Broken symmetry and Yang-Mills theory

“Only by their breaking could the divine configurations be perfected”
Kabbalistic text; Ta’alumoth Chokhmah (The Channels of Wisdom) 1629
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Abstract
From its inception in statistical physics to its role in the construction and in the development of the asymmetric Yang-Mills phase in quantum field theory, the notion of spontaneous broken symmetry permeates contemporary physics. This is reviewed with particular emphasis on the conceptual issues.

1 Contribution to “Fifty years of Yang Mills theory”, editor G. ’t Hooft, World Scientific.
1 Introduction

Physics, as we know it, attempts to interpret the diverse natural phenomena as particular manifestations of general laws. The impressive success of this enterprise in the first half of the twentieth century made it conceivable that all phenomena at the atomic level and at larger distance scales be governed solely by the known laws of classical general relativity and quantum electrodynamics.

The vision of a world ruled by general testable laws is relatively recent. Basically it was initiated by the Galilean inertial principle. The subsequent rapid development of large-scale physics is certainly tributary to the fact that gravitational and electromagnetic forces are long-range and hence can be perceived directly without the mediation of highly sophisticated technical devices.

The discovery of subatomic structures and of the concomitant weak and strong short-range forces raised the question of how to cope with short-range forces in relativistic quantum field theory. The Fermi theory of weak interactions, formulated in terms of a four Fermi point-like current-current interaction, was well-defined in lowest order perturbation theory and successfully confronted many experimental data. However, it is clearly inconsistent in higher orders because of uncontrollable divergent quantum fluctuations. In order words, in contradistinction to quantum electrodynamics, the Fermi theory is not renormalizable. This difficulty could not be solved by smoothing the point-like interaction by a massive, and therefore short-range, charged vector particle exchange (the so-called $W^+$ and $W^-$ bosons): theories with massive charged vector bosons are not renormalizable either. In the early nineteen sixties, there seemed to be insuperable obstacles to formulating a consistent theory with short-range forces mediated by massive vectors.

It is the notion of spontaneous broken symmetry as adapted to gauge theory that provided the clue to the solution.

The notion of spontaneous broken symmetry (SBS) finds its origin in the statistical physics of phase transitions [1]. There, the low temperature ordered phase of the system can be asymmetric with respect to the symmetry principles that govern its dynamics. This is not surprising since more often than not energetic considerations dictate that the ground state or low lying excited states of a many body system become ordered. A collective variable such as magnetization picks up expectation value, which define an order parameter that otherwise would vanish by virtue of the dynamical symmetry (isotropy in the aforementioned example). More surprising was the discovery by Nambu that the vacuum and the low energy excitations of a relativistic field theory...
may bare the mark of SBS. Broken chiral symmetry due to a spontaneous generation of hadron mass induces massless pseudoscalar modes (identified with a massless limit of pion fields), which at infinite wavelength generate rotation of the chiral phase. In absence of gauge field, such massless Nambu-Goldstone (NG) modes and the concomitant vacuum degeneracy in the coset \( \mathcal{G}/\mathcal{H} \), where \( \mathcal{G} \) is the symmetry group and \( \mathcal{H} \) the unbroken subgroup, are general features of spontaneous broken symmetry of a continuous group. The occurrence of SBS, of either a continuous or a discrete group, is also marked by fluctuations of the order parameter described by generically massive scalar excitations. Introducing gauge fields renders local in space-time the otherwise global dynamical symmetry \( \mathcal{G} \) and leads to dramatic effects. While the massive scalar excitations survive, the massless NG bosons disappear as such but provide a longitudinal polarization for the gauge bosons living in the coset, which become massive. The essential degeneracy of the vacuum is removed and local gauge invariance is preserved despite the gauge boson masses. Thus, the apparent global broken symmetry from \( \mathcal{G} \) to \( \mathcal{H} \) is now hiding a true unbroken local gauge symmetry.

This way of obtaining massive vectors and hence short-range forces out of a fundamental massless Yang-Mills Lagrangian was proposed in 1964 independently by Brout and Englert in quantum field theoretic terms [4] and by Higgs in the equations of motion formulation [5]. The preserved gauge invariance was the cornerstone, as in quantum electrodynamics although in a much more sophisticated way, of the proof by ‘t Hooft and Veltman that the mechanism of Brout, Englert and Higgs (BEH) yields a renormalizable theory [6]. The renormalizability made entirely consistent the electroweak theory [7], proposed by Weinberg in 1967, related to a group theoretical model of Glashow and to the dynamics of the BEH mechanism.

I shall review the basic concepts leading to the construction of gauge vector boson masses, discuss further developments and their role in contemporary physics.

2 Spontaneous broken global symmetry

2.1 Broken symmetry in statistical physics

Consider a condensed matter system, whose dynamics is invariant under a continuous symmetry. As the temperature is lowered below a critical one, the symmetry may be reduced by the appearance of an ordered phase. The breakdown of the original symmetry is always a discontinuous event at the phase transition point but the order parameters may set in continuously as a function of temperature. In the latter case the phase transition is second order. Symmetry breaking
by a second order phase transition occurs in particular in ferromagnetism, superfluidity and superconductivity. I first discuss in detail the ferromagnetic phase transition which illustrates three general features of global SBS: ground state degeneracy, the appearance of a “massless mode” when the dynamics is invariant under a continuous symmetry, and the occurrence of a “massive mode”.

\[ V = \lim_{N \to \infty} \frac{G}{N} \]

![Fig.1. Effective potential of a Heisenberg ferromagnet.](image)

In absence of external magnetic fields and of surface effects, a ferromagnetic substance below the Curie point displays a global orientation of the magnetization, while the dynamics of the system is clearly rotation invariant. This is SBS. The features of SBS are neatly illustrated by taking the Heisenberg model with spin 1/2 defined by the Hamiltonian

\[ H = H_0 - \sum_{i=1}^{N} S_i^z h^z \]

in the limit \( N \to \infty \). Here \( v_{ij} \) is the exchange potential between the spins \( \vec{S}_i, \vec{S}_j \) located at the lattice sites \( i \) and \( j \). \( \vec{S}_i = \vec{\sigma}_i/2 \) where the components of \( \vec{\sigma}_i \) are the Pauli matrices \( \{\sigma_i^x, \sigma_i^y, \sigma_i^z\} \) ([\( \vec{\sigma}_i, \vec{\sigma}_j \) = 0 for \( i \neq j \)]) and \( \vec{h} = h^z \vec{1}_z \) an applied magnetic field in the \( z \)-direction. Define the average magnetization \( \vec{M} = M^z \vec{1}_z \), \( M^z = \lim_{N \to \infty} (1/N) \langle \sum_i S_i^z \rangle_{T, h^z} \). The effective potential \( V \) is the Gibbs potential per spin \( G/N = (E - TS)/N + \vec{M} \cdot \vec{h} \). It is given by \( V = -kT \lim_{N \to \infty} (1/N) \ln Z_N + \vec{M} \cdot \vec{h} \), where \( Z_N = \text{Tr} \exp(-H/kT) \). Its behavior is characteristic of second order phase transitions with spontaneous broken symmetry [1]. Above the Curie point,
V has a single minimum at $\vec{M} = 0$. This minimum flattens at at $T = T_c$ and two symmetric minima appear for $T < T_c$ in the $VM^z$-plane. This would be the whole story for a system with discrete symmetry, such as the Ising model obtained from the Hamiltonian Eq. (2.1) by retaining only the $z$-component of the spin. The discrete $Z_2$ symmetry of the action would be spontaneously broken below the Curie point when, as $h^z \to 0$, the system ends in one of the equivalent minima in the $VM^z$-plane exhibited in Fig 1.

But, when $\vec{h} = 0$, the Hamiltonian $H_0$ is invariant under the full rotation group $SO(3)$. This continuous symmetry implies that the thermodynamics of the ferromagnetic phase does not depend on the orientation of the magnetization. The effective potential $V(T, M)$ only depends on the norm $M$ of the magnetization vector $\vec{M}$. Hence the equivalent minima are not only doubly degenerate but span the full 2-sphere, that is the coset space of $SO(3)/U(1)$. By selecting an orientation at a given minimum, the system in the ferromagnetic phase spontaneously breaks the $SO(3)$ symmetry down to $U(1)$.

