A Brownian particle having a fluctuating mass

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We focus on the dynamics of a Brownian particle whose mass fluctuates. First we show that the behaviour is similar to that of a Brownian particle moving in a fluctuating medium, as studied by Beck [Phys. Rev. Lett. 87 (2001) 180601]. By performing numerical simulations of the Langevin equation, we check the theoretical predictions derived in the adiabatic limit, and study deviations outside this limit. We compare the mass velocity distribution with truncated Tsallis distributions [J. Stat. Phys. 52 (1988) 479] and find excellent agreement if the masses are chi-squared distributed. We also consider the diffusion of the Brownian particle by studying a Bernoulli random walk with fluctuating walk length in one dimension. We observe the time dependence of the position distribution kurtosis and find interesting behaviours. We point out a few physical cases where the mass fluctuation problem could be encountered as a first approximation for agglomeration-fracture non equilibrium processes.

I. INTRODUCTION

It is well known that, while the Maxwell-Boltzmann distribution takes place in any system at equilibrium, non-equilibrium systems present in general qualitative and quantitative deviations from the former. A case of particular interest is that of distributions characterized by a power law tail, and therefore by an over-population for high energy as compared to the Gaussian. Indeed, such behaviours occur in a large number of physical systems, going from self-organized media to granular gases, and may have striking consequences due to the large probability of extreme/rare events.

The ubiquity of these fat tail distributions in nature has motivated several attempts in the literature to construct a general formalism for their description, one of the most recent being the Tsallis thermodynamic formalism [1, 2]. The latter is based on a proper extremization of the non extensive entropy

$$ S_q = k \left( 1 - \frac{\sum p_i^q}{q-1} \right) $$

that leads to generalized canonical distributions, often called Tsallis distributions

$$ f_q(x) = \frac{e_q^{-\beta' x}}{Z} $$

In this expression, $x$ denotes the state of the system, and $e_q^x$ is the $q$-exponential function defined by

$$ e_q^x \equiv (1 + (1 - q)x)^{1/(1-q)} $$

This definition implies that $e_q^1 = e^x$, and that the energy is canonically distributed in the classical limit $q \rightarrow 1$, as expected. Let us also note that this formalism draws a direct parallelism with the equilibrium theory, where $\beta'$ plays the role of the inverse of a temperature, and $Z$ that of a partition function.

Tsallis distributions have been observed in numerous fields [3] but their fundamental origin is still debated due to the lack of simple model and exact treatment justifying the formalism. It is therefore important to study simple statistical models in order to show in which context Tsallis statistics apply. There are several microscopic ways to justify Tsallis statistics. One of them has been introduced some years ago by Beck [1]. Another has been recently introduced by Thurner [4]. Beck theory was first initiated by considering the case of a Brownian particle moving in a specially thermally fluctuating medium, i.e. the inverse temperature $\beta$ is a chi-squared distributed random variable. Yet, Thurner [4] has shown that one can derive Tsallis distributions in a general way without Beck ”chi-square assumption”.

In view of the above considerations, the motion of a Brownian particle whose mass is a random variable seems to be a paradigmatic example. Moreover many physical cases are concerned by such situations. A sharp mass or volume variation of entities can be encountered in many non-equilibrium cases: ion-ion reaction [5, 6, 7], electrodeposition [8], granular flow [9, 10, 11], formation of planets through dust aggregation [12, 13, 14], film deposition [15], traffic jams [16, 17], and even stock markets [18, 19] (in which the volume of exchanged shares fluctuates and the price undergoes some random walk). It should be noticed that a priori fluctuating mass problems differ from temperature fluctuations. Two masses can be added to each other; this is hardly the case for temperatures. An exposé of such a generalized Brownian motion and the distinction between masses and temperatures will be emphasized in Sect.II and Sect. III.

By supposing two time scales, i.e. assuming that relax-
ation processes for the particle are faster than the character-
istic times for the mass fluctuation, it will be shown
that its asymptotic velocity distribution is in general non-
Maxwellian. Moreover, for some choice of the mass prob-
bility distribution, the Brownian particle velocities are
Tsallis distributed. We verify this result by performing
simulations of the corresponding Langevin equation.

