Efficient Score Computation and Expectation-Maximization Algorithm in Regime-Switching Models

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Abstract

This study proposes an efficient algorithm for score computation for regime-switching models, and derived from which, an efficient expectation-maximization (EM) algorithm. Different from existing algorithms, this algorithm does not rely on the forward-backward filtering for smoothed regime probabilities, and only involves forward computation. Moreover, the algorithm to compute score is readily extended to compute the Hessian matrix.

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1 Introduction

Regime-switching models have been applied extensively since Hamilton (1989) to study how time-series patterns change across different underlying economic states.

A set of tools involving filtering algorithm, EM algorithm, and an algorithm for score calculation, is popular in the estimation and inference of regime-switching models. Hamilton (1989) introduces algorithms that compute filtered probabilities of latent regimes given observations until the current period and smoothed probabilities of regimes given observations until the last period. Hamilton (1990) introduces the EM algorithm to maximum likelihood estimation (MLE). Hamilton (1996) presents the algorithm to compute the score, the derivative of log-likelihood function with respect to parameters, to implement specification tests. Notwithstanding the apparent success in earlier works, these methods entail limitations that can preclude them from being useful in large and computationally intensive problems. Both the EM algorithm of Hamilton (1990) and the score algorithm of Hamilton (1996) are built on the smoothed probability. They require the computation and storage of the filtered probabilities and the computation of the smoothed probabilities. Moreover, the introduced score algorithm does not generalize to the computation of the Hessian matrix, the second derivative of the log-likelihood function with respect to parameters. The Hessian matrix is often used to construct an estimate of the asymptotic variance of the MLE. Therefore the existing algorithms do not present a unified solution to the Hessian-based inferential exercises.

This study proposes an efficient recursive algorithm of score computation, and derived from which, an efficient EM algorithm. These algorithms do not require the forward-backward computation of smoothed probabilities, and only involve forward computation, and thus generate speedups in computation time and reduce storage burden. Moreover, the score algorithm is readily extended to compute the Hessian matrix, thereby providing a unified solution to Hessian-based inference.

At the center of the score algorithm is a time-recursive procedure we propose to compute derivatives of the product of a sequence of simpler objects. In the score computations, the likelihood function can be written as the summation of complete likelihood functions with both observation and regimes over regimes. The complete likelihood functions can be written as the products of complete likelihood functions in each period. Because the derivatives of complete likelihood functions
in each period are simple to compute, what remains is to apply the recursion we established. This algorithm naturally extends to higher other derivatives such as the Hessian matrix. Moreover, we show that the quantities involved in the expectation step of the EM algorithm can be expressed in a format that can be computed in a similar fashion to score calculation.

The rest of this paper is organized as follows. Section 2 describes the model and notation. Section 3 presents the algorithm to compute the score and Hessian matrix. Subsection 3.1 introduces the technique to compute the derivatives of the product of a sequence of factors. Subsection 3.2 presents our first algorithm to compute the score and Hessian. This algorithm, though fast in computation speed, entails an occasional numerical nuisance if the machine precision is not properly prescribed in practice. To avoid the possible problems caused by machine precision, we propose a second algorithm in Subsection 3.3. We propose a third algorithm in Subsection 3.4 that combines the first two algorithms so that it encompasses the computation simplicity of the first algorithm and also avoids the problem caused by machine precision. In Section 4, we show the algorithm in Section 3 can be applied in the expectation step of the EM algorithm.

2 Model

The conditional distribution of the observable process \((Y_t)\) is governed by an latent regime process \((S_t)\), and it admits a density

\[ g_{\theta}(Y_t|Y_{t-1}, \ldots, Y_{t-p}, S_t, S_{t-1}, \ldots, S_{t-p}, X_t), \]

where \(X_t\) is a predetermined variable (vector), \(g_{\theta}\) is a family of density functions indexed by \(\theta\). The regime transition probability is

\[ q_{\theta}(S_t|S_{t-1}, \ldots, S_{t-p}, Y_{t-1}, \ldots, Y_{t-p}, X_t), \quad (1) \]

where \(q_{\theta}\) is a family of probabilities indexed by \(\theta\). We allow the regime transition probability (1) to include information from observations \(Y_{t-1}, \ldots, Y_{t-p}, X_{t}\). For short notation, we define
\( \mathbf{Y}_t \triangleq (Y_t, \ldots, Y_{t-p+1})' \) for \( t \geq 0 \), \( \mathbf{Y}_m' \triangleq (Y_n, \ldots, Y_m) \), and \( \mathbf{X}_m' \triangleq (X_n, \ldots, X_m) \), for \( m \geq n \).

