TURÁN DETERMINANTS OF BESSEL FUNCTIONS

ÁRPÁD BARICZ AND TIBOR K. POGÁNY

Dedicated to Agata, Boróka and Koppány

Abstract. In this paper first we survey the Turán type inequalities and related problems for the Bessel functions of the first kind. Then we extend the known higher order Turán type inequalities for Bessel functions of the first kind to real parameters and we deduce new closed integral representation formulae for the second kind Neumann type series of Bessel functions of the first kind occurring in the study of Turán determinants of Bessel functions of the first kind. At the end of the paper we prove a Turán type inequality for the Bessel functions of the second kind.

1. Introduction

Consider the Bessel function of the first kind and order \( \nu \), defined by \([51, p. 40]\)

\[
J_{\nu}(x) = \sum_{n \geq 0} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left( \frac{x}{2} \right)^{2n+\nu},
\]

which is a particular solution of the second-order homogeneous Bessel differential equation \([51, p. 38]\)

\[
x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.
\]

In this paper our aim is to survey the Turán type and related inequalities for the Bessel functions of the first kind. The Turán type inequalities have a long history and because of some applications, recently many authors have deduced new Turán type inequalities for some special functions (e.g. Krätzel functions, Gauss and Kummer hypergeometric functions, generalized hypergeometric functions, modified Bessel functions, Euler gamma function). For more details we refer to the papers \([3–12], [15, 17, 23, 24, 31, 34, 46]\) and to the references contained therein. Motivated by the vivid interest on Turán type inequalities in this paper we would like to show how a simple and beautiful inequality, like Turán inequality, can occur in many problems of analysis, especially classical analysis. This survey treats interesting and nice results of different authors.

The paper is organized as follows: in section 2 we survey the Turán type inequalities for Bessel functions of the first kind and some related problems. The results are presented in the chronological order and relevant connections between them are pointed out.

2000 Mathematics Subject Classification. Primary 33C15, Secondary 26D07.

Key words and phrases. Bessel functions of the first kind, Bessel functions of the second kind, Turán type inequalities, Laguerre type inequalities, Turán and Hankel determinants, entire functions, Laguerre-Pólya class, Neumann series of Bessel functions of the first kind, recurrence relations, integral representations, bounds.
2.1. Turán type inequalities for Bessel functions of the first kind. Let us start with von Lommel’s formula \[51\] p. 152 for Bessel functions of the first kind
\[
x^2 \left[ J'_\nu(x) - J_{\nu-1}(x)J_{\nu+1}(x) \right] = \sum_{n \geq 0} (\nu + 1 + 2n) J'_\nu+1+2n(x),
\]
which is valid for all \(x \in \mathbb{R}\) and \(\nu > -1\). Then clearly we obtain the following Turán type inequality for all \(x \in \mathbb{R}\) and \(\nu > -1\)
\[
\Delta_\nu(x) = J'_\nu(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq 0.
\]
The above simple and beautiful inequality had attracted many mathematicians and has been proved in several ways. In 1950 and 1951 Szász \[48, 49\] deduced (2.2) and its sharper form
\[
\sum_{n \geq 0} (\nu + 1 + 2n) J'_\nu+1+2n(x),
\]
where \(x \in \mathbb{R}\) and \(\nu > 0\), by using the recurrence relations for the Bessel functions of the first kind. We note that, since
\[
\lim_{x \to 0} \frac{\Delta_\nu(x)}{J'_\nu(x)} = 1 - \lim_{x \to 0} \frac{J_{\nu-1}(x)J_{\nu+1}(x)}{J'_\nu(x)} = \frac{1}{\nu + 1},
\]
the constant \(1/(\nu + 1)\) on the right-hand side of (2.3) is the best possible. See \[11\] or the last subsection of this section for more details. In 1951 Thiruvenkatachar and Nanjundiah \[50\] proved also (2.2) by using recurrence relations and deduced (2.3) by means of the following formula
\[
\nu J'_\nu(x) - (\nu + 1)J_{\nu-1}(x)J_{\nu+1}(x) = J'_\nu(x) + \nu [J_{\nu+1}(x) - J_\nu(x)J_{\nu+2}(x)].
\]
The constant \(1/(\nu + 1)\) on the right-hand side of (2.3) is the best possible. See \[11\] or the last subsection of this section for more details. In 1951 Thiruvenkatachar and Nanjundiah \[50\] proved also (2.2) by using recurrence relations and deduced (2.3) by means of the following formula
\[
\nu J'_\nu(x) - (\nu + 1)J_{\nu-1}(x)J_{\nu+1}(x) = J'_\nu(x) + \nu [J_{\nu+1}(x) - J_\nu(x)J_{\nu+2}(x)].
\]
It is interesting to note here that the sharp inequality (2.2) in view of (2.3) is actually a consequence of the weaker result (2.2). With other words, for \(\nu > 0\) the Turán type inequalities (2.2) and (2.3) are equivalent. Note also that by using (2.5) Thiruvenkatachar and Nanjundiah \[50\] proved the following formula
\[
\frac{\Delta_\nu(x)}{J'_\nu(x)} = 1 - \lim_{x \to 0} \frac{J_{\nu-1}(x)J_{\nu+1}(x)}{J'_\nu(x)} = \frac{1}{\nu + 1},
\]
which clearly implies the sharp Turán type inequality (2.3). Moreover, it should be mentioned that in view of (2.6) the Turán type inequality (2.3) can be improved as follows
\[
J'_\nu(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq \frac{1}{\nu + 1} J'_\nu(x) + \frac{2}{\nu + 2} J_{\nu+1}(x),
\]
where \(x \in \mathbb{R}\) and \(\nu \geq 0\). Observe that the Turán type inequality (2.7) is sharp as \(x = 0\) or \(\nu = 0\). This can be seen easily by using that
\[
\lim_{x \to 0} \frac{2\nu \sum_{n \geq 2} J'_{\nu+n}(x)}{J_{\nu+1}(x)} = 0
\]
and by noting that \(J_{-1}(x) = J_1(x)\) for all real \(x\).

Another proof for the Turán type inequality (2.2) was given by Skovgaard \[17\] in 1954. Skovgaard’s proof is somewhat similar to the proof of Thiruvenkatachar and Nanjundiah \[50\], however in \[17\] there is used in addition the infinite product representation \[51\] p. 498 of the Bessel function of the first kind
\[
2^\nu \Gamma(\nu + 1)x^{-\nu} J_\nu(x) = \prod_{n \geq 1} \left( 1 - \frac{x^2}{J_{\nu+2n}(x)} \right).
\]

2.2. Turán and Hankel determinants of Bessel functions of the first kind. A generalization of the Turán type inequality (2.2) has been proved by Karlin and Szegő in their mammoth work \[30\] p. 130 in 1960. They proved that for all \(x > 0\), \(\nu > -1\) and \(n\) even natural number the following inequalities are valid for the Turán and Hankel determinants of Bessel functions
\[
(-1)^\frac{n}{2} \begin{vmatrix}
J_\nu(x) & J_{\nu+1}(x) & \cdots & J_{\nu+n-1}(x) \\
J_{\nu+1}(x) & J_{\nu+2}(x) & \cdots & J_{\nu+n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
J_{\nu+n-1}(x) & J_{\nu+n}(x) & \cdots & J_{\nu+2n-2}(x)
\end{vmatrix} \geq 0.
\]
and

\[
(-1)^\frac{m}{2} \begin{vmatrix}
\tilde{J}_\nu(x) & \tilde{J}_\nu'(x) & \cdots & \tilde{J}_\nu^{(n-1)}(x) \\
\tilde{J}_\nu'(x) & \tilde{J}_\nu''(x) & \cdots & \tilde{J}_\nu^{(n)}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{J}_\nu^{(n-1)}(x) & \tilde{J}_\nu^{(n)}(x) & \cdots & \tilde{J}_\nu^{(2n-2)}(x)
\end{vmatrix} \geq 0,
\]

where \( \tilde{J}_\nu(x) = x^{-\frac{\nu}{2}}J_\nu(2\sqrt{x}) \). Moreover, Karlin and Szegö [31, p. 131] pointed out that in the above Turán determinant we can cancel out the factors of powers of \( x \) to have

\[
(-1)^\frac{m}{2} \begin{vmatrix}
J_\nu(2\sqrt{x}) & J_{\nu+1}(2\sqrt{x}) & \cdots & J_{\nu+n-1}(2\sqrt{x}) \\
J_{\nu+1}(2\sqrt{x}) & J_{\nu+2}(2\sqrt{x}) & \cdots & J_{\nu+n}(2\sqrt{x}) \\
\vdots & \vdots & \ddots & \vdots \\
J_{\nu+n-1}(2\sqrt{x}) & J_{\nu+n}(2\sqrt{x}) & \cdots & J_{\nu+2n-2}(2\sqrt{x})
\end{vmatrix} \geq 0,
\]

which can be rewritten also as

\[
(-1)^\frac{m}{2} \begin{vmatrix}
J_\nu(x) & J_{\nu+1}(x) & \cdots & J_{\nu+n-1}(x) \\
J_{\nu+1}(x) & J_{\nu+2}(x) & \cdots & J_{\nu+n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
J_{\nu+n-1}(x) & J_{\nu+n}(x) & \cdots & J_{\nu+2n-2}(x)
\end{vmatrix} \geq 0.
\]

