Constructions of graphs and trees with partially prescribed spectrum

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It is shown how a connected graph and a tree with partially prescribed spectrum can be constructed. These constructions are based on a recent result of Salez that every totally real algebraic integer is an eigenvalue of a tree. Our result implies that for any (not necessarily connected) graph $G$, there is a tree $T$ such that the characteristic polynomial $P(G, x)$ of $G$ can divide the characteristic polynomial $P(T, x)$ of $T$, i.e., $P(G, x)$ is a divisor of $P(T, x)$.

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## 1 Introduction

Graph eigenvalues have been studied intensively [1, 2, 3], and they are very special real numbers. Indeed, they are roots of monic integral polynomials with only real roots, i.e., they are totally real algebraic integers. It is natural for one to wonder whether

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the converse is true. Forty years ago, Hoffman [5] conjectured that this is true, which eventually was confirmed by Estes in 1992 [4].

**Theorem 1.1.** [4] Every totally real algebraic integer is an eigenvalue of a (connected) graph.

Recently, Salez [6] strengthened the result with a simpler proof.

**Theorem 1.2.** [6] Every totally real algebraic integer is an eigenvalue of a tree.

The next natural question is which collection of totally real algebraic integers forms the spectrum of a graph. Of course, there are many more necessary conditions on such collections. Below, we list just a few.

**Lemma 1.3.** If \( S = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\} \) is the spectrum of a graph of order \( n \), then

1. \( S \) contains all the conjugates of each \( \lambda_i \),
2. \( \lambda_1 + \cdots + \lambda_n = 0 \),
3. \( \lambda_1^2 + \cdots + \lambda_n^2 \leq n(n-1) \),
4. \( \lambda_1 \leq n-1 \),
5. \( |\lambda_n| \leq \lambda_1 \). ■

Unfortunately, these conditions are far from being sufficient, as the next example shows.

**Example 1.4.** \( \{2, 1, -1, -2\} \) satisfies all conditions listed in Lemma 1.3, but it is not the spectrum of any graph of order 4.

**Proof:** Suppose that there is a graph \( G \) such that \( Spec(G) = \{2, 1, -1, -2\} \). Then \( G \) is bipartite because \( Spec(G) \) is symmetric about 0. Hence the number of
edges of $G$ is at most 4, because $G$ is a bipartite graph of order 4. On the other hand, the number of edges of $G$, computed by means of the eigenvalues, would be $\frac{1}{2} [2^2 + 1^2 + (-1)^2 + (-2)^2] = 5$, contradiction! ■

The problem of finding necessary and sufficient conditions for a set of totally real algebraic integers to be the spectrum of a graph seems intractable! Instead, we tackle the following modified problem.

**Problem 1.5.** Construct a connected graph such that its spectrum contains a given set of totally real algebraic integers.

In Section 2, we accomplish such a construction via Knönecker product of matrices. In Section 3, we strengthen the result by constructing a tree via an appropriate graph operation.

## 2 Construction of connected graphs

Recall some facts about Kronecker product of matrices:

Fact 1. $\text{Spec}(A \otimes B) = \{\alpha \beta : \alpha \in \text{Spec}(A), \beta \in \text{Spec}(B)\}$

Fact 2. $\text{Spec}(A \otimes I + I \otimes B) = \{\alpha + \beta : \alpha \in \text{Spec}(A), \beta \in \text{Spec}(B)\}$

Fact 3. If $A$ and $B$ are adjacency matrices, then $A \otimes B$ is also an adjacency matrix.

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In view of Facts 3 and 4, we introduce two graph products as follows:
Definition 2.1. Given two graphs $G$ and $H$, define a new graph $G + H$ such that its adjacency matrix is given by $A(G + H) = A(G) \otimes I + I \otimes A(H)$.

Definition 2.2. Given two graphs $G$ and $H$, define a new graph $G \times H$ such that its adjacency matrix is given by $A(G \times H) = A(G) \otimes A(H)$.

Moreover, if $G$ and $H$ are connected, then $G + H$ is also connected. It is well-known that $G \times H$ is connected if and only if both $G$ and $H$ are connected and one of $G$ and $H$ contains a cycle of odd length, i.e., one of them is non-bipartite.

