A general quantum information model for the contextual dependent systems breaking the classical probability law

Masanari Asano*, Irina Basieva†, Andrei Khrennikov‡, Masanori Ohya* and Ichiro Yamato*

Abstract

There exist several phenomena (systems) breaking the classical probability laws. Such systems are contextual dependent adaptive systems. In this paper, we present a new mathematical formula to compute the probability in those systems by using the concepts of the adaptive dynamics and quantum information theory – quantum channels and the lifting. The basic examples of the contextual dependent phenomena can be found in quantum physics. And recently similar examples were found in biological and psychological sciences. Our novel approach is motivated by traditional quantum probability, but it is general enough to describe aforementioned phenomena outside of quantum physics.

Keywords: quantum information and probability, quantum channel, lifting, interference, two slit experiment, cognitive science, cell’s biology

1 Introduction

There exist several phenomena (systems) breaking the classical probability laws such as quantum interference, e.g., [1], [2], (two slit experiment), quantum-like interference in cognitive science, the game of prisoner’s dilemma (PD game), the lactose-glucose interference in E. coli growth. The quantum-like statistical models in psychology and cognitive science has been discussed in [3]–[17]. The PD game was considered by taking

*Department of Information Sciences, Tokyo University of Science, Yamasaki 2641, Noda-shi, Chiba, 278-8510 Japan
†Institute of Information Security, Russian State University for Humanities, Moscow, Russia
‡International Center for Mathematical Modeling in Physics and Cognitive Sciences, Linnaeus University, S-35195, Växjö, Sweden
account of the players’ minds [17] [18]. The lactose-glucose interference is studied as the quantum-like interference [19].

These phenomena (systems) will require us a change of classical probability law, e.g., [2]. One of our trials is to create a new mathematical model which will describe in the unified framework both “traditional quantum phenomena” and recently found quantum-like phenomena outside of physics, cf. [21]. A new general rule of our probabilistic model is the updating the Bayesian law [20]. It is important to notice that these phenomena are contextual dependent, so that they are adaptive to the context of the surroundings.

In such systems, the conditional probability can not be defined in usual mathematical framework. It is well known that in quantum systems the conditional probability does not exist (see the section 3) in the sense of classical systems, so that the naive total probability law should be reconsidered. Same situation is occurred even in non-quantum systems.

Let us consider a simple and intuitive example: When one takes sugar S and chocolate C and he is asked whether it is sweet (1) or not so (2). Then the simple classical probability law may not be satisfied, that is,

\[ P(C = 1) \neq P(C = 1|S = 1)P(S = 1) + P(C = 1|S = 2)P(S = 2) \]

because the LHS \( P(C = 1) \) will be very close to 1 but the RHS will be less than \( \frac{1}{2} \). After taking very sweet sugar, he will taste the chocolate is not so sweet. Taking sugar changes his taste, i.e., the situation of the tongue changes. The conditional probability should be defined on the basis of such a change, so that it is observable-adaptive quantity. The \( P(C = *|S = *) \) should be written as \( P_{\text{adap}}(C = *|S = *) \) and its proper mathematical description (definition) should be given, that is, we will give a mathematical formula to compute the LHS and the RHS above.

In this paper we apply the concept of the adaptive dynamics to make a mathematical framework for the study of these contextual dependent systems. We present adaptive dynamics in the framework of quantum information theory by operating with quantum channels and liftings of input/output states.

We also remark that application of quantum information theory outside of quantum physics, e.g., for macroscopic biological systems, wakes up again the long debate on a possibility to combine the realistic and quantum descriptions, cf. [22], [23], [24]. At the moment we are not able to present a consistent interpretation for coming applications of
quantum information theory outside of quantum physics; we can only keep close to the operational interpretation of quantum information theory, e.g., [25], [26]. In applications of quantum probability outside of physics, the Bayesian approach to quantum probability and information interpretation of the quantum state [27], [28] are the most natural.

2 Adaptive Dynamics

The idea of the adaptive dynamics has implicitly appeared in series of papers [30, 31, 33, 34, 35, 36, 37, 38, 39, 40] for the study of compound dynamics, chaos ansd the SAT algorithm. The name of the adaptive dynamics was deliberately used in [40].The AD has two aspects, one of which is the "observable-adaptive" and another is the "state-adaptive".

The observable-adaptive dynamics is a dynamics characterized as follows: (1) Measurement depends on how to see an observable to be measured. (2) The interaction between two systems depends on how a fixed observable exists, that is, the interaction is related to some aspects of observables to be measured or prepared.

The state-adaptive dynamics is a dynamics characterized as follows: (1) Measurement depends on how the state to be used exists, as same as the observable. (2) The correlation between two systems interaction depends on how the state of at least one of the systems at one instant exists e.g., the interaction Hamiltonian depends on the state at that.

The idea of observable-adaptivity comes from studying chaos. We claimed that any observation will be unrelated or even contradicted to mathematical universalities such as taking limits, sup, inf, etc. Observation of chaos is a result due to taking suitable scales of, for example, time, distance or domain, and it will not be possible in the limiting cases. Examples of the observable-adaptivity are used to understand chaos [31, 38] and examine the violation of Bell’s inequality, namely the chameleon dynamics of Accardi [41]. The idea of the state-adaptivity is implicitly started in constructing a compound state for quantum communication [29, 30, 32, 33]. Examples of the state-adaptivity are seen in an algorithm solving NP complete problem, i.e., a pending problem for more than 30 years asking whether there exists an algorithm solving a NP complete problem in polynomial time, as discussed [35, 36, 39].

We will discuss in the section 5 how we can apply the adaptive dynamics to a bio-system or a psycho-system. The concept of the adaptivity is naturally existed in such systems. Our formulation here contains some treatments shown in the book [36] to understand the evolution of HIV-1, the brain function and the irrational behavior of prisoners.
2.1 Conditional probability and joint probability in quantum systems

The conditional probability and the joint probability do not generally exist in quantum systems, which is an essential difference from classical systems. First of all, let us fix the notations to be used throughout in this paper. We will review these facts for the sequel uses.

Let $\mathcal{H}$, $\mathcal{K}$ be the Hilbert spaces describing the system of interest, $\mathcal{S}(\mathcal{H})$ be the set of all states or probability measures on $\mathcal{H}$, $\mathcal{O}(\mathcal{H})$ be the set of all observables or events on $\mathcal{H}$ and $\mathcal{P}(\mathcal{H}) \subset \mathcal{O}(\mathcal{H})$ be the set of projections in $\mathcal{O}(\mathcal{H})$.

In classical probability, the joint probability for two events $A$ and $B$ is

$$\mu(A \cap B)$$

and the conditional probability is defined by

$$\frac{\mu(A \cap B)}{\mu(B)}.$$

In quantum probability, if the von Neumann-Lüder projection rule is correct, after a measurement of $F \in \mathcal{P}(\mathcal{H})$, a state $\rho$ is considered to be

$$\rho_F = \frac{F \rho F}{\text{tr}\rho F}.$$

When we observe an event $E \in \mathcal{P}(\mathcal{H})$, the expectation value becomes

$$\text{tr}\rho_F E = \frac{\text{tr}F \rho FE}{\text{tr}\rho F} = \frac{\text{tr}F \rho FEF}{\text{tr}\rho F}.$$  \hspace{1cm} (1)

This expectation value can be a candidate of the conditional probability in QP (quantum probability).

There is another candidate for the conditional probability in QP, which is a direct generalization of CP (classical probability).

