TWO-LOOP GAP EQUATIONS
FOR THE MAGNETIC MASS

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Abstract

One-loop gap equations have recently been used by several authors to estimate the non-perturbative mass gap in a 3-dimensional gauge theory. I extend the method to two loops and demonstrate, that the resulting gap equation has a real and positive solution \( m \simeq 0.34g^2 \), which is in good agreement with the one-loop results and lattice data.

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In the high-temperature phase of the Standard Model one naively expects a vanishing Higgs vacuum expectation value and vector boson mass. This leads to the well-known breakdown of the perturbative expansion due to severe infrared divergences in the magnetic sector of the theory. The infrared problem may be cured by a non-vanishing magnetic mass, which acts as a cutoff regularizing these divergences. Its inverse, the magnetic screening length, determines the size of non-perturbative effects in the symmetric phase and is closely related to the confinement scale of the effective 3-dimensional theory which describes the high-temperature limit of the 4-dimensional finite temperature field theory. In an apparently massless 3-dimensional Yang-Mills theory the gauge coupling $g^2$ carries the dimension of mass, thereby providing a natural mass scale. A popular framework for calculating the dynamically generated mass in a 3-dimensional $SU(2)$ gauge theory has recently been via gap equations.

Up to now, several attempts have been made to evaluate the size of this mass gap, all of them at the one-loop level. In order to check, whether the whole approach is consistent, it is crucial to extend the method to two loops. To my knowledge this letter contains the first treatment of gap equations in next-to-leading order. The corresponding calculation is rather involved at the technical level.

After briefly presenting the idea behind the gap equation approach and giving an overview of the one-loop results, I will give reasons for the necessity of a two-loop calculation, the results of which will be extensively discussed.

**General Strategy**

All the models considered so far start form a massless Yang-Mills action $S_G$ in 3-dimensional Euclidean space and then add and subtract some gauge-invariant mass term $S_m$, where the subtracted term enters the perturbation theory at one loop higher than the added term. This can be formalized by introducing a loop-counting parameter $l$ in the following manner: one rescales all the fields by $\sqrt{l}$ and calculates with the modified action

$$S_{\text{eff}} = \frac{1}{l} \left( S_G(\sqrt{l}W) + S_m(\sqrt{l}W) \right) - S_m(\sqrt{l}W)$$

in a formal $l$-expansion. Perturbative calculations are no longer done to a fixed order of the gauge coupling $g$, but as a power series in $l$, resulting in a rearranged or resummed perturbation series. Out of the many choices, which are possible for $S_m$, I will concentrate on the non-linear $\sigma$-model for reasons specified below.

The gap equation is a self-consistent condition for the vector boson mass. The goal is to find the particular size of the tree-level mass-term $m = Cg^2$ leading to a convergent perturbation series. In other words, the pole of the transverse part of the (Euclidean) vector boson propagator should remain at $p^2 = -m^2$ to any loop-order, i.e.
\[ D_T(p^2) = \frac{1}{p^2 + m^2 - \Pi_T(p^2)} \]
\[ \sim \frac{Z}{p^2 + m^2} \quad \text{for} \quad p^2 \sim -m^2, \quad (2) \]

with some residue \( Z \). With eq. (2) one obtains the desired gap equation for the self-energy in resummed perturbation theory

\[ \Pi_T(p^2 = -m^2) \left( 1 + \frac{\partial \Pi_T}{\partial p^2}(p^2 = -m^2) \right) = 0. \quad (3) \]

In \( n \)-th order of resummed perturbation theory one calculates eq. (3) up to \( l^n \) and solves the gap equation for \( m \).

At one-loop eq. (3) reduces to

\[ \Pi_T^{1\text{-loop}}(p^2 = -m^2) = 0. \quad (4) \]

In theories with a BRS-symmetry the position of the pole of the propagator and therefore eq. (3) is gauge-independent on mass-shell \([7]\). The self-energy itself is not gauge-invariant on mass-shell except at one-loop level.

**One-Loop Calculations**

In recent years, several authors have proposed models to extract a gap mass at one-loop level. When studying the electroweak phase transition Buchmüller and Philipsen could obtain a non-vanishing vector boson mass in the symmetric phase of a linear 3-dimensional \( SU(2) \)-Higgs model \([3]\). A mass resummation was supplemented by a vertex resummation in order to get a BRS-invariant resummed tree-level action resulting in a gauge-independent gap equation. Deeply in the symmetric phase the value for the gap mass is approximately the same as the one obtained in a non-linear \( \sigma \)-model. This suggests to investigate first the two-loop effects in this simpler model.

