I. INTRODUCTION

Spacetimes of higher dimensions ($D > 4$) have become much studied as a result of their appearance in theories of unification, such as string/M theory. Of such spacetimes, one important class is a sequence of black-hole metrics of greater and greater generality in higher dimensions that have been discovered over the years.

The first such higher-dimensional black-hole spacetime was the metric for a nonrotating black hole in $D > 4$ (the generalization of the 1916 Schwarzschild metric in four dimensions $\mathbb{R}^4$, found in 1963 by Tangherlini [4]). Next was the metric for a rotating black hole in higher dimensions (the generalization of the 1963 Kerr metric in four dimensions $\mathbb{R}^4$), discovered in 1986 by Myers and Perry $\mathbb{R}^D$ in the case with zero cosmological constant. Then in 1998 Hawking, Hunter, and Taylor-Robinson $\mathbb{R}^D$ found the general $D = 5$ version of the $D = 4$ rotating black hole with a cosmological constant (often called the Kerr–(anti-)de Sitter metric) that had been found in 1968 by Carter $\mathbb{R}^5$. In 2004 Gibbons, Lü, Page, and Pope $\mathbb{R}^D$ discovered the general Kerr–de Sitter metrics in all higher dimensions, and in 2006 Chen, Lü, and Pope $\mathbb{R}^D$ put these into a simple form similar to Carter’s and were able to add a NUT $\mathbb{R}^D$ parameter (though not charge) to get the general Kerr–NUT–(A)dS metrics for all $D$.

II. HIGHER-DIMENSIONAL BLACK-HOLE SPACETIMES

The general Kerr–NUT–(anti-)de Sitter spacetime discovered by Chen, Lü, and Pope may, after a suitable Wick rotation of the radial coordinate, be written $\mathbb{R}^D$

$$g = \sum_{\mu=1}^{n} \left[ \frac{d\varphi^2}{Q_\mu} + Q_\mu \left( \sum_{k=0}^{n-1} A^{(k)}_\mu \, d\psi_k \right)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \left( \sum_{k=0}^{n} A^{(k)}_\mu \, d\psi_k \right)^2,$$

with $n = \lfloor D/2 \rfloor$ and $\varepsilon = D - 2n$. Here, $Q_\mu = X_\mu/U_\mu$.

$$U_\mu = \prod_{\nu \neq \mu}^{n} (x^2_{\nu} - x^2_\mu), \quad X_\mu = \sum_{k=\varepsilon}^{n} c_{\nu} x^2_{\nu} - 2b_{\mu} x^1_\mu - \frac{\varepsilon c}{x^2_\mu},$$

$$A^{(k)}_{\mu} = \sum_{\nu_1 < \cdots < \nu_k}^{n} x^{\nu_1}_\mu \cdots x^{\nu_k}_\mu, \quad A^{(k)} = \sum_{\nu_1 < \cdots < \nu_k}^{n} x^{\nu_1}_\mu \cdots x^{\nu_k}_\mu. \quad (2)$$

The coordinates $x_\mu$ ($\mu = 1, \ldots, n$) correspond to radial and latitude directions, $\psi_k$ ($k = 0, \ldots, n - 1 + \varepsilon$) to temporal and azimuthal directions. The parameter $c_n$ is proportional to the cosmological constant, and the remaining
constants $c_k$, $c$ and $b_\mu$ are related to the rotation parameters, the mass and the NUT parameters. Hamamoto, Houri, Oota and Yasui \cite{14} derived explicit formulas for the curvature and demonstrated that in all dimensions this metric obeys the Einstein equations

$$R_{ab} = (-1)^n(D-1)c_n g_{ab}.$$  

Besides the obvious spacetime isometries generated by the $D-n$ Killing vectors $\partial_\psi$, the spacetime possesses a whole set of hidden symmetries \cite{1,2}, which can be generated from the principal (rank-2 closed) conformal Killing–Yano tensor discovered by Kubizňák and Frolov \cite{15}. These hidden symmetries play the crucial role for the integrability of the geodesic motion.

The metric \cite{11} can be diagonalized. Let us introduce the orthonormal basis one-forms

$$e^\mu = Q^{\mu}_{\nu}^{-1/2} dx_\nu,$$

$$e^\nu = e^{n+\mu} = Q^{\mu}_{\nu}^{1/2} \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k,$$

$$e^{2n+1} = (-c/A)^{(n)/2} \sum_{k=0}^n A^{(k)} d\psi_k.$$  

Then we have

$$g = \sum_{a=1}^D e^a e^a = \sum_{\mu=1}^n (e^{\mu} e^\mu + e^\nu e^{\hat{\nu}}) + \varepsilon e^{2n+1} e^{2n+1},$$  

