About one method of analytical calculations of reaction amplitudes with fermions

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I. INTRODUCTION

Now a comparison the consequences of quantum field theories with the results of experiments in a high-energy physics requires calculation of observed quantities with a high scale of precision. The standard way to obtain a cross section with the fermions of a spin 1/2 in perturbative quantum field theories is to reduce the squared amplitude to a trace from products of $\gamma$-matrices. The study of multi-particle final states in reactions and the investigation of polarization effects have required a new effective method of calculations, because the standard method involves the calculation of a great number of traces with a lot of Dirac $\gamma$-matrices.

The method of direct calculation of matrix elements became the alternative to the standard method. The idea of calculating amplitudes has a long enough history. In 1949 it was suggested in [1] to calculate a matrix element by means of explicit form of $\gamma$-matrices and Dirac bispinors (more detailed bibliography on the problem can be found in [2,3]).

Now there have been worked out a lot of methods of calculating the reaction amplitudes with fermions of the spin 1/2. That’s why one can say that the method is becoming a standard method in high-energy physics for obtaining cross sections and decay rates.

The methods of matrix elements calculations can be divided into two major classes. The first class includes the methods of direct numerical calculation of the Feynman diagrams (see, for example, [4]). The second class includes the methods of analytical calculations of amplitudes with the subsequent numerical calculation of cross sections. It should be noted that there are methods of calculating cross sections without the Feynman diagrams [5]-[7].

The analytical methods of Feynman amplitudes calculation can be divided into two groups. The first group involves the analytical methods that reduce the calculation of $S$-matrix element to trace calculation. The procedure of trace calculation underlies a lot of methods (see, [3],[8]-[15] etc.). In the methods matrix element is reduced to the combination of scalar products of four-vectors and their contraction with the Levi-Civita tensor.

The second group involves the analytical methods that practically do not use the operations with traces from products of $\gamma$-matrices. The method of the CALCUL group which was used for the calculations of the reactions with massless fermions is the most famous among [16]-[18]. In this method the matrix element is reduced to spinor products of bispinors i.e. $\mathcal{U}_\lambda(p)\mathcal{U}_{-\lambda}(k)$. The
spinor products are calculated through momentum components by means of traces. However, the operation of matrix element reduction is not so simple as the calculation of traces. It requires the use of Chisholm spinor identities (see [17]). Also it takes the representation of contraction $\not p = p^\mu \gamma_\mu$ with four-momenta $p^\mu$ and polarization vectors of external photons through bispinors. For gauge massive bosons the additional mathematical constructions are needed [17].

There are generalizations of the method for massive Dirac particles both for special choices of the fermion polarization ([17, 19, 20]) and for an arbitrary fermion polarization [21, 22]. We call the polarization states of fermions in Ref.[17] as Kleiss-Stirling or KS-states.

In Ref. [20] the original algorithm of reduction to spinor products for helicity massive fermions with the help of the Weyl representation for $\gamma$-matrices was offered. It should be noted that for massless fermions we can obtain amplitude in terms of the scalar products of four-momentum vectors and current-like constructions of the type $J^\mu \sim \gamma^\mu U_\lambda (k)$. The components of $J^\mu$ are calculated through momentum components $p, k$ (so called $E$-vector formalism, see [23]).

For KS-states Ref.[24] presents the iterative scheme of calculation that reduces expression for the fermion line $\overleftarrow{U}_\lambda (p) QU_\lambda (k)$ to the combination of spinor products $\overleftarrow{U}_\lambda (p) U_\lambda (k)$ and (or) $\overleftarrow{U}_\lambda (p) \gamma^\mu (g_V + g_A \gamma_5) U_\lambda (k)$ by means of inserting the complete set of non-physical states of bispinors (with $p^2 < 0$) into the fermion line.

It goes without saying that both the spinor products and current constructions in all the methods were calculated by means of traces and then used as a universal function similar to the expression of scalar product of four-vectors through their components.

It is very difficult to estimate the efficiency of different methods of matrix element calculation, as their application often comes from the physical problem that must be solved. So, for example, the methods of calculation by means of traces are universal and they don’t require any additional constructions of polarization vectors of particles as in spinor techniques. However, in the trace method the number of terms to be evaluated increases even quadratically with the number of diagrams.

The efficiency of spinor techniques is that the evaluation of matrix element is expressed through the spinor products that have been calculated before. It decreases the number of terms and it is convenient when we pass over to numerical calculations immediately. However, simple process $e^+ e^- \rightarrow W^+ W^-$ with polarized $W$-bosons for spinor technique is more difficult to calculate than it is in trace method.

The disadvantages of other methods are that they use a specific choice of polarization vectors and accordingly require additional calculations if it is necessary to calculate other polarization configurations of fermions.

Many analytical methods underlie both universal programs of matrix element calculations and cross sections CompHEP [25], GRACE [26], FeynCalc [27], (see also [28]) and special-purpose programs. The detailed list of such programs can be found in Ref.[29].

The aim of this paper is to present a new method for calculating the amplitudes of processes involving both massive fermions of an arbitrary polarization and massless fermions. This method is based on the use of an isotropic tetrad in Minkowski space and basis spinors connected with it. Here we don’t use an explicit form of bispinors and $\gamma$-matrices or operation of trace calculations.

In this method as well as in the trace methods the matrix element of Feynman amplitudes is reduced to the combination of scalar products of momenta and polarization vectors. Unlike spinor technique in different variants [17]-[19], this method doesn’t use either Chisholm identities, or the presentation of the contraction $\not p$ with four vector $p$ and of the polarization vector of bosons through the bispinors. The minimum number of operations and the simple algorithm of the method make it very efficient and not complicated in calculations.

As the suggested method is based on the active use of basis spinors connected with isotropic tetrad vectors we shall call it the method of basis spinors (MBS).
The structure of the paper is as follows: in Sec. II we define an isotropic tetrad and complete set of spinors connected with it. We give the main formulas that underlie the suggested method. In Sec. III we consider the decomposition coefficients of bispinors with an arbitrary polarization vector on basis spinors. We give decomposition coefficients for fermion with the most frequently used polarization states. In Sec. IV we render the method of basis spinors and give a brief comparison of the suggested scheme of calculations with methods of spinor techniques and trace method. Sec. V contains the calculation of the amplitudes of the reaction $e^- e^+ \rightarrow f \bar{f}$ with massive fermions, the process $e^- e^+ \rightarrow W^- W^+$ with polarized $W$-bosons and one of possible Feynman diagrams of reaction $e^+ e^- \rightarrow e^+ e^- e^+ e^- e^+ e^-$ (for massless fermions) as the illustration of the MBS. The last section traditionally contains the concluding remarks. Appendix A includes some relations for matrix elements with massless fermions.

II. ISOTROPIC TETRAD AND BASIS SPINORS. SOME RELATIONS

We use the metric and matrix convention as in the book by Bjorken and Drell [30] i.e. the Levi-Civita tensor is determined as $\epsilon_{0123} = 1$ and the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Let us introduce the orthonormal four-vector basis in Minkowski space which satisfies the relations:

$$l_0^\mu = l_1^\mu - l_2^\mu - l_3^\mu = g^{\mu
u}, \quad l_0^2 = - l_1^2 = - l_2^2 = - l_3^2 = 1, \quad \mu, \nu = 0, 1, 2, 3. \quad (1)$$

In Eq. (1) $g$ is the metric tensor. We fix the basis orientation by the relation

$$\epsilon_{\mu
u\rho\sigma} l_0^\mu l_1^\nu l_2^\rho l_3^\sigma = -\epsilon_{0123} = - 1. \quad (2)$$

An arbitrary four-vector $p$ can be written as

$$p = (p \cdot l_0) \cdot l_0 - (p \cdot l_1) \cdot l_1 - (p \cdot l_2) \cdot l_2 - (p \cdot l_3) \cdot l_3. \quad (3)$$

