Nekrasov and Argyres-Douglas theories in spherical Hecke algebra representation

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Abstract

AGT conjecture connects Nekrasov instanton partition function of 4D quiver gauge theory with 2D Liouville conformal blocks. We re-investigate this connection using the central extension of spherical Hecke algebra in q-coordinate representation, q being the instanton expansion parameter. Based on AFLT basis together with interwiners we construct gauge conformal state and demonstrate its equivalence to the Liouville conformal state, with careful attention to the proper scaling behavior of the state. Using the colliding limit of regular states, we obtain the formal expression of irregular conformal states corresponding to Argyres-Douglas theory, which involves summation of functions over Young diagrams.

1 Introduction

Liouville conformal block is a useful tool to understand $SU(2)$ Nekrasov partition function of 4D quiver gauge theory due to AGT conjecture [1] and is generalized to Toda theory [2] which represents $SU(N)$ Nekrasov partition function. The Virasoro conformal state is soon generalized in [3] where a new conformal state is constructed. The new state is related with asymptotically free $SU(2)$ quiver gauge theories, which reproduce irregular singularities of the Seiberg-Witten curve corresponding to the Argyres-Douglas theory [4, 5]. The new state is a kind of coherent (rather than primary) state and is called Gaïotto state in the physics community. Among mathematicians, however, the state is known as Whitaker state [6] in earlier stage. We will call the new state “irregular conformal state”, and the conformal state corresponding to the Nekrasov partition function “regular conformal state”.

The irregular state is of interest because the irregular conformal block is given as the inner product of two irregular states. For example, two states $|\Delta, \Lambda^2\rangle$ and $|\Delta, \Lambda, m\rangle$ provide such inner products: $\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle$ produces the partition function of $SU(2)$ with $N_f = 0$, and $\langle \Delta, \Lambda^2 | \Delta, \Lambda, m \rangle$ produces that of $SU(2)$ with $N_f = 1$. The special feature of these irregular states is that, they have non-vanishing eigenvalues under the action of certain Virasoro positive modes. For example, the states above considered have the property for $k \geq 1$, $L_k |\Delta, \Lambda^2\rangle = \delta_{k,1} \Lambda^2 |\Delta, \Lambda^2\rangle$ and $L_k |\Delta, \Lambda, m\rangle = (\delta_{k,1} m \Lambda + \delta_{k,2} \Lambda^2) |\Delta, \Lambda, m\rangle$.

Systematic construction of the irregular state is first given in terms of a limiting process in [7]. Four point conformal block provides the irregular state $|\Delta, \Lambda^2\rangle = \sum_{\gamma} \Lambda^{2|\gamma|} Q_{\Delta}^{-1} ([1^{|\gamma'|}], Y) L_{-Y} |\Delta\rangle$ using the Shapovalov form $Q_{\Delta} (Y, Y') = \langle \Delta | L_{Y'} L_{-Y} | \Delta \rangle$, where $L_{-Y} |\Delta\rangle = L_{-k_d} \cdots L_{-k_2} L_{-k_1} |\Delta\rangle$ represents descendant with proper ordering of the Young diagram $Y = \{ k_1 \geq k_2 \geq \cdots \geq k_d > 0 \}$.

Likewise, $|\Delta, \Lambda, m\rangle = \sum_{\gamma} \sum_p m^{|\gamma|-2p} \Lambda^{|\gamma|} Q_{\Delta}^{-1} ([2p, 1^{|\gamma|-2p}], Y) L_{-Y} |\Delta\rangle$. This representation is generalized into

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the simultaneous eigenvector of two generators $L_1$ and $L_n$ in [8]. In addition, Virasoro irregular state of higher rank $n$ (simultaneous eigenstate of $L_k$ with $n \leq k \leq 2n$) is suggested in [9], while some of the coefficients for the representation are not fixed.

Colliding limit, a limiting procedure to obtain the irregular state from the regular state is clarified in [10]. The decoupling limit in [7] and also in the matrix model [11] is a special case of the colliding limit. The colliding limit turns out to be a very efficient tool to investigate the irregular state to find the correct representation of the irregular state of rank greater than 1. Indeed, the coefficients undetermined in [9] are fixed by irregular matrix model in [12] which obeys consistency conditions of Virasoro generators of lower positive modes $L_k$ with $0 \leq k < n$. The irregular matrix model analysis is extended to $W$-symmetry in [13].

The irregular matrix model analysis, however, provides indirect information because the partition function of the matrix model is equivalent to the inner-product of two states. Direct process to find the irregular state is more desirable. For this goal, we resort to the representation of spherical double degenerate affine Hecke algebra (spherical DDAHA or SH for short). DDAHA is generated by $z_i$ and $D_i = z_i \nabla_i + \sum_{j<i} \sigma_{ij} (i = 1, \cdots, N)$ where $\nabla_i$ is the Dunkl operator and $\sigma_{ij}$ is the transposition of $z_i$ to $z_j$. Spherical DDAHA (SH) is restricted to the symmetric part of product of $z_i$’s and $D_i$’s. SH also allows central extension, which is considered in this text and is still denoted as SH instead of $SH^+$ for simplicity. More details refer to [14]. The algebraic elements and their commutation relations are given in section 2.

In this paper, we elaborate and generalize the procedure presented in [15] to irregular Virasoro state of arbitrary rank $m$ using SH algebra based on AFLT orthonormal basis [10] and interwiners [17, 18]. For this purpose we construct the regular conformal state $|T_m\rangle$ in $q$-coordinates ($q$ being the instanton expansion parameter), which is the counter-part of Liouville conformal state $|R_m\rangle$ used in [10]. The equivalence relation is manifest after the proper scaling behavior is compensated. After this, one can find the irregular state $|I_m\rangle$ of rank $m$ using the colliding limit.

This paper is organized as follows: In section 2 we briefly introduce the spherical Hecke algebra, AFLT basis and the interwiner. Based on these elements, we construct the $q$-representation of the gauge conformal state, counterpart of holomorphic representation of Liouville conformal state. In section 3, we investigate the $q$-representation of the Heisenberg and Virasoro representation using the SH algebra, and find the equivalence relation of the gauge conformal state with the Liouville conformal state. The equivalence is established according to AGT dictionary as the consequence of the proper scaling of the $q$-basis. In section 4, we find the formal solution of irregular state using the colliding limit. Section 5 is the conclusion and details of Hecke algebra calculations are collected in the appendix.

## 2 Spherical Hecke algebra and its representation

In this section, we construct regular states using the spherical Hecke algebra with central extension, based on the AFLT basis with interwiners.

### 2.1 Spherical Hecke generators

We summarize the property of Spherical Hecke algebra, the details of which can be found in [19]. The SH algebra has generators $D_{r,s}$ with $r$ integer and $s$ non-negative integer. The first index $r$ is called degree and the second one $s$ order of generator.

The commutation relations between generators of degree $\pm 1, 0$ are the defining relations [14],

$$[D_{0,l}, D_{1,k}] = D_{1,l+k-1}, \quad l \geq 1,$$

(2.1)
\[ |D_{0,t}, D_{-1,k}| = -D_{-1,t+k-1}, \ l \geq 1, \]  
\[ |D_{-1,k}, D_{1,l}| = E_{k+l}, \ l,k \geq 0, \]  
\[ |D_{0,1}, D_{0,k}| = 0, \ k,l \geq 0, \]  

where \( E_k \) is a nonlinear combination of \( D_{0,k} \), determined by a generating function.\(^1\)

\[
1 - \epsilon_+ \sum_{l \geq 0} E_l s^l = \exp(\sum_{l \geq 0} (-1)^{l+1} c_l \pi_l(s)) \exp(\sum_{l \geq 0} D_{0,l+1} \omega_l(s)).
\]  

Here \( \pi_l(s) = s^l G_l(1 - \epsilon_+ s) \) and \( \omega_l(s) = \sum_q (-c_1, -c_2, \epsilon_+) \ s^l (G_q(1 - qs) - G_q(1 + qs)) \). We use notations \( G_0(s) = -\log(s) \), \( G_l(s) = (s^{-l} - 1) / l \) for \( l \geq 1 \) and \( \epsilon_+ = \epsilon_1 + \epsilon_2 \). \( c_l \) \((l \geq 0)\) is the central extension and plays an essential role in comparing with the conformal algebra. Some of the explicit expressions of \( E_l \) are given as follows:

\[
E_0 = c_0, \\
E_1 = -c_1 - c_0 (c_0 - 1) \epsilon_+ / 2, \\
E_2 = c_2 - c_1 (1 - c_0) \epsilon_+ + c_0 (c_0 - 1) (c_0 - 2) \epsilon_+^2 / 6 - 2 \epsilon_1 \epsilon_2 D_{0,1}.
\]  

Other generators \( D_{\pm,r,l} \) for \( l \geq 0, r > 1 \) are defined recursively as:

\[
D_{l+1,0} = \frac{1}{l} [D_{1,1}, D_{0,l}] , \quad D_{-l-1,0} = \frac{1}{l} [D_{-l,0}, D_{1,1}] , \\
D_{r,l} = [D_{0,l+1}, D_{r,0}] , \quad D_{-r,l} = [D_{-r,0}, D_{0,l+1}] .
\]  

It is noted that SH contains the Heisenberg and Virasoro algebras whose generators are identified as \([14]\),

\[
J_n = (-\sqrt{-c_1 c_2})^{-|n|} D_{-n,0} \quad \text{for } n \neq 0 ,
\]

\[
L_n = (-\sqrt{-c_1 c_2})^{-|n|} D_{-n,1}/|n| - (1 - |n|) c_0 \epsilon_+ J_n/2 \quad \text{for } n \neq 0
\]

Zero mode \( J_0 \) is defined using \( E_1 \), \((2.6)\),

\[
J_0 = E_1/(-\epsilon_1 \epsilon_2) ,
\]  

and \( L_0 \) is derived from \( L_0 = [L_1, L_{-1}]/2 \),

\[
L_0 = E_2/(-2 \epsilon_1 \epsilon_2).
\]  

The commutation relations among these Heisenberg and Virasoro generators are,

\[
[J_n, J_m] = \frac{n c_0}{\beta} \delta_{n+m,0} ,
\]

\[
[L_n, J_m] = -m J_{n+m} ,
\]

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0} ,
\]

with the central charge \( c = (c_0 \epsilon_2^2 - c_0 \epsilon_1 \epsilon_+ + c_0 \epsilon_+^2 - c_0^3 \epsilon_+^2 \epsilon_+)/(\epsilon_1 \epsilon_2) \).\(^1\)

\(^1\)we follow the notation in \([18]\) where the omega background parameters \( \epsilon_1, \epsilon_2 \) are used instead of the CFT parameter \( \beta = -\epsilon_1/\epsilon_2 \) in \([15]\). The comparison between the two are given as: \( D_{0,n+1} = (-\epsilon_2)^n \tilde{D}_{0,n+1} \), \( D_{\pm 1,n} = (-\epsilon_2)^n \tilde{D}_{\pm 1,n} \), \( E_n = (-\epsilon_2)^n \tilde{E}_n \), and \( c_n = (-\epsilon_2)^n \tilde{c}_n \). Tilde is used for the ones in \([15]\).
2.2 Gauge conformal state for Nekrasov partition function

