial v} = \xi \frac{\partial \delta f}{\partial v} \left( \frac{k_B T}{m} \frac{\partial \delta f}{\partial v} + \delta f v \right), \quad (37)$$

\(^3\) The steady states of the mean field Smoluchowski equation are the critical points (minima, maxima, saddle points) of the free energy $F[\rho]$ at fixed mass. It can be shown \(^{31}\) that a critical point of free energy is dynamically stable with respect to the mean field Smoluchowski equation if, and only if, it is a (local) minimum. Maxima are unstable for all perturbations so they cannot be reached by the system. Saddle points are unstable only for certain perturbations so they can be reached if the system does not spontaneously generate these dangerous perturbations. The same comments apply to the mean field Kramers equation \(^3\).
\[ \delta \Phi(r, t) = \int u(r - r') \delta \rho(r', t) \, dr'. \] (38)

This equation may be rewritten as
\[ \mathcal{L} \delta f \equiv \frac{\partial \delta f}{\partial t} + v \cdot \frac{\partial \delta f}{\partial r} - \xi \frac{\partial}{\partial \mathbf{v}} \left( \frac{k_B T}{m} \frac{\partial \delta f}{\partial \mathbf{v}} + \delta f \mathbf{v} \right) = \frac{\partial f}{\partial \mathbf{v}} \cdot \int \nabla u(r - r') \delta f(r', \mathbf{v}', t) \, dr' \, d\mathbf{v}', \] (39)

where \( \mathcal{L} \) is the ordinary Kramers operator. To solve this equation we shall use the method of Green functions that has been introduced in similar problems \[35\]. The Green function of the ordinary Kramers operator is defined by
\[ \mathcal{L} G(r - r_0, \mathbf{v}, \mathbf{v}_0, t) = \delta(r - r_0) \delta(\mathbf{v} - \mathbf{v}_0) \delta(t), \] (40)

if \( t \geq 0 \) and \( G(r - r_0, \mathbf{v}, \mathbf{v}_0, t) = 0 \) if \( t < 0 \). It depends only on the space variables \( r \) and \( r_0 \) through the difference \( x = r - r_0 \). The solution of the initial value problem for the ordinary Kramers equation is therefore
\[ \delta f(r, \mathbf{v}, t) = \int G(r - r_0, \mathbf{v}, \mathbf{v}_0, t) \delta f(r_0, \mathbf{v}_0, 0) \, dr_0 \, d\mathbf{v}_0. \] (41)

The Green function of the linearized mean field Kramers equation \[39\] is defined by
\[ \mathcal{L} \tilde{g}(r - r_0, \mathbf{v}, \mathbf{v}_0, t) - \frac{\partial}{\partial \mathbf{v}} \cdot \int \nabla u(r - r') \tilde{g}(r' - r_0, \mathbf{v}', \mathbf{v}_0, t) \, dr' \, d\mathbf{v}' = \delta(r - r_0) \delta(\mathbf{v} - \mathbf{v}_0) \delta(t), \] (42)

if \( t \geq 0 \) and \( g(r - r_0, \mathbf{v}, \mathbf{v}_0, t) = 0 \) if \( t < 0 \). It obeys the integral equation
\[ g(x, \mathbf{v}, \mathbf{v}_0, t) = G(x, \mathbf{v}, \mathbf{v}_0, t) + \int G(x - x', \mathbf{v}, \mathbf{v}', t - t') \frac{\partial}{\partial \mathbf{v}'} \cdot \nabla' u(x' - x'') g(x'', \mathbf{v}'', \mathbf{v}_0, t') \, dx'' \, d\mathbf{v}'' \, dx' \, d\mathbf{v}' \, dt', \] (43)

as may be checked by applying the operator \( \mathcal{L} \). In Eq. \[43\] we must have \( t - t' \geq 0 \) and \( t' \geq 0 \) (otherwise the Green functions vanish) so that \( 0 \leq t' \leq t \). This integral equation can then be solved by applying the convolution theorem. To that purpose, we introduce the Fourier-Laplace transform
\[ \tilde{g}(k, \mathbf{v}, \mathbf{v}_0, \omega) = \int \frac{d\mathbf{r}}{(2\pi)^d} \int_0^{1-\infty} dt \, e^{-i(k \cdot r - \omega t)} g(x, \mathbf{v}, \mathbf{v}_0, t). \] (44)

This expression for the Laplace transform is valid for \( \text{Im}(\omega) \) sufficiently large. For the remaining part of the complex \( \omega \) plane, it is defined by an analytic continuation. The inverse transform is
\[ g(x, \mathbf{v}, \mathbf{v}_0, t) = \int d\omega \int_C \frac{dk}{2\pi} e^{i(k \cdot r - \omega t)} \tilde{g}(k, \mathbf{v}, \mathbf{v}_0, \omega), \] (45)

where the Laplace contour \( C \) in the complex \( \omega \) plane must pass above all poles of the integrand. Taking the Fourier-Laplace transform of Eq. \[43\] we get
\[ \tilde{g}(k, \mathbf{v}, \mathbf{v}_0, \omega) = \tilde{G}(k, \mathbf{v}, \mathbf{v}_0, \omega) + i(2\pi)^d \int \tilde{G}(k, \mathbf{v}, \mathbf{v}', \omega) \frac{\partial}{\partial \mathbf{v}'} \hat{u}(k) \tilde{g}(k, \mathbf{v}'', \mathbf{v}_0, \omega) \, d\mathbf{v}'' \, d\mathbf{v}'. \] (46)

Defining
\[ \tilde{H}(k, \mathbf{v}, \omega) = i(2\pi)^d \hat{u}(k) \int \tilde{G}(k, \mathbf{v}, \mathbf{v}', \omega) \frac{\partial}{\partial \mathbf{v}'} \, d\mathbf{v}', \] (47)

and
\[ \tilde{q}(k, \mathbf{v}_0, \omega) = \int \tilde{g}(k, \mathbf{v}, \mathbf{v}_0, \omega) \, d\mathbf{v}, \] (48)

the foregoing equation may be rewritten as
\[ \tilde{g}(k, \mathbf{v}, \mathbf{v}_0, \omega) = \tilde{G}(k, \mathbf{v}, \mathbf{v}_0, \omega) + \tilde{H}(k, \mathbf{v}, \omega) \tilde{q}(k, \mathbf{v}_0, \omega). \] (49)

Integrating over \( \mathbf{v} \), we obtain
\[ \tilde{q}(k, \mathbf{v}_0, \omega) = \tilde{Q}(k, \mathbf{v}_0, \omega) + \tilde{P}(k, \omega) \tilde{q}(k, \mathbf{v}_0, \omega), \] (50)
where we have defined

$$\hat{Q}(k, v_0; \omega) = \int \tilde{G}(k, v, v_0; \omega) \, dv,$$  \hspace{1cm} (51)

and

$$\hat{P}(k; \omega) = \int \tilde{H}(k, v; \omega) \, dv.$$  \hspace{1cm} (52)

Solving Eq. (50), we get

$$\hat{g}(k, v_0; \omega) = \frac{\hat{Q}(k, v_0; \omega)}{1 - \hat{P}(k; \omega)}.$$  \hspace{1cm} (53)

Substituting this expression in Eq. (19), we finally obtain

$$\hat{g}(k, v, v_0; \omega) = \tilde{G}(k, v, v_0; \omega) + \frac{\tilde{H}(k, v, \omega) \hat{Q}(k, v_0, \omega)}{1 - \hat{P}(k; \omega)}.$$  \hspace{1cm} (54)

This is the resolvent, i.e. the Fourier-Laplace transform of the Green function. It connects $\delta \tilde{f}(k, v, \omega)$ to the initial value. Indeed, the evolution of the perturbed distribution function is given by

$$\delta f(r, v, t) = \int g(r - r_0, v, v_0; t) \delta f(r_0, v_0, 0) \, dr_0 dv_0.$$  \hspace{1cm} (55)

Taking the Fourier-Laplace transform of this expression, we get

$$\delta \tilde{f}(k, v, \omega) = (2\pi)^d \int \hat{g}(k, v, v_0; \omega) \delta \tilde{f}(k, v_0, 0) \, dv_0,$$  \hspace{1cm} (56)

where $\delta \tilde{f}(k, v_0, 0)$ is the Fourier transform of the initial perturbed distribution function. Substituting Eq. (54) in Eq. (56), we obtain

$$\delta \tilde{f}(k, v, \omega) = (2\pi)^d \int \tilde{G}(k, v, v_0; \omega) \delta \tilde{f}(k, v_0, 0) \, dv_0 + (2\pi)^d \frac{\tilde{H}(k, v, \omega)}{1 - \hat{P}(k; \omega)} \int \hat{Q}(k, v_0, \omega) \delta \tilde{f}(k, v_0, 0) \, dv_0.$$  \hspace{1cm} (57)

This is the solution of the initial value problem in Fourier-Laplace space. Integrating over the velocity, we find that the Fourier-Laplace transform of the perturbed density is given by

$$\delta \hat{\rho}(k, \omega) = (2\pi)^d \frac{1}{\epsilon(k, \omega)} \int \hat{Q}(k, v_0, \omega) \delta \tilde{f}(k, v_0, 0) \, dv_0,$$  \hspace{1cm} (58)

where we have introduced the dielectric function

$$\epsilon(k, \omega) = 1 - \hat{P}(k; \omega).$$  \hspace{1cm} (59)

This expression shows that $\hat{P}(k; \omega)$ is the polarization function. The perturbed density $\delta \hat{\rho}(k, \omega)$ given by Eq. (58) appears as a product of two factors: a “universal” factor $\epsilon(k, \omega)^{-1}$ and an integral involving the initial condition $\delta \tilde{f}(k, v_0, 0)$. The first factor is due to collective effects. As we shall explain below, this term can produce damped, steady, or growing oscillations. On the other hand, the integral corresponds to the excess density produced by an initial disturbance in a Brownian gas of non-interacting particles (i.e., for which $\tilde{u}(k) = 0$ or $\epsilon(k, \omega) = 1$). It is typical of an individual particle behavior. The effect of this term disappears for late times since the usual Kramers equation relaxes towards the Maxwell-Boltzmann distribution. For dissipationless systems ($\xi = 0$) this term is responsible for the phenomenon of “phase mixing” associated with the Vlasov equation even in the absence of interaction (see Sec. [x]).

The temporal evolution of the Fourier modes of the density perturbation is given by the inverse Laplace transform

$$\delta \hat{\rho}(k, t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \delta \hat{\rho}(k, \omega).$$  \hspace{1cm} (60)
The “universal” poles of $\delta \hat{\rho}(k, \omega)$ correspond to the complex pulsations $\omega_\alpha(k)$ for which the dielectric function vanishes: $\epsilon(k, \omega_\alpha(k)) = 0$. This defines the dispersion relation. The evolution of the perturbation depends on the position of the zeros of the dielectric function in the complex plane. Using the Cauchy residue theorem, we have

$$\delta \hat{\rho}(k, t) = -i \sum_\alpha e^{-i\omega_\alpha(k)t} \left[ \text{Res} \delta \hat{\rho}(k, \omega) \right]_{\omega = \omega_\alpha(k)},$$

(61)

where the sum runs over the whole set of poles and we have assumed, for simplicity, that the singularities are simple poles. In the following, we shall omit the subscript $\alpha$ for brevity. If at least one zero $\omega$ of the dielectric function lies on the upper half plane (i.e. $\omega_i > 0$), the system is unstable, and the perturbation grows exponentially rapidly with the rate $(\omega_i)_{\text{max}}$ corresponding to the zero with the largest value of the imaginary pulsation. If all the zeros $\omega$ of the dielectric function strictly lie on the lower half-plane (i.e. $\omega_i < 0$), the system is stable, and the perturbation decays to zero exponentially rapidly with the rate $|\omega_i|_{\text{min}}$ corresponding to the zero with the smallest value of the imaginary pulsation in absolute value. If some zero(s) lie(s) on the real axis (i.e. $\omega_i = 0$) while the others lie on the lower half-plane, the system is marginally stable and the perturbation displays an oscillating behavior around zero with the pulsation(s) $\omega_i$. If the integrand has a pole at $\omega = 0$ while the other zeros lie on the lower half-plane, the perturbation tends to a steady state for $t \to +\infty$. Finally, if the integrand has a pole at $\omega = 0$ while other zeros lie on the real axis and the rest on the lower half-plane, the perturbation oscillates about a steady state. For more details, we refer to [5, 6].

