DONALDSON-THOMAS INVARIENTS FROM TROPICAL DISKS

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Abstract. We prove that the quantum DT invariants associated to quivers with genteel potential can be expressed in terms of certain refined counts of tropical disks. This is based on a quantum version of Bridgeland’s description of cluster scattering diagrams in terms of stabilitiy conditions, plus a new version of the description of scattering diagrams in terms of tropical disk counts. We also show via explicit counterexample that Hall algebra broken lines do not result in consistent Hall algebra theta functions, i.e., they violate the extension of a lemma of Carl-Pumperla-Siebert from the classical setting.

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1. Introduction

In [GHKK14], Gross-Hacking-Keel-Kontsevich used scattering diagrams to construct canonical bases for cluster algebras. Several articles [Rei10, GPI10, KS, Nag13, Kel11] have developed connections between DT-invariants and various scattering diagrams or cluster transformations. Building off these ideas, Bridgeland [Bri17] constructed Hall algebra scattering diagrams whose classical integrals often recover the cluster scattering diagrams (cf. our Prop. 3.13 for the quantum analog). On the other hand, [GPS10, CPS, FS15, Man] show how to express various scattering diagrams in terms of certain (refined) counts of tropical curves or disks. By extending and combining these ideas, we obtain new expressions for quantum DT invariants in terms of refined counts of tropical disks.

1.1. Quantum DT invariants from tropical ribbons. Let $Q$ be a finite quiver without loops or oriented 2-cycles, with vertex set denoted $Q_0$ and arrows $Q_1$. Let $W$ be a potential on $Q$ (i.e., a finite linear combination of cycles). Let $\text{rep}(Q, W)$ be the category of finite-dimensional representations of $(Q, W)$, and let $M$ be the moduli stack of objects in $\text{rep}(Q, W)$. Let $N = \mathbb{Z}^{Q_0}$, with $\{e_i\}_{i \in Q_0}$ denoting the natural basis. Let $M_\mathbb{R} := \text{Hom}(N, \mathbb{R})$. For $\theta \in M_\mathbb{R}$, let $1_{\text{ss}}(\theta)$ denote the space of $\theta$-semistable objects in $\text{rep}(Q, W)$ (Def. 3.7). We say $\theta \in M_\mathbb{R}$ is general if it is not in the intersection of two distinct hyperplanes of the form $n^\perp$ for $n \in N \setminus \{0\}$ (cf. Remark 3.9). Let $B$ denote the skew-symmetric

The second author was supported by the National Science Foundation RTG Grant DMS-1246989, and later by the Starter Grant “Categorified Donaldson-Thomas Theory” no. 759967 of the European Research Council.
Euler form on $N$, cf. (2), and define $p^* : N \to M$, $p^*(n) = B(n, \cdot)$. We let $\mathcal{I}_t$ denote the quantum integration map taking varieties over $M$ to elements of a quantum torus algebra $\mathbb{C}[\mathbb{N}^\mathbb{R}]$, cf. (2.6) (if $W = 0$, then $\mathcal{I}_t$ is the generalized Poincaré polynomial).

Let $w$ be a weight-vector, i.e., a tuple $w = (w_1)$, where each $w_i = (w_{ij})_{j=1,...,l_i}$ consists of positive integers $w_{1i} \leq w_{2i} \leq \cdots \leq w_{li}$. Let $l(w) = \sum_i l_i$. Let $\text{Aut}(w)$ be the group of automorphisms of the second indices of the $w_i$'s which act trivially on $w$. Let $R_w := \prod_i \frac{(-1)^{q_i-1}}{w_{ij}(q_i)^{i-1}}$. We say that a tropical disk $h : \Gamma \to M_R$ (cf. §4.1) has degree $\Delta_w$ if the unbounded edges $E_{ij}$ are labelled by the indices of $w$, and if the weighted outgoing direction of $h(E_{ij})$ equals $w_{ij} p^*(e_i)$. Let $A_w$ be a collection of affine subspaces $\{A_{ij} \subset M_R\}$, with $A_{ij}$ a generic translate of $e_i^\perp$. For $\delta > 0$, we say that a tropical disk matches the constraints $\delta A_w$ if $h(E_{ij}) \subset \delta A_{ij}$ for each $i, j$.

Given $\theta \in M_R$, let $\mathcal{I}_w(\theta)$ denote the set of tropical disk types $\tau$ such that, for each $\epsilon > 0$, there exists for all sufficiently small $\delta > 0$ a tropical disk of degree $\Delta_w$, type $\tau$, matching the constraint $\delta A_w$, and with endpoint $V_\infty$ mapping into an $\epsilon$-neighborhood of $\theta$. Let $\tilde{\mathcal{I}}_w(\theta)$ denote the corresponding space of tropical ribbon types, i.e., tropical disk types plus the additional data of a cyclic ordering of the edges at vertex. Let $\mathcal{B}$ denote the form $\mathcal{B}(p^*(n_1), p^*(n_2)) = B(n_1, n_2)$ on $p^*(N)$. For $\tilde{\tau} \in \tilde{\mathcal{I}}_w(\theta)$, let $\nu(\tilde{\tau})$ denote $(-1)$ to the power of the number of vertices of $\tilde{\tau}$ where the ribbon structure does not agree with the orientation induced by $\mathcal{B}$, cf. §4.2.2.

The ribbon structure induces an ordering $E_{i_1,j_1}, \ldots, E_{i_l,j_l}$ on the $E_{ij}$'s. Given such a tropical ribbon type $\tilde{\tau}$, let $\mathfrak{flag}(\tilde{\tau})$ denote the variety over $M$ whose fiber over an (a stacky) point corresponding to a representation $M$ is the space of composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{\sum_{i,j} w_{ij}} = M$$

such that the first $w_{1i,j}$ quotients $M_i/M_{i-1}$ are isomorphic to the simple representation $S_{i_1}$, then the next $w_{2i,j}$ quotients $M_i/M_{i-1}$ are isomorphic to the simple representation $S_{i_2}$, and so on. The following is the quantum integral case of Theorem 4.9.

**Theorem 1.1.** Suppose $W$ is genteel (cf. §3.2.2). Then for general $\theta \in M_R$,

$$\mathcal{I}_t(\log(1_w(\theta))) = \sum_w \frac{1}{|\text{Aut}(w)|} \sum_{\tilde{\tau} \in \tilde{\mathcal{I}}_w(\theta)} \nu(\tilde{\tau}) \mathcal{I}_t(R_w \mathfrak{flag}(\tilde{\tau})) \ .$$

Genteelness of $W$ is known to at least hold for acyclic quivers with $W = 0$ [Bri17, Lem. 11.5]. We use this genteelessness to say that two a priori different scattering diagrams actually agree. Lang Mou [Mou] and Fan Qin [Qin] Thm. 1.2.2] show that this agreement also applies for quivers with non-degenerate potential which admit a green-to-red sequence, so Theorem 1.1 applies in these cases as well.

The same statement applies with $\mathcal{I}_t$ replaced by the classical integration map $\mathcal{I}$ (i.e., the $t \to 1$ limit, i.e., taking generalized Euler characteristics), and similarly for $\mathcal{I}_t$ replaced by other projections $\mathcal{I}'$ of the Hall algebra defined in §3.2.1. For the classical version though, one should view $\mathcal{I}(R_w \mathfrak{flag}(\tilde{\tau}))$ as living in $\mathfrak{A}^{cl}$, a logarithmic version of the Weyl algebra, cf. Example 4.1. These factors $\mathcal{I}_t(R_w \mathfrak{flag}(\tilde{\tau}))$ and $\mathcal{I}(R_w \mathfrak{flag}(\tilde{\tau}))$ can be more easily computed as products in the quantum torus algebra or Weyl algebra, respectively, cf. Remark 4.10.

Alternatively, one can replace the sum over tropical ribbons with a sum over tropical disks, and then the tropical ribbon multiplicities $\nu(\tilde{\tau}) \mathcal{I}_t(R_w \mathfrak{flag}(\tilde{\tau}))$ are replaced with Block-Göttsche [BGI] style refined tropical disk multiplicities $R'_w \prod_i [\text{Mult}(V)]_i$, and similarly for the classical cases, cf. Remark
again. Our intention in this paper though is to give a representation-theoretic description of the
tropical multiplicities, which is why we state Theorem 1.1 in terms of moduli of flags.

See Example 4.11 for a sample computation of a term on the right-hand side of (1).

1.2. Hall algebra broken lines violate the \textbf{CPS} lemma. One might hope (as we had hoped) that Theorem 1.1 holds without applying the integration maps, i.e., as an identity in the Hall algebra. Unfortunately, this fails as a result of the fact that elements of the Hall algebra with parallel dimension vectors need not commute (although we see that the result does hold after modding out by the ideal generated by these commutators). In §5 we show that similar issues cause problems for theta functions.

As in \cite{CPS, GHK15, GHS}, the construction of theta functions in \cite{GHKK14} is based on enumerating broken lines (an abridged version of tropical disks). This enumeration depends on the designated endpoint of the broken lines, but according to \cite{CPS, §4}, different choices of endpoint are related by path-ordered product, essentially meaning that these choices glue to give well-defined global functions on the mirror. \cite{Man, Thm. 2.14} gives a refined version of this \textbf{CPS} result, implying that the analogous gluing property holds for quantum theta functions (cf. Lemma 5.2). Refining further, \cite{Che16} defines Hall algebra broken lines, and from these one might hope to define Hall algebra theta functions. Unfortunately, this is not a well-behaved notion:

\textbf{Proposition 1.2} (Prop. 5.4 in the main text). \textit{The \textbf{CPS} lemma does not hold for Hall algebra broken lines.}

Our proof is via the explicit construction of a counterexample for an $A_3$-quiver, cf. §5.3

1.3. Motivation. When $B$ has rank 2, the tropical disk counts of Theorem 1.1 can be replaced with tropical curve counts, cf. \cite{GPS10, Thm. 2.8} and \cite{FS15, Cor. 4.9}. In higher-dimensions this is only the case for certain limits of choices of $\theta$, cf. \cite{Man, Thm 3.7}. The tropical curve versions are nice because in the classical limit they can be related via \cite{NS06} to log Gromov-Witten invariants, cf. \cite{GPS10, Prop. 5.3}. As the authors have learned from Mark Gross, the classical versions of our tropical disk counts should also have an algebraic Gromov-Witten theoretic meaning: according to the announced result \cite{GS18, Thm. 2.14}, they should be related to certain \textit{punctured} Gromov-Witten invariants (one also expects the existence of corresponding holomorphic disk counts defined from the perspective of open Gromov-Witten theory, e.g., as in \cite{Lin} for the case of K3 surfaces). One expects DT/GW correspondence results to follow from \cite{GS18, Thm. 2.14} combined with \cite{Bri17, Lem. 11.4].

On the other hand, the quantum tropical curve counts in rank 2 are Block-Götsche invariants \cite{BG16}, which have been related to higher-genus Gromov-Witten invariants in \cite{Bou19} and to real curve counts in \cite{Mik17}. Upcoming work of the second author will extend the correspondence of \cite{Mik17} to higher-dimensions, although an extension to tropical disks is still more distant. Still, we hope that Theorem 1.1 will lead to new refined DT/GW correspondence results, and we further hope that this correspondence will be enriched by our interpretation of tropical ribbon multiplicities in terms of moduli of composition series.

A version of Theorem 1.1 for bipartite quivers was previously observed in \cite{FS15, Thm. 5.3}. Their argument was based on the observation that in these cases, (1) is equivalent to a representation-theoretic formula of Manschot-Pioline-Sen \cite{MPS11}. We therefore hope that our result may be related to some generalization of this MPS formula. With this in mind, we strongly suspect that our tropical
counts are closely related to the attractor flow trees studied by physicists, cf. \cite{AP} in particular, as well as \cite{KST14}.

We note that \cite{LMY}, which appeared immediately after this paper was first posted, deals with similar problems on scattering diagrams and tropical disks using a differential-geometric perspective.

1.4. Outline of the paper. In \S\S\ 2.1-2.2, we review Joyce’s construction \cite{Joy07} of the Hall algebra associated to a quiver with potential, following \cite[\S\S\ 4-5]{Bri17}. Then in \S\S\ 2.3-2.4, we use \cite[Lem. 4.4]{Bri12} to describe certain products in the Hall algebra in terms of moduli of composition series. We review the quantum and classical integration maps in \S\S\ 2.5-2.6.

We review the definition of scattering diagrams in \S\ 3.1, and in Theorem 3.5 we generalize previously known results about initial scattering diagrams uniquely determining consistent scattering diagrams. We then we review Bridgeland’s Hall algebra scattering diagrams (and some variants) in \S\ 3.2. If the potential $W$ is genteel, then the Hall algebra scattering diagram is determined by an initial scattering diagram which we describe explicitly in \S\ 3.3.

We review the notion of tropical disks in \S\ 4.1, and in \S\ 4.2 we focus on the tropical ribbons and multiplicities associated to an initial scattering diagram. The description of scattering diagrams in terms of tropical disks (Theorem 4.3) is given in \S\ 4.3 and proven in \S\ 4.4. This is applied to the Hall algebra scattering diagram in \S\ 4.5 to prove our main results, Theorems 4.8 and 4.9.

We turn our attention to theta functions in \S\ 5. We review the definitions of broken lines and theta functions in \S\ 5.1, explaining how these apply to various flavors of cluster varieties in \S\ 5.2. Finally, in \S\ 5.3, we work out an explicit counterexample to show that a foundational result of \cite{CPS} (cf. Lemma 5.2) does not extend to the Hall algebra setting (Proposition 5.4).

Acknowledgements. The authors are very grateful to Ben Davison for patiently and repeatedly explaining the definition of the quantum integration map and for checking the corresponding part of our draft. We also thank Tom Bridgeland, Lang Mou, and Tom Sutherland for helpful conversations.

2. The motivic Hall algebra of a quiver with potential

2.1. Preliminaries on quivers with potential and their representations. Let $Q$ be a finite quiver without loops or oriented 2-cycles. Denote the sets of vertices and arrows of $Q$ as $(Q_0, Q_1)$. Let $\mathbb{C}Q$ denote the path algebra of $Q$. Suppose that $Q$ is equipped with a finite potential, i.e., a finite linear combination of cycles, denoted $W \in \mathbb{C}Q$. Define a two-sided ideal $I_W \subseteq \mathbb{C}Q$ by $I_W = (\partial_{a}W : a \in Q_1)$.

Here, if $b_1 \ldots b_k$ is a cycle of arrows in $Q$, then

$$\partial_{a}(b_1 \ldots b_k) = \sum_{i=1}^{k} \delta_{ab_i} b_{i+1} \ldots b_k b_1 \ldots b_{i-1},$$

where $\delta_{ab_i}$ is 1 if $a = b_i$ and 0 otherwise. Then the Jacobi algebra for $(Q, W)$ is the quotient algebra $\mathbb{C}Q/I_W$. Let $\text{rep}(Q, W) := \text{mod}\mathbb{C}Q/I_W$ be the abelian category of finite-dimensional representations of the quiver with potential $(Q, W)$, i.e., finite-dimensional left $\mathbb{C}Q/I_W$-modules.

Set $N = \mathbb{Z}Q_0$, $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\{e_i\}_{i \in Q_0}$ be the natural basis indexed by the vertices of $Q$. Denote $N^\oplus := \{ \sum_{i} a_i e_i \in N | a_i \in \mathbb{Z}_{\geq 0} \ \forall i \}$, and $N^+ := N^\oplus \setminus \{0\}$. There is a group homomorphism

$$\text{dim} : K_0(\text{rep}(Q, W)) \rightarrow N.$$
sending a representation to its dimension vector. For vertices \(i, j \in Q_0\), let \(a_{ij}\) denote the number of arrows from \(i\) to \(j\). Let \(B\) denote the integral skew-symmetric bilinear form on \(N\) determined by setting
\[
B(e_i, e_j) = a_{ji} - a_{ij}.
\]
We will also use a second \(\mathbb{Z}\)-valued bilinear form \(\chi\) on \(N\) given by
\[
\chi(e_i, e_j) := \delta_{ij} - a_{ij}.
\]
Note that \(B(a, b) := B(e_i, e_j)\) for a smooth projective variety \(X\) over \(\mathbb{C}\) to its Poincaré polynomial
\[
\sum_{k=1}^{2d} \dim_{\mathbb{C}} H^k(X_{an}, \mathbb{C})(-t)^k \in \mathbb{C}[t],
\]
where \(q := t^2\) and \(H^k(X_{an}, \mathbb{C})\) denotes singular cohomology. For \(X \in K(\text{St} / \mathbb{C})\), we will often denote
\[
|X| := \Upsilon(X).
\]
Let
\[
K_\Upsilon(\text{St} / \mathcal{M}) := K(\text{St} / \mathcal{M}) \otimes_{K(\text{St} / \mathbb{C})} \mathbb{C}(t).
\]
As a \(\mathbb{C}(t)\)-module, the (motivic) Hall algebra \(H(Q, W)\) is \(K_\Upsilon(\text{St} / \mathcal{M})\). To define the multiplication, the convolution product, on \(H(Q, W)\) and make it into a \(\mathbb{C}(t)\)-algebra, we consider the stack
\[ M^{(2)} \] of short exact sequences in \( \text{rep}(Q,W) \). There is a diagram

\[
\begin{array}{ccc}
M^{(2)} & \xrightarrow{b} & M \\
\downarrow^{(a_1,a_2)} & & \downarrow \\
M \times M,
\end{array}
\]

where \( a_1, a_2, b \) sends a short exact sequence

\[
0 \to A_1 \to B \to A_2 \to 0
\]
to \( A_1, A_2, \) and \( B \) respectively. The convolution product is defined to be

\[
m = b \circ (a_1, a_2)^* : H(Q,W) \times H(Q,W) \to H(Q,W).
\]

This product can be expressed as

\[
[X_1 \xrightarrow{f_1} M] \ast [X_2 \xrightarrow{f_2} M] = [Z \xrightarrow{b \circ h} M],
\]

where \( Z \) and \( h \) are defined by the Cartesian square

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & M^{(2)} \\
\downarrow & & \downarrow^{(a_1,a_2)} \\
X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & M \times M
\end{array}
\]

The following is due to Joyce [Joy07, Thm. 5.2], see also [Bri12, Thm. 4.3].

