MACROSCOPIC DIMENSION OF THE $\ell^p$-BALL WITH RESPECT TO THE $\ell^q$-NORM

MASAKI TSUKAMOTO

ABSTRACT. We show estimates of the “macroscopic dimension” of the $\ell^p$-ball with respect to the $\ell^q$-norm.

1. Introduction

1.1. Macroscopic dimension. Let $(X,d)$ be a compact metric space, $Y$ a topological space. For $\varepsilon > 0$, a continuous map $f : X \to Y$ is called an $\varepsilon$-embedding if $\text{Diam} f^{-1}(y) \leq \varepsilon$ for all $y \in Y$. Following Gromov [2, p. 321], we define the “width dimension” $\text{Widim}_{\varepsilon} X$ as the minimum integer $n$ such that there exist an $n$-dimensional polyhedron $P$ and an $\varepsilon$-embedding $f : X \to P$. When we need to make the used distance $d$ explicit, we use the notation $\text{Widim}_{\varepsilon}(X,d)$. If we let $\varepsilon \to 0$, then $\text{Widim}_{\varepsilon}$ gives the usual covering dimension:

$$\lim_{\varepsilon \to 0} \text{Widim}_{\varepsilon} X = \text{dim} X.$$ 

$\text{Widim}_{\varepsilon} X$ is a “macroscopic” dimension of $X$ at the scale $\geq \varepsilon$ (cf. Gromov [2, p. 341]). It discards the information of $X$ “smaller than $\varepsilon$”. For example, $[0,1] \times [0,\varepsilon]$ (with the Euclidean distance) macroscopically looks one-dimensional ($\varepsilon < 1$):

$$\text{Widim}_{\varepsilon}[0,1] \times [0,\varepsilon] = 1.$$ 

Using this notion, Gromov [2] defines “mean dimension”. And he proposed open problems about this $\text{Widim}_{\varepsilon}$ (see [2 pp. 333-334]). In this paper we give (partial) answers to some of them.

In [2, p. 333], he asks whether the simplex $\Delta^{n-1} := \{x \in \mathbb{R}^n | x_k \geq 0 (1 \leq k \leq n), \sum_{k=1}^{n} x_k = 1\}$ satisfies

$$(1) \quad \text{Widim}_{\varepsilon} \Delta^{n-1} \sim \text{const}_{\varepsilon} n.$$ 

Our main result below gives the answer: If we consider the standard Euclidean distance on $\Delta^{n-1}$, then (1) does not hold.

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In [2, p. 333], he also asks what is \( \text{Widim}_\varepsilon B_{\ell^p}(\mathbb{R}^n) \) with respect to the \( \ell^q \)-norm, where (for \( 1 \leq p \))
\[
B_{\ell^p}(\mathbb{R}^n) := \left\{ x \in \mathbb{R}^n \mid \sum_{k=1}^{n} |x_k|^p \leq 1 \right\}.
\]
Our main result concerns this problem. For \( 1 \leq q \leq \infty \), let \( d_{\ell^q} \) be the \( \ell^q \)-distance on \( \mathbb{R}^n \) given by
\[
d_{\ell^q}(x, y) := \left( \sum_{k=1}^{n} |x_k - y_k|^q \right)^{1/q}.
\]
We want to know the value of \( \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \). Especially we are interested in the behavior of \( \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \) as \( n \to \infty \) for small (but fixed) \( \varepsilon \). When \( q = p \), we have (from “Widim inequality” in [2, p. 333])
\[
(2) \quad \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^p}) = n \quad \text{for all } \varepsilon < 1.
\]
(For its proof, see Gromov [2, p. 333], Gournay [1, Lemma 2.5] or Tsukamoto [7, Appendix A].) More generally, if \( 1 \leq q \leq p \leq \infty \), then \( d_{\ell^p} \leq d_{\ell^q} \) and hence
\[
(3) \quad \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) = n \quad \text{for all } \varepsilon < 1.
\]
I think this is a satisfactory answer. (For the case of \( \varepsilon \geq 1 \), there are still problems; see Gournay [1].) So the problem is the case of \( 1 \leq p < q \leq \infty \). Our main result is the following:

