Abstract We argue by explicit computations that, although the area product, horizon radii product, entropy product and \textit{irreducible mass product} of the event horizon and Cauchy horizon are universal, the \textit{surface gravity product}, \textit{surface temperature product} and \textit{Komar energy product} of the said horizons do not seem to be universal for Kerr-Newman (KN) black hole space-time. We show the black hole mass formula on the \textit{Cauchy horizon} following the seminal work by Smarr[1] for the outer horizon. We also prescribed the \textit{four} laws of black hole mechanics for the \textit{inner horizon}. New definition of the extremal limit of a black hole is discussed.

1 Introduction

The intriguing features of a stationary axially symmetric black hole is that the product of the horizon areas are often be independent of the mass of the black hole space-time. Rather such products depend on the charge and angular momentum of the black hole space-time. They may also be formulated in terms of the proper radii of the Cauchy horizon (Cauchy horizon) and event horizon (event horizon).

It is also known that every regular axially symmetric and stationary space-time of Einstein-Maxwell system with surrounding matter has a regular Cauchy horizon \(\mathcal{H}^-\) inside the event horizon \(\mathcal{H}^+\) if and only if the angular momentum \(J\) and charge \(Q\) do not both vanish. In contrast, the Cauchy horizon becomes singular and approaches a curvature singularity in the limit \(J \to 0, Q \to 0\) [2,3,4].

Department of Physics
Vivekananda Satabarshiki Mahavidyalaya
Manikpara, West Midnapur
West Bengal-721513, India.
E-mail: pppradhan77@gmail.com
The presence of the Cauchy horizon implies that in Boyer-Lindquist coordinates, the stationary and axi-symmetric Einstein-Maxwell electro-vacuum equations are hyperbolic in nature with in the interior vicinity of $\mathcal{H}^+$. The two horizons $\mathcal{H}^+$ and $\mathcal{H}^-$ are described by the future and past boundary of this hyperbolic region. Remarkably, if the inner Cauchy horizon exists (i.e. if $J$ and $Q$ do not vanish simultaneously), then the product of the area $A_{\pm}$ of the horizons $\mathcal{H}^\pm$ for KN family are expressed by the relation $[2,3,4]$

$$A_+A_- = (8\pi)^2 \left( J^2 + \frac{Q^4}{4} \right).$$

(1)

which is remarkably independent of the mass ($M$). $J$ and $Q$ are the angular momentum and charge of the black hole, respectively.

From the various string theoretic models and holographic principle followed by an observation suggests that the products of the certain Killing horizon areas are in fact independent of the black hole mass. From the idea of statistical mechanics based on microscopic models counting BPS states determined that this product of areas are sometimes quantized. Thus in the super-symmetric extremal limit, one obtains $[5,6,7,8,9,10,11]$

$$A_+A_- = \left( 8\pi \ell_{pl}^2 \right)^2 N, \quad N \in \mathbb{N}.$$  

(2)

where $\ell_{pl}$ is the Planck length. When one moves away from the extremality and super-symmetry, the area products are discretized $[12]$ in terms of the Planck area and the fine structure constant i.e.

$$A_+A_- = \left( 8\pi \ell_{pl}^2 \right)^2 \left[ \ell(\ell + 1) + \frac{\alpha^2 q^2}{4} \right].$$

(3)

which implies that the quantization rules are break down, only because the fine structure constant ($\alpha$) is not an integer. Here $\alpha \in \mathbb{N}$ and $q \in \mathbb{N}$.

The fact is that, the Cauchy horizon is an “infinite blue-shift” region and classically unstable due to the linear perturbation. Thus, when an observer crosses the Cauchy horizon $r = r_-$, he/she may speculated all of the events which occur at “Region-I” and also seen the electromagnetic and gravitational field oscillations at infinite frequency which is actually occur at finite frequency in the “Region-I” $[13]$.

Despite the above fact that the Cauchy horizon is an interesting venue where we study the following features of both the charged rotating space-time and the rotating space-time.

– We prove that in section $[2]$, like the area and entropy product, the surface gravity product, surface temperature or black hole temperature product and Komar energy product of both inner horizon and outer horizon do not shows any global properties due to the mass dependence. Such products are not universal in nature.
We explicitly show that in section (3), the black hole mass or ADM mass can be expressed as in terms of the area of Cauchy horizon $H^-$:

$$M^2 = \frac{A_-}{16\pi} + \frac{4\pi J^2}{A_-} + \frac{Q^2}{2} + \frac{\pi Q^4}{A_-}.$$  \tag{4}

and we prove that the mass can be expressed as sum of the surface energy, rotational energy and electro-magnetic energy of the Cauchy horizon $H^-$:

$$M = E_{s-} + E_{r-} + E_{em-}. \tag{5}$$

Also we find that in section (4), the Christodoulou-Ruffini \[14\] mass formula may be expressed as in terms of the area of the Cauchy horizon $H^-$:

$$M^2 = (M_{irr-} + \frac{Q^2}{4M_{irr-}})^2 + \frac{J^2}{4(M_{irr-})^2}. \tag{6}$$

We further investigate the laws of the black hole mechanics for inner horizon in section (5). We also point out that the product of Christodoulou’s irreducible mass of inner horizon (Cauchy horizon) and outer horizon (event horizon) are independent of mass. i.e.

$$M_{irr+}M_{irr-} = \sqrt{\frac{A_+ A_-}{16\pi}} = \sqrt{\frac{J^2 + Q^4}{4}}. \tag{7}$$

We also shortly derive an identity $K_{\chi^{\mu-}} = 2S_{T^-}$ on the Cauchy horizon in section (6).

