The translation invariant massive Nelson model: III. Asymptotic completeness below the two-boson threshold

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May 3, 2014

Abstract

We show asymptotic completeness of two-body scattering for a class of translation invariant models describing a single quantum particle (the electron) linearly coupled to a massive scalar field (bosons). Our proof is based on a recently established Mourre estimate for these models. In contrast to previous approaches, it requires no number cutoff, no restriction on the particle-field coupling strength, and no restriction on the magnitude of total momentum. Energy, however, is restricted by the two-boson threshold, admitting only scattering of a dressed electron and a single asymptotic boson. The class of models we consider include the UV-cutoff Nelson and polaron models.
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1 Introduction

The last two decades witnessed substantial progress in our understanding of asymptotic completeness (AC) in Quantum Field Theory (QFT). On the relativistic side first examples of massive and massless theories with complete particle interpretation have been constructed in [34, 12]. On the side of non-relativistic QFT, far-reaching insights have been obtained by application of methods from many-body quantum mechanics [14, 7, 27, 33, 8, 25]. AC of systems describing a confined quantum-mechanical particle (the electron) interacting with second-quantized Bose fields is well under control in the case of massive field quanta (bosons) [31, 32, 9, 10, 20, 2] and there is rapid progress on the massless side [46, 24, 6, 16, 17]. However, the case of translation invariant quantum-mechanical systems coupled to quantum fields is far from being fully understood, even if the bosons are massive. The main difficulty here is the phenomenon of the electron mass renormalization, familiar from relativistic QFT. In the existing works this difficulty is overcome only at a cost of technical assumptions on the coupling strength, total momentum of the system and dispersion relations of the electron and bosons [21, 22] or by means of a number cutoff [26]. In the present paper we show that all these restrictions can be eliminated, at least at the level of two-body scattering: We show AC below the two-boson threshold in a class of translation invariant massive QFT under very general assumptions, including the massive Nelson model [40] and the Fröhlich polaron model [18] with physical (infrared-singular) coupling function. We stress that in the case of the polaron model, with constant dispersion relation of bosons, the physical picture of propagating particles is not self-evident, not to speak of AC. It comes to light only after taking the electron mass renormalization properly into account and extracting the effective dynamics of the electron-boson system. This is achieved for the first time in the present work.

We consider a class of models describing a free quantum-mechanical particle, e.g. a non-relativistic electron, linearly coupled to a UV-cutoff massive scalar field, e.g. longitudinal optical phonons or massive relativistic bosons. The isolated energy-momentum spectrum, i.e. the region below the one-boson threshold, is under our assumptions an analytic variety. It consists of the ground state mass shell, which is non-degenerate for all total momenta, and possibly excited isolated mass shells that may cross each other. To each mass shell one can associate a distinct dressed electron species. They have different dispersion relations, hence different masses, and some may even have group velocity in a direction opposite to momentum (non-increasing dispersion). Incoming and outgoing states are of the form $\Psi \otimes \eta$, where $\Psi$ is a dressed electron state (or superpositions thereof) and $\eta$ is a vector in Fock space describing a collection of free asymptotic bosons. We note in passing that during a scattering process the outgoing dressed electron may differ from the incoming dressed electron i.e. the dressed electron species may not be conserved by collisions with bosons. The central objects of our investigation are the (conventional) wave operators, defined in (1.3) below, which map incoming/outgoing states to states in the physical Hilbert space. In particular $\Psi \otimes |0\rangle$, where $|0\rangle$ is the vacuum vector, is mapped into the dressed electron state $\Psi$. For general $\eta$, vectors from the ranges of the wave operators describe scattering states of dressed electrons and bosons. As usual, AC is defined as unitarity of the wave operators, which means that all states of physical interest belong to their ranges.

Existence of the wave operators is known for the Nelson model [9], but not for the polaron model. In the present paper we construct the wave operators and prove AC under rather natural assumptions which cover both the Nelson and polaron case: We employ no number cutoff, hence a dressed electron consists of a bare electron accompanied by an infinite virtual boson cloud. There are no restrictions on the electron-field coupling strength and no
limitations on the magnitude of total momentum. The energy is only restricted by the (total momentum dependent) two-boson threshold which defines the largest spectral subspace on which only single-boson scattering processes take place. Above this threshold, we are not – yet – able to handle the plethora of scattering channels available.

To explain the novel strategy of our proof of AC, we recall several standard concepts, which will be defined precisely in Section 2 and Appendix A. We use the $\Gamma$-functor notation of Segal for constructions of spaces and operators in the context of second quantization. The Hilbert spaces of incoming and outgoing configurations are given by $\mathcal{H}_\pm := \mathcal{H}_{\text{bnd}} \otimes \mathcal{F}$, where $\mathcal{H}_{\text{bnd}}$ contains the dressed electron states and $\mathcal{F}$ is the bosonic Fock space over the single-boson space $\mathfrak{h}$. The extended Hamiltonian and momentum operators are defined as

$$H^{\text{ex}} := H \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega) \quad \text{and} \quad P^{\text{ex}} := P \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k),$$

where $\omega$ is the dispersion relation of the bosons and $(P, H)$ denote the total energy-momentum operators of our system. We recall that $H$ acts on $\mathcal{H}_{\text{bnd}}$ as a direct sum of multiplication operators, one for each dressed electron species. For any pair of bounded operators $q_0, q_\infty$ on $\mathfrak{h}$ we define the map $\tilde{\Gamma}(q_0, q_\infty)^*$, from a domain in $\mathcal{H}_\pm$ to $\mathcal{H}$, by the relation:

$$\tilde{\Gamma}(q_0, q_\infty)^*(\Psi \otimes a^*(h_1) \cdots a^*(h_n)|0\rangle) = a^*(q_\infty h_1) \cdots a^*(q_\infty h_n) \Gamma(q_0) \Psi.$$  

(1.2)

The goal of our investigation is to establish the existence and unitarity of the wave operators

$$\Omega^\pm = s- \lim_{t \to \pm \infty} e^{itH} \tilde{\Gamma}(1, 1)^* e^{-itH^{\text{ex}}},$$

(1.3)

below the two-boson threshold (in the joint spectrum of $(H^{\text{ex}}, P^{\text{ex}})$). For reasons which will become clear below, we divide this region of the spectrum into small subsets $\mathcal{O} \subset \mathbb{R}^r \times \mathbb{R}$. For each $\mathcal{O}$ we construct a localized right inverse of $\Omega^\pm$ on the corresponding spectral subspace of $(P, H)$. As noted in [9], a natural candidate has the form

$$W_{\mathcal{O}}^\pm = s- \lim_{t \to \pm \infty} e^{itH^{\text{ex}} \tilde{\Gamma}(q_0^t, q_\infty^t)} e^{-itH}$$

(1.4)

where $q_0^t, q_\infty^t$ are some time-dependent families of operators s.t. $q_0^t + q_\infty^t = 1$ so that one can exploit the relation $\tilde{\Gamma}(1, 1)^* \tilde{\Gamma}(q_0^t, q_\infty^t) = 1$. One important difference between our approach and previous work on asymptotic completeness in QFT consists in the construction of the operators $q_0^t, q_\infty^t$.

Before we explain this construction, we recall that the Hamiltonian $H$ has a direct integral decomposition into fiber Hamiltonians $H(\xi)$ at fixed momentum $\xi$. As shown in [33], and stated precisely in Theorem 2.2 below, if $\mathcal{O}$ is sufficiently small (and localized outside of some sets of measure zero) we can choose $(\xi_0, \epsilon_0) \in \mathcal{O}$, a neighbourhood $\mathcal{J}_0$ of $\lambda_0$, and $\epsilon_m > 0$ s.t.

$$1_{\mathcal{J}_0}(H(\xi))|i[H(\xi), \mathrm{d}\Gamma(a_{\xi_0})]|1_{\mathcal{J}_0}(H(\xi)) \geq \epsilon_m 1_{\mathcal{J}_0}(H(\xi)),$$

(1.5)

$$1_{\mathcal{J}_0}(H^{(1)}(\xi))|i[H^{(1)}(\xi), 1 \otimes a_{\xi_0}]|1_{\mathcal{J}_0}(H^{(1)}(\xi)) \geq \epsilon_m 1_{\mathcal{J}_0}(H^{(1)}(\xi)),$$

(1.6)

where $H^{(1)} := H \otimes 1 + 1 \otimes \omega$ acts on $\mathcal{H}^{(1)} = \mathcal{H} \otimes \mathfrak{h}$. The estimates hold true for $\xi$ belonging to a small neighbourhood of $\xi_0$, such that the Cartesian product of this neighbourhood with $\mathcal{J}_0$ contains $\mathcal{O}$. The operator $a_{\xi_0}$ has the form

$$a_{\xi_0} = \frac{1}{2} \{ v_{\xi_0} \cdot i\nabla_k + i\nabla_k \cdot v_{\xi_0} \},$$

(1.7)
where $i\nabla_k$ is the boson position operator and $v_{\xi_0}$ is a vector field in momentum space, which carries information about the dispersion relations of incoming/outgoing dressed electrons present in the energy-momentum region $\mathcal{O}$. Now we define $\tilde{a}_{\xi_0} := \frac{1}{2}\{v_{\xi_0} \cdot z + z \cdot v_{\xi_0}\}$, where $z = i\nabla_k - y$ is the relative distance between the electron and the boson, and set 

$$q_0^j := q_0(\tilde{a}_{\xi_0}/t) \quad \text{and} \quad q_{\infty}^j := q_{\infty}(\tilde{a}_{\xi_0}/t), \quad (1.8)$$

where $q_0$, $q_{\infty}$ are smooth approximate characteristic functions of $(-\infty, c_0]$, $[c_0, \infty)$, $c_0 > 0$ is smaller than $c_m$, and $q_0 + q_{\infty} = 1$. With such a choice of $q_0^j$, $q_{\infty}^j$, closely tied to Mourre theory, strong convergence in $\mathcal{L}_1^0$ can be established using the positive commutator estimates $1.3, 1.4$. We note that this convergence result holds only in the spectral subspace of $\mathcal{O}$. Indeed, only in this subspace the definition $1.5$ holds with the operator $v_{\xi_0}$, which entered into the definitions $1.8$. The fact that $W_0^{\pm*}$ has to be defined for each region $\mathcal{O}$ separately is, however, not an obstacle, since we use this operator only as a tool to show the existence and unitarity of the wave operators $\Omega^\pm$, which do not contain any information about the (non-canonical) operators $v_{\xi_0}$.

A large part of our paper is devoted to the proof of strong convergence of the localized inverse of the wave operator in $\mathcal{L}_1^0$ with the help of the Mourre estimates. An important intermediate step here is a novel minimal-velocity propagation estimate (See Proposition 4.1 below). As our proof of this propagation estimate differs significantly from the arguments available in the literature, let us state here its special case and outline the proof: Let $j_0, j_{\infty}$ be smooth approximate characteristic functions of $(-\epsilon, \epsilon)$, $\mathbb{R}\setminus(-\epsilon, \epsilon)$ s.t. $j_0^2 + j_{\infty}^2 = 1$ and let $j^j := (j_0(a_{\xi_0}/t), j_{\infty}(a_{\xi_0}/t))$. Then there exists $c > 0$ such that for all $\Psi \in \mathcal{F}$:

$$\int_{-1}^{\infty} dt \frac{1}{t}(\Psi_t, \hat{\Gamma}(j^j)^* \chi^{(1)}(1 \otimes q^j(a_{\xi_0}/t))\chi^{(1)}\hat{\Gamma}(j^j)\Psi_t) \leq c\|\Psi\|^2, \quad (1.9)$$

where $q^j$ is a smooth approximate characteristic function of $\mathcal{I} := [-R, -\epsilon] \cup [\epsilon, c_0]$, $\Psi_t := e^{-itH(\xi)t}\Psi$, $\chi^{(1)} := \chi(H^{(1)}(\xi))$ and $\chi \in C^\infty_0(\mathbb{R})$ is supported below the two-boson threshold. Proceeding to the proof of $1.9$, we consider a propagation observable $\Phi(t) := \chi d\Gamma(q^j^0)\chi$ where $\chi := \chi(H(\xi))$, $q(\lambda) := \int_{-1}^{\lambda} q^j(s)ds$ and $q^j := q(q_{\xi_0}/t)$. In the standard proofs of propagation estimates in non-relativistic QFT [9, 21] one computes to the leading order in $t$ the Heisenberg derivative

$$D\Phi(t)$D\Phi(t) = \partial_t \Phi(t) + i[H(\xi), \Phi(t)] \quad (1.10)$$

making use of the concrete expression $2.9$ for the Hamiltonian $H(\xi)$. In the presence of the electron mass renormalization this strategy breaks down for large coupling strength, because it introduces into the analysis the bare dispersion relation $\Omega$ of the electron, appearing in $2.9$. To extract the correct physical dynamics of the electron-boson system we proceed differently: Making use of the fact that $\hat{\Gamma}(j^j)^* \hat{\Gamma}(j^j) = 1$, we write

$$D\Phi(t) = \hat{\Gamma}(j^j)^* \hat{\Gamma}(j^j)\chi Dd\Gamma(q^j)\chi = \hat{\Gamma}(j^j)^* \chi^{(1)}D^{(1)}(1 \otimes q^j)\chi^{(1)}\hat{\Gamma}(j^j) + O(t^{-2}), \quad (1.11)$$

where $D^{(1)}$ is the Heisenberg derivative w.r.t. the Hamiltonian $H^{(1)}(\xi)$ and $O(t^{-2})$ denotes a term bounded in norm by $ct^{-2}$. The last step in $1.11$, justified in Proposition 3.3 consists in commuting $\hat{\Gamma}(j^j)$ to the right and showing that the resulting rest-terms are of order $O(t^{-2})$. Here we only indicate how to exchange $\hat{\Gamma}(j^j)$ with $\chi$, since it contains the essence of the argument: First, we make use of the fact that $\hat{\Gamma}(j^j)\chi(H(\xi)) = \chi(H^{ex}(\xi))\hat{\Gamma}(j^j) + O(t^{-1})$ (Lemma 2.3). Next, we exploit that $\chi$ is localized below the two-boson threshold to write $\chi(H^{ex}(\xi)) = \chi(H(\xi)) \oplus \chi(H^{(1)}(\xi))$ (Lemma 2.4). Finally, we show that the first term in
this direct sum gives rise to expressions of order \( O(t^{-2}) \) if \( j_0 \) is supported outside of \( I \). Given expression (1.11), we estimate the commutator \( i[H^{(1)}(\xi), (1 \otimes q^t)] \) from below, using the Mourre estimate (1.6) and, by integrating both sides of the resulting expression along the time evolution, we obtain the propagation estimate (1.9).

It is clear from the above discussion that our proof of AC is very different from the standard arguments used in the absence of the electron mass renormalization \([27, 9]\) or in the weak coupling regime \([21]\). In particular, our argument does not rely on the phase-space propagation estimate, which is problematic in the presence of level crossings in the isolated spectrum. By our methods we can handle a large class of electron and boson dispersion relations and, due to the fact that \( v_\xi_0 \) can be chosen to vanish for small momenta, we can cover the infrared-singular physical coupling of the polaron model. In addition, no smallness conditions on the coupling strength are involved. Thus, similarly to the classical results on asymptotic completeness in quantum mechanics \([7, 27, 43]\), our result applies to a very large class of models which contains experimentally realizable physical systems (e.g. the polaron). We are convinced that our analysis provides a solid foundation for future developments of scattering theory in QFT.

Going beyond the two-boson threshold for the models studied here will be a challenging task requiring more involved constructions of propagation observables, due to the more complicated channel structure. While we do have some ideas as to how to proceed, there are technical obstructions requiring new insights to overcome. Another promising direction of future research concerns the spectral and scattering theory of many-body dispersive systems. The methods developed in this paper, combined with those of \([38]\), can be viewed from a broader perspective as a general strategy to deal with such systems. We hope – in fact expect – that one can study many body Schrödinger operators, with relativistic kinetic energy, as well as spin-wave scattering, i.e. the magnon model, with the aid of the techniques developed here. See \([23, 28, 47]\), where both of these long-standing open problems are discussed. Finally, we would like to point out that collision theory of dispersive systems is an important intermediate step towards the problem of asymptotic completeness in local relativistic QFT, as for example the \( P(\phi)_2 \) models. This observation has recently been exploited in \([13]\) to show the existence of certain asymptotic observables in these theories. Thus an application of the methods of the present paper in the local relativistic setting is another promising – and tractable – research direction. We recall that partial results on asymptotic completeness in \( P(\phi)_2 \) models can be found in \([3, 44]\). For recent progress on relativistic scattering theory we refer to \([11, 12, 34]\).

This paper is organized as follows: In Section 2 we define the class of models under study, summarize the known facts concerning their spectrum, including Mourre theory, and state the main results of this paper. In Section 3 we derive convenient representations for the Heisenberg derivatives of certain propagation observables which are then combined with Mourre estimates in Section 4 to derive minimal velocity propagation estimates. These propagation estimates are the key input to the proofs of existence of the relevant asymptotic observables in Section 5 including the localized inverses of the wave operators of the form (1.4). In Section 6 we establish properties of these operators which are then used in Section 7 to prove the existence and unitarity of the (conventional) wave operators (1.3). More technical steps of our investigation are postponed to appendices.

Acknowledgment: This project started in collaboration with Morten Grud Rasmussen, who contributed to a proof of AC for the polaron model with a short-range condition. This different proof, which preceded the present argument, will be published in a separate paper by the present authors and Morten Grud Rasmussen.
The authors thank Jan Dereziński, Christian Gérard, and Herbert Spohn for useful discussions we had during the course of this work. We acknowledge financial support of the Danish Council for Independent Research grant no. 09-065927 "Mathematical Physics", and hospitality of the Hausdorff Research Institute for Mathematics, Bonn. Moreover, W.D. is grateful for the support of the Lundbeck Foundation and the German Research Foundation (DFG), the latter within the grant SP181/25–2 and stipend DY107/1–1.

2 Preliminaries and Results

2.1 Hamiltonian

Let \( K = L^2(\mathbb{R}^\nu) \) be the Hilbert space of a quantum mechanical particle moving in \( \mathbb{R}^\nu \), whose position is denoted by \( y \) and momentum by \( D_y := -i\nabla_y \). Let \( \mathfrak{h} = L^2(\mathbb{R}^s_k) \) be the Hilbert space of a single boson, whose dispersion relation will be denoted \( \omega(k) \). The Hilbert space for the Bose field is the Fock space

\[
\mathcal{F} = \Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)},
\]

(2.1)

where \( \mathcal{F}^{(n)} = \Gamma^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\otimes n} \) is the symmetric tensor product of the single-boson spaces and the vacuum vector will be denoted by \( |0\rangle \). The boson creation and annihilation operators are denoted by \( a^*(k) \), \( a(k) \) and satisfy the canonical commutation relations

\[
[a(k), a^*(k')] = \delta(k - k') \quad \text{and} \quad [a(k), a(k')] = [a^*(k), a^*(k')] = 0.
\]

The total energy and momentum operators of the bosons are given by

\[
H_{\text{ph}} := d\Gamma(\omega) = \int_{\mathbb{R}^\nu} dk \omega(k)a^*(k)a(k),
\]

(2.2)

\[
P_{\text{ph}} := d\Gamma(k) = \int_{\mathbb{R}^\nu} dk ka^*(k)a(k).
\]

(2.3)

The Hilbert space of the system consisting of the electron and the bosons is \( \mathcal{H} = K \otimes \mathcal{F} \). The dynamics is governed by the Hamiltonian

\[
H = \Omega(D_y) \otimes 1 + 1 \otimes H_{\text{ph}} + \phi(G_y),
\]

(2.4)

where the interaction term is given by

\[
\phi(G_y) := \int_{\mathbb{R}^\nu} dk \left( e^{-iky}G(k)1 \otimes a^*(k) + e^{iky}G(k)1 \otimes a(k) \right).
\]

(2.5)

Under the minimal conditions on \( \Omega, \omega \) and \( G \), specified below following [38], this Hamiltonian is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^\nu) \otimes \mathcal{C} \), where \( \mathcal{C} := \Gamma_{\text{fin}}(C_0^\infty(\mathbb{R}^\nu)) \) is defined in Appendix A.

Condition 1. (Minimal Conditions). There exists \( s_\Omega \in [0, 2] \) and \( C > 0 \) s.t. the dispersion relation \( \omega \) and the coupling function \( G \) satisfy:

(MC1) \( \omega \in C(\mathbb{R}^\nu), \Omega \in C^2(\mathbb{R}^\nu), \langle k \rangle^6 G \in L^2(\mathbb{R}^\nu) \), \( \langle k \rangle = \sqrt{k^2 + 1} \).

(MC2) \( m := \inf_{k \in \mathbb{R}^\nu} \omega(k) > 0. \)

(MC3) \( \forall k \in \mathbb{R}^\nu \) we have \( \omega(k) \leq C\langle k \rangle, \Omega(k) \geq C^{-1}\langle k \rangle^{s_\Omega} - C. \)

(MC4) \( |\partial^\alpha \omega(k)| \leq C\langle k \rangle^{s_\Omega-|\alpha|}, \) for all multiindices \( \alpha \) with \( 0 \leq |\alpha| \leq 2. \)
\((\text{MC5})\) \(\forall k_1, k_2 \in \mathbb{R}^\nu\) we have \(\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2)\).

\((\text{MC6})\) Either \(\lim_{|k| \to \infty} \omega(k) = \infty\) or: \(\sup_{k \in \mathbb{R}^\nu} \omega(k) < \infty\) and \(\lim_{|k| \to \infty} \Omega(k) = \infty\).

We note that \((\text{MC1})\) is stronger than in \([38]\).

We recall that the Hamiltonian \((2.4)\) commutes with the total momentum operators given by

\[ P = D_y \otimes 1 + 1 \otimes P_{ph}, \tag{2.6} \]

thus it has a fiber decomposition. More precisely, using the unitary transform of Lee-Low-Pines \([35]\)

\[ I_{\text{LLP}} := (F \otimes 1) \circ \Gamma(e^{ik\cdot y}), \tag{2.7} \]

where \(F\) is the Fourier transform in the electron position variable and \(\Gamma\) the second quantization functor (cf. Appendix \(A\)), we obtain

\[ I_{\text{LLP}} H I_{\text{LLP}}^* = \int_{\mathbb{R}^\nu} d\xi H(\xi). \tag{2.8} \]

The fiber Hamiltonians have the form

\[ H(\xi) = \Omega(\xi - P_{ph}) + H_{ph} + \phi(G), \tag{2.9} \]

where \(\phi(G) := \phi(G_y)|_{y=0}\), and are essentially self-adjoint on \(\mathcal{C}\). The joint spectrum of the family of commuting self-adjoint operators \((P, H)\) is given by

\[ \Sigma = \{ (\xi, \lambda) \in \mathbb{R}^{\nu+1} \mid \lambda \in \sigma(H(\xi)) \}. \tag{2.10} \]

It can be decomposed into the pure-point, absolutely continuous and singular continuous parts

\[ \Sigma = \Sigma_{pp} \cup \Sigma_{ac} \cup \Sigma_{sc} \tag{2.11} \]

defined as \(\Sigma_i = \{ (\xi, \lambda) \in \mathbb{R}^\nu \times \mathbb{R} \mid \lambda \in \sigma_i(H(\xi)) \}, \) where \(i \in \{pp, ac, sc\}\). We denote the bottom of the spectrum of the fiber Hamiltonians by

\[ \Sigma_0(\xi) := \inf \sigma(H(\xi)) \tag{2.12} \]

and the bottom of the spectrum of the full operator by \(\Sigma_0 := \inf_{\xi \in \mathbb{R}^\nu} \Sigma_0(\xi)\). Moreover, we introduce

\[ \Sigma_0^{(n)}(\xi, \underline{k}) := \Sigma_0(\xi - \sum_{j=1}^n k_j) + \sum_{j=1}^n \omega(k_j) \tag{2.13} \]

and define the \(n\)-boson thresholds

\[ \Sigma_0^{(n)}(\xi) := \inf_{\underline{k} \in \mathbb{R}^{\nu n}} \Sigma_0(\xi, \underline{k}). \tag{2.14} \]

By the HVZ Theorem \([19, 36, 37, 45]\),

\[ \sigma_{\text{ess}}(H(\xi)) = \Sigma_0^{(1)}(\xi, \infty). \tag{2.15} \]
and below $\Sigma_0^{(1)}(\xi)$ the spectrum consists of locally finitely many eigenvalues of finite multiplicity, which can only accumulate at $\Sigma_0^{(1)}(\xi)$. Due to the subadditivity assumption \([MC2]\) on $\omega$, we have

$$\Sigma_0^{(n)}(\xi) \geq \Sigma_0^{(m)}(\xi)$$

(2.16)

for any $n > m$. The inequality is strict if $\lim_{|k| \to \infty} \omega(k) = \infty$. If $M = \sup_{k \in \mathbb{R}^n} \omega(k) < \infty$, then the inequality is also strict if $2 \lim \inf_{|k| \to \infty} \omega(k) > M$, which is satisfied by the constant polaron relation \([37]\). In these cases the region $\mathcal{E}^{(1)}$, where

$$\mathcal{E}^{(1)} = \{(\xi, \lambda) \in \mathbb{R}^{n+1} \mid \lambda \in \mathcal{E}^{(1)}(\xi)\},$$

$$\mathcal{E}^{(1)}(\xi) = \{\lambda \in \mathbb{R} \mid \Sigma_0^{(1)}(\xi) \leq \lambda < \Sigma_0^{(2)}(\xi)\},$$

(2.17)

is non-empty.

### 2.2 Extended Hamiltonian

The formalism of extended Hilbert space, which we present in this section and in Appendix \([A.2]\) was introduced in \([9]\) and used later on in \([2, 10, 20, 36, 37]\) in the context of spectral and scattering theory. Let us define the extended Fock space and the extended physical Hilbert space as follows

$$\mathcal{F}^{\text{ex}} = \mathcal{F} \otimes \mathcal{F} = \mathcal{F} \oplus \bigoplus_{\ell=1}^{\infty} \mathcal{F} \otimes \mathcal{F}^{(\ell)} \simeq \mathcal{F} \oplus \bigoplus_{\ell=1}^{\infty} L^2_{\text{sym}}(\mathbb{R}^\nu; \mathcal{F}),$$

(2.18)

$$\mathcal{H}^{\text{ex}} = \mathcal{H} \otimes \mathcal{F} = \mathcal{H} \oplus \bigoplus_{\ell=1}^{\infty} \mathcal{H} \otimes \mathcal{F}^{(\ell)},$$

(2.19)

where we made use of the identification $\mathcal{F} \otimes \mathcal{F}^{(\ell)} \simeq L^2_{\text{sym}}(\mathbb{R}^\nu; \mathcal{F})$. The extended Hamiltonian and extended total momentum operators are given by

$$H^{\text{ex}} = H \otimes 1 + 1 \otimes d\Gamma(\omega) = H \oplus \bigoplus_{\ell=1}^{\infty} H^{(\ell)},$$

(2.20)

$$P^{\text{ex}} = P \otimes 1 + 1 \otimes d\Gamma(k) = P \oplus \bigoplus_{\ell=1}^{\infty} P^{(\ell)}.$$  

(2.21)

Here

$$H^{(\ell)} = H \otimes 1 + 1 \otimes d\Gamma^{(\ell)}(\omega), \quad P^{(\ell)} = P \otimes 1 + 1 \otimes d\Gamma^{(\ell)}(k).$$

(2.22)

The operators $(H^{\text{ex}}, P^{\text{ex}})$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes C^{\text{ex}}$, where $C^{\text{ex}} := \mathcal{C} \otimes \mathcal{C}$. Similarly, $(H^{(\ell)}, P^{(\ell)})$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu) \otimes C^{(\ell)}$, where $C^{(\ell)} := \mathcal{C} \otimes C_0^\infty(\mathbb{R}^\nu)^{\otimes \ell}$. Since $(H^{\text{ex}}, P^{\text{ex}})$ as well as $(P^{(\ell)}, H^{(\ell)})$, for $\ell \in \mathbb{N}$, form commuting families of self-adjoint operators, we can introduce their joint spectral resolutions $E^{\text{ex}}(\cdot)$ and $E^{(\ell)}(\cdot)$. We use extended Lee-Low-Pines transformations to perform fiber decompositions of $H^{\text{ex}}$ and $H^{(\ell)}$ w.r.t. the total momentum. They have the form

$$I^{\text{LLP}}_\text{LLP} := (F \otimes 1) \circ \Gamma^{\text{ex}}(e^{ik_\cdot y}) = I^{\text{LLP}} \oplus \bigoplus_{\ell=1}^{\infty} I^{(\ell)}_{\text{LLP}},$$

(2.23)
where \( F \) is the Fourier transform in the electron position variable, \( \Gamma_{\text{ex}}(e^{ik \cdot y}) \) is defined as explained in Section 1.2 of [38] and \( I_{\text{LLP}}^{(\ell)} := (I_{\text{LLP}}^{\text{ex}})_{\mu \otimes \mathcal{F}}. \) There holds

\[
H_{\text{ex}} = I_{\text{LLP}}^{\text{ex}} \left( \int_{\mathbb{R}^\nu} d\xi P_{\text{ex}}(\xi) \right) I_{\text{LLP}}^{\text{ex}}, \quad H^{(\ell)} = I_{\text{LLP}}^{(\ell)*} \left( \int_{\mathbb{R}^\nu} d\xi P^{(\ell)}(\xi) \right) I_{\text{LLP}}^{(\ell)}. \tag{2.24}
\]

The fiber Hamiltonians \( H_{\text{ex}}(\xi) \) are essentially self-adjoint on \( \mathcal{C}^{\text{ex}} \) and have the form

\[
H_{\text{ex}}(\xi) = \Omega(\xi - d\Gamma_{\text{ex}}(k)) + d\Gamma_{\text{ex}}(\omega) + \phi(G) \otimes 1, \tag{2.25}
\]

where \( d\Gamma_{\text{ex}}(\cdot) \) is defined in Appendix A.2. The extended fiber Hamiltonians \( H_{\text{ex}}(\xi) \) can be decomposed just as for \( H_{\text{ex}} \), cf. (2.20), and we get as expected

\[
H_{\text{ex}}(\xi) = H(\xi) \oplus \left( \bigoplus_{\ell=1}^{\infty} H^{(\ell)}(\xi) \right). \tag{2.26}
\]

Since there is no interaction in the second tensor component of \( H^{(\ell)}(\xi) \), which is simply a multiplication operator, we can decompose further into a direct integral over momenta from \( \mathbb{R}^\nu \):

\[
H^{(\ell)}(\xi) = \int_{\mathbb{R}^\nu} dk H^{(\ell)}(\xi, k), \tag{2.27}
\]

\[
H^{(\ell)}(\xi; k) = H(\xi - \sum_{j=1}^{\ell} k_j) + \left( \sum_{j=1}^{\ell} \omega(k_j) \right) 1. \tag{2.28}
\]

In our investigation we will often make use of the following simple fact:

**Lemma 2.1.** Let \( \chi: \mathbb{R} \to \mathbb{R} \) be a bounded Borel function, with essential support in the set \((-\infty, \Sigma_0^{(n)}(\xi)) \). Then

\[
\chi(H_{\text{ex}}(\xi)) = \chi(H(\xi)) \oplus \left( \bigoplus_{\ell=1}^{n-1} \chi(H^{(\ell)}(\xi)) \right). \tag{2.29}
\]

**Proof.** Let \( \ell \geq n \). We recall that

\[
\Sigma_{0}^{(\ell)}(\xi) = \inf_{k \in \mathbb{R}^\nu} \left( \Sigma_0(\xi - \sum_{j=1}^{\ell} k_j) + \sum_{j=1}^{\ell} \omega(k_j) \right). \tag{2.30}
\]

Consequently,

\[
H^{(\ell)}(\xi) = \int_{\mathbb{R}^\nu} dk \left( H(\xi - \sum_{j=1}^{\ell} k_j) + \sum_{j=1}^{\ell} \omega(k_j) \right) 1 \geq \Sigma_0^{(\ell)}(\xi) 1. \tag{2.31}
\]

Since \( \Sigma_0^{(\ell)}(\xi) \geq \Sigma_0^{(n)}(\xi) \), and \( \chi \) is supported below \( \Sigma_0^{(n)}(\xi) \), only the first \( n-1 \) terms of the expansion

\[
\chi(H_{\text{ex}}(\xi)) = \chi(H(\xi)) \oplus \left( \bigoplus_{\ell=1}^{\infty} \chi(H^{(\ell)}(\xi)) \right) \tag{2.32}
\]

are non-zero. \( \square \)
2.3 Structure of the spectrum

To continue our discussion of the spectrum of $H$ we need more restrictive assumptions. Following [38], we state:

**Condition 2. (Spectral Theory).** We impose:

**(ST1)** $\Omega$ and $\omega$ are real analytic functions.

**(ST2)** $G$ admits 2 distributional derivatives with $\partial_k^2 G \in L^2_{\text{loc}}(\mathbb{R}^\nu \setminus \{0\})$, for all $1 \leq |\alpha| \leq 2$.

**(ST3)** For all orthogonal matrices $O \in O(\nu)$ and all $k \in \mathbb{R}^\nu$ we have $\omega(Ok) = \omega(k)$, $\Omega(Ok) = \Omega(k)$, and $G(Ok) = G(k)$ almost everywhere.

**(ST4)** $\sup_{k \in \mathbb{R}^\nu} |\partial_k^2 \omega(k)| < \infty$ for all $|\alpha| \geq 1$ and $|\partial_k^3 \Omega(k)| \leq C_3(k)^{s_\Omega - |\beta|}$ for $|\beta| \geq 2$.

$s_\Omega \in [0, 2]$ appeared in Condition 1.

We note that (ST2) coincides with the corresponding condition from [38] for $n_0 = 2$. (ST4) is stronger than in [38].

Making use of Kato’s analytic perturbation theory [33] we obtain a description of the isolated part of the spectrum (cf. (2.15) above):

$$\Sigma_{\text{iso}} = \{(\xi, E) \in \Sigma \mid E < \Sigma_{0}^{(1)}(\xi)\}. \quad (2.33)$$

This spectrum consists of analytic mass shells and level crossings. The set of level crossings is defined as

$$\mathcal{X} := \{(\xi, E) \in \Sigma_{\text{iso}} \mid \forall n \in \mathbb{N} : \Sigma_{\text{iso}} \cap B_{1/n}(\xi, E) \text{ is not a graph}\}. \quad (2.34)$$

The connected components of $\mathcal{X}$ are $S^{\nu-1}$-spheres. They have the form $\partial B(0; R) \times \{E\}$, or, in the degenerate case, $\{0\} \times \{E\}$. They can accumulate either at infinity or at the bottom of the essential spectrum. The level crossings are connected in $\Sigma_{\text{iso}}$ by shells which are real-analytic manifolds. Each shell is a pair $(A, S)$, where $A = \{\xi \in \mathbb{R}^\nu \mid r < |\xi| < R\}$, $0 \leq r < R$, is an open annulus or an open ball centred at zero. The function $S : A \to \mathbb{R}$ is real analytic and rotation invariant.

The structure of the continuous spectrum in $E^{(1)}$, cf. (2.17), was studied in [38] with the help of Mourre theory. As these results are very relevant for the present investigation we summarize them here. For any $\xi \in \mathbb{R}^\nu$ the conjugate operator has the form $A_\xi = d\Gamma(a_\xi)$, where

$$a_\xi = \frac{1}{2}\{v_\xi \cdot i\nabla_k + i\nabla_k \cdot v_\xi\}, \quad (2.35)$$

and $v_\xi \in C_0^\infty(\mathbb{R}^\nu \setminus \{0\}; \mathbb{R}^\nu)$ is a suitable vector field constructed in [38]. It is easily seen that $a_\xi$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$ and $A_\xi$ is essentially self-adjoint on $C$. In [38] one can find a construction of the threshold sets $T^{(1)}(\xi) \subset \mathbb{R}$, $\xi \in \mathbb{R}^\nu$, which carry information about the structure of the isolated spectrum, and exceptional sets

$$\text{Exc}(\xi) = (0, \omega(0)) + \Sigma_{\text{iso}}(\xi), \quad \xi \in \mathbb{R}^\nu, \quad (2.36)$$

which account for a possible singularity of the coupling function $G$ at zero. (We recall that in [38], formula (1.35), $\text{Exc}(\xi)$ was defined to be empty for $G$ regular at zero. Here it is always given by (2.36)). The main result of [38] can be summarized as follows:
Theorem 2.2. Assume Conditions 1 and 2. Let $\xi \in \mathbb{R}^\nu$ unless stated otherwise. Then the following properties hold true:

(a) The sets $E^{(1)}(\xi) \cap T^{(1)}(\xi)$ and $E^{(1)}(\xi) \cap \text{Exc}(\xi)$ are locally finite with possible accumulation points only at $\Sigma^{(2)}_0(\xi)$.

(b) All eigenvalues in $\sigma_{pp}(H(\xi)) \cap E^{(1)}(\xi) \setminus (T^{(1)}(\xi) \cup \text{Exc}(\xi))$ have finite multiplicity.

(c) The set $\sigma_{pp}(H(\xi)) \cap E^{(1)}(\xi)$ is at most countable, with accumulation points at most in $T^{(1)}(\xi) \cup \text{Exc}(\xi) \cup \{\Sigma^{(2)}_0(\xi)\}$.

(d) Let $(\xi_0, \lambda_0) \in E^{(1)}$ be s.t. $\lambda_0 \in E^{(1)}(\xi_0) \setminus (T^{(1)}(\xi_0) \cup \text{Exc}(\xi_0) \cup \sigma_{pp}(H(\xi_0)))$. Then there exist a neighbourhood $N_0$ of $\xi_0$, a neighbourhood $\mathcal{J}_0$ of $\lambda_0$, and a constant $c_m > 0$ s.t. for any $\xi \in N_0$:

$$1_{\mathcal{J}_0}(H(\xi))i[H(\xi), A_{\xi_0}]1_{\mathcal{J}_0}(H(\xi)) \geq c_m 1_{\mathcal{J}_0}(H(\xi)), \quad (2.37)$$

$$1_{\mathcal{J}_0}(H^{(1)}(\xi))i[H^{(1)}(\xi), 1 \otimes a_{\xi_0}]1_{\mathcal{J}_0}(H^{(1)}(\xi)) \geq c_m 1_{\mathcal{J}_0}(H^{(1)}(\xi)). \quad (2.38)$$

(e) The fiber Hamiltonians have no singular continuous spectrum below the two-boson threshold:

$$\sigma_{sc}(H(\xi)) \cap (-\infty, \Sigma^{(2)}_0(\xi)) = \emptyset. \quad (2.39)$$

2.4 Results

We begin by introducing some notation. First of all, the space of bound states $\mathcal{H}_{\text{bnd}}$ of the system is the closure of the span of all states of the form $I_{\text{LLP}}^{(2)} \int_{\mathbb{R}^\nu} \psi(\xi) d\xi$, where $\mathbb{R}^\nu \ni \xi \rightarrow \psi(\xi)$ is compactly supported, measurable and $\psi_{\xi}$ is an eigenvector for $H(\xi)$ for a.e. $\xi$. Expressed concisely in terms of the joint spectral resolution $E$ for the vector of commuting operators $(P, H)$ this amounts to

$$\mathcal{H}_{\text{bnd}} = E(\Sigma_{pp})\mathcal{H}. \quad (2.40)$$

Incoming scattering states prepared at $t = -\infty$, as well as outgoing scattering states at $t \rightarrow +\infty$, consist of a superposition of interacting dressed electrons and a collection of free bosons. That is, the incoming and outgoing spaces are

$$\mathcal{H}_\pm = \mathcal{H}_{\text{bnd}} \otimes \mathcal{F}. \quad (2.41)$$

The asymptotic dynamics on incoming and outgoing spaces are generated by the restriction of $H^{\text{ex}}$ to $\mathcal{H}_\pm$. In the light of our discussion in the preceding two subsections, $H^{\text{ex}}_{|\mathcal{H}_\pm}$ is a direct sum of operators of the form

$$I_{\text{LLP}}^{(2)} \int_{\mathbb{R}^\nu} d\xi \int_{\mathbb{R}^\nu} dk S(\xi - k_1 - \cdots - k_{\ell}) + \omega(k_1) + \cdots + \omega(k_{\ell}) \right) I_{\text{LLP}}^{(2)} \quad (2.42)$$

where $\ell \in \mathbb{N}_0$ and $(A, S)$ are shells in $\Sigma_{\text{iso}}$. Moreover, it is an easy consequence of the HVZ theorem, cf. [37, Theorem 2.1], that $(P, H)$ and $(P^{\text{ex}}, H^{\text{ex}}_{|\mathcal{H}_\pm})$ have identical energy-momentum spectra.

Let us recall that the asymptotic creation operators of bosons are usually defined as follows:

$$a_{\pm}^*(h)\psi := \lim_{t \rightarrow \pm\infty} e^{itH} a_{\pm}^* e^{itH^*} e^{-itH} \psi, \quad (2.43)$$
where \( h \in \mathfrak{h} \) and \( \Psi \) belongs to the dense domain \( \mathcal{D} \) of vectors of bounded energy (i.e. \( \mathcal{D} := \bigcup_{K \in \mathbb{R}^{d+1}} \text{Ran} \, E(K) \), where the union extends over all compact sets). It is well known \([30]\) and easy to see that the limit exists in the case of the massive Nelson model (i.e. \( G \in S(\mathbb{R}^d) \) and \( \omega(k) = \sqrt{k^2 + m^2} \), where \( m > 0 \)). As a consequence, in this case there exist mappings \( \tilde{\Omega}^\pm \) defined on \( \Psi' \in \mathcal{D} \otimes \Gamma_{\text{fin}}(\mathfrak{h}) \) by

\[
\tilde{\Omega}^\pm \Psi' = \lim_{t \to \pm \infty} e^{itH_0} \tilde{\Gamma}(1,1)^* e^{-itH_{\text{ex}}} \Psi', \tag{2.44}
\]

where the scattering identification map \( \tilde{\Gamma}(1,1)^* \) is defined in \((1.2)\) and in Appendix \(A.2\). The restrictions of \( \tilde{\Omega}^\pm \) to \( \mathcal{H}_\pm \), denoted by \( \Omega^\pm \), are usually called the (conventional) wave operators. They were introduced first in \([32]\). The associated (conventional) scattering operator \( S: \mathcal{H}_- \to \mathcal{H}_+ \) is then given by \( S = (\Omega^-)^* \Omega^+ \). Observe that restricted to the subspace \( \mathcal{H}_{\text{bnd}} \otimes \mathbb{C} \subset \mathcal{H}_\pm \), the wave operators trivially exist and act as injections

\[
\forall \Psi \in \mathcal{H}_{\text{bnd}} : \quad \Omega^\pm (\Psi \otimes |0\rangle) = \Psi. \tag{2.45}
\]

To serve as an acceptable wave operator, \( \Omega^\pm \) should be isometric. At small coupling strength such a result seems to be within reach of methods present in the literature \([21, 3]\). However, at arbitrary couplings, in the possible presence of eigenvalues embedded in the continuous spectrum, nothing is known to date about this problem. Not to speak of the problem of asymptotic completeness in the massive Nelson model, which is the question of isometry of the adjoints of the wave operators.

We note that in the case of the polaron model (i.e. \( G = \tilde{G}(k)/|k|, \ \tilde{G} \in S(\mathbb{R}) \) and \( \omega(k) = m \), where \( m > 0 \)), which is also covered by our assumptions, problems start already at the level of existence of the wave operators. Since the boson dispersion relation gives only a phase factor, it might even seem that the wave operators \((2.44)\) do not exist!

