Non-orientable Boundary Superstring Field Theory with Tachyon Field

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We use the BSFT method to study the non-orientable open string field theory (type I). The partition function on the Möbius strip is calculated. We find that, at the one-loop level, the divergence coming from planar graph and unoriented graph cancel each other as expected.
1 Introduction

Tachyon condensation is an interesting issue in string theory [1, 2, 3]. To study it, Witten’s Boundary String Field Theory (BSFT) [4]-[6] has proven to be a useful method. Tachyon condensation at tree level has been studied first by A. Gerasimov and S. Shatashvili [7] and D. Kutasov, M. Marino, G. Moore [8], then followed by many authors [9]-[18]. The BSFT method was also extended to the case of superstrings in [11], [19]-[23]. Recently, tachyon condensation at one-loop level has been discussed by many authors who have studied it by different methods and got similar results [24]-[37].

However, all of these considerations focus on type II superstring theories which are oriented string theories, i.e. including only annulus or cylinder diagrams at one-loop level. In this work, we will study type I superstring theory. In fact, with Chan-Paton charges, open-string theory in supersymmetric 10 dimensional space-time (type I) has to be non-orientable [38]. To study the non-orientable open-string theory, we need to include the Möbius strip at the one-loop level.

It is well known that the gauge group in type I superstring theory has to be $SO(32)$. There are several ways to verify this by anomaly or divergence cancellation. In [38, 44], the on-shell $n$ strings amplitudes for different one-loop graphs are calculated. It was shown that the divergences, due to the integration over the modulus parameter, coming from planar and Möbius graphs are cancelled each other only for $SO(32)$ gauge group. By using BSFT, we will show that, off-shell, this divergences cancellation also holds only for $SO(32)$ gauge group.

In the next section, we briefly recall the analysis of powers of the coupling constant in non-orientable string graphs. In the section 3, the partition function on the Möbius strip is calculated by the method of BSFT. In the section 4, we include Chan-Paton factors by considering an unstable $Dp$-brane in $N$ $D9$-branes background, then show that the divergences due to the integration over the modulus parameter only cancel each other for $N = 32$, namely $SO(32)$ gauge group. We summarize our work in the section 5.
2 Powers of the Coupling Constant in String Graphs

Boundary string theory is defined on all compact two dimensional manifolds [4]. At each order the string coupling constant enters the calculation as:

\[ g^{-\chi}. \] (1)

where \( g \) is open string coupling and \( \chi \) is the Euler number of the manifold.

Any compact orientable two-manifold is topologically equivalent to the direct sum of a sphere with \( h \) handles and \( b \) holes, and the corresponding Euler number is

\[ \chi = 2 - 2h - b. \] (2)

Similarly, any compact non-orientable two-manifold is topologically equivalent to a direct sum of a sphere with \( h \) handles, \( b \) holes and \( c \) cross-caps. The corresponding Euler number is

\[ \chi = 2 - 2h - b - c. \] (3)

It is well known that Type I superstring theory has to be a unoriented open string theory. We will use the string coupling power counting to decide what two-manifolds, which can couple to the D-brane, we need.

The lowest order \((g^{-2})\) is the sphere for which \((h, b, c) = (0, 0, 0)\). We will not consider it because it can not couple to the D-brane. At the next order \((g^{-1})\) there are two graphs, the \(RP_2\) with \((h, b, c) = (0, 0, 1)\) and the disc with \((h, b, c) = (0, 1, 0)\). For the same reason as for the sphere, we will only consider the disc. At the one-loop order \((g^0)\) there are four graphs, which are the torus with \((h, b, c) = (1, 0, 0)\), the annulus with \((h, b, c) = (0, 2, 0)\), the Klein bottle with \((h, b, c) = (0, 0, 2)\) and the Möbius strip with \((h, b, c) = (0, 1, 1)\). We will only consider the annulus and the Möbius strip, the two of them have boundaries. (Fig. [1])

The partition function on the disc has been studied by many authors. We will concentrate on the one-loop graphs in this work.
Figure 1: compact non-orientable two-manifold
For open string theory, there are three types of one-loop diagrams [38]. The first type is the planar diagram, which is an annulus with only one boundary coupled to the D-brane. The second type is the unoriented diagram, which is a Möbius strip. Third type is the non-planar diagram, which is an annulus with both boundaries coupled to the D-brane.

The first and the third types have been considered in [26] and other papers. In the next section, we will use the same method as in [26] to calculate the partition function on the Möbius strip, then write down the effective tachyon action for the unoriented superstring.

