On the period of Lehn, Lehn, Sorger, and van Straten’s symplectic eightfold

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Abstract

For the irreducible holomorphic symplectic eightfold $Z$ associated to
a cubic fourfold $Y$ not containing a plane, we show that a natural
Abel–Jacobi map from $H^4_{\text{prim}}(Y)$ to $H^2_{\text{prim}}(Z)$ is a Hodge isometry.
We describe the full $H^2(Z)$ in terms of the Mukai lattice of the K3
category $\mathcal{A}$ of $Y$. We give numerical conditions for $Z$ to be birational
to a moduli space of sheaves on a K3 surface or to $\text{Hilb}^4(K3)$. We
propose a conjecture on how to use $Z$ to produce equivalences from $\mathcal{A}$
to the derived category of a K3 surface.

Introduction

Beauville and Donagi [6] showed that if $Y \subset \mathbb{P}^5_C$ is a smooth cubic fourfold,
then the variety $F$ of lines on $Y$ is an irreducible holomorphic symplectic
fourfold, deformation-equivalent to the Hilbert scheme of two points on a K3
surface. They showed moreover that the universal line $P \subset Y \times F$ induces
a Hodge isometry

$$[P]_{\ast} : H^4_{\text{prim}}(Y, \mathbb{Z}) \simrightarrow H^2_{\text{prim}}(F, \mathbb{Z}),$$

with the intersection pairing on the left-hand side and the opposite of the
Beauville–Bogomolov–Fujiki pairing on the right.

The first author later refined this result by describing the full $H^2(F, \mathbb{Z})$
in terms of the Mukai lattice of the K3 category of $Y$, and characterized the
cubics for which $F$ is birational to the Hilbert scheme of two points on a K3
surface, or more generally to a moduli space of sheaves on a K3 surface [1].

In his survey paper [11], Hassett remarked that it would be useful to have
analogous results for the symplectic eightfold $Z$ constructed from twisted
cubics on $Y$ by Lehn, Lehn, Sorger, and van Straten [19]. This is the subject
of the present paper. We begin with a statement on primitive cohomology.
Theorem 1. Let $Y$ be a smooth cubic fourfold not containing a plane. Let $u: M \to Z$
be the contraction from the ten-dimensional space of generalized twisted cubics on $Y$ to the LLSvS symplectic eightfold. Let $C \subset Y \times M$ be the universal curve. Then the pullback
$$u^*: H^2(Z, \mathbb{Z}) \to H^2(M, \mathbb{Z})$$
is injective, and the map
$$[C]_*: H^4(Y, \mathbb{Z}) \to H^2(M, \mathbb{Z})$$
restricts to a Hodge isometry
$$[C]_*: H^4_{\text{prim}}(Y, \mathbb{Z}) \simto u^*(H^2_{\text{prim}}(Z, \mathbb{Z})),$$
with the intersection pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.

For a statement about the full $H^2(Z, \mathbb{Z})$, we must recall Kuznetsov’s K3 category $A = \langle O_Y, O_Y(h), O_Y(2h) \rangle \perp \subset D^b(Y)$, the Mukai lattice introduced in [3, §2],
$$K_{\text{top}}(A) = \{ [O_Y], [O_Y(h)], [O_Y(2h)] \} \perp \subset K_{\text{top}}(Y),$$
and the two special classes $\lambda_1, \lambda_2 \in K_{\text{top}}(A)$. For a recent account of the lattice theory of cubic fourfolds, see Huybrechts’ survey paper [16].

Theorem 2. Continue the notation of Theorem 1. Let
$$\Phi^K: K_{\text{top}}(A) \subset K_{\text{top}}(Y) \to K_{\text{top}}(M)$$
be the map on topological K-theory induced by the Fourier–Mukai kernel $I_C^Y(-3h) \in D^b(Y \times M)$. Then the map
$$c_1 \circ \Phi^K: K_{\text{top}}(A) \to H^2(M, \mathbb{Z})$$
restricts to a Hodge isometry
$$\langle \lambda_2 - \lambda_1 \rangle \simto u^*(H^2(Z, \mathbb{Z})),$$
with the Euler pairing on the left-hand side and the opposite of the Beauville–Bogomolov–Fujiki pairing on the right.
One of the main ideas of [2] is that $Z$ is a moduli space of complexes in $\mathcal{A}$ containing the projections of $\mathcal{O}_y$ for points $y \in Y$; the K-theory class of these complexes is $\lambda_2 - \lambda_1$, so (4) is natural in light of O’Grady’s description of the period of a moduli space of sheaves on a K3 surface [25]. Li, Pertusi, and Zhao developed this idea further using Bridgeland stability conditions in [21]; note that their class $2\lambda_1 + \lambda_2$ is related to our $\lambda_2 - \lambda_1$ by an auto-equivalence of $\mathcal{A}$. Statements similar to Theorem 2 appear in [21, Prop. 5.2] and [4, Thm. 29.2(2)]. We offer the present paper nonetheless because our proofs require less machinery, the statement in Theorem 1 is very classical and explicit, and we thought it worthwhile to articulate Theorem 3 and Conjecture 4 below.\textsuperscript{1}

