Risk of Phase Incoherence in Wide Area Control of Synchronous Power Networks With Time-Delayed and Corrupted Measurements

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Abstract—We develop a framework to quantify systemic risk measures in a class of wide-area-control (WAC) laws in power networks in the presence of noisy and time-delayed sensory data. A closed-form calculation of the risk of phase incoherence in interconnected power networks is presented, and the effect of network parameters, information flow in WAC architecture, statistics of noise, and time delays are characterized. We show that in the presence of time-delay and noise, a fundamental tradeoff between the best achievable performance (via tuning feedback gains) and value-at-risk emerges. The significance of our results is that they provide a guideline for developing algorithmic design tools to enhance the coherency and robustness of closed-loop power networks simultaneously. Finally, we validate our theoretical findings through extensive simulation examples.

Index Terms—Delay systems, networked control systems, risk analysis, smart grids, stochastic systems.

I. INTRODUCTION

MODERN power networks have been struggling with evermore narrow stability margins under the ongoing deregulation of energy markets and the growth of highly volatile sources of renewable energy as well as variable load endpoints [1], [2]. These systems may be steered to undesirable contingency events under continued stressful operating conditions. The 1996 Western American blackout [3], the 2003 blackouts in North-East USA, Canada [4], and Italy [5] are examples of failures due to outdated stabilization techniques or naive interconnectivity [6], [7].

The synchronous power systems are considered robust if they remain in or return to synchronism after experiencing a fault event. The concept of transient stability measures this property, where it characterizes the extent to which generators remain in phase as they recover from a nontrivial disturbance [8], [9]. The transient dynamics in power systems are handled via local power system stabilizer (PSS) modules. These are controllers that perform feedback stabilization in the event of generator excitation [10]. However, PSSs perform poorly in the absence of coordinating authority that may result in grid instability [11]. Furthermore, the existing proposed solutions require careful tuning with high-gain feedback control laws [12] that suffer from scalability issues [13]. A different approach utilizes wide-area-control (WAC) methods [13], [14], [15], where network stabilization relies on remote measurements from the entire grid. These methods enjoy the advantage of being technologically feasible due to high-bandwidth phasor measurement units and flexible ac transmission system devices [16]. Furthermore, the theory of multiagent systems is reaching the level of maturity to provide scalable algorithms for control of modern grids [13], [17], [18].

On the downside, measurements have to be transmitted over some cyber data layer [19]. Therefore, in contrast to local control, WAC is vulnerable to asynchronous propagation of information or measurement noise that corrupts actual data. Moreover, their simultaneous interplay may severely impact WAC capabilities on stabilizing power networks [13]. In addition to the aforementioned references, some other related works investigate stability problems in power systems with nonlinear models [9], [20] and time-delay mitigation in WAC loops [21], [22], [23], [24]. The latter work focuses on the worst-case scenarios for robust control, which usually yields conservative control policies.

In this work, we consider the problem of phase incoherence in linear transient dynamics of a power network excited by variable load forces modeled as exogenous noise. We assess the performance of a specific class of WAC policies that seeks to control the power grid through a virtual communication network. The network is prone to asynchronous processing and propagation of information. Moreover, information transmitted between stations is corrupted by measurement noise that captures the impact of imperfect sensors and/or communication protocols. A schematic diagram of the problem setup is illustrated in Fig. 1. The robustness of the closed-loop network is assessed
through the notion of the value-at-risk measure [25], which is a systemic risk measure and adopted from finance literature [26], and has been recently utilized by the authors in the context of multiagent control protocols, including the vehicle platooning and multiagent rendezvous [27], [28], [29], [30], [31] as a surrogate for robustness that scales with high-dimensional dynamical systems.

We examine the role of WAC in reducing the risk of undesirable behaviors in power networks in the presence of measurement noise and time-delay. We introduce a notion of nested systemic events to characterize various levels of experiencing undesirable events. The value-at-risk of phase incoherence between a given pair of synchronous generators is calculated using the corresponding nested systemic events. It is shown that the systemic risk measure depends on the spectral properties of the underlying graph of the network, the WAC gains, statistics of noise, and time-delay. In the presence of time-delay, fundamental limits emerge between the best (lowest) achievable levels of risk and highest possible values for the feedback gains in the WAC. This limit suggests that while higher feedback gains may result in better performance, e.g., smaller total resistive power loss [32], [33], it will increase the risk of phase incoherence in the network. This article is an outgrowth of [34] and [35], which contains several new technical contributions with respect to its conference versions. More specifically, we expand our framework to time-delayed data with corrupted measurements. We investigate the interplay between heterogeneous sources of noise and the way they affect risk measures. Also, we establish new fundamental limits of risk beyond the time-delayed ones. This version includes several examples that offer analysis and synthesis of networks based on systemic risk. Finally, we provide proofs of all the theoretical results, including a thorough analysis of stability region of our closed-loop system dynamics, as well as analysis of asymptotic properties of underlying spectral based functions in the (discussion in the Appendix).

II. MATHEMATICAL NOTATIONS

We denote the nonnegative orthant of the Euclidean space $\mathbb{R}^n$ by and $\mathbb{R}_{+}^n$, $n \geq 1$. The $n \times n$ diagonal matrix is denoted by $D = \text{diag}\{d^{(i)}\}$, the $n \times r$ zero matrix is $0_{n,r}$, and $0_n$ when $r = n$. The $n \times n$ identity matrix is denoted by $I_n$. Finally, by $1_n$, we understand the $n \times 1$ vector of all ones.

Algebraic graph theory: A undirected weighted graph is defined by $G = (V, E, \omega)$, where $V$ is the set of nodes, $E$ is the set of edges (feedback links), and $\omega : V \times V \to \mathbb{R}_{+}$ is the weight function that assigns a nonnegative number (feedback gain) to every link. Two nodes are directly connected if and only if $(i, j) \in E$.

The Laplacian matrix of $G$ is a $n \times n$ matrix $L = [l_{ij}]$ with elements

$$l_{ij} := \begin{cases} -\omega(i, j) & \text{if } i \neq j \\ \omega(i, 1) + \ldots + \omega(i, n) & \text{if } i = j \end{cases}$$

where $k_{i,j} := \omega(i, j)$. Laplacian matrix of a graph is symmetric and positive semidefinite. If graph $G$ is connected, the smallest Laplacian eigenvalue is zero with algebraic multiplicity one. The spectrum of $L$ can be ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$.

Probability theory: Let $\mathcal{L}^2(\mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued random vectors $z = [z^{(1)}, \ldots, z^{(n)}]^T$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moments. A normal random variable $y \in \mathbb{R}^n$ with mean $\mu \in \mathbb{R}^n$ and $n \times n$ covariance matrix $\Sigma$ is represented by $y \sim \mathcal{N}(\mu, \Sigma)$. We employ standard notation $\text{d}t_\nu$ for the formulation of stochastic differential equations. Finally, with $\nu_\varepsilon$, we denote the unique solution of $\int_{-\nu_\varepsilon}^{\nu_\varepsilon} e^{-\frac{t^2}{2\varepsilon}} \text{d}t = \sqrt{2\pi}(1 - \varepsilon)$ for $\varepsilon \in (0, 1)$.

III. TIME-DELAYED CONTROL IN POWER NETWORKS

Let us consider a network of $n$ synchronous generators connected over $m$ transmission lines. The $i$th generator is defined through the (static) triplet $(J_i, \beta_i, E_i)$ and (dynamic) state vector $(\theta^{(i)}_t, \omega^{(i)}_t)$. The triplet of fixed parameters consists of the rotational inertia, the damping coefficient, and the voltage magnitude, respectively. The state variables at time instant $t$ are the rotor phase $\theta^{(i)}_t$ and the rotor frequency $\omega^{(i)}_t = \frac{d}{dt}\theta^{(i)}_t$. The states of the synchronous generators can be stacked as

$$\theta_t = \left[\theta^{(1)}_t, \ldots, \theta^{(n)}_t\right]^T$$

and

$$\omega_t = \left[\omega^{(1)}_t, \ldots, \omega^{(n)}_t\right]^T$$

to define the network’s state vector. Interactions in the power network are characterized through the reduced complex admittance matrix $Y = [Y_{ij} \angle \phi_{ij}]$, where the angles $\phi_{ij} \in [0, \frac{\pi}{2}]$ are the phase shifts that characterize the energy loss due to the transfer conductance. Throughout this article, any transfer conductance is assumed zero, i.e., $\phi_{ij} = \frac{\pi}{2}$ for all $i \neq j$. This makes $Y_{ij}$ the normalized susceptance of the transmission line connecting the $i$th and $j$th generators. The effective power input $p_i$ of generator

![Diagram of WAC in power networks with heterogeneous sources of noise. Data transmitted over the communication network are time-delayed.](image-url)
\( i \) is defined as the difference of the mechanical power and the electrical internal voltage power, i.e., \( p_i = P_i^m - E_i^2 \Re \{ Y_{ii} \} \).

We consider the following benchmark model \([9]\) of the dynamic phase of the \( i \)th generator:

\[
J \dot{\theta}_i^{(i)} = -\beta_i \dot{\theta}_i^{(i)} + \sum_{j=1}^{n} E_i E_j Y_{ij} \sin(\theta_j^{(j)} - \theta_i^{(i)}) + p_i
\]

for \( i = 1, \ldots, n \). Although (1) has been criticized as overly simple \([36]\), it has proven to be a reliable model for exploring control policies in power networks \([21]\). The central objective of transient stability is to investigate conditions under which (1) will converge to an operating equilibrium point. For fixed voltage magnitudes, admittances, and power inputs, the equilibrium point belongs to the manifold

\[
\mathbb{S} = \{ (\theta, \omega) \in \mathbb{R}^{2n} | \omega = 0 \text{ and } |\dot{\theta}^{(i)} - \dot{\theta}^{(j)}| < \frac{\pi}{2} \text{ with } \}
\]

\[
p_i = \sum_{j=1}^{n} E_i E_j Y_{ij} \sin(\theta_j^{(j)} - \theta_i^{(i)}), \ i, j = 1, \ldots, n \}. \]

The condition that geodesic distance between phase angles is restricted in \((-\pi/2, \pi/2)\) allows for the consistency of angle differences; we refer to \([37]\) for more details.

