Geometry of null hypersurfaces

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Abstract

We review some basic natural geometric objects on null hypersurfaces. Gauss-Codazzi constraints are given in terms of the analog of canonical ADM momentum which is a well defined tensor density on the null surface. Bondi cones are analyzed with the help of this object.

1 Introduction

In Synge’s festshrift volume [10] Roger Penrose distinguished three basic structures which a null hypersurface $N$ in four-dimensional spacetime $M$ acquires from the ambient Lorentzian geometry:

- the degenerate metric $g|_N$ (see [9] for Cartan’s classification of them and the solution of the local equivalence problem)
- the concept of an affine parameter along each of the null geodesics from the two-parameter family ruling $N$
- the concept of parallel transport for tangent vectors along each of the null geodesics

Using all three concepts on $N$ one can define several natural geometric objects which we shall review in this article.

In Section 2 we remind the structures which are presented in [1]. In the next section we give solutions, which are mostly based on [2], to the following questions:

- What is the analog of canonical ADM momentum for the null surface?
- What are the "initial value constraints"?
- Are they intrinsic objects?

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More precisely, we remind the construction of external geometry in terms of tensor density which is a well defined intrinsic object on a null surface. We already developed some applications of these object to the following subjects:

- Dynamics of the light-like matter shell from matter Lagrangian which is an invariant scalar density on \( N \)[3].
- Dynamics of gravitational field in a finite volume with null boundary and its application to black holes thermodynamics [6] (see also in this volume).
- Geometry of crossing null shells [4].

In the last section we apply our construction to Bondi cones.

2 Natural geometric structures on \( TN/K \)

We remind some standard constructions on null hypersurfaces (see [1]):

- time-oriented Lorentzian manifold \( M \) with signature \((-\,\,\,\,+,\,\,+\,\,\,\,\,+,\,\,\,\,\,+)\).
- null hypersurface \( N - \) submanifold with codim=1 and degenerate induced metric \( g|_N \) \((0\,\,\,\,\,\,\,\,\,+\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,
null second fundamental form $B_K$ (bilinear form associated to $b_K$ via $h$)

\[ B_K : T_pN/K \times T_pN/K \rightarrow \mathbb{R} \]

$B_K(\mathbf{x}, \mathbf{y}) = h(b_K(\mathbf{x}), \mathbf{y}) = g(\nabla K \mathbf{x}, \mathbf{y})$

Moreover, $b_K$ is self-adjoint with respect to $h$ and $B_K$ is symmetric.

$N$ is totally geodesic (i.e. restriction to $N$ of the Levi-Civita connection of $M$ is an affine connection on $N$, any geodesic in $M$ starting tangent to $N$ stays in $N$) $\iff B = 0$ (non-expanding horizon is a typical example).

null mean curvature of $N$ with respect to $K$

\[ \theta := \text{tr} b = \sum_{i=1}^{2} b_K(e_i) \cdot e_i = \sum_{i=1}^{2} g(\nabla e_i K, e_i) \]

where $e_i$ is an orthonormal basis for $T_pN/K$, $e_i$ is an orthonormal basis for $T_pS$ in the induced metric on $S$ which is a two-dimensional submanifold of $N$ transverse to $K$.

We assume now that $K$ is a geodesic vector field i.e. $\nabla_K K = 0$. Let us denote by prime covariant differentiation in the null direction:

\[ \mathbf{Y}' := \nabla_K \mathbf{Y}, \quad b'(\mathbf{Y}) := b(\mathbf{Y}') - b(\mathbf{Y}) \]

From Riemann tensor we build the following curvature endomorphism

\[ R : T_pN/K \rightarrow T_pN/K, \quad R(\mathbf{X}) = \text{Riemann}(\mathbf{X}, K)K \]

and we get a Ricatti equation

\[ b' + b^2 + R = 0. \tag{1} \]