The entropy of the Cauchy horizon may be expressed in the form $S_{-} = \frac{E_{-}}{2T_{-}}$ is described in section (7).

### 2 Charged Rotating Black Hole

The complete gravitational collapse of a charged body always produces Kerr-Newman (KN) black hole \[15\], which is the most general class among the classical black hole solutions. It is also uniquely described by the electro vacuum black hole solutions of the Einstein-Maxwell system. It can be specified by three parameters: the black hole mass $M$, the charge $Q$ and the angular momentum per unit mass $a = J/M$. As long as $M^2 \geq Q^2 + a^2$ the KN metric describes a black hole, otherwise it has a naked ringlike singularity. It possess two horizons namely event horizon ($H^+$) or outer horizon and Cauchy horizon ($H^-$) or inner horizon. The proper radii of event horizon and Cauchy horizon are

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \text{ and } r_+ > r_- \tag{8}$$

whose product is

$$r_+r_- = a^2 + Q^2. \tag{9}$$
It is speculated that, it does not depend on mass but depends on charge and Kerr parameter \([16]\).

Then the area\([17,18]\) of both the horizons \(\mathcal{H}^\pm\) are

\[
A^\pm = \int \int \sqrt{\det g_{\theta \theta}} = 4\pi (r^2_\pm + a^2) .
\] (10)

The angular velocity of \(\mathcal{H}^\pm\) are

\[
\Omega^\pm = \frac{a}{r^2_\pm + a^2} .
\] (11)

The semiclassical Bekenstein-Hawking entropy of \(\mathcal{H}^\pm\) reads (in units in which \(G = h = c = 1\))

\[
S^\pm = \frac{A^\pm}{4} = \pi (r^2_\pm + a^2) .
\] (12)

The surface gravity of \(\mathcal{H}^\pm\) is given by

\[
\kappa^\pm = \frac{r^\pm - r_\mp}{2(r^2_\pm + a^2)} \text{ and } \kappa_+ > \kappa_- .
\] (13)

and the black hole temperature or Hawking temperature of \(\mathcal{H}^\pm\) reads as

\[
T^\pm = \frac{\kappa^\pm}{2\pi} = \frac{r^\pm - r_\mp}{4\pi (r^2_\pm + a^2)} .
\] (14)

It should be noted that event horizon is hotter than the Cauchy horizon i.e. \(T_+ > T_-\).

The Komar energy for \(\mathcal{H}^\pm\) is given by (which will be discuss elaborately on the section (7))

\[
E^\pm = 2S^\pm T^\pm = \pm \sqrt{\mathcal{M}^2 - a^2 - Q^2} .
\] (15)

Finally, the horizon Killing vector field may be defined for \(\mathcal{H}^\pm\) are

\[
\chi^\pm a = (\partial_t)^a + \Omega^\pm (\partial_\phi)^a .
\] (16)

If, in addition, the black hole is non-extremal (i.e., if there exists the trapped surface interior of the outer horizon) then the following relations are hold:

\[
A_+ > \sqrt{(8\pi)^2 \left( J^2 + \frac{Q^4}{4} \right)} > A_- .
\] (17)

Also the product of entropy is given by

\[
S_+ S_- = (2\pi)^2 \left( J^2 + \frac{Q^4}{4} \right) .
\] (18)
It is also independent of mass ($\mathcal{M}$). The entropy of the non-extremal cases satisfied the following inequality:

$$S_+ > \sqrt{(2\pi)^2 \left( J^2 + \frac{Q^4}{4} \right)} > S_- .$$

(19)

Similarly, we can compute the product of surface gravity of $\mathcal{H}^\pm$ is given by

$$\kappa_+ \kappa_- = -\frac{(r_+-r_-)^2}{4(r_+^2 + a^2)(r_-^2 + a^2)} = -\frac{\mathcal{M}^2 - a^2 - Q^2}{(r_+^2 + a^2)(r_-^2 + a^2)}. \quad (20)$$

The product of surface temperature of $\mathcal{H}^\pm$ reads

$$T_+ T_- = \frac{(r_+-r_-)^2}{(4\pi)^2(r_+^2 + a^2)(r_-^2 + a^2)} = \frac{\mathcal{M}^2 - a^2 - Q^2}{(2\pi)^2(r_+^2 + a^2)(r_-^2 + a^2)}. \quad (21)$$

and the product of Komar energy of $\mathcal{H}^\pm$ is

$$E_+ E_- = (2S_+ T_+)(2S_- T_-) = -(\mathcal{M}^2 - a^2 - Q^2). \quad (22)$$

It seems that these products are not universal.

