MECHANICS AND EQUILIBRIUM GEOMETRY OF BLACK HOLES, MEMBRANES, AND STRINGS.

B. Carter.

Dept. of Relativistic Astrophysics and Cosmology,
C.N.R.S., Observatoire de Paris,
92 Meudon, France.

Abstract. *This course is designed to give a mathematically coherent introduction to the classical theory of black holes and also of strings and membranes (which are like the horizon of a black hole in being examples of physical systems based on a dynamically evolving world sheet) giving particular attention given to the study of the geometry of their equilibrium states.*

Preface.

The purpose of this course is to provide a mathematically coherent introduction to the classical theory of black holes and also to the related and more recently developed topic of the classical theory of relativistic strings and membranes for which many of the same techniques are required. The strategy of the course will be to concentrate on general results rather than special examples, and to distinguish as clearly as possible what has been completely proved from what has only been partly established or merely conjectured so as to give some idea of the main open problems for future research. The discussion is developed on the basis of a chain of key results for which it has been possible to provide reasonably complete and self contained mathematical proofs without resort to disproportionate technical complication. The level of previous knowledge required corresponds to what is obtainable from the relevant sections of a textbook such as that of Misner, Thorne and Wheeler\(^1\) (whose notation will be used as far as possible) or, in a less encyclopedic but more conveniently accessible (and up to date) form, that of Wald\(^2\). There are already several textbooks specifically devoted to various aspects of black hole theory \(^3[4][5][6]\); attention is particularly to be drawn...
that of Hawking and Ellis\textsuperscript{[7]} for advanced mathematical background reading, and to that of Novikov and Frolov\textsuperscript{[8]} for an exceptionally comprehensive survey of the published literature including more than 600 references.

The organisation of the course is as follows.

Section 1 provides a brief astrophysical introduction consisting essentially of a simple explanation\textsuperscript{[9]} of the orders of magnitude that are relevant to the conventional idea of the formation of “ordinary” black holes by stellar collapse (no such simple and clear picture being available for the more exotic phenomenon of the giant black holes that are commonly believed to be at the heart of active galactic nuclei).

After this physical introduction, the main part of the course is more essentially mathematical in nature following more or less the same lines as my previous reviews\textsuperscript{[10][11][12]} though with the omission, except for the necessary references, of certain parts in order to make way for the inclusion of new results. Section 2 presents some of the main results of the theory of exactly spherical gravitational collapse, which is the only case for which a precise dynamical analysis is available. Section 3 gives a brief account of what little is known about dynamical formation of black holes in more realistic situations where spherical symmetry is broken by effects such as rotation. Section 4 deals with the theory of stationary rotating black hole equilibrium states in the general case for which externally orbiting matter rings may be present. Section 5 deals more specifically with the uniqueness theorem that is available when no external sources are present. Section 6 concludes the course on black hole theory by describing some of the rather miraculous special properties of the ensuing Kerr Newman metrics, whose stability is one of the most important topics that (for lack of time and space) has not been included in this course: for the most complete result, going a long way towards confirming that these equilibrium solutions can indeed be considered to be stable, the interested reader is referred to the recent work of Whiting\textsuperscript{[13]}

Section 7 moves on to present a covariant formulation\textsuperscript{[14]} of the basic mechanical principles of classical brane theory meaning the subject that includes the theories of point particles, strings, membranes and continua as special cases. Section 8 deals more specifically with the theory of spatially isotropic branes, a category that includes all classical string models and in particular those representing “superconducting cosmic strings”.

Finally in a purely mathematical appendix, some of the most important tensorial quantities (which are useful for black hole theory and indispensable for brane theory) characterising the different kinds of curvature of an
imbedding are presented in a readily utilisable form\textsuperscript{[15]} that is not yet readily available elsewhere.

1. The astrophysical context of Black Hole formation.

The study of black holes in general, and of black hole equilibrium states in particular, arises as a natural offshoot of the study of stellar equilibrium states whose theoretical foundations were established by workers such as Eddington and Chandrasekhar in the years following the elucidation of the basic principles of quantum mechanics. In terms of the fundamental Plank type unit system that will be used throughout this course (in which the speed of light \(c\), Newton’s gravitational constant \(G\), the Dirac - Plank constant \(\hbar\) and the Boltzman constant \(k\) are all simultaneously set equal to unity) the dominant physical mechanisms governing the situation can be described\textsuperscript{[9][16]} in crude order of magnitude terms (give or take a power of ten here or there) in terms of just three particularly important dimensionless parameters, namely the masses \(m_e\) and \(m_p\) of the electron and the proton, and the magnitude \(e\) of their electric charge, which are expressible as the moderately small “fine structure” coupling constant \(e^2 \approx 1/137\) the considerably smaller mass ratio \(m_e/m_p \approx 1/1800\) and the extremely small gravitational coupling constant \(m_p^2 \approx 10^{-39}\).

In the low temperature limit, the equilibrium states of small, medium, and even moderately large bodies, on scales ranging from single molecules through sand grains up to entire planets, are characterisable in crude order of magnitude by a typical density \(\rho\) given by

\[
\rho \approx e^6 m_e^3 m_p
\]  

which works out (by no means accidentally) to be very roughly of the order of unity in “ordinary” units, \(gm/cm^3\) (which have of course been deliberately normalised to give such a result). Taking account of the fact that in all such states the mean mass per baryon is given to a very good (within one per cent) accuracy by \(m_p\), so that \(\rho \approx m_p n\) where \(n\) is the baryon number density, the relation (1.1) expresses the condition that the mean separation \(\lambda \approx n^{-1/3}\) between baryons will be of the same order as that between the (within a factor of two equally numerous) electrons, and therefore of the order of the Bohr radius, \(\lambda \approx 1/e^2 m_e\), which is the result that is obtainable from the consideration that the equilibrium is determined by the balance between Fermi
(exclusion principle) repulsion between electrons and electrostatic attraction between negatively charged electrons and positively charged ions.

Although applicable to bodies on scales ranging from that of a hydrogen atom to that of the earth, the formula (1.1) loses its validity for bodies so large that the long range cumulative effect of the (individually very weak) gravitational attraction forces becomes stronger than the effect of the electrostatic attraction forces (which of course only act locally because of the long range cancellation resulting from overall electric neutrality). For a body of mass $M$, mean density $\rho$ and hence characteristic mean radius $R \approx (M/\rho)^{1/3}$ resistance to collapse under the influence of gravitational self attraction requires a mean central pressure $P$ given according to the well known “virial theorem” by

$$P \approx M^{2/3} \rho^{4/3},$$

which expresses a balance between the typical radial pressure gradient, of order $P/R$, and the gravitational force density, of order $\rho M/R^2$.

The pressure contribution resulting from the application of the Fermi exclusion principle to the electrons is of the order of the corresponding kinetic energy density, and therefore will be given - in the non relativistic limit - roughly by

$$P \approx \frac{1}{m_e} \left(\frac{\rho}{m_p}\right)^{5/3},$$

in view of the fact that the mean momentum per electron will just be the inverse $\lambda^{-1}$ of the corresponding De Broglie wavelength, which will itself be of the same order of magnitude as the mean separation, $\lambda \approx n^{-1/3}$, where $n \approx \rho/m_p$. So long as the mass $M$ is small compared with a critical value given roughly by $M \approx e^3/m_p^2$, the virial pressure requirement (1.2) is small compared with the Fermi energy density (1.3) at the “ordinary” matter density (1.1) which means that the gravitational compression effect will be unimportant, but but beyond this critical mass (which is of the order of that of the giant planet Jupiter) the long range gravitational attraction will dominate over the short range electrostatic binding so that the corresponding equilibrium states will be of white dwarf type, with the central pressure determined by direct equation of (1.2) and (1.3) which means that the characteristic mean central density $\rho$ will be given as a function of the mass $M$ by an order of magnitude relation of the form

$$\rho \approx m_e^3 m_p^5 M^2.$$
The range of validity of the relation (1.4) is of course limited to that of the non relativistic degenerate electron gas pressure formula (1.3) from which it is derived. When the relevant DeBroglie wavelength \( \lambda \simeq n^{-1/3} \) becomes short compared with the Compton wavelength \( \lambda \simeq m_e \), the kinetic energy per electron is no longer given by \( 1/\lambda^2 m_e \) but just by \( 1/\lambda \), so that the non relativistic formula (1.3) must then be replaced by the corresponding relativistic degenerate gas pressure formula

\[
P \approx \left( \frac{\rho}{m_p} \right)^{4/3}.
\]

(1.5)

The (by now generally accepted) recognition that the theory of black holes must be taken seriously as something whose implications are directly relevant and testable in observational astrophysics derives from the startling (1930) discovery by Chandrasekhar\[17\] that substitution of (1.5) instead of (1.4) in the virial equilibrium condition (1.2) does not just give a modified version of the functional relation (1.4) for the equilibrium density \( \rho \) as a function of the mass \( M \), but instead gives an absolute cut off at a critical mass

\[
M \approx \frac{1}{m_p^2}
\]

(1.6)

above which no ordinary cold equilibrium state is possible at all!

The existence of this upper mass limit does not of course mean that there are no cold equilibrium states beyond the critical density \( \rho \simeq m_e^3 m_p \) at which the white dwarf range (1.4) reaches the Chandrasekar limit (1.6), since it is also possible to have high density states in which the electrons are combined with protons to form neutrons for which the relevant analogue of the non relativistic degenerate gas pressure formula (1.3) is

\[
P \approx \frac{1}{m_p} \left( \frac{\rho}{m_p} \right)^{5/3}.
\]

(1.7)

However the resulting range of neutron star equilibrium states, with density given, by substitution of (1.7) in (1.2), as

\[
\rho \approx m_p^8 M^2,
\]

(1.8)

will be cut off by an upper mass limit that is still given\[18][19\] by the same crude order of magnitude formula (1.6) as before, because the relativistic degenerate
gas pressure has the same form (1.5) for neutrons as for electrons. More exact calculations (whose results are still subject to a considerable uncertainty due to the imprecision of our present understanding of the detailed physical properties of neutron star matter) indicate that the upper mass limit for neutron stars is somewhat larger (though only by a modest factor not much in excess of two), than that for white dwarfs: this conclusion is of great astrophysical importance, and would appear to have been observationally confirmed by the discovery of pulsars (since if the exact upper mass limit for neutron stars had turned out to be smaller than that for white dwarfs then the formation of neutron stars by gravitational collapse would have been rendered virtually impossible).

As the astronomical community belatedly recognised (after more than thirty years of general indifference or incredulity) Chandrasekhar’s discovery\[^{[17]}\] made it absolutely necessary to take the possibility of runaway gravitational collapse - and ensuing formation of massive or ultramassive black holes - very seriously as a phenomenon of potentially crucial relevance to many directly observable phenomena. This contrasts with the situation that still applies to speculation on the subject of microscopic black holes (for which quantum phenomena such as Hawking radiation\[^{[20]}\] are significant) whose relevance to anything actually observable remains subject to reasonable doubt. Nevertheless the existence of several categories of observational “black hole candidates” (of which the most famous prototype example is the galactic X-ray source Cygnus X-1) does not yet amount to a firm confirmation of that black holes with the properties described in the following sections of this course really do exist. The most numerically numerous (and perhaps ultimately most astrophysically important) category of observationally detected “candidates” is that of nuclei of “active” galaxies, but such (ultramassive) objects are all too fuzzy and far away to have been of any use so far from the point of view of verification of the basic physical theory. As far as the more conveniently tractable candidates within our galaxy are concerned, the awkward fact to be faced is that sixty years after our colleague Chandra’s precocious and revolutionary theoretical discovery, and more than twenty years after the collapse of psychological resistance to the notion of a “black hole” following the coining of the term itself (by John Wheeler) and the (approximately simultaneous and no less psychologically significant) experimental discovery (by Jocelyne Bell and Tony Hewish) of the pulsars whose identification as neutron stars has long been unquestionable, there are still disappointingly few observationally discovered objects that can plausibly be interpreted as
“ordinary” (moderate sized) black holes.

The relative scarcity of black holes in the mass range immediately above the Chandrasekhar limit (which is about one and a half times the mass of the sun) might at first seem surprising in view of the fact that this particular mass range is precisely that of the most numerous subclass of the ordinary stars that are visible at night to the naked eye. However from a theoretical point of view this apparent paradox can easily be understood as follows.

The non-existence of any material “ground state” configurations, i.e. minimum energy (cold, static, absolutely stable) equilibrium states, above a critical mass value given in order of magnitude by (1.6) does not of course exclude the existence of more massive “excited” (and therefore in principle ultimately unstable) equilibrium states whose support depends on having more than minimal energy in thermal or other forms (such as that involved in differential rotation) which in the long run are subject to dissipation and loss by radiation but which in practice may be preserved over astrophysically or even cosmologically long time scales. The most important examples of such excited states are of course ordinary hot stars (including most notably those of the main, i.e. hydrogen burning, sequence) which are characterisable by a mean central temperature $\Theta$ say, in terms of which the pressure $P$ will be given by the sum of a radiation contribution $P \simeq \Theta^4$ and of a non-relativistic gas contribution $P \simeq n\Theta$ where the relevant number density of non relativistic particles will be of the same order as the baryon number density $n \simeq \rho/m_p$. Evidently the radiation contribution will be dominant for $\Theta^3 > n\Theta$ while the non relativistic gas contribution will be dominant for $\Theta^3 << n$. On substituting these formulae in the virial equilibrium condition (1.2) it can seen that the only criterion for radiation dominance is the mass $M$ of the star, the critical value (which was first worked out by Eddington[21]) being again given in order of magnitude by the inverse of the gravitational coupling constant $m_p^2$ i.e. by the same formula (6.1) as was obtained a few years later by Chandrasekhar for the more sharply definable upper mass limit for cold equilibrium: below this limit the dominant pressure contribution is that of the non-relativistic particles, whose substitution in the virial equilibrium condition (6.2) leads to a formula giving the characteristic density corresponding to a given characteristic central temperature $\Theta$ in the form

$$\rho \approx \left( \frac{\Theta^3}{m_p^3 M^2} \right)$$

(1.9)
Fig. 1 – Logarithmic plot of characteristic orders of magnitude for density $\rho$ against mass $M$ relative to the standard values given by (1.1) and (1.6), which roughly characterise the Sun.
whereas when the stellar mass exceeds the critical value (1.6) it can be seen that it is the radiation gas pressure contribution that will be dominant so that instead of (1.9) one will obtain a less strongly mass dependent result expressible by

$$\rho \approx \left( \frac{\Theta^3}{M^{1/2}} \right)$$

(1.10)

The explanation for the fact that the typical masses of ordinary observable stars turn out to be comparable with the critical Eddington - Chadrasekhar mass value given by (1.6) is to be found in terms of the criterion for stability with respect to adiabatic variations, in which the pressure will vary as a function of density according to an approximately polytropic law of the form $P \propto \rho^\Gamma$ where the index value is $\Gamma = 5/3$ in the non-relativistic limit to which (1.9) applies, but where $\Gamma = 4/3$ in the radiation dominated limit to which (1.10) applies. It is immediately obvious from the form of the virial condition (1.2) that it is necessary for stability that the effective polytropic index should exceed the critical value 4/3, i.e. precisely the same value that is characterises adiabatic perturbations of a radiation dominated gas. This means that stellar configurations above the critical mass (1.2) can at best be stabilised only marginally by their relatively small non-relativistic gas pressure contribution, and that for mass values a long way (more than two powers of ten) above the critical value $M \approx 1/m_m^2$ stable equilibrium will in practice be extremely difficult to achieve. One would therefore expect that (in accordance with what is actually observed) formation of stars by gravitational condensation (with central heating according to the law $\Theta \propto \rho^{1/3}$ that is obtained from both (1.9) and (1.10)) from initially diffuse gas clouds would inevitably produce objects below or not too far above the critical mass (1.6).

As well as being limited in mass the conceivable range for ordinary stellar equilibrium states is of course also limited in temperature, which must exceed the Rydberg energy value, $\Theta \approx e^4 m_e$, that is the threshold for the ionisation of the gas that accounts for the opacity needed to delay the radiation loss of the thermal energy : on substitution of this minimal Rydberg temperature in (1.9) and comparison with the white dwarf equilibrium condition (1.4) it can be seen (see figure 1) that the smallest possible mass for an ionised stellar configuration with thermal pressure support is the same as the maximum possible value, $M \approx e^3/m_p^2$ for a cold planetary configuration, i.e. about the mass of Jupiter which is situated just at the lower end of the cold white dwarf range. At the opposite extreme the upper cut off to the
conceivable range of temperatures for ordinary stellar configurations is given by the electron positron pair creation temperature \( \Theta \approx m_e \) beyond which there is longer any possibility of stabilisation by a non relativistic electron gas contribution: on substitution of this pair creation temperature in (1.10) and comparison with the white dwarf equilibrium condition (1.4) it can be seen (see figure 1) that the highest possible characteristic density \( \rho \) for an ionised stellar configuration with thermal pressure support is obtained for a mass of the order of the Chandrasekhar limit value (1.6) and is the same as the maximum possible value, \( \rho \approx m_e^3 m_p \), that is obtained at the upper end of the white dwarf range.

Although energy loss by radiation from the outer (“chromospheric”) surface layers prevents them from lasting indefinitely, the “excited” stellar equilibrium states in the range delimited by the considerations of the preceding paragraphs can nevertheless survive over astrophysically long timescales whose minimum value is determined by the minimal opacity contribution that results from Thompson scattering of photons by electrons with effective cross section given in order of magnitude by \( \sigma \approx (e^2 / m_e)^2 \), which leads to a minimal evolution timescale \( \tau \), for stars in the radiative mass range \( M \gg 1/m_p^2 \) to which (1.10) applies, that will be given by

\[
\tau \approx \frac{\varepsilon e^4}{m_e^2 m_p^2}, \tag{1.11}
\]

where \( \varepsilon \) is the efficiency of conversion of rest mass into thermal energy by nuclear reactions. The most efficient thermonuclear energy production process is of course hydrogen burning which yeilds almost one per cent, \( \varepsilon \approx 10^{-2} \) at a “main sequence” temperature \( \Theta \) at which stars spend most of the lifetime allowed by (1.11), which works out at about \( 10^7 \) years. For smaller stars with masses near or below the Eddington - Chandrasekhar critical value (1.6), other mechanisms come into play which increase the opacity and diminish the rate of energy loss by radiation, giving timescales that for the smallest main sequence stars can greatly exceed even the present age of the universe which is of the order of \( 10^{10} \) years. The value of the relevant main sequence central temperature is derivable (by consideration of the probability of coulomb barrier tunnelling by the ionic reactants) as given in order of magnitude by the protonic analogue of the electronic Rydberg energy, i.e. \( \Theta \approx e^4 m_p \) which is logarithmic between the minimal (ordinary electronic) Rydberg ionisation temperature \( \Theta \approx e^4 m_e \) and of the maximal pair creation temperature \( \Theta \approx m_e \).
The foregoing considerations lead to the prediction (in full agreement with observation) not only that formation of stars in that mass range just above the the Eddington-Chandrasekhar critical value should have been relatively common, but also that most such moderately massive stars should already have passed the ends of the thermonuclear lifetimes and so been already obliged to face the issue of runaway gravitational collapse to densities in excess of the critical Michell Laplace limit\textsuperscript{[22][23]} value
\[ \rho \approx \frac{1}{M^2} \] (1.12)
beyond which any description in Newtonian terms must be expected to break down, the usual formula for the scalar gravitational potential \( \varphi \approx \frac{M}{R} \) with \( R \approx (M/\rho)^{1/3} \) giving a result greater than unity, meaning that the gravitational energy is greater than the rest mass energy and hence that the escape velocity is greater than the speed of light, a situation that corresponds, in the General Relativistic formulation described in the following sections, to the light trapping mechanism that is the essence of the phenomenon that is commonly referred to as the formation of a black hole. Since the speed of light is normally supposed to represent an upper bound on the rate of propagation of causal influences of any kind, the infalling matter within the “horizon” (that is presumed to define the boundary of the region from which no light escapes) will become causally decoupled from the outside region, which thereby acquires the freedom to attain an equilibrium state of a new, essentially non-material “black hole” type, whose investigation will be the subject of the discussion in the following sections. Assuming that the density would retain its usual order of magnitude, i.e. that given by (1.1) Michell and Laplace estimated that light trapping would require a minimum mass of the order of \( 10^7 \) times that of the sun (a value so gigantic that it was not taken seriously until, following up a suggestion by Lynden-Bell\textsuperscript{[24]}, its potential relevance to exotic quasar type phenomena in active galactic nuclei was pointed out by Hills\textsuperscript{[25]} who noticed that it represents a threshold value for tidal disruption of ordinary stars\textsuperscript{[26]}).

The paradox is that black holes in the relatively moderate mass range on which (by allowing for compressibility) our attention has so naturally been focussed by the line of astrophysical reasoning developed above i.e. a few times the Chandrasekhar limit (1.6), would appear in practice to be very rare, despite the high (predicted and observed) abundance of potential precursor stars. If these very common massive main sequence stars do not
become black holes, what happens instead?

One much discussed idea which emphatically does not give the correct explanation is that at the more than nuclear energy density that is attained at the upper limit, \( \rho \approx m_p^4 \), of the neutron star density range, the equation of state deviates from the form (characterised by (1.7) and (1.5)) on which the above reasoning is based in such a way as to allow the existence of ordinary material equilibrium states above the Chandrasekhar mass after all. The theoretical objection to this idea is that it would require the pressure to increase with density at a rate that would be incompatible with the causality condition that presumably requires the corresponding sound (compression wave) speed with square given by \( dP/d\rho \) to be less than unity. Since \( P \) is small compared with \( \rho \) in the physically well understood low density regime, respect for this causality requirement that we should have \( P < \rho \) throughout the entire range so that satisfaction of the virial equilibrium condition for a given mass \( M \) entails that the density \( \rho \) cannot exceed the Michell limit value given by (1.12). This consideration does not rule out the possibility of exotic ultra dense (e.g. quark nugget type) material equilibrium states with \( \rho >> m_p^4 \) and correspondingly with \( M << 1/m_p^2 \), but it does rule out their existence for higher mass values. A devil’s advocate might still try to argue that one could still get round Chandrasekhar’s upper mass limit by postulating some appropriately unorthodox relativistic gravitation theory for which the virial condition (1.2) itself would be suitably modified, but such theoretical gymnastics would seem to be pointless in view of the complete absence of the slightest shred of observational evidence in favour of the existence of any such weird states as would be produced that way. The conclusion to be drawn from the intense astronomical activity of recent years is not just that plausible black hole candidates in the mass range just above the Chandrasekhar limit are comparatively rare, but also that there is no sign whatsoever of any alternative non-black hole type of cold (as opposed to hot stellar) equilibrium state at all in this mass range.

We thus get back to the basic question of what actually has happened to the numerous stars in the moderately massive range \( M > 1/m_p^2 \) that due to of the comparative shortness of the timescale (1.11) must have already burned out by now. The answer, which is implicit in the physics described in the proceeding paragraphs, can be presented in terms of several successive steps. To start with, since they are never far from instability, the radiation dominated stars in question will always tend to lose matter from their surface in the form of an outgoing winds which can carry away a very
significant fraction of the original mass during the last stages of the thermonuclear lifetime. Secondly the dense burned out material that will accumulate in the core of the star will ultimately tend to evolve on its own almost independently of the comparatively diffuse (even if much more massive) outer envelope layers. As soon as there is a degenerate central core in excess of the Chandrasekhar limit it can be expected to collapse by itself without waiting for the outer layers to be ready to follow. Surprised and shocked, these outer layers will thus be vulnerable to being blown away in a supernova type explosion by the energy released by the core collapse. The fact, referred to above, that the neutron star mass limit is rather larger, perhaps about double, that for the degenerate electron supported core, means that the core collapse can be expected to be halted, with formation of the shock that acts back on the outer layers, when the central density reaches that of neutron star matter. The conclusion (which is of course supported by a large amount of detailed numerical calculations by many workers) is that while small main sequence stars can obviously be expected to end up in white dwarf states, ones that are initially much more massive can be expected to end up by forming only slightly more massive neutron star remnants, the remainder of the mass being dispersed in the form of a continuous wind followed by an explosive burst, a picture whose broad outline is fully consistent with what is actually observed. Although the details are complicated and still highly controversial, it is easy to see that formation of a black hole is likely to be of more exceptional occurrence, due to partial failure in a restricted parameter range of the supernova mass ejection process, or to subsequent accretion from a binary partner onto the neutron star remnant.