It turns out that the situation is much better than outlined above, at least in the energy-momentum regime below the two-boson threshold i.e. in the region \( \mathcal{R} := \{ (\xi, \lambda) \in \mathbb{R}^{d+1} | \lambda < \Sigma_0^{(2)}(\xi) \} \). Indeed, in this region our main result resolves all the problems mentioned in the two paragraphs above:

**Theorem 2.3.** Assume Conditions 1 and 2. The wave operators \( \Omega^\pm_R : E^{\text{ex}}(\mathcal{R})\mathcal{H}_\pm \to \mathcal{H} \) exists in the sense of the strong limits

\[
\Omega^\pm_R := s-\lim_{t \to \pm \infty} e^{itH_0} \tilde{\Gamma}(1,1)^* e^{-itH_{\text{ex}}}, \tag{2.46}
\]

where \( \tilde{\Gamma}(1,1)^* \) is defined in Appendix \(A.2\). The operators \( \Omega^\pm_R \) are unitary as maps from \( E^{\text{ex}}(\mathcal{R})\mathcal{H}_\pm \) to \( E(\mathcal{R})\mathcal{H} \). More precisely:

\[
\Omega^\pm_R \Omega^\mp_R = E^{\text{ex}}(\mathcal{R})|\mathcal{H}_\pm \quad \text{and} \quad \Omega^\pm_R \Omega^\pm_{\text{ex}} = E(\mathcal{R}). \tag{2.47}
\]

Finally, the scattering operator \( S_R = (\Omega^-)^* \Omega^+_R : E^{\text{ex}}(\mathcal{R})\mathcal{H}_- \to E^{\text{ex}}(\mathcal{R})\mathcal{H}_+ \) is unitary.

In the energy-momentum regime \( \mathcal{R} \), scattering only happens between bound states associated to the isolated part \( \Sigma_{\text{iso}} \) of \( \Sigma_{\text{pp}} \). For this reason a special role is played by the bound states pertaining to isolated mass shells, for which we use the notation

\[
\mathcal{H}_{\text{iso}} = E(\Sigma_{\text{iso}})\mathcal{H}. \tag{2.48}
\]

Let us introduce the terminology that a state \( \Psi \in \mathcal{H}_{\text{bnd}} \) and a smearing function \( h \in \mathfrak{h} \) are \( \mathcal{R} \)-compatible if there exists a Borel set \( S \subset \mathbb{R}^d \times \mathbb{R} \) such that \( \Psi \in E(S)\mathcal{H}_{\text{bnd}} \) and
\{(\xi + k, E + \omega(k)) \mid (\xi, E) \in \mathcal{S}, k \in \text{supp} h \} \subset \mathcal{R}. \text{ By supph, we understand } h \text{'s essential support. Note that by energy-momentum considerations we can always choose } \mathcal{S} \subset \Sigma_{\text{iso}}, \text{ such that in fact } \Psi \in \mathcal{H}_{\text{iso}}. \nolinebreak 

With the terminology just introduced, } \mathcal{E}_{\text{iso}}(\mathcal{R})/\mathcal{H}_{+} \text{ is the direct sum of states of the form } \Psi \otimes |0\rangle, \text{ with } \Psi \in \mathcal{E}(\mathcal{R})/\mathcal{H}_{\text{bnd}}, \text{ and states from the closure of the span of states of the form } \Psi \otimes a^*(h)|0\rangle, \text{ where } \Psi \in \mathcal{H}_{\text{iso}} \text{ and } h \in \mathcal{R} \text{ are } \mathcal{R} \text{-compatible. See Lemma } [\text{MC2}] \text{ for a proof.} \nolinebreak 

For any } \Psi \in \mathcal{H}_{\text{iso}} \text{ and } h \in \mathfrak{h} \text{ which are } \mathcal{R} \text{-compatible we define the corresponding scattering state as follows }

\[ a_+^*(h) \Psi := \Omega_+^*(\Psi \otimes a^*(h)|0\rangle). \quad (2.49) \]

Theorem 2.3 has the following corollary:

**Corollary 2.4.** Let } a_+^*(h) \Psi, a_+^*(h') \Psi' \text{ be scattering states and } \Psi'' \in \mathcal{E}(\mathcal{R})/\mathcal{H}_{\text{bnd}}. \text{ There hold the following properties:} 

(a) **Tensor product structure:** 

\[ \langle a_+^*(h) \Psi, a_+^*(h') \Psi' \rangle = \langle h, h' \rangle \langle \Psi, \Psi' \rangle \quad \text{and} \quad \langle a_+^*(h) \Psi, \Psi'' \rangle = 0. \quad (2.50) \]

(b) **Asymptotic completeness:** 

\[ \mathcal{E}(\mathcal{R})/\mathcal{H} = \text{Span}\{a_+^*(h) \Psi, \Psi'' \mid \Psi, h \text{ are } \mathcal{R}-\text{compatible, } \Psi'' \in \mathcal{E}(\mathcal{R})/\mathcal{H}_{\text{bnd}} \}. \quad (2.51) \]

Note that for the particular case of the polaron model, the notion of } \Psi \text{ and } h \text{ being } \mathcal{R} \text{-compatible is completely trivial. Here } \mathcal{R} = \{(\xi, E) \in \mathbb{R}^{\nu+1} \mid E < \Sigma_0 + 2m\}, \text{ where } m \text{ is the phonon mass, cf. } [\text{MC2}] \text{ and } \Sigma_0 \text{ is the bottom of the spectrum of } \mathcal{H}. \text{ That is, } \mathcal{R} \text{ is just a half-space. Being } \mathcal{R} \text{-compatible thus reduces to } \Psi \in \mathcal{E}(\mathbb{R}^\nu \times (-\infty, \Sigma_0 + m))\mathcal{H}_{\text{bnd}} = \mathcal{H}_{\text{iso}}, \text{ with no condition on } h. \text{ Hence, in this the polaron case we have:} 

\[ \mathcal{E}(\mathcal{R})/\mathcal{H} = \text{Span}\{a_+^*(h) \Psi, \Psi'' \mid \Psi \in \mathcal{H}_{\text{iso}}, h \in \mathfrak{h} \text{ and } \Psi'' \in \mathcal{E}(\mathcal{R})/\mathcal{H}_{\text{bnd}} \}. \quad (2.52) \]

### 3 Heisenberg derivatives

As usual in investigations of the problem of asymptotic completeness, we are interested in the existence of asymptotic observables, which are strong limits as } t \to \infty \text{ of time dependent families of observables of the form 

\[ \mathbb{R} \ni t \to e^{\imath H(\xi)} \Phi(t) e^{-\imath H(\xi)}, \quad (3.1) \]

where the propagation observable } \mathbb{R} \ni t \to \Phi(t) \in B(\mathcal{F}) \text{ is uniformly bounded in time. Since we are going to proceed via Cook’s method, we are interested in the Heisenberg derivatives of propagation observables, defined a priori in the sense of forms on } D(H(\xi)) \text{ as} 

\[ D \Phi(t) = \partial_t \Phi(t) + \imath [H(\xi), \Phi(t)]. \quad (3.2) \]

In Propositions 3.3 and 3.4 below we will express such derivatives by Heisenberg derivatives of some propagation observables } \mathbb{R} \ni t \to \Phi^{(1)}(t) \in B(\mathcal{F} \otimes \mathcal{F}^{(1)}), \text{ given by} 

\[ D^{(1)} \Phi^{(1)}(t) = \partial_t \Phi^{(1)}(t) + \imath [H^{(1)}(\xi), \Phi^{(1)}(t)]. \quad (3.3) \]

Before we state and prove these propositions, which provide the technical basis for our investigation, we need the following definition:
Definition 3.1. Let \( j_0, j_\infty \in C^\infty(\mathbb{R}) \) be s.t. \( j_0', j_\infty' \in C^\infty_0(\mathbb{R}) \), \( 0 \leq j_0, j_\infty \leq 1 \), \( j_0 = 1 \) in a neighbourhood of zero. We set \( j_0' := \frac{j_0(a/t)}{j_0}, j_\infty' := \frac{j_\infty(a/t)}{j_\infty} \), and \( j' := (j_0', j_\infty') \) as a map \( h \mapsto h \oplus h \) defined by \( j'h := (j_0'h, j_\infty'h) \).

Remark 3.2. In Section \[3\] and in Appendices \[4\] \( a := \frac{1}{2} \{ v \cdot i\nabla_k + i\nabla_k \cdot v \} \), where \( v \in C^\infty_0(\mathbb{R}^\nu \setminus \{0\}; \mathbb{R}^\nu) \) is an arbitrary vector field. Unless stated otherwise, in the remaining part of the paper \( a := a_{\xi_0} = \frac{1}{2} \{ v_{\xi_0} \cdot i\nabla_k + i\nabla_k \cdot v_{\xi_0} \} \) is the observable appearing in Theorem \[2.2\] associated with some neighbourhoods \( N_0 \) and \( J_0 \).

Proposition 3.3. Let \( \xi \in \mathbb{R}^\nu \) and \( \chi \in C^\infty(\mathbb{R}) \) be supported in \((\infty, \Sigma_0^{(2)}(\xi))\). Let \( q \in C^\infty(\mathbb{R}) \) be s.t. \( 0 \notin \text{supp} q \) and \( q' \in C^\infty_0(\mathbb{R}) \) (in particular \( q \) is bounded). Let \( j_0, j_\infty \) be as specified in Definition \[3.1\] and s.t. \( j_0^2 + j_\infty^2 = 1 \). Then

\[
D(\chi d\Gamma(q^j)) = \Gamma(j_0')D(\chi d\Gamma(q^j))\Gamma(j_0') + \Gamma^{(1)}(j^j)^*\chi^{(1)}D^{(1)}(1 \otimes q^j)\chi^{(1)}\Gamma^{(1)}(j^j) + O(t^{-2}),
\]

where we set \( \chi := \chi(H(\xi)), \chi^{(\ell)} := \chi(H^{(\ell)}(\xi)) \) and \( q^j := q(a/t) \). Moreover, for \( \text{supp} j_0 \cap \text{supp} q = \emptyset \) we have

\[
\Gamma(j_0')\chi Dd\Gamma(q^j)\chi \Gamma(j_0') = O(t^{-2}).
\]

Proof. We write \( j := j^j \), \( q := q^j \). The Heisenberg derivative of the asymptotic observable \( \Phi(t) := \chi d\Gamma(q)\chi \) is given by

\[
D\Phi(t) = \chi d\Gamma(\partial_t q)\chi + i\chi[H(\xi), d\Gamma(q)]\chi.
\]

We consider the first term on the r.h.s. above:

\[
\chi d\Gamma(\partial_t q)\chi = \tilde{\Gamma}(j)^*\chi^{ex} d\Gamma^{ex}(\partial_t q)\chi^{ex}\tilde{\Gamma}(j) + O(t^{-2}),
\]

where we applied Proposition \[G.2\] and Lemma \[D.3\] and set \( \chi^{ex} := \chi(H^{ex}(\xi)) \). We use the decomposition \[2.26\] of \( H^{ex}(\xi) \) and, by Lemma \[2.1\] it suffices to consider the terms \( \ell = 0 \) and \( \ell = 1 \). The \( \ell = 0 \) term has the following form

\[
\Gamma^{(0)}(j)^*\chi^{(0)}d\Gamma^{(0)}(\partial_t q)\chi^{(0)}\Gamma^{(0)}(j) = \Gamma(j_0)\chi d\Gamma(\partial_t q)\chi \Gamma(j_0).
\]

If \( j_0 \) is supported outside of the support of \( q \), this contribution is of order \( O(t^{-2}) \) by Proposition \[G.1\] Otherwise it contributes to the first term on the r.h.s. of \[3.4\].

The \( \ell = 1 \) term has the form

\[
\Gamma^{(1)}(j)^*\chi^{(1)} \left( d\Gamma(\partial_t q) \otimes 1 + 1 \otimes (\partial_t q) \right) \chi^{(1)}\Gamma^{(1)}(j).
\]

By Corollary \[G.4\] we obtain

\[
\Gamma^{(1)}(j)^*\chi^{(1)}(d\Gamma(\partial_t q) \otimes 1)\chi^{(1)}\Gamma^{(1)}(j) = O(t^{-2}).
\]

So we are left with

\[
\Gamma^{(1)}(j)^*\chi^{(1)}(1 \otimes \partial_t q)\chi^{(1)}\Gamma^{(1)}(j),
\]

which contributes to the expression on the r.h.s. of \[3.4\].

\[1\]One can weaken this assumption to \( \text{supp} j_0 \cap \text{supp} q' = \emptyset \) at a cost of additional complications in Appendix \[G\]. This is, however, not needed in the following.
Now we proceed to the second term on the r.h.s. of (3.6). From Proposition G.5, we obtain
\[
\chi[H(\xi), d\Gamma(q)]\chi = \tilde{\Gamma}(j)^*\chi^e[H^e(\xi), d\Gamma^e(q)]\chi^e\tilde{\Gamma}(j) + O(t^{-2}).
\] (3.12)
Making use of the decomposition (2.26), we get
\[
\tilde{\Gamma}(j)^*\chi^e[H^e(\xi), d\Gamma^e(q)]\chi^e\tilde{\Gamma}(j) = \tilde{\Gamma}(j)^*\left( \bigoplus_{\ell=0}^{\infty} \chi^{(\ell)}[H^{(\ell)}(\xi), d\Gamma^{(\ell)}(q)]\chi^{(\ell)} \right)\tilde{\Gamma}(j).
\] (3.13)
By Lemma 2.11 it suffices to consider \(\ell = 0\) and \(\ell = 1\) terms: The \(\ell = 0\) contribution is the following:
\[
\tilde{\Gamma}^{(0)}(j)^*\chi^{(0)}[H^{(0)}(\xi), d\Gamma^{(0)}(q)]\chi^{(0)}\tilde{\Gamma}^{(0)}(j) = \Gamma(j_0)\chi[H(\xi), d\Gamma(q)]\chi\Gamma(j_0).
\] (3.14)
If \(j_0\) is supported outside of the support of \(q\), this contribution is of order \(O(t^{-2})\) by Proposition G.1. Otherwise it contributes to the first term on the r.h.s. of (3.4).
Let us now consider the contribution with \(\ell = 1\):
\[
\tilde{\Gamma}^{(1)}(j)^*\chi^{(1)}[H^{(1)}(\xi), d\Gamma^{(1)}(q)]\chi^{(1)}\tilde{\Gamma}^{(1)}(j).
\] (3.15)
We recall that \(d\Gamma^{(1)}(q) = d\Gamma(q) \otimes 1 + 1 \otimes q\) and obtain
\[
\tilde{\Gamma}^{(1)}(j)^*\chi^{(1)}[H^{(1)}(\xi), d\Gamma^{(1)}(q)]\chi^{(1)}\tilde{\Gamma}^{(1)}(j)
= \tilde{\Gamma}^{(1)}(j)^*\chi^{(1)}[H^{(1)}(\xi), d\Gamma(q) \otimes 1]\chi^{(1)}\tilde{\Gamma}^{(1)}(j) + \tilde{\Gamma}^{(1)}(j)^*\chi^{(1)}[H^{(1)}(\xi), 1 \otimes q]\chi^{(1)}\tilde{\Gamma}^{(1)}(j).
\] (3.16)
The first term on the r.h.s. above is of order \(O(t^{-2})\) by Lemma G.6. The second term contributes to the expression from the statement of the proposition.
Thus, together with (3.11), we get
\[
\mathbf{D}\Phi(t) = \Gamma(j_0)\mathbf{D}\Phi(t)\Gamma(j_0)
+ \tilde{\Gamma}^{(1)}(j)^*\chi^{(1)}(1 \otimes \partial_t q + i[H^{(1)}(\xi), 1 \otimes q])\chi^{(1)}\tilde{\Gamma}^{(1)}(j) + O(t^{-2}),
\] (3.17)
and the first term on the r.h.s. contributes to \(O(t^{-2})\) for \(j_0\) supported outside of the support of \(q\). This concludes the proof. \[\Box\]

**Proposition 3.4.** Let \(\xi \in \mathbb{R}^\nu\) and \(\chi \in C_0^\infty(\mathbb{R})\) be supported in \((-\infty, \Sigma^{(2)}_0(\xi))\). Let \(q \in C^\infty(\mathbb{R})\) be s.t. \(q' \in C_0^\infty(\mathbb{R})\), \(0 \leq q \leq 1\), \(q = 1\) on a neighbourhood \(\Delta\) of zero. Let \(j_0, j_\infty\) be as specified in Definition 2.7 s.t. \(j_0^2 + j_\infty^2 = 1\) and \(j_0\) is supported in \(\Delta\). Then
\[
\chi\mathbf{D}\Gamma(q^t)\chi = \Gamma^{(1)}(j_0)^*\chi^{(1)}(\Gamma(q^t) \otimes 1)\mathbf{D}^{(1)}(1 \otimes q^t)\chi^{(1)}\Gamma^{(1)}(j_0) + O(t^{-2}),
\] (3.18)
where we set \(\chi := \chi(H(\xi)), \chi^{(\ell)} := \chi(H^{(\ell)}(\xi))\) and \(q^t := q(a/t)\). Consequently,
\[
\chi\mathbf{D}\Gamma(q^t)\chi = \frac{1}{t} \Gamma^{(1)}(j^t)^*\chi^{(1)}C_t(1 \otimes (q^t)^t)\chi^{(1)}\Gamma^{(1)}(j^t) + O(t^{-2}),
\] (3.19)
where \(\{C_t\}_{t \in \mathbb{R}}\) is a family of bounded operators on \(\mathcal{F} \otimes \mathcal{F}^{(1)}\) which satisfies
\[
C_t(N + 1) = O(1) \quad \text{and} \quad [C_t, 1 \otimes p^t] = O(t^{-1}),
\] (3.20)
for any \(p \in C^\infty(\mathbb{R})\) with \(p' \in C_0^\infty(\mathbb{R})\).
Proof. We set \( q := q^t, j := j^t \) and compute the Heisenberg derivative:

\[
\chi \text{D} \Gamma(q) \chi = \chi(d \Gamma(q, \partial_t q) + i[H(\xi), \Gamma(q)])\chi.
\] (3.21)

Making use of Proposition [H.2] we obtain

\[
\chi d \Gamma(q, \partial_t q) \chi = \hat{\Gamma}(j)^* \chi^\text{ex} d \Gamma^\text{ex}(q, \partial_t q) \chi^\text{ex} \hat{\Gamma}(j) + O(t^{-2}),
\] (3.22)

where we set \( \chi^\text{ex} := \chi(H^\text{ex}(\xi)) \) and \( d \Gamma^\text{ex}(\cdot, \cdot) \) is defined by formula (A.15). Inserting decomposition (2.26) of \( H^\text{ex}(\xi) \), we get

\[
\hat{\Gamma}(j)^* \chi^\text{ex} d \Gamma^\text{ex}(q, \partial_t q) \chi^\text{ex} \hat{\Gamma}(j) = \hat{\Gamma}(j)^* \left( \bigoplus_{\ell=0}^\infty \chi^{(\ell)} d \Gamma^{(\ell)}(q, \partial_t q) \chi^{(\ell)} \right) \hat{\Gamma}(j).
\] (3.23)

By Lemma 2.1, it suffices to consider \( \ell = 0 \) and \( \ell = 1 \) terms. For \( \ell = 0 \), we get

\[
\Gamma(j_0) \chi d \Gamma(q, \partial_t q) \chi \Gamma(j_0) = O(t^{-2})
\] (3.24)

by Proposition [H.1]. The \( \ell = 1 \) term is given by

\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)} (d \Gamma(q, \partial_t q) \otimes q + \Gamma(q) \otimes \partial_t q) \chi^{(1)} \hat{\Gamma}^{(1)}(j).
\] (3.25)

We note that, by Corollary G.4,

\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)} (d \Gamma(q, \partial_t q) \otimes q) \chi^{(1)} \hat{\Gamma}^{(1)}(j) = O(t^{-2}).
\] (3.26)

So we are left with

\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)} (\Gamma(q) \otimes \partial_t q) \chi^{(1)} \hat{\Gamma}^{(1)}(j),
\] (3.27)

which contributes to the r.h.s. of (3.18). Next, we choose \( \tilde{\chi} \in C^\infty_0(\mathbb{R}) \) s.t. \( \chi \tilde{\chi} = \chi \) and make use of Lemma [G.3] to write

\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)} (\Gamma(q) \otimes \partial_t q) \chi^{(1)} \hat{\Gamma}^{(1)}(j)
\] (3.28)

where \( \tilde{q} = -(a/t) f(a/t), f \in C^\infty(\mathbb{R}) \) is equal to one on the support of \( q' \) and vanishes outside of a slightly larger set. The operator \( C_{1,t} := (\Gamma(q) \otimes \tilde{q}) \tilde{\chi}^{(1)} \) is the first contribution to \( C_t \) appearing in (3.19). It is obvious that \( C_{1,t} \) satisfies the first property in (3.20), and the second property in (3.20) follows from Lemma [G.3].

Let us now consider the second contribution to the Heisenberg derivative. From Proposition [H.2] we obtain

\[
\chi[H(\xi), \Gamma(q)] \chi = \hat{\Gamma}(j)^* \chi^\text{ex}[H^\text{ex}(\xi), \Gamma^\text{ex}(q)] \chi^\text{ex} \hat{\Gamma}(j) + O(t^{-2}).
\] (3.29)

By inserting the decomposition (2.26), we get:

\[
\hat{\Gamma}(j)^* \chi^\text{ex}[H^\text{ex}(\xi), \Gamma^\text{ex}(q)] \chi^\text{ex} \hat{\Gamma}(j) = \hat{\Gamma}(j)^* \left( \bigoplus_{\ell=0}^\infty \chi^{(\ell)}[H^{(\ell)}(\xi), \Gamma^{(\ell)}(q)] \chi^{(\ell)} \right) \hat{\Gamma}(j).
\] (3.30)

As before, it is enough to consider \( \ell = 0 \) and \( \ell = 1 \) terms. As for the \( \ell = 0 \) term,

\[
\Gamma(j_0) \chi[H(\xi), \Gamma(q)] \chi \Gamma(j_0) = O(t^{-2}),
\] (3.31)
by Proposition H.1. The \( \ell = 1 \) term is given by
\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)}(H^{(1)}(\xi), \Gamma(\xi) \otimes q) \chi^{(1)} \hat{\Gamma}^{(1)}(j)
= \hat{\Gamma}^{(1)}(j)^* \chi^{(1)}(H^{(1)}(\xi), \Gamma(\xi) \otimes 1) \chi^{(1)} \hat{\Gamma}^{(1)}(j) + \hat{\Gamma}^{(1)}(j)^* \chi^{(1)}(\Gamma(\xi) \otimes 1) [H^{(1)}(\xi), 1 \otimes q] \chi^{(1)} \hat{\Gamma}^{(1)}(j). \tag{3.32}
\]
The first term on the r.h.s. above is \( O(t^{-2}) \) by Proposition H.3 and the second term contributes to \( (3.18) \). This concludes the proof of \( (3.18) \).

Let us now complete the proof of \( (3.19) \): First, we note that by Proposition \( \text{F.2} \)
\[
[H^{(1)}(\xi), 1 \otimes q] \chi^{(1)} = O(t^{-1}). \tag{3.33}
\]
Making use of this fact and of Lemma \( \text{F.6} \) we can write
\[
\hat{\Gamma}^{(1)}(j)^* \chi^{(1)} \bar{\chi}^{(1)}(\Gamma(\xi) \otimes 1) [H^{(1)}(\xi), 1 \otimes q] \chi^{(1)} \hat{\Gamma}^{(1)}(j)
= \hat{\Gamma}^{(1)}(j)^* \chi^{(1)}(\Gamma(\xi) \otimes 1) \bar{\chi}^{(1)} [H^{(1)}(\xi), 1 \otimes q] \chi^{(1)} \hat{\Gamma}^{(1)}(j) + O(t^{-2}). \tag{3.34}
\]
Next, we note that by Proposition \( \text{F.1} \),
\[
\bar{\chi}^{(1)} [H^{(1)}(\xi), 1 \otimes q] \chi^{(1)} = \frac{1}{t} \bar{\chi}^{(1)} C (1 \otimes (q')^t) \chi^{(1)} + O(t^{-2})
= \frac{1}{t} \bar{\chi}^{(1)} C \bar{\chi}^{(1)} (1 \otimes (q')^t) \chi^{(1)} + O(t^{-2}), \tag{3.35}
\]
where \( C \) is a bounded operator on \( \mathcal{F} \otimes \mathcal{F}^{(1)} \), which satisfies \( [C, 1 \otimes p'] = O(t^{-1}) \) for any \( p \in C^\infty(\mathbb{R}) \) s.t. \( p' \in C^\infty_0(\mathbb{R}) \) and in the second step in \( (3.35) \) we made use of Lemma \( \text{G.3} \). The second contribution to \( C_t \) is thus given by
\[
C_{2,t} := (\Gamma(\xi) \otimes 1) \bar{\chi}^{(1)} C \bar{\chi}^{(1)}.
\tag{3.36}
\]
Again, it is obvious that \( C_{2,t} \) satisfies the first property in \( (3.20) \), and the second property in \( (3.20) \) follows from \( [C, 1 \otimes p'] = O(t^{-1}) \) and Lemma \( \text{G.3} \). \( \square \)

4 Propagation estimates

In this section we use the expressions for Heisenberg derivatives of propagation observables, established in Section 3, to prove suitable minimal velocity propagation estimates. We will use these estimates in Section 5 to verify the existence of the relevant asymptotic observables.

**Proposition 4.1.** Let \( \chi \in C^\infty_0(\mathbb{R}) \) be supported in \( \mathcal{J}_0 \) and \( \xi \in N_0 \). Fix \( 0 < \varepsilon < c_0 < c_m \), where \( c_m \) appeared in the Mourre estimate \( \text{H.3} \), and \( R > \varepsilon \).

(a) Let \( \mathcal{I}_0 = [R, c_0] \). Then there exists \( c > 0 \) such that for all \( \Psi^{(1)} \in \mathcal{F} \):
\[
\int_1^\infty dt \frac{1}{t} \langle \Psi^{(1)}_t, \chi^{(1)}(1 \otimes \mathbf{1}_{\mathcal{I}_0}(a_{\xi_0}/t)) \chi^{(1)} \Psi^{(1)}_t \rangle \leq c \| \Psi^{(1)} \|^2, \tag{4.1}
\]
where \( \Psi^{(1)}_t := e^{-itH^{(1)}(\xi)} \Psi^{(1)} \) and \( \chi^{(1)} := \chi(H^{(1)}(\xi)) \).

(b) Let \( j_0 \), \( j_\infty \) be as specified in Definition 3.1 and s.t. \( j_0^2 + j_\infty^2 = 1 \). Let \( \mathcal{I} = [-R, -\varepsilon] \cup [\varepsilon, c_0] \). Then there exists \( c > 0 \) such that for all \( \Psi \in \mathcal{F} \):
\[
\int_1^\infty dt \frac{1}{t} \langle \Psi_t, \hat{\Gamma}^{(1)}(j)^* \chi^{(1)}(1 \otimes \mathbf{1}_{\mathcal{I}}(a_{\xi_0}/t)) \chi^{(1)} \hat{\Gamma}^{(1)}(j) \Psi_t \rangle \leq c \| \Psi \|^2, \tag{4.2}
\]
where \( \Psi_t := e^{-itH(\xi)} \Psi \).
Proof. We set $a := a_{\xi_0}$ and start with a brief consideration which is relevant for both parts of the proposition. Let $q \in C^{\infty}(\mathbb{R})$ be s.t. $q' \in C^{\infty}_{0}(\mathbb{R})$, $q' \geq 0$, $\sqrt{q'} \in C^{\infty}_{0}(\mathbb{R})$ and supp$q' \subset [-R-1,c_0]$ for some $c_0 < c'_0 < c_m$. Let us consider the propagation observable

$$\Phi^{(1)}(t) = \chi^{(1)}(1 \otimes q)\chi^{(1)},$$

where we set $q := q(a/t)$. Its Heisenberg derivative gives

$$D^{(1)}\Phi^{(1)}(t) = \chi^{(1)}(-\frac{1}{t}1 \otimes (a/t)q' + i\chi^{(1)}[H^{(1)}(\xi), 1 \otimes q]\chi^{(1)}),$$

where we chose some function $\tilde{\chi} \in C^{\infty}_{0}(\mathbb{R})_{\mathbb{R}}$, supported in $\mathcal{J}_0$, s.t. $\chi\tilde{\chi} = \chi$. Next, making use of Proposition [11] we can write

$$\chi^{(1)}[H^{(1)}(\xi), 1 \otimes q]\chi^{(1)} = \frac{1}{t}(1 \otimes \sqrt{q'})\chi^{(1)}[H^{(1)}(\xi), 1 \otimes a]_{\circ}^{\circ}\chi^{(1)}(1 \otimes \sqrt{q'}) + O(t^{-2})$$

$$\geq \frac{c_m}{t}(1 \otimes \sqrt{q'})\chi^{(1)}(1 \otimes \sqrt{q'}) + O(t^{-2})$$

$$= \frac{c_m}{t}\chi^{(1)}(1 \otimes q')\chi^{(1)} + O(t^{-2}),$$

where in the second step we made use of the Mourre estimate (2.38) and in the last step of Lemma [G.3] (The notation $[\ , \ ]_{\circ}^{\circ}$ is explained in Appendix [C]). On the other hand

$$-\frac{1}{t}1 \otimes (a/t)q' \geq -\frac{c'_0}{t}1 \otimes q'.$$

Thus we obtain from (4.5) and (4.6) that

$$D^{(1)}\Phi^{(1)}(t) \geq \frac{c}{t}\chi^{(1)}(1 \otimes q')\chi^{(1)} + O(t^{-2}),$$

where $c := c_m - c'_0 > 0$.

Now we are ready to prove part [a] of the proposition. By choosing $q$ s.t. $q' = 1$ on $\mathcal{I}_0 = [-R,c_0]$, we obtain from (4.7) that

$$D^{(1)}\Phi^{(1)}(t) \geq \frac{c}{t}\chi^{(1)}(1 \otimes 1_{\mathcal{I}_0}(a/t))\chi^{(1)} + O(t^{-2}),$$

By integrating this expression along the time evolution we obtain (4.11).

Proceeding to part [b] of the proposition we choose $q'$ s.t. supp$q' \subset [-R-1,-\varepsilon/2] \cup [\varepsilon/2,c'_0]$ for $c_0 < c'_0 < c_m$ and $q' = 1$ on $[-R,-\varepsilon] \cup [\varepsilon,c_0]$. We also require that

$$q(\lambda) = \int_0^\lambda q'(s)ds$$

(4.9)

to ensure that $q$ vanishes in a neighbourhood of zero. We consider the propagation observable

$$\Phi(t) = \chi d\Gamma(q)\chi.$$ (4.10)

Proposition [5.3] gives that

$$D\Phi(t) = \Gamma(j_0)D\Phi(t)\Gamma(j_0) + \Gamma^{(1)}(j)^*D^{(1)}\Phi^{(1)}(t)\Gamma^{(1)}(j) + O(t^{-2}),$$

where we set $j := j^t$. As for the second term on the r.h.s. above, we obtain

$$\Gamma^{(1)}(j)^*\chi^{(1)}D^{(1)}\Phi^{(1)}(t)\chi^{(1)}\Gamma^{(1)}(j)$$

$$\geq \frac{c}{t}\Gamma^{(1)}(j)^*\chi^{(1)}(1 \otimes 1_{\mathcal{I}}(a/t))\chi^{(1)}\Gamma^{(1)}(j) + O(t^{-2}),$$

(4.12)
where we made use of (4.7). Let us now estimate the first term on the r.h.s. of (4.11). We choose \( \tilde{j} \) as specified in Definition 3.3, s.t. \( \int_0^\infty \tilde{j}_0^2 + \tilde{j}_\infty^2 = 1 \) and supp \( \tilde{j}_0 \) does not intersect with the support of \( q \). Then, making use again of Proposition 3.3 and of formula (4.12), we obtain

\[
\Gamma(j_0)D\Phi(t)\Gamma(j_0) = \Gamma(j_0)\Gamma^{(1)}(j)^*D^{(1)}\Phi^{(1)}(t)\Gamma^{(1)}(j)\Gamma(j_0) + O(t^{-2}) \geq O(t^{-2}),
\]

i.e. this term is bounded from below by an integrable contribution.

Making use of (4.12) and (4.13), we obtain

\[
D\Phi(t) \geq \frac{c}{t} \Gamma^{(1)}(j)^*\chi^{(1)}(1 \otimes 1_I(a/t))\chi^{(1)}\Gamma^{(1)}(j) + O(t^{-2}),
\]

where \( c > 0 \). By integrating both sides of this inequality along the time evolution and making use of the fact that \( \Phi(t) \) is bounded uniformly in time, we conclude the proof. \( \Box \)

**Proposition 4.2.** Let \( \chi \in C_0^\infty(\mathbb{R}_\mathbb{R}) \) be supported in \( \mathcal{J}_0 \) and \( \xi \in \mathcal{N}_0 \). Then there exist \( c > 0 \) and \( 0 < \varepsilon_0 < c_m/2 \), where \( c_m \) appeared in the Mourre estimate (2.38), s.t. for any \( R > 0 \) and \( \Psi \in \mathcal{F} \):

\[
\int_1^\infty \| \Gamma(1_{[-R,R]}(a_{\xi}/t))\chi(H(\xi))\Psi_t \| \frac{2dt}{t} \leq c\|\Psi\|^2,
\]

where \( \Psi_t = e^{-itH(\xi)}\Psi \).

**Proof.** We set \( a := a_{\xi_0} \) and \( A := d\Gamma(a_{\xi_0}) \). Let \( q \in C_0^\infty(\mathbb{R}) \) be s.t. \( 0 \leq q \leq 1 \). Suppose that \( q \) is supported in \( [-R, 1, 2\varepsilon_0] \) and \( q = 1 \) on \( [-R, \varepsilon_0] \) for some \( \varepsilon_0 > 0 \) to be specified later. Moreover, suppose that \( q' = q_+ - q_- \), where \( q_\pm \geq 0 \), \( \sqrt{q_\pm} \in C_0^\infty(\mathbb{R}) \), supp \( q_+ \subset [-R-1, -R] \), supp \( q_- \subset [\varepsilon_0, 2\varepsilon_0] \).

We set \( q' := q(a/t) \) and introduce the propagation observable

\[
\Phi_\xi(t) = \chi(H(\xi))\Gamma(q'\frac{A}{t})\Gamma(q)\chi(H(\xi)).
\]

Note that by Corollary 4.11 we have \( \Gamma(q')\chi(H(\xi))\mathcal{F} \subset D(H(\xi)) \cap D(A) \), such that the computation above – as well as the one to follow – is meaningful. It can easily be shown that \( \Phi_\xi \) is bounded uniformly in time. Let us now study the Heisenberg derivative of \( \Phi_\xi \): We set \( q := q' \), \( \chi := \chi(H(\xi)) \) and write:

\[
D\Phi_\xi(t) = \chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi + \chi D(\Gamma(q))\frac{A}{t}\Gamma(q)\chi + \chi\Gamma(q)\frac{A}{t}D(\Gamma(q))\chi.
\]

As for the first term on the r.h.s. above, we obtain:

\[
\chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi = -\frac{1}{t}\chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi + \frac{1}{t}\chi\Gamma(q)i[H(\xi), A]\Gamma(q)\chi.
\]

Concerning the first term on the r.h.s. of (4.13), we note the bound

\[
\frac{1}{t}\chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi = \frac{1}{t}\chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi + O(t^{-2})
\]

\[
\leq \frac{1}{t}\chi\Gamma(q)\frac{A}{t}\Gamma(q)\chi + O(t^{-2})
\]

\[
\leq c\varepsilon_0\frac{1}{t}\chi\Gamma(q)^2\chi + O(t^{-2}),
\]

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where \( c \) is independent of \( \varepsilon_0 \) and \( R \). Here \( \tilde{\chi} \in C_0^\infty(\mathbb{R}) \) is s.t. \( \tilde{\chi}\chi = \chi \) and \( \tilde{\chi} \) is supported in \( \mathcal{J}_0 \). We also chose functions \( q_1, q_2 \in C_0^\infty(\mathbb{R}) \), \( 0 \leq q_1, q_2 \leq 1 \) s.t. \( q_1 \chi = \chi \), supp \( q_1 \subset [-R-2, 3\varepsilon_0] \), supp \( q_2 \subset [-3\varepsilon_0, 3\varepsilon_0] \) and \( q_1(s)s \leq q_2(s)s \) for all \( s \in \mathbb{R} \). In (4.19) we made use of Lemma [F.6] the fact that \( \Gamma(q_1)\Gamma(a) = d\Gamma(q_1, q_1a) \), and

\[
\|d\Gamma(q_1, (a/t)q_2)(1 + N)^{-1}\| \leq \|(a/t)q_2\| \leq 3\varepsilon_0. \tag{4.20}
\]

In view of this bound it is clear that the constant \( c = 3\| (1 + N)^{-1} \| \| \tilde{\chi} \| \), appearing in (4.19), is independent of \( R, \varepsilon_0 \). As for the second term on the r.h.s. of (4.18), we write

\[
\frac{1}{t}\chi\Gamma(q)\psi[H(\xi), A]^{\circ}\Gamma(q)\chi = \frac{1}{t}\tilde{\chi}\Gamma(q)\chi[H(\xi), A]^{\circ}\chi\Gamma(q)\tilde{\chi} + O(t^{-2})
\]

\[
\geq c_m\frac{1}{t}\tilde{\chi}\Gamma(q)\chi^2\Gamma(q)\tilde{\chi} + O(t^{-2})
\]

\[
= c_m\frac{1}{t}\chi\Gamma(q)^2\chi + O(t^{-2}). \tag{4.21}
\]

The first step above follows from Lemma [F.6] and from the fact that

\[
\|H(\xi), A]^{\circ}\Gamma(q)\chi| < \infty \quad \text{and} \quad \|H(\xi), A]^{\circ}\chi\| < \infty. \tag{4.22}
\]

These bounds are simple consequences of Corollary [F.11] and Lemma [I.2] after rewriting

\[
[H(\xi), A]^{\circ}\Gamma(q)\chi = \{H(\xi), A]^{\circ}(N + 1)^{-1}\Gamma(q)(H_0(\xi) + 1)^{-3}\}
\]

\[
\times \{(N + 1)(H_0(\xi) + 1)^{-1}\fy(\xi)\}
\]

and recalling Corollary [E.4]. In the second step of (4.21) we used (2.37) and in the last step once more Lemma [F.6]. Summing up, we got

\[
\chi\Gamma(q)\mathcal{D}(\frac{A}{t})\Gamma(q)\chi \geq (c_m - c\varepsilon_0)\frac{1}{t}\chi\Gamma(q)^2\chi + O(t^{-2}). \tag{4.24}
\]

Let us now consider the remaining two terms on the r.h.s. of (4.17). First, we note that

\[
\chi\mathcal{D}(\Gamma(q))\frac{A}{t}\Gamma(q)\chi = \chi\mathcal{D}(\Gamma(q))\frac{\tilde{\chi}}{t}\Gamma(q)\chi + \chi\mathcal{D}(\Gamma(q))[d\Gamma(q, (a/t)q), \tilde{\chi}]\chi
\]

\[
= \chi\tilde{\chi}\mathcal{D}(\Gamma(q))\frac{\tilde{\chi}}{t}\Gamma(q)\chi + O(t^{-2}). \tag{4.25}
\]

Here in the second step we applied Lemma [I.3] and Proposition [F.5] which ensures that \( \chi\mathcal{D}(\Gamma(q)) = O(t^{-1}) \). As for the first term on the r.h.s. above, we obtain from Proposition [3.4]

\[
\chi\mathcal{D}(\Gamma(q))\tilde{\chi} = \mathcal{D}(\tilde{\chi}\Gamma(q)\tilde{\chi}) = \frac{1}{t}\Gamma^{(1)}(j)^*\tilde{\chi}(1 \otimes q'_j)\chi^{(1)}\tilde{\Gamma}^{(1)}(j) + O(t^{-2}), \tag{4.26}
\]

where we set \( j := j' \) and \( q' := (q'_j)^t \). Thus, recalling that \( q' = q_+ - q_- \) and \( \sqrt{q_\pm} \in C_0^\infty(\mathbb{R}) \), we can write

\[
\mathcal{D}(\tilde{\chi}\Gamma(q)\tilde{\chi}) = \sum_{\sigma \in \{\pm\}} 1\sigma\frac{1}{t}\Gamma^{(1)}(j)^*\tilde{\chi}(1 \otimes \sqrt{q_\sigma})C_j(1 \otimes \sqrt{q_\sigma})\chi^{(1)}\tilde{\Gamma}^{(1)}(j) + O(t^{-2}), \tag{4.27}
\]

where we exploited the second property in (3.20). Thus we get

\[
\chi\tilde{\chi}\mathcal{D}(\Gamma(q))\frac{\tilde{\chi}}{t}\Gamma(q)\chi
\]

\[
= \sum_{\sigma \in \{\pm\}} 1\sigma\frac{1}{t}\chi\Gamma^{(1)}(j)^*\tilde{\chi}(1 \otimes \sqrt{q_\sigma})C_jd\Gamma^{(1)}(q, (a/t)q)(1 \otimes \sqrt{q_\sigma})\chi^{(1)}\tilde{\Gamma}^{(1)}(j)\chi
\]

\[
+ O(t^{-2}), \tag{4.28}
\]
where we made use of the fact that \( \hat{\Gamma}^{(1)}(j) A \Gamma(q) = d \Gamma^{(1)}(q, (a/t)q) \hat{\Pi}^{(1)}(j) \) and then of Lemma 4.3 to exchange \( d \Gamma^{(1)}(q, (a/t)q) \) with \( \hat{\Pi}^{(1)}(j) \). Since \( C_t d \Gamma^{(1)}(q, (a/t)q) = O(1) \), by the first part of property (3.20), we obtain for any \( \Psi \in \mathcal{F} \):

\[
|\langle \Psi_t, \chi \hat{\Pi} \Gamma(q) \rangle| \leq \sum_{\sigma \in \{\pm\}} \frac{C}{t} \|\langle 1 \otimes \sqrt{q}\sigma \rangle \hat{\Pi}^{(1)}(j) \chi \Psi_t \|^2 + O(t^{-2})\|\Psi\|^2. \tag{4.29}
\]

This expression is integrable, uniformly in \( \Psi \) from the unit ball in \( \mathcal{F} \), by the Cauchy-Schwarz inequality and Proposition 4.4. (To apply this latter proposition we assume that \( 2\varepsilon_0 < c_m \)). The last term on the r.h.s of (4.17) is treated analogously.

Altogether, we have obtained that

\[
D \Phi(t) \geq (c_m - c\varepsilon_0) \frac{1}{t} \chi \Gamma(q)^2 \chi + B(t) + O(t^{-2}), \tag{4.30}
\]

where \( c \) is independent of \( \varepsilon_0 \) and \( R \), and \( B(t) \) is integrable along the time-evolution provided that \( 2\varepsilon_0 < c_m \). By choosing \( \varepsilon_0 \) sufficiently small, we conclude the proof. \( \square \)

5 Existence of some asymptotic observables

As usually in the time-dependent approach to the problem of asymptotic completeness, the central question is the existence of suitable asymptotic observables as strong limits, as time goes to infinity, of their approximating sequences. In this section we answer this question with the help of the propagation estimates established in Section 4. With this information at hand, the proof of asymptotic completeness, completed in Sections 6 and 7, is relatively straightforward.

