ON SINGULARITY TYPES OF DEL PEZZO SURFACES WITH RATIONAL DOUBLE POINTS IN POSITIVE CHARACTERISTIC

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Abstract. In this paper, we prove that a pair of the minimal resolution of a del Pezzo surface with rational double points whose general anti-canonical member is smooth and its exceptional divisor lifts to the Witt ring. We also classify a del Pezzo surface with rational double points whose anti-canonical members are all singular. As a corollary, we determine all singularity types of del Pezzo surfaces with rational double points which only appear in positive characteristic.

1. Introduction

Let $X$ be a del Pezzo surface with rational double points, that is, normal projective surface over an algebraically closed field which has only rational double points and the ample anti-canonical divisor. In this paper, we investigate a configuration of rational double points in $X$ in the sense of the corresponding Dynkin diagrams. We call this configuration as the singularity type of $X$. For example, we say that the singularity type of $X$ is $3A_1 + D_4$ if $X$ has three $A_1$-singularities and one $D_4$-singularity. In characteristic zero, the singularity type of $X$ is determined by Furushima [6] when the Picard rank $\rho(X)$ is equal to one. Later, Ye [16] classifies singularity types up to $A_n$-singularities when $\rho(X) \geq 2$. In characteristic $p > 3$, Lacini [12, Theorem B.7] very recently proved that the singularity type of $X$ is realizable in characteristic zero when $\rho(X) = 1$. On the other hand, when $p = 2$, Keel–McKernan [11, end of Section 9] constructed a del Pezzo surface with $7A_1$-singularities of the Picard rank one, whose singularity type does not appear in characteristic zero (see also [13, Example 8.2], [2, Section 2.2]).

In this paper, we determine all singularity types of del Pezzo surfaces with rational double points which only appear in positive characteristic. For this, we first focus on the smoothness of a general anti-canonical member. We prove that if a general anti-canonical member of $X$ is smooth, then the singularity type of $X$ is realizable in characteristic zero.

Theorem 1.1 (Theorem 3.1 and Theorem 3.4). Let $X$ be a del Pezzo surface with rational double points over an algebraically closed field $k$ of characteristic $p > 0$. Let $\pi : Y \to X$ be the minimal resolution with a reduced exceptional divisor $E$. Suppose that a general anti-canonical member of $X$ is smooth. Then $(Y, E)$ lifts to the Witt ring $W(k)$. Moreover, there exists a del Pezzo surface with rational double points over $2010$ Mathematics Subject Classification. 14J26, 14J45, 14G17.

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the complex number $\mathbb{C}$ which has the same singularity type, the same Picard rank, and the same degree as $X$.

We remark that the assumption of Theorem 1.1 is satisfied when $p > 3$. In the proof of Theorem 1.1, we prove the vanishing of the cohomology $H^2(Y, T_Y(- \log E))$ by using a smooth anti-canonical member. Then by using a $W(k)$-lifting of the pair $(Y, E)$, we construct a del Pezzo surface with rational double points over $\mathbb{C}$ which has the same singularity type, the same degree and the same Picard rank as $X$. By Theorem 1.1, we can conclude that to determine pathological singularity types in positive characteristic, it suffices to consider the case where all anti-canonical members are singular. In section 4, we then determine del Pezzo surfaces with rational double points whose anti-canonical members are all singular by using the classification of rational quasi-elliptic fibrations by Ito ([8], [9]).

From these arguments, we obtain the following theorem.

**Theorem 1.2.** Let $X$ be a del Pezzo surface with rational double points over an algebraically closed field $k$ of characteristic $p > 0$. Let $d := (K_X^2)$ be the degree and $\rho(X)$ be the Picard rank of $X$. Then there exists a del Pezzo surface with rational double points over the complex number $\mathbb{C}$ which has the same singularity type, the same Picard rank and the same degree as $X$ except when the following cases.

| $p$ | $\rho(X)$ | $d$ | The singularity type of $X$ |
|-----|-----------|-----|-----------------------------|
| 2   | 1         | 2   | $7A_1$                      |
|     |           | 1   | $8A_1$                      |
|     |           | 1   | $4A_1 + D_4$               |

Moreover, for each condition in Table 1, there exists a del Pezzo surface with rational double points over $k$ which satisfies the condition.

**Notation.** In this paper, a variety means an integral separated scheme of finite type over an algebraically closed field. Throughout this paper, we also use the following notation:

- $k$: an algebraically closed field of characteristic $p > 0$.
- $E_f$: the reduced exceptional divisor of a birational morphism $f$.
- $\rho(X)$: the Picard rank of a projective variety $X$.

2. Preliminaries

2.1. Del Pezzo surfaces with rational double points. In this subsection, we gather basic results of del Pezzo surfaces with rational double points.

**Definition 2.1.** Let $X$ be a normal projective surface over an algebraically closed field. We say that $X$ is a del Pezzo surface with rational double points if $-K_X$ is ample and $X$ has only rational double points.

**Lemma 2.2.** Let $X$ be a del Pezzo surface with rational double points over $k$ and $d := (K_X^2)$. Then the followings hold.
Proof. We refer to [1, Propositions 2.10, 2.12 and 2.14] and [10, Proposition 4.6] for the proof.

2.2. Liftability to the Witt ring. We denote by $W(k)$ (resp. $W_n(k)$) the Witt ring (resp. the Witt ring of the length $n$) of $k$.

**Definition 2.3.** Let $T$ be a spectrum of $W(k)$ or $W_n(k)$ for some $n > 0$. Let $Y$ be a smooth separated scheme over $T$ and $E := \sum_{i=1}^r E_i$ be a reduced divisor on $Y$, where each $E_i$ is an irreducible component of $E$. We say that $E$ is simple normal crossing over $T$ if, for any subset $J \subseteq \{1, \ldots, r\}$ such that $\bigcap_{i \in J} E_i \neq \emptyset$, the scheme-theoretic intersection $\bigcap_{i \in J} E_i$ are smooth over $T$ of relative dimension $\dim Y - \dim T - |J|$.