2 The Example of Spherical Collapse.

The collection of more or less well defined and physically plausible qualitative notions - unstoppable collapse with formation of an event horizon hiding the ensuing singularities - that constitute what may be referred to as the black hole paradigm was originally derived from the relatively tractable example provided by the spherically symmetric case, whose analysis will be the subject of the present section. In the case of actual equilibrium states, a considerable amount is now known about the non spherical generalisations that will be the main subject of later sections, but as far as dynamical evolution is concerned, although we can draw a few general qualitative conclusions (such as the Hawking area theorem to be described in the next section) there
is still very little quantitative knowledge about what happens beyond the immediate neighbourhood of spherical symmetry. This makes it necessary to rely rather heavily on the spherical example despite of (and indeed even because of) the fact that even today it is still far from clear to what extent the lessons provided by the spherical model are generically valid.

The mathematical analysis in this and all the following sections will be based on the standard Einstein theory of gravity as formulated in terms of a spacetime manifold with local coordinates $x^\mu$ say ($\mu = 0, 1, 2, 3$) and Lorentz signature metric field $g_{\mu\nu}$ (used for index lowering) that is governed by dynamical equations of the form

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi T^{\mu\nu}$$

where $R_{\mu\nu}$ (with trace $R = R^\mu_{\mu}$) is the Ricci tensor of the spacetime metric $g_{\mu\nu}$ (see the appendix for definitions and notation conventions) and $T^{\mu\nu}$ is an appropriately chosen stress-momentum-energy density tensor whose form will depend on the kind of matter under consideration but which, for consistency with (2.1), must of course must always obey the “covariant conservation” law

$$\nabla_\mu T^{\mu\nu} = 0$$

where $\nabla_\mu$ is the standard operator of Riemannian covariant differentiation as defined with respect to $g_{\mu\nu}$.

As explained in the appendix we shall use an underline whenever necessary to distinguish quantities defined with respect to the geometry of an imbedded surface under consideration from the analogous quantities as defined with respect to the background geometry. As far as this present section is concerned the relevant imbedded surfaces are to be understood as consisting of the congruence of compact spacelike 2-surfaces generated by the spherical symmetry action, whose intrinsic Ricci curvature scalar will therefore, in accordance with this convention, be denoted by $\underline{R}$ to distinguish it from the background Ricci curvature scalar $R$. This allows the specification of what we shall refer to as the Misner Sharp mass function, $M^\sharp$, by the formula

$$(2R)^{3/2} M^\sharp = 2\underline{R} - K^\mu K_\mu$$

where $K_\mu$ is the extrinsic curvature vector of the spacelike two-surface, as defined in the appendix. This definition (whose right hand side is proportional to the mean of Christodoulou’s mutually conjugate “mass aspect”
functions\textsuperscript{[27][28]} has the advantage of being manifestly covariant and giving a result that is well defined as a strictly local field for arbitrary (not necessarily spherical) spacelike two surfaces, (in contrast with the related but only semi-local Hawking mass\textsuperscript{[28]}, which involves surface integration over the two-surface, but which, like (2.3), was chosen so as to agree with the original mass specification of Misner and Sharp\textsuperscript{[29]} in the spherical limit with which we are concerned here).

The specially convenient feature of the scalar field defined by (2.3) is that in the spherically symmetric case its derivative is directly related to the Einstein tensor of the gravitational field equations by (using square brackets to denote antisymmetrisation) the identity

\begin{equation}
(2R^3)^{1/2} \nabla_\mu M^z = 2\bar{G}^\rho_{\mu} K_\rho , \quad \bar{G}_{\mu\nu} = \bar{g}_{\mu}^\rho \bar{g}_{\nu}^\sigma G_{\rho\sigma} ,
\end{equation}

using \(\perp\) to indicate surface orthogonal projection (so that \(\bar{g}_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}\), where \(\bar{g}_{\mu\nu}\) is the fundamental tensor of the spherical two surfaces - see appendix). This identity was first derived (though not in such a manifestly covariant form) by Misner and Sharp\textsuperscript{[29]}. It can be given a rather more explicit form by introducing the usual circumferencial radius function \(r\) of the spheres, in terms of which their extrinsic curvature vector \(K^\mu\) and Ricci scalar \(R\) (whose surface integral is of course \(8\pi\) by the Gauss Bonnet identity) work out to be given simply by

\begin{equation}
K_\mu = -\frac{2}{r} \nabla_\mu r , \quad R = \frac{2}{r^2} .
\end{equation}

so that (2.4) reduces to the form

\begin{equation}
\nabla_\mu M^z = r^2 G^\rho_{\nu} \bar{g}^\nu_{[\mu} \nabla_{\rho]} r ,
\end{equation}

whose derivation will now be described.

One of the convenient features of a spherical (as opposed to more general) spacetime geometry is that it is possible to describe it in terms of an orthonormal tetrad of covectors \(\theta^\Lambda_{\mu}, \Lambda = 0, 1, 2, 3\), that is fully integrable in the sense that each one is proportional to the gradient of a corresponding preferred coordinate. These may be taken to be a provisionally unspecified space and time coordinate, \(x^0\) and \(x^1\) together with the usual spherical angle coordinates \(\theta\) and \(\phi\), so that (using brackets to distinguish frame indices from coordinate indices) one has

\begin{align*}
\theta^0_{(\mu} dx^\mu = \varphi_0 dx^0 , \quad \theta^1_{(\nu} dx^\nu = \varphi_1 dx^1 ,
\end{align*}
\[ \theta^{(2)}_{\mu} dx^\mu = r \, d\theta, \quad \theta^{(1)}_{\nu} = r \sin \theta \, d\phi, \]  
where \( \theta^{(2)} \) and \( \theta^{(1)} \) are the coordinates for the "outer" and "inner" frame vectors, respectively, and \( r \) is the radial coordinate. The spherical symmetry is expressed by the condition that the three unknown metric coefficients \( r, \varphi_0, \varphi_1 \) are all functions of \( x^0 \) and \( x^1 \) only. In terms of these quantities and of the fixed Minkowski frame metric \( g_{\lambda\phi} \) with signature (-1,1,1,1) the metric form

\[ ds^2 = g_{\lambda\phi} \theta^\lambda \theta^\phi dx^\mu dx^\nu = \frac{1}{y_{\mu\nu}} dx^\mu dx^\nu + \frac{1}{\bar{g}_{\mu\nu}} dx^\mu dx^\nu \]  

where the "outer" part is given by

\[ \frac{1}{g_{\mu\nu}} dx^\mu dx^\nu = -\varphi_0^2 dx^0^2 + \varphi_1^2 dx^1^2 \]

while the metric within each two-sphere is given by the standard expression

\[ \frac{1}{g_{\mu\nu}} dx^\mu dx^\nu = r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]  

Although the method most commonly given in textbooks proceeds by working out all the (forty) Christoffel components, the quickest way of evaluating the curvature tensor of a metric such as this is to use the Cartan technique\(^{[30]} \) of proceeding via the calculation of the connection forms \( \varpi_{\mu^\lambda} \), which are got by solving the equations

\[ \nabla_{[\mu} \theta^\lambda_{\nu]} = \theta^\phi_{[\mu} \varpi_{\nu]}^\lambda \phi , \quad \varpi_{\mu(\lambda\phi)} = 0 \],

(2.11)

(2.11)

using square and round brackets to indicate index antisymmetrisation and symmetrisation respectively, and with the understanding that the fixed Minkowski metric is used for lowering and raising of frame indices, the trick being that due to the antisymmetrisation there is no need to know the Christoffel connection components to carry out the covariant differentiation operations, the result being obtainable simply by replacing the covariant differentiation operators \( \nabla_\mu \) by the corresponding partial differentiation operators \( \partial_\mu \). The next step is to use this same (exterior differentiation) trick again in evaluating the corresponding curvature form

\[ R_{\mu\nu^\lambda\phi} = 2\nabla_{[\mu} \varpi_{\nu]}^\lambda \phi + 2\varpi_{[\mu}^\lambda\phi \varpi_{\nu]}_{\sigma\phi} \]

(2.12)

It will be convenient for what follows to use a systematic shorthand notation whereby a suffix \((0)\) or \((1)\) is used to indicate the effect of differentiation
with respect to proper length in the space or time direction respectively, i.e.
to indicate the corresponding frame components of the covariant derivative when acting on a scalar, so that in particular, for the circumferential radius function \( r \) itself we have

\[
    r_{(0)} = \frac{1}{\varphi_0} \frac{\partial r}{\partial x^0}, \quad r_{(1)} = \frac{1}{\varphi_1} \frac{\partial r}{\partial x^1}.
\]  

(2.13)

In terms of this notation scheme the six independent connection forms are found from (2.15) to be expressible as

\[
    \varpi_{\mu}^{(0)} (1) = \frac{\varphi_0^{(1)}}{\varphi_0} \theta_{\mu}^{(0)} + \frac{\varphi_1^{(0)}}{\varphi_1} \theta_{\mu}^{(1)}, \quad \varpi_{\mu}^{(2)} (3) = -\frac{\cot \theta}{r} \theta_{\mu}^{(3)},
\]

\[
    \varpi_{\mu}^{(0)} (2) = \frac{r_0^{(0)}}{r} \theta_{\mu}^{(2)}, \quad \varpi_{\mu}^{(0)} (3) = \frac{r_0^{(0)}}{r} \theta_{\mu}^{(3)},
\]

\[
    \varpi_{\mu}^{(1)} (2) = -\frac{r_1^{(1)}}{r} \theta_{\mu}^{(2)}, \quad \varpi_{\mu}^{(1)} (3) = -\frac{r_1^{(1)}}{r} \theta_{\mu}^{(3)}.
\]  

(2.14)

The Cartan formula (2.16) can now be used for the direct evaluation of the tetrad components \( R_{\mu \nu \lambda \Phi} \) of the Riemann tensor, the only ones that are independent (bearing in mind that the spherical symmetry ensures that they are invariant under interchange of the indices (2) and (3)) being

\[
    R_{(0)(1)}^{(0)} (1) = \frac{\varphi_0^{(1)}}{\varphi_1} - \frac{\varphi_0^{(1)(1)}}{\varphi_0}, \quad R_{(2)(3)}^{(2)} (3) = \frac{1}{r^2} \left( 1 + r_0^{(2)} - r_1^{(2)} \right),
\]

\[
    R_{(0)(2)}^{(0)} (2) = \frac{r_0^{(0)}}{r} - \frac{r_1^{(0)}}{r} \varphi_0, \quad R_{(1)(2)}^{(1)} (2) = \frac{r_0^{(1)}}{r} \varphi_0^{(1)} - \frac{r_1^{(1)}}{r} \varphi_0,
\]

\[
    R_{(1)(2)}^{(0)} (2) = \frac{r_0^{(1)(0)}}{r} - \frac{r_1^{(1)}}{r} \varphi_0^{(1)} = \frac{r_0^{(1)(0)}}{r} - \frac{r_0^{(1)}}{r} \varphi_0^{(1)}. \quad (2.15)
\]

The corresponding frame components of the Einstein tensor will be given in terms of these by

\[
    G_{(0)(0)} = 2R_{(1)(2)}^{(1)} (2) + R_{(2)(3)}^{(2)} (3), \quad G_{(1)(1)} = -2R_{(0)(2)}^{(0)} (2) - R_{(2)(3)}^{(2)} (3),
\]

\[
    G_{(0)(1)} = -2R_{(1)(2)}^{(0)} (2), \quad G_{(0)(2)} = G_{(1)(2)} = G_{(2)(3)} = 0,
\]

\[
    G_{(2)(2)} = -R_{(0)(1)}^{(0)} (1) - R_{(0)(2)}^{(0)} (2) - R_{(1)(2)}^{(1)} (2). \quad (2.16)
\]

The resulting system can be considerably simplified by imposing that the coordinates be \textit{comoving} with respect to the flow congruence determined
by the eigenvector of the energy momentum tensor, which is equivalent to
the condition that the frame be such as to diagonalise the Einstein tensor, i.e.
\[ G_{(0)(1)} = 0 \iff \varphi_{r(0)(1)} = r_{(1)} \varphi_{r(0)}, \] (2.17)
Subject to this requirement, which it is to be emphasized is not a physical rest-
striction but just a gauge condition, the only two Einstein tensor components
still needed for the gravitational field equations can be seen to be expressible
directly in terms of the Misner Sharp mass function specified by (2.3) whose
explicit form is
\[ M^2 = r \left( 1 + r_{(0)}^2 - r_{(1)}^2 \right), \] (2.18)
as
\[ G_{(0)(0)} = \frac{2M^2_{(1)}}{r^2 r_{(1)}}, \quad G_{(1)(1)} = -\frac{2M^2_{(0)}}{r^2 r_{(0)}}. \] (2.19)
This can be seen to be just the frame component translation of the covariant
version (2.6) of the Misner Sharp identity, whose derivation is thus completed.
It is to be remarked that \( G_{(2)(2)} \), the only remaining independent Einstein
tensor component not given by (2.6), is not needed, because the equation in
which it is involved will automatically hold as an identity whenever the other
Einstein equations and the consistency condition (2.2) are satisfied.

In considering the contributions to the right hand side of the Einstein
equations (2.1) it is commonly convenient to work with a decomposition of
the form
\[ T^{\mu\nu} = T_M{}^{\mu\nu} + T_F{}^{\mu\nu} \] (2.20)
in which \( T_M{}^{\mu\nu} \) is a “strictly material” contribution and \( T_F{}^{\mu\nu} \) is an electro-
magnetic field contribution given by
\[ T_F{}^{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu}{}^{\rho} F_{\nu}{}^{\rho} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} \right) \] (2.21)
in terms of an electromagnetic gauge curvature field
\[ F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]} \] (2.22)
(where square brackets indicate antisymmetrisation of the included indices)
with (again necessarily) conserved source current,
\[ J^{\mu} = \frac{1}{4\pi} \nabla_{\nu} F^{\mu\nu}, \quad \nabla_{\nu} J^{\mu} = 0. \] (2.23)
This formulation makes it possible to characterise the important “electrovac” case as that in which the source contributions $T_M^{\mu\nu}$ and $J^{\mu}$ both vanish, the strict vacuum case being that in which the field $F_{\mu\nu}$ also vanishes.

For many purposes, including those of the present section, it is sufficient to use a treatment in which the source contributions are not necessarily restricted to vanish but in which they are postulated to have the particularly simple form describable as that of a non conducting perfect fluid, meaning that there is a preferred timelike unit vector $u^\mu$ and associated orthogonal projection tensor $\gamma^{\mu\nu}$ as characterised by

$$ \gamma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu , \quad u^\mu u^\nu = -1 $$

with respect to which the material and electromagnetic source fields are spatially isotropic, meaning that they satisfy

$$ T_M^{\mu\nu} = \rho u^\mu u^\nu + P \gamma^{\mu\nu} , \quad J^{[\mu u^\nu]} = 0 , \quad (2.25) $$

where the eigenvalues $\rho$ and $P$ are to be interpreted as the precise local values of the mass density and pressure whose characteristic mean values were the subject of discussion in the crude Newtonian order of magnitude treatment of the previous section.

It will be sufficient for the purpose of the present section to further restrict our attention to the case of an adiabatic model, for which $P$ is determined as a (not necessarily uniform) function of $\rho$ along each world line. As far as spherical applications are concerned there will be no further loss of generality in taking the model to be characterised by a pair of conserved number currents

$$ s^\mu = n u^\mu , \quad n^\mu = n u^\mu , \quad \nabla_\mu s^\mu = 0 , \quad \nabla_\mu n^\mu = 0 \quad (2.26) $$

with $n$ interpretable as the baryon number density and $s$ as an entropy density, in terms of which the mass density $\rho$ is specified by a (uniform) equation of state function whose derivatives, interpretable as the effective temperature $\Theta$ say and the effective mass-energy per particle or chemical potential $\mu$ say, determine the corresponding pressure function $P$ by the familiar relation

$$ P = s \Theta + n \mu - \rho , \quad \Theta = \frac{d\rho}{ds} , \quad \mu = \frac{d\rho}{dn} . \quad (2.27) $$

In terms of the corresponding thermal and particle four-momentum covectors,

$$ \Theta_\mu = \Theta u_\mu , \quad \mu_\rho = \mu u_\rho , \quad (2.28) $$
and subject to the conservation laws (2.26) the perfect fluid equations of motion obtained from (2.2) are expressible just as the momentum transport equation

\[ 2u^\rho (s\nabla_\rho \Theta + n\nabla_\rho \mu) = F_{\sigma\rho} J^\rho . \]  

(2.29)

From a computational point of view, this latter formulation has the advantage (as compared with (2.2)) that, as in (2.11) and (2.12), the antisymmetrised “exterior” nature of the derivation involved makes it possible to work it out directly, by direct substitution of the partial differentiation operator \( \partial_\mu \equiv \partial / \partial x^\mu \) in place of the Riemannian operation \( \nabla_\mu \), thereby making it possible to avoid having to go to the trouble of working out the Christoffel connection components.

As a prerequisite to applying the formulae (2.19) in this perfect fluid case, it is necessary to impose the gauge restriction (2.17) to the effect that the coordinates should be comoving, which means that the timelike frame vector \( \theta^{(0)}_\mu \) is to be identified with the unit flow vector \( u^\mu \) of the fluid as introduced in (2.24), so that by (2.25) and (2.26) the particle number and electric source currents will be given by

\[ J^\mu = en^\mu , \quad n_\mu = n\theta^{(0)}_\mu , \]  

(2.30)

where the parameter \( e \) represents the electric charge per particle. It follows that the particle, entropy, and charge conservation laws (2.26) and (2.23) will be expressible simply by

\[ (nr^2 \varphi_i)_\rangle = 0 , \quad \left( \frac{s}{n} \right)_\rangle = 0 , \quad e = 0 \]  

(2.31)

or equivalently by

\[ N = 0 , \quad S = 0 , \quad Q = 0 , \]  

(2.32)

where \( N, S \) and \( Q \) are fields respectively representing the total particle number, entropy, and electric charge within the the corresponding sphere, which will be given as integrals over the interior of the sphere at a fixed value of the coordinate time \( x^0 \) by

\[ \frac{N}{4\pi} = \int r^2 n \varphi_i dx^1 , \quad \frac{S}{4\pi} = \int r^2 s \varphi_i dx^1 , \quad \frac{Q}{4\pi} = \int r^2 en \varphi_i dx^1 . \]  

(2.33)

The corresponding frame components of the total energy momentum tensor are then then obtainable from from (2.21) and (2.25) as

\[ T_\langle(0)\rangle = \rho + \frac{E^2}{8\pi} , \quad T_{(1)(1)} = P - \frac{E^2}{8\pi} . \]  

(2.34)
where the appropriate electric field magnitude, \( E = F_{(1)(0)} \), is obtainable by direct integration from the source equation (2.22) in the form

\[
E = \frac{Q}{r^2} .
\]  

(2.35)

It can be seen from (2.21), by a similar integration, that the corresponding magnetic field component \( F_{(2)(3)} \) is necessarily zero, i.e. there can be no magnetic monopole moment, on the assumption that (initially at least) there is a well behaved spherical centre from which the integrals in (2.33) are understood to be taken. It is to be remarked that the other (crossed) field components \( F_{(0)(2)} \), \( F_{(0)(3)} \), \( F_{(1)(2)} \), \( F_{(1)(3)} \) all vanish trivially as a local requirement for spherical symmetry.

Just as the classically familiar Coulombian form of the relation (2.35) is due to an judicious choice of definitions of the variables involved, so also the particularly astute Misner Sharp choice\[^{[29]}\] for the definition of the mass function \( \Phi^\sharp \) leads to a pseudo Newtonian form for the integral relation expressing the spacial constraint resulting from the first of the Einstein equations obtained from (2.19) which gives

\[
\Phi^\sharp = 4\pi \int r^2 (\rho + \frac{E^2}{8\pi}) dr ,
\]  

(2.36)

in which, as in (2.33) it is to be understood that the integral is taken over the interior of the relevant sphere at a fixed value of the comoving time coordinate, (starting from a central origin that is assumed to be regular, at least initially) which means that the radial variation will be expressible in terms of that of the space coordinate \( x^i \) by the relation \( dr = r^{(1)} \Psi \, dx^i \).

Subject to the constraint (2.36) and the shift condition (2.17) which can be rewritten in the symmetrically equivalent form

\[
\varphi_0^{r(1)(0)} = r^{(0)} \varphi_0^{(1)} ,
\]  

(2.37)

the only other Einstein equation that is needed is the dynamical equation

\[
\Phi^\sharp_{(0)} = (\frac{E^2}{2} - 4\pi P)r^2 r^{(0)} .
\]  

(2.38)

These two equations are to be solved in conjunction with the constraint obtained from the momentum transport equation (2.12 which takes the form

\[
(\rho + P) \varphi_0^{(1)} = \varphi_0 \left( E \epsilon_n - P^{(1)} \right)
\]  

(2.39)

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in which the spacial pressure gradient is of course given by \( P^{(1)} = n\mu^{(1)} + s\Theta^{(1)}. \)

It is apparent at this stage that it will be convenient to introduce a modified, combined (electromagnetic as opposed to purely gravitational) mass function \( M \) say, given in terms of the original Misner Sharp mass function \( M \) by

\[
M = M^\sharp + \frac{Q^2}{2r} \quad (2.40)
\]

or equivalently by the more direct relation

\[
1 - \frac{2M}{r} = r^2(K_\mu K^\mu - E^2) \quad (2.41)
\]

It follows from the Misner Sharp identity (2.4) in conjunction with the Einstein equations (2.1) that the gradient of this combined mass function will be given in terms of the purely “material” contribution \( T_{M}^{\mu \nu} \) in the decomposition (2.20) by the manifestly covariant expression

\[
\nabla_\mu M = \frac{Q}{r} \nabla_\mu Q + 8\pi r^2 T_{M}^{\rho \nu} g^{\nu \mu} \nabla_\rho r \quad . \quad (2.42)
\]

For the case of a nonconducting perfect fluid in the comoving frame, the constraint (2.36) and the dynamical equation (2.38) can be rewritten in terms of this new combined mass variable as

\[
M = 4\pi \int r^2 (\rho + neE) \frac{r}{r^{(1)}} dr \quad , \quad (2.43)
\]

and

\[
M^{(0)} = -4\pi Pr^2 r^{(0)} \quad . \quad (2.44)
\]

of which the latter has the advantage of having the same simple form as that to which the original version (2.38) would reduce if no electromagnetic effects were present. An analogous remark applies also to (2.44) in any external region where the charge density \( ne \) vanishes.

Whichever formulation is used, the solution of the system will in general require numerical computation, as in the pioneering attempt at an astrophysically realistic calculation by May and White\[23\] or the important investigation of the possibility of naked singularity formation by Eardley and Smarr\[34\]. It is however possible to obtain analytic solutions in special cases of which the most obvious are those in which the circumstances are such that
the right hand side of the constraint equation (2.39), and hence also that of (2.37) is zero so that we obtain
\[ \varphi_0 = 1 , \quad r_{(1)(0)} = 0 , \]  
(2.45)
for a suitable normalisation of the time coordinate \( x^0 \) which in this particular case is adjustable to agree with the proper time along the flow lines. Such a possibility obviously occurs in layers of matter that are uncharged and for which the pressure gradient \( P_{(1)} \) is zero, either because the configuration is homogeneous as in the classic prototype collapse calculation of Oppenheimer and Snyder\(^{35}\) or because the matter has a pressure free (so called “dust”) equation of state. A fluid of this latter uncharged dust type (for which the flow will simply be geodesic) is characterisable by
\[ e = 0 , \quad \rho = mn , \quad m_{(0)} = 0 \]  
(2.46)
where \( m \) is a constant mass per particle, so that in a layer of this type \( Q \) will be constant not just in time but also in space, while the combined mass function \( M \) will at least be constant in time:
\[ Q_{(1)} = 0 , \quad M_{(0)} = 0 . \]  
(2.47)
This means that the radial evolution equation, which by the definition of \( M \) will always have the form
\[ r_{(0)}^2 = r_{(1)}^2 - 1 + \frac{2M}{r} - \frac{Q^2}{r^2} , \]  
(2.48)
will in this case be independently integrable for each flow line, since by (2.32), (2.45), and (2.47) the quantities \( Q, r_{(1)}, \) and \( M \) appearing on the right will all be constants along each separate flow line. The simplest possibility is the “parabolic” case corresponding to zero radial velocity in the large radius limit which is got by taking
\[ r_{(1)} = 1 , \quad x^0 = c^0 - \frac{(Mr + Q^2)\sqrt{2Mr - Q^2}}{3M^2} , \quad c^0_{(0)} = 0 \]  
(2.49)
where \( c^0 \) like \( M \) is an initially arbitrary constant along each flow line, i.e. a function only of \( x^1 \). This latter comoving space variable can now be replaced
(except in the special case for which both $c^o$ and $M$ are spacially uniform) by $r$ in the “outer” part of the metric (2.8) which thereby acquires the form

$$\frac{1}{g_{\mu\nu}}dx^\mu dx^\nu = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) (dx^o)^2 + 2\sqrt{(\frac{2M}{r} - \frac{Q^2}{r^2})} dx^o dr + dr^2 \quad (2.50)$$

with the combined mass variable $M$ now determined implicitly through its functional dependence on $c^o$ by the relation (2.49).

