Convergences of Alternating Projections

in $\text{CAT}(\kappa)$ Spaces

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Abstract

We establish the asymptotic regularity and the $\Delta$-convergence of the sequence constructed by the alternating projections to closed convex sets in a $\text{CAT}(\kappa)$ space with $\kappa > 0$. Furthermore, the strong convergence of the alternating von Neumann sequence is presented under certain regularity or compactness.

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1 Introduction

Alternating projection algorithm is one of the most simple and important algorithm for computing a point in the intersection of some convex sets, which is called the convex feasibility problem. More precisely, for given two closed convex subsets $A$ and $B$ with
the corresponding projections $P_A$ and $P_B$, the alternating projection method produces the sequence:

$$b_0 = x_0, \quad a_n = P_A(b_{n-1}), \quad b_n = P_B(a_n), \quad n \in \mathbb{N},$$

where $x_0$ is a given starting point. The alternating projection algorithm for the case of closed subspaces $A$ and $B$ of a Hilbert space was introduced by von Neumann [24], and so it is also called the von Neumann’s alternating projection algorithm. In this case, the sequences $\{a_n\}$ and $\{b_n\}$ are referred to as von Neumann sequences, and $\{b_0, a_1, b_1, a_2, b_2, \ldots\}$ is referred to as an alternating von Neumann sequence, which is denoted by the sequence $\{x_n\}_{n=0}^\infty$, i.e.,

$$x_0 = b_0, \quad x_{2m} = b_m = P_B(a_m), \quad x_{2m-1} = a_m = P_A(b_{m-1}), \quad m \in \mathbb{N}.$$ (1.1)

In 1933, von Neumann proved that the alternating projections defined as in (1.1) converges in norm to $P_{A \cap B}(x_0)$, when $A$ and $B$ are closed subspaces of a Hilbert space (see [24, 6]). In [8], Brègman proved that the alternating projections defined as in (1.1) converges weakly to a point in the intersection of closed convex sets of a Hilbert space, if the intersection is non-empty. In [17], Hundal proved that the alternating projections between two closed convex intersecting sets does not always converge in norm, by providing a sequence of alternating projections which converges weakly, but does not converge in norm.

The convex feasibility problem has been studied by many authors, e.g., [24, 8, 5, 12, 0, 17, 1, 3] and references cited therein. In particular, Bauschke and Borwein [5] studied the alternating projection algorithm to solve the convex feasibility problem. In [5], we also can find several applications of the alternating projection algorithm to the best approximation theory, discrete and continuous image reconstruction, and subgradient algorithms. In [26], Zarikian used the alternating projection algorithm to solve a variety of operator-theoretic problems, e.g., deciding complete positivity, computing completely bounded norms, etc. By many authors, the alternating projection method has been extended to general metric spaces for solving the convex feasibility problem. In [4], Bačák et al. studied the sequence of alternating projections in a CAT(0) space or alternatively Hadamard space and then proved that the sequence converges weakly (equivalently, $\Delta$-converges) to a point in the intersection of closed convex subsets of the CAT(0) space.

Recently, in [3], by developing unified treatment of convex minimization problems, Ariza-Ruiz et al. studied the asymptotic behavior of the sequence constructed by the iterative method for a finite family of firmly nonexpansive maps in the setting of $p$-uniformly convex spaces, and then the asymptotic regularity has been applied to study the common fixed points of the finite family of firmly nonexpansive maps and also to study the convex feasibility problem. In [3], the CAT(0) spaces and CAT($\kappa$) spaces with $\kappa > 0$ were considered as examples of $p$-uniformly convex spaces.

The notion of the CAT($\kappa$) spaces [7, 9], a generalization of Riemannian manifolds of sectional curvature bounded above, has been introduced by Gromov [14], where CAT stands for Cartan-Alexandrov-Toponogov. The group of unitary matrices and the $n$-dimensional sphere have nonnegative sectional curvature (see [7]), indeed, the group $SU(2)$ is a CAT(1) space (see Example 2.2 (ii)). Also, the vector (pure) state space of the space of $2 \times 2$ matrices $M_2(\mathbb{C})$ is a CAT(1) space (see Example 2.3). Every CAT(0) space is CAT($\kappa$) space for all $\kappa > 0$, indeed, a CAT($\kappa$) space is a CAT($\kappa'$) space for
all $\kappa, \kappa' \in \mathbb{R}$ with $\kappa \leq \kappa'$, and Hilbert spaces and classical hyperbolic spaces are typical examples of CAT(0) spaces. Therefore, the study of CAT($\kappa$) spaces is getting more interesting to overcome several difficulties appearing in the estimation theory by using geometric structures.

Main purpose of this paper is to study the sequence constructed by the alternating projection method in CAT($\kappa$) spaces with $\kappa > 0$. Then we prove the asymptotic regularity and the $\Delta$-convergence of the alternating von Neumann sequence. Also, we prove the strong convergence of the sequence under certain regularity or compactness condition on closed convex sets. By comparing the results in [3], it is emphasized that the main results in this paper are proved without any non-expensiveness condition.