3 Partition Function on Möbius Strip

Möbius strip can be described as an annulus with a twist (a of Fig. [2]). But it is difficult to give it global coordinates. Another description is mapping the Möbius strip to the upper half of an annulus with the upper semicircle identified with the lower in an anti-parallel way (b of Fig. [2]). This description has been widely used to calculate the Green function on the Möbius strip world sheet [38, 40, 41]. But we will have problems when trying to embed the Möbius strip in spacetime and attach to the D-brane. The third description is to use a ”cross-cap” (c of Fig. [2]). In this description, the Möbius strip looks like an annulus, but one of the boundaries is replaced by a cross-cap. We will use the cross-cap description to calculate the Green function on the Möbius strip in the following.

Now the world sheet Σ is a Möbius strip with a rotationally invariant flat metric

$$ds^2 = d\sigma_1^2 + d\sigma_2^2,$$  \hspace{1cm} (4)

the complex variable $z = \sigma_1 + i\sigma_2$, with $a \leq |z| \leq b$. Here we choose $\rho = a$ as the boundary of the Möbius strip, and $\rho = b$ is the cross-cap.

Next, we take the tachyon profile as

$$T(X) = uX; \hspace{1cm} \rho = a,$$  \hspace{1cm} (5)

the boundary term can then be written as

$$I_{\text{bndy}} = \frac{y}{8\pi} \int_0^{2\pi} d\theta \left( X^2 + \frac{1}{\partial \theta} \psi \frac{1}{\partial \theta} \bar{\psi} + \bar{\psi} \frac{1}{\partial \theta} \psi \right) \Big|_{\rho=a}.$$  \hspace{1cm} (6)
Figure 2: Three descriptions of the Möbius strip

where $y \equiv u^2$.

The boundary conditions for the Green function of the bosonic fields are

$$
(z\partial + \bar{z}\bar{\partial} - y_a)G_B(z, w)|_{\rho=a} = 0, \\
G_B(z, w)|_{\rho=b} - G_B(-z, w)|_{\rho=b} = 0,
$$

(7)

where the condition at the boundary $\rho = a$ is same as the condition for an annulus, while at the boundary $\rho = b$ we have the symmetric condition for the cross-cap.

To solve for $G_B(z, w)$, we start with the ansatz,

$$
G_B(z, w) = -\ln|z-w|^2 + C_1 \ln |z|^2 \ln |w|^2 + C_2(\ln |z|^2 + \ln |w|^2) + C_3 \\
+ \sum_{-\infty}^{\infty} a_k [(zw)^k + (\bar{z}\bar{w})^k] + \sum_{-\infty}^{\infty} b_k \left[ \left( \frac{z}{w} \right)^k + \left( \frac{\bar{z}}{\bar{w}} \right)^k \right]
$$

(8)

One can easily verify that these are solutions of

$$
\partial_z \bar{\partial}_z G(z, w) = -2\pi \delta^{(2)}(z - w),
$$

(9)
on a Möbius strip. $C_1$, $C_2$, $C_3$, $a_r$’s and $b_r$’s are coefficients to be determined by the boundary conditions (7). In the appendix, we derive the Green function and find that the only differences consist of replacing $\left( \frac{z^2}{|w|^2} \right)$ by $\left( -\frac{\bar{z}^2}{|w|^2} \right)$ and setting $y_b = 0$. 

5
The fermionic Green functions satisfy the equations\[1\]:

\[
\begin{align*}
\bar{\partial} G_F(z, w) &= -i\sqrt{zw}\delta(2)(z - w), \\
\partial \tilde{G}_F(\bar{z}, \bar{w}) &= +i\sqrt{\bar{z}\bar{w}}\delta(2)(\bar{z} - \bar{w})
\end{align*}
\]  \tag{10}

with the boundary conditions

\[
\begin{align*}
(1 - iy_a \frac{1}{\partial_b}) G_F \big|_{\rho=a} &= (1 + iy_a \frac{1}{\partial_b}) \tilde{G}_F \big|_{\rho=a}, \\
G_F(z, w) \big|_{\rho=b} &= \tilde{G}_F(-\bar{z}, \bar{w}) \big|_{\rho=b}.
\end{align*}
\]  \tag{11}

where, similarly, the condition at the boundary \(\rho = a\) is same as the condition for an annulus, the condition at the boundary \(\rho = b\) is the symmetric condition for the cross-cap.

As in the bosonic case, we find that the fermionic Green function on a Möbius strip is the same as that on an annulus by replacing \(\left(\frac{a^2}{b^2}\right)\) by \(\left(-\frac{a^2}{b^2}\right)\) and setting \(y_b = 0\).