and the principal conformal Killing–Yano tensor $h$ which obeys the equations

$$(D-1)\nabla_a h_{bc} = g_{ab} \zeta_c - g_{ac} \zeta_b,$$  

$$\xi_a = \nabla_c h^c_a,$$  

takes the extremely simple form

$$h = \sum_{\mu=1}^n x_\mu e^{\mu} \wedge e^{\hat{\mu}}.$$  

In what follows we shall also use the conformal Killing tensor

$$Q = -hh,$$  

i.e.,

$$Q_{ab} = h_{ac} h_{bd} g^{cd},$$  

which takes the explicit form

$$Q = \sum_{\mu=1}^n x_\mu^2 (e^{\mu} e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}),$$  

and satisfies $\nabla_a Q_{bc} = g(a b Q_c),$ where

$$Q_a = \frac{1}{D+2} (2\nabla_c Q^c_a + \nabla_a Q^c_c).$$  

### III. Constants of Motion

In \cite{1} we have claimed that in the spacetime \cite{6} there are $D$ independent constants of geodesic motion, given by the following quantities: (a) $n-1$ observables $C_j$, $j = 1, \ldots, n-1$, given by traces of powers of the projection $F$ of the principal conformal Killing–Yano tensors $h$ (cf. Eqs. (15) and (16) below)

$$C_j = \text{tr} \left[ \left(-w^{-1} F^2\right)^j \right],$$

(b) $D-n$ observables $p_j$, $j = 0, \ldots, D - n - 1$, given by symmetries of the spacetime

$$p_j = u \cdot \partial_\psi,$$  

and (c) the square $w$ of the (unnormalized) velocity $u$

$$w = u \cdot u = u^a u_a.$$  

Moreover, these quantities commute in the sense of Poisson brackets on the phase space. Here we want to elucidate and prove these properties in more detail.

We understand all mentioned quantities as observables (i.e., functions) on the phase space $\Gamma = T^* M$. It is well known that the cotangent space $T^* M$ has a natural phase space structure (cf. the Appendix or \cite{16}). Since we investigate the relativistic theory and $M$ is a spacetime manifold describing also the physical temporal direction, the phase space $\Gamma = T^* M$ is an unphysical phase space which is, however, well suited for an investigation of the geodesic motion. Doing canonical mechanics on it allows us to solve the geodesic motion in an external time which can be identified at the end with the affine parameter of the studied geodesic.

We denote the momentum variable on the cotangent space as $u$. Indeed, since the geodesic motion is governed by the Lagrange function $L = \frac{1}{2} u \cdot u = \frac{1}{2} u^a u^b g_{ab}$, the canonical momentum can be (up to a position of the tensor index) identified with the (unnormalized) velocity $u$. The Hamiltonian then is

$$H = \frac{1}{2} w = \frac{1}{2} u \cdot u = \frac{1}{2} u^a u_b g^{ab}.$$  

We easily realize that $p_j$ defined in (12) are the special components of momentum and that they are constants of motion since $\partial_\psi$ are Killing vectors. The quantities $C_j$, eq. (14), are constants of motion because the tensor $F$, defined as

$$F_{ab} = (h_{ab} u_c + h_{bc} u_a + h_{ca} u_b) u^c,$$  

and the square $w$ of the velocity $u$, are covariantly conserved along the geodesic. Indeed, thanks to (9), for $u^c \nabla_c u^a = 0$ we have $u^a \nabla_c F_{ab} = 0$.

Next we express the constants of motion $C_j$ in terms of the quantities related to the principal conformal Killing–Yano tensor $h$. The components of the tensor (15) can be rewritten as

$$w^{-1} F = PhP,$$  

where
where $P$ is the projector orthogonal to the velocity $u$, $P = I - p$, i.e., $P_a^a = \delta_a^a - p_a^a$. Here we also introduced the projector $p$, 
\[ p_b^a = w^{-1}u^au_b , \] (17)
on the direction $u$. Using the cyclic property of the trace we thus have 
\[ C_j = (-1)^j w^j \text{tr} [(hP)^{2j}] . \] (18)

The trace of the matrix product could be viewed diagrammatically as a loop formed by joined vertices (each with two ‘legs’) corresponding to matrices in the product. In our case the loop is formed by alternating $h$ and $P$ vertices. Substituting $P = I - p$ we get a sum over all possible loops in which $P$ is replaced either by $I$ or by $-p$. In the case of the identity $I$ the corresponding vertex is effectively eliminated, and in the case of the one-dimensional projector $p = w^{-1}uu$ the loop splits into disconnected pieces. Namely, we can use the identity 
\[ \text{tr} (h^kp h^k\cdots h^k) = \text{tr} (h^kp) \text{tr} (h^k) \cdots \text{tr} (h^k) . \] (19)