2.3. Generalized Turán expression of Bessel functions. Now, we are going to present a generalization of von Lommel’s result (2.9). In 1961 Al-Salam [2] proved the following result:

\[
\frac{4^n(2m)!}{x^{2m}m!(m-1)!} \sum_{k \geq 0} (\nu + m + 2k)(k+1)_{m-1}(\nu + k + 1)_{m-1}J_{\nu+m+2k}^2(x)
\]

(2.9)

\[= \sum_{n=-m}^m (-1)^n \left( \frac{2m}{m-n} \right) J_{\nu-n}(x)J_{\nu+n}(x),\]

where \((a)_0 = 1\) for \( a \neq 0 \) and \((a)_n = a(a+1)(a+2)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a)\) for \( n \in \{1,2,\ldots\} \) is the well-known Pochhammer (or Appell) symbol in terms of the Euler gamma function. For reader’s convenience we note that there is a small typographical mistake in the original formula of Al-Salam: the term \((n+k-1)_{m-1}\) in [2, Eq. (1.2)] should be written as \((n+k+1)_{m-1}\). We also note that in [2] the author does not discussed the range of validity of (2.9), however, a close inspection of the proof of (2.9) reveals that this formula holds for all \( m \in \{0,1,\ldots\} \), \( x \in \mathbb{R} \) and \( \nu > -1 \). Furthermore, (2.9) implies the extended Turán type inequality

\[
\sum_{n=-m}^m (-1)^n \left( \frac{2m}{m-n} \right) J_{\nu-n}(x)J_{\nu+n}(x) \geq 0,
\]

where \( m \in \{0,1,\ldots\} \), \( x \in \mathbb{R} \) and \( \nu > -1 \). Observe that when \( m = 1 \) in (2.9) and (2.10), then we reobtain (2.11) and (2.2). Moreover, if we choose \( m = 2 \) in (2.9), then we obtain

\[
192x^{-4} \sum_{k \geq 0} (\nu + 2k + 2)(k+1)_{2}(\nu + k + 1)_{2}J_{\nu+2k+2}^2(x)
\]

\[= 6J_{\nu}^2(x) - 8J_{\nu-1}(x)J_{\nu+1}(x) + 2J_{\nu-2}(x)J_{\nu+2}(x)
\]

\[= 6\Delta_\nu(x) + 2[J_{\nu-2}(x)J_{\nu+2}(x) - J_{\nu-1}(x)J_{\nu+1}(x)],\]

which immediately implies the new Turán type inequality

\[3\Delta_\nu(x) \geq J_{\nu-1}(x)J_{\nu+1}(x) - J_{\nu-2}(x)J_{\nu+2}(x),\]

where \( x \in \mathbb{R} \) and \( \nu > -1 \).

2.4. Properties of the relative minima and maxima of Turán expression. In 1961 Lakshmana Rao [38] proved the following result concerning the Turán expression \( \Delta_\nu(x) \):

**Theorem 1.** Let \( \nu > 0 \) and \( x \in \mathbb{R} \). The relative maxima (denoted by \( M_{\nu,n} \)) of the function \( x \mapsto \Delta_\nu(x) \) occur at the zeros of \( J_{\nu-1} \) and the relative minima (denoted by \( m_{\nu,n} \)) occur at the zeros of \( J_{\nu+1} \). The values \( M_{\nu,n} \) and \( m_{\nu,n} \) can be expressed as

\[M_{\nu,n} = \Delta_\nu(j_{\nu-1,n}) = J_{\nu}^2(j_{\nu-1,n}),\]

\[m_{\nu,n} = \Delta_\nu(j_{\nu+1,n}) = J_{\nu}^2(j_{\nu+1,n}).\]
and
\[ m_{\nu,n} = \Delta_{\nu}(j_{\nu+1,n}) = J_{\nu}^{2}(j_{\nu+1,n}). \]
Since the quantities \( M_{\nu,n} \) and \( m_{\nu,n} \) are positive for all values of \( \nu \), we have the Turán type inequality \[2.2\]. Moreover, the sequences \( \{M_{\nu,n}\}_{n \geq 1} \) and \( \{m_{\nu,n}\}_{n \geq 1} \) are decreasing for \( \nu \) sufficiently large and in addition, for each \( \nu \) we have \( M_{\nu,n} > m_{\nu,n} \). Finally, for a fixed value of \( \nu \), the functions \( \nu \mapsto M_{\nu,n} \) and \( \nu \mapsto m_{\nu,n} \) are decreasing.

### 2.5. Log-concavity of Bessel functions of the first kind with respect to their order.

In 1978 Ismail and Muldoon \[26\] Lemma 2.3], by using von Neumann’s formula for product of two Bessel functions with different order, proved that for each fixed \( \beta \), with different order, proved that for each fixed \( \nu \), Ismail and Muldoon \[26, Lemma 2.3\], by using von Neumann’s formula for product of two Bessel functions, proved that for each fixed \( \beta \), the function \( \nu \mapsto J_{\nu+\beta}(x)/J_\nu(x) \) is decreasing on \([-(\beta + 1)/2, \infty)\). As it was pointed out later in 1997 by Muldoon \[41\], this actually implies that for fixed \( x > 0 \) and \( \nu \neq j_{\nu,n}, \ n \in \{1, 2, \ldots\} \), the function \( \nu \mapsto J_{\nu}(x) \) is logarithmically concave on \((-1, \infty)\). And this clearly yields the Turán type inequality \[2.2\] for \( \nu > 0 \) and \( x > 0 \) such that \( x \neq j_{\nu,n}, \ n \in \{1, 2, \ldots\} \). We note that the remained part of the Turán type inequality \[2.2\] when \( x = j_{\nu,n}, \ n \in \{1, 2, \ldots\} \), can be verified easily by using the recurrence relations
\[ \begin{align*}
x J_{\nu}(x) + \nu J_{\nu}(x) &= x J_{\nu-1}(x) \\
x J_{\nu}(x) - \nu J_{\nu}(x) &= -x J_{\nu+1}(x).
\end{align*} \]

More precisely, in view of the formulas \[2.11\] and \[2.12\] for all \( n \in \{1, 2, \ldots\} \) and all \( \nu > -1 \) we have
\[ \Delta_{\nu}(j_{\nu,n}) = -J_{\nu-1}(j_{\nu,n})J_{\nu+1}(j_{\nu,n}) = [J_{\nu}^2(j_{\nu,n})] > 0. \]

We note that some of the results (and proofs) of Szász \[49\] concerning the Turán type inequalities \[2.2\] and \[2.3\] have been reobtained later in 1991 by Joshi and Bissu \[28\]. Moreover, it should be mentioned that in 1997 Bustoz and Ismail \[18\] proved also the Turán type inequality \[2.3\] by showing the positivity of Turán determinants of symmetric Pollaczek polynomials, von Lommel polynomials and \( q \)-Bessel functions. Finally, for applications of the Turán type inequality \[2.3\] and \[2.8\] in the extension of some trigonometric inequalities (like Mitrinović-Adamović and Wilker) to Bessel functions we refer to the papers \[6\] and \[16\].

### 2.6. Zeros of the Turán expression \( \Delta_{\nu} \) in the complex plane.

In 2001 Kravanja and Verlinden \[33\] applied the Turán type inequality \[2.2\] in the study of the zeros of \( \Delta_{\nu} \). More precisely, the Turán-type inequality \[2.2\] implies that no zero exists on the positive real axis, and, by symmetry, neither on the negative real axis. Moreover, Kravanja and Verlinden \[33\] proved the following result:

**Theorem 2.** Except for \( z = 0 \), the function \( z \mapsto \Delta_{\nu}(z) \) has no zeros on the imaginary axis. Moreover, all the zeros of \( z \mapsto \Delta_{\nu}(z) \) that lies in \( \mathbb{C} \setminus \{0\} \) are simple.

The zeros of the function \( z \mapsto \Delta_{\nu}(z) \) play an important role in certain physical applications. For example, MacDonald \[38\] used the zeros of \( \Delta_{1} \) to plot representative streamlines for the steady motion of a viscous fluid in a long tube, of constant radius, which rotates about its axis with an angular velocity that changes discontinuously at \( z = 0 \) from one constant value to another of the same sign.

It is also worth mentioning here that the Turán expression \( \Delta_{\nu}(z) \) appears in other physical applications. More precisely, the series
\[ \sum_{n=-\infty}^{\infty} [\Delta_{n}(x)]^2 \]
and
\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Delta_{n}(x)\Delta_{m}(x)\Delta_{n-m}(\mu x), \]
where \( \mu \) is a constant, appear in the investigation into multiple scattering of acoustic waves by random configurations of penetrable circular cylinders. For more details see \[39\], where the asymptotic behavior for large positive \( x \) of the above series were investigated.
2.7. Lower bounds for the logarithmic derivative of Bessel functions of the first kind. In this subsection our aim is to deduce some new inequalities for the logarithmic derivative of $J_\nu$, and for the quotient $J_\nu/J_{\nu-1}$ by using the Turán type inequalities (2.2) and (2.3). Observe that combining (2.2) with (2.11) and (2.12) we immediately get the inequality

$$[xJ'_\nu(x)]^2 + (x^2 - \nu^2)J^2_\nu(x) \geq 0,$$

which holds for all $\nu > -1$ and $x \in \mathbb{R}$. Moreover, combining (2.13) with (2.11) we arrive at

$$xJ^2_\nu(x) - 2\nu J_{\nu-1}(x)J_\nu(x) + xJ^2_{\nu-1}(x) \geq 0,$$

where $\nu > -1$ and $x \in \mathbb{R}$. The inequality (2.13) implies the inequality

$$\left[\frac{xJ'_\nu(x)}{J_\nu(x)}\right]^2 \geq \nu^2 - x^2,$$

where $\nu > -1$ and $x \neq j_{\nu,n}$, $n \in \{1, 2, \ldots\}$. While (2.11) implies the following

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \leq \frac{\nu + \sqrt{\nu^2 - x^2}}{x},$$

where $\nu > 0$, $x \in (0, \nu]$ such that $x \neq j_{\nu-1,n}$, $n \in \{1, 2, \ldots\}$.