The Nekrasov partition function of $U(N)^\otimes n$ linear quiver gauge theory on a Riemann sphere is given in terms of $n + 3$ punctures, and its instanton part is given by

$$Z_{\text{inst}}^{(n+3)-\text{point}}(q_1, \ldots, q_n) = \sum_{\vec{Y}_i} \prod_{i=1}^{n-1} q_i^{\vec{Y}_i} Z_{\text{vect}}(\vec{a}^{(i)}, \vec{Y}_i) \prod_{i=1}^{n-1} Z_{\text{bif}}(\vec{a}^{(i)}, \vec{Y}_i; \vec{a}^{(i+1)}, \vec{Y}_{i+1} | \nu_i)$$

$$\times \prod_{i=1}^{N} Z_{\text{fund}}(\vec{a}^{(i)}, \vec{Y}_i; \vec{\mu}_I) Z_{\text{afid}}(\vec{a}^{(n)}, \vec{Y}_n; \vec{\mu}_I),$$

(2.15)

where $Z_{\text{vect}}$, $Z_{\text{bif}}$, $Z_{\text{fund}}$ and $Z_{\text{afid}}$ denote vector multiplet, bifundamental hypermultiplet, fundamental hypermultiplet and anti-fundamental hypermultiplet, respectively, whose explicit expressions are given in the appendix. $q = e^{\pi i \tau}$ is the instanton expansion parameter. $\vec{a}$ has $N$ complex components and represents the diagonalized vacuum expectation value of vector multiplets. $\mu_I (\vec{\mu}_I, \nu_i)$ represents the mass of anti-fundamental (fundamental, bi-fundamental hypermultiplet). $\vec{Y}$ denotes the $N$-tuple Young diagram $\vec{Y} = (Y_1, \ldots, Y_N)$.

It is observed in [18] that the instanton partition function can be rewritten as an expectation value

$$Z_{\text{inst}}^{(n+3)-\text{point}} = \langle G, \vec{a}, \vec{\mu}_I | \prod_{k=1}^{n-1} \left( q_k^{D} V_{k,k+1} \right) \rangle q_n^{D} | G, \vec{a}^{(n)}, \vec{\mu}_I; \vec{M}(n) \rangle$$

(2.16)

where $D$ is an operator which counts the number of boxes in Young diagrams $|\vec{Y}|$, and $V_{k,k+1}$ is the interwiner

$$V_{k,k+1}(\vec{a}, \vec{a}^{(k+1)}) = \sum_{\vec{Y}_k, \vec{Y}_{k+1}} Z_{\text{bif}}(\vec{a}^{(k)}, \vec{Y}_k; \vec{a}^{(k+1)}, \vec{Y}_{k+1} | \nu_k) \vec{Y}_k(\vec{a}^{(k+1)} + \nu_k \vec{e}; \vec{Y}_{k+1}).$$

(2.17)

The bra and ket in (2.17) are the AFLT bases which satisfy the orthogonality and completeness [16]

$$\langle \vec{a}, \vec{Y} | \vec{a}, \vec{Y'} \rangle = \delta_{\vec{Y}', \vec{Y}}, \quad 1 = \sum_{\vec{Y}} |\vec{a}, \vec{Y}\rangle \langle \vec{a}, \vec{Y}|.$$  

(2.19)

In addition, the brackets in (2.16) are defined on the AFLT basis

$$| G, \vec{a}^{(n)}; \vec{\mu}_I; \vec{M}(n) \rangle = \sum_{\vec{Y}} \sqrt{Z_{\text{vect}}(\vec{a}, \vec{Y})} Z_{\text{afid}}(\vec{a}, \vec{Y}; \vec{\mu}_I) |\vec{a} + \vec{M}(n), \vec{Y}\rangle,$$

(2.20)

$$\langle G, \vec{a}, \vec{\mu}_I | = \sum_{\vec{Y}} \sqrt{Z_{\text{vect}}(\vec{a}, \vec{Y})} Z_{\text{fund}}(\vec{a}, \vec{Y}; \vec{\mu}_I) \langle \vec{a}, \vec{Y} |.$$  

(2.21)

Here, $\vec{M}(n) = \vec{e} \sum_{i=1}^{n-1} \nu_i$, with $\vec{e} = (1, 1, \ldots, 1)$.

One may evaluate the action of SHF generators on the basis $|\vec{a}, \vec{Y}\rangle$ based on the defining relations:

$$D_{\pm 1,n}|\vec{a}, \vec{Y}\rangle = \sum_{x \in A/R(\vec{Y})} (\phi_x)^n A_x(\vec{Y}) |\vec{a}, \vec{Y} \pm x\rangle, \quad D_{0,n+1}|\vec{a}, \vec{Y}\rangle = \sum_{x \in \overline{R}} (\phi_x)^n |\vec{a}, \vec{Y}\rangle$$

(2.22)

$$\langle \vec{a}, \vec{Y}| D_{\pm 1,n} = \sum_{x \in A/R(\vec{Y})} (\phi_x)^n A_x(\vec{Y}) |\vec{a}, \vec{Y} \mp x|, \quad \langle \vec{a}, \vec{Y}| D_{0,n+1} = \sum_{x \in \overline{R}} (\phi_x)^n \langle \vec{a}, \vec{Y}|$$

(2.23)

where the sets $A(\vec{Y})$ ($R(\vec{Y})$) contain all the boxes that can be added to (removed from) the Young diagram $\vec{Y}$ (see Figure 1). It is noted that the generator of degree $\pm 1$ adds/removes a box from the $N$-tuple Young diagram, which
is denoted as $\tilde{Y} \pm x$ following the convention used in [21, 18]. The added/removed box $x$ is characterized by a triplet of indices $(\ell; i, j)$ where $\ell = 1 \cdots N$ and $(i, j) \in Y_\ell$ gives the position of the box in the $\ell$th Young diagram. To each box $x$ is associated with a complex number

$$\phi_x = a_\ell + (i - 1)\epsilon_1 + (j - 1)\epsilon_2$$

(2.24) and

$$\left\{\Lambda_x(\tilde{Y})\right\}^2 = \prod_{y \in A(\tilde{Y}) \setminus x} \frac{\phi_x - \phi_y + \epsilon_+}{\phi_x - \phi_y} \prod_{y \in R(\tilde{Y}) \setminus x} \frac{\phi_x - \phi_y - \epsilon_+}{\phi_x - \phi_y}.$$

(2.25)

The consistent condition of the action of the generators on AFLT basis results in the central charge of the form

$$c_l = \sum_{p=1}^N (a_p + \epsilon_+)^l$$

[19]. This identification shows that the central charge $c$ in (2.14) is given as

$$c = N - N(N^2 - 1)\epsilon_+^2 / (-\epsilon_1\epsilon_2).$$

(2.26)

In addition, $J_0$ in (2.10) has an effect on $|\vec{a}, \vec{Y}\rangle$ as,

$$J_0|\vec{a}, \vec{Y}\rangle = \frac{1}{(-\epsilon_1\epsilon_2)} \left\{ - \sum_{p=1}^N (a_p + \epsilon_+) - N(N - 1)\epsilon_+ / 2 \right\} |\vec{a}, \vec{Y}\rangle.$$

(2.27)

The Virasoro generator $L_0$ given in (2.11) is defined in terms of $D_{0,1}$. According to (2.22) one may consider $D_{0,1}$ as the operator $D$ counting the number $|\tilde{Y}|$ of boxes: $D|\vec{a}, \vec{Y}\rangle = |\tilde{Y}| |\vec{a}, \vec{Y}\rangle$. In this case, $L_0$ given in (2.11) will have the form $L_0 = D + \Omega_0$ where

$$\Omega_0 = \left\{ \sum_{p=1}^N (a_p + \epsilon_+)^2 - \sum_{p=1}^N (a_p + \epsilon_+)(1 - N)\epsilon_+ + N(N - 1)(N - 2)\epsilon_+^2 / 6 \right\} / (-2\epsilon_1\epsilon_2).$$

(2.28)

This shows that $|\alpha, 0\rangle$ represents the primary state with conformal dimension $\Omega_0$ and $|\vec{a}, \vec{Y}\rangle$ a linear combination of Heisenberg+Virasoro descendants of total level $|\tilde{Y}|$.

In order to compare later with the Liouville state, we define a modified AFLT basis,

$$|\vec{a}, \vec{Y}, \delta_0\rangle = q_0^\delta_0|\vec{a}, \vec{Y}\rangle = |\vec{a}, \vec{Y}\rangle \otimes |\delta_0\rangle,$$

(2.29) with $q_0 \to 0$, and introduce an operator $D_0 = q_0^{-1} \delta_0$, so that

$$D_0|\vec{a}, \vec{Y}; \delta_0\rangle = \delta_0|\vec{a}, \vec{Y}; \delta_0\rangle,$$

$$D|\vec{a}, \vec{Y}; \delta_0\rangle = |\tilde{Y}| |\vec{a}, \vec{Y}; \delta_0\rangle.$$

(2.30)
Then, we shift $L_0 \to L_0 + D_0$ so that

$$L_0|\vec{a}, \vec{Y}; \delta_0\rangle = (|\vec{Y}| + \delta_0 + \Omega_0)|\vec{a}, \vec{Y}; \delta_0\rangle. \quad (2.31)$$

$|\vec{a}, \vec{Y}; \delta_0\rangle$ is the primary state with conformal dimension $(\delta_0 + \Omega_0)$ and will have an important role in investigating the AGT conjecture. We do not need shift $L_n$ for $n > 0$, since the corresponding $q_0^{n+1} \frac{\partial}{\partial q_0}$ term vanishes after taking $q_0 \to 0$.

Using the $q$-basis $|\vec{a}, \vec{Y}; \delta_0\rangle$, one may redefine the states shown in (2.16). For example, one may define $|G, \vec{a}, \mu; \delta_0\rangle$ as in (2.20), where $|\vec{a}, \vec{Y}\rangle$ is replaced with the modified AFLT basis $|\vec{a}, \vec{Y}; \delta_0\rangle$

$$|G, \vec{a}, \mu; \delta_0\rangle = \sum_{\vec{Y}} \sqrt{\mathcal{Z}_{\text{vect}}(\vec{a}, \vec{Y})} \prod_{I=1}^{N} \mathcal{Z}_{\text{add}}(\vec{a}, \vec{Y}, \mu_I) |\vec{a}, \vec{Y}; \delta_0\rangle. \quad (2.32)$$

and

$$\langle G, \vec{a}, \mu; \delta_0 | = \sum_{\vec{Y}} \sqrt{\mathcal{Z}_{\text{vect}}(\vec{a}, \vec{Y})} \prod_{I=1}^{N} \mathcal{Z}_{\text{fund}}(\vec{a}, \vec{Y}, \mu_I) \langle \vec{a}, \vec{Y} | q_0^{\delta_0}. \quad (2.33)$$

And one may convert this state using $q_{1}^{D}$:

$$|T_1; \delta_0\rangle \equiv q_{1}^{D} |G, \vec{a}, \mu; \delta_0\rangle = \sum_{\vec{Y}} q_{1}^{D} \sqrt{\mathcal{Z}_{\text{vect}}(\vec{a}, \vec{Y})} \prod_{I=1}^{N} \mathcal{Z}_{\text{add}}(\vec{a}, \vec{Y}, \mu_I) |\vec{a}, \vec{Y}; \delta_0\rangle. \quad (2.34)$$

Then the 4-punctured instanton partition function (2.16) is written in terms of the new $q$-state;

$$Z_{\text{inst}}^{4-\text{point}} = \langle G, \vec{a}, \mu; \delta_0 | T_1; \delta_0\rangle = \langle G, \vec{a}, \mu; q_1^{D} | G, \vec{a}, \mu\rangle. \quad (2.35)$$

As far as the partition function is concerned, the $q_0$ dependence canceled away. This means the partition function has the freedom to define the $q$-phase on the AFLT basis. Actually these 4 points corresponds to the positions 0, $q_0$, 1 and $\infty$. Or equally, $q_0$, 1, and $q_0^{-1}$, with $q_0 \to 0$. That’s why we have explicit $q_0$ and $q_0^{-1}$ (and a hidden 1) terms in (2.32) and (2.33).

