The mass of the dark matter particle from theory and observations

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Abstract

In order to determine (as best as possible) the nature of the dark matter particle (mass and decoupling temperature) we compare the theoretical evolution of density fluctuations computed from first principles since the end of inflation till today to the observed properties of galaxies as the effective core density, core radius and surface density. We match the theoretically computed surface density to its observed value in order to obtain: (i) the decreasing of the phase-space density since equilibration till today (ii) the mass of the dark matter particle and the decoupling temperature (iii) the type of density profile (core or cusp). The dark matter particle mass turns to be between 1 and 2 keV and the decoupling temperature turns to be above 100 GeV. keV dark matter particles necessarily produce cored density profiles while wimps (m \textasciitilde 100 GeV, T_\text{d} \textasciitilde 5 GeV) inevitably create cusped profiles at scales about 0.003 pc. We compute in addition the halo radius \(r_0\), the halo central density \(\rho_0\) and the halo velocity \(\sqrt{V_{\text{halo}}^2}\); they all reproduce the observed values within one order of magnitude. These results are independent of the particle model and vary very little with the statistics of the dark matter particle. Moreover, our results for the surface density using typical CDM wimps agree with the values obtained from CDM simulations (which are five orders of magnitude larger than the surface density observed values). DM particles with mass in the keV scale well reproduce the observed properties of galaxies as the effective core density, cored profiles and the value of the surface density. The linear framework presented here turns to apply both to keV and to GeV mass scale DM particles. keV scale particles reproduce all observed galaxy magnitudes within one order of magnitude while GeV mass particles disagree with observations in up to eleven orders of magnitude.

Keywords: cosmology: dark matter, galaxies: halos, galaxies: kinematics and dynamics

1. Introduction

Since several years and more recently \cite{30,12,16,36} it has been stressed that basic galaxy parameters as mass, size, baryon-fraction, central density, are not independent from each other but in fact all of them do depend on one parameter that works as a galaxy identifier. In fact there exist functional relations that constrain the different galaxy parameters in such a way that the galaxy structure depends essentially on one parameter (\cite{31} and references therein).

These functional relations may play for galaxies the r\'ole that the equations of state play in thermodynamical systems.

First, let us remind that the density of DM halos around galaxies is usually well reproduced by means of dark halos with a cored distribution (author?) \cite{8,32}, where \(r_0\) is the core radius, \(\rho_0\) is the central density \(\lim_{r \to 0} \rho(r) = \rho_0\) and \(\rho(r)\) for \(r < r_0\) is approximately constant. Recent findings highlight the quantity \(\mu_0 \equiv r_0 \rho_0\) proportional to the halo central surface density defined as

\[2 \int_0^{\rho_0} \rho(0,0,x_3) dx_3 \quad \text{where } \vec{x} = (x_1,x_2,x_3)\]

where \(x_3\) goes along the line of sight. The quantity \(\mu_0\) is found nearly constant and independent of luminosity in different galactic systems (spirals, dwarf irregular and spheroidals, elliptics) spanning over 14 magnitudes in luminosity and over different Hubble types. More precisely, all galaxies seem to have the same value for \(\mu_0\), namely \(\mu_0 \simeq 120 \ M_\odot/pc^2\) \cite{26,14,34}. It is remarkable that at the same time other important structural quantities as \(r_0\), \(\rho_0\), the baryon-fraction and the galaxy mass vary orders of magnitude from one galaxy to another.

The constancy of \(\mu_0\) is unlikely to be a coincidence and probably has a deep physical meaning in the dynamics of galaxy formation. It must be stressed that \(\mu_0\) is the only dimensionful quantity which is constant among galaxies.

By analogy with the theory of phase transitions in statistical physics we find useful to call ‘universal’ those quantities which take the same value for a large set of galaxies while non-universal quantities vary orders of magnitude from one galaxy to another. In this context the quantities called universal take the same value up to \(\pm 20\%\) for different galaxies. That is, they are only approximately ‘universal’. Other known universal quantity in the above sense is the shape of the density profile when expressed as a function of \(r/r_0\) and normalized to unit at \(r = 0\).

We use here the linear approximation to the Boltzmann-Vlasov equation \cite{13,25} to compute the DM density profile and from it the surface density \(\mu_0\).

In this paper, from first principles and using the primordial
power spectrum we follow the evolution of the gravitational collapse of a perturbation of mass \( M \sim 3 \times 10^{12} M_\odot \) and derive the resulting linear halo density profile. This reproduces the phase of fast accretion found in N-body simulations. As a result we obtain robust predictions for the properties of DM halos from first principles. In the case of \( \Lambda \)CDM our results agree with the N-body \( \Lambda \)CDM simulations and in the case of AWDM our results agree with the observations.

We obtain a very good fit of the linear profile to the Burkert profile which determines the relation between \( r_0 \) and the free-streaming length. This shows that our linear profile can appropriately fit halo density data. The linear approximation to the Boltzmann-Vlasov equation improves for large halo radius \( r_0 \) (see fig. 7). Since universal quantities take the same value for large or small galaxies one can expect good results from the linear approximation which improves for large and less compact galaxies.

We also compute non-universal galaxy quantities as the halo radius, galaxy mass, halo central density and squared halo velocity. We find that the linear approximation provides halo central densities smaller than or in the range of the observations, and halo velocities larger than the observed ones by a factor between 1 and 10. Notice that the determination of these non-universal galaxy quantities is unrelated to the computation of the DM particle mass. Hence, our linear calculations amply suffice for our specific goals to determine properties of the DM particles.

Clearly, for non-universal quantities one has to include the nonlinear effects (including for instance mergers) to exactly reproduce the observations. Notice that general arguments based on the Boltzmann-Vlasov equation show that cores cannot become cusps through mixing and mergers and that cusps do not become steeper neither shallower through mixing and mergers [3].

Moreover, recent N-body \( \Lambda \)CDM simulations (Acquarius) have found that the DM halos form in a sort of “monolithic” way. Their inner regions, that contain the visible galaxies, are found to be stable since early times and contrary to previous believes, major mergers (i.e. those with progenitor mass ratios greater than 1:10) are found to contribute little to their total mass growth [39]. This indicates that nonlinearities (i.e. mergers) have a reduced importance. Minor mergers, secondary infall, rare major mergers are certainly important for details, but the gross features of DM halos are determined during the fast-accretion phase of their gravitational collapse, as the history of the quasar-galaxies coevolution also seems to indicate [20].

The halo formation essentially consists of two main phases: A first fast accretion phase (that can be treated by the linear approximation), and a second subsequent slow accretion phase with mergers and infalls, that have a random character and that can only be described by numerical simulations. This second phase does not have an essential influence in the shape of the halo profile. Thus, in order to explain the observed halo profiles one just needs to describe the first phase of halo formation, as we do here in this paper.

Therefore, the cored and cusped character we find for the linear profiles depending on the DM particle mass considered (keV and GeV mass scale, respectively) should remain valid after mixing and mergers are taken into account.

Clearly, to build the theory of galaxy formation simulations are needed using the correct linear primordial power spectrum as input. Our work is only indicative in that respect.

We combine in this paper observed properties of galaxies as the effective core density and the core radius with the theoretical linear evolution of density fluctuations computed from first principles since the end of inflation till today. We compute the galaxy surface density and match it with the observed values. Within the same scheme, we derive analytically the halo radius \( r_0 \) and the factor \( Z \) characterizing the reduction of the phase-space density since equilibration till today. In this way, we find the value of the mass of the dark matter particle which turns to be between 1 and 2 keV, and the number of ultrarelativistic degrees of freedom of the dark matter coupling at decoupling \( g_d \), or similarly, the decoupling temperature \( T_d \) which turns to be above 100 GeV. Evidence based on the phase space density pointing towards a DM particle mass in the keV scale was presented in refs. [5, 10].

We consider in this paper the whole range of galaxy virial masses going from 5 to \( 300 \times 10^{11} M_\odot \).

Non-universal quantities are reproduced in the linear approximation within a factor of ten.

