Quantum quench dynamics of some exactly solvable models in one dimension

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The dynamics of the Luttinger model and the sine-Gordon model (at the Luther-Emery point as well as in the semiclassical approximation) after a quantum quench is studied. We compute in detail one and two-point correlation functions for different types of quenches: from a non-interacting to an interacting Luttinger model and vice-versa, and from the gapped to the gapless phase of the sine-Gordon model and vice-versa. A progressive destruction of the Fermi gas features in the momentum distribution is found in the case of a quench into an interacting state in the Luttinger model. The critical exponents for spatial correlations are also found to be different from their equilibrium values. Correlations following a quench of the sine-Gordon model from the gapped to the gapless phase are found in agreement with the predictions of Calabrese and Cardy [Phys. Rev. Lett. \textbf{96} 136801 (2006)]. However, correlations following a quench from the gapped to the gapless phase at the Luther-Emery and the semi-classical limit exhibit a somewhat different behavior, which may indicate a break-down of the semiclassical approximation or a qualitative change in the behavior of correlations as one moves away from the Luther-Emery point. In all cases, we find that the correlations at infinite times after the quench are well described by a generalized Gibbs ensemble [M. Rigol et al. Phys. Rev. Lett. \textbf{98}, 050405 (2007)], which assigns a momentum dependent temperature to each eigenmode.

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\textbf{I. INTRODUCTION}

Most of the theoretical effort in the field of strongly correlated quantum systems over the past few decades has focused on understanding the equilibrium properties of these fascinating systems. For instance, achieving a complete understanding of the phase diagram of rather “simple” models like the two-dimensional fermionic Hubbard model still remains a huge challenge. Nevertheless, however important these endeavors are, understanding the phase diagram and the equilibrium properties of the phases of strongly correlated systems will not certainly exhaust the possibilities for finding new and surprising phenomena in these complex systems, especially out of equilibrium.

In classical systems, the existence of steady states out of equilibrium is well known. Very often, however, the properties of such states have very little to do with the equilibrium properties of the systems where they occur. Moreover, also very often their existence cannot be inferred from any previous knowledge about the equilibrium phase diagram: They are \emph{emergent} phenomena. One good example is provided by the appearance of Rayleigh-Benard convection cells when fluid layer is driven out of equilibrium by the application of a temperature gradient. Although dissipation plays an important role in the formation of those classical non-equilibrium states, one may also wonder if also non-equilibrium steady states can appear when quantum systems are driven out of equilibrium. Differently from classical systems, dissipation in quantum systems causes decoherence, which usually destroys any quantum interference effects that could lead to quantum non-equilibrium states without classical analog. However, it is known that decoherence, due to coupling with the environment, is always present in most quantum many-body systems for large enough collections of particles. This may explain why the study of non-equilibrium phenomena in quantum many-body systems has been regarded, until very recently, as a subject of mostly academic interest.

The recent availability of highly controllable systems such as mesoscopic heterostructures (e.g. quantum dots) and especially ultra-cold atomic gases has finally provided the largely lacking experimental motivation for the study of non-equilibrium phenomena, leading to an explosion of theoretical activity.\(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29\)

Cold atomic gases are especially interesting because they are very weakly coupled to the environment, thus remaining fully quantum coherent for fairly long times (compared to the typical duration of an experiment). At the same time, it is relatively easy to measure the coherent evolution in time of observables such as the density or the momentum distribution. Thus, theorists can now begin to pose questions such as: Assuming that a many-body system is prepared in a given initial state that is not an eigenstate of the Hamiltonian, how will it evolve in time? And, more specifically, will it reach a stationary or quasi-stationary state? If so, what will be the properties of such a state? How much memory will
the system retain of its initial conditions?

From another point of view, the problem described in the previous paragraph can be formulated as the study of the response of a system to a sudden perturbation in which the Hamiltonian is changed over a time scale much shorter than any other characteristic time scales of the system. In what follows, we shall refer to this type of experiment as a quantum quench. Quantum quenches are also of particular interest to the ‘quantum engineering’ program for cold atomic gases. The reason is that, if we intend to use these highly tunable and controllable systems as quantum simulators of models of many-body Physics (such as the 2D fermionic Hubbard model mentioned above), it is utterly important to understand to what extent the final state of the quantum simulator depends on the state in which it was initially prepared. In particular, one is interested in finding out whether the observables in the final state can be obtained from a standard statistical ensemble (say, the microcanonical, or the canonical ensemble at an effective temperature). If this is so, one would speak of thermalization. If this does not happen, then how much memory does the system retain about its initial state beyond the average energy $E = \langle H \rangle$?

We would like to emphasize that the above questions are not a merely academic. Indeed, cold atomic systems allow for the study of non adiabatic dynamics when the system is driven between two quantum phases such as a superfluid and a Mott insulator. If this does not happen, then how much memory does the system retain about its initial state beyond the average energy $E = \langle H \rangle$?

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In this article, we will not try to answer the difficult questions posed in the previous paragraph. Instead, we focus on analyzing the quench dynamics of two relatively well-known one dimensional models: the Luttinger model (for which a brief account of the results has been already published elsewhere) and the sine-Gordon model. For the latter, we present results in two solvable limits in which the Hamiltonian can be reduced to a quadratic form of creation and destruction operators. The long time behavior of the single particle Green’s function after a quench for general quadratic Hamiltonians of this form
has been recently studied by Barthel and Scholwöck. These authors provided some general conditions for the appearance of dephasing and steady non-thermal states. This question has also been taken up recently by Kollar and Eckstein. However, since the models taken up in this article may be relevant for the experiments with cold atomic gases (see Sect. VI) or numerical simulations, it is important to obtain analytical results for them. The simplicity of these models also allows us to test in detail a number of general results.

The rest of this article is organized as follows. In Sect. II we describe the how the dynamics after a quantum quench can be obtained as a time-dependent canonical transformation for fairly general quadratic Hamiltonians. The solution is used extensively in Sects. III and IV to obtain the evolution of observables and correlation functions of the Luttinger and sine-Gordon models. For the Luttinger model we consider the case where the interaction between the fermions is suddenly switched on (for which some of the results where briefly reported in Ref. II) and the reverse situation, when the interaction in suddenly switched off. For the sine-Gordon model, we study the situations when the system is quenched between the gapped and the gapless phases of the model, and vice versa. We show that in the latter case (i.e. when the system is quenched from the gapless to the gapped phase), the results obtained at the Luther-Emery point (where the model can be exactly mapped to a quadratic fermionic Hamiltonian) and in the so-called semiclassical approximation are qualitatively different. The origin of this difference is still not well understood, and we cannot discard that it is indeed an artifact of the quasi-classical approximation (which, however, yields results in agreement with those obtained at the Luther-Emery point for the case when the system is quenched form the gapped to the gapless phase). In Sect. VI we show the long time behavior of some of the correlation functions in these models can be obtained from a generalized Gibbs ensemble (along with instances where it fails). The experimental relevance of our results is briefly discussed in Sect. VII along with other conclusions of this work. Finally, the details of some of the most lengthy calculations are provided in the appendices.

II. QUADRATIC HAMILTONIANS

As an illustrative calculation, let us first study the case of a quantum quench in a model described by a quadratic Hamiltonian:

\[
H(t) = \sum_q \hbar [\omega_0(q) + m(q, t)] b_q^\dagger b_q + \frac{1}{2} \sum_q \hbar g(q, t) [b_q b_{-q} + b_{-q}^\dagger b_q^\dagger (-q)],
\]

where \([b_q, b_{q'}^\dagger] = \delta_{q,q'},\) commuting otherwise. We will assume that the quench takes place at \(t = 0,\) so that, within the sudden approximation, the system is described by \(H_t = H(t \leq 0)\) for \(t < 0\) and by \(H_t = H(t > 0)\) for \(t > 0.\) Furthermore, in order to simplify the analysis, we assume that \(m(q, t \leq 0) = g(q, t \leq 0) = 0,\) and \(m(q, t > 0) = m(q)\) and \(g(q, t) = g(q).\) Notice that the initial Hamiltonian is diagonal in the \(b\)-operators:

\[
H_t = H_0 = \sum_q \hbar \omega_0(q) b_q^\dagger b_q.
\]

In order to obtain the time evolution of operators \(O = \mathcal{O}\{b_q^\dagger(b_q), b_{q'}^\dagger b_{q'}\}\) after the quench, we recall that, in the Heisenberg picture, \(\mathcal{O}(t > 0) = e^{-iH_t/\hbar} \mathcal{O} e^{iH_0/\hbar} = \mathcal{O}\{b_{q,t}^\dagger, b_q\},\) and therefore all that is needed to solve the above quench problem is to obtain the time evolution of \(b_q\) for \(t > 0.\) For Hamiltonians like \(\mathcal{H}^\dagger\) this can be done exactly because \(H_t = H(t > 0)\) can be diagonalized by means of the canonical Bogoliubov (“squeezing”) transformation:

\[
a(q) = \cos \beta(q) b_q + \sinh \beta(q) b_q^\dagger (-q).
\]

Upon choosing

\[
\tanh 2\beta(q) = \frac{g(q)}{\omega_0(q) + m(q)},
\]

the Hamiltonian at \(t > 0\) is rendered diagonal:

\[
H_t = H \equiv E_0 + \sum_q \hbar \omega_q a_q^\dagger a_q,
\]

where \(E_0\) is the energy of the ground state of \(H\) (relative to the ground state energy of \(H_0\)) and

\[
\omega(q) = \sqrt{[\omega_0(q) + m(q)]^2 - |g(q)|^2}
\]

the dispersion of the excitations about the ground state of \(H_t.\) The evolution of the \(a(q,t) = e^{iH_0/\hbar} a(q) e^{-iH_t/\hbar}\) is \(e^{-i\omega(q)t} a(q).\) By application of a direct and reverse Bogoliubov transformation, one can obtain the time evolution of \(b(q):\)

\[
b(q, t) = f(q, t) b_q + g^*(q, t) b_q^\dagger (-q),
\]

where

\[
f(q, t) = \cos \omega(q)t - i \sin \omega(q)t \cosh 2\beta(q),
\]

\[
g(q, t) = i \sin \omega(q)t \sinh 2\beta(q).
\]

It is easy to check that \(b(q)\) obeys the initial condition, \(b(q, t = 0) = b(q),\) and also respects the equal-time commutation rules,

\[
[b(q, t), b_{q'}^\dagger, t] = (f(q, t) g^*(q, t)
\]

\[
- g^*(q, t) f(q, t)) \delta_{q,-q} = 0,
\]

\[
[b(q, t), b^\dagger (q', t')] = (|f(q, t)|^2 - |g(q, t')|^2) \delta_{q,q'} = \delta_{q,q'}
\]

(10)
Thus, a quantum quench described by a quadratic Hamiltonian can be solved by means of a time-dependent canonical transformation.

When the quench is reversed, i.e., when the case with \( m(q,t \geq 0) = g(q,t \geq 0) = 0 \), and \( m(q,t < 0) = m(q) \) and \( g(q,t < 0) = g(q) \) is considered, the roles played by the initial and final Hamiltonians are also reversed: the final Hamiltonian is now diagonal in the \( b \)'s, \( H_f = H_0 \), whereas the transformation of Eq. (3) renders the initial Hamiltonian, \( H_i = H_f \), diagonal. Therefore, in this case the evolution of the \( b \)-operators is trivial: \( H_i: b(q,t) = e^{-i\omega_0(q)t} \), whereas the evolution of the \( a \)'s is given by

\[
a(q,t) = f_0(q,t) a(q) + g_0(q,t) a^\dagger(-q),
\]

where

\[
f_0(q,t) = \cos \omega_0(q)t - i \sin \omega_0(q)t \cosh 2\beta(q),
\]

\[
g_0(q,t) = -i \sin \omega_0(q)t \sinh 2\beta(q).
\]

III. THE LUTTINGER MODEL

The Luttinger model (LM) is a one-dimensional (1D) system of interacting fermions with linear dispersion. It was first described by Luttinger\(^{51}\) but its complete solution was later obtained by Mattis and Lieb.\(^{52}\) who showed that the elementary excitations of the system are not fermionic quasi-particles. Instead, they introduced a set of bosonic operators describing collective density modes (phonons) of the system, which are the true elementary low-energy excitations of the model. The methods of Mattis and Lieb bear strong resemblance to the early work of Tomonaga\(^{51}\) on the one-dimensional electron gas. Extending the work of Tomonaga, Mattis and Lieb, Luther and Peschel\(^{53}\) computed the equilibrium one and two-particle correlation functions, showing that all correlation functions exhibit (at zero temperature) a non-universal power-law behavior at long distances, which signals the absence of long-range order.

Later, Haldane\(^{54,55}\) conjectured that these properties (i.e., collective elementary excitations exhausting the low-energy part of the spectrum as well as power-law correlations) are a distinctive feature of a large class of gapless interacting one-dimensional systems, which he termed (Tomonaga-)‘Luttinger liquids’. Thus, the Luttinger model can be understood as a fixed point of the renormalization-group for a large class of gapless many-body systems in one dimension. Therefore, the thermal equilibrium properties of many 1D system are universal in the sense that they can be accurately described by the Luttinger model. However, in this work we shall be concerned with the non-equilibrium properties of the LM, and because non-equilibrium phenomena can involve highly excited states, we shall make no claim for universality. The precise conditions conditions under which the results obtained here apply to real systems that are the Tomonaga-Luttinger liquids should be investigated carefully for each particular system (see also discussion in Sect. [VI].

The Hamiltonian of the Luttinger model (LM) can be written as follows:

\[
H_{\text{LM}} = H_0 + H_2 + H_4,
\]

\[
H_0 = \sum_{p,\alpha=r,l} \hbar v_F p : \psi_\alpha^\dagger(p) \psi_\alpha(p) :, \quad (15)
\]

\[
H_2 = \frac{2\pi\hbar}{L} \sum_q g_2(q) : J_r(q) J_l(q) :, \quad (16)
\]

\[
H_4 = \frac{\pi\hbar}{L} \sum_{q,\alpha=r,l} g_4(q) : J_\alpha(q) J_\alpha(-q) :, \quad (17)
\]

The index \( \alpha = r, l \) refers to the chirality of the fermion species, which can be either right (\( r \)) or left (\( l \)) moving; the symbol \( : \ldots : \) stands for normal ordering prescription for fermionic operators, which is needed to remove from the expectation values the infinite contributions arising from the ground state.\(^{54}\) The latter is a Dirac sea, namely state where all single-particle fermion levels with \( p < 0 \) are occupied for both chiralities. This defines a stable ground state (at least at the non-interacting level), which will be denoted by \( |0\rangle \).