Likewise, we define a $q$-state $|T_2; \delta_0\rangle$ including one interwinder;

$$|T_2; \delta_0\rangle \equiv q_{1}^{D} V_{12}(\vec{a}, \hat{b}, \nu) q_{2}^{D}(G, \vec{b}, \mu; \vec{M}; \delta_0(2)) \quad (2.36)$$

$$= \sum_{\vec{Y}, \vec{W}} q_{1}^{D} q_{2}^{D} \sqrt{\mathcal{Z}_{\text{vect}}(\vec{a}, \vec{Y}) \mathcal{Z}_{\text{bif}}(\vec{a}, \vec{Y}, \vec{b}, \vec{W}) \mathcal{Z}_{\text{vect}}(\vec{b}, \vec{W}) \prod_{I=1}^{N} \mathcal{Z}_{\text{add}}(\vec{b}, \vec{W}, \mu_I) |\vec{b}, \vec{Y}; \delta_0(2)\rangle}$$

where $\delta_0(2)$ is put instead of $\delta_0$ to emphasize that a different $q$-phase is used. Including $m - 1$ interwiners, one has $|T_m; \delta_0\rangle$

$$|T_m; \delta_0\rangle = \left\{ \prod_{k=1}^{m-1} \left( q_{k}^{D} V_{k,k+1}(\vec{a}^{(k)}, \vec{a}^{(k+1)} | \nu_k) \right) \right\} q_{m}^{D}(G, \vec{a}(m), \mu; \vec{M}(m); \delta_0(m)). \quad (2.37)$$

The instanton partition function is simply given as $Z_{\text{inst}}^{(m+3)-\text{point}} = \langle G, \vec{a}, \mu; \delta_0 | T_m; \delta_0\rangle$. We will choose the $q$-phase $\delta_0(m)$ as following (see section 3.3 for details):

$$\delta_0(m) = \frac{1}{\epsilon_{1} \epsilon_{2}} \left( \mu_1 \mu_2 + (\epsilon_+ - \sum_{i=1}^{m-1} \nu_i)(\mu_1 + \mu_2) + \frac{5}{2} \epsilon_+^2 + \sum_{r=1}^{m-1} (\epsilon_+ - \nu_r)(3 \epsilon_+ - \nu_r - 2 \sum_{i=1}^{r-1} \nu_i) \right). \quad (2.38)$$

From now on, we will skip the notation $\delta_0$ for simplicity, assuming the AFLT basis is the $q$-basis and call $|T_m\rangle$ gauge conformal state of rank $m$, the counter part of the Liouville conformal state of rank $m$ which will be considered in section 3.3.
3 Construction of regular and irregular conformal states

3.1 Action of SH generators on $|T_m\rangle$

We summarize the results of actions of SH generators on $|T_m\rangle$ using the $q$-differential representation, the detailed calculation of which is shown in the appendix. The non-trivial but a rather simple representation for $D_{0,1}$ is obtained if one identifies $D_{0,1} = D + D_0$. For any state $|T_m\rangle$ of rank $m$, one has

$$D_{0,1}|T_m\rangle = (D + D_0)|T_m\rangle = \left[ q_1 \frac{\partial}{\partial q_1} + \delta_0(m) \right]|T_m\rangle .$$  \hspace{1cm} (3.1)

A few operators of order 0 and 1 are also shown, which appear in the defining relations in (2.1)-(2.4). For rank 1 case, one has the representation (same as the one in [15])

$$D_{-1,0}|T_1\rangle = \left\{ q_1 \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \sum_p (a_p + \mu_p)|T_1\rangle, \right. \right.
$$ \hspace{1cm} (3.2)

$$D_{-1,1}|T_1\rangle = \left\{ \sqrt{-\epsilon_1 \epsilon_2} q_1 \frac{\partial}{\partial q_1} + \frac{1}{2} q_1 q_2 \sum_p \left( (a_p)^2 - (\mu_p)^2 \right) + \left( \sum_p (a_p + \mu_p)^2 \right) \right\}|T_1\rangle . \hspace{1cm} (3.3)

$$D_{-2,0}|T_1\rangle = q_1^2 \sum_p (a_p + \mu_p)|T_1\rangle , \hspace{1cm} (3.4)

$$D_{-2,1}|T_1\rangle = \left\{ -2\epsilon_1 \epsilon_2 q_1^3 \frac{\partial}{\partial q_1} + q_1^2 \sum_p \left( (a_p)^2 - (\mu_p)^2 \right) + 2 \left( \sum_p (a_p + \mu_p)^2 \right) \right\}|T_1\rangle . \hspace{1cm} (3.5)

For rank 2, we have

$$D_{-1,0}|T_2\rangle = \left\{ q_1 \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \sum_p (a_p - b_p + \epsilon_+ - \nu) + q_1 q_2 \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \sum_p (b_p + \mu_p) \right\}|T_2\rangle \hspace{1cm} (3.6)

$$D_{-1,1}|T_2\rangle = \left\{ \sqrt{-\epsilon_1 \epsilon_2} q_1 q_2 \left( \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} \right) + \frac{1}{2} q_1 q_2 \sum_p \left( (a_p - b_p - \epsilon_+ + \nu)^2 - (\mu_p - \nu)^2 \right) + \left( \sum_p (a_p - b_p + \epsilon_+ - \nu)^2 \right) \right\}|T_2\rangle , \hspace{1cm} (3.7)

$$D_{-2,0}|T_2\rangle = q_1^2 \sum_p (a_p - b_p + \epsilon_+ - \nu) + q_1^2 q_2^2 \sum_p (b_p + \mu_p) \}, |T_2\rangle \right\} \right. \hspace{1cm} (3.8)

$$D_{-2,1}|T_2\rangle = \left\{ -2\epsilon_1 \epsilon_2 q_1^3 \left( \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} \right) + q_1^2 \sum_p \left( (a_p - b_p - \epsilon_+ + \nu)^2 - (\mu_p - \nu)^2 \right) + 2 \left( \sum_p (a_p - b_p + \epsilon_+ - \nu)^2 \right) - 2\epsilon_1 \epsilon_2 q_1^2 q_2^2 \left( \frac{\partial}{\partial q_2} \right) + q_1^2 q_2^2 \sum_p \left( (b_p + \nu)^2 - (\mu_p - \nu)^2 \right) + 2 \left( \sum_p (b_p + \mu_p)^2 \right) \right\}|T_2\rangle . \hspace{1cm} (3.9)

For rank $m$, we find

$$D_{-1,1}|T_m\rangle = \sum_{k=1}^{m-1} q_{k+1} \frac{\partial}{\partial q_{k+1}} \left\{ -\epsilon_1 \epsilon_2 (q_k \frac{\partial}{\partial q_k} - q_{k+1} \frac{\partial}{\partial q_{k+1}}) \right\} + \frac{1}{2} \sum_{p} \left( (a_p^{(k)} + \sum_{i=1}^{k-1} \nu_i)^2 - (a_p^{(k+1)} - \epsilon_+ + \sum_{i=1}^{k} \nu_k)^2 \right) + 2 \left( \sum_p (a_p^{(k)} - a_p^{(k+1)} + \epsilon_+ - \nu_k)^2 \right) \} \right. \hspace{1cm} (3.10)$$
Therefore, using the results in section 3.1, we have on the state of rank 1
\[
D_{-2,1}|T_m\rangle = \sum_{k=1}^{m-1} (q_1 \cdots q_k)^2 \left\{ -2 \epsilon_1 \epsilon_2 (q_k \partial_{q_k} - q_{k+1} \partial_{q_{k+1}}) + \sum_{p=1}^{k-1} \left( \sum_{i=1}^{\nu_i} (a^{(k)}_p - a^{(k+1)}_p + \epsilon_+ - \nu_k) \right) \right\} |T_m\rangle.
\]

### 3.2 Virasoro action on |T_m⟩

In this section, we provide the q-representation of the Virasoro generators. For this purpose, we restrict ourselves to the gauge group SU(2) for which we put \( N = 2 \) and further require \( \sum_p (a^{(k)}_p) = 0 \).

The \( L_0 \) defined in (2.11) has the q-differential representation on the gauge conformal state of rank \( m \)
\[
L_0|T_m\rangle = \left( q_1 \frac{\partial}{\partial q_1} + \delta_0 + \Omega_0 \right) |T_m\rangle,
\]
where \( \Omega_0 \) defined in (2.28) has a simple form \( \Omega_0 = \left( \frac{1}{2} \sum p a^2_p + 2 \epsilon^2_+ \right) / (-\epsilon_1 \epsilon_2) \).

According to (2.9), we have \( L_1 = -\frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} D_{-1,1} \) and \( L_2 = -\frac{1}{\epsilon_1 \epsilon_2} D_{-2,1} - \frac{\epsilon_1}{\epsilon_1 \epsilon_2} D_{-2,0} \) in terms SH generators. Therefore, using the results in section 3.1, we have on the state of rank 1
\[
L_1|T_1\rangle = q_1^2 \frac{\partial}{\partial q_1} + \frac{q_1}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} \sum p (a^2_p) + \mu_1 \mu_2 \right\},
\]
\[
L_2|T_1\rangle = q_1^2 \frac{\partial}{\partial q_1} + \frac{q_1^2}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} \sum p (a^2_p) + \mu_1 \mu_2 + \frac{1}{2} (\mu_1 + \mu_2)^2 + \epsilon_+ (\mu_1 + \mu_2) \right\}.
\]

On |T_2⟩ we have
\[
L_1|T_2\rangle = \left\{ q_1 (q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2}) + \frac{q_1}{-\epsilon_1 \epsilon_2} \left[ \frac{1}{2} \sum p \left( a^2_p - b^2_p \right) + (\epsilon_+ - \nu)^2 \right] \right\} |T_2\rangle,
\]
\[
L_2|T_2\rangle = \left\{ q_1^2 \left( q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} \right) + \frac{q_1^2}{-\epsilon_1 \epsilon_2} \left[ \frac{1}{2} \sum p \left( b^2_p + \mu_1 \mu_2 + \nu (\mu_1 + \mu_2) \right) \right] + 3(\epsilon_+ - \nu)^2 + 2 \epsilon_+ (\epsilon_+ - \nu) \right\} |T_2\rangle.
\]

