Khasminskii-type theorems for stochastic differential delay equations driven by G-Brownian motion

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ABSTRACT

The Khasminskii theorem have come to play an important role in the nonexplosion solutions of stochastic differential equations (SDEs) without the linear growth condition. In this paper, by using Peng’s G-expectation theory, we establish an even more general Khasminskii-type test for stochastic differential delay equations driven by G-Brownian motion (G-SDDEs) that cover a wide class of highly nonlinear G-SDDEs.

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1. Introduction

Stochastic differential delay equations (SDDEs) have been widely applied in many fields, such as neural networks, automatic control, economics, ecology, etc. Stability is one of the most important topics in the study of SDDEs. Mao and Shah (1997) studied both \( p \)th moment and almost sure exponential stability of the stochastic differential delay equation. Yuan and Mao (2004) investigated the almost surely asymptotic stability for the nonlinear stochastic differential delay equations with Markovian switching, and some sufficient criteria on the controllability and robust stability for linear SDDEs with Markovian switching. Recently, Fei, Hu, Mao, and Shen (2018) investigated a new Lyapunov function in order to deal with the effects of different structures as well as those of different parameters within the same modes. Moreover, the stochastic differential delay equations with Brownian motion is investigated by Mao and Yuan (2006), Mao (2008), Hu, Mao, and Yi (2013), Fei, Hu, Mao, and Shen (2017), Shen, Fei, Mao, and Yong (2018).

However, the classical stochastic differential equations with Brownian motion does not take an ambiguous factor into consideration. By using Peng’s theory of G-expectation (see Peng, 2010), the research of the probability model with ambiguity makes a significant progress. Under the G-framework, Peng (2007) introduced the G-Gaussian distribution, G-Brownian motion and related stochastic calculus of Itô type. Since then, more and more scholars studied the related problems under the G-framework. Denis, Hu, and Peng (2011) obtained some important properties of several typical Banach spaces of functions of G-Brownian motion paths induced by a sublinear expectation. Zhang and Chen (2012) investigated the sufficient conditions of the exponential stability and quasi sure exponential stability for stochastic differential equations driven by G-Brownian motion (G-SDEs). Fei and Fei (2013) investigated the exponential stability of paths for a class of stochastic differential equations disturbed by a G-Brownian motion in the sense of quasi surely (q.s.). Gao (2009) established the solutions of stochastic differential equations with Lipschitzian coefficients driven by G-Brownian motion. Li, Lin, and Lin (2016) studied the solvability and the stability of G-SDEs under Lyapunov-type conditions. Lin (2013) investigated the solvability of the scalar valued G-SDEs with reflecting boundary conditions.

Recently, many interesting works have been done on the G-SDEs and G-SDDEs (2019) explored the stability and boundedness of solutions to highly nonlinear G-SDEs. Ren, Jia, and Sakthivel (2016) discussed the \( p \)-th moment stability of solutions to impulsive G-SDEs. Moreover, some other important properties of G-SDEs have been investigated by many researchers (see Deng, Fei, & Mao, 2019; Faizullah, 2016; In Press; Fei, & Mao, 2018; Hu, Lin, & Hima, 2018; Li & Yang, 2018; Luo & Wang, 2014; Ren et al., 2016; Ren, Yin, & Sakthivel, 2018; Yin, Cao, & Ren, 2019; Yin & Ren, 2017). Mao (2002); Mao and Rassias (2005) established a Khasminskii-type test for SDDEs. Motivated by the above discussion, this paper will establish an even more general Khasminskii-type test
for G-SDDEs that cover a wide class of highly nonlinear G-SDDEs.

The paper is organized as follows. In Section 2, we introduce some preliminaries and notations on sublinear expectations and G-Brownian motions. In Section 3, we give the Khasminskii-type theorems for G-SDDEs. Next, we characterize the moment estimations in Section 4. Finally, the conclusion appears in Section 5.

2. Preliminaries

In this section, we recall some preliminary results of G-expectation, which are needed in the sequel. The reader interested in more detailed description of these notions is referred to Gao (2009), Peng (2010) and Denis et al. (2011).

