ON BETHE VECTORS IN THE $sl_{N+1}$ GAUDIN MODEL

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Abstract. The note deals with the Gaudin model associated with the tensor product of $n$ irreducible finite-dimensional $sl_{N+1}$-modules marked by distinct complex numbers $z_1, \ldots, z_n$. The Bethe Ansatz is a method to construct common eigenvectors of the Gaudin hamiltonians by means of chosen singular vectors in the factors and $z_j$’s. These vectors are called Bethe vectors.

The question if the Bethe vectors are non-zero vectors is open. By the moment, the only way to verify that was based on a relation to critical points of the master function of the Gaudin model, and non-triviality of a Bethe vector was proved only in the case when the corresponding critical point is non-degenerate ([ScV, MV1]). However degenerate critical points do appear in the Gaudin model ([ReV, Section12]).

We believe that the Bethe vectors never vanish, and suggest an approach that does not depend on non-degeneracy of the corresponding critical point. The idea is for a Bethe vector to choose a suitable subspace in the weight space and to check that the projection of the Bethe vector to this subspace is non-zero. We apply this approach to verify non-triviality of Bethe vectors in new examples.

1. Introduction

We study the Gaudin model of statistical mechanics associated with the Lie algebra $sl_{N+1}(\mathbb{C})$. The space of states of the model is the tensor product

$$L = L_{\Lambda(1)} \otimes \cdots \otimes L_{\Lambda(n)},$$

where $L_{\Lambda(j)}$ is a finite-dimensional irreducible $sl_{N+1}$-module with highest weight $\Lambda(j)$, $1 \leq j \leq n$. For the standard notions of representation theory see [FuH].

In the Gaudin model, the modules $L_{\Lambda(1)}, \ldots, L_{\Lambda(n)}$ are the spin spaces of $n$ particles located at distinct points $z_1, \ldots, z_n \in \mathbb{C}$. Write $z = (z_1, \ldots, z_n)$. The Gaudin hamiltonians $H_1(z), \ldots, H_n(z)$ are mutually commuting linear operators in $L$ which are defined as follows,

$$H_j(z) = \sum_{i \neq j} \frac{C_{ij}}{z_j - z_i}, \quad 1 \leq j \leq n,$$

here $C_{ij}$ acts as the Casimir operator on factors $L_{\Lambda(i)}$ and $L_{\Lambda(j)}$ of the tensor product and as the identity on all other factors.

One of the main problems in the Gaudin model is simultaneous diagonalization of the operators $H_1(z), \ldots, H_n(z)$. The Gaudin hamiltonians commute with the diagonal action of $sl_{N+1}$ in $L$, therefore it is enough to find common eigenvectors and the eigenvalues in the subspace of singular vectors of a given weight, for every weight.
The algebraic Bethe Ansatz is a method to construct such vectors. The idea is to find some function $v = v(t)$ taking values in the weight subspace ($t$ is a multidimensional auxiliary variable) and to determine a certain special value of its argument, $t^{(0)}$, in such a way that $v(t^{(0)})$ is a common eigenvector of the hamiltonians. The equations on $t$ which determine these special values of the argument are called the Bethe equations, and $v(t^{(0)})$ is called the Bethe vector. For the Gaudin model, the Bethe equations and the function $v(t)$ are written in [FeFR, ReV, SV]. On Bethe vectors in the Gaudin model see also [C, FaT, Re].

It was believed that for generic $z$ one can find an eigenbasis in the subspace of singular vectors consisting of Bethe vectors only. This is indeed the case for the tensor products of $sl_2(\mathbb{C})$-modules and for the tensor products of several copies of first and last fundamental $sl_{N+1}$-modules ([ScV, MV1]). Recent results of [MV2] show however that generically other eigenvectors have to be present in eigenbases as well. These other vectors are in some sense “more degenerate” than Bethe vectors, see [F1] and especially [F2, Section 5.5] discussing the “degeneracies”.

In the Bethe Ansatz, two problems naturally arise: to find solutions of the Bethe equations, and to check non-triviality of the corresponding Bethe vectors. Both problems are open and seem to be difficult ones. On solutions to the Bethe equations in some particular cases, see [V, ScV, MV1, Sc].

The present note is devoted to the question if Bethe vectors are non-zero vectors. By the moment, the only known way to verify that was extremely non-direct, via the so-called master function. Namely, it appeared that the Bethe equations in the Gaudin model form the critical point system of a certain function $S(t; z)$, here $t$ is a multidimensional variable and $z = (z_1, \ldots, z_n)$ is fixed, [ReV]. Moreover, the norm of the Bethe vector $v(t^{(0)})$ with respect to some (degenerate) bilinear form on the tensor product turned out to be the Hessian of $S(t; z)$ at the critical point $t^{(0)}$; hence the Bethe vectors corresponding to non-degenerate critical points of the function $S(t; z)$ appeared to be non-zero vectors, [V, MV1]. In this way, the non-triviality of Bethe vectors has been checked for generic $z$ in the case of tensor products of $sl_2(\mathbb{C})$-modules and in the case of tensor products of several copies of first and last fundamental $sl_{N+1}$-modules, [ScV, MV1].

