The Mixing Time of Glauber Dynamics for Colouring Regular Trees∗
(Revised Version)

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Abstract

We consider Metropolis Glauber dynamics for sampling proper $q$-colourings of the $n$-vertex complete $b$-ary tree when $3 \leq q \leq b/(2 \ln b)$. We give both upper and lower bounds on the mixing time. Our upper bound is $n^{O(b/\log b)}$ and our lower bound is $n^{\Omega(b/(q \log b))}$, where the constants implicit in the $O()$ and $\Omega()$ notation do not depend upon $n$, $q$ or $b$.

1 Introduction

This paper proves both upper and lower bounds on the mixing time of Glauber dynamics for colourings of regular trees. Specifically, we consider Metropolis Glauber dynamics for sampling proper $q$-colourings of the $n$-vertex complete $b$-ary tree when $3 \leq q \leq b/(2 \ln b)$. We give both upper and lower bounds on the mixing time, providing reasonably precise information on the dependence of the mixing time on $n$, $b$ and $q$. For given $q$ and $b$, our upper bound is $n^{O(b/\log b)}$ and our lower bound is $n^{\Omega(b/(q \log b))}$, where the constants implicit in the $O()$ and $\Omega()$ notation do not depend upon $n$, $q$ or $b$. Since $b/(q \log b)$ and $b/\log b$ are both bounded away from zero, one could equivalently say that the mixing time is sandwiched between $n^{c_1 b/(q \ln b)}$ and $n^{c_2 b/\ln b}$, where $c_1, c_2 > 0$ are constants.

At the time of publication, the results presented here have been sharpened in two stages. Lucier and Molloy, and the same authors with Peres [12], have established the asymptotically correct exponent in the mixing time. (It is our lower bound that is tight.) They also generalised the setting for the upper bounds to non-regular trees. Very recently, Tetali, Vera, Vigoda and Yang [20] have pinned down the constant in the exponent and located the phase transition between “nearly linear” (i.e, $n^{1+o(b)}$) and superlinear mixing time.

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2 Previous work

There has been quite a bit of work on Markov chains for sampling the proper $q$-colourings of an input graph. Much of this work focuses on Glauber dynamics, which is a general term for a Markov chain which updates the colour of one vertex at a time. Proper colourings correspond to configurations in the zero-temperature Potts model from statistical physics, and there is a close connection between the mixing time of Glauber dynamics and the qualitative properties of the model. In particular, rapid mixing, specifically $O(n \log n)$ mixing for an $n$-vertex sub-graph of an infinite graph, often coincides with the case in which the model has a unique infinite-volume Gibbs measure on the infinite graph. See Weitz’s PhD thesis [21] and Martinelli’s lecture notes [13] for an exposition of this material.

Martinelli, Sinclair and Weitz [14] consider Glauber dynamics on the complete $n$-vertex tree with branching factor $b$. They show that for $q \geq b+3$, Glauber dynamics for sampling proper $q$-colourings mixes in $O(n \log n)$ time for arbitrary boundary conditions. This result is optimal in the sense that for $q \leq b+2$ there are boundary conditions for which Glauber dynamics is not even ergodic.

It is also interesting to determine whether Glauber dynamics is rapidly mixing for smaller $q$ in the absence of boundary conditions. Berger, Kenyon, Mossel and Peres [2, Prop. 2.5] prove that the mixing time of Glauber dynamics for quite general spin systems on complete trees is polynomial. Their result includes the case of Glauber dynamics on proper $q$-colourings, for any $q \geq 3$. Our upper bound applies specifically to $q$-colourings, but has the advantage of providing an explicit expression for the exponent of the mixing time in terms of $b$.

Perhaps the main contribution of this paper lies in the lower bound on mixing time. Bhatnagar, Vera and Vigoda [3] consider reconstruction on the complete tree with branching factor $b$. They show that for $C = 2$ and $q > C(b + 1)/\ln(b + 1)$ non-reconstruction holds, meaning that, over random colourings of the leaves, the expected influence on the root is vanishing. It is known [15] that the expected influence is non-vanishing for a sufficiently large $q$ satisfying $q \leq (1 - \varepsilon)(b + 1)/\ln(b + 1)$ for some $\varepsilon > 0$. This non-vanishing influence implies [2, 3] that the mixing time $\tau(\delta)$ of Glauber dynamics cannot be $O(n(\log n + \log \delta^{-1}))$. Here we provide a much stronger polynomial bound that holds when $q$ is just a modest constant factor smaller than this. Again, the exponent is given explicitly in terms of $b$ and $q$.

It is worth mentioning that the possibility of polynomial mixing time in the regime $q < b + 1$ (interpreting $b + 1$ as vertex degree) has been studied in situations other than complete trees. Goldberg, Martin and Paterson [9], and Luby, Randall and Sinclair [11] have treated 3-colourings of finite subregions of the 2-dimensional integer lattice. Hayes Vera and Vigoda [10], and Dyer, Flaxman, Frieze and Vigoda [6] have investigated $q$-colourings in planar graphs and random graphs, respectively.
3 Proof techniques

The upper bound argument is based on canonical paths. The lower bound argument is based on conductance. Essentially, the argument is that it takes a while to move from a colouring in which the colour of the root is forced to be one colour by the induced colouring on the leaves to a colouring in which the colour of the root is forced to be another colour. This is similar to the recursive majority idea [2] used to prove a lower bound for the Ising model.

