Regularity in time along the coarse scale flow for the incompressible Euler equations

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Abstract

One of the most remarkable features of known nonstationary solutions to the incompressible Euler equations is the phenomenon that coarse scale averages of the velocity carry the fine scale features of the flow. In this paper, we study time-regularity properties of Euler flows which are connected to this phenomenon and the observation that each frequency level has a natural time scale when it is viewed along the coarse scale flow. We assume only that our velocity field is Hölder continuous in the spatial variables, which is well-motivated by problems related to turbulence, but which precludes the application of Lagrangian methods or local well-posedness theory.

We show that, for any $0 < \alpha < 1$, periodic Euler flows with $C_t C^\alpha_x$ regularity in space must also possess the same $C_t C^\alpha_x$ regularity in space and time. Furthermore, the pressure has essentially twice as much regularity in both space and time (being almost $C^2_t C^\alpha_x$ for $\alpha$ near 1), and we show that the total kinetic energy of the solution has even better Hölder regularity $C^2_t C^\alpha_x$ in time for $\alpha < 1/3$, even though it might fail to be conserved in view of Onsager’s conjecture. We give several illustrations of improved regularity for advective derivatives, including a proof that higher advective derivatives $D^r \partial_t v = D^{r-1} \partial_t \nabla p$ are well-defined and continuous for all $0 \leq r < \alpha/(1 - \alpha)$. Hence, we recover in our context the celebrated result of Chemin that the particle trajectories of classical solutions to Euler are smooth, and we also prove existence of smooth trajectories in any case where the velocity field $v \in \cap_{\alpha < 1} C_t C^\alpha_x$ has borderline regularity.

The analysis demonstrates that many of the main analytic features of solutions constructed by convex integration methods are consequences of the Euler equations rather than artifacts of the constructions. The proof proceeds by estimating frequency increments associated to various physical quantities of interest. Several types of commutator estimates play a role in the proof, including the commutator estimate of Constantin, E and Titi for the relevant Reynolds stress and a more flexible proof of this estimate.

1 Introduction

The present paper is devoted to studying the regularity in time of solutions to the incompressible Euler equations

$$\begin{cases}
\partial_t v + \text{div} \, v \otimes v + \nabla p = 0 \\
\text{div} \, v = 0
\end{cases}$$

(E)

which describe the motion of an ideal, incompressible fluid with velocity given by the vector field $v(t, x)$ and pressure given by the scalar function $p(t, x)$. We assume only that our velocity field $v$ is Hölder continuous in the spatial variables $v \in C_t C^\alpha_x$, and therefore interpret the system (E) in the sense of distributions, although our results will also hold for classical solutions as well. We will work in the

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periodic setting, so that \( v: I \times \mathbb{T}^n \to \mathbb{R}^n \) and \( p: I \times \mathbb{T}^n \to \mathbb{R} \). Taking the divergence of \( \text{(E)} \), one sees that the pressure is determined up to the addition of a constant depending on time \( C(t) \) which we will normalize so that \( \int_{\mathbb{T}^n} p(t, x) \, dx = 0 \) and hence \( p = \Delta^{-1} \partial_j (v^j v^d) \).

1.1 Background

The study of Hölder continuous weak solutions to the Euler equations is motivated by theories and experimental observations of turbulence in fluids, including the phenomenon of anomalous dissipation of energy. Turbulent flows are modeled as solutions to the 3D Navier-Stokes equations at high Reynolds number, meaning that the viscosity parameter \( \nu \) is small relative to the characteristic velocity and length scale of the fluid. A celebrated prediction of Kolmogorov’s theory of hydrodynamic turbulence states that the differences in velocity for nearby particles in turbulent flows obey on average a universal scaling law corresponding to the Hölder exponent 1/3:

\[
<|v(x + \Delta x) - v(x)|^p>^{1/p} \sim C_p \varepsilon^{1/3} |\Delta x|^{1/3}.
\]

The law \( \text{(1)} \) is derived by dimensional analysis from the basic principles of Kolmogorov’s theory. This theory asserts that the statistical properties of turbulent flows are governed by the rate of energy dissipation \( \varepsilon = -\frac{1}{2} \int |v|^2 \, dx \) together with the viscosity parameter \( \nu \), and that the coarse scale properties of the flow should be independent of viscosity [Kol41]. A basic postulate in this theory is the hypothesis of “anomalous dissipation”, one form of which states that the rate of energy dissipation \( \varepsilon \) remains strictly positive even in the 0 viscosity limit of Navier-Stokes. The law \( \text{(1)} \) is purported to hold in the “inertial regime” of length scales \( |\Delta x| \gtrsim \left( \frac{v^3}{\varepsilon} \right)^{4} \) where viscosity is supposed to play a minor role; thus, \( \text{(1)} \) extends to all length scales in the limit \( \nu \to 0 \). Experimental measurements of turbulent fluid flows affirm the presence of anomalous dissipation and suggest that the 1/3 law \( \text{(1)} \) holds at least for \( p = 2, 3 \), while moments of fourth order and higher tend to be measurably larger than predicted by \( \text{(1)} \) due to a phenomenon known as intermittency [Fri80, VAP72]. We refer to [Fri95] for more general discussion of turbulence.

Related to anomalous dissipation, there is a well-known conjecture of Onsager which states that weak solutions to the Euler equations with spatial Hölder regularity below 1/3 may exhibit decreasing energy [Ons49] despite the fact that regular solutions to \( \text{(E)} \) must conserve energy\(^1\). As we will discuss further below, significant progress towards this conjecture has been made recently involving the construction of \( C^\alpha \) solutions which fail to conserve energy [DLS12], [Ise12] and [BDLS13]. These examples of Hölder continuous solutions exhibit special time regularity properties which motivate several results in the present paper. Our work shows that many of these properties are consequences of the Euler equations, as opposed to biproducts of the constructions. At the same time, part of the present work aims to clarify how this regularity in time poses an obstruction to improving the progress towards Onsager’s conjecture for the current methods of construction. This obstruction will be elaborated in the concluding remarks of the paper.

Onsager also stated that an Euler flow with Hölder regularity \( v \in C^\alpha \) in space must conserve energy if \( \alpha > 1/3 \). This statement has been proven in [CET94] after a slightly weaker result was obtained in [Eyi94] following Onsager’s original computations. In [CCFS08], the proof of energy conservation was extended to the critical Besov space \( B^{1/3}_{3,\infty}(\mathbb{T}^n) \) (which corresponds to a regularity just slightly better than the law \( \text{(1)} \) with \( p = 3 \)); the power law \( \text{(1)} \) with \( p = 3 \) corresponds instead to the

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\(^1\)For \( p = 2 \), the law \( \text{(1)} \) carries a special significance as it gives a physical space expression of the Kolmogorov 5/3-law for the energy spectrum.

