Restless Bandits in Action: Resource Allocation, Competition and Reservation

Jing Fu, Bill Moran and Peter G. Taylor

Abstract

We study a resource allocation problem with different requests, and with resources of limited capacity shared by multiple requests. We model this problem as a set of heterogeneous Restless Multi-Armed Bandit Problems (RMABPs) connected by the constraints imposed by the resource capacity. Following the idea of Whittle relaxation, and the asymptotic optimality proof of Weber and Weiss, we propose a simple policy for our problem and prove its asymptotic optimality when the RMABPs are weakly coupled with each other. To the best of our knowledge, this is the first work providing asymptotic optimality results for such a resource allocation problem and such a combination of multiple RMABPs.

In applying these, we consider the asymptotic optimality in the case of a fixed number of active restless Bandit Processes (BPs) for each RMABP, rather than a fixed ratio of the number of active BPs to the total number of BPs as considered by Weber and Weiss. In particular, we scale the transition rates of the active bandits in proportion to the total number of BPs.

Index Terms

restless bandits; resource sharing; Markov decision process

I. INTRODUCTION

A. Overview and Motivations

The Restless Multi-Armed Bandit Problem (RMABP) was proposed by Whittle [1]. Our mise-en-scène involves $K$ competing Bandit Processes (BPs) with random states, and our aim is to optimize the revenue gained by dynamically activating these BPs. In this paper, we consider

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a stochastic optimization problem that can be modeled as a set of RMABPs coupled by linear inequalities involving random state and action variables. It serves as a general model for resource allocation problems in communication networks and high-performance server farms. Modern techniques enable Internet components such as routers, computing servers, cables, etc. to be abstracted from the physical layer to a virtual layer, which facilitates a quick response to Internet demands for setting up communication networks or processing computation jobs. To manage efficiently and control such a complicated network platform at a large scale, theoretical studies on optimizing resource allocation problems become increasingly important.

We study a resource allocation problem in which \( J \) resource classes, each made up of finite numbers of unit resources, await allocation to incoming requests of \( L \) types. Each resource class is potentially shared and competed for many requests, but reservation of unit resources for still-to-arrive requests is allowed. When a request has been accommodated by a resource class, an appropriate number of unit resources of this type are occupied by the request until it leaves the system. A request is permitted to occupy unit resources of more than one resource class simultaneously. We refer to the number of unit resources of a resource class as its capacity. In this context, the number of requests of the same type that are accommodated by a group of resource classes varies according to a birth-and-death process, where the transition rates are affected by the resource allocation policy employed. In addition, several such processes associated with the same resource class are coupled by its limited capacity.

Such a problem can be easily applied to a rich set of classic models, such as loss networks in telecommunications, resource allocation for logistics systems, and job assignment in parallel computing.

This resource allocation problem with small \( L \) and \( J \) can be modeled by Markov decision process (MDP) and be solved through dynamic programming, however, real-world applications exhibit large \( L \) and \( J \), yielding high dimensionality of the state and action spaces.

Kelly [2] published a comprehensive analysis of the loss network, where network traffic can be re-routed onto alternative paths comprising links when the original path fails or is full. In this work, a list of alternative paths as choices of resources are given for each call/request and will be selected after the preceding offered paths fail. This is slightly different from a typical resource allocation problem in that the optimizer is potentially able to dynamically change the priorities of paths; this is a key concept in this paper.
B. Main Contributions

1) Asymptotic Optimality: We prove that a simple index policy achieves asymptotic optimality for the resource allocation problem, if the offered traffic for the entire system is heavy and the RMABPs in our system are weakly coupled. We rigorously define these two hypotheses as Conditions (a) and (b) in Section V, and write this result as Theorem 1 in Section V. Condition (a) requires the blocking probabilities of all request types to be positive all the time and Condition (b) implies that there is at most one class of unit resources associated with each type of request that is shared with requests of other types. For general cases where multiple classes of unit resources are required by each request, Condition (b) is relatively strong. On the other hand, for the special case where each request requires only one class of unit resources to serve it, exempli gratia, a mobile phone needs connection from only one Internet access point, Condition (b) naturally holds.

The index policy with asymptotic optimality exhibits remarkably reduced computational complexity, compared to conventional optimizers, and is appropriate for large-scale systems where computational power is a scarce commodity. This index policy reduces to the Whittle index policy if \( L = 1 \) or the capacities of all resource classes tend to infinity.

If Conditions (a) and (b) hold and each request requires the same number of unit resources from each class, we prove that an index policy with closed-form indices is asymptotically optimal. This result is also presented in Theorem 1 in Section V.

To the best of our knowledge, no existing work has proved asymptotic optimality in the resource allocation problem, where resource competition and reservation are potentially permitted, nor has there been a previous analysis of such a combination of multiple, different RMABPs, resulting in much higher dimensionality of the state space.

2) Practical Asymptotic Regime: The proof relevant to asymptotic optimality mainly follows the ideas of [1] and [3]. However, our proof differs in terms of two aspects:

(i) The limited capacities for resource classes prevent us from directly applying Whittle relaxation technique.

(ii) Most existing work about asymptotic optimality, including the proof in [3], assumes that the number of active BPs will be scaled proportionately into infinity, so that the proportion of active BPs stays fixed. We consider a fixed number of active BPs for each RMABP that is independent from the scale of the system. Instead, we scale the transition rates between
states of each RMABP and the limited capacities of shared resource classes proportionately for the asymptotic regime.

The remainder of the paper is organized as follows. In Sections II and III, we describe the resource allocation problem and the index policies, respectively. In Section IV, we relax the resource allocation problem by applying the Whittle relaxation technique. In Section V, we prove the asymptotic optimality of an index policy under some conditions: in this policy, the indices are given in closed form. A numerical example is provided in Section VI to illustrate an index policy that approaches asymptotic optimality. In Section VII, we present some conclusions.

C. Relation to the Literature

The conventional Multi-Armed Bandit Problem (MABP) is a stochastic optimization problem in which only one BP among $K$ BPs is active at any one time, while all the other $K - 1$ BPs are frozen: an active BP randomly changes its state, while state transitions will not happen to the frozen BPs. In 1974, Gittins and Jones published the famous index theorem for the MABP [4], and in 1979, Gittins [5] proved the optimality of a simple index policy for the MABP, subsequently referred to as the Gittins index policy. Under the Gittins index policy, an index value, referred to as Gittins index, is associated with each state of each BP, and the BP with the largest index value is selected to be active, while all the other BPs are frozen. More details about Gittins indices can be found in [6, Chapter 2.12] (and references therein).

The optimality of the Gittins index policy for the conventional MABP, or non-restless MABP, fails for the general case where the $K - 1$ BPs that are not selected to be active can also change their states randomly; that is, the RMABP. The word restless alludes to the feature that the state of a BP can change even when it is not activated. Papadimitriou and Tsitsiklis [7] proved that the optimization of RMABP is PSPACE-hard in general. The Restless Multi-Armed Bandit Problem (RMABP) was proposed by Whittle [1], together with a heuristic policy as an approximation for optimality, referred to as the Whittle index policy. The RMABP allowed $M$ ($M = 1, 2, \ldots, K$) BPs to be active simultaneously. In a similar vein to the Gittins index policy, Whittle assigned a state-dependent index value, referred to as the Whittle index, to each BP and always activated the $M$ BPs with the highest indices. The Whittle indices are calculated from a relaxed version of the original RMABP obtained by randomizing the action variables as a relaxation. Whittle [1] defined in a property of a RMABP, referred to as the indexability, under which the Whittle index policy exists. Whittle conjectured in [1] that the Whittle index policy,
if it exists, is asymptotically optimal. Weber and Weiss [3] proved the asymptotic optimality of Whittle index policy under mild conditions.

Niño-Mora [8] proposed a Partial Conservation Law (PCL) for the optimality of RMABP; this is an extension of the General Conservation Law (GCL) published in [9]. Later, Niño-Mora [10] defined a group of problems that satisfies PCL-indexibility and proposed a new index policy that improved the Whittle index. The new index policy was proved to be optimal for problems with PCL-indexibility. PCL-indexibility led to (and was stronger) than Whittle indexibility. A detailed survey about the optimality of bandit problems can be found in [11].

Linear Programming (LP) based relaxation has been studied and applied to resource allocation problems where networked resources are prepared to accept incoming jobs/requests, such that the accumulated revenue over a finite time horizon is maximized in [12, 13, 14, 15, 16, 17]. The LP-based relaxation technique was compared with Lagrangian relaxation approaches for weakly coupled dynamic problems by [12] who proved that the LP-based relaxation produced tighter bounds on the performance deviation from optimality, while the Lagrangian relaxation led to lower computational complexity. In [13], an LP-based relaxation approach was utilized to maximize accumulated revenue by accepting/rejecting requests which consume a determined set of resources. In [14, 15, 16, 17], LP-based relaxation was studied in similar problems and selections of potentially different resources were considered for each request. However, this stream of work prohibited release of occupied resources. In contradistinction, our problem requires bidirectional state transitions for each BP, which models the cases where unit resources are occupied and released respectively.

Stolyar [18] studied a switching model with multiple parallel queues and a server based on a discrete time Markov chain, and proposed a MaxWeight discipline under heavy traffic with an assumption of Resource Pooling which ensures the stability of the multi-queue system. The MaxWeight discipline is a special case of so-called Generalized $c_\mu$-rule (G$c_\mu$-rule) proposed in [19], and has been proved in [18] to asymptotically minimize the cumulative holding cost during a finite time interval. Mandelbaum and Stolyar [19] proved that a simple G$c_\mu$-rule asymptotically minimizes instantaneous and cumulative holding costs in a queueing system with multiple-parallel flexible servers and multi-class jobs when the system is in heavy traffic and a stability condition is satisfied. Nazarathy and Weiss [20] proposed a method for a multi-server queueing system to minimize its accumulated cost over a finite time horizon. The optimal solution of the same problem in the fluid regime was obtained using a Separated Continuous
Linear Program (SCLP), and coincided with the proposed heuristic method; that is, the heuristic method asymptotically minimizes the total cost over a finite time horizon. However, the model discussed in [19, 20] assumed that one server cannot serve more than one job of the same class simultaneously, which does not necessarily hold in this paper.

Verloop [21] proved the asymptotic optimality of the Whittle index policy in an extended version of RMABP, where BPs randomly arrive and depart the system. She also proposed an index policy that was not restricted to Whittle indexable models and numerically demonstrated its near-optimality. Larrañaga et al. [22] applied this extended RMABP to a patient-queueing problem assuming convex, non-decreasing functions for both holding costs and measured values of people’s impatience. More results on asymptotic optimality of index-like polices can be found in [23, Chapter IV].

II. A Resource Allocation Problem

We use $\mathbb{N}_+$ and $\mathbb{N}_0$ as the set of positive and non-negative integers, and for any $N \in \mathbb{N}_+$, let $[N]$, represent the set $\{1, 2, \ldots, N\}$. Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_0$ be the set of all, positive and non-negative reals, respectively.

Recall that there are $L$ types of requests and $J$ classes of resources that are potentially different. Resource class $j \in [J]$ has a capacity $C_j$ unit resources. By serving a request of type $\ell \in [L]$ and occupying a unit resource of class $j$ for one unit of time, we gain expected reward $r_\ell$ and pay expected cost $\varepsilon_j$, respectively. There is a controller who wishes to maximize the revenue, which is the difference between expected reward and cost, by efficiently utilizing the limited amount of resources.