Next we recall Hassett’s Noether–Lefschetz divisors $C_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds, indexed by an integer $d$ satisfying

$$d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod{6}. \quad (*)$$

Again we refer to the survey papers of Huybrechts [16] and Hassett [11]. From [3], [4], and [1] we know that the K3 category $\mathcal{A}$ is equivalent to the derived category of coherent sheaves on a K3 surface, and the fourfold $F$ is birational to a moduli space of sheaves on a K3 surface, if and only if the cubic $Y \in C_d$ for some $d$ satisfying

$$d/2 \text{ is not divisible by } 9 \text{ or any prime } p \equiv 2 \pmod{3}, \quad (**)$$

or equivalently,

$$d \text{ divides } 2n^2 + 2n + 2 \text{ for some } n \in \mathbb{Z}. \quad (**)$$

From [15] and [4] we know that the K3 category $\mathcal{A}$ is equivalent to the derived category of twisted sheaves on a K3 surface, and $F$ is birational to a moduli space of twisted sheaves, if and only if $d$ satisfies the weaker condition

In the prime factorization of $d/2$, primes $p \equiv 2 \pmod{3}$ appear with even exponents.\textsuperscript{1}’

On the other hand, from [1] we know that $F$ is birational to the Hilbert scheme of two points on a K3 surface if and only if $d$ satisfies the stronger condition

$$d \text{ is of the form } \frac{2n^2 + 2n + 2}{a^2} \text{ for some } n, a \in \mathbb{Z}. \quad (***)$$

\textsuperscript{1}After the first version of this paper was posted, Li, Pertusi, and Zhao informed us that the submitted version of [21] includes a statement analogous to our Theorem 3(c).
Let us introduce a new condition

\[ d \text{ is of the form } \frac{6n^2 + 6n + 2}{a^2} \text{ for some } n, a \in \mathbb{Z}. \quad (***)' \]

It is strictly stronger than (**) but incomparable to (***): the first few \( d \)s satisfying (***') are 14, 38, 62, 74, and 86, whereas the first few \( d \)s satisfying (***') are 14, 26, 38, 42, 62, and 86.

**Theorem 3.** Let \( Y \) be a cubic fourfold not containing a plane — that is, not lying in \( C_8 \) — so the symplectic eightfold \( Z \) is defined.

(a) \( Z \) is birational to a moduli space of sheaves on a K3 surface if and only if \( Y \in C_d \) for some \( d \) satisfying (**).

(b) \( Z \) is birational to a moduli space of twisted sheaves on a K3 surface if and only if \( Y \in C_d \) for some \( d \) satisfying (**').

(c) \( Z \) is birational to the Hilbert scheme of four points on a K3 surface if and only if \( Y \in C_d \) for some \( d \) satisfying (**')

From [5, Thm. 1.2(c)] we know that if \( Z \) is birational to a moduli space of sheaves on a K3 surface, then \( Z \) is isomorphic to a moduli space of stable complexes on the same surface for some Bridgeland stability condition. This suggests an alternative approach to proving that \( \mathcal{A} \) is equivalent to the derived category of a K3 surface if and only if \( d \) satisfies (**).

**Conjecture 4.** Suppose that \( Y \in C_d \) for some \( d \) satisfying (**), so \( Z \) is isomorphic to a moduli space of Bridgeland-stable complexes on some K3 surface \( S \). Let \( \alpha \in \text{Br}(Z) \) be the obstruction to the existence of a universal complex on \( Z \times S \), and let \( U \in D^b(Z \times S, \alpha \boxtimes 1) \) be a twisted universal complex. Recall that \( Y \) is naturally embedded in \( Z \), and consider the restriction \( U|_{Y \times S} \), which can be untwisted because \( \text{Br}(Y) = 0 \). Then the functor \( \mathcal{A} \to D^b(S) \) induced by \( U|_{Y \times S} \) is an equivalence.

We also expect a similar statement with (**') and twisted K3 surfaces. To prove Conjecture 4 would require a careful analysis of the “wrong-way slices” \( U|_{Z \times x} \) for \( x \in S \), and of their restrictions to \( Y \).

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2On the other hand, if one assumes this fact (that \( \mathcal{A} \) is geometric if and only if \( d \) satisfies (**)), then Conjecture 4 seems to follow from [21, Prop. 5.7].
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1 Proof of Theorems 1 and 2

1.1 Recollections and reductions

For a smooth cubic fourfold $Y$ not containing a plane, Lehn, Lehn, Sorger, and van Straten considered the irreducible component $M \subset \text{Hilb}^{3n+1}(Y)$ containing twisted cubics, which is a smooth tenfold [19, Thm. A]. They produced a contraction $u: M \to Z$ onto an irreducible holomorphic symplectic eightfold, and a copy of $Y$ naturally embedded in $Z$ [ibid., Thm. B]. Addington and Lehn showed that $Z$ is deformation-equivalent to the Hilbert scheme of four points on a K3 surface [2].

Lemma 5. The pullback map $u^*: H^2(Z, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is injective.