A. Bosonization solution of the LM

In this section we briefly review the solution of the LM. The Hamiltonian in Eqs. (15) to (18) can be written as a quadratic Hamiltonian in terms of a set of bosonic operators.\(^{54}\) First note that the density (current) operators \( J_\alpha(q) = \sum_p : \psi_\alpha^\dagger(p + q) \psi_\alpha(p) : \) obey the following commutation rules:

\[
[J_\alpha(q), J_\beta(q')] = \left( \frac{qL}{2\pi} \right) \delta_{q+q',0} \delta_{\alpha\beta},
\]

which can be transformed into the Heisenberg algebra of bosonic operators by introducing:

\[
b(q) = -i \left( \frac{2\pi}{|q|L} \right)^{1/2} [\vartheta(q) J_r(-q) - \vartheta(-q) J_l(q)].
\]

where \( \vartheta(q) \) is the step function. Note that the \( q = 0 \) components (sometimes calle zero modes) require a separate treatment since \( J_\alpha(0) = N_\alpha \) is the deviation (relative to the ground state) in the number of fermions of chirality \( \alpha = r, l \). Rather than working with \( N_r \) and \( N_l \), it is convenient to introduce:

\[
N = N_r + N_l \quad J = N_r - N_l,
\]

which, since \( N_r \) and \( N_l \) are integers, must obey the following selection rule (\( -1 \))\(^N \) = (\( -1 \))^\( J \) when the Fermi fields obey anti-periodic boundary conditions (\( \psi_\alpha(x + L) = -\psi_\alpha(x) \) \( L \) is the length of the system), and therefore \( \psi_\alpha(x) = L^{-1/2} \sum_p e^{-a_0|pl|} e^{ipx} \psi_\alpha(p) \), with \( p = 2(n - \frac{1}{2})\pi/L \), \( n \) being an integer, and \( a_0 \rightarrow 0^+ \).\(^{54}\)
The Hamiltonian $H_{LM}$ can be expressed in terms of the bosonic operators introduced in Eq. (20):

$$H_0 = \sum_{q \neq 0} \hbar v_F |q| b^\dagger(q)b(q) + \frac{\hbar \pi v_F}{2L} \left(N^2 + J^2\right),$$

(22)

$$H_2 = \frac{1}{2} \sum_{q \neq 0} g_2(q)|q| \left[b(q)b(-q) + b^\dagger(q)b^\dagger(-q)\right] + \frac{\hbar \pi g_2(0)}{2L} \left(N^2 - J^2\right),$$

(23)

$$H_4 = \sum_{q \neq 0} \hbar g_4(q)|q| b^\dagger(q)b(q) + \frac{\hbar \pi g_4(0)}{2L} \left(N^2 + J^2\right).$$

(24)

Ignoring, for the moment, the zero mode part (i.e. $q = 0$ terms, involving $J$ and $N$), the above Hamiltonian has the form of Eq. (1), with the following identifications: $\omega_0(q) = v_F|q|$, $m(q,t) = g_4(q)|q|$, and $g(q,t) = g_2(q)|q|$, and it can be therefore be brought into diagonal form by means of the canonical transformation of Eq. (3). Hence, the Hamiltonian takes the form of Eq. (3) with $\omega(q) = v(q)|q|$, being $v(q) = \left\{ |v_F + g_4(q)|^2 - |g_2(q)|^2\right\}^{1/2}$, and $q \neq 0$. As to the zero mode contribution:

$$H_{ZM} = \frac{\hbar \pi v_N}{2L} N^2 + \frac{\hbar \pi v_J}{2L} J^2,$$

(25)

where $v_N = v_F + g_4(0) + g_2(0)$ and $v_J = v_F + g_4(0) - g_2(0)$. This defines the equilibrium solution of the LM. In the following sections we shall be concerned with the quench dynamics of this model.

### B. Suddenly turning-on the interactions

Although it is possible to solve the general quench problem between two interacting versions of the Luttinger model, we shall focus here on the interactions described by $H_2$ and $H_4$ are suddenly switched on (this section), and (next section) switched off. Thus, in this section, we shall assume that we have made the replacements $g_{2,4}(q) \rightarrow g_{2,4}(q,t) = g_{2,4}(q)\theta(t)$ in Eqs. (17,18). The Hamiltonian at times $t > 0$ is therefore the interacting LM. In other words, in the notation introduced in Sect. 11, $H_t = H_0 + H_2 + H_4 = H_{LM}$, whereas the initial Hamiltonian (for $t \leq 0$) is $H_t = H_0$. However, we note that, since both zero modes, $J$ and $N$, are conserved by $H_1 = H_0$ and $H_t = H_{LM}$, their dynamics can be factored out, and we shall assume henceforth that we work within the sector of the Hilbert space where $J = N = 0$ (this sector contains the non-interacting ground state, $|0\rangle$). Thus, from now on, we shall omit $H_{ZM}$ in all discussions.

As to the initial state, we shall consider that, within the spirit of the sudden approximation, at $t = 0$ the system is prepared in a Boltzmann ensemble of eigenstates of $H_1$, at a temperature $T$:

$$\rho_0 \equiv \rho(t = 0) = Z_0^{-1} e^{-H_1/T},$$

(26)

where $Z_0 = \text{Tr} e^{-H_1/T}$. We shall further assume that the contact with the reservoir is removed at $t = 0$, and that, after the quench, the system evolves unitarily in isolation.

Whereas Eq. (1) defines the solution to the interaction quench in terms of the modes that annihilate the initial ground state $|0\rangle$, the solution itself is not particularly illuminating. To gain some insight into the properties of the system following the quench, let us compute a few observables. Amongst them, we first turn our attention to the instantaneous momentum distribution, which is the Fourier transform of the one-particle density matrix:

$$C_{\psi,\alpha}(x,t) = \langle \psi^\dagger x,\psi(0) e^{-iH_1t/H}\rangle_0,$$

(27)

where $\langle \cdots \rangle_0$ means that the expectation value is taken over the ensemble described by $\rho_0$ (cf. Eq. (26)). The time dependence of the operators is dictated by $H_t$, as described in Section 11. Notice that, since in general $[H_t, \rho_0] \neq 0$, time translation invariance is broken, and the above correlation function is explicitly time-dependent.

The time evolution of $\psi_\alpha(x)$ can be obtained using the bosonization formula for the field operator,

$$\psi_\alpha(x) = \frac{\eta_\alpha e^{i\pi x/4}}{\sqrt{2\pi a}} e^{is_\alpha \phi_\alpha(x)},$$

(28)

being $\eta_\alpha, \eta_\beta$ two Majorana operators (also known as Klein factors, which in the present case reduce to two Pauli matrices) that obey $\{\eta_\alpha, \eta_\beta\} = 2\delta_{\alpha\beta}$, thus ensuring the anticommutation of the left- and right-moving Fermi fields; we have also introduced the index $s_\alpha = 1$ for $\alpha = r$ and $s_\alpha = -1$ for $\alpha = l$. The bosonic fields

$$\phi_\alpha(x) = s_\alpha \varphi_\alpha + \frac{2\pi x}{L} N_\alpha + \Phi^\dagger_\alpha(x) + \Phi_\alpha(x),$$

(29)

where $[N_\alpha, \varphi_{\beta}] = i\delta_{\alpha\beta}$, and, in terms of Fourier modes,

$$\Phi_\alpha(x) = \lim_{q \to 0} \sum_{q > 0} \frac{2\pi}{qL} e^{-q a_\alpha/2} e^{i s_\alpha q x} b(s_\alpha q).$$

(30)

The details of the calculations of $C_{\psi,\alpha}(x,t)$ have been relegated to the Appendix A. In this section we will mainly describe the results. However, a number of remarks about how the calculations were performed are in order before proceeding any further. We first note that interactions in the Luttinger model are assumed to be long ranged. This can be made explicit in the interaction couplings by writing $g_{2,4}(q) = g_{2,4}(q R_0)$, where the length scale $R_0 \ll L$ is the interaction range. Thus, just like system size $L$ plays the role of a cut-off for ‘infrared’ (that is, long wave-length) divergences, the interaction range, $R_0$ plays the role of an ‘ultra-violet’ cut-off that regulates the short-distance divergences of the model. The results given below were derived assuming a particular form of the interaction (or regularization scheme) where the Bogoliubov parameter (cf. Eq. 4) is
chosen such that \(\sinh 2 \beta(q) = \gamma e^{-|qR_0|/2}\). Furthermore, we replaced \(v(q)\) by \(v = v(0)\). Indeed, these approximations are fairly similar to the ones used to compute the time-dependent correlation functions in equilibrium\(^9\) given that the expressions that we obtain for the out-equilibrium correlators are fairly similar as well (see Appendix A for details). This regularization scheme greatly simplifies the calculations while not altering in a significant way (except perhaps for pathological cases, like the Coulomb interaction, where both \(v(q)/|q|\) and \(\sinh 2 \beta(q)\) are singular at \(q = 0\)) the asymptotic behavior of the correlators for distances much larger than \(R_0\).

Returning to the one-particle density matrix defined above in Eq. (27), we note that it can be written as can be written as the product of two factors: \(C_{\psi,}(x, t) = C_{\psi,}^{(0)}(x) h_r(x, t)\), where \(C_{\psi,}^{(0)}(x)\) is the noninteracting one-particle density matrix, and thus \(h_r(x, t)\) accounts for deviations due to the interactions. Hence, this factorization allows us to write the instantaneous momentum distribution function as a convolution:

\[
f(p, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f^{(0)}(p - k) h_r(k, t),
\]

where \(f^{(0)}(p) = (e^{pv_F/T} + 1)^{-1}\) is the Fermi-Dirac distribution, and \(h_r(k, t)\) is the Fourier transform of \(h_r(x, t)\) in \(x\) variable.

Before presenting the results for the expression of the one-particle density matrix as well as the momentum distribution at finite temperatures, it is worth considering (the much simpler looking) zero-temperature expression. We first discuss finite-size effects. For a system of size \(L\) we obtain\(^9\)

\[
C_{\psi,}(x|L) = C_{\psi,}^{(0)}(x|L) \left[ \frac{R_0}{d(x|L)} \right]^{\gamma^2} \times \left| \frac{d(x - 2vt|L)d(x + 2vt|L)}{d(2vt|L)d(-2vt|L)} \right|^{\gamma^2/2},
\]

where \(d(z|L) = L|\sin(\pi z/L)|\) is the cord function and \(C_{\psi,}^{(0)}(x|L) = i \{2L \sin[\pi(x + ia_0)/L]\}^{-1} (a_0 \to 0^+)\) the noninteracting one-particle density matrix. Notice that this result is valid only asymptotically, that is, for \(d(x|L), d(x \pm 2vt) \gg R_0\). Thus, we see that \(C_{\psi,}(x, t|L)\) is a periodic function of time with period equal to \(\tau_0 = L/2v\). This is in agreement with the general expectation that correlations in finite-size systems exhibit time recurrences because its energy spectrum is discrete. Although the recurrence time generally depends on the details of the energy spectrum, in the present version of the LM, the spectrum is linear \(\omega(q) \approx v|q|\) and thus, the energy spacing between (non-degenerate) many-body states is \(\Delta_n \approx 2\pi v/\hbar L\). Hence, the recurrence time \(\tau_0 \sim 2\pi/\Delta_0\) follows (the extra factor of \(\frac{1}{2}\) is explained by the so-called light-cone effect, see further below). The periodic behavior exhibited by the one-particle density matrix (27) implies that, after the quench, the system will not reach a stationary state with time independent properties as \(t \to +\infty\). A similar conclusion is reached by analyzing, e.g. the finite-size version of the density correlation function\(^9\)

\[
C_{J_r}(x, t|L) = \langle e^{iHt/\hbar} J_r(x) J_r(0) e^{-iHt/\hbar} \rangle_0
= -\frac{(1 + \gamma^2)}{4\pi^2} \left( \frac{\gamma^2}{8\pi^2} \right)^2 + \frac{1}{|d(x|L)|^2} \frac{1}{d(x - 2vt|L)|^2} \frac{1}{|d(x + 2vt|L)|^2},
\]

where

\[
J_r(x) = \frac{1}{L} \sum_q e^{iqx} J_r(q) =: \psi_r^1(x) \psi_r(x) :.
\]

is the density (also referred to as current) operator in real space.

However, in the thermodynamic limit \(L \to \infty\), the recurrence time \(\tau_0 = L/2v \to +\infty\), and the system does reach a time-independent steady state. In this limit, \(d(x|L) \to |x|\), and the asymptotic form of the single particle density matrix becomes\(^9\)

\[
C_{\psi,}(x, t > 0) = C_{\psi,}^{(0)}(x) \left[ \frac{R_0}{x} \right]^{\gamma^2} \left| \frac{x^2 - (2vt)^2}{(2vt)^2} \right|^{\gamma^2/2}.
\]

Hence,

\[
h_r(x, t) \simeq Z(t) = \left( \frac{R_0}{2vt} \right)^{\gamma^2},
\]

which can be interpreted as a time-dependent ‘Landau quasi-particle’ renormalization constant in an effective time-dependent Fermi-liquid description of the system. In other words, as time evolves after the quench, we could imagine that the quasi-particle weight at the Fermi level is reduced from its initial value \((Z(t = 0) = 1)\) to \((Z(t > 0) < 1)\). At zero temperature, this time-dependent renormalization of the quasi-particle weight reflects itself in a reduction of the discontinuity of the momentum distribution \(f(p, t)\) at the Fermi level (which corresponds to \(p = 0\) in our notation). Therefore, at any finite time, the system behaves as as if it was a Fermi liquid and therefore it keeps memory of the initial state (a non-interacting Fermi gas).

Yet, for \(t \to +\infty\), \(h_r(x, t)\) becomes a power-law\(^9\)

\[
\lim_{t \to \infty} h_r(x, t) = \left[ \frac{R_0}{x} \right]^{\gamma^2}.
\]
Interestingly, this time-dependent reduction of the quasiparticle weight after quenching into the interacting state has been also found in Ref. 19 when studying a similar quench in the Hubbard model in the limit of infinite dimensions. In this case, however, the discontinuity remains finite even for $t \to +\infty$, which is different from the behavior of the LM, which is known to be a non-Fermi liquid system in equilibrium. These non-Fermi-liquid features also persist in the quench dynamics, as we have found above.

The result of Eq. (38) is similar to the zero temperature result in equilibrium. However, the exponent of $C_{\psi}(x,t \to \infty)$ is equal to $1 + \gamma^2$, and, even for an infinitesimal interaction ($i.e. \gamma \ll 1$), it is always larger than the one that governs the asymptotic ground state ($i.e. \text{equilibrium}$) correlations.\ref{52,54} $\gamma^2 = \sinh^2 2\beta(0) > \gamma_{\text{eq}}^2 = 2\sinh^2 \beta(0)$. The reason for the larger exponent can be understood from two facts: i) By the variational theorem, the initial state ($i.e. \text{the ground state of the non-interacting Hamiltonian } H_0$), is a complicated excited state of the system. ii) Both $H_0$ and the Hamiltonian that performs the time-evolution ($H_t = H_{\text{LM}}$) are critical ($i.e. \text{scale free, apart from the cut-off } R_0$), thus, the system is likely to remain critical. Before going further in the discussion of other results, we should note that, in the literature on Tomonaga-Luttinger liquids, it is customary to introduce the dimensionless parameter $K = e^{-2\beta(0)}$, in terms of which $\gamma^2 = (K - K^{-1})^2/4$ (whereas $\gamma_{\text{eq}}^2 = (K + K^{-1} - 2)/2$). We shall use the parameter $K$ in other expressions below.

