Non-isotopic monotone Lagrangian submanifolds of $\mathbb{C}^n$

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**Abstract.** Let $P$ be a Delzant polytope in $\mathbb{R}^k$ with $n+k$ facets. We associate a closed Lagrangian submanifold $L$ of $\mathbb{C}^n$ to each Delzant polytope. We prove that $L$ is monotone if and only if the polytope $P$ is Fano. Also, we pose the “Lagrangian version of Delzant theorem”.

Let $n$ and $p$ be even integers. Assume $p$ is greater than 3, $n > 2p$. We construct $\frac{p}{2}$ monotone Lagrangian embeddings of $S^{p-1} \times S^{n-p-1} \times T^2$ into $\mathbb{C}^n$, no two of which are related by Hamiltonian isotopies. Some of these embeddings are smoothly isotopic and have equal minimal Maslov numbers, but they are not Hamiltonian isotopic.

Also, we construct infinitely many non-monotone embeddings of $S^{2p-1} \times S^{2p-1} \times T^2$ into $\mathbb{C}^{4p}$, no two of which are related by Hamiltonian isotopies.

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1. Introduction

In papers [1], [2], Chekanov and Schlenk found examples of Lagrangian tori in $\mathbb{C}^n$ that are not Hamiltonian isotopic to the standard torus $S^1 \times \ldots \times S^1$. It was proved by Vianna that $\mathbb{C}P^2$ contains infinitely many non-isotopic monotone Lagrangian tori (see [3], [4]). In [5], Auroux constructed infinitely many Lagrangian monotone tori in $\mathbb{C}^3$, no two of which are related by Hamiltonian isotopies. Also, many interesting examples are found by Mikhalkin in [6].

In papers [7], [8], new monotone Lagrangian submanifolds of $\mathbb{C}^n$ are constructed and their minimal Maslov numbers are found. Using Oh’s spectral sequence, some restrictions on minimal Maslov numbers of the constructed Lagrangians are obtained. Moreover, in certain cases these examples realize all possible minimal Maslov numbers.

In this paper we construct non-isotopic Lagrangians (monotone and non-monotone) submanifolds of $\mathbb{C}^n$. Let $P$ be a Delzant polytope in $\mathbb{R}^k$ with $n + k$ facets. We associate a closed Lagrangian submanifold of $\mathbb{C}^n$ to each Delzant polytope. The Lagrangian $L$ is diffeomorphic to the total space of fiber bundle over $T^k$, where the fiber is the so-called real moment-angle manifold associated to $P$. In some cases real moment-angle manifolds are diffeomorphic to connected sums of sphere products, but in general real moment-angle manifolds define a rich family of smooth manifolds and their topology is far from being completely understood (see [9], [10], and [11]). We prove the following theorem:

**Theorem** (section 4). Let $P$ be a Delzant polytope and $L \subset \mathbb{C}^n$ be the corresponding Lagrangian. Then $L$ is monotone if and only if the polytope $P$ is Fano.

As we know there are many Delzant polytopes (infinitely many even in dimension 2). Also, there are many Delzant Fano polytopes. As a result we obtain a large family of monotone Lagrangian submanifolds. Moreover, many polytopes provide the same diffeomorphism type of $L$. Also, by construction, our Lagrangians come as preimages of real parts of toric manifolds (section 6). By studying many examples of the constructed Lagrangians, we pose the following “Lagrangian version of Delzant theorem”.

**Conjecture 1.1.** Let $P_1$, $P_2$ be Delzant Fano polytopes in $\mathbb{R}^k$ with $n + k$ facets. Let $L_1$, $L_2$ be the corresponding embedded monotone Lagrangian submanifolds of $\mathbb{C}^n$. Then $L_1$ is Hamiltonian isotopic to $L_2$ if and only if $P_1 = g \cdot P_2$, where $g \in SL(k, \mathbb{Z})$.

Removing monotonicity assumption results in a stronger conjecture, which we also expect to be hold:

**Conjecture 1.2.** Let $P_1$, $P_2$ be Delzant polytopes in $\mathbb{R}^k$ with $n + k$ facets. Let $L_1$, $L_2$ be the corresponding embedded Lagrangian submanifolds of $\mathbb{C}^n$. Then $L_1$ is Hamiltonian isotopic to $L_2$ if and only if $P_1 = g \cdot P_2$, where $g \in SL(k, \mathbb{Z})$.

We prove these conjectures in particular cases.

Let us consider product of two simplices $\Delta^p \times \Delta^{n-p}$ defined by inequalities

$$
\begin{align*}
&\begin{cases}
  x_i + 1 \geq 0 & i = 1, \ldots, n \\
  -x_1 - \ldots - x_p + 1 \geq 0 \\
  -x_1 - \ldots - x_k - x_{p+1} - \ldots - x_n + 1 \geq 0
  \end{cases} & n \geq 2p, \ k < p
\end{align*}
$$

We prove these conjectures in particular cases.
These polytopes are Delzant and Fano. We prove that if \( n, p, \) and \( k \) are even numbers, then the corresponding Lagrangians are diffeomorphic to \( S^{p-1} \times S^{n-p-1} \times T^2 \). As we see, the diffeomorphism type is independent of \( k \). As a result, we have \( \frac{p}{2} \) embedded monotone Lagrangians \( L_k \) with minimal Maslov numbers \( \gcd(p,n-p+k) \), where \( \gcd \) stands for the greatest common divisor. Some of these manifolds can be distinguished by the minimal Maslov numbers but not all of them. If we assume that \( p = 6, n = 14 \), then all minimal Maslov numbers are equal to 2. Moreover, the minimal Maslov numbers are equal to 2, but classes of index 2 cannot be represented by holomorphic discs. So, we need more subtle invariants. Actually, embeddings of our Lagrangians are very explicit and we can “find all holomorphic discs”. We find a class \([h_k] \in \pi_2(\mathbb{C}^n, L_k)\) of index \( 2(n-p+k) \) that can be represented by a holomorphic disc. Let us consider a subset of \( \{2 \times \text{primitive elements of } \mathbb{Z}^2 = \pi_2(\mathbb{C}^n, L_k)\} \) that can be represented by holomorphic discs of indices greater than \( 2(n-p) - 1 \). Let us denote this subset by \( K \). It turns out that minimal index among elements of \( K \) is equal to \( 2(n-p+k) \). Then, we prove that all constructed objects are independent of almost complex structure. Hence, varying \( k \) we get Lagrangians, no two of which are related by Hamiltonian isotopies. Also, we can construct smoothly isotopic embeddings with equal minimal Maslov numbers and prove that they are not Hamiltonian isotopic. As a result we have the following theorem:

**Theorem** (section 4). Let \( n, p, \) and \( k \) be even positive integers. Assume \( p \) is greater than 3, \( n > 2p \), and \( 0 \leq k < p-1 \). Then there are at least \( \frac{p}{2} \) monotone Lagrangian embeddings of \( S^{p-1} \times S^{n-p-1} \times T^2 \) into \( \mathbb{C}^n \), no two of which are related by Hamiltonian isotopies. The minimal Maslov numbers are equal to \( \gcd(p,n-p+k) \). Some of these embeddings are smoothly isotopic and have equal minimal Maslov numbers, but they are not Hamiltonian isotopic.

Then we consider non-monotone Lagrangians. Let us take a family of trapezoids (depend on \( k \)) defined by

\[
\begin{align*}
x_1 & \geq 0 \\
x_2 & \geq 0 \\
-x_2 + 1 & \geq 0 \\
-x_1 - kx_2 + k + 1 & \geq 0
\end{align*}
\]

Note that the toric manifolds corresponding to these trapezoids are the Hirzebruch surfaces. By our conjecture these trapezoids provide infinitely many embedded Lagrangians \( L_k \) \((k \text{ is an arbitrary number greater than } 2)\) in \( \mathbb{C}^4 \), no two of which are related by Hamiltonian isotopies. It is easy to prove that if \( k \) is even, then \( L_k \) is diffeomorphic to \( T^4 \).

