ON THE LOCAL GEOMETRY OF
MODULI SPACES OF VECTOR BUNDLES

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In [R1] the author gave a canonical construction of the universal deformation of a compact complex manifold. Here we shall give an analogous construction for the universal deformation of a simple vector bundle (over a fixed base); this construction differs somewhat from that of [R1], featuring a systematic use of the Jacobi complexes of a module. The resulting construction of universal or Poincaré bundle is applicable in other problems as well, such as deformations of manifold, where it yields a simplification of the method employed in [R1] to construct the structure sheaf of the total space of the universal deformation.

As an application of our method, we shall prove the closedness of trace forms—these are $H^2(\mathcal{O}_X)$–valued 2–forms on the moduli space of bundles, $E$, on $X$ induced by the trace or Killing form on $\mathcal{E}\text{nd}(E)$—

$$\tau : H^1(\mathcal{E}\text{nd}(E)) \otimes H^1(\mathcal{E}\text{nd}(E)) \to H^2(\mathcal{E}\text{nd}(E)) \to H^2(\mathcal{O}_X).$$

For $X$ a surface with trivial canonical bundle, $\tau$ is in fact, by Serre duality, a non–degenerate scalar valued 2–form, hence, being closed, yields a symplectic structure on the moduli space, a result first proven by Mukai [M]; see also Kobayashi’s book [K] for an analytic approach.

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1. Calculus.

1.1. Enveloping Algebras and Differential Operators.

Let $\mathfrak{g}$ be a Lie algebra, say over a field $\mathbb{C}$ of characteristic 0, and $E$ a faithful $\mathfrak{g}$–module. As is well known, the universal enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ may be identified as vector space, though not as algebra, with $\bigoplus_{i=0}^{\infty} S^i(\mathfrak{g})$, and this identification takes $\mathcal{U}^m \subset \mathcal{U}$, the elements of order $\leq m$, to $\bigoplus_{i=0}^{m} S^i(\mathfrak{g})$. The algebra structure on $\mathcal{U}(\mathfrak{g})$ may be described as follows. Let $S^* = S^*(E) = \bigoplus_{i=0}^{\infty} S^i(E)$ be the symmetric algebra on $E$ and note that elements of $\mathfrak{g}$ acting on $E$ extend to graded (i.e. homogeneous of degree zero) derivations on $S^*$; in fact it is easy to see that the Lie algebra of all graded derivations coincides with $\mathfrak{gl}(E) = \text{End}(E)$. Therefore if we denote by $\mathcal{D}$ (resp. $\mathcal{D}^m$) the algebra (resp. $\mathfrak{g}$–module) of all (resp. all degree
\[ \leq m \) graded \( \mathbb{C} \)-linear differential operators on \( S^* \), we obtain an injective filtration preserving algebra and \( g \)-module homomorphism

\[ \rho : U(g) \to D. \]

Moreover it is easy to see that for \( g = \mathfrak{gl}(E) \), \( \rho \) is in fact an isomorphism.

We want now to identify \( \text{im}(\rho) \) in case \( g = \mathfrak{sl}(E) \), the traceless endomorphisms (assuming \( E \) is finite–dimensional). To this end, note that a graded endomorphism \( \varphi \) of \( S^* \) admits a graded trace \( \text{tr}(\varphi) \), which itself is a graded endomorphism of \( S^* \) given by

\[ \text{tr}^i(\varphi) = \text{image of } (\varphi^{i+1} \in S^{i+1}(E) \otimes S^{i+1}(E^*) \subset S^i E \otimes S^i(S(E^*) \otimes E \otimes E^*) \]

in \( S^i(E) \otimes S^i(E^*) \) under \( \text{id} \otimes \text{id} \otimes \text{tr} \).

In other words, \( \text{tr} \) is the natural extension of \( \text{tr} : E \otimes E^* \to \mathbb{C} \) as a derivation of degree \(-1\) on the algebra of all graded endomorphisms of \( S^* \). Clearly \( \text{tr}(\varphi) = 0 \) iff for all \( e_1, \ldots, e_i \in E \), \( e_1^*, \ldots, e_i^* \in E^* \), acting on \( S^* \) in the standard manner (interior multiplication), the trace (in the usual sense) of \( e_1^* \cdots e_i^* \circ \varphi^{i+1} \circ e_1 \cdots e_i \) is zero. From this it is easy to see that \( \rho(U(\mathfrak{sl}(E))) \) coincides with the sub–algebra \( SD \subset D \) of traceless operators.

