Organizing boundary RG flows

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Abstract

We show how a large class of boundary RG flows in two-dimensional conformal field theories can be summarized in a single rule. This rule is a generalization of the 'absorption of the boundary spin'-principle of Affleck and Ludwig and applies to all theories which have a description as a coset model. We give a formulation for coset models with arbitrary modular invariant partition function and present evidence for the conjectured rule. The second half of the article contains an illustrated section of examples where the rule is applied to unitary minimal models of the A- and D-series, in particular the 3-state Potts model, and to parafermion theories. We demonstrate how the rule can be used to compute brane charge groups in the example of $N = 2$ minimal models.

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1 Introduction

The study of renormalization group (RG) flows in two-dimensional quantum field theories is an important subject in condensed matter physics and statistical mechanics, and it also plays a vital role in string theory. In systems with boundaries or defects, there are flows generated by boundary fields which only affect the boundary condition and leave the theory in the bulk unchanged. In string theory, such flows describe the dynamics of D-branes in a given closed string background.

How do we find boundary RG flows? For a given boundary perturbation of a boundary conformal field theory (BCFT), we have various tools at our disposal. In some cases when we perturb by a field which is only ‘slightly’ relevant, we can apply perturbation theory [1]. If the perturbation is integrable, we may use exact integral equation techniques (like the Thermodynamic Bethe Ansatz). A method which can always be employed is the Truncated Conformal Space Approach where we truncate the Hilbert space to a finite-dimensional space and compute RG flows numerically.

All these tools have helped to get a substantial knowledge about boundary RG flows. To organize the informations, we need general, model-independent principles. One such principle is the ‘g-conjecture’ of Affleck and Ludwig [1] which states that the boundary entropy $g$ always decreases along a RG flow. Although very important, the ‘g-conjecture’ is not a constructive principle: it only tells us which flows are possible and which are not.

In the case of WZNW models, we have a constructive principle at hand, namely the ‘absorption of the boundary spin’-principle of Affleck and Ludwig [2]. This rule is easy to formulate and describes a large class of flows.

A generalization of this rule to fixed-point free coset models was proposed in [3]. The formulation there was for coset models with a charge-conjugated modular invariant partition function and boundary conditions of Cardy type. Here, we shall present a formulation that is applicable for all maximally symmetric boundary conditions in coset models with any modular invariant. Furthermore we shall work out some arguments supporting the proposal, and employ the rule in a number of examples.

The structure of the paper is as follows: We start with an introduction to coset models and their maximally symmetric boundary conditions in section 2. Although this is essentially a review of [1] [5], we hope to clarify the role of the modular in-
variant when we relate coset boundary conditions to those of WZNW models. In section 3, we formulate the ‘absorption of boundary spin’-principle and its generalization to coset models and discuss the relation to perturbative calculations and the compatibility with the g-conjecture. Through a number of examples, we present the rule at work in section 4. We shall make extensive use of a geometric interpretation of the boundary conditions as ‘branes’ to visualize the RG flows. Whenever we are aware of results on boundary RG flows in the specific examples, we compare them to the predictions of the rule. At the end of section 4 the rule is used to determine the charge group of branes in $N = 2$ minimal models. In the appendix we collect the complete results for the critical and tricritical Ising model as well as the 3-state Potts model.

2 Boundary Conditions in coset models

The coset construction \[\tilde{g}/\tilde{h}\] allows to access a great variety of rational conformal field theories. Boundary conditions in these models have been investigated in the past. Most of the work was concentrated on maximally symmetric boundary conditions, i.e. those where the boundary conformal field theory admits the action of the coset chiral algebra $\hat{g}/\hat{h}$. Untwisted boundary conditions in coset models with charge-conjugated modular invariant partition function are already covered by the seminal paper of Cardy on boundary conditions in rational CFTs [7]. The generalization to twisted boundary conditions and more general modular invariants has been worked out in [4, 5]. Symmetry breaking boundary conditions in coset models have been first considered in [8, 9] relying on previous work in WZNW models [10, 11]. In the $\sigma$-model approach, boundary conditions in gauged WZNW-models have been studied in [12, 13, 14]. Recently there has been also some work on boundary conditions in asymmetric cosets [15, 16, 9].

We give a short introduction to coset models to set up our notation. Subsequently, we discuss some general properties of maximally symmetric boundary conditions. The section ends with a review on the construction of boundary states from known boundary conditions in the product theory with chiral algebra $\hat{g} \oplus \hat{h}$. 
2.1 Coset construction

From now on let \( \mathfrak{h} \subset \mathfrak{g} \) denote some simple subalgebra of the simple Lie algebra \( \mathfrak{g} \) (the generalization to semi-simple Lie algebras is straightforward). We want to study the associated \( \widehat{\mathfrak{g}}/\widehat{\mathfrak{h}} \) coset model. A more precise formulation of this theory requires a bit of preparation (more details can be found e.g. in [17]).

Induced from the embedding of \( \mathfrak{h} \) in \( \mathfrak{g} \), there is an embedding of the affine Lie algebra \( \widehat{\mathfrak{h}}_{k'} \) into \( \widehat{\mathfrak{g}}_k \). The level \( k' \) is related to \( k \) by the embedding index \( x_e \), \( k' = k x_e \). We shall label the sectors \( \mathcal{H}_{k'}^l \) of the affine Lie algebra \( \widehat{\mathfrak{h}}_{k'} \) with labels \( l' \in \text{Rep}(\widehat{\mathfrak{h}}_{k'}) \). Note that the sectors of the numerator theory carry an action of the denominator algebra \( \widehat{\mathfrak{h}}_{k'} \subset \widehat{\mathfrak{g}}_k \) and under this action each sector \( \mathcal{H}_{k'}^l \) decomposes according to

\[
\mathcal{H}_{k'}^l = \bigoplus_{l'} \mathcal{H}^{(l,l')} \otimes \mathcal{H}_{k'}^{l'}.
\]

Here we have introduced the infinite dimensional spaces \( \mathcal{H}^{(l,l')} \) which we want to interpret as sectors of the coset chiral algebra. The latter is usually hard to describe explicitly, but at least it is known to contain a Virasoro field with modes

\[
L_n = L_n^0 - L_n^b.
\]

One may easily check that they obey the usual exchange relations of the Virasoro algebra with central charge given by \( c = c^a - c^b \).

Note that some of the spaces \( \mathcal{H}^{(l,l')} \) may be trivial simply because a given sector \( \mathcal{H}_{k'}^{l'} \) of the denominator theory may not appear as a subsector in a given \( \mathcal{H}_{k'}^l \). This allows to introduce the set

\[
\mathcal{E} = \{ (l, l') \in \text{Rep}(\widehat{\mathfrak{g}}_k) \times \text{Rep}(\widehat{\mathfrak{h}}_{k'}) \mid \mathcal{H}^{(l,l')} \neq 0 \}.
\]

Furthermore, some of the coset spaces labeled by different pairs \((l, l')\) and \((m, m')\) correspond to the same sector of the coset theory. Therefore we label coset sectors by equivalence classes \([l, l']\) of pairs.

There is an elegant formalism to describe these selection and identification rules which is applicable in almost all coset models\(^1\). It involves the so-called identification

\(^1\)The known exceptions all appear at low levels of the involved affine Lie algebras, see e.g. the Maverick cosets [18].
group \( \mathcal{G}_{\text{id}} \) which contains pairs \((\mathcal{J}, \mathcal{J}')\) of simple currents. It is a subgroup of the direct product of the simple current groups of \( \hat{\mathfrak{g}}_k \) and \( \hat{\mathfrak{h}}_{k'} \). A simple current \( \mathcal{J} \) of \( \hat{\mathfrak{g}}_k \) is an element in \( \text{Rep}(\hat{\mathfrak{g}}_k) \) which has the property that the fusion product of \( \mathcal{J} \) with any other representation \( l \) contains exactly one sector \( m =: \mathcal{J}l \in \text{Rep}(\hat{\mathfrak{g}}_k) \),

\[
N_{\mathcal{J}l}^m = \delta_{m, \mathcal{J}l} .
\]

To formulate the selection rules in coset models, we introduce the monodromy charge \( Q_{\mathcal{J}}(l) \) of \( l \) with respect to \( \mathcal{J} \) in terms of conformal weights,

\[
Q_{\mathcal{J}}(l) = h_\mathcal{J} + h_l - h_{\mathcal{J}l} \mod \mathbb{Z} .
\]

The monodromy charge appears when a simple current \( \mathcal{J} \) acts on the modular \( S \)-matrix,

\[
S_{\mathcal{J}l m} = e^{2\pi i Q_{\mathcal{J}}(m)} S_{lm} .
\]

We are now prepared to formulate selection and identification rules in terms of the identification group \( \mathcal{G}_{\text{id}} \) of simple currents:

- A pair \((l, l')\) is allowed, i.e. \((l, l') \in \mathcal{E}\), if \( Q_{\mathcal{J}}(l) = Q_{\mathcal{J}'}(l') \) for all \((\mathcal{J}, \mathcal{J}') \in \mathcal{G}_{\text{id}}\)

- Two pairs \((l, l')\) and \((\mathcal{J}l, \mathcal{J}'l')\) label the same sector, i.e.

\[
\mathcal{H}^{(l,l')} \cong \mathcal{H}^{(\mathcal{J}l, \mathcal{J}'l')} .
\]

At this point we want to make one assumption, namely that all the equivalence classes we find in \( \mathcal{E} \) contain the same number \( N_0 = |\mathcal{G}_{\text{id}}| \) of elements, in other words, \( \mathcal{G}_{\text{id}} \) acts fixed-point free. This holds true for many important examples and it guarantees that the sectors of the coset theory are simply labeled by the equivalence classes\(^2\), i.e. \( \text{Rep}(\hat{\mathfrak{g}}/\hat{\mathfrak{h}}) = \mathcal{E}/\mathcal{G}_{\text{id}} \). It is then also easy to spell out explicit formulas for the fusion rules and the \( S \)-matrix of the coset model. These are given by

\[
N_{[j,j'][k,k']}^{[l,l']} = \sum_{(m,m') \sim (l,l')} N_j^g m N_{j'k'}^h m' ,
\]

\[
S_{[l,l'][m,m']} = N_0 S_{lm}^g S_{l'm'}^h .
\]

where the bar over the second \( S \)-matrix denotes complex conjugation.

\(^2\)For more general cases, there are further sectors that cannot be constructed within the sectors of the numerator theory.
2.2 Maximally symmetric boundary conditions

Let us turn now to coset models with a boundary and summarize some general properties of maximally symmetric boundary conditions to be prepared for the concrete analysis in section 2.3.

We want to impose conditions along the boundary gluing left moving and right moving fields together with a suitable automorphism $\omega$ of the coset chiral algebra. The corresponding set of elementary boundary conditions is denoted by $B^\omega_{g/h}$.

