The Partially Truncated Euler–Maruyama Method for super-linear Stochastic Delay Differential Equations with variable delay and Markovian switching

Yuhao Cong\textsuperscript{a}  Weijun Zhan\textsuperscript{a}\textsuperscript{*}
\textsuperscript{a}Department of Mathematics, Shanghai University, Shanghai 200444, China

Abstract

A class of super-linear stochastic delay differential equations (SDDEs) with variable delay and Markovian switching is considered. The main aim of this paper is to develop the partially truncated Euler–Maruyama (EM) method for the super-linear SDDEs with variable delay and Markovian switching, and investigate the convergence and stability properties of the numerical solution under the generalized Khasminskii-type condition.

Key words: partially truncated Euler–Maruyama, stochastic delay differential equations, variable delay, Markovian switching, super-linear.

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1 Introduction

Systems in many branches of science and industry do not only depend on the present state but also the past ones. Therefore, stochastic delay differential equations with constant delay have been widely used to model such systems \cite{14, 20, 24}. However, it is more realistic for the SDDEs models include varying-time. On the other hand, these systems may often experience abrupt changes by their structure and parameters and continuous-time Markovian chains. Then, these abrupt be used to stochastic differential equations (SDEs) \cite{15, 22, 23}. Hence SDDEs with variable delay and Markovian switching has more meaningful for practice. The sufficient conditions of existence and uniqueness theorems of the exact solution for SDDEs with Markovian were given under the local Lipschitz condition in \cite{12, 15}. Without the local Lipschitz condition and the linear growth condition, Wang et al. \cite{30} investigated the existence and uniqueness theorem of solutions to the nonlinear SDDEs with Markovian switching. These new condition show that the coefficients are polynomial or controlled by polynomial functions.

Since in general both the explicit solutions or the probability distribution of solutions are not known, approximation solution for SDDEs have recently received a great deal of attention \cite{2, 3, 17, 21, 25, 29}. There are many literatures concerned with numerical approximation to SDDEs with Markovian switching, we just mention a few of them. On the one hand, under the global Lipschitz condition or local Lipschitz condition plus linear growth condition, Mao and Yuan \cite{31} proved...
that Euler-Maruyama can convergence the true solution, as well as in [11] presented the EM can preserver the convergence for the linear SDDEs with Markovian switching. Li et al. [12] designed the EM can presever the convergence under the local Lipschitz conditions and Lyapunov-type condition. Further, Wang et al. [1] investigated the numerical solutions of SDDEs with Markovian switching under the drift and diffusion coefficients are Taylor approximations. On the other hand, some other methods has been considered to deal with the SDDEs with Markovian switching. For example, when the linear SDDEs, Liu et al. [10] designed semi-implicit Euler methods for linear SDDEs. Further, when the drift coefficient obey the onesided-Lipschitz condition and polynomial growth condition, and the diffusion coefficient is polynomial growth, convergence in probability of BEM solution was proved in [35]. Moreover, under the non-uniform Lipschitch and non-linear growth, Yeol et al. [5, 9] investigated the p-th moment boundeds of carathodory approxiamtion.

Meanwhile, stability theory of numerical solution is one of central problems in numerical analysis. Mao et al. [26, 32] showed that exponential stability of the exact solution with local Lipschitz and linear growth condition. For the nonlinear SDDEs with Markovian switching, Mao [13] given the asymptotic stability under some specified condition. Further, Zhu et al. [34] investigated some novel sufficient condition to p-th moment exponential stability by using Lapunov function and generalized Halanay inequality.

To our knowledge, there is few literature concerned with the explicit method applied to super-linear SDDEs with Markovian switching in strong sense. The main aim of this article is to provide a, to the best of our knowledge, first contribution towards this goal. Recently, Mao [18, 19] developed a new explicit numerical method, called the truncated EM method, for SDEs under the Khasminskii-type condition plus the local Lipschitz condition and established the strong convergence theory. Guo et al. [7] modified the truncated EM by using a partially truncated technique so that the numerical solution can preserve the mean square exponential stability and asymptotic boundedness of analytical solution to the underlying SDEs. Moreover, Guo et al. [8] and Zhang et al. [33] discussed the convergence of SDDE with constant delay by truncated EM and partially truncated EM under the Khaminskii-type condition, respectively. However, they did not concerned with the SDDEs with variable time and Markovian switching, and investigates the convergence rate and stability. In this paper, we will develop the partially truncated EM method for the super-linear SDDEs with variable time and Markovian switching under the generalized Khasminskii-type condition, meanwhile we give the strong convergence rate and almost surely stability of the numerical solution.

This paper is organized as follows: we will introduce necessary notion, state the generalized Khasminskii-type condition and define the partially truncated EM numerical solution for SDDEs with variable delay and Markovian switching in section 2. We will establish the strong convergence theory and convergence rate, meanwhile illustrate them theory by examples in sections 3. In section 4, we will point out the almost sure exponential stability theory and give example.

## 2 The Partially Truncated Euler-Maruyama Method

Throughout this paper, unless otherwise specified, we use the following notation. Let $|·|$ be the Euclidean norm in $\mathbb{R}^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\mathbb{R}_+ = [0, \infty)$ and $\tau > 0$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Moreover, for two real numbers $a$ and $b$, we use $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$. If $G$ is a set, its indicator function
is denoted by $I_G$, namely $I_G(x) = 1$ if $x \in G$ and 0 otherwise. If $a$ is a real number, we denote by $[a]$ the largest integer which is less or equal to $a$, e.g., $[-1.2] = -2$ and $[2.3] = 2$.

Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N \times N}$ gives by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij} + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ij} + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while

$$\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}.$$ We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$.

Consider a nonlinear stochastic differential equation with variable delay and Markovian switching of the form

$$dx(t) = f(x(t), x(t - \delta(t)), r(t))dt + g(x(t), x(t - \delta(t)), r(t))dB(t), \quad t \geq 0, \quad (2.1)$$

$$x(t) = \xi(t), \quad t \in [-\delta, 0] \quad (2.2)$$

with the initial conditions $x(0) = x_0 \in S$. Here $f: \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n \times m$ are measurable mapping, $\delta(t): [0, \infty) \to [0, \delta]$ is a Borel measurable function. We assume that the coefficients $f$ and $g$ can be decomposed as

$$f(x, y, i) = F_1(x, y, i) + F(x, y, i) \quad \text{and} \quad g(x, y, i) = G_1(x, y, i) + G(x, y, i).$$

Moreover, let $C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x, i)$ on $\mathbb{R}^n \times S$ which are continuously twice differentiable in $x$. For each $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$, define an operator $LV$ from $C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$ to $\mathbb{R}$ by

$$LV(x, y, i) = V_x(x, i)f(x, y, i) + \frac{1}{2}\text{trace}[g^T(x, y, i)V_{xx}(x, i)g(x, y, i)] + \sum_{j=1}^{N} \gamma_{ij}V(x, j)$$

where

$$V_x(x, i) = \left(\frac{\partial V(x, i)}{\partial x_1}, \ldots, \frac{\partial V(x, i)}{\partial x_n}\right), \quad V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x, i)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

The discrete Markovian chain $\{r^\Delta_k, k = 0, 1, 2, \ldots\}$ can be simulated as follows: Let $r^\Delta_0 = i_0$ and generate a random number $\xi_1$ which is uniformly distributed in $[0, 1]$. Define

$$r^\Delta_1 = \begin{cases} i_1 & \text{if } i_1 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\
N & \text{if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{i=1}^{0} P_{i0,j}(\Delta) = 0$ as usual. Generate independently a new random number $\xi_2$ which is again uniformly distributed in $[0, 1]$ and then define

$$r^\Delta_2 = \begin{cases} i_2 & \text{if } i_2 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} P_{i0,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} P_{i0,j}(\Delta), \\
N & \text{if } \sum_{j=1}^{N-1} P_{i0,j}(\Delta) \leq \xi_2. \end{cases}$$
After explaining how to simulate the discrete Markovian Chain, we can now define the partially truncated EM numerical solutions. We first choose a strictly increasing continuous function $x/\mu$ where we set $x, y$ for a continuous-time step process.

For a given step size $\Delta \in \mathbb{R}_+$ we also choose a constant $\Delta^* \in (0, 1]$ and a strictly decreasing function $h : (0, \Delta^*) \to (0, \infty)$ such that

$$h(\Delta^*) \geq \mu(1), \quad \lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \leq 1, \quad \forall \Delta \in (0, \Delta^*].$$

For a given step size $\Delta \in (0, \Delta^*], let us define a mapping $\pi_\Delta$ from $\mathbb{R}^n$ to the closed ball $\{ x \in \mathbb{R}^n : |x| \leq \mu^{-1}(h(\Delta)) \}$ by

$$\pi_\Delta(x) = (|x| \wedge \mu^{-1}(h(\Delta))) \frac{x}{|x|},$$

where we set $x/|x| = 0$ when $x = 0$. That is, $\pi_\Delta$ will map $x$ to itself when $|x| \leq \mu^{-1}(h(\Delta))$ and to $\mu^{-1}(h(\Delta))x/|x|$ when $|x| > \mu^{-1}(h(\Delta))$. We then define the truncated functions

$$F_\Delta(x, y, i) = F(\pi_\Delta(x), \pi_\Delta(y), i) \quad \text{and} \quad G_\Delta(x, y, i) = G(\pi_\Delta(x), \pi_\Delta(y), i)$$

for $x, y \in \mathbb{R}^n$. It is easy to see that

$$|F_\Delta(x, y, i)| \leq |G_\Delta(x, y, i)| \leq \mu(1) h(\Delta), \quad \forall x, y \in \mathbb{R}^n. \quad (2.5)$$

That is, both truncated functions $F_\Delta$ and $G_\Delta$ are bounded although $F$ and $G$ may not.