It is known however, that for some values of $z$ the master function does have degenerate critical points; an example can be found in [ReV, Section 12]. Notice that in that example the corresponding Bethe vector is a non-zero vector as well. We believe that the Bethe vectors are always non-trivial.

**Conjecture.** In the $sl_{N+1}(\mathbb{C})$ Gaudin model, every Bethe vector is non-zero, for any $z$. For some values of $z$ the number of Bethe vectors (i.e. of solutions to the Bethe equations, i.e. of critical points of the master function) may decrease, but the Bethe vectors still are non-zero.

We suggest a more direct approach that does not depend on non-degeneracy of the corresponding critical point. The idea is to project a Bethe vector to a suitable subspace in the space of singular vectors of a given weight and to check that the projection is non-zero.
We exploit this idea in some examples of tensor products of irreducible finite-dimensional $\mathfrak{sl}_{N+1}$-modules. The case of the tensor product of $n = 2$ modules is special. First of all, in this case all values of $z$ are generic. Indeed, as it was pointed out in [ReV, Section 5], for any fixed $z_1 \neq z_2$ the linear change of variables $u = (t - z_1)/(z_2 - z_1)$ turns the Bethe system on $t$ with $z = (z_1, z_2)$ into the Bethe system on $u$ with $z = (0, 1)$. Next, the Gaudin Hamiltonians $H_1(0, 1) = -H_2(0, 1)$ are reduced to the Casimir operator, and hence act in any irreducible submodule of the tensor product by multiplication by a constant, i.e. any singular vector is their common eigenvector. Finally, non-triviality of a Bethe vector for $n > 2$ in many cases can be deduced from non-triviality of a certain set of Bethe vectors corresponding to $n = 2$ and $z = (0, 1)$, by means of iterated singular vectors introduced in [ReV]; see [Sc] for a more detailed explanation.

Let $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$ be the tensor product of two irreducible finite-dimensional $\mathfrak{sl}_{N+1}$-modules, where $L_{\Lambda(1)}$ is marked by $z_1 = 1$ and $L_{\Lambda(0)}$ by $z_0 = 0$. Denote simple positive roots of $\mathfrak{sl}_{N+1}$ by $\alpha_1, \ldots, \alpha_N$.

In our first example (Section 4.1), we consider arbitrary integral dominant weights $\Lambda(1)$, $\Lambda(0)$ and assume $v_k$ to be a Bethe vector of the weight $\Lambda(1) + \Lambda(0) - k\alpha_1$.

In another example (Section 4.3), we restrict $L_{\Lambda(0)}$ to be a symmetric power of the standard $\mathfrak{sl}_{N+1}$-representation, and assume $v_{k,1,1}$ to be a Bethe vector in $L$ of the weight $\Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3$.

**Theorem.** Bethe vectors $v_k$ and $v_{k,1,1}$ are non-trivial.

For $N = 1$, any Bethe vector is of the form $v_k$, therefore the example 4.1 implies that for $N = 1$ and $n = 2$ the Bethe vectors never vanish. Moreover, this example admits an immediate generalization to $n > 2$, see Theorem 3 in Section 4.2. As a corollary we obtain that if $L_{\Lambda(0)} = m\lambda_1$ and $\Lambda(k_1, k_2) = \Lambda(1) + \Lambda(0) - k_1\alpha_1 - k_2\alpha_2$ is the highest weight of an irreducible component of $L$, then Bethe vectors of the weight $\Lambda(k_1, k_2)$ do not vanish. In particular, if $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$ are $\mathfrak{sl}_3$-representations, then all Bethe vectors in $L$ do not vanish.

In the examples 4.1, 4.3 the subspace of singular vectors is one-dimensional, therefore a Bethe vector, if it exists, gives an eigenbasis of the Gaudin Hamiltonians in the corresponding weight subspace. A way to solve the Bethe equations in the example 4.3 is explained in [Sc].

The key ingredient of our proof is to write the Bethe equations and projections of vector $v(t)$ in terms of symmetric functions in $t$, see Section 3.3 and Section 4. Our calculations are based on funny relations between symmetric rational functions which generalize the “Jacobi identity”

\[
\frac{1}{(x - y)(x - z)} + \frac{1}{(y - x)(y - z)} = \frac{-1}{(z - x)(z - y)},
\]

see Theorems 1 and Corollary 1 in Section 2.
Plan of the note. Section 2 is devoted to the “Jacobi-like” identities; Section 3 contains a description of the Bethe equations and Bethe vectors; in Section 4 we verify non-triviality of Bethe vectors in the examples.

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2. Identities

For a function \( g(t_1 \ldots t_k) \), define \( \text{symmetrization} \) as follows,

\[
\text{Sym}_k[g] := \sum_{\pi \in S_k} g(\pi(t_1), \ldots, \pi(t_k)),
\]

here the sum runs over the group \( S_k \) of all permutations \( \pi \) of variables \( t_1 \ldots t_k \).

