Dynamical Domain Wall and Localization

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Based on

Y. Toyozato, K. Bamba and S. Nojiri, “Scalar Domain Wall as the Universe”, Phys. Rev. D 87 (2013) 6, 063008 [arXiv:1202.5375 [hep-th]].

M. Higuchi and S. Nojiri, “Reconstruction of Domain Wall Universe and Localization of Gravity”, Gen. Rel. Grav. 46 (2014) 11, 1822 [arXiv:1402.1346 [hep-th]].

Y. Toyozato, M. Higuchi and S. Nojiri, “Dynamical Domain Wall and Localization”, arXiv:1510.01099.

Construction of the dynamical domain wall universe, where the four dimensional FRW universe is realized on the domain wall in the five dimensional space-time.

Localization of the fields (graviton, chiral spinor, vector, ...)

– Typeset by FoilTEX –
Introduction

Our universe could be a brane or domain wall embedded in a higher dimensional space-time.

K. Akama, “An Early Proposal of ’Brane World’,” Lect. Notes Phys. 176 (1982) 267 [hep-th/0001113].

V. A. Rubakov and M. E. Shaposhnikov, “Do We Live Inside a Domain Wall?,” Phys. Lett. B 125 (1983) 136.

$D$-brane in string theories.

J. Dai, R. G. Leigh and J. Polchinski, “New Connections Between String Theories,” Mod. Phys. Lett. A 4 (1989) 2073.

J. Polchinski, “Tasi lectures on D-branes,” hep-th/9611050.
Brane world scenario.

L. Randall and R. Sundrum, “A Large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. 83, 3370 (1999) [hep-ph/9905221].

L. Randall, R. Sundrum, “An Alternative to compactification,” Phys. Rev. Lett. 83, 4690-4693 (1999). [hep-th/9906064].

Dynamics of brane?
Brane: a limit where the thickness of domain wall vanishes
⇒ Dynamics of domain wall
Domain wall model with two scalar fields

General FRW in the five dimensional space-time, the metric is given by

\[ ds^2 = dw^2 + L^2 e^{u(w,t)} ds^2_{\text{FRW}}. \]

\[ ds^2_{\text{FRW}} = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\}. \]

Action with two scalar fields \( \phi \) and \( \chi \):

\[ S_{\phi\chi} = \int d^5 x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - \frac{1}{2} A(\phi, \chi) \partial_\mu \phi \partial^\mu \phi - B(\phi, \chi) \partial_\mu \phi \partial^\mu \chi \right. \]
\[ \left. - \frac{1}{2} C(\phi, \chi) \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi) \right\}. \]
Energy-momentum tensor

\[ T_{\mu\nu}^{\phi\chi} = g_{\mu\nu} \left\{ -\frac{1}{2} A(\phi, \chi) \partial_\rho \phi \partial^\rho \phi - B(\phi, \chi) \partial_\rho \phi \partial^\rho \chi - \frac{1}{2} C(\phi, \chi) \partial_\rho \chi \partial^\rho \chi - V(\phi, \chi) \right\} \\
+ A(\phi, \chi) \partial_\mu \phi \partial_\nu \phi + B(\phi, \chi) \left( \partial_\mu \phi \partial_\nu \chi + \partial_\nu \phi \partial_\mu \chi \right) + C(\phi, \chi) \partial_\mu \chi \partial_\nu \chi. \]

Field equations read

\[ 0 = \frac{1}{2} A_\phi \partial_\mu \phi \partial^\mu \phi + A \nabla^\mu \partial_\mu \phi + A_\chi \partial_\mu \phi \partial^\mu \chi + \left( B_\chi - \frac{1}{2} C_\phi \right) \partial_\mu \chi \partial^\mu \chi \\
+ B \nabla^\mu \partial_\mu \chi - V_\phi, \]

\[ 0 = \left( -\frac{1}{2} A_\chi + B_\phi \right) \partial_\mu \phi \partial^\mu \phi + B \nabla^\mu \partial_\mu \phi + \frac{1}{2} C_\chi \partial_\mu \chi \partial^\mu \chi + C \nabla^\mu \partial_\mu \chi \\
+ C_\phi \partial_\mu \phi \partial^\mu \chi - V_\chi. \]

\[ A_\phi = \partial A(\phi, \chi) / \partial \phi, \text{ etc.} \]