The partition function, then, can be obtained as (see Appendix for the detailed derivation)

\[
Z_M(a^2/b^2) = Z'_4 y \frac{Z_B^2(y, -a^2/b^2)}{Z_B(2y, \sqrt{-a^2/b^2})},
\]  \tag{12}

where \(Z'\) is the integration constant, and

\[
Z_B(y; a, b) = \frac{1}{\sqrt{y}} \cdot \prod_{k=1}^{\infty} \left[ \left(1 + \frac{y}{k}\right) - \left(1 - \frac{y}{k}\right) \left(-\frac{a^2}{b^2}\right)^k \right]^{-1}
\]  \tag{13}

is the bosonic partition function on the Möbius strip.

Comparing this result with that for the planar graph \[26\], we have, up to a constant,

\[
Z_M \left( y, \frac{a^2}{b^2} \right) = Z_P \left( y, \frac{a^2}{b^2} \right).
\]  \tag{14}

\[1\] For simplification, in this work, we only consider the NS-NS sector with spin structures \(G_{++}\) and \(G_{--}\) (see \[33\] for detail). The other spin structures \(G_{+-}\) and \(G_{-+}\) do not contribute to the zero mode which the divergence cancellation depends on. Moreover, the R-R section does contribute to the zero mode, but after we set \(y_b = 0\) (for planar graph), it does not affect the cancellation.
It is convenient to take the outer radius \( b = 1 \) in the following. Integrating over the modulus \( d\lambda = da/a \), we obtain
\[
Z_M(y) = \int d\lambda Z_M(y, a^2) = \sqrt{2} Z' 4^y \int_0^1 \frac{da}{a \sqrt{y}} \prod_{k=1}^{\infty} \frac{1 + \frac{2y}{k} - (1 - \frac{2y}{k})(\sqrt{-a^2})^k}{(1 + \frac{y}{k}) - (1 - \frac{y}{k})(-a^2)^k}. \tag{15}
\]

Expanding (15) in powers of \( y \), the expression of \( Z_M \) takes the form
\[
Z_M(y) = \sqrt{2} Z' \int_0^1 \frac{da}{a} \prod_{k=1}^{\infty} \frac{1 - (\sqrt{-a^2})^k}{1 - (a^2)^k} \cdot \left( \sqrt{\frac{1}{y}} + \left[ 2 \ln 2 - 4 \sum_{n=1}^{\infty} \ln(1 - (\sqrt{a^2})^{2n-1}) \right] \sqrt{y} + \cdots \right). \tag{16}
\]

4 Divergence Cancellation for Type I Superstring Theory

In this section, we will consider the type I superstring theory. To study the tachyon condensation in the type I theory with gauge group \( SO(N) \), we can have two kinds of unstable \( D \)-brane systems:

- An unstable\(^2\) \( Dp \)-brane in \( N \) stable \( D9 \)-branes.
- \( N_1 D9 \)-branes and \( N_2 \bar{D}9 \)-branes with \( N_1 - N_2 = N \).

By brane-antibrane creation and annihilation, these two cases are equivalent to each other \[42, 43\]. It is enough to consider one of them. We will focus on the first case in this paper.

Type I superstring theory is a non-orientable string theory, therefore instead of a \( U(N) \) group, \( N \) \( D9 \)-branes lead to a nontrivial Chan-Paton \( SO(N) \) space in \( d \) dimensional space.

\(^2\)To be unstable in Type I theory, \( p \neq 1, 5 \) or 9.
time. It is easy to include Chan-Paton factors by taking the boundary action to be
\[ e^{-S_{\text{bndy}}} = \text{Tr} P \exp \left\{ -\frac{1}{8\pi} \int d\theta \left[ T^2(X) + (\psi^\mu \partial_\mu T) \frac{1}{\partial_\theta} (\psi^\nu \partial_\nu T) \right] \right\}. \quad (17) \]

We can generalize the tachyon profile to the form as in \[19\]
\[ T(X) = \sum_{\alpha=1}^{2n} T^\alpha(X) \gamma^\alpha \quad (18) \]
where we set \( N = 2^n \) and expand the \( 2^n \times 2^n \) matrix tachyon field in real Hermitian matrices \( \gamma^\alpha \)'s, which form a Clifford algebra. The theory has \( SO(2^n) \) gauge invariance, of which a \( SO(2n) \) subgroup is manifest. \[23\]

In the following, we will consider the linear tachyon profiles of the forms
\[ T^\alpha(X) = \sum_{i=0}^{p} u^\alpha_i X^i, \quad (19) \]
where \( i = 0, \cdots, p \) are indices of world-volume of \( Dp \)-brane, and \( \alpha = 1, \cdots, 2n \) are indices of spinor representation of \( SO(2n) \) group.