This alternative expression of joint probability and the conditional probability in QP are expressed as

$$\varphi(E \wedge F)$$

and

$$\frac{\varphi(E \wedge F)}{\varphi(F)},$$  \hspace{1cm} (2)

where $\varphi$ is a state (a measure) and $\wedge$ is the meet of two events (projections) corresponding to $\cap$ in CP, and for the state describing by a density operator, we have

$$\varphi(\cdot) = \text{tr}\rho(\cdot).$$
We ask when the above two expressions (1) and (2) in QP are equivalent. From the next proposition, \( \varphi(\cdot \wedge F)/\varphi(F) \) is not a probability measure (state) on \( \mathcal{P}(H) \).

**Proposition 1** (1) When \( E \) commutes with \( F \), the above two expressions are equivalent, namely,

\[
\frac{\varphi(\text{EFF})}{\varphi(F)} = \frac{\varphi(E \wedge F)}{\varphi(F)}.
\]

(2) When \( EF \neq FE \), \( \frac{\varphi(\cdot \wedge F)}{\varphi(F)} \) is not a probability on \( \mathcal{P}_H \), so that the above two expressions are not equivalent.

**Proof.** (1) \( EF = FE \) implies \( E \wedge F = EF \) and \( FEF = EFF = EF^2 = EF \), so that

\[
\frac{\varphi(E \wedge F)}{\varphi(F)} = \frac{\varphi(\text{EFF})}{\varphi(F)} = \frac{\varphi(\text{EF})}{\varphi(F)}.
\]

(2) Put \( K_\varphi(E \mid F) \equiv \frac{\varphi(E \wedge F)}{\varphi(F)} \) and put \( z \in \text{linsp \{x, y\}} \), \( z \neq x, y \) for any \( x, y \in H \). Take the projections \( P_x = |x\rangle \langle x| \), \( P_y = |y\rangle \langle y| \), \( P_z = |z\rangle \langle z| \) such that \( (P_x \lor P_y) \cap P_z = P_z \) and \( P_x \cap P_z = 0 = P_y \cap P_z \). Then

\[
K_\varphi((P_x \lor P_y) \cap P_z \mid F) = K_\varphi(P_z \mid F) \neq 0,
\]

\[
K_\varphi(P_x \cap P_z \mid F) + K_\varphi(P_y \cap P_z \mid F) = 0.
\]

Therefore

\[
K_\varphi((P_x \lor P_y) \cap P_z \mid F) \neq K_\varphi(P_x \cap P_z \mid F) + K_\varphi(P_y \cap P_z \mid F)
\]

so that \( K_\varphi(\cdot \mid F) \) is not a probability measure on \( \mathcal{P}_H \). ■

In CP, the joint distribution for two random variables \( f \) and \( g \) is expressed as

\[
\mu_{f,g}(\Delta_1, \Delta_2) = \mu \left( f^{-1}(\Delta_1) \cap g^{-1}(\Delta_2) \right)
\]

for any Borel sets \( \Delta_1, \Delta_2 \in B(\mathbb{R}) \). The corresponding quantum expression is either

\[
\varphi_{A,B}(\Delta_1, \Delta_2) = \varphi(E_A(\Delta_1) \wedge E_B(\Delta_2))
\]

or

\[
\varphi(E_A(\Delta_1) \cdot E_B(\Delta_2))
\]
for two observables $A$, $B$ and their spectral measures $E_A(\cdot)$, $E_B(\cdot)$ such that

$$A = \int a E_A(da), \quad B = \int b E_B(da).$$

It is easily checked that neither one of the above expressions satisfies neither the condition of probability measure nor the marginal condition unless $AB = BA$, so that they can not be the joint quantum probability in the classical sense.

Let us explain the above situation, as an example, in a physical measurement process. When an observable $A$ has a discrete decomposition like

$$A = \sum_k a_k F_k, \quad F_i \perp F_j \quad (i \neq j),$$

the probability obtaining $a_k$ by measurement in a state $\rho$ is

$$p_k = \text{tr} \rho F_k$$

and the state $\rho$ is changed to a (conditional) state $\rho_k$ such that

$$\rho_k = \frac{F_k \rho F_k}{\text{tr} \rho F_k} = P_\rho(\cdot | F_k).$$

After the measurement of $A$, we will measure a similar type observable $B$ (i.e., $B = \sum_j b_j E_j$, $(E_i \perp E_j \quad (i \neq j))$ and the probability obtaining $b_j$ after we have obtained the above $a_k$ for the measurement of $A$ is given by

$$p_{jk} = (\text{tr} \rho F_k)(\text{tr} \rho_k E_j)$$
$$= \text{tr} \rho F_k E_j F_k$$
$$= P_\rho(E_j | F_k) \text{tr} \rho F_k. \quad (3)$$

This $p_{jk}$ satisfies

$$\sum_{j,k} p_{jk} = 1,$$
$$\sum_j p_{jk} = \text{tr} \rho F_k = p_k, \quad (4)$$

but not

$$\sum_k p_{jk} = \text{tr} \rho E_j$$

unless $E_j F_k = F_k E_j \quad (\forall j, k)$ so that $p_{jk}$ is not considered as a joint quantum probability distribution. More intuitive expression breaking the usual classical probability law is the following:
Therefore we conclude in quantum system that the above two candidates can not satisfy the properties of both conditional and joint probabilities in the sense of classical system.

The above discussion shows that the order of the measurement of two observables $A$ and $B$ is essential and it gives us a different expectation value, hence the state change.

### 3 Lifting and joint probability

In order to partially solve the difficulty of the nonexistence of joint quantum distribution, the notion of compound state \([30]\) satisfying the marginal condition is useful. In this section we discuss a bit general notion named "lifting" \([39]\) to discuss new scheme of probability containing both classical and quantum.

**Definition 2** Let $\mathcal{A}_1, \mathcal{A}_2$ be C*-algebras and let $\mathcal{A}_1 \otimes \mathcal{A}_2$ be a fixed C*-tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. A lifting from $\mathcal{A}_1$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a weak $\ast$-continuous map

$$\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \rightarrow \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$$

If $\mathcal{E}^*$ is affine and its dual is a completely positive map, we call it a linear lifting; if it maps pure states into pure states, we call it pure.

The algebras $\mathcal{A}_1, \mathcal{A}_2$ can be considered as two systems of interest, for instance, $\mathcal{A}_1$ is an objective system for a study and $\mathcal{A}_2$ is the subjective system or the surrounding of $\mathcal{A}_1$.

Note that to every lifting from $\mathcal{A}_1$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ we can associate two channels: one from $\mathcal{A}_1$ to $\mathcal{A}_1$, defined by

$$\Lambda^* \rho_1(\mathcal{A}_1) \equiv (\mathcal{E}^* \rho_1)(\mathcal{A}_1 \otimes 1) ; \quad \forall A_1 \in \mathcal{A}_1$$

another from $\mathcal{A}_1$ to $\mathcal{A}_2$, defined by

$$\Lambda^* \rho_1(\mathcal{A}_2) \equiv (\mathcal{E}^* \rho_1)(1 \otimes \mathcal{A}_2) ; \quad \forall A_2 \in \mathcal{A}_2$$

In general, a state $\varphi \in \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ such that

$$\varphi \mid_{\mathcal{A}_1 \otimes 1} = \rho_1 ; \quad \varphi \mid_{1 \otimes \mathcal{A}_2} = \rho_2$$
is called a compound state of the states $\rho_1 \in \mathcal{S}(A_1)$ and $\rho_2 \in \mathcal{S}(A_2)$. Remark here that the above compound state is nothing but the joint probability in CP.

The following problem is important in several applications: Given a state $\rho_1 \in \mathcal{S}(A_1)$ and a channel $\Lambda^* : \mathcal{S}(A_1) \rightarrow \mathcal{S}(A_2)$, find a standard lifting $E^* : \mathcal{S}(A_1) \rightarrow \mathcal{S}(A_1 \otimes A_2)$ such that $E^*\rho_1$ is a compound state of $\rho_1$ and $\Lambda^*\rho_1$. Several particular solutions of this problem have been proposed by Ohya, Ceccini and Petz, however an explicit description of all the possible solutions to this problem is still missing, which might be related to find a new scheme of probability theory.

