A 2 PN/RM metric of General Relativity

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We present a derivation of 2PN/RM metric field equations from the Einstein field equation in General Relativity. We use the exponential parametrization and the $e^{-3}$ isotropic spatial coordinates such as in IAU2000 recommendations.

Keywords: post newtonian relativistic motion; post newtonian; harmonic gauge; TIPO; ASTROD; laser telemetry in the solar system

I. INTRODUCTION

In a foreseeable future, telemetry will reach a new accuracy using laser links and high accuracy clocks (such as the TIPO experiments which is in study [1]). At such experiments accuracies, the effective 1.5 PN/RM (acronym defined in section II) metric defined from the 1PN IAU2000 [2] metric will be no longer enough to fit data (2 PN/RM effects lead to terms of about $10^{-18}$ for a flying time of about 1000s when tagging units accuracy reach at least $10^{-11}$). Hence, it is necessary to develop the metric up to the 2 PN/RM order. Our work is completely based on General Relativity (GR) and hence we don’t consider any alternative theory in this paper. This means that our result would have a practical usefulness in telemetry only if the GR theory is the "true" theory of gravity or if alternative theory deviates from GR only by sufficiently small post-Newtonian numerical parameters in the metric. This is compatible with present available data.

We start our work from the "exponential parametrization" such as in DSX [3] or in the IAU2000 resolutions [2]. Hence, our notations come from [3]. Other works on the subject have already be done ([4,5]) but they didn’t write their metric in the convenient exponential parametrization and these works have been done in harmonic gauge from the start to the end. This is not our case since we fix the gauge only at the end when we want peculiar solutions in some peculiar gauges.

In section II we introduce a new terminology before recalling some well known results from the DSX paper [3] in section III. Then we derive Einstein’s equations in our metric’s parametrization and finally we give formal solutions in any gauge that respect the exponential parametrization – with a particular attention to the harmonic gauge.

II. TERMINOLOGY : DEFINITION OF THE PN/BM AND PN/RM METRICS

The PN approximation is based on the assumption of a weak gravitational field and weak velocities (ie. of the $\sqrt{GM/r}$ or less, $M$ being some caracteristic mass of the system) for both the sources and the (test) body. It formally consists in looking for solutions under the form of an expansion in powers of $1/c$. The usually so-called nPN order terms in the metric, leading to $e^{-2n}$ terms in the equation of motion of a body describing a bounded orbit, are terms of orders $e^{-2n-2}$ in $g_{00}$, $e^{-2n-1}$ in $g_{0i}$ and $e^{-2n}$ in $g_{ij}$. In this paper, a metric developped this way will be refered as the nPN/BM metric (BM meaning "Bounded Motion" for test particles). It is particularly well-adapted for studying bounded motions in systems made by non-relativistic massive bodies, as the Solar System is.

However, since we are interested in the propagation of light, we are lead to relax the hypothesis on the velocity of the test particle whose motion is considered. Of course, this doesn’t change the full metric, but the terms to be considered in the metric components are not the same as in the PN/BM problem. Indeed, the terms leading to $e^{-2n}$ terms in the equation of motion of a test particle moving with relativistic velocity are terms of order $e^{-2n}$ in $g_{\alpha\beta}$ (ie. in both $g_{00}$, $g_{0i}$ and $g_{ij}$). In this paper, a metric developped this way will be refered as the nPN/RM metric (RM meaning "Relativistic Motion" for test particles). A PN/RM metric is particularly well-suited for studying relativistic motions of test bodies (for instance, the propagation of light) in systems made by non-relativistic bodies, as the Solar System is.

PN/BM and PN/RM orders are illustrated in figure [1].

The present paper deals with the PN/RM problem since we are concerned in propagation of light.

III. 1,5 PN/BM METRIC

We recall here results from DSX [3] on the 1,5PN/BM metric. First, the "exponential parametrization" with the spatial isotropy condition makes the metric writes

\begin{align}
  g_{00} &= -e^{-2w/c^2} \\
  g_{0i} &= -\frac{4}{c^2} w^i \\
  g_{ij} &= \gamma_{ij} e^{2w/c^2}.
\end{align}
IV. 2 PN/RM METRIC

At the 2PN/RM order, we get 6 new degrees of freedom in the metric. As we know, from the 10 degrees of freedom in the equation of GR, there are only 6 cleared by the 10 equations since 4 degrees of freedom correspond to a gauge invariance. Indeed, the metric is determined by the field equations modulo the knowing of the used coordinates. Since, there are four coordinates, there are 4 degrees of freedom not reduced by the 10 field equations. As seen in the previous section, to the 1.5PN/BM approximation correspond 4 metric field equations and the gauge invariance, characterized by $\lambda$, keep 1 degree of freedom. Then, we expect from the 2PN/RM order 6 equations with a gauge invariant field (characterized by a 3-vector) which (the equations) reduce the 6 degrees of freedom to 3. Let us fix the covariant form of the metric

\begin{align}
&g_{00} = -\frac{e^{-2w/c^2}}{c^2} \\
g_{0i} = -\frac{4}{c^3} w_i \\
g^{ij} = \delta^{ij} e^{-2w/c^2} + O(c^{-4}).
\end{align}

