CROUZEIX-RAVIART TRIANGULAR ELEMENTS ARE INF-SUP STABLE

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dedicated to Michel Crouzeix

ABSTRACT. The Crouzeix-Raviart triangular finite elements are inf-sup stable for the Stokes equations for any mesh with at least one interior vertex. This result affirms a conjecture of Crouzeix-Falk from 1989 for $p = 3$. Our proof applies to any odd degree $p \geq 3$ and concludes the overall stability analysis: Crouzeix-Raviart triangular finite elements of degree $p$ in two dimensions and the piecewise polynomials of degree $p - 1$ with vanishing integral form a stable Stokes pair for all positive integers $p$.

Keywords. Stokes problem, inf-sup stability, nonconforming, Crouzeix-Raviart, $p = 3$, odd degree, arbitrary $p$

AMS. 65N30, 65N12, 65N15

1. INTRODUCTION

The Crouzeix-Raviart finite element method was invented to approximate the solution to the Stokes equations for polynomial degree $p \in \mathbb{N}$ in [10]. The realization of the nonconforming schemes for a higher polynomial degree $p$ is less obvious in 2D [1, 2, 5, 6, 7, 20] and is still in its infancy in 3D [8]. The case $p = 1$ is classical [10] and the 2D version of $p = 2$ is known as the Fortin-Soulie element [12]. The Crouzeix-Raviart finite element spaces are significantly different for even and odd polynomial degrees [7].

The important cubic case $p = 3$ was the topic of the seminal contribution [9] that establishes, in particular, the macro-element technology for the proof of the Stokes stability for nonconforming finite element methods. It also guarantees inf-sup stability up to exceptional configurations of four edge-connected triangles characterized by some geometric quantity $D = 0$ in [9, Prop. 3.1] and ends with a conjecture: We conjecture that $\text{CR}_0^p(\mathcal{T}; \mathbb{R}^2) \times P_{p-1}(\mathcal{T})/\mathbb{R}$ forms a stable Stokes pair for any triangulation ... satisfying the minimal angle condition and containing an interior vertex (quoted, in modified notation, from the Remark in [9]). Theorem 2.1 below in this paper for $p = 3$ resolves this Crouzeix-Falk conjecture from 1989 and is recycled to cover all the open remaining cases $p = 3, 5, 7, \ldots$ simultaneously; the accompanying paper [5] further clarifies the role of the singular functions for $p = 5, 7, 9, \ldots$. 

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The stability of the Crouzeix-Raviart finite element method in the 2D Stokes equations for \( p = 4, 6, 8, \ldots \) is shown in \([2]\) with a discussion of singular geometries.

The cubic case is the hardest, because combinatorial proofs and the concept of singular points do not fully characterize the kernel of the piecewise divergence operator (cf. \([18]\) Rem. 3.1) inside the macro-element methodology \([3, 9, 19]\). It is emphasized that singular vertices do not play any role in this paper unlike, e.g., in \([2, 5, 14, 18]\).

The remaining parts of this paper are organised as follows. Theorem 2.1 and its necessary notation follow in Section 2. Section 3 reduces the analysis to four parameters per triangle, before Section 4 introduces the geometry of an interior vertex patch. Section 5 determines the coefficient matrix that equivalently describes the assertion by its kernel vectors. The key point follows in Section 6 with the proof that the constant pressure is the only kernel vector. The linear algebra of some kernel vectors is postponed to Section 7. Some remarks on the macro-element technique in Section 8 conclude the proof and the paper.

2. Setting

Given a (shape-) regular triangulation \( \mathcal{T} \) of the planar polygonal bounded Lipschitz domain \( \Omega \subset \mathbb{R}^2 \) into triangles with set of edges \( E \) (resp. edges on the boundary \( E(\partial \Omega) \)), let \( P_p(\mathcal{T}; \mathbb{R}^2) \equiv P_p(\mathcal{T}) \times P_p(\mathcal{T}) \) denote the piecewise polynomials of (total) degree at most \( p \geq 1 \) in two components. The nonconforming velocity space

\[
\text{CR}_p^0(\mathcal{T}; \mathbb{R}^2) := \{ \mathbf{v}_h \in P_p(\mathcal{T}; \mathbb{R}^2) : \mathbf{v}_h \text{ is continuous at Gauss points of } E \in \mathcal{E} \text{ and vanishes at Gauss points of any boundary edge } E \in \mathcal{E}(\partial \Omega) \}
\]

is considered with the piecewise polynomial pressure space \( P_{p-1}(\mathcal{T})/\mathbb{R} \). The pressure space \( P_{p-1}(\mathcal{T}) \cap L^2_0(\Omega) \equiv P_{p-1}(\mathcal{T})/\mathbb{R} \) is the quotient space of the piecewise polynomials \( P_{p-1}(\mathcal{T}) \) of (total) degree at most \( p-1 \) divided by the constants \( \mathbb{R} \) and identified with the representations in \( L^2_0(\Omega) := \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \} \) with integral zero. The Gauss points of an edge \( E \) are the affine images of the zeros in the interval \((-1, 1)\) of the \( p \)-th order Legendre polynomial \( L_p : [-1, 1] \to \mathbb{R} \) normalized by \( L_p(1) = 1 \). The continuity at the Gauss points in the above definition of the velocity space \( \text{CR}_p^0(\mathcal{T}; \mathbb{R}^2) \) along an edge \( E \) is equivalent to the orthogonality of the jump \( [\mathbf{v}_h]_E \) of \( \mathbf{v}_h \in P_p(\mathcal{T}; \mathbb{R}^2) \) onto the one-dimensional polynomials \( P_{p-1}(E; \mathbb{R}^2) \) in \( L^2(E; \mathbb{R}^2) \).

The main result in this paper is the Stokes stability: The piecewise divergence

\[
\text{div}_{pc} : \text{CR}_0^p(\mathcal{T}; \mathbb{R}^2) \to P_{p-1}(\mathcal{T})/\mathbb{R}
\]

is surjective and allows for a bounded linear right-inverse with an \( h \)-independent operator norm \( C_p \). Standard notation on Lebesgue and Sobolev spaces
and their (semi-) norms applies throughout the paper. The pressure space is endowed with the $L^2$ norm $\| \bullet \|_{L^2(\Omega)}$ in $\Omega$ and the nonconforming velocities are endowed with the piecewise $H^1$ seminorm $\| \bullet \|_{pw}$, defined by

$$\| v_h \|_{pw} = \sqrt{\sum_{T \in T} |v_h|_{H^1(\text{int}(T))}^2}$$

for $v_h \in \text{CR}_0^p(T; \mathbb{R}^2)$, which is a norm owing to the discrete Friedrichs inequality $[4, p. 299]$; $\text{int}(T)$ is the interior of a (compact) triangle $T$ (the convex hull of its three vertices). The reciprocal $\beta_p = 1/C_p > 0$ of the operator norm of the right-inverse of (2.1) is known as an inf-sup constant

$$\beta_p = \inf_{q_h \in P_{p-1}(T) \setminus \mathbb{R}} \sup_{v_h \in \text{CR}_0^p(T; \mathbb{R}^2)} \frac{\int_{\Omega} q_h \text{div} v_h \, dx}{\|q_h\|_{L^2(\Omega)} \| v_h \|_{pw}}.$$  

The point is that a positive lower bound of the inf-sup constant $\beta_p$ is independent of the mesh-size in the triangulation. Two universal positive constants $\varepsilon$ and $M$ describe the dependency of the positive lower bound of $\beta_p$ on the triangulation $T$ in a set of admissible triangulations $T$.

**Definition 1 (admissible triangulations).** Given the positive constants $\varepsilon$ and $M$, let $T$ be the class of all regular triangulations $T$ (of the domain $\Omega$ into triangles) with at least one interior vertex and all interior angles in a triangle $K \in T$ are bounded from below by $\varepsilon > 0$. Furthermore, any triangle $K \in T$ has either at least one vertex in the interior of the domain $\Omega$ (then $M := 0$), or there is a finite sequence of at most $M \geq 1$ triangles $K_0, K_1, \ldots, K_k \in T$ such that $k \leq M$, $\partial K_j \cap \partial K_{j+1}$ is a common edge of $K_j$ and $K_{j+1}$ for all $j = 0, \ldots, k - 1$, $K = K_0$, and $K_k$ has at least one vertex in the interior of $\Omega$.

