An Extreme-Value Approach for Testing the Equality of Large U-Statistic Based Correlation Matrices

Cheng Zhou∗, Fang Han†, Xinsheng Zhang‡ and Han Liu§

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Abstract

There has been an increasing interest in testing the equality of large Pearson’s correlation matrices. However, in many applications it is more important to test the equality of large rank-based correlation matrices since they are more robust to outliers and nonlinearity. Unlike the Pearson’s case, testing the equality of large rank-based statistics has not been well explored and requires us to develop new methods and theory. In this paper, we provide a framework for testing the equality of two large U-statistic based correlation matrices, which include the rank-based correlation matrices as special cases. Our approach exploits extreme value statistics and the Jackknife estimator for uncertainty assessment and is valid under a fully nonparametric model. Theoretically, we develop a theory for testing the equality of U-statistic based correlation matrices. We then apply this theory to study the problem of testing large Kendall’s tau correlation matrices and demonstrate its optimality. For proving this optimality, a novel construction of least favourable distributions is developed for the correlation matrix comparison.

Keyword: Extreme value type I distribution; U-statistics; Hypothesis testing; Kendall’s tau; Jackknife variance estimator.

1 Introduction

Let $X = (X_1, \ldots, X_d)^T$ and $Y = (Y_1, \ldots, Y_d)^T$ be two $d$-dimensional random vectors. We denote $X_1, \ldots, X_{n_1}$ to be $n_1$ independent samples of $X$ and $Y_1, \ldots, Y_{n_2}$ to be $n_2$ independent samples of $Y$. Letting $n := \max\{n_1, n_2\}$, we aim to test the equality of the U-statistic based correlation matrices (e.g., Kendall’s tau or Spearman’s rho) of $X$ and $Y$. We consider the high dimensional regime that $d, n \to \infty$ and $d/n$ does not necessarily go to zero as $n \to \infty$. This problem has important

∗Department of Statistics, Management School, Fudan University, China; e-mail: chengzhmike@gmail.com. The work was carried out during a visit in the SMiLe lab at the Princeton University.
†Department of Biostatistics, Johns Hopkins University, Baltimore, MD 21205, USA; e-mail: fhan@jhu.edu
‡Department of Statistics, Management School, Fudan University, China; e-mail: xszhang@fudan.edu.cn
§Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA; e-mail: hanliu@princeton.edu
applications, including portfolio selection (Markowitz, 1991), high dimensional discriminant analysis (Han et al., 2013; Mai and Zou, 2013) and gene selection (Ho et al., 2008; Hu et al., 2009, 2010).

When $d/n \to 0$, Anderson (2003) and Muirhead (2009) study the problem of testing the equality of two Pearson’s correlation matrices. Major test criteria include the likelihood ratio (Anderson, 2003), spectral norm of difference (Roy, 1957) and Frobenius norm of difference (Nagao, 1973). When $d/n \not\to 0$, the likelihood ratio test and the tests in Roy (1957) and Nagao (1973) perform poorly, as the Pearson’s sample correlation matrices no longer converge to their population counterparts under the spectral norm (Bai and Yin, 1993). A line of research aims to correct the aforementioned tests or proposing new methods. For the likelihood ratio test, Bai et al. (2009) introduce a corrected LRT test which works when $d/n \to c \in (0,1)$, and Jiang et al. (2012) generalize it to the case when $d < n$ and $c = 1$. As a generalization of Nagao’s proposal, Schott (2007) and Li and Chen (2012) propose new test statistics based on an unbiased estimator of the Frobenius norm of the matrix difference, and Srivastava and Yanagihara (2010) propose another test statistic based on the difference of two Frobenius norms. Recently, Cai et al. (2013) propose a method based on the sup-norm of the matrix difference and prove its rate optimality under a sparse alternative.

In many applications, it is more meaningful to test the equality of two rank-based correlation matrices but instead of the Pearson’s correlation matrices. In particular, Embrechts et al. (2003) point out that the Pearson’s correlation coefficient “might prove very misleading” in measuring the dependence and advocate the usage of rank correlation coefficients, such as Kendall’s tau (Kendall, 1938) or Spearman’s rho (Spearman, 1904). Though testing the equality of high dimensional rank-based correlation matrices is of fundamental importance, there has been very little work in this area. To bridge this gap, this paper proposes a unified framework for testing the equality of two large U-statistic based correlation matrices $U_1$ and $U_2$, which includes the rank-based correlation matrices as special examples. More specifically, let $U_1 = (u_{1,ij})$ be a type of correlation matrix of $X$ and all the elements of $U_1$ be U-statistics. Similarly to $U_1$, we define $U_2 = (u_{2,ij})$ to be the same kind of U-statistic based correlation matrix of $Y$. In this paper, we aim to test the hypothesis

$$
H_0 : U_1 = U_2 \ v.s. \ H_1 : U_1 \neq U_2.
$$

(1.1)

Testing (1.1) plays an important role in many fields. For example, testing the equality of two Kendall’s tau correlation matrices $U_{1}^\tau$ and $U_{2}^\tau$,

$$
H_0^\tau : U_{1}^\tau = U_{2}^\tau \ v.s. \ H_1^\tau : U_{1}^\tau \neq U_{2}^\tau,
$$

(1.2)

can be used to test the model of copula discriminant analysis (Han et al., 2013; Mai and Zou, 2013).

There are 4 major contributions of this paper. First, for the first time in the literature, we develop a unified framework for testing the equality of two large U-statistic based correlation matrices. This framework builds upon a fully nonparametric model and enables us to conduct homogeneity tests using a wide range of correlation measures. Secondly, as a special example, we examine the problem of testing the equality of two large Kendall’s tau matrices and prove the minimax optimality of the proposed method. Thirdly, we further propose alternative approaches for testing $U_{1}^\tau = U_{2}^\tau$, which attain better empirical performance than the Jackknife based one.

1Such U-statistic based correlation measures are quite general. For example, $u_{1,ij}$ can represent the Kendall’s tau correlation coefficient between $X_i$ and $X_j$. 

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Finally, to develop a theory of testing the equality of general U-statistic based correlation matrices, we develop an upper bound of Jackknife variance estimation error, which enables us to obtain the explicit rate of convergence. For Kendall’s tau matrices, we prove an upper bound of the traditional plug-in variance estimation error and an upper bound of the variance difference between two Kendall’s tau correlation coefficients. These upper bounds allow us to exploit extreme value theory under the dependent setting to prove theorems in this paper. Their constructions are nontrivial and are of independent technical interest. To prove the optimality of the proposed testing methods for Kendall’s tau matrices, we construct a collection of least favorable distributions with regard to the test hypothesis. This construction technique is novel and tailored for testing the equality of correlation matrices. In contrast, the construction in Cai et al. (2013) only perturbs the diagonal elements of the covariance, which does not affect the resulting correlation matrices.

1.1 More Related Works

Apart from the Pearson’s correlation coefficient and general U-statistic based correlation measurements studied in this paper, existing literature also considers other measures of dependence. These include the distance correlation (Székely et al., 2007) and randomized dependence coefficient (Lopez-Paz et al., 2013). To the best of our knowledge, there is no work discussing testing the equality of dependence structure with regard to these dependent measures.

Our work is closely related to the random matrix theory on rank correlation matrices. Bai and Zhou (2008), Zhou (2007) and Bao et al. (2013) study the theoretical properties of large Spearman’s rho correlation matrix \( U^\rho \). Specifically, Bai and Zhou (2008) prove the Marchenko-Pastur law for the limiting spectral distribution of \( U^\rho \). Zhou (2007) prove the extreme value type I distribution for \( \max_{k \neq k'} |U_{jk}^\rho| \), and Bao et al. (2013) derive the limiting distributions of traces of all higher moments of \( U^\rho \). Most of these results hold only under the independence setting, i.e., the entries of \( X \) are independent of each other. In contrast, our work focuses on the dependent setting.

Our work is also related to the robust testing, where the test statistics are robust estimators of the Pearson’s covariance/correlation coefficients. These include S-estimators and some robust dispersion estimators. We refer to O’Brien (1992), Aslam and Rocke (2005) and the references therein for details. Our work is also related to the adaptive estimation of a large correlation/covariance matrix (Cai and Liu, 2011) or a large Gaussian (copula) graphical model. See, for example, Bickel and Levina (2008), Zhao et al. (2014), Ravikumar et al. (2011) and Liu et al. (2012).

1.2 Notation

We denote \( \|v\|_2 = (\sum_{j=1}^d v_j^2)^{1/2} \) as the Euclidean norm of a vector \( v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d \). For a matrix \( A = (a_{ij}) \in \mathbb{R}^{d \times q} \), we define its spectral norm \( \|A\|_2 := \sup_{\|x\|_2 \leq 1} \|Ax\|_2 \) and Frobenius norm \( \|A\|_F := \sqrt{\sum_{i,j} a_{ij}^2} \). We define the matrix entrywise sup-norm as \( \|A\|_{\text{max}} := \max \{|a_{ij}|\} \). We use \( \text{Rank}(A) \) to denote the rank of \( A \). If \( A \) is a square matrix, we define \( \text{Diag}(A) \) to be a diagonal matrix with the same main diagonal as \( A \). We use \( I_d \) to denote an identity matrix of size \( d \). For two sequences of real numbers \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n = O(b_n) \) if there exists a constant \( C \) such that \( |a_n| \leq C |b_n| \) holds for all sufficiently large \( n \), write \( a_n = o(b_n) \) if \( a_n/b_n \to 0 \), and write \( a_n \asymp b_n \) if there exist constants \( C > c > 0 \) such that \( c |b_n| \leq |a_n| \leq C |b_n| \) for all sufficiently large \( n \).
For a square matrix $\Sigma \in \mathbb{R}^{d \times d}$, we use $\lambda_{\min}(\Sigma)$ and $\lambda_{\max}(\Sigma)$ to denote the minimal and maximal eigenvalues of $\Sigma$. For a set $B$, we use $|B|$ to denote its cardinality.

1.3 Paper Organization

The rest of this paper is organized as follows. Section 2 formalizes the problem, describes a general testing procedure and analyzes the theoretical properties (e.g., size and power) of the proposed test. In Section 3 we focus on testing large Kendall’s tau matrices, for which we consider two models: a fully nonparametric model and a semiparametric Gaussian copula model. Under certain modeling assumptions, we propose additional tests for the Kendall’s tau matrices which have better empirical performance compared to the general testing procedure. Section 4 provides thorough numerical results on both simulated and real equity data. In Section 5 we discuss potential future work. All proofs are collected in the appendix and supplementary materials.

2 A General Procedure for Testing U-Statistic Based Matrices

This section presents a generic testing method for U-statistic based matrix comparison. In Section 2.1 we describe the proposed testing procedure. In Section 2.2 we analyze its asymptotic size and power. In Section 2.3 we consider comparing a row or column of U-statistic based matrices.

Let $X = (X_1, \ldots, X_d)^T$ and $Y = (Y_1, \ldots, Y_d)^T$ be two $d$-dimensional random vectors independent of each other. We denote $X_1, \ldots, X_{n_1}$ to be random samples from $X$ with $X_i = (X_{i1}, X_{i2}, \ldots, X_{id})^T$. Similarly, $Y_1, \ldots, Y_{n_2}$ are random samples from $Y$ with $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{id})^T$.

For $i, j = 1, \ldots, q$, let $\Phi_{ij}$ be a $m$-th order kernel function defined as

$$\Phi_{ij} : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ with the symmetric property: } \Phi_{ij} = \Phi_{ji}^2. \quad (2.1)$$

Thus, we have a family of functions $\left\{ \Phi_{ij}, 1 \leq i, j \leq q \right\}$. Furthermore, each $\Phi_{ij}$ is a symmetric Borel measurable function with kernel order $m$ fixed\(^3\). We assume that $\Phi_{ij}$ is uniformly bounded. Many useful U-statistics satisfy these conditions. We set

$$\hat{u}_{1,ij} := \left( \frac{n_1}{m} \right)^{-1} \sum_{1 \leq \ell_1 < \cdots < \ell_m \leq n_1} \Phi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}),$$

$$\hat{u}_{2,ij} := \left( \frac{n_2}{m} \right)^{-1} \sum_{1 \leq \ell_1 < \cdots < \ell_m \leq n_2} \Phi_{ij}(Y_{\ell_1}, \ldots, Y_{\ell_m}).$$

We then define the following U-statistic based matrices $\hat{U}_a \in \mathbb{R}^{q \times q}$ for $a = 1, 2$:

$$\hat{U}_1 := (\hat{u}_{1,ij})_{1 \leq i, j \leq q} \quad \text{and} \quad \hat{U}_2 := (\hat{u}_{2,ij})_{1 \leq i, j \leq q}. \quad (2.2)$$

Correspondingly, we use $U_a := (u_{a,ij})_{1 \leq i, j \leq q}$ to denote the expectation of $\hat{U}_a$, i.e., $u_{a,ij} = \mathbb{E}[\hat{u}_{a,ij}]$.

We can view $U_1$ and $U_2$ as a type of correlation matrices of $X$ and $Y$. We are interested in\(^2\) the symmetry requirement can be easily relaxed. Here we pose this constraint merely for presentation clearness.

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\(^3\) We assume each $\Phi_{ij}$ has the same fixed kernel order $m$ for presentation clearness. It is straightforward to extend to the setting that $m$’s are uniformly bounded.
testing the equality of $\mathbf{U}_1$ and $\mathbf{U}_2$, which includes testing the equality of two large Kendall’s tau or Spearman’s rho correlation matrices.

We note that $q$ is the row and column number of $\mathbf{U}_a$ and $\hat{\mathbf{U}}_a$, while $d$ is the dimension of $\mathbf{X}$ and $\mathbf{Y}$. $q$ and $d$ can be different. Therefore, the framework considered in this paper is quite general. For example, it allows $\mathbf{U}_a$ to represent the dependence structure on a dimension reduced data, where the dimension reduction step is incorporated in the kernel function $\{\Phi_{ij}, 1 \leq i, j \leq q\}$.

**Remark 2.1.** We can relax $\Phi_{ij}$ to be an asymmetric kernel function without loss of generality. Specifically, an asymmetric kernel $\Phi(\cdot)$ gives a U-statistic

$$\hat{u} = \frac{1}{m!} \binom{n}{m}^{-1} \sum \Phi(\mathbf{x}_{\ell_1}, \ldots, \mathbf{x}_{\ell_m}),$$

where the summation is taken over all combinations of distinct elements $\{\ell_1, \ldots, \ell_m\}$ from $\{1, \ldots, n\}$. Using Hoeffding’s method (Hoeffding, 1948), $\hat{u}$ is also a U-statistic of the symmetric kernel $\Phi^0(\cdot)$:

$$\Phi^0(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \frac{1}{m!} \sum \Phi(\mathbf{x}_{\alpha_1}, \ldots, \mathbf{x}_{\alpha_m}),$$

where the summation is taken over all permutations of $\{1, \ldots, m\}$.

The Kendall’s tau matrix is an example of the U-statistic based matrix defined in (2.2). More specifically, we set

$$\Phi_{ij}(\mathbf{X}_k, \mathbf{X}_\ell) = \text{sign}(X_{ki} - X_{\ell i})\text{sign}(X_{kj} - X_{\ell j}), \quad \Phi_{ij}(\mathbf{Y}_k, \mathbf{Y}_\ell) = \text{sign}(Y_{ki} - Y_{\ell i})\text{sign}(Y_{kj} - Y_{\ell j}),$$

and $q = d$. The Kendall’s tau sample correlation coefficients $\hat{\tau}_{1,ij}$ and $\hat{\tau}_{2,ij}$ are then defined as

$$\hat{\tau}_{1,ij} := \frac{2}{n_1(n_1 - 1)} \sum_{1 \leq k < \ell \leq n_1} \text{sign}(X_{ki} - X_{\ell i})\text{sign}(X_{kj} - X_{\ell j}),$$

$$\hat{\tau}_{2,ij} := \frac{2}{n_2(n_2 - 1)} \sum_{1 \leq k < \ell \leq n_2} \text{sign}(Y_{ki} - Y_{\ell i})\text{sign}(Y_{kj} - Y_{\ell j}).$$

Their population counterparts are $\tau_{a,ij} := E[\hat{\tau}_{a,ij}]$ for $a = 1, 2$. We then write the empirical and population Kendall’s tau matrix as

$$\hat{\mathbf{U}}^\tau_a = (\hat{\tau}_{a,ij}) \quad \text{and} \quad \mathbf{U}_a^\tau = (\tau_{a,ij}), \quad (2.3)$$

where $a = 1, 2$. In Section 3, we aim to test the equality of two large Kendall’s tau matrices.

### 2.1 A General Testing Procedure

For testing (1.1) in high dimensions, we use the sup-norm criterion. Such a choice is motivated by the fact that the sup-norm is very sensitive to perturbations on a small number of entries compared to the null hypothesis. Accordingly, we propose the test statistic:

$$M_n := \max_{1 \leq i, j \leq q} M_{ij} \quad \text{with} \quad M_{ij} := \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2}{\hat{\sigma}^2(\hat{u}_{1,ij}) + \hat{\sigma}^2(\hat{u}_{2,ij})} \quad \text{for} \quad 1 \leq i, j \leq q. \quad (2.4)$$
In (2.4), \( \hat{\sigma}^2(\hat{u}_{1,ij}) \) is a Jackknife estimator of \( \hat{u}_{1,ij} \)'s variance and is defined as
\[
\hat{\sigma}^2(\hat{u}_{1,ij}) := \frac{m^2(n_1 - 1)}{n_1(n_1 - m)^2} \sum_{\alpha=1}^{n}(q_{1\alpha,ij} - \hat{u}_{a,ij})^2,
\]
with
\[
q_{1\alpha,ij} := \left( \frac{n_1 - 1}{m - 1} \right)^{-1} \sum_{1 \leq \ell_j < \ell_{m-1} \leq n_1}^{\ell_j \neq a, j} \Phi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}).
\]

The definition of \( \hat{\sigma}^2(\hat{u}_{2,ij}) \) is similar for \( Y \).

For a given significance level \( 0 < \alpha < 1 \), we construct the test to be
\[
T_\alpha := 1 \{ M_n \geq G^-(\alpha) + 4 \log q - \log(\log q) \},
\]
where \( G^-(\alpha) := -\log(8\pi) - 2 \log( -\log(1 - \alpha)) \). We reject \( H_0 \) in (1.1) if and only if \( T_\alpha = 1 \).

In some applications, our interest is to compare a particular row or column of matrices, i.e., we aim at testing the hypothesis:
\[
H_{0,i} : u_{1,i*} = u_{2,i*} \text{ v.s. } H_{1,i} : u_{1,i*} \neq u_{2,i*},
\]
where \( u_{1,i*} \) and \( u_{2,i*} \) are the \( i \)-th rows of \( U_1 \) and \( U_2 \). To test this hypothesis, we construct a similar test statistic \( M_{n,i} = \max_{1 \leq j \leq q} M_{i,j} \) and the according test is
\[
T_{\alpha,i} := 1 \{ M_{n,i} > G'^-(\alpha) + 2 \log q - \log \log q \},
\]
where \( G'^-(\alpha) := -\log(\pi) - 2 \log(-\log(1 - \alpha)) \). We reject \( H_{0,i} \) if and only if \( T_{\alpha,i} = 1 \).

### 2.2 Theoretical Properties

Our main theoretical result is to characterize the limiting null distribution of \( M_n \). We further analyze the power of the proposed test under a sparse alternative.

We introduce three assumptions that will be used later. Assumption (A1) specifies the sparsity of \( U = U_1 = U_2 \). Assumption (A2) specifies the scaling of \( q, n \). Assumption (A3) is a technical condition that we impose for obtaining the limiting distribution of \( M_n \). In Section 3.2, we will show that Assumption (A3) can be further relaxed under a semiparametric Gaussian copula model.

In detail, for a fixed constant \( \alpha_0 > 0 \), we define
\[
\text{supp}_j(\alpha_0) := \{ 1 \leq i \leq q : |u_{1,ij}| \geq (\log q)^{-1-\alpha_0} \text{ or } |u_{2,ij}| \geq (\log q)^{-1-\alpha_0} \}.
\]
\( \text{supp}_j(\alpha_0) \) is the set of indices \( i \) such that either the \( i \)-th variable of \( X \) is highly correlated \(|u_{a,ij}| > (\log q)^{-1-\alpha_0}\) with the \( j \)-th variable of \( X \), or the \( i \)-th variable of \( Y \) is highly correlated with the \( j \)-th variable of \( Y \). We then introduce Assumption (A1) as follows.

(A1). We assume that there exits a subset \( \Gamma \subset \{1, 2, \ldots, q\} \) with \( |\Gamma| = o(q) \) and a constant \( \alpha_0 > 0 \) such that for all \( \gamma > 0 \), we have
\[
\max_{1 \leq j \leq q} |\text{supp}_j(\alpha_0)| = o(q^\gamma).
\]
Before stating Assumption (A2), we first introduce some additional notations. We define
\[ \Psi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}) := \Phi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}) - u_{1,ij}. \]
For \( \ell = 1, \ldots, n_1 \), we also denote \( g_{ij}(X_{\ell}) \) and \( h_{ij}(X_{\ell}) \) as
\[ g_{ij}(X_{\ell}) := \mathbb{E}[\Phi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m})|X_{\ell}], \quad h_{ij}(X_{\ell}) := \mathbb{E}[\Psi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m})|X_{\ell}], \quad (2.9) \]
where \( \{\ell_1, \ldots, \ell_m\} \) is an arbitrary subset of \( \{1, \ldots, n_1\} \) with distinct elements and contains \( \ell \). \( g_{ij}(Y_{\ell}) \) and \( h_{ij}(Y_{\ell}) \) are similarly defined for \( \ell = 1, \ldots, n_2 \). We then denote \( \zeta_{1,ij} \) to be the variance of \( g_{ij}(X_{\ell}) \), i.e.,
\[ \zeta_{1,ij} := \mathbb{E}[\mathbb{E}[\Psi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m})|X_{\ell}]^2] = \text{Var}(g_{ij}(X_{\ell})). \quad (2.10) \]
Similarly, we define \( \zeta_{2,ij} := \text{Var}(g_{ij}(Y_{\ell})). \)

With these introduced notations, we are now ready to state Assumption (A2).

(A2). We assume \( n_1 \asymp n_2 \asymp n \). We also assume \( \log q = O(n^{1/3-\epsilon}) \) for an arbitrary \( \epsilon > 0 \) and \( \zeta_{a,ij} > r_a > 0 \) for \( a = 1, 2 \), where \( r_1 \) and \( r_2 \) are constants which are irrelevant to \( i \) or \( j \).

The condition that \( \zeta_{a,ij} > r_a > 0 \) is mild. It is used to exclude the degenerate cases of U-statistics and has been widely used for analyzing U-statistics.

To describe Assumption (A3), we write
\[ S = \{(i,j) : 1 \leq i, j \leq q\} \quad \text{and} \quad S_0 = \{(i,j) : 1 \leq i \leq q, i \in \text{supp}(\alpha_0)\}. \quad (2.11) \]
By the definition of \( S_0 \), for any \( (i,j) \in S \setminus S_0 \), we have \( |u_{a,ij}| \leq (\log q)^{-1-\alpha_0} \). Moreover, we use \( u_{1,ijk\ell} \) and \( u_{2,ijk\ell} \) to denote \( \mathbb{E}[g_{ij}(X_{\ell})g_{k\ell}(X_{\ell})] \) and \( \mathbb{E}[g_{ij}(Y_{\ell})g_{k\ell}(Y_{\ell})] \).

(A3). For any \( (i,j) \) and \( (k,\ell) \) in \( S \setminus S_0 \), we assume \( u_{a,ijk\ell} = O((\log q)^{-1-\alpha_0}) \) for \( a = 1 \) or \( 2 \).

Under fully nonparametric models, we note that \( u_{a,ijk\ell} \) is estimable (Klüppelberg and Kuhn, 2009). Thus it is possible to verify Assumption (A3) in applications. When we test the equality of two Kendall’s tau correlation matrices \( U^1_\tau \) and \( U^2_\tau \), under a semiparametric Gaussian copula model, Assumption (A3) can be replaced by a simplified condition which is easier to be verified. More details are provided in Section 3.2.

Under the above assumptions, our main theoretical result quantifies the limiting distribution of the extreme value statistic \( M_n \).

**Theorem 2.2.** Assuming (A1), (A2), (A3) hold, under \( H_0 \) of (1.1), we have
\[ P(M_n - 4 \log q + \log(\log q) \leq x) \rightarrow \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right)\right), \quad (2.12) \]
for any \( x \in \mathbb{R} \), as \( n, q \rightarrow \infty \). Furthermore, (2.12) holds uniformly for all random vectors \( X \) and \( Y \) satisfying Assumptions (A1), (A2) and (A3).
Proof. We list a sketch of this proof. The detailed proof is in Appendix C.1. The proof proceeds in three steps.

**Step (i) (Sketch).** We set \( \hat{\sigma}^2(\hat{u}_{a,ij}) \) as the Jackknife estimator of \( \hat{u}_{a,ij} \) and \( \sigma^2(\hat{u}_{a,ij}) \) as the true variance of \( \hat{u}_{a,ij} \). We then analyze the estimation error of Jackknife variance estimator by providing an upper bound of \( |n_a \hat{\sigma}^2(\hat{u}_{a,ij}) - m^2 \zeta_{a,ij}| \), where \( \zeta_{a,ij} \) is defined in (2.10). The central limit theorem for U-statistics (Lemma F.3) implies that \( m^2 \zeta_{a,ij} \) is the limit of \( n_a \sigma^2(\hat{u}_{a,ij}) \) as \( n_a \) goes to infinity. This motivates us to define

\[
M_{ij} := \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2}{\hat{\sigma}^2(\hat{u}_{1,ij}) + \hat{\sigma}^2(\hat{u}_{2,ij})} \quad \text{and} \quad \tilde{M}_{ij} := \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2}{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}.
\]

In \( \tilde{M}_{ij} \), we use \( m^2 \zeta_{a,ij}/n_1 \) to replace \( \hat{\sigma}^2(\hat{u}_{a,ij}) \) of \( M_{ij} \).

Using the obtained upper bound of \( |n_a \hat{\sigma}^2(\hat{u}_{a,ij}) - m^2 \zeta_{a,ij}| \), we then prove that \( \max_{1 \leq i, j \leq q} M_{ij} \) and \( \max_{1 \leq i, j \leq q} \tilde{M}_{ij} \) have the same limiting distribution, i.e., to obtain Theorem 2.2, it suffices to prove that

\[
\lim_{n, q \to \infty} \mathbb{P}(\tilde{M}_n - 4 \log q + \log(\log q) \leq x) = \exp \left( - \exp(-x/2)/\sqrt{8\pi} \right),
\]

where \( \tilde{M}_n := \max_{1 \leq i, j \leq q} \tilde{M}_{ij} \).