Consider now the system at $T = 0$. The magnetization vector $\vec{M}$ has a quantum mechanical interpretation. As $h^z \to 0$ the magnetization of the ground state points in the $z$-direction. It is the symmetric “all spin up” state $|0\rangle = |+++\cdots\rangle$ where the normalized spin states of the individual spins $|+\rangle$ are quantized in the $z$-direction. One easily verifies that $|0\rangle$ is an eigenstate of $H$ and one has

$$S^z \equiv \sum_i S_i^z \quad \langle 0|S^z|0\rangle = \langle 0|S^y|0\rangle = 0 \quad \langle 0|S^z|0\rangle = NM = \frac{N}{2}. \tag{2.2}$$

The operators $S^\alpha (\alpha = x, y, z)$ obey $[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma$ and are generators of the rotation group. One may construct the rotated ground states from them. The state $|\theta\rangle$ obtained from $|0\rangle$ by rotating an angle $\theta$ about the $x$-axis is

$$|\theta\rangle = e^{iS^x\theta}|0\rangle. \tag{2.3}$$

The states $|\theta\rangle$ and $|0\rangle$ are degenerate since $[\mathcal{H}, S^z] = 0$ and one gets from the commutation relations

$$\langle \theta|S^x|\theta\rangle = 0 \quad \langle \theta|S^y|\theta\rangle = NM \sin \theta \quad \langle \theta|S^z|\theta\rangle = NM \cos \theta. \tag{2.4}$$

In this way, the classical notion of the “arrow” in $\vec{M}$ is given by the expectation value of the operator $\vec{S}$ in the different rotated ground states.

Consider now the two distinct ground states, $|0\rangle$ and $|\theta\rangle$. One has

$$\langle 0|\theta\rangle = \langle 0|e^{iS^x\theta}|0\rangle = \langle 0|\prod_{i=1}^N e^{i(\sigma_i^x/2)\theta}|0\rangle.$$

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It is easy to verify that the orthogonality in the limit $N \to \infty$ still holds between excited states involving finite numbers of “wrong spins” and hence the Hilbert space of the system splits into an infinite number of orthogonal Hilbert subspaces built upon the degenerate ground states labeled by $\vec{M}$. If $N$ is large but finite, the orthogonality condition remains approximatively valid if $\theta > O(1/\sqrt{N})$. This is the expected range of quantum fluctuations around a classical configuration of $N$ aligned spins. One may thus interpret the stability of a particular ground state as due to its classical nature, as corroborated by the computation of the expectation values. This fact will be important for the understanding of the difference between global SBS and its local counterpart.

A feature related to ground state degeneracy under the rotation group is the onset of a normal mode whose energy vanishes at zero wavevector $\vec{q}$. To see this, let us rewrite the Hamiltonian Eq.(2.1) in terms of Fourier components. Defining

\[
\vec{S}(\vec{q}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \vec{S}_i e^{i\vec{q} \cdot \vec{r}_i} \quad v(\vec{q}) = \frac{1}{N} \sum_{i \neq j} v_{ij} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)} ,
\]

Eq.(2.1) yields, at $\hbar z = 0$,

\[
H_0 = -2 \sum_{\vec{q}} v(\vec{q}) \vec{S}(\vec{q}) \cdot \vec{S}(-\vec{q}) .
\]

Defining $S^\pm = S^x \pm iS^y$ and taking into account the relation $S^z(\vec{q})|0\rangle = \sqrt{N} \delta_{\vec{q},0}|0\rangle$ one gets, using the commutation relations of the rotation generators

\[
[H_0, S^-(\vec{q})]|0\rangle = 2[v(0) - v(\vec{q})]S^-(\vec{q})|0\rangle .
\]

Eq.(2.8) reveals a spin wave with energy $\omega$ related to the wavevector $\vec{q}$ by the dispersion relation

\[
\omega = 2[v(0) - v(\vec{q})].
\]

Its energy vanishes in the limit $\vec{q} \to 0$. This is a consequence of the ground state degeneracy. The excitation is indeed created by the operator $S^-(\vec{q})$ acting on the state $|0\rangle$, which in the limit $\vec{q} \to 0$ is proportional to generators rotating the degenerate ground states, and therefore cannot carry energy. In relativistic field theory, an excitation whose energy vanishes as $\vec{q} \to 0$ characterizes a massless mode and the spin wave may be viewed here as the “massless” mode associated with spontaneous broken rotational invariance. It is the ancestor of the NG boson.
that will be discussed in the next section in the context of field theory. Note that if the external magnetic field $h^z$ is non-zero, Eq. (2.9) gets an additional term in the RHS linear in $h^z$, and hence a “mass” term.

The effective potential below the Curie point, depicted in Fig.1, summarizes the essential features of SBS. At a given minimum, say, $\vec{M} = M z \vec{1}$, the curvature of the effective potential measures the inverse susceptibility which determines the energy for infinite wavelength fluctuations, in other words, the “mass”. The inverse susceptibility is zero in directions transverse to the order parameter and positive in the longitudinal direction. One thus recovers, even at non-zero temperature, the massless transverse mode characteristic of broken continuous symmetry and we learn that there is also a (possibly unstable) “massive” longitudinal mode which corresponds to fluctuations of the order parameter and which is present in any spontaneous broken symmetry, continuous or even discrete. Such generically massive mode characterize any ordered structure, be it the broken symmetry phase in statistical physics, the vacuum of the global SBS in field theory presented in Section 2.2 or of the Yang-Mills asymmetric phase discussed in Section 3. The “SBS mass” of the longitudinal mode measures the rigidity of the ordered structure.

Consider now some other second order phase transitions.

Superfluidity in He4 occurs when below a critical temperature a condensate forms out of zero momentum states of the bosonic atoms. This phenomenon is related to the Bose-Einstein condensation of a free boson gas, although interactions reduce the number of particles in the ground state condensate to a finite fraction of the $N$ atoms of the system. The condensation can be described by giving to the creation (or destruction) operator $a_0^+$ at zero momentum a non vanishing expectation value. The $U(1)$ symmetry of the quantum phase is then spontaneously broken by selecting a phase. As in ferromagnetism, this results in a degeneracy of the ground state and in the existence of a concomitant massless mode which here are superfluid sound waves.

A $U(1)$ broken symmetry also occurs in superconductivity through condensation of bosonic Cooper pairs bound states of zero momentum spin singlets $b_{k}^{\uparrow} = a_{k}^{\uparrow} a_{-k}^{\downarrow}$. These are formed because of an attractive force in the vicinity of the electron Fermi surface induced by phonon exchange. Cooper pair condensation leads to a gap at the Fermi surface \[8\]. For neutral superconductors, this gap hosts a massless mode and one recovers the general features of SBS. But the presence of the long-range coulomb interaction modifies the picture. The massless mode disappears by being absorbed by electron density oscillations, namely by the longitudinal massive plasma mode \[9\] \[10\]. The penetration depth $1/m_v$ of a magnetic field is a manifestation
of a transverse mass \( m_v \). The field is either localized at the boundary in the Meissner effect if \( m_s < O(m_v) \) (Type I superconductors), or channeled into flux tubes if \( m_s > O(m_v) \) (Type II superconductors). Here \( m_s \) is the SBS mass which measures the rigidity of the condensate. The appearance of these masses are precursors of the asymmetric Yang-Mills phase presented in Section 3. In superconductivity the transverse and longitudinal masses are different and of different dynamical origin. While the transverse mass is due to the condensate, the longitudinal one uses the total electron density and is also present in the normal phase. In the relativistic theory of Section 3 there will be, due to the Lorentz invariance, only one photon mass.

