Noncommutative Coordinate Picture of the Quantum Phase Space

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Abstract

We illustrate an isomorphic representation of the observable algebra for quantum mechanics in terms of the functions on the projective Hilbert space, and its Hilbert space analog, with a noncommutative product in terms of explicit coordinates and discuss the physical and dynamical picture. The isomorphism is then used as a base for the translation of the differential symplectic geometry of the infinite dimensional manifolds onto the observable algebra as a noncommutative geometry. Hence, we obtain the latter from the physical theory itself. We have essentially an extended formalism of the Schrödinger versus Heisenberg picture which we describe mathematically as like a coordinate map from the phase space, for which we have presented argument to be seen as the quantum model of the physical space, to the noncommutative geometry coordinated by the six position and momentum operators. The observable algebra is taken essentially as an algebra of formal functions on the latter operators. The work formulates the intuitive idea that the noncommutative geometry can be seen as an alternative, noncommutative coordinate, picture of familiar quantum phase space, at least so long as the symplectic geometry is concerned.

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I. INTRODUCTION

In classical physics, physical quantities or observables are modeled by the real valued variables and physical states are identified as points of the phase space which is a geometric structure modeled locally on a finite Cartesian product of the real number lines. Such geometric space are called manifolds. In fact, a state can be described by the values of its coordinate variables, the position and momentum observables. These are the basic observables, combinations of which (like as smooth functions) essentially give all the others. From the mathematical point of view, there is a duality correspondence between the algebra of observable and the geometric structure. The algebra of observables of a classical theory is a commutative algebra, to be identified as functions on the corresponding commutative geometric space (the phase space), a symplectic manifold. For a quantum theory the algebra of observables is a noncommutative (operator) one. In the spirit of noncommutative geometry, one expects a dual geometric structure which is noncommutative. Intuitively, one would naturally think of the latter as having noncommutative coordinate observables which can be given by the position and momentum operators. We seek a point of view based on physics to understand such a noncommutative geometric structure beyond real number manifolds, at least beyond the finite dimensional ones.

In the case of simple quantum mechanics, physicists however have a well established picture of the phase space as a symplectic geometry. It is the Hilbert space, an infinite dimensional complex vector space, or the projective Hilbert space, an infinite dimensional, actually curved, manifold. The latter, as the manifold of the pure states [1–3], is also a geometric structure dual to the noncommutative algebra of observables. Actually, for $C^*$-algebras, which are the class of algebras considered to be the proper setting for the mathematics of noncommutative geometry [4] and the general idea of an observable algebra in a physical theory [5, 6], the space of pure states essentially always has the more familiar commutative geometric structure of Kähler manifolds or Kähler bundles [3, 7]. Typically, they are infinite dimensional. A key perspective here is that a noncommutative observable/quantity can be modeled by an infinite number of commutative variables. A noncommutative coordinate in particular can be described as an infinite set of the real or complex coordinates. The noncommutative geometry of the observable algebra for quantum mechanics may be taken as nothing more than an alternative or better, more intuitive, picture of the infinite dimen-
sional symplectic geometry of the quantum phase space. It may be somewhat similar to the intrinsic description of a curved manifold versus its extrinsic description as part of an Euclidean space.

The idea that a noncommutative observable has the information content of an infinite number of real or complex numbers is easy to appreciate, though it has not been taken seriously enough in our opinion. A quantum operator can be thought of as a matrix on the infinite dimensional Hilbert space characterized by the matrix elements in a chosen basis, i.e. a system of coordinates. In its eigenstate basis in particular, it is described by the set of eigenvalues. The latter, though often considered, is not the most convenient description of the structure for the full algebra. We have introduced the notion of a noncommutative value of an observable which plays a key role in the explicit identification of the six position and momentum operators as coordinates for the quantum phase space \([a, q]\). For a fixed state, the definite noncommutative value carries mathematically the full information about the observable the theory contains. It is experimentally accessible, at least in principle.

The (Kählerian) geometrical picture of quantum mechanics has a slow development. Only after a major part of the last century we have a more comprehensive picture of it available, given in Ref. [10]. The paper also gives the extremely important result of an isomorphic description of the observable algebra as an algebra of the so-called Kählerian functions on the projective Hilbert space which generates Hamiltonian flows preserving the Kähler structure of the manifold, therefore also the metric. The metric is the one of a constant holomorphic sectional curvature fixed by the Planck constant \(\hbar\). Therefore, quantum noncommutativity can be seen as a curvature. A quantum observable as an operator on the Hilbert space can be matched to the set of complex values of the corresponding Kählerian function of all the points. We will present below all those features in terms of explicit coordinates to make them more easily appreciated even by the readers less accustomed to the use of more abstract mathematics. We will use the isomorphic algebraic structure to look at the differential geometric structure for the observable algebra, in terms of the noncommutative coordinates, matching to those of the Hilbert or the projective Hilbert space. That is like defining the notions of noncommutative geometry from the physics of quantum mechanics. The work formulates the idea that the noncommutative geometry of the observable algebra for quantum mechanics can be seen as an alternative, noncommutative coordinate, picture of quantum phase space, at least so long as the symplectic geometry is concerned.
In the next two sections, we review the geometric pictures of quantum mechanics on the Hilbert space and the projective Hilbert space, mostly in terms of explicit coordinates. We intend to give an optimal formulation of the known results, paying due attention to the proper physical dimensions of the various quantities. Our key references are Refs. [10–12]; other references consulted include Refs. [13–29]. The presentation sets the background for the following sections. In Sec. IV we present some details of the Kählerian functions for the noncommutative coordinate observables of \( \hat{x}_i \) and \( \hat{p}_i \) and their complex combinations \( \alpha_i \) and \( \bar{\alpha}_i \) under a convenient choice of coordinates for the Hilbert space and the projective Hilbert space. The explicit identification of a point from the noncommutative values of the coordinates will be illustrated. In Sec. V starts the exploration of the noncommutative differential geometric structure of the observable algebra in line with the above mentioned idea. This will then be extended further in Sec. VI by a kind of coordinate transformation/map between the infinite set of complex coordinates and the six noncommutative coordinates, which can be considered as an extension of the familiar Schrödinger and Heisenberg picture correspondence. The two sections present a complete noncommutative picture of the symplectic differential geometry of the quantum phase space. For background references on noncommutative geometry relevant to our formulation here, we note in particular Refs. [4, 30–35]. Our presentation of that is however to be seen as directly dictated by the physical theory. The last section concludes the paper.

II. GEOMETRIC PARTICLE DYNAMICS ON THE HILBERT SPACE

Let us first recall some basics of the symplecto-geometrical formulation of the quantum mechanics. To start with, the Schrödinger equation, as the equation of motion for a quantum state described by the vector \( |\phi\rangle \), can be casted into the form of the Hamiltonian equations of motion. Take an orthonormal basis \( |n\rangle \) for the Hilbert space \( \mathcal{H} \) (of countably infinite dimension, \( n = 0 \) to \( \infty \)), we have \( |\phi\rangle = \sum_n z^n |n\rangle \) where the complex coordinates \( z^n = \tilde{q}^n + i\tilde{p}^n \) of the state vector have (real coordinates) \( \tilde{q}^n \) and \( \tilde{p}^n \) satisfying

\[
\frac{d\tilde{q}^m}{dt} = \frac{\partial H(\tilde{q}^n, \tilde{p}^n)}{\partial \tilde{p}_m},
\]

\[
\frac{d\tilde{p}^m}{dt} = -\frac{\partial H(\tilde{q}^n, \tilde{p}^n)}{\partial \tilde{q}_m},
\]

(1)
where the Hamiltonian function \( H(\tilde{q}^n, \tilde{p}^n) \) is given by \( \frac{1}{2\hbar} \langle \phi | \hat{H} | \phi \rangle \) for the Hamiltonian operator \( \hat{H} \). Moreover, if we take \( |n\rangle \) to be the eigenstates of \( \hat{H} \) (assuming a discrete spectrum) with \( \hat{H} |n\rangle = \hbar \omega_n |n\rangle \), we have simply
\[
H(\tilde{q}^n, \tilde{p}^n) = \frac{1}{2\hbar} \langle \phi | \hat{H} | \phi \rangle = \frac{1}{2} \omega_n [(\tilde{q}^n)^2 + (\tilde{p}^n)^2] \quad \text{(with summation)} \tag{2}
\]

with
\[
\frac{d\tilde{q}^n}{dt} = \omega_n \tilde{p}^n \quad \text{(no summation)},
\]
\[
\frac{d\tilde{p}^n}{dt} = -\omega_n \tilde{q}^n \quad \text{(no summation).} \tag{3}
\]

Each of the configuration \( \tilde{q}^n \) and the momentum variables \( \tilde{p}^n \) behaves exactly in the same way as those of a harmonic oscillator with frequency \( \omega_n \) and the magnitude and phase of each complex coordinate \( z^n \) serve as an action-angle variable pair of the completely integrable quantum system. The equations of motion are equivalent to
\[
\frac{dz^m}{dt} = -2i \frac{\partial H(z^n, \bar{z}^n)}{\partial \bar{z}_m} = -i \omega_m \bar{z}^m \quad \text{(no summation)},
\]
\[
\frac{dz^m}{dt} = 2i \frac{\partial H(z^n, \bar{z}^n)}{\partial z_m} = i \omega_m z^m \quad \text{(no summation)}, \tag{4}
\]

which are just the conjugates of each other.\(^1\) The analysis above illustrates a couple of basic things very explicitly. From dimensional analysis, the proper physical unit for the coordinates is \( \sqrt{\hbar} \), which is the right unit for the position and momentum when expressed in the same unit. For any choice of \( \hat{H} \) beyond the physical energy observable, the ‘Hamiltonian equations of motion’ are preserved under a scaling of all coordinates with any complex number, suggesting a description with the symmetry reduction. The latter is the formulation on the projective Hilbert space given in the following section. Note that the symmetry of a complex phase rotation of a state vector in particular illustrates the lack of independent meaning of the notion of configuration space and momentum space. We have argued that the correct perspective is for the phase rotation symmetry to be taken as a part of the fundamental (quantum) relativity symmetry for quantum mechanics, which says the quantum phase space is the proper model of the physical space at the quantum level \([36, 37]\).

\[\text{Note that with these coordinates, we always have } z_m = \delta_{mn} z^n \text{ and } \frac{\partial}{\partial z_m} = \delta^{nm} \frac{\partial}{\partial z^n} \text{, independently of the metric; writing the tangent vector } \partial_n = \frac{\partial}{\partial z^n}, \text{ the covector } \bar{\partial}^n \text{ is metric dependent and cannot be taken as } \frac{\partial}{\partial z^n}.\]
The Hilbert space $\mathcal{H}$ can be taken as a Kähler manifold with a trivial metric $G_{\bar{m}n} = \frac{1}{2} \delta_{mn}$, and a symplectic form $\tilde{\omega}_{mn} = i G_{\bar{m}n}$. The tangent space of a vector space can be identified with itself. $G$ and $\tilde{\omega}$ correspond to the real and imaginary part of the inner product, i.e.

$$\langle \psi | \phi \rangle = G(|\psi \rangle , |\phi \rangle) + i \tilde{\omega}(|\psi \rangle , |\phi \rangle) ,$$

with $G(|\psi \rangle , |\phi \rangle) = \tilde{\omega}(|\psi \rangle , i |\phi \rangle) = -\tilde{\omega}(i |\psi \rangle , |\phi \rangle)$. The equations of motion have the standard form

$$\frac{d z^m}{dt} = -2i \delta^{mn} \partial_n H = \tilde{\omega}^{m\bar{n}}(d H)_{\bar{n}} = \tilde{X}_H^m(z) ,$$

for the Hamiltonian function $H$ corresponding to the operator $\hat{H}$, where $\tilde{X}_H$ is the Hamiltonian vector field. Note that $\tilde{\omega}^{m\bar{n}} = -i G^{m\bar{n}} = -2i \delta^{mn}$. The above equation is just a geometrical/coordinate description of the action of the Schrödinger vector field $\tilde{X}_\hat{H} = \frac{1}{i \hbar} \hat{H}$ on a state vector. Actually, we have for a tangent vector $|Y\rangle$ to $\mathcal{H}$ at $|\phi\rangle$

$$dH(|\phi\rangle)(Y) = \left. \frac{d}{dt} H(|\phi\rangle + t |Y\rangle) \right|_{t=0} = \frac{1}{2\hbar} \left( \langle Y|\hat{H}|\phi\rangle + \langle \phi|\hat{H}|Y\rangle \right)$$

$$= G\left( \frac{1}{\hbar} \hat{H} |\phi\rangle , |Y\rangle \right) = \tilde{\omega}(\tilde{X}_{\hat{H}}, Y)(|\phi\rangle) ,$$