It is worth emphasizing that the particular evolution of the asymptotic correlations from Fermi liquid-like at short times to non-Fermi liquid-like at infinite time exhibited by the one-particle density matrix, is also found in other correlation functions. However, the idea that the system 'looks like' an interacting Fermi liquid at any finite $t$ should not be taken too far. In this regard, we should note that for $|x| \gg 2et$, the prefactor of the term $\propto (2\pi x)^{-2}$ of the density correlation function (cf. Eq. (38)), which in equilibrium is proportional to the system compressibility,\ref{50} remains equal to (minus) unity, which is the value that corresponds to a non-interacting Fermi gas (in an interacting Fermi liquid it would deviate from one). For $t \to +\infty$ the same prefactor becomes $(1 + \gamma^2) > 1$, i.e., which does not directly reveal the non-Fermi liquid-like behavior. However, other correlation functions (like the one-particle density matrix discussed above) do. For instance, consider the following:

$$C_{\phi}(x,t) = \langle e^{2im\phi(x,t)} e^{-2im\phi(0,t)} \rangle, \tag{39}$$

$$C_{\theta}(x,t) = \langle e^{im\theta(x,t)} e^{-im\theta(0,t)} \rangle, \tag{40}$$

where $\phi(x) = \frac{1}{2} [\phi_r(x) + \phi_i(x)]$ and $\theta(x) = \frac{1}{2} [\phi_r(x) - \phi_i(x)]$. The spatial derivatives of $\phi$ and $\theta$ are related to the (total) density and current density fluctuations, respectively, in the system.\ref{57} Using exactly
the same methods as above, we find (for \( L \to \infty \)):

\[
\frac{C^m_\phi(x, t)}{C^{(0,m)}_\phi(x)} = \left( \frac{R_0}{2vt} \right)^2 \frac{x^2 - (2vt)^2}{x^2} m^2 (K^2 - 1/2),
\]

(41)

\[
\frac{C^m_\phi(x, t)}{C^{(0,m)}_\phi(x)} = \left( \frac{R_0}{2vt} \right)^2 \frac{x^2 - (2vt)^2}{x^2} n^2 (K^{-2} - 1/8),
\]

(42)

where \( C^{(0,m)}(x) = A_0^m |R_0/x|^{2m^2} \) and \( C^{(0,n)}(x) = A_0^n |R_0/x|^{n^2/2} \) are the non-interacting correlation function (where \( A_0^m \) and \( A_0^n \) are non-universal prefactors). We note that the usual duality relation where \( \phi \to \theta \) and \( K \to K^{-1} \), which one encounters when studying equilibrium correlation functions still holds for these non-equilibrium correlators. Let us next analyze their asymptotic properties. We consider only \( C^m_\phi(x, t) \), as identical conclusions also apply to \( C^m_\psi(x, t) \) by virtue of the duality relations. For \( |x| \gg 2vt \), we have:

\[
C^m_\phi(x, t) \simeq \frac{R_0}{x} m^2 (K^2 - 1).
\]

(43)

Thus, up to the time-dependent pre-factor, correlations take the form of a non-interacting system of Fermions, \( C^{(0,m)}(x) \). However, in the opposite limit \( (|x| \ll 2vt) \), this correlator exhibits a non-trivial power-law:

\[
C^m_\phi(x, t) \simeq \left| \frac{R_0}{x} \right|^{m^2 (K^2 - 1).}
\]

(44)

Notice that this expression also describes the infinite-time behavior, which is controlled by an exponential to \( m^2 (K^2 + 1) \), being again different from the exponent exhibited by the same correlator in equilibrium, which equals \( 2m^2 (\cosh 2\beta(0) - \sinh 2\beta(0)) = 2m^2 K \).

In order to understand why the behavior found in the correlations for \( t \to \infty \) in Eqs. (33-31), also holds for \( |x| \ll 2vt \), let us consider the initial state at zero temperature, \( \rho_0 = |0\rangle \langle 0| \). As mentioned above, this is a rather complicated excited state of the Hamiltonian that performs the time-evolution, \( H_t = H_{LM} \). This means that, initially, there are a large number of excitations of \( H_t \), namely, phonons with dispersion \( \omega(q) = v(q)|q| \). The distribution of the phonons \( \langle b^\dagger(q)b(q) \rangle_0 = \sinh^2 \beta(q) \) is time-independent and peaked at \( q = 0 \). Thus, within the approximation where \( v(q) \simeq v(0) = v \), the excitations propagate between two given points with velocity \( v \). Thus, if we consider the correlations of two points \( A \) and \( B \) separated a distance \( |x| \), the nature of the correlation at a give time \( t \) depends on whether the excitations found initially at, say, point \( A \), have been able to reach point \( B \) or not. This is not the case if \( |x| > 2vt \), and thus correlations retain essentially the properties they had in the initial state. Thus, up to a time-dependent prefactor, \( C_\psi(x, t) \propto C^{(0)}_\psi(x) \). However, if the two points have been able to ‘talk to each other’ through the excitations present in the initial state, then correlations will be qualitatively different. This happens for \( t = t_0 \), when the phonons propagating from \( A \) meet the phonons traveling from point \( B \), that is for \( x - vt_0 = vt_0 \), or \( t_0 = x/2v \) (we assume \( x > 0 \) without loss of generality). Thus, for given separation \( x \) and time \( t \), there is a length scale \( 2vt \), which marks the transition between two different regimes in the correlations. In the instantaneous momentum distribution, this reflects itself in a crossover as a function of time from a momentum distribution \( n(p) \) exhibiting a discontinuous Fermi liquid-like behavior, which is valid i.e. for \( |p| \ll (2vt)^{-1} \) to a power-law behavior of the form \( \sim |pR_0|^{\gamma} sgn(p) \), which applies for \( |p| \gg (2vt)^{-1} \) but \( |p| \ll R_0^{-1} \) for \( |p| \gg R_0^{-1} \) we recover the free particle behavior corresponding to the Fermi-Dirac distribution function at \( T = 0 \). In the \( t \to \infty \) limit, by using the regularization scheme described above, the asymptotic momentum distribution at zero temperature can be obtained with the help of tables. The resulting formula behaves as the non-interacting Fermi-Dirac distribution for \( |p| \gg R_0^{-1} \), whereas for \( |p| \gg R_0^{-1} \) it describes a non-Fermi liquid-like steady state:

\[
f(p, t \to +\infty) = \frac{1}{2} - \frac{pR_0}{2} \left[ K_{\frac{3}{2} + 1}(pR_0) \frac{L_{\frac{3}{2} + 1}(pR_0)}{(pR_0)^{\gamma}} \right] \]

(45)

where \( K_\nu(z), L_\nu(z) \) are the modified Bessel and Struve functions, respectively. This expression yields a power law for \( |pR_0| \ll 1 \), where \( n(p, t \to +\infty) \simeq \frac{1}{2} - \text{const.} \times (|pR_0|)^{\gamma} sgn(p) \). Note that the momentum distribution \( n(p = 0, t) = \frac{1}{2} \), which is given by the invariance of the LM under particle-hole symmetry \( \psi_0(p) \to \psi_0^\dagger(-p) \).

Let us finally present the generalization of the above results for the one-particle density matrix to finite temperatures, \( T > 0 \). For \( T \ll \hbar v R_0 \) (but \( T \gg \Delta_0 = 2\hbar v / L \), so that we can neglect finite-size effects and effective take the thermodynamic limit) \( C_\psi(x, t) \) takes the following asymptotic form:

\[
C_\psi(x, t > 0|T) = C^{(0)}_\psi(x|T) \frac{\pi R_0 / \lambda}{dh(x|T)} \left| \frac{dh(x|T)}{dh(x|T)} \right|^2 \]

(46)

\[
\times \left| \frac{dh(x - 2vt|T)}{dh(x + 2vt|T)} \right|^2 \left| \frac{dh(2vt|T)}{dh(-2vt|T)} \right|^2 \]

where \( C^{(0)}_\psi(x|T) \) and \( dh(x|T) \) can be obtained from \( C^{(0)}_\psi(x|L) \) and \( d(x|L) \) by replacing \( L \sin(\pi x/L) / \pi \) by \( \lambda \sin(\pi x / \lambda) \), where \( \lambda = \hbar v / T \) is the thermal correlation length. At long times, \( h_r(x, t|T) \) reduces to

\[
h_r(x) = \frac{\pi R_0 / \lambda}{\sin(\pi x / \lambda)} \left| \right|^2 \]

(47)

Therefore we again find that \( C_\psi(x, t \to \infty|T) \) has a form similar to the the equilibrium correlation function at finite temperature with a different exponent controlling
the asymptotic exponential decay of correlations. Notice that the exponential decay the correlations for $t > 0$ is a direct consequence of the fact that the initial state has a characteristic correlation length, the thermal correlation length $\lambda_\text{th}$. The exponential decay of correlations at finite $T$ implies that the the steady state will be reached exponentially rapidly approached in a time of the order of $\hbar / T$. It is also worth noting, however, that the above expression depends parametrically on the thermal correlation length $\lambda$, and it is the Fermi velocity, $v_F$, which enters in the expression for $\lambda = \hbar v_F / T$, instead of the (renormalized) phonon velocity which enters in the thermal length $\lambda_\text{th} = \hbar v / T$, characterizing the equilibrium correlations. Thus, since the velocity appears only through the definition of the thermal correlation length $\lambda$, or, in other words, in combination with the temperature, the change from $v$ to $v_F$ can be also understood as an change in the temperature scale. Furthermore, in a system with Galilean symmetry we have that $vK = v_F$ and thus the parameter that controls the temperature scale now is the Luttinger parameter $K$, so that the asymptotic correlations at $t \to \infty$ can be regarded as the equilibrium correlations with a different exponent and an effective temperature, $T_{\text{eff}} = T / K$. Thus, for repulsive interactions (i.e. $K < 1$) we could say that, besides modifying the exponent, quenching the system into the interacting system increases the effective temperature, whereas for attractive interactions (i.e. $K > 1$) the effective temperature is reduced after the quench. This effect has an impact on the momentum distribution at finite temperatures. To demonstrate it, we need to compute the Fourier transform of $h_r(x)$. This can be done by relating it to an integral representation of the associated Legendre function $P_\nu^\mu(z)$ and thus the Fourier transform of $h_r(x)$ can be written as:

$$h_r(p) = \frac{\lambda}{\sqrt{\pi}} \left( \frac{\pi R_0}{\lambda} \right)^{(\gamma^2+1)/2} \left| \Gamma\left( \frac{\gamma^2 + i \lambda p}{2 \gamma} \right) \right|^2 \times P_{\frac{\gamma^2}{2} + \frac{1}{2}} \left[ -\cos \left( \frac{2 \pi R_0}{\lambda} \right) \right]. \quad (48)$$

Hence, the momentum distribution can be obtained by numerically evaluating the convolution with the Fermi-Dirac distribution function (cf. Eq. (31)) of the above expression, Eq. (48). In Figs. 2 and 3 the momentum distribution of the interacting system in the infinite-time limit is displayed for a non-interacting LM that is quenched into an interacting state with repulsive (corresponding to $K = 0.6$) and attractive (corresponding to $K = 1.7$) interactions, respectively.

C. Suddenly turning-off the interactions

Next we briefly consider the opposite situation to the one considered above, namely the case where in the initial state the fermions interact and this interaction suddenly disappears. The fact that the initial state is a highly complicated state of the Hamiltonian that performs the time evolution (in this case $H_i = H_0$, cf. Eq. (15), implies that we cannot expect that a Fermi liquid will emerge asymptotically at long times after the quench. Indeed, at zero temperature a thermodynamically large system approaches a steady state exhibiting equal-time correlations that decay algebraically in space. However, the exponents differ again from the (non-interacting) equilibrium ones. This can be illustrated by, e.g. computing
the following correlation functions:

\[ C_{\phi}^{m}(x,t) = \langle e^{i m \phi(x,t)} e^{-i m \phi(0,t)} \rangle \]
\[ = I_{\phi}^{m}(x) \left( \frac{R_0}{2 v_F t} \right)^{m^2(K^{-1} - K)} \times \frac{x^2 - (2 v_F t)^2}{x^2}^{m^2(K^{-1} - K)/2} \]
\[ C_{\theta}^{m}(x,t) = \langle e^{i n \theta(x,t)} e^{-i n \theta(0,t)} \rangle \]
\[ = I_{\theta}^{n}(x) \left( \frac{R_0}{2 v_F t} \right)^{n^2(K - K^{-1})/2} \times \frac{x^2 - (2 v_F t)^2}{x^2}^{n^2(K - K^{-1})/4} \]  

where

\[ I_{\phi}^{m}(x) = \left( \frac{R_0}{x} \right)^{2m^2 K} \]
\[ I_{\theta}^{n}(x) = \left( \frac{R_0}{x} \right)^{n^2/2K} \]

are the correlation functions in the initial (interacting) ground state \((A_{\phi/\theta} \neq 0)\). We note again that the duality \(\theta \rightarrow \phi\) and \(K \rightarrow K^{-1}\) also holds in this case. The correlations in the stationary state that is asymptotically approached at long times read:

\[ \lim_{t \to \infty} C_{\phi}^{m}(x,t) = A_{\phi} \left( \frac{R_0}{x} \right)^{m^2(K^{-1} + K)} \]
\[ \lim_{t \to \infty} C_{\theta}^{m}(x,t) = A_{\theta} \left( \frac{R_0}{x} \right)^{n^2(K^{-1} + K)/2} \]

However, at short times, \(t \ll |x|/2v_F\), correlations look like those of in the initial state, up to a time-dependent prefactor:

\[ C_{\phi}^{m}(x,t \ll |x|/2v_F) = \left( \frac{R_0}{2 v_F t} \right)^{m^2(K^{-1} - K)} I_{\phi}^{m}(x) \]
\[ C_{\theta}^{m}(x,t \ll |x|/2v_F) = \left( \frac{R_0}{2 v_F t} \right)^{n^2(K - K^{-1})/2} I_{\theta}^{n}(x). \]

In this case the time-dependent prefactor has also a power-law form.

**IV. THE SINE-GORDON MODEL**

Let us next turn our attention towards another class of models in one dimension whose spectrum is not necessarily always critical (i.e. gapless) like the case of the Luttinger model (LM) analyzed previously. This is the case of the sine-Gordon model, which is described by the following Hamiltonian:

\[ H_{sG}(t) = H_0 - \frac{\hbar g(t)}{a_0^2} \int dx \cos 2\phi, \]
\[ H_0 = \frac{\hbar v}{2\pi} \int dx : K^{-1} (\partial_x \phi)^2 + K (\partial_x \theta)^2 : , \]

where \(\ldots\) stands for normal order of the operators, \(a_0\) is a short-distance cut-off, and the phase and density fields \(\phi(x)\) and \(\theta(x)\), which have been introduced in the connection with the LM studied in the previous section, are canonically conjugated in the sense that the obey:

\[ [\phi(x), \partial_x \theta(x')] = i\pi \delta(x - x'). \]  

This model can be regarded as a perturbation to the LM, which still yields an integrable model. In equilibrium, the model is known to have two phases, which, according to the renormalization group analysis and for infinitesimal and positive values of the coupling in front of the cosine term, roughly correspond to \(K < 2\) (gapped phase) and \(K \geq 2\) (gapless phase).

In order to study the non-equilibrium (quench) dynamics, we will consider two different types of quenches: the quench from the gapless to the gapped phase and the reversed process, from the gapped to the gapless . In the first case, we assume that the dimensionless coupling \(g(t)\) is suddenly turned on, i.e. \(g(t) = g \theta(-t)\). With this choice, \(H_t = H_{sG}(t \leq 0)\) is a Hamiltonian whose ground state exhibits a frequency gap, \(m\), to all excitations, whereas \(H_t = H_{sG}(t > 0)\) has gapless excitations. Conversely, in the second case, we consider that \(g(t)\) is suddenly turned off, i.e. \(g(t) = g \theta(t)\). In this case, the ground state of \(H_t\) is gapless whereas the Hamiltonian performing the time evolution, \(H_t\), has gapped excitations. However, although both \(H_t\) and \(H_t\) define integrable field theories, for a general choice of the parameters \(K\) and \(g\), the quench dynamics cannot be analyzed, in general, by the methods discussed above. Nevertheless, in two limits, the Luther-Emery point (which corresponds to \(K = 1\), see Sect. [LV.3] and in the semiclassical limit (that is, for \(K \ll 1\), Sect. [LV.3]), it is possible to study the quench dynamics by the methods of Sect. [LV.3]. However, the statistics of the elementary excitations happens to be different in these two cases.

