Negative phase velocity in a material with simultaneous mirror-conjugated and racemic chirality characteristics

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Abstract. The propagation of electromagnetic plane waves in a material with simultaneous mirror-conjugated and racemic chirality (SMCRC) characteristics is investigated. General conditions for negative phase velocity (NPV) (i.e., phase velocity directed opposite to the time-averaged Poynting vector) are derived for both unirefringent and birefringent propagation. Through numerical studies, it is demonstrated that NPV propagation arises provided that the magnitude of the magnetoelectric constitutive parameter is sufficiently large compared with the magnitudes of dielectric and magnetic constitutive parameters. However, the relative magnitude of the magnetoelectric constitutive parameter has little bearing upon the directions, which support NPV propagation. The propensity for NPV propagation is much enhanced through incorporating dielectric and magnetic constitutive parameters, which are negative-real. A wide range of constitutive parameter values are considered in order to accommodate the possibilities offered by the fabrication of artificial SMCRC materials.

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1. Introduction

The phase velocity of an electromagnetic plane wave is described as negative, if it is directed opposite to its time-averaged Poynting vector. Several interesting consequences follow as a result of negative phase velocity (NPV) propagation [1]. Most prominent amongst these is the phenomenon of negative refraction. Experimental reports of microwave negative refraction in artificial metamaterials [2]–[4] have provoked considerable excitement in the electromagnetics and materials communities within the past few years. Recent efforts have been directed towards achieving negative refraction at higher frequencies, with the ultimate goal being optical negative refraction [5]. In view of the practical difficulties involved in achieving negative refraction in isotropic dielectric-magnetic materials, attention lately has been directed towards more complex materials, such as anisotropic [6, 7] and bianisotropic [8] materials as well as isotropic chiral materials [9]–[11]. We report here on the prospects for NPV propagation in a particular class of anisotropic chiral materials.

The simplest manifestation of chirality arises in isotropic chiral materials (ICMs) [12]. These are specified by frequency-domain constitutive relations of the form

\[ D = \epsilon E + \xi H \quad \text{and} \quad B = \mp \xi E + \mu H, \quad (1) \]

where the material described by the lower signs is the mirror conjugate of the material described by the upper signs. The homogeneous mixture in equal proportions of an ICM and its mirror conjugate, as specified by (1), is called racemic. The racemic mixture is an isotropic dielectric-magnetic medium specified by the constitutive relations:

\[ D = \epsilon E \quad \text{and} \quad B = \mu H, \quad (2) \]

and is therefore similar to the materials commonly investigated for negative refraction [13].

However, it is possible to combine in equal proportions an ICM and its mirror-conjugate in a homogeneous mixture and yet retain optical activity. Such a mixture must be effectively an anisotropic continuum and can be understood as follows. Suppose an artificial ICM is fabricated by randomly dispersing and randomly orienting miniature left-handed springs in some host material [12, 14]. Its mirror-conjugate can be fabricated by similarly dispersing miniature springs...
in an identical host material, with the exception that the springs must be right handed. In a racemic material, left- and right-handed springs must be dispersed in equal proportions [15]. Anisotropic chirality can be induced if all springs in an artificial material are of the same handedness and are aligned parallel to each other, as has been demonstrated satisfactorily [16, 17]. Even more complicated anisotropic chiral behaviour can result if left-handed springs are aligned along a fixed axis, and right-handed springs along some other fixed axis.

Our focus here is on materials which display simultaneous mirror-conjugated and racemic chirality (SMCRC) characteristics. Artificial SMCRC materials can be conceived as containing left-handed springs aligned along the \( x \)-axis, right-handed springs along the \( y \)-axis, and left- as well as right-handed springs in equal proportions aligned along the \( z \)-axis [18]. Other schemes for artificial fabrication also have been offered [19]. We emphasize that the prospect of fabricating artificial SMCRC materials allows a wide range of constitutive parameter values to be contemplated. Very significantly, SMCRC materials occur in nature in the form of certain minerals of the type \( RXO_4 \), where \( R \) is a trivalent rare earth, \( X \) is either vanadium or arsenic or phosphorus and \( O \) is oxygen [20, 21]. The SMCRC characteristics are displayed below the Néel temperatures. Details pertaining to the general propagation properties in SMCRC materials are available elsewhere [18, 19].