The above considerations extend immediately to the case where \( E \) is a vector bundle over a \( \mathbb{C} \)-ringed space \( (X, \mathcal{O}_X) \) and \( g \) is a Lie sub–algebra of \( \mathfrak{gl}(E) = \mathcal{E}_{\text{nd}}\mathcal{O}_X(E) \).

1.2. Functors on \( S \)-Modules.

In [R1] we showed how an Artin local \( \mathbb{C} \)-algebra may be reconstructed from a certain ‘order–symbolic’ or OS structure on the space of \( \mathbb{C} \)-valued differential operators on \( S \). Our purpose here is to note an analogue of this for \( S \)-modules.

Now fix a local \( \mathbb{C} \)-algebra \( S \) with maximal ideal \( m \) and residue field \( S/m = \mathbb{C} \) and put

\[ B_0^i = D_i(S, \mathbb{C}) = \text{Hom}(S_i, \mathbb{C}) = S^*_i, \quad S_i = S/m^{i+1}, \]

and

\[ B_i = D_+(S, \mathbb{C}) = (m/m^{i+1})^*. \]

For an \( S \)-module \( E \), put

\[ B^i(E) = B_0^i \otimes_S E, \]

where \( B_0^i \) is viewed as \( S \)-bi–module and \( E \) as (symmetric) \( S \)-bi–module. At least when \( E \otimes S_i \) is \( S_i \)-free, \( B^i(E) \) may be identified with the right \( S \)-module of differential operators \( D^i(E^\vee, \mathbb{C}) \), \( E^\vee = \text{Hom}_S(E, S) \).

We have a symbol map

\[ \sigma^i : B_0^i \to B_i \otimes_{\mathbb{C}} B_0^{i-1} \]

which factors through \( F_i(B_i \otimes_{\mathbb{C}} B_i^{i-1}) \), where \( F_i \) is the filtration induced by the order filtration on \( B_i \), and this gives rise to a symbol map

\[ \sigma_i : B_i(E) \to B_i \otimes_{\mathbb{C}} B_i^{i-1}(E). \]
which again factors through $F_i(B^i \otimes \mathbb{C} B^{i-1}(E))$. These $\sigma^E_i$, $i \leq m$, together with the obvious maps $B^0(E) \to B^1(E) \to \ldots \to B^m(E)$ are referred to as a ‘modular order–symbolic’ (MOS) structure on $B^m(E)$. Note that $B^m(E)$ itself is a right $S_m$–module, called the $m$–th transpose of $E$.

“Dually,” suppose we are given an MOS structure $G^i$, $G^i$ a right $S_i$–module. We then define an $S_m$–module $C^m(G^i)$, called the module of quasi–scalar homomorphisms $B^m_0 \to C^m(G^i)$, inductively as follows.

$$C^0(G) = G^0, C^i(G) = \text{all right } S_i \text{–linear maps } \varphi^i : B^i_0 \to G^i$$

such that for some $\varphi^i_0 : B^i_0 \to G^i$ the following diagrams commute.

$$\begin{array}{c}
B^i_0 \\
\downarrow \varphi^i \\
B^i \\
\downarrow \sigma^i \\
B^i_0 \\
\end{array} \quad \quad \begin{array}{c}
B^i_0 \\
\downarrow \varphi^i \\
B^i \\
\downarrow \sigma^i \\
\end{array}$$

Note that we have natural maps

$$E \to C^m(B^m(E)) \quad \quad B^m(C^m(G^i)) \to G^i.$$ 

At least when $E$ is $S_m$–free (resp. $G^i$ is ‘co–free’, i.e. a sum of copies of $B^m_0$ with the standard MOS structure), these are isomorphisms.

1.3. Exterior Derivative.

Our purpose here is to give a convenient interpretation of the Cartan formula for exterior derivative. Let $M$ be a manifold with tangent sheaf $T$, and consider the sheaf $B^1(\bigwedge^i T)$, which may be identified as the sheaf of $\mathcal{O}_M$–valued first–order differential operators on the dual $\bigwedge^i T^\vee = \Omega^i$, and fits in an exact sequence

$$0 \to \bigwedge^i T \to B^1(\bigwedge^i T) \to T \otimes \bigwedge^i T \to 0.$$ 