This paper is organized as follows. In Section 2, we review briefly the basic notions in the setting of CAT($\kappa$) spaces and the $\Delta$-convergence in CAT($\kappa$) spaces. In Section 3, we first prove the asymptotic regularity of the sequence constructed by the alternating projection method (Theorem 3.4), and secondly we prove the $\Delta$-convergence of the alternating von Neumann sequence in CAT($\kappa$) spaces (Theorem 3.7). Furthermore, we prove the strong convergence of the alternating von Neumann sequence by assuming certain regularity or compactness condition on convex sets (Corollary 3.9).

2 Preliminary

2.1 CAT($\kappa$) spaces

Let $(M, d)$ be a metric space. A geodesic (path) joining $x \in M$ and $y \in M$ is a map $\gamma : [0, 1] \to M$ satisfying that $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(t_1), \gamma(t_2)) = d(x, y)|t_1 - t_2|$ for all $t_1, t_2 \in [0, 1]$. The image of the geodesic $\gamma$ joining $x$ and $y$ is called a geodesic segment joining $x$ and $y$. If for any $x, y \in M$, there exists a unique geodesic joining $x$ and $y$, then the unique geodesic is denoted by $[x, y]$.

A metric space $(M, d)$ is called a geodesic space ($D$-geodesic space) if for any two points $x, y \in M$ (for any two points $x, y \in M$ with $d(x, y) < D$), there exists a geodesic $\gamma$ joining $x$ and $y$. A subset $C$ of $M$ is said to be convex if

(C) any two points $x, y \in C$ can be joined by a geodesic in $M$ and the geodesic segment of every such geodesic is contained in $C$.

Remark 2.1 In some literature, a subset $C$ of $M$ satisfying the condition (C) is said to be totally convex. In this case, a subset $C$ of $M$ is said to be convex if any two points of $C$ are joined by a geodesic belonging to $M$.

The $n$-dimensional sphere $S^n$ is the set

$$\{x = (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. Let $\rho : S^n \times S^n \to \mathbb{R}$ be the function that assigns to each pair $(x, y) \in S^n \times S^n$ the unique real number $\rho(x, y) \in [0, \pi]$ such that

$$\cos \rho(x, y) = \langle x, y \rangle.$$

Then it is well known fact that $(S^n, \rho)$ is a geodesic metric space, and if $\rho(x, y) < \pi$ for $x, y \in S^n$, then there is only one geodesic segment joining $x$ and $y$. 

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From now on, we always assume that $\kappa > 0$, and put $D_\kappa := \pi/\sqrt{\kappa}$. Then the model space $M^n_\kappa$ is the metric space obtained from $(\mathbb{S}^n, \rho)$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$. We use the same symbol $\rho$ for the distance function of $M^n_\kappa$. Then it is clear that $M^n_\kappa$ is a geodesic metric space. Note that there is a unique geodesic segment joining $x, y \in M^n_\kappa$ if and only if $\rho(x, y) < D_\kappa$.

Let $(M, d)$ be a geodesic metric space. A geodesic triangle $\Delta := \Delta(x, y, z)$ in the metric space $M$ consists of three points in $M$ and three geodesic segments joining each pair of points. For a geodesic triangle $\Delta = \Delta(x, y, z) \subseteq M$, a geodesic triangle $\bar{\Delta} = \bar{\Delta}(\overline{x, y, z}) \subseteq M^n_\kappa$ is called a comparison triangle for $\Delta$ if

$$d(x, y) = \rho(\overline{x, y}), \quad d(x, z) = \rho(\overline{x, z}) \quad \text{and} \quad d(y, z) = \rho(\overline{y, z}).$$

A point $\overline{p} \in [\overline{x, y}] \subseteq \bar{\Delta}$ is called a comparison point for $p \in [x, y] \subseteq \Delta$ if $d(p, x) = \rho(\overline{p, x})$. Note that for a geodesic triangle $\Delta = \Delta(x, y, z) \subseteq M$, if the perimeter of $\Delta$ is (strictly) less than $2D_\kappa$, i.e., $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, then a comparison triangle for $\Delta$ always exists (see [9]). For a geodesic triangle $\Delta = \Delta(x, y, z) \subseteq M$ of perimeter less than $2D_\kappa$, we say that $\Delta$ satisfies CAT($\kappa$) inequality if for any $p, q \in \Delta$ and their comparison points $\overline{p}, \overline{q} \in \bar{\Delta}$, it holds that

$$d(p, q) \leq \rho(\overline{p, q}). \quad (2.1)$$

A metric space $(M, d)$ is called a CAT($\kappa$) space if $(M, d)$ is a $D_\kappa$-geodesic space and all geodesic triangles in $M$ of perimeter less than $2D_\kappa$ satisfy the CAT($\kappa$) inequality.

For a non-empty subset $C$ of a metric space $(M, d)$, the diameter of $C$ is defined by

$$\text{diam}(C) = \sup\{d(x, y) : x, y \in C\}.$$ 

If $(M, d)$ is a CAT($\kappa$) space with diam$(M) < D_\kappa/2$, then for any geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, any $z \in M$ and $t \in [0, 1]$, there exists a constant $c_M \in (0, 1)$ such that the following inequality holds:

$$d(z, \gamma(t))^2 \leq (1 - t)d(z, x)^2 + td(z, y)^2 - c_M t(1 - t)d(x, y)^2, \quad (2.2)$$

(see [19] [22]).