Let $\Omega$ be a given nonempty set and $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$. We suppose that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$.

**Definition 2.1:** A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

1. Monotonicity: if $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$;
2. Constant preserving: $\hat{E}[c] = c$, for all $c \in \mathbb{R}$;
3. Sub-additivity: for each $X, Y \in \mathcal{H}$, $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
4. Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2:** Let $(\Omega, \mathcal{H}, \hat{E})$ be a sublinear expectation space, a random vector $Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}$, is said to be independent under $\hat{E}$ from another random vector $X = (X_1, \ldots, X_n), X_i \in \mathcal{H}$, if for each test function $\varphi \in C_{\text{Lip}}(\mathbb{R}^{m+n})$ we have

$$\hat{E}[\varphi(X,Y)] = \hat{E}[\varphi(X,Y)]_{X=x}.$$

**Definition 2.3:** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, and $X \in \mathcal{H}$ with

$$\tilde{\sigma}^2 = \hat{E}[X^2], \tilde{\sigma}^2 = \hat{E}[-X^2],$$

is said to be G-normal distribution, if for $a, b \geq 0$, we have $aX + b\tilde{X} \sim \sqrt{\tilde{\sigma}^2 + b^2} \tilde{X}$, for each $\tilde{X} \in \mathcal{H}$, which is independent to $X$ and $\tilde{X} \sim \sim$.

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths with $\omega_0 = 0$ equipped with the norm

$$\rho(\omega^1, \omega^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \max_{t \in [0,1]} |\omega^1_t - \omega^2_t| \wedge 1 \right).$$

**Definition 2.4:** A $d$-dimensional process $(B(t))_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a G-Brownian motion if the following properties are satisfied:

1. $B_0(\omega) = 0$;
2. for each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(0 \times s\Sigma)$-distributed and is independent from $(B(t_1), B(t_2), \ldots, B(t_n))$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

**Remark 2.5:** When $\tilde{\sigma}^2 = \sigma^2 = 1$, $(B(t))_{t \geq 0}$ is the classical Brownian motion.

For simplicity, let $(B(t))_{t \geq 0}$ be a $d$-dimensional G-Brownian motion with $G(a) = \frac{1}{2} \hat{E}[aB(1)^2] = \frac{1}{2}(\tilde{\sigma}^2a^+ - \sigma^2a^-)$, where $\hat{E}[B(1)^2] = \sigma^2, \hat{E}[B(1)^2] = \tilde{\sigma}^2, 0 < \sigma \leq \tilde{\sigma} < \infty$, where lower expectation $\hat{E}[X] = -\hat{E}[-X]$ for each $X \in \mathcal{H}$.

**Definition 2.6:** (1) Let $p \geq 1$ be fixed. Define the space $\mathcal{M}_G^{p,0}([0, T])$ of all processes by $\mathcal{M}_G^{p,0}([0, T]) = \{|\eta(t, \omega) = \sum_{k=0}^{N-1} \xi_k(\omega)l_{(t_k, t_{k+1})}(t)\;\xi_k \in L^p_G(\Omega_{t_k}), \; k = 0, 1, \ldots, N-1\}$.

2. We denote by $\mathcal{M}_G^{p,0}([0, T])$ the completion of $\mathcal{M}_G^{p,0}([0, T])$ under the norm

$$\|\eta\|_{\mathcal{M}_G^{p,0}([0, T])} := \left\{\hat{E}\left[\left(\int_0^T |\eta(t)|^p dt\right)^{1/p}\right]\right\} < \infty.$$

**Definition 2.7:** For each $\eta \in \mathcal{M}_G^{2,0}([0, T])$ of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)l_{(t_k, t_{k+1})}(t),$$

we define

$$l(\eta) = \int_0^T \eta_t dB_t := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}} - B_{t_k}).$$

For more properties of the Itô integral, one can see Peng (2010).

