Tracer Dispersion in a Self-Organized Critical System

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We have studied experimentally transport properties in a slowly driven granular system which recently was shown to display self-organized criticality [Frette et al., Nature 379, 49 (1996)]. Tracer particles were added to a pile and their transit times measured. The distribution of transit times is a constant with a crossover to a decaying power law. The average transport velocity decreases with system size. This is due to an increase in the active zone depth with system size. The relaxation processes generate coherently moving regions of grains mixed with convection. This picture is supported by the measurement of transit times. The distribution of transit times is essentially constant for small transit times and decays as a power law for large transit times. A finite-size scaling analysis shows, that the average velocity of tracer particles $\langle V \rangle \propto L/\langle T \rangle$ decreases with system size. This implies the existence of long-range correlations in the rice pile. Correlations in the sequence of transit times decay exponentially with a correlation time that increases with system size but have a crossover to a power-law decay. This indicates that the system has coherently moving regions, the size of which increases with system size. A 1D cellular automaton model, related to the experiment, displays SOC behavior and the distribution of transit times agrees qualitatively with the experimental findings. The scaling of the average transit time with system size is reproduced and hence the decrease of the average velocity with increasing system size. This phenomenon is related to an increase in the active zone depth as the system size is increased. Finally, the numerical and experimental re-
sults for the correlations in the transit times are very similar.

The experimental system consisted of a rice pile confined between two 5 mm thick glass plates supported by 15 mm thick 100 cm × 120 cm polymethylmethacrylate plates. Aluminum rods were inserted between the glass plates to form a vertical wall on one side and a variable base with length \( L \) of the quasi one-dimensional pile. The other side was open, allowing grains to fall off the pile. Grains of “Geisha Naturris” from Stabburet a.s. (Trollåsen, Norway) with length \( \delta = 7.6 \pm 0.9 \) mm and width 2.0 ± 0.1 mm were slowly fed into the gap between the plates close to the vertical wall using an Övrum Nibex 500 (Övrum, Sweden) single seed machine. We used a plate separation of \( d = 6 \) mm and system sizes \( \mathcal{L} = 15, 30, 60 \), and 85.7 cm, which, expressed in units of the grain length \( \delta \), are \( \mathcal{L}/\delta = L = 20, 39, 79 \), and 113.

The injection rate was 2–3 uncolored grains every 4.7 s or, on average, 20 grains/min. When the pile had reached the stationary state, color coded particles were added to the pile by hand and the transit times measured. A total number of 400 tracer particles for the two smaller systems and 800 for the two larger systems were used. Except for the smallest system, where a color coded tracer particle was injected every 2nd minute, the rate was 1 tracer particle every 4th minute. The injection of uncolored grains was continued until all tracers had left the system.

Figure 1 shows a part of the \( L = 113 \) system at a late stage of the experiment. Note that some of the tracer particles are close to the surface layer while others are buried quite deep in the pile. These tracers had to be released by large avalanches in order to move on. However, large avalanches occurred with a very small probability, and hence deeply buried particles tended to stay in the system for a very long time. Figure 2 is a full record of the experimental findings for the \( L = 113 \) rice pile. The projections of the horizontal lines onto the \( x \)-axis represent the time interval each tracer particle spent in the pile, that is, the transit time of the ith tracer \( T(i) = T_{\text{out}}(i) - T_{\text{in}}(i) \), where \( T_{\text{in}}(i) \) and \( T_{\text{out}}(i) \) denote the input and output time measured in units of additions of uncolored grains (1 addition every 7.7 s), respectively. There is a huge variability in the transit times. The distribution functions of transit times \( P(T, L) \) for all system sizes are shown in Fig. 3(a). A data collapse for different system sizes \( L \) is obtained when plotting \( L^\beta P(T, L) \) against the rescaled variable \( T/L^\nu \) when using \( \nu = 1.5 \pm 0.2 \) and \( \beta = 1.4 \pm 0.2 \), see Fig. 3(b). Thus we can write

\[
P(T, L) = L^{-\beta} F(T/L^\nu),
\]

where \( F \) is a scaling function and \( \nu \) a critical exponent expressing how the crossover transit time \( T_c \) scales with system size. The scaling function \( F \) is of the form \( F(x) = \text{constant for } x < 1 \) and \( F(x) \propto x^{-\alpha} \) for \( x > 1 \), where \( \alpha = 2.4 \pm 0.2 \). Since \( \alpha > 1 \), it follows from the form of the scaling function \( F \) and the normalization constraint that \( \beta = \nu \). The power-law tail does not contribute to the mean transit time \( \langle T \rangle \) if \( \alpha > 2 \), and \( \langle T \rangle \propto T_c \propto L^\nu \).

