Deformed logarithms and entropies$^1$

G. Kaniadakis$^a$ M. Lissia$^b$ A.M. Scarfone$^{b,a}$

$^a$Istituto Nazionale di Fisica della Materia and Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy
$^b$Istituto Nazionale di Fisica Nucleare and Dipartimento di Fisica, Università di Cagliari, I-09042 Monserrato, Italy

Abstract

By solving a differential-functional equation imposed by the MaxEnt principle we obtain a class of two-parameter deformed logarithms and construct the corresponding two-parameter generalized trace-form entropies. Generalized distributions follow from these generalized entropies in the same fashion as the Gaussian distribution follows from the Shannon entropy, which is a special limiting case of the family. We determine the region of parameters where the deformed logarithm conserves the most important properties of the logarithm, and show that important existing generalizations of the entropy are included as special cases in this two-parameter class.

Key words: Deformed logarithms and exponential, generalized entropies, generalized statistical mechanics.

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1 Introduction

Many experiments in various fields of nuclear and condensed matter physics suggest the inadequacy of the Boltzmann-Gibbs statistics and the need for the introduction of new statistics [1–11]. In particular a large class of phenomena show power-law distributions with asymptotic long tails. Typically, these phenomena arise in presence of long-range forces, long-range memory, or when dynamics evolves in a non-euclidean multi-fractal space-time. In the last

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decade many authors pursued new statistical mechanics theories that mimic the structure of the Boltzmann-Gibbs theory, while capturing the emerging experimental anomalous behaviors.

An interesting approach generalizes the Shannon entropy using suitable modifications of the logarithm. This idea has generated interesting new entropic forms. Important examples are the Tsallis-entropy [8], the Abe-entropy [9] and the $\kappa$-entropy [10,11]. Remarkably, all these entropies belong to a two-parameter class, see Eq. (5.1), introduced a quarter of century ago by Mittal [12] and Sharma-Taneja [13,14], and recently studied by Borges and Roditi [15].

Let us consider the following class of trace-form entropies (in this work $k_B = 1$)

$$S(p) = -\sum_{i=1}^{N} p_i \Lambda(p_i) \ ,$$  

(1.1)

where $p \equiv \{p_i\}_{i=1}^{N}$ is a discrete probability distribution, and $\Lambda(x)$ is an analytical function that generalizes the logarithm. The main properties of a very large class of generalized logarithms have been considered in Refs. [16–18]. The canonical distribution $p$ is obtained by maximizing the entropy in Eq. (1.1) for fixed normalization and energy,

$$\sum_{i=1}^{N} p_i = 1 \ , \quad \sum_{i=1}^{N} E_i p_i = U ,$$  

(1.2)

obtaining the equation

$$\frac{d}{dp_j} \left[ p_j \Lambda(p_j) \right] = -\beta (E_j - \mu) ,$$  

(1.3)

where $\beta$ and $-\beta \mu$ are the Lagrange multipliers associated to the constraints (1.2). After introducing the inverse function of the generalized logarithm, \textit{i.e.}, the generalized exponential $\mathcal{E}(x) = \Lambda^{-1}(x)$, we require that

$$p_j = \alpha \mathcal{E} \left( -\frac{\beta}{\lambda} (E_j - \mu) \right) ,$$  

(1.4)

where $\alpha$ and $\lambda$ are two arbitrary, real and positive constants. Then Eq. (1.3) becomes the differential-functional equation [11]:

$$\frac{d}{dp_j} \left[ p_j \Lambda(p_j) \right] = \lambda \Lambda \left( \frac{p_j}{\alpha} \right) .$$  

(1.5)
The choice $\lambda = 1$ and $\alpha = e^{-1}$ yields an important special case of this class of equations: the solution of Eq. (1.5) with boundary conditions $\Lambda(1) = 0$ and $d\Lambda(x)/dx|_{x=1} = 1$ is $\Lambda(p_j) = \ln p_j$ and entropy (1.1) reduces to the Shannon-entropy.

Note that, since in general $\Lambda(x) \neq -\Lambda(1/x)$, we could use the different definition of entropy

$$S(p) = \sum_{i=1}^{N} p_i \Lambda \left( \frac{1}{p_i} \right), \quad (1.6)$$

however, we would obtain the same family of deformed logarithms: a simple mapping of the parameters brings results obtained from Eq. (1.1) into those obtained from Eq. (1.6).

In the present contribution, by solving Eq. (1.5), we obtain a class of two-parameter generalized logarithms which leads to the entropy considered in Refs. [12–15].