In the following sections, we derive conditions for NPV propagation in an SMCRC material. Both unirefringent and birefringent propagation are considered for generally dissipative materials. The analytic expressions developed are explored numerically by means of representative examples.

In the notation adopted, vectors are underlined, whereas dyadics are double underlined. Unit vectors are denoted by the \( \hat{} \) symbol, while \( \mathbf{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \) is the identity dyadic. The complex conjugate of a complex-valued quantity \( q \) is represented by \( q^* \), whereas its real and imaginary parts are written as \( \text{Re}\{q\} \) and \( \text{Im}\{q\} \), respectively. All electromagnetic field phasors and constitutive parameters depend implicitly on the circular frequency \( \omega \) of the electromagnetic field and an \( \exp(-i\omega t) \) time-dependence is implicit. The free-space (i.e., vacuum) wavenumber is denoted by \( k_0 = \omega \sqrt{\epsilon_0 \mu_0} \) with \( \epsilon_0 \) and \( \mu_0 \) being the permittivity and permeability of free space, respectively.

2. Analysis

2.1. Preliminaries

The propagation of plane waves with field phasors

\[
E(r) = E_0 \exp(i\mathbf{k} \cdot \mathbf{r}) \quad \text{and} \quad H(r) = H_0 \exp(i\mathbf{k} \cdot \mathbf{r})
\]

is considered in an SMCRC material, whose frequency-domain constitutive relations [18]

\[
\mathbf{D}(r) = \epsilon \cdot E(r) + \xi \cdot H(r) \quad \text{and} \quad \mathbf{B}(r) = -\xi \cdot E(r) + \mu \cdot H(r)
\]

employ the constitutive dyadics

\[
\begin{align*}
\epsilon &= \epsilon_1 \hat{x}\hat{x} + \epsilon_2 \hat{y}\hat{y} + \epsilon_3 \hat{z}\hat{z}, \\
\xi &= \xi_1 \hat{x}\hat{x} - \xi_2 \hat{y}\hat{y}, \\
\mu &= \mu_1 \hat{x}\hat{x} + \mu_2 \hat{y}\hat{y} + \mu_3 \hat{z}\hat{z}.
\end{align*}
\]

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Examples of natural materials described by (4) are GdVO$_4$ and HoPO$_4$ [20, 21]. Generally, the constitutive parameters are complex-valued, but if dissipation is neglected, we have $\epsilon_{1,3} \in \mathbb{R}$, $\mu_{1,3} \in \mathbb{R}$ and $i\xi \in \mathbb{R}$.

The wavenumber $k$ in (3) is also complex-valued in general; i.e.,

$$k = k_R + i k_I \quad (k_R, k_I \in \mathbb{R}).$$

It may be calculated from the planewave dispersion relation

$$\det[L_i (ik \hat{u})] = 0,$$

(7)

wherein

$$L_i (a) = (a \times \mathbb{I} + i \omega \xi) \cdot \mu^{-1} \cdot (a \times \mathbb{I} + i \omega \xi) - \omega^2 \epsilon.$$

(8)

Of particular interest is the orientation of the phase velocity, as specified by the direction of $k_R \hat{u}$, relative to the direction of power flow given by the time-averaged Poynting vector

$$\langle P(r) \rangle_t = (1/2) \operatorname{Re} \{ E(r) \times H^*(r) \}.$$ The combination of the constitutive relations (4) with the source-free Maxwell curl postulates yields

$$\langle P(r) \rangle_t = \frac{1}{2} \exp(-2k_I \hat{u} \cdot r) \operatorname{Re} \left\{ E_0 \times \left[ (\mu^{-1})^* \cdot \left( \frac{k^*}{\omega} \hat{u} \times E_0^* + \xi^* \cdot E_0^* \right) \right] \right\}$$

(9)

for plane waves (3). By definition, NPV propagation occurs when

$$k_R \hat{u} \cdot \langle P(r) \rangle_t < 0.$$

(10)

2.2. General propagation

For planewave propagation in an arbitrary direction, i.e.,

$$\hat{u} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (0 \leq \theta \leq \pi; 0 \leq \phi \leq 2\pi),$$

(11)

the $[L]_{lm}$ components of $L_i (ik \hat{u})$ are given as

$$[L]_{11} = \frac{k^2 \sin^2 \theta \sin^2 \phi}{\mu_3} + \frac{k^2 \cos^2 \theta - \omega^2 (\epsilon_1 \mu_1 + \xi^2)}{\mu_1},$$