### A. Deviation From Synchronous States

Let us denote the desired equilibrium point by \((\theta_0, 0) \in \mathbb{S}\). Due to exogenous stochastic disturbances, the phase and frequency states tend to fluctuate around the equilibrium state. This allows us to investigate behavior of the system (1) in the presence of additive noise using its linearization around the equilibrium point\(^1\) by considering the error dynamics

\[
\begin{bmatrix}
\dot{\theta}_i \\
\dot{\omega}_i
\end{bmatrix} =
\begin{bmatrix}
0 & I_n \\
-L & -D
\end{bmatrix}
\begin{bmatrix}
\theta_i \\
\omega_i
\end{bmatrix} dt +
\begin{bmatrix}
0_n \\
H
\end{bmatrix}
\begin{bmatrix}
\xi_t \\
u_t
\end{bmatrix}
\]

where \(\xi_t = [\xi_t^{(1)}, \ldots, \xi_t^{(n)}]^T\). The corresponding Laplacian matrix \(L = [l_{ij}]\) is defined elementwise as

\[
l_{ij} = \begin{cases} J^{-1} E_i E_j Y_{ij} \cos(\theta_i^{(i)} - \theta_j^{(j)}) & \text{if } i \neq j \\ -\sum_{k \neq i} l_{ik} & \text{if } i = j \end{cases}
\]

and the diagonal matrix \(D = \text{diag}(\beta_1, \ldots, \beta_n)\) contains the damping over inertia ratios of power machines. In this model, the stochastic fluctuations in effective power inputs \(p_i\) for \(i = 1, \ldots, n\) are considered as one source of exogenous disturbance, where the effects of mismatch between power generation and load demand are modeled as mutually independent white noise processes \(\xi_t^{(1)}, \xi_t^{(2)}, \ldots, \xi_t^{(n)}\). The diffusion coefficient for the \(i\)th generator is \(\eta J^{-1}\). Therefore, the diffusion matrix is given by \(H = \eta J^{-1}\), where \(J = \text{diag}(J_1, \ldots, J_n)\).

\(^1\)Observe that \((\theta_0, 0) \in \mathbb{S}\) implies \((\theta_0 + 1, \omega, 0) \in \mathbb{S}\) for any \(t \in \mathbb{R}\). Evidently, stabilization of (1) around a point in \(\mathbb{S}\) is equivalent to stabilization over the corresponding disagreement subspace of \(\mathbb{R}^n\). To keep our notations simple, the state variables \(\theta_i\) and \(\omega_i\) have been recycled to denote the error variables \(\theta_i - \eta_0\) and \(\omega_i - \omega_0\).

### B. Virtual State Feedback Control

In (2), the term involving \(d\xi_t\) induces stochastic fluctuation for the state vector around \((\theta_0, 0) \in \mathbb{S}\). The possible approach to mitigate the effect of the exogenous disturbance is to adjust power voltage on admittance parameters, i.e., to probe into elements of \(L\) and \(D\). However, this is a costly, if not infeasible, solution. For large-scale power networks, the distributed WAC with the aid of a communication network of sensors/controllers is a tractable alternative. The idea is for actuators to be installed on the virtual network and use measurements of their adjacent remote peers \([13, 15]\). To this end, we consider introducing the state feedback control in (2), such that

\[
\begin{bmatrix}
\dot{\theta}_t \\
\dot{\omega}_t
\end{bmatrix} =
\begin{bmatrix}
0_n & I_n \\
-L & -D
\end{bmatrix}
\begin{bmatrix}
\theta_t \\
\omega_t
\end{bmatrix} dt +
\begin{bmatrix}
0_n \\
H
\end{bmatrix}
\begin{bmatrix}
\xi_t + \sum_{j=1}^{n} m_{ij} (\theta_t^{(i)} - \eta_0 + \theta_t^{(j)} + \xi_t^{(j)}) \\
\theta_t^{(i)} - \eta_0 + \theta_t^{(j)} + \xi_t^{(j+n)}
\end{bmatrix}
\]

where the control input \(u_t\) utilizes information of the virtual network from the power network. The communicating sensors are assumed to operate in parallel with the electric grid and established over multiple spatially distributed local controllers.

As is the case with every transmitted and processed information, time-delays in signals are ubiquitous and they account for severe impacts on the closed-loop system.

**Assumption 1:** The control mechanism over the communication network processes state information with a constant and uniform time-delay \(\tau > 0\).

The main sources of time-lagged information are generic latency in measurements, communication, and actuation of the control policies. In this work, we regard \(\tau > 0\) as a nominal constant that all time-delay sources are lumped into.\(^2\) The virtual state feedback control for generator \(i\) is given by

\[
u_t^{(i)} = -\sum_{j=1}^{n} m_{ij} (\theta_t^{(i)} + \xi_t^{(j+n)} + k_{ij} (\theta_t^{(i)} + \xi_t^{(j+n)} + \theta_t^{(j+2n)}))
\]

Real data are usually corrupted by measurement noise due to, e.g., imperfect sensors, simultaneous casting, or other conditions that may occur at other levels of communication \([38]\). The effect of such uncertainties on the rotor phase and frequency in the virtual feedback control law are modeled as \(\xi_t^{(j+n)}\) and \(\xi_t^{(j+2n)}\), for \(j = 1, \ldots, n\), respectively, where the diffusion coefficient \(\eta\) models the noise magnitude.

The feedback gains, which characterize interactions among the communicating controllers over the virtual network, are assumed to be symmetric, i.e., \(m_{ij} = m_{ji}\) and \(k_{ij} = k_{ji}\). By denoting the corresponding interaction matrices by \(M = [m_{ij}]\) and \(K = [k_{ij}]\), the overall closed-loop network can be represented in the following compact form:

\[
\begin{bmatrix}
\dot{\theta}_t \\
\dot{\omega}_t
\end{bmatrix} =
\begin{bmatrix}
\theta_t \\
\omega_t
\end{bmatrix} dt +
\begin{bmatrix}
M & M \\
0 & K
\end{bmatrix}
\begin{bmatrix}
\theta_t - \eta_0 \\
\omega_t - \omega_0
\end{bmatrix} dt +
\begin{bmatrix}
H & H
\end{bmatrix}
\begin{bmatrix}
\xi_t \\
u_t
\end{bmatrix}
\]

\(^2\)In reality, time-delay occurs for various reasons, and its amount ought to be considered stochastic, time dependent, or heterogeneous, which is beyond the scope of our work.
in which
\[
A = \begin{bmatrix}
0_n & I_n \\
-L & -D
\end{bmatrix}, \quad K = \begin{bmatrix}
0_n & 0_n \\
-M & -K
\end{bmatrix}
\]
and, with a little abuse of notation, the vector \( d\xi_t = [d\xi_t^{(1)}, \ldots, d\xi_t^{(n)}] \) in (5), and thereafter, stacks the three noise sources as they appear in (2) and (4). Summing up perturbation input and communication noise, the diffusion matrix can be written as
\[
H = \begin{bmatrix}
0_n & 0_n & 0_n \\
\eta J^{-1} & -iM & -iK
\end{bmatrix} \in \mathbb{R}^{2n \times 3n}.
\]

The diffusion matrix includes two types of uncertainties, i.e., the exogenous noise and the measurement noise, that affect the dynamics of the network simultaneously, but independently. We note that \( H \) incorporates, similar in mathematical form but of interpretation sources. Submatrix \( H = \eta J^{-1} \), introduced in (2), carries white noise vector \([d\xi_t^{(1)}, \ldots, d\xi_t^{(n)}]\) and characterizes, through parameter \( \eta \), the effect of load volatility in the swing stability of the network. This modeling approach was first adopted in [37]. The parameter \( \iota \) introduced in (4) carries noise vector \([d\xi_t^{(n+1)}, \ldots, d\xi_t^{(3n)}]\) and models the effect of communication/measurement deficiencies. We will see in the analysis to follow that both \( \eta \) and \( \iota \) play a nontrivial role in the risk evaluation of the grid; although in real-world applications, one should expect \( \eta \) and \( \iota \) to be scaled differently.

Assumption 2: For all \( i = 1, \ldots, n \), the damping and inertia parameters of generators are assumed to be identical, i.e., \( \forall i \in \{1, \ldots, n\} \)
\[
\beta_i = \beta > 0 \quad \text{and} \quad \lambda_i = \lambda > 0.
\]
This condition implies that all generators have the same damping over inertia ratios. This is a strong assumption that has been widely addressed in the literature [36, 37, 39, 40] and will facilitate the risk analysis and enable the possibilities to obtain explicit results.

C. Problem Statement

In the error dynamics (5), the states of all generators in the power network will fluctuate around the operating equilibrium point. Hence, some of the generators may experience phase incoherence and deterioration in robustness. Our goal is to quantify value-at-risk of phase incoherence in the power network as a function of the underlying graph Laplacian, communication time-delay in virtual control, and noise statistics. Furthermore, we characterize fundamental limits and tradeoffs that emerge from the existence of time-delay and noise in the system.

IV. INTERNAL STABILITY AND MEASUREMENT STATISTICS

In order to calculate systemic risk measures, one needs to develop some intermediate results about the stability and statistics of the steady-state variables, which constitute the subject of this section. First, we establish explicit necessary and sufficient conditions for asymptotic stability of the unperturbed system, i.e., in (5) when \( \eta = \iota = 0 \), where the stability region depends on system and feedback control parameters and the time-delay \( \tau > 0 \). Then, based on the derived stability window, we will calculate the steady-state statistics of the network.

A. Simultaneous Diagonalizable Feedback Control

To allow derivation of analytical results, the feedback gain matrices, which are of interest in this work, satisfy the following diagonalizability condition.

Assumption 3: The feedback gain matrices \( M \) and \( K \) are designed such that each pair out of \( L, M, \) and \( K \) commutes. A standard result [41] asserts that this assumption is equivalent to the existence of a unitary matrix \( Q \) such that \( QTUQ \) is diagonal for every \( U \in \{L,M,K\} \) (cf., [41, Th. 4.1.6]). Therefore, by virtue of Assumption 3, it is assumed that there exists matrix \( Q \) such that \( QTQ = \Lambda_L, QTMQ = \Lambda_M, \) and \( QTQK = \Lambda_K \), where \( \Lambda_L, \Lambda_M, \) and \( \Lambda_K \) are diagonal matrices. In particular, columns of \( Q \) can be arranged so that \( \Lambda_L = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) with elements sorted as \( 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \). Similarly, the \( j \)th diagonal elements of \( \Lambda_M \) and \( \Lambda_K \) are \( \mu_j \) and \( \kappa_j \), respectively.