Over the sub–sheaf $\bigwedge^{i+1} T \subset T \otimes \bigwedge^i T$, this sequence admits a canonical splitting $\varphi_i : \bigwedge^{i+1} T \to B^1(\bigwedge^i T)$ given by

$$(*) \quad \varphi_i(v_0 \wedge \ldots \wedge v_i)(\omega) = \sum (-1)^j v_j \omega(v_0, \ldots, \hat{v}_j, \ldots, v_i) + \sum_{j \neq k} (-1)^{j+k} \omega([v_j, v_k], v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_k, \ldots, v_i)$$

for $\omega \in \Omega^i$. The Cartan formula says in particular that $\varphi_i$ is $\mathcal{O}_M$–linear. Now given an $i$–form, $\omega$, viewed as a map $\bigwedge^i T \to \mathcal{O}_M$, it extends to a map $B^1(\bigwedge^i T) \to B^1(\mathcal{O}_M) = \mathcal{O}_M \oplus T$. Projecting on the $\mathcal{O}_M$ factor, we get a map

$$\tilde{\omega} : B^1(\bigwedge^i T) \to \mathcal{O}_M.$$
Cartan’s formula then says that the exterior derivative \( d\omega \) coincides with \( \tilde{\omega} \circ \varphi_i \).
Note that \( \text{im}(\varphi_1) \) coincides with the kernel of the natural map
\[
{^t}d_0: B^1(T) \to T^2 = D^2_+(\mathcal{O}, \mathcal{O})
\]
\[
{^t}d_0(a) = a \circ d_0
\]
where \( a: \Omega^1 \to \mathcal{O} \) is a first–order differential operator and \( d_0: \mathcal{O} \to \Omega^1 \) is exterior derivative. Indeed we have an exact sequence
\[
\bigwedge^2 T
\]
\[
\varphi_1 \downarrow
\]
\[
0 \to T \to B^1(T) \to T \otimes T \to 0
\]
\[
\| \quad {^t}d_0 \downarrow \quad \downarrow
\]
\[
0 \to T \to T^2 \to S^2 T \to 0
\]
Similarly, using the inclusion \( B^1(\bigwedge^i T) \subset B^1(T \otimes \bigwedge^{i-1} T) = B^1(T) \otimes_\mathcal{O} \bigwedge^{i-1} T \), we may identify \( \text{im}(\varphi_i) \) as the kernel of the natural map \( B^1(\bigwedge^i T) \overset{{^t}d_0 \otimes \text{id}}{\to} T^2 \otimes \bigwedge^{i-1} T = B^2(\bigwedge^{i-1} T)/\Omega^{i-1} =: B^2_+(\bigwedge^{i-1} T) \), which in fact coincides with the composite
\[
B^1(\bigwedge^i T) \overset{{^t}d_0}{\longrightarrow} B^2(\bigwedge^{i-1} T) \to B^2_+(\bigwedge^{i-1} T).
\]

2. Jacobi Complexes and Universal Deformations.

Fix a base space \( X \) which for convenience we assume to be a compact complex space (although the construction works more generally), and a simple vector bundle \( E \) on \( X \). Let \( \mathfrak{g} = \mathfrak{sl}(E) \), the Lie algebra of traceless endomorphisms of \( E \), so that \( H^0(\mathfrak{g}) = 0 \). As in [R1], we have Jacobi complexes \( J_m(\mathfrak{g}) \) which may be described as follows. Let \( X_{<m>} \) be the \( m \)-fold very symmetric product of \( X \), i.e. the space of subsets of \( X \) of cardinality \( \in [1, m] \), with the topology induced by the natural map \( X^m \to X_{<m>} \). For \( i \leq m \) let \( \lambda^i(\mathfrak{g}) \) be the image of the exterior alternating product of \( \mathfrak{g} \), supported on \( X_{<i> \subset X_{<m>}} \). Then the bracket on \( \mathfrak{g} \) gives rise to a map \( \lambda^i(\mathfrak{g}) \to \lambda^{i-1}(\mathfrak{g}) \), and these fit together to form the complex \( J_m(\mathfrak{g}) \)
\[
\lambda^m(\mathfrak{g}) \to \lambda^{m-1}(\mathfrak{g}) \to \ldots \to \lambda^1(\mathfrak{g}) = \mathfrak{g},
\]
in which we put \( \lambda^i(\mathfrak{g}) \) in degree \(-i\). Similarly, the action of \( \mathfrak{g} \) on \( E \) gives rise to a complex \( J_m(\mathfrak{g}, E) \) on \( X_{<m>} \times X \)
\[
\lambda^m(\mathfrak{g}) \boxtimes E \to \ldots \to \mathfrak{g} \boxtimes E \to E,
\]
in degrees \( \in [-m, 0] \), where the last term \( E \) is supported on the diagonal in \( X_{<1>} \times X = X \times X \). The natural maps
\[
J_i(\mathfrak{g}) \to J_m(\mathfrak{g}), \quad i \leq m,
\]
\[
J_i(\mathfrak{g}) \to E \otimes (J_{m-i}(\mathfrak{g}) \boxtimes J_{m-i}(\mathfrak{g})),
\]
whose associated coboundary maps
\[ G^m = \mathbb{R}^0 p_2^*(J_m(g, E)). \]
This forms a sheaf of \( \mathcal{O}_X \otimes R_m \)-modules, and the natural map
\[ J_m(g, E) \to F_m(J_m(g) \otimes J_{m-1}(g, E)) \]
endows \( G^m \) with a MOS structure compatible with the OS structure on \( V^m \), whence as in § 1.2 a sheaf
\[ \mathcal{P}_m(E) := C^m(G^m). \]
As \( G^m \) is locally isomorphic to \( (V_m \oplus \mathbb{C}) \otimes \mathcal{O}_X \), clearly \( \mathcal{P}_m(E) \) is a locally free \( R_m \otimes \mathcal{O}_X \) module, and it turns out to be the Poincaré bundle for \( E \). (The last construction may be applied in other situations, such as deformations of manifolds, where it yields a simplification of the construction in [R1] of the structure sheaf of the universal deformation of a manifold \( X \).)

