A stochastic volatility model with jumps

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Abstract

We consider a stochastic volatility model with jumps where the underlying asset price is driven by the process sum of a 2-dimensional Brownian motion and a 2-dimensional compensated Poisson process. The market is incomplete, resulting in infinitely many Equivalent Martingale Measures. We find the set equivalent martingale measures, and we hedge by minimizing the variance using Malliavin calculus.

Keywords: stochastic volatility model, jumps, European options, incomplete markets, Malliavin calculus, mean-variance hedging.

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1 Introduction

In the pioneer work of Black and Scholes (1973), the financial asset prices are modeled by the Brownian motion, which is a continuous process. The black and Scholes model does not take into account the jumps which can occur at any time and randomly. Three years later, Merton (1976) suggested a model with jumps. Since then, the study of financial mathematical models have attracted the interest of many mathematicians.

The recent international financial crisis and its effects on global stock markets have showed once again the importance of adding jumps to financial modeling for stock prices. Unlike the continuous case, discontinuous models assume this powerful hypothesis: at any moment, a financial price can jump to decrease (increase) and attain,
in a negligible time, a significant lower (higher) value. In other words, these models can simulate financial crisis, thus their importance.

On the other hand, the models in Black and Scholes (1973) and Merton (1976) assume a deterministic volatility. Later on, new models with stochastic volatility have been suggested to take into account the so called *smile* effect. Most of the works on these models assume -for simplification- the continuity of the asset price trajectories (driven by a Brownian motion). For continuous models with stochastic volatility, We refer the reader to Heston (1993), Hull and White (1987), Stein and Stein (1991) and Hagan, Kumar, Lesniewski and Woodward (2002).

We need more realistic models where the stochastic process describing the price trajectories involves jumps. And in the same models, volatility should be stochastic not deterministic in order to consider the *smile* effect. Several papers on stochastic volatility models including jumps have been done. These works show clearly that the stochastic volatility models combined with jump-diffusions are the best for modeling stock prices. Nevertheless they are still not well explored. This is due to their complication. Actually, they generate the incompleteness of the market, i.e., not every contingent claim can be hedged. For instance, while in Bates (1996) the stock price dynamics includes jumps, the stochastic volatility is still considered continuous. In Duffie, Pan, and Singleton (2000) or Broadie, Chernov, and Johannes (2005) the stochastic volatility contains jumps. However, these papers do not deal with the problem of equivalent martingale measure nor with the problem of hedging strategies for options.

In this work we are interested in a more general framework for discontinuous dynamics for the asset price with discontinuous stochastic volatility. The main contribution of this work is solving two problems: finding the equivalent martingale measure minimizing the entropy and finding hedging strategies under a general framework for jump-diffusions markets combined with stochastic volatility.

Assume that we have a market with two assets: a risky asset $S$ which is related to a European Call option and a risk free one with price’s process $A := (A_t)_{t \in [0,T]}$ where

\[\text{the list is not exhaustive.}\]
\[ dA_t = r_tA_t dt, \quad t \in [0, T], \quad A_0 > 0, \] and \( r_t \) is a deterministic function denoting the interest rate. Formally, let the underlying asset price of \( S \) be

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t(1) dW_t^{(1)} + a_t^{(3)} dM_t^{(1)}], \quad t \in [0, T], \quad S_0 = x > 0,
\]

with

\[
\frac{dY_t}{Y_t} = \mu_t^Y dt + \sum_{i=1}^{2} \sigma_t^{(i)} [a_t^{(i)} dW_t^{(i)} + a_t^{(i+2)} dM_t^{(i)}], \quad Y_0 = y \in \mathbb{R},
\]

where \( W = (W^{(1)}, W^{(2)}) \) is a 2-dimensional Brownian motion and \( M = (M^{(1)}, M^{(2)}) \) is a 2-dimensional compensated Poisson process with independent components and deterministic intensity \( (\int_0^t \lambda_s^{(1)} ds, \int_0^t \lambda_s^{(2)} ds) \). We assume that for \( 1 \leq i \leq 4, a^{(i)} : [0, T] \rightarrow \mathbb{R} \) is a deterministic function.

The most serious problem in a stochastic volatility model is incompleteness. These models involve the existence of infinitely many equivalent martingale measures (E.M.M.) i.e probabilities equivalent to the historical one under which the discounted prices are martingales. First we characterize the set of E.M.M.. We show that a probability \( Q \) equivalent to the historical probability \( P \) is specified by its Radon-Nikodym density w.r.t \( P \)

\[
\rho_T = \prod_{i=1}^{2} \exp \left( \int_0^T \mu_t^{(i)} dW_t^{(i)} - \frac{1}{2} \int_0^T (\beta_s^{(i)})^2 ds \right) \exp \left( \int_0^T \ln(1 + \beta_s^{(i+2)}) dM_t^{(i)} + \int_0^T \lambda_s^{(i)} \left[ \ln(1 + \beta_s^{(i+2)}) - \beta_s^{(i+2)} \right] ds \right),
\]

where \((\beta_t)_{t \in [0,T]} \) is a \( \mathbb{R}^4 \)-valued predictable process such that \( \beta^{(3)}, \beta^{(4)} > -1 \). If \( Q \) is a \( P \)-E.M.M., \( \beta^{(1)} \) and \( \beta^{(3)} \) are related by

\[
\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) = 0,
\]

see Proposition 3.1.

The process \( \left( -\frac{\mu_t - r_t}{a_t^{(1)} \sigma(t, Y_t)}, 0, 0, 0 \right) \) is an example of a \( \mathbb{R}^4 \)-valued predictable process satisfying the above equation, and it defines a \( P \)-E.M.M. This means that the set of \( P \)-E.M.M. is not empty. Moreover, since \( \beta^{(2)} \) and \( \beta^{(4)} \) do not appear in the last equation, so they can be chosen arbitrarily, and thus there exists infinitely many \( P \)-E.M.M..
Mean-variance hedging

We hedge using the mean-variance hedging approach initiated by Föllmer and Son-dermann (1986), and we find the strategy by applying Malliavin calculus.