Using Facts 1 and 2, we have

$$Spec(G + H) = Spec(G) + Spec(H),$$

and

$$Spec(G \times H) = Spec(G) \cdot Spec(H).$$

Since the path $P_5$ of order 5 has $Spec(P_5) = \{-\sqrt{3}, -1, 0, 1, \sqrt{3}\}$ and the cycle $C_3$ of order 3 has $Spec(C_3) = \{-1, -1, 2\}$, we have that the graph $F = P_5 + C_3$ has $0 = 1 + (-1)$ and $1 = (-1) + 2$ as its eigenvalues. Obviously, $F$ is connected and non-bipartite since it contains an odd cycle $C_3$. As will be seen in the following, we only need non-bipartite graphs $F$ that have 0 and 1 as its eigenvalues. The above says the existence of such graphs. Actually, there are such graphs of small order and size. For example, the graph obtained by attaching two pendant vertices and a 2-vertex path to the same vertex of the triangle. This graph has 7 vertices.

Lemma 2.3. Given a connected graph $G$ such that $\alpha \in Spec(G)$. Then there is a connected graph $H$ such that $0, \alpha \in Spec(H)$.

Proof: Take $H = F \times G$. Then $H$ is connected since $F$ is non-bipartite. Moreover, since $F$ contains 0 and 1 as its eigenvalues, we have $0, \alpha \in Spec(H)$.
Theorem 2.4. Let \( \alpha_1, \ldots, \alpha_p \) be totally real algebraic integers. Then there is a connected graph \( H \) such that \( \{\alpha_1, \ldots, \alpha_p\} \subseteq \text{Spec}(H) \).

Proof: We prove, by induction on \( p \), a stronger statement: there is a connected graph \( H \) such that \( \{0, \alpha_1, \ldots, \alpha_p\} \subseteq \text{Spec}(H) \).

Consider \( p = 1 \). By Theorem 1.1, there is a graph \( G \) such that \( \alpha_1 \in \text{Spec}(G) \). Without loss of generality, we can assume that \( G \) is connected. Now, by Lemma 2.3, there is a connected graph \( H \) such that \( 0, \alpha_1 \in \text{Spec}(H) \).

Consider \( p > 1 \). By the induction assumption, there is a connected graph \( K \) such that \( \{0, \alpha_1, \ldots, \alpha_{p-1}\} \subseteq \text{Spec}(K) \). Applying the case \( p = 1 \), we have a connected graph \( G \) such that \( 0, \alpha_p \in \text{Spec}(G) \). Take \( H = K + G \). Then \( H \) is connected because both \( K \) and \( G \) are connected. Moreover,

\[
0, \alpha_1, \ldots, \alpha_{p-1}, \alpha_p \in \{0, \alpha_1, \ldots, \alpha_{p-1}\} + \{0, \alpha_p\} \subseteq \text{Spec}(K) + \text{Spec}(G) = \text{Spec}(H) \]

3 Construction of trees

Lemma 3.1: Let \( A \) and \( B \) be square matrices. Then

\[
\text{Spec} \left( \begin{bmatrix} A & F & F \\ E & B & 0 \\ E & 0 & B \end{bmatrix} \right) = \text{Spec}(B) \bigcup \text{Spec} \left( \begin{bmatrix} A & 2F \\ E & B \end{bmatrix} \right).
\]

Proof: Note that

\[
\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix}^{-1}.
\]

Then the following matrix identity is in fact a similarity transformation:

\[
\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix} \begin{bmatrix} A & F & F \\ E & B & 0 \\ E & 0 & B \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & I \end{bmatrix} = \begin{bmatrix} A & 2F & F \\ E & B & 0 \\ 0 & 0 & B \end{bmatrix}.
\]

Hence \( \begin{bmatrix} A & F & F \\ E & B & 0 \\ E & 0 & B \end{bmatrix} \) and \( \begin{bmatrix} A & 2F & F \\ E & B & 0 \\ 0 & 0 & B \end{bmatrix} \) have the same spectrum, and so the conclusion follows. \( \blacksquare \)
Given disjoint graphs $G$, $H_i$, and $H'_i$ such that $H_i$ and $H'_i$ are isomorphic for $i = 1, 2, \ldots, p$. Let $x_i, i = 1, 2, \ldots, p$, be vertices of $G$ (not necessarily different). Let $v_i$ be a vertex of $H_i$, and $v'_i$ a vertex of $H'_i$. Construct a graph $G \circ [H_1, \cdots, H_p]$ by connecting $x_i$ to both $v_i$ and $v'_i$ with new edges, for $i = 1, 2, \ldots, p$.