Moreover, we also consider the case when the mass fluc-
tuation is not adiabatically slow. We show in Sect. IV
that the mass dependence of the relaxation rate may have
non-negligible consequences on the velocity distribution
of the Brownian particle. In order to fit the resulting
distributions, and to describe the deviations from Tsallis
statistics, we introduce truncated Tsallis distributions.
Finally, in Sect. V, we study the diffusive properties of
the Brownian particle. To do so, we model the motion of
the particle by a random walk with time-dependent
jump probabilities, associated with the mass fluctuations
of the particle. The model exhibits standard diffusion,
i.e. \( < x^2 > \sim t \), but the shape of the scaling position
distribution is anomalous and may exhibit quasi-stationary
features. Sect. VI serves as a summary and conclusions.

II. GENERALIZING THE BROWNIAN MOTION TO ONE IN A FLUCTUATING MEDIUM

For setting the framework of the present study, let us recall one microscopic way to justify Tsallis statistics\(^1\).
Beck theory, later called Superstatistics, was first initi-
ated by considering the case of a Brownian particle mov-
ing in a fluctuating medium, i.e. an ensemble of macro-
sopic particles evolving according to Langevin dynamics

\[
m \partial_\tau v = -\lambda v + \sigma L(t) \tag{4}
\]

where \( \lambda \) is the friction coefficient, \( \sigma \) describes the strength
of the noise, and \( L(t) \) is a Gaussian white noise\(^{23}\). Con-
trary to the classical Brownian motion, however, one may
consider that the features of the medium may fluctuate temporally and/or spatially, namely the quantity \( \beta = \frac{1}{T} \),
i.e. a quantity which plays the role of the inverse of tem-
perature, changes temporally on a time scale \( \tau \), or on the
spatial scale \( L \); see also\(^{23}\). E.g. in his original paper,
Beck assumed that this quantity fluctuates adiabatically slow-
ly, namely that the time scale \( \tau \) is much larger than
the relaxation time for reaching local equilibrium. In
that case, the stationary solution of the non-equilibrium
system consists in Boltzmann factors \( e^{\beta x} \) that are aver-
eraged over the various fluctuating inverse temperatures
\( \beta \)

\[
f_{Beck}(x) = \frac{1}{K} \int d\beta \ g(\beta) \ e^{-\beta x} \tag{5}
\]

where \( K \) is a normalizing constant, and \( g(\beta) \) is the prob-
bability distribution of \( \beta \). Let us stress that ordi-
inary statistical mechanics are recovered in the limit
\( g(\beta) \to \delta(\beta - \beta_E) \). In contrast, different choices for the
statistics of \( \beta \) may lead to a large variety of probability
distributions for the Brownian particle velocity.

Several forms for \( g(\beta) \) have been studied in the liter-
ature\(^{21}\), but one functional family of \( g(\beta) \) is partic-
ularly interesting. Indeed, the generalized Langevin model
Eq.(4) generates Tsallis statistics for the velocities of the
Brownian particle if \( \beta \) is a chi-squared random variable

\[
g(\beta) = \frac{1}{b^c (c-1)!} \ e^{-\frac{\beta^2}{b^2}} \tag{6}
\]

where \( b \) and \( c \) are positive real parameters which account
for the average and the variance of \( \beta \). Let us stress that
a chi-squared distribution derives from the summation
of squared Gaussian random variables \( X_i, \beta = \sum_{i=1}^{2c} X_i^2 \),
where the \( X_i \) are independent, and \( < X_i > = 0 \). By intro-
ducing Eq.(4) into Eq.(5), it is straightforward to show
that the velocity distribution of the particle is Eq.(2), if
one identifies \( c = \frac{1}{q-1} \) and \( bc = \beta' \).

For completeness, let us also mention a study of Eq.(4)
when \( \lambda \) fluctuates\(^{22}\).