**Example 2.2** (i) A complete simply connected Riemannian manifold with constant sectional curvature $\leq \kappa$ is a CAT($\kappa$) space: the $n$-dimensional sphere is a CAT(1) space (see [15]).

(ii) The group $SU(2)$, consisting of $2 \times 2$ unitary matrices with determinant 1 is a CAT(1) space. Indeed, there are bijective functions from $SU(2)$ to the 3-dimensional sphere $\mathbb{S}^3$. Furthermore, for a compact, simply connected Lie group $G$, if $G$ admits a left invariant Riemannian structure with strictly positive sectional curvature, then $G$ is Lie isomorphic with $SU(2)$ (see [25] Theorem 2.1).

**Example 2.3** Let $H$ be a Hilbert space and $\mathcal{B}(H)$ the space of all bounded linear operators from $H$ into itself. A linear functional $\tau$ from $\mathcal{B}(H)$ into $\mathbb{C}$ is said to be positive if $\tau(X^*X) \geq 0$ for any $X \in \mathcal{B}(H)$. The linear functional $\tau$ is normalized if $\tau(I) = 1$, where $I$ is the identity map. A normalized positive linear functional $\tau$ from $\mathcal{B}(H)$ into $\mathbb{C}$ is called a state. A state $\tau$ is said to be normal if

$$\sup_{\alpha} \tau(X_\alpha) = \tau(\sup_{\alpha} X_\alpha).$$
for any positive bounded net \( \{X_\alpha\} \). In particular, for each unit vector \( \xi \), the state 
\[ \tau_\xi : \mathcal{B}(H) \ni X \mapsto \langle X\xi, \xi \rangle \in \mathbb{C} \]
is said to be the vector state on \( \mathcal{B}(H) \) determined by \( \xi \). Let \( \mathcal{S} \) be the space of all normal states on \( \mathcal{B}(H) \). Note that the extreme points \( \partial \mathcal{S} \) of \( \mathcal{S} \) consists of all vector states \( \tau_\xi \) with unit vector \( \xi \) in \( H \). If \( \dim H = 2 \), then \( \mathcal{B}(H) \) becomes the vector space of all \( 2 \times 2 \) matrices \( M_2(\mathbb{C}) \). Also we can identify the normal state space \( \mathcal{S} \) of \( M_2(\mathbb{C}) \) with the convex set of all positive trace one matrix in \( M_2(\mathbb{C}) \), and we can identify \( \partial \mathcal{S} \) with \( \mathbb{S}^2 \) (see \([1][2]\)). Therefore, the vector (pure) state space of \( M_2(\mathbb{C}) \) is a CAT(1) space.

For a non-empty subset \( C \) of a metric space \( (M,d) \), the distance function of \( C \) is defined by 
\[ d(x,C) = \inf \{d(x,c) : c \in C\}, \quad \text{for } x \in M. \]

**Proposition 2.4** \([9][13][3]\) Let \( (M,d) \) be a complete CAT(\( \kappa \)) space and \( x \in M \) be given. Let \( C \subseteq M \) be a non-empty closed convex set such that \( d(x,C) < D_\kappa/2 \). Then for given \( x \in M \), there exists a unique point \( P_C(x) \in C \) such that 
\[ d(x,P_C(x)) = d(x,C). \quad (2.3) \]

Let \( (M,d) \) be a complete CAT(\( \kappa \)) space with \( \text{diam}(M) < D_\kappa/2 \) and \( C \) be a non-empty closed convex subset of \( M \). Then from Proposition 2.4 we define the map 
\[ P_C : M \ni x \mapsto P_C(x) \in C, \]
where for each \( x \in M \), \( P_C(x) \) is the unique element in \( C \) satisfying (2.3). The map \( P_C \) is called the (metric) projection onto \( C \). For more detailed study of CAT(\( \kappa \)) spaces, we refer to \([9][13]\).

Now, we review some notions for \( p \)-uniformly convex spaces. Let \( 1 < p < \infty \). Then a metric space \( (M,d) \) is called \( p \)-uniformly convex with parameter \( c > 0 \) if \( (M,d) \) is a geodesic space and for any \( x,y,z \in M \) and \( t \in [0,1] \),
\[ d(z,\gamma_{x,y}(t))^p \leq (1-t)d(z,x)^p + td(z,y)^p - \frac{c}{2}t(1-t)d(x,y)^p, \quad (2.4) \]
where \( \gamma_{x,y} \) is a geodesic joining \( x \) and \( y \) such that \( \gamma_{x,y}(0) = x \) and \( \gamma_{x,y}(1) = y \).

It is well-known that every \( L^p(\Omega,\mu) \) \((1 < p < \infty)\) over a measure space \((\Omega,\mu)\) is \( p \)-uniformly convex. Also, for \( \kappa > 0 \), every CAT(\( \kappa \)) space is \( p \)-uniformly convex, see \([22]\).