**Theorem 1.1.** Let \( 1 \leq p < q \leq \infty \) (\( q \) may be \( \infty \)). We define \( r (\geq p) \) by \( 1/p - 1/q = 1/r \). For any \( \varepsilon > 0 \) and \( n \geq 1 \), we have
\[
(4) \quad \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \leq \min(n, \lceil (2/\varepsilon)^r \rceil - 1),
\]
where \( \lceil (2/\varepsilon)^r \rceil \) denotes the smallest integer \( \geq (2/\varepsilon)^r \). Note that the right-hand-side of (4) becomes constant for large \( n \) (and fixed \( \varepsilon \)). Therefore \( \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \) becomes stable as \( n \to \infty \).

This result makes a sharp contrast with the above (3). For the simplex \( \Delta^{n-1} \subset \mathbb{R}^n \) we have
\[
\text{Widim}_\varepsilon \Delta^{n-1} \leq \text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}) \leq \lceil (2/\varepsilon)^2 \rceil - 1.
\]
Therefore (1) does not hold. Actually this result means that the “macroscopic dimension” of \( \Delta^{n-1} \) becomes constant for large \( n \).

When \( q = \infty \), we can prove that the inequality (4) actually becomes an equality:

**Corollary 1.2.** For \( 1 \leq p < \infty \),
\[
\text{Widim}_\varepsilon (B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) = \min(n, \lceil (2/\varepsilon)^p \rceil - 1).
\]
This result was already obtained by A. Gournay [1, Proposition 1.3]; see Remark 1.6 at the end of the introduction. For general $q > p$, I don’t have an exact formula. But we can prove the following asymptotic result as a corollary of Theorem 1.1.

**Corollary 1.3.** For $1 \leq p < q \leq \infty$,

\[
\lim_{\varepsilon \to 0} \left( \lim_{n \to \infty} \log \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})/|\log \varepsilon| \right) = r = \frac{pq}{q-p}.
\]

Note that the limit $\lim_{n \to \infty} \log \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ exists because $\text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q})$ is monotone non-decreasing in $n$ and has an upper bound independent of $n$.

**Remark 1.4.** Gournay [1, Example 3.1] shows $\text{Widim}_\varepsilon(B_{\ell^1}(\mathbb{R}^2), d_{\ell^p}) = 2$ for $\varepsilon < 2^{1/p}$.

1.2. **Mean dimension theory.** Theorem 1.1 has an application to Gromov’s mean dimension theory. Let $\Gamma$ be an infinite countable group. For $1 \leq p \leq \infty$, let $B_{\ell^p}(\Gamma) \subset \ell^p(\Gamma)$ be the $\ell^p$-space, $B_{\ell^p}(\Gamma)$ the unit ball (in the $\ell^p$-norm). We consider the natural right action of $\Gamma$ on $\ell^p(\Gamma)$ (and $B_{\ell^p}(\Gamma)$):

\[(x \cdot \delta)_\gamma := x_{\delta \gamma} \text{ for } x = (x_\gamma)_{\gamma \in \Gamma} \in \ell^p(\Gamma) \text{ and } \delta \in \Gamma.\]

We give the standard product topology on $\mathbb{R}^\Gamma$, and consider the restriction of this topology to $B_{\ell^p}(\Gamma) \subset \mathbb{R}^\Gamma$. (This topology coincides with the restriction of weak topology of $\ell^p(\Gamma)$ for $p > 1$.) Then $B_{\ell^p}(\Gamma)$ becomes compact and metrizable. (The $\Gamma$-action on $B_{\ell^p}(\Gamma)$ is continuous.) Let $d$ be the distance on $B_{\ell^p}(\Gamma)$ compatible with the topology. For a finite subset $\Omega \subset \Gamma$ we define a distance $d_\Omega$ on $B_{\ell^p}(\Gamma)$ by

\[d_\Omega(x,y) := \max_{\gamma \in \Omega} d(x_\gamma,y_\gamma).\]