We can generalize this construction and get similar polytopes in higher dimensions. The corresponding higher dimensional Lagrangians are

\[
L_k = S^{2p-1} \times S^{2p-1} \times T^2 \subset \mathbb{C}^{4p}.
\]

As in the case of Hirzebruch surfaces, we have some additional parameter \( k \) and diffeomorphism type of \( L_k \) is independent of \( k \) (when \( k \) is even). We can find a class \([h_k] \in \pi_2(\mathbb{C}^{4p}, L_k)\) of index \( 2p(k+2) \) that can be represented by some holomorphic disc. Let \( K \subset \{2 \times \text{primitive elements of } \mathbb{Z}^2 = \pi_2(\mathbb{C}^{4p}, L_k)\} \) be a set of elements that can be represented by holomorphic discs of index greater than \( 4p \). We prove that minimal index
among elements of $K$ is equal to $2p(k + 2)$. Also, we prove that all numbers are independent of almost complex structure. As a result we see that varying $k$ we get infinitely many Lagrangians $L_k$, no two of which are related by Hamiltonian isotopies ($k$ is even number).

**Theorem** (section 5). If $k$ is even and $p > 1$, then all Lagrangians $L_k \subset \mathbb{C}^{4p}$ are diffeomorphic to each other but they are not Hamiltonian isotopic. So, we have infinitely many non-monotone Lagrangian embeddings of $S^{2p-1} \times S^{2p-1} \times T^2$ into $\mathbb{C}^{4p}$, no two of which are related by Hamiltonian isotopies.

2. Polytopes and intersection of quadrics

In this section we discuss toric topology and its applications. For more details we refer our reader to paper of Panov [13] (chapters 2,3,12). Much more details can be found in book [14].

A convex polyhedron $P$ is an intersection of finitely many halfspaces in $\mathbb{R}^k$. Bounded polyhedra are called polytopes.

A supporting hyperplane of $P$ is a hyperplane $H$ which has common points with $P$ and for which the polyhedron is contained in one of the two closed half-spaces determined by $H$. The intersection $P \cap H$ with a supporting hyperplane is called a face of the polyhedron. Zero-dimensional faces are called vertices, one-dimensional faces are called edges, and faces of codimension one are called facets.

Consider a system of $n$ linear inequalities defining a convex polyhedron in $\mathbb{R}^k$

\[ P_{A,b} = \{ x \in \mathbb{R}^k : \langle a_i, x \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, n \}, \]

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^k$, $a_i \in \mathbb{R}^k$, and $b_i \in \mathbb{R}$. By $b$ denote a vector $b = (b_1, \ldots, b_n)^T$, $x = (x_1, \ldots, x_k)^T$ and by $A$ the $k \times n$ matrix whose columns are the vectors $a_i$. Then, our polyhedron can be written in the following form:

\[ P_{A,b} = \{ x \in \mathbb{R}^k : (A^T x + b)_i \geq 0 \text{ for } i = 1, \ldots, n \}. \]

We say that (2.1) is simple if exactly $k$ facets meet at each vertex. We say that (2.1) is generic if for any $x \in P$ the normal vectors $a_i$ of the hyperplanes containing $x$ are linearly independent. Let us assume that $\mathbb{Z} < a_1, \ldots, a_n >$ defines a lattice. A Polyhedron $P$ is called Delzant if it is simple and for any vertex $x \in P$ the vectors $a_i$ normal to the $k$ facets meeting at $x$ form a basis for the lattice $\mathbb{Z} < a_1, \ldots, a_n >$. A Delzant polytope is called Fano if it can be defined by

\[ P_{A,b} = \{ x \in \mathbb{R}^k : \langle a_i, x \rangle + C \geq 0 \text{ for } i = 1, \ldots, n \}, \]

i.e. $C = b_1 = \ldots = b_n$.

Also, (2.1) gives us a linear map from $\mathbb{R}^k$ to $\mathbb{R}^n$

\[ i_{A,b} : \mathbb{R}^k \rightarrow \mathbb{R}^n, \]

\[ i_{A,b}(x) = A^T x + b = (\langle a_1, x \rangle + b_1, \ldots, \langle a_n, x \rangle + b_n)^T. \]

Then, the image $i_{A,b}(\mathbb{R}^k)$ can be written as

\[ i_{A,b}(\mathbb{R}^k) = \{ u \in \mathbb{R}^n : \Gamma u = \Gamma b \}, \]

\[ \Gamma A^T = 0, \quad u = (u_1, \ldots, u_n)^T, \]
where $\Gamma$ is $(n - k) \times n$-matrix whose rows form a basis of linear relations between the vectors $a_i$. We have

\begin{equation}
(2.4) \quad i_{A,b}(P) = i_{A,b}(\mathbb{R}^k) \cap \mathbb{R}^n_+.
\end{equation}

Let us describe the correspondence between the intersection of quadrics and polyhedra. Replacing $u_i$ by $u_i^2$ in (2.3) we get $(n - k)$ quadrics which define a subset in $\mathbb{R}^n$.

Now assume that we have

\begin{equation}
(2.5) \quad \mathcal{R}_{\Gamma,\delta} = \{ u \in \mathbb{R}^n : \gamma_{11}u_1^2 + \ldots + \gamma_{nn}u_n^2 = \delta_i, \quad i = 1, \ldots, n - k, \}.
\end{equation}

The coefficients of the quadrics define $(n - k) \times n$ matrix $\Gamma = (\gamma_{jk})$. The group $\mathbb{Z}_2^n$ acts on $\mathcal{R}_{\Gamma,\delta}$ by

$$
\varepsilon \cdot (u_1, \ldots, u_n) = (\pm u_1, \ldots, \pm u_n),
$$

The quotient $\mathcal{R}_{\Gamma,\delta}/\mathbb{Z}_2^n$ can be identified with the set of nonnegative solutions of the system

$$
\begin{bmatrix}
\gamma_{11}u_1 + \ldots + \gamma_{n1}u_n = \delta_1 \\
\vdots \\
\gamma_{1(n-k)}u_1 + \ldots + \gamma_{n(n-k)}u_k = \delta_{n-k}
\end{bmatrix}
$$

And we get the same system as in (2.3) and (2.4). Solving the homogeneous version of the system above we get the matrix $A$. We have that rows of matrix $\Gamma$ form a basis of linear relations between the vectors $a_i$. Then, we can construct a polytope $\{2.1\}$, where $b = (b_1, \ldots, b_n)$ is an arbitrary solution of the linear system above.

We obtain that a polyhedron defines an intersection of quadrics and an intersection of quadrics defines a polyhedron.

It may happen that some of the inequalities can be removed from the presentation without changing $P_{A,b}$. Such inequalities are called redundant. A presentation without redundant inequalities is called irredundant.

**Theorem 2.1.** (Theorem 3.5 and Chapter 12, [16].) The intersection of quadrics $\mathcal{R}_{\Gamma,\delta}$ is nonempty and nondegenerate if and only if the presentation $P_{A,b}$ is generic.

Assume that we have two polytopes $P_{A,b}, P'_{A',b'}$ and corresponding intersection of quadrics $\mathcal{R}_{\Gamma,\delta}, \mathcal{R}_{\Gamma',\delta'}$. If $P'_{A',b'}$ is obtained from $P_{A,b}$ by adding $m$ redundant inequalities, then $\mathcal{R}_{\Gamma',\delta'}$ is homeomorphic to a product of $\mathcal{R}_{\Gamma,\delta}$ and $\mathbb{Z}_2^m$, i.e. $\mathcal{R}_{\Gamma',\delta'}$ is disjoint union of $2^m$ copies of $\mathcal{R}_{\Gamma,\delta}$.

Faces of $P$ are partially ordered by inclusion. Therefore, faces form partially ordered set and it turns out that topological type of $\mathcal{R}_{\Gamma,\delta}$ depends only on the combinatorics of $P$. We say that two polytopes $P, P'$ are combinatorially equivalent if they are equivalent as partially ordered sets.

**Theorem 2.2.** (Proposition 4.3). Assume that $P$ and $P'$ are combinatorially equivalent simple polytopes. If both presentations are irredundant, then the corresponding manifolds $\mathcal{R}_{\Gamma,\delta}$ and $\mathcal{R}_{\Gamma',\delta'}$ are diffeomorphic.
3. **Lagrangian submanifolds of \( \mathbb{C}^n \)**

A construction explained in this section was discovered by Mironov in [15]. Later, the construction was studied by Panov and Kotelskiy in [16] and [17]. They applied their method to study minimal and Hamiltonian-minimal Lagrangians of toric manifolds. It was noticed by the author in [7] that the new method can be used to construct monotone Lagrangian submanifolds. The construction allows us to find the Maslov class of Lagrangians. Moreover, in certain cases our examples realize all possible minimal Maslov numbers (see [8] for more details).