Taking the trace of (1) we obtain well-known Raychaudhuri equation:

\[ \theta' = -\text{Ricci}(K, K) - B^2, \quad B^2 = \sigma^2 + \frac{1}{2} \theta^2 \tag{2} \]

where $\sigma$ is a shear scalar corresponding to the trace free part of $B$. A standard application of the Raychaudhuri equation gives the following

**Proposition 1.** Let $M$ be a spacetime which obeys the null energy condition, i.e. $\text{Ricci}(X, X) \geq 0$ for all null vectors $X$, and let $N$ be a smooth null hypersurface in $M$. If the null generators of $N$ are future geodesically complete then $N$ has nonnegative null mean curvature i.e. $\theta \geq 0$.

## 3 Canonical momentum on null surface

For non-degenerate hypersurface we define the canonical ADM momentum:

\[ P^{kl} := \sqrt{\det g_{mn}} (g^{kl} g_{ij} \mathcal{K}^{ij} - \mathcal{K}^{kl}), \tag{3} \]

where $\mathcal{K}^{kl}$ is the second fundamental form (external curvature) of the imbedding of the hypersurface into the spacetime $M$. 

Gauss-Codazzi equations for non-degenerate hypersurface are the following:

\[ P_i^l = \sqrt{\det g_{mn}} G_i^\mu n^\mu \ (= 8\pi \sqrt{\det g_{mn}} T_i^\mu n^\mu) , \]  

(4)

\[(\det g_{mn}) R - P_k^l P_{kl} + \frac{1}{2} (P_k^l g_{kl})^2 = 2(\det g_{mn}) G_{\mu \nu} n^\mu n^\nu \]  

(5)

where \( R \) is the (three-dimensional) scalar curvature of \( g_{kl} \), \( n^\mu \) is a four-vector normal to the hypersurface, \( T_{\mu \nu} \) is an energy-momentum tensor of the matter field, and the calculations have been made with respect to the non-degenerate induced three-metric \( g_{kl} \) ("\( | \)" denotes covariant derivative, indices are raised and lowered with respect to that metric etc.)

A null hypersurface in a Lorentzian spacetime \( M \) is a three-dimensional submanifold \( N \subset M \) such that the restriction \( g_{ab} \) of the spacetime metric \( g_{\mu \nu} \) to \( N \) is degenerate.

We shall often use adapted coordinates, where coordinate \( x^3 \) is constant on \( N \). Space coordinates will be labeled by \( k, l = 1, 2, 3 \); coordinates on \( N \) will be labeled by \( a, b = 0, 1, 2 \); finally, coordinates on \( S \) will be labeled by \( \alpha, \beta, \mu, \nu \). Spacetime coordinates will be labeled by Greek characters \( \alpha, \beta, \mu, \nu \).

We will show in the sequel that null-like counterpart of initial data \( (g_{kl}, P_k^l) \) consists of the metric \( g_{ab} \) and tensor density \( Q^a_b \) which is a mixed (contravariant-covariant) tensor density.

The non-degeneracy of the spacetime metric implies that the metric \( g_{ab} \) induced on \( N \) from the spacetime metric \( g_{\mu \nu} \) has signature \((0, +, +)\). This means that there is a non-vanishing null-like vector field \( K^a \) on \( N \), such that its four-dimensional embedding \( K^a \) to \( M \) (in adapted coordinates \( K^3 = 0 \)) is orthogonal to \( N \). Hence, the covector \( K_\nu = K^a g_{a \nu} = K^\mu g_{\mu \nu} \) vanishes on vectors tangent to \( N \) and, therefore, the following identity holds:

\[ K^a g_{ab} \equiv 0 . \]  

(6)

It is easy to prove that integral curves of \( K^a \), after a suitable reparameterization, are geodesic curves of the spacetime metric \( g_{\mu \nu} \). Moreover, any null hypersurface \( N \) may always be embedded in a one-parameter congruence of null hypersurfaces.