Let \( C \) be a non-empty subset of a \( p \)-uniformly convex space \((M,d)\). A map \( T : C \rightarrow X \) is said to be firmly nonexpansive if 
\[ d(Tx,Ty) \leq d(\gamma_{x,y}(t),\gamma_{y,y}(t)) \quad (2.5) \]
for all \( x,y \in C \) and \( t \in [0,1] \). The notion of firmly nonexpansive map has been introduced by Browder \([10]\) (see, also \([11]\)). For more study of firmly nonexpansive maps, we refer to \([3]\).

For any non-empty closed convex subset \( C \) of a complete CAT(0) space \((M,d)\), the metric projection map \( P_C : M \rightarrow C \subseteq M \) is firmly nonexpansive. But, in general, if \( \kappa > 0 \), then the metric projection map \( P_C \) for a non-empty closed convex subset \( C \) of a complete CAT(\( \kappa \)) space need not be nonexpansive. However, more interesting examples of firmly nonexpansive map can be found in \([3]\).
Let $C$ be a non-empty subset of a metric space $(M, d)$. A mapping $T : C \to M$ is said to satisfy property (P1) if $\text{Fix}(T) \neq \emptyset$ and there exist $\ell, \beta > 0$ such that
\[
d(Tx, u)^\ell \leq d(x, u)^\ell - \beta d(Tx, x)^\ell
\]
for all $x \in C$ and $u \in \text{Fix}(T)$ (see Definition 2.5 in [3]).

**Proposition 2.5** [3] Let $(M, d)$ be a complete CAT($\kappa$) space with $\text{diam}(M) < D_\kappa/2$ and $C \subseteq M$ be a non-empty closed convex set. Then for all $x \in M$ and $z \in C$, it holds that
\[
d(z, P_C(x))^2 + c_M d(x, P_C(x))^2 \leq d(x, z)^2,
\]
where $c_M$ is given as in (2.2).

Let $(M, d)$ be a complete CAT($\kappa$) space with $\text{diam}(M) < D_\kappa/2$ and $C$ be a non-empty closed convex subset of $M$. Then from (2.7), it is obvious that the projection map $P_C : M \to C \subseteq M$ satisfies the property (P1).

### 2.2 $\Delta$-convergence in CAT($\kappa$) spaces

For our purpose to study the convergence of the alternating projections in a metric space without any linear structure, motivated of the studies by Bregman [8] and Hundal [17], we need a modification of the notion of the weak convergence in a normed space. One of such modification is called the $\Delta$-converge which was first introduced by Lim [20]. In [18], the authors studied the $\Delta$-convergence in CAT(0) spaces and the convergence was applied to study the fixed point theory. In [13], the authors studied the $\Delta$-convergence in CAT($\kappa$) spaces and the authors proved that, in CAT(0) spaces, the $\Delta$-convergence coincides with the modified $\phi$-convergence introduced by Sosov [23] (see [13, Proposition 5.2]). Those convergences are generalizations to CAT(0) spaces of the notion of the weak convergence in Hilbert spaces. In fact, the two different notions of convergence introduced by Sosov [23] coincide with the $\Delta$-convergence and the weak convergence in Hilbert spaces.

Let $(M, d)$ be a complete CAT($\kappa$) space and $\{x_n\} \subseteq M$ be a (bounded) sequence. For a given point $x \in M$, set
\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n),
\]
and then, the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by
\[
r(\{x_n\}) = \inf_{x \in M} r(x, \{x_n\}).
\]
and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as
\[
A(\{x_n\}) := \{ x \in M \mid r(x, \{x_n\}) = r(\{x_n\}) \}.
\]
Note that $z \in A(\{x_n\})$ if and only if $\limsup_{n \to \infty} d(z, x_n) \leq \limsup_{n \to \infty} d(x, x_n)$ for any $x \in M$.

Now, we recall the notion of $\Delta$-convergence in CAT($\kappa$) spaces. Let $x \in M$. A sequence $\{x_n\}$ is said to $\Delta$-converge to $x$ if for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, the point $x$ is the unique asymptotic center of $\{x_{n_k}\}$, and then $x$ is called the $\Delta$-limit of $\{x_n\}$. A point $x$ in $M$ is called a $\Delta$-cluster point of a sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ $\Delta$-converges to $x$. 
Remark 2.6 In some literatures, the authors used the notion of weak convergence instead of the $\Delta$-convergence, see [4].

Proposition 2.7 [13, 16] Let $M$ be a complete CAT($\kappa$) space and $\{x_n\} \subseteq M$ be a sequence with $r(\{x_n\}) < D_\kappa/2$. Then the following facts hold.

(i) $A(\{x_n\})$ has only one point.

(ii) $\{x_n\}$ has a $\Delta$-convergent subsequence, i.e., $\{x_n\}$ has a $\Delta$-cluster point $x \in M$.