A pattern, denoted by $i$, $i \in [I]$ with postulated integer $I$, consists of a set of resource classes, $\mathcal{J}_i$, $\mathcal{J}_i \in 2^{[J]}$, and a weight vector $w_i$, $w_i \in \mathbb{N}_0^J$. Request type (RT) $\ell$, $\ell \in [L]$, is defined as a non-empty set of patterns $\mathcal{P}_\ell \in 2^{[I]}$. There is a dummy pattern $i_0^\ell$ which models accommodating rejected/blocke requests for RT $\ell$. We consider that $i_0^\ell \in \mathcal{P}_\ell$ for all $\ell \in [L]$. Then, there are $|\mathcal{P}_\ell|$ choices of patterns for each RT $\ell$ and a request comprises of a RT and its time attributes. We consider $I = \sum_{\ell \in [L]} |\mathcal{P}_\ell|$. Requests of RT $\ell$ arrive at the system sequentially, following a Poisson process, with postulated rates $\lambda_\ell$ and occupation times that are exponentially distributed with parameter $\mu_{\ell,i}$. We assume that neither the request nor the system knows the lifespan of this request until it is accomplished and departs the system.
When a request of RT $\ell$ is accommodated by pattern $i \in P_\ell$, an instantiation is generated by the controller of the system, which contains information of: a) this request; b) and the pattern decided to serve it. Once the request departs the system, the associated instantiation will be removed from controller’s memory. Numbers of unit resources of RTs related to the pattern of an instantiation are occupied when the instantiation is generated, and are released when the instantiation is removed. Then, the number of instantiations that is associated with pattern $i$ in the controller’s memory is a birth-and-death process. For our optimization problem, we refer to this process as a restless Bandit Process (BP). We say a BP for/associated with pattern $i$ provided that this BP is the stochastic process formed by the number of instantiations associated with pattern $i$.

Thus a pattern $i$ corresponds to a BP mentioned in Section I. In total, we have $I$ BPs, each of which has state space $\mathcal{N}_i$, $i \in [I]$, which is the set of all numbers of instantiations associated with pattern $i$ present in controller’s memory.

A state of the entire system is represented by $n \in \mathcal{N}$, where $n_i$ gives the number of instantiations with pattern $i$ that are present. Let

$$\mathcal{N} := \prod_{i \in [I]} \mathcal{N}_i.$$ 

Here, $\prod$ represents Cartesian product. We associate action variable $a_\phi^i(n)$ with the BP for pattern $i$ under policy $\phi$ when the system is in state $n$, and define $a_\phi(n) = (a_\phi^i(n) : i \in [I])$. We say the BP associated with pattern $i$ is active if $a_\phi^i(n) = 1$; and passive otherwise. If the BP for pattern $i$ is active, then the numbers of unit resources of the classes related to pattern $i$ are reserved for an incoming request for pattern $i$; and not reserved if this BP is passive. Then, being active/passive affects the transition probabilities of BPs, and therefore the revenue gained in the future.

In this context, we define a policy $\phi$ as a mapping: $\mathcal{N} \rightarrow \mathcal{A}$, where

$$\mathcal{A} := \prod_{\ell \in [L]} \{0, 1\}^{|P_\ell|},$$

and the set of all such mappings as $\Phi$. An index policy is a special case of a policy requiring that a given sequence of indices be assigned to all states in $\mathcal{N}_i$ for all BPs. A rigorous definition of an index policy is given in Section III.
For each RMABP, only one pattern will be selected to accommodate a request type. Thus for any \( \phi \in \Phi \),
\[
\sum_{i \in \mathcal{P}_\ell} a^i_\phi(N^\phi(t)) = 1, \quad \forall \ell \in [L], \quad \forall N^\phi(t) \in \mathcal{N},
\]
(1)
where \( N^\phi_i(t) \) represents the number of instantiations with pattern \( i \) at time \( t \) under policy \( \phi \). Thus, the birth rate of BP for pattern \( i \in \mathcal{P}_\ell, \ell \in [L], \) is \( \lambda_\ell \) if it is active; and zero if it is passive.

For our resource allocation problem, we introduce a set of additional linear constraints accounting for the limited capacities on shared resource classes. A unit resource of class \( j \) can be

1) occupied by an accommodated request
2) or reserved by a still-to-arrive request

of type \( i \), which occupies \( w_{j,i} \) unit resources of type \( j \). We formulate the capacity constraint for the \( J \) resource classes as
\[
\mathcal{W} \left( N^\phi(t) + \mathcal{E} \alpha^\phi(N^\phi(t)) \right) \leq \mathcal{C},
\]
(2)
where \( \mathcal{W} \) is the \( J \times I \) matrix with entries \( w_{j,i} \), \( \mathcal{C} \in \mathbb{N}_+^J \) is a vector with entries \( C_j \) and \( \mathcal{E} \) is a diagonal matrix of size \( I \) with \( e_{i,i} \in \{0,1\} \) for \( i \in [I] \). The \( L \) RMABPs are coupled by (2). When \( e_{i,j} = 1 \), if \( w_{j,i} \) unit resources of class \( j \) for an incoming request of type \( \ell \) with \( i \in \mathcal{P}_\ell \) are reserved, then these reserved unit resources cannot be reserved by other requests. When \( e_{i,j} = 0 \), the reserved unit resources can possibly be reserved by other requests. When a unit resource is reserved by multiple incoming requests, it will be occupied by the first arriving request among all these incoming requests. There is an upper bound, \( \min_{j \in [J]} \lceil C_j/w_{j,i} \rceil \), for the number of instantiations of pattern \( i \), which is also a boundary state where we force this pattern to be passive; that is, \( \mathcal{N}_i = \{0,1,\ldots,\min_{j \in [J]} \lceil C_j/w_{j,i} \rceil \} \) and \( |\mathcal{M}_i| = \min_{j \in [J]} \lceil C_j/w_{j,i} \rceil + 1 < +\infty \).

A special case of resource competition occurs when each unit resource cannot be reserved by more than one request simultaneously; that is, in (2), \( \mathcal{E} \) is assumed to be an identity matrix. Then (2) becomes
\[
\mathcal{W} \left( N^\phi(t) + \alpha^\phi(N^\phi(t)) \right) \leq \mathcal{C}, \quad \forall N^\phi(t) \in \mathcal{N}.
\]
(3)
We assume that there is no row and exactly \( L \) columns in \( \mathcal{W} \) with all zero entries. Each of these zero-columns stands for the dummy pattern \( i_0^\ell \) that models the case where requests of
type $\ell \in [L]$ are blocked. We refer to the time proportion that $a_{i\ell}^\phi(N^\phi(t)) = 1$ as the blocking probability of requests of type $\ell$.

The objective is to maximize the long-run average revenue which exists in the Markov case with finitely-many system states; that is, with $U = \text{diag}(\mu)$, we wish to find

$$ R = \max_{\phi \in \Phi} \lim_{t \to +\infty} \mathbb{E} \left[ (ru - \varepsilon w)N^\phi(t) \right], \quad (4) $$

subject to (1) and (3).

III. An Index Policy

An index policy is a priority-style policy in a RMABP, which gives higher priorities to the bandits with higher indices. In our problem with not only resource allocation but also resource competition, the priorities of bandits that are competing for resource classes are also given by their indices.

At each decision making epoch, we refer to the BPs that we have not yet determined to be active or passive as the non-determined BPs, while the other BPs are determined BPs. Consider a given vector of indices $\nu \in \prod_{i=1}^I [R^{J_i}]$, in our system with $I$ BPs, an index policy based on $\nu$ will check the states of all BPs at each decision making epoch, and:

1) for a non-determined BP where there are not enough available resources to accommodate a new arrival, set the BP to be passive and consider it to be determined;
2) set the BP with the highest index among all non-determined BPs with enough available resources to be active and consider this BP to be determined;
3) reserve the unit resources of the resource classes used by this active BP by removing them from the pool of available resources;
4) set all the other BPs that are related to the same request type as the active BP to be passive and label them as determined BPs;
5) go back to step 1 until all BPs have become determined.

We use $\nu_{i,n}$ to indicate the index value of state $n \in \mathcal{N}_i$ for pattern $i$ when the vector of indices is given by $\nu$. For $a \in \mathcal{A}$, $\ell \in [L]$, $n \in \mathcal{N}$, we define a function

$$ P(a, n) := \left\{ i \in \mathcal{P}_\ell \mid \forall j \in \mathcal{J}_i, \sum_{i' = 1}^I w_{j,i'}n_{i'} + \sum_{i' = 1}^I w_{j,i'}a_{i'} + \sum_{i' = 1}^I w_{j,i'}a_{i'} + w_{j,i} \leq C_j \right\}, \quad (5) $$
Algorithm 1 Index Policy $\sigma^1(\nu)$

**Require:** a vector of real values $\nu \in \prod_{i=1}^{I} \mathbb{R}^{I_i}$.

**Ensure:** a policy $\sigma^1(\nu) \in \Phi$ determined by action vectors $a^{\sigma^1(\nu)}(n)$ for $n \in \mathcal{N}$.

1: function $\text{INDEXACTION}(n, \nu)$
2: $a_i(n) \leftarrow 0$ for all $i \in [I]$  
3: $\mathcal{L} \leftarrow [L]$  
4: while $\mathcal{L} \neq \emptyset$ do  
5: for $\ell \in \mathcal{L}$ do  
6: $i_\ell(n) \leftarrow \arg \max_{i \in \mathcal{P}_\ell(a(n), n)} \nu_{i,n}$ \hfill $\triangleright$ Tie case breaks arbitrarily  
7: end for  
8: $i^* \leftarrow \arg \max_{\ell \in \mathcal{L}} \nu_{i^*(n), n_{i^*}(n)}$ \hfill $\triangleright$ Tie case breaks arbitrarily  
9: $a_{i^*} \leftarrow 1$  
10: $\mathcal{L} \leftarrow \mathcal{L} / \{\ell^*\}$ with the $\ell^*$ satisfying $i^* \in \mathcal{P}_{\ell^*}$  
11: end while  
12: return $a(n)$  
13: end function  

14: procedure $\text{INDEXPOLICY}(\nu)$  
15: for $n \in \mathcal{N}$ do  
16: $a^{\sigma^1(\nu)}(n) \leftarrow \text{INDEXACTION}(n, \nu)$  
17: end for  
18: end procedure

With the help of $\mathcal{P}^\ell(\cdot, \cdot)$, we provide the pseudo-code in Algorithm 1 for a process of constructing an index policy. The vector of indices $\nu$ is the input to this algorithm and its output is an index policy $\sigma^1(\nu)$ that indicates an action vector $a^{\sigma^1(\nu)}(n) \in \mathcal{A}$ for each of the state vectors of the entire system $n \in \mathcal{N}$. Then, we define the set of index policies

$$
\Phi^1 := \left\{ \phi \in \Phi \left| \exists \nu \in \prod_{i=1}^{I} \mathbb{R}^{I_i}, \phi = \sigma^1(\nu) \right. \right\}.
$$

We provide an example to briefly explain how the indices, $\nu$, lead to priorities of states of BPs in Algorithm 1.
Example 1. Consider a system with two RTs \((L = 2)\) and two resource classes \((J = 2)\) with capacities \(C = (2, 2)\), two choices of patterns for RT 1 \((\mathcal{P}_1 = \{1, 2, 4\})\) and one pattern for RT 2 \((\mathcal{P}_2 = \{3, 5\})\), where Patterns 4 and 5 represents the dummy patterns blocking a request. Each of the patterns occupies only one resource class: \(J_1 = \{1\}\), \(J_2 = \{2\}\), and \(J_3 = \{1\}\). Each request of these RTs requires only one unit resource to accommodate it \((w_{j,i} = 1)\) for all \(j \in J_i\), \(i = 1, 2, 3\). In this context, \(N_i = \{0, 1, 2\}\) for all \(i = 1, 2, 3\).

