Remark about T-duality of Dp-branes

Josef Klusoň

Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37, Brno, Czech Republic

E-mail: klu@physics.muni.cz

ABSTRACT: This note is devoted to the analysis of T-duality of Dp-brane when we perform T-duality along directions that are transverse to world-volume of Dp-brane.

KEYWORDS: D-branes, String Duality

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1 Introduction and summary

It is well known that T-duality is central property of string theory, for review, see for example [1]. Generally, if we consider string sigma model in the background with metric $G_{MN}$ and NSNS two form $B_{MN}$ together with dilation $\phi$ and this background possesses an isometry along $d-$directions we find that it is equivalent to string sigma model in T-dual background with dual fields $\tilde{G}_{MN}, \tilde{B}_{MN}$ and $\tilde{\phi}$ that are related to the original fields by famous Buscher’s rules [2, 3] for generalization to more directions, see for example [4, 5].

It is well known that string theories also contain another higher dimensional objects that transform non-trivially under T-duality. In this note we focus on Dp-branes [6, 7], for more recent review, see [10]. Originally Dp-brane was defined with the open string description where the string embedding coordinates obey $p+1$ Neumann boundary conditions and $9-(p+1)$-Dirichlet ones at the boundary of the string world-sheet [6]. It was also shown by Polchinski that Dp-brane transforms into D$(p+1)$-brane when T-duality is performed along direction transverse to world-volume of Dp-brane and Dp-brane transforms to D$(p-1)$-brane in case when T-duality is performed along direction that Dp-brane wraps. In other words Dp-brane transforms with very specific way under T-duality transformations.

On the other hand it is remarkable that many aspects of Dp-brane dynamics can be described by its low energy effective action which is famous Dirac-Born-Infeld action [6]. Then one can ask the question whether this description of Dp-brane dynamics could give correct description of T-duality transformation of Dp-brane. This situation is relatively straightforward when we perform T-duality along directions which Dp-brane wraps. This property is known as covariance of Dp-brane action under T-duality transformations as was previously studied in [8, 9]. We generalize this approach to the T-duality along more longitudinal directions in the next section.

It is important to stress that in order to show full covariance of Dp-brane action with respect to T-duality transformation we should also study how Dp-brane effective action changes when we perform T-duality along transverse directions to its world-volume. The goal of this paper is to perform such an analysis. Our approach is based on previous works...
that consider description of $N$ Dp-branes on the circle [12], for review see [11]. It was shown there that such a configuration should be described by infinite number of Dp-branes on $\mathbb{R}$ which is covering space of $S^1$ when we impose appropriate quotient conditions [11]. Since this description was performed in the context of Matrix theory [13] the low energy effective action describing dynamics of $N$-- Dp-branes was Super Yang-Mills theory (SYM) defined on $p + 1$ dimensional world-volume. Then it was shown in [12] that this theory transforms under T-duality into (SYM) defined on $p + 2$ dimensional world-volume in T-dual background.

The goal of this paper is to generalize this analysis to the case of full DBI action for Dp-brane in the general background when we study T-duality along transverse directions. It is well known that such a T-duality can be defined when the target space-time fields do not depend on these coordinates explicitly. Since, following previous works, we should consider generalization of DBI action that describes infinite number of Dp-branes in covering space. Such an action is non-abelian generalization of DBI action that was introduced in [14]. Then we follow very nice analysis performed in [15]. Explicitly, we introduce quotient conditions and solve them in the same way as in [15]. We show that non-abelian action for infinite number of Dp-branes transforms to the action for D($p+d$)-brane where $d$-- is number of T-dual directions in the T-dual background where T-dual background fields are related to the original one by generalized Buscher’s rules [4, 5].

Let us outline our results. We study how Dp-brane transforms under T-duality we consider T-duality either along longitudinal or transverse directions to Dp-brane’ world-volume. We show that in the first case it transforms do D($p-d$)-brane while in the second one it transforms to D($p+d$)-brane when all background fields transform according to generalized Buscher’s rules. This fact nicely shows covariance of Dp-brane under T-duality transformations.

