Obtaining One-loop Gravity Amplitudes Using Spurious Singularities

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Abstract

The decomposition of a one-loop scattering amplitude into elementary functions with rational coefficients introduces spurious singularities which afflict individual coefficients but cancel in the complete amplitude. These cancellations create a web of interactions between the various terms. We explore the extent to which entire one-loop amplitudes can be determined from these relationships starting with a relatively small input of initial information, typically the coefficients of the scalar integral functions as these are readily determined. In the context of one-loop gravity amplitudes, of which relatively little is known, we find that some amplitudes with a small number of legs can be completely determined from their box coefficients. For increasing numbers of legs, ambiguities appear which can be determined from the physical singularity structure of the amplitude. We illustrate this with the four-point and $\mathcal{N} = 1,4$ five-point (super)gravity one-loop amplitudes.

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I. INTRODUCTION

In general, the one-loop amplitudes of a quantum field theory can be expressed as a sum over Feynman diagrams

\[ A_{n}^{1\text{-loop}} = \sum_{a \in \mathcal{L}} D_{a} \]

where the summation is over all diagrams constructible from the vertices and propagators of the theory. In a gauge theory the number of diagrams grows exponentially with the number of external legs. The one-loop diagrams will involve the integral over a loop momentum

\[ D_{a} = \int d^{D} \ell \frac{P(\ell)}{\prod(\ell - K_{i})^{2}} \]

where the \((\ell - K_{i})^{-2}\) are the propagators attached to the loop and \(P(\ell)\) is a polynomial of Lorentz invariants constructed by contracting the loop momentum with the momenta and polarisations of the external states. If \(r\) is the number of propagators in the loop, in a Yang–Mills theory \(P(\ell)\) is a polynomial of degree \(r\) and in a gravity theory \(P(\ell)\) is a polynomial of degree \(2r\). The integrals are regularised by calculating with \(D = 4 - 2\epsilon\).

Powerful and well-established integral reduction methods [1] allow an \(r\)-point, rank-\(p\) one-loop integral to be expressed as a sum of \(r - 1\) point integrals, to \(O(\epsilon)\),

\[ I_{r}[P^{p}[\ell]] = \sum d_{i} I_{r-1}^{(i)}[P^{p-1}[\ell]], \quad r > 4. \]  

The set of diagrams in the composition is those where one propagator of the parent is collapsed. For \(r = 3, 4\) the decomposition is

\[ I_{r}[P^{p}[\ell]] = I_{r}[1] \sum d_{i} I_{r-1}^{(i)}[P^{p-1}[\ell]] \]

and for \(r = 2\),

\[ I_{2}[P^{p}[\ell]] = I_{2}[1] + R \]

where \(R\) is a rational function of the Lorentz invariants. The end result is that any amplitude in a massless theory can be expressed as

\[ A_{n}^{1\text{-loop}} = \sum_{i \in \mathcal{C}} a_{i} I_{4}^{i} + \sum_{j \in \mathcal{D}} b_{j} I_{3}^{j} + \sum_{k \in \mathcal{E}} c_{k} I_{2}^{k} + R_{n} + O(\epsilon), \]

where the \(I_{r}\) are \(r\)-point scalar integral functions and the \(a_{i}\) etc. are rational coefficients. \(R_{n}\) is a purely rational term.

Dividing the amplitude into integral functions with rational coefficients has been very fruitful: a range of specialised techniques have been devised to determine the rational coefficients, many based on unitarity techniques rather than Feynman diagrams [2,13]. Progress has been made both via the two-particle cuts and using generalisations of unitarity [5] where, for example, triple [8, 10–12] and quadruple cuts [6] are utilised to identify the triangle and box coefficients respectively.

However there is a cost in the division: the rational coefficients do not have the symmetries or singularity structure of the full amplitude. In particular they may acquire “spurious singularities”. In general the Passarino-Veltman reduction coefficients \(d_{i}\) of (1.4) contain a factor of \(\Delta^{-1}\) where \(\Delta\) is the Gram determinant of \(I_{r}\). The vanishing of \(\Delta\) does not necessarily correspond to any physical singularity of the amplitude. Such singularities arising from the
reduction must cancel between the various contributions to the complete amplitude. In this article we explore the web of cancellations which link the “cut-constructible” parts of the amplitude to the rational terms. This has been explored in Yang-Mills theories as part of the bootstrapping process \cite{15–17} where the spurious singularities are combined within integral functions. Here, the spurious singularities arising in the “cut-constructible” parts of the amplitude and cancelled by rational terms constructed from the full integral coefficients and modifications of the integral functions. In gravity we find that the spurious singularities occur with higher powers which consequentially place stronger constraints on the rational terms. We also find in the $\mathcal{N} = 1$ case, it is not practical to simultaneously combine all such singularities into integral functions leaving the coefficients unchanged but instead adopt an approach where we identify the different singularities with a single coefficient and cancel these iteratively in a specified order.

This is useful since supergravity amplitudes are relatively difficult to calculate with only a small number of one-loop helicity amplitudes available for study. For $\mathcal{N} = 8$ supergravity the one-loop structure is relatively well understood: the expansion is purely in terms of scalar box integrals as demonstrated in the explicit calculations of the four-point MHV amplitude \cite{19}, the $n$-point MHV amplitude \cite{20} and the six and seven-point NMHV amplitudes \cite{21,22}. For $\mathcal{N} = 6$ the $n$-point MHV amplitude is also known \cite{23}. For $\mathcal{N} < 6$ very few amplitudes are known. All the four-points amplitudes have been calculated \cite{24} but only the five-point MHV $\mathcal{N} = 4$ is available. For pure gravity, the entirely rational “all-plus” $n$-point amplitude is known \cite{20}, the four, five and six-point “one-minus” and the four-point MHV are known \cite{24–26}.

In this article, we first discuss the physical singularities expected in amplitudes before illustrating the spurious singularities with examples in Yang-Mills amplitudes. We then examine one-loop graviton scattering amplitudes first with the known examples of scalar four-point and the five-point $\mathcal{N} = 4$ amplitudes. Finally we use spurious singularity cancellation and physical factorisation to derive the previously unknown $\mathcal{N} = 1$ five-point amplitude. Spurious singularities have been used previously to constrain and determine coefficients in amplitudes in special cases in Yang-Mills theories \cite{11,27}.

### II. PHYSICAL SINGULARITIES OF SCATTERING AMPLITUDES

Physical singularities correspond to physical factorisations. The factorisation of one-loop massless amplitudes is described in ref. \cite{28},

$$
A_{n}^{\text{1-loop}}(k_{1}, \ldots, k_{n-1}; K) \rightarrow \sum_{\lambda = \pm} A_{n+1}^{\text{1-loop}}(k_{1}, \ldots, k_{n}, K^{\lambda}) \frac{i}{K^{2}} A_{n+1}^{\text{tree}}((-K)^{-\lambda}, k_{n+1}, \ldots, k_{n})
$$

$$
+ A_{n+1}^{\text{tree}}(k_{1}, \ldots, k_{n}; K^{\lambda}) \frac{i}{K^{2}} A_{n+1}^{\text{1-loop}}((-K)^{-\lambda}, k_{n+1}, \ldots, k_{n})
$$

$$
+ A_{n+1}^{\text{tree}}(k_{1}, \ldots, k_{n}; K^{\lambda}) \frac{i}{K^{2}} A_{n+1}^{\text{tree}}((-K)^{-\lambda}, k_{n+1}, \ldots, k_{n}) F_{n}(K^{2}; k_{1}, \ldots, k_{n})
$$

(II.1)

where the one-loop ‘factorisation function’ $F_{n}$ is helicity-independent.

When the momentum $K$ consists of just two external momenta, $K = k_{a} + k_{b}$, the limit is subtle because the three-point tree amplitude vanishes. In a Yang–Mills theory the amplitude has collinear singularities of the form $\langle a b \rangle^{-1}$ and/or $[a b]^{-1}$ rather than $s_{ab}^{-1}$. Note that $| \langle a b \rangle | = | [a b] | = | s_{ab} |^{1/2}$. Gravity amplitudes are not singular in the collinear limit, but take
a form that is specified in terms of amplitudes with one less external leg. If \( k_a \rightarrow z K \)
and \( k_b \rightarrow (1 - z) K \),
\[
M_n(\cdots, a^{h_a}, b^{h_b}) \longrightarrow \sum_{K'} \text{Split}_{-h'}(z, a^{h_a}, b^{h_b}) M_{n-1}(\cdots, K^{h'})
\] (II.2)

where the \( h \)'s denote the various helicities of the gravitons. The “splitting functions” are
\[
\text{Split}_+(z, a^+, b^+) = 0,
\] (II.3)
\[
\text{Split}_-(z, a^+, b^+) = -\frac{[a b]}{z(1 - z) \langle a b \rangle},
\] (II.4)
\[
\text{Split}_+(z, a^-, b^+) = -\frac{z^3 [a b]}{(1 - z) \langle a b \rangle}.
\] (II.5)

As usual, we are using a spinor helicity formalism with the usual spinor products \( \langle j l \rangle = \langle j^+|l^- \rangle = \bar{u}_-(k_j) u_+(k_l) \) and \([j l] \equiv (j^+|l^-) = \bar{u}_+(k_j) u_-(k_l)\), and where \([i|K_{abc}|j^+]\) denotes \( \langle i^+|K_{abc}|j^+ \rangle \) with \( K_{abc} = k^\mu_a + k^\mu_b + k^\mu_c \) etc. Also \( s_{ab} = (k_a + k_b)^2 \), \( t_{abc} = (k_a + k_b + k_c)^2 \), etc.