Examples of simultaneously diagonalizable structures with respect to \( L \) appear when \( M \) and \( K \) are proportional to identity matrix \( I_n \), graph Laplacian \( L \), centering matrix \( M_n \), or matrix \( B_nB_n^T \), where \( B_n \) is the incidence matrix. Another case is when \( M \) and \( K \) are proportional to each other and \( M \) commutes with \( L \). While Assumption 3 may impose some restrictions on admissible WAC policies, one can still form a feasible WAC policy. The reason is that feedback gains \( M \) and \( K \) are only parameters of the virtual network for which it is reasonable to assume that the designer has enough synthesis authority to choose admissible gains, whereas the power network matrices \( L \) and \( D \) represent physical components of the grid, hence they cannot be modified as easy.

B. Internal Stability

For general matrices \( M \) and \( K \), sufficient conditions for the asymptotic behavior of the unperturbed closed-loop system can be inferred using standard methods in the stability theory of delay differential equations [42]. These conditions are usually conservative and of little use for further analysis. However, when \( M \) and \( K \) satisfy Assumption 3, one can explicitly characterize the stability region of the unperturbed system. For the exposition of the following result, we introduce the sets \( \{W_t(s; k) \mid (s; k) \in \mathbb{R}_+ \times \mathbb{R}^2 \}_{i=0}^3 \) that are defined in Table II of Appendix B. Also, let \( \{\lambda_j\}_{j=1}^n \) and \( \{\kappa_j\}_{j=1}^n \) be the eigenvalues of Laplacian matrix \( L \) introduced in (2), and matrices \( M \) and \( K \) as introduced in (5), respectively. The spectra of \( L, M, \) and \( K \) are enumerated in the same order. Finally, we consider the following conditions that associate the aforementioned spectra, the damping ratio \( \bar{d} \), and the time-delay \( \tau > 0 \):

\[
\begin{align*}
C_1 &= \{\mu_1 = 0\} \cap \{\bar{d}\tau, 0; 0, \kappa_1\tau\} \in \mathbb{W}_0 \\
C_2 &= \{\mu_1 \neq 0\} \cap \{\bar{d}\tau, \mu_1\tau^2, \kappa_1\tau\} \in \mathbb{W}_2 \\
C_3 &= \{\bar{d}\tau, \lambda_j\tau^2; \mu_j\tau^2, \kappa_j\tau\} \in \mathbb{W}_r, j = 2, \ldots, n.
\end{align*}
\]
Theorem 1: Suppose that Assumptions 2 and 3 hold. Consider the solution $(\theta_t, \omega_t)$ of (3) with $\mathbf{H} = 0_{2n \times 3n}$. There exists $\theta_\infty \in \mathbb{R}$, such that $\lim_{t \to \infty} \left[ \frac{\theta_t}{\omega_t} \right] = \left[ 1_{n} \theta_\infty \right]$, if and only if $\{C_1 \cap C_3 \} \cup \{C_2 \cap C_3 \}$ holds true. Moreover, if $\{C_2 \cap C_3 \}$ is true, then $\theta_\infty = 0$, whereas if $\{C_1 \cap C_3 \}$ is true, then $\theta_\infty = \frac{1}{n(d+\kappa_1)} \int_{-\infty}^{\infty} [\phi(0) + \phi^\omega(0) - \kappa_1 \int_{-\infty}^{0} \phi^\omega(s) \, ds]$. 

C. Measurement Statistics

Theorem 1 describes the parameteric set that guarantees the asymptotic convergence of the unperturbed system to $\mathbb{S}$. The infused uncertainty into the network generates stochastic fluctuations of the state variables around the equilibrium. It is convenient to study these solutions using the associated transition matrix $\Phi(t)^3$ that retains the stability properties of the system according to Theorem 1. The overall process is formulated as the sum of a deterministic and a stochastic term

$$\left[ \frac{\theta_t}{\omega_t} \right] = T_t + \int_0^t \Phi(t-s) \left[ \begin{array}{c} 0_{n \times 3n} \\ H \end{array} \right] \, d\zeta, \quad \text{(7)}$$

The network (5) is initialized with arbitrary, but fixed, functions $\phi(t) = (\phi^\theta(t), \phi^\omega(t))$ over time interval $t \in [-\tau, 0]$ that are statistically independent of $d\omega_0$. Initial function $\phi(0)$ is encapsulated in transient term $T_t := \int_0^t \Phi(t-s-\tau) \phi(s) \, d\mu(s)$ that is represented as a Stieltjes integral for some appropriate measure $\mu$ with bounded variation; see [44]. Equation (7) illustrates the statistics of states of the error dynamics. For $t < \infty$, the system’s state admits a normal random vector distribution as

$$\left[ \frac{\theta_t}{\omega_t} \right] \sim \mathcal{N} \left( T_t, \int_0^t \Phi(s) \left[ \begin{array}{c} 0_{n} \\ 0_{n} \\ 0_{n} \\ H \end{array} \right] \Phi^T(s) \, ds \right).$$

The quantity of interest for risk assessment is the phase incoherence between two generators, i.e., the elements of the vector $y_t = B_n \theta_t$. One way to obtain insight on the manner with which the power grid, the controllers, the time-delay, and the noise contribute to phase incoherence is to consider the long-term statistics of $y_t$, i.e., $\overline{y} := \lim_{t \to +\infty} y_t$.

Theorem 2: Suppose that Assumptions 1 and 3 hold for the solution $(\theta_t, \omega_t)$ of system (5), and let us consider a function $f : \bigcup_{t=1}^{3} \mathbb{W}_r \to \mathbb{R}_+$ that is defined in Appendix C. The output vector $y_t$, converges, in distribution, to

$$\overline{y} \sim \mathcal{N} \left( 0, \frac{1}{2n} B_n Q \text{ diag } \{ f_l \} Q^T B_n^T \right)$$

where $f_l = f_l((s; k)_l)$ for $l = 1, \ldots, n$ to be defined as

$$f_l = \begin{cases} 0 & \text{if } l = 1 \\ \left( \frac{(k_2)^2}{\tau_2^2} \right) & \text{if } l > 1 \end{cases}$$

and $f = f((s; k)_l)$ is a spectral function defined in (21) and evaluated at $(s; k)_l := (s_1, s_2; k_1, k_2)_l$ in which

$$(s_1, s_2; k_1, k_2)_l := \left( \frac{\ell}{\tau_2}, \frac{\mu_1}{\tau_2}, \kappa_1 \right) \in \bigcup_{r=1}^{3} \mathbb{W}_r.$$

The spectral function $f(s; k)$ is implicitly defined using an improper integral, and its basic properties are discussed in Appendix C. Using the statistics of $\overline{y}$, we will calculate the risk of phase incoherence in the error dynamics (5).

V. Risk of Phase Incoherence

The stability of (1) is challenged when the error dynamics exceed certain margins, which may be the case for certain types and magnitudes of $\mathbf{H}$ in the first-order approximation. Higher order terms may then prevail and steer the system away from the nominal equilibrium. This section aims to quantify “how much uncertainty” can the linearized dynamics sustain before they are steered into unsafe regions. This uncertainty is quantified through the notion of systemic sets.

A. Systemic Set

The $(i, j)$th element of the column vector $y_t$ measures the relative coherency of two synchronous generators $i$ and $j$, i.e., $\theta_{ij}^{(i,j)} - \theta_{ij}^{(i,j)}$. We consider systemic sets on the range of $|\theta_{ij}^{(i,j)} - \theta_{ij}^{(i,j)}|$ as follows. Evidently, the closer the elements of $y_t$ are to zero, the safer the performance of the grid. Let us choose $\zeta > 0$ as the lower limit for which $|y_{ij}^{(i,j)}| > \zeta$ should not happen and any value $|y_{ij}^{(i,j)}| \in (\zeta, \infty)$ is flagged as unsafe. An additional checkpoint between 0 and $\zeta$ is denoted as the permissible, presumably harmless, magnitude of fluctuations, i.e., $\zeta/c \in (0, \zeta)$, with $c > 1$, which introduces a trichotomy of characteristic ranges for $|y_{ij}^{(i,j)}|$. The underlying assumptions behind this construction are given as follows.

1) When $|y_{ij}^{(i,j)}| \in [0, \zeta/c]$, there is no risk of encountering phase incoherence and the risk value is assigned as zero.

2) When $|y_{ij}^{(i,j)}| \in [\zeta/c, \zeta]$, it will trigger a positive value of risk, as an index of safety margin between phase incoherence and undesirable behavior.

3) For consistency reasons, as phase incoherence approaches $\zeta$, the risk increases and becomes $+\infty$ for any value $|y_{ij}^{(i,j)}| \in (\zeta, \infty)$.

Example 1: The motivation behind the $(\zeta, c)$ trichotomy stems from standard transient stability analysis [9]. Let us consider a two-machine system with coupled dynamics (1) with nominal (constant) internal voltages $E_1$ and $E_2$ that satisfies the uniformity Assumption 2. If $|p_1 - p_2| < 2E_1E_2 Y_{12}$, the phase difference dynamics attains a family of equilibria $(0, 0, -\theta^2_2)$, for which $\sin(\theta^1_2 - \theta^2_2) = \frac{(p_1-p_2) E_1 Y_{12}}{2 |p_1-p_2| E_1 Y_{12}}$, and are ofinterlacing stability. In Fig. 2, we depict three such fixed points: $(0, \rho_1)$ (unstable), $(0, \rho_2)$ (stable), and $(0, \rho_3)$ (unstable). Separatrices that connect the unstable equilibria essentially determine the stability region of $(0, \rho_3)$. As explained in [9], the transient stability assesses the ability of postfault dynamics to remain...
The risk of steady-state phase incoherence

\[ \| \Delta \|_{\infty} \leq \frac{\sigma}{c} \]

is if \( \sigma \) and \( \zeta \) are as in (8). The next result provides a closed-form expression on risk in phase incoherence. For its exposition, we recall the root value \( \nu_c \) of \( \int_{-\nu_c}^{\nu_c} e^{-\nu^2/2} \, dt = \sqrt{2\pi}(1-\epsilon) \).