Now, we are going to improve the above inequalities. Let $\mu = \nu/(\nu + 1)$. Then combining (2.3) with (2.11) and (2.12) we immediately get the inequality

$$[xJ'_\nu(x)]^2 + (\mu x^2 - \nu^2)J^2_\nu(x) \geq 0,$$

which holds for all $\nu > -1$ and $x \in \mathbb{R}$. Moreover, combining (2.17) with (2.11) we obtain the following

$$\mu xJ^2_\nu(x) - 2\nu J_{\nu-1}(x)J_\nu(x) + xJ^2_{\nu-1}(x) \geq 0,$$

where $\nu > -1$ and $x \in \mathbb{R}$. The inequality (2.17) implies the inequality

$$\left[\frac{xJ'_\nu(x)}{J_\nu(x)}\right]^2 \geq \nu^2 - \mu x^2,$$

where $\nu > -1$ and $x \neq j_{\nu,n}$, $n \in \{1, 2, \ldots\}$. While (2.13) implies the following

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} \leq \frac{\nu + \sqrt{\nu^2 - \mu x^2}}{\mu x},$$

where $\nu > 0$, $x \in (0, \sqrt{\nu(\nu + 1)}]$ such that $x \neq j_{\nu-1,n}$, $n \in \{1, 2, \ldots\}$.

Observe that since $\nu > -1$ we have $\mu < 1$ and consequently

$$\nu^2 - \mu x^2 > \nu^2 - x^2$$

and

$$\frac{\nu + \sqrt{\nu^2 - \mu x^2}}{\mu x} > \frac{\nu + \sqrt{\nu^2 - x^2}}{x}$$

hold for all $\nu, x > 0$. These in turn imply that (2.19) improves (2.15) and (2.20) improves (2.16).

Finally, it is important to note here that similar inequalities to those given in (2.18) - (2.20) for modified Bessel functions of the first and second kinds $I_\nu$ and $K_\nu$ were given recently by Baricz [9], Laforgia and Natalini [35], and Segura [46]. For more details see also [15] and the references therein.

2.8. On a conjecture for Bessel functions of the first kind. In this subsection we recall a recent conjecture of the first author [11]. Consider the function $\Phi : (0, \infty) \setminus \Xi \to (0, \infty)$, defined by (see [11])

$$\Phi_\nu(x) = \frac{\Delta_\nu(x)J_{\nu+1}(x)}{J^2_\nu(x)} = 1 - \frac{J_{\nu-1}(x)J_{\nu+1}(x)}{J^2_\nu(x)} = \sum_{n \geq 1} \frac{4J^2_{\nu,n}}{(x^2 - J^2_{\nu,n})^2},$$

where $\Xi = \{j_{\nu,n} : n \geq 1\}$ is the set of the zeros of the Bessel function of the first kind of order $\nu$. Based on numerical experiments we conjecture (see [11]) that Szász’s result (2.23) can be improved as follows.

**Conjecture.** Let $\nu > 0$. Then the equation $\Phi_\nu'(x) = 0$ has infinitely many roots. Denoting with $\alpha_{\nu,n}$ these roots, where $\alpha_{\nu,n} \in (j_{\nu,n}, j_{\nu,n+1})$ for all $n \in \{1, 2, \ldots\}$, the Turán type inequality (2.23) can be improved as follows

$$J^2_\nu(x) - J_{\nu-1}(x)J_{\nu+1}(x) > \beta_{\nu,n}J^2_\nu(x),$$

where $x \in (j_{\nu,n}, j_{\nu,n+1})$, $\beta_{\nu,n} = \Phi_\nu(\alpha_{\nu,n})$ for all $n \in \{1, 2, \ldots\}$ and the sequence $\{\beta_{\nu,n}\}_{n \geq 0}$ is strictly increasing, where $\alpha_{\nu,0} = j_{\nu,0} = 0$ and $\beta_{\nu,0} = \Phi_\nu(\alpha_{\nu,0}) = 1/(\nu + 1)$. 

Let \( n \in \{1, 2, \ldots\} \) be fixed. Since
\[
\Phi'(j_{\nu-1,n}) = \frac{2\nu}{j_{\nu-1,n}} > 0 \quad \text{and} \quad \Phi'(j_{\nu+1,n}) = -\frac{2\nu}{j_{\nu+1,n}} < 0,
\]
we obtain that \( \alpha_{\nu,n} \in (j_{\nu+1,n}, j_{\nu-1,n}+1) \). Now, since (according to Theorem 1) the function \( x \mapsto \Delta_\nu(x) \) is increasing on \((j_{\nu+1,n}, j_{\nu-1,n}+1)\), we obtain that
\[
\Phi'_\nu(\alpha_{\nu,n}) = (2/\alpha_{\nu,n}) J_{\nu-1}(\alpha_{\nu,n}) J_{\nu+1}(\alpha_{\nu,n})
= (2/\alpha_{\nu,n}) J^2_\nu(\alpha_{\nu,n}) (1 - \Phi_\nu(\alpha_{\nu,n})) > 0,
\]
which in turn implies that \( \beta_{\nu,n} = \Phi_\nu(\alpha_{\nu,n}) < 1 \) for all \( \nu > 0 \) and \( n \in \{1, 2, \ldots\} \). It is also worth to mention here that the relative minima \( \beta_{\nu,n} \) does not occur in the zeros \( j_{\nu,n} \) of \( J_\nu' \), although it seems from graphics. For example, for \( \nu = 2 \) we have \( \alpha_{2,1} = 6.690090363, \alpha_{2,2} = 9.965082278, \alpha_{2,3} = 13.1685359, \alpha_{2,4} = 16.3465786 \) and \( J_{2,2}' = 6.706133194, J_{2,3}' = 9.969467823, J_{2,4}' = 13.1703708, J_{2,5}' = 16.3475223 \).

3. Higher order Turán type inequalities for Bessel functions of the first kind

This section is devoted to the study of higher order Turán type inequalities and related problems for the Bessel functions of the first kind. The results stated here complement naturally those presented in the previous section.

3.1. The Laguerre-Pólya class of entire functions and necessary conditions. By definition an entire function is a function of a complex variable which is holomorphic in the entire complex plane and can be represented by an everywhere convergent power series
\[
f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots\]
An entire function \( f \) is called real if it is real on the real axis, or equivalently, if it has only real coefficients in its power series. Now, by definition the real entire function \( \varphi \), defined by
\[
\varphi(x) = \varphi(x; t) = \sum_{n=0}^{\infty} b_n(t) \frac{x^n}{n!},
\]
is said to be in the Laguerre-Pólya class (denoted by \( \mathcal{LP} \)), if \( \varphi(x) \) can be expressed in the form
\[
\varphi(x) = cx^d e^{-ax^2 + \beta x} \prod_{i=1}^{\omega} \left(1 - \frac{y}{x_i}\right) e^{\frac{y}{x}}, \quad 0 \leq \omega \leq \infty,
\]
where \( c \) and \( \beta \) are real, \( x_i \)’s are real and nonzero for all \( i \in \{1, 2, \ldots, \omega\} \), \( \alpha \geq 0 \), \( d \) is a nonnegative integer and \( \sum_{i=1}^{\omega} x_i^{-2} < \infty \). If \( \omega = 0 \), then, by convention, the product is defined to be 1. For the various properties of the functions in the Laguerre-Pólya class we refer to [19, 20, 21, 22] and to the references contained therein. We note that in fact a real entire function \( \varphi \) is in the Laguerre-Pólya class if and only if \( \varphi \) can be uniformly approximated on disks around the origin by a sequence of polynomials with only real zeros. This in turn implies that the class \( \mathcal{LP} \) is closed under differentiation, that is, if \( \varphi \in \mathcal{LP} \), then \( \varphi^{(n)} \in \mathcal{LP} \) for all \( n \) nonnegative integer.

Recall that if a real entire function \( \varphi \) belongs to the Laguerre-Pólya class \( \mathcal{LP} \) then satisfies the Laguerre type inequalities
\[
\left[ \varphi^{(n)}(x) \right]^2 - \varphi^{(n-1)}(x) \varphi^{(n+1)}(x) \geq 0, \quad n \in \{1, 2, \ldots\}
\]
and
\[
\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} \varphi^{(n+j)}(x) \varphi^{(n+2k-j)}(x) \geq 0, \quad n, k \in \{0, 1, \ldots\}.
\]
The inequality (3.2) is due to Skovgaard [17], while (3.3) for \( n = 0 \) has been proved first by Jensen [27] and for \( n \in \{0, 1, \ldots, \} \) by Patrick [42].

