On ‘Useful’ R-norm Relative Information Measures and Applications

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Abstract
In this communication a new ‘useful’ R-norm relative information measure is introduced and characterized axiomatically. Its in equalities with particular cases are described. This new information measure has also been applied to study the status of the companies with regard to their loss and profit and that has been illustrated by considering empirical data and drawing figures. Ad joint of the relative information measure is also defined with the illustration of its application in share market with examples.

Introduction
Information theory as a separate subject is about 70 years old. Since information is energy, therefore it is measured, managed, regulated and controlled for the sake of welfare of humanity. The role of information function is to remove uncertainty and the amount of uncertainty removed is a measure of information.

The concept of information proved to be very important and universally useful. These days language used in telephones, business management, and cybernetics falls under the name “Information Processing”. In addition to this, information theory particularly measures of information have applications in physics, statistical inference, data processing and analysis, accountancy, psychology, etc.

Shannon was the first who developed a measure of uncertainty. He was interested in communicating information across the channel.

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in which some information is lost in the process of communication and that was called a noisy channel. His objective was to measure the amount of information lost. He defined a measure of uncertainty of a probability distribution as given below:

\[ H(P) = -k \sum P_i \log P_i, \]  

\[ \text{(1.1)} \]

where \( k \) is an arbitrary positive constant. The measure (1.1) was called entropy. Thereafter, Shannon’s entropy was characterized by various researchers like Khinchin,\(^{16}\) Fadeev,\(^{8}\) Teverberg,\(^{25}\) Chandy and McLeod,\(^{6}\) Kendal,\(^{19}\) Lee,\(^{20}\) Berges,\(^{2}\) Cziszar,\(^{7}\) Cheng,\(^{6}\) etc. on using different sets of postulates.

The quantity (1.1) measures the amount of information of probability distribution \( P \) when effectiveness or importance of the events is not taken into account. In addition to this; some probabilistic problems also play important role. Considering effectiveness of the outcomes, Belis and Guaisu\(^{1}\) introduced \( U = (u_1, u_2, \ldots, u_n) \) as a utility distribution, where \( u_i > 0 \), is the usefulness of an event having probability of occurrence \( p_i \) and consequently, ‘self useful information’ is defined as given below:

\[ H(u_i, P) = -u_i \log p_i, \]  

\[ \text{(1.2)} \]

The measure (1.2) is based two postulated as given below:

P1. In case all the events of an random experiment have the same utility \( u > 0 \), then the self used information generated by the product of two statistical independent events \( E_1 \) and \( E_2 \) can be expressed as the sum of the self-useful information provided by \( E_1 \) and \( E_2 \) individually i.e.

\[ H(p_1, p_2; u) = H(p_1; u) + H(p_2; u), \]  

\[ \text{(2.1)} \]

where \( p_1, p_2 \) is the probability of \( E_1 \) and \( E_2 \), respectively.

P2. \( H(p_i; \delta u) = \delta H(p_1; u) \) for \( \delta > 0. \)

\[ \text{(2.2)} \]

Further, Belis and Guaisu\(^{1}\) gave the following qualitative and quantitative information measure:

\[ H(U; P) = -\sum u_i p_i \log p_i, \]  

\[ \text{(1.5)} \]

Longo\(^{21}\) called (1.5) as ‘useful’ information and Guiasu and Picard\(^{6}\) called weighted entropy.

In this communication, the ‘useful’ relative information measure is defined and characterized axiomatically in section 2. The new measure thus introduced is generalized in section 3 with its and its particular cases are studied in section 4. The applications of new R-norm information measure are described in section 5. In section 6 its ad joint by taking empirical data is studied with its illustration graphically. In the end the conclusion is given along with an exhaustive list of references.

**Useful’ Relative Information Measure**

Let \( X \) be a random variable in an experiment and \( \{ P = (p_1, p_2, \ldots, p_n) \text{ where } p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1 \} \) be its probability distribution having \( U = (u_1, u_2, \ldots, u_n) \) as a utility distribution, where \( u_i > 0 \) for each \( i \), is the utility of an event having probability \( p_i \).

