Parametrised Homotopy Theory and Gauge Enhancement

LMS/EPSRC Durham Symposium on Higher Structures in M-Theory

Vincent Braunack-Mayer

We review a first-principles derivation of Type IIA D-brane charges from M-theory degrees of freedom in the approximation of super rational homotopy theory.

In the article, we apply two universal constructions in super rational homotopy theory in order to solve the problem of gauge enhancement of super M-branes (recalled below). We prove that the topological charge structure of fundamental super M-branes produces twisted K-theory charges of Type II A D-branes under double dimensional reduction when all torsion effects are ignored. Our result is based upon Sati’s conjecture that the topological reduction when all torsion effects are ignored reduces twisted K-theory charges of Type II A D-branes under double dimensional reduction when all torsion effects are ignored.

1. Super p-Branes and Cocycles

We recall that a fundamental (or Green–Schwarz) p-brane propagating in a spacetime X is described by a (p + 1)-dimensional Green–Schwarz sigma model defined on the space of maps

\[ \Sigma_{p+1} \rightarrow X. \]  

(1)

In this schematic, \( \Sigma_{p+1} \) is the abstract worldvolume of the p-brane; a (p + 1)-manifold encoding a p-dimensional extended object dynamically evolving in time. For fundamental super p-branes propagating on supergravity backgrounds, the worldvolume \( \Sigma_{p+1} \) is taken to be a (p + 1)-dimensional supermanifold. Compatibility with local supersymmetry on the background X enforces a strict topological constraint: the super volume form on \( \Sigma_{p+1} \) must locally trivialise a non-trivial (p + 2)-cocycle \( \mu_{p+2} \) in the supersymmetry super Lie algebra cohomology of super-spacetime. In this manner, fundamental super p-branes are in one-to-one correspondence with such super (p + 2)-cocycles. Analysing such Green–Schwarz sigma models from the point of view of their controlling super Lie algebra cocycles, hence from the point of view of super rational homotopy theory, provides a powerful mathematical toolbox for elucidating many previously elusive aspects of M-theory.

In the case of \( d = 11, \mathcal{N} = 1 \) supergravity, the local structure of super-spacetime is controlled by the supertranslation super Lie algebra \( \mathbb{R}^{10,1\vert 32} \) (a classical treatment using different terminology is in [12]). The even piece of this super Lie algebra is spanned by the standard basis \( \{ \epsilon_i \}_{i=0}^{10} \) of Minkowski space \( \mathbb{R}^{10,1} \), the odd piece is determined by the irreducible real \( \text{Spin}(10,1) \)-representation 32, and the only non-trivial component of the Lie bracket is the odd-odd superbracket

\[ 32 \otimes 32 \rightarrow \mathbb{R}^{10,1}, \]

\[ (\psi, \phi) \mapsto (\overline{\psi} \Gamma^a \phi)\epsilon_a, \]  

(2a)

(2b)
determined by the \( \text{Spin}(10,1) \)-invariant spinor-to-vector pairing (here, as throughout this article, summation over repeated indices is understood). The corresponding Chevalley–Eilenberg algebra \( CE(\mathbb{R}^{10,1\vert 32}) \) is the super differential graded algebra with underlying algebra structure free on the even degree-1 generators \( \{ \epsilon_i \}_{i=0}^{10} \) and odd degree-1 generators \( \{ \psi^a \}_{a=1}^{32} \), corresponding to dual bases of \( \mathbb{R}^{10,1} \) and 32 respectively. The differential \( d \) on \( CE(\mathbb{R}^{10,1\vert 32}) \) is obtained by dualising the super Lie bracket; it is defined on generators by

\[ d: \left\{ \begin{array}{ll}
\epsilon_a & \mapsto -\overline{\psi} \Gamma^a \psi \\
\psi^a & \mapsto 0
\end{array} \right. \]  

(3)

and extended to all of \( CE(\mathbb{R}^{10,1\vert 32}) \) as a graded derivation.