We are interested in the growth behavior of $\text{Widim}_\varepsilon(B_{\ell^p}(\Gamma), d_\Omega)$ as $|\Omega| \to \infty$. In particular, if $\Gamma$ is finitely generated and has an amenable sequence $\{\Omega_i\}_{i \geq 1}$ (in the sense of [2, p. 335]), the mean dimension is defined by (see [2, pp. 335-336])

\[\dim(B_{\ell^p}(\Gamma) : \Gamma) = \lim_{\varepsilon \to 0 i \to \infty} \text{Widim}_\varepsilon(B_{\ell^p}(\Gamma), d_{\Omega_i})/|\Omega_i|.\]

As a corollary of Theorem 1.1 we get the following:

**Corollary 1.5.** For $1 \leq p < \infty$ and any $\varepsilon > 0$, there is a positive constant $C(d,p,\varepsilon) < \infty$ (independent of $\Omega$) such that

(5) \[\text{Widim}_\varepsilon(B_{\ell^p}(\Gamma), d_\Omega) \leq C(d,p,\varepsilon) \text{ for all finite set } \Omega \subset \Gamma.\]

Namely, $\text{Widim}_\varepsilon(B_{\ell^p}(\Gamma), d_\Omega)$ becomes stable for large $\Omega \subset \Gamma$. In particular, for a finitely generated infinite amenable group $\Gamma$

(6) \[\dim(B_{\ell^p}(\Gamma) : \Gamma) = 0.\]

(6) is the answer to the question of Gromov in [2, p. 340]. Actually the above (5) is much stronger than (6).
Remark 1.6. This paper is a revised version of the preprint [5]. A referee of [5] pointed out that the above (6) can be derived from the theorem of Lindenstrauss-Weiss [4, Theorem 4.2]. This theorem tells us that if the topological entropy is finite then the mean dimension becomes 0. We can see that the topological entropy of $B(\ell^p(\Gamma))$ (under the $\Gamma$-action) is 0. Hence the mean dimension also becomes 0. I am most grateful to the referee of [5] for pointing out this argument. The essential part of the proof of Theorem 1.1 (and Corollary 1.2 and Corollary 1.3) is the construction of the continuous map $f: \mathbb{R}^n \to \mathbb{R}^n$ described in Section 3. This construction was already given in the preprint [5]. When I was writing this revised version of [5], I found the paper of A. Gournay [1]. [1] proves Corollary 1.2 ([1, Proposition 1.3]) by using essentially the same continuous map as mentioned above. I submitted the paper [5] to a certain journal in June of 2007 before [1] appeared on the arXiv in November of 2007. And [5] is quoted as one of the references in [1].

2. PRELIMINARIES

Lemma 2.1. For $s \geq 1$ and $x, y, z \geq 0$, if $x \geq y$, then

$$x^s + (y + z)^s \leq (x + z)^s + y^s.$$ 

Proof. Set $\varphi(t) := (t + z)^s - t^s (t \geq 0)$. Then $\varphi'(t) = s((t + z)^{s-1} - t^{s-1}) \geq 0$. Hence $\varphi(y) \leq \varphi(x)$, i.e., $(y + z)^s - y^s \leq (x + z)^s - x^s$. □

Lemma 2.2. Let $s \geq 1$ and $c, t \geq 0$. If real numbers $x_1, \cdots, x_n$ ($n \geq 1$) satisfies

$$x_1 + \cdots + x_n \leq c, \quad 0 \leq x_i \leq t \ (1 \leq i \leq n),$$

then

$$x_1^s + \cdots + x_n^s \leq c \cdot t^{s-1}.$$ 

Proof. First we suppose $nt \leq c$. Then $x_1^s + \cdots + x_n^s \leq nt^s \leq c \cdot t^{s-1}$.

