BV-PACKING INTEGRAL IN $\mathbb{R}^n$

KRISTÝNA KUNCOVÁ

Abstract. We introduce new integrals (called packing $\mathcal{R}$ and $\mathcal{R}^*$ integrals) which combine advantages of integrals developed by Pfeffer [17], Mařík [12], Kuncová and Mařík [10] and Mařík and Pfeffer [13]. We prove Gauss-Green theorem in generality of the new integrals and provide comparison with the integrals mentioned above and some others (like $MC_\alpha$ by Ball and Preiss [2]).

Contents

1. Introduction 1
2. Notation and Preliminaries 3
3. BV sets and charges 4
4. Packing $\mathcal{R}$ integral 7
5. Packing $\mathcal{R}^*$ integral 7
6. $\mathcal{R}$ integral 20
7. $\mathcal{GR}$ integral 24
8. $\mathcal{R}^*$ integral 24
9. Henstock-Kurzweil-Stieltjes integral 26
10. $MC$ and $MC_\alpha$ integrals 26
11. Summary of relations 29
Acknowledgements 30
References 30

1. Introduction

The Gauss-Green divergence theorem

$$\int_A \text{div} u(x) \, dx = \int_{\partial_\ast A} u \cdot \nu_A \, d\mathcal{H}^{n-1}$$

holds whenever $A \subset \mathbb{R}^n$ is a bounded BV set (or, in another terminology, a bounded set of finite perimeter) and $u \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Here, $\partial_\ast A$ is the essential boundary and $\nu_A$ is the measure-theoretic unit exterior normal. This setting and its history can be found e.g. in [1]. If we want to allow discontinuous derivatives, routine approximation arguments give (1.1) if $u \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\text{div} u(x) \in L^1(\mathbb{R}^n)$. Beyond Lebesgue integrability of $\text{div} u(x)$, a natural idea is to consider the divergence in the sense of distributions. Particularly deep results have been obtained for divergence measure vector fields, see e.g. Chen, Torres and Ziemer [5], Ziemer [22] or Šilhavý [19, 20, 21].

We pursue another direction. If $u$ is differentiable, the divergence formula still holds even if the divergence is not Lebesgue integrable. This phenomenon indicates
that the $L^1$ setting is not the ultimate generality if we want to consider the divergence as a pointwise function. Such a divergence still plays the role of divergence in the sense of distributions, but the task is to what extent non-absolutely integrable pointwise functions can be represented as distributions. The problem exists already in the one-dimensional case where it has been solved by the Denjoy-Perron integral. The multidimensional case has been treated by many authors, among the most important contribution we mention [9, 14, 7]. The most important progress in this direction has been done by Pfeffer [17], who developed a theory which can be used for the divergence theorem on $BV$ sets. In his setting, indefinite integral is a function on $BV$ sets, so that the definite integral on the left of (1.1) is the evaluation of the indefinite integral at $A$. An interesting extension has been introduced by Pfeffer and Mály in [13]. Their effort leads to the $R^*$ integral, which is stable under reasonable operations and has a rich family of integrable functions. In particular, the $R^*$ integral includes Pfeffer’s $R$ integral [17] and the 1-dimensional Henstock-Kurzweil integral.

In a series of papers [10, 12, 8], a new non-absolutely convergent integral with respect to distributions, called packing integral, has been introduced. Since main motivation comes from the divergence theorem and related results again, it is natural to ask on comparison of this integral with Pfeffer’s approach. In its original setting, the indefinite packing integral is a functional on smooth (or Lipschitz) test functions and its evaluation at $BV$ sets does not make sense. Therefore, the definite integral on the left of (1.1) is the evaluation of the indefinite integral of $\chi_A \text{div} \, u$ at a test function which is 1 on a neighborhood of $\partial A$.

The Pfeffer integral (one of the equivalent versions) is based on Riemann-type sums

$$\sum_{i=1}^{m} \left| F(E_i) - f(x_i)\mathcal{L}(E_i) \right|$$

where $E_i \subset \mathbb{R}^n$ are disjointed $BV$ sets, $x_i \in \mathbb{R}^n$ are tags, $\mathcal{L}$ is Lebesgue measure and $F$ is the candidate for the indefinite integral. In our setting, we also use sums

$$(1.2) \sum_{i=1}^{m} q_{x_i,r_i} (F - f(x_i)\mathcal{L})$$

where $(q_{x,r})_{x,r}$ is a system of suitable seminorms.

Let $A \subset \mathbb{R}^n$ be a bounded $BV$ set. Suppose that $u \in C(\mathbb{R}^n, \mathbb{R}^n)$ and the indefinite packing integral of a function $f$ is the flux of $u$, so that $f = \text{Div} \, u$ in a general sense. We would be happy to see that

$$\int_A \text{Div} \, u(x) \, dx = \int_{\partial A} u \cdot \nu \, d\mathcal{H}^{n-1},$$

where the integral on the left means the integration of $f\chi_A$. (In other words, the characteristic function of $A$ acts as a multiplier for the integration of $f$.) However, in the setting of [10] it is not clear how to estimate the sums (1.2) (and it is probably impossible without additional hypotheses). It helps if we can omit $x_i$ belonging to a small set, say of $\sigma$-finite $\mathcal{H}^{n-1}$ Hausdorff measure, namely, just $\partial A$. This change of definition requires the indefinite integral to be a charge, a functional on $BV \cap L^\infty$ functions continuous with respect to a convergence specified below. Charges can be represented as functions on $BV$ sets, and by this series of thoughts we recover most ingredients of Pfeffer’s setting.
In this paper we present modifications of the packing integral which contains Pfeffer’s $\mathcal{R}$ integral and Pfeffer’s and Malý’s $\mathcal{R}^*$ integral. We apply the new integrals to obtain more general versions of the divergence theorem. In the end we discuss the relationships between particular integrals including the one-dimensional Henstock-Kurzweil-Stieltjes integral and $MC_\alpha$ integral.

2. Notation and Preliminaries

Notation 2.1. Let $E$ be a subset of $\mathbb{R}^n$. Then $d(E)$ denotes the diameter of $E$, i.e.

$$d(E) = \sup\{|y - x|; x, y \in E\}.$$ 

Let $x \in \mathbb{R}^n$ and $r > 0$. Then $B(x, r)$ denotes the open ball

$$B(x, r) = \{y \in \mathbb{R}^n; |y - x| < r\}$$

and $\bar{B}(x, r)$ denotes the closed ball

$$\bar{B}(x, r) = \{y \in \mathbb{R}^n; |y - x| \leq r\}.$$ 

The Lebesgue measure of $E$ is denoted by $|E|$ or $\mathcal{L}(E)$.

Definition 2.2. We say, that measurable sets $A$ and $B$ are equivalent (or $A$ and $B$ belong to the same equivalence class) if $|A \triangle B| = 0$, where $A \triangle B$ denotes the symmetric difference of the sets $A$ and $B$.

Definition 2.3. Let $s \geq 0$. The $s$-dimensional outer Hausdorff measure of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^s_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E),$$

and $\alpha_s = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2} + 1\right)}$.

Proposition 2.4. Let $A \subset \mathbb{R}^n$ be a set and let $\varphi : A \to \mathbb{R}^n$ be a Lipschitz mapping. Then $\mathcal{H}^{n-1}(\varphi(A)) \leq (\text{Lip } \varphi)^{n-1}\mathcal{H}^{n-1}(A)$.

Proof. For the proof and further details see [6, Section 2.4.1].

Definition 2.5. Let $A \subset \mathbb{R}^n$ be a measurable set and let $x \in \mathbb{R}^n$. Then we define the lower density of $A$ at $x$ as

$$\underline{\Theta}(A, x) := \liminf_{r \to 0^+} \frac{|A \cap B(x, r)|}{|B(x, r)|}$$

and the upper density of $A$ at $x$ as

$$\overline{\Theta}(A, x) := \limsup_{r \to 0^+} \frac{|A \cap B(x, r)|}{|B(x, r)|}.$$

The essential closure $\text{cl}_* A$, essential interior $\text{int}_* A$ and essential boundary $\partial_* A$ are then defined as

$$\text{cl}_* A = \{x \in \mathbb{R}^n; \overline{\Theta}(A, x) > 0\},$$

$$\text{int}_* A = \{x \in \mathbb{R}^n; \underline{\Theta}(A, x) = 1\}$$

and

$$\partial_* A = \text{cl}_* A \setminus \text{int}_* A.$$
Definition 2.6. We say that a measurable set $A \subset \mathbb{R}^n$ is admissible if $\text{int}_* A \subset A \subset \text{cl}_* A$.

Remark 2.7. Our definition of admissible set differs from that used by Malý and Pfeffer in [13], according to which $\partial A$ is required to be compact.

Remark 2.8. Let $A$, $A'$ be measurable sets such that $|A \Delta A'| = 0$. Then $\text{cl}_* A = \text{cl}_* A'$, $\text{int}_* A = \text{int}_* A'$ and $\partial_* A = \partial_* A'$.

Hence, for every bounded measurable set $A$ we can find an admissible set $A'$ such that $|A \Delta A'| = 0$.

3. $BV$ sets and charges

In this section we will present some basic facts about spaces of sets of bounded variation ($BV$ sets) and about charges which will be essential in further definitions. For details see [17], [16] and [4].

Definition 3.1. Let $U \subset \mathbb{R}^n$ be an open set. For a measurable set $E \subset \mathbb{R}^n$ we define the perimeter of $E$ in $U$ as

$$P(E, U) = \sup \left\{ \int_{U \cap E} \text{div} \phi : \phi \in C^1_c(U), \| \phi \|_{\infty} \leq 1 \right\}.$$  

If $P(E, U) < \infty$ then the distributional gradient $D\chi_E$ of $\chi_E$ in $U$ is a vector-valued Radon measure and $P(E, U)$ is exactly its total variation. By the De Giorgi–Federer theorem, we can compute $P(E, U)$ as

$$P(E, U) = \mathcal{H}^{n-1}(\partial E \cap U).$$

The particular choice $U = \mathbb{R}^n$ gives the perimeter of $E$

$$P(E) = \| E \| = \mathcal{H}^{n-1}(\partial E).$$

If $A \subset \mathbb{R}^n$ is just measurable, we define also the relative perimeter of $E$ in $A$ as

$$P(E, \text{in} A) = \mathcal{H}^{n-1}(\partial E \cap \text{int}_* A).$$

There is a distinction between $P(E, \text{in} U)$ and $P(E, U)$ if $U$ is open, see Example 3.2 below.

We say that a measurable set $E$ is a locally $BV$ set, if $P(E, A) < \infty$ for each bounded open set $A$. A measurable set $E$ is called a $BV$ set, if $|E| + \| E \| < \infty$.

The family of all $BV$ sets and all locally $BV$ sets is denoted by $BV$ and $BV_{\text{loc}}$, respectively. The family of all bounded $BV$ sets is denoted by $\mathcal{BV}$.

Example 3.2. Let $E = B(0, 1)$ and $A = B(0, 2) \setminus \{ x \in \mathbb{R}^2 : |x| = 1 \}$ be subsets of $\mathbb{R}^2$. Then $P(E, A) = \mathcal{H}^{n-1}(\emptyset) = 0$, whereas $P(E, \text{in} A) = \mathcal{H}^{n-1}(\partial (B(0, 1))) = 2\pi$.

Remark 3.3. If $n = 1$, each $BV$ set $E$ is equivalent to a set $\bigcup_{i=1}^k (a_i, b_i)$, where $a_1 < b_1 < \cdots < a_k < b_k$ are real numbers. In this case, $\| E \| = 2k$.

Definition 3.4. Let $A$ be a locally $BV$ set. Then we define the critical boundary of $A$ as

$$\partial_c A = \left\{ x \in \mathbb{R}^n; \limsup_{r \to 0^+} \frac{P(A, B(x, r))}{r^{n-1}} > 0 \right\}.$$

The critical interior $\text{int}_c A$ and critical exterior $\text{ext}_c A$ are then defined as

$$\text{int}_c A = \text{int}_* A \setminus \partial_c A, \quad \text{ext}_c A = \text{ext}_* A \setminus \partial_c A.$$
In the following, we will define the regularity of a BV set. This concept has been first introduced by Kurzweil, Mawhin and Pfeffer in [11]. In this article, we use the modification established by Pfeffer in [16].

**Definition 3.5.** Let \( E \subset \mathbb{R}^n \) be a bounded BV set and let \( x \in \mathbb{R}^n \). The *regularity* of the set \( E \) is the number

\[
 r(E) = \begin{cases} 
 \frac{|E|}{d(E \cup \{x\}) \|E\|} & \text{if } |E| > 0, \\
 0 & \text{if } |E| = 0.
\end{cases}
\]

The regularity of the pair \((E, x)\) is the number

\[
 r(E, x) = r(E \cup \{x\}) = \begin{cases} 
 \frac{|E|}{d(E \cup \{x\}) \|E\|} & \text{if } |E| > 0, \\
 0 & \text{if } |E| = 0.
\end{cases}
\]

Let \( \varepsilon > 0 \). We say that the set \( E \) and the pair \((E, x)\) are \( \varepsilon \)-regular if \( r(E) > \varepsilon \) and \( r(E, x) > \varepsilon \), respectively. A system \( P = \{(A_1, x_1), \ldots, (A_m, x_m)\} \), \( A_i \subset \mathbb{R}^n \) and \( x_i \in \mathbb{R}^n \), is called \( \varepsilon \)-regular if \( r(A_i, x_i) > \varepsilon \) for \( i = 1, \ldots, m \).

Let us note that every \( \varepsilon \)-regular BV set is bounded.

**Remark 3.6.** For every bounded BV set \( E \) we have the estimate \( r(E) \leq 1/(2^n) \). Especially, the regularity of a ball is equal to \( 1/(2^n) \) (see [17, Chapter 2.3]).

