COMPUTING NONCOMMUTATIVE GLOBAL DEFORMATIONS OF D-MODULES

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Abstract. Let \((X, \mathcal{D})\) be a D-scheme in the sense of Beilinson and Bernstein, given by an algebraic variety \(X\) and a morphism \(\mathcal{O}_X \to \mathcal{D}\) of sheaves of rings on \(X\). We consider noncommutative deformations of quasi-coherent sheaves of left \(\mathcal{D}\)-modules on \(X\), and show how to compute their pro-representing hulls. As an application, we compute the noncommutative deformations of the left \(\mathcal{D}_X\)-module \(\mathcal{O}_X\) when \(X\) is any elliptic curve.

Introduction

Let \(k\) be an algebraically closed field of characteristic 0, and let \(X\) be an algebraic variety over \(k\), i.e. an integral separated scheme of finite type over \(k\). A \(D\)-algebra in the sense of Beilinson, Bernstein \([2]\) is a sheaf \(\mathcal{D}\) of associative rings on \(X\), together with a morphism \(i : \mathcal{O}_X \to \mathcal{D}\) of sheaves of rings on \(X\), such that the following conditions hold: (i) \(\mathcal{D}\) is quasi-coherent as a left and right \(\mathcal{O}_X\)-module via \(i\), and (ii) for any open subset \(U \subseteq X\) and any section \(P \in \mathcal{D}(U)\), there is an integer \(n \geq 0\) such that

\[[\ldots [[P, a_1], a_2], \ldots, a_n] = 0\]

for all sections \(a_1, \ldots, a_n \in \mathcal{O}_X(U)\), where \([P, Q] = PQ - QP\) is the commutator in \(\mathcal{D}(U)\). When \(\mathcal{D}\) is a \(D\)-algebra on \(X\), the ringed space \((X, \mathcal{D})\) is called a \(D\)-scheme.

Let us denote the sheaf of \(k\)-linear differential operators on \(X\) by \(\mathcal{D}_X\), and for any Lie algebroid \(g\) of \(X/k\), let us denote the enveloping \(D\)-algebra of \(g\) by \(U(g)\). We see that \(\mathcal{D}_X\) and \(U(g)\) are examples of noncommutative \(D\)-algebras on \(X\), and that \(\mathcal{O}_X\) is an example of a commutative \(D\)-algebra on \(X\).

Let us define a \(D\)-module on a \(D\)-scheme \((X, \mathcal{D})\) to be a quasi-coherent sheaf of left \(\mathcal{D}\)-modules on \(X\). In Eriksen \([3]\), we developed a noncommutative global deformation theory of \(D\)-modules that generalizes the usual (commutative) global deformation theory of \(D\)-modules, and the noncommutative deformation theory of modules in the affine case, due to Laudal. In section 1 - 2, we review the essential parts of this theory, including the global Hochschild cohomology and the global obstruction calculus, all the time with a view towards concrete computations.

The purpose of this paper is to show how to apply the theory in order to compute noncommutative global deformations of interesting \(D\)-modules. In section 3 we consider the noncommutative deformation functor \(\text{Def}_{\mathcal{O}_X} : \mathfrak{a}_1 \to \text{Sets}\) of \(\mathcal{O}_X\) considered as a left \(\mathcal{D}_X\)-module when \(X\) is any elliptic curve over \(k\). Recall that in this case, a quasi-coherent \(\mathcal{D}_X\)-module structure on \(\mathcal{O}_X\) is the same as an integrable connection on \(\mathcal{O}_X\), and according to a theorem due to André Weil, see Weil \([7]\) and also Atiyah \([1]\), a line bundle admits an integrable connection if and only if it has degree zero.
We show that the noncommutative deformation functor $\text{Def}_{D_X} : \mathfrak{a}_1 \to \text{Sets}$ has pro-representing hull $H = k[\langle t_1, t_2 \rangle]/(t_1 t_2 - t_2 t_1) \cong k[[t_1, t_2]]$ that is commutative, smooth and of dimension two. We also compute the corresponding versal family in concrete terms, and remark that it does not admit an algebraization.

1. Noncommutative Global Deformations of $D$-modules

Let $(X, D)$ be a $D$-scheme, and let $\QCoh(D)$ be the category of quasi-coherent sheaves of left $D$-modules on $X$. This is the full subcategory of $\Sh(X, D)$, the category of sheaves of left $D$-modules on $X$, consisting of quasi-coherent sheaves. We recall that a sheaf $F$ of left $D$-modules on $X$ is quasi-coherent if for every point $x \in X$, there exists an open neighbourhood $U \subseteq X$ of $x$, free sheaves $\mathcal{L}_0, \mathcal{L}_1$ of left $D|_U$-modules on $U$, and an exact sequence $0 \to F|_U \to \mathcal{L}_0 \to \mathcal{L}_1$ of sheaves of left $D|_U$-modules on $U$. We shall refer to the quasi-coherent sheaves of left $D$-modules on $X$ as $D$-modules on the $D$-scheme $(X, D)$.