**Definition 2.4.** Let $Y$ be a smooth separated scheme over $\text{Spec} W_n(k)$ and $E$ be a simple normal crossing divisor over $\text{Spec} W_n(k)$ on $Y$. We write $E = \sum_{i=1}^r E_i$, where each $E_i$ is an irreducible component of $E$. We say that the pair $(Y, E)$ is liftable to the Witt ring $W(k)$ (resp. $W_m(k)$, where $m > n$) if there exist

- a smooth and projective morphism $\mathcal{Y} \to \text{Spec} W(k)$ (resp. $\text{Spec} W_m(k)$),
- effective divisors $\mathcal{E}_1, \ldots, \mathcal{E}_r$ on $\mathcal{Y}$ such that $\mathcal{E} := \sum_{i=1}^r \mathcal{E}_i$ is simple normal crossing over $\text{Spec} W(k)$ (resp. $\text{Spec} W_m(k)$),

such that $\mathcal{Y} \otimes_{W(k)} W_n(k) = Y$ and $\mathcal{E}_i \otimes_{W(k)} W_n(k) = E_i$ (resp. $\mathcal{Y} \otimes_{W_m(k)} W_n(k) = Y$ and $\mathcal{E}_i \otimes_{W_m(k)} W_n(k) = E_i$) for $i = 1, 2, \ldots, r$.

**Lemma 2.5.** Let $f : \mathcal{X} \to \text{Spec} W(k)$ be a morphism from a scheme to the spectrum of the Witt ring. If $f_n : \mathcal{X} \otimes_{W(k)} W_n(k) \to W_n(k)$ is flat for all $n > 0$, then $f$ is flat. In addition, if $f$ is proper and $f_1 : \mathcal{X} \otimes_{W(k)} k \to \text{Spec} k$ is smooth of relative dimension $d$, then $f$ is smooth of relative dimension $d$.

**Proof.** The flatness of $f$ follows from [15, Lemma 10.98.11]. Let us assume that $f$ is proper and $f_1 : \mathcal{X} \otimes_{W(k)} k \to \text{Spec} k$ is smooth. Then we can take an open subscheme $U \subset X$ such that $\mathcal{X} \otimes_{W(k)} k \subset U$ and $f|_U$ is smooth. Since $f$ is proper, $f(X - U)$ is a closed subset of $\text{Spec} W(k)$. Then we have $f(X - U) = \emptyset$ because the unique closed point in $\text{Spec} W(k)$ is not contained in $f(X - U)$. Thus we conclude that $U = X$.

The following lemma is a log version of [4, Theorem 8.5.9]. It seems to be well-known for experts, but we include the sketch of proof for the readers’ convenience.

**Lemma 2.6** (cf. [5, Theorem (A1)]). Let $Y$ be a smooth projective variety over $k$ and $E$ a simple normal crossing divisor on $Y$. If $H^2(Y, T_Y(- \log E)) = H^2(Y, O_Y) = 0$, then $(Y, E)$ is liftable to $W(k)$. 
Proof. Suppose that \((Y^n, E^n)\) is a lifting of \((Y, E)\) over \(W_n(k)\). We first see that \((Y^n, E^n)\) is liftable to \(W_{n+1}(k)\). Since \((Y^n, E^n)\) is simple normal crossing over \(W_n(k)\), we can take an affine open covering \(\{U_i\}\) of \(Y\) such that \((U_i, E_i | U_i)\) is liftable to \(W_{n+1}(k)\). Then for each \(i\) and for any open subset \(U\) of \(U_i\), the set of equivalence classes of such liftings is a torsor under the action of \(T_U(-\log E) := \text{Hom}(\Omega_U(\log E), \mathcal{O}_U)\). We refer to the arguments of [3, Section 8] for the details. Then by a similar argument as in [4, Theorem 8.5.9 (b)], the obstruction for the lifting of \((Y^n, E^n)\) over \(W_{n+1}(k)\) is contained in \(H^2(Y, T_Y(-\log E))\). Thus the vanishing of \(H^2(Y, T_Y(-\log E))\) gives a lift of \(Y\) and \(E_i\) as formal schemes. Since \(H^2(Y, \mathcal{O}_Y) = 0\), they are algebraizable and we get a projective scheme \(\mathcal{Y}\) over \(W(k)\) and a closed subscheme \(\mathcal{E} := \sum_{i=1}^r \mathcal{E}_i\) on \(\mathcal{Y}\) such that \(\mathcal{Y} \otimes_{W(k)} W_n(k) = Y^n\) and \(\mathcal{E}_i \otimes_{W(k)} W_n(k) = E_i^n\) for each \(n\) and \(i\) by [4, Corollary 8.4.5]. We take a subset \(J \subseteq \{1, \ldots, r\}\). Since \(\bigcap_{i \in J} \mathcal{E}_i \otimes_{W(k)} W_n(k) = \bigcap_{i \in J} E_i^n\) is smooth over \(W_n(k)\) for all \(n > 0\) and \(\mathcal{X}\) is projective over \(W(k)\), Lemma 2.5 now shows that \(\bigcap_{i \in J} \mathcal{E}_i\) is smooth of relative dimension \(\dim \mathcal{Y} - \dim W(k) - |J|\). Therefore, \((\mathcal{Y}, \mathcal{E} = \sum_{i=1}^r \mathcal{E}_i)\) is a lifting of \((Y, E)\) over \(W(k)\). \(\square\)

Lemma 2.7. Let \(X\) be a projective variety over \(k\) and let \(f : X \to \text{Spec} W(k)\) be a lifting of \(X\) to \(W(k)\). Then \(f_* \mathcal{O}_X = \mathcal{O}_{\text{Spec} W(k)}\). In particular, the generic fiber of \(f\) is geometrically connected.