However it is not sure that one can resolve the difficulty of quantum probability if one can solve this problem. The compound state corresponds to the joint probability in classical systems, but there is still ambiguity to define the conditional state in quantum systems. As pointed out in Introduction, the usual conditional probability meets an inadequacy to interpret a certain phenomenon, in which it is important not to manage to set the conditional state by mimicking the classical one but to make a mathematical rule to set new treatment of probabilistic aspects of such a phenomenon.

**Definition 3** A lifting from $A_1$ to $A_1 \otimes A_2$ is called nondemolition for a state $\rho_1 \in \mathcal{S}(A_1)$ if $\rho_1$ is invariant for $\Lambda^*$ i.e., if for all $a_1 \in A_1$

\[(E^*\rho_1)(a_1 \otimes 1) = \rho_1(a_1)\]

The idea of this definition being that the interaction with system 2 does not alter the state of system 1.

**Definition 4** A transition expectation from $A_1 \otimes A_2$ to $A_1$ is a completely positive linear map $E : A_1 \otimes A_2 \rightarrow A_1$ satisfying

\[E(1_{A_1} \otimes 1_{A_2}) = 1_{A_1} \cdot \]

Let an initial state (resp. input signal) is changed (resp. transmitted) to the final state resp. output state) due to a dynamics $\Lambda^*$ (resp. channel). Here $A_1$ (resp. $A_2$) is interpreted as the algebra of observables of the input (resp. output) system and $E^*$ describes the interaction between the input and the output. If $\rho_1 \in \mathcal{S}(A_1)$ is the initial state, then the state $\rho_2 = \Lambda^*\rho_1 \in \mathcal{S}(A_2)$ is the output state.

In several important applications, the state $\rho_1$ of the system before the interaction (preparation, input signal) is not known and one would like to know this state knowing only $\Lambda^*\rho_1 \in \mathcal{S}(A_2)$, i.e., the state of the apparatus after the interaction (output signal). From a mathematical
point of view this problem is not well posed, since the map $\Lambda^*$ is usually not invertible. The best one can do in such cases is to acquire a control on the description of those input states which have the same image under $\Lambda^*$ and then choose among them according to some statistical criterion.

Let us show some important examples of liftings and channels below

**Example 5 : Isometric lifting.**

Let $V : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ be an isometry

$$V^*V = 1_{\mathcal{H}_1}.$$ 

Then the map

$$\mathcal{E} : x \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to V^*xV \in \mathcal{B}(\mathcal{H}_1)$$

is a transition expectation in the sense of Accardi, and the associated lifting maps a density matrix $w_1$ in $\mathcal{H}_1$ into

$$\mathcal{E}^*w_1 = Vw_1V^*$$

in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Liftings of this type are called isometric. Every isometric lifting is a pure lifting. In this case the channel $\Lambda^* : \mathcal{H}_1 \to \mathcal{H}_1$ is given by $\text{tr}_{\mathcal{H}_2}\mathcal{E}^*$.

It is the particular isometric lifting characterized by the properties.

$$\mathcal{H}_1 = \mathcal{H}_2 = : \Gamma(\mathbb{C}) \text{ (Fock space over } \mathbb{C}) = L^2(\mathbb{R})$$

$$V : \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C}) \otimes \Gamma(\mathbb{C})$$

is characterized by the expression

$$V |\theta\rangle = |\alpha \theta\rangle \otimes |\beta \theta\rangle$$

where $|\theta\rangle$ is the normalized coherent vector parametrized by $\theta \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ are such that

$$|\alpha|^2 + |\beta|^2 = 1$$

Notice that this liftings maps coherent states into products of coherent states. So it maps the simplex of the so called classical states (i.e., the convex combinations of coherent vectors) into itself. Restricted to these states it is of convex product type explained below, but it is not of convex product type on the set of all states. Denoting, for $\theta \in \mathbb{C}$, $\omega_\theta$ the coherent state on $\mathcal{B}(\Gamma(\mathbb{C}))$, namely,

$$\omega_\theta(b) = \langle \theta, b\theta \rangle ; b \in \mathcal{B}(\Gamma(\mathbb{C}))$$
then for any \( b \in \mathcal{B}(\Gamma(\mathbb{C})) \)

\[
(E^* \omega_b)(b \otimes 1) = \omega_{\alpha \theta}(b),
\]

so that this lifting is not nondemolition. These equations mean that, by the effect of the interaction, a coherent signal (beam) \(|\theta\rangle\) splits into 2 signals (beams) still coherent, but of lower intensity, but the total intensity (energy) is preserved by the transformation.

Finally we mention two important beam splitting which are used to discuss quantum gates and quantum teleportation.

(1) Superposed beam splitting:

\[
V_s |\theta\rangle \equiv \frac{1}{\sqrt{2}}(|\alpha \theta\rangle \otimes |\beta \theta\rangle - i|\beta \theta\rangle \otimes |\alpha \theta\rangle)
\]

(2) Beam splitting with two inputs and two output: Let \(|\theta\rangle\) and \(|\gamma\rangle\) be two input coherent vectors. Then

\[
V_d (|\theta\rangle \otimes |\gamma\rangle) \equiv |\alpha \theta + \beta \gamma\rangle \otimes |\beta \theta + \alpha \gamma\rangle,
\]

where \(|\alpha|^2 + |\beta|^2 = 1\). These extend linearly to isometry, and their isometric liftings are neither of convex product type nor nondemolition type.

**Example 6** Quantum measurement: If a measuring apparatus is prepared by a positive operator valued measure \(\{Q_n\}\) then the state \(\rho\) changes to a state \(\Lambda^* \rho\) after this measurement, \(\rho \rightarrow \Lambda^* \rho = \sum_n Q_n \rho Q_n\).

**Example 7** Reduction (Open system dynamics): If a system \(\Sigma_1\) interacts with an external system \(\Sigma_2\) described by another Hilbert space \(\mathcal{K}\) and the initial states of \(\Sigma_1\) and \(\Sigma_2\) are \(\rho_1\) and \(\rho_2\), respectively, then the combined state \(\theta_t\) of \(\Sigma_1\) and \(\Sigma_2\) at time \(t\) after the interaction between two systems is given by

\[
\theta_t \equiv U_t (\rho_1 \otimes \rho_2) U_t^*,
\]

where \(U_t = \exp(-itH)\) with the total Hamiltonian \(H\) of \(\Sigma_1\) and \(\Sigma_2\). A channel is obtained by taking the partial trace w.r.t. \(\mathcal{K}\) such as

\[
\rho_1 \rightarrow \Lambda^* \rho_1 \equiv \text{tr}_\mathcal{K} \theta_t.
\]

**Example 8 : The compound lifting.**
Let $\Lambda^* : S(A_1) \rightarrow S(A_2)$ be a channel. For any $\rho_1 \in S(A_1)$ in the closed convex hull of the extremal states, fix a decomposition of $\rho_1$ as a convex combination of extremal states in $S(A_1)$

$$\rho_1 = \int_{S(A_1)} \omega_1 d\mu$$

where $\mu$ is a Borel measure on $S(A_1)$ with support in the extremal states, and define

$$E^* \rho_1 \equiv \int_{S(A_1)} \omega_1 \otimes \Lambda^* \omega_1 d\mu$$

Then $E^* : S(A_1) \rightarrow S(A_1 \otimes A_2)$ is a lifting, nonlinear even if $\Lambda^*$ is linear, and it is a nondemolition type.