The class of solutions specified by (2.49) and (2.50) is by no means simple and it is only comparatively recently (with the work of Eardley and Smarr\[34\] on the uncharged $Q = 0$ case) that they have started to be examined seriously from the point of view of questions such as naked singularity formation. They do however include the genuinely simple electrovac case for which the mass coefficient $m$ in (2.46) is set equal to zero, which implies the constancy in space as well as comoving time of the combined mass variable $M$ (but therefore not of the original Misner Sharp mass variable $M^\#$ except in the $Q = 0$ case for which they coincide). In this special electrovac case, as characterised by

$$m = 0, \quad M^{(\#)} = 0, \quad (2.51)$$

the flow just represents a geodesic test particle congruence, so there in favor of the radius variable $r$ so that there is no further loss of generality in imposing the parabolicity condition (2.49). This means that the form (2.50) with not just $Q$ but now also $M$ taken to be constant in space as well as time represents the most general spherical electrovac solution (apart from the exceptional Robinson-Bertotti case for which $c^o$ is uniform, so that\[10\] one obtains a tubular universe with constant radius $r$ throughout). This solution can be seen to be automatically stationary since all dependence on $x^o$ has dropped out, which in the spherical case means more particularly that it must be static, i.e. that it is unaffected not only by displacements but also by reversals of a certain preferred time coordinate, $t$ say, that is determined (modulo a constant of integration) by the differential relation

$$dt = dx^o - \frac{r\sqrt{2Mr - Q^2}}{r^2 - 2Mr + Q^2}. \quad (2.52)$$

Replacement of the comoving proper time coordinate $x^o$ by this preferred time coordinate $t$ leads to the manifestly static form

$$\frac{1}{g_{\mu\nu}}dx^\mu dx^\nu = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \quad (2.53)$$

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that was originally derived by Reissner and Nordstrom on the basis of the postulate of staticity at the outset. The present approach, showing how staticity is obtained automatically as a consequence of spherical symmetry in the source free case, amounts to a demonstration of what is known as Birkhoff’s theorem.

Our parabolically infalling version (2.50) has the significant advantage over the algebraically simpler historic form (2.53) that it remains well behaved on the “Killing horizons”\[36\], i.e. stationary null hypersurfaces that occur, whenever \( Q^2 \leq M^2 \), at the roots

\[
    r = r_\pm , \quad r_\pm = -M \pm \sqrt{M^2 - Q^2} \tag{2.54}
\]

whereas the manifestly static version (2.53) is singular there. However although it is sufficient for describing the outside of a collapsing spherical charged or neutral star model, even the more sophisticated version (2.50) has the limitation of being geodesically incomplete even when extended over the full coordinate range \( 0 \leq r < \infty, \ -\infty < x^a < \infty \). The geometrically complete manifold was first described in the pure vacuum case, \( Q = 0 \) by Kruskal and Szekeres\[37\] and in the generic case \( Q^2 < M^2 \) by Graves and Brill\[38\], while for the special “maximally charged” limit case \( Q^2 = M^2 \) the corresponding construction was first carried out rather later by myself\[39\].

It was for the purpose of describing such extensions that I first introduced \[39\][40] the representational technique of \textit{conformal projection} (the space time analogue of the beloved Mercator projection of terrestrial navigators) that has since been generally adopted as a standard tool for understanding the topological and causal structure of any timelike two dimensional manifold or submanifold, the idea being to first convert the metric into null coordinate form

\[
    g_{\mu\nu}dx^\mu dx^\nu = -\Psi du^+ du^- \tag{2.55}
\]

which is always locally possible for some conformal factor \( \Psi \) determined as a function of the null coordinates \( u^+ \) and \( u^- \), and then to take advantage of the fact that this null form is preserved, only the functional dependence of the conformal factor being altered, by a transformation \( u_+ \mapsto \tilde{u}^+, \ u_- \mapsto \tilde{u}^- \), \( \Psi \mapsto \tilde{\Psi} \) whereby each of the null coordinates is replaced by an arbitrary function only of itself, which one is free to choose in such a way as to cover what from a metric or affine point of view might be an infinite region by a finite coordinate range which can thus be plotted directly as a diagram.
Fig. 2 – Facsimiles of the original C.P. diagrams for the limiting cases $Q^2 = 0$, i.e. Kruskal-Schwarzschild (including disconnected branch representing analytic extension to region $r < 0$), and $Q^2 = M^2$, i.e. “maximally charged” Reissner-Nordstrom.
The use of this method of representation of event horizons in two-dimensional manifolds was suggested to me by the example from a related but rather different context (namely the study of the distant asymptotically flat outer regions) of Penrose’s conformal boundary procedure procedure\textsuperscript{[41]} (as used for setting up what is known to the initiated by the term pronounced as “Scri”). However the concepts of a conformal boundary and a conformal projection should not be confused. The Penrose conformal boundary (Scri) concept is not restricted to two dimensions, but on the other hand it is dependent on rather severe asymptotic regularity conditions that may fail in many relevant applications, whereas the conformal projection (C.P.) technique is limited to two dimensional sections but not otherwise restricted, so that in particular it is very useful for the analysis of boundaries that may be singular.

In the present case the most obvious choice to start with is to take the the incoming and outgoing stationarity preserving null coordinates which are defined (modulo an arbitrary choice of origin) by

\[
 du^\pm = dt \pm \frac{r^2 \, dr}{r^2 - 2Mr + Q^2} \tag{2.56}
\]

whose separate substitution gives the forms

\[
 g_{\mu\nu} dx^\mu dx^\nu = - \frac{r^2 - Mr + Q^2}{r^2} (du^\pm)^2 \pm du^\pm dr . \tag{2.57}
\]

These null inflowing and outflowing coordinate forms (like our original parabolically inflowing form (2.53)) are locally well behaved on the Killing horizons at \( r = r_\pm \) but nevertheless still incomplete. When both are substituted together, with the radius variable now considered no longer as a coordinate in its own right but just as a function of the null coordinates, the metric acquires the required doubly null form (2.55) with the conformal factor given by

\[
 \Psi = \frac{r^2 - 2Mr + Q^2}{r^2} \tag{2.58} .
\]

In the generic case \( Q^2 < M^2 \), the integration of (2.56) gives the explicit expressions

\[
 u^\pm = t \pm r \pm \frac{1}{2\kappa_+} \ln |r - r_+| \pm \frac{1}{2\kappa_-} \ln |r - r_-| , \tag{2.59}
\]
from which, in terms of the decay constants of the Killing horizons which are
given by
\[ \kappa_{\pm} = \pm \frac{(M^2 - Q^2)^{1/2}}{r_{\pm}^2}, \tag{2.60} \]
one obtains the functional dependence of the variable \( r \) in (2.59) in the im-
plicit form
\[ 2r + \frac{1}{\kappa_+} \ln|r - r_+| - \frac{1}{\kappa_-} \ln|r - r_-| = u^+ - u^-, \tag{2.61} \]
whose unambiguous solution requires the specification that \( r \) should lie in
some particular one of the three possible ranges characterised by the condi-
tion that neither, just one, or both of the quantities \( r - r_{\pm} \) be positive. Except
for the first of these three possibilities, which includes the value \( r = 0 \) that
corresponds to an irremovable geometric singularity, the resulting conformal
factor \( \Psi \) will be regular over the full coordinate range \( \infty < u^\pm < \infty \), but
from the point of view of completeness the metric version given by (2.58)
and (2.61) is no improvement on the traditional manifestly stationary ver-
sion (2.53) : all that has been achieved is to push the Killing horizons out of
the coordinate chart, but not to regularise them. However a genuine regular-
isation is now easily obtainable by a conformal coordinate transformation to
a a new null coordinate form
\[ \frac{1}{\tilde{g}_{\mu\nu}} dx^\mu dx^\nu = \tilde{\Psi} d\tilde{u}^+ d\tilde{u}^- . \tag{2.62} \]
Depending on whether it is an “outer” Killing horizon at \( r = r_+ \) or an “inner”
one at \( r = r_- \) that one wishes to cover, it suffices to take
\[ \tilde{\Psi} = \frac{\pm e^{-2\kappa_{\pm}} (r - r_{\pm})}{\kappa_{\pm}^2 r^2 |r - r_+|^{|\kappa_{\mp}/\kappa_+|}} , \tag{2.63} \]
where \( r \) is now given implicitly by
\[ \pm (r_{\pm} - r) e^{2\kappa_{\pm} r} |r - r_{\pm}|^{|\kappa_{\mp}/\kappa_+|} = \tilde{u}^+ \tilde{u}^- . \tag{2.64} \]
This relation shows in particular how, at the outer horizon the decay param-
eter \( \kappa_+ \) is interpretable as measuring the exponential relation between the
affine time parameter \( \tilde{u}^- \) and the group parameter \( u^- \).

The unambiguous solution of (2.64) requires the specification that \( r \)
should lie in one or other of just two possible ranges characterised respectively
by \( r < r_\pm \) and \( r > r_\pm \), and except for the irremovable geometric singularity at \( r = 0 \) in the first of these two ranges, the new conformal factor \( \tilde{\Psi} \) will be regular over the full coordinate range \( \infty < \tilde{u}^\pm < \infty \), of the new null coordinates, including the locus \( r = r_\pm \) which can be seen to consist of two intersecting Killing horizons characterised in the new coordinates by \( \tilde{u}^+ = 0 \) and \( \tilde{u}^- = 0 \) respectively. The stationarity group transported coordinates \( u^+ \) and \( u^- \) of the original system in the more restricted patches on either side of these now regularized horizons are given in terms of the new ones, which can be seen to be characterised by the property of measuring affine distance along the regularised horizon at \( r = r_\pm \), by relations of the simple exponential form

\[
\kappa_\pm u^+ = \ln|\tilde{u}^+|, \quad \kappa_\pm u^- = -\ln|\tilde{u}^-|. \tag{2.64}
\]

The use of transformations of the simple form (2.64) allows us, according to choice, to cover either the locus \( r = r_+ \) or the locus \( r = r_- \) with a regular coordinate chart, but it does not allow us to cover both at once. Nevertheless since the alternative kinds of chart overlap (in the intermediate range \( r_- < r < r_+ \) where both are perfectly regular) they can be used as successive patches to build up a maximally extended manifold in the manner first described (in terms of a somewhat different system) by Graves and Brill\[38\]. It is for the purpose of visualising the final result of such successive extensions that the C.P. (conformal projection) technique\[39\][40] is particularly useful. If one is willing to sacrifice the desideratum of having a simple analytic expression such as (2.64), there is no obstacle in principle to the introduction of further modified null coordinates \( \tilde{u}^\pm \) say whose range covers the entire maximally extended manifold.

An invaluable practical feature of a C.P. diagram of the kind obtained by plotting such coordinates directly (traditionally with a diagonal orientation) on a flat screen or page (as in figures 2 and 3 ) is that, as far as the essential causal and topological features are concerned, it does not matter whether or not one knows the precise functional form of the functional relation between the original (restricted) and new (extended) null coordinate systems : provided the linear (diagonal) representation of the null congruences is preserved, any smooth (not necessarily analytic) deformations are admissible. This means that (provided it is not restricted by the inclusion of too much detail) any C.P. diagram that has been constructed as a rough free-hand sketch has the beautiful feature of being interpretable post facto as an
Fig. 3 – Facsimile of the original C.P. diagram\textsuperscript{[40]} for the Graves - Brill extension of Reissner Nordstrom for $0 < Q^2 < M^2$ together with previously unpublished “new look” version in which complete compactification is achieved by letting the scale for successive universes tend to zero at the extremities of the chain.
accurate representation in terms of null coordinates whose precise specification (if one were interested) could in principle be found out later by carrying out empirical measurements on the sketch.
3. Qualitative theory of non-spherical Black Hole formation.

Whereas a considerable amount is known about non spherical black hole equilibrium states (to which the subsequent sections will be devoted) as also about non stationary states of spherical collapse (the subject of the previous section) the subject of generic nonspherical gravitational collapse and black hole formation still consists mainly of a few vague, qualitative, and for the most part far from rigourously established notions, that are largely inspired by the spherical example. The question of the extent to which various features of spherical collapse scenarios may be taken over to more general situations has long been and still remains a subject of animated debate. The unreliability of the spherical example as a guide to more general cases is shown by the case of Birkhoff’s theorem, to the effect that (as was demonstrated in the previous section) the source free (strict or electromagnetic) vacuum outside a collapsing spherical object must necessarily be static (i.e. not only time independent but even time reversal invariant) whereas in the non spherical case it need not even be stationary (i.e. time independent) in view of the possibility of gravitational and electromagnetic radiation whose absence, exceptionally, in the spherical case is due to the absence of any scalar part of either the electromagnetic field which is purely vectorial or the gravitational field which is purely (i.e. tracelessly) tensorial, at least in Einstein’s theory to which our discussion here is restricted.

Among the features that are generally thought to survive the breaking of spherical symmetry of the collapse, some of the most important may be listed as follows:

(1) The ultimate formation of a *singularity* of some kind (not necessarily just a simple density singularity, but something sufficient to prevent affine completeness) was shown by the work of Penrose and Hawking to occur very generally, but its generic nature is still not well established.

(2) The phenomenon for which Penrose coined the term *cosmic censorship*, whereby the singularities are hidden from the outside asymptotically flat universe behind a regular *event horizon* bounding the region for which Wheeler coined the term *Black Hole* would appear to be stable against moderate perturbations from spherical symmetry and from the uniformity of the homogeneous Oppenheimer Snyder collapse scenario that provides its simplest example. Nevertheless much recent work has made it clear that sufficiently (one might be tempted to say unnaturally) large deviations from uniformity can bring about the occurrence of non-trivial *naked* singular-
Fig. 4 – *Illustration of the standard black hole paradigm by a C.P. diagram representing a two dimensional radial section through a collapsing body with cross hatched shading of the D.O.C. (domain of outer communications).*

ities, i.e. ones from which light can escape to large asymptotic distances, so although the regular black hole scenario, as governed by the cosmic censorship postulate, may plausibly provide a generic description of astrophysically realistic collapses, its mathematical generality would seem to be more severely circumscribed than was once thought.

(3) Although the vacuum region outside a generic collapsing body will not become immediately static (as it must, by Birkhoff’s theorem in the spherical case) it is nevertheless to be expected that the energy of non stationary oscillatons will ultimately be radiated away so that in the end the vacuum region outside an (isolated) collapsing body will settle down asymptotically towards an ultimate *equilibrium* that is stationary at least in the weak sense of being invariant under the action of a Killing vector field that is timelike at large distances even if not everywhere outside the horizon.

Experience with the Schwarzshild and Reissner Nordstrom examples (as described in the previous section) shows however that whereas the physical collapse situation may be regular in the past, starting with an ordinary well behaved asymptotically flat Cauchy initial value hypersurface, the asymptotically approached equilibrium metric may have a “white hole” region including singularities in the past, so the strongest regularity condition it can be expected to satisfy is asymptotic predictability, meaning that
there exists a partial Cauchy surface (a not necessarily complete globally spacelike, i.e. achronal, hypersurface) extending in from outer infinity to the black hole horizon and at governing (i.e. intercepting all sufficiently extended past directed timelike lines from) not necessarily all of its future (as would be required for a strict Cauchy surface) but at least a part consisting of a regular asymptotically flat domain of outer communications with inner bound on a regular black hole horizon.

The future boundary of the region governed by the partial Cauchy surface (which in the standard Reissner Nordstrom example occurs at $r = r_-$) is called its Cauchy horizon, and is an example of what is commonly described as a local “future event horizon”, whereas a black hole horizon (which in the Reissner Nordstrom example occurs at $r = r_+$), i.e. the boundary of the region from which a future directed timelike line can be extended to the outer
asymptotically flat region, is analogously describable as being a local “past event horizon”. Both kinds obviously belong to the category of “achronal boundaries”, meaning boundaries of which no two points are connectable by a strictly timelike curves. Local future and past event horizons are characterised more particularly by the property (which was first systematically exploited by Penrose \cite{42,43} and Hawking \cite{47,7}) of being generated locally by by null geodesics with no respectively past or future end points (which means that they are ordinary null hypersurfaces wherever they are smooth, but that there null generators may reach caustics when extrapolated to respectively the future of the past). In the case of a black hole horizon this property (see figure) is not just local but global i.e. its null generators can never reach a future end pont no matter how far they are extrapolated.

To draw quantitative conclusions from these considerations it is necessary to recapitulate some of the standard kinematic properties of generating congruences. To start with we recall that for any vector field $\ell_\mu$ has an (unnormalised) acceleration $\dot{\ell}^\mu$ that is related to its Lie derivative with respect to itself by

$$\ell \mathcal{L} \ell_\mu = \dot{\ell}_\mu + \frac{1}{2} \nabla_\mu (\ell^\nu \ell_\nu) , \quad \dot{\ell}^\mu = \ell^\nu \nabla_\nu \ell_\mu . \quad (3.1)$$

In order for the field to be normal to a hypersurface, it must satisfy the Frobenius integrability condition

$$\ell_\mu [\nabla_\nu \ell_\rho] = 0 , \quad (3.2)$$

which implies

$$\ell_\mu [\ell \mathcal{L} \ell_\rho] = \ell^\nu \ell_\nu \nabla_\mu [\ell_\rho] . \quad (3.3)$$

It is apparent from (3.3) that in the particular case of a hypersurface that is locally null, i.e. whose normal satisfies

$$\ell^\nu \ell_\nu = 0 ,$$

this normal must automatically satisfy the geodesic equation

$$\ell_\mu [\dot{\ell}_\rho] = 0 . \quad (3.4)$$

Let us introduce a second null vector $\tilde{\ell}^\mu$ say, transverse to the null hypersurface and normalised so that

$$\tilde{\ell}^\nu \ell_\nu = 0 , \quad \tilde{\ell}^\nu \ell_\nu = -1 . \quad (3.5)$$
Such a vector can be uniquely specified by the condition that it be normal to some given spacelike 2-surface $S$ say in the horizon at the point under consideration, in which case the corresponding rank 2 projection tensor

$$\overline{g}^\mu_\nu = g^\mu_\nu + \ell^\mu \tilde{\ell}_\nu + \tilde{\ell}^\mu \ell_\nu, \quad \overline{g}^\nu_\nu = 2$$

will be interpretable (see appendix) as the (first) fundamental tensor of the spacelike 2-surface.

With respect to any previously specified normalisation, corresponding to a time parametrisation such that $\ell^\mu = dx^\mu/dt$, we can define a corresponding affine time parametrisation $\tau$ say whose relation to the original time parameter $\tau$ specifies a corresponding decay coefficient $\kappa$ in terms of which the non affine geodesic equation (3.4) takes the form

$$\dot{\ell}_\mu = \kappa \ell_\mu$$

where explicitly

$$\kappa = \frac{(\ln \dot{\tau})^\prime = \dot{\tau}/\dot{\tau}}{\dot{\tau}^2} = -\tilde{\ell}^\nu \dot{\ell}_\nu.$$ (3.7)

The usual way of defining the divergence $\theta$ and the (automatically real) magnitude $\sigma$ of the (automatically symmetric) shear rate tensor $\sigma_{\mu\nu}$ of the null generators is via the projection

$$\theta_{\mu\nu} = \overline{g}^\rho_\mu \overline{g}^\sigma_\nu \nabla_\rho \ell_\sigma, \quad \theta = \theta^{\nu}_\nu, \quad \sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{2} \theta \overline{g}_{\mu\nu}, \quad \sigma^2 = 2 \sigma_{\mu\nu} \sigma^{\mu\nu}. \quad (3.9)$$

Taking the contraction of the defining identity of the Riemann tensor, which for any field $\ell_\mu$ whatsoever gives the consequent identity

$$\ell^\mu \nabla_\nu (\nabla_\mu \ell^\nu) = \nabla_\mu (\ell^\nu \nabla_\nu \ell^\mu) - (\nabla^\mu \ell_\nu) \nabla_\mu \ell_\nu - R_{\mu\nu} \ell^\mu \ell^\nu,$$ (3.10)

one obtains, for the case of a congruence of null geodesic generators as characterised above, the famous Penrose[42] null version

$$\dot{\theta} - \kappa \theta = -\frac{1}{2} (\theta^2 + \sigma^2) - R_{\mu\nu} \ell^\mu \ell^\nu,$$ (3.11)

of the equation whose analogue for a timelike congruence was first brought to attention by Raychaudhuri[48], where a dot denotes differentiation with respect to an arbitrary time parametrisation, whose adjustment to be affine can be used to get rid of the $\kappa$ term on the left hand side. The special importance of this equation comes from the fact that one is then left with a right
hand side that is strictly non positive provided the Ricci tensor is determined by the Einstein equations (2.1) either for a vacuum or at least with an energy tensor $T^{\mu \nu}$ that, as is the case for all the usual macroscopic matter models, is such as to satisfy the appropriate energy inequality $T^{\mu \nu} \ell_\mu \ell_\nu = 0$.

The Penrose inequality

$$\left( \theta^{-1} \right) \geq \frac{1}{2} \quad (3.12)$$

that is obtained for the rate of affine variation under such conditions can be immediately used to see that if $\theta$ is ever negative then there will inevitably be a caustic where it diverges to infinity within an affine distance bounded above by $2/|\theta|$ in the future. Penrose’s original exploitation of this result was for the purpose of demonstrating the inevitability of some kind of singularity formation to the future of any closed trapped surface on a well behaved initial hypersurface by showing the affine boundedness (which would be impossible in the absence of a singularity) of the future event horizon bounding the future of the closed trapped surface, where this term is understood to mean a compact spacelike topologically spherical 2-surface $S$ for which the divergence $\theta$ of the null normals is everywhere negative.

In terms of the background tensor curvature formalism$^{[15]}$ described in the appendix, it can be seen (from (A9)) that the specifications (3.9) for the divergence and shear of the outgoing null congruence $\ell^\mu$ can be rewritten in terms of the second fundamental tensor $K_{\mu \nu}^{\rho}$ (which is equivalent to what is referred to by Hartley and Tucker$^{[49]}$ as the shape tensor) of the spacelike 2-surface $S$ and of the corresponding curvature vector $K^\mu$ and trace free (and conformally invariant) conformation tensor $C_{\mu \nu}^{\rho}$ as

$$\theta_{\mu \nu} = -K_{\mu \nu}^{\rho} \ell_\rho \quad , \quad \theta = -K^\nu \ell_\nu \quad , \quad \sigma_{\mu \nu} = -C_{\mu \nu}^{\rho} \ell_\rho \quad . \quad (3.13)$$

Similarly for the ingoing null congruence as specified by $\tilde{\ell}^\mu$ (for which we are still assuming the normalisation condition (3.5)) the corresponding divergence and shear will be given by

$$\tilde{\theta}_{\mu \nu} = -K_{\mu \nu}^{\rho} \tilde{\ell}_\rho \quad , \quad \tilde{\theta} = -K^\nu \tilde{\ell}_\nu \quad , \quad \tilde{\sigma}_{\mu \nu} = -C_{\mu \nu}^{\rho} \tilde{\ell}_\rho \quad . \quad (3.14)$$

The usual situation for an approximately spherical 2-surface at approximately constant time in an approximately flat background is to have ingoing null normals that converge, $\theta < 0$ but outgoing ones that diverge,
\( \theta > 0 \) so that the product, which can be seen from (3.6) to be expressible in manifestly the normalisation independent form

\[
\tilde{\theta} = -\frac{1}{2} K^\nu K^\nu,
\]

(3.15)

will be negative, whereas if \( \theta \) changes sign and becomes negative, with \( \tilde{\theta} \) still also negative, then the product will change sign also, i.e. the curvature vector \( K^\mu \) will change from being spacelike to being timelike. A special interest applies to the marginally trapped case characterised by \( \theta = 0 \) everywhere, for which Hawking\(^{[47][50][7]} \) has introduced the term apparent horizon. Evidently such a marginally trapped surface may be described as one for which the curvature vector \( K^\mu \) is null.