Each triangulation $T \in T$ in the uniformly shape-regular class of admissible triangulations $T$ has interior vertices and not too many triangles in $T$ have all three vertices on the boundary $\partial \Omega$ of the domain $\Omega$.

**Theorem 2.1.** For all $p \in \mathbb{N}$ there exists a constant $\beta > 0$ such that, for all $T \in T$, the piecewise divergence operator (2.1) is surjective and has a bounded inverse in the sense that

$$\forall g \in P_{p-1}(T) / \mathbb{R} \exists v_h \in \text{CR}_0^p(T; \mathbb{R}^2) \ g = \text{div}_{pw} v_h \quad \text{and} \quad \beta_p \| v_h \|_{pw} \leq \| g \|_{L^2(\Omega)}.$$  

(The constant $\beta_p$ exclusively depends on $p$, $\varepsilon$, and $M$ in the definition of $T$.)

The discussion in the introduction about the existing references [2, 5, 10, 12] shows that the remaining critical case $p = 3$ is examined below.

### 3. Elimination of interior degrees of freedom

The analysis of conforming finite elements for the 2D Stokes problem simplifies if one eliminates the contributions from the interior degrees of freedom [14]. The latter are associated to the cubic bubble-functions multiplied with vectors of polynomials of (total) degree at most $p - 3$. We suppose $p \geq 3$ in
the sequel and abbreviate the vector-valued bubble-functions, that vanish on the boundary $\partial T$ of the triangle $T$, by

$$B_p(T) := \mathbb{P}_{p-1}(T; \mathbb{R}^2) \cap H^1_0(\text{int}(T); \mathbb{R}^2).$$

The remaining degrees of freedom are four (individually) for each triangle $T$ and any polynomial $g \in P_{p-1}(T)$ of (total) degree at most $p-1$ as follows: The evaluation $g(y)$ at the three vertices $y$ of $T$ and the integral $\int_T g \, dx$. Let $\Lambda_T : P_{p-1}(T) \to \mathbb{R}^4$ denote the vector of those four values.

**Proposition 3.1** (Guzman-Scott). Any polynomial $g \in P_{p-1}(T)$ is the divergence of a bubble-function in $B_p(T)$ (i.e., $g = \text{div } b$ pointwise in $T$ for some $b \in B_p(T)$) if and only if $\Lambda_T(g) = 0$.

**Proof.** This is essentially known since [23], allows a 3D version [17], and is given with a combinatorial proof in [14, Prop. 2]. The cubic case also follows from elementary considerations with $\text{div } B_3(T)$. \hfill $\Box$

The proposition allows a reduction to interpolating vertex values [14] in this paper as follows: First we consider a patch $\omega(z) := \text{int}(\bigcup T(z))$, covered by the $m$ neighboring triangles $T(z) := \{ T \in \mathcal{T} : z \in \mathcal{V}(T) \}$ of an interior vertex $z \in \mathcal{V}(\Omega)$ as depicted in Figure 1 (Here and throughout the paper $\mathcal{V}(T)$ denotes the set of the three vertices of the triangle $T$ and $\mathcal{V}(\Omega)$ denotes the set of all interior vertices in the triangulation $\mathcal{T} \in \mathcal{T}$.)

![Figure 1. Nodal patch $\omega(z)$](image-url)
Second we investigate (2.1) by the selection of an edge-oriented basis of a linear subspace \( V_h(z) \) of \( \text{CR}_0^p(\mathcal{T}(z); \mathbb{R}^2) \) defined below in (1.2) of dimension \( 5m \); the precise definition of \( V_h(z) \) is not important in this section as long as it only intersects trivially with the bubble-functions \( B_p(\mathcal{T}(z)) \) below. Recall that \( \Lambda_T \) denotes the above four degrees of freedom (the evaluation at the three vertices of plus the integral over the triangle) for each of the \( m \) triangles \( T \in \mathcal{T}(z) \) and let \( \Lambda : P_{p-1}(\mathcal{T}(z)) \to \mathbb{R}^{4m} \) denote their patchwise version: Identify \( \mathbb{R}^{4m} \equiv (\mathbb{R}^4)^m \) and let \( (\Lambda g)|_T := \Lambda_T(g|_T) \in \mathbb{R}^4 \) for all \( T \in \mathcal{T}(z) \) and \( g \in P_{p-1}(\mathcal{T}(z)) \). The homogeneous boundary conditions (in the sense of Crouzeix-Raviart at the Gauss points of the edges along the boundary \( \partial \omega(z) \)) of the discrete velocities \( v_h \in V_h(z) \subset \text{CR}_0^p(\mathcal{T}(z); \mathbb{R}^2) \) lead to \( \int_{\omega(z)} \text{div}_p v_h dx = 0 \), i.e., \( \text{div}_p : V_h(z) \to P_{p-1}(\mathcal{T}(z))/\mathbb{R} \). Hence the composition \( \Lambda \circ \text{div}_p : V_h(z) \to \mathbb{R}^{4m} \) is a linear operator and its range is a subspace of \( \mathbb{R}^{4m} \) orthogonal to

\begin{equation}
(3.1) \quad s := (0, 0, 0, 1, 0, 0, 0, 1, \ldots, 1) = \sum_{j=1}^{m} e_{4j} \in \mathbb{R}^{4m}
\end{equation}

in terms of the \( j \)-th canonical unit vector \( e_j \in \mathbb{R}^{4m} \). In fact, given \( v_h \in V_h(z) \) with \( x := \Lambda \text{div}_p v_h \in \mathbb{R}^{4m} \), the component \( x_{4j} = \int_{T(j)} \text{div} v_h dx \) is the integral of \( \text{div} v_h \) over the triangle \( T(j) \) of number \( j = 1, \ldots, m \). Then

\begin{equation}
(3.2) \quad x \cdot s = \sum_{j=1}^{m} \int_{T(j)} \text{div} v_h dx = \int_{\omega(z)} \text{div}_p v_h dx = 0
\end{equation}

is the vanishing integral over the patch \( \omega(z) \). Let \( B_p(\mathcal{T}(z)) \equiv \bigoplus_{j=1}^{m} B_p(T(j)) \subset \text{CR}_0^p(\mathcal{T}; \mathbb{R}^2) \) denote the piecewise bubble-functions in the patch \( \mathcal{T}(z) \). This and Proposition 3.1 clarify that the piecewise divergence operator

\begin{equation}
(3.3) \quad \text{div}_p : V_h(z) \oplus B_p(\mathcal{T}(z)) \to P_{p-1}(\mathcal{T}(z))/\mathbb{R}
\end{equation}

is a linear map \( \text{into} \) the pressure space \( P_{p-1}(\mathcal{T}(z))/\mathbb{R} \) with zero integral.

**Lemma 3.2.** Suppose \( V_h(z) \) is some linear subspace of \( \text{CR}_0^p(\mathcal{T}(z); \mathbb{R}^2) \) such that the linear map \( \Lambda \circ \text{div}_p : V_h(z) \to s^\perp \) is surjective onto the \((4m-1)\)-dimensional orthogonal complement \( s^\perp \) of the vector \( s \) in \( \mathbb{R}^{4m} \). Then the piecewise divergence operator \( (3.3) \) is surjective onto \( P_{p-1}(\mathcal{T}(z))/\mathbb{R} \).

