Open-Loop Control Design via Parametrization 
Applied in a Two-Level Quantum System Model

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Abstract—In the design of quantum computing devices of the future the basic element is the qubit. It is a two-level quantum system which may describe population transfer from one steady-state to another controlled by a coherent laser field. A four-dimensional real-variable differential equation model is constructed from the complex-valued two-level model describing the wave function of the system. The state transition matrix of the model is constructed via the Wei-Norman technique and Lie algebraic methodology. The idea of parametrization using flatness-based control, is applied to construct feasible input–output pairs of the model. This input drives the state of the system from the given initial state to the given final state in a finite time producing the corresponding output of the pair. The population transfer is obtained by nullifying part of the state vector via careful selection of the parameter functions. A preliminary simulation study completes the paper.

I. INTRODUCTION

In quantum mechanical framework deterministic bits "1" and "0" are substituted by the qubit [22]. The qubit is a composition of the pure states "1" and "0". This composition means that the actual state of the qubit is not exactly "1" or "0" but a combination of these. In the measurement, however, the outcome is always one of the two possibilities "1" or "0". The qubit can be represented as a point on the surface of a sphere, so-called Bloch sphere, see e.g. [22]. If one wants to save information into a qubit, then the key problem is to drive the qubit from one state to another. Then one arrives at the description of the qubit as a dynamic differential equation system, the controls of which are the parameters of the driving laser field.

In quantum computation the qubit forms a basic element for building up multi-qubit computing elements of future quantum computers, see [16]. Then a key problem is to drive the qubit from one stable level to another.

Molecular excitation, i.e. driving of an ensemble of molecules from one locally stable steady state to another is one alternative for a qubit structure. This type of systems are controlled by using coherent light. Based on laser technology shorter and shorter coherent pulses can be generated for controlling molecular excitation, see [16]–[19]. The goal is to direct molecular reactions towards unprobable but desirable direction [3]–[5]. Then nonlinear and more and more sophisticated control methods are needed for properly designing durations and forms of the control pulses. In classical N-level problems the system to be controlled can be modelled by using ordinary 2N-dimensional differential equation systems. Due to femto- and picosecond scale pulses feedback is not in general applicable in the control design for these systems. Flatness-based control, see [6]–[9] & [11], is then an ideal methodology for open-loop design relevant in quantum control problems.

Due to the fact that the bilinear quantum control systems are not controllable in the whole Euclidian space $\mathbb{R}^{2N}$ the methodology applied here is called parametrization.

This two-level quantum control problem and some related studies have been carried out by several authors earlier, too, see [1], [10], & [20]. Especially, in [20] a very similar approach as ours is used.

However, we start from the basic definition of differential flatness. The system

$$\frac{dx}{dt} = f(x, u); \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m$$  \hspace{1cm} (1)

is called differentially flat if there exists algebraic functions $\{A, B, C\}$, and finite integers $\alpha, \beta, \gamma$ such that for any pair $(x,u)$ of inputs and controls, satisfying the dynamics (1), there exists a function $z$, called a flat (or linearizing) output, such that the following equations are satisfied

$$\begin{align*}
x(t) &= A(z, \dot{z}, \ldots, z^{(\alpha)}) \\
u(t) &= B(z, \dot{z}, \ldots, z^{(\beta)}) \\
z(t) &= C(x, u, \dot{u}, \ldots, \dot{u}^{(\gamma)}). \hspace{1cm} (2)
\end{align*}$$

The actual output $y$, which is not present in the definition of flatness, may have the dependence

$$y(t) = h(x(t), u(t))$$

for some given output function $h$. In parametrization procedure, due to uncontrollability, the last equation in (2) for $z(t)$ is neither constructed nor applied.

From the standard finite-state Schrödinger equation of two energy levels a four-dimensional real-variable differential equation model is obtained. The Wei-Norman technique is used in the construction according to [21]. The exponential representation of the transition matrix of the system includes three base functions, two of which serve as the parameter functions. In this framework the initial and final states can be defined corresponding to the two levels of the original system model. Then parametrization design is applied for explicitly calculating the parameter functions, which in turn give the desired input–output pairs.