Remark: Although we have considered the Kramers operator for illustration, we emphasize that the results of this section are actually valid for any linear operator $L$. Indeed, the formal solution of the problem only involves the Green function of the operator $L$ and the steady distribution $f(v)$. In this sense, the preceding formalism is very general.

V. INITIAL VALUE PROBLEM FOR THE LINEARIZED VLASOV EQUATION

A. The dielectric function

If we take $\xi = 0$ in Eq. [34], we obtain the Vlasov equation. In that case, we can consider any steady state of the form $f = f(v)$, not only the Maxwellian. Let us check that the general formalism developed previously returns the classical results for the initial value problem of the linearized Vlasov equation. The Green function of a free particle ($\xi = 0$) is simply

$$G(r - r_0, v, v_0, t) = \delta(v - v_0) \delta(r - r_0 - v_0 t).$$

(62)

Its Fourier transform is

$$\hat{G}(k, v, v_0, t) = \frac{1}{(2\pi)^d} \delta(v - v_0) e^{-i k \cdot v_0 t},$$

(63)

and its Fourier-Laplace transform is

$$\hat{G}(k, v, v_0, \omega) = \frac{1}{(2\pi)^d} \delta(v - v_0) \frac{1}{\omega - k \cdot v_0}.$$  

(64)

From this expression, we obtain

$$\hat{H}(k, v, \omega) = (2\pi)^d \hat{u}(k) \frac{k \cdot \frac{\partial f}{\partial v}}{k \cdot v - \omega}, \quad \hat{Q}(k, v_0, \omega) = i \frac{1}{(2\pi)^d} \frac{1}{\omega - k \cdot v_0};$$

(65)

$$\hat{P}(k, \omega) = (2\pi)^d \hat{u}(k) \int \frac{k \cdot \frac{\partial f}{\partial v}}{k \cdot v - \omega} dv, \quad \epsilon(k, \omega) = 1 - (2\pi)^d \hat{u}(k) \int \frac{k \cdot \frac{\partial f}{\partial v}}{k \cdot v - \omega} dv.$$  

(66)

Using Eqs. [47], [52] and [63] we find that the temporal evolution of the polarization function is (see also [36]):

$$\hat{P}(k, t) = -(2\pi)^d \hat{u}(k) k^2 t e^{-\frac{\omega^2 t^2}{2}}.$$  

(67)

According to Eq. [57], the solution of the initial value problem is

$$\delta \hat{f}(k, v, \omega) = \frac{\delta \hat{f}(k, v, 0)}{\omega - k \cdot v} + i (2\pi)^d \hat{u}(k) \frac{k \cdot \frac{\partial f}{\partial v}}{\epsilon(k, \omega) \omega - k \cdot v} \int \frac{\delta \hat{f}(k, v_0, 0)}{\omega - k \cdot v_0} dv_0.$$  

(68)
The resolvent operator that connects $\delta \tilde{f}(k, v, \omega)$ to the initial value through Eq. (55) is

$$\hat{g}(k, v, v_0, \omega) = \frac{1}{(2\pi)^d} \frac{\delta(v - v_0)}{i(k \cdot v - \omega)} + \frac{k \cdot \frac{\partial f}{\partial v}}{k \cdot v - \omega} \frac{\delta \tilde{f}(k, v, 0)}{i(k \cdot v - \omega)} \frac{1}{i(k \cdot v_0 - \omega)}.$$  \hspace{1cm} (69)

The density perturbation is given by

$$\delta \tilde{\rho}(k, \omega) = i \frac{1}{\epsilon(k, \omega)} \int \frac{\delta \tilde{f}(k, v_0, 0)}{\omega - k \cdot v_0} dv_0.$$  \hspace{1cm} (70)

This returns the results obtained by directly taking the Fourier-Laplace transform of the linearized Vlasov equation corresponding to Eqs. (37) and (38) with $\xi = 0$. Indeed, they give

$$\delta \tilde{f}(k, v, \omega) = \frac{k \cdot \frac{\partial f}{\partial v}}{k \cdot v - \omega} \delta \tilde{\Phi}(k, \omega) + \frac{\delta \tilde{f}(k, v, 0)}{i(k \cdot v - \omega)},$$  \hspace{1cm} (71)

$$\delta \tilde{\Phi}(k, \omega) = (2\pi)^d \hat{u}(k) \delta \tilde{\rho}(k, \omega).$$  \hspace{1cm} (72)

Integrating Eq. (71) over the velocity and using Eq. (72), we get Eq. (70). Substituting this result back into Eq. (71), we recover Eq. (68).

The integral in Eq. (70) corresponds to the excess density produced by an initial disturbance in a gas of non-interacting particles. The effect of this term disappears for late times. Indeed, it can be shown that this integral produces damped oscillations (i.e. its poles are in the lower half-plane). Therefore, the density perturbation $\delta \tilde{\rho}(k, t)$ decays to zero although the Vlasov equation is time reversible. By contrast, the perturbed distribution function (68) has an additional real pole $\omega = k \cdot v$. It produces an undamped oscillation $\exp(-i k \cdot v t)$ whose pulsation is proportional to the velocity $v$ of the particles. Therefore the distribution function does not decay to zero but generates small-scale filaments. However, if we consider the perturbed density $\delta \tilde{\rho}(k, t) = \int \delta \tilde{f}(k, v, t) dv$ obtained by integrating the perturbed distribution function $\delta \tilde{f}(k, v, t)$ over the velocity, the various velocities produce destructive interferences of the oscillations, and this is why the density perturbation decays: this is the phenomenon of phase mixing. This is an irreversible homogenization process in which the interactions play no role. The other poles of Eq. (68) correspond to the zeros of the dielectric function. They depend on $k$ but not on $v$. They describe the collective behavior of the system. They produce damped or growing oscillations that “resist” the integration over $v$.

B. The dispersion relation and the stability criterion

Although we can study the dispersion relation of the linearized Vlasov equation for any steady distribution $f(v)$, we restrict ourselves here to the the case of the Maxwellian because we ultimately want to compare the results obtained from the linearized Vlasov equation to the results obtained from the linearized mean field Kramers equation that are valid only for the Maxwellian. Other steady states of the Vlasov equation are considered in [18, 36, 37] and in classical textbooks of plasma physics.

For the Maxwell-Boltzmann distribution, we can write the dielectric function in the form

$$\epsilon(k, \omega) = 1 + (2\pi)^d \hat{u}(k) \rho \beta m W \left( \sqrt{\beta m \omega} \right),$$  \hspace{1cm} (73)

where

$$W(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{x - z} dx,$$  \hspace{1cm} (74)

is the plasma dispersion function. The integration has to be performed along the Landau contour $L$ [13]. For any complex $z$, we have

$$W(z) = 1 - z e^{-z^2/2} \int_{0}^{\infty} e^{y^2/2} dy + i \sqrt{\frac{\pi}{2}} z e^{-z^2/2}.$$  \hspace{1cm} (75)

The dispersion relation $\epsilon(k, \omega) = 0$ can be written as

$$1 + (2\pi)^d \hat{u}(k) \rho \beta m W \left( \sqrt{\beta m \omega} \right) = 0.$$  \hspace{1cm} (76)
The neutral mode corresponds to $\omega = 0$. Using $W(0) = 1$ we get the condition $1 + (2\pi)^d \hat{u}(k) \rho \beta m = 0$. Using the Nyquist theorem \[3, 18, 37\], we can show that the system is stable with respect to a perturbation of wavenumber $k$ when

$$1 + (2\pi)^d \hat{u}(k) \rho \beta m > 0,$$

and unstable otherwise. For repulsive potentials for which $\hat{u}(k) > 0$, the system is always stable. For attractive potentials for which $\hat{u}(k) < 0$, the system is always stable when $T > T_c = (\rho m/k_B)(2\pi)^d \max_k |\hat{u}(k)|$ while it is unstable to some modes (corresponding to the converse of Eq. (77)) when $T < T_c$. This stability criterion can also be obtained from the condition of formal nonlinear dynamical stability (see Appendix B).

We look for solutions of the dispersion relation (76) in the form $\omega = i \omega_i$ where $\omega_i$ is real. When $\omega_i > 0$, the perturbation grows exponentially rapidly and when $\omega_i < 0$ is decays exponentially rapidly (without oscillating). The growth or decay rate $\omega_i$ is given by

$$1 + (2\pi)^d \hat{u}(k) \rho \beta m w \left( \frac{\sqrt{\beta m \omega_i}}{2 k} \right) = 0,$$

with

$$w(x) = 1 - \sqrt{\pi} x e^{-x^2} \text{erfc}(x),$$

where erfc is the complementary error function defined by

$$\text{erfc}(x) = 1 - \text{erf}(x), \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

We note that $W(ix) = w(x/\sqrt{2})$ for any real $x$. This function has the asymptotic behaviors

$$w(x) \approx 1 - \sqrt{\pi} x \quad (x \to 0),$$

$$w(x) \approx \frac{1}{2x^2} \left( 1 - \frac{3}{2x^2} \right) \quad (x \to +\infty),$$

$$w(x) \approx 2\sqrt{\pi} |x| e^{-x^2} \quad (x \to -\infty).$$

The asymptotic behaviors of the inverse function are

$$w^{-1}(y) \approx \frac{1}{\sqrt{\pi}} (1 - y) \quad (y \to 1),$$

$$w^{-1}(y) \approx \frac{1}{\sqrt{2y}} \left( 1 - \frac{3}{2y} \right) \quad (y \to 0),$$

$$w^{-1}(y) \approx - (\ln y)^{1/2} \quad (y \to +\infty).$$

In the unstable case ($\omega_i > 0$), using the Nyquist theorem \[3, 18, 37\], we can show that the purely imaginary pulsation $\omega = i \omega_i$ determined by Eq. (78) is the only solution of the dispersion relation (76). In the stable case ($\omega_i < 0$), there exist other solutions of the form $\omega = \omega_r + i \omega_i$ with $\omega_r \neq 0$.

C. Application to the HMF model, self-gravitating systems, and plasmas

For illustration, we apply the preceding results to the attractive and repulsive HMF models, self-gravitating systems, and plasmas. For the definition of these models and for the notations we refer to \[18, 36, 37\].