Let \( \Gamma \) be a matrix with columns \( \gamma_j \in \mathbb{Z}^{n-k}, j = 1, \ldots, n \). Let \( \mathcal{R} \) be a \( k \)-dimensional submanifold of \( \mathbb{R}^n \) defined by

\[
\begin{align*}
\{ \ & \gamma_{1,i} u_1^2 + \ldots + \gamma_{n,i} u_n^2 = \delta_i, \quad i = 1, \ldots, n-k, \\
\ & \delta_j \in \mathbb{R}, \quad \gamma_{i,j} \in \mathbb{Z} \}
\end{align*}
\]

Denote by \( T_\Gamma \) an \((n-k)\)-dimensional torus

\[
T_\Gamma = (e^{i\pi \langle \gamma_1, \varphi \rangle}, \ldots, e^{i\pi \langle \gamma_n, \varphi \rangle}) \subset \mathbb{C}^n,
\]

where \( \varphi = (\varphi_1, \ldots, \varphi_{n-k}) \in \mathbb{R}^{n-k} \) and \( \langle \gamma_j, \varphi \rangle = \gamma_{j,1} \varphi_1 + \ldots + \gamma_{j,n-k} \varphi_{n-k} \).

Consider a map

\[
\tilde{\psi}: \mathcal{R} \times T_\Gamma \to \mathbb{C}^n,
\]

\[
\tilde{\psi}(u_1, \ldots, u_n, \varphi) = (u_1 e^{i\pi \langle \gamma_1, \varphi \rangle}, \ldots, u_n e^{i\pi \langle \gamma_n, \varphi \rangle}).
\]

Let us show that \( \tilde{\psi} \) is not a nice map. We assumed that \( \dim \mathcal{R} = k \) and this implies that the equations in system (3.1) are linearly independent. Therefore, the integer vectors

\[
\gamma_j = (\gamma_{j,1}, \ldots, \gamma_{j,n-k})^T \in \mathbb{Z}^{n-k}, \quad j = 1, \ldots, n
\]

are linearly independent and vectors (3.3) form a lattice \( \Lambda \) in \( \mathbb{R}^{n-k} \) of maximum rank. The dual lattice \( \Lambda^* \) is defined by

\[
\Lambda^* = \{ \lambda^* \in \mathbb{R}^{n-k} | \langle \lambda^*, \lambda \rangle \in \mathbb{Z}, \lambda \in \Lambda \},
\]

where \( \langle \lambda^*, \lambda \rangle \) is the standard Euclidian product on \( \mathbb{R}^{n-k} \). Let us define a group

\[
D_\Gamma = \Lambda^*/2\Lambda^* \approx \mathbb{Z}_2^{n-k}.
\]

Let \( \gamma \in D_\Gamma \) be a nontrivial element. We see that if \( (u_1, \ldots, u_n) \in \mathcal{R} \), then

\[
(u_1 \cos \pi \langle \gamma, \gamma_1 \rangle, \ldots, u_n \cos \pi \langle \gamma, \gamma_n \rangle) \in \mathcal{R}
\]

because \( \cos \pi \langle \gamma, \gamma_i \rangle = \pm 1 \). We get that

\[
\tilde{\psi}(u_1, \ldots, u_n, \varphi) = \tilde{\psi}(u_1 \cos \pi \langle \gamma, \gamma_1 \rangle, \ldots, u_n \cos \pi \langle \gamma, \gamma_n \rangle, \varphi + \gamma).
\]

This shows that many points of \( \mathcal{R} \times T_\Gamma^{n-k} \) have the same images under \( \tilde{\psi} \). To fix this we need to take quotient of \( \mathcal{R} \times T_\Gamma \) by the group \( D_\Gamma \)

\[
(u_1, \ldots, u_n, \varphi) \sim (u_1 \cos \pi \langle \gamma, \gamma_1 \rangle, \ldots, u_n \cos \pi \langle \gamma, \gamma_n \rangle, \varphi + \gamma),
\]

\[
\mathcal{N} = \mathcal{R} \times D_\Gamma T_\Gamma.
\]
The action of $D\Gamma$ is free, since it is free on the second factor. Hence, $N$ is a smooth $n$-manifold. So, we have a map

$$\psi : N \to \mathbb{C}^n,$$

(3.5)

$$\psi(u_1, ..., u_n, \varphi) = (u_1 e^{i\pi \langle \gamma_1, \varphi \rangle}, ..., u_n e^{i\pi \langle \gamma_n, \varphi \rangle}).$$

As we noticed $D\Gamma$ acts freely on $T\Gamma$. Therefore, the projection $N = R \times D\Gamma \to T\Gamma/D\Gamma = T^{n-k}$ onto the second factor defines a fiber bundle over $T\Gamma/D\Gamma = T^{n-k}$, where the fiber is $R$.

Let us note that we need only system (3.1) to define $\psi$, $T\Gamma$, and $N$. We discussed in the previous section that there exists a polytope $P$ associated to system (3.1). We have the following theorem:

**Theorem 3.1.** (see [15], [16]). $\psi(N)$ is an immersed Lagrangian. The map $\psi$ is an embedding if and only if the polyhedron $P$ corresponding to system (3.1) is Delzant.

We see from (3.4) that the forms $d\varphi_1, ..., d\varphi_{n-k}$ are closed invariant forms, therefore they are elements of $H^1(N, \mathbb{R})$. The lattice of the torus $T\Gamma/D\Gamma = T^{n-k}$ is formed by generators of $\Lambda^*$. Let $\varepsilon_1, ..., \varepsilon_{n-k}$ be a basis for $\Lambda^*$. Cycles $\varepsilon_i = s\varepsilon_i$ are elements of $H_1(N, \mathbb{Z})$, where $s \in [0, 2]$ and $u^2_j = \text{const}$ for all $j$ (see formula (3.5)). By $\varepsilon_{i,p}$ denote the $p$th coordinate of $\varepsilon_i$. Let $I_\omega$ be the symplectic area homomorphism of $\psi(N) \subset \mathbb{C}^n$.

Consider an $(n-k)$-dimensional vector

$$\gamma_1 + ... + \gamma_n = (t_1, ..., t_{n-k})^T.$$

**Lemma 3.2.** (see [7] sections 2 and 4). We have

(3.6)

$$I_\omega(\varepsilon_i) = \pi \sum_{p=1}^{n-k} \varepsilon_{i,p} \delta_p,$$

where $\delta_p$ are defined in (3.4). Moreover, the Maslov class $I_\mu$ is given by

(3.7)

$$\mu = t_1 d\varphi_1 + ... + t_{n-k} d\varphi_{n-k},$$

where $\varphi_i$ are coordinates on the torus, as in (3.2).

4. **Non-isotopic monotone Lagrangian submanifolds of $\mathbb{C}^n$**

It is shown in the previous section that any simple polytope $P$ corresponds to the smooth manifold $R$, where $R$ is defined by the system of quadrics. Also, the system of quadrics corresponds to the immersed Lagrangian submanifold $L = \psi(N) \subset \mathbb{C}^n$. If $P$ is Delzant, then $L$ is embedded. In fact, we have the following theorem:

**Theorem 4.1.** Let $P$ be a Delzant polytope and $L \subset \mathbb{C}^n$ be the corresponding embedded Lagrangian. Assume that $P$ is irredundant (or equivalently $R$ is connected) and $R$ is simply connected. Then $L$ is monotone if and only if the polytope $P$ is Fano.
Remark. This theorem works when $\mathcal{R}$ is not simply connected too. But for simplicity assume that $\mathcal{R}$ is simply connected.