In case of pure Einstein gravity (with out Maxwell field), the above relations are reduces to:

For the proper radii product of $\mathcal{H}^\pm$:

$$r_+ r_- = a^2. \quad (23)$$

For the area product of $\mathcal{H}^\pm$:

$$A_+ A_- = (8\pi J)^2. \quad (24)$$

For the entropy product of $\mathcal{H}^\pm$:

$$S_+ S_- = (2\pi J)^2. \quad (25)$$

For the surface gravity product of $\mathcal{H}^\pm$:

$$\kappa_+ \kappa_- = -\frac{(r_+-r_-)^2}{4(r_+^2 + a^2)(r_-^2 + a^2)} = -\frac{\mathcal{M}^2 - a^2}{(r_+^2 + a^2)(r_-^2 + a^2)}. \quad (26)$$

For the temperature product of $\mathcal{H}^\pm$:

$$T_+ T_- = \frac{(r_+-r_-)^2}{(4\pi)^2(r_+^2 + a^2)(r_-^2 + a^2)} = \frac{\mathcal{M}^2 - a^2}{(2\pi)^2(r_+^2 + a^2)(r_-^2 + a^2)}. \quad (27)$$

For the Komar energy product of $\mathcal{H}^\pm$:

$$E_+ E_- = (2S_+ T_+)(2S_- T_-) = -(\mathcal{M}^2 - a^2). \quad (28)$$

So, the product of the area and entropy of the both horizons are proportional to the square of the spin parameter $J$. Surface gravity product, surface temperature product and Komar energy product depends on mass. Thus, we may conclude that they are not universal except the area product and entropy product.
3 Smarr Formula for Cauchy horizon ($\mathcal{H}^-$)

In the original paper by Larry Smarr [1], the area for the charged rotating black hole is described by the following relation

$$ A = 4\pi \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2} \right). \quad (29) $$

It is indeed constant over the exterior horizon. We suggest here that there are two horizons, so correspondingly both areas must be a constant i.e. the area can be expressed as

$$ A_\pm = 4\pi \left( 2M^2 - Q^2 \pm 2\sqrt{M^4 - J^2 - M^2Q^2} \right). \quad (30) $$

Inverting the above relation one can obtain the black hole mass or ADM mass can be expressed as in terms of area of both horizons $\mathcal{H}^\pm$,

$$ M^2 = \frac{A_\pm}{16\pi} + \frac{4\pi J^2}{A_\pm} + \frac{Q^2}{2} + \frac{\pi Q^4}{A_\pm}. \quad (31) $$

It is remarkable that the mass can be expressed as in terms of both area of $\mathcal{H}^+$ and $\mathcal{H}^-$. Now what happens for the mass differential. It is indeed expressed as three physical invariants of both $\mathcal{H}^+$ and $\mathcal{H}^-$,

$$ dM = T_\pm dA_\pm + \Omega_\pm dJ + \Phi_\pm dQ. \quad (32) $$

where

$$ T_\pm = \frac{1}{M} \left( \frac{1}{32\pi} \left( \frac{2\pi J^2}{A_\pm} - \frac{\pi Q^4}{2A_\pm^2} \right) \right), $$

$$ \Omega_\pm = \frac{4\pi J}{MA_\pm}, $$

$$ \Phi_\pm = \frac{1}{M} \left( \frac{Q}{2} + \frac{2\pi Q^3}{A_\pm} \right). \quad (33) $$

where

$T_\pm$ = Effective surface tension for $\mathcal{H}^+$ and $\mathcal{H}^-$

$\Omega_\pm$ = Angular velocity for $\mathcal{H}^\pm$

$\Phi_\pm$ = Electromagnetic potentials for $\mathcal{H}^\pm$
The effective surface tension may be rewritten as

$$T_{\pm} = \frac{1}{M} \left( \frac{1}{32\pi} - \frac{2\pi J^2}{A_{\pm}} - \frac{\pi Q^2}{2A_{\pm}} \right)$$

(34)

$$= \frac{1}{32\pi M} \left( 1 - \frac{16\pi^2(4J^2 + Q^2)}{A_{\pm}^2} \right)$$

$$= \frac{1}{16\pi M} \left( 1 - \frac{(2M^2 - Q^2)}{r_{\pm}^2 + a^2} \right)$$

$$= \pm \frac{\sqrt{M^2 - a^2 - Q^2}}{8\pi(r_{\pm}^2 + a^2)}$$

$$= \frac{r_{\pm} - M}{8\pi(r_{\pm}^2 + a^2)} = \kappa_{\pm}$$

(35)

where $\kappa_{\pm}$ are the surface gravity of $H_{\pm}$ as previously defined.

Thus the mass can be expressed in terms of these quantities both for $H_{\pm}$ as a simple bilinear form

$$M = 2T_{\pm}A_{\pm} + 2J\Omega_{\pm} + \Phi_{\pm}Q.$$  

(36)

This has been derived from the homogeneous function of degree $\frac{1}{2}$ in $(A_{\pm}, J, Q^2)$. Remarkably, $T_{\pm}$, $\Omega_{\pm}$ and $\Phi_{\pm}$ can be defined and are constant on the $H_{\pm}$ for any stationary, axisymmetric space-time. Since the $dM$ is perfect differential, one may choose freely any path of integration in $(A_{\pm}, J, Q)$ space. Thus the surface energy $E_{s_{\pm}}$ for $H_{\pm}$ can be defined by

$$E_{s_{\pm}} = \int_{0}^{A_{\pm}} T(\tilde{A}_{\pm}, 0, 0)d\tilde{A}_{\pm};$$

(37)

the rotational energy for $H_{+}$ and $H_{-}$ can be defined by

$$E_{r_{\pm}} = \int_{0}^{J} \Omega_{\pm}(A_{\pm}, \tilde{J}, 0)d\tilde{J}, \; A_{\pm} \text{ fixed};$$