Consider an option with payoff \( f(S_T) \), where \((S_t)_{t \in [0,T]} \) is the asset price with maturity \( T \). We work with a \( P \)-E.M.M \( \hat{Q} \). Let \((\hat{\eta}_t, \hat{\zeta}_t)_{t \in [0,T]} \) be a self-hedging strategy and \((\hat{V}_t)_{t \in [0,T]} \) be the portfolio value process. Using the chaotic calculus, we conclude that the strategy minimizing the variance \( E_{\hat{Q}} \left[ (f(S_T) - \hat{V}_T)^2 \right] \) is given by

\[
\hat{\eta}_t = \frac{a_t^{(1)} E[D^{\hat{W}(1)} f(S_T) \mid \mathcal{F}_t] + \lambda_t^{(1)} (1 + \hat{\beta}_t^{(3)}) a_t^{(3)} E[D^{\hat{N}(1)} f(S_T) \mid \mathcal{F}_t]}{((a_t^{(1)})^2 + \lambda_t^{(1)} (1 + \hat{\beta}_t^{(3)}) (a_t^{(3)})^2) e^{\int_0^t \rho_s ds} \sigma(t, Y_t) S_t},
\]

where \( \hat{W}(1) = W^{(1)}_t - \int_0^t \hat{\beta}(1) ds \), and the operators \( D^{\hat{W}(1)} \) and \( D^{\hat{N}(1)} \) are respectively the Malliavin derivative in the direction of the one dimensional Brownian motion \( \hat{W}(1) \) and the Malliavin operator in the direction of the Poisson process \( \hat{N}(1) \).

This paper is organized as follows: In Section 2, we present some necessary formulas. In the third section we introduce the model. The fourth one is devoted to the hedging by minimizing the variance via Malliavin calculus. In the last section, we characterize the E.M.M. minimizing the entropy, which allows us to establish explicit formulae for the strategy.

2 Preliminary

Let \( W = (W^{(1)}, W^{(2)}) \) be a 2-dimensional Brownian motion and \( N = (N^{(1)}, N^{(2)}) \) be a 2-dimensional Poisson process with independent components and deterministic intensity \( (\int_0^t \lambda_s^{(1)} ds, \int_0^t \lambda_s^{(2)} ds) \). We work in a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\), where \((\mathcal{F}_t)_{t \in [0,T]} \) is the natural filtration generated by \( W \) and \( N \). We denote by \( M = (M^{(1)}, M^{(2)}) \) the associated compensated Poisson process, i.e. for \( i = 1, 2 \) and \( t \in [0,T] \), we have \( dM_t^{(i)} = dN_t^{(i)} - \lambda_t^{(i)} dt \). Both \((\mathcal{F}_t)_{t \in [0,T]}\)-martingales \( W \) and \( M \) are independent.

**Definition 2.1** We denote by \( \Gamma \) be the set of all \( \mathcal{F}_t \)-predictable processes \((\gamma_t)_{t \in [0,T]}\)
with values in $\mathbb{R}^4$ such that
\[
\sum_{i=1}^{2} E_P \left[ \int_{0}^{t} (\gamma_s^{(i)})^2 ds \right] + \sum_{i=1}^{2} E_P \left[ \int_{0}^{t} (\gamma_s^{(i+2)})^2 \lambda_s^i ds \right] < \infty, \quad t \in [0, T].
\]

For a semi-martingale $X$ with $X_0 = 0$, the Doléans-Dade exponential $\mathcal{E}(X)_t$ is the unique solution of the stochastic differential equation
\[
Z_t = 1 + \int_{0}^{t} Z_s \cdot dX_s.
\]

We have (Theorem 36 of Protter (1990))
\[
\mathcal{E}(X)_t = \exp \left( X_t - \frac{1}{2} [X_t, X_t] \right) \prod_{s \leq t} (1 + \Delta X_s) \exp(-\Delta X_s). \tag{2.0.1}
\]

**Remark 2.1** Notice that for $\gamma \in \Gamma$ such that $\gamma^{(3)}, \gamma^{(4)} > -1$ and for $i = 1, 2$
\[
\mathcal{E}(\gamma^{(i)}W^{(i)})_t = \exp \left( \int_{0}^{t} \gamma_s^{(i)} dW_s^{(i)} - \frac{1}{2} \int_{0}^{t} (\gamma_s^{(i)})^2 ds \right),
\]
\[
\mathcal{E}(\gamma^{(i+2)}M^{(i)})_t = \exp \left( \int_{0}^{t} \ln(1 + \gamma_s^{(i+2)}) dM_s^{(i)} \right)
\]
\[
+ \int_{0}^{t} \lambda_s^{(i)} \left[ \ln(1 + \gamma_s^{(i+2)}) - \gamma_s^{(i+2)} \right] ds.
\]

The next lemma is the martingale representation theorem (Jacod (1979)).

**Lemma 2.1** Let $Z = (Z_t)_{t \in [0, T]}$ be a $\mathcal{F}_t$-martingale. There exists a predictable process $\gamma \in \Gamma$ such that
\[
dZ_t = \sum_{i=1}^{2} \gamma_t^{(i)} dW_t^{(i)} + \sum_{i=1}^{2} \gamma_t^{(i+2)} dM_t^{(i)}, \quad t \in [0, T].
\]

### 3 The model

Consider a market with two assets: a risky asset which is related to a European call option and a riskless one. The maturity is $T$ and the strike is $K$. The price of the riskless asset is given by
\[
dA_t = r_t A_t dt, \quad t \in [0, T], \quad A_0 = 1,
\]

where \( r_t \) is deterministic and denotes the interest rate. The price of the risky asset has a stochastic volatility and is given by

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma(t, Y_t)[a_t^{(1)} dW_t^{(1)} + a_t^{(3)} dM_t^{(1)}], \quad t \in [0, T], \quad S_0 = x > 0,
\]

\[
dY_t = \mu_Y^t dt + \sum_{i=1}^{2} \sigma_i^t [a_t^{(i)} dW_t^{(i)} + a_t^{(i+2)} dM_t^{(i)}], \quad t \in [0, T], \quad Y_0 = y \in \mathbb{R},
\]

where for \( 1 \leq i \leq 4, a^{(i)}_t : [0, T] \rightarrow \mathbb{R} \) is a deterministic function. We assume that \( \sigma(t, \cdot) \neq 0 \), and \( 1 + \sigma(t, \cdot)a^{(3)}_t > 0, \quad t \in [0, T] \).

