Quantum anharmonic oscillators: a new approach

F J Gómez and J Sesma
Departamento de Física Teórica, Facultad de Ciencias, 50009, Zaragoza, Spain

Abstract. The determination of the eigenenergies of a quantum anharmonic oscillator consists merely in finding the zeros of a function of the energy, namely the Wronskian of two solutions of the Schrödinger equation which are regular respectively at the origin and at infinity. We show in this paper how to evaluate that Wronskian exactly, except for numerical rounding errors. The procedure is illustrated by application to the $gx^2 + x^{2N}$ ($N$ a positive integer) oscillator.

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Quantum anharmonic oscillators have been frequently used in different branches of Physics to simulate a great variety of situations and to explain multitude of phenomena. Apart from this, since the publication of the seminal papers by Bender and Wu [1] and by Simon and Dicke [2] showing the failure of the Rayleigh-Schrödinger perturbation method, they have served to test plenty of approximate methods of solution of the Schrödinger equation. Papers dealing with the most recently proposed methods [3] contain references to older ones, that we omit for brevity. It seems, however, to have been passed unnoticed that an exact procedure exists to obtain a quantization condition that gives the eigenenergies as zeros of an easily calculable function. The idea is the same one exploited in the solution of the Schrödinger equation for the harmonic oscillator or the Coulomb potential, although the procedure is slightly more tricky than in those two simple cases.

The starting point is the Schrödinger equation written as a second order differential equation free of first-derivative terms. Such equation is satisfied by the wave function, in the case of an even one-dimensional anharmonic oscillator, or by the reduced wave function, if one is considering an isotropic D-dimensional oscillator. A solution \( u_{\text{reg}} \) of the differential equation physically acceptable at the origin can be immediately obtained as a power series of the variable. Two other solutions \( u^{(1)} \) and \( u^{(2)} \), characterized by their behavior at large distances, can also be considered. To be specific, let \( u^{(1)} \) represent the solution going exponentially to zero as the variable increases, whereas \( u^{(2)} \) corresponds to an exponentially diverging one. Since \( u^{(1)} \) and \( u^{(2)} \) are independent, \( u_{\text{reg}} \) can always be written as a linear combination of them with coefficients, called connection factors, which depend on the energy and on the parameters of the potential. For a generic value of the energy, both connection factors are different from zero and \( u_{\text{reg}} \) is not a physical solution because of its behavior at infinity. The eigenenergies are then determined by requiring the cancellation of the connection factor multiplying \( u^{(2)} \).

The connection problem for a differential equation with two singular points (let us say, one at the origin and the other at infinity) was discussed by Naundorf [4]. He considered the case of one of the singular points (that at infinity, for instance) being irregular of integer rank \( R > 0 \) and the other one being either irregular of integer rank \( r > 0 \) or regular \( (r = 0) \). Here we are interested in this last case. Naundorf gave a procedure consisting in obtaining \( 2R \) independent formal power series, with an integer index running from \( -\infty \) to \( +\infty \), having well defined asymptotic behaviors, and whose coefficients can serve as a basis in the \( 2R \)-dimensional space of solutions of the recurrence obeyed by the coefficients of \( u_{\text{reg}} \). To obtain such basis, Naundorf replaces, in the known expressions of \( u^{(1)} \) and \( u^{(2)} \) [5], the exponential term determining their respective asymptotic behaviors by \( R \) independent formal expansions of the type of the Heaviside’s exponential series. Multiplication of those formal expansions by the Taylor series of the rest of the exponential terms and the descending power series in \( u^{(1)} \) and \( u^{(2)} \) produces \( 2R \) formal expansions whose coefficients obey the above mentioned recurrence, i. e., the required basis. Comparison of \( 2R \) consecutive coefficients of the power series expression of \( u_{\text{reg}} \) with the analogous coefficients of the elements of the basis leads to
a system of $2R$ linear equations whose solution allows one to obtain the connection factors. That procedure has been applied to the solution of several physical problems, like the hydrogen atom with fine structure [6], the quarkonium [7], the spherical Stark effect in the hydrogen atom [8] or the quartic and sextic anharmonic oscillators [9].