(38)

and the electromagnetic energy for $H_{+}$ and $H_{-}$ are

$$E_{em_{\pm}} = \int_{0}^{Q} \Phi_{\pm}(A_{\pm}, J, \tilde{Q})d\tilde{Q}, \; A_{\pm}, J \text{ fixed};$$

(39)

We may rewritten the Eq. (36) as

$$M = \frac{\kappa_{\pm}}{4\pi}A_{\pm} + 2J\Omega_{\pm} + \Phi_{\pm}Q.$$  

(40)

or

$$M - 2J\Omega_{\pm} - \Phi_{\pm}Q = \frac{\kappa_{\pm}}{4\pi}A_{\pm}.$$  

(41)

or

$$M - 2J\Omega_{\pm} - \Phi_{\pm}Q = \frac{T_{\pm}}{2}A_{\pm}.$$  

(42)
or

\[ \frac{M}{2} = T_{\pm} S_{\pm} + J \Omega_{\pm} + \frac{\Phi_{\pm} Q}{2}. \]  

(43)

This is recognize as a generalized Smarr-Gibbs-Duhem relation for \( \mathcal{H}^\pm \). Here ‘\( + \)’ indicate for event horizon which was already discussed in the literature[19].

We have derived the above relation here for Cauchy horizon only and for our record we have also mentioned here both the horizons.

Now defining a new parameter set \((\eta_{\pm}, \beta_{\pm}, \epsilon_{\pm})\) which is related to the quantities \((A_{\pm}, J, Q)\) to study the intrinsic geometry for the Cauchy horizon \( \mathcal{H}^- \) of a charged rotating black hole is given by

\[ \eta_{\pm} = \sqrt{r_{\pm}^2 + a^2} = \sqrt{\frac{A_{\pm}}{4\pi}}. \]  

(44)

\[ \beta_{\pm} = \frac{a}{\sqrt{r_{\pm}^2 + a^2}} = \frac{a}{\eta_{\pm}}. \]  

(45)

\[ \epsilon_{\pm} = \frac{Q}{\eta_{\pm}}. \]  

(46)

Therefore the integrated mass formula for \( \mathcal{H}^\pm \) is found to be

\[ M = \frac{\eta_{\pm}(1 + \epsilon_{\pm}^2)}{2\sqrt{1 - \beta_{\pm}^2}}. \]  

(47)

\[ E_{s\pm} = \frac{\eta_{\pm}}{2}. \]  

(48)

\[ E_{r\pm} = \frac{\eta_{\pm}}{2} \left[ \frac{1}{\sqrt{1 - \beta_{\pm}^2}} - 1 \right]. \]  

(49)

\[ E_{em\pm} = \frac{\eta_{\pm} \epsilon_{\pm}^2}{2\sqrt{1 - \beta_{\pm}^2}}. \]  

(50)

with

\[ M = E_{s\pm} + E_{r\pm} + E_{em\pm}. \]  

(51)

Interestingly, mass can be expressed as sum of surface energy, rotational energy and electro-magnetic energy of both the horizons \( \mathcal{H}^\pm \). We have already seen the above discussion in [1] for event horizon. We derived here the above relation for Cauchy horizon only and for the sake of completeness we have mentioned both the cases.
4 Christodoulou’s Irreducible Mass for Cauchy horizon

Besides the black hole event horizon, there exist a second horizon inside the black hole - the Cauchy horizon or inner horizon ($H^\pm$). It is defined as the future boundary of the domain of dependence of the ($H^\pm$). What happens the Christodoulou-Ruffini [13] mass formula for Cauchy horizon. This is an important issue that we will discuss now.

The irreducible mass can be defined as

$$M_{irr\pm} = \sqrt{\frac{r^2 \pm a^2}{2}} = \sqrt{\frac{A\pm}{16\pi}}. \quad (52)$$

where + indicate for $H^+$ and – indicate for $H^-$. The area and angular velocity can be expressed as in terms of $M_{irr\pm}$:

$$A\pm = 16\pi(M_{irr\pm})^2 = 4\pi\rho^2\pm. \quad (53)$$

and

$$\Omega\pm = \frac{a}{r^2 \pm a^2} = \frac{a}{4(M_{irr\pm})^2}. \quad (54)$$

Interestingly, the product of the irreducible mass of $H^\pm$ are universal.