The 1,5PN/BM approximation correspond 4 metric field equations and the gauge invariance keep 1 degree of freedom (indeed $\sigma^\alpha$ is approximately divergence free). This gauge invariance is characterized by the arbitrary differentiable function $\lambda$ in DSX.

\begin{align}
w' &= w - \frac{1}{c^2} \partial_t \lambda \\
w'_i &= w_i + \frac{1}{c^2} \partial_i \lambda.
\end{align}

The IAU2000 resolution then corresponds to a peculiar choice of gauge in order to fix the last degree of freedom: the harmonic gauge. This condition writes, in the 1.5PN/BM approximation $g^{\alpha\beta} \Gamma^0_{\alpha\beta} = O(c^{-5})$, $g^{\alpha\beta} \Gamma^i_{\alpha\beta} = O(c^{-4})$. The four metric field equations then reduce to

\begin{align}
\Box_m w &= -4\pi G\sigma + O(c^{-4}) \\
\Delta w_i &= -4\pi G\sigma^i + O(c^{-2})
\end{align}

where $\Box_m = \Delta - \partial_t^2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{orders.png}
\caption{General scheme of orders taken into account in PN/BM and PN/RM metrics.}
\end{figure}

With $\gamma_{ij} = \delta_{ij} + O(c^{-4})$. From which follows

\begin{align}
g^{00} &= -\frac{e^{2w/c^2}}{c^2} + O(c^{-6}) \\
g^{0i} &= -\frac{4}{c^3} w_i + O(c^{-5}) \\
g^{ij} &= \delta^{ij} e^{-2w/c^2} + O(c^{-4}).
\end{align}

The Einstein field equation then reduces to four equations on $w$ and $w_i$

\begin{align}
\Delta w + \frac{3}{c^2} \partial_t^2 w + \frac{4}{c^2} \partial_i \partial_j w_j &= -4\pi G\sigma + O(c^{-4}) \\
\Delta w_i - \partial_j^2 w_j - \partial_t \partial_i w &= -4\pi G\sigma^i + O(c^{-2}).
\end{align}

Where $\sigma = c^{-2} (T^{00} + T^{kk})$, $\sigma^i = c^{-1} T^{0i}$ and $\Delta = \vec{\partial} \cdot \vec{\partial}$. Hence, to the 1.5PN/BM approximation correspond 4 equations of the GR equation and the gauge invariance keep 1 degree of freedom (indeed $\sigma^\alpha$ is approximately divergence free). This gauge invariance is characterized by the arbitrary differentiable function $\lambda$ in DSX.

\begin{align}
w' &= w - \frac{1}{c^2} \partial_t \lambda \\
w'_i &= w_i + \frac{1}{c^2} \partial_i \lambda.
\end{align}

Then we can derive the 2PN/RM field equations from the Einstein field equation $R^\alpha\beta = \frac{8\pi G}{c^4} (T^\alpha\beta - \frac{1}{2} g^\alpha\beta g_{\rho\sigma} T^{\rho\sigma})$.

\begin{align}
\frac{2}{c^4} \Theta_{ij}(\tau_{kl}) + \Xi_{ij}(w, w_k) &= \\
&= \frac{8\pi G}{c^4} \left( \frac{1}{2} \eta^{ij} T^{00} + T^{ij} - \frac{1}{2} \eta^{ij} T^{kk} - \frac{2}{c^2} \eta^{ij} T^{00} \right).
\end{align}

Where

\begin{align}
\Theta_{ij}(\tau_{kl}) &= \partial^2_{ki} \tau_{kj} + \partial^2_{kj} \tau_{ik} - \Delta \tau_{ij} - \partial^2_{ij} \tau_{kk}, \\
\Xi_{ij}(w, w_k) &= \\
&= \frac{\delta_{ij}}{c^2} \Box_m w + \frac{2}{c^3} \left( \partial^2_{0i} w_j + \partial^2_{0j} w_i - \frac{2}{c^2} \partial_t w_i \partial_j w + \frac{4}{c^4} \delta_{ij} w \Delta w \right).
\end{align}

Then, considering $w$ and $w_k$ as source terms and identifying the relation $-\frac{c^4}{2} \left( \frac{4}{c^4} \eta^{ij} w \Delta w \right) = \frac{8\pi G}{c^4} \eta^{ij} T^{00} + O(c^{-2})$ from [11], we get
\[ \Theta_{ij}(\tau_{kl}) = 4\pi G \sigma^{ij} - \left[ \partial^2_{ij}w_j + \partial^2_{ij}w_i - \partial_iw\partial_jw + 2\eta^{ij}\left( \partial^2_{ij}w + \partial^2_{ij}w_k \right) \right]. \]  

(22)