2 The deformed logarithm

Introducing the function $\Lambda(p_j) \equiv (1/p_j) f(\lambda \alpha \ln p_j)$ and performing the change of variable $p_j = \exp(\frac{t}{(\lambda \alpha)})$, Eq. (1.5) becomes an homogeneous differential-difference equation of the first order, belonging to the class of the delay equations [19]

$$\frac{df(t)}{dt} - f(t - t_o) = 0 , \quad (2.1)$$

where $t_o = \lambda \alpha \ln \alpha$. Its general solution is

$$f(t) = \sum_{i=1}^{n} \sum_{j=0}^{m_i-1} a_{ij}(s_1, \ldots, s_n) t^j e^{s_i t} , \quad (2.2)$$

with $n$ the number of different solutions $s_i$ of the characteristic equation

$$s_i - e^{-t_o s_i} = 0 , \quad (2.3)$$

and $m_i$ their multiplicity. In general the integration constants $a_{ij}$ depend on the parameters $s_i$. Eq. (2.3) admits $n = 0, 1$ or 2 real solutions, depending on the value of $t_o$: (a) for $t_o \geq 0$, $n = 1$ and $m = 1$; (b) for $-1/e < t_o < 0$, $n = 2$ and $m_i = 1$; (c) for $t_o = -1/e$, $n = 1$ and $m = 2$ (two coincident
solutions); finally, for \( t_0 < -1/e \) there exists no real solution. The case (a) gives the trivial solution

\[ \Lambda(p_i) = a p_i^b, \quad (2.4) \]

with \( b = \lambda \alpha s_i - 1 \). Clearly, this solution can not be used to define a generalized logarithm.

The general solution for case (b) is

\[ \Lambda(p_i) = A_1(\kappa_1, \kappa_2) p_i^{\kappa_1} + A_2(\kappa_1, \kappa_2) p_i^{\kappa_2}, \quad (2.5) \]

where \( \kappa_i = \lambda \alpha s_i - 1 \) and \( A_i(\kappa_1, \kappa_2) \) are integration constants. The boundary conditions

\[ \Lambda(1) = 0, \quad \text{and} \quad \frac{d\Lambda(x)}{dp_i} \bigg|_{p_i=1} = 1, \quad \forall \kappa_1, \kappa_2, \quad (2.6) \]

imply, respectively, \( A_1 = -A_2 \) and \( \kappa_1 A_1 + \kappa_2 A_2 = (\kappa_1 - \kappa_2) A_1 = 1 \); then Eq. (2.5) becomes

\[ \Lambda(p_i) = \frac{p_i^{\kappa_1} - p_i^{\kappa_2}}{\kappa_1 - \kappa_2}. \quad (2.7) \]

In the following we introduce a different parametrization \( \kappa = (\kappa_1 - \kappa_2)/2 \) and \( r = (\kappa_1 + \kappa_2)/2 \) and write Eq. (2.6) as:

\[ \ln_{(\kappa, r)}(p_i) = p_i^r \frac{p_i^\kappa - p_i^{-\kappa}}{2 \kappa}, \quad (2.8) \]

where the notation \( \ln_{(\kappa, r)}(p_i) \equiv \Lambda(p_i) \) has been used. The boundary condition \( \Lambda(0^+) < 0 \) implies \( r \leq |\kappa| \). From the characteristic equation (2.3) we obtain the system

\[ \begin{cases} 
1 + r + \kappa = \lambda \alpha^{-r-\kappa}, \\
1 + r - \kappa = \lambda \alpha^{-r+\kappa}, 
\end{cases} \quad (2.9) \]

which can be solved for the two constants \( \alpha \) and \( \lambda \)

\[ \alpha = \left( \frac{1 + r - \kappa}{1 + r + \kappa} \right)^{1/2 \kappa}, \quad \lambda = \left( \frac{1 + r - \kappa}{1 + r + \kappa} \right)^{(r+\kappa)/2 \kappa}. \quad (2.10) \]
In the following, we call the solution (2.8) deformed or \((\kappa, r)\)-logarithm. We remark that Eqs. (2.8) and (2.10) are invariant under \(\kappa \to -\kappa\) and that Eq. (2.8) reduces to the standard logarithm in the \((\kappa, r) \to (0, 0)\) limit.