(7) for Hermitian $\hat{H}$. We have been only passing between the geometrical language and the one of the operators and state vectors on $\mathcal{H}$. Each vector is a point of the space; an observable, as an operator, is completely characterized by its values on all possible states and should be seen as a function on the space as suggested by the symplectic formulation. How to properly think about those values and the issue of noncommutativity and Heisenberg uncertainties is a question we will address carefully below. For any Hermitian operator $\hat{K}$, we can introduce the Hamiltonian function for the symplectic geometry, $K(|\phi\rangle) = \frac{1}{2\hbar} \left( \langle \phi|\hat{K}|\phi\rangle \right)$. The Poisson bracket between two functions $H$ and $K$ is given by $\{H, K\}_\omega = \tilde{\omega}(\tilde{X}_H, \tilde{X}_K)$, defined and considered for smooth complex valued functions, though the Hamiltonian function for a (Hermitian) Hamiltonian operator is real. Extending the observable algebra to allow the complex linear combinations of Hermitian operators and consider it as having a $C^*$-algebra structure is the generally adopted approach to the algebraic or noncommutative geometric formulation of quantum mechanics. Such non-Hermitian operators are not quite any less ‘observable’ compared to the Hermitian parts in the linear combination. A Hamiltonian function of such an operator is a complex function the real and imaginary parts of which
are Hamiltonian functions for Hermitian operators. Notice though, not all Hamiltonian functions correspond to operators, Hermitian or otherwise, in the observable algebra. The functions that do are called Kählerian functions, and have the Hamiltonian flows preserving the Kähler structure and therefore the metric, giving isometries. Geometrically, we have

\[ \{H, K\} = (dH)_{m} \tilde{\omega}^{mn} (dK)_{n} + (dH)_{n} \tilde{\omega}^{mn} (dK)_{m} = -2i \left( \partial_{m} H \delta^{mn} \bar{\partial}_{n} K - \partial_{m} K \delta^{mn} \bar{\partial}_{n} H \right), \]  

(8)

where \( \bar{\partial}_{m} = \frac{\partial}{\partial z_{m}} = \frac{\partial}{\partial \bar{z}_{m}} \). Using the coordinates and the matrix elements of the operators, one can easily obtain

\[ \partial_{m} H G^{mn} \partial_{n} K (|\phi\rangle) = \frac{1}{2\hbar^{2}} \left\langle \phi | \hat{H} \hat{K} | \phi \right\rangle . \]  

(9)

This is a very simple but remarkable result, the key result behind the whole analysis. The first application of it gives

\[ \frac{d}{dt} K(|\phi\rangle) = \{K, H\} \omega(|\phi\rangle) = \frac{1}{2i\hbar^{2}} \left\langle \phi | [\hat{K}, \hat{H}] | \phi \right\rangle . \]  

(10)

The latter is equivalent to the Heisenberg equation of motion under the Hamiltonian \( \hat{H} \), namely \( \frac{d}{dt} \hat{K} = \frac{1}{i\hbar} [\hat{K}, \hat{H}] \). The symplecto-geometrical form, however, works also for the complex functions, which suggests to include the non-Hermitian operators in the Heisenberg equation of motion. In terms of the corresponding Schrödinger vector fields (for Hermitian operators), one can get the same equation as

\[ \{K, H\} \omega(|\phi\rangle) = \omega \left( \frac{1}{i\hbar} \hat{K}, \frac{1}{i\hbar} \hat{H} \right) (|\phi\rangle) = \frac{1}{\hbar^{2}} \frac{1}{2i} \left( \left\langle \hat{K} \phi | \hat{H} \phi \right\rangle - \left\langle \hat{H} \phi | \hat{K} \phi \right\rangle \right) . \]  

(11)

Notice that the Schrödinger vector field expressions from Eq.(7) is valid only for Hermitian operators. Denote the Hamiltonian function for an operator \( \beta(\hat{p}_{i}, \hat{x}_{i}) \), for the three position and three momentum operators with \([\hat{x}_{i}, \hat{p}_{j}] = i\hbar \delta_{ij} \), by \( H_{\beta} \) and the Hamiltonian vector field by \( \tilde{X}_{\beta} \). Note that real functions correspond to Hermitian operators and the Hermitian conjugate of \( \beta \) is otherwise given by \( \bar{\beta} \). Moreover \( H_{\bar{\beta}} = \bar{H}_{\beta} \). The last part of Eq.(10) can be written as

\[ H_{[\beta, \gamma]} = \frac{1}{2\hbar} \left\langle \phi | [\beta, \gamma] | \phi \right\rangle = i\hbar \{H_{\beta}, H_{\gamma}\} \omega . \]  

(12)

While a generic symplectic manifold may not possess a Riemannian metric, with the Kähler structure, however, we have the latter being intimately connected to the symplectic structure. In fact, a symplectic form together with a compatible complex structure on a
manifold uniquely fixes the metric. We have seen that the Poisson algebra of $H_\beta$ functions gives an isomorphic description of the Poisson algebra of operators $\beta$ with the commutator, multiplied by $\frac{1}{i\hbar}$, taken as the Poisson bracket. Our derivation here starts with the remarkable result of Eq. (9), expressed in terms of the metric. The result actually gives a full isomorphism between the algebra of $H_\beta$ functions and the observable algebra.

One can define the Riemann bracket

$$\{H_\beta, H_\gamma\}_G\langle\phi\rangle := \frac{1}{2\hbar^2} \langle\phi|[\beta, \gamma]_+\phi\rangle = \frac{1}{\hbar}H_{[\beta, \gamma]_+}$$

in terms of the anticommutator. The latter can be seen as the Riemann bracket for the observable algebra. Note that the Jordan algebraic product, called Jordan bracket, is exactly half the anticommutator. We write

$$(\beta, \gamma)_J = \frac{1}{2}[\beta, \gamma]_+ .$$

It is exactly the nonassociative Jordan product for the operators which is half the anticommutator. Furthermore, one can write the so-called Kähler product on the space of $H_\beta$ functions in the simple form

$$H_\beta \star K H_\gamma = H_{\beta\gamma} ;$$

that is to say, the Kähler product given in terms of $\star_K$ matches the structure of the operator product as the basic product between observables. It is exactly given by Eq. (9), i.e. $\hbar \partial m H_\beta \mathcal{G}^{m\bar{n}} \partial n H_\gamma$. Obviously, we have

$$H_\beta \star_K H_\gamma = \frac{\hbar}{2}\{H_\beta, H_\gamma\}_G + \frac{i\hbar}{2}\{H_\beta, H_\gamma\}_\omega = H_{([\beta, \gamma]_J)} + \frac{1}{2}H_{[\beta, \gamma]_+} + \frac{H_{[\beta, \gamma]_+} + H_{[\beta, \gamma]}}{2} ,$$

as $\beta\gamma = (\beta, \gamma)_J + \frac{1}{2}[\beta, \gamma]$, which is just the splitting of the operator product into the symmetric and antisymmetric parts.

III. GEOMETRIC PARTICLE DYNAMICS ON THE PROJECTIVE HILBERT SPACE

The linearity of the Schrödinger equation says that all the state vectors differing by a nonzero constant factor behave in exactly the same way. The zero vector, however, does not

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\(^2\) We have the even more suggestive form $H_\beta \star_K H_\gamma = H_{\beta\gamma}$, for a formulation of the observable algebra as the Moyal star product algebra of functions of the real variables $\beta(p_i, x_i)$ and $\gamma(p_i, x_i)$.

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correspond to a sensible physical state. This suggests a symmetry reduction of the symplectic system to one of one lower complex dimension, to the projective Hilbert space $\mathcal{P}$ with each ray of vectors $[\phi]$ identified as a point. The latter space is an infinite dimensional complex projective space ($\mathbb{C}P^\infty$), still a Kähler manifold. As the space of pure states, it is a geometric structure dual to the noncommutative $C^*$-algebra as the algebra of observables. Any set of $z^n$ serves as a set of homogeneous coordinates. Natural atlas of affine coordinates is given in the form $w^n = \frac{\tilde{z}^n}{|z|^2}$ with $\tilde{n}$ counting from 1 to $\infty$. Points corresponding to $[\phi]$ with vanishing $z^0$ all have $w^n$ as infinity, although $\mathcal{P}$ is actually compact. If fact, one only has to switch to the another similar coordinate chart, for example one obtained by a swapping the $z^n$ coordinates first, to give such points finite coordinate values. $\{z^n\}$ as a system of coordinates on $\mathcal{P}$ with redundancy has the benefit of being globally applicable. Besides, the Hilbert space picture of quantum mechanics is more than a convenient redundant description. Mathematically, it is the natural structure to arrive at from the point of view of the representation theory of the observable algebra, or that of the fundamental symmetry behind, which can or should be identified as the (quantum) relativity symmetry $[36, 37]$. Physically, the notion of the Berry’s phase clearly indicates that there are nontrivial dynamical issues involving (the changes in) the $\theta$ coordinate for an overall phase factor $[27, 28]$ which cannot be described on $\mathcal{P}$ alone.

From the geometry of the complex projective spaces $\mathcal{P}$, we have the standard Fubini-Study metric given by

$$ds^2 = 2 \tilde{g}_{\tilde{m}\tilde{n}} d\tilde{w}^\tilde{m} d\tilde{w}^\tilde{n} = \frac{2\hbar}{1 + |w|^2} \left( \delta_{\tilde{m}\tilde{n}} - \frac{\tilde{w}_{\tilde{m}} \tilde{w}_{\tilde{n}}}{1 + |w|^2} \right) d\tilde{w}_{\tilde{m}} d\tilde{w}_{\tilde{n}}. \quad (17)$$

with $|w|^2 = \bar{w}_n w^n$. Note that we have supplemented the mathematical result by an $\hbar$ to keep the right physical dimension for $ds^2$, since $w^n$, unlike $z^n$, has no length dimension, and adopted the factor of 2, which fits in with the physics results presented below most nicely. The symplectic form $\omega$ is given by the Kähler form, with $\omega_{\tilde{m}\tilde{n}} = ig_{\tilde{m}\tilde{n}}$. We have also the inverse

$$g_{\tilde{m}\tilde{n}} = \frac{1}{\hbar} (1 + |w|^2) \left( \delta_{\tilde{m}\tilde{n}} + \bar{w}_{\tilde{n}} w_{\tilde{m}} \right). \quad (18)$$

In terms of the $\{z^n\}$ set of the homogeneous coordinates, we can write the Fubini-Study metric as

$$ds^2 = 2\tilde{g}_{\tilde{m}\tilde{n}} dz^m d\bar{z}^n = \frac{2\hbar}{|z|^2} \left( \delta_{mn} - \frac{z_m \bar{z}_n}{|z|^2} \right) dz^m d\bar{z}^n. \quad (19)$$
Note that $\text{det} \tilde{g} = 0$; the metric is hence formally degenerate. Of course $g_{\tilde{m}\tilde{n}}$ is in itself not degenerate. One can describe a point in $\mathcal{P}$ as the equivalent class $[\phi]$ of the Hilbert space vectors $|\phi\rangle$, each being a constant multiple of the others. We have

$$ds^2 = 2\hbar \frac{\langle \delta \phi | \delta \phi \rangle}{\langle \phi | \phi \rangle} - 2\hbar \frac{\langle \delta \phi | \phi \rangle \langle \phi | \delta \phi \rangle}{\langle \phi | \phi \rangle^2},$$

which corresponds to a distance between the two state vectors as given by

$$s(|\phi\rangle, |\phi'\rangle) = \sqrt{2\hbar \cos^{-1}\sqrt{\frac{||\phi|\phi'\rangle|^2}{||\phi\rangle|\phi\rangle||\phi'|\phi'\rangle||\phi|\phi\rangle}}}.$$  

(21)

It depends, of course, only on $[\phi]$ and $[\phi']$ and is the geodesic distance between the two points in $\mathcal{P}$ as the quantum model of the physical space. In the conventional picture of quantum mechanics, it characterizes the distinguishability of the physical states. The maximum value of $s$ is given by $\pi \sqrt{\frac{1}{2}}$, realized between any two orthogonal state vectors.