In order to minimize the amount of diagrams in a two-loop calculation I follow a suggestion by Jackiw and Pi \([9]\). The functional integral for the partition function in the non-linear \( \sigma \)-model, where \( S_m = S_\sigma \), reads

\[ Z = \int DW D\pi \Delta \exp - \frac{1}{l} \left( S_G + S_\sigma + S_{GF} - lS_\sigma \right), \quad (5) \]

with \( S_{GF} \) being some gauge-fixing term, which depends only on \( W_\mu \), and \( \Delta \) the corresponding Fadeev-Popov determinant. They integrated out the Goldstone and ghost fields exactly (in an arbitrary gauge) and arrived at a massive Yang-Mills theory without any additional gauge fixing terms,

\[ Z \propto \int DW \exp \left[ -\frac{1}{l} \left( S_G(\sqrt{I}W) + m^2 \text{Tr} \int d^3x \sqrt{I}W_\mu \sqrt{I}W_\mu - l m^2 \text{Tr} \int d^3x \sqrt{I}W_\mu \sqrt{I}W_\mu \right) \right]. \quad (6) \]
One can also view this massive Yang-Mills theory as a non-linear $\sigma$-model (with $R_\xi$-gauge fixing) in unitary gauge. A calculation of the self-energy to one-loop in both theories indeed yields the same result on mass-shell and therefore the same gap mass. The off-shell self-energies coincide only in the limit $\xi \to \infty$. In unitary gauge the result for the off-shell transverse self-energy is

$$\Pi_{T}^{1-\text{loop}}(p^2) = -\frac{1}{16\pi} \log^2 m \left[ \left( \frac{p^6}{4m^6} - \frac{2p^4}{m^4} - \frac{10p^2}{m^2} + 8 \right) \frac{2m}{p} \arctan \frac{p}{2m} + \frac{p^4}{m^4} - \frac{4p^2}{m^2} - 8 \right]. \quad (7)$$

The gap equation for the massive Yang-Mills theory to one-loop is then a linear equation for $m$ and reads

$$lm_{SM}^2 - \frac{1}{16\pi} \left( \frac{63}{4} \ln 3 - 3 \right) \log^2 m_{SM} = 0, \quad (8)$$

or

$$m_{SM} \simeq 0.28 g^2. \quad (9)$$

Note that in unitary gauge the longitudinal part of the self-energy vanishes for all external momenta, $\Pi_L(p^2) = 0$.

For the 3-dimensional $SU(2)$ gauge theory also other gap equations have been considered, which are based on the Chern-Simons eikonal and on the non-local action

$$S_m^{(JP)} = m^2 \text{Tr} \int d^3 x F_\mu \frac{1}{D^2} F_\mu, \quad (10)$$

where $F_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F_{\nu\rho}$. Interestingly, the one-loop gap equation of Alexanian and Nair yields a magnetic mass closely related to $m_{SM}$,

$$m_{AN} = \frac{4}{3} m_{SM} \simeq 0.38 g^2. \quad (11)$$

Jackiw and Pi obtain a complex magnetic mass with the mass term of eq. (10), which, however, can be modified such that the generated mass gap becomes real. Another attempt was recently made by Cornwall. His pinch-technique gap equation led to a mass gap of

$$m_C \simeq 0.25 g^2. \quad (12)$$

It is also very encouraging, that these analytically calculated gap masses are consistent with the propagator mass obtained in a numerical lattice simulation in Landau gauge, $m_{SM}^L = 0.35(1) g^2$. \quad (13)

**Two-Loop Gap Equation**

Why is it crucial to perform a two-loop calculation?

- The loop expansion does not correspond to an expansion in a small parameter. Nevertheless, it might very well be, that the one-loop results provide reasonable approximations of the true mass gap. This can only be clarified by a two-loop calculation.
• If the whole method is consistent, the numerical values for the mass gap in the different models should converge at higher loop-orders, since to all orders they describe the same Yang-Mills theory. In this case one expects the two-loop correction to be of order $m_{AN} - m_{SM}$.

• The two-loop gap equation is quadratic in $m$, whereas at one loop it is linear. The existence of a positive solution is a non-trivial check of the whole approach.