**Lemma 3.2.** $Spec(H_1 \cup \cdots \cup H_p) \subseteq Spec(G \circ [H_1, \cdots, H_p])$.

*Proof:* Let $H = H_1 \cup \cdots \cup H_p$ and $H' = H'_1 \cup \cdots \cup H'_p$. Since $H_i$ and $H'_i$ are isomorphic, $H$ and $H'$ are also isomorphic. Hence, by a suitable labeling, we have $A(H) = A(H')$ and

$$A(G \circ [H_1, \cdots, H_p]) = \begin{bmatrix} A(G) & E^T & E^T \\ E & A(H) & 0 \\ E & 0 & A(H) \end{bmatrix}.$$  

Consequently, by Lemma 3.1,

$$Spec(H_1 \cup \cdots \cup H_p) = Spec(A(H)) \subseteq Spec(A(G \circ [H_1, \cdots, H_p])) = Spec(G \circ [H_1, \cdots, H_p]).$$

**Theorem 3.3.** Let $\alpha_1, \ldots, \alpha_p$ be totally real algebraic integers. Then there is a tree $T$ such that $\{\alpha_1, \ldots, \alpha_p\} \subseteq Spec(T)$.

*Proof:* For each totally real algebraic integer $\alpha_i$, by Theorem 1.2, there is a tree $T_i$ whose spectrum contains $\alpha_i$. Take $G$ to be any tree (say, just a singleton). By Lemma 3.2, $Spec(T_1 \cup \cdots \cup T_p) \subseteq Spec(G \circ [T_1, \cdots, T_p])$ and so

$$\{\alpha_1, \ldots, \alpha_p\} \subseteq Spec(G \circ [T_1, \cdots, T_p]).$$

Moreover, $T = G \circ [T_1, \cdots, T_p]$ is a tree because $G$ and $T_i$ are all trees.

**Example 3.4.** Note that $Spec(K_2) = \{-1, 1\}$, and $Spec(K_{1,4}) = \{-2, 0, 0, 2\}$. Hence, by the construction in the proof of Theorem 3.4, $E_1 \circ [K_2, K_{1,4}]$ is a tree whose spectrum contains $\{-2, -1, 1, 2\}$.
Let $G$ be a graph. A $k$-matching of $G$ is a set of $k$ edges such that any two distinct edges in the set do not share a common vertex. The matching polynomial $m(G, x)$ of $G$ is defined as

$$m(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k},$$

where $m(G, k)$ denotes the number of $k$-matchings in $G$, and $m(G, 0) = 1$ by convention.

For matching polynomials, we know from [2] that for any (not necessarily connected) graph $G$, all the roots of $m(G, x)$ are totally real algebraic integers, and moreover, there is a tree $T$ such that $m(G, x)$ is a divisor of $m(T, x)$. The next result says that the same thing holds for characteristic polynomials of graphs.

**Corollary 3.5.** For any (not necessarily connected) graph $G$, there is a tree $T$ such that the characteristic polynomial $P(G, x)$ of $G$ can divide the characteristic polynomial $P(T, x)$ of $T$, i.e., $P(G, x)$ is a divisor of $P(T, x)$.

**Proof:** Since all the roots of $P(G, x)$ are totally real algebraic integers, by Theorem 3.3 there is a tree $T$ whose spectrum contains all the roots of $P(G, x)$, and hence the conclusion follows. ■

A real polynomial is unimodal if the sequence of the coefficients of the polynomial is unimodal, i.e., first increasing, and then decreasing, with only one peak.

**Corollary 3.6.** For any totally real algebraic polynomial $f(x)$, there is another totally real algebraic polynomial $g(x)$ such that $f(x)g(x)$ is unimodal.

**Proof:** From Theorem 3.3, we know that $f(x)$ is a divisor of the characteristic polynomial of a tree. It is well-known that the characteristic polynomial of any bipartite graph, and therefore, any tree, is unimodal. The conclusion follows immediately. ■
This result means that any totally real algebraic polynomial can be unimodalized. For example, the characteristic polynomial of an arbitrary graph is usually not unimodal, but it can be unimodalized by another totally real algebraic polynomial. It could be an interesting question to think about how to unimodalize a totally real algebraic polynomial by using another totally real algebraic polynomial with a degree as small as possible.

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