**Theorem 2:** The risk of steady-state phase incoherence \( \| \Delta \|_{\infty} = \lim_{t \to +\infty} |\theta_t^{(i)} - \theta_t^{(j)}| \) between generators \( i \) and \( j \) with error dynamics (5) is

\[ \mathcal{R}_\epsilon^{(i,j)} = \begin{cases} 0 & \text{if } \sigma_{ij} \leq \frac{\zeta_c}{c} \\ \frac{\zeta_c}{c} - \sigma_{ij} \nu_c & \text{if } \frac{\zeta_c}{c} < \sigma_{ij} < \frac{\zeta}{\nu_c} \\ +\infty & \text{if } \sigma_{ij} \geq \frac{\zeta}{\nu_c} \end{cases} \]

where \( \sigma_{ij} = \sqrt{\frac{1}{\pi} \sum_{l=2}^{\infty} (q_{il} - q_{jl})^2} \), and \( \{ f_l \}_{l=2}^{\infty} \) as in Theorem 2.

The abovementioned theorem calculates the value-at-risk of phase incoherence for pairs of power generators. One can also construct the risk profile of the entire network by stacking the risk values for all pairs into one vector as

\[ \mathcal{R}_\epsilon = \left( \mathcal{R}_\epsilon^{(1,2)}, \mathcal{R}_\epsilon^{(1,3)}, \ldots, \mathcal{R}_\epsilon^{(n-1,n)} \right)^T. \]

**C. Special Cases**

We conclude this section by considering the risk in two simplified scenarios for the communication network.

1) **Synchronous State-Feedback** (\( \tau = 0 \)): When there is no time-delay in the feedback control, contrary to (9), the risk formula for \( \tau = 0 \) admits a quite simple form that is worth reporting.

**Corollary 1:** The risk of steady-state phase incoherence \( \| \Delta \|_{\infty} = \lim_{t \to +\infty} |\theta_t^{(i)} - \theta_t^{(j)}| \) between generators \( i \) and \( j \) with synchronous (\( \tau = 0 \)) closed-loop control error dynamics (5) is given by (9) with

\[ \sigma_{ij} = \sqrt{\frac{1}{\pi} \sum_{l=2}^{\infty} (q_{il} - q_{jl})^2} \]

The abovementioned result illustrates how the risk of systemic events is associated with the power network properties, the two sources of disturbances, and the Laplacian eigenvalues of the feedback gain matrices for the simultaneously diagonalizable type of control that concerns the present work.

2) **Noise-Free Measurements** (\( \epsilon = 0 \)): When we neglect sensor imperfections, the remaining source of uncertainty is the exogenous input in the power network that models the load volatility, which is parameterized by \( \eta \). In this case, one also obtains a simple risk formula that follows directly from Theorems 2 and 3.

**Corollary 2:** The risk of steady-state phase incoherence \( \| \Delta \|_{\infty} = \lim_{t \to +\infty} |\theta_t^{(i)} - \theta_t^{(j)}| \) between generators \( i \) and \( j \) with perfect measurement data to control the error dynamics (5) is

\[ \mathcal{R}_\epsilon(y) = \inf \left\{ \delta > 0 : \mathbb{P}(\| y \| \leq U_{\delta}) < \epsilon \right\}. \]
given by (9) with
\[ \sigma_{ij} = \frac{\tau^{3/2} \eta}{J \sqrt{2\pi}} \left[ \sum_{l=1}^{n} (q_{il} - q_{jl})^2 f((s; k)_l) \right] \]
in which the spectral function \( f \) is evaluated as in Theorem 2.

VI. FUNDAMENTAL LIMITS AND TRADEOFFS

The value-at-risk analysis reveals the vulnerability of WAC polices in the presence of time-delay and corrupted measurements. This section explores these limitations separately, establishes universal design bounds in mitigating the risk of power network phase incoherence, and draws remarks on their combinatorial effect.

A. Delay-Induced Limitations

The emergence of delay-induced fundamental lower limits of systemic risk has been previously reported in the context of multiagent systems in [30] and [31]. We show that a nontrivial lower bound on the steady-state variance in the presence of time-delay and exogenous noise imposes a lower limit on the networked control law to mitigate the risk of systemic events. For the exposition of this result, we assume noiseless measurement, i.e., \( \xi = 0 \). Our analysis of the spectral function \( f \) that is conducted in Appendix C helps us identify the existence of a lower bound for \( f(s_l; k) \) for fixed \( s = s_l \) \( (d\tau, \lambda_l \tau^2) \), where \( \lambda_l \) is the \( l \)th eigenvalue of \( L \). Then, let us define
\[ f_l := \min_{(s_l; k_l) \in \mathcal{U}_{\lambda_l = 2} W_r} f((s_l; k)_{l}). \]
The following result highlights how pairwise standard deviation is always lower bounded by the load uncertainty parameter and the time-delay.

**Proposition 1:** For any pair of generators, the variance of phase differences is lower bounded by
\[ \sigma_{ij} \geq \sigma_\ast := \frac{\tau^{3/2} \eta}{J \sqrt{2\pi}} \min_{(s_l; k_l) \in \mathcal{U}_{\lambda_l = 2} W_r} \left( \sum_{l=1}^{n} (q_{il} - q_{jl})^2 f_l \right). \]

B. Noise-Induced Limitations

The measurement noise also imposes hard limits. This limit is independent of time-delay and can be deduced from the form of \( \sigma_{ij} \) in Theorem 2. It can be shown that the pair standard deviation \( \sigma_{ij} \) is a function of eigenvalues of feedback matrices \( K \) and \( \mu \) and that \( \{ \mu_l, \kappa_l \}_{l=2}^{n} \) can be optimized to a global minimum, which depends on the basic characteristics of the power network and noise parameters. This suggests very strong evidence of hard limitation on the feedback control strategy.\(^5\)

The existence of time-delay and measurement noise in our model results in severe limitations on WAC policies, which can be studied via function \( f \) from Theorem 2. Such analysis is algebraically tedious and beyond the scope of this work. Nevertheless, it is straightforward to conclude that the existence of minimum values of \( \sigma_{ij} \) characterizes limits on risk-aware control design. We provide the following result on lower bound of the risk measure for the class of simultaneously diagonalizable feedback controllers.

**Corollary 3:** Given systemic set parameters \( \zeta \) and \( c \), and the acceptance level \( \epsilon \in (0, 1) \), one of the following statements must hold.

1) When \( \sigma_\ast \leq \frac{\zeta}{\zeta - \epsilon^2} \), the gain matrices \( M \) and \( K \) can be tuned to make \( \mathcal{R}_L \) arbitrarily small.

2) When \( \frac{\zeta}{\zeta - \epsilon^2} < \sigma_\ast < \frac{\zeta}{\zeta - \epsilon^2} \), the least achievable risk value cannot be reduced beyond
\[ \mathcal{R}_L \geq \left( \sigma_\ast \epsilon \mu - \zeta \right) \left( \zeta - \sigma_\ast \mu \epsilon \right) \]
for \( r = \frac{n(n-1)}{2} \). \( \mathcal{R}_L \)

3) When \( \sigma_\ast \geq \frac{\zeta}{\zeta - \epsilon^2} \), any choice of \( M \) and \( K \) results in infinite risk of phase incoherence.

C. Limits and Tradeoffs in Consensus WAC

A particular type of state feedback controller that satisfies Assumption 3 is when \( M = \mu L \) and \( K = \kappa L \), where \( L \) is the power grid Laplacian matrix, and \( \kappa > 0 \) and \( \mu > 0 \) are scaling factors. This type of feedback controller obtains a consensus structure since \( M \) and \( K \) are Laplacian matrices. The sensor network is now a consensus network that applies time-delayed feedback control to the grid. The connectivity of the network can be quantified via the notion of the effective resistance \( \Xi \) [46] and its convenient spectral form that for our initial power network graph reads \( \Xi_L = \sum_{l=2}^{n} \lambda_l^{-1} \). It follows that
\[ \Xi_L = \sum_{l=2}^{n} \frac{1}{\kappa_l} = \sum_{l=2}^{n} \frac{1}{\mu_l} \sum_{l=2}^{n} \frac{1}{\kappa_l} \]
where \( \Xi_M \) and \( \Xi_K \) are the corresponding effective resistances of phase and frequency consensus control of the power network. It is known that the stronger the interconnectivity of a graph, the smaller its effective resistance. The stability region \( \bigcup_{l=1}^{n} \mathcal{W}_r \) introduced in Section IV imposes lower bounds on both \( \Xi_K \) and \( \Xi_M \) that we can express as follows. Let \( \lambda_n = \|L\| \) be the maximum Laplacian eigenvalue of the underlying graph and define the set \( Q = \{\mu, \kappa \} \subseteq \mathbb{R}^2 \|\| \|L\| \|\mu \\| \|\kappa \| \kappa^2 \|\|L\| \kappa \tau \} \subset \partial \{\cup_{l=1}^{n} \mathcal{W}_r \} \). The effective resistances of \( K \) and \( M \) are lower bounded by
\[ \Xi_K = \sum_{l=2}^{n} \frac{1}{\kappa_l} > \max \left\{ \kappa \left( \kappa, \mu \right) \in Q \right\} \|L\| \]
\[ \Xi_M = \sum_{l=2}^{n} \frac{1}{\mu_l} > \max \left\{ \mu \left( \kappa, \mu \right) \in Q \right\} \|L\|. \]
Contrary to the lower limits of the systemic risk [30], [31], the abovementioned limitations on the graph structure of WACs are only due to the time-delay. At this point, the risk of phase incoherence and the effective resistance are restricted by independent factors. Following arguments in [30] and [31], one can calculate
a common limit for the product of the systemic risk and the effective resistance. Then, a tradeoff relation can be derived similarly to risk/connectivity tradeoff conditions detailed in [30] and [31] for the first- or second-order consensus dynamics. In particular, given systemic set parameters ζ and c, and the acceptance level ε ∈ (0, 1), there exists a common limit for the product of the systemic risk and the effective resistance

\[ 9 R_t \cdot \sqrt{\Xi_K + \Xi_M} \geq 1, \Omega \]

where \( r = \frac{n(n-1)}{2} \), and \( \Omega > 0 \) is a universal constant that depends on the grid properties, time-delay, and the intensities \( \iota \) and \( \eta \). The abovementioned remark asserts that it is impossible to design a feedback control that simultaneously minimizes the risk of systemic events and maximizes the networked-control connectivity, which are measured by \( \Xi_K \) and \( \Xi_M \), beyond a specific threshold. It can be shown that the total resistive power loss for the linearized model \( 5 \) depends directly to the network parameters as well as \( \Xi_K \) and \( \Xi_M \) [32], [33]. A detailed discussion of such tradeoffs are beyond the scopes of this article as in this article we only consider transmission lines with purely imaginary admittance, i.e., zero resistive losses.