### 2.2 Broken continuous symmetry in field theory

Spontaneous symmetry breaking was introduced in relativistic quantum field theory by Nambu and Jona-Lasinio in analogy with the BCS theory of superconductivity. The problem studied by Nambu \cite{nambu1961} and Nambu and Jona-Lasinio \cite{nambu1961b} is the spontaneous breaking of chiral symmetry induced by a fermion condensate \( \langle \bar{\psi} \psi \rangle \neq 0 \). They consider massless fermions interacting through the four Fermi interaction \( g[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] \) that is invariant under the \( U(1) \) chiral group transformation \( \psi \to \exp(i\gamma_5\alpha)\psi \). This is a global symmetry as \( \alpha \) is constant in space-time. Although no fermion mass may arise perturbatively, summing up an infinity of diagrams allow the self-energy to acquire self-consistently a non-zero contribution from \( \langle \bar{\psi}\psi \rangle \). This yields a fermion mass \( m \) and an eigenvalue equation \( g = f(m/\Lambda) \) where \( \Lambda \) is a ultraviolet cut-off. The eigenvalue equation in turn implies the existence of a massless pseudoscalar mode coupled to the axiovector current. This is a consequence of the chiral Ward identity relating the axial vector vertex \( \Gamma_{\mu 5} \) to the self energy \( \Sigma = A(p^2)\gamma^\mu p_\mu + M(p^2) \) in a chiral invariant theory,

\[
q^\mu \Gamma_{\mu 5} = \gamma_5 \Sigma(p + \frac{q}{2}) + \Sigma(p - \frac{q}{2}) \gamma_5. \tag{2.10}
\]

As \( q_\mu \to 0 \), the form factor \( A(p^2) \) drops out of Eq.(2.10) and

\[
\lim_{q_\mu \to 0} q_\mu \Gamma_{\mu 5} = 2M(p^2)\gamma_5 \quad \Gamma_{\mu 5} \to \frac{2M(p)\gamma_5 q_\mu}{q^2}. \tag{2.11}
\]

The pole at \( q^2 = 0 \) in Eq.(2.11) signals the appearance of the pseudoscalar boson.

The model is not intended to be realistic but sets the scene for more general considerations. The pion is identified with the massless mode of spontaneous broken chiral invariance. It gets its tiny mass (on the hadron scale) from a small explicit breaking of the symmetry, just as a small external magnetic field \( h_z \) imparts a small gap in the spin wave spectrum. This interpretation of the pion mass constituted a breakthrough in our understanding of strong interaction physics.
General features of SBS in relativistic quantum field theory were further analyzed by Goldstone, Salam and Weinberg \[11\] \[12\]. Here, symmetry is broken by non vanishing vacuum expectation values of scalar fields. The method is designed to exhibit the appearance of a massless mode out of the degenerate vacuum and does not really depend on the significance of the scalar fields. The latter could be elementary or represent collective variables of more fundamental fields, as would be the case in the original Nambu model. Compositeness affects details of the model considered, such as the behavior at high momentum transfer and the stability of the SBS massive scalar, but not the existence of the massless excitations encoded in the degeneracy of the vacuum.

Let us illustrate the occurrence of this massless Nambu-Goldstone boson in a simple model of a complex scalar field with $U(1)$ symmetry \[11\]. The Lagrangian density,

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - V(\phi^* \phi) \quad \text{with} \quad V(\phi^* \phi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 , \quad \lambda > 0 , \quad (2.12)$$

is invariant under the $U(1)$ group $\phi \to e^{i\alpha} \phi$. The global $U(1)$ symmetry is broken by a vacuum expectation value of the $\phi$-field given, at the classical level, by the minimum of $V(\phi^* \phi)$. Writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, one may choose $\langle \phi_2 \rangle = 0$. Hence $\langle \phi_1 \rangle^2 = \mu^2/\lambda$ and we select, say, the vacuum with $\langle \phi_1 \rangle$ positive. The potential $V(\phi^* \phi)$ is depicted in Fig.2. It is similar to the effective potential below the ferromagnetic Curie point shown in Fig.1 and leads to similar consequences.

![Fig.2. Spontaneous breaking of a continuous symmetry by scalar fields.](image)

In the unbroken vacuum the field $\phi_1$ has negative mass and acquires a positive mass $2\mu^2$ in the broken vacuum where the field $\phi_2$ is massless. The latter is the NG boson of broken $U(1)$
symmetry and is the analog of the massless spin wave mode in ferromagnetism, corresponding to the vanishing of the inverse transverse susceptibility. The massive scalar describes the fluctuations of the order parameter \( \langle \phi_1 \rangle \) and is the analog of the SBS massive mode in the ordered phase of a many-body system, encoded in a non-vanishing inverse longitudinal susceptibility.

The origin of the massless NG boson is, as in the ferromagnetism phase, a consequence of the vacuum degeneracy. The vacuum characterized by the order parameter \( \langle \phi_1 \rangle \) is rotated into an equivalent vacuum by an operator proportional to the field \( \phi_2 \) at zero space momentum. Such rotation costs no energy and thus the field \( \phi_2 \) at space momenta \( q \to 0 \) has \( q_0 = 0 \) in the equations of motion, and hence zero mass.

This Goldstone theorem can be formalized and generalized by noting that the conserved Noether current \( J_\mu = \phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1 \) gives a charge \( Q = \int J_0 d^3x \). The operator \( \exp (i\alpha Q) \) rotates the vacuum by an angle \( \alpha \). In the classical limit, this charge is, around the chosen vacuum, \( Q = \int \langle \phi_1 \rangle \partial_0 \phi_2 d^3x \). More generally, \( \langle [Q, \phi_2] \rangle = i\langle \phi_1 \rangle \) is non zero in the chosen vacuum. This implies that the propagator \( \partial_\mu \langle T J_\mu(x) \phi_2(x') \rangle \) cannot vanish at zero four-momentum \( q \) because its integral over space-time is precisely \( \langle [Q, \phi_2] \rangle \). Expressing the propagator in terms of Feynman diagrams we indeed see that the \( \phi_2 \)-propagator must have a pole at \( q^2 = 0 \). The field \( \phi_2 \) is the massless NG boson. The proof is immediately extended to spontaneous global symmetry breaking of a semi-simple Lie group \( G \) to \( H \). Let \( \phi^A \) be scalar fields spanning a representation of \( G \) generated by the (antihermitian) matrices \( T^{aBA} \). If the \( G \)-invariant potential has minima for non vanishing \( \phi^A \)'s, the global symmetry is broken and the vacuum is degenerate under \( G \)-rotations. The conserved charges are \( Q^a = \int \partial_\mu \phi^B T^{aBA} \phi^A d^3x \). The propagators of the fields \( \phi^B \) such that \( \langle [Q^a, \phi^B] \rangle = T^{aBA} \langle \phi^A \rangle \neq 0 \) have a NG pole at \( q^2 = 0 \) and the NG bosons live in the coset \( G/H \).

### 3 The asymmetric Yang-Mills phase

#### 3.1 Global to local symmetry

The global \( U(1) \) symmetry in Eq.(2.12) is extended to a local one \( \phi(x) \to e^{i\alpha(x)} \phi(x) \) by introducing a vector field \( A_\mu(x) \) transforming as \( A_\mu(x) \to A_\mu(x) + (1/e)\partial_\mu \alpha(x) \). The corresponding Lagrangian density is

\[
\mathcal{L} = D^\mu \phi^* D_\mu \phi - V(\phi^* \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

(3.13)

with covariant derivative \( D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

Local invariance under a semi-simple Lie group \( G \) is realized by extending the Lagrangian
Eq. (3.13) to incorporate non-abelian Yang-Mills vector fields $A_\mu^a$

$$\mathcal{L}_G = (D^\mu \phi)^A (D\mu \phi)^A - V - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

\begin{equation}
(D\mu \phi)^A = \partial_\mu \phi^A - e A_\mu^a T^{AB} \phi^B 
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e f^{abc} A^b_\mu A^c_\nu.
\end{equation}

Here, $\phi^A$ belongs to the representation of $\mathcal{G}$ generated by $T^{AB}$ and the potential $V$ is invariant under $\mathcal{G}$.

The local abelian or non-abelian gauge invariance of Yang-Mills theory hinges apparently upon the massless character of the gauge fields $A_\mu$, hence on the long-range character of the forces they transmit, as the addition of a mass term for $A_\mu$ in the Lagrangian Eq. (3.13) or (3.14) destroys gauge invariance. But short-range forces such as the weak interaction forces, seem to be as fundamental as the electromagnetic ones despite the apparent departure from exact conservation laws. To reach a basic description of such forces one is tempted to link this fact to gauge fields masses arising from spontaneous broken symmetry. However the problem of SBS is different for global and for local symmetries.