Gravity amplitudes also have soft-limit singularities as \( k_n \rightarrow 0 \),
\[
M_n(\cdots, n - 1, n^h) \longrightarrow \text{Soft}(n^h) M_{n-1}(\cdots, n - 1)
\] (II.6)

where the “soft factor” is given by
\[
\text{Soft}(n^+) = -\frac{1}{\langle 1 n \rangle \langle n n - 1 \rangle} \sum_{j=2}^{n-2} \frac{\langle 1 j \rangle \langle j n - 1 \rangle [j n]}{\langle j n \rangle}.
\] (II.7)

The factorisation arguments used above have all implicitly involved real momenta. There is considerably more information available if we consider complex momenta, but the factorisation properties of the amplitudes are not so well understood. Also, double complex poles arise in some amplitudes, for example in amplitudes with a single negative helicity leg, both in Yang–Mills theory and gravity. These double poles are understood to arise from diagrams of the form illustrated in fig. 1.

Here there is one \( \langle a b \rangle^{-1} \) factor from the loop integral associated with the all-plus triangle and a second from the propagator joining the triangle to the tree amplitude. If the tree is non-vanishing these give rise to contributions of the form
\[
\frac{1}{\langle a b \rangle^2} \times \cdots
\] (II.8)

---

The normalisation of the physical amplitude \( \mathcal{M}^{\text{tree}} = (\kappa/2)^{n-2} \mathcal{M}^{\text{tree}} \), \( \mathcal{M}^{\text{1-loop}} = (\kappa/2)^n \mathcal{M}^{\text{1-loop}} \).
which are double poles, but only in complex momentum space. Explicitly, the three-point 1-loop all-plus amplitudes for Yang–Mills theory and gravity are proportional to \[26, 30, 31\]

\[
A_{3}^{(1)Y-M} = \frac{[ab][aK][Kb]}{s_{ab}}, \quad M_{3}^{(1)grav} = \frac{[ab][aK]^2[Kb]^2}{s_{ab}},
\]

(II.9) respectively, which vanish in the \([ab] \to 0\) limit and diverge in the \([ab] \to 0\) limit.

These complex double poles are not present in all amplitudes. For example, in the case of five-point MHV amplitudes the tree amplitudes in fig. 4 have only one positive helicity leg and so vanish. We can therefore insist that there are no higher-order complex poles in \((ab)\) in the five-point MHV amplitudes we consider. Similar arguments preclude any other complex higher-order poles in these amplitudes.

III. EXAMPLES OF SPURIOUS AND MULTIPLE SINGULARITIES

In this section we illustrate the types of unphysical singularities that arise when we perform a reduction procedure and describe how they cancel in the full amplitude.

As an example involving spurious singularities we consider one of the six-point NMHV amplitudes in Yang–Mills. (As usual we organise the amplitudes according to external helicity. Amplitudes with exactly two-negative helicity legs are termed MHV, while those with exactly three negative helicity legs are next-to-MHV, or NMHV.) At six points there are three independent colour-ordered NMHV amplitudes: \(A_{6}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)\), \(A_{6}(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)\) and \(A_{6}(1^-, 2^+, 3^-, 4^-, 5^-, 6^+)\). Additionally, we organise according to the matter content circulating in the loop, it being most convenient to take a supersymmetric decomposition and consider three components corresponding to a \(\mathcal{N} = 4\) multiplet, a \(\mathcal{N} = 1\) matter multiplet, and a scalar particle circulating in the loop. (See ref. 32 for an overview of this amplitude.) The amplitude \(A_{6}^{N=1, matter}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)\) is given in ref. 33 as

\[
A_{6}^{N=1, matter}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{A_{\text{tree}}}{2} \left(I_{2}(s_{61}) + I_{2}(s_{34})\right)
- \frac{i}{2} \left[c_{1} \frac{L_{0}[t_{345}/s_{61}]}{s_{61}} + c_{2} \frac{L_{0}[t_{234}/s_{34}]}{s_{34}} + c_{3} \frac{L_{0}[t_{234}/s_{61}]}{s_{61}} + c_{4} \frac{L_{0}[t_{345}/s_{34}]}{s_{34}}\right]
\]

(III.1)

where the \(c_i\) are

\[
c_{1} = \frac{[6][3][2][6][k_{2}, K][K][3]}{[2][5][6][1][1][2][3][4][4][5][K][K][1]}; \quad K = K_{345};
\]

\[
c_{2} = \frac{[6][3][2][6][k_{2}, K][K][1]}{[2][5][23][3][4][5][6][1][1][2][3][4][4][5][K][K][1]}; \quad K = K_{234};
\]

\[
c_{3} = \frac{[6][3][2][6][k_{2}, K][K][1]}{[2][5][23][3][4][5][6][1][1][2][3][4][4][5][K][K][1]}; \quad K = K_{234};
\]

\[
c_{4} = \frac{[6][3][2][6][k_{2}, K][K][3]}{[2][5][6][1][1][2][3][4][4][5][K][K][1]}; \quad K = K_{345}.
\]

We have omitted an overall dimensional regularisation factor \((\mu^2)^{r_{T}}c_{T}\) from the amplitude, where

\[
c_{T} = \frac{r_{T}}{(4\pi)^{2-\epsilon}}, \quad r_{T} = \frac{\Gamma(1+\epsilon)^{2}(1-\epsilon)}{\Gamma(1-2\epsilon)}.
\]

(III.3)
The function $L_0$ is defined by
\[
L_0[r] = \frac{\ln(r)}{1 - r}
\]
so that
\[
\frac{L_0[s/s']}{s'} = \frac{1}{s - s'} (I_2(s) - I_2(s'))
\]
where the scalar bubble function is
\[
I_2(s) = \frac{1}{\epsilon} + 2 - \ln(-s) + O(\epsilon).
\]

The coefficient of $I_2(s_{34})$ is thus
\[
\frac{1}{2} A_{\text{tree}} - i \frac{1}{2 t_{34} - s_{34}} \frac{[4|K_{234}|1]^2[4|[K_{234}, k_2]K_{234}|1]}{[2|K_{234}|5][23][34][56](61)t_{234}} - i \frac{1}{2 t_{345} - s_{34}} \frac{[6|K_{345}|3]^2[6|[K_{345}, k_5]K_{345}|3]}{[2|K_{345}|5][61][12][34][45]t_{345}}
\]

This coefficient contains a variety of singularities. There are physical singularities of the forms
\[
\frac{1}{K^2}, \frac{1}{\langle a b \rangle}, \frac{1}{[a b]}.
\]

There are also spurious singularities:
\[
\frac{1}{(t_{234} - s_{34})}, \frac{1}{(t_{345} - s_{34})}, \frac{1}{[2|K_{34}|5]}
\]

which do not correspond to any physical singularity so must vanish in the entire amplitude.

The first of these, $(t_{234} - s_{34})^{-1}$ arises as a Gram determinant in the reduction of the two-mass tensor triangle integral with massless leg $k_2$ and a massive leg $k_3 + k_4$. This becomes a singularity when $k_2 \cdot (k_3 + k_4) = 0$. For real momenta this occurs, provided $k_3$ and $k_4$ have opposite energy in a two-parameter subspace which may be characterised by $k_2 = \alpha(\lambda_3 + e^{i\theta}\lambda_4)(\lambda_3 - e^{-i\theta}\lambda_4)$. At this singularity $s_{23} - t_{234} \rightarrow 0$ so that $\ln(-s_{23}) \rightarrow \ln(-t_{234})$. The singularity is present in the coefficients $c_2$ of (III.2) which contributes to the coefficient of both $I_2(s_{34})$ and $I_2(t_{234})$. It is the cancellation between the two contributions when the integral functions degenerate into each other that leaves the full amplitude finite. The form of the amplitude (III.1) makes this simple to see since, as $r \rightarrow 1$,
\[
L_0[r] = \frac{\ln(r)}{1 - r} \rightarrow -1 - \frac{1}{2}(1 - r) + O((1 - r)^2)
\]
which is finite.

