Exact solution of a differential problem in analytical fluid dynamics: use of Airy’s functions

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Abstract

Treating a boundary value problem in analytical fluid dynamics, translation of 2D steady Navier-Stokes equations to ordinary differential form leads to a second order equation of Riccati type. In the case of a compressible fluid with constant kinematic viscosity along streamlines, it is possible to find an exact solution of the differential problem by rational combination of Airy’s functions and their derivatives.

1 Ordinary differential form of Navier-Stokes equations

Suppose to have a 2D steady flow of a fluid. Let

\[ \Phi : s \rightarrow (\phi_1(s), \phi_2(s)) = (x, y) \]  

(1)

an admissible parameterization \([5]\) for each streamline, with \(\Phi : [a, b] \rightarrow \mathbb{R}^2\) for some suitable values \(a\) and \(b\) such that \(\Phi(a)\) is the initial point of the streamline, \(\Phi(b)\) is the end point in the considered geometrical domain. Then, if \(\mathbf{v} = (v_1, v_2)\) is the flow velocity field, by definition of streamline there is a scalar function \(f = f(s)\) such that

\[ (v_1, v_2) = f(s)(\dot{\phi}_1(s), \dot{\phi}_2(s)) \]  

(2)
where \( \dot{\phi} \) is the derivative of \( \phi \).

**Remark.** The author is developing a more general model of parameterization for Navier-Stokes equations, in the case 3D too, under less restrictive hypothesis than those formulated in this work.

Suppose that \( f(s) \neq 0 \), \( \dot{\phi}_1(s) \neq 0 \) for all \( s \) along the streamline, that is \( v_1 \neq 0 \), and let \( u = v \circ \phi \). Then

**Proposition 1** Along a streamline \( s \mapsto \phi(s) \) the following relation holds:

\[
\mathbf{v} \cdot \nabla v_1 = \frac{1}{\dot{\phi}_1} u_1 \ddot{v}_1
\]  

(3)

**Dim.** Using chain rule we have

\[
\dot{u}_1 = \frac{\partial v_1}{\partial x} \dot{\phi}_1 + \frac{\partial v_1}{\partial y} \dot{\phi}_2
\]

(4)

But from (2)

\[
\dot{\phi}_2 = \frac{v_2}{v_1} \dot{\phi}_1
\]

(5)

therefore

\[
\dot{u}_1 = \frac{\dot{\phi}_1}{v_1} \left[ v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} \right]
\]

(6)

Then (3) follows from the fact that along the streamline \( u_1(s) = v_1(\phi(s)) \). \( \Box \)

Now let \( \phi \) invertible, that is \( \exists g : \phi([a,b]) \rightarrow [a,b] \) such that \( s = g(\phi(x,y)) \) when \( (x,y) = \phi(s) \). Then

**Proposition 2** Along a streamline the following relation holds:

\[
\Delta v_1 = (\nabla g \cdot \nabla g) \ddot{u}_1 + (\Delta g) \dddot{u}_1
\]

(7)

**Dim.** From definition, we have \( \mathbf{v} = u \circ g \), therefore \( \nabla v_1 = \ddot{u}_1 \nabla g \). Then

\[
\frac{\partial^2 v_1}{\partial x^2} = \dddot{u}_1 \left( \frac{\partial g}{\partial x} \right)^2 + \ddot{u}_1 \left( \frac{\partial^2 g}{\partial x^2} \right)
\]

(8)

and
\[
\frac{\partial^2 v_1}{\partial y^2} = \ddot{u}_1 \left( \frac{\partial g}{\partial y} \right)^2 + \dot{u}_1 \left( \frac{\partial^2 g}{\partial y^2} \right)
\]  
(9)

therefore the thesis follows by summing the two previous expressions. □

The following Proposition holds for a scalar function \( p = p(x, y) \):

**Proposition 3** Let \( q = p \circ \phi \). Along a streamline:

\[
\Delta p = \dot{q}(\Delta g)
\]  
(10)

*Dim.* From \( q = p \circ \phi \) follows \( p = q \circ g \). Then, using chain rule,

\[
\frac{\partial p}{\partial x} = \frac{dq}{ds} \frac{\partial g}{\partial x}
\]  
(11)

and analogous equation for \( y \)-differentiation. □

Now let \( \rho \) the density, \( \mu \) the dynamic viscosity and \( f \) the body force per unit volume of fluid. Then the Navier-Stokes equations for the steady flow are (see [1])

\[
\rho(v \cdot \nabla v) = -\nabla p + \rho f + \mu \Delta v
\]  
(12)