Let us now form the discrete-time partially truncated EM solutions. Define $t_k = k\Delta$ for $k = -M, -(M-1), \cdots, 0, 1, 2, \cdots$, where $M = \lfloor \delta(k\Delta)/\Delta \rfloor + 1$. Set $X_\Delta(t_k) = \xi(t_k)$ for $k = -M, -(M-1), \cdots, 0$ and then form

$$\begin{align*}
X_{k+1} - X_k &= f_\Delta(X_k, X_k - \lfloor \delta(k\Delta)/\Delta \rfloor, r_k)\Delta + G_\Delta(X_k, X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}, r_k) DB_k, \\
&= \left[F_1(X_k, X_k - \lfloor \delta(k\Delta)/\Delta \rfloor, r_k) + F_\Delta(X_k, X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}, r_k)\right] \Delta \\
&\quad + \left[G_1(X_k, X_k - \lfloor \delta(k\Delta)/\Delta \rfloor, r_k) + G_\Delta(X_k, X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}, r_k)\right] DB_k.
\end{align*}$$

For $k = 0, 1, 2, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. In our analysis, it is more convenient to work on the continuous-time approximations. There are two continuous-time versions. One is the continuous-time step process $z_1(t), z_2(t)$ and $\bar{r}(t)$ on $t \in [-\delta, \infty)$ defined by

$$z_1(t) = \sum_{k=0}^{\infty} X_k I_{[k\Delta, (k+1)\Delta)}(t), \quad z_2(t) = \sum_{k=0}^{\infty} X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor} I_{[k\Delta, (k+1)\Delta)}(t),$$

$$\bar{r}(t) = \sum_{k=0}^{\infty} r_k I_{[k\Delta, (k+1)\Delta)}(t).$$

The other one is the continuous-time continuous process $x_\Delta(t)$ on $t \in [-\delta, \infty)$ defined by $x_\Delta(t) = \xi(t)$ for $t \in [-\delta, 0)$ while for $t \geq 0$

$$\begin{align*}
x_\Delta(t) &= \xi(0) + \int_0^t [F_1(z_1(s), z_2(s), \bar{r}(s)) + F_\Delta(z_1(s), z_2(s), \bar{r}(s))] ds \\
&\quad + \int_0^t [G_1(z_1(s), z_2(s), \bar{r}(s)) + G_\Delta(z_1(s), z_2(s), \bar{r}(s))] dB(s). \quad (2.7)
\end{align*}$$
We see that \(x_\Delta(t)\) is an Itô process on \(t \geq 0\) with its Itô differential

\[
dx_\Delta(t) = [F_1(z_1(t), z_2(t), \bar{r}(t)) + F_\Delta(z_1(t), z_2(t), \bar{r}(t))]dt + [G_1(z_1(t), z_2(t), \bar{r}(t)) + G_\Delta(z_1(t), z_2(t), \bar{r}(t))]dB(t). \tag{2.8}
\]

It is useful to know that \(X_k = x_\Delta(k\Delta)\) for every \(k \geq -M\), namely they coincide at \(k\Delta\).

To analyze the partially truncated Euler-Maruyama method as well as to simulate the approximate solution, we will need the following lemma (see [1]). And we impose those standing hypotheses.

**Lemma 2.1** Given \(\Delta > 0\) let \(r^\Delta_k = r(k\Delta)\) for \(k \geq 0\), Then \(\{r^\Delta_k, k = 0, 1, 2, \ldots\}\) is a discrete Markov chain with the one-step transition probability matrix

\[
P(\Delta) = (P_{\bar{ij}}(\Delta))_{N \times N} = e^{\Delta \Gamma}.
\]

**Assumption 2.2** There are constant \(K > 0\) and \(\rho \geq 0\) such that

\[
|F_1(x, y, i) - F_1(\bar{x}, \bar{y}, i)|^2 \vee |G_1(x, y, i) - G_1(\bar{x}, \bar{y}, i)|^2 \leq K_1(|x - \bar{x}|^2 + |y - \bar{y}|^2) \tag{2.9}
\]

and

\[
|F(x, y, i) - F(\bar{x}, \bar{y}, i)|^2 \vee |G(x, y, i) - G(\bar{x}, \bar{y}, i)|^2 \leq K_1(1 + |x|^\rho + |y|^\rho + |\bar{x}|^\rho + |\bar{y}|^\rho)(|x - \bar{x}|^2 + |y - \bar{y}|^2) \tag{2.10}
\]

for those \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^n\) and \(\forall i \in S\).

We can derive from (2.9) that the coefficients \(F_1\) and \(G_1\) satisfy the linear growth condition that there exists a constant \(K_1\) such that

\[
|F_1(x, y, i)| \vee |G_1(x, y, i)| \leq K_1(1 + |x| + |y|) \tag{2.11}
\]

for all \((x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S\).

We can derive from Assumption 2.2 that \(f\) and \(g\) also satisfy

\[
|f(x, y, i) - f(\bar{x}, \bar{y}, i)|^2 \vee |g(x, y, i) - g(\bar{x}, \bar{y}, i)|^2 \leq K_1(1 + |x|^\rho + |y|^\rho + |\bar{x}|^\rho + |\bar{y}|^\rho)(|x - \bar{x}|^2 + |y - \bar{y}|^2) \tag{2.12}
\]

for those \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^n\) and \(\forall i \in S\).

**Assumption 2.3** There is a pair of constants \(\bar{p} \geq 2\) and \(K_2 > 0\) such that

\[
x^TF(x, y, i) + \frac{\bar{p} - 1}{2}|G(x, y, i)|^2 \leq K_2(1 + |x|^2 + |y|^2) \tag{2.13}
\]

for all \((x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S\).

The following Lemma show that the truncated functions \(F_\Delta\) and \(G_\Delta\) preserve the generalized Khasminskii-type condition for any \(\Delta \in (0, \Delta^*]\) as shown Lemma 4.2 in [3] and we state it here as a Lemma for the use of this paper.

**Lemma 2.4** Let Assumption 2.3 hold. Then, for every \(\Delta \in (0, \Delta^*]\) and \(\bar{p} \geq 2\), we have

\[
x^TF_\Delta(x, y, i) + \frac{\bar{p} - 1}{2}|G_\Delta(x, y, i)|^2 \leq 2K_2(1 + |x|^2 + |y|^2) \tag{2.14}
\]

for all \((x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S\).
Following a very similar approach used for (2.8) in [7], we can show the following Lemma.

**Lemma 2.5** Let Assumptions 2.2 and 2.3 hold. Then, for every $\Delta \in (0, \Delta^*)$, we can show that for any $p \in [2, \bar{p})$

$$x^T f(x, y, i) + \frac{p - 1}{2} |g(x, y, i)|^2 \leq K_3 (1 + |x|^2 + |y|^2)$$

for all $(x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$. where

$$K_3 = 2K_1 + K_2 + \frac{K_2^2 (p - 1)(\bar{p} - 1)}{2(\bar{p} - p)}.$$

In the same way as lemma 2.5 was proved, we can show that for any $p \in [2, \bar{p})$,

$$x^T f_\Delta(x, y, i) + \frac{p - 1}{2} |g_\Delta(x, y, i)|^2 \leq K_4 (1 + |x|^2 + |y|^2)$$

(2.15)

for all $x, y \in \mathbb{R}^n$, where

$$K_4 = 2K_1 + 2K_2 + \frac{K_2^2 (p - 1)(\bar{p} - 1)}{2(\bar{p} - p)}.$$

### 3 Convergence

We can therefore state a known result (see [15]) as a lemma for the use of this paper.

**Lemma 3.1** Let Assumptions 2.2 and 2.3 hold. Then for any given initial data (2.2), there is a unique global solution $x(t)$ to equation (2.1) on $t \in [-\delta, \infty)$. Moreover, the solution has the property that

$$\mathbb{E} |x(t)|^2 < \infty, \quad \forall t > 0.$$

The following Lemma gives an upper bound, independent of $\Delta$, for the $p$-th moment.

**Lemma 3.2** Let Assumptions 2.2 and 2.3 hold. Then for any $p \in [2, \bar{p})$, we have

$$\sup_{0 < \Delta \leq \Delta^*} \sup_{0 \leq t \leq T} \mathbb{E} |x_\Delta(t)|^p \leq C,$$

(3.1)

where, and from now on, $C$ stands for generic positive real constants dependent on $T, p, \xi$ and $K_1$ etc. as well in the next sections but independent of $\Delta$ and its values may change between occurrences.