The trace in (18) thus leads to 
\[ \text{tr} [(hP)^{2j}] = \text{tr} (h^{2j}) 
+ \sum_{c=1}^{2j} \sum_{k_1 \leq \cdots \leq k_c, k_1 + \cdots + k_c = 2j} (-1)^c N_{k_1 \cdots k_c}^c \prod_{l} \text{tr} (h^l) . \] (20)

The sum over $c$ is the sum over the number of ‘splits’ of the loop, the indices $k_l$ are the ‘lengths’ of the split pieces, and the combinatorial factor $N_{k_1 \cdots k_c}^c$ gives the number of ways in which the loop of the length $2j$ can be split to $c$ pieces of lengths $k_1, \ldots, k_c$. From the fact that the tensor $h$ is antisymmetric, it follows that traces of odd powers of $h$ (optionally multiplied by a projector) are zero. Setting $k_l = 2l$, and introducing the rank-2 conformal Killing tensor $Q$ from [5], eq. (20) thus reduces to 
\[ \text{tr} [(-hPh)^{2j}] = \text{tr} (Q^{2j}) 
+ \sum_{c=1}^{j} \sum_{l_1, l_2, \cdots, l_c} (-1)^c N_{I_1 \cdots I_c}^{I} \prod_{l} \text{tr} (Q^l) , \] (21)

where we used $N_{2l_1 \cdots 2l_c}^{2j} = 2N_{I_1 \cdots I_c}^{I}$ which follows from the definition of the $N$’s. If we define the quantities 
\[ w_j = w \text{tr} (Q^j) = u_{ab} Q_{a_1}^{I} Q_{a_2}^{I} \cdots Q_{a_{j-1}}^{I} u^{a_j} , \] (22)

we finally obtain 
\[ C_j = w^j \text{tr} [(-hPh)^{j}] 
= w^j \text{tr} (Q^j) + 2 \sum_{c=1}^{j} \sum_{l_1, l_2, \cdots, l_c} (-1)^c N_{I_1 \cdots I_c}^{I} w^j w \cdots w^{j-c} \prod_{l} w_{l} , \] (23)

which is eq. (17) of [1].

Let us note that by the same argument as that leading to eq. (24), we can derive the relation for the trace of a power of $QP$, 
\[ \text{tr} [(QP)^{j}] = \text{tr} (Q^j) 
+ \sum_{c=1}^{j} \sum_{l_1, l_2, \cdots, l_c} (-1)^c N_{I_1 \cdots I_c}^{I} \prod_{l} \text{tr} (Q^l) , \] (24)

Comparing with eq. (24), we see that we have proved the relation (16) of [1].

\[ \text{tr} [(-hPh)^{2j}] + \text{tr} (Q^j) = 2 \text{tr} [(QP)^{j}] . \] (25)

The relation (23) and an algorithm for computing the coefficients $N_{I_1 \cdots I_c}^{I}$ can be derived also in a different way. It was mentioned in [2] that the constants $C_j$ can be generated from the generating function $Z(\beta) = \log W(\beta)$:

\[ Z(\beta) = \sum_{j=0}^{\infty} (\frac{1}{2j+1} - \frac{1}{2j}) \beta^{2j} C_j = -\sum_{j=1}^{\infty} \frac{1}{2j} \beta^{j} \text{tr} [(hP)^{2j}] 
= \text{tr} \log (I - \sqrt{\beta} hP) = \log \text{det} (I - \sqrt{\beta} hP) . \] (26)

The third equality follows from the antisymmetry of $h$. Using properties of the determinant, the antisymmetry of $h$, $I = P + p$, and the fact that the projector $p$ is one-dimensional, we can split $Z(\beta)$ into two pieces (cf. eq. (2.7) and (2.8) of [2]):

\[ Z(\beta) = \log W_0(\beta) + \log \Sigma(\beta) , \]
\[ W_0(\beta) = \det (I - \sqrt{\beta} h) = \det (I + \beta Q) , \]
\[ \Sigma(\beta) = \det (P + (I - \sqrt{\beta} h)^{-1}p) 
= \text{tr} ((I - \sqrt{\beta} h)^{-1}p) = \text{tr} ((I + \beta Q)^{-1}p) . \] (27)

Equation (23) then corresponds to the term proportional to $\beta$ in the power expansion of $Z(\beta)$. The first term of (23) is obtained from $\log W_0(\beta)$, and the sum over all possible splittings of the loop corresponds to the $\beta^j$ term of $\log \Sigma(\beta)$. Clearly, the $j$-th derivative of $\log \Sigma(\beta)$ (evaluated at $\beta = 0$) contains the sum over all possible products of $l$-th derivatives $\Sigma^{(l)}(0)$ which are proportional to $w_l$ defined in (22). The factors $N_{I_1 \cdots I_2}$ can thus be obtained by the explicit computation of the derivatives of the generating function $\log \Sigma(\beta)$:

\[ C_j = w^j \text{tr} (Q^j) 
- \frac{2(-w)^j}{(j-1)!} \frac{d^j}{d\beta^j} \log \left( 1 + \sum_{k=1}^{j} (-1)^k w_k w^{j-k} \right) \bigg|_{\beta=0} . \] (28)