The method here suggested is related to the Naundorf’s one insofar as it also rests on the vanishing of one of the connection factors and makes use of Heaviside’s exponential series to obtain formal expansions, but differs from the Naundorf’s method in the procedure of computation: instead of following the steps detailed in the preceding paragraph, we benefit from the fact that the connection factor multiplying $u^{(2)}$ is given by the quotient of Wronskians $W[u_{\text{reg}}, u^{(1)}]/W[u^{(2)}, u^{(1)}]$ and, since the denominator does not vanish, the quantization results from the fulfillment of the condition

$$W[u_{\text{reg}}, u^{(1)}] = 0.$$  

To implement this condition, we need suitable expressions of $u_{\text{reg}}$ and $u^{(1)}$. The series expansion mentioned above is adequate to represent the first of these solutions. For the second one a closed expression does not exist, in general, but a formal (asymptotic) expansion can be easily obtained by substitution in the differential equation. Then it is trivial to write a formal expression of the Wronskian and to require its cancellation.

To illustrate the method, let us apply it to the determination of the eigenenergies of the one-dimensional anharmonic oscillator represented by the potential

$$V(x) = gx^2 + x^{2N}, \quad N \text{ a positive integer}. \quad (2)$$

This problem has been tackled by several authors [10, 11, 12] by using different approximations. We discard the trivial case $N = 1$. The cases $N = 2$ (usually referred to as quartic oscillator) and $N = 3$ (sextic) can be easily solved following the steps we are going to detail, but the resulting equations do not fit in the general form given below. Therefore, we assume $N \geq 4$. The Schrödinger equation (in adequate units for the variable $x$ and the energy $E$)

$$\left(-\frac{d^2}{dx^2} + gx^2 + x^{2N}\right)u(x) = Eu(x), \quad (3)$$

admits solutions, regular at the origin, of the form

$$u_{\text{reg}}(x) = \sum_{n=0}^{\infty} a_n x^{n+\nu}, \quad a_0 \neq 0,$$  

with $\nu = 0$ (even states) or 1 (odd states). Alternatively, two independent solutions, $u^{(1)}$ and $u^{(2)}$, with asymptotic expansions (for $x \to +\infty$)

$$u^{(j)}(x) \sim \exp \left[\frac{\alpha^{(j)}}{N+1} x^{N+1}\right] x^{\mu^{(j)}} \sum_{m=0}^{\infty} h_m^{(j)} x^{-m}, \quad h_{0}^{(j)} \neq 0, \quad j = 1, 2,$$  

can also be considered. Substitution of this formal expansion in (3) gives for the exponents

$$\alpha^{(1)} = -1, \quad \mu^{(1)} = \mu \equiv -N/2;$$

$$\alpha^{(2)} = +1, \quad \mu^{(2)} = \mu \equiv -N/2.$$  

(6)
and for the coefficients
\[ 2\alpha^{(j)} m h_m^{(j)} = (m-N/2)(m-N/2-1) h_{m-N-1}^{(j)} + E h_{m-N+1}^{(j)} - g h_{m-N+3}^{(j)}. \tag{7} \]

Instead of computing directly the left hand side of (1), let us introduce two auxiliary functions
\[ v_{\text{reg}}(x) = \exp\left(\frac{x^{N+1}}{(N+1)}\right) u_{\text{reg}}(x), \tag{8} \]
\[ v^{(1)}(x) = \exp\left(\frac{x^{N+1}}{(N+1)}\right) u^{(1)}(x), \tag{9} \]
which obey the differential equation
\[ \frac{d^2v}{dx^2} - 2x^N \frac{dv}{dx} + \left(E - g x^2 - N x^{N-1}\right) v = 0, \tag{10} \]
and whose Wronskian satisfies
\[ W[v_{\text{reg}}, v^{(1)}] = \exp\left(\frac{2x^{N+1}}{(N+1)}\right) W[u_{\text{reg}}, u^{(1)}]. \tag{11} \]

Now, by using the series expansion
\[ v_{\text{reg}}(x) = \sum_{n=0}^{\infty} b_n x^{n+\nu}, \quad b_0 \neq 0, \tag{12} \]
with coefficients given by the recurrence
\[ (n+\nu)(n+\nu-1) b_n = -E b_{n-2} + g b_{n-4} + 2(n-N/2-1+\nu) b_{n-N-1}, \tag{13} \]
and the asymptotic expansion
\[ v^{(1)}(x) \sim \sum_{m=0}^{\infty} h_m^{(1)} x^{-m+\mu}, \tag{14} \]
one obtains for the left hand side of (11) a formal expansion
\[ W[v_{\text{reg}}, v^{(1)}] \sim \sum_{k=-\infty}^{\infty} \gamma_k x^{k-1+\nu+\mu}, \tag{15} \]
with coefficients
\[ \gamma_k = \sum_{m=0}^{\infty} (-2m - k - \nu + \mu) b_{m+k} h_m. \tag{16} \]