**A. The Luther-Emery point**

Let us start by considering the sine-Gordon model, Eq. [LV.3], for \(K = 1\), which is the so-called Luther-Emery point. It is convenient to introduce rescaled density and phase fields, which will be denoted as \(\varphi(x) = K^{-1/2} \phi(x)\) and \(\bar{\varphi}(x) = K^{1/2} \theta(x)\). Thus, the Hamiltonian in Eq.
becomes:
\[
H_{\text{SC}}(t) = \frac{\hbar v}{2\pi} \int dx : (\partial_x \varphi)^2 + (\partial_x \tilde{\varphi})^2 : - \frac{\hbar v g(t)}{\pi a_0^2} \int dx \cos \kappa \varphi, \tag{59}
\]
where \( \kappa = 2\sqrt{K} \). At the Luther-Emery point \( \kappa = 2 \) (i.e. \( K = 1 \)) and the model can be rewritten as a one-dimensional model of massive Dirac fermions with mass by using the bosonization formula for the Fermi field operators, Eq. (28). To this end, we set \( \phi_r(x) = \varphi(x) + \tilde{\varphi}(x) \) and \( \phi_l(x) = \varphi(x) - \tilde{\varphi}(x) \). Furthermore, for computational convenience, we choose the Majorana fermions in Eq. (28) to be \( \eta_r = \sigma_x \) and \( \eta_l = i\sigma_y \). In addition, we note that the gradient terms in Eq. (59) can be written as the kinetic energy of free massless Dirac fermions in one dimension.\cite{24,56,58}

\[
H_0 = -i\hbar v \int dx : \psi_r^\dagger(x)\partial_x \psi_r(x) - \psi_l^\dagger(x)\partial_x \psi_l(x) : \tag{60}
\]
As far as the cosine operator is concerned, the bosonization formula, Eq. (28), implies that:
\[
\begin{align*}
\psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x) &= \frac{\Gamma}{\pi a_0} \cos 2\varphi(x) \\
&= \frac{\Gamma}{\pi} \cos 2\varphi(x) \tag{61}
\end{align*}
\]
\[
\psi_r^\dagger(x)\partial_x \psi_l(x) - \psi_l^\dagger(x)\partial_x \psi_r(x) = \frac{\Gamma}{\pi} \cos 2\varphi(x). \tag{62}
\]
where \( \Gamma = i\sigma_x \sigma_y \). This is almost the cosine term in the sine-Gordon model (cf. Eq. 59) except for the presence of the operator \( \Gamma \). However, we note that \( \Gamma^2 = 1 \) and that this operator also commutes with \( H_0 \) and with operator in the left hand-side of Eq. 61.

The first property implies that the eigenvalues of \( \Gamma \) are \( \pm 1 \) whereas the second property implies that \( H_{\text{SC}} = H_0 + \hbar v g(t) \int dx \left( \psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x) \right) \) and \( \Gamma \) can be diagonalized simultaneously. Upon choosing the eigenvalue where \( \Gamma = -1 \), we obtain that
\[
H_{\text{SC}}(t) = -i\hbar v \int dx : \psi_r^\dagger(x)\partial_x \psi_r(x) - \psi_l^\dagger(x)\partial_x \psi_l(x) : + \hbar v g(t) \int dx \left( \psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x) \right), \tag{63}
\]
is equivalent to Eq. (59) when \( \kappa = 2 \).

To gain some insight into the phases described by the sG model, let us first consider the Luther-Emery Hamiltonian, \( H_{\text{LE}} \) in two (time-independent) situations: i) \( g(t) = 0 \) (the gapless free fermion phase, which coincides with the LM for \( K = 1 \)), and ii) \( g(t) = g > 0 \), i.e. a time-independent constant (which corresponds to the gapped phase). In order to diagonalize the Hamiltonian in these cases, it is convenient to work in Fourier space and write the fermion field operator as:
\[
\psi_\alpha(x) = \frac{1}{L} \sum_p e^{-a_0 |p|} e^{i p x} \psi_\alpha(p). \tag{64}
\]
where \( \alpha = r, l \). The limit where the cut-off \( a_0 \to 0^+ \) should be formally taken at the end of the calculations, but in some cases we shall not do it in order to regularize certain short-distance divergences of the model. It is also useful to introduce a spinor whose components are the right and left moving fields, and which will make the notation more compact:
\[
\Psi(p) = \begin{bmatrix} \psi_r(p) \\ \psi_l(p) \end{bmatrix}, \quad \mathcal{H}(p) = \begin{bmatrix} \hbar \omega_0(p) & 0 \\ 0 & -\hbar \omega_0(p) \end{bmatrix}, \tag{65}
\]
Thus the Hamiltonian for the gapless phase, \( H_0 \), reads:
\[
H_0 = \sum_p \Psi^\dagger(p) \cdot \mathcal{H}(p) \cdot \Psi(p) :, \tag{66}
\]
where \( \omega_0(p) = vp \) is the fermion dispersion. However, the Hamiltonian of the gapped phase, corresponding to \( g(t) = g > 0 \), \( H \), is not diagonal in terms of the right and left moving Fermi fields. In the compact spinor notation it reads:
\[
H = \sum_p \Psi^\dagger(p) \cdot \mathcal{H}(p) \cdot \Psi(p) :, \tag{67}
\]
where
\[
\mathcal{H}(p) = \begin{bmatrix} \hbar \omega_0(p) & \hbar m \\ \hbar m & -\hbar \omega_0(p) \end{bmatrix}, \tag{68}
\]
being \( m = v g \). Nevertheless, \( H \) can be rendered diagonal by means of the following unitary transformation:
\[
\tilde{\Psi}(p) = \begin{bmatrix} \psi_r(p) \\ \psi_l(p) \end{bmatrix} = \begin{bmatrix} \cos \theta(p) & \sin \theta(p) \\ -\sin \theta(p) & \cos \theta(p) \end{bmatrix} \begin{bmatrix} \psi_r(p) \\ \psi_l(p) \end{bmatrix}, \tag{69}
\]
being
\[
\tan 2\theta(p) = \frac{m}{\omega_0(p)}. \tag{70}
\]
Thus the Hamiltonian of the gapped phase, in diagonal form, reads (we drop an unimportant constant that amounts to the ground state energy):
\[
H = \sum_p \hbar \omega(p) : \psi_r^\dagger(p) \psi_r(p) - \psi_l^\dagger(p) \psi_l(p) :: \tag{71}
\]
where \( \omega(p) = \sqrt{\omega_0(p)^2 + m^2} \). We associate \( \psi_r^\dagger(p) \) \( \psi_l^\dagger(p) \) with the creation operator for particles in the valence (conduction) band.

Before considering quantum quenches, let us briefly discuss some of the the properties of the ground states of the Hamiltonians \( H_0 \) and \( H \). In what follows, these states will be denoted as \( \Phi_0 \) and \( \Phi \), respectively. As mentioned above, the spectrum of \( H_0 \) is gapless, and the fermion occupancies in the ground state \( \Phi_0 \) are:
\[
\begin{align*}
n_r(p) &= \langle \Phi_0 | \psi_r^\dagger(p) \psi_r(p) | \Phi_0 \rangle = \theta(-p), \tag{72} \\
n_l(p) &= \langle \Phi_0 | \psi_l^\dagger(p) \psi_l(p) | \Phi_0 \rangle = \theta(p). \tag{73}
\end{align*}
\]
That is, all single-particle levels with negative momentum are filled, as described in Sect. [III] (recall that $H_0$ is just the Luttinger model with $K = 1$, i.e., $g_2 = g_4 = 0$). However, $H$ has a gapped spectrum and, therefore, when constructing the ground state, $\{\Phi\}$, only the levels in the valence band (which have negative energy) are filled, whereas the levels in the conduction band remain empty:

$$n_v(p) = \langle \Phi | \psi^\dagger_v(p) \psi_v(p) | \Phi \rangle = 1, \quad \text{(74)}$$

$$n_c(p) = \langle \Phi | \psi^\dagger_c(p) \psi_c(p) | \Phi \rangle = 0. \quad \text{(75)}$$

1. Quench from the gapped to the gapless phase

The first situation we shall consider is when $g(t) = g \theta(-t)$ in Eq. [45], so that the spectrum of the Hamiltonian abruptly changes from gapped to gapless (i.e., quantum critical). As we did in Sect. [III] we denote $H_t = H_{L_E}(t < 0) = H_0$ and $H_{t+} = H_{L_E}(t > 0) = H_t$. Although the expressions presented below can be computed for finite temperature, $T > 0$, where the initial state corresponds to $\rho_i = e^{-H_i/T}/Z_i$, we shall restrict ourselves to the zero temperature case, where the initial state $\rho_i = \{\Phi\} | \Phi \rangle$. Notice in this state, $\{\Phi\} \cos 2\phi(x) | \Phi \rangle = \langle \Phi | \cos 2\varphi(x) | \Phi \rangle = \Re (\{\Phi| e^{-2i\varphi(x)} | \Phi \rangle) = -\langle \psi^\dagger_r(x) \psi_l(x) \rangle \neq 0$ (the minus sign stems from the eigenvalue of the operator $\Gamma = \eta^c \eta$, whereas in the ground state of $H_0$ the expectation value of the same operator vanishes. Therefore, it behaves like an order parameter in equilibrium, and we can expect that it exhibits interesting dynamics out of equilibrium. Indeed,

$$e^{-2i\varphi(x,t)} = -\frac{1}{L} \sum_p (\psi^\dagger_r(p,t) \psi_l(p,t)) = -\frac{1}{L} \sum_p e^{-2i\omega_0(p)t} \sin 2\theta(p), \quad \text{(77)}$$

where, in the last expression, we have already taken the $T \to 0$ limit and set $\langle \psi^\dagger_r(p) \psi_l(p) \rangle = -\frac{1}{2} \sin 2\theta(p)$, as it follows from Eqs. (69, 74, 75). The above expression can be readily evaluated by recalling that $\sin 2\theta(p) = m/\omega(p)$, which yields, in the $L \to \infty$ limit,

$$e^{-2i\varphi(x,t)} = -m \int_0^{+\infty} dp \cos 2\omega_0(p)t \frac{1}{\sqrt{\omega_0^2(p) + m^2}} \quad \text{(78)}$$

$$= \left(\frac{m}{2\pi v}\right)^2 K_0(2mt) \simeq \frac{1}{4} \frac{m}{\pi t} e^{-2mt}, \quad \text{(79)}$$

where $K_0$ is the modified Bessel function. Thus we see that the ‘order parameter’ $\cos 2\varphi(0,t)$ decays exponentially at long times at $T = 0$. The decay rate is proportional to the gap between the ground state (the initial state) and the first excited state of the initial Hamiltonian $H_1 = H$. The existence of this gap means, in particular, that correlations in the initial state between degrees of freedom of the system are exponentially suppressed beyond a distance of the order $\xi_c \approx v/m$. Since the system is quenched into a situation where the Hamiltonian performing the time evolution is critical, that is, characterized by excitations that propagate along light cones $x \pm vt$, the light-cone argument discussed in previous sections applies and translates the correlation length scale in the initial state into an exponential decay in time of the order parameter. The exponential decay found in the present case (a quench from a gapped to a gapless or critical system) is also found in the semi-classical approximation to the sine-Gordon model (see Sect. [IVB] below), and it is in agreement with the results of Calabrese and Cardy obtained using a mapping to boundary conformal field theory (BCFT).

Next we consider the (equal-time) two-point correlation function of the same object:

$$G(x,t) = \langle e^{-2i\varphi(x,t)} e^{-2i\varphi(0,t)} \rangle = \frac{1}{L^2} \sum_{p_1, p_2, p_3, p_4} e_i(p_1 - p_2) x$$

$$\times e^{-i|\omega_0(p_1) + \omega_0(p_2) - \omega_0(p_3) - \omega_0(p_4)|t} \times \langle \psi^\dagger_r(p_1) \psi_l(p_2) \psi^\dagger_l(p_3) \psi_r(p_4) \rangle. \quad \text{(80)}$$

Applying Wick’s theorem, there are three different contractions of the above four fermion expectation value, which can be evaluated using Eqs. (69, 74, 75). This yields:

$$\langle \psi^\dagger_r(p) \psi^\dagger_l(p) \rangle = \langle \psi_l(p) \psi_r(p) \rangle = 0 \quad \text{(81)}$$

$$\langle \psi^\dagger_r(p) \psi_r(p) \rangle = \langle \psi_l(p) \psi_l(p) \rangle = -\frac{1}{2} \sin 2\theta(p), \quad \text{(82)}$$

$$\langle \psi^\dagger_r(p) \psi_l(p) \rangle = \langle \psi_r(p) \psi_l(p) \rangle = \sin^2 \theta(p), \quad \text{(83)}$$

$$\langle \psi_l(p) \psi_l(p) \rangle = \cos^2 \theta(p). \quad \text{(84)}$$

Hence, for $x \neq 0$, we obtain

$$G(x,t) = \left(\frac{m}{2\pi v}\right)^2 \left[ K_0(2mt) \right]^2 + K_1 \left(\frac{m|x|}{v}\right)^2. \quad \text{(85)}$$

Let us examine the behavior of this correlation function in the asymptotic limit where $|x| \gg \xi_c \approx v/m$ and $2vt \gg \xi_c$. Since the Bessel functions decay exponentially for large values of their arguments, the asymptotic behavior depends on whether $t < |x|/2v$ or $t > |x|/2v$:

$$G(x,t) \approx \begin{cases} \frac{m}{16\pi v^2} e^{-2m|x|/v} & t > |x|/2v, \\ \frac{m}{32\pi v^2} e^{-4mt} & t < |x|/2v. \end{cases} \quad \text{(86)}$$

These results are in agreement with those obtained by Calabrese and Cardy for quantum quenches from a non-critical into a critical state.

2. Quench from the gapless to gapped phase

We next consider the reversed situation to the one discussed in the previous subsection. In this case, we set
of the order parameter. The calculation yields:

Using Eqs. (88) to (90), we can now compute the decay commutation relations characteristic of Fermi statistics. The first term is a non-universal constant that depends on the energy cut-off \( \theta \) (gapless) phase into the gapped phase, the order parameter exhibits an oscillatory decay towards a non-universal constant value, \( A(ma_0) \) (which corresponds to small \( K \) limit in the original notation of Eq. (57)). In this limit, we can expand the cosine term in \( \varphi(x) \) about one of its minima, e.g. \( \varphi = 0 \). Retaining only the leading quadratic term yields the following quadratic Hamiltonian for the boson field \( \varphi(x) \):

\[
H_{SG} \simeq H_{sec} = \frac{\hbar v}{2\pi} \int dx \left[ (\partial_x \varphi(x))^2 + K(\partial_x \varphi(x))^2 \right]
\]

Within this approximation, the problem of studying a quantum quench in the sine-Gordon model becomes akin to the general problem studied in Sect. II. To see this, we first introduce the expansion in Fourier components of \( \varphi(x) \):

\[
\varphi(x) = \frac{\phi_0}{\sqrt{K}} + \frac{\pi x}{\sqrt{KL}} \delta N
\]

Next we obtain the behavior of this function for \( t \to +\infty \), in which case, the term \( \cos 2\omega(p)t \) oscillates very rapidly and therefore can be dropped. Upon performing the momentum integral, we obtain the following result for large \( |x| \):

\[
H(x, t \to +\infty) \simeq \frac{i}{2\pi m} + \frac{4v^2}{2\pi m^2}.
\]

This result is clearly different from the equilibrium behavior of the same correlation function in the gapped phase, where it decays exponentially to a constant. Indeed, as we have just obtained, both the order parameter and its two-point correlations (Eqs. (95) and (97), exhibit instead an algebraic decay to constant values.