$$M_{irr+}M_{irr-} = \sqrt{\frac{A_+A_-}{16\pi}} \quad (55)$$

$$= \sqrt{\frac{J^2 + Q^4}{4}}. \quad (56)$$

The rest mass of a rotating charged black hole is defined by the Christodoulou-Ruffini mass formula (which may be expressed as in term of its irreducible mass, angular momentum $J$ and charge $Q$) reads as:

$$M^2 = (M_{irr\pm} + \frac{Q^2}{4M_{irr\pm}})^2 + \frac{J^2}{4(M_{irr\pm})^2}. \quad (57)$$

$$= (M_{irr\pm} + \frac{Q^2}{2\rho\pm})^2 + \left(\frac{J}{\rho\pm}\right)^2 \quad (58)$$

$$= (m_{r\pm})^2 + (p\pm)^2 \quad (59)$$

$$= (\gamma m_{r\pm})^2. \quad (60)$$

where $p\pm = \frac{J}{\rho\pm}$ is an effective momentum, and the effective rest mass $m_{r\pm}$ can be defined as

$$m_{r\pm} = M_{irr\pm} + \frac{Q^2}{2\rho\pm} = M_{irr\pm} + \frac{Q^2}{4M_{irr\pm}}. \quad (61)$$
Also the corresponding gamma factor is given by
\[ \gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2}} = \frac{1}{\sqrt{1 - \frac{a^2}{4(M_{\text{irr}}^\pm)^2}}} . \] (62)

It may be noted that \((v_{\pm})^2\) is a strange product of the angular velocity and angular momentum (the conjugate momentum variable) divided by the mass of the black hole.

\[ (v_{\pm})^2 = a \Omega_{\pm} = \frac{J}{M} \Omega_{\pm} . \] (63)

One may compare various formulas for the black hole space-time and certain formulas from mechanics and electromagnetism by rewritten the formula either in \(M_{\text{irr}}^\pm\) or \(\rho_{\pm}\) in geometric units which corresponds to a mass or length variable.

A reversible process is characterized by an unchanged irreducible mass. Whereas an irreversible process is characterized by an increase in irreducible mass of a black hole. It should be noted that there exist no process which will decrease the \(M_{\text{irr}}\) for Cauchy horizon.

This mass decompose into irreducible mass \(M_{\text{irr}}\) and a rotational energy \(M - M_{\text{irr}}\) for Kerr black hole as shown by Christodoulou [20]. For Kerr-Newman black hole the mass can be written as both for \(\mathcal{H}^+\) and \(\mathcal{H}^-\) in the small angular momentum limit, one has

\[ M = M_{\text{irr}}^\pm + \frac{Q^2}{4M_{\text{irr}}^\pm} + K_{\pm} . \] (64)

where,

\[ K_{\pm} = \frac{1}{2} \left( M_{\text{irr}}^\pm + \frac{Q^2}{4M_{\text{irr}}^\pm} \right) (v_{\pm})^2 . \] (65)

looks like an expression for kinetic energy in classical mechanics.

The effective speed \(v_{\pm}\) is given by

\[ v_{\pm} = \frac{p_{\pm}}{M} = \frac{J}{M \rho_{\pm}} = \frac{a}{2M_{\text{irr}}^\pm} = \rho_{\pm} \Omega_{\pm} . \] (66)

where, \(\rho_{\pm} = 2M_{\text{irr}}^\pm. \) Thus the equation (65) reduces to

\[ K_{\pm} = \frac{1}{2} m_{\text{irr}} (v_{\pm})^2 = \frac{J^2}{2I_{\pm}} = \frac{1}{2} I_{\pm} \Omega_{\pm}^2 . \] (67)

where, \(I_{\pm} = m_{\text{irr}} (\rho_{\pm})^2\) playing the role of a moment of inertia in this limit. The above discussion for event horizon can be found in [21]. We have derived the above formula here for CH only.
When the Penrose process \[22,23\] are taking into account, which led to the following exact differential relationship between mass and angular momentum which also characterized by reversible transformations, as described by Christodoulou and Ruffini

$$dM = \frac{a dJ + r_\pm Q dQ}{r_\pm + a^2}.$$  \hspace{1cm} (68)

After integration we obtain the Christodoulou and Ruffini mass formula (57), when the following condition is satisfied:

$$\frac{J^2}{4(M_{\text{irr}}\pm)^2} + \frac{Q^4}{16(M_{\text{irr}}\pm)^4} \leq 1$$ \hspace{1cm} (69)

5 The Four Laws of Black Hole Mechanics on the EH (\(\mathcal{H}^+\)) and CH (\(\mathcal{H}^-\))

Following the remarkable discovery by Carter, Hawking and Bardeen \[24\], we reformulate the black hole thermodynamics both for the event horizon and the Cauchy horizon which is analogous to the classical laws of thermodynamics as follows:

– The Zeroth Law: The surface gravity, \(\kappa_\pm\) of a stationary black hole is constant over both the event horizon (\(\mathcal{H}^+\)) and Cauchy horizon (\(\mathcal{H}^-\)) respectively.

– The First Law: Any perturbation of a stationary black holes, the change of mass (change of energy) is related to change of mass, angular momentum, and electric charge by:

$$dM = \frac{\kappa_\pm}{8\pi} dA_\pm + \Omega_\pm dJ + \Phi_\pm dQ.$$ \hspace{1cm} (70)

It can be seen that \(\frac{\kappa_\pm}{8\pi}\) is analogous to the temperature of \(\mathcal{H}^\pm\) in the same way that \(A_\pm\) is analogous to entropy. It should be noted that \(\kappa_\pm\) and \(A_\pm\) are distinct from the temperature and entropy of the black hole.

The above expression \(\frac{\kappa_\pm}{8\pi}\) can be derived from the Eq. (33) in the following way.