\(^2\)Rather than being based on dimensional analysis, Onsager’s derivation of the exponent 1/3 was based on the dynamical notion of a “frequency cascade” in which the primary mechanism behind the energy dissipation is the movement of energy to arbitrarily high wavenumbers (or small scales), made possible by the nonlinear term in \( \text{(E)} \). We refer to [ESBG, DLS] for further discussion and a review of Onsager’s computations.
Besov space $B^{1/3}_{3,\infty}$, where the proof of energy conservation fails to go through, and the authors show by example that the energy flux may be nonzero for vector fields in this class. For this reason, energy dissipating solutions to Euler with $B^{1/3}_{3,\infty}$ spatial regularity have been considered as a natural setting for exhibiting anomalous dissipation and for containing the turbulent solutions which may arise in the 0 viscosity limit (see for example [CS14] for a recent Littlewood-Paley approach to the theory of intermittency; see also the proof of Proposition 2.2 as well as Section 3 below for more discussion).

In the conclusion of the paper, we discuss a conjecture for the Euler equations related to the above ideas and to Theorem 1.5 below, which offers an explanation as to why energy dissipation should be a nongeneric and unstable phenomenon for solutions with regularity below 1/3. This conjecture suggests that the slightest departure from the 1/3 scaling law will generically lead to the failure of energy dissipation.

The analysis of the present work is focused on the time regularity properties of general solutions to the Euler equations, which hold even for weak solutions with Hölder regularity despite the dramatic failure of well-posedness in this regime. The main observation driving the results of the paper is an improved regularity in the advective derivative $(\partial_t + v \cdot \nabla)$ as compared to the time derivative $\partial_t$, and moreover a specific pattern regarding the transport of fine scale features of the fluid flow by coarse scale averages of the velocity field. Namely, we show that each frequency level of the solution moves along local, coarse scale averages of the velocity field at a specific time scale that is dictated by the regularity of the velocity field. This pattern is made precise in the analysis through estimates derived using Littlewood-Paley theory. The estimates of the paper can be viewed as a quantitative expression for the Euler equations of the well known Taylor hypothesis, which assumes that microstructure in turbulent flow must be convected by the large scale average of the velocity field. The case when the velocity field has 1/3-Hölder or Besov type regularity has particular interest in view of the preceding discussion of turbulence. Being consistent with dimensional analysis, our estimates at the 1/3 regularity give a time scale for the motion of each frequency level $\lambda$ which coincides with the time scale predicted by Kolmogorov’s theory for the turnover time of eddies at the length scale $\lambda^{-1}$. This time scale is reflected in improved estimates for the advective derivative $(\partial_t + v \cdot \nabla)$; in contrast, the estimates for the stationary time derivative $\partial_t$ are substantially worse, being comparable to those of spatial derivatives.

With this background and motivation in hand, we state the main results of the paper.

1.2 Statement of Results

We state in this Section the main results of the paper, which concern the regularity in time of solutions to the incompressible Euler equations in the class $v \in C^1_t C^\alpha_x$, and the improved regularity of the advective derivative $\partial_t + v \cdot \nabla$. Our first theorem states that a solution in the class $v \in C^1_t C^\alpha_x$ for $0 < \alpha < 1$ must also have the same Hölder regularity in time, and that the pressure must be essentially twice as regular in both space and time.

**Theorem 1.1.** Let $0 < \beta < \alpha < 1$ and suppose that $(v, p)$ solve the incompressible Euler equations (E) in the sense of distributions $I \times \mathbb{T}^n$ for some torus $\mathbb{T}^n$, $n \geq 2$ and some open interval $I$. Suppose also that the norm

$$\|v\|_{C^1_t C^\alpha_x} = \sup_{t \in I} \|v(t, \cdot)\|_{C^\alpha}$$

is finite and that the pressure is normalized to have integral 0. Then if $\beta \leq 1/2$, we have

$$v(t, x) \in C^\alpha_{t,x}, \quad p(t, x) \in C^{2\beta}_{t,x}$$

We do not need to assume that $v$ is continuous in $t$ with values in $C^\alpha$ for Theorems 1.1-1.5 and can just as well assume $v \in L^\infty_t C^\alpha_x$ in all of our estimates.
and if $1/2 < \beta < \alpha$, we have
\[ v(t, x) \in C^{\alpha}_{t,x}, \quad (\partial_t p, \nabla p) \in C^{2\beta-1}_{t,x} \] (3)

Assuming further regularity on the velocity field leads to further regularity of advective derivatives, as the following theorem illustrates.

**Theorem 1.2.** If $\alpha > 2/3$, then $\frac{\partial^2}{\partial t^2} v = \frac{\partial}{\partial t} \nabla p = \partial_t \nabla p + \text{div} (v \otimes \nabla p)$, which is well-defined as a distribution by Theorem (1.1), is also continuous.

The time regularity properties for the velocity field stated in Theorems 1.2 and 1.1 are particular examples of the following, more general theorem.

**Theorem 1.3.** If $0 < \alpha < 1$ and $r$ is an integer satisfying $0 \leq r < \frac{\alpha}{1-\alpha}$, then for $\sigma = \alpha - r(1-\alpha)$
\[ \frac{D}{Dt}^s \partial_t^s v \in C^{\beta}_{t,x}, \] (4)
where the material derivative of a continuous tensor field $T$ is defined as the distribution $\frac{D}{Dt} T = \partial_t T + \partial_j(v^j T)$.

In particular, Theorem 1.3 shows that the velocity field has continuous advective derivatives of every order whenever the norm $\| v \|_{C^{\alpha}_{t,C^2}}$ is bounded for all $\alpha < 1$, which includes in particular the case where the vorticity is uniformly bounded. In contrast, $C^1$ regularity in space does not even imply continuity of the first time derivative $\partial_t$. The proof of Theorem 1.3 provides sharp estimates on advective derivatives, which are of interest also in the case of classical solutions.

We prove furthermore that the pressure possesses better regularity in its advective derivatives, which turns out to be more subtle compared to the velocity field. Regularity in time for the pressure may seem surprising given that, a priori, the pressure is only determined by the equations up to a constant depending on time. However, when the pressure is properly normalized to have integral $0$, we obtain Theorem 1.4 below. (Here we use the notation $(y)_+ = \max\{y, 0\}$.)

**Theorem 1.4.** Under the same assumptions, for any non-negative integer $s$ satisfying $s(1-\alpha) - 1 + (1-2\alpha)_+ < 0$ we have
\[ \frac{D}{Dt}^s \partial_t^s p \in C^{\beta}_{t,x} \]
for all $\beta < 1 - (1-2\alpha)_+ - s(1-\alpha)$. In particular, if $\alpha > 1/3$, the distribution $\frac{D}{Dt} p = \partial_t p + \text{div} (pv)$ is continuous, and the pressure associated to a uniformly Lipschitz Euler flow will have continuous advective derivatives of all orders.

In relation to the ideas of anomalous dissipation and Onsager’s conjecture, we are further motivated to consider the regularity in time of the energy profile
\[ e(t) = \int \frac{|v|^2}{2} (t,x)dx \]
The following theorem shows that the energy profile turns out to possesses much better regularity in time than the solution itself.