The analytic expressions we derived for the density profile, and the mass of the dark matter particle also imply that keV dark matter particles always produce cored density profiles while heavy dark matter particles as wimps (\( m = 100 \) GeV, \( T_d = 5 \) GeV) inevitably produce cusped profiles at scales of 0.03 pc. These results are independent of the particle model, vary very little with the statistics of the dark matter particle and are all obtained in the linear framework.

Analytic methods have been used since some time to derive galaxy properties using the primordial power of the density fluctuations [see for example [21, 29]] and using the spherical model (author?) [1, 2].

The theoretical treatment presented here captures many essential features of dark matter, allowing to determine its nature. Our treatment also applies to CDM: if we use the CDM surface density value obtained from CDM simulations [22], we determine [sec. 8] a dark matter particle mass in the wimps mass scale (GeV), fully consistent with CDM simulations.

This paper is organized as follows: Sec. 2 presents galaxy data and empirical formulas relating basic galaxy parameters; sec. 3 deals with the phase-space density; sec. 4 contains our theoretical results for the density profile from the linearized Boltzmann-Vlasov equation. In sec. 5 we derive the DM particle mass and the decoupling temperature from the theoretical and observed galaxy surface density, in sec. 6 we compute non-universal galaxy properties and in sec. 7 we derive the profiles for keV scale DM particles and for wimps (cored vs. cusped profiles). In sec. 8 we present our conclusions.
the surface density is a constant over a large number of galaxies of different kinds.

Notice that the surface density of ordinary matter in luminous galaxies is about a factor 4 larger than the surface density value for dark matter \[17\]. Clusters of galaxies, exhibit a dark matter surface density about a factor 4 or 5 times larger than that of dwarf, elliptical and spiral galaxies \[13\]. Such difference could be due to a baryons effect, which study is beyond the scope of this paper. For clusters of galaxies \(r_0\) is 4 to 50 times larger than for the galaxies in Table 1 and the masses are 100 to 4000 larger than the masses of the galaxies in Table 1. Namely, the variation of \(\mu_0\) from galaxies to clusters of galaxies is a much smaller factor than the change in \(r_0\) and in the total mass. We choose for the present work the data from galaxies in Table 1 (further discussion on clusters of galaxies is given in sec. 6).

3. The invariant phase-space density of DM galaxy halos

The invariant phase-space density is defined by \[23\] and references therein)

\[ Q \equiv \frac{\rho}{\sigma^3} \]  

(4)

where

\[ \sigma^2 \equiv \frac{1}{3} < v^2 > \]  

(5)

is the velocity dispersion. \(Q\) is invariant under the expansion of the universe and decreases due to self-gravity interactions \[15\] from its primordial value \(Q_p\) to the volume average value \(Q_{\text{halo}}\) of the galaxies today:

\[ Q_{\text{halo}} = \frac{1}{Z} Q_p, \]  

(6)

where

\[ Q_{\text{halo}} \equiv \frac{\rho_{\text{halo}}}{\sigma_{\text{halo}}^3}, \quad Q_p \equiv \frac{\rho_{\text{prim}}}{\sigma_{\text{prim}}^3}. \]  

(7)

This equation defines the factor \(Z\) \[10\]. \(Z\) is larger than unity and its value depends on the galaxy considered.

Notice that \(Q_p\) only depends on the properties of the DM particle and its primordial distribution function [see eq.\[12\] below].

The squared halo velocity \(v_{\text{halo}}^2(r)\) follows from the virial theorem combined with the Burkert profile eq.\[1\] with the result \[31\]

\[ v_{\text{halo}}^2(r) = 2 \pi G \frac{\rho_0}{r} r^3 \left[ \ln(1 + x) - \arctan x + \frac{1}{2} \ln(1 + x^2) \right]. \]  

(8)

\(Q_{\text{halo}}\) follows by averaging \(\rho(r)\) and \(v_{\text{halo}}^2(r)\) over the volume using the density itself \(\rho(r)\) as weight function (see Appendix \[A\]). From eqs. \[1\], \[5\], \[12\] and \[8\] we obtain [see eq.\[A.3\]]

\[ Q_{\text{halo}} = \frac{0.069}{G^2 \sqrt{\rho_0 r_0}}, \]  

(9)
Table 1: The observed core radius $r_0$ and the observed surface density $\mu_{0, \text{obs}}$.  

| $r_0$ in kpc | $\mu_{0, \text{obs}}$ in MeV$^3$ |
|--------------|---------------------------------|
| 4.8          | $0.63 \times 10^4$             |
| 6.1          | $0.64 \times 10^4$             |
| 7.9          | $0.63 \times 10^4$             |
| 10.2         | $0.62 \times 10^4$             |
| 13.3         | $0.61 \times 10^4$             |
| 17.3         | $0.60 \times 10^4$             |
| 22.6         | $0.60 \times 10^4$             |
| 29.4         | $0.59 \times 10^4$             |
| 38.7         | $0.57 \times 10^4$             |
| 51.8         | $0.55 \times 10^4$             |

Figure 2: The common logarithm of the phase-space density $Q_{\text{halo}}$ obtained from eq. (9) using the data in Table 1 vs. the virial mass of the galaxy $m_v$. Notice that the virial mass of the galaxy $m_v$ is related to the halo radius $r_0$ through eq. (10).

The primordial invariant phase-space density $Q_p$ can be evaluated in the radiation dominated (RD) era with the result [10]

$$Q_p = \frac{3}{2 \, \pi^2} \, \frac{I_2^2 \, m^4}{G^2 \, \hbar^3},$$

(12)

where $I_2$ and $I_4$ are the dimensionless momenta of the particle DM primordial distribution function, $g$ is the number of internal degrees of freedom of the DM particle ($g = 2$ for Dirac fermions). For example, for Dirac fermions of mass $m$ that decoupled ultrarelativistically at thermal equilibrium we have,

$$Q_p = 0.020395 \, \frac{m^4}{\hbar^3}. \quad (13)$$

Similar expressions and values are obtained for bosons and for particles decoupling ultrarelativistically out of thermal equilibrium [10].

The covariant decoupling temperature $T_d$ can be expressed in terms of the number of ultrarelativistic degrees of freedom at decoupling $g_d$ by using entropy conservation [4]

$$T_d = \left( \frac{2}{g_d} \right)^{\frac{1}{4}} T_\gamma. \quad (14)$$

g_d can be expressed as [10]

$$g_d = \frac{2^\frac{3}{2}}{3^\frac{1}{2} \, \pi^2} \, \frac{T_\gamma^2}{\Omega_{DM}} \, \frac{\rho_c}{\rho_\gamma} \, Q_p \, [I_2 \, I_4]^{\frac{3}{4}}, \quad (15)$$

where $T_\gamma$ is the CMB temperature today, $\Omega_{DM}$ the DM cosmological fraction and $\rho_c$ the critical density of the universe:

$$T_\gamma = 0.2348 \, \text{meV}, \quad \Omega_{DM} = 0.228, \quad \rho_c = (2.518 \, \text{meV}^4 / (\hbar^3 \, c^5), \quad (16)$$

here $1 \, \text{meV} = 10^{-3} \, \text{eV}$. 

For a NFW profile,

$$\rho(r) = \frac{\rho_s}{r_s \left( 1 + r/r_s \right)^2}, \quad (10)$$

we get for $Q_{\text{halo}}$ following the same steps as for the Burkert profile

$$Q_{\text{halo}} = \frac{0.324}{G^2 \, \sqrt{\rho_s \, r_s^4}}. \quad (11)$$

Both results eqs. (9) and (11) are of the same order of magnitude and differ by a factor $\sim 5$. Since $Q \sim m^4$ as shown below in eq. (12), using the cuspy NFW profile instead of the cored Burkert profile only may change the DM particle mass by a factor $\sim 1.5$ keeping its order of magnitude.
4. Theoretical results. The linear Boltzmann-Vlasov equation.