The same method applies to |T_m⟩.
\[
L_1|T_m\rangle = \sum_{k=1}^{m-1} \frac{q_1 \cdots q_k}{-\epsilon_1 \epsilon_2} \left\{ - \epsilon_1 \epsilon_2 (q_k \partial_{q_k} - q_{k+1} \partial_{q_{k+1}}) + (\epsilon_+ - \nu_k) + \sum_{i=1}^{k-1} \nu_i (\epsilon_- + \nu_k) \right\} |T_m\rangle,
\]
\[
+ \frac{1}{2} \sum p (a^{(k)}_p)^2 - \frac{1}{2} \sum (a^{(k+1)}_p)^2 \right\} + \frac{q_1 \cdots q_m}{-\epsilon_1 \epsilon_2} \left\{ - \epsilon_1 \epsilon_2 q_m \frac{\partial}{\partial q_m} + \frac{1}{2} \sum p (a^{(m)}_p)^2 + \mu_1 \mu_2 + \sum_{i=1}^{m-1} \nu_i \right\} |T_m\rangle.
\]
\[
L_2|T_m\rangle = \sum_{r=1}^{m} (q_1 \cdots q_r)^2 \frac{\partial}{\partial (q_1 \cdots q_r)} |T_m\rangle
\]
We will use the (imaginary) Liouville vertex operator as a primary field with
as follows. The background charge in (3.20) is given as

\[ L_{\mu} \]

In addition, the actions of \( |R_{m}\rangle \) on the conformal state

\[ |\Delta_0 \rangle \]

is compatible with (3.21) if one requires

\[ \langle 0 | \kappa_r (r) = 1 \]

where the component notation for \( \kappa_r (r) \) is used and \( Q_{\rho} \) is the background charge. The primary state is defined as \( |\Delta_0 \rangle = \lim_{z_0 \to 0} \Psi_{\Delta_0} (z_0) |0\rangle \). Then, the Virasoro generator \( L_k \) with \( (k \geq -1) \) has the holomorphic representation on the conformal state

\[ L_k |R_m\rangle = \sum_{r=0}^{m} z_r^k \left( z_r \frac{\partial}{\partial z_r} + (k + 1) \Delta_r \right) |R_m\rangle . \]

The holomorphic state \( |R_m\rangle \) and the gauge conformal state \( |T_m\rangle \) have similar structures and their parameters are identified with each other. If one compares the Virasoro action on \( |R_1\rangle \) with \( |T_1\rangle \) using the relations given in (3.12), (3.13), (3.14) and (3.21), one can equate \( z_1 \) with \( q_1 \). However, there is a slight mismatch between \( |R_1\rangle \) and \( |T_1\rangle \). To fix this, one needs to modify \( |T_1\rangle \) by multiplying a function of \( q_1 \) and finds

\[ |K_1\rangle = q_1^{-F_1(1)} |T_1\rangle \]

where \( F_1(1) = \left( \frac{1}{2} \sum_p (a_p)^2 + \mu_1 \mu_2 - (\mu_1 + \mu_2)^2 - 2\epsilon_+ (\mu_1 + \mu_2) \right) / (\epsilon_1 \epsilon_2) \). Then, using (3.22) and (3.12) one finds \( L_0 \) on \( |K_1\rangle \),

\[ L_0 |K_1\rangle = \left\{ q_1 \frac{\partial}{\partial q_1} + \delta_0 + \Omega_0 + F_1(1) \right\} |K_1\rangle . \]

This is compatible with (3.21) if one requires \( \delta_0 \) to have the form

\[ \delta_0 = \Delta_1 + \Delta_0 - (F_1(1) + \Omega_0) . \]

In addition, the actions of \( L_1 \) and \( L_2 \) on \( |K_1\rangle \) provide the relations between other parameters which is summarized as follows. The background charge in (3.20) is given as

\[ Q = -\sqrt{2} \epsilon_+ / \sqrt{-\epsilon_1 \epsilon_2} \]

so that \( \rho_1 = -\rho_2 = -\frac{1}{2} \). This shows that the central charge in (2.26) is given as \( c = 1 - 3Q^2 \) for SU(2) gauge group. Holomorphic coordinates are identified as \( z_1 = q_1 \) and \( z_0 = 0 \) and conformal dimensions are given as

\[ \kappa_1^{(1)} = -\kappa_2^{(1)} = \mu_1 + \mu_2 / \sqrt{-2\epsilon_1 \epsilon_2} , \quad \kappa_0^{(0)} = -\kappa_2^{(0)} = \frac{Q}{2} + \frac{\mu_1 - \mu_2}{\sqrt{-2\epsilon_1 \epsilon_2}} . \]

\[ \Delta_1 = \frac{1}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 + \mu_2)^2 + \epsilon_+ (\mu_1 + \mu_2) \right\} , \quad \Delta_0 = \frac{1}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 - \mu_2)^2 - \frac{1}{2} \epsilon_+^2 \right\} . \]
This parameter identification leads to \( \delta_0 = \delta_0(1) \) as given in (3.28) where we use the relation \( F_1(1) + \Omega_0 = \left( -\mu_1 \mu_2 + (\mu_1 + \mu_2)^2 + 2\varepsilon_+ (\mu_1 + \mu_2) + 2\varepsilon_+^2 \right)/(-\epsilon_1 \epsilon_2) \).

Note that \( L_1 \) and \( L_2 \) are enough to generate the full (positive) Virasoro algebra by commutation relations and all of \( L_k \)'s action on \( | T_1 \rangle \) or \( | K_1 \rangle \) are fixed. This demonstrates that the state \( | K_1 \rangle \) constructed from the gauge theory side is equivalent to the state \( | R_1 \rangle \) constructed from the Liouville vertex operators.

This identification procedure can be generalized to higher rank case. In the same way as rank 2, we find that (3.12), (3.17) and (3.18) are consist with their Liouville conformal counterparts (3.21), as long as \( | K_2 \rangle \) is identified with \( | T_2 \rangle \) with a prefactor,

\[
| K_2 \rangle = q_1^{-F_1(2)} q_2^{-F_2(2)} | T_2 \rangle, \tag{3.28}
\]

\[
F_1(2) = \frac{1}{\epsilon_1 \epsilon_2} \left( \frac{1}{2} \sum_p (a_p)^2 - \frac{1}{2} \sum_p (b_p)^2 - 3(\varepsilon_+ - \nu)^2 - 4\varepsilon_+ (\varepsilon_+ - \nu) \right) + F_2(2),
\]

\[
F_2(2) = \frac{1}{\epsilon_1 \epsilon_2} \left( \frac{1}{2} \sum_p (b_p)^2 + \mu_1 \mu_2 - (\mu_1 + \mu_2)^2 - 2\varepsilon_+ (\mu_1 + \mu_2) + \nu (\mu_1 + \mu_2) \right).
\]

The prefactor allows the differential representation

\[
L_0 | K_2 \rangle = \left( q_1 \frac{\partial}{\partial q_1} + \delta_0 + \Omega_0 + F_1(2) \right) | K_2 \rangle. \tag{3.29}
\]

Noting the relations \( z_1 = q_1, z_2 = q_1 q_2 \) and \( z_0 = 0 \), we have

\[
\delta_0 + \Omega_0 + F_1(2) = \Delta_0 + \Delta_1 + \Delta_2. \tag{3.30}
\]

If one incorporates \( L_1 \) and \( L_2 \), one has \( \delta_0 = \delta_0(2) \) as in (2.38). The background charge \( Q \) is the same as that in (3.26) and conformal dimensions are given as

\[
\kappa_1^{(2)} = -\kappa_2^{(2)} = \frac{\mu_1 + \mu_2}{\sqrt{2\epsilon_1 \epsilon_2}}, \quad \kappa_1^{(1)} = -\kappa_2^{(1)} = -Q - \frac{\sqrt{3}\nu}{\sqrt{-\epsilon_1 \epsilon_2}}, \quad \kappa_1^{(0)} = -\kappa_2^{(0)} = \frac{Q}{2} + \frac{\mu_1 - \mu_2}{\sqrt{-2\epsilon_1 \epsilon_2}}, \tag{3.31}
\]

\[
\Delta_2 = \frac{1}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 + \mu_2)^2 + \varepsilon_+ (\mu_1 + \mu_2) \right\},
\]

\[
\Delta_0 = \frac{1}{-\epsilon_1 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 - \mu_2)^2 - \frac{1}{2} \varepsilon_+^2 \right\}, \quad \Delta_1 = \frac{1}{-\epsilon_1 \epsilon_2} \left\{ 2(\varepsilon_+ - \nu)^2 + 2\varepsilon_+(\varepsilon_+ - \nu) \right\}. \tag{3.32}
\]

It is straight-forward to compare (3.12), (3.17) and (3.18) with (3.21) once the prefactor is found.

\[
| K_m \rangle = \prod_{r=1}^{m-1} (q_1 \cdots q_r)^{-1/2} \left( \frac{1}{4} \sum_p (a_p^r)^2 - \frac{1}{2} \sum_p (a_p^{r+1})^2 + (3\nu_\epsilon - 7\epsilon_+ + 2 \sum_{i=1}^{m-1} \nu_i) (\varepsilon_+ - \nu) \right)
\]

\[
\times (q_1 \cdots q_m)^{-1/2} \left( \frac{1}{4} \sum_p (a_p^{m})^2 + (\mu_1 + \mu_2)(-2\epsilon_+ + \sum_{i=1}^{m-1} \nu_i) + \mu_1 \mu_2 - (\mu_1 + \mu_2)^2 \right) | T_m \rangle. \tag{3.33}
\]

The holomorphic coordiante are identified with the \( q \)-coordinates

\[
z_r = q_1 \cdots q_r, \quad (r = 1, \cdots m); \quad z_0 = 0. \tag{3.34}
\]
The background charge $Q$ is the one in \([3.25]\) and $\delta_0 = \delta_0(m)$ in \([2.38]\). Conformal dimensions are given as
\[
\kappa_r^{(r)} = -\kappa_2^{(r)} = -Q - \sqrt{2\nu r} \left/ \sqrt{-\epsilon_3 \epsilon_2} \right., \quad (1 \leq r \leq m - 1)
\]
\[
\kappa_1^{(m)} = -\kappa_2^{(m)} = \frac{\mu_1 + \mu_2}{\sqrt{-2\epsilon_3 \epsilon_2}}, \quad \kappa_1^{(0)} = -\kappa_2^{(0)} = \frac{Q}{2} + \frac{\mu_1 - \mu_2}{\sqrt{-2\epsilon_3 \epsilon_2}}.
\]
\[
\Delta_r = \frac{1}{\epsilon_3 \epsilon_2} \left\{ 2(\epsilon_+ - \nu r)^2 + 2\epsilon_+ (\epsilon_+ - \nu r) \right\}, \quad (1 \leq r \leq m - 1),
\]
\[
\Delta_m = \frac{1}{\epsilon_3 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 + \mu_2)^2 + \epsilon_+ (\mu_1 + \mu_2) \right\}, \quad \Delta_0 = \frac{1}{\epsilon_3 \epsilon_2} \left\{ \frac{1}{2} (\mu_1 - \mu_2)^2 - \frac{1}{2} \epsilon_+^2 \right\}.
\]
These parameter relations are exactly the AGT dictionary, which translates the CFT parameters to their gauge counterparts. The (imaginary) Liouville CFT side is based on one boson construction, with the vertex operator
\[
\Psi_{\Delta_r} (z_r) = e^{i\alpha^{(r)}} (\psi(z_r)) \quad \text{and} \quad \text{conformal dimension} \; \Delta_r \equiv \alpha^{(r)}(\alpha^{(r)} - Q) \quad \text{if one uses the relation with our two boson construction} \; \psi = \frac{1}{2} \varphi_1 - \frac{1}{2} \varphi_2 \quad \text{and} \quad \alpha_r = \kappa_1^{(r)} = -\kappa_2^{(r)}.
\]

