L-functions of noncommutative tori

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Abstract
We introduce an analog of the $L$-function for noncommutative tori. It is proved that such a function coincides with the Hasse–Weil $L$-function of an elliptic curve with complex multiplication. As a corollary, one gets a localization formula for the noncommutative tori with real multiplication.

Keywords Elliptic curves · Noncommutative tori

Mathematics Subject Classification 11G15 · 46L85

1 Introduction

The Hasse–Weil function $L(E, s)$ is an elegant and powerful invariant of elliptic curves $E$ [9, Chapter 2, Section 10]. The function $L(E, s)$ encodes important arithmetic information about the elliptic curve $E$. For instance, the famous Birch and Swinnerton–Dyer Conjecture says that the rank of $E$ is equal to the order of zero of $L(E, s)$ at the point $s = 1$ [11, p. 198]. Such a rank is always finite by the Mordell Theorem. The Hasse–Weil $L$-functions are also critical in the Langlands Program [5].

Recall that the Sklyanin algebra $S(\alpha, \beta, \gamma)$ is a free $\mathbb{C}$-algebra on four generators $\{x_1, \ldots, x_4\}$ satisfying six quadratic relations: $x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3)$, $x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3$, $x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4)$, $x_1x_3 + x_3x_1 = x_4x_2 - x_2x_4$, $x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2)$, $x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2$, where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ [10, Example 8.5]. The algebra $S(\alpha, \beta, \gamma)$ is twisted homogeneous coordinate ring of an elliptic curve $E \subset \mathbb{C}P^3$ given in the Jacobi form $u^2 + v^2 + w^2 + z^2 = \frac{1-\alpha}{1+\beta} v^2 + \frac{1+\alpha}{1-\gamma} w^2 + z^2 = 0$; we refer the reader to Stafford and van den Bergh [10] for the missing definitions and details. We consider

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a self-adjoint representation \( \rho : S(\alpha, \beta, \gamma) \to \mathcal{B}(\mathcal{H}) \), where \( \mathcal{B}(\mathcal{H}) \) is the ring of bounded linear operators on a Hilbert space \( \mathcal{H} \). The norm-closure of \( \rho(S(\alpha, \beta, \gamma)) \) is a noncommutative torus \( A_\theta \), i.e. a \( C^* \)-algebra generated by two unitary operators \( u \) and \( v \) satisfying the commutation relation \( vu = e^{2\pi i \theta} uv \) for a real constant \( \theta \). The map \( E \mapsto A_\theta \) defines a functor, \( F \), on the category of elliptic curves, such that if \( E \) and \( E' \) are isomorphic, then \( A_\theta \) and \( A_{\theta'} \) are Morita equivalent, i.e. \( A_\theta \otimes K \cong A_{\theta'} \otimes K \), where \( K \) is the \( C^* \)-algebra of compact operators \( \theta \) [8, Section 1.3]. Moreover, if \( E_{CM} \) is an elliptic curve with complex multiplication \( \theta \) [9, pp. 95–96], then \( F(E_{CM}) = A_{RM} \), where \( A_{RM} \) has real multiplication, i.e. \( \theta \) is a quadratic irrationality \( \theta \) [8, Theorem 6.1.2].

The aim of our note is an \( L \)-function for noncommutative tori \( A_{RM} = F(E_{CM}) \), such that:

\[
L(A_{RM}, s) \equiv L(E_{CM}, s) \quad \text{for all} \quad s \in \mathbb{C}. \tag{1.1}
\]

Denote by \( E_{CM}(\mathfrak{p}) \) a localization of elliptic curve \( E_{CM} \) at the prime ideal \( \mathfrak{p} \) over a prime number \( p \) [9, p. 171]. Since the cardinals \( |E_{CM}(\mathfrak{p})| \) generate the \( L(E_{CM}, s) \), it is clear that (1.1) implies a localization formula for the algebra \( A_{RM} \) at \( p \), see Corollary 1.3 and Remark 1.4. A localization theory of non-commutative rings is an active area of research.

To define an \( L \)-function of the \( A_{RM} \), let \( \theta \) be a quadratic irrationality corresponding to the \( A_{RM} \). Denote by \((a_1, \ldots, a_k)\) the minimal period of a continued fraction of \( \theta \). We shall use the following matrix:

\[
A = \begin{pmatrix}
a_1 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
a_2 & 1 \\
1 & 0
\end{pmatrix} \ldots \begin{pmatrix}
a_k & 1 \\
1 & 0
\end{pmatrix}. \tag{1.2}
\]

For each prime \( p \), one gets an integer matrix:

\[
L_p = \begin{pmatrix}
\text{tr} A^{\pi(p)} & p \\
-1 & 0
\end{pmatrix}, \tag{1.3}
\]

where \( \pi(n) \) an integer-valued function defined in Sect. 2.2. Consider an endomorphism of the \( A_{RM} \) given by the action of \( L_p \) on the generators \( u \) and \( v \). The crossed product \( C^* \)-algebra \( A_{RM} \rtimes L_p \mathbb{Z} \) is Morita equivalent to the Cuntz–Krieger algebra \( O_{L_p} \) [1, Section 10.11.9]. For \( z \in \mathbb{C} \) and \( \alpha \in \{-1, 0, 1\} \) we define a local zeta function of the \( A_{RM} \)

\[
\xi_p(A_{RM}, z) := \exp \left( \sum_{n=1}^{\infty} \frac{|K_0(O_{\varepsilon_n})|}{n} z^n \right), \quad \varepsilon_n = \begin{cases}
L^n, & \text{if } p \nmid \text{tr}^2(A) - 4 \\
1 - \alpha^n, & \text{if } p \mid \text{tr}^2(A) - 4,
\end{cases}
\]

where \( K_0(O_{\varepsilon_n}) \) is the \( K_0 \)-group of the \( C^* \)-algebra \( O_{\varepsilon_n} \) [1, Chapter III].

