The N=4 Quantum Conformal Algebra

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Abstract

We determine the spectrum of currents generated by the operator product expansion of the energy-momentum tensor in N=4 super-symmetric Yang-Mills theory. Up to the regular terms and in addition to the multiplet of the stress tensor, three current multiplets appear, $\Sigma$, $\Xi$ and $\Upsilon$, starting with spin $0$, $2$ and $4$, respectively. The OPE’s of these new currents generate an infinite tower of current multiplets, one for each even spin, which exhibit a universal structure, of length 4 in spin units, identified by a two-parameter rational family. Using higher spin techniques developed recently for conformal field theories, we compute the critical exponents of $\Sigma$, $\Xi$ and $\Upsilon$ in the $TT$ OPE and prove that the essential structure of the algebra holds at arbitrary coupling. We argue that the algebra closes in the strongly coupled large-$N_c$ limit. Our results determine the quantum conformal algebra of the theory and answer several questions that previously remained open.

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Despite many efforts, low-energy QCD is still not well-understood. Recently, we planned to approach this problem via a detour, a deeper investigation of the conformal window. Indeed, it seems that this region is rich of phenomena that we can describe, often rigorously, at least in the super-symmetric domain, mainly because it is not so far as QCD from the perturbative region. The idea is that the lower limit of the conformal window should exhibit at least some features of QCD. Something similar does happen in super-symmetric theories.

The conformal window of asymptotically free gauge field theories demands a better understanding and this perspective turns on the interest for many related aspects, that are sometimes more abstract, in the sense that one is lead to consider also theories conceptually far from reality. In the present paper we study N=4 super-symmetric Yang-Mills theory under this perspective, to our knowledge the simplest family of conformal field theories in four dimensions. We apply the ideas of, i.e. techniques similar to those used in the theory of deep inelastic scattering. This allows us to answer some problems that remained open in and set up a basis for future N=1 investigations (particularly in SQCD). The hope is that this work will provide a conceptual basis for the physical problem that we plan to study and will not make our detour longer than it should be.

Conformal field theories in four dimensions contain many unanswered questions, although it is true that today we know much more than a couple of years ago. We feel that the investigation of this subject is only at the very beginning. It is not necessary to stress once more that there are enormous differences between conformal field theories in two and four dimensions. Yet, some similarities persist. We just mention here the very important role played by higher spin currents in two dimensional integrable models. Nevertheless, in four dimensions it seems that the most interesting feature of higher spin currents is not their conservation, but the violation of the conservation condition, in particular the critical exponent associated with this violation. This exponent is also the value of the slope of the beta-function associated with the perturbation generated by the currents.

The plan of the paper is as follows. We start for the analysis of the spin-2 terms of the \( TT \) OPE (the spin-0 and spin-1 terms being fully described in and describe the simplifications implied by N=4 supersymmetry on the general OPE structure studied in). We discuss orthogonality of the multiplets and compute their anomalous dimensions. We then proceed to higher spin and realize that the structure of the current multiplets is universal, determined by a simple transformation rule (an SU(4)-invariant bosonic projection of N=4 supersymmetry). This structure encodes the N=4 quantum conformal algebra. The length of the multiplets is 4 in spin units. The lowest components (even- and odd-spin) have a common form, while the highest components belong to a two-parameter rational family. We compute the critical exponents of the \( TT \) OPE, using a technique that reduces the effort to pure algebra and prove that no conserved current other than the stress-tensor appears. Finally, we study the phenomena that occur when the value of the coupling becomes stronger and prove that the essential structure of the algebra holds at arbitrary coupling. We argue that the algebra closes in the strongly coupled large-\( N_c \) limit.

In it was shown that for spin higher than one there are three independent even-spin
currents (scalar, spinor and vector) and two independent odd-spin currents (spinor and vector) in the $TT$ OPE. This fact determines the length of each current multiplet $\Lambda$. It contains three even-spin currents ($\Lambda_s$, $\Lambda_{s+2}$ and $\Lambda_{s+4}$) and two odd-spin currents ($\Lambda_{s+1}$ and $\Lambda_{s+3}$). Only the stress tensor is exceptional, its multiplet containing a single current. We recall that we focus on $SU(4)$-invariant currents, which simplifies the discussion enormously and is sufficient to our purpose. For a complete discussion of the full $N=4$ multiplets, in particular the multiplet of the Konishi current, we point out the recent references [5, 6].

We will also discuss certain properties that hold in generality (i.e. in absence of supersymmetry) and others that hold in the most general super-symmetric theory. For example, we show that if one of the spin-2 currents appearing at the level of the stress tensor is anomalous dimensioned, than the other one is also, and that if a higher-spin current is conserved, than infinitely many are.

The possibility of a higher-spin flavour symmetry in four dimensions is still open and it is not in contrast with the Coleman-Mandula no-go theorem [7] that classifies the symmetries of the S-matrix. Indeed, the higher-spin symmetry would exist only at the fixed points and the theorem does not apply to interacting conformal field theories, which cannot be described in terms of a theory of scattering. This is the same reason why one focuses on the vacuum-to-vacuum amplitudes of conserved currents and their OPE’s.

In summary, it could be that higher-spin symmetry is the proper notion to classify conformal field theories in four dimensions. The $N=4$ family could then be identified by the property of having no higher-spin symmetry at all, the opposite of a free-field theory.

The techniques of [1] are used throughout the paper, even if often it is not explicitly mentioned. Our investigation is a good opportunity to see the ideas of [1] at work (in nontrivial conformal field theories) and also a severe check of the amplitudes of [1], in particular the numerical factors of the higher-spin two-point functions.

**The spin-2 level of the OPE.**

In super-symmetric theories the number $2N_F$ of Majorana fermions equals $N_V + N_S/2$ and therefore it is always possible to define a spin-2 tensor current $\Xi_{\mu\nu}$ orthogonal to the stress tensor $T_{\mu\nu}$ in the free-field limit, with coefficients that do not depend on $N_V$ and $N_S$. The result is unique and reads

$$\Xi_{\mu\nu} = \frac{1}{6}T^V_{\mu\nu} - \frac{2}{3}T^F_{\mu\nu} + T^S_{\mu\nu}.$$  

Here $T_{\mu\nu} = T^V_{\mu\nu} + T^F_{\mu\nu} + T^S_{\mu\nu}$ is the stress tensor, equal-weight-sum of its vector, spinor and scalar contributions. Factorizing out a common factor $(\frac{1}{4\pi})^2 \prod_{\mu\nu\rho\sigma} (\frac{1}{\sqrt{2}})$, one has $< T^V_{\mu\nu}(x) T^V_{\rho\sigma}(0) > = N_V/5$, $< T^F T^F > = N_F/10$ and $< T^S T^S > = N_S/60$ (indices and spacetime points will be understood from now on).