Following the Penrose application of (3.12) to the future of a closed trapped surface, Hawking\(^{[47][50][17]} \) pointed out that a very powerful result can be obtained by applying it to the black hole horizon itself, using the condition that the future generators of a black hole horizon can never terminate, which implies that the generators of such a horizon can never have negative divergence \( \theta \). Noting that \( \theta \) is interpretable as specifying the rate of variation of the measure of a 2-surface element dragged along by the generators according to the formula

\[
(dS) = \theta dS,
\]

(3.16)

and applying this to the integrated area

\[
\mathcal{A} = \oint dS
\]

(3.17)

of a 2 dimensional spacelike section through the black hole horizon, Hawking obtained the (now famous) law to the effect that the horizon must evolve with time according to the inequalities

\[
\dot{\mathcal{A}} \geq \oint \theta dS \geq 0,
\]

(3.18)

the extra inequality on the right being to allow for the fact that in addition to the area increase resulting from smooth expansion there is also the possibility of an additional increase due to the branching off of new generators from a caustic (see diagram). In particular if two black holes with areas \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) merge to form a combined black hole with area \( \mathcal{A}_3 \) then we must have the strict inequality

\[
\mathcal{A}_3 > \mathcal{A}_1 + \mathcal{A}_2.
\]

(3.19)
Before closing this section and going on to restrict our attention to states of stationary equilibrium, it is to be remarked that though the locality of a marginally trapped surface or “aparent horizon”, as characterised locally by \( K_\mu K^\mu = 0 \), is merely an inner bound on the location of (teleologically defined) true horizon, it may nevertheless give a very good approximation to the localisation of the true horizon in the almost stationary limit when the situation is not too strongly different from its ultimate equilibrium state, in which case the approximate stationarity will determine a corresponding approximately well defined and in general non affine time parametrisation on the horizon, so that there will be a correspondingly well defined decay constant \( \kappa = \ddot{\tau}/\dot{\tau} \) where \( \tau \) is the corresponding proper time. Under such conditions the ratio between the values \( dS_0 \) and \( dS_1 \) of the measure of a generator transported surface element \( dS \) between times \( t_0 \) and \( t_1 \) can be seen\[11\][12] to be given approximately by an expression of the non-teleological form

\[
\ln\left(\frac{dS_1}{dS_0}\right) = \int_{t_0}^{t_1} \theta dt \approx \frac{8\pi}{\kappa} D dt
\]

which gives the Hartle Hawking formula\[51\]

\[
\dot{\mathcal{A}} \approx \oint \frac{D}{\kappa} dS ,
\]

where the rate of effective dissipation is given by

\[
D = \frac{\sigma^2 + \theta^2}{16\pi} + T^{\mu\nu}\ell_\mu \ell_\nu .
\]

The discovery\[47\][51] of the laws (3.18) and (3.22) suggested an obvious thermodynamic analogy, with \( \mathcal{A} \) proportional to the entropy and \( \kappa \) to the temperature. The deeper significance of this analogy was first guessed by Beckenstein\[53\] and later established by the discovery of Hawking radiation\[20\], which is associated with a temperature given exactly by \( \kappa/2\pi \) in Plank units, corresponding to an entropy given exactly by \( \mathcal{A}/4 \). The crux of the analogy is constituted by the “zeroth law” that will be established in the next section.
4. Rotating Equilibrium States: Zamos and local properties of a Killing horizon.

The concept of stationarity that is relevant to the theory of black hole equilibrium states means the condition the spacetime is invariant under the action generated by an asymptotically timelike Killing vector

\[ k^\mu \leftrightarrow \frac{\partial}{\partial t}, \quad \nabla_{(\mu} k_{\nu)} = 0 \]  \hspace{1cm} (4.1)

It is to be noticed that this definition is slightly weaker than the one commonly used in other contexts where it is stipulated that the stationarity Killing vector be timelike not just at large asymptotic distances but throughout, which would exclude the existence of the “ergorgions” which are of importance not just in black hole theory but even in the theory of ultrarapidly rotating stars.

All that follows will be based on the postulate that I decided to adopt when I first looked into this area of work in the 1960’s, namely that the stationary spacetime under consideration is also characterised by axisymmetry, meaning that it is also invariant under the action generated by a spacelike Killing vector

\[ m^\mu \leftrightarrow \frac{\partial}{\partial \phi}, \quad \nabla_{(\mu} m_{\nu)} = 0 \]  \hspace{1cm} (4.2)

whose action is periodic, with closed circular (or, on the axis itself, fixed point) trajectories along which the group parameter \( \phi \) is therefore also periodic, with period \( 2\pi \) for the standard normalisation. Relaxation of the requirement that \( m^\mu \) be spacelike is mathematically conceivable but physically inappropriate since, in view of the periodicity, it obviously violates the causality requirement that there exist no closed timelike or null lines. In an asymptotically flat background it is inevitable\(^\text{[54]} \) that this second symmetry commutes with the first, i.e.

\[ k^\nu \nabla_\nu m^\mu - m^\nu \nabla_\nu k^\mu = 0. \]  \hspace{1cm} (4.3)

Just as it is plausible that a isolated system with or without a central black hole should tend towards a stationary equilibrium state so also it is plausible, particularly in a context where gravitational radiation needs to be taken into account, that under natural conditions the stationary state would also have to be axisymmetric. It is of course possible mathematically to construct artificial counterexamples (such as the Dedekind ellipsoids to
which Chandrasekhar\textsuperscript{[55]} has drawn attention), but under natural astrophysical conditions it is very hard to imagine stationary black hole scenarios for which the axisymmetry assumption would be in danger of failing. (For the case of of an isolated black hole with a vacuum or simple gaseous exterior considerable effort has been invested, most notably by Hawking\textsuperscript{[47][50][7]}, in attempts to prove that axisymmetry of equilibrium states is not just physically plausible but mathematically necessary. However the crucial result, describable as Hawking’s “Strong Rigidity Theorem”, to this effect is itself based on a postulate of analyticity that is also physically plausible but whose mathematically justification requires assuming the axisymmetry one wanted to prove in the first place, so that as a mathematical challenge the problem remains wide open.)

In any study of stationary axisymmetric systems an important role is played by the Killing vector invariants

\[ V = -k^{\mu}k_{\mu} , \quad W = k^{\mu}m_{\mu} , \quad X = m^{\mu}m_{\mu} \quad (4.4) \]

and by the determinant

\[ \varpi^2 = VX + W^2 = \frac{1}{2} \varpi_{\mu\nu} \varpi^{\mu\nu} , \quad \varpi_{\mu\nu} = 2k_{[\mu}m_{\nu]} . \quad (4.5) \]

and the ratio

\[ \omega = -\frac{W}{X} , \quad X > 0 \quad (4.6) \]

which is well defined wherever \( X \) is strictly positive, which by the causality condition that \( m^{\mu} \) is spacelike, will hold everywhere except on the symmetry axis itself where \( m^{\mu} \) reduces to a zero vector so that \( X \) and \( W \) both vanish, making \( \omega \) undefined.

Other contractions of interest are the energy, \( E \) say, and angular momentum \( L \) say, of a particle with momentum covector \( p_{\mu} \), as given by

\[ E = -k^{\nu}p_{\nu} , \quad L = m^{\nu}p_{\nu} \quad (4.7) \]

which are of course conserved for free orbits:

\[ u^{\nu}\nabla_{\nu}p_{\mu} = 0 , \quad p^{\mu} = mu_{\mu} , \quad u^{\mu}u_{\mu} = -1 \Rightarrow u^{\nu}\nabla_{\nu}E = 0 , \quad u^{\nu}\nabla_{\nu}L = 0 . \quad (4.8) \]

The quantities \( \varpi \) and \( \omega \) defined by (4.5) and (4.6) are of particular interest in the context of circular flow, meaning flow along trajectories on
circles generated by the Killing vectors, i.e. with unit flow vector $u^\mu$ of the form
\begin{equation}
\begin{split}
u^\mu = \alpha(k^\mu + \Omega m^\mu)
\end{split}
\end{equation}
where the coefficient $\Omega = d\phi/dt$ is the *angular velocity* of the trajectory. The important “Keplerian” special class of circular trajectories consists of those that are free in the sense of (4.8), a possibility which typically will exist only in a restricted equatorial plane. A more generally defined class that is of more immediate (though mathematical rather than physical) interest for our present purposes consists of what Bardeen\cite{56} has called “zamo” trajectories (short for zero angular momentum orbiters) which are characterised by the (obviously non Keplerian) condition of having $L = 0$, which can be seen to be equivalent to the condition that their angular velocity be given directly by $\Omega = \omega$. The obvious interest of $\varpi$ as defined by (4.5) in this context is that its reality, i.e. the positivity condition
\begin{equation}
\begin{split}
\varpi^2 > 0
\end{split}
\end{equation}
is evidently the necessary and sufficient condition for the existence of a strictly timelike zamo at the position in question. The importance of this purely local condition is that subject to very weak hypotheses it can also be shown\cite{36}\cite{10}\cite{57}\cite{12}, as described later on below, to characterize the domain of outer communications, whose (globally defined) boundary at the surfaces of the black hole region, will be characterisable (locally, this is what is so convenient) as a “zamosphere”, where $\varpi = 0$, on which the zamo’s become null.

A more frequently discussed\cite{98} but in the final instance less important analogue of the zamosphere is the “ergosphere”, where $V = 0$, i.e. on which the stationarity generator $k^\mu$ becomes null. The interest of this is that it bounds the “ergoregion” characterised by $V < 0$ within which a free particle energy $E$, as defined by (4.7) can become negative, whereas outside the ergo region, i.e. wherever $k^\mu$ is timelike, the free particle energy is bounded below by the condition $E \geq m\sqrt{V}$. The existence of an ergoregion (unless confined within the horizon as in non rotating case) makes possible the extraction of energy from the background by the mechanisms such as the Penrose process\cite{44} whereby a particle coming in with energy $E_1$ splits into a part with negative energy, $E_2 < 0$, and an outgoing part with energy $E_3$ which, by conservation of the sum, $E_2 + E_3 = E_1$ must exceed the initial energy, $E_3 > E_1$. 

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For charged orbits, as given by
\[ u^\nu \nabla_\nu p_\mu = F_{\mu \nu} u^\nu , \] (4.11)
in a stationary field,
\[ \bar{k} \mathcal{L} A_\mu = 0 \iff F_{\mu \nu} k^\nu + \nabla_\mu \Phi = 0 , \quad \Phi = -k^\nu A_\nu \] (4.12)
one gets conservation not of the ordinary energy \( E \) but of a generalised
gauge dependent generalisation \( \mathcal{E} \) say, constructed from the gauge dependent
generalised momentum covector
\[ \pi_\mu = p_\mu + e A_\mu , \] (4.13)
i.e. one gets
\[ u^\nu \nabla_\nu \mathcal{E} = 0 , \quad \mathcal{E} = -h^\nu \pi_\nu = E + e \Phi , \] (4.14)
This generalised energy can be negative even outside of the ergosphere, where \( V > 0 \) so that the lower bound will be given by \( \mathcal{E} \geq m \sqrt{V} + e \Phi \). It is apparent
that there will be an extended electric ergoregion\(^{58}\)^{12}\(^{59}\) characterised for a
given charge to mass ratio by the possibility of negative energy for at least
one sign of the charge \( \pm e \), for which the condition is just
\[ V < \left( \frac{e}{m} \right)^2 \Phi^2 . \] (4.15)
The significance of this relation is of course dependent on how the energy
is calibrated, the usual asymptotic specification being not the only one of
interest: another possibility of particular interest\(^{58}\)^{12}\(^{60}\) for the specialised
theory of non rotating black holes\(^{7}\)^{61}\(^{62}\)^{63}\(^{64}\)^{65}\(^{66}\) is to calibrate with re-
spect to the horizon which is possible in that case because of its uniform
potential condition which will be demonstrated below.

The properties of a stationary axisymmetric system simplify enor-
mously under conditions of what I call \textit{circularity} which in practice are al-
most sure to be satisfied in the applications that are relevant to black holes.
In the case of an electromagnetic source current \( j^\rho \) this means just that it
should be a linear combination of the Killing vectors which is equivalent to
the requirement
\[ j^{[\mu} \omega^\rho \sigma ] = 0 , \] (4.16)
while for a gravitational source it means in the case of a perfect fluid just that
the corresponding flow vector \( u^\mu \) should satisfy the ananlogue of (16) which
is equivalent to the condition (4.9) for the corresponding flow trajectories to
be simple circular orbits. For a more general material source (for which a
preferred reference vector $u^\mu$ might not be defined) circularity is to be under-
stood as meaning that the relevant material energy and angular momentum
flux vectors $k_\nu T_M^{\nu \mu}$ and $m_\nu T_M^{\nu \mu}$ should have the same property, this general
circularity condition being expressible as

$$
 k_\nu T_M^{\nu [\mu | \sigma \rho ]} = 0 , \quad m_\nu T_M^{\nu [\mu | \sigma \rho ]} = 0 .
$$

(4.17)

This condition can in principle fail in a star that is partly solid\[99\] (as
in the case of a neutron star crust ) or even in a strictly perfect inviscid fluid
star where there is convection, but for the more plausibly relevant case of
viscous fluid the possibility of other than circular motion can be ruled out
because it would inevitably produce thermal dissipation and thus violate the
requirement of strict stationarity.

The crucial simplification that one gets in such circumstances is pro-
vided by the circularity theorem\[36\][\[10\][\[12\] which (generalising a result first
demonstrated in the case of a pure vacuum by Papapetrou\[68\] and for an un-
charged perfect fluid by Kundt and Trumper\[69\]) tells us that the system will be
orthogonally transitive, meaning that the circular trajectories generated
by the two Killing vectors will be orthogonal to a congruence of two dimensin-
ional surfaces which (must obviously be spacelike where, and only where, the
zamos are timelike) throughout any continuous region connected to the rota-
tion axis within which the source circularity conditions (4.16) and (4.17) are
satisfied. Thus if the source circularity conditions are satisfied everywhere
one gets orthogonal transitivity everywhere. The well known Frobenius con-
dition for such orthogonal transitivity is expressible as the requirement that
the twist vectors

$$
 \omega^\mu = \frac{1}{2} \varepsilon^{\mu \rho \sigma \nu} k_\nu \nabla_\rho k_\sigma , \quad \psi^\mu = \frac{1}{2} \varepsilon^{\mu \rho \sigma \nu} m_\nu \nabla_\rho m_\sigma ,
$$

(4.18)

be orthogonal to the Killing vectors, i.e. that we should have

$$
 \omega^\mu m_\mu = 0 , \quad \psi^\mu k_\mu = 0 .
$$

(4.19)

Since these contractions must both vanish identically on the axis where $m^\mu$
is zero, it is sufficient to obtain the required result that these contractions
should have the property of uniformity, i.e. that the should be constants,
over the region in question.
The proof the circularity theorem to this effect uses the fact that the Killing equations (4.1) and (4.2) imply corresponding higher order conditions
\[ \nabla_\nu \nabla_\mu k^{\mu} = -R^{\mu}_{\nu} k^{\nu} , \quad \nabla_\nu \nabla_\mu m^{\mu} = -R^{\mu}_{\nu} m^{\nu} , \] (4.20)
from which, with the aid of (4.3) one gets
\[ \nabla_\mu (\omega^{\nu} m_\nu) = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} k^{\rho} m^{\sigma} R^{\alpha}_{\beta} k^{\gamma} , \quad \nabla_\mu (\psi^{\nu} k_\nu) = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} m^{\nu} k^{\rho} R^{\alpha}_{\beta} k^{\gamma} . \] (4.21)
In the absence of electromagnetic source contributions the material circularity conditions (4.17) alone are sufficient to ensure that the right hand sides of the foregoing pair of equations vanishes, which is evidently sufficient to establish the required uniformity. To show that the result remains valid in the presence of electromagnetic effects requires a little more work, starting from the group invariance conditions
\[ k^{\nu} \nabla_\nu F_{\rho \sigma} = 2 F_{\nu [\rho} \nabla_\alpha] k^{\nu} , \quad m^{\nu} \nabla_\nu F_{\rho \sigma} = 2 F_{\nu [\rho} \nabla_\alpha] m^{\nu} . \] (4.22)
These conditions, together with the Maxwellian field equations (?.?) imply that the field vectors
\[ E_\mu = F_{\mu \nu} k^{\nu} , \quad B^{\mu} = \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} k_\nu F_{\rho \sigma} , \] (4.23)
will satisfy
\[ \nabla_\mu (E_\nu m^{\nu}) = 0 , \quad \nabla_\mu (B^{\nu} m_\nu) = 4 \pi \varepsilon_{\mu \nu \rho \sigma} k^{\nu} m^{\sigma} j^{\rho} , \] (4.24)
in which not only the first but also the second of the right hand sides will obviously vanish wherever the current singularity condition (4.16) is satisfied, with the implication that \( E_\nu m^{\nu} \) and \( B^{\nu} m_\nu \) will also both be uniform and therefore vanish
\[ E_\nu m^{\nu} = 0 , \quad B^{\mu} m_\mu = 0 \] (4.25)
since they both obviously must vanish on the axis. The conditions (4.25) can appropriately be described as field circularity conditions, since they are sufficient for the corresponding electromagnetic contribution to the gravitational source to satisfy the analogue of the material circularity condition (4.17) so that its effect does not invalidate the conclusion that \( \omega^{\nu} m_\nu \) and \( \psi_\nu k^{\nu} \) will vanish also.

The orthogonal transitivity property that is established in this way means that it will be possible, except where \( \varpi = 0 \), to choose the spacetime
coordinates in such a way as to express the metric in the standard Papetrou form:

\[ ds^2 = g_{ij}dx^i dx^j + X\{(d\phi - \omega dt)^2 - \varpi^2 dt^2\}, \]

(4.26)

where the coefficients \( X, \omega, \varpi \) (as defined by (4.5) and (4.6) are independent of the "ignorable" coordinates \( \varphi \) and \( t \), but functions of the two other coordinates \( x^i, i = 1, 2 \) whose locci of constancy are the orthogonal two surfaces predicted by the theorem. It can be seen that under these conditions the zamo trajectories themselves are orthogonal to the hypersurfaces on which \( t \) is constant, this hypersurface orthogonality condition (which, like (4.26) itself, would fail if the circularity conditions (4.16) and (4.17) were not satisfied) is what is meant by the statement that the zamos congruence is irrotational.

To understand what happens on the zamosurface where the zamo worldlines become null so that \( \varpi \) vanishes and the metric form (4.22) becomes singular, we use the fact that the Frobenius conditions (4.19) on which it depends imply

\[ 2\varpi_{\mu[\nu} \nabla_{\rho]} \varpi_{\sigma\tau} = \varpi_{\sigma\tau} \nabla_{[\rho} \varpi_{\sigma\tau]} \].

(4.27)

This gives an equation for the gradient of \( \varpi^2 \) orthogonal to the Killing vector surfaces of transitivity that is somewhat analogous to the one obtained for the gradient of the zamo angular velocity \( \omega \) directly from its definition, i.e.

\[ \varpi_{[\sigma\tau} \nabla_{\mu]} \varpi^2 = \varpi^2 \nabla_{[\mu} \varpi_{\sigma\tau]} \], \quad X^2 \varpi_{[\sigma\tau} \nabla_{\mu]} \omega = 2\varpi^2 m_{[\mu} \nabla_{\sigma} m_{\tau]} \],

(4.28)

the noteworthy thing about both these equations being that their right hand sides vanish on the zamosurface where \( \varpi^2 = 0 \). Since both \( \varpi \) and \( \omega \) are invariant under the group action their gradients everywhere must be orthogonal to the surfaces of transitivity generated by the Killing vectors, whereas according to (4.28) they must actually be aligned with these surfaces of transitivity on the zamosphere, conditions which can only be reconciled if they are both aligned with the unique combination of Killing vectors that is null on the zamosphere, i.e. with the zamo direction itself, and hence that they are aligned with each other. Provided the gradient \( \nabla_{\mu} \varpi^2 \) is non zero, and so defines the direction normal it the zamosphere it obviously follows that (as can be shown with a little more care to be true in any case\[36]) the the zamosurface is a null hypersurface and that the zamo angular velocity \( \omega \) has a uniform value \( \Omega^H \) say on it. This is the result that I refer to as the weak rigidity theorem\[70] (the qualification weak being because it is bases on

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a line of argument assuming axisymmetry at the outset, in contrast with the stronger rigidity theorem of Hawking\cite{47}\cite{7} based only on an assumption of analyticity).

The uniform angular momentum value whose existence is thus established can be extrapolated off the zamosurface to a uniform value throughout space,

\[ \nabla_\mu \Omega^\mu = 0 , \tag{4.29} \]

in terms of which can construct a unique Killing vector combination

\[ \ell^\mu = k^\mu + \Omega^\mu m^\mu , \quad \nabla (\mu \ell_\nu) = 0 \tag{4.30} \]

which is characterised by the property of becoming aligned with the zamo direction where this direction becomes null, i.e. on the zamosurface which we now know to be itself null. This shows that (unlike an ergosurface which is typically timelike, and subject of course to the postulate that the circularity conditions (4.16) and (4.17) are satisfied) the zamosurface is automatically what I have called a *Killing horizon*\cite{36}, i.e. a null hypersurface whose null generator coincides with the generator of an isometry.

Before going on to consider the global question of the identification of the locally defined zamosurface Killing horizon with the globally defined black hole event horizon, there are some further local properties of Killing horizons that can logically be derived at this stage. To start with it is apparent from the Penrose Raychaudhuri equation (3.11) for the null generator that since all the other terms vanish we must also have

\[ R^\mu_{\nu\lambda} \ell^\nu \ell^\lambda = 0 \tag{4.31} \]

which subject to the material energy positivity postulate means that we must separately have

\[ T^\mu_{\nu\lambda} \ell^\nu \ell^\lambda = 0 , \quad T^\mu_{\nu\lambda} \ell^\nu = 0 \tag{4.32} \]

where the electromagnetic part is given by

\[ T^\mu_{\nu\lambda} \ell^\nu = \frac{1}{8\pi} (E^{\dagger}_\mu E^{\mu} + B^{\dagger}_\mu B^{\mu}) , \quad E^{\dagger}_\mu = F_{\nu\mu} \ell^\nu , \quad B^{\dagger}_\mu = \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} F_{\nu\rho} \ell^\sigma , \tag{4.33} \]

where a dagger symbol is used to distinguish quantities defined with respect to the *corotating* Killing vector field (4.30) from their analogues as defined with respect to the ordinary (asymptotically timelike) stationarity Killing vector (4.1) (a distinction that is not necessary in the static case for which
they both coincide). If the material contribution is simply of perfect fluid type (2.25) subject to the inequalities \( \rho \geq 0 \), \( P \geq 0 \) the first of the conditions (4.28) can be seen to give

\[
\rho + P = 0 \quad \Rightarrow \quad \rho = 0 \ , \ P = 0 , \quad (4.34)
\]
i.e. there must be a vacuum at the horizon, while since both \( E^\dagger_{\mu} \) and \( B^\dagger_{\mu} \) are both by construction orthogonal to \( \ell^\mu \) they cannot be timelike on the Killing horizon, so the reconciliation of (4.32) with (4.33) requires that they both be null there and hence proportional to the generator itself, i.e.

\[
E^\dagger_{[\mu} \ell_{\nu]} = 0 , \quad B^\dagger_{[\mu} \ell_{\nu]} = 0 , \quad (4.35)
\]
of which the first tells us\(^{[10]}\) that the horizon is like a conductor whose equilibrium requires uniformity of the corresponding potential, an analogy that, since it was first noticed, has been developed in considerable detail\(^{[6]}\)\(^{[100]}\)\(^{[101]}\).