VII. CASE STUDIES

In this section, we consider two example structures, for which their stability are widely investigated in the literature of power network systems, to illustrate the effectiveness of the proposed risk measure. The first example to be studied is a two-machine structure, which is usually addressed in literature as suppressed multimachine networks or single-machine infinity bus systems [47]. The second case involves risk-aware analysis and synthesis in the IEEE-39 standard [47], [48].

A. Two-Machine System

Two synchronous generators are connected by a pure reactance \( X = 0.3 \) per unit, as shown in Fig. 3. The generators’ transient reactances are \( X_{d1} = 0.16 \) per unit and \( X_{d2} = 0.20 \) per unit, respectively. Furthermore, the generators’ inertia is set to \( J = 2(MJ)/(MVA) \) and the damping torque \( \beta = 0.15 \). The system is operating in the steady-state with \( E_1 = 1.2 \) and \( E_2 = 2 \) per unit, at the equilibrium point \((0,0) \) \( \in \mathbb{S} \). In this case, the open-loop linearized swing equations read

\[
\begin{align*}
2 \dot{\theta}_1^{(1)} &= -0.15 \dot{\theta}_1^{(1)} + 1.584 (\dot{\theta}_2^{(2)} - \dot{\theta}_1^{(1)}) + \text{distr} \Phi_1 \\
2 \dot{\theta}_1^{(2)} &= -0.15 \dot{\theta}_2^{(2)} + 1.584 (\dot{\theta}_1^{(1)} - \dot{\theta}_2^{(2)}) + \text{distr} \Phi_2.
\end{align*}
\]

The load volatility is modeled via white noise with standard deviation \( \eta = 0.7 \). Within our framework, let us define

\( \mathcal{L} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \)

and express (13) in the form of (3) with

\[
\begin{bmatrix}
0_{2 \times 2} & I_2 \\
-0.792 \mathcal{L} & -0.757 I_2
\end{bmatrix}, \quad
K = \begin{bmatrix}
0_{2 \times 2} & 0_{2 \times 2} \\
-\mu \mathcal{L} & -\kappa \mathcal{L}
\end{bmatrix}, \quad
H = \begin{bmatrix}
0_{2 \times 2} & 0_{2 \times 2} \\
0.35 I_2 & -\mu \mathcal{L} & -\kappa \mathcal{L}
\end{bmatrix}.
\]

Parameter \( \iota > 0 \) represents the standard deviation of the sensor error, and \( \mu \) and \( \kappa \) are the phase and the frequency feedback control gains, respectively. The phase incoherence is defined as \( \theta_i := \theta_i^{(1)} - \theta_i^{(2)} \forall t > 0 \). In the open-loop architecture, \( \theta \) converges to normal distribution with zero mean and standard deviation 1.0155.

Stability analysis: The stability bounds of the closed-loop system are determined from Theorem 1. Our control law attains, by design, \( \kappa_1 = \mu_1 = 0 \), and \( \kappa_2 = \kappa \) and \( \mu_2 = \mu \). This means that \( \mathcal{C}_2 \) is never satisfied. Theorem 1 asserts stability of the closed-loop system if and only if condition \( \mathcal{C}_1 \cup \mathcal{C}_3 \) holds true. Since condition \( \mathcal{C}_3 \) is always true, we are looking for parameters \( \kappa \) and \( \mu \) so that \( \mathcal{C}_3 \) is satisfied, i.e., for fixed \( \tau > 0 \)

\[
\left\{(\mu, \kappa) \in \mathbb{R}^2 : (0.075 \tau, 1.5840 \tau^2, \mu \tau^2, \kappa \tau) \in \cup_{\tau=1}^{3} \mathbb{W}_\tau\right\}.
\]

The stability set is illustrated in Fig. 4.

Next, we consider a few scenarios as an application of Theorem 2 and Corollaries 1 and 2. The results are all illustrated in Fig. 5.

\( \tau = 0 \) Here, we assume that WAC sensors have no time-delay, but only corrupted measurements as additive white noise with \( \iota = 0.5 \). We leverage Corollary 1 as follows: First, we attempt to control the system using phase and frequency state feedback. The dependence of steady-state standard deviation concerning \( \mu \) and \( \kappa \) gains is illustrated in Fig. 5(a). We observe that phase state feedback performs very poorly, and it only manages to decrease the standard deviation by 5.55% (from 1.0155 to 0.9591). By increasing \( \mu \) beyond 0.87, the closed-loop system performs worse than the open-loop system. On the other hand, the frequency state feedback decreases the initial standard deviation by 65.3%. Finally, the joint phase and frequency state feedback achieve a decrease by 68.66%, as illustrated in Fig. 5(b). Solution realizations are provided in Fig. 5(c) for the visual inspection of corresponding dynamics.

\( \iota = 0 \) The second round of simulations regards perfect measurement recordings but lumped time-delay parameter as \( \tau = 0.1 \). The narrative is similar to the previous case. Frequency state feedback control outperforms the phase state feedback control significantly, and the joint control outperforms both. Results are illustrated in Fig. 5(d)–(f). The maximum decrease
from the open-loop system is 6.58% for the phase state feedback control, and 92.45%, and 96.96% for the frequency state feedback control and the joint control.

Risk-aware synthesis: Assume the systemic sets $U_d = \left( \frac{\pi}{3}, \frac{1+\delta}{3+2+\delta}, +\infty \right)$. The corresponding risk of phase incoherence is zero if the absolute value of phase difference is below $2\frac{\pi}{3}$, nonnegative if it is up to $\frac{\pi}{3}$ and infinite, otherwise. The risk is calculated with the acceptance level $\varepsilon = 0.1$, i.e., we evaluate the risk with the probability of staying outside the systemic set, which is at least 90%. Fig. 5(a) and (d) illustrates cutoff values of these systemic sets. Following the risk formulae in Corollary 1 or Corollary 2, we can extract the feedback gain margins (for the individual control scenario) that ensure a zero, positive, or infinite risk. We observe that, in general, the phase state feedback control fails to mitigate the risk of our systemic event at the imposed level of confidence. On the other hand, frequency and joint control perform considerably better. For example, in Fig. 5(a), we see that frequency control achieves zero risk for gain in the interval (0.41,2.76), and similarly for case of $\tau = 0$.

B. IEEE-39 Standard

Also known as the ten-machine New-England Power System, the IEEE-39 standard is a small-scale grid with ten generators and 39 buses. It is initially introduced in [48], and its parameters are taken from [47]. The system’s diagram is illustrated in Fig. 6(a). The equivalent system created using the network reduction technique is illustrated in Fig. 6(b), and it presents an exact reproduction of the transfer impedances of the reduced system as seen from its generator buses. Each nondiagonal element represents the admittance between each pair of generator buses. The bus admittance matrix of the IEEE-39 bus system is reduced to only include the generator nodes in the network. Therefore, all of the off-diagonal elements represent the effective admittance between each generator node in the network. Thus, the width of every edge is proportional to the magnitude of admittance between connected generators.

We assume that the system is operating in a steady-state equilibrium $(\theta^*, \omega^*) = (0, 0) \in \mathbb{S}$ with some nominal voltages $\{E_{i1}\}_{i=1}^{10}$, and the admittance matrix $Y$ is derived after reduction with system parameters from [47]. All generators have the same damping and inertia parameters as in Example 1, and load volatility has a standard deviation $\eta = 1.1$. The sensor parameters are with $\varepsilon = 0.2$ and $\tau = 0.03$. We will examine the feasibility of synthesizing simultaneously diagonalizable controllers mitigating the risk of phase incoherence below deviation for various $\zeta$ limits of the systemic set with a probability of at least 95%. Following steps of Theorems 1–3, we calculate the optimal gains with nonzero eigenvalue of $L$, which is presented in Table I. For $Q$, the eigenvector matrix of $L$ (aligned with eigenvalues as in Table I), the phase and frequency optimal controllers are $M^* = Q \text{Diag}\{\mu_j^*\} Q^T$ and $K^* = Q \text{Diag}\{\kappa_j^*\} Q^T$, respectively.

Results are depicted in Fig. 7. The upper left plot illustrates the extreme risk values with and without optimal controllers $K^*$ and $M^*$ for various systemic set margins. We observe that as we decrease parameter $\zeta$ of the systemic set, the risk increases. In the open-loop case, all pairs of generators fall into incoherence concerning $R_{0.05}$ for values of roughly $\zeta = 0.7$ and below. For the case of optimized control, we do not see the divergence of any pair of generators for values of $\zeta$ larger than 0.58, whereas even for ultimately conservative strict systemic sets, some pairs are still risk free. In the remaining three plots, we illustrate
Fig. 5. Risk-aware control for the two-machines System from Example 1. The first row depicts the case where adding measurement noise ($\eta = 0.3$) to sensors, but the information is processed without time-delay. The second row illustrates the case where sensors have no measurement noise but lagged information processing with $\tau = 0.1$. We compare single-phase and frequency controls with joint phase and frequency, where the latter outperforms the former ones. (a) Separate $\omega$ and $\theta$ controls. (b) Joint ($\omega, \theta$) control. (c) Solution realizations. (d) Separate $\omega$ and $\theta$ controls. (e) Joint ($\omega, \theta$) control. (f) Solution realizations.

Fig. 6. IEEE-39 Test-System for Example 2. (a) Single line diagram. (b) Kron reduced graph. The systemic risk distribution over all pairs of generators. We observe that the optimal controllers can mitigate the risk of phase incoherence efficiently. We also note that the minimal risk controllers are following Corollary 3 when it comes to relative values of minimal standard deviation and systemic set. The absence of open-loop risk values for the first two plots indicates infinite risk in all of its pairs of generators’ phases. On the contrary, the absence of optimal risk values in the bottom right plot indicates that the optimal controllers achieve risk-free fluctuations for the systemic set with $\zeta = \pi/4$, also indicated in the first subplot of Fig. 7.

Fig. 7. Risk measures on Example 2. (a) Subplot of maximum and minimum risk concerning systemic set parameters. (b)–(d) Subplots of steady-state risk distribution over all 45 pairs of generators for the optimal case $\zeta = \pi/8$ and $\zeta = \pi/6$, and for the open-loop case at $\zeta = \pi/4$, respectively.
simplifies risk analysis and reduces the time complexity to that of decomposing a $n \times n$ symmetric matrix, which is $O(n^3)$.

