EFFICIENCY OF EQUILIBRIA IN RANDOM BINARY GAMES

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ABSTRACT. We consider normal-form games with \( n \) players and two strategies for each player where the payoffs are Bernoulli random variables. We define the average social utility associated to a strategy profile as the sum of the payoffs of all players divided by \( n \). We assume that payoff vectors corresponding to different profiles are i.i.d., and the payoffs within the same profile are conditionally independent given some underlying random parameter. Under these conditions we examine the asymptotic behavior of the average social utilities that correspond to the optimum, to the best and to the worst pure Nash equilibrium. We perform a detailed analysis of some particular cases showing that these random quantities converge, as \( n \to \infty \), to some function of the models’ parameters. Moreover, we show that these functions exhibit some interesting phase-transition phenomena.

1. INTRODUCTION

The concept of Nash equilibrium (NE) is central in game theory. Nash (1950, 1951) proved that every finite game admits mixed Nash equilibria (MNE). In general, pure Nash equilibria (PNE) may fail to exist. Given that the concept of pure Nash equilibrium is epistemically more clearly understood than the one of MNE, it is important to understand how rare it is to have games without PNE. One way to address the problem is to consider games in normal form whose payoffs are random. In a random game the number of PNE is also a random variable, whose distribution is interesting to study.

It is known that this distribution depends on the assumptions made on the distribution of the random payoffs. The simplest case that has been considered in the literature deals with i.i.d. payoffs having a continuous distribution function. This implies that ties happen with probability zero. Even in this simple case, although it is easy to compute the expected number of PNE, the characterization of their exact distribution is non-trivial. Asymptotic results exist as either the number of players or the number of strategies for each player diverge. In both cases the number of PNE converges to a Poisson distribution with parameter 1.

Generalizations of the simple case can be achieved either by removing the assumptions that all payoffs are independent or by allowing for discontinuities in their distribution functions, or both. In both cases the number of PNE diverges and some central limit theorem (CLT) holds.

To the best of our knowledge, the literature on this topic has focused on the distribution of the number of PNE but not on their social utility (SU), i.e., the sum of the payoffs of each player. The issue of efficiency of equilibria and its measure has received an attention for more than a century and, at the end of the last millennium, has led to the definition of the price of anarchy (PoA) as a pessimistic measure of inefficiency (Koutsoupias and Papadimitriou, 1999, Papadimitriou, 2001a), followed by the price of stability (PoS) as its optimistic counterpart (Schulz and Stier Moses, 2003, Anshelevich et al., 2008). The PoA is the ratio of the optimum SU over the SU of the worst equilibrium. The PoS is the ratio of the optimum SU over the SU of the best equilibrium. It is interesting to study how these three quantities behave in a random game.

1.1. Our contribution. We consider a model with \( n \) players and two strategies for each player. Payoffs are assumed to be random. To be more precise the payoff vectors corresponding to each
strategy profile are assumed to be i.i.d. and payoffs within the same strategy profile \( s \) to be conditionally i.i.d. Bernoulli random variables, given a parameter \( \Phi(s) \) distributed according to the probability law \( \pi \) on \([0, 1]\). A model with a similar dependence structure was considered in \text{Rinott and Scarsini} (2000), but there the payoffs have a Gaussian distribution.

We will study the asymptotic behavior of the average social utilities (ASUs) in this game as \( n \to \infty \). In particular, we focus our analysis on the optimal ASU, on the ASUs of the best, the worst, and the typical PNE.

As a preliminary step, we will consider the asymptotic behavior of the random number of PNE.

We consider three relevant cases for the measure \( \pi \). First we look at the case where the support of \( \pi \) is the whole interval \([0, 1]\) and we show that the asymptotic behavior of the number of PNE does not depend on \( \pi \). Moreover we show that in this case the asymptotic behavior of the ASU of the optimum, and the best equilibrium coincide and have maximal ASU, i.e., equal to 1. On the other hand, we show that efficiency of the worst PNE depends on \( \pi \) only through its mean.

The same analysis is performed for the case in which \( \pi \) is the Dirac mass at \( p \in (0, 1) \), which corresponds to i.i.d. payoffs.

Finally we deal with a model where the dependence within the profile depends on a single parameter \( q \) and perform the same asymptotic analysis as a function of \( p \) and \( q \).

For each of these models we analyze the behavior of the best and worst equilibria as a function of the relevant parameters, showing some interesting irregularities.

The techniques we use in this paper are standard in the probabilistic literature, and amount mostly to first and second moment analysis, large deviations and calculus. Nonetheless, a refined analysis of a perturbation of the large deviation rate of binomial random variables is required to provide precise asymptotic results on the phase-transition mentioned in the abstract.

1.2. Related literature. The distribution of the number of PNE in games with random payoffs has been studied for a number of years. Many papers assume the random payoffs to be i.i.d. from a continuous distribution. Under this hypothesis, several papers studied the asymptotic behavior of random games, as the number of strategies grows. For instance, \text{Goldman} (1957) showed that in zero-sum two-person games the probability of having a PNE goes to zero. He also briefly dealt with the case of payoffs with a Bernoulli distribution. \text{Goldberg et al.} (1968) studied general two-person games and showed that the probability of having at least one PNE converges to \( 1 - e^{-1} \). \text{Dresher} (1970) generalized this result to the case of an arbitrary finite number of players. Other papers have looked at the asymptotic distribution of the number of PNE, again when the number of strategies diverges. \text{Powers} (1990) showed that, when the number of strategies of at least two players goes to infinity, the distribution of the number of PNE converges to a \text{Poisson}(1). She then compared the case of continuous and discontinuous distributions. \text{Stanford} (1995) derived an exact formula for the distribution of PNE in random games and obtained the result in \text{Powers} (1990) as a corollary. \text{Stanford} (1996) dealt with the case of two-person symmetric games and obtained Poisson convergence for the number of both symmetric and asymmetric PNE.

In all the above models, the expected number of PNE is in fact 1. Under different hypotheses, this expected number diverges. For instance, \text{Stanford} (1997, 1999) showed that this is the case for games with vector payoffs and for games of common interest, respectively. \text{Rinott and Scarsini} (2000) weakened the hypothesis of i.i.d. payoffs; that is, they assumed that payoff vectors corresponding to different strategy profiles are i.i.d., but they allowed some dependence within the same payoff vector. In this setting, they proved asymptotic results when either the number of players or the number of strategies diverges. More precisely, if each payoff vector has a multinormal exchangeable distribution with correlation coefficient \( \rho \), then, if \( \rho \) is positive, the number of PNE diverges and a central limit theorem holds. \text{Rać} (2003) used Chen-Stein method to bound the distance between the distribution of the normalized number of PNE and a normal distribution. His result is very
general, since it does not assume continuity of the payoff distributions. Takahashi (2008) considered the distribution of the number of PNE in a random game with two players, conditionally on the game having nondecreasing best-response functions. This assumption greatly increases the expected number of PNE. Daskalakis et al. (2011) extended the framework of games with random payoffs to graphical games. Strategy profiles are vertices of a graph and players’ strategies are binary, like in our model. Moreover, their payoff depends only on their strategy and the strategies of their neighbors. The authors studied how the structure of the graph affects existence of PNE and they examined both deterministic and random graphs. Amiet et al. (2019) showed that in games with $n$ players and two actions for each player, the key quantity that determines the behavior of the number of PNE is the probability that two different payoffs assume the same value. They then studied the behavior of best-response dynamics in random games.

The issue of solution concepts in games with random payoffs has been explored by various authors in different directions. For instance, Cohen (1998) studied the probability that Nash equilibria (both pure and mixed) in a finite random game maximize the sum of the players’ payoffs. This bears some relation with what we do in this paper.

The fact that selfish behavior of agents produces inefficiencies goes back at least to Pigou (1920) and has been studied in various fashions in the economic literature. Measuring inefficiency of equilibria in games has attracted the interest of the algorithmic-game-theory community around the change of the millennium. Efficiency of equilibria is typically measured using either the PoA or the PoS. The PoA, i.e., the ratio of the optimum SU over the SU of the worst equilibrium, was introduced by Koutsoupias and Papadimitriou (1999) and given this name by Papadimitriou (2001b). The PoS, i.e., the ratio of the optimum SU over the SU of the best equilibrium, was introduced by Schulz and Stier Moses (2003) and given this name by Anshelevich et al. (2008). The reader is referred for instance to Roughgarden and Tardos (2007) for the basic concepts related to inefficiency of equilibria.

1.2.1. Connections with random Constraint Satisfaction Problems (CSP) and partially-oriented percolation. A CSP amounts to find an initialization for a set of $n$ variables taking value in a finite alphabet, say \( \{0, 1\} \), subject to a certain number of constraints. Example of problems in this class are classical in the computer science literature, e.g. SAT, graph coloring, independent set, etc. See, among others, Coja-Oghlan (2009), Mezard and Montanari (2009). Clearly, a binary game can be phrased as a CSP by considering pure Nash equilibria as the solution concept.

Random CSP have attracted a lot of attention in the physics community, where a number of deep conjectures on the behavior of the solution set have been developed, and only part of them have been recently rigorously proved by mathematicians (see, e.g., Achlioptas and Peres (2004), Abbe and Montanari (2014), Ding et al. (2015)). Given a law on the space of instances of a CSP, the first problem lies in the analysis of the size of the solution set, which is a random subset of \( \{0, 1\}^n \).

In Amiet et al. (2019) the authors noticed that a random binary game can be phrased as a marked partially oriented percolation on the hypercube. Strategy profiles represent vertices of the hypercube, each vertex has an array mark, which corresponds to the utilities of the players under the corresponding strategy profile. We place an oriented arc between two profiles if and only if they are neighbors in the hypercube and the mark in the differing coordinate is strictly larger in the arrival vertex. In this framework, the set of Nash equilibria coincide with the set of vertices having out-degree equal to zero, i.e., sinks.

In the physicists’ language, in Amiet et al. (2019) the authors computed the quenched free-energy of the model, see Eq. (2.20). In this work we consider a closely related CSP, in which we enlarge the set of constraints: a “solution” is a pure Nash equilibrium with a certain social utility. In the percolation representation of the problem, we aim at controlling the number of sinks with a given
sum of the entries in the mark. We will see how this additional constraint affects the free-energy and, in general, we refine the analysis of the solution set in Amiet et al. (2019) under the binary-payoff assumption. We stress that, by our analysis, in the case of binary random games a “vanilla” second-moment argument (see Achlioptas and Peres (2004)) is sufficient to control the quenched free-energy of the random CSP.

1.3. Organization of the paper. The rest of the paper is organized as follows. Section 2 is devoted to a second moment analysis under no assumption on the distribution \( F \). In section Sections 3–5 we present a precise analysis of the efficiency of equilibria for three different specific choices for \( F \). Finally, Section 6 is devoted to proofs.

