AN OPTIMAL ALGORITHM FOR CONFLICT-FREE COLORING FOR TREE OF RINGS

Einollah Pira
The Business Training Center of Tabriz, Iran
pira_ep2006@yahoo.com

ABSTRACT
An optimal algorithm is presented about Conflict-Free Coloring for connected subgraphs of tree of rings. Suppose the number of the rings in the tree is $|T|$ and the maximum length of rings is $|R|$. A presented algorithm in [1] for a Tree of rings used $O(\log|T|\log|R|)$ colors but this algorithm uses $O(\log|T|+\log|R|)$ colors. The coloring earned by this algorithm has the unique-min property, that is, the unique color is also minimum.

KEYWORDS
Conflict-Free Coloring, Tree, Tree of Rings

1. INTRODUCTION
A vertex coloring of graph $G=(V,E)$ is an assignment of colors to the vertices such that two adjacent vertices are assigned different colors. A hypergraph $H=(V,E)$ is a generalization of a graph for which hyperedges can be arbitrary-sized non-empty subsets of $V$. A vertex coloring $C$ of hypergraph $H$ is called conflict-free if in every hyperedge there is a vertex whose color is unique among all other colors in the hyperedge. Suppose the hypergraph $H=(V,D)$ of a graph $G=(V,E)$ be defined as follows: The set of vertices $V$ of $H$ is the same as that of $G$ and the set of hyperedges $D$ consists of all possible subsets of $V$ that induce connected subgraphs of $G$. Another possible generalization [5] is the following one:

Definition 1. A vertex coloring of a hypergraph $H=(V,D)$ is called conflict-free if in every hyperedge $e$ there exists at least one vertex which has a unique color among all other colors used for vertices in that hyperedge.

A vertex coloring of a hypergraph such that the minimum (maximum) color of any vertex of a hyperedge is unique (assigned to only one vertex in this hyperedge) is conflict-free and is called unique-min (resp. unique-max) (conflict-free) coloring. The problems of computing a unique-min coloring is equivalent to computing a unique-max coloring since we can replace every color $i$ by $c_{max} - i + 1$, where $c_{max}$ is the maximum color among all vertices [1].

In this paper, first I study unique-min (conflict-free) coloring in chain, ring and tree, second, present a new algorithm for a tree of rings.

Conflict-free coloring have various applications. For Example in [2] consider the following scenario: vertices represent base stations of a cellular network interconnected through a backbone. Mobile client connect to the network by radio links and the reception range of each agent is a connected subgraph of the base stations graph. Then it may be desirable that in each agent's range there is a base station transmitting in a unique frequency, in order to avoid interference. The problem of minimizing the number of necessary frequencies is equivalent to Connected Subgraphs Conflict-Free Coloring.
Related work. The study of conflict-free coloring was initiated in [2] as a geometric problem with applications to cellular networks. Some of the problems proposed in that paper can be defined as hypergraph conflict-free coloring problems. The algorithm that uses $O(\log^2 n)$ colors (where $n$ is the number of vertices) is given in [1] about CF-coloring for trees and trees of rings. Some of the problems presented in [2] can be defined as hypergraph conflict-free coloring problems. In [3,4] the conflict-free coloring was studied for grids. In [6] the conflict-free coloring of $n$ points with respect to (closed) disks were studied and were proved a lower bound of $\Omega(\log n)$ colors. In [7] the conflict-free coloring of $n$ points with respect to axis-parallel rectangles were studied. Various other conflict-free coloring problems have been considered in very recent papers [8,12,13,14,15,16,17,18].

The problem becomes more interesting when the vertices are given online by an adversary. For example, at every given time step $i$, a new vertex $v_i$ is given and the algorithm must assign $v_i$ a color such that the coloring is a conflict-free coloring of the hypergraph that is induced by the vertices $V = \{v_1, v_2, ..., v_i\}$. Once $v_i$ is assigned a color, that color cannot be changed in the future. This is an online setting, so the algorithm has no knowledge of how vertices will be given in the future. In [5] there is the online version of conflict-free coloring of a hypergraph.