A ‘useful’ directed divergence measure was defined by Bhaker and Hooda\(^{3}\) and characterized as given below:

\[ D(U; P; Q) = \frac{\sum u_i p_i \log \left( \frac{p_i}{q_i} \right)}{\sum u_i p_i}, \]  

\[ \text{(2.1)} \]

where \( p_i \geq 0 \text{ and } q_i > 0 \text{ s.t. } \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1 \).

If we consider a uniform probability \( Q = \left\{ \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right\} \) in (2.1), it reduces to \( \log n \cdot (U; P) \).

It may be noted that ‘useful’ directed divergence measure \( D(U; P; Q) \) satisfies the following conditions:

1. \( D(U; P; Q) > 0 \)
2. \( D(U; P; Q) = 0 \text{ iff for each } i, p_i = q_i \).
3. \( D(U; P; Q) \) is a convex function of \( q_1, q_2, \ldots, q_n \), as well as \( p_1, p_2, \ldots, p_n, \)
Further, it is observed that (1.1) is not symmetric in P and Q, since \( D(U;P:Q) = D(U;Q:P) \).

Later on, a symmetric 'useful' J-divergence measure was introduced by Hooda and Ram [2] as given below:

\[
J(U; P : Q) = D(U; P : Q) + D(U; Q : P)
\]

\[
= \sum_{i=1}^{n} u_i p_i \log \left( \frac{p_i}{q_i} \right) + \sum_{i=1}^{n} u_i q_i \log \left( \frac{q_i}{p_i} \right)
\]

Later on, a symmetric 'useful' J-divergence measure was introduced by Hooda and Ram [2] as given below:

\[
J(U; P : Q) = \frac{\sum_{i=1}^{n} u_i p_i (p_i - q_i) \log \left( \frac{p_i}{q_i} \right)}{\sum_{i=1}^{n} u_i p_i}, \quad \text{...(2.3)}
\]

where \( \sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i \). In case \( u_i = 1 \) for \( \forall \ i\), then (2.3) reduces to

\[
J(P : Q) = \sum_{i=1}^{n} (p_i - q_i) \log \left( \frac{p_i}{q_i} \right), \quad \text{...(2.4)}
\]

where (2.4) is a divergence measure due to Jeffrey and thus it is called as J-divergence.

Bhaker and Hooda [3] also characterized a 'useful' relative information measure of order \( \alpha \) as given below.

\[
D_{\alpha}(P : Q; U) = (\alpha - 1)^{-1} \log \left( \frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \right), \quad \text{...(2.5)}
\]

Further Hooda and Ram [10] characterized non-additive 'useful' relative information of degree \( \beta \) as given below:

\[
D^\beta(P : Q; U) = \frac{1}{2^\beta - 1} \left[ \sum_{i=1}^{n} u_i p_i^{\beta-1} q_i^{-\beta} \right]^{1/\beta - 1} \left[ \sum_{i=1}^{n} u_i p_i \right]^{1/\beta}, \quad \text{...(2.6)}
\]

The measure (2.6) reduces to (2.1) when, while in case \( u_i = 1 \) then (2.6) reduces to Kullback-Leibler’s [17] 'useful' relative information measure.

Further, it is observed that (1.1) is not symmetric in P and Q, since \( D(U;P:Q) = D(U;Q:P) \).

Further Boekee and Lubbe [4] defined R-norm information of the distribution P as

\[
H_{R}(P) = \frac{R}{R-1} \left[ 1 - \sum_{i=1}^{n} p_i^{R} \left( \frac{1}{n} \right)^{1/R} \right] \quad \text{...(2.7)}
\]

Kumaret al. [18] also defined the following 'useful' R-norm relative measure:

\[
D_{\alpha}(P : Q; U) = \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i^{\alpha(R-\alpha)} q_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i} \right]^{1/\alpha}, \quad \text{...(2.8)}
\]

There are many other generalizations of (2.8) also and one of them is

\[
D^\beta(P : Q; U) = \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i^{\beta-\beta_1} q_i^\beta}{\sum_{i=1}^{n} u_i p_i} \right]^{1/\beta}, \quad \text{...(2.9)}
\]

where \( \sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} u_i q_i \).

A 'Useful' R-Norm Relative Information Measure.