II. SYSTEM MODELS

Population transfer in a two-level quantum system, see [4], can be described by the time-dependent Schrödinger
equation, i.e. by the dynamics
\[
\frac{d\tilde{\psi}}{dt} = \tilde{H}(t) \tilde{\psi}, \quad \tilde{H}(t) = \begin{bmatrix} E_1 & \Omega(t) \\ \Omega^*(t) & E_2 \end{bmatrix},
\]
where the modified Planck’s constant \( h = \frac{\hbar}{2\pi} \) has been scaled to \( h = 1 \), and \( i = \sqrt{-1} \). The wavefunction \( \tilde{\psi} : \mathbb{R} \to \mathbb{C}^2 \) has the probabilistic interpretation, in the sense that
\[
||\tilde{\psi}(t)||^2 = ||\tilde{\psi}_1(t)||^2 + ||\tilde{\psi}_2(t)||^2 = 1, \quad \forall t \in \mathbb{R},
\]
where \( \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2) \). The control is given by \( \Omega : \mathbb{R} \to \mathbb{C} \), and \( \Omega^* \) is the complex conjugate of \( \Omega \). \( E_1 \) and \( E_2 \) are the energy levels. The unitary transformation \( \tilde{\psi} \mapsto \psi \) and \( \Omega \mapsto u \) by
\[
\tilde{\psi}(t) = U(t) \psi(t),
\]
\[
U(t) = \begin{bmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{bmatrix}
\]
\[
u(t) = e^{-i(E_2 - E_1)t} \Omega(t)
\]
transforms (3) to
\[
\frac{d\psi}{dt} = H(t) \psi,
\]
\[
H(t) = \begin{bmatrix} 0 & u(t) \\ u^*(t) & 0 \end{bmatrix}.
\]
The componentwise representation
\[
\psi(t) = \psi_1(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \psi_2(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
converts (8) to the dynamics
\[
\dot{\psi}_1 = -i u \tilde{\psi}_2,
\]
\[
\dot{\psi}_2 = -i u^* \tilde{\psi}_1.
\]
By using the real-valued decompositions
\[
\begin{cases}
\dot{\psi}_1 = x_1 + i x_2 \\
\dot{\psi}_2 = x_3 + i x_4 \\
\psi_1 = u_1 + i u_2
\end{cases}
\]
one obtains a state-variable representation
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = \begin{bmatrix}
x_4 & x_3 \\
x_2 & x_1 \\
-x_1 & -x_2
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]
or in another form
\[
\frac{dx}{dt} = (u_1 F_1 + u_2 F_2) x,
\]
where the constraint (4) is converted into the form
\[
\sum_{k=1}^{4} x_k^2 = 1.
\]

**Remark 1:** The matrices \( F_1 \) and \( F_2 \) together with their Lie product \( 2F_3 = [F_1, F_2] = F_1 F_2 - F_2 F_1 \) form a Lie algebra. This can be used as a basis for differential geometric considerations of the control system (13). However, the elementary approach applied in this paper is sufficient for our parametrization purposes.

### III. WEI-NORMAN REPRESENTATION

The Lie algebra of the matrices \( F_1, F_2, \) and \( F_3 \) is three-dimensional with the relations
\[
[F_1, F_2] = 2F_3,
\]
\[
[F_2, F_3] = 2F_1,
\]
\[
[F_3, F_1] = 2F_2,
\]
\[
F_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}
\]
Due to the linear structure of the system model (14) with respect to the state \( x \), the state transition matrix of the system, denoted by \( \Phi \), and which relates the values of the state according to
\[
x(t) = \Phi(t, 0) x(0)
\]
can be written as a product of exponentials
\[
\Phi(t, 0) = e^{g_1 F_1} e^{g_2 F_2} e^{g_3 F_3},
\]
where the exponentials are defined by the absolutely converging infinite series
\[
e^{g_i F_i} = \sum_{k=0}^{\infty} \frac{g_i^k F_i^k}{k!}, \quad i = 1, 2, 3.
\]
The state transition matrix satisfies the following initial-value problem (IVP1)
\[
\frac{\partial}{\partial t} \Phi(t, 0) = F(t) \Phi(t, 0); \quad \Phi(0, 0) = I,
\]
\[
F(t) = u_1(t) F_1 + u_2(t) F_2 + 0 \cdot F_3.
\]
The technique we are using is nowadays called Wei-Norman technique according to the paper of Wei and Norman [21]. Substitution of the (24) to the IVP1 gives
\[
\frac{\partial}{\partial t} \Phi = \dot{g}_1 F_1 + \dot{g}_2 e^{g_1 F_1} F_2 e^{-g_1 F_1} \Phi + \dot{g}_3 e^{g_1 F_1} e^{g_2 F_2} F_3 e^{-g_2 F_2} e^{-g_1 F_1} \Phi.
\]
By using (several times) the Campbell-Baker-Hausdorff formula for square matrices \( A \) and \( B \) of the same dimension
\[
e^{A} B e^{-A} = B + [A, B] + [A, [A, B]]/2! + \cdots
\]
Then we have

