Recovering part of the quantum boundary from information causality

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Recently, the principle of information causality has appeared as a good candidate for an information-theoretic principle that would single out quantum correlations among more general non-signalling models. Here we present results going in this direction; namely we show that part of the boundary of quantum correlations actually emerges from information causality.

I. INTRODUCTION

Non-locality is a central feature of quantum mechanics (QM) and a powerful resource for processing information. However, as Tsirelson [1] first proved, the amount of non-locality allowed by QM is limited. In a seminal paper, Popescu and Rohrlich [2] showed that this limitation is not a consequence of relativity. Indeed, there exist theories which are more non-local than QM yet do not allow for superluminal signalling. Identifying the physical principles underlying the limits to quantum non-locality is now a central problem in foundational QM.

Recently, several works have studied the physical and information-theoretic properties of general non-signalling models. Surprisingly, it appears that these models have numerous properties in common with QM, such as no-cloning [3, 4], no-broadcasting [5], monogamy of correlations [3] and information-disturbance trade-offs [6]. General non-signalling models also allow for secure key distribution [7, 8] as well as quantum-like dynamical processes [9]. Therefore, none of these properties, usually thought of as being typically quantum, are useful for separating quantum from post-quantum correlations.

On the other hand, it is known that some particular post-quantum correlations have extremely powerful communication properties. For instance, the availability of PR boxes - the paradigmatic example of post-quantum correlations - makes communication complexity trivial [10]. However, communication complexity is not trivial in QM [11], and it is strongly believed not to be trivial in nature. Therefore correlations which collapse communication complexity, such as PR box correlations, appear unlikely to exist. More recently, a similar conclusion has been shown to hold for two classes of noisy PR boxes [12, 13]. However, there is a large class of post-quantum correlations for which it is still unknown whether communication complexity collapses or not.

In parallel, non-locality has also been studied from the point of view of non-local computation [14, 15]. Remarkably, here Tsirelson’s bound (of quantum non-locality) naturally appears, since all post-quantum correlations violating this bound offer an advantage over classical and quantum correlations. It is also known that part of the quantum boundary emerges from non-locality swapping [5, 15] (an analogue of entanglement swapping), although the origin of this connection is still not understood. Finally Tsirelson’s bound also appears in theories with relaxed uncertainty relations [16].

More recently, Pawlowski et al. [17] have introduced a new physical principle, the principle of information causality (IC), which is satisfied by both classical and quantum correlations. The essence of IC is that the communication of m classical bits can cause a potential information gain of at most m bits. As is the case for non-local computation, Tsirelson’s bound naturally emerges, since all correlations exceeding Tsirelson’s bound violate the principle of IC. Therefore IC is a potential candidate for separating quantum from post-quantum correlations. However, Tsirelson’s bound identifies only one point on the boundary of the set of quantum correlations. There are also post-quantum correlations which lie below Tsirelson’s bound. Thus, while the emergence of Tsirelson’s bound from IC is a remarkable feature, it is not sufficient for singling out quantum correlations. More generally, one aims at finding a principle underlying the full quantum boundary.

In the present paper, we show that part of the quantum boundary actually emerges from IC. More precisely, we show that in two 2-dimensional slices of the binary-input/binary-output non-signalling polytope, the IC criterion analytically coincides with the quantum boundary.

The organisation of the paper is the following. In Section II we review the geometrical approach to non-signalling correlations, while in Section III we review IC. In Section IV, we study the link between IC and the quantum boundary.

