Discreteness of point charge in nonlinear electrodynamics

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Abstract

We consider two point charges in electrostatic interaction between them within the framework of a nonlinear model, associated with QED, that provides finiteness of their field energy. We argue that if the two charges are equal to each other the repulsion force between them disappears when they are infinitely close to each other, but remains as usual infinite if their values are different. This implies that within any system to which such a model may be applicable the point charge is fractional, it may only be $2^n$-fold of a certain fundamental charge, $n = 0, 1, 2, ...$

We find the common field of the two charges in a dipole approximation, where the separation between them is much smaller than the observation distance.

1 Introduction

Introduction

Recently a class of nonlinear electrodynamic models was proposed [1] wherein the electrostatic field of a point charge is, as usual, infinite in the point where the charge is located, but this singularity is weaker than that of the Coulomb field, so that the space integral for the energy stored in the field converges. In contrast to the Born-Infeld model, the models from the
class of Ref. [1] refer to nonsingular Lagrangians that follow from the Euler-Heisenberg (E-H) effective Lagrangian [2] of QED truncated at any finite power of its Taylor expansion in the field. This allows us to identify the self-coupling constant of the electromagnetic field with a definite combination of the electron mass and charge and to propose that such models may be used to extend QED to the extreme distances smaller than those for which it may be thought of as a perfectly adequate theory. More general models based on the Euler-Heisenberg Lagrangian, but fit also for considering non-static nonlinear electromagnetic phenomena, where not-too-fast-varying in space and time fields are involved, received attention as well. Among the nonlinear effects studied, there are the linear and quadratic electric and magnetic responses of the vacuum with a strong constant field in it to an applied electric field [3], with the emphasis on the magneto-electric effect [4, 5, 6] and magnetic monopole formation [7]. Also self-interaction of electric and magnetic dipoles was considered with the indication that the electric and magnetic moments of elementary particles are subjected to a certain electromagnetic renormalization [8] after being calculated following a strong interaction theory, say, QCD or lattice simulations. Interaction of two laser beams against the background of a slow electromagnetic wave was studied along these lines, too [9].

In the present paper we are considering the electrostatic problem of two point charges that interact following nonlinear Maxwell equations stemming from the Lagrangian of the above [1] type, their common field not being, of course, just a linear combination of the individual fields. The problem is outlined in the next Section 2. Once the field energy is finite we are able to define the attraction or repulsion force between charges as the derivative of the field energy with respect to the distance $R$ between them. Contrary to the standard linear electrodynamics, this is evidently not the same as the product of one charge by the field strength produced by the other! Based on the permutational symmetry of the problem that takes place in the special case where the values of the two charges are exactly the same we establish that the repulsion force between equal charges disappears when the distance between them is zero. This may shed light to the ever-lasting puzzle of whether a point-like electric charge may exist without flying to pieces due to mutual repulsion of its charged constituents. The optional answer proposed by the present consideration might be that after admitting that these exists a certain fundamental charge $q$, every other point charge should be fractional, equal to $2^n q$, with $n$, being zero or positive integer. In Section 3 we are developing the procedure of finding the solution to the above static two-body problem in the leading approximation with respect to the ratio of the distance $R$, to the coordinate of the observation point $r$, where this ratio is small – this makes the dipole-like approximation of Sub-
The simplifying circumstance that makes this approximation easy to handle is that it so happens that one needs, as a matter of fact, to solve only the second Maxwell equation, the one following from the least action principle, while the first one, $\nabla \times \mathbf{E} = 0$, is trivially satisfied. The above general statement concerning the nullification of the repulsion force at $R = 0$ for equal charges is traced at the dynamical level of Subsection 3.1

2 Nonlinear Maxwell equations

2.1 Nonlinear Maxwell equations as they originate from QED

It is known that QED is a nonlinear theory due to virtual electron-positron pair creation by a photon. The nonlinear Maxwell equation of QED for the electromagnetic field tensor $F_{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ designates its dual tensor $\tilde{F}^{\mu\nu} = (1/2) \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ produced by the classical source $J_\mu(x)$ may be written as, see e.g. [3].

$$\partial^\nu F_{\nu\mu}(x) - \partial^\tau \left[ \frac{\delta L(\mathfrak{F}, \mathfrak{G})}{\delta \mathfrak{F}(x)} F_{\tau\mu}(x) + \frac{\delta L(\mathfrak{F}, \mathfrak{G})}{\delta \mathfrak{G}(x)} \tilde{F}_{\tau\mu}(x) \right] = J_\mu(x).$$  \hspace{1cm} (1)

Here $L(\mathfrak{F}, \mathfrak{G})$ is the effective Lagrangian (a function of the two field invariants $\mathfrak{F} = 1/4 F^{\mu\nu} F_{\mu\nu}$ and $\mathfrak{G} = (1/4) \tilde{F}^{\mu\nu} F_{\mu\nu}$), of which the generating functional of one-particle-irreducible vertex functions, called effective action [10], is obtained by the space-time integration as $\Gamma[A] = \int L(x) \, d^4 x$.

