A Generalization of the Bargmann’s Theory of Ray Representations

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Abstract

The paper contains a complete theory of factors for ray representations acting in a Hilbert bundle, which is a generalization of the known Bargmann’s theory. With the help of it we have reformulated the standard quantum theory such that the gauge freedom emerges naturally from the very nature of quantum laws. The theory is of primary importance in the investigations of covariance (in contradistinction to symmetry) of a quantum theory which possesses a nontrivial gauge freedom. In that case the group in question is not any symmetry group but it is a covariance group only – that case which has not been deeply investigated. It is shown in the paper that the factor of its representation depends on space and time when the system in question possesses a gauge freedom. In the nonrelativistic theories the factor depends on the time only. In the relativistic theory the Hilbert bundle is over the spacetime and in the nonrelativistic one it is over the time.

We explain two applications of this generalization: in the theory of a quantum particle in the gravitational field in the nonrelativistic limit and in the quantum electrodynamics.

1 Introduction

In the standard Quantum Mechanics (QM) and the Quantum Field Theory (QFT) the spacetime coordinates are pretty classical variables. Therefore the question about the general covariance of QM and QFT emerges naturally just like in the classical theory:

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what is the effect of a changing of the spacetime coordinates in QM and QFT when the changing does not form any symmetry transformation?

It is a commonly accepted believe that there are no substantial difficulties if we refer the question to the wave equation. We simply treat the wave equation, and do not say why, in such a manner as if it was a classical equation. The only problem arising is to find the transformation rule \( \psi \rightarrow T_r \psi \) for the wave function \( \psi \). This procedure, which on the other hand can be seriously objected, does not solve the above stated problem. The heart of the problem as well as of QM and QFT lies in the Hilbert space of states and just in finding the representation \( T_r \) of the covariance group in question. The trouble gets its source in the fact that the covariance transformation changes the form of the wave equation such that \( \psi \) and \( T_r \psi \) do not belong to the same Hilbert space, which means that \( T_r \) does not act in the ordinary Hilbert space. This is not compatible with the paradigm worked out in dealing with symmetry groups.

We show that covariance group acts in a Hilbert bundle \( \mathcal{R} \triangle \mathcal{H} \) over the time in the nonrelativistic theory and in a Hilbert bundle \( \mathcal{M} \triangle \mathcal{H} \) over the spacetime \( \mathcal{M} \) in the relativistic case. The waves are the appropriate cross sections of the bundle in question. The exponent \( \xi(r, s, p) \) in the formula

\[
T_r T_s = e^{i \xi(r, s, p)} T_{rs},
\]

depends on the point \( p \) of the base of the bundle in question: that is, \( \xi \) depends on the time \( t \) in the nonrelativistic theory and on spacetime point \( p \) in the relativistic theory if there exists a nontrivial gauge freedom.

Moreover, we argue that the bundle \( \mathcal{M} \triangle \mathcal{H} \) is more appropriate for treating the covariance as well as the symmetry groups then the Hilbert space itself. Namely, we show that from the more general assumption that the representation \( T_r \) of the Galilean group acts in \( \mathcal{R} \triangle \mathcal{H} \) and has an exponent \( \xi(r, s, t) \) depending on the time \( t \) we reconstruct the nonrelativistic Quantum Mechanics. Even more, in the less trivial case of the theory with nontrivial time-dependent gauge describing the spin less quantum particle in the Newtonian gravity we are able to infer the wave equation and prove the equality of the inertial and gravitational masses.

In doing it we apply extensively the classification theory for exponents \( \xi(r, s, t) \) of \( T_r \) acting in \( \mathcal{R} \triangle \mathcal{H} \) and depending on the time.

The main task of this paper is to construct the general classification theory of spacetime dependent exponents \( \xi(r, s, p) \) of representations acting in \( \mathcal{M} \triangle \mathcal{H} \). On the other hand the presented theory can be viewed as a generalization of the Bargmann’s classification theory of exponents \( \xi(r, s) \) of representations acting in ordinary Hilbert spaces, which are independent of \( p \in \mathcal{M} \).

In the presented theory which is slightly more general then the standard one the gauge freedom emerges from the very nature of the fundamental laws of Quantum Mechanics. By this it opens a new perspective in solving the troubles in QFT caused by the gauge freedom.

In section 2 we present the physical motivation in detail. In section 3 we present the generalization of the ordinary state vector ray and operator ray
introduced by H. Weyl. In sections 3 and 4 we present the continuity assumption from which the strong continuity of the exponent ξ follows and generalize the ordinary notion of the exponent ξ of a ray representation. In section 5 we analyze the local exponents of Lie groups. In section 6 we introduce algebras which are the important tools for the classification theory of local exponents presented in the section 7. In section 8 we investigate the globally defined exponents and classify them in some special cases. In the section 9 we present examples. The first example is the Galilean group. We analyze the group from the point of view of the generalized theory. As the second example the exponents of the Milne group, the covariance group relevant in the theory of nonrelativistic particle in the gravitational field, are analyzed.

The proof of differentiability of the (generalized) exponent and the first three Lemmas goes in an analogous way as those presented in the Bargmann’s work [1]. However, it is not trivial that they are also true in this generalized situation. We present the proof of them explicitly for the reader’s convenience. The rest of our reasoning is not a simple analogue of [1] and proceeds another way.

2 Setting for the Motivation

In this subsection we carry out the general analysis of the representation $T_r$ of a covariance group and compare it with the representation of a symmetry group. We describe also the correspondence between the space of wave functions $\psi(\vec{x}, t)$ and the Hilbert space. We carry out the analysis in the nonrelativistic case, but it can be derived as well in the relativistic quantum field theory.

Before we give the general description, it will be instructive to investigate the problem for the free particle in the flat Galilean spacetime. The set of solutions $\psi$ of the Schrödinger equation which are admissible in Quantum Mechanics is precisely given by

$$\psi(\vec{x}, t) = (2\pi)^{-3/2} \int \varphi(\vec{k}) e^{-i\frac{p \cdot \vec{x}}{m}} d^3k,$$

where $p = \hbar \vec{k}$ is the linear momentum and $\varphi(\vec{k})$ is any square integrable function. The functions $\varphi$ (wave functions in the "Heisenberg picture") form a Hilbert space $\mathcal{H}$ with the inner product

$$(\varphi_1, \varphi_2) = \int \varphi_1^*(\vec{k})\varphi_2(\vec{k}) d^3k.$$ 

The correspondence between $\psi$ and $\varphi$ is one-to-one.

But in general the construction fails if the Schrödinger equation possesses a nontrivial gauge freedom. We explain it. For example the above construction fails for the nonrelativistic quantum particle in the curved Newton-Cartan spacetime. Beside this, in this spacetime we do not have plane wave, see [13]. So, there does not exist any natural counterpart of the Fourier transform. However,
we need not to use the Fourier transform. What is the role of the Schrödinger equation in the above construction of $\mathcal{H}$? Note, that in general

$$||\psi||^2 \equiv \int \psi^*(\vec{x},0)\psi(\vec{x},0) \, d^3x = (\varphi, \varphi) =$$

$$= \int \psi^*(\vec{x},t)\psi(\vec{x},t) \, d^3x.$$ 

This is in accordance with the Born interpretation of $\psi$. Namely, if $\psi^*\psi(\vec{x},t)$ is the probability density, then

$$\int \psi^*\psi \, d^3x$$

has to be preserved in time. In the above construction the Hilbert space $\mathcal{H}$ is isomorphic to the space of square integrable functions $\varphi(\vec{x}) \equiv \psi(\vec{x},0)$ – the set of square integrable space of initial data for the Schrödinger equation, see e.g. [5]. The connection between $\psi$ and $\varphi$ is given by the time evolution $U(0,t)$ operator (by the Schrödinger equation):

$$U(0,t)\varphi = \psi.$$ 

The correspondence between $\varphi$ and $\psi$ has all formal properties such as in the above Fourier construction. Of course, the initial data for the Schrödinger equation do not cover the whole Hilbert space $\mathcal{H}$ of square integrable functions, but the time evolution given by the Schrödinger equation can be uniquely extended on the whole Hilbert space $\mathcal{H}$ by the unitary evolution operator $U$.

The construction can be applied to the particle in the Newton-Cartan space-time. As we implicitly assumed, the wave equation is such that the set of its admissible initial data is dense in the space of square integrable functions (we need it for the uniqueness of the extension). Because of the Born interpretation the integral

$$\int \psi^*\psi \, d^3x$$

has to be preserved in time. Denote the space of the initial square integrable data $\varphi$ on the simultaneity hyperplane $t(X) = t$ by $\mathcal{H}_t$. The evolution is, then, an isometry between $\mathcal{H}_0$ and $\mathcal{H}_t$. But such an isometry has to be a unitary operator, and the construction is well defined, i.e. the inner product of two states corresponding to the wave functions $\psi_1$ and $\psi_2$ does not depend on the choice of $\mathcal{H}_t$. Let us mention, that the wave equation has to be linear in accordance with the Born interpretation of $\psi$ (any unitary operator is linear, so, the time evolution operator is linear). The space of wave functions $\psi(\vec{x},t) = U(0,t)\varphi(\vec{x})$ isomorphic to the Hilbert space $\mathcal{H}_0$ of $\varphi$’s is called in the common "jargon" the "Schrödinger picture".

However, the connection between $\varphi(\vec{x})$ and $\psi(\vec{x},t)$ is not unique in general, if the wave equation possesses a gauge freedom. Namely, consider the two states
\( \varphi_1 \) and \( \varphi_2 \) and ask the question: when the two states are equivalent and by this indistinguishable? The answer is as follows: they are equivalent if

\[
| (\varphi_1, \varphi) | \equiv \left| \int \psi_1^*(\vec{x}, t) \psi(\vec{x}, t) d^3x \right| = | (\varphi_2, \varphi) | \equiv \left| \int \psi_2^*(\vec{x}, t) \psi(\vec{x}, t) d^3x \right|,
\]

for any state \( \varphi \) from \( \mathcal{H} \), or for all \( \psi = U \varphi \) (\( \psi_1 \) are defined to be \( = U(0, t) \varphi_1 \)). Substituting \( \varphi_1 \) and then \( \varphi_2 \) for \( \varphi \) and making use of the Schwarz’s inequality one gets: \( \varphi_2 = e^{i\alpha} \varphi_1 \), where \( \alpha \) is any constant\(^1\). The situation for \( \psi_1 \) and \( \psi_2 \) is however different. In general the condition (1) is fulfilled if

\[
\psi_2 = e^{i\xi(t)} \psi_1
\]

and the phase factor can depend on time. Of course it has to be consistent with the wave equation, that is, together with a solution \( \psi \) to the wave equation the wave function \( e^{i\xi(t)}\psi \) also is a solution to the appropriately gauged wave equation. \textit{A priori} one can not exclude the existence of such a consistent time evolution. This is not a new observation, it was noticed by John von Neumann\(^2\), but it seems that it has never been deeply investigated (probably because the ordinary nonrelativistic Schrödinger equation has a gauge symmetry with constant \( \xi \)). The space of waves \( \psi \) describing the system cannot be reduced in the above way to any fixed Hilbert space \( \mathcal{H} \) with a fixed \( t \). So, the existence of the nontrivial gauge freedom leads to the

**Hypothesis.** The two waves \( \psi \) and \( e^{i\xi(t)}\psi \) are quantum-mechanically indistinguishable.

Moreover, we are obliged to use the whole Hilbert bundle \( \mathcal{R} \Delta \mathcal{H} : t \rightarrow \mathcal{H}_t \) over the time instead of a fixed Hilbert space \( \mathcal{H}_t \), with the appropriate cross sections as the waves \( \psi \), see the next section for details.

Consider now an action \( T_r \) of a group \( G \) in the space of waves \( \psi \). Before we infer some consequences of the assumption that \( G \) is a symmetry group we need to state a postulate:

**Classical-like postulate.** The group \( G \) is a symmetry group if and only if the wave equation is invariant under the transformation \( x' = rx, r \in G \) of independent variables and the transformation \( \psi' = T_r \psi \) of the wave function.

The above postulate is indeed commonly accepted in Quantum Mechanics even when the gauge freedom is not excluded. But it is a mere application

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\(^1\)This gives the conception of the ray, introduced to Quantum Mechanics by Hermann Weyl [H. Weyl, \textit{Gruppentheorie und Quantenmechanik}, Verlag von S. Hirzel in Leipzig (1928)]: a physical state does not correspond uniquely to a normed state \( \varphi \in \mathcal{H} \), but it is uniquely described by a ray, two states belong to the same ray if they differ by a constant phase factor.

\(^2\)J. v. Neumann, \textit{Mathematical Principles of Quantum Mechanics}, University Press, Princeton (1955). He did not mention about the gauge freedom on that occasion. But the gauge freedom is necessary for the equivalence of \( \psi_1 \) and \( \psi_2 = e^{i\xi(t)}\psi_1 \).
of the symmetry definition for a classical field equation applied to the wave equation without any change. The wave \( \psi \), is not any classical quantity, like the electromagnetic intensity. The above Hypothesis is not true for classical field and we have to be careful in forming the appropriate postulate for the wave equation compatible with the Hypothesis. Namely, the two wave equations differing by a mere gauge are indistinguishable. We call them gauge-equivalent. It is therefore natural to assume

**Quantum postulate.** The group \( G \) is a symmetry group if and only if the transformation \( x' = rx, r \in G \) of independent variables and the transformation \( \psi' = T_r \psi \) of the wave function transform the wave equation into a gauge-equivalent one.

Note, that not all possibilities admitted by the Hypothesis are included in **Classical-like postulate.**

From the Classical-like postulate it follows that \( \psi \) as well as \( T_r \psi \) are solutions to exactly the same wave equation, in view of the invariance of the equation. Therefore, \( \psi \) and \( T_r \psi \) belong to the same "Schrödinger picture", so that

\[
T_r T_s \psi = e^{i \xi(r,s)} T_{rs} \psi,
\]

with \( \xi = \xi(r,s) \) independent of the time \( t \). This is in accordance with the known theorem that

**Theorem 1** If \( G \) is a symmetry group, then the phase factor \( \xi \) should be time independent.

But if we start from the Quantum postulate we obtain instead

\[
T_r T_s \psi = e^{i \xi(r,s,t)} T_{rs} \psi
\]

and get the

**Theorem 1’** If \( G \) is a symmetry group, then the phase factor \( \xi = \xi(r,s,t) \) is time-dependent in general.

In this paper we propose to accept Quantum postulate, which is compatible with the Hypothesis, and is more in spirit of Quantum Mechanics than the Classical-like postulate. It should be noted that in the special case when the gauge freedom degenerates to the constant phase the Quantum postulate is equivalent to to the Classical-like postulate.

Acceptation of the Quantum postulate gives a new perspective for solving the two very difficult problems:

(a) generally covariant formulation of Quantum Mechanics,

(b) the troubles in the Quantum Field Theory caused by the gauge freedom.