Now, we list some other necessary conditions for a real entire function to belong to \( \mathcal{LP} \). For the inequality (3.2) see [19, 22], the inequality (3.3) is due to Dimitrov [23] Theorem 1, while (3.4) can be found in Patrick’s paper [42] Theorem 2. We note that when \( k = 1 \) the inequality (3.6) reduces to (3.4).
Theorem 3. Let \( \{b_n(t)\}_{n \geq 0} \) be a sequence of real functions which for certain values of \( t \) have a generating function of the type (3.1) and suppose that the real entire function \( \varphi \), defined by (3.1), is in the class \( \mathcal{LP} \). Then for those values of \( t \) the following Turán type and higher order Turán type inequalities hold

(3.4)
\[
b_n^2(t) - b_{n-1}(t)b_{n+1}(t) \geq 0,
\]
(3.5)
\[
4 \left[ b_n^2(t) - b_{n-1}(t)b_{n+1}(t) \right] \left[ b_{n+1}^2(t) - b_n(t)b_{n+2}(t) \right] - b_n(t)b_{n+1}(t) - b_{n-1}(t)b_{n+2}(t))^2 \geq 0,
\]
(3.6)
\[
\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} b_{n+j}(t)b_{n+2k-j}(t) \geq 0, \quad n, k \in \{0, 1, \ldots \},
\]
where in (3.4) and (3.5) it is assumed that \( n \in \{1, 2, \ldots \} \).

3.2. Higher order Turán type inequalities for Bessel functions of natural order. Now, we are going to point out other results on Bessel functions of the first kind. Some of these results are known, however we have included here for the sake of completeness. We start with the formula

(3.7)
\[
J_0 \left( \sqrt{t^2 - 2xt} \right) = \sum_{n \geq 0} J_n(t) \frac{x^n}{n!}, \quad t \in \mathbb{R},
\]
which is actually a special case of Bessel’s formula (proved later also by von Lommel) [51, p. 140]
\[
(z + h)^{-\nu/2}J_\nu \left( \sqrt{z + h} \right) = \sum_{n \geq 0} \frac{(-h/2)^n}{n!} z^{-(\nu+n)/2} J_{\nu+n} \left( \sqrt{z} \right)
\]
rewritten in the form

(3.8)
\[
t^{\nu/2}(t^2 - 2xt)^{-\nu/2}J_\nu \left( \sqrt{t^2 - 2xt} \right) = \sum_{n \geq 0} J_{\nu+n}(t) \frac{x^n}{n!}, \quad t \in \mathbb{R}.
\]

Since the Bessel function \( J_0 \) belongs to the Laguerre-Pólya class \( \mathcal{LP} \) (see [45, p. 123]), by using the particular Bessel formula (3.7), we can see easily that the Bessel functions of the first kind \( J_n \) satisfies the inequalities (3.4), (3.5) and (3.6), that is, for all \( t \in \mathbb{R} \) we have the inequalities

(3.9)
\[
\Delta_n(t) \geq 0, \quad n \in \{1, 2, \ldots \},
\]
(3.10)
\[
4 [\Delta_n(t) - J_n(t)J_{n+1}(t) - J_{n-1}(t)J_{n+2}(t)]^2 \geq 0, \quad n \in \{1, 2, \ldots \},
\]
(3.11)
\[
\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} J_{n+j}(t)J_{n+2k-j}(t) \geq 0, \quad n, k \in \{0, 1, \ldots \}.
\]

The Turán type inequality (3.9) is well-known, it has been discussed in details in the previous section. The inequality (3.10) has been deduced by Patrick [47, and 3.10] seems to be new. We note that, as it was pointed out by Skovgaard [47] in 1954 and later by Patrick [42] in 1973, the inequalities (3.9) and (3.10) are also satisfied by the first derivative \( J'_n \) of the Bessel function of the first kind \( J_n \). More precisely, since the series
\[
\sum_{n \geq 0} J'_n(t) \frac{x^n}{n!}
\]
is uniformly convergent for \( t \in \mathbb{R} \), differentiating both sides of (3.7) with respect to \( t \) and by using the relation \( J'_0(z) = -J_1(z) \) we get
\[
(x - t)(t^2 - 2xt)^{-1/2}J_1 \left( \sqrt{t^2 - 2xt} \right) = \sum_{n \geq 0} J'_n(t) \frac{x^n}{n!}.
\]
Now taking into account that \( J_0 \in \mathcal{LP} \) and the Laguerre-Pólya class is closed under differentiation, the left-hand side of the above relation also belongs to the Laguerre-Pólya class \( \mathcal{LP} \) and consequently (in view of Theorem 3) for all \( t \in \mathbb{R} \) we have the following inequalities

(3.12)
\[
[J'_n(t)]^2 - J'_{n-1}(t)J'_{n+1}(t) \geq 0,
\]
(3.13)
\[
4 [J'_n(t)]^2 - J'_{n-1}(t)J'_{n+1}(t) [J'_{n+1}(t)]^2 - J'_n(t)J'_{n+2}(t)] - [J'_n(t)J'_{n+1}(t) - J'_{n-1}(t)J'_{n+2}(t)]^2 \geq 0,
\]
The inequality (3.12) is due to Skovgaard [47] and holds for all \( n \in \{1, 2, \ldots \} \), while the inequality (3.14) is due to Patrick [42]. The inequality (3.13) appears to be new and holds for all \( n \in \{1, 2, \ldots \} \). Moreover, as we can see below, it can be shown that in fact the above inequalities for \( J_n \) and \( J'_n \) can be extended to real parameters.

### 3.3. Higher order Turán type inequalities for Bessel functions of real order.

In order to extend the inequalities (3.9), (3.10), (3.11), (3.12), (3.13) and (3.14) to the Bessel function of the first kind \( J_\nu \) and its derivative \( J'_\nu \), where \( \nu > -1 \), we shall use the following result of Krasikov [32] and of Dimitrov and Ben Cheikh [25].

**Lemma.** The function \( J_\nu : \mathbb{R} \to (-\infty, 1] \), defined by \( J_\nu(x) = 2^{\nu} \Gamma(\nu + 1)x^{-\nu}J_\nu(x) \), belongs to the Laguerre-Pólya class \( \mathcal{LP} \) when \( \nu > -1 \).

Note that Krasikov’s [32] argument is simply and use only the infinite product representation (2.8), by noticing that the exponential factors in (2.8) are canceled because of the symmetry of the zeros \( j_{\nu, n} \) with respect to the origin. The proof of the above lemma given by Dimitrov and Ben Cheikh [25] by using the above lemma reobtained von Lommel’s celebrated result: the zeros of the Bessel function \( J_\nu \) are all real when \( \nu > -1 \). Moreover, they proved that the Jensen polynomials associated with the Bessel function \( J_\nu \), properly normalized, are exactly the Laguerre polynomials. More precisely, in [25] it was shown that the only Jensen polynomials associated with an entire function in the Laguerre-Pólya class \( \mathcal{LP} \) that are orthogonal are the Laguerre polynomials.

Now, we are in the position to state and prove the following inequalities.

**Theorem 4.** Let \( \nu > -1 \) and \( t \in \mathbb{R} \). Then the following Laguerre type inequalities

\[
(3.15) \quad [J_\nu^{(n)}(t)]^2 - J_\nu^{(n-1)}(t)J_\nu^{(n+1)}(t) \geq 0, \quad n \in \{1, 2, \ldots \},
\]

and Turán type inequalities

\[
(3.16) \quad \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} J_\nu^{(n+j)}(t)J_\nu^{(n+2k-j)}(t) \geq 0, \quad n, k \in \{0, 1, \ldots \},
\]

and

\[
(3.17) \quad J_{\nu+1}^2(t) - J_\nu(t)J_{\nu+2}(t) \geq 0,
\]

\[
(3.18) \quad 4 \left[ J_{\nu+1}^2(t) - J_\nu(t)J_{\nu+2}(t) \right] \left[ J_{\nu+2}^2(t) - J_{\nu+1}(t)J_{\nu+3}(t) \right]
- \left[ J_{\nu+1}(t)J_{\nu+2}(t) - J_\nu(t)J_{\nu+3}(t) \right]^2 \geq 0,
\]

\[
(3.19) \quad \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} J_{\nu+j}(t)J_{\nu+2k-j}(t) \geq 0, \quad k \in \{0, 1, \ldots \},
\]

and

\[
(3.20) \quad [J_{\nu+1}^2(t)]^2 - J_\nu^2(t)J_{\nu+2}^2(t) \geq 0,
\]

\[
(3.21) \quad 4 \left[ [J_{\nu+1}^2(t)]^2 - J_\nu^2(t)J_{\nu+2}^2(t) \right] \left[ [J_{\nu+2}^2(t)]^2 - J_{\nu+1}^2(t)J_{\nu+3}^2(t) \right]
- [J_{\nu+1}^2(t)J_{\nu+2}^2(t) - J_\nu^2(t)J_{\nu+3}^2(t)]^2 \geq 0,
\]

are valid.
Proof. Inequalities (3.15) and (3.16) follow immediately from Lemma and the Laguerre type inequalities (3.2) and (3.3). By using Lemma, the left-hand side of (3.8) belongs to the Laguerre-Pólya class \( \mathcal{LP} \) and consequently Theorem 3 guarantees that the inequalities (3.9), (3.10) and (3.11) hold true for \( \nu + n \) instead of \( n \). Now, choosing in the modified form of (3.9), (3.10) the value \( n = 1 \) and in the modified form of (3.11) the value \( n = 0 \), we get the inequalities (3.17), (3.18) and (3.19).

