Extensions of the Art Gallery Theorem

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Abstract. Several domination results have been obtained for maximal outerplanar graphs (mops). The classical domination problem is to minimize the size of a set $S$ of vertices of an $n$-vertex graph $G$ such that $G - N[S]$, the graph obtained by deleting the closed neighborhood of $S$, contains no vertices. In the proof of the Art Gallery Theorem, Chvátal showed that the minimum size, called the domination number of $G$ and denoted by $\gamma(G)$, is at most $n/3$ if $G$ is a mop. Here we consider a modification by allowing $G - N[S]$ to have a maximum degree of at most $k$. Let $\iota_k(G)$ denote the size of a smallest set $S$ for which this is achieved. If $n \leq 2k + 3$, then trivially $\iota_k(G) \leq 1$. Let $G$ be a mop on $n \geq \max\{5, 2k + 3\}$ vertices, $n_2$ of which are of degree 2. Upper bounds on $\iota_k(G)$ have been obtained for $k = 0$ and $k = 1$, namely $\iota_0(G) \leq \min\{\frac{n}{4}, \frac{n+n_2}{5}, \frac{n-n_2}{3}\}$ and $\iota_1(G) \leq \min\{\frac{n}{5}, \frac{n+n_2}{6}, \frac{n-n_2}{3}\}$. We prove that $\iota_k(G) \leq \min\{\frac{n}{k+4}, \frac{n+n_2}{k+5}, \frac{n-n_2}{k+2}\}$ for any $k \geq 0$. For the original setting of the Art Gallery Theorem, the argument presented yields that if an art gallery has exactly $n$ corners and at least one of every $k+2$ consecutive corners must be visible to at least one guard, then the number of guards needed is at most $n/(k+4)$. We also prove that $\gamma(G) \leq \frac{n-n_2}{2}$ unless $n = 2n_2$, $n_2$ is odd, and $\gamma(G) = \frac{n-n_2+1}{2}$. Together with the inequality $\gamma(G) \leq \frac{n+n_2}{4}$, obtained by Campos and Wakabayashi and independently by Tokunaga, this improves Chvátal’s bound. The bounds are sharp.

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1. Introduction

Unless stated otherwise, we use small letters such as $x$ to denote integers or elements of a set, and capital letters such as $X$ to denote sets or graphs. The set of positive integers is denoted by $\mathbb{N}$. For $m, n \in \{0\} \cup \mathbb{N}$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$. We abbreviate $[1, n]$ to $[n]$. Note
that \([0]\) is the empty set \(\emptyset\). For a set \(X\), the set of \(k\)-element subsets of \(X\) is denoted by \((X)_k\). Arbitrary sets are assumed to be finite.

Let \(G\) be a simple graph with vertex set \(V(G)\) and edge set \(E(G)\). We may represent an edge \(\{v, w\}\) by \(vw\). The order of \(G\) is \(|V(G)|\). We say that \(G\) is an \(n\)-vertex graph if its order is \(n\). The open neighborhood \(N_G(v)\) of a vertex \(v\) of \(G\) is the set of neighbors of \(v\), that is, \(N_G(v) = \{w \in V(G) : vw \in E(G)\}\). The degree \(d_G(v)\) of \(v\) is \(|N_G(v)|\). The maximum degree \(\Delta(G)\) of \(G\) is \(\max\{d_G(v) : v \in V(G)\}\). The closed neighborhood \(N_G[v]\) of \(v\) is the set \(N_G(v) \cup \{v\}\). For \(U \subseteq V(G)\), the closed neighborhood \(N_G[U]\) of \(U\) is \(\bigcup_{v \in U} N_G[v]\). The subgraph of \(G\) induced by \(U\) is denoted by \(G[U]\), that is, \(V(G[U]) = U\) and \(E(G[U]) = E(G) \cap \binom{U}{2}\). The subgraph of \(G\) obtained by deleting the vertices in \(U\) from \(G\) is denoted by \(G - U\), that is, \(G - U = G[V(G) \setminus U]\). We may abbreviate \(G - \{v\}\) to \(G - v\). We use the standard notation \(K_k\), \(P_k\), \(C_k\), and \(K_{1,k}\) for the \(k\)-vertex complete graph, the \(k\)-vertex path, the \(k\)-vertex cycle, and the \((k+1)\)-vertex star \(E(K_{1,k}) = \{uv : v \in V(K_{1,k}) \setminus \{u\}\}\) for some \(u \in V(K_{1,k})\), respectively. If \(G\) is a \(k\)-vertex path and \(E(G) = \{v_i v_{i+1} : i \in [k-1]\}\), then we represent \(G\) by \(v_1 v_2 \ldots v_k\). If \(G\) is a \(k\)-vertex cycle and \(E(G) = \{v_i v_{i+1} : i \in [k-1]\} \cup \{v_k v_1\}\), then we represent \(G\) by \(v_1 v_2 \ldots v_k v_1\).

A subset \(S\) of \(V(G)\) is a dominating set of \(G\) if each vertex in \(V(G) \setminus S\) is adjacent to at least one vertex in \(S\) (that is, \(N_G[S] = V(G)\)). The classical domination problem is to minimize the size of a dominating set (see [11, 12, 17–20]). Caro and Hansberg [7] introduced an appealing generalization that has also been treated in [2–6, 15, 27–30]. They relaxed the domination condition by considering a subset \(S\) of \(V(G)\) such that \(G - N_G[S]\) contains no forbidden subgraph. Given a graph \(H\), \(S\) is called an \(H\)-isolating set of \(G\) if \(G - N_G[S]\) does not contain a copy of \(H\). The \(H\)-isolation number of \(G\) is the size of a smallest \(H\)-isolating set of \(G\) and is denoted by \(\iota(G, H)\). The domination number of \(G\) is the size of a smallest dominating set of \(G\) and is denoted by \(\gamma(G)\). Note that \(S\) is a dominating set if and only if it is a \(K_1\)-isolating set; thus, \(\gamma(G) = \iota(G, K_1)\). We are interested in the case \(H = K_{1,k+1}\). Note that, for \(k \geq 0\), \(S\) is a \(K_{1,k+1}\)-isolating set of \(G\) if and only if \(\Delta(G - N_G[S]) \leq k\). We abbreviate \(\iota(G, K_{1,k+1})\) to \(\iota_k(G)\). Since \(K_{1,0}\) is the graph \(K_1\), we have \(\gamma(G) = \iota_{-1}(G)\).

A triangulated disc is a plane graph whose interior faces are triangles and whose exterior face (the unbounded face) is bounded by a simple cycle. A maximal outerplanar graph, or a mop, is a triangulated disc \(G\) such that the boundary of the exterior face of \(G\) contains all the vertices of \(G\). O’Rourke [25] pointed out that every mop has a unique Hamiltonian cycle. Thus, the Hamiltonian cycle of a mop is the boundary of the mop. This paper’s notation and terminology on mops follows that of [22]; in particular, each edge of the Hamiltonian cycle of a mop is called a Hamiltonian edge, while any other edge of the mop is called a diagonal. For \(n \geq 3\), the fan \(P_n\) is the mop obtained from \(P_{n-1}\) by adding a new vertex \(v\) and joining it to every vertex of \(P_{n-1}\). We say that \(v\) is the center of \(P_n\).

Domination in mops has been extensively studied since 1975. Chvátal’s proof of his classical result referred to as the Art Gallery Theorem (AGT) [10]
established that the domination number of any \( n \)-vertex mop is at most \( n/3 \). This also follows from Fisk’s elegant proof [16] (included in [1]) of AGT, and was proved directly by Matheson and Tarjan [23]. For results on other types of domination in mops, we refer the reader to [9, 13, 14, 21]. Caro and Hansberg [7] proved that the \( K_{1,1} \)-isolation number of a mop of order \( n \geq 4 \) is at most \( n/4 \). Borg and Kaemawichanurat [6] proved that the \( K_{1,2} \)-isolation number of a mop of order \( n \geq 5 \) is at most \( n/5 \).

**Theorem 1.** Let \( G \) be an \( n \)-vertex mop.

(a) [10] If \( n \geq 3 \), then \( \gamma(G) \leq \frac{n}{3} \).
(b) [7] If \( n \geq 4 \), then \( \iota_0(G) \leq \frac{n}{4} \).
(c) [6] If \( n \geq 5 \), then \( \iota_1(G) \leq \frac{n}{5} \).

Moreover, the bounds are sharp.

When we say that a bound is sharp, we mean that there are infinitely many values of \( n \) for which the bound is attained. For each of the bounds in Theorem 1, the bound is attained for each \( n \) such that the bound is an integer.

The following sharp upper bounds in terms of the order and the number of vertices of degree 2 have also been established.

**Theorem 2.** If \( G \) is a mop of order \( n \geq 3 \) and has exactly \( n_2 \) vertices of degree 2, then

(a) [8, 26] \( \gamma(G) \leq \frac{n+n_2}{4} \),
(b) [27] \( \iota_0(G) \leq \frac{n+n_2}{5} \),
(c) [6] \( \iota_1(G) \leq \frac{n+n_2}{6} \).

Moreover, the bounds are sharp.

**Theorem 3.** If \( G \) is a mop of order \( n \geq 5 \) and has exactly \( n_2 \) vertices of degree 2, then

(a) [27] \( \iota_0(G) \leq \frac{n-n_2}{3} \),
(b) [6] \( \iota_1(G) \leq \frac{n-n_2}{3} \).