2. General model

We consider a game with \( n \) players. We use the symbol \([n]\) for the set of players. Each player can choose one action in \( \{0, 1\} \). Then the set \( \Sigma \) of strategy profiles is the Cartesian product \( \times_{i \in [n]} \{0, 1\} \). As usual, the symbol \( \oplus \) will denote the binary XOR operator, defined as
\[
1 \oplus 0 = 0 \oplus 1 = 1, \quad 0 \oplus 0 = 1 \oplus 1 = 0.
\]
Therefore, \( \oplus \)-adding 1 changes one action into the other. Moreover, for every \( s = (s_1, \ldots, s_n) \in \Sigma \) we let the symbol \( s_{-i} \) denote the strategy profile in which the action of player \( i \) is unspecified, so that \( s = (s_{-i}, s_i) \), for all \( s \in \Sigma \) and \( i \in [n] \).

Let \( \text{NE} \) denote the set of Nash equilibria, i.e.,
\[
\text{NE} := \{ s \in \Sigma \mid u_i(s_{-i}, s_i \oplus 1) \geq u_i(s_{-i}, s_i), \quad \forall i \in [n] \}.
\]
For \( i \in [n] \), \( u_i: \Sigma \to \mathbb{R} \) denotes player \( i \)'s payoff function. We further assume that the payoffs are binary, in the sense that
\[
u_i(s) \in \{0, 1\}, \quad \forall i \in [n], \ \forall s \in \Sigma.
\]
We will refer to such games with the name of binary games.

We will be interested in the behavior of the following quantities:

\[
\text{(2.4) social utility (SU)} \quad \text{SU}(s) := \sum_{i \in [n]} u_i(s),
\]

\[
\text{(2.5) average social utility (ASU)} \quad \text{ASU}(s) := \frac{1}{n} \text{SU}(s).
\]

In particular, we will focus on the extremes of the social utility, in the sense that we consider the following objects

\[
\text{(2.6) social utility of the socially optimum (SO)} \quad \text{SO}(s) := \max_{s \in \Sigma} \text{SU}(s),
\]

\[
\text{(2.7) social utility of the best equilibrium (BEq)} \quad \text{Beq}(s) := \max_{s \in \text{NE}} \text{SU}(s),
\]

\[
\text{(2.8) social utility of the worst equilibrium (WEq)} \quad \text{Weq}(s) := \min_{s \in \text{NE}} \text{SU}(s).
\]

In what follows, we will consider binary games with random payoffs. More precisely, for every choice of \( n \in \mathbb{N} \) we will consider a probability measure on the set of binary games with \( n \) players as follows. Consider a random potential function, \( \Phi: \Sigma \to [0, 1] \), such that
\[
\text{(2.9) } (\Phi(s))_{s \in \Sigma}, \quad \text{i.i.d. } \Phi(s) \sim \pi,
\]
for some probability measure \( \pi \) with \( \text{supp}(\pi) \subseteq [0, 1] \). Notice that considering the common-interests game with payoffs
\[
\text{(2.10)} \quad u_i(s) = \Phi(s), \quad \forall i \in [n], \ \forall s \in \Sigma
\]
we have a potential game. In our model, instead, we consider a discrete perturbation of the potential structure, in the sense that we use the potential $\Phi$ just to model dependences between payoffs of different players under the same profile. More precisely, given the value of the potential at a given profile $s$, i.e. $\Phi(s)$, the utility of the players are $n$ i.i.d. Bernoulli random variables of parameter $\Phi(s)$. Moreover, we assume independence of payoffs vectors under different profiles.

We will call $p = E[\Phi(s)]$. Notice that, marginally,

\[(2.11) \quad P(u_i(s) = 1) = p.\]

Eq. (2.11) implies that the marginal distribution of the payoffs does not depend on the specific choice of $\pi$, but only on its expectation.

In the following section we will present precise results concerning three specific but significant examples:

- **Fully supported potential**: $\pi$ is fully supported in the whole interval $[0, 1]$.
- **Dirac potential**: $\pi$ is the Dirac mass at $p$. Notice that in this case the sequence $(u_i(s))_{i \in [n], s \in \Sigma}$ is i.i.d.. For this reason we will refer to this model as the independent case.
- **Dichotomous potential**: For some $q \in [0, 1]$, $\pi$ is the convex combination of two Dirac masses, i.e., for every $I \subseteq [0, 1]$,

\[(2.12) \quad \pi(I) = (1 - p)\delta_{(1-q)p}(I) + p\delta_{q+(1-q)p}(I).\]

Notice that if $q = 0$ we are back to the independent case, while if $q = 1$ we have a.s. a common-interests game.

We stress that an interpolation of the techniques used in what follows are in principle sufficient to study the general model with arbitrary distribution of the potential. In fact, in this first section we will investigate the first and the second moment of the set of solutions, i.e., the set of equilibria, without any assumption on the measure $\pi$. As we will see, the expected number of equilibria grows exponentially with the number of players, regardless of the specific form of $\pi$. Moreover, the independence of the payoffs across different profiles is sufficient to ensure that the random number of equilibria is well approximated by its expectation.

**Proposition 1.** For any probability measure $\pi$ with mean $p$ we have

\[(2.13) \quad E[|NE|] = \int_0^1 (2 - 2p(1 - x))^n d\pi(x),\]

and

\[(2.14) \quad \lim \inf_{n \to \infty} \frac{1}{n} \log E[|NE|] \geq \log \frac{3}{2}.\]

**Proposition 2.** For any probability measure $\pi$ we have

\[(2.15) \quad \lim_{n \to \infty} \frac{E[|NE|^2]}{E[|NE|^2]} = 1.\]

The next corollary follows immediately by Chebyshev’s inequality and Propositions 1 and 2.

**Corollary 1.** For any probability measure $\pi$, if exists some $c \in [\log(3/2), \log(2)]$ such that

\[(2.16) \quad \lim_{n \to \infty} \frac{1}{n} \log E[|NE|] = c,\]

then

\[(2.17) \quad \frac{1}{n} \log |NE| \xrightarrow{p} c.\]
For the independent model Eq. (2.13) reads

\[(2.18) \quad E[|\text{NE}|] = (1 + \alpha(p))^n,\]

where

\[(2.19) \quad \alpha(p) := p^2 + (1 - p)^2 \geq \frac{1}{2}.\]

In fact, the independent model is a particular instance of the more general one introduced in Amiet et al. (2019), where the authors show that

\[(2.20) \quad \frac{1}{n} \log |\text{NE}| \xrightarrow{P} \log(1 + \alpha(p)).\]

In fact, the analogue of Eq. (2.20) can be proved for other models, as it is stated by Eq. (2.17).

The same phenomenon occurs for the set of equilibria with a certain social utility, as soon as the expected size of this set grows exponentially in \(n\). More precisely, if we call

\[(2.21) \quad W_k = \{s \in \Sigma \mid \text{SU}(s) = k\}, \quad Z_k = \{s \in \text{NE} \mid \text{SU}(s) = k\} \subset W_k,\]

the following proposition holds.

**Proposition 3.** Let \(Q = |Z_k|\) or \(Q = |W_k|\) for some \(k \in \{0, \ldots, n\}\). Then, for any probability measure \(\pi\) for which

\[(2.22) \quad \lim_{n \to \infty} \frac{1}{n} \log E[Q] = c > 1,\]

we have

\[(2.23) \quad \lim_{n \to \infty} \frac{E[Q^2]}{E[Q]^2} = 1,\]

and, consequently,

\[(2.24) \quad \frac{1}{n} \log Q \xrightarrow{P} c.\]

### 3. Fully supported potential

In this section we focus on the case in which \(\pi\) is fully supported in \([0, 1]\). We will show that, under this assumption, the number of equilibria grows at the maximal possible rate.

**Theorem 1** (Number of equilibria and typical efficiency). If \(\pi\) is fully supported in \([0, 1]\), then

\[(3.1) \quad \lim_{n \to \infty} \frac{1}{n} \log |\text{NE}| \xrightarrow{P} \log 2.\]

Moreover, if \(\widehat{\text{NE}}_\varepsilon\) is the set of equilibria having average social utility greater than \(1 - \varepsilon\), then for all \(\varepsilon > 0\),

\[(3.2) \quad \lim_{n \to \infty} \frac{|\widehat{\text{NE}}_\varepsilon|}{|\text{NE}|} \xrightarrow{P} 1.\]

Notice that when \(u_i(s) = 1\) for all \(i\) then the profile \(s\) is automatically a pure equilibrium. On the other hand, if the social utility of \(s\) is \(x\) for some \(x \in (0, 1)\), then the probability that \(s\) is an equilibrium is exponentially small, with a rate depending only on \(x\) and \(p\), more precisely,

\[(3.3) \quad P(s \in \text{NE} \mid \text{SU}(s) = xn) = \left[ (1 - p)^{(1-x)} \right]^n.\]

The rationale underlying Theorem 1 is that—given that \(\pi\) is fully supported—for all \(\varepsilon > 0\) there exists a fraction \(\delta\) of strategy profiles with average social utility larger than \(1 - \varepsilon\). For those profiles, the probability of being an equilibrium has a small exponential cost. In other words, the proof of
Theorem 1 shows that in this framework best equilibria are optimal and have average social utility equal to 1. Moreover, most of the equilibria share this property.

On the other hand, for the behavior of the worst equilibria, we need to analyze the exponential rate in Eq. (3.3). The following Theorem 2 shows that, if \( p < \frac{1}{2} \), then arbitrary bad equilibria exist. On the other hand, if \( p \) is sufficiently large, the worst equilibria have a typical average social utility, which depends only on \( p \).

**Theorem 2.** If \( \pi \) is fully supported in \([0, 1]\), then

\[
\frac{1}{n} \left( \text{SO}, \text{Beq}, \text{Weq} \right) \xrightarrow{P} (1, 1, h(p)),
\]

where \( h : (0, 1) \to [0, 1] \) is the non-decreasing continuous function defined as

\[
h(p) := \begin{cases} 
0 & \text{if } p \leq \frac{1}{2}, \\
\frac{\log(2(1-p))}{\log(1-p)} & \text{if } p > \frac{1}{2}.
\end{cases}
\]

![Figure 1. Plot of the function \( h(p) \) defined in Eq. (3.5).](image)

4. INDEPENDENT PAYOFFS

In this section we analyze the independent model which, as mentioned above, is a particular instance of the model in Amiet et al. (2019). In this framework, the study of the behavior of the random variable \( \text{SO} \) is somehow classical in the probabilistic literature. In fact, the latter can be thought of as the maximum of \( 2^n \) independent random variables with law \( \text{Bin}(n, p) \). Therefore, the analysis of \( \text{SO} \) relies on the study of the large deviation rate of a sequence of Binomial trials and has been performed in details, e.g., in Durrett (1979). Clearly, when one focuses on \( \text{Beq} \) (\( \text{Weq} \)) the analysis is more complicated, due to the fact that dependencies arise when restricting the maximization (minimization) to the random domain \( \text{NE} \). In this context, the behavior of \( \text{Beq} \) and \( \text{Weq} \) can be determined by a precise analysis of the interplay of two different factors: the exponential cost needed to have a large average social utility (i.e., equal to some \( x > p \)) and the exponential cost of being an equilibrium given an average social utility equal to \( x \). Such a competition realizes in the phenomena described in the forthcoming Theorems 3 and 4.