Theorem 3.1

Let P and Q be two probability distributions attached with a utility distribution U, then the following holds:

\[
\left( \sum_{i=1}^{n} u_i p_i^{\beta-\beta_1} q_i^{\beta} \right)^{\frac{1}{\beta}} \geq 1 \quad \text{according to } R \geq \beta
\]

under the condition \( \sum_{i=1}^{n} u_i p_i = \sum_{i=1}^{n} q_i u_i \).

Proof: We know from Holder’s inequality as

\[
\left( \sum_{i=1}^{n} x_i^R \right)^{\frac{1}{R}} \leq \left( \sum_{i=1}^{n} x_i^R \right)^{\frac{1}{R}} \leq \sum_{i=1}^{n} x_i \quad \text{where } R > \beta \quad \text{...(3.1)}
\]

Setting \( x_i = u_i^{\beta-\beta_1} p_i^{\beta_1} q_i^{-\beta} \) and \( y_i = u_i^{\beta_1} q_i^{\beta_1} q_i^{-\beta} \),

and \( \frac{1}{p} + \frac{1}{q} = 1 \quad \text{with } p = \frac{1}{R-\beta+1} \quad q = \frac{1}{\beta-R} \quad x_i, y_i \geq 0 \).

In Holder’s inequality (3.1), we have

\[
\sum_{i=1}^{n} \left( u_i^{\beta-\beta_1} p_i^{\beta_1} q_i^{-\beta} \right)^{\frac{1}{\beta}} \leq \left( \sum_{i=1}^{n} \left( u_i^{\beta_1} q_i^{\beta_1} q_i^{-\beta} \right)^{\frac{1}{\beta}} \right) \leq \left( \sum_{i=1}^{n} u_i p_i^{\beta_1} q_i^{-\beta} \right)^{\frac{1}{\beta}}
\]
On simplification we have
\[
\sum_{i=1}^{n} u_i P_i \frac{\beta - 1}{\alpha} q_i^{\beta - 2} \geq \left( \sum_{i=1}^{n} u_i \right) \left( \sum_{i=1}^{n} q_i \right)^{-\beta - 2} \geq \sum_{i=1}^{n} u_i P_i.
\]

since \( \sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i \)
or
\[
\sum_{i=1}^{n} u_i P_i \frac{\beta - 1}{\alpha} q_i^{\beta - 2} \geq \sum_{i=1}^{n} u_i P_i \geq 1.
\]

...(3.2)

Since \( \frac{1}{R - \beta + 1} < 1 \), therefore (3.2) can be written as
\[
\left( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \right)^{\frac{1}{R - \beta + 1}} \geq 1, R > \beta.
\]

...(3.3)

Similarly, we can prove that
\[
\left( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \right)^{\frac{1}{R - \beta + 1}} \leq 1, R < \beta.
\]

and \( \frac{1}{R - \beta + 1} > 1 \).

...(3.4)

For \( R = \beta \),
\[
\left( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \right)^{\frac{1}{R - \beta + 1}} = 1 \text{ for all}
\]

probability distributions and if \( R \neq \beta \) and \( p_i = q_i \) for each \( i \), i.e. \( P = Q \), then we have
\[
\left( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \right)^{\frac{1}{R - \beta + 1}} = 1.
\]

...(3.5)

**Theorem 2.2**

\( D_{\alpha\beta}^\beta ( U; P; Q) \) is a convex function of \( P \) and \( Q \).

**Proof**

Let \( K = \sum_{i=1}^{n} u_i P_i^{\beta - 1} q_i^{\beta - 2} \).

On differentiating \( K \) with \( \sum_{i=1}^{n} u_i P_i \) regard to \( p_i \) with all \( q_i \) and \( u_i \) having fixed value, we have \( \sum_{i=1}^{n} u_i P_i \) fixed and consequently,
\[
\sum_{i=1}^{n} u_i P_i \geq \sum_{i=1}^{n} u_i q_i \text{ is constant.}
\]

Hence we can write \( K = T \sum_{i=1}^{n} u_i P_i^{\beta - 1} q_i^{\beta - 2} \),
where \( T = \frac{1}{\sum_{i=1}^{n} u_i P_i} = \text{Constant.} \)

It implies that
\[
\frac{\partial K}{\partial p_i} = T u_i (R - \beta + 1) p_i^{\beta - 2} q_i^{\beta - 2} \]

or
\[
\frac{\partial^2 K}{\partial p_i^2} = T u_i (R - \beta + 1) (R - \beta) p_i^{\beta - 2} q_i^{\beta - 2}
\]

or \( R - \beta > 0 \), (2.6) is positive.