First, let us proof the simple lemma, which will be used later. As in the previous section, let $\Lambda$ be the lattice generated by columns $\gamma_1, \ldots, \gamma_n$ in $\mathbb{R}^{n-k}$ and let $\Lambda^*$ be the dual lattice. Without loss of generality, assume that the vectors $\gamma_1, \ldots, \gamma_{n-k}$ form basis for $\Lambda$. Let $\varepsilon_1, \ldots, \varepsilon_{n-k}$ be the vectors dual to $\gamma_1, \ldots, \gamma_{n-k}$. If $u_1, \ldots, u_n$ are constants, then maps

$$
e_i : I \to L$$

$$e_i(s) = \psi(u_1, \ldots, u_n, 2s\varepsilon_i),$$

$I = [0, 1], \ s \in [0, 1], \ u_1, \ldots, u_n = \text{const}$ represent 1-cycles (see (3.5)) for all $i = 1, \ldots, n - k$.

**Lemma 4.2.** Cycles $[e_i]$ are equal to $2[r_i]$ for some primitive elements $r_i \in H_1(\mathcal{N}, \mathbb{Z}) = H_1(T^{n-k}, \mathbb{Z}) = \mathbb{Z}^{n-k}$.

**Proof.** Consider a path $r_i(s) \subset \mathcal{N}$ such that

$$r_i(0) = (u_1, \ldots, u_n, 0, \ldots, 0), \ r_i(1) = (u_1 \cos \pi \langle \varepsilon_i, \gamma_1 \rangle, \ldots, u_n \cos \pi \langle \varepsilon_i, \gamma_n \rangle, \varepsilon_i),$$

The path above exists because $\mathcal{R}$ is connected. Note that $r_i(1) = \varepsilon_i \cdot r_i(0)$, hence $r_i(s)$ is a loop. Let $\sigma$ be the projection $\mathcal{N} = \mathcal{R} \times D_\Gamma \rightarrow T_\Gamma/D_\Gamma = T^{n-k}$. We see that cycles $\sigma_*([r_1]), \ldots, \sigma_*([r_{n-k}])$ generate $\pi_1(T_\Gamma/D_\Gamma)$. Also, $\sigma_*[e_i] = 2\sigma_*([r_i])$ for all $i = 1, \ldots, n - k$. \hfill $\Box$

**Proof of Theorem 4.1**

**Proof.** If $P$ is Fano, then $P$ is given by

$$P_{A,b} = \{ x \in \mathbb{R}^k : < a_i, x > + C \geq 0 \ \text{for} \ i = 1, \ldots, n \}.$$ 

In other words (see Section 2 for definitions)

$$b = (C, \ldots, C)^T.$$ 

From (2.3) we see that the corresponding system of quadrics has the form

$$\Gamma u = \Gamma b.$$ 

From (3.6), (3.7) we have that symplectic area homomorphism $I_\omega$ and the Maslov class $I_\mu$ are given by

$$I_\omega(e_i) = C\pi \sum_{p=1}^{n-k} \varepsilon_{i,p}(\gamma_{1,i} + \ldots + \gamma_{n,i}) = C\pi \sum_{p=1}^{n-k} \varepsilon_{i,p}t_p,$$

$$I_\mu(e_i) = \int_0^2 (t_1\varepsilon_{i,1} + \ldots + t_n\varepsilon_{i,n})ds = 2\sum_{p=1}^{n-k} \varepsilon_{i,p}t_p.$$ 

We see $I_\omega(e_i) = \frac{2}{\pi C}I_\mu(e_i)$ for all $i$. From Lemma 4.2 we have $[e_i] = 2[r_i]$ and

$$I_\mu(r_i) = \frac{1}{2}I_\mu(e_i), \ \ I_\omega(r_i) = \frac{1}{2}I_\omega(e_i).$$
Finally, we obtain
\[ I_\mu = \frac{2}{\pi C} I_\omega. \]
This means that \( L \) is monotone.

Now let us assume that \( L \) is monotone. Hence
\[ I_\mu(r_i) = \sum_{p=1}^{n-k} \varepsilon_{i,p} t_p = C \sum_{p=1}^{n-k} \varepsilon_{i,p} \delta_p = C I_\omega(r_i), \]
where \( C \) is a positive real number. Then
\[ \sum_{p=1}^{n-k} \varepsilon_{i,p} (C \delta_p - t_p) = 0 \Leftrightarrow C \delta_p = t_p = \gamma_{1,i} + \ldots + \gamma_{n,i}. \]
Let us recall that \( \Gamma \) is the matrix with columns \( \gamma_j \). Then from formula \( \Gamma b = \delta \) we get that \( b = (\frac{1}{C}, \ldots, \frac{1}{C}) \) solves the equation above. \( \square \)

Finally, let us construct non-isotopic monotone Lagrangians. Denote by \( \gcd(m, n) \) the greatest common divisor of \( m \) and \( n \).

**Theorem 4.3.** Let \( n, p, \) and \( k \) be even positive integers. Assume \( p \) is greater than 3, \( n > 2p \), and \( 0 \leq k < p - 1 \). Then there are at least \( \frac{p^2}{2} \) monotone Lagrangian embeddings of \( S^{p-1} \times S^{n-p-1} \times T^2 \) into \( \mathbb{C}^n \), no two of which are related by Hamiltonian isotopies. The minimal Maslov numbers are equal to \( \gcd(p, n - p + k) \). Some of these embeddings are smoothly isotopic and have equal minimal Maslov numbers, but they are not Hamiltonian isotopic.

**Proof.** Let us consider a polytope \( P \) defined by inequalities
\[
\begin{align*}
  &x_i + 1 \geq 0 \quad i = 1, \ldots, n \\
  &-x_1 - \ldots - x_p + 1 \geq 0 \\
  &-x_1 - \ldots - x_k - x_{p+1} - \ldots - x_n + 1 \geq 0 \\
\end{align*}
\]
\( n \geq 2p, \ k < p - 1 \)

The polytope \( P \) is combinatorially equivalent to product of \( p \)-simplex \( \Delta^p \) and \((n - p)\)-simplex \( \Delta^{n-p} \). In fact, any polytope in \( \mathbb{R}^n \) with \( n + 2 \) vertices is combinatorially equivalent to product of two simplices. Denote by \( L_k \) the Lagrangian submanifold of \( \mathbb{C}^n \) associated to the polytope \( P \). We see that the polytope \( P \) is Delzant and Fano, hence \( L_k \) is embedded monotone Lagrangian. The corresponding matrix \( \Gamma \) and the system of quadrics have the following forms:
\[
\Gamma = \begin{pmatrix}
  1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
  1 & \ldots & 1 & 0 & \ldots & 0 & 1 & \ldots & 1
\end{pmatrix}
\]

(4.1)

\[
\begin{align*}
  &u_1^2 + \ldots + u_p^2 = p \\
  &u_1^2 + \ldots + u_k^2 + u_{k+1}^2 + \ldots + u_n^2 = n - p + k \\
\end{align*}
\]
\( n \geq 2p, \ k < p - 1 \)

The system above is equivalent to
\[
\begin{align*}
  &2u_1^2 + \ldots + 2u_k^2 + u_{k+1}^2 + \ldots + u_n^2 = n + k \\
  &(n - 2p + k)u_1^2 + \ldots + (n - 2p + k)u_k^2 + (n - p + k)u_{k+1}^2 + \ldots + (n - p + k)u_n^2 - \\
  &-pu_{p+1}^2 - \ldots - pu_n^2 = 0
\end{align*}
\]
(4.2)
The second equation of the system defines a cone over the product of two ellipsoids of dimensions \( p - 1 \) and \( n - p - 1 \). By intersecting it with ellipsoid of dimension \( n - 1 \), defined by the first equation, we obtain that the system (4.1) defines

\[
\mathcal{R} = S^{p-1} \times S^{n-p-1} \subset \mathbb{R}^n.
\]

Also, this can be obtained in another way. Our polytope \( P \) is combinatorially equivalent to \( \Delta^p \times \Delta^{n-p} \), hence \( \mathcal{R} = S^{p-1} \times S^{n-p-1} \).