II. GEOMETRY OF NON-SIGNALLING BOXES

It will be convenient to describe bipartite non-signalling correlations in terms of black boxes shared between two parties, Alice and Bob. Alice and Bob input variables x and y at their ends of the box respectively, and receive outputs a and b. The behaviour of a given correlation box is fully described by a set of joint probabilities P(ab|xy). We focus on the case of
binary inputs and outputs \((a, b, x, y \in \{0, 1\})\), for which
\[
P(ab|xy) = \frac{1}{4} \left[ 1 + (-1)^a C_x + (-1)^b C_y + (-1)^{a+b} C_{xy} \right]
\]
where \(\oplus\) is addition modulo 2, and the correlators are given by
\[
C_{xy} = \sum_{a' = b'} P(a'b'|xy) - \sum_{a' \neq b'} P(a'b'|xy),
\]
and the marginals by
\[
C_x = \sum_{b'} [P(0b'|x=0) - P(1b'|x=0)], \quad C_y = \sum_{a'} [P(a'0|y=0) - P(a'1|y=0)].
\]
In this case, which corresponds to the famous Clauser-Horne-Shimony-Holt (CHSH) [18] scenario, the full set of non-signalling boxes forms an 8-dimensional polytope [19] which has 24 vertices: 8 extremal non-local boxes and 16 local deterministic boxes. The extremal non-local correlations have the form:
\[
P_{NL}^{\mu \nu \sigma}(ab|xy) = \begin{cases} 
\frac{1}{2} & \text{if } a \oplus b = xy \oplus \mu x \oplus \nu y \oplus \sigma \\
0 & \text{otherwise}
\end{cases}
\]
where \(\mu, \nu, \sigma \in \{0, 1\}\), and the canonical PR box corresponds to \(P = I_{NL}^{00}\). Similarly, the local deterministic boxes are described by
\[
P_L^{\mu \nu \sigma}(ab|xy) = \begin{cases} 
1 & \text{if } a = \mu x \oplus \nu \quad b = \sigma y \oplus \tau \\
0 & \text{otherwise}
\end{cases}
\]
The set of local boxes forms a subpolytope of the full non-signalling polytope, and has facets which correspond to Bell inequalities - here the CHSH inequality
\[
C_{00} + C_{01} + C_{10} - C_{11} \leq 2,
\]
and its symmetries. Note that there are 8 symmetries of the CHSH inequality (any odd number of terms on the left hand side of (1) can have a minus sign), and that each CHSH inequality is violated by one of the extremal non-local boxes.

The set of quantum boxes, i.e. correlations obtainable by performing local measurements on a quantum state (of any dimension), is sandwiched between the local polytope and the full non-signalling polytope. In particular, quantum correlations satisfy a variant of inequality (1), where the right hand side is replaced by \(2\sqrt{2}\), a value known as Tsirelson’s bound. The quantum set is a convex body, although it is not a polytope. Thus, its boundary is described by a smooth curve. For binary inputs and outputs, Tsirelson, Landau and Masanes (TLM) [20] have (independently) derived a necessary and sufficient criterion for a set of correlators \(C_{xy}\), to admit a quantum description. In the form of Landau, \(C_{xy}\) must satisfy:
\[
|C_{00}C_{10} - C_{01}C_{11}| \leq \sum_{j=0,1} \sqrt{(1 - C_{0j}^2)(1 - C_{1j}^2)}
\]  
(2)

However, when considering the full probability distribution (including the marginals), this criterion remains necessary but is no longer sufficient. Recently, a refinement of (2) has been derived by Navascues, Pironio, and Acin (NPA) [21]. Their work improves (2) in that it incorporates the marginals of the probability distribution. The NPA criterion reads:
\[
|\text{asin}D_{00} + \text{asin}D_{01} + \text{asin}D_{10} - \text{asin}D_{11}| \leq \pi,
\]
where \(D_{xy} = (C_{xy} - C_{x0}C_{y0})/\sqrt{(1 - C_{x0}^2)(1 - C_{y0}^2)}\). Note that for vanishing marginals, (3) is equivalent to (2). Note also that (3) is in general not sufficient for a probability distribution to be quantum-realizable; to determine whether a probability distribution is quantum or not, one has to test a hierarchy of semi-definite programming conditions [22].