Assumptions 1 and 3 have facilitated our analysis to obtain several analytical formulae. Both lumped time-delay $\tau > 0$ and simultaneous diagonalization helped us fully classify the stability region of the resulting closed-loop system as well as deriving closed-form expressions for the risk measure, although with excessively complicated formulae. One can relax either or both conditions and explore generalized setups within the context of systemic risk by following the same analysis steps outlined in this work. The price to pay is that it will be impossible to obtain a closed form and the exact representation of stability regions and risk expressions. General feedback matrices and heterogeneous time-delays typically are handled with standard textbook techniques [42] that provide sufficient and conservative conditions. Then, the risk measure can probably be approximated with an upper bound. Despite this drawback, generalized universal limitations and tradeoffs can still be derived along the lines described in this article.

Appendix A

Proofs of Technical Results

A. Theorem 1

Given the structure of $S$ regarding the phase of generators, we explore conditions for convergence of the zero-input system (5) with respect to $(q_n \theta, 0)$ for some $\theta \in \mathbb{R}$. Based on Assumptions 2 and 3, it is convenient to work with the transformed coordinates

$$\bar{\eta}_t := Q^T \eta_t$$

The desirable type of convergence is equivalent to $\bar{\eta}_t^{(1)} \to \theta_s$ and $\bar{\eta}_t^{(j)} \to 0$ for $j > 1$, whereas $\bar{\omega}_t^{(j)} \to 0$ for all $j \geq 1$. The transformed dynamics

$$\begin{bmatrix} \bar{v}_t \\ \bar{\omega}_t \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -\Lambda_L & -\Lambda_D \end{bmatrix} \begin{bmatrix} \bar{v}_t \\ \bar{\omega}_t \end{bmatrix} + \begin{bmatrix} 0_n \\ -\Lambda_M -\Lambda_K \end{bmatrix} \begin{bmatrix} \bar{v}_{t-\tau} \\ \bar{\omega}_{t-\tau} \end{bmatrix}$$

are fully decomposed and facilitate separated analysis for $j = 1$ and $j > 1$. By linearity, we seek solutions of the form $e^{\eta t}$ with $\eta \in \mathbb{C}$. Condition $\text{Re}\{\eta\} = 0$ signifies the onset of marginal stability and, in most cases, instability of the zero solution. Locating such solutions is accomplished with the study of the characteristic equation $c_j(\eta) = 0$ of the $j$th pair $(\bar{\eta}_t^{(j)}, \bar{\omega}_t^{(j)})$, that reads

$$c_j(\eta) = \eta^2 - d_j + \kappa_j \eta e^{-\eta \tau} + \lambda_j + \mu_j e^{-\eta \tau} = 0 \quad (14)$$

Analytical results on the solutions of $c_j(\eta) = 0$ for $\tau \geq 0$ can be found in standard textbooks of the subject, such as [49], from where we draw results to establish our proof. Heuristically speaking, $c_j(\eta) = 0$ may have zero, one, or two distinct imaginary roots. The stability of the time-delayed system dynamics is then associated with the stability properties for $\tau = 0$. In the case of no imaginary roots for $c_j(\eta) = 0$, the stability properties of decomposed subsystem are retained for all time-delays, e.g., if it is unstable at $\tau = 0$, the system will remain unstable for all $\tau \geq 0$. When $c_j(\eta)$ attains precisely one imaginary roots,
the instability at \( \tau = 0 \) will be retained for all \( \tau \geq 0 \). Stability at \( \tau = 0 \) will switch only once at a critical time-delay, beyond which the system is unstable. For the two distinct imaginary roots of \( c_j(\eta) = 0 \), there will be a finite number of switches between instability and stability before the termination into the final instability. For more details, we refer to [49, Th. 3.1 in Sec. 3.3]. Since the requirements for \( j = 1 \) differ from \( j > 1 \), we distinguish between the two cases. Finally, all results will be expressed after suppressing time-delay \( \tau > 0 \) into the rest parameters for convenience

\[
\dot{d} \leftrightarrow \dot{\tau}, \lambda_j \leftrightarrow \lambda_j \tau^2, \mu_j \leftrightarrow \mu_j \tau^2, \kappa_j \leftrightarrow \kappa_j \tau. \tag{15}
\]

**Case \( j = 1 \):** Here, we have \( \lambda_1 = 0 \). The subsystem \( j = 1 \) becomes decoupled in dynamics of \( \varpi_t^{(1)} \)

\[
\dot{\varpi}_t^{(1)} = \varpi_t^{(1)}, \quad \dot{\varpi}_t^{(1)} = -\ddot{\varpi}_t^{(1)} - \kappa_1 \varpi_{t-\tau}^{(1)} - \mu_1 \varpi_{t-\tau}. \tag{15}
\]

When \( \mu_1 \neq 0 \), the convergence of \( \varpi_t^{(1)} \) in \( \mathbb{R} \) is associated with the asymptotic stability of \( \varpi_t^{(1)} \) to zero. If \( \varpi_t^{(1)} \to 0 \), this will occur exponentially fast so that \( \varpi_t^{(1)} \to 0 \) as \( \tau \to \infty \). Convergence of \( \varpi_t^{(1)} \) is achieved if and only if \( -\ddot{d} < \kappa_1 < \ddot{d} \), or \( |\kappa_1| > \ddot{d} \) with \( \tau \in (\delta_k, \infty) \), which is critical \( \tau_{s;k} \) as in (18). In addition, for \( \kappa_1 = 0 \), we can completely solve the system and calculate the converging point \( \theta \) as in the statement of Theorem 1. Inducing transformation (15), the stability region is equivalent to the set \( \mathbb{W}_2(s;k) \), evaluated at \((\ddot{d}, 0; \kappa_1)\).

When \( \mu_1 \neq 0 \), \( c_1(\eta) = 0 \) can have only one imaginary root (see the proof in [49, Th. 3.1]). The stability at \( \tau = 0 \) implies the stability for the range of \( \tau \in [0, \tau_{s;k}^{0,1}) \), where \( \tau_{s;k}^{0,1} \) as in (19). After time-delay suppression with the new coordinates (15), the range of parameters is covered by \( \mathbb{W}_2(s;k) \), which is evaluated at \((\ddot{d}, 0; \mu_1 \tau^2, \kappa_1 \tau)\).

**Case \( j > 1 \):** Here, we have \( \lambda_j > 0 \). The subsystem \( j = 2, \ldots, n \) defined by \( (\varpi_t^{(j)}, \varpi_t^{(j)}) \) satisfies

\[
\dot{\varpi}_t^{(j)} = \varpi_t^{(j)}, \quad \dot{\varpi}_t^{(j)} = -\ddot{\varpi}_t^{(j)} - \kappa_j \varpi_{t-\tau}^{(j)} - \lambda_j \varpi_t^{(j)} - \mu_j \varpi_{t-\tau}^{(j)}. \tag{15}
\]

For these cases \( j > 1 \), we require asymptotic stability with respect to zero. The first step is to verify this for \( \tau = 0 \). Applying the Hurwitz criterion, the stability holds if and only if [49]

\[
\kappa_j + \ddot{d} > 0 \quad \text{and} \quad \lambda_j + \mu_j \geq 0. \tag{16}
\]

1. The equation \( c_j(\eta) = 0 \) attains no imaginary roots if and only if \( \lambda_j^2 > \kappa_j^2 \) and \( \lambda_j^2 + 2\lambda_j \ddot{d}^2 < 2\sqrt{\lambda_j^2 - \mu_j^2}. \) The stability is equivalent to the transformed coordinates (15), which belong to the set \( \mathbb{W}_2(s;k) \).

2. The equation \( c_j(\eta) = 0 \) attains exactly one imaginary root if and only if \( \lambda_j^2 \leq \mu_j^2 \). The asymptotic stability is guaranteed for a finite range of time-delay \( \tau \geq 0 \) if and only if the condition (16) holds and the maximum allowed time-delay is \( \tau_{s;k}^{0,1} \), as defined in (19). Implementing (15), the parameter region that fulfills this condition belongs to \( \mathbb{W}_2(s;k) \).

3. The equation \( c_j(\eta) = 0 \) attains two distinct roots with the initial stability at \( \tau = 0 \). As time-delay is increasing, there will be a finite number of stability switches before the eventual instability. The stability region for time-delay \( \tau \) is determined with set \( \mathbb{T}_{k,d} \) as in (20). The rest of the parameter values ought to be the members of the set \( \mathbb{W}_2(s;k) \) (see [49, Th. 3.1]).

For this we revisit the dynamics \( \varpi_t \) and \( \varpi_t \), and observe that for \( \mu_1 \neq 0 \), all other stability conditions drive \( (\varpi_1^{(1)}, \varpi_1^{(1)}) \) to zero for all \( j = 1, \ldots, n \). This implies that for \( \mu_1 \neq 0 \), stability conditions imply \( \theta = 0 \). The case \( \mu_1 = 0 \) offers a different equilibrium for \( \varpi_1^{(1)} \). Observe that \( \varpi_1^{(1)} = \varpi_1^{(1)} \) satisfies

\[
\frac{d}{dt} \varpi_1^{(1)} = -\ddot{\varpi}_1^{(1)} - \kappa_1 \varpi_{t-\tau}^{(1)},
\]

that we rewrite as

\[
\varpi_1^{(1)} = -(\ddot{d} + \kappa_1) \varpi_1^{(1)} + \kappa_1 \int_{t-\tau}^t \varpi_1^{(1)}(s) \, ds.
\]

Equivalently

\[
\frac{d}{dt} \left[ \varpi_1^{(1)} - \kappa_1 \int_{t-\tau}^t \varpi_1^{(1)}(s) \, ds \right] = -(\ddot{d} + \kappa_1) \frac{d}{dt} \varpi_1^{(1)}.
\]