The most general lifting, mapping $S(A_1)$ into the closed convex hull of the extremal product states on $A_1 \otimes A_2$ is essentially of this type. This nonlinear nondemolition lifting was first discussed by Ohya to define the compound state and the mutual entropy for quantum information communication [30, 32]. The above is a bit more general because we shall weaken the condition that $\mu$ is concentrated on the extremal states used in [30].

Therefore once a channel is given, by which a lifting of convex product type can be constructed. For example, the von Neumann quantum measurement process is written, in the terminology of lifting, as follows: Having measured a compact observable $A = \sum_n a_n P_n$ (spectral decomposition with $\sum_n P_n = I$) in a state $\rho$, the state after this measurement will be

$$\Lambda^* \rho = \sum_n P_n \rho P_n$$

and a lifting $E^*$, of convex product type, associated to this channel $\Lambda^*$ and to a fixed decomposition of $\rho$ as $\rho = \sum_n \mu_n \rho_n$ ($\rho_n \in S(A_1)$) is given by:

$$E^* \rho = \sum_n \mu_n \rho_n \otimes \Lambda^* \rho_n.$$

**Example 9** Amplifier channel: To recover the loss in the course of a quantum communication, we need to amplify the signal (photon). In quantum optics, a linear amplifier is usually expressed by means of annihilation operators $a$ and $b$ on $\mathcal{H}$ and $\mathcal{K}$, respectively:

$$c = \sqrt{G} a \otimes I + \sqrt{G^{-1}} I \otimes b^*$$

where $G(\geq 1)$ is a constant and $c$ satisfies CCR (i.e., $[c, c^*] = I$) on $\mathcal{H} \otimes \mathcal{K}$. This expression however is not convenient to compute several information quantities, like entropy. The lifting expression of the amplifier

$$c = \sqrt{G} a_1 \otimes I_1 + \sqrt{G^{-1}} I_1 \otimes b_1^*$$

where $G(\geq 1)$ is a constant and $c_1$ satisfies CCR (i.e., $[c_1, c_1^*] = I_1$) on $\mathcal{H}_1 \otimes \mathcal{K}_1$. This expression however is not convenient to compute several information quantities, like entropy. The lifting expression of the amplifier
is good for such uses and it is given as follows: Let \( c = \mu a \otimes I + \nu I \otimes b^* \) with \( |\mu|^2 - |\nu|^2 = 1 \) and \( |\gamma\rangle \) be the eigenvector of \( c : c |\gamma\rangle = \gamma |\gamma\rangle \). For two coherent vectors \( |\theta\rangle \) on \( \mathcal{H} \) and \( |\theta'\rangle \) on \( \mathcal{K} \), \( |\gamma\rangle \) can be written by the squeezing expression: \( |\gamma\rangle = |\theta \otimes \theta' ; \mu, \nu\rangle \) and the lifting is defined by an isometry
\[
V_{\theta'} |\theta\rangle = |\theta \otimes \theta' ; \mu, \nu\rangle
\]
such that
\[
\mathcal{E}^* \rho = V_{\theta'} \rho V_{\theta'}^* \quad \rho \in \mathcal{S}(\mathcal{H})
\]
The channel of the amplifier is
\[
\Lambda^* \rho = \text{tr}_{\mathcal{K}} \mathcal{E}^* \rho.
\]

Finally we note that a channel is determined by a lifting and conversely a lifting is constructed by a channel.

4 New views of probability both in classical and quantum systems

In this section we will discuss how to use the concept of lifting to explain phenomena breaking the usual probability law.

Let \( \mathcal{A}, \mathcal{B} \) be \( \mathbb{C}^* \)-algebras describing the systems for a study, more specifically, let \( \mathcal{A}, \mathcal{B} \) be the sets of all observables in Hilbert spaces \( \mathcal{H}, \mathcal{K} \); \( \mathcal{A} = \mathcal{O}(\mathcal{H}), \mathcal{B} = \mathcal{O}(\mathcal{K}) \). Let \( \mathcal{E}^* \) be a lifting from \( \mathcal{S}(\mathcal{H}) \) to \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \), so that its dual map \( \mathcal{E} \) is a mapping from \( \mathcal{A} \otimes \mathcal{B} \) to \( \mathcal{A} \). There are several liftings for various different cases to be considered: (1) If \( \mathcal{K} \) is \( \mathbb{C} \), then the lifting \( \mathcal{E}^* \) is nothing but a channel from \( \mathcal{S}(\mathcal{H}) \) to \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \). (2) If \( \mathcal{H} \) is \( \mathbb{C} \), then the lifting \( \mathcal{E}^* \) is a channel from \( \mathcal{S}(\mathcal{H}) \) to \( \mathcal{S}(\mathcal{K}) \). Further \( \mathcal{K} \) or \( \mathcal{H} \) can be decomposed as \( \mathcal{K} = \otimes_i \mathcal{K}_i \) (resp. \( \oplus_i \mathcal{K}_i \)), and so for \( \mathcal{H} \), so that \( \mathcal{B} \) can be \( \otimes_i \mathcal{B}_i \) (resp. \( \oplus_i \mathcal{B}_i \)) and so for \( \mathcal{A} \).

The adaptive dynamics is considered that the dynamics of a state or an observable after an instant (say the time \( t_0 \)) attached to a system of interest is affected by the existence of some other observable and state at that instant. Let \( \rho \in \mathcal{S}(\mathcal{H}) \) and \( A \in \mathcal{A} \) be a state and an observable before \( t_0 \), and let \( \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) and \( Q \in \mathcal{A} \otimes \mathcal{B} \) be a state and an observable to give an effect to the state \( \rho \) and the observable \( A \). In many cases, the effect to the state is dual to that to the observable, so that we will discuss the effect to the state only. This effect is described by a lifting \( \mathcal{E}^*_\sigma Q \), so that the state \( \rho \) becomes \( \mathcal{E}^*_\sigma Q \rho \) first, then it will be \( \text{tr}_K \mathcal{E}^*_\sigma Q \rho = \rho_{\sigma Q} \). The adaptive dynamics is the whole process such as

Adaptive Dynamics: \( \rho \Rightarrow \mathcal{E}^*_\sigma Q \rho \Rightarrow \rho_{\sigma Q} = \text{tr}_K \mathcal{E}^*_\sigma Q \rho \)
That is, what we need is how to construct the lifting for each problem to be studied, that is, we properly construct the lifting $\mathcal{E}_{\sigma Q}^*$ by choosing $\sigma$ and $Q$ properly.

The state change discussed in Section 2 is a naive example of the adaptive dynamics, in which the lifting is given as follows: $Q = A = \sum_k a_k F_k \in A, F_i \perp F_j \ (i \neq j)$ and $\mathcal{E}_{\sigma Q}^* = \{ \mathcal{E}_{F_k A}^* \}$ (this case the state $\sigma$ is not needed) such that for any $B \in \mathcal{A}$

$$P(B \mid A = a_k) = tr B \mathcal{E}_{F_k A}^* \rho = tr BF_k \rho F_k / tr F_k \rho.$$  

In this case the lifting is a channel from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H})$, the case (1) above. It is true that we do not know whether this "projection rule" can describe almost all probabilistic phenomena in nature or not.