**Theorem 1.** For any fixed \( s_1, s_2 \) and \( s \), we have

(I) \[
\text{Sym}_k \left[ \frac{1}{(s_1-t_1)(t_1-t_2) \ldots (t_{k-1}-t_k)(t_k-s_2)} \right] = \frac{(-1)^k \cdot (s_1-s_2)^{k-1}}{(s_1-t_1) \ldots (s_1-t_k) \cdot (s_2-t_1) \ldots (s_2-t_k)},
\]

(II) \[
\text{Sym}_k \left[ \frac{1}{(s-t_1)(t_1-t_2) \ldots (t_{k-1}-t_k)t_k} \right] = \frac{s^{k-1}}{(s-t_1) \ldots (s-t_k) \cdot t_1 \ldots t_k},
\]

(III) \[
\text{Sym}_k \left[ \frac{1}{(t_1-t_2)(t_2-t_3) \ldots (t_{k-1}-t_k)(t_k-s)} \right] = \frac{(-1)^k}{(s-t_1) \ldots (s-t_k)}.
\]

**Proof.** We prove the first and the third identities by induction in \( k \). The second identity can be obtained from the first one by substitution \( s_1 = s \) and \( s_2 = 0 \).

The first identity for \( k = 1 \) becomes

\[
\text{Sym}_1 \left[ \frac{1}{(s_1-t_1) \cdot (t_1-s_2)} \right] = \frac{1}{(s_1-t_1) \cdot (t_1-s_2)} = \frac{-1}{(s_1-t_1) \cdot (s_2-t_1)},
\]

and is true. Suppose that the identity (I) holds for \( k - 1 \) and prove it for \( k \). Consider the subgroup \( S_{k-1} \subset S_k \) of permutations acting on the first \( k - 1 \) variables. Every summand in the symmetrization of our fraction has a form

\[
\frac{1}{(s_1-t_{i_1})(t_{i_1}-t_{i_2}) \ldots (t_{ik-1}-t_{ik})(t_{ik}-s_2)},
\]
Combine together all the summands with a fixed value of \( i_k \), say \( i_k = j \), and factor out the last multiplier \( 1/(t_j - s_2) \). Then we can write

\[
Sym_k \left[ \frac{1}{(s_1 - t_1)(t_1 - t_2) \ldots (t_{k-1} - t_k)(t_k - s_2)} \right] = \\
= \sum_{j=1}^{k} Sym_{k-1} \left[ \frac{1}{(s_1 - t_{i_1})(t_{i_1} - t_{i_2}) \ldots (t_{i_{k-1}} - t_j)} \right] \cdot \frac{1}{t_j - s_2},
\]

here the values of \( i_1, \ldots, i_{k-1} \) are different from \( j \) and the group \( S_{k-1} \) acts by the permutations which keep \( t_j \). By the induction hypothesis this is

\[
\sum_{j=1}^{k} \frac{(-1)^{k-1} \cdot (s_1 - t_j)^{k-2}}{(s_1 - t_1) \ldots (s_1 - t_j) \ldots (s_1 - t_k) \cdot (t_j - t_1) \ldots (t_j - t_j) \ldots (t_j - t_k)} \cdot \frac{1}{t_j - s_2},
\]

where the “hat” means that the corresponding factor is omitted. Multiplying this expression with \( (s_1 - t_1) \ldots (s_1 - t_k) \cdot (s_2 - t_1) \ldots (s_2 - t_k) \) we get

\[
(-1)^k \sum_{j=1}^{k} \left( \frac{s_1 - t_j}{t_j - t_1} \cdot \frac{(s_2 - t_1) \ldots (s_2 - t_j) \ldots (s_2 - t_k)}{(t_j - t_1) \ldots (t_j - t_j) \ldots (t_j - t_k)} \right).
\]

This is nothing but the Lagrange interpolation formula for a polynomial of degree \( k-1 \) in a variable \( s_2 \) that takes the value \((-1)^k(s_1 - t_j)^{k-1}\) at the point \( s_2 = t_j \) for every \( j = 1, \ldots, k \). Therefore it is equal to \((-1)^k(s_1 - s_2)^{k-1}\).

The third identity is obvious for \( k = 1 \),

\[
Sym_1 \left[ \frac{1}{(t_1 - s)} \right] = \frac{1}{(t_1 - s)} = \frac{-1}{(s - t_1)}.
\]

Suppose that the identity (II) holds for \( k - 1 \) and prove it for \( k \). As before, combining together all the summands of the left hand side with a fixed variable \( t_j \) at the last factor of the denominator we get

\[
Sym_k \left[ \frac{1}{(t_1 - t_2) \ldots (t_{k-1} - t_k)(t_k - s)} \right] = \\
= \sum_{j=1}^{k} Sym_{k-1} \left[ \frac{1}{(s_1 - t_{i_1})(t_{i_1} - t_{i_2}) \ldots (t_{i_{k-1}} - t_j)} \right] \cdot \frac{1}{t_j - s},
\]

where the values of \( i_1, \ldots, i_{k-1} \) are different from \( j \) and the group \( S_{k-1} \) acts by the permutations that keep \( t_j \). By the induction hypothesis this is equal to

\[
\sum_{j=1}^{k} \frac{(-1)^{k-1}}{(t_j - t_{i_1})(t_j - t_{i_2}) \ldots (t_j - t_{i_{k-1}})} \cdot \frac{1}{t_j - s}.
\]
Multiplying this expression with \((s - t_1) \ldots (s - t_k)\) we get
\[
(-1)^k \sum_{j=1}^{k} \frac{(s - t_{i_1})(s - t_{i_2}) \ldots (s - t_{i_{k-1}})}{(t_j - t_{i_1})(t_j - t_{i_2}) \ldots (t_j - t_{i_{k-1}})},
\]
where the indices \(i_1, \ldots, i_{k-1}\) in every summand are the integers between 1 and \(k\) different form \(j\). Recognizing in the last expression the Lagrange interpolation formula we conclude that this is exactly \((-1)^k\).