Eq. (1) is the realization of the least action principle

$$\frac{\delta S[A]}{\delta A^\mu(x)} = \partial^\nu F_{\nu\mu}(x) + \frac{\delta \Gamma[A]}{\delta A^\mu(x)} = J_\mu(x),$$  \hspace{1cm} (2)

where the full action $S[A] = S_{\text{Max}}[A] + \Gamma[A]$ includes the standard classical, Maxwellian, electromagnetic action $S_{\text{Max}}[A] = -\int \mathfrak{F}(x) \, d^4 x$ with its Lagrangian known as $L_{\text{Max}} = -\mathfrak{F} = -\frac{1}{4} (E^2 - B^2)$ in terms of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$.

Eq. (1) is reliable only as long as its solutions vary but slowly in the space-time variable $x_\mu$, because we do not include the space and time derivatives of $\mathfrak{F}$ and $\mathfrak{G}$ as possible arguments of the functional $\Gamma[A]$ treated approximately as local. This infrared, or local approximation shows itself as

\footnote{Throughout the paper, Greek indices span Minkowski space-time, Roman indices span its three-dimensional subspace. Boldfaced letters are three-dimensional vectors, same letters without boldfacing and index designate their lengths, except the coordinate vector $\mathbf{x} = r$, whose length is denoted as $r$. The scalar product is $(\mathbf{r} \cdot \mathbf{R}) = x_i R_i$, the vector product is $\mathbf{C} = [\mathbf{r} \times \mathbf{R}]$, $C_i = \epsilon_{ijk} x_j R_k$.}
a rather productive tool [3]– [9]. The calculation of one electron-positron loop with the electron propagator taken as solution to the Dirac equation in an arbitrary combination of constant electric and magnetic fields of any magnitude supplies us with a useful example of $\Gamma [A]$ known as the E-H effective action [2]. It is valid to the lowest order in the fine-structure constant $\alpha$, but with no restriction imposed on the the background field, except that it has no nonzero space-time derivatives. Two-loop expression of this local functional is also available [11].

The dynamical Eq. (1), which makes the ”second pair” of Maxwell equations, may be completed by postulating also their ”first pair”

$$\partial_{\nu} F^{\nu \mu} (x) = 0,$$

(3)

whose fulfillment allows using the 4-vector potential $A^\nu (x)$ for representation of the fields: $F^{\nu \mu} (x) = \partial^\nu A^\mu (x) - \partial^\mu A^\nu (x)$. This representation is important for formulating the least action principle and quantization of the electromagnetic field. From it Eq. (3) follows identically, unless the potential has singularity like the Dirac string peculiar to magnetic monopole.

In the present paper we keep to Eq. (3), although its denial is not meaningless, as discussed in Ref. [7], where a magnetic charge is produced in nonlinear electrodynamics.

We want now to separate the electrostatic case. This may be possible if the reference frame exists where all the charges are at rest, $J_0 (x) = J_0 (r)$. (We denote $r = x$). Then in this ”rest frame” the spacial component of the current disappears, $J (x) = 0$, and the purely electric time-independent configuration $F_{ij} (r) = 0$ would not contradict to equation (1). With the magnetic field equal to zero, the invariant $\mathcal{G} = (E \cdot B)$ disappears, too. In a theory even under the space reflection, to which class QED belongs, also we have

$$\left. \frac{\partial L (F, \mathcal{G})}{\partial \mathcal{G} (x)} \right|_{\mathcal{G}=0} = 0,$$

since the Lagrangian should be an even function of the pseudoscalar $\mathcal{G}$. Then we are left with the equation for a static electric field $E_i = F_{i0} (x)$

$$\partial_i F_{i0} (r) - \partial_i \frac{\delta L (\mathcal{G},0)}{\delta \mathcal{G} (r)} F_{i0}(r) = J_0 (r).$$

(4)

