About the classification of the holonomy algebras of Lorentzian manifolds

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Abstract

The classification of the holonomy algebras of Lorentzian manifolds can be reduced to the classification of irreducible subalgebras \( h \subset \mathfrak{so}(n) \) that are spanned by the images of linear maps from \( \mathbb{R}^n \) to \( h \) satisfying an identity similar to the Bianchi one. T. Leistner found all such subalgebras and it turned out that the obtained list coincides with the list of irreducible holonomy algebras of Riemannian manifolds. The natural problem is to give a simple direct proof to this fact. We give such proof for the case of semisimple not simple Lie algebras \( h \).

Keywords: holonomy algebra, Lorentzian manifold, Berger algebra, weak-Berger algebra, Tanaka prolongation.

1 Introduction

M. Berger [1, 2] classified possible connected irreducible holonomy groups \( H \subset \text{SO}(n) \) of not locally symmetric Riemannian manifolds using the representation theory. It turned out that these groups act transitively on the unite sphere of the tangent space. J. Simens [3] and recently in a simple geometric way C. Olmos [4] proved this result directly.

The classification of the holonomy algebras (i.e. the Lie algebras of the holonomy groups) of Lorentzian manifolds can be reduced to the classification of irreducible weak-Berger subalgebras \( h \subset \mathfrak{so}(n) \), i.e. subalgebras \( h \subset \mathfrak{so}(n) \) that are spanned by the images of linear maps from the space

\[
\mathcal{P}(h) = \{ P \in (\mathbb{R}^n)^* \otimes h | (P(X)Y, Z) + (P(Y)Z, X) + (P(Z)X, Y) = 0, \ X, Y, Z \in \mathbb{R}^n \}.
\]

It is easy to see that if \( h \subset \mathfrak{so}(n) \) is the holonomy algebras of a Riemannian manifold, then it is a weak-Berger algebra. The inverse statement is absolutely not obvious, nevertheless it is true and it is proven by Th. Leistner in [5].

If \( n \) is even and \( h \subset \mathfrak{so}(n) \) is of complex type, i.e. \( h \subset \mathfrak{u}(\frac{n}{2}) \), then it can be shown that \( \mathcal{P}(h) \simeq (h \otimes \mathbb{C})^{(1)} \), where \( (h \otimes \mathbb{C})^{(1)} \) is the first prolongation of the subalgebra \( h \otimes \mathbb{C} \subset \mathfrak{gl}(\frac{n}{2}, \mathbb{C}) \) (cf. [5] and [6]). Using that and the classification of irreducible representations with non-trivial prolongation, Leistner showed that if \( h \subset \mathfrak{u}(\frac{n}{2}) \) is a weak-Berger subalgebra, then it is the holonomy algebra of a Riemannian manifold.

The situation when \( h \subset \mathfrak{so}(n) \) is of real type (i.e. not of complex type) is much more difficult. In this case Leistner considered the complexification \( h \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C}) \), which is irreducible. He
used the classification of irreducible representations of complex semisimple Lie algebras, found a criteria in terms of weights for such representation $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ to be a weak-Berger algebra and considered case by case simple Lie algebras $\mathfrak{h} \otimes \mathbb{C}$, and then semisimple Lie algebras (the problem is reduced to the semisimple Lie algebras of the form $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$, where $\mathfrak{k}$ is simple, and again different possibilities for $\mathfrak{k}$ were considered).

We consider the case of semisimple not simple irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ with irreducible complexification $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$. In a simple way we show that it is enough to treat the case when $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$, where $\mathfrak{k} \subset \mathfrak{sp}(2m, \mathbb{C})$ is a proper irreducible subalgebra, and the representation space is the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^{2m}$. We show that in this case $\mathcal{P}(\mathfrak{h})$ coincides with $\mathbb{C}^2 \otimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is the first Tanaka prolongation the non-positively graded Lie algebra $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0$,

where $\mathfrak{g}_{-2} = \mathbb{C}$, $\mathfrak{g}_{-1} = \mathbb{C}^{2m}$, $\mathfrak{g}_0 = \mathbb{C} \oplus \mathbb{C}$ id$_{\mathbb{C}^{2m}}$, and the grading is defined by the element $-\text{id}_{\mathbb{C}^{2m}}$. We prove that if $\mathcal{P}(\mathfrak{h})$ is non-zero, then $\mathfrak{g}_1$ is isomorphic to $\mathbb{C}^{2m}$, the second Tanaka prolongation $\mathfrak{g}_2$ is isomorphic to $\mathbb{C}$, and $\mathfrak{g}_3 = 0$. Then, the full Tanaka prolongation defines the simple $|2|$-graded complex Lie algebra $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

It is well known that such Lie algebra defines (up the duality) a simply connected Riemannian symmetric space; the holonomy algebra of this space coincides with $\mathfrak{h} \subset \mathfrak{so}(n)$. Thus, if the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is semisimple and not simple, and $\mathcal{P}(\mathfrak{h}) \neq 0$, then we indicate a Riemannian manifold with the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$.