Next we suppose $nt > c$. Let $m := \lfloor c/t \rfloor$ be the maximum integer satisfying $mt \leq c$. We have $0 \leq m < n$ and $c - mt < t$. Using Lemma 2.1 we have

$$x_1^s + \cdots + x_n^s \leq t^s + \cdots + t^s \underbrace{(c - mt)^s}_{m} \leq mt^s + t^{s-1}(c - mt) \leq c \cdot t^{s-1}. \quad □$$

3. PROOF OF THEOREM 1.1

Let $S_n$ be the $n$-th symmetric group. We define the group $G$ by

$$G := \{\pm 1\}^n \rtimes S_n.$$ 

The multiplication in $G$ is given by

$$((\varepsilon_1, \cdots, \varepsilon_n), \sigma) \cdot ((\varepsilon'_1, \cdots, \varepsilon'_n), \sigma') := ((\varepsilon_1\varepsilon_{\sigma^{-1}(1)}, \cdots, \varepsilon_n\varepsilon_{\sigma^{-1}(n)}), \sigma\sigma').$$
where $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon'_1, \ldots, \varepsilon'_n \in \{\pm 1\}$ and $\sigma, \sigma' \in S_n$. $G$ acts on $\mathbb{R}^n$ by

$$(\varepsilon_1, \ldots, \varepsilon_n, \sigma) \cdot (x_1, \ldots, x_n) := (\varepsilon_1 x_{\sigma^{-1}(1)}, \ldots, \varepsilon_n x_{\sigma^{-1}(n)})$$

where $((\varepsilon_1, \ldots, \varepsilon_n, \sigma)) \in G$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The action of $G$ on $\mathbb{R}^n$ preserves the $l^p$-ball $B_{l^p}(\mathbb{R}^n)$ and the $l^p$-distance $d_{l^p}(\cdot, \cdot)$.

We define $\mathbb{R}^n_{\geq 0}$ and $A_n$ by

$$\mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \ (1 \leq i \leq n)\},$$

$$A_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

The following can be easily checked:

**Lemma 3.1.** For $\varepsilon \in \{\pm 1\}^n$ and $x \in \mathbb{R}^n_{\geq 0}$, if $\varepsilon x \in \mathbb{R}^n_{\geq 0}$, then $\varepsilon x = x$. For $\sigma \in S_n$ and $x \in A_n$, if $\sigma x \in A_n$, then $\sigma x = x$. For $g = (\varepsilon, \sigma) \in G$ and $x \in A_n$, if $gx \in A_n$, then $gx = \varepsilon (\sigma x) = \sigma x = x$.

Let $m, n$ be positive integers such that $1 \leq m < n$. We define the continuous map $f_0 : A_n \to A_n$ by

$$f_0(x_1, \cdots, x_n) := (x_1 - x_{m+1}, x_2 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, 0, \cdots, 0).$$

The following is the key fact for our construction:

**Lemma 3.2.** For $g \in G$ and $x \in A_n$, if $gx \in A_n$ ($\Rightarrow gx = x$), then we have

$$f_0(gx) = gf_0(x).$$

**Proof.** First we consider the case of $g = \varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \{\pm 1\}^n$. If $x_{m+1} = 0$, then

$$f_0(\varepsilon x) = (\varepsilon_1 x_1, \cdots, \varepsilon_m x_m, 0, \cdots, 0) = \varepsilon f_0(x).$$

If $x_{m+1} > 0$, then $\varepsilon_i = 1$ ($1 \leq i \leq m+1$) because $\varepsilon_i x_i = x_i \geq x_{m+1} > 0$ ($1 \leq i \leq m+1$). Hence

$$f_0(\varepsilon x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0) = f_0(x) = \varepsilon f_0(x).$$