**Theorem 5.1.** Let \( \chi \in C_0^\infty(\mathbb{R}) \) be supported in \( J_0 \) and \( \xi \in N_0 \). Let \( q \in C_0^\infty(\mathbb{R}) \) be s.t. \( 0 \leq q \leq 1 \), \( q' \in C_0^\infty(\mathbb{R}) \) and \( \supp q' \subset (-\infty, c_m) \setminus [-\varepsilon, \varepsilon] \), for some \( 0 < \varepsilon < c_m \), where \( c_m \) appeared in (2.37). Then the following strong limit exists

\[
Q^+(H(\xi))\chi := s-\lim_{t \to \infty} e^{iH(\xi)t} \Gamma(q')e^{-iH(\xi)t} \chi, \tag{5.1}
\]

and commutes with bounded Borel functions of \( H(\xi) \). (Here we set \( \chi := \chi(H(\xi)) \)). Moreover, if \( \supp q \subset (-\infty, \varepsilon_0) \), where \( \varepsilon_0 \) appeared in Proposition 4.2, then \( Q^+(H(\xi))\chi = 0 \).

**Proof.** Let \( q := q' \) and define \( \Phi(t) = \chi \Gamma(q) \chi \). Making use of Proposition 3.4 we obtain

\[
D \Phi(t) = \frac{1}{t} \Gamma^{(1)}(j)^* \hat{\Pi}^{(1)}(j) C_t (1 \otimes q') \hat{\Pi}^{(1)}(j) + O(t^{-2}), \tag{5.2}
\]

where we set \( j := j' \). Let \( \bar{q} \in C_0^\infty(\mathbb{R}) \) be supported in \( (-\infty, c_m) \setminus [-\varepsilon, \varepsilon] \) and equal to one on the support of \( q' \). Then, making use of the second property in (3.20), we can write

\[
D \Phi(t) = \frac{1}{t} \Gamma^{(1)}(j)^* \bar{q} \hat{\Pi}^{(1)}(j) C_t (1 \otimes q') \hat{\Pi}^{(1)}(j) + O(t^{-2}). \tag{5.3}
\]

Since \( C_t = O(1) \), we obtain

\[
|\langle \Psi_{1,t}, D \Phi(t)\Psi_{2,t} \rangle| \leq \frac{C}{t} \|\langle 1 \otimes \bar{q} \rangle \hat{\Pi}^{(1)}(j) \Psi_{1,t} \| \|\langle 1 \otimes q' \rangle \hat{\Pi}^{(1)}(j) \Psi_{2,t} \|
\]

\[
+ O(t^{-2})\|\Psi_1\|\|\Psi_2\|, \tag{5.4}
\]
where $\Psi_i \in \mathcal{F}$ and $\Psi_{i,t} = e^{-itH(\xi)}\Psi_i$, $i \in \{1, 2\}$. By integrating both sides of this inequality over some time interval, applying the Cauchy-Schwarz inequality to the integral of the first term on the r.h.s. of (5.4), taking supremum over $\Psi_1$ s.t. $||\Psi_1|| \leq 1$ and exploiting Proposition 4.1 we obtain strong convergence in (5.1) by the Cook method. Now we choose $\tilde{\chi} \in C_0^\infty(\mathbb{R})_R$, supported in $\mathcal{J}_0$ and s.t. $\tilde{\chi}\chi = \chi$. Lemma G.3 gives

$$e^{itH(\xi)}\Gamma(q)e^{-itH(\xi)}\chi = e^{itH(\xi)}\tilde{\chi}\Gamma(q)e^{-itH(\xi)}\chi + O(t^{-1}). \quad (5.5)$$

The second term on the r.h.s. above converges strongly by the above consideration. By a computation analogous to (5.5) one shows that $Q^+(H(\xi))\chi$ commutes with $H(\xi)$. This concludes the proof of (5.1).

Let us now show the last statement of the theorem, i.e. that for $q$ s.t. supp$q \subset (-\infty, \varepsilon_0)$ there holds

$$Q^+(H(\xi))\chi = 0. \quad (5.6)$$

Let $q_R \in C_0^\infty(\mathbb{R})$, $0 \leq q_R \leq 1$, be s.t. $q_R(s) = q(s)$ for $s \in (-R, \infty)$ and $q_R = 0$ for $s < -R - 1$, for some $R > c_m$. Then, coming back to the explicit notation $q^t = q(a/t)$ and $q_R^t = q_R(a/t)$, we obtain from Proposition 4.2 and from (5.1) that

$$s - \lim_{t \to \infty} e^{itH(\xi)}\Gamma(q_R^t)e^{-itH(\xi)}\chi = 0. \quad (5.7)$$

On the other hand, Lemma K.1 gives that

$$||\Gamma(q_R^t) - \Gamma(q^t)||e^{-itH(\xi)}\chi\Psi|| = O(R^{-1}) \quad (5.8)$$

uniformly in $t$ for $\Psi$ from some dense domain in $\mathcal{F}$. This concludes the proof of (5.6). \hspace{1cm} \Box

**Theorem 5.2.** Let $\chi \in C_0^\infty(\mathbb{R})_R$ be supported in $\mathcal{J}_0$ and $\xi \in N_0$. Let $p \in C^\infty(\mathbb{R})$ be s.t. $0 \leq p \leq 1$, $p' \in C_0^\infty(\mathbb{R})$ and supp$p' \subset (-\infty, c_m)$, where $c_m$ appeared in [2.37]. Then the following strong limit exists

$$Q^+(H^{ex}(\xi))_\infty \chi^{ex} := s - \lim_{t \to \infty} e^{itH^{ex}(\xi)}(1 \otimes p)\chi^{ex} \quad (5.9)$$

and commutes with bounded Borel functions of $H^{ex}(\xi)$. (Here we set $\chi^{ex} := \chi(H^{ex}(\xi))$.

If, in addition, $p = 1$ on $[c_m, \infty)$, then

$$Q^+(H^{ex}(\xi))_\infty \chi^{ex} = \chi^{ex}. \quad (5.10)$$

**Proof.** Concerning the proof of (5.9), we set $p := p^t$, choose $\tilde{\chi} \in C_0^\infty(\mathbb{R})_R$, supported in $\mathcal{J}_0$ and s.t. $\tilde{\chi}\chi = \chi$. We note the relation

$$[\tilde{\chi}^{ex}, 1 \otimes \Gamma(p)]\chi^{ex} = [\chi^{(1)}, 1 \otimes p]\chi^{(1)} = O(t^{-1}), \quad (5.11)$$

which is a consequence of the decomposition (2.26), Lemma 2.1 (which ensures that only $\ell = 0$ and $\ell = 1$ terms survive in this expansion), and of Lemma G.3. Thus it suffices to prove strong convergence of $t \to e^{itH^{ex}(\xi)}\chi^{ex}(1 \otimes \Gamma(p))\chi^{ex}e^{-itH^{ex}(\xi)}$ for any $\chi \in C_0^\infty(\mathbb{R})_R$ supported in $\mathcal{J}_0$. We apply decomposition (2.26) and Lemma 2.1 to this expression. The $\ell = 0$ component gives $\chi(\xi)^2$ which is time-independent. The $\ell = 1$ component has the form

$$t \to \chi^{(1)}e^{it(\xi)^1}(1 \otimes p)e^{-it(\xi)^1}\chi^{(1)}. \quad (5.12)$$
We consider the propagation observable \( \Phi_\infty(t) := \chi^{(1)}(1 \otimes p)\chi^{(1)} \). To prove the strong convergence of (5.12) we will show integrability of the Heisenberg derivative

\[
D^{(1)} \Phi_\infty(t) = \chi^{(1)} \left( -\frac{1}{t} 1 \otimes (a/t)p' + i[H^{(1)}(\xi), 1 \otimes p] \right) \chi^{(1)}. \tag{5.13}
\]

By Proposition 1.1

\[
\chi^{(1)}[H^{(1)}(\xi), 1 \otimes p] \chi^{(1)} = \frac{1}{t} \chi^{(1)} C(1 \otimes p') \chi^{(1)} + O(t^{-2}), \tag{5.14}
\]

where \( C \) is a bounded operator on \( \mathcal{F} \otimes \mathcal{F}^{(1)} \), which satisfies

\[
[C, 1 \otimes p_1'] = O(t^{-1}) \tag{5.15}
\]

for any \( p_1 \in C^\infty(\mathbb{R}) \) s.t. \( p'_1 \in C^\infty_0(\mathbb{R}) \). Let \( \tilde{p} \in C^\infty_0(\mathbb{R}) \) be supported in \((-\infty, c_m)\) and be equal to one on the support of \( p' \). Then, due to (5.15), we obtain

\[
D^{(1)} \Phi_\infty(t) = \frac{1}{t} \chi^{(1)}(1 \otimes \tilde{p}) \tilde{C}_t(1 \otimes p') \chi^{(1)} + O(t^{-2}), \tag{5.16}
\]

where \( \tilde{C}_t = -1 \otimes (a/t)\tilde{p} + C \) is a family of operators which is uniformly bounded in \( t \). Thus we can write

\[
|\langle \Psi_{1,t}, \Phi_\infty(t) \Psi_{2,t} \rangle| \leq c \frac{t}{t} \| (1 \otimes \tilde{p}) \chi^{(1)} \Psi_{1,t} \| \| (1 \otimes p') \chi^{(1)} \Psi_{2,t} \| + O(t^{-2})\| \Psi_1 \| \| \Psi_2 \|, \tag{5.17}
\]

where \( \Psi_i \in \mathcal{F} \otimes \mathcal{F}^{(1)} \), \( \Psi_{1,t} = e^{-iH^{(1)}(\xi)}\Psi_i, i \in \{1, 2\} \). With the help of this bound, the Cauchy-Schwarz inequality and Proposition 1.1 we obtain strong convergence of (5.12) by the Cook method. This completes the proof of (5.9). To show that the limit commutes with bounded functions of the Hamiltonian, one proceeds analogously as in relation (5.11) above.

Let us now proceed to the proof of (5.10). We come back to the explicit notation \( p' = p(a/t) \). As we have shown above (cf. formula (5.12))

\[
e^{iH^{(1)}(\xi)}(1 \otimes \Gamma(p'))e^{-iH^{(1)}(\xi)} \chi^{(1)} = \chi(H(\xi)) \oplus \tilde{\chi}^{(1)}e^{iH^{(1)}(\xi)}(1 \otimes p')e^{-iH^{(1)}(\xi)} \tilde{\chi}^{(1)} \chi^{(1)} + O(t^{-1}). \tag{5.18}
\]

Thus it suffices to show that for \( \Psi \in \mathcal{F} \otimes \mathcal{F}^{(1)} \) there holds

\[
\lim_{t \to \infty} \langle \Psi_t, \chi^{(1)}(1 \otimes p(a/t)) \chi^{(1)} \Psi_t \rangle = \langle \Psi, (\chi^{(1)})^2 \Psi \rangle. \tag{5.19}
\]

Setting \( q := 1 - p \), this is equivalent to

\[
\lim_{t \to \infty} \langle \Psi_t, \chi^{(1)}(1 \otimes q(a/t)) \chi^{(1)} \Psi_t \rangle = 0. \tag{5.20}
\]

We note that \( \text{supp } q \subset (-\infty, c_m) \). Let us choose a function \( q_R \in C^\infty_0(\mathbb{R}), 0 \leq q_R \leq 1 \), which coincides with \( q \) on \((-R, \infty)\), but is equal to zero on \((-\infty, -R - 1)\) for some \( R > 1 \). We obtain from (4.41) that

\[
\lim_{t \to \infty} \langle \Psi_t, \chi^{(1)}(1 \otimes q_R(a/t)) \chi^{(1)} \Psi_t \rangle = 0, \tag{5.21}
\]

where we exploited the first part of this proposition to obtain convergence.

Now let \( \Psi \) be an element of the domain of \( 1 \otimes a \). Then \( \Psi \) belongs to the domain of \( (1 \otimes a)\chi^{(1)} \), since \( H^{(1)}(\xi) \) is of class \( C^1(1 \otimes a) \). Cf. [38] Lemma 2.2, Proposition 2.8.
Furthermore, the operator representing the commutator form $i[H^{(1)}(\xi), 1 \otimes a]$ is given by $i[H^{(1)}(\xi), 1 \otimes a] = -\nabla \Omega(\xi - d\Gamma^{(1)}(k)) \cdot (1 \otimes v) + 1 \otimes \nabla \omega \cdot v$, which is $H^{(1)}(\xi)$-bounded. Consequently, the group $e^{-itH^{(1)}(\xi)}$ preserves $D(1 \otimes a) \cap D(H^{(1)}(\xi))$. Now we set $q_R := q - q_R$ and compute

\[
\langle \Psi_t, \chi^{(1)}(1 \otimes q_R(a/t)) \rangle \chi^{(1)}(\Psi_t) = \langle \Psi_t, \chi^{(1)}(1 \otimes q_R(a/t)) e^{-itH^{(1)}(\xi)} e^{itH^{(1)}(\xi)} \chi^{(1)}(\Psi_t) \rangle e^{-itH^{(1)}(\xi)}(1 \otimes a) \chi^{(1)}(\Psi_t) \]

Hence, making use of the fact that \(\|H^{(1)}(\xi) \cdot (1 \otimes a)^{\alpha} \chi^{(1)}\| < \infty\), (cf. Lemma I.2), we obtain

\[
|\langle \Psi_t, (1 \otimes q_R(a/t)) \chi^{(1)}(\Psi_t) \rangle| \leq c_R \|\Psi\|^2 + c_R t \|\Psi\| \|1 \otimes a\| \chi^{(1)}(\Psi)|. \tag{5.23}
\]

Since this expression can be made arbitrarily small, uniformly in $t$, by choosing $R$ sufficiently large, we have proven (5.20) for $\Psi$ in the domain of $(1 \otimes a)$, which is dense. This concludes the proof. \(\square\)

**Theorem 5.3.** Let $\chi \in C^\infty_0(\mathbb{R})$ be supported in $J_0$ and $\xi \in N_0$. Let $j_0, j_\infty$ be as specified in Definition 3.1, s.t. $j_0^2 + j_\infty^2 = 1$, supp $j_0 \subset (-\infty, c_m)$ and hence supp $j_\infty \subset (-\infty, c_m)$, where $c_m$ appeared in (2.37). Let $q = (q_0, q_\infty) := (j_0^2, j_\infty^2)$ (in particular $q_0 + q_\infty = 1$). Then the following strong limits exist:

\[
W^+(q^t)(\xi)\chi^{ex} := -\lim_{t \to \infty} e^{it\Gamma(\xi)} \hat{\Gamma}(q^t) e^{-it\Gamma^{ex}(\xi)} \chi^{ex}, \tag{5.24}
\]

\[
W^+(q^t)(\xi)^* \chi := -\lim_{t \to \infty} e^{it\Gamma^{ex}(\xi)} \hat{\Gamma}(q^t) e^{-it\Gamma(\xi)} \chi, \tag{5.25}
\]

where we set $\chi := \chi(H(\xi))$ and $\chi^{ex} := \chi(H^{ex}(\xi))$. These operators intertwine (bounded Borel functions of) $H(\xi)$ and $H^{ex}(\xi)$.

**Proof.** We set $q := q^t$, $j := j^t$ and consider the asymptotic observable $\Phi(t) = \chi^{ex} \hat{\Gamma}(q)\chi$. Its non-symmetric Heisenberg derivative is given by

\[
\mathring{\Phi}(t) = \chi^{ex}(d\hat{\Gamma}(q, \partial_q) + iH^{ex}(\xi)\hat{\Gamma}(q) - i\hat{\Gamma}(q)H(\xi)\chi. \tag{5.26}
\]

The first term on the r.h.s. above can be rearranged as follows

\[
\chi^{ex} d\hat{\Gamma}(q, \partial_q) = 2\chi^{ex} d\hat{\Gamma}^{ex}(j_0, \partial_j) \hat{\Gamma}(j) = 2\chi^{ex} (d\Gamma(j_0, \partial j_0) \otimes \Gamma(j_\infty) + \Gamma(j_0) \otimes d\Gamma(j_\infty, \partial j_\infty)) \hat{\Gamma}(j), \tag{5.27}
\]

where $\partial j := \text{diag}(j_0, j_\infty)$, $\partial j := \text{diag}(\partial j_0, \partial j_\infty)$ are propagation observables on $\mathfrak{h} \oplus \mathfrak{h}$ and in the last step we made use of Lemma A.3. As for the remaining terms on the r.h.s. of (5.26), we obtain from Lemma 3.1 that

\[
\chi^{ex}(H^{ex}(\xi)\hat{\Gamma}(q) - \hat{\Gamma}(q)H(\xi)) \chi = 2\chi^{ex}[H^{ex}(\xi), \Gamma^{ex}(\partial j) \hat{\Gamma}(j)\chi + O(t^{-2}). \tag{5.28}
\]
Thus, altogether, we get

\[ \mathbf{D}\Phi(t) = 2\mathbf{\chi}^{ex}(d\Gamma^{ex}(j, \partial j) + i[H^{ex}(\xi, \Gamma^{ex}(j))]\Gamma(j)\chi + O(t^{-2}) \]  
(5.29)

\[ = 2\mathbf{\chi}^{ex}(d\Gamma^{ex}(j, \partial j) + i[H^{ex}(\xi, \Gamma^{ex}(j))]\chi^{ex}\Gamma(j)\chi + O(t^{-2}), \]  
(5.30)

where in the last step we chose \( \tilde{\chi} \in C_{0}^{\infty}(\mathbb{R}) \), supported in \( \mathcal{J}_{0} \) and s.t. \( \tilde{\chi}\chi = \chi \). To exchange \( \Gamma(j)\tilde{\chi} \) with \( \chi^{ex}\Gamma(j) \), we made use of the fact that \( \tilde{\chi} \sim C_{0}^{\infty}(\mathbb{R}) \).

Thus, altogether, we get

\[ \mathbf{\chi}^{ex}[H^{ex}(\xi, \Gamma^{ex}(j))] = O(t^{-1}), \]  
(5.31)

which follows from Proposition F.5.

Now we apply decomposition (2.26) of \( H^{ex}(\xi) \). As for the \( \ell = 0 \) component, we obtain from (5.29)

\[ \mathbf{\chi}(2d\Gamma(j_{0}, \partial j_{0}) + 2i[H(\xi, \Gamma(j_{0})])\Gamma(j_{0})\chi + O(t^{-2}) \]  
(5.32)

\[ = \mathbf{\chi}(d\Gamma(q_{0}, \partial q_{0}) + i[H(\xi, \Gamma(q_{0})])\chi + O(t^{-2}). \]  
(5.32)

To justify the second step above we make use of the relations

\[ d\Gamma(j_{0}, \partial j_{0}) = 2d\Gamma(j_{0}, \partial j_{0})\Gamma(j_{0}), \]  
(5.33)

\[ \chi[H(\xi), \Gamma(q_{0})]\chi = 2\chi[H(\xi), \Gamma(j_{0})]\Gamma(j_{0})\chi + \chi[\Gamma(j_{0}), [H(\xi), \Gamma(j_{0})]]\chi, \]  
(5.34)

and of the fact that the last term on the r.h.s. of (5.34) is \( O(t^{-2}) \) by Lemma F.2. We note that the first term on the r.h.s. of (5.32) is the Heisenberg derivative of \( \Phi_{0}(t) := \chi\Gamma(q_{0})\chi \) that follows from Corollary G.4 and (5.37) is a consequence of Proposition H.3.

Let us proceed to the \( \ell = 1 \) component: Let \( \chi^{(1)} := \chi(H^{(1)}(\xi)) \). From (5.30) we obtain

\[ \mathbf{\chi}(1)(d\Gamma(j_{0}, \partial j_{0}) \otimes j_{\infty}) + \Gamma(j_{0}) \otimes \partial j_{\infty} \]  
(5.35)

\[ + i[H^{(1)}(\xi), \Gamma(j_{0}) \otimes j_{\infty}]\Gamma^{(1)}(j)\chi + O(t^{-2}). \]  
(5.35)

We note that

\[ \chi^{(1)}(d\Gamma(j_{0}, \partial j_{0}) \otimes j_{\infty})\Gamma^{(1)}(j)\chi = O(t^{-2}), \]  
(5.36)

\[ \chi^{(1)}(H^{(1)}(\xi), \Gamma(j_{0}) \otimes 1)(1 \otimes j_{\infty})\Gamma^{(1)}(j)\chi = O(t^{-2}), \]  
(5.37)

where (5.36) follows from Corollary G.4 and (5.37) is a consequence of Proposition H.3.

Thus, altogether, we get

\[ \mathbf{\chi}(1)(\Gamma(j_{0}) \otimes 1)D^{(1)}(1 \otimes j_{\infty})\Gamma^{(1)}(j)\chi + O(t^{-2}) \]  
(5.38)

\[ = \mathbf{\chi}(1)(\Gamma(j_{0}) \otimes 1)\chi^{(1)}D^{(1)}(1 \otimes j_{\infty})\Gamma^{(1)}(j)\chi + O(t^{-2}), \]  
(5.38)

where in the last step we made use of the fact that \( [\Gamma^{(1)}(j_{0}) \otimes 1] = O(t^{-1}) \), which follows from Lemma F.6 and of the estimate \( D^{(1)}(1 \otimes j_{\infty})\chi^{(1)} = O(t^{-1}) \), which is a consequence of Proposition F.2. Proceeding as in (5.13)–(5.16) above, we obtain that

\[ \mathbf{\chi}(1)D^{(1)}(1 \otimes j_{\infty})\chi^{(1)} = \frac{1}{t}\mathbf{\chi}(1)\tilde{j}(1 \otimes j_{\infty})\tilde{\chi}(1) + O(t^{-2}), \]  
(5.39)
where $\tilde{j}_{\infty} \in C^\infty_0(\mathbb{R})_\mathbb{R}$ is supported in $(-\infty, c_m)$ and is equal to one on the support of $j'_{\infty}$, and $t \to \tilde{C}_t$ is a family of operators, which is uniformly bounded in $t$. Thus we get

$$
\left| \langle \Psi_{1,t}, \tilde{\mathbf{D}} \Phi^{(1)}(t) \Psi_{2,t} \rangle \right|
\leq \frac{c}{t} \left\| (1 \otimes \tilde{j}_{\infty}) \tilde{\chi}^{(1)}(t) \Psi_{1,t} \right\| \left\| (1 \otimes |j'_{\infty}|) \tilde{\chi}^{(1)}(t) \tilde{\Gamma}(j) \chi \Psi_{2,t} \right\| + O(t^{-2}) \left\| \Psi_1 \right\| \left\| \Psi_2 \right\|,
$$

where $\Psi_{1,t} = e^{-itH^{(1)}(\xi)} \Psi_1$ and $\Psi_{2,t} = e^{-it\mathcal{H}(\xi)} \Psi_2$ for some arbitrary vectors $\Psi_1 \in \mathcal{F} \otimes \mathcal{F}^{(1)}$, $\Psi_2 \in \mathcal{F}$. Due to the support properties of $\tilde{j}_{\infty}$ and the fact that $\text{supp} j'_{\infty} \subset (-\infty, c_m) \setminus [-\varepsilon, \varepsilon]$, (since $j_0 = 1$ on $[-\varepsilon, \varepsilon]$ and $j_0^2 + j_2^2 = 1$), we can apply Proposition 6.1 to show integrability of (5.40).

Thus we obtained that both $t \to \Phi(t)$ and $t \to \Phi^*(t)$ converge strongly. Now the result follows by an application of Lemma 6.10 which also gives the intertwining property.

6 Localized wave operators

In this section we construct localized wave operators, defined on a small neighbourhood $\mathcal{O}$ of any point $(\xi_0, \lambda_0) \in \mathcal{E}^{(1)} \setminus (\mathcal{T}^{(1)} \cup \text{Exc} \cup \Sigma_{pp})$. The adjective ‘localized’, used to describe the wave operators constructed in this section, requires a brief clarification: On the one hand it alludes to their construction in an energy-momentum spectral subspace of the small set $\mathcal{O}$. On the other hand it refers to the Sigal-Soffer type localization onto a spectral subspace, constructed using the one-body propagation observable $\tilde{a}_{\xi_0}$ (cf. expression (6.1) below) and describing classically permitted scattering configurations. The fact that these localized wave operators turn out to coincide with the conventional wave operators is due to the Mourre estimate preventing scattering states from occupying classically forbidden configurations in the large time limit.

Definition 6.1. We set $\mathcal{O}_0 = N_0 \times J_0$, where $N_0$ and $J_0$ appeared in Theorem 2.2 and choose an open bounded neighbourhood $\mathcal{O}$ of $(\xi_0, \lambda_0)$, whose closure is contained in the interior of $\mathcal{O}_0$.

We recall from Theorem 2.2 that with the set $\mathcal{O}_0$ we can associate the observable $a_{\xi_0} = \frac{1}{2}\{v_{\xi_0} \cdot i\nabla_k, v_{\xi_0} \cdot i\nabla_k \}$ which enters into the Mourre estimates. We define the following counterpart of this observable

$$
\tilde{a}_{\xi_0} := \frac{1}{2} \{(1 \otimes v_{\xi_0}) \cdot z + z \cdot (1 \otimes w_{\xi_0})\},
$$

where $z := 1 \otimes x - y \otimes 1$ on $\mathcal{K} \otimes \mathfrak{h}$ is the relative distance between the electron and the boson and we set $x := i\nabla_k$. In the remaining part of the section we will set $v := v_{\xi_0}$, $a := a_{\xi_0}$ and $\tilde{a} := \tilde{a}_{\xi_0}$, unless stated otherwise.

Remark 6.2. We will make use of an extension of the expression $\Gamma(q)$ to contractions $q$ on $\mathcal{K} \otimes \mathfrak{h}$, which was discussed in [33, Remark 1.1]. We leave it to the reader to check that this remark applies, whenever we meet second quantization in the extended sense discussed here. Furthermore we will also need to work with such operators $q$ viewed as acting in $\mathcal{K} \otimes \mathcal{F} \otimes \mathfrak{h}$, but skipping over the middle $\mathcal{F}$-component. This is what is meant $\tilde{q}_\infty(\tilde{a}/t)$ in Theorem 6.6.

Finally, we warn the reader that we will be abusing notation, in particular in Proposition 6.3 by writing $1 \otimes \Gamma(p_\delta(\tilde{a}/t))$, for the operator $\mathcal{E}(\Gamma(p_\delta(\tilde{a}/t)) \otimes 1) \mathcal{E}$, where $\mathcal{E} : \mathcal{H}^{\text{ex}} \to \mathcal{H}^{\text{ex}}$ is the exchange operator defined on simple tensors by $\mathcal{E}(f \otimes \eta \otimes \eta') = f \otimes \eta' \otimes \eta$.

Before we proceed to the construction of asymptotic objects in $\mathcal{O}$, we need one more definition:
Definition 6.3. Let $0 < c_0 < \varepsilon_0$, where $\varepsilon_0$ appeared in Proposition 4.2. Let $q \in C^\infty(\mathbb{R})$ be s.t. $0 \leq q \leq 1$, $q(s) = 1$ for $s \leq c_0/2$ and $q(s) = 0$ for $s > c_0$. Furthermore, $q$ is a non-decreasing function. We write $q_\delta(s) = q(s/\delta)$ and $q_\delta^*(s) = q(s/\delta)$ for $0 < \delta \leq 1$.

Proposition 6.4. Let $q$ be as specified in Definition 6.3. Then the following strong limit exists

$$Q^+_\delta(H) := s- \lim_{t \to \infty} e^{itH} (q_\delta(a/t)) e^{-itH} E(O \cup \Sigma_{iso}),$$

and equals $E(\Sigma_{iso})$. In particular, $Q^+_\delta(H) = Q^+_\delta(H)$ is independent of $\delta$.

Proof. Let $\chi \in C^\infty(\mathbb{R}^{v+1})_\mathbb{R}$ be equal to one on $O$ and be supported in $O_0$. Then $E(O) = \chi(P,H)E(O)$ and we can write

$$Q^+_\delta(H)E(O) = e^{itH} \Gamma(q_\delta(a/t)) e^{-itH} \chi(P,H)E(O)$$

$$= I^{LLP}_\delta \left( \int_{\mathbb{R}^v} d\xi e^{itH(\xi)} \Gamma(q_\delta(a/t)) e^{-itH(\xi)} \chi(\xi, H(\xi)) \right) I^{LLP}_\delta E(O),$$

where we denoted by $Q^+_\delta(H)$ the approximants on the r.h.s. of (6.2). It follows from Theorem 6.1 the dominated convergence theorem and the properties of $q$ specified in Definition 6.3 that this expression converges to zero as $t \to \infty$.

Let us consider now $Q^+_\delta(H)E(\Sigma_{iso})$. We recall from Section 2.3 that $\Sigma_{iso}$ is a union of graphs of at most countably many analytic functions $p: N \to \mathbb{R}$, where $N \subset \mathbb{R}^v$ are open sets. Let $G$ be a graph of one of these functions. Then we obtain

$$Q^+_\delta(H)E(G)\Psi = I^{LLP}_\delta \int_{\mathbb{R}^v} d\xi e^{itH(\xi)} \Gamma(q_\delta(a/t)) \Psi, \tag{6.4}$$

where $\mathbb{R}^v \ni \xi \to \Psi_{\xi} \in F$ is a square-integrable Borel function representing $\Psi$. Now by the dominated convergence theorem $\lim_{t \to \infty} Q^+_\delta(H)E(G)\Psi = E(G)\Psi$. \hfill \Box

Proposition 6.5. Let $q$ and $1 - p$ be as specified in Definition 6.3. Then the following strong limits exist

$$Q^+_\delta(H^\text{ex})_0 := s- \lim_{t \to \infty} e^{itH^\text{ex}} (\Gamma(q_\delta(a/t)) \otimes 1) e^{-itH^\text{ex}} E^\text{ex}(O), \tag{6.5}$$

$$Q^+_\delta(H^\text{ex})_\infty := s- \lim_{t \to \infty} e^{itH^\text{ex}} (1 \otimes \Gamma(p_\delta(a/t))) e^{-itH^\text{ex}} E^\text{ex}(O), \tag{6.6}$$

$$Q^+_\delta(H^\text{ex}) := s- \lim_{t \to \infty} e^{itH^\text{ex}} (\Gamma(q_\delta(a/t)) \otimes \Gamma(p_\delta(a/t))) e^{-itH^\text{ex}} E^\text{ex}(O), \tag{6.7}$$

and are independent of $\delta$ (thus we can omit the subscript $\delta$). Moreover there holds

$$Q^+(H^\text{ex})_0 = (E(\Sigma_{iso}) \otimes 1) E^\text{ex}(O), \tag{6.8}$$

$$Q^+(H^\text{ex})_\infty = E^\text{ex}(O), \tag{6.9}$$

$$Q^+(H^\text{ex}) = (E(\Sigma_{iso}) \otimes 1) E^\text{ex}(O). \tag{6.10}$$

Proof. Let $\chi \in C^\infty(\mathbb{R}^{v+1})_\mathbb{R}$ be equal to one on $O$ and be supported in $O_0$. Then $E^\text{ex}(O) = \chi(P^\text{ex}, H^\text{ex})E^\text{ex}(O)$ and we can write

$$Q^+_\delta(H^\text{ex})_0 = s- \lim_{t \to \infty} e^{itH^\text{ex}} (\Gamma(q_\delta(a/t)) \otimes 1) e^{-itH^\text{ex}} E^\text{ex}(O)$$

$$= s- \lim_{t \to \infty} (e^{itH} \Gamma(q_\delta(a/t)) e^{-itH} E(\Sigma_{iso}) \otimes 1) E^\text{ex}(O)$$

$$+ s- \lim_{t \to \infty} (e^{itH} \Gamma(q_\delta(a/t)) e^{-itH} E(O) \otimes |0\rangle \langle 0|) E^\text{ex}(O)$$

$$= (E(\Sigma_{iso}) \otimes 1) E^\text{ex}(O), \tag{6.11}$$

27
where in the first step we made use of Lemma \[6.1\] and in the second step of Proposition \[6.4\] to obtain the existence of the limit. This proves \[6.5\] and \[6.8\].

Making use of Theorem \[5.2\], we obtain that there exists the limit

\[
Q_{\delta}^{+}(H_{\text{ex}})_{\infty} = s- \lim_{t \to \infty} e^{itH_{\text{ex}}} \left(1 \otimes \Gamma(p_{\delta}(\tilde{a}/t))\right) e^{-itH_{\text{ex}}} E_{\text{ex}}(O)
\]

\[
= s- \lim_{t \to \infty} P_{\text{LLP}}^{\text{ex}} \int_{\mathbb{R}^{\nu}} d\xi e^{itH_{\text{ex}}(\xi)} \left(1 \otimes \Gamma(p_{\delta}(a/t))\right) e^{-itH_{\text{ex}}(\xi)} \chi(\xi, H_{\text{ex}}(\xi)) P_{\text{LLP}}^{\text{ex}} E_{\text{ex}}(O)
\]

which equals $E_{\text{ex}}(O)$. This proves \[6.9\] and \[6.10\].

Existence of the limit \[6.7\] and relation \[6.11\] are obvious consequences of the facts proven above.

In the following theorem we construct the localized wave operator $W_{\Omega,\delta}^{+}$, associated with the region $O$ specified in Definition \[6.1\]. We also show that its adjoint is a strong limit of its approximating sequence.

**Theorem 6.6.** Let $j_{0}, j_{\infty}$ be as specified in Definition \[6.4\], s.t. $j_{0}^{2} + j_{\infty}^{2} = 1$ and, in addition, let $j_{0}$ and $1 - j_{\infty}$ satisfy the conditions from Definition \[6.3\]. Let $q = (q_{0}, q_{\infty}) := (j_{0}^{2}, j_{\infty}^{2})$ (in particular $q_{0} + q_{\infty} = 1$). Then the following strong limits exist:

\[
W_{\Omega,\delta}^{+} = s- \lim_{t \to \infty} e^{itH} \tilde{\Gamma}(q_{\delta}(\tilde{a}/t))^{*} e^{-itH_{\text{ex}}} E_{\text{ex}}(O),
\]

\[
W_{\Omega,\delta}^{+*} = s- \lim_{t \to \infty} e^{itH^{\text{ex}}} \tilde{\Gamma}(q_{\delta}(\tilde{a}/t)) e^{-itH} E(O),
\]

and intertwine $\chi(P, H)$ with $\chi(P^{\text{ex}}, H_{\text{ex}})$ for any $\chi \in C_{0}^{\infty}(\mathbb{R}^{\nu+1})_{\mathbb{R}}$. (Consequently, $W_{\Omega,\delta}^{+*}$ is the adjoint of $W_{\Omega,\delta}^{+}$). Moreover, these limits are independent of $\delta$, for sufficiently small $\delta$, thus we can omit the subscript $\delta$.

**Proof.** Let $\chi \in C_{0}^{\infty}(\mathbb{R}^{\nu+1})_{\mathbb{R}}$ be equal to one on $O$ and be supported in $O_{0}$. We write

\[
W_{\Omega,\delta}^{+} = s- \lim_{t \to \infty} P_{\text{LLP}}^{\text{ex}} \int_{\mathbb{R}^{\nu}} d\xi e^{itH(\xi)} \Gamma(q_{\delta}(a/t))^{*} e^{-itH_{\text{ex}}(\xi)} \chi(\xi, H_{\text{ex}}(\xi)) P_{\text{LLP}}^{\text{ex}} E_{\text{ex}}(O),
\]

\[
W_{\Omega,\delta}^{+*} = s- \lim_{t \to \infty} P_{\text{LLP}}^{\text{ex}} \int_{\mathbb{R}^{\nu}} d\xi e^{itH^{\text{ex}}(\xi)} \Gamma(q_{\delta}(a/t)) e^{-itH(\xi)} \chi(\xi, H(\xi)) P_{\text{LLP}}^{\text{ex}} E(O).
\]

The existence of these limits and the intertwining property follows from Theorem \[5.3\] by the dominated convergence theorem.

Let us now show that $W_{\Omega,\delta}^{+}$ is independent of $\delta$ for sufficiently small $\delta$. First, we note that

\[
W_{\Omega,\delta}^{+} = W_{\Omega,\delta}^{+\prime}(Q_{\varepsilon}^{+}(H) \otimes 1)
\]

for $\varepsilon/2 > \delta$. However, by Proposition \[6.3\], $Q_{\varepsilon}^{+}(H) = E(\Sigma_{\text{iso}})$ is independent of $\varepsilon$, hence the relation holds also for $\varepsilon/2 < \delta$. Let us make $\varepsilon$ even smaller, so as to ensure that $\varepsilon/2 < \delta/4$.

Then we can write

\[
W_{\Omega,\delta}^{+}(Q_{\varepsilon}^{+}(H) \otimes 1) = W_{\Omega,\delta}^{+} Q_{\varepsilon}^{+}(H_{\text{ex}}) = W_{\Omega,\varepsilon}^{+} Q_{\delta}^{+}(H_{\text{ex}}).
\]

But the r.h.s. above is independent of $\delta$ by Proposition \[6.5\]. Thus both $W_{\Omega,\delta}^{+}$ and $W_{\Omega,\delta}^{+*}$ are independent of $\delta$. \[\]

\[28\]
Now we proceed to the proof of an isometry property of $W_{O}^+$. An important role in the proof is played by the map $\hat{\Gamma}(1,1)$ whose adjoint is the scattering identification operator from [9] 322.

**Theorem 6.7.** The localized wave operators, defined as in Theorem 6.6, satisfy

$$W_{O}^+W_{O}^{++} = E(O).$$  \hfill (6.19)

**Proof.** Let $q_{\delta} := (q_{0,\delta}(\bar{a}/t), q_{\infty,\delta}(\bar{a}/t))$ be as specified in Theorem 6.6 and abbreviate $q_{\delta} := \text{diag}(q_{0,\delta}(\bar{a}/t), q_{\infty,\delta}(\bar{a}/t))$, the corresponding family of observables on $K \otimes (\mathfrak{h} \oplus \mathfrak{h})$. We set $W^+ := W_{O}^+$ and write

$$W^+W^{++} = W_{O}^+W_{O}^{++} = s- \lim_{t \to \infty} e^{itH}\hat{\Gamma}(1,1)^*\Gamma_{ex}(q_{\delta})e^{-itH}W^+\Psi$$

$$= s- \lim_{t \to \infty} e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})e^{-itH}W^+\Psi$$

$$= s- \lim_{t \to \infty} e^{itH}\hat{\Gamma}(1,1)^*\Gamma_{ex}(q_{\delta})e^{-itH}W^{++}.$$  \hfill (6.20)

Making use of the fact that $O$ is localized below the two-boson threshold, we obtain that $W^+ = \hat{P}W^{++}$, where $\hat{P} := 1 \otimes (P_0 + P)$ acts on $\mathcal{H}_{ex} = \mathcal{K} \otimes \mathcal{F}_{ex}$ and $P_0 : \mathcal{F}_{ex} \to \mathcal{F} \otimes \mathcal{F}(\nu)$ are natural restriction maps. Let $\Psi$ be a vector in the range of $E(O)$. Then, setting $R := (i + H)^{-1}$ and $R_{ex} := (i + H_{ex})^{-1}$ we write

$$e^{itH}\hat{\Gamma}(1,1)^*\Gamma_{ex}(q_{\delta})e^{-itH}W^+\Psi$$

$$= e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})(R_{ex})^{2}e^{-itH}W^{++}R^{-2}\Psi$$

$$= e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})(R_{ex})^{3}e^{-itH}W^{++}R^{-3}\Psi$$

$$+ e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})e^{-itH}e^{itH}R_{ex}(q_{\delta})e^{-itH}W^{++}R^{-1}\Psi.$$  \hfill (6.21)

By Proposition 6.5 and property A.18 the term involving the commutator above is of order $O(t^{-1})$. As for the second term, we note that by Proposition 6.5, the fact that $Q^+(H) = E(\Sigma_{iso})$ and property (A.18),

$$s- \lim_{t \to \infty} e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})(R_{ex})^{3}e^{-itH}W^{++}R^{-3}\Psi = 0.$$  \hfill (6.22)

Since $(Q^+(H) \otimes 1)W^{++} = W^{++}$, (cf. formula (6.17) above), we obtain that the r.h.s. of (6.21) equals (up to terms that tend to zero in the limit $t \to \infty$)

$$e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})e^{-itH}W^{++}R^{-1}\Psi.$$  \hfill (6.23)

This is asymptotic in the limit $t \to \infty$ to the expression

$$e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})e^{-itH}R^{-1}\Psi$$

$$= e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})\chi_{ex} \hat{\Gamma}(j_{\delta})e^{-itH}R^{-1}\Psi + O(t^{-1})$$

$$= e^{itH}\hat{\Gamma}(1,1)^*\hat{\Gamma}_{ex}(q_{\delta})H_{ex}^{\delta} \hat{\Gamma}(j_{\delta})e^{-itH}R^{-1}\Psi$$

$$+ e^{itH}\hat{\Gamma}(j_{\delta})^{*}H_{ex}^{\delta} \hat{\Gamma}(j_{\delta})e^{-itH}R^{-1}\Psi + O(t^{-1}) = \Psi + O(t^{-1}).$$  \hfill (6.24)

Here in the first step we used the identity $\hat{\Gamma}(q_{\delta}) = \Gamma_{ex}(j_{\delta})\hat{\Gamma}(j_{\delta})$ and introduced a function $\chi \in C_{0}^{\infty}(\mathbb{R}^{d+1})_{R}$, supported in $O_{0}$ and equal to one on $O$ so that $\chi(P,H)\Psi = \Psi$. Making use of Lemma [F.10] we got

$$\hat{\Gamma}(j_{\delta})\chi = \chi_{ex}(j_{\delta}) + O(t^{-1}),$$  \hfill (6.25)
where we set \( \chi := \chi(P, H) \) and \( \chi^{ex} := \chi(P^{ex}, H^{ex}) \). In the second step we commuted \( \Gamma^{ex}(I_{\Delta}) \) to the left and used the fact that \( \hat{P}\chi(P^{ex}, H^{ex}) = \chi(P^{ex}, H^{ex}) \). In the third step we exploited Proposition \[F.3\] to show that the resulting commutator is \( O(t^{-1}) \). In the last step we made use again of Lemma \[F.10\].

**Lemma 6.8.** Let \((A, S)\) be an analytic shell in \( \Sigma_{iso} \) (cf. Section \[2.3\]). Let \( \Delta \) be a Borel subset of the graph \( \mathcal{G}_S \) of this shell. Then the operator

\[
\begin{align*}
B & := (1_{\Delta}(P, H) \otimes v_{\xi_0}) \cdot (1 \otimes \nabla \omega - \nabla S(P) \otimes 1) \\
\end{align*}
\]

(6.26)

satisfies

\[
\hat{P}B\hat{P} \geq c_m\hat{P},
\]

(6.27)

where \( \hat{P} := 1_{\mathcal{O}_0}(P^{(1)}, H^{(1)})(1_{\Delta}(P, H) \otimes 1) \). We recall that \( \mathcal{O}_0 \) appeared in Definition \[6.1\].