**Step (ii) (Sketch).** We use the Hoeffding decomposition (Lemma F.4) to decompose the U-statistic \( \tilde{u}_{a,ij} := \hat{u}_{a,ij} - u_{a,ij} \). By the definition of \( \tilde{u}_{a,ij} \), we have \( \mathbb{E}[\tilde{u}_{a,ij}] = 0 \). By Hoeffding decomposition, we decompose \( \tilde{u}_{a,ij} \) into two pieces. One is the sum of independent and identically distributed (i.i.d.) random variables and the other is the residual term. More specifically, we decompose \( \tilde{u}_{a,ij} \) as

\[
\tilde{u}_{1,ij} = \frac{m}{n_1} \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha) + \left( \frac{n_1}{m} \right)^{-1} \Delta_{n_1,ij} \quad \text{and} \quad \tilde{u}_{2,ij} = \frac{m}{n_2} \sum_{\alpha=1}^{n_2} h_{ij}(Y_\alpha) + \left( \frac{n_2}{m} \right)^{-1} \Delta_{n_2,ij},
\]

where we set

\[
\Delta_{n_1,ij} = \sum_{1 \leq \ell_1 < \ell_2 < \ldots < \ell_m \leq n_1} (\Phi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}) - u_{1,ij} - \sum_{k=1}^{m} h_{ij}(X_{\ell_k})),
\]

\[
\Delta_{n_2,ij} = \sum_{1 \leq \ell_1 < \ell_2 < \ldots < \ell_m \leq n_2} (\Phi_{ij}(Y_{\ell_1}, \ldots, Y_{\ell_m}) - u_{2,ij} - \sum_{k=1}^{m} h_{ij}(Y_{\ell_k})).
\]

Apparently, \( m \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha)/n_1 \) and \( m \sum_{\alpha=1}^{n_2} h_{ij}(Y_\alpha)/n_2 \) are terms for the sum of i.i.d. random variables and \( \left( \frac{n_1}{m} \right)^{-1} \Delta_{n_1,ij} \) is the residual term. \( m \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha)/n_1 \) and \( m \sum_{\alpha=1}^{n_2} h_{ij}(Y_\alpha)/n_2 \) are approximations of \( \tilde{u}_{1,ij} \) and \( \tilde{u}_{2,ij} \). This motivates us to define

\[
T_{ij} := \frac{\sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha)/n_1 - \sum_{\alpha=1}^{n_2} h_{ij}(Y_\alpha)/n_2}{\sqrt{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2}} \quad \text{and} \quad T_n := \max_{1 \leq i, j \leq q} (T_{ij})^2.
\]

We then prove that the small residual term \( \left( \frac{n_1}{m} \right)^{-1} \Delta_{n_1,ij} \) is negligible for our theorem, i.e., to obtain Theorem 2.2, it suffices to prove that as \( n, q \to \infty \), we have

\[
\mathbb{P}(T_n - 4 \log q + \log(\log q) \leq x) \to \exp \left( - \exp(-x/2)/\sqrt{8\pi} \right).
\]
Step (iii) (Sketch). In the last step, we derive the limiting distribution of $T_n$ to prove (2.17). $T_n$ is the maximum of $(T_{ij})^2$ over $\{1 \leq i, j \leq q\}$ and $T_{ij}$ is not independent of each other. Therefore, we cannot straightforwardly exploit the extreme value theorem under the independent setting to obtain the limiting distribution of $T_n$. To solve this problem, we exploit the normal approximation to get the extreme value distribution of $\{(T_{ij})^2\}_{1 \leq i, j \leq q}$ under the setting that $T_{ij}$ can be dependent of each other. The detailed proof is in Appendix C.1.

Theorem 2.2 justifies the size of the proposed test $T_\alpha$ in (2.6). It shows that under $H_0$ of (1.1), $M_n - 4 \log q + \log(\log q)$ converges weakly to an extreme value Type I distribution with the distribution function $F(t) = \exp\left(-\exp\left(t/2\right)/\sqrt{8\pi}\right)$.

Remark 2.3. Theorem 2.2 provides a unified framework for testing the equality of two large U-statistic based matrices, which include ranked-based correlation matrices as special examples. Our test method exploits the Jackknife strategy and extreme value statistics, and it works under a fully nonparametric model. Technically, for proving Theorem 2.2, we develop a set of tools for analyzing the Jackknife variance estimator defined in (2.5), which is technically nontrivial and is of independent interest for analyzing U-statistics in more general settings.

Next, we analyze the power of $T_\alpha$. To this end, we first introduce an alternative hypothesis characterized by the following set of matrix pairs

$$A(C) = \left\{ (U_1, U_2) : \max_{1 \leq i, j \leq q} \frac{|u_{1,ij} - u_{2,ij}|}{\sqrt{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}} \geq C \sqrt{\log q} \right\},$$

where $C > 0$ is a constant. The setting that only one entry of $U_1$ and $U_2$ differentiates large enough will make $(U_1, U_2) \in A(C)$ for some constant $C$. The next theorem shows that the null hypothesis is asymptotically distinguishable from $A(4)$ by $T_\alpha$. In other words, we reject $H_0$ in (1.1) using $T_\alpha$ with an overwhelming probability if $(U_1, U_2) \in A(4)$.

**Theorem 2.4.** (Power of the Test $T_\alpha$) If (A2) is satisfied, as $n, q \to \infty$ we have

$$\inf_{(U_1, U_2) \in A(4)} \mathbb{P}(T_\alpha = 1) \to 1. \quad (2.18)$$

Remark 2.5. From the above theorem, only one entry of $U_1 - U_2$ has a magnitude more than $C \sqrt{\log q/n}$ is enough for the test $T_\alpha$ to correctly reject $H_0$ of (1.1). We don’t impose Assumptions (A1) and (A3) to obtain such results.

### 2.3 Testing Rows or Columns of Two U-statistic Based Matrices

In some applications, instead of testing the equality of two full matrices, we are interested in testing the equality of a particular row or column of the given matrix pair. This requires us to test the hypothesis in (2.7). For simplicity, we only present the result for the row comparison in this subsection. The application to the column comparison is straightforward.

To test the hypothesis in (2.7), we define the test statistic as

$$M_{n,i} = \max_{1 \leq j \leq q} M_{ij}.$$

The following theorem derives the limiting distribution of $M_{n,i}$ under the null hypothesis.
Theorem 2.6. If the null hypothesis $H_{0,i}$ in (2.7) and conditions in Theorem 2.2 hold, we have
\[ P(M_{n,i} - 2 \log q + \log \log q \leq x) \to \exp \left( -\frac{1}{\sqrt{\pi}} \exp \left( -\frac{x}{2} \right) \right), \] for any given $x \in \mathbb{R}$, as $n, q \to \infty$.

The above theorem can be proved in a similar way to Theorem 2.2.

Remark 2.7. For analyzing the power of $T_{\alpha,i}$, we define the following set of vector pairs,
\[ A_{i*}(C) = \left\{ (u_{1,i*}, u_{2,i*}) : \max_{1 \leq j \leq q} \frac{|u_{1,ij} - u_{2,ij}|}{\sqrt{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}} \geq C \sqrt{\log q} \right\}. \]
This allows us to yield a similar result to Theorem 2.4.

3 Applications to Testing Large Kendall’s tau Correlation Matrix

In this section, we focus on testing the equality of two Kendall’s tau matrices $U_1^\tau$ and $U_2^\tau$. This section contains two parts. In the first part, we assume the samples are from a fully nonparametric model. Under this model, in addition to the general Jackknife-based approach outlined in the last section, we introduce two additional methods for testing (1.2) and analyze their theoretical properties (e.g., size and power). In the second part, we assume the samples are generated from a Gaussian copula model, under which we can relax Assumption (A3) to a much simplified form.

Kendall’s tau provides a way to describe the nonlinear relationship between two random variables. As it is rank-based, it is especially suitable to analyze data from heavy-tailed distributions. In this section, we aim to test the equality of two Kendall’s tau matrices. More specifically, we set
\[ \Phi_{ij}(X_k, X_\ell) := \text{sign}(X_{ki} - X_{\ell i})\text{sign}(X_{kj} - X_{\ell j}), \]
\[ \Phi_{ij}(Y_k, Y_\ell) := \text{sign}(Y_{ki} - Y_{\ell i})\text{sign}(Y_{kj} - Y_{\ell j}), \]
and $q = d$. We aim to test whether $U_1^\tau = U_2^\tau$.

3.1 Methods and Theory under Fully Nonparametric Models

Section 3.1 contains two parts. The first part introduces two additional test procedures tailored for testing the equality of Kendall’s tau matrices $U_1^\tau$ and $U_2^\tau$. The second part proves the theoretical properties of the three tests. In addition, we further prove the rate-optimality of the proposed tests. Our technical contributions include providing an upper bound of the traditional plug-in variance estimation error, which enables us to establish the explicit rate of convergence of the plug-in variance estimator. We also prove an upper bound of the variance difference between two Kendall’s tau correlation coefficients. These bounds allow us to derive the limiting distribution of two additional test statistics. The construction of these bounds requires the nontrivial usage of special structures of variance estimators and is of independent interest themselves. Moreover, for proving our test methods’ optimality for the Kendall’s tau matrix comparison, we construct a collection of least favourable multivariate normal distributions with regard to the test hypothesis.
This novel construction technique is developed for the correlation matrix comparison and is one of our technical contributions.

Recall that $U_a^\tau$ and $\hat{U}_a^\tau$, defined in (2.3), are symmetric and we have $\text{Diag}(U_a^\tau) = \text{Diag}(\hat{U}_a^\tau) = I_d$. Therefore, we don’t need to compare the main diagonals of $U_a^\tau$. Hence, we reset $S = \{(i,j) : 1 \leq i < j \leq d\}$ for testing the equality of large Kendall’s tau correlation matrices. The Jackknife-based statistic $M_n$ in Section 2.1 then becomes

$$
M_n^{\tau,\text{jack}} := \max_{(i,j) \in S} \left( \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\hat{\sigma}_2^2(\hat{\tau}_{1,ij}) + \hat{\sigma}_2^2(\hat{\tau}_{2,ij})} \right).
$$

(3.1)

Here, we still use $\hat{\sigma}^2(\cdot)$ to denote the jackknife variance estimator. Accordingly, we obtain $T_n^{\tau,\text{jack}}$:

$$
T_n^{\tau,\text{jack}} := 1 \{ M_n^{\tau,\text{jack}} \geq G^{-}(\alpha) + 4 \log d - \log(\log d) \}.
$$

### 3.1.1 Three Procedures to Compare Kendall’s tau Matrices

In this section, we present two additional methods for comparing two Kendall’s tau matrices. We start with the introduction of a plug-in method, which directly estimates the variances of $\{\hat{\tau}_{a,ij}\}_{a=1,2}$ and plugs them into the test statistic. For this, recall that the Kendall’s tau sample correlation between two random variable $U$ and $V$ is calculated by

$$
\hat{\tau} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \text{sign}(U_i - U_j) \text{sign}(V_i - V_j),
$$

where $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ are $n$ random samples from $U$ and $V$. Let $\Pi_c$ be the probability of the event that among two members drawn from the sample without replacement, they are concordant with each other. In other words, we have

$$
\Pi_c = \mathbb{P}((U_2 - U_1)(V_2 - V_1) > 0).
$$

Kruskal (1958) proves that the variance of $\hat{\tau}$ can be written as

$$
\frac{8}{n(n-1)} \Pi_c (1 - \Pi_c) + \frac{16}{n} \frac{n-2}{n-1} (\Pi_{cc} - \Pi_c^2),
$$

(3.3)

where $\Pi_{cc}$ is the probability of the event that among three members drawn from the sample without replacement, the second and third are concordant with the first. In other words, we have

$$
\Pi_{cc} = \mathbb{P}\left( [(U_2 - U_1)(V_2 - V_1) > 0] \cap [U_3 - U_1)(V_3 - V_1) > 0] \right).
$$

(3.4)

As $n \to \infty$, the quantity in (3.3) multiplied by $n$ has the limit $16(\Pi_{cc} - \Pi_c^2)$. Motivated by this result, we propose the following plug-in estimator

$$
\hat{\sigma}^2_{\text{plug}}(\hat{\tau}) = \frac{16}{n} \left( \hat{\Pi}_{cc} - \hat{\Pi}_c^2 \right),
$$

(3.5)

as an alternative to the Jackknife based one, for estimating the variance of $\hat{\tau}$. Here $\hat{\Pi}_{cc}$ and $\hat{\Pi}_c$ are the corresponding U-statistics to estimate $\Pi_{cc}$ and $\Pi_c$. We replace $\hat{\sigma}^2(\cdot)$ in $M_n^{\tau,\text{jack}}$ with $\hat{\sigma}^2_{\text{plug}}(\cdot)$ to construct $M_n^{\tau,\text{plug}}$:

$$
M_n^{\tau,\text{plug}} := \max_{1 \leq i < j \leq d} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\hat{\sigma}^2_{\text{plug}}(\hat{\tau}_{1,ij}) + \hat{\sigma}^2_{\text{plug}}(\hat{\tau}_{2,ij})}.
$$

(3.6)
Accordingly, we construct the plug-in type test $T_{\alpha,\text{plug}}^\tau$ as follows:

$$T_{\alpha,\text{plug}}^\tau := 1 \left\{ M_{n,\text{plug}}^\tau \geq G^{-}(\alpha) + 4 \log d - \log(\log d) \right\}. $$

In Section 3.1.2, we will provide the theoretical justification for this plug-in procedure.

Both the theoretical and numerical results indicate that the variance estimation error is a key factor influencing the test statistics’ powers. Up to now, we consider two kinds of variance estimation procedures (Jackknife based and plug-in based) for testing the equality of two Kendall’s tau matrices. To exploit the sparsity of $U_\tau$, we next propose to use the exact variance under the uncorrelated condition ($\tau = 0$). We name this procedure as “pseudo method”. It calculates the variance of $\hat{\tau}_{a,ij}$ by assuming $\tau_{a,ij} = 0$. We set $\tilde{\sigma}^2_{1,\text{ps}}$ and $\tilde{\sigma}^2_{2,\text{ps}}$ as the variances of $\sqrt{n_1} \hat{\tau}_1$ and $\sqrt{n_2} \hat{\tau}_2$, under $\tau_1 = 0$ and $\tau_2 = 0$. We also set

$$ \sigma_{a,\text{ps}} := \lim_{n_a \to \infty} \tilde{\sigma}_{a,\text{ps}} \quad \text{for} \quad a = 1, 2. $$

The test statistic becomes

$$M_{n,\text{ps}}^\tau := \max_{1 \leq i < j \leq d} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\sigma_{1,\text{ps}}^2/n_1 + \sigma_{2,\text{ps}}^2/n_2}. $$

Similarly, we construct the test $T_{\alpha,\text{ps}}^\tau$:

$$T_{n,\text{ps}}^\tau := 1 \left\{ M_{n,\text{ps}}^\tau \geq G^{-}(\alpha) + 4 \log d - \log(\log d) \right\}. $$

For example, if $X$ and $Y$ are generated from continuous Gaussian copula model, we have

$$ \tilde{\sigma}_{1,\text{ps}}^2 = \frac{2(2n_1 + 5)}{9(n_1 - 1)} \quad \text{and} \quad \tilde{\sigma}_{2,\text{ps}}^2 = \frac{2(2n_2 + 5)}{9(n_2 - 1)}. $$

Correspondingly, we obtain

$$ \sigma_{1,\text{ps}}^2 = \sigma_{2,\text{ps}}^2 = \frac{4}{9}. $$

Remark 3.1. As long as $|\tilde{\sigma}_{2,\text{ps}}^2 - \sigma_{2,\text{ps}}^2| = o((\log d)^{-1-\epsilon})$ with an arbitrary $\epsilon > 0$, we can show that replacing $\sigma_{a,\text{ps}}^2$ with $\tilde{\sigma}_{a,\text{ps}}^2$ still gives a valid test. More details are provided in Theorem 3.3.

3.1.2 Theoretical Properties of Three Testing Procedures

We now present the theoretical properties (size, power, and optimality) of the three tests introduced in the former sections. More specifically, we prove their validity under the null hypothesis and conduct power analysis similarly to Theorems 2.2 and 2.4. Furthermore, we show that these tests are rate optimal against the sparse alternative.

In the beginning, the following theorem gives the limiting distribution for plug-in and Jackknife based test statistics.

Theorem 3.2. Assuming (A1), (A2) and (A3) hold, under $H_0^\tau$ of (1.2), we have

$$ \mathbb{P} \left( M_{n,\text{jack}}^\tau - 4 \log d + \log(\log d) \leq x \right) \to \exp \left( - \frac{1}{\sqrt{8\pi}} \exp \left( - \frac{x}{2} \right) \right), $$

$$ \mathbb{P} \left( M_{n,\text{plug}}^\tau - 4 \log d + \log(\log d) \leq x \right) \to \exp \left( - \frac{1}{\sqrt{8\pi}} \exp \left( - \frac{x}{2} \right) \right), $$

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for any $x \in \mathbb{R}$, as $n,d \to \infty$. Furthermore, the results hold uniformly for all $X$ and $Y$ satisfying (A1), (A2) and (A3).

The following theorem gives the limiting distribution of the pseudo method. It holds under an additional meta-elliptical (defined in Appendix A) distributional assumption on the data.

**Theorem 3.3.** We assume $X$ and $Y$ belong to the meta-elliptical distribution (Fang et al., 2002)\(^4\). If Assumptions (A1), (A2) and (A3) hold, under $\mathcal{H}_0$ in (1.2), we have

$$
P(M_{n,ps}^{\tau} - 4 \log d + \log(\log d) \leq x) \to \exp \left( - \frac{1}{\sqrt{8\pi}} \exp \left( - \frac{x}{2} \right) \right),$$

(3.12)

for any $x \in \mathbb{R}$, as $n,d \to \infty$. Furthermore, the result holds uniformly for all $X$ and $Y$ satisfying (A1), (A2), (A3).

We now analyze the powers of $T_{\alpha}^{\tau,\text{jack}}$, $T_{\alpha}^{\tau,\text{plug}}$ and $T_{\alpha}^{\tau,ps}$. Similarly to Theorem 2.4, we define

$$
\mathbb{U}(C) = \left\{ \begin{pmatrix} U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2 \end{pmatrix} : \max_{1 \leq i < j \leq d} \frac{|\tau_1,ij - \tau_2,ij|}{\sqrt{4\xi_{1,ij}/n_1 + 4\xi_{2,ij}/n_2}} \geq C\sqrt{\log d} \right\},
$$

$$
\mathbb{V}(C) = \left\{ \begin{pmatrix} U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2 \end{pmatrix} : \max_{1 \leq i < j \leq d} \frac{|\tau_1,ij - \tau_2,ij|}{\sqrt{\sigma_1^{2,ps}/n_1 + \sigma_2^{2,ps}/n_2}} \geq C\sqrt{\log d} \right\}.
$$

They are Kendall’s tau versions of $\Lambda(C)$ in Theorem 2.4.

**Theorem 3.4.** (Power Analysis) Assuming (A2) holds, we have

$$
\inf_{(U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2) \in \mathbb{U}(4)} \mathbb{P}(T_{\alpha}^{\tau,\text{jack}} = 1) \to 1,
$$

(3.13)

$$
\inf_{(U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2) \in \mathbb{U}(4)} \mathbb{P}(T_{\alpha}^{\tau,\text{plug}} = 1) \to 1,
$$

(3.14)

as $n,d \to \infty$. If $X$ and $Y$ belong to the meta-elliptical family and (A1), (A2) are satisfied, as $n,d \to \infty$, we have

$$
\inf_{(U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2) \in \mathbb{V}(4)} \mathbb{P}(T_{\alpha}^{\tau,ps} = 1) \to 1.
$$

(3.15)

Theorem 3.4 implies that just one entry of $U_{1} \tau_1 \n_1 \tau_1 \n_2 \tau_2 \n_1 \tau_2 \n_2$ has a magnitude no smaller than $C\sqrt{\log d}/n$ is enough for the test $T_{\alpha}^{\tau}$ to correctly reject $\mathcal{H}_0$.

Next, we show that all the three proposed methods are rate optimal by matching the obtained rates of convergence to a lower bound for correlation matrix comparison. We adopt the general framework used in Baraud (2002) for testing the equality of correlation matrices. The core of the proof is the construction of collections of least favourable multivariate normal distributions with regard to the test hypothesis. Our work is related to Cai et al. (2013) which prove the lower bound for testing the equality of covariance matrices. However, their construction technique is developed for covariance matrices but not the correlation matrices. Specifically, they only perturb the diagonal elements of the covariance, which does not affect the resulting correlation matrices. To test correlation matrices, we need to develop a novel construction by perturbing the off-diagonal elements of the correlation matrices. More details are provided in the proof of Theorem 3.5.

\(^4\)Detailed introduction of the meta-elliptical distribution family is provided in Appendix A.
Theorem 3.5. Let $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assuming that $\log d/n = o(1)$, there exits a sufficiently small positive number $c_0$, such that for any distribution family that contains Gaussian as a subfamily, and all large enough $n$ and $d$, we have

$$\inf_{T_\alpha \in T_\alpha} \sup_{(U_1^n, U_2^n) \in \mathcal{U}(a)} \mathbb{P}(T_\alpha = 0) \geq 1 - \alpha - \beta,$$

where $T_\alpha$ represents all level $\alpha$ tests for testing the equality of two matrices.

Cai et al. (2013) give a similar result for testing the equality of two covariance matrices. They show that the rate $C\sqrt{\log d/n}$ is optimal for comparing covariance matrices under conditions that $X$ and $Y$ have sub-Gaussian-type or polynomial-type tails. In comparison, the lower bound result in Theorem 3.5 illustrates that our proposed methods are rate optimal under the general nonparametric model. In particular, we don’t impose assumptions on the marginal distributions.

3.2 Methods and Theory under Semiparametric Gaussian Copula Models

In this section, we assume that $X$ and $Y$ are $d$-dimensional random vectors from the Gaussian copula with latent correlation matrices $\Sigma_a = (\sigma_{a,ij})$, $a = 1, 2$ and $\text{Diag}(\Sigma_a) = I_d$. Under the Gaussian copula model, the technical assumption (A3) in Section 2.2 can be replaced by a much simplified condition. Specifically, for $r \in (0, 1)$, we define

$$\Omega(r) := \{1 \leq i \leq d : |\tau_{1,ij}| > r \text{ or } |\tau_{2,ij}| > r \text{ for some } j \neq i\}.$$  (3.17)

We describe the technical assumption (A4) as follows:

(A4). For some $r < 1$ and a sequence of numbers $\Omega_d, r$, we have $|\Omega(r)| \leq \Omega_{d,r} = o(d)$.

Assumptions (A4) and (A3) are highly related. However, Assumption (A3) cannot be straightforwardly implied by Assumptions (A1), (A2) and (A4). In fact, the relationship between (A3) and (A4) is complicated. To see the exact relationship, we need some additional definitions.

First, we have $S = \{(i,j) : 1 \leq i < j \leq d\}$. We then define

$$C_0 = \{(i,j) : i \in \Omega(r) \bigcup \Gamma \} \bigcup \{(i,j) : j \in \Omega(r) \bigcup \Gamma \} \quad \text{and} \quad B_0 = S_0 \bigcup C_0,$$

where $\Gamma$ is defined in in Assumption (A1) and $S_0$ is defined in (2.11). Furthermore, we denote $A$ to be the biggest subset of $S \setminus B_0$, such that any two pairs $(i,j), (k, \ell) \in A$ must satisfy a condition (*). More detailed description of condition (*) will be provided in the proof of Theorem 3.6. Essentially, it specifies that, for any $(i,j), (k, \ell) \in S \setminus B_0$, there exits an $i_1 \in \{i, j, k, \ell\}$ such that for any $j_1 \in \{i, j, k, \ell\} \setminus i_1$, we have $|\tau_{a,i_1j_1}| = O((\log d)^{-1-\alpha_0})$. We also define $\tau_{a,ijkl}$ as the Kendall’s tau version of $u_{a,ijkl}$ in Assumption (A3).

Under Assumptions (A1), (A2) and (A4), we can prove that for any $(i, j), (k, \ell) \in A$, we have $|\tau_{a,ijkl}| = O((\log d)^{-1-\alpha_0})$, which is essentially Assumption (A3) with $u_{a,ijkl}$ replaced by $\tau_{a,ijkl}$. The only difference is that these conditions hold on $A$ but instead of $S \setminus S_0$ as in Assumption (A3). Theorem 3.6 below specifics that Assumptions (A1), (A2) and (A4) can be used to replace Assumptions (A1), (A2) and (A3) when we test the equality of Kendall’s tau correlation matrices under the Gaussian copula model.

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\(^5\)Detailed definition of the Gaussian copula is put in Appendix A.
Theorem 3.6. Let $X$ and $Y$ be Gaussian copula random vectors with latent correlation matrices $\Sigma_a$, $a = 1$ or $2$ and $\text{Diag}(\Sigma_a) = I_d$. We assume that the smallest eigenvalue of any 4 by 4 principal sub-matrix of $\Sigma_a$ is bounded away from 0. Assuming (A1), (A2) and (A4) hold, under $H_0$ of (1.2), we have

\[
P(M_{n, \text{jack}}^{\tau} - 4 \log d + \log(\log d) \leq x) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x}{2}\right)\right),
\]

\[
P(M_{n, \text{plug}}^{\tau} - 4 \log d + \log(\log d) \leq x) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x}{2}\right)\right),
\]

\[
P(M_{n, \text{ps}}^{\tau} - 4 \log d + \log(\log d) \leq x) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x}{2}\right)\right),
\]

for any $x \in \mathbb{R}$, as $n, d \to \infty$. Furthermore, these limiting results hold uniformly for all $X$ and $Y$ satisfying (A1), (A2), and (A4).

Proof. Recall that $M_{n, \text{jack}}^{\tau}$, $M_{n, \text{plug}}^{\tau}$ and $M_{n, \text{ps}}^{\tau}$ in (3.1), (3.6) and (3.8) are defined by taking the maximum over $S$. The main idea to prove Theorem 3.6 is to show that it is sufficient to use a version of these quantities taking the maximum over the smaller set $A$ as defined before. The proof is technical and is left to Appendix C.7. \hfill \Box

Remark 3.7. In Appendix A, we show that $U_a^{\tau}$ and $\Sigma_a$ are related in terms of $\sigma_{a,ij} = \sin(\tau_{a,ij}\pi/2)$. Hence, testing (1.2) is equivalent to testing

$$H_0 : \Sigma_1 = \Sigma_2 \text{ v.s. } H_1 : \Sigma_1 \neq \Sigma_2,$$

under the Gaussian copula model.

Remark 3.8. To test the row or column of Kendall’s tau matrices, if any of the conditions of Theorem 3.2, 3.3 and 3.6 are satisfied, we get the same limiting distribution as in (2.19).

4 Experiments

We demonstrate numerical performance of the proposed methods on simulated and real datasets. In particular, we compare the proposed methods with the state-of-the-art method in the literature.

4.1 Numerical Simulations

We compare the proposed methods with the sample covariance based method (denoted by $T_{CLX}^{\alpha}$) in Cai et al. (2013). Under the null hypothesis, we sample $n_1 + n_2$ data points from the following 4 models:

- **Model 1** (Gaussian distribution with blocked $\Sigma$) We set $\Sigma^* = (\sigma_{1,ij})$ with $\sigma_{1,ii} = 1$, $\sigma_{1,ij} = 0.6$ for $5(k-1) + 1 \leq i \neq j \leq 5k$, $k = 1, \ldots, \lfloor d/5 \rfloor$ and $\sigma_{1,ij} = 0$ otherwise. In this model, under the null hypothesis, we generate $n_1 + n_2$ random vectors from $N(0, \Sigma^*)$.

- **Model 2** (Gaussian copula with marginal $t$-distribution) We use the same data generating mechanism as in **Model 1**, except that the marginal distributions are transformed into $t(\nu)$ with density $\Gamma((\nu + 1)/2)(1 + t^2/\nu)^{-(\nu+1)/2}/\sqrt{\nu\pi\Gamma(\nu/2)}$. We set $\nu = 3$. 

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• **Model 3** (Gaussian copula with marginal Cauchy distribution) We use the same data generating mechanism as in Model 1, except that the marginal distributions are transformed into Cauchy(μ, s) with density $s/\pi(s^2 + (x - \mu)^2)$. We set $\mu = 0$ and $a = 1$.

• **Model 4** (Multivariate $t$-distribution) We generate the data according to $\mu + Z/\sqrt{W/\nu}$, where we set $W \sim \chi^2(\nu)$ and $Z \sim N(0, \Sigma^*)$ with $W$ and $Z$ independent of each other. We set $\mu = 0$ and $\nu = 3$.

Under the above models, the two populations of $X$ and $Y$ have the same correlation matrices. We use them to show that the proposed methods are valid given a fixed size. For the power analysis, we introduce a random matrix $\Delta = (\delta_{k\ell}) \in \mathbb{R}^{d \times d}$ with exactly 8 nonzero entries. Among these 8 nonzero entries, 4 entries are randomly selected from the upper triangle of $\Delta$, with a magnitude generated from Unif(0,1). The remaining 4 entries are selected in the lower triangle and are determined by symmetry. Under the alternative hypothesis, we use a pair of matrices $\tilde{\Sigma}_1, \tilde{\Sigma}_2$ to generate samples for $X$ and $Y$ in place of $\Sigma^*$. We set $\tilde{\Sigma}_1 = \Sigma^* + \delta I$ and $\tilde{\Sigma}_2 = \Sigma^* + \Delta + \delta I$ with $\delta = \min\{\lambda_{\min}(\Sigma_1 + \Delta), \lambda_{\min}(\Sigma_1)\} + 0.05$.

