THE BEST BOUNDS OF HARMONIC SEQUENCE

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Abstract. For any natural number $n \in \mathbb{N}$,

$$\frac{1}{2n + \frac{1}{2} - \gamma} - 2 \leq \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{3}{2}}, \quad (1)$$

where $\gamma = 0.57721566490153286 \cdots$ denotes Euler’s constant. The constants $\frac{1}{2n} - 2$ and $\frac{1}{2}$ are the best possible.

As by-products, two double inequalities of the digamma and trigamma functions are established.

1. Introduction

Let $n$ be a natural number, then we have

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n}, \quad (2)$$

where $\gamma = 0.57721566 \cdots$ is Euler’s constant.

The inequality (2) is called in literature Franel’s inequality [5, Ex. 18]. Because of the well known importance of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$, there exists a very rich literature on inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$. For example, [1, 3], [4, pp. 68–78] and references therein.

L. Tóth and S. Mare in [6, p. 264] proposed the following problems:

1. Prove that for every positive integer $n$ we have

$$\frac{1}{2n + \frac{1}{2}} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n + \frac{3}{2}}, \quad (3)$$

where $\gamma$ is Euler’s constant.

2. Show that $\frac{2}{3}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In [10], basing on improving of Euler-Maclaurin summation formula, some general inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$ are established, including recovery of inequality (3). In 1999, Sh.-R. Wei and B.-Ch. Yang in [8] verified inequality (3) again by calculus.

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In 1997, K. Wu and B.-Ch. Yang in [9] proved the second problem due to Tóth and Mare by using the following

\[
\sum_{i=1}^{n} \frac{1}{i} = \gamma + \ln n + \frac{1}{2n} - \sum_{i=1}^{q} \frac{B_{2i}}{2in^{2i}} - \int_{n}^{\infty} \frac{B_{2q}(x)}{x^{2q}} \, dx
\]

(4)

and

\[
\int_{n}^{\infty} \frac{B_{2q-1}(x)}{x^{2q}} \, dx < \frac{(-1)^{q}B_{2q}}{2q^{3}n^{2}}.
\]

(5)

where \(B_{i}(x)\) is Bernoulli’s polynomial, \(B_{2i} = B_{2i}(n)\) is Bernoulli’s number for \(i \in \mathbb{N}\), and \(n\) and \(q\) are positive integers.

In this short note, through establishing, by exploiting the well know first Binet’s formula, two double inequalities of the digamma and trigamma functions and by utilizing the approximating expansion of digamma function \(\psi\), the best lower and upper bounds of the sequence \(\sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma\) are given. These results refine inequality (3) and solve affirmatively the second problem mentioned above.

**Theorem 1.** For any natural number \(n \in \mathbb{N}\), we have

\[
\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}},
\]

(6)

where \(\gamma = 0.57721566490153286 \ldots\) denotes Euler’s constant. The constants \(\frac{1}{1-\gamma} - 2\) and \(\frac{1}{3}\) are the best possible.

2. **Lemma**

In order to prove inequality (3), the following lemma is necessary.

**Lemma 1.** For \(x > 0\), we have

\[
\frac{1}{2x} - \frac{1}{12x^{2}} < \psi(x + 1) - \ln x < \frac{1}{2x}
\]

(7)

and

\[
\frac{1}{2x^{2}} - \frac{1}{6x^{3}} < \frac{1}{x} - \psi'(x + 1) < \frac{1}{2x^{2}} - \frac{1}{6x^{3}} + \frac{1}{30x^{5}};
\]

(8)

where \(\psi = \Gamma'\) is the logarithmic derivative of the gamma function

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t} \, dt.
\]

(9)

**Proof.** It is a well known fact ([1] and [7, p. 103]) that for \(x > 0\) and a nonnegative integer \(m\),

\[
\psi(x + 1) = \psi(x) + \frac{1}{x}
\]

(10)

and

\[
\frac{m!}{x^{m+1}} = \int_{0}^{\infty} t^{m}e^{-xt} \, dt.
\]

(11)

The first Binet’s formula ([1] and [7, p. 106]) states that for \(x > 0\)

\[
\ln \Gamma(x) = \left( x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} - \int_{0}^{\infty} \left( \frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}} \right) \frac{e^{-xt}}{t} \, dt.
\]

(12)
Differentiating (12), integrating by part and using formulas (11) and (10), it is deduced that

\[ \psi(x + 1) - \ln x = \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} \, dt. \quad (13) \]