**Definition 3.7.** A *dyadic cube* is an interval

\[
 \prod_{i=1}^n \left[ \frac{k_i}{2^m}, \frac{k_i + 1}{2^m} \right],
\]

where \( m, k_1, \ldots, k_n \) are integers. A dyadic cube \( C' \) is called the *mother* of a dyadic cube \( C \) if \( C' \) is the smallest (with respect to inclusion) dyadic cube properly containing \( C \).

A finite (possibly empty) union of nondegenerate compact intervals in \( \mathbb{R}^n \) is called a *figure*. A *dyadic figure* is a figure that is a union of finitely many dyadic cubes.

**Definition 3.8.** Let \( B \) be a bounded BV set. We say that a sequence \( \{B_i\} \subset BV \) converges to \( B \) in BV if

1. \( \bigcup_{i=1}^{\infty} B_i \) is a bounded set,
2. \( \lim_{i \to \infty} |B_i \triangle B| = 0 \) and \( \sup_i \|B_i\| < \infty \).

**Lemma 3.9.** Let \( A \) be a bounded BV set. Then there exists a sequence \( \{A_i\} \) of dyadic figures which converges to \( A \) in BV.

**Proof.** See [17, Proposition 1.10.3]. \( \square \)

**Definition 3.10.** We say that a function \( \mathcal{F} : BV \to \mathbb{R} \) is a *charge* if \( \mathcal{F} \) satisfies the following conditions:

1. \( \mathcal{F}(A \cup B) = \mathcal{F}(A) + \mathcal{F}(B) \) for each disjoint bounded BV sets \( A \) and \( B \).
2. Given \( \varepsilon > 0 \) there exists an \( \eta > 0 \) such that \( |\mathcal{F}(C)| < \varepsilon \) for each BV set \( C \subset B(0, 1/\varepsilon) \) with \( \|C\| < 1/\varepsilon \) and \( |C| < \eta \).

**Remark 3.11.** Let \( E \) be a bounded BV set and \( \mathcal{F} \) be a charge. Since \( \mathcal{F} \) is additive and vanishes on bounded negligible sets, \( \mathcal{F}(E) \) depends only on the equivalence class of the set \( E \).
Notation 3.12. Let $E$ be a locally $BV$ set and $\mathcal{F}$ be a charge. Then $\mathcal{F}|_E$ denotes the charge $\mathcal{F}|_E(A) := \mathcal{F}(A \cap E)$, $A \in \mathcal{B}V$.

Definition 3.13. Let $E$ be a locally $BV$ set and let $\mathcal{F}$ be a charge. We say that $\mathcal{F}$ is a charge in $E$ if $\mathcal{F} = \mathcal{F}|_E$.

Proposition 3.14. An additive function $\mathcal{F}$ on $BV$ is a charge if and only if either of the following conditions is satisfied.

1. For given $\varepsilon$ there is a $\theta > 0$ such that for every $BV$ set $B \subset B(1/\varepsilon)$ we have
   $$|\mathcal{F}(B)| < \theta|B| + \varepsilon(\|B\| + 1).$$

2. $\lim \mathcal{F}(A_i) = 0$ for each sequence $\{A_i\}$ with $A_i \to \emptyset$ in $BV$.

Proof. See [17, Proposition 2.2.6, Proposition 2.1.2].

Definition 3.15. Let $A$ be a locally $BV$ set. We say that an additive function $\mathcal{F} : BV \to \mathbb{R}$ is a flux in $A$ of a vector field $u \in C(\bar{A}, \mathbb{R}^n)$, if for each $E \in BV$ we have
   $$\mathcal{F}(E) = \int_{\partial_+(E \cap A)} u \cdot \nu_{E \cap A} \, dH^{n-1},$$
where $\nu_{E \cap A}$ denotes the unit exterior normal of $E \cap A$.

In the case $A = \mathbb{R}^n$ we say that $\mathcal{F}$ is just a flux of $u$.

Examples 3.16. (1) Let $n = 1$. Since every bounded set $E \subset \mathbb{R}$ is equivalent to a finite disjoint union of compact intervals $\bigcup_{i=1}^k [a_i, b_i]$, each additive function $\mathcal{F}$ on $BV$ can be written as
   $$\mathcal{F}(E) = \sum_{i=1}^k (u(b_i) - u(a_i)),$$
where $u : \mathbb{R} \to \mathbb{R}$. The additive function $\mathcal{F}$ is a charge if and only if $u$ is continuous (see [17, Remark 2.1.5]). In other words, $\mathcal{F}$ can be represented as the distributional derivative of a continuous function $u$.

2. Let $\mathcal{F}$ be a flux in $A$ of a continuous vector field $u \in C(\bar{A}, \mathbb{R}^n)$, where $A$ is a locally $BV$ set. Then $\mathcal{F}$ is a charge (see [17, Example 2.1.4]). On the other hand, a charge needs not to be of this form. For an example see [17, Example 2.1.10].

3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a function. Then the function $\mathcal{F} : BV \to \mathbb{R}$ defined as
   $$\mathcal{F}(A) = \int_A f \, d\mathcal{L}$$
is a charge. (See [17, Example 2.1.3].)

4. Let $A$ be a measurable set with $\mathcal{H}^{n-1}(A) > 0$. Then the function $\mathcal{F} : BV \to \mathbb{R}$ defined as $\mathcal{F}(E) = \mathcal{H}^{n-1}(E \cap A)$ is not a charge.

Without loss of generality we may assume $A$ to be bounded. At first let us suppose that $\mathcal{H}^{n-1}(A) < \infty$. Then there is a constant $c$ such that for every $k \in \mathbb{N}$ we can find a sequence of balls $\{B_i\}$ with $A \subset \bigcup_{i=1}^\infty B_i$, diam $B_i < 1/k$ and $\sum_{i=1}^\infty \|B_i\| < c$. Then for $E_k := \bigcup_i B_i$ we have $E_k \subset \bigcup_{x \in A} B(x, 1)$, $\|E_k\| < c$ and $|E_k| < \frac{c}{k}$.

It follows that $E_k \to \emptyset$ in $BV$, whereas $\mathcal{F}(E_k) = \mathcal{H}^{n-1}(E_k \cap A) = \mathcal{H}^{n-1}(A) > 0$. By Proposition 3.14 $\mathcal{F}$ cannot be a charge.

It is easy to check that $\mathcal{F}$ is not a charge if $\mathcal{H}^{n-1}(A) = \infty$.  

4. Packing $\mathcal{R}$ integral

In this section we set up concept of the packing $\mathcal{R}$ integral, which will be further developed in the next section.

**Definition 4.1.** A pairwise disjoint finite system of balls $(B(x_i, r_i))_{i=1}^k$ in $\mathbb{R}^n$ is called a packing.

A function $\delta : E \rightarrow [0, \infty)$, where $E \subset \mathbb{R}^n$, is called a gage if the set $N = \{x; \delta(x) = 0\}$ is of $\sigma$-finite $\mathcal{H}^{n-1}$ Hausdorff measure.

We say that a system $P = \{(A_1, x_1), \ldots, (A_k, x_k)\}$, $A_i \subset \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$, is $\delta$-fine if $d(A_i \cup x_i) < \delta(x_i)$. Let us remark that we do not require $x_i \in A_i$.

Especially, a packing $(B(x_i, r_i))_{i=1}^k$ is $\delta$-fine if and only if $2r_i < \delta(x_i)$ for $i = 1, \ldots, k$.

**Notation 4.2.** Let $x \in \mathbb{R}^n$, $r, \varepsilon > 0$ and $\mathcal{F}$ be a charge. Then we will use the seminorms

$$\bar{p}_{x,r}^\varepsilon(\mathcal{F}) = \sup \{|\mathcal{F}(E)|; E \subset\subset B(x, r), E \in \mathcal{BV}, (E, x) \text{ is } \varepsilon\text{-regular}\}.$$ 

**Definition 4.3.** Let $A \subset \mathbb{R}^n$ be a locally $BV$ set. We say that a charge $\mathcal{F}$ in $A$ is an indefinite packing $\mathcal{R}$ integral of a function $f : \cl A \rightarrow \mathbb{R}$ in $A$ with respect to a charge $\mathcal{G}$ if there exists $\tau \in (0, 1]$ such that for every $\varepsilon > 0$ there exists a gage $\delta : \cl A \rightarrow [0, \infty)$ such that for every $\delta$-fine packing $(B(x_i, r_i))_{i=1}^k$, $x_i \in \cl A$, we have

$$\sum_{i=1}^k \bar{p}_{x_i,r_i}^\varepsilon(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon.$$ 

**Remark 4.4.** In the previous definition, as well as in forthcoming Definitions 5.3, 5.21, 5.27, 6.2, 6.4 and 8.2 it is possible to consider a function $f$ defined only on $\cl A \setminus T$, where $T$ is of $\sigma$-finite $\mathcal{H}^{n-1}$ Hausdorff measure. The integral is well defined since we can consider gages $\delta$ with $\delta = 0$ on $T$. For the same reason, the indefinite packing $\mathcal{R}$ integral with respect to any charge $\mathcal{G}$ does not depend on values of $f$ on a set of $\sigma$-finite $\mathcal{H}^{n-1}$ Hausdorff measure.

**Remark 4.5.** The uniqueness of the indefinite packing integral of $f$ in $A$ will be discussed later.

**Remark 4.6.** The indefinite packing $\mathcal{R}$ integral is linear with respect to a function $f$.

5. Packing $\mathcal{R}^*$ integral

Let us continue with so called packing $\mathcal{R}^*$ integral. We will prove its uniqueness, basic properties and finally we will formulate and prove the Gauss-Greene theorem. Its definition relies on the concept of an $\varepsilon$-isoperimetric set, which was introduced by Mály and Pfeffer in [13]. We will be inspired by their work also further in this section.

**Definition 5.1.** Let $\varepsilon > 0$ and $E \subset \mathbb{R}^n$ be a bounded $BV$ set. We say that $E$ is $\varepsilon$-isoperimetric if for each $T \in \mathcal{BV}$

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{1}{\varepsilon} P(T, \text{in } E).$$

Since $P(T, \text{in } E) = P(E \cap T, \text{in } E)$, it is enough to consider only $T \subset E$. (See [13, Lemma 2.1].)
Notation 5.2. Let \( x \in \mathbb{R}^n, r, \varepsilon > 0 \) and \( F \) be a charge. Then we will use the seminorms
\[
q_{x,r}^\varepsilon(F) = \sup\{|F(E)|; E \subset \subset B(x,r), E \in BV, x \in cl, E, (E,x) \text{ is } \varepsilon\text{-regular and } E \text{ is } \varepsilon\text{-isoperimetric}\}.
\]

Definition 5.3. Let \( A \subset \mathbb{R}^n \) be a locally BV set. We say that a charge \( F \) in \( A \) is an indefinite packing \( \mathcal{R}^* \) integral of a function \( f : cl, A \to \mathbb{R} \) in \( A \) with respect to a charge \( \mathcal{G} \) if there exists \( \tau \in (0,1] \) such that for every \( \varepsilon > 0 \) there exists a gage \( \delta : cl, A \to [0,\infty) \) such that for every \( \delta \)-fine packing \( (B(x_i,r_i))_{i=1}^k, x_i \in cl, A, \) we have
\[
\sum_{i=1}^k q_{x_i,\tau r_i}^\varepsilon(F - f(x_i)\mathcal{G}) < \varepsilon.
\]
In the case \( A = \mathbb{R}^n \) we say that \( F \) is just an indefinite packing \( \mathcal{R}^* \) integral of \( f \) with respect to \( \mathcal{G} \).

The family of all functions packing \( \mathcal{R}^* \) integrable with respect to a charge \( \mathcal{G} \) is denoted by \( \mathcal{P}\mathcal{R}^*(\mathcal{G}) \).

Lemma 5.4. Let \( \tau \in (0,1] \) and \( \varepsilon > 0 \). Then there exists a constant \( c_T \) (depending only on \( \tau \) and \( n \)) with the following property: for each function \( \Phi : \mathbb{R} \to (0,\infty) \), \( x \in \mathbb{R}^n \) and \( R > 0 \) there exists \( 0 < r < R \) such that
\[
\Phi(10r) + \varepsilon |B(x,10r)| \leq c_T (\Phi(\tau r) + \varepsilon |B(x,\tau r)|).
\]

Proof. See [10, Lemma 3.7]. \( \square \)

Lemma 5.5. Let \( 0 < \varepsilon \leq 1/(2n) \) and \( Q = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_n] \) be an \( \varepsilon \)-regular interval. Then
\[
\max\{a_1,\ldots,a_n\} \leq \frac{1}{\varepsilon} \min\{a_1,\ldots,a_n\}.
\]

Proof. For simplicity, let us suppose that \( a_1 \leq a_2 \leq \cdots \leq a_n \). Since \( Q \) is \( \varepsilon \)-regular, we can estimate
\[
a_n(a_2 \cdots a_n) \leq d(Q)\|Q\| \leq \frac{1}{\varepsilon} \|Q\| = \frac{1}{\varepsilon} a_1 a_2 \cdots a_n.
\]
Dividing by \( a_2 \cdots a_n \) we obtain \( a_n \leq \frac{1}{\varepsilon} a_1 \), which establishes the formula. \( \square \)

Lemma 5.6. Let \( \varepsilon > 0, Q \) be an \( \varepsilon \)-regular interval and \( T \in BV, T \subset Q \) satisfying \( |T| \leq |Q|/2 \). Then there exists a constant \( \gamma = \gamma(\varepsilon,n) \) such that
\[
\mathcal{H}^{n-1}(\partial Q \cap \partial_T) \leq \gamma \mathcal{H}^{n-1}(\text{int } Q \cap \partial_T).
\]

Proof. At first let \( Q \) be a cube. By [18, Lemma 6.7.2] there exists a constant \( \eta \) such that
\[
\mathcal{H}^{n-1}(\partial Q \cap \partial_T) \leq \eta \mathcal{H}^{n-1}(\text{int } Q \cap \partial_T). \tag{5.1}
\]

Further, let \( Q \) be an \( \varepsilon \)-regular interval. We can suppose \( Q = [0,a_1] \times [0,a_2] \times \cdots \times [0,a_n], a_1 \leq a_2 \leq \cdots \leq a_n \). Let \( L : \mathbb{R}^n \to \mathbb{R}^n \) be a linear mapping represented by the diagonal matrix
\[
\begin{pmatrix}
a_n/a_1 & 0 & \cdots & 0 \\
0 & a_n/a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]
Then $L(Q)$ is a cube and $|L(T)| \leq |L(Q)|/2$. Moreover, int $L(Q) \cap \partial_* L(T) = L(\text{int} Q \cap \partial_* T)$. Further, we can estimate the Lipschitz constant of $L$ as Lip$(L) = \max\{a_i/a_n\} \leq \frac{1}{\varepsilon}$, which follows from Lemma 5.5. Since $L^{-1}$ can be represented by the matrix

$$
\begin{pmatrix}
  a_1/a_n & 0 & \cdots & 0 \\
  0 & a_2/a_n & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{pmatrix},
$$

we have Lip$(L^{-1}) = 1$.