This study gives the algorithms to compute the score and hessian matrix of the log-likelihood function given the initial observations \( \mathbf{Y}_0 \), \( \ell_{n, \nu}(\theta) = \log p_{\theta, \nu}(Y_1, \ldots, Y_n|\mathbf{Y}_0, \mathbf{X}_1^n) \), where \( \nu \) is the short notation of the initial distribution of regimes \( \nu_\theta(S_0) \triangleq p_\theta(S_0|\mathbf{Y}_0) \). The log-likelihood function can be expressed as

\[
\ell_{n, \nu}(\theta) = \log \sum_{S_0} \sum_{S_1^n} p_\theta(Y_1, \ldots, Y_n, S_1, \ldots, S_n|\mathbf{Y}_0, \mathbf{S}_1^n) \nu_\theta(S_0),
\]

where the period likelihood function

\[
f_\theta(S_t, Y_t|\mathbf{Y}_{t-1}, \mathbf{S}_{t-1}, X_t) \triangleq p_\theta(S_t, Y_t|\mathbf{Y}_{t-1}, \mathbf{S}_{t-1}, X_t)
\]

\[
= g_\theta(Y_t|S_t, \mathbf{S}_{t-1}, \mathbf{Y}_{t-1}, X_t) q_\theta(S_t|\mathbf{S}_{t-1}, \mathbf{Y}_{t-1}, X_t).
\]

(2) follows because conditional on \( \{X_t\}_{t \geq 1} \), \( (Y_t, S_t) \) is a Markov chain of order \( p \) with transition density \( f_\theta(S_t, Y_t|\mathbf{Y}_{t-1}, \mathbf{S}_{t-1}, X_t) \). For the simplicity of notation, we use \( f_{\theta,t}(S_t, \mathbf{S}_{t-1}) \) as a short notation of \( f_\theta(S_t, Y_t|\mathbf{Y}_{t-1}, \mathbf{S}_{t-1}, X_t) \) and \( p_{\theta,t} \) as a short notation of the likelihood \( p_{\theta,\nu}(Y_1, \ldots, Y_t|\mathbf{Y}_0, \mathbf{X}_1^n) \). We abuse the notation and use \( f_{\theta,0}(S_0, \mathbf{S}_{-1}) \) to denote \( \nu(S_0) \), with \( \mathbf{S}_{-1} \triangleq (S_{-1}, \ldots, S_{-p+1}) \). The score function can be expressed as

\[
\nabla_\theta \ell_{n, \nu}(\theta) = \frac{s_{\theta,n}}{p_{\theta,n}},
\]

where

\[
p_{\theta,t} = \sum_{S_{t-p+1}^{t}} \left( \prod_{k=0}^{t} f_{\theta,k}(S_k, \mathbf{S}_{k-1}) \right)
\]

\[
s_{\theta,t} \triangleq \nabla_\theta p_{\theta,t} = \sum_{S_{t-p+1}^{t}} \sum_{k=0}^{t} (\nabla_\theta f_{\theta,k}(S_k, \mathbf{S}_{k-1}) \times \prod_{0 \leq \ell \leq t, \ell \neq k} f_{\theta,\ell}(S_t, \mathbf{S}_{\ell-1})),
\]

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$t = 0, 1, \ldots, n$. The Hessian matrix can be expressed as

$$\nabla^2_{\theta} \ell_{n,t}(\theta) = \frac{p_{\theta,t} \nabla^2_{\theta} p_{\theta,t} - \nabla_{\theta} p_{\theta,t} \nabla_{\theta}^T p_{\theta,t}}{p_{\theta,t}^2},$$

where $\nabla^2_{\theta} p_{\theta,t} = H_{\theta,t} + h_{\theta,t} + h_{\theta,t}^T$ and

$$H_{\theta,t} \triangleq \sum_{S_{t-p+1}}^{t} \sum_{k=0}^{S_{t-p+1}} (\nabla^2_{\theta} f_{\theta,k}(S_k, S_{k-1}) \times \prod_{0 \leq t \leq t, t \neq k} f_{\theta,t}(S_t, S_{t-1})),$$

$$h_{\theta,t} \triangleq \sum_{S_{t-p+1}}^{t} \sum_{0 \leq t_1 < t_2 \leq t} (\nabla_{\theta} f_{\theta,t_1}(S_{t_1}, S_{t_1-1}) \nabla_{\theta}^T f_{\theta,t_2}(S_{t_2}, S_{t_2-1}) \prod_{0 \leq k \leq t, k \neq t_1, t_2} f_{\theta,k}(S_k, S_{k-1})).$$

Next section lists three algorithms to compute the score and Hessian matrix.