Given the asymptotic behavior of \( \varpi_1^{(1)} \), we can integrate the last equation and apply the initial conditions to conclude in the form of \( \theta = \frac{1}{\sqrt{n}} \lim_{t \to \infty} \varpi_1^{(1)} \).
Now, since \( B_n 1_n = 0 \), \( B_n Q \) is an \( r \times n \) matrix with zero first column. This will, in turn, nullify the first element of diagonal submatrices \( \hat{\Phi}_{\partial \theta} \) and \( \hat{\Phi}_{\partial \varpi} \), which refer to possibly marginally stable projections of \( \theta_1 \) and \( \varpi_1 \). The remaining dynamics are, in the view of Theorem 1 exponentially stable, validating the claim on \( C_{T_i} \). Consequently, the nonconstant elements of \( \Sigma_i \) are also exponentially stable, yielding integrable integrands and well posedness of the covariance matrix as \( t \to +\infty \). Now, the integrand of \( \Sigma_i \) equals

\[
\begin{align*}
\mathcal{H} & = \begin{bmatrix} 0_n \\ 0_n \end{bmatrix} \\
H & = \begin{bmatrix} 0_n \\ 0_n \\ \frac{3}{2} I_2 + \mathcal{J} (MM^T + KK^T) \end{bmatrix},
\end{align*}
\]

The integrand of \( \Sigma_i \) equals

\[
B_n Q \hat{\Phi}_{\partial \varpi} (s) Q^T H H^T Q \hat{\Phi}_{\partial \varpi} (s) Q^T B_n^T =
\]

\[
= \text{diag} \left\{ \frac{\eta_{ij}^2}{J^2} + \tau^2 (\mu^2 + \kappa^2) \right\} B_n Q \hat{\Phi}_{\partial \varpi} (s) Q^T B_n^T.
\]

Using the similar argumentation, the first element of diagonal matrix has no contribution as it is nullified by \( B_n Q \). Eventually, \( \lim_{t \to +\infty} \Sigma_i \) is equal to

\[
\text{diag} \left\{ \frac{\eta_{ij}^2}{J^2} + \tau^2 (\mu^2 + \kappa^2) \right\} \int_0^\infty B_n Q \hat{\Phi}_{\partial \varpi} (s) Q^T B_n^T ds.
\]

Now, let us note that \( \hat{\Phi}_{\partial \varpi} (t) = \text{diag} \{ \hat{\Phi}_{\partial \varpi}^{(ij)} (t) \} \) is a transition matrix of the decomposed dynamics. In addition to being a diagonal matrix, all but the first nonzero elements are exponentially stable. From Parseval’s theorem,

\[
\int_0^\infty \left[ \hat{\Phi}_{\partial \varpi}^{(ij)} (t) \right]^2 dt = \frac{1}{2\pi} \int_\mathbb{R} |c_j(i\gamma)|^2 d\gamma
\]

for \( c_j(\eta) \) is the characteristic equation of the \( j \)th decomposed subsystem from (14) and \( i^2 = -1 \). Tedium algebra with the transformation (15) yields

\[
\int_0^\infty \left[ \hat{\Phi}_{\partial \varpi}^{(ij)} (t) \right]^2 dt = \frac{\tau^3}{2\pi} f \left( \hat{d} \tau, \lambda, \tau^2; \mu, \tau^2, \kappa \right)
\]

for \( j = 2, \ldots, n \), and function \( f(s; k) \) is defined in (21). Then, the result follows with simple algebra. 

**1) Theorem 3:** Let us denote by \( \hat{\varphi}^{(ij)} \) the element of \( \hat{\varphi}_{ij} \) that describes the steady-state phase difference between the \( i \)th and \( j \)th generators. By virtue of Theorem 2, \( \hat{\varphi}^{(ij)} \) is normally distributed with zero mean and variance

\[
\sigma_{\varphi}^2 = \frac{1}{2\pi} \sum_{k=2}^n (e_i - e_j)^T B_n Q \text{diag} \{ f \} Q^T B_n^T (e_i - e_j)
\]

We calculate the risk of phase incoherence at \( \varphi^{(ij)} \) concerning \( \{ U_\delta \}_{\delta > 0} \) as follows: Given \( \varepsilon \in (0, 1) \) and \( \delta > 0 \), we write

\[
\mathbb{P} \left( |\varphi^{(ij)}| \in U_\delta \right) < \varepsilon \iff \mathbb{P} \left( \frac{1 + \delta}{\varepsilon} < \frac{1 + \delta}{\varepsilon} < |\varphi^{(ij)}| < \varepsilon \right)
\]

which is equivalent to

\[
\mathbb{P} \left( \frac{1}{1 + \delta} < \frac{1 + \delta}{\varepsilon} < \frac{1 + \delta}{\varepsilon} \right) \geq 1 - \varepsilon.
\]

Clearly, \( \mathcal{R}^{(ij)} = \mathcal{R}(\varphi^{(ij)}) \) is equal to \( \delta > 0 \) where

\[
\inf \left\{ \delta > 0 : \frac{1}{\sqrt{2\pi} \sigma_{\varphi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \delta^2} dt > 1 - \varepsilon \right\}
\]

equivalently,

\[
\inf \left\{ \delta > 0 : \frac{1}{\sqrt{2\pi} \sigma_{\varphi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \delta^2} dt > 1 - \varepsilon \right\}. \quad (17)
\]

Let us denote by \( \nu_\varepsilon > 0 \) the solution of

\[
\int_{-\infty}^{\nu_\varepsilon} e^{-\frac{1}{2} \delta^2} dt = \sqrt{2\pi}(1 - \varepsilon)
\]

a simple monotonicity argument suffices to explain the following three branches of \( \mathcal{R}^{(ij)} \).

If \( \frac{1}{\sigma_{\varphi}} \varepsilon \geq \nu_\varepsilon \), then clearly the infimum \( \delta > 0 \) that satisfies (17) is \( \delta = 0 \).

If \( \frac{1}{\sigma_{\varphi}} \varepsilon \leq \nu_\varepsilon \), we have the other extreme, where no finite value of \( \delta \) satisfies (17). In this case, we have \( \delta = +\infty \).

If \( \frac{1}{\sigma_{\varphi}} \varepsilon < \nu_\varepsilon \), there is unique \( \delta^* > 0 \) such that

\[
\frac{1 + \delta^*}{\varepsilon} \sigma_{\varphi} = \nu_\varepsilon.
\]

The result follows after solving for \( \delta^* \). 

**2) Proposition 1:** Recall from the properties of function \( f(s; k) \in \mathcal{U}_{1=1} \mathcal{W}_1 \) discussed in Appendix C, which function \( f(s; k) \) is positive and finite. In addition, \( f \) diverges on the boundary of \( \mathcal{U}_{1=1} \mathcal{W}_1 \). Consequently, \( f \) must attain a global minimum in the interior of \( \mathcal{U}_{1=1} \mathcal{W}_1 \). With a little abuse of notation, the global minimum in the interior of \( \mathcal{U}_{1=1} \mathcal{W}_1 \) will also reveal \( f_1(k_1, k_2) := f(d\tau, \lambda_1 \tau; k_1 \tau^2, k_2 \tau) \), as well as every finite (and

---

**TABLE II**

| Parameter Sets of Stability Area |
|----------------------------------|
| \( \mathcal{W}_0(s; k) = \{ s \in \mathbb{R}^+ \times k \in \mathbb{R}^3 : s_2 = k_1 = 0, \{ |k_2| < s_1 \} \cup \{ k_2 > s_1, \sqrt{k_2^2 - s_1^2} < \pi - \arccos(s_1/\sqrt{k_2^2 - s_1^2}) \} \} \) |
| \( \mathcal{W}_1(s; k) = \{ s \in \mathbb{R}^+ \times k \in \mathbb{R}^3 : s_2^2 > k_1^2, k_2 + s_1 > 0, k_1 + s_2 > 0, k_2^2 + 2s_2 - s_1^2 \leq 2\sqrt{s_2^2 - k_1^2} \} \) |
| \( \mathcal{W}_2(s; k) = \{ s \in \mathbb{R}^+ \times k \in \mathbb{R}^3 : s_2^2 \leq k_1^2, k_2 + s_1 > 0, k_1 + s_2 > 0, \gamma_\ell(s; k) < \varphi_\ell(s; k) \} \) |
| \( \mathcal{W}_3(s; k) = \{ s \in \mathbb{R}^+ \times k \in \mathbb{R}^3 : s_2^2 > k_1^2, k_2 + s_1 > 0, k_1 + s_2 > 0, k_2^2 + 2s_2 - s_1^2 > 2\sqrt{s_2^2 - k_1^2}, (\gamma_\ell(s; k), \varphi_\ell(s; k)) \in \mathcal{F}_{s; k} \} \) |
positive) weighted sum of functions \( \{f_i(k_1, k_2)\}_{i=0}^{n-2} \). Consequently, \( \sigma_{ij}^2 \) will attain a minimum over all \( i \neq j \).

**3) Corollary 3:** The proof follows immediately from Theorem 3 and Proposition 1.

### APPENDIX B

**CONSTRUCTION OF SETS \( \mathbb{W}_i \), \( i = 0, 1, 2, 3 \).**

For the exposition of Theorem 1, we will define the sets that constitute the stability area of dynamics. To this end it is helpful to introduce notation \( (s; k) \), for \( s = (s_1, s_2) \in \mathbb{R}_+^2 \) and \( k = (k_1, k_2) \in \mathbb{R}^2 \). Examples of their union set \( \mathbb{W} \) as function of \( k_1, k_2 \) is provided in Fig. 8. This way we separate the system quantities in power system (i.e., \( s_1 \) and \( s_2 \)) from the control quantities (i.e., \( k_1, k_2 \)). The first cutoff limit to be defined is

\[
\tau_{s;k} = \frac{1}{\sqrt{k_2^2 - s_1^2}} \arccos\left( \frac{s_1}{\sqrt{k_2^2 - s_1^2}} \right)
\]

whenever \( k_2^2 > s_1^2 \). Next, for \( \Delta_{s;k} = k_2^2 + 2s_2 - s_1^2 \), define \( \gamma_2^2 = \gamma_2^2(s; k) \) as

\[
\gamma_2^2 = \frac{1}{2} \left\{ \Delta_{s;k} \pm \sqrt{\Delta_{s;k}^2 - 4(s_2^2 - k_1^2)} \right\}.
\]

If \( s_2^2 \leq k_1^2 \), there is only one positive solution \( \gamma_+ > 0 \); otherwise, if \( s_2^2 > k_1^2 \), there are two positive solutions \( \gamma_+ > \gamma_- > 0 \). Whenever solutions are defined, the next quantity to be considered are the angles \( \phi_\pm = \phi_\pm(s; k) \) as trigonometric numbers for \( \Delta_{s;k} \) by

\[
\cos(\phi_\pm) = -\frac{s_1 k_1 \gamma_\pm + k_1 (s_2 - \gamma_\pm^2)}{k_2 \gamma_\pm^2 + k_1^2}
\]

\[
\sin(\phi_\pm) = \frac{s_1 k_1 \gamma_\pm - k_2 \gamma_\pm (s_2 - \gamma_\pm^2)}{k_2 \gamma_\pm^2 + k_1^2}.
\]