It implies that \( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \) and \( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \)
are convex functions of \( P \) in view of \( \frac{1}{R - \beta + 1} < 1 \).

Similarly for \( \frac{R - \beta}{R - \beta} < 0 \), \( \frac{R}{R - \beta} > 0 \)
\[
\left[ \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \right]^{\frac{1}{R - \beta + 1}}
\]
is also convex function of \( P \).

For \( R - \beta < 0 \), (3.6) is negative, so \( \sum_{i=1}^{n} u_i P_i^{\beta - 1} q_i^{\beta - 2} \)
and \( \frac{\sum_{i=1}^{n} u_i P_i^{\beta - 1}}{\sum_{i=1}^{n} u_i P_i} \)
are concave functions.
of P, since \( \frac{1}{R - \beta + 1} > 1 \).

Hence for \( R_\beta > 0 \) and \( R_\beta < 0 \), \( D_\beta^R(P: U; P: Q) \) is also a convex function of \( P \).

On the same lines, we can prove that \( D_\beta^R(U; P: Q) \) is a convex function of \( Q \) for \( R_\beta < 0 \) and \( R_\beta > 0 \) provided

\[
\sum_{i=1}^{n} u_i p_i \geq \sum_{i=1}^{n} u_i q_i.
\]

**A Generalized ‘Useful’ R-norm Relative Information Measure of Degree \( \beta \)**

We consider the following function:

\[
D_\beta^R(P; Q; U) = \frac{R}{\beta - R} \left[ F(1) - F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \right].
\]

...(4.1)

where \( F(x) \) is a monotonic increasing function of \( x \).

In view of (3.2), \( R_\beta > 0 \) and \( \frac{1}{R - \beta + 1} < 1 \), we have

\[
\left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \geq 1.
\]

...(4.2)

It implies

\[
F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \geq F(1).
\]

...(4.3)

Or

\[
\frac{R}{\beta - R} \left[ F(1) - F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \right] \geq 0.
\]

...(4.4)

Multiplying (4.3) by \( \frac{R}{\beta - R} < 0 \), we get

\[
\frac{R}{\beta - R} F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \geq 0.
\]

...(4.5)

or

\[
D_\beta^R(P; Q; U) \geq 0.
\]

...(4.6)

It implies that

\[
F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \leq F(1).
\]

...(4.7)

or

\[
D_\beta^R(P; Q; U) \leq 0.
\]

...(4.8)

Multiplying (4.8) by \( \frac{R}{\beta - R} < 0 \), we get

\[
\frac{R}{\beta - R} F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) \leq 0.
\]

...(4.9)

or

\[
D_\beta^R(P; Q; U) \geq 0.
\]

...(4.10)

Hence from (4.5) and (4.6) together give

\[
D_\beta^R(P; Q; U) \geq 0.
\]

...(4.11)

It may be noted that (4.11) vanishes when \( p_i = q_i \) for each \( i \).

It implies that

\[
\frac{R}{\beta - R} F \left( \frac{\sum_{i=1}^{n} u_i p_i^{\beta - \frac{1}{\beta}} q_i^{\beta - R}}{\sum_{i=1}^{n} u_i p_i} \right) = 0, \text{ when } p_i = q_i
\]

for each \( i \).
or
\[ F \left( \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R-\beta - 1}} = F(1) \]

or
\[ \left( \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R-\beta - 1}} = 1 \text{ when } p_i = q_i \text{ for each } i. \quad \ldots(4.12) \]

In particular when \( \beta = 1 \) and \( R \to 1 \), then \( D_1(P ; Q ; U) \)

\[ \sum_{i=1}^{n} u_i p_i \log \frac{p_i}{q_i} \]

reduces to
\[ \frac{1}{\sum_{i=1}^{n} u_i p_i} \]

which is (2.1)

**Particular Cases**

When \( C = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \), then (4.11) reduces to

\[ D(P ; C ; U) = \sum_{i=1}^{n} u_i p_i \log \frac{p_i}{n} \geq 0, \quad \ldots(4.13) \]

and in case \( P = C ; D(P ; C ; U) = 0 \).