(12)

$$[L]_{12} = -\frac{k^2 \sin^2 \theta \sin \phi \cos \phi}{\mu_3},$$

(13)

$$[L]_{13} = -\frac{k \sin \theta (k \cos \theta \cos \phi + \omega \xi \sin \phi)}{\mu_1},$$

(14)

$$[L]_{21} = [L]_{12}. $$

(15)
\[ [L]_{22} = \frac{k^2 \sin^2 \theta \cos^2 \phi}{\mu_3} + \frac{k^2 \cos^2 \theta - \omega^2(\epsilon_1 \mu_1 + \xi^2)}{\mu_1}, \]  
\[ [L]_{23} = -\frac{k \sin \theta(k \cos \theta \sin \phi + \omega \xi \cos \phi)}{\mu_1}, \]  
\[ [L]_{31} = -\frac{k \sin \theta(k \cos \theta \cos \phi - \omega \xi \sin \phi)}{\mu_1}, \]  
\[ [L]_{32} = -\frac{k \sin \theta(k \cos \theta \sin \phi - \omega \xi \cos \phi)}{\mu_1}, \]  
\[ [L]_{33} = \frac{k^2 \sin^2 \theta}{\mu_1} - \omega^2 \epsilon_3, \]  
and the dispersion relation (7) reduces to the \( k^2 \)-quadratic polynomial
\[ k^4[(\epsilon_1 \sin^2 \theta + \epsilon_3 \cos^2 \theta)(\mu_1 \sin^2 \theta + \mu_3 \cos^2 \theta) + \xi^2 \sin^2 \theta \cos^2 2\phi] - k^2 \omega^2(\epsilon_1 \mu_1 + \xi^2) \]
\[ \times [(\epsilon_3 \mu_1 + \epsilon_1 \mu_3) \sin^2 \theta + 2\epsilon_3 \mu_3 \cos^2 \theta] + \epsilon_3 \mu_3 \omega^4(\epsilon_1 \mu_1 + \xi^2)^2 = 0. \]  
(21)

The \( k \)-roots of (21) provide the four wavenumbers \( k = k_1, k_2, k_3 \) and \( k_4 \) as
\[ k_1 = \omega \sqrt{\frac{(\epsilon_1 \mu_1 + \xi^2)[2\epsilon_3 \mu_3 \cos^2 \theta + \sin^2 \theta(\epsilon_3 \mu_1 + \epsilon_1 \mu_3 - \sqrt{(\epsilon_3 \mu_1 - \epsilon_1 \mu_3)^2 - 4\epsilon_3 \mu_3 \xi^2 \cos^2 2\phi})]}{2[(\epsilon_1 \sin^2 \theta + \epsilon_3 \cos^2 \theta)(\mu_1 \sin^2 \theta + \mu_3 \cos^2 \theta) + \xi^2 \sin^2 \theta \cos^2 2\phi]}. \]  
(22)
\[ k_2 = -k_1, \]  
(23)
\[ k_3 = \omega \sqrt{\frac{(\epsilon_1 \mu_1 + \xi^2)[2\epsilon_3 \mu_3 \cos^2 \theta + \sin^2 \theta(\epsilon_3 \mu_1 + \epsilon_1 \mu_3 + \sqrt{(\epsilon_3 \mu_1 - \epsilon_1 \mu_3)^2 - 4\epsilon_3 \mu_3 \xi^2 \cos^2 2\phi})]}{2[(\epsilon_1 \sin^2 \theta + \epsilon_3 \cos^2 \theta)(\mu_1 \sin^2 \theta + \mu_3 \cos^2 \theta) + \xi^2 \sin^2 \theta \cos^2 2\phi]}. \]  
(24)
\[ k_4 = -k_3. \]  
(25)

In consideration of the rate of energy flow, as provided by the time-averaged Poynting vector (9), we remark that only two of the four \( k \)-roots correspond to propagating plane waves, for each propagation direction \{\theta, \phi\}. We denote the wavenumbers of the propagating modes by \( k_a \) and \( k_b \) where
\[ k_a = \begin{cases} k_1 & \text{if } \text{Im}\{k_1\} \geq 0, \\ k_2 & \text{if } \text{Im}\{k_2\} > 0, \end{cases} \]  
(26)
\[ k_b = \begin{cases} k_3 & \text{if } \text{Im}\{k_3\} \geq 0, \\ k_4 & \text{if } \text{Im}\{k_4\} > 0. \end{cases} \]  
(27)