Finally, we discuss case (c). This case corresponds to the limit of case (b) \(\kappa_1 \to \kappa_2\) and then, from Eq. (2.8) we obtain

\[
\ln_{(\kappa, r)}(p_i) = p_i^r \ln p_i ,
\]

which does not satisfy the condition \(\ln_{(\kappa, r)}(0^+) < 0\).

### 3 Properties of the \((\kappa, r)\)-logarithm

Naudts [16] gives a list of properties that a deformed logarithm must satisfy for the corresponding entropy to be physical. In this Section we determine the region of the parameter space \((\kappa, r)\) where the logarithm (2.8) verifies the following properties:

\[
\ln_{(\kappa, r)}(x) \in C^\infty(\mathbb{R}^+) ,
\]

\[
\frac{d}{dx} \ln_{(\kappa, r)}(x) > 0 ,
\]

\[
-|\kappa| \leq r \leq |\kappa| ,
\]

\[
\frac{d^2}{dx^2} \ln_{(\kappa, r)}(x) < 0 ,
\]

\[
-|\kappa| \leq r \leq \frac{1}{2} - \frac{1}{2} \left|\kappa\right| ,
\]

\[
\ln_{(\kappa, r)}(1) = 0 ,
\]

\[
\int_0^1 \ln_{(\kappa, r)}(x) dx = -\frac{1}{(1 + r)^2 - \kappa^2} ,
\]

\[
1 + r > |\kappa| ,
\]

\[
\int_0^1 \ln_{(\kappa, r)}(\frac{1}{x}) dx = \frac{1}{(1 - r)^2 - \kappa^2} ,
\]

\[
1 - r > |\kappa| .
\]

The \((\kappa, r)\)-logarithm is an analytical function for all \(x \geq 0\) and for all \(\kappa, r \in \mathbb{R}\), (3.1); is a strictly increasing function for \(-|\kappa| \leq r \leq |\kappa|\), (3.2); is concave for \(-|\kappa| \leq r \leq 1/2 - 1/2 - |\kappa|\), (3.3); verifies the boundary conditions (2.6); has at most integrable divergences at \(x = 0\) and \(x = +\infty\), (3.5) and (3.6). All these conditions (3.2)–(3.6) select the region of the parameter space

\[
\mathbb{R}^2 \supset \mathcal{R} = \begin{cases} 
-|\kappa| \leq r \leq |\kappa| & \text{if } 0 \leq |\kappa| < \frac{1}{2} , \\
|\kappa| - 1 < r < 1 - |\kappa| & \text{if } \frac{1}{2} \leq |\kappa| < 1 . 
\end{cases}
\]
which is shown in Fig. 1. Let us remark that this region includes value of
the parameters for which the deformed logarithm is finite for $x \to 0^+$ or
$x \to +\infty$, that $\kappa \to 0$ implies $r \to 0$, and that the restriction $|\kappa| < 1$ was
already obtained in Ref. [11] for the case $r = 0$.

The asymptotic behaviors of $\ln_{(\kappa,r)}(x)$, for $x \to 0^+$ and $x \to +\infty$ are

\[
\ln_{(\kappa,r)}(x) \sim \begin{cases} 
\frac{1}{2|\kappa|} \frac{1}{x^{1+|\kappa|}} & \text{for } x \to 0^+ \\
\frac{1}{2|\kappa|} x^{1+|\kappa|} & \text{for } x \to +\infty
\end{cases}, \quad \ln_{(\kappa,r)}(x) \sim \frac{1}{2|\kappa|} x^{1+|r|}, \tag{3.8}
\]

these divergences, which are integrable inside the region (3.7), become finite
values for specific choices on its boundary.

Finally, note that the property of the standard logarithm $\ln(1/x) = -\ln x$ is
satisfied by the $(\kappa, r)$-logarithm only for $r = 0$ [11], otherwise

\[
\ln_{(\kappa,r)}(1/x) = -\ln_{(\kappa,-r)}(x), \tag{3.9}
\]

showing that the entropies (1.6) are the same as the ones defined by Eq. (1.1)
with the re-parametrization $r \to -r$.