**Definition 1.1** By an \( L \)-function of the \( A_{RM} \) we understand the product of the local zetas taken over all primes:

\[
L(A_{RM}, s) = \prod_p \xi_p(A_{RM}, p^{-s}), \quad s \in \mathbb{C}. \tag{1.4}
\]
Theorem 1.2 \( L(\mathcal{A}_{\text{RM}}, s) = L(\mathcal{E}_{\text{CM}}, s). \)

The following corollary is a localization formula for the algebra \( \mathcal{A}_{\text{RM}}. \)

Corollary 1.3 \( \mathcal{E}_{\text{CM}}(\mathbb{F}_{p^n}) \cong K_0(\mathcal{O}_{s_n}). \)

Proof It is proved in Sect. 3, that \( \zeta_p(\mathcal{E}_{\text{CM}}, z) \equiv \zeta_p(\mathcal{A}_{\text{RM}}, z) \) for all \( p. \) Hence \( |\mathcal{E}_{\text{CM}}(\mathbb{F}_{p^n})| = |K_0(\mathcal{O}_{s_n})|. \) We leave it to the reader to prove that the last equality implies an isomorphism of two finite abelian groups. \( \square \)

Remark 1.4 Corollary 1.3 says that the crossed product \( \mathcal{A}_{\text{RM}} \rtimes_{L_p} \mathbb{Z} \) is a non-commutative analog of localization of the polynomial rings at a prime ideal \( \mathfrak{P}. \)

The structure of the article is as follows. In Sect. 2, we introduce notation and review preliminary facts. Theorem 1.2 is proved in Sect. 3.

2 Preliminaries

We briefly review complex multiplication, function \( \pi(n) \) and Cuntz–Krieger algebras. For a detailed account, we refer the reader to Silverman [9], Hasse [7] and Cuntz and Krieger [3], respectively.

2.1 Complex multiplication

Let \( \Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \) be a lattice in the complex plane \( \mathbb{C}. \) Recall that \( \Lambda \) defines an elliptic curve \( E(\mathbb{C}) : y^2 = 4x^3 - g_2x - g_3 \) via the complex analytic map \( \mathbb{C}/\Lambda \to E(\mathbb{C}) \) given by the formula \( z \mapsto (\wp(z, \Lambda), \wp'(z, \Lambda)), \) where \( g_2 = 60 \sum_{\omega \in \Lambda} \omega^{-4}, g_3 = 140 \sum_{\omega \in \Lambda} \omega^{-6}, \) \( \Lambda^\times = \Lambda - \{0\} \) and \( \wp(z, \Lambda) = z^2 + \sum_{\omega \in \Lambda^\times} ((z - \omega)^{-2} - \omega^{-2}) \) is the Weierstrass \( \wp \) function. We shall further identify the elliptic curves \( E(\mathbb{C}) \) with the complex tori \( \mathbb{C}/\Lambda. \) If \( \tau = \omega_2/\omega_1 \) (a complex modulus), then \( E_\tau(\mathbb{C}), E_{\tau'}(\mathbb{C}) \) are isomorphic whenever \( \tau' \equiv \tau \mod SL_2(\mathbb{Z}). \) Recall, that if \( \Lambda \) is a lattice in the complex plane \( \mathbb{C}, \) then the endomorphism ring \( \text{End}(\Lambda) \) is isomorphic either to \( \mathbb{Z} \) or to an order, \( \mathcal{R}, \) in the imaginary quadratic number field \( k [9]. \) In the second case, the lattice is said to have a complex multiplication. We shall denote the corresponding elliptic curve by \( \mathcal{E}_{\text{CM}}. \) Consider the cubic \( E_\lambda : y^2 = x(x - 1)(x - \lambda), \lambda \in \mathbb{C} - \{0, 1\}. \) The \( j \)-invariant of \( E_\lambda \) is given by the formula \( j(E_\lambda) = 2^6(\lambda^2 - \lambda + 1)^3\lambda^{-2}(\lambda - 1)^{-2}. \) To find \( \lambda \) corresponding to the \( \mathcal{E}_{\text{CM}}, \) one has to solve the polynomial equation \( j(E_{\text{CM}}) = j(E_\lambda) \) with respect to \( \lambda. \) Since \( j(E_{\text{CM}}) \) is an algebraic integer [9, p. 38, Prop. 4.5 b], the \( \lambda_{\text{CM}} \in K, \) where \( K \) is an algebraic extension (of degree at most six) of the field \( \mathbb{Q}(j(E_{\text{CM}})). \) Thus, each \( \mathcal{E}_{\text{CM}} \) is isomorphic to a cubic \( y^2 = x(x - 1)(x - \lambda_{\text{CM}}) \) defined over the field \( K. \) We shall write this fact as \( \mathcal{E}_{\text{CM}} \cong E(K). \)