**Theorem 2.1.** The product \( m \) gives \( H(Q,W) \) the structure of an associative unital algebra over \( \mathbb{C}(t) \).

The unit element is \( 1 = [M_0 \subset M] \).

We note that the decomposition \((4)\) of \( M \) induces an \( N^\oplus \)-grading

\[
H(Q,W) = \bigoplus_{d \in N^\oplus} H(Q,W)_d,
\]

where \( H(Q,W)_d \) is the submodule of \( K\tau(\text{St} / M) \) generated by objects of the form \([X \to M_d \subset M]\).

**2.3. k-fold products.** We will also need a description of the \( k \)-fold product \( m_k : H(Q,W)^\otimes_k \to H(Q,W) \). For this we follow [Bri12, §4.1-4.2]. Let \( M^{(k)} \) denote the algebraic moduli stack of \( k \)-flags. That is, the objects of \( M^{(k)} \) over a scheme \( S \) are isomorphism classes of \( k \)-tuples of objects \((E_1, \rho_1), \ldots, (E_k, \rho_k) \) of \( M(S) \), together with monomorphisms

\[
0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_k
\]

respecting the maps \( \rho_i \) and such that each factor \( F_i := E_i / E_{i-1} \) is flat over \( S \). Given another scheme \( T \), an object \((E'_1, \rho'_1), \ldots, (E'_k, \rho'_k) \) over \( T \), and a morphism \( f : T \to S \), a morphism in \( M^{(k)} \) lying over \( f \) is a collection of isomorphisms of sheaves \( \Phi_i : f^*(E_i) \to E'_i \) respecting the maps \( \rho_i \) and the maps in the sequences of monomorphisms as in \((8)\).

For each \( i = 1, \ldots, k \), we have a morphism of stacks \( a_i : M^{(k)} \to M \) taking an object as in \((8)\) to its \( i \)-th factor \( F_i = E_i / E_{i-1} \). We also have another morphism \( b : M^{(k)} \to M \) taking the object as in \((8)\) to the final term \((E_k, \rho_k) \) of the sequence. One easily sees that the stack \( M^{(2)} \), together with
these morphisms $a_1, a_2, b$, is equivalent to the data we had when defining $\mathcal{M}^{(2)}$ as the stack of short exact sequences above. We now obtain a diagram generalizing (6):

$$
\begin{array}{ccc}
\mathcal{M}^{(k)} & \xrightarrow{b} & \mathcal{M} \\
\downarrow & & \downarrow \hspace{1cm} \downarrow_{(a_1, \ldots, a_k)} \\
\mathcal{M}^k
\end{array}
$$

**Lemma 2.2** ([Bri12], Lemma 4.4). The $k$-fold product $m_k : H(Q, W)^{\otimes k} \to H(Q, W)$ is given by

$$
m_k := b \circ (a_1, \ldots, a_k)^*.
$$

2.4. $H_{\text{reg}}$ and the composition algebra. Next, recalling the notation $q = t^2$, let

$$
C_{\text{reg}}(t) := \mathbb{C}[t, t^{-1}][(1 + q + q^2 + \ldots + q^k)^{-1} : k \geq 1] \subset \mathbb{C}(t).
$$

Let $H_{\text{reg}}(Q, W)$ be the $C_{\text{reg}}(t)$-submodule of $H(Q, W)$ generated by elements of the form

$$
[f : X \to \mathcal{M}]
$$

such that $X$ is a variety over $\mathbb{C}$ (so in particular, $X \in \text{St}/\mathbb{C}$, and so we can apply $T$ to $X$).

**Lemma 2.3** ([Bri17], Thm. 5.2). $H_{\text{reg}}(Q, W)$ is closed under the Hall algebra product and thus forms an $N^\oplus$-graded $C_{\text{reg}}(t)$-subalgebra. Furthermore, $H_{\text{reg}}(Q, W)$ forms a Poisson algebra under the bracket

$$
\{a, b\} := (t - t^{-1})^{-1}[a, b].
$$

Now, for any representation $A \in \text{ob}(\text{rep}(Q, W))$, let $p_A$ denote the corresponding (stacky) point in $\mathcal{M}$, and let $\delta_A$ be the element of $H(Q, W)$ corresponding to the inclusion $[p_A \hookrightarrow \mathcal{M}]$. Let

$$
\kappa_A := |\text{Aut}(A)|\delta_A \in H(Q, W)
$$

be the element $[\text{Spec } \mathbb{C} \to p_A \in \mathcal{M}]$. Clearly, $\kappa_A$ is in $H_{\text{reg}}(Q, W)$.

Given a collection of objects $A_1, \ldots, A_k, M \in \text{ob}(\text{rep}(Q, W))$, let $\widetilde{\text{Flag}}(A_1, \ldots, A_k; M)$ denote the space of filtrations

$$
0 = M_0 \subset M_1 \subset \ldots \subset M_k = M
$$

of $M$ such that $M_i/M_{i-1} \cong A_i$ for each $i$. We also consider the quotient stack $\text{Flag}(A_1, \ldots, A_k; M)$ in which the identification of $M_k$ with $M$ is no longer part of the data of an object. This has the effect of enlarging the automorphism groups since now automorphisms of $M$ induce automorphisms of flags, so

$$
\text{Flag}(A_1, \ldots, A_k; M) = [\widetilde{\text{Flag}}(A_1, \ldots, A_k; M)/\text{Aut}(M)].
$$

By Lemma 2.2, we have the following:

**Lemma 2.4.** Given a collection of objects $A_1, \ldots, A_k \in \text{ob}(\text{rep}(Q, W))$, let $d = \sum_{j=1}^k \dim(A_j) \in N^\oplus$. The product $\kappa_{A_1} \cdots \kappa_{A_k}$ is represented by a complex variety $\widetilde{\text{Flag}}(A_1, \ldots, A_k) \to \mathcal{M}_d$ whose fiber over a point $p_M$ is $\widetilde{\text{Flag}}(A_1, \ldots, A_k; M)$. Equivalently, the fiber of $\widetilde{\text{Flag}}(A_1, \ldots, A_k)$ over the geometric point $[\text{Spec } \mathbb{C} \to p_M \in \mathcal{M}_d]$ is $\text{Flag}(A_1, \ldots, A_k; M)$. 
For each vertex $i \in Q_0$, we have an associated simple representation $S_i \in \text{rep}(Q,W)$ of dimension vector $e_i$. We denote $\delta_i := \delta_{e_i}$ and $\kappa_i := \kappa_{S_i}$. More generally, for each $k \in \mathbb{Z}_{\geq 0}$, we will write the semisimple representation $S_i^{\otimes k}$ as $S_{ki}$, and we will write $\delta_{ki} := \delta_{S_{ki}}$ and $\kappa_{ki} := \kappa_{S_{ki}}$. As in [Joy07] Ex. 5.20, we define the composition algebra $C(Q,W)$ to be the subalgebra of $\text{H}_{\text{reg}}(Q,W)$ generated by the elements $\kappa_{ki}$ for $i \in Q_0$. By Lemma 2.4, products of the elements $\kappa_{ki}$ are given in terms of spaces of composition series.

Example 2.5. For $i \in Q_0$, let us apply Lemma 2.4 to $\kappa_{ki}^k$. The only point in $\mathcal{M}_{ki}$ is the one corresponding to the semisimple representation $S_{ki}$. Furthermore, $\text{Flag}(S_i; S_1, S_2, \ldots; S_k)$ ($S_i$ occurring $k$ times before the semicolon) contains only one (stacky) point—all maximal flags of $S_{ki}$ are related by automorphisms of $S_{ki}$. The stabilizer group for this point (i.e., the space of automorphisms of $C^k$ which fix a maximal flag) is the unipotent group $U_k(\mathbb{C})$. Thus,

\[ \kappa_{ki} = [U_k(\mathbb{C})]|_{\kappa_i^k} = q^{k(k-1)/2} \kappa_i^k. \]

Using (10) and the fact that

\[ |\text{Aut}(S_{ki})| = |\text{GL}_k(\mathbb{C})| = q^{k(k-1)/2} \prod_{j=1}^{k}(q^j - 1), \]

we can re-express (12) as

\[ \delta_{ki} = \frac{1}{\prod_{j=1}^{k}(q^j - 1)} \kappa_i^k. \]

Alternatively, this could be realized directly as

\[ \delta_{ki} = \kappa_i^k / |\text{Flag}(S_i; S_1, S_2; S_{ki})| \]

(S, again appearing $k$ times before the semicolon). (13) will be useful in §3.3.

2.5. The quantum torus algebra. Let $\mathbb{C}[N^\oplus]$ denote the quantum torus algebra, by which we mean the $N^\oplus$-graded algebra defined by:

\[ \mathbb{C}[N^\oplus] := \mathbb{C}_{\text{reg}}[t]|z^n : n \in N^\oplus]/\langle z^{n_1}z^{n_2} = t^{B(n_1, n_2)}z^{n_1+n_2} : n_1, n_2 \in N^\oplus \rangle \]

(the monomials $z^n$ adjoined here are non-commuting). This forms a Poisson algebra under the bracket

\[ \{a, b\} := \frac{[a, b]}{t - t^{-1}}. \]

Note that

\[ \{z^{n_1}, z^{n_2}\} = [B(n_1, n_2)]tz^{n_1+n_2}, \]

where for any $a \in \mathbb{Z}$,

\[ [a]_t := \frac{t^a - t^{-a}}{t - t^{-1}} = \text{sgn}(a)(t^{-|a|} + t^{-|a|+1} + \ldots + t^{-|a|+|a|-2} + t^{-|a|}). \]

The usual commutative algebra $\mathbb{C}[N^\oplus]$ also forms a Poisson algebra, with bracket defined by

\[ \{z^{n_1}, z^{n_2}\} := B(n_1, n_2)z^{n_1+n_2}. \]

Note that there is a surjective homomorphism of Poisson algebras defined by

\[ \pi_{t \mapsto 1} : \mathbb{C}_t[N^\oplus] \to \mathbb{C}[N^\oplus], \quad t \mapsto 1, \quad z^n \mapsto z^n. \]
Remark 2.6. Note that $\mathbb{C}_t[N^\oplus]$ viewed as a Lie algebra with its Poisson bracket is isomorphic as a Lie algebra to $(t-t^{-1})^{-1} \cdot \mathbb{C}_t[N^\oplus]$ with its commutator bracket via the map

$$x \mapsto \frac{x}{t-t^{-1}}.$$  

We may thus view $\pi_{t^{-1}}$ as a Lie algebra homomorphism $(t-t^{-1})^{-1} \cdot \mathbb{C}_t[N^\oplus] \rightarrow \mathbb{C}[N^\oplus]$. Similarly, as noted in [Bri17, §5.9], $H_{\text{reg}}(Q,W)$ with the bracket from [Bri17] is isomorphic as a Lie algebra to $(t-t^{-1})^{-1} \cdot H_{\text{reg}}(Q,W)$ with its commutator bracket. In [Bri17, §5.9] we will discuss the “integration map” $I = \pi_{t^{-1}} \circ I_t$ as a homomorphism of Poisson algebras $H_{\text{reg}}(Q,W) \rightarrow \mathbb{C}[N^\oplus]$, but this can also be viewed as a homomorphism of Lie algebras $(t-t^{-1})^{-1} \cdot H_{\text{reg}}(Q,W) \rightarrow \mathbb{C}[N^\oplus]$. Similarly, we may view the quantum integration map $I_t : H_{\text{reg}}(Q,W) \rightarrow \mathbb{C}_t[N^\oplus]$ as a Lie algebra homomorphism $(t-t^{-1})^{-1} \cdot H_{\text{reg}}(Q,W) \rightarrow (t-t^{-1})^{-1} \mathbb{C}_t[N^\oplus]$.

2.6. The integration map. There are several constructions of (quantum) integration maps in the literature, i.e., homomorphisms (of algebras, Lie algebras, or Poisson algebras) from $H(Q,W)$ or $H_{\text{reg}}(Q,W)$ to the (quantum) torus algebra. Reineke [Rei03, Lem. 6.1] first constructed the analog of such a quantum integration map for finitary Hall algebras associated to quivers without potential. Joyce [Joy07, §6] then constructed classical and quantum integration maps with domain $H_{\text{reg}}(Q,0)$. The classical version of Joyce’s map (of Lie algebras) was generalized to quivers with potential in [JS12, §7] (cf. [Bri17, Thm. 11.1] for an interpretation as a map of Poisson algebras). On the other hand, a very general construction of algebra homomorphisms from a full Hall algebra to the “motivic quantum torus algebra” (which can then be further integrated to the usual quantum torus algebra) has been outlined by Kontsevich and Soibelman [KS, §6]. Making this more precise and more algebraic, in [KS11, §7], Kontsevich and Soibelman defined a (monodromic) mixed Hodge structure (building off Saito’s theory of mixed Hodge modules [Sai90]) on the equivariant cohomology of the vanishing cycle complex, and then [DM12] and [Dav18] built on these ideas to rigorously define a quantum integration map $I_t$.

We give a brief sketch of this integration map

$$I_t : H_{\text{reg}}(Q,W) \rightarrow \mathbb{C}_t[N^\oplus]$$

essentially as in [Dav18, §3.3]. We then use this to compute the integration in the simplest cases. We note that by the definitions of the Poisson structures in [Bri17] and [Dav18], it is clear that $I_t$ being a map of algebras implies it is also a map of Poisson algebras, thus also giving maps of Lie algebras as in Remark 2.6.

Recall that $\mathcal{M}$ is the moduli stack of objects in $\text{rep}(Q,W) := \text{mod } \mathbb{C}Q/I_W$. Let $\mathcal{M}^0$ be the moduli stack of objects in $\text{rep}(Q,0)$. Given an arrow $a \in Q_1$, let $t(a), h(a) \in Q_0$ denote the tail and head of $a$ respectively. For any $i \in Q_0$ and $d \in N^\oplus$, let $d_i$ denote the corresponding component of $d$. Denote

$$\widetilde{\mathcal{M}}^0_d := \prod_{a \in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^d_{t(a)}, \mathbb{C}^d_{h(a)}),$$

and

$$\text{GL}_d := \prod_{i \in Q_0} \text{GL}_{d_i}(\mathbb{C}).$$

Then $\mathcal{M}^0 = \bigsqcup_{d \in N^\oplus} \mathcal{M}_d$, where $\mathcal{M}_d$ is the stack-theoretic quotient

$$\mathcal{M}_d = \widetilde{\mathcal{M}}^0_d / \text{GL}_d,$$

(17)
where the action by $GL_d$ is the one induced by the conjugation action of $GL_d(C)$ on $C^{d_1}$ for each $i \in Q_0$.

Viewing elements of $\tilde{M}_d^a$ as modules over the path-algebra $CQ$, we see that multiplication by $W$ gives an endomorphism of $\tilde{M}_d^a$. Since the trace is invariant under the action of $GL_d$, we obtain a function

$$\text{Tr}(W) : M^\circ \to \mathbb{C},$$

the critical locus of which recovers $M$:

$$M = \text{crit}(\text{Tr}(W)) \subset M^\circ.$$

Let $Y$ be a smooth complex variety and let $f : Y \to \mathbb{C}$ be a regular function on $Y$. The corresponding vanishing cycle functor $\varphi_f$ is defined as follows (following [Dav18 §3.1], also cf. [KST11 §7.2]). Let $Y_0 := f^{-1}(0)$, and let $Y_{\leq 0} := f^{-1}(\mathbb{R}_{\leq 0})$. For a sheaf $F$ on $Y$ and $U$ an analytic open subset of $Y$, define

$$
\Gamma_{X_{\leq 0}}F(U) := \ker (F(U) \to F(U \setminus (U \cap X_{\leq 0}))).
$$

The stacks $M_d^a$ for $d \in N^\oplus$ are not quite smooth complex varieties, but each is a quotient of a smooth complex variety by the action of an algebraic group, cf. [17]. One can thus extend the definition of $\varphi_f$ to regular functions $f$ on $M_d^a$ using an equivariant version of the vanishing cycle construction as in [DMa §2.2].

Let $\underline{Q}_d$ denote the constant sheaf on $M_d^a$. For each $u \in C^*$, we can define $\varphi_{\text{Tr}(W)/u}\underline{Q}_d$. Now consider $[X \to M_d] \in H_{\text{reg}}(Q,W)$. Composing with the inclusion $M_d \subset M_d^a$, we can consider the pullback

$$
\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X := (X \to M_d^a)^* \varphi_{\text{Tr}(W)/u}\underline{Q}_d.
$$

This sheaf $\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X$ on $X$ in fact has the structure of a mixed Hodge module on $X$, and so the cohomology $H_c(X, \varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X)$ has a mixed Hodge structure. This induces a double-grading $H_{p,k}^{X_{\text{Tr}(W)/u}\underline{Q}_d}(X,\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X)$ on $H_c(X, \varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X) \otimes \mathbb{C}$.