Identifying not derived functions \( h(x, y) \) with their composition \( h \circ \phi \) and using simple substitutions, from previous Propositions and [2] we can state the *ordinary differential form* of Navier-Stokes equations:

\[
\rho \frac{1}{\phi_1} u_1 \ddot{u}_1 = \rho f_1 - q \frac{\partial g}{\partial x} + \mu(\nabla g \cdot \nabla g) \ddot{u}_1 + \mu(\Delta g) \dot{u}_1
\]  
(13)

\[
u_2 = \frac{\dot{\phi}_2}{\dot{\phi}_1} u_1
\]

Note that, if the streamlines \( \phi = (\phi_1, \phi_2) \) are known, from equation (13) it could be possible calculate \( u_1 \). But what about the physical meaning of this solution? We make now some considerations about conservation of mass.
2 About continuity equation

In the case of incompressible flow, the differential form of this equation is simply $\nabla \cdot \mathbf{v} = 0$ ([1] or [6]). In this case, from $\mathbf{v} = \mathbf{u} \circ g$, follows that $\partial_x v_1 = u_1 \partial_x g$ and $\partial_y v_2 = u_2 \partial_y g$, therefore along a streamline the continuity equation is

$$\dot{u} \cdot \nabla g = 0$$  \hspace{1cm} (14)

But it is possible, in the hypothesis $\dot{\phi}_1 \neq 0$ along the streamline, to rewrite this equation using only $u_1$:

**Proposition 4** If $\dot{\phi}_1(s) \neq 0 \ \forall s$, the continuity equation for incompressible flow is

$$\dot{u}_1 + \left[ \ddot{\phi}_2 - \frac{\dot{\phi}_2}{\dot{\phi}_1} \dot{\phi}_1 \right] \frac{\partial g}{\partial y} u_1 = 0$$  \hspace{1cm} (15)

*Dim.* Differentiating the identity $g(\phi(s)) = s$ respect to variable $s$, we have $\nabla g \cdot \dot{\phi} = 1$. Then

$$\partial_x g = \frac{1}{\dot{\phi}_1} - \frac{u_2}{u_1} \partial_y g$$  \hspace{1cm} (16)

Substituting this formula in the continuity equation (14) and using $u_1 \dot{\phi}_2 = u_2 \dot{\phi}_1$, we obtain the new identity

$$\dot{u}_1 + [\dot{\phi}_1 \ddot{u}_2 - \dot{\phi}_2 \dot{u}_1] \partial_y g = 0$$  \hspace{1cm} (17)

Differentiating the relation $u_2 = \frac{\dot{\phi}_2}{\dot{\phi}_1} u_1$, follows that

$$\dot{u}_2 = \frac{\ddot{\phi}_2 u_1 + \dot{\phi}_2 \ddot{\phi}_1}{\dot{\phi}_1} - \frac{\dot{\phi}_1 \ddot{\phi}_2 u_1}{\dot{\phi}_1^2}$$  \hspace{1cm} (18)

from which the relation (15) is obtained eliminating $\dot{u}_2$ in (17). $\square$

**Remark.** Suppose the streamlines are straight lines expressed by the parameterization $\phi_1(s) = s$, $\phi_2 = k$, with $k$ real constants. Then $\dot{\phi}_1 = 1$ and $\dot{\phi}_2 = 0$, and from (15) the continuity equation is simply $\dot{u}_1 = 0$. Then the velocity field is constant along a streamline, as known for incompressible rectilinear flow (see e.g. [6]).
As we would investigate existence of non trivial solutions to Navier-Stokes equations (13) in the simple case \( x = \phi_1(s) = s \), we have \( g_x = 1 \) and \( g_y = 0 \), therefore in the incompressible case from previous Proposition follows that \( u_1 \) is constant. For a more interesting and realistic solution we assume that flow is steady but with spatially variable density.