The metric in Eq.(19) is exactly that of Eq.(20) expressed in terms of the coordinates of $\mathcal{H}$, and the one in Eq.(17) in terms of the affine coordinates of $\mathcal{P}$. One may then think of $\mathcal{H} - \{0\}$ as a complex line bundle over the base space $\mathcal{P}$ and of the degenerate metric as one for the whole bundle which vanishes on the vertical tangent vectors. Hence, the distance between the points within the same fiber is always zero. The idea that each $[\phi]$, rather than an individual $\phi$, represents a physical state suggests that the Fubini-Study metric, rather than the trivial Hilbert space metric, is the proper metric for the distance between physical states, or state vectors, in quantum mechanics. The $\{r, \theta, \bar{w}^\alpha, \bar{w}^\bar{\alpha}\}$ set with $r = |z|$ and $z^a = re^{i\theta}(1 + |w|^2)^{-1/2}$, as a full set of coordinates for $\mathcal{H}$ may be used best to illustrate that. The $\theta$ coordinate is an overall phase factor for a state vector and $r$ its magnitude. A transformation in the $\theta$ coordinate maintains the inner product of two vectors, hence also the symplectic form and the metric. The coordinate is therefore somewhat redundant or irrelevant to the geometric structure as well as to the dynamics. The identity operator as a ‘Hamiltonian’ operator generates exactly a change in all state vectors by a ‘translation’ of their $\theta$ coordinates by the value $-\frac{\alpha}{\hbar} t$ producing the circle action of the group of $\theta$ transformations. $H_1 = \frac{r^2}{2\hbar}$ is the corresponding Hamiltonian function. We have a standard case of symmetry reduction of a circle action with $\mathcal{P}$ being isomorphic to the quotient of any regular level set of nonzero value of $H_1$ and the circle $S^1$, or $(\mathcal{P}, \omega)$ the corresponding symplectic quotient of $(\mathcal{H}, \tilde{\omega})$ at the constant $r$. That is to say, all the Hamiltonian flows generated by Hermitian operators stay on spheres of fixed radius $r$, with

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the dynamical picture on the each sphere of nonzero radius essentially identical. Note that all \( H_s \) are \( \theta \)-independent. They actually have the form \( r^2 \) times a function of \( \bar{w}^n \) and \( \bar{w}^\bar{n} \). The latter is essentially the reduced Hamiltonian function on \( P \) which is the focus of this section.

Going back to the metric structure, the trivial metric of \( H \) can be written in the form
\[
d s_{(r)}^2 = d r^2 + \frac{r^2}{2 \hbar} d s_{(S)}^2 ,
\]
where \( d s_{(S)}^2 \) is the metric on the sphere at \( r^2 = 2 \hbar \), or the \( r \)-independent part of the full metric. The \( 2 \hbar \) factor is essentially the same one as in the case of the Fubini-Study metric. The vector field \( \partial_r \) is vertical, or orthogonal to the sphere. The metric tensor on \( S \) is hence given by
\[
\frac{2 \hbar}{r^2} (G - \partial_r \otimes \partial_r) - \frac{2 \hbar}{r^4} \partial_\theta \otimes \partial_\theta ,
\]
which is exactly the metric given in Eq.\ref{eq:19} upon substituting \( \partial_r = \frac{z^m}{r} \partial_m + \frac{\bar{z}^m}{r} \partial_{\bar{m}} \) and \( \partial_\theta = i z^m \partial_m - i \bar{z}^m \partial_{\bar{m}} \). One can also take \( r(\partial_r) \) together with \( \partial_\theta \) to be the Killing vector fields for the conformal metric \( \tilde{G}_{m\bar{n}} = \frac{2 \hbar}{r^2} G_{m\bar{n}} \) on \( H - \{0\} \) for a direct reduction and obtain the same result. The procedure is a powerful one, allowing us to get the other corresponding tensors on \( P \), including the useful ‘inverse metric’, which can be obtained as
\[
\tilde{g}^{m\bar{n}} = \tilde{G}^{m\bar{n}} - \frac{1}{2 \hbar} r^2 \partial_r m \partial_{\bar{r}} - \frac{1}{2 \hbar} \partial_\theta m \partial_{\bar{\theta}} = \frac{1}{\hbar^2} (|z|^2 \delta^{m\bar{n}} - z^m \bar{z}^n) .
\] (22)

Note that the singular or degenerate metric \( \tilde{g}_{m\bar{n}} \) in Eq.\ref{eq:19} cannot be inverted. We will apply the procedure extensively in our analysis. More details on the involved mathematics are given in the Appendix, for readers’ convenience.

The Fubini-Study metric on \( P \), besides having the similar role as the metric on \( H \) for defining a Kähler product among functions representing the operators as observables, has also an important role to play in relation to the quantum covariance or the Heisenberg uncertainty, as illustrated below.

Functions on \( P \) can be defined in terms of functions on \( H \) which are independent of the \( r \) and \( \theta \) coordinates. In particular, we consider the so-called Kählerian functions on \( P \) given by
\[
f_\beta([\phi]) = 2 \hbar \frac{H_\beta(|\phi\rangle)}{\langle \phi | \phi \rangle} = \frac{\langle \phi | \beta | \phi \rangle}{\langle \phi | \phi \rangle} ,
\] (23)
each corresponding to the function of the expectation values for the operator $\beta$. Note that we introduced the $\hbar$ factor in the above definition so that $f_\beta$ has the same physical dimension as $H_\beta$ and the operator $\beta$ or its matrix elements. The factor 2 is just for the convenience of having the corresponding functions of the constant operators as the same constants. An extra constant factor in the definition of $f_\beta$ otherwise does not really matter and it may seem to be natural to omit the factor of two so that $f_\beta$ agrees with $H_\beta$ for a normalized $\phi$ (up to $\hbar$). Our choices of exact forms of the $H_\beta$ and $f_\beta$ functions result in a $\frac{r^2}{2\hbar}$ factor difference, i.e. $r^2 = 2\hbar$ is where the two functions have the same value, and the $2\hbar$ factor shows up as the natural way to obtain the various metrics discussed above in the preferred form. Using either the $w^\tilde{n}$ coordinates and $g^\tilde{m}\tilde{n}$ or the $z^n$ with $\tilde{g}^{\tilde{m}\tilde{n}}$, we can easily see that for the Kähler product is defined by

$$f_\beta \ast \kappa f_\gamma = f_\beta f_\gamma + \hbar \partial_{\tilde{m}} f_\beta g^{\tilde{m}\tilde{n}} \partial_{\tilde{n}} f_\gamma = f_\beta f_\gamma + \hbar \partial_{\tilde{m}} f_\beta \tilde{g}^{\tilde{m}\tilde{n}} \partial_{\tilde{n}} f_\gamma ,$$  \hspace{1cm} (24)

to give

$$f_\beta \ast \kappa f_\gamma = f_\beta f_\gamma .$$  \hspace{1cm} (25)

Again, from the antisymmetric and symmetric parts, we have the Poisson and Riemann brackets

$$\{f_\beta, f_\gamma\}_\bar{\omega} = \{f_\beta, f_\gamma\}_\omega = \frac{1}{i\hbar} f_{[\beta,\gamma]} ,$$

$$\{f_\beta, f_\gamma\}_{\tilde{g}} = \{f_\beta, f_\gamma\}_g = \frac{1}{\hbar} f_{[\beta,\gamma]_g} - \frac{2}{\hbar} f_{\beta} f_{\gamma} ,$$  \hspace{1cm} (26)

with

$$f_\beta \ast \kappa f_\gamma = f_\beta f_\gamma + \frac{\hbar}{2} \{f_\beta, f_\gamma\}_g + \frac{i\hbar}{2} \{f_\beta, f_\gamma\}_\omega = \frac{f_{[\beta,\gamma]_g} + f_{[\beta,\gamma]_\omega}}{2}$$  \hspace{1cm} (27)

(and the Jordan bracket $f_{[\beta,\gamma]_J} = \frac{1}{2} f_{[\beta,\gamma]_g}$). In addition, we have $\{f_\beta, f_\beta\}_g = \frac{2}{\hbar} (\Delta \beta)^2$, from the Heisenberg uncertainty $(\Delta \beta)^2 = \frac{\langle \phi | (\beta - \langle \beta \rangle)^2 | \phi \rangle}{\langle \phi | \phi \rangle} - \left( \frac{\langle \phi | \beta | \phi \rangle}{\langle \phi | \phi \rangle} \right)^2$ for the operator $\beta$. More directly, one can introduce the quantum covariance between the two operators as

$$\text{Cov}(\beta, \gamma) = \langle (\beta - \langle \beta \rangle, \gamma - \langle \gamma \rangle) \rangle = \frac{\hbar}{2} \{f_\beta, f_\gamma\}_g ,$$

in relation to which we have the inequality

$$(\Delta \beta)^2 (\Delta \gamma)^2 \geq \left( \frac{\hbar}{2} \{f_\beta, f_\gamma\}_\omega \right)^2 + \left( \frac{\hbar}{2} \{f_\beta, f_\gamma\}_g \right)^2 ,$$

12
as a stronger version of the Heisenberg uncertainty principle. All this illustrates the key role of the Riemann bracket or the metric on \( \mathcal{P} \) in quantum mechanics compared to which the metric \( G \) on \( \mathcal{H} \) is much less physically relevant.

We see here another isomorphic description of the observable algebra, namely the \( f_\beta \) functions with the Kähler product. To compare results here with those of the \( H_\beta \) functions with \( H_\beta = \frac{r^2}{2\hbar} f_\beta \), the first point to note is that

\[
\{ H_\beta, H_\gamma \}_\tilde{\omega} = \frac{i}{\hbar} \frac{r^2}{2\hbar} \{ f_\beta, f_\gamma \} \tilde{\omega} = \frac{r^2}{2\hbar} \{ f_\beta, f_\gamma \} \omega ,
\]

where we have exact equality of the two Poisson brackets for \( r^2 = |z|^2 = 2\hbar \). This is in line with the view of \( \mathcal{P} \) as the symplectic reduction of \( \mathcal{H} \). In fact, applying explicitly the coordinate transformation from \((z^n, \bar{z}^\alpha)\) to \((r, \theta, w^\alpha, \bar{w}^\bar{\alpha})\) onto the Hamiltonian vector field \( \tilde{X}_{H_\beta} (\equiv \tilde{X}_\beta) \) gives

\[
\tilde{X}_{H_\beta} = \frac{r^2}{2\hbar} \tilde{X}_\beta + \frac{f_\beta}{2\hbar} \tilde{X}_r = \tilde{X}_\beta + \frac{f_\beta}{2\hbar} \tilde{X}_r = X_\beta + \tilde{X}_\beta \partial_\theta + \frac{f_\beta}{2\hbar} \tilde{X}_r ,
\]

with \( \tilde{X}_\beta \) being the horizontal lift of \( X_\beta \), from the perspectives of the Killing reduction \[38\] or sub-Riemannian structure \[28\] as the relation between \( \mathcal{P} \) and \( \mathcal{H} - \{0\} \). (More results on the Killing reduction are presented in the Appendix.) We have

\[
\tilde{X}_r = \tilde{X}_{H_2} = -2\partial_\theta
\]

(the Hamiltonian vector field for the operator \( 2\hbar I \)), and \( \tilde{X}_\beta \) is the Hamiltonian vector field of \( f_\beta \), taken as a Hamiltonian function on \((\mathcal{H}, \tilde{\omega})\) and \( X_\beta (\equiv X_{H_\beta}) \) the Hamiltonian vector field on \((\mathcal{P}, \omega)\). That is in exact correspondence with Eq.(28) since \( \partial_\theta (H_\beta) = \partial_\theta (f_\beta) = 0 \) for any operator \( \beta \). None of the vector fields has a \( \partial_r \) component, which is to be expected from the unitary flow point of view. It may also be interested to note that

\[
\tilde{X}_\beta^\theta = \frac{1 + |w|^2}{\hbar} (w^\alpha \partial_\alpha f + w^{\bar{\alpha}} \partial_{\bar{\alpha}} f) = \left( \frac{1}{z^\beta} \partial_\beta f + \frac{1}{\bar{z}^{\bar{\beta}}} \partial_{\bar{\beta}} f \right) .
\]

It is more convenient to focus on the covectors dual to the Hamiltonian vector fields for which we have the expressions of the universal form given by the example \( \tilde{X}_{\beta n} = i\partial_n f_\beta \), \( i.e. \) components are given by the coordinate derivatives of Hamiltonian function multiplied by the imaginary unit \( i \). Moreover, their covariant derivatives satisfy

\[
\nabla_{\tilde{m}} \tilde{X}_{\beta n} = -\nabla_{\tilde{n}} \tilde{X}_{\beta m} = -i\partial_m \partial_\beta f_\beta \quad \text{and} \quad \nabla_{\tilde{m}} \tilde{X}_{\beta n} = \nabla_{\tilde{n}} \tilde{X}_{\beta m} = 0 .
\]

The form in term of the corresponding Hamiltonian functions is common to all Kähler manifolds. For the case at hand, with the function being Kählerian,
the first and the second derivatives are all the independent derivatives\(^{[11, 12, 39]}\), hence their values, together with that of the zeroth order derivative, at a point give a local representation of the full function as the Taylor series. The collection for all Kählerian functions can be seen as another isomorphic description of the observable algebra\(^{[11, 12]}\). Since there is such a local representation for the algebra for each state, we think it should be interpreted as the noncommutative algebra of the values of the observables\(^{[8, 9]}\).