**Proof.** Fix $\Delta \in (0, \Delta^*)$ and the initial data $\xi$ arbitrarily. By the general Itô formula (2.3), we
derive from (2.15) that for $0 \leq t \leq T$,

$$
\mathbb{E}|x_\Delta(t)|^p - |\xi(0)|^p
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta^T(s)f_\Delta(z_1(s), z_2(s), \bar{r}(s)) + \frac{p-1}{2} |g_\Delta(z_1(s), z_2(s), \bar{r}(s))|^2)ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(z_1^T(s)f_\Delta(z_1(s), z_2(s), \bar{r}(s)) + \frac{p-1}{2} |g_\Delta(z_1(s), z_2(s), \bar{r}(s))|^2)ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}K_4(1 + z_1(s))^2 + |z_2(s)|^2)ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}K_4(1 + z_1(s))^2 + |z_2(s)|^2)ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^T f_\Delta(z_1(s), z_2(s), \bar{r}(s))ds
= J_1 + J_2,
$$

(3.2)

where

$$
J_1 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}K_4(1 + z_1(s))^2 + |z_2(s)|^2)ds,
$$

$$
J_2 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^T f_\Delta(z_1(s), z_2(s), \bar{r}(s))ds.
$$

By Young inequality $a^{p-2}b \leq \frac{p-2}{p} a^p + \frac{p}{2} b^{p/2}, \forall a, b \geq 0$ and elementary $(a+b+c)^p \leq 3^{p-1}(a^p + b^p + c^p)$, we then have

$$
J_1 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}K_4(1 + z_1(s))^2 + |z_2(s)|^2)ds
\leq (p-2)\mathbb{E} \int_0^t |x_\Delta(s)|^pds + \frac{p}{2} \mathbb{E} \int_0^t (K_4(1 + z_1(s))^2 + |z_2(s)|^2)^{p/2}ds
\leq (p-2)\mathbb{E} \int_0^t |x_\Delta(s)|^pds + \frac{p}{2} K_4^{p/2} 3^{p-1} \mathbb{E} \int_0^t (1 + z_1(s))^p + |z_2(s)|^p)ds
\leq C \int_0^t (1 + \mathbb{E}|x_\Delta(s)|^p + \mathbb{E}|z_1(s)|^p + \mathbb{E}|z_2(s)|^p)ds.
$$

(3.3)

However, $f_\Delta(z_1(s), z_2(s), \bar{r}(s)) = F_1(z_1(s), z_2(s), \bar{r}(s)) + F_\Delta(z_1(s), z_2(s), \bar{r}(s))$, therefore,

$$
J_2 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^T f_\Delta(z_1(s), z_2(s), \bar{r}(s))ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^TF_1(z_1(s), z_2(s), \bar{r}(s))ds
\leq \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^TF_\Delta(z_1(s), z_2(s), \bar{r}(s))ds
= J_3 + J_4,
$$

where

$$
J_3 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^TF_1(z_1(s), z_2(s), \bar{r}(s))ds,
$$

$$
J_4 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^TF_\Delta(z_1(s), z_2(s), \bar{r}(s))ds.
$$
Similarly, by (2.11), we also show that
\[ J_3 \leq C \int_0^t (1 + \mathbb{E}|x_\Delta(s)|^p + \mathbb{E}|z_1(s)|^p + \mathbb{E}|z_2(s)|^p)ds. \]  

(3.4)

Moreover, according to (2.5) and Young’s inequality, we have
\[ J_4 = \mathbb{E} \int_0^t p|x_\Delta(s)|^{p-2}(x_\Delta(s) - z_1(s))^TF_\Delta(z_1(s), z_2(s), \bar{r}(s))ds \]
\[ \leq (p - 2)\mathbb{E} \int_0^t |x_\Delta(s)|^p ds \]
\[ + \frac{p}{2} \mathbb{E} \int_0^t |x_\Delta(s) - z_1(s)|^{p/2}|F_\Delta(z_1(s), z_2(s), \bar{r}(s))|^{p/2}ds \]
\[ \leq (p - 2)\mathbb{E} \int_0^t |x_\Delta(s)|^p ds + \frac{p}{2}h^{p/2}(\Delta) \int_0^t \mathbb{E}|x_\Delta(s) - z_1(s)|^{p/2}ds. \]  

(3.5)

On the other hand, for any \( s \in [0, T] \), there is a unique \( k \geq 0 \) such that \( k\Delta \leq s < (k + 1)\Delta \). By (2.5), (2.11), element inequality and Itô isometry, we then derive from (2.7) that
\[ \mathbb{E}|x_\Delta(t) - z_1(t)|^{p/2} = \mathbb{E}|x_\Delta(t) - x_\Delta(k\Delta)|^{p/2} \]
\[ \leq 4^{p/2-1} \left[ \mathbb{E} \int_{k\Delta}^t F_\Delta(z_1(s), z_2(s), \bar{r}(s))ds |^{p/2} + \mathbb{E} \int_{k\Delta}^t F_1(z_1(t), z_2(t), \bar{r}(t))ds |^{p/2} \right] \]
\[ + \mathbb{E} \int_{k\Delta}^t G_\Delta(z_1(s), z_2(s), \bar{r}(s))dB(s) |^{p/2} + \mathbb{E} \int_{k\Delta}^t G_1(z_1(s), z_2(s), \bar{r}(s))dB(s) |^{p/2} \]
\[ \leq 4^{p/2-1} \left[ \Delta^{p/2-1} \mathbb{E} \int_{k\Delta}^t |F_\Delta(z_1(s), z_2(s), \bar{r}(s))|^{p/2}ds + \Delta^{p/2-1} \mathbb{E} \int_{k\Delta}^t |F_1(z_1(t), z_2(t), \bar{r}(t))|^{p/2}ds \right] \]
\[ + \Delta^{p/4-1} \mathbb{E} \int_{k\Delta}^t |G_\Delta(z_1(s), z_2(s), \bar{r}(s))|^{p/2}ds + \Delta^{p/4-1} \mathbb{E} \int_{k\Delta}^t |G_1(z_1(s), z_2(s), \bar{r}(s))|^{p/2}ds \]
\[ \leq 4^{p/2-1} \left[ 2\Delta^{p/4}h^{p/2}(\Delta) \right] \]
\[ + \Delta^{p/2-1} \mathbb{E} \int_{k\Delta}^t |F_1(z_1(t), z_2(t), \bar{r}(t))|^{p/2}ds + \Delta^{p/4-1} \mathbb{E} \int_{k\Delta}^t |G_1(z_1(s), z_2(s), \bar{r}(s))|^{p/2}ds \]
\[ \leq C\Delta^{p/4}(h^{p/2}(\Delta) + 1 +\mathbb{E}|z_1(s)|^{p/2} + \mathbb{E}|z_2(s)|^{p/2}). \]  

(3.6)

Substituting this into (3.5) and recalling (2.4), we obtain
\[ J_4 \leq (p - 2)\mathbb{E} \int_0^t |x_\Delta(s)|^p ds + Ch^{p/2}(\Delta)\Delta^{p/4} \int_0^t (1 + h^{p/2}(\Delta) + \mathbb{E}|z_1(s)|^{p/2} + \mathbb{E}|z_2(s)|^{p/2})ds \]
\[ \leq C(1 + \int_0^t (1 + \mathbb{E}|x_\Delta(s)|^p + \mathbb{E}|z_1(s)|^p + \mathbb{E}|z_2(s)|^p)ds). \]  

(3.7)

Substituting (3.3)-(3.7) into (3.2) yields
\[ \mathbb{E}|x_\Delta(t)|^p \leq C(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^p ds). \]

As this holds for any \( t \in [0, T] \), while the sum of the right-hand-side (RHS) terms is non-decreasing in \( t \), we then see
\[ \sup_{0 \leq u \leq t} \mathbb{E}|x_\Delta(u)|^p \leq C(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^p ds). \]
The well-known Gronwall inequality yields that
\[
\sup_{0 \leq u \leq T} \mathbb{E}|x_\Delta(u)|^2 \leq C.
\]
As this holds for any \( \Delta \in (0, \Delta^*] \), while \( C \) is independent of \( \Delta \), we obtain the required assertion \( \Box \).

**Lemma 3.3** Let Assumption 2.2 and 2.3 hold, then for any \( \Delta \in (0, \Delta^*] \), we have
\[
\mathbb{E}|x_\Delta(t) - z_1(t)|^p \leq C\Delta^{p/4}h^p(\Delta), \quad \forall t \geq 0,
\]
(3.8)

**Proof.** By Lemma 3.2 there is a \( \Delta \in (0, \Delta^*] \) such that
\[
\sup_{0 \leq \Delta \leq \Delta^*} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^p \leq C.
\]
(3.9)

Now, fix any \( \Delta \in (0, \Delta^*] \). For any \( t \in [0, T] \), there is a unique \( k \geq 0 \) such that \( k\Delta \leq t \leq (k+1)\Delta \). In the same way as (3.6) was proved, we can then show
\[
\mathbb{E}|x_\Delta(t) - z_1(t)|^p \leq C\Delta^{p/4}(1 + \mathbb{E}|z_1(s)|^{p/2} + \mathbb{E}|z_2(s)|^{p/2} + h^{p/2}(\Delta))
\]
By (3.9), we therefore have
\[
\mathbb{E}|x_\Delta(t) - z_1(t)|^p \leq C\Delta^{p/2}h^p(\Delta).
\]
The proof is complete. \( \Box \)

**Assumption 3.4** We assume that delay function \( \delta(\cdot) \) is bounded and differentiable, moreover, its derivative is bounded by a constant \( \delta \in [0, 1) \), that is
\[
\frac{d\delta}{dt} \leq \delta \quad \text{and} \quad \tau := \sup_{t \geq 0} \delta(t) < \infty.
\]
(3.10)

According this Assumption 3.4, we can obtain that there exists a constants \( K_5 > 1 \) such that
\[
|\delta(a) - \delta(b)| \leq K_5|a - b|.
\]
(3.11)

**Lemma 3.5** Let Assumption 3.4 hold, then, for any \( \Delta \in (0, \Delta^*] \) and \( p \geq 2 \), we have
\[
\mathbb{E}|x_\Delta(t - \delta(t)) - z_2(t)|^p \leq C\Delta^{p/2}h^p(\Delta).
\]

**Proof.** In the same way to the (3.6), we can obtain
\[
\mathbb{E}|x_\Delta(t - \delta(t)) - z_2(s)|^p = \mathbb{E}|x_\Delta(t - \delta(t)) - x_\Delta(k\Delta - In[\delta(k\Delta)/\Delta]\Delta)|^p \\
\leq C(t - \delta(t) - k\Delta - |\delta(k\Delta)/\Delta|\Delta)^{p/2}(1 + \mathbb{E}|z_1(t)|^p + \mathbb{E}|z_2(t)|^p + h^p(\Delta)).
\]
(3.12)
Noting
\[
\delta(k\Delta) - \Delta \leq |\delta(k\Delta)/\Delta|\Delta \leq \delta(k\Delta),
\]
and (3.11), we derive that
\[
|\delta(t) - |\delta(k\Delta)/\Delta|\Delta| \leq \begin{cases} \delta(t) - \delta(k\Delta) + \Delta \leq (K_5 + 1)\Delta & \text{if} \quad \delta(t) > \delta(k\Delta), \\
\delta(k\Delta) - \delta(t) \leq K_5\Delta & \text{if} \quad \delta(t) < \delta(k\Delta) - \Delta,
\end{cases}
\]
and (3.12), we derive that
In other words, we always have
\[ |\delta(t) - [\delta(k\Delta)/\Delta]\Delta| \leq (K_5 + 1)\Delta. \]