B. The semiclassical approximation

A controlled approximation to the sine-Gordon model (cf. Eq. (59) can be obtained in the limit where \( \kappa \ll 1 \) (which corresponds to small \( K \) limit in the original notation of Eq. (57)). In this limit, we can expand the cosine term in \( \varphi(x) \) about one of its minima, e.g. \( \varphi = 0 \). Retaining only the leading quadratic term yields the following quadratic Hamiltonian for the boson field \( \varphi(x) \):

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\[
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\]
to the thermodynamic limit \((L \to \infty)\) and therefore neglect the dynamics of those zero modes. Introducing \(\tilde{H}\) into \((18)\), the Hamiltonian takes the general form of Eq. \((1)\) with \(g(q, t) = m(q, t) = 2\nu g(t)\kappa^2 / |q|d_0^{-2} - \kappa^2 / 2\). Following the procedure described in Sec. \((1)\) this Hamiltonian can be diagonalized by means of the canonical transformation of Eq. \((3)\). Introducing \(m^2 = 4\nu^2\kappa^2 / a_0^{-2} - \kappa^2 / 2\), which plays the role of the gap in the frequency spectrum of the (gapped) Hamiltonian, and setting \(m(q) = g(q) = m^2 / 2\omega_0(q)\) in Eq. \((3)\) yields:

\[
\tanh 2\beta(q) = \frac{m^2 / 2}{\omega_0^2(q) + m^2 / 2}
\]

The dispersion of the excitations in the gapped phase is:

\[
\omega(q) = \sqrt{\omega_0(q)^2 + m^2},
\]

As we did when analyzing the sine-Gordon model at the Luther-Emery point, in what follows, we shall consider the cases of a quench from the gapped to gapless phase of the above quadratic Hamiltonian, \(H\), Eq. \((18)\), and the reverse situation, from gapless to gapped.

1. **Quench from the gapped to the gapless phase**

Let us begin the situation where the initial state is the ground state of \(H_1 = H_{sc}\) and at \(t = 0\) the Hamiltonian is changed to \(H_t = H_0\). In this case, the evolution of the expectation value of the order parameter operator \(e^{-i\phi(x)} = e^{-ix\varphi(x)}\) or the associated correlation functions can be obtained from the knowledge of the two-point (equal time) correlation function out of equilibrium for the boson field \(\varphi(x)\), i.e. \(C(x, t) = \langle \varphi(x, t)\varphi(0, t) \rangle - \langle \varphi^2(0, t) \rangle\). To compute this object when Hamiltonian changes from \(H_1 = H\) (gapped spectrum) to \(H_t\) (gapless spectrum), we first insert the Fourier expansion of \(\varphi(x)\), Eq. \((99)\), and use that the time evolution (as dictated by \(H_t = H_0\)) of the operators \(b(q, t) = e^{-i\omega(q)t}b(q)\). Finally, we use the Bogoliubov transformation, Eq. \((3)\), to relate the \(b\)-to the \(a\)-bosons, the basis where \(H_t\) is diagonal, and obtain the following expression:

\[
\langle \varphi(x, t)\varphi(0, t) \rangle = -\frac{1}{4} \sum_{q \neq 0} \left( \frac{2\pi v}{\omega_0(q)|L|} \right) \left[ (\sinh 2\beta(q) \times \cos (qx - 2\omega_0(q)t)(2n_B(q) + 1) - 2\cosh 2\beta(q) \cos qx n_B(q) - e^{iqx}\sinh 2\beta(q) - e^{-iqx}\cosh 2\beta(q) \right],
\]

where \(n_B(q) = (e^{\omega_0(q)/v} - 1)^{-1}\) is the Bose factor (we have assumed that the initial state is given by the density matrix \(\rho_1 = e^{-H_1(T)/Z}\).

Next let us consider the behavior of the expectation value of the order parameter. Taking into account that \(\langle e^{-2i\phi(0, t)} \rangle = \langle e^{-ix\varphi(0, t)} \rangle = e^{-\frac{q^2}{2}\omega_0(0)}\), we see that \(\langle \varphi^2(0, t) \rangle\) must be evaluated in closed form using Eq. \((103)\). Before performing any manipulation of this expression, it is convenient to subtract the constant \(\langle \varphi^2(0, 0) \rangle\), which is formally infinite (i.e. it depends on the short distance cut-off, \(a_0\)). Thus, \((T = 0\) and \(L \to \infty)\) we have:

\[
\langle \varphi^2(0, t) \rangle - \langle \varphi^2(0, 0) \rangle = \frac{1}{2} \int_{0}^{+\infty} \frac{d(qv)}{\omega(q)} \times \left[ \left( \frac{\omega(q)}{\omega_0(q)} \right)^2 - 1 \right] \sin^2 \omega_0(q)t.
\]

Inserting the expressions for \(\omega(p)\) and \(\omega_0(p)\) in the above equation, we obtain:

\[
\langle \varphi^2(0, t) \rangle - \langle \varphi^2(0, 0) \rangle = -f(2mt/\hbar),
\]

where \(f(z)\) is defined as:

\[
f(z) = 1 + \frac{1}{2} G_{13}^{21} \left( \frac{z^2}{4} \mid 0 \mid 1 \mid 1/2 \right),
\]

being \(G_{13}^{21}\) the Meijer G function. Using the asymptotic expansion for this function, \(f(z) \approx 1 - \frac{\pi |z|}{2}\), and hence the long-time behavior of the order parameter is:

\[
\langle e^{-2i\phi(0, t)} \rangle = \langle e^{-2i\phi(0, 0)} \rangle e^{\frac{q^2}{2}(1 - \pi mt)},
\]

We next examine the behavior of the two-point correlation function of the same operator,

\[
G(x, t) = \langle e^{2i\phi(x, t)} e^{-2i\phi(0, t)} \rangle = e^{\frac{q^2}{2} C(x, t)},
\]

where we have defined \(C(x, t) = \langle \varphi(x, t)\varphi(0, t) \rangle - \langle \varphi^2(0, t) \rangle\). At zero temperature, with the help of Eq. \((103)\), we find that

\[
C(x, t) - C(x, 0) = -\frac{m^2}{4} \int_{0}^{\infty} \frac{d(vq)}{\omega_0(q)^2 \omega(q)} \times (1 - \cos qx)(1 - \cos 2\omega_0(q)t),
\]

where

\[
C(x, 0) = -\frac{1}{2} \int_{0}^{\infty} \frac{d(vq)}{\omega(q)} (1 - \cos qx) e^{-qa_0},
\]

being \(a_0\) is the short-distance cut-off. Evaluating the integrals:

\[
C(x, t) - C(x, 0) = f(mx/v) + f(2mt) - \left[ f\left[ m(x/v + 2t) \right] + f\left[ m(x/v - 2t) \right] \right],
\]

where \(f(z)\) has been defined in Eq. \((106)\). Thus, asymptotically \((\text{for max}\{x/2v, t\} \gg m^{-1})\):

\[
G(x, t) = e^{\frac{q^2}{2} G(x, 0)} \times \begin{cases} e^{-\kappa^2 \pi m|x|/2v}, & \text{for } t > |x|/2v \\ e^{-\kappa^2 \pi m t}, & \text{for } t < |x|/2v, \end{cases}
\]
where \( \mathcal{G}(x,0) \) describes the correlations in the initial (gapped ground) state, and exhibits the following asymptotic behavior:

\[
\mathcal{G}(x,0) \simeq B(a_0) \left(1 - \kappa^2 \frac{\pi \nu}{m \lambda^2} e^{-m|x|/\nu} \right)
\]

where \( B(a_0) \) is a non-universal prefactor. Thus we see that the asymptotic form of \( \mathcal{G}(x,t) \) (Eq. [112]), as well as that of the order parameter, Eq. [102], have the same form as the results found at the Luther-Emery point, and also agree with the results of Calabrese and Cardy based on BCFT [12,15].

2. Quench from the gapless to the gapped phase

In this case, the system finds itself initially in the ground state of \( H_i = H_0 \), and suddenly (at \( t = 0 \)) the Hamiltonian is changed to \( H_t = H_{sc} \). For this situation convenient to obtain the evolution of the observables from the time-dependent canonical transformation of Eq. (7), where \( \beta(q) \) and \( \omega(q) \) are given by Eq. [101] and Eq. [102], respectively. In this case,

\[
\langle \varphi(x,t)\varphi(0,t) \rangle = \frac{1}{4} \sum_{q \neq 0} \left( \frac{2\pi \nu}{\omega(q) L} \right) \left[ \sinh 2\beta(q) \times \cos (qx - 2\omega(q)t) \left(2n_B(q) + 1\right) + 2 \cosh 2\beta(q) \cos qx n_B(q) + e^{i\omega t} \sinh^2 \beta(q) + e^{-i\omega t} \cosh^2 \beta(q) \right],
\]

(114)

where \( n_B(q) = (e^{\omega(q)/T} - 1)^{-1} \) is the Bose factor in the initial state (i.e. the gapless phase).

As in the previous case, \( \langle e^{-2i\varphi(x)} \rangle = e^{-\frac{2\kappa^2}{\nu}\langle \varphi^2(x,t) \rangle} \), and using Eq. [114], we find that:

\[
\langle \varphi^2(x,t) \rangle - \langle \varphi^2(0,0) \rangle = \int_0^\infty \frac{d(vq)}{\omega_0(q)} \left[ \left( \frac{\omega(q)}{\omega_0(q)} \right)^2 - 1 \right] \sin^2 \omega(q)t
\]

(115)

Note, interestingly, that this result can be obtained from Eq. [114] by exchanging \( \omega(q) \) and \( \omega_0(q) \). However, when evaluating the integral we find that \( \langle \varphi^2(0,t) \rangle = +\infty \), for all \( t > 0 \), due to the presence of infrared divergences that are not cured by the existence of a gap in the spectrum of \( H_i = H_{sc} \). Thus, we conclude that \( \langle e^{-2i\varphi(x)} \rangle = e^{-\frac{2\kappa^2}{\nu}\langle \varphi^2(x,t) \rangle} \) vanishes at all \( t > 0 \).

The above result for the evolution of the order parameter seems to indicate that the system apparently remains critical after the quench. This conclusion is also supported by the behavior of the two-point correlation function of the operator \( e^{2i\varphi(x,t)} \). Let \( \mathcal{G}(x,t) = \langle e^{2i\varphi(x,t)}e^{-2i\varphi(0,t)} \rangle = e^{2\kappa^2 \mathcal{C}(x,t)} \), where \( \mathcal{C}(x,t) = \langle \varphi(x,t)\varphi(0,t) \rangle - \langle \varphi^2(0,t) \rangle \). Using Eq. [11] and Eq. [99], we arrive at the following result (at zero temperature, and for \( L \to +\infty \)):

\[
\mathcal{C}(x,t) - \mathcal{C}(x,0) = \frac{m^2}{4} \int_0^\infty \frac{d(vq)}{\omega_0(q)} \frac{1}{1 - \cos qx} \times (1 - \cos 2\omega(q)t)
\]

(116)

where

\[
\mathcal{C}(x,0) = -\int_0^\infty \frac{d(vq)}{2\omega_0(q)}(1 - \cos qx)e^{-\nu_{aq}}
\]

(117)

To illustrate the above point about the apparent criticality of the non-equilibrium state, we can analyze the behavior of the two-point correlation function, \( \mathcal{G}(x,t) \), in two limiting cases, for \( t = 0 \) and \( t \to +\infty \). At \( t = 0 \), the correlation function, as obtained from Eq. [117], reads:

\[
\mathcal{G}(x,0) = A'(a_0) \left( \frac{a_0}{x} \right)^{2\kappa^2}
\]

(118)

where \( A'(a_0) \) depends on the short-distance cut-off \( a_0 \). Thus, the correlations are power-law because the initial state is critical. In the limit where \( t \to +\infty \), the part of the integral in Eq. [118] containing the term \( \cos 2\omega(q)t \) oscillates very rapidly and upon integration averages to zero. The remaining integral can be done with the help of tables [12] yielding:

\[
\mathcal{C}(x,t \to \infty) - \mathcal{C}(x,0) = -\sqrt{\pi} G_{01}^{22} \left( \frac{m^2 \lambda^2}{4v^2} \right) \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1/2 \end{array} \right]
\]

(119)

where \( G_{01}^{22} \) is a Meijer function. Using the asymptotic behavior of the Meijer function [12], we obtain:

\[
\lim_{t \to +\infty} \mathcal{G}(x,t) = B'(a_0) \left( \frac{2v}{m \lambda^2} \right)^{2\kappa^2}
\]

(120)

\( B'(a_0) \) being a non-universal constant. Thus, although initially the system is critical and therefore correlations at equilibrium decay as a power law with exponent \( 2\kappa^2 \), when the system is quenched into a gapped phase (where equilibrium correlations exhibit an exponential decay characterized by a correlation length \( \xi_c \approx v/m \)), the correlations remain power-law, within the semiclassical approximation. The exponent turns out to be smaller, equal to \( \kappa^2 \), which is half the exponent in the initial (gapless) state. In other words, within this approximation, it seems that the system keeps memory of its initial state, and behaves as if it was critical also after the quench. This behavior seems somewhat different from the results obtained for the same type of quench at the Luther-Emery point, where both the order parameter and the correlations for \( t \to +\infty \) approach a constant value, \( A(m\alpha_0) \) (unless the non-universal amplitude \( A(m\alpha_0) = 0 \), which seems to require some fine-tuning). Whether the differences found here between the Luther-Emery point and the semi-classical approximation are due to a break-down of the quasi-classical approximation (which neglects the
existence of solitons and anti-solitons in the spectrum of the sine-Gordon model), or to a qualitative change in the dynamics as one moves away from the Luther-Emery point, it is not clear at the moment. To clarify this issue, further investigation of this issue will be required in the future.

V. LONG-TIME DYNAMICS AND THE GENERALIZED GIBBS ENSEMBLE

Recently, Rigol and coworkers\textsuperscript{16} observed that, at least for observables like the momentum distribution or the ground state density, the asymptotic (long-time) behavior of an integrable system following a quantum quench can be described by adopting the maximum entropy (also called ‘subjective’) approach to Statistical Mechanics, pioneered by Jaynes.\textsuperscript{45,46} Within this approach, the equilibrium state of a system is described by a density matrix that extremizes the von-Neumann entropy, \( S = -\text{Tr} \rho \ln \rho \), subject to all possible constraints provided by the integrals of motion of the Hamiltonian of the system. In the case of an integrable system, if \( \{ I_m \} \) is a set of certain (but not all of the possible) independent integrals of motion of the system, this procedure leads to a ‘generalized’ Gibbs ensemble, described by the following density matrix:

\[
\rho_{gG} = \frac{1}{Z_{gG}} e^{-\sum \lambda_m I_m},
\]

where \( Z_{gG} = \text{Tr} e^{-\sum \lambda_m I_m} \). The values of the Lagrange multipliers, \( \lambda_m \), must be determined from the condition that

\[
\langle I_m \rangle_{gG} = \text{Tr} [\rho_0 I_m] = \langle I_m \rangle.
\]  

(121)

where \( \rho_0 \) describes the initial state of the system, and \( \langle \cdots \rangle_{gG} \) stands for the average taken over the generalized Gibbs ensemble, Eq. (121). Although \( \rho_0 = |\Phi(t = 0)\rangle\langle \Phi(t = 0)| \) in the case of a pure state, as was first used in Ref.\textsuperscript{10}, nothing prevent us from taking \( \rho_0 \) to be an arbitrary mixed state and in particular a thermal state characterized by an absolute temperature, \( T \). In this case, the Lagrange multipliers will depend on \( T \) or any other parameter that defines the initial state.