Moreover, with the help of the Hypothesis we can see that both (a) and (b) are deeply connected. Consider the standard treatment in which the Hypothesis
is not taken into account and $H_0$ is assumed to be the Hilbert space of all states and **Classical-like postulate** is accepted. Then a troubles arise if we intend to formulate a representation theory of a covariance group in contradistinction to a symmetry group. The troubles have their source in the fact that the covariance group transforms solution of the wave equation into a solution of the transformed wave equation, but the transformed equation is different in form in comparison to the initial one. That is $T_r\psi$ does not belong to the same "Schrödinger picture" as $\psi$, and $T_r$ does not act in the Hilbert space $H_0$ of states. In view of the paradigm that any reasonable treating of action of any group in Quantum Mechanics reduces to a unitary representation of the group in the Hilbert space of states, there was no natural way for treating the covariance group. The difficulty disappear if we start from the **Hypothesis** and the **Quantum postulate**. Now the states are the appropriate cross sections in the bundle $\mathcal{R} \triangle \mathcal{H}$ and $T_r$ transforms unitarily fibre $H_t$ onto a fibre $H_{r-1}$ and acts in the same space $\mathcal{R} \triangle \mathcal{H}$ as the symmetry group. Note that the space of states degenerates to a fixed fibre $H_0$ over $t = 0$ if the gauge freedom degenerates to a constant phase.

Before we explain connection to the problem (b), we should resolve a paradox and then make a comment concerning the Quantum Field Theory. Namely, a natural question arises: why the phase factor $e^{i\xi}$ in (2) is time independent for the Galilean group (even when the Galilean group is considered as a covariance group)? The explanation of the paradox is as follows. The Galilean covariance group $G$ induces the representation $T_r$ in the space $\mathcal{R} \triangle \mathcal{H}$ and fulfills (2). But, as we will show later on, the structure of $G$ is such that there always exists a function $\xi(r, t)$ continuous in $r$ and differentiable in $t$ with the help of which one can define a new equivalent representation $T'_r = e^{i\xi(r, t)}T_r$ fulfilling

$$T'_rT'_s = e^{i\xi(r, s, p)}T'_{rs}$$

with a time independent $\xi$. The representations $T_r$ and $T'_r$ are equivalent because $T'_r\psi$ and $T_r\psi$ are equivalent for all $r$ and $\psi$. However this is not the case in general, when the exponent $\xi$ depends on the time and this time dependence cannot be eliminated in such a way as for the Galilean group. We have such a situation when we try to find the most general wave equation for a nonrelativistic quantum particle in the Newton-Cartan spacetime. The relevant covariance group in this case is the Milne group which possesses representations with time dependent $\xi$ not equivalent to any representations with a constant (in time) $\xi$. Moreover, the only physical representations of the Milne group are those with time dependent $\xi$.

We make a general comment concerning the relation (2). There is a physical motivation to investigate representations $T_r$ fulfilling (2) with $\xi$ depending on spacetime point $p$:

$$T_rT_s = e^{i\xi(r, s, p)}T_{rs}. \quad (3)$$

Namely, in the Quantum Field Theory the spacetime coordinates of $p \in \mathcal{M}$ play the role of parameters such as the time plays in the nonrelativistic theory (recall that, for example, the wave functions $\psi$ of the Fock space of the quantum
electromagnetic field are functions of the Fourier components of the field, the spacetime coordinates playing the role of parameters like the time $t$ in the nonrelativistic Quantum Mechanics). By this the two wave functions $\psi$ and $\psi' = e^{i\xi(p)}\psi$ are indistinguishable in the sense that they give the same transition probabilities: $|\langle\psi(p), \phi(p)\rangle| = |\langle\psi'(p), \phi(p)\rangle|$, for any $\phi$. In an analogous way we get the Hilbert bundle $\mathcal{M}\Delta\mathcal{H}$ over the spacetime $\mathcal{M}$ and appropriate cross sections as the wave functions $\psi$, see the next section for details.

Now, we return to the problem (b). It should be mentioned at this place that the troubles in QFT generated by the gauge freedom are of general character, and are well known. For example, there do not exist vector particles with helicity $= 1$, which is a consequence of the theory of unitary representations of the Poincaré group, as was shown by Jan Lopuszański [8]. This is apparently in contradiction with the existence of vector particles with helicity $= 1$ in nature – the photon, which is connected with the electromagnetic four-vector potential. The connection of the problem with the gauge freedom is well known [8]. We omit however the difficulty if we allow the inner product in the ”Hilbert space” to be not positively defined, see [6], or [2]. Due to [8], the vector potential (promoted to be an operator valued distribution in QED) cannot be a vector field, if one wants to have the inner product positively defined – together with the coordinate transformation the gauge transformation has to be applied, which breaks the vector character of the potential. Practically it means that any gauge condition which brings the theory into the canonical form such that the quantization procedure can be consequently applied (with the positively defined inner product in the Hilbert space) breaks the four-vector character of the electromagnetic potential, the Coulomb gauge condition is an example. To achieve the Poincaré symmetry of Maxwell equations with such a gauge condition (the Coulomb gauge condition for example), it is impossible to preserve the vector character of the potential – together with the coordinate transformation a well defined (by the coordinate transformation) gauge transformation $f$ has to be applied:

$$A_\mu \rightarrow A'_\mu = \frac{\partial x'^{\nu}}{\partial x^{\mu}}(A_\nu + \partial_\nu f).$$

This means that the electromagnetic potential can form a ray representation $T_r$ (in the sense of [3]) of the Poincaré group at most, with the spacetime-dependent factor $e^{i\xi}$ if the scalar product is positively defined. One may ask: how possible is it if the Poincaré group is not only a covariance group but at the same time a symmetry group? The solution of this paradox on the grounds of the existing theory is rather obscure. We propose the following solution. The Theorem 1 is true for the symmetry group but under the assumption that the fundamental space describing the states of a quantum system is the ordinary Hilbert space and the Classical-like postulate is true. But we have presented serious objections to this assumption. Namely, the nonrelativistic quantum theory can be reconstructed from the more general assumption about the space of quantum mechanical states saying that it compose the space of appropriate cross sections of the Hilbert bundle $\mathcal{R}\Delta\mathcal{H}$ over time $t \in \mathcal{R}$. The Schrödinger equation can be uniquely reconstructed from the generalized ray representations of the Galilean
group. We watch for also a more fundamental justification of this assumption in the presumption that the time is a purely classical variable in the nonrelativistic quantum mechanics or so to speak a parameter. The most general unitary representation of the locally compact commutative group of the time real line acts in a Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over the time, see Mackey [9]. So, the assumption about the "classicity" of the time $t$ fixes the structure of space of quantum states to be a subset of cross sections of a Hilbert bundle over the time. This is the peculiar property of the Galilean group structure that the whole construction degenerates as if we were started from the ordinary ray representation over the ordinary Hilbert space and the Theorem 1 is true in this case, but only accidentally. The generalization to the relativistic case is natural. First we postulate the spacetime coordinates to be classical commutative variables, which leads to the Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over the space-time manifold $\mathcal{M}$. The factor of the representation of the Poincaré group acting in the bundle $\mathcal{M}\triangle\mathcal{H}$ has not to be a constant with respect to space-time coordinates even when it is a symmetry group.

To realize the above program consequently we are forced to generalize the Bargmann’s theory of factors to embrace the spacetime-dependent factors of representations acting in a Hilbert bundle over the space-time (time).

### 3 Generalized Wave Rays and Operator Rays

In this section we give the strict mathematical definitions to the notions of the preceding section and formulate the problem stated there in the strict way. From the pure mathematical point of view the analysis of spacetime dependent $\xi(r,s,p)$ is more general, so we confine ourselves to this case at the outset, but we mark the place at which important difference arises between the two cases\footnote{It becomes clear in the further analysis that the group $G$ in question has to fulfill the consistency condition that for any $r \in G$, $rt$ is a function of time only in the case of the nonrelativistic theory with [2].}.

Let us remind some definitions, compare e.g. [9]. Let $\mathcal{M}$ be a set and $G$ be a group. A function $p, g \rightarrow pg$ from $\mathcal{M} \times G$ to $G$ will be said to convert the set $\mathcal{M}$ into a $G$-space if the following two conditions are satisfied

(a) $g_2(g_1p) = (g_2g_1)p$, for all $p \in \mathcal{M}$, $g_1, g_2 \in G$.

(b) $ep = p$, for all $p \in \mathcal{M}$, where $e$ is the identity of $G$.

We say that $G$ acts on the left. If we write $pg$ and assume $(pg_1)g_2 = p(g_1g_2)$ instead, we say that $G$ acts on the right. If the function $\mathcal{M} \times G \rightarrow G$ is smooth then we say that $G$ acts smoothly in $\mathcal{M}$.

By a Hilbert bundle over $\mathcal{M}$ or a Hilbert bundle with base $\mathcal{M}$ we shall mean an assignment $\mathcal{H} : p \rightarrow \mathcal{H}_p$ of a Hilbert space $\mathcal{H}_p$ to each $p \in \mathcal{M}$. The set of all pairs $(p, \psi)$ with $\psi \in \mathcal{H}_p$ will be denoted by $\mathcal{M}\triangle\mathcal{H}$ and called the space of the bundle. By a cross section of our bundle we shall mean an assignment
ψ : p → ψ_p of a member of \( \mathcal{H}_p \) to each \( p \in \mathcal{M} \). If \( \psi \) is a cross section and \((p_0, \phi_0)\) a point of \( \mathcal{M} \triangle \mathcal{H} \), we may form a scalar product \((\phi_0, \psi_{p_0})\). In this way every cross section \( \psi \) defines a complex-valued function \( f_\psi \) on \( \mathcal{M} \triangle \mathcal{H} \). By a Borel Hilbert bundle we shall mean a Hilbert bundle together with an analytic Borel structure in \( \mathcal{M} \triangle \mathcal{H} \) such that the following conditions are fulfilled

(1) Let \( \pi(p, \psi) = p \). Then \( E \subseteq \mathcal{M} \) is a Borel set if and only if \( \pi^{-1}(E) \) is a Borel set in \( \mathcal{M} \triangle \mathcal{H} \).

(2) There exist countably many cross sections \( \psi^1, \psi^2, \ldots \) such that

(a) the corresponding complex-valued functions on \( \mathcal{M} \triangle \mathcal{H} \) are Borel functions,

(b) these Borel functions separate points in the sense that no two distinct points \((p_i, \phi_i)\) of \( \mathcal{M} \triangle \mathcal{H} \) assign the same values to all \( \psi^j \) unless \( \phi_1 = \phi_2 = 0 \), and

(c) \( p \rightarrow (\psi^i(p), \psi^j(p)) \) is a Borel function for all \( i \) and \( j \).

A cross section is said to be a Borel cross section if the function on \( \mathcal{M} \triangle \mathcal{H} \) defined by the cross section is a Borel function. All Borel cross sections compose a linear space under the obvious operations, see \([9]\). Now let \( \mu \) be a measure on \( \mathcal{M} \). The cross section \( p \rightarrow \varphi_p \) is said to be square summable with respect to \( \mu \) if

\[
\int_{\mathcal{M}} (\varphi_p, \varphi_p) \, d\mu(p) < \infty.
\]

The space \( \mathcal{L}^2(\mathcal{M}, \mu, \mathcal{H}) \) of all equivalence classes of square summable cross sections, where two cross sections \( \varphi \) and \( \varphi' \) are in the same equivalence class if \( \varphi_p = \varphi'_p \) for almost all \( p \in \mathcal{M} \), forms a Hilbert space with the inner product given by

\[
(\varphi, \theta) = \int_{\mathcal{M}} (\varphi_p, \theta_p) \, d\mu(p),
\]

see \([9]\). It is called the direct integral of the \( \mathcal{H}_p \) with respect to \( \mu \) and is denoted by \( \int_{\mathcal{M}} \mathcal{H}_p \, d\mu(p) \).

The identification with the previous section is partially suggested by the notation. We make the identification more explicit. The set \( \mathcal{M} \) plays the role of the spacetime or the real line \( \mathcal{R} \) of the time \( t \) respectively. The unitary representation of the commutative group of coordinates in the spacetime \( \mathcal{M} \) acts precisely in the distinguished subset \( \mathcal{L}^2(\mathcal{M}, \mu, \mathcal{H}) \) of all Borel cross sections. We refer the reader to \([9]\) and literature therein for a detailed description of this representation. The wave functions \( \psi \) of the preceding section are the Borel cross sections of \( \mathcal{M} \triangle \mathcal{H} \) but they do not belong to the subset \( \mathcal{L}^2(\mathcal{M}, \mu, \mathcal{H}) \) of cross sections which are square integrable. Rather the separate Hilbert spaces \( \mathcal{H}_p \) with their inner products play a role in experiments than the inner product in the direct integral product of them. We have also used interchangeably \( \psi(p) \) and \( \psi_p \) as well as \((\psi_p, \theta_p)\) and \((\psi, \theta)_p\).
The physical interpretation ascribed to the cross section \( \psi \) is as follows. Each experiment is, out of its very nature, a spatiotemporal event. To each act of measurement carried out at the spacetime point \( p_0 \) we ascribe a self-adjoint operator \( Q_{p_0} \) acting in the Hilbert space \( H_{p_0} \) and ascribe to the spectral theorem for \( Q_{p_0} \) the standard interpretation. Hence, assuming for simplicity that \( Q_{p_0} \) is bounded, if \( \phi_0 \in H_{p_0} \) and \( \lambda_0 = \lambda_0(p_0) \) is a characteristic vector and its corresponding characteristic value of \( Q_{p_0} \) respectively then we have the following statement. If the experiment corresponding to \( Q_p \) was performed at the spatiotemporal event \( p_0 \) on a system in the state described by the cross section \( \psi \), then the probability of the measurement value to be \( \lambda_0(p_0) \) and the system to be found in the state described by \( \phi \) such that \( \phi(p_0) = \phi_0 \) after the experiment is given by the square of the absolute value of the Borel function \( |f_\psi(p_0, \phi_0)|^2 = |(\phi_0, \psi|_{p_0})|^2 \) induced by the cross section \( \psi \). In the nonrelativistic case the above statement is a mere rephrasing of the well established knowledge.

