Conformally flat Lorentzian manifolds with special holonomy groups

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Abstract. We obtain a local classification of conformally flat Lorentzian manifolds with special holonomy groups. The corresponding local metrics are certain extensions of Riemannian spaces of constant sectional curvature to Walker metrics.

Bibliography: 28 titles.

Keywords: Lorentzian manifold, special holonomy group, Walker manifold, conformally flat manifold, Nordström’s gravity.

§1. Introduction and main results

It is known [1] that a conformally flat Riemannian manifold is either a product of two spaces of constant sectional curvature, or it is a product of a space of constant sectional curvature with an interval, or its restricted holonomy group is the identity component of the orthogonal group. The last condition represents the general case and among various manifolds satisfying the last condition one can select only the spaces of constant sectional curvature.

One says that the connected holonomy group of an indecomposable pseudo-Riemannian manifold is special if it is different from the connected component of the pseudo-orthogonal group [2]. In this case the holonomy group defines a special geometry on the manifold. For example, a pseudo-Riemannian manifold of signature \((r, s)\) is pseudo-Kählerian if and only if its holonomy group is contained in \(U(r/2, s/2)\). It is clear that there are no conformally flat Riemannian manifolds with special holonomy groups.

In the case of pseudo-Riemannian manifolds, the holonomy group can be weakly irreducible (this means that it does not preserve any nondegenerate proper vector subspace of the tangent space) but not irreducible at the same time, that is, it may preserve a degenerate vector subspace of the tangent space.

The main result of the present paper is a complete local description of conformally flat Lorentzian manifolds \((M, g)\) with weakly irreducible but not irreducible holonomy groups, which are the only special holonomy groups of Lorentzian manifolds. Let \(\dim M = n + 2 \geq 4\). The holonomy algebra, that is, the Lie algebra of the holonomy group \(g \subset so(1, n + 1)\) of such a manifold preserves an isotropic line of the tangent space, which is identified with the Minkowski space \(\mathbb{R}^{1,n+1}\). Hence

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\( g \) is contained in the maximal subalgebra of \( \mathfrak{so}(1, n+1) \) preserving an isotropic line. This algebra is denoted by \( \mathfrak{sim}(n) \), and it admits the decomposition
\[
\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.
\]
The classification of the Lorentzian holonomy algebras \( g \subset \mathfrak{sim}(n) \) can be found in [3]. Any manifold \((M, g)\) with such a holonomy algebra (locally) admits a distribution \( \ell \) of isotropic lines. Such manifolds are called Walker manifolds [4]. On any such manifold \((M, g)\) there exist local coordinates \( v, x^1, \ldots, x^n, u \) such that the metric \( g \) has the form
\[
g = 2 \, dv \, du + h + 2A \, du + H(du)^2, \tag{1.1}
\]
where \( h = h_{ij}(x^1, \ldots, x^n, u) \, dx^i \, dx^j \) is a \( u \)-dependent family of Riemannian metrics, \( A = A_i(x^1, \ldots, x^n, u) \, dx^i \) is a \( u \)-dependent family of one-forms, and \( H \) is a local function on \( M \). The vector field \( \partial_v \) defines the parallel distribution of isotropic lines. An important class of Walker manifolds are pp-waves, which are given locally by (1.1) with \( A = 0, h = \sum_{i=1}^n (dx^i)^2 \) and \( \partial_v H = 0 \); see [5]. Pp-waves are exactly Walker manifolds with the commutative holonomy algebra \( g \subset \mathbb{R}^n \subset \mathfrak{sim}(n) \).

We use Einstein summation for the indices \( i, j, k = 1, \ldots, n \); we use the notation \( \dot{f} = \partial_u f \) for any function \( f \).

On a Walker manifold \((M, g)\) we define the canonical function \( \lambda \) from the equality
\[
\text{Ric} \, p = \lambda p,
\]
where \( p \) is any local vector field tangent to the distribution \( \ell \) and \( \text{Ric} \) is the Ricci operator. If the metric \( g \) is written in the form (1.1), then \( \lambda = (1/2) \partial_v^2 H \), and the scalar curvature of \( g \) satisfies
\[
s = 2\lambda + s_0,
\]
where \( s_0 \) is the scalar curvature of \( h \). The form of a conformally flat Walker metric depends on the vanishing of the function \( \lambda \). In the general case we obtain the following result.

**Theorem 1.** Let \((M, g)\) be a conformally flat (that is, with zero Weyl curvature tensor) Walker manifold of dimension \( n + 2 \geq 4 \). Then in a neighbourhood of each point of \( M \) there exist coordinates \( v, x^1, \ldots, x^n, u \) such that
\[
g = 2 \, dv \, du + \Psi \sum_{i=1}^n (dx^i)^2 + 2A \, du + (\lambda(u) v^2 + vH_1 + H_0)(du)^2,
\]
where
\[
\Psi = \frac{4}{(1 - \lambda(u) \sum_{k=1}^n (x^k)^2)^2},
\]
\[
A = A_i \, dx^i, \quad A_i = \Psi \left( -4C_k(u)x^k x^i + 2C_i(u) \sum_{k=1}^n (x^k)^2 \right),
\]
\[
H_1 = -4C_k(u)x^k \sqrt{\Psi} - \partial_u \ln \Psi + K(u),
\]
and
Theorem 2. Let $(M, g)$ be a conformally flat Walker Lorentzian manifold of dimension $n + 2 \geq 4$.

1) If the function $\lambda$ is nonvanishing at a point, then in a neighbourhood of this point there exist coordinates $v, x^1, \ldots, x^n, u$ such that

$$g = 2 \, dv \, du + \Psi \sum_{i=1}^{n} (dx^i)^2 + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{(1 - \lambda(u))^2},$$

$$H_1 = -\partial_u \ln \Psi, \quad H_0 = \sqrt{\Psi} \left( a(u) \sum_{k=1}^{n} (x^k)^2 + D_k(u)x^k + D(u) \right).$$

2) If $\lambda \equiv 0$ in a neighbourhood of a point, then in a neighbourhood of this point there exist coordinates $v, x^1, \ldots, x^n, u$ such that

$$g = 2 \, dv \, du + \sum_{i=1}^{n} (dx^i)^2 + 2A \, du + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i \, dx^i, \quad A_i = C_i(u) \sum_{k=1}^{n} (x^k)^2, \quad H_1 = -2C_k(u)x^k,$$

$$H_0 = \sum_{k=1}^{n} (x^k)^2 \left( \frac{1}{4} \sum_{k=1}^{n} (x^k)^2 \sum_{k=1}^{n} C_k(u)^2 - (C_k(u)x^k)^2 + \hat{C}_k(u)x^k + a(u) \right) + D_k(u)x^k + D(u).$$

In particular, if all $C_i \equiv 0$, then the metric can be rewritten in the form

$$g = 2 \, dv \, du + \sum_{i=1}^{n} (dx^i)^2 + a(u) \sum_{k=1}^{n} (x^k)^2(du)^2. \quad (1.2)$$
Thus Theorem 2 gives the local form of a conformally flat Walker metric in a neighbourhood of any point where $\lambda$ is nonzero or constantly zero. Such points represent a dense subset of the manifold. Theorem 1 also describes the metric in neighbourhoods of points at which the function $\lambda$ vanishes, but it is not locally zero, for example, in a neighbourhood of isolated zero points of $\lambda$.

Next, we find the holonomy algebras of the obtained metrics and check when the metrics are decomposable.

**Theorem 3.** Let $(M, g)$ be as in Theorem 1.

1) The manifold $(M, g)$ is locally indecomposable if and only if there exists a coordinate system with one of the properties:

   (i) $\dot{\lambda} \neq 0$;
   (ii) $\dot{\lambda} \equiv 0$, $\lambda \neq 0$, that is, $g$ can be written as in part 1) of Theorem 2, and $\sum_{k=1}^{n} D_k^2 + (a + \lambda D)^2 \neq 0$;
   (iii) $\lambda \equiv 0$, that is, $g$ can be written as in part 2) of Theorem 2, and $\sum_{k=1}^{n} C_k^2 + a^2 \neq 0$.