For any $D$-scheme $(X, D)$, $\QCoh(D)$ is an Abelian $k$-category, and we consider noncommutative deformations in $\QCoh(D)$. For any finite family $\mathcal{F} = \{F_1, \ldots, F_p\}$ of quasi-coherent left $D$-modules on $X$, there is a noncommutative deformation functor $\text{Def}^{qc}_{\mathcal{F}} : \mathfrak{a}_p \to \text{Sets}$ of $\mathcal{F}$ in $\QCoh(D)$, generalizing the noncommutative deformation functor of modules introduced in Laudal [5]. We shall provide a brief description of $\text{Def}^{qc}_{\mathcal{F}}$ below; see Eriksen [3] for further details.

We recall that the objects of the category $\mathfrak{a}_p$ of $p$-pointed noncommutative Artin rings are Artinian rings $R$, together with structural morphisms $f : k^p \to R$ and $g : R \to k^p$, such that $g \circ f = \text{id}$ and the radical $I(R) = \ker(g)$ is nilpotent. The morphisms are the natural commutative diagrams. For any $R \in \mathfrak{a}_p$, there are $p$ isomorphism classes of simple left $R$-modules, represented by $\{k_1, k_2, \ldots, k_p\}$, where $k_i = 0 \times \cdots \times k \times \cdots \times 0$ is the $i$th projection of $k^p$ for $1 \leq i \leq p$.

We remark that any $R \in \mathfrak{a}_p$ is a $p \times p$ matrix ring, in the sense that there are $p$ indecomposable idempotents $\{e_1, \ldots, e_p\}$ in $R$ with $e_1 + \cdots + e_p = 1$ and a decomposition $R = \bigoplus R_{ij}$, given by $R_{ij} = e_i R e_j$, such that elements of $R$ multiply as matrices. We shall therefore use matrix notation, and write $R = (R_{ij})$ when $R \in \mathfrak{a}_p$, and $(V_{ij}) = \bigoplus V_{ij}$ when $\{V_{ij} : 1 \leq i, j \leq p\}$ is any family of vector spaces.

For any $R \in \mathfrak{a}_p$, a lifting of $\mathcal{F}$ to $R$ is a sheaf $\mathcal{F}_R$ in $\QCoh(D)$ with a compatible right $R$-module structure, together with isomorphisms $\eta_i : \mathcal{F}_R \otimes_R k_i \to \mathcal{F}_i$ in $\QCoh(D)$ for $1 \leq i \leq p$, such that $\mathcal{F}_R(U) \cong (\mathcal{F}_i(U) \otimes_k R_{ij})$ as right $R$-modules for all open subsets $U \subseteq X$. We say that two liftings $(\mathcal{F}_R, \eta_i)$ and $(\mathcal{F}'_R, \eta'_i)$ are equivalent if there is an isomorphism $\tau : \mathcal{F}_R \to \mathcal{F}'_R$ of $D$-$R$ bimodules on $X$ such that $\eta'_i \circ (\tau \otimes_k k_i) = \eta_i$ for $1 \leq i \leq p$, and denote the set of equivalence classes of liftings of $\mathcal{F}$ to $R$ by $\text{Def}^{qc}_{\mathcal{F}}(R)$. This defines the noncommutative deformation functor $\text{Def}^{qc}_{\mathcal{F}} : \mathfrak{a}_p \to \text{Sets}$.

2. Computing Noncommutative Global Deformations

Let $(X, D)$ be a $D$-scheme, and let $U$ be an open affine cover of $X$ that is finite and closed under intersections. We shall explain how to compute noncommutative deformations in $\QCoh(D)$ effectively using the open cover $U$.

We may consider $U$ as a small category, where the objects are the open subsets $U \subseteq V$, and the morphisms from $U$ to $V$ are the (opposite) inclusions $U \supseteq V$. There is a natural forgetful functor $\QCoh(D) \to \PreSh(U, D)$, where $\PreSh(U, D)$ is the Abelian $k$-category of (covariant) presheaves of left $D$-modules on $U$. For
any finite family \( \mathcal{F} \) in \( \text{QCoh}(\mathcal{D}) \), this forgetful functor induces an isomorphism of noncommutative deformation functors \( \text{Def}_p^\mathcal{F} \to \text{Def}_p^\mathcal{F} \), where \( \text{Def}_p^\mathcal{F} : a_p \to \text{Sets} \) is the noncommutative deformation functor of \( \mathcal{F} \) in \( \text{QCoh}(\mathcal{D}) \) defined in section \([II]\) and \( \text{Def}_p^\mathcal{F} : a_p \to \text{Sets} \) is the noncommutative deformation functor of \( \mathcal{F} \) in \( \text{PreSh}(U, \mathcal{D}) \), defined in a similar way; see Eriksen \([III]\) for details.