Applying Lemma 2.4, inequality (5.2) and properties of $L$ we obtain

$$
\mathcal{H}^{n-1}(\partial Q \cap \partial_* T) = \mathcal{H}^{n-1}(L^{-1}(\partial Q \cap \partial_* T)) \leq \mathcal{H}^{n-1}(\partial L(Q) \cap \partial_* L(T)) \\
\leq \eta \mathcal{H}^{n-1}(\text{int} L(Q) \cap \partial_* L(T)) = \eta \mathcal{H}^{n-1}(L(\text{int} Q \cap \partial_* T)) \\
\leq \frac{\eta}{\varepsilon^{n-1}} \mathcal{H}^{n-1}(\text{int} Q \cap \partial_* T).
$$

Hence (5.1) holds with $\gamma(\varepsilon, n) := \frac{\eta}{\varepsilon^{n-1}}$.

\[\square\]

**Lemma 5.7.** For every $n \in \mathbb{N}$ there exists an increasing function $\beta : (0, \infty) \to \mathbb{R}$ such that every $\varepsilon$-regular interval $Q \subset \mathbb{R}^n$ is $\beta(\varepsilon)$-isoperimetric.

**Proof.** We set $\beta(\varepsilon) = 1/(1 + \gamma(\varepsilon, n))$, the constant $\gamma(\varepsilon, n)$ being as in Lemma 5.6. Now let us fix an $\varepsilon$-regular interval $Q$ and a set $T \in BV$, $T \subset Q$. We need to show that

$$
\min\{P(Q \cap T), P(Q \setminus T)\} \leq \frac{1}{\beta(\varepsilon)} P(T, \text{in} Q),
$$

Let us assume $|T| \leq |Q|/2$. Since $Q$ is an interval, we have int $Q = \text{int}_* Q$. Then by Lemma 5.6 there exists a $\gamma = \gamma(\varepsilon, n)$ such that

$$
P(T) \leq \mathcal{H}^{n-1}(\text{int} Q \cap \partial_* T) + \mathcal{H}^{n-1}(\partial Q \cap \partial_* T) \leq (1 + \gamma) \mathcal{H}^{n-1}(\text{int} Q \cap \partial_* T) \\
\leq (1 + \gamma) P(T, \text{in} Q) = \frac{1}{\beta(\varepsilon)} P(T, \text{in} Q).
$$

In the case $|T| > |Q|/2$ we have $|Q \setminus T| < |Q|/2$ and then we obtain

$$
P(Q \setminus T) = P(Q \cap (Q \setminus T)) \leq (1 + \gamma) P(Q \setminus T, \text{in} Q) = \frac{1}{\beta(\varepsilon)} P(T, \text{in} Q).
$$

\[\square\]

**Lemma 5.8.** Let $r > 0$, $x \in \mathbb{R}^n$ and $Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be an interval such that $Q \subset B(x, 2r)$ and $r = \min\{|b_i - a_i|\}$. Then $(Q, x)$ is $\rho$-regular, where $\rho = \rho(n) = \frac{1}{n^{2n-2}}$. 


Proof. Let us denote \( s := \min \{|b_l - a_l|\} \) and \( w := \max \{|b_l - a_l|\} \). Since \( \frac{r}{2\sqrt{n}} \leq s \), \( w \leq 4r \) and \( \text{diam}(Q \cup \{x\}) \leq 4r \), we can estimate the regularity of \( Q \) as
\[
\rho(Q, x) = \frac{|Q|}{\text{diam}(Q \cup \{x\})} \geq \frac{s^{n-1}w}{4r \cdot 2nw^{n-1}} \geq \left( \frac{r}{2\sqrt{n}} \right)^{n-2} = \frac{1}{n^{\frac{n+1}{2}}2^{2n-2}} = \rho(n).
\]

Lemma 5.9. Let \( \mathcal{F} \) be a charge and \( B(x, r) \subset \mathbb{R}^n \), \( x = (x_1, x_2, \ldots, x_n) \), be a ball. Further, let \( Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \) be an interval such that \( Q \subset B(x, 2r) \) and \( \frac{r}{2\sqrt{n}} \leq \min \{|b_l - a_l|\} \). Then
\[
|\mathcal{F}(Q)| \leq 2^m q_x^{\varepsilon, 2r}(\mathcal{F}),
\]
where \( m = \#\{l; x_l \notin [a_l, b_l]\} \), \( \varepsilon = \min \{\beta(\rho), \rho\} \) and \( \beta \) and \( \rho \) are as in Lemma 5.7 and Lemma 5.8.

Proof. The proof proceeds by induction on \( m \). First, for \( m = 0 \) we have \( x \in Q \). Since \( Q \subset B(x, 2r) \) and \( \frac{r}{2\sqrt{n}} \leq \min \{|b_l - a_l|\} \), \( Q \) is \( \rho(n) \)-regular. Furthermore, by Lemma 5.7 we obtain \( Q \) is also \( \beta(\rho) \)-isoperimetric. Then we can estimate
\[
|\mathcal{F}(Q)| \leq q_x^{\varepsilon, 2r}(\mathcal{F}).
\]

Now let us fix \( m \geq 1 \) and suppose that (5.3) holds for \( m - 1 \). Without loss of generality we can assume that \( x_l \notin [a_l, b_l] \) for \( l = 1, \ldots, m \).

Our next purpose is to define an auxiliary interval
\[
\bar{Q} = [\bar{a}_1, \bar{b}_1] \times \cdots \times [\bar{a}_{m-1}, \bar{b}_{m-1}] \times [\bar{a}_m, \bar{b}_m] \times [a_{m+1}, b_{m+1}] \times \cdots \times [a_n, b_n],
\]
where \( [\bar{a}_m, \bar{b}_m] \) is defined as follows:

In the case \( x_m < a_m \) let us set \( \bar{a}_m = x_m - (b_m - x_m) = b_m \) and \( \bar{b}_m = b_m. \) If \( x_m > b_m, \)
let us set \( \bar{a}_m = a_m, \bar{b}_m = x_m + (x_m - a_m). \)

We see that \( Q \subset \bar{Q} \subset B(x, 2r) \) and \( x \in \bar{Q} \). For simplicity, let us assume \( x_m < a_m. \) Then
\[
Q = \bar{Q} \setminus \bar{Q}',
\]
where
\[
\bar{Q}' = [a_1, b_1] \times \cdots \times [a_{m-1}, b_{m-1}] \times [\bar{a}_m, a_m] \times [a_{m+1}, b_{m+1}] \times \cdots \times [a_n, b_n].
\]

In the following, we need to estimate the regularity of subintervals \( \bar{Q} \) and \( \bar{Q}' \). Since \( \min \{|b_l - a_l|\} \geq \frac{r}{2\sqrt{n}} \) and \( \bar{Q} \subset B(x, 2r) \), by Lemma 5.8 we obtain \( \bar{Q} \) is \( \rho(n) \)-regular. Analogously we obtain the regularity of \( \bar{Q}' \).

By Lemma 5.7 we have \( \bar{Q} \) and \( \bar{Q}' \) are \( \beta(\rho) \)-isoperimetric. Using the additivity of \( \mathcal{F} \) and the inductive assumption we obtain
\[
|\mathcal{F}(Q)| \leq |\mathcal{F}(\bar{Q})| + |\mathcal{F}(\bar{Q}')| \leq 2 \cdot 2^m q_x^{\varepsilon, 2r}(\mathcal{F}) = 2^m q_x^{\varepsilon, 2r}(\mathcal{F}),
\]
which completes the proof.
Theorem 5.10 (Uniqueness of the integral). Let \( f \) be a function and \( \mathcal{G} \) be a charge. Then there exists at most one indefinite packing \( \mathcal{R}^* \) integral of \( f \) with respect to \( \mathcal{G} \).

\( \square \)

Proof. Let \( \mathcal{F}_1, \mathcal{F}_2 \) be indefinite packing \( \mathcal{R}^* \) integrals of \( f \) with respect to \( \mathcal{G} \). Then \( \mathcal{F}_1 - \mathcal{F}_2 \) is the integral of 0 with respect to \( \mathcal{G} \). So it is sufficient to show that if \( \mathcal{F} \) is an indefinite packing \( \mathcal{R}^* \) integral of \( f \equiv 0 \), then \( \mathcal{F} \equiv 0 \).

By Lemma 3.9 it is enough to prove that \( \mathcal{F}(K) = 0 \) for each dyadic cube \( K \). Let \( \tau \) be as in Definition 5.3. Now, let us fix a dyadic cube \( \tau \).

STEP 1.

By Lemma 5.4, applied to \( \Phi(r) := \bar{q}_{x,r}(\mathcal{F}) \), we can find a constant \( c_T \) such that for every \( x \) there exists \( r(x) < \delta(x) \), \( 10r(x) < 1 \), with the following properties:

\( \sum_{i=1}^{h} \bar{q}_{x_i,\tau r_i}(\mathcal{F}) < \varepsilon. \)

STEP 2.

In this step we construct the covering of the set \( K \setminus N \), where \( N = \{ x; \delta(x) = 0 \} \).

By Lemma 5.4, applied to \( \Phi(r) := \bar{q}_{x,r}(\mathcal{F}) \), we can find a constant \( c_T \) such that for every \( x \) there exists \( r(x) < \delta(x), 10r(x) < 1 \), with the following properties:

\( 20r(x) < a_0 \)

and

\( \bar{q}_{x,10r(x)}(\mathcal{F}) + \varepsilon |B(x, 10r(x))| \leq c_T (\bar{q}_{x,r(x)}(\mathcal{F}) + \varepsilon |B(x, r(x))|). \)

Now, let us consider the covering \( \mathcal{C} = \{ \bar{B}(x, r(x)); x \in K \setminus N \} \). By the Vitali theorem we can construct a pairwise disjoint subsystem \( \mathcal{C'} \subset \mathcal{C} \), such that \( \bigcup_{B(x,R) \in \mathcal{C'}} B(x,R) \supset K \setminus N \), where \( \mathcal{C'} = \{ B(x, 5r); B(x, r) \in \mathcal{C'} \} \).

STEP 3.

Now we will cover the set \( N \).

Since \( N \) is of \( \mathcal{H}^{n-1} \) measure, we can write out \( N = \bigcup_{s=1}^{\infty} N_s \), where \( \mathcal{H}^{n-1}(N_s) = c_s < \infty \) for every \( s = 1, 2, \ldots \). Let us fix \( s \in \mathbb{N} \) and \( \varepsilon_s \in (0, \varepsilon) \) such that

\( \varepsilon_s (c_1 c_n 2^{n-1} (c_s + \varepsilon) + 1) < 2^{-s} \varepsilon, \)

where \( c_1 = 2^{n} n^{(3-n)/2} \) and \( c_c = \alpha_n 2^{2n} n^{n/2} \). By Lemma 3.14, with \( \varepsilon_s \) we can associate \( \theta_s \) such that for every \( BV \) set \( E \subset B(1/\varepsilon) \) we have

\( |\mathcal{F}(E)| < \theta_s |E| + \varepsilon_s (\|E\| + 1). \)

Furthermore, there exist \( \zeta_s < 1/2 \) and a system of balls \( N^* = \{ B(x^*_i, R^*_i) \} \) covering \( N^* \) such that \( R^*_i \leq \zeta_s \),

\( 4R^*_i < a_0, \)

\( c_2 \zeta_s \theta_s (c_s + \varepsilon) < 2^{-s} \varepsilon \alpha_{n-1} \)
Note that
\[ c_\theta < c + \varepsilon. \]

(5.10) \[
\sum_{B(x_i^*, R_i^*) \in \mathcal{N}^s} \alpha_{n-1} \left( \frac{\operatorname{diam} B(x_i^*, R_i^*)}{2} \right)^{n-1} \leq (\alpha_{n-1} + 1) \sum_{B(x_i^*, R_i^*) \in \mathcal{N}^s} (R_i^*)^{n-1} < c + \varepsilon.
\]

Note that
\[ c_\theta < c + \varepsilon. \]

(5.11) \[
c_\theta \sum_{B(x_i^*, R_i^*) \in \mathcal{N}^s} (R_i^*)^{n-1} < 2^{-s} \varepsilon,
\]
where \( c_\theta = \alpha_n c \varepsilon_2. \)

Let us denote \( \mathcal{N} := \bigcup_j \mathcal{N}^s. \)

Now, let us consider the covering \( \mathcal{V} := \mathcal{C}'' \cup \mathcal{N}. \) Since \( \mathcal{V} \) covers the compact set \( K, \) we can choose a finite system of balls \( B(x_i, R_i) \in \mathcal{V}, i = 1, \ldots, k, \) covering \( K. \)

Without loss of generality we can assume that \( B(x_1, R_1), \ldots, B(x_h, R_h) \in \mathcal{C}'' \) and \( B(x_h+1, R_{h+1}), \ldots, B(x_k, R_k) \in \mathcal{N}. \)

STEP 4.

