FUJITA DECOMPOSITION AND HODGE LOCIS

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Abstract. This paper contains two results on Hodge loci in $M_g$. The first concerns fibrations over curves with a non-trivial flat part in the Fujita decomposition. If local Torelli theorem holds for the fibers and the fibration is non-trivial, an appropriate exterior power of the cohomology of the fiber admits a Hodge substructure. In the case of curves it follows that the moduli image of the fiber is contained in a proper Hodge locus. The second result deals with divisors in $M_g$. It is proved that the image under the period map of a divisor in $M_g$ is not contained in a proper totally geodesic subvariety of $A_g$. It follows that a Hodge locus in $M_g$ has codimension at least 2.

Keywords: variations of Hodge structure; Hodge loci; Fujita decomposition

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1. Introduction

This paper contains two results concerning Hodge loci.

The first one relates Hodge loci to the second Fujita decomposition. Let $\bar{X}$ be a complex projective manifold of dimension $n+1$ and let $\bar{f} : \bar{X} \to \bar{B}$ be a fibration onto a smooth projective curve $\bar{B}$. Denote by $B$ the set of regular values of $\bar{f}$.

Fujita decomposition says roughly that the Hodge bundle splits as a direct sum of an ample vector bundle and a unitary flat bundle, see [7–9, 21, 28] and §3. Let $d$ be the rank of the flat summand in $M_g$.

Our first result is as follows.

**Theorem 1.1** (See Theorem 3.10). Assume that $d > 0$ and that for generic $b \in B$ the IVHS map $T_b B \otimes H^{n,0}(X_b) \to H^{n-1,1}(X_b)$ is non-zero. Then for any $b \in B$ the Hodge structure $\Lambda^d H^n(X_b)_{\text{prim}}$ admits a proper substructure.

We notice that for $n$ odd the Hodge structure $\Lambda^d H^n(F)_{\text{prim}}$ always splits as a non-trivial direct sum of Hodge substructures. For odd $n$ the substructure provided by our result lies in one particular piece of $\Lambda^d H^n(F)$, which we denote by $E_d$, see 2.15.

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When the fibers are curves we deduce the following.

**Theorem 1.2** (See Theorem 3.14). If \( n = 1 \) and \( \tilde{f} : \tilde{X} \to \tilde{B} \) is a non-isotrivial fibration with \( d > 0 \), then the image of \( B \) in \( \M_g \) is contained in a proper Hodge locus of \( \M_g \).

The Hodge locus containing the moduli image of \( B \) is defined by a substructure of \( \Lambda^d H^1 \). It would be interesting to investigate the structure of such loci. In the case \( d = 1 \) these loci have been studied for example in [10, 11, 15, 38].

A similar result holds for complete intersections with ample canonical bundle, see Theorem 3.15.

These results rely on some important theorems on variations of Hodge structure due to Deligne and Schmid. These are recalled together with some preliminary facts in §2. The proofs of the above results are contained in §3.

In §4 we describe the Hodge substructure provided by the theorem in two examples due to Catanese and Dettweiler [9].

The second part of the paper deals with Hodge loci from another perspective. Let \( j : \M_g \to \A_g \) be the period map. A Hodge locus of \( \M_g \) is nothing else than \( j^{-1}(Z) \) for a Hodge locus \( Z \subset \A_g \). Hodge loci in \( \A_g \) have an important property: they are totally geodesic subvarieties of \( \A_g \), when \( \A_g \) is endowed with the Siegel metric, i.e., it is considered as a locally symmetric orbifold.

The main result of §5 is the following.

**Theorem 1.3** (See Theorem 5.11). If \( g \geq 3 \) and \( Y \subset \M_g \) is an irreducible divisor, then there is no proper totally geodesic subvariety of \( \A_g \) containing \( j(Y) \).

In particular we have the following corollary:

**Corollary 1.4** (See Corollary 5.14). If \( g \geq 3 \), any Hodge locus of \( \M_g \) has codimension at least 2.

The proof of Theorem 1.3 is based on a result of independent interest (Theorem 5.8) that describes the behavior of a divisor in \( \M_g \) at the boundary. This result is a variation on an argument in [29]. It allows to use induction on \( g \). The case \( g = 3 \) follows, as a very special case, from a theorem by Berndt and Olmos [3] on the codimension of totally geodesic submanifolds in symmetric spaces, see 5.10. The inductive step depends on simple Lie theoretic computations showing that \( \H_k \times \H_{g-k} \) is a maximal totally geodesic submanifold of \( \H_g \), see Proposition 5.6.

2. Preliminaries and notation

We start by recalling the main definitions related to Hodge theory needed in the paper. Let \( H \) be a rational vector space of finite dimension. Set \( H_\mathbb{R} := H \otimes_{\mathbb{Q}} \mathbb{R} \) and \( H_\mathbb{C} := H \otimes_{\mathbb{Q}} \mathbb{C} \).

**Definition 2.1.** A rational Hodge structure of weight \( n \) is the datum of (1) a \( \mathbb{Q} \)-vector space \( H \) and (2) a decomposition \( H_\mathbb{C} := \bigoplus_{p+q=n} H^{p,q} \), such that \( H^{q,p} = \overline{H^{p,q}} \).
Set $S := \{ A = (a_{ij}) \in \text{GL}(2) : a_{11} = a_{22}, a_{12} + a_{21} = 0 \}$. $S$ is an algebraic group defined over $\mathbb{Q}$. The map $A \mapsto z := a_{11} + ia_{21}$, is a group isomorphism $S(\mathbb{R}) \cong \mathbb{C}^\ast$. The maps $\varphi \pm : S(\mathbb{C}) \rightarrow \mathbb{C}^\ast$, defined by $\varphi \pm (A) := a_{11} \pm ia_{21}$, are characters of $S(\mathbb{C})$ and $f = (\varphi_+, \varphi_-)$ is an isomorphism of $S(\mathbb{C})$ onto $\mathbb{C}^\ast \times \mathbb{C}^\ast$. It follows that every character of $S(\mathbb{C})$ is of the form $\varphi \pm \cdot \varphi \pm$. If $G_m$ denotes the multiplicative group scheme, then $w : G_m \rightarrow S$, $w(a) := aI_2$ is an injective morphism defined over $\mathbb{Q}$.

2.2. A $\mathbb{Q}$-Hodge structure of weight $n$ is equivalent to the datum of a $\mathbb{Q}$-vector space $H$ with a representation $\rho : S(\mathbb{R}) \rightarrow \text{GL}(H_{\mathbb{R}})$ such that $\rho \circ w(a) = a^n$. Indeed $\rho^C : S(\mathbb{C}) \rightarrow \text{GL}(H_{\mathbb{C}})$ splits as a sum of eigenspaces $H^{p,q}$, on which the action is multiplication by the character $\varphi_+ \cdot \varphi_-$. Since $\rho^C(z, z) = \rho^C \circ w^C(z)$, it follows that $H^{p,q} \neq \{0\}$ only if $p + q = n$. It is easy to check that $H^{p,q} = H^{q,p}$. Thus we get a $\mathbb{Q}$-Hodge structure.

Definition 2.3. A polarization on a Hodge structure $(H, H^{p,q})$ of weight $n$ is a bilinear form $Q : H \times H \rightarrow \mathbb{Q}$ with the following properties.

1. $Q(a, b) = (-1)^n Q(b, a)$.
2. If $h : H_{\mathbb{C}} \times H_{\mathbb{C}} \rightarrow \mathbb{C}$ is the Hermitian form $h(a, b) := i^n Q(a, \bar{b})$, then $h(H^{p,q}, H^{p',q'}) = 0$ if $(p, q) \neq (p', q')$.
3. The restriction of $h$ to $H^{p,q}$ is definite of sign $(-1)^q (-1)^n(n-1)/2$.

If $H^* \otimes H^*$ is endowed with the induced Hodge structure, $Q \in H^* \otimes H^*$ is an element of type $(-n, -n)$.

2.4. Let $\rho : S(\mathbb{R}) \rightarrow \text{GL}(H_{\mathbb{R}})$ be a Hodge structure of weight $n$. The Mumford–Tate group of $H$, denoted $\text{MT}(H)$, is the smallest $\mathbb{Q}$-algebraic subgroup of $\text{GL}(H)$ whose real points contain $\text{im} \rho$. The main property of the Mumford–Tate group is the following: given multi-indices $d, e \in \mathbb{N}^n$ consider

$$T^{d,e}(H) := \bigoplus_{j=1}^m H^\otimes d_j \otimes (H^*)^\otimes e_j.$$ 

This space is a sum of pure Hodge structures. A (rational) vector $v \in T^{d,e}(H)$ is invariant by the natural action of $\text{MT}(H)$ if and only if it is a Hodge class of type $(0, 0)$. See [25, 33, 44, 46] for more details.