By using the idea of Skovgaard, since the series
\[
\sum_{n \geq 0} J'_{\nu+n}(t) \frac{x^n}{n!}
\]
is uniformly convergent for \( t \in \mathbb{R} \), differentiating both sides of (3.8) with respect to \( t \) we get
\[
\left[t^\nu(i^2 - 2xt)^{-\nu/2} J_\nu \left( \sqrt{t^2 - 2xt} \right) \right]' = \sum_{n \geq 0} J'_{\nu+n}(t) \frac{x^n}{n!}, \quad t \in \mathbb{R}.
\]

Now, because of Lemma the left-hand side of (3.8) belongs to \( \mathcal{LP} \), the left-hand side of the above relation belongs also to \( \mathcal{LP} \). This together with Theorem 3 imply that the Turán type inequalities (3.12), (3.13) and (3.14) hold true for \( \nu + n \) instead of \( n \). Similarly, as above, by choosing in the modified form of (3.12), (3.13) the value \( n = 1 \) and in the modified form of (3.14) the value \( n = 0 \), we get the Turán type inequalities (3.20), (3.21) and (3.22).

Remark. We note that when \( k = 1 \) the inequality (3.19) reduces to (3.17), while the inequality (3.22) becomes (3.20). The inequality (3.17) is exactly (2.2) for \( \nu > 0 \) and the inequality (3.19) bears resemblance of (2.10). We also note here that the inequality (3.11) for \( n = 0 \) was one of the crucial facts in Krasikov’s [32] study in order to obtain sharp uniform bounds for the Bessel functions of the first kind \( J_\nu \).

It is worth to mention here that the Turán type inequality (2.3) can be rewritten in terms of \( J_\nu \) as
\[
J^2_\nu(x) - J_{\nu-1}(x) J_{\nu+1}(x) \geq 0,
\]
where \( \nu > 0 \) and \( x \in \mathbb{R} \). A counterpart of the above Turán type inequality was derived by Szász [49] in 1951 (and also by Joshi and Bissu [28] in 1991) as follows
\[
(\nu + 1) J^2_\nu(x) - \nu J_{\nu-1}(x) J_{\nu+1}(x) \leq 1,
\]
which also holds for all \( \nu > 0 \) and \( x \in \mathbb{R} \). These inequalities together show that the quantity
\[
(\nu + 1) J^2_\nu(x) - \nu J_{\nu-1}(x) J_{\nu+1}(x)
\]
belongs to the interval \([0, 1]\) for all \( x \in \mathbb{R} \) and \( \nu > 0 \). Moreover, by means of the differentiation formulae
\[
J'_\nu(x) = -\frac{x}{2(\nu + 1)} J_{\nu+1}(x),
\]
\[
J''_\nu(x) = -\frac{1}{2(\nu + 1)} J_{\nu+1}(x) + \frac{x^2}{4(\nu + 1)(\nu + 2)} J_{\nu+2}(x),
\]
the Laguerre type inequality (3.15) for \( n = 1 \) becomes
\[
\frac{x^2}{2(\nu + 1)} J^2_{\nu+1}(x) + J_\nu(x) J_{\nu+1}(x) \geq \frac{\nu^2}{2(\nu + 2)} J_\nu(x) J_{\nu+2}(x)
\]
which is equivalent to
\[
J^2_{\nu+1}(x) - J_\nu(x) J_{\nu+2}(x) \geq -\frac{1}{x} J_\nu(x) J_{\nu+1}(x), \quad x \neq 0, \nu > -1.
\]

4. Integral representations for second kind Neumann type series of Bessel functions

The applications of the Neumann series of Bessel functions
\[
\mathfrak{N}_\nu(x) = \sum_{n \geq 1} a_n J_{\nu+n}(x)
\]
in science, engineering and technology has been scattered widely in the literature, see for example [44] and the references therein. Recently, Pogány and Suli [44] deduced closed integral expression for \( \mathfrak{N}_\nu(x) \) and with this initiated considerable interest to closed integral expressions for Neumann series of another kind of Bessel functions, see e.g. the results by Baricz et al. [13, 14].
In this section our aim is to deduce a closed integral representation formula for the second kind Neumann type series of Bessel functions defined in the form

\[(4.1) \quad \varphi_{\mu,\nu}^{a,b}(x) := \sum_{n \geq 1} \theta_n J_{\mu+an}(x)J_{\nu+bn}(x), \quad \mu, \nu, a, b \in \mathbb{R}.
\]

This is motivated by the fact that \(\varphi_{\mu,\nu}^{a,b}(x)\) stands for the right-hand side series in von Lommel’s expression (2.1) and for the Al–Salam series (2.2) appearing in (2.9), while \(\varphi_{\mu,\nu}^{a,b}(x)\) covers the series considered by Thiruvenkatachar and Nanjundiah [50] in (2.6); finally, Neumann type finite sum close to \(\varphi_{\mu,\nu}^{a,b}(x)\) appears in (2.9), and another finite sum of type \(\varphi_{\mu,\nu}^{a,b}(x)\) takes place in (6.11).

In order to obtain the integral representation formula for (4.1) we shall use the main idea from [44], that is, the Laplace integral representation of the associated Dirichlet series will be the main tool here. Thus, we take \(x \in \mathbb{R}_+\) and assume in the sequel that the behavior of \(\{\theta_n\}_{n \geq 1}\) ensures the convergence of the series \((4.1)\) over \(\mathbb{R}_+\). Throughout \([a]\) and \(\{a\} = a - [a]\) denote the integer and fractional part some real \(a\) respectively and \(\chi_S\) will stands for the characteristic function of the set \(S \subset \mathbb{R}\). Moreover, let us consider the real-valued function \(x \mapsto a_x = a(x)\) and suppose that \(a \in C^1[k,m], k, m \in \mathbb{Z}, k < m\). The classical Euler–Maclaurin summation formula states that

\[
\sum_{j=k}^{m} a_j = \int_k^{m} a(x)dx + \frac{1}{2} (a_k + a_m) + \int_k^{m} \left( x - \frac{1}{2} \right) a'(x)dx.
\]

The following condensed form of the Euler–Maclaurin formula holds [44]:

\[(4.2) \quad \sum_{j=k}^{m} a_j = \int_k^{m} (a(x) + \{x\}a'(x))dx = \int_k^{m} \varphi_x a(x) dx,
\]

where

\[\varphi_x := 1 + \{x\} \frac{dx}{dx}.
\]

Also, we need in the sequel a tool to estimate Bessel functions of the first kind. Landau [37] gave in a sense the best possible uniform bound for the first kind Bessel function \(J_\nu\) with respect to \(x, \nu > 0\):

\[(4.3) \quad |J_\nu(x)| \leq b_L \nu^{-1/3}, \quad b_L = \sqrt[3]{\pi} \sup_{x \in \mathbb{R}_+} \text{Ai}(x),
\]

where \(\text{Ai}(\cdot)\) stands for the familiar Airy function [11 p. 447]. In fact Krasikov [32] pointed out that this inequality is sharp only in the transition region, i.e. for \(x\) around \(J_{\nu,1}\), the first positive zero of \(J_\nu\). For further reading about and detailed discussions consult [11] and [33] where another fashion upper bounds are given.

**Theorem 5.** Let \(\theta \in C^1(\mathbb{R}_+), \theta|_{\mathbb{N}} = \{\theta_n\}_{n \geq 1}\) and assume that series \(\sum_{n \geq 1} \theta_{n} n^{-2/3}\) is absolutely convergent. Then, for all \(x, a, b > 0\) such that

\[(4.4) \quad x \in \left(0, 2 \min \left\{ 1, \frac{1}{e} \left( \frac{a^{a+b} b^b}{\theta_{\phi}^{a+b}} \right)^{1/(a+b)} \right\} \right) = I_, \quad \min(\mu + a, \nu + b) > 0,
\]

where

\[(4.5) \quad \rho_{\phi}^{a,b} = \limsup_{n \to \infty} \frac{|\theta_n|^{1/n}}{n^{a+b}},
\]

we have that

\[(4.6) \quad \varphi_{\mu,\nu}^{a,b}(x) = - \int_1^\infty \int_0^{|u|} \frac{\partial}{\partial u} \left( \frac{\theta_{\nu}(u)}{\Gamma(\mu + au + 1)\Gamma(\nu + bu + 1)} J_{\mu+au}(x)J_{\nu+bu}(x) \right)
\]

\[
\times \varphi_x \left( \frac{\theta_{\nu}(u)}{\Gamma(\mu + au)\Gamma(\nu + bu)} \right) du \, dv.
\]

**Proof.** Applying Landau’s bound (4.3) we estimate Bessel functions of the first kind such that consists \(\varphi_{\mu,\nu}^{a,b}(x)\)

\[|\varphi_{\mu,\nu}^{a,b}(x)| \leq b_L^2 \sum_{n \geq 1} \frac{|\theta_n|}{(\mu + an)(\nu + bn)} \sim b_L^2 \frac{\sqrt{ab}}{\sqrt{\nu}} \sum_{n \geq 1} \frac{|\theta_n|}{n^{2/3}}.
\]
hence $G_{\mu, \nu}(x)$ absolutely and uniformly converges for $x > 0$. Now, by substituting the integral representation formula [31, p. 48]

$$J_{\nu}(x) = \frac{2(x/2)^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^1 \cos(xt)(1 - t^2)^{\nu - \frac{1}{2}} dt, \quad x \in \mathbb{R}, \nu > -1/2,$$

into (4.1) we conclude

(4.7) $G_{\mu, \nu}(x) = \frac{4}{\pi} \frac{(x/2)^{\mu+\nu}}{2^{\mu+\nu}} \int_0^1 \int_0^1 \cos(xt) \cos(xs)(1 - t^2)^{\mu - \frac{1}{2}}(1 - s^2)^{\nu - \frac{1}{2}} D_{\theta}(t, s) dt \, ds$,

where the Dirichlet series

$$D_{\theta}(t, s) = \sum_{n \geq 1} \frac{\theta_n (x/2)^{a+b} (1 - t^2)^a (1 - s^2)^b) n}{\Gamma(\mu + an) \Gamma(\nu + bn)}$$

$x$-domain of convergence is our first goal. To express Dirichlet series in Laplace integral form it is necessary to have positive Dirichlet parameter, that is,

$$- \ln \left(\frac{x}{2}\right)^{a+b} (1 - t^2)^a (1 - s^2)^b > 0,$$

which holds for all $|x| < 2$ when $a + b > 0$. Also $D_{\theta}(t, s)$ is equiconvergent to the auxiliary power series

$$\sum_{n \geq 1} \frac{\theta_n (x/2)^{a+b} (1 - t^2)^a (1 - s^2)^b) n}{2^a + b a^a b^b}$$

with radius of convergence

$$\rho_{\theta}^{a,b} = \left(\lim_{n \to \infty} \frac{\theta_n}{n^{a+b}}\right)^{-1}.$$

This yields the convergence region $T_{\theta}$ described in (4.4).