It is convenient to write identities on functions which are symmetric with respect to variables \(t_1, \ldots, t_k\) in terms of the elementary symmetry functions.

**Notation.**
\[
T(x) = (x - t_1) \ldots (x - t_k) = x^k - \tau_1 x^{k-1} + \ldots + (-1)^k \tau_k,
\]
that is \(\tau_i\) is the \(i\)-th elementary symmetric function in \(t_1, \ldots, t_k\) for \(1 \leq i \leq k\); we set \(\tau_0 = 1\).

With this notation, the identities of Theorem 1 take the form
\[
(I') \quad \text{Sym}_k \left[ \frac{1}{(s_1 - t_1)(t_1 - t_2) \ldots (t_{k-1} - t_k)(t_k - s_2)} \right] = \frac{(-1)^k \cdot (s_1 - s_2)^{k-1}}{T(s_1) \cdot T(s_2)},
\]
\[
(II') \quad \text{Sym}_k \left[ \frac{1}{(s - t_1)(t_1 - t_2) \ldots (t_{k-1} - t_k)t_k} \right] = \frac{s^{k-1}}{T(s) \tau_k},
\]
\[
(III') \quad \text{Sym}_k \left[ \frac{1}{(t_1 - t_2)(t_2 - t_3) \ldots (t_{k-1} - t_k)(t_k - s)} \right] = \frac{(-1)^k}{T(s)}.
\]

**Corollary 1.** We have
\[
(IV) \quad \text{Sym}_k \left[ \frac{1}{(t_1 - t_2) \ldots (t_{i-2} - t_{i-1})(t_{i-1} - s)(s - t_i)(t_i - t_{i+1}) \ldots (t_{k-1} - t_k)t_k} \right] = \\
\quad \quad \quad \quad \quad = \frac{(-1)^{i-1} s^{k-i} \tau_{i-1}}{T(s) \tau_k}, \quad \text{for} \quad 1 \leq i \leq k.
\]

**Proof.** The formula (IV) for \(i = 1\) is exactly the identity (II'). For \(2 \leq i \leq k\), first let us take the sum over the subgroup \(S_{i-1} \times S_{k+1-i} \subset S_k\), i.e. combine together the summands corresponding to permutations of the first \(i - 1\) and of the last \(k + 1 - i\) variables \(t_j\)’s. Applying the identities (III) and (II) we get
\[
\frac{(-1)^{i-1}}{(s - t_1) \ldots (s - t_{i-1})} \times \frac{s^{k-i}}{(s - t_i) \ldots (s - t_k) \cdot t_i \ldots t_k}.
\]
Now we take the sum over the cosets of the subgroup $S_{i-1} \times S_{k+1-i} \subset S_k$ and collect look like terms. Every denominator is the same, $(s - t_1) \ldots (s - t_k) \cdot t_1 \ldots t_k = T(s) \cdot \tau_k$, whereas the numerators contain all possible products of $i - 1$ of variables $t_j$'s.

**Remarks.**

1. Let us consider $t_1, \ldots, t_k, s_1$ as fixed numbers, and $s, s_2$ as variables. Then the left-hand side of every identity is nothing but a partial fraction decomposition of the function from the right-hand side. This interpretation, indicated by V. Lin, leads to another proof of the identities.

2. As A. Varchenko pointed out, our identity (III) for $s = 0$ follows from the coincidence of the forms $\Omega^{sl_2}$ and $\tilde{\Omega}^{sl_2}$ from [RStV, page 2]. Notice that for arbitrary $s$ the identity (III) can be obtained from this particular case by substitution $t \mapsto t - s$. Similarly, the substitution $s \mapsto s_1 - s_2, t \mapsto t - s_2$ transforms the identity (II) into the identity (I).

3. A remark of the referee is that the identity (III) could be deduced from the identity (I). Indeed if we consider (I) as a function of a complex variable $s_1$ and take the residues of both sides at infinity, then we get (III).

3. **Bethe vectors**

Here we recall the constructions for the tensor product of $n = 2$ modules corresponding to points $z_0 = 0$ and $z_1 = 1$. For $n > 2$ (and for any simple Lie algebra), see [FeFr, ReV].