**Theorem 1.5.** Under the conditions of Theorem 1.1, the energy profile $e(t)$ satisfies a Hölder condition
\[ |e(t + \Delta t) - e(t)| \leq C |\Delta t|^{\frac{\alpha}{1-\alpha}} \] (5)
for some $C$ depending on $v$ and $\alpha$. 4
Theorem 1.5 and its proof should be compared with the results of [CE194, CCFS08] concerning the positive direction of Onsager’s conjecture on the conservation of energy at regularity above 1/3 (see Section 5 below). We conjecture that Theorem 1.5 is sharp, and that moreover the energy profile of generic solutions with Hölder regularity below 1/3 will possess no better regularity than the Hölder estimate (5). In particular, the energy profile of such flows should fail to have bounded variation (and thus fail to be monotonic) on every open interval. In this sense, we expect that the property of energy dissipation will generically fail upon the slightest departure from the 1/3 law. We will resume discussion of this conjecture in the concluding remarks.

Our last demonstration of time regularity concerns the smoothness of trajectories of incompressible Euler flows. Theorem 1.3 recovers in the periodic setting the well-known result that the particle trajectories of classical solutions to the Euler equations (more precisely, solutions in the class $C_t C_x^\alpha$ with $\alpha > 1$) are smooth curves. This fact was first proven by Chemin in the setting of the full space [Che91, Che92] and generalized to bounded domains in [Kat00]. In dimension 2, the smoothness of trajectories is known under weaker regularity assumptions such as bounded vorticity [Ser95b, Gam94, Sue11]. In fact, starting with the investigations of [Ser95a], it has been shown for classical solutions that the particle trajectories and even the motion of a rigid body immersed in an incompressible fluid are analytic in time [Gam94, GST12, Sue11]. We refer to [Sue11] for a summary of this activity, and also note the more recent proofs of [Shn12] and [FZ14].

The above works on the smoothness of trajectories all rely on the existence and local well-posedness theory for the Euler equations and in many cases proceed in Lagrangian coordinates. None of these ingredients are available in our setting, as we consider $v \in C_t C_x^\alpha$ with fractional regularity $\alpha < 1$. For this class of velocities, the particle trajectories may fail to be unique and it is not known whether a sensible flow map can be defined or whether uniqueness holds for the initial value problem. In dimensions 3 and higher, the existence and local well-posedness theory for Euler is restricted to velocity fields which have regularity in space that is strictly better than $C^1$ (see [PP04] for a critical result in this direction). In fact, an example of [BT10] shows that solutions to the 3D Euler equations can immediately lose regularity in $C^\alpha$ for any $\alpha < 1$ (even when the zero viscosity limit is unique), and nonuniqueness of solutions to the initial value problem is known for $\alpha < 1/5$ [Ise12].

Our approach to Theorem 1.4, in contrast, is completely from an Eulerian point of view, and entirely independent of the existence and local well-posedness theories. Furthermore, the proof reveals a specific physical phenomenon that is responsible for this improved regularity: namely that there is a pattern in the time scales of motion for the high frequency components of the solution, and this time scale becomes uniform over frequency once one approaches Lipschitz regularity. As an application of our approach to Theorem 1.4, we establish the smoothness of trajectories in a case of borderline regularity that lies below the well-posedness threshold.

**Theorem 1.6.** [Existence of Smooth Trajectories] Suppose that the solution satisfies $v \in C_t C_x^\alpha$ for all $\alpha < 1$. Let $t_0 \in I$ and $x_0 \in \mathbb{T}^n$ be given. Then there exists $X(t) : I \to \mathbb{T}^n$ satisfying

$$\frac{d}{dt} X(t) = v(t, X(t)), \quad X(t_0) = x_0$$

such that $X(t)$ is of class $C^\infty(I)$ and satisfies $\frac{d^{r+1}}{dt^{r+1}} X(t, x_0) = \frac{D^r}{dt^r} v(t, X(t, x_0))$ for any $r \geq 0$.

Note that the trajectory in Theorem 1.6 may not be unique. Theorem 1.4 can also be extended to construct particle trajectories $X(t, x_0)$ that have improved $C^r$ regularity in time when we only assume $v \in C_t C_x^\alpha$ for some positive $\alpha$ strictly less than 1. We believe it is an interesting question to determine whether there exist particle trajectories failing to exhibit this improved regularity.
1.3 Idea of the Proof

Theorems 1.1-1.6 all relate to the phenomenon that the material derivative in general has better regularity than the stationary time derivative. One basic example which lies at the heart of this phenomenon is the following bound on the material derivative of a Littlewood-Paley projection of the solution

**Lemma 1.1.** There is a universal constant $C$ such that if $(v, p)$ solve the Euler equations with $v \in C_t C^\alpha_x$ then each Littlewood-Paley projection $P_k v$ satisfies

$$
\|(\partial_t + v \cdot \nabla) P_k v\|_{C^0} \leq C 2^{(1-2\alpha)k} \|v\|_{C_t C^\alpha_x}^2
$$

where the homogeneous Hölder seminorm of a vector field on $\mathbb{T}^n$ is defined by

$$
\|v\|_{C_t C^\alpha_x} = \sup_t \sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|v(t, x + h) - v(t, x)|}{|h|^{\alpha}}
$$

The constant $C$ does not depend on the torus and the bound holds on $\mathbb{R}^n$ as well. Actually, the bound we establish directly below is

$$
\|(\partial_t + P_{\leq k} v \cdot \nabla) P_{k+1} v\|_{C^0} \leq C 2^{(1-2\alpha)k} \|v\|_{C_t C^\alpha_x}^2
$$

which is basically equivalent to (7) but is more robust. Here $C^0 = C^0_{t,x}$ is a supremum over time and space. The estimates in our paper hold also with the supremum in $C_t$-based norms replaced by the essential supremum in $L^\infty_t$-based norms, and admit many generalizations to other function spaces.

Observe that the estimate (7) is consistent with dimensional analysis of the Euler equations (both sides having dimensions of length time$^{-2}$ where $2^k$ is an inverse length) and also remains invariant under Galilean transformations. In contrast, from the identity

$$
\partial_t = (\partial_t + P_{\leq k} v \cdot \nabla) - P_{\leq k} v \cdot \nabla,
$$

we can only expect a weaker estimate

$$
\|\partial_t P_k v\|_{C^0} \leq C (2^{(1-2\alpha)k} \|v\|_{C_t C^\alpha_x}^2 + 2^{(1-\alpha)k} \|v\|_{C^0} \|v\|_{C_t C^\alpha_x})
$$

$$
\|\partial_t P_k v\|_{C^0} \leq C_{t,x} 2^{(1-\alpha)k} \|v\|_{C_t C^\alpha_x}^2
$$

for the (stationary) time derivative $\partial_t P_k v$ of a Littlewood-Paley piece (which obviously fails to be Galilean invariant). One can therefore interpret (7) and its proof (along with many other bounds in the present paper) as a demonstration that the fine scale features of an ideal incompressible flow must move along the coarse scale flow.