Consider \(\nu_1,n = 2\), \(\nu_2,n = 1\) and \(\nu_3,n = 3\) for all \(n \in \{0, 1, 2\}\) as the indices for states of BPs associated with the three patterns. Here, for three different state vectors, \(n_1, n_2, n_3 \in N\):

(A) \(n_1 = (0, 0, 0, 0, 0)\);
(B) \(n_2 = (1, 0, 0, 0, 0)\);
(C) \(n_3 = (2, 0, 0, 0, 0)\);

we consider the action vectors under \(\sigma^1(\nu)\).

For Case (A), \(a^{\sigma^1(\nu)}(n_1) = (1, 0, 1, 0, 0)\). For Case (B), \(a^{\sigma^1(\nu)}(n_2) = (0, 1, 1, 0, 0)\), because the single unit resource of class 1 has been reserved by Pattern 3 for RT 2 which has a higher index value. For Case (C), \(a^{\sigma^1(\nu)}(n_3) = (0, 1, 0, 0, 1)\), because there is no unit resource of class 1 for any of the RTs.

IV. WHITTLE RELAXATION

We provide in this section the theoretical analysis of the optimization problem defined by (1), (3) and (4), following the idea of Whittle relaxation [1].

By randomizing the action variable \(a^{\phi}(\cdot)\), we relax constraints defined by (1) and (3) to

\[
\lim_{t \to +\infty} \mathbb{E} \left[ \sum_{i \in \mathcal{P}_\ell} a^{\phi}_i(N^{\phi}(t)) \right] = 1, \quad \forall \ell \in [L],
\]

and

\[
\lim_{t \to +\infty} \mathbb{E} \left[ W(N^{\phi}(t) + a^{\phi}(N^{\phi}(t))) \right] \leq C,
\]

respectively. Constraints (6) and (7) are taking long-run expectations for both sides of constraints (1) and (3) in the original problem. The randomized control variable \(a^{\phi}_i(n)\) takes either value from \(\{0, 1\}\) with probabilities determined by the policy \(\phi\). We refer to the problem defined in (4), (6) and (7) with such randomized control variables as the relaxed problem. In particular, the domain of the policies that provide probability distributions of values of \(a^{\phi}_i(n)\) in the relaxed problem is represented by \(\tilde{\Phi}\).
For \( n_i \in \mathcal{N}_i, \phi \in \Phi, i \in [I] \), we define

\[
\alpha_i^\phi(n_i) := \lim_{t \to +\infty} \mathbb{E} \left[ a_i^\phi(N_i(t)) \mid N_i^\phi(t) = n_i \right],
\]

and vector \( \alpha_i^\phi := (\alpha_i^\phi(n_i) : n_i \in \mathcal{N}_i) \);

- the steady state probability that \( N_i^\phi(t) = n_i \) under policy \( \phi \) for pattern \( i \) is \( \pi_i^\phi \), and the vector \( \pi_i^\phi := (\pi_i^\phi : n_i \in \mathcal{N}_i) \).

Let \( (\mathcal{N}_i) \) represent the vector \((0,1,\ldots,|\mathcal{N}_i| - 1)\). Constraints (6) and (7) can be expressed in the form

\[
\sum_{i \in \mathcal{P}_i} \pi_i^\phi \cdot \alpha_i^\phi = 1, \forall i \in [I],
\]

and

\[
\mathcal{W} \left( \Pi_n^\phi + \Pi_a^\phi \right) \leq C,
\]

where \( \Pi_n^\phi = \left( \pi_i^\phi \cdot (\mathcal{N}_i) : i \in [I] \right)^T \), and \( \Pi_a^\phi = \left( \pi_i^\phi \cdot \alpha_i^\phi : i \in [I] \right)^T \).

We obtain the Lagrange function

\[
\Lambda(\phi, \gamma, \nu, \tilde{\nu}) := (r \mathcal{U} - \varepsilon \mathcal{W}) \Pi_n^\phi - \sum_{\ell = 1}^L \nu_\ell \left( \sum_{i \in \mathcal{P}_\ell} \pi_i^\phi \cdot \alpha_i^\phi - 1 \right) - \gamma \cdot (\mathcal{W}(\Pi_n^\phi + \Pi_a^\phi) - C) - \sum_{i \in [I]} \tilde{\nu}_i \alpha_i^\phi(|\mathcal{N}_i| - 1),
\]

(8)

where \( \phi \in \Phi, \nu \in \mathbb{R}^L \) and \( \gamma \in \mathbb{R}_0^J \) are Lagrangian multiplier vectors for Constraints (6) and (7), and \( \tilde{\nu} \in \mathbb{R}^I \) is the Lagrange multiplier vector for the constraint

\[
\alpha_i^\phi(|\mathcal{N}_i| - 1) = 0, \forall i \in [I].
\]

Constraint (9) is used to model the case of blocking requests in the relaxed problem represented by equations (4), (6) and (7). It corresponds to Constraint (3) in the original problem represented by (4), (1) and (3).

For \( i \in [I] \) and the unique \( \ell \) such that \( i \) is in \( \mathcal{P}_\ell \), let

\[
\Lambda_i(\phi, \gamma, \nu, \tilde{\nu}) := (r \cdot \mu_i - \varepsilon \cdot w_i) \pi_i^\phi(\mathcal{N}_i) - \nu_\ell \pi_i^\phi \cdot \alpha_i^\phi - \gamma \cdot \left( w_i(\pi_i^\phi \cdot (\mathcal{N}_i) + \pi_i^\phi \cdot \alpha_i^\phi) \right) - \tilde{\nu}_i \alpha_i^\phi(|\mathcal{N}_i| - 1),
\]

(10)

where we recall that \( w_i \) is the weight vector of pattern \( i \) given by the \( i \)th column vector of \( \mathcal{W} \).

The Lagrangian dual function is

\[
g(\gamma, \nu, \tilde{\nu}) := \max_{\phi \in \Phi} \Lambda(\phi, \gamma, \nu, \tilde{\nu}) = \max_{\phi \in \Phi} \sum_{i \in [I]} \Lambda_i(\phi, \gamma, \nu, \tilde{\nu}) + \sum_{\ell \in [L]} \nu_\ell + \gamma \cdot C.
\]

(11)
Each \( \Lambda_i(\phi, \nu, \gamma, \tilde{\nu}) \) is dependent on \( \phi \) through the random vector \( a_i^\phi = (a_i^\phi(n) : n \in \mathcal{N}) \) and there is no constraint for the maximization of \( \sum_{i \in [L]} \Lambda_i(\phi, \nu, \gamma, \tilde{\nu}) \) in (11). From the idea presented in [1], if one can operate the underlying BP for pattern \( i \) without any constraint, then it can yield a maximized average revenue determined by \( \max_{a_i^\phi \in \{0,1\}^{\mathcal{N}}} \Lambda_i(\phi, \nu, \gamma, \tilde{\nu}) \), where the maximizing alternative \( a_i^\phi(n) \in \{0,1\} \) indicates that this BP is either active or passive in state \( n \in \mathcal{N}_i \). That is,

\[
g(\gamma, \nu, \tilde{\nu}) = \sum_{i \in [L]} \max_{a_i^\phi \in \{0,1\}^{\mathcal{N}}} \Lambda_i(\phi, \gamma, \nu, \tilde{\nu}) + \sum_{\ell \in [L]} \nu_\ell + \gamma \cdot C. \tag{12}
\]

For the maximization for pattern \( i \) in equation (12), a policy \( \phi \) is determined by the action vector \( \alpha_i^\phi \). We refer to the maximization of \( \Lambda_i(\phi, \gamma, \nu, \tilde{\nu}) \) over \( \alpha_i^\phi \in \{0,1\}^{\mathcal{N}_i} \) as the sub-problem for pattern \( i \). By slightly abusing notation, we refer to the policy \( \phi \) determined by such an action vector \( \alpha_i^\phi \) as the policy for pattern \( i \), and define \( \Phi_i \) as the set of all policies for pattern \( i \).

**Proposition 1.** For given \( \nu, \gamma \) and sufficiently large \( \tilde{\nu} \), the policy \( \varphi \in \Phi_i \), determined by action vector \( \alpha_i^\varphi \in \{0,1\}^{\mathcal{N}_i} \), of the sub-problem for pattern \( i \) is given by, for \( n \in \mathcal{N}_i \),

\[
\alpha_i^\varphi(n) = \begin{cases} 
1 & \text{if } \nu_\ell < \lambda_\ell (r_\ell - \frac{1}{\mu_{\ell,i}} \sum_{j \in \mathcal{J}_i} \varepsilon_{j}w_{j,i}) - (1 + \frac{\lambda_\ell}{\mu_{\ell,i}}) \sum_{j \in \mathcal{J}_i} w_{j,i} \gamma_{j} \text{ and } n < |\mathcal{N}_i| \tag{13} \\
0 \text{ or } 1 & \text{if } \nu_\ell = \lambda_\ell (r_\ell - \frac{1}{\mu_{\ell,i}} \sum_{j \in \mathcal{J}_i} \varepsilon_{j}w_{j,i}) - (1 + \frac{\lambda_\ell}{\mu_{\ell,i}}) \sum_{j \in \mathcal{J}_i} w_{j,i} \gamma_{j} \text{ and } n < |\mathcal{N}_i| \tag{14} \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( \ell \) satisfies \( i \in \mathcal{P}_\ell \).

**Proof.** The justification is given in Appendix A. \( \square \)

From Proposition 1, for all \( n \in \mathcal{N}_i \backslash \{\mathcal{N}_i - 1\}, \ i \in \mathcal{P}_\ell \), we define the special value of \( \nu_\ell \) in (14) as

\[
\nu_{i,n} := \nu_{i,n}^* := \lambda_\ell \left( r_\ell - \frac{1}{\mu_{\ell,i}} \sum_{j \in \mathcal{J}_i} \varepsilon_{j}w_{j,i} \right) - \left( 1 + \frac{\lambda_\ell}{\mu_{\ell,i}} \right) \sum_{j \in \mathcal{J}_i} w_{j,i} \gamma_{j}, \tag{16}
\]

for which there is no difference to set \( a_i^\varphi(n) \) to be 0 or 1. Following the idea of Whittle relaxation, we refer to this \( \nu_{i,n} \) as the index of the resource allocation problem for state \( n \) of the BP for pattern \( i \); in particular, the Whittle index for a standard RMABP. If \( C \to +\infty \) (that is, \( \gamma = 0 \)), then the resource allocation problem reduces to \( L \) independent RMABPs and Proposition 1 leads to the fact that each of them has a Whittle index policy.

In this context, if \( C \to +\infty \) (that is, \( \gamma = 0 \)), then the Whittle index policy is asymptotically optimal in the sense defined in (ii) of Section I for each of the \( L \) independent policies [24].
V. ASYMPTOTIC OPTIMALITY

With a parameter \( h \in \mathbb{N}_+ \), let \( C := hC^0 \), \( C^0 \in \mathbb{N}_+^I \), and the arrival rates scaled as \( \lambda := h\lambda^0 \), \( \lambda^0 \in \mathbb{R}_+^L \). We refer to this parameter \( h \) as the scaling parameter, and the asymptotic regime as the limiting case with \( h \to +\infty \).