### Table 1: Empirical sizes and powers for Model 1, 2 with $\alpha = 0.05$, based on 5000 replications.

|   | Model 1 |   | Model 2 |   |
|---|---------|---|---------|---|
|   | 10      | 30 | 50      | 80 | 100 | 150 | 10      | 30 | 50      | 80 | 100 | 150 |
| $n$ | $d$ | Empirical size | $\tau$, $\alpha$-plug | $\tau$, $\alpha$-jack | $\tau$, $\alpha$-ps | $T_{\alpha}^{\text{CLX}}$ | $\tau$, $\alpha$-plug | $\tau$, $\alpha$-jack | $\tau$, $\alpha$-ps | $T_{\alpha}^{\text{CLX}}$ |
| 100 |   | 0.05 | 0.09 | 0.12 | 0.15 | 0.17 | 0.20 | 0.05 | 0.09 | 0.12 | 0.16 | 0.17 | 0.20 |
|   |   | 0.04 | 0.06 | 0.08 | 0.10 | 0.10 |   | 0.03 | 0.06 | 0.07 | 0.09 | 0.10 | 0.11 |
|   |   | 0.02 | 0.03 | 0.03 | 0.04 | 0.04 |   | 0.02 | 0.03 | 0.03 | 0.04 | 0.04 | 0.04 |
|   |   | 0.03 | 0.04 | 0.04 | 0.04 | 0.05 |   | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |   |
| 200 |   | 0.04 | 0.07 | 0.08 | 0.08 | 0.09 | 0.10 | 0.04 | 0.07 | 0.07 | 0.09 | 0.09 | 0.10 |
|   |   | 0.04 | 0.05 | 0.06 | 0.06 | 0.06 | 0.07 | 0.03 | 0.05 | 0.06 | 0.07 | 0.06 | 0.07 |
|   |   | 0.02 | 0.04 | 0.04 | 0.03 | 0.04 | 0.04 | 0.02 | 0.03 | 0.04 | 0.04 | 0.04 | 0.04 |
|   |   | 0.04 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
|   |   | Empirical power |   |   |   |   |   |   |   |   |   |   |   |
|   |   | $\tau$, $\alpha$-plug | 0.93 | 0.85 | 0.82 | 0.78 | 0.77 | 0.74 | 0.94 | 0.86 | 0.81 | 0.78 | 0.76 | 0.74 |
|   |   | $\tau$, $\alpha$-jack | 0.92 | 0.81 | 0.77 | 0.72 | 0.70 | 0.66 | 0.92 | 0.83 | 0.76 | 0.73 | 0.70 | 0.65 |
|   |   | $\tau$, $\alpha$-ps | 0.87 | 0.73 | 0.68 | 0.60 | 0.57 | 0.51 | 0.87 | 0.75 | 0.66 | 0.61 | 0.57 | 0.50 |
|   |   | $T_{\alpha}^{\text{CLX}}$ | 0.65 | 0.56 | 0.52 | 0.45 | 0.42 | 0.35 | 0.27 | 0.19 | 0.13 | 0.10 | 0.08 | 0.05 |
| 200 |   | $\tau$, $\alpha$-plug | 0.99 | 0.98 | 0.97 | 0.96 | 0.96 | 0.95 | 0.99 | 0.97 | 0.97 | 0.95 | 0.95 | 0.95 |
|   |   | $\tau$, $\alpha$-jack | 0.99 | 0.98 | 0.96 | 0.95 | 0.95 | 0.94 | 0.99 | 0.97 | 0.96 | 0.95 | 0.95 | 0.94 |
|   |   | $\tau$, $\alpha$-ps | 0.98 | 0.97 | 0.95 | 0.94 | 0.94 | 0.92 | 0.99 | 0.97 | 0.95 | 0.94 | 0.94 | 0.92 |
|   |   | $T_{\alpha}^{\text{CLX}}$ | 0.93 | 0.94 | 0.92 | 0.90 | 0.90 | 0.87 | 0.61 | 0.61 | 0.53 | 0.46 | 0.42 | 0.36 |

We set $n_1 = n_2 = n$ with $n = 100, 200$ and $d = 10, 30, 50, 80, 100, 150$. The nominal significance level is 0.05. Tables 1 and 2 present the empirical performances of these methods. We see that $T_{\alpha}^{\tau, \text{ps}}$ always attains the desired size. When $d$ is significantly larger than $n$, $T_{\alpha}^{\tau, \text{plug}}$ and $T_{\alpha}^{\tau, \text{jack}}$ suffer from the size distortion. When $d$ approximates $n$, $T_{\alpha}^{\tau, \text{jack}}$ is still valid but $T_{\alpha}^{\tau, \text{plug}}$ fails.

Examining the empirical powers of Tables 1 and 2, the three proposed methods outperform $T_{\alpha}^{\text{CLX}}$. In particular, for distributions with heavy tails or strong tail dependence, $T_{\alpha}^{\text{CLX}}'$s powers
Table 2: Empirical sizes and powers for Models 3, 4 with $\alpha = 0.05$, based on 5000 replications.

| $n$  | $d$  | Model 3 |               | Model 4 |               |
|------|------|---------|---------------|---------|---------------|
|      |      |         | Empirical size |         | Empirical size |
| 100  | 10   | $T^\tau,\text{plug}_\alpha$ | 0.05 | 0.09 | 0.12 | 0.15 | 0.15 | 0.21 | 0.06 | 0.11 | 0.12 | 0.15 | 0.18 | 0.21 |
|      | 30   | $T^\tau,\text{jack}_\alpha$ | 0.04 | 0.05 | 0.07 | 0.09 | 0.08 | 0.11 | 0.04 | 0.07 | 0.08 | 0.08 | 0.11 | 0.12 |
|      | 50   | $T^\tau,\text{ps}_\alpha$ | 0.02 | 0.03 | 0.03 | 0.04 | 0.03 | 0.04 | 0.02 | 0.03 | 0.03 | 0.04 | 0.04 | 0.04 |
|      | 80   | $T^{\text{CLX}}_\alpha$ | 0.00 | 0.00 | 0.01 | 0.01 | 0.01 | 0.02 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|      | 100  | $T^\tau,\text{plug}_\alpha$ | 0.04 | 0.06 | 0.07 | 0.09 | 0.09 | 0.09 | 0.04 | 0.07 | 0.07 | 0.09 | 0.09 | 0.10 |
|      | 150  | $T^\tau,\text{jack}_\alpha$ | 0.03 | 0.05 | 0.06 | 0.06 | 0.07 | 0.06 | 0.03 | 0.05 | 0.06 | 0.07 | 0.07 | 0.07 |
|      | 200  | $T^\tau,\text{ps}_\alpha$ | 0.02 | 0.03 | 0.04 | 0.04 | 0.04 | 0.04 | 0.02 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 |
| 200  | 10   | $T^{\text{CLX}}_\alpha$ | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.02 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|      | 30   | $T^\tau,\text{plug}_\alpha$ | 0.94 | 0.86 | 0.81 | 0.78 | 0.76 | 0.74 | 0.87 | 0.75 | 0.68 | 0.65 | 0.63 | 0.61 |
|      | 50   | $T^\tau,\text{jack}_\alpha$ | 0.93 | 0.82 | 0.76 | 0.71 | 0.69 | 0.66 | 0.85 | 0.70 | 0.63 | 0.58 | 0.55 | 0.51 |
|      | 80   | $T^\tau,\text{ps}_\alpha$ | 0.88 | 0.75 | 0.67 | 0.59 | 0.56 | 0.49 | 0.76 | 0.59 | 0.49 | 0.41 | 0.37 | 0.32 |
|      | 100  | $T^{\text{CLX}}_\alpha$ | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 | 0.02 | 0.09 | 0.03 | 0.03 | 0.01 | 0.01 | 0.01 |
|      | 150  | $T^\tau,\text{plug}_\alpha$ | 0.99 | 0.98 | 0.96 | 0.96 | 0.96 | 0.94 | 0.98 | 0.95 | 0.93 | 0.91 | 0.91 | 0.89 |
|      | 200  | $T^\tau,\text{jack}_\alpha$ | 0.99 | 0.98 | 0.96 | 0.95 | 0.95 | 0.93 | 0.98 | 0.94 | 0.92 | 0.90 | 0.90 | 0.87 |
|      |      | $T^\tau,\text{ps}_\alpha$ | 0.98 | 0.97 | 0.95 | 0.94 | 0.94 | 0.92 | 0.97 | 0.94 | 0.90 | 0.87 | 0.86 | 0.83 |
|      |      | $T^{\text{CLX}}_\alpha$ | 0.01 | 0.00 | 0.00 | 0.01 | 0.01 | 0.01 | 0.22 | 0.13 | 0.10 | 0.07 | 0.06 | 0.04 |

decrease dramatically, making $T^{\text{CLX}}_\alpha$ inappropriate for such applications. Under the Gaussian distribution, $T^\tau,\text{ps}_\alpha$ has greater power than $T^{\text{CLX}}_\alpha$ since it does not need to estimate the variance.

These finite sample results suggest that $T^\tau,\text{plug}_\alpha$ is useful only when $d$ is significantly smaller than $n$. With $d$ approximates $n$, we recommend to use $T^\tau,\text{jack}_\alpha$ because it has averagely higher power. When $d$ is significantly larger than $n$, we recommend to use $T^\tau,\text{ps}_\alpha$.

4.2 Real Data Example

In this section we analyze the dependence structure of the residual vectors, $e_t := (e_{t1}, e_{t2}, \ldots, e_{td})^T \in \mathbb{R}^d$, of the Fama-French three-factor model (Fama and French, 1993). The Fama-French three-factor model is

$$r_{it} - \gamma_f = \alpha_i + \beta_3(r_{mt} - \gamma_f) + b_{si}\text{SMB}_t + b_{vi}\text{HML}_t + e_{it},$$

where $r_{it}$ is the random return of the $i$th portfolio at the time $t$, $\gamma_f$ is the risk free rate, $r_{mt}$ is the return of the market portfolio, SMB$_t$ is the portfolio return related to the size factor, and HML$_t$ is the portfolio return related to the book-to-market factor.

We use 30 industry value-weighted portfolios from Kenneth R. French’s Data Libray\(^6\). According to the financial theory, $e_{it}$ represents the individual risk owned by the $i$th industry. Heuristically speaking, most of correlations between different industries should be small after adjusting common market factors, as $e_{it}$ represents each industry’s risk. We emphasize that our methods allow the

\(^6\)The website for the data library is http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.
Figure 1: The figure illustrates the p-value sequences for testing the equality of residual vectors’ dependence structure from 1991 to 2012. We conduct the tests every half year to compare the covariance or correlation matrices during a one-year period. The tests are based on returns for every two weeks from the stock market and the significance lever $\alpha$ is 0.05.

existence of strong correlations between some industries after adjusting factors, as long as the number of such industries is small. Therefore, although it is not reasonable to assume the sparsity of Pearson’s or Kendall’s tau correlation matrix of $r_t = (r_{1t}, \ldots, r_{dt})^T$, we can accept the sparsity of them after adjusting the effects of common factors.

We use the regression described in Fama and French (1993) to estimate $\beta_{3i}$, $b_{si}$, $b_{vi}$ and test on the residuals $e_{ti}$. We use returns of every two weeks to compare the dependence structure of residual vectors for a one-year period and test about every half year from 1991 to 2012. Since financial data are heavy tailed, it is natural to test the equality of Kendall’s tau rank correlation matrices as conducted by $T^{\tau,\text{jack}}_\alpha$ and $T^{\tau,\text{ps}}_\alpha$. Since the number of portfolios $d$ is close to the number of samples $n$, we adopt $T^{\tau,\text{jack}}_\alpha$ and $T^{\tau,\text{ps}}_\alpha$ with $T^{\text{CLX}}_\alpha$. We show the results in Figure 1.

Figure 1 illustrates that $T^{\text{CLX}}_\alpha$ sometimes diverges from $T^{\tau,\text{jack}}_\alpha$ and $T^{\tau,\text{ps}}_\alpha$. For example, at November of 2009, the p-value of $T^{\text{CLX}}_\alpha$ is 0.0002, while p-values of $T^{\tau,\text{jack}}_\alpha$ and $T^{\tau,\text{ps}}_\alpha$ are 0.2604 and 0.2904. This phenomenon indicates at this time point the fluctuation of variance rather than the correlation causes the rejection of $T^{\text{CLX}}_\alpha$. From Figure 1 we find that $T^{\tau,\text{ps}}_\alpha$ is more conservative than $T^{\tau,\text{jack}}_\alpha$ and we also observe that some fluctuations of the dependence structure coincide with the economic or market environment changes. For example, from the year 2007 to the year 2008, when a global financial crisis happens, $T^{\tau,\text{jack}}_\alpha$ takes the p-values 0.0342, 0.0156, 0.2657, 0.0113. Tests around the year 2000 reveal a similar pattern, which may be related to the “dot-com bubble”. However, from the known financial crises, not all crises lead to the fluctuations of the dependence structure. This phenomenon illustrates financial crises may influence the equity market differently.
5 Summary and Discussion

This paper considers the problem of testing the equality of high-dimensional U-statistic based matrices. We provide a lower bound for testing the equality of correlation matrices and prove the proposed methods’ optimality. Based on thorough numerical comparisons, $T_{\alpha}^{\text{plug}}$ performs well only when $d$ is significantly smaller than $n$. When $d$ is close to $n$, we recommend to use $T_{\alpha}^{\text{jack}}$ since it has a greater power. When $d$ is much larger than $n$, we recommend to use $T_{\alpha}^{\text{ps}}$. In addition, $T_{\alpha}^{\text{ps}}$ performs quite well for distributions with heavy tails or strong tail dependence. Therefore, $T_{\alpha}^{\text{ps}}$ is potentially more useful for financial applications in which heavy-tailness is a common phenomenon. There are many possible future directions of this work. For example, instead of two-sample problems, it is interesting to generalize the idea to $k$-sample testing problems ($k > 2$). This may require a nontrivial extension of theoretical analysis.

For testing Kendall’s tau matrices, we show that the variance estimation error is a key factor influencing a test procedure’s power. In fact, the test $T_{\alpha}^{\text{ps}}$, which exploits the exact value of variance under the uncorrelated condition ($\tau = 0$), achieves a better finite-sample performance especially when $d$ is much larger than $n$. We can generalize such idea to many other applications. We also provide an upper bound of the Jackknife variance estimation error in the proof of Theorem 2.2. This result is also useful for proving other properties of U-statistics.

Next, we discuss the imposed assumptions. We note that the sparsity assumption (A1) plays a key role for obtaining the limiting extreme value distribution. It is not clear on whether this assumption is necessary, but it is satisfied in many high-dimensional applications. When (A1) is not satisfied, it is possible to exploit the bootstrap method to construct a test statistic. This is left as for future investigation. Regarding (A2), we note that Cai et al. (2013) assume a stronger scaling assumption: $\log(d) = o(n^{1/5})$. We strengthen this scaling by assuming $\log(d) = O(n^{1/3-\epsilon})$ for an arbitrary $\epsilon > 0$. This is due to the fact that the U-statistics studied in this paper are assumed to have bounded kernels.

In the simulation studies, we use $T_{\alpha}^{\text{CLX}}$ as a comparison benchmark. In Appendix B, we provide another heuristic test (denoted by $T_{\alpha}^{\text{R}}$) for testing the equality of Pearson’s correlation matrices. The performances of $T_{\alpha}^{\text{CLX}}$ and $T_{\alpha}^{\text{R}}$ are similar.

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A Introduction to the Meta-Elliptical Distribution

This section introduces the meta-elliptical distribution, which is essentially the elliptical copula family. It does not require the latent correlation matrix must be positive definite.
Definition A.1. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ with $\text{Rank}(\Sigma) = q \leq d$. A $d$-dimensional random vector $X$ has an elliptical distribution, denoted by $X \sim EC_d(\mu, \Sigma, \xi)$, if it has a stochastic representation: $X = \mu + \xi A U$, where $U$ is a random vector uniformly distributed on the unit sphere in $\mathbb{R}^q$ and $\xi \geq 0$ is a scalar random variable independent of $U$. $A \in \mathbb{R}^{d \times q}$ is a deterministic matrix such that $AA^T = \Sigma$.

To define the meta-elliptical distribution, we define a set of symmetric matrices:

$$\mathcal{R}_d = \{ \Sigma^\ell \in \mathbb{R}^{d \times d} : (\Sigma^\ell)^T = \Sigma^\ell, \text{Diag}(\Sigma^\ell) = I_d, \Sigma^\ell \succeq 0 \}.$$ 

Definition A.2. A continuous random vector $X = (X_1, \ldots, X_d)^T \in \mathbb{R}^d$ follows a meta-elliptical distribution, denoted by $X \sim ME_d(\Sigma^\ell; f_1, \ldots, f_d)$, if there exist univariate strictly increasing functions $f_1, \ldots, f_d$ such that

$$(f_1(X_1), \ldots, f_d(X_d))^T \sim EC_d(0, \Sigma^\ell, \xi),$$

where $\Sigma^\ell = (\sigma^\ell_{ij}) \in \mathcal{R}_d$. We call $\Sigma^\ell$ the latent generalized correlation matrix.

The following theorem illustrates an important relationship between the population Kendall’s tau coefficient matrix $U^T$ and the latent generalized correlation matrix $\Sigma^\ell$.

Theorem A.3. (Invariance property of Kendall’s tau (Han and Liu, 2012)). If we let $X = (X_1, \ldots, X_d)^T \sim ME_d(\Sigma^\ell; f_1, \ldots, f_d)$, and denote $\tau_{ij}$ to be the population Kendall’s tau between $X_i$ and $X_j$, we have $\sigma^\ell_{ij} = \sin(\tau_{ij} \pi/2)$.

Theorem A.3 shows that if $X$ and $Y$ follow the meta-elliptical distribution, testing (1.2) is equivalent to testing

$$H_0 : \Sigma^\ell_1 = \Sigma^\ell_2 \text{ v.s. } H_1 : \Sigma^\ell_1 \neq \Sigma^\ell_2.$$

If $X$ and $Y$ follow multivariate Gaussian distributions, $\Sigma^\ell_1$ and $\Sigma^\ell_2$ are Pearson’s correlation coefficient matrices of $X$ and $Y$.

B More Simulation Results

This section provides another heuristic test (denoted by $T^R_\alpha$) for testing the equality of Pearson’s correlation matrices. The performances of $T^{\text{CLX}}_\alpha$ and $T^R_\alpha$ are similar and both outperformed by the proposed rank-based methods.

We first explain the construction of $T^R_\alpha$. We set $\bar{X} := (\bar{X}_1, \ldots, \bar{X}_d)^T$ and $\bar{Y} := (\bar{Y}_1, \ldots, \bar{Y}_d)^T$, where $\bar{X}_j = (X_{1j} + X_{2j} + \ldots + X_{nj})/n_1$ and $\bar{Y}_j = (Y_{1j} + Y_{2j} + \ldots + X_{nj})/n_2$. We use $\tilde{s}_{ij}$ and $\tilde{s}_{2j}$ to denote the sample standard deviations of $X_j$ and $Y_j$ for $1 \leq j \leq d$. We then construct $\tilde{X}_i := (\tilde{X}_{i1}, \ldots, \tilde{X}_{id})^T$ and $\tilde{Y}_i := (\tilde{Y}_{i1}, \ldots, \tilde{Y}_{id})^T$, where $\tilde{X}_{ij} = (X_{ij} - \bar{X}_j)/\tilde{s}_{ij}$ and $\tilde{Y}_{ij} = (Y_{ij} - \bar{Y}_j)/\tilde{s}_{2j}$. $T^R_\alpha$ uses the same testing procedure as $T^{\text{CLX}}_\alpha$ in Cai et al. (2013) except that it is built on the rescaled samples $\{\tilde{X}_i\}_{1 \leq i \leq n_1}$ and $\{\tilde{Y}_i\}_{1 \leq i \leq n_2}$ rather than $\{X_i\}_{1 \leq i \leq n_1}$ and $\{Y_i\}_{1 \leq i \leq n_2}$. Table 3 shows that the performances of $T^R_\alpha$ and $T^{\text{CLX}}_\alpha$ are similar. We note that ideas similar to $T^R_\alpha$ have also been considered in Cai and Zhang (2014) for correlation estimation.
Table 3: This table compares $T_R^\alpha$'s empirical sizes and powers with those of $T_{CLX}^\alpha$ and the three proposed tests under the Gaussian distribution in Model 1. The result shows that the performances of $T_{CLX}^\alpha$ and $T_R^\alpha$ are similar.

| $n$  | $d$ | $T_{R,plug}^\alpha$ | $T_{R,jack}^\alpha$ | $T_{R,ps}^\alpha$ | $T_{CLX}^\alpha$ | $T_R^\alpha$ |
|------|-----|---------------------|---------------------|------------------|------------------|-------------|
| 100  | 10  | 0.05                | 0.04                | 0.02             | 0.03             | 0.02        |
|      | 30  | 0.09                | 0.06                | 0.03             | 0.04             | 0.03        |
|      | 50  | 0.12                | 0.08                | 0.03             | 0.04             | 0.04        |
|      | 80  | 0.15                | 0.10                | 0.03             | 0.04             | 0.04        |
|      | 100 | 0.17                | 0.10                | 0.04             | 0.05             | 0.04        |
|      | 150 | 0.20                |                     |                  |                  |             |
| 200  | 10  | 0.04                | 0.04                | 0.02             | 0.04             | 0.02        |
|      | 30  | 0.07                | 0.06                | 0.03             | 0.04             | 0.03        |
|      | 50  | 0.08                | 0.06                | 0.03             | 0.04             | 0.04        |
|      | 80  | 0.09                | 0.06                | 0.04             | 0.05             | 0.04        |
|      | 100 | 0.10                |                     |                  |                  |             |
|      | 150 |                     |                      |                  |                  |             |

C Proofs of Main Theoretical Results

This section collects proofs of the main theorems. In the sequel, we use $C$, $C_1$, $C_2$, . . . , to denote constants that do not depend on $n$, $d$, $q$ and can vary from place to place.

C.1 Proof of Theorem 2.2

Proof. As explained in the proof sketch, our analysis proceeds in three steps described as below.

Step (i). In this step, we prove that it is sufficient to establish (2.14) for proving the theorem. For this, we need to sharply characterize the estimation error of the Jackknife variance estimator of U-statistics. For this, we introduce the following lemma.

**Lemma C.1.** Let $\hat{\sigma}^2(\hat{u}_{a,ij})$ be the Jackknife estimator of $\hat{u}_{a,ij}$ and $\sigma^2(\hat{u}_{a,ij})$ be the variance of $\hat{u}_{a,ij}$. Recalling the definition of $h_{ij}$ in (2.9), $\zeta_{1,ij}$ and $\zeta_{2,ij}$ are the variances of $h_{ij}(X_I)$ and $h_{ij}(Y_I)$. We have that $m^2\zeta_{a,ij}$ is the limit of $n_a\sigma^2(\hat{u}_{a,ij})$ as $n_a$ goes to infinity. We also have that $m^2\zeta_{a,ij}$ is the limit of $n_a\sigma^2(\hat{u}_{a,ij})$ as $n_a$ goes to infinity. Moreover, under Assumption (A2), as $n, q \to \infty$ we have

$$\mathbb{P}\left(\max_{1 \leq i,j \leq q} \left| n_a\hat{\sigma}^2(\hat{u}_{a,ij}) - m^2\zeta_{a,ij}\right| \geq C\frac{\varepsilon_n}{\log q}\right) = o(1),$$  \hspace{1cm} (C.1)

where $\varepsilon_n = o(1)$ and $a = 1, 2$.

The detailed proof of Lemma C.1 is in the supplementary Appendix D.1. Lemma C.1 implies that both of the following two events

$$\mathcal{E}_1 := \left\{ \max_{1 \leq i,j \leq q} \left| n_1\hat{\sigma}^2(\hat{u}_{1,ij}) - m^2\zeta_{1,ij}\right| < C\frac{\varepsilon_n}{\log q} \right\}, \quad \mathcal{E}_2 := \left\{ \max_{1 \leq i,j \leq q} \left| n_2\hat{\sigma}^2(\hat{u}_{2,ij}) - m^2\zeta_{2,ij}\right| < C\frac{\varepsilon_n}{\log q} \right\},$$

happen with probability going to one as $n, q \to \infty$. Under $\mathcal{E}_1$ and $\mathcal{E}_2$, by $\zeta_{a,ij} \geq r_a > 0$ (Assumption (A2)), we have

$$\left| n_1\hat{\sigma}^2(\hat{u}_{1,ij})/(m^2\zeta_{1,ij}) - 1 \right| < C\varepsilon_n/\log q \quad \text{and} \quad \left| n_2\hat{\sigma}^2(\hat{u}_{2,ij})/(m^2\zeta_{2,ij}) - 1 \right| < C\varepsilon_n/\log q.$$
We set \( M_n := \max_{1 \leq i, j \leq q} M_{ij} \) and \( \tilde{M}_n := \max_{1 \leq i, j \leq q} \tilde{M}_{ij} \). By the definition of \( M_n \) and \( \tilde{M}_n \), we calculate the relative difference of \( M_{ij} \) and \( \tilde{M}_{ij} \) as

\[
\left| \frac{M_{ij} - \tilde{M}_{ij}}{\tilde{M}_{ij}} \right| \leq \left| \frac{\hat{\sigma}^2(\hat{u}_{1,ij}) - m^2 \zeta_{1,ij}/n_1}{\hat{\sigma}^2(\hat{u}_{1,ij})} \right| + \left| \frac{\hat{\sigma}^2(\hat{u}_{2,ij}) - m^2 \zeta_{2,ij}/n_2}{\hat{\sigma}^2(\hat{u}_{2,ij})} \right| \leq C_{\epsilon} \frac{\epsilon_n}{\log q}.
\]  

(C.2)

Therefore, we have \( |M_{ij} - \tilde{M}_{ij}| \leq C \epsilon_n \tilde{M}_{ij} / \log q \), which implies that

\[
|M_n - \tilde{M}_n| \leq \max_{1 \leq i, j \leq n} |M_{ij} - \tilde{M}_{ij}| \leq C \tilde{M}_n \epsilon_n / \log q.
\]  

(C.3)

Combining \( \tilde{M}_n / \log q = O_p(1) \) and \( \epsilon_n = o(1) \), to prove Theorem 2.2 it suffices to show that as \( n, q \to \infty, (2.14) \) holds for any \( x \in \mathbb{R} \).

**Step (ii).** In this step, we use the Hoeffding decomposition (Lemma F.4) to decompose U-statistics. We then prove the residual term \( \Delta_{n,ij}/(n_m) \) is negligible, i.e., to prove the theorem it is sufficient to prove (2.17) as \( n, q \to \infty \).

For notational simplicity, we set

\[
\tilde{N}_{ij} := (\hat{u}_{1,ij} - \hat{u}_{2,ij}) / \sqrt{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}.
\]  

(C.4)

Recall that in (2.13) and (2.16) we define \( \tilde{M}_{ij} \) and \( T_{ij} \) as

\[
\tilde{M}_{ij} := \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2}{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2} \quad \text{and} \quad T_{ij} := \frac{\sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha)/n_1 - \sum_{\alpha=1}^{n_2} h_{ij}(Y_\alpha)/n_2}{\sqrt{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2}}.
\]  

(C.5)

By the definition of \( \tilde{M}_{ij} \), we have \( \tilde{M}_{ij} = (\tilde{N}_{ij})^2 \). Combining the definition of \( T_{ij} \) and (2.15), we have

\[
\tilde{N}_{ij} = T_{ij} + \frac{(n_m)^{-1} \Delta_{n,ij} - (n_m)^{-1} \Delta_{n,ij}}{\sqrt{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}}.
\]  

(C.6)

Then, we introduce the following lemma to analyze the difference of \( \tilde{N}_{ij} \) and \( T_{ij} \).

**Lemma C.2.** As \( n, q \to \infty \), we have

\[
\max_{1 \leq i, j \leq q} (\tilde{N}_{ij})^2 - \max_{1 \leq i, j \leq q} (T_{ij})^2 = o_p(1).
\]  

(C.7)

The detailed proof of Lemma C.2 is in the supplementary Appendix D.2. This lemma illustrates that \( \max_{1 \leq i, j \leq q} \tilde{N}_{ij} \) and \( T_n := \max_{1 \leq i, j \leq q} T_{ij} \) have the same limiting distribution. Hence, to prove Theorem 2.2 it suffices to show (2.17) as \( n, q \to \infty \).