In the experiments, the angle of repose was independent of system size. Thus the average velocity of tracer particles scales like \( \langle V \rangle \propto L/T \propto L^{1-\nu} = L^{-0.5 \pm 0.2} \). It is quiet surprising, that the average velocity is not a constant but decreases with increasing system size. The (tracer) particles have information on the system size! This can only happen if correlations exists throughout the system. In the statistically stationary state, the steady influx of particles is balanced with a steady outflux. During the time \( \Delta t \), each particle moves, on average, a distance \( \Delta t \langle V \rangle \). Since the feeding rate was the same for all system sizes, the number of particles that crossed the outlet in this time interval was constant, that is, \( \Delta t \langle V \rangle \lambda_L = \text{constant}, \) where the active zone depth

\[
\lambda_L = \sqrt{\frac{1}{L} \sum_{x=1}^{L} \langle \left( h_x(t) - \langle h_x(t) \rangle \right)^2 \rangle_t},
\]

\( h_x(t) \) being the height at position \( x \) at time \( t \) and \( \langle \cdot \rangle_t \) denotes the temporal average. This implies that the average velocity is inversely proportional to the active zone depth, \( \langle V \rangle \propto 1/\lambda_L \). An increase in \( \lambda_L \) with system size would be consistent with the experimental finding of a decreasing average velocity with increasing system size.

Further insight into the correlations that exist in this dynamical state can be extracted from the data. We notice that, once in a while, many tracers dropped out of the system at the same time since they were part of a large avalanche that reached the rim of the pile. The corresponding “steps” in the diagrams, like the one shown in Fig. 2, have many different sizes and are interwoven in a complex way. The correlation function

\[
c(\tau) = \langle I(i, i + \tau) \rangle,
\]

where

\[
I(i, i + \tau) = \begin{cases} 1 & \text{if } |T_{\text{out}}(i) - T_{\text{out}}(i + \tau)| < \delta t \\ 0 & \text{otherwise} \end{cases}
\]

is the indicator function of simultaneously drop out and \( \langle \cdot \rangle \) denotes average over all \( i \), gives the probability that tracer \( i \) and \( i + \tau \) left the system simultaneously. Using \( \delta t = 10 \), this quantity is found to behave as \( c(\tau) \propto \exp(-\tau/\tau_c(L)) \) for small \( \tau \), where \( \tau \) is the time difference in units of 4 minutes and \( \tau_c(L) = 5.8 \pm 1.3, 8.7 \pm 2, 19 \pm 4, \) and \( 34 \pm 4 \) is the correlation time for system sizes \( L = 20, 39, 79 \), and 113, respectively. In Fig. 2, we see that \( \tau_c(L = 113) = 34 \) corresponds to the average size of the steps. Since the injection rate was 20 grains/min, the characteristic number of grains spanned by these correlated sequences of tracers were 476, 705, 1539, and 2754.
The solid-like motion of domains described in Ref. 1 is probably one aspect of this coherent motion. Using \( \lambda_L \propto \text{volume}/L \) and disregarding the smallest system, we obtain \( \lambda_L \propto L^{0.25 \pm 0.10} \), consistent within error bars with the result above. However, there is significant dispersion. Grains are being transferred between different coherently moving areas, see Fig. 2.

Inspired by the experiments, we considered a refined version of a simple 1D cellular automaton studied in Ref. 8. In a system of size \( L \), an integer variable \( h_x \) gives the height of the pile at site \( x \). The local slope \( z_x \) at site \( x \) is given by \( z_x = h_x - h_{x+1} \), and we impose \( h_{L+1} = 0 \). The addition of a grain at the wall increases the slope by one at \( x = 1 \), that is, \( z_1 \rightarrow z_1 + 1 \). We proceed by dropping grains at the wall until the slope \( z_1 \) exceeds a critical value, \( z_1 > z_1^c \), then the site topples by transferring one grain to its neighboring site on the right, \( x = 2 \). If \( z_x > z_x^c \), this site topples in turn according to

\[
\begin{align*}
z_x &\rightarrow z_x - 2 \\
z_{x+1} &\rightarrow z_{x+1} + 1
\end{align*}
\]

(unless at the rightmost site where the grains fall off the pile) generating an avalanche. During the avalanche, no grains are added to the pile. Thus the two time scales involved in the dynamic evolution of the pile are separated. The injection rate of grains is low compared to the duration of the relaxation processes. The avalanche stops when the system reaches a stable state with \( z_x \leq z_x^c \) \( \forall x \) and grains are added at the wall until a new avalanche is initiated and so on. The critical slopes \( z_x^c \) are dynamical variables chosen randomly to be 1 or 2 every time site \( x \) has toppled. This is a simple way to model the changes in the local slopes observed in the rice pile experiment. Thus the model differs from the trivial 1D Bak, Tang, Wiesenfeld model where \( z_x^c = 1 \) is a constant and where grains are added on randomly chosen sites. The randomness in the BTW model is external. In our model, the randomness is internal and inherent in the dynamics. Starting with, say, \( z_x = 0 \) and \( z_x^c = 1 \) \( \forall x \), the system reaches a stationary state where the avalanche sizes are power-law distributed with an exponent of \(-1.55 \pm 0.10\) and a cutoff in the power-law distribution that scales with system size as \( L^{2.25 \pm 0.10} \).