**Theorem 1.** Let \((X, \mathcal{D})\) be a \( D \)-scheme, and let \( \mathcal{F} = \{ \mathcal{F}_1, \ldots, \mathcal{F}_n \} \) be a finite family in \( \text{QCoh}(\mathcal{D}) \). If the global Hochschild cohomology \( (\text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}_j, \mathcal{F}_i))) \) has finite \( k \)-dimension for \( n = 1, 2 \), then the noncommutative deformation functor \( \text{Def}_p^\mathcal{F} : a_p \to \text{Sets} \) has a pro-representing hull \( H = H(\mathcal{F}) \), completely determined by \( (\text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}_j, \mathcal{F}_i))) \) for \( n = 1, 2 \) and their generalized Massey products.

In fact, there is a constructive proof of the fact that \( \text{Def}_p^\mathcal{F} : a_p \to \text{Sets} \) of \( \mathcal{F} \) in \( \text{PreSh}(U, \mathcal{D}) \) has a pro-representing hull; see Eriksen \([III]\) for details. The construction in this proof uses the global Hochschild cohomology \( (\text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}_j, \mathcal{F}_i))) \) for \( n = 1, 2 \), and the obstruction calculus of \( \text{Def}_p^\mathcal{F} \), which can be expressed in terms of generalized Massey products on these cohomology groups. We give a brief description of the global Hochschild cohomology and the obstruction calculus below.

### 2.1. Cohomology

For any presheaves \( \mathcal{F}, \mathcal{G} \) of left \( \mathcal{D} \)-modules on \( U \), we recall the definition of the global Hochschild cohomology \( \text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) \) of \( \mathcal{D} \) with values in the bimodule \( \text{Hom}_k(\mathcal{F}, \mathcal{G}) \) on \( U \). For any (opposite) inclusion \( U \supseteq V \) in \( U \), we consider the Hochschild complex \( \text{HC}^*(D(U), \text{Hom}_k(\mathcal{F}(U), \mathcal{G}(V))) \) of \( \mathcal{D}(U) \) with values in the bimodule \( \text{Hom}_k(\mathcal{F}(U), \mathcal{G}(V)) \). We define the category \( \text{Mor} \) \( U \) to have (opposite) inclusions \( U \supseteq V \) in \( U \) as its objects, and nested inclusions \( U' \supseteq U \supseteq V \supseteq V' \) in \( U \) as its morphisms from \( U \supseteq V \) to \( U \supseteq V' \). It follows that we may consider the Hochschild complex

\[
\text{HC}^*(\mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) : \text{Mor} \ U \to \text{Compl}(k)
\]
as a functor on \( \text{Mor} \ U \). The global Hochschild complex \( \text{HC}^*(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) \) is the total complex of the double complex \( \text{D}^{**} = \text{D}^*(U, \text{HC}^*(\mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G}))) \), where \( \text{D}^*(U, -) : \text{PreSh}(\text{Mor} \ U, k) \to \text{Compl}(k) \) is the resolving complex of the projective limit functor; see Laudal \([I]\) for details. Finally, we define the global Hochschild cohomology \( \text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) \) to be the cohomology of the global Hochschild complex \( \text{HC}^*(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) \).

We note that \( \text{H}^n(\text{HC}^*(\mathcal{D}(U), \text{Hom}_k(\mathcal{F}(U), \mathcal{G}(V)))) \cong \text{Ext}^n_{\mathcal{D}(U)}(\mathcal{F}(U), \mathcal{G}(V)) \) for any \( U \supseteq V \) in \( U \) since \( k \) is a field. Hence there is a spectral sequence converging to the global Hochschild cohomology \( \text{HH}^n(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}, \mathcal{G})) \) with

\[
E_2^{pq} = \text{H}^p(U, \text{Ext}^q_{\mathcal{D}(U)}(\mathcal{F}(U), \mathcal{G}(V))),
\]
where \( \text{H}^p(U, -) = \text{H}^p(\text{D}^*(U, -)) \) and we consider \( \text{Ext}^q_{\mathcal{D}(U)}(\mathcal{F}(U), \mathcal{G}(V)) \) as a functor on \( \text{Mor} \ U \), given by \( \{ U \supseteq V \} \mapsto \text{Ext}^q_{\mathcal{D}(U)}(\mathcal{F}(U), \mathcal{G}(V)) \) for all \( q \geq 0 \).