Proposition 2.8 [13, 16] Let $M$ be a complete CAT($\kappa$) space and let $z \in M$. If a sequence $\{x_n\} \subseteq M$ satisfies that $r(z, \{x_n\}) < D_\kappa/2$ and that $\{x_n\}$ $\Delta$-converges to $x \in M$, then

$$x \in \bigcap_{k=1}^{\infty} \text{conv} \left( \{x_k, x_{k+1}, \cdots \} \right),$$

where $\text{conv}(A) = \bigcap \{B \subseteq M \mid A \subseteq B \text{ and } B \text{ is closed and convex} \}$, and

$$d(x, z) \leq \liminf_{n \to \infty} d(x_n, z).$$

Remark 2.9 Under the same assumptions as in Proposition 2.8 if $r(z, \{x_n\}) < D_\kappa/2$ for any $z$ in a subset $C$ of $M$, then

$$d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \text{ for all } z \in C.$$

3 Alternating Projections in CAT($\kappa$) Spaces

For our study, we review the notion of Fejér monotone sequences. Let $C$ be a non-empty subset of a metric space $(M, d)$ and let $\{x_n\}$ be a sequence in $M$. Then $\{x_n\}$ is said to be Fejér monotone with respect to $C$ if for any $z \in C$ and $n \in \mathbb{N}$, it holds that

$$d(x_{n+1}, z) \leq d(x_n, z).$$

A sequence $\{x_n\}$ converges linearly to a point $x \in M$ if there exists $K \geq 0$ and $\alpha \in [0, 1)$ such that

$$d(x_n, x) \leq K\alpha^n, \quad n \in \mathbb{N}.$$ 

In this case, $\alpha$ is called a rate of the linear convergence.

The following proposition is from Proposition 3.3 in [4].

Proposition 3.1 Let $C$ be a non-empty closed convex subset of a complete metric space $(M, d)$ and let $\{x_n\}$ be a sequence in $M$. Suppose that $\{x_n\}$ is Fejér monotone with respect to $C$. Then the following properties hold.

(i) $\{x_n\}$ is a bounded sequence.

(ii) $d(x_{n+1}, C) \leq d(x_n, C)$ for all $n \in \mathbb{N}$.

(iii) $\{x_n\}$ converges to some $x \in C$ if and only if $d(x_n, C) \to 0$ as $n \to \infty$. 
(iv) If there exists $\beta \in [0,1)$ such that $d(x_{n+1}, C) \leq \beta d(x_n, C)$ for each $n \in \mathbb{N}$, then \( \{x_n\} \) converges linearly to some point $x \in C$.

**Proof.** The proofs of (i) and (ii) are obvious and the proof of (iv) is same as in the proof of Proposition 3.3 in [4]. Therefore, we prove only (iii). Suppose that $d(x_n, C) \to 0$ as $n \to \infty$. Then for any $n, k \in \mathbb{N}$ and $c \in C$, by the triangle inequality and the Fejér monotonicity, we have

$$d(x_{n+k}, x_n) \leq d(x_{n+k}, c) + d(x_n, c) \leq 2d(x_n, c),$$

and the by taking infimum, we have

$$d(x_{n+k}, x_n) \leq 2d(x_n, C), \quad (3.1)$$

which implies that \( \{x_n\} \) is a Cauchy sequence in $M$ and so \( \{x_n\} \) converges to some $x \in M$. Also, for any $n \in \mathbb{N}$, it holds that

$$d(x, C) \leq d(x, x_n) + d(x_n, C),$$

which implies that $x \in C$. The converse is obvious.

Let $A$ and $B$ be closed convex subsets of a complete CAT($\kappa$) space $(M, d)$. The alternating projection method produces a sequence \( \{x_n\} \) by

$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}), \quad n \in \mathbb{N}, \quad (3.2)$$

where $x_0$ is a given starting point.

**Lemma 3.2** [16] Let $(M, d)$ be a complete CAT($\kappa$) space and let $C \subset M$ be a non-empty set. Suppose that the sequence \( \{x_n\} \subset M \) is Fejér monotone with respect to $C$ and satisfies that $r(\{x_n\}) < D_\kappa/2$. Suppose also that any $\Delta$-cluster point $x$ of $\{x_n\}$ belongs to $C$. Then $\{x_n\}$ $\Delta$-converges to a point in $C$.

The following lemma is a CAT($\kappa$) space analogue of Lemma 3.4 in [4].

**Lemma 3.3** Let $(M, d)$ be a complete CAT($\kappa$) space with $\text{diam}(M) < D_\kappa/2$. Let $A$ and $B$ be non-empty convex closed subsets of $M$ with $A \cap B \neq \emptyset$. Then the sequence $\{x_n\}$ given as in (3.2) constructed by the alternating projection method with a starting point $x_0$ is Fejér monotone with respect to $A \cap B$.

**Proof.** Let $c \in A \cap B$. For fixed $n \in \mathbb{N}$, without loss of generality we assume that $x_n \in A$. Note that $x_{n+1} = P_B(x_n)$. If $x_{n+1} = c$, then the proof is clear, i.e.,

$$d(x_{n+1}, c) = 0 \leq d(x_{n+1}, c).$$

If $x_{n+1} \neq c$, by Proposition [2.5] we have

$$c_M d(x_n, x_{n+1})^2 + d(x_{n+1}, c)^2 \leq d(x_n, c)^2,$$

which implies that $d(x_{n+1}, c) \leq d(x_n, c)$.