Moreover, the bounds are sharp.

2. Main Results

In this paper, we mostly establish sharp upper bounds on the \( K_{1,k+1} \)-isolation number of a mop in terms of its order and the number of vertices of degree 2 for any \( k \geq -1 \). Our new results are presented in this section and in Sect. 5, and proved in Sects. 4 and 5, respectively. The first three results have Theorems 1 and 2 for \( 0 \leq k \leq 1 \), and Theorem 3(b), as special cases.

**Theorem 4.** If \( k \geq 0 \) and \( G \) is a mop of order \( n \geq k+4 \), then

\[ \iota_k(G) \leq \frac{n}{k+4}. \]

Moreover, the bound is sharp.
In Sect. 5, we present bounds that follow by the same argument used in the proof of Theorem 4. We address a relaxation of the condition in the Art Gallery Theorem that the whole polygon needs to be guarded. We show that if \( k \geq -1 \), a polygon has exactly \( n \geq k + 4 \) corners, and at least one of every \( k + 2 \) consecutive corners must be visible to at least one guard, then the number of guards needed is at most \( n/(k + 4) \) (see Theorem 11).

**Theorem 5.** If \( k \geq 0 \), \( G \) is a mop of order \( n \geq k + 3 \), and \( n_2 \) is the number of vertices of \( G \) of degree 2, then

\[
\iota_k(G) \leq \frac{n + n_2}{k + 5}.
\]

Moreover, the bound is sharp.

**Theorem 6.** If \( k \geq 1 \), \( G \) is a mop of order \( n \geq 2k + 3 \), and \( n_2 \) is the number of vertices of \( G \) of degree 2, then

\[
\iota_k(G) \leq \frac{n - n_2}{k + 2}.
\]

Moreover, the bound is sharp.

Note that, surprisingly, the sharp bound for \( k = 0 \) given by Theorem 3 is not of the general form for \( k \geq 1 \) given by Theorem 6. Unlike Theorems 1 and 2, Theorem 3 provides no bound for \( \gamma(G) \) similar to those in its parts (a) and (b). The missing bound is provided by our next theorem.

**Definition 1.** If \( t \geq 3 \), \( x_1x_2\ldots x_t x_1 \) is the unique Hamiltonian cycle \( C \) of a mop \( M \) contained by a mop \( G \), \( y_1,\ldots,y_t \) are distinct vertices of \( G \) in the exterior face of \( M \), \( N_G(y_i) = \{x_i, x_{i+1}\} \) for each \( i \in [t-1] \), \( N_G(y_t) = \{x_t, x_1\} \), \( V(G) = V(M) \cup \{y_i: i \in [t]\} \), and \( E(G) = E(M) \cup \bigcup_{i=1}^t \{y_i x: x \in N_G(y_i)\} \), then we call \( G \) a \( t \)-extreme mop or simply an extreme mop.

Clearly, a mop \( G \) as in Definition 1 can be constructed for any \( t \geq 3 \), and its Hamiltonian cycle is \( x_1 y_1 x_2 y_2 \ldots x_t y_t x_1 \).

For positive integers \( a \) and \( b \), let

\[
1(a, b) = \begin{cases} 1 & \text{if } b \text{ is odd and } a = 2b, \\ 0 & \text{otherwise}. \end{cases}
\]

**Theorem 7.** If \( G \) is a mop of order \( n \geq 4 \) and has exactly \( n_2 \) vertices of degree 2, then

\[
\gamma(G) \leq \frac{n - n_2 + 1(a, n_2)}{2}.
\]

Moreover, equality holds if \( G \) is extreme or \( n = 4 \).

Therefore, unlike Theorem 6 for \( k \geq 1 \), Theorems 3(a) and 7 yield \( \iota_k(G) \leq \frac{n - n_2 - k1(n, n_2)}{k + 3} \) for \(-1 \leq k \leq 0 \) (recall that \( \gamma(G) = \iota_{-1}(G) \)) and \( n \geq k + 5 \).

Theorem 7 enables us to improve the classical bound of Chvátal in Theorem 1(a), using Theorem 2(a). The improved bound is given in Theorem 8, which also specifies a necessary condition and a sufficient condition for an \( n \)-vertex mop \( G \) to attain Chvátal’s upper bound \( \frac{n}{3} \) on \( \gamma(G) \).
Definition 2. If \( t \geq 2, x_1x_2\ldots x_{2t}x_1 \) is the unique Hamiltonian cycle \( C \) of a mop \( M \) contained by a mop \( G \), \( y_1, \ldots, y_t \) are distinct vertices of \( G \) in the exterior face of \( M \), then \( N_G(y_1), \ldots, N_G(y_t) \) are distinct edges of \( C \), \( x_i \in N_G[\{y_1, \ldots, y_t\}] \) for each \( i \in [2t] \) with \( d_M(x_i) = 2 \), \( V(G) = V(M) \cup \{y_i: i \in [t]\} \), and \( E(G) = E(M) \cup \bigcup_{i=1}^{t} \{y_ix \in N_G(y_i)\} \), then we call \( G \) a \( t \)-special mop or simply a special mop. If \( N_G(y_1), \ldots, N_G(y_t) \) partition \( V(C) \) (that is, their union is \( V(C) \) and no two of them intersect), then we call \( G \) an extra \( t \)-special mop or simply an extra special mop.

Clearly, an extra \( t \)-special mop can be constructed for any \( t \geq 2 \).

Theorem 8. If \( G \) is a mop of order \( n \geq 4 \) and has exactly \( n_2 \) vertices of degree 2, then

\[
\gamma(G) \leq \begin{cases} 
\frac{n+n_2}{4} < \frac{n}{3} & \text{if } n_2 < \frac{n}{3}, \\
\frac{n+n_2}{4} = \frac{n-n_2+1(n,n_2)}{2} = \frac{n}{3} & \text{if } n_2 = \frac{n}{3}, \\
\frac{n-n_2+1(n,n_2)}{2} = \frac{n}{3} & \text{if } (n, n_2) = (6, 3), \\
\frac{n-n_2+1(n,n_2)}{2} < \frac{n}{3} & \text{if } n_2 > \frac{n}{3} \text{ and } (n, n_2) \neq (6, 3).
\end{cases}
\]

Moreover, the following assertions hold:

(a) The bound is sharp.

(b) If \( \gamma(G) = \frac{n}{3} \), then \( G \) is \( n_2 \)-special or 3-extreme.

(c) If \( G \) is extra \( n_2 \)-special, then \( \gamma(G) = \frac{n}{3} \).

We have \( |V(K_{1,k+1})| = k + 2 \). It immediately follows that, for any \( n \)-vertex mop \( G \), \( \iota_k(G) = 0 \) if \( n \leq k + 1 \), and \( \iota_k(G) \leq 1 \) if \( n \leq \max\{k+4,2k+3\} \). Theorems 1–7 immediately give us the following complete solution for any other value of \( n \).

Theorem 9. Let \( G \) be an \( n \)-vertex mop, and let \( n_2 \) be the number of vertices of \( G \) of degree 2.

(a) If \( k \geq 1 \) and \( n \geq 2k+3 \), then

\[
\iota_k(G) \leq \begin{cases} 
\frac{n+n_2}{k+5} & \text{if } n_2 \leq \frac{n}{k+4}, \\
\frac{n}{k+4} & \text{if } \frac{n}{k+4} \leq n_2 \leq \frac{2n}{k+4}, \\
\frac{n-n_2}{k+2} & \text{if } n_2 \geq \frac{2n}{k+4}.
\end{cases}
\]

(b) If \( -1 \leq k \leq 0 \) and \( n \geq k + 5 \), then

\[
\iota_k(G) \leq \begin{cases} 
\frac{n+n_2}{k+5} & \text{if } n_2 \leq \frac{n}{k+4}, \\
\frac{n-n_2-k1(n,n_2)}{k+3} & \text{if } n_2 \geq \frac{n}{k+4}.
\end{cases}
\]

Moreover, the bound is sharp.
3. Extremal Constructions for Theorems 4–6

We now show that the bounds in Theorems 4–6 are attainable. Theorems 1–3 already establish this for $k \leq 1$, so we settle $k \geq 2$.

When $n/(k+4) < n_2 < 2n/(k+4)$, we have $n/(k+4) < \min\{(n+n_2)/(k+4), (n-n_2)/(k+2)\}$, that is, the bound in Theorem 4 is better than those in Theorems 5 and 6 for this range. Therefore, we will first show that the bound $n/(k+4)$ is attained in cases where $n/(k+4) < n_2 < 2n/(k+4)$. For an integer $t \geq 1$, let $F_{k+4}^1, F_{k+4}^2, \ldots, F_{k+4}^{2t}$ be $2t$ vertex-disjoint fans of order $k+4$. For $i \in [2t]$, let $x_{i0}, x_{i1}, \ldots, x_{ik}$ be the vertices of $F_{k+4}^i$ with $x_{i0}$ being the center and with $x_{i1}$ and $x_{ik}$ being the vertices of degree $2$. We extend the union of $F_{k+4}^1, F_{k+4}^2, \ldots, F_{k+4}^{2t}$ to a mop $A_{k,t}$ by adding edges on the $4t$ vertices $x_{i0}, x_{i1}, \ldots, x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i2}, x_{i3}, \ldots, x_{i3}$.