To pinpoint the difference, let us break the symmetries explicitly. To the Lagrangian Eq. (2.12) we add, in analogy with the magnetic field $h^z$ in Eq. (2.1), the term

\begin{equation}
\phi^* h + \phi^* h,
\end{equation}

where $h, h^*$ are constant in space time. Let us take $h$ real. The presence of the field $h$ breaks explicitly the global $U(1)$ symmetry and the field $\phi_1$ develops an expectation value. When $h \to 0$, the symmetry of the action is restored but, when the symmetry is broken by a minimum of $V(\phi \phi^*)$ at $|\phi| \neq 0$, we still have $\langle \phi_1 \rangle \neq 0$. The tiny $h$-field has simply picked up one of the degenerate vacua in perfect analogy with the infinitesimal magnetic field which orients the magnetization of a ferromagnet. As in the latter, the chosen vacuum is stable because it is defined by a classical configuration of the fields and the Hilbert space breaks up into an infinite set of disjoint spaces. The degeneracy of the vacuum can be put into evidence by changing the phase of $h$; in this way, we can reach in the limit $h \to 0$ any $U(1)$ rotated vacuum.

When the symmetry is extended from global to local, one can still break the gauge symmetry by an external “magnetic” field, say a mass term. However in the zero mass limit, no preferred vacuum exists. In contradistinction to the global symmetry case, no energy is needed to change the relative orientation of neighboring “spins”, that is of neighboring configurations of the scalar fields in group space. As a consequence, no classical configuration is available to protect a degenerate vacuum against quantum fluctuations.

This fact has two related consequences.
First, the vacuum is generically non degenerate and points in no particular direction in group space\(^1\). In this sense, local gauge symmetry cannot be spontaneously broken\(^2\).

As a consequence, there cannot be massless NG bosons. These correspond to relative orientation of neighboring “spins” and are now simply gauge transformations. A formal proof of the failure of the Goldstone theorem in presence of gauge fields, in relativistic quantum field theory, was given by Higgs\[^14\].

Recalling that the explicit presence of a gauge vector mass breaks gauge invariance, we are thus faced with a dilemma. How can gauge fields acquire mass without breaking the local symmetry?

In perturbation theory, gauge invariant quantities are evaluated by choosing a particular gauge. One imposes the gauge condition by adding to the action a gauge fixing term and the corresponding Fadeev-Popov ghosts, and gauge invariance is ensured by summing over subsets of graphs satisfying the Ward Identities.

Consider the Yang-Mills theory defined by the Lagrangian Eq. (3.14). To exhibit the similarities and the differences between spontaneous breaking of a global symmetry and its local symmetry counterpart, it is convenient to choose a gauge which preserves Lorentz invariance and a residual global \(G\) symmetry. This can be achieved by adding to the Lagrangian a gauge fixing term \((2\eta)^{-1}\partial_\mu A^{a}\partial_\nu A^{a}\nu\). The gauge parameter \(\eta\) is arbitrary and is not observable.

In such gauges, the global symmetry can be spontaneously broken for suitable potential \(V\), by non zero expectation values \(\langle \phi^A \rangle\) of scalar fields. In Fig.3 we have represented motions of this parameter in the spatial \(q\)-direction and in a direction \(B\) of the coset space \(G/H\) where \(H\) is the unbroken subgroup. Fig.3a pictures the spontaneously broken vacuum of the gauge fixed Lagrangian. Fig.3b and Fig.3c mimic motions in the coset with decreasing wavelength \(\lambda\). Clearly, as \(\lambda \to \infty\), such motions can only induce global rotations in the internal space. In absence of gauge fields, they would give rise, as in spontaneously broken global continuous symmetries, to massless NG modes generating the coset in the limit \(\lambda = \infty\). In a gauge theory, transverse fluctuations of \(\langle \phi^A \rangle\) are just local rotations in the internal space and are unobservable gauge motions. Hence the would-be NG bosons induce only gauge transformations and their excitations disappear from the physical spectrum.

\(^1\)Note that for global symmetry breaking, one can always choose a linear combination of degenerate vacua which is invariant under, say, the \(U(1)\) symmetry. This choice has no observable consequences because of the splitting into orthogonal Hilbert spaces.

\(^2\)For a detailed proof, see reference [13].
Fig. 3. The disappearance of the massless NG boson in a gauge theory.

But what makes local internal space rotations unobservable in a gauge theory is precisely the fact that they can be absorbed by the Yang-Mills fields. The absorption of the NG fields renders massive the gauge fields living in the coset $G/H$ by transferring to them their degrees of freedom which become longitudinal polarizations.

We shall see in the next sections how these considerations are realized in relativistic quantum field theory and give rise to vector masses in the coset $G/H$, leaving long-range forces only in a subgroup $H$ of $G$. Despite the unbroken local symmetry, the group $G$ appears broken to its subgroup $H$ in the asymptotic state description of field theory, and I shall therefore often term SBS or asymmetric such a Yang-Mills phase. The onset of SBS will be described in detail mostly in lowest order perturbation theory around the self-consistent vacuum, both in the field-theoretic [4] and in the equation of motion [5] formulations. This contains already the basic ingredients of the phenomenon and comparison between the two methods gives some insight on the renormalization issue.

3.2 The field theoretic approach

$\alpha$) Breaking by scalar fields

Let us first examine the abelian case as realized by the complex scalar field $\phi$ exemplified in Eq. (3.13).

In the covariant gauges, the free propagator of the field $A_\mu$ is

$$ D^0_{\mu\nu} = \frac{g_{\mu\nu} - q_\mu q_\nu / q^2}{q^2} + \eta \frac{q_\mu q_\nu / q^2}{q^2}, \quad (3.17) $$

where $\eta$ is the gauge parameter.
In absence of symmetry breaking, the lowest order contribution to the self-energy, arising from the covariant derivative terms in Eq. (3.13), is given by the one-loop diagrams of Fig. 4. The self-energy (suitably regularized) takes the form of a polarization tensor

$$\Pi_{\mu\nu} = (g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2),$$  \hspace{1cm} (3.18)

where the scalar polarization $\Pi(q^2)$ is regular at $q^2 = 0$, leading to the gauge field propagator

$$D_{\mu\nu} = \frac{g_{\mu\nu} - q_\mu q_\nu / q^2}{q^2[1 - \Pi(q^2)]} + \eta \frac{q_\mu q_\nu / q^2}{q^2}. \hspace{1cm} (3.19)$$

The polarization tensor in Eq. (3.18) is transverse and hence does not affect the gauge parameter $\eta$. The transversality of the polarization tensor reflects the gauge invariance of the theory and, as we shall see below, the regularity of the polarization scalar signals the absence of symmetry breaking. This guarantees that the $A_\mu$-field remains massless.

![Fig.4. Lowest order vacuum polarization graphs in absence of SBS. Abelian gauge theory.](image)

Symmetry breaking adds tadpole diagrams to the previous ones. To see this write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \hspace{1cm} \langle \phi_1 \rangle \neq 0. \hspace{1cm} (3.20)$$

The scalar field $\phi_1$, whose expectation value plays the role of an order parameter in the gauge considered and whose fluctuations have a gauge invariant SBS mass, is often called the Higgs field and its fluctuations the Higgs boson. This massive mode is not a specific property of the BEH mechanism but is a necessary concomitant of any SBS structured vacuum, as pointed out in Section 2.1. The would-be NG-field is $\phi_2$. The additional diagrams are depicted in Fig. 5.

---

3The transversality of polarization tensors is a consequence of the Ward Identities alluded to in the preceding section.
The polarization scalar \( \Pi(q^2) \) in Eq. (3.18) acquires a pole from the tadpole contribution

\[
\Pi(q^2) = \frac{e^2 \langle \phi_1 \rangle^2}{q^2}, \tag{3.21}
\]

and, in lowest order perturbation theory, the gauge field propagator becomes

\[
D_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu / q^2}{q^2} + \eta q_\mu q_\nu / q^2, \tag{3.22}
\]

which shows that the \( A_\mu \)-field gets a mass

\[
\mu^2 = e^2 \langle \phi_1 \rangle^2. \tag{3.23}
\]

The generalization of Eqs. (3.18) and (3.21) to the non-abelian case described by the action Eq. (3.14) is straightforward. One gets from the graphs depicted in Fig.6,

\[
\Pi^{ab}_{\mu\nu} = (g_{\mu\nu} q^2 - q_\mu q_\nu / q^2) \Pi^{ab}(q^2), \tag{3.24}
\]

\[
\Pi^{ab}(q^2) = \frac{e^2 \langle \phi^*B \rangle T^a BC T^b CA \langle \phi^A \rangle}{q^2}. \tag{3.25}
\]