### 3 A steady compressible flow

Consider a 2D steady flow where streamlines are parameterizable by the following expressions

\[
    x = \phi_1(s) = s, \quad y = \phi_2(s) = \phi_2(x)
\]

where \( \phi_2 \) is invertible on an interval \([0, L]\), that is for each streamline exists a function \( g: \phi_2([0, L]) \rightarrow [a, L] \) such that \( s = g(x, y) = x \). The function \( \phi_2 \) and \( g \) can depend on the single streamline. Note that \( \dot{\phi}_1 = 1 \), \( \nabla g = (1, 0) \) and \( \Delta g = 0 \), therefore from (13) the Navier-Stokes equations along a streamline become

\[
    \rho u_1 \dot{u}_1 = \rho f_1 - \dot{q} + \mu \ddot{u}_1, \quad u_2 = \dot{\phi}_2 u_1
\]

Suppose that \( x \)-component \( f_1 \) of body force \( f \) is constant. Along a streamline we can made the realistic hypothesis that \( \rho(s) \neq 0 \) \( \forall s \); dividing the two members by \( \rho \) we obtain

\[
    u_1 \dot{u}_1 = f_1 - \frac{\dot{q}}{\rho} + \nu \ddot{u}_1
\]

where \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity (see [2]). At this point we suppose that, along a single streamline, \( \nu \) and the quantity \( \frac{\dot{q}}{\rho} \) are constant, with values depending on streamline. The latter, equivalent to \( p = k_1 \int \rho + k_2 \), can be view as a constitutive equation about the fluid.

### 4 An analytical resolution

In this section we try to find a general exact solution of the non linear differential equation (21). Integrating on the independent variable \( s \) and then dividing by \( \nu \), we have

\[
    \dot{u}_1 = \frac{1}{2\nu} u_1^2 + \frac{1}{\nu} \left( \frac{\dot{q}}{\rho} - f_1 \right) s + \frac{c}{\nu}
\]
where \( c \) is an arbitrary constant. This is a first order differential equation of Riccati type (see [2] or [3]). Applying the transformation
\[
\dot{u}_1 = -2\nu \frac{\dot{z}}{z}
\] (23)
we translate previous equation into (see [2])
\[
\ddot{z} + \frac{1}{2\nu^2} \left[ \left( \frac{\dot{q}}{\rho} - f_1 \right) s + c \right] z = 0
\] (24)
This is a form of Airy’s type equation (see [3] and [4]) and its general solution is (see [4])
\[
z(s) = c_1 \text{Ai}(t) + c_2 \text{Bi}(t)
\] (25)
where
\[
t = -\frac{as + b}{(-a)^{\frac{1}{3}}}
\] (26)
\[
a = \frac{1}{2\nu^2} \left( \frac{\dot{q}}{\rho} - f_1 \right)
\] (27)
\[
b = \frac{c}{2\nu^2}
\] (28)
while \( \text{Ai}(t) \) and \( \text{Bi}(t) \) are Airy’s functions (see e.g. [7]), linearly independent solutions of Airy’s equation \( \ddot{y} - ty = 0 \) which appears in optics and quantum mechanics phenomena. Using transformation (23), the exact solution of Navier-Stokes equation (22) along a streamline is
\[
\dot{u}_1 = 2\nu(-a)^{\frac{1}{3}} \frac{c_1 \dot{\text{Ai}}(t) + c_2 \dot{\text{Bi}}(t)}{c_1 \text{Ai}(t) + c_2 \text{Bi}(t)}
\] (29)
where \( \text{Ai}'(t) \) and \( \text{Bi}'(t) \) are the derivative of \( \text{Ai} \) and \( \text{Bi} \).
Note that this solution has physical meaning only if \( a < 0 \). Assuming that inflow zone is at \( s = 0 \), usually pressure drops down towards outflow, so that we can assume \( \dot{q} < 0 \). Also, we assume \( f_1 > 0 \), hence the condition \( a < 0 \) is satisfied.
Also, note that with our assumptions \( \dot{u}_1 \) doesn’t depend on the streamline parametric representation \( (x, y) = \phi(s) \); the flow velocity field depends on \( \phi \) through \( \dot{u}_2 \) component by relation \( \dot{u}_2 = \dot{\phi} u_1 \).
5 A boundary value problem

The general solution (29) depends on three constants of integration: \( c_1 \), \( c_2 \) and, from (22), \( c \). Suppose we want to resolve a boundary value problem for equation (21) with \( u_1(0) = u_{10}, \dot{u}_1(0) = \dot{u}_{10} \) and \( u_1(L) = u_{1L} \). The first two conditions, using (22) at \( s = 0 \), give the value of \( c \). Then, noting that for \( t \in \mathbb{R} \) Airy’s functions \( Ai(t) \) and \( Bi(t) \) have real values (see [7]), the other two constants \( c_1 \) and \( c_2 \) can be found solving the algebraic system \( u_1(0) = u_{10}, u_1(L) = u_{1L} \).

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