**Proof.** We set \( v := v_{\xi_0} \) and \( a := a_{\xi_0} \) as abbreviated already below definition \[6.1\] above. Let \( \chi \in C^0_0(\mathbb{R}) \) be supported in \( J_0 \). We recall from Theorem \[2.2\] that for \( \xi \in N_0 \) (where \( N_0 \) appeared in Theorem \[2.2\])

\[
\chi(H^{(1)}(\xi))i[H^{(1)}(\xi), 1 \otimes a]\chi(H^{(1)}(\xi)) \geq c_m\chi(H^{(1)}(\xi))^2.
\]

(6.28)

Let us set \( \tilde{I}_\Delta(\xi) := \int_{\Delta} dk 1_{\Delta}(\xi - k, H(\xi - k)) \), note that

\[
I_{L_{LLP}}^{(1)}1_{N_0}(P^{(1)})(1_{\Delta}(P, H) \otimes 1)I_{L_{LLP}}^{(1)*} = \int_{N_0} d\xi \tilde{I}_\Delta(\xi)
\]

(6.29)

and set \( \Delta_\xi := \{ k \in \mathbb{R} | (\xi - k, \lambda) \in \Delta, \text{ for some } \lambda \in \mathbb{R} \} \). Then we get

\[
\tilde{I}_\Delta(\xi) \left( \int_{\Delta_\xi} dk \chi(H^{(1)}(\xi; k))i[S^{(1)}(\xi; k), 1 \otimes a]\chi(H^{(1)}(\xi; k)) \right) \tilde{I}_\Delta(\xi)
\]

\[
\geq c_m\chi(H^{(1)}(\xi))^2 \tilde{I}_\Delta(\xi),
\]

(6.30)

and consequently

\[
\tilde{I}_\Delta(\xi) \left( \int_{\Delta_\xi} dk \chi(H^{(1)}(\xi; k))v(k) \cdot \nabla_k S^{(1)}(\xi; k)\chi(H^{(1)}(\xi; k)) \right) \tilde{I}_\Delta(\xi)
\]

\[
= \chi(H^{(1)}(\xi))^2 \tilde{I}_\Delta(\xi) \int_{\Delta_\xi} dk v(k) \cdot \nabla_k S^{(1)}(\xi; k) \geq c_m\chi^2(H^{(1)}(\xi))\tilde{I}_\Delta(\xi),
\]

(6.31)

where we set \( S^{(1)}(\xi; k) = S(\xi - k) + \omega(k) \). By approximating with functions \( \chi \) the characteristic function of \( J_0 \), taking the direct integral of both sides over \( \xi \in N_0 \), and conjugating with \( I_{L_{LLP}}^{(1)} \), we obtain

\[
(1_{\Delta}(P, H) \otimes v) \cdot (1 \otimes \nabla \omega - \nabla S(P) \otimes 1)\hat{P} \geq c_m\hat{P},
\]

(6.32)

where we made use of \[6.29\]. This concludes the proof. \( \square \)

**Theorem 6.9.** The localized wave operators, defined as in Theorem \[6.6\], satisfy

\[
W_{\mathcal{O}}^{+*}W_{\mathcal{O}}^+ = E^{ex}(\mathcal{O})(E(\Sigma_{iso}) \otimes 1).
\]

(6.33)
Proof. Let us set $W^+ := W_O^+$ and let $Ψ ∈ Ran E^{ex}(O)$. By Lemma [M.4] $Ψ$ belongs to the closed span of vectors of two types. The first type are vectors of the form

$$Ψ_1 ⊗ |0⟩,$$  \hspace{1cm} (6.34)

where $Ψ_1 ∈ E(O)H$. Such vectors are elements of the kernel of $W^+$ due to the fact that

$$W^+(Ψ_1 ⊗ |0⟩) = \lim_{t→∞} e^{itH} \hat{Γ}(q_{δ}(a/t))∗(e^{-itH}Ψ_1 ⊗ |0⟩)$$

$$= \lim_{t→∞} e^{itH} Γ(q_{0,δ}(a/t))e^{-itH}Ψ_1 = Q^+(H)E(O)Ψ_1 = 0,$$  \hspace{1cm} (6.35)

where we made use of Proposition [6.4]. This proves relation (6.33) on vectors of type (6.34).

Vectors of the second type that span $Ran E^{ex}(O)$, provided by Lemma [M.1] have the form

$$Ψ_2 ⊗ a^∗(h)|0⟩,$$  \hspace{1cm} (6.36)

where $h ∈ C_0^∞(R^n)$ and $Ψ_2 ∈ E(Δ)H$ are s.t. $Δ ⊂ Σ_{iso}$ is a bounded Borel set and $Δ + (k, ω(k)) ⊂ O$ for all $k ∈ supp h$. For such vectors we obtain

$$W^+(Ψ_2 ⊗ a^∗(h)|0⟩) = \lim_{t→∞} e^{itH} \hat{Γ}(q_{δ}(a/t))∗(e^{-itH}Ψ_2 ⊗ a^∗(h_t)|0⟩)$$

$$= \lim_{t→∞} e^{itH} a^∗(q_{∞,δ}(a/t)h_t)e^{-itH} Γ(q_{0,δ}(a/t))e^{-itH}Ψ_2$$

$$= \lim_{t→∞} e^{itH} a^∗(q_{∞,δ}(a/t)h_t)e^{-itH} Q^+(H)Ψ_2$$

$$= \lim_{t→∞} e^{itH} a^∗(q_{∞,δ}(a/t)h_t)e^{-itH}Ψ_2,$$  \hspace{1cm} (6.37)

where $h_t = e^{-iωt}h$ and in the last step we made use of Proposition [6.4] and the fact that $Ψ_2$ belongs to the range of $E(Σ_{iso})$. In view of the discussion of the isolated spectrum in Section [2.3], we can assume that $Ψ_2$ belongs to the range of $E(Δ)$, where $Δ$ is a subset of the graph $G_S$ of an analytic shell $(A, S)$. Here we used that level crossings sit above a set of momenta, a union of spheres, with zero Lebesgue measure. Let $Ψ_2 ⊗ a^∗(h)|0⟩$ be another vector of the form (6.36), s.t. $Ψ_2$ belongs to the range of $E(Δ)$, where $Δ$ is a subset of the graph of some other shell $S$ which may, but does not have to coincide with $S$. Now we obtain from (6.37)

$$⟨W^+(Ψ_2 ⊗ a^∗(h)|0⟩), W^+(Ψ_2 ⊗ a^∗(h)|0⟩)⟩$$

$$= \lim_{t→∞} ⟨Ψ_2, e^{iH} a(q_{∞,δ}(a/t)h_t)a^∗(q_{∞,δ}(a/t)h_t)e^{-iH}Ψ_2⟩.$$  \hspace{1cm} (6.38)

By commuting the annihilation operator to the right, we get

$$⟨Ψ_2, e^{iH} a(q_{∞,δ}(a/t)h_t)a^∗(q_{∞,δ}(a/t)h_t)e^{-iH}Ψ_2⟩$$

$$= ⟨Ψ_{2,t} ⊗ h_t, q_{∞,δ}^2((1 ⊗ (v/2)) · (1 ⊗ x − y ⊗ 1)/t + h.c.)(Ψ_{2,t} ⊗ h_t)⟩$$

$$+ ⟨a(q_{∞,δ}(a/t)h_t)e^{-iS(P)}Ψ_2, a(q_{∞,δ}(a/t)h_t)e^{-iS(P)}Ψ_2⟩,$$  \hspace{1cm} (6.39)

where $Ψ_{2,t} = e^{-iS(P)}Ψ_2, Ψ_{2,t} = e^{-iS(P)}Ψ_2$. We recall that $v := v_{ξ_0}$ and $x = i∇_k$ is the position operator of the boson.

Let us first show that the last term on the r.h.s. of (6.39) tends to zero: Due to the fact that $Ψ_2 ∈ E(Δ)H$, where $Δ$ is bounded, it is easy to see that

$$∥(1 + H_{ph})a(q_{∞,δ}(a/t)h_t)e^{-iS(P)}Ψ_2∥ ≤ c$$  \hspace{1cm} (6.40)
for some $c$ independent of $t$. (See Lemmas 6.4 and 6.2). On the other hand, as shown in Proposition 6.4 for $\delta'$ sufficiently small,

$$
\Psi_2 = \lim_{t \to \infty} e^{i H (q_0, \delta')(\tilde{a}(t)/t)} e^{-it H} \Psi_2.
$$

(6.41)

Proceeding similarly as in the proof of [21, Lemma 14], we get

$$
(1 + H_{ph})^{-1} a(q_{\infty, \delta}(\tilde{a}(t)/t)) e^{-it H} \Psi_2 \\
= (1 + H_{ph})^{-1} a(q_{\infty, \delta}(\tilde{a}(t)/t)) \Gamma(q_{0, \delta'}(\tilde{a}(t/t))) e^{-it H} \Psi_2 + o(1) \\
= (1 + H_{ph})^{-1} \Gamma(q_{0, \delta'}(\tilde{a}(t/t))) a((q_{0, \delta'}q_{\infty, \delta})(\tilde{a}(t/t))t) e^{-it H} \Psi_2 + o(1),
$$

(6.42)

where $o(1)$ denotes a term which tends in norm to zero as $t \to \infty$. Here we made use of the fact that $(1 + H_{ph})^{-1} a(q_{\infty, \delta}(\tilde{a}(t)/t))$ is bounded, uniformly in $t$. Noting that for $\delta'$ sufficiently small $q_{0, \delta'}q_{\infty, \delta} = 0$, we obtain that the r.h.s. above tends to zero and therefore the last term on the r.h.s. of (6.39) tends to zero.

Let us now consider the limit of the first term on the r.h.s. of (6.39):

$$
\lim_{t \to \infty} \langle \hat{\Psi}_2, t \otimes \hat{h}, q_{\infty, \delta}^2 ((1 \otimes (v/2)) \cdot (1 \otimes x - y \otimes 1)/t + \text{h.c.}) (\Psi_2, t \otimes h) \rangle \\
= \lim_{t \to \infty} \langle \hat{\Psi}_2 \otimes \hat{h}, q_{\infty, \delta}^2 ((1 \otimes (v/2)) \cdot (1 \otimes x - y \otimes 1)/t + \text{h.c.}) e^{it(\hat{S}(P) \otimes \hat{S}(P))} (\Psi_2 \otimes h) \rangle \\
= \lim_{t \to \infty} \langle \hat{\Psi}_2 \otimes \hat{h}, q_{\infty, \delta}^2 (1 \otimes (v) \cdot (1 \otimes \nabla \omega - \nabla \hat{S}(P) \otimes 1)) e^{it(\hat{S}(P) \otimes S(P))} (\Psi_2 \otimes h) \rangle.
$$

(6.43)

Here in the second step we made use of the strong resolvent convergence of the sequence of operators in the argument of $q_{\infty, \delta}^2$. Since $\hat{S}$ is only defined on a subset of $\mathbb{R}^\nu$, the symbol $\nabla \hat{S}(P)$ is to be understood as $\nabla \hat{S}_t(P)$, where $\hat{S}_t$ is the restriction of $\hat{S}$ to the spectral support of the vector $\Psi_2$, which is then extended by zero to $\mathbb{R}^\nu$. Clearly, the last expression on the r.h.s. of (6.43) is equal to zero if $S \neq \hat{S}$, since the argument of $q_{\infty, \delta}^2$ commutes with the spectral projection $E(G_S) \otimes 1$ and $E(G_S) E(G_{\hat{S}}) = 0$ in this case. Thus we have verified (6.33) for $S \neq \hat{S}$ and we can assume that $S = \hat{S}$. We can also assume that both $\Psi_2$ and $\hat{\Psi}_2$ belong to the range of $E(\Delta)$ for some bounded Borel subset $\Delta \subset S$. (Again, since level crossings live on a subset of momentum space with Lebesgue measure zero, we can exclude them from this discussion). Then the last term on the r.h.s. of (6.43) equals

$$
\langle \hat{\Psi}_2 \otimes \hat{h}, q_{\infty, \delta}^2 ((1 \otimes (v) \cdot (1 \otimes \nabla \omega - \nabla S(P) \otimes 1)) (\Psi_2 \otimes h) \rangle \\
= \langle \hat{\Psi}_2 \otimes \hat{h}, q_{\infty, \delta}^2 (1 \otimes v \cdot (1 \otimes \nabla \omega - \nabla S(P) \otimes 1)) (\Psi_2 \otimes h) \rangle \\
= \langle \hat{\Psi}_2 \otimes \hat{h}, q_{\infty, \delta}^2 (\hat{P} \hat{B} \hat{P}) (\Psi_2 \otimes h) \rangle,
$$

(6.44)

where $B$ and $\hat{P}$ were defined in Lemma 6.8. Next, we note that

$$
s - \lim_{\delta \to 0} q_{\infty, \delta}^2 (\hat{P} \hat{B} \hat{P}) = 1_{(0, \infty)}(\hat{P} \hat{B} \hat{P}) = 1 - 1_{\{0\}}(\hat{P} \hat{B} \hat{P}),
$$

(6.45)

where we exploited the fact that $\hat{P} \hat{B} \hat{P} \geq 0$. Next, we observe that

$$
\langle \hat{\Psi}_2 \otimes \hat{h}, 1_{\{0\}}(\hat{P} \hat{B} \hat{P}) (\Psi_2 \otimes h) \rangle = 0.
$$

(6.46)

In fact, if the l.h.s. above was different from zero, then $\Psi_0 := 1_{\{0\}}(\hat{P} \hat{B} \hat{P}) (\Psi_2 \otimes h) \neq 0$. Now inequality (6.27) gives that $\hat{P} \Psi_0 = 0$. Since $\hat{P}$ commutes with $B$, and $\Psi_2 \otimes h$ belongs to
tends strongly to zero, the last expression equals \( \Gamma(1) \) where we use the notation \( \tilde{\Gamma} \)

Proof. Moreover, \( \hat{\Omega} \) Let us now proceed to the construction of the conventional wave operators and to the proof of their completeness below the two-boson threshold. It will be convenient to work with wave operators \( \hat{\Omega} = \hat{\Omega}^+ \) introduced in (7.1) below, which are defined on the entire Hilbert space \( \mathcal{H}^\text{ex} \). As we show in the proof of Theorem 2.3 given below, their restrictions to \( E^\text{ex}(\mathcal{R}) \mathcal{H}_+ \) coincide with the wave operators \( \Omega = \Omega^+ \) defined in (2.46). We construct them first in the small regions \( \mathcal{O} \) of the energy-momentum spectrum, in which we constructed the localized wave operators.

Proposition 7.1. Let \( \mathcal{O} \) be as specified in Definition 6.7. Then there exists the limit

\[
\hat{\Omega}^+ \mathcal{O} := \lim_{t \to \infty} e^{itH} \tilde{\Gamma}(1,1)^* e^{-itH} (E(\Sigma_{\text{iso}}) \otimes 1) E^\text{ex}(\mathcal{O}).
\]

Moreover, \( \hat{\Omega}^+ \mathcal{O} = W^+ \mathcal{O} \).

Proof. We set \( W^+ := W \mathcal{O}^+ \) and \( E^\text{ex} := (i + H^\text{ex})^{-1} \). Theorem 6.9 gives us that \( (E(\Sigma_{\text{iso}}) \otimes 1) E^\text{ex}(\mathcal{O}) = W^+ W^+ \). Thus we can write

\[
e^{itH} \tilde{\Gamma}(1,1)^* e^{-itH} (E(\Sigma_{\text{iso}}) \otimes 1) E^\text{ex}(\mathcal{O}) = e^{itH} \tilde{\Gamma}(1,1)^* \tilde{P} E^\text{ex} e^{-itH} W^+ W^+ (E^\text{ex})^{-1} E^\text{ex}(\mathcal{O}),
\]

where we use the notation \( \tilde{P} = 1 \otimes (P_0 + P_1) \) from the proof of Theorem 6.7. By property (6.18), \( \tilde{\Gamma}(1,1)^* \tilde{P}(i + H^\text{ex})^{-1} \) is a bounded operator. Thus, up to an error term which tends strongly to zero, the last expression equals

\[
e^{itH} \tilde{\Gamma}(1,1)^* \tilde{P} E^\text{ex} \tilde{\Gamma}(j_0) e^{-itH} W^+ (E^\text{ex})^{-1} E^\text{ex}(\mathcal{O})
\]

These steps are justified exactly as in the discussion after formula (6.24) above, in particular \( \chi \in C^\infty_0(\mathbb{R}^{n+1}) \) is a function supported in \( \mathcal{O}_0 \) and equal to one in \( \mathcal{O} \) and we set \( \chi^\text{ex} := \chi(S^\text{ex}, H^\text{ex}) \). Relation (7.3) proves the existence of \( \hat{\Omega}^+ \mathcal{O} \) and the fact that \( \hat{\Omega}^+ \mathcal{O} = W^+ \mathcal{O} \).

In the following two theorems we state and prove our main results. We recall that \( \mathcal{R} = \{(\xi, E) \in \mathbb{R}^{n+1} \mid E < \Sigma(2)(\xi) \} \).
Theorem 7.2. There exists the wave operator \( \tilde{\Omega}^+_R : \mathcal{H}^{ex} \to \mathcal{H} \) given by
\[
\tilde{\Omega}^+_R := \lim_{t \to \infty} e^{it\hat{H}(1,1)^* e^{-itH^{ex}} (E(\Sigma_{pp}) \otimes 1) E^{ex}(\mathcal{R})}.
\] (7.4)

The wave operator satisfies
\[
\hat{\Omega}^+_R \hat{\Omega}^+_R = (E(\Sigma_{pp}) \otimes 1) E^{ex}(\mathcal{R}).
\] (7.5)

Moreover, for any \( \chi \in C_0^\infty(\mathbb{R}^{n+1})_R \),
\[
\hat{\Omega}^+_R \chi(P^{ex}, H^{ex}) = \chi(P, H)\hat{\Omega}^+_R.
\] (7.6)

Proof. Let us recall that \( \mathcal{R} \cap \Sigma = \mathcal{E}^{(1)}(\mathcal{R}) \) and the union is disjoint. Since the lower boundary of the joint spectrum of \( (P^{(1)}, H^{(1)}) \) is \( \xi \to \Sigma_0^{(1)}(\xi) \), we note that
\[
E^{ex}(\mathcal{R}) = E(\mathcal{R}) \oplus E^{(1)}(\mathcal{E}^{(1)}),
\] (7.7)

where \( E^{(1)}(\cdot) \) is the joint spectral resolution of \( (P^{(1)}, H^{(1)}) \). As for the first component, we obtain that for any \( \Psi \in \mathcal{H} \)
\[
e^{it\hat{H}(1,1)^* e^{-itH^{ex}} (E(\Sigma_{pp} \cap \mathcal{R}) \Psi \otimes |0\rangle)} = E(\Sigma_{pp} \cap \mathcal{R}) \Psi,
\] (7.8)

thus \( \hat{\Omega}^+_R \) trivially exists and is an isometry on this subspace.

Let us now consider the second component of the direct sum (7.7). Let \( K \subset \mathcal{E}^{(1)} \) be a compact set. Let us show that \( \hat{\Omega}^+_R \) exists on the range of \( E^{(1)}(K) \). We set \( \mathcal{T} := (\mathcal{T}^{(1)} \cup \text{Exc} \cup \Sigma_{pp}) \) and pick a vector \( \Psi \in E^{(1)}(K)(\mathcal{H} \otimes \mathcal{F}^{(1)}) \). Now we choose open sets \( G_n \supset \mathcal{T} \) s.t.
\[
\langle \Psi, E^{(1)}(G_n \setminus \mathcal{T}) \Psi \rangle \leq \frac{1}{n}.
\] (7.9)

Such sets exist by the regularity of the spectral measure. We note that \( K_n := K \setminus G_n \) are compact sets. By Proposition 7.1 for any \( (\xi_0, \lambda_0) \in K_n \) there exists a neighbourhood \( \mathcal{O} \), specified in Definition 6.1, s.t.
\[
\hat{\Omega}^+_R = \hat{\Omega}^+_R E^{(1)}(\mathcal{O})
\] (7.10)

exists. Such sets \( \mathcal{O} \) form a covering of \( K_n \) from which we can choose a finite sub-covering \( \{\mathcal{O}_j\}_{j=1}^{N_0} \). By taking intersections of the sets in this sub-covering, we can find a family of disjoint Borel sets \( \{B_i\}_{i=1}^N \), whose union coincides with \( K_n \) and s.t. each \( B_i \) is contained in some set \( \mathcal{O}_{j_i} \), as specified above. Thus we can write
\[
e^{it\hat{H}(1,1)^* e^{-itH^{ex}} (E(\Sigma_{pp}) \otimes 1) E^{(1)}(K) \Psi}
= \sum_{i=1}^N e^{it\hat{H}(1,1)^* e^{-itH^{ex}} (E(\Sigma_{pp}) \otimes 1) E^{(1)}(B_i) \Psi + O(1/n)}
\] (7.11)

In the first step above we made use of the relation
\[
E^{(1)}(K) = E^{(1)}(K_n) + E^{(1)}(G_n \cap K)
= E^{(1)}(K_n) + E^{(1)}((G_n \setminus \mathcal{T}) \cap K) + E^{(1)}(\mathcal{T} \cap K).
\] (7.12)
The second term on the r.h.s above, together with the bound (7.9) and property (A.18), gives rise to the term \(O(1/n)\) on the r.h.s. of (7.11), i.e. a term whose norm is bounded by \(c/n\) for some \(c\) independent of \(t\). (Here we exploit compactness of \(K\)). The last term on the r.h.s. of (7.12) is zero due to the relation

\[
E^1(T \cap \mathcal{E}^1) = \mathcal{I}_{\text{LLP}}^{(1)} \left( \int d\xi E^1_\xi (T(\xi) \cap \mathcal{E}^1(\xi)) \right) \mathcal{I}_{\text{LLP}}^{(1)},
\]

the fact that the set \(T(\xi) \cap \mathcal{E}^1(\xi)\) is countable for any \(\xi\) (Theorem 2.2) and therefore \(E^1_\xi (T(\xi) \cap \mathcal{E}^1(\xi)) = E^1_\xi (\Sigma_{\text{pp}}(\xi) \cap \mathcal{E}^1(\xi))\) which is equal to zero except for \(\xi\) from some set of zero Lebesgue measure (Lemma M.2). Now relation (7.11) and Proposition 7.1 give the existence of 

\[
\hat{\Omega}_R^+ + K : \hat{\Omega}_R^+ + \mathcal{E}(1) \to \mathcal{H} \otimes \mathcal{F}(1)
\]

by the Cauchy criterion.

Let us now show that \(\hat{\Omega}_K^+\) is isometric on the range of \(\mathcal{E}(1)(K)\). We obtain from (7.11) that

\[
\hat{\Omega}_K^+ \Psi = \sum_{i=1}^N W_{\hat{\Omega}_K^+}^+ E^1(B_i) \Psi + O(1/n),
\]

where we made use of Proposition 7.1 to replace the conventional wave operators with the localized wave operators \(W_{\hat{\Omega}_K^+}^+\). Recalling that the sets \(B_i\) are disjoint and the localized wave operators intertwine \((P,H)\) with \((P^\text{ex},H^\text{ex})\), we can write

\[
\|\hat{\Omega}_K^+ \Psi\|^2 = \sum_{i=1}^N \|W_{\hat{\Omega}_K^+}^+ E^1(B_i) \Psi\|^2
\]

\[
+ 2 \text{Re} \langle (\hat{\Omega}_K^+ \Psi - O(1/n)), O(1/n) \rangle + \langle O(1/n), O(1/n) \rangle.
\]

The first term on the r.h.s. above satisfies

\[
\sum_{i=1}^N \|W_{\hat{\Omega}_K^+}^+ E^1(B_i) \Psi\|^2 = \sum_{i=1}^N \langle \Psi, E^1(B_i) \Psi \rangle = \langle \Psi, E^1(K_n) \Psi \rangle = \|\Psi\|^2 + O(1/n),
\]

where in the first step we made use of Theorem 6.9, Lemma M.1 and the fact that \(B_i \subset O_j\). In the second step we used that the union of \(B_n\) coincides with \(K_n\) and in the last step we exploited formula (7.12) and the subsequent discussion. The last two terms on the r.h.s. of (7.15) and the last term on the r.h.s. of (7.16) can be made arbitrary small by taking \(n\) sufficiently large. Thus we have shown that

\[
\|\hat{\Omega}_K^+ \Psi\| = \|\Psi\|,
\]

i.e. \(\hat{\Omega}_K^+\) is isometric on the range of \(E^1(K)\).

Now let \(K^n \subset \mathcal{E}^1\) be an increasing family of compact sets s.t. \(\bigcup_{n \geq 0} K_n = \mathcal{E}^1\). Then

\[
\mathcal{D} := \bigcup_{n \geq 0} E^1(K^n)(\mathcal{H} \otimes \mathcal{F}^1)
\]

is a dense domain in \(E(\mathcal{E}^1)(\mathcal{H} \otimes \mathcal{F}^1)\). (Here we exploit the inner regularity of the spectral measure). It follows from our above considerations that \(\hat{\Omega}_K^+\) is well defined on \(\mathcal{D}\) and is an isometry on this domain. Thus \(\hat{\Omega}_K^+\) extends to an isometry on \(E(\mathcal{E}^1)(\mathcal{H} \otimes \mathcal{F}^1)\).
We conclude that $\hat{\Omega}_R^+$, as defined in (7.4), exists. In view of relation (7.8), to complete the proof of (7.5), it suffices to show that for any $\Psi_0 \in \mathcal{D}(\Sigma_\text{pp} \cap \Sigma) \mathcal{H}$ and $\Psi \in \mathcal{D}(1)(K) (\mathcal{H} \otimes \mathcal{F}(1))$ as specified above, there holds

$$\langle \Psi_0, \hat{\Omega}_R^+ \Psi \rangle = 0. \quad (7.19)$$

To this end we make use again of relation (7.14) and of the intertwining property of the localized wave operators to write

$$\langle \Psi_0, \hat{\Omega}_R^+ \Psi \rangle = \sum_{i=1}^N (E(B_i) \Psi_0, W_{O_i}^+ E(1)(B_i) \Psi) + \langle \Psi_0, O(1/n) \rangle. \quad (7.20)$$

Now we note that $E(B_i) \Psi_0 = 0$, since the sets $B_i$ do not intersect with the point spectrum of $(P, H)$. The last term on the r.h.s. above can be made arbitrarily small by choosing large $n$. This concludes the proof of (7.6).

Finally, let us show the intertwining property (7.6). In view of the decomposition (7.7), it suffices to check (7.6) first on vectors of the form $\Psi_0 \otimes |0\rangle$, $\Psi_0 \in E(\Sigma_\text{pp} \cap \Sigma) \mathcal{H}$ and then on vectors $\Psi \in \mathcal{D}(1)(K) (\mathcal{H} \otimes \mathcal{F}(1))$ as specified above. In the first case we get

$$e^{it\hat{\Gamma}(1,1)} e^{-it\hat{H}} \chi(P_{ex}, H_{ex})(\Psi_0 \otimes |0\rangle) = e^{it\hat{\Gamma}(1,1)} e^{-it\hat{H}} \chi(P, H) \Psi_0 \otimes |0\rangle$$

$$= \chi(P, H) \Psi_0 = \chi(P, H) e^{it\hat{\Gamma}(1,1)} e^{-it\hat{H}} \chi(P, H) \Psi_0 \otimes |0\rangle. \quad (7.21)$$

As for the vectors of the second type, we make use of relation (7.14):

$$\hat{\Omega}_R^+ \chi(P_{ex}, H_{ex}) \Psi = \sum_{i=1}^N W_{O_i}^+ \chi(P_{ex}, H_{ex}) E(1)(B_i) \Psi + O(1/n)$$

$$= \chi(P, H) \sum_{i=1}^N W_{O_i}^+ E(1)(B_i) \Psi + O(1/n)$$

$$= \chi(P, H) \hat{\Omega}_R^+ \Psi + O(1/n), \quad (7.22)$$

where we made use of the intertwining relation for the localized wave operators shown in Theorem 6.6. Since $O(1/n)$ can be made arbitrarily small, this concludes the proof.

**Theorem 7.3.** The wave operator $\hat{\Omega}_R^+$, defined in (7.4), satisfies

$$\text{Ran} \hat{\Omega}_R^+ = E(\Sigma) \mathcal{H}. \quad (7.23)$$

**Proof.** We recall that $\Sigma \cap \Sigma = \Sigma_\text{iso} \cup \mathcal{E}(1)$. We note that for any $\Psi \in \mathcal{D}(\Sigma_\text{iso}) \mathcal{H} = \mathcal{H}_\text{iso}$ there holds

$$\Psi = \hat{\Omega}_R^+ (\Psi \otimes |0\rangle), \quad (7.24)$$

so $\text{Ran} \mathcal{D}(\Sigma_\text{iso}) \subset \text{Ran} \hat{\Omega}_R^+$. Next, let us choose a compact set $K \subset \mathcal{E}(1)$. We denote $\mathcal{T} := (\mathcal{T}(1) \cup \text{Exc} \cup \Sigma_{pp})$ and choose a vector $\Psi \in E(K) \mathcal{H}$. We select open sets $G_n \subset \mathcal{T}$ s.t.

$$\langle \Psi, E(G_n \setminus \mathcal{T}) \Psi \rangle \leq \frac{1}{n}. \quad (7.25)$$

Similarly as in the proof of Theorem 7.2, such sets exist by the regularity of the spectral measure. We define sets $K_n := K \setminus G_n$, which are compact. Now we write

$$E(K) = E(K_n) + E(G_n \cap K) = E(K_n) + E((G_n \setminus \mathcal{T}) \cap K) + E(\mathcal{T} \cap K). \quad (7.26)$$
Due to relation (7.25), the second term on the r.h.s. above satisfies
\[ \|E((G_n \setminus T) \cap K)\Psi\| \leq \frac{1}{n}. \]  
(7.27)
As for the last term on the r.h.s. of (7.26), we note that
\[ E(T \cap E^{(1)}) = I_{\text{LLP}}^* \left( \int d\xi E_\xi (T(\xi) \cap E^{(1)}(\xi)) \right) I_{\text{LLP}} = E(\Sigma_{pp} \cap E^{(1)}), \]  
(7.28)
where we made use of the fact that, by Theorem 2.2, \( T(\xi) \cap E^{(1)}(\xi) \) is countable for any \( \xi \). Thus \( E(T \cap K)\Psi \) belongs to the range of \( E(\Sigma_{pp} \cap E^{(1)}) \) and hence belongs to the range of the wave operator. In fact, for any \( \Psi \in E(\Sigma_{pp} \cap E^{(1)})H \) we have
\[ \hat{\Omega}_R^+(\Psi \otimes |0\rangle) = \Psi, \]  
(7.29)
similarly as in formula (7.24) above.

Let us now consider the first term on the r.h.s. of (7.26). For any \((\xi_0, \lambda_0) \in K_n\) there exists a neighbourhood \( \mathcal{O} \) as specified in Definition 6.1. Such neighbourhoods form a covering of \( K_n \) from which we can choose a finite sub-covering \( \{\mathcal{O}_j\}_{j=1}^{N_0} \). By taking intersections of these sets, if necessary, we can find a finite family of disjoint Borel sets \( \{B_i\}_{i=1}^{N} \) and their union coincides with \( K_n \). Thus, by Theorem 6.7 and Proposition 7.1, we can write
\[ E(K_n) = \sum_{i=1}^{N} E(B_i) = \sum_{i=1}^{N} W_{\mathcal{O}_i}^+ W_{\mathcal{O}_i}^{+*} E(B_i) = \sum_{i=1}^{N} \hat{\Omega}_R^+ W_{\mathcal{O}_i}^{+*} E(B_i), \]  
(7.30)
where \( W_{\mathcal{O}_i}^+ \) are the localized wave operators and we made use of the fact that they intertwine (\( P^\text{ex}, H^\text{ex} \)) with (\( P, H \)). Thus the range of \( E(K_n) \) is contained in the range of \( \hat{\Omega}_R^+ \).

Summing up, for any \( \Psi \in E(K)H \) there exists a sequence of vectors \( \Psi_n \in \text{Ran} \hat{\Omega}_R^+ \) s.t.
\[ \|\Psi - \Psi_n\| \leq \frac{1}{n}. \]  
(7.31)
Note that by construction \( \hat{\Omega}_R^+ \) vanishes on \( E^\text{ex}(R)H^\text{ex} \odot (E(\Sigma_{pp}) \otimes 1)E^\text{ex}(R)H^\text{ex} \). Hence, by relation (7.31), \( \text{Ran} \hat{\Omega}_R^+ \) is closed and we obtain that \( \Psi \in \text{Ran} \hat{\Omega}_R^+ \). This completes the proof of the fact that \( \text{Ran} \hat{\Omega}_R^+ \supset E(R) \). The opposite inclusion follows from the intertwining relation (7.1).

We are now in a position to extract our main theorem from Subsection 2.4 as well as its corollary. In proofs we will make use of the identity
\[ E^\text{ex}(R)H_+ = E^\text{ex}(R)(E(\Sigma_{pp}) \otimes 1)H^\text{ex} = (E(\Sigma_{pp}) \otimes 1)E^\text{ex}(R)H^\text{ex}, \]  
(7.32)
which follows from the definition \( H_+ = (E(\Sigma_{pp}) \otimes 1)H^\text{ex} \) of the outgoing Hilbert space.

**Proof of Theorem 2.3** By (7.32) and Theorem 7.2 we conclude the existence of \( \Omega_R^+ = (\hat{\Omega}_R^+)_{E^\text{ex}(R)H_+} \) and the property
\[ \Omega_R^{+*}\Omega_R^+ = E^\text{ex}(R)|_{H_+}. \]  
(7.33)
By construction of \( \hat{\Omega}_R^+ \), we observe that \( \hat{\Omega}_R^+ \Psi = 0 \), for \( \Psi \in E^\text{ex}(R)H^\text{ex} \odot E^\text{ex}(R)H_+ \). Hence, Theorem 7.3 gives \( \text{Ran} \Omega_R^+ = E(R)H \). Together with (7.33) this implies unitarity of \( \Omega_R^+: E^\text{ex}(R)H_+ \to E(R)H \). That is, \( \Omega_R^+\Omega_R^{+*} = E(R) \), which concludes the proof. \qed
Proof of Corollary 2.4. To prove part (a) we recall that \( a^*_\pm(h)\Psi := \Omega^+_R(\Psi \otimes a^*(h)|0\rangle) \), where \( \Psi \in E(\mathcal{R})\mathcal{H}_{bnd} \) and \( h \) are \( \mathcal{R} \)-compatible. Now we compute, making use of (7.33)

\[
\langle a^*_\pm(h)\Psi, a^*_\pm(h')\Psi' \rangle = \langle \Omega^+_R(\Psi \otimes a^*(h)|0\rangle), \Omega^+_R(\Psi' \otimes a^*(h')|0\rangle) \rangle = \langle \Psi, \Psi' \rangle \langle h, h' \rangle.
\]

Similarly, for \( \Psi'' \in E(\mathcal{R})\mathcal{H}_{bnd} \),

\[
\langle a^*_\pm(h)\Psi, \Psi'' \rangle = \langle \Omega^+_R(\Psi \otimes a^*(h)|0\rangle), \Omega^+_R(\Psi'' \otimes |0\rangle) \rangle = 0,
\]

where we made use of the fact that \( \Omega^+_R(\Psi'' \otimes |0\rangle) = \Psi'' \) (cf. relation (7.8)) and of (7.33).

To prove part (b) of the corollary, we recall that \( \text{Ran }\Omega^+_{\mathcal{R}} = E(\mathcal{R})\mathcal{H} \) and therefore, any vector \( \Psi_1 \in E(\mathcal{R})\mathcal{H} \) can be written as \( \Psi_1 = \Omega^+_R \Psi_2 \), where \( \Psi_2 \in E^{ex}(\mathcal{R})\mathcal{H}_+. \) By (M.1) in Lemma (M.1) applied with \( O = \mathcal{R} \), we find that \( E^{ex}(\mathcal{R})\mathcal{H}_+^{ex} = E(\mathcal{R})\mathcal{H} \oplus E^{(1)}(\mathcal{R})(\mathcal{H}_{iso} \otimes \mathfrak{h}) \).

Since the second summand is already sitting inside \( E^{ex}(\mathcal{R})\mathcal{H}_+ \), cf. (7.32), we find that

\[
E^{ex}(\mathcal{R})\mathcal{H}_+ = E(\mathcal{R})\mathcal{H}_{bnd} \oplus E^{(1)}(\mathcal{R})(\mathcal{H}_{iso} \otimes \mathfrak{h}).
\]

The claim (b) now follows from (M.2), applied with \( O = \mathcal{R} \). \( \Box \)

A  Fock space combinatorics

A.1  Fock space

Let \( \mathfrak{h} \) be the single-particle space and \( \Gamma(\mathfrak{h}) \) be the symmetric Fock space over \( \mathfrak{h} \) given by

\[
\Gamma(\mathfrak{h}) := \bigoplus_{n \geq 0} \Gamma^{(n)}(\mathfrak{h}),
\]

where \( \Gamma^{(n)}(\mathfrak{h}) = \mathfrak{h}^\otimes_n \). \( \Gamma^{(0)}(\mathfrak{h}) \) is spanned by the vacuum vector denoted by \( |0\rangle \). (If the single-particle space \( \mathfrak{h} \) is fixed, we use a shorter notation \( \mathcal{F} := \Gamma(\mathfrak{h}) \) and \( \mathcal{F}^{(n)} := \Gamma^{(n)}(\mathfrak{h}) \).)

For any set \( D \subset \mathfrak{h} \) we set \( \Gamma^{(n)}(D) = D^\otimes_n \) and define \( \Gamma_{\text{fin}}(D) \) as the space of finite linear combinations of vectors from \( \Gamma^{(n)}(D) \), \( n = 0, 1, 2 \ldots \).

Let \( D \subset \mathfrak{h} \) be a dense domain and \( a : D \to \mathfrak{h} \) a linear map. Then \( d\Gamma(a) \) is defined on \( \Gamma^{(n)}(D) \), \( n \geq 1 \), by

\[
d\Gamma(a) := \sum_{i=1}^{n} 1 \otimes \cdots 1 \otimes a \otimes 1 \otimes \cdots 1. \tag{A.2}
\]

and extended to \( \Gamma_{\text{fin}}(D) \) by linearity and the relation \( d\Gamma(a)|0\rangle = 0 \). In particular, \( N := d\Gamma(1) \) is called the number operator. We recall that if \( a \) is closable then so is \( d\Gamma(a) \). Moreover, if \( a \) is essentially self-adjoint on \( D \), then \( d\Gamma(a) \) is essentially self-adjoint on \( \Gamma_{\text{fin}}(D) \). Finally, if \( \mathfrak{h} \) is a quadratic form on \( D_1 \times D_2 \), where \( D_1, D_2 \) are dense domains in \( \mathfrak{h} \), then one can also define \( d\Gamma(b) \) as a quadratic form on \( \Gamma_{\text{fin}}(D_1) \times \Gamma_{\text{fin}}(D_2) \).

Let \( \mathfrak{h}_1, \mathfrak{h}_2 \) be two single-particle spaces and let \( D_1 \subset \mathfrak{h}_1 \) be a dense domain. For any linear map \( q : D_1 \to \mathfrak{h}_2 \) we define a map \( \Gamma(q) \) on \( \Gamma^{(n)}(D_1) \), \( n \geq 1 \), by

\[
\Gamma(q) := q \otimes \cdots \otimes q \tag{A.3}
\]

and extend it to \( \Gamma_{\text{fin}}(D_1) \) by linearity and the relation \( \Gamma(q)|0\rangle = |0\rangle \). If \( q \) is a contraction (i.e. \( \|q\| \leq 1 \)) then \( \Gamma(q) \) extends to a contraction \( \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2) \). We recall that for a contraction \( q \) acting on \( \mathcal{K} \otimes \mathfrak{h} \) one can define \( \Gamma(q) \) as a contraction on \( \mathcal{K} \otimes \Gamma(\mathfrak{h}) \). See [38, Remark 1.1].
Let \( q, a_1, \ldots, a_m \) be operators \( D_1 \rightarrow \mathfrak{h}_2 \) defined on some common dense domain \( D_1 \subset \mathfrak{h}_1 \). Then we define \( d\Gamma(q, a_1, \ldots, a_m) \) on \( \Gamma^{(n)}(D_1), n \geq m \), by
\[
d\Gamma(q, a_1, \ldots, a_m) := \sum_{i_1, \ldots, i_m \atop \forall k \neq i} (q \otimes \cdots \otimes q \otimes a_1 \otimes q \otimes \cdots \otimes q \otimes a_m \otimes q \otimes \cdots \otimes q), \tag{A.4}
\]
and extend it to \( \Gamma_{\text{fin}}(D_1) \) by linearity and by setting \( d\Gamma(q, a_1, \ldots, a_m) = 0 \) on \( D_1^{\otimes n} \), where \( 0 \leq n < m \).

We note the following simple relation between the objects introduced above: Let \( q, p \) be bounded operators on \( \mathfrak{h} \) which commute and let \( a \) be a self-adjoint operator on some domain \( D \subset \mathfrak{h} \). Then
\[
[\Gamma(q), d\Gamma(p, a)] = d\Gamma(qp, [q, a]) \tag{A.5}
\]
in the sense of quadratic forms on \( \Gamma_{\text{fin}}(D) \times \Gamma_{\text{fin}}(D) \).