Let \( K \) be a number field and \( E(K) \) an elliptic curve over \( K. \) For each prime ideal \( \mathfrak{P} \) of \( K, \) let \( \mathbb{F}_{\mathfrak{P}} \) be a residue field of \( K \) at \( \mathfrak{P} \) and \( q_{\mathfrak{P}} = N_K^Q \mathfrak{P} = \# \mathbb{F}_{\mathfrak{P}}, \) where \( N_K^Q \) is the norm of the ideal \( \mathfrak{P}. \) If \( E(K) \) has a good reduction at \( \mathfrak{P}, \) one defines \( a_{\mathfrak{P}} = q_{\mathfrak{P}} + 1 - \# E(\mathbb{F}_{\mathfrak{P}}), \) where \( \widehat{E} \) is a reduction of \( E \) modulo the prime ideal \( \mathfrak{P}. \) If
$E$ has good reduction at $\mathfrak{P}$, the polynomial $L_{\mathfrak{P}}(E(K), T) = 1 - a_\mathfrak{P}T + q_\mathfrak{P}T^2$, is called the local $L$-series of $E(K)$ at $\mathfrak{P}$. If $E$ has bad reduction at $\mathfrak{P}$, the local $L$-series are $L_{\mathfrak{P}}(E(K), T) = 1 - T$ (resp. $L_{\mathfrak{P}}(E(K), T) = 1 + T$; $L_{\mathfrak{P}}(E(K), T) = 1$) if $E$ has split multiplicative reduction at $\mathfrak{P}$ (if $E$ has non-split multiplicative reduction at $\mathfrak{P}$; if $E$ has additive reduction at $\mathfrak{P}$). The global $L$-series defined by the Euler product $L(E(K), s) = \prod_{\mathfrak{P}} L_{\mathfrak{P}}(E(K), q_{\mathfrak{P}}^{-s})^{-1}$, is called a Hasse–Weil $L$-function of the elliptic curve $E(K)$.

Let $A_K^*$ be the idele group of the number field $K$. A continuous homomorphism $\psi : A_K^* \to \mathbb{C}^*$ with the property $\psi(K^*) = 1$ is called a Grössencharacter on $K$. (The asterisk denotes the group of invertible elements of the corresponding ring.) The Hecke $L$-series attached to the Grössencharacter $\psi : A_K^* \to \mathbb{C}^*$ is defined by the Euler product $L(s, \psi) = \prod_{\mathfrak{P}} (1 - \psi(\mathfrak{P})q_{\mathfrak{P}}^{-s})^{-1}$, where the product is taken over all prime ideals of $K$.

Let $\mathcal{E}_{CM} \cong E(K)$ be an elliptic curve with complex multiplication by the ring of integers $R$ of an imaginary quadratic field $k$, and assume that $K \supset k$. Let $\mathfrak{P}$ be a prime ideal of $K$ at which $E(K)$ has a good reduction. If $\tilde{E}$ is a reduction of $E(K)$ at $\mathfrak{P}$, we let $\phi_\mathfrak{P} : \tilde{E} \to \tilde{E}$ be the associated Fröbenius map. Finally, let $\psi_{E(K)} : A_K^* \to k^*$ be the Grössencharacter attached to the $\mathcal{E}_{CM}$, see [9, p. 168]. The following diagram is known to be commutative:

$$
\begin{array}{ccc}
E(K) & \overset{\psi_{E(K)}(\mathfrak{P})}{\longrightarrow} & E(K) \\
\downarrow & & \downarrow \\
\tilde{E} & \overset{\phi_\mathfrak{P}}{\longrightarrow} & \tilde{E}
\end{array}
$$

see Silverman [9, p. 174]. In particular, $\psi_{E(K)}(\mathfrak{P})$ is an endomorphism of the $E(K)$ given by the complex number $a_{E(K)}(\mathfrak{P}) \in R$. By $\overline{\psi_{E(K)}(\mathfrak{P})}$ one understand the conjugate Grössencharacter viewed as a complex number. The Deuring Theorem says that the Hasse–Weil $L$-function of the $E(K)$ is related to the Hecke $L$-series of the $\psi_{E(K)}$ by the formula $L(E(K), s) \equiv L(s, \psi_{E(K)})L(s, \overline{\psi_{E(K)}})$.

2.2 Function $\pi(n)$

Let $\mathfrak{A} = \mathbb{Q}(\sqrt{D})$ be a real quadratic number field and $O_\mathfrak{A}$ its ring of integers. For rational integer $n \geq 1$ we shall write $\mathfrak{R}_n \subseteq O_\mathfrak{A}$ to denote an order (i.e. a subring containing 1) of $O_\mathfrak{A}$. The order $\mathfrak{R}_n$ has a basis $\{1, n\omega\}$, where

$$
\omega = \begin{cases} 
\frac{\sqrt{D}+1}{2} & \text{if } D \equiv 1 \mod 4, \\
\frac{\sqrt{D}}{2} & \text{if } D \equiv 2, 3 \mod 4.
\end{cases} \quad (2.1)
$$

In other words, $\mathfrak{R}_n = \mathbb{Z} + (n\omega)\mathbb{Z}$. It is clear, that $\mathfrak{R}_1 = O_\mathfrak{A}$ and the fundamental unit of $O_\mathfrak{A}$ we shall denote by $\varepsilon$. Each $\mathfrak{R}_n$ has its own fundamental unit, which we shall write as $\varepsilon_n$; notice that $\varepsilon_n \neq \varepsilon$ unless $n = 1$.}