$$\frac{\partial \Phi}{\partial t} = [f_1(t)F_1 + f_2(t)F_2 + f_3(t)F_3] \Phi$$  \hspace{1cm} (30)

$$f_1(t) = \dot{g}_1 + \dot{g}_3 \sin(2g_2)$$  \hspace{1cm} (31)

$$f_2(t) = \dot{g}_2 \cos(2g_1) - \dot{g}_3 \cos(2g_2) \sin(2g_1)$$  \hspace{1cm} (32)

$$f_3(t) = \dot{g}_2 \sin(2g_1) + \dot{g}_1 \cos(2g_2) \cos(2g_1)$$  \hspace{1cm} (33)

By comparing the coefficients of the \( F_i \)'s in (30) and (27) one finally obtains a differential relation between the \( g_i \)'s and the controls \( u_1 \) and \( u_2 \) in the form of a matrix equation

$$\begin{bmatrix}
1 & 0 & \sin(2g_2) \\
0 & \cos(2g_1) & -\cos(2g_2) \sin(2g_1) \\
0 & \sin(2g_1) & \cos(2g_2) \cos(2g_1)
\end{bmatrix}
\begin{bmatrix}
\dot{g}_1 \\
\dot{g}_2 \\
\dot{g}_3
\end{bmatrix} =
\begin{bmatrix}
u_1 \\
u_2 \\
0
\end{bmatrix}$$  \hspace{1cm} (34)

the coefficient matrix being the same as in [2], Eq. (3.7).

The relation \( g \leftrightarrow u \) is invertible if the determinant of the coefficient matrix denoted by \( D \) is different from zero

$$|D| = \cos(2g_2) \neq 0.$$  \hspace{1cm} (35)

Then we have

$$D^{-1} = \frac{1}{\cos(2g_2)} \times$$  \hspace{1cm} (36)

$$\begin{bmatrix}
\cos(2g_2) & -\sin(2g_1) \sin(2g_2) & \cos(2g_1) \sin(2g_2) \\
0 & \cos(2g_1) \cos(2g_2) & \sin(2g_1) \cos(2g_2) \\
0 & -\sin(2g_1) & \cos(2g_2)
\end{bmatrix}
\begin{bmatrix}
\dot{g}_1 \\
\dot{g}_2 \\
\dot{g}_3
\end{bmatrix} =
\begin{bmatrix}
u_1 \\
u_2 \\
0
\end{bmatrix}$$

$$g = D^{-1} \tilde{u},$$  \hspace{1cm} (37)

where \( \tilde{u} \) and \( u \) are defined by

$$\tilde{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \end{bmatrix}.$$  \hspace{1cm} (38)

**IV. MODEL PARAMETRIZATION**

Because the system has two (scalar) controls we can choose two of the three base functions \( g_i \) freely corresponding to free selection of the two controls. The third base function has to be determined from the last equation of (23). Parametrization actually means that the input–output pairs can be determined from the parameter functions without explicitly solving of the system equations according to Fig. 1. Due to the flatness-based design idea, computation of the third base function as well as of the controls must not include integrations as given by the equations (2). Only differentiations are allowed. Consequently, based on the third equation in (23), the base functions \( g_2 \) and \( g_3 \) are chosen as parameter functions. Then these are also so-called flat outputs, see [8], denoted by \( z = (z_1, z_2) = (g_2, g_3) \). The