The idea that the high frequency oscillations of the velocity field should move along the coarse scale average flow has played an important role in other parts of the analysis of fluids and the Euler equations. This idea can be viewed as a more precise version of the classical Taylor hypothesis of frozen turbulence, which asserts that microstructures in turbulent fluids are convected along the large-scale average velocity of the fluid. The Taylor hypothesis is a key assumption in many experimental measurements of turbulent flows and has also played a role in the applied literature, where the assumption of “convected fluid microstructure” forms the basic hypothesis underlying multiscale analysis approaches to modelling fluid turbulence. We refer to [HJK+05, HT12] and the references therein for more on these topics. The transport of high frequency waves by a low frequency velocity field has also been a key idea in the construction of energy-dissipating Euler flows used in the recent progress towards Onsager’s conjecture [Ise12, BDL13].
Our estimates for coarse scale advective derivatives obey a pattern which indicates that the motion of each frequency component takes place at a time scale that is naturally dictated by the spatial regularity and dimensional analysis. This time scale agrees with the statistical theory of turbulence when $\alpha = 1/3$. Specifically, when we compare (8) to the bound $\|P_{k+1}v\|_{C^0} \leq C_2^2 - \alpha k \|v\|_{C_2^0}$, we see that the coarse scale advective derivative $(\partial_t + P_{\leq k} v \cdot \nabla)$ costs a factor $2^{(1-\alpha)k} \|v\|_{C_2^0}$ in the estimate. This cost is a general feature in the structure of the estimates in the paper. These bounds can be interpreted informally as expressing that the oscillations at frequency $\lambda$ move with a natural time scale of $\lambda^{-(1-\alpha)} \|v\|_{C_2^0}$ when they are observed along the coarse scale flow. Being consistent with dimensional analysis, this time scale agrees in the case $\alpha = 1/3$ with the time scale $\varepsilon^{-1/3} \lambda^{-2/3}$ that is predicted by the scaling considerations in Kolmogorov’s theory for the turnover time of turbulent eddies with length scale $\lambda^{-1}$. (See also Section 3 and in particular inequality (62) below for more explicit comparisons between the energy variation $\varepsilon$ and the homogeneous norms $\|v\|_{C_2^0}$.) In contrast, the time scale corresponding to a stationary frame of reference, as reflected in the bounds for the stationary time derivative $\partial_t$, is much more rapid, being of the order $\lambda^{-1}$.

The regularity statements in Theorems 1.1 and 1.5 as well as Lemma 1.1 are motivated by the properties demonstrated for the weak solutions to Euler constructed in [DLS12], [Ise12] and [BDLS13] as well as a conjecture studied in [Ise12]. These papers make progress towards Onsager’s conjecture by exhibiting various, different constructions of Hölder continuous, periodic Euler flows which fail to conserve energy, although the exponent $1/3$ remains out of reach. These solutions are shown to have many of the properties stated in Theorems 1.1 and 1.5; however, these properties are all proven using the explicit form of the building blocks of the constructions (see in particular Remark 1.2 of [DLS12] regarding the improved regularity in space for the pressure). Theorems 1.1-1.4 and inequality 7 show that despite allowing for a great amount of flexibility, the Euler equations impose nontrivial constraints (beyond simply the conservation of momentum) even on low regularity solutions. In particular, these bounds impose constraints on what kind of scheme one can use to attack Onsager’s conjecture, as they demonstrate that an improvement in spatial regularity of solutions must be accompanied by improvements in the regularity in time within the construction (see Section 9 for further discussion).

On the other hand, the above results are still consistent with the conjectural “ideal case scenario” studied in [Ise12], which would imply Onsager’s conjecture if the convex integration scheme for Euler could be sufficiently improved.

A key ingredient in the proofs of Theorems 1.1-1.4 and Lemma 7 is the fundamental commutator estimate for the Reynolds stress used in the proof of energy conservation in [ET94]. We give a new proof of this estimate below which is more robust in that it allows us to take advective derivatives as well as spatial derivatives. Several techniques in the proof below are inspired by analogous estimates for solutions constructed by convex integration in [Ise12], and we will comment on these analogies in the course of the proof.

### 1.4 Organization of the paper

The main results of the paper are Theorems 1.3 and 1.4, which are proven by an induction on the number of material derivatives. Sections 2 - 6.3 focus on estimates and applications for the first two material derivatives, which form a base case for the overall induction and provide the clearest setting to demonstrate many of the main ideas. Along the way in Section 3 we prove Theorem 1.5 on the regularity of the energy profile. The notation of the paper is summarized in Sections 2.1 and 7.1. The remainder of Section 4 is devoted to the full proof of Theorems 1.3 and 1.4. In Section 8 we use the results of Section 7 to establish Theorem 1.6. We conclude the paper in Section 9 by discussing the relationship of the present results to convex integration and some questions which are motivated by the present work, including a conjecture related to Theorem 1.5 and the idea of anomalous dissipation.
1.5 Acknowledgements

The author thanks P. Constantin, C. De Lellis and V. Vicol for conversations related to Theorem 1.1 and for discussions regarding an earlier draft of the paper. The author also thanks S.-J. Oh for discussions which motivated the proof of the endpoint regularity for $v$. The author acknowledges the support of the NSF Graduate Research Fellowship Grant DGE-1148900. This material is based upon work supported by the National Science Foundation under Award No. DMS-1402370.

2 The proof

We consider solutions to the Euler equations, written here using the summation convention

$$\begin{cases}
\partial_t v^l + \partial_j (v^j v^l) + \partial^l p = 0 \\
\partial_j v^j = 0,
\end{cases}$$

(9)

for any torus $\mathbb{T}^n = \mathbb{R}^n / \Gamma$. Many of the estimates proven below have universal constants which do not depend on the torus. In some cases when low frequencies play a large role in the estimates, the bounds depend on the torus, generally through the size of the lowest frequency on the dual of $\mathbb{T}^n$ (which can be regarded as a characteristic inverse length for the flow). For the first part of the argument, we state along the way which estimates depend on the torus.

The starting point for our proof is the fact that any solution to the Euler equations has Littlewood-Paley components obeying the bound 1.1 for their material derivatives. The material derivative estimate in Lemma 1.1 follows from the commutator estimate of Constantin, E and Titi together with the following estimate for the Littlewood-Paley components of the pressure.

**Lemma 2.1.** If $(v, p)$ solve the incompressible Euler equations and $0 < \alpha \leq 1$, then the Littlewood-Paley projections of the pressure satisfy

$$\|\nabla P_k p\|_{C^{\alpha}} \leq C 2^{(1-2\alpha)k} \|v\|_{C_t^1 C_x^2}^2$$

(10)

We begin the proof by fixing notation, much of which is standard.

2.1 Notation

In what follows we will always regard distributions on the torus as being periodic distributions on the whole space $\mathbb{R}^n$. All convolutions will refer to convolutions in the spatial variables at a fixed time unless otherwise stated.