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There exists the well-known formula, which relates $\varepsilon_n$ to the fundamental unit $\varepsilon$, see e.g. Hasse [7, p. 297]. Denote by $\mathcal{G}_n := U(O_t/nO_t)$ the multiplicative group of invertible elements (units) of the residue ring $O_t/nO_t$; clearly, all units of $O_t$ map (under the natural mod $n$ homomorphism) to $\mathcal{G}_n$. Likewise let $\mathcal{G}_n := U(\mathcal{R}_n/n\mathcal{R}_n)$ be the group of units of the residue ring $\mathcal{R}_n/n\mathcal{R}_n$; it is not hard to prove [7, p. 296], that $\mathcal{G}_n \cong U(\mathbb{Z}/n\mathbb{Z})$ the “rational” unit group of the residue ring $\mathbb{Z}/n\mathbb{Z}$. Similarly, all units of the order $\mathcal{R}_n$ map to $\mathcal{G}_n$. Since units of $\mathcal{R}_n$ are also units of $O_t$ (but not vice versa), $\mathcal{R}_n$ is a subgroup of $\mathcal{G}_n$; in particular, $|\mathcal{G}_n|/|\mathcal{R}_n|$ is an integer number and $|\mathcal{G}_n| = \varphi(n)$, where $\varphi(n)$ is the Euler totient function. In general, the following formula is true

$$
\frac{|\mathcal{G}_n|}{|\mathcal{R}_n|} = n \prod_{p_i|n} \left( 1 - \left( \frac{D}{p_i} \right) \frac{1}{p_i} \right), \quad (2.2)
$$

where $\left( \frac{D}{p_i} \right)$ is the Legendre symbol, see Hasse [7, p. 351]. We shall write $\pi(n)$ to denote the least integer number dividing $|\mathcal{G}_n|/|\mathcal{R}_n|$ and such that $\varepsilon^{\pi(n)}$ is a unit of $\mathcal{R}_n$ (i.e. belongs to $\mathcal{G}_n$); the following Satz XIII’ of Hasse’s book says that the unit is the fundamental unit of order $\mathcal{R}_n$.

**Lemma 2.1** [7, p. 298] $\varepsilon_n = \varepsilon^{\pi(n)}$.

In what follows, we deal with the special case $n = p$ is a prime number; in this case formula (2.2) becomes

$$
\frac{|\mathcal{G}_p|}{|\mathcal{R}_p|} = p - \left( \frac{D}{p} \right). \quad (2.3)
$$

Notice, that Lemma 2.1 asserts existence of the number $\pi(n)$ (as one of the divisors of $|\mathcal{G}_n|/|\mathcal{R}_n|$) yet no analytic formula for $\pi(n)$ is given; it would be rather interesting to have such a formula.

### 2.3 Cuntz–Krieger algebras

A Cuntz–Krieger algebra, $O_B$, is the $C^*$-algebra generated by partial isometries $s_1, \ldots, s_n$ that act on a Hilbert space in such a way that their support projections $Q_i = s_i^*s_i$ and their range projections $P_i = s_is_i^*$ are orthogonal and satisfy the relations $Q_i = \sum_{j=1}^n b_{ij} P_j$, for an $n \times n$ matrix $B = (b_{ij})$ consisting of 0’s and 1’s [3]. The notion is extendable to the matrices $B$ with the non-negative integer entries [3, Remark 2.16]. It is known, that the $C^*$-algebra $O_B$ is simple, whenever matrix $B$ is irreducible (i.e. a certain power of $B$ is a strictly positive integer matrix). It was established in Cuntz and Krieger [3], that $K_0(O_B) \cong \mathbb{Z}^n/(I - B^t)\mathbb{Z}^n$ and $K_1(O_B) = \text{Ker} (I - B^t)$, where $B^t$ is a transpose of the matrix $B$. It is not difficult to see, that whenever $\det (I - B^t) \neq 0$, the $K_0(O_B)$ is a finite abelian group and $K_1(O_B) = 0$. The both groups are invariants of the stable isomorphism class of the Cuntz–Krieger algebra.
3 Proof of Theorem 1.2

Let \( p \) be such, that \( \mathcal{E}_{CM} \) has a good reduction at \( \mathfrak{p} \); the corresponding local zeta function
\[
\zeta_p(\mathcal{E}_{CM}, z) = (1 - \text{tr} (\psi_{E(K)}(\mathfrak{p}))z + pz^2)^{-1},
\]
where \( \psi_{E(K)} \) is the Grössencharacter on \( K \) and \( \text{tr} \) is the trace of algebraic number. We have to prove, that
\[
\zeta_p(\mathcal{E}_{CM}, z) = \zeta_p(A_{RM}, z) := (1 - \text{tr}(A\pi(p))z + pz^2)^{-1},
\]
The last equality is a consequence of definition of \( \zeta_p(A_{RM}, z) \).

Let \( \mathcal{E}_{CM} \sim= \mathbb{C}/L_{CM} \), where \( L_{CM} = \mathbb{Z} + \mathbb{Z} \tau \) is a lattice in the complex plane [9, pp. 95–96]; let \( K_0(A_{RM}) \sim= \Lambda_{1RM} \), where \( \Lambda_{RM} = \mathbb{Z} + \mathbb{Z} \theta \) is a pseudo-lattice in \( \mathbb{R} \). Roughly speaking, we construct an invertible element (a unit) \( u \) of the ring \( \text{End}(\Lambda_{RM}) \) attached to pseudo-lattice \( \Lambda_{RM} = F(L_{CM}) \), such that:

\[
\text{tr} (\psi_{E(K)}(\mathfrak{p})) = \text{tr}(u) = \text{tr}(A\pi(p)). \tag{3.1}
\]

The latter will be achieved with the help of an explicit formula connecting endomorphisms of lattice \( L_{CM} \) with such of the pseudo-lattice \( \Lambda_{1RM} \) [8, p. 142]:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{End}(L_{CM}) \mapsto \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \in \text{End}(\Lambda_{RM}). \tag{3.2}
\]

We shall split the proof into a series of lemmas, starting with the following simple

\textbf{Lemma 3.1} Let \( A = (a, b, c, d) \) be an integer matrix with \( ad - bc \neq 0 \) and \( b = 1 \). Then \( A \) is similar to the matrix \( (a + d, 1, c - ad, 0) \).