Rigol and coworkers tested numerically the above conjecture by studying the quench dynamics of a 1D lattice gas of hard-core bosons (see Ref.\textsuperscript{10,13} for more details). The question that naturally arises then is whether the family of integrable models studied in this work (see Eqs. (67) and (68)) relax in agreement with the mentioned conjecture. In other words, does the average \( \langle O \rangle(t) \) at long times relax to the value \( \langle O \rangle_{gG} = \text{Tr} \rho_{gG} O \) for any of the correlation functions considered previously? In this section we shall illustrate some of these questions by using the Luttinger model discussed in Sect.\textsuperscript{III}. We shall first show for what kind of operators the generalized Gibbs ensemble fails to reproduce their expectation values. Moreover, by considering the correlation function of the current operators (i.e. \( O = J_r(x) \), cf. Eq. (10)), we will illustrate why it works. Calculations of other observables and consideration of the other models as can be found in the appendices.

Let us define the generalized Gibbs ensemble for the Luttinger model (LM). Since the final Hamiltonian (in the \( N = 0 \) and \( J = 0 \) sector) is diagonal in the \( b \)-boson basis, i.e. \( H_{LM} = \sum_{q \neq 0} \hbar \omega(q) b^\dagger(q) b(q) \), a natural choice for the set of integrals of motion is \( I_m \rightarrow I(q) = b^\dagger(q) b(q) \) for all \( q \neq 0 \) (a more complete version of the ensemble should also include \( N \) and \( J \), but this will not necessary as we focus on the thermodynamic limit here).

Thus, for the quench from the non-interacting to the interacting state (cf. Sect.\textsuperscript{III}), where the initial state is \( |\Phi(t = 0)\rangle = |0\rangle \), the Lagrange multipliers are determined by Eq. (122), which yields:

\[
\langle I(q) \rangle_{gG} = \langle n(q) \rangle_{gG} = \sinh^2 \beta(q) = \frac{1}{e^{\lambda(q)} - 1}.
\]

(123)

Indeed, this result can be quickly established by realizing that \( \rho_{gG} \) has the same form as the density matrix of a peculiar canonical ensemble where the temperature on each eigenmode of the final Hamiltonian depends on the wave-vector \( q \), that is, \( T(q) = \hbar \omega(q) / \lambda(q) \). Alternatively, one can also regard it as an ensemble where the effective Hamiltonian that defines the Boltzmann weight is given by \( H_{eff} / T_{eff} = \sum_{q \neq 0} \lambda(q) n(q) \). However, it is worth noting that \( \rho_{gG} \) is diagonal in \( n(q) \), and therefore it does not capture the correlations existing in the initial state between the \( q \) and \(-q\) modes. Mathematically,

\[
\langle n(q) n(-q) \rangle = \sinh^2 \beta(q) \cosh 2 \beta(q) \neq \langle n(q) n(-q) \rangle_{gG} \]

\[
= \langle n(q) \rangle_{gG} \langle n(-q) \rangle_{gG} = \sinh^4 \beta(q)
\]

(124)

As matter of fact, since \( n(q) n(-q) \) commutes with \( H \), we conclude from the above that \( \langle n(q) n(-q) \rangle_{gG} \) does not relax to the value predicted by \( \rho_{gG} \). Although this defect of \( \rho_{gG} \) can be fixed by enlarging the set of integrals of motion to include the set \( I'(q) = n(q) n(-q) \) as well, we shall show below by explicit calculation that this is not needed for the calculation of the simplest correlators and observables in the thermodynamic limit. However, one important exception to this case are the squared fluctuations of the energy:

\[
\sigma^2 = \langle H^2 \rangle - \langle H \rangle^2 = \sum_{p,q} \hbar \omega(p) \hbar \omega(q) \times [\langle \hat{n}(p) \hat{n}(q) \rangle - \langle \hat{n}(p) \rangle \langle \hat{n}(q) \rangle]
\]

(125)

which yields \( \sigma^2 = 2 \sigma_V^2 = \sum_q \sinh^2 2 \beta(q) \hbar^2 \omega(q)^2 \).

Again, since the operator \( H^2 \) is conserved, \( \sigma^2 \) violates the relaxation hypothesis. However, it is tempting to argue, because that \( \sigma^2 \) (as well as \( \langle H \rangle \)) is a non-universal property of the LM model, this violation is less problematic than a violation in the asymptotic behavior of the correlation functions would be, as the latter tends to be
more universal. Similar results can be obtained for the other models considered in this work.

Let us now consider observable correlations in the LM model. This requires that the operator $O$ must be hermitian, which is the case for the current operator $J_r(x) = \partial_x \phi_r(x)/2 \pi$ but not for the field operator $\psi_r(x)$ (in this case, one has to consider the momentum distribution, as we have done in Sect. 111B). In the case of the current operator, we shall study the following two-time correlation function (no time ordering is implied):

$$C_{J_r}(x, t, \tau) = \langle J_r(x, t + \tau/2) J_r(0, t - \tau/2) \rangle_T,$$  \hspace{1cm} (126)

where $\langle \ldots \rangle_T$ stands for average over other the thermal ensemble described by $\rho_i = e^{-H_i/T}/Z_0$, with $H_i = H_0$ (cf. Eq. 135). Using (29), and (30), we obtain

$$C_{J_r}(x, t, \tau) = \frac{1}{(2\pi)^2} \sum_{q > 0} \left( \frac{2\pi q}{L} \right) e^{-q \omega_0} \left\{ e^{iqx} f(q, t + \tau/2) f(q, t - \tau/2) [1 + n_B(q)] + e^{iqx} g(q, t + \tau/2) g(q, t - \tau/2) n_B(q) + e^{-iqx} f^*(q, t + \tau/2) f(q, t - \tau/2) n_B(q) + e^{-iqx} g^*(q, t + \tau/2) g(q, t - \tau/2) [1 + n_B(q)] \right\},$$  \hspace{1cm} (127)

being $n_B(q) = (e^{-\hbar \omega_0(q)/T} - 1)^{-1}$ ($\omega_0(q) = v_F |q|$) the initial Bose distribution of modes. In the following we shall argue that, in the limit $t \to +\infty$ the above expression reduces to the following correlator in the Generalized Gibbs ensemble:

$$C_{J_r}^G_{L}(x, \tau) = \text{Tr} [\rho_{G}(T) J_r(x, \tau) J_r(0, 0)],$$  \hspace{1cm} (128)

where $\rho_{G}(T)$ is the extension to an initial thermal state of the Generalized Gibbs ensemble introduced above (notice that since $[H_1, J_1] = 0$ and therefore $[H_1, \rho_{G}] = 0$, it is in principle possible to define time-dependent correlation functions on this ensemble, just as we defined the for standard equilibrium states). For this (thermal) initial condition (122) fixes the values of $\lambda(q)$ which now depend on $\beta(q)$ and the temperature:

$$\sin^2 \beta(q) [1 + n_B(q)] + \cosh^2 \beta(q) n_B(q) = \frac{1}{e^{\lambda(q)} - 1}. \hspace{1cm} (129)$$

Introducing this result into the mode expansion for Eq. (125),

$$C_{J_r}^G_{L}(x, \tau) = \frac{1}{(2\pi)^2} \sum_{q > 0} \left( \frac{2\pi q}{L} \right) e^{-q \omega_0} \left\{ e^{iq(x-\nu \tau)} \cosh^2 \beta(q) [1 + \langle n(q) \rangle] + e^{iq(x+\nu \tau)} \sin^2 \beta(q) \langle n(q) \rangle + e^{-iq(x-\nu \tau)} \cosh^2 \beta(q) \langle n(q) \rangle + e^{-iq(x+\nu \tau)} \sin^2 \beta(q) [1 + \langle n(q) \rangle] \right\},$$  \hspace{1cm} (130)

we see that it coincides with the $t \to +\infty$ limit of (127), where the rapidly oscillating terms that depend only on $t$ can be dropped.

Let us close this section with brief consideration of higher order correlation functions. In particular, if we consider computation objects like the four point current correlation function, $\langle J_{r}(x_1, t_1) J_{r}(x_2, t_2) J_{r}(x_3, t_3) J_{r}(x_4, t_4) \rangle_T$, we would encounter terms involving $\langle n(q) n(-q) \rangle$, which are not described by the generalized Gibbs ensemble (in its simpler form where $f(q) = n(q)$). However, it is not hard to convince oneself that in the thermodynamic limit, $L \to \infty$ the contribution from such terms vanishes. This is because momentum conservation when computing $\langle J_{r}(x_1, t_1) J_{r}(x_2, t_2) J_{r}(x_3, t_3) J_{r}(x_4, t_4) \rangle_T$ leaves (in most cases) with independent two wave numbers (out of four) that must be summed over. When taking the thermodynamic limit, the $L^2$ resulting from this two sums exactly cancels the $L^{-2}$ factors coming from the mode expansions of $J_{r}(x)$, thus yielding a finite contribution. However, terms involving $\langle n(q) n(-q) \rangle$ require the four wave numbers to be equal, thus resulting in only one momentum summation, which therefore cannot cancel $L^{-2}$ factor. This justifies the use a Wick’s theorem when computing higher order correlations in the thermodynamic limit, but in finite size systems, this procedure as well as the simplest version of the generalized Gibbs ensemble would miss the contribution stemming from $\langle n(q) n(-q) \rangle$ correlations.

VI. RELEVANCE TO EXPERIMENTS

As we described in the introduction, cold atom systems is the ideal arena to study quench dynamics. This is because they are, to a large extent, entirely isolated systems. Furthermore, as far as one dimensional systems are concerned, there are already a number of experimental realizations, including experiments where quench dynamics has been already probed. Thus, in this section we would like to discuss the possible experimental relevance of the results obtained in previous sections. As mentioned above, this must be done with great care, as our results have been obtained using effective field theory models that can be regarded as ‘caricatures’ of the Hamiltonians that describe real experiments. We must emphasize that the situation in the case of quantum quenches, in particular, and of non-equilibrium dynamics, in general, is very different from the analysis of low-temperature phenomena in equilibrium. In the latter case, the experimental relevance of effective field theories is well established using renormalization-group arguments and it has been, over the years, widely tested using a variety of numerical and also (when possible) analytical methods. By contrast, here we travel through a largely uncharted land, and much needs to be studied in order to achieve a similar level of rigor as in the equilibrium case. Thus, it is convenient to regard this models as ‘toy models’,
which can provide us valuable lessons and insights into the non-equilibrium dynamics of strongly correlated systems. This this cautionary remarks, we can proceed to discuss some experimental systems to which the above results could of some relevance.

The results presented in Section III were obtained for the particular case of the exactly solvable LM. The LM is the exactly solvable model describing the fixed point of a general class of interacting one-dimensional models, known as Tomonaga-Luttinger liquids. This class includes systems such as the one-dimensional Bose gas interacting via a Dirac-delta potential (which is solvable by the Bethe-ansatz), as well as many other systems of interacting Bose gases with repulsive interactions (such as dipolar) or Fermi gases with both attractive and repulsive interactions. With the caveats of the previous section, it would be interesting to test the results obtained using the LM in one of these systems. However, the dynamics may be strongly modified by the fact that higher energy states will be also excited following a quantum quench such as the one described here which involves switching on (or off) the interactions. Such higher energy states are not correctly described by the LM, and the situation is expected to worsen in the case of short range interactions. Thus, one possible way around is to study either experimentally or numerically quenches where the interactions are longer range, as the latter case provides us with a much more faithful realization of the LM. One such system is a single-species dipolar 1D Fermi gas confined to one dimension in a strongly anisotropic trap (e.g. 59). Since p-wave interactions are weak (away from a p-wave Feshbach resonance), the dominant interaction is dipolar, which, when the dipoles are all aligned by an external (electric or magnetic, depending on whether the dipole is electric, like in hetero-nuclear molecules, or magnetic, like in Chromium). When confined to 1D, the dipolar interaction between the atoms can be approximated by the potential:

$$V_{\text{dip}}(x, \theta) = \frac{1}{4\pi\epsilon_0} \frac{D^2\lambda(\theta)}{(x^2 + R_0^2)^{3/2}}$$

(131)

where $D$ is the dipolar momentum of the atoms, $\theta$ is the angle subtended by the direction of the atomic motion and the polarizing field, and $\lambda(\theta) = (1 - 3\cos^2\theta)$. Since in this case $g_2(q) = g_4(q) \propto \lambda(\theta)$, a sudden change in the interactions can be produced by a sudden change in alignment of the field with the direction of motion, that is, a change in $\theta$. In particular, a change in $\theta$ away from the value $\theta_m = \cos^{-1}(\frac{1}{3})$ would produce lead to a sudden switching of the interactions amongst the Fermions.

At zero temperature, the momentum distribution $f(p, t)$ (which can be probed by time of flight measurements) following the quench into the interacting system would evolve as described in Sect. III B (cf. Fig. 1), with the discontinuity at the Fermi level dying out as $t^{-\gamma^2}$. However, currently atomic gases are produced at temperatures $T \sim 10\%$ to $20\%$ of the Fermi energy, and this would complicate the observation of this effect. If much lower temperatures could be reached in experiments, so that the application of effective field theory is much more reliable, we expect that in a time of the order of $\hbar/T$ the quenched dipolar gas reaches a stationary state characterized by a momentum distribution that differs from the thermal one. However, the calculations of $f(p, t)$ presented in Sect. III B (cf. Figs. 2 and 3) show that the differences between the non-equilibrium and equilibrium results in the stationary state may be well below the current experimental resolution. Alternatively, instead of measuring the momentum distribution, one can try to determine the non-equilibrium exponents measuring noise correlations in the time-of-flight images or through interferometry.

Finally, we shall remark that the sine-Gordon model is an effective field-theory description of the Mott insulator to superfluid transition (MI to SF) in 1D or, when the field $\varphi$ is interpreted as the (relative) phase of two coupled 1D Bose gas, it describes the Josephson coupling of two 1D Bose gases. Thus, the results presented here could be of some relevance when interpreting the evolution of such systems following a quantum quench. In particular, in the case of the MI to SF transition, the order parameter $e^{-2\varphi(x)}$ describes the time evolution oscillatory part of the density in the system, whereas the correlations of the order parameter are related to the time evolution of the static structure factor.