Substituting (18) and (19) into (17), we find (see Appendix A.3)
\[ e^{-S_{\text{bndy}}} = 2^n \exp \left\{ -\frac{y_{ij}}{8\pi} \int d\theta \left[ X^i X^j + \psi^i \frac{1}{\partial_\theta} \psi^j + \tilde{\psi}^i \frac{1}{\partial_\theta} \tilde{\psi}^j \right] \right\}, \quad (20) \]
where \( y_{ij} \equiv \sum_{\alpha, \beta=1}^{2n} u^\alpha_i u^\beta_j \).

The partition function on the Möbius strip will simply be
\[ Z_M(y, a^2) = -2^n \prod_{i=0}^{p} Z'_M Z_M(y_i, a^2), \quad (21) \]
where \( Z'_M \) is a integration constant and \( y_i \equiv y_{ii} \). The overall negative sign comes from the twist operator for \( SO(2^n) \) group \[38\].

\[3\]Later we will set \( d=10 \) for Type I superstring theory.

\[4\]Because \( G^{ij} \sim \delta^{ij} \), the terms of \( y_{ij} \) with \( i \neq j \) does not contribute to the partition function.
Combining the unoriented graph and the planar graph, we obtain:

$$Z_{M+P}(y) = (2^n)^2 \int_0^1 \frac{da}{a} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) - 2^n \int_0^1 \frac{da}{a} \prod_{i=0}^p Z'_M Z_M(y_i, a^2)$$

$$= (2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) - 2^n \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_M Z_M(y_i, a^2)$$

$$= (2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) - 2^n \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_M Z_M(y_i, -a^2)$$

$$= (2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) + 2^n \int_{-1}^0 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_M Z_P(y_i, a^2).$$

To proceed, we need the exact relationship between the two integration constants $Z'_P$ and $Z'_M$. This can be done by taking the ratio of two partition functions in some limit and compare them to the direct calculation. Taking the limit

$$\lim_{a \to 0} \lim_{\beta \to 0} \frac{Z'_P Z_P(y_i, a^2)}{Z'_M Z_M(y_i, a^2)} = \frac{Z'_P}{Z'_M}. \quad (23)$$

On the other hand

$$\lim_{a \to 0} \lim_{\beta \to 0} \frac{Z'_P Z_P(y_i, a^2)}{Z'_M Z_M(y_i, a^2)} = \lim_{a \to 0} \int dX e^{I^{P}_{\text{bulk}}} = \lim_{a \to 0} \int dX e^{I^{M}_{\text{bulk}}} = \lim_{a \to 0} Z^{(0)}_P = Z^{(0)}_M, \quad (24)$$

where $Z^{(0)}_P$ and $Z^{(0)}_M$ are the non-tachyon-interaction partition functions for planar and unoriented graphs.

Fortunately, the non-tachyon-interaction partition functions for planar and unoriented graphs have been calculated, they are:

$$Z^{(0)}_P = \left( \frac{1}{8\pi (8\pi^2 \alpha')^{2}} \right)^{\frac{d}{2}} \int_0^\infty ds [2(d-2) + O(e^{-2s})]$$

$$Z^{(0)}_M = \left( \frac{2^d}{8\pi (8\pi^2 \alpha')^{2}} \right)^{\frac{d}{2}} \int_0^\infty ds [2(d-2) + O(e^{-2s})]. \quad (25)$$

---

5The $(2^n)^2$ in front of the planar partition function comes from the two boundaries of an annulus.
Therefore \[^6\]

\[
\frac{Z'_P}{Z'_M} = \lim_{s \to \infty} \left( \frac{2\sqrt{2}}{8\pi(8\pi^2\alpha')^{1/2}} \right)^{1/2} \int_0^\infty ds \left[ 2(d - 2) + O(e^{-2s}) \right] \]

\[
\int_0^\infty ds \left[ 2(d - 2) + O(e^{-2s}) \right] = \frac{1}{\sqrt{2}} \quad (26)
\]

Then

\[
Z_{P+M}(y) = (2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) + 2^n \int_{-1}^0 \frac{da^2}{2a^2} \prod_{i=0}^p \sqrt{2} Z'_P Z_P(y_i, a^2). \quad (27)
\]

It is convenient to regularize the volume divergence of the remaining coordinates as in \[^8\] \[^9\] by periodic identification

\[
X^\mu \sim X^\mu + R^\mu, \mu = p + 1, \cdots, d - 1. \quad (28)
\]

To determine the correct normalization of the \(X\) zero mode, we use the same method as in \[^19\]. We find that the normalization of the \(X\) zero mode is \(1/\sqrt{\pi}\) for the Möbius strip

\[
\int \frac{dX}{\sqrt{\pi}} \exp \left( - \frac{y}{8\pi} \int_0^{2\pi} d\theta X^2 \right) = \sqrt{2} \cdot \sqrt{\frac{2}{y}}, \quad (29)
\]

instead of \(1/\sqrt{2\pi}\) for the annulus

\[
\int \frac{dX}{\sqrt{2\pi}} \exp \left( - \frac{y}{8\pi} \int_0^{2\pi} d\theta X^2 \right) = \sqrt{\frac{2}{y}}. \quad (30)
\]