Because of the specific gluing conditions, the annulus partition function involving the boundary conditions $\alpha, \beta \in B^\omega_{g/h}$ decomposes into coset characters,

$$Z_{\alpha \beta}(q) = \sum_{[l,l']} n_{[l,l']}^{\alpha \beta} \chi^{(l,l')}(q)$$

with non-negative integers $n_{[l,l']}^{\alpha \beta}$. For a complete set of boundary conditions $B^\omega_{g/h}$, these numbers are known to form a representation of the fusion algebra $[19, 20]$,

$$\sum_{\beta} n_{[l,l']}^{\alpha \beta} n_{[j,j']}^{\beta \gamma} = \sum_{[k,k']} N_{[l,l'][j,j'][k,k']} n_{[k,k']}^{\alpha \gamma}.$$ 

The integers $n$ have the further properties

$$n_{[0,0]^{\alpha \beta}} = \delta_{\alpha \beta}$$

and

$$n_{[l,l']^{\alpha \beta}} = n_{[l',l]^{\beta \alpha}}$$

where $l^+$ labels the representation conjugate to $l$.

2.3 Boundary conditions from WZNW models

In the last subsection we have been rather general. It is possible to relate the analysis of boundary conditions in coset models to the investigation of boundary conditions in the product theory with chiral algebra $\hat{g} \oplus \hat{h} [21, 4, 5]$. Before we enter the detailed description, let us sketch our general procedure: we specify a modular-invariant partition function of the coset model and from that we construct a partition function for the product theory. In the resulting theory we impose gluing conditions involving a gluing automorphism $\omega^g \times (\omega^h)^{-1}$ where we assume that $\omega^g$ restricts to
\( h \) and \( \omega^h = \omega^0|_h \). The corresponding boundary conditions of the product theory can be projected to boundary conditions in the coset model by certain selection rules.

\[
\begin{array}{|c|c|c|}
\hline
\text{partition function } Z^{g/h} & \rightarrow & \text{partition function } Z^{g\oplus h} \\
\hline
\text{gluing automorphism} & \omega^g \times (\omega^h)^{-1} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{set of boundary conditions } \mathcal{B}^\omega_{g/h} \\
\downarrow \text{selection rules} \\
\text{set of boundary conditions } \mathcal{B}^\omega_{g\oplus h} \\
\end{array}
\]

Let us become more specific. We start with a coset model with the partition function

\[
Z^{g/h}(q, \bar{q}) = \sum_{[l,l'],[m,m']} Z_{[l,l'],[m,m']} \chi^{[l,l']}(q) \chi^{[m,m']}(\bar{q})
\]

with some non-negative integers \( Z_{[l,l'],[m,m']} \). To this model we associate a product theory \( g \oplus h \) with partition function (following [22])

\[
Z^{g\oplus h}(q, \bar{q}) = \sum_{l,l',m,m'} Z_{[l,l'],[m,m']} \chi^l(q) \chi^{m'}(\bar{q}) \chi^{m}(\bar{q}) \chi^{l'}(\bar{q}) .
\]

Note the exchange of the \( h \)-labels \( l' \) and \( m' \) which is necessary to guarantee modular invariance of the product theory.

In this theory we want to analyze maximally symmetric boundary conditions. To this end, we glue the left- and right-moving currents \( J(z), \bar{J}(\bar{z}) \) of the \( \hat{g} \) and \( \hat{h} \) theory with a gluing automorphism \( \Omega \) induced from \( \omega^g \times (\omega^h)^{-1} \) along the boundary,

\[
J(z) = \Omega(\bar{J})(\bar{z}) \quad \text{for } z = \bar{z} .
\]

Assume now that we have solved the problem of finding all maximally symmetric boundary conditions in the theory, i.e. we have a set of boundary conditions \( \alpha \in \mathcal{B}^\omega_{g\oplus h} \) specified by the boundary couplings \( \psi^{(l,l';\lambda)}_{\alpha} \). The corresponding boundary state is

\[
|\alpha\rangle = \sum_{(l,l';\lambda)} \frac{\psi^{(l,l';\lambda)}_\alpha}{\sqrt{S_{l_0}S_{l'_0}}} \langle (l,l';\lambda)\rangle .
\]
Here, \((l, l'; \lambda)\) labels an Ishibashi state in the sector \((l, l')\) and \(\lambda\) is an additional multiplicity index in the range
\[
\lambda = 1, \ldots, Z_{[l, \omega^{-1}(l')]|\omega(l'), l']}
\]

At this point we want to make an important assumption. We assume that we can find a basis of Ishibashi states s.t. the action of a simple current on the boundary couplings \(\psi\) is given by a pure phase factor which only depends on \(\alpha\),
\[
\psi_{\alpha}^{(\mathcal{J}, \omega(\mathcal{J}'))|\lambda} = e^{2\pi i Q_{(\mathcal{J}, \omega(\mathcal{J}'))}(\alpha)} \psi_{\alpha}^{(l, l'; \lambda)} \quad \text{for } (\mathcal{J}, \mathcal{J}') \in \mathcal{G}_{\text{id}}
\]
This is certainly true in all examples that we considered, but it is unclear whether this assumption holds in general (this problem has already been mentioned in [4, 5]).

We are now prepared to write down a set of boundary conditions for the coset model. For any \(\alpha \in B_{\omega g \oplus h}\) satisfying the selection rule
\[
Q_{(\mathcal{J}, \omega(\mathcal{J}'))}(\alpha) = 0 \quad \text{for all } (\mathcal{J}, \mathcal{J}') \in \mathcal{G}_{\text{id}}, \quad (5)
\]
we define a boundary condition in the coset model which we also label by \(\alpha\),
\[
\psi_{\alpha}^{(l, l'; \lambda)} := \sqrt{N_{0}} \psi_{\alpha}^{(l, \omega(l'); \lambda)} \quad . \quad (6)
\]
The Ishibashi states of the coset model are labeled by equivalence classes of pairs \([l, l']\) together with a multiplicity index \(\lambda\) running from 1 to \(Z_{[l, \omega(l')|\omega(l'), l']})\). Note that the multiplicity of the coset Ishibashi state \([l, l']\) and the product Ishibashi state \((l, \omega(l'))\) coincide so that we can use the same label \(\lambda\) on both sides of eq. (6).

It is straightforward to verify that the \(\psi_{\alpha}^{(l, l'; \lambda)}\) fulfill the completeness conditions
\[
\sum_{\alpha} \psi_{\alpha}^{(l, l'; \lambda)} \overline{\psi_{\alpha}^{(m, m'; \mu}}} = \delta_{[l, l'], [m, m']} \delta_{\lambda, \mu} \quad . \quad (7)
\]
and
\[
\sum_{(l, l'; \lambda)} \psi_{\alpha}^{(l, l'; \lambda)} \overline{\psi_{\beta}^{(l, l'; \lambda)}} = \delta_{\alpha, \beta} \quad . \quad (8)
\]
Furthermore, Cardy’s condition which says that the annulus coefficients
\[
M_{[l, l']|\alpha}^{[0, 0]} \beta = \sum_{(j, j'; \lambda)} \psi_{\alpha}^{(j, j'; \lambda)} \overline{\psi_{\beta}^{(j, j'; \lambda)}} S_{[l, l']|j, j'}^{[0, 0]} S_{[j, j']|0, 0}^{[l, l']}
\]
are non-negative integers, is satisfied, and
\[
M_{[l, l']|\alpha}^{[0, 0]} \beta = M_{(l, \omega(l'))|\alpha}^{[0, 0]} \beta \quad . \quad (9)
\]
2.4 The factorizing case

In this section we want to deal with the situation of a ‘factorizing’ modular invariant partition function in the coset theory. We should state more clearly what we mean by ‘factorizing’, namely that the associated modular invariant of the product theory is a \( \tilde{\mathcal{G}}_{id} \) simple current orbifold of a direct product of a \( \hat{\mathfrak{g}} \) and a \( \hat{\mathfrak{h}} \) modular invariant,

\[
Z^{\mathfrak{g} \oplus \mathfrak{h}} = \sum_{(J,J') \in \tilde{\mathcal{G}}_{id}} \sum_{l,l',m,m'} Z^\mathfrak{g}_{lJm} Z^\mathfrak{h}_{l'J'm'} \chi^l(q) \chi^{l'}(\bar{q}) \chi^m(q) \chi^{m'}(\bar{q})
\]

where \( \tilde{\mathcal{G}}_{id} = \{(J,J')|(J,J') \in \mathcal{G}_{id}\} \). The boundary conditions \( \alpha \in \mathcal{B}_{\mathfrak{g} \oplus \mathfrak{h}} \) in the simple current orbifold can be obtained from pairs \( (L, L') \) of boundary conditions \( L \in \mathcal{B}_{\mathfrak{g}} \) and \( L' \in \mathcal{B}_{\mathfrak{h}} \) of the \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{h}} \) theory, respectively. These pairs are subject to identification rules, and fixed-points can occur (even if \( \mathcal{G}_{id} \) acts fixed-point free on the sectors). These orbifold fixed-points can be easily resolved\(^3\). We shall label the boundary conditions by equivalence classes of pairs \( \alpha = [L, L'] \), always remembering the possible orbifold fixed-point resolution.

When we want to obtain boundary conditions in the coset model along the lines of section 2.3, we only have to impose in addition the selection rules (5) on the boundary conditions \( \alpha = [L, L'] \).

We arrive at the final conclusion that – in the factorizing case – boundary conditions of the coset model are obtained from pairs of boundary conditions of numerator and denominator theory by suitable identification and selection rules. This result has been formulated first in [4]. There, the authors took a direct way not involving boundary conditions in the simple current orbifold. In practice this can simplify things: it is not always necessary to do the fixed-point resolution in the orbifold step, because it may happen that many of the resolved boundary conditions do not survive the selection rules. Still we think that the detour via the simple current orbifold has conceptual advantages. Firstly, it shifts all problems with fixed-point resolution to the orbifold step. Secondly, it fits in the general framework of section 2.3 which is also applicable in the non-factorizing case.

In all our examples in section 4, we shall encounter the factorizing case. The formulation of the rule for boundary RG flows in section 3, however, is more general and can be also used in the non-factorizing case.

\(^3\)In string theory these resolved boundary conditions are called fractional branes.
3 RG flows: a simple rule

3.1 Generalized Affleck-Ludwig rule

To motivate the rule for boundary RG flows in coset models, we shall first review shortly the original proposal of Affleck and Ludwig for the absorption of the boundary spin in the Kondo model.

The Kondo model is designed to understand the effect of magnetic impurities on the low temperature conductivity of a conductor. Usually a decreasing temperature will result in an increasing conductivity, because the scattering with phonons is reduced (Matthiesen’s rule). In some cases, however, when magnetic impurities are present, the conductivity reaches a maximum and starts to decrease again. This phenomenon is explained by the coupling of the electrons to the magnetic impurities. The electrons tend to screen the impurity, and this coupling increases when temperatures become low.

Let us say that the conductor has electrons in $k$ conduction bands. We can build several currents from the basic fermionic fields: the charge current, the flavor current, and the spin current $\vec{J}(y)$. The latter gives rise to a $\hat{su}(2)_k$ current algebra. The coordinate $y$ measures the radial distance from a spin $S$ impurity at $y = 0$ to which the spin current couples\(^4\). This coupling is

$$H_{\text{pert}} = \lambda R_\alpha J^\alpha(0) .$$

(10)

where $R_\alpha$ ($\alpha = 1, 2, 3$) is a $2S + 1$ dimensional irreducible representation of $su(2)$, $\lambda$ is the coupling constant.

The operator $H_{\text{pert}}$ acts on the tensor product $V^S \otimes \mathcal{H}$ of the $2S + 1$-dimensional quantum mechanical state space of our impurity with the Hilbert space $\mathcal{H}$ for the unperturbed theory described by a Hamiltonian $H_0$.