Up to isomorphism, the sheaf $\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X$ is independent of $u$. However, there may be non-trivial monodromy $\mu$ on $H_c(X, \varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X)$ as $u$ travels around the origin in $C$ (most properly, we have the structure of a monodromic mixed Hodge module on $X$, cf. [Dav18 §3 for this viewpoint]). This $\mu$ is quasi-unipotent, i.e., the eigenvalues are roots of unity. Let $H_{p,k}^{X_{\text{Tr}(W)/u}\underline{Q}_d}(X,\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X)$ denote the locus on which $\mu$ is unipotent, and let $H_{c,\neq 1}^{X_{\text{Tr}(W)/u}\underline{Q}_d}(X,\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X)$ denote the direct sum of the generalized eigenspaces for all eigenvalues of $\mu$ other than 1. Finally, the quantum integration map $\mathcal{I}_t$ is defined by taking the Serre polynomial (cf. [DL01, §3.1.3], also [KS, p. 69]) defined as follows:

$$
\mathcal{I}_t([X \to M_d]) := t^\chi(d,d) \sum_{p \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^p \left( \dim(H_{c,1}^{X_{\text{Tr}(W)/u}\underline{Q}_d}(X,\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X))(-t)^k + \dim(H_{c,\neq 1}^{X_{\text{Tr}(W)/u}\underline{Q}_d}(X,\varphi_{X_{\text{Tr}(W)/u}\underline{Q}_d}^X))(-t)^{k+1} \right),
$$

where $\chi$ is defined as in [3].

**Proposition 2.7** ([Dav18, Prop. 3.13]. $\mathcal{I}_t : H_{\text{reg}}(Q,W) \to \mathbb{C}_t[N^\oplus]$ is a homomorphism of $\mathbb{C}_t$-algebras.
This construction simplifies quite a bit for \([f : X \to M_d] \) with \(\text{Tr}(W)|_{M_d} = 0\). In this case, \(\varphi_{\text{Tr}}(W) = \text{Id}\), and so \(H^p_{\mathcal{E}}(X, \varphi_{\text{Tr}}(W)^*\mathcal{Q}_d)\) becomes \(H^p(X, \mathcal{Q}_d)\), i.e., the usual degree \(p\) Betti cohomology with compact support of \(X\). If, furthermore, \(X\) is a smooth projective variety, the cohomology is pure and concentrated in even degree, and so the summations in \((18)\) become just the Poincaré polynomial. Recalling the definition of \(\Upsilon\) from \((5)\), we have thus recovered the following:

**Proposition 2.8.** If \([f : X \to M_d] \in H_{\text{reg}}(Q, W)\) and \(\text{Tr}(W)|_{M_d} = 0\), then

\[
I_t([f : X \to M]) = \Upsilon(X)t^{\chi(d,d)}z^d. \tag{19}
\]

In particular, if \(W = 0\) (e.g., for \(Q\) acyclic), \(I_t\) equals the quantum integration map of \([Joy07, \S 6]\).

**Example 2.9.** Recall \(\kappa_{ki} := [\text{Spec} \mathbb{C} \to p_{S^\oplus k} = M_k e_i]\). We have \(W|_{M_k^{e_k}} = 0\), \(\Upsilon(\kappa_{ki}) = 1\) (the Poincaré polynomial of a point), and \(\chi(k e_i, k e_i) = k^2\). Hence, \(I_t(\kappa_{ki}) = t^{k^2}z^{k e_i}\). In particular,

\[
I_t(\kappa_{ki}) = tz^{e_i}. \tag{20}
\]

As a check, one can use \((12)\) to confirm that \(I_t(\kappa_{ki}) = I_t(\kappa_i)^k\).

Composing \(I_t\) with \(\pi_{t \to 1}\) induces the classical integration map:

\[
I := \pi_{t \to 1} \circ I_t : H_{\text{reg}}(Q, W) \to \mathbb{C}[N^{\oplus}].
\]

The classical integration maps of \([JS12, \S 7]\) and \([Bri17, \text{Thm. 11.1}]\) are always (even for nonzero \(W\)) given by the \(t \to 1\) limit of \((19)\), i.e., by taking Euler characteristics. Note that \((20)\) is sufficient to completely determine the restrictions of \(I_t\) and \(I\) to the composition algebra \(\mathcal{C}(Q, W)\) in which all our computations will lie. Since \(I\) agrees with the classical integration maps of \([JS12, \S 7]\) and \([Bri17, \text{Thm. 11.1}]\) on the generators \(\kappa_i\), the maps necessarily agree on all of \(\mathcal{C}(Q, W)\).

### 3. Scattering diagrams from Hall algebras

#### 3.1. Background on scattering diagrams

Here we review the basic definitions and properties of scattering diagrams from the perspective useful for understanding the Hall algebra scattering diagrams of \([Bri17]\).

Let \(\Lambda\) denote a finite-rank lattice equipped with a \(\mathbb{Z}\)-valued skew-symmetric form \(\{\cdot, \cdot\}\). Let \(\Lambda^\vee := \text{Hom}(\Lambda, \mathbb{Z})\) be the dual lattice, and let \(\langle \cdot, \cdot \rangle : \Lambda \oplus \Lambda^\vee \to \mathbb{Z}\) denote the dual pairing. We have a map

\[
p^\ast : \Lambda \to \Lambda^\vee
\]

\[
n \mapsto \{n, \cdot\}. \tag{21}
\]

Fix a strictly convex rational polyhedral cone \(\sigma_{\Lambda^\oplus} \subset \Lambda^\oplus\). Let \(\Lambda^\oplus := \sigma_{\Lambda^\oplus} \cap \Lambda\), and let \(\Lambda^+ := \Lambda^\oplus \setminus \{0\}\).

Let \(\mathfrak{g} := \bigoplus_{n \in \Lambda^+} \mathfrak{g}_n\) be a Lie algebra graded by \(\Lambda^+\), meaning that \([\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subseteq \mathfrak{g}_{n_1 + n_2}\). We say that \(\mathfrak{g}\) is skew-symmetric with respect to \(\{\cdot, \cdot\}\) if

\[
[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0 \text{ whenever } \{n_1, n_2\} = 0. \tag{22}
\]

For each \(k \in \mathbb{Z}_{\geq 1}\), let

\[
k\Lambda^+ := \{n_1 + \ldots + n_k \in \Lambda^+ | n_i \in \Lambda^+ \text{ for each } i = 1, \ldots, k\}.
\]

Let \(\mathfrak{g}^{\geq k} := \bigoplus_{n \in k\Lambda^+} \mathfrak{g}_n\). Note that \(\mathfrak{g}^{\geq k}\) is a Lie subalgebra of \(\mathfrak{g}\). Let \(\mathfrak{g}_k\) denote the nilpotent Lie algebra \(\mathfrak{g}/\mathfrak{g}^{\geq k}\), and let \(\widehat{\mathfrak{g}} := \varprojlim \mathfrak{g}_k\). We have corresponding Lie groups \(G := \exp \mathfrak{g}\), \(G_k := \exp \mathfrak{g}_k\), and \(\widehat{G} := \exp \widehat{\mathfrak{g}} = \varprojlim G_k\).
For each $n \in \Lambda^+$, we have a Lie subalgebra $\mathfrak{g}^n := \prod_{k \in \mathbb{Z}_{\geq 1}} \mathfrak{g}_{kn} \subset \hat{\mathfrak{g}}$. We say that $\mathfrak{g}$ has Abelian walls if each $\mathfrak{g}^n$ is Abelian. In particular, $\mathfrak{g}$ has Abelian walls whenever $\mathfrak{g}$ is skew-symmetric. Let $G^n_\mathfrak{g} := \exp(\mathfrak{g}^n) \subset \hat{G}$.

The Abelian walls condition is usually assumed to hold when working with scattering diagrams, but when defining Hall algebra scattering diagrams, one needs a slight generalization as in [Bri17, §2].

**Definition 3.1.** A wall in $\Lambda^\vee_{\mathfrak{g}}$ over $\hat{\mathfrak{g}}$ is data of the form $(\mathfrak{d}, g_\mathfrak{d})$, where:

- $g_\mathfrak{d} \in \mathfrak{g}^n_{g_{\mathfrak{d}}}$ for some primitive $n_{\mathfrak{d}} \in \Lambda^+$. The element $-p^*(n_{\mathfrak{d}})$ is called the direction of the wall. We call $g_\mathfrak{d}$ the scattering function associated to the wall.
- $\mathfrak{d}$ is a closed, convex (but not necessarily strictly convex), rational-polyhedral, codimension-one affine cone in $\Lambda^\vee_{\mathfrak{g}}$, parallel to $n_{\mathfrak{d}}^\perp$. We call $\mathfrak{d}$ the support of the wall.

A scattering diagram $\mathcal{D}$ over $\hat{\mathfrak{g}}$ is a set of walls in $\Lambda^\vee_{\mathfrak{g}}$ over $\hat{\mathfrak{g}}$ such that for each $k > 0$, there are only finitely many $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathcal{D}$ with $g_{\mathfrak{d}}$ not projecting to 0 in $\mathfrak{g}_k$. If $(\mathfrak{d}_1, g_{\mathfrak{d}_1})$ and $(\mathfrak{d}_2, g_{\mathfrak{d}_2})$ are two walls of $\mathcal{D}$, and if $\text{codim}_{\Lambda^\vee_{\mathfrak{g}}} (\mathfrak{d}_1 \cap \mathfrak{d}_2) = 1$, then we require that $[g_{\mathfrak{d}_1}, g_{\mathfrak{d}_2}] = 0$ (note that this is automatic for Abelian walls).

A wall with direction $-v$ is called incoming if it contains $v$. Otherwise, the wall is called outgoing.

We will sometimes denote a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ by just $\mathfrak{d}$. Denote $\text{Supp}(\mathcal{D}) := \bigcup_{\mathfrak{d} \in \mathcal{D}} \mathfrak{d}$, and

$$\text{Joints}(\mathcal{D}) := \bigcup_{\mathfrak{d} \in \mathcal{D}} \partial \mathfrak{d} \cup \bigcup_{(\mathfrak{d}_1, \mathfrak{d}_2) \in \mathcal{D}} \mathfrak{d}_1 \cap \mathfrak{d}_2.$$  

Note that for each $k > 0$, a scattering diagram $\mathcal{D}$ over $\hat{\mathfrak{g}}$ induces a finite scattering diagram $\mathcal{D}^k$ over $\mathfrak{g}_k$ with walls corresponding to the $\mathfrak{d} \in \mathcal{D}$ for which the projection of $g_{\mathfrak{d}}$ to $\mathfrak{g}_k$ is nonzero.

Consider a smooth immersion $\gamma : [0, 1] \to \Lambda^\vee_{\mathfrak{g}} \setminus \text{Joints}(\mathcal{D})$ with endpoints not in $\text{Supp}(\mathcal{D})$ which is transverse to each wall of $\mathcal{D}$ it crosses. Let $(\mathfrak{d}_i, g_{\mathfrak{d}_i})$, $i = 1, \ldots, s$, denote the walls of $\mathcal{D}^k$ crossed by $\gamma$, and say they are crossed at times $0 < t_1 < \ldots < t_s < 1$, respectively.\(^1\)

Define

$$\Phi_{\mathfrak{d}_i} := \exp(g_{\mathfrak{d}_i})^{\text{sgn}(n_{\mathfrak{d}_i}, -\gamma'(t_i))} \in G_k.$$  

Let $\Phi_{\gamma, \mathcal{D}}^k := \Phi_{\mathfrak{d}_1} \cdots \Phi_{\mathfrak{d}_i} \in G_k$, and define the path-ordered product:

$$\Phi_{\gamma, \mathcal{D}} := \lim_{k \to \infty} \Phi_{\gamma, \mathcal{D}}^k \in \hat{G}.$$  

**Definition 3.2.** Two scattering diagrams $\mathcal{D}$ and $\mathcal{D}'$ are equivalent if $\Phi_{\gamma, \mathcal{D}} = \Phi_{\gamma, \mathcal{D}'}$ for each smooth immersion $\gamma$ as above. $\mathcal{D}$ is consistent if each $\Phi_{\gamma, \mathcal{D}}$ depends only on the endpoints of $\gamma$.

We say that $x \in \Lambda^\vee_{\mathfrak{g}}$ is general if it is contained in at most one hyperplane of the form $n^\perp$ for $n \in \Lambda$. For $\mathcal{D}$ a scattering diagram over $\hat{\mathfrak{g}}$ and $x \in \Lambda^\vee_{\mathfrak{g}}$ general, denote

$$g_{x, \mathcal{D}} := \sum_{\mathfrak{d} \ni x} g_{\mathfrak{d}} \in \hat{\mathfrak{g}},$$  

where the sum is over all walls $(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathcal{D}$ with $\mathfrak{d} \ni x$. One easily sees the following standard fact (cf. [GHKK14] Lem. 1.9)\(^1\):

**Lemma 3.3.** Two scattering diagrams $\mathcal{D}$ and $\mathcal{D}'$ over $\hat{\mathfrak{g}}$ are equivalent if and only if $g_{x, \mathcal{D}} = g_{x, \mathcal{D}'}$ for all general $x \in \Lambda^\vee_{\mathfrak{g}}$.\(^1\)

\(^1\)If $t_i = t_{i+1}$, then $\mathfrak{d}_i$ must be Abelian, and so the ordering of the corresponding walls does not affect $\Phi_{\gamma, \mathcal{D}}^k$.\(^1\)
Theorem 3.5. Let $\d \in \mathfrak{g}_{\ast}$ such that $\d$ is consistent, $\d \supset \d_m$, and $\d$ consists only of outgoing walls. Then replacing a wall $(\d, g_i) \in \mathfrak{d}$ with a pair of walls $(\d_1, g_{i_1})$, $i = 1, 2$, such that $\d_1 \cup \d_2 = \d$ and codim$_{\mathfrak{g}_k} (\d_1 \cap \d_2) = 2$ produces an equivalent scattering diagram.

Example 3.4.

(1) Consider a collection of walls $\{ (\d, g_i) \in \mathfrak{d} \parallel \mathfrak{g}_{\ast} \} \in \mathcal{D} \mid i \in S$, where $S$ is some countable index set, and $n_k$ and $\d$ are independent of $i$. Suppose that $[g_i, g_j] = 0$ for each $i, j \in S$. Then replacing this collection of walls with a single wall $(\d, \sum_{i \in S} g_i)$ produces an equivalent scattering diagram.

(2) Replacing a wall $(\d, g_i) \in \mathfrak{d}$ with a pair of walls $(\d_1, g_{i_1})$, $i = 1, 2$, such that $\d_1 \cup \d_2 = \d$ and codim$_{\mathfrak{g}_k} (\d_1 \cap \d_2) = 2$ produces an equivalent scattering diagram.

The following theorem is fundamental to the study of scattering diagrams. The 2-dimensional version was first proved in [KS06], and this was generalized to higher dimensions in [GS11] §3 for scattering diagrams over the module of log derivations. The higher-dimensional version for scattering diagrams over skew-symmetric Lie algebras follows from [KS14, Prop. 3.2.6, 3.3.2] (cf. [GHKK14, Lem. C.7] for a review of this argument from our viewpoint). As pointed out to us by Lang Mou, this result had not previously been proven in the presence of non-Abelian walls.

Theorem 3.5. Let $\mathfrak{g}$ be a $\Lambda^+\!$-graded Lie algebra, and let $\mathfrak{D}_m$ be a finite scattering diagram over $\hat{\mathfrak{g}}$ whose walls are of the form $(n_i^+, g_i)$ for various primitive $n_i \in \mathbb{N}^+$. Expand $g_i$ as $\sum_{w \geq 1} g_i,w \in \mathfrak{g}_{n_i}$, for $g_{i,w} \in \mathfrak{g}_{w n_i}$. Assume that for each $i$, the terms $g_{i,w}$ of $g_i$ pairwise commute. Then there is a unique-up-to-equivalence scattering diagram $\mathcal{D}$ such that $\mathcal{D}$ is consistent, $\mathcal{D} \supset \mathcal{D}_m$, and $\mathcal{D} \setminus \mathcal{D}_m$ consists only of outgoing walls.

Note that the assumption that the $g_{i,w}$'s commute for each $i$ is automatic in the case of Abelian walls. This assumption is used for our existence proof, but not for the uniqueness proof.

Proof. The existence of such a $\mathcal{D}$ will actually be proved in §4.4.2, so for now we just assume existence and focus on the uniqueness. The argument here is inspired by that of [GHKK14, Lem. C.7].

Let $\mathcal{D}, \mathcal{D}'$ be two consistent scattering diagrams over $\hat{\mathfrak{g}}$ with incoming walls $\mathcal{D}_m$ as in the statement of the theorem. We shall prove by induction on $k$ that $\mathcal{D}^k$ and $(\mathcal{D}')^k$ are equivalent over $\mathfrak{g}_k$ for each $k$, and then the equivalence of $\mathcal{D}$ and $\mathcal{D}'$ follows. Note that $\mathcal{D}^1$ and $(\mathcal{D}')^1$ are both equivalent to the trivial scattering diagram, hence to each other.

Now suppose that $\mathcal{D}^k$ and $(\mathcal{D}')^k$ are equivalent over $\mathfrak{g}_k$. Let $\mathcal{D}''$ be a scattering diagram over $\mathfrak{g}_{k+1}$ such that

$$g_x,\mathcal{D}'' = g_x,\mathcal{D}^{k+1} - g_x,(\mathcal{D}')^{k+1}$$

for each general $x \in \Lambda^+_{\mathfrak{g}}$. Since $\mathcal{D}^k$ and $(\mathcal{D}')^k$ are equivalent over $\mathfrak{g}_k$, we must have $g_x,\mathcal{D}'' \in \mathfrak{g}^{\parallel \mathfrak{g}} \setminus \mathfrak{g}^{\parallel \mathfrak{g}-1}$, hence $g_x,\mathcal{D}''$ is central in $\mathfrak{g}_{k+1}$. Hence, $(\mathcal{D}')^k \cup \mathcal{D}''$ is a well-defined scattering diagram over $\mathfrak{g}_{k+1}$, and by Lemma 3.3 it is equivalent to $\mathcal{D}^k$. Our goal now is to show that $\mathcal{D}''$ is equivalent to the trivial scattering diagram.