The parametrization obtained in this way for \( g_1 \) and the controls are given by

$$g_1 = \frac{1}{2} \arctan \left[ -\cos(2g_2) \frac{\dot{g}_3}{\dot{g}_2} \right]$$  \hspace{1cm} (39)

$$u_1 = \dot{g}_1 + \dot{g}_3 \sin(2g_2)$$  \hspace{1cm} (40)

$$u_2 = \sqrt{\dot{g}_2^2 + \dot{g}_3^2 \cos^2(2g_2)}.$$  \hspace{1cm} (41)

The state variables are calculated by using the state transition matrix equations (23) and (24)

$$x(t) = \Phi(t, 0)x(0) = e^{g_1F_1} e^{g_2F_2} e^{g_3F_3} x(0).$$

**V. CONTROL OBJECTIVE**

In population transfer problems from the level 1 corresponding to the situation

$$|\psi_1(0)|^2 = x_1(0)^2 + x_2(0)^2 = 0$$  \hspace{1cm} (42)

to the level 2, where

$$|\psi_2(T)| = x_3(T)^2 + x_4(T)^2 = 0,$$  \hspace{1cm} (43)

where \( T \) is the transfer time, we can parametrize the partial trajectory by using a sufficiently smooth, but otherwise arbitrarily chosen, parametrization \( x_1, x_2 \) with the boundary conditions

$$x_1(0)^2 + x_2(0)^2 = 0,$$  \hspace{1cm} (44)

$$x_1(T)^2 + x_2(T)^2 = 1.$$  \hspace{1cm} (45)

By dividing the state vector into two parts

$$x(t) = (w(t), v(t))$$  \hspace{1cm} (46)

$$w(t) = (x_1(t), x_2(t))$$  \hspace{1cm} (47)

$$v(t) = (x_3(t), x_4(t))$$  \hspace{1cm} (48)
we can represent the task of driving the state from the initial one to the final one in a finite time $T$ as follows

$$x(0) = \begin{bmatrix} 0 \\ x_{30} \\ x_{40} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} x_{1T} \\ x_{2T} \\ 0 \end{bmatrix} = x(T) \quad \text{(49)}$$

$$\begin{bmatrix} 0 \\ 0 \\ \sin \alpha \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \cos \beta \\ \sin \beta \\ 0 \end{bmatrix} \quad \text{(50)}$$

We have chosen a specific parametrization for the initial and final values of the state, because the sum of the squares of the nonzero state components must be equal to 1 at the both ends of the planned trajectory.

**VI. PARAMETRIZATION DESIGN**

The state transition equation $x(T) = \Phi(T, 0)x(0)$ can now be written in the form

$$\begin{bmatrix} w_T \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ v_0 \end{bmatrix} \quad \text{(51)}$$

$$\therefore \begin{bmatrix} w_T = Bv_0 \\ 0 = Dv_0 \end{bmatrix} \quad \text{(52)}$$

where $A, B, C,$ and $D$ are $2 \times 2$-blocks of the $4 \times 4$-dimensional state transition matrix $\Phi(T, 0)$.

For the state transition matrix $\Phi(t, 0) = e^{g_1(t)F_1}e^{g_2(t)F_2}e^{g_3(t)F_3}$

where the exponentials are defined by the series

$$e^{g_i F_i} = \sum_{k=0}^{\infty} \frac{1}{k!} g_i^k F_i^k, \quad i = 1, 2, 3 \quad \text{(54)}$$

we obtain the series representations in closed form

$$e^{g_i F_i} = \cos g_i I + \sin g_i F_i \quad \text{(55)}$$

due to the fact that $F_i^2 = -I$, $i = 1, 2, 3$, where $I$ is $4 \times 4$ identity matrix. Then the product of the three exponent functions is of the form

$$\Phi = (c_1 I + s_1 F_1)(c_2 I + s_2 F_2)(c_3 I + s_3 F_3) \quad \text{(56)}$$

$$c_i = \cos g_i, \quad s_i = \sin g_i, \quad i = 1, 2, 3. \quad \text{(57)}$$

Now the $D$-part and $B$-part of the transfer matrix $\Phi$ are given by

$$D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \quad \text{(58)}$$

$$= c_1 c_2 \begin{bmatrix} c_3 & s_3 \\ -s_3 & c_3 \end{bmatrix} - s_1 s_2 \begin{bmatrix} s_3 & -c_3 \\ c_3 & s_3 \end{bmatrix}, \quad \text{(59)}$$