Proof. The map $u$ factors as the blowup $Z' \to Z$ along $Y \subset Z$, and a $\mathbb{P}^2$-fibration $M \to Z'$, both of which induce injections on cohomology. Alternatively, $u$ is surjective, so $u^*: H^*(Z, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ is injective by [28, Lem. 7.28], and $H^2(Z, \mathbb{Z})$ is torsion-free because $Z$ is simply-connected. \qed

The map (1) is clearly a map of Hodge structures; let us argue that (3) is as well. For basics about the map on topological K-theory induced by a Fourier–Mukai kernel, and its compatibility with the more usual map on rational cohomology, we refer to [3, §2.1]. The Hodge structure on $K_{\text{top}}(A) \subset K_{\text{top}}(Y)$ is pulled back via the Mukai vector $v: K_{\text{top}}(Y) \to \bigoplus H^{2i}(Y, \mathbb{Q})$. We have a commutative diagram

$$
\begin{array}{ccc}
K_{\text{top}}(A) & \xrightarrow{\Phi^K} & K_{\text{top}}(M) \\
v \downarrow & & v \downarrow \\
H^*(Y, \mathbb{Q}) & \xrightarrow{\Phi^H} & H^*(M, \mathbb{Q}),
\end{array}
$$

(5)

where $\Phi^K$ and $\Phi^H$ are the maps induced by $I^*_C(-3h) \in D^b(Y \times M)$. Thus we see that $v \circ \Phi^K$ is a map of Hodge structures. To get the map $c_1 \circ \Phi^K$ of (3), we first do $v \circ \Phi^K$, then multiply by $(\text{td } M)^{-1/2}$ which preserves Hodge type, then take the degree-2 part.
Now it is a topological question whether the restrictions (2) and (4) take values in the subgroups claimed, and whether they are isometries, so it is enough to prove the claims for a single cubic \( Y \). In a bit more detail, let \( U \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))) = \mathbb{P}^{55} \) be the Zariski open set of cubic fourfolds that are smooth and do not contain a plane. The construction of \( M \) and \( Z \) works in families, as discussed in the introduction to [19], so we get smooth families \( \mathcal{Y} \to U, \mathcal{M} \to Z \to U, \) and \( \mathcal{C} \subset \mathcal{M} \times_U \mathcal{Y} \). Then (2) and (4) are specializations of maps of local systems over \( U \), so to prove the claims about them it is enough to do so at one point \( Y \in U \).

In §§1.2–1.3 we do this for (4) when \( Y \) is a Pfaffian cubic fourfold whose associated K3 surface \( X \) has Picard rank 1. In §1.4 we do it for (2) when \( Y \) does not belong to any Noether–Lefschetz divisor.

1.2 Proof of Theorem 2 for a very general Pfaffian cubic

To a generic choice of six two-forms on \( \mathbb{C}^6 \), Beauville and Donagi [6] associate a “Pfaffian” cubic fourfold \( Y \subset \mathbb{P}^5 \) and a K3 surface \( X \subset \mathbb{P}^8 \) that parametrizes a complete family of quartic scrolls on \( Y \). In [2] the family is called \( \Gamma \subset X \times Y \); it is generically 4-to-1 over \( Y \). We will recall more details in the next section, but we do not need them yet.

Assuming that \( Y \) does not contain a plane, we continue to let \( M \) be the space of generalized twisted cubics on \( Y \), \( u: M \to Z \) the contraction onto the LLSvS eightfold, and \( C \subset Y \times M \) the universal curve.

We will finish proving Theorem 2 under the assumption that \( X \) has Picard rank 1, which by our discussion above is enough to prove the theorem in general. We will construct a commutative diagram of Fourier–Mukai functors between \( D^b(X) \) and \( D^b(M) \), and deduce from it a commutative diagram of maps between lattices, most of which are isomorphisms, and ultimately compare the map (3) with the composition

\[
[I_\xi] \subset K_{\text{top}}(X) \xrightarrow{E} K_{\text{top}}(X[4]) \hookrightarrow H^2(X[4], \mathbb{Z}),
\]

which is known to be a Hodge isometry; here \( \xi \subset X \) is a subscheme of length 4 and \( \Xi \subset X \times X[4] \) is the universal subscheme. For standard facts about Fourier–Mukai kernels, their adjoints, induced maps on cohomology, etc. we refer to Huybrechts’ book [13, Ch. 5].

We consider the convolution

\[
T := I_\Gamma \circ I_C(2h) \in D^b(X \times M).
\]

and recall some facts proved in [2, §3], with a few improvements:
(a) There is a non-empty, Zariski open set $M_0 \subset M$ such that the restriction $T[1]|_{X \times M_0}$ is quasi-isomorphic to an $M_0$-flat family of ideal sheaves of length-4 subschemes of $X$. We take $M_0$ as big as possible.

(b) There is an open set $Z_0 \subset Z$ such that $M_0 = u^{-1}(Z_0)$.

Proof: From [2, Prop. 2] we know that $M_0$ is a union of fibers of $u$, so $M_0 = u^{-1}(Z_0)$ for some $Z_0 \subset Z$, maybe not open a priori. But $u$ is surjective, so $Z \setminus Z_0 = u(M \setminus M_0)$, and $u$ is proper, hence closed, so $Z \setminus Z_0$ is closed.

(c) The classifying map $t': M_0 \to X^4$ descends to an open immersion $t: Z_0 \to X^4$.

Proof: From [2, §3] we know that $t'$ descends to a map $t$ that is injective. But an injective holomorphic map between complex manifolds of the same dimension is an open immersion by the proof of [10, Prop. on p. 19]; see also [26]. (So the smaller open set $Z_1 \subset Z_0$ of [2, §3] is unnecessary.)