Now, we recall the notion of asymptotically regularity for a sequence. Let $(M, d)$ be a metric space. A sequence $\{x_n\}$ in $M$ is said to be **asymptotically regular** if
\[
\lim_{n \to 0} d(x_n, x_{n+1}) = 0. \text{ A rate of convergence of } \{d(x_n, x_{n+1})\} \text{ towards 0 will be called a rate of asymptotic regularity.}
\]

The next theorem gives us a rate of asymptotic regularity of the sequence given as in (3.2) constructed by the alternating projection method in a CAT(κ) space. We refer to Theorem 5.2 in [21] for a rate of asymptotic regularity of the alternating projections in a CAT(0) space. For the proof of Theorem 3.4 we will follow and refine the proof of Theorem 5.2 in [21].

**Theorem 3.4** Let \((M, d)\) be a complete CAT(κ) space with \(\text{diam}(M) < D_\kappa/2\). Let \(A\) and \(B\) be non-empty convex closed subsets of \(M\) with \(A \cap B \neq \emptyset\). Let \(x_0\) be a starting point and \(\{x_n\}\) be the sequence given as in (3.2). Then for any \(\epsilon > 0\), there exists \(N(\epsilon) \geq 0\) such that for any \(n \geq N(\epsilon)\), it holds that

\[
d(x_n, x_{n+1}) \leq \epsilon.
\]

More precisely, we can take \(N(\epsilon)\) by

\[
N(\epsilon) = \begin{cases} \left\lceil \frac{D_\kappa^2}{4c_M\epsilon} \right\rceil, & \text{for } \epsilon \leq D_\kappa \\ 0, & \text{otherwise}, \end{cases}
\]

where \(c_M\) is given as in [22] and \([a]\) is the largest integer less than or equal to \(a\).

**Proof.** Let \(c \in A \cap B\). Then by Lemma 3.3 we have \(d(x_{n+1}, c) \leq d(x_n, c)\) for all \(n \in \mathbb{N}\). Since by assumption for \(M\),

\[
d(x_n, x_{n+1}) \leq d(x_n, c) + d(c, x_{n+1}) < D_\kappa.
\]

Hence the case of \(\epsilon \geq D_\kappa\) is clear. Suppose that \(\epsilon < D_\kappa\) and set

\[
N = N(\epsilon) := \left\lceil \frac{D_\kappa^2}{4c_M\epsilon} \right\rceil. \tag{3.3}
\]

For fixed \(n \in \mathbb{N}\), without loss of generality, we assume that \(x_n \in A\) and \(x_{n+1} \notin A \cap B\). Note that \(x_{n+1} = P_B(x_n)\). By Proposition 2.5, we have for \(c \in A \cap B\)

\[
c_Md(x_n, x_{n+1})^2 \leq d(x_n, c)^2 - d(x_{n+1}, c)^2. \tag{3.4}
\]

If we assume that \(d(x_n, x_{n+1}) > \epsilon\) for all \(n = 1, \ldots, N\), then by (3.3), we have

\[
c_M \sum_{i=1}^N d(x_i, x_{i+1})^2 \leq d(x_1, c)^2 - d(x_{N+1}, c)^2 < D_\kappa^2/4,
\]

which implies that \(c_M \epsilon N < D_\kappa^2/4\). This contradicts to (3.3). Therefore, there exists \(n \leq N\) such that \(d(x_n, x_{n+1}) \leq \epsilon\). But, the sequence \(\{d(x_{n+1}, x_n)\}\) is non-increasing. Indeed, for fixed \(n \in \mathbb{N}\), again without loss of generality, we assume that \(x_n \in A\). Then since \(x_{n+2} = P_A(x_{n+1})\), by Proposition 2.3, we have

\[
d(x_{n+1}, x_{n+2}) = d(x_{n+1}, P_A(x_{n+1})) = d(x_{n+1}, A) \leq d(x_{n+1}, x_n).
\]

Therefore, the proof is completed.
Lemma 3.6 Let $(M, d)$ be a complete CAT($\kappa$) space with $\text{diam}(M) < D_\kappa/2$. Let $A$ and $B$ be non-empty convex closed subsets of $M$ with $A \cap B \neq \emptyset$. Let $x_0$ be a starting point and $\{x_n\}$ be the sequence given as in (3.2) constructed by the alternating projection method. Then for any $n \in \mathbb{N}$, it holds that
\[
\max \{d(x_n, A)^2, d(x_n, B)^2\} \leq \frac{1}{c_M} \left( d(x_n, A \cap B)^2 - d(x_{n+1}, A \cap B)^2 \right), \tag{3.5}
\]
where $c_M$ is given as in (2.2).

**Proof.** The proof is a modification of the first part of the proof of Theorem 4.1 in [4]. For fixed $n \in \mathbb{N}$, without loss of generality, we assume that $x_n \in A \not\in A \cap B$ and $x_{n+1} = P_B(x_n) \not\in A \cap B$. Then by using Proposition 2.3 for any $z \in B$, we have
\[
d(x_n, z)^2 \geq d(z, P_B(x_n))^2 + c_M d(x_n, P_B(x_n))^2
= d(z, x_{n+1})^2 + c_M d(x_n, x_{n+1})^2,
\]
from which, by taking $z = P_{A\cap B}(x_n) \in A \cap B \subset B$, we have
\[
d(x_n, P_{A\cap B}(x_n))^2 \geq d(P_{A\cap B}(x_n), x_{n+1})^2 + c_M d(x_n, x_{n+1})^2.
\]
Therefore, by Proposition 2.3 it holds that
\[
d(x_n, A \cap B)^2 \geq d(A \cap B, x_{n+1})^2 + c_M d(x_n, B)^2.
\]
Similarly, in case of $x_n \in B$,
\[
d(x_n, A \cap B)^2 \geq d(A \cap B, x_{n+1})^2 + c_M d(x_n, A)^2.
\]
Hence, the proof is completed.