**Step (iii).** In this step, we aim to prove (2.17). In (2.17), \( T_n \) is the maximum of \( T_{ij} \) over \( S := \{(i,j) : 1 \leq i, j \leq q\} \) and these \( T_{ij} \)’s are not independent of each other. Therefore, we cannot straightforwardly exploit the extreme value theorem under the independent setting to obtain the limiting distribution of \( T_n \). To solve this problem, we construct normal approximation to obtain the extreme value distribution of \( (T_{ij})_{1 \leq i, j \leq q} \) under the setting that \( T_{ij} \) can be dependent of each other. The construction of such normal approximation requires most correlations of different \( T_{ij} \).
to be small. Correlations between different $T_{ij}$’s are related to the correlations of entries of $X$ and $Y$. Assumption (A1) specifies sufficient conditions on the correlations of entries of $X$ and $Y$.

To obtain more insight of Assumption (A1), we introduce the following notations. We use $S_0$ to denote pairs of $(i, j)$ such that $X_i$ and $X_j$ are highly correlated ($|u_{1,ij}| > (\log q)^{-1-\alpha_0}$) or $Y_i$ and $Y_j$ are highly correlated ($|u_{2,ij}| > (\log q)^{-1-\alpha_0}$). In detail, we define

$$S_0 := \{(i, j) : 1 \leq i \leq q, \ i \in \text{supp}_j(\alpha_0)\},$$

where we set $\text{supp}_j(\alpha_0)$ as

$$\text{supp}_j(\alpha_0) := \{1 \leq i \leq q : |u_{1,ij}| \geq (\log q)^{-1-\alpha_0} \text{ or } |u_{2,ij}| \geq (\log q)^{-1-\alpha_0}\}.$$

Assumption (A1) implies that the number of highly correlated ($|u_{a,ij}| > (\log q)^{-1-\alpha_0}$) entries of $X$ and $Y$ is small. More specifically, Assumption (A1) assumes $|S_0| = o(q^2)$.

We can prove that correlations between $T_{ij}$’s on $S \setminus S_0$ are all small. We then use the Bofferton inequality (Lemma 1 of Cai et al. (2013)) and normal approximation to obtain the limiting distribution of $\max_{(i,j) \in S \setminus S_0} (T_{ij})^2$ so as to prove (2.17).

We then present the detailed proof of (2.17). Firstly, we prove that it suffices to take the maximum of $T_{ij}$ over $S \setminus S_0$ but instead of over $S$ as in (2.17). By setting $y_q = x + 4 \log q - \log(\log q)$, we have

$$\left|\mathbb{P}\left(\max_{(i,j) \in S} (T_{ij})^2 \geq y_q\right) - \mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq y_q\right)\right| \leq \mathbb{P}\left(\max_{(i,j) \in S} (T_{ij})^2 \geq y_q\right).$$

The next lemma implies that, as $n, q \to \infty$, we have $\mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq y_q\right) \to 0$.

**Lemma C.3.** Under Assumptions (A1) and (A2), as $n, q \to \infty$, we have

$$\mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq y_q\right) \to 0.$$

The detailed proof of Lemma C.3 is in the supplementary Appendix D.3. By Lemma C.3, we have $\mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq y_q\right) \to 0$ as $n, q \to \infty$. Moreover, by (C.9), we have that $\mathbb{P}\left(\max_{(i,j) \in S} (T_{ij})^2 \geq y_q\right)$ and $\mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq y_q\right)$ have the same limit value as $n, q \to \infty$. Therefore, to obtain (2.17), it suffices to prove

$$\mathbb{P}\left(\max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \geq 4 \log q + \log(\log q) \leq x\right) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right)\right),$$

as $n, q \to \infty$. The problem is then reduced to prove (C.10).

For simplicity, by rearranging the two-dimensional indices $\{(i, j) : (i, j) \in S \setminus S_0\}$ in any order, we set them as $\{(i_k, j_k) : 1 \leq k \leq h\}$ with $h = |S \setminus S_0|$. If we denote $T_k := T_{i_kj_k}$, (C.10) becomes

$$\mathbb{P}\left(\max_{1 \leq k \leq h} (T_k)^2 \geq 4 \log q + \log(\log q) \leq x\right) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right)\right).$$

Secondly, we exploit normal approximation to obtain the limiting distribution of $\max_{1 \leq k \leq h} (T_k)^2$. This normal approximation is useful for getting the extreme value distribution of weakly dependent data. By excluding all the pairs in $S_0$, correlations between $T_k$’s are all small. Therefore, we can
use this normal approximation to get the limiting distribution of \( \max_{1 \leq k \leq h}(T_k)^2 \). In detail, we first use the Boferroni inequality to obtain both lower and upper bounds of \( \mathbb{P}( \max_{1 \leq k \leq h}(T_k)^2 \geq y_q) \). The obtained lower and upper bounds can then be shown to have the same limiting distribution, which is the extreme value distribution with the cumulative distribution function of \( \exp \left( - (8\pi)^{-1/2} \exp(-x/2) \right) \).

To describe the procedure of normal approximation, we need some additional notations. We introduce
\[
\begin{align*}
\tilde{Z}_{\beta,ij} &= n_2 h_{ij}(X_{\beta})/n_1 & \text{for} & & 1 \leq \beta \leq n_1, \\
\tilde{Z}_{\beta,ij} &= -h_{ij}(Y_{\beta-n_1}) & \text{for} & & n_1 + 1 \leq \beta \leq n_1 + n_2,
\end{align*}
\]
where \( h_{ij} \) is defined in (2.9). Moreover, by the definition of \( T_{ij} \) in (2.16), we have
\[
T_k := T_{ik,jh} = \sum_{\beta=1}^{n_1+n_2} \tilde{Z}_{\beta,ik,jh}/\sqrt{n_2^2 \zeta_{1,ik,jh}/n_1 + n_2 \zeta_{2,ik,jh}}.
\] (C.13)

After introducing these notations, we explain how to use normal approximation to get the extreme value distribution of \( \max_{1 \leq k \leq h}(T_k)^2 \). Firstly, by the Boferroni inequality (Lemma 1 of Cai et al. (2013)), for any integer \( M \) with \( 0 < M < \lfloor h/2 \rfloor \), we have
\[
\sum_{\ell=1}^{2M} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_\ell \leq h} \mathbb{P} \left( \bigcap_{j=1}^{\ell} E_{k_j} \right) \leq \mathbb{P}( \max_{1 \leq k \leq h}(T_k)^2 \geq y_q) \leq \sum_{\ell=1}^{2M-1} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_\ell \leq h} \mathbb{P} \left( \bigcap_{j=1}^{\ell} E_{k_j} \right),
\] (C.14)
where we set \( E_{k_j} = \{(T_k)^2 \geq y_q\} \). In the next step, we simplify \( \mathbb{P}( \bigcap_{j=1}^{\ell} E_{k_j} ) \). For this, we define
\[
\tilde{Z}_{\beta,k} = \tilde{Z}_{\beta,ik,jh}/(n_2 \zeta_{1,ik,jh}/n_1 + \zeta_{2,ik,jh})^{1/2} \quad \text{and} \quad W_\beta = (\tilde{Z}_{\beta,k_1}, \ldots, \tilde{Z}_{\beta,k_h})^T,
\] (C.15)
for \( 1 \leq k \leq h \) and \( 1 \leq \beta \leq n_1 + n_2 \). Therefore, we have
\[
T_{k_j} = (n_2)^{-1} \sum_{\beta=1}^{n_1+n_2} \tilde{Z}_{\beta,k_j}.
\]
Define \( \|v\|_{\min} = \min_{1 \leq i \leq \ell} |v_i| \) for any vector \( v \in \mathbb{R}^\ell \). With these notations, we rewrite \( \mathbb{P}( \bigcap_{j=1}^{\ell} E_{k_j} ) \) as
\[
\mathbb{P}( \bigcap_{j=1}^{\ell} E_{k_j} ) = \mathbb{P}( \|n^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta\|_{\min} \geq y_q^{1/2} )
\]
Secondly, we use a normal vector \( \mathbf{N}_\ell \) to approximate \( n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta \). In detail, we set \( \mathbf{N}_\ell \) as a normal vector with the same mean vector and the same covariance matrix as \( n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta \). More specifically, we have
\[
\mathbf{N}_\ell := (N_{k_1}, \ldots, N_{k_h})^T \quad \text{with} \quad \mathbb{E}[\mathbf{N}_\ell] = 0 \quad \text{and} \quad \text{Var}(\mathbf{N}_\ell) = n_1 \text{Var}(W_1)/n_2 + \text{Var}(W_{n_1+1}).
\] (C.16)

The following lemma uses \( \mathbf{N}_\ell \) to rewrite the upper and lower bounds in (C.14).
Lemma C.4. Under Assumption (A2), as \( n, q \to \infty \), we have

\[
\mathbb{P}\left( \max_{1 \leq k \leq h} (T_k)^2 \geq y_q \right) \leq \sum_{\ell=1}^{2M-1} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq h} \mathbb{P}\left( \|N_\ell\|_{\min} \geq y_q^{1/2} - \epsilon_n (\log q)^{-1/2} \right) + o(1), \quad (C.17)
\]

\[
\mathbb{P}\left( \max_{1 \leq k \leq h} (T_k)^2 \geq y_q \right) \geq \sum_{\ell=1}^{2M} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq h} \mathbb{P}\left( \|N_\ell\|_{\min} \geq y_q^{1/2} + \epsilon_n (\log q)^{-1/2} \right) - o(1). \quad (C.18)
\]

The detailed proof of Lemma C.4 is in the supplementary Appendix D.4. At last, to complete the proof, we need to prove that the right hand sides of (C.17) and (C.18) have the same limit value \( 1 - \exp \left( - \left( \sqrt{8\pi} \right)^{-1} \exp(-x/2) \right) \) as \( n, q \to \infty \). To calculate the limit value, we need the following lemma.

Lemma C.5. Under Assumption (A3), for any integer \( \ell \geq 1 \) and \( x \in \mathbb{R} \), we have

\[
\sum_{1 \leq k_1 < \cdots < k_{\ell} \leq h} \mathbb{P}\left( \|N_\ell\|_{\min} \geq y_q^{1/2} \pm \epsilon_n (\log q)^{-1/2} \right) = \frac{1}{\ell!} \left( \frac{1}{\sqrt{8\pi}} \exp(-x/2) \right)^\ell (1 + o(1)). \quad (C.19)
\]

The detailed proof of Lemma C.5 is in the supplementary Appendix D.5. By plugging (C.19) into (C.17) and (C.18), we construct the following inequities:

\[
\limsup_{n, q \to \infty} \mathbb{P}\left( \max_{1 \leq k \leq h} (T_k)^2 \geq y_q \right) \leq \sum_{\ell=1}^{2M-1} (-1)^{\ell-1} \frac{1}{\ell!} \left( \frac{1}{\sqrt{8\pi}} \exp(-x/2) \right)^\ell,
\]

\[
\liminf_{n, q \to \infty} \mathbb{P}\left( \max_{1 \leq k \leq h} (T_k)^2 \geq y_q \right) \geq \sum_{\ell=1}^{2M} (-1)^{\ell-1} \frac{1}{\ell!} \left( \frac{1}{\sqrt{8\pi}} \exp(-x/2) \right)^\ell,
\]

for any positive integer \( M \). Letting \( M \to \infty \), we prove (C.11). Therefore, we finish the proof of Theorem 2.2.

C.2 Proof of Theorem 2.4

Proof. In Theorem 2.4, we aim to prove that, under Assumption (A2), as \( n, q \to \infty \), we have

\[
\inf_{(U_1, U_2) \in A(4)} \mathbb{P}(M_n \geq G^-(\alpha) + 4 \log q - \log(\log q)) \to 1, \quad (C.20)
\]

where \( G^-(\alpha) := - \log(8\pi) - 2 \log \left( - \log(1 - \alpha) \right) \). In (C.20), we set \( U_a \) and \( A(C) \) as

\[
U_a = (u_{a, ij}) \in \mathbb{R}^{q \times q} \quad \text{and} \quad A(C) = \left\{ (U_1, U_2) : \max_{(i, j) \in S} \frac{|u_{1, ij} - u_{2, ij}|}{\sqrt{m^2 \zeta_{1, ij}/n_1 + m^2 \zeta_{2, ij}/n_2}} \geq C \sqrt{\log q} \right\},
\]

where \( S = \{(i, j) : 1 \leq i, j \leq q \} \). We use \( A(C) \) to characterize the set of alternative hypotheses for analyzing the power of \( T_a \) in (2.6).

To see the meaning of \( A(C) \), for any constant \( C \), if one entry of \( U_1 - U_2 \) has a magnitude large enough, we have \( (U_1, U_2) \in A(C) \). Therefore, small perturbations on a few entries compared to
the null hypothesis $U_1 = U_2$ makes the pair $(U_1, U_2)$ belong to $A(C)$ for some constant $C$. In
Theorem 2.4, we require that $(U_1, U_2) \in A(4)$, which implies
\begin{equation}
\max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{m^2 \xi_{1,ij}/n_1 + m^2 \xi_{2,ij}/n_2} \geq 16 \log q. \tag{C.21}
\end{equation}
Under the alternative hypothesis, $u_{1,ij} = u_{2,ij}$ cannot hold for all $(i, j) \in S$. This motivates us to define
\begin{equation}
M_n^1 := \max_{(i,j) \in S} \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij} - u_{1,ij} + u_{2,ij})^2}{\bar{\sigma}^2(\hat{u}_{1,ij}) + \bar{\sigma}^2(\hat{u}_{2,ij})} \quad \text{and} \quad M_n := \max_{(i,j) \in S} \frac{(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2}{\sigma^2(\hat{u}_{1,ij}) + \sigma^2(\hat{u}_{2,ij})}, \tag{C.22}
\end{equation}
because $M_n^1$ and $M_n$ are different under the alternative hypothesis.

To prove (2.17), using the inequality $(a \pm b)^2 \leq 2a^2 + 2b^2$, we get
\[(u_{1,ij} - u_{2,ij})^2 \leq 2(\hat{u}_{1,ij} - \hat{u}_{2,ij} - u_{1,ij} + u_{2,ij})^2 + 2(\hat{u}_{1,ij} - \hat{u}_{2,ij})^2.\]

Therefore, by the definitions of $M_n$ and $M_n^1$ in (C.22), we get
\begin{equation}
\max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{\bar{\sigma}^2(\hat{u}_{1,ij}) + \bar{\sigma}^2(\hat{u}_{2,ij})} \leq 2M_n^1 + 2M_n. \tag{C.23}
\end{equation}

Under $(U_1, U_2) \in A(4)$, to prove $\mathbb{P}(M_n \geq G^-(\alpha) + 4 \log q - \log(\log q)) \to 1$, we need the following two lemmas.

**Lemma C.6.** Under Assumption (A2), as $n, q \to \infty$, we have
\[
\mathbb{P}(M_n^1 \leq 4 \log q - \frac{1}{2} \log(\log q)) \to 1. \tag{C.24}
\]

The detailed proof of Lemma C.6 is in the supplementary Appendix D.6.

**Lemma C.7.** Under Assumption (A2), as $n, q \to \infty$, we have
\[
\mathbb{P}
\left(
\max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{\bar{\sigma}^2(\hat{u}_{1,ij}) + \bar{\sigma}^2(\hat{u}_{2,ij})} \geq 16 \log q \right) \to 1. \tag{C.25}
\]
uniformly for all $(U_1, U_2) \in A(4)$.

The detailed proof of Lemma C.7 is in the supplementary Appendix D.7.

Combining (C.23), (C.24) and (C.25), as $n, q \to \infty$, with probability going to one, we have
\[
M_n \geq \frac{1}{2} \max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{\bar{\sigma}^2(\hat{u}_{1,ij}) + \bar{\sigma}^2(\hat{u}_{2,ij})} - M_n^1 \geq 4 \log q + \frac{1}{2} \log(\log q).
\]

Therefore, as $n, q \to \infty$, we have
\[
1 \geq \mathbb{P}(M_n \geq G^-(\alpha) + 4 \log q - \log(\log q)) \geq \mathbb{P}(M_n \geq 4 \log q + \frac{1}{2} \log(\log q)) \to 1.
\]

Hence, we finish the proof of Theorem 2.4. \qed
C.3 Proof of Theorem 3.2

Proof. In Theorem 3.2, we aim to prove (3.10) and (3.11). (3.10) is an application of Theorem 2.2 for Kendall’s tau matrices. Therefore, we only need to prove (3.11). (3.11) is the same as (3.10) except that in (3.11) we use \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{a,ij}) \) (defined in (3.5)) to replace the Jackknife variance estimator \( \hat{\sigma}^2(\hat{\tau}_{a,ij}) \) in (3.10).

We prove (3.11) in the same three steps as those for Theorem 2.2 except that we need to verify whether \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{a,ij}) \) satisfies the following equation:

\[
P \left( \max_{1 \leq i < j \leq d} |\hat{\Sigma}_{a,ij} - \Sigma_{a,ij}| > \varepsilon_n \frac{\sigma}{\log d} \right) = o(1),
\]

where \( \varepsilon_n = o(1) \). (C.26) is the same as (C.1) except that in (C.26) we use \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{a,ij}) \) to replace \( \hat{\sigma}^2(\hat{\tau}_{a,ij}) \) in (C.1). Once (C.26) is obtained, by the same argument in the first step of the proof of Theorem 2.2, we obtain (3.11). (3.11) is the same as (3.10) except that in (3.11) we use \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{a,ij}) \) (defined in (3.5)) to replace the Jackknife variance estimator \( \hat{\sigma}^2(\hat{\tau}_{a,ij}) \) in (3.10).

Following the last two steps of the proof of Theorem 2.2, we obtain (3.11). The proof of (C.26) proceeds in two steps. We define \( \hat{\Sigma}_{c,ij} \) and \( \Sigma_{c,ij} \) in (3.4) and (3.2). In the first step, we rewrite (C.26) as

\[
P \left( \max_{1 \leq i < j \leq d} \left| (\hat{\Pi}_{cc,ij} - \hat{\Pi}_{c,ij})^2 - (\Pi_{cc,ij} - (\Pi_{c,ij})^2) \right| > \varepsilon_n \frac{\sigma}{\log d} \right) = o(1),
\]

where \( \hat{\Pi}_{cc,ij} \) and \( \hat{\Pi}_{c,ij} \) are U-statistics and they estimate \( \Pi_{cc,ij} \) and \( \Pi_{c,ij} \). In the second step, we obtain upper bounds of \( \max_{1 \leq i < j \leq d} |\hat{\Pi}_{cc,ij} - \Pi_{cc,ij}| \) and \( \max_{1 \leq i < j \leq d} |(\hat{\Pi}_{c,ij})^2 - (\Pi_{c,ij})^2| \). Using the obtained upper bounds, we prove (C.27).

Step (i). For notational simplicity, we ignore the subscript \( a \) (\( a = 1 \) or 2) for \( X \) and \( Y \). To rewrite (C.26) as (C.27), we need \( 4\zeta_{ij} = 16(\Pi_{cc,ij} - (\Pi_{c,ij})^2) \) and \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{ij}) = 16(\hat{\Pi}_{cc,ij} - \hat{\Pi}_{c,ij}^2)/n \). By the definition of \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{ij}) \) in (3.5), we have \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{ij}) = 16(\hat{\Pi}_{cc,ij} - \hat{\Pi}_{c,ij}^2)/n \). We then need to prove \( 4\zeta_{ij} = 16(\Pi_{cc,ij} - (\Pi_{c,ij})^2) \). By (3.3), we have that the variance of \( \hat{\tau}_{ij} \) is

\[
\frac{8}{n(n-1)} \Pi_{c,ij}(1 - \Pi_{c,ij}) + 16 \frac{n-2}{n(n-1)} (\Pi_{cc,ij} - \Pi_{c,ij}^2).
\]

Therefore, \( 16(\Pi_{cc,ij} - (\Pi_{c,ij})^2) \) is the asymptotic variance of \( \sqrt{n} \hat{\tau}_{ij} \) as \( n \to \infty \). Lemma C.1 implies that \( 4\zeta_{ij} \) is also the asymptotic variance of \( \sqrt{n} \hat{\tau}_{ij} \) as \( n \to \infty \). Hence, we have

\[
16(\Pi_{cc,ij} - (\Pi_{c,ij})^2) = 4\zeta_{ij},
\]

because both sides of (C.28) are the asymptotic variance of \( \sqrt{n} \hat{\tau}_{ij} \). Therefore, combining \( \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{ij}) = 16(\hat{\Pi}_{cc,ij} - \hat{\Pi}_{c,ij}^2)/n \) and (C.28), to prove (C.26), it suffices to prove (C.27) as \( n, q \to \infty \).

Step (ii). We aim to prove (C.27), in which \( \hat{\Pi}_{cc,ij} \) is a U-statistic estimator of \( \Pi_{cc,ij} \) (defined in (3.4)). Therefore, we use the exponential inequality (Lemma F.2) to obtain

\[
P \left( \max_{1 \leq i < j \leq d} |\hat{\Pi}_{cc,ij} - \Pi_{cc,ij}| > t \right) \leq C_1 d^2 \exp(-C_2 nt^2).
\]

(C.29)

\( \hat{\Pi}_{c,ij} \) is also a U-statistic which estimates \( \Pi_{c,ij} \) (defined in (3.2)). Similarly to (C.29), we use the exponential inequality (Lemma F.2) to obtain

\[
P \left( \max_{1 \leq i < j \leq d} |\hat{\Pi}_{c,ij} - \Pi_{c,ij}| > t \right) \leq C_1 d^2 \exp(-C_2 nt^2).
\]

(C.30)
By the definitions of $\hat{\Pi}_{c,ij}$ and $\Pi_{c,ij}$, we have $|\hat{\Pi}_{c,ij}| \leq 2$ and $|\Pi_{c,ij}| \leq 2$. Therefore, we have $|\hat{\Pi}_{c,ij} + \Pi_{c,ij}| \leq 4$. Considering $d = O(n^{1/3-\epsilon})$ (see Assumption (A2)), (C.29) and (C.31), by setting $\varepsilon_n = 1/(\log d)^{\kappa_0}$ with $\kappa_0 > 0$ sufficiently small, we have

$$
P\left( \max_{1 \leq i < j \leq d} |(\hat{\Pi}_{c,ij} - \Pi_{c,ij})^2| > C\frac{\varepsilon_n}{\log d} \right) = o(1),
$$

(C.32)

By the triangle inequality, we have

$$
P\left( \max_{1 \leq i < j \leq d} |(\hat{\Pi}_{c,ij} - \Pi_{c,ij})^2| > C\frac{\varepsilon_n}{\log d} \right) \leq P\left( \max_{1 \leq i < j \leq d} |(\hat{\Pi}_{c,ij} - \Pi_{c,ij})^2| > 0.5C\frac{\varepsilon_n}{\log d} \right) + P\left( \max_{1 \leq i < j \leq d} |(\hat{\Pi}_{c,ij})^2 - (\Pi_{c,ij})^2| > 0.5C\frac{\varepsilon_n}{\log d} \right).
$$

(C.34)

Combining (C.32), (C.33) and (C.34), we prove (C.27). This completes the proof.

\[\square\]

### C.4 Proof of Theorem 3.3

**Proof.** By setting

$$
M_{ij}^{\tau,ps} := \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2}
$$

and

$$
M_n^{\tau,ps} := \max_{(i,j) \in S} M_{ij}^{\tau,ps} = \max_{(i,j) \in S} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2},
$$

(C.35)

where $S = \{(i,j) : 1 \leq i < j \leq d\}$, we aim to prove that as $n, d \to \infty$, we have

$$
P\left( M_n^{\tau,ps} - 4 \log d + \log(\log d) \leq x \right) \to \exp\left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x}{2}\right) \right).
$$

(C.36)

(C.36) is the same as (3.10) except that in $M_n^{\tau,ps}$ we use $\sigma_{a,ps}^2/n_a$ to replace $\hat{\sigma}^2(\tau_{a,ij})$ in (3.10).

We first present the sketch of the proof. For notational simplicity, we introduce

$$
\begin{align*}
g_{ij}^{\tau}(X_a) & := \mathbb{E}[\text{sign}(X_{ai} - X_{bi})\text{sign}(X_{aj} - X_{bj})|X_a], \\
g_{ij}^{\tau}(Y_a) & := \mathbb{E}[\text{sign}(Y_{ai} - Y_{bi})\text{sign}(Y_{aj} - Y_{bj})|Y_a],
\end{align*}
$$

(C.37)

with $\beta \neq \alpha$. The proof proceeds in three steps. In the first step, we use the Hoeffding method (Hoeffding, 1948) to decompose Kendall’s tau as

$$
\hat{\tau}_{1,ij} = \frac{2}{n_1} \sum_{a=1}^{n_1} h_{ij}^\tau(X_a) + \frac{2}{n_1(n_1 - 1)} \Delta_{n_1,ij}^\tau \quad \text{and} \quad \hat{\tau}_{2,ij} = \frac{2}{n_2} \sum_{a=1}^{n_2} h_{ij}^\tau(Y_a) + \frac{2}{n_2(n_2 - 1)} \Delta_{n_2,ij}^\tau,
$$

(C.38)

where $h_{ij}^\tau(X_a) := g_{ij}^\tau(X_a) - \tau_{1,ij}$, $h_{ij}^\tau(Y_a) := g_{ij}^\tau(Y_a) - \tau_{2,ij}$ and

$$
\begin{align*}
\Delta_{n_1,ij}^\tau & := \sum_{1 \leq k \leq \ell \leq n_1} \left( \text{sign}(X_{ki} - X_{\ell i})\text{sign}(X_{kj} - X_{\ell j}) - \tau_{1,ij} - h_{ij}^\tau(X_k) - h_{ij}^\tau(X_\ell) \right), \\
\Delta_{n_2,ij}^\tau & := \sum_{1 \leq k \leq \ell \leq n_2} \left( \text{sign}(Y_{ki} - Y_{\ell i})\text{sign}(Y_{kj} - Y_{\ell j}) - \tau_{2,ij} - h_{ij}^\tau(Y_k) - h_{ij}^\tau(Y_\ell) \right).
\end{align*}
$$
2\sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha) / n_1 and 2\sum_{\alpha=1}^{n_1} h_{ij}(Y_\alpha) / n_2 are terms for the sum of i.i.d random variables and 
2\Delta_{\alpha=1}^{n_1} h_{ij}(X_\alpha) / n_1(n_a - 1) is a small residual term. Similar to (C.5), we define \( \hat{T}_{ij} \) as

\[
\hat{T}_{ij} := \left( \sum_{\alpha=1}^{n_1} h_{ij}^\tau(X_\alpha) / n_1 - \sum_{\alpha=1}^{n_2} h_{ij}^\tau(Y_\alpha) / n_2 \right) / \sqrt{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2}.
\]

We then prove that the residual term \( \Delta_{\alpha=1}^{n_1} h_{ij}(X_\alpha) / n_1(n_a - 1) \) is negligible, i.e., to obtain Theorem 3.3, it suffices to prove that as \( n, d \to \infty \), we have

\[
P\left( \max_{(i,j)\in S}(\hat{T}_{ij})^2 - 4\log d + \log(\log d) \leq x \right) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right). \tag{C.39}
\]

In the second step, we prove that it is sufficient to take the maximum of \( (\hat{T}_{ij})^2 \) over \( S \setminus S_0 \) but instead of over \( S \) as in (C.39), i.e., to obtain Theorem 3.3, it suffices to prove that as \( n, d \to \infty \), we have

\[
P\left( \max_{(i,j)\in S_0}(\hat{T}_{ij})^2 - 4\log d + \log(\log d) \leq x \right) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right). \tag{C.40}
\]

In the last step, we prove that it is sufficient to replace \( \hat{T}_{ij} \) in (C.40) with \( T_{ij} \), i.e., to obtain Theorem 3.3, it suffices to prove that as \( n, d \to \infty \), we have

\[
P\left( \max_{(i,j)\in S\setminus S_0}(T_{ij})^2 - 4\log d + \log(\log d) \leq x \right) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right), \tag{C.41}
\]

where \( T_{ij} \) and \( S_0 \) are defined in (C.5) and (C.8). Because (C.41) and (C.10) are the same, we prove (C.41) by following the same proof of (C.10). Hence, we complete the proof of Theorem 3.3. We then present the proof in detail.