Since both $\mathcal{D}$ and $\mathcal{D}'$ were assumed to be consistent, and the scattering functions of $\mathcal{D}''$ are all central in $\mathfrak{g}_{k+1}$, $\mathcal{D}''$ must also be consistent (over $\mathfrak{g}_{k+1}$). Furthermore, this consistency plus centrality of the scattering functions implies that, up to equivalence, the support of every wall of $\mathcal{D}''$ is an entire affine hyperplane in $\Lambda^+_{\mathfrak{g}}$. But then all walls of $\mathcal{D}''$ (up to equivalence) are incoming, and since the incoming walls of $\mathcal{D}$ and $\mathcal{D}'$ are the same, this implies that $\mathcal{D}''$ is equivalent to the trivial scattering diagram over $\mathfrak{g}_{k+1}$, as desired.
A scattering diagram playing the role of $D_{in}$ in Theorem 3.5 will be referred to as an **initial scattering diagram**. The consistent scattering diagram $\mathcal{D}$ (up to equivalence) produced by the theorem will be denoted $\text{Scat}(D_{in})$.

**Example 3.6.** Consider $\Lambda = \mathbb{Z}^2$. Equip $\Lambda$ with the skew-symmetric form $\{\cdot, \cdot\}$ represented by
\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]
and consider the quantum torus algebra $\mathbb{C}_t[\Lambda]$ as in 2.5. Take $g$ to be the Lie subalgebra (with respect to Poisson bracket) with basis $\{z^n : n \in \Lambda^+\}$. Let
\[
D_{in} := \{(e_1^\perp, -\text{Li}(-z^{e_1}; t)), (e_2^\perp, -\text{Li}(-z^{e_2}; t))\},
\]
where $\text{Li}(x, t)$ denotes the quantum dilogarithm as in (25) below. Then $D := \text{Scat}(D_{in})$ is obtained by adding a single outgoing wall $(\mathbb{R}_{\geq 0}(1, -1), -\text{Li}(-z^{(1,0)}; t))$, cf. Figure 3.1.

![Figure 3.1. The quantum A_2 scattering diagram.](image)

The consistency of this scattering diagram is equivalent to a version of the quantum pentagon identity of [FK94]. The classical limit is essentially the $\ell_1 = \ell_2 = 1$ case of [GPS10, Ex. 1.6] (with some small changes in sign conventions). We will see in Example 3.10 that this is the scattering diagram obtained when applying the quantum integration map to the Hall algebra scattering diagram associated to the $A_2$-quiver.

### 3.2. Hall algebra scattering diagrams.

**3.2.1. Setup for Hall algebra scattering diagrams and their variants.** We now take $\Lambda = N$, $\Lambda^0 = N^0$, and $\{\cdot, \cdot\} = B$. Recall that $H(Q,W)$ admits a grading by $N^\circ$ as in (7). In particular, we can write $H_{\text{reg}}(Q,W) = H_{\text{reg}}(Q,W)_0 \oplus H_{\text{reg}}(Q,W)_{>0}$ for $H_{\text{reg}}(Q,W)_{>0} := \bigoplus_{d \in N^+} H_{\text{reg}}(Q,W)_d$. Let $g^{\text{Hall}} := (t - t^{-1})^{-1} \cdot H_{\text{reg}}(Q,W)_{>0}$, viewed as a Lie algebra using the commutator bracket as in Remark 2.6.

The Lie algebra $g^{\text{Hall}}$ typically is not skew-symmetric and does not have Abelian walls. To get around this issue, let $i^{\text{skew}}$ denote the Lie ideal of $g^{\text{Hall}}$ generated by the commutators we wish to vanish, i.e.,
\[
i^{\text{skew}} := \langle [g_d^{\text{Hall}}, g_d^{\text{Hall}}] : d_1, d_2 \in N^+, \{d_1, d_2\} = 0 \rangle.
\]
Here, for $S$ a subset of $g_{\text{reg}}$, $\langle S \rangle$ denotes the Lie ideal generated by $S$, i.e., the intersection of all Lie ideals of $g^{\text{Hall}}$ which contain $S$. Then for any Lie ideal $i$ which contains $i^{\text{skew}}$, we define
\[
g^i := g^{\text{Hall}}/i.
\]
Note that for any Lie algebra ideal \( \mathfrak{i} \) of \( \mathfrak{g}^{\text{Hall}} \), \( \mathfrak{g}^{\text{Hall}}/\mathfrak{i} \) is skew-symmetric if and only if \( \mathfrak{i} \supset \mathfrak{i}^{\text{skew}} \). Since the commutator bracket on the quantum torus algebra makes it into a skew-symmetric Lie algebra, we in particular have

\[
\ker(I) \supset \mathfrak{i}^{\text{skew}}.
\]

The resulting Lie algebra \( \mathfrak{g}^\theta := \mathfrak{g}^{\ker(I)} \) is just the quantum torus algebra \((t - t^{-1})^{-1} \cdot \mathbb{C}_t[N^\oplus] \) with its commutator bracket as in \([14]\). Similarly, \( \ker(I) \supset \mathfrak{i}^{\text{skew}} \), and \( \mathfrak{g}^{\text{cl}} := \mathfrak{g}^{\ker(I)} \) is just \( \mathbb{C}[N^\oplus] \) together with its Poisson bracket as in \([16]\). In general, let \( I^i : \mathfrak{g}^{\text{Hall}} \to \mathfrak{g}^i \) denote the projection.

For \( \mathfrak{g} \) equal to \( \mathfrak{g}^{\text{Hall}}, \mathfrak{g}^i, \mathfrak{g}^q \), or \( \mathfrak{g}^{\text{cl}} \), we denote the corresponding Lie group \( G \) by \( G^{\text{Hall}}, G^i, G^q \), or \( G^{\text{cl}} \), respectively. The notation for the associated completions and scattering diagrams will be similarly obvious except for sometimes using “Hall” instead of “reg.”

### 3.2.2. The Hall algebra scattering diagram.

**Definition 3.7.** Given \( \theta \in M_{\mathbb{R}} \), an object \( E \in \text{rep}(Q,W) \) is said to be \( \theta \)-semistable if

- \( \theta(E) = 0 \),
- every subobject \( B \subset E \) satisfies \( \theta(B) \leq 0 \). If, furthermore, this inequality is strict, then we say that \( E \) is \( \theta \)-stable.

The scattering diagram defined in the following theorem of Bridgeland is what one calls the Hall algebra scattering diagram.

**Theorem 3.8.** \([Bri17, \text{Theorem 6.5}]\) There exists a consistent scattering diagram \( \mathfrak{D}^{\text{Hall}} \) in \( M_{\mathbb{R}} \) over \( \mathfrak{g}^{\text{Hall}} \) such that:

1. The support \( \text{Supp}(\mathfrak{D}^{\text{Hall}}) \) consists of those \( \theta \in M_{\mathbb{R}} \) for which there exist \( \theta \)-semistable objects in \( \text{rep}(Q,I) \);  
2. For \( \theta \subset \text{Supp}(\mathfrak{D}^{\text{Hall}}) \setminus \text{Joints}(\mathfrak{D}^{\text{Hall}}) \), there is a unique wall \( (\mathfrak{d},g_{\mathfrak{d}}) \in \mathfrak{D}^{\text{Hall}} \) for which \( \mathfrak{d} \ni \theta \). For this wall, we have \( \exp(g_{\mathfrak{d}}) = 1_{\mathfrak{m}}(\theta) \in G^{\text{Hall}} \).

**Remark 3.9.** We say \( \theta \in M_{\mathbb{R}} \) is general if it is not in the intersection of two distinct hyperplanes of the form \( n^k \) for \( n \in N \setminus \{0\} \). Since the joints of \( \mathfrak{D}^{\text{Hall}} \) are codimension 2 subsets of \( M_{\mathbb{R}} \) and have rational slope, Theorem 3.8 gives the scattering functions of \( \mathfrak{D}^{\text{Hall}} \) at all general points \( \theta \in M_{\mathbb{R}} \). Alternatively, we could use a more refined notion of general. Call \( \theta \in M_{\mathbb{R}} \) special if at least one of the following holds:

- There exists a pair of \( \theta \)-semistable objects with non-parallel dimension vectors;
- Some \( E \in \text{rep}(Q,W) \) is \( \theta \)-semistable, but for \( 0 < \epsilon \ll 1 \), \( E \) is either not \( (\theta + \epsilon p^*(\dim(E))) \)-semistable or not \( (\theta - \epsilon p^*(\dim(E))) \)-semistable.

The former condition accounts for joints where two walls of different slopes intersect, while the latter accounts for intersections of walls with the same slope. That is, \( \theta \in \text{Joints}(\mathfrak{D}^{\text{Hall}}) \) if and only if \( \theta \) is special. Theorem 1.1 will still hold and will be slightly stronger if we define general to mean not special.

---

Note that we could define the Hall algebra scattering diagram using the full Hall algebra \( H(Q,W) \) in place of \( H_{\text{reg}}(Q,W) \) (as is done in \([Bri17]\)), or alternatively using just the composition algebra \( C(Q,W) \). The advantage of using \( H_{\text{reg}}(Q,W) \) or \( C(Q,W) \) instead of \( H(Q,W) \) is just for convenience when we talk about applying integration maps.
Note that we obtain new scattering diagrams $D^i$, $D^q$, and $D^{cl}$ over $g^i$, $g^q$, and $g^{cl}$, respectively, by applying $I^i$, $I^q$, or $I^{cl}$ to $D^{Hall}$. The scattering diagram $D^{cl}$ is what Bridgeland calls the stability scattering diagram. We call $D^q$ the quantum stability scattering diagram.

**Example 3.10.** Let us consider the $A_2$ quiver $1 \leftarrow 2$ with $W = 0$. The corresponding matrix $B$ is

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

as in Example 3.6. Let us explicitly describe the Hall algebra scattering diagram $D^{Hall}$ from Theorem 3.8 in this case. Note that there are 3 indecomposable representations of $A_2$ up to isomorphism: $C \leftarrow 0$, $0 \leftarrow C$, and $C \leftarrow C$ (the last map being nonzero). Consider $\mathfrak{d} = (1, 0)^\perp$.

For any point $\theta \in \mathfrak{d}$, one can see that the representations $(C \leftarrow 0)^\oplus k$ are $\theta$-semistable for any positive integer $k$, and we find $1_{ss}(\theta) = \sum_{k \geq 0}(C \leftarrow 0)^\oplus k$. We similarly compute that for $\theta \in (0, 1)^\perp$, $1_{ss}(\theta) = \sum_{k \geq 0}(0 \leftarrow C)^\oplus k$, and for $\theta \in \mathbb{R}_{\geq 0}(1, -1)$, we have $1_{ss}(\theta) = \sum_{k \geq 0}(C \leftarrow C)^\oplus k$. Note that $(C \leftarrow C)$ contains $(C \leftarrow 0)$ as a subrepresentation, and so $(C \leftarrow C)^\oplus k$ is not $(-\alpha, \alpha)$-semistable for $\alpha \in \mathbb{R}_{>0}$. There are no other $\theta$-semistable representations for any $\theta$ in this example, so the Hall algebra scattering diagram is as in Figure 3.2. Note that $D^q$, obtained from applying the quantum integration map $I$ to the scattering functions of $D^{Hall}$ (cf. §3.3 for such computations) yields the consistent scattering diagram of Example 3.6.

![Figure 3.2. The $A_2$ Hall algebra scattering diagram.](image-url)

As in [Bri17 §11.5], an object $E \in \text{rep}(Q, W)$ is called self-stable if it is stable with respect to the stability condition $-p^*(E)$. A quiver with potential $(Q, W)$ is called genteel if the only self-stable objects are the simple objects, cf. [Bri17 Def. 11.3]. For example, [Bri17 Lem. 11.5] states that if $Q$ is acyclic and $W = 0$, then $(Q, W)$ is genteel.

The condition of being genteel exactly means that the only incoming walls of $D^{Hall}$ are

$$
D^{Hall}_{in} := \{e_i^+, \log 1_{ss}(p^*(e_i))\}.
$$

In any case (whether $W$ is genteel or not), we define

$$
D^{Hall}_{Scat} := \text{Scat}(D^{Hall}_{in})
$$

for $D^{Hall}_{in}$ as in (24).

**Example 3.11.** For the $A_2$-quiver of Example 3.10, the simple representations are $C \leftarrow 0$ and $0 \leftarrow C$. Thus $D^{Hall}_{in}$ would be as in Figure 3.3. By the uniqueness of the consistent scattering diagram from Theorem 3.5 we have that $D^{Hall}_{Scat}$ agrees with $D^{Hall}$ from Figure 3.2.
Proposition 3.12. If \((Q, W)\) is genteel, then \(D^\text{Hall} \) and \(D^\text{Scat} \) are equivalent.

The analogous statements for any \(D^i\) and \(D^i_{\text{Scat}} := \text{Scat}(I^i(D^\text{Hall}_\text{in})) = I^i(D^\text{Hall}_{\text{Scat}})\) follows as a corollary. As previously noted, [Bri17, Lem. 11.5] states that genteelessness holds for acyclic quivers. Fan Qin [Qin, Thm. 1.2.2] has extended the equivalence of \(D^\text{cl} \) and \(D^\text{cl}_{\text{Scat}} \) to other cases as well, e.g., to quivers with non-degenerate potential which admit a green-to-red sequence, and upcoming work of Lang Mou [Mou] will prove the equivalence of \(D^g \) and \(D^g_{\text{Scat}} \) in these cases.

3.3. The initial Hall algebra scattering diagrams. We next wish to better understand the scattering functions of (24). For each \(i \in S\), we will find a nice expression for \(\log \log 1_{ss}(p^*(e_i))\) in terms of powers of \(\kappa_i\). We will need the quantum dilogarithm

\[
\Psi_t(x) := \sum_{k=0}^{\infty} \frac{t^{-k(k-1)/2}x^k}{(t-t^{-1})(t^2-t^{-2}) \cdots (t^k-t^{-k})} = \sum_{k=0}^{\infty} \frac{(tx)^k}{\prod_{j=1}^{k} (t^{2j} - 1)},
\]

and the standard fact that \(\log \Psi_t(x) = -L_i(-x; t)\), where

\[
(25) \quad L_i(x; t) := \sum_{k=1}^{\infty} \frac{x^k}{k(t^k - t^{-k})}.
\]

Denote

\[
f_i := 1_{ss}(p^*(e_i)) = \sum_{k=0}^{\infty} \delta_{ki}.
\]

By (13), we can rewrite \(f_i\) as\(^3\)

\[
f_i = \sum_{k=0}^{\infty} \frac{\kappa_i^k}{\prod_{j=1}^{k} (q^j - 1)} = \Psi_t \left( \frac{\kappa_i}{t} \right).
\]

\(^3\)This expression seems to be well-known to experts, cf. [KS §6.4].
Hence, using that \( \log \Psi_t(x) = -\text{Li}(-x; t) \), we find

\[
\log f_i = -\text{Li}(-\kappa_i/t; t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \kappa_i^k t^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \kappa_i^k}{k(q^k - 1)}. 
\]

We denote

\[
R_k := \frac{(-1)^{k-1}}{k(q^k - 1)} 
\]

so \( \log f_i \) can be written as

\[
\log f_i = \sum_{k=1}^{\infty} R_k \kappa_i^k. 
\]

It follows immediately from (26), (20), and Theorem 3.5 that applying \( \mathcal{I}_t \) to \( D_{\text{HallScat}} \) produces the quantum cluster scattering diagrams of [Man, §4.2]:

**Proposition 3.13.** Applying \( \mathcal{I}_t \) to \( D_{\text{HallScat}} \) produces the scattering diagram \( D_{\text{Scat}}^q := \text{Scat}(D_{\text{in}}^q) \) over the quantum torus algebra, where

\[
D_{\text{in}}^q := \{ e_i^\perp, -\text{Li}(-z^{e_i}, t) \}. 
\]

Applying \( \pi_{t \to 1} \), it follows that \( \mathcal{I} \) applied to \( D_{\text{HallScat}} \) yields \( \text{Scat}(D_{\text{in}}^q) \), where

\[
D_{\text{in}}^q := \{ e_i^\perp, -\text{Li}(-z^{e_i}) \}. 
\]

Here, \( \text{Li}(x) := \sum_{k=1}^{\infty} \frac{x^k}{k} \) is the classical dilogarithm. This is precisely [Bri17, Lem. 11.4].

**Remark 3.14.** Instead of viewing \( g_{\text{Hall}} \) as \( (t - t^{-1})^{-1} \mathcal{H}_{\text{reg}}(Q, W) \) with its commutator bracket, one might try to view it as simply \( \mathcal{H}_{\text{reg}}(Q, W) \) with the Poisson bracket of \( g \), cf. Remark 2.6. In this version, instead of having \( g = \log(1_{s^\perp}(\theta)) \) in Theorem 3.8, one has \( g = (t - t^{-1}) \log(1_{s^\perp}(\theta)) \). In (28), this corresponds to redefining \( R_k \) to be

\[
\frac{(-1)^{k-1}}{k(t+q^2+...+q^{k-1})}. 
\]

4. **Scattering diagrams in terms of tropical disks**

4.1. **Tropical disks.** We now introduce the tropical disks whose enumerations will be related to the scattering diagrams of [8]. For now, our tropical disks will live in \( L_{\mathbb{R}} := L \otimes \mathbb{R} \) for an arbitrary finite-rank lattice \( L \) (later we will take \( L = \Lambda^\vee = M \)).

