THE RADIAL DEFOCUSING ENERGY-SUPERCRITICAL NLS IN
DIMENSION FOUR

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Abstract. We consider the radial defocusing nonlinear Schrödinger equations
\[ iu_t + \Delta u = |u|^p u \]
with supercritical exponent \( p > 4 \) in four space dimensions, and prove that any radial solution that remains bounded in the critical Sobolev space must be global and scatter.

Key Words: Nonlinear Schrödinger equation; scattering; long-time Strichartz estimate; energy supercritical; concentration-compactness.

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1. Introduction

This paper is devoted to study the initial-value problem for defocusing nonlinear Schrödinger equations of the form
\[
\begin{aligned}
(i\partial_t + \Delta)u &= |u|^p u, \\
(\partial_t + \Delta)u &= u_0(x) \in H^{s_c}(\mathbb{R}^4),
\end{aligned}
\]
where \( u(t,x) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C} \), and \( s_c = 2 - \frac{2}{p} \) with \( p > 4 \). We prove that for \( s > s_c \) (i.e. \( p > 4 \)), any radial maximal-lifespan solution that remains uniformly bounded in \( \dot{H}^{s_c}(\mathbb{R}^4) \) must be global and scatter. In [10, 23], the authors proved the analogous statement for (1.1) with \( 2 < p \leq 4 \) and non-radial initial data. In this paper, we treat the remaining case \( p > 4 \) but for radial initial data.

The equation (1.1) has the scaling invariance symmetry:
\[
u_\lambda(x,t) = \lambda^{\frac{2}{s_c}} u(\lambda x, \lambda^2 t),
\]
in the sense that both the equation and the \( \dot{H}^{s_c}(\mathbb{R}^4) \)-norm are invariant under the scaling transformation:
\[
\|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}(\mathbb{R}^4)} = \|u(\lambda^2 t,\cdot)\|_{\dot{H}^{s_c}(\mathbb{R}^4)}.
\]
This scaling defines a notion of criticality for (1.1). If we take \( u_0 \in \dot{H}^{s_c}(\mathbb{R}^4) \), then for \( s = s_c \) we call the problem (1.1) critical. For \( s > s_c \) we call the problem subcritical, while for \( s < s_c \) we call the problem supercritical.

If the solution \( u \) of (1.1) has sufficient decay at infinity and smoothness, it conserves mass and energy:
\[
\begin{aligned}
M(u) &= \int_{\mathbb{R}^4} |u(t,x)|^2 dx = M(u_0), \\
E(u) &= \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u|^2 dx + \frac{1}{p+2} \int_{\mathbb{R}^4} |u(t,x)|^{p+2} dx = E(u_0).
\end{aligned}
\]
As similarly explained in [2], the above quantities are also conserved for the energy solutions \( u \in C^0_t(\mathbb{R}, H^1(\mathbb{R}^4)) \). We call \( \dot{H}^1_x(\mathbb{R}^4) \) the energy space.
There is a lot of works on the global well-posedness and scattering for Schrödinger equations

\[ i\partial_t u - \Delta u = \pm|u|^p u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \quad \text{(NLS)} \]

especially for mass-critical \((p = 4/d)\) or energy-critical \((p = 4/(d-2), \ d \geq 3)\) (NLS), most notably by Bourgain [1], Colliander, Keel, Staffilani, Takaoka and Tao [4], Kenig and Merle [14] and Killip and Visan [15],蔬川, Inui, Shiraishi, Tatsuta and Tsuchiizu [17, 21, 22], and Dodson [5, 6, 7, 8, 9] for the energy-critical case and Tao, Visan and Zhang [29], Killip, Tao and Visan [16], Killip, Visan and Zhang [21] and Dodson [5, 6, 7, 8] for the mass-critical case.

So far, there is no technology for treating large-data (NLS) without some a priori control of a critical norm other than the energy-critical (NLS) and mass-critical (NLS) at the present moment. Kenig-Merle [15] first showed that if the radial solution \( u \) to NLS is such that \( u \in L^3_t (H^{s_c}(\mathbb{R}^3)) \) with \( s_c = 1/2 \), then \( u \) is global and scatters. They were able to handle this case by making use of their concentration compactness technique (as in [14]), together with the Lin-Strauss Morawetz inequality which scales like \( \dot{H}^1_x(\mathbb{R}^3) \). Lately, such result has been extended to high dimensional and inter-critical cases by Murphy [24, 25, 26]. In [17], Killip–Visan proved some cases of the energy-supercritical regime. In particular, they deal with the case of a cubic nonlinearity for \( d \geq 5 \), along with some other cases for which \( s_c > 1 \) and \( d \geq 5 \). The restriction to high dimensions stems from the so-called double Duhamel trick; see [17] for more details. Recently, by making use of the tool “long time Strichartz estimate” à la Dodson [5], and a frequency-localized interaction Morawetz inequality, the authors [10, 23] treat the case \( 2 < p \leq 4 \) (i.e. \( 1 < s_c \leq 3/2 \)) in four space dimensions. In this paper, we address the remaining cases \( p > 4 \) but for radial initial data, where the techniques in [10, 23] break down.

Now, let us make the notion of a solution more precise. A function \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) on a non-empty time interval \( I \ni 0 \) is a solution to (1.1) if it belongs to \( C_t L^3_{t,x}(J \times \mathbb{R}^4) \cap \dot{L}^p_{t,x}(J \times \mathbb{R}^4) \) for any compact interval \( J \subset I \) and obeys the Duhamel formula

\[ u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} F(u(s)) \ ds \quad \text{(1.4)} \]

for each \( t \in I \). We call \( I \) the lifespan of \( u \). We say that \( u \) is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. We call \( u \) global if \( I = \mathbb{R} \).

For a solution \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) to (1.1), we define the scattering size of \( u \) on \( I \) by

\[ S_I(u) := \int_I \int_{\mathbb{R}^4} |u(t,x)|^{3p} \ dx \ dt. \quad \text{(1.5)} \]

If there exists \( t_0 \in I \) such that \( S_{[t_0, \sup I]}(u) = \infty \), we say that \( u \) blows up forward in time. If there exists \( t_0 \in I \) such that \( S_{[\inf I, t_0]}(u) = \infty \), we say that \( u \) blows up backward in time.

If \( u \) is a global and obeys \( S_\mathbb{R}(u) < \infty \), then standard arguments show that \( u \) scatters in the sense that there exist unique \( u_\pm \in \dot{H}^{s_c}(\mathbb{R}^4) \) such that

\[ \lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_\pm\|_{\dot{H}^{s_c}(\mathbb{R}^4)} = 0. \]

Our main result is the following
Theorem 1.1. Let \( s_c > 3/2 \), i.e. \( p > 4 \). Suppose \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) is a radial maximal-lifespan solution to (1.1) such that

\[
u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^4).
\]

Then \( u \) is global and scatters, with

\[
S_{R}(u) \leq C(\|u\|_{L_t^\infty(\mathbb{R}, \dot{H}_x^{s_c}(\mathbb{R}^4))})
\]

for some function \( C : [0, \infty) \to [0, \infty) \).

Let us turn now to an outline of the arguments we will use to establish Theorem 1.1.

1.1. Outline of the proof of Theorem 1.1. We will argue by contradiction. For any \( 0 \leq E_0 < +\infty \), we define

\[
L(E_0) := \sup \left\{ S_I(u) : u : I \times \mathbb{R}^4 \to \mathbb{C} \text{ such that } \sup_{t \in I} \|u\|_{\dot{H}_x^{s_c}(\mathbb{R}^4)}^2 \leq E_0 \right\},
\]

where the supremum ranges all solutions \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) to (1.1) satisfying \( \|u\|_{\dot{H}_x^{s_c}(\mathbb{R}^4)}^2 \leq E_0 \). Thus, \( L : [0, +\infty) \to [0, +\infty) \) is a non-decreasing function. Moreover, from the small data theory (via Strichartz estimates and contraction mapping, cf. [2, 20]), one has

\[
L(E_0) \lesssim E_0^\frac{4}{8} \quad \text{for} \quad E_0 \leq \eta_0^2,
\]

where \( \eta_0 = \eta(d) \) is the threshold from the small data theory.