### A.2 Extended Fock space

Extended Fock space is defined by \( \Gamma^{\text{ex}}(\mathfrak{h}) := \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \) (or, in a shorter notation, \( \mathcal{F}^{\text{ex}} := F \otimes F \)). Let \( U : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \rightarrow \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \) be the canonical identification, defined by the relations
\[
U a^*(h) = (a^*(h) \otimes 1 + 1 \otimes a^*(h))U \quad \text{and} \quad U |0\rangle = |0\rangle \otimes |0\rangle. \tag{A.6}
\]
We will use \( U \) to transport objects defined in the previous subsection to the extended Fock space: Let \( q_0, q_\infty \) and \( a_0, a_\infty \) be operators on \( \mathfrak{h} \) defined on a dense domain \( D \subset \mathfrak{h} \). Let \( q := \text{diag}(q_0, q_\infty) \) and \( a := \text{diag}(a_0, a_\infty) \) be operators on \( \mathfrak{h} \oplus \mathfrak{h} \) defined on the domain \( D \oplus D \). Then we introduce the following operators on the extended Fock space:
\[
\Gamma^{\text{ex}}(q) := U \Gamma(q) U^* = \Gamma(q_0) \otimes \Gamma(q_\infty), \tag{A.7}
\]
\[
d\Gamma^{\text{ex}}(q) := U d\Gamma(q) U^* = d\Gamma(q_0) \otimes 1 + 1 \otimes d\Gamma(q_\infty), \tag{A.8}
\]
\[
d\Gamma^{\text{ex}}(q, a) := U d\Gamma(q, a) U^* = d\Gamma(q_0, a_0) \otimes \Gamma(q_\infty) + \Gamma(q_0) \otimes d\Gamma(q_\infty, a_\infty), \tag{A.9}
\]
which are defined on \( \Gamma_{\text{fin}}(D) \otimes \Gamma_{\text{fin}}(D) \) and in the last equality we used Lemma \( A.3 \) stated below. We note that if \( q_0 \) and \( q_\infty \) are contractions, then \( \Gamma^{\text{ex}}(q) \) is also a contraction. In this situation we set
\[
\Gamma^{(n)}(q) := \Gamma^{\text{ex}}(q)|\Gamma^{(n)}(\mathfrak{h}) \otimes \Gamma^{(n)}(\mathfrak{h})\rangle, \tag{A.10}
\]
\[
d\Gamma^{(n)}(q) := d\Gamma^{\text{ex}}(q)|\Gamma_{\text{fin}}(D) \otimes \Gamma^{(n)}(D)\rangle, \tag{A.11}
\]
\[
d\Gamma^{(n)}(q, a) := d\Gamma^{\text{ex}}(q, a)|\Gamma_{\text{fin}}(D) \otimes \Gamma^{(n)}(D)\rangle. \tag{A.12}
\]
In the special case where \( q_0 = q_\infty =: q, a_0 = a_\infty =: a \) we will drop the underlining and write
\[
\Gamma^{\text{ex}}(q) = \Gamma(q) \otimes \Gamma(q), \tag{A.13}
\]
\[
d\Gamma^{\text{ex}}(a) = d\Gamma(a) \otimes 1 + 1 \otimes d\Gamma(a), \tag{A.14}
\]
\[
d\Gamma^{\text{ex}}(q, a) = d\Gamma(q, a) \otimes \Gamma(q) + \Gamma(q) \otimes d\Gamma(q, a), \tag{A.15}
\]
which is the standard notation. Since extended objects are unitarily equivalent to operators on \( \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \), the properties of closedness and essential self-adjointness are naturally inherited from the single-particle level, as discussed after formula \( A.2 \) above.
Now let \( c_0, c_\infty \) be bounded operators on \( \mathfrak{h} \). We define \( c : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \), which acts on \( h \in \mathfrak{h} \)
by \( ch = (c_0 h, c_\infty h) \), and is s.t. \( \|c^* c\| = \|c_0^* c_0 + c_\infty^* c_\infty\| \leq 1 \). Then

\[
\hat{\Gamma}(c) := U\Gamma(c)
\]

is a mapping \( \mathcal{F} \to \mathcal{F}^\text{ex} \) of norm one \([9]\). We also define \( \hat{\Gamma}^{(n)}(c) := P_n \hat{\Gamma}(c) \), where \( P_n : \mathcal{F}^\text{ex} \to \mathcal{F} \otimes \mathcal{F}^{(n)} \) is the natural restriction map. Next, given a linear map \( a : D \to \mathfrak{h} \oplus \mathfrak{h} \), where \( D \subset \mathfrak{h} \), we set

\[
d\hat{\Gamma}(c, a) := Ud\Gamma(c, a),
\]

which is a mapping \( \Gamma_{\text{fin}}(D) \to \mathcal{F}^\text{ex} \). We also define \( d\hat{\Gamma}^{(n)}(c, a) = P_n d\hat{\Gamma}(c, a) \).

Let us denote by \((1, 1)\) the map \( \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \) which acts by \( (1, 1)h = (h, h) \), where \( h \in \mathfrak{h} \). We note that \( \|(1, 1)^* (1, 1)\| = \sqrt{2} \) and define \( \hat{\Gamma}(1, 1) \) as an unbounded operator on \( \Gamma_{\text{fin}}(\mathfrak{h}) \).

As stated in \([9]\), the following operators

\[
\hat{\Gamma}(1, 1)^* ((N + 1)^{-\frac{2}{3}} \otimes 1_{\{n\}}(N))
\]

are bounded for any \( n \in \mathbb{N} \).

### A.3 Useful lemmas

In this subsection we collect some simple relations between operators on Fock space, which are used repetitively in the paper. Most of these relations are well known (see e.g. \([9]\) Section 2) for \((A.19)\) and \((A.20)\).

**Lemma A.1.** Let \( q, p \) be bounded operators and \( h \in \mathfrak{h} \). Then the following equalities hold in the sense of quadratic forms on \( \Gamma_{\text{fin}}(\mathfrak{h}) \times \Gamma_{\text{fin}}(\mathfrak{h}) \):

\[
d\Gamma(q, p) a^*(h) = a^*(ph)\Gamma(q) + a^*(qh)d\Gamma(q, p), \quad (A.19)
\]

\[
a(h) d\Gamma(q, p) = \Gamma(q)a(p^* h) + d\Gamma(q, p)a(q^* h). \quad (A.20)
\]

**Proof.** Note the identity \( \Gamma(q + sp)a^*(h) = a^*((q + sp)h)\Gamma(q + sp) \) valid for any \( s \in \mathbb{R} \). By computing the matrix elements of this expression between vectors from \( \Gamma_{\text{fin}}(\mathfrak{h}) \) and differentiating them w.r.t. \( s \) at \( s = 0 \) we conclude the proof. \( \square \)

**Lemma A.2.** Let \( \omega, a_1, \ldots, a_n \) be operators defined on a common domain \( D \subset \mathfrak{h} \), whose adjoints are defined on a common domain \( D^* \subset \mathfrak{h} \). Let \( j \) be a bounded operator on \( \mathfrak{h} \). Then, in the sense of quadratic forms on \( \Gamma_{\text{fin}}(D^*) \times \Gamma_{\text{fin}}(D) \)

\[
[d\Gamma(\omega), d\Gamma(j, a_1, \ldots, a_n)]
\]

\[
= d\Gamma(j, [\omega, j], a_1, \ldots, a_n) + \sum_{i=1}^n d\Gamma(j, a_1, \ldots, [\omega, a_i], \ldots, a_n). \quad (A.21)
\]

Now suppose that \( j : \mathfrak{h}_1 \to \mathfrak{h}_2 \) is s.t. \( \|j\| \leq 1 \) and \( a_1, \ldots, a_n : \mathfrak{h}_1 \to \mathfrak{h}_2 \) are bounded operators. Then

\[
\|d\Gamma(j, a_1, \ldots, a_n)(1 + N)^{-n}\| \leq C\|a_1\| \ldots \|a_n\|. \quad (A.22)
\]

**Proof.** Relation \((A.21)\) can easily be seen by differentiating the function

\[
(s, s_1, \ldots, s_n) \to \langle \Psi_1, [\Gamma(1 + s\omega), \Gamma(j + s_1 a_1 + \cdots + s_n a_n)]\Psi_2 \rangle,
\]

where \( \Psi_1 \in \Gamma_{\text{fin}}(D^*) \) and \( \Psi_2 \in \Gamma_{\text{fin}}(D) \), w.r.t. each of the parameters separately and then setting \((s, s_1, \ldots, s_n) = 0\). The bound \((A.22)\) follows immediately from definition \((A.4)\). \( \square \)
Lemma A.3. Let \( q_0, q_∞ \) be bounded operators on \( \mathfrak{h} \), and let \( p_0, p_∞ \) be defined on a domain \( D \subset \mathfrak{h} \). We define the following operators on \( \mathfrak{h} \oplus \mathfrak{h} \):

\[
\begin{bmatrix}
q & 0 \\
0 & q_∞
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
p_0 & 0 \\
0 & p_∞
\end{bmatrix}.
\tag{A.24}
\]

There holds the following identity on vectors from \( \Gamma_\text{fin}(D) \otimes \Gamma_\text{fin}(D) \):

\[
Ud\Gamma(q,p)U^* = (d\Gamma(q_0,p_0) \otimes \Gamma(q_∞) + \Gamma(q_0) \otimes d\Gamma(q_∞,p_∞)).
\tag{A.25}
\]

Proof. Note that \( U\Gamma(q + sp) = (\Gamma(q_0 + sp_0) \otimes \Gamma(q_∞ + sp_∞))U \) for \( s \in \mathbb{R} \). By computing the matrix elements of this expression between vectors from the specified domains and differentiating them w.r.t. \( s \) at \( s = 0 \) we conclude the proof. \( \square \)

Lemma A.4. Let \( \omega, c_i, c_i,∞ \), \( 1 \leq i \leq n \), be operators defined on a common domain \( D \subset \mathfrak{h} \), whose adjoints are defined on a common domain \( D^* \subset \mathfrak{h} \). We define \( \bar{\omega} := \text{diag}(\omega, \omega) \) as an operator on \( \mathfrak{h} \oplus \mathfrak{h} \) with a domain \( D \oplus D \). Now let \( j_0, j_∞ \) be bounded operators on \( \mathfrak{h} \). We define \( c_i := (c_{i0}, c_i,∞) \) as maps \( D \to \mathfrak{h} \oplus \mathfrak{h} \) and \( j := (j_0, j_∞) \) as a map \( \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \). Then the following relation holds in the sense of quadratic forms on \( \Gamma_\text{fin}(D^* \oplus D^*) \times \Gamma_\text{fin}(D) \):

\[
d\Gamma(\bar{\omega})d\Gamma(j, c_1, \ldots, c_n) - d\Gamma(j, c_1, \ldots, c_n)d\Gamma(\bar{\omega}) = d\Gamma(j, [\bar{\omega}, j], c_1, \ldots, c_n) + \sum_{i=1}^n d\Gamma(j, c_1, \ldots, [\bar{\omega}, c_i], \ldots, c_n),
\tag{A.26}
\]

where \( [\bar{\omega}, j] := \bar{\omega}j - j\bar{\omega} = ([\bar{\omega}, j_0], [\bar{\omega}, j_∞]) \).

Proof. The relation follows by differentiating the function

\[
(s, s_1, \ldots, s_n) \to \langle \Psi_1, (\Gamma(1 + s\bar{\omega})j + \sum_{i=1}^n s_ic_i) - \Gamma(j + \sum_{i=1}^n s_ic_i)\Gamma(1 + s\bar{\omega})\rangle \Psi_2,
\tag{A.27}
\]

where \( \Psi_1 \in \Gamma_\text{fin}(D^* \oplus D^*) \) and \( \Psi_2 \in \Gamma_\text{fin}(D) \), w.r.t. \( s \) and \( s_i \) separately and setting \( (s, s_1, \ldots, s_n) = 0 \). \( \square \)

Lemma A.5. Let \( a, b \) be operators defined on some common domain \( D \subset \mathfrak{h} \), whose adjoints are defined on some common domain \( D^* \subset \mathfrak{h} \). We define \( \bar{a} := \text{diag}(a, a) \) and \( \bar{b} := \text{diag}(b, b) \) as operators on \( \mathfrak{h} \oplus \mathfrak{h} \) with domains \( D \oplus D \). Let \( q, j_0, j_∞ \in B(\mathfrak{h}) \) and suppose that \( [q, j_0] = 0 \) and \( [q, j_∞] = 0 \). We define \( j := (j_0, j_∞) \) to be a map \( \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \) and we specify \( q := \text{diag}(q, q) \) to be an operator on \( \mathfrak{h} \oplus \mathfrak{h} \). Then, in the sense of quadratic forms on \( \Gamma_\text{fin}(D^* \oplus D^*) \times \Gamma_\text{fin}(D) \)

\[
\Gamma(j)d\Gamma(q, a) - d\Gamma(q, [\bar{a}, q])\Gamma(j) = d\Gamma(jq, [\bar{a}, q]),
\tag{A.28}
\]

\[
\Gamma(j)d\Gamma(q, a, b) - d\Gamma(q, [\bar{a}, b])\Gamma(j) = d\Gamma(jq, [\bar{a}, b]) + d\Gamma(jq, [\bar{a}, q], b) + d\Gamma(jq, [\bar{a}, q], b) + d\Gamma(jq, [\bar{a}, q], b)
\tag{A.29}
\]

where \( [\bar{a}, q] = qa - qa \).

Proof. The relations follow by differentiating the matrix elements of the functions

\[
s \to \Gamma(j)\Gamma(q + sa) - \Gamma(q + sa)\Gamma(j),
\tag{A.30}
\]

\[
(s, s_1) \to \Gamma(j)\Gamma(q + sa + s_1b) - \Gamma(q + sa + s_1b)\Gamma(j),
\tag{A.31}
\]

in each argument separately, and setting \( s = 0 \), respectively \( (s, s_1) = 0 \). \( \square \)

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\section{Commutator expansions}

Commutator expansions for functions of several commuting observables, which we need in the present work, were established in \cite{[41]}. To state this result, we need first several definitions: For \( \rho \in \mathbb{R} \), we define the class of functions \( S^\rho(\mathbb{R}^\nu) \subset C^\infty(\mathbb{R}^\nu) \), s.t.

\[
|\partial^\alpha f(x)| \leq C_\alpha(x)^{\rho-|\alpha|}, \tag{B.1}
\]

for any multiindex \( \alpha \). In the definition below we use the notation \( \delta_j := (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^\nu \), with 1 on the \( j \)-th entry.

**Definition B.1.** Let \( A = (A_1, \ldots, A_\nu) \) be a vector of commuting self-adjoint operators with domains \( D(A_j) \subset \mathcal{H} \), and \( B \) a bounded operator on \( \mathcal{H} \). We say that \( B \in C^1(A) \), or \( B \) is of class \( C^1(A) \), if the commutator forms \( [A_j, B] \), a priori defined as quadratic forms on \( D(A_j) \), extend by continuity to bounded operators \( [A_j, B] =: \text{ad}_{A_j}(B) = \text{ad}_A^\delta(B) \). For \( n_0 > 1 \) we define the class \( C^{n_0}(A) \) and iterated commutators \( \text{ad}_A^{n_0}(B) \) recursively: We say that \( B \in C^{n_0}(A) \) if \( B \in C^{n_0-1}(A) \) and for any \( |\alpha| < n_0 \) and \( j \in \{1, \ldots, \nu\} \), the commutator forms \( [A_j, \text{ad}_A^{n_0}(B)] \) extend by continuity to bounded operators \( \text{ad}_A^{n_0+\delta_j}(B) \).

**Remark B.2.** In the case \( \nu = 1 \) the above definition reduces to a more standard one: \( B \in B(\mathcal{H}) \) belongs to \( C^{n_0}(A) \) if for any \( \Psi \in \mathcal{H} \) the map

\[
\mathbb{R} \ni s \to e^{isA}Be^{-isA}\Psi
\]

is \( n_0 \) times continuously differentiable in the norm topology. We also recall that this definition can be naturally extended to (possibly unbounded) self-adjoint operators: In this case we say that \( B \in C^{n_0}(A) \) if \( (B-z)^{-1} \in B(\mathcal{H}) \) is in \( C^{n_0}(A) \) for some – hence all – \( z \in \mathbb{C} \setminus \sigma(H) \) \cite{[38]}. Finally, we remind the reader that if \( B \in C^1(A) \) we have \( BD(A) \subset D(A) \).

Now we are ready to state the main result of \cite{[41]}.

**Lemma B.3.** Let \( B \in B(\mathcal{H}) \) and \( A = (A_1, \ldots, A_\nu) \) be a family of self-adjoint, pairwise commuting operators. Assume that \( B \in C^{n_0}(A) \) for some \( n_0 \geq n + 1 \geq 1 \), 0 \( \leq t_1 \leq n + 1 \) and \( 0 \leq t_2 \leq 1 \) and that \( f \in S^\rho(\mathbb{R}^\nu) \) for some \( \rho \in \mathbb{R} \) s.t. \( t_1 + t_2 + \rho < n + 1 \). Then

\[
[f(A), B] = \sum_{\alpha:1 \leq |\alpha| \leq n} (-1)^{|\alpha|+1} \frac{1}{\alpha!} \partial^\alpha f(A)\text{ad}_A^{n_0}(B) + R_{n+1}(f, A, B), \tag{B.3}
\]

as an identity on \( D((A)^\rho) \), where \( R_{n+1}(f, A, B) \in B(\mathcal{H}) \) and there exists a constant \( c_n(f) \), independent of \( A, B, s.t.

\[
\|\langle A \rangle^{t_1}R_{n+1}(f, A, B)\langle A \rangle^{t_2}\| \leq c_n(f) \sum_{\alpha:|\alpha| = n+1} \|\text{ad}_A^{n_0}(B)\|. \tag{B.4}
\]

**Remark B.4.** One can of course read the commutator expansion in Lemma \ref{lem:commutator-expansion} as a form identity on \( D((A)^\rho) \). We wish to argue in this remark, that one can also read it is an operator identity on \( D((A)^\rho) \). Suppose \( B \in C^{n_0}(A) \), with \( n_0 \in \mathbb{N} \). Assume \( B \in C^{n_0}(A) \). We wish to prove by induction after \( n_0 \) that

\[
\text{ad}_A^{n_0}(B)D((A)^{n_0}) \subset D((A)^{n_0-|\alpha|}), \tag{B.5}
\]

for multiindices \( \alpha \) with \( |\alpha| \leq n_0 \). From \ref{lem:extension}, using \( \rho < n_0 \), it follows by interpolation that \( \text{ad}_A^{n_0}(B)D((A)^\rho) \subset D((A)^{\rho-|\alpha|}) \). Hence the expansion in Lemma \ref{lem:commutator-expansion} is meaningful as an operator identity.
Lemma B.6. Let $n$ for some independent of $A$ as an identity on $D$. Proposition C.1. Let $B$. Lemma B.5. The claim now follows by the two induction hypotheses.

In our investigations we will use two special cases of Lemma B.3 which we now state explicitly:

**Lemma B.5.** Let $B \in B(\mathcal{H})$ and $A = (A_1, \ldots, A_\nu)$ be a family of self-adjoint, pairwise commuting operators. Assume that $B \in C^3(A)$ and that $f \in S^2(\mathbb{R}^\nu)$. Then

$$[f(A), B] = \sum_{\alpha: |\alpha| = 1} \partial^\alpha f(A) \text{ad}_{A}^\alpha(B) + R(f, A, B),$$

**B.6**

as an identity on $D(\langle A \rangle^2)$, where $R(f, A, B) \in B(\mathcal{H})$ and there exists a constant $c(f)$, independent of $A$ and $B$, s.t.

$$\|R(f, A, B)\| \leq c(f) \sum_{\alpha: 2 \leq |\alpha| \leq 3} \|\text{ad}_{A}^\alpha(B)\|. \tag{B.7}$$

**Lemma B.6.** Let $B \in B(\mathcal{H})$ and $A$ be a self-adjoint operator. Assume that $B \in C^{n_0}(A)$ for some $n_0 \geq n + 1 \geq 1$, and that $f \in S^0(\mathbb{R}^\nu)$. Then

$$[f(A), B] = \sum_{j=1}^{n} (-1)^{j-1} \frac{1}{j!} f^{(j)}(A) \text{ad}_{A}^j(B) + R_{n+1}(f, A, B), \tag{B.8}$$

as an identity on $\mathcal{H}$, where $R_{n+1}(f, A, B) \in B(\mathcal{H})$ and there exists a constant $c_n(f)$, independent of $A$, $B$, s.t.

$$\|R_{n+1}(f, A, B)\| \leq c_n(f) \|\text{ad}_{A}^{n+1}(B)\|. \tag{B.9}$$

**C  Commutator bounds in $L^2(\mathbb{R}^\nu)$**

Let $A$ and $H$ be self-adjoint operators on a Hilbert space $\mathcal{H}$, defined on domains $D(A)$ and $D(H)$, and s.t. $H$ is of class $C^1(A)$ (cf. Definition B.1 above). We recall that the natural domain $D(A) \cap D(H)$ of the commutator form $i[H, A]$ is dense in $D(H)$ in the topology given by the norm $\|\Psi\|_H := \|H\Psi\| + \|\Psi\|$. We will write $i[H, A]$ for the extension by continuity of the commutator form from $D(A) \cap D(H)$ to $D(H)$ (and also for the associated operator from $D(H)$ to $D(H)^*$). If, furthermore, $i[H, A]$ extends by continuity to an element of $B(\mathcal{H})$, as is sometimes the case below, then $[H, A]$ will denote this extension.

First, we recall the following abstract result from [39]:

**Proposition C.1.** Let $H$ and $A$ be self-adjoint operators that satisfy

(a) $D(A) \cap D(H)$ is a core for $H$.

(b) $e^{itA}D(H) \subset D(H)$ and for each $\Psi \in D(H)$ we have $\lim_{\|\| < \infty}$.
(c) There is a set $S \subset D(A) \cap D(H)$ which is a core for $H$ and is invariant under $e^{itA}$. The form $i[H, A]$ on $S$ is bounded below and closable, and the associated self-adjoint operator $i[H, A]_S^g$ satisfies $D(i[H, A]_S^g) \supset D(H)$.

Then, for all $\Phi, \Psi \in D(A) \cap D(H)$

$$\langle \Phi, i[H, A] \Psi \rangle = \langle \Phi, i[H, A]_S^g \Psi \rangle.$$  \hspace{1cm} (C.1)

Making use of the above proposition, we prove the following technical lemma:

**Lemma C.2.** Let $g \in C^\infty(\mathbb{R}^\nu)$ and let $v \in C^\infty_0(\mathbb{R}^\nu; \mathbb{R}^\nu)$. Let $a := \frac{1}{2}(v \cdot i\nabla_k + i\nabla_k \cdot v)$, which defines a self-adjoint operator in $L^2(\mathbb{R}^\nu)$. Using the same notation for real-valued functions and their associated self-adjoint multiplication operators we have: $g$ is of class $C^1(a)$ and the operator $i[a, g]^\circ$ extends by continuity from $D(g)$ to a bounded operator on $L^2(\mathbb{R}^\nu)$ given by

$$i[a, g]^\circ = -v \cdot \nabla g.$$ \hspace{1cm} (C.2)

Moreover, $(z-a)^{-1}$ leaves $D(g)$ invariant for any $z \in \mathbb{C}\setminus \mathbb{R}$. More precisely, for any $u \in D(g)$ there holds

$$\|g(z-a)^{-1}u\|_2 \leq \frac{c}{|\text{Im} z|} (\|u\|_2 + \|gu\|_2),$$ \hspace{1cm} (C.3)

for some $c \geq 0$ independent of $u$ and $z$.

**Proof.** We set in Proposition [C.1] $A = a$ and $H = g$ and verify the assumptions: As for (a) we note that $C^\infty_0(\mathbb{R}^\nu)$, which is a core for $g$, is a subset of $D(a) \cap D(g)$.

To prove (b) we follow [30, 38]: We recall that $w_t := e^{ita}$ is closely related to the flow $\psi_t$ of the equation $\psi_t = v(\psi_t)$ with the initial condition $\psi_0(k) = k$. Let $J_t$ be the determinant of the Jacobi matrix $D_k\psi_t$. There holds

$$(w_t u)(k) = \sqrt{J_t(k)} u(\psi_t(k)),$$ \hspace{1cm} (C.4)

where $u \in D(g)$ and $J_t$ is uniformly bounded in $k$ as a consequence of the Liouville formula:

$$J_t = e^{\int_0^t ds \text{Tr} Dv(\psi_s(k))}.$$ \hspace{1cm} (C.5)

Making use of the boundedness of $v$ we obtain the property of finite propagation speed of $\psi_t$

$$\sup_{k \in \mathbb{R}^\nu} \|\psi_t(k) - k\| \leq \sup_{k \in \mathbb{R}^\nu} \int_0^t ds \|v(\psi_s(k))\| \leq t \|v\|_\infty.$$ \hspace{1cm} (C.6)

Equation (C.4) gives

$$(|g(\psi_t(k))| + 1)(|g|w_t u)(k) = |g(k)| \sqrt{J_t(k)} (u(\psi_t(k)) + (|g|u)(\psi_t(k))),$$ \hspace{1cm} (C.7)

and consequently,

$$(|g|w_t u)(k) = \frac{|g(k)|}{(|g(\psi_t(k))| + 1)} ((w_t u)(k) + (w_t |g|u)(k)).$$ \hspace{1cm} (C.8)

We note that the factor $|g(k)||(|g(\psi_t(k))| + 1)^{-1}$ is bounded. This follows from the relation

$$g(\psi_t(k)) = g(k) + \int_0^t ds v(\psi_s(k)) \cdot \nabla g(\psi_s(k))$$ \hspace{1cm} (C.9)
and the fact that \( v \) is compactly supported. Thus we obtain from formula (C.8) that \( w_t u \) is in the domain of \( g \) and

\[
\|(gw_t u)\|_2 \leq c(1 + |t|)(\|u\|_2 + \|gu\|_2) \tag{C.10}
\]

for some \( c \geq 0 \) independent of \( u \) and \( t \). This concludes the proof of property (b).

As for (c) we set \( S = C_0^\infty(\mathbb{R}^\nu) \) and conclude from the finite propagation speed property (C.6) that \( w_t \) leaves \( S \) invariant. On \( S \) we easily obtain that

\[
i[a, g] = -v \cdot \nabla g \tag{C.11}
\]

and the r.h.s. is bounded due to the compact support of \( v \). Thus \( i[a, g]_S = -v \cdot \nabla g \) is defined on the entire Hilbert space, which concludes our verification of (c).

Now we obtain from Proposition C.1 that equality (C.11) holds in the sense of forms on \( D(a) \cap D(g) \), and that \( i[a, g] \) can be extended to a bounded, self-adjoint operator \( i[a, g]^\circ \) which coincides with \( -v \cdot \nabla g \).

Now let us show that \( g \in C^1(a) \). We set \( g_0(k) = (z - g(k))^{-1} \) and note that (C.11) applies to the real and imaginary parts of this function. Thus, by (C.11) \( g_0 \) leaves \( D(a) \) invariant and \( i[a, g_0] = g_0 i[a, g] g_0 \), defined first as an operator on \( D(a) \), extends to a bounded operator on \( L^2(\mathbb{R}^\nu) \). Thus Lemma 2.2 of [38] gives that \( g \in C^1(a) \).

Next, we show estimate (C.3). Let us assume that \( \text{Im} z > 0 \). Then

\[
(z - a)^{-1} = -i \int_0^\infty dt e^{iat} e^{itz} \tag{C.12}
\]

and property (C.3) follows from (C.10). For \( \text{Im} z < 0 \) the argument is analogous.

\[ \square \]

Lemma C.3. Let \( g_1, \ldots, g_n \in C^\infty(\mathbb{R}^\nu)_\mathbb{R} \), and let \( f \in S^0(\mathbb{R})_\mathbb{R} \). Then \( f(a/t) \in C^n(g) \), where \( g = (g_1, \ldots, g_n) \) is a family of commuting self-adjoint operators (functions of \( k \)). More precisely: Let \( \hat{g}_i(k) := \chi(k)g(k) \), where \( \chi \in C_0^\infty(\mathbb{R})_\mathbb{R} \) is equal to one on the support of \( v \) and vanishes outside of a slightly larger set. We define the following bounded operators for \( n \in \mathbb{N} \)

\[
\hat{I}_0 := f(a/t), \tag{C.13}
\]

\[
\hat{I}_n := i^n[\hat{g}_n, \ldots, [\hat{g}_1, f(a/t)] \ldots)], \quad n \geq 1. \tag{C.14}
\]

Then, in the sense of quadratic forms on \( D(g_n) \),

\[
i[g_n, \hat{I}_{n-1}] = \hat{I}_n. \tag{C.15}
\]

Consequently, \( \hat{I}_{n-1} \) leaves \( D(g_n) \) invariant and \( \hat{I}_n \) is the unique bounded operator which coincides with \( i[g_n, \hat{I}_{n-1}] \) on \( D(g_n) \) (i.e. \( i[g_n, \hat{I}_{n-1}] = \hat{I}_n \)).

Proof. Proceeding similarly as in [15], we write \( \tilde{h}(x) = f(x)(x + i)^{-1} \) and choose an almost-analytic extension \( \tilde{h} \in C^\infty(\mathbb{C}) \) of \( h \), which satisfies

\[
|\partial_z \tilde{h}(z)| \leq C_N \langle z \rangle^{2-N}|y|^N, \tag{C.16}
\]

where \( z = x + iy \). We set \( \hat{a} := a/t \) and write

\[
f(\hat{a}) = \frac{i}{2\pi} \int_\mathbb{C} \partial_z \tilde{h}(z)(i + \hat{a})(z - \hat{a})^{-1} dz \wedge d\bar{z} \tag{C.17}
\]
as a strong integral on $D(a)$. Let us show that $f(\hat{a}) \in C^1(g_1)$: Making use of Lemma [C.2] and formula [C.17], we can write for $u_1, u_2 \in D(g_1) \cap D(a)$

$$
\langle u_1, [g_1, f(\hat{a})]u_2 \rangle = \frac{-i}{t} \int_C \partial_z \hat{h}(z) \langle u_1, v \cdot \nabla g_1(z - \hat{a})^{-1}u_2 \rangle dz \wedge d\bar{z}.
$$

(C.18)

Due to property [C.16] and the relations $\|(z - \hat{a})^{-1}\| = |\text{Im} z|^{-1}$, $\|\hat{a}(z - \hat{a})^{-1}\| \leq 1 + |z|/|\text{Im} z|$ we conclude that the integrals are convergent. Moreover, we note that we can replace $g_1$ in this formula by the compactly supported function $\tilde{g}_1 = g_1 \chi$. Thus we obtain

$$
\langle u_1, [g_1, f(\hat{a})]u_2 \rangle = \langle u_1, [\tilde{g}_1, f(\hat{a})]u_2 \rangle.
$$

(C.19)

Since $g_1 \in C^1(a)$ by Lemma [C.2], $D(g_1) \cap D(a)$ is dense in $D(g_1)$. Hence the form $\text{Im} i[g_1, f(\hat{a})]$ is bounded on $D(g_1)$ and therefore $f(\hat{a}) \in C^1(g_1)$ (Cf. Lemma 2.2 of [38]). As a consequence, $f(\hat{a})$ preserves $D(g_1)$ and we can write

$$
\text{Im} i[g_1, f(\hat{a})]^o = \text{Im} i[\tilde{g}_1, f(\hat{a})].
$$

(C.20)

Thus we have proved the lemma for $n = 1$.

Let us now consider the case of $n > 1$. In the sense of quadratic forms on $D(g_n)$ we can write

$$
i[g_n, I_{n-1}] = i^n [g_{n-1}, [\ldots, [\tilde{g}_1, i[g_n, f(a/t)]]] \ldots]] = i^n [g_{n-1}, [\ldots, [\tilde{g}_1, i[g_n, f(a/t)]]] \ldots]] = I_{n-1},
$$

(C.21)

where in the second step we made use of [C.19], which holds on $D(g_n)$ as we justified above. Now the proof can be completed as in the case $n = 1$.

Let us now proceed to the decay properties of commutators constructed in the above lemma:

**Lemma C.4.** Let $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$, $f, j_1, \ldots, j_m \in S^0(\mathbb{R})$ and let us set $j_i^t := j_i(a/t)$. Then $f(a/t) \in C^n(g)$, where $g = (g_1, \ldots, g_n)$ is a family of self-adjoint commuting operators (functions of $k$) and the following relations hold:

$$
i[g_1, f(a/t)]^o = \frac{1}{t} v \cdot \nabla g_1 f(a/t) + O(t^{-2}),
$$

(C.22)

$$
i^m [j_1^t, [\ldots, [j_m^t, g_1^o] \ldots]] = O(t^{-m}),
$$

(C.23)

$$
i^{n+m} [j_1^t, [\ldots, [j_m^t, g_1, \ldots, [g_n, f(a/t)^o] \ldots]]^o]'] = O(t^{-n-m}),
$$

(C.24)

Moreover, if $h \in C^\infty_0(\mathbb{R})$ is s.t. $\text{supp} f \cap \text{supp} h = \emptyset$, then, for any $\tilde{n} \in \mathbb{N}$ (independent of $n$)

$$
i^n [g_1, [\ldots, [g_n, f(a/t)^o] \ldots]]^o h(a/t) = O(t^{-\tilde{n}}).
$$

(C.25)

**Proof.** We recall from Lemma [C.2] that the form $i[a, g_1]$, defined first on $D(a) \cap D(g_1)$, has a unique extension to a bounded operator $i[a, g_1]^o$ which satisfies

$$
i[a, g_1]^o = -v \cdot \nabla g_1.
$$

(C.26)
Thus we can define $i^n \text{ad}_a^n(g_1)$ iteratively: Suppose that $i^{n-1} \text{ad}_a^{n-1}(g_1)$ is a bounded operator which coincides with $(-v \cdot \nabla)^{n-1} g_1$. Then we define $i[a, i^{n-1} \text{ad}_a^{n-1}(g_1)]$ as a quadratic form on $D(a)$ and set $i^n \text{ad}_a^n(g_1) := i[a, i^{n-1} \text{ad}_a^{n-1}(g_1)]^\circ$. It is clear from relation (C.26) that

$$i^n \text{ad}_a^n(g_1) = (-v \cdot \nabla)^n g_1. \tag{C.27}$$

Now we recall from Lemma (C.3) that $i[j_1^I, g_1]^\circ = i[j_1^I, \tilde{g}_1]$, where $\tilde{g}_1$ is a compactly supported function of the momentum operator. Since $\tilde{g}_1$ belongs to $C^n(a)$ for any $n \in \mathbb{N}$, by relation (C.27), we conclude from (B.8) that formula (C.28) gives

$$(a/t, g_1) = \sum_{\ell=1}^{n'n} (-1)^{\ell-1} \frac{1}{\ell!} (j_1^{(\ell)})^\epsilon \text{ad}_{a/t}^\epsilon (g_1) + O(t^{-n'}), \tag{C.28}$$

where we used (C.27) and (B.9). This proves (C.22) and (C.23) for $m = 1$. To prove (C.23) for arbitrary $m$, we proceed by induction. Suppose that (C.23) holds for $m < n'$. Then formula (C.28) gives

$$[j_1^I, \ldots, [j_1^{n'}, \tilde{g}_1] \ldots] = \sum_{\ell=1}^{n'n} (-1)^{\ell-1} \frac{1}{\ell!} (j_1^{(\ell)})^\epsilon \frac{1}{t^\ell} [j_2^I, \ldots, [j_{n'}^I, (iv \cdot \nabla)^\ell \tilde{g}_1] \ldots] + O(t^{-n'}), \tag{C.29}$$

which is $O(t^{-n'})$ by the induction hypothesis.

Now we proceed to the proof of (C.24). Similarly as in the proof of Lemma (C.3) we set $\hat{a} = a/t$ and write

$$f(\hat{a}) = \frac{i}{2\pi} \int_C \partial_z \tilde{h}(z)(i + \hat{a})(z - \hat{a})^{-1} dz \wedge d\bar{z}, \tag{C.30}$$

as a strong integral on $D(a)$, where $\partial_z \tilde{h}$ satisfies (C.16). We recall from Lemma (C.3) that

$$i^n [g_1, \ldots, g_n, f(a/t)]^\circ = i^n [\tilde{g}_1, \ldots, \tilde{g}_n, f(a/t)], \tag{C.31}$$

where $\tilde{g}_i$ are compactly supported functions of $k$. With the help of (C.30) we can compute the commutator on the r.h.s. of (C.31) as a quadratic form on $D(a)$ (here we make use of the fact that $\tilde{g}_i \in C^1(a)$ and thus they preserve $D(a)$). The result is a finite linear combination of terms of two types

$$\tilde{I}_{0,n} := \frac{i}{in} \frac{i}{2\pi} \int_C \partial_z \tilde{h}(z)(i + \hat{a})(z - \hat{a})^{-1} \prod_{i=1}^n \{v \cdot \nabla \tilde{g}_{\sigma(i)}(z - \hat{a})^{-1}\} dz \wedge d\bar{z}, \tag{C.32}$$

$$\tilde{I}_{j,n} := \frac{i}{in} \frac{i}{2\pi} \int_C \partial_z \tilde{h}(z)v \cdot \nabla \tilde{g}_j(z - \hat{a})^{-1} \prod_{i=1}^{n-1} \{v \cdot \nabla \tilde{g}_{\delta(i)}(z - \hat{a})^{-1}\} dz \wedge d\bar{z}, \tag{C.33}$$

where $\sigma$ is some permutation of $(1, \ldots, n)$ and $\delta$ is some permutation of $(1, \ldots, \tilde{j}, \ldots, n)$. Making use of properties (C.16), and of the relations $\|(z - \hat{a})^{-1}\| = |\text{Im } z|^{-1}$, $\|\hat{a}(z - \hat{a})^{-1}\| \leq 1 + |z|/|\text{Im } z|$ we conclude that

$$|\langle u_1, \tilde{I}_{j,n} u_2 \rangle| \leq \frac{c}{i^n} \|u_1\| \|u_2\|, \tag{C.34}$$
for \( u_1, u_2 \in D(a) \) and \( i \in \{0, \ldots, n\} \). This gives (C.24) for \( m = 0 \) and also verifies that the r.h.s. of (C.31) coincides with a linear combination of bounded operators \( \tilde{I}_{i,n} \) on the entire Hilbert space. Let us now proceed to the case \( m > 0 \). We note that

\[
[j_1^t, \ldots, [j_m^t, \tilde{I}_{i,n}] \ldots] \tag{C.35}
\]

is again a linear combinations of terms of the form (C.32) and (C.33), except that some of the insertions \( v \cdot \nabla \tilde{g}_j \) are replaced with

\[
[j_{i_1}^t, \ldots, [j_{i_{m'}}^t, v \cdot \nabla \tilde{g}_j] \ldots] \tag{C.36}
\]

for some \( i_1, \ldots, i_{m'} \in \{1, \ldots, m\} \). Since (C.36) is of order \( O(t^{-m'}) \) by (C.28), the proof of (C.24) can now be completed as in the case \( m = 0 \).

To prove (C.25), we proceed by induction: For \( n = 1 \) it follows from (the adjoint of) formula (C.28). Now we define a sequence \( \tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots) \) and write for any \( n \in \mathbb{N} \)

\[
[\tilde{g}_1, \ldots, [\tilde{g}_n, f(a/t)] \ldots] = \text{ad}^{\alpha_n}_{\tilde{g}}(f(a/t)), \tag{C.37}
\]

where \( \alpha_n \) is a multiindex s.t. \( \alpha_n(j) = 1 \) for \( 1 \leq j \leq n \) and \( \alpha_n(j) = 0 \) for \( j > n \). Now suppose that (C.25) holds for \( n < n' \). We obtain

\[
\text{ad}^{\alpha_n}_{\tilde{g}}(f(a/t)) h(a/t) = [\text{ad}^{\alpha_{n'}}_{\tilde{g}}(f(a/t)) h(a/t)]
\]

\[
= \text{ad}^{\alpha_{n'}}_{\tilde{g}}(f(a/t)) [h(a/t), \tilde{g}_{n'}] + O(t^{-\tilde{n}}), \tag{C.38}
\]

where we made use of the induction hypothesis. The first term on the r.h.s. above is \( O(t^{-\tilde{n}}) \) by the induction hypothesis and formula (C.28).

\[\square\]

## D Admissible and regular propagation observables

\begin{definition}
Let \( \mathbb{R} \ni t \to b(t) \in B(\mathfrak{h}) \) be a propagation observable, which is bounded, uniformly in \( t \). Let \( j_i \in S^0(\mathbb{R}) \) and \( g_i \in C^\infty(\mathbb{R}^l) |_{\mathbb{R}} \), \( i, l \in \mathbb{N} \) and let us set \( j_i^t := j_i(a/t) \). Suppose that \( b(t), b(t)\ast \in C^n(g) \) for any \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \), where \( g = (g_1, \ldots, g_n) \) is a family of commuting self-adjoint operators understood as functions of \( k \). (Cf. Definition B.1).

(a) We say that \( b \) is admissible, if for any \( m, n \)

\[
[j_1^t, \ldots, [j_m^t, g_1, \ldots, [g_n, b(t)]^{\ast} \ldots]^{\ast} \ldots] = O(t^{-n-m}). \tag{D.1}
\]

(b) We say that \( b \) is regular, if there exists some neighbourhood of zero \( \Delta \), s.t. for any \( h_\Delta \in C_0^\infty(\mathbb{R}) \), supported in \( \Delta \), and any \( n, \tilde{n} \in \mathbb{N} \)

\[
[g_1, \ldots, [g_n, b(t)]^{\ast} \ldots]^{\ast} h_\Delta(a/t) = O(t^{-\tilde{n}}), \tag{D.2}
\]

and the same relation holds for \( b \) replaced with \( b^\ast \). We will call \( \Delta \) the regularity region of \( b \).

\begin{lemma}
Let \( G \) be the function appearing in the interaction term of the Hamiltonian (2.7) and let \( b \) be a regular propagation observable. Then, for any \( n \in \mathbb{N} \), with \( n \leq 6 \), there exists a \( C_n \) s.t.

\[
||b(t)\omega^n G||_2 \leq C_n/t^2. \tag{D.3}
\]
\end{lemma}
Proof. By \([\text{MC1}]\) and \([\text{MC3}]\) we have \(\|\omega^n G\|_2 < \infty\) for \(n \leq 6\). We recall that \(a = \frac{1}{\pi \{v \cdot i \nabla_k + i V_k \cdot v\}}, v\) is compactly supported and vanishes in a neighbourhood of zero, and by \([\text{ST2}]\) \(\partial_k^2 G\) is locally square-integrable away from zero for \(|\alpha| \leq 2\). Hence, \(\|a^2 \omega^n G\|_2 < \infty\).

Now, exploiting regularity of \(b\), we write

\[
\|b(t) \omega^n G\|_2 \leq \|b(t) h_\Delta(a/t)\| \|\omega^n G\|_2 + \|h_{R \setminus \Delta}(a/t) (a/t)^{-2}\| \frac{1}{t^2} \|a^2 \omega^n G\|_2 \leq C/t^2,
\]

(4.4)

where \(\Delta\) appeared in Definition \([\text{D.1}]\) and \(h_\Delta, h_{R \setminus \Delta}\) form a smooth partition of unity s.t. \(h_\Delta\) is supported in \(\Delta\) and equal to one on a neighbourhood of zero. □

**Lemma D.3.** Let \(q \in C^\infty(\mathbb{R})\) be.s.t. \(q = 0\) on some neighbourhood \(\Delta\) of zero and \(q' \in C^\infty_0(\mathbb{R})\). Then the propagation observables

\[
\mathbb{R} \ni t \mapsto q', \quad \mathbb{R} \ni t \mapsto t \partial_t q
\]

are admissible and regular with the regularity region \(\Delta\). (Here \(q' = q(a/t)\)).

Proof. Follows immediately from Lemma \([\text{C.4}]\) and the assumed support properties of \(q\). □

### E Auxiliary Hamiltonian and energy bounds

In this section we prove higher-order bounds for \(H(\xi)\) w.r.t. the free Hamiltonian \(H_0(\xi)\) (and their counterparts for the auxiliary Hamiltonians introduced in Definition \([\text{E.3}]\) below).

We cannot rely on standard higher-order bounds for \(H(\xi)\) w.r.t. the free boson Hamiltonian \(H_{ph}\) (see e.g. \([20, \text{Lemma 31}]\) and \([21, \text{Lemma 8}]\)), since they do not suffice in the case of the polaron model.