Therefore
\[ |t - \delta(t) - k\Delta - [\delta(k\Delta)/\Delta]\Delta|^{p/2} \leq ((K_5 + 2)\Delta)^{p/2}. \]

Substituting this into (3.12) gives
\[ \mathbb{E}|x_\Delta(t - \delta(t)) - z_2(s)|^p \leq C\Delta^{p/2} \left( 1 + \mathbb{E}|z_1(t)|^p + \mathbb{E}|z_2(t)|^p + h^p(\Delta) \right). \]

By Lemma 3.2 we therefore have
\[ \mathbb{E}|x_\Delta(t - \delta(t)) - z_2(s)|^p \leq C\Delta^{p/2}h^p(\Delta). \]

Then the proof is complete. \(\square\)

**Lemma 3.6** Let Assumption 2.2, 2.3 and 3.4 hold. For any real number \( R > |x(0)| \), define the stopping time
\[ \tau_R = \inf\{ t \geq 0 : |x(t)| \geq R \}, \]
where throughout this paper we set \( \inf \emptyset = \infty \) (and as usual \( \emptyset \) denotes the empty set). Then
\[ \mathbb{P}(\tau_R \leq T) \leq \frac{C}{R^p}. \quad (3.13) \]

**Proof.** By the general Itô formula, Young inequality and Lemma 2.5 we derive that for \( 0 \leq t \leq T \),
\[ \mathbb{E}|x(t \wedge \tau_R)|^p - |x(0)|^p \]
\[ \leq \mathbb{E} \int_0^{t \wedge \tau_R} p|x(s)|^{p-2}(x^T(s)f(x(s), x(s - \delta(s)), r(s)) \]
\[ + \frac{p-1}{2}|g(x(s), x(s - \delta(s)), r(s))|^2)ds \]
\[ \leq K_3 \mathbb{E} \int_0^{t \wedge \tau_R} p|x(s)|^{p-2}(1 + |x(s)|^2 + |x(s - \delta(s))|^2)ds \]
\[ \leq C \mathbb{E} \int_0^{t \wedge \tau_R} (1 + |x(s)|^p + |x(t - \delta(t))|^p)ds \]
\[ \leq CT + C \mathbb{E} \int_0^{t \wedge \tau_R} |x(s)|^p ds + C \mathbb{E} \int_0^{t \wedge \tau_R} |x(s - \delta(s))|^p ds \]
\[ \leq CT + C \mathbb{E} \int_0^{t \wedge \tau_R} |x(s)|^p ds + \frac{C}{1-\delta} \mathbb{E} \int_{\delta(0)}^{(t - \delta(t)) \wedge \tau_R} |x(s)|^p ds \]
\[ \leq CT + \frac{C}{1-\delta} \mathbb{E} \int_{\delta(0)}^{t \wedge \tau_R} |x(s)|^p ds \]
\[ \leq CT + \frac{C\tau}{1-\delta} |\xi|^p + \frac{C}{1-\delta} \mathbb{E} \int_0^{t \wedge \tau_R} |x(s)|^p ds \]
\[ \leq C + \frac{C}{1-\delta} \int_0^t \mathbb{E}|x(s \wedge \tau_R)|^p ds. \]

The Gronwall inequality shows
\[ \mathbb{E}|x(t \wedge \tau_R)|^p \leq C. \]
This implies, by the Chebyshev inequality,

\[ R^p \mathbb{P}(\tau_R \leq T) \leq \mathbb{E}|x(t \wedge \tau_R)|^p \leq C \]

and the assertion (3.13) follows. \( \square \)

The follows Lemma can be proved in the same way as lemma 3.6 was proved.

**Lemma 3.7** let Assumption and hold. For any real number \( R > |x(0)| \), define the stopping time

\[ \rho_{\Delta,R} = \inf\{t \geq 0 : |x_{\Delta}(t)| \geq R\}, \]

Then

\[ \mathbb{P}(\rho_{\Delta,R} \leq T) \leq \frac{C}{R^p}. \] (3.14)

In order obtain our main convergence rate theorem, we need some addition condition.

**Assumption 3.8** There is a pair of constants \( K_6 > 0 \) and \( v \in (0, 1] \) such that the initial data \( \xi \) satisfies

\[ |\xi(a) - \xi(b)| \leq K_6|a - b|^v, \quad -\tau \leq a < b \leq 0. \]

**Assumption 3.9** Assume that there is a positive constant \( K_7 \) and \( \bar{q} > 2 \) such that

\[ (x - \bar{x})^T (F(x, y, i) - F(\bar{x}, \bar{y}, i)) + \frac{q - 1}{2}|G(x, y, i) - G(\bar{x}, \bar{y}, i)|^2 \leq K_7(|x - \bar{x}|^2 + |y - \bar{y}|^2) \]

for all \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \).

In the same way as performed in the proof of Lemma 2.5 and according to the Assumption 2.2, we can shows the following Lemma.

**Lemma 3.10** Let Assumptions 2.2 and 3.8 holds, then for any \( \Delta \in (0, \Delta^*) \), we have for any \( q \in (2, \bar{q}) \)

\[ (x - \bar{x})^T (f(x, y, i) - f(\bar{x}, \bar{y}, i)) + \frac{q - 1}{2}|g(x, y, i) - g(\bar{x}, \bar{y}, i)|^2 \leq K_7(|x - \bar{x}|^2 + |y - \bar{y}|^2) \] (3.15)

for all \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \).

The following Lemma will play a key role in the proof of the convergence rate.

**Lemma 3.11** Let Assumptions 2.2, 2.5, 3.8 and 3.9 hold and assume that \( q > 2 \) and \( p \geq p \) Let \( R > |x(0)| \) be a real number and let \( \Delta \in (0, \Delta^*) \) be sufficiently small such that \( \mu^{-1}(h(\Delta)) \geq R \). Then

\[ \mathbb{E}|x(t \wedge \theta_{\Delta,R}) - x_{\Delta}(t \wedge \theta_{\Delta,R})|^2 \leq C(\Delta^{2v} \vee \Delta h^2(\Delta)). \] (3.16)

**Proof.** Let \( \tau_R \) and \( \rho_{\Delta,R} \) be the same as before. Let

\[ \theta_{\Delta,R} = \tau_R \wedge \rho_{\Delta,R} \quad \text{and} \quad e_{\Delta}(t) = x_{\Delta}(t) - x(t). \]

and we write \( \theta_{\Delta,R} = \theta \) for simplicity. We observe that for \( 0 \leq s \leq t \wedge \theta \),

\[ |x(s)| \vee |x_{\Delta}(s - \delta(s))| \vee |z_1(s)| \vee |z_2(s)| \leq R. \]
Recalling the definition of the truncated functions $F_\Delta$ and $G_\Delta$ as well as (2.3), we hence have that
\[
F_\Delta(z_1(s), z_2(s), i) = F(z_1(s), z_2(s), i), \quad G_\Delta(z_1(s), z_2(s), i) = G(z_1(s), z_2(s), i)
\]
for $0 \leq s \leq t \wedge \theta$. Then
\[
f_\Delta(z_1, z_2, i) = F_1(z_1, z_2, i) + F_\Delta(z_1, z_2, i) = F_1(z_1, z_2, i) + F(z_1, z_2, i) = f(z_1, z_2, i)
\]
and
\[
g_\Delta(z_1, z_2, i) = g(z_1, z_2, i).
\]
The Itô formula and (3.15) shows that
\[
\begin{align*}
\mathbb{E}[e_\Delta(t \wedge \theta)]^2 & = 2\mathbb{E} \int_0^{t \wedge \theta} \left((x(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f_\Delta(z_1(s), z_2(s), \bar{r}(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g_\Delta(z_1(s), z_2(s), \bar{r}(s))|^2 \right) ds \\
& = 2\mathbb{E} \int_0^{t \wedge \theta} \left((x(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \right) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} ((z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))]) ds \\
& = 2\mathbb{E} \int_0^{t \wedge \theta} ((x(s) - z_1(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \right) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))]) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \right) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))]) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \right) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))]) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \\
& \quad + \frac{1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \right) ds \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T[f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))]) ds.
\end{align*}
\]
Noting
\[\frac{1}{2} \left( g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), \bar{r}(s)) \right)^2 \]
\[\leq \frac{1}{2} \times \left( (1 + \frac{q - 2}{1}) |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \
+ (1 + \frac{1}{q - 2}) |g(z_1(s), z_2(s), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \right) \]
\[= \frac{q - 1}{2} \left( |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \
+ \frac{q - 1}{2(q - 2)} |g(z_1(s), z_2(s), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \right). \]