The final singularity in this amplitude, $[2|K_{34}|5]^{-1}$, occurs when $K_{34}$ is co-planar with $k_2$ and $k_5$, i.e. $K_{34} = \alpha k_2 + \beta k_5$. This singularity corresponds to the Gram determinant of a two-mass-easy scalar box with massless legs $k_2$ and $k_5$ together with massive legs $K_{34}$ and $K_{61}$. At this point,
\[
t_{234}t_{345} - s_{34}s_{61} \rightarrow 0
\]

or equivalently
\[
\frac{s_{34}}{t_{234}} \rightarrow \frac{t_{561}}{s_{61}}
\]
so the logarithms in \( L_0[t_{34}/s_{34}] \) and \( L_0[t_{56}/s_{56}] \) may cancel. In this case we are seeing a cancellation between all four of the bubble integral functions

\[
c_1 I_2(s_{34}) + c_2 I_2(t_{23}) + c_3 I_2(s_{61}) + c_4 I_2(t_{345}) \to 0. \tag{III.13}
\]

The other type of singularity we consider are those that appear at the same phase-space points as the physical singularities but are of higher order. These do not correspond to any singularity arising in any Feynman diagram. To distinguish this type of singularity from the spurious singularities discussed above, we refer to them as “higher-order physical” poles.

To illustrate how these arise and cancel we can consider one-loop \( n \)-point MHV Yang–Mills amplitudes. The boxes depicted in fig. 2 have non-vanishing coefficients. Denoting the two negative helicities as \( m_1 \) and \( m_2 \) and considering the box with two massless legs \( a \) and \( b \), the leading colour coefficients of the box integrals are \([3, 4, 34, 35]\)

\[
a^{N=4} = (st - M_2^2 M_4^2) A_{\text{tree}},
\]

\[
a^{N=1} = (st - M_2^2 M_3^2) A_{\text{tree}} \times B_{ab}^{m_1 m_2},
\]

\[
a^{[0]} = (st - M_2^2 M_3^2) A_{\text{tree}} \times (B_{ab}^{m_1 m_2})^2,
\]

where

\[
B_{ab}^{m_1 m_2} = \frac{\langle a m_1 \rangle \langle a m_2 \rangle \langle b m_1 \rangle \langle b m_2 \rangle}{\langle a b \rangle^2 \langle m_1 m_2 \rangle^2}, \tag{III.15}
\]

and

\[
s = (k_a + K_M)^2, \quad M_2^2 = K_M^2, \quad t = (K_M + k_b)^2, \quad M_4^2 = K_N^2. \tag{III.16}
\]

These coefficients contain \( (a b)^{-n} \) singularities. Near these singularities the box integral functions can be expanded as

\[
\frac{(st - M_2^2 M_4^2)}{2\Gamma_s} I_4 = \sum_{s_i = s, t, M_2^2, M_4^2} \left( \pm \frac{1}{c^2} (-s_i)^{-\epsilon} + s_{ab} P_i \ln(-s_i) \right) + s_{ab}^2 P_R, \tag{III.17}
\]

where the \( P_i \) and \( P_R \) are rational functions of the momentum invariants and specifically are polynomials in \( s_{ab} \). Thus, as we approach the singularity these box contributions \textit{degenerate},
the first term of the sum cancels with the one- and two-mass triangle contributions as discussed in more detail in the next section, leaving logarithmic and rational descendants of the box. The logarithmic descendants combine with the logarithms in the bubble contributions to cancel the higher-order physical poles in their coefficients. Similarly, any higher-order physical poles in the rational descendants cancel against the rational piece of the amplitude, \( R_n \).

**IV. ONE-LOOP GRAVITY AMPLITUDES**

A one-loop graviton scattering amplitude can receive contributions from a range of particle types circulating in the loop. It is convenient to perform a supersymmetric decomposition and compute the contributions from entire matter supermultiplets circulating in the loop. The specific particle contributions are then simply obtained as linear combinations of the supersymmetric contributions:

\[
M^{[1/2]} = M^{N=1} - 2M^{[0]}
\]

\[
M^{[1]} = M^{N=4} - 4M^{N=1} + 2M^{[0]}
\]

\[
M^{[3/2]} = M^{N=6} - 6M^{N=4} + 9M^{N=1} - 2M^{[0]}
\]

\[
M^{[2]} = M^{N=8} - 8M^{N=6} + 20M^{N=4} - 16M^{N=1} + 2M^{[0]}
\]

(IV.1)

where the superscript \([s]\) denotes a particle of spin \(s\) circulating in the loop. We will sometimes refer to the contribution from a real scalar, \(M^{[0]}\), as \(M^{N=0}\).

In general, amplitudes with greater supersymmetry have simpler structure and are easier to compute. The \(N=8\) one-loop amplitudes have a particularly simple form consisting only of box-functions [20–22, 36], a feature shared with \(N=4\) Yang–Mills and related to the possible finiteness of maximal supergravity. This simplicity arises from cancellations between diagrams that reduce the effective degree, \(d_{\text{eff}}\), of the loop momentum polynomial \(P(\ell)\) in (I.2). The traditional expectation for \(m\)-point supergravity amplitudes is that cancellation between particle types within a supermultiplet reduces \(d_{\text{eff}}\) from \(2m\) to \(2m - r\), where \(r\) depends upon the degree of supersymmetry.

For \(N=8\) supergravity, that \(r=8\) is manifest term by term within the “string-based rules” method [24, 37]. However the no-triangle hypothesis indicates that further cancellations arise, resulting in \(d_{\text{eff}} = m - 4\). This suggests a degree of \(m+4\) (rather than \(2m\)) for pure gravity, reduced by a further 8 for \(N=8\) supersymmetry. For \(N=6\) supergravity \(d_{\text{eff}} = m - 3\) and for \(N=4\) supergravity \(d_{\text{eff}} = m\) [23, 38]. In the former case the amplitudes have box and triangle contributions only, while in the latter we also have bubbles and purely rational terms. The rational terms are not four dimensional cut-constructible.

It is conventional to express the \(N=6\) and \(N=4\) amplitudes in terms of a basis involving truncated box functions and three-mass triangle functions. While this packaging can be motivated by IR arguments, singularity arguments provide an alternative explanation.

In a basis involving box and triangle functions the \(N=6\) one-loop amplitudes take the form

\[
M^{\text{1-loop},N=6}_n = \sum_{i \in C} a_i I^i_4 + \sum_{j \in T} b_j I^j_3
\]

(IV.2)

For the MHV configuration, the sum of boxes is fairly restrictive, consisting only of boxes with two massive and two massless legs where the massive legs are non-adjacent. The two negative helicities must lie on the opposite clusters of legs denoted \(M\) and \(N\) in fig. 2 (We
also include in the sum the degenerate one-mass case where one of the negative helicity legs is on its own.) The box-coefficient is given by

\[
\frac{(-1)^n}{8} \left( \frac{1}{2} \langle 1 2 \rangle^8 \left( \frac{1}{a b^2 (1 2)^2} \right) \right) h(a, M, b) h(b, N, a) \text{tr}^2[a M b N] \tag{IV.3}
\]

where \( \text{tr}[a M b N] = \text{tr}[k_a k_M k_b k_N] \). The \( h(a, M, b) \) are the “half-soft” functions of ref. [20],

\[
h(a, \{1, 2, \ldots, n\}, b) \equiv \frac{\langle 1 2 \rangle \langle 3 4 \rangle \cdots \langle n - 1, n \rangle \langle a 1 \rangle \langle a 2 \rangle \langle a 3 \rangle \cdots \langle a n \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle n 1 \rangle} + \mathcal{P}(2, 3, \ldots, n). \tag{IV.4}
\]

As we can see this coefficient contains a higher-order physical singularity, \( \langle a b \rangle^{-2} \). Since this is unphysical it must cancel within the amplitude. Using the explicit forms of the one- and two-mass triangle integral functions and their coefficients we can repackage the expansion:

\[
M_{4}^{1\text{-loop}, \mathcal{N}=6} = \sum_{i \in C} a_i I^4_i + \sum_{j \in D} b_j I^3_j \tag{IV.5}
\]

where \( I^4_i, \text{trunc} \) is of order \( s_{ab} \) near \( s_{ab} = 0 \). The explicit form of the one and “two-mass easy” truncated boxes is given in appendix B.