**IV. ON THE NONCOMMUTATIVE COORDINATE FUNCTIONS**

So far, the coordinate systems we use on \(\mathcal{H}\) and \(\mathcal{P}\) are quite generic. Now, we take a specific coordinate system to be used for some explicit results of the basic observables \(\alpha_i = \hat{x}_i + i\hat{p}_i, \bar{\alpha}_i = \hat{x}_i - i\hat{p}_i\). Intuitively, these are the noncommutative coordinates of the ‘phase space’ for a quantum particle with the other observables to be seen as functions of them. Such functions will be denoted by Greek letters, \(\alpha, \beta, \gamma\). As functions of the noncommutative coordinate observables, the position and momentum operators, they are of course operators. The latter fact readers should bear in mind when we talk about them as functions, and we discuss their derivatives with respect to the operator coordinate variables. And the term function is used here only in the formal sense in relation to those variables, without the connotation of functional values, certainly not one as numbers. Recall that we are talking about the observable algebra in the extended, mathematical, sense, as a \(C^*\)-algebra. Instead of including only the ‘physical observables’ as Hermitian operators, that includes also like their complex linear combinations, starting from the basic ‘complex’ coordinates variables \(\alpha_i\) and \(\bar{\alpha}_i\), which we choose to use instead of the position and momentum operators. All structure can be described equally in terms of the latter, and the simple linear relation between them and the ‘complex’ ones guarantees the translation between most of the results put in terms of the different coordinate set, though some physicists would only find results in terms of \(\hat{x}_i\) and \(\hat{p}_i\) easily comprehensible. Our main task is to illustrate how that noncommutative ‘phase space’ and usual familiar quantum phase space can be identified as the same symplectic geometry.

Consider \(\hat{N}_i = \frac{1}{2\hbar} \bar{\alpha}_i \alpha_i = \frac{1}{2\hbar} (\hat{x}_i^2 + \hat{p}_i^2) - \frac{1}{2},\) with \([\hat{N}_i, \alpha_j] = -\delta_{ij} \alpha_j\) and \([\hat{N}_i, \bar{\alpha}_j] = \delta_{ij} \bar{\alpha}_j\). Take the simultaneous eigenstates \(|n_1, n_2, n_3\rangle\) of the number operators \(\hat{N}_i, i = 1\) to \(3\) (i.e. satisfying \(\hat{N}_i |n_1, n_2, n_3\rangle = n_i |n_1, n_2, n_3\rangle\) for nonnegative integers \(n_i\)) as a countable orthonormal basis of
the Hilbert space. A state is then described by the complex coordinates \( z^{(n_1, n_2, n_3)} \), i.e.

\[
|\phi\rangle = \sum_{n_1, n_2, n_3} z^{(n_1, n_2, n_3)} |n_1, n_2, n_3\rangle .
\]  

(32)

The coordinates constitute a set of homogeneous coordinates (of the type \( z^n \)) on the projective Hilbert space from which the set of affine coordinates can be chosen as \( w^{(n_1, n_2, n_3)} = \frac{z^{(n_1, n_2, n_3)}}{z(0,0,0)} \). Let us introduce the short hand index notation \([n]\) standing in for \((n_1, n_2, n_3)\), and further define \( z_{i\pm}^{[n]} = z^{(n_1, n_2, n_3)} \), where \( z_{1\pm}^{(n_1, n_2, n_3)} \equiv z^{(n_1 \pm 1, n_2, n_3)} \ldots \), etc. We have then the compact expressions

\[
H_{\alpha_i} = \frac{1}{2\hbar} \sum_{[n]} \sqrt{2\hbar m_{i\pm} z_{i\pm}^{[n]} z_{i\pm}^{[n]}} = \frac{1}{2\hbar} \sum_{[n]} \sqrt{2\hbar(n_i + 1)} z_{i\pm}^{[n]} z_{i\pm}^{[n]} = H_{\alpha_i} ,
\]

and

\[
\tilde{f}_{\alpha_i} = \sum_{[n]} \frac{\sqrt{2\hbar m_{i\pm} \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n]}}{|\tilde{z}|^2} = \sum_{[n]} \frac{\sqrt{2\hbar(n_i + 1) \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n]}}{|\tilde{z}|^2} = \frac{\sqrt{2\hbar m_{i\pm} \tilde{w}_{i\pm}[n] \tilde{w}_{i\pm}[n]}}{1 + |\tilde{w}|^2} = \tilde{f}_{\alpha_i} .
\]

(33)

(34)

Note that we have used \([n]\) here including the expression in terms of the \( w^{[n]} \) coordinates where we use \( w^{[0]} \equiv 1 \). We have for the components of the covectors

\[
\tilde{X}_{[n]}^{\alpha_j} = \frac{i}{\sqrt{2\hbar m_{i\pm}}} \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n] = \tilde{X}_{[n]}^{\alpha_j} ,
\]

\[
\tilde{X}_{[n]}^{\alpha_j} = \frac{i}{|\tilde{z}|^2} (\sqrt{2\hbar n_{i\pm} \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n]}) = \tilde{X}_{[n]}^{\alpha_j} ,
\]

\[
X_{[n]}^{\alpha_j} = \frac{i}{(1 + |\tilde{w}|^2)} (\sqrt{2\hbar n_{i\pm} \tilde{w}_{i\pm}[n] \tilde{w}_{i\pm}[n]}) = X_{[n]}^{\alpha_j} \quad ([n] \neq [0]) ,
\]

\[
\tilde{X}_{[n]}^{\alpha_j} = -\frac{i}{\sqrt{2\hbar m_{i\pm}}} \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n] = \tilde{X}_{[n]}^{\alpha_j} ,
\]

\[
\tilde{X}_{[n]}^{\alpha_j} = -\frac{i}{|\tilde{z}|^2} (\sqrt{2\hbar(n_{i\pm} + 1) \tilde{z}_{i\pm}[n] \tilde{z}_{i\pm}[n]}) = \tilde{X}_{[n]}^{\alpha_j} ,
\]

\[
X_{[n]}^{\alpha_j} = \frac{i}{(1 + |\tilde{w}|^2)} (\sqrt{2\hbar(n_{i\pm} + 1) \tilde{w}_{i\pm}[n] \tilde{w}_{i\pm}[n]}) = X_{[n]}^{\alpha_j} \quad ([n] \neq [0]) ,
\]

(35)
and the those for the vectors

\[
\tilde{X}^{[n]}_{\alpha_i} = -i \sqrt{\frac{2(n_j + 1)}{\hbar}} z^{[n]}_{j+} = \overline{X}^{[n]}_{\alpha_i},
\]

\[
\tilde{X}^{[n]}_{\bar{\alpha}_j} = -i \sqrt{\frac{2(n_j + 1)}{\hbar}} \bar{z}^{[n]}_{j+} + i \frac{1}{\hbar} z^{[n]} f_{\alpha_j} = \overline{X}^{[n]}_{\bar{\alpha}_j},
\]

\[
X^{[n]}_{\alpha_i} = -i \sqrt{\frac{2(n_j + 1)}{\hbar}} \bar{w}^{[n]}_{j+} + i \frac{2}{\hbar} w^{[0]}_{j+} w^{[n]} = X^{[n]}_{\alpha_j} \quad ([n] \neq [0]),
\]

\[
\tilde{X}^{[n]}_{\alpha_i} = i \sqrt{\frac{2n_j}{\hbar}} \bar{z}^{[n]}_{j-} = \overline{X}^{[n]}_{\alpha_i},
\]

\[
\tilde{X}^{[n]}_{\bar{\alpha}_i} = i \sqrt{\frac{2n_j}{\hbar}} \bar{z}^{[n]}_{j-} - i \frac{1}{\hbar} \bar{z}^{[n]} f_{\alpha_j} = \overline{X}^{[n]}_{\bar{\alpha}_j},
\]

\[
X^{[n]}_{\bar{\alpha}_i} = i \sqrt{\frac{2n_j}{\hbar}} \bar{w}^{[n]}_{j-} = X^{[n]}_{\alpha_j} \quad ([n] \neq [0]).
\]

(36)

Some of the expressions here involve the undefined ‘coordinate’ \(z^{[0]}_{j-}\) which is an abuse of notation. It always goes along with a \(\sqrt{0}\) factor and the terms vanish. The results will be useful in analyses involving the noncommutative coordinates.

Let is also emphasize explicitly that the vector space of states we are working is one of countable infinite dimensions as the span of the Fock states of the harmonic oscillator problem, as used above. In terms of the Schrödinger wavefunctions, it is the space of the rapidly decreasing functions rather than just square-integrable ones. \(\hat{x}_i\) and \(\hat{p}_i\) are well defined Hermitian operators, and their Kählerian functions essentially as expectation value functions are well defined. Linearity gives the well defined Kählerian functions for the \(\alpha_i\) and \(\bar{\alpha}_i\) which have been explicitly given. For the sake of definiteness, we can at this point take the observable algebra simply as the Weyl algebra of polynomials in \(\hat{x}_i\) and \(\hat{p}_i\), or \(\alpha_i\) and \(\bar{\alpha}_i\). Any element of it then has the as expectation value functions well defined, for example through the Kähler products of those for the coordinate observables. For full mathematical rigor, there are issues on topological completeness to be concerned and the full observable algebra should include tempered distributions instead of just the smooth functions. However, our treatment here is considered good enough for the illustration of the key picture.

At the end of the last section, we mentioned that the full Kählerian function, which is a representation of the corresponding operator as a quantum observable, is locally determined by the functional value and the values of the coordinate derivatives of the first two orders. Here above, we have given explicit results for the Kählerian functions of the noncommutative coordinate operators, such as \(H_{\alpha_i}\) and \(f_{\alpha_i}\) of the operator coordinates \(\alpha_i\). Seeing \(\alpha_i\) and \(\bar{\alpha}_i\) as
functions of $\hat{x}_i$ and $\hat{p}_i$ or otherwise, they are intuitive coordinate observables. The Kählerian functions $H_{\alpha_i}$ and $H_{\bar{\alpha}_i}$, or $f_{\alpha_i}$ and $f_{\bar{\alpha}_i}$ functions, are hence in a way noncommutative coordinates for the $\mathcal{H}$ and $\mathcal{P}$. The noncommutativity is of course to be seen with the Kähler products. We note also that the Kählerian functions are not ‘symbols’ of the corresponding operators. Though the Kähler product share some similarity with the Moyal star product, they are quite different things and not to be confused with one another. The only place we touch on the Moyal star product and the ‘symbols’ is footnote 2. They are otherwise not used in the main text.

The noncommutative value of an observable $\beta$ is an element of a noncommutative algebra each of which has a representation by an infinite number of complex numbers as the values of the independent derivatives of $f_{\beta}$ or $H_{\beta}$. Explicitly, we have

$$[\phi](\beta) = \{ f_{\beta}(\phi), X_{[\beta]}(\phi), X_{[\beta]}(\phi), \nabla_m X_{[\beta]}(\phi) \} .$$

It can be taken, for a normalized state vector $|\phi\rangle$ ($|z|^2 = \hbar$), as

$$[\phi](\beta) = \{ H_{\beta}(\phi), \tilde{X}_{[\beta]}(\phi), \tilde{X}_{[\beta]}(\phi), \nabla_m \tilde{X}_{[\beta]}(\phi) \} ,$$

bearing in mind an unphysical overall phase factor ambiguity. We use here the latter expression for simplicity. Looking at the noncommutative values for $\alpha_i$ and $\bar{\alpha}_i$, we have quite an amazing story. For example, the set of values for $\tilde{X}_{[\beta]}$, for any one $\bar{\alpha}_j$, is really like the full set of $z^{[n]}$ coordinate values. Knowing their values can completely fix the state vector. Of course that is based on our knowledge of the factors $i \sqrt{\frac{n}{2\hbar}}$ which are really the constant values of the nonvanishing second derivatives, i.e. a $\nabla_m \tilde{X}_{[\beta]}(\phi)$. Furthermore, the simple results is special to the coordinate system, or basis of $\mathcal{H}$ adopted, the definition of which involved all $\alpha_i$ and $\bar{\alpha}_i$, or $\hat{x}_i$ and $\hat{p}_i$. Hence, if we can determine all values of such components of a covector for a normalized state vector up to an overall phase, we could have all values of the $z^{[n]}$ coordinates up to the undetermined phase, and hence determine the exact physical state and all the information about the local representations of the full set of noncommutative coordinate observables, i.e. the Taylor series expansions of their corresponding Kählerian functions. The calculations involved are more tedious, but the story is essentially the same when we look at the set of $\tilde{X}_{[\beta]}$ or $X^{[\beta]}$ values, except that we also need the $f_{\bar{\alpha}_j}$ value. For example, we start with $z^{[n]} = \frac{1}{f_{\bar{\alpha}_j}} \tilde{X}_{[\beta]}^{[\bar{\alpha}_j]}$ for all $[n]$ with $n_j = 0$ and recursively each $z^{[n]}$ with increasing $n_j$ is given by $\frac{1}{f_{\bar{\alpha}_j}} (\tilde{X}_{[\beta]}^{[\bar{\alpha}_j]} + \sqrt{\frac{n_j}{2\hbar}} z^{[n-1]}_{\bar{\alpha}_j})$. So, the set of ‘noncommutative
values’ for the noncommutative coordinate operators can be seen as carrying a lot of redundant information about the state. The matter certainly worth further careful studies, but here we show what we want to focus on, that the knowing the noncommutative values of the noncommutative coordinates in terms of their representations as the infinite numbers of complex numbers allows us to determine the exact physical state or its coordinates in the (projective) Hilbert space.