Thus, the SMCRC material is generally birefringent. However, there are exceptions: the chosen material is unirefringent with respect to propagation along the racemic (i.e., \( z \)) axis. That is, for \( \theta = 0 \), only one independent wavenumber emerges from the dispersion relation (21).
Furthermore, pathological unirefringence [22] arises when

$$\epsilon_3 \mu_1 - \epsilon_1 \mu_3 = 2\sqrt{\epsilon_3 \mu_3 \xi} \cos 2\phi. \quad (28)$$

2.3. Unirefringent propagation along the racemic axis

For $\hat{u} = \hat{z}$, the dyadic $L(ik \hat{u})$ has the diagonal components

$$\begin{align*}
[L]_{11} &= \frac{k^2 - \omega^2(\epsilon_1 \mu_1 + \xi^2)}{\mu_1}, \quad (29) \\
[L]_{22} &= [L]_{11}, \quad (30) \\
[L]_{33} &= -\omega^2 \epsilon_3, \quad (31)
\end{align*}$$

while $[L]_{\ell m} = 0$ for $\ell \neq m$. Only two $k$-roots emerge from the dispersion relation (21); thus,

$$k_1 = k_3 = \omega \sqrt{\epsilon_1 \mu_1 + \xi^2}. \quad (32)$$

Combining the wavenumbers (32) and $L(ik \hat{z})$ components (29)–(31) with the equation $L(ik \hat{z}) \cdot E_0 = 0$, we see that the $\hat{z}$ component of the electric field phasor is null-valued; i.e., $\hat{z} \cdot E_0 = E_{0z} = 0$ and $E_0 = E_{0x} \hat{x} + E_{0y} \hat{y}$. The time-averaged Poynting vector (9) is given by

$$\langle P(r) \rangle_t = \frac{\exp(-2k_iz)}{2\omega} \text{Re} \left\{ \frac{k^2(|E_{0x}|^2 + |E_{0y}|^2) - \omega \xi^* (E_{0x}E_{0}^* + E_{0x}^*E_{0y})}{\mu_1^*} \right\} \hat{z}; \quad (33)$$

hence

$$k_R \hat{z} \cdot \langle P(r) \rangle_{t|k=k_1} = \frac{\omega \exp(-2k_iz)}{2} \left[ (|E_{0x}|^2 + |E_{0y}|^2) \text{Re} \left\{ \frac{(\sqrt{\epsilon_1 \mu_1 + \xi^2})^*}{\mu_1^*} \right\} \right]$$

$$- (E_{0x}E_{0y}^* + E_{0x}^*E_{0y}) \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \text{Re} \left\{ \sqrt{\epsilon_1 \mu_1 + \xi^2} \right\}. \quad (34)$$

$$k_R \hat{z} \cdot \langle P(r) \rangle_{t|k=k_2} = \frac{\omega \exp(-2k_iz)}{2} \left[ (|E_{0x}|^2 + |E_{0y}|^2) \text{Re} \left\{ \frac{(\sqrt{\epsilon_1 \mu_1 + \xi^2})^*}{\mu_1^*} \right\} \right]$$

$$+ (E_{0x}E_{0y}^* + E_{0x}^*E_{0y}) \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \text{Re} \left\{ \sqrt{\epsilon_1 \mu_1 + \xi^2} \right\}. \quad (35)$$
Figure 1. Maps of the wavenumbers (normalized with respect to $k_0$) projected onto the $\theta, \phi \in (0, \pi /2)$ surface of the unit sphere. Constitutive parameter values: $\epsilon_1 = (4.8 + i0.1)\epsilon_0, \epsilon_3 = (1.15 + i0.5)\epsilon_0, \xi = (0.15 + i4.55)\sqrt{\epsilon_0}\mu_0, \mu_1 = (1.35 + i0.7)\mu_0$ and $\mu_3 = (3.2 + i0.12)\mu_0$. (a) The real (top) and imaginary (bottom) parts of the wavenumber $k_a/k_0$. (b) The real (top) and imaginary (bottom) parts of the wavenumber $k_b/k_0$.