### 3 Algorithms to compute score and Hessian

We can compute the score and Hessian matrix if we know how to compute (4)–(7). (6) can be computed as (5) by replacing $\nabla_{\theta} f_{\theta,t}(S_t, S_{t-1})$ with $\nabla^2_{\theta} f_{\theta,t}(S_t, S_{t-1})$. Thus, the main challenge is to compute (4), (5), and (7) efficiently. To explain the idea of the algorithm, we first consider the simple model where there is no regime switching in Subsection 3.1.

#### 3.1 Simple model without regime switching

In the simple model without regime switching, (4), (5), and (7) are simplified to

$$p_{\theta,t} = \prod_{k=0}^{t} f_{\theta,k},$$

$$s_{\theta,t} = \sum_{k=0}^{t} (f_{\theta,0} \cdots f_{\theta,k-1} \times \nabla_{\theta} f_{\theta,k} \times f_{\theta,k+1} \cdots f_{\theta,t})$$

$$h_{\theta,t} = \sum_{0 \leq t_1 < t_2 \leq t} (\nabla_{\theta} f_{\theta,t_1} \nabla_{\theta}^T f_{\theta,t_2} \prod_{1 \leq k \leq t, k \neq t_1, t_2} f_{\theta,k})$$

$$= s_{\theta,0} \nabla_{\theta}^T f_{\theta,1} f_{\theta,2} \cdots f_{\theta,t} + s_{\theta,1} \nabla_{\theta}^T f_{\theta,2} f_{\theta,3} \cdots f_{\theta,t} + \ldots + s_{\theta,t-1} \nabla_{\theta}^T f_{\theta,t}.$$

They can be computed according to the following algorithm.
1. Initialization: for $t = 1$

$$p_{\theta,1} = f_{\theta,0}f_{\theta,1}, \quad s_{\theta,1} = \nabla_{\theta}f_{\theta,0}f_{\theta,1} + f_{\theta,0}\nabla_{\theta}f_{\theta,1}, \quad h_{\theta,1} = \nabla_{\theta}f_{\theta,0}\nabla_{\theta}^{T}f_{\theta,1}.$$ 

2. Recursion: for $t = 2, \ldots, n$,

$$p_{\theta,t} = p_{\theta,t-1} \times f_{\theta,t},$$

$$s_{\theta,t} = s_{\theta,t-1} \times f_{\theta,t} + p_{\theta,t-1} \times \nabla_{\theta}f_{\theta,t} \tag{10}$$

$$h_{\theta,t} = h_{\theta,t-1} \times f_{\theta,t} + s_{\theta,t-1}\nabla_{\theta}^{T}f_{\theta,t} \tag{11}$$

To see why we update as (10), notice that each summand in (8) is the product of $t + 1$ terms, with one being the the derivative and the rest being the period likelihood. The summand in $s_{\theta,t-1}$ already contains one derivative, so we multiply it with the period likelihood function $f_{\theta,t}$, which brings the first term in (10). We add the second term in (10) to include the term with derivative at time $t$. To write it more specifically,

$$t = 2, \quad s_{\theta,2} = s_{\theta,1} \times f_{\theta,2} + p_{\theta,1} \times \nabla f_{\theta,2} = (\nabla_{\theta}f_{\theta,0}f_{\theta,1} + f_{\theta,0}\nabla_{\theta}f_{\theta,1})f_{\theta,2} + f_{\theta,0}f_{\theta,1}\nabla f_{\theta,2}$$

$$\vdots \quad \vdots$$

$$t = n, \quad s_{\theta,n} = s_{\theta,n-1} \times f_{\theta,n} + p_{\theta,n-1} \times \nabla f_{\theta,n}$$

$$= (\nabla_{\theta}f_{\theta,0}f_{\theta,1} \cdots f_{\theta,n-1} + \ldots + f_{\theta,0} \cdots f_{\theta,n-2} \nabla_{\theta}f_{\theta,n-1})f_{\theta,n} + f_{\theta,0} \cdots f_{\theta,n-1} \nabla_{\theta}f_{\theta,n}$$

$$= \nabla_{\theta}f_{\theta,0}f_{\theta,1} \cdots f_{\theta,n} + f_{\theta,0}\nabla_{\theta}f_{\theta,1}f_{\theta,2} \cdots f_{\theta,n} + \ldots + f_{\theta,0} \cdots f_{\theta,n-1} \nabla_{\theta}f_{\theta,n}.$$ 