Proof. We take the Stein factorization of \(f\) and let \(g : \text{Spec} R \to \text{Spec} W(k)\) be a finite part of this Stein factorization. Since \(R\) is a finite extension of a complete discrete valuation ring, \(R\) is also a discrete valuation ring. The reducedness of a closed fiber \(X\) gives \(mR = mR\), where \(m\) and \(mR\) denote the maximal ideal of \(W(k)\) and \(R\), respectively. Since \(k\) is an algebraically closed field, the Nakayama’s lemma shows that \(g^\# : W(k) \to R\) is an isomorphism. Thus \(f_* \mathcal{O}_X = \mathcal{O}_{\text{Spec} W(k)}\). By localizing at the generic point of \(\text{Spec} W(k)\), we have \(H^0(X_K, \mathcal{O}_{X_K}) = K\) and by base change, we get \(H^0(X_K, \mathcal{O}_{X_K}) = \overline{K}\), where \(K\) and \(\overline{K}\) denote the factional filed of \(W(k)\) and its algebraic closure, respectively. Thus \(X_{\overline{K}}\) is connected. \(\square\)

3. DEL PEZZO SURFACES WITH RATIONAL DOUBLE POINTS WHOSE GENERAL ANTI-CANONICAL MEMBER IS SMOOTH

In this section, \(X\) denotes a del Pezzo surface with rational double points over \(k\) and \(\pi : Y \to X\) denotes the minimal resolution. We show that if a general member of \(|-K_X|\) is smooth, then its singularity type is realizable in characteristic zero. For this, we first show that the smoothness of a general anti-canonical member implies a liftability to \(W(k)\) of \((Y, E\pi)\).

Theorem 3.1. Suppose that a general member of \(|-K_X|\) is smooth. Then \((Y, E\pi)\) is liftable to \(W(k)\).

Proof. Since \(Y\) is rational, we have \(H^2(Y, \mathcal{O}_Y) = 0\). Then it suffices to show the vanishing of \(H^2(Y, T_Y(-\log E\pi)) \simeq H^0(Y, \mathcal{O}_Y(\log E\pi) \otimes \omega_Y)\) by Lemma 2.6. This follows from essentially the same arguments as in [10, Theorem 4.8], but we include the proof for completeness. For the sake of a contradiction, we assume that there exists an injective map \(\omega_Y^{-1} \hookrightarrow \omega_Y(\log E\pi)\). By pushing forward by \(\pi\), we get \(s : \omega_X^{-1} \hookrightarrow \omega_X(\log E\pi)\).
\[ \pi_*\Omega_Y(\log E_\pi) \subset \Omega_X^{[1]}, \text{ where } \Omega_X^{[1]} \text{ is the reflexive hull of } \Omega_X. \] Let \( C \in | -K_X | \) be a general member. By assumption, \( C \) is a smooth elliptic curve. By restricting \( s \) on \( C \), we have an injective map, \( s_C: \omega_X^{-1}|C \hookrightarrow \Omega_X^1|C \). The generality of \( C \) shows that \( s_C \) is injective, \( \omega_X^{-1}|C \) is ample Cartier and \( \Omega_X^{[1]}|C = \Omega_X^1|C \). Let \( t: \omega_X^{-1}|C \to \omega_C = \mathcal{O}_C \) be the composition of \( s|C: \omega_X^{-1}|C \hookrightarrow \Omega_X^1|C \) and the canonical map \( \Omega_X^1|C \to \omega_C \).

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_C(-C) \\
\downarrow{s|C} & & \downarrow{\omega_X^{-1}|C} \\
& \longrightarrow & \Omega_X^1|C \\
& & \longrightarrow \omega_C \\
& & \longrightarrow 0.
\end{array}
\]

Then \( t \) is the zero map since \( \omega_X^{-1}|C \) is ample. Hence an injective map \( \omega_X^{-1}|C \hookrightarrow \mathcal{O}_C(-C) \) is induced by the conormal exact sequence. This is a contradiction because \( \mathcal{O}_C(-C) = \mathcal{O}_C(K_X) \) is anti-ample.

**Remark 3.2.** In characteristic zero, the vanishing of \( H^2(Y, T_Y(-\log E_\pi)) \) follows from Bogomolov–Sommese vanishing. Together with Theorem 3.4, we can see that del Pezzo surfaces with rational double points in Table 1 of Theorem 1.2 violate Bogomolov–Sommese vanishing. More precisely, the reflexive cotangent bundle \( \Omega_X^{[1]} \) contains an ample line bundle \( \omega_X^{-1} \) and the logarithmic cotangent bundle \( \Omega_Y(\log E) \) contains a nef and big line bundle \( \omega_Y^{-1} \). We refer [13, Section 8] and [7, Section 11] for counterexamples of Bogomolov–Sommese vanishing in positive characteristic.

**Lemma 3.3.** If \((Y, E_\pi)\) is liftable to \( W(k) \), then there exists a del Pezzo surface with rational double points over the complex number \( \mathbb{C} \) which has the same singularity type, the same Picard rank and the same degree as \( X \).

**Proof.** We denote \( E_\pi := \sum_{i=1}^{r} E_i \), where \( E_i \) is an irreducible component for each \( i \). Let \((\mathcal{Y}, \mathcal{E} := \sum_{i=1}^{r} \mathcal{E}_i)\) be a \( W(k) \)-lifting of \((Y, E_\pi)\). For a field \( F \) which contains a fractional field \( K \) of \( W(k) \), we denote \( Y_F := \mathcal{Y} \otimes_{W(k)} F \) and \( E_{i,F} := \mathcal{E}_i \otimes_{W(k)} F \) for each \( i \). Then \( Y_K \) and \( E_{i,K} \) are connected by Lemma 2.7, where \( K \) is the algebraic closure of \( K \). In particular, each \( E_{i,K} \) is a \((-2)\)-curve. Since \( E_{i,K} := \sum_{i=1}^{r} E_{i,K} \) has the same intersection matrix as \( E_\pi \), we have a contraction \( \pi_K: Y_K \to X_K \) of \( E_{i,K} \) and \( X_K \) has the same singularity types as \( X \). We first prove that \(-K_{X_K} \) is ample. For the sake of contradiction, we assume that there exists an integral curve \( C_0 \subset Y_K \) such that \( C_0 \) is not contained in \( E_{i,K} \) and \((-K_{Y_K} \cdot C_0) \leq 0 \). We take a finite extension \( L \) of \( K \) such that \( C_0 \) is defined over \( L \) and we define an effective divisor \( C \) over \( K \) as \( C := \sum_{\sigma \in \Gal(L/K)} \sigma(C_0) \).