**Step (i).** For notational simplicity, we introduce

\[
N_{ij}^{\tau,ps} := \frac{\hat{T}_{ij} - \tilde{T}_{ij}}{\sqrt{\sigma_{1,ps}/n_1 + \sigma_{2,ps}^2/n_2}} = \hat{T}_{ij} + \tilde{W}_{ij} \quad \text{with} \quad \tilde{W}_{ij} := \frac{\Delta_{\alpha=1}^{n_1} h_{ij}(X_\alpha) / n_1(n_a - 1) - \Delta_{\alpha=1}^{n_2} h_{ij}(Y_\alpha) / n_2(n_a - 1)}{\sqrt{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2}}. \tag{C.42}
\]

By (C.35) and (C.42), we have \( M_{ij}^{\tau,ps} := (N_{ij}^{\tau,ps})^2 \). We then introduce the following lemma to finish the proof of this step.

**Lemma C.8.** As \( n, d \to \infty \), under Assumption (A2) we have

\[
\left| \max_{1\leq i<j\leq d} (N_{ij}^{\tau,ps})^2 - \max_{1\leq i<j\leq d}(\hat{T}_{ij})^2 \right| = o_p(1). \tag{C.43}
\]

The detailed proof of Lemma C.8 is in the supplementary Appendix D.8. By Lemma C.8, we obtain that \( \max_{1\leq i<j\leq d}(N_{ij}^{\tau,ps})^2 \) and \( \max_{1\leq i<j\leq d}(\hat{T}_{ij})^2 \) have the same limiting distribution as \( n, d \to \infty \). Therefore, for obtaining Theorem 3.3, it suffices to prove (C.39) as \( n, d \to \infty \).

**Step (ii).** In the second step, we aim to prove that (C.40) is enough to prove the theorem. By setting \( y_d = x + 4\log d - \log(\log d) \), we have

\[
\left| P\left( \max_{(i,j)\in S}(\hat{T}_{ij})^2 > y_d \right) - P\left( \max_{(i,j)\in S\setminus S_0}(\hat{T}_{ij})^2 > y_d \right) \right| \leq P\left( \max_{(i,j)\in S_0}(\hat{T}_{ij})^2 > y_d \right).
\]

We then introduce an additional lemma.
Lemma C.9. Under Assumptions (A1) and (A2), as \( n, d \to \infty \), we have
\[
\mathbb{P}\left( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > y_d \right) = o(1). \tag{C.44}
\]
The detailed proof of Lemma C.9 is in the supplementary Appendix D.9. By Lemma C.9, we obtain that as \( n, d \to \infty \), \( \mathbb{P}\left( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > y_d \right) \) and \( \mathbb{P}\left( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 > y_d \right) \) have the same limit. Hence, we complete the proof of the second step.

Step (iii). In this step, we aim to prove that it is sufficient to have (C.41) as \( n, d \to \infty \). For this, we need the following lemma.

Lemma C.10. If \( X \) and \( Y \) follow the meta-elliptical distribution, we have
\[
|4\zeta_{a,ij} - \sigma_{a,ps}^2| \leq C|\tau_{a,ij}|, 
\]
where \( C \) is a constant irrelevant to \( i \) or \( j \). Definitions of \( \zeta_{a,ij} \) and \( \sigma_{a,ps} \) are in (2.10) and (3.7).

The detailed proof of Lemma C.10 is in the supplementary Appendix D.10. By the definition of \( S_0 \) in (C.8), for any \((i, j) \in S \setminus S_0\), we have \( |\tau_{a,ij}| \leq (\log d)^{-1-\alpha_0} \). By Lemma C.10, we have \( |4\zeta_{a,ij} - \sigma_{a,ps}^2| \leq C|\tau_{a,ij}| \). Hence, for \((i, j) \in S \setminus S_0\), we have \( |4\zeta_{a,ij} - \sigma_{a,ps}^2| \leq C|\tau_{a,ij}| \leq C(\log d)^{-1-\alpha_0} \).

Considering \( \zeta_{a,ij} \geq r_a > 0 \) (see Assumption (A2)), for \((i, j) \in S \setminus S_0\), we obtain
\[
|\sigma_{a,ps}^2/4\zeta_{a,ij} - 1| \leq C(\log d)^{-1-\alpha_0} \quad \text{and} \quad |4\zeta_{a,ij}/\sigma_{a,ps}^2 - 1| \leq C(\log d)^{-1-\alpha_0}. \tag{C.45}
\]

To prove the sufficiency of (C.41), we need to prove \( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 - \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 = o_p(1) \). For this, we calculate the relevant difference of \( (\hat{T}_{ij})^2 \) and \( (T_{ij})^2 \) as
\[
\left| (\hat{T}_{ij})^2 - (T_{ij})^2 \right| \leq \left| \frac{\sigma_{1,ps}^2 - 4\zeta_{1,ij}}{\sigma_{1,ps}^2} \right| + \left| \frac{\sigma_{2,ps}^2 - 4\zeta_{2,ij}}{\sigma_{2,ps}^2} \right|.
\]

By (C.45), we then have \( |(\hat{T}_{ij})^2 - (T_{ij})^2| \leq C(T_{ij})^2(\log d)^{-1-\alpha_0} \) for \((i, j) \in S \setminus S_0\). Hence, we have
\[
\max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 - \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \leq \max_{(i,j) \in S \setminus S_0} \left| (\hat{T}_{ij})^2 - (T_{ij})^2 \right| \leq C(\log d)^{-1-\alpha_0} \max_{(i,j) \in S \setminus S_0} (T_{ij})^2.
\]

Considering \( \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 = O_p(\log d) \), we have \( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 - \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 = o_p(1) \). Therefore, it suffices to prove (C.41) as \( n, d \to \infty \). Because (C.41) and (C.10) are the same, we prove (C.41) by following the same proof of (C.10). Hence, we prove Theorem 3.3. \( \square \)

C.5 Proof of Theorem 3.4

Proof. To obtain Theorem 3.4, we aim to prove (3.13), (3.14) and (3.15). The proof of (3.13) is the same as Theorem 2.4, as Kendall’s tau is a special kind of U-statistic with a bounded kernel. Hence, we only need to prove (3.14) and (3.15).

First, we prove (3.14). Under the alternative hypothesis, \( \tau_{1,ij} = \tau_{2,ij} \) cannot hold for all \((i, j) \in S\), where \( S = \{(i, j) : 1 \leq i < j \leq d\} \). This motivates us to define
\[
M_{n,\text{plug}} := \max_{(i,j) \in S} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij} - \tau_{1,ij} + \tau_{2,ij})^2}{\hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{1,ij}) + \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{2,ij})}, \quad M_{n,\tau,\text{plug}} := \max_{(i,j) \in S} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{1,ij}) + \hat{\sigma}_{\text{plug}}^2(\hat{\tau}_{2,ij})}. \tag{C.46}
\]
because $M_n^{1,\text{plug}}$ and $M_n^{\tau,\text{plug}}$ are different under the alternative hypothesis. Similarly to Theorem 2.4, to prove (3.14), using the inequality $(a \pm b)^2 \leq 2a^2 + 2b^2$, we obtain

$$(\tau_{1,ij} - \tau_{2,ij})^2 \leq 2(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij} - \tau_{1,ij} + \tau_{2,ij})^2 + 2(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2.$$ 

By the the definition of $M_n^{\tau,\text{plug}}$ and $M_n^{1,\text{plug}}$ in (C.46), we get

$$\max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma^2_{\text{plug}}(\hat{u}_{1,ij}) + \sigma^2_{\text{plug}}(\hat{u}_{2,ij})} \leq 2M_n^{1,\text{plug}} + 2M_n^{\tau,\text{plug}}. \quad \text{(C.47)}$$

To complete the proof of (3.14), we need two additional lemmas.

**Lemma C.11.** Under Assumption (A2), as $n, d \to \infty$, we have

$$\mathbb{P}(M_n^{1,\text{plug}} \leq 4 \log d - \frac{1}{2} \log(\log d)) \to 1. \quad \text{(C.48)}$$

The detailed proof of Lemma C.11 is in the supplementary Appendix D.11.

**Lemma C.12.** Under Assumption (A2), as $n, d \to \infty$, we have

$$\mathbb{P}\left( \max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma^2_{\text{plug}}(\hat{u}_{1,ij}) + \sigma^2_{\text{plug}}(\hat{u}_{2,ij})} \geq 16 \log d \right) \to 1, \quad \text{(C.49)}$$

uniformly for $(U_1^T, U_2^T) \in U(4)$.

The detailed proof of Lemma C.12 is in the supplementary Appendix D.12. Combining (C.47), Lemmas C.11 and C.12, under $(U_1^T, U_2^T) \in U(4)$, as $n, d \to \infty$, with probability going to one, we have

$$M_n^{\tau,\text{plug}} \geq \frac{1}{2} \max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma^2_{\text{plug}}(\hat{u}_{1,ij}) + \sigma^2_{\text{plug}}(\hat{u}_{2,ij})} - M_n^{1,\text{plug}} \geq 4 \log d + \frac{1}{2} \log(\log d).$$

Therefore, under $(U_1^T, U_2^T) \in U(4)$, as $n, d \to \infty$, we have

$$1 \geq \mathbb{P}(M_n^{\tau,\text{plug}} \geq G^-(\alpha) + 4 \log d - \log(\log d)) \geq \mathbb{P}(M_n^{\tau,\text{plug}} \geq 4 \log d + \frac{1}{2} \log(\log d)) \to 1,$$

where $G^-(\alpha) := -\log(8\pi) - 2 \log\left( -\log(1 - \alpha) \right)$. Hence, we prove (3.14).

Secondly, we aim to prove (3.15). Under the alternative hypothesis, $\tau_{1,ij} = \tau_{2,ij}$ cannot hold for all $(i,j) \in S$. This motivates us to define

$$M_n^{1,\text{ps}} := \max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij} - \tau_{1,ij} + \tau_{2,ij})^2}{\sigma^2_{\text{ps}}/n_1 + \sigma^2_{\text{ps}}/n_2} \quad \text{and} \quad M_n^{\tau,\text{ps}} := \max_{(i,j) \in S} \frac{(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2}{\sigma^2_{\text{ps}}/n_1 + \sigma^2_{\text{ps}}/n_2}, \quad \text{(C.50)}$$

as $M_n^{1,\text{ps}}$ and $M_n^{\tau,\text{ps}}$ are different under the alternative hypothesis.

We then introduce an additional lemma.

**Lemma C.13.** Under Assumptions (A1) and (A2), as $n, d \to \infty$, we have

$$\mathbb{P}(M_n^{1,\text{ps}} \leq 4 \log d - \frac{1}{2} \log(\log d)) \to 1. \quad \text{(C.51)}$$
The detailed proof of Lemma C.13 is in the supplementary Appendix D.13. To prove (3.15), using \((a \pm b)^2 \leq 2a^2 + 2b^2\), we obtain
\[
(\tau_{1,ij} - \tau_{2,ij})^2 \leq 2(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij} - \tau_{1,ij} + \tau_{2,ij})^2 + 2(\hat{\tau}_{1,ij} - \hat{\tau}_{2,ij})^2.
\]
By definitions of \(M_n^{r,ps}\) and \(M_n^{1,ps}\) in (C.50), we have
\[
\max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2} \leq 2M_n^{1,ps} + 2M_n^{r,ps}.
\]  \(\text{(C.52)}\)

Under \((U_1^1, U_2^1) \in \mathbb{V}(4)\), we have
\[
\max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2} \geq 16 \log d.
\]  \(\text{(C.53)}\)

Combining Lemma C.13, (C.52) and (C.53), under \((U_1^1, U_2^2) \in \mathbb{V}(4)\), as \(n, d \to \infty\), with probability going to one, we have
\[
M_n^{r,ps} \geq \frac{1}{2} \max_{(i,j) \in S} \frac{(\tau_{1,ij} - \tau_{2,ij})^2}{\sigma_{1,ps}^2/n_1 + \sigma_{2,ps}^2/n_2} - M_n^{1,plug} \geq 4 \log d + \frac{1}{2} \log(\log d).
\]

Therefore, under \((U_1^1, U_2^2) \in \mathbb{V}(4)\), as \(n, d \to \infty\), we have
\[
1 \geq \mathbb{P}(M_n^{r,ps} \geq G^-(\alpha) + 4 \log d - \log(\log d)) \geq \mathbb{P}(M_n^{r,ps} \geq 4 \log d + \frac{1}{2} \log(\log d)) \to 1,
\]
where \(G^-(\alpha) := -\log(8\pi) - 2\log(-\log(1-\alpha))\). Hence, we prove (3.15). This completes the proof of Theorem 3.4.

\(\blacksquare\)

### C.6 Proof of Theorem 3.5

**Proof.** It suffices to take \(T_{\alpha}\) to be the set of level \(\alpha\) tests over the normal distributions with covariance matrix \(\Sigma\), where \(\text{Diag}(\Sigma) = I_d\), since it contains all the \(\alpha\)-level tests over the collection of the assumed distributions. For these normal distributions, we define
\[
\mathbb{H}(c') = \{(\Sigma_1, \Sigma_2) : \sigma_{1,ii} = 1, \sigma_{2,ii} = 1, \|\Sigma_1 - \Sigma_2\|_{\max} \geq c' \sqrt{\log d/n}\}.
\]

By \(\sigma_{ij} = \sin(\tau_{ij} \pi/2)\) in Theorem A.3, we have \(|\sigma_{1,ij} - \sigma_{2,ij}| \leq |\tau_{1,ij} - \tau_{2,ij}| \pi/2\). Therefore, for any \(c'\), there is a \(c_0\) such that \(\mathbb{H}(c') \subset \mathbb{U}(c_0)\). For simplicity, we set \(n_1 = n_2 = n\).

Let’s consider the Gaussian setting and define
\[
\mathcal{F}(\rho) = \{\Sigma = I_d + \rho e_1 e_j^T + \rho e_1 e_k^T, \text{ where } e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \text{ for } 1 \leq k \leq d, \text{ and } j = 2, \ldots, d\},
\]
where \(\rho = c' \sqrt{\log d/n}\) and \(\mu_{\rho}\) is the uniform measure on \(\mathcal{F}(\rho)\). For simplicity, under the null hypothesis, we set \(\Sigma_1 = \Sigma_2 = I_d\). Under the alternative hypothesis, we set \(\Sigma_1 = \Sigma \sim \mu_{\rho}\) and \(\Sigma_2 = I_d\). Therefore, under the alternative hypothesis, there is a \(c_0\) such that we have \((\Sigma_1, \Sigma_2) \in \mathbb{H}(c') \subset \mathbb{U}(c_0)\).

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Let $\mathbb{P}_\Sigma$ denote the probability measure of samples for $X \sim N(0, \Sigma_1)$ and $Y \sim N(0, \Sigma_2)$ with $\Sigma_1 = \Sigma$ and $\Sigma_2 = I_d$. We set $\mathbb{P}_\mu = \int \mathbb{P}_\Sigma d\mu(\Sigma)$. In particular, let $\mathbb{P}_0$ denote the probability measure of samples for $X \sim N(0, \Sigma_1)$ and $Y \sim N(0, \Sigma_2)$ with $\Sigma_1 = \Sigma_2 = I_d$. We then have

$$\inf_{T_\alpha \in T_\alpha} \sup_{\Sigma \in \mathcal{F}(\rho)} \mathbb{P}_\Sigma(T_\alpha = 0) \geq 1 - \alpha - \frac{1}{2} \|\mathbb{P}_\mu - \mathbb{P}_0\|_{TV},$$

where $\| \cdot \|_{TV}$ denotes the total variation norm. By setting

$$L_{\mu_\rho}(z) := \frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(z),$$

considering the Jensen’s inequality, we have

$$\|\mathbb{P}_\mu - \mathbb{P}_0\|_{TV} = \int \left| L_{\mu_\rho}(z) - 1 \right| d\mathbb{P}_0(z) = \mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}(Z) - 1 \leq (\mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}^2(Z) - 1)^{1/2}.$$

Therefore, as long as $\mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}^2(Z) = 1 + o(1)$, we have

$$\inf_{T_\alpha \in T_\alpha} \sup_{\Sigma \in \mathcal{F}(\rho)} \mathbb{P}_\Sigma(T_\alpha = 0) \geq 1 - \alpha - o(1) > 0,$$

which is the desired result. We then aim to prove $\mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}^2(Z) = 1 + o(1)$. By construction, we have

$$L_{\mu_\rho} = \frac{1}{d-1} \sum_{\Sigma \in \mathcal{F}(\rho)} \left( \prod_{i=1}^n \frac{1}{|\Sigma|^{1/2}} \exp \left( -\frac{1}{2} Z_i^T (\Omega - I_d) Z_i \right) \right),$$

where $\Omega := \Sigma^{-1}$ and $\{Z_i\}$ are independent random vectors with $Z_i = (Z_{i1}, Z_{i2}, \ldots, Z_{id})^T \sim N(0, I_d)$. Therefore, we have

$$\mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}^2 = \frac{1}{(d-1)^2} \sum_{\Sigma_1, \Sigma_2 \in \mathcal{F}(\rho)} \mathbb{E} \left( \prod_{i=1}^n \frac{1}{|\Sigma_1|^{1/2}} \frac{1}{|\Sigma_2|^{1/2}} \exp \left( -\frac{1}{2} Z_i^T (\Omega_1 + \Omega_2 - 2I_d) Z_i \right) \right),$$

where $\Omega_i = \Sigma_i^{-1}$ for $i = 1, 2$. By setting

$$A = \frac{\rho}{1 - \rho^2} \begin{pmatrix} 2\rho & -1 & -1 \\ -1 & \rho & 0 \\ -1 & 0 & \rho \end{pmatrix}, \quad B = \frac{2\rho}{1 - \rho^2} \begin{pmatrix} \rho & -1 \\ -1 & \rho \end{pmatrix},$$

$Z_{i,\{1,2\}} = (Z_{i1}, Z_{i2})^T$ and $Z_{i,\{1,2,3\}} = (Z_{i1}, Z_{i2}, Z_{i3})^T$, we have

$$\mathbb{E}_{\mathbb{P}_0} L_{\mu_\rho}^2 = \frac{d-2}{d-1} \prod_{i=1}^n \left( \frac{1}{1 - \rho^2} \mathbb{E} \exp \left( -\frac{1}{2} Z_{i,\{1,2,3\}}^T A Z_{i,\{1,2,3\}} \right) \right)_{A_3} + \frac{1}{d-1} \prod_{i=1}^n \left( \frac{1}{1 - \rho^2} \mathbb{E} \exp \left( -\frac{1}{2} Z_{i,\{1,2\}}^T B Z_{i,\{1,2\}} \right) \right)_{A_4},$$

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where $A_3$ represents the sum of terms with $\Sigma_1 \neq \Sigma_2$ in (C.54) and $A_4$ represents the sum of terms with $\Sigma_1 = \Sigma_2$ in (C.54). For $A_3$, by the standard argument in calculating moment generating function of the Gaussian quadratic form (Baldessari, 1967), we have

$$A_3 = \frac{d - 2}{d - 1} \cdot \frac{1}{(1 - \rho^2)^n} \left((1 + \lambda_1(A))(1 + \lambda_2(A))(1 + \lambda_3(A))\right)^{-n/2}.$$ 

Moreover, we have $(1 + \lambda_1(A))(1 + \lambda_2(A))(1 + \lambda_3(A)) = |A + I_3| = (1 - \rho^2)^{-2}$. Therefore, we have $A_3 = (d - 2)/(d - 1) = 1 + o(1)$. For $A_4$, it is easy to get $\lambda_1(B) = 2\rho/(1 - \rho)$ and $\lambda_2(B) = -2\rho/(1 + \rho)$. Accordingly, similarly to the calculation of $A_3$, we have $A_4 = (d - 1)^{-1} \cdot (1 - \rho^2)^{-n}$. Considering $\rho = c'\sqrt{\log d/n}$, as long as $c' < 1$, we have

$$A_4 \leq \frac{1}{d - 1} \cdot (1 - c'^2 \log d/n)^{-n} = (d - 1)^{-1} \exp(c'^2 \log d)(1 + o(1)) = o(1),$$

as $n, d \to \infty$. Combining $A_3 = 1 + o(1)$ and (C.55), we prove $E_{\mathbb{P}_0} L_{\mu,\nu}^2 (Z) = 1 + o(1)$. This completes the proof. \hfill \square

### C.7 Proof of Theorem 3.6

We first introduce two additional lemmas. Given these lemmas, we prove Theorem 3.6.

**Lemma C.14.** Suppose that $(X_1, X_2, X_3, X_4)^T \sim N(0, \Sigma_{\text{full}})$ is Gaussian distributed with

$$\Sigma_{\text{full}} = \begin{bmatrix} 1 & g^T \\ g & \Sigma \end{bmatrix} \quad \text{and} \quad \text{Diag}(\Sigma) = I_3,$$

where $g = (\rho_1, \rho_2, \rho_3)^T$. We have

$$\left|E(\Phi(X_1) - 1/2)(\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2)\right| \leq \left(\frac{1}{8\pi} + \frac{1}{4\sqrt{2\pi}}\right) \sqrt{g^T \Sigma^{-1} g},$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal distribution.

The detailed proof of Lemma C.14 is in the supplementary Appendix D.14.

**Lemma C.15.** Suppose that $(X_1, X_2, X_3, X_4)^T \sim N(0, \Sigma_{\text{full}})$ is Gaussian distributed with

$$\Sigma_{\text{full}} = \begin{bmatrix} 1 & \rho_1 & a_1 & a_2 \\ \rho_1 & 1 & a_3 & a_4 \\ a_1 & a_3 & 1 & \rho_2 \\ a_2 & a_4 & \rho_2 & 1 \end{bmatrix}.$$

Then, when $|\rho_1|, |\rho_2|, |a_1|, \ldots, |a_4| \leq r < 1$, we have

$$C_r := \sup_{|\rho_1|, |\rho_2|, |a_1|, \ldots, |a_4| \leq r} |\text{Corr}\{(\Phi(X_1) - 1/2)(\Phi(X_2) - 1/2), (\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2)\}| < 1.$$

Moreover, we have $C_r = 1$ only when $r = 1$ and the set $\{\rho_1, \rho_2, a_1, \ldots, a_4\}$ attains the boundary.
The detailed proof of Lemma C.15 is in the supplementary Appendix D.15.

Proof. To introduce the set $B_0$ and its properties, we set

$$C_0 = \{(i, j) : i \in \Omega(r) \bigcup \Gamma \} \bigcup \{(i, j) : j \in \Omega(r) \bigcup \Gamma \} \quad \text{and} \quad B_0 = S_0 \bigcup C_0,$$

where $\Omega(r)$, $\Gamma$ and $S_0$ are defined in (3.17), Assumption (A4) and (C.8). For Kendall’s tau matrices, we set $S = \{(i, j) : 1 \leq i < j \leq d\}$ and $q = d$. By the definition of $B_0$, we have $|\tau_{a, k\ell}| \leq r < 1$ for any $k \neq \ell \in \{i_1, j_1, i_2, j_2\}$, where $(i_1, j_1) \in S \setminus B_0$ and $(i_2, j_2) \in S \setminus B_0$. For any $(i, j) \in S \setminus B_0$, we also have $|\tau_{a, ij}| \leq (\log d)^{-1-\alpha_0}$, as $S_0$ is a subset of $B_0$. By Assumptions (A1) and (A4), we have $|B_0| = o(d^2)$.

We then prove Theorem 3.6 similarly as the proofs of Theorems 3.2 and 3.3 except that we replace $S_0$ with $B_0$. However, as we don’t require Assumption (A3) in Theorem 3.6, we don’t have Lemma C.5. Therefore, we only need to prove (C.19) under Assumptions (A1), (A2) and (A4).

We then begin to prove (C.19) under Assumptions (A1), (A2) and (A4). As we use $B_0$ to replace $S_0$ in the proofs of Theorems 3.2 and 3.3, we need to redefine some notations. After rearranging the two-dimensional indices $\{(i, j), (i, j) \in S \setminus B_0\}$ in any order, we set them as $\{(i_k, j_k) : 1 \leq k \leq h\}$ with $h = |S \setminus B_0|$. We denote $T_k = T_{i_k, j_k}$, where the definition of $T_{ij}$ is the same as (C.5) except that we use Kendall’s tau as the U-statistic in (C.5). We only need to check whether (C.19) is correct for $\{(i_k, j_k) : 1 \leq k \leq h\}$ under Assumptions (A1), (A2) and (A4). The definition of $N_\ell$ is same as (C.16). By $N_\ell$’s definition in (C.16), the entry in the $a$-th row, $b$-th column of $\text{Var}(N_\ell)$ is

$$\frac{n_2 \text{Cov} \left( h_{i_k, j_k} (X_1), h_{i_b, j_b} (X_1) \right) / n_1 + \text{Cov} \left( h_{i_a, j_a} (Y_1), h_{i_k, j_k} (Y_1) \right)}{\sqrt{n_2 \zeta_{1, i_k, j_k} / n_1 + \zeta_{a, i_a, j_a} \sqrt{n_2 \zeta_{1, i_b, j_b} / n_1 + \zeta_{a, i_b, j_b}}}},$$

(C.58)

where $h_{ij}$ is defined in (2.9). Apparently, when $a = b$, (C.58) equals one.

The proof of (C.19) proceeds in three steps. In the first step, for any $(i, j), (k, \ell) \in S \setminus B_0$, we define the condition $(\ast)$. This condition specifies that, for any $(i, j), (k, \ell) \in S \setminus B_0$, there exits an $i_1 \in \{i, j, k, \ell\}$ such that for any $j_1 \in \{i, j, k, \ell\} \setminus i_1$, we have $|\tau_{a, i_1 j_1}| = O((\log d)^{-1-\alpha_0})$. In the second step, we prove that for $(i, j), (k, \ell) \in S \setminus B_0$ satisfying $(\ast)$, as $n, d \to \infty$, we have

$$\frac{n_2 \text{Cov} \left( h_{ij} (X_1), h_{k\ell} (X_1) \right) / n_1 + \text{Cov} \left( h_{ij} (Y_1), h_{k\ell} (Y_1) \right)}{\sqrt{n_2 \zeta_{1, ij} / n_1 + \zeta_{2, ij} \sqrt{n_2 \zeta_{1, k\ell} / n_1 + \zeta_{2, k\ell}}}} = O((\log d)^{-1-\alpha_0}).$$

(C.59)

In the third step, we prove that for $(i, j), (k, \ell) \in S \setminus B_0$ dissatisfying $(\ast)$, we have

$$\left| \frac{n_2 \text{Cov} \left( h_{ij} (X_1), h_{k\ell} (X_1) \right) / n_1 + \text{Cov} \left( h_{ij} (Y_1), h_{k\ell} (Y_1) \right)}{\sqrt{n_2 \zeta_{1, ij} / n_1 + \zeta_{2, ij} \sqrt{n_2 \zeta_{1, k\ell} / n_1 + \zeta_{2, k\ell}}}} \right| \leq C < 1.$$ 

(C.60)

as $n, d \to \infty$.

By the proof of Lemma 5 in Cai et al. (2013), (C.59) and (C.60) are sufficient conditions for obtaining (C.19). Therefore, we prove (C.19) which finishes the proof of Lemma 3.6.