Thus, $A_{k,t}$ is a mop of order $n = 2(k+4)t$ and has exactly $n_2 = 3t$ vertices of degree 2. We have $n_2 = 3n/(2(k+4))$, satisfying $n/(k+4) < n_2 < 2n/(k+4)$. Clearly, if $S$ is a $K_{1,k+1}$-isolating set of $A_{k,t}$, then $|S \cap V(F_{k+4}^i)| \geq 1$ for each $i \in [2t]$. Thus, $\mu_k(A_{k,t}) \geq 2t$. Since $\{x_{i0} : i \in [2t]\}$ is a $K_{1,k+1}$-isolating set of $A_{k,t}$, we obtain $\mu_k(A_{k,t}) = 2t = n/(k+4)$.

When $n_2 < n/(k+4)$, we have $[(n+n_2)/(k+5)] \leq \min\{(n/(k+4)), (n-n_2)/(k+2)\}$. To see the sharpness of the bound $[(n+n_2)/(k+5)]$, let $F_{k+6}^1, \ldots, F_{k+6}^t$ be $t$ copies of $F_{k+6}$ and let $F_{k+4}^1, \ldots, F_{k+4}^t$ be $t$ copies of $F_{k+4}$, where $F_{k+6}^1, \ldots, F_{k+6}^t, F_{k+4}^1, \ldots, F_{k+4}^t$ are vertex-disjoint and $(k+4)/2 \leq t \leq k$. For $i \in [t]$, let $x_{i0}, x_{i1}, \ldots, x_{ik}$ be the vertices of $F_{k+6}^i$ with $x_{i0}$ being the center and $F_{k+6}^i - \{x_{i0}\} = x_{i1} x_{i2}, \ldots, x_{ik}$. Let $y_{i1}, \ldots, y_{ik+3}$ be the vertices of $F_{k+4}^i$ with $y_{i0}$ being the center and $F_{k+4}^i - \{y_{i0}\} = y_{i1} y_{i2}, \ldots, y_{ik+3}$, and let $T_{2k+10}$ be the mop obtained by adding the edges $x_{i0}, x_{i1}, x_{i2}, x_{i3}$ to the union of $F_{k+6}^i$ and $F_{k+4}^i$. We extend the union of $T_{2k+10}^1, \ldots, T_{2k+10}^t$ to a mop $H_{k,t}$ by adding edges on the $2t$ vertices $x_{i1}, x_{i2}, \ldots, x_{i1}, x_{i2}$. The graph $H_{k,t}$ is illustrated in Fig. 1. It is a mop of order $n = (2k + 10)t$ and has exactly $n_2 = t < 2t < (2k + 10)t/(k+4) = n/(k+4)$ vertices of degree 2. Clearly, if $S$ is a $K_{1,k+1}$-isolating set of $H_{k,t}$, then, for each $i \in [t]$, $|S \cap V(F_{k+6}^i)| \geq 1$ and $|S \cap V(F_{k+4}^i)| \geq 1$. Thus, $\mu_k(H_{k,t}) \geq 2t$. Since $\{x_{i0} : i \in [t]\} \cup \{y_{i0} : i \in [t]\}$ is a $K_{1,k+1}$-isolating set of $H_{k,t}$, we obtain $\mu_k(H_{k,t}) = 2t$. Since $t > k+5$, $\mu_k(H_{k,t}) = [2t + t/(k+5)] = [(n+n_2)/2(k+5)]$. In addition, since $t \geq (k+4)/2$, $[(n/(k+4)) = [t(2k+10)/(k+4)] = [2t + 2t/(k+4)] \geq 2t \geq [(n+n_2)/(k+5)]$ and $[(n-n_2)/(k+2)] = [(2k+10)t - t)/(k+2)] = [2t + 5t/(k+2)] \geq [2t + 5(k+4)/(k+2)] \geq 2t + 2 > [(n+n_2)/(k+5)]$.

We now show that the actual bound $(n+n_2)/(k+5)$ in Theorem 5 is attained for $n_2 = n/(k+4)$. For an integer $t \geq 1$, let $F_{k+4}^1, \ldots, F_{k+4}^t$ be $t$ vertex-disjoint fans of order $k+4$. For $i \in [t]$, let $x_{i0}, x_{i1}, \ldots, x_{ik}$ be the vertices of $F_{k+4}^i$ with $x_{i0}$ being the center and with $x_{i1}$ and $x_{ik}$ being the vertices of degree 2. We extend the union of $F_{k+4}^1, \ldots, F_{k+4}^t$ to a mop $T_{k,t}$ by adding edges on the $2t$ vertices $x_{i1}, x_{i2}, \ldots, x_{i1}, x_{i2}$. Thus, the order $n$ of $T_{k,t}$ is $(k+4)t$, and $x_{ik+3}, \ldots, x_{ik}$ are the vertices of $T_{k,t}$ of degree 2. Clearly, if $S$ is a $K_{1,k+1}$-isolating set of $T_{k,t}$, then $|S \cap V(F_{k+4}^i)| \geq 1$ for each $i \in [t]$. Thus,
\(\tau_k(T_{k,t}) \geq t.\) Since \(\{x'_i, i \in [t]\}\) is a \(K_{1,k+1}\)-isolating set of \(T_{k,t}\), \(\tau_k(T_{k,t}) = t.\) Since \(n = (k + 4)t\) and \(n_2 = t\), it follows that \(\tau_k(T_{k,t}) = n/(k + 4) = t = (n + n_2)/(k + 5).\)

When \(n_2 > 2n/(k + 4)\), we have \((n - n_2)/(k + 2) < \min\{n/(k + 4), (n + n_2)/(k + 5)\}\). We demonstrate that the bound \((n - n_2)/(k + 2)\) is sharp. Let \(x\) be the center of \(F_{k+2}\) and let \(y_1, \ldots, y_{k+1}\) be \(k + 1\) distinct isolated vertices. Let \(uv\) be a Hamiltonian edge of \(F_{k+2}\) with \(d_{F_{k+2}}(u) = d_{F_{k+2}}(v) = 3\), and let \(z_1, z_1', \ldots, z_{k+1}z'_{k+1}\) be the remaining Hamiltonian edges of \(F_{k+2}\). Let \(R_{2k+3}\) be the graph with vertex set \(V(F_{k+2}) \cup \{y_1, \ldots, y_{k+1}\}\) and edge set \(E(F_{k+2}) \cup \{y_1z_1, y_1z_1', \ldots, y_{k+1}z_{k+1}, y_{k+1}z'_{k+1}\}\). Clearly, \(R_{2k+3}\) is a \((2k + 3)\)-vertex mop with exactly \(k + 1\) vertices of degree 2. Let \(R_{2k+3}^1, \ldots, R_{2k+3}^t\) be \(t\) vertex-disjoint copies of \(R_{2k+3}\). For \(i \in [t]\), let \(x_i, u_i, v_i\) be the vertices of \(R_{2k+3}^i\) corresponding to the vertices \(x, u, v\) of \(R_{2k+3}\), respectively. We extend the union of \(R_{2k+3}^1, \ldots, R_{2k+3}^t\) to a mop \(S_{k,t}\) by adding edges on the \(2t\) vertices \(u_1, v_1, \ldots, u_t, v_t\) (see Fig. 2). Thus, the order \(n\) of \(S_{k,t}\) is \((2k + 3)t\), and the number \(n_2\) of vertices of \(S_{k,t}\) of degree 2 is \((k + 1)t\). We have \(n_2 = (k + 1)n/(2k + 3) > 2n/(k + 4)\) as \(k \geq 2.\) Clearly, if \(D\) is a \(K_{1,k+1}\)-isolating set of \(S_{k,t}\), then \(|D \cap V(R_{2k+3}^i)| \geq 1\) for each \(i \in [t]\). Thus, \(\tau_k(S_{k,t}) \geq t.\) Since \(\{x_i, i \in [t]\}\) is a \(K_{1,k+1}\)-isolating set of \(S_{k,t}\), \(\tau_k(S_{k,t}) = t = (n - n_2)/(k + 2).\)

4. Proofs of Theorems 4–8

In this section, we prove Theorems 4–8. We apply results of O’Rourke [24] in computational geometry that were used in a new proof by Lemańska et al. [22] of an upper bound by Dorfling et al. [13] on the size of a smallest total dominating set (a set \(S\) of vertices such that each vertex of the graph is
adjacent to a vertex in $S$) of a mop. Before stating these results, we make a related straightforward observation that we will also use.

Given three mops $G$, $G_1$, and $G_2$, we say that a diagonal $d$ of $G$ partitions $G$ into $G_1$ and $G_2$ if $G$ is the union of $G_1$ and $G_2$, $V(G_1) \cap V(G_2) = d$, and $E(G_1) \cap E(G_2) = \{d\}$. The following is a straightforward well-known fact.

**Lemma 1.** If $d$ is a diagonal of a mop $G$, then $d$ partitions $G$ into two mops $G_1$ and $G_2$.

**Lemma 2.** If $r \geq 0$ and $G$ is a mop of order $n \geq 2r + 4$, then $G$ has a diagonal $d$ that partitions it into two mops $G_1$ and $G_2$ such that $G_1$ has exactly $\ell$ Hamiltonian edges of $G$ for some $\ell \in [r + 2, 2r + 2]$.