**Theorem 3 (Convergence).** There exists three explicit functions

\[
x_{\text{opt}}, x_{\text{beq}}, x_{\text{weq}} : [0, 1] \to [0, 1],
\]

such that

\[
\frac{1}{n} \left( \text{SO}, \text{Beq}, \text{Weq} \right) \xrightarrow{P} \left( x_{\text{opt}}(p), x_{\text{beq}}(p), x_{\text{weq}}(p) \right).
\]
The limit quantities for $SO$, $Beq$ and $Weq$—seen as a functions of $p$—display an interesting behavior. In particular, there exists some threshold for the value of $p$ before/after which the functions stay constant.

Theorem 4 (Phase transitions). The functions $x_{opt}$ and $x_{beq}$ are both increasing on the interval $(0, 1/2)$ and are identically equal to 1 on the interval $[1/2, 1]$. The function $x_{weq}$ is identically 0 on the interval $(0, 1 - \sqrt{2}/2)$ and is increasing on the interval $[1 - \sqrt{2}/2, 1]$.

We stress that in this model efficiency can always be “nearly achieved” at equilibrium, in the sense that the ratio $x_{opt}/x_{beq}$ is near 1 for all the value of $p \in (0, 1)$, see Fig. 2. Notice that by choosing $p = 1/2$ we are considering the uniform measure on the space of binary games with $n$ players. In other words, properties that hold with high probability in the model with potential distribution $\pi(\cdot) = \delta_{1/2}(\cdot)$ are shared by a fraction of binary games that approaches 1 as $n$ grows to infinity. Therefore, if $p = 1/2$, we can rephrase Theorem 4 as a counting problem and obtain the following result.

Corollary 2. For all $\varepsilon > 0$ consider the set $\mathcal{G}_n$ of all the binary games with $n$ players and the subset $\tilde{\mathcal{G}}_{n,\varepsilon}$ of binary games having $SO, Beq \in [1 - \varepsilon, 1]$ and $Weq \in [x_{weq}(1/2) - \varepsilon, x_{weq}(1/2) + \varepsilon]$. Then, for all $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{|\tilde{\mathcal{G}}_{n,\varepsilon}|}{|\mathcal{G}_n|} = 1.$$ 

Roughly, Corollary 2 states that asymptotically almost every binary game has $\text{PoS} \approx 1$ and $\text{PoA} \approx 4.4034$.

5. UNDERLYING DICHTOMOUS POTENTIAL

We now consider the dichotomous potential case, which can be equivalently defined as follows. For every $s \in \Sigma$ consider an auxiliary sequence of random variables $(X(s))_{s \in \Sigma}$ i.i.d. $\text{Bern}(p)$, a sequence $(R_i(s))_{i \in [n], s \in \Sigma}$ of i.i.d. $\text{Bern}(q)$ and, finally, a sequence $(Y_i(s))_{i \in [n], s \in \Sigma}$ of i.i.d. $\text{Bern}(p)$. Moreover, we assume all these sequences to be independent. We then define the game as follows:

$$u_i(s) = \begin{cases} X(s) & \text{if } R_i(s) = 1 \\ Y_i(s) & \text{if } R_i(s) = 0 \end{cases} \quad \forall \ s \in \Sigma, \forall i \in [n].$$
Roughly, with probability \(q\) the payoff of player \(i\) under \(s\) copies the random potential \(X(s)\), while, with the remaining probability, it is an independent \(\text{Bern}(p)\) random variable.

Our next theorem shows that the exponential size of \(|NE|\) increases monotonically with the correlation parameter \(q\). Moreover, asymptotically the efficiency of almost every equilibrium corresponds to the value that optimizes the competition of the exponential costs mentioned in Section 4. We will give an explicit expression for this value and we will show that it is increasing in \(q\) for every fixed \(p \in (0, 1)\). Therefore, the presence of correlation not only increases the number of Nash equilibria, but also their typical efficiency.

**Theorem 5** (Number of equilibria and typical efficiency). For all \((p, q) \in [0, 1] \times [0, 1]\), we have

\[
\frac{1}{n} \log |NE| \xrightarrow{P} \log (1 + \alpha(p) + 2qp(1 - p)).
\]

Moreover, given any \(\varepsilon > 0\) and defining

\[
\widehat{NE}_\varepsilon = \{ s \in NE : \left| \frac{1}{n} \text{SU}(s) - x^+(p, q) \right| < \varepsilon \},
\]

where

\[
x^+(p, q) = \frac{q + p(1 - q)}{1 - p(1 - q) + p^2(1 - q)},
\]

we have

\[
\frac{\widehat{NE}_\varepsilon}{|NE|} \xrightarrow{P} 1.
\]

**Remark 1.** Since \(x^+(1/2, 0) = 2/3\), an immediate consequence of Theorem 5 is that, in the same spirit of Corollary 2, a uniformly sampled equilibrium in a uniformly random binary game has an average social utility of 2/3.

We will now establish the analogue of Theorem 3 for the general model with \(q \geq 0\). Notice that the limit functions in this case depend on the interplay of the two parameters \(p\) and \(q\).

**Theorem 6** (Convergence). There exists three functions

\[
x_{\text{opt}}, x_{\text{beq}}, x_{\text{weq}} : (0, 1) \times [0, 1] \to [0, 1],
\]

such that

\[
\frac{1}{n} (\text{SO, Beq, Weq}) \xrightarrow{P} \left(x_{\text{opt}}(p, q), x_{\text{beq}}(p, q), x_{\text{weq}}(p, q)\right).
\]

Given Eq. (5.7), it is natural to analyze the behavior of the limit quantities as functions of the two parameters \(p\) and \(q\), in the same vein of Theorem 4. In this case, we will fix the parameter \(p \in (0, 1)\) and vary the correlation parameter \(q \in [0, 1]\). We now show that these functions exhibit different kinds of irregularity depending on the choice of \(p\).

**Theorem 7** (Phase transitions). For all \((p, q) \in (0, 1) \times [0, 1]\) the function \(x_{\text{weq}}(p, q)\) has the following properties

(i) If \(p \in \left[0, 1 - \frac{\sqrt{2}}{2}\right]\), then \(x_{\text{weq}}(p, q) = 0\) for every \(q \in [0, 1]\).

(ii) If \(p \in \left[1 - \frac{\sqrt{2}}{2}, 1\right]\), then \(x_{\text{weq}}(p, \cdot)\) is continuous in \([0, 1]\). Moreover, calling

\[
\rho(p) := \frac{4p - 2p^2 - 1}{2(1 - p)p},
\]

\(x_{\text{weq}}(p, \cdot) \in C^1((0, \rho(p)))\) with

\[
\frac{d}{dq} x_{\text{weq}}(p, q) < 0,
\]
and \( x_{\text{weq}}(q,p) = 0 \) for every \( q > \rho(p) \).

(iii) If \( p > \frac{1}{2} \), \( x_{\text{weq}}(p,q) > 0 \) for every \( q \in [0,1] \).

(iv) There exist some critical \( p_c \in (1/2,1) \) (\( p_c \approx 0.732 \)) such that
   - If \( p \in (p_c,1) \), then \( x_{\text{weq}}(p,q) \) is continuous for \( q \in [0,1] \).
   - If \( p \in (1/2,p_c) \), then \( x_{\text{weq}}(p,q) \) is continuous for \( q \in [0,1] \setminus \{ q^*(p) \} \), where

\[
q^*(p) := \frac{1 - 2p + 2p^2}{2p^2}.
\]

Moreover,

\[
\lim_{q \downarrow q^*(p)} x_{\text{weq}}(p,q) < \lim_{q \uparrow q^*(p)} x_{\text{weq}}(p,q).
\]

As we remarked above, in the independent model the efficiency is approximatively achieved at equilibrium, in the sense that, for all \( p \in (0,1) \), \( x_{\text{opt}} \) and \( x_{\text{beq}} \) are not far apart. The following proposition shows that the same is true in the model with dependent payoffs.

**Proposition 4.** For all \( p,q \in (0,1) \) the functions \( x_{\text{opt}}(p,q) \) and \( x_{\text{beq}}(p,q) \) satisfy,

\[
x_{\text{opt}}(p,q) = x_{\text{opt}}(q + (1-q)p,0), \quad x_{\text{beq}}(p,q) \geq x_{\text{beq}}(q + (1-q)p,0).
\]

6. Proofs

In this section we present the main proofs of the results of Sections 2–5. In particular, in Section 6.1 we deal with the moments results in Section 2. In Section 6.2 we prove the convergences in Theorems 3 and 6. In Section 6.2 we prove the phase transition outlined in Theorems 4 and 7. Finally, in Section 6.3 we will prove the results in Section 3.
We will adopt the following notation. For every \( n \in \mathbb{N} \), \( (\Omega^{(n)}, P^{(n)}) \) denotes the probability space introduced above, where \( \pi \) is the probability law of the potential and \( n \) is number of players. Since we are interested in the asymptotic scenario in which the number of players grows to infinity, we will usually drop the dependence on \( n \) in the notation. Moreover, when the choice of \( \pi \) is clear by the context, we will also drop the dependence on \( F \). We say that a sequence of real random variables \((X_n)_{n \in \mathbb{N}}\) in the product probability space \( \times_{n \in \mathbb{N}} (\Omega^{(n)}, P^{(n)}) \) converges to \( \ell \in \mathbb{R} \) in probability (denoted by \( X_n \xrightarrow{P} \ell \)), if
\[
\forall \varepsilon > 0, \quad \lim_{n \to \infty} P_{\pi}^{(n)}(|X_n - \ell| < \varepsilon) = 1.
\]

6.1. **Moments estimates.** We start the subsection by proving Proposition 1, namely the expected size of the set of equilibria.

