NAVIER-STOKES AND STOCHASTIC NAVIER-STOKES EQUATIONS VIA LAGRANGE MULTIPLIERS

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Abstract. We show that the Navier-Stokes as well as a random perturbation of this equation can be derived from a stochastic variational principle where the pressure is introduced as a Lagrange multiplier. Moreover we describe how to obtain corresponding constants of the motion.

1. Introduction. Navier-Stokes equation describes the velocity of incompressible viscous fluids. We consider this equation with periodic boundary conditions; namely, if \( v = v(t, x) \), \( x \in \mathbb{T} \), denotes this velocity at time \( t \), with \( \mathbb{T} \) being the \( d \)-dimensional flat torus that we identify with \([0, 2\pi]^d\), it reads

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = \nu \Delta v - \nabla p, \quad \text{div } (v) = 0,
\]

where \( \nu \) is a positive constant (the viscosity coefficient) and \( t \in [0, T] \). The function \( p = p(t, x) \) denotes the pressure and is also an unknown in the equation.

Lagrange’s point of view consists in describing positions of particles: it concerns the flows driven by the velocity fields. Lagrangian trajectories for the Euler equation (the case where there is no viscosity term) have been identified as minimisers of the kinetic energy defined on the space of diffeomorphisms by V. Arnold in [3]. In other words they are geodesics for a \( L^2 \) metric on such space of curves. This geometric approach to the Euler equation was developed in the fundamental paper by D. Ebin and J. Marsden ([9]) and gave rise to many subsequent works. It is well known that the pressure in an incompressible fluid acts like a Lagrange multiplier and one can, indeed, derive the Euler equation from a variational principle with such a multiplier (cf. for example [6, 11, 16]).

Navier-Stokes equation, being a dissipative physical system, does not correspond to analogous deterministic variational principles. Nevertheless, by replacing the Lagrangian flows by stochastic ones, we may still derive this equation from a (stochastic) variational principle associated with the energy. Then the velocity field is identified with the drift of the Lagrangian diffusion process, which is a time derivative after conditional expectation of the paths. Inspired by [15] and [19], such a stochastic variational principle was proved in [5] (cf. also [10]). More recently it was generalised in the context of Lie groups in [1] and many other dissipative systems can be derived using the same kind of ideas (cf. [4, 8]). Moreover stochastic partial differential equations were also obtained by variational principles,

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corresponding to random perturbations of the action functionals, in [4]. We refer to [12] and other subsequent works from the same author, where a different variational approach to stochastic fluid dynamics is developed (in order to obtain stochastic partial differential equations).

In this paper we show that it is possible to derive the Navier-Stokes equation from a (stochastic) variational principle with a Lagrange multiplier expressed in terms of the pressure. Although we consider here a flat case, the principle can be extended to general manifolds following the construction in [2]. For the general theory of stochastic differential equations on manifolds we refer for example to [13].

Stochastic Noether’s theorem was introduced in [17], [18] in the context of stochastic processes associated with the heat equation. A conserved quantity corresponds there to a martingale. In the spirit of this theorem as well as of [7], we present a result about conserved quantities associated to our stochastic variational principle. The main difference with the one of [7] is that we consider here the Lagrangian motion as a stochastic flow (with respect to its initial values $x$) and in the notion of symmetry we integrate with respect to the variable $x$.

It should be stressed that in our derivation of the Navier-Stokes equation no random perturbation is added. What we advocate is an approach where the presence of the Laplacian in Navier-Stokes equation is interpreted as the underlying presence of diffusion processes, used afterwards for studying (1) in probabilistic terms. In the last section we show how to derive a variational approach to a randomly perturbed Navier-Stokes equation ((19)).

2. A stochastic variational principle. On a fixed standard probability space $(\Omega, \mathbb{P}, \mathbb{P})$ endowed with an increasing filtration $\mathbb{F}_t$ that satisfies the standard assumptions, we consider $\xi$ to be a semimartingale with values in $T$, namely

$$d\xi_t(x) = dM_t(x) + D_t\xi_t(x)dt, \quad \xi_0(x) = x, \tag{2}$$

where $x \in T$, $M_t$ is the martingale part in the decomposition of $\xi_t$ and $D_t\xi$ its drift (for simplicity we do not write the probability parameter $\omega \in \Omega$ in the formulae).