For the attractive HMF model, using $\hat{u}_n = \frac{1}{N} (2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1})$ and $\rho = 1/(2\pi)$, and considering the modes $n = \pm 1$ (the modes $n \neq \pm 1$ cannot propagate), the dispersion relation (76) can be written as

$$1 - \frac{1}{2T} W \left( \frac{\omega}{\sqrt{T}} \right) = 0.$$
According to Eq. (77), the system is stable if $T > T_c = 1/2$ and unstable with respect to the modes $n = \pm 1$ if $T < T_c$. Assuming that $\omega = i\omega_i$, we get

$$\omega_i = \sqrt{2T} w^{-1}(2T).$$

We have the asymptotic behaviors

$$\omega_i \simeq \frac{1}{\sqrt{2}} (1 - 3T) \quad (T \to 0),$$

$$\omega_i \simeq \frac{1}{\sqrt{\pi}} (1 - 2T) \quad (T \to T_c),$$

$$\omega_i \sim -\sqrt{2T \ln T} \quad (T \to +\infty).$$

For self-gravitating systems, using $(2\pi)^d \hat{u}(\mathbf{k}) = -S_d G/k^2$ and making the Jeans swindle (see [6, 18] for more details), the dispersion relation (76) can be written as

$$1 - k^2 J_k^2 W \left( \frac{\omega}{\omega_G} \frac{k_j}{k_j} \right) = 0,$$

where we have introduced the Jeans wavenumber $k_J = (S_d G \rho \beta m)^{1/2}$ and the gravitational pulsation $\omega_G = (S_d G \rho)^{1/2}$ (the inverse of the dynamical time $t_D = 1/\omega_G$). According to Eq. (77), the system is stable if $k > k_J$ and unstable if $k < k_J$. Assuming that $\omega = i\omega_i$, we get

$$\frac{\omega_i}{\omega_G} = \sqrt{2} \frac{k}{k_J} w^{-1} \left( \frac{k^2}{k_J^2} \right),$$

We have the asymptotic behaviors

$$\frac{\omega_i}{\omega_G} \simeq 1 - \frac{3}{2} \frac{k^2}{k_J^2} \quad (k \to 0),$$

$$\frac{\omega_i}{\omega_G} \simeq \sqrt{\frac{2}{\pi}} \left( 1 - \frac{k^2}{k_J^2} \right) \quad (k \to k_J),$$

$$\frac{\omega_i}{\omega_G} \sim -2 \frac{k}{k_J} \sqrt{\ln \left( \frac{k}{k_J} \right)} \quad (k \to +\infty).$$

For the repulsive HMF model, using $\hat{u}_n = -\frac{1}{2N} (2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1})$ and $\rho = 1/(2\pi)$, and considering the modes $n = \pm 1$ (the modes $n \neq \pm 1$ cannot propagate), the dispersion relation (97) can be written as

$$1 + \frac{1}{2T} W \left( \frac{\omega}{\sqrt{T}} \right) = 0.$$ 

According to Eq. (77), the system is always stable. There is no solution of the dispersion relation (97) of the form $\omega = i\omega_i$. However, some asymptotic solutions of Eq. (97) can be obtained [37]. In the limit $T \to 0$ we have $\omega = \omega_r + i\omega_i$ with $\omega_i \ll \omega_r$ and the solution of the dispersion relation is

$$\omega_i^2 \simeq \frac{1}{2} + 3T + \ldots, \quad \omega_i \sim -\frac{1}{8} \sqrt{\frac{\pi}{2}} \frac{1}{T^{3/2}} e^{-\frac{1}{4T}} \quad (T \to 0).$$

At $T = 0$, the perturbation oscillates with the pulsation $\omega_r = 1/\sqrt{2}$. For $T > 0$, it also experiences a weak Landau damping $\omega_i < 0$. In the limit $T \to +\infty$ we have $\omega = \omega_r + i\omega_i$ with $\omega_i \gg \omega_r$ and the solution of the dispersion relation is

$$\omega_r \sim \pi \sqrt{\frac{T}{2 \ln T}}, \quad \omega_i \sim -\sqrt{2T \ln T} \quad (T \to +\infty).$$
In that case, the perturbation exhibits heavily damped oscillations.

For Coulombian plasmas, using \((2\pi)^d \hat{u}(k) = S_d e^2/m^2 k^2\), the dispersion relation \((100)\) can be written as

\[
1 + \frac{k_D^2}{k^2} W \left( \frac{\omega}{\omega_P} \frac{k_D}{k} \right) = 0,
\]

where we have introduced the Debye wavenumber \(k_D = (S_d e^2 \beta/m)^{1/2}\) and the plasma pulsation \(\omega_P = (S_d e^2/m^2)^{1/2}\) (the inverse of the dynamical time \(t_D = 1/\omega_P\)). According to Eq. \((17)\) the system is always stable. There is no solution of the dispersion relation \((100)\) of the form \(\omega = \omega_p\). However, some asymptotic solutions of Eq. \((100)\) can be obtained \([3]\). For \(k \ll k_D\) (long wavelengths) we have \(\omega = \omega_r + i\omega_i\) with \(\omega_i \ll \omega_r\) and the solution of the dispersion relation is

\[
\omega^2_r \simeq \omega_p^2 + \frac{3k_B T}{m} k^2 + \ldots, \quad \omega_i \sim -\frac{\pi}{8} \omega_p \left( \frac{k_D}{k} \right)^3 e^{-\frac{k_D^2}{4k^2}} (k \to 0).
\]

For \(k = 0\), the perturbation oscillates with the plasma pulsation \(\omega_p\). For \(k \gg k_D\) (small wavelengths) we have \(\omega = \omega_r + i\omega_i\) with \(\omega_i \gg \omega_r\) and the solution of the dispersion relation is

\[
\omega_r \sim \frac{\pi}{2} \omega_p \frac{k}{k_D} \frac{1}{\sqrt{\ln(k/k_D)}}, \quad \omega_i \sim -2\omega_p \frac{k}{k_D} \sqrt{\ln(k/k_D)} (k \to +\infty).
\]

In that case, the perturbation exhibits heavily damped oscillations.

**VI. INITIAL VALUE PROBLEM FOR THE LINEARIZED MEAN FIELD SMOLUCHOWSKI EQUATION**

**A. The dielectric function by a direct approach**

The mean field Smoluchowski equation writes

\[
\frac{\partial \rho}{\partial t} = \frac{1}{\xi} \nabla \cdot (\nabla \rho + \rho \nabla \Phi),
\]

\[
\Phi(r, t) = \int u(r - r')\rho(r', t) \, dr'.
\]

For the sake of generality, we consider an arbitrary barotropic equation of state \(p = p(\rho)\). This leads to the generalized mean field Smoluchowski equation \([22, 31–34]\). For example, the polytropic equation of state \(p = K \rho^\gamma\) can account for anomalous diffusion like in porous media. The usual Smoluchowski equation corresponds to the isothermal equation of state \([31]\) leading to normal diffusion with the diffusion coefficient \([16]\).

Considering a small perturbation \(\delta \rho(r, t) \ll \rho\) about a spatially homogeneous steady state, we obtain the linearized mean field Smoluchowski equation

\[
\frac{\partial \delta \rho}{\partial t} = \frac{1}{\xi} \nabla \cdot (c_s^2 \nabla \delta \rho + \rho \nabla \delta \Phi),
\]

\[
\delta \Phi(r, t) = \int u(r - r')\delta \rho(r', t) \, dr',
\]

where \(c_s^2 = p'(\rho)\) is the velocity of sound. Taking the Fourier-Laplace transform of these equations, we obtain

\[
-\xi \delta \hat{\rho}(k, 0) - i\xi \omega \delta \hat{\rho}(k, \omega) = -c_s^2 k^2 \delta \hat{\rho}(k, \omega) - \rho k^2 \delta \hat{\Phi}(k, \omega),
\]

\[
\delta \hat{\Phi}(k, \omega) = (2\pi)^d \hat{u}(k) \delta \hat{\rho}(k, \omega),
\]

where \(\delta \hat{\rho}(k, 0)\) is the Fourier transform of the initial perturbation \(\delta \rho(r, 0)\). Solving these equations, we get

\[
\delta \hat{\rho}(k, \omega) = \frac{\xi \delta \hat{\rho}(k, 0)}{-i\xi \omega + c_s^2 k^2 + (2\pi)^d \hat{u}(k) \rho k^2}.
\]
Taking the Fourier-Laplace transform of Eq. (118) we get
\[ \hat{\delta \rho}(k, \omega) = \frac{1}{\epsilon(k, \omega)} \frac{\delta \hat{\rho}(k, 0)}{c_s^2 k^2 - i\omega}, \]

where we have introduced the dielectric function \[ \epsilon(k, \omega) = \frac{1}{\epsilon(k, \omega)} - \frac{(2\pi)^d \hat{u}(k) \rho k^2}{i\omega - c_s^2 k^2}. \]

Taking the inverse Laplace transform of Eq. (109) and using the Cauchy residue theorem, we find that the temporal evolution of the Fourier components of the density perturbation is
\[ \delta \hat{\rho}(k, t) = \delta \hat{\rho}(k, 0)e^{-[(c_s^2 + (2\pi)^d \hat{u}(k) \rho)]k^2 t/\xi}. \]

Actually, this result may be directly obtained by taking the Fourier transform of Eqs. (105) and (106) which leads to the first order equation in time
\[ \frac{d\delta \hat{\rho}}{dt} + \frac{1}{\xi} [c_s^2 + (2\pi)^d \hat{u}(k) \rho] k^2 \delta \hat{\rho} = 0. \]

Integrating this equation, we obtain Eq. (112).

B. The dielectric function by using the Green function

It is instructive to recover these results by using the same method as in Sec. IV. The linearized mean field Smoluchowski equation may be rewritten as
\[ \mathcal{L} \delta \rho \equiv \frac{\partial \delta \rho}{\partial t} - D_s \Delta \delta \rho = \frac{1}{\xi} \rho \int \Delta u(r - r')\delta \rho(r', t') \, dr', \]

where \( \mathcal{L} \) is the ordinary diffusion operator with diffusion coefficient \( D_s = c_s^2/\xi \). The Green function of the ordinary diffusion operator is defined by
\[ \mathcal{L}G(r - r_0, t) = \delta(r - r_0)\delta(t), \]

if \( t \geq 0 \) and \( G(r - r_0, t) = 0 \) if \( t < 0 \). It depends only on the space variables \( r \) and \( r_0 \) through the difference \( x = r - r_0 \). The solution of the initial value problem for the ordinary diffusion equation is therefore
\[ \delta \rho(r, t) = \int G(r - r_0, t)\delta \rho(r_0, 0) \, dr_0. \]

The Green function of the linearized mean field Smoluchowski equation is defined by
\[ \mathcal{L}g(r - r_0, t) = \frac{1}{\xi} \rho \int \Delta u(r - r')g(r' - r_0, t) \, dr' = \delta(r - r_0)\delta(t), \]

if \( t \geq 0 \) and \( g(r - r_0, t) = 0 \) if \( t < 0 \). It obeys the integral equation
\[ g(x, t) = G(x, t) + \frac{1}{\xi} \rho \int G(x - x', t - t')\Delta' u(x' - x'')g(x'', t') \, dx'' \, dx' \, dt', \]

as may be checked by applying the operator \( \mathcal{L} \). In Eq. (118) we must have \( t - t' \geq 0 \) and \( t' \geq 0 \) (otherwise the Green functions vanish) so that \( 0 \leq t' \leq t \). This integral equation can then be solved by applying the convolution theorem. Taking the Fourier-Laplace transform of Eq. (118) we get
\[ \hat{g}(k, \omega) = \frac{\hat{G}(k, \omega)}{1 + \frac{1}{\xi} \rho(2\pi)^d \hat{u}(k) k^2 \hat{G}(k, \omega)}. \]
This is the resolvent operator, i.e. the Fourier-Laplace transform of the Green function. It connects $\delta \hat{\rho}(k, \omega)$ to the initial value. Indeed, the evolution of the perturbed density is given by

$$\delta \rho(r, t) = \int g(r - r_0, t) \delta \rho(r_0, 0) \, dr_0. \tag{120}$$

Taking the Fourier-Laplace transform of this expression, we get

$$\delta \hat{\rho}(k, \omega) = (2\pi)^d \hat{g}(k, \omega) \delta \hat{\rho}(k, 0). \tag{121}$$