Certainly this paper can be extended in many directions. It would be certainly interesting to analyse transverse T-duality transformations in case of the Wess-Zumino term for Dp-brane that determines coupling of Dp-brane to Ramond-Ramond forms. Clearly we should start non-abelian generalization of this term given in [14] when we consider infinite number of Dp-branes on the covering space. This problem is currently under investigation. It would be also nice to analyse non-abelian T-duality on the world-volume of Dp-brane. We hope to return to this problem in future.

This paper is organized as follows. In the next section 2 we introduce Dp-brane action and study T-duality along longitudinal directions. Then in section 3 we consider T-duality along dimensions transverse to Dp-brane world-volume.

## 2 Longitudinal T-duality

In this section we review T-duality transformation of Dp-brane when we perform T-duality along $d$-- longitudinal directions. Explicitly, let us consider DBI action in the general background with the metric $G_{MN}, B_{MN}$ and dilaton $\phi$. This action has the form

\[
S = -T_p \int d^{p+1} \xi e^{-\phi} \sqrt{-\det(G_{\alpha \beta} + B_{\alpha \beta} + \lambda F_{\alpha \beta})}
\]  

(2.1)
where
\[ \lambda = 2\pi \alpha', \quad T_p = \frac{2\pi}{\lambda(p+1)/2}, \quad \] (2.2)
where we also defined pull back of \( G_{MN} \) and \( B_{MN} \) defined as
\[ G_{\alpha\beta} = G_{MN} \partial_\alpha x^M \partial_\beta x^N, \quad B_{\alpha\beta} = B_{MN} \partial_\alpha x^M \partial_\beta x^N, \quad \] (2.3)
where \( \xi^\alpha, \alpha = 0,1,\ldots, p \) label world-volume directions of Dp-brane and where \( x^M, M = 0,1,\ldots, 9 \) parametrize embedding of DBI action in the target space-time.