We will now evolve the density fluctuations from the end of inflation till today in the standard model of the Universe. This evolution provides the phase-space density $Q_{halo}$ and the surface density $\rho_{0}$ today. The density fluctuations follow from the distribution function which evolves according to the non-linear Boltzmann-Vlasov equation. The evolution is practically linear in the RD era and in the MD era before structure formation. That is, we can use the linear Boltzmann-Vlasov for redshift $z \gtrsim 30$. For $z \lesssim 30$ non-linearities are relevant and one should use the non-linear Boltzmann-Vlasov equation or, alternatively, perform $N$-body simulations.

It must be noticed that the resolution of the linearized Boltzmann-Vlasov equation from the end of inflation till today provides a good approximated picture of the structures today [11]. From this linear evolution of the dark matter fluctuations $\Delta(k, z)$ we obtained the linearized density profile $\rho_{lin}(r)$ [11]. We followed the density fluctuations in the RD era according to the results in [13] and [24]. It is convenient to recast the linearized Boltzmann-Vlasov equation in the matter dominated (MD) era as an integral equation, the Gilbert equation [18]. We solved the Gilbert equation [11,6] to obtain the density fluctuations $\Delta(k, z)$ today with the result

$$\Delta(k, z) \approx \frac{3}{5} T(k) (1 + z_{eq}) \Delta(k, z_{eq}) .$$

(17)

Here the subindex $eq$ refers to equilibration, the beginning of the MD era, $1 + z_{eq} = 3200$ and $T(k)$ is the transfer function which takes into account the evolution of the density fluctuations during the matter dominated era. $T(k)$ enjoys the properties $T(0) = 1$ and $T(k \to \infty) = 0$. Namely, the transfer function $T(k)$ suppresses the large $k$ (small scale) modes.

It is convenient to introduce the dimensionless variable

$$\gamma \equiv k r_{lin} \quad \text{where} \quad r_{lin} \equiv \frac{l_{fj}}{\sqrt{3}} = \frac{\sqrt{2}}{k_{fj}} ,$$

(18)

$l_{fj}$ and $k_{fj}$ stand for the free-streaming length and free-streaming wavenumber respectively,

$$r_{lin} = 2 \sqrt{1 + z_{eq}} \left( \frac{3 M_{Pl}^{2}}{H_{0}^{2} \Omega_{DM} Q_{p}} \right)^{\frac{1}{3}} ,$$

(19)

$H_{0}$ stands for the Hubble constant today and $M_{Pl}$ for the Planck mass,

$$H_{0} = 1.503 \times 10^{-33} \text{ eV}, \quad M_{Pl} = 2.43534 \times 10^{18} \text{ GeV} .$$

(20)

$r_{lin}$ is the characteristic length scale in the linear regime.

We plot in fig. 3 the transfer function $T(\gamma)$ for Fermions (FD) and Bosons (BE) decoupling ultrarelativistically, and for particles decoupling non-relativistically [Maxwell-Boltzmann statistics, (MB)]. We see from fig. 3 that the transfer function $T(\gamma)$ decreases by an amount of order one for $\gamma$ increasing by unit. Therefore, $T(k)$ decreases by an amount of order one when $k$ increases by an amount of the order of the wavenumber $k_{fj}$ [see eq.(18)]. As we see from fig. 3 $T(\gamma)$ shows little variation with the statistics of the DM particles.

4.1. Obtaining the primordial phase density by matching the observed and the theoretical surface density

We match in this section the observed surface density (Table 1) with the theoretical surface density computed here below. This gives as a result eq.(36) which determines the primordial phase density.

We first compute theoretically the linear profile and the halo radius obtained from it to obtain the theoretical surface density.

The linearized density profile follows from the Fourier transform of the density fluctuations today [11]

$$\rho_{lin}(r) = \frac{1}{2 \pi^{2} r} \int_{0}^{\infty} k dk \sin(k r) \Delta(k, z = 0) .$$

(21)

More explicitly, from eq.(17) the linear profile density $\rho_{lin}(r)$ turns to be the Fourier transform of the density fluctuations $\Delta(k, z_{eq})$ by the end of the RD era times the transfer function $T(k)$ with the result:

$$\rho_{lin}(r) = \frac{108 \sqrt{2}}{5 \pi} \frac{\Omega_{DM} M_{Pl}^{2}}{H_{0}} (1 + z_{eq}) A |\Delta_{0}|$$

$$\times b_{1} b_{l} \frac{k_{0}^{2 - n_{s}/2}}{r_{lin}^{2 - n_{s}/2}} \int_{0}^{\infty} dy N(\gamma) \sin\left( \frac{y}{r_{lin}} \right) ,$$

(22)

where $|\Delta_{0}|$ stands for the primordial power amplitude, $n_{s}$ is the primordial spectral index, $k_{0}$ is the pivot wavenumber used by WMAP to fit the primordial power, $k_{eq}$ the horizon wavenumber by equilibration and

$$N(\gamma) \equiv \gamma^{n_{s}/2 - 1} \ln \left( \frac{k_{0} \gamma}{k_{eq} r_{lin}} \right) T(\gamma) .$$

(23)
The numerical values of the cosmological parameters entering in eq. (22) are

$$|\Delta_0| \approx 4.94 \times 10^{-5}, \quad n_s \approx 0.964, \quad k_0 = 2 \text{ Gpc}^{-1},$$

$$k_{eq} = 9.88 \text{ Gpc}^{-1}, \quad c_0 = 0.11604.$$ (24)

All fluctuations with \( k > k_{eq} \) that were inside the horizon by equilibration are relevant here [11]. This introduces in eq. (22) the comoving horizon volume by equilibration

$$\frac{b_1}{k_{eq}^2} \approx \frac{b_1 b_0}{H_0^2},$$ (25)

where \( b_1 \sim 1 \) [actually, \( b_1 = 1 \) in [11]] and \( b_0 \approx 3.669 \times 10^{-3} \) [13].

The initial power fluctuations are multiplied by a Gaussian random field \( g(\vec{k}) \) with unit variance

$$\langle g(\vec{k}) g^*(\vec{k}') \rangle = \delta(\vec{k} - \vec{k}'),$$ (26)

which describes the random quantum character of the primordial fluctuations.

Each realization of the random field \( g(\vec{k}) \) with unit variance and zero average produces a DM configuration in the linear regime (a ‘galaxy’). The simplest one is obtained for \( g(\vec{k}) = 1 \). The presence of \( g(\vec{k}) \) will produce a large variety of non-spherically symmetric galaxy configurations in a large range of masses and sizes. For simplicity we restrict ourselves here to the case \( g(\vec{k}) = 1 \) and leave the inclusion of \( g(\vec{k}) \neq 1 \) to future work. The profile \( \rho_{lin}(r) \) with \( g(\vec{k}) = 1 \) bears the universal properties of the galaxies, that is to say, the general properties common to all (or most) galaxies. This is why the simplest profile \( \rho_{lin}(r) \) with \( g(\vec{k}) = 1 \) is very appropriate and useful to extract these universal properties.

From the results eqs. (22)-(24) we compute and analyze the surface density and the density profile.

We see from eq. (22) that \( \rho_{lin}(r) \) decreases with \( r \) having \( r_{lin} \) as characteristic scale since it depends on \( r/r_{lin} \) being the Fourier transform of a function of \( y \) that decreases with unit characteristic scale in \( y \) [see fig. 3].

We plot in fig. [B.1] the ratio

$$\frac{\rho_{lin}(r)}{\rho_{lin}(0)} = \Psi(y) = \int_0^y N(y) \sin(y y) \, dy \, \text{where} \, y \equiv \frac{r}{r_{lin}},$$ (27)

for Fermions (FD) and Bosons (BE) decoupling ultrarelativistically and for particles decoupling non-relativistically [Maxwell-Boltzmann statistics (MB)]. \( \Psi(y) \) mainly depends on known cosmological parameters and fundamental constants and has a weak logarithmic dependence on the DM particle mass.