   Otherwise, the metric can be written in the form

   $$g = \Psi \sum_{k=1}^{n} (dx^k)^2 + 2 dv du + \lambda v^2 (du)^2, \quad \lambda \in \mathbb{R}.$$  

   The holonomy algebra of this metric is trivial if and only if $\lambda = 0$. Otherwise, the holonomy algebra is isomorphic to $\mathfrak{so}(n) \oplus \mathfrak{so}(1,1)$.

2) Suppose that $(M, g)$ is locally indecomposable. Then its holonomy algebra is isomorphic to $\mathbb{R}^n \subset \mathfrak{sim}(n)$ if and only if $\lambda^2(u) + \sum_{k=1}^{n} C_k^2(u) \equiv 0$ for all coordinate systems. In this case $(M, g)$ is a pp-wave, and $g$ is given by (1.2). If $\lambda^2 + \sum_{k=1}^{n} C_k^2 \neq 0$ in some system of coordinates, then the holonomy algebra is isomorphic to $\mathfrak{sim}(n)$.

The paper is organized as follows. In §2 the result of Kurita [1] is extended to the case of pseudo-Riemannian manifolds. In §3, an expression is given for the curvature tensor and the Weyl conformal curvature tensor $W$ for a Walker metric. It seems that these expressions are given here for the first time. §§3–6 are devoted to the proofs of the main theorems. We rewrite the equation $W = 0$ as a system of partial differential equations and find appropriate systems of coordinates such that the complete solution of this system can be found. In §7, the Ricci operator for the obtained metrics is computed.

In §8, the case of dimension 4 is considered. Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds were also classified in [6]. The metric (1.2) in dimension 4 was given in [7]. In [6] the problem was posed of constructing an example of a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in [6] by $R_{14}$). An attempt to construct such a metric was made in [8]. We show that the metric constructed there is in fact decomposable and its holonomy algebra is $\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)$. Thus in this paper we get conformally flat metrics with the holonomy algebra $\mathfrak{sim}(n)$ for the first time, and even more, we find all such metrics.

The field equations of Nordström’s theory of gravitation, which appeared before Einstein’s theory, are the following:

$$W = 0, \quad s = 0;$$
see [9]–[11]. All metrics from Theorem 1 in dimension 4 and metrics from part 2) of Theorem 2 in higher dimensions provide examples of solutions of these equations. Thus we have found all solutions to Nordström’s gravity with holonomy algebras contained in sim$(n)$. Similarly, the Einstein equation on Lorentzian manifolds with such holonomy algebras was studied in [12]. In this case it is impossible to obtain the complete solution, but the examples of solutions have interesting physical interpretations [12]. An important fact is that a simply connected conformally flat spin Lorentzian manifold admits the space of conformal Killing spinors of maximum dimension [13].

It would be interesting to obtain examples of conformally flat Lorentzian manifolds satisfying some global geometric properties; for example, globally hyperbolic Lorentzian manifolds with special holonomy groups are important [14].

The projective equivalence of 4-dimensional conformally flat Lorentzian metrics with special holonomy algebras was studied recently in [14]. There are many interesting works about conformally flat (pseudo-)Riemannian, and in particular Lorentzian, manifolds. We mention [16]–[22].

The results of this paper are used in [23] for the classification of Lorentzian manifolds satisfying the condition $\nabla^2 R = 0$.

§2. Decomposability of conformally flat pseudo-Riemannian manifolds

In [1] Kurita proved the following theorem for the case of Riemannian manifolds.

**Theorem 4.** Let $(M, g)$ be an $n$-dimensional conformally flat Riemannian manifold. Then its local restricted holonomy group $H_x$ ($x \in M$) is in general $SO(n)$.

If $H_x \neq SO(n)$, then for some coordinate neighbourhood $U$ of $x$ one of the following holds:

1) $H_x$ is trivial and the metric is flat in $U$;
2) $H_x = SO(k) \times SO(n-k)$ and $U$ is a direct product of a $k$-dimensional manifold of constant sectional curvature $K$ and an $(n-k)$-dimensional manifold of constant sectional curvature $-K$ ($K \neq 0$);
3) $H_x = SO(n-1)$ and $U$ is a direct product of a straight line (or a segment) and an $(n-1)$-dimensional manifold of constant sectional curvature.

We generalize this theorem for the case of pseudo-Riemannian manifolds. We also make it more precise.

**Theorem 5.** Let $(M, g)$ be a conformally flat pseudo-Riemannian manifold of signature $(r, s)$ with the restricted holonomy group $\text{Hol}^0(M, g)$. If $(M, g)$ is not flat, then one of the following holds:

1) $\text{Hol}^0(M, g) = SO(r, s)$;
2) $\text{Hol}^0(M, g)$ is weakly irreducible but not irreducible (in particular, it preserves a degenerate subspace of the tangent space);
3) $\text{Hol}^0(M, g) = SO(r_1, s_1) \times SO(r - r_1, s - s_1)$ and each point $x \in M$ has a neighbourhood that is either flat or is a product of a pseudo-Riemannian manifold of constant sectional curvature $K$ and signature $(r_1, s_1)$ and a pseudo-Riemannian manifold of constant sectional curvature $-K$ ($K \neq 0$) and signature $(r-r_1, s-s_1)$;
4) $\text{Hol}^0(M, g) = SO(r - 1, s)$ (resp., $\text{Hol}^0(M, g) = SO(r, s - 1)$) and each point $x \in M$ has a neighborhood that is either flat or is a product of a pseudo-Riemannian
manifold of constant sectional curvature and signature \((r - 1, s)\) (resp., \((r, s - 1)\)) and the space \((L, -(dt)^2)\) (resp., \((L, (dt)^2)\)), where \(L\) is a straight line or a segment.

**Proof.** Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((r, s)\) and dimension \(d = r + s\). The vector bundle \(\mathfrak{so}(TM)\) of skew-symmetric endomorphisms of the tangent bundle \(TM\) can be identified with the space of bivectors \(\wedge^2 TM\) in such a way that
\[
(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X
\]
for all vector fields \(X, Y, Z\) on \(M\). The Weyl tensor \(W\) of the pseudo-Riemannian manifold \((M, g)\) is defined by the equality
\[
W = R + R_L, \tag{2.1}
\]
where the tensor \(R_L\) has the form
\[
R_L(X, Y) = LX \wedge Y + X \wedge LY, \tag{2.2}
\]
where \(L = \frac{1}{d-2} \left( \text{Ric} - \frac{s}{2(d-1)} \text{id} \right)\) is the Schouten tensor and \(s\) is the scalar curvature.

Suppose that the restricted holonomy group \(\text{Hol}^0(M, g)\) is not weakly irreducible. The Wu decomposition theorem [24] states that each point of \(M\) has a neighbourhood \(U\) such that \((U, g|_U)\) is a product \((M_1 \times M_2, g_1 + g_2)\) of two pseudo-Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\). Let \(d_1\) and \(d_2\) be the dimensions of these manifolds. For the curvature tensors, Ricci operators and scalar curvatures we have
\[
R = R_1 + R_2, \quad \text{Ric} = \text{Ric}_1 + \text{Ric}_2, \quad s = s_1 + s_2.
\]
First suppose that \(d \geq 4\). In this case \(W = 0\) and we get
\[
R_1 + R_2 = -R_L. \tag{2.3}
\]
Assume that \(d_1 \geq d_2\) and \(d_1 \geq 2\). The curvature tensor \(R_1\) can be written in the form \(R_1 = W_1 - R_{L_1}\). Considering (2.3) restricted to \(TM_1\), we get that \(W_1 = 0\) and
\[
\frac{1}{d_1 - 2} \left( \text{Ric}_1 - \frac{s_1}{2(d_1 - 1)} \text{id} \right) = \frac{1}{d - 2} \left( \text{Ric}_1 - \frac{s_1 + s_2}{2(d - 1)} \text{id} \right). \tag{2.4}
\]
If \(d_2 \geq 2\), then taking the trace in (2.4) we get
\[
\frac{s_1}{d_1(d_1 - 1)} = -\frac{s_2}{d_2(d_2 - 1)}.
\]
Since \(s_1\) is a function on \(M_1\) and \(s_2\) is a function on \(M_2\), both functions must be constant. Substituting the last equality back into (2.4) we obtain
\[
\text{Ric}_1 = \frac{s_1}{d_1} \text{id}. \tag{2.5}
\]
Next,\[ R_1(X, Y) = \frac{s_1}{d_1(d_1 - 1)} X \wedge Y. \tag{2.6} \]

The same holds for the second manifold. For the sectional curvatures,
\[ k_1 = \frac{s_1}{d_1(d_1 - 1)} = -\frac{s_2}{d_2(d_2 - 1)} = -k_2. \]

If \( d_2 = 1 \), then (2.4) is equivalent to (2.5), and this implies (2.6). From this and the Schur theorem it follows that \( k_1 \) is constant. If \( d_1 = 2 \), then the curvature tensor \( R_1 \) satisfies \( R_1(X, Y) = fX \wedge Y \) for some function \( f \) on \( M_1 \). The proof in this case is the same.