4 The problem

Fix $b \geq 2$ and $q \geq 3$. Let $[q] = \{0, \ldots, q - 1\}$. Suppose $T$ is a complete $b$-ary tree of height $H$ — meaning that there are $H$ edges on a path from the root $r(T)$ to any leaf. Let $V$ be the set of vertices of $T$ and $n = |V|$. Let $L$ be the set of leaves of $T$. Note that

$$n = \frac{b^{H+1} - 1}{b - 1},$$

so

$$H = \frac{\log((b - 1)n + 1)}{\log b} - 1. \tag{1}$$

The height $h(v)$ of a vertex $v \in V$ is the number of edges on a path from $v$ down to a leaf. So a vertex $v \in L$ has $h(v) = 0$ and $h(r(T)) = H$.

For any vertex $v$ of $T$, $T_v$ denotes the subtree of $T$ rooted at $v$. For any subtree $T_v$, let $V(T_v)$ be the set of vertices of $T_v$ and let $L(T_v)$ be the set of leaves. A proper $q$-colouring of $T_v$ is a labelling of the vertices with elements of $[q]$ such that neighbouring vertices receive different colours. Let $\Omega(T_v)$ be the set of proper $q$-colourings of $T_v$ and $\Omega = \Omega(T_{r(T)})$ be the set of proper $q$-colourings of $T$.

For a colouring $x \in \Omega$, let $x(T_v)$ denote the restriction of $x$ to the vertices in the subtree $T_v$. Similarly, for a set $U \subseteq V(T_v)$ and a colouring $x \in \Omega(T_v)$, $x(U)$ denotes the restriction of $x$ to $U$.

Let $\mathcal{M}$ be the Metropolis Glauber dynamics for sampling from $\Omega$. To move from one colouring to another, this chain selects a vertex $v$ and a colour $c$ uniformly at random. The vertex $v$ is re-coloured with $c$ if and only if this results in a proper colouring. If $q \geq 3$ then the set of proper colourings is connected and $\mathcal{M}$ converges to the uniform distribution on $\Omega$, which we call $\pi$. The goal is to study the mixing time of $\mathcal{M}$ as a function of $n$, $b$ and $q$. Let $P$ be the transition matrix of $\mathcal{M}$. The variation distance between distributions $\theta_1$ and $\theta_2$ on $\Omega$ is

$$||\theta_1 - \theta_2|| = \frac{1}{2} \sum_i |\theta_1(i) - \theta_2(i)| = \max_{A \subseteq \Omega} |\theta_1(A) - \theta_2(A)|.$$

For a state $x \in \Omega$, the mixing time of $\mathcal{M}$ from starting state $x$ is

$$\tau_x(\mathcal{M}, \delta) = \min \{t > 0 : ||P^t(x, \cdot) - \pi(\cdot)|| \leq \delta \text{ for all } t' \geq t\}.$$
The mixing time of $\mathcal{M}$ is given by

$$\tau(\mathcal{M}, \delta) = \max_x \tau_x(\mathcal{M}, \delta).$$

Our results are as follows, where lg denotes the base-2 logarithm and ln denotes the natural logarithm.

**Theorem 1.** Suppose $q \geq 3$. Let $\mathcal{M}$ be the Metropolis Glauber dynamics for sampling proper $q$-colourings of the $n$-vertex complete $b$-ary tree. Then for fixed $q$ and $b$ the mixing time $\tau(\mathcal{M}, 1/(2e)) = n^{O(b/\log b)}$ where the constant implicit in the $O()$ notation does not depend upon $n$, $q$ or $b$. In particular,

$$\tau(\mathcal{M}, 1/(2e)) \leq 3bq^2(1 + \lg(n))n^{3 + 3b/\ln b}.$$ 

**Theorem 2.** Suppose $3 \leq q \leq b/(2 \ln b)$. Let $\mathcal{M}$ be the Metropolis Glauber dynamics for sampling proper $q$-colourings of the $n$-vertex complete $b$-ary tree. Then for fixed $q$ and $b$ the mixing time $\tau(\mathcal{M}, 1/(2e)) = n^{\Omega(b/(q \log b))}$ where the constant implicit in the $\Omega()$ notation does not depend upon $n$, $q$ or $b$. In particular,

$$\tau(\mathcal{M}, 1/(2e)) \geq \left(\frac{1}{2} - \frac{1}{2e}\right) \frac{2}{9} n^{\frac{b}{\ln b} \frac{\ln b}{\ln n}}.$$ 

5 Bounds on $H$

The calculations arising in the derivation of Theorems 1 and 2 involve $H$. It is clear from Equation (1) that $H = \Theta(\log(n)/\log b)$. Since we give explicit bounds in the statement of the theorems, we also require upper and lower bounds on $H$. We record these here. Note that the bounds can be improved, but we prefer to avoid the complication.