In this step we construct a partition of the cube \( K \) in the sense that we look for a finite system of nonoverlapping cubes whose union is \( K. \)

Recall that \( Q \) denotes the mother cube of a cube \( Q. \) Let \( K \) denote the family of all dyadic subcubes of \( K. \) For fixed \( i \in \{1, \ldots, k\} \) set
\[ \tilde{\mathcal{Q}}_i = \{ Q \in K; Q \cap B(x_i, R_i) \neq \emptyset, Q \subset B(x_i, 2R_i) \text{ and } Q' \not\subset B(x_i, 2R_i) \}. \]

We show that the union \( \tilde{\mathcal{Q}} = \bigcup_{i=1}^k \tilde{\mathcal{Q}}_i \) is all of \( K. \) Choose \( y \in K. \) Consider a sequence \( P_i \) of dyadic cubes such that \( P_0 = K, P_{i-1} = P_i \) for \( l = 1, 2, \ldots \) and \( \{y\} = \bigcap_{l=0}^\infty P_l. \) There exists \( i \in \{1, \ldots, k\} \) such that \( y \in B(x_i, R_i). \) Since \( \operatorname{diam} P_l \searrow 0, \) there exists \( l \) such that \( P_l \subset B(x_i, 2R_i). \) We find the smallest \( l \) such that \( P_l \subset B(x_i, 2R_i). \) By (5.5) and (5.9), \( l \geq 1. \) We easily verify that \( y \in P_l \in \tilde{\mathcal{Q}}.

Next we show that the system \( \tilde{\mathcal{Q}} \) is finite. Let us fix \( Q \in \tilde{\mathcal{Q}}_i \) and denote the side length of \( Q \) by \( a. \) The length of the diagonal can be expressed as \( \sqrt{n}a. \) Since \( Q \) intersects both \( B(x_i, R_i) \) and \( B(x_i, 2R_i) \), we obtain
\[ R_i/2 < \sqrt{n}a. \]

Hence the side length of all cubes in \( \tilde{\mathcal{Q}}_i \) is bounded from below. Therefore, the systems \( \tilde{\mathcal{Q}}_i \) and hence the system \( \tilde{\mathcal{Q}} \) are finite.

Now we can define the system of cubes
\[ \mathcal{Q} = \tilde{\mathcal{Q}} \setminus \{ Q \in \tilde{\mathcal{Q}}; \exists P \in \tilde{\mathcal{Q}} \text{ such that } P \supseteq Q \}. \]

Since two dyadic cubes are either in inclusion or nonoverlapping, \( \mathcal{Q} \) is a finite partition of \( K; \) we enumerate it as \( \mathcal{Q} = \{ Q_j, j = 1, \ldots, m \}. \) Finally, let us define the systems
\[ \mathcal{Q}_i = \{ Q \in \mathcal{Q} \cap \tilde{\mathcal{Q}}_i; Q \not\subset \bigcup_{l<i} \tilde{\mathcal{Q}}_l \}. \]

Let us fix \( Q_j \in \mathcal{Q}_i \) and denote it side length by \( a_j. \) Recall that the length of the diagonal can be expressed as \( \sqrt{n}a_j \) and since \( Q_j \) is included in \( B(x_i, 2R_i), \) we have \( \sqrt{n}a_j < 4R_i. \) Hence we can estimate the perimeter of \( Q_j: \)
\[ (5.13) \quad \|Q_j\| = 2na_j^{n-1} \leq 2n \left( \frac{4R_i}{\sqrt{n}} \right)^{n-1} = 2^n n^{(n-1)/2} R_i^{n-1} c_1 2^{n-1} R_i^{n-1}. \]
Let us estimate the number of the cubes $Q_j \in \mathcal{D}_i$. Applying (5.12), we obtain

$$\alpha_n(2R_i)^n = |B(x_i, 2R_i)| \geq \left| \bigcup_{Q_j \in \mathcal{D}_i} Q_j \right| \geq \#\mathcal{D}_i \left( \frac{R_i}{2\sqrt{n}} \right)^n.$$ 

Hence $\#\mathcal{D}_j \leq c_c$. (Let us remind that $c_c = \alpha_n 2^{2n} n^{n/2}$.)

**STEP 5.**

Firstly let us suppose that $i \in \{1, \ldots, h\}$. Then $B(x_i, R_i) \in C''$. Let us fix a pair $(Q_j, B(x_i, R_i))$. Since $q_j \geq \frac{R_i}{2\sqrt{n}}$, we can apply Lemma 5.9 and obtain

$$|F(Q_j)| \leq 2^n q_i^{\varepsilon}, \sum_{i=1}^{h} c_2^n q_i^{\varepsilon}(\varepsilon + |B(x_i, \tau r_i)|)$$

Using the fact that $\#Q_j \leq c_c$, the system $\{B(x_1, r_1), \ldots, B(x_h, r_h)\}$ is a $\delta$-fine packing and applying (5.14), (5.6) and (5.4) we can estimate

$$\left| F \left( \bigcup_{i=1}^{h} Q_j \right) \right| = \sum_{i=1}^{h} \sum_{Q_j \in \mathcal{D}_j} |F(Q_j)|$$

Secondly, let us fix $s \in \mathbb{N}$ and set $A^s := \{i \in \{h+1, \ldots, k\}; B(x_i, R_i) \in N^s\}$. Then, applying the fact that $\#\mathcal{D}_j \leq c_c$ and inequalities (5.8), (5.10), (5.7), (5.13) and (5.11) we obtain

$$\left| F \left( \bigcup_{i \in A^s} Q_j \right) \right| \leq \theta_s \sum_{i \in A^s} \left| \bigcup_{Q_j \in \mathcal{D}_i} Q_j \right| + \varepsilon \left( \sum_{i \in A^s} \sum_{Q_j \in \mathcal{D}_i} \|Q_j\| + 1 \right)$$

Since the union $\bigcup_{i=1}^{\infty} \bigcup_{Q_j \in \mathcal{D}_i} Q_j$ has only finite number of nonempty elements, we can use the additivity of $F$ and we obtain

$$|F(K)| < c_c 2^n c T \varepsilon (1 + |K_0|) + \varepsilon \sum_{s=1}^{\infty} 2^{-s+1} = \varepsilon c_c 2^n (1 + |K_0|) + 2.)$$
which completes the proof. □

**Remark 5.11.** The indefinite packing $\mathcal{R}^*$ integral of a function $f$ with respect to a charge $\mathcal{G}$ depends linearly on $f$.

In the preceding, we were concerned with an indefinite packing $\mathcal{R}^*$ integral of a function $f : \mathbb{R}^n \to \mathbb{R}$. Now we will concentrate on a packing $\mathcal{R}^*$ integral in $A$, where $A$ is a locally $BV$ set.

**Theorem 5.12.** Let $A \subset \mathbb{R}^n$ be a locally $BV$ set and let a charge $\mathcal{F}$ be an indefinite packing $\mathcal{R}^*$ integral of a function $f : \text{cl}_c A \to \mathbb{R}$ in $A$ with respect to a charge $\mathcal{G}$. Then $\mathcal{F}$ is also an indefinite packing $\mathcal{R}^*$ integral of $f$ in $A$ with respect to $\mathcal{G}$.

**Proof.** The proof follows from the fact that $\bar{q}_{x,r} \leq \bar{p}_{x,r}$. □

**Notation 5.13.** Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be a function. Then $\bar{f}_A$ denotes the zero extension of $f$:

$$\bar{f}_A = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The two following lemmas with proofs can be found in [13, Lemma 2.5 and 3.7].

**Lemma 5.14.** Let $\mathcal{F}$ be a charge. Then for $\varepsilon > 0$ there is an absolutely continuous Radon measure $\mu$ in $\mathbb{R}^n$ such that for each $BV$ set $E \subset B(0,1/\varepsilon)$,

$$|\mathcal{F}(E)| \leq \mu(E) + \varepsilon P(E).$$

**Lemma 5.15.** Let $A \in BV_{loc}$ and $\varepsilon > 0$. For each $x \in \text{ext}_c A$, there is $\delta > 0$ such that every strongly $\varepsilon$-regular set $E$ with $x \in \text{cl}_c E$ and $d(E) < \delta$ satisfies

$$P(E \cap A) \leq P(E \setminus A).$$

The proof of the next theorem follows the lines of the proof in [13, Lemma 3.8].

**Theorem 5.16.** Let $\mathcal{F}$ be a charge and $A$ be an admissible locally $BV$ set. For given $\tau \in (0,1]$ and $\varepsilon > 0$ there is a gage $\delta : \mathbb{R}^n \to [0, \infty)$ such that

$$\sum_{x_i \in A} \bar{q}_{x_i,\tau r_i} (\mathcal{F}|_{A^c}) < \varepsilon$$

and

$$\sum_{x_i \notin A} \bar{q}_{x_i,\tau r_i} (\mathcal{F}|_A) < \varepsilon$$

for each $\delta$-fine packing $(B(x_i, r_i))_{i=1}^k$.

**Proof.** At first let us suppose that $A$ is bounded. Let us fix $\varepsilon > 0$ such that $A \subset B := B(0,1/\varepsilon')$, where

$$\varepsilon' = \frac{\varepsilon^2}{P(A)}.$$  

By Lemma 5.14, there is an absolutely continuous Radon measure $\mu$ in $\mathbb{R}^n$ such that

$$|\mathcal{F}(E)| \leq \mu(E) + \varepsilon' P(E)$$

for each $E \in BV$, $E \subset B$. Then there exists a compact $K$ such that $K \subset B \setminus A$ and

$$\mu((B \setminus A) \setminus K) < \frac{1}{2} \varepsilon.$$
Applying Lemma 5.15 to $A^c$, for each $x \in B \cap \text{ext}_c A^c = B \cap \text{int}_c A$ we can find $\delta_x > 0$ such that $B(x, \delta_x) \subset B$, and

$$(5.17) \quad P(E \setminus A) \leq P(E \cap A)$$

for each strongly $\varepsilon$-regular set $E$ with $x \in \text{cl}_c E$ and $d(E) < \delta_x$.

Making $\delta_x$ smaller, we may assume that $K \cap B(x, \delta_x) = \emptyset$ for $x \in \text{int}_c A$. Since $A$ and is an admissible set, it follows that also $A^c$ is admissible and hence $\text{int}_c A^c \subset A^c$ and $A^c \cap \text{ext}_c A^c = \emptyset$. Let us set $N := \partial_c A^c = \partial_c A$. Then $N$ is of $\sigma$-finite Hausdorff measure $\mathcal{H}^{n-1}$, which follows from the criterion for finite perimeter [6, p. 222].

Now we can define a gage $\tilde{\delta}$ on $\mathbb{R}^n$ in the following way:

$$\tilde{\delta}(x) = \begin{cases} 
0 & \text{if } x \in N, \\
1 & \text{if } x \in \text{ext}_c A, \\
\delta_x & \text{if } x \in \text{int}_c A.
\end{cases}$$

Let us fix a $\tilde{\delta}$-fine packing $(B(x_i, r_i))_{i=1}^k$ and sets $E_i$, where $E_i \subset B(x_i, \tau r_i)$, $E_i \in BV$, $(E_i, x_i)$ is $\varepsilon$-regular and $E_i$ is $\varepsilon$-isoperimetric for each $i = 1, \ldots, k$. By the $\varepsilon$-regularity of $E_i$, inequality (5.17) and definition of $\tilde{\delta}$, we obtain

1. $x_i \notin N$ for $i = 1, \ldots, k$;
2. $E_i \setminus A \subset (B \cap A^c) \setminus K$ when $x_i \in \text{int}_c A$;
3. $P(E_i \setminus A) \leq (1/\varepsilon) P(A^c, \text{in } E_i)$ when $x_i \in \text{int}_c A$.

Hence, using the inequality (5.16) and the fact that packing is pairwise disjoint, we can estimate

$$\sum_{x_i \in A} |\mathcal{F}(E_i \setminus A)| = \sum_{x_i \in \text{int}_c A} |\mathcal{F}(E_i \setminus A)| \leq \sum_{x_i \in \text{int}_c A} \mu(E_i \setminus A) + \frac{\varepsilon^2}{2P(A)} P(E_i \setminus A) \leq \mu(B \cap A^c \setminus K) + \frac{\varepsilon}{2P(A)} \sum_{x_i \in \text{int}_c A} P(A^c, \text{in } E_i) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Passing to the supremum we obtain

$$\sum_{x_i \in A} \tilde{\delta}_{x_i, \tau r_i}(\mathcal{F}|_{A^c}) < \varepsilon,$$

which we needed.

We now turn to the case $A$ is unbounded. Let us consider a sequence of balls $\{B_m\}$ which forms a locally finite covering of $\mathbb{R}^n$. Choose $\varepsilon > 0$. Let us fix $m \in \mathbb{N}$ and set $A_m := A \cap B_m$. Then $A_m$ is a bounded admissible locally $BV$ set and we can use the previous step to find $\varepsilon_m \leq 2^{-m} \varepsilon$ and a gage $\delta_m : \mathbb{R}^n \to [0, \infty)$ such that

$$\sum_{x_i \in A_m} \tilde{\delta}_{x_i, \tau r_i}(\mathcal{F}|_{A_m^c}) < \varepsilon_m$$

for every $\delta_m$-fine packing $((B(x_i, r_i))_{i=1}^k$.

Further, let us set

$$\delta(x) := \min\{\delta_m(x) : x \in B_m\}.$$
It is easily seen that \( \tilde{\delta} \) is a gage. Let us fix a \( \tilde{\delta} \)-fine packing \( ((B(x_i, r_i))_{i=1}^k) \). Then
\[
\sum_{x_i \in A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c}) \leq \sum_{m=1}^{\infty} \sum_{x_i \in A_m} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A_m^c}) < \sum_{m=1}^{\infty} 2^{-m} \varepsilon = \varepsilon,
\]
which establishes the formula.