Therefore
\[\mathbb{E}|e_{\Delta}(t \wedge \theta)|^2 \]
\[\leq 2 \mathbb{E} \int_0^{t \wedge \theta} \left( (x(s) - z_1(s))^T [f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \right. \]
\[+ \frac{q - 1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \bigg) ds \]
\[+ 2 \mathbb{E} \int_0^{t \wedge \theta} (x(s) - z_1(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] ds \]
\[+ 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_{\Delta}(s))^T [f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] ds \]
\[+ 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_{\Delta}(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] ds \]
\[+ \mathbb{E} \int_0^{t \wedge \theta} \frac{q - 1}{q - 2} |g(z_1(s), z_2(s), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \bigg) ds \]
\[= H_1 + H_2 + H_3 + H_4 + H_5, \quad (3.17) \]

where
\[H_1 = 2 \mathbb{E} \int_0^{t \wedge \theta} \left( (x(s) - z_1(s))^T [f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] \right. \]
\[+ \frac{q - 1}{2} |g(x(s), x(s - \delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 \bigg) ds, \]
\[H_2 = 2 \mathbb{E} \int_0^{t \wedge \theta} (x(s) - z_1(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] ds, \]
\[H_3 = 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_{\Delta}(s))^T [f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] ds, \]
\[H_4 = 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_{\Delta}(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] ds \]

and
\[H_5 = \mathbb{E} \int_0^{t \wedge \theta} \frac{q - 1}{q - 2} |g(z_1(s), z_2(s), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 ds \]

13
According to the Young’s inequality and the (3.15), we have

\[
H_1 = 2\mathbb{E} \int_0^{t \wedge \theta} (x(s) - z_1(s))^T [f(x(s), x(s-\delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] ds \\
+ \frac{q-1}{2} |g(x(s), x(s-\delta(s)), r(s)) - g(z_1(s), z_2(s), r(s))|^2 ds
\]

\[
\leq 2K_7 \mathbb{E} \int_0^{t \wedge \theta} (|x(s) - z_1(s)|^2 + |x(s-\delta(s)) - z_1(s)|^2) ds
\]

\[
\leq 2K_7 \mathbb{E} \int_0^{t \wedge \theta} (|x(s) - x_\Delta(s)|^2 + |x_\Delta(s) - z_1(s)|^2 + |x(s-\delta(s)) - x_\Delta(s-\delta(s))|^2 + |x_\Delta(s-\delta(s)) - z_2(s)|^2) ds
\]

\[
\leq 2K_7 \mathbb{E} \int_0^{t \wedge \theta} |x(s) - x_\Delta(s)|^2 ds + 2K_7 \int_0^T \mathbb{E} |x_\Delta(s) - z_1(s)|^2 ds \\
+ 2K_7 \int_0^T \mathbb{E} |x_\Delta(s-\delta(s)) - z_2(s)|^2 ds + 2K_7 \mathbb{E} \int_{-\delta}^0 |\xi(s/\Delta) - \xi(s)|^2 ds.
\]

By Lemma 3.3 3.5 and Assumption 4.1, we have

\[
H_1 \leq C \left( \int_0^T \mathbb{E} |x(s \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + \Delta^{2\nu} \right). \tag{3.18}
\]

By the Young’s inequality and elementary inequality, we can obtain

\[
H_2 = 2\mathbb{E} \int_0^{t \wedge \theta} (x(s) - z_1(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(s))] ds \\
\leq \mathbb{E} \int_0^{t \wedge \theta} |x(s) - z_1(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \theta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(s))|^2 ds
\]

\[
= \mathbb{E} \int_0^{t \wedge \theta} |x(s) - x_\Delta(s) + x_\Delta(s) - z_1(s)|^2 ds \\
+ \mathbb{E} \int_0^{t \wedge \theta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(s))|^2 ds
\]

\[
\leq \mathbb{E} \int_0^{t \wedge \theta} |x_\Delta(s) - x(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \theta} |x_\Delta(s) - z_1(s)|^2 ds \\
+ \mathbb{E} \int_0^{t \wedge \theta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(s))|^2 ds
\]

\[
\leq \mathbb{E} \int_0^{t \wedge \theta} |x_\Delta(s) - x(s)|^2 ds + \mathbb{E} \int_0^{t \wedge \theta} |x_\Delta(s) - z_1(s)|^2 ds \\
+ \mathbb{E} \int_0^T |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(s))|^2 ds. \tag{3.19}
\]
Let \( j \) be the integer part of \( T/\Delta \). Then
\[
\mathbb{E} \int_0^T |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))|^2 ds
\]
\[
= \sum_{k=0}^j \mathbb{E} \int_{k\Delta}^{(k+1)\Delta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(k\Delta))|^2 ds
\]
\[
\leq 2 \sum_{k=0}^j \mathbb{E} \int_{k\Delta}^{(k+1)\Delta} \left[ |f(z_1(s), z_2(s), r(s))|^2 + |f(z_1(s), z_2(s), r(k\Delta))|^2 \right] I_{\{r(s) \neq r(k\Delta)\}} ds
\]
\[
\leq 2 \sum_{k=0}^j \int_{k\Delta}^{(k+1)\Delta} \mathbb{E}\mathbb{E}[(1 + |z_1(s)|^q + |z_2(s)|^2 + h^2(\Delta))I_{\{r(s) \neq r(k\Delta)\}}] r(k\Delta) || ds. \tag{3.20}
\]
where in the last step, we use the fact that \( z_1(s) \) and \( z_2(s) \) are conditionally independent of \( I_{\{r(s) \neq r(k\Delta)\}} \) given the \( \sigma \)-algebra generated by \( r(k\Delta) \). But, by the Markov property
\[
\mathbb{E}[I_{\{r(s) \neq r(k\Delta)\}} | r(k\Delta)]
\]
\[
= \sum_{i \in S} I_{\{r(k\Delta) = i\}} P(r(s) \neq i | r(k\Delta) = i)
\]
\[
= \sum_{i \in S} I_{\{r(k\Delta) = i\}} \sum_{j \neq i} (\gamma_{ij}(s - t_k) + o(s - t_k))
\]
\[
\leq \left( \max_{i \leq i \leq N} (\gamma_{ii} - o(\Delta)) \right) \sum_{i \in S} I_{\{r(k\Delta) = i\}}
\]
\[
\leq C\Delta + o(\Delta).
\]
So, by lemma 3.2
\[
\mathbb{E} \int_{k\Delta}^{(k+1)\Delta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(k\Delta))|^2 ds
\]
\[
\leq (C\Delta + o(\Delta)) \int_{k\Delta}^{(k+1)\Delta} [1 + \mathbb{E}|z_1(s)|^q + \mathbb{E}|z_2(s)|^2 + h^2(\Delta)] ds
\]
\[
\leq h^2(\Delta)\Delta(C\Delta + o(\Delta)).
\]
Substituting this into (3.20) gives
\[
\mathbb{E} \int_0^T |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))|^2 ds
\]
\[
\leq h^2(\Delta)(C\Delta + o(\Delta)).
\]
This implies that
\[
\mathbb{E} \int_0^{t\wedge \theta} |f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), r(k\Delta))|^2 ds
\]
\[
\leq h^2(\Delta)(C\Delta + o(\Delta)).
\]
Substituting this into (3.19) and lemma 3.3 we have
\[
H_2 \leq C \left( \int_0^t \mathbb{E}|x(s \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + o(\Delta) \right). \tag{3.21}
\]
Moreover, using the (2.12) and Lemma 3.3 we have

\[ H_3 = 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T [f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))] ds \]

\[ \leq \mathbb{E} \int_0^{t \wedge \theta} |z_1(s) - x_\Delta(s)|^2 + \mathbb{E} \int_0^{t \wedge \theta} |f(x(s), x(s - \delta(s)), r(s)) - f(z_1(s), z_2(s), r(s))|^2 ds \]

\[ \leq \mathbb{E} \int_0^{t \wedge \theta} |z_1(s) - x_\Delta(s)|^2 + K_1 \mathbb{E} \int_0^{t \wedge \theta} [1 + |x(s)|^\rho + |x(s - \delta(s))|^\rho + |z_1(s)|^\rho + |z_2(s)|^\rho] \times (|x(s) - z_1(s)|^2 + |x(s - \delta(s)) - z_2(s)|^2) ds \]

\[ \leq C \left( \mathbb{E} \int_0^t |x(s) \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + \Delta^{2+} \right). \]

Therefore

\[ H_3 \leq C \left( \int_0^t \mathbb{E}|x_\Delta(s \wedge \theta) - x(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + \Delta^{2+} \right). \quad (3.22) \]

Similarly to \( H_2 \), we can show

\[ H_4 = 2 \mathbb{E} \int_0^{t \wedge \theta} (z_1(s) - x_\Delta(s))^T [f(z_1(s), z_2(s), r(s)) - f(z_1(s), z_2(s), \bar{r}(s))] ds \]

\[ \leq C \left( \int_0^t \mathbb{E}|x(s \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + o(\Delta) \right). \quad (3.23) \]

And

\[ H_5 = \mathbb{E} \int_0^{t \wedge \theta} \frac{q - 1}{q - 2} |g(z_1(s), z_2(s), r(s)) - g(z_1(s), z_2(s), \bar{r}(s))|^2 \]

\[ \leq C \left( \int_0^t \mathbb{E}|x(s \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + \Delta h^2(\Delta) + o(\Delta) \right). \quad (3.24) \]

Substituting (3.18), (3.21), (3.22), (3.23) and (3.24) into (3.17), we obtain that

\[ \mathbb{E}|x(t \wedge \theta) - x_\Delta(t \wedge \theta)|^2 \leq C \left( \int_0^t \mathbb{E}|x(s \wedge \theta) - x_\Delta(s \wedge \theta)|^2 ds + (\Delta h^2(\Delta) \vee \Delta^{2+}) \right). \]

By the well-known Gronwall inequality yields that

\[ \mathbb{E}|x(t \wedge \theta) - x_\Delta(t \wedge \theta)|^2 \leq C(\Delta h^2(\Delta) \vee \Delta^{2+}). \]

Then the proof is complete.

Let us now state our first result on the convergence rate.