We assume that topologically we have \( N = \mathbb{R}^1 \times S^2 \). Since our considerations are purely local, we fix the orientation of the \( \mathbb{R}^1 \) component and assume that null-like vectors \( K \) describing degeneracy of the metric \( g_{ab} \) of \( N \) will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate \( x^0 \) increases in the direction of \( K \), i.e. inequality \( K(x^0) = K^0 > 0 \) holds. In these coordinates degeneracy fields are of the form \( K = f(\partial_0 - n^A \partial_A) \), where \( f > 0 \), \( n_A = g_{0 A} \) and we rise indices with the help of the two-dimensional matrix \( \tilde{g}^{AB} \), inverse to \( g_{AB} \).

If by \( \lambda \) we denote the two-dimensional volume form on each surface \( \{x^0 = \text{const.}\} \):

\[ \lambda := \sqrt{\det \tilde{g}_{AB}} , \]  

(7)
then for any degeneracy field $K$ of $g_{ab}$ the following object

$$v_K := \frac{\lambda}{K(x^a)}$$

is a well defined scalar density on $N$. This means that

$$v_K := v_K dx^0 \wedge dx^1 \wedge dx^2$$

is a coordinate-independent differential three-form on $N$. However, $v_K$ depends upon the choice of the field $K$.

It follows immediately from the above definition that the following object:

$$\Lambda = v_K K$$

is a well defined (i.e. coordinate-independent) vector density on $N$.

Obviously, it does not depend upon any choice of the field $K$:

$$\Lambda = \lambda (\partial_0 - n^A \partial_A) \quad (8)$$

and it is an intrinsic property of the internal geometry $g_{ab}$ of $N$. The same is true for the divergence $\partial_a \Lambda^a$ which is, therefore, an invariant, $K$-independent, scalar density on $N$. Mathematically (in terms of differential forms) the quantity $\Lambda$ represents the two-form:

$$L := \Lambda^a (\partial_a \int dx^0 \wedge dx^1 \wedge dx^2) ,$$

whereas the divergence represents its exterior derivative (a three-from):

$$dL := (\partial_a \Lambda^a) dx^0 \wedge dx^1 \wedge dx^2 .$$

In particular, a null surface with vanishing $dL$ is the non-expanding horizon.

Both objects $L$ and $v_K$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $p \in N$ the tangent space $T_pN$ may be quotiented with respect to the degeneracy subspace spanned by $K$. The quotient space $T_pN/K$ carries a non-degenerate Riemannian metric $h$ and, therefore, is equipped with a volume form $\omega$ (its coordinate expression would be: $\omega = \lambda dx^1 \wedge dx^2$).

The two-form $L$ is equal to the pull-back of $\omega$ from the quotient space $T_pN/K$ to $T_pN$:

$$\pi : T_pN \rightarrow T_pN/K , \quad L := \pi^* \omega .$$

The three-form $v_K$ may be defined as a product:

$$v_K = \alpha \wedge L ,$$

where $\alpha$ is any one-form on $N$, such that $\langle K, \alpha \rangle \equiv 1$.

We have

$$dL = \theta v_K$$

where $\theta$ is a null mean curvature of $N$.

The degenerate metric $g_{ab}$ on $N$ does not allow to define via the compatibility condition $\nabla g = 0$, any natural connection, which could be applied to generic tensor fields on $N$. Nevertheless, there is one exception: the degenerate metric defines uniquely a certain covariant, first
order differential operator. The operator may be applied only to mixed (contravariant-covariant) tensor density fields $H^a{}_b$, satisfying the following algebraic identities:

$$H^a{}_b K^b = 0, \quad H^a{}_b = H^b{}_a ,$$

(9)

where $H^a{}_b := g_{ac} H^a{}_c$. Its definition cannot be extended to other tensorial fields on $N$. Fortunately, the extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor densities of this type.