Step (i). In this step, we define the condition $(\ast)$. For this, we set graph $G_{ijk\ell} = (V_{ijk\ell}, E_{ijk\ell})$, where $V_{ijk\ell} = \{i, j, k, \ell\}$ is the set of vertices and $E_{ijk\ell}$ is the set of edges. We say that there is
an edge between \( a \neq b \in \{i, j, k, \ell\} \) if and only if \( |\tau_{ab}| \geq (\log d)^{-1-\alpha_0} \). If the number of different vertices in \( V_{ijk\ell} \) is 3, we say that \( G_{ijk\ell} \) is a three vertices graph \((3-G)\). If the number of different vertices in \( V_{ijk\ell} \) is 4, we say that \( G_{ijk\ell} \) is a four vertices graph \((4-G)\). If there is no edge connected to a vertex in \( G_{ijk\ell} \), we say that it is isolated. For any \( 1 \leq k_a \neq k_b \leq h \), we have that \( G_{i_{ka}j_{ka}i_{kb}j_{kb}} \) is either \( 3-G \) or \( 4-G \). For any \( 1 \leq k_a \neq k_b \leq h \), we say \( G := G_{i_{ka}j_{ka}i_{kb}j_{kb}} \) satisfies \((*)\), if a graph \( G \) has the property:

\[(*) \text{ If } G_{i_{ka}j_{ka}i_{kb}j_{kb}} \text{ is } 4-G, \text{ there is at least one isolated vertex in } G_{i_{ka}j_{ka}i_{kb}j_{kb}}; \text{ otherwise } G_{i_{ka}j_{ka}i_{kb}j_{kb}} = \emptyset.\]

**Step (ii).** In this step, we prove that for \((i,j), (k,\ell) \in S \setminus B_0 \) satisfying the condition \((*)\), \((C.59)\) is correct. To prove \((C.59)\), by \( \zeta_{a,ij} \geq r_a > 0 \) (see Assumption \((A2)\)), it suffices to prove

\[
\text{Cov} (h_{ij}(X_1), h_{\ell k}(X_1)) = O((\log d)^{1-\alpha_0}) \quad \text{and} \quad \text{Cov} (h_{ij}(Y_1), h_{\ell k}(Y_1)) = O((\log d)^{1-\alpha_0}). \quad (C.61)
\]

For simplicity, we only show the proof of \( X \). We treat Kendall’s tau as a special kind of U-statistics. By \((2.9)\), we have \( h_{ij}(X_1) = g_{ij}(X_1) - \tau_{1,ij} \). Therefore, we have

\[
\text{Cov} (h_{ij}(X_1), h_{\ell k}(X_1)) = \mathbb{E}[g_{ij}(X_1)g_{\ell k}(X_1)] - \tau_{1,ij}\tau_{1,\ell k}. \quad (C.62)
\]

By \((i,j), (k,\ell) \in S \setminus B_0\), we have

\[
\tau_{1,ij} = O((\log d)^{1-\alpha_0}) \quad \text{and} \quad \tau_{1,\ell k} = O((\log d)^{1-\alpha_0}). \quad (C.63)
\]

Combining \((C.62)\) and \((C.63)\), for getting \((C.61)\), it suffices to show

\[
\mathbb{E}[g_{ij}(X_1)g_{\ell k}(X_1)] = O((\log d)^{1-\alpha_0}). \quad (C.64)
\]

To prove \((C.64)\), by the definition of \( g_{ij} \) in \((C.37)\), we know that for Kendall’s tau, we have

\[
g_{ij}(X_1) = \mathbb{P}((X_{1i} - X_{2i})(X_{1j} - X_{2j}) > 0|X_{1i}, X_{1j}) - \mathbb{P}((X_{1i} - X_{2i})(X_{1j} - X_{2j}) < 0|X_{1i}, X_{1j}). \quad (C.65)
\]

Genz and Bretz (2009) show that for a two-dimensional normal vector \((N_1, N_2)^T \sim N(0, \Sigma)\) with

\[
\tilde{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
\]

we have

\[
\mathbb{P}(N_1 > a_1, N_2 > a_2) = \Phi(-a_1)\Phi(-a_2) + \frac{1}{2\pi} \int_{0}^{\sin^{-1}(\rho)} \exp \left( -\frac{a_1^2 - 2a_1a_2\sin(\theta) + a_2^2}{2\cos^2(\theta)} \right) d\theta. \quad (C.66)
\]

Noticing that \( X_1 \) and \( X_2 \) follow the Gaussian copula distribution, by \((C.65)\), \((C.66)\) and Theorem A.3, we have

\[
g_{ij}(X_1) = (2F_i(X_{1i}) - 1)(2F_j(X_{1j}) - 1) + O(\tau_{1,ij}). \quad (C.67)
\]

where \( F_i \) is the cumulative distribution function of \( X_{1i} \). Combining \((C.63)\) and \((C.67)\), to obtain \((C.64)\), we only need to prove

\[
2^4\mathbb{E}\left[ (F_i(X_{1i}) - 1/2)(F_j(X_{1j}) - 1/2)(F_k(X_{1k}) - 1/2)(F_\ell(X_{1\ell}) - 1/2) \right] = O((\log d)^{1-\alpha_0}). \quad (C.68)
\]
As \( \mathbf{X} \) belongs to the Gaussian copula family, for simplicity, we assume that \((X_{1i}, X_{1j}, X_{1k}, X_{1\ell})^T \sim N(0, \Sigma_{\text{full}})\) is Gaussian distributed with
\[
\text{Diag}(\Sigma) = \mathbf{I}_3 \quad \text{and} \quad \Sigma_{\text{full}} = \begin{bmatrix} 1 & \varrho^T \\ \varrho & \Sigma \end{bmatrix}.
\]
Accordingly, we have \( F_t(\cdot) = \Phi(\cdot) \) for \( t = 1, \ldots, d \), where \( \Phi(\cdot) \) is the cumulative distribution function of a standard normal distribution. Hence, to obtain (C.68), it suffices to prove that as \( n, d \to \infty \), we have
\[
\mathbb{E}[(\Phi(X_{1i}) - 1/2)(\Phi(X_{1j}) - 1/2)(\Phi(X_{1k}) - 1/2)(\Phi(X_{1\ell}) - 1/2)] = O((\log d)^{-1-\alpha_0}). \quad (C.69)
\]
If \( G_{ijk\ell} \) is \( 4 - G \) and satisfies (\( \ast \)), without loss of generality, we suppose that \( X_{1i} \) is the isolated vertex. Hence, by Theorem A.3, we have \( \|\varrho\|_2 = O((\log d)^{-1-\alpha_0}) \). Noticing that the smallest eigenvalue of any 4 by 4 principal sub-matrix of \( \Sigma_\alpha \) is bounded away from 0, by Lemma C.14, we have (C.69). Hence (C.59) is correct. If \( G_{ijk\ell} \) is \( 3 - G \) and satisfies (\( \ast \)), we have (C.69) similarly. This completes the proof of \textbf{Step (ii)}.

\textbf{Step (iii).} For \((i, j), (k, \ell) \in S \setminus B_0\), to get (C.60) as \( n, d \to \infty \), we only need to prove
\[
|\text{Corr}(h_{ij}(X_1), h_{k\ell}(X_1))| \leq C < 1. \quad (C.70)
\]
By (C.67) and (C.63), to prove (C.70), we only need to prove that as \( n, d \to \infty \), we have
\[
|\text{Corr}((F_i(X_{1i}) - 1/2)(F_j(X_{1j}) - 1/2), (F_k(X_{1k}) - 1/2)(F_\ell(X_{1\ell}) - 1/2))| \leq C < 1.
\]
For simplicity, we suppose \((X_{1i}, X_{1j}, X_{1k}, X_{1\ell})^T \sim N(0, \Sigma_{\text{full}})\) is Gaussian distributed with
\[
\text{Diag}(\Sigma) = \mathbf{I}_3 \quad \text{and} \quad \Sigma_{\text{full}} = \begin{bmatrix} 1 & \varrho^T \\ \varrho & \Sigma \end{bmatrix}.
\]
We then need to prove that as \( n, d \to \infty \), we have
\[
|\text{Corr}((\Phi(X_{1i}) - 1/2)(\Phi(X_{1j}) - 1/2), (\Phi(X_{1k}) - 1/2)(\Phi(X_{1\ell}) - 1/2))| \leq C < 1. \quad (C.71)
\]
By Lemma C.15, we have (C.71). Therefore, we prove (C.60). By following the proof of Lemma 5 in Cai et al. (2013), we prove (C.19) under Assumptions (\textbf{A1}), (\textbf{A2}) and (\textbf{A4}). This completes the proof of Theorem 3.6.

\textbf{C.8 Proof of Theorem 2.6 and Remark 3.8}

\textit{Proof.} The proof of Theorem 2.6 is similar and simpler than the proof of Theorem 2.2. The proof of Remark 3.8 is similar to the proofs of Theorem 3.2, 3.3 and 3.6. Due to the close similarity, the proof is omitted.

\textbf{D Proofs of Lemmas of Appendix C}

In this appendix, we present proofs of lemmas in Appendix C. In the sequel, we use \( C, C_1, C_2, \ldots \), to denote constants that do not depend on \( n, d, q \) and can vary from place to place.
D.1 Proof of Lemma C.1

Proof. By Lemma F.3, we have that \( m^2 \zeta_{a,ij} \) is the limiting variance of \( \sqrt{n_a} \tilde{u}_{a,ij} \) as \( n, q \to \infty \). Combining Lemma F.3 and the definition of \( h_{ij} \) in (2.9), we also have that \( \zeta_{1,ij} \) and \( \zeta_{2,ij} \) are variances of \( h_{ij}(X_\ell) \) and \( h_{ij}(Y_\ell) \).

In Lemma C.1, we aim to prove (C.1). For simplicity, we only provide the proof of \( \mathbf{X} \). By the definition of Jackknife variance estimator \( \hat{\sigma}^2(\tilde{u}_{a,ij}) \) in (2.5), we rewrite (C.1) as

\[
\mathbb{P}\left( \max_{1 \leq i,j \leq q} \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (q_{1\alpha,ij} - \tilde{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right) = o(1), \tag{D.1}
\]

where \( q_{1\alpha,ij} \) is defined as

\[
q_{1\alpha,ij} := \left( \frac{n_1 - 1}{m - 1} \right)^{-1} \sum_{1 \leq \ell_1 < \cdots < \ell_{m-1} \leq n_1} \Phi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}).
\]

To prove Lemma C.1, we also need the centralized version of \( q_{1\alpha,ij} \). By \( \Psi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}) = \Phi_{ij}(X_{\ell_1}, \ldots, X_{\ell_m}) - u_{1,ij} \), we define the centralized version of \( q_{1\alpha,ij} \) as

\[
\tilde{q}_{1\alpha,ij} := \left( \frac{n_1 - 1}{m - 1} \right)^{-1} \sum_{1 \leq \ell_1 < \cdots < \ell_{m-1} \leq n_1} \Psi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}). \tag{D.2}
\]

To bound the left hand side of (D.1), it is easy to obtain

\[
\begin{align*}
\mathbb{P}\left( \max_{1 \leq i,j \leq q} \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (q_{1\alpha,ij} - \tilde{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right) &
\leq q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (q_{1\alpha,ij} - \tilde{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right).
\end{align*}
\]

We then replace \( q_{1\alpha,ij} \) and \( \tilde{u}_{1,ij} \) with their centralized versions \( \tilde{q}_{1\alpha,ij} \) and \( \tilde{u}_{1,ij} \) to obtain

\[
q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (q_{1\alpha,ij} - \tilde{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right)
= q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (\tilde{q}_{1\alpha,ij} - \tilde{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right).
\]

Considering that \( \zeta_{1,ij} \) is the variance of \( h_{ij}(X_\ell) \), by setting \( \tilde{h}_{1,ij} := \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha) / n_1 \), we use \( \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \tilde{h}_{1,ij})^2 / n_1 \) to approximate \( \zeta_{1,ij} \). Therefore, we insert the term \( \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \tilde{h}_{1,ij})^2 / n_1 \) to approximate \( \zeta_{1,ij} \).
\[ (\bar{h}_{1,ij})^2/n_1 \] and use the triangle inequality to obtain
\[
q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n} (\bar{q}_{1,\alpha,ij} - \bar{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right)
\leq q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n} (\bar{q}_{1,\alpha,ij} - \bar{u}_{1,ij})^2 - \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 \right| \geq \frac{1}{2} C \frac{\varepsilon_n}{\log q} \right)
+ q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq \frac{1}{2} C \frac{\varepsilon_n}{\log q} \right).
\]

We then introduce an additional lemma to complete the proof.

**Lemma D.1.** Under Assumption (A2), as \( n, q \to \infty \), we have
\[
q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n} (\bar{q}_{1,\alpha,ij} - \bar{u}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right) = o(1),
\]
and
\[
q^2 \max_{1 \leq i,j \leq q} \mathbb{P}\left( \left| \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 - m^2 \zeta_{1,ij} \right| \geq C \frac{\varepsilon_n}{\log q} \right) = o(1),
\]
where \( \varepsilon_n = o(1) \). Results for \( Y \) are the same.

The detailed proof of Lemma D.1 is in Appendix E.1. By Lemma D.1, we have \( A_1 = o(1) \) and \( A_2 = o(1) \). Hence, we prove Lemma C.1. \( \square \)

**D.2 Proof of Lemma C.2**

**Proof.** In Lemma C.2, we aim to prove (C.7). For notational simplicity, we set
\[
W_{ij} = \frac{(n_1)^{-1}}{\sqrt{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2}} \Delta_{n_1,ij} \quad \text{and} \quad W_{a,ij} = \sqrt{n_a} \left( \frac{n_a}{m} \right)^{-1} \Delta_{n_a,ij}.
\]
By (C.6), we have \( W_{ij} = \bar{N}_{ij} - T_{ij} \). We also have \( W_{ij} = c_1 W_{1,ij} + c_2 W_{2,ij} \), where
\[
c_1 = 1/\sqrt{m^2 \zeta_{1,ij} + m^2 \zeta_{2,ij}/n_1} \quad \text{and} \quad c_2 = -1/\sqrt{m^2 \zeta_{1,ij} n_2/n_1 + m^2 \zeta_{2,ij}}.
\]
To prove (C.6), by setting \( L_2 := \max_{1 \leq i,j \leq q} (\bar{N}_{ij})^2 - \max_{1 \leq i,j \leq q} (T_{ij})^2 \), it suffices to prove that as \( n, q \to \infty \) we have \( L_2 = o_p(1) \). By the definitions of \( \bar{N}_{ij} \) and \( W_{ij} \) in (C.4) and (D.5), we obtain
\[
L_2 \leq \max_{1 \leq i,j \leq q} |\bar{N}_{ij} - T_{ij}| \cdot \max_{1 \leq i,j \leq q} |\bar{N}_{ij} + T_{ij}| \leq \max_{1 \leq i,j \leq q} |W_{ij}| \left( \max_{1 \leq i,j \leq q} 2 |T_{ij}| + \max_{1 \leq i,j \leq q} |W_{ij}| \right).
\]
Considering \( \max_{1 \leq i,j \leq q} 2 |T_{ij}| \leq \max_{1 \leq i,j \leq q} |W_{ij}| = O_p(\log q) \), to obtain \( L_2 = o_p(1) \), we only need to prove
\[
\mathbb{P}\left( \max_{1 \leq i,j \leq q} |W_{ij}| > 1/\log q \right) \leq q^2 \max_{1 \leq i,j \leq q} \mathbb{P}(|W_{ij}| > 1/\log q) \to 0,
\]
(6.6)
as \( n, q \to \infty \). By Proposition 2.3(c) of Arcones and Gine (1993) and \( W_{ij} = \tilde{N}_{ij} - T_{ij} \), we have

\[
P\left( |W_{ij}| > \frac{1}{\log q} \right) \leq C \exp\left( - \frac{C_1(n^{m/2-1/2}/\log q)^{2/m}}{\sigma^2/m + C_2((n^{m/2-1/2}/\log q)^{1/m}n^{1/2})^{2/(m+1)}} \right).
\]

(D.7)

Combining \( \log q = O(n^{1/3-\epsilon}) \) (see Assumption (A2)) and (D.7), we obtain (D.6). We then have \( L_2 = o_p(1) \). This completes the proof.

\[ \square \]

D.3 Proof of Lemma C.3

Proof. In Lemma C.3, we aim to prove that as \( n, q \to \infty \), we have \( P(\max_{i,j} \in S_0 (T_{ij})^2 \geq y_q) \to 0 \), where \( T_{ij} \) is defined in (C.5) and \( y_q = x + 4 \log q - \log(\log q) \). For notational simplicity, we introduce

\[ g_{ij}(X_\alpha) := \mathbb{E}[\Phi_{ij}(X_{a_1}, \ldots, X_{a_m})|X_\alpha] \quad \text{and} \quad g_{ij}(Y_\alpha) := \mathbb{E}[\Phi_{ij}(Y_{a_1}, \ldots, Y_{a_m})|X_\alpha], \]

where \( \Phi_{ij} \) is the kernel function of the U-statistic \( \tilde{u}_{a,ij} \). By Lemma F.3, we have \( \zeta_{1,ij} = \text{Var}(g_{ij}(X_\alpha)) \) and \( \zeta_{2,ij} = \text{Var}(g_{ij}(Y_\alpha)) \). We then introduce an additional lemma.

Lemma D.2. If \( g_{ij} \) is bounded, under Assumption (A2), we have

\[
P\left( \max_{(i,j) \in \Lambda} \left( \frac{1}{n_1} \sum_{a=1}^{n_1} g_{ij}(X_\alpha) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_{ij}(Y_\alpha) - u_{1,ij} + u_{1,ij}^2 \right)^2 > t^2 \right) \leq C_1 |\Lambda| (1 - \Phi(t)),
\]

uniformly for any \( \Lambda \subseteq S \) and \( t \in [0, O(n^{1/6-\epsilon})] \), where \( \epsilon \) is an arbitrary positive number.

The detailed proof of Lemma D.2 is in the supplementary Appendix E.2. Considering \( \log q = O(n^{1/3-\epsilon}) \) (see Assumption (A2)), by choosing \( \epsilon < \epsilon / 2 \) with \( \epsilon \) small enough, we have \( \sqrt{\log q} \in [0, O(n^{1/6-\epsilon})] \). By Lemma F.5, we have \( 1 - \Phi(t) \leq C \exp(-C_1 t^2) \) as \( \Phi \) is the cumulative distribution function of a standard normal distribution. Considering \( |S_0| = o(q^2) \) (see Assumption (A1)) and \( y_q = x + 4 \log q - \log(\log q) \), by setting \( \Lambda \) in Lemma D.2 as \( S_0 \), we obtain

\[
P\left( \max_{(i,j) \in S_0} (T_{ij})^2 \geq y_q \right) \leq o(q^2) \exp(-C_1 y_q) = o(1).
\]

This completes the proof.

\[ \square \]

D.4 Proof of Lemma C.4

Proof. To prove Lemma C.4, we obtain both upper and lower bounds of \( P(\max_{1 \leq k \leq h} (T_k)^2 \geq y_q) \) using the Boferoni inequality and normal approximation. By the Boferoni inequality (Lemma 1 of Cai et al. (2013)), for any integer \( M \) with \( 0 < M < [h/2] \), we have

\[
\sum_{\ell=1}^{2M} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq h} P\left( \bigcap_{j=1}^{\ell} E_{k_j} \right) \leq P\left( \max_{1 \leq k \leq h} (T_k)^2 \geq y_q \right) \leq \sum_{\ell=1}^{2M-1} (-1)^{\ell-1} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq h} P\left( \bigcap_{j=1}^{\ell} E_{k_j} \right),
\]

(D.8)
where we set \( E_{kj} = \{(T_{kj})^2 \geq y_q\} \). By the definition of \( T_k \) in (C.13), we have
\[
T_k = T_{kjk} = \sum_{\beta=1}^{n_1+n_2} \bar{Z}_{\beta i_k j_k} \sqrt{n_2 \zeta_{1,i_k j_k} / (n_1 + n_2 \zeta_{2,i_k j_k})}.
\]
For any \( \ell \)-tuple \( k_1, \ldots, k_\ell \) satisfying \( 1 \leq k_1 < \ldots < k_\ell \leq h \), we have
\[
\bar{Z}_{\beta k} = \bar{Z}_{\beta i_k j_k} / (n_2 \zeta_{1,i_k j_k} / (n_1 + \zeta_{2,i_k j_k}))^{1/2} \quad \text{and} \quad W_\beta = (\bar{Z}_{\beta k_1}, \ldots, \bar{Z}_{\beta k_\ell})^T,
\]
for \( 1 \leq k \leq h \) and \( 1 \leq \beta \leq n_1+n_2 \). Therefore, \( T_k = (n_2)^{-1/2} \sum_{\beta=1}^{n_1+n_2} \bar{Z}_{\beta k} \). Define \( \|v\|_{\min} = \min_{1\leq i\leq \ell} |v_i| \) for any vector \( v \in \mathbb{R}^\ell \). By \( E_{kj} = \{(T_{kj})^2 \geq y_q\} \), we have
\[
\mathbb{P}\left( \bigcap_{j=1}^\ell E_{kj} \right) = \mathbb{P}\left( \|n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta\|_{\min} \geq y_q^{1/2} \right).
\]
We set \( N_\ell \) as a normal vector with the same mean vector and covariance matrix as \( n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta \).
More specifically, \( N_\ell := (N_{k_1}, \ldots, N_{k_\ell})^T \) is a normal vector with \( \mathbb{E}[N_\ell] = 0 \) and \( \text{Var}(N_\ell) = n_1 \text{Var}(W_1)/n_2 + \text{Var}(W_{n_1+1}) \). Therefore, \( N_\ell \) is the normal approximation of \( n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta \).
We then aim to use \( N_\ell \) to rewrite obtained lower and upper bounds in (D.8). As \( |Z_{\beta k}| \) is bounded, we set \( |\bar{Z}_{\beta k}| \leq K \), where \( K \) is a constant. By Theorem 1 of Zaitsev (1987), we have
\[
\mathbb{P}\left( \|n_2^{-1/2} \sum_{\beta=1}^{n_1+n_2} W_\beta\|_{\min} \geq y_q^{1/2} \right)
\leq \mathbb{P}\left( \|N_\ell\|_{\min} \geq y_q^{1/2} - \epsilon_n \log q^{-1/2} \right) + C_1 \ell^{5/2} \exp\left( - \frac{n_2^{1/2} \epsilon_n}{C_2 \ell^3 K (\log q)^{1/2}} \right),
\]
where \( \epsilon_n \to 0 \), which will be specified later. Considering that \( \ell \) is a fixed integer that does not depend on \( n \) and \( q \), by \( \log q = O(n^{1/3-\epsilon}) \) (see Assumption (A2)), we can let \( \epsilon_n \to 0 \) sufficiently slow such that we have
\[
c_1 \ell^{5/2} \exp\left( - \frac{n_2^{1/2} \epsilon_n}{c_2 \ell^3 K (\log q)^{1/2}} \right) = O(q^{-J}),
\]
for any large \( J > 0 \). Hence, we prove (C.17). The proof of (C.18) is similar. This completes the proof of Lemma C.4.
\[\square\]

D.5 Proof of Lemma C.5

Proof. We aim to prove (C.19) under Assumption (A3). \( g_{ij} \) is defined in (2.9). Assumption (A3) requires
\[
\begin{align*}
\ell_{1,ijk\ell} &= \mathbb{E}[g_{ij}(X_\alpha) g_{k\ell}(X_\alpha)] = O((\log q)^{-1-\alpha_0}), \\
\ell_{2,ijk\ell} &= \mathbb{E}[g_{ij}(Y_\alpha) g_{k\ell}(Y_\alpha)] = O((\log q)^{-1-\alpha_0}),
\end{align*}
\]
for \((i,j),(k,\ell) \in S \setminus S_0\), where \( S = \{(i,j) : 1 \leq i,j \leq q\} \) and \( S_0 \) is defined in (C.8). By definitions of \( S \) and \( S_0 \), for \((i,j) \in S \setminus S_0\), we have \( |u_{\alpha,ij}| \leq O((\log q)^{-1-\alpha_0}) \). \( N_\ell \) is an \( \ell \)-dimensional normal
random vector with zero mean and covariance matrix $V_\ell$. Combining (2.9), (C.58), (D.9) and (D.10), we obtain $||V_\ell - I_\ell||_2 = O((\log q)^{-1-\alpha})$ uniformly for all $k_1, k_2, \ldots, k_\ell$, where $k_1, \ldots, k_\ell$ is any $\ell$-tuple of positive integers satisfying $1 \leq k_1 < \cdots < k_\ell \leq h$. By setting $w_q = y_q^{1/2} + \epsilon_n(\log q)^{-1/2}$, we have

$$P(||N_\ell||_{\min} \geq y_q^{1/2} + \epsilon_n(\log q)^{-1/2}) = P(|N_{k_1}| \geq w_q, |N_{k_2}| \geq w_q, \ldots, |N_{k_\ell}| \geq w_q) = \frac{1}{(2\pi)^{\ell/2}|V_\ell|^{1/2}} \int_{||x||_{\min} \geq w_q} \exp \left( -\frac{1}{2} x^T V_\ell^{-1} x \right) dx.$$ 

We then calculate the integral to obtain

$$\frac{1}{(2\pi)^{\ell/2}|V_\ell|^{1/2}} \int_{||x||_{\min} \geq w_q} \exp \left( -\frac{1}{2} x^T V_\ell^{-1} x \right) dx = \frac{1}{(2\pi)^{\ell/2}|V_\ell|^{1/2}} \int_{||x||_{\min} \geq w_q, ||x||_{\max} \leq (\log q)^{1/2 + \alpha_0/4}} \exp \left( -\frac{1}{2} x^T V_\ell^{-1} x \right) dx + O(\exp((-\log q)^{1+\alpha_0/2}/4)).$$

Considering that $||V_\ell - I_\ell||_2 = O((\log q)^{-1-\alpha})$ holds uniformly for all $(k_1, k_2, \ldots, k_\ell)$, we obtain

$$\frac{1}{(2\pi)^{\ell/2}|V_\ell|^{1/2}} \int_{||x||_{\min} \geq w_q, ||x||_{\max} \leq (\log q)^{1/2 + \alpha_0/4}} \exp \left( -\frac{1}{2} x^T V_\ell^{-1} x \right) dx + O(\exp((-\log q)^{1+\alpha_0/2}/4)) = \frac{1+O((\log q)^{-\alpha_0/2})}{(2\pi)^{\ell/2}} \int_{||x||_{\min} \geq w_q, ||x||_{\max} \leq (\log q)^{1/2 + \alpha_0/4}} \exp \left( -\frac{1}{2} x^T V_\ell^{-1} x \right) dx + O(\exp((-\log q)^{1+\alpha_0/2}/4)) = \frac{1+O((\log q)^{-\alpha_0/2})}{(2\pi)^{\ell/2}} \int_{||x||_{\min} \geq w_q} \exp \left( -\frac{1}{2} x^T x \right) dx + O(\exp((-\log q)^{1+\alpha_0/2}/4)).$$

We then calculate the integral to get

$$\frac{1+O((\log q)^{-\alpha_0/2})}{(2\pi)^{\ell/2}} \int_{||x||_{\min} \geq w_q} \exp \left( -\frac{1}{2} x^T x \right) dx + O(\exp((-\log q)^{1+\alpha_0/2}/4)) = (1 + o(1)) \left( \frac{2}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right)^\ell q^{-2\ell}.$$ 

Therefore, as $n, q \to \infty$, we have

$$P(||N_\ell||_{\min} \geq y_q^{1/2} + \epsilon_n(\log q)^{-1/2}) = (1 + o(1)) \left( \frac{2}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right)^\ell q^{-2\ell}.$$ 

(D.11)

Considering $C_\ell^h = h!/(\ell!(h-\ell)!)$ and $2h/(q^2) = 1 + o(1)$, by (D.11), we prove (C.19). Hence, we complete the proof.

D.6 Proof of Lemma C.6

Proof. By Theorem 2.2, under Assumptions (A1), (A2) and (A3), we have

$$P(M_n^4 - 4\log q + \log(\log q) \leq x) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right).$$ 

(D.12)
However, in Lemma C.6, we don’t assume (A1) and (A3), so that we cannot get (C.24) from (D.12).