Recall the definition of generalised derivative, that we denote by $D_t$: for $F$ defined in $[0, T] \times T$,

$$D_tF(t, \xi_t(x)) = \lim_{\epsilon \to 0} \mathbb{E}_t \left[ F(t + \epsilon, \xi_{t+\epsilon}(x)) - F(t, \xi_t(x)) \right] \tag{3}$$

when such (a.s.) limit exists, and where $E_t$ denotes the conditional expectation with respect to $\mathbb{F}_t$. This definition justifies in particular the notation used in (2), since the generalised derivative corresponds to the semimartingale’s drift.

If $W_t$ is a $\mathbb{F}_t$-adapted Wiener process, we denote by $g_t(\cdot)$ diffusions on the torus $T$ of the form

$$dg_t(x) = \sqrt{2} \nu dW_t + v(t, g_t(x))dt, \quad g_0(x) = x \tag{4}$$

with $x \in T$, $dW_t$ the Itô differential. The drift function $v$ is assumed to be regular enough so that $g_t(\cdot)$ are diffeomorphisms (cf. [14]).

Note that we do not require, a priori, the vector field $v$ to be divergence free.

For the particular cases $F(t, x) = x$ and $F(t, x) = v(t, x)$, we have, respectively,

$$D_tg_t(x) = v(t, g_t(x))$$
and, using Itô’s formula,
\[ D_t v(t, g_t(x)) = D_t D_t g_t(x) = \left( \frac{\partial}{\partial t} v + (v \nabla) v + \nu \Delta v \right)(t, g_t(x)). \]  
(5)

Let \( \mathbb{H} \) be a linear subspace dense in \( L^2([0, T] \times \mathbb{T}) \). Define the action functional
\[ S(g, p) = \frac{1}{2} E \int_0^T |D_t g_t(x)|^2 dt dx + E \int_0^T \int p(t, g_t(x)) (\text{det} \nabla g_t(x) - 1) dt dx \]  
(6)
\[ := S^1(g, p) + S^2(g, p) \]  
(7)
for \( p \in \mathbb{H} \) and where \( E \) denotes expectation (with respect to \( P \)).

We consider variations
\[ g_t(\cdot) \to g_t^\varepsilon(\cdot) = g_t(\cdot) + \varepsilon h(t, g_t(\cdot)) \]
\[ p(t, \cdot) \to p^\varepsilon(t, \cdot) = p(t, \cdot) + \varepsilon \varphi(t, g_t(\cdot)) \]

with \( h(t, x) \) and \( \varphi(t, x) \) deterministic and smooth in \( x, \varphi \in \mathbb{H} \). We also assume that \( h(T, \cdot) = h(0, \cdot) = 0 \).

We have the following

**Theorem 2.1.** A diffusion \( g_t \) of the form (4) and a function \( p \in \mathbb{H} \) are critical for the action functional (6) iff the drift \( v(t, \cdot) \) of \( g_t \) satisfies the Navier-Stokes equation (without external force)
\[ \partial_t v + (v \nabla) v = \nu \Delta v - \nabla p, \quad \text{div } v(t, \cdot) = 0, \]  
(8)
with \( x \in \mathbb{T}, t \in [0, T] \).