The key to the present proof will be the following fact, whose proof we postpone to the next section:

**Proposition 6.** If the K3 surface $X$ has Picard rank 1, then the open set $Z_0$ contains $Y$, and its complement $Z \setminus Z_0$ has codimension at least 2 in $Z$.

Thus $M \setminus M_0$ has codimension at least 2 in $M$ as well, because $u$ is a $\mathbb{P}^2$-fibration over $Z \setminus Y$; and thus the inclusions $Z_0 \hookrightarrow Z$ and $M_0 \hookrightarrow M$ induce isomorphisms on $H^2$.

The reader may wonder why we need Proposition 6, when we can say immediately that because $Z$ and $X^4$ have nef canonical bundles, the birational map $Z \dasharrow X^4$ is biregular on an open set $W$ containing $Z_0$ whose complement has codimension at least 2 [17, Cor. 3.54]. The reason is that we would not know anything about the restriction of $T$ to $X \times u^{-1}(W)$, which we need in what follows.

Consider the functor $D^b(M) \to D^b(Y)$ induced by $I_C \otimes \mathcal{O}_Y(2h)$, the functor $D^b(Y) \to D^b(X)$ induced by $I_Y$, their composition

$$D^b(M) \xrightarrow{I_C(2h)} D^b(Y) \xrightarrow{I_Y} D^b(X),$$

and its left adjoint

$$D^b(X) \xrightarrow{I_Y^*[2]} D^b(Y) \xrightarrow{I_Y^*(-5h)[4]} D^b(M),$$

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which it will later be convenient to rewrite as

\[ D^b(X) \xrightarrow{I^\vee_{X}[2]} D^b(Y) \xrightarrow{I^\vee_{Y}[3]} D^b(M). \] (7')

We post-compose with the restriction

\[ D^b(X) \xrightarrow{I^\vee_{X}[2]} D^b(Y) \xrightarrow{I^\vee_{Y}[3]} D^b(M) \xrightarrow{\iota^*} D^b(M_0). \] (8)

The first composition (6) is induced by \( T \), so the left adjoint (7) or (7') is induced by \( T^\vee[2] \), so the restriction (8) is induced by \( T^\vee|_{X \times M_0}[2] \).

Because \( T[1]|_{X \times M_0} \) is an \( M_0 \)-flat family of ideal sheaves with classifying map \( t': M_0 \to X[4] \), there is a line bundle \( \mathcal{L} \) on \( M_0 \) such that \( T[1]|_{X \times M_0} = (1 \times t')^* I_\Xi \otimes \mathcal{L} \), where again \( \Xi \subset X \times X[4] \) is the universal subscheme. So the composition (8) agrees with

\[ D^b(X) \xrightarrow{I^\vee_{X}[3]} D^b(X[4]) \xrightarrow{t^*} D^b(M_0) \xrightarrow{\otimes \mathcal{L}^\vee} D^b(M_0). \]

Because \( t' = t \circ u \), this is

\[ D^b(X) \xrightarrow{I^\vee_{X}[3]} D^b(X[4]) \xrightarrow{t^*} D^b(Z_0) \xrightarrow{u^*} D^b(M_0) \xrightarrow{\otimes \mathcal{L}^\vee} D^b(M_0). \]

Passing to topological K-theory [3, \S 2.1], and taking first Chern classes where shown, we get a diagram

\[ K_{\text{top}}(X) \xrightarrow{-I^\vee_{\Xi}} K_{\text{top}}(X[4]) \xrightarrow{t^*} K_{\text{top}}(Z_0) \xrightarrow{u^*} K_{\text{top}}(M_0) \xrightarrow{-i^! \mathcal{L}^\vee} K_{\text{top}}(M_0) \]

\[ \xrightarrow{c_1} \quad \xrightarrow{c_1} \quad \xrightarrow{c_1} \quad \xrightarrow{c_1} \]

\[ H^2(X[4], \mathbb{Z}) \xrightarrow{t^*} H^2(Z_0, \mathbb{Z}) \xrightarrow{u^*} H^2(M_0, \mathbb{Z}) \xrightarrow{-} H^2(M_0, \mathbb{Z}), \] (9)

where there is no well-defined map to fill in the dashed arrow, and the injectivity of the lower \( u^* \) follows from Lemma 5 and Proposition 6.

The map \( D^b(X[4]) \xrightarrow{I^\vee_{\Xi}} D^b(X) \) takes the the skyscraper sheaf of a point to \( I_\xi \), where \( \xi \subset X \) is a subscheme of length 4, so the left adjoint \( I^\vee_{\Xi}[2] \) and its induced map on K-theory

\[ K_{\text{top}}(X) \xrightarrow{I^\vee_{\Xi}} K_{\text{top}}(X[4]) \]

For a careful proof that Hilbert schemes are moduli spaces of stable sheaves when \( H^1(\mathcal{O}) = 0 \), see [18, Lem. B.5.6].
takes classes in $[I] \perp$ to classes of rank 0. Thus in the big diagram (9), if we replace $K_{\text{top}}(X)$ with the subspace $[I] \perp$ then multiplying by $[L^\vee]$ does not affect the first Chern class, so we can forget the last column of the diagram.