We have

\[
S_t = x \exp \left( \int_0^t a_s^{(1)} \sigma(s, Y_s) dW_s^{(1)} + \int_0^t (\mu_s - a_s^{(3)} \lambda_s^{(1)} \sigma(s, Y_s) - \frac{1}{2} (a_s^{(1)})^2 \sigma^2(s, Y_s)) ds \right) \times \prod_{k=N_t}^{k=N_{k+1}} (1 + a^{(3)}_{T_k^{(1)}} \sigma(T_k^{(1)}, Y_{T_k^{(1)}}))
\]

\[0 \leq t \leq T, \text{ where } (T_k^{(1)})_{k \geq 1} \text{ denotes the jump times of } (N_t^{(1)})_{t \in [0, T]}.
\]

### 3.1 Change of probability

Let \( Q \) be a \( P \)-equivalent probability; by the Radon-Nikodym theorem there exists a \( \mathcal{F}_T \)-measurable random variable, \( \rho_T := \frac{dQ}{dP} \), such that \( Q(A) = E_P[\rho_T 1_A], \ A \in \mathcal{P} (\Omega) \). Notice that \( \rho_T \) is strictly positive \( P \)-a.s, since \( Q \) is equivalent to \( P \), and \( E_P[\rho_T] = E_P[\rho_T 1_{\Omega}] = 1 \). Consider now the \( P \)-martingale \( \rho = (\rho_t)_{t \in [0, T]} \) defined by

\[
\rho_t := E_P[\rho_T | \mathcal{F}_t] = E_P \left[ \frac{dQ}{dP} | \mathcal{F}_t \right].
\]

**Definition 3.1** \( \mathcal{H} \) is the set of all \( P \)-E.M.M., i.e \( Q \in \mathcal{H} \) if and only if \( Q \simeq P \) and the discounted prices are \( Q \)-martingales.

The next proposition gives the Radon-Nikodym density w.r.t \( P \) of a \( P \)-E.M.M..

**Proposition 3.1** Let \( Q \in \mathcal{H} \). There exists a predictable process \( (\beta_t)_{t \in [0, T]} \) taking values in \( \mathbb{R}^4 \) such that \( \beta^{(3)}, \beta^{(4)} > -1 \) and the Radon-Nikodym density of \( Q \) w.r.t \( P \) is given by

\[
\rho_T = \prod_{i=1}^{2} \mathcal{E}(\beta^{(i)} W^{(i)})_{T} \mathcal{E}(\beta^{(i+2)} M^{(i)})_{T}
\]
Moreover \( \beta^{(1)} \) and \( \beta^{(3)} \) are related by

\[
\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) = 0. \tag{3.1.2}
\]

**Proof.** We follow Bellamy (1999) for the case of a discontinuous market with deterministic volatility. By the martingale representation theorem (Lemma 2.1) there exists a predictable process \( (\gamma_t)_{t \in [0, T]} \in \Gamma \) such that

\[
d\rho_t = \sum_{i=1}^{2} \gamma_t^{(i)} dW_t^{(i)} + \sum_{i=1}^{2} \gamma_t^{(i+2)} dM_t^{(i)}, \quad t \in [0, T].
\]

We have \( P(\rho_t > 0, t \in [0, T]) = 1 \); assuming \( \beta := \frac{\gamma}{\rho} \), we obtain

\[
\frac{d \rho_t}{\rho_t} = \sum_{i=1}^{2} \beta_t^{(i)} dW_t^{(i)} + \sum_{i=1}^{2} \beta_t^{(i+2)} dM_t^{(i)}, \quad t \in [0, T]. \tag{3.1.1}
\]

(3.1.1) follows from (2.0.1). In addition \( (e^{-\int_0^t r_s ds} S_t)_{t \in [0, T]} \) is a \( Q \)-martingale, in other words, \( (e^{-\int_0^t r_s ds} S_t \rho_t)_{t \in [0, T]} \) is a \( P \)-martingale. The integration by parts formula (Protter (1990)) gives

\[
d(e^{-\int_0^t r_s ds} S_t \rho_t) = \rho_t d(e^{-\int_0^t r_s ds} S_t) + e^{-\int_0^t r_s ds} S_t d\rho_t + d[e^{-\int_0^t r_s ds} S_t, \rho_t],
\]

with

\[
d[e^{-\int_0^t r_s ds} S_t, \rho_t] = \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) dt + \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) dN_t^{(1)},
\]

\[
= \left( \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) \right) dt + \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) dM_t^{(1)}.
\]

Therefore

\[
d(e^{-\int_0^t r_s ds} S_t \rho_t) = \rho_t S_t e^{-\int_0^t r_s ds} \left[ (\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t)) dt \\
+ (\beta_t^{(1)} + \sigma(t, Y_t) a_t^{(1)} dW_t^{(1)} + \beta_t^{(2)} dW_t^{(2)} \\
+ \left( \sigma(t, Y_t) a_t^{(3)} + \beta_t^{(3)} (1 + \sigma(t, Y_t) a_t^{(3)}) \right) dM_t^{(1)} + \beta_t^{(4)} dM_t^{(2)} \right].
\]
Thus $Q$ is a $P$-E.M.M. if
\[
\mu_t - r_t + \beta_t^{(1)} a_t^{(1)} \sigma(t, Y_t) + \lambda_t^{(1)} \beta_t^{(3)} a_t^{(3)} \sigma(t, Y_t) = 0.
\]
\[
\square
\]
Notice that there are no restrictions on $\beta^{(2)}$ and $\beta^{(4)}$, which means that if $\mathcal{H} \neq \emptyset$, then $\mathcal{H}$ contains infinitely many $P$-E.M.M.

## 4 Equivalent Martingale Measure minimizing the entropy

Let $\Gamma^H$ be the set of processes $\beta \in \Gamma$ satisfying (3.1.2). The Radon-Nikodym derivative $\rho_T$ associated to $\beta$ and given by (3.1.1) defines a $P$-E.M.M.. From now on, a $P$-E.M.M. $Q$ in $\mathcal{H}$ will be denoted by $Q^\beta$, where $\beta \in \Gamma^H$. The process $\left(\frac{\mu_t - r_t}{a_t^{(1)} \sigma(t, Y_t)}, 0, 0, 0\right)$ belongs to $\Gamma^H$ and it defines a $P$-E.M.M., so $\mathcal{H} \neq \emptyset$. Thus $\mathcal{H}$ contains infinitely many $P$-E.M.M.. We choose the one that minimizes the relative entropy. Let $Q^\beta \in \mathcal{H}$.