It follows from the stability theory that \( L \) is continuous. Thus, there must exist a unique critical \( E_c \in (0, +\infty) \) such that \( L(E_0) < +\infty \) for \( E_0 < E_c \) and \( L(E_0) = +\infty \) for \( E_0 \geq E_c \). In particular, if \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) is a maximal-lifespan solution to (1.1) such that \( \sup_{t \in I} \|u\|_{\dot{H}_x^{s_c}(\mathbb{R}^4)}^2 < E_c \), then \( u \) is global and moreover,

\[
S_{R}(u) \leq L(\|u\|_{L_t^\infty(\mathbb{R}, \dot{H}_x^{s_c}(\mathbb{R}^4))}).
\]

The proof of Theorem 1.1 is equivalent to show \( E_c = +\infty \). We argue by contradiction. The failure of Theorem 1.1 would imply the existence of very special class of solutions. On the other hand, these solutions have so many good properties that they do not exist. Thus we get a contradiction. While we will make some further reductions later, the main property of the special counterexamples is almost periodicity modulo symmetries:

Definition 1.2 (Almost periodic solutions). Let \( s_c > 0 \). A solution \( u : I \times \mathbb{R}^4 \to \mathbb{C} \) to (1.1) is called almost periodic (modulo symmetries) if \( u \in L_t^\infty \dot{H}_x^{s_c}(I \times \mathbb{R}^4) \) and there exist functions \( N : I \to \mathbb{R}^+ \), \( x : I \to \mathbb{R}^4 \) and \( C : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for all \( t \in I \) and \( \eta > 0 \),

\[
\int_{|x-x(t)| \geq \eta} \left| \nabla \right|^{s_c} u(t, x) \right|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi|^{2s_c} |\hat{u}(t, \xi)|^2 d\xi \leq \eta.
\]

We call \( N(t) \) the frequency scale function, \( x(t) \) the spatial center function, and \( C(\eta) \) the compactness modulus function.

Remark 1.3. (i) As a consequence of radiality, the solutions we consider can only concentrate near the spatial origin. In particular, we may take \( x(t) \equiv 0 \).
for some compact $K$ for all $t$ there exists a radial maximal-lifespan solution an almost periodic solution to Strichartz norms. Theorem 1.4 would imply the existence of the almost periodic solutions as follows.

Moreover, $N(t)$ periodic and blows up in both time directions.

By the same argument as in [10, 20, 23], we can show that the failure of Theorem 1.1 would imply the existence of the almost periodic solutions as follows.

**Theorem 1.4** (Reduction to almost periodic solutions). If Theorem 1.1 fails, then there exists a radial maximal-lifespan solution $u : I \times \mathbb{R}^4 \to \mathbb{C}$ to (1.1) that is almost periodic and blows up in both time directions.

Furthermore, one can adopt the proof of Lemma 5.18 in [20] to prove that the almost periodic solutions satisfy the following local constancy property:

**Lemma 1.5** (Local constancy). Let $u : I \times \mathbb{R}^4 \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to (1.1). Then there exists $\delta = \delta(u) > 0$ such that for all $t_0 \in I$,

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I.$$

Moreover, $N(t) \sim_u N(t_0)$ for $|t - t_0| \leq \delta N(t_0)^{-2}$.

**Definition 1.6** (Interval of local constancy). Let $u : I \times \mathbb{R}^4 \to \mathbb{C}$ be a almost periodic solution. We can divide $I$ into consecutive intervals $J_k$ such that

$$\|u(t, x)\|_{L_t^{2s}([J_k \times \mathbb{R}^4])} = 1, \quad \text{and} \quad N(t) \equiv N_k, \quad t \in J_k.$$

These intervals are called as the intervals of local constancy. If $J \subset I$ is a union of consecutive intervals of local constancy, then we have

$$\sum_{J_k} N(J_k)^{1-2s} \sim \int_J N(t)^{3-2s} dt. \quad (1.9)$$

**Lemma 1.5** provides information about the behavior of the frequency scale at blowup (cf. [20, Corollary 5.19]):

**Corollary 1.7** ($N(t)$ at blowup). Let $u : I \times \mathbb{R}^4 \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to (1.1). If $T$ is a finite endpoint of $I$ then $N(t) \lesssim_{u} |T - t|^{-\frac{1}{2}}$.

We also need the following result, which relates the frequency scale function of an almost periodic solution to its Strichartz norms.
Lemma 1.8 (Spacetime bounds). Let $u : I \times \mathbb{R}^4 \to \mathbb{C}$ be an almost periodic solution to (1.1). Then, there holds

$$\int_I N(t)^2 \, dt \lesssim_{u} \left\| |\nabla|^{s_c} u \right\|_{L^q_t L^r_x(I \times \mathbb{R}^4)}^q \lesssim_{u} 1 + \int_I N(t)^2 \, dt,$$

for $(q, r)$ admissible (see Definition 2.2 below) with $q < \infty$.

Proof. See Lemma 5.21 in [20].

We now refine the class of almost periodic solutions that we consider. By rescaling arguments as in [16, 18, 29], we can guarantee that the almost periodic solutions we consider do not escape to arbitrarily low frequencies on at least half of their maximal lifespan, say $[0, T_{\text{max}})$. Using Lemma 1.5 to divide $[0, T_{\text{max}})$ into characteristic subintervals $J_k$, we arrive at the following theorem.

Theorem 1.9 (Two scenarios for blowup). If Theorem 1.1 fails, then there exists a radial almost periodic solution $u : [0, T_{\text{max}}) \times \mathbb{R}^4 \to \mathbb{C}$ that blows up forward in time and satisfies

$$u \in L^\infty_t \dot{H}^{3/2}_x([0, T_{\text{max}}) \times \mathbb{R}^4).$$

(1.10)

Furthermore, we may write $[0, T_{\text{max}}) = \bigcup_k J_k$, where

$$N(t) \equiv N_k \geq 1 \quad \text{for} \quad t \in J_k, \quad \text{with} \quad |J_k| \sim_{u} N_k^{-2}. \quad (1.11)$$

We classify $u$ according to the following two scenarios: either

$$\int_0^{T_{\text{max}}} N(t)^{3-2s_c} \, dt < \infty \quad \text{(rapid frequency-cascade solution)}, \quad (1.12)$$

or

$$\int_0^{T_{\text{max}}} N(t)^{3-2s_c} \, dt = \infty \quad \text{(quasi-soliton solution)}. \quad (1.13)$$

In view of this theorem, our goal is to preclude the possibilities of all the scenarios in the sense of Theorem 1.9. The quantity appearing in (1.12) and (1.13) is related to the the Lin–Strauss Morawetz inequality of [22], which is given by

$$\int_{I \times \mathbb{R}^4} \frac{|u(t, x)|^{p+2}}{|x|} \, dx \, dt \lesssim \left\| |\nabla|^{1/2} u \right\|_{L^p_t L^2_x(I \times \mathbb{R}^4)}^2, \quad (1.14)$$

Due to the weight $1/|x|$, the Lin–Strauss Morawetz inequality is well suited for preventing concentration near the origin, and hence it is most effective in the radial setting. In fact, it is the use of Lin–Strauss Morawetz inequality that leads to the restriction to the radial setting in Theorem 1.1. We cannot use this estimate directly, however, as the solutions we consider need only belong to $L^\infty_t \dot{H}^{3/2}_x$ (and so the right-hand side of (1.14) need be infinite).

A further manifestation of the minimality of $u$ as a blow-up solution is the absence of the scattered wave at the endpoints of the lifespan $I$; more formally, we have the following Duhamel formula, which is important for showing the additional decay and negative regularity for the rapid frequency cascade. This is a robust consequence of almost periodicity modulo symmetries; see, for example, [4].
Lemma 1.10 (Reduced Duhamel formula). Let \( u : [0, T_{\text{max}}) \times \mathbb{R}^4 \to \mathbb{C} \) be a maximal-lifespan almost periodic solution to (1.1). Then for all \( t \in [0, T_{\text{max}}) \) we have
\[
   u(t) = \lim_{T \to T_{\text{max}}} i \int_t^T e^{i(t-s)\Delta} F(u(s)) \, ds
\] (1.15)
as a weak limit in \( \dot{H}^{s_c}_x(\mathbb{R}^4) \).

We will use this lemma to show that the almost periodic solution enjoys the “long time Strichartz estimate” à la Dodson [5], see Theorem 3.1 below. We will utilize this “long time Strichartz estimate” and the no-waste Duhamel formula to show the rapid frequency cascade solutions admit the negative regularity property. Then making use of a similar method used in \([16, 17, 21]\), we can also show that the mass of the rapid frequency cascade solution is zero and so we get a contradiction.

Finally, to preclude the quasi-soliton solutions, one can show that it admits additional decay in the sprit of no-waste Duhamel formula and the compactness, see [23, Proposition 3.1].