**Proof.** In this proof, all subscripts are taken modulo $n$. Let $x_0x_1 \ldots x_{n-1}x_0$ be the Hamiltonian cycle of $G$. Let $p$ be the smallest integer such that $p \geq r + 2$ and $x_i x_{i+p} \in E(G)$ for some $i \in \{0, 1, \ldots, n - 1\}$ ($p$ exists as $x_0 x_{n-1} \in E(G)$). It suffices to show that $p \leq 2r + 2$. For some $q \in [p - 1]$, $G$ has a triangular face containing $x_i$, $x_{i+q}$, and $x_{i+p}$, so $x_i x_{i+q} x_{i+q} x_{i+p} \in E(G)$. By the choice of $p$, we have $q \leq r + 1$ and $p - q = (i + p) - (i + q) \leq r + 1$. We have $p = p - q + q \leq 2r + 2$.

The case $r = 2$ of Lemma 2 was proved by Chvátal [10] and is restated in [25, Lemma 1.1]. The case $r = 3$ of Lemma 2 was proved by O’Rourke [24].

For a graph $G$ and an edge $uv$ of $G$, the *edge contraction* of $G$ along $uv$ is the graph obtained from $G$ by deleting $u$ and $v$ (and all incident edges), adding a new vertex $x$, and making $x$ adjacent to the vertices in $N_G(\{u, v\}) \setminus \{u, v\}$ only. By looking at polygon corners as vertices, we have that a mop is a triangulation of a simple polygon, meaning that its boundary is the polygon and its interior faces are triangles.

**Lemma 3.** [24] If $G$ is a triangulation of a simple polygon $P$, the order of $G$ is at least $4$, $e$ is a Hamiltonian edge of $G$, and $G'$ is the edge contraction of $G$ along $e$, then $G'$ is a triangulation of some simple polygon $P'$.
If \( G \) is a graph and \( I \subseteq V(G) \) such that \( uv \notin E(G) \) for every \( u, v \in I \), then \( I \) is called an independent set of \( G \).

**Lemma 4.** If \( G \) is a mop of order \( n \geq 4 \), then the set of vertices of \( G \) of degree 2 is an independent set of \( G \) of size at most \( \frac{n}{2} \).

Lemma 4 is given in [6], as are parts (a)–(d) of the next lemma.

**Lemma 5.** If \( G \) is a mop of order \( n \geq 3 \), then the following assertions hold:

(a) Each vertex of \( G \) is of degree at least 2.
(b) \( G \) has at least 2 vertices of degree 2.
(c) If \( n \geq 4 \), then \( G - v \) is a mop for each vertex \( v \) of \( G \) of degree 2.
(d) A graph \( H \) is a mop if \( G = H - w \) for some \( w \in V(H) \) such that \( d_H(w) = 2 \) and \( N_H(w) \) is a Hamiltonian edge of \( G \).
(e) Each vertex of \( G \) has at most two neighbours of degree 2.

**Proof.** For (a)–(d), see [6]. We prove (e). Let \( V_2 \) be the set of vertices of \( G \) of degree 2. Let \( v \in V(G) \). We may label the vertices \( x_1, x_2, \ldots, x_n \) so that \( x_1 x_2 \ldots x_n x_1 \) is the Hamiltonian cycle of \( G \) and \( x_n = v \). If \( i \in [2, n - 2] \) such that \( x_i \in N_G(v) \), then \( x_{i-1}, x_{i+1}, v \in N_G(x_i) \), so \( x_i \notin V_2 \). Thus, \( N_G(v) \cap V_2 \subseteq \{x_1, x_{n-1}\} \).

Lemmas 4 and 5(b) tell us that the number \( n_2 \) of vertices of degree 2 (of a mop) satisfies

\[
2 \leq n_2 \leq \frac{n}{2}. \tag{1}
\]

We mention that both bounds are sharp [6].

The next lemma settles Theorem 4 for \( n \leq 2k + 7 \), and hence allows us to use Lemma 2 in the proof of Theorem 4.

We say that a vertex \( x \) of a mop \( G \) is a diagonal \( K_{1,k+1} \)-isolating vertex of \( G \) if \( \{x\} \) is a \( K_{1,k+1} \)-isolating set of \( G \) and \( x \) is one of the two vertices of a diagonal of \( G \).

**Lemma 6.** If \( k \geq 0 \) and \( G \) is a mop of order \( n \leq 2k + 7 \), then \( \nu_k(G) \leq 1 \).

**Proof.** The result is trivial if \( n \leq 4 \). Suppose \( n \geq 5 \). We show that \( G \) has a diagonal \( K_{1,k+1} \)-isolating vertex.

Suppose \( n = 2k + 7 \). Let \( x_1 x_2 \ldots x_{2k+7} x_1 \) be the unique Hamiltonian cycle of \( G \) and hence the boundary of the exterior face of \( G \). Let \( r = k + 1 \). Thus, \( n > 2r + 4 \). By Lemma 2, \( G \) has a diagonal \( d \) that partitions it into two mops \( G_1 \) and \( G_2 \) such that \( G_1 \) has exactly \( \ell \) Hamiltonian edges of \( G \) for some \( \ell \in [r+2, 2\ell+2] = [k+3, 2k+4] \). We may assume that \( d \) is the edge \( x_1 x_{\ell+1} \) and that \( V(G_1) = \{x_1, x_2, \ldots, x_{\ell+1}\} \). Thus, \( V(G_2) = \{x_1, x_{\ell+1}, x_{\ell+2}, \ldots, x_{2k+7}\} \).

Suppose \( \ell = k + 3 \). Then, \( |V(G_1)| = k + 4 \), \( |V(G_2)| = k + 5 \), \( x_{k+4}, x_n \in N_{G_2}(x_1) \), and \( x_1, x_{k+5} \in N_{G_2}(x_{k+4}) \). Since \( x_1 \) and \( x_{k+4} \) are adjacent in \( G_2 \), Lemma 4 tells us that their degrees in \( G_2 \) cannot be both 2. Thus, \( |N_{G_2}[x_i]| \geq 4 \) for some \( i \in [1, k + 4] \). Clearly, \( |N_{G_1}[x_i]| \geq 3 \). Thus, \( |V(G_1) \setminus N_G[x_i]| \leq k + 1 \) and \( |V(G_2) \setminus N_G[x_i]| \leq k + 1 \), and hence \( G - N_G[x_i] \) contains no copy of \( K_{1,k+1} \). Therefore, \( x_i \) is a diagonal \( K_{1,k+1} \)-isolating vertex of \( G \).
We now proceed by induction on \( \ell \). Thus, we consider \( \ell \geq k+4 \) and assume that, if \( G \) has a diagonal that partitions it into two mops \( H_1 \) and \( H_2 \) such that \( H_1 \) has exactly \( \ell^* \) Hamiltonian edges of \( G \) for some \( \ell^* \in [k + 3, \ell - 1] \), then \( G \) has a diagonal \( K_{1,k+1} \)-isolating vertex. Let \((x_1, x_i, x_{\ell+1})\) be the triangular face of \( G_1 \) containing the Hamiltonian edge \( x_1 x_{\ell+1} \) of \( G_1 \). Thus, \( 2 \leq i \leq \ell \). Let \( \ell' = \ell + 1 - i \) and \( \ell'' = i - 1 \). Suppose \( k + 4 \leq i \leq \ell \). By Lemma 1, \( x_1 x_i \) partitions \( G \) into two mops \( G'_1 \) and \( G'_2 \) such that \( G'_1 \) contains the \( \ell'' \) Hamiltonian edges \( x_1 x_2, x_2 x_3, \ldots, x_{i-1} x_i \) of \( G \). Since \( k + 4 \leq i \leq \ell \), \( k + 3 \leq \ell'' \leq \ell - 1 \). By the induction hypothesis, \( G \) has a diagonal \( K_{1,k+1} \)-isolating vertex. Now suppose \( 2 \leq i \leq k + 3 \). By Lemma 1, \( x_i x_{\ell+1} \) partitions \( G \) into two mops \( G'_1 \) and \( G'_2 \) such that \( G'_1 \) contains the \( \ell' \) Hamiltonian edges \( x_i x_{i+1}, x_{i+1} x_{i+2}, \ldots, x_{\ell} x_{\ell+1} \) of \( G \). If \( 2 \leq i \leq \ell - k - 2 \), then \( k + 3 \leq \ell' \leq \ell - 1 \), and hence, by the induction hypothesis, \( G \) has a diagonal \( K_{1,k+1} \)-isolating vertex. Suppose \( \ell - k - 1 \leq i \leq k + 3 \). Since \( x_1 x_2, x_1 x_{\ell+1} \in E(G_1) \), it follows that \( |\{x_1, x_2, \ldots, x_{i-1}\} \setminus N_{G_1}[x_1]| \leq i - 3 \leq (k + 3) - 3 = k \) and \( |\{x_{i+1}, x_{i+2}, \ldots, x_{\ell+1}\} \setminus N_{G_1}[x_1]| \leq \ell - i \leq \ell - (\ell - k - 1) = k + 1 \). Thus, \( G_1 - N_{G_1}[x_1] \) contains no copy of \( K_{1,k+1} \). Now \( |V(G_2)| = 2k + 8 - \ell \leq k + 4 \) as \( \ell \geq k + 4 \). Since \( x_1, x_{\ell+1}, x_n \in N_{G_2}[x_1] \), \( |V(G_2) \setminus N_{G_2}[x_1]| \leq k + 1 \). Thus, \( G_2 - N_{G_2}[x_1] \) contains no copy of \( K_{1,k+1} \). Therefore, \( x_1 \) is a diagonal \( K_{1,k+1} \)-isolating vertex of \( G \).