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad \text{(60)}$$

$$= c_1 s_2 \begin{bmatrix} c_3 & s_3 \\ -s_3 & c_3 \end{bmatrix} - s_1 c_2 \begin{bmatrix} s_3 & -c_3 \\ c_3 & s_3 \end{bmatrix}. \quad \text{(61)}$$

We must have $D = 0$ due to the requirement $Dv_0 = 0$ for arbitrary $v_0 = (x_{30}, x_{40})$ satisfying the requirement $x_{30}^2 + x_{40}^2 = 1$. Then we have two alternatives in (59):

$$\begin{cases} a) \quad c_1 = s_2 = 0 \\ \therefore \quad D = 0 \quad \Rightarrow \quad Dv_0 = 0. \quad \text{(62)} \end{cases}$$

These conditions are obtained from the two basic alternatives

$$\begin{cases} a) \quad \begin{bmatrix} \cos g_1(T) = 0, \quad g_1(T) = \frac{\pi}{2} \\ \sin g_2(T) = 0, \quad g_2(T) = 0, \end{bmatrix} \\ b) \quad \begin{bmatrix} \sin g_1(T) = 0, \quad g_1(T) = 0 \\ \cos g_2(T) = 0, \quad g_2(T) = \frac{\pi}{2}. \end{bmatrix} \quad \text{(63)} \end{cases}$$

In the case of the first alternative $a)$ we have

$$\begin{cases} s_1 = \sin g_1(T) = 1 \\ c_2 = \cos g_2(T) = 1. \end{cases} \quad \text{(65)}$$

Consequently,

$$B = -s_1 c_2 \begin{bmatrix} s_3 & -c_3 \\ c_3 & s_3 \end{bmatrix}, \quad \text{(66)}$$

$$w_T = Bv_0 = -\begin{bmatrix} \sin g_3 \cos g_3 & -\cos g_3 \\ \cos g_3 \sin g_3 & \sin g_3 \end{bmatrix} \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix} \quad \text{(67)}$$

$$= -\begin{bmatrix} \sin g_3 \sin \alpha - \cos g_3 \cos \alpha \\ \cos g_3 \sin \alpha + \sin g_3 \cos \alpha \end{bmatrix} \quad \text{(68)}$$

$$= \begin{bmatrix} \cos(-g_3 - \alpha) \\ \sin(-g_3 - \alpha) \end{bmatrix} = \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix}. \quad \text{(69)}$$

$$\therefore \quad g_3(T) = -(\alpha + \beta). \quad \text{(70)}$$

In the same way the alternative $b)$ can be solved giving

$$g_3(T) = \frac{\pi}{2} - (\alpha + \beta). \quad \text{(71)}$$

Due to trigonometric functions in the equations there are also other possibilities for the final values of $g_2$ and $g_3$ deviating by the multiples of $\pi$ or $2\pi$. These possibilities need further considerations and are not studied here. We choose the alternative $b)$ for the basis of our control design. So, we have to find sufficiently differentiable parameter functions $g_2$ and $g_3$, which together with the dependent basis function $g_1$ have to satisfy the boundary conditions

$$\begin{cases} g_1(0) = 0, \quad g_1(T) = 0; \\ g_2(0) = 0, \quad g_2(T) = \frac{\pi}{2}; \\ g_3(0) = 0, \quad g_3(T) = \frac{\pi}{2} - (\alpha + \beta). \end{cases} \quad \text{(72)}$$

The final value of $g_1$ depends on the derivatives of $g_2$ and $g_3$. This means that we have to adjust these derivatives via the equation (59) to agree with the requirement $g_1(T) = 0$.