Proposition 1.11 (Additional decay, [23]). Let \( s_c > 1 \). Suppose \( u : [0, T_{\text{max}}) \times \mathbb{R}^4 \to \mathbb{C} \) is an almost periodic solution to (1.1) such that \( \inf_{t \in [0, T_{\text{max}})} N(t) \geq 1 \). Then
\[
   u \in L_t^\infty L_x^{p+1}([0, T_{\text{max}}) \times \mathbb{R}^4). (1.16)
\]
This together with long-time Strichartz estimate allows us to show the localized Morawetz inequality enjoys the bounds as follows.

Proposition 1.12. Let \( u \) be a quasi-soliton solution to (1.1) in the sense of Theorem 1.9. Then for any \( \eta > 0 \), there exists \( N_0 = N_0(\eta) \) such that for all \( N \leq N_0 \)
\[
   K_I \lesssim_u \int_I \int_{\mathbb{R}^4} \frac{|u_N(t, x)|^{p+2}}{|x|} \, dx \, dt \lesssim_u \eta (N^{1-2s_c} + K_I),
\]
for all \( I \subset [0, +\infty) \), where \( K_I := \int_I N(t)^{3-2s_c} \, dt. \)

This leads to a contradiction in the case when \( \int_0^\infty N(t)^{3-2s_c} = +\infty \), since \( K_I \) can be taken arbitrarily large.

The paper is organized as follows. In Section 2, we give some useful lemmas. In Section 3, we show the long time Strichartz estimate. In Section 4, we exclude the existence of rapid frequency cascade solutions in the sense of Theorem 1.9. In Section 5, we show the good upper bound for the localized Morawetz estimate. Finally, we rule out the existence of quasi-soliton solution. Hence we conclude the proof of Theorem 1.1.

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2. Preliminaries

2.1. Some notation. For nonnegative quantities \( X \) and \( Y \), we will write \( X \lesssim Y \) to denote the estimate \( X \leq CY \) for some \( C > 0 \). If \( X \lesssim Y \lesssim X \), we will write \( X \sim Y \). Dependence of implicit constants on the power \( p \) or the dimension will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, \( X \lesssim_u Y \) indicates \( X \leq CY \) for some \( C = C(u) \).
We will use the expression $O(X)$ to denote a finite linear combination of terms that resemble $X$ up to Littlewood–Paley projections, complex conjugation, and/or maximal functions. For example, we will use the expression $X \pm \varepsilon$ for any $\varepsilon > 0$.

For a spacetime slab $I \times \mathbb{R}^4$, we write $L^q_t L^r_x(I \times \mathbb{R}^4)$ for the Banach space of functions $u : I \times \mathbb{R}^4 \to \mathbb{C}$ equipped with the norm
\[
\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^4)} := \left( \int_I \|u(t)\|_{L^r_x(\mathbb{R}^4)}^q \right)^{1/q},
\]
with the usual adjustments when $q$ or $r$ is infinity. When $q = r$, we abbreviate $L^q_t L^q_x = L^q_t L^q_x$. We will also often abbreviate $\|f\|_{L^q_x(\mathbb{R}^4)}$ to $\|f\|_{L^q_x}$. For $1 \leq r \leq \infty$, we use $r'$ to denote the dual exponent to $r$, i.e. the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

We define the Fourier transform on $\mathbb{R}^4$ by
\[
\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{-ix \cdot \xi} f(x) \, dx.
\]
We can then define the fractional differentiation operator $|\nabla|^s$ for $s \in \mathbb{R}$ via
\[
|\nabla|^s f(\xi) := |\xi|^s \hat{f}(\xi),
\]
with the corresponding homogeneous Sobolev norm
\[
\|f\|_{H^s_x(\mathbb{R}^4)} := \||\nabla|^s f\|_{L^2_x(\mathbb{R}^4)}.
\]

2.2. Basic harmonic analysis. We will make frequent use of the Littlewood–Paley projection operators. Specifically, we let $\varphi$ be a radial bump function supported on the ball $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. For $N \in 2\mathbb{Z}$, we define the Littlewood–Paley projection operators by
\[
\widehat{P_{\leq N}} f(\xi) := \mathcal{F}(f_{\leq N})(\xi) := \varphi(\xi/N) \hat{f}(\xi),
\]
\[
\widehat{P_{> N}} f(\xi) := \mathcal{F}(f_{> N})(\xi) := (1 - \varphi(\xi/N)) \hat{f}(\xi),
\]
\[
\widehat{P_N} f(\xi) := \mathcal{F}(f_N)(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{f}(\xi).
\]
We may also define
\[
P_{M < \leq N} := P_{\leq N} - P_{M} = \sum_{M < N \leq N} P_N'
\]
for $M < N$. All such summations should be understood to be over $N' \in 2\mathbb{Z}$.

The Littlewood–Paley operators commute with derivative operators, the free propagator, and the conjugation operation. These operators are self-adjoint and bounded on every $L^p_t$ and $\dot{H}^s_x$ space for $1 \leq p \leq \infty$ and $s \geq 0$. They also obey the following standard Bernstein estimates: For $1 \leq r \leq q \leq \infty$ and $s \geq 0$,
\[
\||\nabla|^{\pm s} P_N f\|_{L^r_t(\mathbb{R}^4)} \sim N^{\pm s} \|P_N f\|_{L^q_t(\mathbb{R}^4)};
\]
\[
\||\nabla|^{s} P_{\leq N} f\|_{L^r_t(\mathbb{R}^4)} \lesssim N^{s} \|P_{\leq N} f\|_{L^r_t(\mathbb{R}^4)};
\]
\[
\|P_{> N} f\|_{L^q(\mathbb{R}^4)} \lesssim N^{-s} \||\nabla|^s P_{> N} f\|_{L^r(\mathbb{R}^4)};
\]
\[
\|P_{\leq N} f\|_{L^q(\mathbb{R}^4)} \lesssim N^{\frac{s}{2} - \frac{1}{r}} \|P_{\leq N} f\|_{L^r(\mathbb{R}^4)}.
\]

We will need the following fractional calculus estimates and paraproduct estimates.
Lemma 2.1. (i) (Fractional product rule, [3]) Let $s \geq 0$, and $1 < r, r_j, q_j < \infty$ satisfy $\frac{s}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, for $i = 1, 2$. Then

$$\ number 1 \$$ \left\| \left\| \nabla^{s} (fg) \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| \leq \left\| \left\| f \right\|_{L_{x}^{r_1}(\mathbb{R}^{4})} \right\| \left\| \left\| \nabla^{s} g \right\|_{L_{x}^{r_2}(\mathbb{R}^{4})} \right\| + \left\| \left\| \nabla^{s} f \right\|_{L_{x}^{r_2}(\mathbb{R}^{4})} \right\| \left\| \left\| g \right\|_{L_{x}^{r_2}(\mathbb{R}^{4})} \right\| \right. \right. \right.

$$ (2.1)$$

(ii) (Paraproduct estimate, [23]) Let $0 < s < 1$. If $1 < r < r_1 < \infty$ and $1 < r_2 < \infty$ satisfy $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} + \frac{s}{4} < 1$, then

$$\ number 2 \$$ \left\| \left\| \nabla^{-s}(fg) \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| \leq \left\| \left\| \nabla^{-s}f \right\|_{L_{x}^{r_1}(\mathbb{R}^{4})} \right\| \left\| \left\| \nabla^{s}g \right\|_{L_{x}^{r_2}(\mathbb{R}^{4})} \right\| \right. \right. \right.

$$ (2.2)$$

(iii) (Basic estimate) From the Hardy inequality and interpolation, we easily get for $0 \leq s \leq 1$

$$\ number 3 \$$ \left\| \left\| \nabla^{s}(\frac{1}{r_2}u) \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| \leq \left\| \left\| \nabla^{s}u \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| \right. \right. \right.

$$ (2.3)$$

2.3. Strichartz estimates. Let $e^{it\Delta}$ be the free Schrödinger propagator, given by

$$\ number 4 \$$ \left\| \left\| e^{it\Delta}f \right\|_{L_{x}^{\infty}(\mathbb{R}^{4})} \right\| \leq \left\| \left\| t^{-\frac{3}{2}}f \right\|_{L_{x}^{2}(\mathbb{R}^{4})} \right\| \right. \right. \right.

$$ (2.4)$$

for $t \neq 0$. From this explicit formula we can read off the dispersive estimate

$$\ number 5 \$$ \left\| \left\| e^{it\Delta}f \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| \leq \left\| \left\| t^{-\frac{3}{2}}f \right\|_{L_{x}^{2}(\mathbb{R}^{4})} \right\| \right. \right. \right.

$$ (2.5)$$

for $t \neq 0$ and $2 \leq r \leq \infty$, where $\frac{1}{q} + \frac{1}{r} = 1$. This estimate implies the standard Strichartz estimates, which we will state below. First, we need to make the following definition:

Definition 2.2 (Admissible pairs). A pair of exponents $(q, r)$ is called Schrödinger admissible if $2 \leq q, r \leq \infty$ and $\frac{2}{q} + \frac{4}{r} = 2$. For a spacetime slab $I \times \mathbb{R}^{4}$, we define the Strichartz norm

$$\ number 6 \$$ \left\| \left\| u \right\|_{S^{0}(I)} \right\| := \sup \{ \left\| \left\| u \right\|_{L_{x}^{q}L_{t}^{r}(I \times \mathbb{R}^{4})} \right\| : (q, r) \text{ Schrödinger admissible} \}.$$

We denote $S^{0}(I)$ to be the closure of all test functions under this norm and write $N^{0}(I)$ for the dual of $S^{0}(I)$.