## 4 Colliding limit and irregular state

Virasoro representation $L_k$ on the irregular state $|I_m\rangle$ of rank $m$ is given in terms of differential operators with respect to the eigenvalue $c_k$ of positive mode of Heisenberg operator $a_k$ with $0 \leq k \leq m$ \([10]\)
\[
L_k = \left\{ \begin{array}{ll} \Lambda_k + \sum_{l=1}^{m-k} c_{l+k} \partial / \partial c_l & \text{for } 0 \leq k \leq 2m \\ 0 & \text{for } k > 2m \end{array} \right.
\]
where $\Lambda_k = \sum_l c_{l+k} - (k + 1)Qc_k$. It is noted that if $L_1$ and $L_2$ are given, then other generators in \(L_k\) are determined from the Virasoro commutation relations. In addition, $L_k$ with $m \leq k \leq 2m$ reduces to the eigenvalue $\Lambda_k$ since there is no $c_k$ with $k > m$. Therefore, when the stress energy tensor applies on the irregular state of rank $m$, one has singular contributions
\[
T_>(y)|I_m\rangle = \left[ \sum_{k=m}^{2m} \frac{\Lambda_k}{y^{k+2}} + \sum_{k=0}^{m-1} \frac{L_k}{y^{k+2}} + \frac{1}{y} L_{-1} \right]|I_m\rangle .
\]
The irregular state is of the form \([9]\)
\[
|I_m\rangle = \sum_{\ell, Y, \ell_p} \Lambda^{\ell/m} \left\{ \prod_{i=1}^{m-1} a_i \right\} \ell^\ell Q_\Delta^{-1} \left( t^\ell y^{\ell} z^{(2m-1)(2m)} Y \right) L_{-\ell} Y^\ell |\Delta\rangle
\]
where $\ell = |Y|$. The eigenvalues are $\Lambda_m = \Lambda t$, $\Lambda_{2n-s} = \Lambda^{(2n-s)/n} a_s$ and $\Lambda_{2n} = \Lambda^2$.

To obtain the colliding limit from the regular state we need to scale away the singular contribution, which is achieved if one defines $|R'_1\rangle$ as
\[
| R'_1 \rangle = z_1^{-2\alpha_1 \alpha_0} | R_1 \rangle
\]
since Virasoro generators has the differential representation on $| R'_1 \rangle$
\[
L_0 | R'_1 (z) \rangle = \left( z_1^{\partial / \partial z_1} + (\alpha_1 + \alpha_0)(\alpha_1 + \alpha_0 - Q) \right) | R'_1 (z) \rangle \]
\[
L_1 | R'_1 (z) \rangle = \left( z_1^{3 \partial / \partial z_1} + 2z_1 \alpha_1 (\alpha_1 + \alpha_0 - Q) \right) | R'_1 (z) \rangle
\]
\[
L_2 | R'_1 (z) \rangle = \left( z_1^{3 \partial / \partial z_1} + z_1^2 \alpha_1 (3 \alpha_1 + 2 \alpha_0 - 3Q) \right) | R'_1 (z) \rangle.
\]
On the other hand, according to (3.23), the gauge conformal state has the form

\[ L_0 | K_1 \rangle = \left\{ q_1 \frac{\partial}{\partial q_1} + \Delta_1 + \Delta_0 \right\} | K_1 \rangle. \]  

(4.6)

However, considering the fusion of two vertex operators at \( z_1 \) and the origin, we need a \( q \)-state \( | K'_1 \rangle \) obeying

\[ L_0 | K'_1 \rangle = \left\{ q_1 \frac{\partial}{\partial q_1} + \Delta_{01} \right\} | K'_1 \rangle \]  

(4.7)

where \( \Delta_{01} = \alpha_{01}(\alpha_{01} - Q) \) with \( \alpha_{01} = \alpha_0 + \alpha_1 \) as given in (4.5). This is achieved if a new \( q \)-representation \( | K'_1 \rangle \) is defined as

\[ \langle K'_1 | = q_1^{(\Delta_0 + \Delta_1) - \Delta_{01}} | K_1 \rangle = q_1^{-2\alpha_{01}} | K_1 \rangle = q_1^{-H_1} | T_1 \rangle, \]  

(4.8)

where \( H_1 = [\Delta_{01} - (\Delta_1 + \Delta_0)] + F_1 = [\Delta_{01} - (\Delta_1 + \Delta_0)] + [\Delta_1 + \Delta_0 - (\delta_0(1) + \Omega_0)] \) and its explicit value is given as

\[ H_1 = \frac{1}{\epsilon_1 \epsilon_2} \left( \sum_p (a_p)^2 - (\mu_1 + \epsilon_+)(\mu_2 - \mu_1)^2 \right). \]

The colliding limit is to put \( \alpha_1 \to \infty \) and \( z_i \to 0 \) while keeping \( c_1 = z_1 \alpha_1 \) and \( c_0 = \alpha_1 + \alpha_0 \) finite and reduces \( | R'_1 \rangle \) to \( | I_1 \rangle \).

\[ L_1 | I_1 \rangle = 2c_1(\alpha_0 - Q) | I_1 \rangle, \quad L_2 | I_1 \rangle = c_1^2 | I_1 \rangle. \]  

(4.9)

Since the actions of \( L_1 \) and \( L_2 \) commute each other, \( L_k = 0 \) when \( k \geq 3 \).

On the same footing, \( | K'_1 \rangle \) becomes the irregular state of rank 1 since \( | K'_1 \rangle \) and \( | R'_1 \rangle \) have the same differential structure when \( z_1 = q_1 \). Therefore, we may obtain \( | I_1 \rangle \) in terms of \( | K'_1 \rangle \) at the colliding limit up to normalization, if \( | \Delta \rangle \) in (4.3) is identified with the newly defined \( q \)-basis \( | \Delta \rangle \equiv \lim_{q_1 \to 0}(\alpha_{01} q_1)^{-H_1} | \vec{a}; \alpha_0 \rangle \). Its descendant \( | \Delta + Y \rangle = L_- Y | \Delta \rangle \) is given as \( c_1^{-H_1} | \vec{a}, \bar{Y}; \delta_0 \rangle \), since \( L_0 | \Delta + Y \rangle = (\Delta + |Y\rangle) | \Delta + Y \rangle \). After this consideration, the irregular conformal state of rank 1 in (4.3) is written in terms of the \( q \)-basis as appeared in [15]

\[ | I_1 \rangle = \sum_{\vec{a}, \bar{Y}} \Lambda^{\vec{a}, \bar{Y}} (Z_{\text{vect}}(\vec{a}, \bar{Y}))^{1/2} Z_{\text{aff}}(\vec{a}, \bar{Y}, \mu_1) | \Delta + Y \rangle \]  

(4.10)

which is equivalent to the one given in [7] when the colliding limit is achieved with \( \mu_2 \to \infty, q \to 0 \) and \( q \mu_2 = \Lambda \) finite. As a result, the inner product \( \langle \Delta | I_1 \rangle = \langle \Delta | \Delta \rangle \), which can be normalized as 1.

The irregular state of rank 2 can be constructed similarly. First, we prepare \( | R'_2 \rangle \) for the colliding limit:

\[ | R'_2 \rangle = z_1^{-2\alpha_0 \alpha_2} z_2^{-2\alpha_2 \alpha_0} (z_1^{-2\alpha_0 \alpha_2} | R_2 \rangle. \]  

(4.11)

The Virasoro representation is given as

\[ L_0 | R'_2(z) \rangle = \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + \Delta_{012} \right) | R'_2(z) \rangle \]  

(4.12)

\[ L_1 | R'_2(z) \rangle = \sum_{a=1,2} \left( z_a \frac{\partial}{\partial z_a} + 2z_a \alpha_0 - Q \right) | R'_2(z) \rangle \]  

\[ L_2 | R'_2(z) \rangle \]  

where \( \Delta_{012} = (\alpha_{012} - Q) \) and \( \alpha_{012} = \alpha_2 + \alpha_1 + \alpha_0 \). At the colliding limit we have finite variables \( c_2 = z_1^{-\alpha_1 + \alpha_2 \alpha_2}, c_1 = z_1 \alpha_1 + z_2 \alpha_2 \) and \( c_0 = \alpha_1 + \alpha_2 + \alpha_0 \) and the Virasoro operation reduces to

\[ L_0 | I_2 \rangle = \left( c_1 \frac{\partial}{\partial c_1} + 2c_2 \alpha_0 \right) | I_2 \rangle \]  

(4.13)

\[ L_1 | I_2 \rangle = \left( c_2 \frac{\partial}{\partial c_1} + 2c_1(c_0 - Q) \right) | I_2 \rangle, \]  

(4.14)

\[ L_2 | I_2 \rangle = \left( 2(c_0 - Q) c_2 + c_1 \right) | I_2 \rangle. \]  

(4.15)
Higher positive non-vanishing generators $L_3$ and $L_4$ are generated from the lower generators $L_1$ and $L_2$ and the irregular state of rank 2 is the simultaneous eigenstate of the three generators $L_2$, $L_3$ and $L_4$.

Similarly for the $q$-state of rank 2, we have

$$L_0|K_2⟩ = \left\{ q_1 \frac{∂}{∂q_1} + ∆_2 + ∆_1 + ∆_0 \right\}|K_2⟩. \tag{4.16}$$

However, we need a state $|K'_2⟩$

$$L_0|K'_2⟩ = \left\{ q_1 \frac{∂}{∂q_1} + ∆_{12} \right\}|K'_2⟩ \tag{4.17}$$

which can be realized if one puts

$$|K'_2⟩ = q_1^{(∆_0+∆_1+∆_2)−∆_{12}} h(q_2)|K_2⟩. \tag{4.18}$$

Considering the identification for the regular conformal state, we may have the form

$$|K'_2⟩ = (q_1)^{-2α_1 α_0}(q_1 q_2)^{-2α_2 α_0}(q_1 − q_1 q_2)^{-2α_1 α_2}|K_2⟩ \tag{4.19}$$

if one uses the relation $z_1 = q_1$ and $z_2 = q_1 q_2$ and (4.11). It is easy to convince that $q_1$ power in (4.18) matches with the one in (4.18):

$$∆_{12} − (∆_0 + ∆_1 + ∆_2) = 2α_1 α_0 + 2α_2 α_0 + 2α_1 α_2. \tag{4.20}$$

The remaining factor $h(q_2)$ in (4.18) can be fixed as

$$h(q_2) = q_2^{-2α_2 α_0}(1 − q_2)^{-2α_1 α_2}. \tag{4.21}$$

Note that the finite parameters at the colliding limit are related with $q_1$ and $q_2$

$$c_0 = α_0 + α_1 + α_2, \quad c_1 = q_1 α_0(1 + q_2 α_2/α_1), \quad c_2 = q_1^2 α_0(1 + q_2^2 α_2/α_1). \tag{4.22}$$

Finite $c_2$ is obtained at the colliding limit as $α_1 \rightarrow ∞$ and $q_1 \rightarrow 0$. Therefore, we may ask if $q_1^2 α_1$ and $q_2^2 α_2/α_1$ are separately finite. This is the case both $q_1$ and $q_2$ go to 0 because as $α_1$ goes to infinity, so does $(α_0 + α_2)$. Therefore, $q_2 \rightarrow 0$ limit ensures that $q_2^2 α_2/α_1$ is finite since the ratio $α_2/α_1$ can be infinite. On the other hand, the limit $q_2 \rightarrow 1$ is not allowed in the colliding limit because as $q_2 \rightarrow 1$ we have $c_1 \rightarrow q_1 α_0(1 + α_2/α_1)$ and $c_2 \rightarrow q_1^2 α_0(1 + α_2/α_1)$ which cannot simultaneously be finite (non-zero) as $q_1 \rightarrow 0$.