### III. INFORMATION CAUSALITY

Let us now briefly review the principle of IC. The authors of [17] considered the following communication task, which is similar to random access coding [23] and oblivious transfer [24, 25]. Alice and Bob, who are separated in space, have access to non-signalling resources such as shared randomness, entanglement or (in principle) PR boxes. Alice receives \(N\) i.i.d. random bits \(\vec{a} = (a_1, a_2, \ldots, a_N)\), while Bob receives a random variable \(b \in \{1, 2, \ldots, N\}\). Alice then sends \(m\) classical bits to Bob, who must output a single bit \(\beta\) with the aim of guessing the value of Alice’s \(b\)-th bit \(a_b\). Their degree of success at this task is measured by
\[
I = \sum_{K=1}^{N} I(a_K : \beta | b = K),
\]
where \(I(a_K : \beta | b = K)\) is the Shannon mutual information between \(a_K\) and \(\beta\). The principle of IC states that physically allowed theories must have \(I \leq m\). Indeed, it was proved in [17] that both classical and quantum correlations satisfy this condition. Moreover, suppose that Alice and Bob share arbitrary binary-input/binary-output non-signalling correlations corresponding to conditional probabilities \(P(ab|xy)\). A condition under which IC is violated was derived in [17] - based on a construction by van Dam [10] and Wolf and Wolf-Schlegle [25] - for a specific realization of the Alice-Bob channel. It goes as follows. Define \(P_I\) and \(P_{II}\):
\[
P_I = \frac{1}{2} \left[ P(a \oplus b = 0|00) + P(a \oplus b = 0|10) \right],
\]
\[
= \frac{1}{2} \left[ 2 + C_{00} + C_{10} \right],
\]
\[
P_{II} = \frac{1}{2} \left[ P(a \oplus b = 0|01) + P(a \oplus b = 1|11) \right].
\]
\[
= \frac{1}{2} \left[ 2 + C_{01} - C_{11} \right].
\]
\(P_I\) and \(P_{II}\) then satisfy
\[
I \equiv \sum_{K=1}^{N} I(a_K : \beta | b = K),
\]
where \(I(a_K : \beta | b = K)\) is the Shannon mutual information between \(a_K\) and \(\beta\). The principle of IC states that physically allowed theories must have \(I \leq m\). Indeed, it was proved in [17] that both classical and quantum correlations satisfy this condition. Moreover, suppose that Alice and Bob share arbitrary binary-input/binary-output non-signalling correlations corresponding to conditional probabilities \(P(ab|xy)\). A condition under which IC is violated was derived in [17] - based on a construction by van Dam [10] and Wolf and Wolf-Schlegle [25] - for a specific realization of the Alice-Bob channel. It goes as follows. Define \(P_I\) and \(P_{II}\):
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= \frac{1}{2} \left[ 2 + C_{00} + C_{10} \right],
\]
\[
P_{II} = \frac{1}{2} \left[ P(a \oplus b = 0|01) + P(a \oplus b = 1|11) \right].
\]
QM. For such correlations, it was not known whether the principle of IC singles out exactly those allowed by quantum physics. We now offer a partial answer to this question.

IV. IC AND THE QUANTUM BOUNDARY

Here we investigate the link between IC and the set of correlations achievable in QM. We would like to determine whether the entire quantum boundary can be recovered from the principle of IC. It will be convenient to re-express the condition \( 5 \) for the violation of IC in terms of the correlators \( C_{xy} \):

\[
(C_{00} + C_{10})^2 + (C_{01} - C_{11})^2 > 4. \tag{6}
\]

Interestingly, this is equivalent to a violation of Uffink’s quadratic inequality \[26\]; note that Uffink’s inequality is known to be strictly weaker than the TLM criterion \[27\].

In the following, we compare \( 6 \) with the TLM and NPA criteria for quantumness. We shall investigate several two-dimensional slices of the non-signalling polytope, which can be grouped into two families. More precisely, we consider noisy PR boxes of the form:

\[
PR_{\alpha,\beta} = \alpha PR + \beta B + (1 - \alpha - \beta) I, \tag{7}
\]

where \( B \) is an extremal non-local box in the first family, and an extremal local deterministic box in the second. Remarkably, in the first family we find two different slices of the polytope where boxes that satisfy IC coincide analytically with the set of quantum boxes. In other words, in these slices IC exactly singles out quantum correlations in that all post-quantum correlations violate IC. Note that because of the symmetry of the polytope, it is sufficient here to focus on non-signalling boxes violating the CHSH inequality \( 1 \), and not those which violate the 7 other symmetries of CHSH; basically, a non-signalling box can never violate more than one symmetry of CHSH.