Now, we recall the notion of regularity of sets in metric spaces (see [4]). Let $(M, d)$ be a metric space and $A, B$ be subsets of $M$. Then we say that

(i) $A$ and $B$ are **boundedly regular** if for any bounded subset $S \subseteq M$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in S$ and $\max\{d(x, A), d(x, B)\} \leq \delta$,
\[
d(x, A \cap B) \leq \epsilon;
\]

(ii) $A$ and $B$ are **boundedly linearly regular** if for any bounded subset $S \subseteq M$, there exists $k > 0$ such that for $x \in S$,
\[
d(x, A \cap B) \leq k \max\{d(x, A), d(x, B)\};
\]

(iii) $A$ and $B$ are **linearly regular** if there exists $k > 0$ such that for any $x \in M$,
\[
d(x, A \cap B) \leq k \max\{d(x, A), d(x, B)\}.$
If the metric space \((M, d)\) is bounded, then boundedly regular and boundedly linearly regular are said to be regular and linearly regular, respectively. Since a \(\text{CAT}(\kappa)\) space with \(\text{diam}(M) < D_\kappa/2\) is bounded, the notions of boundedly linearly regular and linearly regular are same.

The next theorem is the main result in this section.

**Theorem 3.7** Let \((M, d)\) be a complete \(\text{CAT}(\kappa)\) space with \(\text{diam}(M) < D_\kappa/2\). Let \(A\) and \(B\) be non-empty convex closed subsets of \(M\) with \(A \cap B \neq \emptyset\). Let \(x_0\) be a starting point and \(\{x_n\}\) be the sequence given as in (3.2) constructed by the alternating projection method. Then the following properties hold:

(i) \(\{x_n\}\) \(\Delta\)-converges to a point \(x \in A \cap B\).

(ii) If \(A\) and \(B\) are boundedly regular, then \(\{x_n\}\) converges to a point \(x \in A \cap B\).

(iii) If \(A\) and \(B\) are boundedly linearly regular, then \(\{x_n\}\) converges linearly to a point \(x \in A \cap B\).

**Proof.** By Lemma 3.3, the sequence \(\{x_n\}\) is Fejér monotone with respect to \(C := A \cap B\). Therefore, by (ii) in Proposition 3.1, the sequence \(\{d(x_n, C)\}\) is bounded and decreasing sequence in \(\mathbb{R}\), and so \(\{d(x_n, C)\}\) converges to some point in \(\mathbb{R}\). Therefore, by (3.5) in Lemma 3.6, we prove that

\[
\max\{d(x_n, A), d(x_n, B)\} \to 0 \quad (3.6)
\]

as \(n \to \infty\).

(i) Since \(\{x_n\}\) is bounded with \(r(\{x_n\}) < D_\kappa/2\), by (ii) in Proposition 2.7, \(\{x_n\}\) has a \(\Delta\)-cluster point in \(M\). Let \(x \in M\) be a \(\Delta\)-cluster point of \(\{x_n\}\). Then we take a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) which \(\Delta\)-converges to \(x\). Then by Remark 2.9 and (3.6), it holds that

\[
d(x, A) = d(x, B) = 0,
\]

which implies that \(x \in A \cap B\). Therefore, by Lemma 3.2, we conclude that \(\{x_n\}\) \(\Delta\)-converges to a point \(x \in A \cap B\).

(ii) Suppose that \(A\) and \(B\) are boundedly regular. Then since \(\{x_n\}\) is a bounded sequence, by (3.6), we see that

\[
d(x_n, A \cap B) \to 0
\]

as \(n \to \infty\). Therefore, by (iii) in Proposition 3.1, \(\{x_n\}\) converges to a point \(x \in A \cap B\).

(iii) Since \(\{x_n\}\) is a bounded sequence, and \(A\) and \(B\) are boundedly linearly regular, there exists \(k > 0\) such that for all \(n \in \mathbb{N}\),

\[
d(x_n, A \cap B) \leq k \max\{d(x_n, A), d(x_n, B)\}.
\]

By (3.5), we have

\[
d(x_n, A \cap B)^2 \leq \frac{k^2}{c_M} \left( d(x_n, A \cap B)^2 - d(x_{n+1}, A \cap B)^2 \right),
\]
which implies that
\[ d(x_{n+1}, A \cap B) \leq \sqrt{1 - \frac{c_M}{k^2}} d(x_n, A \cap B), \]
where \( c_M \) is given as in (2.2). Therefore, by (iv) in Proposition 3.1 the proof of (iii) is completed. \( \square \)

**Remark 3.8** As same as mentioned in Remark 3.5, the result of Theorem 4.1 in [3] similar to (i) in Theorem 3.7 has been proved with the firmly non-expansiveness of maps.

A metric space \((M, d)\) is said to be a **boundedly compact** if every bounded and closed subset of \(M\) is compact.