III. FLUCTUATING MASS

Let us now consider the diffusive properties of a macro-
sopic cluster such as one arising in granular media\(^3, 10\)
or in traffic\(^{16}\). Such media are composed by a large number
of macroscopic particles. Due to their inelastic
interactions, the systems are composed of very dense re-

gions evolving along more dilute ones. In general, there
is a continuous exchange of particles between the dense
cluster and the dilute region, so that the total mass of the
macroscopic entity is not conserved. As a first ap-
proximation, we have thus considered the simplest ap-
proximation for this dynamics, namely the cluster is
one Brownian-like particle whose mass fluctuates in the
course of time. To mimic this effect, we have assumed
that (i) the distribution of masses is \( a \text{ priori} \) given by
\( g(m) \), and (ii) the mass of the cluster fluctuates with a
characteristic time \( \tau \). By definition, this model evolves
according to the Langevin equation Eq.(4), where \( m \) is
now the random variable.

Given some realization of the random mass, say \( m = m_R \), one easily checks that the velocity distribution of
the particle converges toward the distribution

\[
f_B(v) \to \sqrt{\frac{3m_R}{2\pi}} \ e^{-\frac{m_R v^2}{2}} \tag{7}
\]

This relaxation process takes place over a time scale
\( t_R \sim \frac{m_R}{k_B T} \). Therefore, if the separation of time scales
\( t_R << \tau \) applies, the asymptotic velocity distribution of
the cluster is given by

\[
f_B(v) = \int dm g(m) \sqrt{\frac{3m}{2\pi}} \ e^{-\frac{m v^2}{2}} \tag{8}
\]
Theoretical value

FIG. 1: Time evolution of the \( < v^2 > \) for three values of \( \tau \), characterised by \( s = \frac{t_R}{\tau} = [0.1, 1, 10] \). The possible masses in the system are 1 and 10. Consequently, the \( < v^2 > \) for each species are respectively 1.1 and 0.1. Moreover, in the slow limit (\( \tau \gg t_R \)), the asymptotic velocity fluctuation is, on average, 0.55. When \( \tau \) is small, most of the particle velocities are distributed like the light ones, and their energy is closer to 1 than to 0.55.

Consequently, this leads to a Tsallis distribution if the masses are chi-squared distributed. In that sense there is a direct correspondence between Beck approach and ours. However there is more to see in the latter case because it justifies Tsallis non extensive entropy approach in a more mechanistic way. Subsequently two basic questions can be raised: (i) what are the limits of validity of such a non-equilibrium approach?, and (ii) are the mass fluctuation time scales observable?

IV. RELAXATION MECHANISMS

In this section, we report numerical simulations of the random process Eq.\( \text{(4)} \). The objective is twofold. First, we verify the theoretical prediction Eq.\( \text{(8)} \) in the adiabatic limit. Next, this allows to study systems which are beyond the range of validity of Eq.\( \text{(5)} \), namely systems where the separation of scales \( t_R \ll \tau \) does not apply, thereby investigating the effects of competition between the relaxation to equilibrium and the fluctuating features of this equilibrium state. This program is achieved by considering three different relaxation characteristic time scales for the processes, namely \( s = \frac{t_R}{\tau} = (0.1, 1.0, 10) \). Moreover, we first consider a paradigmatic case when the masses can switch between two different discrete values, each with equal probability:

\[
g(m) = \frac{1}{2}(\delta(m - m_1) + \delta(m - m_2))
\]  

(9)

If Eq.\( \text{(8)} \) applies, i.e. in the slow fluctuation limit, Eq.\( \text{(4)} \) leads to

\[
f_B(v) = \frac{1}{2} \sqrt{\frac{\beta}{2\pi}} \left( m_1 e^{-\frac{\beta v^2 m_1}{2}} + \sqrt{m_2 e^{-\frac{\beta v^2 m_2}{2}}} \right).
\]  

(10)