Let \( \Gamma \) be the topological realization of a finite connected tree without bivalent vertices, and let \( \Gamma \) denote the complement of all but one of its 1-valent vertices. Denote this remaining 1-valent vertex by \( V_\infty \), and denote the edge containing this vertex by \( E_\infty \). Let \( \Gamma^{[0]}, \Gamma^{[1]}, \) and \( \Gamma^{[1]}_{\infty} \) denote the sets of vertices, edges, and non-compact edges of \( \Gamma \), respectively. Let \( e_\infty := \# \Gamma^{[1]}_{\infty} \). Equip \( \Gamma \) with a weighting \( w : \Gamma^{[1]} \to \mathbb{Z}_{>0} \), plus a marking \( \epsilon : S \to \Gamma^{[1]}_{\infty} \) for some index set \( S \) with \( \# S = e_\infty \). For \( s \in S \), we denote \( E_s := \epsilon(s) \).

A **parametrized tropical disk** \( (\Gamma, w, \epsilon, h) \) in \( L_{\mathbb{R}} \) is data \( \Gamma \), \( w \), and \( \epsilon \) as above, plus a proper continuous map \( h : \Gamma \to L_{\mathbb{R}} \) such that:

- For each \( E \in \Gamma^{[1]} \), \( h|_E \) is an embedding into an affine line with rational slope;
• For any vertex $V$ and edge $E \ni V$, denote by $u_{(V,E)}$ the primitive integral vector emanating from $h(V)$ into $h(E)$. For each $V \in \Gamma^{[0]} \setminus \{V_\infty\}$, the following balancing condition is satisfied:

$$
\sum_{E \ni V} w(E) u_{(V,E)} = 0.
$$

For unbounded edges $E_s \ni V$, we may denote $u_{(V,E_s)}$ simply as $u_{E_s}$ or $u_s$. An isomorphism of parameterized tropical disks $(\Gamma, h)$ and $(\Gamma', h')$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights and markings such that $h = h' \circ \Phi$. A tropical disk is then defined to be an isomorphism class of parameterized tropical disks. We will let $(\Gamma, h)$ denote the isomorphism class it represents, and we will often further abbreviate this as simply $\Gamma$ or $h$.

A tropical ribbon $\tilde{\Gamma}$ is a tropical disk $(\Gamma, w, \epsilon, h)$ as above, together with the additional data of a cyclic ordering of the edges at each vertex. A tropical disk or ribbon is called trivalent if every vertex other than $V_\infty$ is trivalent.

The degree $\Delta$ of a tropical disk $(\Gamma, w, \epsilon, h)$ is the map $\Delta : S \to L$ given by

$$
\Delta(s) = w(E_s)u_{E_s}.
$$

Let $\text{Flags}(\Gamma)$ denote the set of flags $(V,E)$, $V \in E$, of $\Gamma$. The type of a tropical disk is the data of $\Gamma$, $w$, and $\epsilon$, along with the data of the map $u : \text{Flags}(\Gamma) \to L$, $(V,E) \mapsto u_{(V,E)}$. Note that the type of a tropical disk determines its degree.

Similarly, the type of a tropical ribbon is the data of the type of the associated tropical disk, plus the data of the ribbon structure, i.e., the data of the cyclic orderings at each vertex.

Let $A := (A_s)_{s \in S}$ be a tuple of affine-linear subspaces $A_s \subset L_R$, each with rational slope. We say a tropical disk $(\Gamma, w, \epsilon, h)$ matches the constraint $A$ if $h(E_s) \subset A_s$ for each $s \in S$.

4.2. Tropical degrees, constraints, and multiplicities associated to a scattering diagram.

We now combine the setup of §4.1 with that of §3.2. Let $L = \Lambda^\vee$. Let $\{e_i\}_{i \in I}$ be a finite collection of vectors in $\Lambda^+$, indexed by a set $I$. Suppose we have an initial scattering diagram $\mathcal{D}_{\text{in}}$ over $\mathfrak{g}$, with $\mathcal{D}_{\text{in}}$ having the form

$$
\mathcal{D}_{\text{in}} = \{(\partial_i, g_i) | i \in I\},
$$

where for each $i$, we have $\partial_i = e_i^+$ and

$$
g_i = \sum_{w \geq 1} g_{i,w} \in \mathfrak{g}_{e_i}^{\parallel},
$$

where $g_{i,w} \in \mathfrak{g}_{we_i}$. Assume as in Theorem 3.5 that for each $i$, the terms $g_{i,w}$ pairwise commute. We denote

$$
v_i = p^*(e_i)
$$

for each $i \in S$, so $-v_i$ is the direction of the wall $\partial_i$.

4.2.1. Degrees and constraints. Let $w := (w_i)_{i \in S}$ be a tuple of weight vectors $w_i := (w_{i,1}, \ldots, w_{i,l_i})$ with $0 < w_{i,1} \leq \ldots \leq w_{i,l_i}$, $w_{ij} \in \mathbb{Z}$. For $\Sigma_{l_i}$ denoting the group of permutations of $\{1, \ldots, l_i\}$, let

$$
\text{Aut}(w) \subset \prod_{i \in S} \Sigma_{l_i}.
$$
be the group of automorphisms of the second indices of the weights \( w \), which act trivially on \( w \). We also define

\[
l(w) = \sum_{i \in S} l_i.
\]

Associated to \( w \), we consider the degree \( \Delta_w : S_w \to L \) given by

\[
S_w := \{(i, j) | i \in I, j \in \{1, \ldots, l_i\}\},
\]

and

\[
\Delta_w((i, j)) = w_{ij}v_i.
\]

For the associated constraints \( A_w = (A_{ij})_{(i, j) \in S_w} \) we take the affine-linear space \( A_{ij} \) to be a generic translate of \( e_i^\perp \). Here, the translates for different pairs \((i, j)\) are generic relative to each other. We fix such a choice of \( A_w \) for each \( w \). Given \( \delta > 0 \), let \( \delta A_w \) denote the constraints obtained from \( A_w \) by multiplying each \( A_{ij} \) by \( \delta \) (i.e., the distance from the origin is multiplied by \( \delta \)).

4.2.2. Multiplicities. For each \((i, j) \in S_w\), we denote

\[
g_{ij} := g_{i, w_{ij}} \in g_{w_{ij}e_i}.
\]

Denote

\[
n_w := \sum_{(i, j) \in S_w} w_{ij}e_i \in \Lambda.
\]

Now consider a trivalent tropical disk \( \Gamma \) of degree \( \Delta_w \). We will denote \( E_{(i, j)} \) simply as \( E_{ij} \). We view \( \Gamma \) as flowing towards the univalent vertex \( V_\infty \), and we use this flow to inductively associate an element \( g_E \in g_{n_E} \subset g \) to each edge \( E \) of \( \Gamma \), where \( n_E \) is an element of \( \Lambda^+ \) such that \( p^*(n_E) \in L \) is the weighted tangent vector to \( h(E) \) pointing in the direction opposite the flow.

To each of the source edges \( E_{ij} \), we associate the element \( g_{ij} \) from (33) above. Now consider a vertex \( V \neq V_\infty \) with \( E_1, E_2 \) flowing into \( V \) and \( E_3 \) flowing out of \( V \), and suppose that for \( i = 1, 2 \), we already have associated elements \( g_{E_i} \in g_{n_{E_i}} \). By the balancing condition, we have \( n_{E_3} = n_{E_1} + n_{E_2} \). Let us assume that the labelling of the edges \( E_1, E_2 \) is such that

\[
\{n_{E_1}, n_{E_2}\} \geq 0
\]

(otherwise we re-label). We then define

\[
g_{E_3} := [g_{E_1}, g_{E_2}] \in g_{n_{E_3}}.
\]

We now define the multiplicity of \( \Gamma \) as

\[
\text{Mult}(\Gamma) := g_{E_\infty} \in g_{n_w}.
\]

Now suppose that \( g \) is a Lie subalgebra of the commutator algebra of a \( \Lambda^+ \)-graded associative algebra \( A \), i.e., we have an associative product such that

\[
[g_1, g_2] = g_1g_2 - g_2g_1.
\]
Example 4.1. For $\mathfrak{g} = \mathfrak{g}^{\text{Hall}}$, Remark 2.6 says that we can take

$$\mathfrak{g}^{\text{Hall}} = (t - t^{-1})^{-1} \cdot H_{\text{reg}}(Q, W) \subset \mathfrak{A}^{\text{Hall}} := H_{\text{reg}}(Q, W)[(t - t^{-1})^{-1}].$$

Similarly, we can take

$$\mathfrak{g} = (t - t^{-1})^{-1}C_t[N^\oplus] \subset \mathfrak{A} := C_t[N^\oplus][(t - t^{-1})^{-1}].$$

Moreover, for any $i$ such that $i_{\text{skew}} \subseteq i \subseteq \ker(I_t)$, since $(t - t^{-1}) \not\in i$, we can take

$$\mathfrak{g}^i = (t - t^{-1})^{-1}H_{\text{reg}}(Q, W)/i \subset \mathfrak{A}^i := (H_{\text{reg}}(Q, W)/i)[(t - t^{-1})^{-1}].$$

However, $t - t^{-1} = 0$ in $\mathfrak{g}^{\text{cl}}$, so we cannot apply this localization in the classical setting. Instead, we take $\mathfrak{A}^{\text{cl}}$ to be the universal enveloping algebra of $\mathfrak{g}^{\text{cl}}$.

Alternatively, the Poisson algebra $\mathbb{C}[N^\oplus]$ can be identified with a subalgebra of the module of log derivations $\Theta(N^\oplus) := \mathbb{C}[N^\oplus] \otimes \mathbb{Z} M$ as in [GPS10] §1.1. Here, $z^n \otimes m$, typically denoted $z^n \partial_m$, is viewed as acting on $\mathbb{C}[N]$ via $z^n \mapsto \langle n, m \rangle z^{n + \langle n, m \rangle} \partial_m$. The commutator of these derivations makes $\Theta(N^\oplus)$ into a Lie algebra with bracket given by

$$[z^n \partial_m, z'^m \partial_{m'}] = z^{n + m + \langle n, m \rangle} \partial_{(n,m) + (m,m') - (n, m) - (n', m')}.$$

Let $\mathfrak{h}$ be the Lie subalgebra spanned by elements of the form $z^n \partial_m$ for $\langle n, m \rangle = 0$. Then $\mathbb{C}[N^\oplus]$ embeds into $\mathfrak{h}$ via $z^n \mapsto z^n \partial_{B(n,0)}$. Hence, instead of taking $\mathfrak{A}^{\text{cl}}$ to be the universal enveloping algebra of $\mathbb{C}[N^\oplus]$, it is reasonable to take it to be the universal enveloping algebra of $\mathfrak{h}$ or $\Theta(N^\oplus)$. The latter is simply a log version of the Weyl algebra in $\text{rank}(N)$ variables. That is, we may view $\mathfrak{A}^{\text{cl}}$ as an algebra of logarithmic differential forms.

We note that the usual classical multiplicities of tropical curves (as in correspondence theorems like those of [NS06]) can similarly be computed via iterated Lie brackets of polyvector fields, cf. [MR]. Also, the quantum ribbon multiplicities computed using $\mathfrak{A}^{\text{reg}}$ are related to certain counts of real curves, cf. [Mik17].

For example, we can always take $\mathfrak{A}$ to be the universal enveloping algebra of $\mathfrak{g}$. Alternatively, for $\mathfrak{g}^{\text{Hall}}$ or $\mathfrak{g}^{\text{reg}}$, we can produce such an $\mathfrak{A}$ using Remark 2.6.

Suppose that $\Gamma$ is equipped with a ribbon structure $\Gamma$. At each vertex $V \neq V_\infty$, let $E_1, E_2$ be the vertices flowing into $V$ and $E_3$ the vertex flowing out of $V$, and assume the cyclic ordering of the labelling $E_1, E_2, E_3$ agrees with the ribbon structure of $\hat{\Gamma}$ at $V$ (otherwise we re-label). We say that the vertex $V \in \hat{\Gamma}^{[0]}$ is positive if the edges $E_1, E_2$, labelled in this way with respect to the ribbon structure, satisfy the condition (34). Otherwise, we say $V$ is negative.

We now describe a method of inductively associating an element of $\mathfrak{g}_{nE} \subset \mathfrak{A}$ to each edge $E$ of $\hat{\Gamma}$, this time denoting the elements by $g_{E}^\gamma$. The vectors $n_E$ will be the same as before, but the elements $g_{E}^\gamma$ will be different and will depend on the ribbon structure. As before, we take $g_{E_i,j}^{\gamma} := g_{ij}$ for the source edges. But now, for $E_1, E_2$ the edges flowing into a vertex $V$, $E_3$ the edge flowing out of $V$, and the labelling agreeing with the ribbon structure at $V$, we define

$$g_{E_3}^{\gamma} := \nu(V)g_{E_1}^{\gamma}g_{E_2}^{\gamma},$$

where

$$\nu(V) := \begin{cases} 1 & \text{if } V \text{ is positive} \\ -1 & \text{if } V \text{ is negative} \end{cases}$$
and $g_{E_1}^\gamma, g_{E_2}^\gamma$ is the associative product in $\mathfrak{A}$. Finally, we define

$$\text{Mult}^\gamma(\hat{\Gamma}) := g_{E_{\text{out}}}^\gamma \in \mathfrak{A}_{n_w}. \quad (35)$$

Alternatively, define

$$\nu(\hat{\Gamma}) := \prod_{V \in \hat{\Gamma}} \nu(V). \quad (36)$$

The ribbon structure induces an ordering of the unbounded edges of $\hat{\Gamma}$, starting with $E_{\text{out}}$ and then continuing with $E_{i_1j_1}, \ldots, E_{i_lj_l(w)}$. Using the associativity of $\mathfrak{A}$, we can rewrite (35) as

$$\text{Mult}^\gamma(\hat{\Gamma}) = \nu(\hat{\Gamma})g_{i_1j_1}g_{i_2j_2} \cdots g_{i_lj_l(w)} \in \mathfrak{A}. \quad (36)$$

One easily sees the following:

**Lemma 4.2.** For each $\Gamma$ as above,

$$\text{Mult}(\Gamma) = \sum \text{Mult}^\gamma(\hat{\Gamma}),$$

where the sum is over all possible tropical ribbons $\hat{\Gamma}$ with underlying tropical curve $\Gamma$.

Note that $\text{Mult}(\Gamma)$ and $\text{Mult}^\gamma(\hat{\Gamma})$ are completely determined by the type $\tau$ of $\Gamma$ or $\hat{\Gamma}$, respectively. We thus define the **multiplicity of a tropical disk or ribbon type** $\tau$ as the multiplicity of any of the tropical disks/ribbons of type $\tau$.

4.3. Tropical ribbon counts and the consistent scattering diagram. We continue with the setup of §4.2. For each weight vector $w$, each $\delta > 0$, and each $\theta \in L_{\mathbb{R}}$, let $\mathfrak{T}_{\mathfrak{w},\delta}(\theta)$ denote the set of types of tropical disks of degree $\Delta_w$ which match the constraint $\delta A_w$ and for which $h(V_\infty) = \theta$. For each $\epsilon > 0$ and $\theta \in L_{\mathbb{R}}$, let $B_\epsilon(\theta)$ denote the open radius $\epsilon$ ball centered at $\theta$ (with respect to the Euclidean metric associated to any fixed choice of basis for $L$). Let $\mathfrak{T}_{\mathfrak{w}}(\theta)$ denote the set of types of tropical disks which, for each $\epsilon > 0$, are in $\mathfrak{T}_{\mathfrak{w},\delta}(\theta)$ for some $\theta \in B_\epsilon(\theta)$ and all sufficiently small $\delta > 0$.

Genericness of $A$ ensures that all such tropical disks are trivalent, so we can define their multiplicities as in §4.2.2. Define

$$N(\theta) := \sum_w \frac{1}{|\text{Aut}(w)|} \sum_{\tau \in \mathfrak{T}_w(\theta)} \text{Mult}(\tau) \in \mathfrak{g}.$$

Let $\mathfrak{T}_w^\gamma(\theta)$ denote the set of tropical ribbon types $\hat{\tau}$ such that the associated tropical disk type $\tau$ is in $\mathfrak{T}_w(\theta)$. By Lemma 4.2, we can express $N(\theta)$ as

$$N(\theta) = \sum_w \frac{1}{|\text{Aut}(w)|} \sum_{\hat{\tau} \in \mathfrak{T}_w^\gamma(\theta)} \text{Mult}^\gamma(\hat{\tau}).$$

Note that for each $n \in \Lambda^+$, the strict convexity of $\Lambda^+$ ensures that there are only finitely many $w$ such that $n = n_w$. Furthermore, for each $w$, there are clearly only finitely many types of tropical disks of degree $\Delta_w$. The well-definedness of $N(\theta)$ follows, assuming that we have already fixed $A_w$. The fact that the generic choice of $A_w$ does not matter is part of the following theorem.

---

4 We believe these can be interpreted as types of “virtual tropical disks” as defined in [CPS, §5], so that the classical case of Theorem 4.8 can be viewed as a special case of [CPS, Prop. 5.14].
**Theorem 4.3.** Assume $\mathfrak{g}$ has Abelian walls. Let $\mathcal{D} = \text{Scat}(\mathcal{D}_{\text{in}})$, and consider $\theta \in L_\mathbb{R} \setminus \text{Joints}(\mathcal{D})$. Up to equivalence, we may assume that $\mathcal{D}$ has at most one wall $(\mathfrak{d}, g_\theta) \in \mathcal{D}$ with $\theta \in \mathfrak{d}$. If there is no such wall, then $N(\theta) = 0$, and otherwise,

$$g_\theta = N(\theta).$$

### 4.4. Proof of Theorem 4.3

Theorem 4.3 is a modified version of [Man, Thm 3.7], or a refinement of some cases of [CPS, Prop. 5.14]. The two-dimensional quantum version is [PS15, Cor. 4.9], and the two-dimensional classical version is [GPS10, Thm. 2.8]. The proof is similar in each case. We repeat the setup here, following [Man, §3.2]. This setup will also be used in §4.4.2 to prove the existence statement from Theorem 4.3.