Where \( \sigma^{ij} = T^{ij} - \eta^{ij}T_{kk} \). As expected, the operator \( \Theta \) is gauge invariant through a relation on any 3-vector field. Indeed, we have \( \Theta_{ij}(\tau_{kl} + \partial_kA_i + \partial_lA_k) = \Theta_{ij}(\tau_{kl}) \), whatever is the 3-vector field \( A_k \). Hence, the 2PN/RM gauge invariance is characterized by both the arbitrary differentiable function \( \lambda \) and the arbitrary differentiable 3-vector \( A_k \).

\[ w' = w - \frac{1}{c^2}\partial_t\lambda \]  

(23)

\[ w'_i = w_i + \frac{1}{4}\partial_i\lambda \]  

(24)

\[ \tau'_{kl} = \tau_{kl} + \partial_kA_l + \partial_lA_k + \frac{1}{2}\delta_{kl}\partial_t\lambda. \]  

(25)

A. General formal solution

We can show that it always exists a gauge transformation \( \tau'_{kl} = \tau_{kl} + \partial_kA_l + \partial_lA_k \) that reduces the operator \( \Theta \) to a Laplacian \( \Theta_{ij}(\tau'_{kl}) = -\triangle \tau'_{ij} \). To be placed on this gauge, \( A_k \) must satisfy \( \triangle A_j = \partial_k\tau_{kj} - \frac{1}{4}\partial_j\tau_{kk} \) which is clearly inversible since we know the laplacian’s green function. Hence, there always exists a peculiar gauge where the solution is

\[ \tau_{ij} = \triangle^{-1}\{-4\pi G \sigma^{ij} + \left[ \partial^2_{ij}w_j + \partial^2_{ij}w_i - \partial_iw\partial_jw + 2\eta^{ij}\left( \partial^2_{ij}w + \partial^2_{ij}w_k \right) \right]\}, \]  

(26)

where

\[ \triangle^{-1}\{f(t,\vec{x})\} = \int \frac{f(t,\vec{x}')}{|\vec{x} - \vec{x}'|}d^3x'. \]  

(27)

Then, from this equation, we can get the solution \( \tau_{kl} \) in any gauge, using any vector field \( V_k \) with \( \tau'_{kl} = \tau_{kl} + \partial_kV_l + \partial_lV_k \).

B. The harmonic gauge

The harmonic gauge condition stands \( g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} = 0 \) (Or \( \partial_\sigma\left(\sqrt{-g}g^{\sigma\gamma}\right) = 0 \) with \( g \) the metric determinant). As seen in DSX \( 12 \), the harmonic 1.5PN/BM condition \( g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma = O(c^{-4}) \) gives the usual harmonic constraint on the "potentials" \( \partial_iw + \partial_kw_k = O(c^{-2}) \). While, the conditions \( g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma = O(c^{-4}) \) give no more information due to the "isotropy condition" which makes spatial coordinates always harmonic modulo \( O(c^{-4}) \). At the 2PN/RM level, the harmonic constraint is \( g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma = O(c^{-5}) \). That gives the harmonic conditions on the "potentials" at the 2PN/RM level

\[ \partial_kw + \partial_kw_k = O(c^{-2}) \]  

(28)

\[ \partial_k\tau_{kk} - \frac{1}{2}\partial_i\tau_{kk} + \partial_iw_i = O(c^{-1}). \]  

(29)

Then \( \Theta_{ij}(\tau_{kl}) \) writes:

\[ \Theta_{ij}(\tau_{kl}) = -\{\triangle \tau_{ij} + \partial_i(\partial_jw_j + \partial_jw_i)\}. \]  

(30)

Hence, considering \( w \) and \( w_i \) as source terms, the solution of the Einstein field equations in the harmonic gauge is :

\[ \tau_{ij} = \triangle^{-1}\{-4\pi G \sigma^{ij} - \partial_iw\partial_jw\}, \]  

(31)

from non-harmonic to harmonic gauge

If \( \tau_{ij} \) is not an harmonic solution, then the gauge transformation \( \tau_{ij} = \epsilon_{ij} + \partial_iA_j + \partial_jA_i \) makes \( \epsilon_{ij} \) an harmonic solution if \( A_j \) satisfies

\[ \triangle A_j = \partial_k\tau_{kj} - \frac{1}{2}\partial_j\tau_{kk} + \partial_iw_j \]  

(32)

where \( w_j \) must be expressed in the harmonic gauge with the 1.5PN/BM field equations \( 12 \). Since it is inversible, this relation always allow us to put the metric in the harmonic gauge, whatever is the gauge from which we start.

V. CONCLUSIONS

In order to fit future inter-planetary telemetry data, we developed a GR metric up to the 2 PN/RM order with the DSX "exponential parametrisation". We gave both general and harmonic formal solutions. A more detailed version of this result can be found in \( 11 \). An explicit solution of these equations and of the (non-massive-)Scalar-Tensor 2PN/RM field equations in the case of the solar system is in preparation.

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