Under the correspondence between cocycles and Green–Schwarz sigma models recalled above, the fundamental M2 and M5-branes correspond respectively to the elements

\[ \mu_{M2} = \frac{i}{2} \overline{\psi} \Gamma_{a_1 a_2} \psi \wedge \epsilon^{a_1} \wedge \epsilon^{a_2}, \]  

(4a)

\[ \mu_{M5} = \frac{1}{3!} \overline{\psi} \Gamma_{a_1 \ldots a_5} \psi \wedge \epsilon^{a_1} \wedge \ldots \wedge \epsilon^{a_5} \]  

(4b)

in the algebra \( CE(\mathbb{R}^{10,1\vert 32}) \). The exceptional Fierz identities for Spin(10, 1) imply the relation

\[ d\mu_{M5} = -\frac{1}{2} \mu_{M2} \wedge \mu_{M2}. \]  

(5)
fundamental branes
in M-theory

\[ \xrightarrow{\text{double dimensional reduction}} \]

fundamental branes
in IIA string theory

Figure 1. A schematic of the process of double dimensional reduction, producing Type IIA branes from M-branes.

In these algebraic terms, Sati’s conjecture on M-brane charge is reflected in the observation that \( \mu_{M2} \) and \( \mu_{M5} \) combine, via (5), to define a map of (super) commutative differential graded algebras

\[
\text{CE}(\mathbb{R}^{10,1}/M2) \xrightarrow{\mu_{M2}/M5} S^1, \tag{6}
\]

where \( S^1 \) is the minimal model of the rational homotopy type of the sphere \( S^1 \) (see [13] for a survey of rational homotopy theory). Viewed through the lens of rational homotopy theory, the combined cocycle \( \mu_{M2}/M5 \) thus encodes the torsion-free part of a map of (super) homotopy types

\[
\mathbb{R}^{10,1}/M2 \xrightarrow{\mu_{M2}/M5} S^1, \tag{7}
\]

controlling the local cohomological charge structure of fundamental M2 and M5-branes propagating in 11-dimensional supergravity.

2. Gauge Enhancement

2.1. Double Dimensional Reduction…

Insofar as it exists, M-theory is supposed to be the joint non-perturbative completion of perturbative string theory and 11-dimensional supergravity. In particular, a complete M-theory must reproduce all of the known structure of perturbative string theory as we pass to various limits. For instance, dimensional reduction of 11-dimensional supergravity along an \( S^1 \)-fibration yields 10-dimensional Type IIA supergravity equipped with fields corresponding to the Fourier modes in the \( S^1 \)-fibre. In the presence of M2 and M5-branes, the brane worldvolumes either extend along (“wrap”) the \( S^1 \)-fibres or they do not. If, say, the M2-brane worldvolume wraps \( S^1 \)-fibres, then it produces a fundamental string in 10-dimensional Type IIA supergravity; whereas if it does not wrap it produces a D2-brane. This is the process of double dimensional reduction; the dimension of the ambient super-spacetime as well as possibly the brane worldvolume is reduced by one. Figure 1 is a schematic of this process.

According to this picture, under double dimensional reduction M-theory readily produces the fundamental string F1, the NS5-brane and the D2 and D4-branes of Type IIA string theory (whereas the D0-brane is also obtained as the Chern class which classifies the 11-dimensional background as an \( S^1 \)-fibration over the 10-dimensional background). We emphasise that this schematic is a heuristic only; although there are many hints for this double dimensional reduction procedure in the literature,[14–17] there is no derivation beyond the rational approximation.[5,6]

The problem of gauge enhancement of super M-branes appears within this context of double dimensional reduction. Perturbative Type IIA string theory admits fundamental \( p \)-branes in dimensions \( p = 0, 1, 2, 4, 5, 6, 8 \); namely the D0, F1, D2, D4, NS5, D6, and D8-branes respectively. Moreover, the Dp-branes are widely expected to carry a combined charge in K-theory twisted by the B-field carried by the fundamental string.[18–22] However, the double dimensional reduction schematic (Figure 1) makes no mention of the missing D6 and D8 branes, nor of the combined D-brane charge in twisted K-theory. The gauge enhancement problem is therefore:

How does M-theory give rise to the fundamental D6 and D8 branes, and how does it exhibit the combined D-brane charge in twisted K-theory?