The operator, which we denote by $\nabla_a$, is defined by means of the four-dimensional metric connection in the ambient spacetime $M$ in the following way:

Given $H^a{}_b$, take any its extension $H^{\mu \nu}$ to a four-dimensional, symmetric tensor density, “orthogonal” to $N$, i.e. satisfying $H^{\perp \nu} = 0$ (”$\perp$” denotes the component transversal to $N$). Define $\nabla_a H^a{}_b$ as the restriction to $N$ of the four-dimensional covariant divergence $\nabla_\mu H^{\mu \nu}$. The ambiguities, which arise when extending three-dimensional object $H^a{}_b$ living on $N$ to the four-dimensional one, cancel finally and the result is unambiguously defined as a covector density on $N$. It turns out, however, that this result does not depend upon the spacetime geometry and may be defined intrinsically on $N$ as follows:

$$\nabla_a H^a{}_b = \partial_a H^a{}_b - \frac{1}{2} H^{ac} g_{ac,b} ,$$

(10)

where $g_{ac,b} := \partial_b g_{ac}$, a tensor density $H^a{}_b$ satisfies identities (9), and moreover, $H^{ac}$ is any symmetric tensor density, which reproduces $H^a{}_b$ when lowering an index:

$$H^a{}_b = H^{ac} g_{ac} .$$

(11)

It is easily seen, that such a tensor density always exists due to identities (9), but the reconstruction of $H^{ac}$ from $H^a{}_b$ is not unique because $H^{ac} + CK^a K^c$ also satisfies (11) if $H^{ac}$ does. Conversely, two such symmetric tensors $H^{ac}$ satisfying (11) may differ only by $CK^a K^c$. Fortunately, this non-uniqueness does not influence the value of (10).

Hence, the following definition makes sense:

$$\nabla_a H^a{}_b := \partial_a H^a{}_b - \frac{1}{2} H^{ac} g_{ac,b} .$$

(12)

The right-hand-side does not depend upon any choice of coordinates (i.e. it transforms like a genuine covector density under change of coordinates).

To express directly the result in terms of the original tensor density $H^a{}_b$, we observe that it has five independent components and may be uniquely reconstructed from $H^0{}_A$ (2 independent components) and the symmetric two-dimensional matrix $H_{AB}$ (3 independent components). Indeed, identities (9) may be rewritten as follows:

$$H^A{}_B = \tilde{g}^{AC} H_{CB} - n^A H^0{}_B ,$$

(13)

$$H^0{}_b = H^0{}_A n^A ,$$

(14)

$$H^B{}_0 = \left( \tilde{g}^{BC} H_{CA} - n^B H^0{}_A \right) n^A .$$

(15)
The correspondence between $H^a{}_b$ and $(H^a{}_{A}, H_{AB})$ is one-to-one.

To reconstruct $H^{ab}$ from $H^a{}_b$ up to an arbitrary additive term $CK^a{}_{K}b$, take the following (coordinate dependent) symmetric quantity:

$$F^{AB} := \tilde{g}^{-1} \tilde{H}_{CD} \tilde{g}^{-1} - n^A \tilde{H}^b_c \tilde{g}^{-1} - n^B \tilde{H}^b_c \tilde{g}^{-1} \tilde{g}^{CA} \ ,$$

(16)

$$F^{0A} := H^0_c g^{-1} =: F^{A0} \ ,$$

(17)

$$F^{00} := 0 \ .$$

(18)

It is easy to observe that any $H^{ab}$ satisfying (11) must be of the form:

$$H^{ab} = F^{ab} + H^{00} K^a{}_{K}b \ .$$

(19)

The non-uniqueness in the reconstruction of $H^{ab}$ is, therefore, completely described by the arbitrariness in the choice of the value of $H^{00}$. Using these results, we finally obtain:

$$\nabla_a H^a_b := \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} = \partial_a H^a_b - \frac{1}{2} F^{ac} g_{ac,b} \ ,$$

$$= \partial_a H^a_b - \frac{1}{2} \left( 2 H^A_b n^A_{a,b} - H_{AC} \tilde{g}^{AC} \right) .$$