The orthogonality condition can be preserved off-criticality, in the sense of perturbation theory, since its implementation is a mere choice of finite local counter-terms. A second operator,
orthogonal to both $T_{\mu\nu}$ and $\Xi_{\mu\nu}$ does not exist, in general (with coefficients independent of $N_V$ and $N_S$), unless a further symmetry establishes a relationship between $N_V$ and $N_S$. This happens for N=4 super-symmetric Yang-Mills theory, where $N_S = 6N_V$ and $N_F = 2N_V$. Then the operator

$$\Sigma_{\mu\nu} = -T^V_{\mu\nu} + \frac{1}{2} T^F_{\mu\nu} + T^S_{\mu\nu}$$

satisfies the required property.

Orthogonality of $\Xi_{\mu\nu}$ and $T_{\mu\nu}$ means that $\Xi$ can freely acquire an anomalous dimension $h_\Xi$ off-criticality. This anomalous dimension further evidentiates the “decoupling” of $\Xi$ and $T$. The same is true for $\Sigma$ in the N=4 theory.

A simple theorem allows us to say that if $h_\Xi \neq 0$ then $h_\Sigma \neq 0$ and vice-versa: these two anomalous dimensions are both zero or both nonzero. The proof uses the OPE projector-invariants identified in [1]. Let us assume, for example, that $\Sigma_{\mu\nu}$ is conserved. Then we can work out the $\Sigma_{\mu\nu}(x) \Sigma_{\rho\sigma}(y)$ OPE in the same way as for the stress tensor. It is sufficient to consider the free-field limit. In particular, let us consider the operator $Q_{\alpha\beta}$ carried by the special invariant $S_{\mu\nu,\rho\sigma,\alpha\beta}(x - y)$ in the $\Sigma\Sigma$ OPE. Given that $T^{V,F,S} T^{V,F,S} \rightarrow T^{V,F,S} \times S_{\mu\nu}$, it is clear that no linear combination $Q_{\alpha\beta}$ of $T^V$, $T^F$ and $T^S$ other than the stress tensor gives itself. Moreover, no linear combination $Q_{\alpha\beta}$ other than the stress tensor gives a linear combination of itself and the stress tensor. Consequently a third independent operator is generated, which has to be conserved, due to the special property of the projector-invariant $S_{\mu\nu,\rho\sigma,\alpha\beta}(x - y)$. Explicitly, $\Sigma\Sigma \rightarrow (T^V + 1/4 T^F + T^S) \times S_{\mu\nu}$ and $T^V + 1/4 T^F + T^S = 7/10 T - 3/14 \Sigma + 18/35 \Xi$.

Now, $\Sigma_{\mu\nu}$ belongs to the same current multiplet as the Konishi current (see below) and therefore its anomalous dimension is non-vanishing [4]. As a consequence of the above theorem, $h_\Xi \neq 0$. Moreover, the explicit computation shows that $h_\Sigma \neq h_\Xi$, which means that the I-degeneracy of [4] is completely removed. This turns out to be correct for all terms in the OPE of the N=4 theory.

A more powerful theorem, that we mention here for completeness although it does not apply to our case, states that if a higher-spin current is conserved, then infinitely many are. The proof is similar to the above one. Suppose that a spin-2 current $\Lambda_s$ is conserved, with $s > 2$, and consider the $\Lambda_s \Lambda_s$ OPE. There is always a special term $S_{\mu\nu,\rho\sigma,\alpha\beta}(x - y)$ in the $\Sigma\Sigma$ OPE. For $s$ even, $S_{\mu\nu,\rho\sigma,\alpha\beta}$ has dimension four (which produces $\delta(x)$ when tracing and $\partial\delta(x)$ when taking the divergence), while for $s$ odd $S_{\mu\nu,\rho\sigma,\alpha\beta}$ has dimension three (like in the case of the OPE of two conserved spin-1 currents). Now, since $\Lambda_s$ has dimension $2 + s$, the operator carried by $S_{\mu\nu,\rho\sigma,\alpha\beta}$ has dimension $2s$ (and therefore spin $2s - 2$) when $s$ is even, while it has spin $2s - 1$ when $s$ is odd. Therefore an $s$ greater than two generates a infinite tower of conserved currents.

This property just proved is simply the known fact, here derived at the quantum level, that any higher spin symmetry algebra is infinite dimensional. It is somewhat opposite to the complete removal of the I-degeneracy (but there could be higher-spin symmetry with no more than one conserved current for each spin).
The form of the N=4 OPE reads (see [1] for the notation)

\[ T_{\mu \nu}(x) T_{\rho \sigma}(y) = \frac{1}{60} \left( \frac{1}{4\pi^2} \right)^2 \prod_{\mu \nu, \rho \sigma}^{(2)} \left( \frac{c_2}{|x - y|^4} \right) + \frac{1}{4\pi^2} T_{\alpha \beta} \tilde{S}P_{\mu \nu, \rho \sigma; \alpha \beta}(x - y), \] (1)

plus anomalous dimensioned operators, descendants and regular terms. Here \( c_2 = 30N_V \). The invariant \( \tilde{S}P_{\mu \nu, \rho \sigma; \alpha \beta}(x - y) \) multiplying the stress-tensor reads

\[ \tilde{S}P_{\mu \nu, \rho \sigma; \alpha \beta}(x - y) = SP_{\mu \nu, \rho \sigma; \alpha \beta}(x - y) + \frac{43}{480} \prod_{\mu \nu, \rho \sigma}^{(2)} \partial_\alpha \partial_\beta \left( |x - y|^2 \ln |x - y|^2 M^2 \right) - \frac{5}{32} \prod_{\mu \nu, \rho \sigma}^{(3)} \left( |x - y|^2 \ln |x - y|^2 M^2 \right). \]

The importance of this invariant is related to its role in the \( < T(x)T(y)T(z) > \) three-point function. Since \( \Sigma_{\mu \nu} \) and \( \Xi_{\mu \nu} \) are orthogonal to \( T \), only this invariant contributes to the three-point functions in this limit. The full correlator is reconstructed uniquely by the invariant \( \tilde{S}P \) and its descendants. Now, the three-point function, and therefore also \( \tilde{S}P \), encodes both the quantity \( a \) and \( c \) [2] and the nonrenormalization theorem says that \( \tilde{S}P \) is independent of the coupling constant.