Let us go back to the discussion using lifting $\mathcal{E}_{\sigma Q}^*$ above. The expectation value of another observable $B \in \mathcal{A}$ or $\mathcal{A} \otimes \mathcal{B}$ in the adaptive state $\rho_{\sigma Q}$ is

$$tr \rho_{\sigma Q} B = tr \mathcal{H} tr \mathcal{K} B \mathcal{E}_{\sigma Q}^* \rho.$$ 

Now suppose that there are two quantum event systems $A = \{ a_k \in \mathbb{R}, F_k \in \mathcal{A} \}$ and $B = \{ b_j \in \mathbb{R}, E_j \in \mathcal{A} \}$, where we do not assume $F_k, E_j$ are projections, but they satisfy the conditions $\sum_k F_k = I, \sum_j E_j = I$ as POVM (positive operator valued measure) corresponding to the partition of a probability space in classical system. Then the "joint-like" probability obtaining $a_k$ and $b_j$ might be given by the formula

$$P(a_k, b_j) = tr E_j \Box F_k \mathcal{E}_{\sigma Q}^* \rho, \quad (5)$$

where $\Box$ is a certain operation (relation) between $A$ and $B$, more generally one can take a certain operator function $f(E_j, F_k)$ instead of $E_j \Box F_k$.

If $\sigma, Q$ are independent from any $F_k, E_j$ and the operation $\Box$ is the usual tensor product $\otimes$ so that $A$ and $B$ can be considered in two independent systems or to be commutative, then the above "joint-like" probability becomes the joint probability. However if not such a case, e.g., $Q$ is related to $A$ and $B$, the situation will be more subtle. Therefore the problem is how to set the operation $\Box$ and how to construct the lifting $\mathcal{E}_{\sigma Q}^*$ in order to describe the particular problems associated to systems of interest. We in the sequel discuss this problem in the contextual dependent systems like bio-systems and psycho-systems mentioned in Introduction. That is, we discuss how to apply the formula (5) to the following three problems breaking the usual probability law: (1) State change of tongue for sweetness, (2) Lactose-glucose interference in E. coli growth, (3) Updating the Bayesian law.
The first problem is not so sophisticated but very simple and common one. As considered in Introduction, when one takes sugar $S$ and chocolate $C$ and he is asked whether it is sweet (1) or not so (2). Then the simple classical probability law may not be satisfied, that is,

$$P(C = 1) \neq P(C = 1|S = 1)P(S = 1) + P(C = 1|S = 2)P(S = 2)$$

because the LHS $P(C = 1)$ will be very close to 1 but the RHS will be less than $\frac{1}{2}$.

Let $e_1$ and $e_2$ be the orthogonal vectors describing sweet and non-sweet states, respectively. The initial state of tongue is neutral such as

$$\rho \equiv |x_0\rangle \langle x_0|,$$

where $x_0 = \frac{1}{\sqrt{2}}(e_1 + e_2)$. Here we start from the neutral pure state $\rho$ because we consider two sweet things. It is enough for us to take the Hilbert space $\mathbb{C}^2$ for this problem, so that $e_1$ and $e_2$ can be set as $(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})$, respectively.

When one takes "sugar", the operator corresponding to taking "sugar" will be given as

$$S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $|\lambda_1|^2 + |\lambda_2|^2 = 1$. This operator can be regarded as the square root of the sugar state $\sigma_S$;

$$\sigma_S = |\lambda_1|^2 E_1 + |\lambda_2|^2 E_2, \quad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(10), \quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(01).$$

Taking sugar, he will taste that it is sweet with the probability $|\lambda_1|^2$ and non-sweet with the probability $|\lambda_2|^2$, so $|\lambda_1|^2$ should be much higher than $|\lambda_2|^2$ for a usual sugar. This comes from the following change of the neutral initial tongue (i.e., non-adaptive) state:

$$\rho \rightarrow \rho_S = \Lambda_S^\ast(\rho) \equiv \frac{S^\ast \rho S}{\text{tr} |S|^2 \rho},$$

which is the state when he takes the sugar. This is similar to the usual expression of state change in quantum dynamics, although it is adaptive for sugar. The subtle point of the present problem is that the state of tongue is neither $\rho_S$ nor $\rho$ at the instant just after taking sugar. Note here that if we kill the subjectivity (personal character?) of one’s tongue, then the state $\rho_S$ can be understood as

$$E_1\rho_S E_1 + E_2\rho_S E_2,$$
which is the unread objective state as usual in quantum measurement. We can use the above two expressions \( \rho_S \), which give us the same result for the computation of the probability.

For some time duration, the tongue becomes dull to sweetness, so the tongue state can be written by means of a certain "exchanging" operator 

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

such that

\[
\rho_{a \rightarrow S} = X \rho_S X,
\]

where "a" means the adaptive change. Then similarly as sugar, when one takes a chocolate, the state will be \( \rho_{a \rightarrow C} \) given by

\[
\rho_{a \rightarrow C} = \Lambda^*_C(\rho_S) \equiv \frac{C^* \rho_S C}{\text{tr}|C|^2 \rho_S^a},
\]

where \( C \) will be given as

\[
C = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}
\]

with \( |\mu_1|^2 + |\mu_2|^2 = 1 \). Common experience tells us that \( |\lambda_1|^2 \geq |\mu_1|^2 \geq |\mu_2|^2 \geq |\lambda_2|^2 \) and the first two are much larger than the last two.

As shown above, the adaptive set \{\sigma, Q\} is the set \{S (= \sigma_S), X, C\}, we introduce the following nonlinear demolition lifting:

\[
\mathcal{E}_{a \sigma \rho}^*(\rho) (= \mathcal{E}^*_{S (= \sigma_S), X C}(\rho)) \equiv \rho_S \otimes \rho_{a \rightarrow C}^* = \Lambda_S^*(\rho) \otimes \Lambda_C^*(X \Lambda_S^*(\rho) X),
\]

which implies the joint probabilities \( P(S = j, C = k) \) \((j, k = 1, 2)\) as

\[
P(S = j, C = k) = \text{tr}E_j \otimes E_k \mathcal{E}_{a \sigma \rho}^* \rho.
\]

The probability that one tastes sweetness of the chocolate after tasting sugar is

\[
P(C = 1, S = 1) + P(C = 1, S = 2) = \frac{|\lambda_2|^2 |\mu_1|^2}{|\lambda_2|^2 |\mu_1|^2 + |\lambda_1|^2 |\mu_2|^2}.
\]

Note that this probability is much less than

\[
P(C = 1) = \text{tr}E_1 \Lambda_C^*(\rho) = |\mu_1|^2,
\]

which is the probability of sweetness tasted by the neutral tongue \( \rho \). In this sense, the usual probability law

\[
P(C = 1) = P(S = 1, C = 1) + P(S = 2, C = 1)
\]

is not satisfied.
6 Activity of lactose operon in E. coli

The lactose operon is a group of genes in E. coli (Escherichia coli), and it is required for the metabolism of lactose. This operon produces β-galactosidase, which is an enzyme to digest lactose into glucose and galactose. There was an experiment measuring the activity of β-galactosidase which E. coli produces in the presence of (I) only 0.4% lactose, (II) only 0.4% glucose, or (III) mixture 0.4% lactose + 0.1% glucose, see [13]. The activity is represented in Miller’s units (enzyme activity measurement condition), and it reaches to 3000 units by full induction. In the cases of (I) and (II), the data of 2920 units and 33 units were obtained. These results make one to expect that the activity in the case (III) will be large, because the number of molecules of lactose is larger than that of glucose. However, the obtained data were only 43 units. This result implies that E. coli metabolizes glucose in preference to lactose. In biology, this functionality of E. coli have been discussed, and it was known that glucose has a property reducing lactose permease provided by the operon. Apart from such qualitative and biochemical explanation, it will be also necessary to discuss a mathematical interpretation such that the biological activity in E. coli is evaluated quantitatively. In the paper [19], it is pointed out that the activity of E. coli violates the total probability law as shown below, which might come from the preference in E. coli’s metabolism. We will explain this contextual behavior by the formula [5].