More details about the holonomy algebras of Lorentzian manifolds can be found in [7, 8].

2 Holonomy algebras of Riemannian manifolds

Irreducible holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(n)$ of not locally symmetric Riemannian manifolds are exhausted by $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$, $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{2}) \oplus \mathfrak{sp}(1)$, $\mathfrak{sp}(\frac{n}{2})$, $G_2 \subset \mathfrak{so}(7)$ and $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$. This list (up to some corrections) obtained M. Berger [1, 2]. Berger classified irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ spanned by the images of the maps from the space

$$\mathcal{R}(\mathfrak{h}) = \{ R \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{h} | R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \text{ for all } X, Y, Z \in \mathbb{R}^n \}$$

of algebraic curvature tensors of type $\mathfrak{h}$ under the condition that the space

$$\mathcal{R}^\nabla(\mathfrak{h}) = \{ S \in (\mathbb{R}^n)^* \otimes \mathcal{R}(\mathfrak{h}) | S_X(Y, Z) + S_Y(Z, X) + S_Z(X, Y) = 0 \text{ for all } X, Y, Z \in \mathbb{R}^n \}$$

of algebraic covariant derivatives of the curvature tensors of type $\mathfrak{h}$ is not trivial. Berger used the classification of irreducible representations of compact Lie groups. The connected Lie subgroups of $\text{SO}(n)$ corresponding to the above subalgebras of $\mathfrak{so}(n)$ mostly exhaust groups of isometries acting transitively on the unite sphere of dimension $n - 1$, and the result of Berger can be reformulated in the following form: if the irreducible holonomy group of a Riemannian manifold $(M, g)$ does not act transitively on the unite sphere of the tangent space, then $(M, g)$ is locally symmetric. A direct proof of this statement obtained in algebraic way J. Simens [3], and recently an elegant geometric proof obtained C. Olmos [4].
The spaces $\mathcal{R}(\mathfrak{h})$ for the irreducible holonomy algebras of Riemannian manifolds $\mathfrak{h} \subset \mathfrak{so}(n)$ are computed by D. V. Alekseevsky in [9]. For $R \in \mathcal{R}(\mathfrak{h})$ define its Ricci tensor by

$$\text{Ric}(R)(X,Y) = \text{tr}(Z \mapsto R(Z,X)Y),$$

$X,Y \in \mathbb{R}^n$. The space $\mathcal{R}(\mathfrak{h})$ admits the following decomposition into $\mathfrak{h}$-modules:

$$\mathcal{R}(\mathfrak{h}) = \mathcal{R}_0(\mathfrak{h}) \oplus \mathcal{R}_1(\mathfrak{h}) \oplus \mathcal{R}'(\mathfrak{h}),$$

where $\mathcal{R}_0(\mathfrak{h})$ consists of the curvature tensors with zero Ricci curvature, $\mathcal{R}_1(\mathfrak{h})$ consists of tensors annihilated by $\mathfrak{h}$ (this space is either zero or one-dimensional), $\mathcal{R}'(\mathfrak{h})$ is the complement to these two spaces. If $\mathcal{R}(\mathfrak{h}) = \mathcal{R}_1(\mathfrak{h})$, then any Riemannian manifold with the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is locally symmetric. Such subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is called a symmetric Berger algebra. The holonomy algebras of irreducible Riemannian symmetric spaces are exhausted by $\mathfrak{so}(n)$, $\mathfrak{u}(\mathbb{H})$, $\mathfrak{sp}(\mathbb{H}) \oplus \mathfrak{sp}(1)$ and by symmetric Berger algebras $\mathfrak{h} \subset \mathfrak{so}(n)$.

It is known that simply connected indecomposable symmetric Riemannian manifolds $(M,g)$ can be reconstructed using its holonomy algebra $\mathfrak{h}$, and only if $\mathfrak{h}$ consists of the curvature tensors with zero Ricci curvature, $\mathfrak{h}$ is the kernel of the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ and $R \in \mathcal{R}(\mathfrak{h})$ of the curvature tensor of $(M,g)$ at a point. For that define the structure of the Lie algebra on the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ in the following way:

$$[A,B] = [A,B]_\mathfrak{h}, \quad [A,X] = AX, \quad [X,Y] = R(X,Y), \quad A,B \in \mathfrak{h}, X,Y \in \mathbb{R}^n.$$  

Then $M = G/H$, where $G$ is the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$, and $H \subset G$ is the connected Lie subgroup corresponding to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