When the boundary spin is large ($2S > k$), the low temperature fixed point of the Kondo model appears only at infinite values of $\lambda$ (‘under-screening’). On the other hand, the fixed point is reached at a finite value $\lambda = \lambda^*$ of the renormalized coupling constant $\lambda$ if $2S \leq k$ (exact- or over-screening resp.). In the latter case, the fixed points are described by non-trivial (interacting) conformal field theories.

\(^4\)We only consider the case of a single isolated impurity.
Affleck and Ludwig \cite{Affleck1989, Ludwig1990} found an elegant rule to determine these strong-coupling fixed-points. The spectrum at the fixed-point is given by

$$\text{tr}_{V^S \otimes \mathcal{H}^l} \left( q^{H_0 + H_{\text{pert}}} \right)_{\lambda = \lambda^*}^{\text{ren}} = \sum_j N_{Sj}^{l} \chi^j(q) .$$

(11)

Here, $H_0 = L_0 + c/24$ is the unperturbed Hamiltonian, and the superscript $\text{ren}$ stands for ‘renormalized’. By $S$ we label a dominant highest-weight representation of $\widehat{su}(2)$. $V^S$ denotes the corresponding module of the finite-dimensional Lie algebra $su(2)$, and $\mathcal{H}^l$ is an irreducible sector of the $\widehat{su}(2)_k$-theory. The formula (11) is the content of the ‘absorption of boundary spin’-principle by Affleck and Ludwig \cite{Affleck1989, Ludwig1990}.

It is straightforward to generalize these considerations to an arbitrary simple Lie algebra $\mathfrak{g}$. The space $\mathcal{H}^l$ can be any of the $\widehat{\mathfrak{g}}$-irreducible subspaces in the physical state space $\mathcal{H}$ of the theory. Formula (11) means that our perturbation with some irreducible representation $S$ interpolates continuously between a building block $\dim(V^S) \chi^l(q)$ of the partition function of the UV-fixed point (i.e. $\lambda = 0$) and the sum of characters on the right hand side of the previous formula,

$$\dim(V^S) \chi^l(q) \rightarrow \sum_j N_{Sj}^{l} \chi^j(q) .$$

(12)

In \cite{Affleck1989} it was proposed to generalize the ‘absorption of boundary spin’-principle to coset models. The suggested rule is

$$\sum_{S', l'} b_{S'S'} N_{S'l'} \chi^{(l', r')}(q) \rightarrow \sum_j N_{Sj}^{l} \chi^{(j, j')}(q) .$$

(13)

Here, $S, l$ and $j'$ label dominant highest-weight representations of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$, respectively. The coefficients $b_{SS'}$ are the branching coefficients describing the decomposition of $V^S$, the corresponding representation of the finite Lie algebra $\mathfrak{g}$, into representations $V^{S'}$ of $\mathfrak{h}$,

$$V^S = \bigoplus b_{SS'} V^{S'} .$$

(14)

The embedding of affine Lie algebras $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ guarantees that these representations can again be identified with highest-weight representations $\mathcal{H}^{S'}$ of $\widehat{\mathfrak{h}}$.

The flows (13) are generated by fields coming from the coset sectors

$$\mathcal{H}^{(0, r')} , \quad \text{where} \quad V^{l'} \subset V^0|_{\mathfrak{h}} .$$

(15)
Here, $\theta$ labels the integrable highest-weight representation which is built from the adjoint representation of the Lie algebra $\mathfrak{g}$. The adjoint representation $l' = \theta'$ of $\mathfrak{h}$ can be omitted from the list (15) if it occurs only once in the decomposition of $\theta$.

To see that (13) is really a generalization of (12), we should recover the flows (12) when specializing to the trivial subgroup \( \{e\} \) of $\mathfrak{g}$. The primed label can then be omitted and the branching coefficient is just the dimension of the representation space $V^S$.

### 3.2 Simple rule for boundary RG flows in coset models

From the stated rule (13) formulated in terms of characters we can infer a rule for a flow between superpositions of boundary conditions. Before we discuss how this is done, let us present the result.

Choose a representation $S \in \text{Rep}(\hat{\mathfrak{g}})$ and a boundary condition $\alpha \in B^S_{\mathfrak{g} \oplus \mathfrak{h}}$ s.t.

\[
Q_{(J, \omega(J', \cdot))}(\alpha) + Q_{(J, \omega(J', \cdot))}(S, 0) = 0 \quad \text{for all } (J, J') \in \mathcal{G}_{\text{id}}.
\]

Then there will be a RG flow between the following coset boundary configurations $X$ and $Y$,

\[
X := (0, S^+|_{\mathfrak{h}}) \hat{\times} \alpha \longrightarrow (S, 0) \hat{\times} \alpha =: Y.
\]

Here, we introduced the shorthand notation $\hat{\times}$ to define a superposition of the form

\[
(l, l') \hat{\times} \alpha := \bigoplus_{\beta} n^{\mathfrak{g} \oplus \mathfrak{h}}_{(l, l')\alpha} (\beta) \cdot
\]

The label $(0, S^+|_{\mathfrak{h}})$ has to be understood as

\[
(0, S^+|_{\mathfrak{h}}) \hat{\times} \ldots := \bigoplus b_{S^+, S'} (0, S') \hat{\times} \ldots
\]

where $b_{S^+, S'}$ denote the finite branching coefficients defined in (14). The flows (16) are generated by fields from the coset sectors (15).

To derive (16) from (13), we introduce an arbitrary ‘spectator’ boundary condition $\beta$. The annulus partition function $Z^{\beta}_X(q)$ can be decomposed into combinations of characters that appear on the left side of (13) just using the fact that the annulus coefficients form a representation of the fusion algebra. The expression that we obtain from applying (13) can then be rewritten as $Z^{\beta}_Y$, i.e. we find the result

\[
Z^{\beta}_X \longrightarrow Z^{\beta}_Y
\]
for arbitrary boundary conditions $\beta$.

In the remainder of this section we want to give two arguments to support our claim. First, we want to relate the rule to results from a perturbative analysis in the limit when some levels are large. Then we shall present evidence that the rule is compatible with the g-conjecture of Affleck and Ludwig.

For a general coset theory with semi-simple numerator and denominator there occur different levels $k_r$ for the simple constituents of the numerator which then determine the levels in the denominator. Assume that we take some of the levels to very large values of the order of a common scale $k \gg 1$. In the limit $k \to \infty$, there are many coset fields whose conformal weight approaches $h = 1$ (the difference to 1 being of the order $1/k$). The RG flows induced by such fields can be studied by perturbative techniques. One way is to use the method of effective actions. Here, the couplings of the boundary fields are combined into matrices $A$ which are interpreted as fields in an effective theory determined by an action $S[A]$. The equations of motion for $A$ are precisely the fixed-point equations $\beta = 0$. In [21, 25] the effective action for untwisted boundary conditions in coset models has been constructed to leading order in $1/k$ building upon earlier works in WZNW models [25]. The generalization to twisted boundary conditions has been worked out in [26] using results of [27].

A special class of solutions for all, untwisted and twisted, boundary conditions [26] has precisely the form (16),

$$(0, S^+|h) \hat{\times} \alpha \longrightarrow (S, 0) \hat{\times} \alpha,$$

but here we have to restrict $S$ to representations s.t. the conformal weight $h_S$ in the $\hat{g}$-theory is of order $1/k$. The rule (16) thus extrapolates the perturbative results to arbitrary values of the levels.

The g-conjecture of Affleck and Ludwig states that the boundary entropy $g$ decreases along a boundary RG flow $X \longrightarrow Y$,

$$g_X > g_Y.$$

The boundary entropy $g_X$ for a superposition $X = \bigoplus \alpha X_\alpha$ (with $X_\alpha \in \mathbb{N}_0$) of boundary conditions $\alpha$ occurring with multiplicity $X_\alpha$ is defined as the sum of the
g-factors of the single boundary conditions,

\[ g_X = \sum_\alpha X_\alpha g_\alpha = \sum_\alpha X_\alpha \frac{\psi^0_\alpha}{\sqrt{S_{00}}} . \]

The ratio \( g_X/g_Y \) for the conjectured flow (16) is given by

\[
\frac{g_X}{g_Y} = \sum_{S'} b_{S'S'} n^{g_{0h}}_{(0,S')_\alpha} g_\beta \sum_\gamma n^{g_{0h}}_{(0,S)} g_\gamma \\
= \frac{\sum_{S'} b_{S'S'} n^{g_{0h}}_{(0,S')_\alpha} \psi^0_\beta}{\sum_\gamma n^{g_{0h}}_{(0,S)} \psi^0_\gamma} .
\]

We simplify this expression by using the fact that the vector \((\psi^0_\beta)\) is an eigenvector of the matrix \((n_{(l,l')})_{\alpha\beta}\) with eigenvalue \(S_{(l,l')0}/S_{00}\) and obtain a result which does only depend on \(S\),

\[
\frac{g_X}{g_Y} = \frac{\sum_{S'} b_{S'S'} S^{g}_{00}S^{h}_{00}}{S^{g}_{S0}S^{h}_{00}} .
\]

Hence, if our conjectured rule (16) and the g-conjecture are correct, we obtain the following inequality for quantum dimensions of \(\hat{\mathfrak{g}}\) and \(\hat{\mathfrak{h}}\) \((S \neq 0)\),

\[
\sum_{S'} b_{S'S'} \frac{S^{g}_{S0}S^{h}_{00}}{S^{g}_{S0}S^{h}_{00}} > \frac{S^{g}_{S0}S^{h}_{00}}{S^{g}_{S0}S^{h}_{00}} . \tag{17}
\]

This inequality can be used to test our proposal.

For diagonal cosets \(\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_{k+t}\) the inequality is satisfied. This follows from the fact that the quantum dimension \(S^{g_{0k}}_{S0}/S^{g_{0k}}_{00}\) of a fixed representation \(S\) is a monotonically increasing function of the level \(k\). Unfortunately, for general coset models we have not found a proof yet. However, numerical checks have been performed in a large number of coset models, all in accordance with the conjecture. Furthermore, when we take some levels to be large, we can confirm the inequality in a perturbative calculation (see appendix A).

This ends our discussion of the general properties of the rule (16). We have given some evidence by showing that the rule is consistent with the perturbative results and, although not completely proven, with the g-conjecture. The next section will provide more evidence coming from specific examples where the rule is able to reproduce a number of known flows.
4 Examples

In this section we shall present the rule (16) at work in a number of examples. In all these examples we shall first give the field content of the theory (coset sectors) and the boundary conditions by specifying identification and selection rules. Then we shall formulate the annulus coefficients for the coset boundary theories. The boundary conditions for the associated product theory are obtained by forgetting the selection rules on them, and the corresponding annulus coefficients are related to the ones from the coset model by (9). We shall introduce a pictorial representation of the boundary conditions as branes in some target space. This is followed by an application of the rule (16) to identify flows which are visualized as ‘brane processes’. For some models we collected the complete results in appendix B.

4.1 Minimal Models, A series

The unitary minimal models can be constructed as diagonal cosets of the form $\hat{su}(2)_k \oplus \hat{su}(2)_1/\hat{su}(2)_{k+1}$ with an integer $k \geq 1$. The modular invariant partition functions for these models are completely classified [28, 29, 30] (see also [17]); in this subsection we shall deal with the A series which is sometimes denoted as $(A_{k+1}, A_{k+2})$.