The composition

$$[I] \perp \subset K_{\text{top}}(X) \xrightarrow{f_*} K_{\text{top}}(X[4]) \xrightarrow{c_1} H^2(X[4], \mathbb{Z})$$

is a Hodge isometry by O’Grady’s classic calculation [25]. Moreover the map $t^*$ is an isomorphism on $H^2$, so the big diagram (9) gives a Hodge isometry from $[I] \perp \subset K_{\text{top}}(X)$ to $u^*(H^2(Z_0, \mathbb{Z})) \subset H^2(M_0, \mathbb{Z})$. Because $M \setminus M_0$ and $Z \setminus Z_0$ have codimension at least 2, this is the same as $u^*(H^2(Z, \mathbb{Z})) \subset H^2(M, \mathbb{Z})$.

If we take (8), pass to topological K-theory, and include first Chern classes, we get a diagram

$$K_{\text{top}}(X) \xrightarrow{f_*} K_{\text{top}}(Y) \xrightarrow{i^*} K_{\text{top}}(M) \xrightarrow{c_1} K_{\text{top}}(M_0) \quad H^2(M, \mathbb{Z}) \xrightarrow{i^*} H^2(M_0, \mathbb{Z}).$$

(10)

In [2, Prop. 3] it is proved that the functor

$$D^b(X) \xrightarrow{f_*(-2h)[4]} D^b(Y)$$

is fully faithful, and its image is $\langle O_Y(-h), O_Y, O_Y(h) \rangle \perp$. In this paper we are taking $A = \langle O_Y, O_Y(h), O_Y(2h) \rangle \perp$, so we should use the equivalence

$$D^b(X) \xrightarrow{f_*(-2h)[4]} A.$$ 

We claim that the induced map on topological K-theory takes $[I]$ to our class $\lambda_2 - \lambda_1$. To see this, first observe that the functor $D^b(Y) \xrightarrow{f(-h)} D^b(X)$ takes the skyscraper sheaf of a point $O_y$ to some $I$, so its right adjoint $D^b(X) \xrightarrow{f(-2h)[4]} D^b(Y)$ takes $I$ to the projection of $O_y$ into $A$. In K-theory we have $[O_y] = [O_\ell(2)] - [O_\ell(1)]$, where $\ell$ is a line on $Y$, so the class of the projection of $O_y$ is $\lambda_2 - \lambda_1$.

Thus in our second big diagram (10), the first map takes $[I] \perp$ isometrically to $(\lambda_2 - \lambda_1) \perp \subset K_{\text{top}}(A) \subset K_{\text{top}}(Y)$. We conclude that the map (3) takes $(\lambda_2 - \lambda_1) \perp$ isometrically to $u^*(H^2(Z, \mathbb{Z}))$, as desired.
1.3 Proof of Proposition 6

Now we recall the construction of the Pfaffian cubic fourfold $Y$, the K3 surface $X$, and the correspondence $\Gamma \subset X \times Y$ in detail. Fix a vector space $V \cong \mathbb{C}^6$ and a generic 6-dimensional subspace $L \subset \Lambda^2 V^*$. The cubic is

$$Y = \left\{ [\varphi] \in \mathbb{P}(L) \mid \text{rank}(\varphi) = 4 \right\} = \left\{ [\varphi] \in \mathbb{P}(L) \mid \varphi^3 = 0 \right\},$$

the K3 surface is

$$X = \left\{ [P] \in \text{Gr}(2, V) \mid \varphi|_P = 0 \text{ for all } \varphi \in L \right\},$$

and the correspondence is

$$\Gamma = \left\{ ([P], [\varphi]) \in X \times Y \mid P \cap \ker \varphi \neq 0 \right\}.$$ 

For a generic choice of $L$, both $X$ and $Y$ are smooth, $X$ does not contain a line, and $Y$ does not contain a plane.

In [2, Lem. 4] it is proved that $\Gamma$ is generically 4-to-1 over $Y$. We improve this as follows:

**Lemma 7.** If $X$ does not contain a $(-2)$-curve then $\Gamma$ is flat over $Y$.

**Proof.** Let $\Gamma_\varphi \subset X$ be the scheme-theoretic fiber of $\Gamma$ over a point $[\varphi] \in Y$. We will argue that $\Gamma_\varphi$ is zero-dimensional; then by the proof of [2, Lem. 4] it has length 4.

Consider the Schubert cycle

$$\Sigma_\varphi = \left\{ [P] \in \text{Gr}(2, V) \mid P \cap \ker \varphi \neq 0 \right\},$$

of which $\Gamma_\varphi$ is a linear section, and the normalization

$$\tilde{\Sigma}_\varphi = \left\{ (l, P) \in \mathbb{P}(\ker \varphi) \times \text{Gr}(2, V) \mid l \subset P \right\}.$$ 

We observe that $\tilde{\Sigma}_\varphi$ is a $\mathbb{P}^4$-bundle over $\mathbb{P}(\ker \varphi) = \mathbb{P}^1$. First we claim that the preimage of $\Gamma_\varphi$ in $\tilde{\Sigma}_\varphi$ meets each $\mathbb{P}^4$ fiber in at most one point, even scheme-theoretically: to see this, note that the $\mathbb{P}^4$s are embedded linearly in $\text{Gr}(2, V) \subset \mathbb{P}(\Lambda^2 V)$, so if a linear section contained more than one point of a $\mathbb{P}^4$ it would contain a line, contradicting our hypothesis on $X$.