We may now state the standard Strichartz estimates in the form that we will need them.

Proposition 2.3 (Strichartz, [11, 13, 27]). Let $s \geq 0$ and suppose $u : I \times \mathbb{R}^{4} \rightarrow \mathbb{C}$ is a solution to $(i\partial_{t} + \Delta)u = F$. Then

$$\ number 7 \$$ \left\| \left\| \nabla^{s}u \right\|_{S^{0}(I)} \right\| \leq \left\| \left\| \nabla^{s}u(t_{0}) \right\|_{L_{x}^{r}(\mathbb{R}^{4})} \right\| + \left\| \left\| \nabla^{s}F \right\|_{N^{0}(I)} \right\| \right. \right. \right.

for any $t_{0} \in I$.

3. Long time Strichartz estimate

In this section we establish a long-time Strichartz estimate for almost periodic solutions to (1.1) as in Theorem 1.9.
Theorem 3.1 (Long-time Strichartz estimate). Let \( s_c > 3/2 \), and let \( u : [0, T_{\text{max}}) \times \mathbb{R}^4 \rightarrow \mathbb{C} \) be an almost periodic solution to (1.1) with \( N(t) = N_k \geq 1 \) on each characteristic \( J_k \subset [0, T_{\text{max}}) \). Suppose

\[
 u \in L^\infty_t (\mathbb{R}^4), \quad H^s (\mathbb{R}^4), \quad \text{for some } s_c - \frac{1}{2} < s \leq s_c. \tag{3.1}
\]

Then on any compact time interval \( I \subset [0, T_{\text{max}}) \), which is a union of characteristic subintervals \( J_k \), and for any \( N > 0 \), we have

\[
 \| |\nabla|^{s_c} u \|_{L^2_t L^4_x (I \times \mathbb{R}^4)} \lesssim_u 1 + N^{\sigma(s)} K_I^{1/2}, \tag{3.2}
\]

where \( K_I := \int_I N(t)^{3-2s_c} \, dt \) and \( \sigma(s) = 2s_c - s - \frac{1}{2} \). In particular, for \( s = s_c \), we have

\[
 \| |\nabla|^{s_c} u \|_{L^2_t L^4_x (I \times \mathbb{R}^4)} \lesssim_u 1 + N^{s_c - \frac{1}{2}} K_I^{1/2}. \tag{3.3}
\]

Moreover, for any \( \eta > 0 \), there exists \( N_0 = N_0(\eta) \) such that for all \( N \leq N_0 \),

\[
 \| |\nabla|^{s_c} u \|_{L^2_t L^4_x (I \times \mathbb{R}^4)} \lesssim_u \eta (1 + N^{s_c - 1/2} K_I^{1/2}). \tag{3.4}
\]

Furthermore, the implicit constants in (3.2) and (3.4) are independent of \( I \).

We fix \( I \subset [0, T_{\text{max}}) \) to be a union of contiguous characteristic subintervals. Throughout the proof, all spacetime norms will be taken over \( I \times \mathbb{R}^4 \) unless explicitly stated otherwise. For \( N > 0 \), we define the quantities

\[
 B(N) := \| |\nabla|^{s_c} u \|_{L^2_t L^4_x (I \times \mathbb{R}^4)} \quad \text{and} \quad B_k(N) := \| |\nabla|^{s_c} u \|_{L^2_t L^4_x (J_k \times \mathbb{R}^4)}. \]

We will prove Theorem 3.1 by induction. For the base case, we have the following

**Lemma 3.2.** The estimate (3.2) holds for \( N \geq N_{\text{max}} := \sup_{J_k \subset I} N_k \).

**Proof.** We have by Lemma 1.8

\[
 \| |\nabla|^{s_c} u \|_{L^2_t L^4_x}^2 \lesssim_u 1 + \int_I N(t)^2 \, dt \lesssim_u 1 + N^{2s_c - 1} \max N_{\text{max}} \int_I N(t)^{3-2s_c} \, dt \lesssim 1 + N^{2\sigma(s)} K_I,
\]

which shows (3.2). \( \square \)

To complete the induction, we establish a recurrence relation for \( B(N) \). To this end, we first let \( \varepsilon_0 > 0 \) and \( \varepsilon > 0 \) be small parameters to be determined later. We use the compact property to find \( c = c(\varepsilon) \) so that

\[
 \| |\nabla|^{s_c} u \|_{L^\infty_t L^2_x} < \varepsilon. \tag{3.5}
\]

The recurrence relations we will use take the following form.

**Lemma 3.3 (Recurrence relations for \( B(N) \)).**

\[
 B(N) \lesssim_u \inf_{\varepsilon \in (0, \varepsilon_0]} \| |\nabla|^{s_c} u \|_{L^2_x} + C(\varepsilon, \varepsilon_0) N^{\sigma(s)} K_I^{1/2} + \varepsilon^p B(N/\varepsilon_0) + \sum_{M > N/\varepsilon_0} (N/M)^{s_c} B(M) \tag{3.6}
\]

uniformly in \( N \) for some positive constant \( C(\varepsilon, \varepsilon_0) \).
We also have the following refinement of \((3.6)\):

\[
B(N) \lesssim g(N) \left(1 + N^{s_- - 1/2} K_{1}^{1/2}\right) + \varepsilon^{p} B(N/\varepsilon_{0}) + \sum_{M > N/\varepsilon_{0}} (\frac{N}{M})^{s_-} B(M),
\]

where

\[
g(N) := \inf_{t \in I} \left\| \nabla^{s_-} u_{\leq N}(t) \right\|_{L_{2}^{2}} + C(\varepsilon, \varepsilon_{0}) \sup_{J_{k} \subset I} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{\infty} L_{2}^{2}(J_{k} \times \mathbb{R}^{4})}.
\]

**Proof.** First, by Strichartz estimate, we get

\[
B(N) \lesssim \inf_{t \in I} \left\| \nabla^{s_-} u_{\leq N}(t) \right\|_{L_{2}^{2}} + \left\| \nabla^{s_-} P_{\leq N} F(u) \right\|_{L_{2}^{1} L_{2}^{4/3}}.
\]

Hence we only need to estimate the nonlinear term. To this end, we decompose

\[
F(u) = F(u_{\leq N/\varepsilon_{0}}) + \left( F(u) - F(u_{\leq N/\varepsilon_{0}}) \right).
\]

Using Bernstein, Hölder, Sobolev embedding, we obtain

\[
\left\| \nabla^{s_-} P_{\leq N} F(u - F(u_{\leq N/\varepsilon_{0}})) \right\|_{L_{2}^{1} L_{2}^{4/3}} \lesssim N^{s_-} \left\| F(u) - F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}} \lesssim N^{s_-} \left\| u \right\|_{L_{2}^{p} L_{2}^{p}} \left\| u_{> N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}} \lesssim \sum_{M > N/\varepsilon_{0}} (\frac{N}{M})^{s_-} B(M). \tag{3.10}
\]

To estimate the term \(\left\| \nabla^{s_-} P_{\leq N} F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}}\), we restrict our attention to a single characteristic interval \(J_{k}\). It is easy to see that

\[
\left\| \nabla^{s_-} P_{\leq N} F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})} \lesssim \left\| u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{p} L_{2}^{p}(J_{k} \times \mathbb{R}^{4})} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})} \lesssim \left\| \nabla^{s_-} P_{c(\varepsilon) N_{k} u_{\leq N/\varepsilon_{0}}} \right\|_{L_{2}^{p} L_{2}^{p}(J_{k} \times \mathbb{R}^{4})} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})} + \left\| \nabla^{s_-} P_{c(\varepsilon) N_{k} u_{\leq N/\varepsilon_{0}}} \right\|_{L_{2}^{p} L_{2}^{p}(J_{k} \times \mathbb{R}^{4})} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})}. \tag{3.11}
\]