**Lemma 3.** $H + 1 \leq \lg(n) + 1$ and $H \leq \ln(n)/\ln b$. If $n \geq b^3$ then $H - 1 \geq \ln(n)/(3 \ln b)$.

**Proof.** For the first upper bound, use Equation (1) to see that

$$H + 1 = \log_b((b - 1)n + 1) \leq \log_b(bn) = 1 + \log_b n \leq 1 + \lg n,$$

since $n \geq 1$ and $b \geq 2$. For the second upper bound, note that

$$H = \frac{\ln(n(b - 1 + 1/n))}{\ln b} - 1 = \frac{\ln n}{\ln b} - \frac{\ln b - \ln(b - 1 + 1/n)}{\ln b} \leq \frac{\ln n}{\ln b}.$$

Finally, for the lower bound, note that

$$H - 1 = \frac{\ln(n(b - 1 + 1/n))}{\ln b} - 2 = \frac{\ln n}{\ln b} + \frac{\ln(b - 1 + 1/n)}{\ln b} - 2.$$

Dropping the non-negative middle term, this is at least $\ln(n)/\ln(b) - 2$, which gives the result since $\ln(n)/(3 \ln b) \geq 1$. $\square$
6 The upper bound

In this section we prove Theorem 1. We will use the canonical paths method of Jerrum and Sinclair [19]. Let \( \mathcal{M}' \) be the trivial Markov chain on \( \Omega \) that moves from a state \( x \) to a new state \( y \) by selecting \( y \) u.a.r. from \( \Omega \). Let \( P' \) be the transition matrix of \( \mathcal{M}' \). Clearly, for any \( \delta' > 0 \), \( \tau(\mathcal{M}', \delta') = 1 \). We will define canonical paths between pairs of colourings in \( \Omega \). These canonical paths will constitute what is called an \((\mathcal{M}, \mathcal{M}')\)-flow. Then Theorem 1 follows from the following proposition (which is Observation 13 in the expository paper [8]) taking \( A(f) \) to be the congestion of the flow and \( c \) to be \( 1/q \). The proof of Proposition 4 combines Diaconis and Saloff Coste’s comparison method [4] with upper and lower bounds on mixing time [1, 5, 18] along lines first proposed by Randall and Tetali [17]. See [8] for details.

Proposition 4. Suppose that \( \mathcal{M} \) is a reversible ergodic Markov chain with transition matrix \( P \) and stationary distribution \( \pi \) and that \( \mathcal{M}' \) is another reversible ergodic Markov chain with the same stationary distribution. Suppose that \( f \) is a \((\mathcal{M}, \mathcal{M}')\)-flow. Let \( c = \min \pi(x,P(x,x)) \), and assume \( c > 0 \). Then, for any \( 0 < \delta' < 1/2 \),

\[
\tau_x(\mathcal{M}, \delta) \leq \max \left\{ \frac{A(f) \left[ \frac{\tau(\mathcal{M}', \delta')}{\ln(1/2\delta')} + 1 \right]}{2c}, \frac{1}{\delta\pi(x)} \right\} \ln \frac{1}{\delta\pi(x)}.
\]

For each pair of distinct colourings \( x, y \in \Omega \) we will construct a path \( \gamma_{x,y} \) from \( x \) to \( y \) using transitions of \( \mathcal{M} \). This gives an \((\mathcal{M}, \mathcal{M}')\)-flow \( f \) with congestion

\[
A(f) = \max_{z,w} \frac{1}{\pi(z)P(z,w)} \sum_{x,y((z,w) \in \gamma_{x,y})} |\gamma_{x,y}| \pi(x)P'(x,y)
\]

\[
= \frac{nq}{|\Omega|} \max_{z,w} \sum_{x,y((z,w) \in \gamma_{x,y})} |\gamma_{x,y}|,
\]

where the maximum is over pairs of distinct states \( z \) and \( w \) in \( \Omega \) with \( P(z,w) > 0 \) (hence, \( P(z,w) = 1/nq \)) and \(|\gamma_{x,y}|\) denotes the length of \( \gamma_{x,y} \), which is the number of transitions on the path. We will prove the following lemma below.

Lemma 5. The canonical paths correspond to an \((\mathcal{M}, \mathcal{M}')\)-flow \( f \) with \( A(f) \leq bq(H + 1)n^2g^{bH} \).

Theorem 1 follows. Combining Proposition 4 with \( \delta' = 1/2c^2 \) and Lemma 5, we get

\[
\tau_x(\mathcal{M}, \delta) \leq bq(H + 1)n^2g^{bH} \left( \frac{1}{2} + 1 \right) \ln(|\Omega|/\delta).
\]

Since \(|\Omega| \leq q^n \),

\[
\tau_x(\mathcal{M}, 1/(2c)) \leq bq(H + 1)n^2g^{bH} \left( \frac{1}{2} + 1 \right) \ln(2eq^n)
\]

\[
\leq (H + 1)bqn^2\frac{3}{2}(2 + n \ln(q))g^{bH}
\]

\[
\leq (H + 1)bq^2n^33e^{3bH}.
\]
Theorem 1 then follows by applying the two upper bounds in Lemma 3.