We compute theoretically the surface density from the linear profile eq. (22) and the halo radius eqs. (19) and (B.1) obtained from first principles evolving the density fluctuations since the end of inflation till today. For

$$\mu_{0,lin} \equiv r_0 \rho_{lin}(0),$$ (28)

we find from eqs. (19)-(22) and (B.1),

$$\mu_{0,lin} = \frac{108}{3 \pi} \frac{\Omega_{DM} |\Delta_0| (1 + \epsilon_{eq})^{1-n_s/4}}{k_0^2 \Omega_{Pl}^2} \int_0^\infty \gamma N(y) \, dy.$$ (29)

The DM linear profile eq. (22) decreases with the characteristic length \( r_{lin} \) which should of the same order of magnitude than the halo radius \( r_0 \) in the emperic density profile eq. (1). We define the coefficient \( \alpha \) as \( \alpha \equiv r_{lin}/r_0 \) and determine it by fitting the linear profile to the Burkert profile in Appendix [Appendix B]. The value of \( \alpha \) turns to be between 0.4 and 0.8 depending on the DM particle statistics (see Table B.1).

Using the numerical values of the parameters eqs. (20) and (24), this theoretical formula takes the form

$$\mu_{0,lin} = 391.064 \frac{(\text{MeV})^3}{h^2 c^8} \frac{b_1}{\alpha} q_p^4 \int_0^\infty \gamma N(y) \, dy,$$ (30)

where

$$q_p \equiv \frac{Q_p}{(\text{keV})^2 \hbar^2 c^8},$$ (31)

and

$$N(y) = \gamma^{n_s/2-1} \ln\left(d_0 q_p^4 T(y)\right), \quad d_0 = 556.6976.$$ (32)

From now on we use the dimensionless primordial density \( q_p \).

We identify the observed surface density \( \mu_{0,obs} \) with the theoretical value obtained in the linear approximation \( \mu_{0,lin} \). We thus obtain the following transcendental equation in the variable \( q_p \):

$$q_p^4 \int_0^\infty \gamma N(y) \, dy = \frac{\alpha}{b_1} \frac{\mu_{0,obs} h^2 c^4}{391.064 (\text{MeV})^3}.$$ (33)
We compute the integrals in eq. (33) using \( N(\gamma) \) eq. (31) [i.e. the transfer function \( T(\gamma) \)] from the solution of the linearized Boltzmann-Vlasov equation obtained in [11, 6]:

\[
\int_0^\infty \gamma^{n/2} T(\gamma) \ln \gamma \, d\gamma = 1.3145 \ldots ,
\]

and hence,

\[
\int_0^\infty \gamma N(\gamma) \, d\gamma = 18.1661 \left(1 + 0.04891 \ln q_e\right) .
\]

(35)

These values correspond to fermions decoupling ultrarelativistically at thermal equilibrium. Bosons and particles obeying the Maxwell-Boltzmann statistics yield similar results as one sees from figs. 3 and B.13.

For fermions decoupling ultrarelativistically at thermal equilibrium, eq. (33) takes then the form:

\[
q_p^{0.161} \left(1 + 0.0489106 \ln q_p\right) = \frac{1}{b_1} \frac{\mu_{0,\text{obs}}^2 \hbar^2 c^4}{10326 \text{ (MeV)}^2} ,
\]

(36)

where we used the numerical values in eqs. (24) and (35). The value of \( b_1 \sim 1 \) which provides the best fit to the halo radius is \( b_1 \approx 0.8 \) (see appendix Appendix B).

We proceed now to solve numerically eq. (36) to obtain the primordial phase-space density \( q_p \) for different values of \( \mu_{0,\text{obs}} \) given in Table 1.

### 5. The DM particle mass and the decoupling temperature from the galaxy surface density

We plot in fig. 4 the solution of eq. (36), \( q_p \) vs. \( m_v \). From eqs. (36) and (37), \( q_p \) can be expressed as

\[
q_p = \frac{Z}{(\text{keV})^2} \frac{Q_{\text{halo}}}{h^1 e^8} .
\]

(37)

Therefore, taking the observed values of the phase-space density \( Q_{\text{halo}} \) (fig. 2) yields the factor \( Z \) as a function of the virial mass \( m_v \) [eq. (3)].

In fig. 5 we plot \( \log_{10} Z \) and \( \log_{10} Q_{\text{halo}}^{-1} \) vs. \( m_v \). We see that \( Q_{\text{halo}} \) decreases with \( m_v \) while \( Z \) increases with \( m_v \) in such a way that the product \( Z Q_{\text{halo}} \) is roughly constant. Moreover, as follows from eqs. (36) and (37), \( Z Q_{\text{halo}} \) gives the DM particle mass

\[
m^4 = 49.032 Z Q_{\text{halo}} .
\]

(38)

We notice in fig. 5 that the factor \( Z \) changes by about two orders of magnitude

\[
2.9 \times 10^5 < Z < 5.4 \times 10^7 ,
\]

over a large range of values of the virial mass. The variation of \( Z \) is relevant in the context of galaxy formation but not for the particle DM determination.
We obtain the DM particle mass \( m \) from eqs. (12)-(13) in terms of the invariant phase-space density \( Q_p \):

\[
m = m_0 \left( \frac{Q_p}{\text{keV}} \right) = m_0 q_p^\frac{1}{4}, \quad m_0 \equiv \left( \frac{2}{3 \pi} \right) ^\frac{1}{4} \left( \frac{T_d}{3} \right) ^\frac{1}{4} \text{keV},
\]

where

\[
m_0 = 2.6462 \text{ keV}/c^2 \quad \text{for Dirac fermions},
\]

\[
m_0 = 2.6934 \text{ keV}/c^2 \quad \text{for scalar Bosons}.
\]

The numerical coefficients here correspond to ultrarelativistic decoupling at thermal equilibrium. For decoupling out of thermal equilibrium the coefficients are of the same order of magnitude \([10]\).

In fig. 6 we plot the number of ultrarelativistic degrees of freedom at decoupling \( q_p \) and \( \mu_{\text{obs}} \) given in Table 1.

We find \( m \approx 2 \) keV (up to \( \pm 10\% \)) for \( b_1 = 0.8 \). More generally, \( m \) is in the keV scale for \( b_1 \sim 1 \).

The variation of the observed surface density \( \mu_{\text{obs}} \) with the core radius \( r_0 \) (fig. 1) is similar to:

- (a) the variation of the DM particle mass \( m \) displayed in fig. 6.
- (b) the variation of the primordial phase-space density \( q_p \) in fig. 4.
- (c) the variation of the density contrast in fig. 9.

Therefore, the precision in the observations of the surface density \( \mu_0 \) translates on the precision in the evaluation of the DM mass \( m \).

From the solution for \( q_p \) eq. (56) and fig. 4, we can also compute the number of ultrarelativistic degrees of freedom at decoupling \( g_d \) and therefore the decoupling temperature \( T_d \) which is a further relevant characteristic magnitude of the DM particle. For Dirac fermions decoupling ultrarelativistically at thermal equilibrium the number of ultrarelativistic degrees of freedom at decoupling can be expressed from eq. (15) as

\[
g_d = 1365.5 \ q_p^\frac{1}{4}.
\]

And from fig. 4

\[
0.14 < q_p < 0.3, \quad 0.61 < q_p^\frac{1}{4} < 0.74.
\]

We thus find that \( g_d \) is in the interval

\[
833 < g_d < 1010,
\]

which correspond to decoupling temperatures \( T_d \) [eq. (14)] above 100 GeV.

6. Non-universal structural galaxy properties.

The characteristic length of the linear profile \( r_{\text{lin}} \) eq. (19) takes the following form in terms of \( q_p \) eq. (31):

\[
r_{\text{lin}} = 21.1 \ q_p^\frac{1}{4} \text{ kpc}.
\]

In fig. 7 we plot \( r_{\text{lin}} \) from eq. (43) and \( \alpha r_0 \) from the data in Table 1 as functions of \( m \).

The halo radius in the linear approximation is given by \( r_{\text{lin}} = \alpha r_0 \) for which DM Dirac fermions becomes

\[
r_0 \equiv \frac{r_{\text{lin}}}{0.688} = 30.7 \ q_p^\frac{1}{4} \text{ kpc},
\]

where we used \( \alpha = 0.688 \) obtained in appendix Appendix B by fitting the Burkert and linear profiles.