V. DIFFERENTIAL GEOMETRIC STRUCTURES FOR THE OBSERVABLE ALGEBRA

As illustrated here in Sec.II and III it is obvious that the commutator multiplied by $\frac{1}{i\hbar}$ is a Poisson bracket on the observable algebra. It is really a trivial one in terms of the symplectic coordinates $\hat{x}^i$ and $\hat{p}^i$, or their complex counterparts $\alpha^i$ and $\bar{\alpha}^i$. Note that these are considered coordinates for which we do not distinguish upper and lower indices. We have then the Hamiltonian vector field $X_\beta$ for $\beta$ given by

$$X_\beta = -\frac{1}{i\hbar}[\beta, \cdot] = -\frac{1}{i\hbar}\text{ad}_\beta.$$  \hspace{1cm} (37)

Hence, it is given in terms of the adjoint action of $\beta$ on elements of the algebra, an inner derivation exactly as expected for the algebra \[4, 30]\]. From the geometrical point of view, we want to think about the Poisson bracket in terms of the derivatives with respect to the noncommutative (operator) coordinates, $\partial_{\hat{x}^i}$ and $\partial_{\hat{p}^i}$. The idea could work well with

$$\partial_{\hat{x}^i} = X_{\hat{p}^i} = -\frac{1}{i\hbar}[\hat{p}^i, \cdot] , \quad \partial_{\hat{p}^i} = -X_{\hat{x}^i} = \frac{1}{i\hbar}[\hat{x}^i, \cdot] ,$$  \hspace{1cm} (38)

giving

$$\partial_i \equiv \partial_{\alpha^i} = -\frac{1}{2i}X_{\bar{\alpha}^i} = -\frac{1}{2\hbar}[\bar{\alpha}^i, \cdot] , \quad \partial_i \equiv \partial_{\bar{\alpha}^i} = \frac{1}{2i}X_{\alpha^i} = \frac{1}{2\hbar}[\alpha^i, \cdot] .$$  \hspace{1cm} (39)

With the algebra formulated for the operators $\beta$ as functions of the coordinate variables, $\hat{x}^i$ and $\hat{p}^i$ or $\alpha^i$ and $\bar{\alpha}^i$, such differentiations can be naturally appreciated. And obviously, the coordinates, of the real or the complex sets, are independent variables. In fact, the Heisenberg commutation relation gives $\text{ad}_{\hat{x}^i}$ as replacing a $\hat{p}^i$ factor in $\beta(\hat{p}^i, \hat{x}^i)$ by $i\hbar$ and $\text{ad}_{\hat{p}^i}$ as replacing a $\hat{x}^i$ factor by $-i\hbar$, at least for the polynomials. That is in perfect agreement with the above expressions. The coordinate derivatives can be seen to be mutually commutative.
explicitly through the adjoint action. We are actually more interested in the expressions and results in terms of the complex (non-Hermitian) coordinates $\alpha^i$ and $\bar{\alpha}^i$ though we will still present some of the corresponding results in terms of $\hat{x}^i$ and $\hat{p}^i$ for the readers’ easy appreciation.

With the coordinate vector fields $\partial_i$ and $\partial_{\bar{i}}$, we next look at the matching 1-forms. The symplectic structure can be used to give an expression for the differential forms $d\beta$ as $d\beta(\mathcal{X}_\gamma) = \langle d\beta, \mathcal{X}_\gamma \rangle = \frac{1}{i\hbar}[\beta, \gamma]$. We can retrieve from that the coordinate 1-forms

$$d\hat{x}^i(\partial_{\hat{x}^j}) = \delta^i_j, \quad d\hat{x}^i(\partial_{\hat{p}^j}) = 0,$$

$$d\hat{p}^i(\partial_{\hat{x}^j}) = \delta^i_j, \quad d\hat{p}^i(\partial_{\hat{p}^j}) = 0; \quad (40)$$

as well as the matching results for the complex coordinates $\alpha^i$ and $\bar{\alpha}^i$. Moreover, we also have $d\alpha^i(\mathcal{X}_\alpha) = (-2i\delta^i_j)\partial_j\beta$ and $d\bar{\alpha}^i(\mathcal{X}_{\bar{\alpha}}) = (2i\delta^i_j)\partial_j\beta$, for which a direct analog to the commutative case would suggest seeing them as components of the $\mathcal{X}_\alpha$ in terms of the basis of the coordinate vector fields. We will see however that the latter idea does not work so well in general. Similarly, we can obtain $d\beta(\partial_i) = \partial_i\beta$ and $d\beta(\partial_{\bar{i}}) = \partial_{\bar{i}}\beta$, the analog of components of the 1-forms in terms of the coordinate 1-forms $d\alpha^i$ and $d\bar{\alpha}^i$. We can go further. A symplectic structure $\Omega$ can be introduced with

$$\Omega(d\beta, d\gamma) = \{\beta, \gamma\}_\alpha = \Omega(\mathcal{X}_\beta, \mathcal{X}_\gamma),$$

where $\{\cdot, \cdot\}_\alpha$ denotes the Poisson bracket, i.e. $\{\beta, \gamma\}_\alpha = \frac{1}{i\hbar}[\beta, \gamma]$. Note that $\mathcal{X}_\alpha(\beta) = \{\beta, \gamma\}_\alpha = -\mathcal{X}_\beta(\gamma)$. For the coordinate vector fields or 1-forms, we can actually write $\Omega^{ij} \equiv \Omega(\alpha^i, \alpha^j) = -2i\delta^{ij}$, and similarly, $\Omega^{ij} = 2i\delta^{ij}$, and $\Omega^{ij} = \Omega^{ij} = 0$, as well as $\Omega_{ij} \equiv \Omega(\partial_i, \partial_j) = \frac{i}{2}\delta_{ij} = -\Omega_{ji}$ and $\Omega_{ij} = \Omega_{ij} = 0$, in exact analog to the commutative case. All the above shows $\hat{x}^i$ and $\hat{p}^i$ behave like three pairs of canonical coordinates of a noncommutative symplectic geometry with an essentially trivial symplectic form.

Up to this point, we have obtained a nice picture of first order differential calculus on the observable algebra, or rather on the noncommutative geometric space behind it, in agreement with the general noncommutative differential geometry simply from identifying the Poisson bracket with $\frac{1}{i\hbar}[\cdot, \cdot]$ and thinking about it in the same way as one on a commutative symplectic manifold given in the differential geometric language. The structure may be considered dictated by the theory of quantum mechanics itself, and is an alternative description of the structure on the algebra of the $H_\beta$ functions or of the $f_\beta$ functions on the Kähler manifolds.
and $\mathcal{P}$, respectively. Hence, it is natural to think about the symplectic geometry behind the operator algebra is really the same one of the Hilbert space or projective Hilbert space. We will look at that with an analysis of like a coordinate transformation in the next section. Note that we have not said anything about the metric tensors yet and therefore do not have the operations of lowering or raising indices, except for the coordinates for which there is no distinction between the two. The closest we get to is only the standard pairing between the coordinate 1-forms and the corresponding coordinate vector fields. Though it is tempting to claim an Kähler structure with a matching metric tensor, we take a special caution against that.

To give a full differential calculus, one has to first put the 1-forms as derivations of the 0-forms under the action of the differential operator $d$. Introducing the ‘Dirac’ operator $D$ with

$$d\beta = [D, \beta] ,$$

we want $d^2 = 0$ and $d(\eta\eta') = d\eta\eta' + (-1)^k \eta d\eta'$ for forms $\eta$ and $\eta'$ with $k$ being the degree of $\eta$. Furthermore, one wants to extend the involution in the observable algebra, the complex conjugation among the $\beta$ functions, to all forms. Naively extending $d = [D, \cdot]$ to all forms does not work. The simple solution is given by

$$d\eta = D\eta - (-1)^k \eta D$$

for a $k$-form $\eta$. It is easy to see that $[D, [D, \cdot]]_+ = [D, [D, \cdot]]_+ = 0$, giving $d^2 = 0$. It is also of interest to note that the Poisson bracket itself can also be naturally extended to the whole differential algebra [31], with $d\eta = i\hbar \{D, \eta\}_\alpha$, by taking the graded commutator in place of the commutator. It is explicitly given by

$$\{\eta, \eta'\}_\alpha = \frac{1}{i\hbar} \left[ \eta\eta' - (-1)^{kk'} \eta'\eta \right] .$$

The involution, as the Hermitian conjugation of the operators, for the observable algebra is just our complex conjugation, $(\beta)^* = \bar{\beta}$. Obviously, $d\hat{x}^i$ and $d\hat{p}^i$, are to be taken as the real. $D$ then has to be taken as purely imaginary, i.e. $D^* = -D$. In addition, we have $(d\eta)^* = (-1)^kd(\eta)^*$. Note that $(\beta\gamma)^* = \bar{\gamma}\bar{\beta}$. There cannot be further nontrivial factors of $-1$ in the conjugate of product of the forms, i.e. in general we simply have $(\eta\eta')^* = (\eta')^*(\eta)^*$ ².

² Note that $(d\eta)^* = (-1)^kd(\eta)^*$ is stated as a convention in Ref.[32], also taken in Ref.[4], while Ref.[30]
We can write $\eta$ for $(\eta)^*$ like in the case of the 0-form. In conclusion, the complex structure on our noncommutative space works in the usual fashion on the differential algebra with only the nontrivial commutation relations to be cared about. Or, the conjugation on products of the forms is exactly in line with taking all the forms as operators on the same Hilbert space, though not necessarily as elements of the observable algebra. However, we do not yet have any explicit definition of the action of $d\beta$ or $D$ on a state vector, not to say forms of the higher degrees, and the presented calculus remains only formal. That we will present in the next section. Note that the ‘Dirac’ operator is itself a 1-form, and an explicit definition of it gives explicit definitions to all forms as operators.

Some checking on the forms and vector fields given above matching well with the general notion in noncommutative geometry is in order. We have the (Hamiltonian) vector fields as differential operators on elements of the algebra given as a derivations. That is quite standard, as discussed in Ref. [30] for example. Note the collection of all such vector fields is not a left module of the algebra, as $\beta \mathcal{X}_\gamma$ is in general not a derivation, and not the Hamiltonian vector field of another element of the algebra. It is a free module over the center of the algebra though. The forms are introduced with a $d$-operator, further expressed through the adjoint action of the ‘Dirac’ operator, satisfying $d\beta(\mathcal{X}_\gamma) = \mathcal{X}_\gamma(\beta)$, and a generic $n$-form would be linear combinations of terms in the form

$$\beta_0 d\beta_1 d\beta_2 \ldots d\beta_n,$$

again all in line with the standard approach [4, 30]. The full differential algebra of all forms then is a two-sided module of the algebra. In particular, $d\beta \gamma$ is given by $d(\beta \gamma) - \beta d\gamma$. It is important to note the general inequality of $d\beta \gamma \neq \gamma d\beta$. For example, while $dx^2 = 2x dx$ for a real variable $x$, we have $d\beta^2 = \beta d\beta + d\beta \beta \neq 2\beta d\beta$ in general, which is true even for our coordinate operators including like $\hat{x}_i$ and $\hat{p}_i$. So, unlike the commutative case, the coordinate 1-forms cannot span a vector space of all 1-forms, an important issue we will get back to below. To get something like a basis for all 1-forms, say if we consider just the collection of all polynomials of $\alpha_i$ and $\bar{\alpha}_i$ casted back as polynomials of $\hat{x}_i$ and $\hat{p}_i$, we would need at least all Weyl ordered monomials of the latter. Though we use notations like $\Omega^{i\bar{j}}$

adopts simply $(d\eta)^* = d(\eta)^*$ from which $(d\beta d\gamma)^* = -(d\gamma)^*(d\beta)^*$ is obtained. In our case here, there is no free choice of convention on the matter and the latter option cannot be taken.
above, we do not have the usual kind of expression of a generic $\Omega(d\beta, d\gamma)$ in terms of $\Omega^{ij}$ and the coordinate derivatives as for the commutative case.