Before focusing on the velocity distributions for the Brownian particles, let us stress that their average energy depends on the speed of the fluctuation mechanism. Indeed, in the slow limit \( s \ll 1 \), the energy of the cluster converges very rapidly toward the equipartition value \( m_i < v^2 > = e \), where \( e \) is the average kinetic energy of
The bath and \( m_i \) is the mass of the cluster at that time. This implies that the fluctuations of the measured velocities are given by \( < v^2 > = \frac{1}{2} \left( \frac{m_i}{m_1} + \frac{m_i}{m_2} \right) \). In contrast, in the faster limit \( s > 1 \), the dependence of the characteristic time \( t_R \sim \frac{1}{m_i} \) cannot be neglected. Indeed, this relation implies that particles with a smaller mass relax faster than particles with a larger mass. Consequently, the lighter particles should have a value close to their equipartition value \( m_i < v^2 > = \epsilon \), while the heavier particles should have an energy larger than their expected value. This property is verified (Fig. 1).

In order to compare the velocity distributions for different time scales \( \tau \), and therefore at different energies, we rescale the velocities so that \( < v > = 1 \). The results, as plotted in Fig. 2, confirm the theoretical predictions Eq. (10) and our description in the previous paragraph. Indeed, in the slow limit \( s = 0.1 \), the velocity distribution converges toward the leptokurtic distribution Eq. (10), i.e. a distribution with a positive kurtosis and an overpopulated tail. In contrast, when the mass dependence of the \( t_R \) has to be taken into account, most of the particles have velocities distributed like those of the light particles, namely are Maxwell-Boltzmann distributed.

In the case of general Tsallis distributions, obtained through chi-squared distributed cluster masses, the problem is more complex due to the continuum of masses in the system, and to the associated continuum of characteristic relaxation times. Moreover, the existence of extreme values for the masses may cause non-realistic numerical problems. Indeed, arbitrary small masses lead to arbitrary high values of the velocities. In the following, we avoid this effect that is responsible for the power law tails observed in Tsallis distributions. This is justified by the fact that any physical system has a minimum size for its internal components. Similarly, we restrict the maximum size of the clusters in order to avoid infinitely slow relaxation processes. These limitations are formalized by using the following truncation for the mass distributions, inspired by truncated Lévy distributions [23]}

\[
g(m;c) = k \chi^2(m;c) \quad \text{if } m > a \text{ and } m < b
\]

\[
g(m;c) = 0 \quad \text{otherwise}
\]

where \( c \) is a parameter characterizing the chi-squared distribution, \( k \) is a normalizing constant, and \( a < b \) are cut-off parameters. In the following, we use \( a = 0.01 \) and \( b = 100.0 \) in the simulations of the Langevin equation. As shown in Fig. 4, this procedure is a natural way to smoothen the tail of the Tsallis velocity distribution while preserving its core. This method, that will be discussed further in a forthcoming paper, should be applicable to large variety of problems (like those mentioned in the introduction) where extreme events have to be truncated for physical reasons.

Numerical simulations of the Langevin equation for chi-squared mass distributions generalize in a straightforward way the results obtained from the 2-level distribution Eq. (9). Indeed, the faster the mass distribution fluctuates the larger deviations from the Tsallis distribution are. Moreover, these deviations have a tendency to underpopulate the tail of the distribution, and therefore to avoid the realization of extreme values of the random process.

It is also important to note the similarities between these asymptotic solutions of the Langevin equation and the truncated Tsallis distributions defined by Eq. (11). They highlight the flexibility of truncated Tsallis distributions in order to describe deviations from the Tsallis distributions, as shown in Fig. 5.
V. DIFFUSION

In this last section, we focus on the diffusive properties of a Brownian particle with fluctuating mass. Therefore, we take into account the hydrodynamic time scale $t_H$, that is associated to the evolution of spatial inhomogeneities. It is well known [24], when the mass of the Brownian particle is constant in time, that separation of time scales $t_H >> t_R$ is required in order to derive the diffusion equation:

$$\partial_t n(x,t) = D \partial_x^2 n(x,t)$$ (12)

where $D$ is fixed by the Einstein formula.