The sectors of the theory are labeled by three integers $[l_1, l_2, l']$ in the range $l_1 = 0, \ldots, k; l_2 = 0, 1; l' = 0, \ldots, k + 1$. Selection rules force the sum $l_1 + l_2 + l'$ to be even, and there is an identification $[l_1, l_2, l'] \sim [k - l_1, 1 - l_2, k + 1 - l']$ between admissible labels. The adjoint field from the sector $[0, 0, 2]$ that induces the flow described by the rule (16) has conformal weight $h = (k + 1)/(k + 3)$.

In the A-series we are in the Cardy case, i.e. the boundary conditions $\alpha$ are labeled by triples $[L_1, L_2, L']$ taking values in the same range as the sectors including selection and identification rules.

The annulus coefficients are just given by the fusion rules $N^{(k)}$ and $N^{(k+1)}$ of $\hat{su}(2)_k$ and $\hat{su}(2)_{k+1}$ resp.,

$$n_{[l_1, l_2, l'][L_1, L_2, L'][J_1, J_2, J'] = N^{(k)}_{l_1 L_1} J_1 N^{(k+1)}_{l_2 L' L'} + N^{(k)}_{k - l_1 L_1} J_1 N^{(k+1)}_{k + 1 - l' L' L'} J' .$$

We now want to give a pictorial representation of the boundary conditions. Coset

5The relation to the usual Kac labels $(r, s)$ is $r = l_1 + 1$ and $s = l' + 1$. 
models can also be formulated as non-linear $\sigma$-models on a background geometry which is essentially given by the space $G/\text{Ad} \, H$ where we divide the group $G$ by the adjoint action of the subgroup $H$. The boundary conditions can then be described by certain subspaces ('branes') onto which the boundary of our two-dimensional world-sheet is mapped \[12,13\]. One should be aware that this geometrical interpretation is only valid for large values of the level. If one, however, views the pictures just as a nice tool to illustrate the boundary conditions, we can profitably employ them for arbitrary levels.

In the case of minimal models we describe the background as a solid cylinder with squeezed ends \[21\]. The boundary conditions are represented by branes, extended objects of dimension 0,1 and 2. Let $x$ be the coordinate along the axis of the cylinder, $z$ the coordinate along the squeezed ends and $y$ a third coordinate perpendicular to the others (see fig. 1). All branes $[L_1, L_2, L']$ are located along surfaces of constant $z$ and are maximally extended in the $y$-direction. In $x$ they stretch between two values $x_{\text{min}}$ and $x_{\text{max}}$. The boundary conditions with $L' = 0, k+1$ are represented as points at the top or bottom of the cylinder, the ones with $L = 0, k$ are one-dimensional objects stretching in $y$-direction (see fig. 1). We shall give explicit formulas for $z, x_{\text{min}}, x_{\text{max}}$. The coordinate $z$ lies in the range $[0, 1]$, and the coordinate $x$ takes values in $[0, k]$.

\[
\begin{align*}
    z &= \begin{cases} 
        \frac{L'}{k+1} & \text{for } L_2 = 1 \\
        1 - \frac{L'}{k+1} & \text{for } L_2 = 0
    \end{cases} \\
    x_{\text{min}} &= \left| L_1 - \frac{k}{k+1} L' \right| \\
    x_{\text{max}} &= \max \{ L_1 + \frac{k}{k+1} L', 2k - L_1 - \frac{k}{k+1} L' \}
\end{align*}
\]
The rule (16) describes a large number of flows for many different starting configurations. We shall concentrate here on two types of starting points: a single boundary condition and a superposition of boundary conditions of the form \([0, L_2, L']\).

Assume that we want to study flows starting from the boundary condition \([L_1, L_2, L']\) with \(1 \leq L' \leq k\). To apply our rule (16), we have to find a boundary condition \(\alpha\) and a 'boundary spin' \(S\) s.t.

\[
(0, S^\prime|_b) \times \alpha = [L_1, L_2, L'] .
\]

On the one hand we can set \(\alpha = [L_1, L_2, 0]\) and \(S = (L', 0)\). This corresponds to the flow

\[
[L_1, L_2, L'] \rightarrow \bigoplus J\ N_{L_1 L'}^k J\ [J, L_2, 0] .
\]

On the other hand, the choice \(\alpha = [k - L_1, 1 - L_2, 0]\) and \(S = (k + 1 - L', 0)\) leads to the flow

\[
[L_1, L_2, L'] \rightarrow \bigoplus J\ N_{L_1 L'}^{k-1} J\ [J, 1 - L_2, 0] .
\]

The number of elementary boundary conditions appearing on the r.h.s. of (18) and (19) can be different depending on the values of \(L_1, L'\). Assume that \(L_1 + L' \leq k\) which we can always achieve by using the identification rules. For \(L_1 < L'\) we find a superposition of \(L_1 + 1\) boundary conditions in both flows (18) and (19). These flows are illustrated in fig. 2. For \(L_1 \geq L'\) there are \(L' + 1\) boundary conditions on the r.h.s. of (18), but in (19) we find a superposition of \(L'\) boundary conditions (see fig. 3).

The first of the flows, (18), has been analyzed in perturbation theory for large \(k\) in [31], the second one, (19), cannot be seen in this limit. Nevertheless, both flows are known to exist [32, 33, 34]. They are generated by the \([0, 0, 2]\) field (in Kac labels \((1, 3)\)) andiffer by the sign of the perturbation. This is in agreement with our general statements (15) on the boundary fields generating the flow.

Now let us choose a superposition of boundary conditions with a trivial first label. We set \(S = (L_1, 0)\) (\(1 \leq L_1 \leq k\)) and \(\alpha = [0, L_2, L']\) in (16) and obtain

\[
\bigoplus J\ N_{L_1 L'}^{k+1} J\ [0, L_2, J] \rightarrow [L_1, L_2, L'] .
\]
Figure 2: A pictorial representation of the flows (18) and (19) for $L_1 < L', L_1 + L' \leq k$: a single brane can flow to a superposition of point-like branes.

Figure 3: A pictorial representation of the flows (18) and (19) for $L_1 \geq L', L_1 + L' \leq k$: a single brane can flow to a superposition of point-like branes.
Figure 4: Pictorial representation of the flows \((20)\) and \((21)\): a superposition of string-like branes can flow to a single brane.

On the other hand, we could choose \(\alpha = [0, L_2, k + 1 - L']\) and \(S = (k + 1 - L_1, 0)\) which leads to

\[
\bigoplus N_{L_1 L'}^{(k+1)J} [0, L_2, J] \rightarrow [L_1 - 1, 1 - L_2, L'] .
\]  \hspace{1cm} (21)

The two flows are illustrated in fig. 4. Again, perturbation theory for large \(k\) can only see the first of these flows \([21, 35]\).

**4.2 Critical Ising model**

The simplest model in the unitary minimal A-series is the critical Ising model with \(k = 1\). There are three boundary conditions, the free one and two with fixed spin (up or down) at the boundary. Our geometrical picture reduces to a cushion-like background where the fixed boundary conditions are point-like objects at the top and bottom whereas the free boundary condition is a string-like object sitting precisely in the middle of the cushion (see fig. 5).

Starting from the free condition, the system can be driven into a theory with fixed spin. These are precisely the two flows \([18], [19]\). They are depicted in fig. 5. Flows starting from a superposition of boundary conditions can be found in appendix B.
4.3 Tricritical Ising model

The second model in the unitary minimal series is the tricritical Ising model with central charge $c = 7/10$. Once more, the flows triggered by the $(1,3)$-field as analyzed in [32] are correctly reproduced by (18) and (19). There are, however, more flows known which correspond to a perturbation with other fields [36]. As the rule depends on the specific coset construction, it is possible to find additional flows by choosing different coset realizations of the same theory. For the tricritical Ising model, such alternative realizations do exist. One is given by $(E_7)_1 \oplus (E_7)_1/(E_7)_2$. When we apply our rule to this coset construction, it reproduces the two known flows caused by the $(3,3)$-field. In Kac labels they read

\[(2,2) \rightarrow (3,1), \quad (2,2) \rightarrow (1,1),\]

and they are depicted together with the other flows in fig. 6. These two flows also appear in higher minimal models [37] where we do not know a coset realization for the $(3,3)$-perturbations. This may be related to the observation that the tricritical Ising model seems to be the only theory in which the considered perturbations are integrable [37]. Nevertheless, recovering flows from the exceptional $E_7$ coset construction can be considered as an important check of the conjectured rule.

There are more realizations of the tricritical Ising model as coset model [38], but only for one of them our rule predicts flows starting from single boundary conditions. This is the construction as a $so(7)_1/(G_2)_1$ coset model. The flows found there coincide with the $(1,3)$-flows [19], i.e. with those flows found in the $SU(2)$ construction that cannot be seen in the perturbative approach.

In appendix B we collected the complete results, i.e. all flows described by the
Figure 6: Boundary RG flows in the tricritical Ising model induced by the (1, 3)-field ($\phi_{13}$) and the (3, 3)-field ($\phi_{33}$).

rule (16) including those that start from a superposition of boundary conditions, for the $su(2)$ and the $E_7$ construction.

### 4.4 Minimal Models, D-Series

For the minimal models with $k \geq 3$, there is in addition to the diagonal modular invariant (A-type) another modular invariant giving rise to the D-series of minimal models. Up to some exceptional values of $k$, these form all possible modular invariants for the minimal models. Depending on $k$ being even or odd, we distinguish between minimal models of type $(D_{k+1}, A_{k+2})$ and $(A_{k+1}, D_{k+2})$, and we shall discuss these two classes of models separately.

**$(D,A)$: $k$ even**

Boundary conditions $\alpha$ in the $(D,A)$ models are labeled by triples $[L_1, L_2, L']$ where $L' = 0, \ldots, k + 1$ and $L_2 = 0, 1$ lie in the usual ranges whereas $L_1$ takes the values $0, 1, \ldots, \frac{k}{2} - 1, \left[\frac{k}{2}, +\right], \left[\frac{k}{2}, -\right]$. The sum $L_1 + L_2 + L'$ of boundary labels has to be
even. We have the identifications \([L_1, L_2, L'] \sim [\hat{L}_1, 1 - L_2, k + 1 - L']\) where

\[
\hat{L}_1 = \begin{cases} 
\frac{k}{2}, \mp & \text{for } L_1 = \left[\frac{k}{2}, \pm\right] \text{ and } k \text{ odd} \\
L_1 & \text{otherwise}
\end{cases}
\]  

(22)

The annulus coefficients can be written as a combination of the annulus coefficients \(n^{D,k}\) for the \(\hat{su}(2)\) model at level \(k\) with D-type partition function and the fusion rules of \(\hat{su}(2)_1\) and \(\hat{su}(2)_{k+1}\),

\[
n^{[J_1,J_2,J']}_{L_1,L_2,L'} = n^{D,k}_{L_1} N^{(1)}_{L_2} N^{(k+1),J'}_{L'_1} + n^{D,k}_{L_1} N^{(1)}_{L_2(1-L_2)} N^{(k+1)}_{L'_1} J'.
\]

Here, \(n^{D,k}\) is given by

\[
n^{D,k}_{L_1} = \begin{cases} 
N^{(k)}_{L_1} J_1 + N^{(k)}_{(k-L_1)1} J_1 & \text{for } J_1, L_1 \neq \frac{k}{2} \\
N^{(k)}_{L_1} & \text{for } J_1 \neq \frac{k}{2}, L_1 = \left[\frac{k}{2}, \pm\right] \\
N^{(k)}_{L_1} \frac{k}{2} & \text{for } J_1 = \left[\frac{k}{2}, \pm\right], L_1 \neq \frac{k}{2} \\
\delta_{l_1} \mod 4 & \text{for } J_1 = \left[\frac{k}{2}, \pm\right], L_1 = \left[\frac{k}{2}, \mp\right] \\
\delta_{l_1-2} \mod 4 & \text{for } J_1 = \left[\frac{k}{2}, \pm\right], L_1 = \left[\frac{k}{2}, \mp\right]
\end{cases}
\]  

(23)

The geometry of the minimal models of the (D,A)-series is the cylinder of the A-series divided by the reflection at the plane at \(x = k/2\) (see fig. 7). The branes which are symmetric with respect to the reflection split into two ‘fractional branes’.