### 2.2. Obstruction calculus

Let \( R \in a_p \) and let \( I = I(R) \) be the radical of \( R \). For any lifting \( \mathcal{F}_R \in \text{Def}_p(R) \) of the family \( \mathcal{F} \) in \( \text{PreSh}(U, \mathcal{D}) \) to \( R \), we have that \( \mathcal{F}_R(U) \cong (\mathcal{F}_i(U) \otimes_k R_{ij}) \) as a right \( R \)-module for all \( U \in U \). Moreover, the lifting \( \mathcal{F}_R \) is completely determined by the left multiplication of \( \mathcal{D}(U) \) on \( \mathcal{F}_R(U) \) for all \( U \in U \) and the restriction map \( \mathcal{F}_R(U) \to \mathcal{F}_R(V) \) for all \( U \supseteq V \) in \( U \). Let us write \( Q^R(U, V) = (\text{Hom}_k(\mathcal{F}_j(U), \mathcal{F}_i(V) \otimes_k R_{ij})) \) and \( Q^R(U) = Q^R(U, U) \) for all \( U \supseteq V \) in \( U \). Then \( \mathcal{F}_R \in \text{Def}_p(R) \) is completely described by the following data:
(1) For all $U \in \mathcal{U}$, a $k$-algebra homomorphism $L(U) : \mathcal{D}(U) \to \mathcal{Q}(U)$ satisfying $L(U)(f_j) = P f_j \otimes \epsilon_j + (\mathcal{F}_i(U) \otimes_k I_{ij})$ for all $P \in \mathcal{D}(U)$, $f_j \in \mathcal{F}_j(U)$.

(2) For all inclusions $U \supseteq V$ in $\mathcal{U}$, a restriction map $L(U, V) \in \mathcal{Q}(U, V)$ satisfying $L(U, V)(f_j) = (f_j|_V) \otimes \epsilon_j + (\mathcal{F}_i(V) \otimes I_{ij})$ for all $f_j \in \mathcal{F}_j(U)$ and $L(U, V) \circ L(U)(P) = L(V)(P|_V) \circ L(U, V)$ for all $P \in \mathcal{D}(U)$.

(3) For all inclusions $U \supseteq V \supseteq W$ in $\mathcal{U}$, we have $L(V, W) = \mathcal{L}(U, W)$ and $L(U, U) = \text{id}$.

A small surjection in $\mathcal{a}_p$ is a surjective homomorphism $u : R \to S$ in $\mathcal{a}_p$ such that $IK = KI = 0$, where $K = \ker(u)$ and $I = I(R)$ is the radical of $R$. To describe the obstruction calculus of $\text{Def}_{\mathcal{F}}$, it is enough to consider the following problem: Given a small surjection $u : R \to S$ and a deformation $\mathcal{F}_S \in \text{Def}_{\mathcal{F}}(S)$, what are the possible liftings of $\mathcal{F}_S$ to $R$? The answer is given by the following proposition; see Eriksen [3] for details:

**Proposition 2.** Let $u : R \to S$ be a small surjection in $\mathcal{a}_p$ with kernel $K$, and let $\mathcal{F}_S \in \text{Def}_{\mathcal{F}}(S)$ be a deformation. Then there exists a canonical obstruction

$$o(u, \mathcal{F}_S) \in (\text{HH}^1(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}_j, \mathcal{F}_i)) \otimes_k I_{ij})$$

such that $o(u, \mathcal{F}_S) = 0$ if and only if there exists a deformation $\mathcal{F}_R \in \text{Def}_{\mathcal{F}}(R)$ lifting $\mathcal{F}_S$ to $R$. Moreover, if $o(u, \mathcal{F}_S) = 0$, then there is a transitive and effective action of $(\text{HH}^1(U, \mathcal{D}, \text{Hom}_k(\mathcal{F}_j, \mathcal{F}_i)) \otimes_k I_{ij})$ on the set of liftings of $\mathcal{F}_S$ to $R$.

In fact, let the deformation $\mathcal{F}_S \in \text{Def}_{\mathcal{F}}(S)$ be given by $L^S(U) : \mathcal{D}(U) \to \mathcal{Q}(U)$ and $L^S(U, V) \in \mathcal{Q}(U, V)$ for all $U \supseteq V$ in $\mathcal{U}$, and let $\sigma : S \to R$ be a $k$-linear section of $u : R \to S$ such that $\sigma(e_i) = e_i$ and $\sigma(R_{ij}) \subseteq R_{ij}$ for $1 \leq i, j \leq p$. We consider $L^R(U) : \mathcal{D}(U) \to \mathcal{Q}(U)$ given by $L^R(U) = \sigma \circ L^S(U)$ and $L^R(U, V) = \sigma(L^S(U, V))$ for all $U \supseteq V$ in $\mathcal{U}$. The obstruction $o(u, \mathcal{F}_S)$ for lifting $\mathcal{F}_S$ to $R$ is given by