Notwithstanding the simple construction of \( \mathcal{P}_m(E) \) and \( R_m \), the proof of their universality involves the somewhat complicated construction of Kodaira–Spencer bi–complexes associated to a given deformation of \( E \). Given such a deformation, i.e. a locally free sheaf \( \mathcal{E} \) on \( X \times \text{Spec}(S) \), \( S \) Artin local, with \( \mathcal{E} \otimes k(0) \to E \), we construct bi–complexes \( W_m ; \), and \( K_m ; \), which fit in exact (up to quasi–isomorphism) sequences (in which \( B^m = D^m_+(S, \mathbb{C}) \) —
\[
\begin{align*}
(2.1) & \quad 0 \to J_m(g) \to W_m \to B^m[1 - m, m] \to 0 & \text{on } X^{<m>}, \\
(2.2) & \quad 0 \to J_m(g, E) \to K_m \to B^m_+(\mathcal{E})[1 - m, m] \to 0 & \text{on } X^{<m>} \times X,
\end{align*}
\]
whose associated coboundary maps
\[
\begin{align*}
\alpha_m : B^m & \to \mathbb{H}^0(J_m(g)) = V_m, \\
\beta_m : B^m_+(\mathcal{E}) & \to \mathbb{R}^0 p_2^*(J_m(g, E)) = G_m
\end{align*}
\]
respect the respective OS structures, hence give rise to a ring homomorphism \( t_{\alpha_m} : R_m \to S_m = S/\mathfrak{m}^m \) and to an \( S_m \)-linear map \( t_{\beta_m} : \mathcal{E}_m := \mathcal{E} \otimes S_m \to \mathcal{P}_m(E) \otimes R_m S_m \), which turns out to be an isomorphism.

We sketch the construction of these bi–complexes. Let \( \mathcal{S}(E) \) be the symmetric algebra on \( E \) and similarly for \( \mathcal{E} \) and \( \mathcal{E}_m \) (as \( \mathcal{O}_X \otimes S \)-modules). A graded, \( \mathcal{O}_X \)-linear map \( \varphi : \mathcal{S}(\mathcal{E}_i) \to \mathcal{S}(\mathcal{E}_j) \) is said to be a differential operator of degree \( \leq k \) if for all \( e^* \in E^* \subset \mathcal{E}_i^* \) and \( m_{e^*} \), the ideal in \( \mathcal{S}(\mathcal{E}_i) \) generated by \( \ker(e^*) \subset \mathcal{E}_i \) and \( m \), we have \( \varphi(m_{e^*}) \subset m_{e^*}^{k} \); similarly for \( \psi : S_i \to S_j \), \( \varphi \) is said to be traceless if it can be locally written as \( \sum \lambda_r \psi_r \), with \( \lambda_r : \mathcal{S}(E) \to \mathcal{S}(E) \) traceless, \( \psi_r : S_i \to S_j \). Define
\[
\begin{align*}
\mathcal{A}^{i,j} = \mathcal{A}^{i,j}(\mathcal{E}) &= \{(\varphi, \psi) | \varphi : \mathcal{S}(\mathcal{E}_i) \to \mathcal{S}(\mathcal{E}_j), \psi : S_i \to S_j, \\
n \mathcal{O}_X \text{–linear graded traceless (resp. } C \text{–linear) diff. op. of order } \leq i - j & \text{ such that } |\varphi| = |\psi| \text{ and } \psi(1) = 0 \},
\end{align*}
\]
Also define

\[ A_{0}^{i,j} = A^{i,j} \oplus \mathbb{C}. \]