By an isomorphism of the Hilbert bundle \( \mathcal{M} \triangle \mathcal{H} \) with the Hilbert bundle \( \mathcal{M}' \triangle \mathcal{H}' \) we shall mean a Borel isomorphism \( T \) of \( \mathcal{M} \triangle \mathcal{H} \) on \( \mathcal{M}' \triangle \mathcal{H}' \) such that for each \( p \in \mathcal{M} \) the restriction of \( T \) to \( p \times \mathcal{H}_p \) has some \( q \times \mathcal{H}'_q \) for its range and is unitary when regarded as a map of \( \mathcal{H}_q \) on \( \mathcal{H}'_q \). The induced map carrying \( p \) into \( q \) is clearly a Borel isomorphism of \( \mathcal{M} \) with \( \mathcal{M}' \) and we denote it by \( T^\pi \). The above defined \( T \) is said to be an automorphism if \( \mathcal{M} \triangle \mathcal{H} = \mathcal{M}' \triangle \mathcal{H}' \). Note that for any automorphism \( T \) we have \( (T\psi, T\phi)|_{T^\pi p} = (\psi, \phi)_p \), but in general \( (T\psi, T\phi)_p \neq (\psi, \phi)_p \). By this any automorphism \( T \) is what is frequently called a bundle isometry.

The function \( r \to T_r \) from a group \( G \) into the set of automorphisms (bundle isometries) of \( \mathcal{M} \triangle \mathcal{H} \) is said to be a general factor representation of \( G \) associated to the action \( G \times \mathcal{M} \ni r, p \to r^{-1}p \in \mathcal{M} \) of \( G \) on \( \mathcal{M} \) if \( T^\pi_r(p) \equiv r^{-1}p \) for all \( r \in G \), and \( T_r \) satisfy the condition [1].

Of course \( T_r \) is to be identified with the one of the preceding section. Our further specializing assumptions partly following from the above interpretation are as follows. We assume \( \mathcal{M} \) to be endowed with the manifold structure inducing a topology associated with the above assumed Borel structure. We confine ourselves to a finite dimensional Lie group \( G \) which acts smoothly and transitively on the spacetime \( \mathcal{M} \), so, a \( G \)-invariant measure \( \mu \) exists on \( \mathcal{M} \).

By a factor representation of a Lie group we mean a general factor representation with the exponent \( \xi(r, s, p) \) differentiable in \( p \in \mathcal{M} \).

Now we define the operator ray \( T \) corresponding to a given bundle isometry operator \( T \) to be set of operators

\[ T = \{ \tau T, p \to \tau(p) \in \mathcal{D} \text{ and } |\tau| = 1 \}, \]

where \( \mathcal{D} \) denotes the set of all differentiable real functions on \( \mathcal{M} \). Any \( T \in T \) will be called a representative of the ray \( T \). The product \( TV \) is defined as the set of all products \( TV \) such that \( T \in T \) and \( V \in V \).

Note that not all Borel sections are physically realizable. Interpreting the discussion of the preceding section in the Hilbert bundle language we see that
the role of the Schrödinger equation is essentially to establish all the physical sections. Any two sections $\psi(p)$ and $\psi'(p) = e^{i\xi(p)}\psi(p)$ are indistinguishable giving the same probabilities $|f_\psi|^2 = |f_{\psi'}|^2$. After this any group $G$ acting in $\mathcal{M}$ induces a ray representation of $G$, i.e. a mapping $r \to T_r$ of $G$ into the space of rays of bundle automorphisms (bundle isometries) of $\mathcal{M} \triangle \mathcal{H}$, fulfilling the condition

$$T_r T_s = T_{rs}.$$ 

To any cross section $\psi$ we define the corresponding ray $\psi = \{e^{i\xi(p)}\psi(p), \xi \in \mathcal{D}\}$. if $\psi$ is a physical cross section then we get the physical ray of the preceding section. Selecting a representative $T_r$ for each $T_r$ we get a factor representation fulfilling (3). Note that $T_r$ transforms rays into rays, and we have $T_r(e^{i\xi(p)}\psi) = e^{i\xi'(r^{-1}p)}T_r\psi$. In the sequel we assume that the operators $T_r$ are such that $\xi'(r^{-1}p)$ denote the the action of $r^{-1} \in G$ on the spacetime point $p \in \mathcal{M}$. Note again, that this is a natural assumption which takes place in practice.

Now we make the last assumption, namely the assumption that all transition probabilities vary continuously with the continuous variation of the coordinate transformation $s \in G$:

For any element $r$ in $G$, any ray $\psi$ and any positive $\epsilon$ there exists a neighborhood $\mathcal{N}$ of $r$ on $G$ such that $d_p(T_s\psi, T_r\psi) < \epsilon$ if $s \in \mathcal{N}$ and $p \in \mathcal{M}$,

where

$$d_p(\psi_1, \psi_2) = \inf_{\psi_i \in \psi_i} \|\psi_1 - \psi_2\|_p = \sqrt{2|1 - |(\psi_1, \psi_2)_p||}.$$ 

Basing on the continuity assumption one can prove the following

**Theorem 2** Let $T_r$ be a continuous ray representation of a group $G$. For all $r$ in a suitably chosen neighborhood $\mathcal{N}_0$ of the unit element $e$ of $G$ one may select a strongly continuous set of representatives $T_r \in T_r$. That is, for any compact set $\mathcal{C} \subset \mathcal{M}$, any wave function $\psi$, any $r \in \mathcal{N}_0$ and any positive $\epsilon$ there exists a neighborhood $\mathcal{N}$ of $r$ such that $\|T_s\psi - T_r\psi\|_p < \epsilon$ if $s \in \mathcal{N}$ and $p \in \mathcal{C}$.

There are many possible selections of such factor representations. But many among them differ by a mere differentiable phase factor and are physically indistinguishable. We call them to be equivalent. Our task is to classify all possible factor representations with respect to this equivalence.

### 4 Local Exponents

The representatives $T_r \in T_r$ selected as in the Theorem 2 will be called admissible and the representation $T_r$ obtained in this way the admissible representation. There are infinitely many possibilities of such a selection of admissible representation $T_r$. We confine ourselves to the local admissible representations defined on a fixed neighborhood $\mathcal{N}_0$ of $e \in G$, as in the Theorem 2.
Let \( T_r \) be an admissible representation. With the help of the phase \( e^{i\zeta(r,p)} \) with a real function \( \zeta(r,p) \) differentiable in \( p \) and continuous in \( r \) we can define

\[
T'_r = e^{i\zeta(r,p)}T_r,
\]

which is a new admissible representation. This is trivial, if one defines in the appropriate way the continuity of \( \zeta(r,p) \) in \( r \). Namely, from the Theorem 2 it follows that the continuity has to be defined in the following way. The function \( \zeta(r,p) \) will be called strongly continuous in \( r \) at \( r_0 \) if and only if for any compact set \( C \subset M \) and any positive \( \epsilon \) there exist a neighborhood \( \mathcal{N}_0 \) of \( r_0 \) such that

\[
|\zeta(r_0,p) - \zeta(r,p)| < \epsilon,
\]

for all \( r \in \mathcal{N}_0 \) and for all \( p \in C \). But the converse is also true. Indeed, if \( T'_r \) also is an admissible representation, then (4) has to be fulfilled for a real function \( \zeta(r,p) \) differentiable in \( p \) because \( T'_r \) and \( T_r \) belong to the same ray, and moreover, because both \( T'_r\psi \) and \( T_r\psi \) are strongly continuous (in \( r \) for any \( \psi \)) then \( \zeta(r,p) \) has to be strongly continuous (in \( r \)).

Let \( T_r \) be an admissible representation, and by this continuous in the sense indicated in the Theorem 2. One can always choose the above \( \zeta \) in such a way that \( T_e = 1 \) as will be assumed in the sequel.

Because \( T_rT_s \) and \( T_{rs} \) belong to the same ray one has

\[
T_rT_s = e^{i\xi(r,s,p)}T_{rs},
\]

with a real function \( \xi(r,s,p) \) differentiable in \( p \). From the fact that \( T_e = 1 \) we have

\[
\xi(e,e,p) = 0.
\]

From the associative law \( (T_rT_s)T_g = T_r(T_sT_g) \) one gets

\[
\xi(r,s,p) + \xi(rs,g,p) = \xi(s,g,r^{-1}p) + \xi(r,sg,p).
\]

The formula (7) is very important and our analysis largely rests on this relation. From the fact that the representation \( T_r \) is admissible follows that the exponent \( \xi(r,s,p) \) is continuous in \( r \) and \( s \). Indeed, take a \( \psi \) belonging to a unit ray \( \psi \), then making use of (4) we get

\[
e^{i\xi(r,s,p)}(T_{rs} - T_{rs'})\psi + (T_{r'}(T_{s'} - T_s)\psi + (T_{r'} - T_r)T_s\psi
\]

\[
= (e^{i\xi(r',s',p)} - e^{i\xi(r,s,p)})T_{rs'}\psi.
\]

Taking norms \( \| \cdot \|_p \) of both sides, we get

\[
|e^{i\xi(r',s',p)} - e^{i\xi(r,s,p)}| \leq \|T_{rs'} - T_{rs}\|_p + \|T_{r'}(T_{s'} - T_s)\psi\|_p + \|(T_{r'} - T_r)T_s\psi\|_p.
\]
From this inequality and the continuity of \( T_r \psi \), the continuity of \( \xi(r, s, p) \) in \( r \) and \( s \) follows. Moreover, from the Theorem 2 and the above inequality the strong continuity of \( \xi(r, s, p) \) in \( r \) and \( s \) follows.

The formula (10) suggests the following definition. Two admissible representations \( T_r \) and \( T'_r \) are called equivalent if and only if \( T'_r = e^{i \zeta(r, p) T_r} \) for some real function \( \zeta(r, p) \) differentiable in \( p \) and strongly continuous in \( r \). So, making use of (10) we get \( T'_r T_s = e^{i \xi(r, s, p) T_{rs}} \), where

\[
\xi'(r, s, p) = \xi(r, s, p) + \zeta(r, p) + \zeta(s, r^{-1}p) - \zeta(rs, p).
\]

Then the two exponents \( \xi \) and \( \xi' \) are equivalent if and only if (10) is fulfilled with \( \zeta(r, p) \) strongly continuous in \( r \) and differentiable in \( p \).

From (10) and (7) immediately follows that

\[
\xi(r, e, p) = 0 \quad \text{and} \quad \xi(e, g, p) = 0,
\]

which is valid for all \( r, s, p \). Moreover, if \( \xi' = \xi + \Delta[\zeta] \) then \( \xi' = \xi + \Delta[\zeta'] \).

The relation (10) between exponents \( \xi \) and \( \xi' \) is reflexive, symmetric and transitive. Indeed, we have: \( \xi = \xi + \Delta[\zeta] \) with \( \zeta = 0 \). Moreover, if \( \xi'' = \xi + \Delta[\zeta] \) then \( \xi'' = \xi + \Delta[\zeta'] \).

The multiplication rule, suggested by the above relation, and will be sometimes denoted by \( \xi' \equiv \xi \). The equivalence relation preserves the linear structure, that is if \( \xi_i \equiv \xi'_i \) (with the appropriate \( \zeta_i \)-s) then \( \lambda_1 \xi_1 + \lambda_2 \xi_2 \equiv \lambda_1 \xi'_1 + \lambda_2 \xi'_2 \) (with \( \zeta = \lambda_1 \zeta_1 + \lambda_2 \zeta_2 \)).

We introduce now the group \( H \), the very important notion for the further investigations. It is evident that all operators \( T_r \) contained in all rays \( T_{r, p} \) form a group under multiplication. Indeed, consider an admissible representation \( T_r \) with a well defined \( \xi(r, s, p) \) in the formula (10). Because any \( T_r \in T_{r, p} \) has the form \( e^{i\theta(p)} T_r \) (with a real and differentiable \( \theta \)), one has

\[
\left( e^{i\theta(p)} T_r \right) \left( e^{i\theta'(p)} T_s \right) = e^{i(\theta(p) + \theta'(r^{-1}p) + \xi(r, s, p)) T_{rs}}.
\]

This important relation suggest the following definition of the local group \( H \) connected with the admissible representation or with the exponent \( \xi(r, s, p) \). Namely, \( H \) consists of the pairs \( \{ \theta(p), r \} \) where \( \theta(p) \) is a differentiable real function and \( r \in G \). The multiplication rule, suggested by the above relation, is defined as follows

\[
\{ \theta(p), r \} \cdot \{ \theta'(p), r' \} = \{ \theta(p) + \theta'(r^{-1}p) + \xi(r, r', p), rr' \}.
\]

The associative law for this multiplication rule is equivalent to (10) (in a complete analogy with the classical Bargmann’s theory). The pair \( \hat{e} = \{ 0, e \} \) plays the
role of the unit element in $H$. For any element $\{\theta(p), r\} \in H$ there exists the inverse $\{\theta(p), r\}^{-1} = \{-\theta(rp) - \xi(r, r^{-1}, rp), r^{-1}\}$. Indeed, from (10) it follows that $\{\theta, r\}^{-1} \circ \{\theta, r\} = \{\theta, r\} \circ \{\theta, r\}^{-1} = e$. The elements $\{\theta(p), e\}$ form an abelian subgroup $N$ of $H$. Any $\{\theta, r\} \in H$ can be uniquely written as $\{\theta(p), r\} = \{\theta(p), e\} \circ \{0, r\}$. Also the same element can be uniquely expressed in the form $\{\theta(p), r\} = \{0, r\} \circ \{\theta(rp), e\}$. So, we have $H = N \ast G = G \ast T$. The abelian subgroup $N$ is a normal factor subgroup of $H$. But this time $G$ does not form any normal factor subgroup of $H$ (contrary to the classical case investigated by Bargmann, when the exponents do not depend on $p$). So, this time $H$ is not direct product $N \otimes G$ of $N$ and $G$, but it is a semidirect product $N \text{S}G$ of $N$ and $G$, see e.g. (10) where the semi-direct product of two continuous groups is investigated in detail. In this case however the theorem that $G$ is locally isomorphic to the factor group $H/N$ is still valid, see (10). Then the group $H$ composes a semicentral extension of $G$ and not a central extension of $G$ as in the Bargmann’s theory.