**Corollary 3.9** Let \((M, d)\) be a complete CAT(\(\kappa\)) space with \(\text{diam}(M) < D_\kappa/2\). Let \(A\) and \(B\) be non-empty convex closed subsets of \(M\) with \(A \cap B \neq \emptyset\). Let \(x_0\) be a starting point and \(\{x_n\}\) be the sequence given as in (3.2) constructed by the alternating projection method. If \(A\) or \(B\) is boundedly compact, then \(\{x_n\}\) converges to a point \(x \in A \cap B\).

**Proof.** The proof is similar to the proof of the second part of Theorem 4.1 in [3]. By (i) in Theorem 3.7 \(\{x_n\}\) \(\Delta\)-converges to a point \(x \in A \cap B\). Since the sequence \(\{x_n\}\) is Fejér monotone with respect to \(A \cap B\), the sequence \(\{d(x_n, x)\}\) is bounded and decreasing in \(\mathbb{R}\), and so \(\{d(x_n, x)\}\) converges to a point in \(\mathbb{R}\). Without loss of generality, we assume that \(A\) is boundedly compact. Then \(A\) is a compact subset in \(M\). Therefore, there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_{2n-1}\} \subset A\) such that \(\{x_{n_k}\}\) converges to a point \(\tilde{x} \in A\). Thus, we have

\[ \lim_{k \to \infty} d(x_{n_k}, \tilde{x}) = 0 \leq \lim_{k \to \infty} d(x_{n_k}, z) \text{ for all } z \in M, \]

which implies that \(\tilde{x} \in A(\{x_{n_k}\})\). By the uniqueness of the asymptotic center, we have \(x = \tilde{x}\). Since \(\{d(x_n, x)\}\) converges, \(\{x_n\}\) converges to \(x \in A \cap B\). \( \square \)

By applying the Hopf-Rinow Theorem (see [3]) and simple modifications of the proof of Corollary 3.9, we have the following corollary.

**Corollary 3.10** c.f. [3] Let \((M, d)\) be a complete CAT(\(\kappa\)) space with \(\text{diam}(M) < D_\kappa/2\). Let \(A\) and \(B\) be non-empty convex closed subsets of \(M\) with \(A \cap B \neq \emptyset\). Let \(x_0\) be a starting point and \(\{x_n\}\) be the sequence given as in (3.2) constructed by the alternating projection method. If \(A\) or \(B\) is locally compact, then \(\{x_n\}\) converges to a point \(x \in A \cap B\).

**Example 3.11** Let \(S = \left\{ \begin{bmatrix} x & -y + iz \\ y + iz & x \end{bmatrix} \right\} | x, y, z \in \mathbb{R} \text{ with } x^2 + y^2 + z^2 = 1 \right\} \)

Then we can identify \(S \subseteq SU(2)\) with \(\mathbb{S}^2\) by the map \(\Phi : \mathbb{S}^2 \to S\) defined as

\[ \Phi(x, y, z) = \begin{bmatrix} x & -y + iz \\ y + iz & x \end{bmatrix}. \]
Consider the following six elements:

\[
M_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \frac{1}{2} & i \frac{\sqrt{3}}{2} \\ i \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} \\ \frac{1}{2} + i \frac{\sqrt{3}}{2} & 0 \end{bmatrix},
\]

\[
M_4 = \begin{bmatrix} \frac{2}{3} & i \frac{2\sqrt{2}}{3} \\ i \frac{2\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix}, \quad M_5 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} + i \frac{2\sqrt{3}}{3} \\ \frac{2}{3} + i \frac{2\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}, \quad M_6 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} + i \frac{2\sqrt{3}}{3} \\ -\frac{2}{3} + i \frac{2\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}
\]
of \(S\). Then the sets

\[
A = \text{conv}\left\{(0,0,1), (1/2,0,\sqrt{3}/2), (0,1/2,\sqrt{3}/2) \in \mathbb{S}^2\right\},
\]

\[
B = \text{conv}\left\{(1/3,0,2\sqrt{2}/3), (2/3,1/3,2/3), (2/3,2/3,1/3) \in \mathbb{S}^2\right\}
\]
can be considered as closed bounded convex sets generated by the subsets

\[
\tilde{A} = \{M_1, M_2, M_3\} \quad \text{and} \quad \tilde{B} = \{M_4, M_5, M_6\}
\]
of \(S \subseteq SU(2)\), respectively, and it is easy to see that the point \((1/3,0,2\sqrt{2}/3)\) is in the geodesic joining \((0,0,1)\) and \((1/2,0,\sqrt{3}/2)\) and so \(A \cap B \neq \emptyset\). Since the 2-dimensional sphere \(\mathbb{S}^2\) is a compact complete metric space, \(A\) and \(B\) are compact subset in \(\mathbb{S}^2\). Thus, by Corollary 3.9 the alternating sequence \(\{x_n\}\) given as in (3.2) converges to a point \(x \in A \cap B\). Therefore, the sequence \(\{\Phi(x_n)\} \subseteq SU(2)\) converges to a point \(\Phi(x) \in \Phi(A) \cap \Phi(B)\) in the sense of a CAT(1) space.

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