This result is well-known and is normally interpreted as a cancellation of spurious IR singularities [4, 9, 35]. Here we wish to note that the truncated boxes can be obtained by requiring the cancellation of higher-order (non-IR) singularities and as such, this is an example where the box-coefficients contain sufficient information to reconstruct the entire amplitude from singularity considerations.

V. FOUR-POINT GRAVITY AMPLITUDES

In this section we discuss the simple example of four-point gravity MHV amplitudes, where the entire amplitude can be constructed from the box-coefficients. The MHV amplitude \( M_{4}(1^{-}, 2^{-}, 3^{+}, 4^{+}) \) contains all three four-point boxes for the \( \mathcal{N} = 8 \) multiplet but for \( \mathcal{N} < 8 \) the \( s = s_{12} \) unitarity cut vanishes identically and we deduce the amplitude has the form

\[
a_4 I^4_i, \text{trunc}(s_{13}, s_{23}) + c_1 I^2_2(s_{23}) + c_2 I^2_2(s_{13}) + R \tag{V.1}
\]

with only the box with ordering of legs 1324 appearing. As discussed previously, we have combined the box with triangle contributions \( I^3_3(s_{23}) \) and \( I^3_3(s_{13}) \) into a truncated box functions. The coefficient of the box could easily be derived using quadruple cuts [6] and is

\[
a_4 = \left( \frac{s_{12} s_{23}}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^2 \times \left( \frac{s_{23} s_{13}}{s_{12}^2} \right)^4 \tag{V.2}
\]
with \( A = 0, 1, 2, 4 \) for the \( \mathcal{N} = 8, 4, 1 \) and 0 multiplets respectively. These amplitudes have an increasingly high order singularity when \( s_{12} \to 0 \) if we allow ourselves to consider complex momenta where \([1 2] \to 0 \) but \([1 2] \neq 0 \).

Setting \( s = s_{12}, t = s_{23} \) and \( u = s_{13} \), and suppressing a factor of \( ic \), the amplitude in the \( \mathcal{N} = 0 \) case (with a real scalar in the loop) takes the form

\[
\left( \frac{st \langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^2 \left( -\frac{t^3 u^3}{s^8} \left( \ln^2(t/u) + \pi^2 \right) + a(t, u) \left( -\frac{1}{\epsilon} - 2 + \ln(-t) \right) + a'(t, u) \left( -\frac{1}{\epsilon} - 2 + \ln(-u) \right) + b(t, u) \right) \tag{V.3}
\]

where, from the symmetry in the amplitude, \( a'(t, u) = a(u, t) \) and \( b(u, t) \) must be symmetric in \((u, t)\). The \( \epsilon^{-1} \) infra-red singularity vanishes in this amplitude \([39]\) so \( a'(t, u) = -a(t, u) \) and the amplitude takes the form

\[
\left( \frac{st \langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^2 \left( -\frac{t^3 u^3}{s^8} \left( \ln^2(t/u) + \pi^2 \right) + a(t, u) \ln(t/u) + b(t, u) \right) \tag{V.4}
\]

with \( a(t, u) = -a(u, t) \).

We are interested in the behaviour of the amplitude as \([1 2] \to 0 \). As \( s_{12} \to 0 \) momentum conservation requires \( s_{34} \to 0 \). If we utilise a shift to approach the singular point \([40]\),

\[
\tilde{\lambda}_1 \to \lambda_1 - z \lambda_3, \quad \lambda_3 \to \lambda_3 + z \lambda_1, \tag{V.5}
\]

then \([3 4]\) is unshifted and \( \langle 3 4 \rangle \) must vanish along with \([1 2]\). The first factor in \((V.4)\) is finite in this limit and the leading singularity is order \([1 2]^{-8}\).

If we expand about \( s = 0 \), using \( s + u + t = 0 \) then

\[
\ln(t/u)^2 = -\pi^2 + 2i\pi \frac{s}{u} + \cdots. \tag{V.6}
\]

The first term can be cancelled by \( b(u, t) \), while the second must be cancelled by the bubble contributions.

Using the expansion,

\[
\ln(t/u) = i\pi + \frac{s}{u} + \cdots \tag{V.7}
\]

we see that \( a(t, u) \) must contain a factor of \( s^{-7} \) and must be anti-symmetric in \( t \) and \( u \). Since it is rational it must therefore contain a factor of \((t - u)\). We can thus take

\[
a(t, u) = \frac{(t - u)(\alpha u^4 + \beta u^3 t + \gamma u^2 t^2 + \beta u t^3 + \alpha t^4)}{s^7} \tag{V.8}
\]

Requiring the cancellation of the \( s^{-7} \) singularity imposes one constraint on the parameters:

\[
2\alpha - 2\beta + \gamma - 1 = 0. \tag{V.9}
\]

If this constraint is satisfied, the cancellation of the \( i\pi s^{-6} \) singularity is automatic. There are four further constraints arising from demanding the cancellation of the \( i\pi s^{-5} \) through \( i\pi s^{-2} \)
singularities. Fortunately only two of these constraints are independent and we have a well posed problem. Solving the system of constraints gives

\[ \alpha = \frac{1}{30}, \quad \beta = \frac{9}{30}, \quad \gamma = \frac{46}{30}. \]  

(C.10)

Cancellations involving \( a(t, u) \) remove all the higher-order poles that have an \( i\pi \) factor, but singularities with no \( i\pi \) factor remain. We therefore add the most general form for \( b(t, u) \) consistent with the symmetries discussed above:

\[ b(t, u) = \frac{\delta u^4 + \eta u^3 t + \zeta u^2 t^3 + \eta u t^3 + \delta t^4}{s^6} \]  

(C.11)

Requiring that the \( s^{-6} \) through \( s^{-2} \) singularities cancel again gives just three independent constraints. Solving these gives

\[ \delta = \frac{2}{180}, \quad \eta = \frac{23}{180}, \quad \zeta = \frac{222}{180}. \]  

(C.12)

The full one-loop amplitude is then

\[
M_{n=4}^{(1^-), (2^-, 3^+, 4^+, \ldots, n^+)} = i\Gamma \left( \frac{st \langle 1 2 \rangle^4}{(1 2) \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^2 \times \left( -\frac{t^3 u^3}{s^8} (\ln^2(t/u) + \pi^2) + \frac{(t - u)(t^4 + 9ut^3 + 46u^2t^2 + 9u^3t + u^4)}{30s^7} \ln(t/u) + \frac{2t^4 + 23ut^3 + 222u^2t^2 + 23u^3t + 2u^4}{180s^6} \right)
\]  

(V.13)

This exactly matches the amplitude previously calculated using the string-based rules for gravity [24, 37]. The corresponding analyses for the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 4 \) amplitudes are progressively simpler and again higher-order pole constraints are sufficient to determine the amplitudes completely.

VI. \( \mathcal{N} = 4 \) FIVE POINTS ONE-LOOP AMPLITUDES

The \( n \)-graviton MHV amplitude for a \( \mathcal{N} = 4 \) matter multiplet is

\[
M_{n=4}^{(1^-), (2^-, 3^+, \ldots, n^+)} = \left( \frac{-1}{8} \right)^n \left( \frac{1}{2} \right)^8 \sum_{2 \leq a < b \leq n} \sum_{1 \in M, 2 \in N} \left( -\frac{\langle a \rangle \langle 2 a \rangle \langle 1 b \rangle \langle 2 b \rangle}{(a b)^2 \langle 1 2 \rangle^2} \right) \frac{h(a, M, b) h(b, N, a) \text{tr}[a M b N]}{I_{4}^{A MbN,\text{trunc}}} + \sum_{1 \in A, 2 \in B} c_2(A; B) I_2(K_{A}^2) + R_n,
\]  

(VI.1)

where the sets \( A \) and \( B \) contain at least one positive helicity leg. As discussed previously, we have combined the box and triangle contributions to the amplitude to leave a sum over the box coefficients multiplied by truncated box functions. For the five-point amplitude the box-coefficients are all equivalent up to relabelling and, for example, the coefficient of \( I_2^{A MbN} \) reduces to,

\[
-\frac{1}{2} \langle 1 2 \rangle^4 \times \frac{1}{\langle 3 4 \rangle^4} \times \frac{[2 5] s_{1}^2 s_{13}^2 \langle 2 3 \rangle \langle 2 4 \rangle}{\langle 2 5 \rangle \langle 3 5 \rangle \langle 4 5 \rangle}
\]  

(VI.2)
which clearly has a higher-order pole: \(\langle 34 \rangle^{-4}\). The limit \(\langle 34 \rangle \to 0\) corresponds to \(u \to 0\) in Mandelstam notation. Close to \(u = 0\) we can expand the truncated box functions:

\[
I_{4}^{\text{trunc}}(s,t,u,m^2) = + \frac{2u}{(st)^2} \left( m^2 \ln(-m^2) - s \ln(-s) - t \ln(-t) \right) - \frac{u^2}{(st)^3} \left( m^4 \ln(-m^2) - s^2 \ln(-s) - t^2 \ln(-t) + (st) \right) + O(u^3)
\]

The \(O(u^3)\) terms combine with the coefficient to yield a \(\langle 34 \rangle^{-1}\) singularity and so only contribute to physical singularities.