Explicitly we have

\[
E^\dagger_{\mu} = \nabla_\mu \Phi^\dagger , \quad \Phi^\dagger = A_\mu \ell^\mu \quad (4.36)
\]
everywhere, with the potential \( \Phi^\dagger \) necessarily uniform over the horizon. Formally, for any tangent vector \( \xi^\mu \) to the Killing horizon we have

\[
\xi^\mu \ell_\mu = 0 \quad \Rightarrow \quad \xi^\mu \nabla_\mu \Phi^\dagger = 0 . \quad (4.37)
\]

This uniformity property of the angular velocity \( \Omega^\mu \) and of the potential \( \Phi^\dagger \) on a Killing horizons are prototypes\(^{[36]}\)\(^{[70]}\)\(^{[10]}\) for a less intuitively obvious uniformity\(^{[10]}\)\(^{[50]}\)\(^{[56]}\)\(^{[52]}\), namely that of the decay parameter \( \kappa \) whose definition by the general formula (3.7) is unambiguous now that the normalisation of \( \ell^\mu \) is fixed by (4.30). The fact that \( \ell^\mu \) must satisfy the Frobenius orthogonality condition (3.2) on the horizon can be seen to mean that there must exist some vector \( q_\mu \) on the horizon such that

\[
\nabla_\nu \ell_\mu = 2q_{[\nu} \ell_{\mu]} , \quad \tilde{\ell}^\nu q_\nu = 0 \quad (4.38)
\]
where \( \tilde{\ell}^\mu \) is an ingoing nullvector as introduced in (3.5). It is easy to see from the expression (4.38) that for any vectors \( \xi^\mu , \eta^\mu \) lying in the horizon we shall have

\[
\xi^\mu \ell_\mu = 0 \ , \quad \eta^\mu \ell_\mu = 0 \quad \Rightarrow \quad \xi^\mu \eta^\nu \nabla_\nu \ell_\mu = 0 \quad \Rightarrow \quad \ell^\mu \eta^\nu \nabla_\nu \xi_\mu = 0 , \quad (4.39)
\]
which shows that the Killing horizon is extrinsically flat (geodesically generated) since it shows that $\eta^\nu \nabla_\nu \xi^\mu$ will automatically be tangential to the horizon. A further derivation now leads (using $\ell_\nu \nabla_\rho \xi^\nu = \xi^\nu q_\nu \ell_\rho$) to

$$\xi^\mu \eta^\nu \nabla_\rho \nabla_\mu \ell_\nu = - (\nabla_\mu \xi_{\nu u}) (\xi^\nu \nabla_\rho \eta^\nu + \eta^\nu \nabla_\rho \eta^\nu \rho_{\nu\eta} \xi^\mu) = 0 \quad (4.40)$$

Since the Killing vector property (4.30) by itself implies

$$\nabla_\rho \nabla_\mu \ell_\nu = R_{\mu \nu \rho \tau} \ell_\nu \ell_\mu \ell_\tau , \quad (4.41)$$

we end up with

$$R_{\mu \nu \rho \tau} \ell_\nu \xi^\rho \eta^\tau = 0 \quad \Rightarrow \quad \overline{g}^{\rho \tau} R_{\mu \nu \rho \tau} \ell^{\mu} \xi^\rho = 0 . \quad (4.42)$$

where $\overline{g}^{\rho \tau}$ is the projection tensor given by ((3.6), whose substitution then gives

$$R_{\mu \nu \rho \tau} \ell^{\mu} \xi^\rho = R_{\mu \nu \rho \tau} \ell^{\mu} \xi^\rho . \quad (4.43)$$

Since the definition (3.7) is clearly equivalent to

$$\kappa = - \tilde{\ell}^\mu \nabla_\nu \ell_\mu = \ell^\nu q_\nu \quad (4.44)$$

direct differentiation gives

$$\xi^\nu \nabla_\nu \kappa = - \xi^\nu \left( \tilde{\ell}^\mu \ell^\nu \nabla_\rho \ell_\mu + \tilde{\ell}^\nu (\nabla_\rho \ell^\mu) \ell_\mu + \kappa \ell_\nu \nabla_\rho \tilde{\ell}^\mu \right) \quad (4.45)$$

in which the last two terms cancel since they are respectively equal and opposite to $\kappa \xi^\nu q_\nu$, so using (4.43) one finally obtains the simple result

$$\xi^\nu \nabla_\nu \kappa = - R_{\mu \nu \ell} \ell^\nu \xi^\nu . \quad (4.46)$$

The reasoning up to this point has been purely kinematic. If we now invoke the Einstein equations we immediately obtain the required uniformity condition that for an arbitrary tangent vector $x^\mu$ to the horizon

$$\xi^\nu \nabla_\nu \kappa = 0 \quad (4.47)$$

in the pure vacuum case. This “zeroth law of black hole mechanics” result[52] can easily be seen to remain valid for a source free Einstein Maxwell vacuum[10][11] since in that case, although $R_{\mu \nu}$ will not be zero $R_{\mu \nu} \ell^\mu$ will be be proportional to the null tangent covector $\ell_\mu$ which is all that is required to get (4.47) from (4.46).
5. Rotating Equilibrium States: the global problem.

The kind of stationary equilibrium state towards which, subject to the cosmic censorship hypothesis, it seems reasonable to suppose an isolated gravitationally collapsing system would evolve, and that we shall understand to be meant by the qualification “well behaved black hole equilibrium state” is a stationary spacetime whose domain of outer communications (D.O.C.) is bounded to the future by a well behaved black hole event horizon.

In the concrete example of the Schwarzschild and Riessner Nordstrom solutions we have seen that the D.O.C. is also bounded to the past by a well behaved “white hole horizon”, but the latter corresponds to nothing that exists in a dynamical system that collapses from well behaved initial condition (being merely an artefact of analytic extrapolation to the past) whereas the black hole horizon in the stationary state really does correspond to the limit of the black hole horizon of the dynamically collapsing state. This is why only the latter can appropriately be postulated to exist as a defining characteristic of a well defined black hole equilibrium state, the existence of any other horizon in the past being something to be proven (if it can be) subsequently, but not to be postulated in advance. If it is already known in some particular case not only that both past and future event horizons exist but also that they have a well behaved Kruskal type crossover on a spacelike two surface then one can prove a result such as the “zeroth law” obtained at the end of Section 4 by a much shorter argument than was given there (see the accompanying lectures of Wald) but it is important for our present purpose to have shown that the result (i.e. the uniformity of $\kappa$) can be established independently of any such assumption since it is needed as an an intermediate step in the line of reasoning that ultimately shows (at least in the generic vacuum case, the more general question remaining open) that the assumed crossover really does occur.

In order to have the right to utilise the local properties established for a zamosurface Killing horizon in the previous section we must show that the globally defined black hole event horizon forming the future boundary of the D.O.C. really is of this type. As a step towards this it, is convenient to use a lemma^{12}[10][57] giving a pseudo local characterisation of the D.O.C. whose original global definition is expressible for a stationary state as the specification that it consists of the intersection of the past and the future of the outer region where the trajectories of the stationary generator, $\partial/\partial t = k^\mu \partial/\partial x^\mu$ are actually timelike. The pseudo local characterisation is based on
consideration of where these generators are globally bradyonic, using this term to qualify any curve in spacetime with the property that any point \( x \) within it determines corresponding points \( x_+ \) and \( x_- \) such that the part of the curve preceding \( x_- \) lies entirely in the past of \( x \) and the part of the curve subsequent to \( x_+ \) lies entirely in the future of \( x \). This condition is satisfied trivially by an ordinary timelike curve (for which \( x_- \) and \( x_+ \) can be identified with the original point \( x \) itself) but it also includes a curve which, though locally spacelike, turns back towards itself in such a way that its evolution is effectively timelike in the long run.

It is not difficult to see\(^{12}\)\(^{10}\)\(^{57}\) that although there may be an ergoregion in which the Killing vector \( k^\mu \) ceases to be timelike, its trajectories must always remain globally bradyonic throughout the D.O.C. It is even more obvious that in any connected region such that the trajectories there are all globally bradyonic, any one of them can be connected to any other by a timelike line with either orientation. This leads to a lemma\(^{12}\)\(^{10}\)\(^{57}\) characterising the D.O.C. as the maximal connected extension of the outer region where the stationary trajectories generated by \( k^\mu \) are timelike such that the stationary generators remain at least globally bradyonic. As an immediate corollary it follows that \( k^\mu \) can never be timelike on the boundary (including the black hole horizon) of the D.O.C., while on the other hand the stationarity generator \( k^\mu \) can never vanish nor have any closed trajectory within the D.O.C. The latter conclusion means that \( k^\mu \) must always be linearly independent of \( m^\mu \) in the D.O.C. and hence that within the D.O.C. the Killing bivector \( \varpi^{\rho\sigma} \) can never be degenerate except on the axis \( m^\rho = 0 \), which means that \( \varpi^2 \) can vanish only on the axis or where the bivector, and thus also the corresponding zamo direction, becomes null.

Let us now designate by \( \mathcal{Z} \) the maximal connected extension of the outer region where the stationary trajectories generated by \( k^\mu \) are timelike such that the local condition that the zamo trajectories be timelike within it is satisfied. Our aim is to show, subject to very weak assumptions that this locally defined domain will be identifiable with the globally defined D.O.C. What is obvious is that, since any connecting curve within \( \mathcal{Z} \) will be continuously deformable into a timelike curve (by the group action along the zamo direction at each point) \( \mathcal{Z} \) must certainly lie entirely within the D.O.C. The definition of \( \mathcal{Z} \) means, according to (4.10), that \( \mathcal{Z} \) is characterised by \( \varpi^2 > 0 \) except on the rotation axis where \( \varpi^2 = 0 \) and that we must have \( \varpi^2 = 0 \) everywhere on the boundary \( \mathcal{Z} \) of \( \mathcal{Z} \). By the conclusion of the previous paragraph, this implies that except on the axis \( \mathcal{Z} \) lies on the locus where the
zamo direction is null, and hence by the results of the previous section that provided the circularity postulate is satisfied; the connected components of $\mathcal{Z}$ must be Killing horizon and thus \textit{null hypersurface} segments, each, by continuity, with uniform time orientation.

To complete the demonstration that $\mathcal{Z}$ can be identified with the D.O.C., we must now invoke the further postulate that the latter be \textit{simply connected}, which means that a (hypothetical) connected component $\mathcal{D}$ say of the complement of $\mathcal{Z}$ in the D.O.C. must have a boundary $\partial \mathcal{D}$ within the D.O.C. that is itself connected. It follows that $\partial \mathcal{D}$ would have to be a null hypersurface segment with uniform time orientation which means that $\mathcal{D}$ could be reached from $\mathcal{Z}$ only by future directed timelike lines, or only by past directed ones, but not by both kinds as would be required for $\mathcal{D}$ to lie within the D.O.C. so in order to avoid a contradiction it must be concluded that $\mathcal{D}$ is empty, and thus that $\mathcal{Z}$ covers the whole of the D.O.C. as required.

The foregoing demonstration to the effect that the Killing bivector $\omega^{\rho\sigma}$ is timelike throughout the D.O.C. means that the two surfaces orthogonal to the Killing vectors that were shown to exist by the circularity theorem of the previous section will correspondingly be strictly spacelike there. Moreover the reinforcement of the circularity postulate introduced in Section 4 section by the simple connectivity postulate introduced in the previous paragraph implies that these orthogonal two surfaces will be constructible not just locally but globally. Since any such surface differs from flatness only by a locally variable conformal factor, $\Sigma$ say, it follows that the space coordinates in the general Papaptrou form (4.26) may be chosen more specifically to be cylindrical type coordinates $\rho, z$ say, in such a way that the metric will be expressible in the form

$$ ds^2 = \Sigma(d\rho^2 + dz^2) + X\{(d\phi - \omega dt)^2 - \omega^2 dt^2\} \quad (5.1) $$

which will be \textit{globally} valid over the entire D.O.C. except for the familiar degeneracy on the axis where $X$ and $\omega^2$ vanish, their values elsewhere being strictly positive, as is the value of $\Sigma$ everywhere. An analogous form is obtainable for the vector potential using the invariance conditions (4.22) which imply the existence locally, and hence by the simple connectivity postulate globally, of scalars $\Phi$ and $B$ such that

$$ F_{\rho\sigma}k^\sigma = \nabla_\rho \Phi, \quad F_{\rho\sigma}m^\sigma = \nabla_\rho B, \quad (5.2) $$

in terms of which, using the consequence (4.25) of the circularity postulate,
it can be verified that the gauge may be chosen in such a way that
\[ A_\mu dx^\mu = \Phi dt + B d\phi. \]  
(5.3)

It is of course to be understood here that the new coefficients \( \Sigma \Phi, B \), like \( X, \omega \), and \( \varpi \) as introduced previously, are all functions only of \( \rho \) and \( z \) only, i.e. the stationarity and axisymmetry is made manifest by their independence of the ignorable coordinates \( \phi \) and \( t \).

Up to this stage the analysis has been sufficiently general to cover a wide range of conceivable black hole configurations with external matter rings for which explicit analytic solutions are not available, but from this point on we shall restrict our attention to the globally source free case for which it has long been known\(^7^1\)\(^7^2\)\(^4^0\)\(^7^3\)\(^7^4\) that the Kerr Newman class of solutions provide explicit examples. The purpose of the systematic step by step approach whose development I have been describing is to solve the problem of whether there can exist any others. It will be shown below that subject to the preceding assumptions, including notably that of a simply connected topology for the D.O.C., it can be shown, by an argument that has been able to be made completely watertight only comparatively recently\(^9^2\)\(^9^3\)\(^9^4\), that these known solutions are indeed the only ones for the strictly source free Einstein Maxwell equations. However the problem remains wide open\(^7^5\) for the slightly more general case of solutions of the source free Einstein Maxwell equations with cosmological \( \Lambda \) term: such solutions cannot of course be asymptotically flat, but I have discovered\(^7^6\)\(^7^7\)\(^1^0\) a wide class of asymptotically De Sitter solutions (one of the first cases for which the C.P. technique described in Section 2 proved quite indispensable for providing an understandable global description\(^1^0\)\(^7^8\). The problem that remains unsolved is the extent to which these known asymptotically De Sitter black rotating black be whole solutions are unique.

The reason why the results that follow have not yet been generalised to allow for a cosmological \( \Lambda \) term in the generalised source free Einstein Maxwell system
\[ R^\mu_\nu - \frac{1}{2} R g^\mu_\nu = 8 \pi T^\mu_\nu + \Lambda g^\mu_\nu. \]  
(5.4)

is that the next step uses the trace, not over the full four dimensional system (5.4) but over its restriction to the two dimensional surface of transitivity generated by the Killing vectors for which due to the circularity condition the electromagnetic contribution cancels out so that substitution of the form
\( (5.1) \) simply gives
\[
-\frac{1}{\sigma \varpi} \nabla^2 \varpi = 2\Lambda ,
\]
where \( \nabla^2 \) is the Laplacian that is defined with respect to of the flat two dimensional metric \( d\rho^2 + dz^2 \).

In the absence of the cosmological term, \( (5.5) \) tells us the \( \varpi \) is a harmonic function on the conformally flat spacelike two surfaces which means that using the freedom to make conformal adjustments one can choose the coordinate system so as to identify it with \( \rho \):
\[
\Lambda = 0 \Rightarrow \varpi = \rho .
\]

In such a coordinate system \( (5.1) \) reduces to the specialised Papapetrou form
\[
ds^2 = \Sigma (d\varpi^2 + dz^2) + X \{(d\phi - \omega dt)^2 - \varpi^2 dt^2\}.
\]

On the other hand the presence of a \( \Lambda \) term suffices to block the apparently innocent but actually crucial step from \( (5.1) \) to \( (5.7) \), without which none of the work that follows can be carried through, the discovery of an alternative route being thus left as a challenge for future work.

As has been well known since early work\cite{79}\cite{80} on stationary axisymmetric systems in other contexts, after the variable \( \varpi \) has been thus taken out of the list of unknown variables by its promotion to the status of a “known” coordinate variable, the system of source free field equations reduces to a decoupled system just for the two metric variables \( X \) and \( \omega \) together with the two electromagnetic potentials \( \Phi \) and \( B \), together with a separate equation that can be solved afterwards to obtain the remaining variable, i.e. the conformal factor \( \Sigma \) by a direct quadrature, with the constant of integration fixed by the condition \( \Sigma \to 1 \) at large asymptotic distance. This means that the four dimensional black hole equilibrium problem from which we started reduces now to a two dimensional boundary problem for the fields \( X, \omega, \Phi, B \) as functions of the independent variables \( \varpi \) and \( z \).

The conclusion\cite{10} that \( (5.7) \) is not just valid locally but covers the entire D.O.C. with \( \varpi \) (ranging from 0 to \( \infty \)) and \( z \) (ranging from \( -\infty \) to \( \infty \)) as globally well behaved cylindrical type coordinates, depends on the simple connectivity postulate and the use of a specialisation\cite{81} of Morse theory to exclude the possibility of critical points of \( \varpi \). Ordinary Morse theory establishes that the number of maxima plus the number of minima minus the number of minimaxes is a topological invariant subject to fixed
boundary conditions and the assumption that no degenerate critical points occur. In the harmonic case it is possible to make a stronger statement since maxima and minima cannot occur while degenerate critical points do not need to be assumed to be absent since their presence can easily be allowed for by labelling them with an appropriate positive degeneracy index. The resulting theorem states that the index weighted sum over all critical points including possible degenerate ones is a topological invariant which in the present application can be seen (by considering any special case such as the Schwarzschild solution) to be zero. Since the index is always of the same sign in the harmonic case no cancellation is possible, so the fact that the total is zero in the case under consideration makes it possible to deduce with certainty that there are no critical points at all, degenerate or otherwise.

Having got to this point we introduce a further topological simplification postulate to the effect that we are dealing with only a single topologically spherical black hole. Our simple connectivity postulate has already excluded conceivable toroidal black hole configurations but has left open the possibility of having several topologically spherical black holes lined up on a common rotation axis. A certain amount of work has been carried out, particularly by Hawking and Gibbons towards showing that such configurations are impossible except in the extreme Papapetrou Majumdar limit of maximally charged non rotating configurations in which electromagnetic repulsion balances gravitational attraction, but we shall not go into the study of such exotic topological possibilities here. Assuming then that the black hole topology is of simple spherical type we can fix the cylindrical coordinate system symmetrically with respect to the black hole by taking the poles at which the horizon meets the rotation axis to be given by opposite values of \(z\) which, it can easily be seen, must be given explicitly by in terms of the area and decay constant of the horizon by

\[
\begin{align*}
z & = \pm c, \\
c & = \frac{\kappa A}{4\pi}.
\end{align*}
\]  

(5.8)

Leaving aside the awkward and still only partially understood special case for which the horizon is degenerate in the sense of having \(\kappa = 0\) (corresponding to zero temperature in the thermodynamic limit) we can conveniently proceed by replacing the cylindrical coordinates \(\varpi, z\) by ellipsoidal type coordinates \(\lambda, \mu\) according to the specifications

\[
\begin{align*}
z & = \lambda \mu, \\
\varpi^2 & = (\lambda^2 - c^2)(1 - \mu^2).
\end{align*}
\]  

(5.9)
which are such as to arrange that the horizon is now given by the limit \( \lambda \to c \) while the two disconnected (“north” and “south”) parts of the symmetry axis in the D.O.C. are given respectively by \( \mu \to \pm 1 \), the whole D.O.C. being covered by the coordinate range \( c < \lambda < \infty, -1 \leq \mu \leq \mu \).

In this system the metric takes the form

\[
ds^2 = \Xi \hat{d}s^2 + Xd\phi^2 + 2Wd\phi dt - Vdt^2
\]

for a conformally flat space metric given by

\[
\hat{d}s^2 = \frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1 - \mu^2}
\]

with

\[
\Xi = (\lambda^2 - c^2\mu^2)\Sigma \quad W = X\omega \quad V = X^{-1}(\omega^2 - W^2).
\]

In terms of the two dimensional covariant differentiation operator \( \hat{\nabla} \) defined in terms of the known conformally flat metric (5.11), the system of independent source free Einstein Maxwell equations reduces to a pair of Maxwell equations

\[
\hat{\nabla} \left\{ \frac{X}{\omega}(\hat{\nabla}\Phi - \omega\hat{\nabla}B) \right\} = 0,
\]

\[
\hat{\nabla} \left\{ \frac{\omega}{X}\hat{\nabla}B + \frac{\omega}{\omega}(\hat{\nabla}\Phi - \omega\hat{\nabla}B) \right\} = 0,
\]

together with a pair of Einstein equations

\[
\hat{\nabla} \left\{ \frac{X^2}{\omega}\hat{\nabla}\omega + \frac{4B}{\omega}(\hat{\nabla}\Phi - \omega\hat{\nabla}B) \right\} = 0
\]

\[
\hat{\nabla} \left\{ \frac{\omega}{X}\hat{\nabla}X \right\} + \frac{|\hat{\nabla}\omega|^2}{\omega} + \frac{2X}{\omega}|\hat{\nabla}\Phi - \omega\hat{\nabla}B|^2 + \frac{2\omega}{X}|\hat{\nabla}B|^2 = 0.
\]

Although this system is singular on the axis where \( X \) and \( \omega \) both vanish, and also on the horizon where \( \omega \) also vanishes, it is guaranteed to be regular everywhere within the half plane under consideration where we have

\[
\omega > 0, \quad X > 0
\]

the latter inequality being derived from the causality postulate. This is the motivation for having used a formulation giving the leading role to the axisymmetry Killing vector \( m^\mu \) rather than using the more traditional approach
giving the leading role to the stationarity Killing vector $k^\mu$, which would have given an analogous system but with $V$ turning up instead of $X$ in the denominators, which would have the seriously inconvenient consequence of making the system singular on the ergosurface that generically occurs within the D.O.C.

The foregoing system can be made more tractable by performing the analogue of the transformation introduced originally for the traditional formulation based on $k^\mu$ rather than $m^\mu$ by Ernst\cite{80}. This is done by first using the Maxwellian equation (5.13) to justify the introduction of a stream function type electric potential $E$ given by

$$X \left( \frac{\partial \Phi}{\partial \lambda} - \omega \frac{\partial B}{\partial \lambda} \right) = (1 - \mu^2) \frac{\partial E}{\partial \mu}, \quad X \left( \frac{\partial \Phi}{\partial \mu} - \omega \frac{\partial B}{\partial \mu} \right) = -(\lambda^2 - c^2) \frac{\partial E}{\partial \lambda},$$

(5.18)

and by using the Einstein equation (5.15) to justify the introduction of an analogous rotation potential $Y$ given by

$$X^2 \frac{\partial \omega}{\partial \lambda} = (1 - \mu^2) \left\{ \frac{\partial Y}{\partial \mu} + 2E \frac{\partial B}{\partial \mu} - 2B \frac{\partial E}{\partial \mu} \right\},$$

$$X^2 \frac{\partial \omega}{\partial \mu} = -(\lambda^2 - c^2) \left\{ \frac{\partial Y}{\partial \lambda} + 2E \frac{\partial B}{\partial \lambda} - 2B \frac{\partial E}{\partial \lambda} \right\}. \quad (5.19)$$

Using the new potentials $E$ and $Y$ to replace $\Phi$ and $\omega$ one obtains the Maxwell equations in the form

$$\nabla \left\{ \frac{\omega}{X} \nabla B \right\} + \frac{\omega}{X^2} \left\{ \nabla Y + 2E \nabla B - 2B \nabla E \right\} \cdot \nabla E = 0,$$

$$\nabla \left\{ \frac{\omega}{X} \nabla E \right\} - \frac{\omega}{X^2} \left\{ \nabla Y + 2E \nabla B - 2B \nabla E \right\} \cdot \nabla B = 0, \quad (5.20)$$

$$\nabla \left\{ \frac{\omega}{X^2} \nabla X \right\} + \frac{\omega}{X^3} \left\{ |\nabla X|^2 + |\nabla Y + 2E \nabla B - B \nabla E|^2 \right\} + \frac{2\omega}{X^2} \left\{ |\nabla E|^2 + |\nabla B|^2 \right\} = 0. \quad (5.21)$$

In terms of this new system the asymptotic boundary conditions for regularity at large distance, i.e. as $\lambda \to \infty$, are more complicated than in the traditional approach, being obtainable\cite{10} as

$$\lambda^{-2}X = (1 - \mu^2) + O(\lambda^{-1}), \quad E = -Q\mu + O(\lambda^{-1}),$$

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\[ Y = 2J\mu(3 - \mu^2) + O(\lambda^{-1}) , \quad B = O(\lambda^{-1}) . \quad (5.22) \]

where \( J \) is the asymptotically measured angular momentum about the rotation axis while \( Q \) is the total charge, and where the requirement that the total magnetic monopole should vanish has been taken into account. The asymptotic mass \( M \) does not appear explicitly, but it is implicitly fixed by the overall scale which is determined by the choice of the parameter \( c \). In compensation for this rather inhabitual degree of complication in the familiar large distance limit, we get extremely simple boundary conditions in the less familiar limit at the horizon as \( \lambda \to c \), the only condition here being that the unknowns \( E, B, X, Y \) should be regular as differentiable functions of the ellipsoidal type coordinates \( \lambda \) and \( \mu \). The most mathematically delicate boundary conditions (for which however no physical considerations or parameter values are involved) are those for geometrical regularity on the rotation axis \( \mu = \pm 1 \), i.e. for the limit \( (1 - \mu^2) \to 0 \), which are given by

\[
\frac{\partial E}{\partial \lambda} = O(1 - \mu^2) , \quad \frac{\partial B}{\partial \lambda} = O(1 - \mu^2) , \\
\frac{\partial Y}{\partial \lambda} = O(1 - \mu^2) , \quad \frac{\partial Y}{\partial \mu} + 2E \frac{\partial B}{\partial \mu} - 2B \frac{\partial E}{\partial \mu} = O(1 - \mu^2) , \\
X = 0(1 - \mu^2) , \quad \frac{(\mu^2 - 1)}{2X} \frac{\partial X}{\partial \mu} = 1 + O(1 - \mu^2) . \quad (5.23)
\]

After I first obtained this system I succeeded deriving a pure vacuum “no hair theorem”\(^{[91]}\)\(^{[10]}\) which Robinson was able soon after to generalise to the full electromagnetic case\(^{[90]}\), establishing that the solutions belong to discrete families each depending continuously only on the three relevant physical parameters involved in the boundary conditions namely \( Q, J, \) and \( C \), of which the latter fixes the overall scale and thus implicitly the mass \( M \). Our method was to equate a certain divergence to a positive definite function of the infinitesimal difference between nearby solutions for the same parameter values and hence (using the boundary conditions) to show that the latter must vanish. One such family consisted of course of the already known Kerr solutions (subject to the condition \( M^2 > Q^2 + J^2/M^2 \)) and in view of various restrictions on special limits such as that of spherical symmetry it seemed unlikely from the outset that any others existed. Nevertheless it was necessary to wait several years before it was established beyond doubt that they do not.
The way that Robinson and I had constructed the divergence with the miraculously positive definite form we needed was based on a purely trial and error approach whose success in the electromagnetic case\[^90\] required a veritable algebraic tour de force. Robinson even succeeded in using the trial and error method to construct a finite difference generalisation\[^91\] that completely established the uniqueness of the original Kerr black hole solutions, with \(Q = 0, M^2 > J^2/M^2\), for the pure vacuum case, but the trial and error method was never able to cope with the finite difference case in its full electromagnetic generality, and so the complete solution had to wait the introduction of new and more sophisticated techniques by Bunting\[^92\][93\] and Mazur\[^94\]. The Bunting method is of great interest in its own right, being potentially useful for much more general problems\[^95\]. I shall however restrict myself here to the description of the Mazur method which is more specialised but more explicit.