Using software for algebraic manipulation we easily get the first five constants (sufficient for the integrability of
geodesic motion up through $D = 13$):

$$
C_1 = w \text{ tr } Q - 2w_1 ,
$$
$$
C_2 = w^2 \text{ tr } Q^2 - 4w_2 w_2 + 2w_4^2 ,
$$
$$
C_3 = w^3 \text{ tr } Q^3 - 6w_3^2 w_3 + 6w_1 w_1 w_1 - 2w_4^3 ,
$$
$$
C_4 = w^4 \text{ tr } Q^4 - 8w_3^3 w_4 + w^2 (4w_2^2 + 8w_1 w_3) - 8w_2 w_4 w_2 + 2w_4^3 ,
$$
$$
C_5 = w^5 \text{ tr } Q^5 - 10w_4^4 w_4 + w^3 (10w_2 w_3 + 10w_1 w_4) - w^2 (10w_1 w_2^2 + 10w_1 w_3^2 + 10w_1 w_4^2 - 2w_4^5) .
$$

Taking into account the facts that the eigenvalues of the principal conformal Killing-Yano tensor $h$ are given by the coordinates $x_{\mu}$, cf. eq. (7), respectively, that the eigenvalues of $Q$ are $x_{\mu}^2$, see eq. (9), we can write down an explicit form for $\text{tr} Q^j$ and $w_j$:

$$
\text{tr} Q^j = 2 \sum_{\mu=1}^{n} x_{\mu}^{2j} ,
$$

$$
w_j = \sum_{\mu=1}^{n} x_{\mu}^{2j} (u_{\mu}^2 + u_{\mu}^4) .
$$

Let us also point out that on the level of the generating functions the relation (28) corresponds to

$$
\det(I - \beta hPhP) \det(I + \beta Q) = \det^2(I + \beta QP) .
$$

It was realized in [2] that the generating function $W(\beta) = \exp Z(\beta) = W_0(\beta)\Sigma(\beta)$ actually generates another set of conserved quantities $c_j$ by

$$
W(\beta) = \frac{1}{w} \sum_{j=0}^{\infty} c_j \beta^j ,
$$

which are quadratic in the velocity $u$. (That they are quadratic can be seen from the fact that $W_0$ does not depend on the velocity, from eq. (31), and from $w_\Sigma(\beta) = \sum_{j=0}^{\infty} (-1)^j w_j \beta^j$.) The relation between $W(\beta)$ and $Z(\beta)$ implies that

$$
C_j = \frac{2(-w)^j}{(j-1)!} \frac{d^j}{d\beta^j} \log\left( w + \sum_{k=1}^{j} c_k \beta^k \right) \bigg|_{\beta=0} ,
$$

and in particular:

$$
C_1 = 2c_1 ,
$$
$$
C_2 = -4w_2 c_2 + 2c_2^2 ,
$$
$$
C_3 = 6w_2^2 c_3 - 6w_1 c_1 c_2 + 2c_3^3 ,
$$
$$
C_4 = -8w_3 c_4 + 8w_2^2 c_1 c_3 + 4w_2 c_2^2 - 8w_1 c_1 c_2 + 2c_4^3 ,
$$
$$
C_5 = 10w_4^2 c_5 - 10w_3^3 c_4 - 10w_2^2 c_1 c_3 + 10w_2^2 c_1 c_3 + 10w_2 c_1 c_3 + 2c_5^3 ,
$$

which are the inverse of the relations (3.19) of [2].

IV. INDEPENDENCE OF CONSTANTS OF MOTION

Now we can demonstrate that the quantities $w$, $p_j$ and $C_j$ are independent at a generic point of the phase space $\Gamma = T^* M$. This means that their gradients on the phase space are linearly independent. To prove this, it is sufficient to show that these gradients are independent in the vertical direction of the cotangent bundle $T^* M$, i.e., that the derivatives of these quantities with respect to the momentum $u$ are linearly independent. To achieve this we will study the wedge product of the ‘vertical’ derivatives. Let us, instead of $w$ and $C_j$, consider the equivalent set of observables $(j = 1, \ldots, n - 1)$

$$
2\hat{C}_j = -\frac{1}{2j} w^{1-j} C_j = -\frac{1}{2j} \text{ tr } Q^j + w^{j+1} ,
$$

$$
2\hat{C}_0 = w ,
$$

where dots in the first expression denote terms which contain $w_k$ with $k < j$, cf. eqs. (23), (29).