Now we would like to perform T-duality along last \( d \)–directions when we presume that there are directions which Dp-brane wraps. The fact that these directions are longitudinal mean that Dp-brane world-volume coordinates coincide with the target space ones. Explicitly we have
\[ x^m = \xi^m, \quad m = 9-d,\ldots,9. \] (2.4)
Then we presume that all world-volume fields do not depend on \( \xi^\alpha \) only where \( \hat{\alpha} = 0,1,\ldots, p - d \). Let us also introduce matrix \( E_{MN} = G_{MN} + B_{MN} \). Then we have
\[ E_{\alpha\beta} + \lambda F_{\alpha\beta} = \begin{pmatrix} E_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}} & E_{\hat{\alpha}n} + \lambda \partial_{\hat{\alpha}} A_n \\ E_{m\hat{\beta}} - \lambda \partial_{\hat{\beta}} A_m & E_{mn} \end{pmatrix}, \quad \] (2.5)
where \( E_{\hat{\alpha}\hat{\beta}} = E_{\mu\nu} \partial_{\hat{\alpha}} x^\mu \partial_{\hat{\beta}} x^\nu, \quad \mu, \nu = 0,1,\ldots, 9 - d \). Then performing standard manipulation with determinant we obtain
\[ \det(E_{\alpha\beta} + \lambda F_{\alpha\beta}) = \det \left( E_{\hat{\alpha}\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}} - (E_{\hat{\alpha}m} + \lambda \partial_{\hat{\alpha}} A_m) E_{mn}(E_{n\hat{\beta}} - \lambda \partial_{\hat{\beta}} A_n) \right) \det E_{mn} \]
\[ = \det \left( E_{\hat{\alpha}\hat{\beta}} - E_{\hat{\alpha}m} E_{mn} E_{n\hat{\beta}} + \lambda F_{\hat{\alpha}\hat{\beta}} + \lambda E_{\hat{\alpha}m} E_{mn} \partial_{\hat{\beta}} A_n - \lambda \partial_{\hat{\alpha}} A_m E_{mn} E_{n\hat{\beta}} + \lambda^2 \partial_{\hat{\alpha}} A_m E_{mn} \partial_{\hat{\beta}} A_n \right) \det E_{mn}, \quad \] (2.6)
and where \( \tilde{E}^{mn} \) is inverse to \( E_{mn} \) in the sense that \( \tilde{E}^{mn} E_{nk} = \delta^k_m \). As the next step we define T–dual coordinates
\[ \tilde{x}_m \equiv \lambda A_m. \quad \] (2.7)
Then we can write
\[ E_{\hat{\alpha}\hat{\beta}} - E_{\hat{\alpha}m} \tilde{E}^{mn} E_{n\hat{\beta}} + \partial_{\hat{\alpha}} x_m \tilde{E}^{mn} \partial_{\hat{\beta}} x^n = \partial_{\hat{\alpha}} x^\mu (E_{\mu\nu} - E_{\mu m} \tilde{E}^{mn} E_{n\nu}) \partial_{\hat{\beta}} x^\nu + \partial_{\hat{\alpha}} \tilde{x}_m \tilde{E}^{mn} \partial_{\hat{\beta}} \tilde{x}_n, \]
\[ E_{\hat{\alpha}m} \tilde{E}^{mn} \partial_{\hat{\beta}} \tilde{x}_n = \partial_{\hat{\alpha}} x^\mu E_{\mu m} E^{mn} \partial_{\hat{\beta}} x^\nu, \]
\[ -\partial_{\hat{\alpha}} \tilde{x}_m \tilde{E}^{mn} E_{n\hat{\beta}} = -\partial_{\hat{\alpha}} \tilde{x}_m \tilde{E}^{mn} E_{m\nu} \partial_{\hat{\beta}} x^\nu, \quad \] (2.8)
that can be interpreted as an embedding of D(p-d)-brane in T-dual background with the background fields
\[ \tilde{E}_{\mu\nu} = E_{\mu\nu} - E_{\mu m} \tilde{E}^{mn} E_{n\nu}, \]
\[ \tilde{E}_{\mu}^m = E_{\mu m} \tilde{E}^{mn}, \quad \tilde{E}_{\nu}^m = -\tilde{E}^{mn} E_{n\nu}, \]
\[ e^{-\phi} = e^{-\phi} \det E_{mn}. \] (2.9)
Explicitly, the D(p-d)-brane action in T-dual background has the form

\[
S = -T_{p-d} \int_0^{\sqrt{\lambda}} d^d \xi \int d^{p-d+1} \xi \sqrt{-\det(\tilde{E}_{\alpha \beta} + \lambda \Phi_{\alpha \beta})},
\] (2.10)

where

\[
\tilde{E}_{\alpha \beta} = \tilde{E}_{\mu \nu} \partial_\alpha x^\mu \partial_\beta x^\nu + \tilde{E}_{\mu}^m \partial_\alpha x^\mu \partial_\beta x_m + \tilde{E}_m^\alpha \partial_\alpha x_m \partial_\beta x^n + \partial_\alpha \tilde{x}_m \tilde{E}^{mn} \partial_\beta \tilde{x}_n,
\] (2.11)

and where we defined tension for D(p-d)-brane in the form

\[
T_{p-d} = T_p \int_0^{\sqrt{\lambda}} d^d \xi = \lambda^{d/2} T_p.
\] (2.12)

Note that the transformation rules for T-dual fields given in (2.9) coincide with the results derived previously in [4, 5] and which are now derived independently using covariance of Dp-brane under T-duality transformations.

However in order to see consistency of T-duality covariance of Dp-branes we should also consider opposite situation when we consider Dp-brane in general background and perform T-duality along directions that are transverse to the world-volume of Dp-brane.