Next, the Laplace integral expression (cf. [29]) for Dirichlet series $D_{\theta}(t, s)$ becomes

(4.8) $D_{\theta}(t, s) = \sum_{n \geq 1} \frac{\theta_n \Gamma(\mu + an) \Gamma(\nu + bn)}{\Gamma(\mu + an) \Gamma(\nu + bn)} \exp\left\{-n \ln \frac{(x/2)^{a+b} (1 - t^2)^a (1 - s^2)^b) n}{2^a + b a^a b^b}\right\}$

$$\times \left(\sum_{n=1}^{\infty} \frac{\theta_n}{\Gamma(\mu + an) \Gamma(\nu + bn)}\right) du$$

$$= \ln \frac{(x/2)^{a+b} (1 - t^2)^a (1 - s^2)^b) n}{2^a + b a^a b^b} \int_0^\infty \left(\frac{(x/2)^{a+b} (1 - t^2)^a (1 - s^2)^b) n}{2^a + b a^a b^b}\right) u$$

$$\times \left(\sum_{n=1}^{\infty} \frac{\theta_n}{\Gamma(\mu + an) \Gamma(\nu + bn)}\right) du,$$

where the last equality we deduced by virtue of condensed Euler–Maclaurin summation formula (4.2). The last formula in conjunction with (4.7) gives

$$G_{\mu, \nu}(x) = 4 \frac{x}{\pi} \frac{(x/2)^{\mu+\nu}}{2^{\mu+\nu}} \int_0^\infty \int_0^1 \int_0^1 \theta(v) \left(\frac{\Gamma(\mu + an) \Gamma(\nu + bn)}{\Gamma(\mu + an) \Gamma(\nu + bn)}\right) du \, dv$$

where

$$J_{\mu, \nu}(u) = - \frac{x}{2} \frac{(a+b)u}{(a+b)u} \int_0^1 \int_0^1 \frac{(x/2)^{a+b} u}{2^{a+b}} \times \ln \left(\frac{(x/2)^{a+b} u}{2^{a+b}}\right) dt \, ds.$$

Because

$$\int J_{\mu, \nu}(u) \, du = - \frac{x}{2} \frac{(a+b)u}{(a+b)u} I_{\mu,a}(u) I_{\nu,b}(u),$$
where

\[ I_{\mu,a}(u) = \int_0^1 \cos(xt)(1 - t)^{\mu+au-\frac{1}{2}} dt \]

\[ = \frac{\sqrt{\pi}}{2} \left( \frac{x}{2} \right)^{-\mu-au} \Gamma(\mu + au + 1)J_{\mu+au}(x) \]

there holds

\[ J_{t,s}(u) = -\frac{\pi}{4} \left( \frac{x}{2} \right)^{-\mu-s} \frac{\partial}{\partial u} (\Gamma(\mu + au + 1)\Gamma(\nu + bu + 1))J_{\mu+au}(x)J_{\nu+bu}(x) . \]

Now, straightforward transformations yield the asserted formula (4.10). □

In the preliminary part of this section we mentioned the equalities by von Lommel, Thiruvenkatachar and Nanjundiah and Al–Salam, where previous integral expression’s special cases play important roles. By particular choices of \( a \) and \( b \) we have the following cases.

**Corollary 5.1.** If \( \nu > 2 \) and

\[ x \in \left( 0, 2 \min \left\{ 1, 2 \left( \nu^2 \rho a \right)^{-\frac{1}{2}} \right\} \right), \]

then we have that

\[ \mathcal{G}_{\nu-2,\nu-2}(x) = -\int_1^\infty \int_0^{|u|} \frac{\partial}{\partial u} (\Gamma^2(\nu - 1 + 2u)J^2_{\nu-2+2u}(x)) \times d_v \left( \frac{\nu - 2 + 2v}{\Gamma^2(\nu - 2 - 2v)\Gamma(v)\Gamma(\nu + v)} \right) du dv. \]

(4.9)

Moreover, for \( \nu > 0 \) and

\[ x \in \left( 0, 2 \min \left\{ 1, \left( \nu^2 \rho a \right)^{-\frac{1}{2}} \right\} \right) \]

there holds true

\[ \mathcal{G}_{\nu+1,\nu+1}(x) = -\int_1^\infty \int_0^{|u|} \frac{\partial}{\partial u} (\Gamma^2(\nu + u + 2)J^2_{\nu+u+2}(x)) \times d_v \left( \frac{(\nu + v)^{-1}(\nu + v + 2)^{-1}}{\Gamma^2(\nu + v + 1)} \right) du dv. \]

(4.10)

We note that the second kind Neumann series in von Lommel’s formula (2.1) possesses divergent auxiliary series \( \sum_{n \geq 1} \theta_n \cdot n^{-2/3} \), therefore it is not covered by Theorem 5. Also, Al-Salam’s Neumann series, taking place in (2.9), converges only when \( m < 1/3 \), so the integral representation (4.9) of the auxiliary series associated with the Neumann series by Thiruvenkatachar and Nanjundiah in (2.6) is equiconvergent to the Riemannian \( \zeta(8/3) \), thus this case meet Theorem 5 of Neumann type have to be considered separately, since their summation require different approach, being the parameter space of \( \mathcal{G}_{\mu,\nu}^{a,b}(x) \) restricted by the required positivity of upper parameters \( a, b \) in Theorem 5.

5. Turán type inequalities for Bessel functions of the second kind and related problems

5.1. **Turán type inequalities for Bessel functions of the second kind.** The Bessel function of the second kind \( Y_\nu \) is defined by [51] p. 64]

\[ Y_\nu(x) = \frac{J_\nu(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)}, \]

where the right-hand side of this equation is replaced by its limiting value if \( \nu \) is an integer or zero. Recently, Baricz [11] posed the following open problem concerning the Bessel function of the second kind: Is it true that the Turán type inequality

\[ Y_\nu^2(x) - Y_{\nu-1}(x)Y_{\nu+1}(x) > \frac{1}{1 - \nu} Y_\nu^2(x) \]

holds true for all \( \nu \in \mathbb{R} \) and \( x > 0 \)? For those values of \( \nu \) for which (5.1) holds the constant \( 1/(1 - \nu) \) is the best possible?
In this section our aim is to give the solution to the above open problem. Observe that the Bessel function of the second kind $Y_\nu$ has the same recurrence relations as the Bessel function of the first kind $J_\nu$ and hence by using the idea from \[50\] we may obtain the following result.

**Theorem 6.** If $\nu > 1$ and $x > x_{\nu}$, where $x_{\nu} \leq \nu$ is the unique positive root of the equation

$$Y_\nu^2(x) - Y_{\nu-1}(x)Y_{\nu+1}(x) = 0,$$

then the Turán type inequality

(5.2) $$Y_\nu^2(x) - Y_{\nu-1}(x)Y_{\nu+1}(x) > 0$$

is valid. Moreover, (5.2) is reversed for $0 < x < x_{\nu}$.

**Proof.** Let us denote

$$\Delta_\nu(x) = Y_\nu^2(x) - Y_{\nu-1}(x)Y_{\nu+1}(x)$$

and consider the function $\Psi_\nu : (0, \infty) \to \mathbb{R}$, defined by $\Psi_\nu(x) = x^2 \Delta_\nu(x)$. In view of the recurrence relations \[51\] p. 66

(5.3) $$xY_\nu''(x) + \nu Y_\nu(x) = xY_{\nu-1}(x)$$

and

(5.4) $$xY_\nu''(x) - \nu Y_\nu(x) = -xY_{\nu+1}(x),$$

we get

$$\Psi_\nu(x) = x^2 Y^2_\nu(x) - [\nu Y_\nu(x) + xY_\nu'(x)][\nu Y_\nu(x) - xY_\nu'(x)]$$

(5.5) $$= (x^2 - \nu^2) Y^2_\nu(x) + x^2 [Y_\nu'(x)]^2$$

and then

$$\Psi_\nu'(x) = 2x [Y_\nu(x)]^2 + 2Y_\nu'(x) [x^2 Y''_\nu(x) + xY_\nu'(x) + (x^2 - \nu^2) Y_\nu(x)],$$

Since the Bessel function of the second kind $Y_\nu$ satisfies (1.1), we obtain

$$x^2 Y''_\nu(x) + xY_\nu'(x) + (x^2 - \nu^2) Y_\nu(x) = 0,$$

which in turn implies that $\Psi_\nu'(x) = 2x [Y_\nu(x)]^2$, and then the function $\Psi_\nu$ is increasing on $(0, \infty)$.