Next we claim that the preimage of $\Gamma_\varphi$ in $\tilde{\Sigma}_\varphi$ meets at most finitely many $\mathbb{P}^4$ fibers. Otherwise it would meet them all, hence would give a section of this $\mathbb{P}^4$-bundle over $\mathbb{P}^1$, hence a smooth rational curve on $X$, again contradicting our hypothesis.

Thus the preimage of $\Gamma_\varphi$ in $\tilde{\Sigma}_\varphi$ is zero-dimensional, so $\Gamma_\varphi$ is as well. 

\end{proof}
Lemma 8. Suppose that $X$ has Picard rank 1, so in particular $X$ contains no $(-2)$-curve and thus $\Gamma$ induces a regular map $j : Y \to X^{[4]}$. Let $\tilde{H}$ and $B$ be the basis for $\text{Pic}(X^{[4]})$ used in $[5, \S 13]$. Then

\begin{align*}
j^*B &= 9h \\
j^*\tilde{H} &= 14h
\end{align*}

where $h \in \text{Pic}(Y)$ is the hyperplane class.

Proof. By construction of $j : Y \to X^{[4]}$ we have a Cartesian diagram

\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \Xi \\
p \downarrow & & \downarrow p' \\
Y & \longrightarrow & X^{[4]}
\end{array}
\]

By $[20, \text{Lem. 3.7}]$ we have $B = -c_1(p'_*\mathcal{O}_\Xi)$, and thus

\[j^*B = -c_1(p_*\mathcal{O}_\Gamma).\]

Applying the Grothendieck–Riemann–Roch formula to the embedding $i : Y \hookrightarrow \mathbb{P}^5$, we get

\[i_* \text{ch}(p_*\mathcal{O}_\Gamma) = \text{ch}(i_*p_*\mathcal{O}_\Gamma) \cdot \text{td}(\mathcal{O}_{\mathbb{P}^5}(3)).\] \tag{11}

Let $0 \to \mathcal{P} \to \mathcal{O}_X \otimes V \to \mathcal{Q} \to 0$ be the restriction to $X$ of the tautological bundle sequence on $\text{Gr}(2, V)$. In $[2, \S 2]$ it is argued that $(1 \times i)(\Gamma) \subset X \times \mathbb{P}^5$ is the degeneracy locus of a map $\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{P}^5} \to \mathcal{Q}^\vee \boxtimes \mathcal{O}_{\mathbb{P}^5}(1)$, of the expected dimension. Thus with the Eagon–Northcott resolution of $(1 \times i)_*\mathcal{O}_\Gamma$ and a lot of Schubert calculus, we can compute the right-hand side of (11). We carried out this computation by hand and checked it with the Schubert2 package of Macaulay2 [9]; we include our code in the ancillary file Gamma.m2.

The result is

\[i_* \text{ch}(p_*\mathcal{O}_\Gamma) = 12h - 27h^2 + \frac{65}{2}h^3 - \frac{33}{2}h^4 + \frac{19}{8}h^5 \in H^*(\mathbb{P}^5, \mathbb{Q}).\]

Because $i_*(h) = 3h^2$, we read off $c_1(p_*\mathcal{O}_\Gamma) = -9h$, so

\[j^*B = 9h \in H^2(Y, \mathbb{Q}).\]

To find $j^*\tilde{H}$, let $H \subset X$ be a hyperplane section, and let $q' : \Xi \to X$ be the natural projection. Then $p'_*q'^*\mathcal{O}_H$ is a sheaf supported on the locus of 4-tuples of which one is contained in $H$, that is, on $\tilde{H}$; and this sheaf
has generic rank 1 on its support, so its first Chern class is $\tilde{H}$. Using the resolution

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

we find that

$$i_* \text{ch}(p_*(\mathcal{O}_Γ \otimes q^*\mathcal{O}_H)) = 42h^2 - 91h^3 + 56h^4 - \frac{35}{4}h^5.$$ 

Again the code is included in `Gamma.m2`. Taking the coefficient of $h^2$ and dividing by 3, we get $j^*(\tilde{H}) = 14h$ as desired.

**Lemma 9.** Suppose that $X$ has Picard rank 1, so again $Γ$ induces a regular map $j: Y \to X^{[4]}$. Then every effective divisor on $X^{[4]}$ meets $j(Y)$.

**Proof.** First we calculate the movable cone of $X^{[4]}$, applying Bayer and Macrì’s [5, Prop. 13.1] with $d = 7$ and $n = 4$. Then $d(n - 1) = 21$ is not a perfect square; and $3X^2 - 7Y^2 = 1$ has no solution, as we see by reducing mod 3; so we are in case (c). The minimal positive solution to $X^2 - 21Y^2 = 1$ is $X = 55, Y = 12$. This satisfies $X \equiv 1 \pmod{3}$ rather than $X \equiv -1 \pmod{3}$, contradicting [5, eq. (33)], but the latter contains a typo which Debarre corrects in [8, Example 3.20]. Thus the movable cone is spanned by

$$\tilde{H} \quad \text{and} \quad 55\tilde{H} - 84B.$$ 

The same information can in principle be obtained from work of Markman [23], which does not use Bridgeland stability conditions, but [5] is more user-friendly.