2.5. If $n$ is odd, a polarization $Q$ is a symplectic form on $H$, if $n$ is even it is a non-degenerate symmetric form. Let $\text{GSp}(H)$ be the group of symplectic similitudes in case $n$ is odd and let $\text{GO}(H)$ be the group of orthogonal similitudes for $n$ even. They are algebraic subgroups defined over $\mathbb{Q}$. If $Q$ is a polarization for the Hodge structure $H$, then $\text{MT}(H) \subset \text{GSp}(H)$ for $n$ odd and $\text{MT}(H) \subset \text{GO}(H)$ for $n$ even.
Theorem 2.10. See [17, p. 45, note].

(4.2.2.1)–(4.2.2.4). The only non-trivial point is property (4.2.2.4) which is exactly Schmid of variations of Hodge structures on a quasi-projective manifold satisfies the properties $m > 0$ such that $(\Lambda^d G)_{\otimes m} = \mathbb{C}_B$.

This is proved in [17, Corollary 4.2.8 (iii), b)]. The proof uses the fact that the category of variations of Hodge structures on a quasi-projective manifold satisfies the properties (4.2.2.1)–(4.2.2.4). The only non-trivial point is property (4.2.2.4) which is exactly Schmid Theorem 2.10. See [17, p. 45, note].
2.12. We say that a variation of Hodge structure \((B, H, F^\bullet)\) is trivial if both \(H\) and \(F^\bullet\) are product bundles. We say that it is isotrivial if there is a finite branched cover \(f : B' \rightarrow B\) such that \((B', f^* H, f^* F^\bullet)\) is trivial.

2.13. Given a variation of Hodge structure \((B, H, F^\bullet)\) and multi-indices \(d, e \in \mathbb{N}^n\), passing to the universal cover \(p : \hat{B} \rightarrow B\) we have \(p^*(T^{d,e} H) = \hat{B} \times T^{d,e} H_0\) for a fixed \(\mathbb{Q}\)-vector space \(H_0\). A vector \(t \in T^{d,e} H_0\) gives a section \(\tilde{t}\) of \(p^*(T^{d,e} H)\). Consider the locus \(Y(t) := \{\tilde{b} \in \hat{B} : \tilde{t}(\tilde{b}) \in (p^* T^{d,e} H)^{0,0}\}\). It is a countable union of irreducible analytic sets in \(\hat{B}\). A Hodge locus is by definition an irreducible analytic subset \(Z \subset B\) such that there exist \(t_i \in T^{d_i,e_i} H_0\) for \(i = 1, \ldots, k\), such that \(Z\) is an irreducible component of \(p(\bigcap Y(t_1) \cap \cdots \cap Y(t_k)) \subset B\). See [33, 46] for more details.

**Definition 2.14.** The Hodge loci for the natural variation of Hodge structure on \(A_g\) are called special subvarieties or Shimura subvarieties.

2.15. We recall some elementary facts from representation theory that are needed in §3. Let \(W\) be a complex vector space of dimension \(2m\) with a complex symplectic form \(\omega\). The symplectic form induces an isomorphism \(W \cong W^*\). Let \(\omega_p\) be the element of \(\Lambda^2 W\) corresponding to \(\omega \in \Lambda^2 W^*\). For \(2 \leq p \leq m\) set

\[
\varphi_p : \Lambda^p W \rightarrow \Lambda^{p-2} W, \quad \varphi_p(s) := \omega \cdot s,
\]

and \(\varphi_1 \equiv 0\). For \(p \geq 2\) the morphism \(\varphi_p\) is surjective and \(E_p(W) := \ker \varphi_p\) is an irreducible representation of \(\text{Sp}(W)\). Setting \(L : \Lambda^i W \rightarrow \Lambda^{i+2} W, \ L(s) := \omega^b \wedge s\), we have

\[
\Lambda^p W = E_p(W) \oplus L(\Lambda^{p-2} W)
\]

(2.1)
as \(\text{Sp}(W)\)-modules. It is clear that the group \(\text{GSp}(W)\) also preserves \(E_p(W)\), hence we get an irreducible representation

\[
\varepsilon_p : \text{GSp}(W) \rightarrow \text{GL}(E_p(W)).
\]

See e.g., [42, p. 14], [22, p. 260] [5, p. 201] for the proof and more details.

2.16. Let now \((H, Q)\) be a polarized rational Hodge structure of weight \(n\). Assume that \(n\) is odd and positive. For \(p \leq \dim H/2\), \(\Lambda^p H\) is a Hodge structure and \(E_p(H)\) is a substructure. The same holds for \(L(\Lambda^{p-2} H)\). Indeed if \(\rho : S \rightarrow \text{GSp}(H)\) is the representation defining the Hodge structure on \(H\), then \(\varepsilon_p \circ \rho\), which is a summand of \(\Lambda^p \rho\), defines \(E_p(H)\) as a substructure of \(\Lambda^p H\). We notice that if \(n = 1\) and \(H\) corresponds to an abelian variety \(A\), then \(E_p(H) = H^p(A)\)prim.

2.17. In the setting of 2.16 we have

\[
(\Lambda^p H)^{np,0} \subset E_p(H).
\]

Indeed given \(\alpha \in (\Lambda^p H)^{np,0}\), write \(\alpha = \beta + \gamma\), with \(\beta \in E_p(H)\) and \(\gamma \in L(\Lambda^{p-2} H)\). Taking the \((np,0)\)-components we get \(\alpha = \beta^{np,0} + \gamma^{np,0}\) with \(\beta^{np,0} \in E_p(H)\) and
\[ \gamma^{np,0} \in L(\Lambda^{p-2}H), \] since both \( E_p(H) \) and \( L(\Lambda^{p-2}H) \) are substructures. The operator \( L \) is the wedge with \( Q^p \). Since \( Q \in (H^* \otimes H^*)^{-n,n} \), \( Q^p \in (H \otimes H)^{n,n} \). Hence \((L(\Lambda^{p-2}H))^{np,0} = \{0\} \). Therefore \( \alpha = \beta^{np,0} \).

2.18. Let \( W \) be a complex vector space of dimension \( m \) with a non-degenerate symmetric bilinear form \( q \). We recall that if \( m \) is odd then \( \Lambda^p W \) is an irreducible representation of \( O(W,q) \) for any \( p \). If \( m = 2s \), then \( \Lambda^p W \) is an irreducible representation of \( O(W,q) \) for any \( p \neq s \). See [20, p. 287 and p. 295] or [42, p. 16 and p. 18]. It follows that if \( (H, Q) \) is a polarized rational Hodge structure of weight \( n \), with \( n \) even, then \( \Lambda^p H \) is an irreducible representation of \( GO(H) \) for any \( p \neq \dim H/2 \).

3. Fujita decomposition and Hodge loci

3.1. Let \( \tilde{X} \) be a complex projective manifold of dimension \( n + 1 \) and let \( \tilde{f} : \tilde{X} \to \tilde{B} \) be a fibration onto a smooth projective curve \( \tilde{B} \). Denote by \( B \) the set of regular values of \( \tilde{f} \), set \( X := \tilde{f}^{-1}(B) \) and \( f := \tilde{f}|_X \).

Fix a Hodge class \([\omega] \in H^2(\tilde{X}, \mathbb{Z}) \). Consider the fibrewise primitive cohomology with respect to the restriction of this polarization to the smooth fibers:

\[ H := (R^nf_*\mathbb{Q}_X)^\text{prim}. \]

Then \( H \) is a local system of \( \mathbb{Q} \)-vector spaces on \( B \). Its associated vector bundle \( H \otimes \mathcal{O}_B \) is endowed with the Gauss–Manin connection \( \mathcal{V} \) and with a polarization \( Q \) obtained from the intersection form. Denote by \( F^* \) the weight filtration. Then \((B, H, F^*, Q) \) is a polarized variation of Hodge structure on the quasi-projective curve \( B \). Let \( h \) denote the associated Hermitian form: \( h(\alpha, \beta) = i^n Q(\alpha, \tilde{\beta}) \). The sheaf

\[ V := \tilde{f}_*\omega_{\tilde{X}/\tilde{B}} \]

is locally free on \( \tilde{B} \), so we identify it with the corresponding vector bundle \( V \to \tilde{B} \), which is called the Hodge bundle of the fibration. We have \( V|_B = F^n \). The restriction of \( h \) to \( V|_B \) is positive definite, hence we can define the orthogonal projection \( p : H_{\mathbb{C}} \to V|_B \). The Gauss–Manin connection induces a connection \( D := p \nabla \) on \( V|_B \).

**Theorem 3.2** (Fujita). In the setting of 3.1 there is a decomposition

\[ V = A \oplus U, \tag{3.1} \]

where \( A \) in an ample vector bundle on \( \tilde{B} \) and \( U \) is a vector bundle on \( \tilde{B} \), such that \( U|_B \) is the holomorphic vector bundle associated to a subsystem of \( H \). We call (3.1) the (second) Fujita decomposition.

See [20, 21] for background and [7, 8, 28] for the proof. See also [9, 24, 39] for related problems.