If the symmetric space is quaternionic-Kählerian, then $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{f} \subset \mathfrak{so}(4k)$, where $n = 4k$ and $\mathfrak{f} \subset \mathfrak{sp}(k)$. The complexification of $\mathfrak{g} \oplus \mathbb{R}^{4k}$ is equal to $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{t}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k})$, where $\mathfrak{t} = \mathfrak{f} \otimes \mathbb{C} \subset \mathfrak{sp}(2k, \mathbb{C})$. Let $e_1, e_2$ be the standard basis of $\mathbb{C}^2$, and let

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the basis of $\mathfrak{sl}(2, \mathbb{C})$. We obtain the following $\mathbb{Z}$-grading of $\mathfrak{g} \otimes \mathbb{C}$:

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathbb{C}F \oplus e_2 \otimes \mathbb{C}^{2k} \oplus (\mathfrak{t} \oplus \mathbb{C}H) \oplus e_1 \otimes \mathbb{C}^{2k} \oplus \mathbb{C}E.$$  

Conversely, any such simple $\mathbb{Z}$-graded Lie algebra defines up to the duality a simply connected quaternionic-Kählerian symmetric space.

### 3 Weak curvature tensors

The spaces $\mathcal{P}(\mathfrak{h})$ are computed in [6]. Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible subalgebra. There exists the decomposition

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}),$$

where $\mathcal{P}_0(\mathfrak{h})$ is the kernel of the $\mathfrak{h}$-equivariant map

$$\tilde{\text{Ric}} : \mathcal{P}(\mathfrak{h}) \to \mathbb{R}^n, \quad \tilde{\text{Ric}}(P) = \sum_{i=1}^{n} P(e_i)e_i$$

($e_1, \ldots, e_n$ is an orthogonal basis of $\mathbb{R}^n$), and $\mathcal{P}_1(\mathfrak{h})$ is the orthogonal complement of $\mathcal{P}_0(\mathfrak{h})$ in \(\mathcal{P}(\mathfrak{h})\). The space $\mathcal{P}_1(\mathfrak{h})$ is either trivial or it is isomorphic to $\mathbb{R}^n$. If $n \geq 4$, then $\mathcal{P}_0(\mathfrak{h}) \neq 0$ if and only if $\mathcal{R}_0(\mathfrak{h}) \neq 0$. Next, $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ if and only if $\mathcal{R}_1(\mathfrak{h}) \simeq \mathbb{R}$. For the symmetric Berger algebras it holds $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ and $\mathcal{P}_0(\mathfrak{h}) = 0$. 

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4 Tanaka prolongations

Consider a $\mathbb{Z}$-graded Lie algebra of the form

$$g = g_{-2} \oplus g_{-1} \oplus g_0.$$  

For $k \geq 1$, the $k$-th Tanaka prolongation is defined by the induction

$$g_k = \{ u \in (g_{-2}^* \otimes g_{k-2}) \oplus (g_{-1}^* \otimes g_{k-1}) | u([X, Y]) = [u(X), Y] + [X, u(Y)], X, Y \in g_{-2} \oplus g_{-1} \}.$$  

Let $k \geq 1$, and $l \geq 0$. For $u \in g_k$ and $v \in g_l$ define Lie brackets $[u, v] \in g_{k+l}$ by the condition

$$[u, v]X = [[u, X], v] + [u, [v, X]], \quad X \in g_{-2} \oplus g_{-1};$$  

the Lie brackets of $u \in g_k$ and $X \in g_{-2} \oplus g_{-1}$ are defined as $[u, X] = Xu$. This gives the structure of a Lie algebra on the vector space $\oplus_{k=2}^\infty g_k$.

Let $\mathfrak{t} \subseteq \mathfrak{sp}(2m, \mathbb{C})$ be a subalgebra, $m \geq 2$. Consider the Lie algebra

$$g_{-2} \oplus g_{-1} \oplus g_0, \quad g_{-2} = \mathbb{C}F, \quad g_{-1} = \mathbb{C}^{2m}, \quad g_0 = \mathfrak{t} \oplus \mathbb{C}H$$  

with the non-zero Lie brackets

$$[X, Y] = \Omega(X, Y)F, \quad [A, X] = AX, \quad [A, B] = [A, B]_t, \quad [H, X] = -X, \quad [H, F] = -2F,$$  

where $X, Y \in \mathbb{C}^{2m}, A, B \in \mathfrak{t}$, and $\Omega$ is the symplectic form on $\mathbb{C}^{2m}$.

Lemma 1. It holds

$$g_1 = \{ \varphi \in g_{-1}^* \otimes g_0 | \exists A \in g_{-1}, \varphi(X)Y - \varphi(Y)X = \Omega(X, Y)A, X, Y \in g_{-1} \}.$$  

If $\mathfrak{t} \subsetneq \mathfrak{sp}(2m, \mathbb{C})$ is a proper irreducible subalgebra and $g_1 \neq 0$, then $g_1 \simeq \mathbb{C}^{2m}, g_2 \simeq \mathbb{C}$, and $g_3 = 0$. The Lie algebra

$$g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$$  

is simple.