We therefore find that NPV propagation arises provided that

\[
(|\gamma|^2 + 1) \text{Re} \left\{ \frac{(\sqrt{\epsilon_1\mu_1 + \xi^2})^*}{\mu_1^*} \right\} < (\gamma + \gamma^*) \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \text{Re} \left\{ \sqrt{\epsilon_1\mu_1 + \xi^2} \right\} \quad \text{for } k = k_1, \tag{36}
\]

\[
(|\gamma|^2 + 1) \text{Re} \left\{ \frac{(\sqrt{\epsilon_1\mu_1 + \xi^2})^*}{\mu_1^*} \right\} < - (\gamma + \gamma^*) \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \text{Re} \left\{ \sqrt{\epsilon_1\mu_1 + \xi^2} \right\} \quad \text{for } k = k_2, \tag{37}
\]

wherein the complex-valued constant

\[
\gamma = \frac{E_{0x}}{E_{0y}} \tag{38}
\]

has been introduced for the sake of convenience.

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It is instructive to consider the idealized case in which dissipation is neglected. Then, (34) and (35) reduce to

$$ k\hat{z} \cdot \langle P(r) \rangle |_{k=k_1} = k\hat{z} \cdot \langle P(r) \rangle |_{k=k_2} = \frac{\omega (\epsilon_1 \mu_1 + \xi^2)(|y|^2 + 1)}{2 \mu_1} |E_0| \cdot |E_0| \cdot |E_0| $$

(39)

For propagating planewaves, the inequality

$$ \epsilon_1 \mu_1 + \xi^2 > 0 $$

(40)

must be satisfied. Therefore, we infer that unirefringent NPV propagation along the racemic axis is only possible in a nondissipative SMCRC material if both $\epsilon_1 < 0$ and $\mu_1 < 0$. Let us note that a similar situation occurs for nondissipative isotropic dielectric-magnetic mediums: therein NPV propagation develops as a result of the permittivity and permeability being simultaneously negative-valued [5].

2.4. Birefringent propagation

Use of the components of $L(i k \hat{u})$ given in (12)–(20) along with the time-averaged Poynting vector (9) yields

$$ k_R \hat{u} \cdot \langle P(r) \rangle |_r = \frac{k_R \exp(-2k_R \hat{u} \cdot r)}{2} \left( \frac{1}{\omega} \{ |E_{0x}|^2 + |E_{0y}|^2 \} \cos^2 \theta + |E_{0z}|^2 \sin^2 \theta \right) $$

$$ - \left[ (E_{0x}^* E_{0z} + E_{0z}^* E_{0x}) \cos \phi + (E_{0y}^* E_{0z} + E_{0z}^* E_{0y}) \sin \phi \right] \sin \theta \cos \theta \right] \text{Re} \left\{ \frac{k^*}{\mu_1^*} \right\} $$

$$ + \frac{1}{\omega} (E_{0x} \sin \phi - E_{0y} \cos \phi)(E_{0x}^* \sin \phi + E_{0y}^* \cos \phi) \sin^2 \theta \text{Re} \left\{ \frac{k^*}{\mu_3^*} \right\} $$
Figure 3. Maps of the wavenumbers (normalized with respect to $k_0$) projected onto the $\theta, \phi \in (0, \pi/2)$ surface of the unit sphere. Constitutive parameter values are as in figure 1, except that $\xi = (0.3 + i9.1) \sqrt{\epsilon_0/\mu_0}$. (a) The real (top) and imaginary (bottom) parts of the wavenumber $k_a/k_0$. (b) The real (top) and imaginary (bottom) parts of the wavenumber $k_b/k_0$.

\begin{equation}
+ \sin \theta \text{Re} \left\{ (E_{0x}^* E_{0c} \sin \phi + E_{0y}^* E_{0c} \cos \phi) \frac{\xi^*}{\mu_1^*} \right\}
- (E_{0x}^* E_{0y}^* + E_{0x}^* E_{0y}^*) \cos \theta \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\}.
\end{equation}

After introducing the complex-valued parameters

\begin{equation}
\alpha = \frac{E_{0x}}{E_{0c}},
\end{equation}

\begin{equation}
\beta = \frac{E_{0y}}{E_{0c}}
\end{equation}
Figure 4. Regions of NPV propagation, corresponding to the wavevectors of figure 3, mapped onto the $\theta, \phi \in (0, \pi/2)$ surface of the unit sphere. Constitutive parameter values are as in figure 3. NPV propagation for the wavenumber $k_a$ is mapped in green.