Note that \( C_L := C \otimes_K L \) and \( E_L \) have no common components. Indeed, if there exists an integral component \( C_i \) of \( C_L \) such that \( C_i = E_{j,L} \) for some \( j \), then there exists \( \sigma \in \Gal(L/K) \) such that \( C_0 = \sigma(C_i) = \sigma(E_{j,L}) = E_{j,L} \) but this is a contradiction. We denote by \( \overline{C} \) the closure of \( C \) in \( \mathcal{Y} \) and define an effective divisor \( C_k := \overline{C} \otimes_{W(k)} k \). If \( \Supp C_k \subset E_\pi \), we can denote \( C_k = \sum_{a_i > 0} a_i E_i \) for some \( a_i > 0 \). Since \( C_L \) and \( E_L \) have no common components, we have \( (C_k^2) = (C_L \cdot \sum_{a_i > 0} a_i E_i) \geq 0 \), but this contradicts the negative definiteness of \( E_\pi \). Therefore, there exists an integral curve \( C_k' \subset C_k \) such that \( C_k' \) is not contained in \( E_\pi \). Since \(-K_Y \) is nef and \( 0 \leq (-K_Y \cdot C_k') \leq (-K_Y \cdot C_k) = (-K_{Y_K} \cdot C \otimes_K K) = |Gal(L/K)|(-K_{Y_K} \cdot C_0) \leq 0 \), we
have \((-K_Y \cdot C'_k) = 0\). Then \((-K_X \cdot \pi_*(C'_k)) = 0\) and this contradicts the ampleness of \(-K_X\). Finally, we check that \((K^2_X) = (K^2_{X_Y})\) and \(\rho(X) = \rho(X_Y)\). By the crepantness of \(\pi\) and \(\pi_X\), we get \((K^2_X) = (K^2_Y) = (K^2_{X_Y}) = (K^2_{Y_K})\). Also, since \(Y\) and \(Y_K\) is a smooth rational surface, we have \(\rho(Y_K) = 10 - (K^2_{Y_K}) = 10 - (K^2_Y) = \rho(Y)\). Then we obtain \(\rho(X) = \rho(X_Y)\) because \(\pi_K\) contracts the same number of \((-2)\)-curves as \(\pi\). □

Combining Theorem 3.1 and Lemma 3.3, we get the following theorem.

**Theorem 3.4.** Let \(X\) be a del Pezzo surface with rational double points. Suppose that a general anti-canonical member of \(X\) is smooth. Then there is a del Pezzo surface with rational double points over \(\mathbb{C}\) which has the same singularity type, the same Picard rank and the same degree as \(X\).

**Remark 3.5.** By considering the mod \(p\) reduction, all the singularity types in characteristic zero are realized in sufficiently large characteristic.

### 4. Del Pezzo surfaces with rational double points whose anti-canonical members are all singular

In this section, we determine del Pezzo surfaces with rational double points over \(k\) whose anti-canonical members are all singular by using Ito’s classifications of rational quasi-elliptic fibrations ([8], [9]). Making use of this, we prove Theorem 1.2. By Lemma 2.2, such a del Pezzo surface is of degree at most two, and \(p = 2\) or \(3\). First, we treat the case where the degree is one.

**Proposition 4.1.** There is one to one correspondence between the isomorphism classes of del Pezzo surfaces with rational double points of degree one whose anti-canonical members are all singular, and the isomorphism classes of rational quasi-elliptic surfaces.

**Proof.** Let \(X\) be a del Pezzo surface with rational double points of degree one whose anti-canonical members are all singular. Take \(\pi: Y \to X\) as the minimal resolution. Then the base locus of \(|-K_Y|\) consists of one point, say \(y\). The blow-up \(g: Z \to Y\) at \(y\) gives an elimination \(f: Z \to \mathbb{P}^1_k\) of the anti-canonical map. Since a general member of \(|-K_Y|\) is disjoint from \(E_\pi\), it is isomorphic to its image by \(\pi\). Thus all members of \(|-K_Y|\) are also singular. Since any two members of \(|-K_Y|\) intersect transversely with each other at \(y\), each \(f\)-fiber is isomorphic to its image on \(Y\). In particular, \(f: Y \to \mathbb{P}^1_k\) is a quasi-elliptic fibration and \(E_y\) is a \(f\)-section.

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow f & & \downarrow \pi \\
\mathbb{P}^1_k & \xleftarrow{\varphi_{|-K_X|}} & X
\end{array}
\]

Thus we make a correspondence from the isomorphism classes of such del Pezzo surfaces to the isomorphism classes of pairs of quasi-elliptic surfaces and sections. Since the Mordell–Weil group of a quasi-elliptic surface acts on the set of its sections transitively, an isomorphism class of the pair of a quasi-elliptic surface and a section is the same as that of a quasi-elliptic surface. Hence we have the assertion. □
Corollary 4.2. Let $X$ be a del Pezzo surface with rational double points of degree one whose anti-canonical members are all singular. Then the singularity type of $X$ is one of $E_8, A_2 + D_6$ and $4A_2$ when $p = 3$, and one of $E_8, D_8, A_1 + E_7, 2D_4, 2A_1 + D_6, 4A_1 + D_4$ and $8A_1$ when $p = 2$. Conversely, all the singularity types above are realizable over $k$.