3.3. On \( B \) we have \( D(U) \subset U \otimes \Omega^1_B \) and \( D(A) \subset A \otimes \Omega^1_B \), so \( D = D^A \oplus D^U \). Indeed let \( \sigma_U : U \to A \otimes \Lambda^{1,0} \) and \( \sigma_A : A \to U \otimes \Lambda^{1,0} \) be the second fundamental forms of \( U \subset V \) and \( A \subset V \). By [26, p. 20] \( \sigma_A \) is (up to sign) the adjoint of \( \sigma_U \). Moreover \( \sigma_U = 0 \) since \( U \) is preserved by \( \nabla \), hence by \( D \).
The following observation is well-known.

Lemma 3.4. If $(V|_{B}, D)$ is flat, then $V|_{B}$ is preserved by $\nabla$.

Proof. We use the notation of [1, p. 224]. Let $s$ be a local holomorphic section of $V|_{B} = F^n \subset H_C$. Then $i(RVs, s) = i(R_{HC}s, s) - i(\sigma s, \sigma s)$, where $\sigma$ is the second fundamental form of $V|_{B} \subset H_C$. By assumption $RV = 0$. Also $R_{HC} = 0$. So $(\sigma s, \sigma s) = 0$. By Griffiths transversality $\nabla(V|_{B}) = \nabla(F^n) \subset F^{n-1} \otimes \Omega^1_B$. Since $F^{n-1} = F^n \otimes H^{n-1,1}$, we have $\sigma : V|_{B} \rightarrow H^{n-1,1} \otimes \Lambda^{1,0}(B)$. But $h$ is definite on $H^{n-1,1}$, so we conclude that $\sigma s = 0$. Therefore $\sigma = 0$ and $\nabla$ preserves $V|_{B}$.

Our first result is the following.

Theorem 3.5. Let $\tilde{f} : \tilde{X} \rightarrow \tilde{B}$ be as in 3.1 and let $d := \text{rank } U > 0$. Then there is an étale cyclic cover $u : \hat{B} \rightarrow B$ such that $E_d(u^*H)^{\text{inv}} \neq \{0\}$ if $n$ is odd and $(\Lambda^d u^*H)^{\text{inv}} \neq \{0\}$ if $n$ is even.

Proof. By assumption there is a subsystem $G \subset H_C$ such that $U|_{B}$ is the vector bundle associated to $G$. By Theorem 2.11 there is an $m > 0$ such that $(\Lambda^d G)^{\otimes m} \cong \mathbb{C}_B$. Fix $b \in B$ and let $\rho : \pi_1(B, b) \rightarrow \mathbb{C}^* = \text{GL}(\Lambda^d G_b)$ be the monodromy representation of $\Lambda^d G$. The image of $\rho$ is finite cyclic. Let $u : \hat{B} \rightarrow B$ be the unramified covering associated to $\ker \rho \subset \pi_1(B, b)$. It is a cyclic Galois cover. Then $\hat{G} := u^*G$ is a subsystem of $u^*H$. Fix $\hat{b} \in u^{-1}(b)$. The monodromy representation of $\Lambda^d \hat{G} = u^*\Lambda^d G$ at $\hat{b}$ is the composition $\rho \circ u_* : \pi_1(\hat{B}, \hat{b}) \rightarrow \mathbb{C}^*$. Hence $\Lambda^d \hat{G} \cong \mathbb{C}_{\hat{b}}$. It follows that $\Lambda^d \hat{G}$ is contained in the fixed part of $\Lambda^d u^*H$. If $n$ is even we are done. If $n$ is odd, observe that $\Lambda^d \hat{G} \subset (\Lambda^d u^*H)^{\text{prim},0} \subset E_d(u^*H)$ by 2.17. Hence $(E_d(u^*H))^{\text{inv}} \neq \{0\}$.

Remark 3.6. It is important to stress that the variation $u^*\Lambda^d H$ is geometric and that this yields another proof of the Theorem, which avoids Schmid theorem. To see this, we work over $\hat{B}$ instead of $B$. Up to base change we can assume that $\det U$ is trivial. Set $Z := X \times \hat{B} \cdots \times \hat{B} X$ ($d$ times) and let $F : Z \rightarrow \hat{B}$ be the induced fibration: $F(x_1, \ldots, x_d) = f(x_1)$. Over $\hat{B}$ the morphism $F$ is smooth. Denoting by $p_i$ the $i$th projection, the form $\omega := \sum_{i=1}^d p_i^* \omega$ is a Kähler form on $X \times \cdots \times X$. Let $R^{nd} F_* \mathbb{Q}_{Z_{\text{prim}}}$ denote the fibrewise primitive cohomology with respect to $[\omega]$, which is a geometric variation of Hodge structure on $B$. For any $t \in B$, we have $\Lambda^d H^n(X_t)_{\text{prim}} \subset H^{nd}(Z_t)_{\text{prim}}$. Thus $\Lambda^d H = \Lambda^d R^1 f_* \mathbb{Q}_{X_{\text{prim}}}$ is a summand of the geometric variation $R^{nd} F_* \mathbb{Q}_{Z_{\text{prim}}}$ as claimed. Next set

$$W_t := H^{nd}(Z_t)^{\text{inv}} \cap \Lambda^d H^n(X_t)_{\text{prim}},$$

where $H^{nd}(Z_t, \mathbb{Q})^{\text{inv}}$ denotes the fixed part of the variation $R^{nd} F_* \mathbb{Q}_{Z_{\text{prim}}}$. We can deduce that this fixed part is a Hodge substructure, from the Global Invariant Cycle Theorem, which asserts that it coincides with the image of the restriction $H^{nd}(Z) \rightarrow H^{nd}(Z_t)$, see e.g., [47, Theorem 16.24, p. 385]. Thus $W_t$ is a Hodge substructure and $\det U \subset W$, which implies that $W$ is non-trivial.

3.7. To make some of the following statements simpler we introduce the following notation: if $H$ is a polarized rational Hodge structure or a polarized rational variation of
Hodge structure we set

$$K_d(H) := \begin{cases} E_d(H) & \text{if } n \text{ is odd}, \\ \Lambda^d H & \text{if } n \text{ is even}. \end{cases}$$

**Corollary 3.8.** Let $\tilde{f} : \tilde{X} \to \tilde{B}$ be as in 3.1 and let $d := \text{rank } U > 0$. Then either $K_d(H)$ is isotrivial or for any $b \in B$ the Hodge structure $K_d(H_b)$ admits a proper substructure.

**Proof.** Let $\pi : \tilde{B} \to B$ be as in Theorem 3.5. Set $B := K_d(H)$. Consider the variation of Hodge structure $u^* K$ on $\tilde{B}$. By Schmid theorem 2.10 $(u^* K)^{\text{inv}}$ is subvariation of $u^* K$ and we know from Theorem 3.5 that $(u^* K)^{\text{inv}} \neq \{0\}$. If $(u^* K)^{\text{inv}} = u^* K$, then $u^* K$ is trivial and hence $B$ is isotrivial. Otherwise for any $b \in B$ and $b \in \text{inv}(b)$ we have

$$\{0\} \neq (u^* K)^{\text{inv}} \subset \subset (u^* K)_b = K_b. \quad \square$$

**Lemma 3.9.** Let $\tilde{f} : \tilde{X} \to \tilde{B}$ be as in 3.1 and let $d := \text{rank } U > 0$. If $K_d(H)$ is isotrivial, then the Hodge bundle $V|_B$ is flat.

**Proof.** Step 1: $\Lambda^d V|_B$ is preserved by $\Lambda^d \nabla$.

(Here $\Lambda^d \nabla$ denotes the connection induced by $\nabla$ on $\Lambda^d H$.) It is easy to check that $(\Lambda^d H)^{nd,0} = \Lambda^d F^n$. On the other hand it follows from 2.17 that $(\Lambda^d H)^{nd,0} = K^d_b(H)^{nd,0}$. Since $u^* K_d(H)$ is trivial for some base change $u : \tilde{B} \to B$, its $(nd,0)$-component is preserved by $\Lambda^d \nabla$, so also $K_d(H)^{nd,0} = \Lambda^d F^n = \Lambda^d V|_B$ is preserved by $\Lambda^d \nabla$.

Step 2: $(\Lambda^d V|_B, \Lambda^d D)$ is a flat bundle.

Let $h$ denote the Hodge Hermitian product on $H_{\omega}$ defined in 3.1. Recall that $h > 0$ on $F^n$ so the orthogonal projection $p : H_{\omega} \to F^n$ is well-defined. Since $(\Lambda^d h)|_{\Lambda^d F^n} = \Lambda^d (h|_{F^n})$ and the right hand side is positive definite, also $\Lambda^d h > 0$ on $\Lambda^d F^n$. So the orthogonal projection $p' : \Lambda^d H_{\omega} \to \Lambda^d F^n$ is well-defined. Moreover $p'(\Lambda^d \nabla) = \Lambda^d D$, since both are compatible connections on the Hodge bundle $(\Lambda^d F^n, \Lambda^d h|_{\Lambda^d F^n})$. Since $\Lambda^d F^n = \Lambda^d V|_B$ is preserved by $\Lambda^d \nabla$, we conclude that $\Lambda^d D = \Lambda^d \nabla$ on $\Lambda^d V|_B$ and of course $\Lambda^d D$ is flat on $\Lambda^d V|_B$. This proves the claim.