1. $(P, Q) \mapsto L^R(U)(PQ) - L^R(U)(P) \circ L^R(U)(Q)$ for all $U \in \mathcal{U}$, $P, Q \in \mathcal{D}(U)$
2. $P \mapsto L^R(U, V) \circ L^R(U)(P) - L^R(U)(P|_V) \circ L^R(U, V)$ for all $U \supseteq V$ in $\mathcal{U}$, $P \in \mathcal{D}(U)$
3. $L^R(U, V) \circ L^R(U, V) - L^R(U, W)$ for all $U \supseteq V \supseteq W$ in $\mathcal{U}$

We see that these expressions are exactly the obstructions for $L^R(U)$ and $L^R(U, V)$ to satisfy conditions [1] - [3] in the characterization of $\text{Def}_{\mathcal{F}}(R)$ given above.

3. Calculations for $\mathcal{D}$-modules on elliptic curves

Let $X \subseteq \mathbb{P}^2$ be the irreducible projective plane curve given by the homogeneous equation $f = 0$, where $f = y^2z - x^3 - axz^2 - bz^3$ for fixed parameters $(a, b) \in k^2$. We assume that $\Delta = 4a^3 + 27b^2 \neq 0$, so that $X$ is smooth and therefore an elliptic curve over $k$. We shall compute the noncommutative deformations of $\mathcal{O}_X$, considered as a quasi-coherent left $\mathcal{D}_X$-module via the natural left action of $\mathcal{D}_X$ on $\mathcal{O}_X$.

We choose an open affine cover $U = \{U_1, U_2, U_3\}$ of $X$ closed under intersections, given by $U_1 = D_+(y)$, $U_2 = D_+(z)$ and $U_3 = U_1 \cap U_2$. We recall that the open subset $D_+(h) \subseteq X$ is given by $D_+(h) = \{p \in X : h(p) \neq 0\}$ for $h = y$ or $h = z$. It follows from the results in section 2 that the noncommutative deformation functor $\text{Def}_{\mathcal{O}_X} : a_1 \to \text{Sets}$ has a pro-representing hull $H$, completely determined by the global Hochschild cohomology groups and some generalized Massey products on them. We shall therefore compute $\text{HH}^n(U, \mathcal{D}, \text{End}_k(\mathcal{O}_X))$ for $n = 1, 2$.

It is known that $\mathcal{D}_X(U)$ is a simple Noetherian ring of global dimension one and that $\mathcal{O}_X(U)$ is a simple left $\mathcal{D}_X(U)$-module for any open affine subset $U \subseteq X$; see
for instance Smith, Stafford [10]. The functor $\text{Ext}^q_{D_X}(\mathcal{O}_X, \mathcal{O}_X): \text{Mor} U \to \text{Mod}(k)$ is therefore given by $\text{Ext}^q_{D_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$ for $q \geq 2$ and $\text{End}_{D_X}(\mathcal{O}_X) = k$. Since the spectral sequence for global Hochschild cohomology given in section II degenerates,

$$\text{HH}^n(U, D_X, \text{End}_k(\mathcal{O}_X)) \cong H^{n-1}(U, \text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X)) \text{ for } n \geq 1$$

$$\text{HH}^0(U, D_X, \text{End}_k(\mathcal{O}_X)) \cong k$$

We shall compute $\text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X)$ and use this to find $H^{n-1}(U, \text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X))$ for $n = 1, 2$.

Let $A_i = \mathcal{O}_X(U_i)$ and $D_i = D_X(U_i)$ for $i = 1, 2, 3$. We see that $A_1 \cong k[x, z]/(f_1)$ and $A_2 \cong k[x, y]/(f_2)$, where $f_1 = z - x z^2 - b z^3$ and $f_2 = y^2 - x z - ax - b$. Moreover, we have that $\text{Der}_k(A_i) = A_i \partial_i$ and $D_i = A_i \langle \partial_i \rangle$ for $i = 1, 2$, where

$$\partial_1 = (1 - 2axz - 3bz^2) \partial/\partial x + (3x^2 + az^2) \partial/\partial z$$

$$\partial_2 = -2y \partial/\partial x - (3x^2 + a) \partial/\partial y$$

On the intersection $U_3 = U_1 \cap U_2$, we choose an isomorphism $A_3 \cong k[x, y, y^{-1}]/(f_3)$ with $f_3 = f_2$, and see that $\text{Der}_k(A_3) = A_3 \partial_3$ and $D_3 = A_3 \langle \partial_3 \rangle$ for $\partial_3 = \partial_2$. The restriction maps of $\mathcal{O}_X$ and $D_X$, considered as presheaves on $U$, are given by