Using the range of values of \( q_p \) eq. (42) obtained by solving eq. (50) yields

\[
46 \text{ kpc} < r_0 < 59 \text{ kpc}.
\]

which is in the upper range of the observed \( r_0 \) values in Table 1. Namely, the linear approximation for the halo radius give values above or in the range of the observations.

The total mass of the galaxy \( M_{\text{gal}} \) follows by integrating the density profile eq. (22). We find

\[
M_{\text{gal}} \approx 20 \ r_0^3 \ \rho_{\text{lin}}(0) = 20 \ r_0^3 \ \mu_{\text{obs}}.
\]

In fig. 8 we plot \( M_{\text{gal}}/M_{\text{virial}} \) vs. \( m \), where the observed \( m \), and \( M_{\text{virial}} \) are defined by eq. (2).

We see that the ratio \( M_{\text{gal}}/M_{\text{virial}} \) turns to be in the interval,

\[
0.12 < \frac{M_{\text{gal}}}{M_{\text{virial}}} < 5.
\]

The contrast density, that is, the ratio between the maximum DM mass density \( \rho_{\text{lin}}(0) \) and the average DM mass density \( \bar{\rho}_{\text{DM}} \) in the universe results

\[
\text{contrast} \equiv \frac{\rho_{\text{lin}}(0)}{\bar{\rho}_{\text{DM}}},
\]

with \( \bar{\rho}_{\text{DM}} = \Omega_{\text{DM}} \rho_c \) and \( \Omega_{\text{DM}} \) and \( \rho_c \) given by eq. (16). \( \rho_{\text{lin}}(0) \) is given by eq. (28) as

\[
\rho_{\text{lin}}(0) = \frac{\mu_{\text{lin}}}{r_0}.
\]

We plot in fig. 9 the contrast density

\[
\text{contrast} = \frac{\mu_{\text{lin}}}{\Omega_{\text{DM}} \rho_c r_0}
\]

As seen from fig. 9 the ratio obtained is between 1/3 and 1/2 of the observed value \( \sim 3 \times 10^{-5} \) in \([53]\). The values obtained are below the observed values because the linear halo radius \( r_0 = r_{\text{lin}}/0.688 \) is larger than the observed halo radius \( r_0 \) and the density contrast goes as \( 1/r_0 \) eq. (26). This property shows again that the larger and more dilute is the galaxy the better is the linear approximation for non-universal quantities (see Table 3).
Figure 7: The halo radius in the linear approximation \( r_0 \) in kpc from eq. (44) in broken green line, the halo radius \( r_0 \) in kpc from the data in Table 1 in solid red line vs. the virial mass of the galaxy \( m_v \equiv M_{\text{virial}}/10^{11} M_\odot \). \( r_0 \) computed from first principles approaches asymptotically \( r_0 \) for large galaxies.

Figure 8: The common logarithm of the predicted total mass of the galaxy \( M_{\text{gal}} \) given by eq.(45), divided by the observed virial mass \( M_{\text{virial}} \) vs. the virial mass of the galaxy \( m_v \equiv M_{\text{virial}}/10^{11} M_\odot \). The ratio \( M_{\text{gal}}/M_{\text{virial}} \) turns to be in the interval \( 0.12 < M_{\text{gal}}/M_{\text{virial}} < 5.0 \). Notice that the discrepancy of \( M_{\text{gal}} \) with \( M_{\text{virial}} \) is irrelevant to the determination of the DM particle mass.

Figure 9: The ratio \( \rho_{\text{lin}}(0)/\bar{\rho}_{\text{DM}} \) between the maximum DM mass density \( \rho_{\text{lin}}(0) \) and the average mass density in the universe \( \bar{\rho}_{\text{DM}} \) vs. the virial mass of the galaxy \( m_v \equiv M_{\text{virial}}/10^{11} M_\odot \). The ratio \( \rho_{\text{lin}}(0)/\bar{\rho}_{\text{DM}} \) turns to be between 1/3 and 1/2 of the observed value \( \sim 3 \times 10^5 \). Notice that we consider the whole range of galaxy virial masses going from 5 to 300 \( \times 10^{11} M_\odot \). Universal quantities as the surface density stay constant up to \( \pm 20\% \) within this wide range of galaxy masses.

It is relevant to evaluate the halo velocity given by eq. (A.4)

\[
\sqrt{v_{\text{halo}}^2} = 2.316 \, G \, \mu_0 \, r_0 .
\]

Using eq. (44) this equation becomes

\[
\sqrt{v_{\text{halo}}^2} = 6.705 \, \frac{\mu_0}{\text{MeV}^3} \left( \text{km/sec} \right)^2 q_p^{-\frac{1}{3}} .
\]

From Table 1 the input observed surface density takes the value

\[\mu_0 \approx 6000 \, \text{MeV}^3 .\]

Eq. (48) thus becomes

\[
\sqrt{v_{\text{halo}}^2} = \frac{201}{q_p^\frac{1}{3}} \, \text{km/sec} .
\]

The obtained range of values of \( q_p \) eq. (42) yields \( q_p^\frac{1}{3} \approx 0.77 \) and

\[
\sqrt{v_{\text{halo}}^2} \approx 260 \, \text{km/sec} .
\]

This value is to be compared with the values arising from \( \mu_0 \) and eq. (47) and the observed values \( r_0 \) in Table 1.

\[79.3 \, \text{km/sec} < \sqrt{v_{\text{halo}}^2} < 261 \, \text{km/sec} .\]
The halo central density in the linear approximation is given from eqs. (44) and (49) by
\[ \rho_{0,\text{lin}} = \frac{\mu_0}{\bar{r}_0} = 2.90 \times 10^{-25} \frac{g}{\text{cm}^3}. \]

Using the range of values of \( q_p \), eq. (42) obtained by solving eq. (46) yields
\[ 1.18 \times 10^{-25} \frac{g}{\text{cm}^3} < \rho_0 < 1.52 \times 10^{-25} \frac{g}{\text{cm}^3}, \]
which must be compared with the observed values in Table 1.

We see that the linear approximation produces halo central densities smaller or in the range of the observations and halo velocities larger than the observed ones by a factor of order one.

Clusters of galaxies exhibit halo radius \( r_0 \) about 210 kpc [3] well beyond the linear halo radius \( \sim 50 \) kpc. Hence, clusters of galaxies cannot be described by the initial conditions used here. We restricted here for simplicity to initial primordial conditions with \( g(k) \equiv 1 \). Chosing general random fields \( g(k) \neq 1 \) fulfilling eq. (26) will provide general configurations with a large range of masses and sizes. Each realization of the random field \( g(k) \) produces a possible galaxy configuration. The factor \( g(k) \) multiplies the transfer function \( T(k) \) and therefore is to be added in the r. h. s. of eqs. (17), (23) and (32) and inside the \( k^6 \)-integrands [r. h. s. of eqs. (22), (27), (29), (30), (33), (34), (35) and (35)]. We leave the inclusion of the factor \( g(k) \) to future work.

In summary, the solution of the linearized Boltzmann-Vlasov equation presented here provides a satisfactory picture of the general galaxy properties. Although nonlinear effects and baryons are not taken into account, the linear description presented here qualitatively reproduces the main non-universal and general characteristics of a galaxy summarized in Table 2. Moreover, the agreement is even quantitative (approximately) for the linear halo radius \( r_0 \), the galaxy mass \( M_{\text{gal}} \), the linear halo central density \( \rho_0 \) and the halo velocity \( \sqrt{V_{\text{halo}}^2} \) compared to the respected observed values in the limiting case of large galaxies (both \( r_0 \) and \( M_{\text{gal}} \) large). The agreement is very good for universal galaxy quantities as the surface density and the density profile as discussed above.

7. The linear profile density: cores vs. cusps

The properties of the linear profile density \( \rho_{\text{lin}}(r) \) depend crucially on the free streaming length \( \lambda_{\text{in}} \) and therefore on the mass of the DM particle as we discuss here below.