Proof. Let $u = \psi + \varphi$, where $\psi \in g_{-2}^* \otimes g_{-1}$, and $\varphi \in g_{-1}^* \otimes g_0$. The condition $u \in g_1$ is equivalent to the equations

$$[\varphi(X), F] = \Omega(\psi(F), Y)F, \quad \varphi(X)Y - \varphi(Y)X = \Omega(X, Y)\psi(F).$$

The first statement of the lemma is that the second equation implies the first one.

Let us denote $\mathbb{C}^{2m}$ by $V$. First suppose that $\mathfrak{t} = \mathfrak{sp}(V)$. Let us find $g_1$. We have the following isomorphisms of the $\mathfrak{sp}(V)$-modules: $g_{-2}^* \otimes g_{-1} \simeq V$, and

$$g_{-1}^* \otimes g_0 \simeq V \otimes (\mathfrak{sp}(V) \oplus \mathbb{C}) = V \oplus (V \oplus V_{3\pi_1} \oplus V_{\pi_1+\pi_2}),$$  

where $V_\Lambda$ denotes the irreducible $\mathfrak{sp}(V)$-module with the highest weight $\Lambda$. By the definition, the intersection of $g_1$ and $g_{-1}^* \otimes g_0$ coincides with

$$(\mathfrak{sp}(V) \oplus \mathbb{C}H)^{(1)} = (\mathfrak{sp}(V))^{(1)} \simeq \mathbb{C}^3V \simeq V_{3\pi_1}.$$
Clearly, the intersection of $\mathfrak{g}_1$ and $\mathfrak{g}^* \otimes \mathfrak{g}_{-1}$ is trivial. Consequently, if $\mathfrak{g}_1$ is different from $\mathfrak{sp}(V)^{(1)}$, then $\mathfrak{g}_1$ contains a submodule isomorphic to $V$. Any $\mathfrak{sp}(V)$-equivariant map from $V$ to $(\mathfrak{g}^{*}_{-2} \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_0)$ is of the form

$$Z \mapsto \psi^Z + \varphi^Z, \quad \psi^Z(F) = aZ, \quad \varphi^Z(Y) = b\Omega(Z,Y)H + cZ \circ Y,$$

where $a, b, c \in \mathbb{R}$, and $Z \circ Y \in \mathfrak{sp}(V)$ is defined as

$$(Z \circ Y)X = \Omega(Z,X)Y + \Omega(Y,X)Z.$$  

The second equation on $\mathfrak{g}_1$ takes the form

$$-b\Omega(Z,X)Y + b\Omega(Z,Y)X + c(\Omega(Y,Z)X - \Omega(X,Z)Y + 2\Omega(Y,X)Z) = a\Omega(X,Y)Z.$$  

This equation should hold for all $X, Y, Z \in V$, and it is equivalent to $b = -c = -\frac{1}{2}a$ (since $\dim V \geq 4$). The second equation on $\mathfrak{g}_1$ takes the form

$$-2b\Omega(Z,Y) = a\Omega(Z,Y)$$

and it follows from the first one. Thus the orthogonal complement to $(\mathfrak{sp}(2m, \mathbb{C}))^{(1)}$ in $\mathfrak{g}_1$ is isomorphic to $V$, and the isomorphism is given by

$$Z \in V \mapsto \psi^Z + \varphi^Z, \quad \psi^Z(F) = 2Z, \quad \varphi^Z(Y) = -\Omega(Z,Y)H + Z \circ Y, \quad Y \in V.$$  

Let $\mathfrak{t} \subseteq \mathfrak{sp}(V)$ be a proper irreducible subalgebra. It is clear that

$$\mathfrak{g}_1 = ((\mathfrak{g}^{*}_{-2} \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_0)) \cap (\mathfrak{sp}(V) \oplus CH)_{1},$$

and $\mathfrak{h}^{(1)} = (\mathfrak{g}^{*}_{-1} \otimes \mathfrak{g}_0) \cap \mathfrak{sp}(V)^{(1)}$. It is known that $\mathfrak{h}^{(1)} = 0$. Consequently, if $\mathfrak{g}_1 \neq 0$, than $\mathfrak{g}_1$ is isomorphic to $V$ and it is included diagonally into $V \oplus \mathfrak{sp}(V)^{(1)}$.