The string field action evaluated for the tachyon field is then given by

\[
S_{1+\text{loop}}^{P+M}(y) = (2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p Z'_P Z_P(y_i, a^2) \prod_{\mu=p+1}^{d-1} \left( \frac{R^\mu}{\sqrt{2\pi}} \right)
\]

\[
+ 2^n \int_{-1}^0 \frac{da^2}{2a^2} \prod_{i=0}^p \sqrt{2} Z'_P Z_P(y_i, a^2) \prod_{\mu=p+1}^{d-1} \left( \frac{R^\mu}{\sqrt{\pi}} \right)
\]

\[^6\]The limit \(a \to 0\) here is corresponding to the limit \(s \to \infty\) in \[^4\].
\[
(2^n)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p P_0 Z_P(y_i, a^2) \prod_{\mu=p+1}^{d-1} \left( \frac{R^\mu}{\sqrt{2\pi}} \right) \
2^{d/2} \cdot 2^n \int_{-1}^0 \frac{da^2}{2a^2} \prod_{i=0}^p P'_0 Z_P(y_i, a^2) \prod_{\mu=p+1}^{d-1} \left( \frac{R^\mu}{\sqrt{2\pi}} \right). \tag{31}
\]

For \(d = 10\) and \(n = 5\), which correspond to \(SO(2^5) = SO(32)\) gauge group, it can be written as

\[
S_{P+M}^{1-\text{loop}}(y) = (32)^2 \int_0^1 \frac{da^2}{2a^2} \prod_{i=0}^p P'_0 Z_P(y_i, a^2) \prod_{\mu=p+1}^{d-1} \left( \frac{R^\mu}{\sqrt{2\pi}} \right). \tag{32}
\]

Now we see that the divergence due to the integral over modulus \(a\) is cancelled if we consider the integration in (32) as the principal value integral. After performing the integration over \(a\), We can write down (32) in term of tachyon fields up to the first two orders

\[
S_{P+M}^{1-\text{loop}}(y) \simeq C_1 T_p \int d^{10}X e^{-\frac{1}{2}T^2} [C_2 (2 \ln 2) \partial_\mu T(X) \partial^\mu T(X) + 1], \tag{33}
\]

where \(C_1\) and \(C_2\) are constants and \(T_p\) is the \(Dp\)-brane tension.

Now we are ready to evaluate the one-loop effective action of non-orientable open string field theory by adding up the planar, unoriented and oriented non-planar diagrams [26].

\[
S_{1-\text{loop}}^{P+M+T} \simeq C_1 T_p \int d^{10}X e^{-\frac{1}{2}T^2} [C_2 (2 \ln 2) \partial_\mu T \partial^\mu T + 1] \
+ T_p (\Lambda + C_3) \int d^{10}X e^{-\frac{1}{2}T^2} [(2 \ln 2) \partial_\mu T \partial^\mu T + 1] \
= T_p \int d^{10}X e^{-\frac{1}{2}T^2} [(\Lambda + \Lambda_{\text{finite}})(2 \ln 2) \partial_\mu T \partial^\mu T + \Lambda + \Lambda'_{\text{finite}}], \tag{34}
\]

where \(\Lambda\) is the cut-off, \(\Lambda_{\text{finite}}\) and \(\Lambda'_{\text{finite}}\) are constants.

Then the effective tachyon action, up to one-loop level, is

\[
S = S_{\text{tree}}^{\text{Disk}} + S_{1-\text{loop}}^{P+M+T} \
\simeq \frac{T_p}{g} \int d^{10}X e^{-\frac{1}{2}T^2} [(2 \ln 2) \partial_\mu T \partial^\mu T + 1]
\]
where \( \lambda' \) is the renormalized effective string coupling which is defined by

\[
\frac{1}{\lambda'} = \frac{1}{\lambda} + \Lambda, \tag{36}
\]

and

\[
\lambda = e^{-\frac{1}{4}T^2}. \tag{37}
\]

\( T'_p \) is the renormalized \( Dp \)-brane tension which is defined by

\[
T'_p = (1 + \lambda' \Lambda_{\text{finite}})T_p. \tag{38}
\]