Step 3 : $(\Lambda^d - U|_B \otimes A|_B, \Lambda^d D)$ is flat.

$\Lambda^d V|_B$ is a direct sum of various bundles, one of them being $\Lambda^d - U|_B \otimes A|_B$. All these summands are preserved by the connection $\Lambda^d D$, as follows from 3.3. Since $(\Lambda^d V|_B, \Lambda^d D)$ is flat, every summand is flat with this connection.

Step 4: $(A|_B, D^A)$ is flat.

On $\Lambda^d - U|_B \otimes A|_B$ we have $\Lambda^d D = \Lambda^d - D^U \otimes D^A$. Moreover $U$ is flat. So step 3 yields that

$$0 = R_{\Lambda^d - 1 U \otimes A} = R_{\Lambda^d - 1 U} \otimes I_A + I_{\Lambda^d - 1 U} \otimes R_A = I_{\Lambda^d - 1 U} \otimes R_A.$$

So $R_A = 0$ on $B$. Thus $(V|_B, D)$ is flat. \quad \square

Recall that if $b_0 \in B$ and $W \subset B$ is an open contractible neighborhood of $b_0$, then the period mapping $\mathcal{P}^{n,n} : W \to \text{Grass}(H^{n,0}, H^n(X_{b_0}))$ associates to $b \in W$ the subspace $H^{n,0}(X_b) \subset H^n(X_b) \cong H^n(X_{b_0})$, see e.g., [6, 47, p. 229]. The derivative of the period mapping at $b$ can be identified with the *infinitesimal variation of Hodge structure* (IVHS) map $T_b B \otimes H^{n,0}(X_b) \to H^{n-1,1}(X_b)$ obtained by taking the cup product with the Kodaira–Spencer class.
Theorem 3.10. Let \( \tilde{f} : \tilde{X} \to \tilde{B} \) be as in 3.1 and let \( d := \text{rank} U > 0 \). Assume that for generic \( b \in B \) the IVHVS map \( T_{\tilde{b}}B \otimes H^{n,0}(X_b) \to H^{n-1,1}(X_b) \) is non-zero. Then for any \( b \in B \) the Hodge structure \( K_d(H_b) \) admits a proper substructure.

Proof. By Corollary 3.8 either \( K_d(H) \) is isotrivial or for any \( b \in B \) the Hodge structure \( K_d(H_b) \) admits a proper substructure. In the first case the Hodge bundle \( V|_b \) is flat by Lemma 3.9. Hence by Lemma 3.4 \( V|_B \) is preserved by \( \nabla \). But then the map \( T_{\tilde{b}}B \otimes H^{n,0}(X_b) \to H^{n-1,1}(X_b) \) would be trivial for any \( b \in B \), contrary to the assumption. This proves that for any \( b \in B \) the Hodge structure \( K_d(H_b) \) admits a proper substructure.

If \( C \) is a smooth curve we denote by \( \text{MT}(C) \) the Mumford–Tate group of \( H^1(C) \). The following fact is well-known. We recall the proof for the reader’s convenience.

Proposition 3.11. If \([C]\) is very general in \( M_g \), then the Mumford–Tate group \( \text{MT}(C) = \text{GSp}(H^1(C, \mathbb{Q})) \).

Proof. Fix \( m \geq 3 \). Let \( \Gamma_g \) be the mapping class group and let \( \Gamma_g[m] \) be the kernel of the composition \( \Gamma_g \to \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/m) \). The moduli space \( M_g^{(m)} \) of genus \( g \) curves with level \( m \) structure is the quotient of the Teichmüller space \( T_g \) by the properly discontinuous and free action of \( \Gamma_g[m] \). Over \( M_g^{(m)} \) there is a universal family \( \pi : \mathcal{C} \to M_g^{(m)} \) and a corresponding integral variation of Hodge structure on \( \text{Sp}(H) \), see e.g., [18, p. 170]. Since \( t \) is very general, it follows from [46, Corollary 4.12] that \( \text{MT}(C) \) contains a finite index subgroup \( \Gamma' \) of \( \text{im}\rho \). Then \( \Gamma' \) has finite index in \( \text{Sp}(H) \), thus it is an arithmetic subgroup of \( \text{Sp}(H) \). By Borel density theorem (see e.g [4], [40, p. 205]) \( \Gamma' \) is Zariski dense in \( \text{Sp}(H_{\mathbb{Q}}) \). Thus \( \text{MT}(C) \) contains \( \text{Sp}(H_{\mathbb{Q}}) \).

Proposition 3.12. Let \( C \) be a curve with \( \text{MT}(C) = \text{GSp}(H) \). If \( \tilde{f} : \tilde{X} \to \tilde{B} \) is a fibration as in 3.1 with \( C \cong X_t \) for some \( t \in B \), then either the fibration is isotrivial or \( U = \{0\} \) in (3.1).

Proof. Assume that \( d := \text{rank} U > 0 \). Here the weight \( n = 1 \), so \( K_d(H) = E_d(H) \). Since \( \text{MT}(C) = \text{GSp}(H) \), \( E_d(H) \) is an irreducible representation of \( \text{MT}(C) \), see 2.15. So it is an irreducible Hodge structure. Assume by contradiction that \( f \) is not isotrivial. The assumption of Theorem 3.10 is satisfied by Torelli theorem. But Theorem 3.10 implies that the Hodge structure is reducible, which gives a contradiction. Therefore \( f \) is isotrivial.

Theorem 3.13. If \( C \) is a very general curve in \( M_g \) and \( \tilde{f} : \tilde{X} \to \tilde{B} \) is a fibration as in 3.1 with \( C \cong X_t \) for some \( t \in B \), then either the fibration is isotrivial or \( U = \{0\} \) in (3.1).

Proof. By Proposition 3.11 the Mumford–Tate \( \text{MT}(C) \) equals \( \text{GSp}(H) \). The result follows from Proposition 3.12.

The following statement refines the previous one using the notion of Hodge locus.
Theorem 3.14. Let $\tilde{X}$ be a surface and let $\tilde{f} : \tilde{X} \to \tilde{B}$ be a fibration as in 3.1. Assume that $\tilde{f}$ is not isotrivial and that $d := \text{rank } U > 0$. Then the image of $\tilde{B}$ in $\mathcal{M}_g$ is contained in a proper Hodge locus $Z$ of $\mathcal{M}_g$.

Proof. Fix $b \in B$ and let $L \subset E_d(H_b)$ be the Hodge substructure given by Corollary 3.8. The orthogonal complement $L^\perp$ is also a Hodge substructure, so the orthogonal projection $p : E_d(H_b) \to E_d(H_b)$ onto $L$ is a Hodge class in $(\text{End}E_d(H_b))^{0,0}$. Let $Z_b$ be the Hodge locus defined by this class, see 2.13. Thus the image of $\tilde{B}$ is contained in $\bigcup_b Z_b$. Since $\tilde{B}$ is irreducible and the Hodge loci can be at most countable, Baire theorem implies that the image of $\tilde{B}$ is in fact contained in $Z_{b_0}$ for some $b_0 \in B$. It remains to show that $Z = Z_{b_0}$ is a proper subset of $\mathcal{M}_g$. This follows from Proposition 3.11: for a very general curve $C$ the space $E_d(H)$ is an irreducible representation of $\text{MT}(C)$ (see 2.15), thus it is an irreducible Hodge structure. □

Let $\varphi : X \to S$ be the universal family of smooth complete intersections of multidegree $d_1, \ldots, d_k$ in $\mathbb{P}^N$.

Theorem 3.15. Assume that $\sum_i d_i > N + 1$ (i.e., the canonical bundle is ample). Let $s$ be a very general point of $S$. If $\tilde{f} : \tilde{X} \to \tilde{B}$ is a fibration as in 3.1 with $X_t \cong \varphi^{-1}(s)$ for some $t \in B$, then either the fibration is isotrivial or $U = \{0\}$ in (3.1).

Proof. Set $Y = \varphi^{-1}(s)$ and $H := H^n(Y, \mathbb{Z})$. By [31, p. 17–18] $\text{MT}(H) = \text{GO}(H)$ for $n$ even and $\text{MT}(H) = \text{GSp}(H)$ for $n$ odd. Assume $d := \text{rank } U > 0$. Start with the case $d \neq \frac{1}{2} \text{rank } H$. Using 2.15 and 2.18 we deduce that $K_d(H)$ is an irreducible representation of $\text{MT}(H)$, hence an irreducible Hodge structure. If $f$ is non-isotrivial, [35, Theorem 5.4], implies that the IVHS map $T_B \otimes H^{n,0}(X_b) \to H^{n-1,1}(X_b)$ is non-zero for generic $b \in B$. But then Theorem 3.10 yields a contradiction. This proves that $f$ is isotrivial.