Consider the full Tanaka prolongation $\mathfrak{g} = \bigoplus_{i=-2}^{\infty} \mathfrak{g}_i$. Let $\mathfrak{g}^0 = \bigoplus_{i=0}^{\infty} \mathfrak{g}_i \subset \mathfrak{g}$. We claim that $\mathfrak{g}$ is a primitive $\mathbb{Z}$-graded Lie algebra, i.e. $\mathfrak{g}^0 \subset \mathfrak{g}$ is a maximal graded subalgebra and $\mathfrak{g}^0$ contains no graded ideals of $\mathfrak{g}$ except $\{0\}$. Indeed, suppose that there exists a subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ such that $\mathfrak{g}^0 \subset \tilde{\mathfrak{g}}$. Then $aF + X \in \tilde{\mathfrak{g}}$ for some $a \in \mathbb{R}$, $X \in \mathfrak{g}_{-1}$. If $a \neq 0$, then taking $u \in \mathfrak{g}_1$, we get $0 \neq u(F) \in \tilde{\mathfrak{g}} \cap \mathfrak{g}_{-1}$, i.e. we may assume that there exists non-zero $X \in \mathfrak{g}_{-1}$ such that $X \in \tilde{\mathfrak{g}}$. Since $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ irreducible, we get $\mathfrak{g}_{-1} \subset \tilde{\mathfrak{g}}$. Finally, $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, i.e. $\mathfrak{g}_{-2} \subset \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \mathfrak{g}$. Suppose now that $\tilde{\mathfrak{g}} = \bigoplus_{i=0}^{\infty} \tilde{\mathfrak{g}}_i \subset \mathfrak{g}^0$ is a graded ideal. For $X \in \mathfrak{g}_{-1}$ and $\xi \in \mathfrak{g}_0$ it holds $[\xi, X] \in \mathfrak{g}_{-1}$. On the other hand, $[\xi, X] \in \tilde{\mathfrak{g}}$, and we get $[\xi, X] = 0$ for all $X \in \mathfrak{g}_{-1}$. This implies $\mathfrak{g}_0 = 0$. In the same way it can be shown that $\mathfrak{g}_k = 0$ for all $k \geq 2$. Thus, $\mathfrak{g}$ is a primitive $\mathbb{Z}$-graded Lie algebra. If $\mathfrak{g}$ is infinite dimensional, then from [10] Th. 6.1] it follows that $\mathfrak{g}_0 = \mathfrak{sp}(V) \oplus CH$, which gives a contradiction, since we assume that $\mathfrak{t} \subset \mathfrak{sp}(V)$ is a proper subalgebra. Thus, $\mathfrak{g}$ is of finite dimension. Since the element $H \in \mathfrak{g}_0$ defines the $\mathbb{Z}$-grading of $\mathfrak{g}$, any ideal $\mathfrak{t} \subset \mathfrak{g}$ is graded. As in the above claim it can be shown that either $\mathfrak{t} = \mathfrak{g}$ or $\mathfrak{t} = \mathfrak{g}$, i.e. $\mathfrak{g}$ is a simple Lie algebra. For the Killing form of a $\mathbb{Z}$-graded Lie algebra it holds $b(\mathfrak{g}_k, \mathfrak{g}_l) = 0$ for $k \neq -l$. This shows that $\mathfrak{g}_2 \simeq \mathbb{C}$ and $\mathfrak{g}_3 = 0$. The lemma is proved.  

\section{Semisimple not simple weak-Berger algebras}

\textbf{Theorem 1} Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be a semisimple not simple irreducible subalgebra of real type. If $\mathcal{P}(\mathfrak{h}) \neq 0$, then $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a symmetric Riemannian space.
Proof. From the assumption of the theorem it follows that the complexified representation $h \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ is irreducible. Since $h \otimes \mathbb{C}$ is semisimple and not simple, it can be decomposed into the direct sum of two ideals, $h \otimes \mathbb{C} = h_1 \oplus h_2$. The representation of $h_1 \oplus h_2$ on $\mathbb{C}^n$ must be of the from of the tensor product, $\mathbb{C}^n = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$, where $h_1 \subset \mathfrak{gl}(n_1, \mathbb{C})$, $h_2 \subset \mathfrak{gl}(n_2, \mathbb{C})$ are irreducible. Since $h_1 \oplus h_2 \subset \mathfrak{so}(n, \mathbb{C})$, it holds that either $h_1 \subset \mathfrak{so}(n_1, \mathbb{C})$, $h_2 \subset \mathfrak{so}(n_2, \mathbb{C})$, $n_1, n_2 \geq 3$ or $h_1 \subset \mathfrak{sp}(n_1, \mathbb{C})$, $h_2 \subset \mathfrak{sp}(n_2, \mathbb{C})$, $n_1, n_2 \geq 2$. In [6] it is shown in a simple way that $\mathcal{P}(\mathfrak{so}(n_1, \mathbb{C}) \oplus \mathfrak{so}(n_2, \mathbb{C})) \simeq \mathbb{C}^n$, and if $n_1, n_2 \geq 3$, then $\mathcal{P}(\mathfrak{sp}(n_1, \mathbb{C}) \oplus \mathfrak{sp}(n_2, \mathbb{C})) \simeq \mathbb{C}^n$. This implies that if $h_1 \oplus h_2$ is a proper irreducible subalgebra of $\mathfrak{so}(n_1, \mathbb{C}) \oplus \mathfrak{so}(n_2, \mathbb{C})$ or of $\mathfrak{sp}(n_1, \mathbb{C}) \oplus \mathfrak{sp}(n_2, \mathbb{C})$ with $n_1, n_2 \geq 3$, then $\mathcal{P}(h_1 \oplus h_2) = 0$. Note that the holonomy algebras of the Riemannian symmetric spaces