We now begin the analysis of the dimensioned operators. The Lagrangian of the theory is

\[ \mathcal{L} = \text{tr} \left[ \frac{1}{4} F^2 + \frac{1}{2} \lambda_i \gamma_\mu \nabla_\mu \lambda_i + \frac{1}{2} \left( \nabla_\mu A_{ij} \right)^2 + \frac{1}{2} \left( \nabla_\mu B_{ij} \right)^2 + \cdots \right]. \]

The scalar operator appearing in the \( TT \) OPE is

\[ \Sigma_0 = A_{ij}^2 + B_{ij}^2. \]

The axial current appearing in the \( TT \) OPE is the \( SU(4) \) singlet current that belongs to the same super-field as the scalar operator \( \Sigma_0 \),

\[ \Sigma_\mu = \frac{1}{2} \bar{\lambda}_i \gamma_5 \gamma_\mu \lambda_i. \]

The factor 1/2 w.r.t. [1] is due to the Majorana condition. In particular, applying two supersymmetries \( \delta_\zeta \) to \( \Sigma_0 \) with the same fermionic parameter \( \zeta \) and looking at the coefficient of \( \bar{\zeta}_i \gamma_5 \gamma_5 \zeta_j \), one obtains the full set of Konishi currents \( \Sigma^i_{ij} \) in the \( 4 \otimes \bar{4} = 15 \oplus 1 \) representations of \( SU(4) \),

\[ \Sigma^i_{ij} = \frac{1}{2} (\bar{\lambda}_i \gamma_5 \gamma_\mu \lambda_j - \delta_{ij} \bar{\lambda}_k \gamma_5 \gamma_\mu \lambda_k) + 4iA_{ik} \tilde{\nabla}_\mu B_{kj}. \]

The trace \( \Sigma_\mu \) is the spin-1 operator appearing in the \( TT \) OPE. Instead, isolating the coefficient of \( \bar{\zeta}_i \gamma_\mu \zeta_j \) in \( \delta_\zeta^2 \Sigma_0 \), one gets the set of vector currents

\[ V^i_{\mu j} = -i \bar{\lambda}_i \gamma_\mu \lambda_j + 4iA_{ik} \tilde{\nabla}_\mu A_{kj} + 4iB_{ik} \tilde{\nabla}_\mu B_{kj}. \]

\(^2\) In our formulas we convert the Minkowskian notation of Sohnius [3] to the Euclidean framework by the substitutions \( \delta_{\mu \nu} \rightarrow -\delta_{\mu \nu}, T^{V,F,S} \rightarrow -T^{V,F,S}, \varepsilon_{\mu \nu \rho \sigma} \rightarrow -i \varepsilon_{\mu \nu \rho \sigma} \) and \( \gamma_\mu, \gamma_5 \rightarrow -i \gamma_\mu, -i \gamma_5 \). Moreover, we multiply \( A \) and \( B \) by a factor 2.
two-loop graph computation is necessary (done in [4]). We can check the method with Σ have Σ which we expect to be present, since they are present in the divergence of the Konishi current h computation of the lowest order of T T inside the (we recall that by definition the expression of the higher spin non-conserved currents is read

\[ J_i = \frac{3}{2}c_i^{(2)}(g)\delta_{ij} \hat{I}_{\mu\nu,\rho\sigma}(x) |x|^{-8}, \]

where \( \{J_2^{(i)}\} = \{T, \Sigma_2, \Xi_2\}, i = T, \Sigma, \Xi \) and \( c_i^{(2)}(g) \) are the spin-2 central charges, their free-field values being \( \{c_i^{(2)}(0)\} = \{30 N_V, 21 N_V, 35/3 N_V\} \). Moreover, \( \hat{I}_{\mu\nu,\rho\sigma} = \hat{I}_{\mu\rho} \hat{I}_{\nu\sigma} + \hat{I}_{\mu\sigma} \hat{I}_{\nu\rho} - \frac{1}{2} \delta_{\mu\nu} \delta_{\rho\sigma} \) and \( \hat{I}_{\mu\nu} = \delta_{\mu\nu} - 2x_{\mu}x_{\nu}/|x|^2 \) as usual. This implies in turn that

\[ < \partial_\mu J_\mu^{(i)}(x) \partial_\nu J_\nu^{(j)}(0) >= h_i 3 \frac{3}{2\pi^4} c_i^{(2)}(0) \delta_{ij} \frac{\hat{I}_{\rho\sigma}(x)}{|x|^{10}} + O(g^4). \]

This formula exhibits the violation of the conservation condition. Now, given that \( \partial_\mu J_\mu^{(i)} = O(g) \) and \( h_i = O(g^2) \), we see that we can neglect \( O(g^2) \) in \( \partial_\mu J_\mu^{(i)} \) and retain only the Yukawa and gauge terms. Moreover, since the matrix of renormalization constants \( Z_{ij} \) has the form \( \delta_{ij} + O(g^2) \), we can neglect the difference between bare and renormalized operators at this level. The same can be said about any finite factor attached to \( J^{(i)} \) by its very definition (we recall that by definition the expression of the higher spin non-conserved currents is read inside the TT OPE [4], which fixes their finite factors also), as well as about anomalous terms (which we expect to be present, since they are present in the divergence of the Konishi current \( \Sigma_\mu \); nevertheless they begin at least with the order \( g^2 \)). In practice this method reduces the computation of the lowest order of \( h_i \) to pure algebra.

The procedure is completely general, if one excludes the scalar operator \( \Sigma_0 \) for which a two-loop graph computation is necessary (done in [4]). We can check the method with \( \Sigma_\mu \). We have

\[ < \Sigma_\mu(x) \Sigma_\nu(0) >= -\frac{2N_V}{\pi^4} \frac{\hat{I}_{\mu\nu}(x)}{|x|^{6+2\Sigma}} \text{ irrelevant}, \quad < \partial \cdot \Sigma(x) \partial \cdot \Sigma(0) >= -h_\Sigma \frac{16N_V}{\pi^4} \frac{1}{|x|^8} \text{ irr. } \]

The field equations give \( \partial \cdot \Sigma = 2igf_{abc}\lambda^a_i \gamma_5(A^b_{ij} - i\gamma_5 B^b_{ij})\lambda^c_j \) and \( < \partial \Sigma \partial \Sigma > \) gives immediately \( h_\Sigma = 3N_c \alpha/\pi \).

The spin-2 case is more subtle. We first covariantize the free-field operators so as to preserve formal conformality, i.e. conformality at the classical level. Then we extract all traces. A
nontrivial check that this procedure is good is the emergence of the correct stress-tensor, even if we do not apply directly the Noether method. Our method is more general and works also for non-conserved currents. At higher orders the situation is further complicated by the renormalization constants $Z_{ij}$ and the mixing with descendants of the improvement terms [1].