Proof. By Proposition 4.1, the minimal resolution of $X$ is obtained from a rational quasi-elliptic surface by contracting a section. Quasi-elliptic surfaces are classified by [8, Theorem 3.3] and [9, Theorem 5.2] into 10 types, and each type is realizable over $k$. When $p = 3$, we can see that rational quasi-elliptic surfaces of type (1), (2) and (3) as in [8, Theorem 3.3] correspond to del Pezzo surfaces of singularity type $E_8, A_2 + D_6$ and $4A_2$ respectively. When $p = 2$, we can also see that rational quasi-elliptic surfaces of type (a), (b), (c), (d), (e), (f) and (g) as in [9, Theorem 5.2] correspond to del Pezzo surfaces of singularity type $E_8, D_8, A_1 + E_7, 2D_4, 2A_1 + D_6, 4A_1 + D_4$ and $8A_1$ respectively. \[\square\]

Next, we treat the case where the degree is two. The following proposition claims that the anti-canonical double covering must be purely inseparable.

Proposition 4.3. Let $X$ be a del Pezzo surface with rational double points with $(K_X^2) = 2$. Suppose that the anti-canonical double covering $\varphi_{-K_X^1}: X \to \mathbb{P}_k^2$ is separable. Then a general anti-canonical member is smooth.

Proof. Take the minimal resolution $\pi: Y \to X$. Let $t \in \mathbb{P}_k^2$ be a general point and $V \subset |-K_Y|$ the pullback of a pencil of $|\mathcal{O}_{\mathbb{P}_k^2}(1)|$ which consists of all the members passing through $t$. The base locus of $V$ consists of two points, say $y_1$ and $y_2$, such that there is no $(-1)$-curves or $(-2)$-curves passing through $y_1$ or $y_2$ because $t$ is general and there exist only finitely many $(-1)$-curves and $(-2)$-curves on $Y$. Let $g: Z \to Y$ be the blow-up at $y_1$ and $y_2$, and $E_i$ the $g$-exceptional divisor over $y_i$ for $i \in \{1, 2\}$. Then $g$ gives an elimination $f: Z \to \mathbb{P}_k^1$ of the pencil $\varphi_V: Y \to \mathbb{P}_k^1$. Since every two members of $V$ intersect transversely at $y_1$ and $y_2$, a general $f$-fiber is isomorphic to its image on $Y$.

Now let us show that a general member of $|-K_X|$ is smooth. Conversely, suppose that members of $|-K_X|$ are all singular. Then $f: Y \to \mathbb{P}_k^1$ is a quasi-elliptic fibration, and $E_1$ and $E_2$ are $f$-sections by the same arguments as in Proposition 4.1. Since there are no $(-1)$-curves on $Y$ which through $y_1$ or $y_2$, each $(-2)$-curves in $Z$ either intersects with both $E_1$ and $E_2$ or is disjoint from both $E_1$ and $E_2$.

On the other hand, the rational quasi-elliptic surface $Z$ is one of the type (1)-(3) as in [8, Theorem 3.3] and the type (a)-(g) as in [9, Theorem 5.2]. By the configurations of reducible fibers and sections, we can check that such sections $E_1$ and $E_2$ can exist.
only when $Z$ is of type (g). Hence $Z$ has eight reducible fiber. This implies, however, $Y$ contains eight disjoint $(-2)$-curves, a contradiction with $\rho(Y) = 8$. $\square$

**Proposition 4.4.** Let $X$ be a del Pezzo surface with rational double points with $(K_Z^2) = 2$. Suppose that members of $| - K_x |$ are all singular. Then $p = 2$ and its singularity type is one of $E_7$, $A_1 + D_6$, $3A_1 + D_4$ and $7A_1$.

**Proof.** By Proposition 4.3, the anti-canonical double covering $\varphi_{| - K_x |}: X \to \mathbb{P}^2_k$ is purely inseparable. In particular, we have $p = 2$. Take the minimal resolution $\pi: Y \to X$. Let $t \in \mathbb{P}^2_k$ be a general point and $V \subset | - K_Y |$ the pullback of a pencil of $| O_{\mathbb{P}^2_k}(1) |$ which consists of all the members passing through $t$. Then the base locus of $V$ consists of one point, say $y$. By the generality of $t$, there is no $(-1)$-curves or $(-2)$-curves passing through $y$. For general two members $C_1$ and $C_2$ of $V$, they intersect with each other at $y$ with multiplicity two since $| - K_x |$ is a homeomorphism. Moreover, one of them is smooth at $y$ since otherwise $2 = (K_Z^2) = (C_1 \cdot C_2) \geq 4$. Thus general members of $V$ are smooth at $y$, and have the same tangent direction at $y$. Hence there is a point $y'$ infinitely near to $y$ such that the blow-up $g: Z \to Y$ at $y$ and $y'$ gives an elimination $f: Z \to \mathbb{P}^1_k$ of the pencil $\varphi_Y: Y \dashrightarrow \mathbb{P}^1_k$. Since a general member of $V$ is smooth at $y$, a general $f$-fiber is isomorphic to its image on $Y$. In particular, $f: Y \to \mathbb{P}^1_k$ is a quasi-elliptic fibration. By construction, $E_q$ consists of a $(-1)$-curve $E_1$ and a $(-2)$-curve $E_2$. In particular, $E_1$ is a $f$-section and $E_2$ is contained in a reducible $f$-fiber.

Suppose that the reduced structure of the $f$-fiber containing $E_2$ is simple normal crossing. Then there is another $(-2)$-curve $C$ intersecting with $E_2$. Since $C$ and $E_2$ are contained in the same $f$-fiber, $E_1$ is disjoint from $C$. This implies, however, $g_C$ is a $(-1)$-curve passing through $y$, a contradiction with the choice of $y$. Hence $E_2$ is contained in a reducible $f$-fiber whose reduced structure is not simple normal crossing. Combing this fact and [9, Theorem 5.2], we conclude that $E_2$ is contained in a reducible $f$-fiber of type III, where we use Kodaira’s notation. In particular, $Y$ is one of the type (c), (e), (f) and (g) as in [ibid.]. Therefore the singularity type of $X$ is one of $E_7$, $A_1 + D_6$, $3A_1 + D_4$ and $7A_1$. $\square$

Next, let us show that del Pezzo surfaces of degree two whose anti-canonical members are all singular are uniquely determined by their singularity types.