**3.1. Subspace of singular vectors in $L$.** Denote by \( \{ e_i, f_i, h_i \}_{i=1}^N \) the standard Chevalley generators of $sl_{N+1}(\mathbb{C})$,

\[
[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i; \quad [h_i, h_j] = 0, \quad [e_i, f_j] = 0 \text{ if } i \neq j.
\]

Let \( \mathfrak{h} \) be the Cartan subalgebra and \( \mathfrak{h}^\ast \) its dual,

\[
\mathfrak{h}^\ast = \mathbb{C}\{ \lambda_1, \ldots, \lambda_{N+1} \} / (\lambda_1 + \cdots + \lambda_{N+1} = 0),
\]

with the standard bilinear form (\( \cdot, \cdot \)). The simple positive roots are \( \alpha_i = \lambda_i - \lambda_{i+1} \), \( 1 \leq i \leq N \),

\[
(\alpha_i, \alpha_i) = 2; \quad (\alpha_i, \alpha_j) = 0, \text{ if } |i - j| > 1; \quad \text{and} \quad (\alpha_i, \alpha_j) = -1, \text{ if } |i - j| = 1.
\]

Let \( \Lambda(1) \) and \( \Lambda(0) \) be integral dominant weights, and \( \mathbf{k} = (k_1, \ldots, k_N) \) be a vector with nonnegative integer coordinates such that

\[
\Lambda(\mathbf{k}) := \Lambda(1) + \Lambda(0) - k_1\alpha_1 - \cdots - k_N\alpha_N
\]

is an integral dominant weight as well. Denote by

\[
\text{Sing}_k L := \{ \mathbf{v} \in L \mid h_i \mathbf{v} = (\Lambda(\mathbf{k}), \alpha_i) \mathbf{v}, \ e_i \mathbf{v} = 0, \ i = 1, \ldots, N \}
\]

the subspace of singular vectors in $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$ of weight $\Lambda(\mathbf{k})$. 
3.2. **Bethe system associated with** $\text{Sing}_k L$. For every $i = 1, \ldots, N$ introduce a set of $k_i$ auxiliary variables associated with the root $\alpha_i$,

$$t(i) := (t_1(i), \ldots, t_{k_i}(i)),$$

and write $t := (t(1), \ldots, t(N))$.

The **Bethe system** is the following system of equations on variables $t_l(i)$,

$$\sum_{s \neq l} \frac{2}{t_l(i) - t_s(i)} - \sum_{s=1}^{k_i-1} \frac{1}{t_l(i) - t_s(i-1)} - \sum_{s=1}^{k_i+1} \frac{1}{t_l(i) - t_s(i+1)} - \frac{(\Lambda(0), \alpha_i)}{t_l(i)} - \frac{(\Lambda(1), \alpha_i)}{t_l(i) - 1} = 0,$$

here $1 \leq i \leq N, 1 \leq l \leq k_i$.

Every solution $t^{(0)}$ to this system determines a **Bethe vector** $v(t^{(0)}) = v_k(t^{(0)}) \in \text{Sing}_k L$.

The function $v(t)$ is described in Section 3.4.

3.3. **Bethe equations in terms of polynomials** $T_1(x), \ldots, T_N(x)$. We use the notation introduced in Section 2.

**Proposition 1.** Assume all the roots of $T(x)$ to be simple. Then

$$\frac{T'(x)}{T(x)} = \sum_{j=1}^{k} \frac{1}{x - t_j}; \quad \frac{T''(t_i)}{T'(t_i)} = \frac{2}{t_i - t_j}. $$

**Proof.** The first equation is just the logarithmic derivative of $T$.

We have

$$T'(x) = \left( \sum_{j=1}^{k} \frac{1}{x - t_j} \right) \cdot T(x). $$

Derivation of this equation gives

$$T''(x) = \left( \sum_{j=1}^{k} \frac{1}{x - t_j} \right)' \cdot T(x) + \left( \sum_{j=1}^{k} \frac{1}{x - t_j} \right)^2 \cdot T(x). $$

Therefore

$$\frac{T''(x)}{T(x)} = - \sum_{j=1}^{k} \frac{1}{(x - t_j)^2} + \left( \sum_{j=1}^{k} \frac{1}{x - t_j} \right)^2 = 2 \sum_{1 \leq j < l \leq k} \frac{1}{(x - t_j)(x - t_l)}. $$

We have

$$\frac{T''(x)}{T'(x)} = \frac{2}{\sum_{1 \leq j < l \leq k} \frac{1}{(x - t_j)(x - t_l)}} = \frac{2}{\sum_{j=1}^{k} (x - t_1) \cdots (x - t_j) \cdots (x - t_l) \cdots (x - t_k)}. $$
Substitution $x = t_i$ gives
\[
\frac{T''(t_i)}{T'(t_i)} = \frac{2 \sum_{j \neq i} (t_i - t_1) \ldots (t_i - t_j) \ldots (t_i - t_l) \ldots (t_i - t_k)}{(t_i - t_1) \ldots (t_i - t_l) \ldots (t_i - t_k)},
\]
and the division finishes the proof.

Now we can re-write the Bethe system in terms of polynomials $T_1(x), \ldots, T_N(x)$, where
\[
T_i(x) = (x - t_1(i)) \ldots (x - t_{k_i}(i)).
\]
We have
\[
\frac{T''_i(t_i)}{T'_i(t_i)} - \frac{T'_{i-1}(t_i)}{T_{i-1}(t_i)} - \frac{T'_{i+1}(t_i)}{T_{i+1}(t_i)} = \frac{(\Lambda(0), \alpha_i)}{t_i(i)} - \frac{(\Lambda(1), \alpha_i)}{t_i(i) - 1} = 0,
\]
for $1 \leq i \leq N$, $1 \leq l \leq k_i$.