**Proof of Proposition 1.** We start by computing the probability that a given profile \( s \) is an equilibrium by conditioning on the value of the potential at \( s \), namely
\[
P(s \in \text{NE}) = \int_0^1 P(s \in \text{NE} | \Phi(s) = x) \, dF(x)
\]
\[
= \int_0^1 [1 - p(1 - x)]^n \, dF(x),
\]
(6.2)
\[
(6.3)
\]
where we used the fact that
\[
P(s \in \text{NE} | \Phi(s) = x) = P(\exists i \in [n] \text{ s.t. } u_i(s) = 0, u_i(s_{-i}, s_i \oplus 1) = 1)
\]
\[
= \prod_{i \in [n]} (1 - P(u_i(s) = 0, u_i(s_{-i}, s_i \oplus 1) = 1))
\]
\[
= [1 - P(u_1(s) = 0, u_1(s_{-1}, s_1 \oplus 1) = 1)]^n
\]
(6.5)
\[
(6.6)
\]
Therefore, by the linearity of the expectation,
\[
E[|\text{NE}|] = 2^n P(s \in \text{NE}),
\]
(6.7)
(6.8)
from which the first part of the thesis follows. Notice that, regardless of the specific choice of \( p \) the expected number of equilibria grows exponentially in \( n \). In fact, by the fact that \( \pi \) has mean \( p \), there exists some \( \varepsilon > 0 \) such that \( P(\Phi(s) \geq p) > \varepsilon \). Then
\[
E[|\text{NE}|] = \int_0^1 (2 - 2p(1 - x))^n \, d\pi(x)
\]
(6.9)
\[
\geq \int_p^1 (2 - 2p(1 - x))^n \, d\pi(x)
\]
(6.10)
\[
\geq \varepsilon (2 - 2p(1 - p))^n
\]
(6.11)
\[
\geq \varepsilon \left( \frac{3}{2} \right)^n
\]
(6.12)
Hence,
\[
\lim inf_{n \to \infty} \frac{1}{n} \log E[|\text{NE}|] \geq \log \frac{3}{2}.
\]
(6.13)
(6.14)
Proof of Propositions 2 and 3. We now aim at computing the second moment of the quantities \(|Z_k|, |W_k|\), and \(|NE|\). We start the proof with the analysis of the second moment of \(|W_k|\), which is easier to compute. Indeed, for every distinct \(s, s' \in \Sigma\), thanks to the independence of the payoffs vector across profile, we have

\[
P(s, s' \in W_k) = P(s \in W_k)^2.
\]

Therefore,

\[
E[|W_k|^2] \leq E[|W_k|^2] = \sum_{s \in \Sigma} \sum_{s' \in \Sigma} P(s, s' \in W_k)
\]

\[
= \sum_{s \in \Sigma} P(s \in W_k) + \sum_{s \in \Sigma} \sum_{s' \neq s} P(s \in W_k)^2
\]

\[
\leq E[|W_k|] + E[|W_k|^2].
\]

In particular, if there exists some subset of values \(k \in [n]\) and some \(\varepsilon > 0\) for which

\[
\lim \inf_{n \to \infty} \frac{1}{n} \log E[|W_k|] \geq 1 + \varepsilon,
\]

then Eqs. (6.16) and (6.19) ensure that for those \(k\)'s the following asymptotic estimate holds

\[
\frac{E[|W_k|^2]}{E[|W_k|]^2} \geq 1 - \frac{1}{(1 + \varepsilon)^n}.
\]

The above argument fails if the \(W_k\) is replaced by its subset \(Z_k\). This is due to the fact that

\[
P(s, s' \in Z_k) \neq P(s \in Z_k)^2.
\]

Since the proof is identical for both \(|Z_k|\) and \(|NE|\), we will prove the lemma using \(|Z_k|\). The proof for \(|NE|\) is similar.

We claim that for every \(s, s'\) differing in at least three strategies, the events \(\{s \in Z_k\}\) and \(\{s' \in Z_k\}\) are independent. We notice that the event \(\{s \in Z_k\}\) is measurable with respect to the \(\sigma\)-field

\[
\sigma\left(\{u_i(s)\}, \{u_i(s_{-i}, s_i \oplus 1)\} : i \in [n]\right).
\]

We remark that in the independent case, i.e., \(\pi(\cdot) = \delta_p(\cdot)\), if \(s'\) differs from \(s\) in at least two strategies we have that the events \(\{s \in Z_k\}\) and \(\{s' \in Z_k\}\) are measurable with respect to independent \(\sigma\)-fields, hence they are independent. On the other hand, in the general case, the events \(\{s \in Z_k\}, \{s' \in Z_k\}\) are still measurable with respect to the \(\sigma\)-fields of the type in Eq. (6.22), nonetheless, if \(s, s'\) differ in a only one or two strategies, such \(\sigma\)-fields are not independent. Notice that, in particular, \(\{s \in Z_k\}\) is measurable with respect to the \(\sigma\)-field generated by the complete information about the payoffs of all the players in the neighboring profiles, i.e.,

\[
\sigma\left(\{u_i(s)\}, \{u((s_{-i}, s_i \oplus 1))\} : i \in [n]\right).
\]

Clearly, if \(s, s'\) differs in more than two strategies, then they are measurable with respect to independent \(\sigma\)-fields, hence are independent.

We now want to upper bound the probability of the event \(\{s, s' \in Z_k\}\) when the two profiles \(s\) and \(s'\) differs in only one or two strategies. We start by analyzing the case in which there exists a unique \(i \in [n]\) such that

\[
s' = (s_{-i}, s_i \oplus 1).
\]
Then,
\begin{equation}
(6.25) \quad P(s, s' \in Z_k) = P(s \in Z_k) P(s' \in Z_k | s \in Z_k) = P(s \in Z_k) P(s' \in Z_k | u_i(s') \leq u_i(s)) = P(s \in Z_k) \frac{P(s' \in Z_k \cap u_i(s') \leq u_i(s))}{P(u_i(s') \leq u_i(s))} = \frac{1}{1 - p(1 - p)} P(s \in Z_k) P(s' \in Z_k \cap u_i(s') = u_i(s)).
\end{equation}

Notice that the probability of the intersection must satisfy, uniformly in \(k\),
\begin{equation}
(6.26) \quad P^{(n)}(s \in Z_k \cap u_i(s) = u_i(s)) = (1 - p)^2 P^{(n-1)}(s \in Z_k) + 1_{k > 0} p^2 P^{(n-1)}(s \in Z_{k-1}) = \Theta \left(P^{(n)}(s \in Z_k)\right).
\end{equation}

Hence, we can conclude that for all distinct \(s, s' \in \Sigma\) differing in exactly one strategy, uniformly in \(k\),
\begin{equation}
(6.27) \quad P(s, s' \in Z_k) = \Theta \left(P(s \in Z_k^2)\right).
\end{equation}

Let us now consider the case in which \(s\) and \(s'\) are at distance 2. In other words, assume that there are two distinct players \(i\) and \(j\), such that
\begin{equation}
(6.28) \quad s' = (s_{-ij}, s_i \oplus 1, s_j \oplus 1).
\end{equation}

Consider also the intermediate strategies
\begin{equation}
(6.29) \quad s'' := (s_{-j}, s_j \oplus 1) = (s'_{-i}, s_i \oplus 1), \quad s''' := (s_{-i}, s_i \oplus 1) = (s'_{-j}, s_j \oplus 1).
\end{equation}

Arguing as in Eq. (6.25) we get
\begin{equation}
(6.30) \quad P(s, s' \in Z_k) = P(s \in Z_k) P(s' \in Z_k | s \in Z_k) = P(s \in Z_k) P(s' \in Z_k | u_i(s) \geq u_i(s'') \cap u_j(s) \geq u_j(s'')) = \Theta \left(P(s \in Z_k^2)\right).
\end{equation}

We can now compute the second moment. Denote by \(N_{t}(s)\) the set of strategy profiles differing from \(s\) in exactly \(t\) coordinates. By the asymptotic estimates in Eqs. (6.25) and (6.29) we can conclude that there exists some constant \(C = C(p) > 0\), such that
\begin{equation}
E[|Z_k|^2] = \sum_{s \in \Sigma} \sum_{s' \in \Sigma} P(s, s' \in Z_k) = \sum_{s \in \Sigma} \left[ P(s \in Z_k) + \sum_{s' \in N_1(s)} P(s, s' \in Z_k) + \sum_{s' \in N_2(s)} P(s, s' \in Z_k) + \sum_{s' \in N_{\geq 3}(s)} P(s \in Z_k)^2 \right] \leq E[Z_k] + C \cdot 2^n \cdot (n + n(n - 1)) P(s \in Z_k)^2 + 2^n \cdot (2^n - n - n(n - 1) - 1) P(s \in Z_k)^2 = E[Z_k] + (1 + o(1)) 2^n P(s \in Z_k)^2 = E[Z_k] + (1 + o(1)) E[|Z_k|^2].
\end{equation}

Therefore, if \(\liminf_{n \to \infty} \frac{1}{n} \log E[Z_k] > 1 + \varepsilon\) for some \(\varepsilon > 0\),
\begin{equation}
(6.31) \quad \frac{E[|Z_k|^2]}{E[|Z_k|^2]} \leq 1 + \frac{C \cdot 2^n \cdot n^2 P(s \in Z_k)^2}{2^{2n} \cdot P(s \in Z_k)^2} + \frac{1}{E[Z_k]} \leq 1 + \frac{1}{(1 + \varepsilon)^n} \quad \square.
\end{equation}
6.2. Proof of the convergence. Notice that Theorem 3 is a particular case of Theorem 6, with 
$q = 0$. In this section we will use a unified approach, showing directly Theorem 6 and obtaining 
Theorem 3 by taking $q = 0$. The first lemma deals with the expected size of the sets in Eq. (2.21).

We remind to the reader that the entropy of a Bernoulli$(x)$ is defined as $H : (0, 1) \to (0, \infty)$, where

\begin{equation}
H(x) := -x \log(x) - (1 - x) \log(1 - x).
\end{equation}

The following definitions are needed.

**Definition 1.** Consider the following bounded analytic functions

- The function $f_W : (0, 1)^2 \to [0, 2]$ is defined as
  \[ f_W(p, x) := 2p^x(1 - p)^{1-x}e^{H(x)}. \]
- The function $f_Z : (0, 1)^2 \to [0, 2]$ is defined as
  \[ f_Z(p, x) := 2p^x(1 - p)^{2(1-x)}e^{H(x)}. \]
- The function $g_W : (0, 1)^3 \to [0, 2]$ is defined as
  \[ g_W(p, q, x) := \max\{f_W(q + (1 - q)p, x), f_W((1 - q)p, x)\}. \]
- The function $g_W^+ : (0, 1)^3 \to [0, 2]$ is defined as
  \[ g_W^+(p, q, x) = f_W(q + (1 - q)p, x)(1 - p)^{1-x}. \]
- The function $g_W^- : (0, 1)^3 \to [0, 2]$ is defined as
  \[ g_W^-(p, q, x) = f_W((1 - q)p, x)(1 - p)^{1-x}. \]
- The function $g_Z : (0, 1)^3 \to [0, 2]$ is defined as
  \[ g_Z(p, q, x) := (1 - p)^{1-x}g_W(p, q, x) = \max\{g_W^-(p, q, x), g_W^+(p, q, x)\}. \]

**Remark 2.** Notice that, for all $(p, x) \in (0, 1)^2$

\begin{equation}
g_W(p, 0, x) = f_W(p, x), \quad g_Z^-(p, 0, x) = g_W^+(p, 0, x) = g_Z(p, 0, x) = f_Z(p, x).
\end{equation}

Moreover, for all $p \in (0, 1)$, the functions defined in Definition 1 admit the limits $x \uparrow 1$ and $x \downarrow 0$, see Lemmas 1 and 2. Therefore, for all $p \in (0, 1)$ we can extend the functions in Definition 1, as functions of the second variable, to continuous functions in $[0, 1]$.