Conformality at the classical level is implemented by the transformations

$$A_{ij} \to |x|^2 A_{ij} \quad B_{ij} \to |x|^2 B_{ij}, \quad \lambda_i \to |x|^2 \gamma_5 \lambda_i, \quad \bar{\lambda}_i \to |x|^2 \bar{\lambda}_i \gamma_5 \bar{\gamma},$$
$$\partial_{\mu} \to |x|^2 \partial_{\mu}, \quad D_{\mu} \to |x|^2 \partial_{\mu} \partial_{\nu}, \quad F_{\mu\nu} \to |x|^2 \partial_{\mu} \partial_{\nu} F_{\rho\sigma}, \quad (3)$$

and selects the “improved” currents [1]. The transformation of $A_{\mu}$ (as well as $D_{\mu}$ and $F_{\mu\nu}$ in the non-Abelian case) is actually defined only up to gauge-transformations, therefore it is understood that the above rules have to be applied to gauge-invariant operators. Formal conformality goes through the covariantization process with no problem and follows exactly as for $g = 0$ (with $\partial \to D$). The double covariant derivatives do not present any ambiguity, since their indices are always symmetrized. Moreover, they can often be moved away by using the improvement terms (see below). Classical conformality is sufficient for our purposes since the anomalous dimension $h$ is order $g^2$ and its effects can be neglected here.

Therefore we define

$$\Xi_{\mu\nu} = \frac{1}{6} T_{\mu\nu}^V - \frac{2}{3} T_{\mu\nu}^F + T_{\mu\nu}^S - \frac{1}{4} \delta_{\mu\nu} \left( \frac{1}{6} \Theta^V - \frac{2}{3} \Theta^F + \Theta^S \right),$$
$$\Sigma_{\mu\nu} = -T_{\mu\nu}^V + \frac{1}{2} T_{\mu\nu}^F + T_{\mu\nu}^S - \frac{1}{4} \delta_{\mu\nu} \left( -\Theta^V + \frac{1}{2} \Theta^F + \Theta^S \right),$$
$$T_{\mu\nu}^V = T_{\mu\nu}^V + T_{\mu\nu}^F + T_{\mu\nu}^S - \frac{1}{4} \delta_{\mu\nu} \left( \Theta^V + \Theta^F + \Theta^S \right),$$

where $\Theta$ denotes the traces and

$$T_{\mu\nu}^S = D_{\mu} A_{ij}^a D_{\nu} A_{ij}^a - \frac{1}{2} \delta_{\mu\nu} (D_{\alpha} A_{ij}^a)^2 + D_{\mu} B_{ij}^a D_{\nu} B_{ij}^a - \frac{1}{2} \delta_{\mu\nu} (D_{\alpha} B_{ij}^a)^2 - \frac{1}{6} \pi_{\mu\nu} (A^2 + B^2),$$
$$T_{\mu\nu}^F = \frac{1}{4} (\bar{x}_i^b \gamma_4 D_{\nu} x^a_i + \bar{\lambda}_i^b \gamma_4 D_{\nu} \lambda^a_i), \quad T_{\mu\nu}^V = F_{\mu\nu}^a F_{\rho\sigma}^{ac} - \frac{1}{4} \delta_{\mu\nu} F^2,$$

where $\pi_{\mu\nu} = \partial_{\mu} \partial_{\nu} - \Box \delta_{\mu\nu}$. The calculation is a bit lengthy, but straightforward. In this illustrative case, we give the details and in the rest of the paper we present the derivation of our results more schematically. The divergences $\partial_{\mu} J_{\mu}^{(i)}$ are the sums of three operators $u_{\mu}$, $v_{\mu}$ and $z_{\mu}$ that are orthogonal to one another at the free-field level. Precisely,

$$u_{\mu} = \frac{9}{2} f_{abc} \bar{\lambda}_i \gamma_4 \lambda^a_i F_{\mu\nu}^c, \quad v_{\mu} = -g f_{abc} (A_{ij}^a \partial_{\mu} A_{ij}^b + B_{ij}^a \partial_{\mu} B_{ij}^b) F_{\mu\nu}^c,$$
$$z_{\mu} = -g f_{abc} \bar{\lambda}_i \partial_{\mu} A_{ij}^b - i \gamma_5 \partial_{\mu} B_{ij}^b \lambda^a_j + \frac{g}{4} f_{abc} \partial_{\mu} [\bar{\lambda}_i (A_{ij}^b - i \gamma_5 B_{ij}^b) \lambda^a_j].$$

Actually, only two combinations of these three operators appear in $\partial_{\mu} J_{\mu}^{(i)}$, since $T_{\mu\nu}$ is conserved. The other two divergences are

$$\partial_{\mu} \Xi_{\mu\nu} = \frac{3}{2} u_{\mu} - v_{\mu} - \frac{1}{2} z_{\mu}, \quad \partial_{\nu} \Xi_{\mu\nu} = -\frac{5}{3} \left( \frac{1}{2} u_{\mu} + \frac{1}{4} v_{\mu} + z_{\mu} \right). \quad (4)$$
Factorizing out the common factor $g^2 N_c N_V \mathcal{I}_{\mu \nu}(x)/(\pi^0 |x|^{10})$, we have $<uu> = 6$, $<vv> = 9$ and $<zz> = 9/2$, which gives

$$\begin{align*}
h_{\Sigma} &= 3 N_c \frac{\alpha}{\pi}, \\
h_\Xi &= \frac{25}{6} N_c \frac{\alpha}{\pi},
\end{align*}$$

and, as a further nontrivial check, $<\partial_{\mu} \Xi_{\mu \nu}(x) \partial_{\rho} \Sigma_{\rho \sigma}(0)> = 0$.

As expected, $h_\Xi$ is nonzero and different from $h_{\Sigma}$, stressing that $\Xi$ belongs to a new superfield and that the $I$-degeneracy is completely removed.

**The spin-3 level of the OPE.**

Now, we turn to the spin-3 currents. As usual, we apply two supersymmetry transformations $\delta_\xi$ to the spin-2 currents, look at the coefficient of $\bar{\zeta}_i \gamma_\mu \gamma_5 \zeta_j$, and then trace it in $i,j$. One obtains two linear combinations of the spin-3 currents appearing in the $TT$ OPE [1] (vector and spinor). $\delta^2 \delta T_{\mu \nu}$ gives zero, while the other combinations are

$$\begin{align*}
\Sigma_{\mu \nu} &\rightarrow \Sigma_{\mu \nu \rho} = A_{\mu \nu \rho}^F - 8 A_{\mu \nu \rho}^V, \\
\Xi_{\mu \nu} &\rightarrow \Xi_{\mu \nu \rho} = A_{\mu \nu \rho}^F + \frac{16}{3} A_{\mu \nu \rho}^V,
\end{align*}$$

where

$$\begin{align*}
A_{\mu \nu \rho}^F &= \sum_{\text{symm}} -2 D_\mu \bar{\lambda}_i^\alpha \gamma_5 \gamma_\rho D_\nu \lambda_i^\alpha + \frac{2}{5} \partial_\mu \partial_\nu \left( \bar{\lambda}_i^\alpha \gamma_5 \gamma_\rho \lambda_i^\alpha \right) - \text{traces}, \\
A_{\mu \nu \rho}^V &= \frac{1}{3} \left[ F_{\nu \alpha} \overleftrightarrow{D_\mu} F_{\alpha \rho} + F_{\mu \alpha} \overleftrightarrow{D_\nu} F_{\alpha \rho} + F_{\mu \alpha} \overleftrightarrow{D_\rho} F_{\alpha \nu} \right] - \text{traces}.
\end{align*}$$

We have already written the expressions for non-vanishing $g$. It is somewhat convenient to move the double covariant derivatives to the improvement terms, where they become simple derivatives. We shrink the notation by denoting the spin of an element in the current multiplet with a subscript and suppressing indices when possible. So, we write $\Sigma_0, \Sigma_3, \Xi_2, \text{etcetera}$.