(11) is updated with the same reasoning. Each summand in (9) is the product of $t + 1$ terms, with two being the derivatives and the rest being the period likelihood. The summand in $h_{\theta,t-1}$ already contains two derivatives, so we multiply it with $f_{\theta,t}$, which is the first term in (11). For the second term in (11), the summand in $s_{\theta,t-1}$ contains one derivative, so we multiply it with $\nabla_{\theta}^{T}f_{\theta,t}$. Next subsection explains the algorithm to compute the score and Hessian matrix for regime-switching models.
3.2 The first algorithm to compute the score and Hessian matrix

Subsection 3.1 explains the algorithm to compute (4), (5), and (7) in the model without regime switching. For the model with regime switching, we need to modify the algorithm and take summation over regimes at appropriate steps. For each regime \( S_t \), it is involved in \( f_{\theta,k}(S_k, S_{k-1}) \), \( t \leq k \leq t + p \), but not in \( f_{\theta,\ell}(S_t, S_{\ell-1}) \), \( \ell \geq t + p + 1 \). Thus, we can take summation over \( S_t \) at the recursion step of \( t + p \). The complete algorithm can be described as follows.

1. Initialization: for \( t = 1 \)

\[
p_{\theta,1}(S_1) = \sum_{S_{-p+1}} f_{\theta,0}(S_0, S_{-1}) f_{\theta,1}(S_1, S_0)
\]

\[
s_{\theta,1}(S_1) = \sum_{S_{-p+1}} [\nabla \ell f_{\theta,0}(S_0, S_{-1}) f_{\theta,1}(S_1, S_0) + f_{\theta,0}(S_0, S_{-1}) \nabla f_{\theta,1}(S_1, S_0)]
\]

\[
h_{\theta,1}(S_1) = \sum_{S_{-p+1}} \nabla f_{\theta,0}(S_0, S_{-1}) \nabla^T f_{\theta,1}(S_1, S_0)
\]

\[
H_{\theta,1}(S_1) = \sum_{S_{-p+1}} [\nabla^2 f_{\theta,0}(S_0, S_{-1}) f_{\theta,1}(S_1, S_0) + f_{\theta,0}(S_0, S_{-1}) \nabla^2 f_{\theta,1}(S_1, S_0)]
\]

2. Recursion: for \( t = 2, \ldots, n \),

\[
p_{\theta,t}(S_t) = \sum_{S_{t-p}} p_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1})
\]

\[
s_{\theta,t}(S_t) = \sum_{S_{t-p}} [s_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + p_{\theta,t-1}(S_{t-1}) \times \nabla f_{\theta,t}(S_t, S_{t-1})]
\]

\[
h_{\theta,t}(S_t) = \sum_{S_{t-p}} [h_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + s_{\theta,t-1}(S_{t-1}) \times \nabla^T f_{\theta,t}(S_t, S_{t-1})]
\]

\[
H_{\theta,t}(S_t) = \sum_{S_{t-p}} [H_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + p_{\theta,t-1}(S_{t-1}) \times \nabla^2 f_{\theta,t}(S_t, S_{t-1})]
\]

3. Let \( H_{\theta,n} = \sum_{S_n} H_{\theta,n}(S_n) \), \( h_{\theta,n} = \sum_{S_n} h_{\theta,n}(S_n) \), \( s_{\theta,n} = \sum_{S_n} s_{\theta,n}(S_n) \), and \( p_{\theta,n} = \sum_{S_n} p_{\theta,n}(S_n) \).

Compute the score and the Hessian matrix:

\[
\nabla_{\theta} \ell_{n,\nu}(\theta) = \frac{s_{\theta,n}}{p_{\theta,n}}
\]
\[
\n\nabla_\theta^2 \ell_{n, \nu}(\theta) = \frac{H_{\theta, n} + h_{\theta, n} + h_{\theta, n}^T}{p_{\theta, n}} - \frac{s_{\theta, n}}{p_{\theta, n}} \times \frac{s_{\theta, n}^T}{p_{\theta, n}}.
\]

(13)

The preceding algorithm entails an occasional, numerical nuisance if the machine precision is not properly prescribed in practice. When the difference between \(p_{\theta, n}\) and zero is smaller than the machine precision, computation software treats \(p_{\theta, n}\) as zero, and (12) and (13) return infinity or invalid numbers. To avoid the problem, we modify the algorithm in the next section.