The forthcoming Lemmas 1 and 2 establish some easy facts about the behavior of the functions defined in Definition 1, which can be checked by direct computation.

**Lemma 1.** The functions $f_W$ and $f_Z$ have the following properties:

(i) For every $p \in (0, 1)$

\begin{equation}
\frac{\partial}{\partial x} \log f_W(p, x) = \log(\eta(p, x)),
\end{equation}

where

\begin{equation}
\eta(p, x) := \frac{p}{1 - p} \cdot \frac{1 - x}{x}.
\end{equation}

Hence, fixed any $p \in (0, 1)$

\begin{equation}
\frac{\partial}{\partial x} f_W(p, x) = 0 \iff x = p.
\end{equation}

Moreover,

\begin{equation}
f_W(p, p) = 2.
\end{equation}
(ii) For every $x \in (0, 1)$

\[
\frac{\partial}{\partial p} \log f_W(p, x) = \tau(p, x),
\]

where

\[
\tau(p, x) := \frac{x - p}{p(1 - p)}.
\]

Hence, fixed any $x \in (0, 1)$

\[
\frac{\partial}{\partial p} f_W(p, x) = 0 \iff p = x.
\]

(iii) For every $p \in (0, 1)$

\[
\lim_{x \uparrow 1} f_W(p, x) = 2p, \quad \lim_{x \downarrow 0} f_W(p, x) = 2 - 2p.
\]

(iv) For every $p, x \in (0, 1)$

\[
f_W(p, x) > f_Z(p, x).
\]

(v) For every $p \in (0, 1)$

\[
\frac{\partial}{\partial x} \log f_Z(p, x) = \log (\beta(p, x))
\]

where

\[
\beta(p, x) := \frac{p}{(1 - p)^2} \cdot \frac{1 - x}{x}
\]

hence, fixed any $p \in (0, 1)$

\[
\frac{\partial}{\partial x} f_Z(p, x) = 0 \iff x = \hat{x}(p) := \frac{p}{1 - p + p^2},
\]

moreover

\[
f_Z(p, \hat{x}(p)) = 1 + p^2 + (1 - p^2) = 1 + \alpha(p).
\]

(vi) For every $x \in (0, 1)$

\[
\frac{\partial}{\partial p} \log f_Z(p, x) = v(p, x),
\]

where

\[
v(p, x) := \frac{x - p(2 - x)}{p(1 - p)}.
\]

Hence, fixed any $x \in (0, 1)$

\[
\frac{\partial}{\partial p} f_Z(p, x) = 0 \iff p = \frac{x}{2 - x}.
\]

(vii) For every $p \in (0, 1)$

\[
\lim_{x \downarrow 0} f_Z(p, x) = 2(1 - p)^2, \quad \lim_{x \uparrow 1} f_Z(p, x) = 2p.
\]

Lemma 2. The functions $g^+_Z(p, q, x)$ and $g^-_Z(p, q, x)$, defined in Definition 1, have the following properties:
(a) For every $p, q \in (0, 1)$

\[
\frac{\partial}{\partial x} \log g^+_Z(p, q, x) = \log(\beta_+(p, q, x)), \quad \frac{\partial}{\partial x} \log g^-_Z(p, q, x) = \log(\beta_-(p, q, x))
\]

where

\[
\beta_+(p, q, x) := \frac{q + (1 - q)p}{(1 - p)^2(1 - q)} \cdot \frac{1 - x}{x}, \quad \beta_-(p, q, x) := \frac{(1 - q)p}{(1 - p)(1 - p(1 - q))} \cdot \frac{1 - x}{x}.
\]

(b) For all $p, q \in (0, 1)$ there exists two points

\[
x^+(p, q) := \frac{q + p(1 - q)}{1 - p(1 - q) + p^2(1 - q)}, \quad x^-(p, q) := \frac{p(1 - q)}{1 - p + p^2(1 - q)}
\]

such that

\[
\frac{\partial}{\partial x} g^\pm_Z(x) \begin{cases} > 0 & \text{if } x < x^\pm(p, q), \\ = 0 & \text{if } x = x^\pm(p, q), \\ < 0 & \text{if } x > x^\pm(p, q). \end{cases}
\]

(c) For every $p, q, x \in (0, 1)$

\[
\frac{\partial}{\partial q} \log g^+_Z(p, q, x) = \frac{q + p - 1 + x}{(q - 1)(p - q) + q}, \quad \frac{\partial}{\partial q} \log g^-_Z(p, q, x) = \frac{p(q - 1) + x}{(q - 1)(p - q) + 1}.
\]

Moreover,

\[
\frac{\partial}{\partial q} g^\pm_Z(p, q, x) \begin{cases} > 0 & \text{if } x < v^\pm(p, x), \\ = 0 & \text{if } x = v^\pm(p, x), \\ < 0 & \text{if } x > v^\pm(p, x), \end{cases}
\]

where

\[
v^-(p, x) := \frac{p - x}{p}, \quad v^+(p, x) := \frac{x - p}{1 - p}.
\]

(d) For every $p, q \in (0, 1)$

\[
g^+_Z(p, q, x) \begin{cases} < g^-_Z(p, q, x) & \text{if } x < \gamma(p, q), \\ = g^-_Z(p, q, x) & \text{if } x = \gamma(p, q), \\ > g^-_Z(p, q, x) & \text{if } x > \gamma(p, q), \end{cases}
\]

where,

\[
\gamma(p, q) := \frac{\log \theta_1(p, q)}{\log \theta_2(p, q)},
\]

\[
\theta_1(p, q) := \frac{(1 - p)(1 - q)}{1 - p(1 - q)}, \quad \theta_2(p, q) := \theta_1(p, q) \cdot \frac{p(1 - q)}{q - p(1 - q)}.
\]

(e) For every $p, q \in (0, 1)$

\[
\lim_{x \to 0} g^-_Z(p, q, x) = 2(1 - p)(1 - q), \quad \lim_{x \to 0} g^+_Z(p, q, x) = 2(1 - p)^2(1 - q),
\]

and

\[
\lim_{x \to 1} g^-_Z(p, q, x) = 2p(1 - q), \quad \lim_{x \to 1} g^+_Z(p, q, x) = 2q + p(1 - q).
\]
Remark 3. Notice that if \( q = 0 \) we are back to the independent case, indeed \( g^2_Z(p, 0, x) = f_Z(p, x) \) for every \( p, x \in (0, 1) \) and

\[
\beta_\pm(p, 0, x) = \frac{p}{(1-p)^2} \cdot \frac{1-x}{x}, \quad x^\pm(p, 0) = \frac{p}{1-p+p^2} g^2_Z(p, 0, x^\pm(p, 0)) = 1 + o(p).
\]

Lemma 3. For all \((p, q) \in (0, 1) \times [0, 1] \), \( \varepsilon > 0 \), \( k \in [\varepsilon n, (1-\varepsilon)n] \) we have

\[
\frac{1}{n} \log E|W_k| \sim \log g_W(p, q, \frac{k}{n}), \quad \frac{1}{n} \log E|Z_k| \sim \log g_Z(p, q, \frac{k}{n}).
\]

Proof of Lemma 3. By independence of payoffs under different strategy profile, we have

\[
E[|W_k|] = 2^n P(s \in W_k)
\]

\[
= 2^n \cdot P(Bin(n, q + (1-q)p) = k) + (1-p) \cdot P(Bin(n, (1-q)p) = k)
\]

\[
= p \cdot 2^n \cdot \left( \binom{n}{k} (q+(1-q)p)^k (1-q-(1-q)p)^{n-k} \right) +
\]

\[
+ (1-p) \cdot 2^n \cdot \left( \binom{n}{k} ((1-q)p)^k (1-(1-q)p)^{n-k} \right),
\]

where, to get the second equality, we conditioned on the outcome of \( X(s) \), defined at the beginning of Section 5.

Fix \( \varepsilon > 0 \), and pick some \( k \in [\varepsilon n, (1-\varepsilon)n] \). Let \( x = \frac{k}{n} \), and notice that using the asymptotic approximation

\[
\left( \frac{n}{xn} \right) = e^{(1+o(1))H(x)n},
\]

we can estimate

\[
E[|W_k|] = p \cdot \left[ (1+o(1))2e^{H(x)}(q + (1-q)p)^x (1-q-(1-q)p)^{1-x} \right]^n +
\]

\[
+ (1-p) \cdot \left[ (1+o(1))2e^{H(x)}((1-q)p)^x (1-(1-q)p)^{1-x} \right]^n
\]

\[
= p \cdot [(1+o(1))f_W(q + (1-q)p, x)]^n + (1-p) \cdot [(1+o(1))f_W((1-q)p, x)]^n.
\]

Notice that the convex coefficients \( p \) and \( 1-p \) are absorbed in the \( (1+o(1)) \) error within the squared brackets. Hence, if \( k \in [\varepsilon n, (1-\varepsilon)n] \) for any \( \varepsilon > 0 \),

\[
E[|W_k|] = (1+o(1)) \max \{ f_W(q + (1-q)p, k/n), f_W((1-q)p, k/n) \}^n.
\]

By taking the logarithm and normalizing,

\[
\frac{1}{n} \log E|W_k| \sim \max \{ \log f_W(q + (1-q)p, k/n), \log f_W((1-q)p, k/n) \}.
\]

We now compute the expected size of \( Z_k \). By conditioning on the social utility of the profile \( s \in \Sigma \)

\[
E[|Z_k|] = 2^n P(s \in Z_k)
\]

\[
= 2^n P(s \in NE \mid s \in W_k) P(s \in W_k).
\]

Notice that, conditioning on any \( \omega \in \{s \in W_k\} \), the probability that \( s \) is an equilibrium is exactly the probability of the event

\[
\{ u_i(s_{-i}, s_i \oplus 1) = 0, \forall i \text{ s.t. } u_i(s) = 0 \}.
\]

By the independence across \( i \)'s in the events in Eq. (6.66), we conclude that

\[
P \left( \bigcap_{i; u_i(s)=0} u_i(s_{-i}, s_i \oplus 1) = 0 \mid s \in W_k \right) = (1-p)^{n-k}.
\]
Hence, using the same argument as in Eq. (6.64),
\[
\mathbb{E}[|Z_k|] = 2^n \cdot (1 - p)^{n-k} \cdot [p \cdot \mathbb{P}(\text{Bin}(n, q + (1 - q)p) = k) + (1 - p) \cdot \mathbb{P}(\text{Bin}(n, (1 - q)p) = k)]
\]
\[
= \left[ (1 + o(1)) \max \{g^+_Z(p, q, \frac{k}{n}), g^+_Z(p, q, \frac{k}{n}) \} \right]^n.
\]
Therefore, if \( k \in [\varepsilon n, (1 - \varepsilon)n] \) for any \( \varepsilon > 0 \),
\[
\frac{1}{n} \log \mathbb{E}[|Z_k|] \sim \max \{\log g^+_Z(p, q, \frac{k}{n}), \log g^+_Z(p, q, \frac{k}{n})\}.
\]

We can use the result in Lemma 3 to show Theorem 5, which controls the number and typical efficiency of equilibria.