Further, it can be verified that \( D(P ; C ; U) \) is a convex function of \( P \).

Next, it may be noted that (4.13) can be written as

\[ D(P ; C ; U) = \log n + \sum_{i=1}^{n} u_i \log p_i + \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} = D(C ; U) - D(P ; U), \quad \ldots(4.14) \]

Next we consider

\[ H_{1,x}^{(x)}(P ; U) = \frac{R}{\beta - R} \left[ F \left( \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{n-\beta}} \right] - \frac{1}{R} \left[ F \left( \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R-\beta - 1}} \right], \quad \ldots(4.15) \]

and

\[ \lim_{R \to 1} H_{1,x}^{(x)}(C ; U) = F'(1) \log n. \quad \ldots(4.16) \]

Further, if \( F(x) = x^j, (j \geq 1) \), then we have

\[ H_{1,x}^{(x)}(P ; U) = \frac{R}{\beta - R} \left[ \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right]^{\frac{1}{R-\beta - 1}} - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R-\beta - 1} q_i^{n-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R-\beta - 1}}, \quad \ldots(4.17) \]

For \( j = 1 \), we get

\[ H_{1,x}^{(x)} = n R \left[ \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right]^{\frac{1}{R}} - 1, \quad \ldots(4.18) \]

In case \( R \to 1 \), the measure (4.13),(4.17) and (4.5) respectively reduce to

\[ H_{1,x}^{(x)}(P ; U) = - F'(1) \left( \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \right); \beta = 1, \quad \ldots(4.19) \]

and

\[ H_{1,x}^{(x)}(P ; U) = - \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}, \quad \ldots(4.20) \]

and

\[ H_{1,x}^{(x)}(P ; U) = - \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}, \quad \ldots(4.21) \]

It may be noted that (4.21) was defined and characterized by Bhaker and Hooda [3].

In case \( F(x) = \log x \) in (4.1), it reduces to
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\[ H^\beta_{R,F}(P;U) = \frac{R}{\beta - R} \left[ \log \left( \frac{\sum_{i=1}^{n} u_i p_i^{\frac{1}{\beta-1}}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\beta-1}} \right] \]

In case \( u_i = 1 \) for each \( i \) in (4.10), it reduces

\[ H^\beta_{R,F}(P;U) = \frac{1}{1-R} \log \sum_{i=1}^{n} p_i^R, \quad \ldots(4.24) \]

which is well known Renyi’s [23] entropy of order \( R \).

or

\[ H^1_{R,F} = \frac{1}{1-R} \log \left( \sum_{i=1}^{n} u_i p_i^R \right), \quad \text{when} \ \beta = 1. \quad \ldots(4.23) \]

Illustration with an Example

In this section we consider production data of different companies due to Nager and Singh\textsuperscript{22} represented in Table 5.1. We calculate \( D^\beta_{R,F}(P;Q;U) \) in Table 5.2 and represent graphically in fig.5.1.