We find from eqs. (28), (30) and (35) for the linear profile at the origin
\[ \rho_{\text{lin}}(0) = \frac{\mu_0}{\bar{r}_0} = 336.688 \times 10^{-25} \frac{g}{\text{cm}^3} \times 1 + 0.0489106 \ln q_p. \]

We use eqs. (39) and (44) that
\[ q_p = \left( \frac{m}{m_0} \right)^{\frac{2}{3}}, \quad r_{\text{lin}} = 77.23 \text{ kpc} \left( \frac{\text{keV}}{m} \right)^{\frac{2}{3}}, \]
for DM particles decoupling ultrarelativistically at thermal equilibrium with \( m_0 \) given by eq. (40). Then eq. (53) can be written as
\[
\rho_{\text{lin}}(0) = 0.472977 \times 10^{-25} \left( \frac{m}{\text{keV}} \right)^{1.976} \times [1 + 0.241648 \ln \left( \frac{m}{\text{keV}} \right)] \frac{g}{\text{cm}^3},
\]
where \( m = 0.1483698 \times 10^{-26} \frac{g}{\text{cm}^3}. \)

For the DM particle mass value \( m \approx 2 \) keV found in the previous section, \( \rho_{\text{lin}}(0) \) from eq. (55) is two to three times smaller than the observed values (as it is the contrast density, discussed in the previous section). This is not surprising since \( \rho_{\text{lin}}(0) \) is not an universal quantity and given the linear approximation of our theoretical computation.

We derive in Appendix C eq. (C.9) the density profile behaviour for \( r \gg r_{\text{lin}} \) when \( \rho_{\text{lin}} \) is given by eq. (53):
\[
\rho_{\text{lin}}(r) \approx 10^{-26} \frac{g}{\text{cm}^3} \left( \frac{36.447 \text{ kpc}}{r} \right)^{1.482} \ln \left( \frac{7.932 \text{ Mpc}}{r} \right). \]

It should be noticed that this behaviour has only a mild logarithmic dependence on the DM particle mass \( m \). The scales in eq. (56) only depend on known cosmological parameters and not on \( m \).

We present in this paper clear evidences for a DM particle mass around 1 keV. In addition, one can wonder what is the shape of the linear profile and the value of the linear density at the origin for a typical hundred GeV wimp. This value for the wimps density profile at the origin turns to be larger than the observed values by eleven orders of magnitude. This result gives a further clear evidence in favour of DM particles at the keV scale which reproduce very well both the surface density and the density profile at the origin.
The free-streaming length \( r_{\text{lin}} \) is the characteristic scale where \( \rho_{\text{lin}}(r) \) varies (see fig. 13). This length is of the order of hundred kpc for keV mass scale DM particles as shown by eq. (54). For a hundred GeV wimp decoupling at \( T_{d, \text{wimp}} = 5 \) GeV we find from eqs. (43) and (58)

\[
r_{\text{lin}}(m_{\text{wimp}} = 100 \text{ GeV}, T_{d, \text{wimp}} = 5 \text{ GeV}) = 0.0031 \text{ pc} = 639 \text{ AU}.
\]

Therefore, with such small \( r_{\text{lin}} \) for wimps we can use for all relevant galactic scales the asymptotic behaviour of \( \rho_{\text{lin}}(r) \) eq. (61)

\[
\rho_{\text{lin}}(r)_{\text{wimp}} \sim r^{0.003 \text{pc}} \times \left( \frac{0.003 \text{ pc}}{r} \right)^{1.482} \left[ 1 + 0.04616 \ln \left( \frac{0.003 \text{ pc}}{r} \right) \right]^{-0.003 \text{pc}}
\]

This profile clearly exhibits a cusp behaviour for scales \( r \lesssim 0.003 \text{ pc} \). Notice that this asymptotic formula eq. (61) approximatively matches around \( r \sim 0.003 \text{ pc} \) the value of the wimp profile at the origin eq. (59).

In summary, the linear profile \( \rho_{\text{lin}}(r) \) eq. (22) exhibits a cusp around the origin for a wimp DM particle and a core behaviour at \( r = 0 \) for a keV scale DM particle mass.

To illustrate the use of the linear approximation we plot in fig. 10 the density profile \( \rho_{\text{lin}}(r) \) according to eqs. (22) and (27) for DM particle masses of 1 and 2 keV and the Burkert density profile for the largest galaxy \( r_0 = 51.8 \text{ kpc} \) and \( \rho(0) = 1.57 \times 10^{-25} \frac{\text{g}}{\text{cm}^3} \) in Table 1. The linear profile best follows the Burkert profile for a DM particle mass slightly below 2 keV in agreement with fig. 6.

We display in fig. 11 the linear density profile for 100 GeV wimps and the NFW profile for the largest galaxy in Table 1. The linear density profile for 1-2 keV particles in fig. 10 and the linear density profile for wimps in fig. 11 practically coincide for \( r \gtrsim 30 \text{ kpc} \) while they strongly differ at smaller scales \( r \lesssim 30 \text{ kpc} \). The keV mass linear profile exhibits a core like the Burkert profile while the wimp linear profile exhibits a cusp like the NFW profile.

We can also evaluate the halo velocity for wimps from the general formula eq. (50) and the value of \( q_{p, \text{wimp}} \) eq. (58). We obtain

\[
\sqrt{V^2_{\text{halo lj, wimp}}} = 0.0768 \text{ km/sec}
\]

three orders of magnitude below the observed halo velocities eq. (52). Recall that keV scale DM particles yield a halo velocity eq. (51) of the same order of magnitude than the observed
characteristic scale for such a NFW profile. For such a wimp DM particle, the linear density profile is similar to (within a factor 2-3, irrelevant for the aims of this paper) while for cusped as shown in fig. 11. Figs. 10-11 show that the linear density profile is cored as depicted in fig. 11. Figure 11: The linear density profile for 100 GeV wimps (broken green line) and the NFW profile (solid red line) for the same galaxy mass as the Burkert profile in fig. 10. In all cases the densities are in g/cm² and r in kpc. The wimp linear density profile follows eq. (C.8). The wimp linear profile exhibits a cusp like the NFW profile.

It is illuminating to insert in eq. (36) the above value of the CDM surface density \( \mu_0 \) instead of the observed value \( \mu_{0, \text{obs}} \). This gives for the mass of the CDM particle \( m_{\text{CDM}} \sim 60 \gev \) which is a typical wimp mass. Therefore, the linear approximation also provides a consistent value for the mass of the CDM particles from simulations.

These results show that our theoretical treatment captures many essential features of dark matter, allowing to determine its nature. When contrasted to the CDM surface density value obtained from CDM simulations (instead of the surface density value obtained from observations), our approach gives for the dark matter particle mass the typical CDM wimps mass scale (GeV), fully consistent with CDM simulations.

It must be stressed that the linear framework presented here applies to any kind of DM particles: particles with mass in the kpc scale reproduce all observed galaxy magnitudes within one order of magnitude, while wimps (\( m \sim 100 \gev \)) present discrepancies with observations of up to eleven orders of magnitude. This is a robust indication that the DM particle mass is in the kpc scale.

8. Conclusions

Dark matter is characterized by two basic quantities: the DM particle mass \( m \) and the number of ultrarelativistic degrees of freedom at decoupling \( g_d \) (or, alternatively the decoupling temperature \( T_d \)). By computing the evolution of the density fluctuations since the end of inflation till today we obtain the density profiles and theoretical relations between \( m \) and \( g_d \) involving the observable densities \( \rho_{DM} \) and \( \mu_0 \) eqs. (12), (15) and (19). Inserting the observed values of \( \rho_{DM} \) and \( \mu_0 \) in these theoretical relations yields \( m \), \( g_d \) and \( Q_p \) eqs. (19)-(20) and (41), respectively.

We derive the values of the halo radius \( r_0 \) and the values of the factor \( Z \) characterizing the reduction of the phase-space density since the equilibration time till today. For these results we use the observed values of the halo phase-space density \( Q_{\text{halo}} \).

From the observed values of the surface density we present here clear evidence that the mass of the DM particle is about one or two keV. Evidence based on the phase space density pointing towards a DM particle mass in the keV scale was presented in refs. [3, 10].