We split the BP associated with pattern \( i \) into \( h \) identical sub-BPs labeled as \( (i, k) \), \( k \in [h] \). We divide \( N^\phi_i(t) \) the number of instantiations for pattern \( i \) under policy \( \phi \) at time \( t \) into \( h \) parts, and represent the number of instantiations of the \( k \)th part as \( N^\phi_{i,k}(t) \), satisfying
\[
N^\phi_i(t) = \sum_{k=1}^h N^\phi_{i,k}(t).
\]
We then associate a sub-pattern \( (i, k) \) with sub-BP \( (i, k) \), and refer to \( N^\phi_{i,k}(t) \) as the number of instantiations for sub-pattern \( (i, k) \). Similarly, define the state space for the counting process specified by \( N^\phi_{i,k}(t) \) for any \( k \in [h] \), \( i \in [I] \), as \( \mathcal{N}^\phi_i \), where the maximal value of \( N^\phi_{i,k}(t) \) is defined as \( \min_{j \in J_i} [C^0_j/w_{i,j}] \). That is, \( \mathcal{N}^\phi_i := \{0, 1, \ldots, \min_{j \in J_i} [C^0_j/w_{i,j}]\} \). In particular, \( \mathcal{N}^\phi_0 = \{0\} \) for any dummy pattern \( i \) which records that an incoming instantiation will be blocked.

The objective and constraints defined by (4), (1) and (3) still apply to the sums of variables
\[
\sum_{k=1}^h N^\phi_{i,k}(t) = N^\phi_i(t), \quad i \in [I].
\]
We say BP associated with pattern \( i \) has been replaced by the \( h \) sub-BPs associated with sub-patterns \( (i, k) \), \( k \in [h] \). Let \( N^\phi_i(t) = (N^\phi_{i,k}(t) : i \in [I], k \in [h]) \) be the state variable after this replacement.

The optimization problem consisting of the \( hI \) sub-BPs associated with \( hI \) sub-patterns, coupled through constraints defined by (1) and (3) by replacing \( N^\phi(t) \) with \( N^\phi_h(t) \), can be analyzed and relaxed along the same lines as in Section IV. A sub-problem is now associated with a sub-pattern \( (i, k) \), leading to the same optimal solution as original pattern \( i \) that is given by Proposition 1.

Our approach will be to postulate a vector \( \nu^* \in \prod_{i \in [I]} \mathbb{R}^{\mathcal{N}^\phi_i} \), and discuss its values under different conditions.

**Definition 1.** Let \( \varphi^* = \sigma^1(\nu^*) \) represent the index policy based on indices given by \( \nu^* \).

For a vector \( \nu^* \), we order the states \( n \) in \( \mathcal{N}^\phi_i \setminus \{\mathcal{N}^\phi_i - 1\} \) for all \( i \in [I] \) in descending order of \( \nu^*_{i,n} \), where ties break arbitrarily, and continue with the states \( \mathcal{N}^\phi_i - 1 \) for all \( i \in [I] \). For the sake of presentation, \( n \) precedes \( n' \) if \( \nu^*_{i,n} = \nu^*_{i,n'} \), \( n, n' \in \mathcal{N}^\phi_i \) and \( n < n' \), or \( \nu^*_{i,n} = \nu^*_{i,n'} \), \( n \in \mathcal{N}^\phi_i \) and \( n' \in \mathcal{N}^\phi_i \) and \( i < i', i, i' \in [I] \). Then, to emphasize the priorities/orders of these states, we label all these ordered states by their order \( i \in [N] \) with \( N = \sum_{i \in [I]} \mathcal{N}^\phi_i \), and \( n_i \) giving the number of instantiations of the \( i \)th state. State \( n \in \mathcal{N}^\phi_i \), \( i \in [I] \), can thus be uniquely indicated
by label \( \iota \). There exists one and only one \( \ell \) and \( i \) satisfying \( n \in \mathcal{A}_i^0 \) and \( i \in \mathcal{P}_\ell, \ell \in [L], \) for any state \( n \) labeled by \( \iota \). Such an \( \ell \) and \( i \) are denoted by \( i_\iota \) and \( \ell_\iota \), respectively.

Since all sub-BPs in the same state are equivalent to each other: there is no difference between activating one to each other, a controller does not really worry about which sub-BP is active in, for instance, state \( \iota \) but the number of active sub-BPs in this state. Define a random variable \( \mathcal{Z}_t^{\phi,h}(t) \) as the proportion of sub-BPs in state \( \iota \) at time \( t \) under policy \( \phi \) where \( h \) is the scaling parameter; that is,

\[
\mathcal{Z}_t^{\phi,h}(t) = \frac{1}{hI} \left| \left\{ (i, k) \in [I] \times [I] \mid N_{i,k}^\phi(t) = n_i, \ i \in \mathcal{A}_i^0 \right\} \right|.
\]

Let \( \mathcal{Z}^{\phi,h}(t) = (\mathcal{Z}_t^{\phi,h}(t) : \ \iota \in [N]) \) and \( \mathcal{Z} = \{ z \in [0, 1]^N \mid \sum_{i \in [N]} z_i = 1 \} \). Thus, the counting process \( N_{h}^\phi(t), \ t \geq 0, \) is transformed so that it is a process on \( \mathcal{Z}^{\phi,h}(t) \). When the process \( \mathcal{Z}^{\phi,h}(t) \) is in state \( z \in \mathcal{Z} \), it can only transition to states \( z + e_{\iota,i'} \in \mathcal{Z} \) with \( \iota = \iota' \), where \( e_{\iota,i'} \in \mathbb{R}^N \) is a vector with \( \iota \)th element set to be \( +1/hI \), \( \iota' \)th element set to be \( -1/hI \) and all the other elements set to be zero. For our birth-and-death process, a transition will only happen with \( n_{i'} = n_i \pm 1 \) corresponding to the arrival and departure of a request, respectively.

We define \( \psi_t^{\phi,h}(z), \ \iota \in [N], \ z \in \mathcal{Z} \), as the probability that a sub-BP in state \( \iota \) is active and the corresponding sub-pattern is prepared to accept an incoming request, when the proportions of sub-BPs in all states are currently specified by \( z \). Let \( \psi^{\phi,h}(z) = (\psi_t^{\phi,h}(z) : \ \iota \in [N]) \). The variables \( \psi^{\phi,h}(z), z \in \mathcal{Z} \), provide sufficient information for a policy \( \phi \) to make decisions on the counting process \( N_{h}^\phi(t) \).

Let \( \zeta_t^{\phi,h}(z) \) represent the maximal proportion of sub-BPs in state \( \iota \) that can be active if we only consider the constraint defined by (3) (neglecting the constraint defined by (1)) with proportions of sub-BPs in all states specified by \( z \). Let \( \mathcal{A}_i^+, \ \iota \in [N], \) represent the set of states preceding state \( \iota \), we obtain for \( \iota = 1, 2, \ldots, N - L \),

\[
\zeta_t^{\phi^*,h}(z) = \min \left\{ z_{i, \max} \left\{ 0, \min_{j \in \mathcal{L}_i} \frac{1}{w_{j,i}} \left[ \sum_{\iota' = 1}^N w_{j,i} \psi_{t}^{\phi^*,h}(z) \right] \right\} \right\},
\]

and for \( z_{i} > 0 \),

\[
\psi_t^{\phi^*,h}(z) = \frac{1}{z_{i}} \min \left\{ \zeta_t^{\phi^*,h}(z), \ \max \left\{ 0, \frac{1}{hI} - \sum_{\iota' \in \mathcal{A}_i^+} \zeta_t^{\phi^*,h}(z) \right\} \right\},
\]

(17)

(18)
where we recall that $\varphi^*$ is the index policy with indices $\nu^*$ defined in Definition 1. For $z_i = 0$, let $z^*_i = (z_1, z_2, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_N)$,

$$v_i^{\varphi^*,h}(z) = \lim_{x \downarrow 0} v_i^{\varphi^*,h}(z^*). \quad \text{(19)}$$

For a given $z$, $\zeta^{\varphi^*,h}(z)$ and $v_i^{\varphi^*,h}(z)$ can be calculated iteratively from $\iota = 1$ to $\sum_{i \in [I]} (|N_i^0| - 1)$. Let $q_0^{i,h}(\iota, \iota', z)$ and $q_1^{i,h}(\iota, \iota', z)$ represent the transition rates at which a sub-BP in state $\iota$ moves to a state $\iota'$ when passive and active, respectively, where the proportions of sub-BPs in each state are specified by $z \in \mathcal{Z}$. Then the transition rate of a pattern in state $\iota$ transitioning to state $\iota'$ under policy $\phi$ is

$$q^{\phi,h}(\iota, \iota', z) = v_i^{\phi,h}(z)q_1^{i,h}(\iota, \iota', z) + (1 - v_i^{\phi,h}(z))q_0^{i,h}(\iota, \iota', z). \quad \text{(20)}$$

Consider a deterministic process $z^{\phi,h}(t) = (z^{\phi,h}_i(t) : \iota \in [N])$ with $z^{\phi,h}(0) = z_0$ satisfying the differential equation

$$\frac{d}{dt} z_i^{\phi,h}(t) = \frac{1}{hI} \sum_{\iota' \in [N]} \left[ q^{\phi,h}(\iota', \iota, z^{\phi,h}(t)) - q^{\phi,h}(\iota, \iota', z^{\phi,h}(t)) \right] := y_i^{\phi,h}(z^{\phi,h}(t)), \quad \text{(21)}$$

where $y_i^{\phi,h}(z^{\phi,h}(t))$ is a function of $z^{\phi,h}(t)$.

**Definition 2.** Consider the matrix $W = (w_{j,i})$. We say that row $j \in [J]$ is

1) a first-style row if there is at most one $i \in [I]$ with $w_{j,i} > 0$;
2) a second-style row if there is more than one $i \in [I]$ with $w_{j,i} > 0$.

That is, row $j$ is a first-style row if resource class $j$ is not shared by requests of different types; and is a second-style row, otherwise. Recall that the set of resource classes used by pattern $i$ is $J_i = \{j \in [J] \mid w_{j,i} > 0\}$. We then define two conditions:

(a) the blocking probability is positive in the asymptotic regime;
(b) for any $i$, there is at most one $j \in J_i$ with row $j$ being a second-style row;

Condition (a) implies that the offered traffic for the entire system is heavy. Condition (b) implies that there is at most one shared resource class associated with each pattern. In other words, under Condition (b), if pattern $i_1$ shares a resource class $j_{12}$ with pattern $i_2$ and pattern $i_1$ shares a resource class $j_{13}$ with pattern $i_3$ then $j_{12} = j_{13}$. Condition (b) clearly holds when each of the patterns in the system requires only one resource class. For general cases with more than one
Proposition 2.

1) For a given \( z \in \mathcal{Z} \), if Conditions (a) and (b) hold, then there exists an index policy \( \varphi \) and an optimal solution \( \text{OPT} \) for the relaxed problem defined by (4), (6) and (7), such that

\[
\left. \frac{dz^\varphi,h(t)}{dt} \right|_{z^\varphi,h(t)=z} = \left. \frac{dz^{\text{OPT},h}(t)}{dt} \right|_{z^{\text{OPT},h}(t)=z}.
\]

2) For a given \( z \in \mathcal{Z} \), if Conditions (a) and (b) hold, and \( w_{j,i} = w_i \) for any \( j \in \mathcal{J}, i \in [I] \), then there exists a \( \nu^* \) and an optimal solution \( \text{OPT} \) for the relaxed problem defined by (4), (6) and (7), such that

\[
\left. \frac{dz^{\nu^*,h}(t)}{dt} \right|_{z^{\nu^*,h}(t)=z} = \left. \frac{dz^{\text{OPT},h}(t)}{dt} \right|_{z^{\text{OPT},h}(t)=z}.
\]

In particular, this \( \nu^* \) is given by, for \( i \in \mathcal{D}, \ell \in [L], n \in \mathcal{N}_i^0 \)

\[
\nu^*_{i,n} = \frac{\nu^*_i(0)}{w_i(1 + \lambda_{\ell}/\mu_{\ell,i})}.
\]

Proof. The proof for Proposition 2 is given in Appendix B.