Denoting by $I(Q^\beta, P)$ the relative entropy of $Q^\beta$ w.r.t $P$, we have
\[
I(Q^\beta, P) = E_P \left[ \frac{dQ^\beta}{dP} \ln \frac{dQ^\beta}{dP} \right].
\]
Our aim is to minimize $I(P, Q^\beta)$ under $\mathcal{H}$. We have
\[
I(P, Q^\beta) = E_Q^\beta \left[ \frac{dP}{dQ^\beta} \ln \frac{dP}{dQ^\beta} \right].
\]
Therefore the problem is to find a $\hat{\beta}$ which satisfies
\[
I(P, Q^{\hat{\beta}}) = \min_{\beta \in \Gamma^H} -E_P \left[ \ln \frac{dQ^\beta}{dP} \right]. \tag{4.0.1}
\]

**Lemma 4.1** The minimization problem (4.0.1) is equivalent to the minimization of
\[
(\mu_t - r_t + \lambda_t^{(1)} a_t^{(3)} \beta_t^{(3)} \sigma(t, Y_t))^2 - 2\sigma^2(t, Y_t)(a_t^{(1)} \lambda_t^{(1)}) \left[ \ln(1 + \beta_t^{(3)}) - \beta_t^{(3)} \right] \]
\[-2\sigma^2(t, Y_t)\lambda_t^{(2)} \left[ \ln(1 + \beta_t^{(4)}) - \beta_t^{(4)} \right],
\]
under all $\beta = \left(\frac{\mu_t - r_t + \lambda_t^{(1)} a_t^{(3)} \sigma(t, Y_t)\beta_t^{(3)}}{\sigma(t, Y_t)a_t^{(1)}}, 0, \beta^{(3)}, \beta^{(4)}\right) \in \Gamma^H$. 

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Proof. Let $Q^3 \in \mathcal{H}$. By (3.1.1)

\[
I(P, Q^3) = -E_P \left[ \ln \frac{dQ^3}{dP} \right]
\]

\[
= E_P \left[ \int_0^T \sum_{i=1}^2 \frac{1}{2} (\beta_t^{(i)})^2 - \lambda_t^{(i)} \left[ \ln(1 + \beta_t^{(i+2)}) - \beta_t^{(i+2)} \right] \right]
\]

\[
= E_P \left[ \int_0^T \frac{G(\beta_t)}{2\sigma^2(t, Y_t)(a_t^{(1)})^2} dt \right],
\]

where $\beta \in \Gamma^H$, and $G$ is the function defined by

\[
G(\beta_t) = 2\sigma^2(t, Y_t)(a_t^{(1)})^2 \left( \frac{1}{2}(\beta_t^{(1)})^2 + \frac{1}{2}(\beta_t^{(2)})^2 - \lambda_t^{(1)} \left[ \ln(1 + \beta_t^{(3)}) - \beta_t^{(3)} \right]
\]

\[ -\lambda_t^{(2)} \left[ \ln(1 + \beta_t^{(4)}) - \beta_t^{(4)} \right] \right), \quad t \in [0, T].
\]

For a fixed $t$, we have by (3.1.2),

\[
G(\beta_t) = (\mu_t - r_t + \lambda_t^{(1)} a_t^{(3)} \sigma(t, Y_t) \beta_t^{(3)})^2 + \sigma^2(t, Y_t)(a_t^{(1)})^2 \left( (\beta_t^{(2)})^2
\]

\[ -2\lambda_t^{(1)} \left[ \ln(1 + \beta_t^{(3)}) - \beta_t^{(3)} \right] - 2\lambda_t^{(2)} \left[ \ln(1 + \beta_t^{(4)}) - \beta_t^{(4)} \right] \right), \quad t \in [0, T].
\]

Since $\beta_t^{(2)}$ appears only in the term $\sigma^2(t, Y_t)(a_t^{(1)})^2 (\beta_t^{(2)})^2$ which is always positive, $\beta_t^{(2)}$ must be equal to zero.

\[
\square
\]

The following proposition gives the solution to the minimization 4.0.1.

**Proposition 4.1** Consider $(\hat{\beta}_t^{(1)}, \hat{\beta}_t^{(2)}, \hat{\beta}_t^{(3)}, \hat{\beta}_t^{(4)})_{t \in [0, T]} \in \Gamma^H$, with

\[
\hat{\beta}_t^{(2)} = \hat{\beta}_t^{(4)} = 0, \quad \hat{\beta}_t^{(1)} = \begin{cases} \frac{r_t - \mu_t - \lambda_t^{(1)} a_t^{(3)} \sigma(t, Y_t) \beta_t^{(3)}}{\sigma(t, Y_t) a_t^{(1)}} & \text{if } a_t^{(1)} \neq 0, \\ 0 & \text{if } a_t^{(1)} = 0, \end{cases}
\]

and let $\hat{\beta}_t^{(3)}$ be the unique solution of the equation

\[
\lambda_t^{(1)} \sigma(t, Y_t)(a_t^{(3)})^2 x + (a_t^{(1)})^2 \sigma(t, Y_t) \left( \frac{x}{1+x} \right) - a_t^{(3)}(r_t - \mu_t) = 0. \quad (4.0.2)
\]

Then, the P.E.M.M. $\hat{Q}$ defined by its Radon-Nikodym density

\[
\prod_{i=1}^2 \mathbf{E}(\hat{\beta}_t^{(i)} W_t^{(i)})_T \mathbf{E}(\hat{\beta}_t^{(i+2)} M_t^{(i)}),
\]

is the P.E.M.M. minimizing $I(P, Q^3)$.
Proof. By Lemma 4.1, we have to minimize the function $F : [-1, \infty[ \times [-1, \infty[ \rightarrow \mathbb{R}$ defined by

$$F(x, y) = (\mu_t - r_t + \lambda_t^{(1)} a_t^{(3)} \sigma(t, Y_t)x)^2 - 2\sigma^2(t, Y_t)(a_t^{(1)})^2 \left( \lambda_t^{(1)} [\ln(1 + x) - x] \right.$$