(20)

The operator on the right-hand-side of (20) is called the (three-dimensional) covariant derivative of $H^a{}_b$ on $N$ with respect to its degenerate metric $g_{ab}$. It is well defined (i.e. coordinate-independent) for a tensor density $H^a{}_b$ fulfilling conditions (9). One can also show that the above definition coincides with the one given in terms of the four-dimensional metric connection and, due to (10), it equals:

$$\nabla_a H^a_b = \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} = \partial_a H^a_b - \frac{1}{2} H^{ac} g_{ac,b} \ ,$$

(21)

hence, it coincides with $\nabla_a H^a_b$ defined intrinsically on $N$.

To describe exterior geometry of $N$ we begin with covariant derivatives along $N$ of the “orthogonal vector $K$”. Consider the tensor $\nabla_a K^a$. Unlike in the non-degenerate case, there is no unique “normalization” of $K$ and, therefore, such an object does depend upon a choice of the field $K$. The length of $K$ vanishes. Hence, the tensor is again orthogonal to $N$, i.e. the components corresponding to $\mu = 3$ vanish identically in adapted coordinates. This means that $\nabla_a K^a$ is a purely three-dimensional tensor living on $N$. For our purposes it is useful to use the “ADM-momentum” version of this object, defined in the following way:

$$Q^a_b(K) := -s \left( v_K \left( \nabla_b K^a - \delta_b^a \nabla_c K^c \right) + \delta_b^a \partial_c \Lambda^c \right) \ ,$$

(22)

where $s := \text{sgn} g^{00} = \pm 1$. Due to the above convention, the object $Q^a_b(K)$ feels only external orientation of $N$ and does not feel any internal orientation of the field $K$.

**Remark:** If $N$ is a non-expanding horizon, the last term in the above definition vanishes.

The last term in (22) is $K$-independent. It has been introduced in order to correct algebraic properties of the quantity

$$v_K \left( \nabla_b K^a - \delta_b^a \nabla_c K^c \right) \ .$$
One can show that $Q^a_b$ satisfies identities (9) and, therefore, its covariant divergence with respect to the degenerate metric $g_{ab}$ on $N$ is uniquely defined. This divergence enters into the Gauss–Codazzi equations, which relate the divergence of $Q$ with the transversal component $G^b_a$ of the Einstein tensor density $G^a_{b\nu} = \sqrt{|\det g|} (R^a_{\nu} - \delta^a_{\nu} \frac{1}{2} R)$. The transversal component of such a tensor density is a well defined three-dimensional object living on $N$. In coordinate system adapted to $N$, i.e. such that the coordinate $x^3$ is constant on $N$, we have $G^a_{b3} = G^a_{3b}$. Due to the fact that $G$ is a tensor density, components $G^a_{b3}$ do not change with changes of the coordinate $x^3$, provided it remains constant on $N$. These components describe, therefore, an intrinsic covector density living on $N$.

**Proposition 2.** The following null-like-surface version of the Gauss–Codazzi equation is true:

$$\nabla_a Q^a _b (K) + s v_K \partial_b \left( \frac{\partial_c \Lambda^c}{v_K} \right) \equiv -G^b _a.$$ (23)

The proof is given in [3]. We remind the reader that the ratio between two scalar densities: $\partial_c \Lambda^c$ and $v_K$, is a scalar function $\theta$. Its gradient is a covector field. Finally, multiplied by the density $v_K$, it produces an intrinsic covector density on $N$. This proves that also the left-hand-side is a well defined geometric object living on $N$.

The component $K^a G^b_a$ of the equation (23) is nothing but a densitized form of Raychaudhuri equation (2) for the congruence of null geodesics generated by the vector field $K$.

