REALIZATION OF METRIC SPACES AS INVERSE LIMITS, AND BILIPSCHITZ EMBEDDING IN $L_1$

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Abstract. We give sufficient conditions for a metric space to bilipschitz embed in $L_1$. In particular, if $X$ is a length space and there is a Lipschitz map $u : X \to \mathbb{R}$ such that for every interval $I \subset \mathbb{R}$, the connected components of $u^{-1}(I)$ have diameter $\leq \text{const} \cdot \text{diam}(I)$, then $X$ admits a bilipschitz embedding in $L_1$. As a corollary, the Laakso examples, (Geom Funct Anal 10(1):111–123, 2000), bilipschitz embed in $L_1$, though they do not embed in any any Banach space with the Radon–Nikodym property (e.g. the space $\ell_1$ of summable sequences). The spaces appearing the statement of the bilipschitz embedding theorem have an alternate characterization as inverse limits of systems of metric graphs satisfying certain additional conditions. This representation, which may be of independent interest, is the initial part of the proof of the bilipschitz embedding theorem. The rest of the proof uses the combinatorial structure of the inverse system of graphs and a diffusion construction, to produce the embedding in $L_1$.

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1 Introduction

Overview. This paper is part of a series [CK06b, CK06a, CK10a, CK09, CK10b, CKN09, CK13b] which examines the relations between differentiability properties and bilipschitz embeddability in Banach spaces. We introduce a new notion of dimension—Lipschitz dimension—and show that spaces of Lipschitz dimension $\leq k$ admit a representation as a certain kind of inverse limit. We then use this characterization to show that spaces of Lipschitz dimension $\leq 1$ bilipschitz embed in $L_1$. This embedding result applies to several known families of spaces, illustrating the sharpness of earlier nonembedding theorems.

Metric spaces sitting over $\mathbb{R}$. We begin with a special case of our main embedding theorem.

**Theorem 1.1.** Let $X$ be a length space. Suppose $u : X \to \mathbb{R}$ is a Lipschitz map, and there is a $C \in (0, \infty)$ such that for every interval $I \subset \mathbb{R}$, each connected component of $u^{-1}(I)$ has diameter at most $C \cdot \text{diam}(I)$. Then $X$ admits a bilipschitz embedding $f : X \to L_1(Z, \mu)$, for some measure space $(Z, \mu)$.

We illustrate Theorem 1.1 with two simple examples:

**Example 1.2** (Lang–Plaut [LP01], cf. Laakso [Laa00]). We construct a sequence of graphs $\{X_i\}_{i \geq 0}$ where $X_i$ has a path metric so that every edge has length $4^{-i}$. Let $X_0$ be the unit interval $[0, 1]$. For $i > 0$, inductively construct a $X_i$ from $X_{i-1}$ by replacing each edge of $X_{i-1}$ with a copy of the graph $\Gamma$ in Figure 1, rescaled by the factor $4^{-(i-1)}$. The graphs $X_1$, $X_2$, and $X_3$ are shown. The sequence $\{X_i\}$ naturally forms an inverse system,

$$X_0 \leftarrow \pi_0 \leftarrow \cdots \leftarrow X_i \leftarrow \pi_i \leftarrow \cdots,$$

where the projection map $\pi_{i-1} : X_i \to X_{i-1}$ collapses the copies of $\Gamma$ to intervals. The inverse limit $X_\infty$ has a metric $d_\infty$ given by

$$d_\infty(x, x') = \lim_{i \to \infty} d_{X_i}(\pi_i^\infty(x), \pi_i^\infty(x')),$$

(1.3)

where $\pi_i^\infty : X_\infty \to X_i$ denotes the canonical projection. (Note that the sequence of metric spaces $\{X_i\}_{i \geq 0}$ Gromov–Hausdorff converges to $(X_\infty, d_\infty)$.) It is not hard to verify directly that $\pi_0^\infty : (X_\infty, d_\infty) \to [0, 1]$ satisfies the hypotheses of Theorem 1.1; this also follows from the results in Section 3.

**Example 1.4.** Construct an inverse system

$$X_0 \leftarrow \pi_0 \leftarrow \cdots \leftarrow X_i \leftarrow \pi_i \leftarrow \cdots,$$

inductively as follows. Let $X_0 = [0, 1]$. For $i > 0$, inductively define $X'_{i-1}$ to be the result of trisecting all edges in $X_{i-1}$, and let $N \subset X'_{i-1}$ be new vertices added in