We are only going to discuss flows starting from a single boundary condition \([L_1, L_2, L']\) with \(L' \neq 0, k + 1\). Because of the identification rules we are allowed to choose \(L' \leq \frac{k}{2}\). We distinguish three cases:

- \(L_1 + L' < \frac{k}{2}\).

In [16] we choose \(\alpha = [L_1, L_2, 0]\) and \(S = (L', 0)\) and find the flow

\[
[L_1, L_2, L'] \longrightarrow \bigoplus_J N^{(k)}_{L_1 L'} J [J, L_2, 0].
\]  

(24)

\(^6\)When we use \(L_1\) only as a numerical value and not as a label, we forget about the possible signs \(\pm\) from fixed-point resolution.
Figure 7: The geometry of the (D,A)-minimal model is the cylinder of the A-minimal model modulo the reflection (indicated by the arrow) at the $x = k/2$-plane.

Figure 8: The flows (24) and (25) illustrated in the half-cylinder geometry of the (D,A)-minimal models.

Alternatively, we choose $\alpha = [L_1, 1 - L_2, 0]$ and $S = (k + 1 - L', 0)$ and obtain

$$[L_1, L_2, L'] \rightarrow \bigoplus_J N_{L_1, (L'-1)}^{(k)} [J, 1 - L_2, 0]. \quad (25)$$

The flows are illustrated in fig. 8.

- $\frac{k}{2} \leq L_1 + L'$, $L_1 \neq \frac{k}{2}$

As in the previous case, we set $\alpha = [L_1, L_2, 0]$ and $S = (L', 0)$. The rule (16)
Figure 9: The flows (26) and (27) in the (D,A)-minimal model. Some of the branes appear with multiplicity 2. Note that the point-like branes sitting on the fixed-plane come in pairs of fractional branes indicated by + and −.

then leads to

\[ [L_1, L_2, L'] \rightarrow \bigoplus_{J < \frac{k}{2}} \left( N_{L_1 L'_1}^{(k) J} + N_{L_1 L'_1}^{(k) k-J} \right) [J, L_2, 0] \]  (26)

\[ \oplus N_{L_1 L'_1}^{(k) \frac{k}{2}} ([\frac{k}{2}, +], L_2, 0) \]

\[ \oplus N_{L_1 L'_1}^{(k) \frac{k}{2}} ([\frac{k}{2}, -], L_2, 0) \]

We find a second flow for \( \alpha = [L_1, 1 - L_2, 0] \) and \( S = (k + 1 - L') \),

\[ [L_1, L_2, L'] \rightarrow \bigoplus_{J < \frac{k}{2}} \left( N_{L_1 (L'-1)}^{(k) J} + N_{L_1 (L'-1)}^{(k) k-J} \right) [J, 1 - L_2, 0] \]  (27)

\[ \oplus N_{L_1 (L'-1)}^{(k) \frac{k}{2}} [[\frac{k}{2}, +], 1 - L_2, 0] \]

\[ \oplus N_{L_1 (L'-1)}^{(k) \frac{k}{2}} [[\frac{k}{2}, -], 1 - L_2, 0] \]

We have depicted the flows in fig. 9.

- \( L_1 = [\frac{k}{2}, +] \) (analogously for \( [\frac{k}{2}, -] \))

Here we find the flows

\[ [L_1, L_2, L'] \rightarrow \bigoplus_{J < \frac{k}{2}} N_{L'_1}^{(k) \frac{J}{2}} [J, L_2, 0] \oplus \left\{ \begin{array}{ll}
([\frac{k}{2}, +], L_2, 0) & L' = 0 \mod 4 \\
([\frac{k}{2}, -], L_2, 0) & L' = 2 \mod 4 \\
0 & L' \text{ odd}
\end{array} \right. \]  (28)
Figure 10: The flows (28) and (29) for $L' = 2$. In contrast to the flows in figure 9 all branes only appear with multiplicity 1. Note that the sign of the fractional brane at the RG end-point depends on the value of $L'$.

and

$$[L_1, L_2, L'] \longrightarrow \bigoplus_{J \leq \frac{k}{2}} N^{(k)}_{(L',-1)} J \ [J, L_2, 0] \oplus \begin{cases} ([\frac{k}{2}, +], L_2, 0) & L' = 1 \mod 4 \\ ([\frac{k}{2}, -], L_2, 0) & L' = 3 \mod 4 \\ 0 & L' \text{ even} \end{cases}$$

One example for the described flows with $L' = 2$ can be found in figure 10.

**(A,D): $k$ odd**

In the (A,D)-series of the minimal model, we label the boundary conditions by triples $[L_1, L_2, L']$ where $L_1 = 0, \ldots, k$, $L_2 = 0, 1$, and $L' = 0, \ldots, \frac{k-1}{2}, \frac{k+1}{2}, \frac{1}{2} \frac{k+1}{2}, -\frac{1}{2} - \frac{k+1}{2}$. Selection rules force the sum $L_1 + L_2 + L'$ to be even, triples $[L_1, L_2, L']$ and $[k-L_1, 1-L_2, \hat{L}']$ are identified. Here, $\hat{L}'$ is defined as in (22) with $k$ replaced by $k + 1$.

The annulus coefficients are given by the fusion rules of $\hat{s}u(2)$ at level $k$ and 1 and by the annulus coefficients $n^{D,k+1}$ of the $\hat{s}u(2)_{k+1}$ model with D-type modular invariant (see (23)),

$$n^{[J_1,J_2,J']}_{[L_1,L_2,L']} = N^{(k)}_{l_1 L_1} J_1 N^{(1)}_{l_2 L_2} J_2 n^{D,k+1}_{L'} J' + N^{(k)}_{l_1 (k-L_1)} J_1 N^{(1)}_{l_2 (1-L_2)} J_2 n^{D,k+1}_{\hat{L}'} J'.$$

The geometry of the (A,D)-minimal models is obtained from the cylinder geometry of the A-series by dividing out the reflection at the center (see fig. 11). A brane which is symmetric under this reflection splits into two fractional branes.
Figure 11: The (A,D)-minimal model is described by the cylinder geometry of the A-series modulo the reflection at the center (indicated by the arrows).

As in the (D,A)-case we look for flows starting from a single boundary condition $[L_1, L_2, L']$ with $L' \neq 0$. We have to distinguish two cases:

- $L' \neq \frac{k+1}{2}$
  
  When we choose $\alpha = [L_1, L_2, 0]$ and $S = (L', 0)$ in (16), we find
  \[
  [L_1, L_2, L'] \rightarrow \bigoplus J N_{L_1 L'}^k [J, L_2, 0].
  \]
  (30)
  
  Similarly, for $\alpha = [k - L_1, 1 - L_2, 0]$ and $S = (k + 1 - L', 0)$ we obtain
  \[
  [L_1, L_2, L'] \rightarrow \bigoplus J N_{L_1(L'-1)}^k [J, 1 - L_2, 0].
  \]
  (31)
  
  These flows are shown in fig. 12.

- $L' = \left[\frac{k+1}{2}, \pm\right]$ 
  
  By setting $\alpha = [L_1, L_2, 0]$ and $S = (\frac{k+1}{2}, 0)$ we find a flow for a superposition of two boundary conditions,
  \[
  [L_1, L_2, \left[\frac{k+1}{2}, +\right]] \oplus [L_1, L_2, \left[\frac{k+1}{2}, -\right]] \rightarrow \bigoplus J N_{L_1 \frac{k+1}{2}}^{(k)} [J, L_2, 0].
  \]
  (32)
  
  An example for this flow is shown in fig. 13.
Figure 12: The flows (30) and (31) in the (A,D)-minimal model.

Figure 13: The flow (32) in the (A,D)-minimal models starting from a superposition of two fractional branes $+$ and $-$.

4.5 Parafermion series

The parafermion series can be realized by the cosets $\hat{su}(2)_{k}/\widehat{u}_{2k}$. The sectors of the theory are labeled by pairs $[l, l']$ where $l = 0, \ldots, k$ and $l'$ is a $2k$-periodic integer for which we usually choose the range $l' = -k + 1, \ldots, k$. Selection rules force the sum $l + l'$ to be even, and the pairs $[l, l']$ and $[k - l, l' + k]$ are identified. The fields which appear as perturbing fields in our rule have conformal weight $h = (k - 1)/k$.

The maximally symmetric branes in parafermion theories come in two classes: the untwisted branes (A-branes), and the twisted branes (B-branes). The untwisted branes are the usual Cardy branes and carry labels $[L, L']$ from the same set as the sectors. The annulus coefficients are given by

$$n_{[l, l'][L, L']^{[J, J']]} = \delta_{l' + L' - J' \mod 2k} N_{_{LL}}^{(k)J} + \delta_{l' + L' - J + k \mod 2k} N_{_{(k-L)L}}^{(k)J}$$

where $N^{(k)}$ are the fusion rules of $\widehat{su}(2)_{k}$.
In the limit of large levels \( k \), the parafermion models can be described by a non-linear \( \sigma \)-model on a disc with non-trivial metric (sometimes this geometry is called ‘bell’). We want to use this picture to visualize boundary conditions as branes. The untwisted branes \([0, L']\) appear as point-like objects sitting at \( k \) special equidistant points on the boundary of the circle. The other untwisted branes \([L, L']\) are one-dimensional objects that stretch between these points (see fig. 14). These pictures have been introduced in [10].

Let us now apply the rule (16) to untwisted boundary conditions. The general result is

\[
[L, L' - S] \oplus [L, L' - S + 2] \oplus \cdots \oplus [L, L' + S] \longrightarrow \bigoplus_{J} N_{SL}^{(k)J}[J, L']
\]

for any label \( L, L', S \) with \( L + L' + S \) even. An example of a flow with \( L = 2, S = 1 \) is graphically presented in fig. 15. Particularly interesting is the case \( L = 0 \) where the end configuration consists only of a single boundary condition:

\[
[0, L' - S] \oplus [0, L' - S + 2] \oplus \cdots \oplus [0, L' + S] \longrightarrow [S, L'] .
\]

Such a flow is shown in fig. 16.