Being the deformed logarithm (2.8) a strictly monotonic function for $\kappa, r \in \mathcal{R}
(3.2)$, its inverse function exists and we call it deformed exponential $\exp_{(\kappa,r)}(x)$. Its analytical properties readily follow from the ones of the deformed logarithm
(3.1)–(3.6):

\[
\exp_{(\kappa,r)}(x) \in C^\infty(\mathbb{R}), \quad \kappa, r \in \mathcal{R} - \{r = \pm|\kappa|\}, \tag{3.10}
\]
\[
\frac{d}{dx} \exp_{(\kappa,r)}(x) > 0, \quad -|\kappa| \leq r \leq |\kappa|, \tag{3.11}
\]
\[
\frac{d^2}{dx^2} \exp_{(\kappa,r)}(x) > 0, \quad -|\kappa| \leq r \leq \frac{1}{2} - \frac{1}{2} - |\kappa|, \tag{3.12}
\]
\[
\exp_{(\kappa,r)}(0) = 1, \tag{3.13}
\]
\[
\int_{-\infty}^{0} \exp_{(\kappa,r)}(x) dx = \frac{1}{(1+r)^2 - \kappa^2}, \quad 1 + r > |\kappa|, \tag{3.14}
\]
\[
\int_{-\infty}^{0} \frac{dx}{\exp_{(\kappa,r)}(-x)} = \frac{1}{(1-r)^2 - \kappa^2}, \quad 1 - r > |\kappa|. \tag{3.15}
\]

Note that Eq. (3.10) states that for choices $r = \pm|\kappa|$ the deformed exponential
is defined on a reduced domain $x \in \text{Image}[\log_{(\kappa,\pm|\kappa|)}(x)]$ and not for all $x \in \mathbb{R}$,
since the deformed logarithm goes to a finite value, $\mp 1/|2\kappa|$, when $x \to 0^+$ or
Property (3.9) implies
\[
\exp_{(\kappa, r)}(x) \exp_{(\kappa, -r)}(-x) = 1 ,
\]  
(3.16)
which coincides with the one satisfied by the standard exponential, \(\exp(x) \exp(-x) = 1\), only for \(r = 0\). [11].

Finally, the asymptotic power-law behaviors of \(\exp_{(\kappa, r)}(x)\) are
\[
\exp_{(\kappa, r)}(x) \sim x \rightarrow \pm \infty |2 \kappa x|^{1/(r \pm |\kappa|)} .
\]  
(3.17)

4 Special examples of deformed logarithms

Different authors have introduced one-parameter families of logarithms in the context of generalized statistical mechanics. In the following we show how some important instances belong the two-parameter class under exam for suitable choices of the parameters.

4.1 Tsallis’ logarithm

If we impose \(r = \pm |\kappa|\) (dashed lines in Fig. 1) and introduce the parameter \(q = 1 \pm 2 |\kappa|\), Eq. (2.8) becomes Tsallis’ logarithm [8]
\[
\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} ,
\]  
(4.1)
which is an analytical, increasing and concave function for all \(x \geq 0\); property (3.5) becomes \(\int_0^1 \ln_q(x) \, dx = -1/(2 - q)\). The relation (3.9) becomes
\[
\ln_q \left(\frac{1}{x}\right) = - \ln_{2-q} (x) .
\]  
(4.2)

The asymptotic behaviors are
\[
\ln_q(x) \sim x \rightarrow 0^+ - \frac{1}{1 - q} \quad \text{and} \quad \lim_{x \rightarrow +\infty} \ln_q(x) = \frac{x^{1-q}}{1 - q}
\]  
(4.3)
for \(q \in (0, 1)\); behaviors for \(q \in (1, 2)\) follow from Eq. (4.2).
We remark that the relation between Tsallis’ logarithm and entropy is obtained from Eq. (1.1) with the substitution \( q \to 2 - q \).

The \( q \)-exponential is

\[
\exp_q(x) = [1 + (1 - q) x]^{1/(1-q)}, \tag{4.4}
\]

which is an analytic, monotonic and convex function in \( x \in [-1/(1-q), +\infty) \) for \( q \in (0, 1) \) and in \( x \in (-\infty, 1/(q-1)] \) for \( q \in (1, 2) \). When \( q \in (0, 1) \), the exponential (4.4) diverges as

\[
\exp_q(x) \sim x^{(1-q)/q-1} \quad \text{as} \quad x \to +\infty, \tag{4.5}
\]

while it approaches to zero for \( x \to -1/(1-q) \). As for the logarithm, the behavior for \( q \in (1, 2) \) can be obtained from the symmetry \((x, q) \to (-x, 2-q)\).

4.2 Abe logarithm

The general logarithm (2.8) with the constraint \( r = \sqrt{1+\kappa^2} - 1 \) (dash-dotted line in Fig. 1) becomes the logarithm associated to the entropy introduced by Abe [9]

\[
\ln_{q_A}(x) = \frac{x^{(q_A^{-1})-1} - x^{q_A^{-1}}}{q_A^{-1} - q_A}, \tag{4.6}
\]

where \( q_A = \sqrt{1+\kappa^2} + \kappa \); the standard logarithm is recovered for \( q_A \to 1 \). The fact that the Abe-logarithm is a special case of a two-parameter generalized logarithm has already been noticed [15].