With the help of the vectors $l_\lambda$ ($A = 0, 1, 2, 3$) we can define lightlike vectors, which form the isotropic tetrad in Minkowski space (about isotropic tetrad, see [31]):

$$b_\rho = l_0 + \rho l_3, n_\lambda = l_1 + i\lambda l_2, \quad \rho, \lambda = \pm 1. \quad (4)$$

From Eqs. (1), (4) it follows that

$$b_\rho \cdot b_{-\lambda} = 2\delta_{\lambda,\rho}, \quad (n_\lambda \cdot n_{-\rho}) = - 2\delta_{\lambda,\rho}, \quad (b_\rho \cdot n_\lambda) = 0, \quad (5)$$

$$1/2 \sum_{\lambda = -1}^{1} [l_\lambda^\mu \circ b_{-\lambda}^\nu - n_\lambda^\mu \circ n_{-\lambda}^\nu] = g^{\mu\nu}. \quad (6)$$

Therefore, Dirac matrix $\gamma^\mu$ and contracting $\slashed{p} = p_\mu \gamma^\mu$ with four-vector $p_\mu$ can be written as

$$\gamma^\mu = 1/2 \sum_{\lambda = -1}^{1} \left[ b_{-\lambda}^\mu b_\lambda^\nu - n_{-\lambda}^\mu n_\lambda^\nu \right], \quad (7)$$

$$\slashed{p} = 1/2 \sum_{\lambda = -1}^{1} \left[ (b_{-\lambda} \cdot p) \slashed{n}_\lambda - (n_{-\lambda} \cdot p) \slashed{n}_\lambda \right]. \quad (8)$$

With the help of isotropic tetrad (4) we define basis spinors $U_\lambda (b_{-1})$ and $U_\lambda (b_1)$

$$U_\lambda (b_{-1}) U_\lambda (b_{-1}) = \omega_\lambda U_\lambda (b_{-1}), \quad (9)$$
\[ U_\lambda (b_1) \equiv \frac{\nu_1}{2} U_{-\lambda} (b_{-1}) . \]  
\[ (10) \]

\[ \omega_\lambda U_\lambda (b_{\pm 1}) = U_\lambda (b_{\pm 1}) \]
\[ (11) \]

with \( \omega_\lambda = 1/2 (1 + \lambda \gamma_5) \).

If we introduce raising and lowering spin operators

\[ \frac{\lambda}{2} \gamma_{\lambda} U_{-\nu} (b_{-1}) = \delta_{\lambda,\nu} U_\lambda (b_{-1}) \]
\[ (12) \]

we can fix the phases of the spinors \( U_\lambda (b_{-1}) \) and \( U_\lambda (b_1) \).

By using the properties of lightlike vectors (4) and \( \gamma \)-matrix algebra it is possible to determine that

\[ \frac{\nu_{-1}}{2} U_{-\lambda} (b_1) = U_\lambda (b_{-1}) , \]
\[ (13) \]

\[ \frac{\lambda}{2} \gamma_{\lambda} U_\nu (b_1) = -\delta_{\lambda,\nu} U_{-\lambda} (b_1) . \]
\[ (14) \]

The important property of basis spinors (9), (10) is the completeness relation

\[ \frac{1}{2} \sum_{\lambda, A = -1}^{1} U_\lambda (b_A) \overline{U}_{-\lambda} (b_{-A}) = I, \]
\[ (15) \]

which follows from Eqs.(4)-(11). Thus, the arbitrary bispinor can be decomposed in terms of basis spinors \( U_\lambda (b_A) \).

With the help of Eqs.(7),(12)-(14) we can obtain that

\[ \gamma^\mu U_\lambda (b_{-1}) = b_1^\mu U_{-\lambda} (b_{-1}) + \lambda \gamma_5 U_{-\lambda} (b_{-1}) , \]
\[ (16) \]

\[ \gamma^\mu U_\lambda (b_1) = b_1^\mu U_{-\lambda} (b_{-1}) + \lambda \gamma_5 U_{-\lambda} (b_1) . \]
\[ (17) \]

Eqs.(16) and (17) can be rewritten in a general form

\[ \gamma^\mu U_\lambda (b_A) = b_A^\mu U_{-\lambda} (b_{-A}) + \lambda \gamma_5 U_{-\lambda} (b_A) . \]
\[ (18) \]

Another important property of basis spinors is that spinor products are simple and are similar to scalar products of isotropic tetrad vectors

\[ \overline{U}_\lambda (b_C) U_\rho (b_A) = 2 \delta_{\lambda, -\rho} \delta_{C, -A}, \quad C, A = \pm 1, \lambda, \rho = \pm 1. \]
\[ (19) \]

Eqs.(18), (19), and also

\[ \omega_\lambda U_\rho (b_A) = \delta_{\lambda, \rho} U_\rho (b_A) \]
\[ (20) \]

underlie the suggested method (MBS).

Let us consider some properties of construction

\[ \Gamma^{[\alpha, \beta, \ldots, \mu]}_{C; A; \sigma; \rho} \equiv \overline{U}_\sigma (b_C) \gamma_\alpha \gamma_\beta \ldots \gamma_\mu U_\rho (b_A) . \]
\[ (21) \]
By means of relations (18) and (19) it is easy to calculate \( \Gamma^{\alpha,\beta,\ldots,\mu}_{C,A;\sigma,\rho} \) in terms of the isotropic tetrad vectors. For example, the current-like construction \( \Gamma^{(\mu)}_{C,A;\sigma,\rho} \) has the following form

\[
\Gamma^{(\mu)}_{C,A;\sigma,\rho} = 2\delta_{\sigma,\rho} \left( \delta_{C,A} b^\mu_A + \rho n^\mu_A \delta_{C,A} \right). \tag{22}
\]

Eq. (22) can be rewritten in matrix form

\[
\Gamma^{(\mu)}_{C,A;\sigma,\rho} = 2\delta_{\sigma,\rho} \left( \begin{array}{c} b^\mu_1 \\ \rho n^\mu_1 \\ b^\mu_{-1} \end{array} \right). \tag{23}
\]

With the help of completeness relation (15) we can receive recursion formula for \( \Gamma_{-C,A;\sigma,\rho} \)

\[
\Gamma^{(\alpha,\beta,\ldots,\mu)}_{-C,A;\sigma,\rho} = 1/2 \sum_{D=-1}^{1} \Gamma^{(\alpha)}_{-C,D;\sigma,\sigma} \Gamma^{(\beta,\ldots,\mu)}_{-D,A;\sigma,\rho} \]

\[
= 1/4 \sum_{D,B=1}^{1} \Gamma^{(\alpha)}_{-C,D;\sigma,\sigma} \Gamma^{(\beta)}_{-D,B;\sigma,\sigma} \Gamma^{(\mu)}_{-B,A;\sigma,\rho}. \tag{24}
\]

By means of the additional construction

\[
\Gamma^{a}_{C,A;\sigma,\rho} = k_{1}^{a} k_{2}^{\beta} \ldots k_{n}^{\mu} \Gamma^{(\alpha,\beta,\ldots,\mu)}_{C,A;\sigma,\rho}, \tag{25}
\]

where \( k_{1}, k_{2}, \ldots, k_{n} \) are the arbitrary four-vectors, recursion relation (24) can been rewritten in matrix form

\[
\Gamma^{a}_{-C,A;\sigma,\rho} = 2\delta_{\sigma,(-1)^{n-1} \rho} \prod_{j=1}^{n} \left( (-1)^{j-1} \rho \left( n_{(-1)^{j-1} \rho} \cdot k_{j} \right) \right) \left( b_{1} \cdot k_{j} \right) \left( b_{-1} \cdot k_{j} \right) \left( b_{1} \cdot k_{j} \right) \left( b_{-1} \cdot k_{j} \right) \ldots . \tag{26}
\]

Thus, the construction \( \Gamma^{a,\beta,\ldots,\mu} \) can be easily calculated for a great number of \( \gamma \)-matrices with the help of recursive relation (26). These calculations are an important part of the method of basis spinors. The algorithm of the method of basis spinors will be explained in Sec.[V]

### III. DECOMPOSITION OF THE BISPINORS IN TERMS OF BASIS SPINORS

The major part of my method is the calculation of decomposition coefficients of an arbitrary bispinor on basis spinors (1)- (10). The opportunity of deriving these coefficients is founded on that an arbitrary bispinor of a fermion can be determined through the basis spinor \( U_{\rho} (b_{-1}) \) (or \( U_{\rho} (b_{1}) \)) with the help of projection operators.