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APPENDIX A: DETAILS OF THE CALCULATION OF THE ONE-PARTICLE DENSITY MATRIX IN THE LUTTINGER MODEL

In this Appendix, we shall provide the details of the calculation of non-equilibrium one-particle density matrix:

$$C_{\psi_i}(x, t) = \langle e^{iH_f t/\hbar}\psi_i(x)\psi_i(0)e^{-iH_f t/\hbar}\rangle$$

(A1)
To this end, the formula (28) is used. In normal ordered form:

$$\psi_\alpha(x) = \frac{e^{-\phi_*}}{\sqrt{L}} : e^{i\phi_*}(x) : = e^{i\phi(x)/\sqrt{L}} ,$$

(A2)

where the normal order is defined:

$$\langle e^{i\phi_*}e^{\phi_*} e^{2\pi i N \phi} e^{-i\phi}(x) e^{i\phi_*}(x) \rangle .$$

(A3)

The boson field $\Phi_\alpha(x)$ is given by Eq. (30). Hence,

$$e^{-i\phi_*(x)} : e^{i\phi_*(0)} : = e^{-2i\pi N \phi} e^{i\Phi_\alpha(x)} \times e^{-i\phi_*(x)-\phi_*(0)} : ,$$

(A4)

where we have used the identity $e^{Ae^B} = e^{[A,B]} e^B e^A$, which holds provided $[A,B]$ is a c-number. Using that $(a_0 \to 0^+)$ is the short-distance cut-off:

$$\langle [\Phi_\alpha(x), \Phi_\alpha^\dagger(0)] \rangle = \sum_{q>0} \frac{2\pi}{qL} e^{-qa_0} e^{iqx}$$

(A5)

$$= -\ln \left[ 1 - e^{-2\pi a_0/L} e^{2\pi x/L} \right] ,$$

(A6)

we arrive at the following expression for $C_\psi(x,t)$:

$$C_\psi(x,t) = G_r^{(0)}(x) \langle e^{-iF_r^\dagger(x,t)} e^{iF_r(x,t)} \rangle ,$$

(A7)

where

$$G_r^{(0)}(x) = \frac{1}{2L} \sin \left[ \frac{\pi}{L} (x+ia) \right]$$

(A8)

$$F_r(x,t) = e^{iHt/h} [\Phi_\alpha(x) - \Phi_\alpha(0)] e^{-iHt/h}$$

(A9)

$$= \sum_{q>0} \frac{2\pi}{qL} \left( e^{iqx} - 1 \right)$$

(A10)

$$\times \left[ f(q,t)b(q) + g^*(q,t)b^\dagger(-q) \right] .$$

(A11)

To derive the last expression we have used Eq. (7). Employing the identities $e^A e^B = e^{[A,B]} e^B e^A$ (provided $[A,B]$ is a c-number) and that $\langle e^D \rangle = e^{\langle D \rangle + \frac{1}{2}\langle (D^2) \rangle}$, we obtain:

$$C_\psi(x,t) = G_r^{(0)}(x) \langle e^{-(F_r^\dagger(x,t) F_r(x,t))} \rangle ,$$

(A12)

where we have used that $\langle [F_r^\dagger(x,t), F_r(x,t)] \rangle = 0$ because $\langle [b(q) b^\dagger(q)] \rangle = \langle [b(q)] \rangle = \langle [b^\dagger(q)] \rangle = 0$, and since the commutator $[F_r^\dagger(x,t), F_r(x,t)]$ is a c-number, it can be safely replaced by $\langle [F_r^\dagger(x,t), F_r(x,t)] \rangle$. Note that for $t = 0$ $F_r(x,t)$ contains only $b(q)$ and thus the average $\langle F_r^\dagger(x,t) F_r(x,t) \rangle = 0$ at $T = 0$. In Eq. (A12) the exponent can be expanded to yield:

$$\langle F_r^\dagger(x,t) F_r(x,t) \rangle = \sum_{q>0} \frac{2\pi}{qL} e^{-qa_0} e^{iqx} - 1 \rangle^2$$

$$\times \left[ f(q,t)^2 n_0(q) + g(q,t)^2 (n_0(q)+1) \right] ,$$

(A13)

being $n_0(q) = (e^{-\lambda|q|}-1)^{-1}$ the distribution of Tomonaga bosons in the initial state (which has been assumed to be a mixed thermal state), and $\lambda = h\nu T$ is the thermal correlation length. We next evaluate explicitly the above result in several limiting cases.

1. Zero temperature and finite length

Let us now consider the $T = 0$ limit, where $n_0(q) = 0$, and thus, using (3) and (9), Eq. (A13) simplifies to:

$$\langle F_r^\dagger(x,t) F_r(x,t) \rangle_{T=0} = \sum_{q>0} \left( \frac{2\pi}{qL} \right) e^{-qa_0} \sin^2[2\beta(q)]$$

$$\times (1 - \cos qx) \left[ 1 - \cos 2 \nu q | q | t \right] .$$

(A14)

To make further progress, we shall assume that $\sin 2\beta(q) = e^{-\gamma q R_0}/2$ where $R_0$ is the range of the interaction. Furthermore, we shall replace $\nu(q)$ by $\nu(0)$, what allows us to safely take the limit $a_0 \to 0+$. Next, in order to simplify the computation, we introduce the quantity

$$\mathcal{E}_r(z) = \sum_{q>0} \left( \frac{2\pi}{qL} \right) e^{-R_0 q \cos qz} ,$$

(A15)

which can be readily computed to give

$$\mathcal{E}_r(z) = -\ln \left[ \frac{\pi}{L} L \left( z + i R_0 | L \right) \right] + \frac{\pi R_0}{L} - \ln 2 ,$$

(A16)

where $d(z|L) = L | \sin \pi z/L | / \pi$ is the cordon function. Using this result into Eq. (A14), yields the following expression for the one-particle density matrix:

$$C_\psi(x,t > 0 | L) = G_r^{(0)}(x) \left[ \frac{d(x+iR_0|L)}{d(x+iR_0|L)} \right]^2 \frac{d(x+2\nu t+iR_0|L)}{d(2\nu t+iR_0|L)} \sin^2[2\beta(q)] .$$

(A17)

Taking into account that $R_0/L \ll 1$ and, in the scaling limit, one recovers the result quoted in the main text, Eq. (32).

2. Thermodynamic limit and finite temperature

We next consider Eq. (A12) for $L \to \infty$ and finite temperature, $T$. Equation (A13) can be recast as:

$$\langle F_r^\dagger(x,t) F_r(x,t) \rangle_T = \langle F_r(x,t) F_r(x,t) \rangle_{T=0} + \mathcal{H}(x) + \mathcal{G}(x,t)$$

(A18)

where we have introduced the following functions:

$$\mathcal{H}(x) = 2 \int_0^\infty \frac{dq}{q} e^{q^2} \langle 1 - \cos qx \rangle n_0(q) ,$$

(A19)

$$\mathcal{G}(x,t) = 2 \gamma^2 \int_0^\infty \frac{dq}{q} e^{-q R_0} \langle 1 - \cos qx \rangle$$

(A20)

$$\left[ 1 - \cos (2\nu q t) \right] n_0(q) ,$$

(A21)

which hold in the thermodynamic limit and upon replacing $\nu(q)$ by $\nu(0) = 0$ and $\sin 2\beta(q) = e^{-\gamma q R_0}/2$.
as we did in the previous section. We next define the function
\[
g(u; r) = 2 \int_0^{+\infty} dq \frac{e^{-q r}}{q} \frac{(1 - \cos qu)}{e^{\lambda q} - 1}. \tag{A22}
\]
which can be evaluated to yield\cite{27}
\[
g(u; r) = 2 \ln \left| \frac{\Gamma(1 + \lambda^{-1} r)}{\Gamma(1 + \lambda^{-1}(r + iu))} \right|, \tag{A23}
\]
where \(\Gamma(z)\) is the Gamma function. In the limit where \(r \ll u\), and using that \(\Gamma(z) \Gamma(1 - z) = \pi / \sin(\pi z)\), the above expression reduces to
\[
g(u; r) = -\ln \left| \frac{dh(i r | T)}{dh(u + i r | T)} \right| \left( \frac{u + i r}{r} \right). \tag{A24}
\]

In the previous expression we have defined:
\[
dh(z | T) = \frac{\lambda}{\pi} |\sinh(\pi \lambda^{-1} z)|. \tag{A25}
\]

Combining this result with Eqs. (A18)–(A21) and (A12), it is seen that the second term in Eq. (A24) exactly cancels the contributions from \(G^{(0)}(x | L)\) and \((F_r(x; t) F_r(x, t))_{T=0}\) in the thermodynamic limit, and therefore,
\[
C_{\psi_r}(x, t | T) = G_r^{(0)}(x | T) \left[ \frac{dh(i R_0 | T)}{dh(x + i R_0 | T)} \right]^{\gamma^2} \times \left[ \frac{dh(x + 2vt + i R_0 | T)}{dh(2vt + i R_0 | T)} \right] \frac{\pi \lambda^{-1}}{2 \pi \sinh(\pi \lambda^{-1} (x + i a_0))}. \tag{A26}
\]

where
\[
G_r^{(0)}(x | T) = \frac{i}{2 \pi \sinh(\pi \lambda^{-1} x)} \left( \frac{\pi \lambda^{-1}}{T} \right). \tag{A27}
\]

We note that the result of Eq. (A26) can be obtained from Eq. (A17) upon making the replacement \(L \sin(\pi L^{-1} x) / \pi \) by \(\lambda \sin(\pi \lambda^{-1} x)\). Taking into account that \(R_0 / L\), and in the scaling limit, we retrieve the result quoted in the main text, Eq. (46).

**APPENDIX B: ONE-BODY DENSITY MATRIX OF THE LUTTINGER MODEL IN THE GENERALIZED GIBBS ENSEMBLE**

Next we take up the calculation of the one-body density matrix in the generalized Gibbs ensemble for the Luttinger model discussed in Sect. IV. That is, we shall evaluate the expression at \(T = 0\).

\[
C_{\psi_r}^G(x) = \text{Tr} \left[ \rho_G \psi_r^\dagger(x) \psi_r(0) \right] \tag{B1}
\]

Using the bosonization identity, Eq. (28), we can write the expression as follows:
\[
C_{\psi_r}^G(x) = G_r^{(0)}(x) \langle e^{-i \varphi_r(x) - \varphi_r(0)} \rangle g_G \tag{B2}
\]
\[
= G_r^{(0)}(x) \langle e^{-i F_r(x) \psi_r(x)} \rangle g_G \tag{B3}
\]

Taking into account that
\[
\tilde{F}_r(x) = \Phi_r(x) - \Phi_r(0) \tag{B4}
\]
\[
= \sum_{q > 0} \left( \frac{2 \pi}{q L} \right)^{1/2} e^{-q a/2(\pi q x - 1)} \times \left[ \cosh(\beta q a(q) - \sinh(\beta q a)(q)) \right]. \tag{B5}
\]

The expression for \(C_{\psi_r}^G(x)\) can be easily computed by using the trick of regarding \(\rho_G\) as a canonical ensemble with \(g\)-dependent temperature. Thus, following the same steps as in the previous section we arrive at:

\[
C_{\psi_r}^G(x) = G_r^{(0)}(x) e^{-\langle \tilde{F}_r(x) \rangle_{g_G}} \tag{B7}
\]

Given that
\[
\langle \tilde{F}_r(x) \rangle_{g_G} = \sinh^2 \beta \left( D_r(x) - D_r(0) \right), \tag{B8}
\]
where
\[
D_r(x) = \text{Re} \left\{ \sum_{q \geq 0} \left( \frac{2 \pi}{q L} \right) e^{-q a/2(\pi q x - 1)} \right\} \tag{B9}
\]
\[
= -\ln \left[ \frac{\sin(\pi (x + i R_0))}{\Gamma(1 + \lambda^{-1} x)} \right] - 2 \frac{\pi R_0}{L}. \tag{B10}
\]

Hence, taking the thermodynamic limit
\[
C_{\psi_r}^G(x) = \frac{i}{2 \pi (x + i a)} \left( \frac{R_0}{x} \right)^{\gamma^2}. \tag{B11}
\]

Thus we see that one recovers the same results as \(\lim_{x \to +\infty} C_{\psi_r}(x, t)\), Eq. (55).

**APPENDIX C: THE SINE-GORDON MODEL AND THE GENERALIZED GIBBS MODEL**

1. **Quench from the gapped to the phase at the Luther-Emery point**

In this case, the type evolution of the system is performed by \(H_0\) (cf. Eq. (60)), which is diagonal in the operators \(n_\alpha(p) = \langle \psi_\alpha^\dagger(p) \psi_\alpha(p) \rangle\) : \(\alpha = r, l\). Thus, the generalized Gibbs ensemble can be defined by the following set of integrals of motion \(I_m \to I_\alpha(p) = n_\alpha(p)\). We see immediately that the fact this ensemble is diagonal in \(n_\alpha^G(p)\) means that the order parameter, \(\langle e^{-2i \varphi_r(x)} \rangle_{g_G} = \langle \psi_r(x) \psi_l(x) \rangle_{g_G} = 0\), which agrees with the \(t \to +\infty\) limit of the order parameter, which was shown in Sect. IV A to exhibit an exponential decay to zero. However, the two-point correlator of \(e^{2i \varphi(x)}\) has a non-vanishing limit for
Introducing the last expression into Eq. (C1) yields:

\[ \langle e^{-2iϕ(x)}e^{2iϕ(0)} \rangle_{G} = \langle ψ_{r}^{1}(x)ψ_{l}(x)ψ_{l}^{1}(0)ψ_{r}(0) \rangle_{G} \]

\[ = ∑_{p_{1},p_{2},p_{3},p_{4}} \frac{e^{i(p_{1}-p_{2})x}}{L^{2}} \langle ψ_{1}^{1}(p_{1})ψ_{1}(p_{2})ψ_{1}^{1}(p_{3})ψ_{1}(p_{4}) \rangle_{G} \]

(C1)

Since the ensemble is diagonal in the chirality index, \( α \), as well as momentum, \( p \), we evaluation of the above expression can be carried out by noting that:

\[ \langle ψ_{α}^{1}(p)ψ_{α}(p') \rangle_{G} = \frac{\text{Tr} e^{-∑_{p,α'\neq α} λ_{α'}(p)I_{α'}(p)ψ_{α}(p)} }{\text{Tr} e^{-∑_{p,α'\neq α} λ_{α'}(p)I_{α'}(p)}} \]

\[ = \frac{δ_{p,p'}}{e^{λ_{−}(p)+1}}. \]  

(C2)

where the Lagrange multipliers \( λ(q) \) can be related to the values of the same expectation values in the initial states by imposing their conservation, that is,

\[ \langle ψ_{1}^{1}(p)ψ_{l}(p) \rangle_{G} = \frac{1}{e^{λ_{1}(p)+1}} \]  

(C3)

\[ = \langle ψ_{1}^{1}(p)ψ_{l}(p) \rangle = \cos^{2} θ(p) \]  

(C4)

\[ \langle ψ_{r}^{1}(p)ψ_{r}(p) \rangle_{G} = \frac{1}{e^{λ_{r}(p)+1}} \]  

(C5)

\[ = \langle ψ_{r}^{1}(p)ψ_{r}(p) \rangle = \sin^{2} θ(p). \]  

(C6)

Hence,

\[ \langle ψ_{1}^{1}(p_{1})ψ_{1}(p_{2})ψ_{l}^{1}(p_{3})ψ_{l}(p_{4}) \rangle_{G} = \langle ψ_{1}^{1}(p_{1})ψ_{r}(p_{4}) \rangle_{G} \]

\[ × \langle ψ_{1}(p_{2})ψ_{l}^{1}(p_{3}) \rangle_{G} \]

\[ = δ_{p_{1},p_{4}}δ_{p_{2},p_{3}} \sin θ(p_{1})(1-\cos^{2} θ(p_{2})) \]

\[ = δ_{p_{1},p_{4}}δ_{p_{2},p_{3}} \sin^{2} θ(p_{1})\sin^{2} θ(p_{2}). \]  

(C7)

Introducing the last expression into Eq. (C1) yields:

\[ \langle e^{-2iϕ(x)}e^{2iϕ(0)} \rangle_{G} = \left| ∑_{p} e^{ipx} \sin^{2} θ(p) \right|^{2} \]

\[ = \frac{1}{L} \sum_{p} e^{ipx} \sin^{2} θ(p) \]

(C8)

and using that \( \sin^{2} θ(p) = (1-\cos 2θ(p))/2 \) and \( \cos 2θ(p) = ω_{0}(p)/\sqrt{ω_{0}^{2}(p)+m^{2}} \), we find (for \( x \neq 0 \)),

\[ \langle e^{-2iϕ(x)}e^{2iϕ(0)} \rangle_{G} = \left( \frac{m}{2πv} \right)^{2} \left[ K_{1} \left( \frac{m|x|}{v} \right) \right]^{2}, \]

(C9)

which equals to the \( t \rightarrow +∞ \) limit of Eq. (55).