**Proposition 2.8:** There exists a weakly compact family of probability measures $P$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[X] = \max_{P \in \mathcal{P}} E_P[X], \quad \text{for all } X \in \mathcal{H},$$

where $E_P[\cdot]$ is the linear expectation with respect to $P$.

For this $P$, we define the associated G-upper capacity $\mathbb{V}(\cdot)$ and G-lower capacity $\mathbb{V}(\cdot)$ by:

$$\mathbb{V}(A) = \sup_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{B}(\Omega),$$

$$\mathbb{V}(A) = \inf_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{B}(\Omega).$$
Proposition 2.9 (Peng, 2010): For any $\zeta(t) \in M_0^2(0, T)$, we have
\[
\mathbb{E}\left[\int_0^T \zeta(t) \, dB(t) \right] = 0,
\]
\[
\mathbb{E}\left[\int_0^T |\zeta(t)|^2 \, dt \right] \leq \int_0^T \mathbb{E}\left[|\zeta(t)|^2 \, dt \right],
\]
\[
\mathbb{E}\left[\int_0^T \zeta(t)^2 \, dB(t) \right]^2 \leq \mathbb{E}\left[\int_0^T |\zeta(t)|^2 \, dt \right] \mathbb{E}\left[\int_0^T |\zeta(t)|^2 \, dB(t) \right] \leq \sigma^2\mathbb{E}\left[\int_0^T |\zeta(t)|^2 \, dt \right].
\]

3. The Khasminskii-type theorems for G-SDDES

Let $(B(t))_{t \geq 0}$ be an $n$-dimensional G-Brownian motion defined on a generalized filtered sublinear expectation space $(\Omega, \mathcal{H}, \{\mathcal{H}_t\}, \mathbb{E}, \mathbb{V})$, where $\mathcal{H}_t := \sigma(B(s); s \leq t)$. Let $\cdot$ be the Euclidean norm in $\mathbb{R}^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$, its trace norm is denoted by $|A| = \text{trace}(A^TA)$. Let $\tau > 0$ and $C([-\tau, 0], \mathbb{R}^n)$ denote the family of all bounded continuous $\mathbb{R}^n$-valued functions on $[-\tau, 0]$. Consider a nonlinear G-SDDE

\[
dX(t)
= f(X(t), X(t - \tau), t) \, dt + g(X(t), X(t - \tau), t) \, dB(t)
+ h(X(t), X(t - \tau), t) \, d\mathcal{B}(t), \quad t \geq 0
\]
with initial data
\[
|X(\theta) - \xi(\theta)| : -\tau \leq \theta \leq 0
= \xi \in C([-\tau, 0], \mathbb{R}^n),
\]
and $f, g, h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $f, g, h \in M_0^2(0, T)$.

Assumption 3.1: Assume that for any $b > 0$, there exists a positive constant $L_b$ such that
\[
|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 \vee |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \vee
|h(x_1, y_1, t) - h(x_2, y_2, t)|^2 \leq L_b(|x_1 - x_2|^2 + |y_1 - y_2|^2),
\]
for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq b$ and all $t \in [0, T]$.

We denote $C^{2,1}(\mathbb{R}^n \times [-\tau, \infty) ; \mathbb{R}_+)$ as the family of nonnegative functions $V(x, t)$ defined on $\mathbb{R}^n \times [-\tau, \infty)$,