**Proof of Lemma 5**

**Defining the canonical paths: a special case**

We start by defining paths between colourings \( x \) and \( y \) for the special case in which, for all \( v \in V \), \( y(v) = x(v) + 1 \pmod{q} \). The sequence of colourings on the path is defined to be the sequence of colourings visited by procedure \( Cycle^+ \) below when it is called with the input \( T \), which is initially coloured \( x \).

Here is the description of procedure \( Cycle^+(\hat{T}) \), where \( \hat{x} \) is a global variable, representing the current colouring of tree \( T \), and the input parameter \( \hat{T} \) may be any of the subtrees \( T_v \).

1. Let \( \hat{T}_1, \ldots, \hat{T}_b \) be the subtrees rooted at the children of \( r(\hat{T}) \) and let \( S = \{ i : x(r(\hat{T}_i)) + 1 \neq x(r(\hat{T})) \pmod{q} \} \),
2. For each \( i \in S \) do \( Cycle^+(\hat{T}_i) \).
3. Recolour the root \( r(\hat{T}) \) so that \( \hat{x}(r(\hat{T})) = x(r(\hat{T})) + 1 \pmod{q} \).
4. For each \( i \notin S \) do \( Cycle^+(\hat{T}_i) \).

Since \( q \geq 3 \), we are guaranteed that \( x(r(\hat{T})) + 1 \neq \hat{x}(r(\hat{T})) \pmod{q} \), for all \( i \), after line 2; this ensures that the root can be recoloured in line 3.

The procedure \( Cycle^+ \) is useful in the construction of more general canonical paths. When recolouring a non-leaf vertex \( v \) it will generally be necessary to change the colours of some of the children of \( v \) to avoid clashes. Applying the procedure \( Cycle^+ \) to the subtrees beneath \( v \) provides a systematic means to achieve this.

**Analysis of the special case**

Suppose we observe a transition at some point during the execution of the procedure \( Cycle^+(\hat{T}) \), in which the colouring \( \hat{x} \) is transformed by adding 1 to the colour of some vertex \( v \) (modulo \( q \)). How many initial colourings \( x(\hat{T}) \) (and hence how many final colourings \( y(\hat{T}) \)) are consistent with this observed transition?

We will let \( s(h) \) denote the maximum number of consistent initial colourings \( x(\hat{T}) \), maximised over all trees \( \hat{T} \) of height \( h \) and over all possible transitions. We will compute an upper bound on \( s(h) \).

**Case 1:** Suppose that \( v = r(\hat{T}) \).

The subtrees \( \hat{T}_i \) with \( i \in S \) have already been processed by the time that the transition takes place, so \( \hat{x}(T_i) = y(T_i) \) for these trees. The subtrees with \( i \notin S \) are yet to be processed, so for these trees we have \( \hat{x}(T_i) = x(T_i) \). However, we do not know the set \( S \) from observing the transition from \( \hat{x} \). Thus, as many as \( 2^b \) initial colourings \( x(\hat{T}) \) may be consistent with the observed transition from \( \hat{x} \).
Case 2: Otherwise, \( v \) is in one of the subtrees \( \hat{T}_k \) rooted at one of the children of \( r(\hat{T}) \). Then, by the argument of Case 1, there are two choices for the initial colouring \( x(T_i) \) of every subtree with \( i \neq k \); also there are two possibilities for \( x(r(\hat{T})) \), since we don’t know whether line (3) has been executed at the point of the transition. Then \( s(h) \) satisfies the recurrence \( s(h) \leq \max\{2^b, 2^b s(h-1)\} \) with initial condition \( s(0) = 1 \). Solving the recurrence, we discover that at most
\[
s(h) \leq 2^{bh}
\]
initial colourings \( x(\hat{T}) \) are consistent with the observed transition, so there are at most \( s(H) \leq 2^{bh} \) initial colourings \( x \) of \( T \) consistent with an observed transition of the procedure \( \text{Cycle}^+(T) \)

Defining the canonical paths: the general case

Let \( \text{Cycle}^- \) be defined analogously to \( \text{Cycle}^+ \) but implementing the permutation of colours that subtracts 1 (modulo \( q \)) from every colour; that is, \( y(v) = x(v) - 1 \) (mod \( q \)) for all \( v \in V \).

Let \( F \subseteq [q] \) be a set of “forbidden colours” of size at most two. Given \( \text{Cycle}^+ \) and \( \text{Cycle}^- \) it is easy to implement a procedure \( \text{Cycle}(\hat{T}, F) \) that systematically recolours the tree \( \hat{T} \) so that the new colour assigned to \( r(\hat{T}) \) avoids the forbidden colours \( F \): simply apply \( \text{Cycle}^+ \) or \( \text{Cycle}^- \) or neither in order to bring a colour not in \( F \) to the root of \( \hat{T} \). If we observe a transition during the execution of \( \text{Cycle}(\hat{T}, F) \) we can tell whether it comes from \( \text{Cycle}^+ \) or from \( \text{Cycle}^- \).

The recursive procedure \( \text{Recolour} \), to be described presently, provides a systematic approach to transforming an arbitrary initial colouring \( x \) to an arbitrary final colouring \( y \) using single-vertex updates. In doing so, it defines canonical paths between arbitrary pairs of proper colourings \( x \) and \( y \) of \( T \). The sequence of colourings on the path \( \gamma_{x,y} \) is defined to be the sequence of colourings visited by procedure \( \text{Recolour} \) when it is called with the input \( T \) (which is initially coloured \( x \)) and with colouring \( y \).