Substituting Eq. (119) in Eq. (121), we obtain

$$\delta \hat{\rho}(k, \omega) = \frac{1}{(2\pi)^d} \frac{1}{\epsilon(k, \omega)} \hat{G}(k, \omega) \delta \hat{\rho}(k, 0), \tag{122}$$

where we have introduced the dielectric function

$$\epsilon(k, \omega) = 1 - \hat{P}(k, \omega) = 1 + \frac{1}{\xi} \rho(2\pi)^d \hat{u}(k) k^2 \hat{G}(k, \omega). \tag{123}$$

The Green function of the ordinary diffusion equation is

$$G(r - r_0, t) = \frac{1}{(4\pi D^* t)^{d/2}} e^{-\frac{|r-r_0|^2}{4D^* t}}. \tag{124}$$

Its Fourier transform is

$$\hat{G}(k, t) = \frac{1}{(2\pi)^d} e^{-D^* k^2 t}, \tag{125}$$

and its Fourier-Laplace transform is

$$\hat{G}(k, \omega) = \frac{1}{(2\pi)^d} \frac{1}{D^* k^2 - i\omega}. \tag{126}$$

Substituting Eq. (126) in Eqs. (122) and (123), we recover Eq. (110). From Eqs. (123) and (126) the temporal evolution of the polarization function is given by

$$\hat{P}(k, t) = -\frac{1}{\xi} (2\pi)^d \hat{u}(k) \rho k^2 e^{-c^2 k^2 t/\xi}. \tag{127}$$

C. The dispersion relation and the stability criterion

The dispersion relation $\epsilon(k, \omega) = 0$ can be written as

$$i\xi \omega = c_s^2 k^2 + (2\pi)^d \hat{u}(k) \rho k^2. \tag{128}$$

The complex pulsation is purely imaginary: $\omega = i\omega_i$. The perturbation grows exponentially rapidly when $\omega_i > 0$ and it decays exponentially rapidly when $\omega_i < 0$ (without oscillating). The neutral mode corresponds to $\omega = 0$. We get the condition $c_s^2 + (2\pi)^d \hat{u}(k) \rho = 0$. The system is stable with respect to a perturbation with wavenumber $k$ when

$$c_s^2 + (2\pi)^d \hat{u}(k) \rho > 0, \tag{129}$$

and unstable otherwise. This stability criterion can also be obtained from the study of the second order variations of the free energy (see Appendix B).

D. Application to the BMF model, self-gravitating systems, and plasmas

For the attractive BMF model, using $\hat{u}_n = \frac{1}{2\pi} (2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1})$ and $\rho = 1/(2\pi)$, the dispersion relation (128) can be written as

$$i\xi \omega = c_s^2 n^2, \quad (n \neq \pm 1), \quad i\xi \omega = c_s^2 - \frac{1}{2}, \quad (n = \pm 1). \tag{130}$$
The modes \( n \neq \pm 1 \) are damped exponentially rapidly (stable). The modes \( n = \pm 1 \) are damped exponentially rapidly if \( c_s^2 > 1/2 \) (stable) and they grow exponentially rapidly if \( c_s^2 < 1/2 \) (unstable).

For self-gravitating systems, using \((2\pi)^d \hat{u}(k) = -S_d G/k^2\), the dispersion relation \([128]\) can be written as

\[
i \xi \omega = c_s^2 k^2 - S_d G \rho.
\] (131)

The system is stable if \( k > k_f \) and unstable if \( k < k_f \).

For the repulsive BMF model, using \( \hat{u}_n = -\frac{1}{2}\delta_{n,0}(2\delta_{n,1} - \delta_{n,-1}) \) and \( \rho = 1/(2\pi) \), the dispersion relation \([128]\) can be written as

\[
i \xi \omega = c_s^2 n^2, \quad (n \neq \pm 1), \quad i \xi \omega = c_s^2 + \frac{1}{2}, \quad (n = \pm 1).
\] (132)

The system is always stable.

For Coulombian plasmas, using \((2\pi)^d \hat{u}(k) = S_a e^2/m^2 k^2\), the dispersion relation \([128]\) can be written as

\[
i \xi \omega = c_s^2 k^2 + \omega_p^2.
\] (133)

The system is always stable.

VII. INITIAL VALUE PROBLEM FOR THE LINEARIZED MEAN FIELD KRAMERS EQUATION

A. The dielectric function

We now consider the mean field Kramers equation \([33]\) which contains the Vlasov equation and the mean field Smoluchowski equation as particular cases. The Green function of the ordinary Kramers equation has been computed by Chandrasekhar \([39]\). It can be written as

\[
G(r - r_0,v,v_0,t) = \frac{1}{(2\pi)^d (FG - H^2)^{d/2}} e^{-\frac{i}{2}(FG - H^2)k^2 R^2} e^{2\pi i GR H^2 S + FS^2},
\] (134)

with

\[
F = \frac{D}{\xi^3} (2\xi t - 3 + 4e^{-\xi t} - e^{-2\xi t}), \quad G = \frac{D}{\xi} (1 - e^{-2\xi t}), \quad H = \frac{D}{\xi^2} (1 - e^{-\xi t})^2,
\] (135)

\[
R = r - r_0 - \frac{1}{\xi} v_0 (1 - e^{-\xi t}), \quad S = v - v_0 e^{-\xi t}.
\] (136)

Its Fourier transform is

\[
\hat{G}(k,v,v_0,t) = \frac{1}{(2\pi)^d (2\pi G)^{d/2}} e^{-\frac{i}{2}(FG - H^2)k^2} e^{\frac{i}{2} \xi^2 k^2 v_0 (1 - e^{-\xi t})} e^{-\frac{i}{2} \xi^2 e^{-\xi t} k S}.
\] (137)

We note that its Fourier-Laplace transform is not available in a simple form contrary to the Fourier-Laplace transforms \([63]\) and \([126]\) of the Green function of a free particle (\( \xi = 0 \)) and of an overdamped particle (\( \xi \to \infty \)). Using Eq. \([137]\), we obtain after some calculations

\[
\hat{Q}(k,v_0,t) = -\frac{1}{(2\pi)^d} e^{i\xi F k^2} e^{-\frac{i}{2} \xi^2 k v_0 (1 - e^{-\xi t})},
\] (138)

and

\[
\hat{P}(k,t) = -\frac{1}{\xi} (2\pi)^d \hat{u}(k) \rho k^2 (1 - e^{-\xi t}) e^{-\frac{i}{2} \xi^2} e^{-\frac{i}{2} \xi^2 (1 - e^{-\xi t})}.
\] (139)

For \( \xi \to 0 \) we recover Eq. \([67]\) and for \( \xi \to +\infty \) we recover Eq. \([127]\). The Laplace transform of \( \hat{P}(k,t) \) is

\[
\hat{P}(k,\omega) = \int_0^{+\infty} e^{i\omega t} \hat{P}(k,t) \, dt.
\] (140)
Substituting Eq. (139) in Eq. (140) and making the change of variables \( s = e^{-\xi t} \) for \( \xi > 0 \) we can express the integral in terms of the incomplete Gamma functions
\[
\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt.
\] (141)
We find
\[
\tilde{P}(k, \omega) = -(2\pi)^d \hat{u}(k) \rho \beta m F \left( \frac{Dk^2}{\xi^3} - \frac{i\omega}{\xi}, \frac{Dk^2}{\xi^3} \right),
\] (142)
where we have defined
\[
F(\alpha, x) = \frac{e^{x}}{x^{\alpha-1}} \left[ \gamma(\alpha, x) - \frac{1}{x^{\alpha}} \gamma(\alpha + 1, x) \right].
\] (143)
Some properties of this function are given in Appendix A. The dielectric function can finally be written as
\[
\epsilon(k, \omega) = 1 + (2\pi)^d \hat{u}(k) \rho \beta m F \left( \frac{Dk^2}{\xi^3} - \frac{i\omega}{\xi}, \frac{Dk^2}{\xi^3} \right).
\] (144)
In the strong friction limit \( \xi \to +\infty \), using \( F(\alpha, x) \sim x/\alpha (\alpha + 1) \) for \( x \to 0 \) (see Appendix A), we recover the dielectric function (111) associated with the linearized Smoluchowski equation. In order to take the no friction limit, it seems preferable to come back to the expression (139) of the polarization function which reduces to Eq. (67) when \( \xi \to 0 \). Substituting this expression in Eq. (140) and using Eq. (59), we get Eq. (C1) which is equivalent to the dielectric function (66-b) associated with the linearized Vlasov equation (see Appendix C).

**B. The dispersion relation and the stability criterion**

The dispersion relation \( \epsilon(k, \omega) = 0 \) can be written as
\[
1 + (2\pi)^d \hat{u}(k) \rho \beta m F \left( \frac{Dk^2}{\xi^3} - \frac{i\omega}{\xi}, \frac{Dk^2}{\xi^3} \right) = 0.
\] (145)
The neutral mode corresponds to \( \omega = 0 \). Using \( F(x, x) = 1 \) which immediately results from Eq. (A1), we get the condition \( 1 + (2\pi)^d \hat{u}(k) \rho \beta m = 0 \). It can be shown that the system is stable with respect to a perturbation with wavenumber \( k \) when
\[
1 + (2\pi)^d \hat{u}(k) \rho \beta m > 0,
\] (146)
and unstable otherwise. This stability criterion can also be obtained from the study of the second order variations of the free energy (see Appendix B). We stress that the stability criterion (146) does not depend on the friction coefficient \( \xi \) while, of course, the evolution of the perturbation in the stable and unstable regimes (i.e. the value of the complex pulsations that are the solution of the dispersion relation) depend on it in a non trivial manner.

**Remark:** The non-interacting limit corresponds to \( \hat{u}(k) = 0 \). In that case, it is necessary that \( F(\cdot) \to +\infty \) in Eq. (145). According to Eq. (A1), this implies that \( \omega = i\omega_i \) with \( \omega_i = -Dk^2/\xi^2 - n\xi \) where \( n \geq 0 \) is any positive integer. We therefore recover the well-known proper pulsations of the usual Kramers equation [30].

**C. Application to the BMF model, self-gravitating systems, and plasmas**

For the attractive BMF model, using \( \hat{u}_n = \frac{1}{2N} (2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1}) \) and \( \rho = 1/(2\pi) \), and considering the modes \( n = \pm 1 \) (the modes \( n \neq \pm 1 \) evolve with the proper pulsations of the usual Kramers equation), the dispersion relation (145) can be written as
\[
1 - \frac{1}{2T} F \left( \frac{T}{\xi^2} - \frac{i\omega}{\xi}, \frac{T}{\xi^2} \right) = 0.
\] (147)
The system is stable if $T > T_c = 1/2$ and unstable (with respect to the modes $n = \pm 1$) if $T < T_c$. For $\xi \gg 1$, using Eq. (A2), we get
\begin{equation}
 i\xi \omega \simeq \left(1 - \frac{1}{2\xi^2}\right) \left( T - \frac{1}{2} \right),
\end{equation}
(148)
This is the first order correction to the Smoluchowski limit $\xi \rightarrow +\infty$. We now assume that $\omega = i\omega_i$ (for $T < T_c$, this is the only solution of the dispersion relation with $\omega_i > 0$). For $T \rightarrow 0$, using Eq. (A2), we obtain
\begin{equation}
 \omega_i(T, \xi) \simeq \omega_i(0, \xi) + b(\xi) T + \ldots
\end{equation}
(149)
Close to the neutral mode $\omega_i = 0$, i.e. for $T \rightarrow T_c = 1/2$, using Eq. (A4) we obtain
\begin{equation}
 \omega_i(T, \xi) = \frac{\xi}{G} \left( \frac{1}{2\xi^2} \right) (1 - 2T).
\end{equation}
(150)
For $\xi \rightarrow 0$ and $\xi \rightarrow +\infty$, Eqs. (149) and (150) return the results of Sects. V and VI respectively. The fundamental pulsation $\omega_i$ is plotted as a function of the temperature $T$ in Fig. II for different values of the friction parameter $\xi$.