In fact, the limit $q_2 \rightarrow 1$ corresponds to the t-channel limit $z_1 \sim z_2 \rightarrow 0$ and is not allowed in the colliding limit. In contrast, the limiting procedure $q_1, q_2 \rightarrow 0$ corresponds to the limit $|z_2| < |z_1| \rightarrow 0$. This concludes that the colliding limit allows only the s-channel limiting procedure and the hierarchical behavior $z_2 < z_1 \rightarrow 0$ (or $q_1, q_2 \rightarrow 0$) should be present in the final result. More explicitly, this s-channel limit shows that $(1 + q_2 α_2/α_1) \rightarrow O(q_1)$ considering $c_1$. In fact, we find $α_1 \sim c_2/q_1^2$ and $α_2 \sim −c_2/(q_1^2 q_2)$, so $q_1^2 α_1$ and $q_2 α_2/α_1$ is finite.

At the colliding limit $|K'_2⟩$ is reduced to $|I_2⟩$ if the primary state $|Δ⟩$ has the conformal dimension $Δ_{12}$ which can be defined in terms of the $q$-state

$$|Δ_{012}⟩ = \lim_{q_1, q_2 \rightarrow 0} q_1^{−(Δ_{012}−(Δ_0+Δ_1+Δ_2)−F_1(2))} (1 − q_2)^{−2α_1 α_2}|a, 0; δ_0(2)⟩ \tag{4.23}$$

where $Δ_0 + Δ_1 + Δ_2 − F_1(2) = δ_0(2) + Ω_0$ as in (4.30). We put the proper conformal dimension by multiplying the $q_1$ factor and remove the t-channel information by multiplying $(1 − q_2)$ factor. The descendant state $|Δ_{012} + |\bar{Y}⟩⟩$ is obtained if one uses $|a, \bar{Y}; δ_0(2)⟩$ in (4.23). In addition, we can use the freedom to put normalization constant $−(α_2/α_1)^{−(F_2(2)+2α_2 α_0)}$ so that $q_2$ factor in front is 1 as $q_2 \rightarrow 0$. Then the gauge conformal state we are preparing for the colliding limit is given as

$$|K'_2⟩ = \sum_{\bar{Y}} q_1^{\bar{Y}} Z_{\text{vec}}(a, \bar{Y})\left\{ \sum_{\bar{b}, \bar{W}} q_2^{\bar{W}} Z_{\text{vec}}(\bar{b}, \bar{W}) Z_{\text{bif}}(a, \bar{b}; \bar{Y}; \bar{W}|ν) \prod_{l=1,2} Z_{\text{af}}(\bar{b}, \bar{W}, µ_1) \right\}|Δ_{012} + |\bar{Y}⟩⟩. \tag{4.24}$$
Note that $c_2 = \Lambda$ provides the overall scaling parameter. Therefore, we need $(q^2_1\alpha_1)^{1/2}$ in the summation over $\tilde{Y}$. On the other hand, for the summation over $\tilde{W}$, we need $q_2$ dependent quantity. Note that there are three other parameters $a_1, b_1$ and $t$ in (4.23). All the quantities are to be given in $c_0, c_1$ and $c_2$ which is finite at the colliding limit. In fact, $a_1A^{3/2}$ and $t\Lambda$ correspond to the eigenvalue of $L_3$ and $L_2$, respectively. $b_1$ is to be related with the normalization of the irregular state $\|\tilde{W}\|^2$. The candidates of the $q_1$ dependent terms are $c_0$ and the combination $c_1^2/c_2 = \alpha_1(1+q_2\alpha_2/\alpha_1)^2$. Therefore, $c_1^2/c_2$ is very tricky to get because we need the combination of $\alpha_1$ and $q_2\alpha_2/\alpha_1$ at the colliding limit in the summation over $\tilde{W}$ in (4.24).

It is worth to note that at $|\tilde{Y}| = 0$ the coefficient in (4.24) is not 1. Therefore, at the colliding limit one has the inner product

$$\langle \Delta_{012}|K'_2 \rangle = \sum_{\tilde{W}} q_2^{|\tilde{W}|} Z_{\text{vect}}(\tilde{b}, \tilde{W}) Z_{\text{bit}}(\mathbf{\tilde{a}}, \mathbf{\tilde{b}}, \tilde{W} | \nu) \prod_{l=1,2} Z_{\text{af}(\tilde{b}, \tilde{W}, \nu_l)}(\Delta_{012}|\Delta_{012}),$$

(4.25)

which is not a simple constant but should be related with the partition function $Z_{(02)}$ of the irregular matrix model $[12,22]$.

For the general state of rank $m$, one can start with the Liouville state

$$|R'_m\rangle = \prod_{r=1}^{m} (z_r)^{-2\alpha_r - \alpha_0} \times \prod_{i<j} (z_i - z_j)^{-2\alpha_i\alpha_j} |R_m(z)\rangle$$

(4.26)

so that $|R'_m\rangle$ satisfies

$$L_1|\tilde{R}'_m(z)\rangle = \sum_{r=1}^{m} \left( z_r^2 \frac{\partial}{\partial z_r} + 2z_r\alpha_r \left( \sum_{k=0}^{m} \alpha_k - Q \right) \right) |\tilde{R}'_m(z)\rangle$$

(4.27)

$$L_2|\tilde{R}'_m(z)\rangle = \sum_{r=1}^{m} \left( z_r^2 \frac{\partial}{\partial z_r} + z_r^2\alpha_r \left( \alpha_r + 2 \sum_{k=0}^{m} \alpha_k - 3Q \right) \right) + 2m \sum_{i<j} z_iz_j\alpha_i\alpha_j |\tilde{R}'_m(z)\rangle.$$ (4.28)

At the colliding limit we have finite variables $c_k = \sum_{i=1}^{m} z_i^k \alpha_i$ and

$$L_1|I_m(z)\rangle = \sum_{l=1}^{m-1} l\gamma_{l+1} \frac{\partial}{\partial \gamma_l} + 2c_1(Q - c_0) |I_m(z)\rangle$$

(4.29)

$$L_2|I_m(z)\rangle = \sum_{l=1}^{m-2} l\gamma_{l+2} \frac{\partial}{\partial \gamma_l} + (3Q - 2c_0)c_2 - c_1^2 |I_m(z)\rangle.$$ (4.30)

Similarly, we define the $q$-state

$$|K'_m\rangle = q_1^{\sum_{i=0}^{m-1} \Delta_i - \Delta} \langle f(q_{j\neq 1}) |K_m\rangle$$

(4.31)

with $|K_m\rangle$ defined in (3.33). Explicitly, we have

$$|K'_m\rangle = \prod_{r=1}^{m-1} (q_1 \cdots q_r)^{1/2 \gamma_{r+2}} \left( \frac{1}{2} \sum_{p} (a_p^{(r)})^2 + \frac{1}{2} \sum_{p} (a_p^{(r+1)})^2 + \sum_{i=1}^{r-1} \mu_{i,i+1} (e_i - \mu_{r+1}) - 2(\mu_1 - \mu_2)(\epsilon_+ - \mu_{r+1}) \right)$$

(4.32)

$$\times \left( q_1 \cdots q_m \right)^{1/2 \gamma_{m+2}} \left( \frac{1}{2} \sum_{p} (a_p^{(m)})^2 + (\mu_1 + \mu_2)(\epsilon_+ + \sum_{i=1}^{m-1} \mu_{i,i+1}) - \mu_1^2 - 2\mu_2^2 \right)$$

$$\times \prod_{i<j} (q_1 \cdots q_i - q_1 \cdots q_j)^{1/2 \gamma_{j+2}} \left( \sum_{i=1}^{m-1} (\mu_{i,i+1} - \epsilon_+)(\epsilon_+ - \mu_{j,j+1}) \right) \prod_{r=1}^{m-1} (q_1 \cdots q_i - q_1 \cdots q_j)^{1/2 \gamma_{i+2}} \left( 2(\mu_{r+1} - \epsilon_+)(\mu_1 + \mu_2) \right) |T_m\rangle.$$  

If we send $\tilde{b} \to \nu$ together with the colliding limit, we have $|T_2; \delta_0\rangle \sim \sum_{Y, \tilde{W}} q_1^{\|Y\|} q_2^{\|\tilde{W}\|} a_1^{\|\tilde{W}\|} a_2^{\|\tilde{W}\|} |\tilde{W}; Y; \delta_0(2)\rangle$, with $F$ finite. In this case the $\tilde{W}$ related terms are finite.
As noted in rank 2, the holomorphic coordinates should have the hierarchical structure \( z_m < z_{m-1} < \cdots < z_0 \to 0 \) at the colliding limit. This s-channel limit is obtained if all \( q_i \to 0 \). \( q_1 \) dependence takes care of the proper scaling and disappears. In addition, t-channel quantity (powers of \((1 - q_i)\)) should be absorbed into the definition of the primary q-state \(|\Delta_{01\ldots m}\rangle\). Then we are left with \(|K'_m\rangle = |T_m\rangle\) where the factor \( \prod_{r=2}^{m-1}(q_r)^{|r|} \) in (4.32) is normalized as 1 by multiplying the appropriate constant. In this case, the inner product \( \langle \Delta_{01\ldots m}|K'_m\rangle \) at the colliding limit is identified as the partition function \( Z_{0m} \) of irregular matrix which is now given by summing Young diagrams.

It is interesting to apply the Heisenberg algebra to q-state \(|T_m\rangle\). Using \([L_1, J_k] = -kJ_{k+1}\) one finds

\[
J_k|T_m\rangle = \sum_{i=1}^{m-1} \frac{(q_1 \cdots q_i)^k}{\sqrt{-\epsilon_1 \epsilon_2}} \left(N(\epsilon_+ - \nu_i)\right) + \frac{(q_1 \cdots q_m)^k}{\sqrt{-\epsilon_1 \epsilon_2}} \left(\sum_{I} \mu_I\right)|T_m\rangle. \tag{4.33}
\]

At the colliding limit, one has \( J_k|T_m\rangle = 0 \) when \( k > m \) but

\[
J_k|T_m\rangle = c_k|T_m\rangle \tag{4.34}
\]

for \( 1 \leq k \leq m \). Therefore, \(|T_m\rangle\) becomes the coherent state of Heisenberg algebra at the colliding limit.