The logarithms arising from the expansion of \(I_{4}^{\text{trunc}}\) around \(u = 0\) can be combined with those from the bubble contributions. The bubble contributions are presented in general form in appendix [A]. The resulting coefficients of the logarithms have only simple poles in \(u\) and therefore in the \(u \to 0\) limit only produce logarithmic contributions to physical singularities/factorisations.

The rational term or descendent arising from the expansion of \(I_{4}^{\text{trunc}}\) contains a higher-order pole which must be cancelled by \(R_{5}\). We therefore introduce

\[
R_{5}^{a} = -\frac{(12)(34)(25)(23)(24)}{2} \langle 1,2,3,4,5 \rangle + \mathcal{P}(\{1,2,3,4,5\})
\]

where \(\mathcal{P}(\{x_i\}, \{y_i\}, \ldots)\) denotes a sum over the independent permutations of the \(x_i\) and \(y_i\), and express the full rational term as

\[
R_{5} = R_{5}^{a} + R_{5}^{b}.
\]

As \(R_{5}^{a}\) cancels all of the higher-order poles descending from the boxes, \(R_{5}^{b}\) contains only physical singularities. Assuming that any non-standard complex factorisations are restricted to the \(\langle c d \rangle^{-1}\) pieces of collinear limits involving two positive helicity legs, we find that the only singularities that \(R_{5}^{b}\) can have take the form \(\langle c d \rangle^{-1}\) where \(c\) and \(d\) denote positive helicity legs. There is only one combination of spinor products involving only these poles that has the correct spinor and momentum weights. The normalisation of this term can be determined by evaluating real collinear limits, yielding

\[
R_{5}^{b} = -\langle 1,2 \rangle^4 \frac{[34][35][45]}{\langle 34 \rangle \langle 35 \rangle \langle 45 \rangle}.
\]

Explicit computation using string based-rules numerically verifies \(R_{5}\) [23] which has also been deduced using colour–kinematics duality applied to gravity [41].

VII. \(\mathcal{N} = 1\) FIVE POINT ONE-LOOP AMPLITUDE

The pole structure for the five-point \(\mathcal{N} = 1\) amplitude is much richer than the \(\mathcal{N} = 4\) case and contains both higher-order physical poles and spurious singularities. Starting with the contributions of the boxes to the amplitude, the truncated box integral functions have coefficients:

\[
a^{\mathcal{N}=1}[\{a^-, e^+, c^+, b^-, d^+\}] = \frac{(ab)^2 \langle a c \rangle^2 \langle a d \rangle^2 \langle b c \rangle \langle b d \rangle \langle a e \rangle s_{bc} s_{de}}{2 \langle a e \rangle \langle c d \rangle \langle c e \rangle \langle d e \rangle}.
\]

12
Close to the \( \langle cd \rangle = 0 \) pole the box contributions generate logarithms with coefficients containing poles up to \( \langle cd \rangle^{-5} \) and rational descendants with coefficients containing poles up to \( \langle cd \rangle^{-4} \). The higher-order poles in these rational descendants must be cancelled by the rational piece of the amplitude. The rational terms needed to cancel the \( \langle cd \rangle^{-4} \) and \( \langle cd \rangle^{-3} \) poles in the rational descendants are easily obtained from the expansion of the truncated box integral function. For each box we introduce a rational term,

\[
R^{(c,d)^{-4}, (c,d)^{-3}}_{(a^-, e^+, c^+, b^-, d^+)} = \frac{\langle ab \rangle^2 \langle ac \rangle^2 \langle ad \rangle^2 \langle bc \rangle \langle bd \rangle \langle ae \rangle s_{bd}^2}{2 \langle ae \rangle \langle cd \rangle^6 \langle ce \rangle \langle de \rangle} \left( \frac{s_{cd}^2}{s_{bc}^2 s_{bd}^3} - \frac{s_{cd} s_{ae}}{3 s_{bc}^3 s_{bd}} \right). \tag{VII.2}
\]

These rational pieces do not contain additional higher-order poles, but do not cancel the \( \langle cd \rangle^{-2} \) poles in the rational descendants. If the expansion of the truncated box integral function is taken a stage further, the rational term that is naively generated contains double poles in \( s_{bc} \) and \( s_{bd} \). To deal with the \( \langle cd \rangle^{-2} \) poles we must also consider the bubble contributions to the amplitude.

The bubble coefficients are readily evaluated using the canonical basis procedure \[^{14} \text{[14]}\]. The full \( \mathcal{N} = 1 \) bubble coefficient is given in appendix \[^{4} \text{[4]}\]. These coefficients contain a number of higher-order physical and spurious poles:

\[
\mathcal{C}^{\mathcal{N}=1}[[\{a^-, c^+\}, \{b^-, d^+, e^+\}]] \supset \{\langle cd \rangle^{-5}, \langle ce \rangle^{-5}, \langle de \rangle^{-5}, \langle d \cdot K_{ac} \rangle^{-3}, \langle e \cdot K_{ac} \rangle^{-3}\}. \tag{VII.3}
\]

The \( \langle cd \rangle^{-5} \) type poles are precisely those needed to cancel the logarithmic descendants of the boxes on \( \langle cd \rangle \to 0 \) type singularities. In fact, combining the logarithms from the bubbles with those descending from the boxes gives logarithms with simple poles in \( \langle cd \rangle \) as \( \langle cd \rangle \to 0 \). Therefore there are no rational terms descending from the logarithms as we approach this type of singularity.

Each spurious pole occurs in two bubbles as \( d \cdot K_{ac} = -d \cdot K_{be} \) etc. As the logarithms themselves cannot contain a spurious pole, the coefficients of each pair of logarithms must cancel to order \( (d \cdot K_{ac})^{-1} \), so as we approach \( d \cdot K_{ac} = 0 \) the bubble contributions combine to give,

\[
\mathcal{C}^{\mathcal{N}=1} (- \ln(-s_{ac}) + \ln(-s_{bc})) + O \left( (d \cdot K_{ac})^0 \right) = \mathcal{C}^{\mathcal{N}=1} \ln \left( 1 + \frac{d \cdot K_{ac}}{s_{ac}} \right) + O \left( (d \cdot K_{ac})^0 \right) \tag{VII.4}
\]

\[
= \mathcal{C}^{\mathcal{N}=1} \left( \frac{d \cdot K_{ac}}{s_{ac}} - \frac{1}{2} \frac{(d \cdot K_{ac})^2}{s_{ac}^2} \right) + O \left( (d \cdot K_{ac})^0 \right)
\]

The rational descendants of the logarithms on the spurious pole contain both spurious and higher-order physical poles, all of which must be cancelled by \( R_5 \). As the spurious poles do not appear in the box coefficients, it is natural to remove them before we combine the box and bubble induced rational pieces. The full bubble coefficient is quite complicated, but the term containing the leading spurious pole is much simpler:

\[
\mathcal{C}^{\mathcal{N}=1}[[\{a^-, c^+\}, \{b^-, d^+, e^+\}]] = \frac{(ab)^2 (ac)^2 (ad)^3 (bd)^3 [ae] [be] [cd]^3 [de]}{12 \langle cd \rangle^2 \langle de \rangle^2 (2d \cdot K_{ac})^3} + O \left( (d \cdot K_{ac})^{-2} \right). \tag{VII.5}
\]

13
The leading order descendant is simply this multiplied by $d \cdot K_{ac}/s_{ac}$. On the spurious pole $s_{ad} = -s_{cd}$ and $s_{de} = -s_{bd}$, allowing us to construct a factor that has adjustable sub-leading behaviour:
\[
\left( \alpha + (\alpha - 1) \frac{s_{cd}}{s_{ad}} \right) \left( \gamma + (\gamma - 1) \frac{s_{de}}{s_{bd}} \right) = 1 + O \left( d \cdot K_{ac} \right).
\] (VII.6)

For each bubble we introduce a rational term,
\[
R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{spur}} = \frac{\langle ab \rangle^2 \langle ac \rangle^2 \langle ad \rangle^2 \langle bd \rangle^3 [ac][be][cd]^3[de]}{24 \langle cd \rangle^2 \langle de \rangle^2 (2d \cdot K_{ac})^2 s_{ac}}\times
\left( \alpha_{d} + (\alpha_{d} - 1) \frac{s_{cd}}{s_{ad}} \right) \left( \gamma_{d} + (\gamma_{d} - 1) \frac{s_{de}}{s_{bd}} \right) + (d \leftrightarrow e).
\] (VII.7)

The extra factor of $1/2$ arises as each descendant originated from a pair of bubbles. By construction this cancels the leading spurious pole for any values of $\alpha_{d}$, $\alpha_{e}$, $\gamma_{d}$ and $\gamma_{e}$. Additionally, this term is found to cancel the $(d \cdot K_{ac})^{-1}$ poles if,
\[
\gamma_{d} = -2 - \alpha_{e}, \quad \gamma_{e} = -2 - \alpha_{d}.
\] (VII.8)

Thus the spurious poles have been removed and we don’t need to refer to the bubble coefficient again. It is also worth noting that the only double physical poles in $R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{spur}}$ are those that appear explicitly in the expansion of the bubble coefficient.