Now recall the asymptotic formula \[1\] p. 360

$$Y_\nu(x) \sim -\frac{1}{\pi} \Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu},$$

which holds when $\nu > 0$ is fixed and $x \to 0$. Then clearly when $\nu > 1$ is fixed and $x \to 0$ we have

$$\Psi_\nu(x) \sim \frac{2^{2\nu}}{\pi^2} x^{2 - 2\nu} \left[\Gamma^2(\nu) - \Gamma(\nu - 1)\Gamma(\nu + 1)\right],$$

which in turn implies that $\Psi_\nu(x)$ tends to $-\infty$ as $x \to 0$. Here we used the fact that the Euler’s gamma function $\Gamma$ is logarithmically convex on $(0, \infty)$ and then for all $\nu > 1$ we have

$$\Gamma^2(\nu) < \Gamma(\nu - 1)\Gamma(\nu + 1).$$

Note that the above inequality can be realized also by means of the well-known formula $\Gamma(\mu + 1) = \mu \Gamma(\mu)$, $\mu > 0$. More precisely, since $\Gamma(\mu) > 0$ for all $\mu > 0$, we have

$$\Gamma^2(\nu) - \Gamma(\nu - 1)\Gamma(\nu + 1) = -\Gamma(\nu - 1)\Gamma(\nu) < 0.$$

On the other hand, it is known that when $\nu$ is fixed and $x \to \infty$ we have \[1\] p. 364

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left(x - \nu \frac{\pi}{2} - \frac{\pi}{4}\right),$$

which yields

$$\Psi_\nu(x) \sim \frac{2}{\pi} \left[\sin^2 \left(x - \nu \frac{\pi}{2} - \frac{\pi}{4}\right) - \sin \left(x - (\nu + 1) \frac{\pi}{2} - \frac{\pi}{4}\right) \sin \left(x - (\nu - 1) \frac{\pi}{2} - \frac{\pi}{4}\right)\right] = \frac{2}{\pi} x^{\nu},$$

that is, $\Psi_\nu(x)$ tends to $\infty$ as $x \to \infty$.

Summarizing, the function $\Psi_\nu$ is increasing, at zero tends to $-\infty$ and at infinity tends to $\infty$. Thus, there exists an unique $x_{\nu} > 0$ such that $\Psi_\nu(x) < 0$ for $0 < x < x_{\nu}$, $\Psi_\nu(x_{\nu}) = 0$ and $\Psi_\nu(x) > 0$ for $x > x_{\nu}$.

With this the proof of (5.2) is done. Finally, by using \[67\] for all $x$ and $\nu$ real we have

$$\Psi_\nu(x) \geq (x^2 - \nu^2) Y^2_\nu(x).$$
and then
\[ 0 = \Psi_\nu(x_\nu) \geq (x_\nu^2 - \nu^2)Y_\nu^2(x_\nu), \]
which in turn implies that \( x_\nu \leq \nu \) for all \( \nu > 0 \) real. \( \square \)

Observe that for \( \nu > 1 \) and \( x \geq x_\nu \) the inequality (5.2) clearly implies (5.1). Thus, although the expression \( 1 - Y_{\nu-1}(x)Y_{\nu+1}(x)/Y_\nu^2(x) \) tends to \( 1/(1 - \nu) \) if \( \nu > 1 \) is fixed and \( x \) tends to zero, because of the next result, in (5.1) the constant \( 1/(1 - \nu) \) is not the best possible, even if was claimed in [1].

We note also that by using the recurrence relation [51, p. 66]
\[ (5.6) \]
we obtain
\[ \nu Y_\nu(x) [Y_\nu(x) + Y_{\nu+2}(x)] = (\nu + 1)Y_\nu(x) [Y_{\nu-1}(x) + Y_{\nu+1}(x)], \]
which in turn implies that
\[ (\nu + 1)\Delta_\nu(x) - \nu\Delta_{\nu+1}(x) = Y_\nu^2(x) + Y_{\nu+1}^2(x). \]
Consequently, the inequality
\[ (\nu + 1)\Delta_\nu(x) - \nu\Delta_{\nu+1}(x) > 0 \]
is valid for all admissible values of \( \nu \) and \( x \). This inequality in particular implies that for all \( \nu > 0 \) we have \( \Delta_\nu(x_{\nu+1}) \geq 0 \) and \( \Delta_{\nu+1}(x_\nu) \leq 0 \).

By following the proof of Theorem [11] we may prove the following result, which completes Theorem 6.

**Theorem 7.** For \( \nu > 0 \) the relative maxima (denoted by \( \overline{M}_{\nu,n} \)) of the function \( x \mapsto \Delta_\nu(x) \) occur at the zeros of \( Y_{\nu-1} \) and the relative minima (denoted by \( \underline{M}_{\nu,n} \)) occur at the zeros of \( Y_{\nu+1} \). If \( \nu < 0 \), the above statement is reversed. The values \( \overline{M}_{\nu,n} \) and \( \underline{M}_{\nu,n} \) can be expressed as
\[ \overline{M}_{\nu,n} = \Delta_\nu(y_{\nu-1,n}) = Y_{\nu}^2(y_{\nu-1,n}), \]
and
\[ \underline{M}_{\nu,n} = \Delta_\nu(y_{\nu+1,n}) = Y_{\nu}^2(y_{\nu+1,n}), \]
where \( y_{\nu,n} \) denotes the \( n \)th positive zero of the Bessel function \( Y_\nu \). Consequently (5.2) holds true for all \( \nu \neq 0 \) and \( x \geq y_{\nu-1,1} \). If \( \nu = 0 \), then (5.2) holds true for all \( x \neq 0 \).

**Proof.** By using the recurrence relation [51, p. 66]
\[ Y_{\nu-1}(x) - Y_{\nu+1}(x) = 2Y_\nu'(x), \]
and the formula (5.3) for \( \nu - 1 \) instead of \( \nu \) and (5.3) for \( \nu + 1 \) instead of \( \nu \), we obtain
\[ [\Delta_\nu(x)]' = 2Y_{\nu-1}(x)Y_{\nu+1}(x) \]
and then
\[ [\Delta_\nu(x)]'' = \frac{2}{x}Y_{\nu-1}'(x)Y_{\nu+1}(x) + \frac{2}{x}Y_{\nu-1}(x)Y_{\nu+1}'(x) - \frac{2}{x^2}Y_{\nu-1}(x)Y_{\nu+1}(x). \]
Now, by using again (5.4) for \( \nu - 1 \) instead of \( \nu \), and (5.6), we obtain
\[ [\Delta_\nu(x)]'|_{x=y_{\nu-1,n}} = 0 \quad \text{and} \quad [\Delta_\nu(x)]''|_{x=y_{\nu-1,n}} = -\frac{4\nu}{y_{\nu-1,n}^2}Y_{\nu}^2(y_{\nu-1,n}). \]
Similarly, by using (5.4) for \( \nu + 1 \) instead of \( \nu \), and (5.6), we get
\[ [\Delta_\nu(x)]'|_{x=y_{\nu+1,n}} = 0 \quad \text{and} \quad [\Delta_\nu(x)]''|_{x=y_{\nu+1,n}} = \frac{4\nu}{y_{\nu+1,n}^2}Y_{\nu}^2(y_{\nu+1,n}). \]
These in turn imply that indeed for \( \nu > 0 \) (\( \nu < 0 \)) the relative maxima (minima) of the function \( x \mapsto \Delta_\nu(x) \) occur at the zeros of \( Y_{\nu-1} \) and the relative minima (maxima) occur at the zeros of \( Y_{\nu+1} \).

Now, because the quantities \( \overline{M}_{\nu,n} \) and \( \underline{M}_{\nu,n} \) are positive, (5.2) holds true for all \( \nu \neq 0 \) and \( x \geq y_{\nu-1,1} \). Finally, since \( Y_{-1}(x) = -Y_1(x) \), we have \( \overline{M}_0(x) = Y_0^2(x) + Y_1^2(x) \), which is clearly positive. \( \square \)
5.2. **Lower bound for the logarithmic derivative of Bessel functions of the second kind.** In this subsection our aim is to deduce new inequalities for the logarithmic derivative of $Y_{\nu}$ and for the quotient $Y_{\nu}/Y_{\nu-1}$ by using the Turán type inequality (5.2). These results are the analogous of those of subsection 2.7. Observe that by using Theorem 6 and combining (5.2) with (5.3) and (5.4) we immediately obtain the inequality

$$ \frac{xY'_\nu(x)}{Y_\nu(x)} \geq \frac{\nu}{x}, $$

which holds for all $\nu > 1$ and $x \geq x_{\nu}$. Moreover, combining (5.7) with (5.8) we get

$$ xY'_\nu(x) - \nu x Y_{\nu-1}(x) Y_{\nu}(x) + xY^2_{\nu-1}(x) \geq 0, $$

where $\nu > 1$ and $x \geq x_{\nu}$. The inequality (5.7) implies the inequality

$$ \frac{xY'_\nu(x)}{Y_\nu(x)} \geq \nu^2 - x^2, $$

where $\nu > 1$ and $x \geq x_{\nu}$ such that $x \neq y_{\nu,n}$, $n \in \{1, 2, \ldots\}$, while the inequality (5.8) implies the following

$$ \frac{Y_{\nu}(x)}{Y_{\nu-1}(x)} \geq \frac{\nu + \sqrt{\nu^2 - x^2}}{x}, $$

where $\nu > 1$, $x_{\nu} \leq x \leq \nu$ such that $x \neq y_{\nu-1,n}$, $n \in \{1, 2, \ldots\}$.