**Table 5.1: Data due to Nager and Singh\textsuperscript{22}**

| S.No | Company’s Name | 2 Sept 2010 | \( p_i \) | 3 Sept 2010 | \( q_i \) | \( u_i \) |
|------|----------------|-------------|---------|-------------|---------|--------|
| 1    | Reliance Ind.  | 10.46       | 0.1046  | 10.44       | 0.1044  | 30     |
| 2    | Infosys Tech.  | 10.04       | 0.1004  | 9.93        | 0.0993  | 29     |
| 3    | ICICI Bank     | 8.38        | 0.0838  | 8.43        | 0.0843  | 28     |
| 4    | L&T            | 8.37        | 0.0837  | 8.39        | 0.0839  | 27     |
| 5    | ITC            | 6.41        | 0.0641  | 6.47        | 0.0647  | 26     |
| 6    | HDFC           | 6.05        | 0.0605  | 6.13        | 0.0613  | 25     |
| 7    | HDFC Bank      | 6.02        | 0.0602  | 6.10        | 0.061   | 24     |
| 8    | SBI            | 5.40        | 0.054   | 5.35        | 0.0535  | 23     |
| 9    | ONGC           | 3.40        | 0.034   | 3.36        | 0.0336  | 22     |
| 10   | Bharti Airtel  | 3.18        | 0.0318  | 3.14        | 0.0314  | 21     |
| 11   | Tata Consult.  | 3.01        | 0.0301  | 2.96        | 0.0296  | 20     |
| 12   | BHEL           | 2.89        | 0.0289  | 2.85        | 0.0285  | 19     |
| 13   | Tata Steel     | 2.42        | 0.0242  | 2.44        | 0.0244  | 18     |
| 14   | Tata Motors    | 2.29        | 0.0229  | 2.30        | 0.023   | 17     |
| 15   | Hindustan Unilever | 2.14 | 0.0214 | 2.15        | 0.0215  | 16     |
| 16   | Jindal Steel   | 2.03        | 0.0203  | 2.02        | 0.0202  | 15     |
| 17   | M&M            | 1.94        | 0.0194  | 1.94        | 0.0194  | 14     |
| 18   | Hindako        | 1.61        | 0.0161  | 1.61        | 0.0161  | 13     |
| 19   | Sterlite Industry | 1.57 | 0.0157 | 1.59        | 0.0159  | 12     |
| 20   | Wipro          | 1.54        | 0.0154  | 1.54        | 0.0154  | 11     |
| 21   | Tata Power     | 1.48        | 0.0148  | 1.50        | 0.015   | 10     |
| 22   | NTPC           | 1.29        | 0.0129  | 1.28        | 0.0128  | 9      |
| 23   | Maruti Suzuki  | 1.27        | 0.0127  | 1.27        | 0.0127  | 8      |
| 24   | Hero Honda     | 1.19        | 0.0119  | 1.17        | 0.0117  | 7      |
| 25   | Reliance       | 1.18        | 0.0118  | 1.15        | 0.0115  | 6      |
| 26   | Cipla          | 1.12        | 0.0112  | 1.12        | 0.0112  | 5      |
| 27   | Jaiprakash Assoc. | 0.97 | 0.0097 | 1.01        | 0.0101  | 4      |
| 28   | DLF            | 0.85        | 0.0085  | 0.85        | 0.0085  | 3      |
| 29   | Reliance Comm. | 0.83        | 0.0083  | 0.83        | 0.0083  | 2      |
| 30   | ACC            | 0.68        | 0.0068  | 0.68        | 0.0068  | 1      |
Next we compute these values of the generalized 'useful' r-norm relative measure when \( R = 2 \) and \( \beta = 0.5 \) in the following table:

Now, the graphically representation of the new 'useful' R-norm relative information of degree \( \beta \) when \( R=2 \) and \( \beta=0.5 \) is given in fig. 5.1.

| \( p_i \) | \( q_i \) | \( u_i \) | \( D_i^R(p_i q_i u_i) \) |
|-------|-------|-------|-----------------|
| 0.1046 | 0.1044 | 30    | 0.000186868     |
| 0.1004 | 0.0993 | 29    | -0.00036369     |
| 0.0838 | 0.0843 | 28    | -0.001666033    |
| 0.0837 | 0.0839 | 27    | -0.0113086      |
| 0.0641 | 0.0647 | 26    | -0.000976981    |
| 0.0605 | 0.0613 | 25    | 0.000122931     |
| 0.0602 | 0.061  | 24    | 0.00205616      |
| 0.054  | 0.0535 | 23    | 0.004712        |
| 0.034  | 0.0336 | 22    | 0.00409425      |
| 0.0318 | 0.0314 | 21    | 0.00324745      |
| 0.0301 | 0.0296 | 20    | 0.00212248      |
| 0.0289 | 0.0285 | 19    | 0.0001693       |
| 0.0242 | 0.0244 | 18    | -0.00188175     |
| 0.0229 | 0.023  | 17    | -0.00108002     |
| 0.0214 | 0.0215 | 16    | -0.000645739    |
| 0.0203 | 0.0202 | 15    | -0.00006348     |
| 0.0194 | 0.0194 | 14    | -0.000880559    |
| 0.0161 | 0.0161 | 13    | -0.00107502     |
| 0.0157 | 0.0159 | 12    | -0.00129549     |
| 0.0154 | 0.0154 | 11    | 0.000681397     |
| 0.0148 | 0.015  | 10    | 0.000855283     |
| 0.0129 | 0.0128 | 9     | 0.0414387       |
| 0.0127 | 0.0127 | 8     | 0.00353355      |
| 0.0119 | 0.0117 | 7     | 0.004735        |
| 0.0118 | 0.0115 | 6     | 0.00125834      |
| 0.0112 | 0.0112 | 5     | -0.0085101      |
| 0.0097 | 0.0101 | 4     | -0.0139873      |
| 0.0085 | 0.0085 | 3     | 0               |
| 0.0083 | 0.0083 | 2     | 0               |
| 0.0068 | 0.0068 | 1     | 0               |