Next we consider the case of $g = \sigma \in S_n$. $gx \in A_n$ implies $x_{\sigma(i)} = x_i$ ($1 \leq i \leq n$). Set $y := f_0(x)$. Let $r$ ($1 \leq r \leq m+1$) be the integer such that

$$x_{r-1} > x_r = x_{r+1} = \cdots = x_{m+1}.$$  

From $x_{\sigma(i)} = x_i$ ($1 \leq i \leq n$), we have

$$1 \leq i < r \Rightarrow 1 \leq \sigma(i) < r \Rightarrow y_{\sigma(i)} = x_{\sigma(i)} - x_{m+1} = y_i,$$

$$r \leq i \Rightarrow r \leq \sigma(i) \Rightarrow y_{\sigma(i)} = 0 = y_i.$$

Hence we have $f_0(\sigma x) = f_0(x) = \sigma f_0(x)$. 




Finally we consider the case of \( g = (\varepsilon, \sigma) \in G \). Since \( gx \in \Lambda_n \), we have \( gx = \varepsilon(\sigma x) = \sigma x = x \in \Lambda_n \) (see Lemma 3.1). Hence

\[
\sigma x = f_0(x) = \sigma f_0(x) = \varepsilon f_0(\sigma x) = \varepsilon f_0(x) = g f_0(x).
\]

\( \square \)

We define a continuous map \( f : \mathbb{R}^n \to \mathbb{R}^n \) as follows; For any \( x \in \mathbb{R}^n \), there is a \( g \in G \) such that \( gx \in \Lambda_n \). Then we define

\[
f(x) := g^{-1} f_0(gx).
\]

From Lemma 3.2, this definition is well-defined. Since \( \mathbb{R}^n = \bigcup_{g \in G} g\Lambda_n \) and \( f|_{g\Lambda_n} = g f_0 g^{-1} (g \in G) \) is continuous on \( g\Lambda_n \), \( f \) is continuous on \( \mathbb{R}^n \). Moreover \( f \) is \( G \)-equivariant.

**Proposition 3.3.** Let \( 1 \leq p < q \leq \infty \). For any \( x \in B_{lp}(\mathbb{R}^n) \), we have

\[
d_{ev}(x, f(x)) \leq \left( \frac{1}{m+1} \right)^{\frac{1}{p} - \frac{1}{q}}.
\]

Note that the right-hand side does not depend on \( n \).

**Proof.** Since \( f \) is \( G \)-equivariant and \( d_{ev} \) is \( G \)-invariant, we can suppose \( x \in B_{lp}(\mathbb{R}^n) \cap \Lambda_n \), i.e. \( x = (x_1, x_2, \cdots, x_n) \) with \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \). We have

\[
f(x) = (x_1 - x_{m+1}, \cdots, x_m - x_{m+1}, 0, \cdots, 0).
\]

Hence

\[
d_{ev}(x, f(x)) = \left\| (x_{m+1}, \cdots, x_{m+1}, x_{m+2}, \cdots, x_n) \right\|_{ev}.
\]

Set \( t := x_{m+1}^p \) and \( s := q/p \). Since \( x_1^p + \cdots + x_n^p \leq 1 \) and \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \), we have \( t \leq 1/(m+1) \), \( 0 \leq x_k^p \leq t \,(m+1 \leq k \leq n) \) and \( x_{m+2}^p + \cdots + x_n^p \leq 1 - (m+1)t \). Using Lemma 2.2, we have

\[
x_{m+2}^q + \cdots + x_n^q \leq \{1 - (m+1)t\}^{t^{s-1}} = t^{s^{-1}} - (m+1)t^s.
\]

Therefore

\[
d_{ev}(x, f(x))^q = (m+1)x_{m+1}^q + x_{m+2}^q + \cdots + x_n^q \leq t^{s^{-1}} \leq (1/(m+1))^{s^{-1}}.
\]