For self-gravitating systems, using $(2\pi)^d \tilde{u}(k) = -S_d G/k^2$, the dispersion relation (145) can be written as
\begin{equation}
 1 - \frac{k_J^2}{k^2} F \left( N^2 k_J^2 - N^2 \frac{i\omega}{\omega_G}, N^2 \frac{k^2}{k_J^2} \right) = 0.
\end{equation}
(151)
We have introduced the dimensionless number $N = \tau_B/\tau_D = \omega_G/\xi$ corresponding to the ratio between the Brownian time and the dynamical time. The system is stable if $k > k_J$ and unstable if $k < k_J$. For $N \ll 1$, using Eq. (A2), we get
\begin{equation}
 i\xi \omega \simeq (1 - N^2)(c_s^2 k^2 - S_d G \rho).
\end{equation}
(152)
This is the first order correction to the Smoluchowski limit $N \rightarrow 0$. We now assume that $\omega = i\omega_i$ (for $k < k_J$, this is the only solution of the dispersion relation with $\omega_i > 0$). For $k \ll k_J$, using Eq. (A2), we obtain $\omega_i(k/k_J, N)/\omega_G \simeq
FIG. 2: Complex pulsation \( \omega = i\omega_i \) as a function of the wavenumber \( k/k_J \) for different values of \( 1/(\sqrt{2}\mathcal{N}) = 0.1, 0.5, 1, 1.5, 2 \) in the case of self-gravitating Brownian systems described by the mean field Kramers equation. We have plotted only the fundamental pulsation. The dashed line, obtained from Eq. (93), corresponds to the dissipationless limit \( \xi = 0 \) (Vlasov). For \( \xi \to +\infty \) (Smoluchowski) the complex pulsation is given by Eq. (131).

\[
\frac{\omega_i(0,\mathcal{N})}{\omega_G} + B(\mathcal{N})(k/k_J)^2 + \ldots
\]

with

\[
\omega_i(0,\mathcal{N})/\omega_G = -\frac{1}{2\mathcal{N}} \pm \frac{1}{2} \sqrt{1 + \frac{1}{\mathcal{N}^2}}, \quad B(\mathcal{N}) = \mathcal{N} \left( \frac{2}{\pm\mathcal{N} \sqrt{1 + \frac{1}{\mathcal{N}^2}} + \frac{1}{\mathcal{N}^2}} - 1 \right).
\]

(153)

Close to the neutral mode \( \omega_i = 0 \), i.e. for \( k \to k_J \), using Eq. (A4) we obtain

\[
\frac{\omega_i(k/k_J,\mathcal{N})}{\omega_G} = \frac{1}{\mathcal{N}G(\mathcal{N}^2)} \left( 1 - \frac{k^2}{k_J^2} \right).
\]

(154)

For \( \mathcal{N} \to +\infty \) and \( \mathcal{N} \to 0 \), Eqs. (153) and (154) return the results of Sects. V and VI respectively. The fundamental pulsation \( \omega_i \) is plotted as a function of the wavenumber \( k/k_J \) in Fig. 2 for different values of \( \mathcal{N} \). For \( k < k_J \), \( \omega_i > 0 \) so the perturbation grow exponentially rapidly without oscillating. For \( k > k_J \), \( \omega_i < 0 \) so the perturbation decreases exponentially rapidly without oscillating (in that case, other modes exist with a non-vanishing pulsation \( \omega_r \) but they are damped more rapidly). Actually, the description of self-gravitating systems is similar to the description of the attractive BMF model provided that we make the correspondences \( \omega \leftrightarrow \omega/(\sqrt{2\omega_G}), T \leftrightarrow k^2/(2k_J^2) \), and \( \xi \leftrightarrow 1/(\sqrt{2\mathcal{N}}) \).

For the repulsive BMF model, using \( u_n = -\frac{1}{2\mathcal{N}}(2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1}) \) and \( \rho = 1/(2\pi) \), and considering the modes \( n = \pm 1 \) (the modes \( n \neq \pm 1 \) evolve with the proper pulsations of the usual Kramers equation), the dispersion relation (145) can be written as

\[
1 + \frac{1}{2T} F \left( \frac{T}{\xi^2} - \frac{i\omega}{\xi}, \frac{T}{\xi^2} \right) = 0.
\]

(155)

The system is always stable. For \( \xi \gg 1 \), using Eq. (A2), we get

\[
i\xi \omega \simeq \left( 1 + \frac{1}{2\xi^2} \right) \left( T + \frac{1}{2} \right).
\]

(156)

This is the first order correction to the Smoluchowski limit \( \xi \to +\infty \). For \( T \to 0 \), using Eq. (A2), we obtain

\[
\omega(T,\xi) \simeq \omega(0,\xi) + b(\xi)T + \ldots
\]

with

\[
\omega(0,\xi) = -\frac{i\xi \pm \sqrt{2 - \xi^2}}{2}, \quad b(\xi) = \frac{i}{\xi} \left( \pm 3i\xi \sqrt{2 - \xi^2} + 2 - \xi^2 - 1 \right).
\]

(157)

Let us consider the case \( T = 0 \). Eq. (157a) shows that there is a critical friction parameter \( \xi_c = \sqrt{2} \). For \( \xi < \sqrt{2} \), the perturbation oscillates with a pulsation \( \omega_r = \frac{1}{2} \sqrt{2 - \xi^2} \) and is damped at a rate \( \omega_i = -\xi/2 \). For \( \xi > \sqrt{2} \), the
FIG. 3: Real and imaginary parts of the complex pulsation $\omega$ as a function of the temperature $T$ for different values of the friction parameter $\xi = 0.1, 0.5, 1, \sqrt{2}$ ($\xi \leq \xi_c$) in the case of the repulsive BMF model described by the mean field Kramers equation. The ordering of the curves may be seen by considering the case $T = 0$. For $\xi = 0$ (Vlasov), $\omega_r = 1/\sqrt{2}$ and $\omega_i = 0$. For $\xi = \xi_c$, $\omega_r = 0$ and $\omega_i = -1/\sqrt{2}$.

FIG. 4: Real and imaginary parts of the complex pulsation $\omega$ as a function of the temperature $T$ for $\xi = 2$ ($\xi > \xi_c$) in the case of the repulsive BMF model described by the mean field Kramers equation. We find that $\omega_r = 0$ for $T < T_c(\xi)$ and $\omega_r \neq 0$ for $T > T_c(\xi)$. This is similar to a second order phase transition. For $\xi \to +\infty$ (Smoluchowski), the critical temperature $T_c(\xi)$ is rejected to infinity and the complex pulsation is given by Eq. (132-b).

perturbation is damped at a rate $\omega_i = -\xi/2 + \frac{1}{2}\sqrt{\xi^2 - 2}$ without oscillating. For $\xi \to +\infty$, we recover the results of Sect. VI. For $\xi \to 0$, we recover the results of Sect. VI for the real part of the complex pulsation (98-a), but we do not obtain the Landau damping (98-b). Therefore, we conclude that frictional effects erase the Landau damping. We recall that our expansion is valid for fixed $\xi > 0$ and $T \to 0$. On the other hand, for fixed $T$ and $\xi \to 0$, we recover the results of Landau [13] since the dispersion relation coincides with the one obtained from the linearized Vlasov equation (see the comment after Eq. (139)). Therefore, the limits $\xi \to 0$ and $T \to 0$ do not commute (see the Appendix B where we consider $\xi \to 0$ then $T \to 0$ while here we have considered $T \to 0$ then $\xi \to 0$). Indeed, there is an indetermination when both $T$ and $\xi$ go to zero since the ratio $T/\xi$ is not well-defined.
and \( \omega_r \neq 0 \) for \( T > T_*(\xi) \). This is similar to a second order phase transition. The derivatives of \( \omega_r(T, \xi) \) and \( \omega_i(T, \xi) \) presents a discontinuity at \( T = T_*(\xi) \).

For Coulombian plasmas, using \((2\pi)^4 \delta(k) = S_{de} c^2/m k^2\), the dispersion relation can be written as

\[
1 + \frac{k^2}{k_D^2} F \left( \mathcal{N}^2 \frac{k^2}{k_D^2} - \mathcal{N}^2 \frac{i \omega_{p}}{\omega_{p}}, \mathcal{N}^2 \frac{k^2}{k_D^2} \right) = 0. \tag{158}
\]

We have introduced the dimensionless number \( \mathcal{N} = t_B/t_D = \omega_P/\xi \) corresponding to the ratio between the Brownian time and the dynamical time. The system is always stable. For \( \mathcal{N} \ll 1 \), using Eq. (A2), we get

\[
i \xi \omega \simeq (1 + \mathcal{N}^2)(c_s^2 k^2 + \omega_p^2). \tag{159}
\]

This is the first order correction to the Smoluchowski limit \( \mathcal{N} \to 0 \). For \( k \ll k_D \), using Eq. (A2), we obtain

\[
\omega(k/k_D, \mathcal{N})/\omega_p \simeq \omega(0, \mathcal{N})/\omega_p + B(\mathcal{N})(k^2/k_D^2) + \ldots \text{ with}
\]

\[
\omega(0, \xi)/\omega_p = -\frac{1}{4} \pm \frac{\sqrt{4 - \frac{1}{\mathcal{N}^2}}}{2}, \quad B(\mathcal{N}) = i \mathcal{N} \left( \pm \frac{1}{\sqrt{4 - \frac{1}{\mathcal{N}^2}}} + \frac{2}{\mathcal{N}} - 1 \right). \tag{160}
\]

Let us consider the case \( k = 0 \). Eq. (160) shows that there is a critical number \( \mathcal{N}_c = 1/2 \). For \( \mathcal{N} > 1/2 \), the perturbation oscillates with a pulsation \( \omega_r/\omega_p = \frac{1}{2} \sqrt{4 - 1/\mathcal{N}^2} \) and is damped at a rate \( \omega_i/\omega_p = -1/(2\mathcal{N}) \). For \( \mathcal{N} < 1/2 \), the perturbation is damped at a rate \( \omega_i/\omega_p = -1/(2\mathcal{N}) + \frac{1}{2} \sqrt{1/\mathcal{N}^2 - 4} \) without oscillating. For \( \mathcal{N} \to 0 \), we recover the results of Sect. VI. For \( \mathcal{N} \to +\infty \), we recover the results of Sect. VII for the real part of the complex pulsation (101a), but we do not obtain the Landau damping (88b). Therefore, we conclude that frictional effects erase the Landau damping (see footnote 4). The description of plasmas is similar to the description of the repulsive BMF model provided that we make the correspondences \( \omega \leftrightarrow \omega/(\sqrt{2} \omega_p), T \leftrightarrow k^2/(2k_D^2) \), and \( \xi \leftrightarrow 1/(\sqrt{2} \mathcal{N}) \). Therefore, the evolution of the real and imaginary parts of the complex pulsation as a function of the wavenumber \( k/k_D \) for different values of the friction can be easily deduced from Figs. 3 and 4. The same phenomenon of “first order phase transition” occurs at a particular wavenumber \( k_*(\xi) \) when \( \mathcal{N} < 1/2 \).