A similar expansion can be obtained for the right hand side of (11) by recalling the Heaviside’s exponential series
\[ \exp(t) \sim \sum_{n=-\infty}^{\infty} \frac{t^{n+\delta}}{\Gamma(n+1+\delta)}, \tag{17} \]
introduced by Heaviside in the second volume of his Electromagnetic theory (London, 1899) and probed by Barnes \[ \text{[13]} \] to be an asymptotic expansion for arbitrary \( \delta \) and \( |\arg(t)| < \pi \). Let us construct \( N+1 \) expansions
\[ \exp\left(\frac{2x^{N+1}}{(N+1)}\right) \sim \mathcal{E}_L \equiv \sum_{n=-\infty}^{\infty} \frac{(2x^{N+1}/(N+1))^{n+\delta_L}}{\Gamma(n+1+\delta_L)}, \tag{18} \]
of the type (17) with appropriate choices for \( \delta \),
\[ \delta_L = (\nu + \mu + L)/(N+1), \quad L = 0, 1, \ldots, N. \tag{19} \]
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Table 1. The four lowest eigenenergies of the oscillator (2), for \( N = 4 \) and several values of \( g \). The energies \( E_0 \) and \( E_2 \) correspond to even states (\( \nu = 0 \) in (1)); \( E_1 \) and \( E_3 \) to odd ones (\( \nu = 1 \)).

| \( g \)  | \( E_0 \)     | \( E_1 \)     | \( E_2 \)     | \( E_3 \)     |
|---------|---------------|---------------|---------------|---------------|
| -20     | -15.62781790 | -15.60342843  | -1.99759674   | 0.04913769    |
| -10     | -3.89894214  | -3.32541335   | 3.26415045    | 8.82212629    |
| -1      | 0.93527862   | 4.11346827    | 9.4908984     | 16.4913253    |
| -0.1    | 1.19798114   | 4.69299658    | 10.16968229   | 17.25807961   |
| 0       | 1.22582011   | 4.75587441    | 10.24494698   | 17.34308797   |
| 0.1     | 1.25340643   | 4.81845727    | 10.32015025   | 17.42806187   |
| 1       | 1.49101990   | 5.36877806    | 10.99373734   | 18.19110002   |
| 10      | 3.21296474   | 9.8689192     | 17.20007166   | 25.52311499   |
| 20      | 4.48741520   | 13.54543209   | 22.89430780   | 32.78241704   |

It is evident that, for any set of constants \( \beta_L \) (\( L = 0, 1, \ldots, N \)) restricted by the condition

\[
\mathcal{W}[u_{\text{reg}}, u^{(1)}] = \sum_{L=0}^{N} \beta_L,
\]

one has

\[
\exp\left(2x^{N+1}/(N+1)\right) \mathcal{W}[u_{\text{reg}}, u^{(1)}] \sim \sum_{L=0}^{N} \beta_L \mathcal{E}_L. \tag{21}
\]

If, according to Eq. (11), this formal expansion has to coincide with that in (15), the constants \( \beta_L \) must be

\[
\beta_L = \frac{\Gamma(n + 1 + \delta_L)}{\left(2/(N+1)\right)^{n+\delta_L}} \gamma_{k_L}, \quad k_L = n(N + 1) + 1 + L, \tag{22}
\]

where the integer \( n \) can be chosen at will. Substitution of those values in (20) allows one to write the quantization condition (1) in the final form

\[
\sum_{L=0}^{N} \Gamma(n + 1 + \delta_L) ((N + 1)/2)^{L/(N+1)} \gamma_{k_L} = 0. \tag{23}
\]

We have used the last expression of the quantization condition to find the lowest eigenenergies of the anharmonic oscillator (2) for different values of the coupling parameter \( g \) and four different choices of \( N \). In the computation we have used a FORTRAN program with double precision. The results are shown in Tables 1 to 4.