### 3.3 The second algorithm to compute the score and Hessian matrix

To avoid the problem caused by the division in (12) and (13) and the machine precision, we re-scale each value with the period likelihood at each recursion step, instead of making division at the end of the algorithm. More specifically, from \(p_{\theta, t} = \prod_{k=1}^t p_{\theta, \nu}(Y_k | \overline{Y}_{0}^{k-1}, X_1^k)\), (5) and (7) divided by \(p_{\theta, t}\) can be rewritten as

\[
\begin{align*}
\frac{s_{\theta, t}}{p_{\theta, t}} &= \frac{\sum s_{t-p+1}^t \sum_{k=0}^t (\nabla f_{\theta, k}(S_k, S_{k-1}) \prod_{0 \leq \ell \leq t, \ell \neq k} f_{\theta, \ell}(S_\ell, S_{\ell-1}))}{\prod_{k=1}^t p_{\theta, \nu}(Y_k | \overline{Y}_{0}^{k-1}, X_1^k)} \\
\frac{h_{\theta, t}}{p_{\theta, t}} &= \frac{\sum s_{t-p+1}^t \sum_{1 \leq t_1 < t_2 \leq t} (\nabla f_{\theta, t_1}(S_{t_1}, S_{t_1-1}) \nabla f_{\theta, t_2}(S_{t_2}, S_{t_2-1}) \prod_{1 \leq k \leq t, k \neq t_1, t_2} f_{\theta, k}(S_k, S_{k-1}))}{\prod_{k=1}^t p_{\theta}(Y_k | \overline{Y}_{0}^{k-1}, S_0, X_1^k)}.
\end{align*}
\]

Then we can divide \(s_{\theta, t}\), \(h_{\theta, t}\), and \(H_{\theta, t}\) with \(p_{\theta, \nu}(Y_t | \overline{Y}_{0}^{t-1}, X_1^t)\) at the recursion step at time \(t\), instead of making division with \(p_{\theta, n}\) at the last step. The complete algorithm can be described as follows.

1. Initialization: for \(t = 1\),

\[
\begin{align*}
p_{\theta, 1}(S_1) &= \sum_{S_{-p+1}} f_{\theta, 0}(S_0, S_{-1}) f_{\theta, 1}(S_1, S_0) \\
s_{\theta, 1}(S_1) &= \sum_{S_{-p+1}} \nabla f_{\theta, 0}(S_0, S_{-1}) f_{\theta, 1}(S_1, S_0) + f_{\theta, 0}(S_0, S_{-1}) \nabla f_{\theta, 1}(S_1, S_0) \\
h_{\theta, 1}(S_1) &= \sum_{S_{-p+1}} \nabla f_{\theta, 0}(S_0, S_{-1}) \nabla f_{\theta, 1}(S_1, S_0) \\
H_{\theta, 1}(S_1) &= \sum_{S_{-p+1}} \nabla^2 f_{\theta, 0}(S_0, S_{-1}) f_{\theta, 1}(S_1, S_0) + f_{\theta, 0}(S_0, S_{-1}) \nabla^2 f_{\theta, 1}(S_1, S_0)
\end{align*}
\]
In this algorithm, for \( \theta, n \),

\[
\begin{align*}
\tilde{p}_{\theta,t}(S_t) &= \frac{p_{\theta,t}(S_t)}{\sum_{S_t} p_{\theta,t}(S_t)}, \quad \tilde{s}_{\theta,t}(S_t) = \frac{s_{\theta,t}(S_t)}{\sum_{S_t} s_{\theta,t}(S_t)}, \quad \tilde{h}_{\theta,t}(S_t) = \frac{h_{\theta,t}(S_t)}{\sum_{S_t} h_{\theta,t}(S_t)} \quad \text{and} \quad \tilde{f}_{\theta,t}(S_t) = \frac{f_{\theta,t}(S_t)}{\sum_{S_t} f_{\theta,t}(S_t)}.
\end{align*}
\]