The effective surface tension can be rewritten as

$$T_\pm = \frac{\kappa_\pm}{8\pi} = \frac{\partial M}{\partial A_\pm}$$ \hspace{1cm} (71)

and

$$\Omega_\pm = \frac{4\pi J}{M A_\pm} = \frac{\partial M}{\partial J}$$ \hspace{1cm} (72)

$$\Phi_\pm = \frac{1}{M} \left( \frac{Q}{2} + \frac{2\pi Q^3}{A_\pm} \right) = \frac{\partial M}{\partial Q}.$$ \hspace{1cm} (73)
The Second Law: The area $A_{\pm}$ of both event horizons ($H^+$) and Cauchy horizons ($H^-$) never decreases, i.e.

$$dA_{\pm} = \frac{4A_{\pm}}{r_{\pm} - r_\mp} (dM - \Omega_{\pm} dJ - \Phi_{\pm} dQ) \geq 0$$  \hspace{1cm} (74)

or

$$dM_{\text{irr}, \pm} = \frac{2M_{\text{irr}, \pm}}{r_{\pm} - r_\mp} (dM - \Omega_{\pm} dJ - \Phi_{\pm} dQ) \geq 0$$  \hspace{1cm} (75)

The change in irreducible mass of both event horizons ($H^+$) and Cauchy horizons ($H^-$) can never be negative. It follows that immediately from the above equation

$$dM > \Omega_{\pm} dJ + \Phi_{\pm} dQ$$  \hspace{1cm} (76)

The Third Law: It is impossible by any mechanism, no matter how idealized, to reduce, $\kappa_{\pm}$ the surface gravity of both event horizon ($H^+$) and Cauchy horizon ($H^-$) to zero by a finite sequence of operations.

6 Komar Conserved Quantity =
2 × Entropy on $H^\pm$ × Temperature on $H^\pm$ or $K_{\chi^\mu_{\pm}} = 2S_{\pm}T_{\pm}$

It is known that on the $H^+$ the Komar conserved quantity ($K_{\chi^\mu_{\pm}}$) corresponding to a null Killing vector $\chi^\mu_{\pm}$ is equal to the twice of the product of entropy ($S_+$) on $H^+$ and temperature ($T_+$) on $H^+$. Here, we shall derive the similar expression which exists on $H^-$. Thus, we have to prove the identity $K_{\chi^\mu_{\pm}} = 2S_-T_-$ on the $H^-$. Due to the stationarity and axially symmetric nature of Kerr Newman space-time, the space-time has two Killing vectors. These two vectors are $\xi^\mu_{(t)} = (1, 0, 0, 0)$ and $\xi^\mu_{(\phi)} = (0, 0, 0, 1)$ which corresponds to timelike and spacelike at the asymptotic limit. Thus we can define a Killing vector on both the ($H^\pm$) which is the combination of these two vectors.

$$\chi^\mu_{\pm} = \xi^\mu_{(t)} + \Omega_{\pm} \xi^\mu_{(\phi)} = (1, 0, 0, \Omega_{\pm})$$  \hspace{1cm} (77)

It should be noted that on the ($H^\pm$), $\chi^\mu_{\pm} \mid_{r=r_{\pm}} = 0$, then $\xi^\mu$ becomes a null Killing vector. Now, we can define a Komar conserved quantity on ($H^\pm$) which corresponds to the Killing vectors are given by

$$K_{\chi^\mu_{\pm}} = K_{\xi^\mu_{(t)}} + \Omega_{\pm} K_{\xi^\mu_{(\phi)}}$$  \hspace{1cm} (78)

where, $K_{\chi^\mu_{\pm}}$ is the Komar conserved quantity corresponding to the timelike Killing vector which may be defined as

$$K_{\xi^\mu_{(t)}} = \frac{1}{8\pi} \int_{\partial \Sigma} \*d\sigma$$  \hspace{1cm} (79)
whose one form is given by
\[ \sigma = \xi(\mu)dx^\mu = g_{\mu\nu}dx^\mu = g_{tt}dt + g_{t\phi}d\phi . \] (80)
and \( *d\sigma \) is the dual to the two form \( d\sigma \). \( d\Sigma \) is an appropriate boundary surface of a spatial three volume \( (\Sigma) \).