Let us consider massless fermions. An arbitrary bispinor \( U_{\lambda} (p) \) of momentum \( p \) \( (p^2 = 0, (p \cdot b_{-1}) \neq 0) \) and helicity \( \lambda \) is defined in terms of basis spinor (see, for example, 17)

\[
U_{\lambda} (p) = \frac{p}{\sqrt{2 (p \cdot b_{-1})}} U_{-\lambda} (b_{-1}). \tag{27}
\]

As it follows from completeness relation (14), the decomposition coefficients of arbitrary bispinors are the spinor products

\[
D_{\lambda,\rho} (A; p) = \frac{1}{2} U_{\lambda} (b_{A}) U_{\rho} (p). \tag{28}
\]
Using Eqs. (18)-(27) we obtain, that

\[ D_{\lambda,\rho}(A; p) = \frac{\delta_{\lambda,-\rho}}{\sqrt{2}} \left[ \delta_{A,-1} \sqrt{(p \cdot b)} + \delta_{A,1} \frac{\lambda (p \cdot n_\lambda)}{\sqrt{(p \cdot b)}} \right]. \]  

(29)

If \( p = \text{const} b \), the decomposition has the most simple form, as in this case

\[ U_\lambda(p) = \sqrt{\text{const}} \ U_{\lambda}(b_{-1}). \]  

(30)

For numerical calculations, as well as in the case of using spinor techniques, it is convenient to determine the coefficient (29) through the momentum components \( p = (p^0, p^x = p^0 \sin \theta_p \sin \varphi_p, p^y = p^0 \sin \theta_p \cos \varphi_p, p^z = p^0 \cos \theta_p) \)

\[ D_{\lambda,\rho}(A; p) = \frac{\delta_{\lambda,-\rho}}{\sqrt{2}} \left[ \delta_{A,-1} \sqrt{p^+} - \delta_{A,1} \lambda \exp(i \lambda \varphi_p) \sqrt{p^+} \right] 
\[ = \delta_{\lambda,-\rho} \sqrt{p^0} \left[ \delta_{A,-1} \cos \frac{\theta_p}{2} - \delta_{A,1} \lambda \sin \frac{\theta_p}{2} \exp(i \lambda \varphi_p) \right], \]  

(31)

where

\[ p^\pm = p^0 \pm p^z, \quad p^x + i \lambda p^y = \sqrt{(p^x)^2 + (p^y)^2} \exp(i \lambda \varphi_p). \]

The decomposition of the bispinor of antifermion \( V_\lambda(p) \) is received from the ratio

\[ V_\lambda(p) = U_{\lambda}(p). \]  

(32)

Let us consider massive Dirac particles. The bispinors of the massive fermion and anti-fermions with arbitrary polarization vectors are determined by the basic spinor (see the Appendix in [13]).

\[ U_\lambda(p, s_p) = \frac{\tau_u^\lambda(p, s_p)}{\sqrt{(b_{-1} \cdot (p + m_p s_p))}} U_{\lambda}(b_{-1}), \]  

(33)

\[ V_\lambda(p, s_p) = \frac{\tau_v^\lambda(p, s_p)}{\sqrt{(b_{-1} \cdot (p + m_p s_p))}} U_{\lambda}(b_{-1}), \]  

(34)

where the projection operators \( \tau_u^\lambda(p, s_p), \tau_v^\lambda(p, s_p) \) are expressed by the relations

\[ \tau_u^\lambda(p, s_p) = \frac{1}{2} (\not{p} + m_p) (1 + \gamma_5 \not{s_p}), \]  

(35)

\[ \tau_v^\lambda(p, s_p) = \frac{1}{2} (\not{p} - m_p) (1 + \gamma_5 \not{s_p}). \]  

(36)

We obtain

\[ \not{p} U_\lambda(p, s_p) = m_p U_\lambda(p, s_p), \quad \not{p} V_\lambda(p, s_p) = -m_p V_\lambda(p, s_p), \]

\[ \gamma_5 \not{s_p} U_\lambda(p, s_p) = \lambda U_\lambda(p, s_p), \quad \gamma_5 \not{s_p} V_\lambda(p, s_p) = \lambda V_\lambda(p, s_p) \]  

(37)

i.e. the bispinors \( U_\lambda(p, s_p) \) and \( V_\lambda(p, s_p) \) satisfy Dirac equation and spin condition for massive fermion and antifermion. The notation \( s_p \) for bispinors indicates that fermion with momentum \( p \)
has fixed polarization vector $s_p$. We also found, that the bispinors of fermions and antifermions, Eqs. (33), (34), were related by

$$V_\lambda (p, s_p) = -\lambda \gamma_5 U_{-\lambda} (p, s_p), \quad \nabla_\lambda (p, s_p) = U_{-\lambda} (p, s_p) \lambda \gamma_5.$$  \hspace{1cm} (38)

After evaluations with the help of Eqs. (33), (34) we receive, that the decomposition coefficients for a massive fermion with momentum $p$, an arbitrary polarization vector $s_p$ and mass $m_p$ can be written as scalar products of tetrad and physical vectors

$$D_{\lambda,\rho} (A; p, s_p) = \frac{1}{\sqrt{2} (b_{-1} \cdot \xi_i^p)} \left[ \delta_{A,-1} \left\{ \delta_{\lambda,-\rho} (b_{-1} \cdot \xi_1^p) - \frac{\rho \delta_{\lambda,\rho}}{2m_p} ((b_{-1} \cdot \xi_1^p) (n_{-\rho} \cdot \xi_2^p) + (n_{-\rho} \cdot \xi_1^p) (b_{-1} \cdot \xi_2^p)) \right\} + \delta_{A,1} \left\{ -\rho \delta_{\lambda,-\rho} (n_{-\rho} \cdot \xi_1^p) + \frac{\delta_{\lambda,\rho}}{2m_p} ((b_{-1} \cdot \xi_1^p) (b_{1} \cdot \xi_2^p) - (n_{-\rho} \cdot \xi_1^p) (n_{\rho} \cdot \xi_2^p)) \right\} \right],$$  \hspace{1cm} (39)

$$\xi_1^p = p + m_p s_p, \quad \xi_2^p = p - m_p s_p.$$  \hspace{1cm} (40)

Let’s determine decomposition coefficients for helicity and $KS$ polarization states of fermions, as they are the most-used in calculations of matrix elements. The polarization vector of $KS$-states is defined as follows [17, 22, 24]:

$$s_p = \frac{p}{m_p} - m_p \frac{b_{-1}}{(p \cdot b_{-1})}.$$  \hspace{1cm} (41)

In this case, the massive fermion bispinor is related with basis bispinor most simply [17, 24]

$$U_\lambda (p, KS) = \frac{p + m_p}{\sqrt{2} (p \cdot b_{-1})} U_{-\lambda} (b_{-1})$$  \hspace{1cm} (42)

and the decomposition coefficients have a compact form

$$D_{\lambda,\rho} (A; p, KS) = \frac{1}{\sqrt{2}} \left\{ \delta_{\lambda,-\rho} \left[ \delta_{A,-1} \sqrt{(p \cdot b_{-1})} + \delta_{A,1} \frac{\lambda (p \cdot n_A)}{\sqrt{(p \cdot b_{-1})}} \right] + \delta_{\lambda,\rho} \delta_{A,1} \frac{m_p}{\sqrt{(p \cdot b_{-1})}} \right\}.$$  \hspace{1cm} (43)

Choosing

$$s_p = \frac{(p \cdot l_0) p - m_p^2 l_0}{m_p \sqrt{(p \cdot l_0)^2 - m_p^2}}$$  \hspace{1cm} (44)

we find that the polarization state of a fermion is its helicity state.