If $d = \frac{1}{2} \text{rank } H$, then $U = V$ in (3.1), so again by [35, Theorem 5.4], the fibration is isotrivial. □

Also in this case we can refine the previous statement making use of the notion of Hodge locus. Let $\mathcal{M}$ be the moduli space of smooth complete intersections of multidegree $d_1, \ldots, d_k$ in $\mathbb{P}^N$.

Theorem 3.16. Assume that $\sum_i d_i > N + 1$ (i.e., the canonical bundle is ample). Let $\tilde{f} : \tilde{X} \to \tilde{B}$ be a fibration as in 3.1. Assume that $\tilde{f}$ is not isotrivial and that $d := \text{rank } U > 0$. Then the image of $\tilde{B}$ in $\mathcal{M}$ is contained in a proper Hodge locus of $\mathcal{M}$.

The proof is exactly as the one for Theorem 3.14.

4. Examples

In this Section we compute explicitly the Hodge substructure given by Corollary 3.8 in two examples belonging to the infinite family constructed by Catanese and Dettweiler in [9, §§ 3–4] in order to get counterexamples to a question of Fujita. They are families of cyclic covers of $\mathbb{P}^1$. 
In the Fujita decomposition (3.1), we have that $H$ yields a Shimura curve in $\overline{\mathbb{t}}$ and Dettweiler in order to get counterexamples to a question of Fujita.) This family of $\overline{\mathbb{t}}$ is a Hodge substructure as in Corollary 3.8 and clearly genus of $f$ primitive root of unity.

Consider the 1-dimensional family of Example 1. Therefore the notation of 2.15, observe that $H$ and $\omega$ by subspaces $\sigma$ is a natural a representation $k$ $V$. Using [9, equation (3.2)] one easily computes

$$\dim V_1 = 0, \quad \dim V_2 = \dim V_3 = 1, \quad \dim V_4 = 2.$$ 

In the Fujita decomposition (3.1), we have that $U_t = V_4$ for every $t \in B$. In fact, for every $t \in B$, $H^1(X_t, \mathbb{C}) = \bigoplus_{j=1, \ldots, 4} \mathbb{H}_j$, where $\mathbb{H}_j$ is the $\zeta_5^j$-eigenspace for the action of $\mathbb{Z}/5$ on $H^1(X_t, \mathbb{C})$. We have $\mathbb{H}_1 = V_j \oplus V_{5-j}$. Hence

$$\mathbb{H}_1 = V_4, \quad \mathbb{H}_2 = V_2 \oplus V_3, \quad \mathbb{H}_3 = V_3 \oplus V_2, \quad \mathbb{H}_4 = V_4.$$ 

Therefore $(f\omega_{X/B})_t = H^0(X_t, K_{X_t}) = V_2 \oplus V_3 \oplus V_4$ and $V_4 = \mathbb{H}_4 = U_t$, while $A_t = V_2 \oplus V_3$, see [9].

The decomposition $H^1(X_t, \mathbb{C}) = \bigoplus_{j=1, \ldots, 4} \mathbb{H}_j$ is defined over $\mathbb{Q}(\zeta_5)$, that is there are subspaces $\mathbb{F}_j \subset H^1(X_t, \mathbb{Q}(\zeta_5))$ such that

$$H^1(X_t, \mathbb{Q}(\zeta_5)) = \bigoplus_{j=1, \ldots, 4} \mathbb{F}_j,$$

and $\mathbb{H}_j = \mathbb{F}_j \otimes \mathbb{C}$. A generator of the Galois group $G := \text{Gal}(\mathbb{Q}(\zeta_5), \mathbb{Q}) \cong (\mathbb{Z}/5)^*$ is given by $h : \mathbb{Q}(\zeta_5) \to \mathbb{Q}(\zeta_5)$ defined by $h(\zeta_5) = \zeta_5^2$. Since $H^1(X_t, \mathbb{Q}(\zeta_5))$ is defined over $\mathbb{Q}$ there is a natural a representation $\sigma : G \to \text{GL}_Q(H^1(X_t, \mathbb{Q}(\zeta_5)))$. We have $\sigma(h)(\mathbb{F}_j) = \mathbb{F}_{k}$, with $k \equiv 2j \bmod 5$. So $\sigma(h)(\mathbb{F}_1) = \mathbb{F}_2, \sigma(h)(\mathbb{F}_2) = \mathbb{F}_4, \sigma(h)(\mathbb{F}_3) = \mathbb{F}_3, \sigma(h)(\mathbb{F}_3) = \mathbb{F}_1$. Therefore $\bigoplus_{j=1}^4 \Lambda^2 \mathbb{F}_j$ is a $G$-invariant subspace of $\Lambda^2 H^1(X_t, \mathbb{Q}(\zeta_5))$, hence it is defined over $\mathbb{Q}$. Thus

$$H_t := \bigoplus_{i=1, \ldots, 4} \Lambda^2 \mathbb{H}_i = \Lambda^2 \overline{\mathbb{V}_4} \oplus (V_2 \otimes \overline{\mathbb{V}_3}) \oplus (V_3 \otimes \overline{\mathbb{V}_2}) \oplus \Lambda^2 \mathbb{V}_4$$

is a Hodge substructure as in Corollary 3.8 and clearly $\Lambda^2 U_t = \Lambda^2 \mathbb{H}_4 \subset H_t$.

It is also easy to check that $H_t$ is a proper substructure of $E_2(H^1(X_t, \mathbb{Q}))$. In fact using the notation of 2.15, observe that $\omega \in \Lambda^2 H^1(X_t, \mathbb{Q})^*$ is the cup product and $\varphi_2(s) = \omega \wedge s = \omega(s)$. Hence if $s = \alpha \wedge \beta$ with $\alpha \in \mathbb{H}_i$ and $\beta \in \mathbb{H}_j$, then

$$\varphi_2(s) = \int_C \alpha \wedge \beta = \int_C \zeta_5^i (\alpha \wedge \beta) = \zeta_5^{i+j} \int_C \alpha \wedge \beta.$$
Thus if \( i + j \not\equiv 0 \pmod{5} \), \( \varphi_2(s) = 0 \). So \( H_t \subset \ker \varphi_2 \). Next recall that \( E_2(H^1(X_t, \mathbb{Q})) \) is simply the orthogonal space to \( \mathbb{Q} \omega \) inside \( \Lambda^2 H^1(X_t, \mathbb{Q}) \), so \( \dim E_2 = 27 \). Thus clearly \( \{0\} \subsetneq H_t \subseteq E_2(H^1(X_t, \mathbb{Q})) \).

**Example 2.** Consider the 1-dimensional family of \( \mathbb{Z}/7 \)-covers of \( \mathbb{P}^1 \) given by the equation

\[
y^7 = x(x - 1)(x + 1)(x - t)^4.
\]

(This is one of the examples of [9] and also family (17) in [30, Table 2].) The general fiber has genus 6. Using the same notation and by the same analysis as in the previous example one gets

\[
\dim(V_1) = \dim(V_2) = 0, \quad \dim(V_3) = \dim(V_4) = 1, \quad \dim(V_5) = \dim(V_6) = 2, \quad H^1(X_t, \mathbb{C}) = \bigoplus_{j=1}^{6} \mathbb{H}_j, \quad U_t = V_5 \oplus V_6 = \mathbb{H}_5 \oplus \mathbb{H}_6.
\]

A generator of the Galois group \( \text{Gal}(\mathbb{Q}(\zeta_7), \mathbb{Q}) \cong (\mathbb{Z}/7)^* \) is \( h : \mathbb{Q}(\zeta_7) \to \mathbb{Q}(\zeta_7) \), \( h(\zeta_7) = \zeta_7^3 \). Hence \( \sigma(h)(F_j) = F_k \), with \( k \equiv 3j \mod 7 \). Therefore \( \sigma(h)(F_1) = F_3 \), \( \sigma(h)(F_3) = F_2 \), \( \sigma(h)(F_2) = F_6 \), \( \sigma(h)(F_6) = F_4 \), \( \sigma(h)(F_4) = F_5 \), \( \sigma(h)(F_5) = F_1 \). So the subspace

\[
H_t := (\Lambda^2 \mathbb{H}_5 \otimes \Lambda^2 \mathbb{H}_6) \oplus (\Lambda^2 \mathbb{H}_1 \otimes \Lambda^2 \mathbb{H}_4) \oplus (\Lambda^2 \mathbb{H}_3 \otimes \Lambda^2 \mathbb{H}_5) \\
\quad \quad \oplus (\Lambda^2 \mathbb{H}_2 \otimes \Lambda^2 \mathbb{H}_1) \oplus (\Lambda^2 \mathbb{H}_6 \otimes \Lambda^2 \mathbb{H}_3) \oplus (\Lambda^2 \mathbb{H}_4 \otimes \Lambda^2 \mathbb{H}_2)
\]

is a Hodge substructure and \( \Lambda^4 U_t = \Lambda^2 \mathbb{H}_5 \otimes \Lambda^2 \mathbb{H}_6 \subset H_t \). Since \( \omega_{j,s} = 0 \) for any \( s \in \mathbb{H}_i \land \mathbb{H}_j \) if \( i + j \not\equiv 0 \pmod{7} \), one easily checks that \( H_t \subset E_4(H^1(X_t, \mathbb{Q})) \). By a dimension count \( H_t \) is a proper substructure.