Finally, we proceed similarly to find \( \tilde{\delta}^c \) which yields the second inequality and set
\[
\delta = \min\{\tilde{\delta}, \tilde{\delta}^c\},
\]
which gives both inequalities at the same time. \( \square \)

In the proof of the next theorem we are inspired by [13, Proposition 3.9].

**Theorem 5.17.** Let \( \mathcal{G}, \mathcal{F} \) be charges, \( f \in \mathcal{PR}^*(\mathcal{G}) \) and let \( \mathcal{F} \) be an indefinite packing \( \mathcal{R}^* \) integral of \( f \) with respect to \( \mathcal{G} \). If \( A \) is an admissible locally BV set, then \( \chi_{AF} \in \mathcal{PR}^*(\mathcal{G}) \) and \( \mathcal{F}^{A^c} \) is an indefinite packing \( \mathcal{R}^* \) integral of \( \chi_{AF} \) with respect to \( \mathcal{G} \).

**Proof.** Let us fix \( \tau \in (0, 1] \) as in Definition 5.3 and \( \varepsilon > 0 \). By the definition of packing \( \mathcal{R}^* \) integral and Theorem 5.16 there exists a gage \( \delta : \mathbb{R}^n \to [0, \infty) \) such that for every \( \delta \)-fine packing \( ((B(x_i, r_i))_{i=1}^k) \) we have
\[
\sum_{i=1}^k \tilde{q}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon,
\]
\[
\sum_{x_i \in A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c}) < \varepsilon \quad \text{and} \quad \sum_{x_i \notin A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c}) < \varepsilon.
\]
Hence
\[
\sum_{i=1}^k \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c} - f(x_i)\chi_A(x_i)\mathcal{G}) = \sum_{x_i \in A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c} - f(x_i)\mathcal{G}) + \sum_{x_i \notin A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c}) < \sum_{x_i \in A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c} - f(x_i)\mathcal{G}) + \sum_{x_i \notin A} \tilde{q}_{x_i, \tau r_i}(\mathcal{F}^{A^c}) < 3\varepsilon,
\]
which completes the proof. \( \square \)

**Theorem 5.18.** Let \( A \) be an admissible locally BV set. Then the charge \( \mathcal{F} \) in \( A \) is an indefinite packing \( \mathcal{R}^* \) integral of a function \( f : \text{cl}_A \to \mathbb{R} \) with respect to a charge \( \mathcal{G} \) in \( A \) if and only if \( \mathcal{F} \) is an indefinite packing \( \mathcal{R}^* \) integral of \( \tilde{f}_A \) with respect to \( \mathcal{G} \) in \( \mathbb{R}^n \).
Proof. Let us suppose that $F$ in $A$ is the indefinite packing $R^*$ integral of $f$ with respect to $G$ in $A$. Let us fix $\varepsilon > 0$. Now let $\tau \in (0,1]$ and a gage $\delta_1$ on $\text{cl}_A A$ be as in Definition 5.3 and let $\delta_2$ on $\mathbb{R}^n$ be as in Theorem 5.16. Then let us fix a $\delta$-fine packing $(B(x_i, r_i))_{i=1}^k$ and set

$$\delta = \begin{cases} 
\min\{\delta_1(x), \delta_2(x)\} & \text{if } x \in \text{cl}_A A, \\
\delta_2(x) & \text{if } x \in \mathbb{R}^n \setminus \text{cl}_A A.
\end{cases}$$

At first, let us consider the sum over $x_i \in A$. Since $F$ in $A$ is the indefinite packing $R^*$ integral of $f$ in $A$, we have

$$\sum_{x_i \in A} \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) = \sum_{x_i \in A} \tilde{q}_{x_i, \tau}(F - f(x_i)\mathcal{G}) < \varepsilon.$$ 

Further, for the case $x_i \not\in A$, we have by Theorem 5.16 the estimate

$$\sum_{x_i \not\in A} \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) < \varepsilon.$$ 

Therefore we obtain

$$\sum_{i=1}^k \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) = \sum_{x_i \in A} \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) + \sum_{x_i \not\in A} \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) < 2\varepsilon.$$ 

Hence $F$ is the indefinite packing $R^*$ integral of $\tilde{f}_A$ with respect to $\mathcal{G}$.

Conversely, let $F$ be the indefinite packing $R^*$ integral of $\tilde{f}_A$ with respect to $\mathcal{G}$. By Theorem 5.17 it follows that $F|A$ is the indefinite packing $R^*$ integral of $\tilde{f}_A$ with respect to $\mathcal{G}$ in $\mathbb{R}^n$. In other words, for fixed $\varepsilon > 0$ there exists a gage $\delta : \mathbb{R}^n \rightarrow [0, \infty)$ such that $\delta = 0$ on $\text{cl}_A A \setminus A$ and for every $\delta$-fine packing $(B(x_i, r_i))_{i=1}^k$ we have

$$\sum_{i=1}^k \tilde{q}_{x_i, \tau}(F|A - \tilde{f}_A(x_i)\mathcal{G}) < \varepsilon.$$ 

By the uniqueness of packing $R^*$ integral we have $F|A = F$ and hence

$$\sum_{x_i \in \text{cl}_A A} \tilde{q}_{x_i, \tau}(F - f(x_i)\mathcal{G}) = \sum_{x_i \in A} \tilde{q}_{x_i, \tau}(F - \tilde{f}_A(x_i)\mathcal{G}) < \varepsilon,$$

which we needed.

\[ \square \]

**Corollary 5.19.** Let $A$ be an admissible locally BV set and let a charge $F$ be an indefinite packing $R^*$ integral of a function $f : \text{cl}_A A \rightarrow \mathbb{R}$ in $A$ with respect to a charge $G$. Then, by Theorem 5.18, $F$ is the indefinite packing $R^*$ integral of $\tilde{f}_A$ with respect to $\mathcal{G}$, which is unique by Theorem 5.10. Therefore the indefinite packing $R^*$ integral in $A$ is unique as well.

Further, let $A$ be an admissible locally BV set and let a charge $F$ be an indefinite packing $R$ integral of a function $f : \text{cl}_A A \rightarrow \mathbb{R}$ in $A$ with respect to a charge $G$. Then $F$ is also the packing $R^*$ integral of in $A$ with respect to $\mathcal{G}$ by Theorem 5.12. Hence the uniqueness holds also for the indefinite packing $R$ integral in $A$. 


Remark 5.20. Since the function $f$ is defined on $\mathbb{R}$, the requirement that $A$ be admissible might seem to be unnecessary. This is really the case with $\mathcal{G} = \mathcal{L}$, because sets of measure zero (such as $A \triangle \mathbb{R}$) does not play a role in integration with respect to Lebesgue measure. On the other hand, Lebesgue null sets cannot be neglected in general. For example, the classical Cantor set cannot be neglected for integration with respect to the Cantor measure in $\mathbb{R}$, which is a charge by Example 3.16(1).

Definition 5.21. Let $A \in \mathbf{BV}$ be an admissible set, $f : \mathbb{R} \to \mathbb{R}$ be a function and $\mathcal{F}, \mathcal{G}$ be charges. We say that the number $\mathcal{F}(A)$ is a definite packing $\mathcal{R}^*$ integral of $f$ over $A$ with respect to $\mathcal{G}$ if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of $\bar{f}_A$ with respect to $\mathcal{G}$.

More generally: if $A \subseteq \mathbb{R}^n$ is a bounded measurable set and $\mathcal{F}$ is the indefinite packing $\mathcal{R}^*$ integral of $\bar{f}_A$ with respect to $\mathcal{G}$, then the definite packing $\mathcal{R}^*$ integral of $f$ over $A$ with respect to $\mathcal{G}$ is the number $\mathcal{F}(A')$, where $A' \in \mathbf{BV}$, $A' \supset A$ is a bounded admissible set.

The family of all functions packing $\mathcal{R}^*$ integrable with respect to $\mathcal{G}$ over $A$ is denoted by $\mathcal{P}\mathcal{R}^*(A, \mathcal{G})$.

Remark 5.22. The integral does not depend on the choice of $A'$. Indeed, let $A'$ and $A''$ be bounded admissible $\mathbf{BV}$ sets. Since $\bar{f}_A \cdot \chi_{A'} = \bar{f}_{A'} \cdot \chi_{A''}$, by Theorem 5.17 and by the uniqueness of the packing $\mathcal{R}^*$ integral we obtain $\mathcal{F}|_{A'} = \mathcal{F}|_{A''}$. Then $\mathcal{F}(A') = \mathcal{F}|_{A'}(A' \cup A'') = \mathcal{F}|_{A''}(A' \cup A'') = \mathcal{F}(A'').$

Remark 5.23. Let $A \in \mathbf{BV}$ be an admissible set, $\mathcal{G}$ be a charge and $f \in \mathcal{P}\mathcal{R}^*(A, \mathcal{G})$. Let $\mathcal{F}$ be the indefinite packing $\mathcal{R}^*$ integral of $f$ in $A$ with respect to $\mathcal{G}$. Then the definite $\mathcal{P}\mathcal{R}^*$ integral of $f$ over $A$ with respect to $\mathcal{G}$ is just $\mathcal{F}(A)$. This fact follows from Theorem 5.18.

Remark 5.24. If $f$ is a merely an indefinite packing $\mathcal{R}^*$ integrable function, it does not make sense to define the definite integral over unbounded sets in general. If we want to set up the definite integral over an unbounded set, we must suppose some additional limiting behaviour of the indefinite integral at infinity. There are several nonequivalent ways how to do it and we do not pursue this direction.

Remark 5.25. Let $A \subseteq \mathbb{R}^n$ be a bounded measurable set and let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then $\bar{f}_A$ is also a Lebesgue integrable function.

Then there exists an indefinite packing $\mathcal{R}^*$ integral of $\bar{f}_A$ with respect to Lebesgue measure. Hence the definite packing $\mathcal{R}^*$ integral of $f$ over $A$ is well defined.

In the following theorem, we will focus on the convergence of a sequence of sets. The importance of this property will be demonstrated in Section 7. The proof uses ideas from Pfeffer and Malý in [13, Theorem 3.20].

Theorem 5.26. Let $A$ be a bounded admissible $\mathbf{BV}$ set, $\mathcal{G}$ and $\mathcal{F}$ be charges and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let $\{A_j\}_{j=1}^\infty$ be a sequence of bounded admissible $\mathbf{BV}$ sets such that $A_j \subseteq A$ for $j = 1, 2, \ldots$ and $A_j \to A$ in $\mathbf{BV}$. Further, let $f \chi_{A_j} \in \mathcal{P}\mathcal{R}^*$ and $\mathcal{F}|_{A_j}$ be an indefinite packing $\mathcal{R}^*$ integral of $f \chi_{A_j}$ with respect to $\mathcal{G}$ with constants $\tau_j$ as in Definition 5.3. Let $\inf_j \tau_j > 0$. Then there exists an indefinite packing $\mathcal{R}^*$ integral of $f \chi_{A_j}$ with respect to $\mathcal{G}$ and is equal to $\mathcal{F}|_{A}$.

Proof. Let us fix $\tau = \inf_j \tau_j$ and let us denote $N := A \setminus \bigcup_{j=1}^\infty A_j$. Then $N$ is of $\mathcal{H}^{n-1}$ measure (see [17, Cor. 6.2.7]). Let us choose $\varepsilon > 0$. Since $\bar{q}_{\mathcal{L}^n, \tau} \leq \bar{q}_{\mathcal{L}^n, \tau/\tau}$...
for \( \tau \leq \tau' \), we can by the definition of packing \( \mathcal{R}^* \) integral and by Theorem 5.16 for \( j \in \mathbb{N} \) find a gage \( \delta_j \) such that for each \( \delta_j \)-fine packing \( (B(x_i, r_i))_{i=1}^k \) we obtain

\[
(5.18) \quad \sum_{x_i \in A_j} q_{x_i, r_i}^e (\mathcal{F}[A_j - f(x_i)] \mathcal{G}) < \varepsilon 2^{-j}
\]

and

\[
(5.19) \quad \sum_{x_i \in A_j} q_{x_i, r_i}^e (\mathcal{F}[A_j]) < \varepsilon 2^{-j}.
\]

Further, for \( x \in \bigcup_{j=1}^{\infty} A_j \) let us set \( j_x := \min\{ j \in \mathbb{N} ; x \in A_j \} \). Now we can define a gage

\[
\delta(x) = \begin{cases} 
\delta_{j_x}(x) & \text{if } x \in \bigcup_{j=1}^{\infty} A_j, \\
0 & \text{if } x \in \mathcal{N}.
\end{cases}
\]

By Theorem 5.18 it is enough to show that \( \mathcal{F}[A] \) is the indefinite packing \( \mathcal{R}^* \) integral of \( f \) in \( A \) with respect to \( \mathcal{G} \). Let us choose \( \delta \)-fine packing \( (B(x_i, r_i))_{i=1}^k \), \( x_i \in A \), and denote \( j_i := j_{x_i} \). Using the additivity of \( \mathcal{F} \) and estimates (5.18) and (5.19) we can for fixed \( p \in \mathbb{N} \) estimate

\[
\sum_{x_i \in A_j} q_{x_i, r_i}^e (\mathcal{F}[A - f(x_i)] \mathcal{G}) \leq \sum_{x_i \in A_j} q_{x_i, r_i}^e (\mathcal{F}[A_j - f(x_i)] \mathcal{G}) + q_{x_i, r_i}^e (\mathcal{F}[A \setminus A_j]) < \varepsilon 2^{-p+1}.
\]

Summing over \( p \) we obtain

\[
\sum_{p=1}^{\infty} \sum_{x_i : j_i = p} q_{x_i, r_i}^e (\mathcal{F}[A - f(x_i)] \mathcal{G}) < \sum_{p=1}^{\infty} \varepsilon 2^{-p+1} = 2\varepsilon,
\]

which completes the proof.