**Proof of Theorem 5.** In order to compute the expectation of \( |\text{NE}| \) it is sufficient to notice that
\[
P(s \in \text{NE}) = p \cdot P(s \in \text{NE} | X(s) = 1) + (1 - p) \cdot P(s \in \text{NE} | X(s) = 0)
\]
\[
= p \cdot (1 - p(1 - q)(1 - p))^n + (1 - p) \cdot (1 - p(q + (1 - q)(1 - p))^n).
\]
Therefore
\[
\mathbb{E}[|\text{NE}|] = 2^n P(s \in \text{NE}) = p \cdot [2(1 - p(1 - q)(1 - p))]^n (1 + o(1))
\]
\[
= p \cdot [1 + \alpha(p) + 2qp(1 - p)]^n
\]
\[
= \Omega((3/2)^n)
\]
Therefore, by Proposition 2, \( |\text{NE}|/\mathbb{E}[|\text{NE}|] \xrightarrow{P} 1 \), and by Eq. (6.70) we get Eq. (5.2). At this point, in order to show the validity of Eq. (5.5) it is sufficient to notice that for every sufficiently small \( \varepsilon > 0 \)
\[
1 < \frac{1}{n} \log \left( \mathbb{E}[|\text{NE} \setminus \hat{\text{NE}}_{\varepsilon}|] \right) < \log (1 + \alpha(p) + 2qp(1 - p))
\]
and again by Proposition 2, \( |\hat{\text{NE}}_{\varepsilon}|/\mathbb{E}[|\hat{\text{NE}}_{\varepsilon}|] \xrightarrow{P} 1. \)

Notice that if there exists some \( \delta > 0 \) such that \( g_W(p, q, k/n) \geq 1 + \delta \) then the expectation \( \mathbb{E}[W_k] = \omega(1) \), while if \( g_W(p, q, k/n) \leq 1 - \delta \) then the expectation \( \mathbb{E}[W_k] = o(1) \). The main idea of the proof of Theorem 6 goes as follows. If for some triple \((p, q, x)\) we have \( g_Z(p, q, x) < 1 \), then by Markov’s inequality we can infer that asymptotically there are no Nash equilibria with average social utility \( x + o(1) \). On the other hand, if \( g_Z(p, q, x) > 1 \), the set of equilibria with average social utility \( x + o(1) \) is not empty; this will be proved by the control on the second moments in Proposition 3.

We start by using Lemma 3 to control the probability that the set \( Z_k \) is empty. As remarked above, for every \( k \) that is not too close to 0 or \( n \), the expectation of \( |Z_k| \) is exponential of rate \( g_{NE}(p, q, k/n) \in [0, 2] \). Hence, in what follows we will be interested in studying the solution of the following equations
\[
g_W(p, q, x) = 1 \quad \text{and} \quad g_Z(p, q, x) = 1.
\]
See Figs. 4 and 5 for a representation of the functions \( g_W(p, q, x) \) and \( g_Z(p, q, x) \) as a function of the variable \( x \) for some fixed values of \( p \) and \( q \).

For instance, when \( q = 0 \), for every given \( p \in (0, 1) \), the smallest \( x \) for which \( f_Z(p, x) \geq 1 \) is our proxy for the ASU of the worst equilibrium, while the largest \( x \) for which \( f_Z(p, x) \geq 1 \) is our proxy for the ASU of the best equilibrium. Similarly, the optimum ASU can be obtained by looking at the largest \( x \) for which \( f_W(p, x) \geq 1 \). We formalize this intuition in the following proposition, which is the technical version of the more readable Theorem 6.

**Proposition 5.** Fix \( p, q \in (0, 1) \times [0, 1) \) and any \( \varepsilon, \delta > 0 \).
Figure 4. In Blue: $f_Z(p,x)$ as function of $x$. In Orange: $f_W(p,x)$ as function of $x$. The value of $p$ in the four pictures is, respectively, $p = 0.25, 1 - \sqrt{2}/2, 0.4$. The green line is at height 1.

Figure 5. Plot of the functions $g_W(0.5, 0.75, x)$ (in orange) and $g_Z(0.5, 0.75, x)$ (in blue). The fact that the orange curve lies above the line at height one, means that the expectation of the number of strategy with SU $k$ diverges exponentially fast, for all $k \in [0, n]$. On the other hand, the regions where the blue curve lies below the green line denotes values of ASU which will not appear within the strategies in NE.

(a) Call

(6.74) $N_{\epsilon, \delta}^+ := \{k \in [\delta n, (1 - \delta)n] : g_W(p,q,k/n) \geq 1 + \epsilon\},$

(6.75) $N_{\epsilon, \delta}^- := \{k \in [\delta n, (1 - \delta)n] : g_W(p,q,k/n) \leq 1 - \epsilon\},$

we have

(6.76) $\lim_{n \to \infty} P(\bigcup_{k \in N_{\epsilon, \delta}^-} W_k = \emptyset) = 1, \quad \lim_{n \to \infty} P(\forall k \in N_{\epsilon, \delta}^+, W_k \neq \emptyset) = 1.$

(b) Called

(6.77) $M_{\epsilon, \delta}^+ := \{k \in [\delta n, (1 - \delta)n] : g_Z(p,q,k/n) \geq 1 + \epsilon\},$

(6.78) $M_{\epsilon, \delta}^- := \{k \in [\delta n, (1 - \delta)n] : g_Z(p,q,k/n) \leq 1 - \epsilon\},$

we have,

(6.79) $\lim_{n \to \infty} P(\bigcup_{k \in M_{\epsilon, \delta}^-} Z_k = \emptyset) = 1, \quad \lim_{n \to \infty} P(\forall k \in M_{\epsilon, \delta}^+, Z_k \neq \emptyset) = 1.$

Proof. We prove (a); the proof of (b) is similar.

Fix a pair $p, q \in (0, 1) \times [0, 1]$ and $\epsilon, \delta > 0$. Pick some $k \in N_{\epsilon, \delta}^-$. We have

(6.80) $P(\|W_k\| \geq 1) \leq E[\|W_k\|],$

hence, for every sufficiently large $n$

(6.81) $\frac{1}{n} \log P(\|W_k\| \geq 1) \leq 1 - \epsilon,$
where the first is Markov inequality and the second estimate follows from Lemma 3 and holds uniformly in \( k \in \mathbb{N}_{\varepsilon,\delta}^- \). By applying the union bound we get the first limit in Eq. (6.76).

On the other hand, by Proposition 3 and the second moment method, (see Alon and Spencer, 2016, Ch. 4),

\[
P(W_k \neq \emptyset) \geq \frac{\mathbb{E}[|W_k|^2]}{\mathbb{E}[|W_k|^2]} \geq 1 - \frac{1}{(1 + \varepsilon)^n}.
\]

We can therefore deduce the second limit in Eq. (6.76) by applying a union bound over \( k \in \mathbb{N}_{\varepsilon,\delta}^+ \).

The results in Theorems 3 and 6 now follow as a corollary of Proposition 5.

**Proof of Theorem 6.** The claim follows directly by Eqs. (6.76) and (6.79). We show the result for the Weq, since the proofs for Beq and SO are identical. Fix \( \varepsilon, \delta > 0 \). Notice that, by Eq. (6.77),

\[
\frac{1}{n} \min_{s \in \mathcal{N}} S_{\text{SU}}(s) = \frac{1}{n} \min\{k \in [n] : Z_k \neq \emptyset\} \leq \frac{1}{n} \min\{k \in M_{\varepsilon,\delta}^+\} =: \frac{1}{n} k_{\varepsilon,\delta}.
\]

On the other hand, by Eq. (6.79), we have

\[
\frac{1}{n} \min\{k \in [n] : Z_k \neq \emptyset\} \geq \frac{1}{n} \max\{k \in M_{\varepsilon,\delta}^- | k \leq k_{\varepsilon,\delta}\} =: \frac{1}{n} k_{\varepsilon,\delta},
\]

where we define \( k_{\varepsilon,\delta} = 0 \) if the set in its definition is empty. By continuity of \( g_Z(p,q,\cdot) \) and the fact that \( g_Z(p,q,\cdot) \) and its derivative are bounded around 0 and 1 (see Lemma 2), we have that for all \( \eta > 0 \), we can find \( n_0, \varepsilon \) and \( \delta \) such that, for all \( n > n_0 \),

\[
\left| \frac{1}{n} k_{\varepsilon,\delta} - \frac{1}{n} k_{\varepsilon,\delta} \right| < \eta,
\]

and

\[
\left| \frac{1}{n} k_{\varepsilon,\delta} - \inf\{x \in (0,1) : g_Z(p,q,x) \geq 1\} \right| < \eta.
\]

Hence,

\[
\lim_{n \to \infty} P \left( \frac{1}{n} \min_{s \in \mathcal{N}} S_{\text{SU}}(s) - \inf\{x \in (0,1) : g_Z(p,q,x) \geq 1\} \leq \eta \right) = 1,
\]

which concludes the proof.

As a byproduct of the above proof we get the following characterization of the limit functions defined in Theorem 6. By Eq. (6.86), for all \( p,q \in (0,1) \times [0,1] \), the function \( x_{\text{weq}} \) defined in Theorem 6 admits the following implicit representation

\[
x_{\text{weq}}(p,q) = \inf\{x \in (0,1) : g_Z(p,q,x) \geq 1\}.
\]

Similarly, the functions \( x_{\text{opt}} \) and \( x_{\text{beq}} \) admit the representations

\[
x_{\text{opt}}(p,q) = \sup\{x \in (0,1) : g_W(p,q,x) \geq 1\},
\]

\[
x_{\text{beq}}(p,q) = \sup\{x \in (0,1) : g_Z(p,q,x) \geq 1\}.
\]

The phase-transition in Theorem 4 is then a consequence of the analysis of the functions in Eqs. (6.87) and (6.88), Eq. (6.89).