**Proof.** Using the notation \( \delta S(g, p) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} S(g^\varepsilon, p^\varepsilon) \), the variation of the first term in the action gives:
\[ \delta S^1(g, p) = E \int_0^T \int (D_t g_t(x), D_t h(t, g_t(x))) dt dx. \]
The notation \( <\cdot, \cdot> \) stands below for the \( L^2(\mathbb{T}) \) scalar product. By Itô’s formula, the expression
\[ d < D_t g_t, h > = - < D_t D_t g_t, h > dt - < D_t g_t, D_t h > dt - < D_t g_t, dh > \]
where the last term denotes the Itô contraction, is the differential of a martingale (whose expectation vanishes); therefore
\[ D_t < D_t g_t, h > = < D_t D_t g_t, h > + < D_t g_t, D_t h > + < dD_t g_t, dh >. \]  
(9)

We deduce that
\[ \delta S^1 = E < D g_T, h(T, g_T) > - E < D g_0, h(0, g_0) > 
- E \int_0^T \int (D_t D_t g_t(x), h(t, g_t(x))) dt dx 
- E \int_0^T \int (dD_t g_t(x), dh(t, g_t(x))) dx. \]
\[ = - E \int_0^T \int (D_t D_t g_t(x), h(t, g_t(x))) dt dx - 2 \nu E \int_0^T (\nabla v. \nabla h)(t, g_t(x)) dt dx 
= - E \int_0^T \int ((\partial_t v + (v \nabla) v - \nu \Delta v), h)(t, g_t(x)) dt dx \]
where, for the last equality we have used the equality \( D_t D_t g_t(x) = D_t v(t, g_t(x)) = (\partial_t v + (v \nabla) v + \nu \Delta v)(t, g_t(x)) \) and integration by parts.
Concerning the second part of the action functional, we have

$$\delta S^2 = E \int_0^T \int \varphi(t, g_t(x))(\det \nabla g_t(x) - 1) dtdx$$

(10)

$$+ E \int_0^T \int (\nabla p(t, g_t(x)).h(t, g_t(x))(\det \nabla g_t(x) - 1) dtdx$$

(11)

$$+ E \int_0^T \int p(t, g_t(x)) \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \det \nabla g_t(x) + \epsilon h(t, g_t(x)) dtdx$$

(12)

Since $\varphi$ is arbitrary we conclude from the term (10) that critical points of the action are volume-preserving diffeomorphisms ($\det \nabla g_t(x) = 1$) and therefore have divergence-free drifts. It follows immediately that (11) = 0 so we only have to compute (12). We have,

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \det \nabla g_t(x) + \epsilon h(t, g_t(x)) = \det \nabla g_t(x) \text{ tr} \left( (\nabla g_t(x))^{-1} \frac{d}{d\varepsilon} |_{\varepsilon=0} \nabla g'_t(x) \right)$$

$$= \det \nabla g_t(x) \text{ tr} \left( (\nabla g_t(x))^{-1} \nabla (h(t, g_t(x))) \right).$$

Since

$$\partial_i (p(t, g_t))(\nabla g_t)_{ij}^{-1} h(t, g_t)^j$$

$$= \partial_i (p(t, g_t))(\nabla g_t)_{ij}^{-1} h(t, g_t)^j + p(t, g_t)(\nabla g_t)_{ij}^{-1}\partial_i (h^j(t, g_t))$$

$$+ p(t, g_t)h^j(t, g_t)\partial_i (\nabla g_t)_{ij}^{-1}$$

and we are in the periodic case,

$$E \int_0^T \int \left[ \partial_i (p(t, g_t))(\nabla g_t)_{ij}^{-1} + p(t, g_t)\partial_i ((\nabla g_t)_{ij}^{-1}) \right] h^j(t, g_t) \det \nabla g_t \ dt dx.$$

Notice that we already concluded that $\det \nabla g_t = 1$. On the other hand,

$$\sum_i \partial_i (\nabla g_t)_{ij}^{-1} = 0.$$

Indeed, derivating the equality $\det \nabla g_t = 1$, we get

$$\partial_k \det(\nabla g_t) = \text{ tr}((\nabla g_t)^{-1} \partial_k (\nabla g_t)) = \sum_i (\nabla g_t)_{ij}^{-1} \partial_k \partial_i g_t^j = 0.$$