$V_x, V_{xx}, V_t$ are continuous on $\mathbb{R}^n \times [-\tau, \infty)$, where
\[
V_x(x, t) = \left( \frac{\partial V(x, t)}{\partial x_1}, \ldots, \frac{\partial V(x, t)}{\partial x_n} \right),
V_{xx}(x, t) = \left( \frac{\partial^2 V(x, t)}{\partial x_i \partial x_j} \right)_{n \times n},
V_t(x, t) = \frac{\partial V(x, t)}{\partial t}.
\]
We define an operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by
\[
LV(x, y, t) = V_t(x, t) + \langle V_x(x, t), h(x, y, t) \rangle
+ \langle V_{xx}(x, t)g(x, y, t), g(x, y, t) \rangle,
\]
where
\[
\langle V_x(x, t), h(x, y, t) \rangle + \langle V_{xx}(x, t)g(x, y, t), g(x, y, t) \rangle
\]
is the symmetric matrix in $\mathcal{S}^d(\mathbb{R})$, with the form
\[
\langle \langle V_x(x, t), h(x, y, t) \rangle + \langle V_{xx}(x, t)g(x, y, t), g(x, y, t) \rangle \rangle
:= \{(V_x(x, t), h_{ij}(x, y, t) + h_{ij}(y, x, t)) \}
+ \langle V_{xx}(x, t)g(x, y, t), g(x, y, t) \rangle \}_{ij=1}^d.
\]
Let $X(t)$ be a solution of G-SDDE (1), for convention, we use the following notation in the sequel
\[
M^t_x := \int_s^t \left[ V_x(x(u), u), h_{ij}(x(u), x(u - \tau), u) \right]
+ \frac{1}{2} \langle V_{xx}(x(u), u)g(x(u), x(u - \tau), u) \rangle \, du
\]
\[
\times g(x(u), x(u - \tau), u) \rangle \, du
- \int_s^t G(\langle V_x(x(u), u), 2h(x(u), x(u - \tau), u) \rangle)
+ \langle V_{xx}(x(u), u)g(x(u), x(u - \tau), u) \rangle \times g(x(u), x(u - \tau), u) \rangle \, du.
\]
From Peng (2010), we can obtain $\{M^t_x\}_{t \geq s}$ is a G-martingale and $\mathbb{E}[M^T_x | \mathcal{F}_s] = 0$.

Assumption 3.2: There are two functions $V \in C^{2,1}(\mathbb{R}^n \times [-\tau, \infty) ; \mathbb{R}_+) \text{ and } U \in C(\mathbb{R}^n \times [-\tau, \infty) ; \mathbb{R}_+) \text{, such that}$
\[
\lim_{|x| \to \infty} \inf_{0 \leq t < \infty} V(x, t) = \infty,
\]
and
\[
LV(x, y, t) \leq \theta_1 (1 + V(x, t) + V(y, t - \tau) + U(y, t - \tau))
- \theta_2 U(x, t)
\]
for each $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, where $\theta_1$ and $\theta_2$ are positive constants.
Theorem 3.3: Let Assumptions 3.1 and 3.2 hold. Then there is a unique global solution \( x(t) \) to (1) on \( t \in [-\tau, \infty) \) for any given initial data (2). Moreover, the solution has the properties that

\[
\hat{E}V(x(t), t) < \infty \quad \text{and} \quad \mathcal{E} \int_0^t U(x(s), s) \, ds < \infty \tag{6}
\]

for all \( t \in \mathbb{R}_+ \).

**Proof:** Given any initial data \( \{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n) \), there is a unique maximal local solution \( x(t) \) on \( t \in [-\tau, \tau_e) \), where \( \tau_e \) is the explosion time. Let \( m_0 > 0 \),

\[
\frac{1}{m_0} < \min_{-\tau \leq t \leq 0} |x(t)| \leq \max_{-\tau \leq t \leq 0} |x(t)| < m_0.
\]

For any integer \( m \geq m_0 \), we define the stopping time

\[
\tau_m = \inf \{t \in [0, \tau_e) : |x(t)| \notin (1/m, m)\},
\]

where we set \( \inf \emptyset = \infty \) in the paper. Obviously, \( \tau_m \) is increasing as \( m \to \infty \). Set \( \tau_\infty = \lim_{m \to \infty} \tau_m \), whence \( \tau_\infty \leq \tau_e \) a.s. If we can show that \( \tau_\infty = \infty \) a.s., then \( \tau_e = \infty \) a.s.