By \((3.10)\), we have

\[
\left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})} \lesssim \varepsilon^{p} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})}. \tag{3.12}
\]

Note that it suffices to consider the case \(c(\varepsilon) N_{k} \leq N/\varepsilon_{0}\) in \((3.12)\), i.e. \(\frac{N}{c(\varepsilon) N_{k}} \geq 1\). In this case, using Hölder, Bernstein, Sobolev embedding, interpolation, Lemma \(1\) and \(10\), we derive that

\[
\left\| \nabla^{s_-} P_{\leq N} F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})}^{2} \lesssim \left(\frac{N}{c(\varepsilon) N_{k}}\right)^{\frac{3}{2} - \frac{2}{s_{+}}} \left\| \nabla^{s_-} u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{p} L_{2}^{p}(J_{k} \times \mathbb{R}^{4})}^{2} \lesssim \left(\frac{N}{c(\varepsilon) N_{k}}\right)^{\frac{3}{2} - \frac{2}{s_{+}}} \left(\frac{N}{\varepsilon_{0}}\right)^{2(s_{+} - s)} \left\| u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{p} H^{s}(J_{k} \times \mathbb{R}^{4})}^{2}.
\]

Hence,

\[
\left\| \nabla^{s_-} P_{\leq N} F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})}^{2} \lesssim \sum_{J_{k} \subset I} \left\| \nabla^{s_-} P_{\leq N} F(u_{\leq N/\varepsilon_{0}}) \right\|_{L_{2}^{1} L_{2}^{4/3}(J_{k} \times \mathbb{R}^{4})}^{2} \lesssim \varepsilon^{p} B(N/\varepsilon_{0}) + e^{-2(s_{+} - \frac{1}{2}) \frac{N}{\varepsilon_{0}}} \sup_{J_{k} \subset I} \left\| u_{\leq N/\varepsilon_{0}} \right\|_{L_{2}^{p} H^{s}(J_{k} \times \mathbb{R}^{4})}^{2} N^{2s(\varepsilon)} K_{1}.
\]
This together with (3.10) implies Lemma 3.3.

Next, we turn to prove Theorem 3.1.

Proof of Theorem 3.1. From Lemma 3.2 we see that (3.2) holds for $N \geq N_{\text{max}}$. That is, we have

$$B(N) \leq C(u)(1 + N^{\sigma(s)} K_t^{1/2}),$$

(3.13)

for all $N \geq N_{\text{max}}$. Clearly this inequality remains true if we replace $C(u)$ by any larger constant.

We now suppose that (3.13) holds for frequencies above $N$ and use the recurrence formula (3.6) to show that (3.13) holds at frequency $N/2$. Choosing $\varepsilon_0 < 1/2$, we use (3.6) and (3.13) to obtain

$$B(N/2) \leq \tilde{C}(u) \left( 1 + C(\varepsilon, \varepsilon_0)(N/2)^{\sigma(s)} K_t^{1/2} + \varepsilon^p B(N/2\varepsilon_0) + \sum_{M > N/2\varepsilon_0} (\frac{N}{2\varepsilon_0})^{s_c} B(M) \right)$$

$$\leq \tilde{C}(u) \left( 1 + C(\varepsilon, \varepsilon_0)(N/2)^{\sigma(s)} K_t^{1/2} + \varepsilon^p \left( 1 + (N/2\varepsilon_0)^{\sigma(s)} K_t^4 \right) + C(u) \sum_{M > N/2\varepsilon_0} (\frac{N}{2\varepsilon_0})^{s_c} \left( 1 + M^{\sigma(s)} K_t^{1/2} \right) \right)$$

$$\leq \tilde{C}(u) \left( 1 + C(\varepsilon, \varepsilon_0)(N/2)^{\sigma(s)} K_t^{1/2} + \varepsilon^p \left( 1 + (N/2\varepsilon_0)^{\sigma(s)} K_t^4 \right) + C(u) \varepsilon_0^{s_c} + C(u) \varepsilon_0^{s_c - \sigma(s)} (N/2)^{\sigma(s)} K_t^4 \right)$$

$$= \tilde{C}(u) \left( 1 + \varepsilon^p + C(u) \varepsilon_0^{s_c} \right)$$

$$+ \tilde{C}(u) \left( C(\varepsilon, \varepsilon_0) + \varepsilon^p \varepsilon_0^{\sigma(s)} + C(u) \varepsilon_0^{s_c - \sigma(s)} \right) (N/2)^{\sigma(s)} K_t^{1/2},$$

where we use $s_c - \frac{1}{2} < s$ in the third inequality to guarantee the convergence of the sum.

If we now choose $\varepsilon_0$ possibly even smaller depending on $\tilde{C}(u)$; $\varepsilon$ sufficiently small depending on $\tilde{C}(u)$ and $\varepsilon_0$; and $C(u)$ possibly larger such that

$$C(u) \geq \tilde{C}(u) \left( 1 + \varepsilon^p + C(u) \varepsilon_0^{s_c} \right) + \tilde{C}(u) \left( C(\varepsilon, \varepsilon_0) + \varepsilon^p \varepsilon_0^{\sigma(s)} + C(u) \varepsilon_0^{s_c - \sigma(s)} \right),$$

we get

$$B(N/2) \leq C(u) \left( 1 + (N/2)^{\sigma(s)} K_t^{1/2} \right).$$

Thus (3.13) holds at frequency $N/2$, and so we conclude (3.2) by induction.

Next, we will use the recurrence formula (3.7) to show (3.4). To do this, we note that for fixed $\varepsilon, \varepsilon_0 > 0$, we can use the compact property and the fact that $\inf_{t \in I} N(t) \geq 1$ to get

$$\lim_{N \to 0} g(N) = 0,$$

(3.14)

where $g(N)$ is as in (3.3).
4. The rapid frequency-cascade scenario

In this section, we rule out the existence of rapid frequency-cascade solutions, that is, almost periodic solutions as in Theorem 1.9 such that \( \int_0^{T_{\max}} N(t)^{3-2s_c} \, dt < \infty \). The proof will rely primarily on the long-time Strichartz estimate proved in the previous section.

**Theorem 4.1** (No rapid frequency-cascades). Let \( s_c > 3/2 \). Then there are no radial almost periodic solutions \( u : [0, T_{\max}) \times \mathbb{R}^4 \to \mathbb{C} \) to (1.1) with \( N(t) \equiv N_k \geq 1 \) on each characteristic subinterval \( J_k \subset [0, T_{\max}) \) such that
\[
\|u\|_{L_{t,x}^p([0,T_{\max}) \times \mathbb{R}^4)} = +\infty
\]
and
\[
K := \int_0^{T_{\max}} N(t)^{3-2s_c} \, dt < +\infty.
\]

We argue by contradiction. Suppose that \( u \) were such a solution. Then, using (4.2) and Corollary 1.7 we see
\[
\lim_{t \to T_{\max}} N(t) = \infty,
\]
whether \( T_{\max} \) is finite or infinite. Combining this with the compact property, we see that
\[
\lim_{t \to T_{\max}} \| |\nabla|^{s_c} u|_{N(t)}\|_{L_2^2(\mathbb{R}^4)} = 0
\]
for any \( N > 0 \).

**Proposition 4.2** (Lower regularity). Let \( s_c > 3/2 \). Let \( u : [0, T_{\max}) \times \mathbb{R}^4 \to \mathbb{C} \) be an almost periodic solution with (4.2). Suppose
\[
u \in L_t^\infty([0, T_{\max}), \dot{H}^s(\mathbb{R}^4)) \text{ for some } s_c - \frac{1}{2} < s \leq s_c.
\]
Then,
\[
u \in L_t^\infty([0, T_{\max}), \dot{H}^s(\mathbb{R}^4)), \quad \forall \ s - \sigma(s) < \alpha \leq s_c,
\]
where \( \sigma(s) = 2s_c - s - \frac{1}{2} \).