Next we need to know that if $D$ is an effective divisor on $X^{[4]}$ and $M$ is movable, then $q(D, M) \geq 0$, where $q$ is the Beauville–Bogomolov–Fujiki pairing. This follows from the proof of [12, Thm. 7]. In fact a result of Boucksom [7, Prop. 4.4] implies that the pseudo-effective cone is dual to the closure of the movable cone, but we do not need the full strength of this.

Now if an effective divisor $D$ on $X^{[4]}$ does not meet $j(Y)$, then $j^*(D) = 0$, so from Lemma 8 we know that $D$ is a multiple of $9\tilde{H} - 14B$. But we have

$$q(\tilde{H}, \tilde{H}) = 14 \quad q(\tilde{H}, B) = 0 \quad q(B, B) = -6,$$

and thus

$$q(9\tilde{H} - 14B, \tilde{H}) = 126 \quad q(9\tilde{H} - 14B, 55\tilde{H} - 84B) = -126,$$

so no multiple of $9\tilde{H} - 14B$ pairs non-negatively with both walls of the movable cone. □
We conclude this section by deducing Proposition 6 from the lemmas above. From [2, §3] we know that the open immersion \( t: Z_0 \to X^{[4]} \), restricted to \( Y_0 := Y \cap Z_0 \), agrees with the map \( j: Y_0 \to X^{[4]} \) induced by \( I_\Gamma \). If \( X \) has Picard rank 1 then \( Y_0 = Y \) by Lemma 7. The open immersion \( t: Z_0 \to X^{[4]} \) gives a birational map \( Z \to X^{[4]} \). Because \( Z \) and \( X^{[4]} \) are minimal models, this birational map is biregular on an open set \( W \) containing \( Z_0 \) whose complement has codimension at least 2 in both \( Z \) and \( X^{[4]} \). If \( Z \setminus Z_0 \) has any component of codimension 1, then so too does \( W \setminus Z_0 \), and taking its closure in \( X^{[4]} \) we get an effective divisor that does not meet \( j(Y) \), contradicting Lemma 9.

### 1.4 Proof of Theorem 1 for a very general cubic

By our discussion in §1.1, to prove Theorem 1 it is enough to show that for a single cubic \( Y \), the map (1) takes \( H^4_{\text{prim}}(Y, \mathbb{Z}) \) into \( u^*(H^2_{\text{prim}}(Z, \mathbb{Z})) \) and respects the pairings. We deduce this from Theorem 2 when \( Y \) is not in any Noether–Lefschetz divisor.

Because \( Y \) is Noether–Lefschetz general, the transcendental lattice of \( \langle \lambda_2 - \lambda_1 \rangle \subset K_{\text{top}}(\mathcal{A}) \) is \( \langle \lambda_1, \lambda_2 \rangle \). The transcendental lattice of \( H^2(Z, \mathbb{Z}) \) is contained in \( H^2_{\text{prim}}(Z, \mathbb{Z}) \), so the Hodge isometry (4) takes \( \langle \lambda_1, \lambda_2 \rangle \) into \( u^*(H^2_{\text{prim}}(Z, \mathbb{Z})) \); but they are primitive sublattices of the same rank, so (4) gives an isomorphism between them.

So we need to compare the action of (3) on \( \langle \lambda_1, \lambda_2 \rangle \) with the action of (1) on \( H^4_{\text{prim}}(Y, \mathbb{Z}) \). In the diagram (5), we know from [3, Prop. 2.3] that \( v \) takes \( \langle \lambda_1, \lambda_2 \rangle \subset K_{\text{top}}(\mathcal{A}) \) isometrically to \( H^4_{\text{prim}}(Y, \mathbb{Z}) \subset H^*(Y, \mathbb{Q}) \). To finish proving Theorem 1, it remains to show that the maps

\[
H^4_{\text{prim}}(Y, \mathbb{Z}) \to H^2(M, \mathbb{Q})
\]

induced by \( v(I_C(3h)) \in H^*(Y \times M, \mathbb{Q}) \) and \( [C] \in H^6(Y \times M, \mathbb{Z}) \) are the same. To see this, observe that

\[
\text{ch}(I_C) = 1 - [C] + \text{higher-order terms},
\]

\[
\text{ch}(I_C') = 1 + [C] + \text{higher-order terms}.
\]

What happens when we map a class \( \alpha \in H^4_{\text{prim}}(Y) \) to \( H^*(M, \mathbb{Q}) \) using \( v(I_C(3h)) \)? First we multiply by \( (td Y)^{1/2} \) and \( \text{ch}(\mathcal{O}_Y(3h)) \), which are polynomials in the hyperplane class \( h \in H^2(Y, \mathbb{Z}) \) and thus have no effect on \( \alpha \). Then we pull up to \( H^*(Y \times M, \mathbb{Q}) \), multiply with \( \text{ch}(I_C') \), and push down to \( H^*(M, \mathbb{Q}) \), which yields \( [C]_s \alpha + \text{higher-order terms} \). Then we multiply by \( (td M)^{1/2} \), which does not affect the leading term \( [C]_s \alpha \). Then we take the degree-2 part, leaving \( [C]_s \alpha \) as desired.
2 Proof of Theorem 3

We adapt an argument developed in [1] and [15, Prop. 4.1] and polished in [16, §3.2], staying especially close to the latter reference.