In addition to the untwisted (Cardy) boundary conditions there are twisted ones
which involve a non-trivial automorphism $\omega$. In the $u(1)$ part it acts as reflection,

$$\omega(J)(z) = -J(z),$$

on the numerator $su(2)$ it acts only as an inner automorphism. The twisted boundary conditions have first been constructed in [10].

These boundary conditions are labeled by pairs $[L, L']$ where $L = 0, \ldots, k$ is an integer coming from the numerator part, and the sign $L' = \pm$ comes from the twisted $U(1)$. Selection rules force $L$ to be even in combination with the sign $L' = +$, and odd if it comes with $L' = -$. As $L'$ is determined by $L$, we shall often leave it out and write $[L, \cdot]$. Furthermore, there is an identification between pairs, $[L, +] \sim [k - L, (-1)^k]$. For even $k$, the pair $[\frac{k}{2}, L']$ is a fixed-point of this identification, and the corresponding boundary condition has to be resolved into two elementary boundary conditions $[\frac{k}{2}, L'; \pm]$.  

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Figure 17: A generic twisted brane $[L, \pm]$ in the parafermion model. For $L = 0$ the brane becomes point-like. If $k$ is even we find two branes for $L = \frac{k}{2}$ which cover the whole disc.

The annulus coefficients (before fixed-point resolution) are given by

$$n_{[l,l'][L,L'][J,J']} = N^{(k)}_{lL} n_{l'l'}^{J,J'} + N^{(k)}_{k-lL} n_{k+l'L'}^{J,J'}$$

where the coefficients $n_{S^L,J^J}$ for the twisted $U(1)$ read

$$n_{l'-} = n_{l'+} = \begin{cases} 1 & l' \text{ even} \\ 0 & l' \text{ odd} \end{cases}, \quad n_{l'} = n_{l'} = \begin{cases} 0 & l' \text{ even} \\ 1 & l' \text{ odd} \end{cases}.$$

The resolution of the fixed-point for $L = k/2$ is straightforward.

In our geometric picture the brane $[0, +]$ appears as point-like object in the center of the disc, and the branes $[L, \cdot]$ are two-dimensional discs placed at the origin (see fig. 17).

The rule (16) applied to the twisted parafermion branes (ignoring again the fixed-point resolution) describes a flow from a superposition of $S + 1$ identical boundary conditions $[L, \cdot]$ to some other configuration,

$$(S + 1) \ [L, \cdot] \rightarrow \bigoplus_{J} N^{(k)}_{S} J [J, \cdot].$$

An example for $L = 0, S = 1$ is shown in fig. 18.

\footnote{Note that from the point of view of closed strings as derived in [10], the smallest twisted brane is not point-like, but has a small, non-zero radius. We do not want to discuss these differences any further, as the only purpose of the geometrical pictures here is to visualize the boundary conditions and the boundary RG flows.}
Figure 18: Two of the smallest twisted branes condense into a larger disc.

There is a different realization of the parafermion series, namely as diagonal coset models $\widehat{su}(k)_1 \oplus \widehat{su}(k)_3/\widehat{su}(k)_2$. Here, the adjoint field which induces the flows has conformal weight $h = 2/(k + 2)$. In this realization, we can even find flows starting from single boundary conditions, e.g. a one-dimensional brane at the boundary of the disc flows to a point-like one. We leave it at these general words here, but we shall work out the flows for $k = 3$ in section 4.6.

4.6 3-state Potts model

The 3-state Potts model is a square lattice model where at each site $i$ there is an angular variable $\theta_i$ taking values $0, \pm 2\pi/3$. The interaction is given by the classical Hamiltonian

$$\beta H = -c \sum_{(i,j)} \cos(\theta_i - \theta_j),$$

the sum running over nearest neighbor pairs. When the model is at its critical coupling, it can be described by a conformal field theory. Introducing a boundary into the problem, one can show that there are 8 possible boundary conditions \cite{39, 40}. These are the free boundary condition, the three different fixed boundary conditions, three mixed boundary conditions (one of the three spin states is forbidden at the boundary) and one additional boundary condition whose interpretation in the classical Potts model is not as simple as for the others (see \cite{39} for details). We use the nomenclature of \cite{39} and call the boundary conditions $F$, $A$, $B$, $C$, $AB$, $BC$, $AC$ and $N$ (for ‘new’), respectively.

The CFT describing the critical 3-state Potts model is a minimal model of central
charge $c = 4/5$. It can be obtained by various coset constructions: it belongs e.g. to the minimal D-series for $k = 3$ and to the parafermion series also for $k = 3$. In addition to these two realizations, we shall review the construction as a diagonal $su(3)$ coset. In all these realizations we determine flows between boundary conditions using the rule (16). In this section, we shall see the rule in action in examples with twisted boundary conditions.

We start with the construction as a $su(2)_3/\Delta_6$ coset that we already encountered in the discussion of parafermion theories in section 4.5. The untwisted branes are labeled by pairs $[L, L']$ where the labels $L$ and $L'$ lie in the range $L = 0, 1, 2, 3$ and $L' = -2, -1, 0, 1, 2, 3$. Selection rules force the sum $L + L'$ to be even, and the pairs $[L, L']$ and $[3 - L, L' \pm 3]$ label the same brane. These are the usual Cardy branes, and there are six of them in the model. We adopt the geometric interpretation from section 4.5. In this interpretation, three branes are points on the boundary of the disc and correspond to the three fixed boundary conditions $A, B, C$. The other three describe mixed boundary conditions $AB, BC, AC$ and are represented as lines (see fig. 19).

Table 1: Boundary conditions in the 3-state Potts model in three different coset constructions. The g-factors are given in terms of $N^4 = (5 - \sqrt{5})/2$ and $\lambda^2 = (1 + \sqrt{5})/2$.

| $su(2)_3$ | Boundary label from $su(3)_1 \oplus su(3)_1$ | g-factor $N\lambda^2$ | Notation from 39 |
|-----------|-----------------------------------------------|------------------------|------------------|
| $[0, 0]$  | $[0, 0, 0] \oplus [(0, 0), (0, 0), (0, 0)]$    | $N$                    | $A$              |
| $[0, 2]$  | $[0, 0, 2+ \oplus [(0, 0), (0, 1), (2, 0)]$   | $N$                    | $B$              |
| $[0, -2]$ | $[0, 0, 2- \oplus [(0, 0), (1, 0), (0, 2)]$   | $N$                    | $C$              |
| $[1, 1]$  | $[2, 0, 2- \oplus [(0, 0), (1, 0), (1, 0)]$   | $N\lambda^2$          | $AB$             |
| $[1, 3]$  | $[2, 0, 0 \oplus [(0, 0), (0, 0), (1, 1)]$    | $N\lambda^2$          | $BC$             |
| $[1, -1]$ | $[2, 0, 2+ \oplus [(0, 0), (0, 1), (0, 1)]$   | $N\lambda^2$          | $AC$             |
| $[1, -]$  | $[1, 0, 1 \oplus [0, 0, 0; \omega]$          | $N\lambda^2\sqrt{3}$  | $N$              |
| $[0, +]$  | $[3, 0, 1 \oplus [0, 0, 1; \omega]$          | $N\sqrt{3}$           | $F$              |
Figure 19: Pictorial representation of boundary conditions in the 3-state Potts model.

The remaining two boundary conditions can be constructed as twisted branes. They are labeled by pairs \([L, \pm]\) where \(L = 0, \ldots, 3\) is an integer coming from the numerator part, and the sign \(\pm\) comes from the twisted \(U(1)\). Selection and identification rules leave us with the two boundary conditions \([0, +] \sim [3, -]\) and \([1, -] \sim [2, +]\). In our geometric picture the brane \([0, +]\) appears as point-like object in the center of the disc, and the brane \([1, -]\) is a two-dimensional disc placed at the origin (see fig. 19). They are the ‘free’ and the ‘new’ boundary condition, respectively. Table 1 gives an overview of boundary conditions in this particular model.

Now, we want to apply our rule (16) to determine RG flows. We first observe that the rule does not describe flows starting from a single boundary condition. Instead, we shall analyze all possible flows for superpositions of two boundary conditions. In all these cases the boundary spin triggering the flow is \(S = 1\).

We start with untwisted branes. Applying the rule (16) for \(\alpha = [L = 0, L' = 1]\), we find the flow

\[
A \oplus B = [0, 0] \oplus [0, 2] \rightarrow [1, 1] = \overline{AB}.
\]

As one could already infer from symmetry arguments, there are also the flows \(B \oplus\)
\( C \rightarrow BC \) and \( A \oplus C \rightarrow AC \). Starting instead with \( \alpha = [1, 2] \) we find

\[
AB \oplus BC = [1, 1] \oplus [1, 3] \quad \rightarrow \quad [0, 2] \oplus [2, 2] = B \oplus AC ,
\]

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure1.png}
\end{array}
\]

analogous results can be obtained for permutations of the letters \( A, B, C \).

Choosing \( \alpha = [0, -] \) in (16) yields the flow

\[
2 \cdot F = 2 \cdot [0, +] \quad \rightarrow \quad [1, -] = N .
\]

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure2.png}
\end{array}
\]

If we set \( \alpha = [1, +] \), the resulting flow is

\[
2 \cdot N = 2 \cdot [1, -] \quad \rightarrow \quad [0, +] \oplus [2, +] = F \oplus N .
\]

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure3.png}
\end{array}
\]

These are all flows provided by the rule (16) for superpositions of two boundary conditions. The field responsible for the flows comes from the coset sectors \( \mathcal{H}^{(0, \pm 2)} \) and has conformal weight \( h = 2/3 \). This can be concluded from our general prescription in section 3 (see eq. (15)).

We now turn to the description of the Potts model as diagonal \( su(2) \) coset,

\[
\frac{\widehat{su}(2)_3 \oplus \widehat{su}(2)_1}{\widehat{su}(2)_4}
\]

where the modular invariant is obtained from charge-conjugated modular invariants in the numerator, the denominator \( su(2)_4 \) contributes a \( D_4 \) modular invariant. The perturbing field comes from the adjoint sector \([0, 0, 2]\) and has conformal weight \( h = 2/3 \).