Eq. (3.25) defines the mass matrix

\[
(\mu^2)^{ab} = e^2 \langle \phi^*B \rangle T^a BC T^b CA \langle \phi^A \rangle. \tag{3.26}
\]

In terms of its non-zero eigenvalues \( (\mu^2)^a \), the propagators of the massive gauge vectors take the same form as Eq. (3.22),

\[
D^a_{\mu\nu} = \frac{g_{\mu\nu} - q_\mu q_\nu / q^2}{q^2 - (\mu^2)^a} + \eta q_\mu q_\nu / q^2. \tag{3.27}
\]
The gauge invariance is expressed, as it was in absence of symmetry breaking, through the transversality of the polarization tensors Eqs. (3.18) and (3.24). The singular $1/q^2$ contributions to the polarization scalars Eqs. (3.21) and (3.25) preserve transversality and yield gauge invariant masses for the gauge bosons. They stem from the long-range NG boson fields. The latter are, as such, unobservable gauge terms but their absorption in the gauge field propagators transfers the degrees of freedom of the would-be NG bosons to the third degree of polarization of the massive vectors. Indeed, on the mass shell $q^2 = (\mu^2)^a$, one easily verifies that the numerator in the transverse propagator in Eq. (3.27) is

$$g_{\mu\nu} - \frac{g_{\mu\nu}}{q^2} = \sum_{\lambda=1}^{3} e^{(\lambda)}_{\mu}e^{(\lambda)}_{\nu} = (\mu^2)^a,$$

where the $e^{(\lambda)}_{\mu}$ are three polarization vectors orthonormal in the rest frame of the particle.

In this way, the would-be NG bosons generate massive propagators for the gauge fields in $G/H$. Long-range forces only survive in the subgroup $H$ of $G$ which leaves invariant the non-vanishing expectation values $\langle \phi^A \rangle$.

Note that the explicit form of the scalar potential $V$ does not enter the computation of gauge field propagators which depend only on the expectation values at its minimum. This is because trilinear terms arising from covariant derivatives, which yields the second graphs of Fig.5 and Fig.6, can only couple the tadpoles to other scalar fields through group rotations and hence couple them only to the would-be NG bosons. These are the eigenvectors with zero eigenvalue of the scalar mass matrix given by the quadratic term in the expansion of the potential $V$ around its minimum. Hence the massive scalars decouple from the tadpoles at the tree level considered above.

$\beta)$ Dynamical symmetry breaking

The symmetry breaking giving mass to gauge vector bosons may arise from the fermion condensate breaking chiral symmetry. This is illustrated by the following chiral invariant Lagrangian

$$\mathcal{L} = \mathcal{L}^F_0 - e_V \bar{\psi}\gamma_\mu\psi V_\mu - e_A \bar{\psi}\gamma_\mu\gamma_5\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}(V) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}(A).$$

Here $F_{\mu\nu}(V)$ and $F_{\mu\nu}(A)$ are abelian field strength for $U(1) \times U(1)$ symmetry. Chiral anomalies are eventually canceled by adding in the required additional fermions.

As in global SBS, the Ward identity for the chiral current Eq. (2.10) shows that if the fermion self-energy $\gamma^{\mu}p_\mu A(p^2) - M(p^2)$ acquires a non vanishing $M(p^2)$ term, thus a dynamical mass,
the axial vertex $\Gamma_{\mu 5}$ develops a pole at $q^2 = 0$. In leading order in $q$, we get as in Eq. (2.11)
\[ \Gamma_{\mu 5} \to 2M(p)\gamma_5 \frac{q_\mu}{q^2}. \] (3.30)

The pole in the vertex function induces a pole in the suitably regularized gauge invariant polarization tensor $\Pi_{\mu\nu}^{(A)}$ of the axial vector field $A_\mu$ depicted in Fig. 7
\[ \Pi_{\mu\nu}^{(A)} = e^2 (g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi^{(A)}(q^2), \] (3.31)
with
\[ \lim_{q^2 \to 0} q^2 \Pi^{(A)}(q^2) = \mu^2 \neq 0. \] (3.32)
The field $A_\mu$ acquires in this approximation a gauge invariant mass $\mu$.

This example illustrates the fact that the transversality of the polarization tensor used in the quantum field theoretic approach to mass generation is a consequence of a Ward identity. This is true whether vector masses arise through fundamental scalar or through fermion condensate. The generation of gauge invariant masses is therefore not contingent upon the “tree approximation” used to get the propagators Eqs. (3.22) and (3.27). It is a consequence of the $1/q^2$ singularity in the vacuum polarization scalars Eqs. (3.21), (3.24) or (3.32) which comes from the would-be NG boson contribution.

### 3.3 The equation of motion formulation

The BEH mechanism can be understood in terms of equations of motions which illustrate nicely the fate of the NG bosons. This is shown below for the abelian case described by the action Eq. (3.13).

Taking as in Eq. (3.20), the expectation value of the scalar field to be $\langle \phi_1 \rangle$, and expanding the NG field $\phi_2$ to first order, one gets from the action Eq. (3.13) the classical equations of motion
\[ \partial^\mu \{ \partial_\mu \phi_2 - e\langle \phi_1 \rangle A_\mu \} = 0, \] (3.33)
\[ \partial_\nu F^{\nu\mu} = e\langle \phi_1 \rangle \{ \partial^\mu \phi_2 - e\langle \phi_1 \rangle A^\mu \}. \] (3.34)

\[ ^4 \text{The validity of the approximation, and in fact of the dynamical approach, rests on the high momentum behavior of the fermion self energy, but this problem will not be discussed here.} \]
Defining
\[ B_\mu = A_\mu - \frac{1}{e \langle \phi_1 \rangle} \partial_\mu \phi_2 \quad \text{and} \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}, \tag{3.35} \]
one gets
\[ \partial_\mu B^\mu = 0 \quad \partial_\nu G^{\mu\nu} + e^2 \langle \phi_1 \rangle^2 B^\mu = 0. \tag{3.36} \]

Eq. (3.36) shows that \( B_\mu \) is a massive vector field with mass squared \( e^2 \langle \phi_1 \rangle^2 \) in accordance with Eq. (3.23). As pointed out in the previous section, the vector boson mass does not depend explicitly on the scalar potential, but only on the value of \( \langle \phi_1 \rangle \) at its minimum.

The value of \( \langle \phi_1 \rangle \) and the mass of the massive scalar boson are determined by the potential and are of course not affected by the gauging. For the potential Eq. (3.23), one recovers from the equation of motion for the massive scalar
\[ \left\{ \partial^2 - V''(\langle \phi_1 \rangle) \right\} \delta \phi_1 = 0, \tag{3.37} \]
the mass \( 2\mu^2 \) using \( \langle \phi_1 \rangle^2 = \mu^2/\lambda. \)

In this formulation, we see clearly from Eq. (3.35) how the NG boson is absorbed into a redefined massive vector field. The disappearance of the NG boson was further analyzed in reference [15]. In the gauge defined by Eq. (3.36), the field \( B_\mu \), which contains only the physical degrees of freedom of the massive vector, does appear. This is the “unitary gauge” of the theory. In contradistinction, the field theoretic approach introduces a spurious \( 1/q^2 \) pole in the polarization Eq. (3.21), which is not observable. The comparison between these two different approaches to massive gauge vector boson masses contains the germ of the renormalizability of the BEH mechanism, as will now be discussed.

### 3.4 The renormalization issue

The massive vector propagator Eq. (3.27) differs from a conventional free massive propagator in two respects. First the presence of the unobservable longitudinal term reflects the arbitrariness of the gauge parameter \( \eta \). Second the NG pole at \( q^2 = 0 \) in the transverse projector \( g_{\mu\nu} - q_\mu q_\nu / q^2 \) is unconventional. Its significance is made clear by expressing the propagator of the \( A_\mu \) field in Eq. (3.27) as (putting \( \eta \) to zero)
\[ D_{\mu\nu}^a = g_{\mu\nu} - q_\mu q_\nu / q^2 = g_{\mu\nu} - q_\mu q_\nu / (\mu^2)^a + \frac{1}{(\mu^2)^a} q_\mu q_\nu / q^2. \tag{3.38} \]
The first term in the right hand side of Eq. (3.38) is the conventional massive vector propagator. It may be viewed as the (non-abelian generalization of the) free propagator of the \( B_\mu \)-field.
defined in Eq. (3.35), while the second term is a pure gauge propagator due to the NG boson 
\( [(1/e \langle \phi_1 \rangle) \partial_\mu \phi_2 \) in Eq. (3.35) \] which converts the gauge field \( A_\mu \) into the massive vector field \( B_\mu \).