$$x \mapsto xy^{-1}, \ z \mapsto y^{-1}, \ \partial_1 \mapsto \partial_3$$

for the inclusion $U_1 \supseteq U_3$, and the natural localization map for $U_2 \supseteq U_3$. Finally, we find a free resolution of $A_i$ as a left $D_i$-module for $i = 1, 2, 3$, given by

$$0 \leftarrow A_i \leftarrow D_i \leftarrow 0$$

and use this to compute $\text{Ext}^1_{D_i}(A_i, A_j) \cong \text{coker}(\partial_i|_{U_i} : A_j \to A_j)$ for all $U_i \supseteq U_j$ in $U$. We see that $\text{Ext}^1_{D_i}(A_i, A_3) \cong \text{coker}(\partial_3 : A_3 \to A_3)$ is independent of $i$, and find the following $k$-linear bases for $\text{Ext}^1_{D_i}(A_i, A_j)$:

| $a \neq 0$ | $a = 0$ |
|-----------------|-----------------|
| $U_1 \supseteq U_1$ | $1, z, x z, x z^3$ |
| $U_2 \supseteq U_2$ | $1, y^2$ |
| $U_3 \supseteq U_3$ | $x^2 y^{-1}, 1, y^{-1}, y^{-2}, y^{-3}$ |

The functor $\text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X): \text{Mor} U \to \text{Mod}(k)$ defines the following diagram in $\text{Mod}(k)$, where the maps are induced by the restriction maps on $\mathcal{O}_X$:

$$\begin{align*}
\text{Ext}^1_{D_1}(A_1, A_1) & \to \text{Ext}^1_{D_2}(A_2, A_2) \\
\text{Ext}^1_{D_1}(A_1, A_3) & \to \text{Ext}^1_{D_3}(A_3, A_3) \\
\text{Ext}^1_{D_3}(A_3, A_3) & \to \text{Ext}^1_{D_2}(A_2, A_3)
\end{align*}$$

We use that $15y^2 = \Delta y^{-2}$ in $\text{Ext}^1_{D_3}(A_3, A_3)$ when $a \neq 0$ and that $-3byy^{-2} = x$ in $\text{Ext}^1_{D_3}(A_3, A_3)$ when $a = 0$ to describe these maps in the given bases. We compute $H^{n-1}(U, \text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X))$ for $n = 1, 2$ using the resolving complex $D^+(U, -)$; see Lualal [10] for definitions. In particular, we see that $H^0(U, \text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X))$ consists of all pairs $(h_1, h_2)$ with $h_i \in \text{Ext}^1_{D_i}(A_i, A_i)$ for $i = 1, 2$ that satisfies the condition $h_1|_{U_3} = h_2|_{U_3}$. Moreover, $H^1(U, \text{Ext}^1_{D_X}(\mathcal{O}_X, \mathcal{O}_X))$ consists of all pairs $(h_{13}, h_{23})$.
with \( h_{ij} \in \text{Ext}_D^1(A_i, A_j) \) for \((i, j) = (1, 3), (2, 3)\), modulo the pairs of the form \((h_1|U_j - h_3, h_2|U_j - h_3)\) for triples \((h_1, h_2, h_3)\) with \( h_i \in \text{Ext}_D^1(A_i, A_i) \) for \( i = 1, 2, 3 \). We find the following \( k \)-linear bases:

| \( a \neq 0 \) | \( a = 0 \) |
|---|---|
| \( n = 1 \) | \( \xi_1 = (1, 1, 1), \xi_2 = (\Delta x^2, 15y^2, \Delta y^{-2}) \) | \( \xi_1 = (1, 1, 1), \xi_2 = (-3b x z, x, x) \) |
| \( n = 2 \) | \( \omega = (0, 0, 0, 6ax^2y^{-1}) \) | \( \omega = (0, 0, 0, x^2y^{-1}) \) |

We recall that \( \xi_1, \xi_2 \) and \( \omega \) are represented by cocycles of degree \( p = 0 \) and \( p = 1 \) in the resolving complex \( \mathcal{D}(U, \text{Ext}_{D^1}^1(O_X, O_X)) \),

\[
\mathcal{D}^p(U, \text{Ext}_{D^1}^1(O_X, O_X)) = \prod_{U_0 \supseteq \cdots \supseteq U_p} \text{Ext}_{D^1}^1(O_X, O_X)(U_0 \supseteq U_p)
\]

and the product is indexed by \( \{ U_1 \supseteq U_2 \supseteq U_3 \} \) when \( p = 0 \), and \( \{ U_1 \supseteq U_2 \supseteq U_3 \supseteq U_4 \supseteq U_5 \} \) when \( p = 1 \).