Using formulas (11) and (13) and the series expansion of \( e^x \) at \( x = 0 \) yields

\[
\begin{align*}
\psi(x + 1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} &= \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{1}{12} t \right) e^{-xt} \, dt \\
&= \int_0^\infty \frac{12(e^t - 1) - 12t - 6t(e^t - 1) + t^2(e^t - 1)}{12t(e^t - 1)} e^{-xt} \, dt \\
&= \int_0^\infty \left[ \frac{1}{12t(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n \right] e^{-xt} \, dt \\
&> 0
\end{align*}
\]

and

\[
\begin{align*}
\psi(x + 1) - \ln x - \frac{1}{2x} &= \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} \right) e^{-xt} \, dt \\
&= - \int_0^\infty \left[ \frac{1}{2t(e^t - 1)} \sum_{n=3}^\infty \frac{n-2}{n!} t^n \right] e^{-xt} \, dt \quad (15) \\
&< 0.
\end{align*}
\]

Hence, inequality (7) follows.

Differentiation of (13) immediately produces

\[ \frac{1}{x} - \psi'(x + 1) = \int_0^\infty \left( 1 - \frac{t}{e^t - 1} \right) e^{-xt} \, dt. \quad (16) \]

Exploiting formulas (11) and (16) and the series expansion of \( e^x \) at \( x = 0 \) yields

\[
\begin{align*}
\frac{1}{x} - \psi'(x + 1) - \frac{1}{2x^2} + \frac{1}{6x^3} &= \int_0^\infty \left( 1 - \frac{t}{e^t - 1} - \frac{1}{2} t + \frac{1}{12} t^2 \right) e^{-xt} \, dt \\
&= \int_0^\infty \left[ \frac{1}{12(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n \right] e^{-xt} \, dt \\
&> 0
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{x} - \psi'(x + 1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} &= \int_0^\infty \left( 1 - \frac{t}{e^t - 1} - \frac{1}{2} t + \frac{1}{12} t^2 - \frac{1}{720} t^4 \right) e^{-xt} \, dt \\
&= \int_0^\infty \left[ \frac{1}{120(e^t - 1)} \sum_{n=7}^\infty \frac{720}{n!} \left( \frac{360}{(n-1)!} t + \frac{60}{(n-2)!} t - \frac{1}{(n-4)!} t^n \right) \right] e^{-xt} \, dt. \quad (18)
\end{align*}
\]
Noticing that for \( n \geq 7, \)
\[
\frac{720}{n!} - \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} = -\frac{120 + 218(n-7) + 119(n-7)^2 + 22(n-7)^3 + (n-7)^4}{n!} < 0,
\]
we obtain
\[
\frac{1}{x} - \psi'(x + 1) + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < 0.
\] (20)
Therefore, inequality (8) holds. The proof is complete. □

3. Proof of Theorem 1

In [1], [2, p. 593] and [7, p. 104] it is given that \( \psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma. \) Thus, inequality (6) can be rearranged as
\[
\frac{1}{3} < \frac{1}{\psi(n + 1) - \ln n} - 2n \leq \frac{1}{1 - \gamma} - 2.
\] (21)
Define for \( x > 0 \)
\[
\phi(x) = \frac{1}{\psi(x + 1) - \ln x} - 2x.
\] (22)
Differentiating \( \phi \) and utilizing (7) and (8) reveals that for \( x > \frac{12}{5}, \)
\[
(\psi(x + 1) - \ln x)^2 \phi'(x)
= \frac{1}{x} - \psi'(x + 1) - 2(\psi(x + 1) - \ln x)^2
< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2 \left( \frac{1}{2x} - \frac{1}{12x^2} \right)^2
= \frac{12 - 5x}{360x^5} < 0,
\] (23)
and \( \phi(x) \) decreases with \( x > \frac{12}{5}. \)
Straightforward calculation produces
\[
\phi(1) = \frac{1}{1 - \gamma} - 2 = 0.36527211862544155 \cdots,
\] (24)
\[
\phi(2) = \frac{1}{\frac{2}{\gamma} - \ln 2} - 4 = 0.35469600731465752 \cdots,
\] (25)
\[
\phi(3) = \frac{1}{\frac{6}{\gamma} - \ln 3} - 6 = 0.34898948531361115 \cdots.
\] (26)
Therefore, the sequence
\[
\phi(n) = \frac{1}{\psi(n + 1) - \ln n} - 2n, \quad n \in \mathbb{N}
\] (27)
is decreasing strictly, and for \( n \in \mathbb{N} \)
\[
\lim_{n \to \infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1 - \gamma} - 2.
\] (28)
Making use of approximating expansion of \( \psi \) in [1], [2, p. 594], or [7, p. 108] gives
\[
\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \to \infty),
\] (29)
and then

$$\lim_{n \to \infty} \phi(n) = \lim_{x \to \infty} \phi(x) = \lim_{x \to \infty} \frac{1}{1 + O(x^{-1})} = \frac{1}{3}. \quad (30)$$

The proof is complete.

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