If \( d = 3 \), then \( d_1 = 2 \) and \( d_2 = 1 \). We have \( R = R_1 \) and \( R_1(X, Y) = fX \wedge Y \) for some function \( f \) on \( M_1 \). In this case \((M, g)\) is conformally flat if and only if the Cotton tensor \( C \) defined by
\[ C(X, Y, Z) = g((\nabla_Z X, Y) - g((\nabla_Y L)X, Z) \]

is zero. This implies that \( f \) is constant, that is, \((M_1, g_1)\) has constant sectional curvature.

It remains to prove that if \( \text{Hol}^0(M, g) \) is irreducible, then it coincides with \( \text{SO}(r, s) \). Suppose that \( \text{Hol}^0(M, g) \) is irreducible and is different from \( \text{SO}(r, s) \) and \( U(r/2, s/2) \). Then the manifold is an Einstein manifold [2]. Since \((M, g)\) is in addition conformally flat, \((M, g)\) has constant sectional curvature and its connected holonomy group must be either trivial or \( \text{SO}(r, s) \), that is, we get a contradiction. It is known that if a pseudo-Kählerian manifold is conformally flat, then it is flat [25], hence \( \text{Hol}^0(M, g) \neq U(r/2, s/2) \). This proves Theorem 5.

§ 3. The curvature tensor and the Weyl conformal curvature tensor of Walker metrics

In order to prove Theorem 1 we give some information about the curvature tensor of the Walker metric (1.1). For the fixed coordinates \( v, x^1, \ldots, x^n, u \) consider the field of frames
\[ p = \partial_v, \quad X_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2} H \partial_v. \]

Consider the distribution \( E \) generated by the vector fields \( X_1, \ldots, X_n \). The fibres of this distribution can be identified with the tangent spaces to the Riemannian manifolds with the Riemannian metrics \( h(u) \). Denote by \( R_0 \) the tensor corresponding to the family of the curvature tensors of \( h(u) \) under this identification. Similarly denote by \( \text{Ric}(h) \) the corresponding Ricci endomorphism acting on sections of \( E \).

From the results of [26] it follows that the curvature tensor \( R \) of the metric \( g \) can be written in the form
\[ R(p, q) = -\lambda p \wedge q - p \wedge \tilde{v}, \tag{3.1} \]
\[ R(X, Y) = R_0(X, Y) - p \wedge (P(Y)X - P(X)Y), \tag{3.2} \]
\[ R(X, q) = -g(\tilde{v}, X)p \wedge q + P(X) - p \wedge T(X), \quad R(p, X) = 0. \tag{3.3} \]
for all $X, Y \in \Gamma(E)$. Here $\lambda$ is a function, $\vec{v} \in \Gamma(E)$, $T \in \Gamma(\text{End}(E))$ is symmetric, $T^* = T$, $R_0 = R(h)$, and the tensor $P \in \Gamma(E^* \otimes \mathfrak{so}(E))$ satisfies $g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0$ for all $X, Y, Z \in \Gamma(E)$. These elements can be found in terms of the coefficients of the metric (1.1). Let $P(X_k)X_j = P_{jk}^lX_l$ and $T(X_j) = \sum_i T_{ij}X_i$. Then

$$h_{il}P_{jk}^l = g(R(X_k, q)X_j, X_i), \quad T_{ij} = -g(R(X_i, q)q, X_j).$$

By direct computation, we obtain

$$\lambda = \frac{1}{2} \partial_v^2H, \quad \vec{v} = \frac{1}{2} (\partial_v \partial_v H - A_i \partial_v^2 H) \ h^{ij}X_j,$$

$$h_{il}P_{jk}^l = -\frac{1}{2} \nabla_k F_{ij} + \frac{1}{2} \nabla_k \dot{h}_{ij} - \dot{\Gamma}^{kl}_{ij} h_{ki},$$

$$T_{ij} = \frac{1}{2} \nabla_i \nabla_j H - \frac{1}{4} (F_{ik} + \dot{h}_{ik})(F_{jl} + \dot{h}_{jl})h^{kl} - \frac{1}{4} (\partial_v H)(\nabla_i A_j + \nabla_j A_i)$$

$$- \frac{1}{2} (A_i \partial_j \partial_v H + A_j \partial_i \partial_v H) - \frac{1}{2} (\nabla_i \dot{A}_j + \nabla_j \dot{A}_i)$$

$$+ \frac{1}{2} A_i A_j \partial_v^2 H + \frac{1}{2} h_{ij} + \frac{1}{4} \dot{h}_{ij} \partial_v^2 H,$$

where

$$F = dA, \quad F_{ij} = \partial_i A_j - \partial_j A_i,$$

is the differential of the 1-form $A$, and the covariant derivatives are taken with respect to the metric $h$. In the case where $h, A$ and $H$ are independent of $u$, the curvature tensor of the metric is found in [12].

The Ricci operator has the following form:

$$\text{Ric}(p) = \lambda p, \quad \text{Ric}(X) = -g(X, \widetilde{\text{Ric}}P - \vec{v})p + \text{Ric}(h)(X),$$

$$\text{Ric}(q) = -(\text{tr} T)p - \widetilde{\text{Ric}}(P) + \vec{v} + \lambda q,$$

where $\widetilde{\text{Ric}}P = h^{ij}P(X_i)X_j$ [27]. For the scalar curvature we get $s = 2\lambda + s_0$, where $s_0$ is the scalar curvature of $h$. Using this we can compute the tensor $R_L$:

$$R_L(p, X) = \frac{1}{n} p \wedge \left( \text{Ric}(h) + \frac{(n - 1)\lambda - s_0}{n + 1} \text{id} \right) X,$$

$$R_L(p, q) = \frac{1}{n} \left( \frac{2n\lambda - s_0}{n + 1} p \wedge q + p \wedge (\vec{v} - \widetilde{\text{Ric}}P) \right),$$

$$R_L(X, Y) = \frac{1}{n} \left( p \wedge (g(X, \vec{v} - \widetilde{\text{Ric}}P)Y - g(Y, \vec{v} - \widetilde{\text{Ric}}P)X) + \left( \text{Ric}(h) - \frac{s}{2(n + 1)} \right) X \wedge Y + X \wedge \left( \text{Ric}(h) - \frac{s}{2(n + 1)} \right) Y \right),$$

$$R_L(X, q) = \frac{1}{n} \left( (\text{tr} T)p \wedge X + g(X, \vec{v} - \widetilde{\text{Ric}}P)p \wedge q + X \wedge (\vec{v} - \widetilde{\text{Ric}}P) \right.$$ 

$$\left. + \left( \text{Ric}(h) + \frac{(n - 1)\lambda - s_0}{n + 1} \text{id} \right) X \wedge q \right).$$

The Weyl tensor $W$ can be computed using this and (2.1).
§ 4. Proof of Theorem 1

Suppose that \((M, g)\) is a Walker manifold, that is, its holonomy algebra is contained in \(\text{sim}(n)\). The local form of the metric \(g\) is given by (1.1). Suppose that \(g\) is conformally flat, that is, \(W = 0\).