The procedure presented above assumes the capability to compute the \( N + 1 \) coefficients \( \gamma_{k_L} \) (\( L = 0, 1, \ldots, N \)) by summation of the series in (16). We have not yet proved rigorously the convergence of such series, albeit extensive numerical explorations guarantee its convergence for sufficiently large \( k \), i.e., for \( n \), in Eq. (22), above a certain threshold which depends on the values of the coupling parameter \( g \) and on the energy. Moreover, those explorations show that, the larger \( n \) is taken, the faster becomes the convergence. Investigations tending to elucidate that question are currently in progress.

Besides the eigenenergies, our method determines also, in principle, the eigenfunctions. In the example considered, they are given by Eqs. (8) and (12).
Table 2. The four lowest eigenenergies of the Hamiltonian $H_2$, for $N = 5$ and several values of $g$.

| $g$  | $E_0$           | $E_1$           | $E_2$           | $E_3$           |
|------|-----------------|-----------------|-----------------|-----------------|
| –20  | 11.56630147     | 11.45854677     | 0.56494700      | 4.90729085      |
| –10  | 2.83782675      | 1.83075483      | 4.90946147      | 11.94279256     |
| –1   | 1.03205834      | 4.51533389      | 10.48697985     | 18.45464482     |
| –0.1 | 1.27308185      | 5.04058836      | 11.08762465     | 19.11537634     |
| 0    | 1.29884370      | 5.09787653      | 11.15431820     | 19.1880956      |
| 0.1  | 1.32441224      | 5.15495387      | 11.22099452     | 19.26224408     |
| 1    | 1.54626351      | 5.65933772      | 11.81996788     | 19.92310357     |
| 10   | 3.21711708      | 9.93229322      | 17.51589563     | 26.43450876     |
| 20   | 4.48623513      | 13.55329264     | 22.99231828     | 33.19354764     |

Table 3. The four lowest eigenenergies of the Hamiltonian $H_2$, for $N = 6$ and several values of $g$.

| $g$  | $E_0$           | $E_1$           | $E_2$           | $E_3$           |
|------|-----------------|-----------------|-----------------|-----------------|
| –20  | 9.36607177      | 9.13010587      | 2.01035459      | 7.97554684      |
| –10  | 2.24187409      | 0.87004433      | 6.12159677      | 14.16512836     |
| –1   | 1.11369983      | 4.84470202      | 11.28130698     | 19.99987959     |
| –0.1 | 1.33949907      | 5.33347217      | 11.83181276     | 20.59539322     |
| 0    | 1.36377971      | 5.38694202      | 11.89300908     | 20.66163760     |
| 0.1  | 1.38786579      | 5.44024556      | 11.95420520     | 20.7284945     |
| 1    | 1.59799050      | 5.91264617      | 12.50470842     | 21.32474109     |
| 10   | 3.22441873      | 10.00630419     | 17.83164730     | 27.2876498     |
| 20   | 4.48680192      | 13.57082013     | 23.11371663     | 33.63281210     |

Table 4. The four lowest eigenenergies of the oscillator $H_2$, for $N = 7$ and several values of $g$.

| $g$  | $E_0$           | $E_1$           | $E_2$           | $E_3$           |
|------|-----------------|-----------------|-----------------|-----------------|
| –20  | 7.97489149      | 7.59026706      | 3.05916112      | 10.19269195     |
| –10  | 1.8474624       | 0.17159144      | 7.07320094      | 15.87259291     |
| –1   | 1.18393765      | 5.12329191      | 11.93911991     | 21.26204013     |
| –0.1 | 1.39832030      | 5.58552094      | 12.45475050     | 21.81341553     |
| 0    | 1.42143888      | 5.63618503      | 12.51210199     | 21.87477520     |
| 0.1  | 1.44442247      | 5.68671175      | 12.56946066     | 21.93615283     |
| 1    | 1.64542730      | 6.13542477      | 13.08581400     | 22.48930458     |
| 10   | 3.23335919      | 10.08458888     | 18.13465608     | 28.04433038     |
| 20   | 4.48835326      | 13.59428939     | 23.24781210     | 34.07417453     |
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Nevertheless, although the series in (12) converges for all finite \( x \), it cannot be used safely for large values of \( x \), unless a considerable number of digits are maintained in the successive arithmetical operations. Certainly, the asymptotic expansion (13) can be used for sufficiently large values of \( x \) (above about 5 units). But it is not clear the advantage of this procedure over the conventional numerical integration of the Schrödinger equation, especially if one needs the normalized wave function for a large number of points.