**Proof.** Given any \( f \in P_{p-1}(\mathcal{T}(z))/\mathbb{R} \) with \( x := \Lambda f \in \mathbb{R}^{4m} \), \( \int_{\omega(z)} f dx = 0 \) implies \( x \cdot s = 0 \); written \( x \in s^\perp \). The surjectivity of \( \Lambda \circ \text{div}_p : V_h(z) \to s^\perp \) leads to some \( v_h \in V_h(z) \) with \( x = \Lambda \text{div}_p v_h \). Consequently, \( g := f - \text{div}_p v_h \in P_{p-1}(\mathcal{T}(z))/\mathbb{R} \) satisfies \( \Lambda g = 0 \). Proposition 3.1 applies to any \( g|_T \) for \( T \in \mathcal{T}(z) \) and leads to some \( b \in B_p(\mathcal{T}(z)) \) with \( \text{div}_p b = g \) in \( \omega(z) \). Hence \( f = \text{div}_p(v_h + b) \). \( \square \)
4. Cubic polynomials and geometry of an interior patch

Figure 1 displays the closed nodal patch of an interior vertex $z$ in the shape-regular triangulation $T$, sometimes also called star, that consists of $m \geq 3$ triangles $T(z) = \{T(1), \ldots, T(m)\}$ with interior vertex $z$ and the vertices $P(1), \ldots, P(m)$ on the boundary $\partial \omega(z)$; $T(z)$ is a regular triangulation of the open and simply connected domain $\omega(z) \equiv \text{int}(\cup T(z))$ covered by the neighbouring triangles $T(1), \ldots, T(m)$ around $z$. A typical triangle $T(j)$ has the vertices $z, P(j)$, and $P(j+1)$ counted counterclockwise as in Figure 1 such that $\partial T(j-1) \cap \partial T(j) = E(j) = \text{conv}\{z, P(j)\}$ is a common edge of the consecutive triangles $T(j-1)$ and $T(j)$ for all $j = 1, \ldots, m$; here and throughout a cyclic notation is understood with $T(0) \equiv T(m), P(0) \equiv P(m)$, etc. The vertex-oriented $P_1$ nodal basis functions $\varphi_z \in P_1(T(z)) \cap H^1_0(\omega(z))$ and $\varphi_1, \ldots, \varphi_m \in P_1(T(z)) \cap H^1(\omega(z))$ are the nodal basis functions defined by $\varphi_j(P_k) = \delta_{jk}$ (with Kronecker’s $\delta_{jk}$) and $\varphi_j(z) = 0 = \varphi_z(P_k)$ for all $j, k = 1, \ldots, m$, while $\varphi_z(z) = 1$. The Crouzeix-Raviart function for the edge $E(j)$ is defined with the cubic Legendre polynomial $\mathcal{L}_3$ by

\begin{equation}
\psi_j(x) := \mathcal{L}_3(1 - 2\varphi_k(x)) \quad \text{at } x \in T(j) \quad \text{with } k = j + 1
\end{equation}

resp. at $x \in T(j-1)$ with $k = j - 1$ and zero elsewhere. Then the subspace $V_h(z)$ is defined as the span of all the following $5m$ vector-valued functions

\begin{equation}
\varphi_j \varphi_z^2 n(j), \varphi_j \varphi_z^2 t(j), \varphi_j^2 \varphi_z n(j), \varphi_j^2 \varphi_z t(j), \psi_j n(j) \in CR_0^3(T(z); \mathbb{R}^2),
\end{equation}

also enumerated as $b(5j - 4) := \varphi_j \varphi_z^2 n(j), \ldots, b(5j) := \psi_j n(j)$, where $n(j)$ is the unit normal and $t(j) := (z - P(j))/|E(j)|$ the tangential unit vector of the edge $E(j) = \text{conv}\{z, P(j)\}$ of length $|E(j)| = |z - P(j)| > 0$ with number $j = 1, \ldots, m$ as depicted in Figure 1.

Recall the definition of $\Lambda_T$ from Section 3 for any $T \in T(z)$ and enumerate all those $4m$ components also by $(\Lambda_{4j-3}, \ldots, \Lambda_{4j}) := \Lambda_T(j)$ for $j = 1, \ldots, m$. The resulting Vandermonde matrix

\begin{equation}
M = (\Lambda_j(\text{div}_{pw} b(k)))_{k=1, \ldots, 5m}^{j=1, \ldots, 4m} \in \mathbb{R}^{5m \times 4m}
\end{equation}

will be calculated explicitly in the next section. The following duality argument allows an application of Lemma 3.2. Recall $M_s = 0$ from 3.1 - 3.2.

**Lemma 4.1.** *If the kernel of the matrix $M$ is the span of $s$, then the linear map $\Lambda \circ \text{div}_{pw} : V_h(z) \rightarrow s^\perp$ is surjective onto $s^\perp \subset \mathbb{R}^{4m}$.*

**Proof.** Notice that $M$ is already the transpose of the coefficient matrix $M^t = (\Lambda(\text{div}_{pw} b(k)) : k = 1, \ldots, 5m)$ associated to the operator $\Lambda \circ \text{div}_{pw} : \text{span}\{b(1), \ldots, b(5m)\} \rightarrow s^\perp$ and the images $\text{Im}(\Lambda \circ \text{div}_{pw}) = \text{Im} M^t$ coincide. The known orthogonal decomposition $\text{ker} M \perp \text{Im} M^t = \mathbb{R}^{4m}$ into the kernel ker $M$ of the matrix $M$ and the image $\text{Im} M^t$ of its transpose $M^t$ is the fundamental theorem of linear algebra [21] included in textbooks on linear algebra, e.g., in the form $\text{ker} M = (\text{Im} M^t)^\perp$ in [13, Eq. (2.47)]. With $\text{ker} M = \text{span}\{s\}$ this implies the assertion $\text{Im}(\Lambda \circ \text{div}_{pw}) = \text{Im} M^t = s^\perp$. \(\square\)
This section concludes with general observations on the geometry of an interior patch. The inner angles of the triangle $T(j)$ at the respective vertices $z$, $P(j)$, and $P(j + 1)$ are denoted by $\omega(j)$, $\alpha(j)$, and $\beta(j)$. The area $|T(j)|$ of the triangle $T(j)$ and the length $|E(j)|$ of an edge $E(j)$ displayed in Figure 1 are related to certain cotangent sums; namely, for $j = 1, \ldots, m$,

$$\gamma_j^- := \frac{|E(j)|^2}{2|T(j - 1)|} = \cot \omega(j - 1) + \cot \beta(j - 1) > 0,$$

$$\gamma_j^+ := \frac{|E(j)|^2}{2|T(j)|} = \cot \omega(j) + \cot \alpha(j) > 0.$$  

(The cyclic notation applies for $\omega(0) \equiv \omega(m)$ and $\beta(0) \equiv \beta(m)$; the above geometric identities in terms of angles in a triangle follow from elementary geometry in a triangle.) The combination of the above leads to

$$(4.4) \quad \gamma_j := \gamma_j^- + \gamma_j^+ = \kappa_j + \mu_j > 0$$

with real coefficients

$$\kappa_j := \cot \alpha(j) + \cot \beta(j - 1) \quad \text{and} \quad \mu_j := \cot \omega(j - 1) + \cot \omega(j).$$

The cotangent sums $\kappa_j$ and $\mu_j$ may become zero or negative; but $\kappa_j + \mu_j > 0$ is positive: the two quantities cannot vanish simultaneously.

A final observation concerns the situation $\kappa_j = 0$ that may happen repeatedly. But the convex hull of the compact patch $\omega(z)$ contains at least three extreme points and those are convex corners of $\omega(z)$. Hence there exist at least three vertices $P(j)$ with $\kappa_j > 0$.

5. The Coefficient Matrix

This section derives an explicit closed formula for the Vandermonde matrix $M$ from (4.3) in terms of a cyclic block-bi-diagonal matrix

$$(5.1) \quad M = \begin{pmatrix} M_1^+ & & M_1^- \\ M_2^+ & \ddots & \ & M_2^- \\ & \ddots & \ddots & \ \\ M_m^+ & & M_m^- \end{pmatrix} \in \mathbb{R}^{5m \times 4m}.$$  

Empty positions in (5.1) are filled by zero and $M_j^\pm$ is a $5 \times 4$ matrix computed in the subsequent lemma, where the $5 \times 8$ matrix $(M_j^-, M_j^+)$ denotes the composition of $M_j^-$ and $M_j^+$.