We have measured the transit times (in units of added grains) of the added particles after the pile has reached the stationary critical state. Using Eq. (1), a reasonable data collapse is obtained with \( \nu = 1.30 \pm 0.10 \) and \( \beta = 1.35 \pm 0.10 \) as displayed in Fig. 4. We find \( \alpha = 2.22 \pm 0.10 \), that is, the results agree well with the experimental findings. The average depth of tracer particles during the transport through the system as a function of the average transit time is displayed as an inset of Fig. 4, and it shows, that the power-law tail in the distribution of transit times arises from tracer particles that become deeply buried in the pile, while tracer particles that stayed close to the surface of the pile during the transport through the system mainly contributed to the constant part. The average velocity \( \langle V \rangle \propto L/(T) \propto L^{1-\nu} = L^{-0.30 \pm 0.10} \), in fairly good agreement with the experimental result. Furthermore, in the model, we are able to measure directly the scaling of the active zone depth with system size. We find \( \lambda_L \propto L^{0.25 \pm 0.10} \) in agreement with the relation \( \langle V \rangle \propto 1/\lambda_L \). Alternatively, a third measure for the active zone depth is available from the inset in Fig. 4, giving \( \lambda_L \propto \text{depth} \propto L^{0.3 \pm 0.10} \). Moreover, correlation functions calculated from Eqs. (3) and (4) give an exponential decay for short times with a crossover to a decaying power law for large times. Adding grains at the wall and allowing for dynamical critical slopes, we have seen similar results for the nonlocal limited model in Ref. 9. Thus the behavior seems to be universal.

In conclusion, this new direction of research sheds light upon the dynamics of SOC systems in general and granular systems in particular. We find that transport in a self-organized critical granular medium is characterized by an average grain velocity that approaches zero when the system size increases. The dynamics of the system displays comprehensive correlations. These experimental findings agree well with the behavior seen in simple one-dimensional computer models of the self-organized critical pile.

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FIG. 1. A close up photograph of the rice pile with size $L = 113$. The plate separation/grain length ratio $d/\delta = 0.79$ and most of the grains are aligned along the flow direction. Each tracer particle is uniquely color coded; thus it is possible to measure individual transit times. This picture was taken after 638 out of 800 tracer particles had been added. The region shown contains 5 of a total of 73 tracer particles that were inside the pile at this time.

FIG. 2. A record of the tracer experiment in a pile of size $L = 113$ where a total number of 800 tracer particles were added, one every 4th minute. The tilted line connects the injection times for all the tracers. The transit time for each tracer particle is represented by the length of a horizontal line whose projection onto the x-axis of the left (right) endpoint equals the time the particle entered $T_{in}$ ($T_{out}$) the system. The transit time is measured in units of the number of injections of uncolored grains (no. additions), 1 addition every 7.7 s. Note the large variation in the transit times $T = T_{out} - T_{in}$ and that, repeatedly, many tracers left the system at the same time. The horizontal arrow indicates the time at which the photograph in Fig.1 was taken. Furthermore, the correlation time $\tau_c$ (see text related to Eq. (3)) is indicated. The inset is the part of the full record marked by dashed lines and shows that the different sizes of steps are interwoven.

FIG. 3. (a) The experimental results for the normalized distribution of transit times in piles with sizes $L = 20, 39, 79,$ and 113. The data have been averaged over exponentially increasing bins with base 2 in order to reduce the fluctuations in the statistics due to the relatively small number of tracer particles. The functions are essentially constant for small $T$ and have a decaying power-law tail with a slope of $\alpha = 2.4 \pm 0.2$. These large transit times correspond to tracer particles which, during the transport through the systems, become deeply embedded in the pile. (b) Disregarding the smallest system and using Eq. (2), a reasonable data collapse of the three largest systems is obtained with $\nu = 1.5 \pm 0.2$ and $\beta = 1.4 \pm 0.2$.

FIG. 4. A finite-size scaling plot using Eq. (5) with $\nu = 1.30 \pm 0.10$ and $\beta = 1.35 \pm 0.10$ of the normalized distribution of transit times in the numerical model with system sizes $L = 25, 100, 400,$ and 1600. The statistics shown correspond to $10^6$ tracer particles ($10^9$ for $L = 1600$), and the data have been averaged over exponentially increasing bins with base 1.1. The functions are constant for small transit times and decay as power laws with a slope of $\alpha = 2.22 \pm 0.10$. The inset shows that a data collapse of the correlation between the transit times and depth of the tracer particles can be obtained using a finite-size scaling plot analogous to Eq. (4), with $\nu' = 1.20 \pm 0.10$ and $\beta' = -0.30 \pm 0.10$. 

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