From (4.1) we have that the corresponding torus \( T_\Gamma \) and the embedding of \( \mathcal{R} \times_{D_\Gamma} T_\Gamma \) into \( \mathbb{C}^n \) are given by

\[
T_\Gamma = \left( e^{i\pi(\varphi_1 + \varphi_2)}, \ldots, e^{i\pi(\varphi_1 + \varphi_2)}, e^{i\pi\varphi_1}, \ldots, e^{i\pi\varphi_1}, e^{i\pi\varphi_2}, \ldots, e^{i\pi\varphi_2} \right) \subset \mathbb{C}^n,
\]

\[
\psi(u_1, \ldots, u_n, \varphi_1, \varphi_2) = (u_1 e^{i\pi(\varphi_1 + \varphi_2)}, \ldots, u_k e^{i\pi(\varphi_1 + \varphi_2)}, u_{k+1} e^{i\pi\varphi_1}, \ldots, u_p e^{i\pi\varphi_1}, u_{p+1} e^{i\pi\varphi_2}, \ldots, u_n e^{i\pi\varphi_2})
\]

\[
\varphi_1, \varphi_2 \in \mathbb{R}.
\]

We know that \( \mathcal{R} \times_{D_\Gamma} T_\Gamma \to T^2 = T_\Gamma / D_\Gamma \) is a fibration, where the fiber is \( \mathcal{R} \). It is proved in [8] that under our assumptions \((n, p, k)\) are even) the fibration is trivial and is diffeomorphic to \( S^{p-1} \times S^{n-p-1} \times T^2 \). Put

\[
L_k = \psi(\mathcal{R} \times_{D_\Gamma} T_\Gamma).
\]

From (3.7) we have that the Maslov class of \( L_k \) is given by

\[
I_{\mu_k} = p d\varphi_1 + (n - p + k) d\varphi_2
\]

and the minimal Maslov number \( N_{L_k} \) is equal to \( \gcd(p, n - p + k) \) (see [8] for more details).

Let us consider \( 1 \)-cycles

\[
[e_1] = \psi(u_1, \ldots, u_n, 2s, 0), \quad [e_2] = \psi(u_1, \ldots, u_n, 0, 2s), \quad s \in [0, 1].
\]

By Lemma 4.2 we have

\[
[e_1] = 2[r_1], \quad [e_2] = 2[r_2],
\]

where \([r_1], [r_2]\) generate \( \pi_1(L_k) = \mathbb{Z}^2 \).

Let us study classes of \( \pi_2(\mathbb{C}^n, L_k) \) that can be represented by holomorphic discs.

**Lemma 4.4.** If \( \gamma : (D^2, S^1) \to (\mathbb{C}^n, L_k) \) is a holomorphic disc, then \( \mu_k(\gamma) \) equals either \( 2\lambda_1 p + 2\lambda_2(n - p + k) \) or \( 2\lambda_1 p + 2\lambda_2(n - 2p + k) \), where \( \lambda_1, \lambda_2 \) are non-negative integer numbers. If \( \gamma(\partial D^2) \) passes through a generic point, then the index of \( \gamma \) is equal to \( 2\lambda_1 p + 2\lambda_2(n - p + k) \).

**Proof.** Let \( \gamma \) be a holomorphic disc, i.e.

\[
\gamma : (D^2, S^1) \to (\mathbb{C}^n, L_k),
\]

\[
\gamma(z) = (\gamma_1(z), \ldots, \gamma_n(z)),
\]

where \( D^2 \) is unit disc.
From (4.1) we see that at least one of the numbers \( u_{p+1}, \ldots, u_n \) is nonzero (because \( n \geq 2p \)). Therefore, at least one of the functions \( \gamma_{p+1}, \ldots, \gamma_n \) is not equal to zero identically. Assume that \( \gamma_n \) is not equal to zero identically.

1) First, let us assume that at least one of the functions \( \gamma_{k+1}, \ldots, \gamma_p \) is not identically equal to zero. Suppose \( \gamma_p \) is nonzero function. We have

\[
\frac{1}{2i\pi} \int_{S^1} dln(\gamma_p) = \text{number of zeroes} = \text{the winding number of } \gamma_p(\partial D^2) \text{ around the origin} \geq 0.
\]

We see from (4.3) that if \( \gamma_j \) is not a constant, then \( \gamma_p \) and \( \gamma_j \) have equal winding numbers, where \( j = k + 1, \ldots, p - 1 \).

Assume the winding numbers of all functions \( \gamma_{k+1}, \ldots, \gamma_p \) and functions \( \gamma_{p+1}, \ldots, \gamma_n \) are equal to zero. Then from (4.4) we have \( \mu_k(\gamma) = 0 \). But our Lagrangian is monotone and this implies that the symplectic area of \( \gamma \) is equal to zero. As a result we get \( \gamma = \text{const} \).

So, at least one of the functions \( \gamma_{k+1}, \ldots, \gamma_n \) has nonzero winding number. Let us recall that \( L_k \) is a fiber bundle over \( T^2 = T_1/\mathbb{Z}_2^2 \) and from (4.4) we have

\[
\mu_k(\gamma) = 2p \text{ times the winding number of } \gamma_p(\partial D^2) + 2(n - p + k) \text{ times the winding number of } \gamma_n(\partial D^2) = 2p\lambda_1 + 2(n - p + k)\lambda_2,
\]

where \( \lambda_1, \lambda_2 \) are non-negative integers. Moreover, we see that \( \gamma(\partial D^2) \) represents the element

\[
(2\lambda_1, 2\lambda_2) \in \pi_1(L_k) = \mathbb{Z}^2
\]

relative to basis (4.5).

Note, that all these numbers can be realized. Let us consider

\[
\gamma(z) = (0, \ldots, 0, \sqrt{p}z^{\lambda_1}, 0, \ldots, 0, \sqrt{n - p + k}z^{\lambda_2}).
\]

Direct computations show that \( I_{\mu_k}(\gamma) = 2p\lambda_1 + 2(n - p + k)\lambda_2 \).

2) Assume that all functions \( \gamma_{k+1}, \ldots, \gamma_p \) are identically equal to zero. Then,

\[
\begin{cases}
|\gamma_1(\partial D^2)|^2 + \ldots + |\gamma_k(\partial D^2)|^2 = p \\
|\gamma_{p+1}(\partial D^2)|^2 + \ldots + |\gamma_n(\partial D^2)|^2 = n - 2p + k
\end{cases}
\]

From (4.4) and from (4.3) we see that

\[
I_{\mu_k}(\gamma) = 2p(\text{winding number of } \gamma_1(\partial D^2) - \text{winding number of } \gamma_n(\partial D^2)) + 2(n - p + k)(\text{winding number of } \gamma_n(\partial D^2)) = 2\lambda_1p + 2\lambda_2(n - 2p + k).
\]

We see that \( \gamma \) represents the class

\[
(2\lambda_1 - 2\lambda_2, 2\lambda_2) \in \pi_1(L_k) = \mathbb{Z}^2.
\]
These numbers can be realized too. Discs
\[ \gamma(z) = \left( \sqrt{p}z^{\lambda_1}, 0, ..., 0, 0, \sqrt{n - 2p + k}z \right) \]
have index \( 2\lambda_1p + 2\lambda_2(n - 2p + k) \).

We see that if index of \( \gamma \) is equal to \( 2(n - 2p + k) \), then \( \gamma_{k+1}, ..., \gamma_p \equiv 0 \). This means that \( \gamma \) doesn’t pass through a generic point.

**Lemma 4.5.** The previous lemma holds true for any almost complex structure.

**Proof.** Let \( \mathbb{C}^n = (\mathbb{R}^{2n}, J_0) \) be vector space where \( L_k \) is embedded. Let \( J_1 \) be another almost complex structure. Denote by \( J \) a path connecting \( J_0 \) and \( J_1 \). We assume that all this complex structures are \( \omega \)-tame. Our embedding \( \psi: \mathcal{R} \times \mathcal{D} \times T^2 \to (\mathbb{R}^{2n}, J_0) \) has the following form:

\[
\psi(u_1, ..., u_n, \varphi_1, \varphi_2) = \\
\left( u_1 \cos(\pi(\varphi_1 + \varphi_2)), u_1 \sin(\pi(\varphi_1 + \varphi_2)), ..., u_k \cos(\pi(\varphi_1 + \varphi_2)), u_k \sin(\pi(\varphi_1 + \varphi_2)), \\
\quad u_{k+1} \cos(\pi \varphi_1), u_{k+1} \sin(\pi \varphi_1), ..., u_p \cos(\pi \varphi_1), u_p \sin(\pi \varphi_1), \\
\quad u_{p+1} \cos(\pi \varphi_2), u_{p+1} \sin(\pi \varphi_2), ..., u_n \cos(\pi \varphi_2), u_n \sin(\pi \varphi_2) \right)
\]

Let us take \( J_t \)-holomorphic disc
\[
\gamma: (D^2, S^1) \to (\mathbb{R}^{2n}, L_k) \\
\gamma(D^2) = (f_1, g_1, ..., f_n, g_n).
\]

Arguing as in the previous Lemma we can consider the winding numbers of
\[
\partial D^2 \to \mathbb{R}^2 \\
\partial D^2 \to (f_i(\partial D^2), g_i(\partial D^2)).
\]

Let us note that the winding numbers depend continuously on \( t \). All arguments from the previous lemma work for \( J_t \) and in the same way we get analogues results.