Proposition 3. For any \( \delta > 0 \), if \( Z^{\nu^*,h}(0) = z^{\nu^*,h}(0) \),

\[
\lim_{h \to +\infty} \lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}\{|Z^{\nu^*,h}(u) - Z^{\nu^*,h}(u)| > \delta\} du = 0.
\]

Proof. The justification of this proposition is given in Appendix C.

For a state \( \iota \), let \( \iota^+ \) represent the state with \( n_{\iota^+} = n_\iota + 1 \) and \( i_\iota = i_{\iota^+} \). For \( z \in \mathcal{Z} \), we define a neighborhood of \( z \) as

\[
\Theta^h(z) = \left\{ z' \in \mathcal{Z} \left| z' = z + \sum_{i \in [I]} a_i e_{i,i^+} \text{ for all } a_i \in [0, 1], \ i_\iota \right. \right\},
\]

where \( e_{i,i^+} \) is dependent on \( h \), as defined before, and for \( \ell \in [L] \)

\[
\nu^*_\ell(z) = \lim_{h \to +\infty} \min_{\iota \in [N]} \left\{ \iota \in [N] \left| \ell_i = \ell, \sum_{\iota' \in [N], \ell_{\iota'} = \ell} \zeta_{\iota'}(z) \geq \frac{1}{hI} \right. \right\}.
\]

Proposition 4. There exists a unique \( z^{\nu^*} \in \mathcal{Z} \) with a \( z^h \in \Theta^h(z^{\nu^*}) \) satisfying

1)\[
\lim_{h \to +\infty} \left. \frac{dz^{\nu^*,h}(t)}{dt} \right|_{z^{\nu^*,h}(t)=z^h} = 0;
\]
and

2) for $t < t^*_h(z^\varphi^*)$ and any $z^p = pz^\varphi^* + (1 - p)z^h$ with $p \in (0, 1]$, 

$$
\lim_{h \to +\infty} \frac{d}{dt} y^\varphi^*,h(t) \bigg|_{z^\varphi^*,h(t) = z^p} = \lim_{h \to +\infty} y^\varphi^*,h(z) \begin{cases} < 0 & \text{if } z^p > z^h, \\
> 0 & \text{if } z^p < z^h, \\
= 0 & \text{otherwise.} \end{cases}
$$

(28)

Proof. The proof of this proposition is given in Appendix D. \qed

Theorem 1. If Conditions (a) and (b) hold, then there exists an index policy that is asymptotically optimal. In particular, if Conditions (a) and (b) hold and $w_{j,i} = w_{j',i}$ for all $j \in J_i$, $i \in [I]$, then $\varphi^*$ with indices given in “closed form” as defined in (24) is asymptotically optimal.

Proof. If $\lambda^0$ and $\mu_{\ell,i}$, $\ell \in [I]$, $i \in [I]$, are rational numbers, then all the elements of $z^\varphi^*$ are rational numbers. For such a $z^\varphi^*$, there exists $h_0 \in \mathbb{N}_+$ such that $z^\varphi^* h_0 \in \mathbb{N}_+$ for all $t \in [N]$. In other words, $z^\varphi^*$ becomes a possible value of $Z^\varphi^*,h(t)$ when $h = mh_0$, $m \in \mathbb{N}_+$. In this context, we set that $Z^\varphi^*,h(0) = z^\varphi^*,h(0) = z^\varphi^*$, and, from Propositions 3 and 4, we obtain

$$
\lim_{h \to +\infty} \lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P} \{ |Z^\varphi^*,h(u) - z^\varphi^*| > \delta \} du = 0.
$$

This conclusion also applies to the case with non-rational element(s) of $z^\varphi^*$ by picking a rational number in $\Theta^h(z^\varphi^*)$, denoted by $\tilde{z}^\varphi^*$, and setting the initial states $Z^\varphi^*,h(0)$ and $z^\varphi^*,h(0)$ to be $\tilde{z}^\varphi^*$.

As a consequence, from Propositions 2 and 4, we can obtain the asymptotic optimality of $\varphi^*$ as $h \to +\infty$ if Conditions (a) and (b) hold. In particular, if Conditions (a) and (b) hold and $w_{j,i} = w_{j',i}$ for any $j, j' \in J_i$, $i \in [I]$, the indices $\nu^*$ for this asymptotically optimal policy $\varphi^*$ are given in closed form as defined in (24). \qed

VI. A NUMERICAL EXAMPLE

In Example 2 below, Conditions (a) and (b) are satisfied. The numbers were generated randomly.

Example 2. Consider a system with six different resources ($J = 6$) and requests of six different types ($L = 6$). Note that we neglect actions and states of the dummy pattern in this example for convenient presentation.

For the RTs, let
The capacities and cost rates for unit resources of different classes are presented in Table I.

| Capacity $C_j$ | 2   | 2   | 4   | 7   | 4   | 3   |
|----------------|-----|-----|-----|-----|-----|-----|
| Cost Rate $ε_j$ | 2.81056 | 0.912125 | 3.05382 | 0.450702 | 8.06337 | 7.31632 |

- $P_1 = \{1, 2\}, J_1 = \{1\}, J_2 = \{4\}, λ_1 = 8.13197, \mu_{1,1} = \mu_{1,2} = 0.406598, w_1 = 1, r_1 = 1463.06$;
- $P_2 = \{3, 4, 5\}, J_3 = \{2\}, J_4 = \{3\}, J_5 = \{6\}, λ_2 = 8.13197, \mu_{2,3} = \mu_{2,4} = \mu_{2,5} = 0.406598, w_2 = 1, r_2 = 1078.92$;
- $P_3 = \{6, 7\}, J_6 = \{1\}, J_7 = \{4\}, λ_3 = 9.97059, \mu_{3,6} = \mu_{3,7} = 0.498529, w_3 = 1, r_3 = 1199.98$;
- $P_4 = \{8\}, J_8 = \{5\}, λ_4 = 9.97059, \mu_{4,8} = 0.412531, w_4 = 1, r_4 = 1360.74$;
- $P_5 = \{9\}, J_9 = \{2\}, λ_5 = 9.23649, \mu_{5,9} = 0.461825, w_5 = 1, r_5 = 1320.73$;
- and $P_6 = \{10\}, J_{10} = \{5\}, λ_6 = 9.23649, \mu_{6,10} = 0.461825, w_6 = 1, r_6 = 1482.43$.

The capacities and cost rates for unit resources of the six classes are presented in Table I.

Let OPT represent the optimal solution of the relaxed problem satisfying (4), (6) and (7) in...
asymptotic regime. The average revenue under OPT is then an upper bound of the maximized average revenue of the original problem defined by (4), (1) and (3). To compare $\varphi^*$ and OPT, we define the approximation ratio of $\varphi^*$ as the ratio of the average revenue under $\varphi^*$ to that for OPT, and its normalized deviation as the deviation of average revenue between OPT and $\varphi^*$ divided by the average revenue under OPT. In Figures 1(a) and 1(b), we plot the approximation ratio and the normalized deviation of $\varphi^*$ against the scaling parameter $h$, respectively, for the system described in Example 2. Both figures indicate that the average revenues under the index policy $\varphi^*$ and under OPT are close, even when $h$ is relatively small, and those under $\varphi^*$ approach those under OPT with increasing $h$. The observation is consistent with the asymptotic optimality of $\varphi^*$.

The confidence intervals of the simulated average revenues at the 95% level based on the Student’s t-distribution are maintained within 3% of the observed mean.

VII. CONCLUSIONS

We have modeled a resource allocation problem, described by (4), (1) and (3), as a combination of various RMABPs coupled by limited capacities of resource classes, in which the resource classes are shared, competed for and reserved by various requests. It is an optimization problem in a stochastic system aiming to maximize the long-run average revenue by dynamically accommodating requests onto appropriate resource classes. With the ideas of Whittle relaxation [1] and the asymptotic optimality proof of Weber and Weiss [3], we have proposed a resource allocation policy based on state-dependent indices and proved its asymptotic optimality under Conditions (a) and (b) (Theorem 1). To the best of our knowledge, no existing work provides proved asymptotic optimality for a resource allocation problem where dynamic allocation, competition and reservation of resources are potentially permitted. Unlike the proof of Weber and Weiss [3], to match engineering applications, the asymptotic regime of our problem has been defined by scaling the average arrival rates for RTs (the rates of transitions triggered by arrivals of requests for the BPs) to infinity, in proportion to the total number of BPs, with a fixed number (which is set to be one) of active BP(s) for each RMABP.

APPENDIX A

OPTIMIZATION OF THE RELAXED PROBLEM

We now provide the proof of Proposition 1.
where

\( n \in \mathbb{N} \)

rate from \( \alpha \) Equation (29) can be rewritten as \( \alpha \) is no difference between making \( \phi \) number of instantiations generated for pattern \( i \). For any \( \phi \in \Phi_i \) and \( 0 \leq n < n' \leq |\mathcal{N}_i| - 1 \), if \( \alpha_\phi^i(n) = 1 \), there is a positive transition rate from state \( n \) to state \( n + 1 \); otherwise, the transition rate from \( n \) to \( n + 1 \) is zero. The transition rate from \( n \) to \( n - 1 \), if \( n \geq 1 \), is always positive and independent of the policy employed.

We then start our proof. For any \( \phi \in \Phi_i \) and \( 0 \leq n < n' \leq |\mathcal{N}_i| - 1 \), if \( \alpha_\phi^i(n) = 0 \) then there is no difference between making \( \alpha_\phi^i(n') = 0 \) or 1, because we cannot reach state \( n' \) if we start below state \( n \). For any \( \phi \in \Phi_i \) and \( 0 \leq n < n' \leq |\mathcal{N}_i| - 1 \), we assume without loss of generality that if \( \alpha_\phi^i(n) = 0 \) then \( \alpha_\phi^i(n') = 0 \). For \( \phi \in \Phi_i \), we define

\[
m^\phi = \begin{cases} 
0, & \text{if } a_\phi^i(0) = 0, \\
m, & \text{if } a_\phi^i(m - 1) = 1, a_\phi^i(m) = 0, m \in [|\mathcal{N}_i| - 1]. 
\end{cases}
\]

Let \( \pi_n^m \) represent the steady state probability of state \( n \in \mathcal{N}_i \) under policy \( \phi \) with \( m^\phi = m \).

For an optimal solution \( \phi^* \) of (10) among all policies in \( \Phi_i \) with \( i \in \mathcal{P}_t \), we obtain

\[
m^{\phi^*} \in \arg \max_{m \in [|\mathcal{N}_i| - 1]} \left\{ \frac{\pi_n^m}{\pi_0^m} \sum_{n=1}^{m} \frac{(\lambda_{\ell})^n}{n!(\mu_{\ell,i})^n} \left[ r_i(n, \gamma) - \frac{n \mu_{\ell,i}}{\lambda_{\ell}} \left( \nu_\ell + \sum_{j \in J_i} w_{j,i} \gamma_j \right) \right] \right\}
\]

(29)

where

\[
r_i(n, \gamma) = n \mu_{\ell} r_{\ell} - \sum_{j \in J_i} \varepsilon_j w_{j,i} n - \sum_{j \in J_i} \gamma_j w_{j,i} n.
\]

Equation (29) can be rewritten as

\[
m^{\phi^*} \in \arg \max_{m} \left\{ \frac{(\lambda_{\ell})}{\sum_{n=0}^{m-1} \frac{(\lambda_{\ell})^n}{n!(\mu_{\ell,i})^n}} - \frac{(\lambda_{\ell})}{\sum_{n=0}^{m} \frac{(\lambda_{\ell})^n}{n!(\mu_{\ell,i})^n}} \right\}
\]

(30)

where

\[
\tilde{r}_i = r_{\ell} - \frac{1}{\mu_{\ell,i}} \sum_{j \in J_i} \varepsilon_j w_{j,i} - \frac{1}{\mu_{\ell,i}} \sum_{j \in J_i} \gamma_j w_{j,i} - \frac{1}{\lambda_{\ell}} \sum_{j \in J_i} \gamma_j w_{j,i}.
\]

Note that the right hand side of (30) may return a set with more than one element: the optimal value for \( m^{\phi^*} \) is not necessarily unique. If this is the case, we choose any of the possible maxima. From (30), if \( \nu_\ell < \lambda_{\ell} \tilde{r}_i \), then \( m^{\phi^*} = |\mathcal{N}_i| - 1 \); if \( \nu_\ell = \lambda_{\ell} \tilde{r}_i \), then \( m^{\phi^*} \) can be any value in \( \mathcal{N}_i \); otherwise, \( m^{\phi^*} = 0 \). We thus interpret \( \phi^* \) as an index policy where indices for states \( n \in \mathcal{N}_i \backslash \{ |\mathcal{N}_i| - 1 \} \) are given by \( \lambda_{\ell} \tilde{r}_i \).