To facilitate understanding the method, we have chosen above a very simple example: a one-dimensional anharmonic oscillator with only two terms in the potential. The procedure is equally applicable to isotropic D-dimensional oscillators with any number of integer powers of the radial variable in the potential and for any value of the D-dimensional “angular momentum”. It is also applicable, of course, to easier problems. Let us consider, for instance, three anharmonic oscillators algebraically solvable, namely, the Pöschl-Teller, the modified Pöschl-Teller and the Morse potentials. Their exact solution can be found in Ref. [14], whose notation we adopt. In what follows, we concentrate on obtaining, by our procedure, the eigenenergies of the bound states. But, since the reflection and transmission coefficients are trivially related to the connection factors, our method is also useful for calculating phase shifts.

In the case of the Pöschl-Teller potential

\[
V(x) = \frac{1}{2} V_0 \left( \frac{\kappa(\kappa - 1)}{\sin^2(\alpha x)} + \frac{\lambda(\lambda - 1)}{\cos^2(\alpha x)} \right), \quad V_0 = \frac{\hbar^2 \alpha^2}{m}, \quad \kappa, \lambda > 1, \quad (24)
\]

defined in the interval \( x \in [0, \pi/2] \), the Schrödinger equation can be written in the form

\[
y(y - 1)u'' + \left( \frac{1}{2} - y \right) u' + \frac{1}{4} \left( \frac{k^2}{\alpha^2} - \frac{\kappa(\kappa - 1)}{y} - \frac{\lambda(\lambda - 1)}{1 - y} \right) u = 0, \quad (25)
\]
in terms of a new variable

\[
y = \sin^2(\alpha x) \quad (26)
\]
and using, instead of the energy \( E \), the parameter

\[
k^2 = \frac{2mE}{\hbar^2}. \quad (27)
\]
Equation (25) can be written as a hypergeometric one by means of the change of function done in Ref. [14]. Then, it is immediate to write the connection factors and to obtain the quantization condition. Nevertheless, let us ignore this fact and try to apply our method directly to Eq. (25), to be solved between the two regular singular points \( y = 0 \) and \( y = 1 \). The solution regular at \( y = 0 \) can be written as a power series

\[
u_{\text{reg}}(y) = \sum_{n=0}^{\infty} a_n y^{n+\kappa/2}, \quad a_0 \neq 0, \quad (28)
\]
with coefficients given by the recurrence

\[
n \left( n - \frac{1}{2} + \kappa \right) a_n = \left( \left( n - 1 + \frac{\kappa}{2} \right) \left( 2n - \frac{5}{2} + \kappa \right) - \frac{1}{4} \left( \frac{k^2}{\alpha^2} + \kappa(\kappa - 1) - \lambda(\lambda - 1) \right) \right) a_{n-1}
\]
\[
- \left( \left( n - 2 + \frac{\kappa}{2} \right)^2 - \frac{k^2}{4\alpha^2} \right) a_{n-2}. \quad (29)
\]
The solution regular at \( y = 1 \) can be immediately written if one realizes that the differential equation (25) is invariant under the interchange
\[
\begin{aligned}
\{ y, \kappa \} &\leftrightarrow \{ 1 - y, \lambda \} \\
\end{aligned}
\]
and, therefore,
\[
u^{(1)}(y) = \sum_{n=0}^{\infty} b_n (1 - y)^{n + \lambda/2}, \quad b_0 \neq 0,
\]
with coefficients given now by
\[
n \left( n - \frac{1}{2} + \lambda \right) b_n = \left( \left( n - 1 + \frac{\lambda}{2} \right) \left( 2n - \frac{5}{2} + \lambda \right) - \frac{1}{4} \left( \frac{k^2}{\alpha^2} + \lambda(\lambda - 1) - \kappa(\kappa - 1) \right) \right) b_{n-1}
\]
\[
- \left( \left( n - 2 + \frac{\lambda}{2} \right) - \frac{k^2}{4\alpha^2} \right) b_{n-2}.
\]
Both series in Eqs. (28) and (30) are convergent for \( y \in (0, 1) \). We can, therefore, write a convergent (not merely formal, as in the problem considered before) expansion of the Wronskian. At the point \( y = 1/2 \), for instance, one has
\[
W[u_{\text{reg}}, u^{(1)}](y = 1/2) = -\frac{1}{2(\kappa + \lambda/2)} \left( \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{m=0}^{\infty} \frac{(m + \lambda/2)b_m}{2^{m-1}} \right) \right)
\]
\[
+ \left( \sum_{n=0}^{\infty} \frac{(n + \kappa/2)a_n}{2^{n-1}} \right) \left( \sum_{m=0}^{\infty} \frac{b_m}{2^m} \right) .
\]
Giving numerical values to \( \kappa \) and \( \lambda \), one can check that the Wronskian vanishes whenever
\[
\frac{k^2}{\alpha^2} = (\kappa + \lambda + 2n)^2, \quad n = 0, 1, 2, \ldots ,
\]
as it should be.