Let \( h_k \) be a holomorphic disc defined by
\[
\begin{align*}
& h_k: (D^2, S^1) \to (\mathbb{C}^n, L_k), \\
& h_k(z) = (u_1 e^{i\pi(\alpha + \beta)}z, ..., u_k e^{i\pi(\alpha + \beta)}z, u_{k+1} e^{i\pi \alpha}, ..., u_p e^{i\pi \alpha}, u_{p+1} e^{i\pi \beta} z, ..., u_n e^{i\pi \beta} z) \\
& u_1, ..., u_n = \text{const}, \alpha, \beta = \text{const}, u_p \neq 0, u_n \neq 0, \\
& u_1^2 + ... + u_p^2 = p, u_1^2 + ... + u_k^2 + u_{k+1}^2 + ... + u_n^2 = n - p + k.
\end{align*}
\]

We see that \( I_{\mu_k}(h_k) = 2(n - p + k) \).

We say that \( v \in \mathbb{Z}^2 \) is primitive if \( v \) is not equal to \( \lambda u \), where \( u \in \mathbb{Z}^2 \), \( \lambda \in \mathbb{Z} \), and \( \lambda \neq \pm 1 \). Let \( K \) be a subset of \( \{2 \times \text{ primitive elements of } \mathbb{Z}^2 = \pi_2(\mathbb{C}^n, L_k)\} \) such that the following is satisfied:

1. Elements of \( K \) can be represented by holomorphic discs.
2. $\gamma \in K$ if $\gamma(\partial D^2)$ passes through a generic point.
3. $\gamma \in K$ if index of $\gamma$ is greater than $2(n - p) - 1$.

Let $m = \min\{I_\mu(K) \subset \mathbb{Z}\}$. Let us prove that $m = 2(n - p + k)$. First, we know that discs of $K$ pass through a generic point. This means that

$$I_\mu(K) = 2p\lambda_1 + 2\lambda_2(n - p + k).$$

If $\lambda_2 = 0$ and $\lambda_1 > 1$, then we get contradiction with $K \subset \{2 \times \text{primitive elements of } \mathbb{Z}^2\}$. Also, $I_\mu(K) > 2(n - p) - 1$, hence $\lambda_2 > 0$. Therefore, $m = 0p + 2(n - p + k) = 2(n - p + k)$.

Now assume that $\gamma$ is a holomorphic disc of index $2(n - p + k)$. If $n - p + k$ is not divisible by $p$, then from Lemma 4.4 we have that index of $\gamma$ can be obtained in two ways

$$2(n - p + k) = 0p + 2(n - p + k), \quad 2(n - p + k) = 2p + 2(n - 2p + k),$$

i.e. either $\lambda_1 = 0, \lambda_2 = 1$ or $\lambda_1 = 1, \lambda_2 = 1$. From (4.7) and (4.8) we have that $\gamma(\partial D^2)$ represents the element $(0, 2) \in \pi_1(L_k)$ relative to the basis (4.5). If $n - p + k$ is divisible by $p$, then we have an additional way

$$2(n - p + k) = 2\frac{n - p + k}{p},$$

i.e. $\lambda_1 = \frac{n - p + k}{p}, \lambda_2 = 0$. In this case we see from (4.7) that $\gamma(\partial D^2)$ represents the element $(\frac{n - p + k}{p}, 0) \in \pi_1(L_k)$.

As a result we see that there is only one homotopy class $[h_k] \in \pi_2(\mathbb{C}^n, L_k)$ such that $h_k$ belongs to $K$ and has index $m = 2(n - p + k)$.

This means that we can distinguish $L_k$ by $m$. As a result varying $k$ we can get $\frac{p}{2}$ embeddings of $S^{p - 1} \times S^{n - p - 1} \times T^2$ into $\mathbb{C}^n$, no two of which are related by Hamiltonian isotopies.

Let us construct smoothly isotopic examples with equal minimal Maslov numbers and then show that they are not Hamiltonian isotopic.

**Proposition 4.6.** Assume that $p = 14, n = 14m$, where $m > 1$. Then $L_k$ is diffeomorphic to $S^{13} \times S^{14m - 15} \times T^2$ and the minimal Maslov number is equal to $\gcd(14, 14m - 14 + k)$. If $k \neq 0$, then the minimal Maslov numbers of Lagrangians $L_k$ are equal to 2. All Lagrangians $L_k$ are not Hamiltonian isotopic but at least two of them are smoothly isotopic.

**Proof.** The first part easily follows from the theorem proved above.

Let us show that at least two of our submanifolds are smoothly isotopic. It is proved by Haefliger and Hirsch (see [12]) that the isometry classes of smooth embeddings of $n$–dimensional manifold $L$ into $\mathbb{C}^n$ are in bijection with the elements of

$$\begin{cases} H_1(L, \mathbb{Z}) & \text{if } n \text{ is odd} \\ H_1(L, \mathbb{Z}_2) & \text{if } n \text{ is even} \end{cases}$$

In our example $n$ is even and $H_1(L_k, \mathbb{Z}_2) = \mathbb{Z}_2^2$. But we constructed 6 Lagrangians (for $k = 2, 4, 6, 8, 10, 12$) with minimal maslov number 2 and theorem of Haefliger and Hirsch says that at least two of them are smoothly isotopic. \qed
5. Infinitely many non-isotopic Lagrangians in $\mathbb{C}^n$

In this section we consider Lagrangian analogues of Hirzebruch surfaces (and higher dimensional analogues). We consider a family of trapezoids (depend on $k$) defined by

$$
\begin{cases}
x_1 \geq 0 \\
x_2 \geq 0 \\
-x_2 + 1 \geq 0 \\
-x_1 - kx_2 + k + 1 \geq 0
\end{cases}
$$

and the corresponding system of quadrics

$$
\begin{cases}
u_2^2 + u_3^2 = 1 \\
u_2^2 + ku_2^2 + u_4^2 = k + 1
\end{cases}
$$

The polytope above is not Fano but it is Delzant. Hence, by our construction it defines Lagrangian embedded submanifold. The system of quadrics defines $R = T^2$ (trapezoid is combinatorially equivalent to $I^1 \times I^1$) and

$$T_\Gamma = (e^{i\pi \varphi_2}, e^{i\pi (\varphi_1 + k \varphi_2)}, e^{i\pi \varphi_1}, e^{i\pi \varphi_2}).$$

Therefore, from (5.1) we see that embedding of $T^2 \times D_\Gamma$ $T_\Gamma$ is given by

$$
\psi(u_1, u_2, u_3, u_4, \varphi_1, \varphi_2) = (u_1 e^{i\pi \varphi_2}, u_2 e^{i\pi (\varphi_1 + k \varphi_2)}, u_3 e^{i\pi \varphi_1}, u_4 e^{i\pi \varphi_2}).
$$

Denote by $L_k$ the embedded Lagrangian. The Lagrangian $L_k$ is diffeomorphic to the total space of fiber bundle over $T_\Gamma/\mathbb{Z}_2^2$, where the fiber is $T^2$ and transition maps are

$$
(u_1, u_2, u_3, u_4, \varphi_1, \varphi_2) \to (u_1, -u_2, -u_3, u_4, \varphi_1 + 1, \varphi_2), \quad (u_1, u_2, u_3, u_4, \varphi_1, \varphi_2) \to (-u_1, (-1)^k u_2, u_3, -u_4, \varphi_1, \varphi_2 + 1)
$$

If $k$ is even, then

$$
\begin{align*}
(u_1, \cos(t) u_2 - \sin(t) u_3, \sin(t) u_2 + \cos(t) u_3, u_4) \\
(\cos(t) u_1 - \sin(t) u_4, u_2, u_3, \sin(t) u_1 + \cos(t) u_4)
\end{align*}
$$

$$
t \in [0, \pi]
$$

give isotopy between the transition maps and identity maps. Therefore, the fiber bundle is trivial for even $k$ and $L_k$ is diffeomorphic to $4$-torus $T^4$. In other words, if $k$ is even, then diffeomorphism type of $L_k$ is independent of $k$.