3.4. Function $v(t)$. The function $v(t)$ has been obtained in [SV Sections 6,7] in general setting (see also [MV], where it is called the universal weight function). Below we rewrite this function for the weight $\Lambda(k)$ in the tensor product of two modules $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$ corresponding to points $z_1 = 1$ and $z_0 = 0$. Generic case can be found in [FeFRe, ReV].

To begin with, we construct the vectors that generate the subspace $L_{\Lambda(k)} \subset L$ of weight $\Lambda(k)$. In general, their number is greater than the dimension of that subspace, so they are linearly dependent.

Consider all pairs of words $(F_1, F_0)$ in letters $f_1, \ldots, f_N$ subject to the condition that the total number of occurrences of letter $f_i$ in both words is precisely $k_i$. Our vectors will be labeled by these pairs. Namely, we may think about words $F_1$ and $F_0$ as elements of the universal enveloping algebra of $sl_{N+1}$ that naturally act on the spaces $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$, respectively. Fix the highest weight vectors $v_1 \in L_{\Lambda(1)}$, $v_0 \in L_{\Lambda(0)}$. Then the vector
\[
w_{(F_1, F_0)} := F_1 v_1 \otimes F_0 v_0
\]
has weight $\Lambda(k)$, and all such vectors generate the weight space $L_{\Lambda(k)}$.

Now we define $v_k(t)$ as a linear combination
\[
v_k(t) := \sum_{(F_1, F_0)} \omega_{(F_1, F_0)}(t) w_{(F_1, F_0)},
\]
where $\omega_{(F_1, F_0)}(t)$ are certain rational functions. We will construct these functions in two steps described below. Write
\[
F_1 = f_{i_1} \ldots f_{i_{s_1}}, \quad F_0 = f_{j_1} \ldots f_{j_{s_0}}, \quad F_1 F_0 = f_{i_1} \ldots f_{i_{s_1}} f_{j_1} \ldots f_{j_{s_0}}.
\]
The length of the word $F_1 F_0$ equals $s_1 + s_0 = k_1 + \cdots + k_N$.

The first step is to translate $(F_1, F_0)$ into a rational function $g_{(F_1, F_0)}(t)$ of $t$. For every $i = 1, \ldots, N$, we replace the first occurrence (from left to right) of $f_i$ in the word $F_1 F_0$ by the variable $t_1(i)$; the second occurrence by the variable $t_2(i)$; and so on up to the last, $k_i$-th, occurrence, where $f_i$ will be replaced by $t_{k_i}(i)$. We will get a pair of words in $t$. We
augment these two words by 1 and 0, according to the values of \( z_1 \) and \( z_0 \), and thus get the row,

\[
t_{a_1}(i_1) t_{a_2}(i_2) \ldots t_{a_{s_1}}(i_{s_1}) 1, \quad t_{b_1}(j_1) t_{b_2}(j_2) \ldots t_{b_{s_0}}(j_{s_0}) 0,
\]

in which every variable \( t_i(i) \) from \( t \) appears precisely once. This row defines the fraction

\[
g_{(F_1,F_0)}(t) := \frac{1}{(t_{a_1}(i_1) - t_{a_2}(i_2))(t_{a_2}(i_2) - t_{a_3}(i_3)) \cdots (t_{a_{s_1-1}}(i_{s_1}) - t_{a_{s_1}}(i_{s_1}))(t_{a_{s_1}}(i_{s_1}) - 1)} \times \frac{1}{(t_{b_1}(j_1) - t_{b_2}(j_2))(t_{b_2}(j_2) - t_{b_3}(j_3)) \cdots (t_{b_{s_0-1}}(j_{s_0}) - t_{b_{s_0}}(j_{s_0}))(t_{b_{s_0}}(j_{s_0}) - 1)}.
\]

The second step is the symmetrization of \( g_{(F_1,F_0)}(t) \). Let \( S_k \) denote the group of permutations of variables

\[
t = (t_1(1), \ldots, t_{k_1}(1), \ t_1(2), \ldots, t_{k_2}(2), \ldots, \ t_1(N), \ldots, t_{k_N}(N))
\]

that permute variables \( t_1(i), \ldots, t_{k_i}(i) \) within their own, \( i \)-th, set, for every \( i = 1, \ldots, N \). Thus \( S_k \) is isomorphic to the direct product \( S_{k_1} \times S_{k_2} \times \cdots \times S_{k_N} \) of permutation groups.

For a function \( g(t) \) define the symmetrization by the formula

\[
\text{Sym}_k[g] := \sum_{\pi \in S_k} g\left(\pi(t_1(1)), \ldots, \pi(t_{k_N}(N))\right).
\]

Finally we set

\[
\omega_{(F_1,F_0)}(t) := \text{Sym}_k \left[ g_{(F_1,F_0)}(t) \right].
\]

Notice that the universal weight function \( v_k(t) \) is defined for any, not necessarily dominant, weight \( \Lambda(k) \) presented in \( L = L_{\Lambda(1)} \otimes L_{\Lambda(0)} \). However in the Bethe Ansatz it is used only when \( \Lambda(k) \) is the highest weight of an irreducible component of \( L \).