### 3 Transverse T-duality

In this section we consider opposite situation when we study Dp-brane in the background that has isometry along \(d\)-directions in the transverse space to Dp-brane world-volume. The best way how to describe such a Dp-brane is to consider infinite number of Dp-branes on the covering space of torus \(T^d\) which is \(\mathbb{R}^d\) and impose appropriate quotient conditions. Further, we should also consider appropriate action for \(N\) Dp-branes which is famous Myers non-abelian action [14]

\[
S = -T_p \text{Str} \int d^{p+1} \xi e^{-\phi} \sqrt{-\det(P[E_{\alpha \beta}] + P[E_{\alpha r} E^{rs} ((Q^{-1})^s_\alpha - \delta^s_\alpha) E_{l \beta}] + \lambda \Phi_{\alpha \beta}) \det Q_j},
\] (3.1)

where \(i, j, k, l, m, n, r, s, t, \ldots = p + 1, \ldots, 9\) label directions transverse to the world-volume of \(N\) Dp-branes. Note that the location of \(N\) Dp-branes in the transverse space is determined by \(N \times N\) Hermitian matrices \(\Phi^m, m = p + 1, \ldots, 9\) and all background fields depend on them so as for example \(E_{\alpha \beta}(\Phi)\) and so on. We use convention where \(\Phi^m\) are Hermitian matrices and field strength \(F_{\alpha \beta}\) is defined as

\[
F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i [A_\alpha, A_\beta],
\] (3.2)

where \(A_\alpha\) is \(N \times N\) Hermitian matrix corresponding to non-abelian gauge field. Finally, \(P[E_{\alpha \beta}]\) is a pull-back of the background \(E_{MN}(\Phi)\) defined as

\[
P[E_{\alpha \beta}] = E_{\alpha \beta} + D_\alpha \Phi^r E_{r \beta} + E_{\alpha r} D_\beta \Phi^r + D_\alpha \Phi^r E_{r s} D_\beta \Phi^s,
\] (3.3)

where \(D_\alpha \Phi^m\) is covariant derivative

\[
D_\alpha \Phi^m = \partial_\alpha \Phi^m + i [A_\alpha, \Phi^m].
\] (3.4)
Note that non-abelian action for $N$ D$p$-branes is implicitly defined in the static gauge where world-volume coordinates $\xi^a$ coincide with the target space ones $x^a$. Finally Str means symmetrized trace and in order to describe infinite number of D$p$-branes we should divide the action (3.1) by the infinite order the quotient group $\mathbb{Z}^d$.

Further, $Q^i_j$ is defined as

$$Q^i_j = \delta^i_j + i\lambda^{-1}[\Phi^i, \Phi^k]E_{kj}$$

(3.5)

and $(Q^{-1})^j_k$ is its inverse in the sense

$$Q^i_j(Q^{-1})^j_k = \delta^i_k .$$

(3.6)

Finally, $P[E_{ar}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{l\beta}]$ is defined as

$$P[E_{ar}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{l\beta}] = E_{ar}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{l\beta} + D_\alpha\Phi^m E_{mr}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{l\beta}$$

$$+ E_{ar}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{l\beta}D_\beta\Phi^k + D_\alpha\Phi^m E_{mr}E^{rs}((Q^{-1})^l_s - \delta^l_s)E_{tn}D_\beta\Phi^n,$$

(3.7)

where $E^{rs}$ is matrix inverse to $E_{mr}$ defined as

$$E_{mr}E^{rs} = \delta^s_m .$$

(3.8)

In order to implement T-duality along $d$ transverse directions we follow analysis performed in [15] which we generalize to the case of non-linear non-abelian action (3.1). Let us presume that the background fields do not depend on $x^A$ coordinates, where $A = p + 1, \ldots, p + d$ and that these coordinates are periodic with period $\sqrt{2\pi\lambda}$. This is natural if we recognize that all geometrical properties of the background are encoded in the field $E_{MN}$. Then we consider an infinite number of D$p$-branes on compact space with coordinates $x^A$ when we impose following quotient conditions

$$U^{-1}_B\Phi^A U_B = \delta^A_B\sqrt{\lambda} + \Phi^A ,$$

$$U^{-1}_B\Phi^{i'} U_B = \Phi^{i'}, ~ i' = p + d + 1, \ldots, 9 .$$

(3.9)