The remaining higher-order poles involve factors of the form $\langle cd \rangle^{-2}$. These are present in the rational terms we have already introduced and descend from the box contributions in the $\langle cd \rangle \to 0$ limit. Setting $\alpha_{d} = \alpha_{e} = 0$ restricts the higher-order poles to a small number of terms in $R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{spur}}$. Focussing on terms containing $\langle cd \rangle^{-2}$ specifically, it is possible to rewrite the sum of these terms in a form that involves no spurious poles and no other higher-order poles. The $\langle cd \rangle^{-2}$ poles can then be cancelled by introducing a rational term:
\[
R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:cd}} = R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:X}} + R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:Y}},
\] (VII.9)

where
\[
R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:X}} = \frac{5 \langle ab \rangle^2 \langle ac \rangle^2 \langle ad \rangle^2 [ae][cd]^4}{24 \langle cd \rangle^2 \langle de \rangle^2 [bd][ce][cd][bd]} + (a \leftrightarrow b),
\] (VII.10)

and
\[
R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:Y}} = \frac{\langle ab \rangle^2 \langle bc \rangle [cd]}{12 \langle cd \rangle^2 \langle de \rangle [ae][bc]} \left\{ - \langle ab \rangle [bd][ce]^3 + \langle ad \rangle [cd][ce]^2[de]
\right.
\left. + \langle ac \rangle \langle ad \rangle [ae][cd]^2 \left( \frac{\langle ab \rangle \langle ad \rangle [ae] + \langle ae \rangle \langle bd \rangle [ce]}{\langle ae \rangle \langle bc \rangle [de]} \right) \right\} + \mathcal{P}(\{a, b\}, \{c, d\}).
\] (VII.11)

The $\langle c e \rangle^{-2}$ and $\langle d e \rangle^{-2}$ poles can similarly be cancelled by introducing rational terms that are relabellings of $R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:cd}}$.

The full rational piece of the amplitude can now be written as,
\[
R_{5} = \left( \sum_{\text{boxes: } i} R_{i}^{\langle cd \rangle^{-4}, \langle cd \rangle^{-3}} \right) + \left( \sum_{\text{bubbles: } j} R_{j}^{\text{spur}} \right) + R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:cd}} + R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:ce}} + R_{\{a^-, c^+, b^-, d^+, e^+\}}^{\text{quad:de}} + R_{5}^{b},
\] (VII.12)

where $R_{5}^{b}$ contains only simple physical poles.
To determine $R_5^b$ we apply complex factorisation constraints, again assuming that any non-standard factorisations are restricted to the $(c,d)^{-1}$ pieces of limits involving two positive helicity legs. Firstly: there should be no poles as either $(a\ b) \to 0$ or $[ab] \to 0$. The amplitude already satisfies these constraints. Next, we expect no poles of the form $[a \ b]^{-1}$. As such poles are present in the rational terms we have already identified, we introduce a further rational term to cancel them:

$$R^q = \frac{(a\ b)^2 [c\ d]^3 [c\ e] [d\ e] (s_{ce} s_{de} - s_{ae} s_{be})}{12 [a\ c] [a\ d] [b\ c] [b\ d] \langle c\ d \rangle \langle c\ e \rangle \langle d\ e \rangle} + \text{(cyclic perms \{c, d, e\}).}$$  \hspace{1cm} (VII.13)

Note that this term introduces no higher-order physical poles, spurious poles, $(a\ b)$ poles or $[a\ b]$ poles, so it doesn’t disrupt any of the previous cancellations. With the introduction of $R^q$ the $(a\ c)$ type poles in the rational term have the correct coefficients to reproduce the standard factorisations as $k_{a^-} \cdot k_{c^+} \to 0$ etc.

The only remaining poles involve terms of the form $(c\ d)^{-1}$ and $(c\ d)^{-1}$. We expect no factorisations to contribute to the former type of pole and the amplitude has none. This leaves terms involving $(c\ d)^{-1}$ type poles. The only term with the correct spinor and momentum weights involving only these poles is,

$$R^f = -\frac{1}{4} (a\ b)^4 \frac{[c\ d] [c\ e] [d\ e]}{\langle c\ d \rangle \langle c\ e \rangle \langle d\ e \rangle},$$  \hspace{1cm} (VII.14)

where the normalisation is fixed by examining the $k_{c^-} \cdot k_{d^+} \to 0$ collinear limit.

The full rational piece of the amplitude is thus,

$$R_5 = \left( \sum_{\text{boxes: } i} R^{i}_{[c\ d]^{-4},(c\ d)^{-3}} \right) + \left( \sum_{\text{bubbles: } j} R_j^{\text{spur}} \right) + R^{\text{quad}:cd} + R^{\text{quad}:ce} + R^{\text{quad}:de} + R^q + R^f.$$

(VII.15)

With this rational piece, we have an ansatz for the amplitude which a) is free of spurios poles, b) has the correct symmetries, c) has the correct pole structure, collinear and soft limits. Since, any potential ambiguity must vanish in all limits, we expect this ansatz to be correct.

**VIII. CONCLUSIONS**

The absence of spurious singularities from complete one-loop scattering amplitudes introduces a constraining web of relationships between the rational functions arising when the amplitude is expanded in terms of scalar one-loop integrals. In particular, these constraints involve the purely rational pieces of the amplitude as well as the four-dimensional cut-constructible pieces. As the latter are relatively easily determined from unitarity considerations, this web of constraints readily provides information about the purely rational pieces.

In the simplest cases, for example the four-graviton amplitudes considered in this article, these constraints determine the entire one-loop amplitude starting from the coefficients of the box integral functions. For more complicated examples, such as the five-graviton MHV amplitudes, the web of constraints determine a significant portion of the purely rational terms. The remainder has a relatively simple form, which can be determined from the symmetries and the physical factorisation properties of the amplitudes. We have successfully used this to reproduce the one-loop five-point $\mathcal{N} = 4$ supergravity amplitude, and obtain the previously-unknown one-loop five-point amplitude for $\mathcal{N} = 1$ supergravity.

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Appendix A: Bubbles in Supergravity MHV amplitudes

The bubble integral functions $I_2(P^2)$ will have vanishing coefficients for the MHV amplitude unless the momenta $P$ (and hence $-P$) contain exactly one negative helicity leg and at least one positive helicity leg. We can thus take $P$ of the form $\{m_1, a^+_1, a^+_2, \ldots, a^+_n\}$ and the legs on the other side to be $\{m^-_2, b^+_1, b^+_2, \ldots, b^+_n\}$.