5 Conclusions

In this work, we have studied type I superstring field theory, which is a non-orientable open string field theory, with the tachyonic field interaction on the boundary by using the BSFT method. We obtained the tachyon effective action up to one-loop level with the renormalized \( Dp \)-brane tension and renormalized string coupling. The expected cancellation of divergence between the planar graph and the unoriented graph was obtained. We noted that this cancellation is due to the exact relationship between the two integration constants \( Z'_P \) and \( Z'_M \), namely the normalization. There is another method to fix the normalization, by calculating the three tachyon amplitude and comparing it to the same amplitude obtained by the cubic open string field theory. This has been done for both bosonic string field theory \([13]\) and superstring field theory \([10]\) on the disk, but not at the one-loop level. An alternative way to obtain the relationship between \( Z'_P \) and \( Z'_M \) is to directly calculate the partition function by using the boundary states method. We hope that these calculation will confirm our results.
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A Derivation of Green’s functions and the partition function

A.1 Bosonic and Fermionic Green’s functions

In the bosonic case, to solve
\[ \partial_z \bar{\partial}_z G(z,w) = -2\pi \delta^{(2)}(z-w), \]  
(A.1)

with the boundary condition
\[ (z\partial + \bar{z}\partial - y_a)G_B(z,w)|_{\rho=a} = 0, \]
\[ G_B(z,w)|_{\rho=b} - G_B(-z,w)|_{\rho=b} = 0, \]  
(A.2)

we start with the ansatz,
\[ G_B(z,w) = -\ln |z-w|^2 + C_1 \ln |z|^2 \ln |w|^2 + C_2 (\ln |z|^2 + \ln |w|^2) + C_3 
+ \sum_{-\infty}^{\infty} a_k [(z\bar{w})^k + (\bar{z}w)^k] 
+ \sum_{-\infty}^{\infty} b_k \left[ \left( \frac{z}{w} \right)^k + \left( \frac{\bar{z}}{\bar{w}} \right)^k \right] \]  
(A.3)

Inserting this ansatz into the boundary conditions (A.2), expanding it by series and following the procedure as in [24, 26], we get

\[ G_B(z,w) \]
\[ = -\ln |z-w|^2 + \ln |z|^2 \ln |w|^2 + \frac{2 - y \ln a^2}{y} 
- \sum_{n=1}^{\infty} \ln \left[ 1 - (-1)^n \left( \frac{a}{b} \right)^{2n} \frac{z\bar{w}}{a^2} \right] \cdot 1 - (-1)^n \left( \frac{a}{b} \right)^{2n} \frac{b^2}{z\bar{w}} \]  

\[ 13 \]
with the boundary conditions

\begin{align*}
\bar{\partial} G_F(z, w) &= -i\sqrt{zw} \delta^{(2)}(z - w), \\
\partial \tilde{G}_F(\tilde{z}, \tilde{w}) &= +i\sqrt{\tilde{z}\tilde{w}} \delta^{(2)}(\tilde{z} - \tilde{w})
\end{align*}

(A.5)

with the boundary conditions

\begin{align*}
\left(1 - i \frac{y}{\partial_y} \right) G_F \bigg|_{\rho = a} &= \left(1 + i \frac{y}{\partial_y} \right) \tilde{G}_F \bigg|_{\rho = a}, \\
G_F(z, w) \big|_{\rho = b} &= \tilde{G}_F(-\tilde{z}, \tilde{w}) \big|_{\rho = b}.
\end{align*}

(A.6)

we start with the ansatz,

\begin{align*}
iG_F(z, w) &= \frac{\sqrt{zw}}{z - w} + \sum_{r \in \mathbb{Z} + \frac{1}{2}} a_r(z\bar{w})^r + \sum_{r \in \mathbb{Z} + \frac{1}{2}} a'_r\left(\frac{z}{w}\right)^r, \\
i\tilde{G}_F(\tilde{z}, \tilde{w}) &= -\frac{\sqrt{\tilde{z}\tilde{w}}}{\tilde{z} - \tilde{w}} - \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r(\tilde{z}\bar{w})^r - \sum_{r \in \mathbb{Z} + \frac{1}{2}} b'_r\left(\frac{\tilde{z}}{\tilde{w}}\right)^r.
\end{align*}

(A.7)

After a straightforward, albeit lengthy, calculation we get the following results for $G_F(z, w)$ and $\tilde{G}_F(\tilde{z}, \tilde{w})$

\begin{align*}
iG_F(z, w) &= \frac{\sqrt{zw}}{z - w}
\end{align*}
\[-\sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{a^2}{b^2}} \cdot \sqrt{(-1)^n \left( \frac{a}{b} \right)^{2n} \frac{z w}{a^2}} + \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{a^2}{b^2}} \cdot \sqrt{(-1)^n \left( \frac{a}{b} \right)^{2n} \frac{b^2}{z w}} \]

\[-\sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{z}{w}} + \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{z}{w}} \]

\[+ 2 \sum_{r \geq 1} \frac{y a^4 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{z w}{a^2} \right)^r \]