We consider two events $L$ and $E$: $L$: E. coli detects a lactose molecule in his metabolism and $G$: E. coli detects a glucose molecule. In the case of (I) or (II), the probability $P(L) = 1$ or $P(G) = 1$ is given. In the case of (III), $P(L)$ and $P(G)$ are calculated as

$$P(L) = \frac{0.4}{0.4 + 0.1} = 0.8, \quad P(G) = \frac{0.1}{0.4 + 0.1} = 0.2.$$

The events $L$ and $G$ are exclusive each other, so it is assumed that $P(L \cup G) = P(L) + P(G) = 1$. Further, we consider the events $\{+,-\}$ which means that E. coli activates his lactose operon or not. From the experimental data of the cases (I) and (II), the following conditional probabilities are obtained:

$$(I): \ P(+) | L = \frac{2920}{3000},$$

$$(II): \ P(+) | G = \frac{33}{3000}. \ (6)$$

In the case (III), if the total probability law is satisfied, the probability
\( P(\mathcal{P} \cap (L \cup G)) \) is computed as
\[
P(\mathcal{P} \cap (L \cup G)) = P(\mathcal{P}|L \cup G)P(L \cup G) = P(\mathcal{P}|L \cup G)
= P(\mathcal{P}|L) + P(\mathcal{P}|G) = P(\mathcal{P}|L)P(L) + P(\mathcal{P}|G)P(G)
\]
However, from the experimental data, one can estimate
\[
P(\mathcal{P}|L \cup G) = \frac{43}{3000},
\]
so that the total probability law is violated:
\[
P(\mathcal{P}|L \cup G) \neq P(\mathcal{P}|L)P(L) + P(\mathcal{P}|G)P(G).
\]

We will explain how to compute the probabilities above by using the concept of lifting. Firstly, we introduce the initial state \( \rho = |x_0\rangle \langle x_0| \) on Hilbert space \( \mathcal{H} = \mathbb{C}^2 \). The state vector \( x_0 \) is written by
\[
x_0 = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_2.
\]
The basis \( \{e_1, e_2\} \) denote the detection of lactose or glucose by E. coli, that is, the events, \( L \) and \( G \). The E. coli at the initial state \( \rho \) has not recognized the existence of lactose and glucose yet. When the E. coli recognizes them, the following state change occurs;
\[
\rho \leftrightarrow \rho_D = \Lambda_D^*(\rho) \equiv \frac{D \rho D^*}{\text{tr}(|D|^2 \rho)},
\]
where
\[
D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]
with \( |\alpha|^2 + |\beta|^2 = 1 \). Note that \( |\alpha|^2 \) and \( |\beta|^2 \) implies the probabilities for the events \( L \) and \( G \), that is, \( P(L) \) and \( P(G) \). The state \( \sigma_D \equiv DD^* \) means the distribution of \( P(L) \) and \( P(G) \). In this sense, the state \( \sigma_D \) represents the solution of lactose and glucose. We call \( D \) the detection operator and call \( \rho_D \) the detection state. The state determining the activation of the operon in E.coli depends on the detection state \( \rho_D \). We give such state by
\[
\rho_{op} = \Lambda_Q^*(\rho_D) \equiv \frac{Q \rho_D Q^*}{\text{tr}(Q \rho Q^*)},
\]
where the operator \( Q \) is written as
\[
Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
We call $\rho_{op}$ the activation state for the operon and call $Q$ the activation operator. (The components $a, b, c$ and $d$ are discussed later.)

We introduce the lifting

$$\mathcal{E}_{D,Q}(\rho) = \Lambda_Q(\Lambda_D(\rho)) \otimes \Lambda_D(\rho) \in \mathcal{K} \otimes \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2,$$

by which we can describe the correlation between the activity of lactose operon and the ratio of concentration of lactose and glucose. For example, the joint probabilities $P(+ \cap L)$ and $P(- \cap L)$ are given by

$$P(+ \cap L) = \text{tr}(E_1 \otimes E_1 \mathcal{E}_{D,Q}(\rho)),
\quad
P(- \cap L) = \text{tr}(E_2 \otimes E_1 \mathcal{E}_{D,Q}(\rho)).$$

(7)

Here, let us consider the case of $P(L) = |a|^2 = 1$, then the probabilities

$$P(\pm |L \cup G) = P(\pm \cap (L \cup G)) = P(\pm \cap L) + P(\pm \cap G)$$

correspond to $P(\pm |L)$ of Eq. (8)-(I), and the Eq. (7) gives the conditional probabilities as

$$P(+|L) = \frac{|a|^2}{|a|^2 + |c|^2}, \quad P(-|L) = \frac{|c|^2}{|a|^2 + |c|^2}.$$

From these results, we may give the following forms for $a$ and $b$.

$$a = \sqrt{P(+|L)}e^{i\vartheta_L}k_L, \quad c = \sqrt{P(-|L)}e^{i\vartheta_G}k_L$$

Here, $k_L$ is a certain real number. In a similar way, we obtain

$$b = \sqrt{P(+|G)}e^{i\vartheta_G}k_G, \quad d = \sqrt{P(-|G)}e^{i\vartheta_G}k_G$$

for the components $b$ and $d$. To simplify the discussion, hereafter, we assume $\vartheta_L = \vartheta_G$, $\vartheta_L = \vartheta_G$ and denote $e^{i\vartheta_L}k_L$, $e^{i\vartheta_G}k_G$ by $\tilde{k}_L$, $\tilde{k}_G$. Then, the operator $Q$ is decomposed to

$$Q = \left( \begin{array}{cc}
\sqrt{P(+|L)} & \sqrt{P(+|G)} \\
\sqrt{P(-|L)} & \sqrt{P(-|G)}
\end{array} \right) \begin{pmatrix}
\tilde{k}_L \\
0
\end{pmatrix}.$$

(8)

The probability $P(+|L \cap G)$ with this $Q$ is calculated as

$$P(+|L \cap G) = \frac{|\sqrt{P(+|L)}\tilde{k}_L\alpha + \sqrt{P(+|G)}\tilde{k}_G\beta|^2}{|\sqrt{P(+|L))\tilde{k}_L\alpha + \sqrt{P(+|G))\tilde{k}_G\beta|^2} + |\sqrt{P(-|L))\tilde{k}_L\alpha + \sqrt{P(-|G))\tilde{k}_G\beta|^2}|}.$$
Bayesian Updating Biased By Psychological Factor

The Bayesian updating is an important concept in Bayesian statistics, and it is used to describe a process of inference, which is explained as follows: Consider two event systems denoted by $S_1 = \{A, B\}$ and $S_2 = \{C, D\}$, where the events $A$ and $B$ are mutually exclusive, and the same holds for $C$ and $D$. Firstly, a decision-making entity, say Alice, estimates the probabilities $P(A)$ and $P(B)$ for the events $A$ and $B$, which are called the prior probabilities. The prior probability is sometime called “subjective probability” or “personal probability”. Further, Alice knows the conditional probabilities $P(C|A)$ and $P(C|B)$ which are obtained from some statistical data. When Alice sees the occurrence of the event $C$ or $D$ in the system $S_2$, she can change her prior prediction $P(A)$ and $P(B)$ to the following conditional probabilities by Bayes’ rule: When Alice sees the occurrence of $C$ in $S_2$, she can update her prediction for the event $A$ from $P(A)$ to

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|B)P(B)}.$$ 

When Alice sees the occurrence of $D$ in $S_2$, she can update her prediction to

$$P(A|D) = \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B)}.$$ 

These conditional probabilities are called the posterior probabilities. The change of prediction is described as an “updating” from the prior probabilities $P(A)$ to the posterior probability, and it is called the Bayesian updating.