The expression (41) may be written as

$$k_R \hat{u} \cdot \langle P(r) \rangle_t = \frac{k_R \exp(-2ik_a r)}{2} \left( \frac{1}{\omega} \{(|\alpha|^2 + |\beta|^2) \cos^2 \theta + \sin^2 \theta - (\alpha + \alpha^*) \cos \phi \right.$$  
$$+ (\beta + \beta^*) \sin \phi \sin \theta \cos \theta \} \text{Re} \left\{ \frac{k^*}{\mu_1^*} \right\} + \frac{1}{\omega} (\alpha \sin \phi - \beta \cos \phi)$$  
$$\times (\alpha^* \sin \phi - \beta^* \cos \phi) \sin^2 \theta \text{Re} \left\{ \frac{k^*}{\mu_3^*} \right\} + \sin \theta \text{Re} \left\{ (\alpha^* \sin \phi + \beta^* \cos \phi) \frac{\xi^*}{\mu_1^*} \right\}$$  
$$- (\alpha^* \beta + \alpha^* \beta) \cos \theta \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \right) |E_{0z}|^2.$$  

(44)

Hence, we have the following condition for birefringent NPV propagation:

$$k_R \left( \frac{1}{\omega} \{(|\alpha|^2 + |\beta|^2) \cos^2 \theta + \sin^2 \theta - (\alpha + \alpha^*) \cos \phi + (\beta + \beta^*) \sin \phi \sin \theta \cos \theta \} \text{Re} \left\{ \frac{k^*}{\mu_1^*} \right\} \right.$$  
$$+ \frac{1}{\omega} (\alpha \sin \phi - \beta \cos \phi) (\alpha^* \sin \phi - \beta^* \cos \phi) \sin^2 \theta \text{Re} \left\{ \frac{k^*}{\mu_3^*} \right\}$$  
$$+ \sin \theta \text{Re} \left\{ (\alpha^* \sin \phi + \beta^* \cos \phi) \frac{\xi^*}{\mu_1^*} \right\} - (\alpha^* \beta + \alpha^* \beta) \cos \theta \text{Re} \left\{ \frac{\xi^*}{\mu_1^*} \right\} \right) < 0.$$  

(45)
**Figure 5.** Maps of the wavenumbers (normalized with respect to $k_0$) projected onto the $\theta, \phi \in (0, \pi/2)$ surface of the unit sphere. Constitutive parameter values are as in figure 1, except that $\epsilon_1 = (-4.8 + i0.1)\epsilon_0$ and $\epsilon_3 = (-1.15 + i0.5)\epsilon_0$. (a) The real (top) and imaginary (bottom) parts of the wavenumber $k_a/k_0$. (b) The real (top) and imaginary (bottom) parts of the wavenumber $k_b/k_0$.

The quantities $\alpha$ and $\beta$ may be derived from the dyadic $L(ik\hat{u})$ as

$$\alpha = \frac{[L]_{12}[L]_{23} - [L]_{13}[L]_{22}}{[L]_{11}[L]_{22} - [L]_{12}[L]_{21}},$$  

(46)

$$\beta = \frac{[L]_{23}[L]_{11} - [L]_{13}[L]_{21}}{[L]_{12}[L]_{21} - [L]_{11}[L]_{22}}.$$

(47)
3. Numerical results

We explore the NPV inequality (45) by means of numerical examples. For illustrative results, let us consider the material specified by the constitutive parameter values

\[
\begin{align*}
\epsilon_1 &= (4.8 + i0.1)\epsilon_0, \\
\epsilon_3 &= (1.15 + i0.5)\epsilon_0, \\
\xi &= (0.15 + i4.55)\sqrt{\epsilon_0\mu_0}, \\
\mu_1 &= (1.35 + i0.7)\mu_0, \\
\mu_3 &= (3.2 + i0.12)\mu_0.
\end{align*}
\]

While these constitutive parameters are representative of a generic SMCRC material, we stress that the possibilities of fabricating artificial SMCRC materials offer the potential to realize a wide range of constitutive parameter values. The relatively large magnitudes of the magnetoelectric parameters compared with the magnitudes of the permittivity and permeability parameters are not uncommon within the realm of NPV propagation: achiral materials with zero permittivity and permeability [23, 24] have been generalized to isotropic chiral materials with high magnetoelectric parameters in relation to the permittivity and permeability [10, 11, 25].