**Remark 4.1.** The families in the two examples above yield Shimura curves contained in the Torelli locus. This is not the case for all the other examples constructed by Catanese and Dettweiler, thanks to [30]. Nevertheless in all the examples of Catanese and Dettweiler the bundle \( U \) is non-trivial and by computations similar to the previous one, one can describe the Hodge substructure given by Corollary 3.8.

5. **Hodge loci in the moduli space of curves**

We start by recalling some facts concerning totally geodesic subvarieties and Hodge loci in \( A_6 \).

5.1. Let \( \omega \) be the standard symplectic form on \( \mathbb{R}^{2g} \). The **Siegel space** \( \mathfrak{H}_g \) is the set of complex structures on \( \mathbb{R}^{2g} \) that are compatible with \( \omega \), i.e., such that \( J^* \omega = \omega \) and \( \omega(J \cdot, J \cdot) > 0 \). The group \( G := \text{Sp}(2g, \mathbb{R}) \) acts on \( \mathfrak{H}_g \) by conjugation. This action is transitive. If \( J \in \mathfrak{H}_g \), the stabilizer \( G_J \) is the group of unitary transformations of \( (\mathbb{R}^{2g}, J, \omega(\cdot, J \cdot)) \). Fix \( J \in \mathfrak{H}_g \) and a unitary basis \( \{e_1, \ldots, e_g\} \) of \( V^{1,0}(J) \). Set \( e_{g+j} := \tilde{e}_j \). Then \( \{e_1, \ldots, e_{2g}\} \) is a basis of \( \mathbb{C}^{2g} \). In this basis

\[
\mathfrak{g} := \text{Lie}G = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} : X \in \mathfrak{u}(g), \ Y \in \mathfrak{J}^g \right\},
\]

(5.1)
where \( \mathcal{Z}_g \) denotes the space of complex symmetric matrices of order \( g \) (see e.g., [41, p. 78–79]). Denote by \( K = G_J \) the stabilizer of \( J \) for the \( G \)-action. Let \( m \) be the \( \text{Ad} \ K \)-invariant complement of \( \mathfrak{t} := \text{Lie} K \) in \( \mathfrak{g} \). We have the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \). It is easy to check that

\[
\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & \bar{X} \end{pmatrix} \right\} \cong u(g), \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & Y \\ \bar{Y} & 0 \end{pmatrix} \right\} \cong \mathcal{Z}_g.
\]

(5.2)

It follows that \( \mathcal{H}_g \) is a Hermitian symmetric space of the noncompact type. In terms of the identifications (5.2) the isotropy representation of \( \mathfrak{k} \) on \( \mathfrak{m} \) becomes

\[
\text{ad}_{\mathfrak{k}} : \mathfrak{k} \rightarrow \text{gl}(\mathfrak{m}), \quad \text{ad}_{\mathfrak{k}}(X)(Y) = XY - Y\bar{X}.
\]

(5.3)

5.2. The fact that \( \mathcal{H}_g \) is a symmetric space of the noncompact type has important consequences for its totally geodesic submanifolds. On the one hand these are necessarily symmetric spaces of their own. On the other hand the closed totally geodesic submanifolds of \( \mathcal{H}_g \) passing through \( J \) are in bijective correspondence with Lie triple systems \( l \), that is with linear subspaces \( l \subset \mathfrak{m} \) such that \([ [l, l], l] \subset l\), see e.g., [27, p. 237].

5.3. The group \( \Gamma := \text{Sp}(2g, \mathbb{Z}) \) acts properly discontinuously and holomorphically on \( \mathcal{H}_g \). Hence \( \mathcal{A}_g := \Gamma \backslash \mathcal{H}_g \) is a complex analytic global quotient orbifold. The symmetric metric of \( \mathcal{H}_g \) descends to an orbifold Kähler metric on \( \mathcal{A}_g \). We always consider this metric on \( \mathcal{A}_g \). It is a locally symmetric (orbifold) metric.

5.4. In Riemannian geometry a submanifold \( N \) of a Riemannian manifold \( (M, g) \) is called \textit{totally geodesic} if the second fundamental form of \( N \) in \( M \) vanishes identically. We use the same terminology for suborbifolds of a Riemannian orbifold. More precisely, we say that a subset \( Z \subset \mathcal{A}_g \) is a \textit{totally geodesic subvariety}, if it is a closed algebraic subvariety of \( \mathcal{A}_g \) and there is a totally geodesic submanifold \( \tilde{Z} \) of \( \mathcal{H}_g \) such that \( \pi(\tilde{Z}) = Z \).

It has been proved by Mumford that special subvarieties, i.e., Hodge loci of \( \mathcal{A}_g \), are totally geodesic, see [32, 34].

5.5. For \( 1 \leq k \leq g - 1 \) consider the map \( \varphi_k : \mathcal{A}_k \times \mathcal{A}_{g-k} \rightarrow \mathcal{A}_g \),

\[
\varphi_k([A_1, c_1(L_1)], [A_2, c_1(L_2)]) := [A_1 \times A_2, c_1(L_1 \boxtimes L_2)].
\]

(5.4)

We often drop the polarizations from the notation. Set

\[
Z_k := \varphi_k(\mathcal{A}_k \times \mathcal{A}_{g-k}).
\]

**Proposition 5.6.** (a) \( Z_k \) is a totally geodesic subvariety of \( \mathcal{A}_g \).

(b) \( Z_k \) is maximal in the following sense: if \( Z \) is a totally geodesic subvariety of \( \mathcal{A}_g \) and \( Z_k \subset Z \), then either \( Z = Z_k \) or \( Z = \mathcal{A}_g \).

(c) Fix \( [A_0] \in \mathcal{A}_k \) and set \( h : \mathcal{A}_{g-k} \rightarrow \mathcal{A}_g, h([A]) := [A_0 \times A] \). If \( Z \subset \mathcal{A}_g \) is a totally geodesic subvariety, then \( h^{-1}(Z) \) is a totally geodesic subvariety of \( \mathcal{A}_{g-k} \).
Proof. (a) Fix \([J] \in Z_k\). Then \((\mathbb{R}^{2g}/\mathbb{Z}^{2g}, J) \cong A_1 \times A_2\). Set \(V_i = T_0A_i \subset \mathbb{R}^{2g}\). Then \(J(V_i) = V_i\). Moreover there are sublattices \(\Lambda_i \subset \mathbb{Z}^{2g}\), such that \(V_i = \Lambda_i \otimes \mathbb{R}\), \(\Lambda_1 \oplus \Lambda_2 = \mathbb{Z}^{2g}\) and \(\omega|_{\Lambda_i}\) is a principal polarization i.e., a form of type \((1, \ldots, 1)\). Set \(G' := \{a \in \text{Sp}(2g, \mathbb{R}) : a(V_i) = V_i\ \text{for}\ \ i = 1, 2\}\). Fix a basis \(\{e_1, \ldots, e_g\}\) of \(V_i^{1,0}\) and a basis \(\{e_{k+1}, \ldots, e_g\}\) of \(V_2^{1,0}\). Using the basis \(\{e_1, \ldots, e_g\}\) as in 5.1, the matrix representation (5.1) and the identifications (5.2) we see that \(g' = \mathfrak{t}' \oplus \mathfrak{m}'\), where

\[
\mathfrak{t}' = \left\{ X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \ X_1 \in u(k), \ X_2 \in u(g-k) \right\}, \\
\mathfrak{m}' = \left\{ Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}, \ Y_1 \in \mathfrak{g}_k, \ Y_2 \in \mathfrak{g}_{g-k} \right\}.
\]

Thus \(G' \cong \text{Sp}(2k, \mathbb{R}) \times \text{Sp}(2g-2k, \mathbb{R})\) and \(G'_J = U(k) \times U(g-k)\). It follows that \(\tilde{Z}_k := G' \cdot J\) is a totally geodesic submanifold of \(\mathfrak{g}_k\) isometric to \(\mathfrak{g}_k \times \mathfrak{g}_{g-k}\). Clearly \(\pi(\tilde{Z}_k) = Z_k\). This proves (a).