When the mass of the Brownian particle fluctuates, however, we have shown above that there is an additional time scale $\tau$ in the dynamics. As first approximation, we restrict the scope to the limit of very slow fluctuations ($\tau >> t_H$), that is more restrictive than the limit discussed in the previous section ($\tau >> t_R$). In that case, it is possible to show [25] that the Chapman-Enskog procedure leads to Eq. (7), where $D$ is now time-dependent, i.e. a random variable that is a function of the mass of the Brownian particle.

In the following, we investigate the process associated to Eq. (7) with fluctuating diffusion coefficient. To do so, we simplify the analysis by considering a one-dimensional discrete time random walk, where the jump probabilities may fluctuate in time [26]. Namely, the walker located at $x$ performs at each time step a jump of length $l$, with Bernoulli probabilities:

$$P(k)|_l = \frac{1}{2} [\delta(k,l) + \delta(k,-l)]$$ (13)

The quantity $l$ fluctuates between two integer values $l_A < l_B$ that correspond to an heavy/light state for the Brownian particle. When $l_A = l_B$, it is easy to show that the first distance moments $m_i = \langle x^i \rangle$ asymptotically behave like:

$$m_2 = Dt$$

$$m_4 = 3D^2t^2$$ (14)

where the diffusion coefficient $D = l_A^2$. Relation (14) implies that the kurtosis of the distance distribution $\kappa = \frac{m_4}{m_2^2} - 3$ vanishes asymptotically, as required by the central-limit theorem.

Another simple limit consists in a system with $l_A \neq l_B$, and where the mass fluctuation process is infinitely slow. Consequently, the particles do not change mass and the system is composed of two species that diffuse differently.

FIG. 6: Kurtosis of the distance distribution Eq. (16), for two Bernoulli random walkers in one dimension with different walk lengths.

FIG. 7: Kurtosis $\kappa$ of the distance distribution, as a function of time, for the Bernoulli random walker with fluctuating mass (see main text). In (a), we focus on the limiting cases $p = 0.0$ and $p = 1.0$ that converge toward the theoretical prediction, Eq. (16), $\kappa = 1.08$ and $\kappa_G = 0$ (Gaussian) respectively. These asymptotic values are represented by solid lines. In (b), we focus on the intermediate cases $p = 0.01$ and $p = 0.0001$. The state associated to Eq. (16) is stationary during a long time extent, and is followed by a convergence toward the Gaussian.
In that case, the first distance moments read:

\[ m_2 = \frac{1}{2} (D_A + D_B) t \]
\[ m_4 = \frac{3}{2} (D_A^2 + D_B^2) t^2 \]  \hspace{1cm} (15)

where the diffusion coefficients are \( D_A = l_A^2 \) and \( D_B = l_B^2 \). This relation implies that the asymptotic kurtosis is equal to:

\[ \kappa = 6 \frac{(D_A^2 + D_B^2)}{(D_A + D_B)^2} - 3 \]  \hspace{1cm} (16)

In Fig.6 we plot \( \kappa(r) = 6 \left( \frac{l_A^2 + l_B^2}{r^2 + l_A^2} \right) - 3 \), where \( r = \frac{D_A}{D_B} \). The figure shows that \( \kappa \geq 0 \), i.e. the distribution is characterised by an overpopulated tail, except in the usual limit \( r = 1 \). This result is expected, as a system with \( r \neq 1 \) is composed of two species that explore the space at different speeds. Let us stress that despite this anomalous position distribution, the diffusion is standard, i.e. \( <x^2> \sim t \) (see Eq.15).

In order to study intermediate situations, we have performed computer simulations of the random walk. The system is composed of 50000 walkers, that are initially located at \( x = 0 \) and randomly divided in the species \( A/B \). The mass fluctuations are uncorrelated and occur with probability \( p \in [0,1] \) at each time step, i.e. the characteristic time of the fluctuations is \( \tau \sim p^{-1} \). In the simulations, we have used \( l_A = 1 \) and \( l_B = 2 \), so that the prediction for the kurtosis in the limit \( \tau \to \infty \) is \( \kappa_{AB} = 1.08 \). This prediction (Fig.7h) is verified by simulations in a system where the mass of a walker is constant in time. In the limit \( \tau = 1 \), i.e. the walker changes mass at each time step, the kurtosis converges rapidly toward the gaussian value \( \kappa_G = 0 \). In contrast, for higher finite values of \( \tau \) (Fig.7h), one observes a crossover between the two asymptotic behaviours \( \kappa_{AB} \) and \( \kappa_G \). The stability of the state \( \kappa_{AB} \) is longer and longer for increasing values of \( \tau \). This quasi-stationarity \footnote{27} originates from the following reason. Over a long time period \( T \), with \( T \gg \tau \), the particles have suffered so many changes from state \( A \) to \( B \) and forth, that their asymptotic dynamics is to that of a random walk with 2 possible jump lengths \( l_A \) and \( l_B \), with probabilities:

\[ P(k) = \frac{1}{4} [\delta(k,l_A) + \delta(k,-l_A) + \delta(k,l_B) + \delta(k,-l_B)] \]  \hspace{1cm} (17)

Consequently, its asymptotic dynamics is subject to the classical central-limit theorem, and the position distribution is a Gaussian for \( T \gg \tau \).

VI. CONCLUSIONS

In examining an apparently unusual generalization of the old Brownian motion problem, i.e. a particle with fluctuating mass, we have found a very simple example justifying non-extensive thermodynamics. However the simplicity is related to underlying considerations on quite various physical (or other) systems in which some "mass" is evolving with time, sometimes stochastically, as mentioned in the introduction of this paper.

For the sake of generality, we have studied systems with arbitrary mass distribution and time scale of the mass fluctuation. Let us stress that the mass statistics has been chosen \textit{a priori}, and that a more detailed study requires dynamical treatment of the fluctuating mass by Langevin equation. This additional modelling depends on the nature of the considered problem. The velocity distribution of the Brownian particle has been studied by performing simulations that highlight the important role of mass fluctuation time scale. In the case of chi-squared mass distribution, it is shown that truncated Tsallis distributions seem to describe in a relevant way deviations from the Tsallis statistics. Such distributions should be applicable to a large variety of problems where extreme events have to be truncated for physical reasons, e.g. finite size effects, - when there is no infinity! Among these possible applications, let us note their occurrence in airline disasters statistics \footnote{28}.

We have also studied the diffusive properties of the Brownian particle. In the limit of slow mass fluctuation times, the particle motion is modelled by a random walk with time-dependent jump probabilities. Moreover, for the sake of clarity, we restrict the scope to a dichotomous mass distribution. This modelling is a simplification of the complete problem, that has to be justified by a complete analysis starting from the Fokker-Planck equation itself \footnote{28}. Nonetheless, despite its apparent simplicity, the random walk analysis shows non-trivial behaviours, namely the system is characterised by standard diffusion, associated with non-Gaussian scaling distributions and quasi-stationary solutions. These features originate from very general mechanisms that suggest their relevance in various systems with fluctuating "mass" parameters.

In conclusion, we have generalized Brownian motion to the case of fluctuating mass systems, and distinguish them from systems in which there are local temperature fluctuations. We have found that the velocity of such a particle may exhibit various anomalous distributions, including Tsallis distributions and truncated Tsallis distributions. The study of a one dimensional random walk also indicates anomalies in the kurtosis of the position distributions which might indicate physical biases in various processes as those recalled in the introduction. This suggests to look for the value of higher order distribution moments and on their time evolution for understanding properties of the non-equilibrium systems.

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From now on, we restrict our analysis to the one-dimensional case for the sake of writing simplicity, but without much loss of generality.
Tsallis distribution

Probability distribution vs. Rescaled velocities for different values of $a$: $a=0.1$ (+) and $a=1.0$ (×).
Probability distribution versus rescaled velocities for different values of $a$. The curve marked with '+' represents $a=0.1$, and the one marked with '×' represents $a=1.0$. The dotted line indicates the Tsallis distribution.
Probability distribution

Rescaled velocities

Tsallis distribution

s=10
s=1
s=0.1
Probability distribution vs. Rescaled velocities.

- Langevin equation
- Truncated Tsallis distribution
- Tsallis distribution