The rest of this paper is based on the following reasoning (the author was largely inspired by the Bargmann’s work (11)). If the two exponents $\xi$ and $\xi'$ are equivalent, that is $\xi' = \xi + \Delta [\xi]$, then the semicentral extensions $H$ and $H'$ connected with $\xi$ and $\xi'$ are homomorphic. The homomorphism $h : \{\theta, r\} \mapsto \{\theta', r'\}$ is given by

$$\theta'(p) = \theta(p) - \zeta(r, p), \ r' = r. \quad (14)$$

Using an Iwasawa-type construction we show that any exponent $\xi(r, s, p)$ is equivalent to a differentiable one (in $r$ and $s$). We can confine ourselves then to the differentiable $\xi$ and $\xi'$. We show that $\zeta(r, p)$ is also differentiable function of $(r, p)$. Moreover, we show that any $\xi$ is equivalent to the canonical one, that is such $\xi$ which is differentiable and for which $\xi(r, s, p) = 0$ whenever $r$ and $s$ belong to the same one-parameter subgroup. Then we can restrict the investigation to the canonical $\xi$ and we consider the subgroup of all elements $\{\theta(p), r\} \in H$ with differentiable $\theta(p)$. Let us denote the subgroup by the same symbol $H$ for simplicity. We embed the subgroup in an infinite dimensional Lie group $D$ with manifold structure modeled on a Banach space. Then we consider the subgroup $\overline{H}$ which is a closure of $H$ in $D$. After this $\overline{H}$ turns into a Lie group and the homomorphism (14) becomes to be an isomorphism of the two Lie groups. So, the group $\overline{H}$ has the Banach Lie algebra $\mathfrak{H}$. We apply the general theory of analytic groups developed by (31) and (32). From this theory follows that the correspondence between the local $\mathfrak{H}$ and $\mathfrak{S}$ is bi-unique and one can construct uniquely the local group $\overline{\mathfrak{H}}$ from the algebra $\overline{\mathfrak{S}}$ also. As we will see the algebra defines a spacetime dependent antilinear form $\Xi$ on the Lie algebra $\mathfrak{G}$ of $G$, the so called infinitesimal exponent $\Xi$. By this we reduce the classification of local $\xi$’s which define $\overline{\mathfrak{H}}$’s to the classification of $\Xi$’s which define $\overline{\mathfrak{S}}$’s. So, we will simplify the problem of the classification of local $\xi$’s to a largely linear problem.
5 Local Exponents of Lie Groups

Iwasawa construction. Denote by $d^r$ and $d^*r$ the left and right invariant Haar measure on $G$. Let $\nu(r)$ and $\nu^*(r)$ be two infinitely differentiable functions on $G$ with compact supports contained in the fixed neighborhood $\mathcal{U}_0$ of $e$. Multiplying them by the appropriate constants we can always reach:

$$\int_G \nu(r) \, dr = \int_G \nu^*(r) \, d^*r = 1.$$ Let $\xi(r, s, p)$ be any admissible local exponent defined on $\mathcal{U}_0$. We will construct a differentiable (in $r$ and $s$) exponent $\xi''(r, s, p)$ which is equivalent to $\xi(r, s, p)$ and is defined on $\mathcal{U}_0$, in the following two steps:

$$\xi' = \xi + \Delta[\zeta], \quad \text{with} \quad \zeta(r, p) = -\int_G \xi(r, l, p)\nu(l) \, dl,$$

$$\xi'' = \xi' + \Delta[\zeta'], \quad \text{with} \quad \zeta'(r, p) = -\int_G \xi'(u, r, up)\nu^*(u) \, d^*u.$$

A rather simple computation in which we use (8) and (7) and the invariance property of the Haar measures gives:

$$\xi''(r, s, p) = \int \int_G \xi(u, l, ur^{-1}p)\{\nu(s^{-1}l) - \nu(l)\}\{\nu^*(ur^{-1}) - \nu^*(u)\} \, dl \, d^*u.$$

Because $\nu$ and $\nu^*$ are differentiable (up to any order) and $\xi(r, s, p)$ is a differentiable function of $p \in \mathcal{M}$ (up to any order) then $\xi''(r, s, p)$ is a differentiable (up to any order) exponent in all variables $(r, s, p)$.

**Lemma 1** If two differentiable exponents $\xi$ and $\xi'$ are equivalent, that is, if $\xi' = \xi + \Delta[\zeta]$, then $\zeta(r, p)$ is differentiable in $r$.

**Proof.** Clearly, the function $\chi(r, s, p) = \xi'(r, s, p) - \xi(r, s, p)$ is differentiable. Similarly the function $\eta(r, p) = \int_G \chi(r, u, p)\nu(u) \, du$, where $\nu$ is defined as in the Iwasawa construction, is a differentiable function. But the difference $\zeta' = \eta - \zeta$ is equal

$$\zeta'(r, p) = \int_G \{\zeta(u, r^{-1}p) - \zeta(ru, p)\}\nu(u) \, du =$$

$$= \int_G \{\zeta(u, r^{-1}p)\nu(u) - \zeta(u, p)\nu(r^{-1}u)\} \, du$$

and clearly it is a differentiable function. By this $\zeta = \eta - \zeta'$ also is a differentiable function (recall that $\zeta(r, p)$ is differentiable function of $p \in \mathcal{M}$).

**Lemma 2** Every (local) exponent of one-parameter group is equivalent to zero.

**Proof.** We can map such a group $r = r(\tau) \mapsto \tau$ on the real line ($\tau \in \mathcal{R}$) in such a way that $r(\tau)r'(\tau') = r(\tau + \tau')$. Set

$$\vartheta(\tau, \sigma, p) = \frac{\partial \xi(\tau, \sigma, p)}{\partial \sigma}.$$
From (6), (9) and (7) one gets
\[ \xi(0, 0, p) = 0, \quad \xi(\tau, 0, p) = 0, \]  
\[ (15) \]
\[ \xi(\tau, \tau', p) + \xi(\tau + \tau', p) = \xi(\tau', \tau'', r(-\tau)p) + \xi(\tau, \tau' + \tau'', p). \]  
\[ (16) \]
Now we derive the expression with respect to \( \tau'' \) at \( \tau'' = 0 \). This yields
\[ \vartheta(\tau + \tau', 0, p) = \vartheta(\tau', 0, r(-\tau)p) + \vartheta(\tau, \tau'). \]  
\[ (17) \]
Let us define now
\[ \zeta(\tau, p) = \int_{\tau}^{0} \vartheta(\sigma, 0, p) \, d\sigma = \int_{1}^{0} \tau \vartheta(\mu \tau, 0, p) \, d\mu. \]
We have then
\[ -\Delta[\zeta] = \zeta(\tau + \tau', p) - \zeta(\tau, p) - \zeta(\tau', r(-\tau)p) = \]
\[ = \int_{0}^{\tau'} \{ \vartheta(\tau + \sigma, 0, p) - \vartheta(\sigma, 0, r(-\tau)p) \} \, d\sigma. \]
Using now the Eq. (17) and (15) we get
\[ -\Delta[\zeta] = \int_{0}^{\tau'} \vartheta(\tau, \sigma, p) \, d\sigma = \int_{0}^{\tau'} \frac{\partial \xi(\tau, \sigma, p)}{\partial \sigma} \, d\sigma = \xi(\tau, \tau', p) \]
and \( \xi \) is equivalent to 0.

Let us recall that the continuous curve \( r(\tau) \) in a Lie group \( G \) is a one-parameter subgroup if and only if \( r(\tau_1) r(\tau_2) = r(\tau_1 + \tau_2) \) i.e. \( r(\tau) = (r_0)^\tau \), for some element \( r_0 \in G \), note that the real power \( r^\tau \) is well defined on a Lie group (at least on some neighborhood of \( e \)). The coordinates \( \rho^k \) in \( G \) are canonical if and only if any curve of the form \( r(\tau) = \tau \rho^k \) (where the coordinates \( \rho^k \) are fixed) is a one-parameter subgroup (the curve \( r(\tau) = \tau \rho^k \) will be denoted in short by \( \tau a \), with the coordinates of \( a \) equal to \( \rho^k \)). The "vector" \( a \) is called by physicists the generator of the one-parameter subgroup \( \tau a \).

A local exponent \( \xi \) of a Lie group \( G \) is called canonical if \( \xi(r, s, p) \) is differentiable in all variables and \( \xi(r, s, p) = 0 \) if \( r \) and \( s \) are elements of the same one-parameter subgroup.

**Lemma 3** Every local exponent \( \xi \) of a Lie group is equivalent to a canonical local exponent.

**Proof.** Set \( \rho^j \) and \( \sigma^i \) for the canonical coordinates of the two elements \( r, s \in G \) respectively, and define
\[ \vartheta_k = \frac{\partial \xi(r, s, p)}{\partial \sigma^k}. \]
Let us define now
\[
\zeta(r, p) = \int_0^1 \sum_{k=1}^n \rho^k \partial_k (\mu r, 0, p) \, d\mu.
\]

Consider a one-parameter subgroup \( r(\tau) \) generated by \( a \), i.e. \( r(\tau) = \tau a \). Because \( \xi \) is a local exponent fulfilling (6), (7) and (9) then \( \xi_0(\tau, \tau', p) \equiv \xi(\tau a, \tau'a, p) \) fulfills (15) and (16). Repeating now the same steps as in the proof of Lemma 2 one can show that
\[
\xi(\tau a, \tau'a, p) + \Delta[\zeta(\tau a, p)] = 0.
\]

**Lemma 4** Let \( \xi \) and \( \xi' \) be two differentiable and equivalent local exponents of a Lie group \( G \), and assume \( \xi \) to be canonical. Then \( \xi' \) is canonical if and only if \( \xi' = \xi + \Delta[\Lambda] \), where \( \Lambda(r, p) \) is a linear form in the canonical coordinates of \( r \) fulfilling the condition that \( \Lambda(a, (\tau a)p) \) is constant as a function of \( \tau \), i.e. it follows that
\[
\frac{d\Lambda(a, (\tau a)p)}{d\tau} = \lim_{\epsilon \to 0} \frac{\Lambda(a, (\epsilon a)p) - \Lambda(a, p)}{\epsilon} = 0. \tag{18}
\]

The limit in the above expression can be understood in the ordinary pointwise sense with respect to \( p \). But after this the assertion of the Lemma is much stronger than (18). We will use later the fact that (18) is true in any linear topology in the function linear space (with the obvious addition) of \( \theta(p) \) providing that \( p \to \Lambda(a, p) \) is differentiable in the sense of this linear topology.

In the sequel we will use the simple notation
\[
a f(p) = \frac{df((\tau a)p)}{d\tau} \bigg|_{\tau = 0} = \lim_{\epsilon \to 0} \frac{f((\epsilon a)p) - f(p)}{\epsilon},
\]
and \( a f(p) = 0 \) means that \( f(p) \) is constant along the integral curves \( p(\tau) = (\tau a)p_0 \). After this from the condition of Lemma 6 follows that
\[
a\Lambda(a, p) = 0.
\]

**Proof of Lemma 4. 1°.** Necessity of the condition. Because the exponents are equivalent we have \( \xi'(r, s, p) = \xi(r, s, p) + \Delta[\zeta] \). Because both \( \xi \) and \( \xi' \) are differentiable then \( \zeta(r, p) \) also is a differentiable function, which follows from Lemma 1. Suppose that \( r = \tau a \) and \( s = \tau'a \). Because of both \( \xi \) and \( \xi' \) are canonical we have \( \xi(\tau a, \tau'a, p) = \xi'(\tau a, \tau'a, p) = 0 \), such that \( \Delta[\zeta](\tau a, \tau'a, p) = 0 \), i.e.
\[
\zeta((\tau + \tau')a, p) = \zeta(\tau a, p) + \zeta(\tau'a, (-\tau a)p).
\]

Applying recurrently this formula one gets
\[
\zeta(\tau a, p) = \sum_{k=0}^{n-1} \zeta(\frac{\tau}{n} a, (-\frac{k}{n} \tau a)p). \tag{19}
\]
\( \zeta \) is differentiable (up to any order) and we can use the Taylor Theorem. Because in addition \( \zeta(0, p) = 0 \) we get the following formula

\[
\zeta\left(\frac{\tau}{n}, a, p\right) = \left(1 - \frac{\tau}{n}\zeta'(0, p)\right) + \frac{1}{2} \zeta''\left(\theta_2 \frac{\tau}{n}, a, p\right) \left(\frac{\tau}{n}\right)^2,
\]

where \( \zeta' \) and \( \zeta'' \) stand for the first and the second derivative of \( \zeta(x, a, p) \) with respect to \( x \), and \( 0 \leq \theta_2 \leq 1 \). Recall that in the Taylor formula

\[
f(x + h) = f(x) + f'(x)h + \frac{1}{2} f''(x + \theta h)h^2
\]

the \( \theta_2 \) depends on \( h \), which is marked by the subscript \( h \): \( \theta_2 \). Inserting \( \tau = n - 1 \) to the formula and multiplying it by \( \frac{\tau}{n} \) (provided the coordinates \( a \) of an element \( r_0 \) of \( G \) are chosen in such a way that \( r_0 \) lies in the neighborhood \( \mathcal{N}_0 \) on which the exponents \( \xi \) and \( \zeta' \) are defined) one gets

\[
\frac{\tau}{n} \zeta(a, p) = \frac{\tau}{n} \left(\zeta'(0, p) + \frac{1}{2} \zeta''(\theta_1 a, p)\right).
\]

Taking now the difference of the last two formulas we get

\[
\zeta(\tau n, a, p) = \frac{\tau}{n} \left(\zeta(a, p) - \frac{1}{2} \zeta''(\theta_1 a, p)\right).
\]

Inserting this to the formula (19) we get

\[
\zeta(\tau a, p) = \frac{\tau}{n} \sum_{k=0}^{n-1} \left\{ \zeta(a, (-k\frac{\tau}{n} a, p) - \frac{1}{2} \zeta''(\theta_1 a, (-k\frac{\tau}{n} a, p)\right\}
\]

+ \frac{1}{2} \left(\frac{\tau}{n}\right)^2 \sum_{k=0}^{n-1} \zeta''\left(\theta_2 \frac{\tau}{n}, a, (-k\frac{\tau}{n} a, p)\right).
\]

Denote the supremum and the infimum of the function \( \zeta''(x, (-ya)p) \) in the square \((0 \leq x \leq \tau, 0 \leq y \leq \tau)\) by \( M \) and \( N \) respectively. We have

\[
\frac{1}{2} \left(\frac{\tau}{n}\right)^2 nN + \frac{\tau}{n} \sum_{k=0}^{n-1} \left\{ \zeta(a, (-k\frac{\tau}{n} a, p) - \frac{1}{2} \zeta''(\theta_1 a, (-k\frac{\tau}{n} a, p)\right\}
\]

\[
\leq \zeta(\tau a, p) \leq \frac{1}{2} \left(\frac{\tau}{n}\right)^2 nM + \frac{\tau}{n} \sum_{k=0}^{n-1} \left\{ \zeta(a, (-k\frac{\tau}{n} a, p) - \frac{1}{2} \zeta''(\theta_1 a, (-k\frac{\tau}{n} a, p)\right\}.\]

Passing to the limit \( n \to +\infty \) we get

\[
\zeta(\tau a, p) = \int_{0}^{\tau} \left\{ \zeta(a, (-\sigma a, p) - \frac{1}{2} \zeta''(\theta_1 a, (-\sigma a, p)\right\} d\sigma.
\]
Taking into account that the functions $\zeta'$ and $\zeta''$ are independent the general solution $\zeta$ fulfilling $\Delta[\zeta](\tau a, \tau' a, p) = 0$ for any $\tau$, $\tau'$, $a$ and any $p \in \mathcal{M}$, can be written in the following form

$$\zeta(\tau a, p) = \int_0^\tau \zeta(a, (-\sigma a)p) \, d\sigma,$$  \hspace{1cm} (20)

where $\varsigma = \varsigma(r, p)$ is any differentiable function. Differentiate now the expression (20) with respect to $\tau$ at $\tau = 0$. After this one gets

$$\zeta(a, p) = \sum_{k=1}^n \lambda_k(p)a^k, \text{ with } \lambda_k(p) = \frac{\partial \zeta(0, p)}{\partial a^k},$$  \hspace{1cm} (21)

where $a^k$ stands for the coordinates of $a$. So, the function $\zeta(a, p)$ is linear with respect to $a$. Suppose that the spacetime coordinates $X$ are chosen in such a way that the integral curves $p(x) = (xa)p_0$ are coordinate lines, which is possible for appropriately small $\tau$. There are of course three remaining families of coordinate lines beside $p(x)$, which can be chosen in arbitrary way, the parameters of which will be denoted by $y_i$. After this, Eq. (20) reads

$$\zeta(\tau a, x, y_i) = \int_0^\tau \zeta(a, x - \sigma, y_i) \, d\sigma.$$

So, because $\zeta(a, p)$ is linear with respect to $a$, then for appropriately small $a$ one gets

$$\zeta(a, x, y_i) = \frac{1}{\tau} \int_0^\tau \zeta(a, x - \sigma, y_i) \, d\sigma =$$

$$= \frac{1}{\tau} \int_{x-\tau}^x \zeta(a, z, y_i) \, dz,$$

for any $\tau$ (of course with appropriately small $|\tau|$, in our case $|\tau| \leq 1$) and for any (appropriately small) $x$. But this is possible for the function $\zeta(a, x, y_k)$ continuous in $x$ (in our case differentiable in $x$) if and only if $\zeta(a, x, y_k)$ does not depend on $x$. This means that $\zeta(a, x, y_k)$ does not depend on $x$ and the condition of Lemma 6 is proved.