Note that the formal 1-form $d\beta$ and $D$ operators will be given a more rigorous definition below through their action on the Hilbert space. While our picture of the differential calculus is compatible with the general mathematics of noncommutative geometry, our so-called ‘Dirac’ operator $D$ is certainly not that of the Connes’ spectral triple $[4]$. In fact, it is not clear at all the operator is Dirac in the usual sense as in commutative geometries, and there is no spinor space under discussion. We use the name ‘Dirac’ operator only in relation to Eq. (41).

VI. HEISENBERG VERSUS SCHRÖDINGER PICTURE OF THE DIFFERENTIAL GEOMETRIC STRUCTURES AND THE TRANSFORMATION BETWEEN COMMUTATIVE AND NONCOMMUTATIVE COORDINATES

In the description of the time evolution in quantum mechanics, we have the Heisenberg picture of changing observables on a fixed state and the Schrödinger picture of a changing state with fixed observables, identified by the same time dependent expectation values. The key mathematical feature behind is the duality between the observable algebra and the (geometric) manifold of the pure states $\mathcal{P}$. In fact, the usual Schrödinger picture is a description completely in terms of the latter space when the time evolution is given in the symplectic formulation. If that has to be clearly distinguished from the usual description in the language of unitary flow for the state vector, one can call it the Hamilton–Schrödinger picture. We will refer to it simply as the Schrödinger picture in this paper and include in it the description of $\mathcal{H}$ as a symplectic manifold. We see that the dual descriptions, the Heisenberg and the Schrödinger pictures, can be generalized to the matching between the structures on the observable algebra, with $\alpha_i$ and $\bar{\alpha}_i$, or $\hat{x}_i$ and $\hat{p}_i$ as the basic noncommutative coordinate variables on the one hand and the state space $\mathcal{P}$ or $\mathcal{H}$ with real/complex number symplectic coordinate variables on the other. Duality between geometric and algebraic structures, exemplified in the commutative case by a familiar finite dimensional real manifold and its algebra of the smooth functions, is a key perspective in modern mathematics along which the mathematics of noncommutative geometry has been developed for the noncommutative algebras. For the specific case of the noncommutative observable algebra in quantum
mechanics as a physical theory, our notion of Heisenberg versus Schrödinger picture descriptions gives an explicit formulation of that duality which may be seen as providing a physics approach to the formulation of the noncommutative geometry. The perspective is already there implicitly in our description of the symplectic differential geometric structures for the observable algebra in the previous section. An explicit description through the duality will be developed here. We want to push that to the ideal goal of such a duality map being fully based on a transformation between the six noncommutative coordinates and the infinite set of the real/complex number coordinates.

We have discussed above the isomorphic picture of the observable algebra as an algebra of the Kählerian functions $f_\beta$ on $\mathcal{P}$, as well as the corresponding $H_\beta$ functions on $\mathcal{H}$. The isomorphisms are essentially the duality maps. The two are in fact only slightly different ways of writing the same duality map as the $f_\beta$ functions and $H_\beta$ functions are directly related. In particular, their symplectic structures, or the Poisson brackets, match to the same single Poisson bracket and the corresponding symplectic structure on the observable algebra, as presented in the last section. Their metric structures, however, do not. The implication of that has to be looked into very carefully. From the start, each Kähler product algebra of the Kählerian functions is our Schrödinger picture description of the observable algebra though, unlike in the classical case, the algebra is the one of a more limited class of functions. They generate Hamiltonian flows which are the isometries, giving the metric a key role in the dynamics. With the operators $\beta = \beta(\alpha_i, \bar{\alpha}_i)$ formulated as functions of the coordinate operators, the duality fits in with the intuitive idea of the position and momentum operators being the noncommutative position and momentum coordinate variables of a quantum physical/phase space, for which there is a commutative coordinate description requires an infinite number of real/complex number coordinates. We will keep looking at both Schrödinger picture descriptions, on $\mathcal{P}$ and $\mathcal{H}$ (with the $G$ metric), in fact with both the $z$-coordinate and $\tilde{g}$ metric and $w$-coordinate and $g$ metric for $f_\beta$ functions on $\mathcal{P}$, as we have been doing above. Note that the metrics are used in two ways here, namely for raising and lowering indices and for the associated symplectic forms. We describe and discuss most of the results in terms of $z$-coordinates and $\tilde{g}$ metric, whenever explicit mathematical expressions are given. Direct parallel results for the other cases with similar properties are listed in Table 1.

Along the depicted perspective, we think of the Kählerian functions $f_{\xi i}$ and $f_{\bar{\xi} i}$ as a
TABLE I: Matching results for the differential symplectic geometries under the coordinate maps from the projective Hilbert space \( \mathcal{P} \) and the Hilbert space \( \mathcal{H} \) to the observable algebra as a noncommutative geometric object \( \mathcal{P}_{nc} \) with the noncommutative (operator) coordinates. [Note that the \( z^n \) coordinates count from 0 while the \( w^n \) coordinates from 1 and \( \partial_n \) denote \( \frac{\partial}{\partial z^n} \) in the last two columns but \( \frac{\partial}{\partial w^n} \) in the \( w^n \)-coordinate column.]

| coordinate | on \( \mathcal{P}_{nc} \) | \( \hat{f}^* \) to \( \mathcal{P} (w) \) | \( \hat{f}^* \) to \( \mathcal{P} (z) \) | \( \hat{f}^*_n \) to \( \mathcal{H} \) |
|------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \( \alpha^i, \bar{\alpha}^i \) | \( w^n = \frac{z^n}{\bar{z}^n}, \bar{w}^n \) | \( z^n, \bar{z}^n \) | \( z^n, \bar{z}^n \) |
| \( \beta^i(\alpha^i, \bar{\alpha}^i) \) | \( f_{\alpha^i}, f_{\bar{\alpha}^i} = \tilde{f}_{\alpha^i} \) | \( f_{\beta^i}, f_{\bar{\beta}^i} = \tilde{f}_{\beta^i} \) | \( H_{\alpha^i}, H_{\bar{\alpha}^i} = \tilde{H}_{\alpha^i} \) |
| \( \beta \gamma \) | \( \beta_{\gamma} \) | \( \beta_{\gamma} \) | \( \beta_{\gamma} \) |
| \( d\beta = [D, \beta] \) | \( df_{\beta} = f_{d\beta} \) | \( df_{\beta} = f_{d\beta} \) | \( dH_{\beta} = H_{d\beta} \) |
| \( d\alpha^i, d\bar{\alpha}^i \) | \( df_{\alpha^i}, df_{\bar{\alpha}^i} \) | \( df_{\alpha^i}, df_{\bar{\alpha}^i} \) | \( dH_{\alpha^i}, dH_{\bar{\alpha}^i} \) |
| \( \frac{1}{i\hbar}[\beta, \gamma] = \Omega(\mathcal{X}_\gamma, \mathcal{X}_{\beta}) \) | \( \omega(\mathcal{X}_\gamma, \mathcal{X}_{\beta}) \) | \( \omega(\mathcal{X}_\gamma, \mathcal{X}_{\beta}) \) | \( \omega(\mathcal{X}_\gamma, \mathcal{X}_{\beta}) \) |
| \( \partial_i = (\frac{1}{2}) X_{\alpha^i}, \bar{\partial}_i = (\frac{1}{2}) X_{\bar{\alpha}^i} \) | \( \partial_i = (\frac{1}{2}) X_{\alpha^i}, \bar{\partial}_i = (\frac{1}{2}) X_{\bar{\alpha}^i} \) | \( \partial_i = (\frac{1}{2}) X_{\alpha^i}, \bar{\partial}_i = (\frac{1}{2}) X_{\bar{\alpha}^i} \) | \( \partial_i = (\frac{1}{2}) X_{\alpha^i}, \bar{\partial}_i = (\frac{1}{2}) X_{\bar{\alpha}^i} \) |
| \( J^{-1} \) | \( J^{-1} \) | \( J^{-1} \) | \( J^{-1} \) |
| \( J \) | \( J \) | \( J \) | \( J \) |

description of the noncommutative coordinates in terms of the functions of the commutative coordinates, and \( \beta \) and \( f_\beta \) are the expressions of the same observable/function in terms of the noncommutative and the commutative coordinate variables, respectively. The six \( f_{\alpha^i} \) and \( f_{\bar{\alpha}^i} \) complex values at a point on \( \mathcal{P} \) of course cannot be a representation of an infinite number of \( z^n \) coordinates. They are noncommutative objects, the coordinates, in the Kähler product algebra. Their full noncommutative values as rather the noncommutative values of the operators \([8]\) serves as that representation. Going directly to the operators, we can consider
an implicitly defined coordinate transformation map \( \hat{f} : P \to P_{nc} \) with \( \hat{f}(w^n, \bar{w}^n) = (\alpha^i, \bar{\alpha}^i) \), where \( P_{nc} \) denotes the noncommutative manifold of all admissible noncommutative values of \((\alpha^i, \bar{\alpha}^i)\). The algebraic isomorphism sending \( \beta \) to \( f_\beta \) is then simply the pull-back transform on the corresponding functional space, i.e. \( f_\beta = \hat{f}^\ast(\beta) \). The Schrödinger picture of the quantum physics on \( P \) in terms of its commutative symplectic geometry, is to be mapped to the Heisenberg picture on \( P_{nc} \) as the dual description based on the noncommutative coordinate variables. We can also think of the map \( \hat{f} \) as \( \hat{f}(z^n, \bar{z}^n) = (\alpha^i, \bar{\alpha}^i) \), with \( f_\beta \) as \( f_\beta(z^n, \bar{z}^n) \). Similarly, we have \( \hat{f}_H : H \to P_{nc} \) with \( \hat{f}_H(z^n, \bar{z}^n) = (\alpha^i, \bar{\alpha}^i) \) and \( H_\beta = \hat{f}^\ast_H(\beta) \) as the algebraic isomorphism. However, we will see immediately below that for the coordinate maps it makes good sense to consider the Kählerian functions only under the condition \( |z|^2 = 2\hbar \), under which \( f_\beta = H_\beta \) and \( \hat{f}_H \) is to be taken rather as a \( S \to P_{nc} \) map.