The modified Pöschl-Teller potential, defined for \( x \in (-\infty, +\infty) \), reads
\[
V(x) = -\frac{\hbar^2}{2m} \alpha^2 \frac{\lambda(\lambda - 1)}{\cosh^2(\alpha x)}, \quad \lambda > 1.
\]
Instead of the (negative) energy \( E \), we use the parameter
\[
\kappa^2 = \frac{2m(-E)}{\hbar^2}.
\]
Once again, the Schrödinger equation can be written as a hypergeometric one with adequate changes of variable and function. The change of variable used in Ref [14] maps the interval \((-\infty, +\infty)\) for the variable \( x \) onto \([1, +\infty)\) for the new variable. Here we prefer, however, to make a different change of variable, namely
\[
y = \frac{1}{\cosh^2(\alpha x)},
\]
in order to get the mentioned interval, where the differential equation has to be solved, mapped onto \([0, 1]\). The Schrödinger equation turns then into
\[
y^2(1 - y)u'' + y \left( 1 - \frac{3}{2}y \right) u' + \frac{1}{4} \left( -\frac{\kappa^2}{\alpha^2} + \lambda(\lambda - 1)y \right) u = 0.
\]
The regular solution at the regular singular point $y = 0$ can be written as a series
\[ u_{\text{reg}}(y) = \sum_{n=0}^{\infty} a_n y^{n+\kappa/(2\alpha)}, \quad a_0 \neq 0, \] (38)
with coefficients obtainable by means of
\[ n \left( n + \frac{\kappa}{\alpha} \right) a_n = \left( (n-1 + \frac{\kappa}{2\alpha}) \left( n - \frac{1}{2} + \frac{\kappa}{2\alpha} \right) - \frac{\lambda(\lambda-1)}{4} \right) a_{n-1}. \] (39)

Now we need to write the well behaved solution at $y = 1$. Two independent expansions in power series of $1 - y$ of the form,
\[ u(y) = \sum_{n=0}^{\infty} b_n (1 - y)^{n+\mu}, \quad b_0 \neq 0, \] (40)
with
\[ \mu = 0 \quad \text{or} \quad \mu = 1/2 \]
and coefficients obeying the recurrence
\[ (n+\mu) \left( n + \frac{1}{2} - \frac{\mu}{2} \right) b_n = \left( 2(n-1+\mu)^2 + \frac{\kappa^2}{4\alpha^2} - \frac{\lambda(\lambda-1)}{4} \right) b_{n-1} \]
\[ - \left( (n-2+\mu) \left( n - \frac{3}{2} + \mu \right) - \frac{\lambda(\lambda-1)}{4} \right) b_{n-2}, \] (41)
are physically acceptable. Those solutions with $\mu = 0$ and $\mu = 1/2$ correspond, respectively, to even and odd wavefunctions in the variable $x$. Choosing the point $y = 1/2$ to evaluate the Wronskian of $u_{\text{reg}}$ and each one of those functions, one has
\[ \mathcal{W}[u_{\text{reg}}, u](y = 1/2) = -\frac{1}{2^{\kappa/(2\alpha)+\mu}} \left( \sum_{n=0}^{\infty} \frac{a_n}{2^n} \right) \left( \sum_{m=0}^{\infty} \frac{(m+\mu)b_m}{2^{m-1}} \right) \]
\[ + \left( \sum_{n=0}^{\infty} \frac{(n+\kappa/(2\alpha))a_n}{2^{n-1}} \right) \left( \sum_{m=0}^{\infty} \frac{b_m}{2^m} \right). \] (42)