Here we present the bubble coefficients for MHV amplitudes for the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ matter multiplets. There are a variety of techniques available to determine the bubble coefficient from the cut: we will use the method of canonical forms \[14\]. The $\mathcal{N} = 4$ coefficients appear in ref. \[23\]:

\[
c^{\mathcal{N}=4}[\{m_1, a_i\}; \{m_2, b_i\}] = \frac{1}{2} \langle m_1 m_2 \rangle^4 \sum_{P_L, P_R} C_{P_L} C_{P_R} \left( \sum_{x \neq a_1} D_x \langle m_2 a_1 \rangle \langle b_1 a_1 \rangle H^0_2[a_1, x; m_1, m_1; P] + \sum_{x \neq b_1} D_x \langle m_2 b_1 \rangle \langle a_1 b_1 \rangle H^0_2[b_1, x; m_1, m_1; P] + D_{a_1} \langle m_2 a_1 \rangle H^0_2[a_1; m_1, m_1; P] + D_{b_1} \langle m_2 b_1 \rangle H^0_2[b_1; m_1, m_1; P] \right), \tag{A.1}
\]

\[
c^{\mathcal{N}=1}[\{m_1, a_i\}; \{m_2, b_i\}] = \frac{1}{2} \langle m_1 m_2 \rangle^2 (P^2)^2 \sum_{P_L, P_R} C_{P_L} C_{P_R} \left( \sum_{x \neq a_1} D_x \langle m_2 a_1 \rangle \langle b_1 a_1 \rangle H^2_{1,1}[a_1; x; \{B_i\}^{\mathcal{N}=1}; m_1; \{D_i\}^{\mathcal{N}=1}; P] + \sum_{x \neq b_1} D_x \langle m_2 b_1 \rangle \langle a_1 b_1 \rangle H^2_{1,1}[b_1; x; \{B_i\}^{\mathcal{N}=1}; m_1; \{D_i\}^{\mathcal{N}=1}; P] + D_{a_1} \langle m_2 a_1 \rangle H^2_{1,1}[a_1; \{B_i\}^{\mathcal{N}=1}; m_1; \{D_i\}^{\mathcal{N}=1}; P] + D_{b_1} \langle m_2 b_1 \rangle H^2_{1,1}[b_1; \{B_i\}^{\mathcal{N}=1}; m_1; \{D_i\}^{\mathcal{N}=1}; P] \right), \tag{A.2}
\]

where $P_L$ and $P_R$ are permutations of the positive helicity legs $\{a_i\}$ and $\{b_i\}$ respectively,

\[
\{B_i\}^{\mathcal{N}=1} = \{m_1, m_2, m_1\}, \quad \{D_i\}^{\mathcal{N}=1} = \{P|m_1\}, \{P|m_2\},
\]

\[
C_{P_L} = \frac{[n_L m_1]}{\langle n_L m_1 \rangle \prod_{i=1}^{n_L-1} \langle a_i a_{i+1} \rangle}, \quad C_{P_R} = \frac{[n_R m_2]}{\langle n_R m_2 \rangle \prod_{i=1}^{n_R-1} \langle b_i b_{i+1} \rangle}, \tag{A.3}
\]

and

\[
D_x = \frac{\langle m_2 x \rangle \prod_{i=1}^{n_L-1} [a_i \tilde{K}_{i+1} |x\rangle] \prod_{i=1}^{n_R-1} [b_k \tilde{K}'_{k+1} |x\rangle]}{\prod_{y \neq x} (x | y)},
\]

where $\tilde{K}_p = k_p + \cdots k_{a_{n_L}} + k_{m_1}$ and $\tilde{K}'_p = k_p + \cdots k_{b_{n_R}} + k_{m_2}$. The functions in (A.1) and (A.2) of the form $H^N_{\{S\}}$ are the canonical forms \[14\]. The index $N$ indicates the power of loop momenta present in the cut. With increasing $N$ we find increasing complexity and increasing powers of spurious denominators. The simplest canonical form is

\[
H^0_1[A; B; P] = \frac{[A|P|B]}{[A|P|A]}, \tag{A.5}
\]
which is linear in the spurious singularity \([A|P|A] = 2k_A \cdot P\). It is convenient to define extensions,

\[
H^n[H; B; P] = \sum_i \prod_{j=2}^n \frac{\langle B_j A_i \rangle \langle B_1 | P | A_i \rangle}{\prod_{j \neq i} \langle A_j A_i \rangle \langle A_1 | P | A_i \rangle}, \quad \langle A_i A_j \rangle \neq 0. \tag{A.7}
\]

and the special cases where \(A_1 = A_2 = A\),

\[
H^n[1, A; B_1, B_2; P] = \frac{\langle A | P | B_1 \rangle [A | P | B_2]}{[A | P | A]^2}. \tag{A.8}
\]

We also need the \(H^1_1\),

\[
H^1_1[B; D; P] = \frac{1}{2} [D | P | D], \tag{A.9}
\]

\[
\begin{aligned}
H^1_1[A; B_1, B_2; D; P] &= \frac{P^2}{4[A|P|A]^2} (\langle D | A | B_1 \rangle [A | P | B_2] + (B_1 \leftrightarrow B_2)) \\
&+ \frac{1}{4[A|P|A]} ([D | P | B_1] [A | P | B_2] + (B_1 \leftrightarrow B_2)) \tag{A.10}
\end{aligned}
\]

\[
\begin{aligned}
H^1_1[A_1; A_2; B_1, B_2, B_3; D; P] &= \frac{\langle A_1 B_3 \rangle}{\langle A_1 A_2 \rangle} H^1_1[A_1; B_1, B_2; D; P] \\
&+ \frac{\langle A_2 B_3 \rangle}{\langle A_2 A_1 \rangle} H^1_1[A_2; B_1, B_2; D; P] + \frac{\langle B_3 A_2 \rangle}{\langle A_2 A_1 \rangle} \sum_{i \in P} [D \leftrightarrow i] H^n_2[A_2, i, B_2; P] \\
&+ \frac{\langle B_3 A_2 \rangle}{\langle A_2 A_1 \rangle} \sum_{i \in P} [D \leftrightarrow i] H^n_2[A_1, i, B_2; P], \tag{A.11}
\end{aligned}
\]

\[
\begin{aligned}
H^2_2[A; B_1, B_2; B_3; D; P] &= \frac{P^2}{3[A|P|A]^3} ([A|P|B_3][A|P|B_1][D|A|B_2] + (B_1 \leftrightarrow B_2) \\
&- 2 [A|P|B_1][A|P|B_2][D|A|B_3]) \tag{A.12}
&+ \frac{1}{6[A|P|A]^2} ([A|P|B_1][A|P|B_2][D|P|B_3] + \mathcal{P}(\{B_i\})),
\end{aligned}
\]

and the \(H^2_2\)

\[
H^2_2[B_1, B_2; D_1, D_2; P] = \frac{1}{6} [D_1 | P | B_1] [D_2 | P | B_2] + (B_1 \leftrightarrow B_2), \tag{A.13}
\]

\[
\begin{aligned}
H^2_2[A; B_1, B_2, B_3; D_1, D_2; P] &= \frac{(P^2)^2}{18[A|P|A]^3} ([D_1|A|B_1][D_2|A|B_2][A|P|B_3] + \mathcal{P}(\{B_i\})) \\
&+ \frac{(P^2)^2}{36[A|P|A]^2} ([D_1|P|B_1][D_2|A|B_2][A|P|B_3] + \mathcal{P}(\{B_i\}; \{D_i\})) \\
&+ \frac{1}{18[A|P|A]} ([D_1|P|B_1][D_2|P|B_2][A|P|B_3] + \mathcal{P}(\{B_i\})), \tag{A.14}
\end{aligned}
\]

17
\[ H_{1,1}'[A; A_1; B_1, B_2, B_3; C_4; D_1, D_2; P] = \frac{\langle C_A A_1 \rangle}{\langle A_2 A_1 \rangle} H_{1}^{2}[A_1, B_1, B_2, B_3, D_1, D_2, P] \]
\[ + \frac{\langle C_A A_2 \rangle}{\langle A_1 A_2 \rangle} H_{1}^{2}[A_2, B_1, B_2, B_3, D_1, D_2, P] \]
\[ - \frac{\langle C_A A_2 \rangle}{\langle A_2 A_1 \rangle} H_{1}^{2}[A_2, B_2, B_3, P[D_1], D_2, P] \]
\[ - \frac{\langle B_1 A_1 \rangle}{\langle A_2 A_1 \rangle} \langle C_A A_2 \rangle H_{1,1}^{2}[A_1, A_2, B_2, B_3, P[D_1], D_2, P]. \]

\[ H_{2,2}'[A; B_1, B_2, B_3; C_4; D_1, D_2; P] = \]
\[ \frac{(P^2)^2}{9|A|P|A|^4} ([D_1|A|B_1]|D_2|A|B_2|A|P|B_3|A|P|C_4) + \mathcal{P}([B_1]) \]
\[ - \frac{(P^2)^2}{72|A|P|A|^4} ([D_1|A|C_4]|D_2|A|B_1|A|P|B_3|A|P|B_4) + \mathcal{P}([B_1], [D_1]) \]
\[ - \frac{(P^2)^2}{18|A|P|A|^4} ([D_1|A|C_4]|D_2|P|B_1|A|P|B_3|A|P|B_4) + \mathcal{P}([B_1], [D_1]) \]
\[ + \frac{(P^2)^2}{36|A|P|A|^4} ([D_1|P|C_4]|D_2|A|B_1|A|P|B_2|A|P|B_3) + \mathcal{P}([B_1], [D_1]) \]
\[ + \frac{1}{18|A|P|A|^4} ([D_1|P|B_1]|D_2|P|B_2|A|P|B_3|A|P|C_4) + \mathcal{P}([B_1]) \]