\begin{align*}
SO(n_1 + n_2)/(SO(n_1) \times SO(n_2)), & \quad n_1, n_2 \geq 3, \\
\text{Sp}(n_1 + n_2)/(\text{Sp}(n_1) \times \text{Sp}(n_2)), & \quad n_1, n_2 \geq 1
\end{align*}

are respectively $\mathfrak{so}(n_1) \oplus \mathfrak{so}(n_2)$ and $\mathfrak{sp}(n_1) \oplus \mathfrak{sp}(n_2)$ [2].

Thus, we are left with the case $n_1 = 2$, $h_1 = \mathfrak{sl}(2, \mathbb{C})$, and $h_2 \subset \mathfrak{sp}(n_2, \mathbb{C})$. Let $\mathfrak{t} = h_2$. From Proposition [1] below, Lemma [1] and the considerations of Section [2] it follows that $h = \mathfrak{sp}(1) \oplus \mathfrak{f} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(k) \subset \mathfrak{so}(4k)$ is the holonomy algebra of a quaternionic-Kählerian symmetric space. The theorem is true. □

Proposition 1 Let $\mathfrak{t} \subset \mathfrak{sp}(2m, \mathbb{C})$ be an irreducible subalgebra, $m \geq 2$. Then

$$\mathcal{P}(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{t}) \simeq \mathbb{C}^2 \otimes \mathfrak{g}_1,$$

where $\mathfrak{g}_1$ is the first Tanaka prolongation of the Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \mathbb{C} F \oplus \mathbb{C}^{2m} \oplus (\mathfrak{t} \oplus \mathbb{C} H).$$

Proof. Let $V = \mathbb{C}^{2m}$, let $\Omega$, $\omega$ be the symplectic forms on $V$ and $\mathbb{C}^2$, and let $e_1$, $e_2$ be a basis of $\mathbb{C}^2$ such that $\omega(e_1, e_2) = 1$. Let $F, H, E$ be the basis of $\mathfrak{sl}(2, \mathbb{C})$ as above. For a linear map

$$P : \mathbb{C}^2 \otimes V \to \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{t}$$

and $X \in V$ we write

$$P(e_i \otimes X) = \alpha(e_i \otimes X)E + \beta(e_i \otimes X)F + \gamma(e_i \otimes X)H + T(e_i \otimes X), \quad T(e_i \otimes X) \in \mathfrak{t}, \quad i = 1, 2.$$ 

Let us consider the condition $P \in \mathcal{P}(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{t})$. Let $X, Y, Z \in V$. Taking the vectors $e_1 \otimes X$, $e_1 \otimes Y$, $e_1 \otimes Z$, we get

$$\beta(e_1 \otimes X)\Omega(Y, Z) + \beta(e_1 \otimes Y)\Omega(Z, X) + \beta(e_1 \otimes Z)\Omega(X, Y) = 0.$$ 

Since $\dim V \geq 4$, this implies $\beta(e_1 \otimes X) = 0$ for all $X \in V$. Similarly, considering the vectors $e_2 \otimes X$, $e_2 \otimes Y$, $e_2 \otimes Z$, we get $\alpha(e_2 \otimes X) = 0$.

Considering the vectors $e_1 \otimes X$, $e_1 \otimes Y$, $e_2 \otimes Z$, we obtain

$$\gamma(e_1 \otimes X)\Omega(Y, Z) + \Omega(T(e_1 \otimes X)Y, Z) - \gamma(e_1 \otimes Y)\Omega(X, Z) - \Omega(T(e_1 \otimes Y)X, Z) - \beta(e_2 \otimes Z)\Omega(Y, X) = 0.$$ 

Let $A \in V$ be the dual vector to $\beta|_{e_2 \otimes V}$, i.e. $\beta(e_2 \otimes Z) = \Omega(A, Z)$ for all $Z \in V$. We obtain

$$\gamma(e_1 \otimes X)Y + T(e_1 \otimes X)Y - \gamma(e_1 \otimes Y)X - T(e_1 \otimes Y)X + \Omega(X, Y)A = 0.$$ 