As soon as I obtained the system given above I noticed that the field equations (5.21) and (5.22) have the striking feature\[^10\] (which does not apply to the traditional system defined in terms of \(k^\mu\) rather than \(m^\mu\)) of being derivable from a Lagrangian integral

\[
\mathcal{I} = \int \mathcal{L} d\lambda d\mu
\]

that is positive definite with the comparatively simple form

\[
\mathcal{L} = \frac{[\hat{\nabla}X]^2 + [\hat{\nabla}Y + 2E\hat{\nabla}B - 2B\hat{\nabla}E]^2}{2X^2} + 2\frac{[\hat{\nabla}E]^2 + [\hat{\nabla}B]^2}{X}
\]

but neither I nor Robinson had seen how to exploit this directly. The breakthrough by Mazur was based on work by Geroch\[^96\] and Kinnersley\[^97\] who showed that Ernst type systems can be interpreted as belonging to a class of non-linear \(\sigma\) models whose field equations are equivalent to a partially redundant set of ordinary divergence type conservation laws of the form

\[
\hat{\nabla} \mathbf{J} = 0
\]

where \(\mathbf{J}\) is a matrix vector constructed according to the prescription

\[
\mathbf{J} = \varpi^{-1}\Phi^{-1} \cdot \hat{\nabla} \Phi
\]

where \(\varpi\) is the known positive weight function given by (5.9) (which does not appear in traditional \(\sigma\) models but whose appearance here adds no significant complication) and where \(\Phi\) is a hermitian matrix function with the...
important property of being positive definite in the present case, its components being given by

$$\Phi_{ab} = \eta_{ab} + 2\bar{v}_a v_b$$

(5.28)

(using a bar to denote complex conjugation and placing a dot on conjugately transforming indices) where \(\eta_{ab}\) is just the the fixed Minkowski type hermitian metric in diagonal form with signature \((-1, 1, 1)\) for the space of three dimensional complex vectors, in which the field vector \(v_a\) is given in terms of the complex Ernst type variables

$$\varepsilon = -X + iY - \psi \bar{\psi}, \quad \psi = E + iB$$

(5.29)

by

$$(v_0, v_1, v_2) = \frac{1}{2}|X|^{-1/2}(\varepsilon - 1, \varepsilon + 1, 2\psi)$$

(5.30)

which is such as to make \(v_a\) automatically “timelike” in hermitian space, with unit normalisation given by

$$\eta^{ab} v_a \bar{v}_b = -1$$

(5.31)

which is sufficient to guarantee the required positivity of the hermitian matrix given by (5.28).

The preservation of the field equations by the SU(2,1) action (leaving \(\eta_{ab}\) invariant) that Kinnersley thus made manifest can also be seen to extend “off shell” in the sense that our Lagrangian (15.25) is also invariant, as can be seen by rewriting it as

$$L = 2|\eta^{ab} v_a \hat{\nabla} \bar{v}_b|^2 - \eta^{ab} (\hat{\nabla} v_a) \hat{\nabla} \bar{v}_b = \frac{1}{2} \hat{g}_{ij} \text{tr}\{J^i \cdot J^j\}$$

(5.32)

where \(\hat{g}_{ij}\) \((i, j = 1, 2)\) is the positive definite two dimensional metric given by (5.11) that is used for the specification of \(\hat{\nabla}\).

The Mazur method of establishing the uniqueness of the solutions of such a system, subject to appropriate boundary conditions such as are given in the present case, is essentially dependent on the positive definiteness of both \(\Phi\) and \(\hat{g}_{ij}\). The objective is to prove the vanishing of the difference

$$\hat{\Phi} = \Phi_{[1]} - \Phi_{[0]}$$

(5.33)

between any pair of matrices representing conceivably distinct solutions. The vanishing of this difference is evidently equivalent to the vanishing of what I refer to as the deviation matrix,

$$\Delta = \Phi_{[1]} \cdot \Phi_{[0]}^{-1} - 1 = \hat{\Phi} \cdot \Phi_{[0]}^{-1}$$

(5.34)
where 1 is the unit matrix (in the complex 3 space). The gradient of this deviation matrix will evidently be given by

\[ \tilde{\nabla} \Delta = \Phi_{[1]} \cdot \hat{J} \cdot \Phi_{[0]}^{-1} , \quad \hat{J} = J_{[1]} - J_{[0]} . \]  

(5.35)

Taking the difference we obtain

\[ \tilde{\nabla} (\tilde{\nabla} \Delta) = \Phi_{[1]} \cdot \left\{ \tilde{\nabla} \hat{J} + \omega^{-1} \hat{g}_{ij} (J_{[1]}^i \cdot J_{[1]}^j - 2J_{[1]}^i \cdot J_{[0]}^j + J_{[0]}^i \cdot J_{[0]}^j) \right\} \cdot \Phi_{[0]}^{-1} \]  

(5.36)

The next step is to use the hermiticity property

\[ \Phi = \Phi^* \Rightarrow J^* = \Phi \cdot J \cdot \Phi^{-1} \]  

(5.37)

(where the asterisk denotes the complex conjugate of the transpose) to rewrite the quadratic terms in (5.36) as

\[ \Phi_{[1]} \cdot (J_{[1]}^j \cdot \hat{J} - \hat{J} \cdot J_{[0]}^j) \cdot \Phi_{[0]}^{-1} = J_{[1]}^i \cdot J_{[0]}^i - \Phi_{[0]} \cdot J_{[1]}^i \cdot J_{[0]}^i - \Phi_{[1]} \cdot J_{[0]}^i \cdot J_{[0]}^i . \]  

(5.38)

On taking the trace of (5.36) we thus obtain the scalar identity

\[ \tilde{\nabla} \left( \tilde{\nabla} \text{tr}\{\Delta\} \right) - \text{tr}\{\Phi_{[0]}^{-1} \cdot \Phi_{[1]} \cdot \hat{J} \} = \omega^{-1} \hat{g}_{ij} \text{tr}\{\Phi_{[0]}^{-1} \cdot J^* \cdot \Phi_{[1]} \cdot J^j \} . \]  

(5.39)

This Mazur identity includes as special cases the identities found for the linearised or uncharged limits by Robinson and myself\cite{89,10,90,91} using a less systematic approach.

It follows directly from the form (15.26) of the field equations that the current difference satisfies

\[ \tilde{\nabla} \hat{J} = 0 \]  

(5.40)

and hence that the left hand side of the identity (5.39) will reduce to a divergence whose integral can be converted using Green’s theorem to a surface contribution which will vanish,

\[ \oint dS_i \omega \hat{g}_{ij} \tilde{\nabla}_j (\text{tr}\Delta) \rightarrow 0 \]  

(5.41)

subject to the appropriate boundary conditions which can be verified (using particular care for the axis where \( X^{-1} \) diverges) to be in fact satisfied in this case. Under these conditions one can deduce that the right hand side of
(5.39) vanishes since (by the positivity of \( \Phi \)) it is clearly a positive definite function of \( \mathcal{J} \) which must therefore vanish, i.e. using (5.35) we get
\[
\mathcal{J} = 0 \quad \rightarrow \quad \Delta = C
\] (5.42)
where \( C \) is some constant matrix. Since the boundary conditions as \( \lambda \to \infty \) ensure that \( \Delta \to 0 \) there one ends by getting
\[
C = 0 \quad \rightarrow \quad \Phi[1] = \Phi[0]
\] (5.43)
which finally establishes the required uniqueness.

6. Special Properties of the Kerr Newman Vacuum Solutions.

The theorem obtained at the end of Section 5 establishes conclusively that in the source free electrovac case there are no (topologically simple) stationary axisymmetric asymptotically flat black hole solutions with non degenerate \( (\kappa > 0) \) horizon other than those of the Kerr Newman family as restricted by the condition
\[
c^2 > 0 , \quad c^2 = M^2 - a^2 - Q^2 , \quad a = J/M ,
\] (6.1)
where this parameter \( c \) represents the value for these solutions of the quantity introduced more generally by (5.8). This includes, for \( Q = 0 \), the pure vacuum family of Kerr solutions, whose black hole nature, in the allowed parameter range \( M^2 > a^2 \), was first clearly recognised by Boyer [71][72]. (For the degenerate limit for which \( c = 0 \) or equivalently \( \kappa = 0 \), corresponding to a horizon at zero temperature in the thermodynamic analogy, the problem has still not been completely solved, but it is known that in the non rotating case \( J = 0 \) there is a class of electrically balanced solutions with \( M^2 = Q^2 \) that is much more general than the corresponding Reisner Nordstrom subset within the Kerr Newman family.) Taking full advantage of the very special properties that will be briefly surveyed below, a long series of investigations, of which the first was that of Vishveshwara [104] and the most recent that of Whiting [13] confirm that the Kerr solutions specified by (6.1) are effectively stable against the all the most obviously relevant kinds of perturbation.

In terms of ordinary coordinates \( r, \theta \) introduced by
\[
\lambda = r - M , \quad \mu = \cos \theta
\] (6.2)
the explicit solutions for the Ernst type variables are given by

\[
X = \left\{ r^2 + a^2 + \frac{(2MR - Q^2)a^2\sin^2\theta}{r^2 + a^2\cos^2\theta} \right\} \sin^2\theta
\]

\[
Y = \left\{ M(2 + \sin^2\theta) - \frac{\sin^2\theta[Q^2r - Ma^2\sin^2\theta]}{r^2 + a^2\cos^2\theta} \right\} 2a \cos \theta
\]

\[
E = \frac{Q(r^2 + a^2)\cos \theta}{r^2 + a^2\cos^2\theta} , \quad B = \frac{-Qra \sin^2\theta}{r^2 + a^2\cos^2\theta} . \quad (6.3)
\]

Going back to ordinary metric and electromagnetic potential components gives the rather simpler forms

\[
V = 1 - \frac{2Mr - Q^2}{r^2 + a^2\cos^2\theta} , \quad \Phi = \frac{Qr}{r^2 + a^2\cos^2\theta} ,
\]

\[
W = \frac{-Q(r^2 - Q^2)a \sin^2\theta}{r^2 + a^2\cos^2\theta} , \quad \Xi = r^2 + a^2\cos^2\theta . \quad (6.4)
\]

Subject to (6.1) these solutions do in fact have turn out\cite{40,73,74,10} to have the property (which was not assumed in advance in the approach outlined above) of having a well behaved Kruskal type horizon crossover when analytically extended towards the past. When analytically extended to the interior they exhibit many amusing but one presumes physically irrelevant features such as a time machine\cite{74,57} in the region beyond the Cauchy horizon, which (in this more general case, as in the Reissner Nordstrom case discussed in Section 2 and in more detail in the accompanying lectures of Israel) is a sign of instability occurring at \( r = r_+ \), using the standard abbreviation

\[
r_\pm = M \pm c . \quad (6.5)
\]

In these solutions the three quantities whose uniformity over the black hole event horizon at \( r = r_+ \) was guaranteed in advance by the results of Section 4, namely the decay constant \( \kappa \), the limiting value \( \Omega^\mu \) of the zamo angular velocity \( \omega \), and the value \( \Phi^\mu \) of the comoving potential \( \Phi^\dagger = \Phi + \Omega^\mu B \), will be given respectively by

\[
\kappa = \frac{c}{2Mr_+ - Q^2} , \quad \Omega^\mu = \frac{a}{2Mr_+ - Q^2} , \quad \Phi^\mu = \frac{Qr_+}{2Mr_+ - Q^2} . \quad (6.6)
\]
Neither the physically motivated global geometrical approach that we have followed here, nor the even the analytical approach (based on the assumption of a special form for the Weyl tensor) used originally by Kerr and Newman makes it at all obvious in advance that the solutions should have such remarkable special algebraic properties as they actually do. The special simplicity of the metric that is finally obtained can be made most directly manifest by expressing it in tetrad form

$$ds^2 = dx^\mu dx^\nu (-e^0_\mu e^0_\nu + e^1_\mu e^1_\nu + e^2_\mu e^2_\nu + e^3_\mu e^3_\nu)$$  \hspace{1cm} (6.7)$$

terms of a preferred canonical separable tetrad which I found very early in the history of the study of the Kerr Newman solutions when the separablity of the Hamilton jacobi equation for free particle trajectories and of the scalar Klein Gordon wave equation was first brought to light\cite{[74]}\cite{[77]}. However although it is useful generally\cite{[105]}\cite{[106]}\cite{[107]}\cite{[108]} and is even more strongly to be preferred for obtaining separability of higher spin wave equations\cite{[109]}\cite{[110]}\cite{[111]}\cite{[112]} , this canonical tetrad has unfortunately tended to have been neglected by workers on higher spin separability\cite{[113]}\cite{[114]}\cite{[115]} in favor of other tetrads (particularly that of Kinnersley\cite{[116]} which has advantages in other contexts but not in this one) thereby making many unavoidably heavy calculations\cite{[117]} even longer and more complicated than necessary. To get maximum algebraical symmetry it is necessary to replace the geometrically defined time and angle coordinates $t$ and $\phi$ as introduced in the previous section (and which in the specific context of the Kerr solutions are commonly referred to as the coordinates of Boyer Linquist\cite{[73]} in order to distinguish them from the rather different time and angle coordinates used by Kerr himself\cite{[103]} ) by very closely related ignorable coordinates $\tilde{t}$ and $\tilde{\phi}$ say, and to introduce a recalibrated angle coordinate $q$ say in place of $\mu$, according to the specifications

$$\tilde{t} = t - a\phi \hspace{1cm} \tilde{\phi} = a^{-1}\phi \hspace{1cm} q = a \cos \theta \hspace{1cm} (6.8)$$

The canonical maximally symmetric tetrad is then expressible as

$$e^0_\mu dx^\mu = - \left( \frac{\Delta_r}{r^2 + q^2} \right)^{1/2} (q^2 d\tilde{\phi} + d\tilde{t}) \hspace{1cm} e^1_\mu dx^\mu = \left( \frac{r^2 + q^2}{\Delta_r} \right)^{1/2} dr \hspace{1cm} \hspace{1cm}$$

$$e^3_\mu dx^\mu = - \left( \frac{\Delta_q}{r^2 + q^2} \right)^{1/2} (r^2 d\tilde{\phi} - d\tilde{t}) \hspace{1cm} e^2_\mu dx^\mu = \left( \frac{r^2 + q^2}{\Delta_q} \right)^{1/2} dq \hspace{1cm} (6.9)$$
where $\Delta_r$ and $\Delta_q$ are quadratic functions respectively of $r$ only and of $q$ only, the symmetry between these variables being broken only by the presence of a linear mass term in the former but not the latter:

$$\Delta_r = r^2 - 2Mr + a^2, \quad \Delta_q = a^2 - q^2.$$

(6.10)

The electromagnetic field potential is expressible in the even simpler form (proportional to the first of these tetrad forms as

$$A_\mu dx^\mu = \left( \frac{Qr}{r^2 + q^2} \right) (q^2 d\tilde{\phi} + dq)$$

(6.11)

At the expense of violating the asymptotic boundary conditions the algebraic symmetry between $r$ and $q$ could be made complete\textsuperscript{[77]} by including appropriate gravimagnetic and electromagnetic monopole terms. The full symmetry is however already manifest in the corresponding expression\textsuperscript{[12]} for the crucially important Killing Yano 2-form $f_{\mu\nu}$ given by

$$f_{\mu\nu}dx^\mu dx^\nu = qdr \wedge (d\tilde{t} + q^2 d\tilde{\phi}) + rdq \wedge (d\tilde{t} - r^2 d\tilde{\phi}),$$

(6.12)

whose existence, as a solution of the Killing Yano equations

$$f_{\mu\nu} = 0, \quad \nabla_\rho f_{\mu\nu} = \nabla_{[\rho} f_{\mu\nu]},$$

(6.13)

underlies the remarkable hidden symmetries of the Kerr Newman family and is by itself sufficient to characterise them completely among asymptotically flat electrovac solutions.

Why the purely local characterisation given by (6.13) should give the same result as the global boundary conditions for a black hole equilibrium problem remains mysterious, but once it is known that (as was first revealed by the work of Penrose and his collaborators \textsuperscript{[118],[119],[120],[121],[122]} using a spinorial approach) there is a non zero solution of (6.13), most of the other special properties of the Kerr solutions can be obtained by more or less straightforward deduction without the intervention of any further independent miracles, starting with the stationarity, whose generator is given\textsuperscript{[123]} directly by

$$k^\mu = \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} \nabla_\nu f_{\rho\sigma},$$

(6.14)

which, by (6.13) will automatically satisfy the Killing equation (4.1). In addition to this “primary” Killing vector (6.11) also\textsuperscript{[123],[124],[12]} ensures the existence of a “secondary”, generically independent, one given by

$$h^\mu = a^\mu_\nu k^\nu, \quad a^\mu_\nu = f^\mu_{\rho\sigma} f^\rho_\nu f^\sigma_\nu,$$

(6.15)
which again satisfies Killing’s equation as a further automatic consequence of (6.13) and which turns out when evaluated to be given as a linear combination of the “primary”, stationarity generating Killing vector $k^\mu$ and of the axisymmetry generating Killing vector $m^\mu$ (as distinguished by having closed trajectories) that is interpretable\cite{124}\cite{12} as generating rigid rotations with angular velocity $\Omega = a^{-1} = M/J$ about the axis: explicitly it satisfies

$$\nabla_{(\mu} h_{\nu)} = 0 , \quad h^\mu = a^2 k^\mu + a m^\mu . \quad (6.16)$$

The pair of independent Killing vectors thus obtained will of course give rise to a corresponding pair of quantities $k^\nu u_\nu$ and $h^\nu u_\nu$ (or $k^\nu u_\nu$ and $m^\nu u_\nu$) that are conserved along solutions of the geodesic equations (4.8). The “hidden symmetry” corresponding to the existence of the “fourth” constant of motion that is needed to provide a complete set of first integrals of the equations of motion (the third being given trivially just by $u^\mu u_\mu$ itself) is given by the tensor $a^{\mu\nu}$ as defined in (6.15) which is an ordinary symmetric Stackel Killing tensor in the sense that, again as an automatic consequence of (6.13), it satisfies the conditions

$$a_{[\mu\nu]} = 0 , \quad \nabla_{(\mu} a_{\nu)} = 0 \quad (6.17)$$

which evidently suffice to ensure that the quadratic combination $a_{\mu\nu} u^\mu u^\nu$ will indeed be constant along solutions of (4.8), thus providing the “fourth” constant that is needed to make these equations completely integrable\cite{74}\cite{10}. Moreover this tensor $a^{\mu\nu}$ is not just a Killing Stackel tensor in the weak sense of satisfying (6.17) but is automatically, by the integrability conditions\cite{12}\cite{124} for (6.13), is automatically a Killing tensor in the strong sense\cite{125} that it also satisfies

$$a^\rho_{[\mu} R_{\nu\rho]} = 0 \quad (6.18)$$

which is the supplementary condition needed in conjunction with (6.17) to ensure that the corresponding self-adjoint differential operator commutes with Dalembertian (scalar) wave operator, i.e.

$$[\nabla_\mu a^{\rho\nu} \nabla_\nu, \nabla^\rho \nabla_\rho] = 0 , \quad (6.19)$$

Just as the scalar Dalembertian can be thought of as a sort of square of the ordinary first order Dirac operator $\gamma^\mu \nabla_\mu$ acting on 4-spinors, so analogously\cite{126}\cite{30}\cite{12} the operator $\nabla_\mu a^{\mu\nu} \nabla_\nu$ can be thought of as a sort of square of a first order
generalised spinor angular momentum operator $L$ that commutes with the Dirac operator

$$[L, \gamma^\mu \nabla_\mu] = 0, \quad L = i\gamma^\mu (\gamma^5 f_{\mu\nu} \nabla_\nu - k_\mu) ,$$  \hspace{1cm} (6.20)

in the same way as does the ordinary Kosman$^{[127]}$ energy operator $K$ for any Killing vector,

$$[K, \gamma^\mu \nabla_\mu] = 0, \quad K = ik^\mu \nabla_\mu + \frac{1}{4} [\gamma^{[\mu} \gamma^{\nu]} k_{\nu}, i\nabla_\mu] .$$  \hspace{1cm} (6.21)

Just as the commutation law (6.19) is interpretable as resulting from the separability of the scalar Klein Gordon equation$^{[77]}$ so analogously the commutation law (6.20) is interpretable as resulting from the separability of the Dirac equation$^{[108]}$. In addition to these cases of separability for massive particle wave equations, the Kerre Newman solutions are also characterised by analogous separability properties for massless higher spin wave equations, including notably those for the separate electromagnetic and electromagnetic perturbations that are relevant to stability analysis$^{[114][115]}$, though so far not for the case when the electromagnetic and gravitational perturbations are coupled as will be the case for the generic perturbation in the charged Kerr Newman case. The analysis of such wave equations is carried out most conveniently by using not the orthonormal version but the corresponding null version of the canonical tetrad, the latter being given in terms of the former by a transformation of the standard form

$$\ell_\mu = \frac{1}{\sqrt{2}} (e_\mu^0 + e_\mu^1) , \quad \bar{\ell}_\mu = \frac{1}{\sqrt{2}} (e_\mu^0 - e_\mu^1) , \quad z_\mu = \frac{1}{\sqrt{2}} (e_\mu^0 + i e_\mu^1) ,$$  \hspace{1cm} (6.22)

where $\ell_\mu$ and $\bar{\ell}_\mu$ are null and $z_\mu$ is complex. Let us consider together the case of an ordinary complex scalar field $\Psi_0$ say, the case of an ordinary Maxwell field $F_{\mu\nu}$, for which we take a privileged pair of complex tetrad components

$$\Psi_1 = F_{\mu\nu} \ell^\mu z^\nu , \quad \Psi_{-1} = F_{\mu\nu} \bar{z}^\mu \bar{\ell}^\nu ,$$  \hspace{1cm} (6.23)