We are interested in the quantity

$$
J = \partial \hat{C}_0 \wedge \cdots \wedge \partial \hat{C}_{n-1} \wedge \partial p_0 \wedge \cdots \wedge \partial p_{D-1} .
$$

Due to (12) and (36), we have $\partial p_j = \partial \psi_j$, and

$$
\partial \hat{C}_j = -\frac{1}{2j} (\text{tr} Q^j) u + Q^j \cdot u + \cdots ,
$$

where dots denote linear combinations of $Q^k \cdot u$ with $k < j$: $Q^j \cdot u$ represents the vector with components $Q_{a_1}^a Q_{a_2}^{a_1} \cdots Q_{a_j}^{a_{j-1}} u_{a_1}$. From the antisymmetry of the wedge product it follows that

$$
J = u \wedge (Q \cdot u) \wedge \cdots \wedge (Q^{n-1} \cdot u) \wedge \partial \psi_0 \wedge \cdots \wedge \partial \psi_{D-1} .
$$

(Matrices) powers $Q^j$ of the conformal Killing tensor can be written as

$$
Q^j = \sum_{\mu=1}^{n} x_{\mu}^{2j} e_{\mu} e^\mu + \sum_{\mu=1}^{n} x_{\mu}^{2j} e_{\mu} e^\mu .
$$

The second term acts on the subspace of the vectors spanned on $\partial \psi$. Thus, thanks to the $\partial \psi_0 \wedge \cdots \wedge \partial \psi_{D-1}$ term in the wedge product, this part can be ignored in (39). Taking into account that $e_{\mu} e^\mu = \partial x_{\mu} dx_{\mu}$ and $u^\mu = dx_{\mu} \cdot u$, the substitution of (10) into (39) leads to

$$
J = u^1 \cdots u^n U \partial x_1 \wedge \cdots \wedge \partial x_n \wedge \partial \psi_0 \wedge \cdots \wedge \partial \psi_{D-1} ,
$$

1 The derivative $\partial f$ is the vector field on spacetime $M$ with components $\partial f/\partial u_\mu$, cf. the Appendix A (it could be written more explicitly as $\partial f/\partial u_\mu$). The wedge product is, strictly speaking, defined for (antisymmetric) forms. However, we can easily define the wedge product also for the vectors or lower the vector indices with the help of the metric to get 1-forms.
where

\[ U = \sum_{\text{permutations } \sigma} \text{sign } \sigma \ x_1^{2\sigma_1} \cdots x_n^{2\sigma_n} = \prod_{\mu, \nu = 1}^n (x_\mu^2 - x_\nu^2) . \]  

(42)

In a generic point of the phase space we have \( w^j \neq 0 \) and \( x_\mu^2 \neq x_\nu^2 \) (for \( \mu \neq \nu \)) and therefore \( J \neq 0 \) there, thus showing that the constants of motion are independent.

V. POISSON BRACKETS

Finally we show that the observables \( w, C_j \), and \( p_j \) Poisson commute on the phase space.

The Poisson bracket of two functions on the phase space \( \Gamma = T^* M \) can be written as

\[ \{ A, B \} = \nabla A \cdot \partial B - \partial A \cdot \nabla B , \]  

(43)

where \( \nabla F \) represents an arbitrary (torsion-free) covariant derivative which ignores the dependence of \( F \) on the momentum \( u \), and \( \partial B \) is the derivative of \( B \) with respect to the momentum \( u \), cf. the Appendix. \( \nabla F \) is a 1-form and \( \partial F \) a vector field on the spacetime \( M \), and the dot indicates the contraction in spacetime tensor indices. We use naturally the covariant derivative \( \nabla \) generated by the metric on \( M \).

Clearly, the commutation of any observable with the Hamiltonian \( \frac{1}{\hbar} \w \) of the geodesic motion is equivalent to the conservation of the observable, cf. eq. (12), so we have

\[ \{ w, p_j \} = 0 , \quad \{ w, C_j \} = 0 . \]  

(44)

The Poisson bracket between observables \( p_j = u \cdot \partial \psi_j \), reduces to Lie brackets of the Killing vector fields \( \partial \psi_j \), which vanish because \( \partial \psi_j \) are coordinate vector fields:

\[ \{ p_j, p_j \} = [ \partial \psi_j, \partial \psi_j ] \cdot u = 0 \]  

(45)

The Poisson bracket of any observable with the observable \( p = l \cdot u \) linear in momentum leads to the Lie derivative along the vector field \( l \), see eq. (12):

\[ \{ C_l, p \} = l \cdot \partial \psi_j \cdot C_l = 0 . \]  

(46)

Here, the Lie derivative \( L_{\partial \psi_j} C_l \) ignores the dependence of \( C_l \) on the momentum \( u \), cf. the Appendix. It vanishes because \( \partial \psi_j \) is a Killing vector and the definition of \( C_l \) respects the symmetry of the spacetime (it does not depend explicitly on \( \psi_j \)).