**D. Graphical construction to locate the purely imaginary pulsations**

For the attractive and repulsive BMF models\(^5\), the dispersion relation may be written as

\[
\frac{1}{2T} F(T + \omega_i, T) = \pm 1. \tag{161}
\]

To simplify the discussion we have taken \( \xi = 1 \) but we shall explain later how to treat the general case. We have also assumed that \( \omega = i \omega_i \). To understand the structure of the dispersion relation, we have plotted \( \frac{1}{2T} F(T + \omega_i, T) \) as a function of \( \omega_i \) for different values of the temperature in Figs. 47. As explained previously, the function \( F(T + \omega_i, T) \) diverges when \( \omega_i = -T - n \) for any integer \( n \geq 0 \). The zeros \( \omega_i \) correspond to the intersections between the curve \( \frac{1}{2T} F(T + \omega_i, T) \) and the horizontal line +1 in the attractive case or the horizontal line -1 in the repulsive case.

Depending on the value of the temperature, there may be several intersections corresponding to purely imaginary pulsations \( \omega = i \omega_i \). On the other hand, the absence of intersection may reveal that the pulsation has a real part: \( \omega = \omega_r + i \omega_i \) with \( \omega_r \neq 0 \). Of course, the pulsation with the highest imaginary part (fundamental pulsation) is the most relevant.

Let us first consider the attractive case. Since the curve \( \frac{1}{2T} F(T + \omega_i, T) \) tends to +\( +\infty \) when \( \omega_i \to -T \) and to zero when \( \omega_i \to +\infty \) (see Appendix A), we conclude that there is always a solution \( \omega_i = i \omega_i \) with \( \omega_i \geq -T \). This is the fundamental pulsation. For \( T = T_c = 1/2 \) we have \( \omega_i = 0 \) (neutral), for \( T < T_c \) we have \( \omega_i > 0 \) (unstable), and for \( T > T_c \) we have \( \omega_i < 0 \) (stable). This is illustrated in Figs. 47. The evolution of the fundamental pulsation \( \omega_i \) with \( T \) is represented in Fig. 1 for different values of \( \xi \). In Fig. 6 since the temperature \( T = 0.1 \) is small, we have represented by black bullets the values of the pulsation given by the approximate expression (149). We see that they give a good agreement with the numerical (exact) values of the first two pulsations with the highest imaginary part.

\(^5\) We treat here the case of the BMF model but the discussion is similar for self-gravitating systems and plasmas provided that we use the correspondences given in Sec. VII C.
We note that the other pulsations are very close to the asymptotes at $\omega_i = -T - n$ with $n \geq 2$ where $F$ diverges. This explains why we cannot obtain them with the expansion that we have used to obtain Eq. (149). On the other hand, since the temperatures $T = 0.1$ and $T = 1$ in Figs. 6 and 7 are not too far from $T_c = 1/2$ we have represented by a white bullet the value of the pulsation given by the approximate expression (150). Again, we obtain a good agreement with the numerical (exact) value of the fundamental pulsation. We note that if the temperature is sufficiently small, there exist other purely imaginary pulsations. There also exist pulsations with a non-vanishing real part producing damped oscillations. However, these pulsations are less “fundamental” than the pulsation represented in Fig. 1 since they decay more rapidly.

We now consider the repulsive case. Since the function $\frac{1}{2T}F(T + \omega_i, T)$ is positive for $\omega_i > -T$, there is no intersection with the horizontal line $-1$ in that range. We conclude therefore that $\omega_i$ is necessarily negative so that the system is always stable. The fundamental pulsation has its imaginary part $\omega_i$ in the range $-T - 1 \leq \omega_i \leq -T$. For $\xi = 1$, there is no intersection with the horizontal line $-1$ in that range. This implies that the fundamental pulsation has a non-zero real part $\omega_r \neq 0$. This is in agreement with the result (157) valid for $T \to 0$ which shows that the pulsation has a non-vanishing real part when $\xi < \xi_c = \sqrt{2}$. We note that, depending on the temperature, there may
exist purely imaginary pulsations with \( \omega_i \leq -T - 2 \). For \( T \to 0 \), they are very close to the asymptotes at \( \omega_i = -T - n \) with \( n \geq 2 \) which explains why we cannot obtain them with the expansion that we have used to obtain Eq. (157). For \( \xi > \sqrt{2} \), the curve \( \frac{1}{2T} F(T + \omega_i, T) \) intersects the horizontal line \( -1 \) in the range \( -T - 1 \leq \omega_i \leq -T \) provided that the temperature is not too high. In that case, the fundamental pulsation is purely imaginary. For \( T \to 0 \) it is given by Eq. (157). These results can be understood graphically as follows. First, restoring the friction parameter, we note that the dispersion relation may be written as \( \frac{1}{2T} F(T/\xi^2 + \omega_i/\xi, T/\xi^2) = -\xi^2 \). Therefore, the curves of Fig. 7 correspond to the left hand side of this relation provided that \( T \) is interpreted as \( T/\xi^2 \) and \( \omega_i \) is interpreted as \( \omega_i/\xi \). In that case, we have to consider the intersection with these curves and the horizontal line \( -\xi^2 \). For fixed \( \xi \) and \( T \to 0 \), the maximum of the curve in the range \( -1 \leq \omega_i \leq 0 \) is \( -2 \) (the curve is \( 1/[2(\omega_i/\xi)(\omega_i/\xi + 1)] \)). This implies that for \( T = 0 \) the fundamental pulsation is purely imaginary when \( \xi > \sqrt{2} \) (intersection) while it has a non-vanishing real part when \( \xi < \sqrt{2} \) (no intersection). Furthermore, for fixed \( \xi \), we see that the maximum of the curve \( \frac{1}{2T} F(T/\xi^2 + \omega_i/\xi, T/\xi^2) \) in the range \( -T - 1 \leq \omega_i \leq -T \) decreases as \( T \) increases. Therefore, when \( \xi > \sqrt{2} \) the fundamental pulsation is purely imaginary for \( T < T_c(\xi) \) (intersection) while it has a non-vanishing real part when \( T > T_c(\xi) \) (no intersection) in agreement with the discussion of Sec. VII C.

**FIG. 7**: Graphical construction determining the purely imaginary pulsations \( \omega = i\omega_i \) of the BMF model for \( T = 1 > T_c \) and \( \xi = 1 \). In the attractive case, the imaginary part of the fundamental pulsation is \( \omega_i < 0 \) (stable). The white bullet corresponds to the analytical value valid for \( T \to T_c \).

### VIII. THE MEAN FIELD DAMPED EULER EQUATIONS

It is interesting to compare the results obtained from the mean field Kramers equation with those obtained from the mean field damped Euler equations which also include a dissipative term. However, we stress that the damped Euler equations, which rely on a local thermodynamic equilibrium (LTE) assumption, cannot be rigorously derived from the Kramers equation. Therefore, the dispersion relation associated with the linearized mean field Kramers equation is very different from the dispersion relation associated with the mean field damped Euler equations except in particular limits.

#### A. The local thermodynamic equilibrium assumption

The mean field damped Euler equations write

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi - \xi \mathbf{u},
\]

(162)

\[
\Phi(\mathbf{r}, t) = \int \mathbf{u}(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) d\mathbf{r}'.
\]

(163)
For $\xi = 0$, we recover the mean field Euler equations and for $\xi \to +\infty$ we recover the mean field Smoluchowski equation (30). The mean field damped Euler equations with an isothermal equation of state (31) may be obtained by taking the hydrodynamic moments of the mean field Kramers equation and making a LTE assumption

$$f(r, v, t) = \left(\frac{\beta m}{2\pi}\right)^{d/2} \rho(r, t) e^{-\frac{1}{2}\beta m|v-u(r,t)|^2},$$

(164)
to close the hierarchy of equations. A generalization of this procedure to treat systems described by other equations of state is developed in [22, 31, 32]. However, we stress that there is no rigorous justification of the LTE assumption except in the strong friction limit $\xi \to +\infty$ where we obtain the (generalized) mean field Smoluchowski equation.

In the isothermal case, the mean field damped Euler equation satisfies an $H$-theorem for the free energy

$$F[\rho, u] = \int \rho \frac{u^2}{2} dr + \frac{1}{2} \int \rho \Phi dr + k_B T \int \rho m \left( \frac{\rho}{N_m} \right) dr - \frac{1}{2} Nk_B T \ln \left( \frac{2\pi k_B T m}{\rho} \right).$$

(165)
The expression (165) can be obtained from Eq. (11) by using Eq. (164). For an arbitrary barotropic equation of state $p(\rho)$ we have [22, 31, 32]:

$$F[\rho, u] = \int \rho \frac{u^2}{2} dr + \frac{1}{2} \int \rho \Phi dr + \int \rho \int \rho p(\rho') \rho' d\rho.$$

(166)
up to an additional constant. A simple calculation gives [31]:

$$\dot{F} = -\int \xi \rho u^2 dr \leq 0.$$  

(167)
Therefore, $\dot{F} \leq 0$ and $\dot{F} = 0$ if, and only if, $u = 0$ and $\nabla p + \rho \nabla \Phi = 0$ (hydrostatic equilibrium). In the isothermal case, this leads to the mean field Boltzmann distribution (29) with the temperature of the bath $T$. Because of the $H$-theorem, the system converges, for $t \to +\infty$, towards a distribution that is a (local) minimum of free energy at fixed mass. If several minima exist, the selection depends on a notion of basin of attraction. The relaxation time is $t_B \sim 1/\xi$.

B. The dispersion relation and the stability criterion

The dispersion relation associated with the linearized mean field damped Euler equations may be written as [36]:

$$\omega^2 + i\xi \omega - \omega_0^2(k) = 0,$$

(168)
with $\omega_0^2(k) = c_s^2 k^2 + (2\pi)^d \hat{u}(k)k^2 \rho$. The complex pulsations are given by

$$\omega = -\frac{i\xi \pm \sqrt{4\omega_0^2(k) - \xi^2}}{2}.$$  

(169)
The system is stable with respect to a perturbation with wavenumber $k$ when

$$c_s^2 + (2\pi)^d \hat{u}(k) \rho > 0,$$

(170)
and unstable otherwise [36]. This stability criterion can also be obtained from the study of the second order variations of the free energy (see Appendix B).