The first thing to check is orthogonality between $\Sigma_{\mu \nu \rho}$ and $\Xi_{\alpha \beta \gamma}$. Factorizing out the common factor $N_V \left( \frac{1}{4\pi^2} \right)^2 \prod_{\mu \nu \rho, \alpha \beta \gamma} \left( \frac{1}{|x|} \right)$ and keeping into account of the Majorana condition for the spinors $\lambda_i$, the results of [1] give $< A_3^F A_3^F >= 32/105$ and $< A_3^V A_3^V >= 1/84$, wherefrom $< \Sigma_3 \Sigma_3 >= 0$ follows. This is a good check of the numerical factors of [1]. One then finds $< \Sigma_3 \Sigma_3 >= 16/15$ and $< \Xi_3 \Xi_3 >= 32/75$.

Now, let us write the generic form of the spin-3 two-point function,

$$< A_{\mu \nu \rho}^{(i)}(x) A_{\alpha \beta \gamma}^{(j)}(0) > = \left( \frac{1}{4\pi^2} \right)^2 c_i^{(3)} \delta_{ij} \prod_{\mu \nu \rho, \alpha \beta \gamma} \left( \frac{1}{|x|} \right)^4 = c_i^{(3)} \delta_{ij} \left( \frac{1}{4\pi^2} \right)^2 \Pi_{\mu \nu \rho, \alpha \beta}^{(3)} \left( \frac{x}{|x|} \right)^{10},$$

where $\{A_3^{(i)}\} = \{\Sigma_3, \Xi_3\}$, $i = \Sigma, \Xi$ and $\{c_i^{(3)}\} = \{16N_V/15, 32N_V/75\}$. The correlator of the two divergences is

$$< \partial_{\rho} A_{\mu \nu \rho}^{(i)}(x) \partial_{\sigma} A_{\alpha \beta \gamma}^{(j)}(0) > = -\frac{896}{5} h_i \left( \frac{1}{4\pi^2} \right)^2 c_i^{(3)} \delta_{ij} \prod_{\mu \nu \alpha \beta}^{(2)} \left( \frac{1}{|x|} \right)^4. $$
Using this formula we check the value of $h_{\Xi}$. We can restrict ourselves to a single current and, to simplify further the calculation (and to prepare for a new, more difficult one to be done below) we calculate the two-point function of the divergence of the spin-3 vector current

$$A_3^V = \frac{5}{56}(\Xi_3 - \Sigma_3).$$

Then (8) gives

$$< \partial_\mu A_{\mu \rho}(x) \partial_\gamma A_{\alpha \beta \gamma}(0) >= -\frac{10}{7} \left( \frac{1}{4\pi^2} \right)^2 \frac{(c^{(3)}_{\gamma} h_{\Sigma} + c^{(3)}_{\Xi} h_{\Xi})}{|x|^4} \Pi_{\mu \nu,\alpha \beta}^{(2)} \left( \frac{1}{|x|^4} \right),$$

The explicit computation gives indeed

$$< \partial_\rho A_{\mu \rho}(x) \partial_\gamma A_{\alpha \beta \gamma}(0) >= -\frac{64}{9} N_c N_V \left( \frac{1}{4\pi^2} \right)^2 \frac{\alpha}{\pi} \frac{1}{|x|^4} \Pi_{\mu \nu,\alpha \beta}^{(2)} \left( \frac{1}{|x|^4} \right),$$

which perfectly agrees with the values (8) of $h_{\Sigma}$ and $h_{\Xi}$. When performing this computation the following observation helps reducing the effort and improves the precision. The field equations show that the contracted derivatives $E_{\mu} = D_\mu F_{\mu}^{\pm}$ are orthogonal to $F^{\pm}$ (since $E_{\mu}$ contain only gauginos $\lambda_i$ and scalar fields $A, B$). Their two-point function is $< E_{\mu}^{a}(x) E_{\rho}^{b}(0) >= 11g^2 N_c \delta^{ab}/(4\pi^2)^2 \mathcal{I}_{\mu \nu}(x)/|x|^6$. Therefore, the divergence $\partial_\rho A_{\mu \rho}$ is the sum of two terms that are separately conformal: one term contains $E_{\mu}^{a}$; the other term is cubic in the field strength and comes from the commutation of the covariant derivatives. The field equations can be freely used, since, as we know, anomalous contributions are negligible at this level.

**The spin-4 and higher levels of the OPE.**

In order to proceed, we observe that the operation $\delta_{\chi}^2$ that relates the currents of a multiplet is actually universal. To see this, let us normalize the currents as follows,

$$J^V = F_{\mu \rho}^{+} \Omega_{\text{even}} F_{\alpha \nu}^{\pm} + \text{impr.}, \quad J^F = \frac{1}{2} \tilde{\lambda}_i \gamma_{\mu} \Omega_{\text{odd}} \lambda_i + \text{impr.},$$

$$A^V = F_{\mu \rho}^{+} \Omega_{\text{odd}} F_{\alpha \nu}^{\pm} + \text{impr.}, \quad A^F = \frac{1}{2} \tilde{\lambda}_i \gamma_{\mu} \Omega_{\text{even}} \lambda_i + \text{impr.},$$

$$J^S = A_{ij} \Omega_{\text{even}} A_{ij} + B_{ij} \Omega_{\text{even}} B_{ij} + \text{impr.},$$

where $\Omega_{\text{even/odd}}$ denotes an even/odd string of derivative operators $\tilde{\tilde{\partial}}$ and “impr.” stands for the improvement terms [1] (here they are not written explicitly, but they are important for all our computations). This condensed notation is particularly useful, since the operation $\delta_{\chi}^3$ commutes with $\Omega_I$. Therefore, a simple set of basic rules suffices to determine the operation $\delta_{\chi}^3$.