2. Recursion: for \( t = 2, \ldots, n \),

\[
\begin{align*}
\tilde{g}_{\theta,t}(S_t) &= \sum_{S_{t-1}} \left[ \tilde{g}_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + \tilde{p}_{\theta,t-1}(S_{t-1}) \times \nabla^2_{\theta} f_{\theta,t}(S_t, S_{t-1}) \right] \\
\tilde{h}_{\theta,t}(S_t) &= \sum_{S_{t-1}} \left[ \tilde{h}_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + \tilde{s}_{\theta,t-1}(S_{t-1}) \times \nabla_{\theta} f_{\theta,t}(S_t, S_{t-1}) \right] \\
\tilde{s}_{\theta,t}(S_t) &= \sum_{S_{t-1}} \left[ \tilde{s}_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1}) + \tilde{p}_{\theta,t-1}(S_{t-1}) \times \nabla_{\theta} f_{\theta,t}(S_t, S_{t-1}) \right] \\
\tilde{p}_{\theta,t}(S_t) &= \sum_{S_{t-1}} \tilde{p}_{\theta,t-1}(S_{t-1}) \times f_{\theta,t}(S_t, S_{t-1})
\end{align*}
\]

Re-scale: \( \tilde{p}_{\theta,t}(S_t) = \frac{p_{\theta,t}(S_t)}{\sum_{S_t} p_{\theta,t}(S_t)}, \quad \tilde{s}_{\theta,t}(S_t) = \frac{s_{\theta,t}(S_t)}{\sum_{S_t} s_{\theta,t}(S_t)}, \quad \tilde{h}_{\theta,t}(S_t) = \frac{h_{\theta,t}(S_t)}{\sum_{S_t} h_{\theta,t}(S_t)} \quad \text{and} \quad \tilde{f}_{\theta,t}(S_t) = \frac{f_{\theta,t}(S_t)}{\sum_{S_t} f_{\theta,t}(S_t)} \).

3. Let \( H_{\theta,n} = \sum_{S_n} \tilde{g}_{\theta,n}(S_n), \quad h_{\theta,n} = \sum_{S_n} \tilde{h}_{\theta,n}(S_n), \quad s_{\theta,n} = \sum_{S_n} \tilde{s}_{\theta,n}(S_n), \quad \text{and} \quad p_{\theta,n} = \sum_{S_n} \tilde{p}_{\theta,n}(S_n). \) Compute the score and the Hessian matrix:

\[
\begin{align*}
\nabla_{\theta} \ell_{n,\nu}(\theta) &= s_{\theta,n} \\
\nabla^2_{\theta} \ell_{n,\nu}(\theta) &= H_{\theta,n} + h_{\theta,n} + h_{\theta,n}^T - s_{\theta,n} s_{\theta,n}^T.
\end{align*}
\]

In this algorithm, for \( t \geq 2 \), \( p_{\theta,t}(S_t) = p_{\theta,\nu}(Y_t, S_t|Y_0^{t-1}, X_1^t), \sum_{S_t} p_{\theta,t}(S_t) = p_{\theta,\nu}(Y_t|Y_0^{t-1}, X_1^t), \) and \( \tilde{p}_{\theta,t}(S_t) = p_{\theta}(S_t|Y_0^{t}, X_1^t) \). Thus, the recursion in this algorithm extends the standard prediction and update algorithm of likelihood evaluation to compute the score and the Hessian matrix simultaneously.

### 3.4 A combination of the two algorithms

Compared to the algorithm in 3.2, the algorithm in Subsection 3.3 might apply to a wider range of applications because it avoids the problem caused by machine precision in practice. This algorithm,
however, is slower than the one in Subsection 3.2 because of the re-scale step. When the number of unknown parameter in $\theta$ is $|\theta|$, then we need to making division $\left(1 + |\theta| + |\theta|^2 + \frac{|\theta|(|\theta|+1)}{2}\right)$ times at the re-scale step for each $t$, while we only need to make division once in the algorithm of Subsection 3.2.

In this subsection, we give a third algorithm that combines the two algorithms. More specifically, we use the first algorithm if $p_{\theta,t}$ is larger than a specified threshold $B \epsilon$ and the second algorithm if otherwise, where $\epsilon$ is the machine precision and $B$ is an adjusted number set by the user. In this study, we set $B = 1000$. Then this algorithm encompasses the computation simplicity of the first algorithm and also avoids the problem caused by machine precision. We use $\delta$ to denote the machine precision.