**Lemma 1.** The equation \(W = 0\) is equivalent to the following system of equations:

\[
s_0 = -n(n-1)\lambda, \quad R_0 = -\frac{1}{2}\lambda R_{id}, \quad P(X) = \bar{v} \wedge X, \quad T = f \; id_E, \quad (4.1)
\]

where \(X\) is any section of \(E\) and \(f\) is a function. In particular, \(W = 0\) implies that 
\[
\text{Ric} P = -(n-1)\bar{v} \quad \text{and the Weyl tensor } W_0 \text{ of } h \text{ is zero.}
\]

**Proof.** Suppose that \(W = 0\). Then \(R = -R_L\). From (3.3) and (3.9) it follows that 
\[
\text{Ric}(h) = -((n-1)\lambda - s_0)/(n + 1)) \text{id. Taking the trace we get } s_0 = -n(n-1)\lambda.
\]
Hence \(\text{Ric}(h) = (s_0/n) \text{id, that is, the metrics } h \text{ are Einstein and}
\[
R_0 = W_0 - \frac{s_0}{2n(n-1)} R_{id}.
\]
From (3.1) and (3.11) it follows that
\[
R_0 + \frac{s_0}{2n(n-1)} R_{id} = 0.
\]

We conclude that \(W_0 = 0\). Using (3.3) and (3.12) we get

\[
P(X) = -\frac{1}{n} X \wedge (\bar{v} - \text{RicP}).
\]

Applying \(\text{Ric}\) we obtain \(\text{RicP} = -(n-1)\bar{v}\). Consequently, \(P(X) = \bar{v} \wedge X\). From (3.3) and (3.12) it follows that

\[
T(X) = \frac{1}{n}(\text{tr} T) X,
\]
that is, \(T = f \; id\) for some function \(f\). Conversely, (4.1) implies \(W = 0\).

Since \(h\) is independent of \(v\), \(\partial_v \lambda = 0\). From (3.4) it follows that

\[
H = \lambda v^2 + H_1 v + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.
\]
From (3.5) it follows that the components of the tensor \(P\) do not depend on the coordinate \(v\). This, the equation \(P(X) = \bar{v} \wedge X\) and (3.4) imply that \(\partial_i \partial_v^2 H = 0\), that is, \(\partial_v \lambda = 0\); consequently \(s_0\) and \(\lambda\) are functions depending only on \(u\). Note that for \(n \geq 3\) this follows also from \(\text{Ric}(h) = (s_0/2) \text{id}\). We conclude that each metric in the family \(h(u)\) is of constant sectional curvature. There exists a transformation \(\bar{x}^i = \bar{x}^i(x^k, u)\) of the coordinates \(x^1, \ldots, x^n\) depending on the parameter \(u\) such that with respect to the new coordinates the metric \(h\) takes the form

\[
h = \Psi \sum_{k=1}^n (dx^k)^2, \quad \Psi = \frac{4}{(1 - \lambda(u) \sum_{k=1}^n (x^k)^2)^2}.
\]
Considering the transformation $\vec{v} = v, \vec{x}^i = \vec{x}^i(x^k, u), \vec{u} = u$ we may assume that $h$ in (1.1) has the just-obtained form.

Let us consider the equation $T_{ij} = f\delta_{ij}$. From (3.6) it follows that $f = vf_1 + f_0, \partial_v f_1 = \partial_v f = 0$. Applying $\partial_v$ to $T_{ij} = f\delta_{ij}$, we get the equations

$$f_1 \delta_{ij} = \frac{1}{2} \nabla_i \nabla_j H_1 - \lambda(u)\frac{1}{2}(\nabla_i A_j + \nabla_j A_i).$$

These equations are equivalent to

$$\nabla_i Z_i = \nabla_j Z_j, \quad \nabla_i Z_j + \nabla_j Z_i = 0, \quad i \neq j, \quad (4.2)$$

where

$$Z_i = \lambda A_i - \frac{1}{2} \partial_i H_1. \quad (4.3)$$

Note that $\vec{v} = -Z_i h^{ij} X_j$. The Christoffel symbols of the metric $h$ are the following:

$$\Gamma^k_{ij} = \frac{1}{2\Psi} (\delta_{kj} \partial_i \Psi + \delta_{ki} \partial_j \Psi - \delta_{ij} \partial_k \Psi).$$

Using that, the above equations can be rewritten in the form

$$\partial_i \left( \frac{Z_i}{\Psi} \right) = \partial_j \left( \frac{Z_j}{\Psi} \right), \quad \partial_i \left( \frac{Z_j}{\Psi} \right) + \partial_j \left( \frac{Z_i}{\Psi} \right) = 0, \quad i \neq j. \quad (4.4)$$

We will distinguish the cases $n = 2$ and $n \geq 3$.

**Case $n \geq 3$.**

**Lemma 2.** If $n \geq 3$, then the general solution of the system

$$\partial_i f_i = \partial_j f_j, \quad \partial_i f_j + \partial_j f_i = 0, \quad i \neq j,$$

has the form

$$f_i = x^i B_k x^k - \frac{1}{2} B_i \sum_{k=1}^{n} (x^k)^2 + d_{ik} x^k + c x^i + c_i,$$

where $B_i, c_i, c, d_{ik} \in \mathbb{R}, d_{ki} = -d_{ik}$.

**Proof.** Let $i, j, k$ be pairwise distinct; then

$$\partial_i \partial_j f_k = -\partial_i \partial_k f_j = \partial_k \partial_j f_i = -\partial_i \partial_j f_k,$$

that is, $\partial_i \partial_j f_k = 0$. This shows that $f_k = \sum_{i \neq k} C_{ki}(x^i, x^k)$. Then it is not hard to find these functions.

We conclude that

$$Z_i = \Psi \left( x^i B_k(u) x^k - \frac{1}{2} B_i(u) \sum_{k=1}^{n} (x^k)^2 + d_{ik}(u) x^k + c(u) x^i + c_i(u) \right), \quad (4.5)$$

where $B_i(u), c_i(u), c(u), d_{ik}(u)$ are functions of $u$ and $d_{ki}(u) = -d_{ik}(u)$.

The system of equations in Lemma 2 is very similar to the equation of the Killing 1-form:

$$\nabla_i Z_j + \nabla_j Z_i = 0.$$

Let us prove the following lemma.
Lemma 3. Any Killing vector field on the space with the metric $\Psi \sum_{k=1}^{n} (dx^k)^2$ has the following form:

$$X = X^i \partial_i, \quad X^i = f_{ik} x^k - 2\lambda x^i c_k x^k + \lambda c_i \sum_{k=1}^{n} (x^k)^2 + c_i,$$

where $b_i, f_{ik} \in \mathbb{R}$, $f_{ki} = -f_{ik}$.

Depending on the value of $\lambda$, the above metric is the metric of one of the following spaces: the sphere, the Lobachevskian space, the Euclidean space. This description corresponds to the fact that the Lie algebra spaces is isomorphic respectively to $so(n+1)$, $so(1,n)$, $iso(\mathbb{R}^n)$. The symmetric decomposition of the Lie algebra is of the form $\mathfrak{p} = so(n) + \mathbb{R}^n$. The vector fields defined by the numbers $f_{ik}$ correspond to elements of $so(n)$, while the vector fields defined by the numbers $c_i$ correspond to elements of $\mathbb{R}^n$.

Proof of Lemma 3. Consider the Killing 1-form $Z_i = h_{ij} X^j$. In addition to equations (4.2) it satisfies the equations $\nabla_i Z_i = 0$. These equations take the form

$$\partial_i \left( \frac{Z_i}{\Psi} \right) = -\frac{1}{2\Psi^2} \sum_{k=1}^{n} Z_k \partial_k \Psi.$$

This implies that $Z_i = \Psi f_i$, where the $f_i$ are given by Lemma 2 with $c = 0$ and $B_k = -2\lambda c_k$. This proves the lemma.

From (4.3) it follows that

$$\lambda F_{ij} = \partial_i Z_j - \partial_j Z_i = \Psi^{3/2} \left( (B_i - 2\lambda C_i x^j - (B_j - 2\lambda C_j) x^i + 2\lambda x^k (d_{jk} x^i - d_{ik} x^j)) - 2\Psi d_{ij}, \right)$$

This implies that $B_i(u) = \lambda(u) \tilde{B}_i(u)$ for some function $\tilde{B}_i(u)$.