The result is

$$A^V_{2s-1} \rightarrow -4 J^V_s + \frac{1}{4} J^F_s; \quad A^F_{2s-1} \rightarrow -16 J^V_s - 2 J^F_s + 2 J^S_s,$$

$$J^V_s \rightarrow -4 A^V_{2s+1} + \frac{1}{4} A^F_{2s+1}; \quad J^F_s \rightarrow -16 A^V_{2s+1} - 2 A^F_{2s+1},$$

$$J^S_s \rightarrow -6 A^F_{2s+1};$$

9
independently of the spin. The arrow raises the spin by one unit. In particular, we rescale \( \Sigma_2 \) and \( \Xi_2 \) by a factor -4. We normalize currents in the current multiplets by fixing the coefficients of \( \mathcal{J}_S \) and \( \mathcal{A}_F \) to one, for even and odd spin respectively. Note that for spin two we have \( T^V = -2 \mathcal{J}_2^V, T^F = \mathcal{J}_2^F/2 \) and \( T^S = -\mathcal{J}_2^S/4 \).

Let us analyse the four current multiplets of the \( TT \) OPE. The shortest multiplet is the one of the stress tensor, which contains only the stress tensor. The other three current multiplets before the regular terms (the Konishi multiplet \( \Sigma \), the \( \Xi \)-multiplet and a new multiplet \( \Upsilon \)) are

\[
\begin{align*}
\Sigma_0 &= \mathcal{J}_0^S \\
\Sigma_1 &= \mathcal{A}_F^F \\
\Sigma_2 &= -8 \mathcal{J}_2^V - \mathcal{J}_2^F + \mathcal{J}_2^S \\
\Sigma_3 &= -8 \mathcal{A}_3^V + \mathcal{A}_3^F \\
\Sigma_4 &= 8 \mathcal{J}_4^V - 2 \mathcal{J}_4^F + \mathcal{J}_4^S \\
\Xi_2 &= \frac{4}{3} \mathcal{J}_2^V + \frac{4}{3} \mathcal{J}_2^F + \mathcal{J}_2^S \\
\Xi_3 &= \frac{16}{5} \mathcal{A}_3^V + \mathcal{A}_3^F \\
\Xi_4 &= -\frac{120}{7} \mathcal{J}_4^V - \frac{3}{7} \mathcal{J}_4^F + \mathcal{J}_4^S \\
\Xi_5 &= -8 \mathcal{A}_5^V + \mathcal{A}_5^F \\
\Xi_6 &= 8 \mathcal{J}_6^V - 2 \mathcal{J}_6^F + \mathcal{J}_6^S \\
\Upsilon_4 &= \frac{16}{5} \mathcal{J}_4^V + \frac{8}{5} \mathcal{J}_4^F + \mathcal{J}_4^S \\
\Upsilon_5 &= \frac{32}{7} \mathcal{A}_5^V + \mathcal{A}_5^F \\
\Upsilon_6 &= -\frac{120}{7} \mathcal{J}_6^V - \frac{3}{7} \mathcal{J}_6^F + \mathcal{J}_6^S \\
\Upsilon_7 &= -8 \mathcal{A}_7^V + \mathcal{A}_7^F \\
\Upsilon_8 &= 8 \mathcal{J}_8^V - 2 \mathcal{J}_8^F + \mathcal{J}_8^S
\end{align*}
\]

This table (that can continue for arbitrarily high spin) is constructed by combining N=4 supersymmetry, in particular the \( \delta_2^2 \) operation, and orthogonality. The former rule dictates the vertical movement in the table (OPE-singularity direction), the latter prescribes the horizontal one (spin of the lowest component). For example, at each odd-spin \( 2s - 1 \) there are two orthogonal currents. Supersymmetry determines their two spin-2s partners. Then orthogonality determines the third spin-2s current. Afterwards supersymmetry generates two spin-(2s + 1) currents (one spin-2s current being annihilated) and so on.

As a first check of the above table, one can prove that \( < \Sigma_4 \; \Xi_4 > = 0 \), using the results of \( \Xi \) for the two-point functions. Indeed, factorizing out the common factor \( N_V/16\pi^4 \prod(4) \langle 1/|x|^4 \rangle \), the tables of ref. \( \Xi \) give: \( < \mathcal{J}_0^S \; \mathcal{J}_0^S > = 32/105, \; < \mathcal{J}_2^F \; \mathcal{J}_2^F > = 8/63 \) and \( < \mathcal{J}_4^V \; \mathcal{J}_4^V > = 1/252 \). \( \Upsilon_4 \) is then determined by imposing the orthogonality conditions \( < \Sigma_4 \; \Upsilon_4 > = < \Xi_4 \; \Upsilon_4 > = 0 \).

A second check is \( < \Sigma_5 \; \Xi_5 > = 0 \). Indeed according to \( \Xi \) \( < \mathcal{A}_5^F \; \mathcal{A}_5^F > = 64/1155 \) and \( < \mathcal{A}_5^V \; \mathcal{A}_5^V > = 1/660 \), factorizing out the common factor \( N_V/16\pi^4 \prod(5) \langle 1/|x|^4 \rangle \).

It is important to observe that this table can be worked out at the free-field level. To continue it, one would need to generalise the formulas of \( \Xi \) to arbitrary spin (and we hope that our results stimulate the study of this algebraic problem). There is a third direction that can be added to the table, namely turning the interactions on (anomalous dimension), described in Fig. 1. It would be interesting to work out the complete spectrum. The problem is still algebraic to first order and the general formula will be presumably rather simple, due to the "prime-factor-rule". It should be similar to those that one finds in the context of deep inelastic scattering.

We now observe that \( \Xi_{5,6} \) have the same form as \( \Sigma_{3,4} \) and \( \Upsilon_{7,8} \), but different spin. The special combination \( 8 \; \mathcal{J}^V - 2 \; \mathcal{J}^F + \mathcal{J}^S \) is the only one which is annihilated by the \( \delta_2^2 \) operation and therefore it has to be the lowest term of any finite current multiplet. It can come only from
the combination \(-8 \mathcal{A}^V + \mathcal{A}^F\), which, in turn, can come from many. Precisely, the square of the \(\delta^2\) operation acts on \(\{\mathcal{A}^V, \mathcal{A}^F\}\) via the matrix \(N = \begin{pmatrix} 1 & 8 \\ -1/8 & -1 \end{pmatrix}\) and therefore produces \(-8 \mathcal{A}^V + \mathcal{A}^F\) on any linear combination of \(\mathcal{A}^V\) and \(\mathcal{A}^F\). We conclude that the structure of the current multiplets is universal, with a rational two-parameter freedom. Any current multiplet is 4-spin long, apart from the stress-tensor multiplet, and has the form

\[
\begin{align*}
\Lambda_{2s} &= a_s \mathcal{J}_{2s}^V + b_s \mathcal{J}_{2s}^F + c_s \mathcal{J}_{2s}^S \\
\Lambda_{2s+1} &= -16(a_s + 4b_s) \mathcal{A}_{2s+1}^V + (a_s - 8b_s - 24c_s) \mathcal{A}_{2s+1}^F \\
\Lambda_{2s+2} &= 24(a_s + 8b_s + 8c_s) \mathcal{J}_{2s+2}^V - 3(a_s - 8c_s) \mathcal{J}_{2s+2}^F + (a_s - 8b_s - 24c_s) \mathcal{J}_{2s+2}^S \\
\Lambda_{2s+3} &= -8 \mathcal{A}_{2s+3}^V + \mathcal{A}_{2s+3}^F \\
\Lambda_{2s+4} &= 8 \mathcal{J}_{2s+4}^V - 2 \mathcal{J}_{2s+4}^F + \mathcal{J}_{2s+4}^S.
\end{align*}
\]