We now choose \( \phi = t \) and \( \chi = w. \)
We can construct a model to realize the arbitrary metric in the form,
\[ ds^2 = dw^2 + L^2 e^{u(w,t)} ds^2_{\text{FRW}}, \]
\[ ds^2_{\text{FRW}} = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1-k r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right\}, \]
by choosing \((H \equiv (1/a) (da/dt))\)

\[ A = \frac{1}{\kappa^2} \left( \frac{2k}{a^2} - \ddot{u} - 2\dot{H} + \frac{(\dot{u})^2}{2} + \dot{u}H \right), \]

\[ B = -\frac{3u'}{2\kappa^2 L^2 e^u} (\dot{u} + 2H), \]

\[ C = \frac{1}{\kappa^2} \left( -\frac{3}{2} u'' + \frac{2k}{L^2 e^u a^2} - \frac{1}{2e^u} \left( \dddot{u} + 2\dot{H} + (\dot{u})^2 + 5\dot{u}H + 6H^2 \right) \right), \]

\[ V = \frac{1}{\kappa^2} \left( -\frac{3}{4} \left( u'' + 2 \left( u' \right)^2 \right) + \frac{3k}{L^2 e^u a^2} + \frac{1}{4L^2 e^u} \left( 3\dddot{u} + 6\dot{H} + 3 (\dot{u})^2 + 15\dot{u} + 18H^2 \right) \right). \]

Einstein equation is satisfied.
Field equations are nothing but the Bianchi identities.
If any eigenvalue of the matrix \( \begin{pmatrix} A & B \\ B & C \end{pmatrix} \) is negative, there appears a ghost.

**Example without ghost**

\[
a(t) \propto t^{h_0}, \quad e^{u(w,t)} = W(w) T(t),
\]

\[
T(t) = T_1 t^{1-3h_0} + T_2 t^{-2h_0}, \quad W(w) = e^{-\sqrt{\frac{1+w^2}{w_0^2}}},
\]

\[
\Rightarrow A = \frac{3}{2\kappa^2} \left( \frac{T(t)}{T(t)} + \frac{2h_0}{t} \right)^2 > 0, \quad B = 0, \quad C = \frac{3}{2\kappa^2} \left( \frac{\frac{1}{w_0^2}}{\left(1 + \frac{w^2}{w_0^2}\right)^{\frac{3}{2}}} \right) > 0.
\]

\[\cdots \textbf{No ghost.}\]

General FRW universe can be realized by the Brans-Dicke type model.
Localization of graviton

**Perturbation** $g_{\mu \nu} \rightarrow g_{\mu \nu} + h_{\mu \nu}$.

**Assume** $h_{\mu \nu} = 0$ if $\mu$ or $\nu \neq 1, 2, 3$.

**Einstein eq.** $\Rightarrow$ equation for graviton in five dimensions:

$$0 = \left[ \partial_w^2 - u'' - u'^2 + L^{-2} e^{-u} \left( \ddot{u} + \frac{\dot{a} \dot{u}}{a} + \dot{u} \partial_t + 2\frac{\dot{a}}{a} + \frac{\dot{a}}{a} \partial_t - \partial_t^2 + \frac{\Delta}{a^2} \right) \right] h_{ij}.$$

**By assuming** $h_{ij}(w, x) = e^{u(w,t)} \hat{h}_{ij}(x)$,

$$0 = \left( 2\frac{\dot{a} \dot{u}}{a} - \dot{u} \partial_t + 2\frac{\dot{a}}{a} + \frac{\dot{a}}{a} \partial_t - \partial_t^2 + \frac{\Delta}{a^2} \right) \hat{h}_{ij}.$$

If $u$ goes to minus infinity sufficiently rapidly for large $|w|$, $\hat{h}_{ij}(w, x)$ is normalized in the direction of $w$ and therefore there occurs the localization of graviton.
Graviton localized on the domain wall

\[ 0 = \left( 2 \frac{\dot{a} \ddot{u}}{a} - \dot{u} \partial_t + 2 \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) \hat{h}_{ij}. \]

In the four dimensional FRW space-time,

\[ 0 = \left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) h_{ij}. \]

If \( \dot{u} \left( 2 \frac{\dot{a}}{a} - \partial_t \right) \hat{h}_{ij} = 0 \), two expressions coincide with each other.
If \( \dot{u} \neq 0 \), there could appear some corrections proportional to \( \dot{u} \).
Localization of chiral spinor

There was an attempt to localize the fermion on the domain wall has been failed.
D. P. George, M. Trodden and R. R. Volkas, “Extra-dimensional cosmology with domain-wall branes,” JHEP 0902 (2009) 035 [arXiv:0810.3746 [hep-ph]].