**Proof.** Let \( I_n \subset [0, T_{\max}) \) be a nested sequence of compact time intervals, each of which is a contiguous union of characteristic subintervals. We claim that for any \( N > 0 \), we have
\[
\| |\nabla|^{s_c} u|\leq N\|_{L_t^2 L_x^4(I_n \times \mathbb{R}^4)} \lesssim u \inf_{t \in I_n} \| |\nabla|^{s_c} u|\leq N(t)\|_{L_x^2} + N^{\sigma(s)}.
\]
Indeed, defining
\[
B_n(N) := \| |\nabla|^{s_c} u|\leq N\|_{L_t^2 L_x^4(I_n \times \mathbb{R}^4)},
\]
we have by (3.6) and (4.2)
\[
B_n(N) \lesssim u \inf_{t \in I_n} \| |\nabla|^{s_c} u|\leq N(t)\|_{L_x^2} + C(\varepsilon, \varepsilon_0) N^{\sigma(s)} + \varepsilon^p B(N/\varepsilon_0) + \sum_{M/N/\varepsilon_0} (N/M)^{s_c} B_n(M).
\]
Arguing as we did to obtain (3.2), we derive (4.7).

Now, letting \( n \to \infty \) in (4.7) and using (4.4), we obtain
\[
\| |\nabla|^{s_c} u|\leq N\|_{L_t^2 L_x^4([0, T_{\max}) \times \mathbb{R}^4)} \lesssim u N^{\sigma(s)}
\]
for all \( N > 0 \).
We now claim that (4.8) implies
\[
\left\| \nabla^s u_{\leq N} \right\|_{L_t^p L_x^2((0,T_{\text{max}}) \times \mathbb{R}^4)} \lesssim_u N^{2s_c - 1/2}
\] (4.9)
for all \( N > 0 \). To this end, we first use the reduced Duhamel formula and Strichartz to get
\[
\left\| \nabla^s u_{\leq N} \right\|_{L_t^p L_x^2((0,T_{\text{max}}) \times \mathbb{R}^4)} \lesssim \left\| \nabla^s P_{\leq N} F(u) \right\|_{L_t^p L_x^{4/3}((0,T_{\text{max}}) \times \mathbb{R}^4)}.
\] (4.10)
We now decompose \( F(u) \) by
\[
F(u) = F(u_{\leq N}) + (F(u) - F(u_{\leq N}))
\] (4.11)
and estimate the contribution of each piece individually.

We begin by estimating the contribution of the first term in (4.11) to (4.10). Using H"older, the fractional product rule, the fractional chain rule, Sobolev embedding, interpolation, \(\|u\|_{L_t^p L_x^2} \lesssim N^{s_c} \), \(\|u\|_{L_t^p L_x^{4/3}} \lesssim N^{s_c} \), \(\|u\|_{L_t^p L_x^2} \lesssim N^{s_c} \), and \(\|u\|_{L_t^p L_x^{4/3}} \lesssim N^{s_c} \), we estimate
\[
\left\| \nabla^s P_{\leq N} F(u_{\leq N}) \right\|_{L_t^p L_x^{4/3}} \lesssim \left( u \right)^p \left\| u \right\|_{L_t^p L_x^{4/3}} \left\| \nabla^s u_{\leq N} \right\|_{L_t^p L_x^{4/3}}.
\] (4.12)
where \( s_1 = 2 - \frac{2s - s_c + 2}{p} \in (s, s_c) \).

Next, we estimate the contribution of the second term in (4.11) to (4.10). We can use H"older, Bernstein, (2.2), (4.5) and (4.8) to estimate
\[
\left\| \nabla^s P_{\leq N} (F(u) - F(u_{\leq N})) \right\|_{L_t^p L_x^{4/3}} \lesssim N^{s_c} \left\| \nabla^{-(s - s_c)} \int_0^1 F(u_{\leq N} + \lambda u_{> N}) u_{> N} d\lambda \right\|_{L_t^p L_x^{4/3}}.
\]
\[
\lesssim \int_0^1 N^{s_c} \left\| \nabla^{-(s - s_c)} F(u_{\leq N} + \lambda u_{> N}) \right\|_{L_t^p L_x^{4/3}} \left\| \nabla^{-(s - s_c)} u_{> N} \right\|_{L_t^p L_x^{4/3}}\left\| u_{> N} \right\|_{L_t^p L_x^{4/3}}\sum_{M > N} \left( \frac{N}{M} \right)^{s_c} \left\| \nabla^{s_c} u_M \right\|_{L_t^p L_x^{4/3}}
\]
\[
\lesssim u \sum_{M > N} \left( \frac{N}{M} \right)^{s_c} M^{s(s)}
\]
\[
\lesssim u N^{s(s)}
\]
where \( s_1 = 2 - \frac{s - s_c + 2}{p} \in (s, s_c) \). This together with (4.12) and (4.10) implies the claim (4.9).

Finally, using Bernstein inequality, (4.8), and (4.9), we obtain
\[
\left\| \nabla^s u \right\|_{L_t^p L_x^2} \lesssim \left\| \nabla^s u_{\geq 1} \right\|_{L_t^p L_x^2} + \sum_{N \leq 1} N^{s(s) - s} \left\| \nabla^s u_N \right\|_{L_t^p L_x^2}
\]
\[
\lesssim \left\| \nabla^s u \right\|_{L_t^p L_x^2} + \sum_{N \leq 1} N^{s(s)}
\]
\[
\lesssim 1,
\]
where we need the restriction \( s - \sigma(s) < \alpha \leq s_c \).

\[ \square \]

**Theorem 4.3** (Negative Regularity). Let \( u : [0, T_{\text{max}}] \times \mathbb{R}^4 \to \mathbb{C} \) be a solution to (1.1) which is almost periodic modulo symmetries in the sense of Theorem 1.9 with (4.2). And assume that \( \inf_{t \in [0, T_{\text{max}}]} N(t) \geq 1 \), then for any \( 0 < \varepsilon < \frac{1}{2} \)

\[
L_{t}^{\infty}([0, T_{\text{max}}); \dot{H}_{x}^{s_{c}}(\mathbb{R}^4)).
\]

\[ (4.13) \]

**Proof.** First, we use Proposition 4.2 with \( s = s_{c} \) to get

\[
u \in L_{t}^{\infty}([0, T_{\text{max}}); \dot{H}_{x}^{s_{c}}(\mathbb{R}^4)), \quad \forall \frac{1}{2} < \alpha \leq s_{c}.
\]

(4.14)

Applying Proposition 4.2 with \( s = (s_{c} - \frac{1}{2})_{+} \) again, we obtain (4.13).

\[ \square \]

Now, we turn to prove Theorem 4.1. It follows from Theorem 4.3 that \( u \in L_{t}^{\infty}([0, T_{\text{max}}); \dot{H}_{x}^{\varepsilon}(\mathbb{R}^4)) \) with \( 0 < \varepsilon < \frac{1}{2} \). For \( \eta > 0 \), we can interpolate this bound with (1.5) to get

\[
\int_{|\xi| \leq \varepsilon(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi \lesssim_{u} N(t)^{-s_{c} \varepsilon}.
\]

Hence, we obtain by Plancherel

\[
M[u_0] = M[u(t)] = \int_{|\xi| \leq \varepsilon(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi + \int_{|\xi| \geq \varepsilon(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi \lesssim_{u} N(t)^{-s_{c} \varepsilon} + (c(\eta)N(t))^{-2s_{c}} \|\nabla u\|_{L_{t}^{\infty}L_{x}^{2}}^2.
\]

Choosing \( \eta \) small, letting \( t \to T_{\text{max}} \), and (4.1), we can deduce that \( M(u_{0}) = 0 \). Thus, we obtain \( u \equiv 0 \), which contradicts (1.1).

Therefore, we complete the proof of Theorem 4.1.

\[ \square \]

5. The frequency-localized Morawetz inequality

In this section, we establish spacetime bounds for the high-frequency portions of almost periodic solutions to (1.1). We will use these estimates in the next section to preclude the existence of quasi-soliton solutions in the sense of Theorem 1.9.

**Theorem 5.1** (Frequency-localized Morawetz inequality). Let \( s_{c} > 3/2 \) and \( u : [0, T_{\text{max}}] \times \mathbb{R}^4 \to \mathbb{C} \) be an almost periodic solution to (1.1) such that \( N(t) \equiv N_{k} \geq 1 \) on each characteristic subinterval \( J_{k} \subset [0, T_{\text{max}}) \). Let \( I \subset [0, T_{\text{max}}) \) be a compact time interval, which is a contiguous union of characteristic subintervals \( J_{k} \). Then for any \( \eta > 0 \), there exists \( N_{0} = N_{0}(\eta) \) such that for all \( N \leq N_{0} \), we have

\[
\int_{I} \int_{\mathbb{R}^4} |u_{N}(t, x)|^{p+2} \frac{dx dt}{|x|} \lesssim_{u} \eta(N^{1-2s_{c}} + K_{I}),
\]

(5.1)

where \( K_{I} := \int_{I} N(t)^{3-2s_{c}} dt \). Furthermore, \( N_{0} \) and the implicit constants above are independent of the interval \( I \).