$$\left. + \lambda_t^{(2)} [\ln(1 + y) - y] \right),$$

for a fixed $t$ in $[0, T]$. Let $F'_x$ and $F'_y$ denote the first order partial derivatives of $F$. The critical points of $F$ are determined by solving the equations $F'_x(x, y) = F'_y(x, y) = 0$. Let $\hat{x}$ be the solution of (4.0.2). It is unique since the function $x \mapsto 2(\lambda_t^{(1)} \sigma^2(t, Y_t)(a_t^{(3)})^2 x + 2(a_t^{(1)})^2 \lambda_t^{(1)} \sigma^2(t, Y_t) \frac{x}{1 + x} + 2\lambda_t^{(1)} \sigma(t, Y_t) a_t^{(3)} (\mu_t - r_t),$ is strictly increasing from $]-1, \infty[ \to \mathbb{R}$. One can check that $(\hat{x}, 0)$ is the only point which satisfies $F'_x(x, y) = F'_y(x, y) = 0$. Moreover we have

$$(F''_{xy}(\hat{x}, 0))^2 - F''_{xx}(\hat{x}, 0)F''_{yy}(\hat{x}, 0) < 0 \quad \text{and} \quad F''_{xx}(\hat{x}, 0) > 0.$$

Therefore $F$ has a strict local minimum at $(\hat{x}, 0)$. This minimum is global since $F$ goes to infinity when $x$ (y) approaches infinity. \(\square\)

5 Hedging

In this section we are interested in finding an optimal hedging strategy for the model described in Section 3. We find the strategy minimizing the variance using the Malliavin calculus. From now on, we work with $\hat{Q}$: the P-E.M.M. minimizing the entropy given by $\hat{\beta}$ from Proposition 4.1. Consider the two processes $\hat{W} = (\hat{W}^{(1)}, \hat{W}^{(2)})$ and $\hat{M} = (\hat{M}^{(1)}, \hat{M}^{(2)})$ where for $i = 1, 2$

$$\hat{W}_t^{(i)} = W_t^{(i)} - \int_0^t \hat{\beta}_s^{(i)} ds, \quad t \in [0, T], \quad \text{and} \quad \hat{M}_t^{(i)} = M_t^{(i)} - \int_0^t \lambda_s^{(i)} \hat{\beta}_s^{(i+2)} ds, \quad t \in [0, T],$$

by Girsanov theorem (Jacod (1979)) $\hat{W}$ is a $\hat{Q}$-Brownian motion and $\hat{M}$ is a $\hat{Q}$-compensated Poisson process. Under $\hat{Q}$, $(S_t)_{t \in [0, T]}$ satisfies

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, Y_t)[a_t^{(1)} d\hat{W}_t^{(1)} + a_t^{(3)} d\hat{M}_t^{(1)}], \quad t \in [0, T], \quad S_0 = x > 0.$$
5.1 Chaotic calculus

Let us denote by $\hat{X}$ the process

$$(\hat{X}^{(1)}_t, \hat{X}^{(2)}_t, \hat{X}^{(3)}_t, \hat{X}^{(4)}_t) = (\hat{W}^{(1)}_t, \hat{W}^{(2)}_t, \hat{M}^{(1)}_t, \hat{M}^{(2)}_t), \quad t \in [0, T],$$

and let $(\hat{\mathcal{F}}_t)_{t \in [0,T]}$ be the natural filtration generated by $\hat{X}$. We define the multiple stochastic integral and introduce the Malliavin gradient and the Clark-Ocone formula in the multidimensional Brownian-Poisson case (the following definitions and formulas can be extended for the $d-$dimensional case, $d \leq 4$). For more details we refer to Løkka (1999), Nualart (1995), Nualart and Vives (1990), Øksendal (1996) and Pri-vault (1997 a,b). Let $(e_1, e_2, e_3, e_4)$ be the canonical base of $\mathbb{R}^4$. For $g_n \in L^2([0, T]^n)$ we define the $n$-th iterated stochastic integral of the function $f_n e_{i_1} \otimes \ldots \otimes e_{i_n}$, with $1 \leq i_1, \ldots, i_n \leq 4$, by

$$I_n(g_n e_{i_1} \otimes \ldots \otimes e_{i_n}) := n! \int_0^T \int_0^{t_n} \ldots \int_0^{t_2} g_n(t_1, \ldots, t_n) d\hat{X}^{(i_1)}_{t_1} \ldots d\hat{X}^{(i_n)}_{t_n}.$$ 

The iterated stochastic integral of a symmetric function $f_n = (f_n^{(i_1, \ldots, i_n)})_{1 \leq i_1, \ldots, i_n \leq 4} \in L^2([0, T], \mathbb{R}^4)^{\otimes n}$, where $f_n^{(i_1, \ldots, i_n)} \in L^2([0, T]^n)$, is

$$I_n(f_n) := \sum_{i_1, \ldots, i_n = 1}^4 I_n(f_n^{(i_1, \ldots, i_n)} e_{i_1} \otimes \ldots \otimes e_{i_n})$$

$$= n! \sum_{i_1, \ldots, i_n = 1}^4 \int_0^T \int_0^{t_n} \ldots \int_0^{t_2} f_n^{(i_1, \ldots, i_n)}(t_1, \ldots, t_n) d\hat{X}^{(i_1)}_{t_1} \ldots d\hat{X}^{(i_n)}_{t_n}.$$ 

Recall that $\hat{X}$ has the Chaotic Representation Property (CRP) which states that any square-integrable $\hat{\mathcal{F}}_T$-measurable functional can be expanded into a series of multiple stochastic integrals -w.r.t $\hat{X}_t$- of deterministic functions. For $F \in L^2(\Omega)$, there exists a unique sequence $(f_n)_{n \in \mathbb{N}}$ of deterministic symmetric functions $f_n = (f_n^{(i_1, \ldots, i_n)})_{i_1, \ldots, i_n \in \{1, \ldots, 4\}} \in L^2([0, T], \mathbb{R}^4)^{\otimes n}$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n). \quad (5.1.1)$$
Definition 5.1 Let \( l \in \{1, \ldots, 4\} \), the operator \( \hat{D}^{(l)} : \text{Dom} (\hat{D}^{(l)}) \subset L^2(\Omega) \to L^2(\Omega, [0,T]) \) maps \( F \in \text{Dom} (\hat{D}^{(l)}) \) (\( F \) having the representation (5.7.1)) to the process \( (\hat{D}^{(l)}_t F)_{t \in [0,T]} \) given by

\[
\hat{D}^{(l)}_t F := \sum_{n=1}^{\infty} \sum_{i=1}^{4} \sum_{l_1, \ldots, l_n = 1} I_{l_1} \left( f^{(l_1, \ldots, l_n)}(t_1, \ldots, t_l, t_{l+1}, \ldots, t_n) e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes e_{i_n} \right)
\]

\[
= \sum_{n=1}^{\infty} n I_{n-1} (f^l_n(*, t)), \quad dP \times dt - a.e.
\]

with \( f^l_n = (f^{(l_1, \ldots, l_{n-1})}_n e_{i_1} \otimes \cdots \otimes e_{i_{n-1}})_{1 \leq i_1, \ldots, i_{n-1} \leq 4} \).