Also, derivating equality

$$(\nabla g_t)_{ij}^{-1} \partial_k g_t^j = \delta_{ik}$$

we obtain

$$\sum_i \partial_i (\nabla g_t)_{ij}^{-1} \partial_k g_t^j + (\nabla g_t)_{ij}^{-1} \partial_i \partial_k g_t^j = 0;$$

therefore

$$\sum_i \partial_i (\nabla g_t)_{ik}^{-1} = -((\nabla g_t)_{ij}^{-1} \partial_k \partial_i g_t^j)(\nabla g_t)_{ik}^{-1} = 0$$

and

$$E \int_0^T \int (\partial_i (p(t, g_t(x)))(\nabla g_t(x))_{ij}^{-1}) h^j(t, g_t(x)) \det \nabla g_t(x) dtdx$$

$$= -E \int_0^T \int (\nabla p(t, g_t(x)).h(t, g_t(x))) dtdx.$$
Putting together the expressions for $\delta S^1$ and $\delta S^2$, we conclude that $\delta S = 0$ in the class of variations considered, is equivalent to the condition

$$E \int_0^T \int (\partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p(t, g_t(x))) h(t, g_t(x)) dt dx = 0$$

for every test function $h$, together with the incompressibility condition $\det \nabla g_t(x) = 1$.

**Remark 1.** Comparing with [5, 1], the variations we have used here are defined by shifts, since we do not have to work a priori in the class of measure-preserving flows.

**Remark 2.** It is possible to consider subspaces $\mathcal{H}$ which are not dense in $L^2([0,T] \times \mathbb{T})$. In this case the resulting equation of motion is the projection of the Navier-Stokes one in the corresponding space.

3. On conserved quantities. In this section we present a Noether-type result where only transformations in space of the Lagrangian function are considered. A more general study of symmetries for equations obtained by stochastic variational principles will be considered in a forthcoming work.

Let us consider transformations of the following form:

$$g_t(\cdot) \rightarrow g^\alpha_t(\cdot) = g_t(\cdot) + \alpha \eta(t, g_t(\cdot))$$

with $\eta$ smooth, $\eta(0, \cdot) = \eta(T, \cdot) = 0$. We say that the Lagrangian

$$L(g, p) = \frac{1}{2} |D_t g_t(x)|^2 + p(t, g_t(x))(\det \nabla g_t(x) - 1)$$

$$:= L^1(g, p) + L^2(g, p)$$

used in the definition of the action functional (6), is invariant under the transformation associated with $\eta$ if there exits a function $G : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ such that for every $t$, $P$-a.e.,

$$\frac{d}{d\alpha} \bigg|_{\alpha = 0} \int L(g^\alpha_t, p) dx = \int D_t G(t, g_t(x)) dx.$$

Using this definition we have the following,

**Theorem 3.1.** If $L$ is invariant under the transformation associated with $\eta$ then, denoting $L_t = \frac{d}{dt} + (v \cdot \nabla) + \nu \Delta$ where $v(t, \cdot)$ is the solution of the Navier-Stokes equation considered above, the following identity

$$\int (L_t(v \cdot \eta - G))(t, x) dx = 0$$

holds for all $t \in [0, T]$.

**Proof.** Considering the first term in the Lagrangian, we have

$$\frac{d}{d\alpha} \bigg|_{\alpha = 0} L^1(g^\alpha_t, p) = (D_t g_t(x), D_t \eta(t, g_t(x)))$$

and, by the arguments in the proof of last section’s Theorem,

$$\frac{d}{d\alpha} \bigg|_{\alpha = 0} L^2(g^\alpha_t, p) = (\nabla p(t, g_t(x)) \cdot \eta(t, g_t(x))(\det \nabla g_t(x) - 1)$$

$$+ \partial_i(p(t, g_t(x))) \nabla g_t(x)^{-1} \eta^j(t, g_t(x))) - (\nabla p(t, g_t(x)) \cdot \eta(t, g_t(x))).$$

(15)
We know that \((\det \nabla g_t(x) - 1) = 0\) on the critical points of the action functional, therefore the first term in the r.h.s. of last equality vanishes. The second one also vanishes after integration in \(x\), as we consider periodic boundary conditions. We are therefore left with the equality, valid for \(g_t\) critical of the action functional,

\[
\int \left( (D_t g_t(x). D_t \eta(t, g_t(x))) - (\nabla p(t, g_t(x)). \eta(t, g_t(x))) \right) dx = \int D_t G(t, g_t(x)) dx,
\]