2. Quench from the gapless to the gapped phase at the Luther-Emery point

In this case the initial state is the gapless ground state of \( H_{0} \), Eq. (56) whereas the Hamiltonian that performs the time evolution has a gap in the spectrum and it is diagonal in the basis of \( ψ_{r}(p) \) and \( ψ_{l}(p) \) Fermi operators (cf. Eq. (71)). Therefore, the conserved quantities are

\[ I_{r}(p) = n_{r}(p) = ψ_{r}^{†}(p)ψ_{r}(p), \]

\[ I_{l}(p) = n_{l}(p) = ψ_{l}^{†}(p)ψ_{l}(p). \]  

(C10)

(C11)

The associated Lagrange multipliers (at zero temperature), \( λ_{r}(p) \) and \( λ_{l}(p) \) can be obtained upon equating \( \langle I_{r,c}(p) \rangle_{G} = \langle (ψ_{r,l}(p)|Ψ(0)) \rangle \). This yields:

\[ \langle I_{r}(p) \rangle_{G} = \frac{1}{e^{λ_{r}(p)+1}} \]

\[ = δ(−p)\sin^{2} θ(p) + δ(p)\cos^{2} θ(p) \]  

(C12)

\[ \langle I_{l}(p) \rangle_{G} = \frac{1}{e^{λ_{l}(p)+1}} \]

\[ = δ(−p)\cos^{2} θ(p) + δ(p)\sin^{2} θ(p), \]  

(C13)

(C14)

(C15)

where \( δ(p) \) denotes the step function. Using these expressions we next proceed to compute the expectation values of the following observables:

\( a. \) Order parameter

We start by computing the order parameter,

\[ \langle e^{-2iϕ(x)} \rangle_{G} = \langle ψ_{r}^{†}(x)ψ_{l}(x) \rangle \]

\[ = \frac{1}{2L} \sum_{p} \sin 2θ(p) \left[ (I_{r}(p))_{G} - (I_{l}(p))_{G} \right], \]  

(C16)

and upon using Eqs. (C13) and (C15),

\[ \langle e^{-2iϕ(x)} \rangle_{G} = - \frac{1}{L} \sum_{p>0} \sin 2θ(p) \cos 2θ(p) \]

\[ = - \int_{0}^{∞} dp \frac{mω_{0}(p)}{2π} e^{−mω_{0}} \]

\[ = A(mω_{0}), \]

(C17)

(C18)

(C19)

(C20)

where we have used that \( \cos 2θ_{−p} = −\cos 2θ_{p} \); \( A(mω_{0}) \) is the non-universal constant introduced in Sect. IV B 2 . This result agrees with the one obtained in Sect. IV B 2 for the order parameter in the limit \( t \rightarrow +∞ \).

\( b. \) Two-point correlation function

We next consider the two-point correlator of the order parameter, namely

\[ \langle e^{2iϕ(x)}e^{2iϕ(0)} \rangle_{G} = \frac{1}{L^{2}} \sum_{p_{1},p_{2},p_{3},p_{4}} e^{i(p_{1}−p_{2})x} \langle ψ_{r}^{†}(p_{1})ψ_{l}(p_{2}) \]

\[ × ψ_{l}^{†}(p_{3})ψ_{r}(p_{4}) \rangle_{G}. \]  

(C21)

(C22)
The calculation of the average in this case is a bit more involved, but it can be performed by resorting to a factorization akin to Wick’s theorem. This is applicable only in the thermodynamic limit, as it neglects terms the four momenta of the above expectation value coincide. These terms yield contributions of \( O(1/L) \) compared to others. When factorizing as dictated by Wick’s theorem, the only non-vanishing terms are:

\[
\langle \psi^\dagger_r(p_1)\psi(p_2)\psi^\dagger_r(p_3)\psi_r(p_4) \rangle_{\text{G}} = -\delta_{p_1p_4}\delta_{p_2p_3} \\
\times \langle \psi^\dagger_r(p_1)\psi_r(p_4) \rangle_{\text{G}} \langle \psi^\dagger_r(p_3)\psi(p_2) \rangle_{\text{G}} \\
+ \delta_{p_1p_2}\delta_{p_3p_4} \langle \psi^\dagger_r(p_1)\psi_r(p_2) \rangle_{\text{G}} \langle \psi^\dagger_r(p_3)\psi_r(p_4) \rangle_{\text{G}}.
\]

Upon using

\[
\langle \psi^\dagger_r(p)\psi_r(p) \rangle_{\text{G}} = \frac{1}{2}\theta(p)\sin^2\theta(p) \\
+ \theta(-p) \left( 1 - \frac{1}{2}\sin^2\theta(p) \right),
\]

\[
\langle \psi^\dagger_r(p)\psi(p) \rangle_{\text{G}} = \frac{1}{2}\theta(-p)\sin^2\theta(p) \\
+ \theta(p) \left( 1 - \frac{1}{2}\sin^2\theta(p) \right),
\]

\[
\langle \psi^\dagger_r(p)\psi_r(p) \rangle_{\text{G}} = \langle \psi^\dagger_r(p)\psi(p) \rangle_{\text{G}}
= \frac{1}{2}\sin\theta(p)\cos\theta(p)\text{sgn}(p),
\]

the average over the generalized Gibbs ensemble of the four Fermi fields on the right hand-side of Eq. \((C21)\) can be computed and yields the following expression for the two-point correlation function (up to terms of \( O(1/L^2) \)):

\[
\langle e^{2i\phi(x)}e^{-2i\phi(0)} \rangle_{\text{G}} = \left\{ \frac{1}{L} \sum_p e^{ipx} \left[ \frac{1}{2}\theta(p)\sin^2\theta(p) \\
+ \theta(-p) \left( 1 - \frac{1}{2}\sin^2\theta(p) \right) \right]^2 \\
+ \frac{1}{L} \sum_{p>0} \sin\theta(p)\cos\theta(p) \right\}^2
\]

The last term in the above expression in just \( \langle e^{2i\varphi(x)} \rangle_{\text{G}} \) \( \langle e^{-2i\varphi(0)} \rangle_{\text{G}} \) (cf. Eq. \((C18)\)), whereas the first term in the right hand-side can be written as

\[
\left\{ \frac{1}{L} \sum_p e^{ipx} \left[ \theta(-p) + \frac{1}{2}\text{sgn}(p)\sin^2\theta(p) \right]^2 \\
= \left\{ \frac{1}{L} \sum_{p>0} e^{-ipx} + \frac{1}{2} \sum_{p>0} \text{sgn}(p) \frac{m^2}{\omega^2(p)} \right\}^2 \\
= \left\{ \frac{1}{L} \sum_{p>0} e^{-ipx} + \lim_{t\to\infty} \mathcal{H}(x,t) \right\}^2
\]

and coincides with the \( t\to+\infty \) limit of the second term in the right hand-side of Eq. \((C14)\) in Sect. \[V A 2\] (the function \( \mathcal{H}(x,t) \) is defined in Eq. \((B5)\)).

3. Quench from the gapless to the gapped phase in the semi-classical approximation

In this case Hamiltonian performing the time evolution is gapless, and therefore, diagonal in the \( b \)-operators. Hence, the conserved quantities are

\[
(I(q))_{\text{G}} = \frac{1}{e^{\lambda(q)} - 1} = \langle \Phi(0)|b^\dagger(q)b(q)|\Phi(0) \rangle
= \sinh^2\beta(q),
\]

where \( \beta(q) \) is defined by Eq. \((101)\). Hence, using this result we next proceed to compute the order parameter and the two-point correlation function. We first note that the order parameter vanishes in the generalized Gibbs ensemble since \( \langle e^{-2i\phi(x)} \rangle_{\text{G}} = e^{-2i\phi(0)} \rangle_{\text{G}} \) and \( \langle \phi^2(0) \rangle_{\text{G}} = \frac{1}{L} \langle \varphi^2(0) \rangle \) is divergent in the \( L\to+\infty \) limit (see below). This agrees with the result found in Sect. \[V B 1\] where it was found that the order parameter decays exponentially in time. Thus, in what follows we shall be concerned with the the two-point correlation function.

\[\text{a. Two-point correlation function}\]

Since \( \langle e^{-2i\phi(x)}e^{2i\phi(0)} \rangle_{\text{G}} = e^{-\frac{\beta^2}{2}\mathcal{C}^G(x)} \) where \( \mathcal{C}^G(x) = \langle \varphi(x)\varphi(0) \rangle_{\text{G}} - \langle \varphi^2(0) \rangle_{\text{G}} \). In order to obtain this correlator, we introduce the Fourier expansion of \( \varphi(x) \) (neglecting the zero-mode contribution),

\[
\varphi(x) = \frac{1}{2} \sum_{q>0} \left( \frac{2\pi v}{\omega_0(q)L} \right)^{1/2} e^{iqx} \left[ b(q) + b^\dagger(-q) \right],
\]

into the expectation value, and using \((C31)\) to evaluate the averages in the generalized Gibbs ensemble, we find that, in the thermodynamic limit,

\[
\langle \varphi(x)\varphi(0) \rangle_{\text{G}} = \int_0^\infty \frac{d(vq)}{4\omega_0(q)} \cos qx \cosh 2\beta(q),
\]

and therefore,

\[
\mathcal{C}^G(x) = \langle \varphi(x)\varphi(0) \rangle_{\text{G}} - \langle \varphi^2(0) \rangle_{\text{G}},
\]

\[
\mathcal{C}^G(x) = -\int_0^\infty \frac{d(vq)}{\omega_0(q)} \cosh 2\beta(q) (1 - \cos qx)
\]

\[
= \mathcal{C}(x,0) - \frac{m^2}{4} \int_0^{+\infty} \frac{d(vq)}{\omega(q)[\omega_0(q)]^2} (1 - \cos qx)
\]
where $C(x,0) \equiv C(x,t=0)$ is defined in Eq. (110). Upon comparing the last result with Eq. (109) in the limit where $t \to +\infty$, we see they are identical.

4. Quench from the gapless to a gapped phase

In this case the Hamiltonian that performs the time evolution is gapless, whereas the initial state is gapless. Thus, differently from the previous case, the Hamiltonian that performs the evolution is diagonal in the $a$-operators, and therefore, the conserved quantities are $I(q) = a^\dagger(q)a(q)$. The corresponding Lagrange (at zero temperature) are fixed from the condition:

$$\langle I(q) \rangle_{G} = \frac{1}{e^{\lambda(q)} - 1} = \langle \Phi(0)|a^\dagger(q)a(q)|\Phi(0)\rangle$$

$$= \sinh^2 \beta(q),$$

where $\beta(q)$ is given by Eq. (101).

In order to obtain the one and two-point correlation functions of $e^{2i\phi(x)} = e^{2i\varphi(x)}$, we first need to write the $\varphi(x)$ field in terms of the $a$-operators. Upon using the canonical transformation Eq. (3):

$$\varphi(x) = \frac{1}{2} \sum_{q \neq 0} \left( \frac{2\pi}{\omega(q)L} \right)^{1/2} e^{iqx} [a(q) + a^\dagger(-q)].$$

Hence, since $\langle \varphi(x) \rangle_{G} = \langle \varphi(x) \rangle_{G} = e^{-\frac{i\kappa}{2}}(\varphi(0))_{G}$, and $\langle \varphi^2(0) \rangle_{G}$ is logarithmically divergent in the thermodynamic limit (see expressions below), the find that $\langle e^{-i\kappa \varphi(x)} \rangle_{G} = 0$. This result is in agreement with the one found in Sect. IV B 2 for the order parameter.

a. Two-point correlation function

Next we consider the two-point correlation function of the same operator, namely $\langle e^{-2i\phi(x)} e^{2i\phi(0)} \rangle_{G} = e^{-\frac{i\kappa}{2} e^{\tau C(x)}}$, where $C_G(x) = \langle \varphi(x)\varphi(0) \rangle_{G} - \langle \varphi^2(0) \rangle_{G}$. We first obtain:

$$\langle \varphi(x)\varphi(0) \rangle_{G} = \int_0^\infty \frac{q}{4\omega(q)} \cos qx \cos 2\beta(q).$$

Hence,

$$C_G(x) = - \int_0^\infty \frac{dq}{2\omega(q)} \cosh \beta(q) (1 - \cos qx)$$

$$= C(x,0) + \frac{m^2}{4} \int_0^\infty \frac{dq}{\omega(0)\omega(q)^2} (1 - \cos qx)$$

where $C(x,0)$ is defined in Eq. (117). The latter result agrees with Eq. (106) in the $t \to +\infty$ limit.

1 E. Altman and A. Auerbach, Phys. Rev. Lett. 89, 250404 (2002).
2 K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A 69, 053616 (2004).
3 R. A. Barankov and L. S. Levitov, Phys. Rev. Lett. 96, 230403 (2006).
4 E. A. Yuzbashyan, B. L. Altshuler, V. B. Kuznetsov, and V. Z. Enolskii, Phys. Rev. B 72, 220503 (2005).
5 K. Kollath, U. Schollwöck, J. von Delft, and W. Zwerger, Phys. Rev. A 71, 053606 (2005).
6 E. Altman and A. Vishwanath, Phys. Rev. Lett. 95, 110404 (2005).
7 A. Ruschhaupt, A. Campo, and J. G. Muga, Euro. Phys. J. D 40, 399 (2006).
8 E. A. Yuzbashyan and M. Dzero, Phys. Rev. Lett. 96, 230404 (2006).
9 M. A. Cazalilla, Phys. Rev. Lett. 97, 154403 (2006).
10 M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. 98, 050405 (2007).
11 E. Perfetto, Phys. Rev. B 74, 205123 (2006).
12 P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).
13 M. Rigol, M. Olshanii, and A. Muramatsu, Phys. Rev. A 74, 053616 (2006).
14 S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu, Phys. Rev. Lett. 98, 210405 (2007).
15 P. Calabrese and J. Cardy, J. Stat. Mech.: Theor. Exp. p. P06008 (2007).
16 C. Kollath, A. Laeuchli, and E. Altman, Phys. Rev. Lett. 98, 180601 (2007).
17 M. Eckstein and M. Kollar, Phys. Rev. Lett. 100, 120404 (2008).
18 M. Kollar and M. Eckstein, Phys. Rev. A 78, 013626 (2008).
19 M. Moeckel and S. Kehrein, Phys. Rev. Lett. 100, 175702 (2008).
20 M. Cramer, A. Flesch, I. McCulloch, U. Schollwoeck, and J. Eisert, Phys. Rev. Lett. 101, 063001 (2008).
21 A. Flesch, M. Cramer, I. McCulloch, U. Schollwoeck, and J. Eisert, Phys. Rev. A 78, 033608 (2008).
22 M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Nature (London) 452, 854 (2007).
23 S. R. Manmana, S. Wessel, R. M. Noack, and A. Muramatsu (2008), arxiv:0812.0561.
24 C. D. Grandi, R. A. Barankov, and A. Polkovnikov, Phys. Rev. Lett. 101, 230402 (2008).
25 P. Reimann, Phys. Rev. Lett. 101, 190403 (2008).
26 A. Farihiault, P. Calabrese, and J.-S. Caux, J. Stat. Mech.: Theor. Exp. p. P03018 (2009).
27 D. Patanè, A. Silva, L. Amico, R. Fazio, and G. E. Santoro (2008), arxiv:0812.3685.
28 D. Patanè, A. Silva, F. Sols, and L. Amico (2008), arxiv:0812.4717.
29 D. Rossini, A. Silva, G. Mussardo, and G. Santoro (2008),
We shall encounter a similar situation in Sect. IV when analyzing quenches at $T=0$ from a non-critical (that is, gapped) to a critical (that is, gapless) state. In that case, the role of $\lambda$ will be played by the correlation length of the system that is determined by the (inverse of the) energy gap in the initial state.