Dirac equation in five dimensions

\[ \Gamma^M \nabla_M \Psi + \tilde{f}_\chi (w) \Psi = 0. \]

\( \tilde{f}_\chi \): a function of the scalar field \( \chi = w \), general yukawa interaction.
\( \nabla_M \): covariant derivative:

\[ \nabla_M \overset{\text{def}}{=} \partial_M + \frac{1}{4} \omega_{ABM} \Gamma^{AB}, \quad \Gamma^{AB} \overset{\text{def}}{=} \frac{1}{2} [\Gamma^A, \Gamma^B]. \]
\[ \Psi = \eta(t, w) \psi(x) \quad (\psi: \text{four dimensional Dirac spinor}) \]

\[
\Gamma^M \nabla_M \Psi + \tilde{f} \chi(w) \Psi \\
= L^{-1} e^{-u/2} \left\{ \Gamma^0 \partial_\eta + \frac{1}{a(t)} \Gamma^i \partial_i \psi \right\} \eta + \{ \partial_5 \eta + u'(t, w) \eta \} \Gamma^5 \psi \\
+ \tilde{f} \chi(w) \eta \psi + L^{-1} e^{-u/2} \left\{ \partial_\eta + \frac{3}{2} \left( \frac{1}{2} \dot{u} + \frac{\dot{a}}{a} \right) \eta \right\} \Gamma^\eta \psi.
\]

**Let \( \tilde{C}_1(w) \) and \( D_1(t) \) be arbitrary functions.**

\[
\eta(t, w) \overset{\text{def}}{=} \zeta(t) \lambda(w) g(t, w) \\
g(t, w) \overset{\text{def}}{=} \exp \left[ \tilde{C}_1(w) + D_1(t) - u(t, w) \right],
\]
By assuming $\Gamma^5 \psi = \pm \psi$,

$$\Rightarrow \Gamma^M \nabla_M \Psi + \tilde{f} \chi (w) \Psi$$

$$= L^{-1} e^{-u/2} \left\{ \Gamma^0 \partial_0 \psi + \frac{1}{a(t)} \Gamma^\imath \partial_\imath \psi \right\} \eta$$

$$\pm \zeta g \left\{ \partial_5 \lambda (w) + \tilde{C}_1' (w) \lambda (w) \pm \tilde{f} \chi (w) \lambda (w) \right\} \psi$$

$$+ L^{-1} e^{-u/2} \lambda g \left\{ \partial_0 \zeta (t) + \left( \frac{3 \dot{a}}{2a} - \frac{1}{4} \dot{u} (t, w) + \dot{D}_1 (t) \right) \zeta (t) \right\} \Gamma^0 \psi .$$

$e^{u(t,w)} = T(t) W(w)$ and using the Dirac equation,

$$\lambda (w) = C_3 \exp \left[ -\tilde{C}_1 (w) \mp \tilde{f} \int dw \chi (w) \right]$$

$$\zeta (t) = C_4 a^{-3/2} T^{1/4} \exp \left[ -D_1 (t) \right] .$$
\[ \eta(t, w) = C_3 C_4 a^{-3/2} T^{1/4} \exp \left[ -u(t, w) \mp \tilde{f} \int d\omega \chi(w) \right] \]
\[ = C_5 a^{-3/2} T^{-3/4} W^{-1} \exp \left[ \mp \tilde{f} \int d\omega \chi(w) \right], \]
\[
\chi(w) = w \Rightarrow \eta(t, w) = C_5 a^{-3/2} T^{-3/4} W^{-1} \exp \left[ \mp \frac{\tilde{f}}{2} w^2 \right].
\]

**Condition of localization**

\[ I = \int_{-\infty}^{+\infty} d\omega e^{3u/2} |\eta|^2 = C_5^2 a^{-3} \int_{-\infty}^{+\infty} d\omega W^{-1/2} \exp \left[ \mp \tilde{f} w^2 \right] < \infty. \]
Fermion can be localized on the domain wall and the localized fermion can be chiral or anti-chiral.