In the paper [42], we redescribed the process of Bayesian updating in the framework of “quantum-like representation”, where we introduced the following state vector defined on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$;

$$|\Phi\rangle = \sqrt{P(A')}|A'\rangle \otimes (\sqrt{P(C'|A')}|C'\rangle + \sqrt{P(D'|A')}|D'\rangle)$$

$$+ \sqrt{P(B')}|B'\rangle \otimes (\sqrt{P(C'|B')}|C'\rangle + \sqrt{P(D'|B')}|D'\rangle).$$ (9)

We call this vector the prediction state vector. The set of vectors $\{|A'\rangle, |B'\rangle\}$ becomes an orthogonal basis on $\mathcal{H}_1$, and $\{|C'\rangle, |D'\rangle\}$ is another orthogonal basis on $\mathcal{H}_2$. The $A'$, $B'$, $C'$ and $D'$ represent the events defined as

Event $A'$: Alice judges “the event $A$ occurs in the system $S_1$.”
Event $B'$: Alice judges “the event $B$ occurs in the system $S_1$.”

Event $C'$: Alice judges “the event $C$ occurs in the system $S_2$.”

Event $D'$: Alice judges “the event $D$ occurs in the system $S_2$.”

These events are the subjective events (judgments) in Alice’s mentality and the vectors $|A'⟩$, $|B'⟩$, $|C'⟩$ and $|D'⟩$ mean the minds toward the above judgements. The vector $|Φ⟩$ represents that these minds coexists in Alice’s mentality. For example, Alice is conscious of the mind $|A'⟩$ with the weight $\sqrt{P(A')}$, and under the condition of the event $A'$, she gives the weights $\sqrt{P(C'|A')}$ and $\sqrt{P(D'|A')}$ for the minds $|C'⟩$ and $|D'⟩$. Such an assignment of weights implies that Alice feels causality between $S_1$ and $S_2$: The events in $S_1$ are causes and the events in $S_2$ are results. The square of $\sqrt{P(A')}$ is equivalent to a prior probability $P(A)$ in the Bayesian theory. If Alice knows the objective conditional probabilities $P(C|A)$ and $P(C|B)$, Alice can give the weights of $\sqrt{P(C'|A')}$ and $\sqrt{P(C'|B')}$ as $P(C'|A') = P(C|A)$ and $P(C'|B') = P(C|B)$.

The process of the above Bayesian updating is represented in the term of the lifting discussed in the previous section. When Alice has the prediction state $|Φ⟩⟨Φ| ≡ \rho$ and sees the occurrence of the event $C$ in $S_2$, her mind for the event $D'$ is vanished instantaneously. This vanishing is represented as the reduction by the projection operator $M_{C'} = I \otimes |C'⟩⟨C'|$:

$$M_{C'}\rho M_{C'}^\dagger \equiv \rho_{C'}$$

The posterior probability $P(A|C)$ is calculated by

$$\text{tr}(M_{A'}\rho_{C'})$$

where $M_{A'} = |A'⟩⟨A'| \otimes I$.

The inference based on the Bayesian updating is rational from the view point of probability theory. However, we empirically know that our behavior is sometimes irrational in various situations from the mathematical view point and we have psychological factors that frequently disturb rational inferences. Such irrational inference can be mathematically discussed by using the concept of lifting as the examples above. Let us introduce the lifting for this problem from $S(ℋ)$ to $S(ℋ \otimes ℋ)$ by

$$\mathcal{E}^*_{σ,V}(ρ) = V ρ \otimes σV^*.$$  

Here $σ \in S(ℋ)$ is a state specifying a psychological factor which Alice holds in her mentality. The operator $V$ on $ℋ \otimes ℋ$ is isometry and gives a correlation between the prediction state $ρ$ and the psychological factor.
σ, in other words, it specifies a psychological affection to Alice’s rational inference. We call the state defined by
\[ \tilde{\rho} \equiv \text{tr}_K(\mathcal{E}_{\sigma,V}^{\ast}(\rho)), \]
the prediction state biased from the rational prediction \( \rho \). The posterior probability is generally given by
\[ \tilde{P}(A,C) \equiv \text{tr} \left( M_{\ast} M_{C}^{\ast} \tilde{\rho} M_{C}^{\ast} \right), \]
so that
\[ \tilde{P}(A|C) = \text{tr} \left( M_{\ast} \frac{M_{C}^{\ast} \tilde{\rho} M_{C}^{\ast}}{\text{tr}(M_{C}^{\ast} \tilde{\rho})} \right), \]
which is not equal to the original \( P(A|C) \) in the rational inference.

Here, it should be noted that the prior probability from \( \tilde{\rho} \), which is given by
\[ \tilde{P}(A) = \text{tr}_H(M_{\ast} \tilde{\rho}), \]
is not equal to the original \( P(A) = \text{tr}_H(M_{\ast} \rho) \) in general. However, the prior probability is given and fixed before the updating, and if the psychological factor affects only the estimation of posterior probabilities, then the prediction state \( \tilde{\rho} \) can be set to satisfy the condition
\[ \tilde{P}(A) = P(A). \]

In the paper [41], we gave an example of such a state \( \tilde{\rho} \) which is provided by the psychological factor called the reliability of information. This example has the following equalities
\[ P(A) = \tilde{P}(A,C) + \tilde{P}(A,D) = \tilde{P}(A), \]
\[ P(B) = \tilde{P}(B,C) + \tilde{P}(B,D) = \tilde{P}(B), \]
for the joint probabilities given by
\[ \tilde{P}(A,C) = \text{tr}_H(M_{\ast} M_{C}^{\ast} \tilde{\rho}), \quad \tilde{P}(B,C) = \text{tr}_H(M_{B} M_{C}^{\ast} \tilde{\rho}), \]
\[ \tilde{P}(A,D) = \text{tr}_H(M_{\ast} M_{D}^{\ast} \tilde{\rho}), \quad \tilde{P}(B,D) = \text{tr}_H(M_{B} M_{D}^{\ast} \tilde{\rho}). \]
On the other hand, for \( P(C) = \text{tr}_H(M_{C}^{\ast} \rho) \) and \( P(D) = \text{tr}_H(M_{D}^{\ast} \rho) \),
\[ P(C) \neq \tilde{P}(A,C) + \tilde{P}(B,C) = \tilde{P}(C|A)P(A) + \tilde{P}(C|B)P(B), \]
\[ P(D) \neq \tilde{P}(A,D) + \tilde{P}(B,D) = \tilde{P}(D|A)P(A) + \tilde{P}(D|B)P(B), \]
that is, \( P(C) \neq \tilde{P}(C) \) and \( P(D) \neq \tilde{P}(D) \). Note that \( P(C) \) means the probability of the event \( C \) which will be estimated by Alice with \( \rho \). Alice with the biased \( \tilde{\rho} \) will estimate \( \tilde{P}(C) \), and then, the violation of total probability law of Eq. (10) is not occurred actually. However, Eq. (10) represents the violation of Alice’s rationality, and the difference between \( P(C) \) and \( \tilde{P}(C) \) specifies the degree of Alice’s irrationality.
References