Lemma 5.1. The Vandermonde matrix $M$ is the block-diagonal matrix (5.1) with the typical $5 \times 8$ block $(M_j^-, M_j^+)$ for $j = 1, \ldots, m$. The diagonal scaling

$$D_L(j) := \operatorname{diag}(|E(j)|, -|E(j)|, |E(j)|, |E(j)|, |E(j)|/12) \in \mathbb{R}^{5 \times 5},$$

$$D_R(j) := \operatorname{diag}(1, 1, 1, 6/|T(j - 1)|, 1, 1, 1, 6/|T(j)|) \in \mathbb{R}^{8 \times 8},$$

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leads to the $5 \times 8$ matrix $D_L(j)(M_j^-, M_j^+D_R(j)$ that is equal to

$$
\begin{pmatrix}
\cot \omega(j-1) & 0 & 0 & \gamma_j^- - \cot \omega(j) & 0 & 0 & -\gamma_j^+ \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \cot \beta(j-1) & \gamma_j^- & 0 & -\cot \alpha(j) & -\gamma_j^+ \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\gamma_j^- & \gamma_j^- & \gamma_j^- & \gamma_j^- & \gamma_j^+ & -\gamma_j^+ & -\gamma_j^+ & -\gamma_j^+ \\
\end{pmatrix}.
$$

(Recall the cyclic convention $(\omega(0), \beta(0)) \equiv (\omega(m), \beta(m))$ for $j = 1$.)

**Proof.** This outline of the proof guides the reader through the arguments from elementary geometry in triangles. The functional $\Lambda$ from Section 3 consist of point evaluation or integrals of the piecewise divergence of the ansatz functions $b(5j-4) \equiv \varphi_j \varphi_2^2 n(j)$, $\ldots$, $b(5j) \equiv \psi_j n(j)$. The block structure of $M$ results from the support $T(j-1) \cup T(j)$ of those 5 piecewise polynomials.

The first component of the functional $\Lambda$ associated to the triangle $T(j)$ is the point evaluation at the vertex $z$ of a function $\text{div}(\varphi_j \varphi_2^2 v) = v \cdot \nabla (\varphi_j \varphi_2^2)$ resp. $\text{div}(\varphi_j^2 \varphi_2 v)$ for some constant vector $v \in \mathbb{R}^2$ and this is equal to $v \cdot \nabla \varphi_j |_{T(j)}$ resp. zero. The (global) nodal basis functions $\varphi_z$ and $\varphi_j$ are barycentric coordinates in each triangle and their derivatives are known from the geometric fundamentals of the (conforming) $P_1$ FEM: $|E(j)| n(j) \cdot \nabla \varphi_z$ and $|E(j)| n(j) \cdot \nabla \varphi_j$ are equal to $-\cot \alpha(j)$ and $-\cot \omega(j)$ in $T(j)$ (resp. $\cot \beta(j-1)$ and $\cot \omega(j-1)$ in $T(j-1)$). The tangential components $|E(j)| t(j) \cdot \nabla \varphi_z = 1$ and $|E(j)| t(j) \cdot \nabla \varphi_j = -1$ are constant in $T(j-1) \cup T(j)$. This leads to the entries $M^-_j(1,1) = |E(j)|^{-1} \cot \omega(j-1)$ and $M^+_j(1,1) = |E(j)|^{-1} \cot \omega(j)$ in the first position of $M^\pm_j \in \mathbb{R}^{5 \times 4}$ and to $M^-_j(2,1) = -|E(j)|^{-1} = M^+_j(2,1)$.

Similar arguments apply to the point evaluations at the other vertices $P(j+1)$ and $P(j+1)$ and we obtain, e.g., $M^-_j(3,3) = |E(j)|^{-1} \cot \beta(j-1)$, $M^+_j(3,3) = -|E(j)|^{-1} \cot \alpha(j)$, and $M^-_j(4,3) = |E(j)|^{-1} = M^+_j(4,2)$.

The situation is slightly different for the edge-bubble function $\psi_j$ from (1.1): The one-dimensional derivative $L_3'(\pm 1) = 6$ of the cubic Legendre polynomial $L_3(t) = (5t^3 - 3t)/2$ is the same at the end-points of the reference interval $[-1,1]$. Thus the point evaluations of $\text{div}(\psi_j(n)(j)) = n(j) \cdot \nabla \psi_j(x) = -2L_3'(1 - 2\varphi_k(x)) n(j) \cdot \nabla \varphi_k$ at the three vertices of $T(j)$ (resp. $T(j)$) coincide with $-12 n(j) \cdot \nabla \varphi_k$. Since $\varphi_{j+1}$ vanishes along $E(j)$, its gradient $\nabla \varphi_{j+1}|_{T(j)} = n(j)/h_{j,j+1}$ is perpendicular to $t(j)$, i.e., parallel to $n(j)$, and has a length $h_{j,j+1}^- = 2|T(j)|/|E(j)|$ is the height of the vertex $P(j+1)$ onto the edge $E(j)$ in the triangle $T(j)$. Consequently $\text{div}(\psi_j n(j))$ assumes the value $6 |E(j)|/|T(j)| = 12 \gamma_j^- / |E(j)|$ at any vertex of $T(j-1)$ (resp. the value $-6 |E(j)|/|T(j)| = -12 \gamma_j^+ / |E(j)|$ at any vertex of $T(j)$). Given any vector $v \in \mathbb{R}^2$, the integrals over $T(j)$ allow an
integration by parts
\[
\int_{T(j)} \text{div}(\varphi_j \varphi_2^2 v) dx = v \cdot \int_{\partial T(j)} \varphi_j \varphi_2^2 n_{T(j)} ds
\]
\[
= -v \cdot n(j) \int_{E(j)} \varphi_j \varphi_2^2 ds = -v \cdot n(j)|E(j)|/12
\]
(with the exterior unit normal \( n_{T(j)} \) and \( \int_0^1 s^2(1-s) \, ds = 1/12 \). The same argument applies to \(- \int_{T(j)} \text{div}(\psi_j n(j)) dx = |E(j)| = \int_{T(j-1)} \text{div}(\psi_j n(j)) dx. \)

The summary of the aforementioned identities leads to the assertion. □

Lemma 5.1 suggests a global scaling of the matrix \( M \) with the block-diagonal matrices
\[
D_L := \text{diag}(D_L(1), \ldots, D_L(m)) \in \mathbb{R}^{5m \times 5m},
\]
\[
D_R := \text{diag}(D_R(1), \ldots, D_R(m)) \in \mathbb{R}^{4m \times 4m}
\]
with \( D_L(j) \in \mathbb{R}^{5 \times 5} \) as above and \( D_R(j) := \text{diag}(1, 1, 1, 6/|T(j)|) \in \mathbb{R}^{4 \times 4}. \)

The investigation of the kernel of \( M \) is then rewritten in that of the cyclic block-bi-diagonal matrix

\[
(5.2) \quad A := D_L M D_R = \begin{pmatrix}
A_1^+ & \ldots & A_1^- \\
A_2^+ & \ldots & A_2^- \\
& \ldots & \\
A_m^+ & \ldots & A_m^-
\end{pmatrix} \in \mathbb{R}^{5m \times 4m}
\]

with the blocks \( (A_j^+, A_j^-) := D_L(j)(M_j^-, M_j^+)D_R(j) \) displayed in Lemma 5.1.