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The last equation on $P$ can be obtained in the same way and it is of the form
\[ \gamma(e_2 \otimes X)Y - T(e_2 \otimes X)Y - \gamma(e_2 \otimes Y)X + T(e_2 \otimes Y)X + \Omega(X,Y)B = 0, \]
where $B \in V$ is defined by $\beta(e_1 \otimes Z) = \Omega(B, Z)$, $Z \in V$. We conclude that $P \in \mathcal{P}(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k})$ if and only if the maps
\[ \gamma(e_1 \cdot \cdot)H + T(e_1 \cdot \cdot), \quad \gamma(e_2 \cdot \cdot)H - T(e_2 \cdot \cdot) : V \to \mathfrak{k} \oplus \mathbb{C}H \]
belong to $\mathfrak{g}_1$. Thus,
\[ \mathcal{P}(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}) \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathfrak{g}_1, \]
which is an isomorphism of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$-modules. □

6 Further remarks

We are left with the problem to give a direct proof of the fact that if for a real irreducible representation $\mathfrak{h} \subset \mathfrak{so}(n)$ of a simple Lie algebra $\mathfrak{h}$ it holds $\mathcal{P}(\mathfrak{h}) \neq 0$, then $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold. The following two cases should be considered: $\mathcal{P}_0(\mathfrak{h}) \neq 0$ and $\mathcal{P}_1(\mathfrak{h}) \neq 0$. It is necessary to prove that the first condition implies that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a not locally symmetric Riemannian manifold; the second condition implies that $\mathfrak{h} \subset \mathfrak{so}(n)$ may appear as the holonomy algebra of a symmetric Riemannian manifold. It would be useful to give a direct proof to the following statement:

If the connected Lie subgroup $H \subset \text{SO}(n)$ corresponding to an irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ does not act transitively on the unite sphere, then $\mathcal{P}_0(\mathfrak{h}) = 0$.

The relation of the spaces $\mathcal{P}(\mathfrak{h})$ and $\mathcal{R}(\mathfrak{h})$ is the following:
\[ \mathcal{R}(\mathfrak{h}) = \{ S \in \mathbb{R}^{n*} \otimes \mathcal{P}(\mathfrak{h}) | S(X)(Y) = -S(Y)(X) \}. \]

Consider the natural map
\[ \tau : \mathbb{R}^n \otimes \mathcal{R}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h}), \quad \tau(X \otimes R) = R(X, \cdot) \in \mathcal{P}(\mathfrak{h}). \]

Using the results of [3], in [6] it is shown that
\[ \tau(\mathbb{R}^n \otimes \mathcal{R}_0(\mathfrak{h})) = \mathcal{P}_0(\mathfrak{h}) \quad (\text{if } n \geq 4), \quad \tau(\mathbb{R}^n \otimes \mathcal{R}_1(\mathfrak{h})) = \mathcal{P}_1(\mathfrak{h}). \]

It would be useful to get a direct proof of these statements for any irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$.

Suppose that $\mathcal{P}_1(\mathfrak{h}) \neq 0$, i.e. $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$. Then there exists an $\mathfrak{h}$-equivariant linear isomorphism $S : \mathbb{R}^n \to \mathcal{P}_1(\mathfrak{h})$ defined up to a constant multiple. It should be proved that $S(X)(Y) = -S(Y)(X)$, i.e. $S \in \mathcal{R}_1(\mathfrak{h})$.

The space $\mathcal{P}(\mathfrak{h})$ is contained in the tensor product $\mathbb{R}^n \otimes \mathfrak{h}$. A statement form [6] implies that the decomposition of $\mathbb{R}^n \otimes \mathfrak{h}$ into the sum of irreducible $\mathfrak{h}$-modules is of the form
\[ \mathbb{R}^n \otimes \mathfrak{h} = k \mathbb{R}^n \oplus (\oplus \lambda V_\lambda), \]
where $k$ is the number of non-zero labels on the Dynkin diagram for the representation of $\mathfrak{h} \otimes \mathbb{C}$ on $\mathbb{C}^n$, and $V_\lambda$ are pairwise non-isomorphic irreducible $\mathfrak{h}$-modules that are not isomorphic to $\mathbb{R}^n$. If $\mathcal{P}_0(\mathfrak{h}) \neq 0$, then it coincides with the highest irreducible component in $\mathbb{R}^n \otimes \mathfrak{h}$. 

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The space $\mathcal{R}(\mathfrak{h})$ is contained in $\odot^2 \mathfrak{h}$ [9]. If $\mathcal{R}_1(\mathfrak{h}) \neq 0$, then it is spanned by the map $id_{\mathfrak{h}} \in \odot^2 \mathfrak{h} \subset \wedge^2 \mathbb{R}^n \otimes \mathfrak{h}$, note that $id_{\mathfrak{h}}(X, Y) = pr_{\mathfrak{h}}(X \wedge Y)$. Consequently, if $\mathcal{R}_1(\mathfrak{h}) \neq 0$, then

$$\mathcal{P}_1(\mathfrak{h}) = \tau(\mathbb{R}^n \otimes id_{\mathfrak{h}}) = \{pr_{\mathfrak{h}}(X \wedge \cdot) | X \in \mathbb{R}^n\}.$$ 

But it is not clear why if $\mathcal{P}_1(\mathfrak{h}) \neq 0$, then it should coincide with $\tau(\mathbb{R}^n \otimes id_{\mathfrak{h}})$ (such statement would imply $\mathcal{R}_1(\mathfrak{h}) \simeq \mathbb{R}$).