The domain of \( \hat{D}^{(l)} \) is

\[
\text{Dom} (\hat{D}^{(l)}) = \left\{ F : \sum_{n=0}^{\infty} \sum_{l_1, \ldots, i_n = 1} I_n (f^{(l_1, \ldots, l_n)}_n e_{i_1} \otimes \cdots \otimes e_{i_n}) \in L^2(\Omega) : \sum_{l_1, \ldots, i_n = 1}^{n} n! \| f^{(l_1, \ldots, l_n)}_n \|_{L^2([0,T]_n)}^2 < \infty \right\}.
\]

The probabilistic interpretations of \( \hat{D}^{(l)} \) for the Brownian motion and Poisson process cases are respectively given below.

**The Brownian operator** For \( 1 \leq l \leq 2 \), the operator \( \hat{D}^{(l)} \) is, in fact, the Malliavin derivative in the direction of the one dimensional Brownian motion \( \hat{W}^{(l)} \). So, we have for \( 1 \leq l \leq 2 \) and \( F = f \left( \hat{W}_{t_1}, \ldots, \hat{W}_{t_n} \right) \in L^2(\Omega) \), where \( (t_1, \ldots, t_n) \in [0,T]^n \) and \( f(x^{11}, x^{21}, \ldots, x^{1n}, x^{2n}) \in C^\infty_b(\mathbb{R}^{2n}) \)

\[
\hat{D}^{(l)}_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x^{lk}} \left( \hat{W}_{t_1}, \ldots, \hat{W}_{t_n} \right) 1_{[0,t_k]}(t).
\]

To find the Malliavin derivative of an Itô integral, we need the following proposition (see corollary 5.13 of Øksendal (1996)).

**Proposition 5.1** Let \( (u_t)_{t \in [0,T]} \) be a \( \hat{F}_t \)-adapted process such that \( u_t \in \text{Dom} (\hat{D}^{(l)}) \). Then for \( l = 1, 2 \) we have

\[
\hat{D}^{(l)}_t \int_0^T u_t d\hat{W}^{(l)}_s = \int_0^T (\hat{D}^{(l)}_t u_s) d\hat{W}^{(l)}_s + u_t,
\]

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The Poisson operator
For $3 \leq l \leq 4$, $\hat{D}(l)$ is the Malliavin operator in the direction of the Poisson process $N^{(l-2)}$. For $F \in \text{Dom}(\hat{D}(l))$

$$\hat{D}(l)^{F}(\omega^{(1)}, \ldots, \omega^{l}) = \left\{ \begin{array}{ll}
F(\omega^{(1)}, \omega^{(2)}, \omega^{(3)} + 1_{[t, \infty)}, \omega^{(4)}) - F(\omega^{(1)}, \ldots, \omega^{(4)}), & l = 3, \\
F(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)} + 1_{[t, \infty)}) - F(\omega^{(1)}, \ldots, \omega^{(4)}), & l = 4.
\end{array} \right.$$ 

The Clark-Ocone formula is given by the next proposition.

**Proposition 5.2 (The Clark-Ocone formula)** Consider a square-integrable, $\hat{F}$-measurable, functional $F$ such that $F \in \bigcap_{l=1}^{4} \text{Dom}(\hat{D}^{l}(l))$. $F$ has the following predictable representation

$$F = E_{Q}[F] + \sum_{l=1}^{2} \int_{0}^{T} E_{Q}[\hat{D}(l)^{F} | \hat{F}_{t}]d\hat{W}_{t}^{l} + \sum_{l=1}^{2} \int_{0}^{T} E_{Q}[\hat{D}(l+2)^{F} | \hat{F}_{t}]d\hat{M}_{t}^{l}.$$ 

### 5.2 Strategy minimizing the variance

Suppose that we are required to find a portfolio $(\hat{\zeta}_{t}, \hat{\eta}_{t})_{t \in [0,T]}$ which leads to a given value $\hat{V}_{T} = F$. The process $(\hat{V}_{t})_{t \in [0,T]}$ denote the value of the portfolio and $\hat{\zeta}_{t}$ and $\hat{\eta}_{t}$ denote the number of shares invested at time $t$ in the risky and in the riskfree assets respectively. We have for, $t \in [0,T]$, $\hat{V}_{t} = \hat{\zeta}_{t}A_{t} + \hat{\eta}_{t}S_{t}$. The strategy is assumed to be self-financing thus $d\hat{V}_{t} = \hat{\zeta}_{t}dA_{t} + \hat{\eta}_{t}dS_{t}$ and

$$d\hat{V}_{t} = r_{t}\hat{V}_{t}dt + \sigma(t, Y_{t})\hat{\eta}_{t}S_{t}[a^{(1)}_{t}d\hat{W}_{t}^{(1)} + a^{(3)}_{t}d\hat{M}_{t}^{(1)}], \quad t \in [0,T].$$