This proves that the noncommutative deformation functor \( \text{Def}_{O_X} : a_1 \to \text{Sets} \) of the left \( D_X \)-module \( O_X \) has tangent space \( \text{HH}_1(U, D_X, \text{End}_k(O_X)) \cong k^2 \) and obstruction space \( \text{HH}_2(U, D_X, \text{End}_k(O_X)) \cong k \) for any elliptic curve \( X \) over \( k \), and a pro-representing hull \( H = k \ll t_1, t_2 \gg (F) \) for some noncommutative power series \( F \in k \ll t_1, t_2 \gg \).

We shall compute the noncommutative power series \( F \) and the versal family \( \mathcal{F}_H \in \text{Def}_{O_X}^H(H) \) using the obstruction calculus. We choose base vectors \( t_1^l, t_2^l \) in \( \text{HH}_1(U, D, \text{Hom}_k(O_X, O_X)) \), and representatives \( (\psi_l, \tau_l) \in D^1 \oplus D^0 \) of \( t_1^l \) for \( l = 1, 2 \), where \( D^1 = \text{D}(U, \text{HC}_2(D, \text{End}_k(O_X))) \). We may choose \( \psi_l(U_i) \) to be the derivation defined by

\[
\psi_l(U_i)(P_i) = \begin{cases} 0 & \text{if } P_i \in A_i \\ \xi_l(U_i) \cdot \text{id}_{A_i} & \text{if } P_i = \partial_i \end{cases}
\]

for \( l = 1, 2 \) and \( i = 1, 2, 3 \), and \( \tau_l(U_i \supseteq U_j) \) to be the multiplication operator in \( \text{Hom}_{A_i}(A_i, A_j) \cong A_j \) given by \( \tau_1 = 0, \tau_2(U_i \supseteq U_i) = 0 \) for \( i = 1, 2, 3 \) and

| \( a \neq 0 \) | \( a = 0 \) |
|---|---|
| \( \tau_2(U_1 \supseteq U_3) = 0 \) | \( \tau_2(U_1 \supseteq U_3) = x^2y^{-1} \) |
| \( \tau_2(U_2 \supseteq U_3) = -4a^2y^{-1} - 3xy + 9byx^{-1} - 6a^2y^{-1} \) | \( \tau_2(U_2 \supseteq U_3) = 0 \) |

Let \( a_1(n) \) be the full subcategory of \( a_1 \) consisting of all \( R \) such that \( I(R)^n = 0 \) for \( n \geq 2 \). The restriction of \( \text{Def}_{O_X} : a_1 \to \text{Sets} \) to \( a_1(2) \) is represented by \( (H_2, \mathcal{F}_{H_2}) \), where \( H_2 = k(t_1, t_2)/(t_1, t_2)^2 \) and the deformation \( \mathcal{F}_{H_2} \in \text{Def}_{O_X}^H(H_2) \) is defined by \( \mathcal{F}_{H_2}(U_i) = A_i \otimes_k H_2 \) as a right \( H_2 \)-module for \( i = 1, 2, 3 \), with left \( D_i \)-module structure given by

\[
P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2
\]

for \( i = 1, 2, 3 \) and for all \( P_i \in D_i \), \( m_i \in A_i \), and with restriction map for the inclusion \( U_i \supseteq U_j \) given by

\[
m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j) m_i|_{U_j} \otimes t_2
\]

for \( i = 1, 2 \), \( j = 3 \) and for all \( m_i \in A_i \).
Let us attempt to lift the family \( \mathcal{F}_{H_2} \in \text{Def}_{\mathcal{O}_X}(H_2) \) to \( R = k \langle t_1, t_2 \rangle / (t_1, t_2)^3 \).
We let \( \mathcal{F}_R(U_i) = A_i \otimes_k R \) as a right \( R \)-module for \( i = 1, 2, 3 \), with left \( D_i \)-module structure given by

\[
P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2
\]

for \( i = 1, 2, 3 \) and for all \( P_i \in D_i, \, m_i \in A_i \), and with restriction map for the inclusion \( U_i \supseteq U_j \) given by

\[
m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j) m_i|_{U_j} \otimes t_2 + \frac{\tau_2(U_i \supseteq U_j)^2}{2} m_i|_{U_j} \otimes t_2^2
\]

for \( i = 1, 2, j = 3 \) and for all \( m_i \in A_i \). We see that \( \mathcal{F}_R(U_i) \) is a left \( D_X(U_i) \)-module for \( i = 1, 2, 3 \), and that \( t_1t_2 - t_2t_1 = 0 \) is a necessary and sufficient condition for \( D_X \)-linearity of the restriction maps for the inclusions \( U_1 \supseteq U_3 \) and \( U_2 \supseteq U_3 \). This implies that \( \mathcal{F}_R \) is not a lifting of \( \mathcal{F}_{H_2} \) to \( R \). But if we define the quotient \( H_3 = R / (t_1t_2 - t_2t_1) \), we see that the family \( \mathcal{F}_{H_3} \in \text{Def}_{\mathcal{O}_X}(H_3) \) induced by \( \mathcal{F}_R \) is a lifting of \( \mathcal{F}_{H_3} \) to \( H_3 \).