From Verbitsky’s hyperkähler Torelli theorem [27, 14, 22] and Markman’s work on the monodromy of manifolds of K3^[n]-type [23, §9], we know that if n − 1 is a prime power, then two manifolds M and M’ of K3^[n]-type are birational (or bimeromorphic) if and only if there is a Hodge isometry $H^2(M, \mathbb{Z}) \cong H^2(M', \mathbb{Z})$. If n − 1 has two or more prime factors there is a more subtle statement, but this does not concern us since the eightfold Z is of K3^[4]-type.

By [16, Prop. 1.24 and Prop. 1.13], a cubic fourfold Y is in $C_d$ for some d satisfying (**) if and only if the Mukai lattice $K_{top}(A)$ is Hodge-isometric to the Mukai lattice $\tilde{H}(S, \mathbb{Z})$ of a K3 surface, and similarly with (**′) and the Mukai lattice $\tilde{H}(S, \alpha, \mathbb{Z})$ of a twisted K3 surface.

If $\varphi: K_{top}(A) \to \tilde{H}(S, \mathbb{Z})$ is a Hodge isometry, let $v = \varphi(\lambda_2 - \lambda_1)$, which is a primitive vector satisfying $v^2 = 6$. Then for a v-generic polarization $h \in \text{Pic}(S)$, the moduli space $M := M_h(v)$ of $h$-stable sheaves with Mukai vector $v$ is a smooth variety of K3^[4]-type, and satisfies $H^2(M, \mathbb{Z}) \cong v^\perp \subset \tilde{H}(S, \mathbb{Z})$. Thus

$$H^2(Z, \mathbb{Z}) \cong \langle \lambda_2 - \lambda_1 \rangle ^\perp \cong v^\perp \cong H^2(M, \mathbb{Z}).$$

Conversely, any Hodge isometry $\langle \lambda_2 - \lambda_1 \rangle ^\perp \cong v^\perp$ extends to a Hodge isometry $K_{top}(A) \to \tilde{H}(S, \mathbb{Z})$ taking $\lambda_2 - \lambda_1$ to ±v by [24, Cor. 1.5.2]: the discriminant group of the lattice $v^\perp$ is $\mathbb{Z}/6$ whose automorphism group is ±1. With twisted K3 surfaces the argument is similar. This proves parts (i) and (ii) of Theorem 3.

To prove part (iii), by [1, Prop. 5] or by the proof of [16, Prop. 3.4(i)] we see that Z is birational to $S^{[4]}$ for some K3 surface S if and only if the algebraic lattice $K_{num}(A) \subset K_{top}(A)$ contains a copy of the hyperbolic plane $U$ which contains $\lambda_2 - \lambda_1$; or equivalently, if and only if there there is a vector $w \in K_{num}(A)$ with $\chi(w, w) = 0$ and $\chi(w, \lambda_2 - \lambda_1) = 1$.

---

4Huybrechts [16] and many other authors define the Mukai pairing on $K_{top}(A)$ as having the opposite sign from the Euler pairing, in order to agree with the Beauville–Bogomolov pairing on $H^2$ of various hyperkähler varieties, whereas Addington and Thomas [3, 1] define it having the same sign as the Euler pairing. In effect, the first convention regards $K_{top}(A)$ as a Hodge structure of weight 2, the second as a Hodge structure of weight 0. In the calculation that follows, to avoid confusion, we only use Euler pairing and do not mention the Mukai pairing.
If there is such a $w$, let $L = \langle \lambda_1, \lambda_2, w \rangle \subset K_{\text{num}}(A)$, and let $n = \chi(w, \lambda_1)$, so the Gram matrix of the Euler pairing on $L$ is

$$
\begin{pmatrix}
-2 & 1 & n \\
1 & -2 & n + 1 \\
n & n + 1 & 0
\end{pmatrix}
$$

Thus $\text{disc}(L) = 6n^2 + 6n + 2$. Let $M$ be the saturation of $L$ in $K_{\text{num}}(A)$, let $a$ be the index of $L$ in $M$, and let $d = \text{disc}(M)$; then $a^2d = \text{disc}(L)$, so $d = (6n^2 + 6n + 2)/a^2$. By [3, Prop. 2.5] we have $Y \in C_d$.

Conversely, suppose that $a \equiv 1 \pmod{6}$: to see this, observe that $6n^2 + 6n + 2$ is the norm of the primitive vector $(n, -n - 1)$ in the $A_2$ lattice, hence satisfies $(\ast\ast)$ by [16, Prop. 1.13(iii)], so $a$ must be a product of primes $p \equiv 1 \pmod{3}$. Thus $d \equiv 2 \pmod{6}$; write $d = 6k + 2$ and $a = 3m + 1$. By [1, Lem. 9] there is an element $\tau \in K_{\text{num}}(A)$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice on which the Gram matrix of the Euler pairing is

$$
\begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & 2k
\end{pmatrix}.
$$

Then the vector

$$w := (m - n)\lambda_1 + (2m - n)\lambda_2 + a\tau$$

satisfies $\chi(w, w) = 0$ and $\chi(w, \lambda_2 - \lambda_1) = 1$, as desired.

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