A brief observation on the constant pressure values concludes this section. Recall \( Ms = 0 \) from (3.1)-(3.2) so that
\[
v_0 := 6D_R^{-1} s = (0, 0, 0, |T(1)|, 0, 0, 0, |T(2)|, \ldots, 0, 0, 0, |T(m)|)^t \in \mathbb{R}^{4m}
\]
belongs to the kernel of \( A \). If the dimension of the kernel of \( A \) is smaller than or equal to one, then it is one (because \( Av_0 = 0 \) and the kernel \( M \)

6. The kernel of \( A \) is one-dimensional

The characterization of the kernel of \( A \) follows in three steps. The first step reduces the problem to a sub-matrix and concerns the row number two, four, and five of \( A_j^- \) and \( A_j^+ \) for each \( j = 1, \ldots, m \); i.e.,

\[
(6.1) \quad B := A(2, 4, 5, 7, 9, 10, \ldots, 5m; 1, \ldots, 4m) \in \mathbb{R}^{3m \times 4m}
\]

with the \( 3m \) rows of number \( k = 5j - 3, 5j - 1, 5j \) for \( j = 1, \ldots, m \). (Standard notation for sub-matrices from matrix analysis applies throughout this paper: \( A(I; J) \) denotes the (rectangular) sub-matrix of \( A \) with the entries \( A(i, j) \) for \( (i, j) \in I \times J \) specified in the index lists \( I \) and \( J \).) Let \( e_j \) be the \( j \)-th canonical unit vector in \( \mathbb{R}^{4m} \).
Define the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{m+1} \in \mathbb{R}^{4m}$ by
\[
\mathbf{v}_0 = \sum_{j=1}^{m} |T(j)| \mathbf{e}_{4j}, \quad \mathbf{v}_1 := -\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_{4m-1} - \mathbf{e}_{4m}, \\
\mathbf{v}_j := \mathbf{e}_{4j-5} - \mathbf{e}_{4j-4} - \mathbf{e}_{4j-2} + \mathbf{e}_{4j} \text{ for any } j = 2, \ldots, m, \\
\mathbf{v}_{m+1} := \sum_{j=1}^{m} (-1)^{j+1}(\mathbf{e}_{4j-3} - \mathbf{e}_{4j}).
\]

**Lemma 6.1.** The kernel of $\mathbf{B}$ from (6.1) has the dimension $m + 1 + \sigma$ with $\sigma = 0$ if $m = 3, 5, 7, \ldots$ is odd and $\sigma = 1$ if $m = 4, 6, 8, \ldots$ is even. The vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{m+\sigma} \in \mathbb{R}^{4m}$ form a basis of the kernel $\ker \mathbf{B}$ of $\mathbf{B}$.

The proof of Lemma 6.1 by straightforward linear algebra will be postponed to Section 7 in order not to delay the flow of arguments in the proof.

**Step two** of the proof recalls that $\mathbf{v}_0$ is a known kernel vector of $\mathbf{A}$ and so can and will be neglected in the sequel. The aim is a characterization of all remaining kernel vectors $\mathbf{x}$ of $\mathbf{A}$ as a linear combination
\[
\mathbf{x} = \eta \mathbf{v}_{m+1} + \sum_{j=1}^{m} \xi_j \mathbf{v}_j \in \mathbb{R}^{4m}
\]
of the other kernel vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m+\sigma}$ of $\mathbf{B}$ with real coefficients $\xi_1, \ldots, \xi_m$ and $\eta$ and the following convention: $\eta := 0$ if $\sigma = 0$ and $m = 3, 5, 7, \ldots$ is odd, while $\eta \in \mathbb{R}$ if $\sigma = 1$ and $m = 4, 6, 8, \ldots$ is even. Suppose throughout this section that $\mathbf{x}$ from (6.2) satisfies $\mathbf{A}\mathbf{x} = \mathbf{0}$. The claim of this section is that $\mathbf{x} = \mathbf{0}$. Recall $\mathbf{B}\mathbf{x} = \mathbf{0}$ by Lemma 6.1 and consider the remaining equations $0 = (\mathbf{A}\mathbf{x})(5j - 4)$ and $0 = (\mathbf{A}\mathbf{x})(5j - 2)$ of $\mathbf{A}\mathbf{x} \in \mathbb{R}^{5m}$ in the position $5j - 4$ and $5j - 2$ for each $j = 1, \ldots, m$, which are equivalent to
\[
(6.3) \quad (-1)^{j-1}\kappa_j \eta = \gamma_j^{-}\xi_{j-1} - \gamma_j \xi_j + \gamma_j^{+}\xi_{j+1},
\]
\[
(6.4) \quad (-1)^{j} \gamma_j \eta = \gamma_j^{-}\xi_{j-1} - \mu_j \xi_j + \gamma_j^{+}\xi_{j+1}
\]
with (4.2) and the geometric terms from the end of Section 4. (Follow the cyclic convention $\xi_0 \equiv \xi_m$ and $\xi_{m+1} \equiv \xi_1$ where necessary.) Define the convex coefficients $0 < \lambda_j := \gamma_j^{-}/\gamma_j < 1$ and $0 < 1 - \lambda_j = \gamma_j^{+}/\gamma_j < 1$ to rewrite (6.3)-(6.4), for all $j = 1, \ldots, m$, as
\[
(6.5) \quad \lambda_j \xi_{j-1} - \xi_j + (1 - \lambda_j) \xi_{j+1} = (-1)^{j}(\kappa_j/\gamma_j) \eta,
\]
\[
(6.6) \quad \lambda_j \xi_{j-1} - (\mu_j/\gamma_j) \xi_j + (1 - \lambda_j) \xi_{j+1} = (-1)^{j}/\eta.
\]

The aim is to verify that those $2m$ equations (6.5)-(6.6) cannot hold simultaneously unless all the real coefficients $\xi_1, \ldots, \xi_m$ and $\eta$ vanish. The indirect proof assumes that $J := \{j = 1, \ldots, m : \xi_j \neq 0\}$ is non-empty and then leads to a contradiction. (Then $J = \emptyset$ and (6.6) imply $\eta = 0$ as well, whence $\mathbf{x} = \mathbf{0}$.) Let us fix the notation $J^C := \{1, \ldots, m\} \setminus J$ before we distinguish two cases $\eta = 0$ and $\eta \neq 0$ in step three of the proof.
Hence \( \eta \) vanish and the difference of the two equations shows
\[
0 = (1 - \mu / \gamma) \xi_j = (\kappa / \gamma) \xi_j \quad \text{for all } j = 1, \ldots, m.
\]
This provides no new information for \( j \in J^C \), but it implies
\[
\kappa_j = 0 \quad \text{for all } j \in J.
\]

The geometry of the nodal patch of \( z \) enters at this point as an additional argument. Section 4 closes with the existence of at least three vertices \( P(j) \) with \( \kappa_j > 0 \), whence \( J^C \) contains at least three indices. The hypothesis \( J \neq \emptyset \) leads to \( m \geq 4 \) and at least one index in \( J \). Hence we can find \( k \) consecutive indices \( j + 1, \ldots, j + k \in J \) with \( j, j + k + 1 \in J^C \) — interpreted in a cyclic notation (with index 0 identified with \( m \) etc.). Changing the enumeration of the vertices (if necessary) we find \( k \in \{1, \ldots, m - 3\} \) such that (in the new enumeration) \( \xi_m = 0 = \xi_{k+1} \) while \( \xi_1, \ldots, \xi_k \in \mathbb{R} \setminus \{0\} \) do not vanish. The homogeneous equation \( (6.5) \) holds for \( j = 1, \ldots, k \) and this can be written simultaneously as a linear system of \( k \) equations
\[
(6.7) \quad \begin{pmatrix}
1 & \lambda_1 - 1 \\
-\lambda_2 & 1 & \lambda_2 - 1 \\
\ddots & \ddots & \ddots \\
-\lambda_{k-1} & 1 & \lambda_{k-1} - 1 \\
-\lambda_k & 1 & \lambda_k - 1
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_{k-1} \\
\xi_k
\end{pmatrix} = 0
\]
with a (possibly non-symmetric) \( k \times k \) tri-diagonal coefficient matrix. For \( k = 1 \) and \( k = 2 \), \( (6.7) \) exclusively allows the trivial solution \( (\xi_1, \ldots, \xi_k) = 0 \). For \( k \geq 3 \), the tri-diagonal matrix in \( (6.7) \) is irreducible and weakly diagonally dominant, but strictly diagonally dominant in the first and last row. Those matrices are called weakly chained diagonally dominant and known to be regular; cf., e.g., [15, Cor. 6.2.27] or O. Taussky’s paper [22]. It follows that \( (6.7) \) merely allows the trivial solution \( (\xi_1, \ldots, \xi_k) = 0 \). This contradicts the assumption \( 1, \ldots, k \in J \). Hence \( J = \emptyset \) and \( \ker A = \text{span}\{v_0\} \). □