As mentioned above, we first consider the Lagrangian of a massive YM (a resummed non-linear $\sigma$-model in unitary gauge)

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} m^2 W_{\mu}^a W_{\mu}^a - \frac{l}{2} m^2 W_{\mu}^a W_{\mu}^a ,$$ \hspace{1cm} (14)

with

$$F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \sqrt{\text{lg} \epsilon} \epsilon^{a\nu \lambda} W_{\mu}^b W_{\lambda}^c, \ G = SU(2), \ d = 3 - 2\epsilon .$$ \hspace{1cm} (15)

Eq. (3) has to be expanded up to $O(l^2)$, which requires the evaluation of the diagrams depicted in fig. 1. The contribution of diagrams 1, 2 and 3 to the transverse on-shell self-energy was already calculated at one loop, cf. eq. (8). The contributions of diagrams 4 and 5 are easily evaluated,

$$\Pi_{T}^{1\text{-loop} - \text{CT}} (p^2 = -m^2) = \frac{1}{8 \pi} \left( \frac{21}{4} \ln 3 - 9 \right) l^2 g^2 m .$$ \hspace{1cm} (16)

For $\frac{\partial}{\partial p^2} \Pi_T$, which contributes to $O(l^2 g^4)$, only one-loop diagrams are needed in the two-loop gap equation, which can directly be obtained from eq. (7),

$$\frac{\partial}{\partial p^2} \Pi_T^{1\text{-loop}} (p^2 = -m^2) = \frac{1}{32 \pi} \left( 33 - \frac{21}{4} \ln 3 \right) l^2 g^2 m .$$ \hspace{1cm} (17)

Far more work has to be done for the evaluation of the remaining 9 two-loop diagrams, which contribute to $O(l^2 g^4)$. As all propagators are massive and the external momentum does not vanish, the reduction of the scalar integrals to basic integrals with no momenta in the numerators turns out to be the most difficult step in the calculation. For propagator type integrals this task has been achieved only recently by Tarasov [10]. Using his recurrence relations it is possible to reduce the self-energy integrals to a small set of linearly independent basic integrals. For the first time this method achieves a complete reduction and stays on an algebraic level as far as possible. Since the recurrence relations are in some cases quite involved, they have to be implemented into a FORM package [11]. In unitary gauge the situation is even more complex due to the high powers of momenta in the numerator.

In $3 - 2\epsilon$ dimensions, the result of the reduction to basic integrals for the on-shell self-energy in unitary gauge reads

$$\frac{1}{l^2 g^4} \Pi_T^{2\text{-loop}} (p^2 = -m^2) = -\frac{3}{128} m^2 \frac{\circleddash}{\circleddash} - \frac{1329}{64} \frac{\circleddash}{\circleddash} m^2$$
Except for \( -\frac{1}{2} \), which has to be evaluated numerically, there exist analytic expressions for the basic integrals in \( 3 - 2\epsilon \) dimensions [12]. The result of (18) was also obtained using a FORM package written independently by O. Tarasov. Neglecting the resummation counter-terms, which spoil BRS-invariance, I also calculated the self-energy for the non-linear \( \sigma \)-model in Feynman gauge, \( \xi = 1 \). I obtained the same position for the pole of the propagator as in unitary gauge, which constitutes a very stringent test for the algorithm I used.

Two further remarks have to be made concerning the two-loop calculation. First, the longitudinal part of the self-energy in unitary gauge vanishes for all momenta at two-loop level, which is another nice check of the calculation. Second, the non-linear sigma model is non-renormalizable. This is not a problem at one loop, since the 3-dimensional self energy is finite in dimensional regularization. At two loops, however, \( \frac{1}{d} \) and \( \frac{1}{n} \) are UV-divergent, which requires the addition of counter-terms (\( \sim l^2 \)) to the Lagrangian. The explicit calculation in Feynman gauge shows that a mass and wave function renormalization is sufficient to remove the infinities in the self-energy,

\[
\frac{1}{l^2 g^4} \Pi^{2-\text{loop}}_\xi=1 (p^2) = \left( \frac{7}{12} - \frac{1}{60 m^2} \right) \frac{1}{64 \pi^2 \epsilon} + \text{finite}. \tag{19}
\]

Compared to the unitary gauge, where ghosts and Goldstone bosons are integrated out, the Feynman gauge involves the evaluation of many more diagrams, 33 generic two-loop graphs instead of 9. Note also, that the unitary gauge is not suitable for renormalization. Even in renormalizable theories the bad high-energy behavior of the propagator leads to terms \( \left( \frac{m^2}{p^2} \right)^n \frac{1}{\epsilon}, n > 1 \), in the self-energy, which cannot be dealt with by a mass or wave function renormalization. Renormalization in Feynman gauge introduces a renormalization scale \( \mu \). Using the \( \overline{\text{MS}} \)-scheme it turns out that there is almost no numerical dependence of the two-loop gap mass on the scale \( \mu = \mu_{\overline{\text{MS}}} \).