2.2 Model approach

Equation (4) is seen to be the equation of motion stemming directly from the Lagrangian

$$L = -\mathcal{G} + L (\mathcal{G},0)$$

(5)

with the constant external charge $J_0 (r)$. In the rest of the paper we shall be basing on this Lagrangian in understanding that it may originate from
QED as described above or, alternatively, be given *ad hoc* to define a certain model. In the latter case, if treated seriously as applied to short distances near a point charge where the field cannot be considered as slowly varying, in other words, beyond the applicability of the infrared approximation of QED outlined above, the Lagrangian (5) may be referred to as defining an extension of QED to short distances once $L(\mathbf{F},0)$ is the E-H Lagrangian (or else its multi-loop specification) restricted to $\mathbf{F} = 0$.

It was shown in [1] that the important property of finiteness of the field energy of the point charge is guaranteed if $L(\mathbf{F},0)$ in (5) is a polynomial of any power, obtained, for instance, by truncating the Taylor expansion of the H-E Lagrangian at any integer power of $\mathbf{F}$. On the other hand, it was indicated in [12] that a weaker condition is sufficient: if $L(\mathbf{F},0)$ grows with $-\mathbf{F}$ as $(-\mathbf{F})^w$, the field energy is finite provided that $w > \frac{3}{2}$. The derivation of this condition is given in [14] and in [13]. As a matter of fact a more subtle condition suffices: $L(\mathbf{F}) \sim (-\mathbf{F})^\frac{3}{2} \ln^u (-\mathbf{F})$, $u > 2$. In what follows any of these sufficient conditions is meant to be fulfilled.

In the present paper we confine ourselves to the simplest example of the nonlinearity generated by keeping only quadratic terms in the Taylor expansion of the E-H Lagrangian in powers of the field invariant $\mathbf{F}$

$$L(\mathbf{F}(x),0) = \frac{1}{2} \left. \frac{d^2 L(\mathbf{F},0)}{d^2 \mathbf{F}} \right|_{\mathbf{F}=0} \mathbf{F}^2(x),$$

where the constant and linear terms are not kept, because their inclusion would contradict the correspondence principle that does not admit changing the Maxwell Lagrangian $L_{\text{Max}} = -\mathbf{F}$ for small fields. The correspondence principle is laid into the calculation of the E-H Lagrangian via the renormalization procedure.

Finally, we shall be dealing with the model Lagrangian quartic in the field strength

$$L = -\mathbf{F}(x) + \frac{1}{2} \gamma \mathbf{F}^2(x)$$

with $\gamma$ being a certain self-coupling coefficient with the dimensionality of the fourth power of the length, which may be taken as

$$\gamma = \left. \frac{d^2 L(\mathbf{F},0)}{d^2 \mathbf{F}} \right|_{\mathbf{F}=0} = \frac{e^4}{45\pi^2 m^4},$$

where $e$ and $m$ are the charge and mass of the electron, if $L$ is chosen to be the E-H one-loop Lagrangian. We do not refer to this choice henceforward. Generalization to general Lagrangians can be also done in a straightforward way.
The second (4) and the first (3) Maxwell equations for the electric field $\mathbf{E}$ with Lagrangian (6) are

$$
\nabla \cdot \left[ \left( 1 + \frac{\gamma^2}{2} \mathbf{E}^2(\mathbf{r}) \right) \mathbf{E}(\mathbf{r}) \right] = j_0(\mathbf{r}),
$$

(7)

$$
\nabla \times \mathbf{E}(\mathbf{r}) = 0.
$$

(8)

Denoting the solution of the linear Maxwell equations as $\mathbf{E}^{lin}(\mathbf{r})$

$$
\nabla \cdot \mathbf{E}^{lin}(\mathbf{r}) = j_0(\mathbf{r}),
$$

(9)

$$
\nabla \times \mathbf{E}^{lin}(\mathbf{r}) = 0,
$$

(10)

we write the solution of (7), in the following way [3] – [8]

$$
\left( 1 + \frac{\gamma^2}{2} \mathbf{E}^2(\mathbf{r}) \right) \mathbf{E}(\mathbf{r}) = \mathbf{E}^{lin}(\mathbf{r}) + \left[ \nabla \times \mathbf{\Omega}(\mathbf{r}) \right],
$$

(10)

where the vector function $\mathbf{\Omega}(\mathbf{r})$ may be chosen in such a way that $\nabla \cdot \mathbf{\Omega}(\mathbf{r}) = 0$. Imposing equation (8) we get

$$
\mathbf{\Omega}(\mathbf{r}) = \frac{1}{\nabla^2} \left[ \nabla \times \mathbf{E}(\mathbf{r}) \right] = -\frac{1}{4\pi} \int \frac{\left[ \nabla' \times \mathbf{E}(\mathbf{r'}) \right] d\mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|},
$$