#### 4.4.1. Perturbing the scattering diagram.

**Definition 4.4.** For any scattering diagram $\mathcal{D}$ over a Lie algebra $\mathfrak{g}$ with Abelian walls, the **asymptotic scattering diagram** $\mathcal{D}_{\text{as}}$ of $\mathcal{D}$ is defined by replacing every wall $(n + \mathfrak{d}, g_\theta) \in \mathcal{D}$ with the wall $(\mathfrak{d}, g_\theta)$. Here, $\mathfrak{d}$ denotes a rational polyhedral cone (with apex at the origin) and $n \in N_\mathbb{R}$ translates this cone.

Now let $T$ denote the commutative polynomial ring $\mathbb{Z}[t_1 | i \in I]$, and let $T_k := T/\langle t_1^{k+1} | i \in I \rangle$. Let $\mathcal{D}_{\text{in},T_k}$ and $\mathcal{D}_{\text{in},T}$ be the initial scattering diagrams over $\mathfrak{g} \otimes T_k$ and $\mathfrak{g} \otimes T$, respectively, given by replacing each $g_\theta = \sum_{j \geq 1} g_{ij}$ from $\mathcal{D}_{\text{in}}$ with $g'_\theta = \sum_{j \geq 1} t_j^k g_{ij}$. We will show that Theorem 4.3 holds for $\mathcal{D}_{T_k} := \text{Scat}(\mathcal{D}_{\text{in},T_k})$ for all $k$, hence for $\mathcal{D}_T := \text{Scat}(\mathcal{D}_{\text{in},T})$. Taking $t_i = 1$ for each $i$ then recovers the theorem for $\mathcal{D} = \text{Scat}(\mathcal{D}_{\text{in}})$.

We have an inclusion of commutative rings

$$T_k \hookrightarrow T'_k := \mathbb{Z}[\pi_{ij} | i \in I, 1 \leq j \leq k]/\langle \pi_{ij} | i \in I, 1 \leq j \leq k \rangle$$

$$t_i \mapsto \sum_{j=1}^k \pi_{ij}.$$

Using this inclusion to work in $\mathfrak{g} \otimes T'_k$, we have

$$g'_\theta = \sum_{w=1}^k t_w^w g_{iw} = \sum_{w=1}^k \sum_{\# J = w} w! g_{iw} \pi_{i,J},$$

where the second sum is over all subsets $J \subset \{1, \ldots, k\}$ of size $w$, and

$$\pi_{i,J} := \prod_{j \in J} \pi_{ij}.$$

Consider our scattering diagram $\mathcal{D}_{\text{in}} = \{(\mathfrak{d}_i, g_i) | i \in I\}$ with $\mathfrak{d}_i = e^+_i$ and $g_i = \sum_{w \geq 1} g_{i,w} w$ as in (30). For now we do not assume that $\mathfrak{g}$ has Abelian walls, but recall that we require $[g_{i,w_1}, g_{i,w_2}] = 0$ for each $i \in I$ and each $w_1, w_2 \in \mathbb{Z}_{\geq 1}$. Applying the equivalence from Example 3.4(1) in reverse and then perturbing the walls (i.e., translating the walls by some generic amount), we obtain a scattering diagram

$$\mathcal{D}_k^0 := \{(\mathfrak{d}_{i,J}, w! g_{iw} \pi_{i,J}) | 1 \leq w \leq k, J \subset \{1, \ldots, k\}, \# J = w\},$$

where $\mathfrak{d}_{i,J}$ is some generic translation of $\mathfrak{d}_i = e^+_i$. Note that $\text{Scat}(\mathcal{D}_k^0)_{\text{as}} = \mathcal{D}_{\text{in},T_k}$. 

It will be useful for us to refine this setup a bit, working over a different commutative ring $\tilde{T}_k$ defined by

$$\tilde{T}_k := \mathbb{Z}[u_{iJ} | i \in I, J \subset \{1, \ldots, k\}]/(u_{iJ_1} u_{iJ_2} | J_1 \cap J_2 \neq \emptyset).$$

Note that we have a surjective homomorphism

$$(39) \quad \tilde{\pi} : \tilde{T}_k \rightarrow T'_k, \quad u_{iJ} \mapsto u_{iJ}.$$ 

Let $D^0_k$ denote the initial scattering diagram over $\mathfrak{g} \otimes \tilde{T}_k$ defined as in (38), but with the factors $\tilde{u}_{iJ}$ replaced by $u_{iJ}$, i.e.,

$$(40) \quad D^0_k := \{(d_{iJ}, w^!g_{iw}u_{iJ}) | 1 \leq w \leq k, J \subset \{1, \ldots, k\}, \#J = w\}.$$ 

**Example 4.5.** For $D$ in as in Example 3.11, the corresponding $D^0_k$ for $k = 2$ may look like Figure 4.4.

![Figure 4.4](image)

**Figure 4.4.** $D^0_2$ perturbing the $A_2$ initial scattering diagram of Figure 3.3.

Note that $\tilde{\pi}$ takes the scattering functions of $D^0_k$ to those of $D^0_k$, and so the same will be true for the corresponding consistent scattering diagrams and their asymptotic versions. We will write our walls in the form $(d, g_{d} u_{J_d})$, where $g_{d} \in g_{n_d}$ for some $n_d \in N^+$, $J_d$ is a collection of pairwise-disjoint subsets of $I \times \{1, \ldots, k\}$ of the form $(i, J)$ for various $i \in I$ and $J \subset \{1, \ldots, k\}$, and $u_{J_d} := \prod_{(i,J) \in J_d} u_{iJ}$.

$$(41) \quad u_{J_d} := \prod_{(i,J) \in J_d} u_{iJ}.$$ 

We now inductively produce a scattering diagram $D^\infty_k = \text{Scat}(D^0_k)$ from $D^0_k$ as follows: whenever two walls $(d_1, g_{d_1} u_{J_{d_1}})$ and $(d_2, g_{d_2} u_{J_{d_2}})$ intersect and satisfy $u_{J_{d_1}} u_{J_{d_2}} \neq 0$, we add a new wall $\mathfrak{d}(d_1, d_2)$ defined by

$$(42) \quad \mathfrak{d}(d_1, d_2) := ((d_1 \cap d_2) + \mathbb{R}_{\leq 0} p^*(n_{d_1} + n_{d_2}), [g_{d_1}, g_{d_2}] u_{J_{d_1}} u_{J_{d_2}}).$$ 

This indeed terminates in finitely many steps and produces a consistent scattering diagram $D^\infty_k$, cf. [Man §3.2.2] for details.

**Definition 4.6.** If $\mathfrak{d} = \mathfrak{d}(d_1, d_2)$, define $\text{Parents}(\mathfrak{d}) := \{d_1, d_2\}$, and if $\mathfrak{d} \in D^0_k$, define $\text{Parents}(\mathfrak{d}) := \emptyset$. Recursively define $\text{Ancestors}(\mathfrak{d})$ by $\text{Ancestors}(\mathfrak{d}) := \{\mathfrak{d}\} \cup \bigcup_{\mathfrak{d}' \in \text{Parents}(\mathfrak{d})} \text{Ancestors}(\mathfrak{d}')$. Define $\text{Leaves}(\mathfrak{d}) := \{\mathfrak{d}' \in \text{Ancestors}(\mathfrak{d}) | \text{d}' \text{ is the support of a wall in } D^0_k\}$. 

[Man] assumes $\mathfrak{g}$ has Abelian walls, but this is not used anywhere in the proof of the finiteness or consistency of $D^\infty_k$. 

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[^Man]: [Man] assumes $\mathfrak{g}$ has Abelian walls, but this is not used anywhere in the proof of the finiteness or consistency of $D^\infty_k$. 

4.4.2. Proof of the existence from Theorem 3.5. If \( g \) has Abelian walls, then the scattering diagram \( \mathcal{D} = \text{Scat}(\mathcal{D}_n) \) as in Theorem 3.5 can be constructed as follows: for each \( k \), we take \( \mathcal{D}_k = (\mathcal{D}^\infty_\text{as})_k \), and we then take \( \mathcal{D}_k \) to be the scattering diagram over \( g_k \) obtained by setting each \( t_i \) equal to 1. Taking \( k \to \infty \) then yields \( \mathcal{D} \).

The exact same argument holds without Abelian walls, except that we must use the following generalized definition of the asymptotic scattering diagram: Let \( \mathcal{D} \) be an arbitrary finite scattering diagram. Using Example 3.4(2), refine the supports of the walls of \( \mathcal{D} \) so that if \( (n + \vartheta_1, g_0) \) and \( (n + \vartheta_2, g_0) \) are two walls with \( \text{codim}_{\mathcal{D}^\infty_\text{as}}(\vartheta_1 \cap \vartheta_2) = 1 \), then \( \vartheta_1 = \vartheta_2 \). For \( \vartheta \) a cone with apex at the origin which is obtained as the translation of the support of some wall of \( \mathcal{D} \), let \( S_\vartheta \) be the set of walls of (the refinement of) \( \mathcal{D} \) whose supports are translates of \( \vartheta \). Let

\[
g_\vartheta := \log \left( \prod_{(\vartheta, g) \in S_\vartheta} \exp \gamma \right),
\]

where the order of the product is as follows: Let \( n_0 \) be the primitive element of \( \mathbb{N}^+ \) such that \( g_i \in \mathbb{R}^n_0 \) for each \( (\vartheta_i, g_i) \in S_\vartheta \). Let \( \gamma \) be a smooth path which crosses each wall of \( S_\vartheta \) and which satisfies \( \langle n_0, -\gamma'(t) \rangle > 0 \) for all \( t \). Then the product in (43) is ordered from right to left in agreement with the order in which \( \gamma \) crosses the walls of \( S_\vartheta \).

Now, \( \mathcal{D}_\text{as} := \bigcup_\vartheta (\vartheta, g_\vartheta) \), where the union is over all cones \( \vartheta \) with apex at the origin which are translations of supports of walls of \( \mathcal{D} \), and \( g_\vartheta \) is defined as in (43). By design, if \( \gamma \) is a generic smooth closed path, then the path-ordered product \( \Phi_{\mathcal{G}, \mathcal{D}} \) is equal to the path-ordered product \( \Phi_{\mathcal{C} \gamma, \mathcal{D}} \) for \( C \gg 1 \) a constant re-scaling \( \gamma \). Thus, consistency of \( \mathcal{D}_\text{as} \) follows from that of \( \mathcal{D} \), as desired.

4.4.3. The tropical description of \( \mathcal{D}^\infty_k \). We continue to write \( \mathcal{J} \) to denote a collection of pairwise-disjoint subsets of \( I \times \{1, \ldots, k\} \) of the form \( (i, J) \) for various \( i \in I \) and \( J \subset \{1, \ldots, k\} \). Now, as in \( 4.2.1 \) fix a weight vector \( w := (w_i)_{i \in J}, w_i := (w_{i1}, \ldots, w_{il_i}) \) with \( 0 < w_{i1} \leq \ldots \leq w_{il_i} \). Let \( \mathcal{J}_w \) denote the set of all possible \( \mathcal{J} \) which can be written in the form

\[
\mathcal{J} = \{(i, J_{ij}) : i \in I, j = 1, \ldots, l_i\}
\]

with \( \# J_{ij} = w_{ij} \). Note that each \( \mathcal{J} \in \mathcal{J}_w \) corresponds to a set of walls

\[
\mathcal{D}^0_{k, \mathcal{J}} = \{\vartheta_{ij} \}_{(i, J_{ij}) \in \mathcal{J}} \subset \mathcal{D}^0_k,
\]

and two choices of \( \mathcal{J} \) correspond to the same \( \mathcal{D}^0_{k, \mathcal{J}} \) exactly if they are related by an element of \( \text{Aut}(w) \). Given \( \mathcal{J} \), let \( w_{\mathcal{J}} \) denote the corresponding weight vector \( w \) for which \( \mathcal{J} \in \mathcal{J}_w \).

Let \( \mathcal{D}^\infty_{k, \mathcal{J}} \) denote the set of walls in \( \mathcal{D}^\infty_k \) whose leaves are precisely the walls of \( \mathcal{D}^0_{k, \mathcal{J}} \). Note that, for \( \mathcal{J} \in \mathcal{J}_w \) and \( (\vartheta, g_0 u_{\mathcal{J}}) \in \mathcal{D}^\infty_{k, \mathcal{J}} \), we must have \( g_0 \in \mathbb{R}^n_w \). We will write \( \mathcal{T}_{w, \delta}(\theta, A_{\mathcal{J}}) \) to indicate \( \mathcal{T}_{w, \delta}(\theta) \) with the the representatives of the incidence conditions \( A_w \) chosen so that \( A_{ij} = \vartheta_{ij} \).

**Lemma 4.7.** For every wall \( (\vartheta, g_0 u_{\mathcal{J}}) \in \mathcal{D}^\infty_{k, \mathcal{J}} \) and every \( \theta \) in the interior of \( \vartheta \), there exists a unique tropical disk \( h : \Gamma \to L_\vartheta \) in \( \mathcal{T}_{w, \delta}(\theta, A_{\mathcal{J}}) \) with \( h(V_{\text{out}}) = \theta \). Furthermore, we have

\[
g_\vartheta = \text{Mult}(\Gamma) \prod_{ij} |v_{ij}|.
\]

**Proof.** We construct the tropical disk by starting at \( h(V_\infty) = \theta \in \vartheta \) and following \( \vartheta \) in the direction \( p^*(n_0) \) until we reach a point \( p \in \vartheta_1 \cap \vartheta_2 \), where \( \{\vartheta_1, \vartheta_2\} = \text{Parents}(\vartheta) \). The resulting segment is given weight \( |n_0| \) (the index of \( n_0 \), i.e., \( n_0 \) equals \( |n_0| > 0 \) times a primitive vector). From \( p \), extend the
tropical curve in the directions \( n_{a_1} \) and \( n_{a_2} \) with weights \( |n_{a_1}| \) and \( |n_{a_2}| \), respectively, until reaching the boundaries of the walls \( \mathfrak{d}_1 \) and \( \mathfrak{d}_2 \). The balancing condition at \( p \) follows easily from \((42)\) and the fact that commutators in \( \mathfrak{g} \) respect the \( N^+ \)-grading. The process is repeated for each of these branches, and continues until every branch extends to infinity in some leaf. This gives the desired tropical disk. The formula for \( g_0 \) follows easily from \((42)\) and the definition of \( g_r \), noting that the \( \prod w_i \) factor appears because of the fact that \( g_{iw} \) is multiplied by \( w! \) in the definition of \( \mathfrak{D}_k \) in \((40)\), and similarly for the \( u_j \) factor.

4.4.4. Proof of Theorem 4.3. Given a weight vector \( \mathbf{w} \), let \( |\mathbf{w}_i| := \sum_{j=1}^{l_i} w_{ij} \), and let \( t^\mathbf{w} = \prod_{i,j} t_i^{w_{ij}} \). Also, for \( J = \{(i, J_{ij}) \subset I \times \{1, \ldots, k\}\}_{(i,j) \in S} \), let

\[
\tau_J := \prod_{i,j} \tau_{i,j}.
\]

We will use the following formula, cf. \( \text{Man} \) (45):

\[
(45) \quad t^\mathbf{w} = \sum_{J \in J^\mathbf{w}} \left( \prod_{i,j} \tau_{i,j} \right)^{\prod_{i,j} w_{ij}}.
\]

For a scattering diagram \( \mathcal{D} \) and \( \delta \in \mathbb{R}_{>0} \), let \( \delta \mathcal{D} \) denote the scattering diagram obtained by multiplying the supports of the walls of \( \mathcal{D} \) by \( \delta \) (i.e., multiplying their distances from the origin by 0).

Now fix a point \( \theta \in L_{\mathbb{R}} \setminus \text{Joints}(\mathcal{D}_T) \). Recall that \( \mathcal{D}_T = (\mathfrak{f}(\mathcal{D}^\infty_k))_{\text{as}} \). Hence, if \( \theta \notin \text{supp}(\mathcal{D}_T) \), then for sufficiently small \( \delta > 0 \), no walls of \( \delta \mathcal{D}_k^\infty = \text{Scat}(\delta \mathcal{D}_k^0) \) will intersect a small \( \epsilon \)-neighborhood of \( \theta \). So then by Lemma 4.7, no tropical disks representing a type in any \( \mathcal{T}_{\theta, \mathbf{w}, 1}^{\mathbf{w}, 1}(\theta, A_J) \) will intersect such an \( \epsilon \)-neighborhood either, and so we obtain \( \mathcal{N}(\theta) = 0 \).

Now suppose \( \theta \in \text{supp}(\mathcal{D}_k) \), and for convenience, use Example 3.4.1 to combine all walls containing \( \theta \) into a single wall \( \mathfrak{d} \). Then since \( \mathcal{D}_T = (\mathfrak{f}(\mathcal{D}^\infty_k))_{\text{as}} \), we know that \( g_0 = \sum \tau(g_{\mathbf{w}, 1}) \), where the sum is over all walls \((\mathfrak{d}, g_{\mathbf{w}, 1}) \in \mathcal{D}_k^\infty\) such that for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) for which \( \delta \mathfrak{d} \) intersects \( B_{\epsilon}(\theta) \). By Lemma 4.7, this is the same as

\[
\bar{\tau} \left( \sum_{\mathbf{w}} \left[ \frac{1}{|\text{Aut}(\mathbf{w})|} \sum_{J \in J^\mathbf{w}} \sum_{\tau \in \mathfrak{T}_{\mathbf{w}}(\theta, A_J)} \text{Mult}(\tau) u_J w_{ij}! \right] \right),
\]

where here we write \( \mathfrak{T}_{\mathbf{w}}(\theta, A_J) \) to indicate \( \mathfrak{T}_{\mathbf{w}}(\theta) \) for our particular choice of \( A_{\mathbf{w}} \) as \( A_J \) (since a priori \( \mathfrak{T}_{\mathbf{w}}(\theta) \) might depend on this choice). Here we use our observation that two choices of \( J \) correspond to the same \( \mathcal{D}_{k,J}^\infty \), hence the same \( \mathcal{D}_{\mathbf{w}, J}^\infty \), if and only if they are related by an element of \( \text{Aut}(\mathbf{w}) \).