**Lemma E.1.** Let \(F \in C^\infty(\mathbb{R}^\mu, \mathbb{R}), f \in C^\infty(\mathbb{R}^\nu; \mathbb{R}^\mu), \mu \in \mathbb{N},\) and let \(G \in L^2(\mathbb{R}^\nu)\). Then, in the sense of operators on \(\mathcal{C} = \Gamma_{\text{fin}}(C^\infty_0(\mathbb{R}^\nu))\),

\[
a^*(G) F(d\Gamma(f(k))) = \int dp G(p) F(-f(p) + d\Gamma(f(k))) a^*(p),
\]

(E.1)

\[
a(G) F(d\Gamma(f(k))) = \int dp \overline{G}(p) F(f(p) + d\Gamma(f(k))) a(p).
\]

(E.2)

Proof. Let \(\Psi = \{\Psi^{(n)}\}_{n \in \mathbb{N}} \in \mathcal{C}\), and observe that only finitely many \(\Psi^{(n)}\)'s are nonzero. The expression \(F(d\Gamma(f(k)))\) is well-defined as a symmetric operator on \(\mathcal{C}\), where it is also essentially self-adjoint. Then

\[
\{a^*(G) F(d\Gamma(f(k)))\Psi\}^{(n)}(k_1, \ldots, k_n)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^n G(k_i) F(f(k_1) + \cdots + f(k_i) + \cdots + f(k_n)) \Psi^{(n-1)}(k_1, \ldots, \hat{k_i}, \ldots, k_n)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^n \int dp \delta(p - k_i) G(p) F(f(k_1) + \cdots + f(k_i) + \cdots + f(k_n) - f(p))
\]

\[
\times \Psi^{(n-1)}(k_1, \ldots, \hat{k_i}, \ldots, k_n)
\]

\[
= \int dp \left\{ G(p) F(d\Gamma(f(k)) - f(p)) a^*(p) \Psi\right\}^{(n)}(k_1, \ldots, k_n).
\]

(E.3)
This proves (E.1). As for (E.2), we compute

\[\{a(G)F(d\Gamma(f(k)))\Psi\}^{(n)}(k_1, \ldots, k_n)\]

\[= \sqrt{n+1} \int dp G(p)F(f(p) + f(k_1) + \cdots + f(k_n))\Psi^{(n+1)}(p, k_1, \ldots, k_n)\]

\[= \int dp \{G(p)F(f(p) + d\Gamma(f(k)))a(p)\Psi\}^{(n)}(k_1, \ldots, k_n).\]  

(E.4)

This concludes the proof.

We have the following higher-order lemma

**Lemma E.2.** Let \(n_0 \in \mathbb{N}\) and suppose \((k)^{(n_0-1)}\max\{1, n_0\}G \in L^2(\mathbb{R}^v)\). Then \(D(|H(\xi)|^n) = D(H_0(\xi)^n)\) for all \(n \leq n_0\). Furthermore,

\[\sup_{\lambda \geq 0, \xi \in \mathbb{R}^v} \left(\|H_0(\xi + \lambda)^n(H(\xi) - i + \lambda)^{-n}\| + \|H(\xi) + \lambda^nH_0(\xi) - i + \lambda)^{-n}\|\right) < \infty.\]  

(E.5)

**Proof.** It is an easy consequence of the spectral theorem, that it suffices to prove the claimed uniform bound for \(\lambda = 0\) and uniformly in \(\xi\). (This follows by an application of the binomial formula to \((H_0(\xi) + \lambda)^n\) and \((H(\xi) + \lambda)^n\). In fact, in order to take fractional roots we observe that we can replace \(i\) by a point below the bottom of the spectrum of the relevant operator. For this purpose we recall the notation \(\Sigma_0 = \inf \sigma(H)\). We begin by arguing that for \(n \leq n_0\) we have

\[(H(\xi) - \Sigma_0 + 1)^{-n}F \subset D(H_0(\xi)^n)\]  

(E.6)

and

\[\sup_{\xi \in \mathbb{R}^v} \|H_0(\xi)^n(H(\xi) - \Sigma_0 + 1)^{-n}\| < \infty,\]  

(E.7)

for all \(n \leq n_0\). The proof will go by induction in half integer powers \(n\). Clearly, since \(D(H(\xi)) = D(H_0(\xi))\), (E.6) holds true for \(n = 1\) (and hence by interpolation for \(n = 1/2\)). Furthermore, the computation

\[H_0(\xi)(H(\xi) - \Sigma_0 + 1)^{-1} = I - (1 - \Sigma_0 + \phi(G))(H(\xi) - \Sigma_0 + 1)^{-1},\]  

(E.8)

together with \(N^{1/2}\)-boundedness of \(\phi(G)\) and the estimate

\[\|N^{1/2}(H(\xi) - \Sigma_0 + 1)^{-1}\|^2 \leq \frac{1}{m} \|H(\xi) - \Sigma_0 + 1)^{-1}(H_0(\xi) + 1)(H(\xi) - \Sigma_0 + 1)^{-1}\|\]

\[\leq C_1 + C_2\|N^{1/2}(H(\xi) - \Sigma_0 + 1)^{-1}\|\]  

(E.9)

implies (E.7) for \(n = 1\). Hence by interpolation also for \(n = 1/2\).

We now assume \(n \geq 3/2\) (and \(n \leq n_0\)) is an element of \(\mathbb{N}/2\), and by induction we can assume that (E.6) and (E.7) hold with \(n\) replaced with \(n-1/2\). To perform the induction step it suffices to show that

\[\phi(G)(H(\xi) - \Sigma_0 + 1)^{-n}\Psi \in D(H_0(\xi)^{n-1}),\]  

(E.10)

for \(\Psi \in \mathcal{H}\), and

\[\sup_{\xi \in \mathbb{R}^v} \|H_0(\xi)^{(n-1)}\phi(G)(H(\xi) - \Sigma_0 + 1)^{-n}\| < \infty.\]  

(E.11)

The statement (E.10) is implied by showing that we have

\[\phi(G)(H(\xi) - \Sigma_0 + 1)^{-n}\Psi \in D(d\Gamma(\omega)^{n-1}) \cap D(\Omega(\xi - d\Gamma(k))^{n-1})\]  

(E.12)
and statement (E.11) follows from

\[
\sup_{\xi \in \mathbb{R}^v} \|d\Gamma(\omega)^{-n-1} \phi(G)(H(\xi) - \Sigma_0 + 1)^{-n}\| < \infty, \\
\sup_{\xi \in \mathbb{R}^v} \|\Omega(\xi - d\Gamma(k))^{-n-1} \phi(G)(H(\xi) - \Sigma_0 + 1)^{-n}\| < \infty, 
\]

(E.13)

which by induction is known to hold for \(n\) replaced with a half-integer \(n' \leq n - 1/2\), cf. what was done for \(n = 1\).

Let us now prove (E.13). Let \(F_1(r) = r^{-n-1}\), \(f_1(k) = \omega(k)\), \(F_2(k) = \Omega(\xi - k)^{n-1}\), and \(f_2(k) = k\), where \(r \in \mathbb{R}, k \in \mathbb{R}^v\). Write \(\phi(G) = a^*(G) + a(G)\). Below we only deal with the \(a(G)\) contribution, which is the most complicated. The contribution from \(a^*(G)\) is similar but simpler. Compute for \(\Phi, \Psi_1 \in \mathcal{C}\)

\[
\langle F_j(d\Gamma(f_j(k)))\Phi, a(G)\Psi_1 \rangle = \langle a^*(G)F_j(d\Gamma(f_j(k)))(N + 1)^{-\frac{1}{2}}\Phi, N^{\frac{1}{2}}\Psi_1 \rangle. 
\]

(E.14)

Anticipating the use of (E.1) we introduce

\[
\Phi_1 = \int dp'G(p)\frac{F_j(d\Gamma(f_j(k)) - f_j(p))}{1 + F_j(d\Gamma(f_j(k)))}a^*(p)(N + 1)^{-\frac{1}{2}}\Phi. 
\]

(E.15)

The norm of the \(n\)-particle \((n \geq 1)\) contribution is

\[
\|
\Phi_1^{(n)}
\|^2 = \int dk_1 \cdots dk_n \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} G(k_i) \frac{F_j(f_j(k_1) + \cdots + f_j(k_n) - f_j(k_i))}{1 + F_j(f_j(k_1) + \cdots + f_j(k_n))} \right|^2 \times \left| \frac{1}{2} \Phi^{(n-1)}(k_1, \ldots, \tilde{k}_i, \ldots, k_n) \right|^2. 
\]

(E.16)

Observe the bound \(F_j(x - f_j(k_i)) \leq C(1 + F_j(x))(k_i)^{n-1}s\), valid for \(j = 1, 2\) uniformly in \(\xi\), where \(s = \max\{1, s_0\}\). Here we used \([MC3] [MC5] \) This implies

\[
\|
\Phi_1^{(n)}
\|^2 \leq \frac{C^2}{n^2} \int dk_1 \cdots dk_n \left( \sum_{i=1}^{n} (k_i)^{(n-1)s} |G(k_i)\Phi^{(n-1)}(k_1, \ldots, \tilde{k}_i, \ldots, k_n)|^2 \right)^2 \leq C'\|
\langle k \rangle^{(n-1)s}G\|^2\|
\Phi^{(n-1)}\|^2. 
\]

(E.17)

Hence, for some \(\xi\)-independent constant \(C''\) we have

\[
\|
\Phi_1\|
\| \leq C''\|
\langle k \rangle^{(n-1)s}G\||\Phi||. 
\]

(E.18)

Combining (E.1), (E.14), and the above estimate we get

\[
|\langle F_j(d\Gamma(f_j(k)))\Phi, a(G)\Psi_1 \rangle| \leq C''\|
\langle k \rangle^{(n-1)s}G\|\|
\Phi||\|1 + F_j(d\Gamma(f_j(k)))\|N^{\frac{1}{2}}\Psi_1\|
\]

\[
\leq m^{-\frac{1}{2}}C''\|
\langle k \rangle^{(n-1)s}G\|\|
\Phi||\|1 + H_0(\xi)^{n-\frac{1}{2}}\Psi_1\|. 
\]

From the assumption on \(G\) we conclude \(\|
\langle k \rangle^{(n-1)s}G\| < \infty\). Since \(C\) is a core for any power of \(H_0(\xi)\), the estimate extends from \(C\) to \(\Psi_1 \in D(H_0(\xi)^{n-\frac{1}{2}})\), and hence in particular to \(\Psi_1 = (H(\xi) - \Sigma_0 + 1)^{-n}\Psi\). This shows that \(a(G)(H(\xi) + \Sigma_0 + \lambda)^{-n}\Psi \in D(F_j(d\Gamma(f_j(k)))))\), for \(j = 1, 2\), and hence completes the proof of (E.12). The uniform estimates (E.13) follow by the same estimate just derived, since (E.13) holds true with \(n\) replaced by \(n - 1/2\).
In order to establish the reverse claim, that \((H_0(\xi) + 1)^{-n_0} F \subset D(|H(\xi)|^{n_0})\) and that 
\[ \sup_{\xi \in \mathbb{R}^\nu} \|(H(\xi) - \Sigma_0)^{n_0}(H_0(\xi) + 1)^{-n_0}\| < \infty, \]
we argue again by induction after half-powers. The computations 
\[ (H(\xi) - \Sigma_0)^n(H_0(\xi) + 1)^{-n} = \{(H(\xi) - \Sigma_0)^{n-1}(H_0(\xi) + 1)^{-1} - (n-1)\} \]
\[ \times \{(H_0(\xi) + 1)^{-1}(H(\xi) - \Sigma_0)(H_0(\xi) + 1)^{-n}\} \]
and
\[ (H_0(\xi) + 1)^{-1}H(\xi)(H_0(\xi) + 1)^{-n} = H_0(\xi)(H_0(\xi) + 1)^{-1} + (H_0(\xi) + 1)^{-1}\phi(G)(H_0(\xi) + 1)^{-n}, \]
together with the observation that we never used above that the \(n\) resolvents were interacting, conclude the proof of the lemma. 

**Definition E.3.** We define the free and interacting auxiliary Hamiltonians on \(\mathcal{H}_1 = \Gamma(\mathfrak{h} \oplus \mathfrak{h})\) by 
\[ H_{1,0}(\xi) := \Omega(\xi - d\Gamma(k)) + d\Gamma(\omega) \quad \text{and} \quad H_1(\xi) := H_{1,0}(\xi) + \phi(G, 0) \quad \text{ (E.19)} \]
on their domain of essential self-adjointness \(C_1 := \Gamma_{\text{fin}}(C_0^\infty(\mathbb{R}^\nu) \oplus C_0^\infty(\mathbb{R}^\nu))\). The operators \(k = \text{diag}(k_1, k_2, \ldots, k_{1+\nu}), k \in \mathbb{Z}^\nu\) and \(\omega = \text{diag}(\omega_1, \omega_2, \ldots, \omega_{1+\nu}) \) are essentially self-adjoint on \(C_0^\infty(\mathbb{R}^\nu) \oplus C_0^\infty(\mathbb{R}^\nu) \subset \mathfrak{h} \oplus \mathfrak{h}\) and \((G, 0) \in \mathfrak{h} \oplus \mathfrak{h}\). We note that \(H_{1,0}(\xi) = UH_0^{\text{ex}}(\xi)U^*, H_1(\xi) = UH^{\text{ex}}(\xi)U^*\) and \(C_1 = U^{*}\text{C}^{\text{ex}}\).

**Corollary E.4.** Let \(n_0 \in \mathbb{N}\) and suppose \(\langle k \rangle^{(n_0 - 1)\max\{1, s_1\}}G \in L^2(\mathbb{R}^\nu)\). Then for any \(n \leq n_0, \ell \in \mathbb{N}\) and \(\xi \in \mathbb{R}^\nu\) we have \(D(|H^{\text{ex}}(\xi)|^{n}) = D(H_0^{\text{ex}}(\xi)^n), D(|H^{\ell}(\xi)|^{n}) = D(H_0^{\ell}(\xi)^n)\) and \(D(|H_1(\xi)|^{n}) = D(H_{1,0}(\xi)^n)\).

**Proof.** Using the direct integral decomposition 
\[ H_0^{\ell}(\xi)^n(H^{\ell}(\xi) + \Sigma_0 + 1)^{-n} = \int d\xi \int d\xi \int d\xi \int d\xi H_0^{\ell}(\xi)^n(H^{\ell}(\xi) + \Sigma_0 + 1)^{-n}, \]
and similarly with \(H_0\) and \(H\) interchanged, we conclude the corollary from Lemma E.2. 

**Remark E.5.** The auxiliary Hamiltonian \(H_1(\xi)\) is useful when computations and estimates involve multiple Fock space operations, since only one Fock space is involved. However, when one makes manifest use of conservation of asymptotic particle number, the \(H^{\text{ex}}(\xi)\) representation is advantageous.

Having established estimates and identities for \(H_1(\xi)\) we shall by conjugating with the unitary \(U\) obtain analogous results for the extended Hamiltonian \(H^{\text{ex}}(\xi)\). Then, by applying the projection \(P_\nu\) on the subspace \(\mathcal{F} \otimes \mathcal{F}^{\ell} \subset \mathcal{F}^{\text{ex}}\) we obtain analogous results for the Hamiltonians \(H^{\ell}(\xi)\).

**Corollary E.6.** Let \(n_0 \in \mathbb{N}\) and suppose \(\langle k \rangle^{(n_0 - 1)\max\{1, s_1\}}G \in L^2(\mathbb{R}^\nu)\). Then for any \(n \leq n_0, \ell \in \mathbb{N}\) there exists \(C > 0\) such that for all \(z \in \mathbb{C}, \) with \(\text{Im} z \neq 0,\) we have 
\[ \|H_0^{\ell}(\xi)^n(H^{\ell}(\xi) - z)^{-1}(H_0(\xi) + 1)^{-1}\| \leq C|\text{Im} z|^{-1}, \]
\[ \|H_0^{\ell}(\xi)^n(H^{\ell}(\xi) - z)^{-1}(H_0^{\ell}(\xi) + 1)^{-1}\| \leq C|\text{Im} z|^{-1}, \]
\[ \|H_0^{\text{ex}}(\xi)^n(H^{\text{ex}}(\xi) - z)^{-1}(H_0^{\ell}(\xi) + 1)^{-1}\| \leq C|\text{Im} z|^{-1}, \]
\[ \|H_{1,0}(\xi)^n(H_1(\xi) - z)^{-1}(H_{1,0}(\xi) + 1)^{-1}\| \leq C|\text{Im} z|^{-1}. \]
Proof. The corollary follows from Corollary [E.4] Remark [E.5] and the computation:

\[
H_{1,0}(\xi)^n(H_1(\xi) - z)^{-1}(H_{1,0}(\xi) - \Sigma_0 + 1)^{-n} = \left\{ H_{1,0}(\xi)^n(H_1(\xi) - \Sigma_0 + 1)^{-n} \right\}
\times \left\{ (H_1(\xi) - \Sigma_0 + 1)(H_1(\xi) - z)^{-1} \right\} \left\{ (H_1(\xi) - \Sigma_0 + 1)^{n-1}(H_{1,0}(\xi) + 1)^{-n-1} \right\}.
\]

Alternatively, one can repeat the computation above for each pair of free and interacting Hamiltonian, invoking Corollary [E.4] for each of them separately. □

F Commutators with the Hamiltonian

F.1 Commutators involving \(d\Gamma(\cdot, \cdot)\)

In this subsection we will make use of the auxiliary Hamiltonian \(H_1(\xi)\) introduced in Definition [E.3].

Lemma F.1. Let \(q_0, q_\infty \in C^\infty(\mathbb{R})\) be s.t. \(q_0, q_\infty \in C^\infty_0(\mathbb{R})\) and \(0 \leq q_0, q_\infty \leq 1\). Let \(q_j = \text{diag}(q_{0j}, q_{\infty j})\) be the corresponding propagation observable on \(\mathfrak{h} \oplus \mathfrak{h}\). Let \(\mathbb{R} \ni t \mapsto \mathfrak{b}_j(t) \in B(\mathfrak{h})\), \(j \in \{0, \infty\}\), be two families of admissible operators and let \(\mathfrak{b} = \text{diag}(b_0, b_\infty)\) be the corresponding propagation observable on \(\mathfrak{h} \oplus \mathfrak{h}\). Let \(f \in S^{\alpha}(\mathbb{R})\). Then, setting \(R_{1,0} := (1 + H_{1,0}(\xi))^{-1}\), we obtain

\[
[f(d\Gamma(\mathfrak{k})), d\Gamma(q, \mathfrak{b})]R_{1,0}^4 = \nabla f(d\Gamma(\mathfrak{k})) \cdot (d\Gamma(q, [\mathfrak{k}, \mathfrak{b}]^0) + d\Gamma(q, [\mathfrak{k}, q]^0, \mathfrak{b}))R_{1,0}^4 + O(t^{-2})
\]

and each term on the r.h.s. above is bounded and \(O(t^{-1})\).

Proof. Observe first that by Lemma [C.3] and Definition [D.1] \(\mathfrak{k}, q \in C^1(\mathfrak{k}_j)\), for each \(j\). Hence \([\mathfrak{k}_j, \mathfrak{b}]\) and \([\mathfrak{k}_j, q]\) extend from \(D(\mathfrak{k})\) (dense in each \(D(\mathfrak{k}_j)\)) to bounded operators \([\mathfrak{k}_j, \mathfrak{b}]^0\) and \([\mathfrak{k}_j, q]^0\). We write \([\mathfrak{k}_j, \cdot]^0\) for the vector \(([\mathfrak{k}_1, \cdot]^0, \ldots, [\mathfrak{k}_\infty, \cdot]^0)\). By Lemma [C.4] and Definition [D.1] all these bounded operators are \(O(t^{-1})\).

Making use of Lemma [B.5] with \(B = d\Gamma(q, \mathfrak{b})(1 + N_1)^{-4}\), we get

\[
[f(d\Gamma(\mathfrak{k})), d\Gamma(q, \mathfrak{b})](1 + N_1)^{-4} = \nabla f(d\Gamma(\mathfrak{k})) \cdot (d\Gamma(q, [\mathfrak{k}, \mathfrak{b}]^0) + d\Gamma(q, [\mathfrak{k}, q]^0, \mathfrak{b})) (1 + N_1)^{-4} + R(f, d\Gamma(\mathfrak{k}), d\Gamma(q, \mathfrak{b})(1 + N_1)^{-4}),
\]

as a form equality on \(D(d\Gamma(\mathfrak{k})^2)\), where \(N_1\) is the number operator on \(\Gamma(\mathfrak{h} \oplus \mathfrak{h})\). Here we exploited the fact that \(f \in S^2(\mathbb{R}^n)\) and that \(d\Gamma(q, \mathfrak{b})(1 + N_1)^{-4}\) is bounded and belongs to \(C^3(d\Gamma(\mathfrak{k}))\) by the assumed properties of \(q, \mathfrak{b}\) and by Lemma [A.2]. Moreover, we obtain from Lemma [B.5] and Lemma [A.2] that

\[
\|R(f, d\Gamma(\mathfrak{k}), d\Gamma(q, \mathfrak{b})(1 + N_1)^{-4})\|
\leq \sum_{\alpha:2 \leq \alpha \leq 3} \|\text{ad}_{d\Gamma(\mathfrak{k})}^\alpha(d\Gamma(q, \mathfrak{b}))(1 + N_1)^{-4}\| = O(t^{-2}).
\]

(F.3)

Since \((1 + N_1)^\ell(1 + H_{1,0}(\xi))^{-\ell}\) is bounded for any \(\ell \in \mathbb{N}\), we have shown that

\[
[f(d\Gamma(\mathfrak{k})), d\Gamma(q, \mathfrak{b})]R_{1,0}^4 = \nabla f(d\Gamma(\mathfrak{k})) \cdot (d\Gamma(q, [\mathfrak{k}, \mathfrak{b}]^0) + d\Gamma(q, [\mathfrak{k}, q]^0, \mathfrak{b}))R_{1,0}^4 + O(t^{-2}).
\]

(F.4)

Let us show that the term involving \(d\Gamma(q, [\mathfrak{k}, q]^0, \mathfrak{b})\) above is \(O(t^{-1})\). If \(s_\Omega \leq 1\), then it follows from Lemma [A.2] directly, since \(\nabla f\) in this case is bounded. If \(s_\Omega \geq 1\), we can insert \(I =
(dΓ(k) · dΓ(k) + 1)^{-1} (dΓ(k) · dΓ(k) + 1), noting that dΓ(k) · dΓ(k) = \sum_{j=1}^{\nu} dΓ(diag(k_j, k_j))^2, and compute

\[ \nabla f(dΓ(k)) \cdot dΓ(q, [k, q]^\circ, b) R_{1,0}^4 \]

\[ = \nabla f(dΓ(k)) \cdot (dΓ(k) \cdot dΓ(k) + 1)^{-1} [dΓ(q, [k, q]^\circ, b), dΓ(k) \cdot dΓ(k)] R_{1,0}^4 \quad (F.5) \]

\[ + \nabla f(dΓ(k)) \cdot (dΓ(k) \cdot dΓ(k) + 1)^{-1} [dΓ(q, [k, q]^\circ, b)] dΓ(k) \cdot dΓ(k) + 1) R_{1,0}^4. \]

This expression is \( O(t^{-1}) \) by Lemma \( \ref{lem:A.2} \) and the bound \( |\partial f(\eta)| \leq c(\eta) \). Note that to deal with the first term on the right-hand side one should first expand the commutator and write

\[ [dΓ(q, [k, q]^\circ, b), dΓ(k) \cdot dΓ(k)] = \sum_{j=1}^{\nu} dΓ(diag(k_j, k_j)) [dΓ(q, [k, q]^\circ, b), dΓ(diag(k_j, k_j))] \]

\[ + \sum_{j=1}^{\nu} [dΓ(q, [k, q]^\circ, b), dΓ(diag(k_j, k_j))] dΓ(diag(k_j, k_j)). \]

An analogous argument applies to the term involving \( dΓ(q, [k, b]^\circ) \) on the r.h.s of (F.2). \( \square \)

**Proposition F.2.** Let \( q_0, q_\infty \) be as specified in Definition \( \ref{def:3.7} \) and let \( q^t = diag(q_0^t, q_\infty^t) \) be the corresponding propagation observable on \( \mathfrak{h} \oplus \mathfrak{h} \). Let \( \mathbb{R} \ni t \rightarrow b_j(t) \in B(\mathfrak{h}), \ j \in \{0, \infty\}, \) be two families of admissible operators s.t. \( b_0 \) is regular. Let \( b = diag(b_0, b_\infty) \) be the corresponding propagation observable on \( \mathfrak{h} \oplus \mathfrak{h} \). Then, setting \( q = q^t \), \( R_{1,0} := (1 + H_{1,0}(\xi))^{-1} \) and \( R_0 := (1 + H_0(\xi))^{-1} \), we obtain

\[ [H_1(\xi), dΓ(q, b)] R_{1,0}^4 = -\nabla \Omega(\xi - dΓ(k)) \cdot (dΓ(q, [k, b]^\circ) + dΓ(q, [k, q]^\circ, b)) R_{1,0}^4 \]

\[ + (dΓ(q, [\omega, b]^\circ) + dΓ(q, [\omega, q]^\circ, b)) R_{1,0}^4 + O(t^{-2}) \quad (F.6) \]

and consequently

\[ [H(\xi), dΓ(q_0, b_0)] R_0^4 = -\nabla \Omega(\xi - dΓ(k)) \cdot (dΓ(q_0, [k, b_0]^\circ) + dΓ(q_0, [k, q_0]^\circ, b)) R_0^4 \]

\[ + (dΓ(q_0, [\omega, b_0]^\circ) + dΓ(q_0, [\omega, q_0]^\circ, b_0)) R_0^4 + O(t^{-2}). \quad (F.7) \]

Each term on the r.h.s. of relations (F.6) and (F.7) is bounded and \( O(t^{-1}) \).

**Proof.** Observe first that by Lemma \( \ref{lem:C.3} \) and Definition \( \ref{def:D.1} \) \( q \in C^1(k_j) \cap C^1(\omega) \), for each \( j \), and \( b \in C^1(\omega) \). See also the first paragraph in the proof of Lemma \( \ref{lem:F.1} \) for notation and the observation that the bounded operators \( [k, q]^\circ \), \( [\omega, q]^\circ \) and \( [\omega, b]^\circ \) are all \( O(t^{-1}) \).

Let us first prove (F.6). Making use of Lemma \( \ref{lem:F.2} \) and of the fact that \( \Omega \in S^\infty(\mathbb{R}) \), we obtain the identity

\[ [\Omega(\xi - dΓ(k)), dΓ(q, b)] R_{1,0}^4 \]

\[ = -\nabla \Omega(\xi - dΓ(k)) \cdot (dΓ(q, [k, b]^\circ) + dΓ(q, [k, q]^\circ, b)) R_{1,0}^4 + O(t^{-2}) \quad (F.8) \]

in the sense of forms on \( D(H_{1,0}(\xi)) \). All terms on the r.h.s. above are \( O(t^{-1}) \).

As for the second term from the free auxiliary fiber Hamiltonian \( (E.19) \), it suffices to note that Lemma \( \ref{lem:A.2} \) gives

\[ [dΓ(\omega), dΓ(q, b)] R_{1,0}^4 = (dΓ(q, [\omega, q]^\circ, b) + dΓ(q, [\omega, b]^\circ)) R_{1,0}^4 = O(t^{-1}) \quad (F.9) \]

in the sense of forms on \( D(H_{1,0}(\xi)) \). The interaction term in the Hamiltonian gives

\[ [\phi(G, 0), dΓ(q, b)] R_{1,0}^4 = (a^*((1 - q_0)G, 0)dΓ(q, b) - a^*(b_0G, 0)\Gamma(\underline{q}) + \Gamma(q)a(b_0G, 0) + dΓ(q, b)a((q_0 - 1)G, 0)) R_{1,0}^4 = O(t^{-2}), \quad (F.10) \]
where we made use of Lemma A.1 and exploited regularity of \( b_0 \) and \( 1 - q_0 \). This concludes
the proof of (F.6).

Now let us prove relation (F.7). By conjugating (F.6) with the unitary \( U \), we get
\[
[H^{\text{ex}}(\xi), d\Gamma(\bar{q}, \bar{b})](R^{\text{ex}}_0)^4
\]
\[
= -\nabla \Omega(\xi - d\Gamma^{\text{ex}}(\bar{b})) \cdot (d\Gamma^{\text{ex}}(\bar{q}, [\bar{b}, \bar{b}]^0) + d\Gamma^{\text{ex}}(\bar{q}, [\bar{b}, \bar{q}]^0, \bar{b}))(R^{\text{ex}}_0)^4
+ (d\Gamma^{\text{ex}}(\bar{q}, [\bar{b}, \bar{b}]^0) + d\Gamma^{\text{ex}}(\bar{q}, [\bar{b}, \bar{q}]^0, \bar{b}))(R^{\text{ex}}_0)^4 + O(t^{-2}),
\]
where \( R^{\text{ex}}_0 = (1 + H_0^{\text{ex}}(\xi))^{-1} \). By applying the projection \( P_0 \) on the subspace \( \mathcal{F} \otimes \mathcal{F}^{(0)} \subset \mathcal{F}^{\text{ex}} \) to both sides of this equality, we obtain (F.7).

\[\square\]

**Lemma F.3.** Let \( q^t \) and \( h \) be as specified in Proposition F.2. Let \( \chi \in C_0^\infty(\mathbb{R}) \). Then
\[
[H^\text{ex}(\xi), d\Gamma(\bar{q}, \bar{b})] = O(t^{-1}).
\] (F.12)

**Proof.** As above we abbreviate \( R_{1,0} = (1 + H_{1,0}(\xi))^{-1} \). Proposition F.2 gives
\[
[H_{1}(\xi), d\Gamma(\bar{q}, \bar{b})]R^{4}_{1,0} = O(t^{-1}).
\] (F.13)

Now we will use the Helffer-Sjöstrand calculus. (See e.g. [8] Proposition C.2.1.) We choose an almost analytic extension \( \tilde{\chi} \in C_0^\infty(\mathbb{C}) \) of \( \chi \) s.t.
\[
|\partial_z \tilde{\chi}(z)| \leq C_n |\text{Im} z|^n,
\] (F.14)
for \( n \in \mathbb{N} \) and write
\[
[H_{1}(\xi), d\Gamma(\bar{q}, \bar{b})] = \frac{1}{2\pi} \int dz \wedge d\bar{z} \partial_z \tilde{\chi}(z)(z - H_1(\xi))^{-1}[H_1(\xi), d\Gamma(\bar{q}, \bar{b})](z - H_1(\xi))^{-1}R^{3}_{1,0}.
\] (F.15)

Making use of relations (F.14), (F.13), and of the fact that \( \|(1 + H_{1,0}(\xi))^{-1}(z - H_1(\xi))^{-1}(1 + H_{1,0}(\xi))^{-3}\| \leq c/|\text{Im} z| \), proven in Corollary F.6, we show that
\[
[H_{1}(\xi), d\Gamma(\bar{q}, \bar{b})]R^{3}_{1,0} = O(t^{-1}).
\] (F.16)

Next, we choose a function \( \chi \in C_0^\infty(\mathbb{R}) \) s.t. \( \chi \tilde{\chi} = \chi \) and write
\[
[H_{1}(\xi), d\Gamma(\bar{q}, \bar{b})] = [\chi(H_1(\xi)), d\Gamma(\bar{q}, \bar{b})]\tilde{\chi}(H_1(\xi))
+ \chi(H_1(\xi))\tilde{\chi}(H_1(\xi)), d\Gamma(\bar{q}, \bar{b})].
\] (F.17)

Making use of (F.16), we conclude the proof. \( \square \)

**F.2 Commutators involving \( \Gamma(\cdot) \)**

**Lemma F.4.** Let \( j_0, j_\infty \in C^\infty(\mathbb{R}) \), \( j_0', j_\infty' \in C_0^\infty(\mathbb{R}) \) and \( 0 \leq j_0, j_\infty \leq 1 \). Let \( j^t := \text{diag}(j^t_0, j^t_\infty) \) be defined as a propagation observable on \( \mathfrak{h} \oplus \mathfrak{h} \). Let \( f \in S^m(\mathbb{R}^\nu) \). Then, setting \( R_{1,0}(\xi) = (1 + H_{1,0}(\xi))^{-1} \), we obtain
\[
[f(d\Gamma(\bar{k} - \xi), \Gamma(\bar{j}^t))]R_{1,0}(\xi)^3 = \nabla f(d\Gamma(\bar{k} - \xi) \cdot d\Gamma(\bar{j}^t, [\bar{k}, \bar{j}^t]^0)R_{1,0}(\xi)^3 + O(t^{-2}),
\] (F.18)
and the first term on the r.h.s. above is bounded and \( O(t^{-1}) \), with both \( O \)-symbols being uniform in \( \xi \in \mathbb{R}^\nu \).
Proof. Observe first that by Lemma C.3 we have $j^t \in C^1(\mathfrak{k})$. The operator $[k, j^t]^\circ$ is bounded and $O(t^{-1})$ by Lemma C.4.

We set $\dot{j} := j^t$ and write, making use of Lemma B.5,

$$[f(d\Gamma(k) - \xi), \Gamma(\dot{j})](1 + N_1)^{-3} = \nabla f(d\Gamma(k) - \xi) \cdot d\Gamma(\dot{j}, [k, j^t]^\circ)(1 + N_1)^{-3} + R(f, d\Gamma(k) - \xi, \Gamma(\dot{j}))(1 + N_1)^{-3}, \quad (F.19)$$

as a form equality on $D(H_{1,0}(\xi))$. Here $N_1$ is the number operator on $\Gamma(\mathfrak{h} \oplus \mathfrak{h})$ and we used the fact that $f \in \mathcal{S}^2(\mathbb{R}^\nu)$ and that $\Gamma(\dot{j})(1 + N_1)^{-3}$ belongs to $C^3(d\Gamma(k) - \xi)$. To justify this latter property, we note that for $|\alpha| \leq 3$

$$\text{ad}_{d\Gamma(k) - \xi}(\Gamma(\dot{j}))(1 + N_1)^{-3} = O(t^{-|\alpha|}), \quad (F.20)$$

where we made use of Lemma A.2. We obtain from Lemma B.5 that

$$\|R(f, d\Gamma(k) - \xi, \Gamma(\dot{j}))(1 + N_1)^{-3}\| \leq \sum_{\alpha: 2 \leq |\alpha| \leq 3} \|\text{ad}_{d\Gamma(k) - \xi}(\Gamma(\dot{j}))(1 + N_1)^{-3}\| = O(t^{-2}), \quad (F.21)$$

uniformly in $\xi$. In fact, the right-hand side does not actually depend on $\xi$. Since $(1 + N_1)\ell(1 + H_{1,0}(\xi))^{-\ell}$ is bounded uniformly in $\xi$ for any $\ell \in \mathbb{N}$, we have shown that

$$[f(d\Gamma(k) - \xi), \Gamma(\dot{j})]R_{1,0}(\xi)^3 = \nabla f(d\Gamma(k)) \cdot d\Gamma(\dot{j}, [k, j^t]^\circ)R^3_{1,0}(\xi) + O(t^{-2}), \quad (F.22)$$

uniformly in $\xi$ and in the sense of forms on $D(H_{1,0}(\xi))$. To check that the first term on the r.h.s. above is bounded and of order $O(t^{-1})$ uniformly in $\xi$, we can as in the proof of Lemma F.1 assume that $s_\Omega \geq 1$ and argue in the exact same fashion, replacing however $d\Gamma(k) \cdot d\Gamma(k) + 1$ by $(d\Gamma(k) - \xi) \cdot (d\Gamma(k) - \xi) + 1$. We skip the details, which are slightly simpler here since there is only one term. \hfill \Box

Proposition F.5. Let $j_{0}, j_{\infty}$ be as specified in Definition 3.1. Let $j^t := \text{diag}(j_{0}, j_{\infty})$ be defined as an operator on $\mathfrak{h} \oplus \mathfrak{h}$. Then, setting $R_{1,0}(\xi) := (1 + H_{1,0}(\xi))^{-1}$, $R_{0}(\xi) := (1 + H_{0}(\xi))^{-1}$, we get

$$[H_{1}(\xi), \Gamma(\dot{j}^t)]R^3_{1,0}(\xi) = (-\nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(\dot{j}^t, [k, j^t]^\circ) + d\Gamma(\dot{j}^t, [\omega, j^t]^\circ))R^3_{1,0}(\xi) + O(t^{-2}), \quad (F.23)$$

uniformly in $\xi \in \mathbb{R}^\nu$. Consequently

$$[H(\xi), \Gamma(j_{0})]R^3_{0}(\xi) = (-\nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(j_{0}, [k, j_{0}^t]^\circ) + d\Gamma(j_{0}, [\omega, j_{0}^t]^\circ))R^3_{0}(\xi) + O(t^{-2}), \quad (F.24)$$

uniformly in $\xi \in \mathbb{R}^\nu$. The explicit terms on the r.h.s. of relations (F.23), and (F.24) are bounded and $O(t^{-1})$, uniformly in $\xi$.

Proof. Observe first that by Lemma C.3 we have $j^t \in C^1(\mathfrak{k}) \cap C^1(\omega)$. The operators $[k, j^t]^\circ$ and $[\omega, j^t]^\circ$ are bounded and $O(t^{-1})$ by Lemma C.4.

We set $\dot{j} := j^t$. Lemma F.4 gives that

$$[\Omega(\xi - d\Gamma(k)), \Gamma(\dot{j})]R_{1,0}(\xi)^3 = -\nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(\dot{j}, [k, j^t]^\circ)R_{1,0}(\xi)^3 + O(t^{-2}), \quad (F.25)$$

and all terms on the r.h.s. above are $O(t^{-1})$ uniformly in $\xi$. 56
As for the second term from the free auxiliary Hamiltonian (F.19), it suffices to note that uniformly in \( \xi \) we have
\[
[d\Gamma(\omega), \Gamma(j)]R_{1,0}(\xi)^3 = d\Gamma(j, [\omega, j]^0)R_{1,0}(\xi)^3 = O(t^{-1}).
\] (F.26)

The interaction term from the interacting auxiliary Hamiltonian (E.19) contributes to \( O(t^{-2}) \). To show this, we recall the relations
\[
\Gamma(j) a^*(G, 0) = a^*(\tilde{j}(G, 0)) \Gamma(\tilde{j}) \quad \text{and} \quad a(G, 0) \Gamma(j) = \Gamma(j) a(j(G, 0)),
\] (F.27)
which hold on \( \Gamma_{\text{fin}}(\mathfrak{g} \oplus \mathfrak{g}) \) and imply that
\[
[\phi(G, 0), \Gamma(j)]R_{1,0}(\xi)^3 = (\Gamma(j) a((j_0 - 1)G, 0) - a^*((j_0 - 1)G, 0) \Gamma(j)) R_{1,0}(\xi)^3 = O(t^{-2}),
\] (F.28)
uniformly in \( \xi \). In the last step we made use of the fact that \( j_0 - 1 \) is regular and of Lemma D.2. This concludes the proof of (F.23).

Now let us prove relation (F.24). By conjugating (F.23) with the unitary \( U \), we get
\[
[H_{\text{ex}}^\text{ex}(\xi), \Gamma_{\text{ex}}(\tilde{j})] R_{0}^\text{ex}(\xi)^3 = (-\nabla \Omega(\xi - dt^\text{ex}(\tilde{j})) \cdot d\Gamma_{\text{ex}}(\tilde{j}, [\tilde{k}, \tilde{j}]^0) + d\Gamma_{\text{ex}}(\tilde{j}, [\tilde{k}, \tilde{j}]^0)) R_{0}^\text{ex}(\xi)^3 + O(t^{-2}),
\] (F.29)
where \( R_{0}^\text{ex}(\xi) = (1 + H_{0}^\text{ex}(\xi))^{-1} \). By applying the projection \( P_0 \) on the subspace \( \mathcal{F} \otimes \mathcal{F}^{0} \subset \mathcal{F}^\text{ex} \) to both sides of this equality, we obtain (F.24).

**Lemma F.6.** Let \( \chi \in C^\infty_0(\mathbb{R})_R \) and \( j_0, j_\infty \) be as specified in Definition 3.1. Let \( \tilde{j}^t := \text{diag}(j_0^t, j_\infty^t) \) be defined as an operator on \( \mathfrak{h} \oplus \mathfrak{h} \). Then there holds the estimate
\[
[\chi(H_1(\xi) + \lambda), \Gamma(\tilde{j}^t)] = O(t^{-1}),
\] (F.30)
uniformly in \( \xi \in \mathbb{R}^\nu \) and \( \lambda \geq 0 \).

**Proof.** This lemma follows from Proposition F.5 by the method of almost analytic extensions. (Cf. the proof of Lemma F.3). \( \square \)

### F.3 Commutators involving \( \tilde{\Gamma}(\cdot) \)

**Lemma F.7.** Let \( j_0, j_\infty \in C^\infty(\mathbb{R}) \), \( j_0^t, j_\infty^t \in C^\infty_0(\mathbb{R}) \), \( 0 \leq j_0, j_\infty \leq 1 \), and \( j_0^2 + j_\infty^2 \leq 1 \). Let \( j^t := (j_0^t, j_\infty^t) \) be defined as an operator from \( \mathfrak{h} \) to \( \mathfrak{h} \oplus \mathfrak{h} \). Let \( f \in S^\mu(\mathbb{R}^\nu) \). Then, setting \( R_0(\xi) = (1 + H_0(\xi))^{-1} \) and \( R_{0}^\text{ex}(\xi) = (1 + H_{0}^\text{ex}(\xi))^{-1} \), we obtain
\[
(f(d\Gamma^\text{ex}(k) - \xi) \tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t) f(d\Gamma(k) - \xi)) R_0(\xi)^3 = \nabla f(d\Gamma^\text{ex}(k) - \xi) \cdot d\Gamma^\text{ex}(j^t, [\tilde{k}, j^t]^0) R_0(\xi)^3 + O(t^{-2}),
\] (F.31)
\[
R_{0}^\text{ex}(\xi)^3 (f(d\Gamma^\text{ex}(k) - \xi) \tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t) f(d\Gamma(k) - \xi)) = R_{0}^\text{ex}(\xi)^3 \nabla f(d\Gamma^\text{ex}(k) - \xi) \cdot d\Gamma^\text{ex}(j^t, [\tilde{k}, j^t]^0) + O(t^{-2}),
\] (F.32)
uniformly in \( \xi \in \mathbb{R}^\nu \). Furthermore, all explicit terms on the r.h.s. of (F.31) and (F.32) above are bounded and \( O(t^{-1}) \), uniformly in \( \xi \in \mathbb{R}^\nu \).