In [1] we provide a full answer to this question in the torsion-free approximation, using new techniques in super rational homotopy theory from [23]. To sketch this solution we first observe that, similarly to our previous discussion, the Green–Schwarz sigma models for fundamental \( p \)-branes in Type IIA string theory are controlled by elements in the Chevalley–Eilenberg complex \( \text{CE}(\mathbb{R}^{9,1}/M) \) of the \( d = 10, \mathcal{N} = (1, 1) \) supertranslation super Lie algebra. Explicitly, these elements are:

\[
\mu_{F1} = i(\bar{\psi}\gamma_a \Gamma_{10} \psi) \wedge e^a, \tag{8a}
\]

\[
\mu_{D0} = \bar{\psi} \Gamma_{10} \psi, \tag{8b}
\]

\[
\mu_{D2} = \frac{i}{2} (\bar{\psi} \Gamma_{a_1 a_2} \psi) \wedge e^{a_1} \wedge e^{a_2}, \tag{8c}
\]

\[
\mu_{D4} = \frac{1}{4!} (\bar{\psi} \Gamma_{a_1 \ldots a_4} \Gamma_{10} \psi) \wedge e^{a_1} \wedge \ldots \wedge e^{a_4}, \tag{8d}
\]

\[
\mu_{NS5} = \frac{1}{3!} (\bar{\psi} \Gamma_{a_1 \ldots a_5} \psi) \wedge e^{a_1} \wedge \ldots \wedge e^{a_5}, \tag{8e}
\]

\[
\mu_{D6} = \frac{i}{6!} (\bar{\psi} \Gamma_{a_1 \ldots a_6} \psi) \wedge e^{a_1} \wedge \ldots \wedge e^{a_6}, \tag{8f}
\]
\[
\mu_{\text{D}_8} = \frac{1}{8!} (\Psi \Gamma_{a_1} \cdots a_4 \Gamma_{10} \Psi) \wedge e^{a_1} \wedge \ldots \wedge e^{a_6},
\]

controlling the fundamental string, D0, D2, D5, NS5, D6 and D8-branes respectively. These elements satisfy various compatibility conditions amongst themselves:

\[
\begin{align*}
d\mu_{\text{F}_1} &= 0, \\
d\mu_0 &= 0, \\
d\mu_{\text{D}(2p+2)} &= \mu_{\text{F}_1} + \mu_{\text{D}(2p)}, \quad p \in \{0, 1, 2, 3\}, \\
d\mu_{\text{NS5}} &= \mu_0 + \frac{1}{2} \mu_{\text{D}_2} + \mu_{\text{D}_2}.
\end{align*}
\]

In particular, the F1 and D\(p\)-brane cocycles together define a cocycle in the rational image of twisted K-theory.

The first universal construction appearing in our solution of the gauge enhancement problem is an adjunction

\[
\text{Spaces}/B^S \overset{\perp}{\longrightarrow} \text{Spaces} \xrightarrow{\text{Cyc}} \text{Ext}
\]

implementing double dimensional reduction. Here, the left adjoint Ext sends a map of spaces \(\tau = (X \rightarrow B^S)\) to the total space of the \(S^1\)-bundle on \(X\) classified by \(\tau\) (hence the extension of \(X\) by the cocycle \(\tau\)). The right adjoint Cyc sends a space \(Y\) to the (homotopy) quotient of the free loop space of \(Y\) by the action of rigid rotations of loops: \(\text{Cyc}(Y) = [S^1, Y]//S^1\). We implement a version of this adjunction in super rational homotopy theory, where one finds that the \(d = 11, A = 1\) supertranslation super Lie algebra is the central extension of the \(d = 10, A = (1, 1)\) supertranslation super Lie algebra by the D0-brane 2-cocycle:

\[
\mathbb{R}^{10,1}\times \mathbb{S} \cong \text{Ext}(\mu_0 : \mathbb{R}^{9,1}\times \mathbb{R} \rightarrow B^S)
\]