5 Conclusion

We construct gauge conformal state based on AFLT basis and interwiners for the spherical Hecke algebra with central extension. The q-coordinate is the instanton expansion parameter and Hecke algebra has the q-differential representation on the q-state. The q-representation is used to find the exact relation with the Liouville conformal state where conformal scaling is to be carefully matched. The q-state reduces to the irregular conformal state at the colliding limit, which provides the formal structure of the irregular state and its inner product is identified with the partition of the irregular matrix. However, it is not yet clear how to get the explicit summation over Young diagrams.

Our study has been limited to the Virasoro conformal state which is related with SU(2) gauge group. This method can be extended to \( \mathcal{W} \) conformal state without any difficulty. The q-state for SU(N) gauge group can be obtained by extending the Young diagrams. Using \( D_{r,s} \), one has the actions of \( \mathcal{W}_{r,s}^{(s)} \) operators on the SU(N) gauge conformal state. Besides, generalization to 5 dimensions seems natural, using the 5D version of SH [23].

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A Calculation details

All the following holds for SU(N) case.

A.1 The component for the instanton part of Nekrasov Partition function

\[
\mathcal{Z}_{\text{inst}}(a, b; m_{12}) = \prod_{\ell=1}^{N_1} \prod_{\ell'=1}^{N_2} g_{Y_{\ell'}, W_{\ell}}(a_\ell - b_{\ell'} - m_{12}) \tag{A.1}
\]

\[
g_{\lambda, \mu}(x) = \prod_{(i,j) \in \lambda} (x + \epsilon_1 (\lambda_j' - i + 1) - \epsilon_2 (\mu_i - j)) \prod_{(i,j) \in \mu} (-x + \epsilon_1 (\mu_j' - i) - \epsilon_2 (\lambda_i - j + 1)). \tag{A.2}
\]
Here $\lambda_i$ is the height of $i^{th}$ column and $\lambda'_j$ is the length of $j^{th}$ row of Young diagram $\lambda$

$$Z_{\text{fund}}(m; \vec{a}, \vec{Y}) = Z_{\text{bif}}(m, \vec{0}; \vec{a}, \vec{Y}|0). \quad (A.3)$$

$$Z_{\text{afd}}(m; \vec{a}, \vec{Y}) = Z_{\text{bif}}(\vec{a}, \vec{Y}; -m, \vec{0}|0) = Z_{\text{fund}}(-\epsilon_+ - m; \vec{a}, \vec{Y}). \quad (A.4)$$

$$Z_{\text{vect}}(\vec{a}, \vec{Y}) = Z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}|0)^{-1}. \quad (A.5)$$

A.2 Useful formulas

We use the formulas in [18]

$$Z_{\text{bif}}(\vec{a}, \vec{Y} + x; \vec{b}, \vec{W}|\nu) = \prod_{y \in A(\vec{W})} (\phi_x - \phi_y + \epsilon_+ - \nu) \prod_{y \in R(\vec{Y})} (\phi_x - \phi_y - \nu), \quad (A.6)$$

$$Z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}|\nu) = \prod_{y \in A(\vec{W})} (\phi_x - \phi_y - \nu), \quad (A.7)$$

$$Z_{\text{bif}}(\vec{a}, \vec{Y} - x; \vec{b}, \vec{W}|\nu) = \prod_{y \in A(\vec{Y})} (\phi_x - \phi_y + \epsilon_+ - \nu), \quad (A.8)$$

$$Z_{\text{bif}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}|m) = \prod_{y \in R(\vec{Y})} (\phi_x - \phi_y + \nu), \quad (A.9)$$

$$Z_{\text{vect}}(\vec{a}, \vec{Y} + x) = -\frac{1}{\epsilon_1 \epsilon_2} \prod_{y \in A(\vec{Y})} (\phi_x - \phi_y) (\phi_x - \phi_y - \epsilon_+), \quad (A.10)$$

$$Z_{\text{vect}}(\vec{a}, \vec{Y}) = -\frac{1}{\epsilon_1 \epsilon_2} \prod_{y \in A(\vec{Y})} (\phi_x - \phi_y) (\phi_x - \phi_y + \epsilon_+), \quad (A.11)$$

$$\frac{Z_{\text{afd}}(\vec{a}, \vec{Y} + x, \mu_I)}{Z_{\text{afd}}(\vec{a}, \vec{Y}, \mu_I)} = (\phi_x + \mu_I) \quad (A.12)$$

Also we know that for arbitratry $m$ and $z_k = q_1 \cdots q_k$, \quad \(r = 1, \cdots N)\),

$$\left\{ \sum_{k=1}^{N-1} (q_1 \cdots q_k)^m (|\vec{Y}_k| - |\vec{Y}_{k+1}|) + (q_1 \cdots q_N)^m |\vec{Y}_N| \right\}|T_m \rangle \quad (A.13)$$

$$= \left\{ \sum_{k=1}^{N-1} (q_1 \cdots q_k)^m (q_k \frac{\partial}{\partial q_k} - q_{k+1} \frac{\partial}{\partial q_{k+1}}) + (q_1 \cdots q_N)^m (q_N \frac{\partial}{\partial q_N}) \right\}|T_m \rangle$$

$$= \sum_{k=1}^{N} \frac{m+1}{m_k} \frac{\partial}{\partial z_k}|T_m \rangle. \quad (A.14)$$

A.3 Action on $|G, \vec{a}, \mu_I\rangle$

In the following we re-derive the rank 1 case using a method introduced in [18] as an alternative of [15]. First notice that,

$$D_{-1,n} q_1^D |G, \vec{a}, \mu_I\rangle = q_1^{D+1} D_{-1,n} |G, \vec{a}, \mu_I\rangle. \quad (A.14)$$

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Then from (2.22),
\[
D_{-1,n}(G, \bar{a}, \mu_I) = \sum_{\bar{Y}} \sqrt{Z_{\text{vec}}(\bar{a}, \bar{Y})} \prod_{l=1}^{N} Z_{\text{afld}}(\bar{a}, \bar{Y}, \mu_I) \sum_{x \in R(\bar{Y})} (\phi_x)^{n} \Lambda_x(\bar{Y})|\bar{a}, \bar{Y} - x\rangle
\]
\[
= \sum_{\bar{Y}} \sum_{x \in A(\bar{Y})} \sqrt{Z_{\text{vec}}(\bar{a}, \bar{Y} + x)} \left( \prod_{l=1}^{N} Z_{\text{afld}}(\bar{a}, \bar{Y} + x, \mu_I) \right) (\phi_x)^{n} \Lambda_x(\bar{Y} + x)|\bar{a}, \bar{Y}\rangle
\]
using the fact that \(\Lambda_x(\bar{Y} + x)^2 = \Lambda_x(\bar{Y})^2\), \(\forall x \in A(\bar{Y})\),
\[
D_{-1,n}(G, \bar{a}, \mu_I) = \sum_{\bar{Y}} \sum_{x \in A(\bar{Y})} (\phi_x)^{n} \Lambda_x(\bar{Y}) \sqrt{Z_{\text{vec}}(\bar{a}, \bar{Y} + x)} \left( \prod_{l=1}^{N} Z_{\text{afld}}(\bar{a}, \bar{Y} + x, \mu_I) / Z_{\text{vec}}(\bar{a}, \bar{Y}) \right)
\times \sqrt{Z_{\text{vec}}(\bar{a}, \bar{Y})} \left( \prod_{l=1}^{N} Z_{\text{afld}}(\bar{a}, \bar{Y}, \mu_I) \right)|\bar{a}, \bar{Y}\rangle
\]
After some calculation, we find
\[
D_{-1,n}(G, \bar{a}, \mu_I) = \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \sum_{\bar{Y}} \sum_{x \in A(\bar{Y})} (\phi_x)^{n} \prod_{w \in R(\bar{Y})} (\phi_x - \phi_w - \epsilon) \prod_{y \in A(\bar{Y}) \setminus x} (\phi_x - \phi_y) \prod_{l=1}^{N} (\phi_x + \mu_I)
\times \sqrt{Z_{\text{vec}}(\bar{a}, \bar{Y})} \prod_{l=1}^{N} Z_{\text{afld}}(\bar{a}, \bar{Y}, \mu_I)|\bar{a}, \bar{Y}\rangle
\]
By setting \(x_I = \{\phi_y, (y \in A(\bar{Y}^{(I)})\})\) and \(y_J = \{\phi_w + \epsilon, (w \in R(\bar{Y}^{(J)}); -\mu_I, \ldots, -\mu_N\}\), the above can be simplified using the KMZ equation [20],
\[
\sum_{I=1}^{N} (x_I)^{m} \prod_{J=1}^{M} (x_I - y_J) = \sum_{n=0}^{m+1+M-N} f_{m-n+1+M-N}(-y)b_n(x),
\]
where \(f_n(x) = \sum_{I: I_1 \leq \cdots \leq I_n} x_{I_1} \cdots x_{I_n}\), and \(b_n(x) = \sum_{I: I_1 \leq \cdots \leq I_n} x_{I_1} \cdots x_{I_n}\). In details, since
\[
\left( \sum_{I} x_I - \sum_{J} y_J \right) = \sum_{p} a_p + \sum_{I} \mu_I,
\]
and
\[
\frac{1}{2} \left( \sum_{I} x_I^{2} - \sum_{J} y_J^{2} \right) = -\frac{1}{2} \sum_{p} \left( \sum_{k=1}^{f_p} (2a_p + \epsilon_1(r_k + r_{k-1}) + 2\epsilon_2 s_k)(\epsilon_1 r_k - \epsilon_1 r_{k-1}) + \frac{1}{2}(a_p + \epsilon_1 r_f)^2 \right) - \frac{1}{2} \sum_{l=1}^{N} \mu_l^{2},
\]
we find
\[
\sum_{n=0}^{2} f_{2-n}(-y)b_n(x) = \frac{1}{2} \left( \sum_{I} x_I - \sum_{J} y_J \right)^{2} + \frac{1}{2} \left( \sum_{I} x_I^{2} - \sum_{J} y_J^{2} \right)
\]
\[
= -\epsilon_1 \epsilon_2 |\bar{Y}| + \frac{1}{2} \sum_{p} (a_p)^{2} - \frac{1}{2} \sum_{l=1}^{N} \mu_l^{2},
\]
which leads to (3.2) and (3.3) for SU(2) case.
Further for $D_{-2,d}$, we know
\[
D_{-2,0} = [D_{-1,0}, D_{-1,1}] ,
\]
\[
D_{-2,1} = [D_{-1,0}, D_{-1,2}] ,
\]
\[
D_{-2,d} = [D_{-1,0}, D_{-1,d+1}] - [D_{-1,1}, D_{-1,d}] .
\]