Now we are ready to discuss the two-loop gap equation. In an arbitrary gauge of the resummed non-linear \( \sigma \)-model it reads

\[
l m^2 - 0.28455 l g^2 m + f_1(\xi) l^2 g^2 m + f_2(\xi) l^2 g^2 m
- 0.064346 l^2 g^4 + 0.0037995 l^2 g^4 \ln \frac{\mu}{m} = 0, \tag{20}
\]

with

\[
f_1(\xi) = \frac{1}{8\pi} \left( \frac{21}{4} \ln 3 - 9 + \frac{1}{4\sqrt{\xi}} \ln 3 + \sqrt{\xi}(3 - \ln 3) \right),
\]
\[ f_2(\xi) = \frac{1}{8\pi} \left( \frac{33}{4} - \frac{21}{16} \ln 3 + (\xi - \frac{1}{4}) \ln \frac{2\sqrt{\xi} + 1}{2\sqrt{\xi} - 1} - 3\sqrt{\xi} \right), \quad \xi > \frac{1}{4}. \] (21)

The two-loop gap equation is not exactly gauge parameter independent. There remains a weak gauge dependence stemming from diagrams which involve the resummation counter-terms (mass counter-terms for the vector, ghost and Goldstone field). In the unitary gauge \( f_1 \) and \( f_2 \) reduce to eq. (16) and (17). Note that the limit \( \xi \to \infty \) has to be taken before divergent integrals are evaluated [13]. The gap equations in unitary and Feynman gauge turn out to be identical.

Two-Loop Results

The major result is that the gap equation (20) has indeed a real and positive solution for \( \frac{m}{g^2} \). The results are given in table 1 for different values of \( \mu \) and \( \xi \). The two-loop correction to the one-loop gap mass (9) is only 15–20%.

| \( \frac{m}{g^2} \) | \( \xi = 1, \infty \) | \( \xi = 2 \) | \( \xi = 10 \) |
|---|---|---|---|
| 0.3 | 0.343 | 0.345 | 0.350 |
| 1 | 0.335 | 0.336 | 0.342 |
| 3 | 0.327 | 0.328 | 0.334 |

Table 1: Solutions of the two-loop gap equation

One may worry about the dependence of the gap mass on the renormalization scale \( \mu \) and on the gauge parameter \( \xi \). This is an artefact of (resummed) perturbation theory, which is expected to be cancelled at higher orders. Fortunately, the dependence of \( \frac{m}{g^2} \) on \( \mu \) and \( \xi \) is numerically unimportant. This suggests that the solution constitutes a reliable approximation to the exact gluon propagator mass in \( SU(2) \) gauge theory.

A vector boson mass \(~ 0.34g^2\) is not in contradiction with confinement. It is of the same size as the confinement scale given by the string tension which was calculated in [14]. The connection of such a propagator mass to the heavier glueball masses \(~ \mathcal{O}(1)g^2\) [15] in a 3-dimensional \( SU(2) \) gauge theory remains to be clarified.

The two-loop result survives all the crucial tests which have been mentioned above unexpectedly well: the quadratic gap equation has a real and positive solution, which is not far away from the one-loop result. Moreover, it is now in better agreement with Alexanian and Nair’s gap mass and matches perfectly the lattice result obtained by Karsch et. al.

To judge the significance of a calculation in the non-renormalizable non-linear sigma model, it is crucial to perform the whole calculation in the linear Higgs model, which is super-renormalizable. The linear model involves the summation of hundreds of two-loop diagrams.

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have shown that the pole of the Higgs and of the vector boson propagator is gauge-invariant to two loops, which is a very powerful test of Tarasov’s algorithm and my FORM package. More importantly, the two-loop gap equation in the linear Higgs model turned out to be numerically nearly independent of the Higgs mass. Therefore, the non-linear sigma model remains a very good approximation, as it already proved to be the case at one loop. Detailed results of this work will be published in a forthcoming paper \cite{16}.

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Figure 1: Diagrams contributing to the two-loop gap equation in unitary gauge