C. Application to the BMF model, self-gravitating systems, and plasmas

For the attractive BMF model, using $\hat{u}_n = \frac{1}{2\pi} (2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1})$ and $\rho = 1/(2\pi)$, the dispersion relation (168) can be written as

$$\omega(\omega + i\xi) = c_s^2 n^2, \quad (n \neq \pm 1), \quad \omega(\omega + i\xi) = c_s^2 - \frac{1}{2}, \quad (n = \pm 1).$$

(171)
For \( n \neq \pm 1 \), the complex pulsations are given by

\[
\omega = -i\xi \pm \frac{\sqrt{4c_s^2n^2 - \xi^2}}{2}.
\]  

(172)

If \( n^2 < \xi^2/4c_s^2 \), the perturbation decays exponentially rapidly at a rate \( \omega_i = -\xi/2 \pm \sqrt{\xi^2 - 4c_s^2n^2} < 0 \) without oscillating \((\omega_r = 0)\). If \( n^2 > \xi^2/4c_s^2 \), the perturbation oscillates with a pulsation \( \omega_r = \pm (1/2)\sqrt{4c_s^2n^2 - \xi^2} \) and is damped at a rate \( \omega_i = -\xi/2 < 0 \). For \( n = \pm 1 \), the complex pulsations are given by

\[
\omega = -i\xi \pm \frac{\sqrt{4(c_s^2 - 1/2) - \xi^2}}{2}.
\]  

(173)

The system is stable if \( c_s^2 > (c_s^2)_s = \xi^2/4 + 1/2 \) and unstable if \( c_s^2 < 1/2 \). When \( c_s^2 < (c_s^2)_s = \xi^2/4 + 1/2 \), we find \( \omega_r = 0 \) and \( \omega_i = -\xi/2 \pm (1/2)\sqrt{\xi^2 - 4(c_s^2 - 1/2)} \). We have to distinguish two cases. If \( c_s^2 < 1/2 \), the perturbation grows exponentially rapidly at a rate \( \omega_{i1}^{(+) > 0} \) (unstable) without oscillating. If \( c_s^2 > 1/2 \), the perturbation decays exponentially rapidly at a rate \( \omega_{i1}^{(-) < 0} \) (stable) without oscillating. When \( c_s^2 > \xi^2/4 + 1/2 \) the perturbation oscillates with a pulsation \( \omega_r = \pm (1/2)\sqrt{4(c_s^2 - 1/2) - \xi^2} \) and is damped at a rate \( \omega_i = -\xi/2 < 0 \). For the Euler equation \((\xi = 0)\), the dispersion relation reduces to \( \omega^2 = c_s^2n^2 \) for \( n \neq \pm 1 \) and to \( \omega^2 = c_s^2 - 1/2 \) for \( n = \pm 1 \). The modes \( n \neq \pm 1 \) oscillate with a pulsation \( \omega_r = \pm c_s n \). For the modes \( n = \pm 1 \), we have to distinguish two cases. If \( c_s^2 < 1/2 \), the perturbation grows exponentially rapidly at a rate \( \omega_i = \sqrt{1/2 - c_s^2} > 0 \) without oscillating. If \( c_s^2 > 1/2 \), the perturbation oscillates with a pulsation \( \omega_r = \pm \sqrt{c_s^2 - 1/2} \). These results are illustrated in Fig. [S]

For self-gravitating systems, using \((2\pi)^d \hat{u}(k) = -S_d G/k^2\), the dispersion relation [168] can be written as

\[
\omega(\omega + i\xi) = c_s^2k^2 - \omega_G^2.
\]  

(174)

The system is stable if \( k > k_J \) and unstable if \( k < k_J \). The complex pulsations are given by

\[
\frac{\omega}{\omega_G} = \frac{-i}{N^2} \pm \frac{\sqrt{4(k^2/k_J^2 - 1) - 1}}{N^2}.
\]  

(175)

When \((k/k_J)^2 < 1/(4N^2) + 1\) we find \( \omega_r = 0 \) and \( \omega_i/\omega_G = -1/(2N^2) \pm (1/2)\sqrt{1/N^2 - 4(k^2/k_J^2 - 1)} \). We have to distinguish two cases. If \( k < k_J \), the perturbation grows exponentially rapidly at a rate \( \omega_{i1}^{(+) > 0} \) (unstable) without oscillating. If \( k > k_J \), the perturbation decays exponentially rapidly at a rate \( \omega_{i1}^{(-) < 0} \) (stable) without oscillating. When \((k/k_J)^2 > 1/(4N^2) + 1\), the perturbation oscillates with a pulsation \( \omega_r/\omega_G = \pm (1/2)\sqrt{4(k^2/k_J^2 - 1) - 1/N^2} \) and is damped at a rate \( \omega_i/\omega_G = -1/(2N^2) < 0 \). For the Euler equation \((\xi = 0)\), the dispersion relation reduces

FIG. 8: Real and imaginary parts of the complex pulsation \( \omega \) as a function of the temperature \( T \) for the attractive BMF model described by the mean field damped Euler equations (we have taken \( \xi = 1 \)).
to \(\omega^2 = c_s^2k^2 - S_dG\rho\). If \(k < k_J\), the perturbation grows exponentially rapidly at a rate \(\omega_I/\omega_G = \sqrt{1 - k^2/k_J^2} > 0\) without oscillating. If \(k > k_J\), the perturbation oscillates with a pulsation \(\omega_r/\omega_G = \pm \sqrt{k^2/k_J^2 - 1}\).

For the repulsive BMF model, using \(\hat{u}_n = -(2\delta_{n,0} - \delta_{n,1} - \delta_{n,-1})\) and \(\rho = 1/(2\pi)\), the dispersion relation \(168\) can be written as Eq. \((171a)\) for \(n \neq \pm 1\) and as

\[
\omega(\omega + i\xi) = c_s^2 + \frac{1}{2},
\]

for \(n = \pm 1\). The system is always stable. The discussion of the modes \(n \neq \pm 1\) is the same as the one given previously so we consider here the modes \(n = \pm 1\). The complex pulsations are given by

\[
\omega = \frac{-i\xi \pm \sqrt{4(c_s^2 + \frac{1}{2}) - \xi^2}}{2}.
\]

If \(\xi < \xi_c = \sqrt{2}\), the perturbation oscillates with a pulsation \(\omega_r = \pm (1/2)\sqrt{4(c_s^2 + 1/2) - \xi^2}\) and is damped at a rate \(\omega_I = -\xi/2 < 0\). We now assume \(\xi > \sqrt{2}\). If \(c_s^2 < (c_s^2)_s = \xi^2/4 - 1/2\) the perturbation is damped at a rate \(\omega_I = -\xi/2 \pm (1/2)\sqrt{\xi^2 - 4(c_s^2 + 1/2)} < 0\) without oscillating \(\omega_r = 0\). If \(c_s^2 > \xi^2/4 - 1/2\) the perturbation oscillates with a pulsation \(\omega_r = \pm \sqrt{c_s^2 + 1/2}\). These results are illustrated in Fig. 9.

For Coulombian plasmas, using \((2\pi)^d\hat{u}(k) = S_d e^2/m^2k^2\), the dispersion relation \(168\) can be written as

\[
\omega(\omega + i\xi) = c_s^2 k^2 + \omega_P^2.
\]

The system is always stable. The complex pulsations are given by

\[
\frac{\omega}{\omega_P} = \frac{-\frac{1}{\sqrt{N}} \pm \sqrt{4(c_s^2 + 1) - \frac{1}{N^2}}}{2}.
\]

If \(N > 1/2\), the perturbation oscillates with a pulsation \(\omega_r/\omega_P = \pm (1/2)\sqrt{4(k^2/k_D^2 + 1) - 1/N^2}\) and is damped at a rate \(\omega_I/\omega_P = -1/(2N) < 0\). We now assume \(N < 1/2\). If \((k/k_D)^2 < 1/(4N^2) - 1\) the perturbation is damped at a rate \(\omega_I/\omega_P = -1/(2N) \pm (1/2)\sqrt{1/N^2 - 4(k^2/k_D^2 + 1)} < 0\) without oscillating \(\omega_r = 0\). If \((k/k_D)^2 > 1/(4N^2) - 1\) the perturbation oscillates with a pulsation \(\omega_r/\omega_P = \pm (1/2)\sqrt{4(k^2/k_D^2 + 1) - 1/N^2}\) and is damped at a rate \(\omega_I/\omega_P = -1/(2N) < 0\). For the Euler equation \((\xi = 0)\), the dispersion relation reduces to \(\omega^2 = c_s^2k^2 + \omega_P^2\). The perturbation oscillates with a pulsation \(\omega_r/\omega_P = \pm \sqrt{k^2/k_D^2 + 1}\).
Comparing the results of this section with the results of Sec. [VIII] we see that the complex pulsations associated with the linearized mean field damped Euler equations are different from the complex pulsations associated with the linearized mean field Kramers equation except for \( T = 0 \) (for the BMF model) or for \( k = 0 \) (for self-gravitating systems and plasmas). On the other hand, for the attractive BMF model and for self-gravitating systems, the perturbation oscillates at high \( T \) or \( k \) (for \( \xi < +\infty \)) contrary to the case of the mean field Kramers equation where it is purely damped.

IX. CONCLUSION

In this paper, we have solved the initial value problem for the linearized mean field Kramers equation. The corresponding dielectric function has been expressed in terms of incomplete Gamma functions. Although the complex pulsations depend on the friction coefficient, the stability criterion does not depend on it. As an illustration, we have considered the attractive and repulsive BMF models, self-gravitating systems, and plasmas. Previously known results valid for the Vlasov equation (no friction \( \xi = 0 \)) and for the mean field Smoluchowski equation (strong frictions \( \xi \to +\infty \)) have been recovered as particular limits of the present study. For \( \xi > 0 \), the Landau damping is erased by frictional effects. We have also considered the damped mean field Euler equations which include a dissipation term for

\[ F(\alpha, x) \]

\[ G(x) \]

Appendix A: The functions \( F(\alpha, x) \) and \( G(x) \)

The function \( F(\alpha, x) \) is defined in terms of incomplete Gamma functions in Eq. (143). Using \( \gamma(\alpha, x) \sim e^{-x}x^{\alpha}/\alpha \) for \( \alpha \to +\infty \), we find that \( F(\alpha, x) \sim x/\alpha^2 \) for \( \alpha \to +\infty \). Another expression of this function in the form of a series is

\[ F(\alpha, x) = e^x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} (x + n) \frac{x^n}{n + \alpha}. \]  

(A1)

We note that \( F(\alpha, x) \to \pm \infty \) when \( \alpha \to -n \) where \( n \geq 0 \) is any positive integer. The function \( F(\alpha, x) \) can also be written as

\[ F(\alpha, x) = \sum_{n=1}^{+\infty} \frac{n x^n}{\alpha(\alpha + 1)\ldots(\alpha + n)} = \Gamma(\alpha) \sum_{n=1}^{+\infty} \frac{n x^n}{\Gamma(\alpha + n + 1)}. \]  

(A2)

Using \( \gamma(\alpha + 1, x) = \alpha \gamma(\alpha, x) - x^\alpha e^{-x} \) we have

\[ F(\alpha, x) = 1 + e^x(x - \alpha) \gamma(\alpha, x). \]  

(A3)

From Eq. (A1) or from Eq. (A3), we directly obtain \( F(x, x) = 1 \). On the other hand, for \( \epsilon \ll 1 \) we can make the approximation \( F(x + \epsilon, x) \approx 1 - \epsilon G(x) \) where we have defined

\[ G(x) = e^x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{x^n}{n + x}. \]  

(A4)

This function may also be written as

\[ G(x) = e^x \left[ 1 - \frac{x}{x + \epsilon} \right] \]  

(A5)

where

\[ \Gamma(\alpha, x) = \int_x^{+\infty} t^{\alpha-1} e^{-t} \, dt, \quad \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \, dt, \]  

(A6)

are the incomplete and complete Gamma functions. We have the asymptotic behaviors

\[ G(x) \sim \frac{1}{x} \quad (x \to 0), \quad G(x) \sim \sqrt{\frac{\pi}{2x}} \quad (x \to +\infty). \]  

(A7)

In order to obtain the second behavior, we have used the results

\[ \Gamma(x, x) \sim x^{x-1} e^{-x} \sqrt{\frac{\pi x}{2}}, \quad \Gamma(1 + x) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x, \quad (x \to +\infty). \]  

(A8)
Appendix B: Thermodynamical stability of the mean field Maxwell-Boltzmann distribution

The steady states of the mean field Kramers equation \([10]\) correspond to the mean field Maxwell-Boltzmann distribution \([14]\). They are the critical points of the free energy \([11]\) at fixed mass. Using general arguments based on the fact that the free energy is the Lyapunov functional of the mean field Kramers equation, we can show that dynamical and thermodynamical stability coincide \([31]\): the mean field Maxwell-Boltzmann distribution is dynamically stable with respect to the Kramers equation if, and only if, it is a (local) minimum of free energy at fixed mass (thermodynamical stability).