\(P\text{-}a.e.\) We use the identity

\[
D_t(D_t g_t(x). \eta(t, g_t(x))) = (D_t D_t g_t(x). \eta(t, g_t(x))) + (D_t g_t(x). D_t \eta(t, g_t(x))) + (dD_t g_t(x). d\eta(t, g_t(x)).)
\]

From the last two equalities we deduce that

\[
\int D_t \left( (D_t g_t(x). \eta(t, g_t(x)) - G(t, g_t(x))) \right) dx = 0.
\]

We have \(D_t g_t(x) = v(t, g_t(x))\). By the incompressibility condition, the flow \(g_t(\cdot)\) keeps the measure \(dx\) invariant (a.s.) and the result follows from the expression of the operator \(D_t\).

Comparing with the finite-dimensional Noether’s theorem of [17], [18], here we have an extra integration with respect to the space variable \(x\) in the notion of conserved quantity.

4. **A stochastic Navier-Stokes equation.** In this section we show that it is also possible to derive random perturbations of the Navier-Stokes equation from a stochastic variational principle.

Let \(\xi\) be a semimartingale with values in \(\mathbb{T}\) of the form (2). We consider the random action functional

\[
\hat{S}(\xi, p) = \frac{1}{2} \int_0^T \int |D_t \xi_t(x)|^2 dt dx
\]

\[
+ \int_0^T \int (D_t \xi_t(x). dM_t(x)) - \sqrt{2\nu} \int_0^T \int (D_t \xi_t(x). dW_t)
\]

\[
+ \int p(t, \xi_t(x))(\det \nabla \xi_t(x) - 1) dt dx,
\]

with \(p \in \mathbb{H} \subset L^2([0, T] \times \mathbb{T})\). Variations of \(\xi\) and \(p\) are taken as in Section 1, namely

\[
g_t(\cdot) \rightarrow g_t'(\cdot) = g_t(\cdot) + \epsilon h(t, g_t(\cdot))
\]

\[
p(t, \cdot) \rightarrow p'(t, \cdot) = p(t, \cdot) + \epsilon \varphi(t, g_t(\cdot))
\]

except that here we allow \(h\) and \(\varphi\) to be random.

We want to characterise critical points of \(\hat{S}\) of the form

\[
dg_t(x) = \sqrt{2\nu} dW_t + v(t, g_t(x)) dt, \quad g_0(x) = x
\]

now considering the vector field \(v\) to be random. We proceed as in the theorem of section 1. Computations are analogous and we have to add, in the variations of \(S\), those of the second and third new terms of this functional. These terms give,

\[
\int_0^T \int [(h(t, g_t). \sqrt{2\nu} dW_t) + (D_t g_t. (\nabla h(t, g_t). dW_t)) - (h(t, g_t). \sqrt{2\nu} dW_t)] dx
\]
that reduces to
\[
\int_0^T \int v(t, g_t(x)).(\nabla h(t, g_t(x))).dW_t) = \int_0^T v(t, x).dW_t) = -\int_0^T ((\nabla v(t, x), h(t, g_t(x))))dW_t)
\]
equality which holds \(P\)-almost surely.

We therefore conclude that a diffusion process of the form
\[
dg_t(x) = \sqrt{2\nu}dW_t + v(t, g_t(x))dt, \quad g_0(x) = x,
\]
is critical for the action functional \(\tilde{S}\) iff its (random) drift
\[
v(t, \cdot) = 0,
\]
with \(x \in T, t \in [0, T]\).

This stochastic equation can be also regarded as a (Stratonovich) perturbation
of the Euler one. Indeed, denoting by \(\circ dW\) the Stratonovich differential, it can be
written as
\[
dv + (v, \nabla)v = \sqrt{2\nu} \nabla v \circ dW_t - \nabla p, \quad \text{div } v(t, \cdot) = 0.
\]

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