Second case \( \sigma = 1 \) and \( \eta \neq 0 \). Define the coefficients \( \eta_j := (-1)^j \xi_j / \eta \in \mathbb{R} \) and substitute \( \xi_j := (-1)^j \eta \eta_j \) in \( (6.5)-(6.6) \) for all \( j = 1, \ldots, m \). This leads, for all \( j = 1, \ldots, m \), to the two identities
\[
(6.8) \quad -\lambda_j \eta_j - 1 - (1 - \lambda_j) \eta_{j+1} = \kappa / \gamma + \eta_j = 1 + (\mu / \gamma) \eta_j.
\]
With \( (6.4) \) and the geometric notions from the end of Section 4 the second equality in \( (6.8) \) is equivalent to
\[
\gamma_j = \kappa_j (1 + \eta_j) > 0.
\]
Hence \( \eta_j \neq -1 \) and \( \kappa_j \neq 0 \) for all \( j = 1, \ldots, m \). The first equality in \( (6.8) \) can be rewritten as \( \lambda_j \eta_j - 1 + c_j \eta_j + (1 - \lambda_j) \eta_{j+1} = d_j \) with
\[
c_j := \begin{cases}
0 & \text{if } \eta_j = 0, \\
1 + 1/(\eta_j(1 + \eta_j)) & \text{if } \eta_j \neq 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
-1 & \text{if } \eta_j = 0, \\
0 & \text{if } \eta_j \neq 0.
\end{cases}
\]
for any \( j = 1, \ldots, m \). All those \( m \) conditions simultaneously form a system of linear equations

\[
(6.9) \quad T(\eta_1, \ldots, \eta_m)^t = \mathbf{d} := (d_1, \ldots, d_m)^t \in \mathbb{R}^m
\]

with the right-hand side \( \mathbf{d} \) and the cyclic tri-diagonal coefficient matrix

\[
T := \begin{pmatrix}
c_1 & 1 - \lambda_1 & & \\
\lambda_2 & c_2 & 1 - \lambda_2 & \\
& \ddots & \ddots & \ddots \\
& & \lambda_{m-1} & c_{m-1} & 1 - \lambda_{m-1} \\
1 - \lambda_m & & & & c_m
\end{pmatrix}.
\]

The compact row-wise Gershgorin discs \( \overline{B}(c_j, 1) \) in the coefficient matrix \( T \) have the center \( c_j \) and the radius one. If \( j \in J \) and so \( 0 \neq \eta_j \neq -1 \), then \( |c_j| > 1 \) (use \( c_j = -1 + \eta_j / (1 + \eta_j) + (1 + \eta_j) / \eta_j < -1 \) in case \( -1 < \eta_j < 0 \)). Hence the Gershgorin disc \( \overline{B}(c_j, 1) \neq 0 \) does not include zero. A first conclusion is that \( J^C = \emptyset \) implies that \( T \) is regular (zero does not belong to any of the Gershgorin discs and so is not an eigenvalue) and the right-hand side \( \mathbf{d} = 0 \) vanishes. Hence \( (6.8) \) allows only the trivial solution and all coefficients \( \eta_1, \ldots, \eta_m \) vanish. This leads to \( J = \emptyset \). A second conclusion concerns the remaining cases: Suppose, for a contradiction, that \( J \neq \emptyset \neq J^C \) and delete all the rows and columns in the coefficient matrix \( T \) and in the right-hand side \( \mathbf{d} \) in \( (6.9) \) with an index from \( J^C \) to obtain the sub-matrix \( T' := T(J, J) \). Since \( \eta_j \) vanishes for all \( j \in J^C \) and \( d_j = 0 \) for all \( j \in J \), the linear system \( (6.9) \) reduces to the homogeneous linear system \( T'y' = 0 \) with the reduced coefficient vector \( y' := (\eta_j : j \in J) \). The row-wise compact Gershgorin discs \( \overline{B}(c_j, r_j) \) for the cyclic tri-diagonal matrix \( T' \) for any \( j \in J \) have the center \( c_j \) with \( |c_j| > 1 \) as before; but the radius \( 0 \leq r_j \leq 1 \) may be smaller than one in case a corresponding row in \( T \) has been deleted to obtain \( T' \). It follows that zero does not belong to any Gershgorin disc \( \overline{B}(c_j, r_j) \) in \( T' \) for \( j \in J \) and so \( T' \) is a regular matrix. Thus the reduced homogeneous system \( T'y' = 0 \) allows only the trivial solution \( y' = 0 \); whence \( J = \emptyset \).

In all cases, \( J = \emptyset \), but that means a contradiction in \( (6.8) \). Hence the second case with \( \eta \neq 0 \) cannot arise.

\[ \square \]

7. Proof of Lemma 6.1

The submatrix \( \mathbf{B} \) of \( \mathbf{A} \) from \( (6.1) \) is the cyclic block-bi-diagonal matrix

\[
(7.1) \quad \mathbf{B} = \begin{pmatrix}
\mathbf{B}_1^+ \\
\mathbf{B}_2^+ \\
\ddots \\
\mathbf{B}_m^-
\end{pmatrix} \in \mathbb{R}^{3m \times 4m}
\]
with the $3 \times 8$ blocks, for $j = 1, \ldots, m$,

\begin{equation}
(7.2) \quad (B_j^-, B_j^+) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\gamma_j^- & \gamma_j^- & \gamma_j^- & \gamma_j^+ & -\gamma_j^- & -\gamma_j^+ & -\gamma_j^- & -\gamma_j^+
\end{pmatrix}.
\end{equation}

Recall $v_0, \ldots, v_{m+\sigma} \in \mathbb{R}^{4m}$ from Lemma 6.1 in terms of the canonical unit vectors $e_1, \ldots, e_{4m}$ in $\mathbb{R}^{4m}$.

**Claim 1:** $v_0, \ldots, v_{m+\sigma}$ are kernel vectors of $B$. This follows from straightforward calculations of the scalar product of the three rows of $(B_j^-, B_j^+)$ from (7.2) for $j = 1, \ldots, m$ with $v_k := e_{4k-5} - e_{4k-4} - e_{4k-2} + e_{4k}$ for $k = 2, \ldots, m$. The first row of $(B_j^-, B_j^+)$ has two ones only at positions where $v_k$ vanishes; the second row matches for $j = k$ at two positions with $1 - 1 = 0$ (and no interaction otherwise). The third row vanishes because the sum of all components of $v_k$ in the positions $4j - 3, 4j - 2, 4j - 3, 4j$ vanish for each $j = 1, \ldots, m$. Further details are not displayed here with two exceptions.

The first one is that $|T(j - 1)| \gamma_j^- = |T(j)| \gamma_j^+ = |E(j)|^2/2$ and so $Bv_0 = 0$, a rewriting of (3.1) - (3.2). The second detail is $Bv_{m+1} = (1 - (-1)^m)e_1$ to underline that $v_{m+1}$ is a kernel vector of $B$ if and only if $m$ is even.

**Claim 2:** $v_0, \ldots, v_{m+1}$ are linearly independent. Suppose that $\xi_0, \ldots, \xi_{m+1}$ are the real coefficients of the trivial linear combination $0 = \sum_{j=0}^{m+1} \xi_j v_j$.

Three observations complete the proof that all coefficients vanish. The first one is that the scalar product $v_j \cdot (1, 1, 1, 1, 0, \ldots, 0) = 0$ vanishes for any $j = 1, \ldots, m + 1$. This and $v_0 \cdot (1, 1, 1, 1, 0, \ldots, 0) = |T(1)|$ show $0 = \left(\sum_{j=0}^{m+1} \xi_j v_j\right) \cdot (1, 1, 1, 1, 0, \ldots, 0) = \xi_0 |T(1)|$; whence $\xi_0 = 0$.

The second observation is the scalar product $v_j \cdot e_1 = 0$ vanishes for all $j = 1, \ldots, m$, while $v_{m+1} \cdot e_1 = 1$. Thus $\xi_{m+1} = \left(\sum_{j=1}^{m+1} \xi_j v_j\right) \cdot e_1 = 0$.