2°. Sufficiency of the condition is trivial.

6 The Lie Algebras

According to Lemma 3 we can assume that the exponent is canonical. Also we confine ourselves to the subgroup of $\{\theta(p), r\} \in H$ with differentiable $\theta$, and denote this subgroup by the same letter $H$. We embed this subgroup $H$ in an infinite dimensional Lie group with the manifold structure modeled on a Banach space. We will extensively use the theory developed by Birkhoff [3] and Dynkin [4]. For the systematic treatment of manifolds modeled on Banach spaces, see
e.g. [7]. By this embedding we ascribe bi-uniquely a Lie algebra to the group $H$ with the convergent Baker-Hausdorff series.

Note first that the formula

$$H \times L^2(\mathcal{M}, \mu, \mathcal{H}) \ni (\{\theta(p), r\}, \phi) \rightarrow e^{i\theta(p)}T_r\phi$$

together with (12) can be viewed as rule giving the action of $H$ in the direct integral Hilbert space $\int_{\mathcal{M}} H_p \, d\mu(p)$ defined in the 3rd section. Moreover, this is a unitary action, provided $\mu$ is $G$-invariant. In accordance to [3] the group $D$ of all unitary operators of a Hilbert space is an infinite dimensional Lie group. By this $H = N \otimes G$ can be viewed as a subgroup of a Lie group.

We consider now the closure $\overline{H}$ of $H$ in the sense of the topology in $D$.

**Lemma 5** The subgroup $\overline{H}$ also has locally the structure of the semi-direct product $N \otimes G$.

**Proof.** It is the consequence of the following four facts.

1. $N$ is a normal subgroup of $\overline{H} = N \otimes G$.
2. $G$ is finite dimensional, so, $G = G$.
3. Locally (in a neighborhood $O$) the multiplication in $D$ is given by the Baker-Hausdorff formula in the Banach algebra of $D$. Because $N$ is normal in $\overline{H}$, then the above mapping converts locally the multiplication $N \times S$ of $N$ by any subset $S$ of $\overline{H}$ into the sum $N + S$. Because $G$ is finite dimensional, and by this is locally compact, the neighborhood $O$ can be chosen in such a way that locally (in the closure of $O + O$)

$$N + G = N + \overline{G} = \overline{H}.$$  

4. The local $N$ (intersected with $O$) has finite co-dimension in local $N + \overline{G}$ (intersected with $O + O$) and by this it splits locally $N + \overline{G}$. So, we have locally, i.e. in $O + O$,

$$N + \overline{G} = \overline{H} = N \oplus G',$$

where $G' = G$ and $\oplus$ stands for direct sum. From this it follows that $G' = \overline{G}$ locally.

Because $\overline{H} = N \otimes G$ every $h \in \overline{H}$ is uniquely representable in the form $ng$, where $n \in N$ and $g \in G$. Notice now that

$$(n_1g_1)(n_2g_2) = n_1g_1n_2g_1^{-1}g_1g_2 = [n_1(g_1n_2g_1^{-1})](g_1g_2)$$

and that $g_1n_2g_1^{-1} \in N$ because $N$ is normal in $\overline{H}$. Let us denote the automorphism $n \rightarrow ghg^{-1}$ of $N$ by $R_g$. The group $\overline{H}$ can be locally viewed as a topological product of Banach spaces $\overline{N} \times \Theta$ one of which, namely $\Theta$ is finite dimensional and isomorphic to the Lie algebra of $G$. The multiplication in $\overline{H}$ can be written as $(n_1g_1)(n_2g_2) = (n_1R_{g_1}(n_2), g_1g_2)$. Moreover, $\overline{N}$ can be viewed locally as the Banach space $\overline{N}$ with the multiplication law given by the vector addition in $\overline{N}$.
Now, our task is to reconstruct the Lie algebra \( \mathfrak{h} \) corresponding to the subgroup \( H \).

Let \( \lambda \rightarrow \lambda a \) be a one-parameter subgroup of \( G \). The mapping

\[
(\lambda, n) \rightarrow (R_{\lambda a}n, \lambda a)
\]

of the Banach space \( R \times \mathfrak{m} \) into the Banach space \( \mathfrak{m} \times \mathfrak{g} \) is continuous. Indeed. The multiplication law in \( \mathfrak{g} \) is continuous. On the other hand multiplying \((0, \lambda a)\) and \((n, 0)\) we get \((R_{\lambda a}n, \lambda a)\) from which the continuity of the above mapping follows. By this \( R \ni \lambda \rightarrow R_{\lambda a}n \in \mathfrak{m} \) as well as \( \mathfrak{m} \ni n \rightarrow R_{\lambda a}n \) are continuous. By this the function \( \lambda \rightarrow R_{\lambda a}n \) can be integrated over any compact interval and we have

\[
R_{\lambda a} \int_0^\tau R_{\sigma a}n \, d\sigma = \int_0^\tau R_{\lambda a} \circ R_{\sigma a}n \, d\sigma = \int_0^\tau R_{(\lambda a)(\sigma a)}n \, d\sigma = \int_0^\tau R_{(\lambda + \sigma)a}n \, d\sigma.
\]

If one takes it into account, then a straightforward computation shows that

\[
\tau \rightarrow (n_{\tau a}, \tau a) := \left( \int_0^\tau R_{\sigma a}n \, d\sigma, \tau a \right)
\]

is a one-parameter subgroup of \( \mathfrak{h} \). It is not hard to show that to any element \( h \) of \( \mathfrak{h} \) we can construct in this way a subgroup passing through \( h \). Moreover, the element \( \hat{a} \) of the algebra \( \mathfrak{h} \) corresponding to the one-parameter subgroup is equal (compare [3], [4])

\[
\lim_{\tau \to 0} \frac{(n_{\tau a}, \tau a)}{\tau} = \lim_{\tau \to 0} \frac{\left( \int_0^\tau R_{\sigma a}n \, d\sigma, \tau a \right)}{\tau} = (n, a),
\]

because for any Banach-valued continuous function \( \mathcal{R} \ni \sigma \rightarrow F(\sigma) \),

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau F(\sigma) \, d\sigma = F(0)
\]

in the sense of limit induced by the norm in the Banach space.

The Lie bracket \([\hat{a}, \hat{b}]\) in \( \mathfrak{h} \) is uniquely determined by the one-parameter subgroups \( \tau \hat{a} = \{\alpha_{\tau a}, \tau a\} \) and \( \tau \hat{b} = \{\beta_{\tau b}, \tau b\} \) corresponding to \( \hat{a} \) and \( \hat{b} \) respectively (compare [3], [4])

\[
[\hat{a}, \hat{b}] = \lim_{\tau \to 0} \frac{((\tau a)(\tau b))(\tau a)^{-1}(\tau b)^{-1}}{\tau^2},
\]

where the limit is in the sense of the topology induced from the Lie group \( D \), of course.

The elements of \( H \subset \mathfrak{h} \) are representable in the ordinary form \( \{\alpha, r\} \) with differentiable \( \alpha = \alpha(p), p \in \mathcal{M} \), and \( r \in G \). Let \( \lambda \rightarrow \lambda a \) be a one-parameter subgroup of \( G \). Consider the above defined operator \( R_{\lambda a} \). Its restriction to \( H \subset \mathfrak{h} \) is given by (remember that \( \xi \) is canonical)

\[
\alpha(p) \rightarrow (R_{\lambda a}\alpha)(p) = \alpha((\lambda a)^{-1}p).
\]
We compute now explicitly the Lie bracket and the Jacobi identity for all the elements \((\alpha(p), a)\) of the subalgebra \(\mathfrak{H} \subset \mathfrak{F}\) corresponding to the subgroup \(H\). A rather straightforward computations, in which the continuity of \((\sigma a, \alpha) \rightarrow R_{\sigma a} \alpha\) as well as the homomorphism property \(R_{\tau a} R_{\lambda b} = R_{(\tau a)(\lambda b)}\) are used, gives

\[
[\hat{a}, \hat{b}] = \{a, \beta - b \alpha + \Xi(a, b, p), [a, b]\},
\]  

(23)

\[
\Xi(a, b, p) = \lim_{\tau \rightarrow 0} \tau^{-2}\{\xi((\tau a)(\tau b), (\tau a)^{-1}(\tau b)^{-1}, p) + \\
+\xi(\tau a, \tau b, p) + \xi((\tau a)^{-1}, (\tau b)^{-1}, (\tau a)^{-1})\},
\]  

(24)

Let us stress once more that

\[
a \theta = \lim_{\epsilon \rightarrow 0} \frac{\theta((\epsilon a)p) - \theta(p)}{\epsilon},
\]  

(25)

and the limit is in the sense of topology induced from the Lie group \(D\).

From the associative law in \(H\) one gets

\[
((\tau \hat{a})(\tau \hat{b}))(\tau \hat{c}) = (\tau \hat{a})(\tau \hat{b})(\tau \hat{c}).
\]

We divide now the above expression by \(\tau^3\) and then pass to the limit \(\tau \rightarrow 0\). Inserting the explicit values we get

\[
\Xi([a, a'], a'', p) + \Xi([a', a''], a, p) + \Xi([a'', a], a', p) = \\
= a \Xi(a', a'', p) + a' \Xi(a'', a, p) + a'' \Xi(a, a', p),
\]  

(26)

which can be shown to be equivalent to the Jacobi identity

\[
[[\hat{a}, \hat{a'}], \hat{a''}] + [[\hat{a'}, \hat{a''}], \hat{a}] + [[\hat{a''}, \hat{a}], \hat{a'}] = 0.
\]  

(27)

So, we have reconstructed in this way the Lie algebra \(\mathfrak{F}\) giving explicitly \([\hat{a}, \hat{b}]\) for all \(\hat{a}, \hat{b} \in \mathfrak{H} \subset \mathfrak{F}\). Because \(\mathfrak{F}\) is dense in \(\mathfrak{F}\), the local exponent \(\Xi\) determines the algebra \(\mathfrak{F}\) uniquely. But from the theory of Lie groups the correspondence between the algebras \(\mathfrak{F}\) and local Lie groups \(\mathfrak{H}\) is bi-unique, at last locally, see e.g. [3] and [4]. So we get

Corollary 1  The correspondence \(\mathfrak{H} \rightarrow \mathfrak{F}\) between the local group \(\mathfrak{H}\) and the algebra \(\mathfrak{F}\) is one-to-one.

Our method is most effective in the case in which the limit in (25) can be replaced by the ordinary point-wise limit (with respect to the variable \(p \in M\)). Then the operator \(a\) becomes to be an ordinary differential operator. In other words, this is the case when the existence of the limit in \(\mathfrak{F}\) implies the existence of the point-wise limit and the both limits are always equal. We describe the important case of this situation. Suppose, that the subalgebra \(\mathfrak{F}'\) generated from the set of elements \(\{0, a\}\), where \(a\) is any element of algebra of \(G\), is finite dimensional. Then the topology induced in \(\mathfrak{F}'\) from \(\mathfrak{F}\) is equivalent to
any Hausdorff linear topology in $\mathcal{H}'$. In particular it can be the point-wise topology.

The natural question arises, then, when the group $G$ possesses finite dimensional extended algebra $\mathcal{H}'$. We show now that this is always the case in the nonrelativistic case.

In the nonrelativistic theory $\xi = \xi(r, s, t)$ depends on the time. In this case, according to our assumption about $G$, any $r \in G$ transforms simultaneity hyperplanes into simultaneity hyperplanes. So, there are two possibilities for any $r \in G$. First, when $r$ does not change the time: $t(rp) = t(p)$ and the second in which the time is changed, but in such a way that $t(rp) - t(p) = f(t)$. We assume in addition that the base generators $a_k \in \mathfrak{g}$ can be chosen in such a way that only one acts on the time as the translation and the remaining ones do not act on the time. Because we are searching for a finite dimensional extension we can assume that the operators $a$ are the ordinary differential operators. After this the Jacobi identity (26) reads

$$\Xi([a, a'], a'') + \Xi([a', a''], a') + \Xi([a'', a], a') = \partial_t \Xi(a', a''),$$

(28)

if one and only one among $a, a', a''$ is the time translation generator, namely $a$, and

$$\Xi([a, a'], a'') + \Xi([a', a''], a) + \Xi([a'', a], a') = 0,$$

(29)

in all remaining cases. The Jacobi identity (26) and (24) can be treated as a system of ordinary differential linear equations for the finite set of unknown functions $\Xi_{ij}(t) = \Xi(a_i, a_j, t)$, where $a_i$ is the base in the Lie algebra of $G$. Indeed, the identity gives us the only set of nontrivial equations which provides us the tool for the classification of possible $\Xi$-s on $G$, or equivalently the possible algebras $\mathcal{H}$. Some of the unknowns $\Xi$ are not determined by the Jacobi equations (in general), and some $\Xi_{ij}(t)$ are left completely arbitrary. In section 9 we will show that different values of undetermined $\Xi_{ij}$ lead to homomorphic algebras. Then, we can put the undetermined $\Xi_{ij}$ equal to zero, and do not lose any generality. After this we are left with a system of fewer equations for a fewer set of unknowns $\Xi$, which has to be determined. Let us order the fewer set of unknowns $\Xi$ and compose a vector-column $\mathbf{y}$ of unknowns. For a fixed $t$ any $\mathbf{y}$ is an element of a finite dimensional vector space $Y$. Then, the system of linear equations can be written as follows

$$\dot{\mathbf{y}} = A \mathbf{y},$$

(30)

where dot is the time derivation and $A$ is a linear operator in $Y$. From this system of linear equations we see that the time derivative $\partial_t \Xi_{ij}$ is determined by linear combinations of $\Xi_{ij}$. From this follows that $\Xi_{ij}$ compose the base for the algebra $\mathcal{H}$, which shows the finite dimensionality of $\mathcal{H}$. This simplifies the classification theory for time dependent $\xi$, when the the only transformation acting on the time is the time translation. However, the reasoning fails in general and it is an open question if a given group $G$ possess a finite dimensional extension ascribed to the exponent $\Xi$ in question.
7 Classification of Local Exponents of Lie Groups

Because the exponent $\xi$ determines the multiplication rule in $H$ and vice-versa, then from the Corollary 1 of the preceding section it follows

**Corollary 2** The correspondence $\xi \rightarrow \Xi$ between the local $\xi$ and the infinitesimal exponent $\Xi$ is one-to-one.