Like \( \text{Cycle}^+ \), the procedure \( \text{Recolour} \) takes an argument \( \hat{T} \), which is the tree which will be recoloured from \( x(\hat{T}) \) to \( y(\hat{T}) \). It also takes the argument \( y \). As before, \( \hat{x} \) is a global variable representing the current colouring of the tree \( T \), which is initially coloured \( x \). Here is the description of procedure \( \text{Recolour}(\hat{T}, y) \).

1. Let \( \hat{T}_1, \ldots, \hat{T}_b \) be the subtrees rooted at the children of \( r(\hat{T}) \).
2. For each \( i, 1 \leq i \leq b \), do \( \text{Cycle}(\hat{T}_i, \{x(r(\hat{T})), y(r(\hat{T}))\}) \). (This step permutes the colours in a subtree, to allow the root to be recoloured in the following step.)
3. Assign the root \( r(\hat{T}) \) its final colour \( y(r(\hat{T})) \).
4. For each \( i, 1 \leq i \leq b \), do \( \text{Recolour}(\hat{T}_i, y) \).
Analysis of the canonical paths

Suppose we observe a transition at some point during the execution of a procedure call Recolour($\hat{T}, y$) when $\hat{T}$ has height $h$. Let $P(h)$ be an upper bound on the number of pairs $(x(\hat{T}), y(\hat{T}))$ consistent with this transition, maximised over all trees $\hat{T}$ of height $h$ and over all possible transitions. Let

$$C(h) = q(q - 1)^{(b^h+1)-(b-1)-1}$$

be the number of proper colourings of a $b$-ary tree of height $h$. Note that $P(H)$ is an upper bound on the number of canonical paths $\gamma_{x,y}$ using a given transition. In order to compute the congestion $A(f)$ using Equation (2), we need to compute an upper bound on $P(H)$. We will compute an upper bound on $P(h)$ by induction on $h$. The base case is $P(0) = 1$.

Now suppose $h > 0$. Suppose that the transition starts at a colouring $\hat{x}$ and changes the colour of vertex $v$ from $\hat{x}(v)$ to a new colour.

**Case 1:** First, suppose $v = r(\hat{T})$. We start by bounding the number of colourings $x(\hat{T})$ that are consistent with the transition. From the transition, we know the initial colour of the root, $x(r(\hat{T}))$. For each subtree $\hat{T}_i$, we know that the initial colouring $x(\hat{T}_i)$ can be obtained by permuting the colours in $\hat{x}(\hat{T}_i)$. There are three possible permutations (corresponding to adding $-1$, $0$, or $1$ modulo $q$). So the number of possibilities for $x(\hat{T})$ is at most $3^b$. Next we bound the number of consistent colourings $y(\hat{T})$. The colour $y(r(\hat{T}))$ is fixed by the transition, but we know nothing about the colourings of the subtrees $\hat{T}_i$ beyond the fact that they must be consistent with the root being coloured $y(r(\hat{T}))$. Thus there are at most $((q - 1)C(h - 1)/q)^b$ possibilities for $y(\hat{T})$. Overall, we have the upper bound

$$P(h) \leq (3(q - 1)C(h - 1)/q)^b$$

(4)

in the case $v = r(T)$.

**Case 2:** Now suppose $v$ is contained in one of the subtrees $\hat{T}_k$. It could be that the transition under consideration is employed during Step 2 of Recolour (we call such a transition Type A), or in Step 4 (Type B).

**Case 2A:** Consider first pairs of Type A. How many pairs $(x(\hat{T}), y(\hat{T}))$ of initial and final colourings may use the transition? We’ll bound this number by considering separately the pairs $(x(r(\hat{T})), y(r(\hat{T})))$ and $(x(\hat{T}_i), y(\hat{T}_i))$ and multiplying the results. For the root, $x(r(\hat{T})) = \hat{x}(r(\hat{T}))$, while there are $q$ possibilities for $y(r(\hat{T}))$. For $i < k$, there are at most three possibilities for the colouring $x(\hat{T}_i)$, and at most $C(h - 1)$ for $y(\hat{T}_i)$. For $i > k$, $x(\hat{T}_i)$ is fixed by the transition, while there are at most $C(h - 1)$ possibilities for $y(\hat{T}_i)$. Now consider the possibilities for $x(\hat{T}_k)$ and $y(\hat{T}_k)$, starting with $x(\hat{T}_k)$. Given the transition from $\hat{x}(v)$ to its new colour we can tell whether the instance of Cycle($\hat{T}_k$, $(x(r(\hat{T})), y(r(\hat{T})))$) is applying Cycle$^+$ to $\hat{T}_k$ or Cycle$^-$ to $\hat{T}_k$. In either case, (3) guarantees that the number of initial colourings $x(\hat{T}_k)$ that are consistent with the transition is at most $2^{b(h-1)}$. Since the number of possibilities for $y(\hat{T}_k)$ is at most $C(h - 1)$, the number for the pair $(x(\hat{T}_k), y(\hat{T}_k))$ is bounded by $2^{b(h-1)}C(h - 1)$. This gives
an upper bound of \(3^h q(2^{h-1}C(h-1))^h\) on the total number of pairs \((x(\hat{T}), y(\hat{T}))\) such that the given transition is a Type A transition.