\textbf{Proof} Indeed, take a matrix \( (1, 0, d, 1) \in SL_2(\mathbb{Z}) \). The matrix realizes the similarity, i.e.

\[
\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} a + d & 1 \\ c - ad & 0 \end{pmatrix}. \tag{3.3}
\]

\[\square\]

\textbf{Lemma 3.2} The matrix \( A = (a + d, 1, c - ad, 0) \) is similar to its transpose \( A^t = (a + d, c - ad, 1, 0) \).

\textbf{Proof} We shall use the following criterion: the (integer) matrices \( A \) and \( B \) are similar, if and only if the characteristic matrices \( xI - A \) and \( xI - B \) have the same Smith normal form. The calculation for the matrix \( xI - A \) gives:

\[
\begin{pmatrix} x - a - d - 1 \\ ad - c & x \end{pmatrix} \sim \begin{pmatrix} x - a - d \\ x^2 - (a + d)x + ad - c & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & x^2 - (a + d)x + ad - c \end{pmatrix},
\]

where \( \sim \) are the elementary operations between the rows (columns) of the matrix. Similarly, a calculation for the matrix \( xI - A^t \) gives:
\[
\begin{pmatrix}
  x - a - d & ad - c \\
  -1 & x
\end{pmatrix} \sim \begin{pmatrix}
  x - a - d x^2 - (a + d) x + ad - c \\
  -1 & 0
\end{pmatrix} \sim \begin{pmatrix}
  1 & 0 \\
  0 & x^2 - (a + d)x + ad - c
\end{pmatrix}.
\]

Thus, \((xI - A) \sim (xI - A')\) and Lemma 3.2 follows.

**Corollary 3.3** The matrices \((a, 1, c, d)\) and \((a + d, c - ad, 1, 0)\) are similar.

**Proof** It follows from Lemmas 3.1–3.2.

Let \(E_{\text{CM}}\) be elliptic curve with the complex multiplication by an order \(R\) in the ring of integers of the imaginary quadratic field \(k\). Then \(A_{\text{RM}} = F(E_{\text{CM}})\) is a noncommutative torus with real multiplication by the order \(R_n\) of conductor \(n \geq 1\) in the ring of integers \(O_k\) of a real quadratic field \(k\). Let \(\alpha \) be the trace function of a (quadratic) algebraic number field.

**Lemma 3.4** Each \(\alpha \in \mathbb{R}\) goes under \(F\) into an \(\omega \in \mathbb{R}\), such that \(\text{tr} (\alpha) = \text{tr} (\omega)\).

**Proof** Recall that each \(\alpha \in \mathbb{R}\) can be written in a matrix form for a given base \(\{\omega_1, \omega_2\}\) of the lattice \(\Lambda\). Namely,

\[
\begin{align*}
  \alpha \omega_1 &= a\omega_1 + b\omega_2 \\
  \alpha \omega_2 &= c\omega_1 + d\omega_2,
\end{align*}
\]

where \((a, b, c, d)\) is an integer matrix with \(ad - bc \neq 0\) and \(\text{tr} (\alpha) = a + d\).

The first equation implies \(\alpha = a + b\tau\); since both \(\alpha\) and \(\tau\) are algebraic integers, one concludes that \(b = 1\).

In view of corollary 3.3, in a base \(\{\omega'_1, \omega'_2\}\), the \(\alpha\) has a matrix form \((a + d, c - ad, 1, 0)\).

To calculate a real quadratic \(\omega \in \mathbb{R}\) corresponding to \(\alpha\), recall an explicit formula from [8, p. 142]. Namely, every endomorphism \((a, b, c, d)\) of the lattice \(L_{\text{CM}}\) maps to the endomorphism \((a, b, -c, -d)\) of the pseudo-lattice \(\Lambda_{\text{RM}} = F(L_{\text{CM}})\). Thus, one gets a map:

\[
\begin{pmatrix}
  a + d & c - ad \\
  1 & 0
\end{pmatrix} \longmapsto \begin{pmatrix}
  a + d & c - ad \\
  -1 & 0
\end{pmatrix}.
\]

In other words, for a given base \(\{\lambda_1, \lambda_2\}\) of the pseudo-lattice \(\mathbb{Z} + \mathbb{Z}\theta\) one can write

\[
\begin{align*}
  \omega \lambda_1 &= (a + d)\lambda_1 + (c - ad)\lambda_2 \\
  \omega \lambda_2 &= -\lambda_1.
\end{align*}
\]

It is an easy exercise to verify that \(\omega\) is a real quadratic integer with \(\text{tr} (\omega) = a + d\). The latter coincides with the \(\text{tr} (\alpha)\).
\[ \tilde{\omega} = \omega \] is another unit of \( R \), and suppose to the contrary, that \( u = \tilde{\omega} \) is not the fundamental unit of \( R \). Then there exists a prime number, then we let \( \psi \) be the corresponding Artin–Mazur zeta function \[ \zeta_\omega(t) = \exp \left( \sum_{k=1}^{\infty} \frac{\text{tr} (\omega^k)}{k} t^k \right) \] the corresponding Artin–Mazur zeta function of the subshift of finite type is an invariant of shift equivalence, we conclude that \( \zeta_\omega(t) = \zeta_{\tilde{\omega}}(t) \); in particular, \( \text{tr} (\omega) = \text{tr} (\tilde{\omega}) \). Hence the matrix form of \( \omega = (a + d, 1, -1, 0) \) is an automorphism of \( \Lambda_n \). It is easy to see, that \( \text{tr} (u) = a + d = \text{tr} (\omega) \). Lemma 3.5 follows. \( \square \)