This proves Proposition 1.

\( \square \)
APPENDIX B

PROOF OF PROPOSITION 2

Before proving it, we discuss the structure of an optimal policy for the problem presented in equations (4), (6) and (7). Here we consider a case where Conditions (a) and (b) are satisfied. For a given vector of BP proportions \( z \in \mathcal{Z} \) under policy \( \phi \), we define the set of states with fully activated BPs (sub-patterns) as

\[
\mathcal{F}_1^{\phi,h}(z) = \{ \iota \in [N] \mid \upsilon_{\iota}^{\phi,h}(z) = 1 \},
\]

the set of states with partially activated BPs as

\[
\mathcal{F}_{<1}^{\phi,h}(z) = \{ \iota \in [N] \mid 0 < \upsilon_{\iota}^{\phi,h}(z) < 1 \},
\]

and the set of states with passive BPs as

\[
\mathcal{F}_0^{\phi,h}(z) = \{ \iota \in [N] \mid \upsilon_{\iota}^{\phi,h}(z) = 0 \}.
\]

Let

\[
\Delta_i \begin{cases} 
\geq 0 & \text{if } \iota \in \mathcal{F}_1^{\phi^*,h}(z), \\
= 0 & \text{if } \iota \in \mathcal{F}_{<1}^{\phi^*,h}(z), \\
\leq 0 & \text{if } \iota \in \mathcal{F}_0^{\phi^*,h}(z).
\end{cases}
\]

We define conditions:

(c) for an equilibrium vector of BP proportions \( z \in \mathcal{Z} \), there exist \( \nu \in \mathbb{R}^L, \gamma \in \mathbb{R}_{+}^J \), and \( \Delta_i \), such that for all \( i, \iota = i, i \in \mathcal{P}_i, \ell \in [L], (31) \) holds, and

\[
\nu_{\ell} + \left( 1 + \frac{\lambda_{\ell}}{\mu_{\ell,i}} \right) \sum_{j \in \mathcal{F}_i} w_{j,i} \gamma_j + \Delta_i = \nu_{\iota}^{*}(0);
\]

(d) the complementary slackness conditions for the dual problem in (12) and the primary problem defined by (4), (6) and (7) hold.

If Conditions (c) and (d) hold, then the policy \( \varphi^* \) specified by (18) is optimal for the primary problem presented in equations (4), (6) and (7); that is, the strong duality holds for the dual and primary problems.

If a state \( \iota \in \mathcal{F}_{<1}^{\phi^*,h}(z) \cup \mathcal{F}_0^{\phi^*,h}(z) \), then \( \upsilon_{\iota}^{\phi^*,h}(z) \) is restricted by either the capacity limit of a resource class (Constraint (7)) or the maximal proportion of active patterns for the same RT.
(Constraint (6)). In the former case, we refer to this resource class that prevents higher activation of patterns in or after state \( \iota \) as the critical resource of state \( \iota \), denoted by \( j_\iota \).

We construct an \( N \times N \) matrix \( \mathcal{M} = (m_{i,j}) \) and write the linear functions specified by (32) as a linear equation \( \mathcal{M} \mathbf{x} = \mathbf{y} \), \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^N \), where we choose \( y_\iota = \nu^*_\iota(0) \). Our aim is to solve for \( \mathbf{x} \) where the entries \( x_\iota, \iota \in [N] \), and the matrix \( \mathcal{M} \) are defined by:

(i) if state \( \iota \in \mathcal{F}^{\rho^*,h}_1(z) \), then \( x_\iota = \Delta_\iota \), \( m_{\iota,\iota} = 1 \) and \( m_{\iota',\iota} = 0 \), \( \forall \iota' \neq \iota, \iota' \in [N] \);

(ii) if state \( \iota \in \mathcal{F}^{\rho^*,h}_0(z) \cup \mathcal{F}^{\rho^*,h}_1(z) \) with \( \nu^*_\iota(0) > \nu_{\ell_\iota} \), then \( x_\iota = \gamma_{j_\iota} \) and \( m_{\iota',\iota'} = 1 \), \( m_{\ell_\iota,\iota'} = 0 \) for all \( \iota' \in [N] \);

(iii) if state \( \iota = \min\{ \iota' \in \mathcal{F}^{\rho^*,h}_0(z) \cup \mathcal{F}^{\rho^*,h}_1(z) \mid \nu^*_{\iota'}(0) = \nu_{\ell_\iota} \} \), then \( x_\iota = \nu_{\ell_\iota} \), and \( m_{\iota',\iota} = 1 \) if \( \ell_\iota' = \ell_\iota \); \( m_{\iota',\iota} = 0 \) otherwise for all \( \iota' \in [N] \);

(iv) and for all the other states \( \iota \in \mathcal{F}^{\rho^*,h}_0 \) with \( \nu^*_{\iota}(0) \leq \nu_{\ell_\iota} \), \( x_\iota = -\Delta_\iota \), entry \( m_{\iota,\iota} = -1 \) and \( m_{\iota',\iota'} = 0 \), \( \forall \iota' \neq \iota, \iota' \in [N] \).

From the condition of complementary slackeness, we set all the other \( \gamma_j \) that are not involved in case (ii) to be zero. Consequently, equations (32) in Condition (c), for all \( \iota \in [N] \), are expressed as \( \mathcal{M} \mathbf{x} = \mathbf{y} \).

We now provide the proof of Proposition 2.

**Proof.** We provide in this proof a specified \( \nu^* \) and a solution for linear functions \( \mathcal{M} \mathbf{x} = \mathbf{y} \) which satisfy Conditions (c) and (d), when Conditions (a) and (b) hold. Under Condition (b), for each \( i \) and \( j \) with \( w_{j,i} > 0 \), if row \( j \) is the only second-style row for \( i \), we refer to this special \( j \) for \( i \) as \( j_\iota \). We now show that \( \nu^* \) is given by

\[
\nu^*_{i,n} = \begin{cases} 
\frac{\nu^*_i(0)}{w_{j,i}(1 + \frac{\lambda_{ji}}{\mu_{ji})}} & \text{if for all } j \text{ with } w_{j,i} > 0, \text{ row } j \text{ of matrix } \mathcal{W} \text{ is a first-style row.} \\
\frac{\nu^*_i(0)}{w_{j_i,i}(1 + \frac{\lambda_{ji}}{\mu_{ji})}} & \text{otherwise.} 
\end{cases}
\]

In particular, if \( w_{j,i} = w_i \) for all \( j \in \mathcal{F}_i, i \in [I] \), such \( \nu^* \) satisfies (24). For a state \( n \in \mathcal{A}^{(0)}_i \backslash \{|\mathcal{A}^{(0)}_i| - 1\} \) labeled by \( \iota \), we define \( \nu^*_{i,n} := \nu^*_{i,n} \).

We assume the heavy traffic condition (Condition (a)); That is, the dummy patterns \( i_\ell^0 \) for \( \ell \in [L] \) are activated with positive probability, \( \nu_1 = \nu_2 = \ldots = \nu_L = \nu^*_\iota(\cdot) = 0 \). As a consequence, there exist \( \Delta_{N-I-L+1}, \Delta_{N-I-L+2}, \ldots, \Delta_{N-I} \) satisfying (31) and (32), where we recall that states from \( N-I-L+1 \) to \( N-I \) are the states for the \( L \) dummy patterns. Also, for
given $\gamma$ and $\iota = N - I + 1, N - I + 2, \ldots, N$, there always exists $\Delta_\iota$ satisfying (31) and (32). It remains to discuss the functions in $M\mathbf{x} = \mathbf{y}$ associated with states $\iota = 1, 2, \ldots, N - I - L$.

For a given $i$, $i \in [I]$, states $\iota$ with $n_\iota \in N_i \{N_i - 1\}$ share the same linear function (32), when we set $\Delta_i = \Delta_i$ for all such $\iota$. In this context, when we consider a solution for $M\mathbf{x} = \mathbf{y}$ satisfying (32), we only need to consider one of these identical linear functions for states $\iota$ with $n_\iota \in N_i \{N_i - 1\}$ associated with one row and column of $M$. Thus, we remove the other $|N_i| - 2$ linear functions by removing corresponding rows and columns of $M$ and elements of $\mathbf{x}$ and $\mathbf{y}$. We represent the state associated with the remaining linear function for pattern $i$ by $i_\iota$.

For $j \in [J]$, let $\mathcal{I}_j = \{i \in [I] \mid j \in \mathcal{I}_i\}$. Under Condition (b), if $|\mathcal{I}_j| \geq 2$ and (32) holds for $i \in \mathcal{I}_{\iota}^{*, h} \cap N_i$, $\Delta_i = 0$, $i \in \mathcal{I}_j$, then for $i' \in N_i$, $i \neq i'$, we obtain $i' \in \mathcal{I}_{\iota}^{*, h}$ and there exists $\Delta_i \leq 0$, so that (32) also holds for $i'$. For all such $i'$, we then remove the $i'$th row and column from $M$.

Removing these unnecessary rows and columns mentioned above, we reformulate $M\mathbf{x} = \mathbf{y}$ as $\tilde{M}\mathbf{x} = \tilde{\mathbf{y}}$ where $\tilde{M} \in \mathbb{R}^{I \times I'}$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbb{R}^{I'}$.

If $|\mathcal{I}_i| = 1$ for all $i \in [I]$, then there exists $m \in \mathbb{N}_+$, $m_k \in \mathbb{N}_+$, $k \in [m+1]$, and $M_k \in \mathbb{R}^{m_k \times m_k}$ such that $\tilde{M} = \text{diag}(M_k : k \in [m+1])$ with appropriately re-ordered columns of $\tilde{M}$. In particular, $m$ is equal to the number of critical resources and $\sum_{k \in [m+1]} m_k = I$. The rows in $M_k$, $k \in [m]$, are associated with pattern $i$ requiring the $k$th critical resource, denoted by $j_k$, satisfying $j_k \in \mathcal{I}_i$.

For the general case Condition (b) holding, in a similar way, $\tilde{M}$ is an upper triangular matrix with the just defined $M_k$, $k \in [m+1]$, as its diagonal. There is at most one critical component $j_k$ restricting a higher activating probability of sub-patterns identical to pattern $i$, which is associated with a row of $M_k$, $k \in [m]$, referred to as row $\tilde{i}$ of matrix $\tilde{M}$, and at most one other critical resource $j_{k'}$, $k \neq k'$, satisfying $j_{k'} \in \mathcal{I}_i$.