**Theorem 3.12** Let Assumptions 2.2, 2.3, 4.7 and 3.7 hold and assume that \( p \in (2, \bar{p}) \) and \( p > \rho \), If

\[ h(\Delta) \geq \mu \left( (\Delta^{2+} \vee \Delta h^2(\Delta))^{-1/(p-2)} \right), \quad (3.25) \]

then there is a \( \Delta \in (0, \Delta^*) \) such that

\[ \mathbb{E}|x_\Delta(T) - x(T)|^2 \leq C(\Delta^{2+} \vee \Delta h^2(\Delta)). \quad (3.26) \]

16
Proof. Let $\varepsilon > 0$ be arbitrary. Let $\tau_R, \rho_{\Delta,R}, \theta_{\Delta,R}$ and $e_\Delta(T)$ be same as before. For a sufficiently large $R > |x(0)|$, we have that

$$
E|e_\Delta(T)|^2 = E(|e_\Delta(T)|^2 I_{\theta_{\Delta,R}>T}) + E(|e_\Delta(T)|^2 I_{\theta_{\Delta,R}\leq T})
$$

\[ \leq E(|e_\Delta(T)|^2 I_{\theta_{\Delta,R}>T}) + \frac{2\varepsilon}{p} E|e_\Delta(T)|^2 + \frac{p-2}{p\rho^{2/(p-2)}} P(\theta_{\Delta,R}\leq T). \] (3.27)

Applying Lemma 3.1 and 3.2, we can see that

$$
E|e_\Delta(T)|^2 \leq 2E|x(T)|^2 + 2E|x_\Delta(T)|^2 \leq C.
$$ (3.28)

Using the Lemma 3.6 and 3.7, we obtain that

$$
P(\theta_{\Delta,R} \leq T) \leq P(\tau_R \leq T) + P(\rho_{\Delta,R} \leq T) \leq \frac{C}{R^2}. \] (3.29)

Substituting (3.28) and (3.29) into (3.27) and choosing $\varepsilon = \Delta^{2v} \lor \Delta h^2(\Delta)$ and $R = (\Delta^{2v} \lor \Delta h^2(\Delta))^{-1/(p-2)}$, we have that

$$
E(|e_\Delta(T)|^2 I_{\theta_{\Delta,R}\leq T}) \leq C(\Delta^{2v} \lor \Delta h^2(\Delta)).
$$ (3.30)

By the Lemma 3.11, we can show that

$$
E(|e_\Delta(T \lor \theta_{\Delta,R})|^2) \leq C(\Delta^{2v} \lor \Delta h^2(\Delta)).
$$ (3.31)

By the (3.25), we can see that

$$
\mu^{-1}(h(\Delta)) \geq (\Delta^{2v} \lor \Delta h^2(\Delta))^{-1/(p-2)} = R.
$$

Therefore, substituting (3.30) and (3.31) into (3.27) yields (3.26). The proof is therefore complete. \qed

Example 3.13 Consider a nonlinear scalar hybrid SDDE

$$
dx(t) = f(x(t), x(t - \delta(t)), r(t))dt + g(x(t), x(t - \delta(t)), r(t))dB(t) \] (3.32)

Here, $B(t)$ is a scalar Brownian, delay function $\delta(t) = 0.1 \cos(t)$, and $r(t)$ is a Markovian chain on the state space $S = \{1, 2\}$ and they are independent. Let the generator of the Markovian chain that

$$
\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.
$$

Moreover, for $\forall (x, y, i) \in \mathbb{R} \times \mathbb{R} \times S$,

$$
f(x, y, i) = \begin{cases} -6x - x^5 + y & \text{if } i = 1 \\ -6x - x^5 + \frac{y}{1+y} & \text{if } i = 2 \end{cases} \quad \text{and} \quad g(x, y, i) = \begin{cases} x^2 & \text{if } i = 1 \\ \sin x \sin y & \text{if } i = 2 \end{cases}
$$

Step 1. Check the assumptions

It can be seen that

$$
F_1(x, y, i) = \begin{cases} -6x + y & \text{if } i = 1 \\ -6x & \text{if } i = 2 \end{cases} \quad \text{and} \quad G_1(x, y, i) = \begin{cases} 0 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}
$$
For Assumption 3.9, it is straightforward to see that then Assumption 2.2 holds and the delay function $\delta(t) = 0.1 \cos t$ fulfilled Assumption 3.4 clearly. For Assumption 3.9 it is straightforward to see that

\[
(x - \bar{x})^T (F(x, y, 1) - F(\bar{x}, \bar{y}, 1)) + \frac{q-1}{2} |G(x, y, 1) - G(\bar{x}, \bar{y}, 1)|^2
\]

\[
= (x - \bar{x})(-x^5 + \bar{x}^5) + \frac{q-1}{2} |x^2 - \bar{x}^2|^2
\]

\[
= (x - \bar{x})[(-x - \bar{x})(x^4 + x^3\bar{x} + x^2\bar{x} + x\bar{x}^3 + \bar{x}^4)] + \frac{q-1}{2} |(x - \bar{x})(x + \bar{x})|^2
\]

\[
= (x - \bar{x})^2[-(x^4 + x^3\bar{x} + x^2\bar{x} + x\bar{x}^3 + \bar{x}^4)] + \frac{q-1}{2} (x + \bar{x})^2].
\]

However

\[-(x^3\bar{x} + x\bar{x}^3) = -x\bar{x}(x^2 + \bar{x}^2) \leq 0.5(x^2 + \bar{x}^2)^2 = 0.5(x^4 + \bar{x}^4) + x^2\bar{x}^2.\]

Hence

\[
(x - \bar{x})^T (F(x, y, 1) - F(\bar{x}, \bar{y}, 1)) + \frac{q-1}{2} |G(x, y, 1) - G(\bar{x}, \bar{y}, 1)|^2
\]

\[
\leq (x - \bar{x})^2[-0.5(x^4 + \bar{x}^4) + \frac{q-1}{2} (x^2 + \bar{x}^2)](x - \bar{x})^2
\]

\[
\leq [1 + \frac{(q-1)^2}{4}](x - \bar{x})^2.
\]

Moreover,

\[
(x - \bar{x})^T (F(x, y, 2) - F(\bar{x}, \bar{y}, 2)) + \frac{q-1}{2} |G(x, y, 2) - G(\bar{x}, \bar{y}, 2)|^2
\]

\[
= (x - \bar{x})[-x^5 + \frac{y}{1+y^2} - (-\bar{x}^5 + \frac{\bar{y}}{1+\bar{y}^2})] + \frac{q-1}{2} |\sin x \sin y - \sin \bar{x} \sin \bar{y}^2|.
\]

Using the mean value theorem, we set $A(x) = \frac{x}{1+x^2}$, then there is existing a $x^* \in (x, \bar{x})$ such that

\[
|A(x) - A(\bar{x})| = |A'(\xi)(x - \bar{x})| \leq |A'(x^*)| |x - \bar{x}|,
\]

where $A'(x)$ is the derivative and $A'(x) = \frac{1-x^2}{(1+x^2)^2} \leq 1$ therefore

\[
\left| \frac{y}{1+y^2} - \frac{\bar{y}}{1+\bar{y}^2} \right| \leq |y - \bar{y}|.
\]

Meanwhile,

\[
|\sin x \sin y - \sin \bar{x} \sin \bar{y}|^2 = |\sin x \sin y - \sin x \sin \bar{y} + \sin x \sin \bar{y} - \sin \bar{x} \sin \bar{y}|^2
\]

\[
\leq |\sin x \sin y - \sin x \sin \bar{y}|^2 + |\sin x \sin \bar{y} - \sin \bar{x} \sin \bar{y}|^2
\]

\[
\leq |y - \bar{y}|^2 + |x - \bar{x}|^2.
\]

Therefore

\[
(x - \bar{x})^T (F(x, y, 2) - F(\bar{x}, \bar{y}, 2)) + \frac{q-1}{2} |G(x, y, 2) - G(\bar{x}, \bar{y}, 2)|^2
\]

\[
\leq (x - \bar{x})(-0.5(x^4 + \bar{x}^4) + |y - \bar{y}|) + \frac{q-1}{2} (|x - \bar{x}|^2 + |y - \bar{y}|^2)
\]

\[
\leq (x - \bar{x})(|y - \bar{y}|) + \frac{q-1}{2} (|x - \bar{x}|^2 + |y - \bar{y}|^2)
\]

\[
\leq \frac{q}{2} |x - \bar{x}|^2 + \frac{q}{2} |y - \bar{y}|^2.
\]
So, for any $i \in \mathbb{S}$, we have
\[
(x - \bar{x})^T (F(x, y, i) - F(\bar{x}, \bar{y}, i)) + \frac{q - 1}{2} |G(x, y, i) - G(\bar{x}, \bar{y}, i)|^2 \leq \left(\frac{q^2 + 3}{4}\right)(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\]
In other words, Assumption 3.9 is also fulfilled for any $q$. Moreover,
\[
xF(x, y, 1) + \frac{\bar{p} - 1}{2} |G(x, y, 1)|^2 = -x^6 + \frac{\bar{p} - 1}{2} |x|^2
\]
and
\[
xF(x, y, 2) + \frac{\bar{p} - 1}{2} |G(x, y, 2)|^2 = -x^6 + \frac{xy}{1 + y^2} + \frac{\bar{p} - 1}{2} |\sin x \sin y|^2
\]
Therefore for any $i \in \mathbb{S}$, we have
\[
xF(x, y, i) + \frac{\bar{p} - 1}{2} |G(x, y, i)|^2 \leq \frac{(\bar{p} - 1)^2}{2}(1 + |x|^2 + |y|^2)
\]
that is, Assumption 2.3 is satisfied for any $\bar{p}$.