**Remark 3.6** There exists a canonical proof of lemma 3.5 based on the notion of a subshift of finite type \[ 12 \]; we shall give such a proof below, since it generalizes to pseudo-lattices of any rank. Consider a dimension group \[ 1, p. 55 \] corresponding to the endomorphism \( \omega \) of lattice \( \mathbb{Z}^2 \), i.e. the limit \( G(\omega) \):

\[
\mathbb{Z}^2 \xrightarrow{\omega} \mathbb{Z}^2 \xrightarrow{\omega} \mathbb{Z}^2 \xrightarrow{\omega} \ldots
\]  

(3.8)

It is known that \( G(\omega) \cong \mathbb{Z} \langle \frac{1}{\lambda} \rangle \), where \( \lambda > 1 \) is the Perron–Frobenius eigenvalue of \( \omega \). We shall write \( \hat{\omega} \) to denote the shift automorphism of dimension group \( G(\omega) \), Effros \[ 4, p. 37 \] and \( \zeta_\omega(t) = \exp \left( \sum_{k=1}^{\infty} \frac{\text{tr} (\omega^k)}{k} t^k \right) \) and \( \zeta_{\hat{\omega}}(t) = \exp \left( \sum_{k=1}^{\infty} \frac{\text{tr} (\hat{\omega}^k)}{k} t^k \right) \) the corresponding Artin–Mazur zeta functions \[ 12, p. 273 \]. Since the Artin–Mazur zeta function of the subshift of finite type is an invariant of shift equivalence, we conclude that \( \zeta_\omega(t) \equiv \zeta_{\hat{\omega}}(t) \); in particular, \( \text{tr} (\omega) = \text{tr} (\hat{\omega}) \). Hence the matrix form of \( \omega = (a + d, 1, -1, 0) = u \) and, therefore, \( \text{tr} (u) = \text{tr} (\omega) \). Lemma 3.5 follows.

**Lemma 3.7** The automorphism \( u \) is a unit of the ring \( \mathcal{R}_n = \text{End} (\Lambda_n) \); it is the fundamental unit of \( \mathcal{R}_n \), whenever \( n = p \) (a prime number) and \( \text{tr} (u) = \text{tr} (\psi_{E(K)}(\mathcal{P})) \).

**Proof** (i) Since \( \text{deg} (u) = 1 \), the element \( u \) is invertible and, therefore, a unit of the ring \( \mathcal{R}_n \); in general, unit \( u \) is not the fundamental unit of \( \mathcal{R}_n \), since it is possible that \( u = \varepsilon^a \), where \( \varepsilon \) is another unit of \( \mathcal{R}_n \) and \( a \geq 1 \).

(ii) When \( n = p \) is a prime number, then we let \( \psi_{E(K)}(\mathcal{P}) \) be the corresponding Grössencharacter on \( K \) attached to an elliptic curve \( E_{CM} \cong E(K) \), see Sect. 2.1 for the notation. The Grössencharacter can be identified with a complex number \( \alpha \in k \) of the imaginary quadratic field \( k \) associated to the complex multiplication.

Let \( \text{tr} (u) = \text{tr} (\psi_{E(K)}(\mathcal{P})) \) and suppose to the contrary, that \( u \) is not the fundamental unit of \( \mathcal{R}_p \), i.e. \( u = \varepsilon^a \) for a unit \( \varepsilon \in \mathcal{R}_p \) and an integer \( a \geq 1 \). Then there exists a Grössencharacter \( \psi'_{E(K)}(\mathcal{P}) \), such that

\[
\text{tr} (\psi'_{E(K)}(\mathcal{P})) < \text{tr} (\psi_{E(K)}(\mathcal{P})).
\]  

(3.9)
Since \( \text{tr} (\psi_{E(K)}(\mathfrak{P})) = q_{\mathfrak{P}} + 1 - \#\hat{E}(\mathbb{F}_{q_{\mathfrak{P}}}) \), one concludes that \( \#\hat{E}\left(\mathbb{F}'_{q_{\mathfrak{P}}}\right) > \#\hat{E}(\mathbb{F}_{q_{\mathfrak{P}}}) \); in other words, there exists a non-trivial extension \( \mathbb{F}'_{q_{\mathfrak{P}}} \supset \mathbb{F}_{q_{\mathfrak{P}}} \) of the finite field \( \mathbb{F}_{q_{\mathfrak{P}}} \). The latter is impossible, since any extension of \( \mathbb{F}_{q_{\mathfrak{P}}} \) has the form \( \mathbb{F}_{q_{\mathfrak{P}}}^{n} \) for some \( n \geq 1 \); thus \( a = 1 \), i.e. unit \( u \) is the fundamental unit of the ring \( \mathfrak{R}_{p} \). Lemma 3.7 follows. \( \Box \)

**Lemma 3.8** \( \text{tr} (\psi_{E(K)}(\mathfrak{P})) = \text{tr} (A^{\pi(p)}) \).

**Proof** In view of Lemma 2.1, the fundamental unit of the order \( \mathfrak{R}_{p} \) is given by the formula \( \varepsilon_{p} = \varepsilon^{\pi(p)} \), where \( \varepsilon \) is the fundamental unit of the ring \( O_{K} \) and \( \pi(p) \) an integer number. On the other hand, matrix \( A = \prod_{i=1}^{n} (a_{i}, 1, 1, 0) \), where \( \theta = (a_{1}, \ldots, a_{n}) \) is a purely periodic continued fraction. Therefore

\[
A \left( \frac{1}{\theta} \right) = \varepsilon \left( \frac{1}{\theta} \right),
\]

where \( \varepsilon > 1 \) is the fundamental unit of the real quadratic field \( k = \mathbb{Q}(\theta) \). In other words, \( A \) is the matrix form of the fundamental unit \( \varepsilon \). Therefore the matrix form of the fundamental unit \( \varepsilon_{p} = \varepsilon^{\pi(p)} \) of \( \mathfrak{R}_{p} \) is given by matrix \( A^{\pi(p)} \). We apply lemma 3.7 and get

\[
\text{tr} (\psi_{E(K)}(\mathfrak{P})) = \text{tr} (\varepsilon_{p}) = \text{tr} (A^{\pi(p)}).
\]