4. Checking the non-triviality of Bethe vectors in examples

4.1. Example \( \Lambda(1) + \Lambda(0) - k\alpha_1 \). We assume \( \Lambda(1) \) and \( \Lambda(0) \) to be integral dominant weights and \( k \) an integer such that

\[
\Lambda(k, 0) := \Lambda(1) + \Lambda(0) - k\alpha_1
\]

is the highest weight of an irreducible component of \( L = L_{\Lambda(1)} \otimes L_{\Lambda(0)} \). In this case Steinberg’s formula implies that \( f_1^k v_0 \neq 0 \) [Hi Exercise 24.12]. We will show that the universal weight function \( v_k(t) := v_{(k,0,\ldots,0)}(t) \) never vanish. We have

\[
k = (k,0,\ldots,0), \quad t = (t(1)), \quad F_0 = f_1^k, \quad g_{(0,F_0)}(t) = \frac{1}{(t_1 - t_2) \cdots (t_{k-1} - t_k)t_k}.
\]

Simplifying the notation, write

\[
t := (t_1, \ldots, t_k), \quad T(t) = \prod_{i=1}^k (x - t_i), \quad \omega_{k,0}(t) := \omega_{(0,F_0)}(t).
\]

Theorem 2. The projection of the vector \( v_k(t) \) to the subspace of \( L_{\Lambda(k,0)} \) spanned by \( v_1 \otimes f_1^k v_0 \) is a non-zero vector.
Proof. Notice that the domain of the function $v_k(t)$ is given by the inequalities,
$$t_i \neq t_j, \quad t_i \neq 0, \quad 1 \leq i \neq j \leq k.$$  
The considered projection of $v_k(t)$ has the form $\omega_{k,0}(t)v_1 \otimes f_1^k v_0,$ where
$$\omega_{k,0}(t) = \text{Sym}_k \left[ \frac{1}{(t_1 - t_2) \ldots (t_{k-1} - t_k)t_k} \right].$$  
The identity (III) with $s = 0$ gives
$$\omega_{k,0}(t) = \frac{(-1)^k T(0)}{t_1 t_2 \ldots t_k},$$  
and this fraction never vanishes. □

4.2. Generalization to arbitrary $n$. Theorem 2 has the following generalization to the universal weight function $v(t)$ corresponding to the weight
$$\Lambda(k_1, \ldots, k_m) = \sum_{i=1}^n \Lambda(i) - \sum_{i=1}^m k_i \alpha_i$$  
in the tensor product
$$L = L_{\Lambda(1)} \otimes \cdots \otimes L_{\Lambda(n)}$$  
of $n$ highest weight $sl_{N+1}$-representations marked by distinct complex numbers $z_1, \ldots, z_n$.

Theorem 3. Assume that $m \leq \min(n, N)$ and
$$k_i \leq (\Lambda(i), \alpha_i), \quad i = 1, \ldots, m.$$  
Then the universal weight function $v(t)$ corresponding to the weight $\Lambda(k_1, \ldots, k_m)$ does not vanish.

Proof. Fix highest weight vectors $v_i \in L_{\Lambda(i)}, \quad i = 1, \ldots, n$. According to our assumptions, we have
$$f_i^k v_i \neq 0, \quad i = 1, \ldots, m.$$  
Consider the projection of $v(t)$ to the one-dimensional subspace of $L$ spanned by
$$f_1^{k_1} v_1 \otimes \cdots \otimes f_m^{k_m} v_m \otimes v_{m+1} \otimes \cdots \otimes v_n.$$  
Applying the identity (III), one gets that the corresponding coefficient is equal to
$$\frac{(-1)^{k_1+\cdots+k_m}}{T_1(z_1) \ldots T_m(z_m)},$$  
where polynomials $T_i(x)$ are as in Section 3.3 and hence does not vanish. □

In particular, if the Bethe vector of the weight $\Lambda(k_1, \ldots, k_m)$ exists, then it is a non-zero vector.

Returning to $n = 2$, in the case when one of two modules is a symmetric power of the standard representation, we arrive at the following result.
Corollary 2. If $L_{\Lambda(0)} = m\lambda_1$ and $\Lambda(k_1, k_2) = \Lambda(1) + \Lambda(0) - k_1\alpha_1 - k_2\alpha_2$ is the highest weight of an irreducible component of $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$, then the universal weight function $v(t)$ corresponding to the weight $\Lambda(k_1, k_2)$ does not vanish. In particular, if $L_{\Lambda(1)}$ and $L_{\Lambda(0)}$ are $sl_3$-representations, then all Bethe vectors in $L$ do not vanish.

Proof. Elementary considerations with the Pieri formula [FuH, Proposition 15.25] show that the conditions
\[ k_1 \leq (\Lambda(0), \alpha_1), \quad k_2 \leq (\Lambda(1), \alpha_2) \]
are always fulfilled. In the $sl_3$ case all highest weights are clearly of the form $\Lambda(k_1, k_2)$. □

4.3. Example $\Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3$. We assume that $\Lambda(0) = m\lambda_1$, $N \geq 3$, and $k \geq 1$ is an integer such that
\[ \Lambda(k, 1, 1) := \Lambda(1) + \Lambda(0) - k\alpha_1 - \alpha_2 - \alpha_3 \]
is the highest weight of an irreducible component of $L = L_{\Lambda(1)} \otimes L_{\Lambda(0)}$. As before, the Pieri formula [FuH, Proposition 15.25] implies that $f_2f_1^kv_0 \neq 0$ and $f_3v_1 \neq 0$ for any fixed highest weight vectors $v_0 \in L_{\Lambda(0)}$ and $v_1 \in L_{\Lambda(1)}$.

