A Novel Study on Direct Sampling Method for Imaging Multiple Targets

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Abstract. The Direct Sampling Method (DSM) is well known to be a fast, stable, and effective
detection technique in inverse scattering problems, but its analysis is restricted to imaging of a single
target. Motivated by this, we investigate the mathematical structure of DSM for imaging small
multiple dielectric targets by establishing a relationship with the radii and permittivities of targets,
and Bessel function of order zero. This structure explains why DSM can be applied to the imaging of
multiple targets. Numerical results support our findings.

Introduction

One of the interesting research subjects in inverse scattering problem is to retrieve the location of
unknown small inhomogeneities from measured scattered field or far-field patterns. Various
algorithms have been suggested for this in the literature and most of them are based on Newton-type
iteration scheme. Related researches can be found in [1,2,3,4] and references therein. In general, to
obtain good results through such a scheme, the iteration procedure must start with a good initial guess
that is close to the unknown object location. Therefore, these methods additionally demand the
development of fast algorithms to obtain good initial guess. Motivated from this, non-iterative
techniques for retrieving the location of such inhomogeneities have been investigated. Those include
the MultIPLE SIgnal Classification (MUSIC) algorithm [5,6,7], the linear sampling method [8,9,10],
and Kirchhoff and subspace migrations [11,12,13]. Although these techniques are confirmed to be
fast, stable, and effective, they still require observations from a significant number of directions of
the incident and scattered field or far-field data to obtain an acceptable result.

The Direct Sampling Method (DSM) is a non-iterative technique for finding location of small
inhomogeneities, introduced in [14,15,16]. Contrary to the non-iterative techniques mentioned above,
DSM requires an incident field with a single or small number of directions of propagation. It has been
identified that the indicator function of DSM can be represented as the Bessel function of order zero
of the first kind [14,16]. This representation helps explain why small inhomogeneities are detectable
by DSM as well as the presence of ring-shaped artifacts centered at the location of inhomogeneities.
However, it still does not explain unexpected results that occur when the permittivity or size of one
inhomogeneity is smaller than that for the others. To explore the structure of the indicator function of
DSM here we present an expression for the Bessel function of order zero of the first kind and identify
its particular properties.

Direct Scattering Problem and Asymptotic Expansion Formula

Let $\Sigma_m$ be a small inhomogeneity located in two-dimensional space. Throughout this paper, we
assume that every $\Sigma_m$ is a small ball such that

$$\Sigma_m = x_m + \alpha_m B_m,$$

where $x_m$ and $\alpha_m$ denote the location and size of $\Sigma_m$, respectively. Here, $B_m$ is a simple connected
smooth domain containing the origin. For the sake of simplicity, we assume that $|B_m| = 1$. We denote
$\Sigma$ to be the collection of $\Sigma_m$, $\omega$ the applied angular frequency, $k$ the wavenumber, and $\lambda$ the
wavelength such that $\alpha_m \ll \lambda$. We assume that all materials involved are non-magnetic and
characterized by their electric permittivity. Let $\varepsilon_0$ and $\varepsilon_m$ denote the dielectric permittivity of $\Sigma_m$ and
background, respectively. Then, we can define the following piecewise constant
\[ \varepsilon(x) = \begin{cases} 
  \varepsilon_m & \text{for } x \in \Sigma_m \\
  \varepsilon_0 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma. 
\end{cases} \]

In this paper, we consider the following plane-wave illumination: let \( \psi_{\text{inc}}(x) = \exp(ikd \cdot x) \) be the incident field with propagation direction \( d \in S^1 \) and \( \psi(x) \) be the time-harmonic total field that satisfies the Helmholtz equation

\[ \Delta \psi(x) + \omega^2 \varepsilon(x) \psi(x) = 0 \quad \text{in } \mathbb{R}^2 \]

with transmission condition on the boundary of \( \Sigma_m \). Here, \( S^1 \) denotes the two-dimensional unit circle. Let \( \psi_{\text{scat}}(x) \) be the scattered field, which satisfies the Sommerfeld radiation condition

\[ \lim_{|x| \to \infty} \sqrt{|x|} \left( \frac{\partial \psi_{\text{scat}}(x)}{\partial |x|} - ik \psi_{\text{scat}}(x) \right) = 0 \]

uniformly in all directions \( \theta = x/|x| \). The far-field pattern is defined as function \( \psi_{\infty}(\theta, d) \) that satisfies

\[ \psi_{\text{scat}}(x) = \frac{\exp(ik|x|)}{\sqrt{|x|}} \psi_{\infty}(\theta, d) \quad \text{as } |x| \to \infty \]

uniformly in all directions \( \theta = x/|x| \).