In

D. P. George, M. Trodden and R. R. Volkas, “Extra-dimensional cosmology with domain-wall branes,” JHEP 0902 (2009) 035 [arXiv:0810.3746 [hep-ph]]

It was assumed that the warp factor does not depend on the time but we have used the time-dependent warp factor.
Localization of vector field

Action of vector field

\[ S_V = \int d^5 x \sqrt{-g} \left\{ -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} m(\chi)^2 A_M A^M \right\}, \]

\[ F_{MN} = \partial_M A_N - \partial_N A_M. \]

Background

\[ ds^2 = dw^2 + L^2 W(w) T(t) ds^2_{\text{FRW}}, \]

\[ ds^2_{\text{FRW}} = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2. \]
\[ S_V = \int d^5 x \left\{ \frac{1}{2} L^2 W(w) T(t) a(t)^3 F_{50}^2 - \frac{1}{2} L^2 W(w) T(t) a(t) F_{5i}^2 ight. \\
+ \frac{1}{2} a(t) F_{0i}^2 - \frac{1}{4} a(t)^{-1} F_{ij}^2 \\
- \frac{1}{2} m(\chi)^2 \left( L^4 W(w)^2 T(t)^2 a(t)^3 A_5^2 \right) \\
- L^2 W(w) T(t) a(t)^3 A^2_0 + L^2 W(w) T(t) a(t) A_i^2 \right\}. \]
Variation of $A_5$, $A_0$, and $A_i$ ⇒

\[ 0 = L^2 W(w) \partial_0 (T(t) a(t)^3 (\partial_5 A_0 - \partial_0 A_5)) \]
\[- L^2 W(w) T(t) a(t) (\partial_5 \partial_i A_i - \partial_i^2 A_5) \]
\[- m(\chi)^2 L^4 W(w)^2 T(t)^2 a(t)^3 A_5 , \]

\[ 0 = - L^2 T(t) a(t)^3 \partial_5 (W(w) (\partial_5 A_0 - \partial_0 A_5)) \]
\[ + a(t) (\partial_0 \partial_i A_i - \partial_i^2 A_0) + m(\chi)^2 L^2 W(w) T(t) a(t)^3 A_0 , \]

\[ 0 = L^2 T(t) a(t) \partial_w (W(w) (\partial_5 A_i - \partial_i A_5)) - \partial_0 (a (\partial_0 A_i - \partial_i A_0)) \]
\[- a(t)^{-1} (\partial_i \partial_j A_j - \partial_j^2 A_i) - m(\chi)^2 L^2 W(w) T(t) a(t) A_i . \]
Assuming

\[ A_5 = 0, \quad A_\mu = X(w)C_\mu(x^\nu), \quad \mu, \nu = 0, 1, 2, 3, \]

and choosing

\[ m(\chi = w)^2 = \frac{(W(w)X'(w))'}{W(w)X(w)}, \]

\[ \Rightarrow 0 = \partial_5 X(w) \left\{ \partial_0 \left( T(t)a(t)^3 C_0 \right) - T(t)a(t)\partial_i C_i \right\}, \]

\[ 0 = \partial_0 \partial_i C_i - \partial_i^2 C_0, \]

\[ 0 = \partial_0 \left( a(t) \left( \partial_0 C_i - \partial_i C_0 \right) \right) + a(t)^{-1} \left( \partial_i \partial_j C_j - \partial_j^2 C_i \right). \]

Last two eqs. : nothing but the field equations of the vector field in four dimensions.
First eq. : a gauge condition, which is a generalization of the Landau gauge, \( \partial^\mu A_\mu = 0. \)
If $X(w) \to 0$ sufficiently when $|w| \to \infty$, $A_\mu$: normalizable.
⇒ Localization of vector field on the domain wall.

In case

$$(W(w)X'(w))' = 0 \Rightarrow m(\chi) = 0.$$ 

⇒ non-abelian gauge theory?
(In general, non-normalizable)
Summary

- We have constructed the dynamical domain wall, where arbitrary four dimensional FRW universe is realized on the domain wall in the five dimensional space-time.