**Proposition 4.5.** There is a unique del Pezzo surface with $E_7$-singularity whose anti-canonical members are all singular over $k$.

**Proof.** Let $X$ be a del Pezzo surface with $E_7$-singularity whose anti-canonical members are all singular. Let $Y \to X$ be the minimal resolution. Combining Corollary 4.2 and Proposition 4.4, we conclude that $Y$ is given from the unique rational quasi-elliptic surface $Z$ of type (c) as in [9, Theorem 5.2] by blowing-down the connected union of a section $S$ and a $(-2)$-curve in the reducible fiber of type III. Since the Mordell-Weil group of the quasi-elliptic surface $Z$ acts on the set of all sections transitively, the isomorphism class of $X$ is independent of the choice of $S$. We note that a $(-2)$-curve intersecting with $S$ is uniquely determined by [ibid.]. Hence $X$ is also uniquely determined if exists.
Next, let us show the existence of such a del Pezzo surface. Fix coordinates \( \{x, y, z\} \) of \( \mathbb{P}^2_k \) and take a net \( N = \langle x^3 + y^2z, xz^2, z^3 \rangle \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \). Then we can check that members of \( N \) are all singular by the Jacobian criterion. Let \( h_1: Y_1 \to \mathbb{P}^2_k \) be the blow-up at \( [0 : 1 : 0] \) and \( C_1 \subset Y_1 \) the strict transform of \( \{x^3 + y^2z = 0\} \). For \( 2 \leq i \leq 7 \), we define \( h_i: Y_i \to Y_{i-1} \) as the blow-up at \( E_{h_{i-1}} \cap C_{i-1} \) and \( C_i \subset Y_i \) as the strict transform of \( C_{i-1} \) by induction. For \( 1 \leq i \leq 7 \), let \( E_i \) be the total transform of \( E_{h_i} \) in \( Y := Y_7 \). Let \( h = h_1 \circ \cdots \circ h_7 \) be the composition. Then \( | - K_Y | \) is the same as \( h^*N - \sum_{i=1}^7 E_i \), whose members are all singular. On the other hand, \( Y \) contains seven \((-2\)-curves, which are the strict transforms of \( \{z = 0\} \) and \( E_{h_i} \) for \( 1 \leq i \leq 6 \) in \( Y \). Hence the contraction \( Y \to X \) of such \((-2\)-curves gives a desired surface \( X \).

**Proposition 4.6.** There is a unique del Pezzo surface with \( A_1 + D_6\)-singularity whose anti-canonical members are all singular over \( k \).

**Proof.** Let \( X \) be a del Pezzo surface with \( A_1 + D_6\)-singularity whose anti-canonical members are all singular. Let \( Y \to X \) be the minimal resolution. Combining Corollary 4.2 and Proposition 4.4, we conclude that \( Y \) is given from the unique rational quasi-elliptic surface \( Z \) of type (e) as in [9, Theorem 5.2] by blowing-down connected union of a section \( S \) and a \((-2\)-curve \( T \) in a reducible fiber of type III.

Suppose that \( S = Q \) and \( T \) is the irreducible component of the fiber over \( t = 0 \) intersecting with \( Q \) as in Figure 8 of [ibid.]. Then, by blowing-down the nine \( \mathbb{P}^1 \)'s which are drawn as bold lines, we get birational morphisms \( h: Z \to \mathbb{P}^2_k \) and \( h': Y \to \mathbb{P}^2_k \). The image of the fiber over \( t = 0 \) (resp. \( \infty, 1 \)) is a cuspidal cubic \( C \) (resp. the sum of line \( L \) and a conic \( R \), the sum of two lines \( L_1 \) and \( L_2 \)). Moreover, \( h(E_h) \) consists of three points \( p_1, p_2, p_3 \) which satisfy the following:

- \( C \) is singular at \( p_1 \), and \( L_1 \) and \( R \) pass through \( p_1 \). Moreover, \( h^{-1}(p_1) = S \cup T \).
- \( C \) intersects with \( L \) at \( p_2 \) with multiplicity three, with \( L_1 \) at \( p_2 \) transversally, and with \( L_2 \) at \( p_2 \) transversely.
- \( C \) intersects with \( R \) at \( p_3 \) with multiplicity four and with \( L_2 \) with multiplicity two.

By [14, Theorem 3.1], we can choose coordinates \( \{x, y, z\} \) of \( \mathbb{P}^2_k \) such that \( C \) is defined by \( x^3 + y_2z = 0 \). Then \( p_1 = [0 : 0 : 1] \) and \( p_2 = [0 : 1 : 0] \). Since the automorphism \( \varphi: [x : y : z] \mapsto [ax : y : a^3z] \) of \( \mathbb{P}^2_k \) with \( a \in k^* \) makes \( C \) stable, we also may assume that \( p_3 = [1 : 1 : 1] \). Then an easy calculation shows that \( L, Q, L_1 \) and \( L_2 \) are defined by \( z = 0, xz + y^2 = 0, y = 0 \) and \( x + y = 0 \) respectively. Hence \( Y \) is obtained by blowing-up three points on \( C \) infinitely near to \( p_2 \) and four points on \( C \) infinitely near to \( p_3 \). Moreover, the net \( h'|{-K_Y}| \) is generated by \( C, L_1 + 2L_2 \) and \( 3L_2 \), whose members are all singular by the Jacobian criterion. Hence so are members of \( |{-K_X}| \).

When we choose another connected union of a section and an irreducible component of a reducible fiber of type III, we get the same configuration of curves \( C, L, R, L_1 \) and \( L_2 \) after a suitable blow-down by symmetry of the configuration of sections and reducible fibers on \( Z \). Therefore such \( X \) is uniquely determined.