[1] Plotnitsky, A.: Reading Bohr: Physics and Philosophy. Springer, Heidelberg-Berlin-New York (2006).
[2] Plotnitsky, A.: Epistemology and Probability: Bohr, Heisenberg, Schrödinger, and the Nature of Quantum-Theoretical Thinking. Springer, Heidelberg-Berlin-New York (2009).
[3] A. Khrennikov, *Open Systems and Information Dynamics* **11** (3), 267-275 (2004).
[4] A. Khrennikov, *BioSystems* **84**, 225–241 (2006).
[5] K.-H. Fichtner, L. Fichtner, W. Freudenberg and M. Ohya, On a quantum model of the recognition process. *QP-PQ: Quantum Prob. White Noise Analysis* **21**, 64-84 (2008).
[6] J. B. Busemeyer, Z. Wang, and J. T. Townsend, Quantum dynamics of human decision making. *J. Math. Psychology* 50, 220-241 (2006)
[7] J. R. Busemeyer, M. Matthews, and Z. Wang : A Quantum Information Processing Explanation of Disjunction Effects. In: Sun, R. and Myake, N. (eds.) The 29th Annual Conference of the Cognitive Science Society and the 5th International Conference of Cognitive Science (Pp. 131-135) Mahwah, NJ. Erlbaum (2006)
[8] J. R. Busemeyer, E. Santuy, A. Lambert-Mogiliansky, Comparison of Markov and quantum models of decision making. In P. Bruza, W. Lawless, K. van Rijsbergen, D. A. Sofge, B. Coeke, S. Clark (Eds.) Quantum interaction: Proceedings of the Second Quantum Interaction Symposium, pp.68-74. London: College Publications, (2008)
[9] T. Cheon and T. Takahashi, Interference and inequality in quantum decision theory, *Phys. Lett. A* 375 (2010) 100-104.
[10] T. Cheon and I. Tsutsui, Classical and quantum contents of solvable game theory on Hilbert space, *Phys. Lett. A* 348 (2006) 147-152.
[11] L. Accardi, A. Khrennikov, M. Ohya, The problem of quantum-like representation in economy, cognitive science, and genetics. In.: *Quantum Bio-Informatics II: From Quantum Information to Bio-Informatics*. L. Accardi, W. Freudenberg, M. Ohya, eds., p. 1-8, WSP, Singapore (2008).
[12] L. Accardi, A. Khrennikov, M. Ohya, Quantum Markov Model for Data from Shafir-Tversky Experiments in Cognitive Psychology. *Open Systems and Information Dynamics*, **16**, 371E85 (2009).
[13] Conte E., Khrennikov A., Todarello O., Federici A., Zbilut J. P. Mental States Follow Quantum Mechanics during Perception and Cognition of Ambiguous Figures. *Open Systems and Information Dynamics*, **16**, 1-17 (2009).
[14] Khrennikov A., Haven E. Quantum mechanics and violations of the
sure-thing principle: the use of probability interference and other concepts. *Journal of Mathematical Psychology*, **53**, 378-388 (2009).

[15] A. Khrennikov, *Ubiquitous quantum structure: from psychology to finance*, Springer, Heidelberg- Berlin-New York, 2010.

[16] A. Khrennikov, *Contextual approach to quantum formalism* (Fundamental Theories of Physics). Springer, Heidelberg- Berlin-New York, 2009.

[17] M. Asano, M. Ohya and A. Khrennikov, Quantum-Like Model for Decision Making Process in Two Players Game, *Foundations of Physics Volume 41*, Number 3, 538-548, (2010).

[18] M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov and I. Basieva, On application of Gorini-Kossakowski-Sudarshan-Lindblad equation in cognitive psychology, *Open Systems & Information Dynamics Volume: 18*, Issue: 1(2011) pp. 55-69.

[19] I. Basieva, A. Khrennikov, M. Ohya and I. Yamato, Quantum-like interference effect in gene expression: glucose-lactose destructive interference, *Systems and Synthetic Biology*, DOI: 10.1007/s11693-011-9081-8.

[20] M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov and I. Basieva, Quantum-like Representation of Bayesian Updating, *American Institute of Physics Volume 1327*, pp. 57-62 : Proceedings of the International Conference on Advances in Quantum Theory (2011).

[21] A. Plotnitsky, On the Reasonable and Unreasonable Effectiveness of Mathematics in Classical and Quantum Physics. *Foundations of Physics*. 41, 466-491 (2011).

[22] C. Garola and S. Sozzo, Generalized Observables, Bells Inequalities and Mixtures in the ESR Model, *Foundations of Physics*, **41** (2011) (424-449).

[23] C. Garola and S. Sozzo, The ESR Model: A Proposal for a Non-contextual and Local Hilbert Space Extensions of QM. *Europhysics Letters*, **86**, (2009), 20009-20015.

[24] Allahverdyan, A. E., Balian, R., Nieuwenhuizen, Th. M.: The quantum measurement process in an exactly solvable model. In: *Foundations of Probability and Physics-3*, pp. 16-24. American Institute of Physics, Ser. Conference Proceedings 750, Melville, NY (2005).

[25] De Muynck, W. M.: *Foundations of Quantum Mechanics*, an Empiricists Approach. Kluwer Academic Publ., Dordrecht (2002).

[26] G. M. D’Ariano, Operational axioms for quantum mechanics. *Foundations of Probability and Physics-3*, pp. 79–105. American Institute of Physics, Ser. Conference Proceedings 889, Melville, NY (2007).

[27] C. M. Caves, Ch. A. Fuchs, and R. Schack, Quantum probabilities
as Bayesian probabilities. Phys. Rev. A 65, 022305 (2002).
[28] Ch. A. Fuchs and R. Schack, A Quantum-Bayesian Route to Quantum-State Space. Foundations of Physics, 41, 345-356 (2011).
[29] Ohya M.: Note on quantum proability, L.Nuovo Cimento, Vol.38, No.11, 203-206, (1983)
[30] Ohya M.: On compound state and mutual information in quantum information theory, IEEE Trans.Information Theory, 29, pp.770–777 (1983)
[31] Ohya M.: Complexities and their applications to characterization of chaos, International Journal of Theoretical Physics, Vol.37, No.1, 495-505, (1998)
[32] Ohya M.: Some aspects of quantum information theory and their applications to irreversible processes, Rep.Math.Phys., Vol.27, 19-47, (1989)
[33] Accardi L., Ohya M.: Compound Channels, Transition Expectations, and Liftings, Appl. Math. Optim., 39,33-59 (1999)
[34] Inoue K., Ohya M., Sato K.: Application of chaos degree to some dynamical systems, Chaos, Soliton & Fractals, 11, 1377-1385, (2000)
[35] Ohya M., Volovich I.V.: New quantum algorithm for studying NP-complete problems, Rep.Math.Phys., 52, No.1,25-33 (2003)
[36] Ohya M., Volovich I.V.: Mathematical Foundations of Quantum Information and Computation and Its Applications to Nano- and Bio-systems, Springer, (2011)
[37] Inoue K., Ohya M. Volovich I.V.: Semiclassical properties and chaos degree for the quantum baker’s map, Journal of Mathematical Physics, Vol.43, No.1 (2002)
[38] Kossakowski A., Ohya M., Togawa Y.: How can we observe and describe chaos?, Open System and Information Dynamics 10(3): 221-233 (2003)
[39] Accardi L., Ohya M.: A Stochastic Limit Approach to the SAT Problem,Open Systems and Information dynamics, 11,1-16, (2004)
[40] Ohya M.: Adaptive Dynamics and its Applications to Chaos and NPC Problem. QP-PQ: Quantum Probability and White Noise Analysis, Quantum Bio-Informatics 2007, 21: 181-216 (2007)
[41] Accardi L.: Urne e camaleonti: Dialogo sulla realta, le leggi del caso e la teoria quantistica.Il Saggiatore 1997 (English edition, World Scientific 2002; japanese edition, Makino 2002, russian edition, Regular and Chaotic dynamics (2002)
[42] M.Asano, M. Ohya, Y.Tanaka, A. Khrennikov and I. Basieva, Quantum-like Representation of Bayesian Updating, American Institute of physics Volume 1327, pp. 57-62 : Proceedings of the In-
[43] Inada T, Kimata K, Aiba H., Mechanism responsible for glucose-lactose diauxie in Escherichia coli challenge to the cAMP model. Genes and Cells (1996) 1, 293-301.

[44] M. Asano, M. Ohya, Y. Tanaka, A. Khrennikov and I. Basieva, Quantum-like Bayesian Updating, TUS preprint (2011)