The norm $\|X\|_{C^\alpha}$ refers to the $C^\alpha_{t,x}$ norm of $X$ in both time and space. For an operator $T$ acting on $C^0(\mathbb{T}^n)$, we will denote by $\|T\|$ the operator norm as a bounded mapping on $C^0(\mathbb{T}^n)$. Very often our operators will be of a convolution form $Tv(x) = \int_{\mathbb{R}^n} v(x + h)K(h)dh$, so that $\|T\| \leq \|K\|_{L^1(\mathbb{R}^n)}$.

We recall here the basics of Littlewood-Paley theory to fix notation. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a radially symmetric smooth function such that $\hat{\eta}$ is supported in the unit ball of $\mathbb{R}^n$, and such that $\hat{\eta}(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. In particular, $\hat{\eta}(0) = \int_{\mathbb{R}^n} \eta(h)dh = 1$. Now set $\eta_{\leq k}(h) = 2^{nk} \eta(2^k h)$ so that $\eta_{\leq k} \in C^\infty_c(B_{2^k}(\mathbb{R}^n))$ also has integral $\int_{\mathbb{R}^n} \eta_{\leq k}(h)dh = 1$ and define

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4 More precisely, we require a similar estimate for the derivative of the commutator, which appears in [CDLS10].
Definition 2.1 (Littlewood-Paley Projections). For any continuous, vector valued function \( v : \mathbb{T}^n \to \mathbb{R}^m \), set

\[
P_{\leq k}v(t, x) = \int_{\mathbb{R}^n} v(t, x + h)\eta_{\leq k}(h)dh \quad \forall (t, x) \in \mathbb{R} \times \mathbb{T}^n \quad (11)
\]

\[
P_k v(t, x) = P_{\leq k}v(t, x) - P_{\leq k-1}v(t, x) = \eta_k * v(t, x) \quad (12)
\]

\[
\eta_k = \eta_{\leq k} - \eta_{\leq k-1} \quad (13)
\]

We also define \( P_{-\infty} v = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} vdh \) to be the average value of \( v \)

\[
P_{-\infty} v = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} vdx \quad (14)
\]

\[
P_{(-\infty, k)} v = P_{\leq k}v - P_{-\infty}v \quad (15)
\]

For any integers \( k_1 < k_2 \), we likewise define

\[
P_{[k_1, k_2]} = P_{\leq k_2} - P_{\leq k_1} = \eta_{[k_1, k_2]} \quad (16)
\]

Note that the Fourier transform of \( \hat{\eta}_k \) has support in a band \( 2^{-(k-1)} < |\xi| < 2^{(k+1)} \). Our analysis will use heavily the following (rather delicate) properties of Littlewood-Paley projections, which give motivation for using the terminology “projection”

\[
P_{\leq k} = P_{\leq k+2}P_{\leq k}, \quad P_k = P_{[k-2, k+2]}P_k \quad (17)
\]

As we are working in the context of periodic functions, observe that there exists a \( k_0 \) depending on \( \mathbb{T}^n = \mathbb{R}^n/\Gamma \) such that

\[
P_k v = 0, \quad \forall k < k_0(\mathbb{T}^n) \quad (18)
\]

We then have a decomposition

**Proposition 2.1** (Littlewood-Paley decomposition). For any continuous vector-valued function \( v \) on \( \mathbb{T}^n \), we have

\[
v = \sum_{k=k_0(\mathbb{T}^n)}^{\infty} P_k v + P_{-\infty}v \quad (19)
\]

where the summation \((18)\) is interpreted in the sense of distributions.

### 2.2 Bounds on Littlewood-Paley Pieces

Now, assuming that \((v^j, p)\) satisfy the Euler equations, let us study the equation obeyed by the Littlewood-Paley projections of \( v^j \), which we write using the Einstein summation convention as

\[
\begin{align*}
\partial_t P_{\leq k}v^j + \partial_j (P_{\leq k}v^j P_{\leq k}v^j) + \partial^j P_{\leq k}p &= \partial_j R_{\leq k}^j \\
\partial_j P_{\leq k}v^j &= 0 \\
R_{\leq k}^j &= P_{\leq k}v^j P_{\leq k}v^j - P_{\leq k}(v^j v^j) 
\end{align*} \tag{19}
\]

Using the fact that \( P_{\leq k}v^j \) is divergence free, and subtracting \((19)\) for \( P_{\leq k+1}v^j \) and \( P_{\leq k}v^j \) gives

\[
\begin{align*}
\partial_t P_{k+1}v^j + P_{\leq k}v^j \partial_j P_{k+1}v^j + P_{k+1}v^j \partial_j P_{\leq k+1}v^j + \partial^j P_{k+1}p &= \partial_j (R_{\leq k+1}^j - R_{\leq k}^j) \\
\partial_j P_{k+1}v^j &= 0 
\end{align*} \tag{20}
\]

From equations \((19)\) and \((21)\) we will be able to deduce the following bounds
Proposition 2.2 (Bounds on Littlewood-Paley pieces, 1). If \((v, p)\) solve Euler on \(I \times \mathbb{T}^d\) for some open interval \(I\) and \(0 < \alpha \leq 1\), then their Littlewood Paley pieces satisfy the bounds

\[ \|P_k v\|_{C^0} \leq C 2^{-2\alpha (k)} \|v\|_{C^0}^2 \]

\[ \|\nabla P_k v\|_{C^0} \leq C 2^{-(1-2\alpha)k} \|v\|_{C^0}^2 \]

\[ \|\nabla^2 P_k v\|_{C^0} \leq C 2^{-(2-2\alpha)k} \|v\|_{C^0}^2 \]

\[ \|P_{k+1} v\|_{C^0} \leq C 2^{-\alpha k} \|v\|_{C^0} \]

\[ \|\nabla P_{\leq k} v\|_{C^0} + \|\nabla P_{k+1} v\|_{C^0} \leq C 2^{-(1-\alpha)k} \|v\|_{C^0} \]

\[ \|\nabla (\partial_t + P_{\leq k} v \cdot \nabla) P_{k+1} v\|_{C^0} \leq C 2^{-(2-2\alpha)k} \|v\|_{C^0}^2 \]

where \(\nabla\) represents any spatial derivative.

Proof of Proposition 2.2. First note that (20), being a standard estimate, is clear from the expression

\[ P_k v^j(x) = \int v^j(x+h)(\eta_{\leq k}(h) - \eta_{\leq k-1}(h))dh \]

\[ = \int (v^j(x+h) - v^j(x+2h))\eta_{\leq k}(h)dh \]

\[ \|P_k v\|_{C^0} \leq \|v\|_{C^0} \int |h|^\alpha \eta_{\leq k}(h)dh \]

The estimate (20), which is also standard, follows from the formula

\[ \partial_t P_{\leq k} v(x) = \int (v(x+h) - v(x))\partial_t \eta_{\leq k}(h)dh \]

which uses the identity \(\int \partial_t \eta_{\leq k}(h)dh = 0\). The same estimate follows as well for the individual projections \(P_{k+1} v = P_{\leq k+1} v - P_{\leq k} v\).