**Remark F.8.** Here \( [k, j]^0 \) is a \( \nu \)-vector with entries \( [k_i, j_i]^0 \), which is itself a 2-vector \( ([k_i, j_0^0], [k_i, j_\infty^0]) \) with bounded operators, obtained through extension by continuity of the form \( [k_i, j_i] = [k_i, j_i^0] - j_i k_i \) densely defined on \( D(k) \times D(k) \). Recall that by Lemma C.3 we have \( j_0^0, j_\infty^0 \in C^1(k_i) \), for each \( i \). The vector operator \( [k, j]^0 \) is bounded and \( O(t^{-1}) \) by Lemma C.4.
Proof. As in (33), we define
\[ \tilde{\Gamma}^{\text{ex}}(q) = \tilde{\Gamma}(q)P_0 \quad \text{and} \quad d\tilde{\Gamma}^{\text{ex}}(q,p) = d\tilde{\Gamma}(q,p)P_0, \] (F.33)
where \( P_0 : \mathcal{F}^{\text{ex}} \to \mathcal{F} \) is the natural restriction to the subspace \( \mathcal{F} \otimes \mathcal{F}^{(0)} = \mathcal{F} \in \mathcal{F}^{\text{ex}} \). (The notation \( \tilde{\Gamma}^{\text{ex}} \) and \( d\tilde{\Gamma}^{\text{ex}} \) is used only in this proof). After identifying \( \mathcal{F} \otimes \mathcal{F}^{(0)} \) with \( \mathcal{F} \), we can write
\[ (f(d\Gamma^{\text{ex}}(k) - \xi)\tilde{\Gamma}(j^1) - \tilde{\Gamma}(j^1)f(d\Gamma(k) - \xi))R_0(\xi)^3 \]
\[ = [f(d\Gamma^{\text{ex}}(k) - \xi), \tilde{\Gamma}^{\text{ex}}(j^1)]R_0^{\text{ex}}(\xi)^3. \] (F.34)
Next, we set \( j := j^1 \) and write, making use of Lemma [B.5]
\[ [f(d\Gamma^{\text{ex}}(k) - \xi), \tilde{\Gamma}^{\text{ex}}(j)](1 + N^{\text{ex}})^{-3} \]
\[ = \nabla f(d\Gamma^{\text{ex}}(k) - \xi) \cdot d\Gamma^{\text{ex}}(j, [k, j]^0)(1 + N^{\text{ex}})^{-3} \] (F.35)
\[ + R(f, d\Gamma^{\text{ex}}(k) - \xi, \tilde{\Gamma}^{\text{ex}}(j)(1 + N^{\text{ex}})^{-3}), \]
as a form identity on \( D(H^{\text{ex}}_0(\xi)) \). Here \( N^{\text{ex}} \) is the number operator on \( \mathcal{F}^{\text{ex}} \). We used Lemma [A.5] the assumption that \( f \in S^{\alpha_0}(\mathbb{R}^\nu) \) and that \( \Gamma^{\text{ex}}(j)(1 + N^{\text{ex}})^{-3} \) belongs to \( C^3(d\Gamma^{\text{ex}}(k) - \xi) \). To justify this latter property, we note that by Lemma [A.4]
\[ \text{ad}_{d\Gamma^{\text{ex}}(k) - \xi}(\tilde{\Gamma}^{\text{ex}}(j))(1 + N^{\text{ex}})^{-3} = O(t^{-|\alpha|}), \] (F.36)
uniformly in \( \xi \), for \( |\alpha| \leq 3 \) (the expression is in fact \( \xi \)-independent). Thus we obtain from Lemma [B.5] that
\[ \|R(f, d\Gamma^{\text{ex}}(k) - \xi, \tilde{\Gamma}^{\text{ex}}(j))(1 + N^{\text{ex}})^{-3}\| \]
\[ \leq \sum_{\alpha:2 \leq |\alpha| \leq 3} \|\text{ad}_{d\Gamma^{\text{ex}}(k) - \xi}(\tilde{\Gamma}^{\text{ex}}(j))(1 + N^{\text{ex}})^{-3}\| = O(t^{-2}), \] (F.37)
uniformly in \( \xi \). Since \( (1 + N^{\text{ex}})^\ell(1 + H^{\text{ex}}_0(\xi))^{-\ell} \) is bounded uniformly in \( \xi \) for any \( \ell \in \mathbb{N} \), we have shown that uniformly in \( \xi \) we have \( \mathcal{F}^{\text{ex}} \)
\[ [f(d\Gamma^{\text{ex}}(k) - \xi), \tilde{\Gamma}^{\text{ex}}(j)]R_0^{\text{ex}}(\xi)^3 = \nabla f(d\Gamma^{\text{ex}}(k) - \xi) \cdot d\Gamma^{\text{ex}}(j, [k, j]^0)R_0^{\text{ex}}(\xi)^3 + O(t^{-2}), \] (F.38)
\[ R_0^{\text{ex}}(\xi)^3[f(d\Gamma^{\text{ex}}(k) - \xi), \tilde{\Gamma}^{\text{ex}}(j)] = R_0^{\text{ex}}(\xi)^3 \nabla f(d\Gamma^{\text{ex}}(k) - \xi) d\Gamma^{\text{ex}}(j, [k, j]^0) + O(t^{-2}). \] (F.39)
To check that all the terms on the r.h.s. of the above relations are \( O(t^{-1}) \) uniformly in \( \xi \), we can assume that \( s_0 \geq 1 \) and write
\[ (R_0^{\text{ex}})^3 \nabla f(d\Gamma^{\text{ex}}(k)) \cdot d\Gamma^{\text{ex}}(j, [k, j]^0) \]
\[ = (1 + N^{\text{ex}})(R_0^{\text{ex}})^3 \nabla f(d\Gamma^{\text{ex}}(k))(1 + N^{\text{ex}})^{-1} \cdot d\Gamma^{\text{ex}}(j, [k, j]). \] (F.40)
For (F.38) we argue as at the end of the proofs of Lemmata [F.1] and [F.4] inserting \( I = (1 + (d\Gamma(k) - \xi)^2)^{-1}(1 + (d\Gamma(k) - \xi)^2) \) and commuting the second factor onto the resolvent on the right to obtain
\[ \nabla f(d\Gamma^{\text{ex}}(k) - \xi) \cdot d\Gamma^{\text{ex}}(j, [k, j])R_0^{\text{ex}}(\xi)^3 \]
\[ = -\nabla f(d\Gamma^{\text{ex}}(k) - \xi)(1 + (d\Gamma(k) - \xi)^2)^{-1} \cdot [d\Gamma^{\text{ex}}(j, [k, j]^0), (d\Gamma(k) - \xi)^2]R_0^{\text{ex}}(\xi)^3 \]
\[ + \nabla f(d\Gamma^{\text{ex}}(k) - \xi)(1 + (d\Gamma(k) - \xi)^2)^{-1} \cdot d\Gamma^{\text{ex}}(j, [k, j]^0)(1 + (d\Gamma(k) - \xi)^2)R_0^{\text{ex}}(\xi)^3. \] (F.41)
Here \( (d\Gamma(k) - \xi)^2 = (d\Gamma(k) - \xi) \cdot (d\Gamma(k) - \xi) \). Recalling (F.33) and that \( |\partial_\ell f(\eta)| \leq c(\eta) \), and making use of Lemma [A.4] we conclude the proof.
Proposition F.9. Let \( j_0, j_\infty \) be as specified in Definition F.1 and s.t. \( j_0^2 + j_\infty^2 \leq 1 \). Put \( j^t = (j_0^t, j_\infty^t): \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \). Then, setting \( R_0(\xi) = (1 + H_0(\xi))^{-1} \) and \( R_0^\text{ex}(\xi) = (1 + H_0^\text{ex}(\xi))^{-1} \), we obtain uniformly in \( \xi \in \mathbb{R}^\nu \) the asymptotic expansions

\[
(H^\text{ex}(\xi)\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)H(\xi))R_0^3(\xi) = \left( -\nabla_\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\Gamma(j^t, [k, j^t]^0) + d\tilde{\Gamma}(j^t, [\omega, j^t]^0) \right)R_0^3(\xi) + O(t^{-2}) \quad (F.42)
\]

and

\[
R_0^\text{ex}(\xi)^3(H^\text{ex}(\xi)\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)H(\xi)) = R_0^\text{ex}(\xi)^3\left( -\nabla_\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\Gamma(j^t, [k, j^t]^0) + d\tilde{\Gamma}(j^t, [\omega, j^t]^0) \right) + O(t^{-2}), \quad (F.43)
\]

and all explicit terms on the r.h.s. of relations (F.42) and (F.43) are bounded and \( O(t^{-1}) \) uniformly in \( \xi \in \mathbb{R}^\nu \).

Proof. We prove only relation (F.42), as the proof of (F.43) is analogous. Observe again that by Lemma C.3 we have \( j^t \in C^1(\xi) \cap C^1(\omega) \). The operators \([k, j^t]^0 \) and \([\omega, j^t]^0 \) are bounded and \( O(t^{-1}) \) by Lemma C.4. See also Remark F.8 for a more thorough explanation of the notation.

Lemma F.7 gives

\[
(\Omega(\xi - d\Gamma^\text{ex}(k))\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)\Omega(\xi - d\Gamma(k)))R_0(\xi)^3 = -\nabla_\Omega(\xi - d\Gamma^\text{ex}(k))d\Gamma(j^t, [k, j^t]^0)R_0(\xi)^3 + O(t^{-2}), \quad (F.44)
\]

and all terms on the r.h.s. above are \( O(t^{-1}) \) uniformly in \( \xi \). As for the second term in the extended Hamiltonian (2.25), we obtain

\[
(d\Gamma^\text{ex}(\omega)\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)d\Gamma(\omega))R_0(\xi)^3 = d\tilde{\Gamma}(j^t, [\omega, j^t]^0)R_0(\xi)^3, \quad (F.45)
\]

where we made use of Lemma A.5. It is clear that this expression is \( O(t^{-1}) \) uniformly in \( \xi \).

Finally, we consider the interaction term from Hamiltonian (2.25). There holds

\[
((\phi(G) \otimes 1)\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)\phi(G))R_0(\xi)^3 = ((a^*((1 - j^t_0)G) \otimes 1 + 1 \otimes a^*(j^t_\infty G))\tilde{\Gamma}(j^t) + \tilde{\Gamma}(j^t)a((j^t_0 - 1)G))R_0(\xi)^3.
\]

Since \( j_0^t - 1 \) and \( j_\infty^t \) are regular propagation observables, this expression is \( O(t^{-2}) \) (uniformly in \( \xi \)) by Lemma D.2. This concludes the proof.

Lemma F.10. Let \( j_0, j_\infty \) be as specified in Definition F.1 and s.t. \( j_0^2 + j_\infty^2 \leq 1 \). Put \( j^t = (j_0^t, j_\infty^t): \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h} \). There holds the relation

\[
(\chi(H^\text{ex}(\xi) + \lambda)\tilde{\Gamma}(j^t) - \tilde{\Gamma}(j^t)\chi(H(\xi) + \lambda)) = O(t^{-1}), \quad (F.47)
\]

uniformly in \( \xi \in \mathbb{R}^\nu \) and \( \lambda \geq 0 \).

Proof. This lemma follows from Proposition F.9 by the method of almost analytic extensions. (Cf. the proof of Lemma F.3.)
Corollary F.11. Let \( j_0, j_\infty \) be as specified in Definition F.4 and s.t. \( j_0^2 + j_\infty^2 \leq 1 \). Suppose \( q \in C^\infty(\mathbb{R}) \), with \( 0 \leq q \leq 1 \), is bounded with bounded derivatives. Then

\[
\Gamma(q^i) : D(H(\xi)^3) \to D(H(\xi)),
\]

\[
\hat{\Gamma}(j^i) : D(H(\xi)^3) \to D(H^{ex}(\xi)),
\]

\[
\hat{\Gamma}(j^i)^* : D(H^{ex}(\xi)^3) \to D(H(\xi)).
\]

We do not believe the third power in the corollary above is optimal. It is however sufficient for our purpose. A similar result was derived in [36] for localizations in configuration space.

G  Auxiliary results for the proof of Proposition 3.3

Proposition G.1. Let \( \chi \in C^\infty_0(\mathbb{R}) \). Let \( q \in C^\infty(\mathbb{R}) \) be s.t. \( q' \in C^\infty_0(\mathbb{R}) \) and \( 0 \leq q \leq 1 \), and let \( b \) be an admissible and regular propagation observable. Let \( j_0 \) be as specified in Definition 3.1 and s.t. \( \text{supp} j_0 \subset \Delta \), where \( \Delta \) appeared in Definition D.1. Then

\[
\chi \text{d}\Gamma(q^i, b) \chi \Gamma(j_0^i) = O(t^{-1}),
\]

\[
\chi[H(\xi), \text{d}\Gamma(b)] \chi \Gamma(j_0^i) = O(t^{-2}),
\]

where we set \( \chi := \chi(H(\xi)) \).

Proof. To prove (G.1) we write

\[
\chi \text{d}\Gamma(q^i, b) \chi \Gamma(j_0^i) = \chi \text{d}\Gamma(q^i, b)[\chi, \Gamma(j_0^i)] + O(t^{-2}),
\]

where we exploited regularity of \( b \). The first term on the r.h.s. is of order \( O(t^{-1}) \) by Lemma F.6.

To verify (G.2), we make use of Proposition F.2, which gives

\[
\chi[H(\xi), \text{d}\Gamma(b)] \chi \Gamma(j_0^i) = -\chi \nabla \Omega(\xi - \text{d}\Gamma(k)) \cdot \text{d}\Gamma([k, b]^0)[\chi, \Gamma(j_0^i)] + \chi \text{d}\Gamma([\omega, b]^0)[\chi, \Gamma(j_0^i)] + O(t^{-2}),
\]

where we exploited regularity of \( b \). The first two terms on the r.h.s. above are \( O(t^{-2}) \) by admissibility of \( b \), cf. Definition D.1 and Lemma F.6.

Proposition G.2. Let \( q \in C^\infty(\mathbb{R}) \) be s.t. \( q' \in C^\infty_0(\mathbb{R}) \) and \( 0 \leq q \leq 1 \). Let \( b \) be an admissible propagation observable and \( j_0, j_\infty \) be as specified in Definition F.4 with \( j_0^2 + j_\infty^2 = 1 \). Then

\[
\chi(H(\xi) + \lambda) \text{d}\Gamma(q^i, b) \chi(H(\xi) + \lambda) = \hat{\Gamma}(j^i)^* \chi^{ex}(H^{ex}(\xi) + \lambda) \text{d}\Gamma^{ex}(q^i, b) \chi^{ex}(H^{ex}(\xi) + \lambda) \hat{\Gamma}(j^i) + O(t^{-1}),
\]

uniformly in \( \xi \in \mathbb{R}^d \) and \( \lambda \geq 0 \).

Proof. We set \( j := j^i, q := q^i, \chi := \chi(H(\xi) + \lambda) \) and \( \chi^{ex} := \chi(H^{ex}(\xi) + \lambda) \). The reader is asked to keep in mind that \( \chi \) and \( \chi^{ex} \) depend on both \( \xi \) and \( \lambda \). Write

\[
\chi \text{d}\Gamma(q, b) \chi = \hat{\Gamma}(j)^* \chi^{ex}(\hat{\Gamma}(j)^i) \text{d}\Gamma(q, b) \chi + \hat{\Gamma}(j)^* \chi^{ex} \hat{\Gamma}(j)^i \text{d}\Gamma(q, b) \chi.
\]
The first term on the r.h.s. above is of order $O(t^{-1})$ uniformly in $\xi$ and $\lambda \geq 0$ by Lemma \ref{lem:G.10}. The last term on the r.h.s. of \eqref{eq:G.6} can be rearranged as follows

$$
\hat{\Gamma}(j)^* \chi^\text{ex}\hat{\Gamma}(j)\,d\Gamma(q,b)\chi = \hat{\Gamma}(j)^* \chi^\text{ex}d\Gamma^\text{ex}(q,b)\hat{\Gamma}(j)\chi + \hat{\Gamma}(j)^* \chi^\text{ex}d\Gamma(jq,[j,b])\chi \\
= \hat{\Gamma}(j)^* \chi^\text{ex}d\Gamma^\text{ex}(q,b)\hat{\Gamma}(j)\chi + O(t^{-1}),
$$

\hfill \eqref{eq:G.7}

uniformly in $\xi$ and $\lambda \geq 0$. Here we made use of Lemma \ref{lem:A.5} and admissibility of $b$, cf. Definition \ref{defn:D.1}. (Note that $[j,b] = j b - b j$ and $b = \text{diag}(b,b)$ is a bounded propagation observable on $\mathfrak{h} \oplus \mathfrak{h}$). To exchange $\hat{\Gamma}(j)\chi$ with $\chi^\text{ex}\hat{\Gamma}(j)$ in \eqref{eq:G.7} we use again Lemma \ref{lem:F.10}. This concludes the proof. \hfill \Box

**Lemma G.3.** Let $q \in C^\infty(\mathbb{R})_R$ be s.t. $q' \in C^\infty_0(\mathbb{R})$. Let $\chi \in C^\infty_0(\mathbb{R})_R$. Then

$$
[1 \otimes q', \chi(H^{(1)}(\xi))] = O(t^{-1}).
$$

\hfill \eqref{eq:G.8}

**Proof.** We follow the strategy explained in Remark \ref{rem:G.5}. By setting in Lemma \ref{lem:F.3} $q' = \text{diag}(1,1)$, $b = (0,q')$, conjugating formula \eqref{eq:F.12} with the unitary $U$ and applying the projection $P_1$ on the subspace $\mathcal{F} \otimes \mathcal{F}^{(1)} \subset \mathcal{F}^\text{ex}$ we obtain \eqref{eq:G.8}. \hfill \Box

**Corollary G.4.** Let $q,p \in C^\infty(\mathbb{R})$ be s.t. $q', p' \in C^\infty_0(\mathbb{R})$ and $0 \leq q \leq 1$. Let $b$ be an admissible and regular propagation observable. Let $\chi \in C^\infty_0(\mathbb{R})_R$ be supported in $(-\infty, \Sigma^{(2)}_0(\xi))$. Then

$$
\chi(H^{(1)}(\xi))(d\Gamma(q',b) \otimes p')\chi(H^{(1)}(\xi)) = O(t^{-1}).
$$

\hfill \eqref{eq:G.9}

**Proof.** We set $q := q'$, $p = p'$ and choose $\chi_0 \in C^\infty_0(\mathbb{R})$, supported in $(-\infty, \Sigma^{(2)}_0(\xi))$ and s.t. $\chi = \chi_0 \chi$. Then, abbreviating $\chi^{(1)} := \chi(H^{(1)}(\xi))$ and making use of the fact that $\chi^{(1)}(d\Gamma(q,b) \otimes 1) = O(1)$, we obtain from Lemma \ref{lem:G.3} that

$$
\chi^{(1)}(d\Gamma(q,b) \otimes p)\chi^{(1)} = \chi^{(1)}(d\Gamma(q,b) \otimes 1)\chi^{(1)}(1 \otimes p)\chi_0^{(1)} + O(t^{-1}).
$$

\hfill \eqref{eq:G.10}

Thus it suffices to prove \eqref{eq:G.9} with $p = 1$. We rewrite this expression as a direct integral

$$
\int_{\mathbb{R}} dk \chi(H(\xi - k) + \omega(k))d\Gamma(q,b)\chi(H(\xi - k) + \omega(k)).
$$

\hfill \eqref{eq:G.11}

In order to establish that the above expression is $O(t^{-1})$, it suffices to argue that

$$
\chi(H(\xi - k) + \omega(k))d\Gamma(q,b)\chi(H(\xi - k) + \omega(k)) = O(t^{-1}),
$$

\hfill \eqref{eq:G.12}

uniformly in $k \in \mathbb{R}^\nu$.

For $k \in \mathbb{R}^\nu$, define the function $\chi_k(s) := \chi(s + \omega(k))$. It is easily seen that $\chi_k \in C^\infty_0(\mathbb{R})$ is supported in $(-\infty, \Sigma^{(1)}_0(\xi - k))$. Indeed, If $s + \omega(k) \in \text{supp}\chi$, then $s + \omega(k) < \Sigma^{(2)}_0(\xi) \leq \Sigma^{(1)}_0(\xi - k) + \omega(k)$.

Now let $j_0, j_\infty$ be as specified in Definition \ref{defn:D.1} s.t. $j_0^2 + j_\infty^2 = 1$ and $j_0$ is supported in the set $\Delta$ specified in Definition \ref{defn:D.1}. We set $j := j'$ below. Then

$$
\chi_k(H(\xi - k))d\Gamma(q,b)\chi_k(H(\xi - k)) \\
= \hat{\Gamma}(j)^* \chi_k(H^\text{ex}(\xi - k))d\Gamma^\text{ex}(q,b)\chi_k(H^\text{ex}(\xi - k))\hat{\Gamma}(j) + O(t^{-1}) \\
= \Gamma(j_0)\chi_k(H(\xi - k))d\Gamma(q,b)\chi_k(H(\xi - k))\Gamma(j_0) + O(t^{-1}).
$$

\hfill \eqref{eq:G.13}

Here in the first step we made use of Proposition \ref{prop:G.2} which in particular ensures that the asymptotic expansion above is uniform in $k \in \mathbb{R}^\nu$. In the second step we applied the
decomposition (G.20) of $H^{\text{ex}}(\xi - k)$ and observed that, due to the support property of $\chi_k$, only the $l = 0$ term is non-zero. Next, making use of Lemma A.5, we get
\[
\Gamma(j_0)\chi_k(H(\xi - k))d\Gamma(q, b)\chi_k(H(\xi - k)) = [\Gamma(j_0), \chi_k(H(\xi - k))]d\Gamma(q, b)\chi_k(H(\xi - k)) + \chi_k(H(\xi - k))d\Gamma(j_0 q, j_0 b)\chi_k(H(\xi - k)).
\] (G.14)

This expression is of order $O(t^{-1})$, uniformly in $k$, by Lemma F.6 and regularity of $b$. This concludes the verification of (G.12), and hence we have established the corollary.

**Proposition G.5.** Let $b$ be an admissible and regular propagation observable. Let $\chi \in C^\infty_0(\mathbb{R})_\mathbb{R}$ and $j_0, j_\infty$ be as specified in Definition [\ref{def:admissible}] and s.t. $j_0^2 + j_\infty^2 = 1$. Then
\[
\chi[H(\xi), d\Gamma(b)]\chi = \tilde{\Gamma}(j^t)^* \chi^{\text{ex}}[H^{\text{ex}}(\xi), d\Gamma^{\text{ex}}(b)]\chi^{\text{ex}}\tilde{\Gamma}(j^t) + O(t^{-2}),
\] (G.15)
where we set $\chi := \chi(H(\xi))$ and $\chi^{\text{ex}} := \chi(H^{\text{ex}}(\xi))$.

**Proof.** We set $j := j^t$ and write
\[
\chi[H(\xi), d\Gamma(b)]\chi = \tilde{\Gamma}(j)^* \chi^{\text{ex}}\tilde{\Gamma}(j)[H(\xi), d\Gamma(b)]\chi + O(t^{-2}),
\] (G.16)
where we made use of the fact that, by Proposition F.2, $[H(\xi), d\Gamma(b)]\chi = O(t^{-1})$. Next, we will show that
\[
\chi^{\text{ex}}(\tilde{\Gamma}(j)[H(\xi), d\Gamma(b)] - [H^{\text{ex}}(\xi), d\Gamma^{\text{ex}}(b)]\tilde{\Gamma}(j))\chi = O(t^{-2}).
\] (G.17)

In view of Proposition F.2 it is enough to check that
\[
\chi^{\text{ex}}\tilde{\Gamma}(j)\partial_t \Omega(\xi - d\Gamma(k))d\Gamma([k_i, b]^o)\chi = \chi^{\text{ex}}\partial_t \Omega(\xi - d\Gamma(k))d\Gamma^{\text{ex}}([k_i, b]^o)\tilde{\Gamma}(j)\chi + O(t^{-2}),
\] (G.18)
and
\[
\chi^{\text{ex}}\tilde{\Gamma}(j)d\Gamma([\omega, b]^o)\chi = \chi^{\text{ex}}d\Gamma^{\text{ex}}([\omega, b]^o)\tilde{\Gamma}(j) + O(t^{-2}).
\] (G.19)

We prove only (G.18), since the proof of (G.19) is analogous (and simpler). First, we note that,
\[
\chi^{\text{ex}}\tilde{\Gamma}(j)\partial_t \Omega(\xi - d\Gamma(k))d\Gamma([k_i, b]^o)\chi = \chi^{\text{ex}}\partial_t \Omega(\xi - d\Gamma(k))\tilde{\Gamma}(j)d\Gamma([k_i, b]^o)\chi + O(t^{-2}),
\] (G.20)
where we exploited Lemma F.7. Next, making use of Lemma A.5 we obtain
\[
\tilde{\Gamma}(j)d\Gamma([k_i, b]^o)\chi = d\Gamma^{\text{ex}}([k_i, b]^o)\tilde{\Gamma}(j)\chi + \tilde{\Gamma}(j, [j, [k_i, b]^o])\chi = d\Gamma^{\text{ex}}([k_i, b]^o)\tilde{\Gamma}(j)\chi + O(t^{-2}),
\] (G.21)
where we used admissibility of $b$, cf. Definition D.1. Thus we have justified (G.18).

To conclude the proof, it suffices to show that
\[
\chi^{\text{ex}}[H^{\text{ex}}(\xi), d\Gamma^{\text{ex}}(b)](\tilde{\Gamma}(j)\chi - \chi^{\text{ex}}\tilde{\Gamma}(j)) = O(t^{-2}).
\] (G.22)
This follows from Proposition F.2 and Lemma F.10. 

Lemma G.6. Let $\chi \in C^\infty_0(\mathbb{R})_\mathbb{R}$ be supported in $(-\infty, \Sigma_0^{(2)}(\xi))$. Let $b_0$ be an admissible and regular propagation observable. Then

$$\chi(H^{(1)}(\xi))[H^{(1)}(\xi), d\Gamma(b_0) \otimes 1]\chi(H^{(1)}(\xi)) = O(t^{-2}). \quad (G.23)$$

**Proof.** By conjugating formula [F.6] with the unitary $U$, setting $g^t = \text{diag}(1,1)$ and $\bar{b} = \text{diag}(b_0,0)$, we obtain

$$[H^{\text{ex}}(\xi), d\Gamma^{\text{ex}}(\bar{b})] = \left(d\Gamma^{\text{ex}}([\bar{b},\bar{b}]^\circ) - \sum_{i=1}^\nu \partial_i \Omega(\xi - d\Gamma^{\text{ex}}(k))d\Gamma^{\text{ex}}([\bar{k},\bar{k}]^\circ)\right) + O(t^{-2})(H^{\text{ex}}_0(\xi) + 1)^4,$$

as a form identity on $D(H^{\text{ex}}_0(\xi))^4$. Now we apply the projection $P_1$ on the subspace $\mathcal{F} \otimes \mathcal{F}^{(1)} \subset \mathcal{F}^{\text{ex}}$ to both sides of this equality and insert both sides between the operators $\chi^{(1)} := \chi(H^{(1)}(\xi))$. We get

$$\chi^{(1)}[H^{(1)}(\xi), d\Gamma(b_0) \otimes 1]\chi^{(1)} = \chi^{(1)} \left(-\nabla \Omega(\xi - d\Gamma^{(1)}(k)) \cdot (d\Gamma([k,b_0]^\circ) \otimes 1) + (d\Gamma([\omega,b_0]^\circ) \otimes 1)\right) \chi^{(1)} + O(t^{-2}),$$

where we used the higher order domain result in Corollary [E.4] for $H^{\text{ex}}(\xi)$. In view of Lemma [F.3] it suffices to show that

$$\chi^{(1)}(d\Gamma([g,b_0]^\circ) \otimes 1)\chi^{(1)} = O(t^{-2}) \quad \text{(G.26)}$$

for any $g \in C^\infty(\mathbb{R}^\nu)$, whose derivatives (of non-zero order) are bounded. This follows from Corollary [G.3] and the fact that $t \rightarrow t[g,b_0(t)]^\circ$ is admissible and regular. This latter fact is clear from Definition [D.1].

**H Auxiliary results for the proof of Proposition 3.4**

**Proposition H.1.** Let $\chi \in C^\infty_0(\mathbb{R})_\mathbb{R}$. Let $q \in C^\infty(\mathbb{R})$ be s.t. $q^t \in C^\infty_0(\mathbb{R})$, $0 \leq q \leq 1$ and $q = 1$ on a neighbourhood $\Delta$ of zero. Let $j_0$ be as specified in Definition [3.4] and s.t. supp $j_0 \subset \Delta$. Then

$$\chi d\Gamma(q^t, \partial_t q^t) \chi \Gamma(j_0^t) = O(t^{-2}) \quad \text{and} \quad \chi[H(\xi), \Gamma(q^t)] \chi \Gamma(j_0^t) = O(t^{-2}), \quad \text{(H.1)}$$

where $\chi := \chi(H(\xi))$.

**Proof.** We set $q := q^t$, $j_0 := j_0^t$ and write

$$\chi d\Gamma(q, \partial_t q) \chi \Gamma(j_0) = \chi d\Gamma(q, \partial_t q) \chi \Gamma(j_0) = O(t^{-2}), \quad \text{(H.2)}$$

due to the support property of $j_0$, Lemma [F.6] and the fact that $\chi d\Gamma(q, \partial_t q) = O(t^{-1})$.

Proceeding to the proof of the second part of (H.1), we write

$$\chi[H(\xi), \Gamma(q)] \chi \Gamma(j_0) = \chi[H(\xi), \Gamma(q)] \chi \Gamma(j_0) + \chi[H(\xi), \Gamma(q)] \Gamma(j_0) \chi.$$

Here we used Corollary [F.11] to justify the formal computation. The first term on the r.h.s. above is $O(t^{-2})$ by Lemma [F.6] and Proposition [F.5]. As for the second term, we apply Proposition [F.5] again:

$$\chi[H(\xi), \Gamma(q)] \Gamma(j_0) \chi \quad \text{and} \quad \chi[H(\xi), \Gamma(q)] \Gamma(j_0) \chi = \chi(-\nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(q, [k,\omega]^\circ) \Gamma(j_0) + d\Gamma(q, [\omega,\omega]^\circ) \Gamma(j_0)) \chi + O(t^{-2}).$$

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We note that \(d\Gamma(q, [\omega, q]^{\circ})\Gamma(j_0)\chi = O(t^{-2})\) and \(d\Gamma(q, [k, q]^{\circ})\Gamma(j_0)\chi = O(t^{-2})\), since \(1 - q\) is regular with the regularity region \(\Delta\). See (D.2) and Lemma (D.3). This concludes the proof.

**Proposition H.2.** Let \(\chi \in C_0^\infty(\mathbb{R})\). Let \(q \in C^\infty(\mathbb{R})\) be s.t. \(q' \in C_0^\infty(\mathbb{R})\), \(0 \leq q \leq 1\) and \(q = 1\) on a neighbourhood of zero. Let \(j_0, j_{\infty}\) be as specified in Definition \(3.7\) and s.t. \(j_0^2 + j_{\infty}^2 = 1\). Then

\[
\begin{align*}
\chi d\Gamma(q^t, \partial_t q^t)\chi &= \tilde{\Gamma}(j^t)^* \chi^{ex} d\Gamma^{ex}(q^t, \partial_t q^t)\chi^{ex} \tilde{\Gamma}(j^t) + O(t^{-2}), \\
\chi[H(\xi), \Gamma(q^t)]\chi &= \tilde{\Gamma}(j^t)^* \chi^{ex}[H^{ex}(\xi), \Gamma^{ex}(q^t)]\chi^{ex} \tilde{\Gamma}(j^t) + O(t^{-2}),
\end{align*}
\]

where \(\chi := \chi(H(\xi))\) and \(\chi^{ex} := \chi(H^{ex}(\xi))\).

**Proof.** As for (H.5), it follows from Proposition (G.2), Lemma (D.3) and (D.2) applied with \(b(t) = t\partial_t q^t\). Proceeding to the proof of formula (H.6), we set \(j := j^t\), \(q := q^t\), and note that

\[
\tilde{\Gamma}(j)\chi[H(\xi), \Gamma(q)]\chi = \chi^{ex} \tilde{\Gamma}(j)[H(\xi), \Gamma(q)]\chi + O(t^{-2})
\]

by Lemma (E.10) and Proposition (F.5). Note that Corollary (E.11) ensures the validity of the computation above, as well as those to follow. Next, we will show that

\[
\chi^{ex} \left[ \tilde{\Gamma}(j)[H(\xi), \Gamma(q)] - [H^{ex}(\xi), \Gamma^{ex}(q)] \tilde{\Gamma}(j) \right] \chi = O(t^{-2}).
\]

By Proposition (F.5), it suffices to check that

\[
\chi^{ex} \tilde{\Gamma}(j) \nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(q, [k, q]^{\circ}) \chi = \chi^{ex} \nabla \Omega(\xi - d\Gamma^{ex}(k)) \cdot d\Gamma^{ex}(q, [k, q]^{\circ}) \tilde{\Gamma}(j) \chi + O(t^{-2})
\]

and

\[
\chi^{ex} \tilde{\Gamma}(j) \chi^{ex} d\Gamma(q, [\omega, q]^{\circ}) \chi = \chi^{ex} d\Gamma^{ex}(q, [\omega, q]^{\circ}) \tilde{\Gamma}(j) \chi + O(t^{-2}).
\]

We show only (H.9), as the proof of (H.10) is analogous. Making use of Lemma (F.7) and of the fact that \(d\Gamma(q, [k, q]^{\circ})\chi = O(t^{-1})\), we can write

\[
\chi^{ex} \tilde{\Gamma}(j) \nabla \Omega(\xi - d\Gamma(k)) \cdot d\Gamma(q, [k, q]^{\circ}) \chi = \chi^{ex} \nabla \Omega(\xi - d\Gamma^{ex}(k)) \cdot \tilde{\Gamma}(j) d\Gamma(q, [k, q]^{\circ}) \chi + O(t^{-2}).
\]

Next, by exploiting the fact that \(\chi^{ex} \nabla \Omega(\xi - d\Gamma^{ex}(k))\) is bounded, and that Lemma (A.5) gives

\[
\tilde{\Gamma}(j) d\Gamma(q, [k, q]^{\circ}) \chi = d\Gamma^{ex}(q, [k, q]^{\circ}) \tilde{\Gamma}(j) \chi + O(t^{-2}),
\]

we conclude the proof of (H.9).

We still have to show that

\[
\chi^{ex}[H^{ex}(\xi), \Gamma^{ex}(q)](\tilde{\Gamma}(j) \chi - \chi^{ex} \tilde{\Gamma}(j)) = O(t^{-2}).
\]

This follows from Lemma (E.10) and Proposition (F.5).

**Proposition H.3.** Let \(q, \bar{q} \in C^\infty(\mathbb{R})\) be s.t. \(q', \bar{q}' \in C_0^\infty(\mathbb{R})\), \(0 \leq q \leq 1\), \(q = 1\) on a neighbourhood of zero. Let \(\chi \in C_0^\infty(\mathbb{R})\) be supported in \((-\infty, \Sigma_{\theta}(2)(\xi))\). Then

\[
\chi^{ex}(H^{ex}(\xi); \Gamma(q^t) \otimes 1)(1 \otimes \bar{q}^t) \chi^{ex}(1) = O(t^{-2}),
\]

where \(\chi^{(1)} = \chi(H^{(1)}(\xi))\).
Proof. Let us set in Proposition F.5 \( j_0 = q, \ j_\infty = 1 \) and conjugate equation (F.23) with the unitary \( U \). We obtain, as a form identity on \( D(H_0^{\text{ex}}(\xi)^3) \),

\[
[H^{\text{ex}}(\xi), \Gamma^{\text{ex}}(\dot{\xi})] = -\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot d \Gamma^{\text{ex}}(\dot{\xi}, [k, \dot{\xi}]^0) + d \Gamma^{\text{ex}}(\dot{\xi}, [\omega, \dot{\xi}]^0) + O(t^{-2})(H_0^{\text{ex}}(\xi) + 1)^3, \tag{H.15}
\]

where we set \( j := j' \). Let us now apply the projection \( P_1 \) on \( \mathcal{F} \otimes \mathcal{F}(1) \) to both sides of this equality. We get, as a form identity on \( D(H_0^{\text{ex}}(\xi)^3) \),

\[
[H^{\text{ex}}(\xi), \Gamma(q) \otimes 1] = -\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot (d \Gamma(q, [k, q]^0) \otimes 1) + (d \Gamma(q, [\omega, q]^0) \otimes 1) + O(t^{-2})(H_0^{\text{ex}}(\xi) + 1)^3, \tag{H.16}
\]

where we abbreviated \( q := q' \) and made use of relation (A.9). Thus we can write

\[
\chi^{(1)}[H^{\text{ex}}(\xi), \Gamma(q) \otimes 1](1 \otimes \tilde{q})\chi^{(1)} = -\chi^{(1)}\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot (d \Gamma(q, [k, q]^0) \otimes \tilde{q})\chi^{(1)} + \chi^{(1)}(d \Gamma(q, [\omega, q]^0) \otimes \tilde{q})\chi^{(1)} + O(t^{-2}), \tag{H.17}
\]

where we set \( \tilde{q} := q' \). Here we used Corollary [E.4] with \( n = 3 \). Let us consider the first term on the r.h.s. above. We choose a function \( \tilde{\chi} \in C_0^\infty(\mathbb{R})_\mathbb{R} \), supported in \((-\infty, \Sigma_0^{(2)}(\xi)) \) and s.t. \( \tilde{\chi}\chi = \chi \). Then we get

\[
(1 \otimes \tilde{q})\chi^{(1)} = (\tilde{\chi}(1))^2(1 \otimes \tilde{q})\chi^{(1)} + O(t^{-1}) \tag{H.18}
\]

by Lemma [C.3]. Next, we note that

\[
(d \Gamma(q, [k, q]^0) \otimes 1)(\chi^{(1)})^2 = \chi^{(1)}(d \Gamma(q, [k, q]^0) \otimes 1)\tilde{\chi}(1) + O(t^{-2}). \tag{H.19}
\]

Here we made use of Lemma [E.3] (after conjugating expression (F.12) with \( U \) and applying \( P_1 \) as above) and of the fact that \( t \to t[k, q]^0 \) is an admissible and regular propagation observable. This is a consequence of the fact that \( 1 - q \) is admissible and regular by Lemma [D.3].

Thus making use of the fact that \( \chi^{(1)}\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \) is bounded and

\[
\chi^{(1)}\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot (d \Gamma(q, [k, q]^0) \otimes 1) = O(t^{-1}), \tag{H.20}
\]

we obtain

\[
\chi^{(1)}\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot (d \Gamma(q, [k, q]^0) \otimes \tilde{q})\chi^{(1)} = \chi^{(1)}\nabla \Omega(\xi - d \Gamma^{\text{ex}}(k)) \cdot (d \Gamma(q, [k, q]^0) \otimes \tilde{q})\chi^{(1)}(1 \otimes \tilde{q})\chi^{(1)} + O(t^{-2}). \tag{H.21}
\]

Exploiting again the fact that \( t \to t[k, q]^0 \) is admissible and regular, we obtain from Corollary [E.4] that the first term on the r.h.s. above is \( O(t^{-2}) \). The term involving \( d \Gamma(q, [\omega, q]^0) \) on the r.h.s. of (H.17) is treated analogously. \( \square \)

I Auxiliary results for the proof of Propositions 4.1 and 4.2

Proposition I.1. Let \( \chi \in C^\infty_0(\mathbb{R})_\mathbb{R} \) and let \( q \in C^\infty(\mathbb{R})_\mathbb{R} \) be s.t. \( q' \in C^\infty_0(\mathbb{R})_\mathbb{R} \). Then

\[
\chi^{(1)}i[H^{(1)}(\xi), 1 \otimes q']\chi^{(1)} = \frac{1}{t}\chi^{(1)}C(1 \otimes (q')^t)\chi^{(1)} + O(t^{-2}), \tag{I.1}
\]

where \( \chi^{(1)} := \chi(H^{(1)}(\xi)) \) and \( C \) is a bounded operator on \( \mathcal{F} \otimes \mathcal{F}(1) \), which satisfies \([C, 1 \otimes p^t] = O(t^{-1}) \) for any \( p \in C^\infty(\mathbb{R})_\mathbb{R} \) s.t. \( p' \in C^\infty_0(\mathbb{R})_\mathbb{R} \).
If, in addition, \( q' \) is positive and \( \sqrt{q'} \in C_0^\infty(\mathbb{R}) \), then
\[
\chi(1)i[H(1)(\xi), 1 \otimes q']\chi(1) = \frac{1}{t}(1 \otimes \sqrt{(q')^*})\chi(1)i[H(1)(\xi), 1 \otimes a]\chi(1)(1 \otimes \sqrt{(q')^*}) + O(t^{-2}). \tag{1.2}
\]

Proof. We write \( q := q^2 \) and set in Proposition [F.2] \( b_1 = 0 \) and \( b_2 = q' \). Clearly, \( b_2 \) is admissible. By conjugating formula [F.6] with the unitary \( U \), we obtain
\[
[H^{ex}(\xi), d\Gamma(b)] = -\nabla\Omega(\xi - d\Gamma(1)(k)) \cdot d\Gamma^{ex}([k, b]^\circ) + d\Gamma^{ex}([\omega, b]^\circ) + O(t^{-2})(H_0^{ex}(\xi) + 1)^4, \tag{1.3}
\]
as a form identity on \( D(H_0^{ex}(\xi))^4 \).

Now we apply to both sides of this equality the projection \( P_1 \) on the subspace \( \mathcal{F} \otimes \mathcal{F}^{(1)} \subset \mathcal{F}^{ex} \) and multiply by the operators \( \chi(1) \). We get
\[
\chi(1)i[H(1)(\xi), 1 \otimes q]\chi(1) = \chi(1)(-\nabla\Omega(\xi - d\Gamma(1)(k)) \cdot (1 \otimes i[k, q]^\circ) + 1 \otimes i[\omega, q]^\circ)\chi(1) + O(t^{-2}). \tag{1.4}
\]
Here we used Corollary [E.4] with \( n = 4 \). Now we obtain from Lemma C.4 that
\[
i[k, q]^\circ = \frac{1}{t}i[k, a]^\circ q + O(t^{-2}) \quad \text{and} \quad i[\omega, q]^\circ = \frac{1}{t}i[\omega, a]^\circ q + O(t^{-2}), \tag{1.5}
\]
where \( i[k, a]^\circ = v \) and \( i[\omega, a]^\circ = \nabla\omega \cdot v \). Thus we get from relation (1.4) that
\[
\chi(1)i[H(1)(\xi), 1 \otimes q]\chi(1) = \frac{1}{t}\chi(1)(-\nabla\Omega(\xi - d\Gamma(1)(k)) \cdot (1 \otimes v) + 1 \otimes \nabla\omega \cdot v)(1 \otimes q')\chi(1) + O(t^{-2}). \tag{1.6}
\]
We choose \( \hat{\chi} \in C_0^\infty(\mathbb{R}) \) s.t. \( \chi\hat{\chi} = \chi \) and set
\[
C := \hat{\chi}(1)(-\nabla\Omega(\xi - d\Gamma(1)(k)) \cdot (1 \otimes v) + 1 \otimes \nabla\omega \cdot v). \tag{1.7}
\]
It is clear that \( C \) is bounded. The property \( [C, 1 \otimes p] = O(t^{-1}) \) follows from Lemmas C.4, F.1 and G.3. This concludes the proof of the first part of the proposition.