It there are such $i, k, \tilde{i}, k'$, we perform an elementary row operation on $[\tilde{M}|\tilde{y}]$: replacing row $\tilde{i}$ by the difference between row $\tilde{i}$ and row $\tilde{i}'$ of $\tilde{M}$ where $\tilde{i}' = \sum_{k'' \in [k']} m_{k''}$, while replace the $\tilde{i}$th element of $\tilde{y}$ by the difference between this element and the $\tilde{i}'$th element of $\tilde{y}$. The corresponding $\gamma_{jk}$ for row $\tilde{i}$ (associated with pattern $i$) is thus equivalent to $rac{w_{jk} - d(\nu_{i_{\tilde{i}'}} - \nu_{i_{\tilde{i}'}})}{w_{jk}}$, where $i_{\tilde{i}'}$ is the pattern corresponding to row $\tilde{i}'$. Recall that $\nu_{i_{\tilde{i}'}} = \nu_{i_{\tilde{i}'}}$ for any $i_{\tilde{i}'}, i_{\tilde{i}'}$ according to (33). Here, since $\nu_{i_{\tilde{i}'}} - \nu_{i_{\tilde{i}'}} \geq 0$, we have shown that (32) and (31) are satisfied for $\iota$ with $n_\iota \in N_i \{N_i - 1\}$. After this operation, we remove row and column $\tilde{i}$ from $\tilde{M}$ and the $\tilde{i}$th
elements from \( \tilde{x} \) and \( \tilde{y} \) correspondingly. With some abuse of notation, we refer to the resulting matrix and vectors as \( \tilde{M} \), \( \tilde{x} \) and \( \tilde{y} \). As a consequence, when all such rows and columns are removed from \( M \), we write the remaining matrix as \( \tilde{M} = \text{diag}(M_k : k \in [m + 1]) \). In this way, each of the remaining columns in \( \tilde{M} \) is associated with a resource class \( j \) with row \( j \) of matrix \( W \) being a second-style row.

The matrix can be analysed by Cramer’s rule. Let \( \ell(i) \), \( i \in [I] \), represent the \( \ell \in [L] \) satisfying \( i \in P_\ell \) and \( i^*_k \) represent the pattern associated with row \( m_k \) of \( M_k \). We obtain, for \( \ell = \ell(i^*_k) \),

\[
|\tilde{M}| = \prod_{k=1}^{m} |M_k| = \prod_{k=1}^{m} \left[ w_{j^*_k,i^*_k} \left( \frac{\lambda_{\ell}}{\mu_{\ell,i^*_k}} + 1 \right) \right],
\]

where \( j^*_k \) is in fact the \( k \)th critical component associated with \( M_k \).

Let \( \tilde{M}_\tilde{i}, \tilde{i} \in [\sum_{k=1}^{m} m_k] \), be the matrix after replacing its column \( \tilde{i} \) by \( \tilde{y} \). For a column \( \tilde{i} \), \( \tilde{i} \in [\sum_{k=1}^{k} m_{k'}] \setminus [\sum_{k'=1}^{k-1} m_{k'}] \) associated with pattern \( i \in P_\ell, \ell \in [L] \),

\[
|\tilde{M}_\tilde{i}| = \prod_{k'=1}^{m} |M_{k'}| \times \begin{cases} 
\left( w_{j_i,i} \left( \frac{\lambda_{i,i}}{\mu_{i,i}} + 1 \right) \nu_i^* \right. & \text{for } i \in \mathcal{N}_{\tilde{i}}^0 \setminus \{ \mathcal{N}_{\tilde{i}}^0 - 1 \}, i \in [I] \\
\left. w_{j_i,i} \left( \frac{\lambda_{i,i}}{\mu_{i,i}} + 1 \right) \nu_i^* \right. & \text{otherwise,}
\end{cases}
\]

where \( \Delta_i \geq 0 \) and \( \nu_i^* = \nu_{i^*_k}^* \) for any \( i^*_k \) with \( n_{i^*_k} \in \mathcal{N}_{\tilde{i}}^0 \setminus \{ \mathcal{N}_{\tilde{i}}^0 - 1 \}, i \in [I] \).

There always exists a sequence of \( \Delta_i = (\nu_i^* - \nu_{i^*_k}^*)w_{j_i,i}(\lambda_{i,i}/\mu_{i,i},i + 1) \), yielding \( |\tilde{M}_\tilde{i}| \geq 0 \) for all \( \tilde{i} \in [\sum_{k=1}^{m} m_k] \).

From Cramer’s rule, for \( j = j_1, j_2, \ldots, j_m \) and \( \gamma_j = \nu_{i^*_k}^* \), \( k \in [m] \), the \( \nu^* \) defined in (33) leads to a solution \( x \) for linear functions \( Mx = y \) which satisfy Conditions (c) and (d). Hence, the policy \( \varphi^* \) specified by (18) is optimal for the primary problem presented in equations (4), (6) and (7). Namely, for a given \( z \in \mathcal{Z} \), there exists an optimal solution OPT of this primary problem, which satisfies \( i^*_k \varphi^*,h(z) = i^*_k \text{OPT},h(z) \). Substituting (18) into (21), we obtain (23). This proves the existence of an index policy satisfying (23) as the first claim of the proposition. When \( w_{j,i} = w_i \) for all \( j \in J_i, i \in [I] \), (33) reduces to (24), which proves the second claim of the proposition.
APPENDIX C

PROOF OF PROPOSITION 3.

Following the ideas and methods of [3] and [25], we provide a proof of Proposition 3.

Proof. We firstly construct a stochastic process that matches the hypothesis of [25, Chapter 7, Theorem 2.1].

Let \( t^h_{\ell,k} \) and \( t^h_{\ell+i,k} \) be the times of \( k \)th arrival of RT \( \ell, \ell \in [L] \), and \( k \)th departure of instantiations with pattern \( ([i(i - 1)/h] + 1, i - h ([i(i - 1)/h])) \), \( i \in [hI], k \in \mathbb{N}_+ \), respectively. For our network system, the inter-arrival and inter-departure times are positive with probability one and, also with probability one, any two events will not occur at the same time.

Define a random vector \( \xi^h = (\xi^h_{s,t} : s \in [L + hI]) \) as follows: for \( s \in [L + hI] \),

\[
\xi^h_{s,t} = \begin{cases} 
\frac{1}{t^h_{s,k} - t^h_{s,k'}}, & \text{if } \exists h_{s,k} = \min_{k' \in \mathbb{N}_+} \{ t^h_{s,k'} | t^h_{s,k'} > t \}, \exists h_{s',k'} = \max_{k'' \in [L + hI]} \{ t^h_{s',k''} | t^h_{s',k''} < t^h_{s,k'} \}, \\
0, & \text{otherwise.}
\end{cases}
\]

Then, for \( s = 1, 2, \ldots, L \), the number \( \lfloor \int_0^T \xi^h_{s,t} dt \rfloor \) represents the number of arrivals of RT \( s \) by time \( T \); and for \( s = L + 1, L + 2, \ldots, L + hI \), the number \( \lfloor \int_0^T \xi^h_{s,t} dt \rfloor \) is the number of departures associated with pattern \( s - L \) by time \( T \).

We define a function, \( Q^h(\ell, \ell', x, \xi^h) \), on \( \ell, \ell' \in [N], x \in \mathbb{R}^N, \xi^h \in \mathbb{R}^{L+hI} \). For \( z \in \mathcal{Z}, \ell \in [N] \), let \( a_\ell(z) = \lim_{h \to +\infty} v^\ell z \xi^h z \), \( hI \). Note that this limit exists for the index policy \( \varphi^* \in \Phi^1 \) because of (18). For given \( \xi^h \in \mathbb{R}^{L+hI} \) and \( x \in \mathbb{R}^N \), \( Q^h(\ell, \ell', x, \xi^h) \) is defined by

\[
Q^h(\ell, \ell', x, \xi^h) = \begin{cases} 
a_\ell(x) \xi^h_{i_\ell} + f^{0,h}_{i_\ell,a}(x, \xi^h), & \text{if } i_\ell = i_{\ell'}, n_\ell = n_{\ell'} - 1, \\
\sum_{s = [x^-_\ell] + 1}^{[x^+_\ell]} \xi^h_{s + L} + f^h_{i_\ell,a}(x, \xi^h), & \text{if } i_\ell = i_{\ell'}, n_\ell = n_{\ell'} + 1, \\
0, & \text{otherwise}, \end{cases}
\]

where \( x^-_\ell = \sum_{k=1}^\ell x_k \) with \( x^-_0 = 0 \), and \( f^{0,h}_{i_\ell,a}(z, \xi^h) \) and \( f^h_{i_\ell,a}(z, \xi^h) \) are appropriate functions to make \( Q^h(\ell, \ell', x, \xi^h) \) Lipschitz continuous in \( x \) for all given \( \xi^h \) and \( 0 < a < 1 \). For \( x \in \mathbb{R}^N \setminus \mathbb{R}^N_0 \), let \( (x)_+ = ((x)_i)_i \) : \( i \in [N] \), we define \( Q^h(\ell, \ell', x, \xi^h) = Q^h(\ell, \ell', (x)_+, \xi^h) \).

If we set \( f^{0,h}_{i_\ell,a}(z, \xi^h) \) and \( f^h_{i_\ell,a}(z, \xi^h) \) to be zero all the time, then \( Q^h(\cdot, \cdot, x, \cdot) \) is a discontinuous function of \( x \). The idea to make the function smooth in \( x \) is to utilize the fact that the Dirac delta function can be considered as a limit case of a zero-centered normal distribution.
To keep $Q^h(\cdot, \cdot, \cdot, x, \cdot)$ smooth in $x \in \mathbb{R}_0^N$, functions $f_{t,a}^{0,h}(z, \xi_h)$ and $f_{t,a}^{1,h}(z, \xi_h)$ can be constructed by: for $i \in [N]$, let $\Gamma_i(x) = x_i \lim_{h \to +\infty} \xi_i^x \eta^h(x/hI)$ and $\chi_i^h(x) = \sum_{i'' \in N^+} \ell_i'' = \ell_i, \Gamma_i(x) > 0$, with $0 < a < 1$, we define a function of $x \in [0, 1]$ as

$$y_a(x) = \begin{cases} \int_{-\infty}^{-\cot(\pi x)} \frac{1}{\sqrt{\pi}} e^{-(u-\frac{1}{a})^2/a^2} du & \text{if } x \in (0, 1) \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1, \end{cases}$$

where $\cot(\cdot)$ is the cotangent function. This $y_a(x)$ is continuous in $x \in (0, 1)$, right-continuous in $x = 0$ and left continuous in $x = 1$. We then define

$$f_{t,a}^{0,h}(x, \xi_h) = \begin{cases} \xi_i^h y_a(1 - \chi_i^h(x)), & \text{if } \Gamma_i(x) > 0, \text{ and } 0 < \chi_i^h(x) < 1, \\ -\xi_i^h y_a(1 - \chi_i^h(x)), & \text{if } \chi_i^h(x) = 0, \text{ and } 0 < \chi_i^h(x) < 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $x_{i-1} < 0$ and $[x^-] < hI$,

$$f_{t,a}^{1,h}(x, \xi_h) = -\xi_i^h y_a(1 - [x_{i-1}] + x_{i-1}) + \xi_i^h y_a([x_{i-1}] - x_{i-1}) - \xi_i^h y_a([x_{i-1}] - x^-) + \xi_i^h y_a(1 - [x^-] + x^-),$$

for $x_{i-1} = 0$ and $[x^-] < hI$,

$$f_{t,a}^{1,h}(x, \xi_h) = -\xi_i^h y_a(1 - [x_{i-1}] + x_{i-1}) - \xi_i^h y_a([x_{i-1}] - x^-) + \xi_i^h y_a(1 - [x^-] + x^-),$$

for $x_{i-1} > 0$ and $[x^-] = hI$,

$$f_{t,a}^{1,h}(x, \xi_h) = -\xi_i^h y_a(1 - [x_{i-1}] + x_{i-1}) + \xi_i^h y_a([x_{i-1}] - x^-) - \xi_i^h y_a([x^-] - x^-),$$

and for $x_{i-1} = 0$ and $[x^-] = hI$

$$f_{t,a}^{1,h}(x, \xi_h) = -\xi_i^h y_a(1 - [x_{i-1}] + x_{i-1}) - \xi_i^h y_a([x^-] - x^-).$$

With these $f_{t,a}^{0,h}(z, \xi_h)$ and $f_{t,a}^{1,h}(z, \xi_h)$, $Q^h(t, t', x, \xi_h)$ is Lipschitz continuous in $x \in \mathbb{R}_0^N$ for all given $\xi_h \in \mathbb{R}^{L+hI}$ and $0 < a < 1$.