1. Initialization: $B = 1000$. Flag=FALSE. For $t = 1$,

\[
p_{\theta,t} (S_1) = \sum_{S_{-p+1}} f_{\theta,0} (S_0, S_{-1}) f_{\theta,1} (S_1, S_0)
\]

\[
s_{\theta,1} (S_1) = \sum_{S_{-p+1}} \nabla f_{\theta,0} (S_0, S_{-1}) f_{\theta,1} (S_1, S_0) + f_{\theta,0} (S_0, S_{-1}) \nabla f_{\theta,1} (S_1, S_0)
\]

\[
h_{\theta,1} (S_1) = \sum_{S_{-p+1}} \nabla f_{\theta,0} (S_0, S_{-1}) \nabla^T f_{\theta,1} (S_1, S_0)
\]

\[
H_{\theta,1} (S_1) = \sum_{S_{-p+1}} \nabla^2 f_{\theta,0} (S_0, S_{-1}) f_{\theta,1} (S_1, S_0) + f_{\theta,0} (S_0, S_{-1}) \nabla^2 f_{\theta,1} (S_1, S_0)
\]

If $\sum_{S_1} p_{\theta,2} (S_1) \leq B \epsilon$, assign Flag=True and re-scale: $\tilde{p}_{\theta,1} (S_1) = \frac{p_{\theta,1} (S_1)}{\sum_{S_1} p_{\theta,1} (S_1)}$, $\tilde{h}_{\theta,1} (S_1) = \frac{h_{\theta,1} (S_1)}{\sum_{S_1} p_{\theta,1} (S_1)}$, $\tilde{h}_{\theta,1} (S_1) = \frac{H_{\theta,1} (S_1)}{\sum_{S_1} p_{\theta,1} (S_1)}$.

2. Recursion: for $t = 2, \ldots, n$,

If Flag == FALSE,

\[
H_{\theta,t} (S_t) = \sum_{S_{t-p}} \left[ H_{\theta,t-1} (S_{t-1}) \times f_{\theta,t} (S_t, S_{t-1}) + p_{\theta,t-1} (S_{t-1}) \times \nabla^2 f_{\theta,t} (S_t, S_{t-1}) \right]
\]

\[
h_{\theta,t} (S_t) = \sum_{S_{t-p}} \left[ h_{\theta,t-1} (S_{t-1}) \times f_{\theta,t} (S_t, S_{t-1}) + s_{\theta,t-1} (S_{t-1}) \times \nabla^T f_{\theta,t} (S_t, S_{t-1}) \right]
\]

\[
s_{\theta,t} (S_t) = \sum_{S_{t-p}} \left[ s_{\theta,t-1} (S_{t-1}) \times f_{\theta,t} (S_t, S_{t-1}) + p_{\theta,t-1} (S_{t-1}) \times \nabla f_{\theta,t} (S_t, S_{t-1}) \right]
\]
\[ p_{\theta,t}(\mathbf{S}_t) = \sum_{S_{t-p}} p_{\theta,t-1}(\mathbf{S}_{t-1}) \times f_{\theta,t}(S_t, \mathbf{S}_{t-1}) \]

If \( \sum_{\mathbf{S}_t} p_{\theta,t}(\mathbf{S}_t) \leq Be \), assign Flag=TRUE and re-scale: \( \tilde{p}_{\theta,t}(\mathbf{S}_t) = \frac{p_{\theta,t}(\mathbf{S}_t)}{\sum_{\mathbf{S}_t} p_{\theta,t}(\mathbf{S}_t)} \), \( \tilde{h}_{\theta,t}(\mathbf{S}_t) = \frac{h_{\theta,t}(\mathbf{S}_t)}{\sum_{\mathbf{S}_t} p_{\theta,t}(\mathbf{S}_t)} \), \( \tilde{s}_{\theta,t}(\mathbf{S}_t) = \frac{s_{\theta,t}(\mathbf{S}_t)}{\sum_{\mathbf{S}_t} p_{\theta,t}(\mathbf{S}_t)} \).

If Flag == TRUE,

\[ h_{\theta,t}(\mathbf{S}_t) = \sum_{S_{t-p}} \left[ \tilde{h}_{\theta,t-1}(\mathbf{S}_{t-1}) \times f_{\theta,t}(S_t, \mathbf{S}_{t-1}) + \tilde{p}_{\theta,t-1}(\mathbf{S}_{t-1}) \times \nabla^2 \tilde{f}_{\theta,t}(S_t, \mathbf{S}_{t-1}) \right] \]

\[ s_{\theta,t}(\mathbf{S}_t) = \sum_{S_{t-p}} \left[ \tilde{s}_{\theta,t-1}(\mathbf{S}_{t-1}) \times f_{\theta,t}(S_t, \mathbf{S}_{t-1}) + \tilde{p}_{\theta,t-1}(\mathbf{S}_{t-1}) \times \nabla \tilde{f}_{\theta,t}(S_t, \mathbf{S}_{t-1}) \right] \]