Now, note that for each \( \mathbf{w} \), \( (\mathcal{D}_k^0)_{\text{as}} \) is symmetric with respect to permuting the elements of \( J^\mathbf{w} \), i.e., for \( J_1, J_2 \in J^\mathbf{w} \), swapping the supports of \( \mathfrak{d}_{J_1} \) and \( \mathfrak{d}_{J_2} \) in \((40)\) does not affect \( (\mathcal{D}_k^0)_{\text{as}} \). Hence, \( \sum_{\tau \in \mathfrak{T}_{\mathbf{w}}(\theta, A_J)} \text{Mult}(\tau) \) is independent of \( J \in J^\mathbf{w} \), and so we obtain

\[
g_0 = \sum_{\mathbf{w}} \frac{1}{|\text{Aut}(\mathbf{w})|} \sum_{\tau \in \mathfrak{T}_{\mathbf{w}}(\theta)} \text{Mult}(\tau) \left( \sum_{J \in J^\mathbf{w}} \tau_J w_{ij}! \right).
\]

Finally, applying \((45)\) yields the desired result. \( \square \)

4.5. The Main Theorem. We cannot apply Theorem 4.3 directly to \( \mathcal{D}_\text{Scat}^{\text{Hall}} \) because \( g^{\text{Hall}} \) is not skew-symmetric. However, the theorem does apply to any of our \( \mathcal{D}_\text{Scat}^i := \mathcal{T}^i(\mathcal{D}_\text{Scat}^{\text{Hall}}) \) for \( i \supseteq \{\text{skew}\} \). Here, we take the associative algebra \( \mathfrak{A}^i \) to be as in Remark 4.1 with \( \mathfrak{A}^i \) meaning \( \mathfrak{A}^{\text{cl}} \) as Remark 4.1 in the case where \( i = \ker(I) \).
Given a weight vector \( w \), define
\[
R_w := \prod_{i,j} R_{w_{ij}},
\]
where we recall from (27) that \( R_k := \frac{(-1)^k}{k(g^* - 1)} \).

Now, let us fix a quiver with potential \((Q, W)\) and consider the corresponding \( \mathcal{D}^\text{Hall}_{\text{Scat}} \). Consider a weight vector \( w \) and a choice of \( \tau \in \mathfrak{X}_w(\theta) \). Recall that the ribbon structure induces an ordering of the unbounded edges of \( \tau \), starting with \( E_{\text{out}} \) and then continuing with \( E_{i_1,j_1}, \ldots, E_{i_l(j_1)j_l(w)}(w) \). Using (36) and (28), we have
\[
\text{Mult}^\mathcal{Y}(\tau) = \nu(\tau) R_w^{w_{i_1(j_1)} \cdots w_{i_l(j_l)(w)}},
\]
By Lemma 2.4, we have
\[
(46) \quad \mathcal{K}_{i_1}^{w_{i_1(j_1)}} \cdots \mathcal{K}_{i_l(j_l)(w)}^{w_{i_l(j_l)(w)}(w)} = \mathfrak{f}\text{lag}(w_{i_1(j_1)}, \ldots, w_{i_l(j_l)(w)}, S_{i_l(j_l)(w)}) =: \mathfrak{f}\text{lag}(\tau),
\]
where we write \( w_{i_k(j_k)} S_{i_k} \) to indicate that the entry \( S_{i_k} \) appears \( w_{i_k(j_k)} \) times, and we neglect writing the data of the map to \( \mathcal{M}_{n_w} \subset \mathcal{M} \). Finally, applying Theorem 4.3 to the image under \( \mathcal{I}^i \), we obtain:

**Theorem 4.8.** Let \( \mathcal{D} = \mathcal{D}^\text{Scat} \) for \( i \supset 1^\text{kew} \), and consider \( \theta \in L_\mathbb{R} \setminus \text{Joints}(\mathcal{D}) \). Up to equivalence, we may assume that \( \mathcal{D} \) has at most one wall \((\mathfrak{o}, g_\mathfrak{o}) \in \mathcal{D} \) with \( \theta \in \mathfrak{o} \). If there is no such wall, then \( N(\theta) = 0 \), and otherwise, \( g_\mathfrak{o} = N(\theta) \), where \( N(\theta) \) is defined as
\[
N(\theta) := \sum_w \left( \frac{1}{|\text{Aut}(w)|} \sum_{\tau \in \mathfrak{X}_w(\theta)} \nu(\tau) \mathcal{I}^i(R_w \mathfrak{f}\text{lag}(\tau)) \right).
\]
Combining Theorem 4.8 with Theorem 3.3 and Proposition 3.12, we immediately obtain the following:

**Theorem 4.9** (Main result). Let \((Q, W)\) be a quiver with gentle potential. Let \( \theta \in M_\mathbb{R} \) be general. Then for any \( i \supset 1^\text{kew} \), we have
\[
\mathcal{I}^i(\log(1_{\text{aut}}(\theta))) = \sum_w \left( \frac{1}{|\text{Aut}(w)|} \sum_{\tau \in \mathfrak{X}_w(\theta)} \nu(\tau) \mathcal{I}^i(R_w \mathfrak{f}\text{lag}(\tau)) \right).
\]

Theorem 1.1 is the special case where \( i = \ker(\mathcal{I}_.) \). The classical limit is the case where \( i = \ker(\mathcal{I}). \)

**Remark 4.10.** While we find the expression of \( \text{Mult}^\mathcal{Y}(\tau) \) in terms of moduli of flags to be interesting, it is of course not generally simple to compute \( \mathfrak{f}\text{lag}(\tau) \). However, it is not difficult to describe the quantum and classical integrals of the terms \( R_w \mathfrak{f}\text{lag}(\tau) \).

First, for the quantum cases, recall from (46) that \( \mathfrak{f}\text{lag}(\tau) \) arose as a product \( \mathcal{K}_{i_1}^{w_{i_1(j_1)}} \cdots \mathcal{K}_{i_l(j_l)(w)}^{w_{i_l(j_l)(w)}(w)} \).

From (2.9), \( \mathcal{I}_i(\mathcal{K}_{i_j}) = t^e_{c_{i_j}} \in \mathbb{C}[N_+^\mathfrak{o}] \). Hence, defining \( R'_k := \frac{(-1)^k}{k(g^* - 1)} \) and \( R'_w := \prod_{i,j} R'_{w_{ij}} \), we have
\[
(48) \quad \mathcal{I}_i(R_w \mathfrak{f}\text{lag}(\tau)) = R'_w \cdot z^{w_{i_1(j_1)} c_{i_1}} \cdots z^{w_{i_l(j_l)(w)} c_{i_l(w)}(w)} \cdot \mathfrak{g}^\mathfrak{cl} \in \mathfrak{g}^\mathfrak{cl}.
\]

Now for the classical version, recall that the map \( \pi_{t-1} : g^\mathfrak{cl} \to g^\mathfrak{cl} \) takes \( z^n \) to \( z^n \). Let us embed \( g^\mathfrak{cl} \) into the Weyl algebra \( \mathfrak{a}^\mathfrak{cl} \) as in Example 4.1, so \( z^n \) becomes \( z^n \partial_B(n,.) \). Let \( R'^1_k := \frac{(-1)^k}{k} \) and \( R'^1_w := \prod_{i,j} R'^1_{w_{ij}} \). We then obtain
\[
\mathcal{I}(R_w \mathfrak{f}\text{lag}(\tau)) = R'^1_w \cdot z^{w_{i_1(j_1)} c_{i_1}} \partial_B(w_{i_1(j_1)} c_{i_1} \cdots) \cdot z^{w_{i_l(j_l)(w)} c_{i_l(w)}(w)} \partial_B(w_{i_l(j_l)(w)} c_{i_l(w)}(w)) \in \mathfrak{a}^\mathfrak{cl}.
\]
We wrote Theorems 4.8 and 4.9 in terms of tropical disk counts because we do not know a nice moduli-theoretic description of the tropical curve multiplicities. However, there are again nice interpretations for the quantum and classical integrals. Consider a tropical curve type \( \tau \in T_w(\theta) \). For each vertex \( V \) of \( \tau \setminus \{ V_\infty \} \), let \( u_1 \) and \( u_2 \) be any two of the weighted tangent vectors of edges emanating from \( V \). Define \( \text{Mult}(V) := |B(u_1, u_2)| \). Then the classical multiplicity \( \text{Mult}(\tau) \) is given by
\[
\text{Mult}(\tau) = R_\text{cl} \prod_V \text{Mult}(V),
\]
and the quantum multiplicity \( \text{Mult}^q(\tau) \), by which we mean \( \text{Mult}(\tau) \) in the quantum cases, is given by
\[
\text{Mult}^q(\tau) = R'_\text{w} \prod_V [\text{Mult}(V)]_t,
\]
where \([\text{Mult}(V)]_t\) is defined as in (15).

**Example 4.11.** Let us continue our ongoing example of the \( A_2 \) quiver \( 1 \leftarrow 2 \) with \( W = 0 \) as in Example 3.10. Recall that in this case, \( L \cong \mathbb{Z}^2 \), \( I = \{1, 2\} \), and \( B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Consider the weight-vector \( w = ((1, 1), (1, 1)) \), so \( \text{Aut}(w) = 2^2 = 4 \). For \( \theta \in \mathbb{R}_{>0}(1, -1) \), a possible tropical disk type \( \tau \in T_w(\theta) \) is illustrated in Figure 4.5:

![Figure 4.5. A tropical disk (given by the dashed segments) representing a type \( \tau \in T_w(\theta) \). The unbounded segments of the tropical curve are in generically specified lines. The dashed circle around \( \theta \) is an \( \epsilon \)-ball \( B_\epsilon(\theta) \). The solid rays are the support of the scattering diagram.](image)

Since there are 3 vertices, there are \( 2^3 \) possible ribbon structures on this tropical disk. One such ribbon structure \( \tilde{\tau} \) is illustrated in Figure 4.6, namely, the ribbon structure for which \( \nu(V) = -1 \) for each \( V \).

The multiplicity \( \text{Mult}^Y(\tilde{\tau}) \) is given by \( T(\nu(\tilde{\tau})R_w\kappa_2\kappa_2\kappa_1\kappa_1) \). We easily see \( \nu(\tilde{\tau}) = (-1)^3 = -1 \), and \( R_w = \left( \frac{1}{1 - 1} \right)^3 \). The space \( \mathfrak{Flag}(\tilde{\tau}) = \kappa_2\kappa_2\kappa_1\kappa_1 \) is the space of composition series of the following form:
\[
(0 \leftarrow 0) \subset (0 \leftarrow \mathbb{C}) \subset (0 \leftarrow \mathbb{C}^2) \subset (\mathbb{C} \leftarrow \mathbb{C}^2) \subset (\mathbb{C}^2 \leftarrow \mathbb{C}^2).
\]
All maps \( V \leftarrow W \) here, where \( V, W \) are \( \mathbb{C} \)-vector spaces, are necessarily the 0 map, and so by Lemma 2.4 we have \( \mathfrak{Flag}(\tilde{\tau}) = \mathfrak{Flag}(S_2, S_2, S_1, S_1; \mathbb{C}^2 \leftarrow \mathbb{C}^2) \kappa_2\kappa_2\kappa_2\kappa_2 \). As in Example 2.5, since we work modulo
automorphisms of $\mathbb{C}^2 \leftarrow \mathbb{C}^2$, there is only one flag, and its stabilizer group is $U_2(\mathbb{C})^2$ (one copy of the unitary group $U_2(\mathbb{C})$ for each $\mathbb{C}^2$. Hence, we find

$$\text{Flag}(\hat{\tau}) = \frac{1}{|U_2(\mathbb{C})|^2} \kappa_{\mathbb{C}^2},$$

So the contribution of this tropical ribbon $\hat{\tau}$ to (47) is

$$\frac{1}{|\text{Aut}(w)|} \nu(\hat{\tau}) I^i(R_{w} \text{Flag}(\hat{\tau})) = \left(\frac{1}{4}\right)^i (1 - 1) \left(\frac{1}{q - 1}\right)^3 \kappa_{\mathbb{C}^2,0}.$$

In the quantum case $i = \ker(I_t)$, we find using Proposition 2.8 that

$$I_t(\kappa_{\mathbb{C}^2,0}) = \Upsilon(\text{pt}) t^{3/(2,2)}(2,2)z^{(2,2)} = t^{4}z^{(2,2)},$$

and so (49) becomes

$$-\frac{z^{(2,2)}}{4(q - 1)^3}.$$

Alternatively, this may be computed using (48).

5. Broken lines and theta functions

5.1. Definitions of broken lines and theta functions. Recall the notation and setup of §3.1.

Fix a scattering diagram $\mathcal{D}$. Suppose we have a commutative ring $R$ and a $\Lambda$-graded $R$-algebra $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ with $A_0 = R$ on which $\mathfrak{g}$ acts via $\Lambda$-graded $R$-algebra derivations. We say that this action is skew-symmetric if $\mathfrak{g}_n \cdot A_\lambda = 0$ whenever $\{n, \lambda\} = 0$. Let $\hat{A}$ denote the $(\Lambda^+)$-adic completion of $A$. Note that $\hat{G}$ acts on $\hat{A}$ via $\Lambda$-graded $R$-algebra automorphisms.

**Definition 5.1.** Let $\lambda \in \Lambda \setminus \{0\}$, $Q \in \Lambda^{\vee}_0 \setminus \text{Supp}(\mathcal{D})$. A broken line $\gamma$ with ends $(\lambda, Q)$ is the data of a continuous map $\gamma : (-\infty, 0] \to \Lambda^{\vee}_0 \setminus \text{Joints}(\mathcal{D})$, values $-\infty =: t_{-1} \leq t_0 \leq t_1 \leq \ldots \leq t_\ell = 0$, and for each $i = 0, \ldots, \ell$, an associated homogeneous element $a_i \in A_{\lambda_i}$ for some $\lambda_i \in \Lambda \setminus \{0\}$, such that:

(i) $\lambda_0 = \lambda$ and $\gamma(0) = Q$. 

(ii) For $i = 0, \ldots, \ell$, $\gamma'(t) = -p^*(\lambda_i)$ for all $t \in (t_{i-1}, t_i)$.

(iii) $a_0 = z^{\lambda}$.

(iv) For $i = 0, \ldots, \ell - 1$, $\gamma(t_i) \in \text{Supp}(\mathcal{D})$. Let

$$g_i := \prod_{(\vartheta, g_2) \in \mathcal{D}} \exp(g_2)^{\text{sgn}(n_2, p^*(\lambda))} \in \hat{G}.$$ 

I.e., $g_i$ is the $\epsilon \to 0$ limit of the wall-crossing automorphism $\Phi_{\gamma|(t_i, t_{i+1})}$ defined in [23] (using a smoothing of $\gamma$). Then $a_{i+1}$ is a homogeneous term of $g_1 \cdot a_i$, other than $a_1$.

The theta function $\vartheta_{\lambda, \mathcal{Q}} \in \hat{A}$ is defined by

$$\vartheta_{\lambda, \mathcal{Q}} := \sum_{\gamma} a_{\gamma},$$

where the sum is over all broken lines $\gamma$ with ends $(\lambda, \mathcal{Q})$, and $a_{\gamma}$ denotes the element of $A$ associated to the last straight segment of $\gamma$.

If $g$ is skew-symmetric and the action on $A$ is skew-symmetric, then [Man] Thm. 2.14 (a refinement of [CPS, Lemmas 4.7, 4.7]) states the following:

**Lemma 5.2 (The [CPS] Lemma).** Suppose $g$ is skew-symmetric with skew-symmetric action on $A$, and suppose $\mathcal{D} = \text{Scat}(\mathcal{D}_{\text{in}})$ as in Theorem 5.3. Let $\gamma$ be a smooth path in $\Lambda_\mathbb{R}^\vee \setminus \text{Joints}(\mathcal{D})$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$, with $\mathcal{Q}_1, \mathcal{Q}_2 \notin \text{Supp}(\mathcal{D})$. Then for any $\lambda \in \Lambda$,

$$\vartheta_{\lambda, \mathcal{Q}_2} = \Phi_{\gamma, \mathcal{D}}(\vartheta_{\lambda, \mathcal{Q}_1}).$$

In any case, we have a copy $\hat{A}_{\mathcal{Q}}$ of $\hat{A}$ and a collection of elements $\{\vartheta_{\lambda, \mathcal{Q}} | \lambda \in \Lambda \} \subset \hat{A}_{\mathcal{Q}}$ associated to every $\mathcal{Q} \in \Lambda_\mathbb{R}^\vee \setminus \text{Supp}(\mathcal{D})$. If $\mathcal{D}$ is consistent, then the identifications of the $\hat{A}_{\mathcal{Q}}$'s with $\hat{A}$ are all compatible with the path-ordered products. Furthermore, if Lemma 5.2 holds, it says that the elements $\vartheta_{\lambda, \mathcal{Q}} \in \hat{A}_{\mathcal{Q}}$ are also compatible with the path-ordered products, thus giving a canonical collection of elements $\vartheta_{\lambda} \in \hat{A}$. We may therefore simply denote $\vartheta_{\lambda, \mathcal{Q}}$ as $\vartheta_{\lambda}$.