In Lemma C.6, we aim to prove (D.12) under Assumption (A2). By (C.3) and (C.7), we have

\[
\left| M_n^1 - \max_{(i,j) \in S} \left( \frac{\frac{1}{n_1} \sum_{a=1}^{n_1} g_{ij}(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_{ij}(Y_a) - u_{1,ij} + u_{1,ij}}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2} \right) \right| = o_p(1).
\]

(D.13)

By Lemma D.2, we have

\[
\mathbb{P}\left( \max_{(i,j) \in S} \left( \frac{\frac{1}{n_1} \sum_{a=1}^{n_1} g_{ij}(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_{ij}(Y_a) - u_{1,ij} + u_{1,ij}}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2} \right) > t^2 \right) \leq C_1 |S|(1 - \Phi(t)),
\]

where \(\Phi(t)\) is the cumulative distribution function of a standard normal distribution. By Lemma F.5, we have \(1 - \Phi(t) \leq \exp(-t^2/2)/(\sqrt{2\pi})\) for \(t > 0\). Considering \(|S| = q^2\) and (D.14), by setting \(t^2 = 4 \log q - 0.5 \log(\log q)\), under Assumption (A2), we get

\[
\mathbb{P}\left( \max_{(i,j) \in S} \left( \frac{\frac{1}{n_1} \sum_{a=1}^{n_1} g_{ij}(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_{ij}(Y_a) - u_{1,ij} + u_{1,ij}}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2} \right) > 4 \log q - \frac{1}{2} \log(\log q) \right) = o(1).
\]

Combining (D.13) and (D.15), we prove (C.24). This completes the proof.

\[\square\]

**D.7 Proof of Lemma C.7**

Proof. By setting \(S = \{(i, j) : 1 \leq i, j \leq q\}\), Lemma C.1 implies that both of the following two events

\[
\mathcal{E}_1 := \left\{ \max_{(i,j) \in S} \left| n_1 \hat{\sigma}^2(\hat{u}_{1,ij}) - m^2 \zeta_{1,ij} \right| < C \frac{\varepsilon_n}{\log q} \right\}, \quad \mathcal{E}_2 := \left\{ \max_{(i,j) \in S} \left| n_2 \hat{\sigma}^2(\hat{u}_{2,ij}) - m^2 \zeta_{2,ij} \right| < C \frac{\varepsilon_n}{\log q} \right\},
\]

happen with probability going to one as \(n, q \to \infty\). Under \(\mathcal{E}_1\) and \(\mathcal{E}_2\), by \(\zeta_{a,ij} \geq r_a > 0\) (see Assumption (A2)), we have

\[
\left| n_1 \hat{\sigma}^2(u_{1,ij})/(m^2 \zeta_{1,ij}) - 1 \right| < C \varepsilon_n/\log q \quad \text{and} \quad \left| n_2 \hat{\sigma}^2(u_{2,ij})/(m^2 \zeta_{2,ij}) - 1 \right| < C \varepsilon_n/\log q.
\]

Therefore, under \(\mathcal{E}_1\) and \(\mathcal{E}_2\), we have

\[
\left| \max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{\hat{\sigma}^2(\hat{u}_{1,ij}) + \hat{\sigma}^2(\hat{u}_{2,ij})} - \max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2} \right| \\
\leq \max_{(i,j) \in S} \left| \frac{(u_{1,ij} - u_{2,ij})^2}{\hat{\sigma}^2(\hat{u}_{1,ij}) + \hat{\sigma}^2(\hat{u}_{2,ij})} - \frac{(u_{1,ij} - u_{2,ij})^2}{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2} \right| \leq C \frac{\varepsilon_n}{\log q}.
\]

Hence, as \(n, q \to \infty\), we have

\[
\left| \max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{\hat{\sigma}^2(\hat{u}_{1,ij}) + \hat{\sigma}^2(\hat{u}_{2,ij})} - \max_{(i,j) \in S} \frac{(u_{1,ij} - u_{2,ij})^2}{m^2 \zeta_{1,ij}/n_1 + m^2 \zeta_{2,ij}/n_2} \right| = o_p(1).
\]

(D.16)

Considering \((U_1^n, U_2^n) \in \mathcal{A}(4)\), by (D.16), we obtain (C.25). This completes the proof.

\[\square\]
D.8 Proof of Lemma C.8

Proof. Considering the definition of $N_{ij}^{T,ps}$ in (C.8), by (3.8), we have $M_{n,ps}^{T} = \max_{(i,j)\in S}(N_{ij}^{T,ps})^2$. By setting $S = \{(i,j) : 1 \leq i < j \leq d\}$, we aim to prove that as $n, d \to \infty$, we have

$$\left| \max_{(i,j)\in S}(N_{ij}^{T,ps})^2 - \max_{(i,j)\in S}(\hat{T}_{ij})^2 \right| = o_p(1),$$

(D.17)

where we define $\hat{T}_{ij}$ in (C.42). By setting $L_{2,ps}^{T} := \max_{(i,j)\in S}(N_{ij}^{T,ps})^2 - \max_{(i,j)\in S}(\hat{T}_{ij})^2$, we aim to prove $L_{2,ps}^{T} = o_p(1)$. To prove $L_{2,ps}^{T} = o_p(1)$, we construct an upper bound of $L_{2,ps}^{T}$ as

$$L_{2,ps}^{T} \leq \max_{(i,j)\in S}|N_{ij}^{T,ps} - \hat{T}_{ij}| \max_{(i,j)\in S}|N_{ij}^{T,ps} + \hat{T}_{ij}| \leq \max_{(i,j)\in S}|\hat{W}_{ij}|(\max_{(i,j)\in S}2|\hat{T}_{ij}| + \max_{(i,j)\in S}|\hat{W}_{ij}|).$$

Considering $\max_{(i,j)\in S}2|\hat{T}_{ij}| + \max_{(i,j)\in S}|\hat{W}_{ij}| = O_p(\log d)$, to show $L_{2,ps}^{T} = o_p(1)$, we only need to prove

$$\mathbb{P}\left( \max_{(i,j)\in S}|\hat{W}_{ij}| > 1/\log d \right) \leq d^2 \max_{(i,j)\in S}\mathbb{P}(|\hat{W}_{ij}| > 1/\log d) \to 0,$$

(D.18)

as $n, d \to \infty$. By Proposition 2.3(c) of Arcones and Gine (1993), we have

$$\mathbb{P}(|\hat{W}_{ij}| > \frac{1}{\log d}) \leq C \exp\left(- \frac{C_2\sqrt{n}/\log d}{\sigma + C_1(\log d\sqrt{n})^{-1/3}}\right).$$

(D.19)

Considering $\log d = O(n^{1/3-\epsilon})$ (see Assumption (A2)), by (D.18) and (D.19), we obtain

$$\mathbb{P}\left( \max_{(i,j)\in S}|\hat{W}_{ij}| > 1/\log d \right) \to 0,$$

as $n, d \to \infty$. Hence, we complete the proof.

\[\square\]

D.9 Proof of Lemma C.9

Proof. $\sigma_{a,ps}^2$ and $g_{ij}^p$ are defined in (3.7) and (C.37). We aim to prove (C.44). For this, we introduce the following lemma.

Lemma D.3. By setting $S = \{(i,j) : 1 \leq i < j \leq d\}$, under Assumption (A2), we have

$$\mathbb{P}\left( \max_{(i,j)\in \Lambda} \left( \frac{1}{n_1} \sum_{a=1}^{n_1} g_{ij}^a(X) - \frac{1}{n_2} \sum_{a=2}^{n_2} g_{ij}^a(Y) - \tau_{1,ij} + \tau_{1,ij}^2 \right)^2 > \epsilon^2 \right) \leq C|\Lambda|(1 - \Phi(C_1t)),$$

uniformly for any $\Lambda \subseteq S$ and $t \in [0, O(n^{1/6-\epsilon})]$, where $\epsilon$ is an arbitrary positive number. $\Phi(t)$ is the cumulative distribution function of a standard normal distribution.

The detailed proof of Lemma D.3 is in the supplementary Appendix E.3. By Lemma D.3, we have

$$\mathbb{P}\left( \max_{(i,j)\in S_0}(\hat{T}_{ij})^2 > y_d \right) \leq C|S_0|(1 - \Phi(C_1\sqrt{y_d})),$$

(D.20)

where $y_d = x + 4\log d - \log(\log d)$. By Lemma F.5, we have

$$1 - \Phi(t) \leq \exp(-t^2/2)/(t\sqrt{2\pi}) \quad \text{for} \quad t > 0.$$

(D.21)

Considering $|S_0| = o(d^2)$ (see Assumption A1), by (D.20) and (D.21), we have $\mathbb{P}\left( \max_{(i,j)\in S_0}(\hat{T}_{ij})^2 > y_d \right) = o(1)$. This completes the proof.

\[\square\]
D.10 Proof of Lemma C.10

Proof. In Lemma C.10, we aim to prove $|4ζ_{a,ij} − σ^2_{ps}| ≤ C|τ_{a,ij}|$. For simplicity, we ignore the subscript $a$ ($a = 1$ or 2) for $X$ and $Y$. By (C.28), we have $4ζ_{ij} = 16(Π_{cc,ij}^u − (Π_{c,ij}^u)^2)$, where $Π_{cc,ij}$ and $Π_{c,ij}$ are defined in (3.4) and (3.2). By the definition of the meta-elliptical distribution in Definition A.2, we have

$$\frac{(X_{a,i} X_{a,j})}{d} = (R_{a} \cos(Θ_{a}), R_{a} \sin(Θ_{a} + u)), \quad (D.22)$$

where $R_a$ is a positive radial random variable and $Θ_a$ is an independent, uniformly distributed angle on $[-π, π]$ with $α = 1, 2, \ldots, n_1$ and $u = τ_{ij, π} / 2$. By the definition of $σ^2_{ps}$ in (3.7), we have $σ^2_{ps} = 16(Π_{cc,ij}^u − (Π_{c,ij}^u)^2)$, where $Π_{cc,ij}^u$ and $Π_{c,ij}^u$ is the $Π_{cc,ij}$ and $Π_{c,ij}$ under the condition $u = τ_{ij, π} / 2 = 0$.

By the definition of $Π_{cc,ij}$ in (3.4), $Π_{cc,ij}$ is the probability of four parts:

$$E_1 := \{X_{2i} > X_{1i}, X_{2j} > X_{1j}, X_{3i} > X_{1i}, X_{3j} > X_{1j}\},$$

$$E_2 := \{X_{2i} < X_{1i}, X_{2j} < X_{1j}, X_{3i} < X_{1i}, X_{3j} < X_{1j}\},$$

$$E_3 := \{X_{2i} > X_{1i}, X_{2j} > X_{1j}, X_{3i} < X_{1i}, X_{3j} < X_{1j}\},$$

$$E_4 := \{X_{2i} < X_{1i}, X_{2j} < X_{1j}, X_{3i} > X_{1i}, X_{3j} > X_{1j}\},$$

i.e., $Π_{cc,ij} = P(E_1) + P(E_2) + P(E_3) + P(E_4)$. By (D.22), we rewrite the event $E_1$ as

$$E_1 = \{R_2 \cos(Θ_2) > X_{1i}, R_2 \sin(Θ_2 + u) > X_{1j}, R_3 \cos(Θ_3) > X_{1i}, R_3 \sin(Θ_3 + u) > X_{1j}\}.$$

We then define $g_u(x_{1i}, x_{1j}, r_2, r_3)$ as

$$g_u(x_{1i}, x_{1j}, r_2, r_3) := P[E_1 | X_{1i} = x_{1i}, X_{1j} = x_{1j}, R_2 = r_2, R_3 = r_3]. \quad (D.23)$$

By $P(E_1) = E\left[P[E_1 | X_{1i}, X_{1j}, R_2, R_3]\right]$, we have $P(E_1) = E[g_u(X_{1i}, X_{1j}, R_2, R_3)]$. By (D.22) and (D.23), we have

$$|g_u(x_{1i}, x_{1j}, r_2, r_3) − g_{u=0}(x_{1i}, x_{1j}, r_2, r_3)| ≤ Cτ_{ij}, \quad (D.24)$$

where $C$ is a positive constant. By (D.24), we obtain

$$\left| E[g_u(X_{1i}, X_{1j}, R_2, R_3)] − E[g_{u=0}(X_{1i}, X_{1j}, R_2, R_3)] \right| ≤ C|τ_{ij}|.$$

By similar proofs on $E_2$, $E_3$ and $E_4$, we have $|Π_{cc,ij} − Π_{cc,ij}^u| ≤ C|τ_{ij}|$. Considering the boundedness of $Π_{c,ij}$, we also have $|Π_{c,ij}^u|^2 − (Π_{c,ij}^u)^2| ≤ C|τ_{ij}|$. We then complete the proof by $σ^2_{ps} = 16(Π_{cc,ij}^u − (Π_{c,ij}^u)^2)$ and $4ζ_{ij} = 16(Π_{cc,ij} − (Π_{c,ij})^2)$. \hfill □

D.11 Proof of Lemma C.11

Proof. To prove Lemma C.11, we set

$$M_{n,1}^{plug} := \max_{(i,j) \in S} \frac{(\hat{τ}_{1,ij} − \hat{τ}_{2,ij} − τ_{1,ij} + τ_{2,ij})^2}{\hat{σ}_{plug}^2(\hat{τ}_{1,ij}) + \hat{σ}_{plug}^2(\hat{τ}_{2,ij})}, \quad \text{and} \quad \hat{M}_{n,1}^{plug} := \max_{(i,j) \in S} \frac{(\hat{τ}_{1,ij} − \hat{τ}_{2,ij} − τ_{1,ij} + τ_{2,ij})^2}{4ζ_{1,ij}/n_1 + 4ζ_{2,ij}/n_2},$$

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where \( S = \{(i, j) : 1 \leq i < j \leq d\} \). By Theorem 3.2, under Assumptions (A1), (A2) and (A3), we have
\[
P\left( M_n^{1,\text{plug}} - 4 \log d + \log(\log d) \leq x \right) \to \exp\left( -\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{2}\right) \right). \tag{D.25}
\]
However, in Lemma C.11, we don’t assume (A1) and (A3). Therefore, we cannot obtain (C.48) from (D.25).

In Lemma C.11, we aim to prove (C.48) under Assumption (A2). By (C.26) and (C.7), as \( n, d \to \infty \), we obtain
\[
\left| M_n^{1,\text{plug}} - \max_{(i, j) \in S} \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^\tau (X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^\tau (Y_\alpha) - \tau_{1, ij} + \tau_{1, ij} \right| = o_p(1), \tag{D.26}
\]
where \( g_{ij}^\tau \) is defined in (C.37). By Lemma D.2, we have
\[
P\left( \max_{(i, j) \in S} \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^\tau (X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^\tau (Y_\alpha) - \tau_{1, ij} + \tau_{1, ij} > t^2 \right) \leq C|S|(1 - \Phi(t)), \tag{D.27}
\]
where \( \Phi(t) \) is the cumulative distribution function of a standard normal distribution. By Lemma F.5, we have \( 1 - \Phi(t) \leq \exp(-t^2/2)/(t\sqrt{2\pi}) \) for \( t > 0 \). Considering \( |S| = O(d^2) \) and (D.27), by setting \( t^2 = 4 \log d - 0.5 \log(\log d) \), under Assumption (A2), we obtain
\[
P\left( \max_{(i, j) \in S} \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^\tau (X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^\tau (Y_\alpha) - \tau_{1, ij} + \tau_{1, ij} > 4 \log d - \frac{1}{2} \log(\log d) \right) = o(1). \tag{D.28}
\]
Combining (D.26) and (D.28), we prove (C.48). Hence, we complete the proof of Lemma C.11. \( \square \)

### D.12 Proof of Lemma C.12

**Proof.** By setting \( S = \{(i, j) : 1 \leq i < j \leq d\} \), (C.26) implies that both of the following two events
\[
\mathcal{E}_3 := \left\{ \max_{(i, j) \in S} \left| n_1 \sigma_{\text{plug}}^2 (\hat{u}_{1, ij}) - m^2 \hat{\zeta}_{1, ij} \right| < C \frac{\varepsilon_n}{\log q} \right\}, \quad \mathcal{E}_4 := \left\{ \max_{(i, j) \in S} \left| n_2 \sigma_{\text{plug}}^2 (\hat{u}_{2, ij}) - m^2 \hat{\zeta}_{2, ij} \right| < C \frac{\varepsilon_n}{\log q} \right\},
\]
happen with probability going to one as \( n, d \to \infty \). Under \( \mathcal{E}_3 \) and \( \mathcal{E}_4 \), by \( \zeta_{a, ij} \geq r_a > 0 \) (see Assumption (A2)), we have
\[
\max_{1 \leq i < j \leq d} \left| \frac{n_a \sigma_{\text{plug}}^2 (\hat{\tau}_{a, ij})}{m^2 \zeta_{a, ij}} - 1 \right| \leq C \frac{\varepsilon_n}{\log q} \quad \text{and} \quad \max_{1 \leq i < j \leq d} \left| \frac{m^2 \zeta_{a, ij}}{n_a \sigma_{\text{plug}}^2 (\hat{\tau}_{a, ij})} - 1 \right| \leq C \frac{\varepsilon_n}{\log q}.
\]
Therefore, under \( \mathcal{E}_3 \) and \( \mathcal{E}_4 \), as \( n, d \to \infty \), we have
\[
\left| \max_{(i, j) \in S} \frac{(\tau_{1, ij} - \tau_{2, ij})^2}{\sigma_{\text{plug}}^2 (\hat{\tau}_{1, ij}) + \sigma_{\text{plug}}^2 (\hat{\tau}_{2, ij})} - \max_{(i, j) \in S} \frac{(\tau_{1, ij} - \tau_{2, ij})^2}{m^2 \hat{\zeta}_{1, ij}/n_1 + m^2 \hat{\zeta}_{2, ij}/n_2} \right| \leq C \frac{\varepsilon_n}{\log d}.
\]
Hence, as \( n, d \to \infty \), we have
\[
\left| \max_{(i, j) \in S} \frac{(\tau_{1, ij} - \tau_{2, ij})^2}{\sigma_{\text{plug}}^2 (\hat{\tau}_{1, ij}) + \sigma_{\text{plug}}^2 (\hat{\tau}_{2, ij})} - \max_{(i, j) \in S} \frac{(\tau_{1, ij} - \tau_{2, ij})^2}{m^2 \hat{\zeta}_{1, ij}/n_1 + m^2 \hat{\zeta}_{2, ij}/n_2} \right| = o_p(1). \tag{D.29}
\]
Considering \( (U_1^*, U_2^*) \in U(4) \), by (D.29), we obtain (C.49) to complete the proof. \( \square \)
D.13 Proof of Lemma C.13

Proof. We aim to prove (C.51) under Assumptions (A1) and (A2). For Kendall's tau, we have

\[
\hat{T}_{ij} = \frac{1}{n} \sum_{a=1}^{n_1} g_i^a(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_i^a(Y_a) - \tau_{1,ij} + \tau_{1,ij},
\]

\[
T_{ij} = \frac{1}{n} \sum_{a=1}^{n_1} g_i^a(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_i^a(Y_a) - \tau_{1,ij} + \tau_{1,ij}.
\]

(D.30)

(D.31)

Considering the definition of \(M_{1,ps}^1\) in (C.50), by (C.43), we have

\[
\left| M_{1,ps}^1 - \max_{(i,j) \in S} \frac{1}{n} \sum_{a=1}^{n_1} g_i^a(X_a) - \frac{1}{n_2} \sum_{a=1}^{n_2} g_i^a(Y_a) - \tau_{1,ij} + \tau_{1,ij} \right| = o_p(1),
\]

where \(S = \{(i, j) : 1 \leq i < j \leq d\}\). Therefore, to obtain (C.51), it suffices to prove as \(n, d \to \infty\), we have \(\mathbb{P}( \max_{(i,j) \in S} (\hat{T}_{ij})^2 > \bar{y}_d) = o(1)\), where \(\bar{y}_d := 4 \log d - \log(\log d)/2\).

We then aim to prove that as \(n, d \to \infty\), we have

\[
\mathbb{P}( \max_{(i,j) \in S} (\hat{T}_{ij})^2 > \bar{y}_d) - \mathbb{P}( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 > \bar{y}_d) = o(1),
\]

(D.32)

where \(S_0\) is defined in (C.8). To prove (D.32), by

\[
\mathbb{P}( \max_{(i,j) \in S} (\hat{T}_{ij})^2 > \bar{y}_d) - \mathbb{P}( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 > \bar{y}_d) \leq \mathbb{P}( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > \bar{y}_d),
\]

we need to prove \(\mathbb{P}( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > \bar{y}_d) = o(1)\) as \(n, d \to \infty\). By Lemma D.3, we have

\[
\mathbb{P}( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > \bar{y}_d) \leq C|S_0|(1 - \Phi(\sqrt{\bar{y}_d})),
\]

(D.33)

where \(\Phi(t)\) is the cumulative distribution function of a standard normal distribution. Lemma F.5 claims \(1 - \Phi(t) \leq \exp(-t^2/2)/(t \sqrt{2\pi})\) for \(t > 0\). Considering \(|S_0| = o(d^2)\) (see Assumption (A1)), by (D.33), we have \(\mathbb{P}( \max_{(i,j) \in S_0} (\hat{T}_{ij})^2 > \bar{y}_d) = o(1)\). Therefore, we prove that as \(n, d \to \infty\), we have (D.32). To obtain (C.51), by (D.32), it suffices to prove that as \(n, d \to \infty\), we have \(\mathbb{P}( \max_{(i,j) \in S \setminus S_0} (\hat{T}_{ij})^2 > \bar{y}_d) = o(1)\).

Considering the definition of \(S_0\) in (C.8), by Lemma C.10, for any \((i, j) \in S \setminus S_0\), we have \(|4\zeta_{a,ij} - \sigma_{a,ps}^2| \leq C|\tau_{a,ij}| \leq C(\log d)^{-1-\alpha_0}\). By \(\zeta_{a,ij} \geq r_a > 0\) (see Assumption (A2)), for \((i, j) \in S \setminus S_0\), we then have

\[
|\sigma_{a,ps}^2/4\zeta_{a,ij} - 1| \leq C(\log d)^{-1-\alpha_0} \quad \text{and} \quad |4\zeta_{a,ij}/\sigma_{a,ps}^2 - 1| \leq C(\log d)^{-1-\alpha_0}.
\]

(D.34)

Therefore, by (D.34), for \((i, j) \in S \setminus S_0\), we have

\[
\left| \frac{(\hat{T}_{ij})^2 - (T_{ij})^2}{(T_{ij})^2} \right| \leq \left| \frac{\sigma_{1,ps}^2 - 4\zeta_{1,ij}}{\sigma_{1,ps}^2} \right| + \left| \frac{\sigma_{2,ps}^2 - 4\zeta_{2,ij}}{\sigma_{2,ps}^2} \right| \leq C(\log d)^{-1-\alpha_0}.
\]
Therefore, for \((i,j) \in S \setminus S_0\), we have 
\[
|\hat{T}_{ij}^2 - (T_{ij})^2| \leq C(T_{ij})^2 (\log d)^{-1-\alpha_0}.
\]
We then have
\[
\left| \max_{(i,j) \in S \setminus S_0} \hat{T}_{ij}^2 - \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 \right| \leq o_p(1).
\]
\[
\mathbb{P}\left( \max_{(i,j) \in S \setminus S_0} (T_{ij})^2 > \hat{y}_d \right) = o(1),
\]
as \(n, d \to \infty\). By Lemma D.2, we have
\[
\mathbb{P}\left( \max_{(i,j) \in S \setminus S_0} \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^T(X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^T(Y_\alpha) - \tau_{1,ij} + \tau_{1,ij}^2 \right. \left. \xi_{1,ij}/n_1 + \xi_{2,ij}/n_2 > t^2 \right) \leq C_1 |S | (1 - \Phi(t)).
\]
By Lemma F.5, we have \(1 - \Phi(t) \leq \exp(-t^2/2)/(t\sqrt{2\pi})\) for \(t > 0\). Considering \(|S | = O(d^2)\) (see Assumption (A1)) and (D.37), by setting \(t^2 = 4 \log d - 0.5 \log(\log d)\), as \(n, d \to \infty\), we obtain
\[
\mathbb{P}\left( \max_{(i,j) \in S \setminus S_0} \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^T(X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^T(Y_\alpha) - \tau_{1,ij} + \tau_{1,ij}^2 \right. \left. \xi_{1,ij}/n_1 + \xi_{2,ij}/n_2 > \hat{y}_d \right) = o(1).
\]
Therefore, by (D.31), we prove (D.36). Hence, we complete the proof. \(\square\)

\subsection{D.14 Proof of Lemma C.14}

\textit{Proof.} We write \(X_1 = Z_0 + Z_1\) with \(Z_1 = g^T \Sigma^{-1} (X_2, X_3, X_4)^T\) and \(Z_0 = X_1 - Z_1\). We then obtain that \(Z_0\) is independent of \((X_2, X_3, X_4)\), \(Z_1 \sim N(0, g^T \Sigma^{-1} g)\) and \(Z_0 \sim N(0, 1 - g^T \Sigma^{-1} g)\). We set \(\sigma^2 := g^T \Sigma^{-1} g\). Accordingly, we write
\[
\mathbb{E}\left[ (\Phi(X_1) - 1/2)(\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2) \right] = \mathbb{E}\left[ \Phi(X_1) - 1/2 | Z_1 \right] \cdot \mathbb{E}\left[ (\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2) | Z_1 \right],
\]
We first focus on bounding \(\mathbb{E}\left[ \Phi(X_1) - 1/2 | Z_1 \right]\). We use \(\ell_3\) to denote it. Given \(Z_1\), we have \(Z_0 + Z_1 | Z_1 \sim N(Z_1, 1 - \sigma^2)\). Hence, we have \(\ell_3 = \mathbb{E}\left[ \Phi(\xi) - \mathbb{E}\Phi(\eta) | Z_1 \right]\) with \(\xi \sim N(Z_1, 1 - \sigma^2)\) and \(\eta \sim N(0, 1)\). Because the equation holds for any joint distribution of \(N(Z_1, 1 - \sigma^2)\) and \(N(0, 1)\), we have \(\ell_3 = \mathbb{E}\left[ \Phi(Z_1 + \sqrt{1 - \sigma^2} Y_0) - \Phi(Y_0) | Z_1 \right]\), where \(Y_0 \sim N(0, 1)\) is independent of \(Z_1\). Then we have
\[
\ell_3 \leq \mathbb{E}\left[ \Phi(Z_1 + \sqrt{1 - \sigma^2} Y_0) - \Phi(Y_0) | Z_1 \right] \leq \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[ |Z_1 + \sqrt{1 - \sigma^2} Y_0 - Y_0| | Z_1 \right].
\]
Given \(Z_1\), because \(Z_1 + \sqrt{1 - \sigma^2} Y_0 - Y_0 \sim N(Z_1, (1 - \sqrt{1 - \sigma^2})^2)\), denoting \(\sigma_2^2 = (1 - \sqrt{1 - \sigma^2})^2\), we then have
\[
\mathbb{E}\left[ \Phi(X_1) - 1/2 | Z_1 \right] \leq \frac{1}{\sqrt{2\pi}} \left[ \sigma_2 \sqrt{2/\pi} \exp(-Z_1^2/2\sigma_2^2) + Z_1 [1 - 2\Phi(-Z_1/\sigma_2)] \right].
\]
Moreover, it is easy to obtain $|\Phi(\cdot) - 1/2| \leq 1/2$. Therefore, we have

$$
E\left[ (\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2) | Z_1 \right] \leq 1/8. \quad (D.40)
$$

Combing (D.38), (D.39) and (D.40), we then have

$$
\left| E\left[ \Phi(X_1) - 1/2)(\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2) \right] \right|
\leq \frac{\sigma_2}{8\pi} + \frac{1}{8\sqrt{2\pi}} E\left[ Z_1(1 - 2\Phi(-Z_1/\sigma_2)) \right]. \quad (D.41)
$$

Considering $E[Z_1]^2 = \sigma^2$ and $E[\Phi^2(Z_1/\sigma_2)] \leq 1$, we have

$$
E\left[ Z_1(1 - 2\Phi(-Z_1/\sigma_2)) \right] = 2E\left[ Z_1\Phi(Z_1/\sigma_2) \right] \leq 2E[Z_1^2]^{1/2}E[\Phi^2(Z_1/\sigma_2)]^{1/2} \leq 2\sigma. \quad (D.42)
$$

Noticing $\sigma_2 < \sigma$, combining (D.41) and (D.42), we obtain

$$
\left| E(\Phi(X_1) - 1/2)(\Phi(X_2) - 1/2)(\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2) \right| \leq \left( \frac{1}{8\pi} + \frac{1}{4\sqrt{2\pi}} \right) \sigma.
$$

This completes the proof. \hfill \Box

### D.15 Proof of Lemma C.15

**Proof.** First, we prove that only when $r = 1$ and the set $\{\rho_1, \rho_2, a_1, \ldots, a_4\}$ attains the boundary, we have $C_r = 1$. When $C_r = 1$, we have

$$(\Phi(X_1) - 1/2)(\Phi(X_2) - 1/2) = a \cdot (\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2),$$

for some constant $a$. This implies that

$$X_1 = \Phi^{-1}\left( \frac{a \cdot (\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2)}{\Phi(X_2) - 1/2} + 1/2 \right).$$

We have $X_1 \sim N(0, 1)$ if and only if we have

$$\frac{a \cdot (\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2)}{\Phi(X_2) - 1/2} \sim \text{Unif}(-1/2, 1/2).$$

When $X_2 \neq \pm X_3$ and $X_2 \neq \pm X_4$, there is always a possibility such that $X_2$ is very close to zero and both $X_3$ and $X_4$ are away from zero. Under this condition, $(a \cdot (\Phi(X_3) - 1/2)(\Phi(X_4) - 1/2))/(\Phi(X_2) - 1/2)$ is very large and outside $[ -1/2, 1/2 ]$. Therefore, $X_2$ must equal $\pm X_3$ or $\pm X_4$. Equivalently, $\{\rho_1, \rho_2, a_1, \ldots, a_4\}$ attains the boundary. This completes the proof of the first part.