Proceeding to the proof of (1.2), we note that by Lemma C.4
\[
[v, \sqrt{q}] = O(t^{-1}) \quad \text{and} \quad [\nabla\omega \cdot v, \sqrt{q}] = O(t^{-1}). \tag{1.8}
\]
There also holds by Lemma [F.1] (after conjugating it with \( U \) and applying the projection \( P_1 \))
\[
\chi(1)[\nabla\Omega(\xi - d\Gamma(1)(k)), 1 \otimes \sqrt{q}] = O(t^{-1}). \tag{1.9}
\]
On the other hand, Lemma G.3 gives \( [1 \otimes \sqrt{q}, \chi(1)] = O(t^{-1}) \). Observing that
\[
i[H(1)(\xi), 1 \otimes q]^\circ = -\nabla\Omega(\xi - d\Gamma(1)(k)) \cdot (1 \otimes v) + 1 \otimes \nabla\omega \cdot v, \tag{1.10}
\]
we conclude (1.2) by symmetrizing (1.9), with the aid of (1.8) and (1.9). \( \square \)

Lemma I.2. Let \( a_i := \frac{1}{t}(v_i(k) \cdot i\nabla_k + i\nabla_k \cdot v_i(k)) \) for some \( v_i \in C_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n) \), for \( i \in \{1, 2\} \). Let \( \bar{a} = \text{diag}(a_1, a_2) \) be an operator on \( (a \text{ domain in} \) \( h \oplus h \). Then \( H(1)(\xi) \) is of class \( C^1(\text{d}\Gamma(\bar{a})) \) and
\[
[H(1)(\xi), \text{d}\Gamma(\bar{a})]^\circ \in B(D(NH_{1,0}(\xi)); \mathcal{H}) \subset B(D(H_{1,0}(\xi)^2); \mathcal{H}) \tag{1.11}
\]
In particular \( \chi(H(1)(\xi)) \in C^1(\text{d}\Gamma(\bar{a})) \), for any \( \chi \in C_0^\infty(\mathbb{R}) \).
Proof. From Prop. 2.8], and a conjugation by the unitary $U$, we learn that $H_1(\xi)$ is of class $C^1(d\Gamma(q))$. We furthermore find that

$$i[H_1(\xi), d\Gamma(a)]^0 = d\Gamma(v \cdot \nabla \omega) - d\Gamma(v) \cdot \nabla \Omega(\xi - d\Gamma(\xi)) - \phi(ia_1 G, 0), \quad (I.12)$$

where $v := \text{diag}(v_1, v_2)$ is a $\nu$-tuple of operators on $\mathfrak{h} \oplus \mathfrak{h}$ and the expression on the r.h.s. above $N$ is manifestly $NH_{1,0}(\xi)$-bounded. The remaining part of the lemma follows analogously as in the proof of Lemma F.3. \hfill \Box

Lemma I.3. Let $\chi \in C^\infty(\mathbb{R})$, $\tilde{q} \in C^\infty(\mathbb{R})$, $q \in C^\infty(\mathbb{R})$ be s.t. $q' \in C^\infty(\mathbb{R})$, $0 \leq q \leq 1$ and $q = 1$ in some neighbourhood of zero. Let $t \to b(t) = (a/t)\tilde{q}(a/t)$. Let $q^\dagger = (q', q^2)$ and $b(t) = (b(t), b(t))$ be propagation observables on $\mathfrak{h} \oplus \mathfrak{h}$. Then

$$[d\Gamma(q^\dagger, b), \chi(H_1(\xi))] = O(t^{-1}). \quad (I.13)$$

Proof. We set $q := q^\dagger$ and $R_{1,0} = (1 + H_{0,1}(\xi))^{-1}$. We note that $b$ is admissible by Lemma C.3. Let us first estimate the commutator of $d\Gamma(q^\dagger, b)$ with $H_1(\xi)$. As for the first term from the free auxiliary Hamiltonian, cf. (E.19), Lemma F.1 gives

$$[\Omega(\xi - d\Gamma(\xi)), d\Gamma(q, b)]R_{1,0}^4 = O(t^{-1}). \quad (I.14)$$

Concerning the second term from the Hamiltonian, we obtain from Lemma A.2

$$[d\Gamma(\omega), d\Gamma(q, b)]R_{1,0}^4 = (d\Gamma(q, [\omega, q]^0, b) + d\Gamma(q, [\omega, b]^0))R_{1,0}^4 = O(t^{-1}). \quad (I.15)$$

The interaction term from the Hamiltonian gives

$$[\phi(G, 0), d\Gamma(q, b)]R_{1,0}^4 = (a^*((1 - q)G, 0)d\Gamma(q, b) - a^*(bG, 0)\Gamma(q) + \Gamma(q)a(b^*G, 0) + d\Gamma(q, b)a((q - 1)G, 0))R_{1,0}^4 = O(t^{-1}), \quad (I.16)$$

where we made use of Lemma A.1. In the last step we exploited the fact that $\|(1 - q)G\|_2 \leq C/t^2$, since $1 - q$ is regular, and the bound $\|bG\|_2 = \frac{1}{t}\|qaG\|_2 \leq c/t$, which follows from the fact that $G$ is in the domain of $a$. Thus we have shown that

$$[H_1(\xi), d\Gamma(q, b)]R_{1,0}^4 = O(t^{-1}). \quad (I.17)$$

Now one concludes the proof using the method of almost analytic extensions as in the proof of Lemma F.3. \hfill \Box

J Auxiliary results for the proof of Theorem 5.3

In the present appendix we ask the reader to keep Corollary F.11 in mind. It ensures that the statements of results and manipulations in proofs are meaningful.

Lemma J.1. Let $\chi \in C^\infty(\mathbb{R})$, $j_0, j_\infty$ be as specified in Definition 3.1 and s.t. $j_0^2 + j_\infty^2 = 1$, and let $q = (q_0, q_\infty) := (j_0^2, j_\infty^2)$. Then

$$\chi^\text{ex}(H^\text{ex}(\xi)\Gamma(q^\dagger) - \Gamma(q^\dagger)H(\xi))\chi = 2\chi^\text{ex}[H^\text{ex}(\xi), \Gamma^\text{ex}(q^\dagger)]\Gamma(q^\dagger)\chi + O(t^{-2}), \quad (J.1)$$

where $j^\dagger := \text{diag}(j_0^\dagger, j_\infty^\dagger)$ is a propagation observable on $\mathfrak{h} \oplus \mathfrak{h}$ and we set $\chi := \chi(H(\xi))$ and $\chi^\text{ex} := \chi(H^\text{ex}(\xi))$. 

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Proof. We set \( q := q^t, j := j^t \). We note that, by Proposition F.10,

\[
\chi^\text{ex}(H^\text{ex}(\xi)\bar{\Gamma}(q) - \bar{\Gamma}(q)H(\xi))\chi = \chi^\text{ex}\left(\nabla\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\bar{\Gamma}(q, [k, q]^\circ) + d\Gamma(q, [\omega, q]^\circ)\right)\chi + O(t^{-2}),
\]

where \([\omega, q]^\circ\) is the extension by continuity of the form \( \omega q - q \omega \), a priori defined on \((D(\omega) \oplus D(\omega)) \times D(\omega)\). The same remark goes for \([k, q]^\circ\). On the other hand, Proposition F.8 gives

\[
\chi^\text{ex}[H^\text{ex}(\xi), \Gamma^\text{ex}(j)]\bar{\Gamma}(j)\chi = \chi^\text{ex}\left(\nabla\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\Gamma^\text{ex}(j, [k, j]^\circ) + d\Gamma^\text{ex}(j, [\omega, j]^\circ)\right)\bar{\Gamma}(j)\chi + O(t^{-2}),
\]

where we made use of Lemma F.10 to observe \( \bar{\Gamma}(j)\chi = \chi^\text{ex}\bar{\Gamma}(j) + O(t^{-1}) \).

In view of (J.2) and (J.3), to complete the proof of the lemma, it suffices to show that

\[
\chi^\text{ex}\nabla\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\bar{\Gamma}(q, [k, q]^\circ)\chi = 2\chi^\text{ex}\nabla\Omega(\xi - d\Gamma^\text{ex}(k)) \cdot d\Gamma^\text{ex}(j, [k, j]^\circ)\bar{\Gamma}(j)\chi + O(t^{-2})
\]

and

\[
\chi^\text{ex}d\bar{\Gamma}(q, [\omega, q]^\circ)\chi = 2\chi^\text{ex}d\Gamma^\text{ex}(j, [\omega, j]^\circ)\bar{\Gamma}(j)\chi + O(t^{-2}).
\]

Both relations are a consequence of the following fact: Let \( g \in C^\infty(\mathbb{R}) \) be s.t. all its derivatives of non-zero order are bounded. Let \( \varphi := \text{diag}(g, g) \) be the corresponding operator on \( \mathfrak{h} \oplus \mathfrak{h} \). Then

\[2d\Gamma^\text{ex}(j, [g, j]^\circ)\bar{\Gamma}(j)\chi = d\bar{\Gamma}(j, 2[g, j]^\circ)\chi = d\bar{\Gamma}(q, [g, q]^\circ)\chi - d\bar{\Gamma}(q, [g, j]^\circ)\chi = d\bar{\Gamma}(q, [g, q]^\circ)\chi + O(t^{-2}),\]

where in the last step we made use of Lemma C.3. This concludes the proof.

\[\Box\]

Lemma J.2. Let \( \chi \in C^\infty_0(\mathbb{R})_\mathbb{R} \) and let \( j_0 \) be as specified in Definition J.1. Then there holds

\[\chi[\Gamma(j_0), [H(\xi), \Gamma(j_0)]]\chi = O(t^{-2}),\]

where we set \( \chi := \chi(H(\xi)) \).

Proof. We set \( j_0 := j_0^t \) and recall that, by Proposition F.5

\[
[H(\xi), \Gamma(j_0)] = \left(-\nabla\Omega(\xi - d\Gamma(k)) \cdot d\Gamma(j_0, [k, j_0]^\circ) + d\Gamma(j_0, [\omega, j_0]^\circ)\right) + O(t^{-2})(H_{1,0}(\xi) + 1)^3,
\]

in the sense of forms on \( D(H_{1,0}(\xi)^3) \). In view of this relation and the fact that \( \Gamma(j_0)\chi = \chi\Gamma(j_0) + O(t^{-1}) \) (Lemma F.6), it is enough to check that

\[\chi[\Gamma(j_0), \nabla\Omega(\xi - d\Gamma(k)) \cdot d\Gamma(j_0, [k, j_0]^\circ)]\chi = O(t^{-2})\]

and

\[\chi[\Gamma(j_0), d\Gamma(j_0, [\omega, j_0]^\circ)]\chi = O(t^{-2}).\]

Relation (J.10) follows immediately from formula (A.5) which gives

\[\chi[\Gamma(j_0), d\Gamma(j_0, [\omega, j_0]^\circ)]\chi = \chi d\Gamma(j_0^2, [j_0, [\omega, j_0]^\circ])\chi = O(t^{-2}),\]

\[\Box\]
where in the last step we applied Lemma C.4. Formula (J.9) follows from

\[ \left[ \Gamma(j_0), d\Gamma(j_0, [k, j_0]) \right] \chi = O(t^{-2}), \quad (J.12) \]

which is justified as (J.11), from the fact that \( d\Gamma(j_0, [k, j_0]) \chi = O(t^{-1}) \) and from Lemma F.4, which gives

\[ \chi[\Gamma(j_0), \nabla \Omega(\xi - d\Gamma(k))] = O(t^{-1}). \quad (J.13) \]

This concludes the proof.

\[ \square \]

### K Negative spectrum of the conjugate operator

**Lemma K.1.** Let \( \chi \in C^\infty(\mathbb{R}) \) and \( \Psi \in D(d\Gamma(a)) \). There exists a constant \( c > 0 \) such that the following holds true: For any pair of functions \( q, q_R \in C^\infty(\mathbb{R}) \), with \( 0 \leq q, q_R \leq 1 \), \( \text{supp}q \subset (-\infty, \varepsilon) \) for some \( \varepsilon > 0 \), \( q(s) = q_R(s) \) for \( s > -R \) and \( q_R(s) = 0 \) for \( s < -R - 1 \); we have

\[ \sup_{t \geq 1} \| (\Gamma(q(t)) - \Gamma(q'_R(t))) e^{-itH(\xi)} \chi(H(\xi)) \Psi \| \leq \frac{c}{R}. \quad (K.1) \]

**Proof.** Let us denote by \( 1_{\{A \leq -R \}} \) the spectral projection of a self-adjoint operator \( A \) on the interval \(( -\infty, -R \)\). We set \( q := q' \) and \( q_R := q'_R \) and recall that \( \Psi \in D(d\Gamma(a)) \). As a consequence \( \chi(H(\xi)) \Psi \in D(d\Gamma(a)) \) for any \( \chi \in C^\infty(\mathbb{R}) \), since \( H(\xi) \) is of class \( C^1(d\Gamma(a)) \) [38, Proposition 2.5].

Making use of the subsequent Lemma K.2 and abbreviating \( \chi = \chi(H(\xi)) \), we obtain

\[ \| (\Gamma(q(a/t)) - \Gamma(q_R(a/t))) e^{-itH(\xi)} \chi \Psi \| = \| 1_{\{d\Gamma(a/t) \leq -R + \epsilon N \}} (\Gamma(q) - \Gamma(q_R)) e^{-itH(\xi)} \chi \Psi \| \]

\[ = \| 1_{\{d\Gamma(a/t) \leq -R + \epsilon N \}} \| \Gamma(q) - \Gamma(q_R) \| e^{-itH(\xi)} \chi \Psi \| \]

\[ + \| 1_{\{d\Gamma(a/t) \leq -R + \epsilon N \}} \| \Gamma(q) - \Gamma(q_R) \| e^{-itH(\xi)} \chi \Psi \|. \quad (K.2) \]

The first term on the r.h.s. above can be estimated by

\[ \| 1_{\{d\Gamma(a/t) \leq -R(1-\epsilon) \}} (\Gamma(q) - \Gamma(q_R)) e^{-itH(\xi)} \chi \Psi \| \]

\[ = \| 1_{\{d\Gamma(a/t) \leq -R(1-\epsilon) \}} d\Gamma(a/t)^{-1} (\Gamma(q) - \Gamma(q_R)) d\Gamma(a/t) e^{-itH(\xi)} \chi \Psi \| \]

\[ \leq \frac{2}{(1 - \epsilon) R t} \| e^{-itH(\xi)} \chi \Psi \|. \quad (K.3) \]

To estimate the expression on the r.h.s. of (K.3), we proceed similarly as in the proof of [21, Lemma 44]:

\[ \| e^{itH(\xi)} d\Gamma(a) e^{-itH(\xi)} \chi \Psi \| \leq \int_0^t dt' \| e^{it'H(\xi)} \chi \Psi \| \]

\[ + \int_0^t dt' \| d\Gamma(a)^{t'} e^{-it'H(\xi)} \chi \Psi \| + \| d\Gamma(a) \chi \Psi \| \]

\[ \leq \epsilon' \| \chi \Psi \| + \| d\Gamma(a) \chi \Psi \|. \quad (K.4) \]

where we made use of the fact that \( \epsilon' := \| [H(\xi), d\Gamma(a)]^\circ \chi \| < \infty \), by Lemma I.2. Thus we obtain that

\[ \| 1_{\{d\Gamma(a/t) \leq -R(1-\epsilon) \}} (\Gamma(q) - \Gamma(q_R)) e^{-itH(\xi)} \chi \Psi \| \leq \frac{c'}{R} \| \chi \Psi \| + \| \chi \Psi \|, \quad (K.5) \]

\[ \leq \epsilon'' \| \chi \Psi \|. \quad (K.6) \]
where the constant $c''$ does not depend on the choice of $t$, $q$ and $q_R$. As for the second term on the r.h.s. of (K.2), it is bounded by

$$2\|1_{\{N \geq R\}} \chi \Psi\| \leq 2\|1_{\{N \geq R\}} (1 + N)^{-1} \| (1 + N) \chi \Psi\| \leq \frac{c''}{R} \|\Psi\|,$$

(K.6)

where $c'' := 2\| (N + 1) \chi (H(\xi))\| < \infty$. Altogether, we get that

$$\| (\Gamma(q(a/t)) - \Gamma(q_R(a/t))) e^{-itH(\xi)} \chi \Psi\| \leq \frac{c}{R} \|\Psi\| + \frac{c}{Rt} \|\Gamma(a) \chi \Psi\|,$$

(K.7)

where $c$ is independent of $t$, $q$ and $q_R$. This concludes the proof. □

**Lemma K.2.** Let $q, q_R \in C^\infty(\mathbb{R})$ be s.t. $0 \leq q, q_R \leq 1$, supp $q \subset (-\infty, \varepsilon)$ for some $\varepsilon > 0$, $q(s) = q_R(s)$ for $s > -R$ and $q_R(s) = 0$ for $s < -R - 1$. Then, for $\Psi \in \mathcal{H}$,

$$(\Gamma(q(a/t)) - \Gamma(q_R(a/t))) \Psi = 1_{\{d\Gamma(a/t) \leq -R + \varepsilon N\}} (\Gamma(q(a/t)) - \Gamma(q_R(a/t))) \Psi.$$  \hspace{1cm} (K.8)

**Proof.** As all the operators involved commute with the number operator, it is enough to consider the problem in some $n$-particle subspace. We embed $\mathcal{F}^{(n)}$ into the non-symmetrized tensor product of single-particle spaces $\mathfrak{h}^{\otimes n}$. We note that $a \otimes 1 \otimes \cdots \otimes 1, 1 \otimes a \otimes \cdots \otimes 1, \ldots, 1 \otimes \cdots \otimes 1 \otimes a$ is a family of $n$ commuting operators on $\mathfrak{h}^{\otimes n}$. We denote their joint spectral projection-valued measure by $F$. Thus the $n$-particle component of the vector on the l.h.s. of (K.8) is a sum of terms of the form

$$\int q(a_1/t) \ldots q(a_{i-1}/t) (q(a_i/t) - q_R(a_i/t)) q_R(a_{i+1}/t) \ldots q_R(a_n/t) dF(a) \Psi_n,$$

(K.9)

where $\Psi_n$ is the $n$-particle component of $\Psi$. Now, by the assumed properties of $q$ and $q_R$, we obtain that the above expression is equal to

$$\int 1((a_1/t + \cdots + a_n/t) \leq -R + \varepsilon n)$$

$$\times q(a_1/t) \ldots q(a_{i-1}/t) (q(a_i/t) - q_R(a_i/t)) q_R(a_{i+1}/t) \ldots q_R(a_n/t) dF(a) \Psi_n.$$

(K.10)

This proves (K.8) on $\mathfrak{h}^{\otimes n}$. Since both sides of (K.8) leave $\mathcal{F}^{(n)}$ invariant, this completes the proof. □

**L Structure of the isolated spectrum**

We begin by recalling some analytic perturbation theory for isolated eigenvalues following Kato. Suppose that $D \subset \mathbb{C}$ is an open set which intersects with the real line and $D \ni \kappa \mapsto T(\kappa)$ is a holomorphic family of Type A in the sense of Kato. We assume that $T(\kappa)$ is a self-adjoint operator when $\kappa \in D \cap \mathbb{R}$. Suppose $\lambda_0 \in \mathbb{R}$ is an isolated eigenvalue of the self-adjoint operator $T(\kappa_0)$, with $\kappa_0 \in \mathbb{R} \cap D$. Denote by $n_0$ its multiplicity, which we assume to be finite. Let $e > 0$ be such that $\sigma(T(\kappa_0)) \cap J_{2e} = \{\lambda_0\}$, where $J_e := [\lambda_0 - e, \lambda_0 + e]$.

Abbreviate $\sigma_e(\kappa) := \sigma(T(\kappa)) \cap B_e(\lambda_0)$. There exists $r > 0$ such that $B_r(\kappa_0) \subset D$ and for all $\kappa \in B_r(\kappa_0)$ we have $\sigma_{2e}(\kappa) = \sigma_e(\kappa)$. Such an $r$ exists because the set $\{ (\kappa, \lambda) | \lambda \in \sigma(T(\kappa)) \}$ is a closed subset of $D \times \mathbb{C}$.

Denote by $C$ the circle in $\mathbb{C}$ encircling $\lambda_0$ with radius $3e/2$. Then $\sigma_e(\kappa)$ is enclosed by the circle for all $\kappa \in B_r(\kappa_0)$, and accounts for all the spectrum of $T(\kappa)$ inside (or on) the circle. We can thus compute the Riesz projection:

$$P(\kappa) = \frac{1}{2\pi i} \oint_C dz \frac{1}{z - T(\kappa)}.$$
For real $\kappa$ the bounded operator $P(\kappa)$ is the spectral projection onto the spectral subspace pertaining to the spectrum of $T(\kappa)$ inside the cluster $\sigma_e(\kappa)$. In particular $P(\kappa_0) = P_{\lambda_0}(\kappa_0)$, the orthogonal projection onto the $n_0$-dimensional eigenspace of $T(\kappa_0)$, pertaining to the eigenvalue $\lambda_0$. Due to norm-continuity of $\kappa \to P(\kappa)$ we conclude that the set $\sigma_e(\kappa)$ has cardinality at most $n_0$, corresponding to eigenvalues with (algebraic) multiplicities summing up to $n_0$.

Denote by $v_0^1, \ldots, v_0^{n_0}$ an ONB for the range of $P_{\lambda_0}(\kappa_0)$. Then, possibly choosing $r$ smaller, we may assume that $v_j(\kappa) = P(\kappa)v_j^0$ forms a linearly independent analytic set of vectors spanning $\text{Ran}(P(\kappa))$. Using the Gram-Schmidt procedure we can pass to an analytic ONB $u_1(\kappa), \ldots, u_{n_0}(\kappa)$ for $\text{Ran}(P(\kappa))$. Such a basis defines an analytic family of unitary maps $\Pi_\kappa \in \mathbb{C}^{n_0}$, defining $\Pi_\kappa(u_j(\kappa)) = e_j$, the $j$th standard basis vector. We can now construct an analytic family of $n_0 \times n_0$ matrices $A(\kappa) = \Pi_\kappa T(\kappa) \Pi_\kappa^*$. By construction $A(\kappa)$ is self-adjoint for $\kappa \in B_r(\kappa_0) \cap \mathbb{R}$ and $\sigma(A(\kappa)) = \sigma_e(\kappa)$.

By a result of Kato [33, Theorem 6.1], we can identify a number $m_0 \leq n_0$ of real analytic functions $\mu_j : B_r(\kappa_0) \cap \mathbb{R} \to \mathbb{R}$, such that $\sigma_e(\kappa) = \{\mu_1(\kappa), \ldots, \mu_{m_0}(\kappa)\}$. They all coincide with $\lambda_0$ if $\kappa = \kappa_0$ and are otherwise distinct.

The above discussion implies the following result on analytic continuation of shells through crossings.

**Proposition L.1.** Let $A_1$ be a level crossing, which is a sphere of radius $R > 0$. Let $(A_{m_0}^+, S_m)$, $m \in J^-$ and $(A_{m_0}^+, S_m^0)$, $n \in J^+$ be shells approaching this crossing from the inside and outside, respectively. Then, (after suitable identification of the index sets $J_{\pm} =: J$) one can find analytic functions

$$A_n^+ \cup A_1 \cup A_n^- \ni \xi \to S_n(\xi), \quad (L.1)$$

such that $S_n(\xi) = S_n^+(\xi)$, $\xi \in A_n^+$.

*Proof.* Put $T(\kappa) = H(\kappa, 0, \ldots, 0)$, where we exploit the rotation invariance of the spectrum to conclude the proposition from the preceding discussion. \qed

Let $A(r; R) := \{\xi \in \mathbb{R}^n \mid r < |\xi| < R\}$ for some $0 \leq r < R \leq \infty$. Keeping in mind the possibility that the inner or outer boundary of a shell is a subset of the essential spectrum, we obtain from the above proposition that $\Sigma_{iso} \setminus \{0 \times \mathbb{R}\}$ is a union of graphs of an at most countable family of rotation invariant analytic functions

$$A(r_n; R_n) \ni \xi \to S_n(\xi), \quad (L.2)$$

where $n \in J$. (The zero total momentum fiber has been cut out since one may in principle have shells like graphs of the two functions $\xi \to (\xi \pm \xi_0)^2$ crossing analytically at $\xi = 0$ but not naturally occurring as a single-valued rotation invariant function.) These considerations enable a splitting of the isolated bound states $\mathcal{H}_{iso} = E(\Sigma_{iso}) \mathcal{H}$ into dressed electron subspaces:

$$\mathcal{H}_{iso} = \bigoplus_n \mathcal{H}_{iso, n}, \quad \text{where} \quad \mathcal{H}_{iso, n} = \mathcal{H}_{iso, n}, \quad (L.3)$$

$$\tilde{\mathcal{H}}_{iso, n} = \left\{I_{\text{LLP}}^{\vee} \int d\xi \Psi_\xi \mid \Psi \in C_0^0(A(r_n; R_n); \mathcal{F}), \ H(\xi)\Psi_\xi = S_n(\xi)\Psi_\xi \right\}, \quad (L.4)$$

where by $\Psi \in C_0^0(A(r_n; R_n); \mathcal{F})$ it is understood that $\xi \to \Psi_\xi \in \mathcal{F}$ is a continuous function, compactly supported in $A(r_n; R_n)$.

After this preparation we state and prove the following corollary of Proposition L.1.
**Corollary L.2.** Let \( \omega \) be the boson dispersion relation. Then
\[
S_n^{(1)}(\xi; k) := S_n(\xi - k) + \omega(k),
\]
defined for \( k \in \xi - \mathcal{A}(r_n; R_n) \), is a constant function at most for \( \xi \) from some countable set. 

*Proof.* Let us first assume that \( \omega \) is not a constant function. Suppose that
\[
\xi - \mathcal{A}(r_n; R_n) \ni k \rightarrow S_n^{(1)}(\xi; k)
\]
is constant for \( \xi = \xi_0 \) and \( \xi = \xi_0 + k' \) for some \( k' \neq 0 \). (For \( \nu > 1 \) it is enough to assume that there is one such \( k' \) to arrive at a contradiction. For \( \nu = 1 \) we assume that there are uncountably many). Then
\[
\begin{align*}
k \in \xi_0 - \mathcal{A}(r_n; R_n) : & \quad S_n(\xi_0 - k) + \omega(k) = c_{\xi_0}, \\
k \in \xi_0 + k' - \mathcal{A}(r_n; R_n) : & \quad S_n(\xi_0 - k + k') + \omega(k) = c_{\xi_0 + k'}.
\end{align*}
\]
But the latter condition means that \( k - k' \in \xi_0 - \mathcal{A}(r_n; R_n) \), so we can substitute it into the first equality, obtaining the equations
\[
\begin{align*}
k \in \xi_0 + k' - \mathcal{A}(r_n; R_n) : & \quad S_n(\xi_0 - k + k') + \omega(k - k') = c_{\xi_0}, \\
& \quad S_n(\xi_0 - k + k') + \omega(k) = c_{\xi_0 + k'}.
\end{align*}
\]
Consequently,
\[
\omega(k) - \omega(k - k') = c_{\xi_0 + k'} - c_{\xi_0}.
\]
Since this equality holds on an open set, it extends to all \( k \in \mathbb{R}^\nu \) by analyticity. Now let us assume that \( \nu > 1 \). Then, making use of rotation invariance of \( \omega \), we obtain for any \( O \in \text{O}(\nu) \)
\[
\omega(k) - \omega(k - Ok') = c_{\xi_0 + k'} - c_{\xi_0}.
\]
By differentiating this relation w.r.t. one-parameter families of rotations, we obtain \( \nabla \omega(k) \cdot Lk' = 0 \), for any element \( L \) of the Lie algebra of the group of rotations. Recalling that such \( L \) are antisymmetric matrices and choosing coordinates so that \( k' = (c, 0, \ldots, 0) \), we obtain that \( \partial_i \omega(k) = 0 \) for all \( 2 \leq i \leq \nu \). By rotation invariance, this is only possible if \( \omega \) is constant, which is a contradiction.

Let us now go back to formula (L.9) and assume that \( \nu = 1 \). By differentiating this relation w.r.t. \( k \), we obtain that
\[
\nabla \omega(k) = \nabla \omega(k - k')
\]
i.e. \( \nabla \omega \) is a continuous function which has uncountably many periods \( k' \). But this is only possible if \( \nabla \omega \) is a constant function \([3]\). This implies that \( \omega(k) = c_1 k + c_2 \). We note that \( c_1 = 0 \) by reflection invariance. Thus we obtain again that \( \omega \) is a constant function contradicting our assumption.

Finally, let us suppose that \( \omega \) is a constant function. Then \( S_n^{(1)}(\xi; k) = S_n(\xi - k) + \omega(k) \) can only be constant if \( S_n \) is constant. But this is excluded by the following property
\[
\lim_{|\xi| \rightarrow \infty} (\Sigma_0^{(1)}(\xi) - \Sigma_0(\xi)) = 0,
\]
proven in \([37] \) Theorem 2.4], and the fact that for a constant dispersion relation
\[
\Sigma_0^{(1)}(\xi) = \inf_{k \in \mathbb{R}^\nu} (\Sigma_0(\xi - k) + \omega(k)) = \inf_{k \in \mathbb{R}^\nu} \Sigma_0(k) + m = \text{const}.
\]
In the above reasoning we made use of Proposition L.1 to show that any shell \((\mathcal{A}, S)\) s.t. \( S \) is constant extends to a constant shell \( S_n \) on \( \mathcal{A}(0, \infty) \).
\[\Box\]
M Structure of the spectrum of the extended Hamiltonian

For a Borel set $O \subset \mathbb{R} \times \mathbb{R}^{v'}$ we recall the notion of $O$-compatibility from Subsection 2.4. A state $\Psi \in \mathcal{H}_{\text{bnd}}$ and a boson wave packet $h \in \mathfrak{h}$ are called $O$-compatible if there exists a Borel subset $S \subset \mathbb{R}^{v'+1}$ such that: $\Psi \in E(S)\mathcal{H}$ and for any $k$ in the essential support of $h$ and $(\xi, \mu) \in S$, we have $(\xi + k, \mu + \omega(k)) \in O$. As shown in Lemma M.1 below, this property ensures that the simple tensor $\Psi \otimes a^*(h)|0\rangle$ is an element of $E^{(1)}(O)(\mathcal{H}_{\text{bnd}} \otimes \mathfrak{h})$.

Recall that $E^{(1)}$ denotes the joint spectral resolution for the pair $P^{(1)}, H^{(1)}$, cf. [22], as well as the notation $\mathcal{H}_{\text{iso}} = E(S_{\text{iso}})\mathcal{H}$ for the subspace of $\mathcal{H}_{\text{bnd}}$, consisting of isolated bound states [24]. Finally, we remind the reader of the notation $\mathcal{R} \subset \mathbb{R}^{v'+1}$ for the set of points $(\xi, \lambda)$, with $\lambda < \Sigma^{(1)}(\xi)$, i.e. the energy-momentum set below the two-boson threshold. For the purpose of this appendix we write $\mathcal{C}(O) \subset \mathcal{H}_{\text{bnd}} \otimes \mathfrak{h}$ for the set of $O$-compatible pairs $(\Psi, h)$. The following lemma characterizes the incoming and outgoing states below the two-boson threshold. It is similar to [21] Lemma 30.

**Lemma M.1.** Let $O \subset \mathcal{R}$ be an open set. Then

\[ E^{(1)}(O)(\mathcal{H}_{\text{iso}} \otimes \mathfrak{h}) = \text{Span}\{ \Psi \otimes a^*(h)|0\rangle \mid (\Psi, h) \in \mathcal{C}(O) \}, \quad (M.2) \]

\[ E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h}) \subset \mathcal{H}_{\text{iso}} \otimes \mathfrak{h}. \quad (M.3) \]

**Proof.** Let $1_O$ be the characteristic function of $O$. Making use of the decomposition [22], we compute

\[ 1_O(P^{ex}, H^{ex}) = 1_O(P, H) \oplus \bigoplus_{\ell=1}^{\infty} 1_O(H^{(\ell)}) \]. \quad (M.4) \]

Since $O$ is located below the 2-boson threshold $\Sigma^{(2)}$, the contributions from asymptotic particle sectors, with $\ell \geq 2$, are zero. The range of the 0'th summand is $E(O)\mathcal{H}$ and the range of the 1'st summand is $E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h})$. We are thus reduced to establishing the identity \text{(M.2)} and the inclusion \text{(M.3)}.

Abbreviate

\[ V := \text{Span}\{ \Psi \otimes a^*(h)|0\rangle \mid (\Psi, h) \in \mathcal{C}(O) \}. \quad (M.5) \]

Clearly, $V \subset \mathcal{H}_{\text{iso}} \otimes \mathfrak{h}$. In order to prove \text{(M.2)} we need to verify $E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h}) = V$.

In the following we will make repeated use of the direct integral representation

\[ I_{\text{LLP}}^{(1)}(\Psi \otimes a^*(h)|0\rangle) = \int \int d^\infty \xi d^\infty k h(k)\Psi_{\xi-k} \]. \quad (M.6) \]

for simple tensors, with $\Psi \in \mathcal{H}$ and $h \in \mathfrak{h}$. This decomposition is the same as the one in Subsection 2.2; cf. (2.24), (2.27) and (2.28). If $\Psi \in \mathcal{H}_{\text{iso}}, n$, cf. [L.4], we can in particular compute:

\[ I_{\text{LLP}}^{(1)}E^{(1)}(\Psi \otimes a^*(h)|0\rangle) = \int \int d^\infty \xi d^\infty k h(k)1_O(\xi, H^{(1)}(\xi;k))\Psi_{\xi-k} \]

\[ = \int d^\infty \xi \int d^\infty k h(k)1_O(\xi, S_n(\xi-k) + \omega(k))\Psi_{\xi-k}. \quad (M.7) \]

If $\Psi$ and $h$ are $O$-compatible we see that for $k \in \text{supp} h$ and $\xi$ such that $\Psi_{\xi-k} \neq 0$, we must have $(\xi, S_n(\xi - k) + \omega(k)) \in O$ and hence, by \text{(L.3)} and a density argument, we have established that $V \subset E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h})$. 

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We proceed to show that $E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h}) \subset \mathcal{H}_{\text{iso}} \otimes \mathfrak{h}$. For this it suffices to argue that for any Borel set $U \subset \{ (\xi, \lambda) \mid \lambda \geq \Sigma^{(1)}_0(\xi) \}$ and state $\varphi = E^{(1)}(O)(\Psi \otimes a^*(h)|0\rangle)$, with $\Psi \in \mathcal{H}$ and $h \in \mathfrak{h}$, we must have $(E(U) \otimes 1)\varphi = 0$. For this we compute

$$(E(U) \otimes 1)\varphi = (E(U) \otimes 1)I_{\text{LLP}}^{(1)*} \int \, d\xi \int \, dk \, h(k)1_O(\xi, H^{(1)}(\xi; k))\Psi_{\xi-k}$$

$$= I_{\text{LLP}}^{(1)*} \int \, d\xi \int \, dk \, h(k)1_U(\xi-k, H(\xi-k))1_O(\xi, H(\xi-k) + \omega(k))\Psi_{\xi-k}. \quad (M.8)$$

For a point $(\xi-k, \mu)$ to be in $U$ we must have $\mu \geq \Sigma^{(1)}_0(\xi-k)$. Hence, $\mu + \omega(k) \geq \Sigma^{(1)}_0(\xi-k) + \omega(k) \geq \Sigma^{(2)}(\xi)$. Conversely, for a point $(\xi, \mu + \omega(k))$ to be in $O$, we must have $\mu + \omega(k) < \Sigma^{(2)}(\xi)$. Since these two situations cannot occur simultaneously we conclude that $1_U(\xi-k, H(\xi-k))1_O(\xi, H(\xi-k) + \omega(k)) = 0$. This concludes the proof of $(M.3)$.

Consider a state of the form $\chi(P^{(1)}, H^{(1)})(\Psi \otimes a^*(h)|0\rangle)$, $\chi \in C^\infty_0(O)_{\mathbb{R}}$, $\Psi \in \mathcal{H}_{\text{iso}, n}$, and $h \in \mathfrak{h}$ with compact essential support. To conclude the proof it suffices to show that such states can be approximated by elements from $V$. Note that by $(L.3)$, the spectral theorem, and the inclusion $E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h}) \subset \mathcal{H}_{\text{iso}} \otimes \mathfrak{h}$ just proved: any state in $E^{(1)}(O)(\mathcal{H} \otimes \mathfrak{h})$ can be approximated using states of the considered form.

Let $r := d(\text{supp} \chi, \mathbb{R}^{n+1}|O) > 0$. Let $\epsilon > 0$ be given. We may assume $2\epsilon < r$. Using that $\chi$ is uniformly continuous we get a $\delta'$, such that $|\chi(\xi', \mu') - \chi(\xi'', \mu'')| < \epsilon$, for $(\xi', \mu')$, $(\xi'', \mu'')$ with $|\mu' - \mu''| < \delta'$ and $|\xi' - \xi''| < \delta'$. We may take $\delta' < \epsilon$. Let $R > 0$ be so large that $\text{supp} \, h \subset \{ k \in \mathbb{R}^n \mid |k| \leq R \}$. Using that $\omega$ is also uniformly continuous on the ball of radius $R$, we get a $0 < \delta < \delta'$ such that $|\omega(k') - \omega(k'')| < \delta'$ if $|k' - k''| < \delta$.

Cover $B_R(0)$ with finitely many pairwise disjoint Borel sets $B_\ell$ such that $B_\ell \subset B_\delta(k_\ell)$, $\ell = 1, \ldots, L$, for some collection of momenta $k_1, \ldots, k_L$. Write

$$I_{\text{LLP}}^{(1)}(P^{(1)}, H^{(1)})(\Psi \otimes a^*(h)|0\rangle) = \int \, d\xi \int \, dk \, h(k)\chi(\xi, S_n(\xi-k) + \omega(k))\Psi_{\xi-k}$$

$$= \sum_{\ell=1}^{L} \int \, d\xi \int \, dk \, h(k)1_{B_\ell}(k)\chi(\xi, S_n(\xi-k) + \omega(k))\Psi_{\xi-k}. \quad (M.9)$$

For $k \in B_\delta(k_\ell)$ we have $|\chi(\xi; S_n(\xi-k) + \omega(k)) - \chi(\xi-k + k_\ell; S_n(\xi-k) + \omega(k_\ell))| < \epsilon$. Define

$$\psi_\ell := (\chi_\ell(P, H)\Psi \otimes a^*(1_{B_\ell}h)|0\rangle), \quad (M.10)$$

with $\chi_\ell(\xi, \lambda) := \chi(\xi + k_\ell, \lambda + \omega(k_\ell))$. Then $K_\ell := \text{supp} \chi_\ell = \text{supp} \chi - (k_\ell, \omega(k_\ell))$. Note that $K_\ell \cap \Sigma \subset \Sigma_{\text{iso}}$, and hence; $\psi_\ell \in \mathcal{H}_{\text{iso}} \otimes \mathfrak{h}$.

Estimate

$$\left\| I_{\text{LLP}}^{(1)}\psi_\ell - \int \, d\xi \int \, dk \, h(k)1_{B_\ell}(k)\chi(\xi, S_n(\xi-k) + \omega(k))\Psi_{\xi-k} \right\|^2$$

$$= \int \, d\xi \int \, dk \, |1_{B_\ell}(k)||h(k)|^2 |\chi(\xi-k, S_n(\xi-k)) - \chi(\xi, S_n(\xi-k) + \omega(k))|^2 \|\Psi_{\xi-k}\|^2$$

$$\leq \epsilon^2 \|\Psi\|^2 \int \, d\xi \int \, dk \, |1_{B_\ell}(k)||h(k)|^2. \quad (M.11)$$

Due to the fact that $B_\ell \cap B_\ell' = \emptyset$, summing up over $\ell$ yields

$$\|\chi(P^{(1)}, H^{(1)})(\Psi \otimes a^*(h)|0\rangle) - \sum_{\ell=1}^{L} \psi_\ell\| \leq \epsilon \|h\|\|\Psi\|. \quad (M.12)$$
It remains to verify that \( \chi_\ell(P,H)\Psi \) and \( 1_B \cdot h \) are \( O \)-compatible, such that we in fact have \( \psi_\ell \in V \). Let \( k \in \mathcal{B}_\ell \subset B_\delta(\ell) \) and \( (\xi, \mu) \in K_\ell \). Then
\[
(\xi + k, \mu + \omega(k)) = (\xi + k_\ell, \mu + \omega(k_\ell)) + (k - k_\ell, \omega(k) - \omega(k_\ell)) \in \text{supp} \chi + (k - k_\ell, \omega(k) - \omega(k_\ell)).
\] (M.13)

By the choice of \( \delta \) we conclude that \( |(k - k_\ell, \omega(k) - \omega(k_\ell))| < \epsilon < r \) and hence we have \( (\xi + k, \mu + \omega(k)) \in O \). This means that \( \chi_\ell(P,H)\Psi \) and \( 1_B \cdot h \) are \( O \)-compatible, which concludes the proof. □

**Lemma M.2.** Let
\[
\Sigma^{(1)}_{pp} := \{ (\xi, \lambda) \in \mathbb{R}^{\nu+1} \mid \lambda \in \sigma_{pp}(H^{(1)}(\xi)) \}.
\] (M.14)

Then \( E^{(1)}(\Sigma^{(1)}_{pp} \cap \mathcal{E}^{(1)}) = 0 \), hence the set \( \{ \xi \in \mathbb{R}^{\nu} \mid \sigma_{pp}(H^{(1)}(\xi)) \cap \mathcal{E}^{(1)}(\xi) \neq \emptyset \} \) has zero Lebesgue measure.

**Proof.** Let us consider a vector \( \Psi \in E^{(1)}(O)(\mathcal{H} \otimes \mathcal{F}^{(1)}) \), where \( O \subset \mathcal{E}^{(1)} \) is some Borel subset. Let \( 1_O \) be the characteristic function of \( O \). Then, making use of the expansion (2.26), we can write
\[
\Psi = I^{(1)*}_{LLP} \int_\mathbb{R}^\nu d\xi \int_\mathbb{R} \, dk \, 1_O(\xi, H^{(1)}(\xi;k)) \Psi_{-k}.
\] (M.15)

Now suppose that \( \Psi \in E^{(1)}(\Sigma^{(1)}_{pp})(\mathcal{H} \otimes \mathcal{F}^{(1)}) \). We note that \( \Sigma^{(1)}_{pp}(\xi) \cap \mathcal{E}^{(1)}(\xi) \) can be at most countable due to the separability of \( \mathcal{F} \otimes \mathcal{F}^{(1)} \). Then, by [42, Théoréme 21], \( \Sigma^{(1)}_{pp} \cap \mathcal{E}^{(1)} \) is a countable union of graphs of Borel functions from Borel subsets of \( \mathbb{R}^\nu \) to \( \mathbb{R} \). Thus, without loss of generality, we can assume that there exists a Borel function \( p: N \to \mathbb{R} \), defined on a Borel set \( N \), s.t. \( \Psi \in 1_N(P^{(1)})(\mathcal{H} \otimes \mathcal{F}^{(1)}) \) and
\[
H^{(1)}\Psi = p(P^{(1)})\Psi.
\] (M.16)

Suppose, by contradiction, that \( \Psi \neq 0 \) and satisfies (M.16). Since \( 1_O \) is supported below the two-boson threshold, it is easy to see that
\[
\Psi_{\xi} \in E_{\xi}((-\infty, \Sigma^{(1)}_{pp}(\xi)))\mathcal{F},
\] (M.17)
where \( E_{\xi} \) is the spectral measure of \( H(\xi) \). Consequently, \( \Psi \in \mathcal{H}_{iso} \otimes h \). Hence, there exists a shell \( (\mathcal{A}, S) \) in \( \Sigma_{iso} \) s.t.
\[
\Psi' := (1_{\mathcal{G}_S}(P,H) \otimes 1)\Psi \neq 0,
\] (M.18)
where \( \mathcal{G}_S \) is the graph of \( S \). Since \( 1_{\mathcal{G}_S}(P,H) \otimes 1 \) commutes with \( H^{(1)}, P^{(1)} \), we obtain that \( \Psi' \) also satisfies (M.16). Thus we obtain
\[
\int_N d\xi \int dk \, (S(\xi - k) + \omega(k) - p(\xi))^2 \|\Psi'_{-k}\|^2 = 0.
\] (M.19)

Hence the set of \( \xi \) for which
\[
\int dk \, (S(\xi - k) + \omega(k) - p(\xi))^2 \|\Psi'_{-k}\|^2 \neq 0
\] (M.20)
has zero Lebesgue measure. Conversely, the set of \( \xi \) for which the real analytic function \( k \to S(\xi - k) + \omega(k) \) is constant also has zero Lebesgue measure by Corollary [L.2]. Since \( k \to \Psi'_{\xi - k} \) has essential support of positive Lebesgue measure, we conclude that the above integral can only vanish for a set of \( \xi \)'s having zero Lebesgue measure. This is a contradiction, which concludes the proof. □

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