### 4 Initial data on asymptotic Bondi cones

Recall (see [7]) that in Bondi-Sachs coordinates $(u, x, x^A)$ the space-time metric takes the form:

$$^4g = -x V e^{2\beta} du^2 + 2e^{2\beta} x^{-2} du dx + x^{-2} h_{AB} \left( dx^A - U_A^B du \right) \left( dx^B - U_B^A du \right).$$ (24)

Let us derive explicitly canonical data $(g_{ab}, Q^a_b)$ on null surfaces $N := \{u = \text{const.}\}$ which we call *Bondi cones*. The intrinsic coordinates on null surface $N$ are $x^a = (x, x^A)$. We choose null field

$$K := e^{-2\beta} x^2 \partial_x.$$ (25)

The components of the degenerate metric $g_{ab}$ are as follows:

$$g_{AB} = x^{-2} h_{AB}, \quad g_{xA} = 0 = g_{xx}.$$ From (24), (25) and (22) we obtain the following formulae:

$$s Q^a_s(K) = 0$$ (26)

$$s Q^A _B(k) = -\frac{1}{2} \sin \theta \ h^{AC} (x^{-2} h_{CB})_x$$ (27)

$$s Q^x _A(k) = x^{-2} \sin \theta \left( \beta_A + \frac{1}{2} e^{-2\beta} h_{AB} U^{B}_x \right)$$ (28)
If we assume that Bondi cone data is polyhomogeneous and conformally \( C^1 \times C^0 \)-compactifiable, it follows that (cf. [3])

\[
h_{AB} = \tilde{h}_{AB} \left( 1 + \frac{x^2}{4} \chi^{CD} \chi_{CD} + x \chi_{AB} + x^2 \chi_{AB} + x^3 \chi_{AB} + O_{\ln x}(x^4) \right),
\]

where \( \chi_{AB} \) and \( \chi_{AB} \) are polynomials in \( \ln x \) with coefficients which smoothly depend upon the \( x^A \)'s. By definition of the Bondi coordinates we have

\[
\det h = \det \tilde{h} = \sin \theta,
\]

which implies \( \tilde{h}_{AB} \chi_{AB} = \tilde{h}_{AB} \chi_{AB} = 0 \). Further,

\[
\beta = -\frac{1}{32} \chi^{CD} \chi_{CD} x^2 + B x^3 + O_{\ln x}(x^4),
\]

(29)

\[
h_{AB} U^B = -\frac{1}{2} \chi_A \chi_B x^2 + W_A x^3 + O_{\ln x}(x^4),
\]

(30)

where \( B \) and \( W_A \) are again polynomials in \( \ln x \) with smooth coefficients depending upon the \( x^A \)'s, while \( \| \) denotes covariant differentiation with respect to the unit sphere metric \( \tilde{h} \). This leads to the following approximate formulae:

\[
s Q_A^B(K) = x^{-2} \sin \theta \left( x^{-1} \delta_A^B - \chi_A^B + O(x^2) \right)
\]

(31)

\[
s Q_A^A(K) = x^{-2} \sin \theta \left( -\frac{1}{2} x \chi_A^B \chi^B + O(x^2) \right)
\]

(32)

\[
g_{AB} = x^{-2} \left( \tilde{h}_{AB} + x \chi_{AB} + O(x^2) \right)
\]

(33)

It is easy to verify that the asymptotic behaviour of canonical data \((g_{ab}, Q^a)\) is determined by “free data” \( \chi_{AB} \) which agrees with standard Bondi-Sachs approach to the null initial value formulation.

We hope that the variational formula on a truncated cone, which is space-like inside and light-like near \( \text{Scri} \), (proposed in [3]) can be formulated with the help of the object \( Q^a \) for arbitrary hypersurfaces, i.e. without assumption that the null part of the initial surface is a Bondi cone.

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