We will use the following corollary to prove Theorem 5.1.
Lemma 5.2 (Low and high frequencies control). Let \( u : [0, T_{\text{max}}) \times \mathbb{R}^4 \to \mathbb{C} \) be an almost periodic solution to (1.1) with \( N(t) \equiv N_k \geq 1 \) on each characteristic \( J_k \subset [0, T_{\text{max}}) \). Then on any compact time interval \( I \subset [0, T_{\text{max}}) \), which is a union of continuous subintervals \( J_k \), and for any frequency \( N > 0 \), we have

\[
\| u_{\geq N} \|_{L_t^2 L_x^4(I \times \mathbb{R}^4)} \lesssim u N^{-s_c} (1 + N^{2s_c - 1}K_I)^{\frac{1}{2}}. 
\]

(5.2)

For any \( \eta > 0 \), there exists \( N_1 = N_1(\eta) \) such that for all \( N \leq N_1 \), we have

\[
\| |\nabla|^{s_c} u_{\geq N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^4)} \lesssim u \eta N^{-s_c}. 
\]

(5.3)

From (3.4), we know that for any \( \eta > 0 \), there exists \( N_2 = N_2(\eta) \) such that for all \( N \leq N_2 \)

\[
\| |\nabla|^{s_c} u_{\leq N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^4)} \lesssim u (1 + N^{2s_c - 1}K_I)^{\frac{1}{2}}. 
\]

(5.4)

Proof. Throughout the proof, all spacetime norms will be taken over \( I \times \mathbb{R}^4 \).

By Bernstein’s inequality and (3.3), we have

\[
\| u_{\geq N} \|_{L_t^2 L_x^2} \lesssim \sum_{M \geq N} M^{-s_c} \| |\nabla|^{s_c} u_M \|_{L_t^2 L_x^2} 
\]

\[
\lesssim \sum_{M \geq N} M^{-s_c} (1 + M^{s_c - \frac{1}{2}}K_I^{\frac{1}{2}}) 
\]

\[
\lesssim N^{-s_c} + N^{-\frac{1}{2}}K_I^{\frac{1}{2}} 
\]

\[
\lesssim N^{-s_c} (1 + N^{2s_c - 1}K_I)^{\frac{1}{2}}, 
\]

which shows (5.2).

Using Remark 13 (ii) and the fact that \( \inf_{t \in I} N(t) \geq 1 \), we may find \( c(\eta) > 0 \) such that

\[
\| |\nabla|^{s_c} u \|_{L_t^\infty L_x^2} \leq \eta. 
\]

Combining this estimate with Bernstein, we get

\[
N^{-s_c - \frac{1}{2}} \| |\nabla|^{s_c} u_{\geq N} \|_{L_t^\infty L_x^2} \lesssim N^{-s_c - \frac{1}{2}} \| |\nabla|^{s_c} u_{N \leq \leq c(\eta)} \|_{L_t^\infty L_x^2} + N^{-s_c - \frac{1}{2}} \| |\nabla|^{s_c} u_{c(\eta)} \|_{L_t^\infty L_x^2} 
\]

\[
\lesssim \| |\nabla|^{s_c} u_{\leq c(\eta)} \|_{L_t^\infty L_x^2} + \frac{N^{s_c - \frac{1}{2}}}{c(\eta)^{s_c - \frac{1}{2}}} \| |\nabla|^{s_c} u \|_{L_t^\infty L_x^2} 
\]

\[
\lesssim \eta + N^{-s_c - \frac{1}{2}}. 
\]

Taking \( N \) sufficiently small, we get (5.3). \( \square \)

The proof of Theorem 5.1. Throughout the proof, all spacetime norms will be taken over \( I \times \mathbb{R}^4 \).

Let \( 0 < \eta \ll 1 \) and choose

\[
N < \min \{ N_1(\eta), \eta^2 N_2(\eta^{2s_c}) \}, 
\]

where \( N_1 \) and \( N_2 \) are as in Lemma 5.2. In particular, we note that (5.2) gives

\[
\| u_{N \leq \eta} \|_{L_t^2 L_x^2} \lesssim u \eta N^{-s_c} (1 + N^{2s_c - 1}K_I)^{1/2}. 
\]

(5.5)

Moreover, as \( N/\eta^2 < N_2(\eta^{2s_c}) \), we can apply (5.4) to get

\[
\| |\nabla|^{s_c} u_{\leq \eta} \|_{L_t^2 L_x^2} \lesssim u (1 + N^{2s_c - 1}K_I)^{1/2}. 
\]

(5.6)
Define the Morawetz action
\[ \text{Mor}(t) = 2 \, \text{Im} \int_{\mathbb{R}^4} \frac{x}{|x|} \cdot \nabla u_{> N}(t, x) \bar{u}_{> N}(t, x) \, dx. \]

Since \((i\partial_t + \Delta)u_{> N} = P_{> N}(F(u))\), we obtain
\[ \partial_t \text{Mor}(t) \gtrsim \int_{\mathbb{R}^4} \frac{x}{|x|} \cdot \{ P_{> N}(F(u)), u_{> N} \}_P \, dx, \]

where the momentum bracket \(\{\cdot, \cdot\}_P\) is defined by \(\{f, g\}_P := \text{Re}(f \nabla \bar{g} - g \nabla \bar{f})\). Thus, by the fundamental theorem of calculus, we get
\[ \int_{I \times \mathbb{R}^4} \frac{x}{|x|} \cdot \{ P_{> N}(F(u)), u_{> N} \}_P \, dx \lesssim \|\text{Mor}\|_{L^\infty(I)}. \tag{5.7} \]

Noting that \(\{F(u), u\}_P = -\frac{p}{p+2} \nabla(|u|^{p+2})\), we write
\[
\{ P_{> N}(F(u)), u_{> N} \}_P = \{ F(u), u \}_P - \{ F(u), u_{\leq N} \}_P - \{ P_{\leq N}(F(u)), u_{> N} \}_P \\
- \{ F(u) - F(u_{\leq N}), u_{\leq N} \}_P - \{ P_{\leq N}(F(u)), u_{> N} \}_P \\
= -\frac{p}{p+2} \nabla(|u|^{p+2} - |u_{\leq N}|^{p+2}) - \{ F(u) - F(u_{\leq N}), u_{\leq N} \}_P \\
=: I + II + III.
\]

Integrating by parts, we see that \(I\) contributes to the left-hand side of \((5.7)\) a multiple of
\[ \int_{I \times \mathbb{R}^4} \frac{|u_{> N}(t, x)|^{p+2}}{|x|} \, dx \, dt \]
and to the right-hand side of \((5.7)\) a multiple of
\[ \left\| \frac{1}{|x|}(|u|^{p+2} - |u_{> N}|^{p+2} - |u_{\leq N}|^{p+2}) \right\|_{L^1_x}. \tag{5.8} \]

For term \(II\), we use \(\{f, g\}_P = \nabla \mathcal{O}(fg) + \mathcal{O}(f \nabla g)\). When the derivative hits the product, we integrate by parts. We find that \(II\) contributes to the right-hand side of \((5.7)\) a multiple of
\[ \left\| \frac{1}{|x|} u_{\leq N}[ F(u) - F(u_{\leq N}) \right\|_{L^1_x} \tag{5.9} \]
\[ + \left\| \nabla u_{\leq N}[ F(u) - F(u_{\leq N}) \right\|_{L^1_x}. \tag{5.10} \]

Finally, for \(III\), we integrate by parts when the derivative hits \(u_{> N}\). We find that \(III\) contributes to the right-hand side of \((5.7)\) a multiple of
\[ \left\| \frac{1}{|x|} u_{> N} P_{\leq N}(F(u)) \right\|_{L^1_x} \tag{5.11} \]
\[ + \left\| u_{> N} \nabla P_{\leq N}(F(u)) \right\|_{L^1_x}. \tag{5.12} \]