These are \((S_{j}, S_{i})\) bi–modules and admit filtrations \(F_{s}\) and \(G_{s}\) by total order (resp. horizontal order); in particular \(G_{0}(A^{i,j})\) consists of the \(S_{j}\)–linear or ‘relative’ operators on \(E_{j}\) and \(G_{0}(A^{i,j})/mG_{0}A^{i,j} = U_{i-j}(g)\). Put \(\tilde{A}^{i,j} = A^{i,j}/mA^{i,j}_{0}\). As in [R1] define

\[
W_{m}^{i,-j} = \lambda^{j-i-1}(g) \boxtimes F_{m}(\sigma^{j+1}(\tilde{A}^{m,j-1})), \quad (i, j) \neq (m - 2, m)
\]

with \(\sigma^{\cdot} = \) exterior symmetric product (over \(\mathbb{C}\)), and

\[
W_{m-2,-m} = \ker(A^{m,m-1} \boxtimes F_{m}(\sigma^{m-1}(\tilde{A}^{m,m-1})) \to \sigma^{m}(B^{m,m-1}) \to B^{m}).
\]

The let \(W_{m,0}^{i,-j}\) be the analogous complex with \(A^{i,j}\) replaced by \(A_{0}^{i,j}\) and put

\[
K_{m,0}^{i,-j} = W_{m,0}^{i,-j} \boxtimes S_{m}E_{m}, \quad (i, j) \neq (0, 0)
\]

\[ K_{m,0}^{0,0} = E. \]

As in [R1], \(W_{m}^{i,-j}\) fits in an exact sequence as in (2.1), and using the fact that \(E_{m}\) is \(S_{m}\)–flat, \(K_{m}^{i,-j}\) fits in (2.2).

For instance, (2.2) has the form

\[
\begin{array}{cccccc}
0 & \longrightarrow & g \boxtimes E & \longrightarrow & A_{0}^{1,0} \boxtimes S_{1}E & \longrightarrow & B_{1}(E) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
E & & E & & J_{1}(g, E) & & K_{1}^{i,-j}
\end{array}
\]

3. Closed 2–forms.

Let \(M\) be a manifold parameterizing a deformation of a vector bundle \(E\) on \(X\) with \(H^{0}(\mathfrak{sl}(E)) = 0\). For a point \(0 \in M\) we have the Kodaira–Spencer map

\[ \alpha_{1} : T_{0}M \to H^{1}(g), \quad g = \mathfrak{sl}(E) \]

and its higher order analogues. For simplicity we assume \(E\) is unobstructed, which means that the spectral sequence

\[ E_{1}^{p,q} = \mathbb{H}^{q}(J_{m}^{p}(g)) \Rightarrow \mathbb{H}^{q}(J_{m}(g)) \]

degenerates at \(E_{1}\) for \(p + q = 0\) for all \(m\); in particular the map \(S^{2}H^{1}(g) \to H^{2}(g)\) induced by bracket vanishes. Now we have a trace or Killing form

\[ \text{tr} : g \boxtimes g \to \mathcal{O}_{X} \]

\[ \text{tr}(A \cdot B) = \text{tr}(AB) \]
As is well known, this is a non-degenerate symmetric bilinear form and induces an alternating form
\[ \tau: \bigwedge^2 H^1(g) \to H^2(O_X) \]

hence a map
\[ \tau_{M,0}: \bigwedge^2 T_{M,0} \to H^2(O_X) \]

which we view as yielding a 2-form \( \tau_M \) on \( M \) with values in the fixed vector space \( H^2(O_X) \). When \( n = \dim X = 2 \) and \( K_X = O_X \), the isomorphism \( g \to g^* \) induced by \( \text{tr} \) yields an isomorphism \( H^1(g) \to H^1(g^*) = H^1(g)^* \), hence the 2-form \( \tau \) is non-degenerate on the moduli space. In general, any \( \eta \in H^{n,n-2}(X) \) yields via cup-product with \( \eta \), a (scalar-valued) 2-form on \( M \).