Finally, it remains to evaluate the brackets \( \{ C_l, C_j \} \). To simplify the following computation, we will study rescaled observables

\[ C_j = (-1)^j w^j C_j = \text{tr} \left( h \hat{P}^{2j} \right) , \]  

(47)

cf. eq. (18), and we denote

\[ \hat{P} = w \hat{P} = w I - uu . \]  

(48)

Using the cyclic property of the trace, the derivative of \( C_j \) in the spacetime direction is

\[ \nabla_a C_j = 2 j \text{tr} \left[ (\nabla_a h) \hat{P} \hat{P} (\hat{P})^{2j-1} \right] . \]  

(49)

Here \( \nabla_a h \) is the matrix of components \( \nabla_a h^b_c \) of the covariant derivative \( \nabla h \). Substituting for \( \nabla_a h^b_c \) from eq. (43) and using the antisymmetry of \( h \), we obtain

\[ \frac{\partial}{\partial x^j} \nabla_a C_j = \xi_{a b} \hat{P}^{a b} \hat{P} \hat{P} (\hat{P})^{2j-1} \]  

(50)

For the derivative with respect of the momentum \( u \) we get

\[ \frac{\partial}{\partial u^c} C_j = u^c \left( h^{d_1 c_1} \hat{P}^{d_1} h^{d_2 c_2} \hat{P}^{d_2} \cdots \hat{P}^{d_{2j-1} c_{2j-1}} h^{d_{2j} c_{2j}} \right) \]  

(51)

+ \( h^c_c \hat{P}^{d_1} \hat{P}^{d_2} \cdots \hat{P}^{d_{2j} c_{2j}} u^{c_{2j}} \).

Substituting (50) and (51) into expression (12) for \( \{ C_l, C_j \} \) and using \( \hat{P}^{a b} u^b = 0 \), we find

\[ \frac{\partial}{\partial x^j} \{ C_l, C_j \} = 0 . \]  

(52)

We thus proved that the conserved quantities \( w, p_j \), and \( C_j \) Poisson commute with each other. Since the generating function \( Z(\beta) \) is given by power series in \( \beta \) with coefficients given (up to constant factors) by the constants \( C_j \), then also this function (and similarly \( W(\beta) = \exp Z(\beta) \) Poisson commute with \( w \) and \( p_j \), as well as with itself for different choices of \( \beta \):

\[ \{ Z(\beta_1), Z(\beta_2) \} = 0 , \quad \{ W(\beta_1), W(\beta_2) \} = 0 . \]  

(53)

The same is true also for the quantities \( c_j \) generated from \( W(\beta) \) introduced in [2]. Therefore the constants of motion all Poisson commute (are in involution), so the geodesic motion is completely integrable [16, 17].

VI. SUMMARY

We have explicitly proved the complete integrability of geodesic motion in the general higher-dimensional rotating black-hole spacetimes [12]. The ‘nontrivial’ constants of motion are associated with the Killing tensors which we generated from the principal conformal Killing–Yano tensor. Observables \( c_j \) are quadratic in momenta and

\[ \text{scaling} \]  

(17) differs from (9) used in the previous section.
correspond to rank-2 Killing tensors, whereas constants \( C_j \) are of higher order in momenta and correspond to Killing tensors of increasing rank.

The complete integrability of the geodesic motion is related to the issue of separability of the Hamilton-Jacobi equation recently accomplished by Frolov, Krtouš, and Kubizňák [15]. The relation between integrability and separability on a general level has been studied in the series of papers by Benenti and Francaviglia (see, e.g., [19]) where it was demonstrated that the separability is possible only if all the constants of motion, corresponding to Killing vectors and rank-2 Killing tensors, Poisson commute.

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APPENDIX A: COVARIANT CANONICAL FORMALISM ON THE COTANGENT BUNDLE

It is textbook knowledge [16] that the cotangent bundle \( T^* M \) has the natural structure of a phase space, i.e., it possesses a symplectic form \( \Omega \) which defines the Poisson bracket \{ , \}. For a base manifold \( M \) which is equipped with an additional geometric structure, it can be useful to express phase-space quantities and operations with the help of quantities and operations on the base manifold. In this Appendix we shortly review such a procedure.\(^3\)

We call functions on the phase space \( \Gamma = T^* M \) observables, and we write \( F(x, u) \) to emphasize the dependence of \( F \) on the configuration variable \( x \in M \) and on the momentum \( u \in T^*_x M \).