Secondly, it is obvious that there is a one-to-one map between $r$ and $C_r$. Accordingly, as long as $r < 1$, $C_r < 1$ only depends on $r$. \hfill \Box

### E Proofs of Lemmas in Appendix D

In Appendix E, we present proofs of three lemmas introduced in Appendix D.
E.1 Proof of Lemma D.1

Proof. To prove (D.3) and (D.4), we need to analyze the following two terms:

\[ A_1 := \mathbb{P} \left( \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \left( \sum_{\alpha=1}^{n_1} (\tilde{q}_{1\alpha,i,j} - \tilde{u}_{1,i,j})^2 - \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 \right) \geq t \right) , \quad (E.1) \]

\[ A_2 := \mathbb{P} \left( \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 - m^2 \zeta_{1,i,j} \geq t \right) . \quad (E.2) \]

We bound \( A_1 \) and \( A_2 \) separately. \( A_2 \) represents \( \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 \)’s approximation error for \( \zeta_{1,i,j} \). Considering the definition of \( \zeta_{a,i,j} \) in Lemma F.3, by setting \( \tilde{h}_{1,i,j} = \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha) \), we have

\[ \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 = \sum_{\alpha=1}^{n_1} h_{ij}^2(X_\alpha) - n_1(\bar{h}_{1,ij})^2 \quad \text{and} \quad \zeta_{1,i,j} = \mathbb{E}[h_{ij}^2(X_\alpha)]. \quad (E.3) \]

Thus, by using the triangle inequality, we obtain

\[ \mathbb{P} \left( \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha) - \bar{h}_{1,ij})^2 - m^2 \zeta_{1,i,j} \geq t/2 \right) \]

\[ \leq \mathbb{P} \left( \frac{1}{n_1} \sum_{\alpha=1}^{n_1} h_{ij}^2(X_\alpha) - \mathbb{E}[h_{ij}^2(X_\alpha)] \geq C_1 t \right) + \mathbb{P} \left( (\bar{h}_{1,i,j})^2 \geq C_2 t \right) . \]

By Lemma F.1, we use the Hoeffding’s inequality to obtain

\[ \mathbb{P} \left( \left| \frac{1}{n_1} \sum_{\alpha=1}^{n_1} h_{ij}^2(X_\alpha) - \mathbb{E}[h_{ij}^2(X_\alpha)] \right| \geq C_1 t \right) \leq C_3 \exp(-C_4 n_1 t^2) . \quad (E.4) \]

Considering \( h_{ij}(X_\alpha) \) is bounded, by \( \mathbb{E}[h_{ij}(X_\alpha)] = 0 \), we use Hoeffding inequality (Lemma F.1) to obtain

\[ \mathbb{P} \left( (\bar{h}_{1,i,j})^2 \geq C_2 t \right) = \mathbb{P} \left( |\bar{h}_{1,i,j}| \geq \sqrt{C_2 t} \right) \leq C_5 \exp(-C_6 n_1 t) . \quad (E.5) \]

Hence, by (E.4) and (E.5), we have

\[ A_2 \leq C_3 \exp(-C_4 n_1 t^2) + C_5 \exp(-C_6 n_1 t) . \quad (E.6) \]

By (E.6) and \( \log q = O(n^{1/3-\epsilon}) \) (Assumption (A2)), we have (D.4) by setting \( \epsilon_n = 1/(\log q)^{\kappa_0} \), where \( \kappa_0 > 0 \) is sufficiently small.

For \( A_1 \), we use (E.3) to rewrite \( A_1 \) as

\[ A_1 = \mathbb{P} \left( \frac{m^2(n_1 - 1)}{(n_1 - m)^2} \left( \left( \sum_{\alpha=1}^{n_1} (\tilde{q}_{1\alpha,i,j}^2 - n_1(\tilde{u}_{1,i,j})^2) - \left( \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha)^2 - n_1(\bar{h}_{1,ij})^2) \right) \right) \geq t/2 \right) . \quad (E.7) \]
Similarly to $A_2$, we rearrange terms in $A_1$ and use the triangle inequality to obtain
\[
\mathbb{P}\left(\frac{m^2(n_1-1)}{(n_1-m)^2}\left(\sum_{\alpha=1}^{n_1} (\bar{q}_{\alpha,ij})^2 - n_1(\bar{u}_{ij})^2\right) - \left(\sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha))^2 - n_1(\bar{h}_{ij})^2\right) \right) \geq t/2
\]
\[
\leq \mathbb{P}\left(\frac{m^2(n_1-1)}{(n_1-m)^2}\left(\sum_{\alpha=1}^{n_1} (\bar{q}_{\alpha,ij})^2 - \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha))^2\right) \geq t/4\right)_{I_1}
\]
\[
+ \mathbb{P}\left(\frac{m^2n_1(n_1-1)}{(n_1-m)^2}\left(\bar{u}_{ij}^2 - \bar{h}_{ij}^2\right) \geq t/4\right)_{I_2}.
\]

Therefore, to bound $A_1$, we need to bound $I_1$ and $I_2$ separately. By
\[
(\bar{u}_{ij})^2 - (\bar{h}_{ij})^2 = (\bar{u}_{ij} + \bar{h}_{ij})(\bar{u}_{ij} - \bar{h}_{ij}) \quad \text{and} \quad |\bar{u}_{ij} + \bar{h}_{ij}| \leq C,
\]
we use the triangle inequality to bound $I_2$ and get
\[
I_2 \leq \mathbb{P}\left(|\bar{u}_{ij} - \bar{h}_{ij}| \geq C t\right) \leq \mathbb{P}\left(|\bar{u}_{ij}| \geq C t/2\right) + \mathbb{P}\left(|\bar{h}_{ij}| \geq C t/2\right).
\]

We know that $\bar{u}_{ij}$ and $\bar{h}_{ij}$’s kernel functions are bounded. Noticing $\mathbb{E}[\bar{u}_{ij}] = \mathbb{E}[\bar{h}_{ij}] = 0$, we use the exponential inequalities (Lemmas F.1 and F.2) to obtain $I_2 \leq C_1 \exp(-C_2 n_1 t^2)$. Considering $\log q = o(n^{1/3-\epsilon})$ in Assumption (A2), by setting $\varepsilon_n = 1/\log q$ with $\kappa_0 > 0$ sufficiently small, we have
\[
q^2 \max_{1 \leq i, j \leq q} \mathbb{P}\left(\frac{m^2n_1(n_1-1)}{(n_1-m)^2}\left(\bar{u}_{ij}^2 - (\bar{h}_{ij})^2\right) \geq C \frac{\varepsilon_n}{\log q}\right) = o(1). \quad (E.9)
\]

To prove (D.3), by (E.8) and (E.9), we only need to prove
\[
q^2 \max_{1 \leq i, j \leq q} \mathbb{P}\left(\frac{m^2(n_1-1)}{(n_1-m)^2}\left(\sum_{\alpha=1}^{n_1} (\bar{q}_{\alpha,ij})^2 - \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha))^2\right) \geq C \frac{\varepsilon_n}{\log q}\right) = o(1). \quad (E.10)
\]

For this, we need to bound $I_1$. To bound $I_1$, we introduce the following decomposition of $\bar{q}_{\alpha,ij}$. We set
\[
\sum_{\alpha=1}^{n_1} \Psi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}) = Ah_{ij}(X_\alpha) + BS_{1ij} + \Upsilon^{(\alpha)}_{1ij}, \quad (E.11)
\]
with $A = \left(n_1 - \frac{n_1-1}{m-1}\right)$, $B = \left(n_1 - \frac{n_1-1}{m-2}\right)$, $S_{1ij} := \sum_{\beta=1}^{n_1} h_{ij}(X_\beta)$ and
\[
\Upsilon^{(\alpha)}_{1ij} := \sum_{\alpha=1}^{n_1} \left(\Psi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}) - (h_{ij}(X_\alpha) + \sum_{k=1}^{m-1} h_{ij}(X_{\ell_k}))\right). \quad (E.12)
\]

Furthermore, we set
\[
V^2_{1ij} := \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha))^2, \quad \Lambda^2_{1ij} := \sum_{\alpha=1}^{n_1} (\Upsilon^{(\alpha)}_{1ij})^2 \quad \text{and} \quad D = \left(n_1 - \frac{1}{m-1}\right). \quad (E.13)
\]

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By $q_{1\alpha,ij}$’s definition in (D.2), we have $q_{1\alpha,ij} = (Ah_{ij}(X_\alpha) + BS_{1ij} + \Upsilon_{1ij}^{(\alpha)})/D$. By setting $L_1$ as

$$L_1 := \left|\frac{1}{n_1} \left(\frac{1}{D^2} \sum_{\alpha=1}^{n_1} (Ah_{ij}(X_\alpha) + BS_{1ij} + \Upsilon_{1ij}^{(\alpha)})^2 - \sum_{\alpha=1}^{n_1} (h_{ij}(X_\alpha))^2\right)\right|,$$

(E.14)

bounding $I_1$ is equivalent to bounding $\mathbb{P}(L_1 \geq Ct)$. By expanding $\sum_{\alpha=1}^{n_1} (q_{1\alpha,ij})^2$, we get

$$\sum_{\alpha=1}^{n_1} (q_{1\alpha,ij})^2 = \left(\sum_{\alpha=1}^{n_1} \frac{V_{1ij}^2 + \Lambda_{1ij}^2 + (n_1B^2 + 2AB)(S_{1ij})^2 + 2A \sum_{\alpha=1}^{n_1} h_{ij}(X_\alpha) \Upsilon_{1ij}^{(\alpha)} + 2BS_{1ij} \sum_{\alpha=1}^{n_1} \Upsilon_{1ij}^{(\alpha)}}{D^2}\right).$$

We use the Cauchy-Swartz inequality on $\sum_{\alpha=1}^{n_1} \Upsilon_{1ij}^{(\alpha)}$ to get

$$2BS_{1ij} \sum_{\alpha=1}^{n_1} \Upsilon_{1ij}^{(\alpha)}/D \leq 2B|S_{1ij}|\sqrt{n_1} \Lambda_{1ij}/(D^2n_1).$$

This motivates us to set $J_5 := 2B|S_{1ij}|\sqrt{n_1} \Lambda_{1ij}/(D^2n_1)$. By using the triangle inequality on $L_1$, we obtain $L_1 \leq J_1 + J_2 + J_3 + J_4 + J_5$. For $J_4$ and $J_5$, we use the Cauchy-Swartz inequality on $S_{1ij} = \sum_{\beta=1}^{m_1} h_{ij}(X_\beta)$ to yield

$$J_4 + J_5 \leq \left|\frac{2A}{n_1D^2} V_{1ij} \Lambda_{1ij}\right| + \left|\frac{2Bn_1V_{1ij} \Lambda_{1ij}}{n_1D^2}\right| \leq \left|\frac{2A + 2Bn_1}{n_1D^2} V_{1ij} \Lambda_{1ij}\right|.$$  

This motivates us to define $J_6 := |2(A + Bn_1)V_{1ij} \Lambda_{1ij}/n_1D^2|$. Therefore, to bound $L_1$, we only need to bound $J_1$, $J_2$, $J_3$ and $J_6$ separately.

By the definitions of $A$ and $D$, we obtain

$$A = O(n_1^{m-1}), \quad D = O(n_1^{m-1}) \quad \text{and} \quad D - A = \binom{n_1}{m} = O(n_1^{m-2}).$$

Thus, for $J_1$, by the definition of $V_{1ij}$ in (E.13), we use the Hoeffding inequality (Lemma F.1) to have

$$J_1 = \mathbb{P}\left(\frac{1}{n_1^2} V_{1ij}^2 \geq Ct\right) \leq C_1 \exp(-C_2n_1^2t^2 + C_3n_1^2t).$$

(E.15)

Similarly, for $J_3$, we use the Hoeffding inequality (Lemma F.1) to have

$$J_3 = \mathbb{P}\left(\frac{(S_{1ij})^2}{n_1^2} \geq Ct\right) \leq C_1 \exp(-C_2n_1t).$$

(E.16)

To bound $J_2$, by $\Lambda_{1ij}^2 := \sum_{\alpha=1}^{n_1} (\Upsilon_{1ij}^{(\alpha)})^2$, we have

$$\mathbb{P}\left(\frac{\Lambda_{1ij}^2}{n_1^{2m-1}} \geq Ct\right) = \mathbb{P}\left(\sum_{\alpha=1}^{n_1} (\Upsilon_{1ij}^{(\alpha)})^2 \geq Cn_1^{2m-1}t\right) \leq \sum_{\alpha=1}^{n_1} \mathbb{P}\left((\Upsilon_{1ij}^{(\alpha)})^2 \geq Cn_1^{2m-2}t\right).$$

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By the definition of $\Upsilon_{ij}^{(\alpha)}$ in (E.12), given $X_\alpha$, we can treat
\[ \Psi_{ij}(X_\alpha, X_{\ell_1}, \ldots, X_{\ell_{m-1}}) - \left( h_{ij}(X_\alpha) + \sum_{k=1}^{m-1} h_{ij}(X_{\ell_k}) \right), \]
as a symmetric kernel function. Therefore, given $X_\alpha$, $\Upsilon_{ij}^{(\alpha)}/D$ is a U-statistic with a kernel function of zero mean and $m-1$ order. Hence, by Lemma F.2, it follows that we have the following inequality:
\[ \mathbb{P}(\Upsilon_{ij}^{(\alpha)})^2 \geq C n_1^{2m-2} t |X_\alpha) = \mathbb{P}(|\Upsilon_{ij}^{(\alpha)}| \geq C n_1^{m-1} \sqrt{t} |X_\alpha) \leq C_1 \exp(-C_2 n_1 t). \quad (E.17) \]

By taking expectation of (E.17), we have
\[ \mathbb{P}(\Upsilon_{ij}^{(\alpha)})^2 \geq C n_1^2 t) \leq C_1 \exp(-C_2 n_1 t). \]

Hence, for $J_2$, we construct the following bound:
\[ J_2 \leq \mathbb{P}\left( \frac{A_{1ij}^2}{n_1^{m-1}} \geq C t \right) \leq C_1 n_1 \exp(-C_2 n_1 t). \quad (E.18) \]

At last, by $A = O(n_1^{m-1})$, $B = O(n_1^{m-2})$ and $D = O(n_1^{m-1})$, bounding $J_6$ is equivalent to bounding $\mathbb{P}(V_{1ij} A_{1ij}/n_1^{m} \geq C t)$. We then have
\[ J_6 = \mathbb{P}\left( \frac{V_{1ij} A_{1ij}^2}{n_1^{2m-2+2\kappa}} \geq C t^2 \right) \leq \mathbb{P}\left( \frac{V_{1ij}^2}{n_1^{2m-2+2\kappa}} \geq C t \right) + \mathbb{P}\left( \frac{A_{1ij}^2}{n_1^{2m-2+2\kappa}} \geq C t \right), \quad (E.19) \]

where $\kappa > 0$ will be given afterwards. Similarly to (E.15), we obtain
\[ \mathbb{P}\left( \frac{V_{1ij}^2}{n_1^{2m-2+2\kappa}} \geq C t \right) \leq C_1 \exp(-C_2 n_1^{3-4\kappa} t^2 + C_3 n_1^{2-2\kappa} t). \quad (E.20) \]

For the other term in (E.19), we have
\[ \mathbb{P}\left( \frac{A_{1ij}^2}{n_1^{2m-2+2\kappa}} \geq C t \right) \leq \sum_{\alpha=1}^{n_1} \mathbb{P}(\Upsilon_{1ij}^{(\alpha)})^2 \geq C n_1^{2m-3+2\kappa} t = \sum_{\alpha=1}^{n_1} \mathbb{P}(|\Upsilon_{1ij}^{(\alpha)}| \geq C \sqrt{n_1^{2m-3+2\kappa} t}). \quad (E.21) \]

Similarly to (E.17), we have
\[ \mathbb{P}\left( |\Upsilon_{1ij}^{(\alpha)}| \geq C \sqrt{n_1^{2m-3+2\kappa} t} \right) \leq C_4 \exp(-C_5 n_1^{2\kappa} t). \quad (E.22) \]

Therefore, Combing (E.19), (E.20), (E.21) and (E.22), we obtain
\[ J_6 \leq C_1 \exp(-C_2 n_1^{3-4\kappa} t^2 + C_3 n_1^{2-2\kappa} t) + C_4 n_1 \exp(-C_5 n_1^{2\kappa} t). \quad (E.23) \]

Noticing $\log q = O(n^{1/3-\epsilon})$ (see Assumption (A2)), if we set $\kappa = 1/3$, combining (E.15), (E.16), (E.18) and (E.23), we have (E.10) by setting $\varepsilon_n = 1/(\log q)^{\kappa_0}$ with $\kappa_0 > 0$ sufficiently small. This completes the proof. \(\square\)
### E.2 Proof of Lemma D.2

**Proof.** By the definition of $\hat{Z}_{\beta,ij}$ in (C.12), we obtain

$$
\left( \sum_{\alpha=1}^{n_1} \frac{g_{ij}(X_\alpha)}{n_1} - \sum_{\alpha=1}^{n_2} \frac{g_{ij}(Y_\alpha)}{n_2} - \tau_{1,ij} + \tau_{2,ij} \right)^2
\leq C|\Lambda| \left( \frac{n_1+1}{n_2 \tau_{1,ij}/n_1 + \tau_{2,ij}} \right) \leq C|\Lambda| \left( \frac{n_1+1}{n_2 \tau_{1,ij}/n_1 + \tau_{2,ij}} \right),
$$

(E.24)

where $\zeta_{1,ij}$ and $\zeta_{2,ij}$ are variances of $g_{ij}(X_\alpha)$ and $g_{ij}(X_\alpha)$. By the Bernstein’s inequality, for any $M > 0$, we have

$$
P\left( \frac{1}{n_1} \sum_{\beta=1}^{n_1} (\hat{Z}_{\beta,ij})^2 - \frac{n_2}{n_1} \zeta_{1,ij} \geq C \frac{\varepsilon_n}{\log q} \right) = O(q^{-M}),
$$

(E.25)

and

$$
P\left( \frac{1}{n_2} \sum_{\beta=n_1+1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2 - \zeta_{2,ij} \geq C \frac{\varepsilon_n}{\log q} \right) = O(q^{-M}),
$$

by letting $\varepsilon_n$ converge to zero sufficiently slowly. Considering $\zeta_{a,ij} \geq r_a > 0$ (see Assumption (A2)), by (E.25), we have that the two events

$\{ \left| \frac{1}{n_1} \sum_{\beta=1}^{n_1} (\hat{Z}_{\beta,ij})^2 - 1 \right| \leq C \frac{\varepsilon_n}{\log q} \}$

and

$\{ \left| \frac{1}{n_2} \sum_{\beta=n_1+1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2 - 1 \right| \leq C \frac{\varepsilon_n}{\log q} \}$,

happen with probability going to one, as $n, q \to \infty$. Under these two events, we have

$$
\left| \frac{\sum_{\beta=1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2}{n_2 \zeta_{1,ij}/n_1 + \tau_{2,ij}} - 1 \right| \leq \frac{\sum_{\beta=1}^{n_1} (\hat{Z}_{\beta,ij})^2}{n_2 \zeta_{1,ij}/n_1} - 1 + \frac{\sum_{\beta=n_1+1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2}{n_2 \zeta_{2,ij}} - 1 \leq C \frac{\varepsilon_n}{\log q}.
$$

(E.26)

By the self-normalized large deviation for independent variables in Jing et al. (2003), we have

$$
\max_{(i,j) \in S} P\left( \frac{\sum_{\beta=1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2}{n_1} \geq t^2 \right) \leq C(1 - \Phi(t)),
$$

(E.27)

uniformly for $t \in [0, O(n^{1/6-\epsilon})]$ for an arbitrary positive number $\epsilon$. Combining (E.24), (E.26) and (E.27), as $n, q \to \infty$, we have

$$
P\left( \max_{(i,j) \in \Lambda} \frac{1}{n_2} \sum_{\alpha=1}^{n_1} g_{ij}(X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}(Y_\alpha) - u_{1,ij} + u_{1,ij} \geq t^2 \right)
\leq |\Lambda| \max_{(i,j) \in \Lambda} P\left( \frac{\sum_{\beta=1}^{n_1+n_2} (\hat{Z}_{\beta,ij})^2}{n_1} \geq (1 + C \frac{\varepsilon_n}{\log q}) \right) \leq C|\Lambda|(1 - \Phi(t)).
$$

This completes the proof of Lemma D.2.
E.3 Proof of Lemma D.3

Proof. Lemma D.3 is the same as Lemma D.2 except that in Lemma D.3 we use $\sigma_{a,ps}^2/4$ to replace $\zeta_{a,ij}$ used in Lemma D.2. This motivates us to write

$$
\frac{\left( \sum_{\alpha=1}^{n_1} g_{ij}^\tau(X_\alpha)/n_1 - \sum_{\alpha=1}^{n_2} g_{ij}^\tau(Y_\alpha)/n_2 - \tau_{1,ij} + \tau_{2,ij} \right)^2}{\sigma_{1,ps}^2/4n_1 + \sigma_{2,ps}^2/4n_2} = \frac{\left( \sum_{\alpha=1}^{n_1} g_{ij}^\tau(X_\alpha)/n_1 - \sum_{\alpha=1}^{n_2} g_{ij}^\tau(Y_\alpha)/n_2 - \tau_{1,ij} + \tau_{2,ij} \right)^2}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2} \cdot \frac{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2}{\sigma_{1,ps}^2/4n_1 + \sigma_{2,ps}^2/4n_2},
$$

where $g_{ij}^\tau$ is defined in (C.37) and $\zeta_{1,ij}$ and $\zeta_{2,ij}$ are variances of $g_{ij}^\tau(X_\alpha)$ and $g_{ij}^\tau(Y_\alpha)$. By the definitions of $g_{ij}^\tau$ and $\zeta_{a,ij}$, under Assumption (A2), we have $0 < r_a \leq \zeta_{a,ij} \leq 1$. We then have

$$0 < (C_1)^2 \leq \frac{\sigma_{1,ps}^2/4n_1 + \sigma_{2,ps}^2/4n_2}{1/n_1 + 1/n_2} \leq \frac{\sigma_{1,ps}^2/4n_1 + \sigma_{2,ps}^2/4n_2}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2}. \quad (E.28)
$$

Therefore, by Lemma D.2, we have

$$
P\left( \max_{(i,j) \in \Lambda} \frac{\left( \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^\tau(X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^\tau(Y_\alpha) - \tau_{1,ij} + \tau_{1,ij} \right)^2}{\sigma_{1,ps}^2/(4n_1) + \sigma_{2,ps}^2/(4n_2)} > t^2 \right) \leq P\left( \max_{(i,j) \in \Lambda} \frac{\left( \frac{1}{n_1} \sum_{\alpha=1}^{n_1} g_{ij}^\tau(X_\alpha) - \frac{1}{n_2} \sum_{\alpha=1}^{n_2} g_{ij}^\tau(Y_\alpha) - \tau_{1,ij} + \tau_{1,ij} \right)^2}{\zeta_{1,ij}/n_1 + \zeta_{2,ij}/n_2} > (C_1)^2 t^2 \right) \leq C|\Lambda|(1 - \Phi(C_1 t)).
$$

This completes the proof of Lemma D.3. \hfill \Box

F Some Useful Technical Lemmas

In this appendix, we introduce the following technical lemmas that we use in our proofs.

**Lemma F.1.** (Hoeffding’s inequality (Hoeffding, 1963)). $X_1, \ldots, X_n$ are independent random variables and $X_i$ takes its value in $[a_i, b_i]$. Letting $S = \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])$, we have

$$P(S \geq x) \leq \exp \left( - \frac{2x^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right),$$

for any positive $x \in \mathbb{R}$.

To present some useful results on U-statistics, we set $X_1, \ldots, X_n$ to be independent and identically distributed random vectors in $\mathbb{R}^d$. $\Phi(x_1, \ldots, x_m)$ is a kernel function of $m(\leq n)$ vectors $x_\gamma = (x_{\gamma_1}, \ldots, x_{\gamma_d})^T$. The U-statistic is defined as $U := \sum \Phi(X_{i_1}, \ldots, X_{i_m})/n(n-1) \cdots (n-m+1)$, where the summation is taken over all $m$-tuples $i_1, \ldots, i_m$ of distinct positive integers not exceeding $n$. 
Lemma F.2. (Exponential inequality for bounded U-statistics (Hoeffding, 1963)). If the kernel function $\Phi$ is bounded, i.e., $a \leq \Phi(x_1, \cdots, x_m) \leq b$, we have
\[
P(U - \mathbb{E}[U] \geq t) \leq \exp\left(-\frac{2kt^2}{(b-a)^2}\right),
\]
where $k = \lfloor n/m \rfloor$.

We define $U = (U_1, \ldots, U_K) \in \mathbb{R}^K$ as a $K$-dimensional U-statistic with the kernel function $\Phi(\cdot) = (\Phi_1(\cdot), \ldots, \Phi_K(\cdot))^T \in \mathbb{R}^K$, where $\Phi_k(\cdot)$ is defined as
\[
\Phi_k(\cdot) : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R} \quad \text{for} \quad 1 \leq k \leq K.
\]

Letting $\phi_k := \mathbb{E}[\Phi_k(X_1, \ldots, X_{m(k)})]$, we define
\[
\zeta_{k\ell} = \mathbb{E}\left[\mathbb{E}[\Phi_k(X_1, \ldots, X_{m(k)}) - \phi_k | X_1] \mathbb{E}[\Phi_{\ell}(X_1, \ldots, X_{m(\ell)}) - \phi_{\ell} | X_1]\right].
\]

Hence, by setting $\Sigma^U = (m(k)m(\ell)\zeta_{k\ell}) \in \mathbb{R}^{K \times K}$, we introduce the central limit theorem for U-statistics.

Lemma F.3. (Central limit theorem for U-statistics (Hoeffding, 1948)). We assume $\Phi_k(\cdot)$ is a symmetric kernel for $1 \leq k \leq K$. If $\mathbb{E}[\Phi_k^2(X_1, \ldots, X_{m(k)})]$ exists for $k = 1, \ldots, K$, as $n \to \infty$, we have
\[
(\sqrt{n}(U_1 - \phi_1), \ldots, \sqrt{n}(U_K - \phi_K))^T \to N(0, \Sigma^U),
\]
where $N(0, \Sigma^U)$ is the distribution of a normal random vector with mean vector 0 and $K \times K$ covariance matrix $\Sigma^U$.

To get the central limit theorem, we need the Hoeffding decomposition. The Hoeffding decomposition divides a U-statistic into two pieces. One is a sum of i.i.d random variables. The other is a small residual term. For notational simplicity, for $\ell = 1, \ldots, n$, we set
\[
g(X_\ell) = \mathbb{E}[\Phi(X_{\ell_1}, \ldots, X_{\ell_m}) | X_\ell] \quad \text{and} \quad h(X_\ell) = g(X_\ell) - \phi,
\]
where $\phi = \mathbb{E}[g(X_\ell)]$ and $\ell \in \{\ell_1, \ldots, \ell_m\}$.

Lemma F.4. (Hoeffding Decomposition (Hoeffding, 1948)). If $U$ is a U-statistic with a symmetric kernel function $\Phi(x_1, x_2, \ldots, x_m)$, we can decompose $U$ as
\[
U = \frac{m}{n} \sum_{\ell=1}^{n} h(X_\ell) + \left(\frac{n}{m}\right)^{-1} \Delta_n,
\]
where we set
\[
\Delta_n = \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq n} (\Phi(X_{\ell_1}, \ldots, X_{\ell_m}) - \phi - \sum_{k=1}^{m} h(X_{\ell_k})).
\]
In Lemma F.4, \( m \sum_{\ell=1}^{n} h(X_\ell)/n \) is a sum of i.i.d random variables and \( \left( \frac{n}{m} \right)^{-1} \Delta_n \) is a small residual term. Therefore, we can use \( m \sum_{\ell=1}^{n} h(X_\ell)/n \) to approximate \( U \). Combining Lemmas F.3 and F.4, we know that \( U \) and \( m \sum_{\ell=1}^{n} h(X_\ell)/n \) have the same limiting distribution as \( n \to \infty \).

Next, we introduce a lemma on the tail probability of a normal distribution.

**Lemma F.5.** If \( \xi \) follows a standard normal distribution, we have

\[
\frac{t}{t^2 + 1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq P(\xi > t) \leq \frac{1}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2},
\]

for any \( t > 0 \).

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