Similarly, we can define Komar conserved quantity as \( (H^\pm) \) correspond to the spacelike Killing vector is given by
\[ K^\nu_{(\omega)} = -\frac{1}{8\pi} \int_{\partial \Sigma} *d\eta . \] (81)
where the spacelike killing one form is defined as
\[ \eta = \xi(\phi)dx^\mu = g_{\mu\phi}dx^\mu = g_{\phi\phi}d\phi . \] (82)
After some computations, we find
\[ K^\nu_{(\pm)} = M - 2M\frac{a^2}{r^2_\pm + a^2} - \frac{Q^2 r_\pm}{r^2_\pm + a^2} . \] (83)
Using the expressions \( \Omega_\pm \) and \( r_\pm \), we simplify the above equation to rewritten in the compact form
\[ K^\nu_{(\pm)} = \pm \sqrt{M^2 - a^2 - Q^2} = \frac{r_\pm - r_\mp}{2} \] (84)
\[ = 2 \left[ \pi (r^2_\pm + a^2) \right] \frac{r_\pm - r_\mp}{4\pi (r^2_\pm + a^2)} \] (85)
\[ = \frac{A_\pm \kappa_\pm}{4\pi} = 2 \left( \frac{A_\pm}{4} \right) \left( \frac{\kappa_\pm}{2\pi} \right) \] (86)
\[ = 2S_\pm T_\pm . \] (87)
Thus we have obtained on the \( (H^\pm) \) the Komar conserved charge corresponding to the null Killing vector is twice the product of the entropy and the surface temperature of the Kerr Newman black hole. We can connect this quantity with a similar relation which has been derived in the previous section on the \( (\Sigma) \).
\[ E_\pm = 2S_\pm T_\pm . \] (88)
where \( E_\pm \) is the Noether charge of diffeomorphism symmetry on \( (H^\pm) \). It should be noted that the above relation has been derived particularly for a static local Killing horizon \[26\] which may or may not be a black hole event horizon. However Kerr Newman space-time is a stationary and therefore does it valid for any stationary space-time which is not clear, but we have explicitly compute a similar relation on the Cauchy horizon \( H^- \). Introducing the scalar potential on the \( (\Sigma) \), we get
\[ \Phi_\pm = \frac{Qr_\pm}{r^2_\pm + a^2} . \] (89)
Eq. (81) can be rewritten in the form

$$K_{\chi^\pm} = \mathcal{M} - 2J\Omega_{\pm} - \Phi_{\pm}Q .$$  \hspace{1cm} (90)

Again from the Eq. (36) we have

$$\mathcal{M} - 2J\Omega_{\pm} - \Phi_{\pm}Q = \frac{A_{\pm}k_{\pm}}{4\pi} \hspace{1cm} (91)$$

$$= \frac{A_{\pm}T_{\pm}}{2} . \hspace{1cm} (92)$$

This is the well known Smarr formula on the $(\mathcal{H}^\pm)$. In the literature, we have seen the above discussion for the event horizon only. We extend the above relations, particularly for the Cauchy horizon. For the completeness, we compute all the relations or formula for the event horizon also.

### 7 Generalized Smarr Formula for Mass on the Cauchy horizon

In this section, we will derive for a stationary state black hole space-time the entropy can be expressed as $S_\pm = \frac{E_\pm}{2r_\pm}$ on the $(\mathcal{H}_\pm)$, where $T_\pm$ is the Hawking temperature on $(\mathcal{H}^\pm)$ and $E_\pm$ is shown to be the Komar energy on the $(\mathcal{H}^\pm)$. We also derive the generalized Smarr formula for mass on the $(\mathcal{H}^\pm)$. Which is compatible with the relation in Eq. (92). We have already known from [27], the Komar energy for Kerr Newman black hole in a compact form on the $(\mathcal{H}^+)$ is given by

$$2S_+ T_+ = E_+ = \sqrt{\mathcal{M}^2 - a^2 - Q^2}$$

$$= \mathcal{M} - \frac{Q^2}{r_+} - 2J\Omega_+ \left( 1 - \frac{Q^2}{2\mathcal{M}r_+} \right)$$

$$= \mathcal{M} - 2J\Omega_+ - QV_+ . \hspace{1cm} (93)$$

where $V_+ = \frac{Q}{r_+} - \frac{JQ\Omega_+}{\mathcal{M}r_+}$. Similarly, we can obtain easily the Komar energy on the Cauchy horizon for Kerr Newman black hole:

$$2S_- T_- = E_- = -\sqrt{\mathcal{M}^2 - a^2 - Q^2}$$

$$= \mathcal{M} - 2J\Omega_- - QV_- . \hspace{1cm} (96)$$

Remarkably, the energy is negative which also reverify that the Killing vector field is negative inside the $\mathcal{H}^+$. Thus the energy is negative on the $\mathcal{H}^-$ due to this fact.

Again, we can rewrite the Eq. (90) as

$$2S_+ T_+ = E_+ = -\sqrt{\mathcal{M}^2 - a^2 - Q^2} = \mathcal{M} - 2J\Omega_+ - QV_+ . \hspace{1cm} (97)$$
where, $V_- = \frac{Q}{r} - \frac{JQ\Omega}{Mr}$. 

Thus Eq. (95) and Eq. (97) can be rewritten for both the horizons ($\mathcal{H}^\pm$) in a compact form as

$$2S_\pm T_\pm = E_\pm = \pm \sqrt{M^2 - a^2 - Q^2} = M - 2J\Omega_\pm - QV_\pm .$$

(98)

8 Degenerate Black Hole or Extremal Black Hole

Thus one may define an extremal black hole is a black hole, when the radius of event horizons and Cauchy horizons are converging i.e.,

$$r_+ = r_-$$

(99)

or, when the area of two horizons are merging i.e.,

$$A_+ = A_-$$

(100)

or, when the entropy of two horizons are coincident i.e.,

$$S_+ = S_-$$

(101)

or, when the surface gravity of both horizons are equal i.e.,

$$\kappa_+ = \kappa_-$$

(102)

or, when the temperature of both horizons are same i.e.,

$$T_+ = T_-$$

(103)

or, when the angular velocity of both horizons are coincident i.e.,

$$\Omega_+ = \Omega_-$$

(104)

or, when the irreducible mass of both horizons are equal i.e.,

$$M_{irr+} = M_{irr-}$$

(105)