The module $L_{\Lambda(0)}$ is the $m$-th symmetric power of the standard $sl_{N+1}$-representation. Take $v_0 = \epsilon_1^m$, where $\{\epsilon_i\}$ is a basis in the standard representation,
\[ f_if_i = \epsilon_{i+1}, \quad f_i\epsilon_j = 0, \quad i \neq j. \]
The subspace of weight $\Lambda(0) - k\alpha_1 - \alpha_2$ in $L_{\Lambda(0)}$ is one-dimensional and generated by the vector $\epsilon_1^{m-k}\epsilon_2^{k-1}\epsilon_3$.

There are three sets of auxiliary variables. We write
\[ t(1) = (t_1, \ldots, t_k), \quad t(2) = s, \quad t(3) = r, \quad t = (t, s, r) = (t_1, \ldots, t_k, s, r), \quad T(x) = \prod_{i=1}^k (x - t_i). \]
The word $F_0$ can be written as $F_0 = f_1^{i-1}f_2f_1^{k+1-i}$ for $i = 1, \ldots, k + 1$. Notice that $f_1^if_2v_0 = 0$ for our choice of $\Lambda(0)$, therefore we assume that $i$ varies from 1 to $k$ and set
\[ \omega_i(t, s, r) := \omega_{(f_1, f_1^{i-1}f_2f_1^{k+1-i})}(t), \quad i = 1, \ldots, k. \]

Theorem 4. If $T'(s) \neq 0$, then the projection of the vector $v_{k,1,1}(t, s, r)$ to the subspace $L_{\Lambda(k,1,1)}$ spanned by
\[ f_3v_1 \otimes (\epsilon_1^{m-k}\epsilon_2^{k-1}\epsilon_3) \]
is a non-zero vector.

Proof. Notice that the domain of the function $v_{k,1,1}(t, s, r)$ is given by the inequalities,
\[ t_i \neq t_j, \quad t_i \neq s, \quad t_i \neq r, \quad r \neq s, \quad t_i, s, r \neq 0, 1, \quad 1 \leq i \neq j \leq k. \]
The projection of vector $v_{k,1,1}(t, s, r)$ to the chosen subspace has the form
\[ \sum_{i=1}^k \omega_i(t, s, r)w_i, \]
where \( w_i = f_3 v_1 \otimes f_1^{i-1} f_2 f_1^{k+1-i} v_0 \). Here \( f_2 \) stands at the \( i \)-th place from the left, and
\[
\omega_i(t, s, r) = \frac{1}{r - 1} \text{Sym}_k \left[ \frac{1}{(t_1 - t_2) \cdots (t_{i-2} - t_{i-1}) (t_{i-1} - s) (s - t_i) (t_i - t_{i+1}) \cdots (t_{k-1} - t_k) t_k} \right].
\]

An easy calculation shows that
\[
w_i = (k + 1 - i) \cdot m(m - 1) \cdots (m + 1 - k) \cdot f_3 v_1 \otimes \epsilon_1^{m-k} \epsilon_2^{k-1} \epsilon_3,
\]
therefore the projection is
\[
\left( \sum_{i=1}^{k} (k + 1 - i) \cdot \omega_i(t, s, r) \right) \cdot m(m - 1) \cdots (m + 1 - k) \cdot f_3 v_1 \otimes \epsilon_1^{m-k} \epsilon_2^{k-1} \epsilon_3.
\]

It is convenient to use the notation introduced at the end of Section 2. The identity (IV) of Corollary gives
\[
\omega_i(t, s, r) = \frac{(-1)^{i-1} s^{k-i} \tau_{i-1}}{(r - 1) T(s) \tau_k}, \quad 1 \leq i \leq k.
\]
Therefore
\[
\sum_{i=1}^{k} (k + 1 - i) \cdot \omega_i(t, s, r) = \frac{T'(s)}{(r - 1) T(s) \tau_k},
\]
and the statement of the theorem follows.

**Corollary 3.** If the Bethe vector \( v_{k,1,1} \) exists, then it does not vanish.

**Proof.** We show that \( T'(s) \) can not vanish at a solution of the Bethe system. The Bethe equation corresponding to the variable \( s \) is as follows,
\[
\frac{1}{s - r} + \frac{T'(s)}{T(s)} + \frac{(\Lambda(1), \alpha_2)}{s - 1} = 0,
\]
whereas the one corresponding to \( r \) has the form
\[
\frac{1}{r - s} + \frac{(\Lambda(1), \alpha_3)}{r - 1} = 0,
\]
see Section 3.3. Denote \( (\Lambda(1), \alpha_2) = A \) and \( (\Lambda(1), \alpha_3) = B. \) Assuming \( T'(s) = 0 \) one gets the following linear system with respect to \( s \) and \( r, \)
\[
-Ar + (A + 1)s = 1, \quad (B + 1)r - Bs = 1.
\]
The solution to this system is \( r = s = 1 \) and contradicts to the conditions \( r \neq s \neq 1. \)
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