The online version of Connected Subgraphs Conflict-Free Coloring in chains was presented in [8]. Also, in the case of intervals, there are several algorithms [11]. Their randomized algorithm uses $O(\log^2 n \log \log n)$ colors with high probability. Their deterministic algorithm uses $O(\log n)$ colors in the worst case. Recently, randomized algorithms that use $O(\log n)$ colors have been found in [9,10].

2. PRELIMINARIES

The topologies i study during this paper are chain, ring, tree and tree of rings. A graph is a ring when all its vertices $V$ are connected in such a way that they form a cycle of length $|V|$. A tree of rings can be defined recursively in the following manner [18]: it is either a single ring or a ring $R$ attached to a tree of rings $T$ by identifying exactly one vertex of $R$ to one vertex of $T$. An Example of a tree of rings is displayed in Figure 1.

Algorithm for unique-minimum conflict-free coloring in a chain: in [2] there exists an algorithm that uses $\left\lceil \log n \right\rceil+1$ colors for chains. The algorithm for a chain $\{1,2,\ldots,n\}$ as follows:

step 1: Color vertex $\left\lceil \frac{n}{2}\right\rceil$ with color 1

step 2: Color vertices $\left\lceil \frac{n}{2^2}\right\rceil$, $\left\lceil \frac{n}{2^2} + \frac{n}{2}\right\rceil$ with color 2

step 3: Color vertices $\left\lceil \frac{n}{2^3}\right\rceil$, $\left\lceil \frac{n}{2^3} + \frac{n}{2^2}\right\rceil$, $\left\lceil \frac{n}{2^3} + \frac{n}{2^3}\right\rceil$, $\left\lceil \frac{n}{2^3} + \frac{n}{2^2} + \frac{n}{2}\right\rceil$ with color 3

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step i: Color vertices $\left\lceil \frac{n}{2^i}\right\rceil$, $\ldots$, $\left\lceil \frac{n}{2^i} + \frac{n}{2^i} + \frac{n}{2^i} + \ldots + \frac{n}{2^i}\right\rceil$ with color i

Color i is used only if $\left\lceil \frac{n}{2^i}\right\rceil = 1$, so in fact $\left\lceil \log n \right\rceil+1$ colors are used by the algorithm.

For example, if $n=8$, the coloring is 32313234. It is clearly to see that the coloring is unique-minimum conflict-free coloring.
The above algorithm with a small change can be used to solve the unique-minimum conflict-free coloring in a ring. Pick an arbitrary vertex v and color it with a color 1 (not to be reused anywhere else in the coloring). The remaining vertices form a chain that color with the algorithm described above. This algorithm colors a ring of n vertices with \( \lceil \log (n-1) \rceil + 2 \) colors. For example, if n=8, the coloring is 14342434, where `1' is the first unique color used for v. It is not difficult to see that the coloring is conflict-free: All paths that include v are conflict-free colored, and the remaining graph G–v is a chain of n–1 vertices, so paths of G–v are also conflict-free colored.

![Diagram of rings and tree representation]

Figure 1. A tree of rings G and the corresponding tree representation T(G).

An important notion for my algorithm is \( \alpha \)-separator.

**Definition 2.** An \( \alpha \)-separator (\( \alpha < 1 \)) of a graph \( G=(V,E) \) is a vertex u the removal of which partitions G to connected components of size at most \( \alpha |V| \).

It is obvious from the above definition that on a general graph an \( \alpha \)-separator does not always exist. It is a folklore result that in trees a (1/2)-separator always exists; moreover it can be found in polynomial time [19]. In my algorithm i will often make use of (1/2)-separators.