(b) It is enough to check that \(\tilde{Z}_k\) is a maximal totally geodesic submanifold of \(\mathfrak{g}_k\). Totally geodesic submanifolds are in bijective correspondence with Lie triple systems in \(\mathfrak{m}\) see 5.2. Thus it is enough to prove that the Lie triple system \(\mathfrak{m}'\) in (5.5) corresponding to \(\tilde{Z}_k\) is a maximal Lie triple system in \(\mathfrak{m}\). Denote by \(U\) the vector space of complex \((g-k) \times k\) matrices. The orthogonal complement of \(\mathfrak{m}'\) in \(\mathfrak{m}\) with respect to the Killing form (which is a multiple of the trace) is the space

\[
\mathfrak{m}'' = \left\{ Y = \begin{pmatrix} 0 & 1 \xi \\ \xi & 0 \end{pmatrix}, \ \xi \in U \right\}.
\]

Identify \(\mathfrak{m}''\) with \(U\) by the correspondence \(Y \leftrightarrow \xi\). Then, using (5.3), for \(X \in \mathfrak{t}'\) as in (5.5) and \(Y \in \mathfrak{m}'\) as in (5.6) we get

\[
\text{ad}_{\mathfrak{t}}(X)(Y) = X_2 \xi - \xi X_1 = X_2 \xi + \xi \cdot X_1.
\]

Thus the representation of \(\mathfrak{t}'\) on \(\mathfrak{m}'\) reduces to the representation

\[
\text{u}(k) \oplus \text{u}(g-k) \rightarrow \mathfrak{gl}(U), \ (X_1, X_2) \cdot \xi = X_2 \xi + \xi \cdot X_1.
\]

This is an irreducible representation, since it is the outer tensor product of the standard representations of \(\text{u}(k)\) and \(\text{u}(g-k)\), see e.g., [23, p. 197]. Thus \(\mathfrak{m}''\) is an irreducible \(\mathfrak{t}'\)-module. Assume now that \(\mathfrak{m}'''\) is a Lie triple system such that \(\mathfrak{m}' \subset \mathfrak{m}''' \subset \mathfrak{m}\). Set \(q := \mathfrak{m}'' \cap \mathfrak{m}'''\). Then

\[
[\mathfrak{t}', q] = [[\mathfrak{m}', \mathfrak{m}'], q] \subset [[\mathfrak{m}'', \mathfrak{m}''], \mathfrak{m}'''] \subset \mathfrak{m}'''.
\]

Moreover \([\mathfrak{t}', q] \subset [\mathfrak{t}', \mathfrak{m}'] \subset \mathfrak{m}''\). Thus \([\mathfrak{t}', q] \subset q\), i.e., \(q\) is a \(\mathfrak{t}'\)-submodule of \(\mathfrak{m}''\). Since \(\mathfrak{m}''\) is an irreducible \(\mathfrak{t}'\)-module, there are two possibilities: either \(q = \{0\}\) and \(\mathfrak{m}''' = \mathfrak{m}'\), or \(q = \mathfrak{m}''\) and \(\mathfrak{m}''' = \mathfrak{m}\). These possibilities correspond to \(\tilde{Z} = \tilde{Z}_k\) and \(\tilde{Z} = \mathfrak{g}_k\) respectively. Thus \(\tilde{Z}_k\) and \(Z_k\) are indeed maximal.

(c) Set \(W := h(A_{g-k})\). Assume that \(A_0 \cong (V_0/\Lambda_0, J_0, \omega_0)\). Choose complementary sublattices \(\Lambda_i \subset \mathbb{Z}^{2g}\) such that \(\omega|_{\Lambda_i}\) is principal and there is a symplectic isomorphism \(\eta : (\Lambda_0, \omega_0) \rightarrow (\Lambda_1, \omega|_{\Lambda_1})\). Set \(V_i := \Lambda_i \otimes \mathbb{R}\). Set \(\tilde{W} := \{J' \in \mathfrak{g}_g : J'(V_i) = V_i\ \text{for}\ \ i = 1, 2\}.\)
Then \( Z \) an isometric immersion. Next let \( m \) of genus \( g \) stable curves of the form Deligne–Mumford compactification. Let \( \tilde{\eta} \) isometric, have nothing to prove. Otherwise we can choose a totally geodesic submanifold \( \tilde{Z} \subset \tilde{\eta} \) such that \( \pi(\tilde{Z}) = Z \) and \( \tilde{W} \cap \tilde{Z} \neq \emptyset \). This intersection is totally geodesic. Since \( \tilde{h} \) is isometric, \( \tilde{h}^{-1}(\tilde{W} \cap \tilde{Z}) \) is totally geodesic in \( \tilde{\eta} \) and it is a connected component of \( \pi^{-1}(h^{-1}(Z)) \). This proves (c). \( \square \)

**5.7.** Let \( M_g \) and \( \overline{M}_g \) denote the moduli space of curves of genus \( g \) and its Deligne–Mumford compactification. Let \( \Delta^0 \) denote the set of points in \( \overline{M}_g \) that represent stable curves of the form \( E \cup_p C \), where \( E \) is a smooth elliptic curve, \( C \) is a smooth curve of genus \( g-1 \), \( p \in C \) and the notation \( E \cup_p C \) means that \( 0 \in E \) and \( p \in C \) are identified. Then \( \Delta_1 = \Delta^0 \). The map

\[
\psi : M_{1,1} \times M_{g-1,1} \rightarrow \Delta^0_1, \quad \psi([E], [C, p]) := [E \cup_p C].
\]

(5.7)
is an isomorphism. Let \( \pi_1 : M_{1,1} \times M_{g-1,1} \rightarrow M_{1,1} \) and \( \pi_2 : M_{1,1} \times M_{g-1,1} \rightarrow M_{g-1,1} \) be the projections. Set

\[
p_1 := \pi_1 \circ \psi^{-1} : \Delta^0_1 \rightarrow M_{1,1}, \quad p_2 := \pi_2 \circ \psi^{-1} : \Delta^0_1 \rightarrow M_{g-1,1}.
\]

(5.8)

The following result uses the same argument as in [29, §4].

**Theorem 5.8.** Assume \( g \geq 4 \). If \( Y \subset M_g \) is an irreducible divisor and \( \bar{Y} \) is its closure in \( \overline{M}_g \), then \( p_1(\bar{Y} \cap \Delta^0_1) = M_{1,1} \).

**Proof.** Both \( M_g \) and \( \overline{M}_g \) have quotient singularities and are therefore \( \mathbb{Q} \)-factorial. So we can find a line bundle \( L \rightarrow \overline{M} \), a section \( s \in H^0(\overline{M}, L) \) and an integer \( m > 0 \) such that \( m\bar{Y} \) is the zero divisor of \( s \).

A basis for \( \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \) is given by \( \{\lambda, \delta_0, \ldots, \delta_{[g/2]}\} \), where \( \lambda \) denotes the determinant of the Hodge bundle and \( \delta_i \) are the boundary divisors. (These are not line bundles on \( M_g \), but on the moduli stack \( \overline{M}_g \). We are interested in properties that do not change when a divisor/line bundle is multiplied by a positive integer. So this is no harm.)

In \( \text{Pic}(\overline{M}_g) \otimes \mathbb{Q} \) we have \( L \equiv a\lambda + \sum_i d_i \delta_i \) for some \( a, d_i \in \mathbb{Q} \). It is well-known that \( a \neq 0 \). (Given \( x \in Y_{\text{reg}} \) there is a complete curve \( C \subset M_g \) such that \( x \in C \) and \( T_x C \cap T_x Y \), see [29, Remark 4.1, p. 431]. Hence \( \deg(L|_C) = a \cdot \deg(\lambda|_C) > 0 \).)

Since \( g-1 \geq 3 \) we can find a complete curve \( B \subset M_{g-1} \). Next we fix an arbitrary elliptic curve \( E \). For \( b \in B \) denote by \( \Gamma_b \) the smooth curve corresponding to the moduli point \( b \). For \( p \in \Gamma_b \) consider the nodal curve \( \Gamma_b \cup_p E \) obtained by gluing the points \( p \in \Gamma_b \) and \( 0 \in E \). Varying \( p \) and \( b \) we get a complete surface \( S \subset \Delta^0_1 \). Let \( C \subset S \) be the curve obtained by fixing a particular value \( b_0 \in B \). Observe that \( \Delta_i \cap S \neq \emptyset \) only for \( i = 1 \). Thus \( \delta_i|_S = 0 \) for \( i \neq 1 \) and

\[
L|_S \equiv a\lambda|_S + d_1 \delta_1|_S.
\]

Moreover \( \lambda|_C = 0 \), since the Hodge structure does not vary on \( C \). Thus

\[
L|_C \equiv d_1 \delta_1|_C.
\]
Thus the generic point of some irreducible divisor with \( \dim \) such that \( \bar{\lambda}_{|S} \equiv 0 \). Since \( a \neq 0 \), this implies that \( \bar{\lambda}_{|S} \equiv 0 \). But this is false: denote by \( p_3 : \Delta^0_1 \to \bar{M}_{g-1} \) the composition of \( p_2 \) with the obvious projection \( M_{g-1,1} \to \bar{M}_{g-1} \). Then \( \lambda_{|S} \cong (p^*_3 \lambda_{M_{g-1,1}} \otimes p^*_3 \lambda_{M_{g-1}})|_S \) and \( p_3 : S \to B \) has connected fibers, so \( (p_3)_*(\lambda_{|S}) = \lambda_{M_{g-1}}|_B \). Since \( \lambda_{M_{g-1}} \), \( B > 0 \), \( \lambda_{|S} \neq 0 \). We have proved that \( \bar{Y} \cap S \neq \emptyset \). If \( x \in \bar{Y} \cap S \subset \bar{Y} \cap \Delta^0_1 \), then \( p_1(x) = [E] \). Thus \( [E] \in p_1(\bar{Y} \cap \Delta^0_1) \). Since \( E \) is arbitrary the theorem is proved. \( \square \)

The following definition is standard in Riemannian geometry. In some sense it is the Riemannian analogue of the notion of non-degenerate projective variety.