**Proposition 4.7.** There is a unique del Pezzo surface with \( 3A_1 + D_4\)-singularity whose anti-canonical members are all singular over \( k \).
Proof. Let $X$ be a del Pezzo surface with $3A_1 + D_4$-singularity whose anti-canonical members are all singular. Let $Y \rightarrow X$ be the minimal resolution. Combining Corollary 4.2 and Proposition 4.4, we conclude that $Y$ is given from a rational quasi-elliptic surface $Z$ of type (f) as in [9, Theorem 5.2] by blowing-down connected union of a section $S$ and a $(-2)$-curve $T$ in a reducible fiber of type III.

From now on, we follow the notation used in Figure 10 of [ibid.]. Suppose that $S = O$ and $T = \Theta_{0,0}$. Then, by blowing-down $O, \Theta_{0,0}, P_1, P_2, Q_1, Q_2, \Theta_{1,1}, \Theta_{1,2}$ and $\Theta_{1,3}$ in this order, we get birational morphisms $h: Z \rightarrow \mathbb{P}^2_k$ and $h': Y \rightarrow \mathbb{P}^2_k$. Then $L = h_\ast \Theta_{1,4}$, $L_1 = h_\ast \Theta_{\ast,1}$, $L_2 = h_\ast \Theta_{\alpha,1}$ and $L_3 = h_\ast \Theta_{\alpha,2,1}$ are lines in $\mathbb{P}^2_k$. Moreover, $h(E_h)$ consists of five points $p_0, \ldots, p_4$ which satisfy the following:

- $h^{-1}(p_0) = S \cup T$.
- For $1 \leq i \leq 3$, the line $L$ intersects with $L_i$ at $p_i$.
- $L_1, L_2$ and $L_3$ pass through $p_4$.

We can choose coordinates $\{x, y, z\}$ of $\mathbb{P}^2_k$ such that $p_1 = [0 : 0 : 1]$, $p_2 = [0 : 1 : 1]$, $p_3 = [0 : 1 : 0]$ and $p_4 = [1 : 0 : 0]$. Then $L, L_1, L_2$ and $L_3$ are defined by $x = 0$, $x = y = 0$, $y + z = 0$ and $z = 0$ respectively. Hence $Y$ is obtained by blowing-up at two points on $L_i$ infinitely near to $p_i$ for $1 \leq i \leq 3$ and at $p_4$. Moreover, the net $h'_\ast | - K_Y |$ is generated by $2L + L_1, 2L + L_3$ and $L_1 + L_2 + L_3$, whose members are all singular by the Jacobian criterion. Hence so are members of $| - K_X |$.

When we choose another connected union of a section of an irreducible component of a reducible fiber of type III, we get the same configuration of curves $L, L_1, L_2$ and $L_3$ after a suitable blow-down by symmetry of the configuration of sections and reducible fibers on $Z$. Therefore such $X$ is uniquely determined. □

Proposition 4.8. There is a unique del Pezzo surface with $7A_1$-singularity over $k$.

Proof. Let $X$ be a del Pezzo surface with $7A_1$-singularity and $Y \rightarrow X$ the minimal resolution. Since [16] shows that there is no del Pezzo surface over $\mathbb{C}$ with such singularity, members of $| - K_X |$ are all singular by Theorem 3.4. Combining Corollary 4.2 and Proposition 4.4, we conclude that $Y$ is given from a rational quasi-elliptic surface $Z$ of type (g) as in [9, Theorem 5.2] by blowing-down the connected union of a section $S$ and a $(-2)$-curve $T$. Let us compile the configuration of reducible fibers and sections of $Z$ given in [ibid.]. There are eight reducible singular fibers of type III on $Z$, and all the other fiber is irreducible. Hence there are exactly sixteen $(-2)$-curves $\{\Theta_{i,j}\}_{0 \leq i \leq 7, 1 \leq j \leq 2}$ on $Z$ such that $(\Theta_{i,j}, \Theta_{i',j'}) > 0$ if and only if $i = i'$ and $j \neq j'$. On the other hand, there is exactly six sections $\{A_{k,l}\}_{0 \leq k \leq 7, 1 \leq l \leq 2}$ on $Z$ such that $(A_{k,l}, A_{k',l'}) > 0$ if and only if $k = k'$ and $l \neq l'$. We may assume that $S = A_{0,2}$, $T = \Theta_{0,2}$, and $(\Theta_{i,j}, A_{0,2}) = 1$ if and only if $j = 2$. By Figure 11 of [ibid.], we also may assume that $(\Theta_{0,2}, A_{k,l}) = 1$ if and only if $l = 2$. By contracting $A_{0,2}, \Theta_{0,2}$ and $A_{k,l}$ for $1 \leq k \leq 7$, we get a birational morphism $h: Z \rightarrow \mathbb{P}^2_k$.