**Case 2B:** Finally, consider pairs of Type B. For the root, \(x(r(\hat{T}))\) is arbitrary, while \(y(r(\hat{T})) = \hat{x}(r(\hat{T}))\), so there are \(q\) possibilities in all. For \(i < k\), there are at most \(C(h-1)\) possibilities for the colouring \(x(\hat{T}_i)\), while \(y(\hat{T}_i)\) is fixed. For \(i > k\), there are three possibilities for \(x(\hat{T}_i)\), while there are at most \(C(h-1)\) possibilities for \(y(\hat{T}_i)\). Inductively, the number of possibilities for the pair \((x(\hat{T}_k), y(\hat{T}_k))\) is \(P(h-1)\). This gives an upper bound of \(3^h qC(h-1)^bP(h-1)\) on the total number of pairs \((x(\hat{T}), y(\hat{T}))\) such that the given transition is a Type B transition.

**Completing Case 2:** Summing the bounds on the number of pairs \((x(\hat{T}), y(\hat{T}))\) such that the given transition is a Type A or Type B transition we obtain an upper bound of

\[
P(h) \leq 3^h q C(h-1)^{b-1} [2^{(h-1)b} C(h-1) + P(h-1)]
\]

on the total number of canonical paths using a given transition in the case \(v \neq r(\hat{T})\). Notice that (5) always dominates (4) since \(h \geq 1\). Now let \(\chi(h) = P(h)/C(h)\). Since \(q^{b-1} C(h) = (q-1)^b C(h-1)^b\), we have the recurrence:

\[
\chi(h) \leq \left(\frac{3q}{q-1}\right)^b \left[2^{(h-1)b} + \chi(h-1)\right],
\]

with initial condition \(\chi(0) = q^{-1}\). Now note that the recurrence (6) satisfies \(\chi(h) \leq q^{bh}\).

**Completing the Analysis:** Let \(\lambda(h)\) be an upper bound on the number of updates performed by \(\text{Recolour}(\hat{T}, y)\) when \(\hat{T}\) has height \(h\). Thus, \(\lambda(H)\) is an upper bound on the length of a canonical path \(\gamma_{x,y}\).

Now, by Equation (2),

\[
A(f) = \frac{nq}{|\Omega|} \max_{z,w} \sum_{x,y(z,w) \in \gamma_{x,y}} |\gamma_{x,y}| \leq \lambda(H) \frac{nq}{|\Omega|} P(H) = \lambda(H) nq \chi(H),
\]

so to prove Lemma 5 we need an upper bound on \(\chi(h)\).

The subroutine \(\text{Cycle}\) creates paths of length \((b^{h+1} - 1)/(b - 1)\). The recurrence governing \(\lambda(h)\) is thus \(\lambda(h) = (b^{h+1} - 1)/(b - 1) + b\lambda(h-1)\), with initial condition \(\lambda(0) = 1\). Note that \(\lambda(h) \leq (h + 1)b^{h+1}\). This can be verified by induction on \(h\). For the inductive step,

\[
\lambda(h) = \sum_{j=0}^{h} b^j + b\lambda(h-1) \leq \sum_{j=0}^{h} b^j + hb^{h+1},
\]

which is at most \((h + 1)b^{h+1}\) since \(\sum_{j=0}^{h} b^j \leq b^{h+1}\) for \(b \geq 2\). Thus \(\lambda(H) \leq (H + 1)b^{H+1} \leq b(H + 1)n\). Putting it all together, the congestion \(A(f)\) is bounded above by \(qn\chi(H)\lambda(H)\) which proves Lemma 5.
7 The lower bound

Suppose

\[ 2q \leq b/ \ln b. \]  

(7)

The lower bound proof will use the following fact.

**Lemma 6.** If \( q \geq 3 \) and \( 2q \leq b/ \ln b \) then \( b - 2 \geq 2(q - 1) \ln(q - 1) \).

**Proof.** By (7), \( q - 1 \leq q \leq b/(2 \ln b) \) so

\[ 2(q - 1) \ln(q - 1) \leq \frac{b}{\ln b} \ln \left( \frac{b}{2 \ln b} \right) = \frac{b}{\ln b} \left( \ln(b) - \ln(2 \ln b) \right) = b - \frac{b \ln(2 \ln b)}{\ln b} \leq b - 2, \]

where the final inequality holds since \( q \geq 3 \) so \( b \geq 6 \) so \( b \geq 2 \ln(b)/ \ln(2 \ln b) \). \( \square \)

Given a colouring \( x \in \Omega \), define

\[ F(x) = \{ w \in V \mid \forall y \in \Omega(T_w) \text{ with } y(L(T_w)) = x(L(T_w)) \text{ we have } y(w) = x(w) \}. \]

Informally, \( F(v) \) is the set of vertices \( w \) of \( T \) whose colour is forced by \( x(L(T_w)) \). Our lower bound will be based on a conductance argument which shows that it takes a while to move from a colouring \( x \) in which \( r(T) \) is forced to be one colour to a colouring \( y \) in which \( r(T) \) is forced to be another colour. It is useful to note that \( F(x) \) can be defined recursively using the structure of \( T \). If \( w \) is a child of \( v \) we say that \( w \) is \( c \)-permitting for \( v \) in \( x \) if either \( x(w) \neq c \) or \( w \notin F(x) \) (or both).