- Graviton, Chiral Spinor, and Vector can localize on the brane.
Graviton in FRW Universe in Four Dimensions

**Perturbation** $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu} \Rightarrow$

$$
\delta R_{\mu\nu} = \frac{1}{2} \left[ \nabla_\mu \nabla^\rho h_{\nu\rho} + \nabla_\nu \nabla^\rho h_{\mu\rho} - \nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu ( g^{\rho\lambda} h_{\rho\lambda} ) \\
-2 R^\lambda _\nu R^\rho _\mu h_{\lambda\rho} + R^\rho _\mu h_{\mu\nu} + R^\rho _\nu h_{\rho\mu} \right],
$$

$$
\delta R = - h_{\mu\nu} R^{\mu\nu} + \nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 (g^{\mu\nu} h_{\mu\nu}).
$$

**Imposing the gauge condition** $\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0$

**Einstein equation** $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \Rightarrow$

$$
\frac{1}{2} \left[ -\nabla^2 h_{\mu\nu} - 2 R^\lambda _\nu R^\rho _\mu h_{\lambda\rho} + R^\rho _\mu h_{\mu\nu} + R^\rho _\nu h_{\rho\mu} \\
- h_{\mu\nu} R + g_{\mu\nu} R^{\rho\lambda} h_{\rho\lambda} \right] = \kappa^2 \delta T_{\mu\nu}.
$$
By using the formulation in

S. Nojiri and S. D. Odintsov, “Unifying phantom inflation with late-time acceleration: Scalar phantom-non-phantom transition model and generalized holographic dark energy”, Gen. Rel. Grav. 38 (2006) 1285 [hep-th/0506212].

consider scalar field theory

\[ S_\phi = \int d^4x \sqrt{-g} \mathcal{L}_\phi, \quad \mathcal{L}_\phi = -\frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi), \]

\[ \Rightarrow \quad T_{\mu\nu} = -\omega(\phi) \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L}_\phi, \]

\[ \Rightarrow \quad \delta T_{\mu\nu} = h_{\mu\nu} \mathcal{L}_\phi + \frac{1}{2} g_{\mu\nu} \omega(\phi) \partial^\rho \phi \partial^\lambda \phi h_{\rho\lambda}. \]
Because we are now interested in the graviton, we may assume $h_{\mu\nu} = 0$ except the components with $\mu, \nu = 1, 2, 3$. In the FRW universe ($k = 0$), assume $\phi = t$, FRW equations

$$\frac{3}{\kappa^2} \frac{\dot{a}^2}{a^2} = \frac{\omega}{2} + V, \quad \frac{1}{\kappa^2} \left( 2 - \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{\omega}{2} - V,$$

$$\Rightarrow \quad \omega = \frac{1}{\kappa^2} \left( 2 - \frac{\ddot{a}}{a} + 4 \frac{\dot{a}^2}{a^2} \right), \quad V = -\frac{1}{\kappa^2} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right).$$

$$\Rightarrow \quad 0 = \left( 2 - \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \partial_t - \partial_t^2 + \frac{\Delta}{a^2} \right) h_{ij}. $$
Explicit expressions of connections and curvatures in five dimensions

Metric

\[
g_{AB} = \begin{pmatrix}
-L^2 e^{u(w,t)} & L^2 e^{u(w,t)} \frac{a(t)^2}{1-kr^2} \\
\frac{L^2 e^{u(w,t)} a(t)^2}{1-kr^2} & L^2 e^{u(w,t)} a(t)^2 r^2 \\
L^2 e^{u(w,t)} a(t)^2 r^2 \sin^2 \theta & 1
\end{pmatrix}.
\]

Connections

\[
\Gamma_{tt}^t = \frac{1}{2} \dot{u}, \quad \Gamma_{tt}^w = \frac{1}{2} L^2 e^u u', \quad \Gamma_{tt}^r = \Gamma_{tt}^{\theta} = \Gamma_{tt}^{\phi} = \frac{\dot{a}}{a} + \frac{1}{2} \ddot{u}, \quad \Gamma_{tw}^t = \Gamma_{rw}^r = \Gamma_{tw}^{\theta} = \Gamma_{tw}^{\phi} = \frac{1}{2} u', \\
\Gamma_{ij}^t = -L^2 e^{-u} \left( \frac{\dot{a}}{a} + \frac{1}{2} \ddot{u} \right) g_{ij}, \quad \Gamma_{rr}^r = \frac{kr}{1-kr^2}, \quad \Gamma_{ij}^w = -\frac{1}{2} u' g_{ij}, \quad \Gamma_{rr}^{\theta} = \Gamma_{rr}^{\phi} = \frac{1}{r}, \\
\Gamma_{\theta\theta}^r = -r(1-kr^2), \quad \Gamma_{\phi\theta}^{\phi} = \cot \theta, \quad \Gamma_{\phi\phi}^r = -r(1-kr^2) \sin^2 \theta, \quad \Gamma_{\phi\phi}^{\theta} = -\cos \theta \sin \theta.
\]
Ricci curvatures