It reads that
\[
D_{-1,m}D_{-1,n}(G, \vec{a}, \mu_l) = \sum_{\vec{a}, \vec{Y}} \sum_{x \in A(R)} \left( Z_{\text{vec}}(\vec{a}, \vec{Y} + x) \prod_{l=1}^{N} Z_{\text{afd}}(\vec{a}, \vec{Y} + x, \mu_l)(\phi_x)^n \Lambda_x(\vec{Y}) \sum_{t \in R(\vec{Y})} (\phi_t)^m \Lambda_t(\vec{Y}) |\vec{a}, \vec{Y} - t) \right) (A.25)
\]
\[
\times \left( \phi_t - \phi_x \right) |\vec{a}, \vec{Y} \rangle ,
\]
and for $n \geq 1$,
\[
D_{-2,n}(G, \vec{a}, \mu_l) = \sum_{\vec{a}, \vec{Y}} \sum_{x \in A(R)} \left( Z_{\text{vec}}(\vec{a}, \vec{Y} + x) \prod_{l=1}^{N} Z_{\text{afd}}(\vec{a}, \vec{Y} + x, \mu_l)(\phi_x)^n \Lambda_x(\vec{Y} + t) \Lambda_t(\vec{Y}) \right) \times \left( \phi_t - \phi_x \right) |\vec{a}, \vec{Y} \rangle .
\]

The key is to solve the following,
\[
\sum_{x \in A(\vec{Y} + t)} (\phi_x)^n \Lambda_x(\vec{Y} + t) \left( Z_{\text{vec}}(\vec{a}, \vec{Y} + x) \prod_{l=1}^{N} Z_{\text{afd}}(\vec{a}, \vec{Y} + x, \mu_l)(\phi_x)^n \Lambda_x(\vec{Y} + t) \Lambda_t(\vec{Y}) \right) \times (\phi_x - \phi_t) (A.28)
\]
By exchanging $x$ and $t$,

\[
\sum_{t \in A(Y)} \prod_{w \in R(Y)} (\phi_t - \phi_w - \epsilon_+) \prod_{y \not= t} (\phi_x - \phi_y) = \sum_{t \in A(Y)} \prod_{w \in R(Y)} (\phi_t - \phi_w - \epsilon_+) \prod_{y \not= t} (\phi_x - \phi_y) \prod_{I \in A(Y)} (\phi_x + \mu_I) \]

(A.29)

As a result,

\[
D_{-2,n}(G, \vec{a}, \mu) = \]

\[
\prod_{t \in A(Y)} \frac{1}{2(\epsilon_2 - \epsilon_1)} \left\{ 1 \sum_{w \in R(Y)} (\phi_t - \phi_w - \epsilon_+) \prod_{y \not= t} (\phi_x - \phi_y) \prod_{I \in A(Y)} (\phi_x + \mu_I) \right\} \]

(A.30)

with $x_i = \{ \phi_y, (y \in A(Y^{(i)}) \}$ and $y_j = \{ \phi_w + \epsilon_+, (w \in R(Y^{(j)}); -\mu_1, \ldots, -\mu_N \}$,

\[
x_i^{(1)} = \{ x_1; x_1 - \epsilon_1 \} \]

\[
x_i^{(2)} = \{ x_1; x_1 + \epsilon_1 \} \]

\[
x_i^{(3)} = \{ x_1; x_1 - \epsilon_2 \} \]

\[
x_j^{(4)} = \{ x_1; x_1 + \epsilon_2 \} \]

Especially,

\[
D_{-2,0}(G, \vec{a}, \mu_i) = (\sum_i x_i - \sum_j y_j), \quad \text{(A.31)}
\]

and

\[
D_{-2,1}(G, \vec{a}, \mu_i) = 2(\sum_i x_i - \sum_j y_j)^2 + (\sum_i x_i^2 - \sum_j y_j^2). \quad \text{(A.32)}
\]

Thus we obtain (3.4) and (3.5).

### A.4 Action on \(|T_2|\)

The proofs are given below. By definition,

\[
D_{-1,n}|T_2| = D_{-1,n} V_{12} q_2 D_{12} [G, \vec{a}, \mu] = D_{-1,n} V_{12} q_2 D_{12} [G, \vec{a}, \mu] + q_1^D V_{12} q_2 D_{12} D_{-1,n} [G, \vec{a}, \mu] \quad \text{(A.33)}
\]

Since

\[
\sum_{\vec{a}, \vec{b}, \vec{\nu}} \prod_{Y, X \in A(Y)} (\phi_x)^n \Lambda_x(\vec{Y}) \bar{Z}_{\text{bit}}(\vec{a}, \vec{Y}; \vec{b}, \vec{X}^{(n)}|\vec{Y}, \vec{\nu}) \bar{Z}_{\text{bit}}(\vec{a}, \vec{Y}; \vec{b}, \vec{X}^{(n)}|\vec{a}, \vec{Y})(\vec{b} + \vec{\nu}, \vec{X}^{(n)}), \quad \text{(A.34)}
\]

\[\text{For } n = 0 \text{ there is an extra factor of } 1/2 \text{ due to (A.20).} \]
with
\[
\Lambda_x(\bar{Y}) \frac{\tilde{Z}_{\text{bif}}(\bar{a}, \bar{y}; \bar{b}, \bar{x} | \nu)}{Z_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} | \nu)} = \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \frac{\prod_{y \in R(\bar{Y})} (\phi_x - \phi_y - \epsilon_+)}{\prod_{y \in A(\bar{X})} (\phi_x - \phi_y - \nu + \epsilon_+)} .
\] (A.35)

And
\[
- V(\bar{a}, \bar{b}) D_{-1, n} = \sum_{\bar{Y}, \bar{X}} \sum_{x \in A(\bar{X})} (\phi_x + \nu)^n \Lambda_x(\bar{X}) \frac{\tilde{Z}_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} - x | \nu)}{Z_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} | \nu)} \tilde{Z}_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} | \nu)|\bar{a}, \bar{Y}, \nu, \bar{b}| + \nu, \bar{X}| ,
\] (A.36)

with
\[
\Lambda_x(\bar{X}) \frac{\tilde{Z}_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} - x | \nu)}{Z_{\text{bif}}(\bar{a}, \bar{Y}; \bar{b}, \bar{X} | \nu)} = \frac{1}{\sqrt{-\epsilon_1 \epsilon_2}} \frac{\prod_{y \in A(\bar{X})} (\phi_x - \phi_y + \nu)}{\prod_{y \in R(\bar{Y})} (\phi_x - \phi_y + \nu - \epsilon_+)} .
\] (A.37)

So we find
\[
[D_{-1, n}, V_{12}] = \sum_{x \in A(\bar{X})} (\phi_x + \nu)^n \prod_{y \in R(\bar{Y})} (\phi_x - \phi_y - \epsilon_+) \prod_{y \in A(\bar{X})} (\phi_x - \phi_y - \nu + \epsilon_+) \\
+ \sum_{x \in R(\bar{X})} (\phi_x + \nu)^n \prod_{y \in A(\bar{X})} (\phi_x - \phi_y + \epsilon_+) \prod_{y \in R(\bar{Y})} (\phi_x - \phi_y + \nu - \epsilon_+) \\
= \sum_{i=1}^{N} (x_i)^n \prod_{j=1}^{M} x_i - y_j \prod_{j \neq i}^{N} (x_i - x_j) ,
\] (A.38)

with \( \tilde{x}_I = \{ \phi_y, (y \in A(Y^{(i)}); \phi_x + \nu, (w \in R(X^{(i)})) \} \) and \( \tilde{y}_J = \{ \phi_w + \epsilon_+, (w \in R(Y^{(i)})); \phi_y - \epsilon_+ + \nu, (y \in A(X^{(i)})) \} \).

Explicitly,
\[
(\sum_I \tilde{x}_I - \sum_J \tilde{y}_J) = \sum_{k} (a_k - b_k + \epsilon_+ - \nu) ,
\] (A.39)

and
\[
\frac{1}{2} \left( \sum_I \tilde{x}_I^2 - \sum_J \tilde{y}_J^2 \right) = -\epsilon_1 \epsilon_2 (|\bar{Y}| - |\bar{X}|) + \frac{1}{2} \sum_p (a_p)^2 - \frac{1}{2} \sum_p (b_p - \epsilon_+ + \nu)^2 ,
\] (A.40)

thus
\[
[D_{-1, 1}, V_{12}] = \sum_{n=0}^{2} f_{2-n}(-y) b_n(x) \\
= -\epsilon_1 \epsilon_2 (|\bar{Y}| - |\bar{X}|) + \frac{1}{2} \sum_p \left[ a_p^2 - (b_p - \epsilon_+ + \nu)^2 \right] + \frac{1}{2} \left( \sum_p (a_p - b_p + \epsilon_+ - \nu)^2 \right) .
\] (A.41)

So we find \( [\bar{X}, T_2] = 0 \). Similarly,
\[
D_{-2, n} T_2 = q_1 D_{-2, n} V_{12} q_2 G, \bar{a}, \mu_1 + q_1 D_{-2, n} V_{12} q_2 D_{-2, n} G, \bar{a}, \mu_1 .
\] (A.42)

Here
\[
[D_{-2, n}, V_{12}] = \frac{1}{2(\epsilon_1 - \epsilon_2)} \left( \epsilon_1 \sum_{i=1}^{N} (x_i)^n \prod_{j \neq i}^{M} (x_i - y_j)^n \prod_{j \neq i}^{N} y_j^2 - \tilde{x}_j^2 - \tilde{y}_j^2 \right) - \epsilon_2 \sum_{i=3}^{4} \sum_{l=1}^{N} (x_l)^n \prod_{j \neq l}^{M} (x_l - y_j)^n \prod_{j \neq l}^{N} y_j^2 \right) ,
\] (A.43)
with \( \tilde{x}^{(1)}_I = \{ \tilde{x}; \tilde{x}_I - \epsilon_1 \} \) and \( \tilde{y}^{(1)}_I = \{ \tilde{y}; \tilde{y}_I - \epsilon_1 \} \), \( \tilde{x}^{(2)}_I = \{ \tilde{x}; \tilde{x}_I + \epsilon_1 \} \) and \( \tilde{y}^{(2)}_I = \{ \tilde{y}; \tilde{y}_I + \epsilon_1 \} \), \( \tilde{x}^{(3)}_I = \{ \tilde{x}; \tilde{x}_I - \epsilon_2 \} \) and \( \tilde{y}^{(3)}_I = \{ \tilde{y}; \tilde{y}_I - \epsilon_2 \} \), \( \tilde{x}^{(4)}_I = \{ \tilde{x}; \tilde{x}_I + \epsilon_2 \} \) and \( \tilde{y}^{(4)}_I = \{ \tilde{y}; \tilde{y}_I + \epsilon_2 \} \).

Then

\[
[D_{-2,1}, V_{12}] = 2 \left( \sum_I \tilde{x}_I - \sum_J \tilde{y}_J \right)^2 + \left( \sum_I \tilde{x}_I^2 - \sum_J \tilde{y}_J^2 \right)
- 2\epsilon_1 \epsilon_2 (|\tilde{Y}| - |\tilde{X}|) + \frac{1}{N} \sum_p \left[ a_p^2 - (b_p - \epsilon_+ + \nu)^2 \right] + 2 \left( \sum_p (a_p - b_p + \epsilon_+ - \nu) \right)^2.
\]

(A.44)

This leads to (3.9).

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