In fact, we claim that the restriction of \( \text{Def}_{\mathcal{O}_X} : a_1 \rightarrow \text{Sets} \) to \( a_1(3) \) is represented by \( (H_3, \mathcal{F}_{H_3}) \). One way to prove this is to show that it is not possible to find any lifting \( \mathcal{F}_R' \in \text{Def}_{\mathcal{O}_X}(R) \) of \( \mathcal{F}_{H_3} \) to \( R \). Another approach is to calculate the cup products \( < t_i^*, t_j^* > \) in global Hochschild cohomology for \( i, j = 1, 2 \), and this gives

\[
< t_i^*, t_j^* > = o^*, \quad < t_i^*, t_j^* > = -o^*
\]

for \( a \neq 0 \)

\[
< t_i^*, t_j^* > = o^*, \quad < t_i^*, t_j^* > = -o^*
\]

for \( a = 0 \)

where \( o^* \in \text{HH}^2(U, \mathcal{O}_X, \text{End}_k(\mathcal{O}_X)) \) is the base vector corresponding to \( \omega \). Since all other cup products vanish, this implies that \( F = t_1t_2 - t_2t_1 + (t_1, t_2)^3 \).

Let \( H = k \langle t_1, t_2 \rangle / (t_1t_2 - t_2t_1) \). We shall show that it is possible to find a lifting \( \mathcal{F}_H \in \text{Def}_{\mathcal{O}_X}(H) \) of \( \mathcal{F}_{H_3} \) to \( H \). We let \( \mathcal{F}_H(U_i) = A_i \otimes_k H \) as a right \( H \)-module for \( i = 1, 2, 3 \), with left \( D_i \)-module structure given by

\[
P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2
\]

for \( i = 1, 2, 3 \) and for all \( P_i \in D_i, \, m_i \in A_i \), and with restriction map for the inclusion \( U_i \supseteq U_j \) given by

\[
m_i \otimes 1 \mapsto \sum_{n=0}^{\infty} \frac{\tau_2(U_i \supseteq U_j)^n}{n!} m_i|_{U_j} \otimes t_2
\]

for \( i = 1, 2, j = 3 \) and for all \( m_i \in A_i \). This implies that \( (H, \mathcal{F}_H) \) is the pro-representing hull of \( \text{Def}_{\mathcal{O}_X} \), and that \( F = t_1t_2 - t_2t_1 \). We remark that the versal family \( \mathcal{F}_H \) does not admit an algebraization, i.e. an algebra \( H_{\text{alg}} \) of finite type over \( k \) such that \( H \) is a completion of \( H_{\text{alg}} \), together with a deformation in \( \text{Def}_{\mathcal{O}_X}(H_{\text{alg}}) \) that induces the versal family \( \mathcal{F}_H \in \text{Def}_{\mathcal{O}_X}(H) \).

Finally, we mention that there is an algorithm for computing the pro-representing hull \( H \) and the versal family \( \mathcal{F}_H \) using the cup products and higher generalized Massey products on global Hochschild cohomology. We shall describe this algorithm in a forthcoming paper. In many situations, it is necessary to use the full power of this machinery to compute noncommutative deformation functors.
References

1. M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. (1957), 181–207.
2. A. Beilinson and J. Bernstein, *A proof of the Jantzen conjectures*, Adv. Soviet Math. 16 (1993), no. 1, 1–50.
3. E. Eriksen, *Noncommutative deformations of sheaves and presheaves of modules*, ArXiv: math.AG/0405234, 2005.
4. O. A. Laudal, *Formal moduli of algebraic structures*, Lecture Notes in Mathematics, no. 754, Springer-Verlag, 1979.
5. , *Noncommutative deformations of modules*, Homology Homotopy Appl. 4 (2002), no. 2, part 2, 357–396.
6. S. Paul Smith and J. T. Stafford, *Differential operators on an affine curve*, Proc. London Math. Soc. (3) 56 (1988), no. 2, 229–259.
7. A. Weil, *Généralisation des fonctions abéliennes*, J. Math. Pures Appl. 17 (1938), 47–87.

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