In the following, we will show \( \tau_\infty = \infty \) a.s. and assertion (6). We calculate by the \( G \)-Itô formula and condition (5) that

\[
dV(x(t), t) = L V(x(t), x(t - \tau), t) \, dt \\
+ \langle Vx(x(t), t), g(x(t), x(t - \tau), t) \rangle \, dB(t) \\
\leq (\theta_1 [1 + V(x(t), t) + V(x(t - \tau), t - \tau)] \\
+ U(x(t - \tau), t - \tau)) - \theta_2 U(x(t), t) \, dt \\
+ \langle Vx(x(t), t), g(x(t), x(t - \tau), t) \rangle \, dB(t) \tag{7}
\]

for \( t \in [0, \tau_\infty) \). For each \( m \geq m_0 \) and \( t_1 \in [0, \tau] \), we can integrate both sides of (7) from 0 to \( \tau_m \wedge t_1 \) and then take the \( G \)-expectations to derive

\[
\hat{E}V(x(\tau_m \wedge t_1), \tau_m \wedge t_1) - V(x(0), 0) \\
\leq \hat{E} \int_0^{\tau_m \wedge t_1} (\theta_1 [1 + V(x(t), t) + V(x(t - \tau), t - \tau)] \\
+ U(x(t - \tau), t - \tau)) - \theta_2 U(x(t), t) \, dt. 
\]

This implies

\[
\hat{E}V(x(\tau_m \wedge t_1), \tau_m \wedge t_1) \leq V(x(0), 0) \\
+ \hat{E} \int_0^{\tau_m \wedge t_1} \theta_1 [1 + V(x(t), t) + V(x(t - \tau), t - \tau)] \\
+ U(x(t - \tau), t - \tau)) \, dt \\
+ \theta_1 \hat{E} \int_0^{\tau_m \wedge t_1} V(x(t), t) \, dt 
\]

for \( \tau \in [0, \tau_\infty) \). Setting \( \hat{E} \leq \hat{E} \int_0^{\tau_m \wedge t_1} (\theta_1 [1 + V(x(t), t) + V(x(t - \tau), t - \tau)] \\
+ U(x(t - \tau), t - \tau)) - \theta_2 U(x(t), t) \, dt. 
\]

By the Proposition 2.8 and (10), we have

\[
\mu_m V(\tau_m \leq \tau) \leq C_1 e^{\theta_1 \tau}. 
\]

By condition (4), \( \lim_{m \to \infty} \mu_m = \infty \). Letting \( m \to \infty \) in the above inequality, we can obtain that \( V(\tau_\infty \leq \tau) = 0 \), namely

\[
\hat{E}V(x(t_1), t_1) \leq C_1 e^{\theta_1 \tau}, \quad 0 \leq t_1 \leq \tau. \tag{12}
\]

Moreover, letting \( t_1 = \tau \) in (8) then yields that

\[
\hat{E} \int_0^{\tau_\infty} U(x(t), t) \, dt \leq C_1 + \theta_1 \hat{E} \int_0^{\tau_\infty} V(x(t), t) \, dt. 
\]

Letting \( m \to \infty \) and then making use of (11) and (12), we have that

\[
\hat{E} \int_0^{\tau_\infty} U(x(t), t) \, dt \leq \frac{1}{\theta_2} (C_1 + \tau \theta_1 C_1 e^{\theta_1 \tau}) < \infty. \tag{13}
\]

Now we turn to the proof of \( \tau_\infty > 2\tau \) a.s. given that we have shown (11)–(13). For each \( m \geq m_0 \) and \( t_1 \in [0, 2\tau] \),
applying the G-Itô formula to (7) and then taking the G-expectations, we get

\[
\hat{\mathbb{E}}V(x(t_m \wedge t_1), t_m \wedge t_1) 
\leq C_2 + \theta_1 \hat{\mathbb{E}} \int_{0}^{t_m \wedge t_1} V(x(t), t) \, dt 
+ \hat{\mathbb{E}} \left(-\theta_2 \int_{0}^{t_m \wedge t_1} U(x(t), t) \, dt\right),
\]

where

\[
C_2 = V(x(0), 0) 
+ \hat{\mathbb{E}} \int_{0}^{2\tau} \theta_1 (1 + V(x(t - \tau), t - \tau)) \, dt 
+ U(x(t - \tau), t - \tau) \, dt
\]

by (12) and (13). Consequently,

\[
\hat{\mathbb{E}}V(x(t_m \wedge t_1), t_m \wedge t_1) 
\leq C_2 e^{2\theta_1 \tau}, \quad 0 \leq t_1 \leq 2\tau, \quad m \geq m_0.
\]