**Definition 5.27.** Let \( A \subset \mathbb{R}^n \) be a locally \( BV \) set and let \( f : \text{cl}_* A \to \mathbb{R} \) and \( u \in C(\bar{A}, \mathbb{R}^n) \) be functions. Further, let a charge \( \mathcal{F} \) be the flux of \( u \) in \( A \). We say that \( f \) is a **generalized divergence** of \( u \) in \( A \) if \( \mathcal{F} \) is an indefinite packing \( \mathcal{R}^* \) integral of \( f \) in \( A \). The generalized divergence of \( u \) will be denoted by \( \text{Div} u \).

The following three definitions was mentioned by Pfeffer in Chapters 2.3 and 2.5 of [17].

**Definition 5.28.** Let \( A \subset \mathbb{R}^m \) be a measurable set and let \( x \in A \cap \text{int}_* A \). A map \( u : A \to \mathbb{R}^n \) is called **differentiable at \( x \) relative to \( A \)** if there is a linear map \( L : \mathbb{R}^m \to \mathbb{R}^n \) such that for given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that

\[
|u(y) - u(x) - L(y-x)| < \varepsilon |y-x|
\]

for each \( y \in A \cap B(x, \delta) \). The linear map \( L \) is called the **differential of \( u \) at \( x \) relative to \( A \)** and is denoted by \( D_A u(x) \).

Let \( x \in \text{int}_* A \) and \( u : \text{cl}_* A \to \mathbb{R}^m \) be a vector field. Let \( u \) be differentiable at \( x \) relative to \( \text{cl}_* A \). The **divergence of \( u \) at \( x \) relative to \( \text{cl}_* A \)** is the number

\[
\text{div}_* u(x) := \text{tr} D_{\text{cl}_* A} u(x),
\]

where \( \text{tr} D_{\text{cl}_* A} u(x) \) denotes the trace of the matrix representation of the linear transformation \( D_{\text{cl}_* A} u(x) : \mathbb{R}^m \to \mathbb{R}^m \).

By \( \text{div} u \) we will denote the pointwise divergence defined on interior points of \( A \) at which \( u \) is differentiable. Especially, \( \text{div}_* u = \text{div} u \) whenever \( \text{div} u \) is defined.
Definition 5.29. Let \( F \) be a charge and let \( x \in \mathbb{R}^n \). Then for \( \eta \geq 0 \) we define

\[
D_\eta F(x) := \sup_{\delta > 0} \inf_{E} \frac{F(E)}{|E|} \quad \text{and} \quad D^\eta F(x) := \inf_{\eta > 0} \sup_{\delta > 0} \frac{F(E)}{|E|},
\]

where \( E \in \text{BV} \) such that \( d(E \cup \{x\}) < \delta \) and \( r(E, x) > \eta \).

The lower and upper derivative of \( F \) at \( x \) are defined as

\[
D^\pm F(x) := \inf_{\eta > 0} D_\eta F(x) \quad \text{and} \quad D^\pm F(x) := \sup_{\eta > 0} D_\eta F(x).
\]

We say that \( F \) is derivable at \( x \), if

\[
D^\pm F(x) = D^\pm F(x) \neq \pm \infty.
\]

The derivative of \( F \) at \( x \) is then defined as \( D F(x) := D^\pm F(x) = D^\pm F(x) \).

Definition 5.30. Let \( E \) be a locally \( \text{BV} \) set, \( u : \text{cl}_e E \to \mathbb{R}^n \) be a bounded Borel measurable vector field and \( F \) be the flux of \( u \). If \( F \) is derivable at \( x \in \text{int}_e E \), we call the number \( \text{div} \ u(x) := D F(x) \) the mean divergence of \( u \) at \( x \).

Applying the inclusion between \( \mathcal{R} \) integral and packing \( \mathcal{R}^\ast \) integral we can state sufficient conditions for existence of generalized divergence. For further details see [17, Example 2.3.2, Remark 2.5.9, Theorem 5.1.12, Proposition 2.5.7 and Corollary 5.1.13].

Proposition 5.31. Let \( A \) be a locally \( \text{BV} \) set and \( u \in C(\overline{A}, \mathbb{R}^n) \).

1. If \( A = \mathbb{R}^n \) and \( u \) is differentiable in \( \mathbb{R}^n \), then \( \text{div} \ u \) is a generalized divergence of \( u \).
2. If \( u \) is differentiable relatively to \( \text{cl}_e A \) on \( \text{int}_e A \), then \( \text{div} \ u \) is a generalized divergence of \( u \).
3. If \( u \) is differentiable relatively to \( \text{cl}_e A \) on \( \text{int}_e A \), then \( \text{div} \ u \) is a generalized divergence of \( u \).
4. If \( u \) is Lipschitz on \( \text{cl}_e A \setminus T \), where \( T \) is of \( \sigma \)-finite Hausdorff measure \( \mathcal{H}^{n-1} \), then \( \text{div} \ u \) is a generalized divergence of \( u \).

Theorem 5.32 (Gauss-Green divergence theorem). Let \( A \subset \mathbb{R}^n \) be a bounded \( \text{BV} \) set, let \( u \in C(A, \mathbb{R}^n) \). Let us suppose that there exists a generalized divergence \( \text{Div} \ u \) in \( A \). Then

\[
\int_A \text{Div} \ u(x) \, dx = \int_{\partial A} u \cdot \nu_A \, d\mathcal{H}^{n-1},
\]

where the integral on the left is the definite packing \( \mathcal{R}^\ast \) integral.

Proof. Since \( |A \cap \text{cl}_e A| = 0 \), it is enough to show that \( \int_{\text{cl}_e A} \text{Div} \ u(x) \, dx = \int_{\partial A} u \cdot \nu_A \, d\mathcal{H}^{n-1} \) (see Remark 5.20). Let \( F \) denote the indefinite packing \( \mathcal{R}^\ast \) integral of \( \text{Div} \ u \) in \( A \). Since \( F \) is the flux of \( u \) in \( A \), we have \( F(A) = \int_{\partial A} u \cdot \nu_A \, d\mathcal{H}^{n-1} \). By Theorem 5.18 we obtain \( \int_{\text{cl}_e A} \text{Div} \ u(x) \, dx = F(A) \), which completes the proof.

6. \( \mathcal{R} \) integral

In this section we will introduce Pfeffer’s \( \mathcal{R} \) integral described in [17]. For easier comparison of integrals we use the characterization of \( \mathcal{R} \) integral [17, Proposition 5.5.6] rather than original definition.

Definition 6.1. A \( \text{BV} \) partition is a system of couples \( \{(A_1, x_1), \ldots, (A_k, x_k)\} \) of pairwise disjoint bounded \( \text{BV} \) sets \( A_i \) and points \( x_i \in \mathbb{R}^n \) for \( i = 1, \ldots, k \). It is not required \( x_i \in A_i \).
Definition 6.2. Let $A$ be a locally $BV$ set and $\mathcal{F}$, $\mathcal{G}$ be charges in $A$. Let $f$ be a function defined on $cl\ A$. We say that $\mathcal{F}$ is an intrinsic indefinite $\mathcal{R}$ integral of $f$ in $A$ with respect to $\mathcal{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : cl\ A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} |\mathcal{F}(A_i) - f(x_i)\mathcal{G}(A_i)| < \varepsilon$$

for each $\varepsilon$-regular $\delta$-fine $BV$ partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ with $\bigcup_{i=1}^{k} A_i \subset A$ and $x_i \in cl\ A$ for $i = 1, \ldots, k$.

The family of all $\mathcal{R}$ integrable functions in $A$ with respect to $\mathcal{G}$ is denoted by $\mathcal{R}(A, \mathcal{G})$. The family of all $\mathcal{R}$ integrable functions in $A$ with respect to Lebesgue measure is denoted just by $\mathcal{R}(A)$.

Remark 6.3. The intrinsic indefinite $\mathcal{R}$ integral is well defined, unique and linear. For the proof and other properties see [17, p. 211-213].

Definition 6.4. Let $A$ be a locally $BV$ set and $\mathcal{F}$, $\mathcal{G}$ be charges in $A$. Let $f$ be a function defined on $cl\ A$. We say that $\mathcal{F}$ is an indefinite $\mathcal{R}$ integral of $f$ in $A$ with respect to $\mathcal{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : cl\ A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} |\mathcal{F}(A_i) - f(x_i)\mathcal{G}(A_i)| < \varepsilon$$

for each $\varepsilon$-regular $\delta$-fine $BV$ partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ with $x_i \in cl\ A$ for $i = 1, \ldots, k$. (We do not require that $A_i \subset A$.)

The family of all $\mathcal{R}$ integrable functions in $A$ with respect to $\mathcal{G}$ is denoted by $\mathcal{IR}(A, \mathcal{G})$.

Remark 6.5. Let us remark that our terminology slightly differs from that used in [17]. Namely, what we call “intrinsic indefinite $\mathcal{R}$ integral in $A$” is termed simply “indefinite $\mathcal{R}$ integral” in [17]. Furthermore, in [17] it is distinguished between the $\mathcal{R}$ integral (with respect to Lebesgue measure) and $\mathcal{J}$ integral (Stieltjes version; with respect to an arbitrary charge).

Lemma 6.6. Let $\varepsilon > 0$ and $A \subset \mathbb{R}^n$ be an $\varepsilon$-regular bounded $BV$ set. Then $[diam(A)]^n \leq \frac{1}{c\varepsilon} |A|$, where $c = c(n)$ is a constant depending only on $n$.

Proof. Since $A$ is $\varepsilon$-regular, we have $diam(A)P(A) \leq \frac{1}{\varepsilon} |A|$. Further, by the isoperimetric inequality (see [17, Theorem 1.8.7]) we have $|A|^{\frac{n+1}{n}} \leq p(n)P(A)$, where $p(n)$ is a constant depending on $n$. Thus $diam(A)P(A) \leq \frac{1}{\varepsilon} |A| \leq \frac{1}{c\varepsilon} |A|^{\frac{n+1}{n}}P(A)^{\frac{n}{n+1}}$. Hence

$$diam(A)^{n-1}P(A)^{n-1} \leq \frac{1}{c^{n-1}}p(n)^nP(A)^n,$$

$$diam(A)^{n-1} \leq \frac{1}{c^{n-1}}p(n)^nP(A),$$

$$diam(A)^n \leq \frac{1}{c^n}p(n)^nP(A)diam(A) \leq \frac{1}{c^n}c|A|,$$

where $c = p(n)^n$. \qed
Then \( r < \delta \)

Further, let us find a minimal ball

Let us set \( \delta \) such that for every \( r < R \)

We need to show that the system \( \{A_i, x_i\} \) is obviously a \( \delta \)-fine \( BV \) partition. We need to show that the system \( \{(A_i', x_1), \ldots, (A_k', x_k)\} \) is \( \varepsilon' \)-regular.

Let us fix \( i \in \{1, \ldots, k\} \). Since \( (A_i, x_i) \) is \( \varepsilon \)-regular, we have

Further, let us find a minimal ball \( B = B(x_i, r) \) with the property that \( A_i \subset B \). Then \( r < \delta(x) \).

By Lemma 6.6 there exists a constant \( c \) such that

Then applying (6.2) we can estimate

\[
|A_i| \leq |A_i \cap A| + |A_i \setminus A| \\
\leq |A_i \cap A| + |B \setminus A| \\
\leq |A_i \cap A| + \varepsilon^{n+1}|B| \\
\leq |A_i \cap A| + c\varepsilon|A_i|.
\]
Hence

\[(1 - \alpha_n \varepsilon)|A_i| \leq |A_i \cap A|.
\]

Applying (6.1), (6.3) and Lemma 6.6 then gives

\[
diam(A_i \cap A \cup \{x_i\}) P(A_i \cap A) \leq diam(A_i \cup \{x_i\}) [P(A_i) + P(A, B)]
\]

\[
\leq \frac{1}{\varepsilon}|A_i| + diam(A_i \cup \{x_i\})\varepsilon^{n-1}r^{n-1}
\]

\[
\leq \frac{1}{\varepsilon}|A_i| + \varepsilon^{n-1}[diam(A_i \cup \{x_i\})]^n
\]

\[
\leq \left(\frac{1}{\varepsilon} + \frac{c}{\varepsilon}\right)|A_i|
\]

\[
\leq \frac{1 + c}{\varepsilon(1 - \alpha_n \varepsilon)}|A_i \cap A|
\]

\[
= \frac{1}{\varepsilon}|A_i \cap A|
\]

Thus the system \{(A'_1, x_1), \ldots, (A'_{k'}, x_{k'})\} is \(\varepsilon'\)-regular \(BV\) partition. Since \(\mathcal{F}\) and \(\mathcal{G}\) are charges in \(A\), we have \(\mathcal{F}(A_i) = \mathcal{F}(A'_i)\) and \(\mathcal{G}(A_i) = \mathcal{G}(A'_i)\) for \(i = 1, \ldots, k\). Further, since \(\mathcal{F}\) is the intrinsic indefinite \(\mathcal{R}\) integral of \(f\) with respect to \(\mathcal{G}\), we can estimate

\[
\sum_{i=1}^{k} |\mathcal{F}(A_i) - f(x_i)\mathcal{G}(A_i)| = \sum_{i=1}^{k} |\mathcal{F}(A'_i) - f(x_i)\mathcal{G}(A'_i)| < \varepsilon', \varepsilon < \varepsilon,
\]

which completes the proof.

\[\Box\]

**Corollary 6.8.** Let \(A\) be a locally \(BV\) set, \(\mathcal{F}\), \(\mathcal{G}\) be charges in \(A\). Let \(f\) be a function defined on \(cl \ast A\). Then \(\mathcal{F}\) is an intrinsic indefinite \(\mathcal{R}\) integral of \(f\) in \(A\) if and only if \(\mathcal{F}\) is an indefinite \(\mathcal{R}\) integral of \(f\) in \(A\) with respect to \(\mathcal{G}\).