(11)

(this is an algebraic shadow of \(D0\)-brane condensation). Taking the combined M2/5-brane cocycle, we apply the Cyc functor and compose with the unit of the \((\text{Ext} \dashv \text{Cyc})\)-adjunction to obtain a diagram of super rational homotopy types fibred over \(B^S\):

\[
\begin{array}{ccc}
\mathbb{R}^{9,1}\times \mathbb{R} & \xrightarrow{\mu_{\text{M}_2/\text{M}_5}} & \text{Cyc}(B^S) \\
\xrightarrow{\text{Cyc}(\mu_{\text{M}_2/\text{M}_5})} & & \xrightarrow{\text{Cyc}(\text{S}^4)}
\end{array}
\]

By direct calculation, we see that the cocycle \(\mu_{\text{M}_2/\text{M}_5}\) so obtained reproduces precisely the F1, D2, D4 and NS5-brane cocycles. The D0-brane cocycle is encoded in the map \(\mathbb{R}^{9,1}\times \mathbb{R} \rightarrow B^S\), hence as the rational image of the Chern character.

### 2.2. ...and Gauge Enhancement

In order to complete this picture and produce the D6 and D8-brane cocycles, two additional points are in order. Firstly, recent results\(^{[10]}\) provide an equivariant enhancement of the combined M2/5S-brane cocycle at ADE subgroups of \(SU(2)\), establishing a homotopy-theoretic underpinning for the black brane scan. The original setting of gauge enhancement of Chan–Paton factors at \(Z_n\)-orbifold points\(^{[24]}\) corresponds to equivariant enhancement of the \(\mu_{\text{M}_2/\text{M}_5}\) cocycle at A-series subgroups of \(SU(2)\). Considering A-series actions in the limit as \(n \rightarrow \infty\), the \(A_n \sim Z_{n+1}\)-actions exhaust an \(S^1\)-action, so that we naturally led to expect the existence of an \(S^1\)-equivariant cocycle such as the dotted arrow in the diagram:

\[
\begin{array}{ccc}
\mathbb{R}^{10,1}\times \mathbb{S} & \xrightarrow{\mu_{\text{M}_2/\text{M}_5}} & \text{S}^4 \\
\xrightarrow{\text{Cyc}(\mu_{\text{M}_2/\text{M}_5})} & & \xrightarrow{\text{Cyc}(\text{S}^4)}
\end{array}
\]

However, the existence of such an \(S^1\)-equivariant cocycle is obstructed by nonvanishing of the D4-brane cocycle.

This leads us to our second main point: homotopical perturbation theory. Homotopy theory is extremely rich but very computationally demanding. There is a tower of increasingly accurate approximations to full homotopy theory provided by the Goodwillie calculus of functors (see [25] for a review), which is analogous to analysing smooth functions via their Taylor series expansions. The first-order approximation of a space \(X\) is \(\Omega^\infty \Sigma^\infty X\), the underlying space of the free infinite loop space on \(X\). This homotopical linearisation assignment \(X \mapsto \Omega^\infty \Sigma^\infty X\) is analogous to “linearising” a set \(S\) by passing to the free abelian group \(\mathbb{Z}/S\) and then forgetting the group structure. There is a similar homotopical linearisation procedure in the relative setting of fibred spaces, where we have an assignment

\[
(Y \rightarrow X) \mapsto (\Omega^\infty \Sigma^\infty Y \rightarrow X)
\]

(14)

that applies homotopical linearisation fibrewise. Working modulo torsion, the main result of the author’s thesis\(^{[21]}\) provides algebraic models for parametrised stable rational homotopy types. In particular, this allows for straightforward algebraic calculations of homotopical linearisations in the torsion-free approximation.