**Proof of Theorem 4.** We start by analyzing the function \( x_{\text{opt}} : (0,1/2) \to [0,1] \).

By Lemma 1(iii), we have

\[
\lim_{x \to 1} \log f_W(p, x) < 0.
\]

This inequality, together with the continuity of \( f_W(p,x) \), justifies the characterization of \( x_{\text{opt}}(p) \) as the largest solution in \( x \in (0,1) \) of the equation

\[
\log f_W(p, x) = 0.
\]
In order to use implicit function theorem, we need to check that, for every fixed \( p \in (0, 1/2) \), the partial derivative with respect to \( x \) of the function \( f_W(p, x) \) does not vanish at the largest solution of Eq. (6.91). Notice that \( x_{\text{opt}}(p) > p \) since \( f_W(p, p) = 2 \). Moreover, thanks to Lemma 1(i) for every \( p \in (0, 1/2) \) and \( x \in (p, 1) \) we have

\[ \eta(p, x) < 1 \quad \implies \quad \frac{\partial}{\partial x} \log f_W(p, x) < 0, \]

where \( \eta \) is defined as in Eq. (6.34). Relying again on the rough estimate \( x_{\text{opt}}(p) > p \), together with Lemma 1(ii), we get

\[ \tau(p, x) > 0 \quad \implies \quad \frac{\partial}{\partial p} \log f_W(p, x_{\text{opt}}(p)) > 0, \]

where \( \tau \) is defined as in Eq. (6.38). Therefore, by the implicit function theorem, the implicit function \( x_{\text{opt}}(p) \) defined by Eq. (6.91) admits, for all \( p \in (0, 1/2) \), derivative of the form

\[ \frac{d}{dp} x_{\text{opt}}(p) \bigg|_{p_0} = -\frac{\frac{\partial}{\partial p} \log f_W(p, x_{\text{opt}}(p))}{\frac{\partial}{\partial x} \log f_W(p, x_{\text{opt}}(p))} > 0. \]

We proceed similarly for \( x_{\text{beq}}(p) \) in the same interval \((0, 1/2)\). By Lemma 1(vii)

\[ \lim_{x \uparrow 1} \log f_W(p, x) < 0. \]

Hence, by continuity of \( f_Z(p, x) \), the function \( x_{\text{beq}}(p) \) is defined implicitly as the largest solution in \((0, 1)\) of the equation

\[ \log f_Z(p, x) = 0. \]

Notice that, thanks to Lemma 1(v),

\[ \frac{\partial}{\partial x} \log f_Z(p, x) < 0, \quad \forall x > \hat{x}(p) := \frac{p}{1 - p + p^2}. \]

Since

\[ \log f_Z(p, \hat{x}(p)) = \log(1 + \alpha(p)) > 0, \]

it holds that \( x_{\text{beq}}(p) > \hat{x}(p) \). Moreover, by Lemma 1(vi), we have

\[ \frac{\partial}{\partial p} \log f_Z(p, x) > 0 \quad \forall x > \frac{2p}{1 + p}. \]

Since

\[ f_Z \left( p, \frac{2p}{1 + p} \right) = (1 + p)(2 - 2p)^{\frac{2}{p+1}-1} > 1, \]

we have

\[ x_{\text{beq}} > \frac{2p}{1 + p}. \]

In conclusion, by the implicit function theorem, the function \( x_{\text{beq}} \) is \( C_1(0, 1/2) \) and

\[ \frac{d}{dp} x_{\text{beq}}(p) \bigg|_{p_0} > 0. \]

The behavior of the function \( x_{\text{opt}} \) and \( x_{\text{beq}} \) in the interval \((1/2, 1)\) follows by the continuity of \( f_W \) and \( f_Z \) together with Lemma 1(iv) and Lemma 1(vii).
We are left to show the regularity properties of the function $x_{\text{weq}}$. If we restrict to $p \in (1 - \sqrt{2}/2, 1)$, we can characterize $x_{\text{weq}}$ as the smallest solution in $x \in (0, 1)$ of the equation in Eq. (6.96). Notice that by Lemma 1(iii), Eq. (6.100), and the continuity of $f_Z(p, \cdot)$ we deduce that

(6.103) \[ x_{\text{weq}}(p) < \frac{2p}{1+p}. \]

For the same reason, given Eq. (6.98), we have $x_{\text{weq}}(p) < \frac{p}{1-p+p^2}$. By the converse of the inequalities in Eqs. (6.97) and (6.99), we obtain

(6.104) \[ \frac{\partial}{\partial p} \log f_Z(p, x) < 0 \iff x < \frac{2p}{1+p} \]

(6.105) \[ \frac{\partial}{\partial x} f_Z(p, x) > 0 \iff x < \frac{p}{1-p+p^2}. \]

So, we can apply the implicit function theorem and conclude that the function $x_{\text{weq}}(p)$ admits a derivative

(6.106) \[ \frac{d}{dp} x_{\text{weq}}(p) \big|_{p_0} > 0 \quad \forall p_0 \in (1 - 1/\sqrt{2}, 1). \]

To conclude, we need to show that $x_{\text{weq}}$ is constantly zero in the interval $(0, 1 - \sqrt{2}/2)$. This is just a simple consequence of Lemma 1(vii). In fact,

\[ \lim_{x \to 0} \log f_W(p, x) > 0 \iff p < 1 - \frac{\sqrt{2}}{2}. \]

We are now going to prove the phase-transition phenomenon described in Theorem 7. We recall to the reader that here we assume $p$ to be fixed while the moving parameter is the correlation $q$.

**Proof of Theorem 7.** By steps:

(i) By Lemma 2(e) we have

(6.107) \[ \lim_{x \to 0} g_Z(p, q, x) = 2(1-p)(1-p(1-q)) > 1 \iff q > \rho(p). \]

Notice that

(6.108) \[ \rho(p) > 0 \iff p > 1 - \frac{1}{\sqrt{2}}. \]

The result follows immediately.

(ii) We note that $\rho(p) : (1-1/\sqrt{2}, 1/2) \to (0, 1)$ is a bijection. Moreover, by the implicit function theorem, for all $q_0 \in (0, \rho(p))$,

(6.109) \[ \frac{d}{dq} \tilde{x}_{\text{weq}}(p, q) \big|_{q_0} = -\frac{\partial}{\partial q} \log g_Z^-(p, q, x) \big|_{q_0, \tilde{x}_{\text{weq}}(p,q_0)} \]

(6.110) \[ = -\frac{p(1-q)-x}{(1-q)(1-p(1-q)) \log (\beta^-(p, q, x)) \big|_{q_0, \tilde{x}_{\text{weq}}(p,q_0)}}, \]

where $\beta^-$ is defined as in Eq. (6.51). We aim at showing that the signs of numerator and denominator in Eq. (6.110) coincide. Note that we have

(6.111) \[ \log \beta^-(p, q, x_{\text{weq}}(p, q)) > 0, \quad \forall p \in \left(1 - \frac{\sqrt{2}}{2}, \frac{1}{2}\right), \forall q > \rho(p). \]

In fact, the sign of $\log \beta^-(p, q, \cdot)$ is the sign of the partial derivative of $g_Z^-(p, q, x)$ with respect to $x$, which is positive at $x_{\text{weq}}(p, q)$. On the other hand, by definition,

(6.112) \[ (1-q)(1-p(1-q)) > 0. \]
Hence, we are left with showing that
\[(6.113) \quad p(1 - q) - x_{weq}(p, q) > 0, \quad \forall p \in (1 - \frac{\sqrt[2]{2}}{2}, 1/2), \forall q > \rho(p).\]
Since for \( p \in (1 - \sqrt[2]{2}/2, 1/2) \) the quantity \( x_{weq}(p, q) \) is the smallest solution of the equation
\[(6.114) \quad \log g_Z^-(p, q, x) = 0, \]
it is sufficient to check that
\[(6.115) \quad \log g_Z^-(p, q, p(1 - q)) > 0, \quad \forall p \in (1 - \frac{\sqrt[2]{2}}{2}, 1/2), \forall q > \rho(p).\]
We can rewrite Eq. (6.115) as
\[(6.116) \quad \log(2) + (1 - p(1 - q)) \log(1 - p) > 0, \quad \forall p \in (1 - \frac{\sqrt[2]{2}}{2}, 1/2), \forall q > \rho(p).\]
Notice that the following inequality is stronger than Eq. (6.116):
\[(6.117) \quad \log(2) + \log(1 - p) > 0, \quad \forall p \in (1 - \frac{\sqrt[2]{2}}{2}, 1/2), \forall q > \rho(p).\]
and the latter is trivially true.
(iii) By the same argument used to prove (i), we have
\[(6.118) \quad \lim_{x \rightarrow 0} g_Z^+(p, q, x) < 1 \quad \iff \quad q < \rho(p).\]
To conclude, notice that \( \rho(p) > 1 \) if \( p \in (1/2, 1). \)
(iv) For every \( p \in (1/2, 1) \) call \( q^*(p) \) the unique solution of the equation in \( q \)
\[(6.119) \quad \log g_Z^+(p, q, x^-(p, q)) = \log \left( 2 \left( p^2(1 - q) + (1 - p) \right) \right) = 0.\]
For all \( p \in (1/2, 1) \) the function \( g_Z^+(p, q, \cdot) \) is decreasing in \( q \in (0, 1) \), and the unique solution of the equation in Eq. (6.119) is given by
\[(6.120) \quad q^*(p) = \frac{1 - 2p + 2p^2}{2p^2}, \quad \forall p \in (1/2, 1).\]
The existence of a unique solution to Eq. (6.119) implies that, for all \( p \in (1/2) \), there exist a unique value of \( q \) for which the maximum attained by the curve \( g_Z^+(p, q, \cdot) \) is exactly 1.

Having in mind the plots in Figs. 6 and 7, we are interested in understanding whether
\[(6.121) \quad x^-(p, q^*(p)) \leq \inf \{ x \in (0, 1) : g_Z^+(p, q^*(p), x) \geq 1 \} \).
In order to do so, we aim at analyzing the map
\[(6.122) \quad p \mapsto g_Z^+(p, q^*(p), x^-(p, q^*(p))),\]
\[\text{namely, the value assumed by the function } g_Z^+ \text{ at the point in which the function } g_Z^- \text{ attains its maximum height, i.e., } 1. \text{ We start by claiming that the function in Eq. (6.122) is increasing, hence there exists a unique solution in } (1/2, 1) \text{ of the equation in } p\]
\[(6.123) \quad g_Z^+(p, q^*(p), x^-(p, q^*(p))) = 1.\]
We define \( p_c \) the solution of Eq. (6.123); numerically, \( p_c \approx 0.731642 \). As suggested by the plots in Figs. 6 and 7, we expect two different behaviors of the function \( x_{weq}(p, \cdot) \) when \( p \in (1/2, p_c) \) (see Fig. 6), and when \( p \in (p_c, 1) \) (see Fig. 7).
In order to show the monotonicity of the map in Eq. (6.122) it is sufficient to proceed by explicit computation. In fact,
\[(6.124) \quad \log g^+_Z(p, q^*(p), x^-(p, q^*(p))) = \log \left( \frac{3 - 1}{p} - 2p \right) + \left( 2 - \frac{1}{p} \right) \log \left( \frac{1 - (3 - 4p)p}{(2p - 1)^2(1 - p)} \right),\]
Figure 6. Plot of $g_Z(p, q, x)$ (blue) and $g_W(p, q, x)$ (orange) when $p = 0.6 < p_c$ and $q = 0.5, 0.68, 0.74, 0.77$, respectively. Recall that $g_Z(p, q, x) = g_Z^-(p, q, x)$ on the left of the dashed line $x = \gamma(p, q)$, while $g_Z(p, q, x) = g_Z^+(p, q, x)$ on the right. Similarly $g_W(p, q, x) = f_W((1-q)p, x)$ on the left of the dashed line, while $g_W(p, q, x) = f_W(q + (1-q)p, x)$ on the right.