The propagator Eq. (3.27) which appeared in the field theoretic approach contains thus, in
the covariant gauges, the transverse projector \( g_{\mu\nu} - q_\mu q_\nu / q^2 \) in the numerator of the massive
gauge field \( A_\mu \) propagator. This is in sharp contradistinction to the numerator \( g_{\mu\nu} - q_\mu q_\nu / (\mu^2)^a \)
characteristic of the conventional massive vector field \( B_\mu \) propagator. It is the transversality of
the polarization tensor in covariant gauges, which led in the tree approximation to the transverse
projector in Eq. (3.27). As mentioned above, the transversality of the polarization tensor is a
consequence of a Ward identity and therefore does not rely on the tree approximation. This fact
is already clear from the dynamical example Eq. (3.31) but was proven in more general terms in
a subsequent publication\(^5\). The importance of this fact is that transversality in covariant
gauges determines the power counting of irreducible diagrams. It is then straightforward to verify
that the quantum field theory formulation has the required power counting for a renormalizable
field theory. On this basis it was suggested that it indeed was renormalizable \(^16\).

However power counting is not enough to prove the renormalizability of a theory with local
gauge invariance. In addition, to be consistent, the theory must also be unitary, a fact which is
not apparent in “renormalizable” covariant gauges but is manifest in the “unitary gauge” defined
in the free theory by the \( B_\mu \)-field introduced in Eq. (3.35). In the unitary gauge however, power
counting requirements fail. The equivalence between the \( A_\mu \) and \( B_\mu \) free propagators, which is
only true in a gauge invariant theory where their difference is the unobservable NG propagator
appearing in Eq. (3.35), is a clue of the consistency of the BEH theory. It is of course a much
harder and subtler affair to proof that the full interacting theory is both renormalizable and
unitary. This was achieved in the work of ’t Hooft and Veltman \(^6\), which thereby established
the consistency of the BEH mechanism.

4 The unification paradigm

I first review very briefly the basic elements of the electroweak theory, one of the most brilliant
achievements of the twentieth century. Its remarkable success played an important role in the
further quest for unification which has become a paradigm in most of contemporary research on
fundamental interactions.

In the electroweak theory, the gauge group is taken to be \( SU(2) \times U(1) \) with corresponding

\(^5\) The proof given in reference \(^16\) was not complete because closed Yang-Mills loops, which would
have required the introduction of Fadeev-Popov ghosts were not included.
generators and coupling constants \( g A^a_\mu T^a \) and \( g' B_\mu Y' \). The \( SU(2) \) acts on left-handed fermions only. The scalar field \( \phi \) is a doublet of \( SU(2) \) and its \( U(1) \) charge is \( Y' = 1/2 \). Breaking is characterized by \( \langle \phi \rangle = 1/\sqrt{2} \{ 0, v \} \) and \( Q = T^3 + Y' \) generates the unbroken subgroup. \( Q \) is identified with the electromagnetic charge operator. The only residual massless gauge boson is the photon and the electric charge \( e \) is usually expressed in terms of the mixing angle \( \theta \) as 
\[
g = e / \sin \theta, \quad g' = e / \cos \theta.
\]

Using Eqs. (3.23) and (3.26) one gets the mass matrix
\[
|\mu^2| = \frac{v^2}{4}
\begin{bmatrix}
g^2 & 0 & 0 & 0 \\
0 & g^2 & 0 & 0 \\
0 & 0 & g'^2 & -gg' \\
0 & 0 & -gg' & g^2 \\
\end{bmatrix}
\]
whose diagonalization yields the eigenvalues
\[
M^2_{W^+} = \frac{v^2}{4} g^2, \quad M^2_{W^-} = \frac{v^2}{4} g'^2, \quad M^2_Z = \frac{v^2}{4} (g^2 + g'^2), \quad M^2_A = 0. \quad (4.39)
\]
This permits to relate \( v \) to the the four Fermi coupling \( G \), namely \( v^2 = (\sqrt{2}G)^{-1} \).

Although the electroweak theory has been amply verified by experiment, the existence of the massive scalar boson has, as yet, not been confirmed. It should be noted that its physics is, as previously discussed, more sensitive to the dynamical assumptions of the model than the massive vectors \( W^\pm \) and \( Z \), be it a genuine elementary field or a manifestation of a composite due to a more elaborate mechanism. Observation of its mass and width is of particular interest for further understanding of the mechanism at work.

The discovery that confinement could be found in the strong coupling limit of quantum chromodynamics based on the “color” gauge group \( SU(3) \) led to tentative Grand Unification schemes where electroweak and strong interaction could be unified in a simple gauge group \( G \) containing \( SU(2) \times U(1) \times SU(3) \) [17]. Breaking occurs through vacuum expectation values of scalar fields and unification is apparent at high energies because, while the renormalization group makes the small gauge coupling of \( U(1) \) increase logarithmically with the energy scale, the converse is true for the asymptotically free non abelian gauge groups.

Originally the BEH mechanism was conceived to unify the theoretical description of long-range and short-range forces. The success of the electroweak theory made the mechanism a candidate for further unification. Grand unification schemes, where the scale of unification is pushed close to the scale of quantum gravity effects, strengthen the believe in a still larger unification that would include gravity. This trend towards unification received a further impulse from
the developments of string theory and from its connection with eleven-dimensional supergravity. The latter is then often viewed as a classical limit of a hypothetical M-theory into which all perturbative string theories would merge to yield a comprehensive theory of “all” interactions.

Such vision may be premature. Quite apart from obvious philosophical questions raised by a “theory of everything” formulated in the present framework of theoretical physics, the transition from perturbative string theory to its M-theory generalization hitherto stumbles on the treatment of non-perturbative gravity. This might well be a hint that new conceptual elements have to be found to cope with the relation between gravity and quantum theory and which might not be directly related to the unification program.

5  Further developments : conceptual issues

Aside from, or part of, the unification program, the BEH mechanism has put into evidence concepts which may have a profound impact on further research. One of the richest sources of such concepts is the discovery by ’t Hooft and Polyakov of regular monopoles in non-abelian gauge theories. I shall review the underlying features which are present in all semi-simple Lie groups and stress their implications. Also of interest is the geometrical interpretation of the mechanism in the context of the string theory approach.

5.1 Monopoles, electromagnetic duality, confinement

In electromagnetism, monopoles can be included at the expense of introducing a Dirac string \[18\]. The latter creates a singular potential along a string ending at the monopole.

For instance to describe a point-like monopole located at \( \vec{r} = 0 \), one can take the line-singular
This potential has a singularity along the negative $z$-axis ($\theta = \pi$) where the string has been put (see Fig.8). The unobservability of the string implies that its fictitious flux be quantized according to the Dirac condition

$$e g = 2\pi n \quad n \in \mathbb{Z}. \quad (5.41)$$

In contradistinction to the string in the $U(1)$ theory, the Dirac string in non abelian gauge groups can be removed by a gauge transformation for well defined magnetic charge quantization encoded in the global structure of the group. An $SO(3)$ regular monopole was obtained by ’t Hooft and Polyakov \[19\] by breaking the symmetry to $U(1)$ by scalar fields $\phi^a$ belonging to the adjoint representation. In a point-singular limit corresponding to vanishing matter current, denoting by $A^i_a$ the space components of the isovector fields $\vec{A}_a$, it is given by

$$A^i_a = \frac{g}{4\pi} \epsilon^{ijr} r^j r^2 \phi^r_a = \frac{r^a F}{r} \quad e g = 4\pi. \quad (5.42)$$

Here the structure constants in the Yang-Mills Lagrangian are normed to $\epsilon^{abc}$ and the constant $F$ is fixed by the minimum of the scalar potential. The potential $\vec{A}_a$ in Eq.(5.42) is the “spherical” gauge transformed of the solution given in the “abelian” gauge by Eq.(5.40) with $\vec{A} = \vec{A}_3 (\vec{A}_2 = \vec{A}_1 = 0)$ and by a scalar isovector $\phi^3 (\phi^1 = \phi^2 = 0)$ constant in space. In performing the gauge transformation to the spherical gauge the Dirac string has been removed. The point singularity is smeared in solutions with non vanishing matter current to yield a topologically stable ’t Hooft-Polyakov regular monopole \[19\].