Let us show that $h_\ast \Theta_{i,j} \sim O_{\mathbb{P}^2_k}(j)$ for each $1 \leq i \leq 7$ and $1 \leq j \leq 2$. We need only consider the case where $i = 1$ by symmetry and the case where $j = 1$ since $h_\ast (\Theta_{1,1} + \Theta_{1,2}) \sim h_\ast (-K_Y) \sim O_{\mathbb{P}^2_k}(3)$. Conversely, suppose that $h_\ast \Theta_{i,1} \sim O_{\mathbb{P}^2_k}(2)$. Then exactly six of $A_{1,1}, A_{2,1}, \ldots, A_{7,1}$ intersect with $\Theta_{i,1}$ since $(h_\ast \Theta_{1,1})^2 - \Theta_{i,1}^2 = 6$. We may assume that $A_{1,1}$ is disjoint from $\Theta_{1,1}$.
Now fix \( i_0 \in \{2, 3, \ldots, 7\} \). If \( h_* \Theta_{i_0,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(2) \), then at least five of \( A_{1,1}, A_{2,1}, \ldots, A_{7,1} \) intersect with both \( \Theta_{1,1} \) and \( \Theta_{i_0,1} \), which implies \( 4 = (h_* \Theta_{1,1}, h_* \Theta_{i_0,1}) \geq 5 \), a contradiction. Hence \( h_* \Theta_{i_0,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(1) \). Then exactly three of \( A_{1,1}, A_{2,1}, \ldots, A_{7,1} \) intersect with \( \Theta_{i_0,1} \) since \( (h_* \Theta_{i_0,1})^2 - \Theta_{i_0,1}^2 = 3 \). Moreover, \( A_{i,1} \) intersects with \( \Theta_{i_0,1} \) since otherwise we would obtain \( 2 = (h_* \Theta_{1,1}, h_* \Theta_{i_0,1}) \geq 3 \). On the other hand, for \( 2 \leq i_1 < i_2 \leq 7 \), only \( A_{i,1} \) intersects with both \( \Theta_{i_1,1} \) and \( \Theta_{i_2,1} \) among \( \{A_{k,1}\}_{1 \leq k \leq 7} \) since otherwise we would obtain \( 1 = (h_* \Theta_{1,1}, h_* \Theta_{i_0,1}) \geq 2 \). Hence we may assume that \( \Theta_{i,1} \) intersects with \( A_{1,1}, A_{2i-2,1}, A_{2i-1,1} \) for \( 2 \leq i \leq 4 \). It implies, however, that \( 1 = (h_* \Theta_{5,1}, h_* \Theta_{i,1}) \geq 2 \) for some \( 2 \leq i \leq 4 \), a contradiction. Therefore \( h_* \Theta_{i,1} \sim \mathcal{O}_{\mathbb{P}^2_k}(1) \).

Let \( p_i = h(A_{i,1}) \) and \( l_i = h_* \Theta_{i,1} \) for \( 1 \leq i \leq 7 \). Then we have checked that \( \{l_i\}_{1 \leq i \leq 7} \) is a set of lines passing through exactly three of \( \{p_i\}_{1 \leq i \leq 7} \). Hence the set \( \Sigma := \{(i, j) \mid l_i \text{ passes through } p_j\} \) consists of 21 elements. On the other hand, distinct two lines cannot share two points. Combining this fact and \( \sharp \Sigma = 21 \), we conclude that \( \{p_i\}_{1 \leq i \leq 7} \) is a set of points contained in exactly three of \( \{l_i\}_{1 \leq i \leq 7} \).

Next let us show that \( \{p_i\}_{1 \leq i \leq 7} \) contains four points in general position. Changing the indices of \( \{l_i\}_{1 \leq i \leq 7} \) and \( \{p_i\}_{1 \leq i \leq 7} \), we may assume that \( l_1 \) (resp. \( l_2 \)) passes through \( p_1 \) and \( p_2 \) (resp. \( p_1 \) and \( p_3 \)). Since three of \( \{l_i\}_{1 \leq i \leq 7} \) passes through \( p_2 \), it contains the line spanned by \( p_2 \) and \( p_3 \), say \( l_3 \). Then there is a unique point, say \( p_4 \), in \( \{p_i\}_{1 \leq i \leq 7} \) disjoint from \( l_1 \cup l_2 \cup l_3 \). Hence \( p_1, p_2, p_3 \) and \( p_4 \) are in general position.

Changing coordinates of \( \mathbb{P}^2_k \), we may assume that \( p_1 = [1 : 0 : 0] \), \( p_2 = [0 : 1 : 0] \), \( p_3 = [0 : 0 : 1] \) and \( p_4 = [1 : 1 : 1] \). Then we can check that \( p_i \)'s and \( l_i \)'s are all the points and lines defined over \( \mathbb{F}_2 \), respectively. Therefore \( Y \) is the blow-up of all the points in \( \mathbb{P}^2_{\mathbb{F}_2} \), and \( X \) is the contraction of strict transforms of all the lines in \( \mathbb{P}^2_{\mathbb{F}_2} \).

**Remark 4.9.** Cascini–Tanaka [2, Proposition 6.4] proved that some del Pezzo surface constructed by Langer [13, Example 8.2] is actually isomorphic to some del Pezzo surface constructed by Keel–McKernan [11, end of section 9]. Proposition 4.8 gives another proof of this fact. Moreover, Proposition 4.8 says that this surface is also isomorphic to a counterexample of Akizuki–Nakano vanishing in [7, Proposition 11.1 (1)] with \( p = n = 2 \).

Now, we can prove the main theorem.

**Proof of Theorem 1.2.** When \( p > 3 \), the assertion follows from Lemma 2.2 (3) and Theorem 3.4.

Next suppose that \( p = 3 \). Then, combining Theorem 3.4, Corollary 4.2 and Proposition 4.4, we conclude that there exists a del Pezzo surface with rational double points over \( \mathbb{C} \) which has the same singularity type, the same Picard rank and the same degree as \( X \) except possibly when \( \rho(X) = 1 \) and the singularity type of \( X \) is one of \( E_8, A_2 + D_6 \) and \( 4A_2 \). By [16, Theorem 1.2], there is also a desired del Pezzo surface over \( \mathbb{C} \) when the singularity type of \( X \) is one of them.

Finally suppose that \( p = 2 \). Then, combining Theorem 3.4, Corollary 4.2 and Propositions 4.4–4.8, we conclude that there is a desired del Pezzo surface over \( \mathbb{C} \) except possibly when \( \rho(X) = 1 \) and the singularity type of \( X \) is one of \( E_8, D_8, A_1 + E_7, 2D_4, 2A_1 + D_6, 4A_1 + D_4, 8A_1, E_7, A_1 + D_6, 3A_1 + D_4 \) and \( 7A_1 \), Comparing this list and
[16, Theorem 1.2], we deduce that there is a desired del Pezzo surface over \( \mathbb{C} \) if and only if the singularity type of \( X \) is not \( 7A_1, 8A_1 \) or \( 4A_1 + D_4 \).

\[\square\]

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