**Step 2.** We need choose $\mu(\cdot)$ and $h(\cdot)$.

According to (2.3), $|F(x, y, i)| = |x^5 + \frac{y}{1 + y^2}| \leq |x|^5$ and $|G(x, y, i)| = |\sin x \sin y| \leq 1$, then we can set $\mu(w) = w^5$ such that
\[
\sup_{|x| \lor |y| \leq w} (|F(x, y, i)| \lor |G(x, y, i)|) \leq \sup_{|x| \lor |y| \leq w} |x|^5 < w^5, \quad w > 1.
\]
If we set $h(\Delta) = \Delta^{-1/10}$, then all the conditions in (2.4) hold for all $\Delta^* \in (0, 1)$, and obviously we have $\mu^{-1}(h(\Delta)) = \Delta^{-1/50}$.

**Step 3.** Applying (2.3), we can obtain the numerical solution. Since it is hardly to find a true solution of (3.32). We use the numerical solution produced by partially truncated EM method with step size $2 \times 10^{-4}$, $2^2 \times 10^{-4}$, $2^3 \times 10^{-4}$ and $2^4 \times 10^{-4}$ and the step size $10^{-4}$ as the ‘true solution’ at the terminal time $T = 1$, the square roots of the mean square errors are plotted in Figure 1 by simulating 500 paths. We can see that the convergence rate is approximately 1/2.

### 4 Stability

**Assumption 4.1** Assume that there are constants $\theta \in [0, \infty]$ and there is a pair of constant $\lambda_1 > 2\lambda_3 \geq 2(1 - \bar{\delta})\lambda_4 \geq 0, \lambda_2 \geq 0$ and $\bar{\delta} \in (0, 1)$ such that
\[
2x^T F_1(x, y, i) + (1 + \theta)|G_1(x, y, i)|^2 \leq -\lambda_1|x|^2 + \lambda_2(1 - \bar{\delta})|y|^2
\]
and
\[ 2x^T F(x, y, i) + (1 + \theta^{-1})|G(x, y, i)|^2 \leq \lambda_3|x|^2 + \lambda_4(1 - \bar{\delta})|y|^2 \quad (4.2) \]
for all \( x, y \in \mathbb{R}^d \), where throughout the remaining part of this paper. We choose \( \theta = 0 \) and set \( \theta^{-1}|G(x, y)|^2 = 0 \) when there is no \( G(x, y) \) term in \( g(x, y) \), where choose \( \theta = \infty \) and set \( \theta|G(x, y)|^2 = 0 \) when there is no \( G_1(x, y) \) term in \( g(x, y) \).

This can implying that
\[ 2x^T f(x, y, i) + |g(x, y, i)|^2 \leq -\eta_2 (x_1 - \lambda_3)|x|^2 + (1 - \bar{\delta})(\lambda_2 + \lambda_4)|y|^2 \quad (4.3) \]

\textbf{Lemma 4.2} Let Assumption 4.1 hold, then for all \( \Delta \in (0, \Delta^*] \)
\[ 2x^T f_\Delta(x, y, i) + |g_\Delta(x, y, i)|^2 \leq -(\lambda_1 - 2\lambda_3)|x|^2 + (1 - \bar{\delta})(\lambda_2 + \lambda_4)|y|^2. \quad (4.4) \]
The proof is similar to the (4.8) on [7], so we omit it here.

From [27] we known that the SDDEs (2.1) is almost sure exponential stable. To be precise, we cite it as follows.

\textbf{Lemma 4.3} Let Assumption 3.4 and 4.1 holds. Then for any given initial data (2.2), the solution has the properties that
\[ \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}|x(t)| \leq -\frac{\eta}{2} \quad (4.5) \]
and
\[ \limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\frac{\eta}{2} \quad a.s. \quad (4.6) \]
where \( \eta \) is the unique root to the equation
\[ \lambda_1 - 2\lambda_3 = \eta + (\lambda_2 + \lambda_4)e^{\eta \tau}. \quad (4.7) \]
The following theorem shows that the partially truncated EM method can preserve this almost sure exponential stability.
Theorem 4.4 Let Assumption 3.4 and 4.1 holds. Let $\gamma^*$ be the unique positive root of the equation
\[
[(1 - \bar{\delta})(\lambda_2 + \lambda_3) + \varepsilon]e^{\gamma T} = \lambda_1 - 2\lambda_3 - \varepsilon - \gamma
\]  
(4.8)
with a positive number $\varepsilon$ satisfied
\[
\varepsilon < [(1 - \bar{\delta})(\lambda_2 + \lambda_4) - \lambda_1 + 2\lambda_3]/2.
\]  
(4.9)
Then the solution generated by the partially truncated EM method is almost sure exponential stable, i.e.,
\[
\limsup_{k \to \infty} \frac{1}{k\Delta} \log |X_k| \leq \frac{\gamma}{2} + \varepsilon \quad a.s.
\]  
(4.10)
Proof. For any positive constant $C > 1$, we have
\[
C^{(k+1)\Delta}|X_{k+1}|^2 - C^{k\Delta}|X_k|^2 = C^{(k+1)\Delta}(|X_{k+1}|^2 - |X_k|^2) + (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2.
\]
For simple, we let
\[
f_\Delta = f_\Delta(X_k, X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}, r_k^\Delta) \quad \text{and} \quad g_\Delta = g_\Delta(X_k, X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}, r_k^\Delta).
\]
We can easily obtain from (2.6) that
\[
|X_{k+1}|^2 = |X_k|^2 + 2X_k^T f_\Delta \Delta + |g_\Delta|^2 \Delta + |f_\Delta|^2 \Delta^2 + m_k.
\]
where
\[
m_k = 2X_k^T g_\Delta \Delta B_k + 2\Delta f_\Delta g_\Delta \Delta B_k + |g_\Delta|^2((\Delta B_k)^2 - \Delta)
\]
which is a martingale (see (2.9)). By condition (4.1) that
\[
C^{(k+1)\Delta}|X_{k+1}|^2 - C^{k\Delta}|X_k|^2
\]
\[
= C^{(k+1)\Delta}(2X_k^T f_\Delta \Delta + |g_\Delta|^2 \Delta + |f_\Delta|^2 \Delta^2 + m_k) + (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2
\]
\[
\leq -(\lambda_1 - 2\lambda_3)C^{(k+1)\Delta}|X_k|^2 + C^{(k+1)\Delta}(1 - \bar{\delta})(\lambda_2 + \lambda_4)\Delta |X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}|^2
\]
\[
+ C^{(k+1)\Delta}|f_\Delta|^2 \Delta^2 + C^{(k+1)\Delta}m_k + (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2
\]
\[
\leq -(\lambda_1 - 2\lambda_3)C^{(k+1)\Delta}|X_k|^2 + C^{(k+1)\Delta}(1 - \bar{\delta})(\lambda_2 + \lambda_4)\Delta |X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}|^2
\]
\[
+ C^{(k+1)\Delta}\varepsilon \Delta (|X_k|^2 + |X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}|^2) + C^{(k+1)\Delta}m_k + (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2
\]
\[
\leq C^{(k+1)\Delta}(-(\lambda_1 - 2\lambda_3 - \varepsilon)\Delta)|X_k|^2 + C^{(k+1)\Delta}((1 - \bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon\Delta)|X_{k-\lfloor \delta(k\Delta)/\Delta \rfloor}|^2
\]
\[
+ (C^{(k+1)\Delta} - C^{k\Delta})|X_k|^2 + C^{(k+1)\Delta}m_k.
\]  
(4.11)
Applying induction to (4.11) gives
\[
C^{k\Delta}|X_k|^2 - |X_0|^2
\]
\[
\leq -(\lambda_1 - 2\lambda_3 - \varepsilon)\Delta \sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_i|^2
\]
\[
+ [(1 - \bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon\Delta] \sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_{i-\lfloor \delta(i\Delta)/\Delta \rfloor}|^2
\]
\[
+ \sum_{i=0}^{k-1} (C^{(i+1)\Delta} - C^{i\Delta})|X_i|^2 + \sum_{i=0}^{k-1} C^{(i+1)\Delta}m_i
\]
\[
= -(\lambda_1 - 2\lambda_3 - \varepsilon)\Delta \sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_i|^2
\]
\[
+ [(1 - \bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon\Delta] \sum_{i=0}^{k-1} C^{(i+1)\Delta}|X_{i-\lfloor \delta(i\Delta)/\Delta \rfloor}|^2 + \sum_{i=0}^{k-1} C^{(i+1)\Delta}m_i,
\]  
(4.12)
Therefore,

\[
\sum_{i=0}^{k-1} C^{(i+1)\Delta} |X_{i-[\delta(i\Delta)/\Delta]}|^2 \]

\[
= \sum_{i=-[\delta(i\Delta)/\Delta]}^{-1} C^{(i+[\delta(i\Delta)/\Delta]+1)\Delta} |X_i|^2 + \sum_{i=0}^{k-1} C^{(i+[\delta(i\Delta)/\Delta]+1)\Delta} |X_i|^2 \]

\[
- \sum_{i=k-[\delta(i\Delta)/\Delta]}^{k-1} C^{(i+[\delta(i\Delta)/\Delta]+1)\Delta} |X_i|^2 \]

\[
\leq \sum_{i=-m}^{-1} C^{(i+m+1)\Delta} |X_i|^2 + \sum_{i=0}^{k-1} C^{(i+m+1)\Delta} |X_i|^2 \]

\[
- \sum_{i=k-[\delta(i\Delta)/\Delta]}^{k-1} C^{(i+[\delta(i\Delta)/\Delta]+1)\Delta} |X_i|^2. \tag{4.13}
\]

Substituting \((4.13)\) into \((4.12)\), we get

\[
C^{\Delta} |X_k|^2 + [(1-\bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon \Delta] \sum_{i=k-[\delta(i\Delta)/\Delta]}^{k-1} C^{(i+[\delta(i\Delta)/\Delta]+1)\Delta} |X_i|^2 \leq Y_k, \tag{4.14}
\]

where

\[
Y_k = |X_0|^2 + [(1-\bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon \Delta] \sum_{i=-m}^{-1} C^{(i+m+1)\Delta} |X_i|^2 \]

\[
+ \left[ - (\lambda_1 - 2\lambda_3 - \varepsilon)\Delta + (1 - C^{-\Delta}) + [(1-\bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon \Delta] C^{m\Delta} \right] \sum_{i=0}^{k-1} C^{(i+1)\Delta} |X_i|^2 \]

\[
+ \sum_{i=0}^{k-1} C^{(i+1)\Delta} m_i. \]