Now suppose \( n = 2k + 6 \). Let \( uw \) be a Hamiltonian edge of \( G \). By Lemma 5(d), we can obtain a mop \( H \) from \( G \) by inserting a vertex \( w \) in the exterior face of \( G \) and adding the edges \( uw \) and \( wv \). Since \( H \) is a mop of order \( 2k + 7 \), \( H \) has a diagonal \( K_{1,k+1} \)-isolating vertex \( v \). Since \( w \) is not a vertex of a diagonal of \( H \), \( x \neq w \). Thus, \( \{x\} \) is a \( K_{1,k+1} \)-isolating set of \( G \). Let \( C \) be the Hamiltonian cycle of \( G \), and let \( y \) and \( y' \) be the two members of \( N_C(x) \). If \( N_G(x) \setminus N_C(x) \neq \emptyset \), then \( xz \) is a diagonal of \( G \) for each \( z \in N_G(x) \setminus N_C(x) \), so \( x \) is a diagonal \( K_{1,k+1} \)-isolating vertex of \( G \). Suppose \( N_G(x) \setminus N_C(x) = \emptyset \). Then, \( N_G(x) = \{y, y'\} \). Thus, since \( G \) is a mop, the face having \( xy \) and \( xy' \) on its boundary must also have \( yy' \) on its boundary (as all interior faces are triangles), and hence \( yy' \) is a diagonal of \( G \) (as \( n \geq 5 \)). Since \( \{x, y, y'\} = N_G[x] \subseteq N_G[y] \), \( y \) is a diagonal \( K_{1,k+1} \)-isolating vertex of \( G \).

For \( 5 \leq i \leq 2k + 5 \), we obtain the result for \( n = i \) from the result for \( n = i + 1 \) in the same way we obtained the result for \( n = 2k + 6 \) from the result for \( n = 2k + 7 \). \( \square \)

We now prove Theorems 4–8. Recall that the bounds in Theorems 4–6 are sharp by the constructions in Sect. 3, so we now prove the bounds.

**Proof of Theorem 4.** If \( k + 4 \leq n \leq 2k + 7 \), then \( \kappa(G) \leq 1 \leq n/(k + 4) \) by Lemma 6. We now assume that \( n \geq 2k + 8 \) and proceed by induction on \( n \). Let \( x_1 x_2 \ldots x_n x_1 \) be the unique Hamiltonian cycle \( C \) of \( G \) and hence the boundary of the exterior face of \( G \). By Lemma 2 with \( r = k + 2 \), \( G \) has a diagonal \( d \) that partitions it into two mops \( G_1 \) and \( G_2 \) such that \( G_1 \) has exactly \( \ell \) Hamiltonian edges of \( G \) for some \( \ell \in [k + 4, 2k + 6] \). We may assume that \( d = x_1 x_{\ell+1} \) and \( V(G_1) = \{x_1, x_2, \ldots, x_{\ell+1}\} \). Note that \( x_1 x_2, x_2 x_3, \ldots, x_{\ell} x_{\ell+1} \) are the \( \ell \) Hamiltonian edges of \( G \) that belong to \( G_1 \). Let \( (x_1, x_j, x_{\ell+1}) \) be the triangular face of \( G_1 \) that contains the edge \( x_1 x_{\ell+1} \). Then, \( 2 \leq j \leq \ell \).
Suppose $\ell = k + 4$. Let $G'$ be the graph obtained from $G$ by deleting the vertices $x_2, x_3, \ldots, x_{k+4}$ and contracting the edge $x_1x_{k+5}$ to form a new vertex $z$ (see Fig. 3). Thus, $G'$ is obtained from $G_2$ by contracting the edge $x_1x_{k+5}$. By Lemma 3, $G'$ is a $\emptyset$-map. Let $n' = |V(G')|$. Thus, $n' = n - (k + 4) \geq k + 4$.

By the induction hypothesis, $\iota_k(G') \leq n'/(k + 4) = n/(k + 4) - 1$. Let $S'$ be a smallest $K_{1,k+1}$-isolating set of $G'$. Then, $|S'| = \iota_k(G') \leq n/(k + 4) - 1$. Since $|V(G)|\backslash N_{G_1}\{x_1, x_{k+5}\} \leq k+1$, $\{x_1, x_{k+5}\}$ is a $K_{1,k+1}$-isolating set of $G_1$. We have $N_{G_1}(z) \subseteq N_G\{x_1, x_{k+5}\}$. Thus, if $z \in S'$, then $(S'\{z\}) \cup \{x_1, x_{k+5}\}$ is a $K_{1,k+1}$-isolating set of $G$, and hence $\iota_k(G) \leq (|S'| - 1) + 2 \leq n/(k + 4) - 1 + 2 = n/(k + 4)$. Suppose $z \notin S'$. Since $x_1, x_{j-1}, x_j, x_{j+1}, x_{k+5} \in N_{G_1}\{x_j\}$ and $\{x_{j-1}, x_{j+1}\}\{x_1, x_{k+5}\} \neq \emptyset$ (as $2 \leq j \leq \ell = k + 4$), we have $|V(G)\backslash N_{G_1}\{x_j\}| \leq k + 1$, so $\{x_j\}$ is a $K_{1,k+1}$-isolating set of $G_1$. Since $x_1, x_{k+5} \in N_G\{x_j\}$, it follows that $S' \cup \{x_j\}$ is a $K_{1,k+1}$-isolating set of $G$. Therefore, $\iota_k(G) \leq |S'| + 1 \leq n/(k + 4)$.

We now use induction on $\ell$. Thus, we consider $\ell \geq k + 5$ and assume that, if $G$ has a diagonal that partitions it into two mops $H_1$ and $H_2$ such that $H_1$ has exactly $\ell^*$ Hamiltonian edges of $G$ for some $\ell^* \in [k + 4, \ell - 1]$, then $\iota_k(G) \leq n/(k + 4)$. Since $\ell \leq 2k + 6$, $\ell = k + 4 + t$ for some $t \in [k + 2]$.

**Claim 1.** If $j \notin [t + 2, k + 4]$, then $\iota_k(G) \leq n/(k + 4)$.

**Proof.** Let $\ell' = j - 1$ and $\ell'' = \ell + 1 - j$. Suppose $k + 5 \leq j \leq \ell$. Then, $k + 4 \leq \ell' \leq \ell - 1$. By Lemma 1, $x_1x_j$ partitions $G$ into two mops $G'_1$ and $G'_2$ such that $G'_1$ contains the $\ell'$ Hamiltonian edges $x_1x_2, x_2x_3, \ldots, x_{j-1}x_j$ of $G$. By the induction hypothesis, $\iota_k(G) \leq n/(k + 4)$. Now suppose $2 \leq j \leq t + 1$. Since $\ell = k + 4 + t$, $k + 4 \leq \ell'' \leq \ell - 1$. By Lemma 1, $x_jx_{\ell+1}$ partitions $G$ into two mops $G''_1$ and $G''_2$ such that $G''_1$ contains the $\ell''$ Hamiltonian edges $x_jx_{j+1}, x_{j+1}x_{j+2}, \ldots, x_{\ell}x_{\ell+1}$ of $G$. By the induction hypothesis, $\iota_k(G) \leq n/(k + 4)$. \hfill \Box

In view of Claim 1, we now assume that $j \in [t + 2, k + 4]$.

**Claim 2.** $G_1 \backslash N_{G_1}\{x_j\}$ contains no copy of $K_{1,k+1}$.
Proof. Since $x_1, x_{j-1}, x_{j+1}, x_{\ell+1} \in N_G(x_j)$ and $j \in [t+2, k+4]$, there are at most $j - 3 \leq k + 1$ vertices in \{x_1, x_2, \ldots, x_{j-1}\} which are not adjacent to $x_j$, and at most $\ell + 1 - (j + 2) \leq (k + 4 + t + 1) - (t + 4) \leq k + 1$ vertices in \{x_{j+1}, x_{j+2}, \ldots, x_{\ell+1}\} which are not adjacent to $x_j$. Since no vertex in \{x_2, \ldots, x_{j-1}\} is adjacent to a vertex in \{x_{j+1}, \ldots, x_\ell\} (by Lemma 1 as $x_1x_j$ is a diagonal of $G_1$), the claim follows. \qed

Let $G'$ be the graph obtained from $G$ by deleting the vertices $x_2, x_3, \ldots, x_\ell$. Then, $G'$ is the mop $G_2$. Let $n' = |V(G')|$. Then, $n' = n - (\ell - 1) \leq n - (k + 4)$. Suppose $n' \leq k + 3$. Then, $G_2 - \{x_1, x_{\ell+1}\}$ contains no copy of $K_{1, k+1}$. Together with $x_1, x_{\ell+1} \in N_G(x_j)$ and Claim 2, this gives us that \{x_j\} is a $K_{1, k+1}$-isolating set of $G$, so $\nu_k(G) \leq 1 < n/(k + 4)$. Now suppose $n' \geq k + 4$. By the induction hypothesis, $\nu_k(G') \leq n'/(k + 4) \leq n/(k + 4) - 1$. Let $S'$ be a $K_{1, k+1}$-isolating set of $G'$ with $|S'| = \nu_k(G')$. Since $x_1, x_{\ell+1} \in N_G(x_j)$, it follows by Claim 2 that $S' \cup \{x_j\}$ is a $K_{1, k+1}$-isolating set of $G$, so $\nu_k(G) \leq |S'| + 1 \leq n/(k + 4)$. \qed

Proof of Theorem 5. We use an inductive argument similar to that in the proof of Theorem 4.