**Theorem 6.9.** Let \(A\) be an admissible locally \(BV\) set, \(\mathcal{F}\), \(\mathcal{G}\) be charges in \(A\). Let \(f\) be a function defined on \(cl \ast A\). Let \(\mathcal{F}\) be an (intrinsic) indefinite \(\mathcal{R}\) integral of \(f\) in \(A\) with respect to \(\mathcal{G}\). Then \(\mathcal{F}\) is also an indefinite packing \(\mathcal{R}\) integral of \(f\) in \(A\) with respect to \(\mathcal{G}\).

**Proof.** Let us set \(\tau := 1\). Now let us choose \(\varepsilon > 0\) and find a gage \(\delta\) as in Definition 6.4. Let us fix a \(\delta\)-fine packing \((B(x_i, r_i))_{i=1}^{k}, x_i \in cl \ast A\). We need to show that

\[
\sum_{i=1}^{k} \bar{p}_{x_i, r_i}(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon,
\]

where \(\bar{p}_{x, r}(\mathcal{F}) = \text{sup}\{ |\mathcal{F}(E)| ; E \subset B(x, r), E \in BV, (E, x) \text{ is } \varepsilon\text{-regular}\} \).

Now, let us fix test sets \(E_i\) such that \(E_i \subset B(x_i, r_i)\), \(E_i\) are \(BV\) sets and \((E_i, x_i)\) are \(\varepsilon\)-regular for \(i = 1, \ldots, k\). Obviously, the system \\{(E_i, x_i)\}\ is \(\varepsilon\)-regular \(\delta\)-fine \(BV\) partition and hence by Definition 6.4 and Theorem 6.7 we have

\[
\sum_{i=1}^{k} |\mathcal{F}(E_i) - f(x_i)\mathcal{G}(E_i)| < \varepsilon.
\]
Passing to the supremum we obtain
\[
\sum_{i=1}^{k} \tilde{p}_{r_i}(F - f(x_i)g) \leq \varepsilon.
\]

\[\square\]

7. \( GR \) integral

It can happen that a function which is \( R \) integrable in sets \( A_1 \) and \( A_2 \) is not \( R \) integrable in their union. Also, \( R \) integrability is not closed with respect to \( BV \) convergence of sets. To correct this deficiency, Pfeffer [17] extended the definition of the \( R \) integral. Fortunately, the construction based on the closure with respect to \( BV \) convergence of sets solves automatically the problem of additivity. The result of this construction is called \( GR \) integral (the generalized Riemann integral). Using our Theorem 7.5 we show that also this \( GR \) integral is contained in our packing \( R^* \) integral.

**Notation 7.1.** Let \( f \) be a function whose domain contains a locally \( BV \) set \( E \) and let \( F \) be a charge.

Then we denote by \( R(f, F, E) \) the family of all bounded \( BV \) sets \( A \subset E \) such that \( f_{\chi_A} \) belongs to \( R(A) \) and the charge \( F|_A \) is the indefinite \( R \) integral of \( f_{\chi_A} \).

Further, let us denote \( \overline{R}(f, F, E) \) the minimal system of bounded \( BV \) sets containing \( R(f, F, E) \) and closed with respect to convergence in \( BV \).

**Definition 7.2.** Let \( f \) be a function defined on a locally \( BV \) set \( E \). We say that a charge \( F \) is an indefinite \( GR \) integral of \( f \) in \( E \) if \( R(f, F, E) = BV(E) \), where \( BV(E) = \{ A \in BV; A \subset E \} \). The family of all \( GR \) integrable functions in \( E \) is denoted by \( GR(E) \).

**Remark 7.3.** The indefinite \( GR \) integral is well defined, unique and linear. For further details see [17, Sec. 6.3].

The next theorem with proof can be found in [17, Proposition 6.3.12].

**Theorem 7.4.** Let \( E \) be a locally \( BV \) set. Then

1. If \( n = 1 \), then \( R(E) = GR(E) \).
2. If \( n \geq 2 \) and \( \text{int} E \neq \emptyset \), then \( R(E) \subseteq GR(E) \).

**Theorem 7.5.** Let \( E \) be a bounded admissible \( BV \) set. Then \( GR(E) \subset PR^*(E) \).

**Proof.** The proof follows from Theorems 5.12, 6.9 and 5.26. \[\square\]

8. \( R^* \) integral

The \( R^* \) integral was introduced by Malý and Pfeffer in [13]. It is an alternative approach to overcome drawbacks of the \( R \) integral. Moreover, in \( \mathbb{R}^1 \) this integral coincides with the Henstock-Kurzweil integral.

**Definition 8.1.** Let \( \varepsilon > 0 \). We say that an \( \varepsilon \)-regular \( BV \) partition \( \{(A_1, x_1), \ldots, (A_k, x_k)\} \) is strongly \( \varepsilon \)-regular if \( A_i \) is \( \varepsilon \)-isoperimetric and \( x_i \in \text{cl}_* A_i \) for \( i = 1, \ldots, k \).
**Definition 8.2.** Let $A \subset \mathbb{R}^n$ be a locally $BV$ set. We say that a charge $\mathcal{F}$ in $A$ is an *indefinite $R^*$ integral* of a function $f : \text{cl} A \to \mathbb{R}$ in $A$ with respect to a charge $G$ if for given $\varepsilon > 0$ we can find a gage $\delta : \text{cl} A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} |\mathcal{F}(A_i) - f(x_i)G(A_i)| < \varepsilon$$

for each strongly $\varepsilon$-regular $\delta$-fine $BV$ partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$.

The family of all $R^*$ integrable functions in $A$ is denoted by $R^*(A, G)$. The family of all $R^*$ integrable functions in $A$ with respect to Lebesgue measure is denoted just by $R^*(A)$.

**Remark 8.3.** It is easily seen that for an admissible $BV$ set $E$ we have $\mathcal{R}(E) \subset R^*(E)$.

**Theorem 8.4.** Let $A \subset \mathbb{R}^n$ be a locally $BV$ set. Let a charge $\mathcal{F}$ be an indefinite $R^*$ integral of a function $f : \text{cl} A \to \mathbb{R}$ in $A$ with respect to a charge $G$. Then $\mathcal{F}$ is also an indefinite packing $R^*$ integral of $f$ in $A$ with respect to $G$.

**Proof.** Let us set $\tau := 1$. Then let us choose $\varepsilon > 0$ and find a gage $\delta$ as in Definition 8.2.

Let us fix a $\delta$-fine packing $(B(x_i, r_i))_{i=1}^{k}$, $x_i \in \text{cl} A$. We need to show that

$$\sum_{i=1}^{k} \bar{q}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)G) < \varepsilon,$$

where

$$\bar{q}_{x, \tau r}(\mathcal{F}) = \sup \{|\mathcal{F}(E)| ; E \subset B(x, r), E \in \mathcal{BV}, x \in \text{cl} E, (E, x) \text{ is } \varepsilon \text{-regular and } \varepsilon \text{-isoperimetric} \}.$$

Now let us fix test sets $E_i, E_i \subset B(x_i, r_i), x_i \in \text{cl} E_i, E_i$ is $BV$ and $(E, x)$ is $\varepsilon$-regular and $\varepsilon$-isoperimetric for $i = 1, \ldots, k$.

Obviously, the system $\{(E_i, x_i)\}$ is strongly $\varepsilon$-regular $\delta$-fine $BV$ partition and hence by Definition 8.2 we obtain

$$\sum_{i=1}^{k} |\mathcal{F}(E_i) - f(x_i)G(E_i)| < \varepsilon.$$

Passing to the supremum we obtain

$$\sum_{i=1}^{k} \bar{q}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)G) \leq \varepsilon.$$

$\Box$

**Remark 8.5.** Let $E$ be an admissible locally $BV$ set. Then $\mathcal{G}(E) \subseteq R^*(E)$. The inclusion follows from [13, Corollary 3.18] and [13, Theorem 3.20]. An example of function which is $R^*$ integrable but not $\mathcal{G}$ integrable can be found in [15, Example 6.9] and [15, Proposition 10.8].
9. Henstock-Kurzweil-Stieltjes integral

In the next two sections we will investigate packing $\mathcal{R}$ and packing $\mathcal{R}^*$ integral on the real line. For this purpose let us note that a charge $\mathcal{F}$ in $\mathbb{R}^1$ can be identified with an “ordinary” function $F$ through the relation $\mathcal{F}((u, v)) = F(v) - F(u)$. Since for those integral $\mathcal{F}$ is supposed to be a charge, in the view of Example 3.16, $F$ is continuous.

**Definition 9.1.** Let $[a, b] \subset \mathbb{R}^1$ be a compact interval. A finite collection $([a_i, b_i], \xi_i)_{i=1}^k$ of tagged intervals is called a subpartition of $[a, b]$ if intervals $[a_i, b_i]$ are nonoverlapping and $\xi_i \in [a_i, b_i]$ for every $i = 1, \ldots, k$.

A function $\delta : [a, b] \to (0, \infty)$ is called a positive gage. We say that a subpartition is $\delta$-fine if $|b_i - a_i| < \delta(\xi_i)$.

**Definition 9.2.** Let $f, G, F : [a, b] \to \mathbb{R}$ be functions. We say that $F$ is the strong Henstock-Kurzweil-Stieltjes integral of $f$ with respect to $G$ if for every $\varepsilon > 0$, there exists a positive gage $\delta : [a, b] \to (0, \infty)$, so that for every $\delta$-fine subpartition $([a_i, b_i], \xi_i)_{i=1}^k$ we have

$$\sum_{i=1}^k |F(b_i) - F(a_i) - f(\xi_i)(G(b_i) - G(a_i))| < \varepsilon.$$  

In the case $G$ is the identity function we say that $F$ is just the strong Henstock-Kurzweil integral of $f$.

The families of all strongly Henstock-Kurzweil-Stieltjes integrable functions on $[a, b]$ with respect to $G$ and all strongly Henstock-Kurzweil integrable functions on $[a, b]$ are denoted by $HKS([a, b], G)$ and $HK([a, b])$, respectively.

**Definition 9.3.** Let $f, G, F : \mathbb{R} \to \mathbb{R}$ be functions. We say that $F$ is the indefinite Henstock-Kurzweil-Stieltjes integral of $f$ with respect to $G$ if $F$ is the strong Henstock-Kurzweil-Stieltjes integral of $f$ with respect to $G$ on every compact interval $[a, b] \subset \mathbb{R}$.

In the case $G$ is the identity function we say that $F$ is just the indefinite Henstock-Kurzweil integral of $f$.

The families of all Henstock-Kurzweil-Stieltjes integrable functions with respect to $G$ and all Henstock-Kurzweil integrable functions are denoted by $HKS(\mathbb{R}, G)$ and $HK(\mathbb{R})$, respectively.

The proof of the following proposition can be found in [13, Proposition 3.6].

**Proposition 9.4.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is $\mathcal{R}^*$ integrable with respect to Lebesgue measure if and only if $f$ is strongly Henstock-Kurzweil integrable on every compact interval $[a, b] \subset \mathbb{R}$.

Applying Theorem 7.4, Remark 8.5 and Proposition 9.4 we obtain the following theorem.

**Theorem 9.5.** $\mathcal{R}(\mathbb{R}) \subsetneq HK(\mathbb{R})$.

10. $MC$ and $MC_\alpha$ integrals

In this section we will introduce $MC$ and $MC_\alpha$ integrals. The monotonically controlled Stieltjes ($MC$) integral was defined by Bendová and Malý in [3]. The theory of the $MC_\alpha$ integral with respect to Lebesgue measure was further developed.
by Ball and Preiss in [2]. Their ideas will be used in the proofs of Propositions 10.3 and 10.4.

**Definition 10.1.** Let $\alpha > 0$ be a real number and $f, F, G : \mathbb{R} \to \mathbb{R}$ be functions, let $G$ be continuous. We say that $F$ is an indefinite $MC_\alpha$ integral of $f$ with respect to $G$ if there exists a strictly increasing control function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$ we have

$$
\lim_{h \to 0} \frac{F(x + h) - F(x) - f(x)(G(x + h) - G(x))}{\varphi(x + \alpha h) - \varphi(x)} = 0.
$$

The families of all $MC_\alpha$ integrable functions with respect to $G$ and all $MC_\alpha$ integrable functions with respect to identity function are denoted by $MC_\alpha(G)$ and $MC_\alpha$, respectively.

Especially, if $\alpha = 1$, we say that $F$ is an indefinite $MC$ integral of $f$ with respect to $G$. We write $MC(G) = MC_1(G)$ and $MC = MC_1$.

**Remark 10.2.** In Definition 10.1 the control function $\varphi$ can be chosen to be bounded. (See [3, Lemma 1].)

**Proposition 10.3.** Let $\alpha > 0$ and let $f, F, G : \mathbb{R} \to \mathbb{R}$ be functions, let $G$ be continuous. If $F$ is an indefinite $MC_\alpha$ integral of $f$ with respect to $G$, then $F$ is continuous.

**Proof.** Let us fix $\varepsilon > 0$ and $x \in \mathbb{R}$. We need to find $\delta$ such that for every $|h| < \delta$ we have

$$
|F(x + h) - F(x)| < \varepsilon.
$$

Since $G$ is continuous at $x$, we can find $\delta_1$ such that for every $|h| < \delta_1$ we have $|G(x + h) - G(x)| < \varepsilon$. Further, since $F$ is the indefinite $MC_\alpha$ integral of $f$ with respect to $G$, there exists a strictly increasing control function $\varphi : \mathbb{R} \to \mathbb{R}$ and a $\delta < \delta_1$ such that for every $|h| < \delta$ we have

$$
\left| \frac{F(x + h) - F(x) - f(x)(G(x + h) - G(x))}{\varphi(x + \alpha h) - \varphi(x)} \right| < \varepsilon.
$$

Applying Remark 10.2 we can assume that there exists a constant $M$ such that $|\varphi(x)| < M$ for every $x \in \mathbb{R}$.