\[ R_{tt} = \left[ -\frac{1}{2}u'' - u'r^2 + \frac{3}{2}L^{-2}e^{-u} \left( \ddot{u} + \frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a} \right) \right] g_{tt}, \]

\[ R_{ij} = \left[ -\frac{1}{2}u'' - u'r^2 + \frac{1}{2}L^{-2}e^{-u} \left( \ddot{u} + 5\frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + \dot{u}' + 4\frac{\ddot{a}^2}{a^2} + 2\frac{k}{a^2} \right) \right] g_{ij}, \]

\[ R_{ww} = -2u'' - u'r^2 \]

\[ R_{tw} = -\frac{3}{2}\dot{u}'. \]

Scalar curvature

\[ R = -4u'' - 5u'r^2 + 3L^{-2}e^{-u} \left( \ddot{u} + \frac{1}{2}\ddot{u}^2 + 3\frac{\dot{a}\dot{u}}{a} + 2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} \right). \]
Explicit form of spin connection

Vierbein field \( e^\hat{B} \), \( g_{MN} \equiv e^\hat{A}_M \eta_{\hat{A}\hat{B}} e^\hat{B}_N \)

\[
\Rightarrow e^\hat{A}_M = \begin{bmatrix}
Le^{u/2} & 0 & 0 & 0 & 0 \\
0 & Le^{u/2}a & 0 & 0 & 0 \\
0 & 0 & Le^{u/2}a & 0 & 0 \\
0 & 0 & 0 & Le^{u/2}a & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} .
\]

\[
e_{\hat{A}M} \equiv \eta_{\hat{A}\hat{B}} e^\hat{B}_M = \begin{bmatrix}
-Le^{u/2} & 0 & 0 & 0 & 0 \\
0 & Le^{u/2}a & 0 & 0 & 0 \\
0 & 0 & Le^{u/2}a & 0 & 0 \\
0 & 0 & 0 & Le^{u/2}a & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} ,
\]
$e_{\hat{A}}^M \overset{\text{def}}{=} e_{\hat{A}N}g^{MN}$

\[
\begin{bmatrix}
L^{-1}e^{-u/2} & 0 & 0 & 0 \\
0 & L^{-1}e^{-u/2}a^{-1} & 0 & 0 \\
0 & 0 & L^{-1}e^{-u/2}a^{-1} & 0 \\
0 & 0 & 0 & L^{-1}e^{-u/2}a^{-1}
\end{bmatrix}.
\]

Ricci rotation coefficients and spin connections

\[
\Omega_{ABM} \overset{\text{def}}{=} \left( \partial_A e^\hat{C}_B - \partial_B e^\hat{C}_A \right) e_{\hat{C}M} = -\Omega_{BAM},
\]

\[
\omega_{ABM} \overset{\text{def}}{=} -\frac{1}{2} \left( \Omega_{ABM} - \Omega_{BMA} - \Omega_{MAB} \right) = -\omega_{BAM}.
\]
Non-vanishing components

\[
\begin{align*}
\Omega_{050} &= -\Omega_{500} = \frac{1}{2} u' L^2 e^u \\
\Omega_{0ij} &= -\Omega_{i0j} = \left( \frac{1}{2} \dot{u} a^2 + \dot{a} a \right) L^2 e^u \delta_{ij} , \\
\Omega_{5ij} &= -\Omega_{i5j} = \frac{1}{2} u' L^2 e^u a^2 \delta_{ij} , \\
\omega_{050} &= -\omega_{500} = -\frac{1}{2} u' L^2 e^u , \\
\omega_{0ij} &= -\omega_{i0j} = - \left( \frac{1}{2} \dot{u} a^2 + \dot{a} a \right) L^2 e^u \delta_{ij} , \\
\omega_{5ij} &= -\omega_{i5j} = -\frac{1}{2} u' L^2 e^u a^2 \delta_{ij} .
\end{align*}
\]