From (3.5) and (3.4) it follows that the equation $P(X) = \vec{v} \wedge X$ takes the form

$$-\frac{1}{2} \nabla_k F_{ij} - \delta_{ik} \frac{\Psi}{2} \partial_u \partial_j \ln \Psi + \delta_{jk} \frac{\Psi}{2} \partial_u \partial_j \ln \Psi$$

$$= \Psi \left( \frac{1}{2} \partial_j H_1 - \lambda A_j \right) \delta_{ki} - \Psi \left( \frac{1}{2} \partial_i H_1 - \lambda A_i \right) \delta_{kj}.$$ (4.7)

If $i$, $j$, $k$ are pairwise distinct, then

$$0 = \nabla_k F_{ij} = \partial_k F_{ij} - \frac{1}{\Psi} F_{ij} \partial_k \Psi - \frac{1}{2\Psi} F_{kj} \partial_i \Psi - \frac{1}{2\Psi} F_{ik} \partial_j \Psi$$

$$= \Psi \partial_k \left( \frac{F_{ij}}{\Psi} \right) - \lambda F_{kj} x^i \sqrt{\Psi} - \lambda F_{ik} x^j \sqrt{\Psi}.$$

Using this and equation (4.6) we obtain

$$-\partial_k \left( \frac{F_{ij}}{\Psi} \right) = \lambda x^k \Psi \left( (\tilde{B}_j - 2C_j) x^i - (\tilde{B}_i - 2C_i) x^j \right)$$

$$+ 2\sqrt{\Psi} (x^i d_{kj} - x^j d_{ki}) + 2\lambda \Psi x^k (x^j d_{il} - x^i d_{jl}) x^l.$$
Integrating with respect to $x^k$ we get

$$F_{ij} = \Psi^{3/2}((\tilde{B}_i - 2C_i)x^j - (\tilde{B}_j - 2C_j)x^i + 2(d_i x^j - d_j x^i)x^j) - \Psi C_{ij}(x^i, x^j, u) \quad (4.8)$$

for some functions $C_{ij}(x^i, x^j, u)$. Comparing this with $(4.6)$ we conclude that $d_{ij} = \frac{\lambda}{2} C_{ij}$. Equation $(4.7)$ for $k = i \neq j$ is of the form

$$\nabla_i F_{ij} = -\Psi \partial_u \partial_j \ln \Psi + 2\Psi Z_j.$$

Direct computations show that

$$\nabla_i F_{ij} = \Psi^{3/2} \partial_i \left( \frac{F_{ij}}{\Psi^{3/2}} \right) + \lambda F_{ij} x^i \sqrt{\Psi}.$$

Using this, $(4.3)$, $(4.6)$ and $(4.8)$ we get

$$\Psi^{3/2} \left( -2d_{ij} - \partial_i \frac{C_{ij}}{\sqrt{\Psi}} \right) + \Psi \partial_u \partial_j \ln \Psi - \Psi^2 \left( \lambda D_k x^k x^j - \frac{1}{2} D_j \sum_{k=1}^{n} (x^k)^2 + 2c(u) x^j + \frac{1}{2} D_j \right) = 0,$$

where $D_k = \tilde{B}_k + 2C_k$. Using the equalities

$$\partial_j \partial_u \ln \Psi = \dot{x}^j \Psi, \quad \partial_i \left( \frac{C_{ij}}{\sqrt{\Psi}} \right) = -\lambda x_i C_{ij} + \frac{\partial_i C_{ij}}{\sqrt{\Psi}}, \quad d_{ij} = \frac{\lambda}{2} C_{ij},$$

we get

$$-\partial_i C_{ij} - \Psi \left( \lambda D_k x^k x^j - \frac{1}{2} D_j \sum_{k=1}^{n} (x^k)^2 + (2c(u) - \lambda) x^j + \frac{1}{2} D_j \right) = 0.$$

If $\lambda(u) \neq 0$, then the equality $d_{ij} = \frac{\lambda}{2} C_{ij}$ implies $\partial_t C_{ij} = 0$. Consequently, $D_k = 0$ and $c = \dot{\lambda}/2$. If $\lambda(u) = 0$, then we get

$$\partial_j \partial_t C_{ij} + 4(D_j x^j - (2c - \dot{\lambda})) = 0.$$

Here we write $x_j$ instead of $x^j$ in order to avoid the sum over $j$. Since $C_{ij} = -C_{ji}$, we get $D_j = 0$, $c = \frac{\dot{\lambda}}{2}$, and $\partial_t C_{ij} = 0$. Thus $D_k = 0$, $\tilde{B}_k = -2C_k$, $c = \frac{\dot{\lambda}}{2}$ and $d_{ij} = \frac{\lambda}{2} C_{ij}$, where $c = \frac{1}{2} \dot{\lambda}$, and $d_{ij} = \frac{\lambda}{2} C_{ij}$, and where the $C_{ij}$ are functions of $u$. We conclude that

$$F_{ij} = \Psi^{3/2} \left( 4C_j(u)x^i - 4C_i(u)x^j + \lambda(u)(C_{ti}(u)x^j - C_{ij}(u)x^i)x^j \right) - \Psi C_{ij}(u). \quad (4.9)$$

Recall that $F = dA$. In [12] it is noted that the transformation

$$v \mapsto v - \phi(x^1, \ldots, x^m, u)$$

changes $A_i$ to $A_i + \partial_i \phi$. Clearly, this transformation does not change $F$. This allows us to choose any $A$ such that $dA = F$. We take

$$A_i = \Psi \left( -4C_k(u)x^k x^i + 2C_i(u) \sum_{k=1}^{n} (x^k)^2 + \frac{1}{2} C_{ik}(u)x^k \right).$$
Consider the coordinate transformation with the inverse transformation

\[ v = \tilde{v}, \quad x^j = A^j_i(u) \tilde{x}^i, \quad u = \tilde{u}, \]

where \( A^j_i(u) \) is a family of orthogonal matrices. It is easy to check that

\[ \tilde{A}_i = \sum_{k=1}^{n} A_k^k(u)(\partial_u A_i^k(u)) \tilde{x}^k + A_i^k(u) A_i^l. \]

The obtained metric has the same form and

\[ \tilde{C}_i(u) = C_j(u) A_j^i(u), \quad \tilde{C}_{ij}(u) = \sum_{k=1}^{n} A_k^i(u) A_k^j(u) + \frac{1}{2} A_i^r(u) C_{rk}(u) A_j^k(u). \]

Consider the equation \( \tilde{C}_{ij}(u) = 0 \). Since \( \sum_{k=1}^{n} A_k^i(u) A_k^j(u) = \delta^{ij} \), it can be written in the form

\[ \partial_u A_i^k(u) = A_i^j(u) \frac{1}{2} C_{jk}(u). \]

Since \( C_{jk}(u) \) is skew-symmetric, \( (1/2)C_{jk}(u) \) is a curve in the Lie algebra \( so(n) \). Then \( A_i^k(u) \) satisfying the above equation is none other than the development of the curve \( (1/2)C_{jk}(u) \) in the Lie group \( SO(n) \). Thus, applying such transformation, we may assume that \( C_{ij}(u) = 0 \). Next,

\[ \partial_i H_1 = 2\lambda A_i - 2Z_i = -2\Psi \left( 2\lambda C_k(u) x^k x^i - \lambda C_i(u) \sum_{k=1}^{n} (x^k)^2 + \frac{1}{2} \lambda x^i + C_i(u) \right). \]

We conclude that

\[ H_1 = -4C_k(u) x^k \sqrt{\Psi} - \partial_u \ln \Psi + K(u) \]

for some function \( K(u) \).

There remains only one unknown function \( H_0 \). Consider the equations \( T_{ij} = f \delta_{ij} \).