For helicity states the expression for decomposition coefficients in terms of scalar products of physical and isotropic tetrad vectors have more complex form, than for $KS$-states. But, if to consider this expression through components of momentum $p = (p^0, \vec{p} \sin \theta_p \sin \varphi_p, \vec{p} \sin \theta_p \cos \varphi_p, \vec{p} \cos \theta_p)$ we obtain that the decomposition coefficients have a simple form

$$D_{\lambda,\rho} (A; p, Hel) = \frac{1}{\sqrt{2}} \left[ \sqrt{p^0 + \vec{p}^2} \left( \delta_{A,-1} \cos \frac{\theta_p}{2} - \delta_{A,1} \lambda \exp (i \lambda \varphi_p) \sin \frac{\theta_p}{2} \right) \delta_{\lambda,-\rho} + \sqrt{p^0 - \vec{p}^2} \left( \delta_{A,1} \cos \frac{\theta_p}{2} + \delta_{A,-1} \lambda \exp (-i \lambda \varphi_p) \sin \frac{\theta_p}{2} \right) \delta_{\lambda,\rho} \right].$$  \hspace{1cm} (45)
It is clear, if \( m_p = 0 \) Eq. (45) turns into Eq. (31).

The analysis of decomposition coefficients for massive and massless fermions of a spin 1/2 shows, that the decomposition coefficient \( D_{\lambda,\rho} \) for massless case becomes diagonal on spin indices \( \lambda, \rho \). This fact simplifies the calculation of matrix elements with massless fermions.

The decomposition coefficients for an antifermion with bispinor (34) can be easily obtained with the help of expression (38).

**IV. METHOD OF THE BASIS SPINORS**

The amplitude of the Feynman diagram is the product of the expressions of the type

\[
M_{\lambda_p,\lambda_k} (p, s_p; k, s_k) = \mathcal{U}_{\lambda_p} (p, s_p) Q \mathcal{U}_{\lambda_k} (k, s_k).
\]

Equation (46) corresponds to the fermion line with the matrix operator \( Q \), which is expressed as the combination of \( \gamma \)-matrices and their contractions with four-vectors.

In the method based on the use of traces Eq. (46) is rewritten as follows

\[
M_{\lambda_p,\lambda_k} (p, s_p; k, s_k) = Tr \left( \mathcal{U}_{\lambda_k} (k, s_k) \otimes \mathcal{U}_{\lambda_p} (p, s_p) Q \right).
\]

The main problem of the trace approach is to get explicit form of expression \( \mathcal{U}_{\lambda_k} (k, s_k) \otimes \mathcal{U}_{\lambda_p} (p, s_p) \).

A great number of methods of matrix element calculation are directed to the solution of this problem. In our case we can also obtain the expression of the matrix \( \mathcal{U}_{\lambda_k} (k, s_k) \otimes \mathcal{U}_{\lambda_p} (p, s_p) \) by means of Eqs. (33) and (34) for various spin configurations of fermions

\[
\mathcal{U}_{\lambda} (k, s_k) \otimes \mathcal{U}_{\lambda} (p, s_p) = \frac{\tau^u_\lambda (k, s_k) \omega_{-\lambda} \gamma_{-1} \tau^u_\lambda (p, s_p)}{4 \sqrt{b_{-1} \cdot (p + m_p s_p) \sqrt{b_{-1} \cdot (k + m_k s_k)}}},
\]

\[
\mathcal{U}_{-\lambda} (k, s_k) \otimes \mathcal{U}_{\lambda} (p, s_p) = \frac{\lambda \tau^u_{-\lambda} (k, s_k) \gamma_{-\lambda} \omega_{-\lambda} \gamma_{-1} \tau^u_\lambda (p, s_p)}{8 \sqrt{b_{-1} \cdot (p + m_p s_p) \sqrt{b_{-1} \cdot (k + m_k s_k)}}}.
\]

Thus, to calculate (46) it is necessary to evaluate the relations:

\[
M_{\lambda,\lambda} (p, s_p; k, s_k) = \frac{Tr (\tau^u_\lambda (k, s_k) \omega_{-\lambda} \gamma_{-1} \tau^u_\lambda (p, s_p) Q)}{4 \sqrt{b_{-1} \cdot (p + m_p s_p) \sqrt{b_{-1} \cdot (k + m_k s_k)}}},
\]

\[
M_{\lambda,-\lambda} (p, s_p; k, s_k) = \frac{\lambda Tr (\tau^u_{-\lambda} (k, s_k) \gamma_{-\lambda} \omega_{-\lambda} \gamma_{-1} \tau^u_\lambda (p, s_p) Q)}{8 \sqrt{b_{-1} \cdot (p + m_p s_p) \sqrt{b_{-1} \cdot (k + m_k s_k)}}}.
\]

The main inconvenience of the trace method is the considerable increase of terms with the increase of the number of \( \gamma \)-matrices in the operator \( Q \). In particular, the authors of [32] suggest to use KS polarization states to decrease the number of terms in \( \mathcal{U}_{\lambda_k} (k, s_k) \otimes \mathcal{U}_{\lambda_p} (p, s_p) \). It is also necessary to evaluate spin configurations of fermions separately.

In the paper I present the method of calculating expressions (46) without using the above scheme, i.e. without using the traces of type (50), (71). This approach can be realized by using the properties of basis spinors (H), (II). That’s why the method is called the method of basis spinors (MBS).
The essence of the MBS is as follows. With the help of completeness relation (13) we decompose bispinors in (46). As a result the matrix element is

\[
M_{\lambda_p,\lambda_k} (p, s_p; k, s_k) = \frac{1}{4} \sum_{\sigma, \rho = -1} \sum_{A, C = -1} \mathcal{U}_{\lambda_p} (p, s_p) U_\sigma (b_C) \mathcal{U}_{-\sigma} (b_C) Q U_\rho (b_A) \mathcal{U}_{-\rho} (b_A) U_{\lambda_k} (k, s_k)
\]

\[
\{ \mathcal{U}_{-\sigma} (b_C) Q U_\rho (b_A) \} \mathcal{U}_{-\rho} (b_A) U_{\lambda_k} (k, s_k)
\]

\[
= \sum_{\sigma, \rho = -1} \sum_{A, C = -1} D_{\lambda_p, \sigma}^\dagger (C; p, s_p) \Gamma_{-C, A; -\sigma, \rho} (Q) D_{-\rho, \lambda_k} (A; k, s_k) ,
\]

where the coefficients \(D_{\rho, \lambda}\) are the decomposition coefficients of the bispinors, and the construction \(\Gamma\) is defined by the relation

\[
\Gamma_{C, A; \sigma, \rho} (Q) \equiv \mathcal{U}_{\sigma} (b_C) Q U_\rho (b_A) .
\]

Equation (52) can be rewritten in matrix form

\[
M_{\lambda_p,\lambda_k} (p, s_p; k, s_k) = \sum_{\sigma, \rho = -1} \left( \begin{array}{cc}
D_{\lambda_p, \sigma}^\dagger (-1; p, s_p) & D_{\lambda_p, \sigma}^\dagger (1; p, s_p)
\end{array} \right) \left( \begin{array}{cc}
\Gamma_{1, -1; -\sigma, \rho} (Q) & \Gamma_{1, 1; -\sigma, \rho} (Q)
\end{array} \right) \left( \begin{array}{c}
\Gamma_{\sigma, \rho} (1; k, s_k)
\end{array} \right)
\]

Thus in the MBS the problem of calculating (46) involves two steps:

1. The calculation of the decomposition coefficients \(D_{\lambda_p, \sigma} (C; p, s_p)\) and \(D_{\rho, \lambda_k} (A; k, s_k)\)

2. The calculation of the value \(\Gamma_{C, A; \sigma, \rho} (Q)\) with further summing.

The first part of the problem is solved in Sec.III. Evidently, such calculation are made only once and further on the decomposition coefficients are used as ready-made functions.