If $k + 3 \leq n \leq 2k + 7$, then, by Lemmas 5(b) and 6, $\nu_k(G) \leq 1 \leq (n + n_2)/(k + 5)$. We now assume that $n \geq 2k + 8$ and proceed by induction on $n$. Let $x_1x_2 \ldots x_nx_1$ be the unique Hamiltonian cycle $C$ of $G$ and hence the boundary of the exterior face of $G$. By Lemma 2 with $r = k + 2$, $G$ has a diagonal $d$ that partitions it into two mops $G_1$ and $G_2$ such that $G_1$ has exactly $\ell$ Hamiltonian edges of $G$ for some $\ell \in [k + 4, 2k + 6]$. We may assume that $d = x_1x_{\ell+1}$ and $V(G_1) = \{x_1, x_2, \ldots, x_{\ell+1}\}$. Note that $x_1x_2, x_2x_3, \ldots, x_\ell x_{\ell+1}$ are the $\ell$ Hamiltonian edges of $G$ that belong to $G_1$. Let $(x_1, x_j, x_{\ell+1})$ be the triangular face of $G_1$ that contains the edge $x_1x_{\ell+1}$. Then, $2 \leq j \leq \ell$.

Suppose $\ell = k + 4$. Then, $x_1, x_{k+5}, x_n \in N_{G_2}(x_1)$ and $x_1, x_{k+6} \in N_{G_2}(x_{k+5})$. Since $x_1$ and $x_{k+5}$ are adjacent in $G_2$, Lemma 4 tells that the degrees of $x_1$ and $x_{k+5}$ in $G_2$ cannot both be 2. Thus, $d_{G_2}(x_1) + d_{G_2}(x_{k+5}) \geq 5$.

Suppose $d_{G_2}(x_1) + d_{G_2}(x_{k+5}) = 5$. We may assume that $d_{G_2}(x_1) = 3$ and $d_{G_2}(x_{k+5}) = 2$. We have $x_1x_n, x_{k+5}x_{k+6} \in E(C) \cap E(G_2)$. Since $x_1x_{k+5}$, $x_{k+5}x_{k+6} \in E(G_2)$ and $d_{G_2}(x_{k+5}) = 2$, $N_{G_2}(x_{k+5}) = \{x_1, x_{k+6}\}$. Thus, since $G_2$ is a mop, the face having $x_1x_{k+5}$ and $x_{k+5}x_{k+6}$ on its boundary must also have $x_1x_{k+6}$ on its boundary (as all interior faces are triangles), that is, $x_1x_{k+6} \in E(G_2)$ (see Fig. 4). Together with $x_1x_{k+5}, x_1x_n \in E(G_2)$ and $d_{G_2}(x_1) = 3$, this gives us $N_{G_2}(x_1) = \{x_{k+5}, x_{k+6}, x_n\}$. Thus, since $G_2$ is a mop, the face having $x_1x_{k+6}$ and $x_1x_n$ on its boundary must also have $x_{k+6}x_n$ on its boundary, that is, $x_{k+6}x_n \in E(G_2)$. Let $G' = G - \{x_1, x_2, \ldots, x_{k+5}\}$. Then, $G' = G_2 - \{x_1, x_{k+5}\}$. Since $d_{G_2}(x_{k+5}) = 2$, $G_2 - x_{k+5}$ is a mop by Lemma 5(c). Since $d_{G_2}(x_{k+5}) = 2$, $G'$ is a mop by Lemma 5(c). Let $n' = |V(G')|$ and $n'_2 = |\{v \in V(G'): d_G(v) = 2\}|$. We have $n' = n - (k + 5) \geq k + 3$.

By Lemma 4, at most one of $x_{k+6}$ and $x_n$ has degree 2 in $G'$. By Lemma 4, at most one of $x_1$ and $x_{k+5}$ has degree 2 in $G_1$, and hence, by Lemma 5(b), $d_{G_1}(x_h) = 2$ for some $h \in [2, k + 4]$. Since $x_h \in V(G_1)\setminus V(G_2)$, $d_G(x_h) =
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Figure 4. The case \( d_{G_2}(x_1) + d_{G_2}(x_{k+5}) = 5 \)

\( d_{G_1}(x_h) \). Therefore, \( n'_2 \leq n_2 \), and hence \( n' + n'_2 \leq n + n_2 - (k + 5) \). By the induction hypothesis, \( \iota_k(G') \leq (n' + n'_2)/(k + 5) \leq (n + n_2)/(k + 5) - 1 \). Let \( S' \) be a smallest \( K_{1,k+1} \)-isolating set of \( G' \). Clearly, \( |V(G_1 - N_{G_1}[x_j])| \leq k + 1 \), so \( G_1 - N_{G_1}[x_j] \) does not contain a copy of \( K_{1,k+1} \). Since \( x_j \) is adjacent to both \( x_1 \) and \( x_{k+5} \), it follows that \( S' \cup \{x_j\} \) is a \( K_{1,k+1} \)-isolating set of \( G \). Thus, we have \( \iota_k(G) \leq |S'| + 1 = \iota_k(G') + 1 \leq (n + n_2)/(k + 5) \).

Now suppose \( d_{G_2}(x_1) + d_{G_2}(x_{k+5}) \geq 6 \). Let \( G' \) be the graph obtained from \( G \) by deleting the vertices \( x_2, x_3, \ldots, x_{k+4} \) and contracting the edge \( x_1x_{k+5} \) to form a new vertex \( y \). Then, \( G' \) is obtained from \( G_2 \) by contracting \( x_1x_{k+5} \). Thus, \( G' \) is a mop by Lemma 3. Let \( n' = |V(G')| \) and \( n'_2 = |\{v \in V(G') : d_{G'}(v) = 2\}| \). We have \( n' = n - (k + 4) \geq k + 4 \).

Suppose \( d_{G'}(y) \leq 2 \). As noted above, \( x_1x_n, x_{k+5}x_{k+6} \in E(G_2) \). Thus, \( N_{G'}(y) = \{x_{k+6}, x_n\} \). Since \( d_{G_2}(x_1) + d_{G_2}(x_{k+5}) \geq 6 \), we obtain \( N_{G_2}(x_1) = \{x_{k+5}, x_{k+6}, x_n\} \) and \( N_{G_2}(x_{k+5}) = \{x_1, x_{k+6}, x_n\} \). Since \( N_{G_2}(x_1) = \{x_{k+5}, x_{k+6}, x_n\} \), \( x_1x_{k+6} \) is a diagonal of \( G_2 \). By Lemma 1, we obtain \( x_{k+5}x_n \notin E(G_2) \), which contradicts \( N_{G_2}(x_{k+5}) = \{x_1, x_{k+6}, x_n\} \). Therefore, \( d_{G'}(y) \geq 3 \).

Suppose that every vertex that has degree 2 in \( G' \) also has degree 2 in \( G \). As in the proof for the case \( d_{G_2}(x_1) + d_{G_2}(x_{k+5}) = 5 \), \( d_{G_1}(x_h) = 2 \) for some \( h \in [2, k + 4] \), so \( n'_2 \leq n_2 - 1 \). Thus, \( n' + n'_2 \leq n + n_2 - (k + 5) \). By the induction hypothesis, \( \iota_k(G') \leq (n' + n'_2)/(k + 5) \leq (n + n_2)/(k + 5) - 1 \). Let \( S' \) be a smallest \( K_{1,k+1} \)-isolating set of \( G' \). Then, \( |S'| \leq (n + n_2)/(k + 5) - 1 \).

We can continue as in the proof of Theorem 4 for the case \( \ell = k + 4 \) to obtain \( \iota_k(G) \leq |S'| + 1 \leq (n + n_2)/(k + 5) \).