Thus

\[
d_{ev}(x, f(x)) \leq (1/(m+1))^{1/p - 1/q}.
\]

\( \square \)

**Proof of Theorem 1.1.** Set \( m := \min(n, \lceil (2/\varepsilon)^r \rceil - 1) \). We will prove \( \text{Widim}_e(B_{lp}(\mathbb{R}^n), d_{ev}) \leq m \). If \( n = m \), then the statement is trivial. Hence we suppose \( n > m = \lceil (2/\varepsilon)^r \rceil - 1 \).

From \( m + 1 = \lceil (2/\varepsilon)^r \rceil \geq (2/\varepsilon)^r \) and \( 1/r = 1/p - 1/q \),

\[
2 \left( \frac{1}{m+1} \right)^{\frac{1}{p} - \frac{1}{q}} \leq \varepsilon.
\]
We have
\[ f(\mathbb{R}^n) = \bigcup_{g \in G} g f(\Lambda_n). \]
Note that \( f(\Lambda_n) \subset \mathbb{R}^m := \{(x_1, \cdots, x_m, 0, \cdots, 0) \in \mathbb{R}^n\} \). Proposition 3.3 implies that
\[
\left. f \right|_{B_\ell^p(\mathbb{R}^n)} : (B_\ell^p(\mathbb{R}^n), d_\ell^q) \to \bigcup_{g \in G} g \cdot \mathbb{R}^m \text{ is a } 2 \left( \frac{1}{m+1} \right)^{\frac{1}{p} - \frac{1}{q}} \text{-embedding.}
\]
Therefore we get
\[
\text{Widim}_\varepsilon(B_\ell^p(\mathbb{R}^n), d_\ell^q) \leq m.
\]
\[ \square \]

4. Proof of Corollaries 1.2 and 1.3

4.1. Proof of Corollary 1.2. We need the following result. (cf. Gromov [2, p. 332]. For its proof, see Lindenstrauss-Weiss [4, Lemma 3.2] or Tsukamoto [6, Example 4.1].)

Lemma 4.1. For \( \varepsilon < 1 \),
\[ \text{Widim}_\varepsilon([0, 1]^n, d_{\ell^\infty}) = n, \]
where \( d_{\ell^\infty} \) is the sup-distance given by \( d_{\ell^\infty}(x, y) := \max_i |x_i - y_i| \).

From this we get:

Lemma 4.2. Let \( B_{\ell^\infty}(\mathbb{R}^n, \rho) \) be the closed ball of radius \( \rho \) centered at the origin in \( \ell^\infty(\mathbb{R}^n) \) \((\rho > 0)\). Then for \( \varepsilon < 2\rho \)
\[ \text{Widim}_\varepsilon(B_{\ell^\infty}(\mathbb{R}^n, \rho), d_{\ell^\infty}) = n. \]

Proof. Consider the bijection
\[
[0, 1]^n \to B_{\ell^\infty}(\mathbb{R}^n, \rho), \quad (x_1, \cdots, x_n) \mapsto (2\rho x_1 - \rho, \cdots, 2\rho x_n - \rho).
\]
Then the statement easily follows from Lemma 4.1. \( \square \)