By means of Eqs. (19), (20) and (23)-(26) it is easy to calculate the value \(\Gamma (Q)\) as linear combinations of values \(\Gamma_{C, A; \sigma, \rho}^{(\alpha, \beta, \ldots, \mu, \nu)}\) (28). We can obtain Eq.(53) in terms of the scalar products of physical vectors included in \(Q\) and the vectors of an isotropic tetrad (see, Sec. II and Appendix A).

The further procedure of summing is largely simplified because of Kronecker symbols occurring in calculating both decomposition coefficients and value (33).

In the case when operator \(Q\) contains non-contracted Lorentz-indices, the final result of Eq.(46) will make up the corresponding tensor constructed from the vectors of an isotropic tetrad.

As an example of the MBS let’s consider the calculation of Eq.(46) for massless fermions

\[
M_{\lambda_p,\lambda_k} (p, k; Q) \equiv \mathcal{U}_{\lambda_p} (p) Q \mathcal{U}_{\lambda_k} (k)
\]

Using coefficients of decomposition (29) and summing we find expression (55) in general form (for arbitrary \(Q\))

\[
M_{\lambda_p,\lambda_k} (p, k; Q) = D_{\lambda_p, -\lambda_p}^\dagger (1; p) \left( D_{-\lambda_k, \lambda_k} (1; k) \Gamma_{-1, 1; \lambda_p, \lambda_k} (Q) + D_{-\lambda_k, -\lambda_k} (1; k) \Gamma_{1, -1; \lambda_p, \lambda_k} (Q) \right) + D_{\lambda_p, -\lambda_p}^\dagger (-1; p) \left( D_{-\lambda_k, \lambda_k} (1; k) \Gamma_{1, 1; \lambda_p, \lambda_k} (Q) + D_{-\lambda_k, -\lambda_k} (1; k) \Gamma_{1, -1; \lambda_p, \lambda_k} (Q) \right)
\]

\[
= -\lambda_p \frac{(p \cdot n_{\lambda_p})}{2 \sqrt{(p \cdot b_{-1})}} \left( \sqrt{(k \cdot b_{-1})} \Gamma_{1, 1; \lambda_p, \lambda_k} (Q) - \lambda_k (k \cdot n_{-\lambda_k}) \sqrt{(k \cdot b_{-1})} \Gamma_{1, -1; \lambda_p, \lambda_k} (Q) \right) + \frac{(p \cdot b_{-1})}{2} \left( \sqrt{(k \cdot b_{-1})} \Gamma_{1, 1; \lambda_p, \lambda_k} (Q) - \lambda_k (k \cdot n_{-\lambda_k}) \sqrt{(k \cdot b_{-1})} \Gamma_{1, -1; \lambda_p, \lambda_k} (Q) \right) .
\]
For completeness we have included expressions (55) with $Q = \gamma^\mu \gamma^\nu$ and $Q = \gamma^\mu \gamma^\nu \gamma^\alpha$ in Appendix A. With the help of these relations we have an opportunity to gain larger “building” blocks of the Feynman diagrams and to use them as universal functions.

Let’s briefly compare the method of basis spinors with spinor technique of calculating the processes with massless fermions ([17]). We recall some details of the spinor techniques with small modifications. Instead of vectors $k_0 = (1, 1, 0, 0)$ and $k_1 = (0, 0, 1, 0)$ used in Ref.([17]) we use $b_{-1}$ and $n_\lambda$ accordingly.

The CALCUL spinor techniques includes the following operations.

1. An arbitrary massless spinor $U_\lambda (p)$ is determined through the basic spinor by relation (27), i.e.

$$U_\lambda (p) = \frac{\sqrt[p]{\lambda}}{\sqrt[2]{(p \cdot b_{-1})}} U_{-\lambda} (b_{-1}) .$$  \hfill (59)

2. The spinor Chisholm identity

$$\gamma^\mu \left\{ U_\lambda (p) \gamma_\mu U_\lambda (k) \right\} = 2 \left\{ U_\lambda (k) U_\lambda (p) + U_{-\lambda} (p) U_{-\lambda} (k) \right\}$$

is used.

3. The four-vector $p$ with $p^2 = 0$ can be written as the sum of the projection operators

$$\hat{p} = \sum_{\lambda=-1}^{1} U_\lambda (p) \overline{U}_\lambda (p) .$$  \hfill (61)

4. The circular polarization vectors of massless boson $\varepsilon^\mu_\lambda (k)$ with momentum $k$ ($k^2 = 0$) is determined by

$$\varepsilon^\mu_\lambda (k) \sim \overline{U}_\lambda (q) \gamma^\mu U_\lambda (k) .$$  \hfill (62)
With the help of Eqs.(60)-(62) we can reduce the amplitudes of processes to expressions involving only spinor products of the type
\[ s_\lambda (p, k) \equiv U_\lambda (p) U_{-\lambda} (k) = -s_\lambda (k, p) . \]  
(63)

Spinor product (63) due to Eqs.(7),(9) and (59) is reduced to the calculation of trace [17]
\[ s_\lambda (p, k) = \frac{\lambda}{4} \frac{Tr (\omega_{-\lambda} \phi_1 \phi_2 \phi_3 \phi_4)}{(b_1 \cdot p) \sqrt{(b_1 \cdot k)}} \]
\[ = \frac{\lambda}{2} \left[ (p \cdot b_1) (k \cdot n_\lambda) - (k \cdot b_1) (p \cdot n_\lambda) \right] - i \epsilon (b_1, n_\lambda, p, k) \]
(64)

with \( \epsilon (p, r, k, q) = \epsilon^{\alpha\beta\rho\mu} \epsilon_{\alpha r \beta k} \).

Using properties of isotropic tetrad vectors Eq.(64) can be rewritten
\[ s_\lambda (p, k) = \frac{\lambda}{2} \left[ (p \cdot b_1) (k \cdot n_\lambda) - (k \cdot b_1) (p \cdot n_\lambda) \right] \]
\[ \sqrt{(b_1 \cdot p) (b_1 \cdot k)} \]
(65)

The comparative analysis of algorithms of matrix elements reduction (calculation) shows that the spinor techniques and the MBS differ from each other essentially. Neither Chisholm identity (60) nor additional constructions (61), (62) are used in the MBS. The method of basis spinors does not require a special procedure for constructing polarization vectors of massive gauge bosons (see, for example [17]), as all four-vectors in the method are "processed" similarly. Besides, it is necessary to mark that if in Eq.(60) the vector \( p \) or \( k \) coincides with \( b_1 \) or \( b_{-1} \) you should be very accurate using spinor Chisholm identities, because they alter (see [33]).

The MBS can also be supplemented by constructing the polarization vector of massless and massive bosons in the basis of an isotropic tetrad (for photons, see [31]). The final expression for (55) will contain only scalar products of four-vectors of particles and the isotropic tetrad.

It is always possible to construct the basis of an isotropic tetrad by means of physical vectors for particular reaction. The procedure enables to obtain the expression of the Feynman amplitude in terms of the scalar products of the four-momenta and their contraction with Levi-Civita tensor. Thus, we can get the matrix element in explicit Lorentz-covariant form. Besides, the successful construction of the basis can lead to a considerable decrease of the number of terms.

V. APPLICATIONS

Let us consider a series of examples as an illustration and a test of the method of basis spinors. We shall take electron-positron reactions as there are enough examples of analytical calculations of the matrix element for them. For simplification we shall consider initial fermions as massless.