Now let us start with equation (5.1) and increase the dimension of coordinate spaces, i.e. take a system

$$
\begin{cases}
\sum_{l=p+1}^{2p} u_l^2 + \sum_{l=2p+1}^{3p} u_l^2 = 1 \\
\sum_{l=1}^{p} u_l^2 + k \sum_{l=p+1}^{2p} u_l^2 + \sum_{l=3p+1}^{4p} u_l^2 = k + 1
\end{cases}
$$

Let $P$ be the polytope corresponding to system (5.4). In toric topology, there exists an algorithm for finding $P$ (see [18]). Actually, we only need to know that $P$ is Delzant. This
Proof. Let us recall that our proof more difficult and we hope to prove this theorem for all \( C \). \( \lambda \) \( S \) have infinitely many non-monotone Lagrangian embeddings of are diffeomorphic to each other but they are not Hamiltonian isotopic. As a result, we  

\[
\frac{\partial}{\partial D^2} \gamma = 0, \quad \gamma(\partial D^2) \]

and, as in (4.2), this system defines

\[
R = S^{2p-1} \times S^{2p-1}.
\]

Let us denote by \( L_k \) the corresponding embedded Lagrangians. As we know \( L_k \) is the total space of fiber bundle over \( T_1/\mathbb{Z}_2 = T^2 \) with transition maps (see (5.4))

\[
(u_1, \ldots, u_{4p}, \varphi_1, \varphi_2) \rightarrow (u_1, \ldots, u_{4p}, -u_{p+1}, \ldots, -u_{2p}, -u_{2p+1}, \ldots, -u_{3p}, u_{3p+1}, \ldots, u_{4p}, \varphi_1 + 1, \varphi_2),
\]

\[
(u_1, \ldots, u_{4p}, \varphi_1, \varphi_2) \rightarrow (-u_1, \ldots, -u_{4p}, (-1)^k u_{p+1}, \ldots, (-1)^k u_{2p}, u_{2p+1}, \ldots, u_{3p}, -u_{3p+1}, \ldots, -u_{4p}, \varphi_1, \varphi_2 + 1)
\]

Formulas similar to (5.3) give isotopy between transition maps and identity maps (\( k \) is even). We get that the fiber bundle is trivial and \( L_k \) is diffeomorphic to

\[
S^{2p-1} \times S^{2p-1} \times T^2.
\]

We see that diffeomorphism type of \( L_k \) is independent of \( k \). From (5.4) we have that embedding of \( L_k \) is given by

\[
\psi(u_1, \ldots, u_{4p}, \varphi_1, \varphi_2) = (u_1e^{i\varphi_2}, \ldots, u_{4p}e^{i\varphi_2}, u_{p+1}e^{i(\varphi_1+k\varphi_2)}, \ldots, u_{2p}e^{i(\varphi_1+k\varphi_2)}, u_{2p+1}e^{i\varphi_2}, \ldots, u_{3p}e^{i\varphi_1}, u_{3p+1}e^{i\varphi_2}, \ldots, u_{4p}e^{i\varphi_2}).
\]

From (3.7) we have that the Maslov class of \( L_k \) is given by

\[
I_{\mu_k} = 2pd\varphi_1 + p(k + 2)d\varphi_2.
\]

Theorem 5.1. Assume that \( p > 1 \). If \( k > 2 \) and is even, then all Lagrangians \( L_k \subset \mathbb{C}^{4p} \) are diffeomorphic to each other but they are not Hamiltonian isotopic. As a result, we have infinitely many non-monotone Lagrangian embeddings of \( S^{2p-1} \times S^{2p-1} \times T^2 \) into \( S^{4p} \), no two of which are related by Hamiltonian isotopies.

Remark. When \( p = 1 \) we have \( R = T^2 \) and \( R \) is not simply connected. That makes our proof more difficult and we hope to prove this theorem for all \( p \) later.

Proof. Let us recall that \( v \in \mathbb{Z}^2 \) is called primitive if \( v \) is not equal to \( \lambda u \), where \( u \in \mathbb{Z}^2 \), \( \lambda \in \mathbb{Z} \), and \( \lambda \neq \pm 1 \). Let \( K \) be a subset of \( \{2 \times \text{primitive elements of } \mathbb{Z}^2 = \pi_2(\mathbb{C}^{4p}, L_k)\} \) such that the following is satisfied:

1. Elements of \( K \) can be represented by holomorphic discs.
2. \( \gamma \in K \) if \( \gamma(\partial D^2) \) passes through a generic point.
3. \( \gamma \in K \) if index of \( \gamma \) is greater than \( 4p \).
Put \( m = \min\{I_{\mu_k}(K) \subset \mathbb{Z}\} \). The idea of our proof is the following:

1. Prove that \( m = 2p(k+2) \).
2. Prove that \( m \) is independent of almost complex structure.
3. Distinguish Lagrangians \( L_k \) by the number \( m \).

As in the previous section we have 1–cycles
\[
[e_1] = \psi(u_1, \ldots, u_{4p}, 2s, 0), \quad [e_2] = \psi(u_1, \ldots, u_{4p}, 0, 2s), \quad s \in [0, 1].
\]
and by Lemma 4.2 we have
\[
(5.7) \quad [e_1] = 2[r_1], \quad [e_2] = 2[r_2],
\]
where \([r_1], [r_2]\) generate \( \pi_1(L_k) = \mathbb{Z}^2 \).

Let \( \gamma \) be a holomorphic disc
\[
\gamma : (D^2, S^1) \to (\mathbb{C}^{4p}, L_k)
\]
\[
\gamma(z) = (\gamma_1, \ldots, \gamma_{4p})
\]
First, note that \( \gamma_1, \ldots, \gamma_p, \gamma_{3p+1}, \ldots, \gamma_{4p} \) can not be all equal to zero identically. Indeed, if they are zero functions, then from (5.4) we have \( u_{p+1}^2 + \ldots + u_{2p}^2 = \frac{k+1}{k} > 1 \) and this contradicts with the first equation of (5.4). Without loss of generality assume that \( \gamma_1 \) is not identically equal to zero.

1) Let us assume that \( \gamma_{3p} \) is not identically equal to zero. Arguing as in Theorem 4.3 from (5.6) we have
\[
(5.8) \quad I_{\mu_k}(\gamma) = 4p \text{ times the winding number of } \gamma_{3p}(\partial D^2) +
\]
\[
+2p(k+2) \text{ times the winding number of } \gamma_1(\partial D^2) =
\]
\[
= 4p\lambda_1 + 2p(k+2)\lambda_2,
\]
where \( \lambda_1, \lambda_2 \) are non-negative integers. All these indices can be realized by holomorphic discs
\[
(0, \ldots, 0, 0, \ldots, 0, z^{\lambda_2}, 0, \ldots, 0, \sqrt{k+1} z^{\lambda_1}, 0, \ldots, 0).
\]
Also, we see that \( \gamma(\partial D^2) \) represents the class
\[
(5.9) \quad (2\lambda_1, 2\lambda_2) \in \pi_1(L_k) = \mathbb{Z}^2.
\]
relative to basis (5.7).