**Observation 7.** If \( h(v) = 0 \) then \( v \in F(x) \). If \( h(v) > 0 \) then \( v \in F(x) \) if and only if, for every colour \( c \neq x(v) \), there is a child \( w \) of \( v \) which is not \( c \)-permitting for \( v \) in \( x \).

The recursive definition of \( F(x) \) illustrates the connection between our conductance argument and lower-bound arguments based on recursive majority functions [2, 16].

Consider a colouring \( x \) chosen uniformly at random from \( \Omega \). Observe that the events \( v \notin F(x) \), with \( v \) ranging over all vertices at height \( h \), are independent. To see this, think about constructing the colouring downwards from the root, with each vertex choosing a colour uniformly at random from the colours not used by its parent. Now observe that each event \( v \notin F(x) \) is completely determined by decisions made in the subtrees rooted at \( v \). Let \( u_h \) be probability of the event \( v \notin F(x) \), for any vertex \( v \) at height \( h \); note that by symmetry this probability depends only on \( h \).

**Lemma 8.** \( u_h \leq 1/b \).
Proof. The proof is by induction on $h$. Note that $u_0 = 0$. For the inductive step, let $v$ be a vertex at height $h > 0$. Consider a colouring $x$ chosen uniformly at random from $\Omega$. Fix a colour $c \neq x(v)$ and a child $w$ of $v$. The probability that $x(w) = c$ is $1/(q-1)$. Also, the probability that $w \in F(x)$ is $1 - u_{h-1}$ and this is independent of the the event $x(w) = c$. (This can be seen by considering the downwards colouring scheme, as before, noting that $F(x)$ is determined by the restriction of $x$ to the subtrees rooted at the children of $w$.) So the probability that $w$ is $c$-permitting for $v$ in $x$ is $1 - (1 - u_{h-1})/(q-1)$. These events are independent for different children $w$ of $v$ so the probability that every child $w$ is $c$-permitting for $v$ in $x$ is

$$\left(1 - \frac{1 - u_{h-1}}{q-1}\right)^b.$$  

By Observation 7, the event $v \notin F(v)$ occurs when there exists a colour $c \neq x(v)$ such that every child $w$ if $c$-permitting for $v$ in $x$, so by the union bound:

$$u_h = \Pr(v \notin F(x)) \leq (q-1) \left(1 - \frac{1 - u_{h-1}}{q-1}\right)^b$$

$$\leq (q-1) \exp \left(-\frac{b(1 - u_{h-1})}{q-1}\right)$$

$$\leq (q-1) \exp \left(-\frac{b-1}{q-1}\right)$$

$$\leq (q-1)b^{-2}$$

$$\leq b^{-1},$$

where (8) applies the induction hypothesis and (9) uses assumption (7).

Consider a vertex $v$ of $T$ with $h(v) \geq 1$ and a leaf $\ell$ that is a descendant of $v$. Consider $x \in \Omega$. Say that $v$ is $\ell$-loose in $x$ if there is a $c \neq x(v)$ such that every child $w$ of $v$, except possibly the one on the path to $\ell$, is $c$-permitting for $v$ in $x$.

Let $\Psi_{v,\ell}$ be the probability that $v$ is $\ell$-loose in $x$ when $x$ is chosen u.a.r. from $\Omega$. Let $\varepsilon = (q-1) \exp \left(-\frac{b-1}{q-1}\right)$.

**Lemma 9.** Consider a vertex $v$ of $T$ with $h(v) \geq 1$ and a leaf $\ell$ that is a descendant of $v$. Then $\Psi_{v,\ell} \leq \varepsilon$.

**Proof.** The calculation is very similar to the calculation in the proof of Lemma 8, with $b-1$ replacing $b$. Let $h = h(v)$. Then

$$\Psi_{v,\ell} \leq (q-1) \left(1 - \frac{1 - u_{h-1}}{q-1}\right)^{b-1}$$

$$\leq (q-1) \exp \left(-\frac{b-2}{q-1}\right),$$

where we have used the fact $u_{h-1} \leq b^{-1}$. \qed
Lemma 10. \( \Phi(\mathcal{M}) = \min_S \Phi_S(\mathcal{M}) \), where the min is over all \( S \subseteq \Omega \) with \( 0 < \pi(S) < 1 \). The inverse of the conductance of \( \mathcal{M} \) gives a lower bound on the mixing time of \( \mathcal{M} \). In particular,

\[
\tau(\mathcal{M}, 1/(2e)) \geq (1/2 - 1/(2e))/\Phi(\mathcal{M}).
\] (10)

Equation (10) is due to Dyer, Frieze and Jerrum [7]. The formulation used here is Theorem 17 of the expository paper [8].