In particular,

\[
\hat{\mathbb{E}}V(x(t_m \wedge 2\tau), t_m \wedge 2\tau) \leq C_2 e^{2\theta_1 \tau}, \quad \forall \ m \geq m_0.
\]

This implies

\[
\mu_m V(t_m \leq 2\tau) \leq C_2 e^{2\theta_1 \tau}.
\]

Letting \( m \to \infty \), we then obtain that \( V(t_m \leq 2\tau) = 0 \), namely,

\[
V(t_m \geq 2\tau) = 1.
\]

Letting \( m \to \infty \) in (15) yields

\[
\hat{\mathbb{E}}V(x(t_1), t_1) \leq C_2 e^{2\theta_1 \tau}, \quad 0 \leq t_1 \leq 2\tau.
\]

Moreover, setting \( t_1 = 2\tau \) in (14), we have

\[
-\hat{\mathbb{E}} \left(-\theta_2 \int_{0}^{t_m \wedge 2\tau} U(x(t), t) \, dt\right) 
\leq C_2 + \theta_1 \hat{\mathbb{E}} \int_{0}^{t_m \wedge 2\tau} V(x(t), t) \, dt.
\]

Letting \( m \to \infty \) and then making use of (16) and (17), we have that

\[
\mathcal{E} \int_{0}^{2\tau} U(x(t), t) \, dt \leq \frac{1}{\theta_2} (C_2 + 2\theta_1 \tau C_2 e^{2\theta_1 \tau}) < \infty.
\]

Repeating this procedure, we can show that, for any integer \( i \geq 1, \tau_\infty \geq i\tau \) a.s. and

\[
\hat{\mathbb{E}}V(x(t), t) \leq C_i e^{i\theta_1 \tau}, \quad 0 \leq t \leq i\tau,
\]

and

\[
\mathcal{E} \int_{0}^{i\tau} U(x(t), t) \, dt \leq \frac{1}{\theta_2} (C_i + i\theta_1 \tau C_i e^{i\theta_1 \tau}),
\]

where

\[
C_i = V(x(0), 0) 
+ \hat{\mathbb{E}} \int_{0}^{\tau} \theta_1 (1 + V(x(t), t) + U(x(t), t)) \, dt
\]

We must therefore have \( \tau_\infty = \infty \) a.s. as well as the required assertion (6).

**Example 3.4:** For simplicity, setting \( h(X(t), X(t - \tau), t) = 0 \), consider a one-dimensional G-SDDE

\[
dX(t) = \left[ a(t)X^2(t - \tau) - X^3(t) \right] \, dt + b(t)X^2(t - \tau) \, dB(t)
\]

where \( B(t) \) is a one-dimensional Brownian motion and both \( a(t) \) and \( b(t) \) are bounded real-valued functions on \( t \geq 0 \). Let \( V(x, t) = X^4 \). Then the corresponding operator \( LV : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \) has the form

\[
LV(x, y, t) = 4X^3 (a(t)Y^2 - X^3) + 6b^2(t)X^2Y^4.
\]

Similar to Mao and Rassias (2005), there is a unique global solution \( X(t) \) to Equation (19) on \( t \in [-\tau, \infty) \). Moreover, the solution has the properties that

\[
\hat{\mathbb{E}}|X(t)|^4 < \infty \quad \text{and} \quad \mathcal{E} \int_{0}^{t} |X(s)|^6 \, ds < \infty
\]

for any \( t \geq 0 \).