We find four boundary conditions \( L_1 = 0, 1, 2, 3 \) in the \( su(2)_3 \) part and two boundary conditions \( L_2 = 0, 1 \) in the \( su(2)_1 \) part. The \( su(2)_4 \) part has a \( D_4 \) modular invariant. There are four boundary conditions which we label by \( L' = 0, 1, 2+, 2- \). The coefficients of the corresponding boundary states in terms of Ishibashi states can be found e.g. in [20].
Identification and selection rules leave us with eight boundary conditions for the 3-state Potts model. They are given in Table 1. Applying our rule, we observe first that we find the same flows involving superpositions of two boundary conditions that we discussed in the parafermion construction. In addition we find flows relating ‘free’ and ‘new’ boundary conditions with the others, namely (for superpositions of maximally three boundary conditions):

\[ \begin{align*}
\bullet &= F \rightarrow A = \bullet \\
\bullet &= F \rightarrow AB = \bullet \\
\text{\textbf{\textcolor{Gray}{\text{\textbullet}}} } &= N \rightarrow AB = \text{\textcolor{Gray}{\text{\textbullet}}} \\
\text{\textbf{\textcolor{Gray}{\text{\textbullet}}} } &= N \rightarrow AC \oplus B = \text{\textcolor{Gray}{\text{\textbullet}}} \\
\text{\textbf{\textcolor{Gray}{\textbf{\textbullet}}} } &= 2 \cdot F \rightarrow AB = \text{\textcolor{Gray}{\textbf{\textbullet}}} \\
\text{\textbf{\textcolor{Gray}{\textbf{\textbullet}}} } &= 2 \cdot N \rightarrow AC \oplus B = \text{\textcolor{Gray}{\textbf{\textbullet}}} \\
\text{\textcolor{Gray}{\textbf{\textbullet}}} &= A \oplus B \oplus C \rightarrow F = \bullet \\
\text{\textcolor{Gray}{\textbf{\textbullet}}} &= A \oplus B \oplus C \rightarrow N = \text{\textcolor{Gray}{\textbf{\textbullet}}} \\
\text{\textbullet} &= AB \oplus BC \oplus AC \rightarrow N = \text{\textbullet} \\
\text{\textbullet} &= AB \oplus BC \oplus AC \rightarrow F \oplus N = \text{\textbullet}
\end{align*} \]

Let us finally discuss the construction of the Potts model as

\[
\frac{\tilde{su}(3)_1 \oplus \tilde{su}(3)_1}{\tilde{su}(3)_2}
\]

coset. Its sectors are labeled by three \( su(3) \) weights

\[
[(l_1, l_2), (m_1, m_2), (l'_1, l'_2)]
\]
where \( l_i, m_i, l'_i \) are non-negative integers (Dynkin labels) obeying
\[
0 \leq l_1 + l_2 \leq 1 , \quad 0 \leq m_1 + m_2 \leq 1 , \quad 0 \leq l'_1 + l'_2 \leq 2
\]
\[
2(l_1 + m_1 - l'_1) + l_2 + m_2 - l'_2 = 0 \mod 3 .
\]

The sectors are identified according to the field identification
\[
[(l_1, l_2), (m_1, m_2), (l'_1, l'_2)] \sim
[(1 - l_1 - l_2, l_1), (1 - m_1 - m_2, m_1), (2 - l'_1 - l'_2, l'_1)] .
\]

What remains are 6 sectors. According to the standard Cardy construction, these give rise to 6 boundary conditions which are listed in table 1 along with their g-factors. Before we go to construct the remaining two boundary conditions, we want to look for RG flows.

Let us start with the boundary condition \( AB \) and exhibit what flows are ‘predicted’ by (16). We choose the perturbation \( S = ((0, 0), (0, 1)) \) and find the flow
\[
AB = [(0, 0), (1, 0), (1, 0)] \longrightarrow [(0, 0), (0, 0), (0, 0)] = A .
\]

The spin \( S = ((0, 1), (0, 0)) \) leads to
\[
AB = [(0, 0), (1, 0), (1, 0)] \longrightarrow [(0, 0), (0, 1), (2, 0)] = B .
\]

Analogously, we find \( BC \to B, BC \to C \) and \( AC \to A, AC \to C \). These constitute all flows from single boundary conditions described by the rule. For a superposition of two boundary conditions we find flows of the form
\[
AC \oplus B \longrightarrow A .
\]

The two remaining boundary conditions can be obtained from twisted gluing conditions using an automorphism which interchanges the twoDynkin labels of
the $su(3)$ theories. In the $su(3)_1$ there is only one sector left invariant under this automorphism, in the $su(3)_2$ theory there are two. In total we find two twisted boundary conditions

$$[0, 0, 0; \omega] \quad \text{and} \quad [0, 0, 1; \omega] ,$$

there are no selection or identification rules in this example. We can calculate their g-factors (see table I) and identify the two boundary conditions as the ‘new’ and the ‘free’ boundary condition, respectively.
Again, we want to investigate what flows are described by the rule (16). Let us start with the ‘new’ boundary condition and try the perturbation \( S = ((1, 0), (0, 0)) \). This leads to

\[
N = [0, 0, 0; \omega] \rightarrow [0, 0, 1; \omega] = F .
\]

We can identify the field that drives the described flows. From our general prescription (15), we conclude that the perturbing field is \([(0, 0), (0, 0), (1, 1)]\) and has conformal weight \( h = 2/5 \).

Let us compare our results with the work of Affleck et al.\[39\] (see also \[41\]). They find several flows driven by fields of conformal weight \( h = 2/3 \) and \( h = 2/5 \). The flows they find are all reproduced by our rule. For single boundary conditions we find exact coincidence, for superpositions our rule suggests further flows that have not been analyzed in \[39\].

Figure 20 summarizes part of the results for boundary RG flows in the 3-states Potts model obtained by the rule (16). The complete results can be found in appendix B.

### 4.7 \( N = 2 \) Minimal Models

As last example we choose the supersymmetric parafermion theories, the \( N = 2 \) minimal series. They can be constructed as cosets \( \widehat{su}(2)_k \oplus \widehat{u}_4/\widehat{u}_{2k+4} \). The sectors of the theory are labeled by triples \([l_1, l_2, l']\) where\footnote{Usually the sectors are labeled by the triples \((l, m, s)\) where \( l \) corresponds to \( l_1 \), \( m \) to \( l' \) and \( s \) corresponds to \( l_2 \). We choose a different order here and put all labels that belong to the numerator theory to the front.} \( l_1 = 0, \ldots, k \) and \( l' \) is a \((2k+4)\)-periodic integer with standard range \( l' = -k - 1, \ldots, k + 2 \). The third label \( l_2 \) can take the values \( l_2 = -1, 0, 1, 2 \). Selection rules force \( l_1 + l_2 + l' \) to be even and the triples \([l_1, l_2, l']\) and \([k - l_1, l_2 \pm 2, l' \pm (k + 2)]\) label the same sectors.

The discussion of boundary conditions is analogous to the parafermion case (section 4.5). There are untwisted boundary conditions (A-branes), labeled by triples \([L_1, L_2, L']\), and twisted ones (B-branes) \([10]\). As many results can be directly translated from the parafermion case, we are not going to repeat the whole picture here,
but restrict our discussion now to untwisted, ‘even’ \( (L_2 = 0, 2) \) boundary conditions.

Geometrically they can be displayed by straight, oriented lines (orientation depends on the label \( L_2 \)) stretched between \( k + 2 \) special punctures on a disc \([10]\). The smallest lines connecting two neighboring points have a label \( L_1 = 0, k \).

Let us see what flows are described by the rule (16). Let us choose \( L_1, L_2, L', S \) s.t. \( L_1 + L_2 + L' + S \) is even. Then we find the flow

\[
[L_1, L_2, L' - S] \oplus [L_1, L_2, L' - S + 2] \oplus \cdots \oplus [L_1, L_2, L' + S] \quad \rightarrow \quad \bigoplus_j N_{SL_1}^{(k)} [J, L_2, L'] .
\] (33)

We show some examples for \( k = 3 \) in figures [21] and [22].

At the end of this section, we want to show how these results can be used to determine the group of brane charges in the \( N = 2 \) minimal models. We assign

![Diagram of flows in the \( N = 2 \) minimal model for \( k = 3 \).](image)
charges $Q_{[L_1,L_2,L']}$ to the boundary conditions s.t. they are conserved during RG flows. From the rule we see immediately that all charges can be expressed by the charges $Q_{[0,L_2,L']}$ of boundary conditions with $L_1 = 0$ (just set $L_1$ to zero in the flow (33)). This means that for the even untwisted boundary conditions we have at most the charge group $\mathbb{Z}^{2k+4}$, one copy of $\mathbb{Z}$ for every boundary condition with $L_1 = 0$. Now, the rule implies more constraints on the charges. It turns out that in the end we remain with the relations

$$Q_{[0,0,L' - k - 1]} + Q_{[0,0,L' - k + 1]} + \cdots + Q_{[0,0,L' + k + 1]} = 0 \quad (34a)$$

$$Q_{[0,0,L']} = -Q_{[0,2,L']} \quad (34b)$$

To find (34a), it is sufficient to consider the flows with $S = k$ and $L_1 = 1$ in (33) (see e.g. the first flow of figure 22). The second relation (34b) results from flows with $S = k$ and $L_1 = 0$ (see the second flow of figure 22) combined with (34a). One can then show that other flows do not give any further constraints.

To summarize we find that the charge group of even untwisted branes is $\mathbb{Z}^{k+1}$. This result has already been obtained in [10]. It coincides with the computation of RR-charges in [42].
5 Conclusions

In this work we presented a proposal for a simple rule for boundary RG flows in coset models. A specialized version of the rule for untwisted Cardy boundary conditions had been announced earlier [3]. Evidence for the rule comes from three directions. First, the rule is in concordance with perturbative results. Second, there are strong arguments that all flows described by the rule satisfy the general ‘g-theorem’ of Affleck and Ludwig. Third, in a number of examples, including non-trivial exceptional coset constructions, the rule is able to reproduce flows that have been obtained by different means.

In the last section of the paper, we presented the broad range of application of the rule in different coset constructions. Having a rich knowledge of possible boundary RG flows in particular models, we can start to determine quantities that are invariant under RG flows. Interpreting the RG flows as dynamical processes between brane configurations in string theory, the determination of invariants leads to the computation of D-brane charge groups. For WZNW models such a computation has been performed in [43, 44] using the rule of Affleck and Ludwig. We showed how the generalized rule can be used to determine brane charges in $N = 2$ minimal models. The same method can be applied to other coset models, in particular to other Kazama-Suzuki models and compared to the RR-charges that have been calculated in [42].

Let us mention two open issues that one could consider in the future. In the perturbative regime we see a lot more flows than are described by our rule. What happens to them when we take the level to finite values? Another problem which is almost unexplored concerns boundary RG flows for non-maximally symmetric boundary conditions. In the perturbative regime, we have some (though very limited) informations from non-symmetric solutions of the effective action found in [27]. It would be interesting to study these boundary conditions and their relevance for brane charges further.

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A Compatibility with g-conjecture

In this appendix we want to show that the inequality \[ (17) \] which describes the compatibility of our proposed rule with the g-conjecture is satisfied when we take the levels to be large. For convenience, let us rewrite the inequality here and introduce the abbreviations \( g_L \) and \( g_R \) for left and right hand side of the relation,

\[
g_L := \sum_{S'} b_{SS'} \frac{S_{S_0}^h S_{S_0}^g}{S_{00}^h} > \frac{S_{S_0}^g}{S_{00}^g} := g_R. \tag{35}
\]

We shall expand both sides of the relation to the order \( 1/k^2 \) where \( k \) is the level which is sent to large values. Note that we not necessarily have to take all levels to be large as long as the representation \( S \) is trivial w.r.t. the algebras that stay at small levels.

Let us consider the simplest case when \( g \) and \( h \) are simple Lie algebras with level \( k \) and \( k' = k x_e \), respectively. Here, the integer \( x_e \) denotes the embedding index of the embedding \( h \hookrightarrow g \). When we want to expand the expressions in (35), we can make use of the formula

\[
\frac{S_{S_0}^g}{S_{00}^g} = \dim(V^S) \left(1 - \frac{\pi^2}{6 k^2} g^\vee C_S\right) + \mathcal{O}(1/k^3)
\]

which can be found e.g. in [17, eq. (13.175)]. Here, \( g^\vee \) is the dual Coxeter number of \( g \), \( V^S \) is the representation space of the representation labeled by \( S \), and \( C_S \) is the quadratic Casimir,

\[
C_S = \frac{1}{\dim V^S} \text{tr}_{V^S}(T^S)\ .
\]

\( T^S \) are the generators of the Lie algebra \( g \). To relate the Casimir of a \( g \)-representation with that of a representation of \( h \) we use the formula

\[
\text{tr}_{V^S}(T^S) = \frac{1}{x_e \dim h} \dim g \text{tr}_{V^S}(T^S)\, (T^T)^m)
\]

where the \( T^T \) are the generators of \( h \) embedded in \( g \).