\[\frac{y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{b^2}{z w} \right)^r \]

\[+ 2 \sum_{r \geq 1} \frac{(-1)^r y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{z w}{a^2} \right)^r \]

\[\frac{(-1)^r y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{w}{z} \right)^r, \quad (A.8)\]

and

\[i \tilde{G}_F(z, w) = -\frac{\sqrt{z w}}{z - w} \]

\[+ \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{a^2}{b^2}} \cdot \sqrt{(-1)^n \left( \frac{a}{b} \right)^{2n} \frac{z w}{a^2}} - \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{a^2}{b^2}} \cdot \sqrt{(-1)^n \left( \frac{a}{b} \right)^{2n} \frac{b^2}{z w}} \]

\[+ \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{z}{w}} - \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n \frac{z}{w}} \]

\[+ 2 \sum_{r \geq 1} \frac{y a^4 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{z w}{a^2} \right)^r \]

\[\frac{y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{b^2}{z w} \right)^r \]

\[+ 2 \sum_{r \geq 1} \frac{(-1)^r y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{z w}{a^2} \right)^r \]

\[\frac{(-1)^r y a^2 r b^2 r}{(b^2 r - (-1)^r a^2 r)[(r + y)b^2 r - (r - y)(-1)^r a^2 r]} \left( \frac{w}{z} \right)^r \]

15
\[ +2 \sum_{r \geq \frac{1}{2}} \frac{(-1)^r y a^{2r} b^{2r}}{(b^{2r} - (-1)^r a^{2r})[(r + y)b^{2r} - (r - y)(-1)^r a^{2r}]} \left( \frac{\bar{w}}{z} \right)^r. \tag{A.9} \]

### A.2 the Partition function

To evaluate the partition function, we need to calculate the propagators on the boundary, for both the bosonic and the fermionic fields. At the boundary \( \rho = a \), let \( z = a e^{i\theta} \) and \( w = a e^{i(\theta + \epsilon)} \), we have

\[
\lim_{\epsilon \to 0} G_B(y, \epsilon; a, b) \\
\equiv \lim_{\epsilon \to 0} G_B(\alpha e^{i\theta}, \alpha e^{i(\theta + \epsilon)}) \\
= -2 \ln(1 - e^{i\epsilon}) - 2 \ln(1 - e^{-i\epsilon}) + \frac{2}{y} - 8 \sum_{n=1}^{\infty} \ln \left[ 1 - \left( -1 \right)^n \left( \frac{a}{b} \right)^{2n^2} \right] \\
-4 \sum_{k=1}^{\infty} \frac{y a^{4k}}{k(b^{2k} - (-1)^k a^{2k})[(k + y)b^{2k} - (k - y)(-1)^k a^{2k}]} \\
-4 \sum_{k=1}^{\infty} \frac{y b^{4k}}{k(b^{2k} - (-1)^k a^{2k})[(k + y)b^{2k} - (k - y)(-1)^k a^{2k}]} \\
-8 \sum_{k=1}^{\infty} \frac{(-1)^k y a^{2k} b^{2k}}{k(b^{2k} - (-1)^k a^{2k})[(k + y)b^{2k} - (k - y)(-1)^k a^{2k}]} \\
\equiv -2 \ln(1 - e^{i\epsilon}) - 2 \ln(1 - e^{-i\epsilon}) + F(y, -a^2/b^2), \tag{A.10} \]

where \( F(y, -a^2/b^2) \equiv F(y, \epsilon = 0, -a^2/b^2) \) are the terms which are not singular when we take the limit \( \epsilon = 0 \).

We define

\[
G'_F(\epsilon, y) \equiv G'_F(\alpha e^{+i\theta}, \alpha e^{+i(\theta + \epsilon)}) = \langle \psi(\theta) \frac{1}{\partial \theta} \psi(\theta + \epsilon) \rangle, \\
\tilde{G}'_F(\epsilon, y) \equiv \tilde{G}'_F(\alpha e^{-i\theta}, \alpha e^{-i(\theta + \epsilon)}) = \langle \tilde{\psi}(\theta) \frac{1}{\partial \theta} \tilde{\psi}(\theta + \epsilon) \rangle. \tag{A.11} \]

We expand \( G'_F(\epsilon, y) \) and \( \tilde{G}'_F(\epsilon, y) \), then compare them to the expansion of \( G_B(y, \epsilon, a^2/b^2) \).