5.2. Hall algebra, quantum, and classical broken lines. Take $\Lambda = N^{\text{prin}} := N \oplus M$, and take $\Lambda^\oplus := (N^\oplus, 0)$. Denote $M^{\text{prin}} := (N^{\text{prin}})^\vee = M \oplus N$. Take $\{\cdot, \cdot\}$ to be the $\mathbb{Z}$-valued skew-symmetric form $B^{\text{prin}}$ on $N^{\text{prin}}$ defined via

$$B^{\text{prin}}((n_1, m_1), (n_2, m_2)) = B(n_1, n_2) + \langle n_1, m_2 \rangle - \langle n_2, m_1 \rangle.$$ 

We will write

$$\pi^* : N \to M$$

$$n \mapsto B(n, \cdot),$$

while the map $p^*$ of [21] will be denoted by

$$p^{\text{prin}}^* : N^{\text{prin}} \to M^{\text{prin}}$$

$$(n, m) \mapsto B^{\text{prin}}((n, m), \cdot) = (\pi^*(n) - m, n).$$

One can show that $B^{\text{prin}}$ is unimodular, so the map $p^{\text{prin}}^*$ is an isomorphism.

Now for our $\Lambda$-graded algebras, we take the following:

$$A^{\text{Hall, prin}} := H_{\text{reg}}(Q, W) \otimes_{\mathbb{C}_{\text{reg}}(t)} \mathbb{C}_{\text{reg}}(t)[M],$$
\[ A^q,\text{prin} := C_t[N^\oplus] \otimes_{C_{\text{reg}}(t)} C_{\text{reg}}(t)[M], \]

and

\[ A^{\text{cl},\text{prin}} := C[N^\oplus] \otimes_C C[M]. \]

The algebra structure on \( A^{\text{Hall},\text{prin}} \) is determined by specifying that for \( a_d \in H_{\text{reg}}(Q, W)_d \) and \( m \in M \), we have

\[ a_d \ast z^m = q^{(d,m)}z^m \ast a_d. \]

Similarly, for \( A^q,\text{prin} \) we specify that

\[ z^d \ast z^m = q^{(d,m)}z^m \ast z^d. \]

Equivalently, \( A^q,\text{prin} \) is the quantum torus algebra \( C_t[A] \) with respect to the form \( B^{\text{prin}} \). Finally, \( A^{\text{cl},\text{prin}} \) is just given the usual algebra structure, making it into \( C[A] \).

For the action of \( g^\text{Hall} \) on \( A^{\text{Hall},\text{prin}} \) we take the right-hand adjoint action, i.e.,

\[ g \cdot a := [a, g] = ag - ga, \]

using the natural inclusion of \( g^\text{Hall} \) into \( (t - t^{-1})^{-1} \cdot A^{\text{Hall},\text{prin}} \) to make sense of the multiplication. Similarly for the action of \( g^q \) on \( A^q,\text{prin} \). The action of \( g^\text{cl} \) is the right-adjoint action with respect to the Poisson bracket, i.e.,

\[ z^d \cdot z^{(n,m)} := \{ z^{(n,m)}, z^{(d,0)} \} = B^{\text{prin}}((n, m), (d, 0))z^{(n + d, m)}. \]

One sees that the maps \( \mathcal{I}_t, \mathcal{I}, \) and \( \pi_{t \mapsto t+1} \) extend to homomorphisms between these algebras which commute with the corresponding Lie algebra actions.

We note that we could also define \( A^i,\text{prin} \) for any other \( i \supseteq \text{skew} \) by applying \( \mathcal{I}^i \) to \( A^{\text{Hall},\text{prin}} \). The induced \( g^i \)-actions on \( A^i,\text{prin} \) are skew-symmetric, thus yielding new examples of algebras for which Lemma 5.2 holds.

Let \( \Box \) represent Hall, \( i, q, \) or \( \text{cl} \). We can consider scattering diagrams in \( M^\text{prin}_R \) over \( \mathfrak{g}^\Box \). We take \( \mathfrak{D}^\Box_{\text{Scat}} := \text{Scat}(\mathfrak{D}^\Box_{\text{in}}, \text{prin}) \), where \( \mathfrak{D}^\Box_{\text{in},\text{prin}} \) is defined as in [24], but with \( (e_i, 0) \perp \) in place of \( e_i^\perp \), and with \( \log 1_{\mathfrak{m}}(p^*(e_i)) \) replaced with its image under \( \mathcal{I}^i, \mathcal{I}, \text{or} \mathcal{I} \) if \( \Box \) represents \( i, q \) or \( \text{cl} \), respectively. Note that the intersection of \( \mathfrak{D}^\Box \) with \( (M_R, 0) \subset M^\text{prin}_R \) agrees with what we previously called \( \text{Scat}(\mathfrak{D}^\Box_{\text{in}}) \).

Remark 5.3. Note that all scattering walls have supports of the form \( (n, 0) \perp \) for \( n \in N \), so they are invariant under translation by \( (0, N_R) \). It follows that \( \vartheta_{\lambda, Q} \) is invariant under translation of \( Q \) by elements of \( (0, N_R) \), and when enumerating broken lines, it suffices to consider their projections modulo \( (0, N_R) \).

Note that \( g^\Box \) and the action on \( A^\Box,\text{prin} \) are skew-symmetric if \( \Box = i, q \) or \( \text{cl} \), but typically not for \( \Box = \text{Hall} \). With this setup and for \( \Box = \text{Hall} \), the broken lines with ends \( (\lambda, Q) \) with \( Q \in (M \oplus N)_R \) and \( \lambda \in \Lambda \) are precisely the Hall algebra broken lines discussed in [Che16]. These will be examined in 5.3.

We now briefly explain how the above theta functions for \( \Box = \text{cl} \) relate to those of [GHKKJ14]. In the usual cluster algebras language, \( z^m \) for \( m \in M \) gives the \( A \) cluster variables, while \( z^n \) for \( n \in N \) gives the \( X \) cluster variables.\(^\text{6}\) In the principle coefficients setting, we have \( z^{(m,n)} = \prod_i A_i^{m_i} X_i^{n_i} \).

\(^\text{6}\)The \( A \) and \( X \) notation is due to Fock and Goncharov, cf. [FG09], and was also used by [GHKKJ14]. Some other authors use the Fomin-Zelevinsky [FZ07] convention of denoting the \( A \)-variables by \( X \) and the \( X \)-variables by \( Y \).
where \( m = \sum_i m_i e_i^* \) and \( n = \sum_i n_i e_i \). The theta functions on \( A_{\text{prin}} \), \( A \), and \( \lambda^\prime \) are obtained as follows:

- Allowing any \( \lambda \in \Lambda \), the resulting theta functions \( \vartheta_\lambda^\text{prin} := \vartheta_\lambda \) are the theta functions which \([\text{GHKK14}]\) constructs on the cluster variety with principal coefficients \( A_{\text{prin}} \) (or rather, on some formal version of this in general). The theta functions \( \vartheta_\lambda^\text{prin} \) for \( \lambda \in (N^\oplus)\gamma \) (i.e., the positive span of the vectors \( e_i^* \)) are the ones examined by Bridgeland \([\text{Bri17}]\) Thm. 1.4.
- One obtains the theta function \( \vartheta_\lambda^A \) on the cluster \( A \)-variety via the projection \((n, m) \mapsto m \) of \( \vartheta_\lambda^\text{prin} \) (i.e. setting all the \( X \)-variables in \( \vartheta_\lambda \) equal to 1), assuming that this projection is well-defined, i.e., that it converges. The middle cluster algebra of \([\text{GHKK14}]\) is defined to be the span of all the \( \vartheta_\lambda^A \) for which the convergence holds. The corresponding elements \( \lambda \) form a cone \( \Xi \subset M \) which contains the Fock-Goncharov \([\text{FG09}]\) cluster complex \( C \). The \( \vartheta_\lambda^A \) for \( \lambda \in C \) give the cluster monomials.
- By applying a change of variables \( z^{(m,n)} \mapsto z^n \) to \( \{ \vartheta_{(n,m)|m=\pi^\gamma(n)} \} \), one obtains \([\text{GHKK14}]\)’s theta functions \( \vartheta_\lambda^A \) for the \( \lambda \)-space (a formal version thereof). Note that \( m = \pi^\gamma(n) \) implies \( p^\text{prin}(n, m) = (0, n) \), so this change of variables essentially amounts to applying \( p^\text{prin} \).

The theta functions for \( \square = q \) are among those considered in \([\text{Man}]\) and are expected to give bases for the quantum cluster varieties (or formal versions thereof). These will be further explored in upcoming work of the second author with Ben Davison.

### 5.3. Hall algebra theta functions and the CPS lemma

As noted in \([5.2]\), \( g_{\text{Hall}} \) and its action on \( A_{\text{Hall}, \text{prin}} \) typically fail to be skew-symmetric, and so \([\text{Man}]\)’s proof of Lemma 5.2 does not apply to Hall algebra broken lines. In fact, we provide here a counterexample, thus showing that:

**Proposition 5.4.** The analog of Lemma 5.2 does not generally hold for theta functions constructed from Hall algebra broken lines.

We note though that Hall algebra broken lines are still useful for understanding theta functions. For example, by studying Hall algebra broken lines and then integrating, we can understand the quantum or classical broken lines in terms of quiver Grassmannians, cf. \([\text{Che16}]\).

Recall from Remark 5.3 that we may compute theta functions using the images of broken lines under the projection \( M^\text{prin}_R = M_R \oplus N_R \to M_R \). **We will work in this projection throughout this subsection.** Furthermore, the theta function we will consider will be of the form \( \vartheta_{(0,m),Q} \) with \( m \in \pi^\gamma(N) \). Suppose \( a_i \in A_{\lambda_i} \) is the homogeneous element element attached to some straight segment of a broken line contributing to \( \vartheta_{(0,m),Q} \). Then \( \lambda_i \) has the form \( (n_i, m) \) for \( n_i \in N \). Using (52), we see that the projection of \( p^\text{prin}(\lambda_i) \) modulo \( (0, N) \) is \( \pi^\gamma(n_i) - m \in \pi^\gamma(N) \). Hence, by Definition 5.1(ii) (modulo \( (0, N) \)), we have

\[
\gamma'(t) = m - \pi^\gamma(n_i) \in \pi^\gamma(N)
\]

for all \( t \) in the corresponding straight segment of \( \gamma \). We thus obtain the following:

**Lemma 5.5.** For \( m \in \pi^\gamma(N) \), all broken lines contributing to \( \vartheta_{(0,m),Q} \) are contained in \( Q + \pi^\gamma(N_R) \).
For our counterexample, we use the $A_3$-quiver $1 \leftarrow 2 \rightarrow 3$. In the corresponding (standard) basis $e_1, e_2, e_3$ for $N$, the matrix for $B$ is

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

In general, the map $\pi^*: N \to M$ takes $e_i$ to the $i$-th row of $B$. In particular, we see that $\ker(\pi^*)$ is in this case generated by $e_1 - e_3$, and $\text{Image}(\pi^*) = (e_1 - e_3)^\perp$, or the span of the first two rows of $B$ (viewed as vectors in the dual basis).

The walls of the initial scattering diagram $\mathcal{D}_{\text{Hall}}$ (i.e., $\mathcal{D}_{\text{Hall, prin}}$ projected to $M_R$) are $\mathcal{d}_i := (e_i^+, 1_{ss}(\pi^*(e_i)))$. Figure 5.7 depicts a slice of the resulting consistent scattering diagram $\mathcal{D}_{\text{Hall}}$ (which agrees with $\mathcal{D}_{\text{Hall}}$ by Proposition 3.12, parallel to $\pi^*(N_R)$ (taking advantage of Lemma 5.5), with the upward-pointing direction in the figure being parallel to $\pi^*(e_1) = \pi^*(e_3) = (0, 1, 0)$, and the leftward-pointing direction being parallel to $\pi^*(e_2) = (-1, 0, -1)$. In the figure, whenever two walls with attached scattering functions $g_{i_1}, g_{i_2}$ collide with the corresponding $v_{i_1}, v_{i_2}$ satisfying $\{v_{i_1}, v_{i_2}\} > 0$ (otherwise reorder), the picture is locally the same, up to a change of variables, as the picture in Example 3.2. Hence, consistency results in one new wall with attached scattering function given up to first order by $[g_{i_1}, g_{i_2}]$. Recall from (25) that $\text{log}(1_{ss}(\pi^*(e_i))) = (q-1)^{-1}\kappa_i + (\text{higher order terms})$. So in particular, the element of $\mathcal{G}_{\text{Hall}}$ attached to $\mathcal{d}_{12}$ is $(q-1)^{-2}[\kappa_{1}, \kappa_2] + (\text{higher order terms})$.

![Figure 5.7](counterexample.png)

**Figure 5.7.** Counterexample to the [CPS] Lemma for Hall algebra broken lines.

The bold lines in Figure 5.7 represent broken lines with ends $(m, Q_1)$ and $(m, Q_2)$, where $m = \pi^*(e_1) + \pi^*(e_2)$, i.e., $m = (-1, 1, -1)$ in the basis $e_1^*, e_2^*, e_3^*$ for $M$. Here we keep in mind that by (54), $\gamma'(t) = m$ on the first straight segment of $\gamma$.

By inspection, these are the only broken lines contributing to $\vartheta_{m, Q_1}$ or $\vartheta_{m, Q_2}$ whose attached element of $\hat{A}$ is in the $A_{m+e_1+e_2+e_3}$ (the subscript denoting the degree in the $\mathcal{N}^{\text{prin}}$-grading). So for our counterexample, it suffices to check that these two attached elements are different from one another.

Recall from (5.2) that when a broken line with attached element $a$ crosses a wall with attached element $g \in \mathcal{G}_{\text{Hall}}$, the result of the action of $g$ on $a$ is $\exp[a, g]$. In particular, if $g = \sum_{k \geq 1} g_k$ with $g_k \in \mathcal{G}_{\text{Hall}}$ for some $n \in \mathbb{N}^+$, then the action yields $[a, g] + (\text{higher order terms})$. Note that for each of the two broken lines in Figure 5.7, it is only the first-order terms of the scattering functions that contribute. Also note that all the signs in the exponents as in (50) are positive for the two broken...
Similarly, the final attached element for the broken line $\gamma_1$ ending at $Q_1$ is
\[ a_{\gamma_1} = (q-1)^{-3}[[[z^m, \kappa_3], \kappa_1], \kappa_2], \]
Similarly, the final attached element for the broken line $\gamma_2$ ending at $Q_2$ is
\[ a_{\gamma_2} = (q-1)^{-3}[[z^m, \kappa_3], [\kappa_1, \kappa_2]] \]
\[ = (q-1)^{-3}[[z^m, \kappa_3], [\kappa_1, \kappa_2]] + (q-1)^{-3}[\kappa_1, [[z^m, \kappa_3], \kappa_2]], \]
where in the last step we applied the Jacobi identity. So the difference between $a_{\gamma_2}$ and $a_{\gamma_1}$ is
\[ a_{\gamma_2} - a_{\gamma_1} = (q-1)^{-3}[\kappa_1, [[z^m, \kappa_3], \kappa_2]]. \]

After applying $T_i$ for $i \supset i^{skew}$, the skew-symmetry of the brackets implies that $[[[z^m, \kappa_3], \kappa_2]$ will vanish, and so this difference is indeed 0 as implied by Lemma 5.2. But in $A^{Hall, prin}$ this is not the case, as we will now check. Using (55), we compute that
\[ [z^m, \kappa_3] = (1-q^{-1})z^m \kappa_3, \]
and then
\[ [(1-q^{-1})z^m \kappa_3, \kappa_2] = (1-q^{-1})z^m (\kappa_3 \kappa_2 - q \kappa_2 \kappa_3). \]
Let $\kappa_{23,f}$ denote the $\kappa$-element of the Hall algebra corresponding to the representation $(0 \leftarrow \mathbb{C} \xrightarrow{f} \mathbb{C})$. Up to isomorphism, $f$ here is either 0 or Id. Then using Lemma 2.4 we compute
\[ \kappa_3 \kappa_2 = \kappa_{23,0} + (q-1)\kappa_{23,1d} \]
and
\[ \kappa_2 \kappa_3 = \kappa_{23,0}. \]

Now the right-hand side of (55) becomes:
\[ (1-q^{-1})(1-q)z^m (\kappa_{23,0} - \kappa_{23,1d}) \]
As a check, note that $I_i(\kappa_{23,0} - \kappa_{23,1d}) = 0$, so we do not violate Lemma 5.2 after integrating.

Finally, we must check that that
\[ [\kappa_1, z^m (\kappa_{23,0} - \kappa_{23,1d})] \]
is nonzero. Moving $\kappa_1$ past $z^m$ yields
\[ [\kappa_1, z^m (\kappa_{23,0} - \kappa_{23,1d})] = z^m \left(q^{-1}\kappa_1 (\kappa_{23,0} - \kappa_{23,1d}) - (\kappa_{23,0} - \kappa_{23,1d}) \kappa_1\right). \]
Let $\kappa_{123,a,b}$ denote the $\kappa$-element corresponding to the representation $\mathbb{C} \xrightarrow{a} \mathbb{C} \xrightarrow{b} \mathbb{C}$. Rather than completely computing $q^{-1}\kappa_1 (\kappa_{23,0} - \kappa_{23,1d}) - (\kappa_{23,0} - \kappa_{23,1d}) \kappa_1$, let us just look at the coefficient of $\kappa_{123,0,0}$ in the product. The product $q^{-1}\kappa_1 \kappa_{23,0}$ yields a contribution of $q^{-1}\kappa_{123,0,0}$, while the product $-\kappa_{23,0} \kappa_1$ yields a contribution of $-\kappa_{123,0,0}$. Thus, (56) includes a term of the form
\[ z^m (q^{-1} - 1)\kappa_{123,0,0} \]
which will not cancel with any other terms. In particular, (56) is nonzero, as desired. This proves Proposition 5.4. \qed
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