Now we are prepared to check the inequality (35). We expand the l.h.s. according to the formula (36) and obtain

\[
g_L = \sum_{S'} b_{SS'} \dim(V^{S'}) \left(1 - \frac{\pi^2}{6 k^2} g^\vee C_{S'}\right) + \mathcal{O}(1/k^3) \ .
\]
After some manipulations and applying the relation (37) we arrive at
\[ g_L = \dim(V^S) \left( 1 - \frac{\pi^2}{6k'^2} g_0^\vee x_e \frac{\dim h}{\dim g} \right) + O(1/k'^3) . \]

Inserting \( k' = kx_e \) and using the fact that \( \dim g > \dim h \) as well as \( g^\vee > g_0^\vee/x_e \) we can finally conclude that \( g_L > g_R \) in the order \( 1/k^2 \).

The proof for semi-simple Lie algebras can be done essentially along the same lines. Let us sketch the procedure in the example of a coset model of the form \( g_{k_1}^{(1)} \oplus \cdots \oplus g_{k_n}^{(n)}/h_{k'} \) where the level of the denominator is determined by \( k' = k_1x_1 + \cdots + k_nx_n \). Using the expansion formula (36), we reduce our problem to proving the inequality
\[ \sum_{i=1}^{n} \frac{g_i^\vee}{k_i^2} \text{tr}(T_i^2) > \frac{g_0^\vee}{k'^2} \text{tr}(\sum T_i'^2) . \] (38)

The left hand side can be estimated from below as before,
\[ \sum_{i=1}^{n} \frac{g_i^\vee}{k_i^2} \text{tr}(T_i^2) > \frac{1}{\sum k_jx_j} \sum_i \frac{g_i^\vee}{k_i x_i} \text{tr}(T_i'^2) . \] (39)

We show now that this estimate is good enough to prove (38), i.e. that the difference
\[ \eta := \sum_i \frac{\sum_j k_jx_j}{k_i x_i} \text{tr}(T_i'^2) - \text{tr}(\sum T_i'^2) \]
is positive. By introducing \( a_i := \sqrt{k_i x_i/\sum_j k_j x_j} \) we can rewrite \( \eta \) in a manifestly positive form,
\[ \eta = \sum_{i=1}^{n-1} \text{tr} \left[ \frac{a_i}{1 + a_n (T_1' + \cdots + T_{n-1}') + \frac{a_i}{a_n} T_n' - \frac{1}{a_i} T_i'} \right]^2 , \]
which completes the proof.

**B Tables for flows in specific models**

**B.1 Critical Ising model**

Boundary conditions: 0 (free), + (spin up), − (spin down).
• **Coset realization** $\frac{su(2)_1 \oplus su(2)_2}{su(2)_3}$: Perturbing field has $h = 1/2$.

Flows resulting from $\boxed{16}$ starting from

- a **single** boundary condition

\[
0 \rightarrow + \\
\rightarrow -
\]

- a superposition of **two** boundary conditions

\[
+ \oplus - \rightarrow 0 \\
\rightarrow + \\
\rightarrow -
\]

### B.2 Tricritical Ising model

Boundary conditions:

| Symbol | Conf. weight $h$ of corresp. field | g-factor $a$ |
|--------|-----------------------------------|-------------|
| +      | $3/2$                             | $a \approx .5127$ |
| $-$    | 0                                 | $a \approx .5127$ |
| 0      | $7/16$                            | $a\sqrt{2} \approx .7251$ |
| $0+$   | $3/5$                             | $b/a\sqrt{2} \approx .8296$ |
| $-0$   | $1/10$                            | $b/a\sqrt{2} \approx .8296$ |
| $d$    | $3/80$                            | $b/a \approx 1.173$ |

Here, $a^4 = \frac{5 - \sqrt{5}}{40}$, $b^2 = \frac{5 + \sqrt{5}}{2}$.

• **Coset realization** $\frac{su(2)_2 \oplus su(2)_1}{su(2)_3}$: Perturbing field has $h = 3/5$.

Flows resulting from $\boxed{16}$ starting from

- a **single** boundary condition

\[
d \rightarrow 0 \\
0+ \rightarrow 0 \\
-0 \rightarrow 0 \\
\rightarrow + \oplus - \\
\rightarrow + \\
\rightarrow -
\]
– a superposition of two boundary conditions

\[
\begin{align*}
0 \oplus d & \rightarrow d & - \oplus 0^+ & \rightarrow - & + \oplus - & \rightarrow + \\
& \rightarrow 0 & & \rightarrow -0 & & \rightarrow 0^+ \\
& \rightarrow + \oplus - & & \rightarrow d & & \rightarrow d \\
& \rightarrow 0^+ \oplus -0 & & \rightarrow 0 & & \rightarrow 0 \\
\end{align*}
\]

– a superposition of three boundary conditions

\[
\begin{align*}
2 \cdot d \oplus 0 & \rightarrow d & 2 \cdot -0 \oplus + & \rightarrow d & 2 \cdot 0^+ \oplus - & \rightarrow d \\
& \rightarrow -0 \oplus 0^+ & & \rightarrow -0 & & \rightarrow 0^+ \\
\end{align*}
\]

- Coset realization \( \frac{(E_7)_1 \oplus (E_7)_1}{(E_7)_2} \): Perturbing field has \( h = 1/10 \).

Flows resulting from (16) starting from

– a single boundary condition

\[
\begin{align*}
d & \rightarrow + \\
& \rightarrow - \\
\end{align*}
\]

– a superposition of two boundary conditions

\[
\begin{align*}
0^+ \oplus -0 & \rightarrow 0 & 0 \oplus d & \rightarrow 0^+ & \rightarrow -0 \\
\end{align*}
\]

– a superposition of four boundary conditions

\[
\begin{align*}
+ \oplus - \oplus 0^+ \oplus -0 & \rightarrow d & 2 \cdot 0 \oplus 2 \cdot d & \rightarrow 0 \\
& \rightarrow + & & \rightarrow + \\
& \rightarrow - \\
\end{align*}
\]

– a superposition of six boundary conditions

\[
\begin{align*}
+ \oplus - \oplus 2 \cdot 0^+ \oplus 2 \cdot -0 & \rightarrow 0^+ & 2 \cdot 0 \oplus 4 \cdot d & \rightarrow d \\
& \rightarrow -0 \\
\end{align*}
\]
B.3 Three-state Potts model

Boundary conditions: \(A, B, C, AB, BC, AC, F, N\) (see table [I] on page [III]). Note that the model has a \(\mathbb{Z}_3\)-symmetry which acts on the boundary conditions as \(A \rightarrow B \rightarrow C \rightarrow A, AB \rightarrow BC \rightarrow AC \rightarrow AB\), the boundary conditions \(F\) and \(N\) are fixed. We shall only write out flows modulo this symmetry, e.g. the flow \(A \oplus B \rightarrow AB\) stands also for \(B \oplus C \rightarrow BC\) and \(C \oplus A \rightarrow AC\).

- **Coset realization** \(\frac{su(2)_1 \oplus su(2)_3}{su(2)_4}\): Perturbing field has \(h = 2/3\).
  Flows resulting from [I] starting from
  
  - a *single* boundary condition
    
    \[
    \begin{align*}
    F & \rightarrow BC \\
    N & \rightarrow BC \\
    \quad & \rightarrow A \\
    \quad & \rightarrow A \oplus BC
    \end{align*}
    \]

  - a superposition of *two* boundary conditions
    
    \[
    \begin{align*}
    2 \cdot F & \rightarrow N \\
    2 \cdot N & \rightarrow F \oplus N \\
    A \oplus B & \rightarrow AB \\
    \quad & \rightarrow BC \\
    \quad & \rightarrow A \oplus BC \\
    AB \oplus AC & \rightarrow A \oplus BC
    \end{align*}
    \]

  - a superposition of *three* boundary conditions
    
    \[
    \begin{align*}
    A \oplus B \oplus C & \rightarrow N \\
    AB \oplus BC \oplus AC & \rightarrow F \oplus N \\
    \quad & \rightarrow F \\
    \quad & \rightarrow A \oplus BC \\
    \quad & \rightarrow A \\
    \quad & \rightarrow AB \\
    3 \cdot F & \rightarrow N \\
    3 \cdot N & \rightarrow F \oplus N \\
    \quad & \rightarrow F \\
    \quad & \rightarrow N
    \end{align*}
    \]

  - a superposition of *six* boundary conditions
    
    \[
    \begin{align*}
    2 \cdot A \oplus 2 \cdot B \oplus 2 \cdot C & \rightarrow N \\
    2 \cdot AB \oplus 2 \cdot BC \oplus 2 \cdot AC & \rightarrow F \oplus N
    \end{align*}
    \]

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• **Coset realization** $\frac{su(2)}{u(1)}$: Perturbing field has $h = 2/3$.

Flows resulting from (16) starting from

- a superposition of **two** boundary conditions

\[
\begin{align*}
A \oplus B & \rightarrow AB \\
AB \oplus AC & \rightarrow A \oplus BC
\end{align*}
\]

\[
2 \cdot F \rightarrow N
\]

\[
2 \cdot N \rightarrow F \oplus N
\]

- a superposition of **three** boundary conditions

\[
\begin{align*}
A \oplus B \oplus C & \rightarrow AB \\
AB \oplus BC \oplus AC & \rightarrow A \oplus BC
\end{align*}
\]

\[
3 \cdot F \rightarrow N
\]

\[
3 \cdot N \rightarrow F \oplus N
\]

- a superposition of **four** boundary conditions

\[
\begin{align*}
2 \cdot A \oplus B \oplus C & \rightarrow A \\
2 \cdot AB \oplus BC \oplus AC & \rightarrow AB
\end{align*}
\]

\[
4 \cdot F \rightarrow F
\]

\[
4 \cdot N \rightarrow N
\]

• **Coset realization** $\frac{su(3)_1}{su(3)_2}$: Perturbing field has $h = 2/5$.

Flows resulting from (16) staring from

- a **single** boundary condition

\[
\begin{align*}
AB & \rightarrow A \\
& \rightarrow B \\
N & \rightarrow F
\end{align*}
\]

- a superposition of **two** boundary conditions

\[
\begin{align*}
A \oplus BC & \rightarrow AB \\
& \rightarrow AC \\
& \rightarrow A \\
& \rightarrow B \\
& \rightarrow C
\end{align*}
\]

\[
F \oplus N \rightarrow N \\
\rightarrow F
\]

- a superposition of **three** boundary conditions

\[
\begin{align*}
A \oplus 2 \cdot BC & \rightarrow BC \\
& \rightarrow AB \\
& \rightarrow AC
\end{align*}
\]

\[
F \oplus 2 \cdot N \rightarrow N
\]

\[
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\]
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