Firstly, we give an explicit description of differentiation in the Heisenberg picture, which has been given through the formal operator expressions in the previous section, dual to the familiar counterpart in the Schrödinger picture. The two pictures of the time variation can be directly generalized to a generic differentiation. The Schrödinger time evolution is given in terms of \( z^n(t) \), for example, with the observables \( H_\beta(t) = H_\beta(z^n(t), \bar{z}^n(t)) \), while the Heisenberg description of the state in terms of the noncommutative coordinates \( \alpha^i(t) \), with the observables \( \beta(t) = \beta(\alpha^i(t), \bar{\alpha}^i(t)) \). The two pictures are connected via \( H_\beta(t) = H_{\beta(t)} \). For a generic infinitesimal variation, we consider the corresponding relation

\[
\frac{1}{2\hbar} \left( \langle \delta \phi | \beta \phi \rangle + \langle \phi | \beta \delta \phi \rangle \right) = dH_\beta = H_{\delta \beta} = \frac{1}{2\hbar} \langle \phi | d\beta | \phi \rangle , \tag{44}
\]

which can be seen as our physics definition of the 1-form operators \( d\beta \), with \( |\delta \phi \rangle = \sum_n dz^n |n\rangle \) as an infinitesimal state. Equating this definition with the one given in Eq.(41) suggests

\[
|\delta \phi \rangle = -D |\phi \rangle \quad \text{and} \quad \langle \delta \phi \rangle = \langle \phi | D . \tag{45}
\]

Therefore, \( D \) is antihermitian or pure imaginary, in agreement with the results of the previous section. That would, however, imply \( \langle \delta \phi | \phi \rangle = -\langle \phi | \delta \phi \rangle \) giving \( d\langle \phi | \phi \rangle = 0 \). The same can also be obtained by considering \( \beta \) to be a constant operator, for example the identity operator. We want such \( d\beta \) to be zero as an operator, which is not consistent with Eq.(44) unless the \( d|z|^2 = 0 \). In the language of the \( \hat{f}_H \) map, for the pull-back of the 1-forms \( d\beta \) as \( \hat{f}_H^\ast(d\beta) = dH_\beta \), we surely want the pull-back of zero as a 1-form to be zero. Moreover,
\[ \hat{f}^*(d\beta) = df_\beta \] surely pull-backs the zero 1-form to zero, but
\[ df_\beta = \frac{1}{\langle \phi|\phi \rangle} (\langle \delta\phi|\beta|\phi \rangle + \langle \phi|\beta|\delta\phi \rangle) - \hat{f}_\beta \left( \langle \delta\phi|\phi \rangle + \langle \phi|\delta\phi \rangle \right), \]
while
\[ f_{d\beta} = \frac{1}{\langle \phi|\phi \rangle} \langle \phi|d\beta|\phi \rangle. \]

Having consistent \[ dH_\beta = H_{d\beta} \] and \[ df_\beta = f_{d\beta} \] implies Eq. (45) and \[ d|z|^2 = 0. \] In fact, for the identity operator as the constant function \[ \beta(\alpha^i, \bar{\alpha}^\dagger_i) = 1 \] to be represented by the same constant function \[ H_\beta(z^n, \bar{z}^\dagger_n) = 1, \] it requires \[ |z|^2 = 2\hbar. \] So, we can have the exact Heisenberg and Schrödinger picture correspondence only when \[ H_\beta \] is taken with fixed \[ r = |z| \] value, most conveniently taken to be \( \sqrt{2\hbar} \), giving the \( H_\beta \) as a function on \( S \) and \( dH_\beta \) as the 1-form. We can either use the \( z^n \) coordinates under that condition or the set of \{0, \( w^n, \bar{w}^\dagger_n \} (n \neq 0); \) note that we drop the \( \bar{w}^\dagger \) notation in this section. The \( \hat{f}_H \) map now is to be taken as \( S \to \mathcal{P}_{nc}. \) However, in most of our analysis we keep the Hilbert space \( \mathcal{H} \) as its domain and impose the condition \[ |z|^2 = 2\hbar \] in the end. Note that \[ f_\beta = H_\beta \] for \[ |z|^2 = 2\hbar. \]

For any of the coordinate 1-forms \( da \), we may write formally, following the usual case for the \( \alpha \) set as if they are commutative coordinates, \( da^i = dw^n \hat{J}^i_n + dw^n \hat{J}_{\bar{n}}^i \) with \( \hat{J}^i_n = \frac{\partial \alpha^i}{\partial w^n} \) and \( \hat{J}_{\bar{n}}^i = \frac{\partial \alpha_i}{\partial \bar{w}^n} \). Expressions of that kind are, however, something we would not quite know how to deal with. We can look at the pull-back though, written as
\[ \hat{f}^*(da^i) = dw^n J_n^i + dw^n \bar{J}_{\bar{n}}^i, \]
for which we have
\[ J_n^i = \frac{\partial f_n^i}{\partial w^n} = -i(X_\alpha^i)_n, \]
\[ J_{\bar{n}}^i = \frac{\partial f_{\bar{n}}^i}{\partial \bar{w}^n} = i(X_{\bar{\alpha}})_i. \]

A point of paramount importance to note is that \( J_{\bar{n}}^i \), and its complex conjugate \( J_{\bar{\alpha}}^i = \frac{\partial f_{\bar{n}}^i}{\partial \bar{w}^\dagger_n} \), are nonzero. Even \( \hat{J}_{\bar{n}}^i \) does not look like a vanishing quantity. The coordinate map actually cannot be taken as a holomorphic one. That speaks explicitly against taking a Kähler structure with a trivial metric on \( \mathcal{P}_{nc} \). The lack of the correspondence between the complex structures confirms that we cannot take on \( \mathcal{P}_{nc} \) a Kähler structure connecting the symplectic and the metric one.

For a mapping between commutative manifolds, the corresponding map between the tangent spaces is obtained by pushing forward. If one follows naively that formulation, the
vector field on $\mathcal{P}_{nc}$, as the push-forward $\hat{f}_*(X)$ of the vector field $X$ on $\mathcal{P}$, should satisfy the relation

$$\hat{f}_*(X)(\beta) = X(\hat{f}^* (\beta)) = X(f_\beta),$$

and one would have $\hat{f}_*(\frac{\partial}{\partial w^i}) = \hat{J}_n^i \partial_t + \hat{J}_n^i \partial_t$. Again, we avoid that and look at the pull-back. In fact, it does not look like one can identify a vector field on $\mathcal{P}_{nc}$ as $\hat{f}^*(\frac{\partial}{\partial n})$, within the scope of our discussion in the previous section. To get around the difficulty, we first note that we should not be looking at all the functions or vector fields on $\mathcal{P}_{nc}$. We are interested only in the Kählerian functions as only those correspond to the functions of $\mathcal{P}_{nc}$, as pull-backs. With the Poisson bracket on $\mathcal{P}_{nc}$, the noncommutative coordinate vector fields are Hamiltonian vector fields with the matching ones on $\mathcal{P}$. Therefore, we can consider each $X_\beta$ as the pull-back of $\mathcal{X}_\beta$, and the kind of Hamiltonian vector fields look like the only vector fields we need in quantum mechanics. The coordinate vector field $\partial_{\bar{n}}$, for example, may not have a push-forward on $\mathcal{P}_{nc}$ as it is not a Hamiltonian vector field of a Kählerian function.

With the Poisson brackets discussed above, we have

$$X_\beta(\hat{f}^*(\gamma)) = \{ f_\beta, f_\gamma \}_\omega = \hat{f}^* \{ \gamma, \beta \}_\alpha = \hat{f}^*(\mathcal{X}_\beta(\gamma)). \tag{48}$$

The last expression can be considered $\hat{f}^*(\mathcal{X}_\gamma)(\hat{f}^*(\gamma))$. Hence, we have to give $X_\beta = \hat{f}^*(\mathcal{X}_\gamma)$. Then we can write

$$\mathcal{X}_\beta(\gamma) = \hat{f}_*(X_\beta)(\hat{f}_* (f_\gamma)) = \hat{f}_*(X_\beta(f_\gamma))$$

and $\hat{f}_*(X_\beta) = \mathcal{X}_\beta$ as the inverse. With that, we can also write the relation

$$\hat{f}^*(\left< \eta, \hat{f}_* (X_\beta) \right>) = \left< \hat{f}^* (\eta), X_\beta \right>,$$

where $\eta$ is a 1-form on $\mathcal{P}_{nc}$ and $X$ a vector field on $\mathcal{P}$. Again, one can check that

$$\left< da^i, \mathcal{X}_{\alpha^j} \right> = -2i \delta^j_i = \left< df_{a^i}, X_{\alpha^j} \right>$$

and

$$\left< da^i, \mathcal{X}_{\alpha^i} \right> = 0 = \left< df_{a^i}, X_{\alpha^i} \right>. \tag{49}$$

Note that $\left< \eta, \hat{f}_* (X_\beta) \right>$ is in general an operator while $\left< \hat{f}^*(\eta), X_\beta \right>$ is a complex number, which should be the pull-back of the former. We do not have the explicit pull-back expression in the line above only because a constant operator (a multiple of the identity) is pull-back by $\hat{f}^*$ to the same constant as a number. For the parallel results with $\hat{f}^*_R$ taken as the original
\( \mathcal{H} \to \mathcal{P}_{\text{nc}} \) map, however, we have actually \( \left( dH_{\alpha}, \tilde{X}_{\alpha} \right) = \frac{-i}{2} \delta_i^j \). Enforcing the restriction to \( \mathcal{S} \), of course, retrieves the same result as \( \hat{f}^* \). Note that all \( \tilde{X}_\beta \), as the vector fields on \( \mathcal{H} \), have no components in the \( \partial_r \) direction to which they are orthogonal, hence serve as the proper vector fields on \( \mathcal{S} \). Similarly, using \( \hat{f} \) as a map from the \( z^n \) coordinates matched with \( \tilde{\omega} \), the \( df_\beta \) and \( \tilde{X}_\beta \) are elements of the cotangent and tangent bundles of \( \mathcal{H} \), but are horizontal in relation to the vertical directions of the corresponding fiber spaces. The full setup for the differential symplectic geometry on \( (\mathcal{P}_{\text{nc}}, \Omega) \) can be seen to have as the pull-back the differential symplectic geometry on \( (\mathcal{P}, \omega) \), or equivalently, the one on \( (\mathcal{P}, \tilde{\omega}) \) or \( (\mathcal{H}, \tilde{\omega}) \), which should be the proper way to interpret the formulation of the former, given in the previous section.

Applying the pull-back to the noncommutative coordinate vector fields, we have \( \hat{f}^*(\partial_i) = \frac{-1}{2i} X_{\alpha}^i \) and \( \hat{f}^*(\partial_i) = \frac{1}{2i} X_{\alpha}^i \). Therefore, we can write

\[
J^{-1}_j = \frac{-1}{2i} X^i_{\alpha j}, \quad J^{-1}_j = \frac{1}{2i} X^i_{\alpha j},
\]

with

\[
\hat{f}^*(\partial_i) = J^{-1}_{ij} \partial_j + J^{-1}_{ij} \partial_j.
\]

\( J^{-1} \) is obtained as a left inverse, with \( J^{-1}_{ij} J_{ij} = \delta_j^i \) and \( J^{-1}_{ij} J_{ij} + J^{-1}_{ij} J_{ij} = 0 \).

We have also confirmed the last results with explicit expressions for the Hamiltonian vector fields and covectors given in Eqs.(36) and Eqs.(35). The corresponding expressions for \( \hat{f}_f^* \) similarly give \( \tilde{J}^{-1}_{ij} \tilde{J}_{ij} + \tilde{J}^{-1}_{ij} \tilde{J}_{ij} = \frac{|z|^2}{2} \delta_j^i \), the pull-back of \( \delta_j^i (\hat{f}) \). One can also check that, based on the pull-back relation between the symplectic structure, we have

\[
\hat{f}^*(\Omega_{ij}) = J^{-1}_{ij} \omega_{m\bar{n}} J^{-1}_{j} + J^{-1}_{ij} \omega_{m\bar{n}} J^{-1}_{j} = \frac{1}{4} \{ f_\alpha, f_\alpha \} \omega = 0,
\]

\[
\hat{f}^*(\Omega_{ij}) = J^{-1}_{ij} \omega_{m\bar{n}} J^{-1}_{j} + J^{-1}_{ij} \omega_{m\bar{n}} J^{-1}_{j} = \frac{1}{4} \{ f_\alpha, f_\alpha \} \omega = \frac{i}{2} \delta_{ij},
\]

and

\[
\hat{f}^*(\Omega^{ij}) = J^{j}_{[m}] \omega^{m\bar{n}} J^{i}_{\bar{n}] + J^{j}_{m} \omega^{m\bar{n}} J^{i}_{\bar{n}] = \{ f_\alpha, f_\alpha \} \omega = 0,
\]

\[
\hat{f}^*(\Omega^{ij}) = J^{j}_{[m}] \omega^{m\bar{n}} J^{i}_{\bar{n}] + J^{j}_{m} \omega^{m\bar{n}} J^{i}_{\bar{n}] = \{ f_\alpha, f_\alpha \} \omega = 2i \delta^{ji}.
\]
Again, we have the corresponding results for \((\mathcal{P}, \bar{\omega})\) and \((\mathcal{H}, \bar{\omega})\). Expressions like \(J_i^m J_i^{-1[n]} + J_i^\bar{m} J_i^{-1[n]}\) do not appear to be very sensible; and there is really no reason to expect otherwise. Therefore, our coordinate transformation picture works perfectly well for the differential symplectic geometry.

**VII. CONCLUSIONS**

As physicists, we are more interested in a specific case that certainly has a role in our description of the nature rather than the general/formal mathematical considerations. For any new mathematical ideas, we prefer to use those obtainable from the established physical theories and our related thinking. Quantum mechanics is a very well established physical theory. It is also our first realization of the noncommutative nature of physics — the noncommutativity of physical observables. The Heisenberg uncertainty principle clearly indicates that the position and momentum observables do not commute. Intuitively, we can understand it as the noncommutativity of the quantum phase space, or even the quantum model of the physical space. Without the modern mathematical concept of the noncommutative geometry of an operator algebra, that intuitive picture can hardly be formulated. On the other hand, the Hilbert space seems to serve the purpose of describing quantum states well. A lot of progress in the relevant mathematics have come by in the recent decades and it is the time to use the perspectives gained there to review the theory of quantum mechanics, as well as to see how that first noncommutative physical theory informs us about the noncommutative geometry certainly relevant to our description of the nature.