If \( i \neq j \), then \( \nabla_i \nabla_j H_0 = \sqrt{\Psi} \partial_i \partial_j (H_0/\sqrt{\Psi}) \), and using (3.6) we obtain

\[ \partial_i \partial_j \frac{H_0}{\sqrt{\Psi}} = 2\Psi^{3/2} x^i x^j \sum_{k=1}^{n} C_k^2(u). \]

If \( \lambda(u) \neq 0 \), then the function \( (4/\lambda^2(u)) \Psi \sum_{k=1}^{n} C_k^2(u) \) is a partial solution of the system. The general solution is of the form

\[ H_0 = \frac{4}{\lambda^2(u)} \Psi \sum_{k=1}^{n} C_k^2(u) + \sqrt{\Psi} \sum_{k=1}^{n} f_k(x^k, u). \]

The condition \( \nabla_i \nabla_i H_0 = \nabla_j \nabla_j H_0 \) implies \( \partial_i^2 f_i = \partial_i^2 f_j \), hence

\[ f_i(x^i, u) = a(u)(x^i)^2 + D_i(u)x_i + d_i(u) \]

for some functions \( a(u), D_i(u), d_i(u) \). We see that \( H_0 \) is as in the statement of the theorem for the case \( \lambda(u) \neq 0 \). The case \( \lambda(u) = 0 \) is similar.
Case \( n = 2 \). The system of equations (4.4) takes the form

\[
\partial_1 \left( \frac{Z_1}{\Psi} \right) = \partial_2 \left( \frac{Z_2}{\Psi} \right), \quad \partial_1 \left( \frac{Z_2}{\Psi} \right) + \partial_2 \left( \frac{Z_1}{\Psi} \right) = 0, \tag{4.11}
\]

which implies that \( Z_1/\Psi \) and \( Z_2/\Psi \) are the real and imaginary parts of a complex holomorphic function of the variable \( x^1 + ix^2 \). Note that

\[
\nabla_1 F_{12} = \Psi \partial_1 \frac{F_{12}}{\Psi}, \quad \nabla_2 F_{12} = \Psi \partial_2 \frac{F_{12}}{\Psi}.
\]

Using (4.7) we get

\[
Z_1 = -\frac{1}{2} \partial_2 F_{12} \frac{1}{\Psi} + \frac{1}{2} \partial_u \partial_1 \ln \Psi, \quad Z_2 = \frac{1}{2} \partial_1 F_{12} \frac{1}{\Psi} + \frac{1}{2} \partial_u \partial_2 \ln \Psi.
\]

Substituting this into the first equation in (4.11) and using the equality \( \partial_u \partial_i \ln \Psi = \dot{\lambda} x^i \Psi \), we get

\[
\partial_1 \partial_2 \frac{F_{12}}{\Psi^{3/2}} = 0.
\]

This implies

\[
F_{12} = \Psi^{3/2} (f_1(x^1, u) + f_2(x^2, u)).
\]

Substituting this into the second equation in (4.11) we obtain \( \partial_1^2 f_1 = \partial_2^2 f_2 \), that is,

\[
f_1 = t(u)(x^1)^2 + a^1(u)x^1 + b_1(u), \quad f_2 = t(u)(x^2)^2 + a^2(u)x^2 + b_2(u).
\]

We have

\[
\partial_1 H_1 = 2Z_1 + 2\lambda A_1, \quad \partial_2 H_1 = 2Z_2 + 2\lambda A_2.
\]

Consequently,

\[
0 = \partial_2 \partial_1 H_1 - \partial_1 \partial_2 H_1 = -2\lambda F_{12} + 2 \partial_2 Z_1 - 2 \partial_1 Z_2.
\]

Direct computations show that this implies \( t = \lambda b \), where \( b = b_1 + b_2 \). Thus

\[
F_{12} = \Psi^{3/2} (\lambda(u)b(u)((x^1)^2 + (x^2)^2) + a_4(u)x^i(u) + b(u)).
\]

Equality (4.9) for \( n = 2 \) can be written in the form

\[
F_{12} = \Psi^{3/2} \left( -\frac{1}{2} \lambda(u) C_{12}(u)((x^1)^2 + (x^2)^2) - \frac{1}{2} C_{12}(u) + 4C_2(u)x^1 - 4C_1(u)x^2 \right).
\]

This shows that \( F_{ij} \) is the same as for \( n \geqslant 3 \), and the rest of the proof for \( n = 2 \) is the same as for \( n \geqslant 3 \).

Theorem 1 is proved.
§ 5. Proof of Theorem 2

Consider the coordinates $v, x^1, \ldots, x^n, u$ and the metric $g$ as in Theorem 1.

1) Suppose that $\lambda(u)$ is nowhere vanishing. Consider the transformation

$$v \mapsto v - \phi, \quad \phi = \frac{2}{\lambda(u)} C_k(u)x^k\sqrt{\Psi}.$$

Then $A_i$ changes to

$$A_i + \partial_i \phi = \Psi \left(-2C_k(u)x^k x^i + C_i(u) \sum_{k=1}^n (x^k)^2 + \frac{C_i(u)}{\lambda(u)} \right).$$

By Lemma 3, for each $u$, $h^{ik} A_k$ is a Killing vector field of the Riemannian metric $h(u)$.

Following the ideas from [28], we are looking for a coordinate transformation in order to set $A_i$ to zero. Consider the coordinate transformation given by the inverse transformation

$$v = \bar{v}, \quad x^k = x^k(\bar{x}^i, u), \quad u = \bar{u}.$$ We have

$$\bar{A}_k = \frac{\partial x^i}{\partial \bar{x}^k} \left( A_i + h_{ij} \frac{\partial x^j}{\partial u} \right).$$

The equation $\bar{A}_k = 0$ is of the form

$$\frac{\partial x^j(\bar{x}^1, \ldots, \bar{x}^n, u)}{\partial u} = V^j(x^1(\bar{x}^1, \ldots, \bar{x}^n, u), \ldots, x^n(\bar{x}^1, \ldots, \bar{x}^n, u), u),$$

where $V^j = -h^{jk} A_k$. Considering $\bar{x}^1, \ldots, \bar{x}^k$ as parameters and imposing the initial condition

$$x^i(\bar{x}^1, \ldots, \bar{x}^n, u_0) = \bar{x}^i,$$

we see that the above system is a system of ordinary differential equations depending on the initial conditions as parameters. Such a system has a unique solution that gives the required transformation. Since for each $u$, $h^{ik} A_k$ is a Killing vector field of the Riemannian metric $h(u)$, the transformation $x^k = x^k(\bar{x}^i, u)$ is an isometry, hence $h$ remains the same. Thus we get the metric from Theorem 1 with $C_i(u) = 0$.

Applying the transformation $v \mapsto v + (1/(2\lambda(u))) K(u)$, we get $H_1 = -\partial_u \ln \Psi$.

2) Suppose that $\lambda$ is identically zero. The form $F$ is given by

$$F_{ij} = 32(C_j(u)x^i - C_i(u)x^j).$$

As explained in Theorem 1 we can take $A_i = 16C_i(u) \sum_{k=1}^n (x^k)^2$. Considering the transformation $x^i \mapsto x^i/2$ and replacing $2C_i(u)$ by $C_i(u)$ we get $A = C_i(u) \sum_{k=1}^n (x^k)^2$, that is, $A_i = C_i(u) \sum_{k=1}^n (x^k)^2$. Moreover, $h = \sum_{k=1}^n (dx^k)^2$. From the equation $P(X) = \bar{v} \wedge X$ it follows that

$$\partial_j H_1 = -\nabla_i F_{ij} = -2C_j.$$ Hence $H_1 = -2C_k x^k + K(u)$ for some function $K(u)$. 

Suppose that \( \sum_{k=1}^{n} C_{k}^{2} \neq 0 \). Consider the coordinate transformation given by the inverse transformation

\[
v = \tilde{v}, \quad x^{i} = \tilde{x}^{i} + b^{i}(\tilde{u}), \quad u = \tilde{u}
\]

such that \( 2C_{i}(u)b^{i}(u) = K(u) \). We then obtain \( H_{1} = -2C_{i}(u)x^{i} \), that is, we may assume that \( K(u) = 0 \). The equation \( T_{ij} = f\delta_{ij} \) for \( i \neq j \) takes the form

\[
\frac{1}{2} \partial_{i}\partial_{j}H_{0} = x^{i}x^{j} \sum_{k=1}^{n} C_{k} - 2C_{k}x^{k}(C_{j}x^{i} + C_{i}x^{j}) - C_{i}C_{j} \sum_{k=1}^{n} (x^{k})^{2} + (\dot{C}_{j}x^{i} + \dot{C}_{i}x^{j})/2.
\]

The general solution to this system under the condition \( T_{ii} = T_{jj} \) is given in the statement of the theorem.