As an example of our method we shall calculate the amplitudes of the reaction
\[ e^- (p_1, \lambda_1) + e^+ (p_2, \lambda_2) \rightarrow W^- (k_1, \alpha) + W^+ (k_2, \beta) . \]
(66)
The Feynman diagrams are shown in Fig.I. The amplitude of the process can be written
\[ M_{e^+e^-\rightarrow W^+W^-} = M_{\gamma Z} + M_{\nu}, \]
(67)

\[ M_{\gamma Z} = \frac{4\pi\alpha}{P^2} \left[ \overline{U}_{\lambda_2} (p_2) \gamma_\mu U_{\lambda_1} (p_1) - \frac{P^2}{(P^2 - m_Z^2) 2 \sin^2 \theta_W} \times \right. \]
FIG. 1: Feynman diagrams for the process $e^-e^+ \to W^-W^+$.

\[
\nabla_{\lambda_{1}}(p_{2}) \gamma_{\mu} \left( g_{V}^{e} - g_{A}^{e}\gamma_{5} \right) U_{\lambda_{1}}(p_{1}) \right] \Gamma^{\mu\alpha\beta}(P, k_{1}, k_{2}) \varepsilon_{\alpha}(k_{1}) \varepsilon_{\beta}(k_{2}), \quad (68)
\]

\[
M_{\nu} = \frac{\pi \alpha}{2 Q^2 \sin^2 \theta_{W}} \nabla_{\lambda_{2}}(p_{2})\tilde{\varepsilon}(k_{2})(1 - \gamma_{5}) Q \tilde{\varepsilon}(k_{1})(1 - \gamma_{5}) U_{\lambda_{1}}(p_{1}), \quad (69)
\]

where $g_{A}^{e}$, $g_{V}^{e}$ are axial and vector electron couplings accordingly. The tensor $\Gamma^{\mu\alpha\beta}(P, k_{1}, k_{2})$ is determined as

\[
\Gamma^{\mu\alpha\beta}(P, k_{1}, k_{2}) = g^{\alpha\beta}(k_{1} - k_{2})^{\mu} + 2 \left( P_{\alpha} g^{\mu\beta} - P_{\beta} g^{\mu\alpha} \right). \quad (70)
\]

In Eqs. (68) and (69) we use the following designations: $\alpha$ and $\theta_{W}$ are the fine-structure constant and weak mixing angle respectively, the vectors $\varepsilon_{\alpha}(k_{1})$, $\varepsilon_{\beta}(k_{2})$ are the polarization vectors of $W$-bosons and the momenta $P, Q$ are $P = p_{1} + p_{2}$, $Q = p_{1} - k_{1}$.

Let us consider reaction (66) in the center of momentum system $e^+e^-$. Then vectors of isotropic tetrad (4) can be expressed by means of physical vectors $p_{1}, p_{2}, k_{1}$

\[
b_{1} = \frac{\sqrt{2}p_{1}}{(p_{1} \cdot p_{2})}, \quad b_{-1} = \frac{\sqrt{2}p_{2}}{(p_{1} \cdot p_{2})},
\]

\[
n_{\lambda} = \frac{b_{-1}(k_{1} \cdot b_{1}) + b_{1}(k_{1} \cdot b_{-1}) - 2k_{1} + i\lambda [b_{1}, b_{-1}, k_{1}]}{2(b_{-1} \cdot k_{1})(b_{1} \cdot k_{1})} \quad (71)
\]

with $[p, r, k]^{\mu} = \epsilon^{\alpha\beta\rho\mu} p_{\alpha} r_{\beta} k_{\rho}$.

In this case it is easy to find with the help of the MBS that

\[
\nabla_{\lambda_{2}}(p_{2}) \gamma_{\mu} U_{\lambda_{1}}(p_{1}) = \lambda_{1} \delta_{-\lambda_{2}, \lambda_{1}} \sqrt{2 (p_{1} \cdot p_{2}) n_{\lambda_{1}}^{\mu}}, \quad (72)
\]

Using Eq.(72) we obtain the matrix elements with $\gamma$ and $Z^{0}$-boson exchanges in terms of scalar
products of the physical vectors and the isotropic tetrad vectors:

\[ M_{\gamma Z} = 4\pi \alpha \frac{\lambda_1 \delta_{-\lambda_2,\lambda_1}}{\sqrt{2 (p_1 \cdot p_2)}} \left( 1 - \chi \left( P^2 \right) \frac{g^e_{-\lambda_1}}{2 \sin^2 \theta_W} \right) \]

\[ \times (n_{\lambda_1})_\mu \Gamma^{\mu \alpha \beta} (P, k_1, k_2) \varepsilon_{\alpha} (k_1) \varepsilon_{\beta} (k_2) \]

\[ = 4\pi \alpha \frac{\lambda_1 \delta_{-\lambda_2,\lambda_1}}{\sqrt{2 (p_1 \cdot p_2)}} \left( 1 - \chi \left( P^2 \right) \frac{g^e_{-\lambda_1}}{2 \sin^2 \theta_W} \right) \left[ [(\varepsilon (k_1) \cdot \varepsilon (k_2)) ((k_1 - k_2) \cdot n_{\lambda_1}) + \right. \]

\[ \left. 2 ((P \cdot \varepsilon_{\alpha} (k_1)) (\varepsilon (k_2) \cdot n_{\lambda_1}) - (P \cdot \varepsilon (k_2)) (\varepsilon (k_1) \cdot n_{\lambda_1})) \right], \quad (73) \]

where we use the following notations: \( \chi (P^2) = P^2 / (P^2 - m_Z^2) \), \( g^e_\alpha = (g^e_\nu + \lambda g^e_\lambda) \). It should be noted that similar calculation by means of the spinor techniques of the CALCUL group requires much more efforts.

Let us consider the calculation of the diagram with neutrino exchange as an example of using building blocks, which were calculated before by the method of basis spinors. Amplitude (69) can be rewritten in the form

\[ M_\nu = \frac{\pi \alpha (1 - \lambda_1)}{Q^2 \sin^2 \theta_W} \varepsilon_{\mu} (k_2) Q_{\nu} \varepsilon_{\beta} (k_1) \overline{U}_{-\lambda_2} (p_2) \gamma^\mu \gamma^\nu \gamma^\beta U_{\lambda_1} (p_1). \quad (74) \]

Using Eq. (A2) from Appendix A and the definition of isotropic tetrad vectors (71) we obtain

\[ M_\nu = \delta_{-\lambda_2,\lambda_1} \sqrt{2 (p_1 \cdot p_2)} \frac{\pi \alpha (\lambda_1 - 1)}{Q^2 \sin^2 \theta_W} \times \]

\[ [(\varepsilon (k_2) \cdot b_{-1}) ((\varepsilon (k_1) \cdot b_1) (n_{-\lambda_1} \cdot Q) - (\varepsilon (k_1) \cdot n_{-\lambda_1}) (b_1 \cdot Q)) + \]

\[ (\varepsilon (k_2) \cdot n_{-\lambda_1}) ((\varepsilon (k_1) \cdot n_{-\lambda_1}) (n_{\lambda_1} \cdot Q) - (\varepsilon (k_1) \cdot b_1) (b_{-1} \cdot Q))]. \quad (75) \]

To get an explicit Lorentz-covariant form of amplitude (74) it is necessary to substitute the vectors of isotropic tetrad in relations (73),(75) by formulas (71).

Taking into account the kinematics of the process the components of the vectors in the center of momentum system \( e^+ e^- \) are determined by the relations:

\[ p_1 = \frac{\sqrt{s}}{2} (1, 0, 0, 1), \quad p_2 = \frac{\sqrt{s}}{2} (1, 0, 0, -1), \]

\[ k_1 = \frac{\sqrt{s}}{2} (1, \beta_W \sin \theta, 0, \beta_W \cos \theta), \quad k_2 = \frac{\sqrt{s}}{2} (1, -\beta_W \sin \theta, 0, -\beta_W \cos \theta), \]

\[ \varepsilon^\mu_T (k_1) = \frac{1}{\sqrt{2}} (0, \cos \theta, \nu_1 i, -\sin \theta), \quad \varepsilon^\mu_T (k_2) = \frac{1}{\sqrt{2}} (0, \cos \theta, -\nu_2 i, -\sin \theta), \]

\[ \varepsilon^\mu_L (k_1) = \gamma_W (\beta_W, \sin \theta, 0, \cos \theta), \quad \varepsilon^\mu_L (k_2) = \gamma_W (\beta_W, -\sin \theta, 0, -\cos \theta), \quad (76) \]

where \( s = (p_1 + p_2)^2 \), \( \beta_W = \sqrt{1 - 4m_W^2 / s} \), \( \gamma_W = \sqrt{s} / (2m_W) \), \( \nu_1, \nu_2 = \pm 1 \) and the angle \( \theta \) is the scattering angle of \( W^- \)-boson in the center of momentum system. For longitudinally polarized \( W \)-bosons (\( \varepsilon (k_{1,2}) \equiv \varepsilon_L (k_{1,2}) \)) after a series transformations it is not difficult to obtain

\[ M_{\gamma Z}^{L \mu} = 4\pi \alpha \lambda_1 \delta_{-\lambda_2,\lambda_1} \left( 1 - \chi (s) \frac{g^e_{-\lambda_1}}{2 \sin^2 \theta_W} \right) \beta_W (2\gamma_W^2 + 1) \sin \theta, \quad (77) \]

\[ M_{\nu}^{L \mu} = \frac{2\pi \alpha \delta_{-\lambda_2,\lambda_1}}{\beta_W \sin^2 \theta_W} (1 - \lambda_1) \left( \gamma_W^2 - \frac{1}{\gamma_W^2 (1 + \beta_W^2 - 2\beta_W \cos \theta)} \right) \sin \theta. \quad (78) \]