Let us presume that solution of the quotient equation corresponds to operators $U_A$ that commute

$$[U_A, U_B] = 0 .$$

(3.10)

In order to solve (3.9) it is natural to introduce an auxiliary Hilbert space on which $\Phi^A$ and $U_B$ act. The simplest way is to introduce Hilbert space of auxiliary functions living on $d$–dimensional torus taking value in $\mathbb{C}^d$. Then we take $U_A$ as generators of the functions on $d$–dimensional torus

$$U_A = e^{i\lambda^{-1/2}\sigma_A}$$

(3.11)

where $\sigma_A$ are coordinates on the covering space of torus. Then $\Phi^A$ has to be equal to

$$\Phi^A = -i\lambda(\partial^A - iA^A(\sigma_A))$$

(3.12)
since then

\[ U_B^{-1} \Phi^A U_B = \lambda^{1/2} \delta_B^A + \Phi^A. \]  

(3.13)

Using these results we can now proceed to write corresponding action in T-dual background. As the first step we perform following manipulation with the determinant in the action (3.1)

\[
\det(P[E]_{\alpha\beta} - P[E_{ar}E^{rs}E_{s\beta}] + \lambda F_{\alpha\beta} \mathbf{A}_\alpha^m Q_{mn}^{\beta}) \det E_{mn},
\]

(3.14)

where \( Q^{ij} = E^{ij} + i \lambda^{-1} [\Phi^i, \Phi^j] \) and where

\[
\mathbf{A}^m_\alpha = D_\alpha \Phi^k E_{kr} E_{r\beta} + E_{ar} E_{r\beta}, \quad \mathbf{B}^m_\beta = -E^{mk} E_{k\beta} - D_\beta \Phi^m.
\]

(3.15)

First of all we observe that

\[
P[E_{ar}E^{rs}E_{s\beta}] = E_{\alpha\beta} - E_{ar} E^{rs} E_{s\beta}.
\]

(3.16)

To proceed further we use the fact that \( D_\beta \Phi^A \) acting on arbitrary function \( f(\sigma) \) defined on the space labelled by \( \sigma_A \) is equal to

\[
D_\alpha \Phi^A f = \lambda (\partial_\alpha \Phi^A + i[A_\alpha, \Phi^A]) f = \lambda(\partial_\alpha A^A - \partial^A A_\alpha) f \equiv \lambda F^A_\alpha f
\]

(3.17)

and hence we can identify \( D_\alpha \Phi^A \) with \( \lambda F^A_\alpha \). Using this identification we obtain

\[
\mathbf{A}^A_\alpha = \lambda F^A_\alpha + E_{ar} E^{rA}, \quad \mathbf{A}^{i'}_\alpha = D_\alpha \Phi^{i'} + E_{ar} E^{r{i'}}.
\]

\[
\mathbf{B}^A_\beta = -E^{AB} E_{B\beta} - E^{A{i'}} E_{i'{\beta}} - \lambda F^A_\beta, \quad \mathbf{B}^{i'}_\beta = -E^{i'j} E_{j{\beta}} - E^{i'A} E_{A{\beta}} - \partial_\beta \Phi^{i'}
\]

(3.18)

and finally

\[
Q^{AB} = E^{AB} + i \lambda^{-1} [\Phi^A, \Phi^B] = E^{AB} + \lambda F^{AB}, \quad Q^{A{i'}} = E^{A{i'}} + \lambda \partial^A \Phi^{i'}, \quad Q^{i'B} = E^{i'B} - \partial^B \Phi^{i'}, \quad Q^{i'j'} = E^{i'j'},
\]