**4. Moment estimations**

In the previous section, under Assumptions 3.1 and 3.2, we have not only obtained the existence and uniqueness theorem on the global solution for the G-SDDE, but also showed that the solution has the properties that

\[
\hat{\mathbb{E}}V(x(t), t) < \infty \quad \text{and} \quad \mathcal{E} \int_{0}^{t} U(x(s), s) \, ds < \infty, \quad \forall \ t \geq 0.
\]

However, these estimations are not precise enough. Thus, we will replace condition (5) with many specified conditions.
Theorem 4.1: Let Assumptions 3.1 and 3.2 hold except (5), which is replaced by
\[
LV(x, y, t) \leq \gamma_1 (1 + V(x, t) + V(y, t - \tau))
- \gamma_2 (U(x, t) - U(y, t - \tau)),
\]
for all \((x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+\), where \(\gamma_1 \geq 0\) and \(\gamma_2 > 0\). Then for any given initial data (2), the unique global solution \(x(t)\) to the G-SDDEs (1) has the properties that
\[
\hat{E}V(x(t), t) \leq (H_1 + \gamma_1 t) e^{2\gamma_1 t}, \quad \forall t \geq 0 \tag{21}
\]
and for \(\forall t \geq \tau\),
\[
\mathcal{E} \int_{t-\tau}^{t} U(x(s), s) \, ds \leq \frac{1}{\gamma_2} (H_1 + \gamma_1 t + (H_1 + \gamma_1 t) e^{2\gamma_1 t}), \tag{22}
\]
where
\[
H_1 = V(x(0), 0) + \int_{-\tau}^{0} (\gamma_1 V(x(s), s) + \gamma_2 U(x(s), s)) \, ds.
\]
Proof: Obviously, condition (20) implies condition (5) if we set \(\theta_1 = \gamma_1 \wedge \gamma_2\) and \(\theta_2 = \gamma_2\). So for any given initial data (2), Equation (1) has a unique global solution \(x(t)\) on \(t \geq -\tau\), which has the properties (6). By these properties, we can use the G-Itô formula and condition (20) to obtain that for any \(t \geq 0\),
\[
\hat{E}V(x(t), t) \leq V(x(0), 0)
+ \gamma_1 \hat{E} \int_{0}^{t} (1 + V(x(s), s) + V(x(s - \tau), s - \tau)) \, ds
+ \hat{E} \int_{0}^{t} (-\gamma_2 (U(x(s), s) - U(x(s - \tau), s - \tau))) \, ds.
\]
Noting
\[
\int_{0}^{t} V(x(s), s) \, ds = \int_{-\tau}^{0} V(x(s), s) \, ds
\]
and similarly for \(\int_{0}^{t} U(x(s), s) \, ds\), this implies
\[
\hat{E}V(x(t), t) \leq H_1 + \gamma_1 t
+ 2\gamma_1 \hat{E} \int_{0}^{t} V(x(s), s) \, ds
+ \gamma_2 \hat{E} \int_{0}^{t} (-U(x(s), s)) \, ds,
\]
which is assertion (21). Moreover, if \(t \geq \tau\), we obtain from (23) that
\[
\gamma_2 \mathcal{E} \int_{t-\tau}^{t} U(x(s), s) \, ds \leq H_1 + \gamma_1 t + 2\gamma_1 \int_{0}^{t} \hat{E}V(x(s), s) \, ds
\leq H_1 + \gamma_1 t + (H_1 + \gamma_1 t) e^{2\gamma_1 t}
\]
and hence the required assertion (21) follows.

Theorem 4.2: Let Assumptions 3.1 and 3.2 hold except (5), which is replaced by
\[
LV(x, y, t) \leq \gamma_1 (1 + V(x, t) + V(y, t - \tau))
- \gamma_2 U(x, t) + \gamma_3 U(y, t - \tau), \tag{24}
\]
for all \((x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+\), where \(\gamma_1 \geq 0\) and \(\gamma_2 > \gamma_3 \geq 0\). Then for any given initial data (2), the unique global solution \(x(t)\) to the G-SDDEs (1) has property (21) and for \(\forall t \geq 0\),
\[
\mathcal{E} \int_{0}^{t} U(x(s), s) \, ds
\leq \frac{1}{\gamma_2 - \gamma_3} (H_1 + \gamma_1 t + (H_1 + \gamma_1 t) e^{2\gamma_1 t}), \tag{25}
\]
where \(H_1\) is the same as defined in Theorem 4.1.