This analysis can be extended to all semi-simple Lie groups $G$. For a general Lie groups $G$, the possibility of gauging out the Dirac string depends on the global properties of $G$. Namely, the map of an infinitesimal curve surrounding the Dirac string into $G$ must be a curve continuously deformable to zero\(^6\). For sake of brevity I limit here the discussion of this condition to Yang-Mills theories for simple Lie group with scalar matter fields belonging to the adjoint representation of the group.

The full Lagrangian Eqs.(3.14), (3.15) is invariant under the group $\mathcal{G}_A = \tilde{G}/\mathbb{Z}$ where $\tilde{G}$ is the universal covering group of the adjoint group $\mathcal{G}_A$ and $\mathbb{Z}$ its center. Let the potential be such that the symmetry breaks to $\mathcal{H} = \mathcal{T}$ where $\mathcal{T}$ is a maximal abelian subgroup of $\mathcal{G}_A$. It is easily seen that the Lagrangian Eq.(3.14) admit line-singular solutions with the gauge fields $A^a_\mu(x)$ in

\(^6\)Alternatively, one may require that the Wu and Yang potentials \[21\] be gauge equivalent in the overlapping region \[20\].
abelian configurations of the type Eq. (5.40), namely
\[ \vec{A}_a = \frac{g_a}{4\pi} (1 - \cos \theta) \vec{\nabla} \varphi \quad \phi^a = \text{constant} \quad a \in T. \] (5.43)

The condition for the string to be unobservable is
\[ \exp \left(.ie g^a t_a^{(G_A)} \right) = 1, \] (5.44)
where \( t_a^{(G_A)} \) are abelian generators in a faithful representation of \( G_A \). The condition Eq. (5.44) expresses that a closed curve in space is mapped onto a closed curve in the group space of \( G_A \) starting and ending at the unit element. If this curve can be continuously shrunk to zero, the Dirac string can be removed, leaving a point-singular solution. This implies
\[ \exp \left(ie g^a t_a^{(\tilde{G})} \right) = 1, \] (5.45)
where \( t_a^{(\tilde{G})} \) is a faithful representation of \( \tilde{G} \). Except for \( G_2, F_4 \) and \( E_8 \) that have \( Z = 1 \), Eq. (5.45) yields a more stringent condition than Eq. (5.44). The closed curves in \( G_A \) which are homotopic to zero are only those which correspond to the trivial element of \( Z \), or equivalently to closed curves in \( \tilde{G} \).

All eigenvalues of \( t_a^{(\tilde{G})} \) are vectors \( \vec{m} \) of the weight lattice \( \Lambda_W \) of \( \tilde{G} \) of which the root lattice \( \Lambda_R \) generated by the simple roots \( \vec{\alpha}_i \) is a sublattice. The simple roots are normalized by the structure constants used in the Yang-Mills Lagrangian, which generate the adjoint representation of the group. The lattice \( \Lambda_R^\vee \) generated by the coroots \( \vec{\alpha}_i^\vee = 2\vec{\alpha}_i/\vec{\alpha}_i \vec{\alpha}_i \) of the simple roots \( \vec{\alpha}_i \) is dual to \( \Lambda_W \). Hence the point-singular monopoles obey the quantization condition
\[ e\vec{g} = 2\pi n^i \vec{\alpha}_i^\vee \quad n^i = \text{integer}. \] (5.46)
For an \( SO(3) \) theory with structure constants \( e^{abc} \), one recovers from Eq. (5.46) the ‘t Hooft quantization condition for the single component \( g \) of the magnetic charge
\[ eg = 4\pi n^i \quad n^i = \text{integer}. \] (5.47)

One may then in general search for regular monopole solutions by taking non-constant values for the scalar fields and hence admitting non-abelian configurations of the gauge fields.

The lattice \( \Lambda_R^\vee \) generated by the coroots is a root lattice. It is isomorphic to the original root lattice for all simple groups except the \( C_n \) and \( B_n \) series which are interchanged, as pointed out by Goddard, Olive and Nuyts [22]. The transformation \( \alpha \rightarrow \alpha^\vee \) is an involution and one has thus in addition to the previous duality relation \( \Lambda_W^\ast = \Lambda_R^\vee \) the corresponding relation \( \Lambda_R^\ast = \Lambda_W^\vee \).
Solutions of Eq. (5.44) which are not solutions of Eq. (5.45) characterize Dirac monopoles. Solution of Eq. (5.44) are on the lattice dual to $\Lambda_R$, hence their magnetic charges are given by

$$ e\vec{g}_d = 2\pi n^i \vec{m}^\vee_i \quad n^i = \text{integer} \quad , \quad e\vec{g}_d \neq 2\pi n^i \vec{a}^\vee_i . \quad (5.48) $$

If $\mathcal{H} = \mathcal{T}$, the magnetic charges Eq. (5.46) and (5.48) are well defined (up to Weyl reflections) but if $\mathcal{H}$ is larger than $\mathcal{T}$, these solutions can be continuously deformed in $\mathcal{H}$ and only some components can be defined in a topologically invariant way. If the symmetry is fully broken, topological stability is lost. However possibly regular locally stable flux tubes can be formed and retain from the Dirac quantization condition Eq. (5.48) the quantum numbers characterizing the distinct discrete conjugation classes or equivalently the center of the group.

The duality relations between $\Lambda_W$ and $\Lambda_W^\vee$ (and $\Lambda_R$ and $\Lambda_R^\vee$) was interpreted in reference [22] as an electric-magnetic duality between different gauge groups and was generalized to all groups $\mathcal{G}$ locally isomorphic to $\mathcal{G}$. A conjecture of how electromagnetic duality could be realized in a full quantized theory for the BPS limit of regular monopoles [23] was suggested by Montonen and Olive [24] and a form of the Montonen-Olive duality was displayed in $N = 2$ supersymmetric Yang-Mills theory [25].

These results give credence to the old conjecture that confinement is essentially magnetic superconductivity [26]. The BEH mechanism, when $\mathcal{G}$ symmetry is completely broken, is a relativistic analog of superconductivity and may be viewed as a condensation of electric charges. Magnetic fluxes are then channeled into quantized flux tubes. In confinement, it is the electric flux which is channeled into quantized tubes. Electric-magnetic dualities suggest that, at some fundamental level, confinement is a condensation of magnetic monopoles and constitutes the magnetic dual of the BEH mechanism.

### 5.2 Fermions from bosons

The monopole solution Eq. (5.42) and its regular generalization are invariant under simultaneous rotations in space and isospace. This is an invariance under the diagonal subalgebra $so_{\text{diag}}(3) = \text{diag}[so_{\text{space}}(3) \oplus so_{\text{isospace}}(3)]$. It implies that a bound state of a scalar of isospin $1/2$ with the monopole is a space-time fermion [27]. In this way, fermions can be made out of bosons.

In field theory, such transmutations are rather exceptional. But it may be of importance if the nature of space-time emerges from a more basic description as illustrated in the string theory approach to quantum gravity. A suggestion along these lines was first proposed in reference [28]. To see how this might happen, compactify the bosonic closed string on maximal toroids
of the rank 16 group $E_8 \times \widetilde{SO}(16)/Z_g$, where $Z_g$ is a subgroup of the center $Z_2 \times Z_2$ of the universal covering $\widetilde{SO}(16)$ of the rotation group $SO(16)$. This yields four modular invariant bosonic closed string theories $[29]$. Each sector of these closed strings contains ten dimensional space-time fermionic subspaces, which appear by selecting in the light-cone gauge transverse states transforming under the diagonal subalgebra $so_{\text{diag}}(8) = \text{diag}[so_{\text{space}}(8) \oplus so_{\text{internal}}(8)]$. Here $so_{\text{internal}}(8)$ is a subalgebra of the algebra $so(16)$ which emerges as a symmetry of the compactified bosonic string. Consistency of the truncation to these states stems from the non-trivial requirement that the algebra $so_{\text{diag}}(8)$ closes on the Lorentz algebra $so(9,1) [30, 29]$. For spinor representation of $so_{\text{internal}}(8)$, one gets space-time fermions in analogy with the transmutation arising from the diagonal subalgebra $\text{diag}[so_{\text{space}}(3) \oplus so_{\text{isospace}}(3)]$ in the ‘t Hooft-Polyakov monopole.