**Theorem 3.1.** \( \tau_M \) is a closed 2-form on \( M \).

**proof.** It suffices to prove this for the universal second-order deformation. \( \text{tr} \) yields a map
\[ \text{tr}: \sigma^2 g \to O_X \]
which on cohomology gives rise to
\[ \tau: H^2(\sigma^2 g) = \bigwedge^2 H^1(g) \to H^2(O_X). \]

Now we may identify \( B^1(\bigwedge^2 T_M)_0 \) as \( \mathbb{H}^0(J_1(g, \sigma^2 g)) \), i.e. as \( \mathbb{H}^0 \) of the following complex on \( X^{<2>} \times X \):
\[ g \boxtimes \sigma^2 g \to \sigma^2 g \]
\[ A \times (B, C) \mapsto ([A, B], C) + (C, [A, B]) + (B, [A, C]) + ([A, C], B) \]

Now applying \( \text{tr} \) we get
\[ 2\text{tr}(ABC - BAC + BAC - BCA) = 2\text{tr}(A(BC) - (BC)A) = 0, \]
hence \( \text{tr} \) is a map of \( g \)-modules, where \( g \) acts trivially on \( O_X \), hence \( \text{tr} \) extends to a map of complexes
\[ J_1(g, \sigma^2 g) \to J_1(g, O). \]

Now the differential \( g \boxtimes \sigma^2 g \to \sigma^2 g \) clearly vanishes on \( \sigma^3 g \), so that \( \sigma^3 g[2] \) forms a sub-complex of \( J_1(g, \sigma^2 g) \) and we have a diagram
\[ \begin{array}{ccc}
\sigma^3(g) & \to & \sigma^2(g) \\
\downarrow & & \downarrow \\
g \boxtimes \sigma^2 g & \longrightarrow & \sigma^2(g) \\
\downarrow & & \downarrow \\
\lambda^2(g) \boxtimes g & \longrightarrow & g \boxtimes g
\end{array} \]

where the left vertical arrows compose to zero and \( \mathbb{H}^0 \) of the bottom complex is \( T^2 \otimes T^1 \); in fact the bottom complex splits as a direct sum of \( J_1(g)/J_2(g) \) and a
complex $C^r : \sigma^{2,1}(g) \to \sigma^{2}(g)$, $\sigma^{2,1}$ being the mixed tensor power and $J_1^1(g, \sigma^{2}(g)) = \sigma^3(g)[2] \oplus C'$. As discussed in §1.3, $d\tau : \bigwedge^3 H^1(g) \to H^2(O_X)$ coincides with the composite
\[
\bigwedge^3 H^1(g) = H^3(\sigma^3 g) \to \mathbb{H}^2(J_1(g, \sigma^2 g) \to \mathbb{H}^2(g, \mathcal{O}) = H^2(\mathcal{O}) \oplus H^1(g) \otimes H^2(\mathcal{O}) \to H^2(\mathcal{O})
\]
But in view of the diagram
\[
\begin{array}{ccc}
\sigma^3 g[2] & \longrightarrow & J_1(g, \sigma^2 g) \\
\downarrow & & \downarrow \\
g \boxtimes \mathcal{O}[2] & \longrightarrow & J_1(g, \mathcal{O}) \longrightarrow \mathcal{O}[1]
\end{array}
\]
and the fact that the bottom arrows compose to zero, $d\tau$ clearly vanishes. □

Remark. If $X$ admits a symplectic form $\omega$, and is moreover Kähler, it is shown by Mukai [M] for $n = 2$, and Kobayashi [K] in general that the 2–form $\tau \wedge \omega^{n/2-1} \wedge \bar{\omega}^{n/2}$ is non–degenerate (and closed), yielding a symplectic structure on the moduli space.

References.

[K] Kobayashi, S.: ‘Differential Geometry of Complex Vector Bundles’. Iwanami and Princeton University Press, 1987.

[M] Mukai, S.: ‘Symplectic Structure on the Moduli Space of Sheaves on an Abelian or K3 Surface’. Invent. Math. 77 (1984), 101-116.

[R1] Ran, Z.: ‘Canonical Infinitesimal Deformations’. preprint.

[R2] Ran, Z.: ‘Infinitesimal Deformations of Vector Bundles and Their Cohomology Groups’. (in preparation).