For the base manifold \( M \) with a (torsion-free) covariant derivative \( \nabla \) (in our case the spacetime manifold with the metric connection), it is possible to introduce the covariant derivative of an observable \( F(x, u) \) in the horizontal (configurational) direction of the phase space \( \Gamma \). For any base manifold vector \( l \in \mathcal{T} M \) we define

\[
\Gamma^l \nabla_e F(x, u) = \frac{d}{da} F(x(a), u(a)) \bigg|_{a=0}, \quad (A1)
\]

where \( x(\alpha) \) is a curve starting from \( x \) with tangent vector \( l \), and \( u(\alpha) \) is parallel transport of \( u \) along \( x(\alpha) \).

For a 1-form \( p \in T^*_x M \) we can also introduce the derivative of an observable \( F(x, u) \) in the vertical (momentum) direction

\[
p_e \partial^e F(x, u) = \frac{d}{da} F(x, u + \alpha p) \bigg|_{a=0}. \quad (A2)
\]

Thanks to the linearity of \( \partial^e \), this derivative is independent of any additional geometrical structure.

Derivatives \( \Gamma^l \nabla_e F \) and \( p_e \partial^e F \) are derivative operators on \( \Gamma \) and as such they define vector fields on \( \Gamma \), which we denote \( \Gamma^l \nabla_e \) and \( p_e \partial^e \). These derivatives and vector fields depend ultralocally on the base manifold quantities \( l \) and \( p \), respectively, and we can thus introduce differentials \( \nabla_e F \) and \( \partial^e F \) and mixed tensor quantities \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial u} \) by ‘tearing off’ \( l \) and \( p \), respectively, and by ‘tearing off’ the function \( F \).

Clearly, \( \nabla_e F \) is the covariant derivative of the observable \( F(x, u) \) which ‘ignores’ the momentum \( u \) leaving it covariantly constant. On the other side, the derivative \( \partial^e F \) ‘ignores’ the configuration variable \( x \).

For an observable \( F(x, u) \) given by a contraction of a spacetime tensor field \( f \) with several momenta \( u \),

\[
F(x, u) = f^{abc...}(x) u_a u_b u_c \ldots, \quad (A3)
\]

the covariant derivative reduces to the standard base manifold covariant derivative

\[
\nabla_e F(x, u) = \nabla_e f^{abc...}(x) u_a u_b u_c \ldots. \quad (A4)
\]

The momentum derivative leaves \( f \) intact

\[
\partial^e F(x, u) = f^{abc...}(x) u_b u_c \ldots + f^{ace...}(x) u_a u_c \ldots
\]

\[
+ f^{bce...}(x) u_a u_b \ldots + \ldots. \quad (A5)
\]

A general phase space observable can then be written as a (infinite) sum of terms of this type.

The mixed tensor \( \frac{\partial}{\partial x} \) is a vector field on the phase space (from \( \mathcal{T} \)) and a 1-form on the base manifold (from \( T^*_x M \)). It is actually the horizontal lift from \( T^* M \) to \( T \) corresponding to the covariant derivative \( \nabla \). The mixed tensor \( \frac{\partial}{\partial u} \) is a vector field on the phase space (from \( \mathcal{T} \)) and a vector field on the base manifold (from \( T^* M \)). It gives a natural identification of the cotangent fiber \( T^*_x M \) with its vertical tangent space \( \mathcal{T}_x M \).

The inverse symplectic form \( \Omega^{-1} \) and the Poisson bracket can be written as

\[
\Omega^{-1} = \nabla_e \partial^e - \partial^e \nabla_e \quad (A6)
\]

---

\(^3\) Similarly to the main text we type tensors in bold. Optionally, we write here the tensors with abstract indices [21] which help to indicate tensorial operations as, for example, contraction. However, the abstract indices do not refer to any particular choice of coordinates. We use small latin letters for base manifold indices (for tensors from \( T^* M \)), but we do not introduce indices for the phase space tensors (tensors from \( \mathcal{T} \)). We assume implicitly the tensor product, i.e., \( ab = a \otimes b \).