One obtains in this way all the consistent fermionic ten-dimensional closed strings, namely the supersymmetric IIA and IIB and the non supersymmetric OA and OB strings. One gets, solely from bosonic consideration, the spectra and tensions of all their $p$-dimensional $Dp$-branes as well as all their anomaly-free open descendants with the concomitant Chan-Paton factors. All the fermionic strings are interrelated at the level of their bosonic parents through the global properties of the sixteen-dimensional rotation group $[29]$.

Although these results are essentially kinematical in character, they raise the possibility that space-time fermions and perhaps even supersymmetry could arise from bosonic degrees of freedom. In such a perspective no fermionic degrees of freedom might be needed in a fundamental theory of quantum gravity.

5.3 A geometrical view on the BEH mechanism

The BEH mechanism operates within the context of gauge theories. Despite the fact that grand unification schemes reach scales comparable to the Planck scale, there was, a priori, no indication that Yang-Mills fields offer any insight into quantum gravity. The superstring and M-theory approach to quantum gravity did produce theoretical achievements, in particular in the context of a quantum interpretation of the black holes entropies. Of particular interest in that context are the $Dp$-branes. Here I will recall how $Dp$-branes yield a geometrical interpretation of the BEH mechanism.

When $N$ BPS $Dp$-branes coincide, they admit massless excitations from the $N^2$ zero length oriented strings with both end attached on the $N$ coincident branes. There are $N^2$ massless vectors and additional $N^2$ massless scalars for each dimension transverse to the branes. The
open string sector has local $U(N)$ invariance. At rest, BPS $D_p$-branes can separate from each other in the transverse dimensions at no cost of energy. Clearly this can break the symmetry group from $U(N)$ up to $U(1)^N$ when all the branes are at distinct location in the transverse space, because strings joining two different branes have finite length and hence now describe finite mass excitations. The only remaining massless excitations are then due to the zero length strings with both ends on the same brane.

![Fig.9. Breaking $U(N)$ gauge symmetry by Dp-branes.](image)

This symmetry breaking mechanism can be understood as a BEH mechanism from the action describing low energy excitations of $N$ $D_p$-branes. The action is the reduction to $p+1$ dimensions of 10-dimensional supersymmetric Yang-Mills with $U(N)$ gauge fields [31].

The Lagrangian is

$$
\mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} \left( \frac{1}{2} D_\mu A^i D^\mu A^i - \frac{1}{4} [A^i, A^j]^2 \right) + \text{fermions},
$$

(5.49)

where $\mu$ labels the $p+1$ brane coordinates and $i$ the directions transverse to the branes. $F_{\mu\nu} = F^a_{\mu\nu} T_a$, $A^i = A^{i\alpha} T_\alpha$ where $T_\alpha$ is a generator of $U(N)$ in a defining representation.

The states of zero energy are given classically, and hence in general because of supersymmetry, by all commuting $A^i = \{x^i_{mn}\}$ matrices, that is, up to an equivalence, by all diagonal matrices $\{x^i_m\} = \{x^i_m \delta_{mn}\}$. Label the $N^2$ matrix elements of $A_\mu$ by $A_{\mu mn}$. The $(N^2 - N)$ gauge fields given by the non diagonal elements $m \neq n$ acquire a mass

$$
m^2_{mn} \propto (\bar{x}_m - \bar{x}_n)^2,
$$

(5.50)

if $\bar{x}_m \neq \bar{x}_n$, as is easily checked by computing the quadratic terms in $A_{\mu mn}$ appearing in the covariant derivatives $\text{Tr} D_\mu A^i D^\mu A^i$.

This symmetry breaking is induced by the expectation values $\{x^i_m\}$. The gauge invariance is ensured, as usual, by unobservable $(N^2 - N)$ would-be NG bosons. To identify the latter,
consider the scalar potential in Eq. (5.49), namely
\[ V = \text{Tr} \frac{1}{4} [\mathbf{A}^i, \mathbf{A}^j] [\mathbf{A}^i, \mathbf{A}^j] = \frac{1}{4} \sum_{i,j;m,n} \langle m | [\mathbf{A}^i, \mathbf{A}^j] | n \rangle \langle n | [\mathbf{A}^i, \mathbf{A}^j] | m \rangle. \] (5.51)

One writes
\[ \langle m | \mathbf{A}^j | n \rangle = x^j_{\lambda} \delta_{mn} + y^j_{\lambda \mu} n. \] (5.52)

Here the diagonal elements \( \{ x^j_{\lambda} \} \) are the expectation values and the \( y^j_{\lambda \mu} \) (\( = - [y^j_{\mu \lambda}]^* \)) define \( d(N^2 - N) \) hermitian scalar fields \( (y^j_{\lambda \mu})^a (a = 1, 2) \) where \( y^j_{\lambda \mu} = (y^j_{\lambda \mu})^a + i(y^j_{\lambda \mu})^2 \) , \( m > n \), and \( d \) is the number of transverse space dimensions. The mass matrix for the fields \( (y^j_{\lambda \mu})^a \) is

\[ \frac{\partial^2 V}{\partial (y^j_{\lambda \mu})^a \partial (y^j_{\lambda \mu})^b} = \delta^{ab} \left[ (\tilde{x}_m^j - \tilde{x}_n^j)^2 \delta^{kl} - (x^k_m - x^k_n)(x^l_m - x^l_n) \right], \] (5.53)

and has for each pair \( m, n \) \( (m < n) \), two zero eigenvalues corresponding to the eigenvectors \( (y^j_{\lambda \mu})^a \propto (x^l_m - x^l_n) \). These are the required \( (N^2 - N) \) would-be NG bosons, as can be checked directly from the coupling of \( \mathbf{A}^i \) to \( A^\mu \) in the Lagrangian Eq. (5.49).

As mentioned above, the breaking of \( U(N) \) up to \( U(1)^N \) may be viewed in the string picture as due to the stretched strings joining branes separated in the dimensions transverse to the branes. One identifies the \( \{ x^j_{\lambda} \} \) as coordinates transverse to the brane \( m \). The mass of the vector \( A^\mu_{mn} \) is then the mass shift, due to the stretching, of the otherwise massless open string vector excitations. The unobservable NG bosons \( \tilde{y}_{mn} \parallel (\tilde{x}_m - \tilde{x}_n) \) are the field theoretic expression of the unobservable longitudinal modes of the strings joining the branes \( m \) and \( n \). In this way \( Dp \)-branes provide a geometrical interpretation of the BEH mechanism.

An interesting situation occurs when \( p = 0 \). The Lagrangian Eq. (5.49) then describes a pure quantum mechanical system where the \( \{ x^j_{\lambda mn} \} \) are the dynamical variable.\(^7\) The time component \( A^t \), which enters the covariant derivative \( D^t \mathbf{A}^i \) can be put equal to zero, leaving a constraint which amounts to restrict the quantum states to singlets of \( SU(N) \). The \( \{ x^j_{\lambda mn} \} \) which define in string theory D0-brane coordinates (viewed as partons in the infinite momentum frame in reference [32]) are the analog, for \( p = 0 \), of the expectation values in the \( p \neq 0 \) case, although they label now classical collective position variables of the quantum mechanical system. The non-diagonal quantum degrees of freedom \( \tilde{y}_{mn} \perp (\tilde{x}_m - \tilde{x}_n) \) have a positive potential energy proportional to the distance squared between the D0-branes \( m \) and \( n \). Hence they get locked in their ground state when the D0-branes are largely separated from each other. In this way, the D0-brane \( \mathbf{A}^i = \{ x^j_{\lambda mn} \} \) matrices commute at large distance scale and define geometrical degrees of freedom. However these matrices do not commute at short distances where the potential

\(^7\)This Lagrangian first appeared as a description of the supermembrane [33].
energies of the $y^i_{mn}$ go to zero. This, and its aforementioned analog for D$p$-branes ($p > 0$) suggests that the space-time geometry exhibits non commutativity at small distances \[31\], a feature which might be relevant for quantum gravity.

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In the second half of the twentieth century, the progress of our understanding of natural phenomena in rational terms bears the mark of Yang-Mills local gauge invariance. The reconciliation of this large symmetry with the apparent diversity of natural phenomena where symmetry is hidden appears possible through the implementation of a structured vacuum originating in the concept of spontaneous broken symmetry.

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