Hence

$$
|F(x + h) - F(x)|
\leq \left| \frac{F(x + h) - F(x) - f(x)(G(x + h) - G(x))}{\varphi(x + \alpha h) - \varphi(x)} (\varphi(x + \alpha h) - \varphi(x)) \right| + |f(x)(G(x + h) - G(x))| < \varepsilon(2M + f(x)).
$$

**Proposition 10.4.** Let $0 < \alpha < \beta$ be real numbers, $f, F, G : \mathbb{R} \to \mathbb{R}$ be functions and let $G$ be continuous. If $F$ is an indefinite $MC_\alpha$ integral of $f$ with respect to $G$, then $F$ is also an indefinite $MC_\beta$ integral of $f$ with respect to $G$.

**Proof.** The proof follows from the fact that for $0 < \alpha < \beta$ we have $|\varphi(x + \alpha h) - \varphi(x)| \leq |\varphi(x + \beta h) - \varphi(x)|$ for $h \in \mathbb{R}$.

The two following theorems can be found in [2, Theorem 3].
Theorem 10.5. For every $\alpha \geq 2$ there exists a function which is not $MC_\alpha$ integrable but is $MC_\beta$ integrable for every $\beta > \alpha$.

Theorem 10.6. Let $\alpha > 2$. Then $MC$ is a proper subspace of $MC_\alpha$.

For the proof of the next theorem see [2, Theorem 3].

Theorem 10.7. Let $\alpha \in [1, 2]$. Then $MC = MC_\alpha$.

Theorem 10.8. Let $G, F, f : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $G$ is continuous. Then $F$ is an indefinite $MC$ integral of $f$ with respect to $G$ if and only if $F$ is an indefinite Henstock-Kurzweil-Stieltjes integral of $f$ with respect to $G$.

Proof. For the proof and further details see [3, Theorem 3] and [2, Theorem 17]. □

Theorem 10.9. Let $\alpha \geq 1, G, F, f : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $G$ is continuous. Let $F$ be an indefinite $MC_\alpha$ integral of $f$ with respect to $G$. Further, let $\mathcal{F}$ and $\mathcal{G}$ be charges induced by $F$ and $G$ in the sense of Example 3.16. Then $\mathcal{F}$ is also an indefinite integral of $f$ with respect to $\mathcal{G}$.

Proof. First, let us note that $F$ is continuous by Proposition 10.3. Hence it is legitimate to use the term charges for the set functions $\mathcal{F}$ and $\mathcal{G}$ constructed as in Example 3.16.

Let us set $\varepsilon := 1/\alpha$. Further, let us fix $\varepsilon > 0$ and write $\varepsilon' := \varepsilon^2$. Since $f$ is $MC_\alpha$ integrable, there exists a strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following property: for each $x \in \mathbb{R}$ there exists $\delta(x) > 0$ such that for every $|h| < \delta(x)$ we have

\begin{equation}
|F(x + h) - F(x) - f(x)(G(x + h) - G(x))| < \varepsilon' |\varphi(x + \alpha h) - \varphi(x)|.
\end{equation}

Moreover, by Remark 10.2 we can suppose that there exists $M > 0$ such that $|\varphi| \leq M$.

We need to show that for fixed $\delta$-fine packing $(B(x_i, r_i))_{i=1}^k$, we have

\[
\sum_{i=1}^k \delta_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon,
\]

where $\delta_{x_i, \tau r_i}(\mathcal{F}) = \sup \{|\mathcal{F}(E)| : E \subset B(x_i, \tau r_i), E \in \mathcal{B} \mathcal{V}, (E, x_i) \text{ is } \varepsilon\text{-regular}\}$.

Let us fix $i \in \{1, \ldots, k\}$ and a test set $E_i \in \mathcal{B} \mathcal{V}$ such that $E_i \subset B(x_i, \tau r_i)$ and $(E_i, x_i)$ is $\varepsilon$-regular. In other words, $E_i = \bigcup_{j=1}^{l_i} (a_{ij}^j, b_{ij}^j)$ is a finite union of disjoint nondegenerate intervals in $B(x_i, \tau r_i)$ (up to a Lebesgue null set). Moreover, since $E_i$ is $\varepsilon$-regular and $\mathcal{H}^0$ is the counting measure, we estimate

\[
\frac{1}{||E_i||} \geq \frac{|E_i|}{d(E_i \cup \{x_i\}) ||E_i||} > \varepsilon
\]

and

\[
\frac{1}{\varepsilon} \geq ||E_i|| = \mathcal{H}^0(\partial_* E_i) = 2l_i.
\]

Let us set $m$ to be the greatest natural number such that $m \leq 1/(2\varepsilon)$. Then $l_i \leq m \leq 1/(2\varepsilon)$.

Further, since for each $a_{ij}^j$ and $b_{ij}^j$, $i = 1, \ldots, k$ and $j = 1, \ldots, l_i$, we have $|a_{ij}^j - x_i| < \delta(x_i)$ and $|b_{ij}^j - x_i| < \delta(x_i)$, by (10.1) and the fact that $\varphi$ is increasing we have the estimates

\[
|F(b_{ij}^j) - F(x_i) - f(x_i)(G(b_{ij}^j) - G(x_i))| < \varepsilon' |\varphi(x_i + r_i) - \varphi(x_i - r_i)|
\]
and
\[ |F(a_j^i) - F(x_i) - f(x_i)(G(a_j^i) - G(x_i))| < \varepsilon' |\varphi(x_i + r_i) - \varphi(x_i - r_i)|. \]

Moreover, since the system \((B(x_i, r_i))_{i=1}^k\) is pairwise disjoint and \(\varphi\) is strictly increasing and bounded, we have
\[
\sum_{i=1}^k |F(b_j^i) - F(a_j^i) - f(x_i)(G(b_j^i) - G(a_j^i))| \\
\leq \sum_{i=1}^k |F(b_j^i) - F(x_i) - f(x_i)(G(b_j^i) - G(x_i))| \\
+ |F(a_j^i) - F(x_i) - f(x_i)(G(a_j^i) - G(x_i))| \\
\leq \sum_{i=1}^k 2\varepsilon' |\varphi(x_i + r_i) - \varphi(x_i - r_i)| \\
< 2\varepsilon'(\varphi(x_k + r_k) - \varphi(x_1 - r_1)) \\
< 4\varepsilon'M.
\]

Let us denote \(L := \max_i l_i\). For \(j \in \{1, \ldots, L\}\) let \(I_j\) be the set of indices \(i \in \{1, \ldots, k\}\) for which \(l_i \geq j\). Then applying estimates in (10.2) we obtain
\[
\sum_{i=1}^k |\mathcal{F}(E_i) - f(x_i)\mathcal{G}(E_i)| \\
\leq \sum_{i=1}^k \sum_{j=1}^{l_i} |F(b_j^i) - F(a_j^i) - f(x_i)(G(b_j^i) - G(a_j^i))| \\
\leq \sum_{i=1}^k \sum_{j=1}^{l_i} |F(b_j^i) - F(x_i) - f(x_i)(G(b_j^i) - G(x_i))| \\
+ |F(a_j^i) - F(x_i) - f(x_i)(G(a_j^i) - G(x_i))| \\
\leq \sum_{j=1}^L \sum_{i \in I_j} |F(b_j^i) - \mathcal{F}(x_i) - f(x_i)(G(b_j^i) - G(x_i))| \\
+ |F(a_j^i) - \mathcal{F}(x_i) - f(x_i)(G(a_j^i) - G(x_i))| \\
< \sum_{j=1}^L 4\varepsilon'M = 4L\varepsilon^2M \leq \frac{4\varepsilon^2M}{2\varepsilon} = 2M\varepsilon.
\]

Finally, passing to the supremum we obtain
\[
\sum_{i=1}^k \mathcal{X}_{x_i r_i}(\mathcal{F} - f(x_i)\mathcal{G}) \leq 2M\varepsilon,
\]
which we needed.

\[ \square \]

11. Summary of relations

Let \(A \subset \mathbb{R}^n\) be an admissible locally \(BV\) set. The relation between classes of integrable functions in \(A\) is shown in the following diagram.
The strictness of the inclusion $\mathcal{IR} \subset \mathcal{GR}$ holds for $n \geq 2$ and can be found in Theorem 7.4(2) and Corollary 6.8; the case $n = 1$ is discussed below. The fact that $\mathcal{GR} \subsetneq \mathcal{R}^*$ is mentioned in Remark 8.5. Corollary 6.8 shows the equality of $\mathcal{IR}$ and $\mathcal{R}$. The relationship $\mathcal{R} \subseteq \mathcal{PR}$ is described in Theorem 6.9; Theorems 9.5, 10.9 and 10.8 show that this inclusion is strict. The inclusion $\mathcal{PR} \subset \mathcal{PR}^*$ is proved in Theorem 5.12. Theorem 8.4 proves the inclusion $\mathcal{R}^* \subseteq \mathcal{PR}^*$, the fact, that this inclusions is proper follows from Theorems 10.9, 10.8, 10.6 and Proposition 9.4.

In the case $A = \mathbb{R}$, we can compare integrable functions in the following way.

\[
\mathcal{GR} = \mathcal{R} \subseteq HK = MC \subseteq MC_\beta \subseteq \mathcal{PR} \subset \mathcal{PR}^*
\]

\[
\mathcal{R}^* \subseteq MC_\alpha
\]

The equality $\mathcal{GR} = \mathcal{R}$ is described in Theorem 7.4(1) and the inclusion $\mathcal{R} \subseteq HK$ in Theorem 9.5. The fact that $HK$ integral coincides with $\mathcal{R}^*$ integral can be found in 9.4. Theorem 10.8 shows the equality $HK = MC$. Theorem 10.7 proves the equality $MC = MC_\alpha$ for $\alpha \in [1, 2]$. The inclusion $MC \subseteq MC_\beta$ for $\beta > 2$ is proved in Proposition 10.4, the fact, that this inclusion is proper is shown in Theorem 10.6. The relationship $MC_\beta \subseteq \mathcal{PR}$ (not only) for $\beta \geq 2$ is proved in Theorem 10.9 and 10.5. Finally, the inclusion $\mathcal{PR} \subset \mathcal{PR}^*$ is shown in Theorem 5.12.

ACKNOWLEDGEMENTS

The research was supported by the grants GA ČR P201/15-08218S and P201/18-07996S.

The author would also like to express deep gratitude to Jan Malý for many valuable suggestions and helpful comments.

REFERENCES

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[2] T. Ball and D. Preiss. Monotonically controlled integrals. The university of Warwick, Mathematics Institute, 2016.

[3] H. Bendová and J. Malý. An elementary way to introduce a Perron-like integral. *Ann. Acad. Sci. Fenn. Math.*, 36(1):153–164, 2011.

[4] Z. Buczolich, T. De Pauw, and W. F. Pfeffer. Charges, BV functions, and multipliers for generalized Riemann integrals. *Indiana Univ. Math. J.*, 48(4):1471–1511, 1999.

[5] G.-Q. Chen, M. Torres, and W. P. Ziemer. Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Comm. Pure Appl. Math.*, 62(2):242–304, 2009.

[6] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
[7] R. Henstock. Definitions of Riemann type of the variational integrals. *Proc. London Math. Soc. (3)*, 11:402–418, 1961.

[8] P. Honzík and J. Malý. Non-absolutely convergent integrals and singular integrals. *Collect. Math.*, 65(3):367–377, 2014.

[9] J. Jarník, J. Kurzweil, and v. Schwabík. On Mawhin’s approach to multiple nonabsolutely convergent integral. *Časopis Pěst. Mat.*, 108(4):356–380, 1983.

[10] K. Kuncová and J. Malý. Non-absolutely convergent integrals in metric spaces. *J. Math. Anal. Appl.*, 401(2):578–600, 2013.

[11] J. Kurzweil, J. Mawhin, and W. F. Pfeffer. An integral defined by approximating BV partitions of unity. *Czechoslovak Math. J.*, 41(116)(4):695–712, 1991.

[12] J. Malý. Non-absolutely convergent integrals with respect to distributions. *Ann. Mat. Pura Appl. (4)*, 193(5):1457–1484, 2014.

[13] J. Malý and W. F. Pfeffer. Henstock-Kurzweil integral on BV sets. *Math. Bohem.*, 141(2):217–237, 2016.

[14] J. Mawhin. Generalized multiple Perron integrals and the Green-Goursat theorem for differentiable vector fields. *Czechoslovak Math. J.*, 31(106)(4):614–632, 1981.

[15] W. F. Pfeffer. The Gauss-Green theorem. *Adv. Math.*, 87(1):93–147, 1991.

[16] W. F. Pfeffer. *The Riemann approach to integration*, volume 109 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993. Local geometric theory.

[17] W. F. Pfeffer. *Derivation and integration*. Cambridge Tracts in Mathematics 140. Cambridge University Press, Cambridge, 2001.

[18] W. F. Pfeffer. *The divergence theorem and sets of finite perimeter*. Pure and Applied Mathematics (Boca Raton). CRC Press, Boca Raton, FL, 2012.

[19] M. Šilhavý. Divergence measure fields and Cauchy’s stress theorem. *Rend. Sem. Mat. Univ. Padova*, 113:15–45, 2005.

[20] M. Šilhavý. Cauchy’s stress theorem for stresses represented by measures. *Contin. Mech. Thermodyn.*, 20(2):75–96, 2008.

[21] M. Šilhavý. The divergence theorem for divergence measure vectorfields on sets with fractal boundaries. *Math. Mech. Solids*, 14(5):445–455, 2009.

[22] W. P. Ziemer. Cauchy flux and sets of finite perimeter. *Arch. Rational Mech. Anal.*, 84(3):189–201, 1983.

**Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic**

**E-mail address:** kuncova@karlin.mff.cuni.cz