The statement of the following lemma can be checked directly.

**Lemma 2** Let $S : \mathbb{R}^n \rightarrow \mathcal{P}(\mathfrak{h})$ be a linear map. Consider the map

$$T : \wedge^2 \mathbb{R}^n \rightarrow \mathfrak{h}, \quad T(X, Y) = S(X)(Y) - S(Y)(X).$$

Then $T + T^* \in \mathcal{R}(\mathfrak{so}(n))$, where $T^* : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ is given by $(T^*(X, Y)Z, W) = (T(Z, W)X, Y)$.

We are able to show that the condition $\mathcal{P}_1(\mathfrak{h}) \neq 0$ implies $\mathcal{R}_1(\mathfrak{h}) \neq 0$ only under an additional assumption on the representation $\mathfrak{h} \subset \mathfrak{so}(n)$.

**Proposition 2** Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible representation of real type of a simple Lie algebra $\mathfrak{h}$ such that $\mathcal{P}_1(\mathfrak{h}) \neq 0$. If the irreducible representation $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ is given by the Dynkin diagram with only one or two non-zero labels, then $\mathcal{R}_1(\mathfrak{h}) \neq 0$, i.e. $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a symmetric Riemannian space.

**Proof.** If the non-zero label is only one, then the multiplicity of $\mathbb{R}^n$ in the tensor product $\mathbb{R}^n \otimes \mathfrak{h}$ is one, namely the submodule of $\mathbb{R}^n \otimes \mathfrak{h}$ isomorphic to $\mathbb{R}^n$ is equal to $\tau(\mathbb{R}^n \otimes id_{\mathfrak{h}})$, this implies the proof.

Suppose that there are two non-zero labels. Then the multiplicity of $\mathbb{R}^n$ in the tensor product $\mathbb{R}^n \otimes \mathfrak{h}$ is two. One submodule isomorphic to $\mathbb{R}^n$ is equal to $\tau(\mathbb{R}^n \otimes id_{\mathfrak{h}})$. The orthogonal complement to $\tau(\mathbb{R}^n \otimes id_{\mathfrak{h}})$ in $\mathbb{R}^n \otimes \mathfrak{h}$ is the subspace $(\mathbb{R}^n \otimes \mathfrak{h})_0 \subset \mathbb{R}^n \otimes \mathfrak{h}$ consisting of linear maps $\varphi : \mathbb{R}^n \rightarrow \mathfrak{h}$ with $\text{Ric}(\varphi) = 0$ [9]. This space contains a uniquely defined submodule isomorphic to $\mathbb{R}^n$. It is obvious that the projection of $\mathcal{P}_1(\mathfrak{h})$ to $\tau(\mathbb{R}^n \otimes id_{\mathfrak{h}})$ is not trivial. Clearly, the subspace $W \subset \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathfrak{h}$ of elements annihilated by $\mathfrak{h}$ is two-dimensional; it contains the subspace $\mathbb{R} id_{\mathfrak{h}} \subset \odot^2 \mathfrak{h} \subset \wedge^2 \mathbb{R}^n \otimes \mathfrak{h}$. Since $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$, there exists an $\mathfrak{h}$-equivariant isomorphism $S : \mathbb{R}^n \rightarrow \mathcal{P}_1(\mathfrak{h}), S \in W$. If $W \subset \wedge^2 \mathbb{R}^n \otimes \mathfrak{h}$, then $S \in \mathcal{R}_1(\mathfrak{h})$. Otherwise, $W = \mathbb{R} id_{\mathfrak{h}} \oplus \mathfrak{h} \psi$, where $\psi \in \wedge^2 \mathbb{R}^n \otimes \mathfrak{h}$. Since $\mathcal{P}_1(\mathfrak{h}) \not\subset (\mathbb{R} \otimes \mathfrak{h})_0, S \notin \mathfrak{h} \psi$. The element $T \in \wedge^2 \mathbb{R}^n \otimes \mathfrak{h}$ defined in the above lemma belongs to $W$, hence $T = c id_{\mathfrak{h}}$ for some non-zero $c \in \mathbb{R}$. Next, $id_{\mathfrak{h}}^* = id_{\mathfrak{h}}$, and from the lemma it follows that $id_{\mathfrak{h}} \in \mathcal{R}_1(\mathfrak{h})$, i.e. $\mathcal{R}_1(\mathfrak{h}) = \mathbb{R} id_{\mathfrak{h}}$. This proves the proposition. □

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