Note that the words 'local $\xi = \xi(r,s,p)$' mean that $\xi(r,s,p)$ is defined for $r$ and $s$ belonging to a fixed neighborhood $N_0 \subset G$ of $e \in G$, but in our case it is defined globally as a function of the spacetime variable $p \in M$.

Consider the nonrelativistic theory for the moment. Suppose the dimension of $G$ to be $n$. Let $a_k$ with $k \leq n$ be the base in the Lie algebra $\mathfrak{G}$ of $G$. Let us introduce the base $\tilde{a}_j$ in $H$ in the following way: $\tilde{a}_{n+1} = \{\alpha_1(t),0\}, \ldots, \tilde{a}_{n+q} = \{\alpha_q(t),0\}$ and $\tilde{a}_1 = \{0,a_1\}, \ldots, \tilde{a}_n = \{0,a_n\}$. After this we have

$$[\tilde{a}_i, \tilde{a}_j] = c^k_{ij} \tilde{a}_k + \Xi(a_i, a_j), \quad (31)$$

for $i, j \leq n$. It means that, in general, the commutation relations of a ray representation of $G$ are not equal to the commutation relations $[A_i, A_j] = c^k_{ij} A_k$ of $G$, but they are equal to $[A_i, A_j] = c^k_{ij} A_k + \Xi(a_i, a_j, t)$, $1$. The generator $A_i$ corresponding to $a_i$ is defined in the following way4

$$A_i \psi = \lim_{\tau \to 0} \frac{(T_{\tau a_i} - 1)\psi}{\tau}. \quad (25)$$

Now, we pass to describe the relation between the infinitesimal exponents $\Xi$ and local exponents $\xi$. Let us compute first the infinitesimal exponents $\Xi$ and $\Xi'$ given by (24) which correspond to the two equivalent canonical local exponents $\xi$ and $\xi' = \xi + \Delta[\Lambda]$. Inserting $\xi' = \xi + \Delta[\Lambda]$ to the formula (24) one gets

$$\Xi'(a,b,p) = \Xi(a,b,p) + a\Lambda(b,p) - b\Lambda(a,p) - \Lambda([a,b],p). \quad (32)$$

Recall, that according to the Lemma 3, we can confine ourselves to the canonical exponents. According to Lemma 4 $\Lambda = \Lambda(a, (\tau b)p)$ is a constant function of $\tau$ if $a = b$, and $\Lambda(a, p)$ is linear with respect to $a$ (we use the canonical coordinates on $G$). By this $\Xi'(a,b,p)$ is antisymmetric in $a$ and $b$ and fulfills (26) if only $\Xi(a,b,p)$ is antisymmetric in $a$ and $b$ and fulfills (26). This suggests the definition: two infinitesimal exponents $\Xi$ and $\Xi'$ will be called equivalent if and only if the relation (32) holds. For short we write the relation (32) as follows:

$$\Xi' = \Xi + d[\Lambda].$$

The transformation $T_{\tau}$ does not act in the ordinary Hilbert space but in the Hilbert bundle space $\mathcal{R} \Delta \mathcal{H}$, by this we cannot immediately appeal to the Stone and Gårding Theorems. Nonetheless, $T_{\tau}$ induces a unique unitary representation acting in the Hilbert space $\int_{\mathcal{R}} \mathcal{H} d\mu(t)$ and it can be shown that it is meaningful to tell about the generators $A$ of $T_{\tau}$.
Lemma 6 Two canonical local exponents \( \xi \) and \( \xi' \) are equivalent if and only if the corresponding infinitesimal exponents \( \Xi \) and \( \Xi' \) are equivalent.

Proof. (1) Assume \( \xi \) and \( \xi' \) to be equivalent. Then, by the definition of equivalence of infinitesimal exponents \( \Xi' = \Xi + d[\Lambda] \). (2) Assume \( \Xi \) and \( \Xi' \) to be equivalent: \( \Xi' = \Xi + d[\Lambda] \) for some linear form \( \Lambda(a,t) \) such that \( \Lambda(a, (\tau a)p) \) does not depend on \( \tau \). Then \( \xi + \Delta[\Lambda] \rightarrow \Xi' \), and by the uniqueness of the correspondence \( \xi \rightarrow \Xi \) (Corollary 2), \( \xi' = \xi + \Delta[\Lambda] \), i.e. \( \xi \) and \( \xi' \) are equivalent.

At last from Lemma 3 every local exponent is equivalent to a canonical one and by the Corollary 2 to every \( \Xi \) corresponds uniquely a local exponent. So, we can summarize our results in the following

Theorem 3 (1) On a Lie group \( G \), every local exponent \( \xi(r,s,p) \) is equivalent to a canonical local exponent \( \xi'(r,s,p) \) which, on some canonical neighborhood \( \mathcal{N}_0 \), is analytic in canonical coordinates of \( r \) and \( s \) and and vanishes if \( r \) and \( s \) belong to the same one-parameter subgroup. Two canonical local exponents \( \xi, \xi' \) are equivalent if and only if \( \xi' = \xi + \Delta[\Lambda] \) on some canonical neighborhood, where \( \Lambda(r,p) \) is a linear form in the canonical coordinates of \( r \) such that \( \Lambda(r,sp) \) does not depend on \( s \) if \( s \) belongs to the same one-parameter subgroup as \( r \). (2) To every canonical local exponent of \( G \) corresponds uniquely an infinitesimal exponent \( \Xi(a,b,p) \) on the Lie algebra \( \mathfrak{g} \) of \( G \), i.e. a bilinear antisymmetric form which satisfies the identity \( \Xi([a,a'], a'', p) + \Xi([a', a''], a, p) + \Xi(a'', a, a', p) = a\Xi(a', a'', p) + a'\Xi(a'', a, p) + a''\Xi(a, a', p) \). The correspondence is linear. (3) Two canonical local exponents \( \xi, \xi' \) are equivalent if and only if the corresponding \( \Xi, \Xi' \) are equivalent, i.e. \( \Xi'(a,b,p) = \Xi(a,b,p) + a\Lambda(b,p) - b\Lambda(a,p) - \Lambda([a,b],p) \) where \( \Lambda(a,p) \) is a linear form in a on \( \mathfrak{g} \) such that \( \tau \rightarrow \Lambda(a,(\tau b)p) \) is constant if \( a = b \). (4) There exist a one-to-one correspondence between the equivalence classes of local exponents \( \xi \) (global in \( p \in M \)) of \( G \) and the equivalence classes of infinitesimal exponents \( \Xi \) of \( \mathfrak{g} \).

8 Global Extensions of Local Exponents

Theorem 3 provides the full classification of exponents \( \xi(r,s,p) \) local in \( r \) and \( s \), defined for all \( p \in M \). But we will show that if \( G \) is connected and simply connected, then we have the following theorems. (1) If an extension \( \xi' \) of a given local (in \( r \) and \( s \)) exponent \( \xi \) does exist, then it is uniquely determined (up to the equivalence transformation (\( \mathcal{N}_0 \)) (Theorem 4). (2) There exists such an extension \( \xi' \) (Theorem 5), proved for \( G \) which possess finite dimensional extension \( \mathcal{H}' \) only.

In the global analysis the topology of \( H \) induced from \( D \) is not applicable. For we are not able to prove that the homomorphism (\( H \)) is continuous when \( \xi \) is not canonical. Note, that any \( \xi \) is equivalent to a canonical one, but only locally! We introduce another topology. Because of the semidirect structure of \( H = N \otimes G \) it is sufficient to introduce it into \( N \) and \( G \) separately in such a manner that \( G \) acts continuously on \( N \), compare e.g. (\( H \)). From the discussion
of section \(3\) it is sufficient to introduce the Fréchet topology of almost uniform convergence in the function space \(N\). Indeed, from the strong continuity of \(\xi\) and \(\zeta\) in \(1\) it follows that the multiplication rule as well as the homomorphism \(1\) are continuous.

**Theorem 4** Let \(\xi\) and \(\xi'\) be two equivalent local exponents of a connected and simply connected group \(G\), so that \(\xi' = \xi + \Delta[\zeta]\) on some neighborhood, and assume the exponents \(\xi_1\) and \(\xi'_1\) of \(G\) to be extensions of \(\xi\) and \(\xi'\) respectively. Then, for all \(r,s \in G\) \(\xi'_1(r,s,p) = \xi_1(r,s,p) + \Delta[\zeta_1]\) where \(\zeta_1(r,p)\) is strongly continuous in \(r\) and differentiable in \(p\), and \(\zeta_1(r,p) = \zeta(r,p)\), for all \(p \in M\) and for all \(r\) belonging to some neighborhood of \(e \in G\).

**Proof.** The two exponents \(\xi_1\) and \(\xi'_1\) being strongly continuous (by assumption) define two semicentral extensions \(H_1 = N_1 \circ G\) and \(H'_1 = N'_1 \circ G\), which are continuous groups. Note, that the linear groups \(N_1, N'_1\) are connected and simply connected. Because \(H_1\) and \(H'_1\) both are semi-direct products of two connected and simply connected groups they are both connected and simply connected. Eq. \(14\) defines a local isomorphism mapping \(h : \tilde{r} \rightarrow \tilde{r}' = h(\tilde{r})\) of \(H_1\) into \(H'_1\)

\[
h(\tilde{r}) = h(\theta, r) = \{\theta(p) - \zeta(r, p), r\}
\]

on the appropriately small neighborhood of \(e\) in \(G\), on which \(\xi_1 = \xi\) and \(\xi'_1 = \xi'\). Because \(H_1\) and \(H'_1\) are connected and simply connected the isomorphism \(h\) given by \(14\) can be uniquely extended to an isomorphism \(h_1(\tilde{r}) = h(\theta, r) = \tilde{r}'\) of the entire groups \(H_1\) and \(H'_1\) such that \(h_1(\tilde{r}) = h(\tilde{r})\) on some neighborhood of \(H_1\), see \(11\), Theorem 80. The isomorphism \(h_1\) defines an isomorphism of the two abelian subgroups \(N_1\) and \(h_1(N_1)\). By \(14\) \(h_1(\theta, e) = \{\theta, e\}\) locally in \(H_1\), that is for \(\theta\) lying appropriately close to 0 (in the metric sense defined previously). Both \(N_1\) and \(h_1(N_1)\) are connected, and \(N_1\) in addition simply connected, so applying once again the Theorem 80 of \(11\), one can see that \(h_1(\theta, e) = \{\theta, e\}\) for all \(\theta\). Set \(h_1(0,r) = \{-\zeta_1(p), g(r)\}\). Note, that because \(f_1\) is an isomorphism it is continuous in the topology of \(H_1\) and \(H'_1\). By this \(\zeta_1(r,p)\) is strongly continuous in \(r\) and \(g(r)\) is a continuous function of \(r\). The equation \(\{\theta, r\} = \{\theta, e\}\{0, r\}\) implies that \(h_1(\theta(p), r) = \{\theta(p) - \zeta_1(r, p), g(r)\}\). Computing now \(h_1(0,r)h_1(0,s)\) we find that \(g(rs) = g(r)g(s)\). So, \(g(r)\) is an automorphism of a connected and simply connected \(G\), for which \(g(r) = r\) locally, then applying once more the Theorem 80 of \(11\) one shows that \(g(r) = r\) for all \(r\). Thus

\[
h_1(\tilde{r}) = h_1(\theta(p), r) = \{\theta(p) - \zeta_1(r, p), r\},
\]

for all \(\tilde{r} \in H_1\). Finally, \(h_1(0,r)h_1(0,s) = h_1(\xi_1(r,s,p), rs)\). Hence

\[
\{\xi'_1(r,s,p) - \zeta_1(r,p) - \zeta_1(s,r^{-1}p), rs\} = \\
\{\xi_1(r,s,p) - \zeta_1(rs,p), rs\},
\]

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for all $r, s, p$. That is, $\xi_1'(r, s, p) = \xi_1(r, s, p) + \Delta[\zeta_1]$ for all $r, s, p$ and by (14) $\zeta_1(r, p) = \zeta(r, p)$ on some neighborhood of $e$ on $G$.

The following Theorem is proved for the group $G$ having a finite dimensional extended algebra $\mathfrak{g}'$.

**Theorem 5** Let $G$ be connected and simply connected Lie group. Then to every exponent $\xi(r, s, X)$ of $G$ defined locally in $(r, s)$ there exists an exponent $\xi_0$ of $G$ defined on the whole group $G$ which is an extension of $\xi$. If $\xi$ is differentiable, $\xi_0$ may be chosen differentiable.

Because the proof of Theorem 5 is identical as that of the Theorem 5.1 in [1], we do not present it explicitly. Note that the proof largely rests on the global theory of classical (finite dimensional) Lie groups. Namely, it rests on the theorem that there always exists the universal covering group to any finite dimensional Lie group. We can use those methods because there exist a finite dimensional extension $H'$ of $G$.

We have obtained the full classification of time dependent $\xi$ defined on the whole group $G$ for Lie groups $G$ which are connected and simply connected in the nonrelativistic theory. But for any Lie group $G$ there exists the universal covering group $G^*$ which is connected and simply connected. So, for $G^*$ the correspondence $\xi \rightarrow \Xi$ is one-to-one, that is, to every $\xi$ there exists the unique $\Xi$ and vice versa, to every $\Xi$ corresponds uniquely $\xi$ defined on the whole group $G^*$ and the correspondence preserves the equivalence relation. Because $G$ and $G^*$ are locally isomorphic the infinitesimal exponents $\Xi$'s are exactly the same for $G$ and for $G^*$. Because to every $\Xi$ there does exist exactly one $\xi$ on $G^*$, so, if there exists the corresponding $\xi$ on the whole $G$ to a given $\Xi$, then such a $\xi$ is unique. We have obtained in this way the full classification of $\xi$ defined on a whole Lie group $G$ for any Lie group $G$, in the sense that no $\xi$ can be omitted in the classification. The set of equivalence classes of $\xi$ is considerably smaller than the set of equivalence classes of $\Xi$, it may happen that to some local $\xi$ there does not exist any global extension. Therefore, the classification is full in this sense in the relativistic theory also.