(3.19)

where we used (3.12) so that

\[
[\Phi^A, \Phi^B] = -i \lambda^2 F^{AB}, \quad [\Phi^A, \Phi^{i'}] = -i \lambda \partial^A \Phi^{i'}.
\]

(3.20)
Now we return to the first determinant in (3.14) and rewrite it to the form
\[
\text{det} \left( P[E]_{\alpha \beta} - P[E_{\alpha \tau} E_{\tau \delta}] + \lambda F_{\alpha \beta} A^\alpha_\gamma \right) \\
\begin{pmatrix}
B_{\beta}^m \\
Q_{mn}
\end{pmatrix}
= \text{det} \left( \begin{pmatrix}
E_{\alpha \beta} - E_{\alpha \tau} E_{\tau \delta} + \lambda F_{\alpha \beta} A^\alpha_\gamma & A^B_\alpha A^j_\gamma \\\nB^j_\beta & Q^{AB} Q^{Aj'}
\end{pmatrix}
\begin{pmatrix}
A^B_\alpha & A^j_\gamma' \\
Q^{AB} & Q^{Aj'}
\end{pmatrix}
\right)
= \text{det} \left( \begin{pmatrix}
E_{\alpha \beta} - E_{\alpha \tau} E_{\tau \delta} + \lambda F_{\alpha \beta} & A^B_\alpha - A^j_\gamma (Q^{-1})_\gamma j' Q^{j'B} \\\nB^j_\beta - Q^{Ak'} (Q^{-1})_k j' B^\gamma \beta & Q^{AB} - Q^{Ak'} (Q^{-1})_k j' Q^{j'B} \right)
\begin{pmatrix}
0 & 0 \\
0 & Q^{j'j'}
\end{pmatrix}
\right)
\equiv \text{det} \left( \begin{pmatrix}
D^\alpha_\beta & D^B_\alpha \\
D^j_\gamma & D^{AB}
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
Q^{j'j'} & Q^{j'j'}
\end{pmatrix}
\right) \quad (3.21)
\]

Since \(Q^{j'j'} = E^{j'j'}\) it is clear that the matrix inverse \((Q^{-1})_\gamma j'\) is equal to \((Q^{-1})_\gamma j' = \tilde{E}_\gamma j'\) where
\[
\tilde{E}_\gamma k' E^{k'l} = \delta^l_\gamma . \quad (3.22)
\]

Now we explicitly calculate components of the matrix \(D\) as
\[
D_{\alpha \beta} = E_{\alpha \beta} - E_{\alpha \tau} E_{\tau \delta} + \partial_\delta \Phi^i E^{i j'} E^{j' \tau} E_{\tau \beta} + \partial_\beta \Phi^i E^{i j'} E^{j' \tau} \partial_\beta \Phi^j \\
+ E_{\alpha \tau} E^{i j'} E^{j' k} E_{k \beta} + E_{\alpha \tau} E^{i j'} \tilde{E}_\gamma j' \partial_\beta \Phi^j + \lambda F_{\alpha \beta} \\
= E_{\alpha \beta} - E_{\alpha \lambda}(E^{AB} - E^{A j'} \tilde{E}_\gamma j'E^{j'B}) E_{B \beta} \\
+ \partial_\delta \Phi^i E^{i j'} E^{j' \tau} E_{\tau \beta} + \partial_\beta \Phi^i \tilde{E}_\gamma j' \partial_\beta \Phi^j \\
+ E_{\alpha \tau} E^{i j'} \tilde{E}_\gamma j' \partial_\beta \Phi^j + \lambda F_{\alpha \beta} . \quad (3.23)
\]