(11)

where we have introduced the auxiliary electric field as the cubic combination

$$
\mathbf{E}(\mathbf{r}) = \frac{\gamma}{2} \mathbf{E}^2(\mathbf{r}) \mathbf{E}(\mathbf{r}).
$$

(In the case of a general Lagrangian that would be a more complicated function of $\mathbf{E}(\mathbf{r})$, namely $\mathbf{E}(\mathbf{r}) = \frac{\delta L[F,0]}{\delta F(\mathbf{x})} \mathbf{E}(\mathbf{r})$). From (10), (11) it follows that

$$
\mathbf{E}(\mathbf{r}) + \mathbf{E}(\mathbf{r}) = \mathbf{E}^{lin}(\mathbf{r}) + \left[ \nabla \times \mathbf{\Omega}(\mathbf{r}) \right] = \mathbf{E}^{lin}(\mathbf{r}) + \left[ \nabla \times \left[ \nabla \times \mathbf{E}(\mathbf{r}) \right] \right] \nabla^2,
$$

(12)

or, in components,

$$
E_i(\mathbf{r}) = E_i^{lin}(\mathbf{r}) + \frac{\partial_i \partial_j \gamma}{\nabla^2} \frac{1}{2} \mathbf{E}^2(\mathbf{r}) E_j(\mathbf{r}).
$$

(13)

In the centre-symmetric case of a single point charge considered in [1], [12], the projection operator $\frac{\partial_i \partial_j}{\nabla^2}$ in the latter equation is identity ($\mathbf{\Omega}(\mathbf{r}) = 0$), and Eq. (12) is no longer an integral equation. The same will be the case in the cylindric-symmetric problem of two point charges within the approximations to be considered in the next Section. This simplification makes solution possible. In this case it is sufficient present the solution of the differential part of Eq. (7) in the form (10) setting $\mathbf{\Omega}(\mathbf{r}) = 0$ in it, then the first Maxwell equation (8) is fulfilled automatically.
3 Two-body problem

By the two point charge problem we mean the one, where the current \( j_0(r) \) in (7) is the sum of delta-functions centered in the positions \( r = \pm R \) of two charges \( q_1 \) and \( q_2 \) separated by the distance \( 2R \) (with the origin of coordinates \( x_i \) placed in the middle between the charges)

\[
\nabla \cdot \left[ \left( 1 + \frac{\gamma}{2} E^2(r) \right) E(r) \right] = q_1 \delta^3(r - R) + q_2 \delta^3(r + R).
\]

In what follows we shall be addressing this equation accompanied by (8) for the combined field of two charges.

We shall be separately interested in the force acting between them. The force \( F_i = \frac{dP_0}{dR_i} \) should be defined as the derivative of the field energy \( P_0 = \int \Theta^{00} d^3x \) stored in the solution of Eqs. (14), (8) over the distance between them.

The Noether energy-momentum tensor for the Lagrange density (5) is

\[
T^{\rho\nu} = (1 - \gamma \mathcal{F}(x)) F^{\mu\nu} \partial^\rho A_\mu - \eta^{\rho\nu} L(x).
\]

By subtracting the full derivative \( \partial_\mu [(1 - \gamma \mathcal{F}(x) F^{\mu\nu}) A^\nu] \) due to the field equations (11) (without the source and with no dependence on \( \mathcal{G} \)), the gauge-invariant and symmetric under the transposition \( \rho \leftrightarrow \nu \) energy-momentum tensor

\[
\Theta^{\rho\nu} = (1 - \gamma \mathcal{F}(x)) F^{\mu\nu} F_\mu \rho - \eta^{\rho\nu} L(x)
\]

is obtained. This is the expression for the electromagnetic energy proper, without the interaction energy with the source, the same as in the reference book [15]. When there is electric field alone, the energy density is

\[
\Theta^{00} = (1 + \gamma E^2/2) E^2 - E^2/2 \left( 1 + \gamma E^2/4 \right) = \frac{E^2}{2} + \frac{3\gamma E^4}{8}.
\]

The integral for the full energy of two charges \( P_0 = \int \Theta^{00} d^3x \) converges since it might diverge only when integrating over close vicinities of the charges. But in each vicinity the field of the nearest charge dominates, and we know from the previous publication [1] (also to be explained below) that the energy of a separate charge converges in the present model. When the charges are in the same point, \( R = 0 \), they make one charge \( q_1 + q_2 \), whose energy converges, too.