The two-parameter rational set of currents can be represented as a discrete set of points on a plane: if \(\Lambda_s\) is the highest component of a current multiplet, a generic point on the plane \((\mathcal{J}^V, \mathcal{J}^F)\), then the transformation \(\Lambda_{2s} \rightarrow \Lambda_{2s+2}\) is the projection onto the line \(y = -x/16 - 3/2\), while \(\Lambda_{2s+2} \rightarrow \Lambda_{2s+4}\) is the projection of this line onto the “fixed point” \((8, -2)\) (see Fig. 2).

Fig. 1: The N=4 quantum conformal algebra: turning the interaction on.
Via the procedure that we have outlined, the $TT$ OPE determines algorithmically the set $(T, \Lambda_{2s})$, $s = 0, 1, \ldots$, formed by the stress-tensor and one 4-spin-long current multiplet for each even spin. $a_s$, $b_s$ and $c_s$ appear to involve prime factors not larger than $2s + 1$ (see [1] for this number theoretical relationship). Moreover Fig. 2 suggests that, fixing the coefficient of $J^S$ to one, all the points $(x_s, y_s)$ on the $(J^V, J^F)$ plane are located to the above-left of the stress-tensor, i.e. $x_s \leq 8$ and $y_s \geq -2$ $\forall s$.

The currents of the set $(T, \Lambda_{2s})$ are a basis. All the $SU(4)$-invariant currents with a non-vanishing free-field limit, quadratic in the fields, can be expressed in terms of $(T, \Lambda_{2s})$. The singular terms of the OPE’s of $\Sigma$, $\Xi$, $\Upsilon$ and all $\Lambda_{2s}$’s themselves can be expressed via this basis. Our construction determines also the set $(c, c_{2s})$ of higher-spin central charges defined in [1].

The basis $(T, \Lambda_{2s})$ depends on the theory and actually identifies its quantum conformal algebra.

Summarizing, we have defined an algorithm to find the currents of the quantum conformal algebra, starting from the stress-tensor $T$ and the spin-0 component $\Sigma_0$ of the Konishi multiplet and proceeding via a combination of two operations:

i) supersymmetry, that moves “vertically” in the algebra;

ii) orthogonalization of two-point functions, that moves “horizontally”.

In principle, all $\Lambda_{2s}$ can be determined with this procedure. From the practical point of view, however, the task might be rather difficult: in order to work out, say, the currents of the multiplet $\Lambda_{100}$, one has to know all the currents with spin smaller than 100. Some sort of “hierarchy” governs the mathematics of our construction. The problem is to solve this hierarchy explicitly, constructing a generating functional for the multiplets $\Lambda_{2s}$. This goal might be achieved by combining our procedure with techniques used in the domain of higher-spin fields. In particular, some aspects investigated in [8] have points in common with our construction.
an infinite tower of higher-spin currents is generated by a product of two “supersingletons”. This approach might have applications to the question we are adressing and reveal the intrinsic relationship between OPE’s of ordinary quantum field theory and higher-spin fields. Solving the quantum conformal hierarchies is a challenging and intriguing problem.

We now perform the computation of the anomalous dimension of the Υ multiplet. We study the spin-4 vector current

\[ J^V_{\mu \nu \rho \sigma} = \sum_{\text{symm}} -4 D_\mu F^+_{\rho \alpha} D_\nu F^-_{\alpha \sigma} + \frac{6}{7} \partial_\mu \partial_\nu (F^+_{\rho \alpha} F^-_{\alpha \sigma}) - \text{traces}. \]

As before, we have written the terms in a form that does not contain double covariant derivatives. The expressions of \( \Sigma_4 \), \( \Xi_4 \) and \( \Upsilon_4 \) give

\[ J^V_4 = \frac{5}{1848} (11 \Sigma_4 - 18 \Xi_4 + 7 \Upsilon_4). \]  

(10)

The form of the correlator of two operators \( J^i_4 \) = \{ \Sigma_4, \Xi_4, \Upsilon_4 \} is

\[ \langle J^i_4(x) J^j_4(0) \rangle = \left( \frac{1}{4\pi^2} \right)^2 \frac{c_i^{(4)} \delta_{ij}}{|x|^2 \Pi(3)} \left( \frac{1}{|x|^4} \right) \]

where \( \{ c_i^{(4)} \} = \{ 16N_V/15, 88N_V/75, 352N_V/525 \} \). The correlator of the two divergences is

\[ \langle \partial J^i_4(x) \partial J^j_4(0) \rangle = -\frac{1800}{7} \frac{c_i^{(4)} \delta_{ij}}{|x|^4} \Pi(3) \left( \frac{1}{|x|^4} \right), \]

(11)

where-from we get

\[ \langle \partial J^V_4(x) \partial J^V_4(0) \rangle = -\frac{625}{332024} \left( \frac{1}{4\pi^2} \right)^2 \left( 121 c_\Sigma^{(4)} h_\Sigma + 324 c_\Xi^{(4)} h_\Xi + 49 c_\Upsilon^{(4)} h_\Upsilon \right) \Pi(3) \left( \frac{1}{|x|^4} \right), \]

(12)

The explicit computation gives

\[ \langle \partial J^V_4(x) \partial J^V_4(0) \rangle = -\frac{590}{147} \left( \frac{1}{4\pi^2} \right)^3 g^2 N_c N_V \Pi(3) \left( \frac{1}{|x|^4} \right). \]

(13)

The match of the full tensorial structure appearing on the right hand side is a severe check of the calculation. In turn, it is also a confirmation that the higher-spin techniques developed so far for conformal field theories work correctly. The tensorial structure is uniquely fixed by conformality and a finer check is obtained by observing that the divergence \( \partial J^V_4 \) is the sum of two separately conformal terms (a similar remark was made for spin 3), one linear in the field strength and \( E^\alpha_\mu \) and the other cubic in the field strength.