The projective Hilbert space, as the manifold of pure states for the quantum observable algebra, is a dual object to the latter. The former is an infinite dimensional Kähler manifold while the latter can be thought of as a geometric structure with the six noncommutative coordinates. Appreciating that a quantum observable has the information content of infinite number of complex/real numbers, we seek a direct correspondence between the complex number coordinates and the noncommutative coordinates. We use the full duality of the symplectic dynamics in the (Hamilton-)Schrödinger picture and the Heisenberg picture to define the differential geometry of the observable algebra, showing that the full differential symplectic geometric structures on either side match perfectly well, even to the extent of having an implicit coordinate map. Such a map can be thought of as a coordinate
transformation between the two as two picture of the same geometric object.

Our approach to noncommutative geometry is as interesting as it is intuitive. Physicists appreciating some noncommutative geometry would naturally identify with the idea of the position and momentum observables/operators as noncommutative coordinates of the phase space for a quantum particle, though studies mostly focused on a notion of noncommutative spacetime as more like the configuration space. Mathematically, the space as the manifold of pure states has been identified as an object dual to the noncommutative algebra, and at least as one of the three candidates for its geometric picture. However, explicit identification and description of symplectic geometry of the familiar quantum phase space as a noncommutative geometry with the position and momentum operators as coordinates has not otherwise been available. In fact, without the new conceptual notion of their noncommutative values, a consistent picture of the values of the six coordinates determining a point in the projective Hilbert space cannot be a sensible one. Our results on the new perspective of the coordinate transformation between the commutative and noncommutative ones is certainly somewhat short of mathematical rigor. We believe the analysis here serves as a good first step in the direction for going further studies of noncommutative geometries of physical interest.

Our line of work has as one of its goal to get to an intuitive picture of the physics of quantum mechanics with the position and momentum operators as geometric coordinates in the same sense as their classical counterpart, except for the noncommutativity \[9\]. We believe the present study is a good advance in that direction.

A. APPENDIX: ON \( P \) FROM THE KILLING REDUCTION OF \( \mathcal{H} \) \(-\{0\}\) AND THE RESULTS FOR THE RELATED HAMILTONIAN VECTOR FIELDS AND COVECTORS

We look at \( \mathcal{H} \) \(-\{0\}\) as a complex line bundle with \( P \) as the base space. We can take on it the conformal metric given by

\[
\tilde{G}_{m\bar{n}} = \frac{2\hbar}{|z|^2} G_{m\bar{n}}. \tag{53}
\]

It is better to replace \( r \) as a coordinate by \( \tau = \ln r \) to have \( \{\tau, \theta, w^n, \bar{w}^n\} \) as the coordinate system. \( \partial_\tau = r \partial_r \) and \( \partial_\theta \) are Killing vectors the Killing reduction of which gives the
Riemannian manifold \((\mathcal{P}, \tilde{g})\) from \((\mathcal{H} - \{0\}, \tilde{G})\). Notice that we have here

\[
ds^2_{(\mathcal{H} - \{0\})} = 2\hbar d\tau^2 + ds^2_{(\mathcal{S})},
\]

which when restricted to any submanifold of constant \(\tau\) gives exactly the \(ds^2_{(\mathcal{S})}\) metric on the sphere \(\mathcal{S}\). The metric is \(\theta\)-independent. A further projection to \(\mathcal{P}\) induces on the latter the metric given by the \(\hat{g}\) tensor which should only be taken as an expression for the actual metric \(g\) in the homogeneous coordinates, with the advantage of being globally applicable.

The number of indices in \(\hat{g}\) is bigger than the dimension of the tangent space. \(\tilde{g}\) can also be taken as a degenerate metric, \(\det \tilde{g} = 0\), on \(\mathcal{H} - \{0\}\) which vanishes on the vertical part of the tangent space spanned by the two Killing vectors. The vertical tangent space is exactly the tangent space of the fiber manifold. A manifold with a singular Riemannian metric does not have an inverse metric though many of the differential geometric structures of the usual Riemannian manifolds may still be of interest \([40, 41]\). The stationary class among such manifolds, which corresponds to our case at hand, has been a focus of mathematical studies \([40]\). The orthogonal complement is the horizontal tangent space, which can be thought of as the tangent space for \(\mathcal{P}\), or rather the horizontal lift of it. Horizontal tensors are defined accordingly. As a tensor, \(\hat{g}\) is exactly the horizontal lift of \(g\). In the appendix of Ref.\([38]\), it is shown how horizontal tensors orthogonal to the Killing vectors and their covectors can be obtained from a generic tensor by projecting out the vertical parts. For the case at hand, we have

\[
\tilde{g}_{m\bar{n}} = \tilde{G}_{m\bar{n}} - [2\partial_{\tau_l} \partial^l_{\bar{r}}]^{-1} \partial_{\tau_m} \partial_{\bar{r}_n} - [2\partial_{\theta_l} \partial^l_g]^{-1} \partial_{\theta_m} \partial_{\theta_n},
\]

(54)

where \(\partial_{\tau} = z^m \partial_m + \bar{z}^m \partial_{\bar{m}}\) and \(\partial_{\theta} = iz^m \partial_m - i\bar{z}^m \partial_{\bar{m}}\). Notice that \(\partial_{\theta_l}\) is the covector on \((\mathcal{H} - \{0\}, \tilde{G})\) here, which is different from the covector on \((\mathcal{H}, G)\). The structure can also be seen as a sub-Riemannian one \([28]\), characterized by the canonical 1-form \(\frac{1}{\hbar} \text{Im} \langle \phi | d\phi \rangle = \frac{1}{\hbar} (\bar{z}_n dz^n - z_n d\bar{z}^n)\), which reduces to \(-\frac{i}{\hbar} \bar{z}_n dz^n\) on \(\mathcal{S}\). The 1-form is exactly what defines the Berry connection \([27, 28]\). The Killing reduction technique given in Ref.\([38]\) is a powerful tool which, however, has apparently not been much applied in the mathematical studies. One can actually have an ‘inverse metric’ \(\tilde{g}^{m\bar{n}}\), for example, directly from the Killing reduction
of $\tilde{G}^{\hat{m}\hat{n}}$ as

$$\tilde{g}^{\hat{m}\hat{n}} = \tilde{G}^{\hat{m}\hat{n}} - [2\partial_{\eta}^l\partial_{\tau}^l]^{-1}\partial_{\tau}^m\partial_{\tau}^n - [2\partial_{\eta}^l\partial_{\eta}^l]^{-1}\partial_{\eta}^m\partial_{\eta}^n,$$

(55)

which can and has been used above for the Kähler product of the $f_\beta$ functions. The result agrees with that obtained from the affine coordinates with the equivalent metric $g$ for $P$, and that of the $H_\beta$ functions on $(H, G)$. There is also

$$\tilde{g}^{\hat{m}\hat{n}} = \delta^{\hat{m}\hat{n}} - \bar{z}_m z^n \frac{1}{|z|^2} = \tilde{g}_{\hat{m}\hat{n}},$$

(56)

obtainable similarly from $\tilde{G}^{\hat{m}\hat{n}} = \delta^{\hat{m}\hat{n}}$, which is involved in the explicitly Killing reduction expressions for the tensors.

Since the $f_\beta$ functions, as functions on $\mathcal{H} - \{0\}$, are $\tau$ and $\theta$ independent, the corresponding covectors $\partial_{\eta} f_\beta$ are naturally horizontal, meaning they are exactly the covectors on the Killing reduced $P$. For the covector dual to the Hamiltonian vector field $\tilde{X}_\beta^n$, we have $\tilde{X}_{\beta m} = i\partial_{\eta} f_\beta$, the same form as for any Kähler manifold with a non-degenerate metric. It is hence exactly the same as the covector for $\tilde{X}_\beta$ on $\mathcal{H}$ as a Kähler manifold, and actually also identical to the covector for $X_\beta$. In fact, in terms of complex coordinates and the splits of the exterior derivative into holomorphic and antiholomorphic parts, $df_\beta = \partial f_\beta + \bar{\partial} f_\beta$, we always have the covector for a Hamiltonian vector field as $i\partial f_\beta - i\bar{\partial} f_\beta$. In particular, $\tilde{X}_\beta = X_\beta$ as covectors, i.e. we have $\tilde{X}_\beta = X_\beta dw^n + X_{\bar{h}} dw^{\bar{h}}$, and the horizontal covector nicely has no component in the $d\tau$ (or $d\tau$) and $d\theta$ directions, though the $\{z^n, \bar{z}^n\}$ to $\{\tau, \theta, \bar{w}^n, \bar{w}^{\bar{n}}\}$ transformation is not a holomorphic one. Explicitly, a covector $\zeta_n$ is horizontal if

$$z^n \zeta_n = \partial_{\tau} z^n \zeta_n = (-i)\partial_{\theta} z^n \zeta_n = 0.$$

For a horizontal vector field $\tilde{X}_n$, we have

$$\frac{\hbar}{|z|^2} \tilde{z}_n \tilde{X}_n = 0.$$

That is satisfied by the Hamiltonian vector fields $\tilde{X}_\beta$, $\tilde{X}_\beta^n = \tilde{\omega}^{nm}\partial_{\eta} f_\beta$ (with $\tilde{\omega}^{nm} = -i\tilde{g}^{nm}$).

Note that the $w$-coordinate derivatives $\partial_{\eta}$, as vectors are not horizontal, hence neither is $X_\beta$. That is to say, the tangent space of $P$ as a manifold is not exactly the horizontal subspace of the tangent space of $\mathcal{H} - \{0\}$. $\tilde{X}_\beta$ is exactly the horizontal lift of $X_\beta$, and it equals $\frac{|z|^2}{2\hbar} \tilde{X}_\beta$. 

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the Hamiltonian vector field of $f_\beta$ on $(\mathcal{H}, \Omega)$. The difference is in the nonzero $\hat{X}_\beta^\theta$. Another expression of interest is the covariant derivative for the covectors. We have from the Killing reduction

$$\hat{\nabla}_m \hat{X}_\beta^\theta = \hat{g}_m^l \hat{g}_n^\theta \hat{\nabla}_l \hat{X}_\beta^\theta ,$$

(57)

where $\hat{\nabla}_m$ and $\hat{\nabla}_l$ are respectively the covariant derivatives on $(\mathcal{P}, \hat{g})$ and $(\mathcal{H} - \{0\}, \hat{G})$. $\hat{\nabla}_l$ is actually quite nontrivial with nonvanishing Christoffel symbols given by

$$\hat{\Gamma}^l_{mn} = -\frac{1}{2|z|^2} (\delta^l_m \hat{z}_n + \delta^l_n \hat{z}_m),$$

$$\hat{\Gamma}^l_{mn} = -\frac{1}{2|z|^2} (\delta^l_m \hat{z}_n - \delta^l_n \hat{z}_m),$$

$$\hat{\Gamma}^l_{\bar{m}\bar{n}} = -\frac{1}{2|z|^2} (\delta^l_m \hat{z}_n - \delta^l_n \hat{z}_m),$$

(58)

and their complex conjugates as $\hat{\Gamma}^{\bar{l}}_{m\bar{n}}$, $\hat{\Gamma}^{\bar{l}}_{\bar{m}n}$ and $\hat{\Gamma}^{\bar{l}}_{mn}$. Note that $(\mathcal{H} - \{0\}, \hat{G})$ is not a Kähler manifold. The covariant derivative of the Hamiltonian covector, however, reduces to simply

$$\hat{g}_m^l \hat{g}_n^\theta \partial_l \hat{X}_\beta^\theta = -i \partial_m \partial_n f_\beta ,$$

(59)

which is obviously horizontal. Moreover, we have $\hat{\nabla}_n \hat{X}_\beta^\theta = -\hat{\nabla}_m \hat{X}_\beta^\theta$ and $\hat{\nabla}_m \hat{X}_\beta^\theta = \hat{\nabla}_m \hat{X}_\beta^\theta = 0$. The results together with $\hat{\nabla}_m \hat{X}_\beta^\theta = -i \partial_m \partial_n f_\beta$ (=$\partial_m \hat{X}_\beta^\theta$), like $\hat{X}_\beta^\theta = i \partial_n f_\beta$, are generally valid for Kähler manifolds.

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