Next, the critical cutoffs

\[
\tau^{(l)}_+ = \frac{\phi_+ + 2l\pi}{\gamma_+}, \quad \tau^{(l)}_- = \frac{\phi_- + 2l\pi}{\gamma_-}
\]

for \( l = 0, 1, 2, \ldots \), are sorted as

\[
0 < \tau^{(0)}_+ < \tau^{(0)}_- < \tau^{(1)}_+ < \tau^{(1)}_- < \cdots < \tau^{(l-1)}_+ < \tau^{(l-1)}_- < \tau^{(l)}_+ < \tau^{(l)}_- < \tau^{(l+1)}_+ \]

for the minimum positive integer \( l^* \) that satisfies

\[
\frac{\gamma_-(\phi_+ + 2\pi) - \gamma_+ \phi_-}{2\pi(\gamma_+ - \gamma_-)} < l^* < \frac{(2\pi - \phi_-)\gamma_+ + \phi_- \gamma_-}{2\pi(\gamma_+ - \gamma_-)}.
\]

The assortment of critical cutoffs signifies the transition of system (3) from stability to instability and back to stability. Under additional assumptions (explained in Theorem 1), the system is asymptotically stable for \( \tau \in [0, \tau^{(0)}_+] \), unstable for \( \tau \in (\tau^{(0)}_+, \tau^{(0)}_-) \), and stable for \( \tau \in (\tau^{(0)}_-, \tau^{(l)}_-) \), and so forth, up until the last region of stability, that is when \( \tau \in (\tau^{(l-1)}_-, \tau^{(l)}_-) \) (see [49, p. 77]). The union of all stability regions constitute the set \( \mathcal{I} = \mathcal{I}(s; k) \)

\[
\mathcal{I} = \left[ 0, \tau^{(0)}_+ \right) \cup \bigcup_{l=1}^{l^*} (\tau^{(l)}_-, \tau^{(l+1)}_+) \right].
\]

After implementing the transformation (15), the set (20) is, in turn, transformed to set \( \mathcal{J} = \mathcal{J}_{s;k} \)

\[
\mathcal{J}_{s;k} = \{ \gamma_+ < \varphi_+ \} \cup \left\{ \bigcup_{l=1}^{l^*} \{ \varphi_- + 2(l-1)\pi \cap \varphi_+ + 2l\pi \} \right\}.
\]

Based on the aforementioned quantities, we can proceed to define the sets \( \mathcal{W}_i(s; k) \subset \mathbb{R}_+^2 \times \mathbb{R}^2 \) as in Table I.

**Proposition 2:** Let \( s = (s_1, s_2) \in \mathbb{R}_+^2 \) be arbitrary but fixed. Union \( \bigcup_{i=1}^3 \mathcal{W}_i \) parameterized by \( k = (k_1, k_2) \in \mathbb{R}^2 \) contains the origin, it is compact and connected.

**Proof:** First, observe that the origin belongs to \( \mathcal{W}_1 \).

**Compactness:** It is straightforward to show that given \( s_1, s_2 > 0 \), the set \( \mathcal{W}_1 \) is compact: \( |k_1| \leq s_2 \) and \( |k_2| \leq \sqrt{s_1^2 - 2s_2 + 2\sqrt{s_1^2 - s_2^2}} \). Set \( \mathcal{W}_2 \) is also compact: \( k_1 \geq -s_1 \) and \( k_1 \geq -s_2 \), and in the view of monotonicity of \( \gamma_+(k_1, k_2) \), condition \( \gamma_+ \leq \varphi_+ \in [0, 2\pi) \) provides a sharp bound that is always achieved when \( k_1 \) and/or \( k_2 \) are large enough. Finally, the compactness of \( \mathcal{W}_3 \) is proved using similar arguments: \( |k_1| \leq s_2 \) and \( |k_2| \geq -s_1 \). Now, as \( k_2 \) increases, \( \gamma_+ \) increases +∞ and \( \gamma_- \) vanishes monotonically. Elementary analysis reveals that for a fixed \( k_1 \), \( \lim_{k_2 \to +\infty} \varphi_+(k_1, k_2) \in [\pi, \frac{3\pi}{2}] \). Consequently, for any such fixed \( k_1 \) with \( (k_1, k_2) \in \mathcal{W}_3 \), the boundary \( \mathcal{W}_3 \) is achieved for \( l \geq 1 \) and finite \( k_2 \).

**Connectedness:** Clearly, \( \mathcal{W}_1 \cap \mathcal{W}_3 \neq \emptyset \). This concerns all points in \( \mathcal{W}_1 \) on the curve \( k_2^2 + 2s_2 - s_1^2 = 2\sqrt{s_1^2 - k_1^2} \) that satisfies the last condition of \( \mathcal{W}_3 \). Similarly, \( \mathcal{W}_1 \cap \mathcal{W}_2 \neq \emptyset \). This follows after setting \( k_1 = s_2 \), which necessarily yields \( \gamma_+(s_2, k_2) = 0 \) for \( (s_2, k_2) \in \mathcal{W}_1 \).

### APPENDIX C

**SPECTRAL FUNCTION \( f(s; k) \)**

Function \( f \), defined in (21) in the form of improper integral, plays a central role in this work. Its arguments are presented as \( (s; k) \), where \( s = (s_1, s_2) \in \mathbb{R}_+^2 \) correspond to the fixed parameters of nominal system and \( k = (k_1, k_2) \in \mathbb{R}^2 \) corresponds to controllable parameters. An example of its form as function of \( k_1, k_2 \) is provided in Fig. 9. Its domain is the union of sets \( \cup_{s=0}^{n-2} \mathcal{W}_i(s) \subset \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) and its range is \( \mathbb{R}_+ \).

**Proposition 3:** Function \( f \) attains the following properties.

1) For fixed \( s \in \mathbb{R}_+^2 \), \( s; k \in \text{dom} f \), \( f(s; k) \geq f \geq 0 \).

2) For an increasing sequence \( \{s_j\}_{j=1}^{\infty} \) with \( \lim_{j \to +\infty} s_j \to +\infty \), and \( (s_j, s_2; k) \subset \bigcup_{i=0}^3 \mathcal{W}_i \) for \( j \geq 1 \), we have \( \lim_{j \to +\infty} f(s_j, s_2; k) = 0 \).

Here, we have generic \( s_2, k_1 \), and \( k_2 \) instead of \( \lambda_j, \mu_j \), and \( \kappa_j \).
\[
f(s; k) = \int_{\mathbb{R}} \frac{1}{2} \left(2(s_1 k_2 - k_1) r^2 + s_2 k_1 \cos(r) - 2r(k_2 r^2 + s_1 k_1 - k_2 s_2) \sin(r) + r^4 + \left(s_1^2 + k_2^2 - 2s_2\right) r^2 + s_2^2 + k_1^2 \right) dr
\]

(21)

3) For an increasing sequence \( \{s^{(j)}_2\}_{j \geq 1} \) with \( \lim_{j \to +\infty} s^{(j)}_2 \to +\infty \), and \( (s_1, s^{(j)}_2; k) \in \bigcup_{j=1}^{3} \mathbb{W}_j \) for \( j \geq 1 \), we have \( \lim_{j \to +\infty} f(s_k, s^{(j)}_2; k) = 0 \).

**Proof:**

1) Recall that the denominator in the integrand in (21) is the square of magnitude of characteristic function. It is, therefore, an even and nonnegative function for every \( r \). It is not hard to see that the denominator is bounded as follows:

\[
r^4 + \alpha_3 |r|^2 + \alpha_2 r^2 + \alpha_1 |r| + \alpha_0 < \begin{cases} \beta_1 r + \alpha_0, & r \in (0, 1) \\ \beta_2 r^4, & r > 1 \end{cases}
\]

with \( \alpha_3 = 2k_2, \alpha_2 = 2|s_1 k_2 - k_1| + |s_2| + k_2^2 - 2s_2|, \alpha_1 = 2|s_1 k_1 - s_2 k_2|, \alpha_0 = s_2^2 + k_1^2, \beta_1 = \sum_{i=1}^{3} \alpha_i, \text{ and } \beta_2 = \sum_{i=0}^{3} \alpha_i \), all functions of \( s \) and \( k \). Consequently, \( f(s; k) \geq \int_{0}^{1} \frac{2\, dr}{\beta_1 r + \alpha_0} + \int_{1}^{\infty} \frac{2\, dr}{\beta_2 r^4} = \frac{2}{\beta_1} \ln \left(1 + \frac{\alpha_0}{\beta_1} + \frac{2}{3\beta_2} \right) > 0 \).

2) Denote by \( g_r(s; k) \) the integrand of (21). It can be shown that \( \frac{\partial}{\partial s} g_r (s; k) < 0 \) for \( s_1 > |k_1| - k_2 \). Choosing a large enough \( j^* \), to end up with sequence \( g^{(j)}(r) := g_r(s^{(j)}_1, s_2; k) \) for \( j > j^*(r) \), each of which is dominated by \( g_{j^*} \). Also, \( \lim_{j \to +\infty} g_{j^*}(r) = 0 \). The application of the dominated convergence theorem yields \( \lim_{j \to +\infty} f(s^{(j)}_1, s_2; k) = \lim_{j \to +\infty} \int_{0}^{\infty} g_{j^*}(r) \, dr = 0 \).

3) For any \( \varepsilon > 0 \), pick any \( M > \sqrt{\varepsilon \over 2\alpha} \) and choose \( j^* \) large enough such that \( s^{(j)}_2 > \max\left\{ \sqrt{\varepsilon \over 2\alpha} - k_1, M^2 - k_1 - k_2 \right\} \) for \( j > j^* \). Fix such \( j \) and observe that

\[
f(s_1, s^{(j)}_2; k) = 2 \int_{0}^{M} g(r) \, dr + 2\int_{M}^{\infty} g(r) \, dr < 2 \int_{0}^{M} g(r) \, dr + 2\int_{M}^{\infty} \frac{dr}{r^4} < \frac{2M}{2(s_1^2 + k_1) + (s^{(j)}_2)^2 + k_2^4} + \frac{2}{3M^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

The last step is due to the choices of \( M \) and index \( j \). For an arbitrarily small \( \varepsilon > 0 \), the result follows. ■

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