In figure 1, the values of the two independent wavenumbers \(k_a\) and \(k_b\), as computed using (22)–(27), are mapped for propagation directions corresponding to \(0 < \theta < \pi/2\) and \(0 < \phi < \pi/2\). We note that the distribution of wavenumbers is symmetric with respect to the plane \(\phi = \pi/4\). The propagation directions which support NPV propagation are presented in figure 2. The NPV propagation directions are clearly confined to the equatorial regions. Furthermore, for this particular example, NPV only occurs for the planewave propagation mode specified by the wavenumber \(k_a\).

Further numerical investigations revealed that reducing the value of \(|\xi|\), relative to the magnitudes of \(\epsilon_{1,3}\) and \(\mu_{1,3}\), results in NPV propagation being impossible for all propagation modes.
Figure 7. Maps of the wavenumbers (normalized with respect to $k_0$) projected onto the $\theta, \phi \in (0, \pi/2)$ surface of the unit sphere. Constitutive parameter values are as in figure 1, except that $\epsilon_1 = (-4.8 + i0.1)\epsilon_0$, $\epsilon_3 = (-1.15 + i0.5)\epsilon_0$, $\mu_1 = (-1.35 + i0.7)\mu_0$, and $\mu_3 = (-3.2 + i0.12)\mu_0$. (a) The real (top) and imaginary (bottom) parts of the wavenumber $k_a/k_0$. (b) The real (top) and imaginary (bottom) parts of the wavenumber $k_b/k_0$.

The parameter regimes explored in figures 1–4 reflect the scenarios which arise in naturally-occurring SMCRC materials insofar as $\text{Re}\{\epsilon_{1,3}\} > 0$ and $\text{Re}\{\mu_{1,3}\} > 0$. With an eye to the possibility of fabricating artificial SMCRC materials, we now turn to more general parameter
regions. In figure 5, the wavenumbers corresponding to the constitutive parameters given in (48), but with \( \epsilon_1 = (-4.8 + i0.1) \epsilon_0 \) and \( \epsilon_3 = (-1.15 + i0.5) \epsilon_0 \) are presented. The distribution of wavenumbers in figure 5 is quite different from that in figures 1 and 3, albeit the symmetry plane at \( \phi = \pi/4 \) is a common feature. The corresponding distribution of NPV planewave modes is displayed in figure 6. We see that NPV propagation for the \( k_a \) mode occurs in both polar regions and in equatorial regions along the general directions of the \( x \) and \( y \) axes. Furthermore, there is a relatively large non-polar region, that includes equatorial regions along the general directions of the \( x \) and \( y \) axes, for which NPV is supported for both \( k_a \) and \( k_b \) modes.

Finally, we repeated the calculations of figures 5 and 6, but with \( \mu_1 = (-1.35 + i0.7) \mu_0 \) and \( \mu_3 = (-3.2 + i0.12) \mu_0 \). The corresponding wavenumber distributions and NPV distributions are given in figures 7 and 8. Clearly, NPV propagation is now widespread, with both \( k_a \) and \( k_b \) modes supporting NPV propagation for generally polar directions. Indeed, NPV propagation is supported for all propagation directions.

4. Concluding remarks

NPV conditions for both unirefringent and birefringent planewave propagation have been established for materials which exhibit SMCRC behaviour. For scenarios wherein \( \text{Re}\{\epsilon_{1,3}\} > 0 \) and \( \text{Re}\{\mu_{1,3}\} > 0 \)—which correspond to the constitutive parameter regimes observed in naturally occurring SMCRC minerals—we observe that NPV propagation arises only if \( |\xi| \) is sufficiently large compared to \( |\epsilon_{1,3}| \) and \( |\mu_{1,3}| \). These findings are consistent with those reported for NPV studies in isotropic chiral mediums [10] and Faraday chiral mediums [8]. By allowing for the possibility of \( \text{Re}\{\epsilon_{1,3}\} < 0 \) and \( \text{Re}\{\mu_{1,3}\} < 0 \), we find that occurrence of NPV propagation—in terms of propagation modes and their directions—becomes much more widespread. Thus, the potential for NPV propagation in artificially fabricated SMCRC materials is emphasized.
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