Lemma 3.8 follows. \( \Box \)

One can finish the proof of theorem 1.2 by comparing the local \( L \)-series of the Hasse–Weil \( L \)-function for the \( E_{CM} \) with that of the local zeta for the \( A_{RM} \). The local \( L \)-series for \( E_{CM} \) are \( L_{\mathfrak{P}}(E(K), T) = 1 - a_{\mathfrak{P}} T + q_{\mathfrak{P}} T^{2} \) if the \( E_{CM} \) has a good reduction at \( \mathfrak{P} \) and \( L_{\mathfrak{P}}(E(K), T) = 1 - \alpha T \) otherwise; here

\[
\begin{align*}
q_{\mathfrak{P}} &= N_{\mathbb{Q}}^{K} \mathfrak{P} = \#\mathbb{F}_{\mathfrak{P}} = p, \\
a_{\mathfrak{P}} &= q_{\mathfrak{P}} + 1 - \#\hat{E}(\mathbb{F}_{\mathfrak{P}}) = \text{tr} (\psi_{E(K)}(\mathfrak{P})), \\
\alpha &\in \{-1, 0, 1\}.
\end{align*}
\]

Therefore,

\[
L_{\mathfrak{P}}(E_{CM}, T) = \begin{cases} 
1 - \text{tr} (\psi_{E(K)}(\mathfrak{P})) T + p T^{2}, & \text{for good reduction} \\
1 - \alpha T, & \text{for bad reduction.}
\end{cases}
\]

Let now \( A_{RM} = F(E_{CM}) \).

**Lemma 3.9** \( \zeta_{p}^{-1}(A_{RM}, T) = 1 - \text{tr} (A^{\pi(p)}) T + p T^{2}, \) whenever \( p \nmid \text{tr}^{2}(A) - 4 \).

**Proof** By the formula \( K_{0}(O_{B}) = \mathbb{Z}^{2} / (I - B^{t}) \mathbb{Z}^{2} \), one gets:

\[
|K_{0}(O_{L_{p}'})| = \left| \frac{\mathbb{Z}^{2}}{(I - (L_{p}')^{t}) \mathbb{Z}^{2}} \right| = |\text{det}(I - (L_{p}')^{t})| = |\text{Fix}(L_{p}')|,
\]

\( \Box \)
where $\text{Fix}(L^n_p)$ is the set of (geometric) fixed points of the endomorphism $L^n_p : \mathbb{Z}^2 \to \mathbb{Z}^2$. Thus,

$$\zeta_p(\mathcal{A}_{RM}, z) = \exp \left( \sum_{n=1}^{\infty} \frac{|\text{Fix}(L^n_p)|}{n} z^n \right), \quad z \in \mathbb{C}. \quad (3.15)$$

But the latter series is an Artin-Mazur zeta function of the endomorphism $L_p$; it converges to a rational function $\det^{-1}(I - z L_p)$ [6, p. 455]. Thus, $\zeta_p(\mathcal{A}_{RM}, z) = \det^{-1}(I - z L_p)$.

The substitution $L_p = (\text{tr}(A^{\pi(p)}), p, -1, 0)$ gives us:

$$\det(I - z L_p) = \det \left( 1 - \frac{\text{tr}(A^{\pi(p)}) z - p z^2}{z} \right) = 1 - \text{tr}(A^{\pi(p)}) z + p z^2. \quad (3.16)$$

Put $z = T$ and get $\zeta_p(\mathcal{A}_{RM}, T) = (1 - \text{tr}(A^{\pi(p)}) T + p T^2)^{-1}$, which is a conclusion of Lemma 3.9. \qed

Lemma 3.10 $\zeta_p^{-1}(\mathcal{A}_{RM}, T) = 1 - \alpha T$, whenever $p \mid \text{tr}^2(A) - 4$.

Proof Indeed, $K_0(O_{1-\alpha^n}) = \mathbb{Z}/(1 - 1 + \alpha^n)\mathbb{Z} = \mathbb{Z}/\alpha^n\mathbb{Z}$. Thus, $|K_0(O_{1-\alpha^n})| = \text{det}(\alpha^n) = \alpha^n$. By the definition,

$$\zeta_p(\mathcal{A}_{RM}, z) = \exp \left( \sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^n \right) = \exp \left( \sum_{n=1}^{\infty} \frac{(\alpha z)^n}{n} \right) = \frac{1}{1 - \alpha z}. \quad (3.17)$$

The substitution $z = T$ gives a conclusion of Lemma 3.10. \qed

Lemma 3.11 Let $\mathfrak{P} \subset K$ be a prime ideal over $p$; then $E_{CM} = E(K)$ has a bad reduction at $\mathfrak{P}$ if and only if $p \mid \text{tr}^2(A) - 4$.

Proof Let $k$ be a field of complex multiplication of the $E_{CM}$; its discriminant we shall write as $\Delta_k < 0$. It is known, that whenever $p \mid \Delta_k$, the $E_{CM}$ has a bad reduction at the prime ideal $\mathfrak{P}$ over $p$.