Proof of Corollary 1.2. Set \( m := \min(n, \lceil (2/\varepsilon)^p \rceil - 1) \). We already know (Theorem 1.1) \( \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \leq m \). We want to show \( \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \geq m \). Note that for any real number \( x \) we have \( \lceil x \rceil - 1 < x \). Hence \( m \leq \lceil (2/\varepsilon)^p \rceil - 1 < (2/\varepsilon)^p \). Therefore \( m(\varepsilon/2)^p < 1 \). Then if we choose \( \rho > \varepsilon/2 \) sufficiently close to \( \varepsilon/2 \), then \( (m \leq n) \)
\[ B_{\ell^\infty}(\mathbb{R}^m, \rho) \subset B_{\ell^p}(\mathbb{R}^n). \]
(If \( \varepsilon \geq 2 \), then \( m = 0 \) and \( B_{\ell^\infty}(\mathbb{R}^m, \rho) \) is \( \{0\} \).) From Lemma 4.2
\[ \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^\infty}) \geq \text{Widim}_\varepsilon(B_{\ell^\infty}(\mathbb{R}^m, \rho), d_{\ell^\infty}) = m. \]
Essentially the same argument is given in Gournay [1, pp. 5-6]. \( \square \)
4.2. Proof of Corollary 1.3. The following lemma easily follows from (2)

**Lemma 4.3.** Let \( B_{\ell^q}(\mathbb{R}^n, \rho) \) be the closed ball of radius \( \rho \) centered at the origin in \( \ell^q(\mathbb{R}^n) \) (\( 1 \leq q \leq \infty \) and \( \rho > 0 \)). For \( \varepsilon < \rho \),

\[
\text{Widim}_\varepsilon(B_{\ell^q}(\mathbb{R}^n, \rho), d_{\ell^q}) = n.
\]

**Proposition 4.4.** For \( 1 \leq p < q \leq \infty \),

\[
\min(n, \lceil \varepsilon^{-r} \rceil - 1) \leq \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^q}),
\]

where \( r \) is defined by

\[
\frac{1}{r} = \frac{1}{p} - \frac{1}{q}.
\]

**Proof.** We can suppose \( q < \infty \). Set \( m := \min(n, \lceil \varepsilon^{-r} \rceil - 1) \). From Hölder’s inequality,

\[
(|x_1|^p + \cdots + |x_m|^p)^{1/p} \leq m^{1/r}(|x_1|^q + \cdots + |x_m|^q)^{1/q}.
\]

As in the proof of Corollary 1.2, we have \( m \leq \lceil \varepsilon^{-r} \rceil - 1 < \varepsilon^{-r} \), i.e. \( m^{1/r} \varepsilon < 1 \). Therefore if we choose \( \rho > \varepsilon \) sufficiently close to \( \varepsilon \), then \( (m \leq n) \)

\[
B_{\ell^p}(\mathbb{R}^m, \rho) \subset B_{\ell^p}(\mathbb{R}^n).
\]

From Lemma 4.3

\[
\text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^p}) \geq \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^m, \rho), d_{\ell^p}) = m.
\]

\( \square \)

**Proof of Corollary 1.3** From Theorem 1.1 and Proposition 4.4 we have

\[
\lceil \varepsilon^{-r} \rceil - 1 \leq \lim_{n \to \infty} \text{Widim}_\varepsilon(B_{\ell^p}(\mathbb{R}^n), d_{\ell^p}) \leq \lceil (2/\varepsilon)^r \rceil - 1.
\]

From this estimate, we can easily get the conclusion. \( \square \)

5. **Proof of Corollary 1.5**

Let \( 1 \leq p < \infty \) and \( \varepsilon > 0 \). Set \( X := B(\ell^p(\Gamma)) \). To begin with, we want to fix a distance on \( X \) (compatible with the topology). Since \( X \) is compact, if we prove (5) for one fixed distance, then (5) becomes valid for any distance on \( X \). Let \( w : \Gamma \to \mathbb{R}_{>0} \) be a positive function satisfying

\[
\sum_{\gamma \in \Gamma} w(\gamma) \leq 1.
\]

We define the distance \( d(\cdot, \cdot) \) on \( X \) by

\[
d(x, y) := \sum_{\gamma \in \Gamma} w(\gamma)|x_\gamma - y_\gamma| \text{ for } x = (x_\gamma)_{\gamma \in \Gamma} \text{ and } y = (y_\gamma)_{\gamma \in \Gamma} \text{ in } X.
\]