**Definition 5.9.** An analytic subset \( X \subset A_g \) is *full* if there is no proper totally geodesic subvariety \( Z \not\subset A_g \) that contains \( X \).

**5.10.** Let \( \tilde{Z} \subset \mathcal{S}_g \) be a totally geodesic submanifold. Then the real codimension of \( \tilde{Z} \) in \( \mathcal{S}_g \) is at least \( g \). Indeed by a theorem of Berndt and Olmos [3] the real codimension of a totally geodesic submanifold of a Riemannian symmetric space is at least the rank of the symmetric space. Since the rank of \( \mathcal{S}_g \) is \( g \) the result follows immediately.

**Theorem 5.11.** Let \( j : M_g \to A_g \) be the period map. If \( g \geq 3 \) and \( Y \subset M_g \) is an irreducible divisor, then \( j(Y) \) is full in \( A_g \).

**Proof.** Assume first that \( g = 3 \). Then \( \dim M_3 = \dim A_3 = 6 \). If \( Y \subset M_3 \) is a hypersurface, also \( j(Y) \) is a hypersurface. If \( j(Y) \) is contained in a proper totally geodesic subvariety \( Z \), this is also a hypersurface. Then \( \tilde{Z} \subset \mathcal{S}_3 \) would be a totally geodesic submanifold of real codimension 2. This is impossible by the theorem of Berndt and Olmos quoted above.

We proceed by induction on \( g \). Assume that the result holds for \( g - 1 \) and that \( g \geq 4 \). Let \( Z \subset A_g \) be a totally geodesic subvariety and assume by contradiction that there is an algebraic hypersurface \( Y \subset M_g \) such that \( j(Y) \subset Z \). We want to prove that \( Z = A_g \).

The period map \( j \) extends to \( j : \tilde{M}_g - \Delta_0 \to A_g \) and \( j(\bar{Y} - \Delta_0) \subset Z \).

We claim that for any \( [E] \in A_1 \) there is an irreducible divisor \( Y_E \subset \tilde{M}_{g-1} \) such that

\[
([E]) \times j(Y_E) \subset j(\bar{Y} - \Delta_0).
\]

Let \( \psi : A_1 \times M_{g-1,1} \to \Delta^0_1 \subset \tilde{M}_g \) be the map \( \psi([E], [C, p]) = [E \cup_p C] \) as in (5.7). By Theorem 5.8 the intersection \( \bar{Y} \cap \Delta^0_1 \) is non-empty and has dimension \( 3g - 5 \). Since \( \psi \) is an isomorphism, the set \( W := \psi^{-1}(\bar{Y} \cap \Delta^0_1) \) is a closed algebraic subset of \( M_{1,1} \times M_{g-1,1} \) of dimension \( 3g - 5 \). Let

\[
\pi_1 : M_{1,1} \times M_{g-1,1} \to M_{1,1}, \quad \pi_2 : M_{1,1} \times M_{g-1,1} \to M_{g-1,1},
\]

\[
\pi_3 : M_{g-1,1} \to M_{g-1}, \quad q := \pi_1|_W : W \to M_{1,1},
\]

be the obvious projections. Fix \( [E] \in M_{1,1} \). By Theorem 5.8 there is \( [C, p] \in M_{g-1,1} \) such that \( [E \cup_p C] \in \bar{Y} \). Then \( q^{-1}([E]) = ([E]) \times W \) for a closed subset \( W' \subset \tilde{M}_{g-1,1} \) with \( \dim W' \geq 3g - 6 \). Set \( W'' := \pi_3(W') \subset M_{g-1} \). Since \( \dim W'' \geq 3g - 7 \), \( W'' \) contains the generic point of some irreducible divisor \( Y_E \) of \( M_{g-1} \). By construction for any
[C] ∈ W"", there is p ∈ C such that [E ∪p C] ∈ W. Moreover we have j([E ∪p C]) = [E × J(C)] as polarized abelian varieties. Thus [E × J(C)] = j([E ∪p C]) ∈ j(W) ⊂ j(Ŷ − ∆0). This holds in particular for the generic point of the divisor YE. Hence \([E] × j(Y_E) ⊂ j(Ŷ − ∆0)\). The claim is proved.

Now fix \([E] ∈ A_1\) and set

\[h : A_{g−1} \rightarrow A_g, \quad h([A]) := [E × A].\]

Since \(j(Ŷ − ∆0) ⊂ Z\), we have \(h ∘ j(Y_E) = ([E] × j(Y_E) ⊂ Z\). Thus

\[j(Y_E) ⊂ h^{-1}(Z).\]

By Proposition 5.6(c) \(h^{-1}(Z)\) is a totally geodesic subvariety of \(A_{g−1}\) and it contains \(j(Y_E)\). By the inductive hypothesis \(j(Y_E)\) is full in \(A_{g−1}\). So we conclude that \(h^{-1}(Z) = A_{g−1}\). Using the notation of (5.4) this means that \(ϕ_1([E] × A_{g−1}) ⊂ Z\). But \([E] ∈ A_1\) is arbitrary, so we have in fact \(Z_1 = ϕ_1(A_1 × A_{g−1}) ⊂ Z\). By Proposition 5.6(b) \(Z_1\) is a maximal totally geodesic subvariety of \(A_g\). Therefore either \(Z = Z_1\) or \(Z = A_g\). The first possibility is absurd since \(Z\) contains by hypothesis the Jacobians of smooth curves.

Hence we have proved that \(Z = A_g\), i.e., that \(j(Y)\) is full. □

**Remark 5.12.** This result of course implies that the Jacobian locus \(j(M_g)\) itself is full for \(g ≥ 3\). This can be proved directly (and easily) using the same argument.

**Remark 5.13.** The second fundamental form of \(M_g\) in \(A_g\) has been studied in [16] using the Hodge–Gaussian maps (see also [37]). It follows from the analysis in that paper that the second fundamental form is non-zero along Schiffer variations. (See also [12, 13] for related results.) Using this it has been proven in [14, Theorem 4.4] that any totally geodesic submanifold of \(A_g\) that is generically contained in \(j(M_g)\) has dimension at most \(\frac{3}{2}(g − 1)\).

In particular if \(Y ⊂ M_4\) a hypersurface, then \(j(Y)\) is not totally geodesic in \(A_g\). This yields a different proof of the theorem for \(g = 4\). Indeed assume by contradiction that \(Z ⊂ A_g\) is a proper totally geodesic subvariety and that \(j(Y) ⊂ Z\). By the theorem of Berndt and Olmos mentioned in 5.10, \(\dim_Z Z ≤ 16\). Since \(\dim_Y Y = 16\), \(j(Y)\) would be open in \(Z\), so \(j(Y)\) would be totally geodesic. But then we should have \(8 = \dim j(Y) ≤ \frac{3}{2}(g − 1) = \frac{15}{2}\), a contradiction.

**Corollary 5.14.** Let \(g ≥ 3\). Let \(Z ⊂ A_g\) be a special subvariety, i.e., a Hodge locus for the canonical variation of Hodge structure on \(A_g\). Then \(j^{-1}(Z)\) has codimension at least two in \(M_g\).

**Proof.** This follows immediately from Theorem 5.11 using the theorem of Mumford mentioned in 5.4. □

**Remark 5.15.** It is well-known that for a very general \([A] ∈ A_g\) we have \(MT(A) = GSp(H^1(A))\). By Corollary 5.14 if \([C] ∈ M_g\) is very general, then \(j([C])\) belongs to no proper Hodge locus of \(A_g\), hence \(MT(C) = GSp(H^1(C))\). This argument yields another proof of Proposition 3.11.
Remark 5.16. It might be interesting to notice that the previous results give some information about the monodromy along a divisor. Indeed let $Y \subset M_g$ be an irreducible divisor and let $y \in Y$ be a very general point. Let $C_y$ be a curve with moduli point $y$. By the Corollary 5.14 $MT(C_y) = GSp(2g)$. Denote by $\Gamma_y$ the monodromy group at $y$ of the variation of Hodge structure over $Y$. Let $\overline{\Gamma}_y$ be the Zariski closure of $\Gamma_y$ inside $GL(g, \mathbb{Q})$ and let $H_y := \overline{\Gamma}_y^0$ be the connected component of the identity. By a result of André [2, Theorem 1], $H_y$ is a normal subgroup of the commutator subgroup of $MT(y)$. (In André’s paper this result is formulated for variations of mixed Hodge structure. The statement in the pure case is simpler, see [36, Theorem 16].) Since $Sp(g, \mathbb{Q})$ is simple, this means that $H_y = Sp(g, \mathbb{Q})$. Thus a finite index subgroup of the monodromy along $Y$ is Zariski dense in $Sp(g, \mathbb{Q})$.

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