The third observation is that the scalar product of $v_j$ with $e_{4k-2}$ reads

\begin{equation}
(7.3) \quad v_j \cdot e_{4k-2} = -\delta_{jk} \quad \text{for all } j, k = 1, \ldots, m.
\end{equation}

Hence $0 = \left(\sum_{j=1}^{m} \xi_j v_j\right) \cdot e_{4k-2} = -\xi_k$ vanishes for all $k = 1, \ldots, m$. 

**Claim 3:** The kernel of $B$ is included in the span of $v_0, \ldots, v_{m+\sigma}$. Suppose that $x = (x_1, \ldots, x_{4m}) \in \mathbb{R}^{4m}$ is any kernel vector of $B$ and define

$$y := x + \sum_{j=1}^{m} x_{4j-2} v_j \in \mathbb{R}^{4m}.$$
j = 1, \ldots, m \) (with \( y_{4m+1} := y_1 \)). These \( m \) equations are collected in an equivalent matrix form as

\[
(7.4) \quad \begin{pmatrix}
1 & 1 & & \\
1 & 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_5 \\
\vdots \\
y_{4m-3}
\end{pmatrix} = 0 \in \mathbb{R}^m
\]

with an \( m \times m \) coefficient matrix \( 1 + \mathbf{F} \) for the unit matrix \( \mathbf{1} \) and a Frobenius companion matrix \( \mathbf{F} \). The determinant of \( 1 + \mathbf{F} \) is \( 1 - (-1)^m \) (this is well known and elementary to verify) and this motivates the following two cases. If \( m \geq 3 \) is odd and \( \sigma := 0 \), then the linear system \((7.3)\) has only the trivial solution, i.e., \( 0 = y_{4j-3} \) for \( j = 1, \ldots, m \) and we set \( \mathbf{z} := \mathbf{y} \in \mathbb{R}^{4m} \). If \( m \geq 4 \) is even and \( \sigma = 1 \), then \( 1 + \mathbf{F} \) is singular with a one-dimensional kernel spanned by the vector \((1, -1, +1, \ldots, -1) \in \mathbb{R}^m \). Consequently \( y_{4j-3} = (-1)^{j+1}y_1 \) for all \( j = 1, \ldots, m \) and we define

\[
\mathbf{w} := \mathbf{y} - y_1 \mathbf{v}_{m+1} \in \mathbb{R}^{4m}.
\]

A direct calculation shows \( 0 = w_{4j-3} \) for all \( j = 1, \ldots, m \). In summary of the even or odd cases, we have for any \( m \geq 3 \) that the kernel vector \( \mathbf{w} = (w_1, \ldots, w_{4m}) \in \mathbb{R}^{4m} \) satisfies \( \mathbf{x} - \mathbf{w} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{m+\sigma}\} \) and \( 0 = w_{4j-3} = w_{4j-2} = w_{4j-1} \) for all \( j = 1, \ldots, m \) (the vector \( \mathbf{v}_{m+1} \) does not interfere with the components number 2, 3, 6, 7, 10, 11, \ldots, 4m - 1). Thus

\[
\mathbf{w} = \sum_{j=1}^{m} w_{4j} \mathbf{e}_{4j} \in \mathbb{R}^{4m}
\]

has non-zero entries at most in the components 4, 8, \ldots, 4m. The components number 3, 6, 9, \ldots, 3m of \( \mathbf{0} = \mathbf{Bw} \in \mathbb{R}^{3m} \) read \( 0 = \gamma_j^+ w_{4(j+1)} - \gamma_j^- w_{4j} \) for all \( j = 1, \ldots, m \) (with \( w_0 := w_{4m} \)). The resulting \( m \) relations imply \( w_{4j} = w_{4m} \prod_{k=j+1}^{m} (\gamma_k^- / \gamma_k^+) = w_{4m} |T(j)|/|T(m)| \) for all \( j = 1, \ldots, m \). Hence

\[
\mathbf{w} = (w_{4m}/|T(m)|) \mathbf{v}_0.
\]

Claim 3 follows from \( \mathbf{w}, \mathbf{x} - \mathbf{w} \in \text{span}\{\mathbf{v}_0, \ldots, \mathbf{v}_{m+\sigma}\} \).

\[\square\]

8. Finish of the proof of Theorem 2.1

Macro-element methodology. The arguments from \[3\] \[9\] \[19\] are known to lead to Theorem 2.1 and are merely summarized. The calculations in the previous sections characterize the kernel of the matrix \( \mathbf{A} \) from \((5.2)\) and then that of the Vandermonde matrix \( \mathbf{M} \) from \((4.3)\) and \((5.1)\) as the spanker \( \mathbf{M} = \text{span}\{\mathbf{s}\} \) of the vector \( \mathbf{s} \) from \((3.1)\). This and first Lemma 4.1 and second Lemma 3.1 prove that the piecewise divergence operator \((3.3)\) is surjective onto \( P_{0-1}(T(z))/\mathbb{R} \). This allows us to argue as in the proofs of \[9\] Prop. 3.1, 5.1] to conclude that there exists a bounded right-inverse in
the sense that
\[ \forall g \in P_{p-1}(T(z))/\mathbb{R} \quad \exists v_h \in \text{CR}_g^0(T(z); \mathbb{R}^2) \quad g = \text{div}_{pw} v_h \quad \text{and} \]
\[ (8.1) \quad \|v_h\|_{\omega(z), pw} := \sqrt{\sum_{T \in T(z)} \|v_h\|^2_{H^1(\text{int}(T))}} \leq \|g\|_{L^2(\omega(z))}/\gamma. \]

The operator norm of the bounded inverse of (5.3) and so \( \gamma > 0 \) in (8.1) exclusively depend on \( p \) and the shape regularity (i.e., on the constant \( \epsilon \) in the definition of \( \mathbb{T} \)) but not on the mesh-size. Then (8.1) is, in different notation, what is called hypothesis H0 in [9] for a macro-element \( \omega(z) \).

Recall from the definition of \( \mathbb{T} \) that \( M = 0 \) means that all triangles in \( T \) have at least one vertex in the interior of the domain \( \Omega \); the more general case allows at most \( M \) many edge-connected triangles that link interior vertices to all triangles in \( T \).

**Proof for** \( p = 3 \) **for** \( M = 0 \). The point is that any triangle \( K \) in \( T \) belongs to the patch \( T(z) \) \( \ni K \) of at least one interior vertex \( z \in \mathcal{V}(\Omega) \). Hence [9, Thm. 2.1] applies verbatim to the macro-elements \( (\omega(z) : z \in \mathcal{V}(\Omega)) \) and shows that (8.1) \( \equiv \) H0 implies Theorem 2.1. It moreover provides an explicit bound for \( \beta_3 \) in terms of \( \gamma \) and the shape regularity of \( \mathbb{T} \) (and a bounded overlap of \( L = 3 \) of the macro-elements).

**Proof for** \( p = 3 \) **for** \( M \geq 1 \). The macro-elements have to be enlarged in this case to include all triangles; for instance to \( \Omega(z) \) covered by the (regular) sub-triangulation \( \mathcal{T}(z) \) that consists of \( T(z) \) plus \( M \) layers of triangles in \( T \) around it. The proof of H0 for \( \Omega(z) \) requires Lemma 6.1 in [9], which is reformulated as follows for \( p = 3 \).

**Lemma 8.1** (Crouzeix-Falk). Suppose the interior edge \( E = \partial T_+ \cap \partial T_- \) is the common edge of the triangles \( T_+, T_- \in T \) with \( T(E) := \{T_+, T_-\} \) and \( \omega(E) := \text{int}(T_+ \cup T_-) \). Then the piecewise divergence \( \text{div}_{pw} : \text{CR}_0^0(T(E); \mathbb{R}^2) \to P_{p-1}(T(E))/\mathbb{R} \) has a right inverse in \( P_{p-1}(T(E)) \) (the full set of polynomials of degree at most \( p - 1 \) in \( T_+ \)) in the sense that
\[ \forall g \in P_{p-1}(T_+) \quad \exists v_h \in \text{CR}_0^0(T(E); \mathbb{R}^2) \quad g = \text{div}_{pw} v_h \quad \text{in} \quad T_+ \quad \text{and} \]
\[ \|v_h\|_{\omega(E), pw} := \sqrt{\|v_h\|^2_{H^1(\text{int}(T_+))} + \|v_h\|^2_{H^1(\text{int}(T_-))}} \leq \|g\|_{L^2(T_+)}/\gamma. \]
(The constant \( \gamma > 0 \) depends exclusively on \( \epsilon \) and \( p \geq 3 \).)