Take, for example, a Lie subgroup $G$ of the Milne group and its ray representation $T_r$. We have classified in this way all exponents for this $T_r$ and $r \in G$. In general such a $\Xi$ may exists that there does not exist any $\xi$ corresponding to this $\Xi$ if the group $G$ is not connected and simply connected. But this not important for us, the important fact is that no $\xi(r, X)$ can be omitted in this classification.

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5In the proof we consider the finite dimensional extension $H'$ of $G$ instead of the Lie group $H$ in the proof presented in [1]. The remaining replacements are rather trivial, but we mark them here explicitly to simplify the reading. (1) instead of the formula $\vec{r}' = t(\theta)\vec{r} = \vec{r}(\theta)$ of (5.3) in [1] we have $\vec{r}' = t(\theta(\vec{r}^{-1}p))\vec{r} = \vec{r}(\theta(p))$. By this, from the formula $(h_1(r)h_1(s))h_1(g) = h_1(r)(h_1(s)h_1(g))$ (see [1]) follows $(r, s, p) + \xi(rs, g, p) = \xi(s, g, r^{-1}p) + \xi(r, sg, p)$ instead of (5.8) in [1]. (2) Instead of (4.9), (4.10) and (4.11) we use the Iwasawa-type construction presented in this paper. (3) Instead of Lemma 4.2 in [1] we use the Lemma 1.
9 Examples

9.1 Example 1: The Galilean Group

According to the conclusions of section 2 one should *a priori* investigate such representations of the Galilean group $G$ which fulfill the Eq. (2), with $\xi$ depending on the time. The following paradox, then, arises. Why the transformation law $T_r$ under the Galilean group has time-independent $\xi$ in (2) independently of the fact if it is a covariance group or a symmetry group? We will solve the paradox in this subsection. Namely, we will show that any representation of the Galilean group fulfilling (2) is equivalent to a representation fulfilling (2) with time-independent $\xi$. This is a rather peculiar property of the Galilean group not valid in general. For example, this is not true for the group of Milne transformations.

According to section 3 we shall determine all equivalence classes of infinitesimal exponents $\Xi$ of the Lie algebra $\mathfrak{g}$ of $G$ to classify all $\xi$ of $G$. The commutation relations for the Galilean group are as follows

\[ [a_{ij},a_{kl}] = \delta_{jk}a_{il} - \delta_{ik}a_{jl} + \delta_{il}a_{jk} - \delta_{jl}a_{ik}, \quad (33) \]

\[ [a_{ij},b_k] = \delta_{jk}b_i - \delta_{ik}b_j, \quad \{b_i, b_j\} = 0, \quad (34) \]

\[ [a_{ij},d_k] = \delta_{jk}d_i - \delta_{ik}d_j, \quad \{d_i, d_j\} = 0, \quad [b_i, d_j] = 0, \quad (35) \]

\[ [a_{ij},\tau] = 0, \quad \{b_k, \tau\} = 0, \quad \{d_k, \tau\} = b_k, \quad (36) \]

where $b_i, d_i$ and $\tau$ stand for the generators of space translations, the proper Galilean transformations and time translation respectively and $a_{ij} = -a_{ji}$ are rotation generators. Note, that the Jacobi identity (29) is identical to the Jacobi identity in the ordinary Bargmann's Theory of time-independent exponents (see [1], Eqs (4.24) and (4.24a)). So, using (33) – (35) we can proceed exactly after Bargmann (see [1], pages 39,40) and show that any infinitesimal exponent defined on the subgroup generated by $b_i, d_i$, $a_{ij}$ is equivalent to an exponent equal to zero with the possible exception of $\Xi(b_i, d_k, t) = \gamma \delta_{ik}$, where $\gamma = \gamma(t)$. So, the only components of $\Xi$ defined on the whole algebra $\mathfrak{g}$ which can a priori be not equal to zero are: $\Xi(b_i, d_k, t) = \gamma \delta_{ik}$, $\Xi(a_{ij}, \tau, t)$, $\Xi(b_i, \tau, t)$ and $\Xi(d_k, \tau, t)$. We compute first the function $\gamma(t)$. Substituting $a = \tau$, $a' = b_i$, $a'' = d_k$ to (28) we get $d\gamma/dt = 0$, so that $\gamma$ is a constant, we denote the constant value of $\gamma$ by $m$. Inserting $a = \tau$, $a' = a_{ij}^s$, $a'' = a_{ij}$ to (28) and summing up with respect to $s$ we get $\Xi(a_{ij}, \tau, t) = 0$. In the same way, but with the substitution $a = \tau, a' = a_{ij}^s, a'' = b_s$, one shows that $\Xi(b_i, \tau, t) = 0$. At last the substitution $a = \tau, a' = a_{ij}^s, a'' = d_s$ to (28) and summation with respect to $s$ gives $\Xi(d_i, \tau, t) = 0$. We have proved in this way that any time depending $\Xi$ on $\mathfrak{g}$ is equivalent to a time-independent one. In other words, we get a one-parameter family of possible $\Xi$, with the parameter equal to the inertial
mass $m$ of the system in question. Any infinitesimal time-dependent exponent of the Galilean group is equivalent to the above time-independent exponent $\Xi$ with some value of the parameter $m$; and any two infinitesimal exponents with different values of $m$ are inequivalent. As was argued in (Theorems 3 ÷ 5) the classification of $\Xi$ gives the full classification of $\xi$. Moreover, it can be shown that the classification of $\xi$ is equivalent to the classification of possible $\theta$-s in the transformation law

$$T_r \psi(p) = e^{i\theta(r,p)} \psi(r^{-1}p) \tag{37}$$

for the spinless nonrelativistic particle. On the other hand, the exponent $\xi(r,s,t)$ of the representation $T_r$ given by (37) can be easily computed to be equal \(\theta(rs,p) - \theta(r,p) - \theta(s,r^{-1}p)\), and the infinitesimal exponent belonging to $\theta$ defined as $\theta(r,p) = -mv \cdot \bar{x} + \frac{mv^2}{2} t$, covers the whole one-parameter family of the classification (its infinitesimal exponent is equal to that infinitesimal exponent $\Xi$, which has been found above). So, the standard $\theta(r,p) = -mv \cdot \bar{x} + \frac{mv^2}{2} t$, covers the full classification of possible $\theta$-s in (37) for the Galilean group. Inserting the standard form for $\theta$ we see that $\xi$ does not depend on $\mathcal{X}$ but only on $r$ and $s$. By this, any time-depending $\xi$ on $G$ is equivalent to a time-independent one.

This result can be obtained in the other way. Namely, using now the Eq. (31) we get the commutation relations for the ray representation $T_r$ of the Galilean group

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{ik}A_{jl} - \delta_{jl}A_{ik},
[A_{ij}, B_k] = \delta_{jk}B_i - \delta_{ik}B_j, \ [B_i, B_j] = 0,
[A_{ij}, D_k] = \delta_{jk}D_i - \delta_{ik}D_j,
[D_i, D_j] = 0, \ [B_i, D_j] = m\delta_{ij},
[A_{ij}, T] = 0, \ [B_k, T] = 0, \ [D_k, T] = B_k,$$

where the generators $A_{ij}, \ldots$ which correspond to the generators $a_{ij}, \ldots$ of the one-parameter subgroups $r(\sigma) = \sigma a_{ij}, \ldots$ are defined in the following way (compare the 4th footnote)

$$A_{ij}\psi(X) = \lim_{\sigma \to 0} \frac{1}{\sigma}(T_r(\sigma) - 1)\psi(X).$$

$A_{ij}$ is well defined for any differentiable $\psi(p)$. So, we get the standard commutation relations such as in the case when the Galilean group is a symmetry group. The above standard commutation relations for the transformation $T_r$ of the form (37) gives a differential equations for $\theta$. It is easy to show, that they can be solved uniquely (up to an irrelevant function $f(t)$ of time and the group parameters) and the solution has the standard form $\theta(r,p) = -mv \cdot \bar{x} + \frac{mv^2}{2} t$.

Note, that to any $\xi$ (or $\Xi$) there exists a corresponding $\theta$ (and such a $\theta$ is unique up to a trivial equivalence relation). As we will see this is not the case for the Milne group, where such $\Xi$ do exist which cannot be realized by any $\theta$. 

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9.2 Example 2: Milne group as a covariance group

In this subsection we apply the theory of section 3 to the Milne transformations group. We proceed like with the Galilean group in the preceding section. The Milne group $G$ does not form any Lie group, which complicates the situation. We will go on according to the following plan. First, 1) we define the topology in the Milne group. Second, 2) we define the sequence $G(1) \subset \ldots \subset G(m) \subset \ldots$ of Lie subgroups of the Milne group $G$ dense in $G$. 3) Then we compute the infinitesimal exponents and exponents for each $G(m)$, $m = 1, 2, \ldots$, and by this the $\theta$ in (37) for $G(m)$. 4) As we have proved in 3 the (strong) continuity of the exponent $\xi(r,s,t)$ in the group variables follows as a consequence of the Theorem 2. It can be shown that also $\theta(r,p)$ is strongly continuous in the group variables $r \in G$. By this, $\theta(r,p)$ defined for $r \in G(m)$, $m = 1, 2, \ldots$ can be uniquely extended on the whole group $G$. This can be done effectively thanks to the assumption that the wave equation is local.

Before we go further on we make an important remark. The Milne group $G$ is an infinite dimensional group and there are infinitely many ways in which a topology can be introduced in $G$. On the other hand the physical contents of the continuity assumption of section 3 depends effectively on the topology in $G$. By this the assumption is in some sense empty. True, but it is important to stress here, that the whole relevant physical content rests on the Lie subgroup $G(m)$ (see the further text for the definition of $G(m)$) for a sufficiently large $m$, and not on the whole $G$. That is, the covariance condition with respect to $G(m)$ for sufficiently large $m$ is sufficient for us. By this, there are no ambiguities in the continuity assumption. The topology in $G$ is not important from the physical point of view, and the extension of the formula (37) from $G(m)$ to the whole group is of secondary importance. However, we construct such an extension to make our considerations more complete, living the opinion about the "naturality" of this extension to the reader.

1) Up to now the Milne group of transformations

$$(\vec{x}, t) \rightarrow (R\vec{x} + \vec{A}(t), t + b), \quad (38)$$

where $R$ is an orthogonal matrix, and $b$ is constant, has not been strictly defined. The extent of arbitrariness of the function $\vec{A}(t)$ in (38) has been left open up to now. The topology depends on the degree of this arbitrariness. It is natural to assume the function $\vec{A}(t)$ in (38) to be differentiable up to any order. Consider the subgroups $G_1$ and $G_2$ of the Milne group which consist of the transformations: $(\vec{x}, t) \rightarrow (\vec{x} + \vec{A}(t), t)$ and $(\vec{x}, t) \rightarrow (R\vec{x}, t + b)$ respectively. Then the Milne group $G$ is equal to the semidirect product $G_1 \rtimes G_2$, where $G_1$ is the normal factor of $G$. It is sufficient to introduce a topology in $G_1$ and then define the topology in $G$ as the semi-Cartesian product topology, where it is clear what is the topology in the Lie group $G_2$. We introduce a linear topology in the linear group $G_1$ which makes it a Fréchet space, in which the time derivation operator $\frac{d}{dt} : \vec{A} \rightarrow \frac{d\vec{A}}{dt}$ becomes a continuous operator. Let
$K_N, N = 1, 2, \ldots$ be such a sequence of compact sets of $\mathcal{R}$, that

$$K_1 \subset K_2 \subset \ldots \text{ and } \bigcup_N K_N = \mathcal{R}.$$ 

Then we define a separable family of seminorms

$$p_N(\vec{A}) = \max \left\{ |\vec{A}^{(n)}(t)|, t \in K_N, n \leq N \right\},$$

where $\vec{A}^{(n)}$ denotes the $n$-th order time derivative of $\vec{A}$. Those seminorms define on $G_1$ a locally convex metrizable topology. For example, the metric

$$d(\vec{A}_1, \vec{A}_2) = \max_{N \in \mathbb{N}} \frac{2^{-N} p_N(\vec{A}_2 - \vec{A}_1)}{1 + p_N(\vec{A}_2 - \vec{A}_1)}$$

defines the topology.

2) It is convenient to rewrite the Milne transformations (38) in the following form

$$x' = R x + \vec{A}(t) \vec{v}, \quad t' = t + b,$$

where $\vec{v}$ is a constant vector, which does not depend on the time $t$. We define the subgroup $G(m)$ of $G$ as the group of the following transformations

$$x' = R x + \vec{v}(0) + t \vec{v}(1) + \frac{t^2}{2!} \vec{v}(2) + \ldots + \frac{t^m}{m!} \vec{v}(m), \quad t' = t + b,$$

where $R = (R_k^a, \nu_k^{(n)})$ are the group parameters – in particular the group $G(m)$ has the dimension equal to $3m + 7$.

3) Now we investigate the group $G(m)$, that is, we classify their infinitesimal exponents. The commutation relations of $G(m)$ are as follows

$$[a_{ij}, a_{kl}] = \delta_{jk} a_{il} - \delta_{ik} a_{jl} + \delta_{il} a_{jk} - \delta_{il} a_{ik}, \quad (39)$$

$$[a_{ij}, a_k^{(n)}] = \delta_{jk} a_i^{(n)} - \delta_{ik} a_j^{(n)} - [d_i^{(n)}, d_j^{(k)}] = 0, \quad (40)$$

$$[a_{ij}, \tau] = 0, \quad [d_i^{(0)}, \tau] = 0, \quad [d_i^{(n)}, \tau] = d_i^{(n-1)}, \quad (41)$$

where $d_i^{(n)}$ is the generator of the transformation $r(\nu_i^{(n)})$:

$$x'^i = x^i + \frac{t^n}{n!} \nu_i^{(n)},$$

which will be called the $n$-acceleration, especially 0-acceleration is the ordinary space translation. All the relations (39) and (40) are identical with (33) ÷ (35) with the $n$-acceleration instead of the Galilean transformation. So, the same argumentation as that used for the Galilean group gives: $\Xi(a_{ij}, a_{kl}) = 0$, $\Xi(a_{ij}, a_k^{(n)}) = 0$, and $\Xi(d_i^{(n)}, d_j^{(n)}) = 0$. Substituting $a_{i, a_{i, a_{i, \tau}}}$ for $a, d', d''$ into the Eq. (28), making use of the commutation relations and summing up with
respect to $h$ we get $\Xi(a_{ij}, \tau) = 0$. Substituting $a_{k}^{l}, d_{h}^{(l)}, d_{k}^{(n)}$ for $a, a', a''$ into the Eq. (28), we get in the analogous way $\Xi(d_{k}^{(l)}, d_{k}^{(n)}) = \frac{1}{3} \Xi(d_{l}^{(h)}, d_{h}^{(n)}) \delta_{ik}$.