2) Assume all functions \( \gamma_{2p+1}, \ldots, \gamma_{3p} \equiv 0 \). Therefore,
\[
I_{\mu_k}(\gamma) = 4p(\text{winding number of } \gamma_2(\partial D^2) - k \cdot \text{winding number of } \gamma_1(\partial D^2)) +
\]
\[
+2p(k+2) \text{ times the winding number of } \gamma_1(\partial D^2) =
\]
\[
= 4p\lambda_1 + 2p(2-k)\lambda_2,
\]
where \( \lambda_1, \lambda_2 \) are non-negative integers. These discs represent elements
\[
(5.11) \quad (2\lambda_1 - 2k\lambda_2, 2\lambda_2) \in \pi_1(L_k) = \pi_2(\mathbb{C}^{4p}, L_k)
\]
relative to basis (5.7).
As we can see we have holomorphic discs with zero and negative indices \((L_k)\) is not monotone. But index of \(\gamma\) can be negative or zero only if all functions \(\gamma_{2p+1}, \ldots, \gamma_{3p} \equiv 0\). So, only discs of positive indices pass through a generic point.

Assume \(\gamma \in K\). Therefore, \(I_{mu_k}(\gamma) > 4p\) and \(\gamma\) passes through a generic point. This implies that the index of \(\gamma\) is defined by formula \((5.8)\). If \(\lambda_2 = 0\), then \(\lambda_1\) has to be greater than 1 and this contradicts with \(K \subset \{2 \times \text{primitive elements}\}\). So, we see that \(\lambda_2\) has to be greater than 0. Then it is obvious that \(m = 2p(k + 2)\).

Let us consider a holomorphic disc

\[
h_k : (D^2, S^1) \to (\mathbb{C}^n, L_k),
\]

\[
h_k(z) = (u_1 e^{i\pi \beta} z, \ldots, u_p e^{i\pi \beta} z, u_{p+1} e^{i\pi (\alpha + k \beta)} z, \ldots, u_{2p} e^{i\pi (\alpha + k \beta)} z, u_{2p+1} e^{i\pi \alpha} z, \ldots, u_{3p} e^{i\pi \alpha} z, u_{3p+1} e^{i\pi \beta} z, \ldots, u_{4p} e^{i\pi \beta} z),
\]

\[
u_1, \ldots, u_{4p} = \text{const}, \ \alpha, \beta = \text{const}, \ u_{3p} \neq 0.
\]

We see that \(h_k\) belongs to \(K\), has index \(m = 2p(k + 2)\), and represent the element \((0, 2)\) in \(\pi_2(\mathbb{C}^{4p}, L_k)\) relative to basis \((5.7)\).

Arguing as in Lemma \((4.5)\) we can prove that \(m\) and all other objects are independent of almost complex structure. Hence, varying \(k\) we get Lagrangians, no two of which are related by Hamiltonian isotopies.

\[
6. \text{LAGRANGIANS AND REAL PARTS OF TORIC MANIFOLDS}
\]

Let \((W, \omega)\) be a symplectic manifold of dimension \(2n\) and \(T^{n-k}\) be an \((n-k)\)-dimensional torus. Assume that \(T^{n-k}\) acts on \(W\) preserving the symplectic form \(\omega\). Any element \(v \in T_c T^k\) generates a one dimensional subgroup of \(T^{n-k}\) and let us denote by \(X_v\) the corresponding \(T^{n-k}\)-invariant vector field on \(W\). The torus action is called Hamiltonian if for any \(v \in T_c T^k\) there exists a function \(H_v\) such that

\[
\omega(X_v, Y) = dH_v(Y)
\]

for any vector field \(Y\) on \(W\). Let us choose a basis \(e_1, \ldots, e_{n-k}\) for \(T_c T^k = \mathbb{R}^k\) and corresponding Hamiltonians \(H_{e_1}, \ldots, H_{e_{n-k}}\). Then the moment map \(m\) is defined by

\[
m : W \to \mathbb{R}^{n-k}, \ \ m(z) = (H_{e_1}(z), \ldots, H_{e_{n-k}}(z)).
\]

Assume that \(\delta\) is a regular value, then the level set \(m^{-1}(\delta)\) is an \((n+k)\)-dimensional \(T^{n-k}\) invariant submanifold of \(W\). If \(T^{n-k}\) action on \(m^{-1}(\delta)\) is free, then \(m^{-1}(\delta)/T^k\) is \(2k\)-dimensional smooth manifold. The restriction of the symplectic form \(\omega\) to \(m^{-1}(\delta)\) may be degenerate but the quotient manifold \(m^{-1}(\delta)/T^k\) is endowed with a unique symplectic form \(\omega'\)

\[
(6.1) \quad \pi^* \omega' = i^* \omega,
\]

where \(i^* : m^{-1}(\delta) \to W\) is the inclusion and \(\pi : m^{-1}(\delta) \to m^{-1}(\delta)/T^k\) is the projection.

Suppose that \(W = \mathbb{C}^n\) with

\[
\omega = \frac{i}{2} \sum_{i=1}^{n} d\bar{z}_i \wedge dz_i.
\]
Let us consider a torus
\[ T^{n-k} = (e^{i\pi <\gamma_1, \varphi>} \cdots e^{i\pi <\gamma_n, \varphi>}) \subset \mathbb{C}^n, \]
\[ \varphi = (\varphi_1, \ldots, \varphi_{n-k}) \in \mathbb{R}^{n-k}, \quad \gamma_1, \ldots, \gamma_n \in \mathbb{Z}^{n-k}. \]
Then \( T^{n-k} \) acts on \( \mathbb{C}^n \) by coordinate-wise multiplication
\[ (e^{i\pi <\gamma_1, \varphi>} \cdots e^{i\pi <\gamma_n, \varphi>}) \cdot (z_1, \ldots, z_n) = (e^{i\pi <\gamma_1, \varphi>} z_1, \ldots, e^{i\pi <\gamma_n, \varphi>} z_n) \]
and the moment map has the form
\[ m(z) = (H_{\epsilon_1}(z), \ldots, H_{\epsilon_{n-k}}(z)) \]
\[ H_{\epsilon_j} = \sum_{i=1}^{n} \gamma_{j,i} |z_i|^2. \]
Assume that \( \delta \in \mathbb{R}^{n-k} \) is a regular value. The manifold \( m^{-1}(\delta) \) is called the moment-angle manifold and can be obtained as set of solutions of
\[ \sum_{i=1}^{n} \gamma_{j,i} |z_i|^2 = \delta_j \quad j = 1, \ldots, n-k \]
Let us denote \( m^{-1}(\delta) \) by \( Z \).
Also, we can consider real moment-angle manifolds. We can replace \( T^{n-k} \) by \( \mathbb{Z}_{2}^{n-k} = \text{Re}(T^{n-k}) \), \( |z_i|^2 \) by \( u_i^2 \), where \( u_i \in \mathbb{R} \). We get intersection of real quadrics
\[ \sum_{i=1}^{n} \gamma_{j,i} u_i^2 = \delta_j \quad j = 1, \ldots, n-k \]
From section 2.1 we know that there is a polytope \( P \) associated to (6.4). Denote the intersection of quadrics (6.4) by \( \mathcal{R} \). We need the following theorem:

**Theorem 6.1.** ([14], Proposition 6.3.1)
1) The action \( T^{n-k} \) on \( m^{-1}(\delta) \) is free if and only if \( P \) is Delzant
2) \( m^{-1}(\delta) \) is compact if and only if \( P \) is polytope, i.e. \( P \) is bounded.

Assume that \( P \) is Delzant. Let us consider \( M = Z / T^{n-k} \). The manifold \( M \) is called the toric manifold associated to the polytope \( P \). As it is mentioned in the beginning of the section, \( M \) is endowed with a symplectic structure. It is easy to see that \( \text{Re}(M) \) is Lagrangian submanifold of \( M \). The real part of \( M \) can be obtained as \( \mathcal{R} / \mathbb{Z}_{2}^{n-k} \).
Let \( \sigma \) be the projection \( Z \rightarrow M \). Let \( L \) be \( \sigma^{-1}(\text{Re}(M)) \subset Z \subset \mathbb{C}^n \). The following theorem was proved by Kotelskiy:

**Theorem 6.2.** ([17]) \( L \) is Lagrangian submanifold of \( \mathbb{C}^n \) with respect to the standard symplectic structure. Moreover, \( L \) is diffeomorphic to \( \mathcal{R} \times_{D_c} T_{\Gamma} \) and \( L \) is Hamiltonian-minimal (we use notations of section 3).

The theorem above shows that our constructed non-isotopic Lagrangians come as a real parts of toric manifolds.
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