The proof of Lemma 8.1 for \( p \geq 4 \) is postponed and we continue with its application to the cubic case \( p = 3 \). Arguing as in Lemma 6.2 in [9], Lemma 8.1 allows to enlarge \( \omega(z) \) (where H0 holds) successively to the bigger domain \( \Omega(z) \). This verifies H0 for the macro-element \( \Omega(z) \) for each \( z \in \mathcal{V}(\Omega) \): The piecewise divergence \( \text{div}_{pw} : \text{CR}_0^0(T(z); \mathbb{R}^2) \to P_{p-1}(T(z))/\mathbb{R} \) is surjective with a bounded right inverse in the sense that
\[ \forall g \in P_{p-1}(T(z))/\mathbb{R} \quad \exists v_h \in \text{CR}_0^0(T(z); \mathbb{R}^2) \quad g = \text{div}_{pw} v_h \quad \text{in} \quad \Omega(z) \quad \text{and} \]
\[ (8.2) \quad \|v_h\|_{\Omega(z), pw} := \sqrt{\sum_{T \in T(z)} \|v_h\|^2_{H^1(\text{int}(T))}} \leq \|g\|_{L^2(\Omega(z))}/\gamma. \]
holds for a constant $\gamma > 0$ that exclusively depends on $\epsilon, M,$ and $p$. The hypothesis $\|s\|_{H^1} = 0$ in [9] for the macro-elements $(\Omega(z) : z \in V(\Omega))$ allows an application of [9] Thm. 2.1] with a larger overlap constant $L'(\Omega(z))$ that depends on $\varepsilon$ and $M$. This concludes the proof of Theorem 2.1 for $p = 3$. \hfill \Box

**Proof of Theorem 2.1 for $p = 5, 7, \ldots$.** The results in Section 2 hold for any polynomial degree $p \geq 3$ and lead to the design of $5m$ functions in (4.2). The function $\psi_2$ defined in (4.1), however, does not belong to $\text{CR}_0^p(T)$ for odd $p \geq 5$ and, with the notation of Figure 1, may be replaced by

$$ (8.3) \quad \psi_j(x) := \frac{6}{L_p(1)}
\left( L_p(1 - 2\varphi_k(x)) + (5L_p(1) - 30)\varphi_2^2(x)\varphi_j^2(x) \right) $$

at $x \in T(j)$ with $k = j + 1$ resp. at $x \in T(j-1)$ with $k = j - 1$ and zero elsewhere ($L_p'(1) > 0$ denotes the derivative of the Legendre polynomial of degree $p$ at 1); then $\psi_j$ in (8.3) coincides with that in (4.1) for $p = 3$.

The important point is that (8.3) defines a (scalar) Crouzeix-Raviart function $\psi_j \in \text{CR}_0^p(T)$ for all odd $p \geq 3$ with a scaling such that $\text{div}(\psi_j n(j))$ is equal to $-12n(j) \cdot \nabla \varphi_k$ at all three vertices of $T(\ell)$ and shifted in the integral along $E(j)$ such that the integral of $\text{div}(\psi_j n(j))$ over $T(\ell)$ is equal to $(-1)^{\ell+1} E(j)$ for the relevant triangle number $\ell = j - 1, j$. Those values are independent of $p = 3, 5, 7, \ldots$. In other words, for any $T \in \mathcal{T}(\Omega(z))$, the functionals in $\Lambda_T$ from Section 3 evaluated at $\text{div}(\psi_j n(j))$ with $\psi_j$ from (8.3) coincide for all $p = 3, 5, 7, \ldots$. As a consequence the Vandermonde matrix $M$ in (4.3) is the same for all $p = 3, 5, 7, \ldots$; whence ker $M = \{s\}$. This and the macro-element methodology allow the proof analogously to the cubic case; hence further details are omitted.

**Proof of Lemma 8.1 for $p \geq 4$.** Suppose $E := E(2) = \partial T_1 \cap \partial T_2$ is the common edge of the triangles $T_1 := T_1$ and $T_2 := T_2$ in $\mathcal{T}(E) := \{T_1, T_2\}$ in the notation of Figure 1. Let $\psi_2$ denote the function in (8.3) for $j = 2$ with the edge $E \equiv E(2)$ with unit normal $n := n(2)$. The considerations in the proof of Lemma 5.1 and in the previous proof show that the $4 \times 4$ Vandermonde matrix of the four functionals in $\Lambda_{T(1)}$ and the four functions

$$ 12 \varphi_2^2 \varphi_2^2 \nabla \varphi_2 \mid_{T(1)}, 12 \varphi_2^2 \varphi_z \nabla \varphi_z \mid_{T(1)}, \psi_2 n, 30 \varphi_2^2 \varphi_z^2 \in \text{CR}_0^p(T(E); \mathbb{R}^2) $$

reads, for odd $p \geq 3$ and with the abbreviation $\delta := -12n \cdot \nabla \varphi_2 \mid_{T(1)} > 0$, as

$$ \begin{pmatrix}
12 \left| \nabla \varphi_2 \mid_{T(1)} \right|^2 & 0 & 0 & \left| E \right| n \cdot \nabla \varphi_2 \mid_{T(1)} \\
0 & 0 & 12 \left| \nabla \varphi_z \mid_{T(1)} \right|^2 & \left| E \right| n \cdot \nabla \varphi_z \mid_{T(1)} \\
\delta & (-1)^{p+1} \delta & \delta & \left| E \right| \\
0 & 0 & 0 & \left| E \right|
\end{pmatrix}.$$ 

(For even $p$, $L_p(1) = -L_p'(1)$ leads to a change of signs in the second entry of the third row.) The $4 \times 4$ Vandermonde matrix is regular for all $p \geq 4$. In other words, the triple formed by the triangle $T$, the vector space of the four functions, and the four functionals in $\Lambda_{T(1)}$ is a finite element in the sense of
Ciarlet and allows for an interpolation. That means, given any polynomial \( q \in P_{p-1}(T(1)) \), there exists some

\[
v_h \in \text{span}\{\varphi_2 \varphi_z \nabla \varphi_2|_{T(1)}, \varphi_2 \varphi_z \nabla \varphi_z|_{T(1)}, \psi_2 n, \varphi_2 \varphi_z n\}
\]

with \( \Lambda_{T(1)}(q - \text{div} \, v_h) = 0 \). Proposition \( \ref{prop:interpolation} \) shows \( q = \text{div}(v_h + b) \) in \( T(1) \) for some \( b \in B_p(T(1)) \subset CR^0_p(T(E); \mathbb{R}^2) \). Hence the piecewise divergence \( \text{div}_{pw} : CR^0_p(T(E); \mathbb{R}^2) \to P_{p-1}(T(E))/\mathbb{R} \) maps surjective onto \( P_{p-1}(T(1)) \) and so allows a right-inverse as asserted. A scaling argument with a contravariant Piola transform (cf., e.g., \cite{16, p. 79} or \cite{11, p. 106}) shows that \( \gamma > 0 \) depends on \( p \) and the shape regularity of \( T(E) \). \( \square \)

**Minimal Crouzeix-Raviart method.** This paper has identified Crouzeix-Raviart velocity sub-spaces sufficient for the inf-sup stability for \( p = 3, 5, 7, \ldots \) as the five functions in (4.2) for any interior edge \( E(j) \) plus all the vector-valued bubble-functions \( B_p(T) \) in the triangulation \( T \). A practical implementation for odd \( p = 3, 5, 7, \ldots \) may therefore utilize a strict subspace of \( CR^0_p(T(E); \mathbb{R}^2) \), that consists of the conforming Lagrange polynomials of degree \( p \) in two components and the scalar \( p \)-th order Crouzeix-Raviart edge-bubble function in the normal direction for each interior edge.

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