(where for reasons of notational convenience that will become obvious we do not use the traditional counting system which would label these components as $\Phi_0$ and $\Phi_2$), and finally the case of a gravitational perturbations with Weyl tensor $C_{\mu\nu\rho\sigma}$ for which we similarly take the privileged pair of complex tetrad components

$$\Psi_2 = -C_{\mu\nu\rho\sigma} \ell^\mu z^\nu \ell^\rho z^\sigma , \quad \Psi_{-2} = -C_{\mu\nu\rho\sigma} \bar{\ell}^\mu \bar{z}^\nu \bar{\ell}^\rho \bar{z}^\sigma ,$$  \hspace{1cm} (6.24)

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(which in the traditional counting system would be labeled $\Psi_\circ$ and $\Psi_\bullet$). Then the upshot of the many studies referred to above is that\textsuperscript{[109]} corresponding field equations are separable, in terms of our present notational system, by setting

$$(r - iq)^{|s|}\Psi_s = X_s(r)Y_s(q)e^{-i(E\hat{t} - \Phi\hat{\phi})} \quad (6.25)$$

(where the helicity index $s$ runs over the values 0, ±1, ±2) with the resulting separated equations having the form

$$\left\{ \frac{d}{dr}\Delta_r \frac{d}{dr} + \frac{(Er^2 + is(r - M) - \tilde{\Phi})^2}{\Delta_r} + 4isEr \right\} X_s = \tilde{K}X_s$$

$$\left\{ \frac{d}{dq}\Delta_q \frac{d}{dq} + \frac{(Eq^2 + sq + \tilde{\Phi})^2}{\Delta_q} + 4sEq \right\} Y_s = -\tilde{K}Y_s \quad (6.26)$$

where $\tilde{K}$ is the separation constant whose existence expresses the hidden symmetry and where it can be seen from the relation

$$E\tilde{t} - \tilde{\Phi}\tilde{\phi} = Et - \Phi\phi \quad (6.27)$$

that $E$ is interpretable as the ordinary energy associated with the “primary” Killing vector $k^\mu$ while $\tilde{\Phi}$, which is analogously associated with the secondary Killing vector $h^\mu$ is related to the ordinary angular momentum constant $\Phi$ associated with the axial Killing vector $m^\mu$ by the simple relation

$$\tilde{\Phi} = a\Phi - a^2 E \quad (6.28)$$

In the case $s = 0$ the above form agrees directly with what is obtained in the limit of vanishing particle charge and mass, $e = m = 0$ from my original separated form of the Klein Gordon equation, but in the higher spin cases it differs from the forms originally obtained by Teukolsky due to his use of a non canonical (Kinnersley\textsuperscript{[116]} type) tetrad which lead to an unnecessarily complicated form in which the symmetries manifest in the version above (the helicity symmetry between $s$ and $-s$, and the almost perfect algebraic symmetry between $r$ and $iq$) are all spoiled by subjecting (6.9) to a symmetry violating tetrad transformation consisting of a combined boost and rotation of the form

$$\ell^\mu \rightarrow \left(\frac{2(r^2 + q^2)}{\Delta_r}\right)^{1/2} \ell^\mu, \quad \tilde{\ell}^\mu \rightarrow \left(\frac{\Delta_r}{2(r^2 + q^2)}\right)^{1/2} \tilde{\ell}^\mu,$$
\[ z^\mu \rightarrow \frac{r - iq}{\sqrt{r^2 + q^2}} z^\mu . \] 

(6.29)

The dazzling prestige conferred on the Kinnersley tetrad by its successful use in the original discovery of the higher spin separability has unfortunately blinded many workers to the fact that the separation works even more efficiently in terms of the original canonical tetrad, with the result that many published calculations\textsuperscript{117} are at least twice as long as necessary. The canonical tetrad has however tended to come back into use in more recent work, whose achievements include separation of the equations of parallel transport of a tetrad\textsuperscript{107}\textsuperscript{108}\textsuperscript{102} and the equations for the stationary equilibrium of cosmic strings\textsuperscript{128}\textsuperscript{129}\textsuperscript{130} of of certain simple kinds (not just the ordinary Goto Nambu kind but also the more general non dispersive model allowing for the averaged effect of noise or wiggles) whose mechanics will be explained in the following sections.

7. Basic Brane Mechanics.

The purpose of the last part of this course is to give a brief introductory overview (and some illustrative applications in the context of cosmic strings) of the general principles of brane dynamics using a recently developed fully covariant approach\textsuperscript{14}\textsuperscript{15} that avoids the use of excess mathematical bagage (such as the use of distribution theory and specially adapted coordinates for separate subsystems) that may be useful for detailed calculations in specific applications, but that would obscure the simplicity and generality of laws such as the general equation governing the extrinsic motion of any brane, which is expressible in the formalism set up below (using underlining to distinguish quantities defined with respect to a \( p \)-brane under consideration from any higher dimensional analogue that may also be relevant) in terms of its stress momentum energy tensor \( T^{\mu \nu} \), and its second fundamental tensor \( K_{\mu \nu \rho} \) in the form

\[ T^{\mu \nu} K_{\mu \nu \rho} = \underline{T}^\rho , \] 

(7.1)

where \( \underline{T}^\rho \) is the total orthogonally projected force contribution (such as that of the wind on a sail, or of an external electromagnetic field on the current in a cosmic string) from the various external systems (if any) with which the brane may interact.

Following an increasingly popular usage\textsuperscript{131}\textsuperscript{132}, the term \textit{brane} is used here to designate a physical model of the category that includes continuous
media and point particles as extreme cases, with ordinary membranes (from which the term is derived) and strings as the only other possibilities in a 4-dimensional background. Generally, a $(p-1)$ brane is to be understood to be a dynamical system defined in terms of fields with support confined to a $p$ dimensional world sheet surface $S$ in a background (flat or curved) spacetime manifold of dimension $n \geq p$. The extreme case, with $n = p$ is that of a continuous medium for which the confinement condition is redundant.

The use of this concept makes it possible to give a unified description of basic properties that are common to a very wide range of physically diverse phenomena. A simple and very important example is the universal rule that (as a consequence of (7.1) and independently of the nature of any external forces so long as their coupling does not involve gradients of internal field variables) the condition for a (contravariant) vector $\eta^\mu$ say to be an extrinsic bicharacteristic vector, i.e. to be tangent to the direction of “group” propagation of localised wave packets of small extrinsic displacements of the localisation of the world sheet (which of course is meaningful only for $n < p$) with a corresponding characteristic covector $\chi_\mu$ normal to the direction of the associated “brane wave” sheets, will be given\textsuperscript{[123]} simply by

$$\eta^\mu = T^{\mu\nu} \chi_\nu , \quad T^{\mu\nu} \chi_\mu \chi_\nu = 0 . \quad (7.2)$$

The hyperbolicity condition to the effect that the characteristic equation (7.2) should define a real characteristic cone provides a restriction (trivial for a point particle and reducing just to a requirement of positivity of the ordinary tension $T$ in the case of a string\textsuperscript{[133]}) that must always be satisfied as a condition for local stability except of course in the case $p = n$ of a continuous medium for which there is no geometric possibility of extrinsic perturbations, which is why an ordinary perfect fluid with positive pressure $P$ can be stable after all, despite the fact that it is elliptic (with no real roots) as far as the criterion (7.2) is concerned.

In an ordinary spacetime with $n = 4$ a continuous medium (with $p = 4$) counts as a 3-brane, the other possibilities being that of a membrane model (with $p = 3$) which counts as a 2-brane, a string model (with $p=2$) which counts as a 1-brane, and finally at the other extreme, a point particle model (with $p=1$) which counts as a zero brane. Employment of brane models of lower dimension, $p < n$, for which the extrinsic confinement condition and the associated hyperbolicity requirement derived from (7.2) are essential, is often useful for providing an approximate descriptions of higher dimensional case when the the fields characterising the latter are highly concentrated.
in the neighbourhood of a lower dimensional world sheet within a distance that is small compared with the scales characteristic of dynamic variations in directions tangential to the world sheet. Thus for example a point particle model might be useful for describing the motion, with respect to a relatively slowly varying background, of a small loop in a string model that might itself be just a approximation for describing what at a more microscopically accurate level might need the use of a continuum model. The example that has been most important in motivating the development of the relativistic formalism described here is that of the representation (as originally suggested by Kibble\cite{134}, Witten\cite{135} and others) of vortex defects (due to spontaneous symmetry breaking) of the vacuum by ("cosmic") string models as a macroscopic approximation for use in the (cosmologically important) cases in which the vortex thickness can be treated as negligible compared with other relevant length scales. This lead to the introduction of models of variational type in which the action was to be thought of as being derived from the microscopic action of the relevant underlying field theory by integral across the vortex in a local equilibrium state.

Quite generally, in cases where a compound system has a variational formulation in terms of a total action of the form $\sum I$, the action contribution of an individual $p$ brane of the system will be given by a corresponding $p$ surface integral

$$I = \int L dS$$

(7.3)

where $dS$ denotes the induced surface measure and $L$ is a Lagrangian scalar function of whatever internal fields on the world sheet are involved and also of any relevant externally induced fields such as those given by (7.2). In the simplest (non conducting) cosmic string models originally envisaged by Kibble\cite{134} it was sufficient to use a Goto-Nambu\cite{136} action in which (as in the analogous Dirac membrane model\cite{137}) the scalar $L$ is specified trivially as a constant, which is interpretable as the negative of the (spatially isotropic) tension $T$ which in this case is not only uniform but (as an expression of the special property of intrinsic Lorentz invariance which distinguishes these particular models) is also equal in this case to the value of the energy density $U$ say. The quantity $U$ can be defined, for a generic brane model, as the eigenvector corresponding to the timelike eigenvector of its surface stress momentum energy tensor $T^{\mu \nu}$, while in a string model the tension $T$ is unambiguously definable as the other eigenvalue, the case of a membrane being more complicated in far as it admits the possibility of two possibly distinct
tension eigenvalues.

It typically occurs that the approximate macroscopic treatment of a system that is conservative, with a variational formulation, at a microscopic level may require the use of a non conservative macroscopic model involving averaging over microscopic degrees of freedom that are taken into count as entropy. Although it may invalidate the conservative nature of the model as a whole, such an averaging process does not invalidate the local conservation laws obeyed by additive quantities such as energy momentum or electromagnetic charge: what happens is that instead of having the status of Noether identities expressing the invariance properties that hold for the underlying variational model, such conservation laws are to be interpreted in the macroscopic model as constistency conditions for the existence of a corresponding microscopic variational model. The commonly but (not always) appropriate notion that a macroscopic model under consideration is obtainable by integrating out the fine details of a more complicated underlying model makes it seem physically natural to try to preserve some of the spirit of the original finer model by using a description in terms of Dirac distributions. However although very useful for some purposes when used with discretion, use of Dirac distributions can easily become addictive, and is often systematically abused in a manner that hinders clear analysis and provides an archetypical example of the kind of excess mathematical baggage that the present approach is designed to avoid.

The most important example of mathematical machinery that is very helpful for many specific purpose but whose use needs to be avoided (as excess baggage) when one wants to obtain a simple formulation of general principles such as that embodied in the “generalised sail equation”\textsuperscript{[14]} (1.1), is that of the introduction of a system of internal coordinates $\sigma^i$ say, $(i = 0, ..., p - 1)$, on the $p$ dimensional world sheet of the $(p - 1)$ brane under consideration, whose imbedding is thereby describable as a mapping $\sigma^i \mapsto x^\mu$ where the $x^\mu$, $(\mu = 0, 1, ..., n - 1)$ are local coordinates on the $n$ dimensional background spacetime. A possibility that is of considerable practical utility in the intermediate stages of many calculation is the use of what I call “adapted coordinates” meaning a matched system of internal and external coordinates in terms of which the imbedding mapping is characterised by $x^0 = \sigma^0$, ..., $x^{p-1} = \sigma^{p-1}$, $x^p = 0$, ..., $x^{n-1} = 0$, but this obviously can not be done for the simultaneous treatment of intersecting branes (as at the junctions in a cluster of soap bubbles) and it is also obviously incompatible with the freedom to use an objective characterisation of the background
coordinates (e.g. in flat space applications by the requirement that they be Minkowskian) which may be important for the final presentation and utilisability of the results.

One of the uses, as an intermediate step, of a coordinate mapping \( \xi^i \mapsto x^\mu \) is for the explicit construction of the corresponding intrinsic components of the images induced in the imbedding of covariant tensor fields on the background space, such as the electromagnetic potential \( A_\mu \) and most important of all the background space time metric \( g_{\mu\nu} \), whose respective images are given by

\[
A_\mu \mapsto \alpha_i = A_\mu \frac{\partial x^\mu}{\partial \sigma^i}, \quad g_{\mu\nu} \mapsto h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j} .
\]  

(7.4)

In cases where a compound system has a variational formulation as a sum in which each distinct brane contributes a term of the form (7.3), the obvious analogue of the traditional variational specification of the conserved current and stress energy momentum tensor (whose local conservation equations are the Noether identities expressing gauge invariance and general diffeomorphism covariance) will take the form

\[
j^i = \frac{\partial L}{\partial \alpha_i}, \quad t^{ij} = 2 \frac{\partial L}{\partial h_{ij}} + L h^{ij} ,
\]  

subject to the proviso (which is not necessary for the simple and conducting cosmic string models models that will be considered below)that Eulerian variational derivatives are to be used instead of simple partial derivatives if derivatives of the potential and metric are involved. The quantities \( h^{ij} \) appearing in (7.5) are of course the components of the contravariant inverse of the induced metric which is to be used for raising and lowering internal indices.

Whether they are specified variationally, as in (7.5), or whether they are specified in some more empirical way, as would be necessary in a general, non conservative model, the internal current and stress energy momentum tensor will determine corresponding background tensor fields by the natural pull back mapping that is determined directly by the imbedding for any contravariant vector fields, the corresponding coordinate expressions being given by

\[
J^\mu = j^i \frac{\partial x^\mu}{\partial \sigma^i}, \quad T^{\mu\nu} = t^{ij} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j} .
\]  

(7.6)

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The idea of the strategy developed here is that it is more efficient for general theoretical (as opposed to specific computational) purposes not to work with internal tensors such as $j^i$ and $t^{ij}$ but rather to work with the corresponding background spacetime tensors, which in this case are $J^\mu$ (with the underlining as a reminder that it refers to a surface not volume current) and $T^{\mu
u}$. When a variational specification is available it is preferable (particularly for dealing with compound systems involving several mutually interacting branes of diverse dimensions) to bypass the passage via (7.4) and (7.6) through the internal coordinate versions by replacing (7.5) by the equivalent but more direct background coordinate specifications

$$J^\mu = \frac{\partial L}{\partial A_\mu}, \quad T^{\mu
u} = 2 \frac{\partial L}{\partial g_{\mu\nu}} + L g^{\mu\nu}, \quad (7.7)$$

in which the only formal difference from the usual expression for a continuous medium as opposed to lower dimensional brane model is the replacement in the last term of the contravariant version $g^{\mu\nu}$ of the ordinary background metric by what I call the (first) fundamental tensor $\overline{g}^{\mu\nu}$ of the brane world sheet, which is obtained here as the pull back of the contravariant inverse of the induced metric, i.e.

$$\overline{g}^{\mu
u} = h^{ij} \frac{\partial x^{\mu}}{\partial \sigma^i} \frac{\partial x^{\nu}}{\partial \sigma^j}, \quad (7.8)$$

where a double overline is introduced here to denote the surface tangential part of any tensor as defined with respect to the background metric, i.e. the result of contracting all its indices with respect to the mixed rank $p$ projection tensor version $\overline{g}^{\mu}_{\nu}$ of the fundamental tensor itself. It is to be noted that any background tensor that is obtained as the pullback of an intrinsic tensor within the imbedded surface will automatically be equal to its own tangential part, so that in particular we shall have $\overline{J}^\mu = J^\mu$ and $\overline{T}^{\mu\nu} = T^{\mu\nu}$.

The “fundamental” tensor of the imbedding that is thus specified in accordance with (7.8) is of great (but still insufficiently widely recognised) importance as the starting point for the systematic tensorial analysis of imbedding curvature as described in the next section, whose results are applicable not just to a timelike brane world sheet but also to submanifolds that are spacelike (though not to those that are null, i.e. metrically degenerate).

In the particular case of a Goto Nambu string model\cite{136} or a Dirac membrane model\cite{137}, as characterised by an action of the form (7.3) with $L = L_0$ for some fixed value $L_0$, which (see section 3) gives a uniform isotropic
tension \( T \) that is equal to the corresponding energy density \( U \) and opposite to the Lagrangian itself, i.e. \( U = T = -L_0 \), the introduction of the fundamental tensor \( g^{\mu\nu} \) makes it easy to check the well known property that the characteristic propagation speed of extrinsic perturbations is, in this case, that of light (\( c=1 \) in the units used here) by substituting in (7.1) the simple formula whereby the stress momentum energy density for such a (Gotu Nambu or Dirac) model is expressible directly as \( \mathcal{T}^{\mu\nu} = -U g^{\mu\nu} \).

As explained in the appendix, we shall adopt the systematic use of a convention using an overhead parallelism symbol, \( \parallel \), to indicate the effect of projection into the surface, and an overhead perpendicularity symbol, \( \perp \), to indicate the effect of the complementary orthogonal projection operation, so that the surface tangentiality conditions that the surface current and stress momentum energy tensors must satisfy by construction, will simply take the form

\[
\mathcal{T}^\mu = 0, \quad \mathcal{T}^{\mu\nu} = 0,
\]

(7.9)

it can be seen from (7.7) that the variations in a brane Lagrangian \( L \) due to an infinitesimal electromagnetic gauge variation \( A_\mu \mapsto A_\mu + \nabla_\mu \chi \) and an infinitesimal diffeomorphism variation \( g_{\mu\nu} \mapsto g_{\mu\nu} + \nabla_\mu (\xi_\nu) \) of the metric will be expressible respectively as

\[
J^\mu \nabla_\mu \chi = \nabla_\mu (\chi J^\mu) - \chi \nabla_\mu J^\mu,
\]

(7.10)

and

\[
T^{\mu\nu} \nabla_\mu \xi_\nu = \nabla_\mu (\xi_\nu T^{\mu\nu}) - \xi_\nu \nabla_\mu T^{\mu\nu}.
\]

(7.11)

It can be seen that the first term on the right of each of these equations has the form\(^{18}\) that characterises a tangential current divergence within the \( p \) dimensional brane world sheet, and hence that by the appropriate \( p \) dimensional version of Green’s theorem the corresponding surface integral will be expressible as the integral over the brane boundary (if any) of the contraction of the tangential current with the unit world sheet tangent vector normal to, and oriented towards, the boundary.

Let us consider the very large class of situations\(^{14}\) that can be represented by a well behaved brane complex (or “rigging system”) in which direct action of a lower on a higher dimensional brane occurs only when the former forms a smooth boundary segment of the latter (as when a monopole, treated as a point particle, forms the termination of a string, or when a sail forms the boundary between two external wind volumes), subject to dynamic
equations to the effect that the infinitesimal variation of the relevant fields other than the externally determined background fields \( g_{\mu\nu} \) and \( A_\mu \), gives no contribution to the variation of the combined action \( \sum \mathcal{I} \) taken over the various brane constituents of the system, restricting ourselves to cases in which derivatives of the external fields \( g_{\mu\nu} \) and \( A_\mu \) are not involved in the action. (The exclusion of more general derivative couplings merely avoids the extra technical complications that are present in more elaborate, e.g. polarised systems, but the exclusion of direct action except on a smooth boundary is more essential, being needed to avoid the serious divergence difficulties, exemplified by that of the radiation back reaction on a point particle, which would otherwise be involved.) Then it can be seen (by systematically using (7.10) to convert divergences to boundary contributions) that the requirement that this combined action \( \sum \mathcal{I} \) should also be identically invariant under gauge transformations generated by an arbitrary field \( \chi \) is equivalent to the condition that there should be a total current conservation law expressed by the condition\(^{[123]} \) that for each \( p \) brane of the system we should have
\[
\nabla_\mu J^\mu = \sum \lambda_\mu J^\mu, \tag{7.12}
\]
where the summation is taken over the separate \((p+1)\) branes of which the \( p \) brane under consideration forms part of the boundary, and where \( J^\mu \) without underline denotes the value on the boundary segment of the current vector in the higher dimensional sheet while \( \lambda_\mu \) denotes the unit normal from the \( p \) dimensional boundary into the relevant externally attached brane world sheet. Similarly (by analogous systematic use of (7.11) to convert divergences to boundary contributions) it can be seen under the same conditions that the general covariance requirement that the combined action be invariant under diffeomorphisms generated by an arbitrary vector field \( \xi^\mu \) is equivalent to a local energy momentum conservation law to the effect\(^{[1]} \) that for each brane of the system we should have
\[
\nabla_\nu T^{\mu\nu} = f^\mu, \quad f_\mu = \sum \lambda_\nu T^{\nu\mu} + F_{\mu\nu} J^\nu \tag{7.13}
\]
in which the force density is obtained as the sum of contact contributions from the (non underlined) stress momentum energy density tensor \( T^{\mu\nu} \) of each of the attached \( p \) branes (at most two if \( p = n \), but arbitrarily many for \( p < n \)) of which the \((p-1)\) brane under consideration is a boundary segment, together with an external electromagnetic force contribution determined by the Maxwellian field \( F_{\mu\nu} = 2 \nabla_{[\mu} A_{\nu]} \).
Although the foregoing direct derivation starts from a variational postulate, charge and energy momentum conservation laws of the form (7.12) and (7.13) can still be expected to hold for more general dissipative models such as would be obtained by macroscopic averaging over internal degrees of freedom whose net effect would be taken into account in terms of entropy currents. An alternative (for some tastes more intuitive, but mathematically much more awkward) way of deriving (3.4) and (3.5) in such cases would be to consider the brane system as the infinitely thin limit of a continuous medium model where the current \( J^\mu \) and stress energy momentum density \( T^{\mu \nu} \) are no longer continuous fields but have become Dirac distributions, whose coefficients are interpretable as the corresponding smooth world sheet supported fields \( J^\mu \) and \( T^{\mu \nu} \). By whatever route they may have been obtained, the ubiquitous generality of (7.12) and (7.13) - and of the extrinsic equation of motion (7.1) that is obtainable via (7.9) as a direct consequence - cannot be overemphasised. In the particular case of a free motion for which external electromagnetic and contact effects are absent we evidently get

\[
f_\mu = 0 \implies T^{\mu \nu} K_{\mu \nu} = 0 \quad (7.14)
\]

In the case of a variational model with action simply proportional to the world sheet measure, as in the case of a Dirac membrane, a Goto-Nambu string, or an ordinary free point particle, the force free equation of extrinsic (“brane wave”) motion (7.14) obviously reduces to the even simpler (“harmonic”) form \( K^\mu = 0 \) which includes the equation for a geodesic in the one dimensional case.

8. Perfect Brane Models.

For a general brane model, we can always define an energy density scalar, \( U \) say, as the negative of the eigenvalue specified by

\[
T^\mu_\nu u^\nu = -U u^\nu \quad (8.1)
\]

where the corresponding eigenvector \( u^\mu \) is distinguished by the requirement that it be timelike or null. As a widely applicable special case (including the Dirac membrane mentioned above, as well as all point particle and string models) a \((p-1)\) brane may be described as “perfect” if its surface stress momentum energy tensor is isotropic with respect to the other orthogonal directions, which in the generic case for which the eigenvector \( u^\mu \) is strictly
timelike (not null) and hence normalisable to unity, one gets the explicit form
\[ T^\mu_\nu = (U - T)u^\mu u_\nu - T \bar{g}^\mu_\nu , \quad u^\mu u_\mu = -1 , \quad (8.2) \]
where \( T \) (the negative of the other \((p - 1)\) degenerate eigenvalues) is what is interpretable as the tension of the \((p - 1)\) brane.