For \( c \in [q] \), let \( S_c = \{ x \in \Omega \mid (r(T) \in F(x)) \wedge (x(r(T)) = c) \} \). Let \( S_q = \{ x \in \Omega \mid r(T) \notin F(x) \} \). Clearly, \( S_0, \ldots, S_q \) form a partition of \( \Omega \). Let \( S = S_0 \cup \cdots \cup S_{\lfloor q/2 \rfloor - 1} \). Then \( \Phi(\mathcal{M}) \leq \Phi_S(\mathcal{M}) \).

Now by Lemma 8 we have \( 0 \leq \pi(S) \leq 1/b \). Also, by symmetry, \( \pi(S_c) = \pi(S_{c'}) \) for \( c, c' \in [q] \). So

\[
\left( 1 - \frac{1}{b} \right) \frac{\lfloor q/2 \rfloor}{q - 1} \leq \pi(S) \leq \frac{\lfloor q/2 \rfloor}{q - 1}.
\]

Since \( b \geq 6 \) and \( q \geq 3 \) this gives \( \frac{5}{6} \leq \pi(S) \leq \frac{2}{3} \), so \( \pi(S) \pi(S_c) \geq \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \). Thus

\[
\Phi_S(\mathcal{M}) \leq \frac{9}{4} \left( \sum_{x \in S} \sum_{y \in S} \pi(x)P(x, y) + \sum_{x \in S} \sum_{y \in S} \pi(x)P(x, y) \right),
\]

and by reversibility

\[
\Phi(\mathcal{M}) \leq \frac{9}{2} \sum_{x \in S} \sum_{y \in S} \pi(x)P(x, y) \leq \frac{9}{2} \sum_{x, y} \pi(x)P(x, y),
\] (11)

where the summation is over \( x \) and \( y \) for which \( r(T) \in F(x) \) and either \( r(T) \notin F(y) \) or \( x(r(T)) \neq y(r(T)) \). Note that if \( x \) and \( y \) contribute to the summation in (11) then since \( P(x, y) > 0 \), they differ on a single vertex. Since \( r(T) \in F(x) \) we cannot move from \( x \) to a proper colouring \( y \) by changing the colour of \( r(T) \). Thus the only possibility is that \( r(T) \notin F(y) \) and \( x \) and \( y \) differ on a leaf. Also, given the dynamics, we have \( P(x, y) = 1/(nq) \).

Lemma 10. \( \Phi(\mathcal{M}) \leq \frac{9}{2} e^{H-1} \).

Proof. From Equation (11) and the discussion above we have

\[
\Phi(\mathcal{M}) \leq \frac{9}{2} \sum_{x, y} \pi(x)P(x, y)
\]
where the sum is over all colourings $x$ and $y$ for which $r(T) \in F(x)$ and $r(T) \not\in F(y)$ and $x$ and $y$ differ on exactly one leaf, $\ell$. Letting $c = y(\ell)$, we can write

$$\Phi(\mathcal{M}) \leq \frac{9}{2} \sum_{x \in \Omega} \sum_{\ell \in L} \sum_{c \in [q]} 1_{x,\ell,c} \pi(x) \frac{1}{nq},$$

where $1_{x,\ell,c}$ is the indicator for the event that $r(T) \not\in F(y)$ when $y$ denotes the colouring formed from $x$ by recolouring leaf $\ell$ with colour $c$. Multiplying by the $q$ possibilities for $c$ and noting that $\pi(X) = 1/|\Omega|$, we get

$$\Phi(\mathcal{M}) \leq \frac{9}{2} \frac{1}{|\Omega|} q \frac{1}{nq} \sum_{x \in \Omega, \ell \in L} 1_{x,\ell},$$

where $1_{x,\ell}$ is the indicator variable for the event that there is a colour $c$ such that, when $y$ is obtained from $x$ by changing the colour of leaf $\ell$ to $c$, we have $r(T) \not\in F(y)$. This event implies that every vertex $v$ on the path from $\ell$ to $r(T)$ is $\ell$-loose in $x$. When $x$ is chosen uniformly random these events are independent and by Lemma 9 they all have probability at most $\varepsilon$. So

$$\Phi(\mathcal{M}) \leq \frac{9}{2} \frac{1}{|\Omega|} n b^H |\Omega| \varepsilon^{H-1},$$

where $b^H$ is the number of $\ell$ in the summation and $|\Omega|$ is the number of $x$.

Theorem 2 follows from Lemma 10 since, by Equation (10), the lemma implies

$$\tau(\mathcal{M}, 1/(2e)) \geq (1/2 - 1/(2e)) \frac{2}{9} \varepsilon^{-(H-1)}.$$

Also

$$\varepsilon^{-(H-1)} = \left(\frac{1}{(q-1) \exp(-(b-2)/(q-1))}\right)^{H-1} = e^{(H-1)(\frac{b-2}{q-1} - \ln(q-1))}.$$

Using Lemma 6, this is at least

$$e^{(H-1)(\frac{b-2}{2(q-1)})}.$$

Using Lemma 3, this is at least

$$e^{\ln n \left(\frac{b-2}{2(q-1)}\right)} = n^{\frac{b-2}{2(q-1)\ln n}},$$

which gives Theorem 2.
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