As in Section 1, we define the distance \( d_\Omega \) on \( X \) for a finite subset \( \Omega \subset \Gamma \) by

\[
d_\Omega(x, y) := \max_{\gamma \in \Omega} d(x_\gamma, y_\gamma).
\]
For each $\delta \in \Gamma$, there is a finite set $\Omega_\delta \subset \Gamma$ such that
\[
\sum_{\gamma \in \Gamma \setminus \Omega_\delta} w(\delta^{-1}\gamma) \leq \varepsilon/4.
\]
Set $\Omega' := \bigcup_{\delta \in \Omega} \Omega_\delta$. $\Omega'$ is a finite set satisfying
\[
\sum_{\gamma \in \Gamma \setminus \Omega'} w(\delta^{-1}\gamma) \leq \varepsilon/4 \quad \text{for any } \delta \in \Omega.
\]
Set $c := \lceil (4/\varepsilon)^p \rceil - 1$. Let $\pi : X \to B_{\ell^p}(\mathbb{R}^{\Omega'}) = \{ x \in \mathbb{R}^{\Omega'} \mid \|x\|_p \leq 1 \}$ be the natural projection. From Theorem [1] there are a polyhedron $K$ of dimension $\leq c$ and an $\varepsilon/2$-embedding $f : (B_{\ell^p}(\mathbb{R}^{\Omega'}), d_{\ell^\infty}) \to K$. Then $F := f \circ \pi : (X, d_\Omega) \to K$ becomes an $\varepsilon$-embedding; If $F(x) = F(y)$, then $d_{\ell^\infty}(\pi(x), \pi(y)) \leq \varepsilon/2$ and for each $\delta \in \Omega$
\[
d(\delta x, \delta y) = \sum_{\gamma \in \Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma| + \sum_{\gamma \in \Gamma \setminus \Omega'} w(\delta^{-1}\gamma)|x_\gamma - y_\gamma|,
\]
\[
\leq \frac{\varepsilon}{2} \sum_{\gamma \in \Omega'} w(\delta^{-1}\gamma) + 2 \sum_{\gamma \in \Gamma \setminus \Omega'} w(\delta^{-1}\gamma),
\]
\[
\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Hence $d_\Omega(x, y) \leq \varepsilon$. Therefore,
\[
\operatorname{Widim}_\varepsilon(X, d_\Omega) \leq c.
\]
This shows (5). If $\Gamma$ has an amenable sequence $\{ \Omega_i \}_{i \geq 1}$, then $|\Omega_i| \to \infty$ and hence
\[
\lim_{i \to \infty} \operatorname{Widim}_\varepsilon(X, d_{\Omega_i})/|\Omega_i| = 0.
\]
This shows (6).

References

[1] A. Gournay, Widths of $\ell^p$ balls, [arXiv:0711.3081](http://arxiv.org/abs/0711.3081)
[2] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, Math. Phys. Anal. Geom. 2 (1999) 323-415
[3] E. Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, Inst. Hautes Études Sci. Publ. Math. 89 (1999) 227-262
[4] E. Lindenstrauss, B. Weiss, Mean topological dimension, Israel J. Math. 115 (2000) 1-24
[5] M. Tsukamoto, Mean dimension of the unit ball in $\ell^p$, preprint, [http://www.math.kyoto-u.ac.jp/preprint/index.html](http://www.math.kyoto-u.ac.jp/preprint/index.html) (2007)
[6] M. Tsukamoto, Moduli space of Brody curves, energy and mean dimension, preprint, [arXiv:0706.2981](http://arxiv.org/abs/0706.2981)
[7] M. Tsukamoto, Deformation of Brody curves and mean dimension, preprint, [arXiv:0712.0266](http://arxiv.org/abs/0712.0266)

Masaki Tsukamoto
Department of Mathematics, Faculty of Science
Kyoto University
Kyoto 606-8502
Japan

E-mail address: tukamoto@math.kyoto-u.ac.jp