Moreover for any $t \in [0,T]$, we have

$$d\left(e(-\int_{0}^{t} r_{s}ds)\hat{V}_{t}\right) = -r_{t}e(-\int_{0}^{t} r_{s}ds)\hat{V}_{t}dt + e(-\int_{0}^{t} r_{s}ds)d\hat{V}_{t}$$

$$= e(-\int_{0}^{t} r_{s}ds) \left[ -r_{t}\hat{V}_{t}dt + r_{t}\hat{V}_{t}dt + \sigma(t, Y_{t})\hat{\eta}_{t}S_{t}[a^{(1)}_{t}d\hat{W}_{t}^{(1)} + a^{(3)}_{t}d\hat{M}_{t}^{(1)}]\right],$$

therefore

$$e(-\int_{0}^{t} r_{s}ds)\hat{V}_{T} = \hat{V}_{0} + \int_{0}^{T} e(-\int_{0}^{s} r_{u}du)\sigma(t, Y_{t})\hat{\eta}_{t}S_{t}[a^{(1)}_{t}d\hat{W}_{t}^{(1)} + a^{(3)}_{t}d\hat{M}_{t}^{(1)}].$$

\[\text{Notice that, unlike the Brownian case, the Malliavin operator in the Poisson space is not a derivative}\]
and

\[ \hat{V}_T = \hat{V}_0 e^{\int_0^T r_s ds} + \int_0^T e^{\left(\int_0^T r_s ds\right)} \sigma(t, Y_t) \hat{\eta}_t S_t a_t^{(1)} d\hat{W}_t^{(1)} + a_t^{(3)} d\hat{M}_t^{(1)}. \] (5.2.2)

Assuming that \( F \) satisfies the hypothesis of the Proposition 5.2, apply the Clark-Ocone formula to \( F \). Comparing with the equation (5.2.2), we see that the equality \( \hat{V}_T = F \) cannot hold unless

\[ E[D_t \hat{W}^{(2)} F \mid \hat{F}_t] = E[D_t \hat{N}^{(2)} F \mid \hat{F}_t] = 0, \] (5.2.3)

because the expression of \( \hat{V}_T \) in (5.2.2) does not contain an integral term w.r.t. \( d\hat{W}_t^{(2)} \) nor w.r.t. \( d\hat{M}_t^{(2)} \).

Take \( F \) equals to the payoff \( f(S_T) \) of the model in section 3), we see that (5.2.3) is not satisfied, because \( D_t^{(2)} f(S_T) = f'(S_T)D_t^{(2)} S_T \neq 0 \) and \( D_t^{(2)} f(S_T) \neq 0 \). In other words, the payoff \( f(S_T) \) is not attainable. The market is then incomplete.

Next we aim to find the strategy \( (\hat{\zeta}_t, \hat{\eta}_t)_{t \in [0, T]} \) that minimizes the variance

\[ E_{\hat{Q}} \left[ (f(S_T) - \hat{V}_T)^2 \right]. \] (5.2.4)

The next proposition gives the strategy minimizing the variance for our model considered in the Section 3

**Proposition 5.3** The strategy minimizing (5.2.4) in the model of Section 3 is given by

\[ \hat{\eta}_t = a_t^{(1)} E[D_t^{(1)} f(S_T) \mid \hat{F}_t] + \lambda_t^{(1)} (1 + \beta_t^{(3)}) a_t^{(3)} E[D_t^{(1)} f(S_T) \mid \hat{F}_t], \] (5.2.5)

where \( a_t^{(1)} = \frac{E[D_t^{(1)} f(S_T) \mid \hat{F}_t]}{\left(\lambda_t^{(1)} (1 + \beta_t^{(3)}) a_t^{(3)}\right)^2 + \left(\lambda_t^{(1)} (1 + \beta_t^{(3)}) a_t^{(3)}\right)^2 e^{\int_0^T r_s ds} \sigma(t, Y_t) S_t}. \)

**Proof.** Notice that the payoff \( f(S_T) = (S_T - K)^+ \) is \( \hat{F}_T \)-measurable. We approach the function \( x \mapsto f(x) = (x - K)^+ \) or \( (K - x)^+ \) by polynomials on compact intervals and proceed as in Øksendal (1996) pp. 5-13. By dominated convergence, \( f(S_T) \in \bigcap_{l=1}^4 \text{Dom} (D^{(l)}) \). Thus by applying the Clark-Ocone formula to \( f(S_T) \) and using (5.2.2), we obtain

\[ E_{\hat{Q}} \left[ (f(S_T) - \hat{V}_T)^2 \right] = \]

\[ E_{\hat{Q}} \left[ \left( \int_0^T \left( E[D_t^{(1)} f(S_T) \mid \hat{F}_t] - e^{\int_0^T r_s ds} \sigma(t, Y_t) \hat{\eta}_t S_t a_t^{(1)} \right) d\hat{W}_t^{(1)} \right)^2 \right]. \]
\[ + \left( \int_0^T E_{\hat{Q}}[D_t \hat{W}^{(2)} f(S_T) \mid \hat{F}_t]d\hat{W}_t^{(2)} \right)^2 + \left( \int_0^T E_{\hat{Q}}[D_t \hat{N}^{(2)} f(S_T) \mid \hat{F}_t]d\hat{M}_t^{(2)} \right)^2 \\
+ \left( \int_0^T \left( E_{\hat{Q}}[D_t \hat{N}^{(2)} f(S_T) \mid \hat{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) \hat{n}_t S_t \alpha_t^{(3)} \right) d\hat{M}_t^{(1)} \right)^2 \]

\[ = \mathbb{E}_{\hat{Q}} \left[ \int_0^T h_2(\hat{n}_t) dt \right], \]

where

\[ h_2(x) = (E_{\hat{Q}}[D_t \hat{W}^{(2)} f(S_T) \mid \hat{F}_t])^2 + \lambda_t^{(2)} (1 + \hat{\beta}_t^{(4)}) (E[D_t \hat{N}^{(2)} f(S_T) \mid \hat{F}_t])^2 \]

\[ + \left( E_{\hat{Q}}[D_t \hat{W}^{(1)} f(S_T) \mid \hat{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) x S_t \alpha_t^{(1)} \right)^2 \]

\[ + \lambda_t^{(1)} (1 + \hat{\beta}_t^{(3)}) \left( E_{\hat{Q}}[D_t \hat{N}^{(1)} f(S_T) \mid \hat{F}_t] - e^{\int_t^T r_s ds} \sigma(t, Y_t) x S_t \alpha_t^{(3)} \right)^2. \]

It is easily verified that \( h_2 \) is convex, hence its minimum is reached at \( h_2'(x) = 0 \). Therefore the strategy minimizing the variance is given by \( (5.2.5) \).

\[ \square \]

### 5.3 Explicit formulae

In order to derive explicit formulas for the strategy obtained in Proposition 5.3, we consider the following two special cases of the model in Section 3.: a continuous stochastic volatility model with Brownian motion and a pure jumps stochastic volatility model with Poisson process.