By comparison of (12) and (13) one obtains the final result

\[ h_\Upsilon = \frac{49}{10} N_c \frac{\alpha}{\pi}. \]
A further check is provided by the “prime-factor-rule”, according to which the basic algebraic quantities should contain no prime factor higher than $2s + 1$. At most, higher prime factors can appear in the intermediate steps of the calculation. In our case, for example, the prime number 59 appearing in \( (13) \) disappears in the result.

Other violations, relatively under control, come by taking sums of algebraic quantities that obey the rule. An example is provided by the higher-spin central charges $c_i^{(s)}$, where the prime factor $2s + 3$ appears also (check this for $s = 2$ and $s = 4$), despite each contribution (scalar, spinor, vector) to the same quantities obeys the $(2s + 1)$-rule.

General properties at arbitrary coupling.

In this section we describe the phenomena that happen when the valued of the coupling constant is not small.

Our results are consistent with certain theorems that were worked out in the context of the theory of deep inelastic scattering and the light-cone operator product expansions of electromagnetic and weak currents. The Euclidean OPE differs considerably from the light-cone one, in the sense that the terms are differently organized (for example, infinitely many operators, including their descendants, have the same light-cone singularity). Nevertheless, the theorems we are talking about are completely general and follow just from unitarity and dispersion relations.

Our anomalous dimensions are always positive. This is in agreement with the theorem of Ferrara, Gatto and Grillo \[9\], which states that they cannot be negative, as a consequence of unitarity. Moreover, our values increase with the spin and the difference between two consecutive values decreases with the spin. According to a theorem by Nachtmann \[10\], the lowest (most singular) value $h_{2s}$ of the anomalous dimensions of the operators of even spin $2s$, $s \geq 2$, considered as a function of $s$, is increasing and convex:

$$h_{2s} \leq h_{2(s+1)}, \quad h_{2(s+1)} - h_{2s} \leq h_{2s} - h_{2(s-1)}.$$  

The four values $\{h_T, h_\Sigma, h_\Xi, h_\Upsilon\} = \{h_2, h_4, h_6, h_8\} = \left\{ 0, 3, \frac{25}{16}, \frac{49}{14} \right\} N_c \alpha \pi$, do satisfy these properties, illustrated in Fig. 1.

An important consequence is that no conserved current other than the stress-tensor appears in the N=4 quantum conformal algebra, at least generically, i.e. apart from special points $g_*$ such that $h(g_*^2) = 0 \ \forall s$. Moreover, when the value of the coupling constant increases the multiplets move altogether. They never cross one another and the relevant structure of the algebra is the same at an arbitrary magnitude of the interaction: no dramatic phenomenon can occur in the N=4 conformal family.

A richer set of phenomena does occur in general (see \[1\], sect. 4.5, for an example in the SQCD conformal window), but for this to happen, one needs more current multiplets with the same highest and lowest spins (the Nachtmann theorem puts restrictions only to the minimal anomalous dimension of each even-spin level of the OPE). The N=1 and N=2 quantum conformal algebras exhibit such a richer set of phenomena, as we will show in \[1\].
A consequence of the Nachtmann theorem is an alternative derivation the statement, derived in the first section of this paper using OPE considerations, that the existence of a conserved current with spin greater than 2 implies the existence of infinitely many conserved currents (at least one for each even spin greater than 2). Suppose \( h_{2s}(g^2_s) = 0 \) for some \( s > 1 \) and some \( g_s \). Now, \( h_2 \) is always zero for any \( g \), because of the presence of the stress tensor. The only non-negative increasing convex function passing twice by zero is the null function \( h_{2s}(g^2_s) \equiv 0 \).

In the \( N=4 \) case, all the currents are conserved in these points \( g_s \).

In summary, only two remarkable events can take place in the \( N=4 \) quantum conformal algebra, when varying the parameters \( g \) and \( N_c \): i) \( h_{2s} = 0 \) \( \forall s > 0 \) and ii) \( h_{2s} = \infty \) \( \forall s > 0 \). In the former case the theory is free, in the latter case the \( TT \) OPE closes.

At \( N_c \) fixed, \( g \to \frac{1}{g} \) duality arguments suggest that \( h_s(g^2) = h_s(1/g^2) \). \( h_s \) should have a maximum at \( g \sim 1 \) and all \( h_s \)'s should go back to zero in the limit of infinitely strong coupling. Instead, in the limit of large \( N_c \) and \( g^2 N_c \), the functions are expected to tend to infinity, which means that the OPE should close solely with the stress-tensor. This fact was conjectured in [1], sect. 4.5.

The Nachtmann theorem does not apply before spin 2. Therefore, there can be conserved spin-1 currents or even finite scalar operators without having an infinite symmetry. This does not happen in the case that we have considered in this paper, but it is a property, for example, of the \( N=2 \) quantum conformal algebra [11].

With these remarks, we think that we have determined all the salient features of the \( N=4 \) quantum conformal algebra.

**Conclusion.**

In this paper we have worked out the quantum conformal algebra of \( N=4 \) super-symmetric Yang-Mills theory and used it as a technique to study the \( N=4 \) conformal family at a generic magnitude of the interaction. Here we make some simple final comments.

As long as our knowledge deepens, conformal field theories in four dimensions continue to exhibit, at the same time, similarities and differences with respect to conformal field theories in two dimensions. The similarity discussed in [1] and applied here to a concrete model is the importance of higher spin currents. The differences that emerge are the specific role played by higher spin currents and their properties.

In two dimensions, suitable perturbations of conformal field theories create a mass gap and, at the same time, preserve an infinite symmetry, precisely an infinite set of conserved higher spin currents. These properties make the model integrable and the \( S \)-matrix exactly computable. In four dimensions, instead, each (eventually non-conserved) higher spin current is associated with a perturbation of the conformal theory and the anomalous dimension is the slope of the beta function generated by this perturbation. For example, the multiplet of the stress tensor is associated with the marginal deformation (the \( N=4 \) lagrangian, zero anomalous dimension), the Konishi current multiplet \( \Sigma \) is associated with the \( N=1 \) deformation of the superpotential [3]. Similar considerations apply to higher spin currents. It would be interesting to better identify
the deformations associated with Ξ and Υ and check the values $h_Ξ$ and $h_Υ$ by computing the slopes of the corresponding beta-functions.

We believe that these concepts are interesting not only in an abstract sense, enriching our knowledge of new algebraic structures and many related intriguing problems. We expect that these ideas will help us getting a better description of critical points, which, in a long-range perspective, can be useful both to quantum field theory, hopefully QCD, and condensed matter physics, in particular (high temperature) super-conductivity. Our plan is to use these techniques in order to solve the conformal window of QCD.

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