The category of perfect branes includes, as the extreme case \( p = n \), the example of an ordinary “perfect fluid” (with \( U = \rho \), where \( \rho \) is the ordinary volume density of mass-energy, while \( T = -P \) where \( P \) is the ordinary, positive, pressure). In the other cases, i.e. for a \((p - 1)\) brane of lower dimension than the background, i.e. \( p < n \), for which extrinsic displacements are possible (so that the tension must be non negative in order to avoid local instability\(^{[14][133]}\) the extrinsic motion will be governed by (7.1) or in the force free case by (7.14) which, on substitution of (8.2) gives the dynamic equations for a free perfect brane world sheet in the form
\[ c^2 E^\mu_\nu K^\mu = (1 - c^2 E^2) \bar{g}^\mu_\nu \dot{u}^\mu , \quad \dot{u}^\mu = u^\nu \nabla_\nu u^\mu , \quad c_E = \sqrt{\frac{T}{U}} \quad (8.3) \]
where \( \dot{u}^\mu \) is the acceleration vector of the unit eigenvector \( u^\mu \) and \( c_E \) is interpretable as the speed of propagation - relative to the preferred frame specified by \( u^\mu \) - of extrinsic perturbations, as derived from the general characteristic equation (7.2). It is to be noted that in the ultra relativistic case of a Dirac membrane or Goto Nambu string one has \( c_E = 1 \) which means that the right hand side of (8.3) will vanish. On the other hand the strings and membranes that are commonly used (in violins, drums, etc.) by old fashionned non relativistic (i.e. non electronic) orchestras for music generation, will also be describable to a very good approximation by this same equation but with \( c_E << 1 \), which means that the coefficient \( c_E^2 \) will be able to be neglected on the right though not of course on the left.

The extreme case of a “zero brane” with \( p = 1 \), i.e. that of an ordinary (massive) point particle, can be considered as being automatically of the perfect type characterised by (8.2) with \( U = m \) where \( m \) is its mass, and with identically vanishing tension \( T = 0 \) which is consistent with the obvious necessity of having zero relative speed of propagation of any perturbation in this one dimensional case. For a point particle trajectory the first and second fundamental tensors will be given simply by
\[ \bar{g}^\mu_\nu = -u^\mu u_\nu , \quad K^\mu_\nu = \bar{g}^\mu_\nu K^\nu , \quad K^\mu = -\dot{u}^\mu , \quad \dot{u}^\mu = u^\nu \nabla_\nu u^\mu \quad (8.4) \]
while in terms of the particle mass $m$ and charge $e$ say substitution of the appropriate expressions

$$T^{\mu \nu} = -m \bar{g}^{\mu \nu}, \quad J^\mu = e u^\mu,$$

(8.5)

into the general expressions (7.12) and (7.13) gives the dynamical equations in the familiar form

$$u^\mu \nabla_\mu e = 0, \quad u^\mu \nabla_\mu m = 0, \quad -m K_\mu = e F_{\mu \nu} u^\nu,$$

(8.6)

subject of course to the usual proviso (which in this context is to be taken quite literally!) that there are no strings attached, since otherwise corresponding contact contributions on the right of (7.12) and (7.13) could cause variations of the values of the charge and mass scalars, $e$ and $m$ as well as modifying the acceleration equation in (7.14). 

The case of a membrane in 4-dimensions (or more generally of an $(n-2)$ brane in $n$ dimensions) shares with the opposite extreme case of a point particle the property of having comparatively simple kinematic properties, since any timelike hypersurface has first and second fundamental tensors that are expressible in terms of its unit normal $\lambda^\mu$ (as specified by an arbitrary choice of orientation) in the form

$$\bar{g}^{\mu \nu} = g^{\mu \nu} - \lambda^\mu \lambda_\nu, \quad K_{\mu \nu}^\rho = K_{\mu \nu} \lambda^\rho, \quad \lambda^\mu \lambda_\mu = 1.$$

(8.7)

Analogously to the way the first fundamental tensor $\bar{g}^{\mu \nu}$ is specifiable (by (7.8)) as the pull back of the contravariant version of the induced metric, i.e. of what is commonly known as the first fundamental form of the imbedding, so analogously the symmetric tensor $K_{\mu \nu}^\rho$ is the pull back of the contravariant version of what is commonly known as the second fundamental form on the hypersurface, a quantity whose specification, like that of the unit normal $\lambda^\mu$ involves an arbitrary choice of sign. (In addition to its principle advantage of being applicable to imbeddings of arbitrary dimension, not just hypersurfaces, an advantage of our present strategy of working with the three index second fundamental tensor rather than the two index second fundamental form even in the hypersurface case where the latter is available is that unlike that of $K_{\mu \nu}^\rho$ the specification of $K_{\mu \nu}^\rho$ is quite unambiguous.) Whereas the kinematic specifications (8.7) are simpler than their analogues for the lower dimensional case of a string, on the other hand the dynamics of a membrane are generally more complicated. Unlike the case of a string model which must
always, trivially, be perfect in the sense of (8.2) (or of its null limit\cite{14}) the
dulate of “perfection” in this sense is a serious restriction in the case of a
membrane, being satisfied for a Dirac membrane or an ordinary soap bubble
type membrane, (and even as a reasonable approximation to the way musical
drum membranes are most commonly tuned), but it will not be at all valid
for such applications as to a typical ship’s sail.

Between the highpersurface supported case of a membrane and
the curve supported case of a point particle the only intermediate kind of
brane that can exist in 4-dimensions is that of 1-brane, i.e. a string model,
which (for any background dimension \(n\)) will have a first fundamental tensor
that is expressible as the square of the antisymmetric tangential tensor \(\mathcal{E}^{\mu\nu}\)
that is defineable\cite{138} as the pullback of the contravariant version of the in-
duced measure tensor that is specified modulo a choice of orientation by the
imbedding, i.e. we shall have

\[
\mathcal{g}^{\mu\nu} = \mathcal{E}^\mu\rho \mathcal{E}_\rho^{\\nu}, \quad \mathcal{E}^{\mu\nu} = \mathcal{E}\left[\mu\nu\right].
\]  

A special feature distinguishing string models from point particle
models on one hand and from higher dimensional brane models on the oth-
er is the dual symmetry\cite{139,14} that exists at a formal level between the spacelike
and timelike eigenvectors \(u^\mu\) (as already introduced) and \(v^\mu\) that for a generic
case (excluding the null state limit\cite{14}) are characterised modulo a choice of
orientation by the expression

\[
T^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu, \quad v^\mu v_\mu = 1 = -u^\mu u_\mu
\]  

in which the tension \(T\) appears as the dual analogue of the “rest frame”
energy per unit length \(U\). This formal duality can also be made apparent in
the expression for the extrinsic curvature vector of the string, which can be
expressed as

\[
K^\mu = \frac{1}{\mathcal{g}^{\nu\rho}}(v^\nu - u^\nu), \quad v^\mu = v^\nu \nabla_\nu v^\mu, \quad \dot{u}^\mu = u^\nu \nabla_\nu u^\mu
\]  

whose substitution in (8.3) enables the equation of extrinsic motion of a free
string to be expressed in the manifestly self dual form

\[
U \frac{1}{\mathcal{g}^{\nu\rho}} \dot{u}^\rho = T \frac{1}{\mathcal{g}^{\nu\rho}} v^\rho.
\]  

Of course the extrinsic equation of motion, whether of the general
form (7.1) or the free string specialised form (8.11), cannot actually be used to
determine the evolution of the world sheet until the appropriate prescription has been given for evaluating the necessary stress momentum energy tensor components, which in the string case (8.11) can be taken to be just $T$ and $U$. In the simple Goto-Nambu case, for which these eigenvalues are specified in advance to have constant values, $U = T = -L_0$, no further preparation is needed for the integration of (8.12) but in general, for a string model with non trivial intrinsic structure the completion of the system of equations of motion will involve the specification of other differential equations. The simplest non trivial possibility, which is applicable to higher dimensional perfect brane models as well as to strings, is what is known in the specific context of perfect fluid theory as the “barotropic” case, meaning the case in which $T$ is specified (directly or parametrically) as a function only of $U$ by a single equation of state. In this barotropic case (which includes the Witten type conducting cosmic string models\textsuperscript{[135]} whose investigation provided the original motivation for this work) the only differential equations that are needed to supplement the extrinsic equation of motion (8.3) or (8.11) are those that are obtained from the projection into the world sheet of the full local momentum energy conservation equation (7.13), which in the force free case simply gives

$$\nabla_\mu T^{\mu\nu} = 0 \quad (8.12)$$

whose two independent components can be conveniently expressed as a pair of mutually dual surface current conservation laws given by

$$\nabla_\mu (\nu u^\mu) = 0, \quad \nabla_\mu (\mu v^\mu) = 0, \quad (8.13)$$

in terms of an effective number density $\nu$ and an associated effective mass density $\mu$ that obtained from the equation of state as functions of $U$ or equivalently of $T$ by a pair of (mutually dual) integral relations of the form

$$\ln \nu = \int \frac{dU}{U - T}, \quad \ln \mu = \int \frac{dT}{T - U}, \quad (8.14)$$

which fix them modulo a pair of constants of integration of which one is conventionally fixed by imposing the (self dual) restraint condition

$$\mu \nu = U - T. \quad (8.15)$$

Apart from the extrinsic perturbations of the world sheet location itself, which propagate with the “brane wave” speed $c_E$ (relative to the
frame determined by $u^\mu$) as already discussed, the only other kind of perturbation mode that can occur in a barytropic string are longitudinal modes specified by the variation of $U$ or equivalently of $T$ within the world sheet. Such longitudinal perturbations (the analogue of ordinary sound waves in a perfect fluid) can easily be seen \[14][133] to have a relative propagation velocity given by

$$c_L = \sqrt{\frac{\nu d\mu}{\mu d\nu}} = \sqrt{-\frac{dT}{dU}}, \quad (8.16)$$

which must be real in order for the string to be locally stable. Knowledge of whether the longitudinal perturbation speed $c_L$ is greater or less than the extrinsic speed $c_E$ may be critically significant for questions such as the stability of stationary rotating ring equilibrium states \[14][140][141][142] and their deformed generalisations \[130] which for Witten type cosmic strings (as opposed to the ordinary Goto Nambu type for which no such states exist) may be cosmologically important \[141][143]. Most early, and many more recent discussions \[144][145][146][147] of Witten type strings were implicitly based on the use of an equation of state for which the sum $U + T$ remains constant, which implies longitudinal propagation at a speed equal to that of light, $c_L = 1$ which thus necessarily exceeds $c_E$ but more accurate investigations \[148][149] have recently been developed \[150] to a stage at which it is becoming increasingly clear that the opposite is usually the case, i.e. Witten type models would seem to be typified by $c_L < c_E$.

A very special interest attaches to the intermediate non-dispersive, case characterised by $c_E = c_L$, which corresponds to an equation of state for which the eigenvalue product $TU$ is constant, leading to dynamic equations \[151] that I have shown to be explicitly integrable (like those of the degenerate Goto Nambu case) in a flat spacetime background, the general form in an arbitrary curved background being expressible as

$$L_{\pm}^{\nu} \nabla_{\nu} L_{\pm}^{\mu} = 0, \quad L_{\pm}^{\mu} = \frac{\sqrt{U} u^{\mu} \pm \sqrt{T} v^{\mu}}{\sqrt{U - T}}, \quad T = \frac{U_0^2}{U}, \quad (8.17)$$

where $U_0$ is a constant and the (timelike) unit vectors $L_{\pm}^{\mu}$ are directed along the “left” and “right” moving unit characteristic directions, the former being parallel propagated by the latter and vice versa. Another, more recently established special property of the non-dispersive equation of state $UT = U_0^2$ is the remarkable solubility (by separation of the relevant Hamilton Jacobi equation) of the corresponding equations of stationary equilibrium not only
in flat space but in a generalised Kerr black hole background\textsuperscript{[130]}. This special “constant product” string model (which can be recognised\textsuperscript{[14]} as turning up spontaneously in Kaluza Klein theory\textsuperscript{[152][153][154][155]} is not just of purely mathematical interest: my prediction\textsuperscript{[151]} that it should provide a good description of the averaged effect of random noise perturbations on an “ordinary” Goto-Nambu type cosmic string (on the grounds that their presence should not introduce dispersion) has been confirmed by Vilenkin’s more detailed “wiggly string” calculations\textsuperscript{[156]}.

Appendix : Background tensor analysis of the curvature of an imbedding.

In any $n$ dimensional manifold with a non degenerate (Riemannian or pseudo Riemannian) metric tensor with components $g_{\mu\nu}$ (with respect to some local coordinate patch) that is to be used for index lowering and raising, any non-null (strictly spacelike or timelike) $p$-dimentional surface element at a point determines a corresponding decomposition

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{g^{\mu\nu}}$$

where $\bar{g}_{\mu\nu}$ is the fundamental (rank $p$) projection tensor of the surface element, and $\frac{1}{g^{\mu\nu}}$ is the complementary (rank $n-p$) tensor of projection orthogonal to the surface. Consistently with (A1) we shall adopt the systematic use of a convention using an overhead parallelism symbol, $\parallel$, to indicate the effect of projection into the surface, and an overhead perpendicularity symbol, $\perp$, to indicate the effect of the complementary orthogonal projection operation, so that in particular for an arbitrary vector with components $\xi^\mu$, and for the standard operator $\nabla_\mu$ of Riemannian covariant differentiation (as defined with respect to a symmetric connection such that $\nabla_\mu g_{\nu\rho}$ vanishes), we have

$$\xi^\mu \overset{\text{def}}{=} \bar{g}^{\mu\nu} \xi_\nu , \quad \nabla_\mu \overset{\text{def}}{=} \bar{g}^{\mu\nu} \nabla_\nu , \quad \frac{1}{\xi^\mu} \overset{\text{def}}{=} \frac{1}{\bar{g}^{\mu\nu}} \xi_\nu , \quad \nabla_\mu \overset{\text{def}}{=} \frac{1}{\bar{g}^{\mu\nu}} \nabla_\nu . \quad (A2)$$

In terms of this convention, the fundamental tangential and orthogonal projection operators are thus characterised by the conditions that for any vector $u^\mu$ that is tangent to the $p$-surface element, and any vector $\lambda^\mu$ that is orthogonal to the $p$-surface element we must have

$$\bar{u}^\mu = u^\mu , \quad \parallel u^\mu = 0 , \quad \bar{\lambda}^\mu = 0 , \quad \perp \lambda^\mu = \lambda^\mu . \quad (A3)$$
Unlike the full covariant differentiation operator $\nabla_{\mu}$ and its orthogonally projected part $\nabla_{\mu}$, the tangential covariant differentiation operator, $\nabla_{\mu}$, has the property of being well defined not only for (sufficiently smooth) fields defined on an open neighbourhood of the background space but even for fields with support is confined to a (sufficiently smooth) $p$-surface whose tangent surface element specifies the projection. In particular, for any such $p$-surface there will be a well defined second fundamental tensor, $K_{\mu\nu\rho}$ defined[133][14][15] in terms of its first fundamental tensor $\bar{g}^\mu_\nu$ by

$$K_{\mu\nu\rho} \overset{def}{=} \bar{g}^\rho_\nu \nabla_{\mu} \bar{g}^\rho_\sigma,$$  \hspace{1cm} (A4)

which as a trivial algebraic identity is obviously tangential on the first two indices and almost as obviously orthogonal on the last, i.e. for an arbitrary vector $\xi^\mu$ it satisfies

$$K_{\mu\nu\rho} \xi^\rho = \frac{\partial}{\partial x^\nu} \bar{g}^\rho_\mu \xi^\rho = K_{\mu\nu\rho} \bar{g}^\rho_\mu \xi^\rho.$$  \hspace{1cm} (A5)

Such a tensor $K_{\mu\nu\rho}$ is of course definable not only for the fundamental projection tensor of a $p$-surface, but also for any (smooth) field of rank $p$ projection operators $\bar{g}^\mu_\nu$ as specified by a field of arbitrarily orientated $p$-surface elements. What distinguishes the integrable case, i.e. that in which the elements mesh together to form a well defined $p$-surface through the point under consideration, is the condition that the tensor defined by (A5) should also satisfy the Weingarten identity

$$K_{\mu[\nu\rho]}^{\rho} = 0$$  \hspace{1cm} (A6)

(where the square brackets denote antisymmetrisation), this non trivial symmetry property of the second fundamental tensor being derivable[124] as a version of the well known Frobenius theorem.

The second fundamental tensor $K_{\mu\nu\rho}$ has the property of fully determining the tangential derivatives of the first fundamental tensor $\bar{g}^\mu_\nu$ by the formula

$$\nabla_{\mu} \bar{g}^\rho_\nu = 2 K_{\mu(\nu\rho)}$$  \hspace{1cm} (A7)

(using round brackets to denote symmetrisation) and it can be seen to be characterisable by the condition that the orthogonal projection of the acceleration of any tangential vector field $u^\mu$ will be given by

$$\frac{1}{\bar{g}^\mu_\nu} u^\nu \nabla_{\nu} u^\mu = u^\mu u^\nu K_{\mu\nu\rho},$$  \hspace{1cm} (A8)
as well as by the condition (in which the non-trivial role of the symmetry
property (A6) is more apparent) that the tangential projection of the deriva-
tive of any field of surface normal vectors $\lambda^\mu$ should be given by

$$\nabla_\mu \lambda_\nu = -K^\rho_{\mu\nu} \lambda_\rho . \tag{A9}$$

Going on to higher order we can introduce the \textit{third} fundamental
tensor in an analogous manner as

$$\Xi_{\lambda\mu\nu}^\rho = \overline{g}^\rho_{\mu} \overline{g}^\rho_{\nu} \overline{g}^\rho_{\lambda} \nabla_\sigma K^\sigma_{\rho} , \tag{A10}$$

which by construction is obviously symmetric between the second and third
indices and tangential on all the first three indices while being, i.e. (for an
arbitrary vector $\xi^\mu$) it satisfies the trivial identities

$$\Xi_{\lambda[\mu\nu]}^\rho = 0 , \quad \Xi_{\lambda\mu\nu}^\rho \xi_\rho = \Xi_{\lambda\mu\nu}^\rho \xi_\rho = \Xi_{\lambda\mu\nu}^\rho \xi_\rho . \tag{A11}$$

In a spacetime background that is flat (or of constant curvature as is the
case for the DeSitter universe model) this third fundamental tensor is fully
symmetric over all the first three indices by what is interpretable as the
generalised Codazzi identity which is expressible in a background with
arbitrary Riemann curvature $R_{\lambda\mu\nu\sigma}$ as

$$\Xi_{\lambda\mu\nu}^\rho = \Xi_{(\lambda\mu\nu)}^\rho + \frac{2}{3} \overline{g}^\rho_{\lambda} \overline{g}^\rho_{\mu} \overline{g}^\rho_{\nu} R_{\sigma\tau\beta} \overline{g}^\rho_{\alpha} \overline{g}^\rho_{\beta} . \tag{A12}$$

It is very useful for a great many purposes to introduce the \textit{extrinsic curvature vector} $K^\mu$, defined as the trace of the second fundamental
tensor, i.e.

$$K^\mu \overset{\text{def}}{=} K^\nu_{\mu} , \quad \overline{K}^\mu = 0 \tag{A13}$$

The specification of this extrinsic curvature vector for a timelike $p$-surface in
a dynamic theory provides what can be taken as the equations of extrinsic
motion of the $p$-surface (the simplest case being the “harmonic” condition
$K^\mu = 0$ obtained from a simple surface measure variational principle such
as that of the Goto-Nambu string model or the Dirac membrane model). It
is also useful for many purposes to introduce the \textit{extrinsic conformation}
tensor $C_{\mu\nu}^\rho$ defined as the trace free part of the second fundamental tensor by

$$C_{\mu\nu}^\rho \overset{\text{def}}{=} K_{\mu\nu}^\rho - \frac{1}{p} \overline{g}_{\mu\nu} K^\rho , \quad C^\nu_{\nu} = 0 . \tag{A14}$$
which (like the Wey tensor of the background metric) has the noteworthy property of being conformally invariant with respect to conformal modifications of $g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu}$ of the background metric.

The condition of preserving the tangent element to an imbedded $p$-surface at a point breaks down the full $n$ dimensional rotation group preserving the background metric into the product of the restricted $p$ dimensional rotation group preserving the induced metric in the imbedding with the restricted $(n-p)$ dimensional rotation group preserving the induced metric in the orthogonal element. Associated with each of these subgroups there is a corresponding naturally induced connection and covariant differentiation operator acting on the corresponding bundles of tangent vectors $u^\mu$ and orthogonal vectors $\lambda^\mu$ respectively, and for each there will be a corresponding, respectively “inner” and “outer” bundle curvature, which will be represented by a corresponding background tensor, the former “inner” curvature tensor being just the pull-back onto the background by the imbedding mapping of the ordinary Riemann curvature of the intrinsic geometry induced by the imbedding. Explicitly for any vector fields satisfying the appropriate tangentiality and orthogonality conditions (A2), the effects of the corresponding restricted “inner” (tangentially projected) and “outer” (orthogonally projected) differentiation operations will be given respectively by

$$\nabla_{\mu} t^\nu \equiv g_{\nu\rho} \nabla_{\mu} u^\rho = \nabla_{\mu} u^\nu - K_{\mu\rho}^\nu u^\rho, \quad g_{\nu\rho} \nabla_{\mu} \lambda^\rho = \nabla_{\mu} \lambda^\nu + K_{\mu\rho}^\nu \lambda^\rho. \quad (A15)$$

Using the convention that an underline is inserted whenever necessary to distinguish a quantity defined with respect to an imbedding from its higher dimensional background analogue, the corresponding inner curvature tensor $R_{\kappa\lambda}{}^{\mu\nu}$ of the $p$-surface (as distinct from the ordinary background Riemann tensor $R_{\kappa\lambda}{}^{\mu\nu}$) and the corresponding outer curvature tensor $\Omega_{\kappa\lambda}{}^{\mu\nu}$ (for which no background analogue exists, so that there is no need to underline it) are specifiable by the respective conditions that for any tangential vector $u^\mu$ and any orthogonal vector $\lambda^\mu$ to the surface, i.e. for any vectors satisfying (A3) we should respectively have

$$2\nabla_{[\mu} \nabla_{\nu]} u^\rho \equiv 2g^{\sigma\lambda} g^{\nu} \nabla_{\mu} (g^{\rho} \nabla_{\sigma} u_{\tau}) = R_{\mu\nu}{}^{\rho\sigma} u^\sigma, \quad (A16)$$

and

$$2g^{\nu} \lambda g^{\tau} \nabla_{[\nu} (g^{\rho} \nabla_{\lambda]} u_{\rho} \nabla_{\sigma} \lambda_{\tau}) = \Omega_{\mu\nu}{}^{\rho\sigma} \lambda^\rho. \quad (A17)$$
Then it can be verified that the inner curvature tensor is given in terms of the tangential projection of its background analogue by the relation

\[ R_{\mu\nu\rho\sigma} = 2K^\rho_{\ [\mu} K_{\nu]\sigma\tau} + \overline{R}_{\mu\nu\rho\sigma} , \]  

which is the translation into the present scheme of what is well known in other schemes as the generalised Gauss identity. The much less well known analogue for the (identically trace free and conformally invariant) outer curvature, for which the most historically appropriate name is arguably that of Schouten, is given\(^{[15]}\) by the expression

\[ \Omega_{\mu\nu\rho\sigma} = 2C_{[\mu}^{\tau\rho} C_{\nu]\tau\sigma} + \overline{g}_{\mu}^{\lambda} \overline{g}_{\nu}^{\alpha} C_{\kappa\lambda\tau}^{\tau} \overline{g}^{\alpha} \overline{g}^{\rho} \sigma , \]  

where \(C_{\mu\rho\sigma}\) is (trace free conformally invariant) background Weyl tensor, which is definable implicitly for a background of dimension \(n > 2\) by the decomposition of the Riemann tensor into trace and trace free parts as

\[ R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{4}{n-2} g_{[\mu}^{\rho} R_{\nu]} - \frac{2}{(n-1)(n-2)} R g_{[\mu}^{\rho} g_{\nu]} , \]  

where, as usual the background Ricci tensor and Ricci scalar are given by

\[ R_{\mu\nu} = R_{\rho\mu}^{\rho\nu} , \quad R = R^{\nu}_{\nu} . \]  

It can be seen from the form of the identity (A19) that in a flat or conformally flat background (for which it is necessary, and for \( n \geq 4\) sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor \(C_{\mu\rho}^{\nu}\) will be sufficient (independently of the behaviour of the extrinsic curvature vector \(K^{\mu}\)) for vanishing of the outer curvature tensor \(\Omega_{\mu\nu\rho\sigma}\), and hence (by (A17)) for the possibility of constructing fields of orthogonal vectors \(\lambda^{\mu}\) that satisfy the generalised Fermi-Walker propagation condition to the effect that \(\overline{g}^{\rho}_{\mu} \overline{\nabla}_{\nu} \lambda_{\rho}\) should vanish. It can also be shown (taking special trouble for the case \(p = 3\)) that in a conformally flat background (of arbitrary dimension \(n\)) the vanishing of the conformation tensor \(C_{\mu\rho}^{\nu}\) is always sufficient (though by no means necessary) for conformal flatness of the induced geometry in the imbedding.
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