Suppose that \( \sum_{k=1}^{n} C_{k}^{2} \equiv 0 \). It is easy to check that the nonzero components of the curvature tensor are \( R_{iuju} \), \( R_{uiuj} \), \( R_{iuiu} \), \( R_{uiju} \), hence the metric is a pp-wave [3], that is, \( h = \sum_{k=1}^{n}(dx^{k})^{2} \), \( A = 0 \) and \( H = H_{0} \). Moreover,

\[
H_{0} = a(u) \sum_{i=1}^{n} (x^{i})^{2} + D_{i}(u)x^{i} + D(u).
\]

Consider the new coordinates

\[
\tilde{v} = v - \sum_{j} \frac{db^{j}(u)}{du} x^{j} + d(u), \quad \tilde{x}^{i} = x^{i} + b^{i}(u), \quad \tilde{u} = u.
\]

We obtain a metric of the same form with

\[
\tilde{H}_{0} = a(u) \sum_{i=1}^{n} (x^{i})^{2} + \tilde{D}_{i}(u)x^{i} + \tilde{D}(u),
\]

where

\[
\tilde{D}_{j} = -2\left( \frac{db^{j}}{du} \right)^{2} + 2ab^{j} + D_{j}, \quad (5.1)
\]

\[
\tilde{D} = 2\left( \frac{dd(u)}{du} \right) + \sum_{j} \left( \frac{db^{j}}{du} \right)^{2} + a \sum_{j} (b^{j})^{2} + D_{i}b^{i} + D. \quad (5.2)
\]

Equation (5.1) implies the existence of \( b^{j}(u) \) such that \( \tilde{D}_{k} = 0 \). Using the last equation, we can choose \( d(u) \) such that \( \tilde{D} = 0 \). Thus,

\[
H_{0} = a(u) \sum_{i=1}^{n} (x^{i})^{2}.
\]

Theorem 2 is proved.
§ 6. Proof of Theorem 3

We consider the metric \( g \) from Theorem 1. Let \( g \subset \text{sim}(n) \) be the holonomy algebra of \((M, g)\).

Suppose that \( \lambda \equiv 0 \) on \( M \) and that \( \sum_{k=1}^{n} C_k^2 \neq 0 \). Then there exists a point \( x \in M \) such that \( \vec{v}_x \neq 0 \). The condition on the curvature tensor shows that

\[
R_x(p_x, q_x) = -p_x \wedge \vec{v}_x, \quad R_x(X, Y) = p_x \wedge ((X \wedge Y) \vec{v}_x).
\]

This shows that \( p_x \wedge E_x \subset g \). Next,

\[
R_x(\vec{v}_x, q_x) = -g(\vec{v}_x, \vec{v}_x)p_x \wedge q_x - p_x \wedge T_x(\vec{v}_x),
\]

which implies \( \mathbb{R}p_x \wedge q_x \subset g \). Finally,

\[
R_x(X, q_x) = -g(\vec{v}_x, X)p_x \wedge q_x + \vec{v}_x \wedge X - p_x \wedge T_x(X).
\]

Since the bivectors of the form \( \vec{v}_x \wedge X \) generate the Lie algebra \( \mathfrak{so}(E_x) \), we conclude that

\[
g = \mathbb{R}p_x \wedge q_x + \mathfrak{so}(E_x) + p_x \wedge E_x \simeq \text{sim}(n).
\]

If \( \sum_{k=1}^{n} C_k^2 \equiv 0 \) and \( a^2 \neq 0 \), then the metric is given by (1.2), and its holonomy algebra coincides with \( \mathbb{R}^n \). If \( \sum_{k=1}^{n} C_k^2(u) \equiv 0 \) and \( a^2(u) \equiv 0 \), then the metric is flat.

Suppose that \( \lambda \) is a nonzero constant. Then the metric can be written as the first metric from Theorem 2. We have

\[
R(p, q) = -\lambda p \wedge q, \quad R(X, Y) = -\frac{1}{2} \lambda X \wedge Y, \quad X, Y \in \Gamma(E).
\]

This shows that \( g \) contains the subalgebra \( \mathbb{R} \oplus \mathfrak{so}(n) \subset \text{sim}(n) \). Next,

\[
T_{ii} = \lambda \Psi^{3/2} D_k x^k + \Psi(\sqrt{\Psi} - 1)(a + \lambda D).
\]

If \( \sum_{k=1}^{n} D_k^2 + (a + \lambda D)^2 \neq 0 \), then the tensor \( T \) is not identically zero, and the equality \( \tilde{R}(X, q) = -n \text{tr} T p \wedge X \) shows that \( g \) contains \( \mathbb{R}^n \subset \text{sim}(n) \), that is, \( g = \text{sim}(n) \). Otherwise,

\[
g = \Psi \sum_{k=1}^{n} (dx^k)^n + 2dv du + (\lambda v^2 + 2D(u))(du)^2.
\]

We see that \( g \) is decomposable. The metric \( 2dv du + (\lambda v^2 + 2D(u))(du)^2 \) is of constant sectional curvature \( \lambda/2 \), hence it is isometric to the metric \( 2dv du + \lambda v^2(du)^2 \).

Suppose that \( \dot{\lambda} \neq 0 \) for some coordinate system. Then \( \lambda \neq 0 \) on an open subset of the domain of definition of this system, and the metric can be written as the first metric from Theorem 2. We have \( \vec{v} = -(1/2)\dot{\lambda}x^i \partial_i \). Let \( x \) be any point such that \( \vec{v}_x \neq 0 \). Then we have

\[
R_x(p_x, q_x) = -\lambda x p_x \wedge q_x - p_x \wedge \vec{v}_x \in g,
\]

\[
R_x(X, Y) = -\frac{\lambda}{2} X \wedge Y + p_x \wedge ((Y \wedge X) \vec{v}_x) \in g, \quad X, Y \in T_x M.
\]
Taking the Lie bracket of the last two elements of $\mathfrak{g}$, we obtain

$$p_x \wedge ((Y \wedge X)\vec{v}_x) \in \mathfrak{g}.$$ 

This shows that $\mathfrak{g}$ contains a subalgebra isomorphic to $\mathbb{R}^n \subset \mathfrak{g}$. The above two equalities imply that $\mathfrak{g}$ contains the subalgebra $\mathbb{R} \oplus \mathfrak{so}(n) \subset \mathfrak{sim}(n)$. Thus $\mathfrak{g} = \mathfrak{sim}(n)$.

Theorem 3 is proved.

§7. The Ricci operator of the obtained metrics

The Ricci operator of the metric (1.2) has the form

$$\text{Ric} = na(u) \partial_u \otimes du,$$

in particular, $\text{Ric}^2 = 0$.

In [19] conformally flat Lorentzian manifolds $(M, g)$ satisfying the condition

$$[R(X, Y), \text{Ric}] = 0$$

were studied. It was shown that these manifolds are exhausted by the spaces of constant sectional curvature, by the products of two spaces of constant sectional curvature, and by products of spaces of constant sectional curvature with intervals. The Ricci operator of the metric (1.2) satisfies (7.1). Moreover, such a metric is complete, for instance, for $a(u) = 1$, that is, for the Cahen-Wallach spaces. Thus some metrics were omitted in [19]. In [20], pseudo-Riemannian conformally flat manifolds $(M, g)$ satisfying (7.1) were studied. It was shown that in addition to the obvious cases, $(M, g)$ can be a complex sphere or a space satisfying $\text{Ric}^2 = 0$. Various examples of conformally flat manifolds with $\text{Ric}^2 \neq 0$ were constructed in [21].

The Ricci operator of the second metric from Theorem 2 is the following:

$$\text{Ric}(p) = 0, \quad \text{Ric}(X) = ng(X, \vec{v})p, \quad \text{Ric}(q) = nT_{11}p + n\vec{v}, \quad X \in \Gamma(E);$$

here $\vec{v} = -\sum_{k=1}^n C_k(u)X_k$. The function $T_{11}$ can be found using (3.6). We have $\text{Ric}^2 \neq 0$ and $\text{Ric}^3 = 0$. Condition (7.1) is not satisfied. The scalar curvature of this metric is zero.