We recall the well-known calculations above to emphasize that the Littlewood-Paley projections we are employing are simply particular examples of averages\(^5\) or linear combinations of velocity differences. In particular, line (30), which expresses \(P_k v\) as an average of relative velocities concentrated at distance \(|h| \leq 2^{-k}\), shows how bounds on Littlewood-Paley projections can be viewed as estimates on the average size of velocity differences at a given scale. This calculation also provides a starting point for comparing statements about statistically averaged velocity differences to statements within the mathematical formalism of Littlewood-Paley theory, and explains the sense in which the scaling law \(\ref{1}\) corresponds to a Besov-type regularity for the velocity field.

Now, observe that, since \(v\) and all of its Littlewood-Paley projections are divergence free, one can routinely justify the following computation by passing from a smooth approximation

\[ P_k v = \partial_j \partial_{t} \Delta^{-1} P_k (v^j v^l) \]

\[ \Delta^{-1} P_k [v^j v^l] = \partial_j \partial_{t} \partial_{t} \Delta^{-1} P_k (v^j v^l) \]

\[ = \partial_j \partial_{t} \partial_{t} \Delta^{-1} P_k [(v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l) + (v^j - P_{\leq k} v^j) P_{\leq k} v^l \]

\[ + P_{\leq k} v^l (v^j - P_{\leq k} v^j) + P_{\leq k} v^j P_{\leq k} v^l] \]

\[ = \partial_j \partial_{t} \partial_{t} \Delta^{-1} P_k [(v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l)] + \partial_{t} \partial_{t} \Delta^{-1} P_k [(v^j - P_{\leq k} v^j) \partial_j P_{\leq k} v^l] \]

\[ + \partial_{t} \partial_{t} \Delta^{-1} P_k [\partial_{t} P_{\leq k} v^j (v^l - P_{\leq k} v^l)] + \partial_{t} \partial_{t} \Delta^{-1} P_k [\partial_{t} P_{\leq k} v^j \partial_j P_{\leq k} v^l] \]

---

\(^5\)The expression in line \(\ref{30}\) can be viewed as a type of average in the sense that the weight function satisfies \(\int \eta_{\leq k}(h)dh = 1\), even though the function \(\eta_{\leq k}\) takes on both positive and negative values for the Littlewood-Paley projections we employ here.
The bounds (22), (23) and (24) follow when we apply the bounds
\begin{align}
\|v - P_{\leq k}v\|_{C^0} &\leq 2^{-\alpha k} \|v\|_{C_1 C_2^\alpha} \tag{36} \\
\|\nabla P_{\leq k}v\|_{C^0} &\leq 2^{(1-\alpha)k} \|v\|_{C_1 C_2^\alpha} \tag{37}
\end{align}
and we observe, by scaling considerations, that
\[\|\nabla^D \Delta^{-1} P_k\| \leq C D 2^{(D-2)k}\]
as a bounded operator on $C^0$, since the right hand side bounds the $L^1$ norm of the kernel.

To obtain (27) and (28), we use equation (21), the bounds (23) and (24) for the pressure, and the Constantin, E, Titi commutator estimate \[\text{[CET94]},\] which we state in the form
\[\|\nabla^D P_{\leq k}\|_{C^0} \leq C D 2^{(D-2\alpha)k} \|v\|_{C_1 C_2^\alpha}^2 \tag{38}\]
The statement (38) which includes bounds on higher derivatives is proven in \[\text{[CDLS10]}.\] Later on we will give a different proof of (38) which will enable us to prove higher regularity in time.

Let us emphasize again that the bounds in Proposition (2.2) are all consistent with the dimensional analysis of the Euler equations and remain invariant under Galilean transformations. That is, the velocity carries units of length/time and the pressure carries units of length/time² while the factor $k^2$ has the units of an inverse length scale. Furthermore, the bounds involve only velocity differences or derivatives, giving them a Galilean invariance. In contrast, the bound available for the stationary time derivative
\begin{align}
\|\partial_t P_k v\|_{C^0} &\leq \|\partial_t + P_{\leq k-1} v \cdot \nabla P_k v\|_{C^0} + \|P_{\leq k-1} v \cdot \nabla P_k v\|_{C^0} \tag{39} \\
&\leq C(2^{(1-2\alpha)k} \|v\|_{C_1 C_2^\alpha}^2 + 2^{(1-\alpha)k} \|v\|_{C^0} \|v\|_{C_1 C_2^\alpha}) \tag{40} \\
&\leq C \alpha^n 2^{(1-\alpha)k} \|v\|_{C_1 C_2^\alpha} \|v\|_{C_1 C_2^\beta} \tag{41}
\end{align}
(which can be proven directly without the commutator estimate) is weaker and clearly fails to be Galilean invariant. Nonetheless, (41) is enough to imply the Hölder regularity in time of $v \in C^\alpha_{t,x}$ asserted in Theorem (1.1).

As a prelude to establishing Theorem (1.1) we first observe that the regularity $v \in C^\beta_{t,x}$ for all $\beta < \alpha$ can be obtained from the following interpolation argument.
\begin{align}
\|P_k v\|_{C^\beta_{t,x}} &\leq C \|P_k v\|_{C_1 C_2^\beta}^{1-\beta} \|\nabla \partial_t P_k v\|_{C^0}^{\beta} \tag{42} \\
&\leq C \alpha^n \left(2^{-\alpha k} \|v\|_{C_1 C_2^\alpha}\right)^{1-\beta} \left(2^{(1-\alpha)k} \|v\|_{C_1 C_2^\alpha} \|v\|_{C_1 C_2^\alpha}^{1 + \|v\|_{C_1 C_2^\alpha}}\right)^{\beta} \tag{43} \\
&\leq C \alpha^n 2^{(\beta-\alpha)k} \|v\|_{C_1 C_2^\alpha} \|v\|_{C_1 C_2^\beta} \tag{44}
\end{align}
The same interpolation argument using (22) and (23) implies that $p \in C^\beta_{t,x}$ for all $\beta < \alpha \leq 1$.