The energy

\[
P_0 = \int \Theta^{00} d^3x
\]
is rotation-invariant. Hence it may only depend on the length \( R \), in other words, be an even function of \( R \). Then, in the point of coincidence \( R = 0 \), the force \( F_i = \frac{dP}{dv} \) must either disappear – if \( P \) is a differentiable function of \( R \) that point – or be infinite – if not. Crucial to distinguish these cases is the value of the charge difference \( \delta q = q_2 - q_1 \). If the two charges are equal, \( \delta q = 0 \), the solution of equation (14) for the field is an even function of \( R \), since this equation is invariant under the reflection \( R \rightarrow -R \). We shall see in the next subsection that the linear term in the expansion of the solution in powers of the small ratio \( \frac{R}{r} \) is identical zero in this special case, and so is the linear term of \( P \).

### 3.1 Large distance case \( r \gg R \) (dipole approximation)

We shall look for the solution in the form

\[
E = E^{(0)} + E^{(1)} + ...
\]

where \( E^{(0)} \) and \( E^{(1)} \) are contributions of the zeroth and first order with respect to the ratio \( \frac{R}{r} \), respectively.

The zero-order term is spherical-symmetric, because it corresponds to two charges in the same point that make one charge,

\[
E^{(0)} = \frac{r}{R} E^{(0)}(r).
\]  
Eq. (8) is automatically fulfilled for this form.

Let us write the first-order term \( E^{(1)} \) in the following general cylindric-symmetric form, linear in the ratio \( \frac{R}{r} \)

\[
E^{(1)} = r (R \cdot r) a(r) + R g(r),
\]  
where \( a \) and \( g \) are functions of the only scalar \( r \), and the cylindric axis is fixed as the line passing through the two charges. Let us subject (20) to the equation \( \nabla \times E^{(1)} = 0 \). This results in the relation

\[
a(r) = \frac{1}{r} \frac{d}{dr} g(r),
\]
provided that the vectors \( r, R \) are not parallel. We shall see that with the ansatzes (20) and (19) equation (10) can be satisfied with the choice \( \Omega(r) = 0 \):

\[
\left(1 + \frac{2}{2} E^2(r)\right) E(r) = E^{\text{kin}}(r),
\]

namely, we shall find the coefficient functions \( a, g \) from Eq. (22) and then ascertain that the relation (21) is obeyed by the solution.

The inhomogeneity in (22)

\[ E^{\text{lin}}(r) = \frac{q_1}{4\pi r} \frac{r - R}{|r - R|^3} + \frac{q_2}{4\pi} \frac{r + R}{|r + R|^3} \]

satisfies the linear \((\gamma = 0)\) limit of equation (14)

\[ \nabla \cdot E^{\text{lin}}(r) = q_1 \delta^3(r - R) + q_2 \delta^3(r + R) \] (23)

and also (8). The inhomogeneity is expanded in \( r \) as

\[ E^{\text{lin}}(r) = \left( \frac{q_1 + q_2}{4\pi r^2} \right) r + \frac{1}{4\pi} \left( \frac{d \cdot r}{r^3} - \frac{3}{r^5} \right) + ..., \] (24)

where \( d = (q_2 - q_1) R \) is the dipole moment, while the dots stand for the disregarded quadrupole and higher multipole contributions.

The zero-order term satisfies the equation

\[ \left( 1 + \frac{\gamma}{2} E^{(0)2}(r) \right) E^{(0)}(r) = \frac{(q_1 + q_2)}{4\pi r^2}, \] (25)

with the first term of expansion (24) taken for inhomogeneity. This is an algebraic (not differential) equation, cubic in the present model (8), solved explicitly for the field \( E^{(0)} \) as a function of \( r \) in this case, but readily solved for the inverse function \( r(E^{(0)}) \) in any model, which is sufficient for many purposes. Even without solving it we see that for small \( r \ll \gamma \frac{1}{2} \) the second term in the bracket dominates over the unity, therefore the asymptotic behavior in this region follows from (25) to be

\[ E^{(0)}(r) \sim \left( \frac{q_1 + q_2}{2\pi \gamma} \right)^{\frac{1}{2}} r^{-\frac{3}{2}}. \]

This weakened – as compared to the Coulomb field \( \frac{q_1 + q_2}{4\pi r^2} \) – singularity is not an obstacle for convergence of the both integrals in (18), (17) for the proper field energy of the equivalent point charge \( q_1 + q_2 \).