Substituting $a_{k}^{l}, d_{h}^{(n)}$, $\tau$ for $a, a', a''$ into the Eq. (28), making use of commutation relations, and summing up with respect to $h$, we get $\Xi(d_{k}^{(n)}, \tau) = 0$. Now, we substitute $d_{k}^{(n)}, d_{l}^{(0)}, \tau$ for $a, a', a''$ in (28), and proceed recurrently with respect to $n$, we obtain in this way $\Xi(d_{k}^{(0)}, d_{k}^{(n)}) = P^{(0,n)}(t) \delta_{ik}$, where $P^{(0,n)}(t)$ is a polynomial of degree $n - 1$ - the time derivation of $P^{(l,n)}(t)$ has to be equal to $P^{(0,n-1)}(t)$, and $P^{(0,0)}(t) = 0$. Substituting $d_{k}^{(n)}, d_{l}^{(i)}, \tau$ to (28) we get in the same way $\Xi(d_{k}^{(l)}, d_{k}^{(n)}) = P^{(l,n)}(t) \delta_{ki}$, where $d_{k}^{(l)}P^{(1,n)} = P^{(l-1,n)} + P^{(l,n-1)}$.

This allows us to determine all $P^{(l,n)}$ by the recurrent integration process. Note that $P^{(0,0)} = 0$, and $P^{(l,n)} = -P^{(n,l)}$, so we can compute all $P^{(l,n)}$ having given the $P^{(0,n)}$. Indeed, we have $P^{(1,0)} = -P^{(0,1)}$, $P^{(3,1)} = 0$, $dP^{(3,3)}/dt = P^{(0,3)} + P^{(1,2)},$ and after $m - 1$ integrations we compute all $P^{(l,n)}$. Each elementary integration introduces a new independent parameter (the arbitrary additive integration constant). Exactly in the same way we can compute all $P^{(2,n)}$ having given all $P^{(l,n)}$ after the $m - 2$ elementary integration processes. In general the $P^{(l-1,n)}$ allows us to compute all $P^{(l,n)}$ after the $m - l$ integrations. So, $P^{(l,n)}(t)$ are $l + n - 1$-degree polynomial functions of t, and all are determined by $m(m + 1)/2$ integration constants. Because $d[\Delta](d_{k}^{(l)}, d_{k}^{(i)}) = 0$, the exponents $\Xi$ defined by different polynomials $P^{(l,n)}$ are inequivalent. By this the space of inequivalent classes of $\Xi$ is $m(m + 1)/2$-dimensional.

However, not all $\Xi$ can be realized by the transformation $T_{r}$ of the form (37). All the above integration constants have to be equal to zero with the exception of those in $P^{(0,n)}(t)$. By this, all exponents of $G(m)$, which can be realized by the transformations $T_{r}$ of the form (37) are determined by the polynomial $P^{(0,m)}$, that is, by $m$ constants. We show it first for the group $G(2)$, because the case is the simplest one and it suffices to explain the principle of all computations for all $G(m)$. From the above analysis we have $P^{(0,1)} = \gamma_{1}, P^{(0,2)} = \gamma_{1}t + \gamma_{2}, P^{(1,2)} = \frac{1}{3} \gamma_{1}t^{2} + \gamma_{2}t + \gamma_{1,2}$, where $\gamma_{1,2}$ are the integration constants. We will show that $\gamma_{1,2} = 0$. A simple computation gives the following formula $\xi(r,s) = \theta(rs, X) - \theta(r, X) - \theta(s, r^{-1}X)$ for the exponent of the representation $T_{r}$ of the form (37). Inserting this $\xi$ to the Eq. (24) and performing a rather straightforward computation we get the following formula

$$\Xi(d_{k}^{(j)}, d_{k}^{(n)}) = \frac{t^{n}}{n!} \frac{\partial^{2} \theta}{\partial x^{i} \partial v_{i}^{(k)}} - \frac{t^{k}}{k!} \frac{\partial^{2} \theta}{\partial x^{i} \partial v_{i}^{(n)}},$$

for the infinitesimal exponent $\Xi$ of the representation $T_{r}$ given by (37), where the derivation with respect to $v_{(p)}^{q}$ is taken at $v_{(p)}^{q} = 0$. Comparing this $\Xi(d_{j}^{(k)}, d_{j}^{(n)})$ with $P^{(k,n)} \delta_{ij}$ we get the equations

$$\frac{t^{n}}{n!} \frac{\partial^{2} \theta}{\partial x^{i} \partial v_{i}^{(k)}} - \frac{t^{k}}{k!} \frac{\partial^{2} \theta}{\partial x^{i} \partial v_{i}^{(n)}} = P^{(k,n)} \delta_{ij},$$

(42)
Because of the linearity of the problem, we can consider the three cases 1°. \( \gamma(2) = \gamma(1,2) = 0 \), 2°. \( \gamma_1 = \gamma(1,2) = 0 \) and 3°. \( \gamma_1 = \gamma_2 = 0 \), separately. In the case 1°. we have the solution

\[ \theta(r, X) = \gamma_1 \frac{d \tilde{A}}{dt} \cdot \vec{x} + \tilde{\theta}(t), \]

where \( \tilde{\theta}(t) \) is an arbitrary function of time and the group parameters, and \( \tilde{A}(t) \in G(2) \). In the case 2° we have

\[ \theta(r, X) = \gamma_2 \frac{d^2 \tilde{A}}{dt^2} \cdot \vec{x} + \tilde{\theta}(t), \]

with arbitrary function \( \tilde{\theta}(t) \) of time. Consider at last the case 3°. From (42) we have (corresponding to \((k, n) = (0, 1), (0, 2)\) and \((1, 2)\) respectively)

\[ \frac{t^2}{2} \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(0)}} - \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(1)}} = 0 \]

(43)

\[ \frac{t^2}{2} \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(2)}} = 0 \]

(44)

\[ \frac{t^2}{2} \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(1)}} - t \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(2)}} = \gamma(1,2) \delta_{ij}. \]

(45)

From (46) and (44) we get

\[ \frac{t^2}{2} \left\{ \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(1)}} - t \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(0)}} \right\} = \gamma(1,2) \delta_{ij}. \]

(46)

But \( \Xi(d_i^{(0)}, d_j^{(0)}) = 0 = \partial^2 \theta / \partial x^i \partial v_j^{(0)} - \partial^2 \theta / \partial x^i \partial v_j^{(0)} \), so, from (46) and (44) we get

\[ 0 = \frac{\partial^2 \theta}{\partial x^i \partial v_j^{(0)}} \left\{ \frac{t^3}{2} - \frac{t^3}{2} \right\} = \gamma(1,2) \delta_{ij}, \]

and \( \gamma(1,2) = 0 \).

The following

\[ \theta(r, X) = \gamma_1 \frac{d \tilde{A}}{dt} \cdot \vec{x} + \gamma_2 \frac{d^2 \tilde{A}}{dt^2} \cdot \vec{x} + \tilde{\theta}(t) \]

(47)

fulfills all Eqs. (42) with \( k, n \leq 2 \) and its local exponents cover the full classification of \( \Xi \)'s for \( G(2) \) which can be realized by \( T_r \) of the form (37), that is, all \( \Xi \)'s with \( \gamma(1,2) = 0 \). Then, the formula (47) gives the most general \( \theta \) in (37) for \( r \in G(2) \). This is because the classification of \( \Xi \)'s covers the classification of all possible \( \theta \)'s (however we live it without proof).

It can be immediately seen that any integration constant \( \gamma(l,q) \) of the polynomial \( P^{(l,q)}(t) \) has to be equal to zero if \( l, q \neq 0 \), provided the exponent \( \Xi \)
It can be shown that (we live it without proof) from this assumption that the higher derivatives cannot enter into \( \theta \) for \( r \neq 0 \). Eventually of the group parameters. A rather simple computation shows from this and the equations (42) corresponding to \((k,n) = (l,q)\), which gives the result that \( \gamma_{(l,q)} \delta_{ij} \). From the equations (42) corresponding to \((k,n) = (l,q)\) and \((l-1,q)\) we get

\[
\frac{t^q}{q!} \frac{\partial \theta}{\partial x^j \partial \nu^q_{(i-1)}} - \frac{t^{q+1}}{q!} \frac{\partial^2 \theta}{\partial x^j \partial \nu^q_{(i-1)}} = \gamma_{(l,q)} \delta_{ij}.
\]

From this and the equations (42) corresponding to \((k,n) = (q-1,l-1)\) we get

\[
\frac{t^q}{q!} \frac{\partial \theta}{\partial x^j \partial \nu^q_{(l-1)}} - \frac{t^{q+1}}{q!} \frac{\partial^2 \theta}{\partial x^j \partial \nu^q_{(l-1)}} = \gamma_{(l,q)} \delta_{ij}.
\]

From this and the equations (42) corresponding to \((k,n) = (l,q-1)\) one gets

\[
0 = \frac{t^{q+1}}{q!} \frac{\partial \theta}{\partial x^j \partial \nu^q_{(l-1)}} = \gamma_{(l,q)} \delta_{ij},
\]

which gives the result that \( \gamma_{(l,q)} = 0 \).

Consider the \( \theta \), given by the formula

\[
\theta(r,p) = \gamma_1 \frac{d\vec{A}}{dt} + \gamma_2 \frac{d^2 \vec{A}}{dt^2} + \ldots + \gamma_m \frac{d^m \vec{A}}{dt^m} + \tilde{\theta}(t),
\]

for \( r \in G(m) \), where \( \gamma_i \) are the integration constants which define the polynomial \( P^{(0,m)} = \gamma_1 \frac{t^{m-1}}{(m-1)!} + \gamma_2 \frac{t^{m-2}}{(m-2)!} + \ldots + \gamma_m \), and \( \tilde{\theta}(t) \) is any function of the time \( t \) and eventually of the group parameters. A rather simple computation shows that this \( \theta \) fulfills all (42) for \( k,n \leq m \) and that it covers all possible \( \Xi \) which can be realized by (37). That is, the infinitesimal exponents corresponding to the \( \theta \) given by (42) give all possible \( \Xi \) with all integration constants \( \gamma_{(k,n)} = 0 \), for \( k,n \neq 0 \). So, the most general \( \theta(r,p) \) defined for \( r \in G(m) \) is given by (42).

At this place we make use of the assumption that the wave equation is local. It can be shown that (we live it without proof) from this assumption that the \( \theta(r,p) \) can be a function of a finite order derivatives of \( \vec{A}(t) \), say \( k \)-th at most, the higher derivatives cannot enter into \( \theta \). By this, the most general \( \theta(r,p) \) defined for \( r \in G(m) \) has the following form

\[
\theta(r,X) = \gamma_1 \frac{d\vec{A}}{dt} + \ldots + \gamma_k \frac{d^k \vec{A}}{dt^k} + \tilde{\theta}(t),
\]

4) Now, we extend the formula (49) on the whole Milne group \( G \). It is a known fact that the time derivative operator \( d/dt : \vec{A} \rightarrow d\vec{A}/dt \) is a continuous
operator on $G$ in the topology introduced in 1), see e.g. [12]. It remains to show that the sequence $G(m), m \in \mathcal{N}$ is dense in $G$. The proof of this presents no difficulties$^6$. By this the function $\theta(r,p)$ can be uniquely extended on the whole group $r \in G$

$$\theta(r,X) = \gamma_1 \frac{dA}{dt} + \ldots + \gamma_4 \frac{d^4A}{dt^4} \tilde{\theta}(t).$$

It should be stressed here that not only the topology in $G$ is needed to derive the formula, but also the locality assumption is very important. If the coefficients $a, b^1, \ldots, g$ in the wave equation were admitted to be nonlocal, then an infinite number of other solutions for $\theta$ in $G$ would exist.

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References

[1] V. Bargmann, Ann. Math. 59, 1, (1954).

[2] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42, 127, (1975); Ann. Phys. 98, 287, (1976); I. V. Tyutin, Liebiediev Institute preprint N39(1975). See for example: S. Weinberg, The Quantum Theory of Fields, volume II, Univ. Press, Cambridge 1996, where the BRST-formalism is described.

[3] G. Birkhoff, Continuous Groups and Linear Spaces, Recueil Mathématique (Moscow) 1(5), 635, (1935); Analytical Groups, Trans. Am. Math. Soc. 43, 61, (1938).

[4] E. Dynkin, Uspekhi Mat. Nauk 5, (1950), 135; Amer. Math. Soc. Transl. 9(1), (1950), 470.

$^6$It is sufficient to use the following two facts. 1) The Weierstrass Theorem: For any continuous (and by this any differentiable) function $f(t)$ and any compact set $C$ there exist a sequence of polynomial functions $P_n(t)$, uniformly convergent to $f(t)$ on $C$. 2) The following Theorem: Let $\{P_n(t)\}$ be a sequence of functions differentiable in the interval $[a, b]$, convergent at least in one point of this interval. If the sequence $\{P'_n(t)\}$ of derived functions is uniformly convergent in $[a, b]$ to the function $\varphi(t)$, then the sequence of primitive functions $\{P_n(t)\}$ (anti-derivatives of $P'_n$) is uniformly convergent to a differentiable function $\phi(t)$ the derivative $\phi'(t)$ of which is equal to $\varphi(t)$ in $[a, b]$. 

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[5] D. Giulini, *States, Symmetries and Superselection*, in: *Decoherence: Theoretical, Experimental and Conceptual Problems*, (Lecture Notes in Physics, Springer Verlag 2000), page 87.

[6] S. N. Gupta, Proc. Phys. Soc. **63**, 681, (1950); K. Bleuler, Helv. Phys. Acta **23**, 567, (1950).

[7] S. Lang, *Differential Manifolds*, Springer-Verlag, Berlin, Heidelberg, New York (1985).

[8] J. Łopuszański, Fortschrritte der Physik **26**, 261, (1978); *Rachunek spinorów*, PWN, Warszawa 1985 (in Polish).

[9] G. W. Mackey, *Unitary Group Representations in Physics, Probability, and Number Theory*. Addison-Wesley Publishing Company, INC. The Advanced Book Program. Redwood City-California, New York, Amsterdam, Wokingham-UK (1989).

[10] L. Nachbin, *The Haar Integral*. D. Van Nostrad Company I. N. C. Princeton-New Jersey- Toronto-New York-London (1965).

[11] L. Pontrjagin, *Topological groups*, Moscow (1984) (in Russian).

[12] W. Rudin, *Functional Analysis*, second edition, Mc-Graw-Hill, Inc. (1991).

[13] J. Wawrzycki, Int. Jour. of Theor. Phys. **40**, 1595 (2001).