To proceed further we observe that
\[
(E^{AB} - E^{A j'} \tilde{E}_\gamma j'E^{j'B}) E_{BC} = \delta^A_{\gamma j} \quad (3.24)
\]
and hence we can identify expression in the bracket with the matrix inverse \(\tilde{E}^{AB}\) to \(E_{AB}\) so that \(\tilde{E}^{AB} E_{BC} = \delta^A_{\gamma j}\). Further, let us consider following expression
\[
E^{j'} j' - E^{j'} A \tilde{E}^{AB} E_{B j'} \quad (3.25)
\]
and multiply it with \(E^{j' k'}\). Then, after some calculations, we get
\[
(E^{j'} j' - E^{j'} A \tilde{E}^{AB} E_{B j'}) E^{j' k'} = \delta^j_{\gamma k'} \quad (3.26)
\]
so that we can identify expression in the bracket with matrix \(\tilde{E}_{\gamma j'}\)
\[
\tilde{E}_{\gamma j'} = E^{j'} j' - E^{j'} A \tilde{E}^{AB} E_{B j'} . \quad (3.27)
\]
Using these results we obtain following useful expressions

\[ E^{\alpha j'} E_{\alpha j'} = -\tilde{E}^{AB} E_{B j'}, \quad \tilde{E}_{\alpha j'} E^{j'B} = -E_{\alpha j'} E^{CB} \]  \hspace{1cm} (3.28)

and

\[ \tilde{E}_{\alpha j'} E_{j\gamma} = E_{\alpha j'} - E_{\alpha j'} \tilde{E}^{AB} E_{B j'}. \]  \hspace{1cm} (3.29)

Using (3.28) and (3.29) we get

\[
D_{\alpha \beta} = E_{\alpha \beta} - E_{\alpha A} \tilde{E}^{AB} E_{B \beta} + \partial_{\alpha} \Phi^{I'}(E_{\alpha j'} - E_{\alpha j'} \tilde{E}^{AB} E_{B j'}) \\
+ (E_{\alpha A} - E_{\alpha A} \tilde{E}^{AB} E_{B j'}) \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'}(E_{\alpha j'} - E_{\alpha j'} \tilde{E}^{AB} E_{B j'}) \partial_j \Phi^{I'}
\]  \hspace{1cm} (3.30)

and

\[
D^{AB} = E^{AB} + \lambda F^{AB} + \partial_{\alpha} \Phi^{I'}(E_{\alpha j'} - E_{\alpha j'} \tilde{E}^{AB} E_{B j'}) \partial_j \Phi^{I'} + (E_{AB} - E_{AB} \tilde{E}^{CD} E_{D j'}) \partial_j \Phi^{I'},
\]  \hspace{1cm} (3.31)

In the same way we obtain

\[
D^B_{\alpha} = E_{\alpha B} + E_{\alpha A} \tilde{E}^{AB} E_{B C} + \partial_{\alpha} \Phi^{I'} E_{\alpha C} E^{CB} + (E_{\alpha A} - E_{\alpha A} \tilde{E}^{AB} E_{B j'}) \partial_j \Phi^{I'} + (E_{\alpha B} - E_{\alpha B} \tilde{E}^{BC} E_{B j'}) \partial_j \Phi^{I'},
\]  \hspace{1cm} (3.32)

and also

\[
D^{A}_{\beta} = -E^{AB} E_{B \beta} - \lambda F^{AB} E_{B j'} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'}(E_{\alpha B} - E_{\alpha B} \tilde{E}^{BC} E_{B j'}) \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j'} \partial_j \Phi^{I'}.
\]  \hspace{1cm} (3.33)

Finally we consider following combinations of determinants that appear under square root in the action (3.1)

\[
\det E_{mn} \det E^{ij'} = \det(E_{ij'} - E_{ij'} \tilde{E}^{AB} E_{B j'}) \det E_{AB} \det E^{ij'} = \det \tilde{E}_{ij'} \det E^{ij'} \det E_{AB} = \det E_{AB}.
\]  \hspace{1cm} (3.34)