Carefully planned and realized simulations are needed to confirm the feasibility of our parametrization approach.
VII. SIMULATION STUDY

First preliminary simulation results demonstrate that the methodology developed actually drives the state of the system from the given initial state (level 1) to the given final state (level 2). A minimal parametrization for the parameter functions \(g_2\) and \(g_3\) were chosen without any specific optimization procedure. The only requirements are that the given boundary conditions (72) are satisfied, and that the equation which gives the base function \(g_1\) also gives the correct initial and final values for \(g_1\). The following values were used in the simulations

\[
\begin{align*}
\alpha &= -2\pi/3, \\
\beta &= \pi/3, \\
T &= 10.
\end{align*}
\]

Then the final value for \(g_3\) becomes

\[
g_3(T) = \frac{5\pi}{6} \approx 2.62. \tag{74}
\]

Because \(g_2\) has to change from 0 to \(\pi/2\), we chose the linear function

\[
g_2(t) = \frac{\pi}{2} \frac{t}{T} \approx 0.157 t. \tag{75}
\]

The boundary values

\[
\begin{align*}
g_1(0) &= 0, \\
g_1(T) &= 0
\end{align*} \tag{76}
\]

are obtained when we choose

\[
\begin{align*}
g_2(0) &= 0, \\
g_2(T) &= 0.
\end{align*} \tag{77}
\]

Then the third order polynomial suffices

\[
g_3(t) = a_0 + a_1 \frac{t}{T} + a_2 \left( \frac{t}{T} \right)^2 + a_3 \left( \frac{t}{T} \right)^3 . \tag{78}
\]

The coefficients are obtained from the boundary conditions, giving finally

\[
g_3(t) = \gamma \left\{ 3 \left( \frac{t}{T} \right)^2 - 2 \left( \frac{t}{T} \right)^3 \right\} , \quad \gamma = \frac{\pi}{4} - (\alpha + \beta) . \tag{79}
\]

The binding condition

\[
\dot{g}_2 \sin(2g_1) + \dot{g}_3 \cos(2g_2) \cos(2g_1) = 0 \tag{80}
\]

gives the base function \(g_1\) for the given parameter functions (75) and (79). The functions are depicted in Fig. 2 and 3. The controls were calculated by using the formulas (40) and (41). They are depicted in Fig. 4.

The behaviour of the state variables are given in Fig. 5 indicating that the desired final state, where

\[
x_3(T) = x_4(T) = 0
\]

has been obtained. The simulations were carried out and the figures produced by using Mathematica 7 package [25].

\[
\begin{align*}
\dot{x}_3 &= 0, \\
\dot{x}_4 &= 0.
\end{align*}
\]

VIII. CONCLUSIONS

The parametrization idea for constructing open-loop controls for uncontrollable bilinear systems is applied here. We have also earlier studied parametrization of systems described by partial differential equations and pseudo-differential operator models, see [12]–[14]. Flatness-based ideas, originally developed by Michel Fliess and his co-workers [7]–[9] have been developed for open-loop control design. In some quantum control problems, where laser pulses are used for the control, the dynamics is so fast that, at least at the present level of the speed of possible computations, feedback control seems to be impossible to implement even if so-called homodyne detection principles
can be applied to obtain closed-loop controls.

Here we studied a two-level population transfer problem. Without more advanced differential geometric considerations, which might be helpful in understanding quantum phenomena in general, we use the formulation found generally in the literature, to obtain our basic driftless system model of the form \( \dot{x} = g(x)u \), where \( g \) is linear in the state \( x \).

Simulation study was required to confirm the quantum control approach chosen. Then depending on the choice of the alternatives \( a) \) or \( b) \) different state trajectories can be obtained resulting, however, the same final state of the system when the flatness-based control is applied. Our preliminary simulations were based of the alternative \( b) \).

The basic technique applied here is useful also in multi-qubit systems and in controlling entanglement of, say, two or more qubits. Then tensor product formalism in the Euclidian framework is a feasible alternative in the system model design.

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