Now suppose that \( G' \) has a vertex \( z \) such that \( d_{G'}(z) = 2 \neq d_{G}(z) \). As \( z \neq y \), we have \( x_1, x_{k+5} \in N_{G_2}(z) \). For each \( i \in [k + 7, n - 1] \) with \( d_{G'}(x_i) = 2 \), we have \( N_{G'}(x_i) = N_{G}(x_i) = \{x_{i-1}, x_{i+1}\} \), so \( z \neq x_i \). Thus, \( z = x_{k+6} \) or \( z = x_n \). By symmetry, we may assume that \( z = x_{k+6} \). Since \( x_{k+6}x_{k+7} \in E(C) \cap E(G_2) \) and \( x_1, x_{k+5} \in N_{G_2}(x_{k+6}) \), \( N_{G_2}(x_{k+6}) = \{x_1, x_{k+5}, x_{k+7}\} \). Thus, since \( G_2 \) is a mop, the face having \( x_1x_{k+6} \) and \( x_{k+6}x_{k+7} \) on its boundary must also have \( x_1x_{k+7} \) on its boundary (as all interior faces are triangles), meaning that \( x_1x_{k+7} \in E(G_2) \). By Lemma 1, \( x_1x_{k+7} \) partitions \( G \) into two mops \( H_1 \) and
H_2 such that V(H_2) = \{x_1, x_{k+7}, x_{k+8}, \ldots, x_n\}. Let G^* = H_2, n^* = |V(H_2)|, and n_2^* = |\{v \in V(H_2) : d_{H_2}(v) = 2\}|. We have n^* = n - (k + 5) \geq k + 3. By Lemma 4, for each i \in \{1, 2\}, at most one of x_1 and x_{k+7} has degree 2 in \ H_i. By Lemma 5(b), d_{H_1}(x_1) = 2 for some h \in V(H_1) \setminus \{x_1, x_{k+7}\}, and hence d_G(x_h) = 2. Therefore, n_2^* \leq n_2, and hence n^* + n_2^* \leq n + n_2 - (k + 5). By the induction hypothesis, \nu_k(G^*) \leq (n^* + n_2^*)/(k + 5) \leq (n + n_2)/(k + 5) - 1. Let S^* be a smallest K_{1,k+1}-isolating set of G^*. Let x^* = x_{k+5} if j = 2, and let x^* = x_1 otherwise. If 3 \leq j \leq k+4, then x_1, x_2, x_j, x_{k+5}, x_{k+6}, x_{k+7} \in N_{H_1}[x^*]. If j = 2, then x_1, x_2, x_{k+4}, x_{k+5}, x_{k+6} \in N_{H_1}[x^*]. Since x_1, x_{k+5} is a diagonal of G, we have \{x_2, x_3, \ldots, x_{k+4}\} \cap N_G(x_{k+7}) = \emptyset by Lemma 1. Therefore, S^* \cup \{x^*\} is a K_{1,k+1}-isolating set of G. Thus, we have \nu_k(G) \leq |S^*| + 1 = \nu_k(G^*) + 1 \leq (n + n_2)/(k + 5).

Having settled the case \ell = k + 4, we now use induction on \ell. Thus, we consider \ell \geq k + 5 and assume that, if G has a diagonal that partitions it into two mops H_1 and H_2 such that H_1 has exactly \ell^* Hamiltonian edges of G for some \ell^* \in \{k + 4, \ell - 1\}, then \nu_k(G) \leq (n + n_2)/(k + 5). Since \ell \leq 2k + 6, \ell = k + 4 + q for some q \in [k + 2]. By the argument in the proof of Claim 1, if j \notin [t + 2, k + 4], then \nu_k(G) \leq (n + n_2)/(k + 5). Now suppose j \in [t + 2, k + 4]. By the same argument for Claim 2, G_1 - N_{G_1}[x_j] contains no copy of K_{1,k+1}.

Let G' be the graph obtained from G by deleting the vertices x_2, x_3, \ldots, x_\ell. Then, G' is the mop G_2. Let n' = |V(G')| and n'_2 = |\{v \in V(G') : d_G(v) = 2\}|. Then, n' = n - (\ell - 1) \leq n - (k + 4). Suppose n' \leq k + 3. Since x_1, x_{k+1} \in N_{G_1}(x_j) \cap V(G') and G_1 - N_{G_1}[x_j] contains no copy of K_{1,k+1}-isolating set of G, so \nu_k(G) \leq 1 < (n + n_2)/(k + 5). Now suppose n' \geq k + 4. Since x_1x_{\ell+1} is a diagonal of G, Lemma 1 gives us that d_{G'}(v) = d_G(v) for each v \in V(G') \setminus \{x_1, x_{\ell+1}\}. By Lemma 4, at most one of x_1 and x_{\ell+1} has degree 2 in G'. We have x_1, x_{j}, x_\ell \in N_{G_1}(x_{\ell+1}). Since x_1x_{j} is a diagonal of G_1 (as t + 2 \leq j \leq k + 4 < \ell), Lemmas 1, 4, and 5(b) give us that at least one vertex in \{x_2, \ldots, x_{j-1}\} has degree 2 in G, and that at least one vertex in \{x_{j+1}, \ldots, x_\ell\} has degree 2 in G. Thus, n'_2 \leq n_2 - 2, and hence n' + n'_2 \leq n + n_2 - (k + 5). Let S' be a smallest K_{1,k+1}-isolating set of G'. By the induction hypothesis, |S'| \leq (n' + n'_2)/(k + 5) \leq (n + n_2)/(k + 5) - 1. Since x_1, x_{\ell+1} \in N_{G}(x_j) and G_1 - N_{G_1}[x_j] contains no copy of K_{1,k+1}, S' \cup \{x_j\} is a K_{1,k+1}-isolating set of G, so \nu_k(G) \leq |S'| + 1 \leq (n + n_2)/(k + 5).

**Proof of Theorem 6.** By (1), 2 \leq n_2 \leq n/2. Since n - n_2 \geq n/2 \geq (2k + 3)/2, n - n_2 \geq k + 2. Let V_2 be the set of vertices of G of degree 2, let G' = G - V_2, and let n' = |V(G')|. Then, n' = n - n_2. By Lemmas 4 and 5(c), G' is a mop. If k = 1, then let S be a smallest dominating set of G'. If k \geq 2, then let S be a smallest K_{1,k-1}-isolating set of G'. By Theorems 1(a) and 4, |S| \leq n'/(k + 2). By Lemmas 4 and 5(e), V_2 is an independent set of G and, in G, each vertex in V(G') is adjacent to at most two vertices in V_2. Consequently, S is a K_{1,k+1}-isolating set of G, and hence \nu_k(G) \leq |S| \leq (n - n_2)/(k + 2).

**Proof of Theorem 7.** If n = 4, then the result is trivial. Suppose n \geq 5. Let V_2 be the set of vertices of G of degree 2, let G' = G - V_2, and let n' = |V(G')|.
Then, \( n' = n - n_2 \). By Lemma 4, \( n \geq 2n_2 \), so \( n' \geq n/2 \). Since \( n \geq 5 \), \( n' \geq 3 \). By Lemmas 4 and 5(c), \( G' \) is a mop. Let \( x_0x_1\ldots x_{n'-1}x_0 \) be the unique Hamiltonian cycle \( C' \) of \( G' \). Let \( y_1, \ldots, y_{n_2} \) be the vertices in \( V_2 \).

**Lemma 7.** \( N_G(y_1), \ldots, N_G(y_{n_2}) \) are distinct edges of \( C' \).

**Proof.** Consider any \( r \in [n_2] \). By Lemma 4, \( N_G(y_r) \subseteq V(G') \). Since \( y_r \in V_2 \), \( N_G(y_r) = \{ x_h, x_{(h+j) \mod n'} \} \) for some \( h, j \in \{0, 1, \ldots, n' - 1 \} \) with \( j \geq 1 \) (where \( \mod \) is the usual modulo operation). Since \( G \) is a mop, no vertex of \( G \) lies in the interior of a cycle of \( G \). Thus, none of \( y_1, \ldots, y_{n_2} \) lie in the interior of \( C' \). Suppose \( 2 \leq j \leq n' - 2 \). Since \( x_{(h+n'-1) \mod n'} \) does not lie in the interior of the cycle \( x_h x_i (h+1) \mod n' \cdots x_{(h+j) \mod n'} x_h \) of \( G \), we obtain that \( x_{(h+1) \mod n'} \) lies in the interior of the cycle \( x_h y_r x_i (h+j) \mod n' x_{(h+j+1) \mod n'} \cdots x_{(h+n'-1) \mod n'} x_h \) of \( G \), a contradiction. Thus, \( j \in \{ 1, n' - 1 \} \), and hence \( N_G(y_r) \in E(C') \). We may assume that \( j = 1 \). Suppose \( N_G(y_s) = N_G(y_r) \) for some \( s \in [n_2] \setminus \{r\} \). Since \( x_{(h+2) \mod n'} \cdots x_{(h+n' - 1) \mod n'} \) do not lie in the interior of the cycle \( x_h x_i (h+1) \mod n' y_r x_h \) of \( G \) and do not lie in the interior of the cycle \( x_h y_r x_i (h+1) \mod n' y_s x_h \) of \( G \) and \( y_r \) does not lie in the interior of the cycle \( x_h y_s x_i (h+1) \mod n' x_h \) of \( G \), we obtain that \( y_s \) lies in the interior of the cycle \( x_h y_r x_i (h+1) \mod n' x_h \) of \( G \), a contradiction. Thus, \( N_G(y_s) \neq N_G(y_r) \). Therefore, the lemma is proved.

Suppose \( e' \notin \{ N_G(y_1), \ldots, N_G(y_{n_2}) \} \) for some edge \( e' \) of \( C' \). By Lemma 7, \( n_2 < n' \), so \( n > 2n_2 \). We may assume that \( e' = x_1 x_2 \). Then, \( \{x_0\} \cup \{x_{2i+1} : 1 \leq i \leq [n'/2] - 1 \} \) is a dominating set of \( G \), and hence \( \gamma(G) \leq n'/2 \). Now suppose \( e \in \{ N_G(y_1), \ldots, N_G(y_{n_2}) \} \) for each edge \( e \) of \( C' \). By Lemma 7, \( n_2 = n' \), so \( n = 2n_2 \). If \( n_2 \) is even, then \( \{x_{2i-1} : 1 \leq i \leq n'/2 \} \) is a dominating set of \( G \), so \( \gamma(G) \leq n'/2 \). If \( n_2 \) is odd, then \( \{x_0\} \cup \{x_{2i-1} : 1 \leq i \leq (n'-1)/2 \} \) is a dominating set of \( G \), so \( \gamma(G) \leq (n'+1)/2 \).