Collecting these terms together we obtain final form of D(p+d)-brane action in T-dual background in the form

\[
S = -\frac{T_p}{\sqrt{\lambda_{d/2}}} \int d^{p+1} \xi d^d \sigma e^{-\phi} \sqrt{-\det D},
\]  \hspace{1cm} (3.35)

where we also used the relation between trace over infinite dimensional matrices and integration over coordinates \(\sigma\)

\[
\text{Tr} = \frac{1}{\sqrt{\lambda_{d/2}}} \int d^d \sigma,
\]  \hspace{1cm} (3.36)

and where the matrix \(D\) has following components

\[
D_{\alpha \beta} = \tilde{E}_{\alpha \beta} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\beta I'} + E_{\alpha I'} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j'} \partial_j \Phi^{I'} + \lambda F_{\alpha \beta},
\]
\[
D^B_{\alpha} = \tilde{E}^{B}_{\alpha} + \partial_{\alpha} \Phi^{I'} \tilde{E}^{B}_{I'} + E_{\alpha j'} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j'} \partial_j \Phi^{I'} + \lambda F^{B}_{\alpha},
\]
\[
D^{A}_{\beta} = \tilde{E}^{A}_{\beta} + \tilde{E}^{A}_{j} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j} \partial_j \Phi^{I'} - \lambda F^{A}_{\beta},
\]
\[
D^{AB} = \tilde{E}^{AB} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j} \partial_j \Phi^{I'} + \tilde{E}^{A}_{j} \partial_j \Phi^{I'} + \partial_{\alpha} \Phi^{I'} \tilde{E}_{\alpha j} + \lambda F^{AB},
\]  \hspace{1cm} (3.37)
where we have following components of the background matrix $\tilde{E}$

\[
\tilde{E}_{\alpha \beta} = E_{\alpha \beta} - E_{\alpha A} \tilde{E}^{AB} E_{B \beta}, \\
\tilde{E}_{\alpha \beta'} = E_{\alpha \beta'} - E_{\alpha A} \tilde{E}^{AB} E_{B \beta'}, \\
\tilde{E}_{\alpha}^B = E_{\alpha A} \tilde{E}^{AB}, \\
\tilde{E}_{\alpha}^{B'} = E_{\alpha C} \tilde{E}^{CB}, \\
\tilde{E}_{\alpha'}^B = E_{\alpha' A} \tilde{E}^{AB} E_{B \beta'}, \\
\tilde{E}_{\alpha'}^{B'} = E_{\alpha' C} \tilde{E}^{CB},
\]

(3.38)

and where

\[
e^{-\tilde{\phi}} = e^{-\phi} \sqrt{-\det \tilde{E}_{AB}}. \\
\]

(3.39)

It is important to stress that resulting $D(p+d)$-brane propagates in T-dual background where the T-dual background is given by Buscher’s rules that are given in equations in (2.11). More explicitly, note that we perform T-duality along directions labelled by $A, B, \ldots$ where $A = p + 1, \ldots, p + d$ that should be identify with $m, n, \ldots$ given in the section 2. In the same way $\alpha, \beta = 0, \ldots, p$ and $\alpha', \beta' = p + d + 1, \ldots, 9$ should be identified with $\mu, \nu$ again given in section 2. Explicitly, if we denote $\mu, \nu, \ldots = (\alpha, \beta, \alpha', \beta', \ldots)$ and $A, B, C, \ldots = m, n, k, \ldots$, then we can rewrite (3.38) into the form

\[
\tilde{E}_{\mu \nu} = E_{\mu \nu} - E_{\mu m} \tilde{E}^{mn} E_{n \nu}, \\
\tilde{E}_{\mu}^{\nu} = E_{\mu m} \tilde{E}^{mn}, \\
\tilde{E}^{m}_{\nu} = -\tilde{E}^{mn} E_{n \nu},
\]

(3.40)

which exactly match generalized T-duality rules given in (2.11) and hence fully proves covariance of Dp-brane action under T-duality transformations.

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