3. If Flag==FALSE, let \( H_{\theta,n} = \sum_{\mathbf{S}_n} H_{\theta,n} (\mathbf{S}_n) \), \( h_{\theta,n} = \sum_{\mathbf{S}_n} h_{\theta,n} (\mathbf{S}_n) \), \( s_{\theta,n} = \sum_{\mathbf{S}_n} s_{\theta,n} (\mathbf{S}_n) \), and \( p_{\theta,n} = \sum_{\mathbf{S}_n} p_{\theta,n} (\mathbf{S}_n) \). Compute the score and the Hessian matrix:

\[ \nabla_{\theta} \ell_{n,t}(\theta) = \frac{s_{\theta,n}}{p_{\theta,n}} \]

\[ \nabla^2_{\theta} \ell_{n,t}(\theta) = \frac{H_{\theta,n} + h_{\theta,n} + s_{\theta,n}}{p_{\theta,n}} - \frac{s_{\theta,n}}{p_{\theta,n}} \times \frac{s_{\theta,n}^T}{p_{\theta,n}} \]

If Flag==TRUE, let \( H_{\theta,n} = \sum_{\mathbf{S}_n} \tilde{H}_{\theta,n} (\mathbf{S}_n) \), \( h_{\theta,n} = \sum_{\mathbf{S}_n} \tilde{h}_{\theta,n} (\mathbf{S}_n) \), \( s_{\theta,n} = \sum_{\mathbf{S}_n} \tilde{s}_{\theta,n} (\mathbf{S}_n) \), and \( p_{\theta,n} = \sum_{\mathbf{S}_n} \tilde{p}_{\theta,n} (\mathbf{S}_n) \). Compute the score and the Hessian matrix:

\[ \nabla_{\theta} \ell_{n,t}(\theta) = s_{\theta,n} \]

\[ \nabla^2_{\theta} \ell_{n,t}(\theta) = H_{\theta,n} + h_{\theta,n} + s_{\theta,n}^T - s_{\theta,n} s_{\theta,n}^T \]
4 EM algorithm

The algorithm in Section 3 to compute (5) can be generalized to compute

$$\frac{\sum_{S_{n-p+1}}^{S_n} \sum_{t=0}^{n} \left( r_{\theta,t}(S_t, S_{t-1}) \times \prod_{0 \leq k \leq n, k \neq t} f_{\theta,k}(S_k, S_{k-1}) \right)}{\prod_{t=0}^{n} f_{\theta,t}(S_t, S_{t-1})},$$

(14)

where $r_{\theta,k}(S_k, S_{k-1})$ is a short hand notation of $r_{\theta}(S_k, S_{k-1}, Y_k, Y_{k-1}, X_k)$ and is a function that depends on $(S_k, S_{k-1}, Y_k, Y_{k-1}, X_k)$. An example of (14) is the quantities involved in the E-step of the expectation-maximization algorithm. Hamilton (1990) showed the EM estimates can be expressed with quantities of the form $\sum_{t=1}^{n} r_{\theta}(S_t, S_{t-1}, Y_t, Y_{t-1}, X_t) p_{\theta}(S_t, S_{t-1} | Y^n_0, X^n_1)$. To see it can be expressed in the form of (14),

$$\sum_{t=1}^{n} \sum_{S_t}^{S^n_t} r_{\theta}(S_t, S_{t-1}, Y_t, Y_{t-1}, X_t) \prod_{0 \leq k \leq n} f_{\theta,k}(S_k, S_{k-1})$$

$$= \sum_{S_t}^{S^n_t} \sum_{t=1}^{n} r_{\theta}(S_t, S_{t-1}, Y_t, Y_{t-1}, X_t) p_{\theta}(S_t, S_{t-1} | Y^n_0, X^n_1)$$

$$= \sum_{S_t}^{S^n_t} \sum_{t=1}^{n} r_{\theta}(S_t, S_{t-1}, Y_t, Y_{t-1}, X_t) \prod_{k=1}^{n} f_{\theta,k}(S_k, S_{k-1})$$

$$= \sum_{S_t}^{S^n_t} \sum_{t=1}^{n} r_{\theta}(S_t, S_{t-1}, Y_t, Y_{t-1}, X_t) p_{\theta}(Y_1, \ldots, Y_n | Y^n_0, X^n_1)$$

which is in the form of (14).

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