Algorithm for unique-minimum conflict-free coloring in a tree: in [1] there exists an algorithm that uses \( \lceil \log n \rceil \) colors for trees. The algorithm for a tree is displayed in Figure 2.
Algorithm 1: Unique-Min Coloring for a Tree

**Input:** a tree T  
**Output:** a coloring of vertices of T.

1: Set $T_1 := T$, $i := 1$.
2: while $T_i \neq \emptyset$ do
3:    Find $(1/2)$-separators on all connected components of forest $T_i$.
4:    Add these separators to set $V_i$.
5:    Color vertices in $V_i$ with color $i$.
6:    Construct forest $T_{i+1}$ by removing vertices $V_i$ from $T_i$.
7:    Set $i := i + 1$.
8: end while

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3. AN ALGORITHM FOR TREES OF RINGS

In order to present my algorithm for a tree of rings, I will use the notion of tree representation of a tree of rings. Assume a tree of rings $G$ is $R_1, R_2, \ldots, R_p$. Let me first describe how to construct such a representation $T(G)$ of a tree of rings $G$: Connect all vertices together that lied in intersection of rings. An Example of a tree of rings and its tree representation is displayed in Figure 1. The algorithm for a tree of rings is displayed in Figure 3.

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Algorithm 2: Unique-Min Coloring for a Tree of Rings

**Input:** a tree of rings $G$ by names $R_1, R_2, \ldots, R_p$  
**Output:** a coloring of vertices of $G$

1: Construct the tree representation $T(G)$ of the tree of rings $G$.
2: Color the tree $T(G)$ with algorithm 1.
3: for $i := 1$ to $|T| - 1$ do
   Color the vertex in intersection (if exists and before didn’t colour) of the rings $R_i, R_{i+1}$ by color vertex $v_{i,(i+1)}$ in tree $T(G)$.
end for
4: for $i := 1$ to $|T|$ do
5:    set $cm := \text{a max color of the colored vertices of ring } R_i$.
6:    Delete the colored vertices of ring $R_i$ and connect the neighbors of them.
7:    Let $R'_i$ denote the resulting cycle.
8:    Color cycle $R'_i$ with said algorithm in section 2 by using colors from $\{cm + 1, \ldots, cm + \lceil \log |R'_i| \rceil + 2 \}$.
9: end for

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3.1. Analysis of the algorithm

**Lemma 1.** The coloring obtained by Algorithm 2 is a connected-subgraphs unique-min conflict-free coloring.
Proof. Assume that C is a path in G. There are two cases for C. Case 1: C is part of a ring or a ring itself. If C does not contain the common vertices of the rings, C will be colored in a unique-min way because C colored in line 8 from algorithm 2. If C contains the common vertices of the rings, C will be colored in a unique-min way because the coloring of it start from the max of the colors of the common vertices of the rings (see lines 5,8 from algorithm 2). Case 2: C lies on a connected subset of rings, say $R_1, \ldots, R_j$; the corresponding vertices of these rings in $T(G)$, say $v_{i(j+1)}, v_{j-1}$, Since these vertices of $T(G)$ in line 2 from algorithm 2 are colored in a unique-min way, and each ring $R_k$ in C lies between vertices $v_{(k-1)k}, v_{k(k+1)}$ that colored in line 8 from algorithm 2, therefore C has been colored in a unique-min way.

Lemma 2. The Algorithm 2 uses $O(\log|T| + \log|R|)$ colors.

Proof. The number of colors for coloring $T(G)$ equal $\log|T|$. For coloring the rings, in line 5 from algorithm 2, the maximum of cm’s is $\log|T|$, therefore the maximum color is used in line 8 are $\log|T| + 2 + \log|R|$. Thus the Algorithm 2 uses $O(\log|T| + \log|R|)$ colors.

4. Conclusions

I have presented an optimal algorithm for coloring a tree of rings such that each connected subgraph has a vertex with a unique minimum color. Also I have proved this algorithm uses $O(\log|T| + \log|R|)$ colors.

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**Biography**

- M.S. in Computer Engineering, Sharif University of Technology, Tehran, Iran [2000-2002]
- B.S. in Computer Engineering, Tarbiat Moallem University of Technology, Tehran, Iran [1996-2000]
- Diploma in Math. and Physics, Razi High School, ajabshir, Iran [1992-1996]