For the special case with $h = 1$, any given $0 < a < 1$, and $\epsilon > 0$, we define $X^\epsilon_t$ such that

$$\dot{X}^\epsilon_t := b(X^\epsilon_t, \xi^1_{t/\epsilon}) = \sum_{i' \in [N]} Q^1(t', t, X^\epsilon_t, \xi^1_{t'/\epsilon}) - Q^1(t, t', X^\epsilon_t, \xi^1_{t'/\epsilon}).$$
It follows that $b(X_t^\epsilon, \xi_{t/\epsilon})$ satisfies a Lipschitz condition over $X_t^\epsilon$ and $\xi_{t/\epsilon}$.

For any $x \in \mathbb{R}^N$, $h = 1$, $\delta > 0$, there exists $\overline{b}(x)$ satisfying
\[
\lim_{T \to +\infty} \mathbb{P} \left\{ \frac{1}{T} \int_{t}^{t+T} b(x, \xi_s^1) \, ds - \overline{b}(x) > \delta \right\} = 0,
\]
uniformly in $t > 0$. Let $\overline{x}(t)$ be the solution of $\dot{\overline{x}}(t) = \overline{b}(\overline{x}(t))$, $\overline{x}_0 = X_0^\epsilon$.

Now we invoke [25, Chapter 7, Theorem 2.1]: if (36) holds true, and $\mathbb{E} \left| b(x, \xi_t^1) \right|^2 < +\infty$ for all $x \in \mathbb{R}^N$, then, for any $T > 0$, $\delta > 0$,
\[
\lim_{\epsilon \to 0} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left| X_t^\epsilon - \overline{x}(t) \right| > \delta \right\} = 0. \tag{37}
\]
We scale the time line of the stochastic process by $\epsilon$. The effect of $\epsilon$ becoming smaller is to speed up time, so that the stochastic process $\{X_t^\epsilon\}$, driven by the random variable $\xi_{t/\epsilon}$, converges to the deterministic process $\{\overline{x}(t)\}$.

We interpret the scalar $\epsilon$ and the scaling effects in another way. For $x \in \mathbb{R}^N$ and $\xi^h \in \mathbb{R}^{L+hL}$, we define
\[
b^h(x, \xi^h) := \sum_{\iota' \in [N]} Q^{\phi,h}((\iota', i, x, \xi^h)) - Q^h((\iota', i, x, \xi^h)). \tag{38}
\]
If we set $\epsilon = 1/h$, then following the same technique as [24], for any $x \in \mathbb{R}^N$, $h > 0$ and $T > 0$, we observe that $\int_0^T b(x, \xi_{t/\epsilon}^1) dt$ and $\int_0^T (b(hx, \xi_t^1)/h) dt$ are identically distributed. We define $Z_0^h = Z_0^h := x_0/I$, and
\[
\dot{Z}_t^h := \frac{1}{hI} b(hI Z_t^h, \xi_t^1),
\]
and
\[
\dot{Z}_t^\epsilon := \frac{1}{I} b(I Z_t^\epsilon, \xi_{t/\epsilon}).
\]
From (37), for any $T > 0$ and $\delta > 0$, we obtain
\[
\lim_{h \to +\infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left| Z_t^h - \overline{x}(t)/I \right| > \delta \right\} = 0. \tag{39}
\]
Effectively then, scaling time by $\epsilon = 1/h$ is equivalent to scaling the system size by $h$.

Note that $\dot{Z}_t^h$ and $\dot{\overline{x}}(t)$ are dependent on a parameter $\alpha \in (0, 1)$ through functions $f_{i,\alpha}^0(x, \xi^h)$ and $f_{i,\alpha}^1(x, \xi^h)$ that are used in definition (34). Equation (39) holds true for any given $0 < a < 1$. Because of the Lipschitz behavior of $\dot{Z}_t^h$ and $\dot{\overline{x}}(t)$ on $0 < a < 1$, $\lim_{\alpha \to 0} \dot{Z}_t^h/\alpha = 0$ and $\lim_{\alpha \to 0} d\overline{x}(t)/d\alpha = 0$, equation (39) holds for the limit case where $\alpha \to 0$. Also, if $Z_0^h = Z^{\phi^*,h}(0)$, and $\overline{x}(0)/I = z^{\phi^*,h}(0)$, then $\lim_{\alpha \to 0} \overline{x}(t)/I = \lim_{h \to +\infty} z^{\phi^*,h}(t)$ and $\lim_{h \to +\infty} \lim_{\alpha \to 0} Z_t^h = \lim_{h \to +\infty} Z^{\phi^*,h}(t)$. Recall that $Z^{\phi^*,h}(t)$ is the vector of proportions of sub-BPs under policy
\(\varphi^*\) at time \(t\) and \(z^{\varphi^*,h}(t)\) is given by (21) (as defined in Section V). Then, for any \(T > 0\) and \(\delta > 0\),
\[
\lim_{h \to +\infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left| Z^{\varphi^*,h}(t) - z^{\varphi^*,h}(t) \right| > \delta \right\} = 0,
\]
which leads to (25). This proves the proposition.

**APPENDIX D**

**PROOF OF PROPOSITION 4**

**Proof.** For \(i \in \mathcal{N}_i, i' \in \mathcal{N}_{i'}, i \in \mathcal{P}_i, i' \in \mathcal{P}_{i'}, \) and \(\ell, \ell' \in [L],\) the rates at which an active and passive sub-BP in state \(i\) transition to state \(i'\) are,
\[
q^{1,h}(i, i', z) = \begin{cases} 
\lambda_{\ell} i, hI, & \text{if } i = i', \ n_{i'} = n_i + 1, \\
n_{i} \mu_{\ell,i} z, hI, & \text{if } i = i', \ n_{i'} = n_i - 1, \\
0, & \text{otherwise},
\end{cases}
\]
and
\[
q^{0,h}(i, i', z) = \begin{cases} 
n_{i} \mu_{\ell,i} z, hI, & \text{if } i = i', \ n_{i'} = n_i - 1, \\
0, & \text{otherwise},
\end{cases}
\]
respectively. Then the average transition rate for a BP in state \(i\) transitioning to state \(i'\) under an index policy \(\varphi \in \Phi^1\) is,
\[
q^{\varphi,h}(i, i', z) = \nu_{i}(z) q^{1,h}(i, i', z) + (1 - \nu_{i}(z)) q^{0,h}(i, i', z)
\]
\[
= \begin{cases} 
\nu_{i}(z) \lambda_{\ell} i, hI, & \text{if } i = i', \ n_{i'} = n_i + 1, \\
\nu_{i}(z) n_{i} \mu_{\ell,i} z, hI, & \text{if } i = i', \ n_{i'} = n_i - 1, \\
0, & \text{otherwise},
\end{cases}
\]

For given \(t > 0\), from (21), the left hand side of (27) is a function of \(z^{\varphi^*,h}(t)\). Substituting (40) and (21) into the left hand side of (27), there is a unique \(z^{\varphi^*} \in \mathcal{Z}\) and a \(z^h \in \Theta^h(z^{\varphi^*})\) satisfying (27).

We then consider a \(z^h\) satisfying (28).

We note that, under policy \(\varphi^*\), the deterministic process \(z^{\varphi^*,h}(t)\) consists of \(I\) deterministic sub-processes related to \(I\) BPs with \(I\) underlying Markov chains that are isolated from each other. That is, for \(i \in \mathcal{N}_i^0\), the value of \(d z^{\varphi^*,h}(t)/dt\) relays on vector \(z^{\varphi^*,h}(t)\) only through its elements related to the states of pattern \(i\).
Because \( z^h \in \Theta^h(z^{\varphi^*}) \), let \( z^h = z^{\varphi^*} + \sum_{i \in [I]} a_i e_{i, \iota^+}, a_i \in [0, 1], \iota_i \in \mathcal{N}^0_i \setminus \{ \mathcal{N}^0_i - 1 \}, i \in [I] \). Recall that state \( \iota^+ \) for a state \( \iota \) is the state with \( n_{i^+} = n_i + 1 \) and \( i_{i^+} = i_i \). For an \( i \) and \( \iota_i \), if \( z^*_{i_i} > 0 \) and \( z^\varphi_{i_i} > 0 \), then for any \( \iota \in \mathcal{N}_i^0 \),

\[
\lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^{\varphi^*}} = \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^h} = 0. \tag{41}
\]

Thus, we set \( z^h \) to be a specific vector in \( \Theta^h(z^{\varphi^*}) \) which satisfies: a) equation (27); b) and for any \( i \in [I] \), if \( \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^{\varphi^*}} = 0 \) for all \( \iota \) with \( i_i = i \), then \( a_i = 0 \).

We now prove with contradiction that for such \( z^h \), there is not any \( \iota \in [I] \), satisfying \( a_i > 0 \), \( z^*_{i_i} > 0 \) and \( z^\varphi_{i_i} > 0 \). We assume there exists such an \( \iota \in [I] \). If \( z^*_{i_i} > 0 \) and \( z^\varphi_{i_i} > 0 \), then for all \( \iota \in \mathcal{N}_i^0 \), \( \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^{\varphi^*}} = \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^h} \) which is zero. We obtain \( a_i = 0 \) from the definition of the specific \( z^h \). Hence, this assumption cannot hold.

Recall that \( z^p = p z^{\varphi^*} + (1-p) z^h \) with \( p \in (0, 1] \). If \( a_i = 0 \), then for all \( \iota \in \mathcal{N}_i^0 \), \( z^p_{i_i} = z^\varphi_{i_i} = z^h_i \), and equation (41) holds. Also, if \( z^*_{i_i} = 0 \), then \( a_i = 0 \), because all elements of \( z^h \) are non-negative. Accordingly, for all \( \iota \in \mathcal{N}_i^0 \), \( z^p_{i_i} = z^\varphi_{i_i} = z^h_i \), and equation (41) holds in this case.

We mainly discuss the \( i \) with \( a_i > 0 \), \( z^*_{i_i} = 0 \) and \( z^\varphi_{i_i} > 0 \). From (40) and (21),

\[
\lim_{h \to +\infty} \frac{q^{\varphi^*, h}(t^+, \iota_i, z^p)}{hI} = \lim_{h \to +\infty} \frac{q^{\varphi^*, h}(t^+, \iota_i, z^h)}{hI},
\]

and

\[
\lim_{h \to +\infty} \frac{q^{\varphi^*, h}(t_i, t^+, z^p)}{hI} < \lim_{h \to +\infty} \frac{q^{\varphi^*, h}(t_i, t^+, z^h)}{hI}.
\]

So \( \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^p} > 0 \) and \( \lim_{h \to +\infty} d \frac{z^{\varphi^*, h}(t)}{d t}|_{z^{\varphi^*, h}(t) = z^p} < 0 \).

There exists a \( z^h \) satisfying (28) for any \( z^p = p z^{\varphi^*} + (1-p) z^h \) with \( p \in (0, 1] \). Proposition 4 has been proved.

\[ \square \]

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