We have verified the bound in the theorem. It remains to show that the bound is attained if \( G \) is extreme.

Suppose that \( G \) is as in Definition 1. The Hamiltonian cycle of \( G \) is \( x_1 y_1 x_2 y_2 \cdots x_1 y_1 x_1 \). We have \( V_2 = \{ y_i : i \in [t] \} \), \( n_2 = t \), and \( n = 2n_2 \). Let \( s = [(t-1)/2] \). Let \( S \) be a smallest dominating set of \( G \). Let \( D = (S \setminus \{ y_i : i \in [t] \}) \cup \{ x_i : i \in [t], y_i \in S \} \). Then, \( D \subseteq \{ x_i : i \in [t], |D| \leq |S| \}, \) and \( D \) is a dominating set of \( G \). We may assume that \( x_1 \in D \). For each \( i \in [t-1] \), we have \( y_{i} \in N_{G}[D] \), so at least one of \( x_i \) and \( x_{i+1} \) is in \( D \). Thus, since \( D \supseteq \{ x_1 \} \cup \bigcup_{i=1}^{s} \{ D \cap \{ x_2i, x_{2i+1} \} \} \), we obtain \( |D| \geq 1 + \sum_{i=1}^{s} \{ D \cap \{ x_2i, x_{2i+1} \} \} \geq 1 + s \). Since \( \{ x_2i : i \in [s] \} \cup \{ x_i \} \) is a dominating set of \( G \), \( \gamma(G) \leq 1 + s \). Thus, since \( 1 + s \leq |D| \leq |S| = \gamma(G) \), \( \gamma(G) = 1 + s \). We have \( s = \lceil (n_2-1)/2 \rceil \). If \( n_2 \) is odd, then \( \gamma(G) = 1 + (n_2 - 1)/2 = (n - n_2 + 1)/2 \). If \( n_2 \) is even, then \( \gamma(G) = 1 + (n_2 - 2)/2 = (n - n_2)/2 \). \( \square \)

**Proof of Theorem 8.** The bound and part (a) are immediate consequences of Theorems 2 and 7.
Let $V_2$ be the set of vertices of $G$ of degree 2, let $y_1, \ldots, y_{n_2}$ be the members of $V_2$, let $G' = G - V_2$, and let $n' = |V(G')|$. Then, $n' = n - n_2$.

By Lemma 5(b), $n_2 \geq 2$. If $n \geq 5$, then $n' \geq 3$ by Lemma 4, so $G'$ is a mop by Lemmas 4 and 5(c). Let $C'$ be the Hamiltonian cycle of $G'$. By Lemma 7, $N_G(y_1), \ldots, N_G(y_{n_2})$ are distinct edges of $C'$ if $n \geq 5$. For each $x \in V(G')$ with $d_{G'}(x) = 2$, $x \in N_G[V_2]$ as $x \notin V_2$.

Suppose $\gamma(G) = n/3$. By the first part of the result, either $n_2 = n/3$ or $(n, n_2) = (6, 3)$. Thus, $n \geq 6$. If $n_2 = n/3$, then $n' = 2n_2$, so $G$ is $n_2$-special. If $(n, n_2) = (6, 3)$, then $C'$ is a 3-vertex cycle and $N_G(y_1), \ldots, N_G(y_{n_2})$ are its 3 edges, so $G$ is 3-extreme. Thus, (b) is proved.

Suppose that $G$ is extra $n_2$-special. Then, $n = \sum_{i=1}^{n_2} |N_G[y_i]| = 3n_2$. Thus, $n \geq 6$. If $S$ a dominating set of $G$, then $S \cap N_G[y_i] \neq \emptyset$ for each $i \in [n_2]$. Thus, $\gamma(G) \geq n_2$. Since $V_2$ is a dominating set of $G$, $\gamma(G) = n_2$. Since $n = 3n_2$, (c) follows. □

5. An Art Gallery Theorem Relaxation for Guarding Corners

For an integer $k \geq -1$ and a mop $G$, let $P_k(G)$ denote the set of $(k + 2)$-vertex paths of the Hamiltonian cycle of $G$, and let $c_k(G)$ denote the size of a smallest subset $S$ of $V(G)$ such that no member of $P_k(G)$ is a subgraph of $G - N_G[S]$. By Theorem 1(a), $c_{-1}(G) \leq n/3$. By the same argument used in the proof of Theorem 4 (including that of Lemma 6), we have the following theorem.

**Theorem 10.** If $k \geq -1$, $n \geq k + 4$, and $G$ is a mop of order $n$, then

$$c_k(G) \leq \frac{n}{k + 4}.$$ 

It is worth pointing out that the same argument yields stronger results; in particular, we have that there exists a subset $S$ of $V(G)$ such that $|S| \leq n/(k + 4)$ and $G - N_G[S]$ contains no connected graph of order at least $k + 2$.

Theorem 10 gives a result in computational geometry that extends the Art Gallery Theorem. We assume that an art gallery is the closed set of points bounded by a polygon $P$ of $n$ sides (so $P$ has $n$ corners). Two points in $P$ (including the sides and corners of $P$ as the set is closed) are visible to each other if the straight line joining them does not intersect the exterior of $P$. The classical problem solved by Chvátal [10] was to find the minimum number of guards that can be placed in $P$ so that every point in $P$ is visible to at least one guard. We relax the problem by restricting the visibility condition to corners only and allowing the guards to ignore sets of at most $k + 1$ consecutive corners on the perimeter of $P$ in the following sense: more than $k + 1$ corners may be ignored, but at least one of every $k + 2$ consecutive corners needs to be visible to at least one guard. Let $g_k(P)$ denote the minimum number of guards that can be used for this purpose. Note that having $k = -1$ means that we do not allow the guards to ignore any corner. With a slight abuse of notation, let $V(P)$ denote the set of corners of $P$. Using Theorem 10, we obtain the following result.
Theorem 11. If $k \geq -1$, $n \geq k+4$, and $n$ is the number of corners of a polygon $P$, then

$$g_k(P) \leq \frac{n}{k+4}.$$ 

Moreover, for every $t \geq 1$, there exists a polygon $P_{k,t}$ such that $g_k(P_{k,t}) = t = \frac{|V(P_{k,t})|}{k+4}$.

Proof. We may represent $P$ by a cycle $C_P$ drawn on the plane. Thus, the vertices of $C_P$ represent the corners of $P$, and the edges of $C_P$ represent the sides of $P$. We insert non-crossing edges in the interior of $C_P$ (without adding vertices) until we obtain a mop $G_P$. Thus, $C_P$ is the Hamiltonian cycle of $G_P$. By Theorem 10, there exists a set $S$ of at most $n/(k+4)$ vertices of $G_P$ such that $G_P - N_{G_P}[S]$ contains no $(k+2)$-vertex path of $C_P$. By placing guards at the corners of $P$ represented by the vertices in $S$, we obtain that at least one of every $k+2$ consecutive corners of $P$ is visible to at least one guard. Therefore, $g_k(P) \leq n/(k+4)$.

We now show that the bound in the theorem can be attained for any integer value of $n/(k+4)$ by constructing $P_{k,t}$ explicitly. If the interior angle at a corner of a polygon is reflex (that is, more than $180^\circ$), then we call the corner a reflex corner. A reflex chain is a sequence of consecutive reflex corners. A polygon is spiral if it is a triangle or has exactly one reflex chain. Let $t$ be a positive integer. For each $i \in [t]$, let $S_i$ be a spiral polygon of $k+4$ sides, and let $c_1^i, \ldots, c_{k+4}^i$ be the corners of $S_i$, listed in the order they appear in the clockwise direction and such that, if $k \geq 0$, then $c_{2}^i, \ldots, c_{k}^i$ is the reflex chain. If $t = 1$, then we take $P_{k,t}$ to be $S_1$, and we trivially have $g_k(P_{k,t}) = 1 = |V(P_{k,t})|/(k+4)$. Suppose $t \geq 2$. Place $S_1, \ldots, S_t$ consecutively on a plane in such a way that no two intersect, $c_{k+4}^1, c_1^2, c_{k+4}^2, \ldots, c_{k+4}^{t-1}, c_1^{t-1}, c_{k+4}^{t-1}$, and $c_1^t$ are on the same horizontal line $L_1$, and $c_{1}^1$ and $c_{k+4}^t$ are on the same horizontal line $L_2$ slightly below $L_1$. For each $i \in [t]$, remove the side $c_1^i c_{k+4}^i$ of $S_i$. Join $c_1^i$ to $c_{k+4}^i$ by a line segment, and for each $i \in [t-1]$, join $c_{k+4}^i$ to $c_{1}^{i+1}$ by a line segment. Let $P_{k,t}$ be the polygon obtained. The polygon $P_{3,4}$ is illustrated in Fig. 5. We have $|V(P_{k,t})| = t(k+4)$. For each $i \in [t]$, if we place a guard at the corner $c_1^i$ of $P_{k,t}$, then each corner of $S_i$ that is not visible to the guard is one of the $k+1$ consecutive corners $c_3^i, \ldots, c_{k+3}^i$. Consequently, $g_k(P_{k,t}) \leq t$. We take $L_1$ and $L_2$ close enough so that each of the sets $V(S_1), \ldots, V(S_t)$ of consecutive corners of $P_{k,t}$ needs its own guard. Therefore, $g_k(P_{k,t}) = t$. \qed
Figure 5. The polygon $P_{3,4}$

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