**Appendix B: Box Integral Functions**

The scalar box integral is,
\[ I_4 = -i (4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2 (p - K_1)^2 (p - K_1 - K_2)^2 (p + K_1)^2}, \] (B.1)

where \( K_i \) is the sum of the momenta of the legs attached to the \( i \)-th corner. If a single leg is attached then \( K_i \) is null. The form of the integral depends upon the number of the \( K_i \) which are non-null, \( K_i^2 \neq 0 \). We often misname these *massive legs*. The integrals are functions of the non-zero \( K_i^2 \) and the invariants,
\[ S \equiv (K_1 + K_2)^2, \quad T \equiv (K_2 + K_3)^2. \] (B.2)

The scalar box functions needed for our amplitudes, expanded to \( \mathcal{O}(\epsilon^0) \), for the one-mass
(with leg 4 massive) and the two-mass-easy (with legs 2 and 4 massive) are

\[ I_{4}^{1m} = \frac{-2r_{\Gamma}}{ST} \times \left( \frac{1}{\epsilon^{2}} \left[ (-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_{4}^{2})^{-\epsilon} \right] \right. \\
+ \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{S} \right) + \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{T} \right) + \frac{1}{2} \ln^{2} \left( \frac{S}{T} \right) + \frac{\pi^{2}}{6} \right) \\
I_{4}^{2ne} = \frac{-2r_{\Gamma}}{ST - K_{2}^{2}K_{4}^{2}} \times \left( \frac{1}{\epsilon^{2}} \left[ (-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_{2}^{2})^{-\epsilon} - (-K_{4}^{2})^{-\epsilon} \right] \right. \\
+ \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}}{S} \right) + \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}}{T} \right) + \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{S} \right) \\
+ \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{T} \right) - \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}K_{4}^{2}}{ST} \right) + \frac{1}{2} \ln^{2} \left( \frac{S}{T} \right) \right) \quad (B.3) \\
\]

where

\[ r_{\Gamma} = \frac{\Gamma(1 + \epsilon)\Gamma^{2}(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}, \quad (B.4) \]

The truncated box-functions are these with the singularities removed,

\[ I_{4}^{1m,\text{trunc}} = \frac{-2r_{\Gamma}}{ST} \times \left( \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{S} \right) + \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{T} \right) + \frac{1}{2} \ln^{2} \left( \frac{S}{T} \right) + \frac{\pi^{2}}{6} \right) \]
\[ I_{4}^{2ne,\text{trunc}} = \frac{-2r_{\Gamma}}{ST - K_{2}^{2}K_{4}^{2}} \times \left( \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}}{S} \right) + \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}}{T} \right) + \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{S} \right) \\
+ \text{Li}_{2} \left( 1 - \frac{K_{4}^{2}}{T} \right) - \text{Li}_{2} \left( 1 - \frac{K_{2}^{2}K_{4}^{2}}{ST} \right) + \frac{1}{2} \ln^{2} \left( \frac{S}{T} \right) \right) \quad (B.5) \]

The truncated zero-mass box (only necessary for the four-point amplitude) is obtained by setting \( K_{4} = 0 \) in the above expression for \( I_{4}^{1m,\text{trunc}} \).

[1] G. Passarino and M. Veltman, Nucl. Phys. B \textbf{160}, 151, (1979); R. G. Stuart, Comput. Phys. Commun. \textbf{48} (1988) 367; G. J. van Oldenborgh and J. A. M. Vermaseren, Z. Phys. C \textbf{46} (1990) 425; Z. Bern, L. J. Dixon, D. A. Kosower, Nucl. Phys. B\textbf{412} (1994) 751-816. [hep-ph/9306240].
[2] R. E. Cutkosky, J. Math. Phys. \textbf{1} (1960) 429.
[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B \textbf{425} (1994) 217 [hep-ph/9403226].
[4] Z. Bern, L. J. Dixon, D. C. Dunbar, D. A. Kosower, Nucl. Phys. B \textbf{435} (1995) 59-101. [hep-ph/9409265].
[5] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B \textbf{513} (1998) 3 [hep-ph/9708239].
[6] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B \textbf{725} (2005) 275 [hep-th/0412103].
[7] R. Roiban, M. Spradlin and A. Volovich, Phys. Rev. Lett. \textbf{94} (2005) 102002 [hep-th/0412265].
[8] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, Phys. Lett. B 612 (2005) 75 [hep-th/0502028].
[9] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D 72 (2005) 065012 [hep-ph/0503132].
[10] D. Forde, Phys. Rev. D 75, 125019 (2007) arXiv:0704.1835 [hep-ph].
[11] N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, JHEP 0804 (2008) 038 arXiv:0709.2086 [hep-ph].
[12] P. Mastrolia, Phys. Lett. B 644 (2007) 272 [hep-th/0611091].
[13] E. W. Nigel Glover and C. Williams, JHEP 0812 (2008) 067 arXiv:0810.2964 [hep-th].
[14] D. C. Dunbar, W. B. Perkins and E. Warrick, JHEP 0906 (2009) 056 arXiv:0903.1751 [hep-ph].
[15] Z. Bern, L. J. Dixon, D. A. Kosower, Phys. Rev. D 73 (2006) 065013. [hep-ph/0507005].
[16] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, Phys. Rev. D 74, 036009 (2006) [hep-ph/0604195]; Phys. Rev. D 75, 016006 (2007) [hep-ph/0607014].
[17] C. F. Berger et al., Phys. Rev. D 78, 036003 (2008) arXiv:0803.4180 [hep-ph].
[18] M. B. Green, J. H. Schwarz, L. Brink, Nucl. Phys. B198 (1982) 474-492.
[19] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B 546, 423 (1999) [hep-th/9811140].
[20] Z. Bern, N. E. J. Bjerrum-Bohr and D. C. Dunbar, JHEP 0505 (2005) 056 arXiv:hep-th/0501137]. N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, Phys. Lett. B 621, 183 (2005) arXiv:hep-th/0503102.
[21] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, JHEP 0612 (2006) 072 arXiv:hep-th/0610043.
[22] D. C. Dunbar, J. H. Ettle, W. B. Perkins, Phys. Rev. D83 (2011) 065015. [arXiv:1011.5378 [hep-th]].
[23] D.C. Dunbar and P.S. Norridge, Nucl. Phys. B 433, 181 (1995) [hep-th/9408014].
[24] M. T. Grisaru, J. Zak, Phys. Lett. B90 (1980) 237.
[25] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, Phys. Rev. D 74, 036009 (2006) [hep-ph/0604195]; Phys. Rev. D 75, 016006 (2007) [hep-ph/0607014].
[26] C. F. Berger et al., Phys. Rev. D 78, 036003 (2008) arXiv:0803.4180 [hep-ph].
[27] S. D. Badger, JHEP 0901, 049 (2009) arXiv:0806.4600 [hep-ph].
[28] Z. Bern and G. Chalmers, Nucl. Phys. B 447, 465 (1995) [hep-ph/9503236].
[29] F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. B 211, 91 (1988).
[30] Z. Bern, L. J. Dixon, D. A. Kosower, Phys. Rev. D71 (2005) 105013. [hep-th/0501240].
[31] A. Brandhuber, S. McNamara, B. Spence, G. Travaglini, JHEP 0703 (2007) 029 [hep-th/0701187].
[32] D. C. Dunbar, Nucl. Phys. Proc. Suppl. 183 (2008) 122 [arXiv:0901.1202 [hep-ph]].
[33] S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon and D. C. Dunbar, Phys. Lett. B 606 (2005) 189 arXiv:hep-th/0410296.
[34] J. Bedford, A. Brandhuber, B. J. Spence, G. Travaglini, Nucl. Phys. B712 (2005) 59-85. [hep-th/0412108].
[35] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, Phys. Lett. B 608 (2005) 151 arXiv:hep-th/0412023.
[36] N. E. J. Bjerrum-Bohr and P. Vanhove, JHEP 0804 (2008) 065 arXiv:0802.0868 [hep-th]. arXiv:0805.3682 [hep-th].
[37] Z. Bern, D.C. Dunbar and T. Shimada, Phys. Lett. B 312, 277, (1993) [hep-th/9307001].
[38] Z. Bern, J. J. Carrasco, D. Forde, H. Ita, H. Johansson, Phys. Rev. D77 (2008) 025010. arXiv:0707.1035 [hep-th].

20
[39] D.C. Dunbar and P.S. Norridge, Class. Quantum Grav. 14, 351 (1997), \texttt{hep-th/9512084}.
[40] R. Britto, F. Cachazo, B. Feng, E. Witten, Phys. Rev. Lett. 94 (2005) 181602, \texttt{hep-th/0501052}.
[41] Z. Bern, C. Boucher-Veronneau and H. Johansson, \texttt{arXiv:1107.1935} [hep-th].