If any one of the above properties are satisfied then a black hole is said to be an extremal black hole. Thus one gets the area in the extremal limit

$$A_+ = A_- = 8\pi \sqrt{J^2 + \frac{Q^4}{4}}$$

(106)

As a result of (106), the another relation

$$\frac{J^2}{M^2} + Q^2 = MCR^2$$

(107)
of the extremal KN space-time continues to be hold in presence of the surrounding matter in accordance with the fact that KN black holes are degenerate and if they are extremal. $\mathcal{M}_{\text{CR}}$ denotes Christodoulou and Ruffini mass. Also for the entropy one obtains,

$$S_+ = S_- = 2\pi \sqrt{J^2 + \frac{Q^4}{4}}$$

(108)

It is well known that the surface gravity and surface temperature goes to zero at the extremal limit i.e. $\kappa_+ = \kappa_- = 0$ and $T_+ = T_- = 0$.

It may be noted that the Komar energy goes to zero at the extremal limit i.e. $E_+ = E_- = 0$. This may implies that this is the another way to seeing the discontinuity between extremal space-time and non-extremal space-time.

9 Discussions

In this work, we have derived the Smarr formula on the Cauchy horizon ($\mathcal{H}^-$). We have proposed the four laws of black hole mechanics for inner horizon ($\mathcal{H}^-$). We have found, in contrast to some earlier work [9,28] particularly for the first law of inner horizon ($\mathcal{H}^-$), a complete consistency between our results and their results.

We have also demonstrated that the area product, horizon radii product, entropy product and irreducible mass product of the event horizon and Cauchy horizon although are universal, the surface gravity product, surface temperature product and Komar energy product are not universal for Kerr-Newman black hole.

We have also defined the Christodoulou and Ruffini mass on the Cauchy horizon. We have further showed that the identity $K_{\chi\nu_-} = 2S_-T_-$ is valid on the inner horizon ($\mathcal{H}^-$) and also the Komar energy in a compact form $E_- = 2S_-T_-$. Which also relates the generalized Smarr formula $E_- = M - 2J\Omega_- - QV_-$ on the $\mathcal{H}^-$.

Another interesting point we have found that the Komar energy goes to zero at the extremal limit indicates a discontinuity between extremal space-time and non-extremal space-time.

References

1. L. Smarr, Phys. Rev. Lett. 30 71 (1973); 31, 521 (1973).
2. M. Ansorg and J. Hennig, Classical Quantum Gravity 25 222001 (2008).
3. M. Ansorg and J. Hennig, Phys. Rev. Lett. 102 221102 (2009).
4. M. Ansorg, J. Hennig and C. Cederbaum, Gen. Rel. Grav. 43 1205 (2011).
5. P. Larsen, Phys. Rev. D 56 1005 (1997).
6. M. Cvetic, G. W. Gibbons and C. N. Pope, Phys. Rev. Lett. 106 121301 (2011).
7. M. Cvetic and F. Larsen, J. High Energy Phys. 09 088 (2009).
8. B. Chen, S. X. Liu and J. J. Zhang, J. High Energy Phys. 017, 1211 (2012).
9. A. Castro and M. J. Rodriguez, Phys. Rev. D 86 024008 (2012).
10. M. Visser, Phys. Rev. D 88 044014 (2013).
Black Hole Interior Mass Formula

11. V. Faraoni, A. F. Z. Moreno “Are quantization rules for horizon areas universal?”, arXiv: 1208.3814 [hep-th] (2013).
12. M. Visser, J. High Energy Phys. 06 023 (2012).
13. S. Chandrasekhar, The Mathematical Theory of Black Holes, Clarendon Press, Oxford (1983).
14. D. Christodoulou and R. Ruffini, Phys. Rev. D. 4 3552 (1971).
15. E. Newman, K. Chinnaparad, A. Exton, A. Prakash, R. Torrence, J. Math. Phys. 6, 918-919 (1965).
16. R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
17. J. D. Bekenstein, Phys. Rev. D 7 2333 (1973).
18. J. D. Bekenstein, Phys. Rev. D 9 3292 (1974).
19. P. C. W. Davies, Rep. Prog. Phys., Vol. 41, (1978).
20. D. Christodoulou, Phys. Rev. Lett. 25 1596 (1970).
21. D. Bini, R. Jantzen, R. Ruffini, “Reinterpretation of the Mass Formula for Black Hole” (2012).
22. R. Penrose and R. M. Floyd, Nature 229, 177 (1971).
23. R. Penrose, Rev. Nuovo Cimento 1, 252 (1969).
24. J. M. Bardeen, B. Carter, S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
25. S. Modak and S. Samanta, Int. J. Theor. Phys. 51 (2012) 1416-1424.
26. T. Padmanabhan, Class. Quant. Grav. 21 (2004) 4485-4494.
27. R. Banerjee and B. R. Majhi, Phys. Rev. D. 81 124006, (2010).
28. S. Detournay, “Inner Mechanics of 3d Black Holes”, arXiv: 1204.6088 [hep-th] (2012).