Eqs. (77) and (78) coincide with the matrix elements of this process, which are presented in (54).
We shall consider the process with massive fermions \( f, \bar{f} \)

\[
e^- (p_1, \lambda_1) + e^+ (p_2, \lambda_2) \to f (k_1, \nu_1) + \bar{f} (k_2, \nu_2) \quad (f \neq e)
\] (79)
as the following test of the MBS. The Feynman diagrams for this process are shown in Fig. 2. The

![Feynman diagrams for the process \( e^- e^+ \to f \bar{f} \).](image)

amplitude of the process can be written as

\[
M_{e^+ e^- \to f \bar{f}} (\lambda_1, \lambda_2; \nu_1, \nu_2) = 4\pi \alpha / s \left[ M_\gamma (\lambda_1, \lambda_2; \nu_1, \nu_2) + M_{Z^0} (\lambda_1, \lambda_2; \nu_1, \nu_2) \right],
\] (80)

where

\[
M_\gamma (\lambda_1, \lambda_2; \nu_1, \nu_2) = Q_f \nabla_{\lambda_2} (p_2) \gamma_\mu U_{\lambda_1} (p_1) \overline{U}_{\nu_1} (k_1) \gamma^\mu V_{\nu_2} (k_2),
\] (81)

\[
M_{Z^0} (\lambda_1, \lambda_2; \nu_1, \nu_2) = R_Z \left( g_{\mu\nu} - P_\mu P_\nu / m_Z^2 \right)
\nabla_{\lambda_2} (p_2) \gamma_\nu \left( g_V^f - g_A^f \gamma_5 \right) U_{\lambda_1} (p_1) \nabla_{\nu_1} (k_1) \gamma_\mu \left( g_V^f - g_A^f \gamma_5 \right) V_{\nu_2} (k_2)
\] (82)

with \( R_Z = (G_F m_Z^2 s) / \left( 2\sqrt{2}\pi \alpha (s - m_Z^2) \right) \). The values \( g_V^f, g_A^f \) are fermion coupling constants, \( G_F \) is Fermi constant and \( Q_f \) is fermion charge \( f \) in units \( e \).

We shall consider the amplitudes of process (79) for the helicity and \( KS \) polarization states of final fermions. Using (72) and the decomposition coefficients for \( KS \) polarization states (43) it is easy to obtain the expressions both in terms of scalar products and through momentum components respectively by means of the MBS

\[
M_{e^+ e^- \to f \bar{f}}^{KS} (\lambda_1, \lambda_2; \nu_1, \nu_2) = \frac{-8\pi \alpha}{s} \frac{\sqrt{s}\delta_{\lambda_1, -\lambda_2}}{(b_{-1} \cdot k_1) (b_1 \cdot k_2)} \left[ \delta_{\lambda_1, \nu_1} (b_{-1} \cdot k_1)
\right.

\[
\left. \left( \lambda_2 Q_f - R_Z g_e^{-\lambda_1} g_f^{-\nu_1} \right) ((n_{\nu_2} \cdot k_2) \delta_{\nu_1, -\nu_2} + m_f \nu_2 \delta_{\nu_1, \nu_2}) + \nu_2 (b_{-1} \cdot k_2)
\right.

\[
\left. \left( \nu_1 \delta_{\nu_1, -\nu_2} \delta_{-\lambda_1, \nu_1} (n_{\nu_1} \cdot k_1) \left( -Q_f \lambda_2 + R_Z g_e^{-\lambda_1} g_f^{-\nu_1} \right)
\right.

\[
\left. + m_f \left( Q_f \lambda_2 - R_Z g_e^{-\lambda_1} g_f^{-\nu_1} \right) \right),
\] (83)
The method gives the analytical expression of the matrix element for all spin configurations of fermions. The Kronecker symbols occur as the consequence of the used method of calculations. That’s why the calculation of one of the possible diagrams (see fig. 3) for reaction massless fermions as an illustration. Using definition (58) the matrix element of given Feynman phase factor.

Using the MBS matrix element (86) is reduced to compact form.

Using the MBS it is simple obtain the helicity amplitudes with the help of the coefficients of decomposition (13):

\[ M_{e^+e^-\rightarrow e^+e^-}(\lambda_1, \lambda_2; \nu_1, \nu_2) = 4\pi \alpha \delta_{\lambda_1, -\lambda_2} \frac{\sin \theta}{\sqrt{1 - \beta_f^2 \cos^2 \theta}} \left[ \delta_{\nu_1, -\nu_2} \beta_f \left( \lambda_1 Q_f + R_Z g_{e^{-\lambda_1}} g_{f}^{-\nu_1} \right) \right. \]

It should be noted, that \( \delta \)-symbols which occur in (83) and (84), (85) are not artificially introduced. The Kronecker symbols occur as the consequence of the used method of calculations. That’s why the method gives the analytical expression of the matrix element for all spin configurations of fermions at the same time. The condition distinguishes the method from the trace method where the spin configurations of the fermions should be calculated separately. Besides, it is convenient to use chiral (left and right) coupling constants (\( g^\pm_f = g_f^l \pm g_f^R \)) in the MBS, because they occur during calculation (83).

To avoid the impression that the method can calculate only simple binary processes let’s consider the calculation of one of the possible diagrams (see fig. 3) for reaction \( e^+e^- \rightarrow e^+e^- e^-e^- \) with massless fermions as an illustration. Using definition (58) the matrix element of given Feynman diagram can be written as

\[ M_{e^+e^-\rightarrow e^+e^- e^-e^-} = \frac{e^6}{P_{12 Q12 Q2} Q_2^2 Q_1 Q_2} J^\beta_{-\lambda_2, \lambda_1} (p_2, p_1) J^\nu_{\nu_1, \nu_2} (k_1, k_2) \]

Here we use the following notations for four-vectors

\[ Q_1 = -(k_1 + k_2 + k_6), Q_2 = k_3 + k_4 + k_5, \]
\[ Q_3 = k_3 + k_4, Q_{12} = k_1 + k_2, P_{12} = p_1 + p_2. \]