For later use, we record the following estimates, which are related to the bounds in Proposition (2.2)
\begin{proposition}
Under the assumptions of Proposition (2.2) and $0 < \alpha < 1$, we have
\begin{align}
\|\nabla^{2+D} P_{\leq k}\|_{C^0} &\leq C D 2^{(D+2-2\alpha)k} \|v\|_{C_1 C_2^\alpha}^2 \tag{45} \\
\|\nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) P_{\leq k} v\|_{C^0} &\leq C D 2^{(D+2-2\alpha)k} \|v\|_{C_1 C_2^\alpha}^2 \tag{46}
\end{align}
\end{proposition}

\text{[6]Here it seems natural to define the norm $\|v\|_{C_1 C_2^\alpha} = \|v\|_{C^0} + 2^{-\alpha k_0(T^n)} \|v\|_{C_1 C_2^\alpha}$ with $k_0(T^n)$ as defined in Proposition (2.3) in order to obtain a dimensionally consistent estimate in (45) with a constant independent of $T^n$.}
If $\alpha = 1$, we have instead

$$\|\nabla^{2+D} P_{\leq k} p\|_{C^0} \leq C_D (1 + |k - k_0(\mathbb{T}^n)|) 2^{Dk} \|v\|_{C_0}^2$$

(47)

$$\|\nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) P_{\leq k} v\|_{C^0} \leq C_D (1 + |k - k_0(\mathbb{T}^n)|) 2^{Dk} \|v\|_{C_0}^2$$

(48)

The factor $(1 + |k - k_0(\mathbb{T}^n)|)$ can be omitted for $D > 0$.

Here we recall that $k_0(\mathbb{T}^n)$ is an integer such that $2^{k_0(\mathbb{T}^n)}$ is comparable to the lowest frequency on the dual of the torus $\mathbb{T}^n = \mathbb{R}^n / \Gamma$, the latter of which can be regarded as a characteristic inverse length for the flow. In particular, the difference $|k - k_0(\mathbb{T}^n)|$, being the logarithm of a ratio of inverse lengths, is dimensionless.

Proof. The bound (45) for $0 < \alpha < 1$ and $D = 0$ follows by estimating

$$\|\nabla^2 P_{\leq k} p\|_{C^0} \leq \sum_{l = k_0(\mathbb{T}^n)}^k \|\nabla^2 P_l p\|_{C^0}$$

$$\leq C \sum_{l = -\infty}^k 2^{(2-2\alpha)l} \|v\|_{C_1C_2}^2$$

In the case $\alpha = 1$ we obtain instead the bound

$$\|\nabla^2 P_{\leq k} p\|_{C^0} \leq C(1 + |k - k_0(\mathbb{T}^n)|) \|v\|_{C^1C_2}^2$$

The proof for $D > 0$ is identical, but does not involve the loss of a logarithmic factor $|k - k_0(\mathbb{T}^n)|$.

The bound (46) can now be obtained from equation (45) by using the bound (48) together with (45) and the basic estimates for $P_{k+1} v \cdot \nabla P_{\leq k+1} v$. 

\[\Box\]

2.3 On the endpoint regularity

The argument of line (44) establishes that the sequence $P_{\leq k} v$ is Cauchy in the space $C_{t,x}^\beta$ for $\beta < \alpha < 1$, and hence $v \in C_{t,x}^\beta$ for $\beta < \alpha$. Here we show that a more careful argument (which adapts the main idea in the Littlewood-Paley theory characterization of Hölder spaces) establishes the endpoint regularity $v \in C_{t,x}^\alpha$.

Letting $\Delta t \in \mathbb{R}$ be fixed, the main idea is to estimate $|v(t + \Delta t, x) - v(t, x)|$ by summing the estimate

$$|P_k v(t + \Delta t, x) - P_k v(t, x)| \leq \min\{||\partial_t P_k v||_{C^0} |\Delta t|, 2 \|P_k v\|_{C^0}\}$$

(49)

The bound $||\partial_t P_k v||_{C^0} |\Delta t|$ is more useful for low frequencies, which vary less rapidly, while high frequencies have smaller amplitude and the $C^0$ estimate is more useful in this case. We proceed to estimate $|v(t + \Delta t, x) - v(t, x)|$ by decomposing the velocity difference into

$$|v(t + \Delta t, x) - v(t, x)| \leq \sum_{k = k_0(\mathbb{T}^n)}^K ||\partial_t P_k v||_{C^0} |\Delta t| + 2 \sum_{k > K} \|P_k v\|_{C^0}$$

$$\leq C \left( \sum_{k = -\infty}^K |\Delta t| 2^{(1-\alpha)k} \|v\|_{C_1C_2} \|v\|_{C_1C_2} + \sum_{k > K} 2^{-\alpha k} \|v\|_{C_1C_2} \right)$$

$$\leq C \left( 2^{(1-\alpha)K} \|v\|_{C_1C_2} |\Delta t| + 2^{-\alpha K} \|v\|_{C_1C_2} \right)$$
Choosing $K$ so that both terms are roughly equal gives

$$|v(t + \Delta t, x) - v(t, x)| \leq C|\Delta t^\alpha\|v\|_{C_1C_2}^\alpha\|v\|_{C_1C_2}$$  \hspace{1cm} (50)

The estimate (50) is dimensionally correct with a universal constant provided we normalize the $\|v\|_{C_1C_2}$ norm as in the footnote after (41). The same idea applied to the difference $|v(t + \Delta t, x + \Delta x) - v(t, x)|$ establishes Hölder regularity in space and time.

Several endpoint cases for the regularity results in this paper will be established using the above argument, but there are some exceptions. Specifically, we will see that there are logarithmic losses in the estimates for the pressure which prohibit us from obtaining $p \in C_{t,x}^{2,\alpha}$ regularity in time, and the time regularity of the energy profile in the case $\alpha = 1/3$ is slightly more subtle.

### 3 Regularity of the Energy Profile

In this Section, we establish Theorem 1.5 on the Hölder regularity for the energy profile as a function of time, and we conclude with some discussion of this Theorem’s relationship to the discussion of turbulence and anomalous dissipation in the introduction.

Let us define

**Definition 3.1.** The energy increment from frequency $2^{k+1}$ is defined as

$$\delta e(k)(t) = \int_{T^n} (|P_{\leq k+1} v|^2 - |P_{\leq k} v|^2) dx$$  \hspace{1cm} (51)

Then we have

$$\int_{T^n} |v|^2 dx = \int_{T^n} |P_{-\infty} v|^2 dx + \sum_{k=0}^{\infty} \delta e(k)(t)$$

where the first term, which is essentially the square of the total momentum, is conserved in time.

For exponents $\alpha < 1/3$, the Hölder regularity of $\int |v|^2(t, x) dx$ in time follows using the argument of Section 2.3 from the following estimates

**Proposition 3.1.**

$$\|\delta e(k)(t)\|_{C^0_T} \leq C_T 2^{-2\alpha k} \|v\|_{C_1C_2}^2$$  \hspace{1cm} (52)

$$\frac{d}{dt} \|\delta e(k)(t)\|_{C^0_T} \leq C_T 2^{(1-3\alpha)k} \|v\|_{C_1C_2}^3$$  \hspace{1cm} (53)

We begin by proving (52).