In addition, one can wonder what would be the results for heavy wimps. For example, for wimps at \( m_{\text{wimp}} = 100 \gev \) the characteristic scale \( r_{\text{lin}} \) eq. (19) takes the value given by eq. (60). For such small \( r_{\text{lin}} \), the linear profile \( \rho_{\text{lin}}(r)_{\text{wimp}} \) appears as a cusped profile when observed at scales from 0.003 pc 1 pc as shown in fig. (11). Cusped profiles are thus clearly associated to heavy DM particles with a huge mass \( m_{\text{wimp}} \) well above the physical keV scale while cored profiles are associated to DM particles with mass in the kpc scale.

Notice that the linear density profile turns out to be cored or cuspy depending on the DM particle mass \( m \). For \( m \sim \text{keV} \) the resulting linear density profile is cored as depicted in fig. (10) while for \( m \gtrsim \text{GeV} \) the linear density profile to be cusped as shown in fig. (11). Figs. (10) and (11) show that the linear density profiles for a 1-2 keV DM particle are similar to Burkert (within a factor 2-3, irrelevant for the aims of this paper) while for a wimp DM particle, the linear density profile is similar to a NFW profile.

Interestingly enough, it is possible to derive the value of the surface density \( \mu_0 \) from CDM simulations. Values of the product \( r_s \rho_s \) from NFW fits to CDM simulations for galaxies were reported in [23]. From these values of \( r_s \rho_s \) we can derive the surface density \( \mu_0 \), since \( \mu_0 = r_0 = 25 \mu_s \rho_s \) with the result

\[
\mu_0 \sim 10^7 \, M_{\odot}/\text{pc}^2.
\]

(62)

[Notice that \( \rho_s \) in [23] differs by a factor four from eq. (19)].

We see that the surface density from CDM simulations is five orders of magnitude larger than the observed surface density \( \mu_{0, \text{obs}} \sim 120 \, M_{\odot}/\text{pc}^2 \) [26, 14, 34].

P. S. thanks the Observatoire de Paris for the kind hospitality extended to him.
Appendix A. The average phase space-density

$Q_{\text{halo}}$ in sec. 5 follows averaging $\rho(r)$ and $v_{\text{halo}}^2(r)$ over the volume. We define their average using the density $\rho(r)$ eq. (1) as weight function:

$$\bar{\rho} \equiv \frac{\int_0^{R_{\text{vir}}} r^2 \rho(r) \, dr}{\int_0^{R_{\text{vir}}} r^2 \rho(r) \, dr}, \quad \bar{v}_{\text{halo}}^2 \equiv \frac{\int_0^{R_{\text{vir}}} r^2 \rho(r) v_{\text{halo}}^2(r) \, dr}{\int_0^{R_{\text{vir}}} r^2 \rho(r) \, dr}.$$  \hspace{1cm} (A.1)

The virial radius $R_{\text{vir}}$ is defined by the radius where the mass computed from the Burkert profile eq. (1) takes the value $M_{\odot}$. Appendix A. The average phase space-density

$$M(R_{\text{vir}}) \approx 10^{12} M_{\odot} \left( \frac{R_{\text{vir}}}{259 \ \text{kpc}} \right)^3.$$  \hspace{1cm} (A.2)

Here,

$$M(R_{\text{vir}}) = 4 \pi \int_0^{R_{\text{vir}}} r^2 \rho(r) \, dr = 2 \pi \rho_0 r_0^3 \times \left[ \ln(1 + \hat{c}) - \arctan \hat{c} + \frac{1}{2} \ln(1 + \hat{c}^2) \right],$$

and therefore

$$\hat{c} \equiv \frac{R_{\text{vir}}}{r_0}.$$  \hspace{1cm} (A.3)

Eliminating $M(R_{\text{vir}})$ between eqs. (A.2) and (A.3) gives $\hat{c}$ as a function of $\rho_0$ through the transcendental equation

$$\rho_0 = \frac{0.6187 \times 10^{-27} \, \text{g/cm}^3}{\ln(1 + \hat{c}) - \arctan \hat{c} + \frac{1}{2} \ln(1 + \hat{c}^2)}.$$  \hspace{1cm} (A.4)

For the Burkert profile eq. (1), the virial mass takes the form

$$M(R_{\text{vir}}) = 4 \pi \int_0^{R_{\text{vir}}} r^2 \rho(r) \, dr = 4 \pi \rho_s r_s^3 \left[ \ln(1 + c) - \frac{c}{1 + c} \right].$$

and therefore we find for $\rho_s$,

$$\rho_s = \frac{0.30999 \times 10^{-27} \, \text{g/cm}^3}{\ln(1 + \hat{c}) - \frac{c}{1 + c}}.$$  \hspace{1cm} (A.5)

The observations give for $c$ the empirical relation $[33]

$$c = 9.7 \left( \frac{M(R_{\text{vir}})}{10^{12} M_{\odot}} \right)^{0.13}.$$  \hspace{1cm} (A.6)

Therefore, knowing $M(R_{\text{vir}})$ and $R_{\text{vir}}$ we obtain $\rho_s$ and $c$ from eqs. (A.3) and (A.6). For the galaxies in Table 1, we find $23.2 \ \text{kpc} < r_s < 62.5 \ \text{kpc}$, $0.439 \times 10^{-25} \, \text{g/cm}^3 < \rho_s < 1.087 \times 10^{-25} \, \text{g/cm}^3$, $7.64 < c < 13.1$. We use the values of $r_s$ and $\rho_s$ for the larger galaxy to plot the NFW curve in fig. [11] Namely, $r_s = 62.5 \ \text{kpc}$ and $\rho_s = 1.087 \times 10^{-25} \, \text{g/cm}^3$.

Appendix B. The linearized density profile.

Both, the Burkert profile $F_B(r/r_0)$ and the linear profile $\Psi(r/r_{\text{lin}})$, have the same qualitative shape. To make the connection quantitative, we fit the linear profile with a Burkert profile setting

$$x = \alpha y, \quad \text{that is,} \quad r_{\text{lin}} = \alpha r_0.$$  \hspace{1cm} (B.1)

We look for the value of $\alpha$ that gives the best fit by minimizing the sum of squares:

$$[\Psi(y) - F_B(\alpha y)]^2 \quad \text{for} \quad 0 < y < 3.$$  \hspace{1cm} (B.2)

The best fit for each DM particle statistics is obtained for the values of $\alpha$ in Table 2. We display in fig. [B.12] the Burkert profile $F_B(\alpha y)$ and the linear profiles $\Psi(y)$ for Fermi-Dirac, Bose-Einstein and Maxwell-Boltzmann statistics, respectively. We see from fig. [B.12] that the profiles for Bose-Einstein and Fermi-Dirac statistics are better fitted by a Burkert profile than the profile for Maxwell-Boltzmann statistics.

We compute the behaviour of the linear profile $\rho_{\text{lin}}(r)$ eq. (22) for $r \gg r_{\text{lin}}$ in Appendix C. We find that the linear approximation can be used for (see Appendix [C].

$$0 \leq r < r_{\text{max}} \quad \text{where} \quad r_{\text{max}} \approx 8 \ \text{Mpc}.$$  \hspace{1cm} (C.1)

It must be noticed that the maximum radius $r_{\text{max}}$ turns to be independent of the DM mass $m$ and only depends on known cosmological parameters.

We have at the origin $F'_B(0) = -1$ while $\Psi'(0) = 0$ and $\Psi''(0) < 0$. More precisely $\Psi''(0) = -2.74$ for fermionic DM particles. At the origin, the Burkert profile decreases with unit slope while the linear profile has an inverse-parabola shape.