Let us now introduce the function

\[
J(C, \Delta) = [(1-\bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon \Delta] C^{(m+1)\Delta} + (1 - (\lambda_1 - 2\lambda_3 - \varepsilon)\Delta) C^{\Delta} - 1. \tag{4.15}
\]

Choose \(\Delta^*_1 > 0\) such that for any \(\Delta < \Delta^*_1\), \(1 - (\lambda_1 - 2\lambda_3 - \varepsilon)\Delta > 0\), we therefore have for any \(C > 1\),

\[
\frac{d}{dC} J(C, \Delta) = (m + 1)\Delta [(1-\bar{\delta})(\lambda_2 + \lambda_4)\Delta + \varepsilon \Delta] C^{(m+1)\Delta-1} + (1 - (\lambda_1 - 2\lambda_3 - \varepsilon)\Delta) C^{\Delta-1} > 0.
\]

Clearly

\[
J(1) = [(1-\bar{\delta})(\lambda_2 + \lambda_4) + \varepsilon - \lambda_1 + 2\lambda_3 + \varepsilon] \Delta,
\]

since \((4.9)\), we have \(J(1) < 0\), which implies that there exists a unique \(C^*_\Delta > 1\) such that \(J(C^*_\Delta, \Delta) = 0\). We choosing \(C = C^*_\Delta\), we therefore have

\[
Y_k = |X_0|^2 + [(\lambda_2 + 2\lambda_3 + \varepsilon)\Delta] \sum_{i=-m}^{-1} C^{*(i+m+1)\Delta} |X_i|^2 + \sum_{i=0}^{k-1} C^{*(i+1)\Delta} m_i.
\]
Noting that the initial sequence \( X_i < \infty \) for all \( i = -m, \ldots, 0 \), by the semimartingale convergence Lemma, for \( C = C_{\Delta}^{*} \), we have
\[
\lim_{k \to \infty} Y_k < \infty \quad a.s.
\]
By (4.14), we therefore have
\[
\limsup_{k \to \infty} C_{\Delta}^{*k\Delta}|X_k|^2 \leq \limsup_{k \to \infty} (\lambda_2 + 2\lambda_3 + \varepsilon) \Delta \sum_{i=k-[\delta(i\Delta)/\Delta]}^{k-1} C_{\Delta}^{*(i+\delta(i\Delta)/\Delta)+1\Delta}|X_i|^2
\]
\[
\leq \limsup_{k \to \infty} Y_k < \infty \quad a.s.
\]
(4.16)
Noting that \( m\Delta = \tau \), by (4.15)
\[
J_{\Delta}^{*}(C_{\Delta}^{*}\Delta, \Delta) = [(1 - \bar{\delta})(\lambda_2 + \lambda_4) + \varepsilon]C_{\Delta}^{*\tau} + \frac{1}{\Delta}(1 - C_{\Delta}^{*\Delta}) - (\lambda_1 - 2\lambda_3 - \varepsilon) = 0.
\]
(4.17)
Choosing that constant \( \sigma \) such that \( C = e^{\sigma} \) and hence \( 1 - C_{\Delta} = 1 - e^{\sigma \Delta} \). Define
\[
\bar{J}_{\Delta}(\sigma) = [(1 - \bar{\delta})(\lambda_2 + \lambda_4) + \varepsilon]e^{\sigma \tau} + \frac{1}{\Delta}(1 - e^{-\sigma \tau}) - (\lambda_1 - 2\lambda_3 - \varepsilon).
\]
(4.18)
By (4.17) for any \( \Delta < \Delta_{1}^{*} \), we have
\[
\bar{J}_{\Delta}(\sigma_{\Delta}^{*}) = 0.
\]
(4.19)
Noting that \( \lim_{\Delta \to 0}(1 - e^{-\sigma \Delta})/\Delta = \sigma \), we have
\[
\lim_{\Delta \to 0} \bar{J}_{\Delta}(\sigma) = [(1 - \bar{\delta})(\lambda_2 + \lambda_4) + \varepsilon]e^{\sigma \tau} + \sigma - (\lambda_1 - 2\lambda_3 - \varepsilon).
\]
By the definition (4.8) of \( \gamma \), (4.18) and (4.19) yields
\[
\lim_{\Delta \to 0} \sigma_{\Delta}^{*} = \gamma,
\]
which implies that for any positive \( \bar{\varepsilon} \in (0, \gamma/2) \), there exist a \( \Delta_{2}^{*} > 0 \) such that for any \( \Delta < \Delta_{2}^{*} \), we have
\[
\sigma_{\Delta}^{*} > \gamma - 2\bar{\varepsilon}.
\]
Note that (4.16), together with the definition of \( \mu_{\Delta}^{*} \) show that
\[
\limsup_{k \to \infty} e^{\sigma_{\Delta}^{*k\Delta}}|X_k|^2 < \infty \quad a.s.
\]
So there exists a finite random variable \( \gamma \) such that
\[
\limsup_{k \to \infty} e^{\sigma_{\Delta}^{*k\Delta}}|X_k|^2 < \gamma \quad a.s.
\]
Besides for any \( \Delta < \Delta_{1}^{*} \wedge \Delta_{2}^{*} \), we have
\[
0 = \limsup_{k \to \infty} \frac{\log \gamma}{k\Delta} \geq \limsup_{k \to \infty} \frac{\log(e^{\sigma_{\Delta}^{*k\Delta}}|X_k|)}{k\Delta} = \mu_{\Delta}^{*} + \limsup_{k \to \infty} \frac{2\log|X_k|}{k\Delta},
\]
which implies that
\[
\limsup_{k \to \infty} \frac{2\log|X_k|}{k\Delta} \leq -\sigma_{\Delta}^{*} \leq -\gamma + 2\bar{\varepsilon} \quad a.s.,
\]
that is
\[
\limsup_{k \to \infty} \frac{\log |X_k|}{k\Delta} \leq \sigma^*_\Delta \leq -\frac{\gamma}{2} + \varepsilon \quad a.s
\]
as required. The proof is hence complete. 

We demonstrate the process of implementing the partially truncated EM by the following example.

**Example 4.5** Consider a nonlinear scalar hybrid SDDE
\[
dx(t) = f(x(t), x(t - \delta(t)), r(t))dt + g(x(t), x(t - \delta(t)), r(t))dB(t)
\]
(4.20)
Here \(B(t), \delta(t)\) and the Markovian chain as the same as the example 3.32, where,
\[
f(x, y, i) = \begin{cases} -6x - x^5 + y & \text{if } i = 1 \\ -6x - x^5 + \frac{y}{1+y^2} & \text{if } i = 2 \end{cases}
\]
and
\[
g(x, y, i) = \begin{cases} x^2 & \text{if } i = 1 \\ x \sin^2 y & \text{if } i = 2
\end{cases}
\]
It can be seen that
\[
F_1(x, y, i) = \begin{cases} -6x + y & \text{if } i = 1 \\ -6x & \text{if } i = 2 \end{cases}
\]
and
\[
G_1(x, y, i) = \begin{cases} 0 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}
\]
Choosing \(\theta = \infty\), we then have
\[
2x^T F_1(x, y, 1) + |G_1(x, y, 1)|^2 = 2x(-6x + y) = -12x^2 + 2xy \leq -11x^2 + y^2
\]
and
\[
2x^T F_1(x, y, 2) + |G_1(x, y, 2)|^2 = 2x(-6x) = -12x^2.
\]
Therefore, for any \(i \in S\), we have
\[
2x^T F_1(x, y, i) + |G_1(x, y, i)|^2 \leq -11x^2 + y^2,
\]
that is, (4.4) is satisfied with \(\lambda_1 = 11, \lambda_2 = 1\). Moreover,
\[
2x^T F(x, y, 1) + |G(x, y, 1)|^2 = 2x(-x^5) + |x^2|^2 = -2x^6 + x^4,
\]
but
\[
-2x^6 + x^4 = -(2x^6 - x^4 + \frac{1}{8}x^2) + \frac{1}{8}x^2 = -2x^2(x^2 - \frac{1}{4})^2 + \frac{1}{8}x^2 \leq \frac{1}{8}x^2.
\]
Hence
\[
2x^T F(x, y, 1) + |G(x, y, 1)|^2 \leq \frac{1}{8}x^2
\]
and
\[
2x^T F(x, y, 2) + |G(x, y, 2)|^2
= 2x(-x^5 + \frac{y}{1+y^2}) + |x \sin^2 y|^2
\leq -2x^6 + \frac{2xy}{1+y^2} + x^2 = -2x^6 + x^2 + \frac{x^2}{1+y^2} + \frac{y^2}{1+y^2} + x^2
\leq -2x^6 + x^2 + y^2 + x^2 \leq 2x^2 + y^2.
\]
Then for any $i \in \mathbb{S}$ we have

$$2x^T F(x, y, i) + |G(x, y, i)|^2 \leq 2x^2 + y^2. \quad (4.21)$$

In other words, the (4.2) holds with the $\lambda_3 = 2, \lambda_4 = 1$. By the Theorem 4.4, the SDDEs (4.20) is almost sure exponentially stable. The left one of Figure 2 displays the almost surely asymptotic stable behaviour of the numerical solutions for the equations (4.20). The right one of Figure 2 illustrates the almost sure exponentially stability of the numerical solution produced by the partially truncated EM method, 200 sample paths are generated.

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