To solve the minimization problem

\[
F(T) = \min_{\rho} \{ F[\rho] = E[\rho] - TS[\rho] \mid M[\rho] = M \},
\]

we can proceed in two steps (see, e.g., Appendix A of \([41]\)). We first minimize \(F[\rho]\) at fixed normalization and density \(\rho(\mathbf{r})\). This gives

\[
f(\mathbf{r}, \mathbf{v}) = \left( \frac{\beta m}{2\pi} \right)^{d/2} \rho(\mathbf{r}) e^{-\beta m v^2 / 2}.
\]

Using Eq. \([12]\), we can express the free energy \(F[\rho]\) given by Eq. \([11]\) as a functional of the density \(\rho\). This leads to Eq. \([20]\). Finally, the solution of the minimization problem \([31]\) is given by Eq. \([12]\) where \(\rho(\mathbf{r})\) is the solution of the minimization problem

\[
F(T) = \min_{\rho} \{ F[\rho] \mid M[\rho] = M \}.
\]

It can be shown that the minimization problems \([31]\) and \([33]\) are equivalent for global and local minimization \([41]\). If we consider the overdamped model, we immediately arrive at the minimization problem \([33]\).

The critical points of \([33]\) at fixed mass satisfy \(\delta F + \alpha T \delta M = 0\) and they lead to the mean field Boltzmann distribution \([26]\). The second order variations of free energy are given by

\[
\delta^2 F = \frac{1}{2} \int \delta \rho \delta \Phi \, d\mathbf{r} + \frac{k_B T}{m} \int \frac{(\delta \rho)^2}{2\rho} \, d\mathbf{r},
\]

with

\[
\delta \Phi(\mathbf{r}) = \int u(\mathbf{r} - \mathbf{r}') \delta \rho(\mathbf{r}') \, d\mathbf{r}'.
\]

The Boltzmann distribution is a (local) minimum of free energy at fixed mass if, and only if, \(\delta^2 F > 0\) for all perturbations satisfying \(\int \delta \rho \, d\mathbf{r} = 0\). If the critical point of free energy is spatially homogeneous, we can decompose the perturbation \(\delta \rho\) in Fourier modes as in Eq. \([43]\). The second order variations of free energy can then be rewritten as

\[
\delta^2 F = \frac{(2\pi)^d}{2\rho} \int \left[ \frac{k_B T}{m} + (2\pi)^d \hat{u}(k) \rho \right] |\delta \rho(k)|^2 \, dk.
\]

If \(\hat{u}(k) > 0\) for all \(k\) (repulsive interaction), the homogeneous phase is thermodynamically stable. If \(\hat{u}(k) < 0\) for some mode(s) \(k\) (attractive interaction), the homogeneous phase is thermodynamically stable when \(T > T_c = (pm/k_B)(2\pi)^d \max_k |\hat{u}(k)|\) and thermodynamically unstable (with respect to the modes such that \(k_B T/m + (2\pi)^d \hat{u}(k) \rho < 0\)) when \(T < T_c\). This returns the stability criterion \([146]\).

For the Vlasov equation, corresponding to Eq. \([10]\) with \(\xi = 0\), the functional \([11]\) is conserved. It may be interpreted as an energy-Casimir functional. A minimum of this functional is formally nonlinearly dynamically stable with respect to the Vlasov equation \([42]\). In general, this criterion provides just a sufficient condition of dynamical stability. More refined dynamical stability criteria exist (see \([43]\) for details). However, for spatially homogeneous distributions, this criterion can be shown to be both necessary and sufficient \([43]\). This leads to Eq. \([146]\) then to Eq. \([77]\). For distribution functions different from the Maxwell distribution, these results remain valid for the Vlasov equation provided that \(k_B T/m\) is replaced by \(c_s^2\) where \(c_s\) is the velocity of sound in the “corresponding barotropic gas” (see \([18, 24, 37]\) for details).

For the mean field damped Euler equations \([162, 163]\), the free energy \([165]\) plays the role of a Lyapunov functional. Therefore, a steady state of the mean field damped Euler equations is dynamically stable if, and only, if it is a (local) minimum of free energy at fixed mass (thermodynamical stability). This leads to the minimization problem \([133]\).
then to Eq. 136 where $k_B T/m$ is replaced by $c_s^2$, and finally to the stability criterion 170. For the mean field Euler equation ($\xi = 0$), the functional 165 is conserved. It corresponds to the energy functional of a barotropic gas 18, 26, 37. A minimum of this functional at fixed mass is formally nonlinearly dynamically stable with respect to the mean field Euler equations 42. This leads to the stability criterion 170. Since the energy functional and the mass are the only conserved quantities, this criterion provides a necessary and sufficient condition of dynamical stability.

Appendix C: An alternative calculation of $w(x)$

The Fourier-Laplace transform of the polarization function associated with the linearized Vlasov equation is given by Eq. (66-a). Taking its inverse Laplace transform and using the Cauchy residue theorem, we get the expression (67). Taking the Laplace transform of this expression and using Eq. (59), we find that the dielectric function can be written as

$$
\varepsilon(k, \omega) = 1 + (2\pi)^d \hat{u}(k) \rho k^2 \int_0^{+\infty} dt e^{i\omega t} x e^{-\frac{\omega^2 t^2}{4 \beta m}}.
$$

(C1)

With the change of variables $x = kt/\sqrt{\beta m}$, the foregoing equation may be rewritten as Eq. (C3) with

$$
W(z) = \int_0^{+\infty} dx e^{izx} x e^{-x^2/2}.
$$

(C2)

Assuming that $z = iz$ where $x$ is real, integrating by parts, and using the identity

$$
\int_0^{+\infty} dx e^{-\gamma x} e^{-\frac{x^2}{2\pi}} = \sqrt{\pi} e^{\beta^2/2} \text{erfc}(\beta),
$$

(C3)

we obtain $W(ix) = w(x/\sqrt{2})$ where $w(x)$ is given by Eq. (79).

Appendix D: The expansion $\xi \to 0$

According to Eqs. 79 and 139, the dielectric function associated with the linearized mean field Kramers equation may be written as

$$
\varepsilon(k, \omega) = 1 + \frac{1}{\xi} (2\pi)^d \hat{u}(k) \rho \beta m \left[ W\left(\sqrt{\beta m \omega} \frac{k}{\xi}\right) + \xi \beta m \frac{U\left(\sqrt{\beta m \omega} \frac{k}{\xi}\right)}{k} \right] = 0,
$$

(D2)

where $W(z)$ is given by Eq. (C2) and $U(z)$ by

$$
U(z) = \int_0^{+\infty} dx e^{izx} x \left(-\frac{1}{2} x + \frac{1}{6} x^3\right) e^{-x^2/2}.
$$

(D3)

Expanding the integrand for $\xi \to 0$, we find that the dispersion relation at the order $O(\xi)$ is

$$
1 + (2\pi)^d \hat{u}(k) \rho \beta m \left[ W\left(\sqrt{\beta m \omega} \frac{k}{\xi}\right) + \xi \beta m \frac{U\left(\sqrt{\beta m \omega} \frac{k}{\xi}\right)}{k} \right] = 0,
$$

(D2)

where $W(z)$ is given by Eq. (C2) and $U(z)$ by

$$
U(z) = \int_0^{+\infty} dx e^{izx} x \left(-\frac{1}{2} x + \frac{1}{6} x^3\right) e^{-x^2/2}.
$$

(D3)

These functions may be rewritten as

$$
W(z) = 1 - \sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}} \left[-i + \text{erfi}\left(\frac{z}{\sqrt{2}}\right)\right],
$$

(D4)

$$
U(z) = \frac{1}{12} z \left\{ 2i(2 - z^2) + \sqrt{2\pi} z(z^2 - 3) e^{-\frac{z^2}{2}} \left[ 1 + i \text{erfi}\left(\frac{z}{\sqrt{2}}\right)\right] \right\},
$$

(D5)

where erfi($z$) = erf($iz$)/$i$. We note that $U(z) = -\frac{1}{6} z [1 + (z^2 - 3) W(z)]$. 

Considering the attractive BMF model, assuming $\omega = i\omega_i$ and taking the limit $T \to 0$, we obtain after careful calculations to order $O(T)$:

$$\omega_i = \frac{1}{\sqrt{2}} - \frac{\xi}{2} + \left(-\frac{3}{\sqrt{2}} + 4\xi\right) T \quad (T \to 0).$$

We can check that this result agrees with Eq. (149) at the order $O(\xi)$. On the other hand, taking the limit $T \to T_c$ and using $W(z) \simeq 1 + i\sqrt{\pi/2} z + \ldots$ and $U(z) \sim iz/3$ for $z \to 0$, we get

$$\omega_i = \frac{1}{\sqrt{\pi}} \left(1 - \frac{2\xi}{3\sqrt{\pi}} + \ldots\right) (1 - 2T), \quad (T \to T_c).$$

Using $G(x) \simeq \sqrt{\pi/(2x)} + 1/(3x) + \ldots$ for $x \to +\infty$, we can check that Eq. (D7) agrees with Eq. (150) at the order $O(\xi)$.

Considering the repulsive BMF model, and taking the limit $T \to 0$ for which $\omega_i \ll \omega_r$, we obtain after careful calculations to order $O(T)$:

$$\omega_r^2 = 1 + 3T - \frac{\xi}{T^{3/2}} \frac{1}{96} \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2T}} \quad (T \to 0),$$

$$\omega_i = -\frac{1}{8} \sqrt{2} \frac{1}{T^{3/2}} e^{-\frac{\pi}{2T}} - \frac{\xi}{2} (1 + 8T) - \frac{\xi}{T^6} \frac{\pi}{3072} e^{-\frac{\pi}{2T}} \quad (T \to 0).$$

We can check that this result agrees with Eq. (157) at the order $O(\xi)$ except for the exponentially small terms (see footnote 4). When $\xi = 0$ we recover the Landau damping (138) but as soon as $\xi > 0$, the Landau damping for $T \to 0$ becomes negligible (subdominant) with respect to the frictional terms.

The dispersion relation (D2) can also be solved perturbatively by writing $\omega = \omega_0 + \xi \omega_1 + \ldots$ with $\xi \ll 1$. Substituting this expansion in Eq. (D2) we find that $\omega_0$ is given by Eq. (76) and that

$$\omega_1 = \frac{U}{W' \sqrt{\beta m \omega_0}}.$$  \hspace{1cm} (D10)

Using the identity $W'(z) = (1/z - z)W(z) - 1/z$ and expressing $U(z)$ in terms of $W(z)$ we obtain

$$1 + (2\pi)^2 \hat{u}(k) \rho \beta m W(z_0) = 0, \quad \omega_1 = i z_0^3 \frac{1 + (z_0^2 - 3)W(z_0)}{6 (1 - z_0^2)W(z_0) - 1}. \hspace{1cm} (D11)$$

where we have defined $z_0 = \sqrt{\beta m \omega_0}/k$. These results can be obtained by other methods (in preparation). For the attractive and repulsive BMF models, we get

$$1 + \frac{1}{2T} W\left(\frac{\omega_0}{\sqrt{T}}\right) = 0, \quad \omega_1 = i z_0^3 \frac{1 \pm 2(\omega_0^2 - 3T)}{6T \pm 2(T - \omega_0^2) - 1}. \hspace{1cm} (D12)$$

For $T \to 0$ and $T \to T_c$ (in the attractive case), we recover Eqs. (D6), (D9). For $T \to +\infty$ we find

$$\omega_i \simeq -\sqrt{2T \ln T} + \frac{1}{3} \xi \ln T, \hspace{1cm} (D13)$$

in the attractive case and

$$\omega_r \simeq \pi \sqrt{\frac{T}{2 \ln T}} - \frac{\pi}{3} \xi, \quad \omega_i = -\sqrt{2T \ln T} + \frac{1}{3} \xi \ln T, \hspace{1cm} (D14)$$

in the repulsive case.

We have given here the results for the BMF model. The corresponding results for self-gravitating systems and plasmas can be obtained by using the correspondences of Sec. VII C.

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