With the zero-order equation (25) fulfilled, we write a linear equation for the first-order correction \( E^{(1)} \) from (22), to which the second, dipole part in (24) serves as an inhomogeneity

\[ E^{(1)} = \frac{(q_2 - q_1)}{r^2} \left( \frac{R}{r} - \frac{3}{r} \frac{(R \cdot r)}{r^2} \right) - \frac{\gamma}{2} \left[ 2 \left( E^{(1)} \cdot E^{(0)} \right) E^{(0)} + E^{(0)2} E^{(1)} \right]. \]
This equation is linear and it does not contain derivatives. We use (20) as the ansatz. After calculating

$$2 \left( \mathbf{E}^{(1)} \cdot \mathbf{E}^{(0)} \right) \mathbf{E}^{(0)} + \mathbf{E}^{(0)2} \mathbf{E}^{(1)} = r E^{(0)2} \frac{\left( \mathbf{R} \cdot \mathbf{r} \right)}{r^2} \left( 2g + 3r^2 a \right) + \mathbf{R} g E^{(0)2};$$

we obtain two equations, along $\mathbf{R}$ and $\mathbf{r}$, with the solutions ($\delta q = q_2 - q_1$, $Q = q_2 + q_1$):

$$g = \frac{\delta q}{r^3} \frac{1}{1 + \frac{3}{2} E^{(0)2} r^2} = \frac{\delta q}{Q r} E^{(0)}, \quad (26)$$

$$a = -\frac{\delta q}{r^5} \frac{3 + \frac{5}{2} E^{(0)2}}{(1 + \frac{3}{2} E^{(0)2})^2} \left( 1 + \frac{3}{2} E^{(0)2} \right) \quad (27)$$

From (25) we obtain

$$\frac{d}{dr} E^{(0)} = -\frac{2Q}{r^3 \left( 1 + \frac{3}{2} E^{(0)2} \right)} - \frac{\gamma E^{(0)2}}{1 + \frac{3}{2} E^{(0)2}} \frac{d}{dr} E^{(0)}.$$

Hence

$$\frac{d}{dr} E^{(0)} = -\frac{2Q}{r^3 \left( 1 + \frac{3}{2} E^{(0)2} \right)}.$$

(28)

With the help of this relation the derivative of (26) can be calculated to coincide with (27) times $r$. This proves Eq. (21) necessary to satisfy the first Maxwell equation (8).

By comparing this with (27) we see that Eq. (21) necessary to satisfy the first Maxwell equation (8) has been proved.

Finally, relations (26) and (27) as substituted in the general cylindric covariant decomposition (20) give the linear in $\mathbf{R}$ (27) correction $\mathbf{E}^{(1)}$ to the zero-order field $\mathbf{E}^{(0)}$, subject to the equation (25), in terms of $E^{(0)}$, which is explicitly known in our special model. These results may be considered as giving nonlinear correction to the electric dipole field (the second term in (24)) due to nonlinearity.

Coming back to the discussion on the repulsion force we have to analyze the contribution of the found linear term $\mathbf{E}^{(1)}$ into the energy. The contribution of $\mathbf{E}^{(1)}$ into the energy density (17) linear in $\mathbf{R}$ contains the factor $(\mathbf{E}^{(1)} \cdot \mathbf{E}^{(0)}) = (\mathbf{E}^{(1)} \cdot \frac{\mathbf{R}}{r}) E^{(0)}$. According to the result (20) this factor is linear with respect to the scalar product $(\mathbf{R} \cdot \mathbf{r}) = Rr \cos \theta$. It would give zero contribution into the energy (18) due to the angle integration. This does not imply, however, that the force at the point of coincidence $\mathbf{R} = 0$ is zero, because the linear contribution into the integrand in (18) would create divergence of the integral (18) near $r = 0$. The interchange of the integration over $r$ and of the limiting transition $\frac{\mathbf{R}}{r} \to 0$ is not permitted. In
the region $r < R$ of integration the linear approximation in the ratio $\frac{R}{r}$ is not relevant. This region gives the infinite contribution into the repulsion force between two charges when the approach each other infinitely close. The case where these charges are equal, $q_1 - q_2 = 0$, is different. Then the solution for $E^{(1)}$ is just zero, and we confirm the conclusion made above following general argumentation that equal point charges do not repulse when their positions coincide.

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