Returning to the gauge enhancement problem, we obtain a solution by passing to homotopical perturbation theory. We had previously found that passing to the limit over A-series actions does not descend to an \(S^1\)-equivariant cocycle in \(d = 10, A = (1, 1)\) supergravity. However, there is a natural comparison map

\[
S^3 / S^1 \rightarrow \text{Cyc}(S^3)
\]

(15)

that fibres rationally over \(B^2 U(1)\). Passing to fibrewise homotopical linearisations now yields a diagram of super rational homotopy types as in Figure 2 and we can ask if the dotted arrow in that diagram exists. This is equivalent to asking whether \(\mu_{\text{M}_2/\text{M}_5}\) descends to an \(S^1\)-equivariant cocycle up to first order in homo-
topical perturbation theory. Using the algebraic models of [23], we find that

i) there exists a morphism
\[ \tilde{\mu}_{M2/M5} : \mathbb{R}^{0,1|16+16} \rightarrow \Omega^\infty_{B^2 U(1)} \Sigma^\infty_{B^2 U(1)} S^4 \cup S^1 \] (16)
rendering the diagram of Figure 2 commutative;

ii) the rational homotopy type of
\[ \Omega^\infty_{B^2 U(1)} \Sigma^\infty_{B^2 U(1)} S^4 \cup S^1 \] (17)
contains the rational homotopy type of twisted connective K-theory
\[ (\Omega^\infty_{B^2 U(1)} ku) \cup B U(1) \] (18)
as a factor; and

iii) the extended cocycle \( \tilde{\mu}_{M2/M5} \) hits precisely this twisted K-theory factor, and is moreover unique up to homotopy for this specification.

Put differently, by passing to the first-order approximation in the Goodwillie calculus of functors we obtain an unique-up-to-homotopy extension \( \tilde{\mu}_{M2/M5} \) of the cocycle \( \mu_{M2/M5} \) of (12) which

i) recovers the Type IIA D0, D2, F1, D4, D6 and D8-brane cocycles; and

ii) exhibits the combined twisted K-theory charge carried by Type IIA D-branes in the torsion-free approximation.

This is the main result of [1]: the full derivation of gauge enhancement in M-theory, proceeding from M-brane charges in rational cohomotopy.

3. Conclusion

We conclude with some comments on how our result fits into the broader M-theory literature. To begin with, there are all manner of twisted spectra which look like twisted K-theory in the torsion-free approximation of rational homotopy theory. Our result only applies to this rational approximation, so we have not obtained a derivation of twisted K-theory coefficients from M-theory. Indeed, despite widely held beliefs within the string theory community, there is as of yet no definitive mathematically rigorous argument for twisted K-theory being the right coefficients for D-brane charge: the purported derivation of [26] is rather a consistency check on the sign of the partition function over K-theory fields—many other (twisted) cohomology theories could potentially yield the same sign rule; the arguments of [18] are differential form computations, hence cannot capture any torsion information; finally, there are some persistent conceptual issues with the proposal that D-brane charge is classified by twisted K-theory (see for example [27,28]).

The main result of [1] fills an important gap in the literature by providing the first full mathematical derivation of rational D-brane charge from M-theory degrees of freedom. Additionally, since we argue using universal constructions in homotopy theory, this derivation already contains hints to its own completion beyond the rational approximation.

Acknowledgements

V. B.-M. acknowledges the partial support of the RTG 1670 ‘Mathematics inspired by String Theory and Quantum Field Theory’.

Conflict of Interest

The author has declared no conflict of interest.

Keywords

double dimensional reduction, gauge enhancement, Green–Schwarz sigma models, M-branes, parametrised homotopy theory, rational homotopy theory

Received: November 30, 2018
Published online: June 14, 2019

[1] V. Braunack-Mayer, H. Sati, U. Schreiber, Comm. Math. Phys. 2018, 1806.01115 [hep-th], https://www.arxiv.org/abs/1806.01115.

[2] H. Sati, J. Math. Phys. 2018, 59, 062304 [1310.1060 [hep-th]], https://doi.org/10.1063/1.5007185, https://www.arxiv.org/abs/1310.1060.