Using the MBS matrix element (83) is reduced to compact form

\[ M_{e^+e^-\rightarrow e^+e^- e^-e^-} = \frac{\delta_{\nu_1, -\nu_2} \delta_{\nu_3, -\nu_4} \delta_{\nu_5, -\nu_6} \delta_{\lambda_1, -\lambda_2} \sqrt{\lambda_1 e^6}}{(k_5 \cdot b_{-1}) (k_6 \cdot b_{-1})} \]

\[ (\delta_{\lambda_1, -\lambda_2} [(Q_1 \cdot b_1) (k_6 \cdot b_{-1}) (j_2 \cdot n_{-\nu_5}) - (j_2 \cdot b_{-1}) (k_6 \cdot n_{-\nu_5})] + (Q_1 \cdot n_{-\nu_5}) [(j_2 \cdot n_{\nu_5}) (k_6 \cdot n_{-\nu_5}) - (k_6 \cdot b_{-1}) (j_2 \cdot b_1)] \times \]

\[ [(Q_2 \cdot b_{-1}) (k_5 \cdot n_{\nu_5}) (j_3 \cdot n_{-\nu_5}) - (k_5 \cdot b_{-1}) (j_3 \cdot b_1)] + (Q_2 \cdot n_{-\nu_5}) [(k_5 \cdot b_{-1}) (j_3 \cdot n_{\nu_5}) - (j_3 \cdot b_{-1}) (k_5 \cdot n_{\nu_5})] \]
\[ \delta_{\lambda_1,\nu_5} \left[ (k_6 \cdot n_{-\nu_5}) ((j_2 \cdot b_{-1}) (Q_1 \cdot n_{\nu_5}) - (Q_1 \cdot b_{-1}) (j_2 \cdot n_{\nu_5})) + (k_6 \cdot b_{-1}) ((j_2 \cdot b_1) (Q_1 \cdot b_{-1}) - (Q_1 \cdot n_{\nu_5}) (j_2 \cdot n_{-\nu_5})) \right] \times \left[ (Q_2 \cdot b_1) ((k_5 \cdot b_{-1}) (j_3 \cdot n_{\nu_5}) - (j_3 \cdot b_{-1}) (k_5 \cdot n_{\nu_5})) + (Q_2 \cdot n_{\nu_5}) ((j_3 \cdot n_{-\nu_5}) (k_5 \cdot n_{\nu_5}) - (k_5 \cdot b_{-1}) (j_3 \cdot b_1)) \right] \right], \tag{88} \]

where

\[ j_2^\mu \equiv J_{\nu_1,\nu_2}^\mu (k_1, k_2), \quad j_3^\mu \equiv J_{\nu_3,\nu_4}^\mu (k_3, k_4). \tag{89} \]

The scalar products of the vectors \( j_2, j_3 \) with the vectors of an isotropic tetrad are easily obtained with the help of (88).

**VI. CONCLUDING REMARKS**

The suggested computational method of matrix elements from the methodological point of view (but not from the point of view of a method of calculations) is close to the methods offered in [20], [24] for helicity and KS-fermion states. In contrast to them the method of the basis spinors has a simpler algorithm and does require the explicit form of \( \gamma \)-matrices and bispinors. It can also be used for arbitrary fermion polarization.

The calculation of a matrix element in the MBS is simplified through the incorporation of a complete set of massless basis spinors that makes many evaluations trivial. And the main ”laborious” operation is the calculation of decomposition coefficients of physical bispinors on basis ones.

The suggested method combines the advantages of both the trace methods and the computational methods based on spinor technique. The method as well as the method of spinor techniques enables to calculate the blocks of the Feynman diagrams (current-like constructions, spinor products, and
by means of the MBS even more complicated structures) with the help of recursion relation \((26)\) and then to use them in the calculation as universal functions. The method as well as trace methods does not require either the compulsory construction of the polarization vectors of bosons, or the transformation of the slash construction \(\not{p}\) into bispinors.

The MBS can be easily realized in the systems of symbolic calculations (Mathematica, Maple, Reduce). As an example, all calculations have been done with the help of the simple rule-based program in environment “MATHEMATICA”\(\textsuperscript{[35]}\). It should be noted that getting the analytical expression in terms of scalar products on the given matrix element of the reaction on the ordinary computer (Pentium-III) takes from 0.2 to 0.6 second.

In conclusion, I’d like to emphasize that the purpose of my paper is to present the new method of analytical calculations of the matrix element with massless and massive fermions. That is why I didn’t try to stress which method is more efficient (better or worse), since the criteria of the efficiency can be different.

**APPENDIX A: SOME RELATIONS FOR MASSLESS FERMIONS**

By means of Eq. \((24)\) or \((26)\) we can get expressions for \(\Gamma_{C,A;\sigma,\rho}^{\mu,\nu}\) with \(Q = \gamma^\mu \gamma^\nu\) and \(Q = \gamma^\mu \gamma^\nu \gamma^\alpha\):

\[
\Gamma_{C,A;\sigma,\rho}^{\mu,\nu} = 2\delta_{\sigma,-\rho} \left( \rho \left( b_1^\mu n_{-\rho}^\nu - n_{-\rho}^\mu b_1^\nu \right) - \left( n_{-\rho}^\mu b_1^\nu - n_{-\rho}^\nu b_1^\mu \right) \right)_{C,A},
\]

\[
\Gamma_{C,A;\sigma,\rho}^{\mu,\nu,\alpha} = 2\delta_{\sigma,\rho} \left( \rho \left( n_{-\rho}^\mu b_1^\nu - n_{-\rho}^\nu b_1^\mu \right) b_1^\alpha \right) + \rho \left( b_1^\mu b_1^\nu - b_1^\nu b_1^\mu \right) \left( n_{-\rho}^\alpha b_1^\nu - n_{-\rho}^\nu b_1^\alpha \right)_{C,A}.
\]

Using Eq.\((56)\) and the Eqs.\((A1),(A2)\) we get two expressions

\[
M^{\lambda_y,\lambda_k}(p,k;\gamma^\mu \gamma^\nu) \equiv \overline{U}_{\lambda_y}(p) \gamma^\mu \gamma^\nu U_{\lambda_k}(k) = \frac{\lambda_y \delta_{\lambda_y,-\lambda_k}}{\sqrt{(p \cdot b_{-1}) (k \cdot b_{-1})}} \{ (k \cdot n_{\lambda_y}) \left[ (p \cdot n_{\lambda_y}) \left( b_1^\nu b_1^\alpha \mu - b_1^\mu b_1^\nu \alpha \right) + (p \cdot b_{-1}) \left( b_1^\mu b_{-1}^\nu - b_{-1}^\mu n_{-\lambda_y}^\nu \right) \right] + (k \cdot b_{-1}) \left[ (p \cdot n_{\lambda_y}) \left( n_{-\lambda_y}^\mu b_1^\nu - b_{-1}^\mu b_1^\nu \right) + (p \cdot b_{-1}) \left( n_{-\lambda_y}^\mu b_1^\nu - b_{-1}^\mu n_{-\lambda_y}^\nu \right) \right] \},\]

\[
M^{\lambda_y,\lambda_k}(p,k;\gamma^\mu \gamma^\nu \gamma^\alpha) = \frac{\delta_{\lambda_y,-\lambda_k}}{\sqrt{(p \cdot b_{-1}) (k \cdot b_{-1})}} \{ (p \cdot b_{-1}) \left[ (k \cdot b_{-1}) \left( n_{-\lambda_y}^\alpha b_1^\nu \mu - b_{-1}^\alpha n_{-\lambda_y}^\nu \right) + n_{-\lambda_y}^\alpha (n_{-\lambda_y}^\mu b_1^\nu - b_{-1}^\nu n_{-\lambda_y}^\mu) \right] + (k \cdot n_{-\lambda_y}) \left[ (k \cdot n_{-\lambda_y}) \left( b_1^\mu b_{-1}^\nu \alpha - b_1^\nu b_{-1}^\mu \alpha \right) + \left( n_{-\lambda_y}^\mu n_{-\lambda_y}^\nu \right) \right] + (p \cdot n_{\lambda_y}) \left[ (k \cdot n_{-\lambda_y}) \left( n_{-\lambda_y}^\mu b_{-1}^\nu - b_{-1}^\nu n_{-\lambda_y}^\mu \right) + \left( n_{-\lambda_y}^\mu n_{-\lambda_y}^\nu \right) \right] \}.
\]
These relations can be used as universal functions for calculations of processes with massless fermions.

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FIGURE CAPTIONS

Fig. 1 Feynman diagrams for the process $e^- e^+ \rightarrow W^- W^+$

Fig. 2 Feynman diagrams for the process $e^- e^+ \rightarrow f \bar{f}$

Fig. 3 One of the possible Feynman diagrams of $e^+ e^- \rightarrow e^+ e^- e^+ e^- e^+ e^-$. 