**Direct Sampling Method and Its Structure**

In this section, we introduce an indicator function of DSM from the measured far-field pattern. The main idea of DSM is to construct an indicator function that has large and small values inside and outside of an inhomogeneity, respectively. For any search point \( z \), the indicator function of DSM from the measured far-field pattern is given by

\[ \Phi(z) = \frac{\langle \psi_{\infty}(\theta, d), \exp(-ik\theta \cdot z) \rangle_{L^2(S^1)}}{\| \psi_{\infty}(\theta, d) \|_{L^2(S^1)} \| \exp(-ik\theta \cdot z) \|_{L^2(S^1)}} \]

where

\[ \langle E(\theta), F(\theta) \rangle_{L^2(S^1)} = \int_{S^1} E(\theta) \overline{F(\theta)} d\theta. \]

Notice that, based on the following identity (see [16])

\[ \int_{S^1} \left( \frac{1 + i}{4\sqrt{k\pi}} \exp(-ik\theta \cdot x) \right) \left( \frac{1 + i}{4\sqrt{k\pi}} \exp(-ik\theta \cdot z) \right) d\theta \approx CJ_0(k|z - x|), \]

the indicator function can be represented as

\[ \Phi(z) = \sum_{m=1}^{M} J_0(k|z - x_m|). \]

Here, \( J_0 \) denotes the Bessel function of order zero of the first kind and \( C \) is a constant depending on the wavenumber. With this representation, we can observe that the indicator function will have a value of 1 at the location of inhomogeneities and small value (approximately 0) outside of inhomogeneities. This representation also tells us why small inhomogeneities are detectable by DSM and why ring-shaped artifacts centered at the location of inhomogeneities are present. However, this is restricted for the following situation: there is only one inhomogeneity or permittivities and sizes of all inhomogeneities are the same. The above representation does not accurately describe other situations. For this reason, we need further analysis of the structure of the indicator function. Throughout a careful analysis, we can obtain the following result:
\[ \Phi(z) = \frac{|\Psi(z)|}{\max\{\Psi(z) : z \in \mathbb{R}^2\}}. \]

where

\[ \Psi(z) = \sum_{m=1}^{M} (\alpha_m)^2 \left( \frac{\varepsilon_m - \varepsilon_0}{\sqrt{\varepsilon_0}} \right) J_0(k|z - x_m|). \]

Based on this expression, we can observe that the performance of the indicator function is highly dependent on the values of permittivity and size of inhomogeneities and the total number of observation directions \( N \). This means that if the permittivity or the size of one inhomogeneity is larger than that of the others, DSM can identify with high accuracy. Otherwise, identifying the inhomogeneity via DSM will be difficult.

Simulation Results

In this section, we present results of numerical simulations to validate the expression derived for the indicator function. We set \( \lambda = 0.4 \) and one incident direction \( \mathbf{d} = [\cos 45^\circ, \sin 45^\circ] \). The far-field pattern is measured at 30 points uniformly distributed on \( S^1 \). The search area is fixed at \([-1,1] \times [-1,1]\) and the step size between search point is 0.02. Far-field pattern data are generated by means of the Foldy-Lax framework to avoid an inverse crime.

Figure 1 shows the map of the indicator function of DSM for a single inhomogeneity. The location, size, and permittivity of inhomogeneity is set to \([0.5,0.3], 0.1, \) and 5, respectively. Same as previous results in \([14,16]\), location of the inhomogeneity is identified very accurately.

Figure 2 shows the map of the indicator function of DSM for two different inhomogeneities. The radii and permittivities of all inhomogeneities are set to 0.1 and 5, respectively, and locations of inhomogeneities are set to \([0.5,0.3]\) and \([-0.5,-0.3]\). Same as in the previous example and previous results in \([14,16]\), locations of all inhomogeneities are recovered very accurately.
Now, let us consider the identification of two different inhomogeneities when their permittivities are different from each other. The radii of all inhomogeneities are equally set to 0.1 and permittivities of inhomogeneities located at $[0.5, 0.3]$ and $[-0.5, -0.3]$ are given by 5 and 3, respectively. In Figure 3, we can observe that the location of the inhomogeneity with larger permittivity is detected. Although the appearance of artifacts disturbs the recognition, the location of the remaining inhomogeneities can be extracted from the map of the indicator function.

![Figure 3. Map of the indicator function and location of inhomogeneities. White circles denote the inhomogeneities.](image)

Now, let us consider the identification of three different inhomogeneities with the same radii and permittivities as 0.1 and 5, respectively. Locations of inhomogeneities are $[0.7, 0.5]$, $[-0.7, 0.0]$, and $[0.2, -0.5]$. Here, locations of all inhomogeneities can be identified very accurately as well.

![Figure 4. Map of the indicator function and location of inhomogeneities. White circles denote the inhomogeneities.](image)

For the final example, we consider the identification of three different inhomogeneities with different permittivities. The radii and locations of all inhomogeneities are kept same as the previous example. The permittivities of inhomogeneities at $[0.7, 0.5]$, $[-0.7, 0.0]$, and $[0.2, -0.5]$ are 4, 5, and 3, respectively. Since the permittivity of the inhomogeneity centered at $[-0.7, 0.0]$ is larger than the others, this inhomogeneity is identified very clearly. However, the permittivity of the inhomogeneity centered at $[0.2, -0.5]$ is smaller than the others and many artifacts are present in the map of the indicator function so that identification of the location of this inhomogeneity is extremely difficult.

![Figure 5. Map of the indicator function and location of inhomogeneities. White circles denote the inhomogeneities.](image)
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