Proof: Condition (24) implies condition (20) when \(\gamma_2 > \gamma_3\), so assertion (21) follows from Theorem 4.1. To show (25), by the G-Itô formula and condition (24), we have
\[
\hat{E}V(x(t), t) \leq V(x(0), 0)
+ \gamma_1 \hat{E} \int_{0}^{t} (1 + V(x(s), s) + V(x(s - \tau), s - \tau)) \, ds
+ \hat{E} \int_{0}^{t} (-\gamma_2 U(x(s), s)) \, ds
+ \gamma_3 \hat{E} \int_{0}^{t} U(x(s - \tau), s - \tau) \, ds
\leq H_1 + \gamma_1 t + 2\gamma_1 \int_{0}^{t} \hat{E}V(x(s), s) \, ds
+ (\gamma_2 - \gamma_3) \hat{E} \int_{0}^{t} (-U(x(s), s)) \, ds.
\]
This, together with (21), yields
\[
- (\gamma_2 - \gamma_3) \hat{E} \int_{0}^{t} (-U(x(s), s)) \, ds
\leq H_1 + \gamma_1 t + 2\gamma_1 \int_{0}^{t} (H_1 + \gamma_1 s) e^{2\gamma_1 s} \, ds
\leq H_1 + \gamma_1 t + (H_1 + \gamma_1 t) e^{2\gamma_1 t}
\]
and the required assertion (25) follows.
Theorem 4.3: Let Assumptions 3.1 and 3.2 hold except (5), which is replaced by
\[
LV(x,y,t) \leq \gamma_1 - \gamma_2 (V(x,t) - V(y,t - \tau))
- \gamma_3 U(x,t) + \gamma_4 U(y,t - \tau),
\]
for all \((x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+\), where \(\gamma_1, \gamma_2 \geq 0\) and \(\gamma_3 > \gamma_4 \geq 0\). Then for any given initial data (2), the unique global solution \(x(t)\) to the G-SDDEs (1) has properties that
\[
\limsup_{t \to \infty} \frac{\mathbb{E} V(x(t),t)}{t} \leq \gamma_1
\]  
and
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t U(x(s),s) \, ds \leq \frac{\gamma_1}{\gamma_3 - \gamma_4}.
\]

Proof: For any \(t \geq 0\), by the G-Itô formula and condition (26), we easily derive
\[
\begin{align*}
\mathbb{E} V(x(t),t) & \leq V(x(0),0) \\
 & + \mathbb{E} \int_0^t (\gamma_1 - \gamma_2 (V(x(s),s) - V(x(s - \tau),s - \tau))) \, ds \\
 & + \gamma_3 \mathbb{E} \int_0^t (-U(x(s),s)) \, ds \\
 & + \gamma_4 \mathbb{E} \int_0^t U(x(s - \tau),s - \tau) \, ds \\
& \leq H_2 + \gamma_1 t + (\gamma_3 - \gamma_4) \mathbb{E} \int_0^t (-U(x(s),s)) \, ds.
\end{align*}
\]
where
\[H_2 = V(x(0),0) + \int_{-\tau}^0 (\gamma_2 V(x(s),s) \, ds + \gamma_4 U(x(s),s)) \, ds.
\]
From this, the desired assertion (27) and (28) follow.

5. Conclusion
The Khasminskii theorem have come to play an important role in the nonexplosion solutions of SDEs without the linear growth condition. In this paper, by using Peng’s theory of sublinear expectations, we have established an even more general Khasminskii-type test for G-SDDEs that cover a wide class of highly nonlinear G-SDDEs, this test can be applied to many important G-SDDEs.

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