On the other hand, the explicit formula (3.2) applied to the matrix $L_p$ gives us

$$F : (\text{tr}(A^{\pi(p)}), p, -1, 0) \mapsto (\text{tr}(A^{\pi(p)}), p, 1, 0).$$

The characteristic polynomials of the above matrices are $x^2 - \text{tr}(A^{\pi(p)})x + p$ and $x^2 - \text{tr}(A^{\pi(p)})x - p$, respectively. They generate an imaginary (resp., a real) quadratic field $k$ (resp., $\ell$) with the discriminant $\Delta_k = \text{tr}^2(A^{\pi(p)}) - 4p < 0$ (resp., $\Delta_\ell = \text{tr}^2(A^{\pi(p)}) + 4p > 0$). Thus, $\Delta_\ell - \Delta_k = 8p$. It is easy to see, that $p \mid \Delta_\ell$ if and only if $p \mid \Delta_k$. It remains to express the discriminant $\Delta_\ell$ in terms of the matrix $A$. Since the characteristic polynomial for $A$ is $x^2 - \text{tr}(A)x + 1$, it follows that $\Delta_\ell = \text{tr}^2(A) - 4$. \qed

We are prepared now to prove the first part of Theorem 1.2. Note, that a critical piece of information is provided by Lemma 3.8, which says that $\text{tr}(\psi_{E(K)}(\mathfrak{P})) = \text{tr}(A^{\pi(p)})$. Thus, in view of Lemmas 3.9–3.11, $L_{\mathfrak{P}}(E_{CM}, T) = \zeta_p^{-1}(\mathcal{A}_{RM}, T)$. The first part of Theorem 1.2 follows.
3.1 Case \( p \) is a good prime

Let us prove the second part of Theorem 1.2 in the case \( n = 1 \). From the left side:

\[
K_0(\mathcal{O}_{L_p} \rtimes \mathbb{Z}) \cong K_0(\mathcal{O}_{L_p}) \cong \mathbb{Z}^2 / (I - L'_p)^2 \mathbb{Z}^2,
\]
where \( L_p = (\text{tr} (A^\pi(p)), p, -1, 0) \).

To calculate the abelian group \( \mathbb{Z}^2 / (I - L'_p)^2 \mathbb{Z}^2 \), we shall use a reduction of the matrix \( I - L'_p \) to the Smith normal form:

\[
I - L'_p = \begin{pmatrix}
1 - \text{tr} (A^\pi(p)) & 1 \\
-p & 1
\end{pmatrix} \sim \begin{pmatrix}
1 + p - \text{tr} (A^\pi(p)) & 0 \\
-p & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 \\
0 & 1 + p - \text{tr} (A^\pi(p))
\end{pmatrix}.
\]

Therefore, \( K_0(\mathcal{O}_{L_p}) \cong \mathbb{Z}_{1 + p - \text{tr} (A^\pi(p))} \).

From the right side, the \( \mathcal{E}_{CM}(\mathbb{F}_p) \) is an elliptic curve over the field of characteristic \( p \). Recall, that the chord and tangent law turns the \( \mathcal{E}_{CM}(\mathbb{F}_p) \) into a finite abelian group. The group is cyclic and has the order \( 1 + q_\mathfrak{P} - a_\mathfrak{P} \). But \( q_\mathfrak{P} = p \) and \( a_\mathfrak{P} = \text{tr} (\psi_{E(K)}(\mathfrak{P})) = \text{tr} (A^\pi(p)) \) (Lemma 3.8). Thus, \( \mathcal{E}_{CM}(\mathbb{F}_p) \cong \mathbb{Z}_{1 + p - \text{tr} (A^\pi(p))} \); therefore \( K_0(\mathcal{O}_{L_p}) \cong \mathcal{E}_{CM}(\mathbb{F}_p) \).

The general case \( n \geq 1 \) is treated likewise. Repeating the argument of Lemmas 3.1–3.2, it follows that \( L^n_p = (\text{tr} (A^{n\pi(p)}), p^n, -1, 0) \). Then one gets \( K_0(\mathcal{O}_{L^n_p}) \cong \mathbb{Z}_{1 + p^n - \text{tr} (A^{n\pi(p)})} \) on the left side. From the right side, \( |\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 + p^n - \text{tr} (\psi_{E(K)}^n(\mathfrak{P})) \); but a repetition of the argument of Lemma 3.8 yields us \( \text{tr} (\psi_{E(K)}^n(\mathfrak{P})) = \text{tr} (A^{n\pi(p)}) \). Comparing the left and right sides, one gets that \( K_0(\mathcal{O}_{L^n_p}) \cong \mathcal{E}_{CM}(\mathbb{F}_{p^n}) \). This argument finishes the proof of the second part of Theorem 1.2 for the good primes.

3.2 Case \( p \) is a bad prime

From the proof of Lemma 3.10, one gets for the left side \( K_0(\mathcal{O}_{\mathfrak{e}^n}) \cong \mathbb{Z}_{q^n} \). From the right side, it holds \( |\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 + q_\mathfrak{P} - a_\mathfrak{P} \), where \( q_\mathfrak{P} = 0 \) and \( a_\mathfrak{P} = \text{tr} (\mathfrak{e}_n) = \mathfrak{e}_n \). Thus, \( |\mathcal{E}_{CM}(\mathbb{F}_{p^n})| = 1 - \mathfrak{e}_n = 1 - (1 - q^n) = q^n \). Comparing the left and right sides, we conclude that \( K_0(\mathcal{O}_{\mathfrak{e}^n}) \cong \mathcal{E}_{CM}(\mathbb{F}_{p^n}) \) at the bad primes.

Since all cases are exhausted, Theorem 1.2 is proved.

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