\(^4\) We could be more explicit and write them as \( \Gamma^l \frac{\partial}{\partial x} \) and \( p_e \frac{\partial}{\partial u} \). Similarly we could write \( \frac{\partial}{\partial x} \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial u} \frac{\partial}{\partial x} \) for quantities \( \nabla_e F \) and \( \partial^e F \) introduced below. However, we use such an explicit notation only for the mixed tensor fields \( \frac{\partial}{\partial x} \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial u} \frac{\partial}{\partial x} \) (see below) where the notation \( \nabla_e \) and \( \partial^e \) would be too brief.
and
\[ \{A, B\} = \nabla_c A \partial^c B - \partial^c A \nabla_c B . \] (A7)

They do not depend on a choice of the covariant derivative. Indeed, if we choose another torsion-free covariant derivative \( \nabla \) on \( M \), which can be done by specifying the ‘difference’ tensor \( \Gamma_{ac}^b \):
\[ \nabla_a b = \nabla_a a b + \Gamma_{ac}^b a c , \] (A8)

the induced covariant derivative of the phase space observables transforms as
\[ \nabla_a F(x, u) = \nabla_a F(x, u) + u_c \Gamma_{ac}^e(x) \partial^e F(x, u) . \] (A9)
Substituting this into (A7) and using the symmetry \( \Gamma_{ac}^b = \Gamma_{ca}^b \) we find that
\[ \{A, B\} = \nabla_c A \partial^c B - \partial^c A \nabla_c B , \] (A10)
i.e., the Poisson bracket is independent of the choice of the covariant derivative. The argument for the symplectic structure is similar.

The Poisson bracket of an observable of type (A3) with an observable \( p \) linear in momenta \( u \),
\[ p(x, u) = l^r(x) u_c , \] (A11)
leads, with help of (A4) and (A5), to the Lie derivative:
\[ \{F, p\} = l^r \nabla_r F - \partial^r F (\nabla_r l^r) u_c = (l^r \nabla_r f^{ab} \cdots f^{cb} \cdots \nabla_l l^b \cdots - \partial^r \nabla_l l^b \cdots ) u_a u_b \cdots = (l^r f^{ab} \cdots ) u_a u_b \cdots \equiv l^r F . \] (A12)
Here \( l^r f \) is the standard Lie derivative on \( M \) along the vector field \( l \). The last equality then defines the generalized Lie derivative \( l^r F \) of the phase space observable \( F \) along the base manifold vector field \( l \) which effectively ‘ignores’ the dependence of \( F \) on the momentum \( u \). It can be extended to general phase space observables by linearity. It can be also defined similarly to (A1) with \( u(\alpha) = \phi_\alpha u \) given by a flow \( \phi_\alpha \) induced by the vector field \( l \) acting on \( u \). \( l^r F \) can be also viewed as the derivative of the observable \( F \) along the vector field \( l^r \) on \( T^* M \) which is called the complete lift of the vector field \( l \) on \( M \).

Clearly, the Poisson bracket with the Hamiltonian (14) leads to the covariant derivative along the \( u \) direction:
\[ \{F, H\} = u^r \nabla_r F . \] (A13)

Despite the fact that we do not need them in the main text, let us introduce for completeness the mixed tensor fields \( D^e x \) and \( \nabla u_c \) dual to \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial u} \) defined by
\[ \frac{\partial}{\partial x} \cdot D^a x = \delta^a , \quad \frac{\partial}{\partial u} \cdot \nabla u_b = \delta^b , \] (A14)
\[ \frac{\partial}{\partial x} \cdot \nabla u_b = 0 , \quad \frac{\partial}{\partial u} \cdot D^b x = 0 . \]
Here the dot ‘·’ indicates the contraction of the phase space tensor indices.

\( D^e x \) is a vector field on the base manifold \( M \) and a 1-form on the phase manifold \( \Gamma \). It is actually the differential of the bundle projection \( x : T^* M \rightarrow M \). \( \nabla u_c \) is a 1-form both on the base manifold \( M \) and phase space \( \Gamma \).

These phase space ‘forms’ satisfy the completeness relation
\[ \frac{\partial}{\partial x} D^r x + \frac{\partial}{\partial u} \nabla u_c = \delta , \] (A15)
with \( \delta \) being the identity tensor on \( TT \). The symplectic structure \( \Omega \) can be written as
\[ \Omega = D^r x \nabla u_c - \nabla u_c D^r x . \] (A16)

Finally, if we choose the coordinate derivative \( \partial \) associated with a coordinate system \( x^a \) on \( M \),
\[ \partial x^a = 0 , \quad \partial d x^a = 0 , \] (A17)
instead of the covariant derivative \( \nabla \), the relations (A16), (A10), and (A7) reduce to the standard relations in terms of the canonical coordinates \( x^a, u_b \) on \( \Gamma \), namely
\[ \Omega = d x^r d u_c - d u_c d x^r , \] (A18)
\[ \Omega^{-1} = \partial x^a \partial u_c - \partial u_c \partial x^a , \]
and
\[ \{A, B\} = \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial u_c} - \frac{\partial A}{\partial u_c} \frac{\partial B}{\partial x^a} . \] (A19)
All coordinate vectors and 1-forms in (A18) live on the phase space \( \Gamma \).

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