By a straightforward calculation it can be shown that

\[
G'_F(\epsilon, y) + \tilde{G}'_F(\epsilon, y) \\
= G_B(y, \epsilon, -a^2/b^2) - 2G_B(2y, \frac{\epsilon}{2}, \sqrt{-a^2/b^2}). \tag{A.12} \]
Now we are ready to calculate the partition function at the boundary \( \rho = a \).

\[
\frac{d}{dy} \ln Z_M = -\frac{1}{8\pi} \int_0^{2\pi} d\theta (X^2 + \psi \frac{1}{\partial_\theta} \psi + \bar{\psi} \frac{1}{\partial_\theta} \bar{\psi}) \\
\equiv \lim_{\epsilon \to 0} -\frac{1}{8\pi} \int_0^{2\pi} d\theta (X(\theta)X(\theta + \epsilon) + \psi(\theta) \frac{1}{\partial_\theta} \psi(\theta + \epsilon) + \bar{\psi}(\theta) \frac{1}{\partial_\theta} \bar{\psi}(\theta + \epsilon)) \\
= \lim_{\epsilon \to 0} [G_B(y_a, y_b, \epsilon, -a^2/b^2) + G'_B(\epsilon, y) + \bar{G}'_B(\epsilon, y)] \\
= 2 \lim_{\epsilon \to 0} [G_B(y, \epsilon, -a^2/b^2) - G_B(2y, \epsilon, \sqrt{-a^2/b^2})] \\
= -8 \ln 2 + 2F(y, -a^2/b^2) - F(2y, \sqrt{-a^2/b^2}). \quad (A.13)
\]

Integrating over \( y \), up to an integration constant, we get

\[
\ln Z_M(a/b) = (2 \ln 2)y - \frac{1}{2} \ln y \\
+ \sum_{k=1}^{\infty} \left\{ \ln \left[ \left( 1 + \frac{2y_a}{k} \right) \left( 1 + \frac{2y_b}{k} \right) - \left( 1 - \frac{2y_a}{k} \right) \left( 1 - \frac{2y_b}{k} \right) \left( \sqrt{-\frac{a^2}{b^2}} \right)^k \right] \\
- 2 \ln \left[ \left( 1 + \frac{y_a}{k} \right) \left( 1 + \frac{y_b}{k} \right) - \left( 1 - \frac{y_a}{k} \right) \left( 1 - \frac{y_b}{k} \right) \left( \frac{a^2}{b^2} \right)^k \right] \right\}. \quad (A.14)
\]

Comparing this Green function to the one on an annulus, we find that the only differences consist of replacing \( \left( \frac{a^2}{b^2} \right)^k \) by \( \left( -\frac{a^2}{b^2} \right)^k \) and setting \( y_b = 0 \).

The partition function, then, can be obtained as

\[
Z_M(a/b) = Z' 4^y Z_B^2(y, -a^2/b^2) \frac{Z_B(2y, \sqrt{-a^2/b^2})}{Z_B(y, -a^2/b^2)}, \quad (A.15)
\]

where \( Z' \) is the integration constant, and

\[
Z_B(y; a, b) = \frac{1}{\sqrt{y}} \prod_{k=1}^{\infty} \left[ \left( 1 + \frac{y}{k} \right) - \left( 1 - \frac{y}{k} \right) \left( \frac{a^2}{b^2} \right)^k \right]^{-1} \quad (A.16)
\]

is the bosonic partition function on the Möbius strip.
A.3 Chan-Paton Factor

Including the Chan-Paton factors, the boundary action can be generalized to be

\[ e^{-S_{\text{bndy}}} = \text{Tr} P \exp \left\{ -\frac{1}{8\pi} \int d\theta \left[ T^2(X) + (\psi^\mu \partial_\mu T) \frac{1}{\partial_\theta} (\psi^\nu \partial_\nu T) \right] \right\}. \quad (A.17) \]

Substituting the tachyon profile

\[ T(X) = \sum_{i=0}^{p} \sum_{\alpha=1}^{2n} u_i^\alpha X^i \gamma^\alpha \quad (A.18) \]

in to (A.17) and using the symmetry of \( X_i^\alpha X_j^\beta \) and the Clifford relations \( \{ \gamma^\alpha, \gamma^\beta \} = 2\delta^\alpha\beta \), we find that the boundary terms are proportional to the identity matrix

\[ e^{-S_{\text{bndy}}} = \text{Tr} P \exp \left\{ -\frac{y_{ij}}{8\pi} \int d\theta \left[ X_i X_j + \psi_i \frac{1}{\partial_\theta} \psi_j \right] \cdot 1_{2^n \times 2^n} \right\} = 2^n \exp \left\{ -\frac{y_{ij}}{8\pi} \int d\theta \left[ X_i X_j + \psi_i \frac{1}{\partial_\theta} \psi_j \right] \right\}, \quad (A.19) \]

where \( y_{ij} \equiv \sum_{\alpha=1}^{2n} u_i^\alpha u_j^\alpha \).

Thus the boundary action decouples to the abelian ones. It is easy to obtain the partition function as before.

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