The logarithm (4.6) is analytic, monotonic and concave for \( q_A \in (1/2, 2) \) and \( x \geq 0 \); it is invariant under \( q_A \to 1/q_A \). Eq. (3.5) becomes \( \int_0^1 \ln_{q_A}(x) \, dx = -1 \), while its asymptotic behaviors for \( q_A < 1 \) are

\[
\ln_{q_A}(x) \sim x^{q_A^{-1}} - \frac{x^{q_A^{-1}}}{q_A^{-1} - q_A} \quad \text{as} \quad x \to 0^+, \quad \ln_{q_A}(x) \sim \frac{x^{(q_A^{-1})-1}}{q_A^{-1} - q_A} \quad \text{as} \quad x \to +\infty. \tag{4.7}
\]

The corresponding inverse function, the Abe exponential, has all the general properties (3.10)–(3.15) plus the specific ones corresponding to the symmetries of the logarithm. It is not possible to express this inverse in terms of known functions.
4.3 $\kappa$-logarithm

Given the form (2.8) of the logarithms we are studying, it is of special interest the symmetric choice $r = 0$ (solid line in Fig. 1) which yields the $\kappa$-logarithm introduced in Ref. [10]

$$\ln_{(\kappa)}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} \quad -1 < \kappa < 1 . \quad (4.8)$$

In fact this $\kappa$-logarithm is the only member of the family satisfying the relation

$$\ln_{(\kappa)} \left( \frac{1}{x} \right) = -\ln_{(\kappa)}(x) , \quad (4.9)$$

making equivalent the choices $-\log_{(\kappa)}(x)$ or $\log_{(\kappa)}(1/x)$ for constructing the entropy. The main properties of $\ln_{(\kappa)}(x)$ are those in Eqs. (3.1)--(3.6); in particular $\int_0^1 \ln_{(\kappa)}(x) \, dx = -1/(1 - \kappa^2)$, and the asymptotic behaviors are

$$\ln_{(\kappa)}(x) \sim x \to 0^+ \frac{1}{2 |\kappa|} \frac{1}{x^{|\kappa|}} , \quad \ln_{(\kappa)}(x) \sim x \to +\infty \frac{1}{2 |\kappa|} x^{|\kappa|} . \quad (4.10)$$

The inverse function of $\ln_{(\kappa)}(x)$ is the $\kappa$-exponential:

$$\exp_{(\kappa)}(x) = \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{1/\kappa} , \quad (4.11)$$

which is an analytical, monotonic and convex function for all $x \in IR$ that reproduces the standard exponential for $\kappa \to 0$ and that has the asymptotic power-law behaviors

$$\exp_{(\kappa)}(x) \sim x \to \pm\infty |2 \kappa x|^{2|\kappa|} . \quad (4.12)$$

In addition, it verifies the relation

$$\exp_{(\kappa)}(x) \exp_{(\kappa)}(-x) = 1 : \quad (4.13)$$

no ambiguity exists between $\exp_{(\kappa)}(-x)$ or $1/\exp_{(\kappa)}(x)$ as statistical weight.
Fig. 1. Parameter space $(\kappa, r)$ for the logarithm (2.8). The shaded region represents the constraints of Eq. (3.7) on the parameters. The three lines, dashed, dash-dotted and solid, correspond to the Tsallis-, Abe- and $\kappa$-logarithm, respectively, defined by Eqs. (4.1), (4.6), and (4.8).

5 Conclusions

We have studied the bi-parametric family of deformed logarithms (2.8). These logarithms are obtained by solving the differential-functional equation (1.5) arising from the canonical MaxEnt principle with the resulting entropy

$$S_{\kappa, r}(p) = -\sum_{i=1}^{N} p_i^{1+r} \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa}.$$  \hspace{1cm} (5.1)

We have determined the ranges for the deformed parameters $\kappa$ and $r$ (see Fig. 1) for which the $(\kappa, r)$-logarithm verifies those properties that allow Eq. (5.1) to be used as entropy.
Several important one-parameter generalized entropies (Tsallis-entropy, Abe-entropy and $\kappa$-entropy) have been shown to belong to this family. There remains the question of the relevance of each mathematically sound entropy to specific physical situations.

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