Finally note that, according to Theorem 6 the conditions $\nu > 1$ and $x \geq x_{\nu}$ in inequalities (5.7), (5.8) and (5.3) can be replaced to $\nu \neq 0$ and $x \geq y_{\nu-1,1}$, while in (5.10) to $\nu > 0$ and $x \geq y_{\nu-1,1}$.

**Acknowledgments.** The research of Á. Baricz was supported in part by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and in part by the Romanian National Authority for Scientific Research CNCSIS-UEFISCSU, project number PN-II-RU-PD 2012. The work of this author was initiated during his visit in April 2010 to the Indian Institute of Technology Madras, Chennai, India. The visit was supported by a partial travel grant from the Committee on Development and Exchanges, International Mathematical Union. Á. Baricz acknowledges the hospitality from Professor Saminathan Ponnusamy. Both of the authors are grateful to the referee for his/her useful and constructive comments.

**References**

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with formulas. Graphs and Mathematical Tables, Dover Publications, New York, 1965.

[2] W.A. Al-Salam, A generalized Turán expression for Bessel functions, Amer. Math. Monthly 68(2) (1961) 146–149.

[3] H. Alzer, G. Felder, A Turán-type inequality for the gamma function, J. Math. Anal. Appl. 350 (2009) 276–282.

[4] Á. Baricz, Functional inequalities for Galué’s generalized modified Bessel functions, J. Math. Inequal. 1(2) (2007) 183–193.

[5] Á. Baricz, Turán type inequalities for generalized complete elliptic integrals, Math. Z. 256(4) (2007) 895–911.

[6] Á. Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, Expo. Math. 26(3) (2008) 279–293.

[7] Á. Baricz, Turán type inequalities for hypergeometric functions, Proc. Amer. Math. Soc. 136(9) (2008) 3223–3229.

[8] Á. Baricz, Mills’ ratio: Monotonicity patterns and functional inequalities, J. Math. Anal. Appl. 340(2) (2008) 1362–1370.

[9] Á. Baricz, On a product of modified Bessel functions, Proc. Amer. Math. Soc. 137(1) (2009) 189–193.

[10] Á. Baricz, Turán type inequalities for some probability density functions, Studia Sci. Math. Hungar. 47(2) (2010) 175–189.

[11] Á. Baricz, Turán type inequalities for modified Bessel functions, Bull. Aust. Math. Soc. 82 (2010) 254–264.

[12] Á. Baricz, D. Jankov, T.K. Pogány, Turán type inequalities for Krätzel functions, J. Math. Anal. Appl. (in press).

[13] Á. Baricz, D. Jankov, T.K. Pogány, On Neumann series of Bessel functions, Integral Transforms Spec. Funct. (in press).

[14] Á. Baricz, D. Jankov, T.K. Pogány, Integral representations for Neumann-type series of Bessel functions

[15] Á. Baricz, S. Ponnusamy, On Turán type inequalities for modified Bessel functions, Proc. Amer. Math. Soc. (in press).

[16] Á. Baricz, J. Sándor, Extensions of the generalized Wilker inequality to Bessel functions, J. Math. Inequal. 2(3) (2008) 397–406.

[17] R.W. Barnard, M.B. Gordy and K.C. Richards, A note on Turán type and mean inequalities for the Kummer function, J. Math. Anal. Appl. 349(1) (2009) 259–263.

[18] J. Bustoz, M.E.H. Ismail, Turán inequalities for symmetric orthogonal polynomials, Internat. J. Math. Math. Sci. 20 (1997) 1–8.

[19] T. Craven, G. Csordas, Jensen polynomials and the Turán and Laguerre inequalities, Pacific J. Math. 136 (1989) 241–260.

[20] T. Craven, G. Csordas, Composition theorems, multiplier sequences and complex zero decreasing sequences, In: Value Distribution Theory and Its Related Topics, (G. Barsegian, I. Laine and C.C. Yang, eds.), Kluwer Press, 2003.
[21] G. Csordas, T.S. Norfolk, R.S. Varga, The Riemann hypothesis and the Turán inequalities, Trans. Amer. Math. Soc. 296 (1986) 521–541.
[22] G. Csordas, R.S. Varga, Necessary and sufficient conditions and the Riemann hypothesis, Adv. Appl. Math. 11 (1990) 328–357.
[23] D.K. Dimitrov, Higher order Turán inequalities, Proc. Amer. Math. Soc. 126 (1998) 2033–2037.
[24] D.K. Dimitrov, V.P. Kostov, Sharp Turán inequalities via very hyperbolic polynomials, J. Math. Anal. Appl. 376 (2011) 385–392.
[25] D.K. Dimitrov, Y. Ben Cheikh, Laguerre polynomials as Jensen polynomials of Laguerre-Pólya entire functions, J. Comput. Appl. Math. 233 (2009) 703–707.
[26] M.E.H. Ismail, M.E. Muldoon, Monotonicity of the zeros of a cross product of Bessel functions, SIAM J. Math. Anal. 9(4) (1978) 759–767.
[27] J.L.W.V. Jensen, Recherches sur la théorie des équations, Acta Math. 36 (1913) 181–195.
[28] C.M. Joshi, S.K. Bissu, Some inequalities of Bessel and modified Bessel functions, J. Austral. Math. Soc. (Series A) 50 (1991) 333–342.
[29] J. Karamata, Theory and Application of the Stieltjes Integral, Srpska Akademija Nauka, Posebna izdanja CLIV, Matematički institut, Knjiga I, Beograd, 1949. (in Serbian)
[30] S. Karlin, G. Szegő, On certain determinants whose elements are orthogonal polynomials, J. Analyse Math. 8 (1960/61), 1–157, reprinted in R. Askey, ed., “Gábor Szegő Collected Papers”, vol.3, Birkhauser, Boston, 1982, 605–761.
[31] D. Karp, S.M. Sitnik, Log-convexity and log-concavity of hypergeometric-like functions, J. Math. Anal. Appl. 364 (2010) 384–394.
[32] I. Krasikov, Uniform bounds for Bessel functions, J. Appl. Anal. 12(1) (2006) 83–91.
[33] P. Kravanja, P. Verlinden, On the zeros of \( J_n(z) \pm i J_{n+1}(z) \) and \( J_n(z)J_{n+2}(z) \), J. Comput. Appl. Math. 132 (2001) 237–245.
[34] M.E.H. Ismail, A. Laforgia, Monotonicity properties of determinants of special functions, Constr. Approx. 26 (2007) 1–9.
[35] A. Laforgia, P. Natalini, Some inequalities for modified Bessel functions, J. Inequal. Appl. (2010) Art. 253035.
[36] S.K. Lakshmana Rao, On the relative extrema of the Turán expression fo Bessel functions, Proc. Indian Acad. Sci. Sect. A 53 (1961) 239–243.
[37] J. Landau, Monotonicity and bounds on Bessel functions, in Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory (Berkeley, CA, 1999), 147-154 Electron. J. Differ. Equ. Conf., 4, Southwest Texas State Univ., San Marcos, TX, 2000.
[38] D.A. MacDonald, The zeros of \( J_\frac{1}{2} (\zeta) - J_0 (\zeta) J_2 (\zeta) = 0 \) with an application to swirling flow in a tube, SIAM J. Appl. Math. 51 (1991) 40–48.
[39] P.A. Martin, On functions defined by sums of products of Bessel functions, J. Phys. A: Math. Theor. 41 (2008) Art. 015207.
[40] M.E. Muldoon, Convexity properties of special functions and their zeros, in: G.V. Milovanovic (Ed.), Recent progress in inequalities, Kluwer Academic Publishers. Math. Appl., Dordrecht 430 (1998) 309–323.
[41] A.Ya. Olenko, Upper bound on \( \sqrt{f''}_J(x) \) and its applications, Integral Transforms Spec. Funct. 17(6) (2006) 455–467.
[42] M.L. Patrick, Extensions of inequalities of the Laguerre and Turán type, Pacific J. Math. 44 (1973) 675–682.
[43] T.K. Pogány, H.M. Srivastava, Some improvements over Love’s inequality for the Laguerre function, Integral Transforms Spec. Funct. 18 (2007) 351–358.
[44] T.K. Pogány, E. Süli, Integral representation for Neumann series of Bessel functions, Proc. Amer. Math. Soc. 137(7) (2009) 2363–2368.
[45] G. Pólya, Collected Works. Location of Zeros, vol. 2, MIT Press, Cambridge, MA, 1974.
[46] J. Segura, Bounds for ratios of modified Bessel functions and associated Turán-type inequalities, J. Math. Anal. Appl. 374(2) (2011) 516–528.
[47] H. Skovgaard, On inequalities of the Turán type, Math. Scand. 2 (1954) 65–73.
[48] O. Szász, Inequalities concerning ultraspherical polynomials and Bessel functions, Proc. Amer. Math. Soc. 1 (1950) 256–267.
[49] O. Szász, Identities and inequalities concerning orthogonal polynomials and Bessel functions, J. Analyse Math. 1 (1951) 116–134.
[50] V.R. Thiruvenkatachar, T.S. Nanjundiah, Inequalities concerning Bessel functions and orthogonal polynomials, Proc. Indian Acad. Sci. Sect. A. 33 (1951) 373–384.
[51] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1944.

DEPARTMENT OF ECONOMICS, BABES-BOYAI UNIVERSITY, CLUJ-NAPoca 400591, ROMANIA
E-mail address: bariczocs@ yahoo.com

FACULTY OF MARITIME STUDIES, UNIVERSITY OF RIJEKA, RIJEKA 51000, CROATIA
E-mail address: poganj@brod.pfri.hr