Figure 7. Plot of $g_Z(p, q, x)$ (blue) and $g_W(p, q, x)$ (orange) when $p = 0.8 > p_c$ and $q = 0.2, 0.45, 0.52, 0.55$, respectively. The dashed line lies at $x = \gamma(p, q)$.

take the derivative

$$
\frac{d}{dp} \log g_Z^+(p, q^*(p), x^-(p, q^*(p))) = \frac{-8p^4 + 10p^2 - 8p + 3}{p(2p - 1)(4p^2 - 3p + 1)} + \frac{1}{p^2} \log \left( \frac{4p^2 - 3p + 1}{(1-p)(2p-1)^2} \right),
$$
and notice that, for all \( p \in (1/2, 1) \),

\[
(6.125) \quad \frac{d}{dp} \log g_Z^-(p, q^*(p), x^-(p, q^*(p))) > 0.
\]

Therefore, there exists a unique \( p_c \in (1/2, 1) \) for which

\[
(6.126) \quad g_Z^-(p_c, q^*(p_c), x^-(p_c, q^*(p_c))) = g_Z^+(p_c, q^*(p_c), x^-(p_c, q^*(p_c))) = 1.
\]

Moreover, thanks to Lemma 2(d), the value of \( p_c \) can be further characterized as the unique \( p \in (1/2, 1) \) such that

\[
(6.127) \quad x^-(p_c, q^*(p_c)) = \gamma(p_c, q^*(p_c)),
\]

where \( \gamma \) is defined as in Eq. (6.58). In conclusion, if \( p \in (1/2, p_c) \) the function \( x_{\text{weq}}(p, \cdot) \) is discontinuous at \( q^*(p) \).

On the other hand, by Eq. (6.89),

\[
(6.128) \quad q \mapsto \inf \{ x \in (0, 1) : \max \{ \log g_Z^- (p, q, x), \log g_Z^+ (p, q, x) \} = 0 \},
\]

is continuous in \([0, 1]\). In fact, by Eq. (6.119), \( g_Z^-(p, q, x^-(p, q)) \) is decreasing in \( q \), hence, if \( q \in (q^*(p), 1) \), \( x_{\text{weq}}(p, q) = \inf \{ x \in (0, 1) : \log g_Z^- (p, q, x) = 0 \} \), which is clearly continuous. Conversely, if \( q \in (0, q^*(p)) \), then the function \( g_Z^-(p, q, \cdot) \) is increasing in \((0, \gamma(p,q))\), and the function \( g_Z^+(p, q, \cdot) \) is increasing in a neighborhood of \( \gamma(p,q) \). Hence, Eq. (6.128) can be written as

\[
(6.129) \quad q \mapsto \begin{cases} 
\inf \{ x \in (0, 1) : \log g_Z^- (p, q, x) = 0 \} & \text{if } q < q^\Box(p), \\
\inf \{ x \in (0, 1) : \log g_Z^- (p, q, x) = 0 \} & \text{if } q > q^\Box(p),
\end{cases}
\]

where \( q^\Box(p) \) solves

\[
(6.130) \quad g_Z^-(p, q^\Box(p), \gamma(p, q^\Box(p))) = 1.
\]

By definition of \( \gamma \),

\[
(6.131) \quad g_Z^+(p, q^\Box(p), \gamma(p, q^\Box(p))) = 1.
\]

hence, there cannot be any discontinuity when passing from the first to the second branch of Eq. (6.129). \( \square \)

**Proof of Proposition 4.** By Eq. (6.88) and Lemma 1,

(6.132) \quad x_{\text{opt}}(p, q) = \sup \{ x \in (0, 1) : \max (f_W((1-q)p, 0), f_W(q + (1-q)p, x)) \geq 1 \}

(6.133) \quad = \sup \{ x \in (0, 1) : f_W(q + (1-q)p, x) \geq 1 \}

(6.134) \quad = x_{\text{opt}}(q + (1-q)p, 0).

On the other hand, by Eq. (6.89),

(6.135) \quad x_{\text{beq}}(p, q) = \sup \{ x \in (0, 1) : (1-p)^{1-x} \max (f_W((1-q)p, x), f_W(q + (1-q)p, x)) \geq 1 \}

(6.136) \quad = \sup \{ x \in (0, 1) : (1-p)^{1-x} f_W(q + (1-q)p, x) \geq 1 \}

(6.137) \quad \geq \sup \{ x \in (0, 1) : (1-p)^{1-x} f_W(p, x) \geq 1 \}

(6.138) \quad = x_{\text{beq}}(q + (1-q)p, 0).

where the inequality above follow from Lemma 1(ii), which implies that \( \forall (q, x) \in (0, 1)^2 \)

\[
(1-p)^{1-x} f_W(q + (1-q)p, x) \geq (1-p)^{1-x} f_W(p, x).
\] \( \square \)
6.3. **Proof for the fully supported potential.** In this subsection we focus on the case in which \( \pi \) is fully supported in \([0, 1]\). More precisely, we will assume that for every \( I \subset [0, 1] \) with \( \text{Leb}(I) = \epsilon \) there exists some \( \delta = \delta(\epsilon) \) such that

\[
\pi(I) \geq \delta. 
\]

(6.139)

**Proof of Theorem 1.** We start by showing a concentration result for the number of profiles with a large potential. By Eq. (6.139), for any fixed \( \epsilon > 0 \) there exists some \( \delta = \delta(\epsilon) > 0 \) such that

\[
P(\Phi(s) \geq 1 - \epsilon) = \delta.
\]

It follows that the expected number of profiles with potential in the interval \([1 - \epsilon, 1]\) is

\[
E[|\{s : \Phi(s) \geq 1 - \epsilon\}|] = \delta^2 n. 
\]

(6.140)

Moreover, by the Chernoff bound, for any constant \( \gamma \in (0, 1) \)

\[
P(\text{Binomial}(2^n, \delta) < (1 - \gamma)\delta^2 n) \leq \exp(-\Theta(2^n)). 
\]

(6.141)

Hence, considering the family of events

\[
\mathcal{E}_{\epsilon,c} := \{ |\{s : \Phi(s) \geq 1 - \epsilon\}| > c2^n \}, 
\]

there exists some sufficiently small constant \( c = c(\epsilon) > 0 \) for which

\[
\lim_{n \to \infty} P(\mathcal{E}_{\epsilon,c}) = 1. 
\]

(6.142)

Notice now that if \( \Phi(s) \geq 1 - \epsilon \) then the probability that \( SU(s) = n \) can be lower bounded by

\[
P(SU(s) = n | \Phi(s) \geq 1 - \epsilon) \geq (1 - \epsilon)^n. 
\]

(6.144)

Therefore,

\[
E[|Z_n|] \geq E[|Z_n| \mid |\{s : \Phi(s) \geq 1 - \epsilon\}| > c2^n] P(|\{s : \Phi(s) \geq 1 - \epsilon\}| > c2^n) 
\]

\[
\geq c(2(1 - \epsilon))^n. 
\]

(6.145)

By Proposition 3,

\[
\frac{|Z_n|}{E[|Z_n|]} \xrightarrow{P} 1, 
\]

(6.147)

from which, by taking \( \epsilon \to 0 \), follows

\[
\frac{1}{n} \log(|Z_n|) \xrightarrow{P} 2. 
\]

(6.148)

**Proof of Theorem 2.** Fix some \( s \in \Sigma \) and notice that

\[
P(s \in Z_k) = P(SU(s) = k) P(s \in \text{NE} \mid SU(s) = k). 
\]

(6.149)

It is worth noting that the quantity \( P(s \in \text{NE} \mid SU(s) = k) \) depends only on \( p \), regardless of the specific form of \( \pi \). In fact,

\[
P(s \in \text{NE} \mid SU(s) = k) = \text{P}(u_i(s) = 0)^{n-k}. 
\]

(6.150)

\[
= \left( \int_0^1 (1 - x) d\pi(x) \right)^{n-k} 
\]

(6.151)

\[
= (1 - p)^{n-k}. 
\]

(6.152)

We notice further that, if \( \Phi(s) = x \) then, by the law of large numbers, for all \( \epsilon > 0 \)

\[
\lim_{n \to \infty} P\left(\frac{SU(s)}{n} \in [x - \epsilon, x + \epsilon] \mid \Phi(s) \in [x - \epsilon/2, x + \epsilon/2]\right) = 1. 
\]

(6.153)

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Being \( \pi \) fully supported we have
\[
\Pr (\Phi(s) \in [x - \varepsilon/2, x + \varepsilon/2]) = \Omega(1),
\]
and therefore
\[
E [\{s : \Phi(s) \in [x - \varepsilon/2, x + \varepsilon/2]\}] = \Theta(2^n),
\]
so that by the independence of the sequence \( (\Phi(s))_{s \in \Sigma} \) we have
\[
\left| \{s : \Phi(s) \in [x - \varepsilon/2, x + \varepsilon/2]\} \right| \overset{P}{\longrightarrow} 1.
\]
Therefore, by Eq. (6.153) and (6.156) we conclude that for any arbitrarily small interval \([x - \varepsilon, x + \varepsilon] \subseteq [0, 1] \) we have \( \Theta(2^n) \) profiles with such an ASU. Moreover, by Eq. (6.152), a profile with ASU \( x \pm \varepsilon \) is a NE with probability
\[
(1 - p)^{(1-x+\varepsilon)n} \leq \Pr (s \in \text{NE} \mid \text{SU}(s) \in [(x - \varepsilon)n, (x + \varepsilon)n]) \leq (1 - p)^{(1-x-\varepsilon)n}.
\]
Hence,
\[
2(1 - p)^{(1-x+\varepsilon)n} \leq E [\left| \{s : s \in \text{NE}, \text{SU}(s)/n \in [x - \varepsilon, x + \varepsilon]\} \right|] \leq [2(1 - p)^{(1-x-\varepsilon)}]^n.
\]
The theorem follows by letting \( \varepsilon \to 0 \) in Eq. (6.158), by using Proposition 3 and by noting that
\[
2(1 - p)^{(1-x)} = 1 \quad \iff \quad p \geq \frac{1}{2} \quad \text{and} \quad x = h(p).
\]
\[\square\]

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