It can be checked numerically that the right hand side of (42) becomes zero if,
for $n = 0, 1, 2, \ldots$, \quad $0 < \frac{\kappa}{\alpha} = \left\{ \begin{array}{ll} \lambda - 1 - 2n & \text{for even states}, \\ \lambda - 2 - 2n & \text{for odd states}. \end{array} \right.$

We arrive finally to the case of the Morse potential
\[ V(r) = D(\exp(-2\alpha x) - 2 \exp(-\alpha x)), \quad x = (r - r_0)/r_0, \quad \alpha > 0, \] (43)
exactly solvable for angular momentum $l = 0$. By introducing, as in Ref. [14], a new variable
\[ y = \frac{2\gamma}{\alpha} \exp(-\alpha x), \] (44)
and denoting
\[ \beta^2 = -\frac{2mE r_0^2}{\hbar^2}, \quad \gamma^2 = -\frac{2mD r_0^2}{\hbar^2}, \quad \beta, \gamma > 0, \] (45)
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The Schrödinger equation becomes

\[ y^2 u'' + yu' + \left( -\frac{\beta^2}{\alpha^2} + \frac{\gamma}{\alpha} - \frac{1}{4} y^2 \right) u = 0. \]  

(46)

This equation presents a regular singular point at the origin and an irregular one at infinity. The physical solution, however, needs to be defined only between \( y = 0 \), corresponding to \( x \to \infty \) \((r \to \infty)\), and \( y = y_0 \equiv (2\gamma/\alpha) \exp(\alpha) \), corresponding to \( x = -1 \) \((r = 0)\). Such physical solution must be regular at \( y = 0 \) and become zero at \( y = y_0 \). The solution regular at \( y = 0 \) can be given as a series

\[ u_{\text{reg}}(y) = \sum_{n=0}^{\infty} a_n y^{n+\beta/\alpha}, \quad a_0 \neq 0, \]  

(47)

with coefficients obeying

\[ n(n + 2\beta/\alpha) a_n = -\left( \frac{\gamma}{\alpha} \right) a_{n-1} + \left( \frac{1}{4} \right) a_{n-2}. \]  

(48)

The other extreme of the interval of definition of the wave function, \( y = y_0 \), is an ordinary point. There are two independent solutions of the differential equation, both finite at \( y = y_0 \). But only the linear combination of them becoming zero at that point is physically acceptable. Let us call it \( u^{(1)} \). Now, following our procedure, we should impose the cancellation of the Wronskian of \( u_{\text{reg}} \) and \( u^{(1)} \) at any point of \([0, y_0]\). If we choose \( y = y_0 \), it becomes

\[ \mathcal{W}[u_{\text{reg}}, u^{(1)}](y = y_0) = u_{\text{reg}}(y_0) u^{(1)'}(y_0), \]  

(49)

and, since \( u^{(1)'} \) cannot vanish at \( y = y_0 \), the quantization condition reads

\[ u_{\text{reg}}(y_0) = 0, \]  

(50)

an expression that could have been obtained trivially, without having recourse to our method. It is not difficult to see that, if one takes \( a_0 = 1 \) in (47), one has

\[ u_{\text{reg}}(y) = y^{\beta/\alpha} \exp(-y/2) \, _1F_1 \left( \frac{1}{2} + \frac{\beta \alpha}{\alpha} - \frac{\gamma}{\alpha}, 1 + 2 \frac{\beta}{\alpha}; y \right), \]  

(51)

and the quantization condition (50) coincides with that given in Ref. [14].

Unlike what happened in the case of potential (2), the numerical convergence of the power series giving the solutions of the Schrödinger equation in the three last examples is rapid enough as to guarantee an accurate computation of the eigenfunctions. For instance, in the case of the Pöschl-Teller potential, the series in (28) can be used for \( y \leq 1/2 \) and that in (30) for \( y \geq 1/2 \), the coefficients \( a_0 \) and \( b_0 \) being determined by continuity at \( y = 1/2 \) and normalization in the interval \( y \in [0, 1] \).

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