#### 5.3.1 Brownian case

Assume that \( a_t^{(1)} = a_t^{(2)} = 1 \) and \( a_t^{(3)} = a_t^{(4)} = 0 \), so \( (S_t)_{0 \leq t \leq T} \) depends on Brownian information only. Under \( \hat{Q} \), \( (S_t)_{0 \leq t \leq T} \) is given by

\[ S_t = x \exp \left( \int_0^t \left( r_s - \frac{\sigma^2(s, Y_s)}{2} \right) ds + \int_0^t \sigma(s, Y_s) d\hat{W}_s^{(1)} \right), \]

with

\[ Y_t = y + \int_0^t \left( \mu_s^Y + \sigma_s^{(1)} r_s - \frac{\mu_s^Y}{\sigma(s, Y_s)} \right) ds + \int_0^t \sigma_s^{(1)} d\hat{W}_s^{(1)} + \int_0^t \sigma_s^{(2)} dW_s^{(2)}. \]

In the following proposition we compute the Malliavin derivative of the payoff \( (S_T - K)^+ \). We can replace the result in the formula \( (5.2.5) \), and obtain an explicit formula for the strategy.
Proposition 5.4 We have

\[ D_t^W (S_T - K)^+ = 1_{\{S_T > K\}} S_T \left( \sigma(t, Y_t) + \int_t^T \frac{\partial \sigma}{\partial y}(s, Y_s) D_s^W Y_t d\tilde{W}_s \right) \]

\[ - \int_t^T \sigma(s, Y_s) \frac{\partial \sigma}{\partial y}(s, Y_s) D_s^W Y_s ds \]

where

\[ D_t^W Y_s = \sigma_t^{(1)} \exp \left( - \int_t^s \frac{r_u - \mu_u}{\sigma^2(u, Y_u)} du \right) \quad s \in [t, T]. \] (5.3.7)

Proof. By the chain rule for \( D_t^W \) and thanks to Proposition 5.1 we obtain

\[ D_t^W (S_T - K)^+ = 1_{\{S_T > K\}} S_T \left( \sigma(t, Y_t) + \int_t^T \frac{\partial \sigma}{\partial y}(s, Y_s) D_s^W Y_t d\tilde{W}_s \right) \]

\[ - \int_t^T \sigma(s, Y_s) \frac{\partial \sigma}{\partial y}(s, Y_s) D_s^W Y_s ds \]

which gives (5.3.6). Concerning the other derivative, we have for \( 0 \leq t \leq s \leq T \)

\[ D_t^W Y_s = \int_t^s \left( \mu_u + \sigma_u^{(1)} \frac{r_u - \mu_u}{\sigma(u, Y_u)} \right) du + \sigma_t^{(1)} \]

\[ = \sigma_t^{(1)} - \int_t^s \sigma_u^{(1)} \frac{r_u - \mu_u}{\sigma^2(u, Y_u)} D_t^W Y_u du, \]

So for \( t \) fixed in \([0, T]\), the Malliavin derivative of \( Y_s \) for \( s \in [t, T] \), \( (D_t^W Y_s)_{s \in [t, T]} \), satisfies a stochastic differential equation, whose solution is precisely (5.3.7). \( \square \)

5.3.2 The Poisson case

Similarly, as in the Brownian case, we aim to compute the quantity \( D_t^W (S_T - K)^+ \) and replace the result in the expression of the strategy in order to obtain an explicit formula for the Poisson case. Suppose that we are working in the Poisson space with a 2-dimensional Poisson process. The underlying asset price \((S_t)_{0 \leq t \leq T}\) depends on the Poisson process only. Hence we assume that \( a_t^{(3)} = a_t^{(4)} = 1 \) and \( a_t^{(1)} = a_t^{(2)} = 0 \).

Under \( \hat{Q} \), the dynamics of \((S_t)_{0 \leq t \leq T}\) is given by

\[ S_t = x \exp \left( \int_0^t \left( \mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s)} \ln(1 + \sigma(s, Y_s)) \right) ds + \int_0^t \ln(1 + \sigma(s, Y_s)) d\tilde{M}_s \right), \]
for \( t \in [0, T] \). The process \((Y_t)_{t \in [0, T]}\) under \( \hat{Q} \), has the representation
\[
Y_t = y + \int_0^t \left( \mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s)} \right) ds + \int_0^t \sigma_s^{(1)} d\hat{M}_t^{(1)} + \int_0^t \sigma_s^{(2)} dM_s^{(2)}.
\]

**Proposition 5.5**
\[
D_t^{\hat{M}^{(1)}} (S_T - K)^+ = \left( - (S_T - K)^+ + \left( \exp \left( \int_t^T \left[ \mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s + \sigma_t^{(1)})} \right] ds \right) \right) \right) + \int_t^T \ln(1 + \sigma(s, Y_s + \sigma_t^{(1)})) d\hat{M}_s^{(1)} \times S_t(1 + \sigma(t, Y_t + \sigma_t^{(1)})) - K)
\]

**Proof.** Using the probabilistic interpretation of \( D_t^{\hat{M}^{(1)}} \) given below, we obtain
\[
D_t^{\hat{M}^{(1)}} (S_T - K)^+ = (S_T(\omega + 1_{[t,T]}) - K)^+ - (S_T(\omega) - K)^+.
\]

But
\[
S_T(\omega + 1_{[t,T]}) = x \exp \left( \int_0^t \left[ \mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s + \sigma_t^{(1)})} \right] ds \right) \right) + \int_0^t \ln(1 + \sigma(s, Y_s + \sigma_t^{(1)})) d\hat{M}_s^{(1)} \times \exp \left( \int_t^T \left[ \mu_s + \frac{r_s - \mu_s}{\sigma(s, Y_s + \sigma_t^{(1)})} \right] ds \right) + \int_t^T \ln(1 + \sigma(s, Y_s + \sigma_t^{(1)})) d\hat{M}_s^{(1)} \times (1 + \sigma(t, Y_t + \sigma_t^{(1)})),
\]

and
\[
Y_t(\omega + 1_{[t,T]}) = Y_t + \sigma_t^{(1)}, \quad t \in [0, T],
\]
\[
Y_s(\omega + 1_{[t,T]}) = \begin{cases} Y_s & \text{if } s \in [0, t], \\ Y_s + \sigma_t^{(1)} & \text{if } s \in [t, T]. \end{cases}
\]

The proof is complete. \( \square \)

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