Galaxy profiles take an universal form when $p(r)/\rho_0$ is expressed as a function of $r/r_0$. The Burkert profile is a particularly simple formula that satisfactorily reproduces the observations. The linear profile $\Psi(y)$, especially for Fermi-Dirac and Bose-Einstein statistics, fits very well the Burkert profile and therefore, $\Psi(y)$ is also able to well reproduce the observations. Namely, the linear profile $\rho_{\text{lin}}(r)$ is well appropriated for small and intermediate scales

$$0 \leq r < r_{\text{max}}.$$  \hspace{1cm} (C.2)

This means that although the linear approximation cannot capture the whole content of the structure formation, it can well reproduce universal features which are common to all types of galaxies as the density profile. Notice that the linear profile $\Psi(y)$ is universal as a function of $y = r/r_{\text{lin}}$. The values of $r_{\text{lin}}$ and $\rho_{\text{lin}}(0)$ are not universal and change by orders of magnitude according to the halo mass. On the contrary, the surface density
### Table B.3: The values of the parameter $\alpha \equiv r_{\text{lin}}/r_0$ for which the Burkert profile $F_B(\alpha y)$ best fits the linear profile $\Psi(y) \equiv \rho_{\text{lin}}(r)/\rho_{\text{lin}}(0), y = r/r_{\text{lin}}$. $\alpha$ Bose-Einstein 0.805 Fermi-Dirac 0.688 Maxwell-Boltzmann 0.421

---

μ₀ defined by eq. (3) is an universal quantity. Indeed, the theoretical value of $\mu_0$ that follows from the linear profile $\rho_{\text{lin}}(r)$ eq. (22) can reproduce the observed values of $\mu_0$ as it has been shown in [11].

We use this property in section 4.1 to derive the values of the DM particle mass $m$ and the number of ultrarelativistically degrees of freedom at decoupling $g_d$.

As shown above the linear profile and the Burkert profile are the closest for $r_{\text{lin}} = \alpha r_0$ with $\alpha = 0.688$. On the other hand, we know that the linear approximation always gives values for $r_0$ larger than the observed values, namely, the linear approximation improves for large galaxies [11]. Therefore, we require that $r_{\text{lin}}$ tends to $r_0 \equiv 0.688 r_0$ for large galaxies which fixes $b_1$ to be $b_1 \approx 0.8$. In any case the dependence of the results on $b_1$ [which must be anyway $b_1 \sim 1$] is quite mild.
Appendix C. Asymptotic behaviour of the linear density profile.

To derive the asymptotic behaviour of $\rho_{\text{lin}}(r)$ it is convenient to change the integration variable in eq. (22) to

$$\eta \equiv \frac{r}{r_{\text{lin}}}, \quad y = \frac{r}{r_{\text{lin}}},$$

and we obtain

$$\Psi(y) = \frac{\rho_{\text{lin}}(r)}{\rho_{\text{lin}}(0)} = \frac{1}{y^2} \int_0^\infty \gamma N(\gamma) \, d\gamma,$$

$$\times \int_0^\infty N \left( \frac{\eta}{y} \right) \sin \eta \, d\eta \quad \text{(C.2)}$$

In the limit $y = r/r_{\text{lin}} \rightarrow \infty$ we have from eq. (32)

$$N \left( \frac{\eta}{y} \right) \approx \left( \frac{\eta}{y} \right)^{-1} \left[ \ln \left( \frac{c_0}{y} q_p^2 \right) + \ln \eta \right]$$

where we used that $T(0) = 1$.

Therefore eq. (C.2) gives

$$\Psi(y) \approx \frac{1}{y} \int_0^\infty \Gamma \left( \frac{n_s}{2} \right) \sin \left( \frac{\pi}{4} n_s \right) \, d\gamma,$$

$$\times \left[ \ln \left( \frac{c_0}{y} q_p^2 \right) + \psi \left( \frac{n_s}{2} + \frac{\pi}{2} \cotg \left( \frac{\pi}{4} n_s \right) \right) \right].$$

where we used the formulas (13)

$$\int_0^\infty \eta^{\frac{n_s}{2} - 1} \sin \eta \, d\eta = \Gamma \left( \frac{n_s}{2} \right) \sin \left( \frac{\pi}{4} n_s \right),$$

$$\int_0^\infty \eta^{\frac{n_s}{2} - 1} \sin \eta \ln \eta \, d\eta = \Gamma \left( \frac{n_s}{2} \right) \sin \left( \frac{\pi}{4} n_s \right),$$

$$\times \left[ \psi \left( \frac{n_s}{2} + \frac{\pi}{2} \cotg \left( \frac{\pi}{4} n_s \right) \right) \right].$$

We plot in fig. C.14 the asymptotic formula eq. (C.5) and the numerical Fourier transform eq. (27) for $\Psi(y)$. We see that the asymptotic formula well reproduces the linear profile for $y \gtrsim 1$ and not just for $y \gg 1$.

We see that there exists a maximum value $y_{\text{max}}$ (and therefore $r_{\text{max}}$) where the linear profile vanishes:

$$y_{\text{max}} = 102.7 \left( \frac{m}{\text{keV}} \right)^{\frac{1}{3}}, \quad r_{\text{max}} = 7.932 \text{ Mpc},$$

We obtain for DM particles decoupling ultrarelativistically at thermal equilibrium using eqs. (39) and (40),

$$\Psi(y) = 0.770518 \left( \frac{77.23 \text{ kpc}}{r} \right)^{1.482} \left( \frac{\text{keV}}{m} \right)^{1.976},$$

$$\times \left[ 1 + 0.111377 \ln \left( \frac{\text{keV}}{m} \right) \right],$$

We plot in fig. C.14 the asymptotic formula eq. (C.5) and the numerical Fourier transform eq. (27) for $\Psi(y)$. We see that the asymptotic formula well reproduces the linear profile for $y \gtrsim 1$ and not just for $y \gg 1$.

We see that there exists a maximum value $y_{\text{max}}$ (and therefore $r_{\text{max}}$) where the linear profile vanishes:

$$y_{\text{max}} = 102.7 \left( \frac{m}{\text{keV}} \right)^{\frac{1}{3}}, \quad r_{\text{max}} = 7.932 \text{ Mpc},$$

where we used $1 + n_s/2 = 1.482, 2(2 + n_s)/3 = 1.976$.

We plot in fig. C.14 the asymptotic formula eq. (C.5) and the numerical Fourier transform eq. (27) for $\Psi(y)$. We see that the asymptotic formula correctly reproduces for $y \gg 1$ but for all $y \gtrsim 1$.

We see that there exists a maximum value $y_{\text{max}}$ (and therefore $r_{\text{max}}$) where the linear profile vanishes:

$$y_{\text{max}} = 102.7 \left( \frac{m}{\text{keV}} \right)^{\frac{1}{3}}, \quad r_{\text{max}} = 7.932 \text{ Mpc},$$

where we used $1 + n_s/2 = 1.482, 2(2 + n_s)/3 = 1.976$.

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where we used $1 + n_s/2 = 1.482, 2(2 + n_s)/3 = 1.976$.

We plot in fig. C.14 the asymptotic formula eq. (C.5) and the numerical Fourier transform eq. (27) for $\Psi(y)$. We see that the asymptotic formula well reproduces the linear profile for $y \gtrsim 1$ and not just for $y \gg 1$.
We then find for DM particles decoupling ultrarelativistically at thermal equilibrium using eqs. (39) and (40),

\[ \rho_{lin}(r) \gtrsim r_{lin} = 10^{-26} \frac{g}{cm^3} \left( \frac{36.447 \text{ kpc}}{r} \right)^{1.482} \ln \left( \frac{7.932 \text{ Mpc}}{r} \right) \left[ 1 + 0.241648 \ln \left( \frac{m}{\text{keV}} \right) \right], \tag{C.9} \]

where \( r_{lin} \) is given by eq. (54). It should be remarked that this behaviour has only a mild logarithmic dependence on the DM particle mass \( m \). The scales in eqs. (C.8)–(C.9) only depend on known cosmological parameters and not on \( m \).

As noticed in [11], the asymptotic decrease of the linear profile given by eq. (C.8) is in remarkable agreement with the universal empirical behaviour put forward from observations in [38] and from ΛCDM simulations in [37]. For larger scales we would expect that the contribution from small \( k \) modes where nonlinear effects are dominant will give the customary \( r^{-3} \) tail exhibited by the Burkert profile eq. (1).

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