For the first metric from Theorem 2 we have

$$\text{Ric}(p) = \lambda p, \quad \text{Ric}(X) = ng(X, \vec{v})p - (n - 1)\lambda X, \quad \text{Ric}(q) = nT_{11}p + n\vec{v} + \lambda q,$$

where $\vec{v} = -\lambda/2 \Psi x^kX_k$. The Ricci operator is not nilpotent. The scalar curvature equals $2\lambda + s_0 = -(n - 2)(n + 1)\lambda$ and it is zero only in dimension four.

§8. The case of dimension 4

Applying Theorem 5 to a conformally flat, but not flat, Lorentzian manifold $(M, g)$ of dimension 4 with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, 3)$, we obtain that $(M, g)$ must satisfy one of the following conditions:

1) $\mathfrak{g} = \mathfrak{so}(1, 3)$;
2) \( g \subset \text{sim}(2) \), that is, \((M, g)\) is as in Theorem 1 with \( n = 2 \);

3) \( g = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(2) \) and \((M, g)\) is locally isometric either to the product of \((dS_2, cg_{dS_2})\) with \((L^2, cg_{L^2})\), or to the product of \((A dS_2, cg_{A dS_2})\) with \((S^2, cg_{S^2})\);

4) \( g = \mathfrak{so}(1, 2) \) and \((M, g)\) is locally isometric to the product of \((\mathbb{R}, (dt)^2)\) either with \((dS_3, cg_{dS_3})\), or with \((A dS_3, cg_{A dS_3})\); or \( g = \mathfrak{so}(3) \), and \((M, g)\) is locally isometric to the product of \((\mathbb{R}, -(dt)^2)\) either with \((S^3, cg_{S^3})\), or with \((L^3, cg_{L^3})\).

Here \( c > 0 \) is a constant, and \( S^n, L^n, dS_n, A dS_n \) denote, respectively, the sphere, Lobachevskian space, de Sitter space and anti de Sitter space. The standard Friedmann-Robertson-Walker spacetime is conformally flat and has holonomy algebra \( \mathfrak{so}(1, 3) \) [6].

Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds are also classified in [6]. The first metric from Theorem 1 in dimension 4 was given in [7]. In [6], it is stated that it is an open problem to construct a conformally flat metric with the holonomy algebra \( \text{sim}(2) \) (which is denoted in [6] by \( R_{14} \)). An attempt to construct such a metric was made in [8], where the following metric was constructed:

\[
g = 2 \, dx \, dt + 4 \, dy \, dt \, dy - 4 \, dz \, dt \, dz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2. \tag{8.1}
\]

This metric is conformally flat. Applying the transformation

\[
x \mapsto x - y^2 + z^2, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t,
\]

we obtain

\[
g = 2 \, dx \, dt + 2x^2(dt)^2 + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2}. \tag{8.2}
\]

We get that the metric (8.1) is decomposable and its holonomy algebra coincides with \( \mathfrak{so}(1, 1) \oplus \mathfrak{so}(2) \). Thus in this paper we have obtained conformally flat metrics with holonomy algebra \( \text{sim}(2) \) for the first time and, what is more, we have found all such metrics.

Bibliography

[1] M. Kurita, “On the holonomy groups of the conformally flat Riemannian manifold”, Nagoya Math. J. 9 (1955), 161–171.

[2] R. Bryant, “Classical, exceptional, and exotic holonomies: a status report”, Actes de la Table Ronde de Géométrie Différentielle (Luminy 1992), Semin. Congr., vol. 1, Soc. Math. France, Paris 1996, pp. 93–165.

[3] A. Galaev and Th. Leistner, “Holonomy groups of Lorentzian manifolds: classification, examples, and applications”, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich 2008, pp. 53–96.

[4] M. Brozos-Vázquez, E. García-Rio, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo, The geometry of Walker manifolds (San Rafael, CA), Synth. Lect. Math. Stat., vol. 5, Morgan & Claypool Publ., Williston, VT 2009.

[5] Th. Leistner and P. Nurowski, “Ambient metrics for n-dimensional pp-waves”, Comm. Math. Phys. 296:3 (2010), 881–898.
[6] G. S. Hall and D. P. Lonie, “Holonomy groups and spacetimes”, Classical Quantum Gravity 17:6 (2000), 1369–1382.

[7] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact solutions of Einstein’s field equations, Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge 2003.

[8] R. Ghanam and G. Thompson, “Two special metrics with $R_{14}$-type holonomy”, Classical Quantum Gravity 18:11 (2001), 2007–2014.

[9] J. D. Norton, “Einstein, Nordström and the early demise of scalar, Lorentz-covariant theories of gravitation”, Arch. Hist. Exact Sci. 45:1 (1992), 17–94.

[10] N. Ravndal, “Scalar gravitation and extra dimensions”, Proceedings of the Gunnar Nordström Symposium on Theoretical Physics, Comment. Phys. Math., vol. 166, Finn. Soc. Sci. Lett., Helsinki 2004, pp. 151–164.

[11] V. P. Vizgin, Relativistic theory of gravitation (sources and formation, 1900–1915), Nauka, Moscow 1981. (Russian)

[12] G. W. Gibbons and C. N. Pope, “Time-dependent multi-centre solutions from new metrics with holonomy Sim(n − 2)”, Classical Quantum Gravity 25:12 (2008), ID 125015.

[13] H. Baum, “Conformal Killing spinors and the holonomy problem in Lorentzian geometry — a survey of new results”, Symmetries and overdetermined systems of partial differential equations, IMA Vol. Math. Appl., vol. 144, Springer, New York 2008, pp. 251–264.

[14] Ya. V. Bazaliik, “Globally hyperbolic Lorentzian spaces with special holonomy groups”, Sibirsk. Mat. Zh. 50:4 (2009), 721–736; English transl. in Siberian Math. J. 50:4 (2009), 567–579.

[15] G. Hall, “Projective relatedness and conformal flatness”, Cent. Eur. J. Math. 10:5 (2012), 1763–1770.

[16] D. V. Alekseevskii (Alekseevsky) and B. N. Kimel’fel’d, “Classification of homogeneous conformally flat Riemannian manifolds”, Mat. Zametki 24:1 (1978), 103–110; English transl. in Math. Notes 24:1 (1978), 559–562.

[17] V. F. Kirichenko, “Conformally flat and locally conformal Kähler manifolds”, Mat. Zametki 51:5 (1992), 57–66; English transl. in Math. Notes 51:5 (1992), 462–468.

[18] V. V. Slavskii, “Conformally-flat metrics and pseudo-Euclidean geometry”, Sibirsk. Mat. Zh. 35:3 (1994), 674–682; English transl. in Siberian Math. J. 35:3 (1994), 605–613.

[19] M. Erdogan and T. Ikawa, “On conformally flat Lorentzian spaces satisfying a certain condition on the Ricci tensor”, Indian J. Pure Appl. Math. 26:5 (1995), 417–424.

[20] K. Honda, “Conformally flat semi-riemannian manifolds with commuting curvature and Ricci operators”, Tokyo J. Math. 26:1 (2003), 241–260.

[21] K. Honda and K. Tsukada, “Conformally flat semi-Riemannian manifolds with nilpotent Ricci operators and affine differential geometry”, Ann. Global Anal. Geom. 25:3 (2004), 253–275.

[22] K. Honda and K. Tsukada, “Three-dimensional conformally flat homogeneous Lorentzian manifolds”, J. Phys. A 40:4 (2007), 831–851.

[23] D. V. Alekseevsky and A. S. Galaev, “Two-symmetric Lorentzian manifolds”, J. Geom. Phys. 61:12 (2011), 2331–2340.

[24] H. Wu, “On the de Rham decomposition theorem”, Illinois J. Math. 8 (1964), 291–311.
[25] K. Yano, “On pseudo-Hermitian and pseudo-Kählerian manifolds”, Proceedings of the International Congress of Mathematicians, vol. III (Amsterdam 1954), North-Holland, Amsterdam 1954, pp. 190–197.

[26] A.S. Galaev, “The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds”, Differential Geom. Appl. 22:1 (2005), 1–18.

[27] A.S. Galaev, “One component of the curvature tensor of a Lorentzian manifold”, J. Geom. Phys. 60:6–8 (2010), 962–971.

[28] A.S. Galaev and T. Leistner, “On the local structure of Lorentzian Einstein manifolds with parallel distribution of null lines”, Classical Quantum Gravity 27 (2010), ID 225003.

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