Fig.5.1: Graph of the new 'useful' R-norm relative information
The amount of divergence values can be arranged for forecasting the profit maximization in a table as

**Table 5.3: Data of Divergence Values**

| S.No. | Name of Company | Amount of Divergence |
|-------|-----------------|----------------------|
| 1     | Bharti Airtel    | 0.008552             |
| 2     | HDFC Bank        | 0.004735             |
| 3     | Maruti Suzuki    | 0.004712             |
| 4     | NTPC             | 0.004090             |
| 5     | SBI              | 0.003533             |
| 6     | Tata Power       | 0.0032471            |
| 7     | Wipro            | 0.002122             |
| 8     | Hero Honda       | 0.002056             |
| 9     | ONGC             | 0.001443             |
| 10    | HDFC             | 0.00112584           |
| 11    | Tata Consult     | 0.000681397          |
| 12    | ACC              | 0.000186868          |
| 13    | Sterlite Industry| 0.0001693            |
| 14    | Reliance         | 0.000122931          |
| 15    | ICICI Bank       | 0                   |
| 16    | Infosys Tech.    | 0                   |
| 17    | Reliance Ind.    | 0                   |
| 18    | Reliance Comm.   | -0.000036369         |
| 19    | Hindustan Unilever| -0.0000634824      |
| 20    | Jindal Steel     | -0.000645739         |
| 21    | Tata Motors      | -0.000880599         |
| 22    | Cipla            | -0.000976981         |
| 23    | Tata Steel       | -0.00107502          |
| 24    | M&M              | -0.00108002          |
| 25    | BHEL             | -0.00129549          |
| 26    | DLF              | -0.00166033          |
| 27    | Hindalco         | -0.00188175          |
| 28    | ITC              | -0.0038501           |
| 29    | Jaiprakash Assoc.| -0.0113086           |
| 30    | L&T              | -0.0139873           |

**Interpretation**

On the basis of values of generalized divergent 'useful' R-norm relative information of degree β represented in Table 5.3, we can suggest the investor to select the company having maximum divergence value for investment.

**Adjoint of the Generalized Information Measure and its Application**

On taking Q and P after interchanging in (2.9), we get

\[
D^\beta(Q; P, U) = \frac{R}{\beta} \left[ 1 - \frac{\sum_{i=1}^{n} u_i q_i^{1-\beta} p_i^{\beta+1}}{\sum_{i=1}^{n} u_i q_i^{1-\beta} p_i^{\beta+1}} \right]^{\frac{1}{\beta+1}} \quad R > 0; 0 < \beta < 1, 
\]

Thus (6.1) is called the adjoint of (2.9). Similarly, we compute these values of the adjoint of generalized measure at \( R = 2, \beta = 0.5 \) and represented table (6.1) as given below:

**Table 6.1: Values of adjoint of generalized measure**

| \( P_i \)  | \( q_i \)  | \( u_i \)  | \( D^\beta(Q; P, U) \) |
|------------|------------|------------|-------------------------|
| 0.1046     | 0.1044     | 30         | -0.00000089             |
| 0.1004     | 0.0993     | 29         | 0.000242452             |
| 0.0838     | 0.0843     | 28         | 0.00185256              |
| 0.0837     | 0.0839     | 27         | 0.00134892              |
| 0.0641     | 0.0647     | 26         | 0.00123747              |
| 0.0605     | 0.0613     | 25         | 0.000155902             |
| 0.0545     | 0.0535     | 23         | 0.00125012              |
| 0.0348     | 0.0336     | 22         | 0.00036369              |
| 0.0226     | 0.023      | 17         | 0.00187745              |
| 0.0289     | 0.0285     | 19         | 0.000032657             |
| 0.0242     | 0.0244     | 18         | 0.00204114              |
| 0.0229     | 0.023      | 17         | 0.00125012              |
| 0.0214     | 0.0215     | 16         | 0.000843364             |
| 0.0203     | 0.0202     | 15         | 0.00291878              |
| 0.0194     | 0.0194     | 14         | 0.00114291              |
| 0.0161     | 0.0161     | 13         | 0.00139457              |
| 0.0157     | 0.0159     | 12         | 0.00167957              |
| 0.0154     | 0.0154     | 11         | -0.00276932             |
| 0.0148     | 0.015      | 10         | -0.00035104             |
| 0.0129     | 0.0128     | 9          | -0.0036284              |
| 0.0127     | 0.0127     | 8          | -0.00288184             |
| 0.0119     | 0.0117     | 7          | -0.0038725              |
| 0.0118     | 0.0115     | 6          | -0.000224019            |
| 0.0112     | 0.0112     | 5          | 0.0092187               |
| 0.0097     | 0.0101     | 4          | 0.0149516               |
| 0.0085     | 0.0085     | 3          | 0                       |
| 0.0083     | 0.0083     | 2          | 0                       |
| 0.0068     | 0.0068     | 1          | 0                       |

Considering the above Table 6.1, the graph is drawn as given in the following figure 6.1:
Fig. 6.1: Graph of Non-symmetric Adjoint

Table 6.2: Arrangement of Data for Forecasting Profit

| Serial No. | Name of Company   | Divergence values |
|------------|-------------------|-------------------|
| 1          | ITC               | 0.0092187         |
| 2          | Hindalco          | 0.00204114        |
| 3          | DLF.              | 0.00185256        |
| 4          | BHEL              | 0.00167957        |
| 5          | Reliance Infras.  | 0.00155902        |
| 6          | L&T               | 0.00149516        |
| 7          | Tata Steel        | 0.00139457        |
| 8          | Jaiprakash Assoc. | 0.00134892        |
| 9          | M&M               | 0.00125102        |
| 10         | Cipla             | 0.00123747        |
| 11         | Tata Motors       | 0.00114291        |
| 12         | Jindal Steel      | 0.000843364       |
| 13         | Hindustan Unilever| 0.000291878       |
| 14         | Reliance Comm.    | 0.002424432       |
| 15         | Sterlite Industry | 0.000032657       |
| 16         | ICICI Bank        | 0                 |
| 17         | InfosysTech.      | 0                 |
| 18         | Reliance Ind.     | 0                 |
| 19         | ACC               | -0.000000898522   |
| 20         | HDFC              | -0.000224019      |
| 21         | Tata Consult       | -0.000276932      |
| 22         | Bharti Airtel     | -0.000347704      |
| 23         | HeroHonda         | -0.00179842       |
| 24         | Wipro             | -0.00187745       |
| 25         | SBI               | -0.00288184       |
| 26         | Tata Power        | -0.00300699       |
| 27         | ONGC              | -0.0036284        |
| 28         | NTPC              | -0.00386599       |
| 29         | HDFC Bank         | -0.0038725        |
| 30         | Maruti Suzuki     | -0.00451361       |
The data for forecasting the profit maximization is arranged as given below:

**Interpretation**
The adjoint of useful R-norm relative information of degree in decreasing order in the table (6.1 to suggest the investor to make investment in the company of maximum divergence

**Conclusion**
In this paper we have defined and characterized the generalized 'useful' R-norm relative information of degree $\beta$ and discussed its particular cases also. The application of this information measure has been studied. The adjoint of this measure is defined and its application in share market and decision making problems are described graphically. The 'Useful' R-norm relative information measures of degree and its adjoint are defined and studied in this communication can further be generalized parametrically and applied in planning, forecasting, agriculture, etc.

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**Conflict of Interest**
There is no conflict of interest among the authors of this paper.

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