Family 1. We first consider correlations of the form \( 7 \), where \( B = P_{NL}^{\mu\nu\sigma\tau} \), and \( \mu\nu\sigma\tau \) can take any values except for 000 and 001 (since these are co-linear with PR and I). The corresponding correlators are given by \( C_{00} = \alpha + (-1)^{\sigma\beta} \beta \), \( C_{01} = \alpha + (-1)^{\mu\sigma\beta} \beta \), \( C_{10} = \alpha + (-1)^{\mu\beta\sigma} \beta \), and \( C_{11} = -\alpha + (-1)^{\mu\beta\sigma\tau} \beta / 2 \).

We see from \( 4 \) that if boxes of this form are to be quantum-realizable then we require that \( \alpha^2 + \beta^2 < \frac{1}{2} \). Note that here the TLM criteria is necessary and sufficient for quantumness since the probability distribution given by boxes in this family has a specific form \[28\]. On the other hand, we see from \( 6 \) that if \( B = PR_2 = P_{NL}^{010} \), then IC is violated when

\[
\alpha^2 + \beta^2 > \frac{1}{2}. \tag{8}
\]

Thus, in this particular slice of the non-signalling polytope, a box violates IC if and only if it is post-quantum (Fig. 1). Note that here we could have chosen \( B = PR_2 = P_{NL}^{011} \) as well.

The above proof is easily adapted to another slice. By exchanging the roles of Alice and Bob, the same can also be seen to hold in the slice where \( B = PR_4 = P_{NL}^{100} \) (or equivalently \( B = PR_3 = P_{NL}^{101} \)).

Finally, note that in the case where \( B = PR_4 = P_{NL}^{111} \), the criterion for violating IC reduces to \( \alpha > \frac{1}{\sqrt{2}} \). Thus boxes below Tsirelson’s bound are not known to violate IC in this slice (Fig. 2). We stress that this does not imply that there exist post-quantum boxes lying below Tsirelson’s bound which do not violate IC. The fact that boxes which satisfy \( 6 \) also violate IC follows from considering a particular strategy for using the boxes, found in \[17\]. It remains possible that a different strategy could be used to show that all post-quantum correlations violate IC in this slice as well.

Family 2. Next, we consider correlations of the form \( 7 \), where \( B = P_{L}^{\mu\nu\sigma\tau} \) with \( \mu\sigma\nu\tau = 0 \); note that these are the local deterministic boxes sitting on the CHSH facet below IC & QM

$$\text{CHSH} = C_{00} + C_{01} + C_{10} + C_{11} = 4\beta$$

FIG. 1: (Color online) A slice of the non-signalling polytope where correlations violate IC if and only if they are post-quantum. Above the blue dashed curve, IC is violated; below, correlations are quantum realizable.

$$\text{CHSH} = C_{00} + C_{01} + C_{10} + C_{11} = 4\beta$$

FIG. 2: (Color online) A slice of the non-signalling polytope where post-quantum boxes which lie below Tsirelson’s bound (CHSH = \( 2\sqrt{2} \)) are not known to violate IC. The red solid line is the upper limit on quantum correlations, as given by the NPA criteria.
In this case, the correlators are given by

\[ \text{FIG. 3: (Color online) A slice of the non-signalling polytope where IC does not single out quantum correlations.} \]

the PR box. For simplicity, we will focus here on \( B = P^{0000}_L \).

In this case, the correlators are given by \( C_{00} = C_{01} = C_{10} = \alpha + \beta, C_{11} = \beta - \alpha \), and the marginals by \( C'_0 = C'_1 = C'_b = C'_1 = \beta \). It follows from (9) that IC is violated whenever

\[ (\alpha + \beta)^2 + \alpha^2 > 1. \tag{9} \]

However, this does not coincide with the NPA criterion (3). Fig. 3 shows clearly the discrepancy between the quantum boundary, or more precisely the upper bound given by NPA, and the IC condition (9). Let us re-iterate that the bound (9) follows from a particular strategy in [17] for using boxes to violate IC. Thus it might still be the case that a better strategy would single out quantum correlations in this particular slice.

V. CONCLUSION

We have shown that in the binary-input/binary-output non-signalling polytope, part of the quantum boundary emerges from the principle of IC. The central question is now whether this connection can be extended to the full non-signalling polytope, which would establish IC as the information-theoretic principle singling out quantum correlations.

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