Thus, continuing from \((5.7)\), we see that to complete the proof of Theorem 5.1 it will suffice to show that
\[ \|\text{Mor}\|_{L^\infty(I)} \lesssim \eta N^{1-2\epsilon} \tag{5.13} \]
and that the error terms \((5.8)\) through \((5.12)\) are acceptable, in the sense that they can be controlled by \(\eta(N^{1-2\epsilon} + K_I)\).
To prove (5.13), we use Bernstein, (6.3), (2.3) to estimate
\[
\|\text{Mor}\|_{L^\infty_x(t)} \lesssim \|\nabla\|^{1/2} u_{>N} \|L^\infty_xL^2_t\| \|\nabla\|^{1/2} (\frac{u_{>N}}{\|x\|}) \|L^\infty_xL^2_t\|
\lesssim \|\nabla\|^{1/2} u_{>N} \|L^\infty_xL^2_t\| \lesssim \eta N^{1-2s_c}.
\]

We next turn to the estimation of the error terms (5.8) through (5.12). For (5.8), we write
\[
(5.8) \lesssim \|\frac{1}{|x|}(u_{\leq N})^{p+1} u_{>N}\|_{L^1_t x} + \|\frac{1}{|x|}u_{\leq N}(u_{>N})^{p+1}\|_{L^1_t x}.
\]

For (5.14), we use Hölder, Hardy, the chain rule, Bernstein, (1.6), (5.2), and (5.4) to estimate
\[
\|\frac{1}{|x|}(u_{\leq N})^{p+1} u_{>N}\|_{L^1_t x} \lesssim \|\frac{1}{|x|}(u_{\leq N})^{p+1}\|_{L^2_t L^2_x} \|u_{>N}\|_{L^2_t L^4_x}
\lesssim \|\nabla (u_{\leq N})^{p+1}\|_{L^2_t L^2_x} \|u_{>N}\|_{L^2_t L^4_x}
\lesssim \|u\|^{p+1} \|\nabla u_{\leq N}\|_{L^2_t L^{p+2}_x} \|u_{>N}\|_{L^2_t L^4_x}
\lesssim \|\nabla^s u_{\leq N}\|_{L^2_t L^4_x} \|u_{>N}\|_{L^2_t L^4_x}
\lesssim \eta N^{-s_c} (1 + N^{2s_c-1} K_I)
\lesssim \eta N^{1-2s_c} (1 + N^{2s_c-1} K_I),
\]
which is acceptable, where we used Proposition 1.11 and \(u \in L^\infty_t L^2_x\) to get
\[
\|u\|_{L^\infty_t L^\frac{p^2}{p-2} x} \lesssim 1, \quad \frac{4p^2}{3p-2} \in (p + 1, 2p),
\]
and \(N < 1\).

For (5.14), we consider two cases. If \(|u_{\leq N}| \ll |u_{>N}|\), then we can absorb this term into the left-hand side of (5.7), provided we can show
\[
\|\frac{1}{|x|}u_{>N}^{p+2}\|_{L^1_t x} < \infty.
\]

Otherwise, we are back in the situation of (5.14), which we have already handled. Thus, to render (5.13) an acceptable error term it suffices to establish (5.16). To do this, we use Hardy, Sobolev embedding, Bernstein, and Lemma 1.8 to estimate
\[
\|\frac{1}{|x|}u_{>N}^{p+2}\|_{L^1_t x} \lesssim \|\nabla\|^{p+2} u_{>N} \|L^{p+2}_x\| \lesssim \|\nabla\|^{p+2} u_{>N} \|L^{p+2}_x\|
\lesssim \|\nabla\|^{p+2} u_{>N} \|L^{p+2}_x\| \lesssim N^{1-2s_c} \|\nabla^s u\|^{p+2} \|L^{p+2}_x\| \lesssim N^{1-2s_c} \left(1 + \int N(t)^2 dt\right) < \infty.
\]

We next turn to (5.9). Writing
\[
\|\frac{1}{|x|}u_{\leq N}[F(u) - F(u_{\leq N})]\|_{L^1_t x} \lesssim \|\frac{1}{|x|}(u_{\leq N})^{p+1} u_{>N}\|_{L^1_t x} + \|\frac{1}{|x|}u_{\leq N}(u_{>N})^{p+1}\|_{L^1_t x},
\]
we recognize the error terms that we just estimated, namely (5.14) and (5.15). Thus (5.9) is acceptable.
For (5.10), we use Hölder, Proposition 1.11, (1.6), (5.2), and (5.4) to estimate
\begin{equation}
\|\nabla u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
\lesssim \| \nabla u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
\lesssim \| \nabla u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
\lesssim \eta N^{-s_c}(1 + N^{2s_c-1} K_1) \\
\lesssim \eta N^{-s_c}(1 + N^{2s_c-1} K_1),
\end{equation}
which is acceptable.

Finally, for (5.11) and (5.12), we first use Hardy to estimate
\begin{align}
& \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \eta N^{-s_c}(1 + N^{2s_c-1} K_1)^{1/2}.
\end{align}

To this end, we use Hölder, Bernstein, the fractional chain rule, (1.6), (5.5), and (5.6) to estimate
\begin{align}
& \| \nabla u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \| u \|_{L^p_t L^2_x} \| u \|_{L^2_t L^2_x} \| u \|_{L^p_t L^{p/2}_x} \\
& \lesssim \eta N^{-s_c}(1 + N^{2s_c-1} K_1)^{1/2}.
\end{align}
Therefore, we complete the proof of Theorem 5.1.

6. The quasi-soliton scenario

In this section, we rule out the existence of quasi-soliton soutions, that is, solutions as in Theorem 1.9 such that \( \int_0^{T_{\max}} N(t) \, dt = \infty \). The proof will rely primarily on the frequency-localized Morawetz estimate.

**Theorem 6.1** (No quasi-solitons). Let \( s_c > 3/2 \). Then there are no radial almost periodic solutions \( u : [0, T_{\max}] \times \mathbb{R}^3 \to \mathbb{C} \) to (1.1) with \( N(t) \equiv N_k \geq 1 \) on each characteristic subinterval \( J_k \subset [0, T_{\max}] \) that satisfy
\begin{equation}
\| u \|_{L^p_t L^2_x([0,T_{\max}])} = \infty
\end{equation}
and
\begin{equation}
K := \int_0^{T_{\max}} N(t) \, dt = \infty.
\end{equation}

**Lemma 6.2** (Lower bound). Let \( u : [0, T_{\max}] \times \mathbb{R}^3 \to \mathbb{C} \) be a radial almost periodic solution as in Theorem 1.9 with \( s_c > 3/2 \). Let \( I \subset [0, T_{\max}] \). Then there exists \( N_0 > 0 \) such that for any \( N < N_0 \), we have
\begin{equation}
K_I \lesssim \int_I \int_{\mathbb{R}^3} \frac{|u_N(t,x)|^{p+2}}{|x|} \, dx \, dt,
\end{equation}
with \( K_I := \int_I N(t) \, dt \).
Proof. First, by the same argument as (7.3) in [23], we deduce that there exists $C(u)$ sufficiently large and $N_0 > 0$ so that for $N < N_0$

$$\inf_{t \in I} N(t)^{2s_c} \int_{|x| \leq \frac{C(u)}{N(t)^{1/2}}}|u_{> N}(t, x)|^2 \, dx \gtrsim_u 1. \quad (6.3)$$

This together with Hölder’s inequality yields

$$\int \int_{I \times \mathbb{R}^4} \frac{|u_{> N}(t, x)|^{p+2}}{|x|} \, dx \, dt \gtrsim_u \int_{I} N(t) \int_{|x| \leq \frac{C(u)}{N(t)^{1/2}}} |u_{> N}(t, x)|^{p+2} \, dx \, dt$$

$$\gtrsim_u \int_{I} N(t)^{1+2p} \left( \int_{|x| \leq \frac{C(u)}{N(t)^{1/2}}} |u_{> N}(t, x)|^2 \, dx \right)^{\frac{p+2}{2}} \, dt$$

$$\gtrsim_u \int_{I} N(t)^{1+2p} (N(t)^{-2s_c})^\frac{p+2}{2} \, dt \gtrsim_u K_I.$$

Thus, we complete the proof of Lemma 6.2. □

Finally, we turn to prove Theorem 6.1. Suppose $u$ were such a solution. Let $\eta > 0$ and let $I \subset [0, T_{\text{max}})$ be a compact time interval, which is a contiguous union of characteristic subintervals.

Combining (6.1) and (6.2), we find that for $N$ sufficiently large, we have

$$K_I \lesssim_u \eta(N^{1-2s_c} + K_I).$$

Choosing $\eta$ sufficiently small, we deduce $K_I \lesssim_u N^{1-2s_c}$ uniformly in $I$. We now contradict (6.1) by taking $I$ sufficiently large inside of $[0, T_{\text{max}})$. This completes the proof of Theorem 6.1. Therefore, we conclude Theorem 1.1.

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