**Proof of (52).** The first term in

$$\delta e(k)(t) = \int_{T^n} |P_{k+1} v|^2 dx + 2 \int_{T^n} P_{k+1} v \cdot P_{\leq k} v dx$$  \hspace{1cm} (54)

$$= \delta e(k),1 + 2\delta e(k),2$$  \hspace{1cm} (55)

is easily bounded by

$$\delta e(k),1 \leq |T^n| \|P_{k+1} v\|_{C^0}^2$$  \hspace{1cm} (56)

$$\leq C_T 2^{-2\alpha k} \|v\|_{C_1C_2}^2$$  \hspace{1cm} (57)
For $\delta e_{(k),2}$, we have no control over the size of $P_{\leq k}v$, so we exploit the fact that the interaction between $P_{k+1}v$ and $P_{\leq k}v$ takes place on the common frequency $\sim 2^k$. 

$$\delta e_{(k),2} = \int_{\mathbb{T}^n} P_{k+1}v \cdot P_{[k-3,k]}v \, dx$$

(58)

at which point (52) is clear.

A natural approach to estimating $\frac{d}{dt} \delta e_{(k)}(t)$ would be to observe that (since $P_{\leq k}v$ is divergence free) we have

$$\frac{d}{dt} \delta e_{(k)}(t) = \int_{\mathbb{T}^n} (\partial_t + P_{\leq k}v \cdot \nabla) [\|P_{k+1}v\|^2 + P_{k+1}v \cdot P_{[k-3,k]}v] \, dx$$

(59)

Ideally, the cost for a material derivative should always be $2^{(1-\alpha)k}\|v\|_{C_t C_x^\alpha}$, which would give the bound (63). However, one quickly sees that there is no control over the size of the term $P_{\leq k}v$ that remains when the material derivative hits $P_{k+1}v$ (which is related to the fact that $P_{\leq k}v$ is an average of absolute velocities rather than relative velocities). Thus, before differentiating it is useful to first express this interaction term in terms of relative velocities, as in the proof of (52).

On the other hand, when we express the derivative of the energy increment as

$$\frac{d}{dt} \delta e_{(k)}(t) = \int_{\mathbb{T}^n} (\partial_t + P_{\leq k}v \cdot \nabla) [\|P_{k+1}v\|^2 + P_{k+1}v \cdot P_{[k-3,k]}v] \, dx$$

(60)

it follows from (27) and (25) that the material derivative costs $2^{(1-\alpha)k}\|v\|_{C_t C_x^\alpha}$ in the estimate for each term, which is the desired bound and gives (53).

We note here that the divergence free property of $P_{\leq k}v$ is not important in this estimate. Namely, even without incompressibility, the other term which could arise would take the form

$$\int \partial_t P_{\leq k}v^i [\|P_{k+1}v\|^2 + P_{k+1}v \cdot P_{[k-3,k]}v] \, dx$$

after an integration by parts, and the cost of introducing the term $\text{div} P_{\leq k}v$ is exactly the factor

$$\|\partial_t P_{\leq k}v^i\|_{C_0} \lesssim 2^{(1-\alpha)k}\|v\|_{C_t C_x^\alpha}$$

that we desire. \hspace{1cm} \square

For the endpoint case $\alpha = 1/3$, the fact that $e(t)$ is Lipschitz in $t$ follows from the argument of [CET94], which shows that the time derivative of the frequency truncated energy

$$\frac{d}{dt} e_{\leq k}(t) = \frac{d}{dt} \int_{\mathbb{T}^n} |P_{\leq k}v|^2(t,x) \, dx = \int_{\mathbb{T}^n} (\partial_t + P_{\leq k}v \cdot \nabla) |P_{\leq k}v|^2 \, dx$$

$$= - \int_{\mathbb{T}^n} \partial_j(P_{\leq k}v)_i R_{ij}^{(l)} \, dx$$

is bounded uniformly in $k$ by

$$\left| \frac{d}{dt} e_{\leq k}(t) \right| \leq C_{\mathbb{T}^n} 2^{(1-3\alpha)k}\|v\|_{C_t C_x^\alpha}^3$$

(61)

It follows that $\frac{d}{dt} e(t) \in L^\infty$, so that the energy profile is Lipschitz. This calculation concludes the proof of Theorem 1.5.
We conclude this section with some observations concerning the case where the velocity field has Besov-type regularity and some comments related to anomalous dissipation. First, we observe that the same argument shows that \[ \left\| \frac{d}{dt} e(t) \right\|_{L^1_t} \leq k \left\| v \right\|_{L^3_t B^{1/3}_{3,\infty}}^3 \]

for solutions in the class \( v \in L^3_t B^{1/3}_{3,\infty} \). Thus, the energy profile for a solution in the class \( L^3_t B^{1/3}_{3,\infty} \) is of bounded variation, which is precisely the minimal regularity in time implied by anomalous dissipation. Similarly, note also that the property of having a Lipschitz energy profile holds as well for solutions in the class \( v \in L^\infty_t B^{1/3}_{3,\infty} \).

In this case the bound involves a constant \( C \) that is universal, and inequality \( (62) \) bears a close formal resemblance to the \( p = 3 \) case of \( (1) \). It is in this class that the quality of energy dissipation for Euler flows could conceivably be stable under perturbation, as a simple generalization of the above proof shows that the energy variation \( \left\| \frac{d}{dt} e(t) \right\|_{L^\infty_t} \)

is continuous in \( L^\infty_t B^{1/3}_{3,\infty} \) topology. In contrast, we conjecture that energy dissipation or more generally the quality of having an energy profile with bounded variation should be unstable and nongeneric for solutions with lesser regularity (see Section 9 below).

### 4 Material derivative estimates for the pressure increments and commutators

One might hope to prove that the pressure also has H"older regularity \( p \in C^{2a}_{t,x} \) in time by establishing a bound

\[ \left\| (\partial_t + P_{\leq k} v \cdot \nabla) P_k p \right\|_{C^0} \leq C 2^{(1-3\alpha)k} \left\| v \right\|_{L^3_t B^{1/3}_{3,\infty}}^3 \]

such a bound would be consistent with \( (22) \), \( (25) \) and \( (27) \), since the cost of a material derivative should be \( 2^{(1-\alpha)k} \left\| v \right\|_{C^1_t C^2_x} \) (which has the dimensions of an inverse time). However, it is not clear whether \( (63) \) can be proven when \( \alpha \leq 1/3 \): in this case, the high frequencies \( 2^k \) which contribute to the pressure through the nonlinearity move along the more violent flow of \( (\partial_t + P_{\leq I} v \cdot \nabla) \) rather than that of \( (\partial_t + P_{\leq k} v \cdot \nabla) \). After using the Littlewood-Paley calculus to expand

\[ P_k [(v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l)] \approx \sum_{I > k} P_k [P_I v^j P_I v^l] \]

it appears that the most optimistic attempt to bound the derivative

\[ (\partial_t + P_{\leq k} v \cdot \nabla) = (\partial_t + P_{\leq I} v \cdot \nabla) - P_{[k, I]} v \cdot \nabla \]

of \( (64) \) gives rise to a divergence when \( \alpha \leq 1/3 \).

Here we get around this difficulty for \( \alpha \leq 1/3 \) by decomposing the pressure

\[ p = \sum_{k=k_0(\mathbb{Z}^n)}^{\infty} \delta p(k) \]

into increments \( \delta p(k) \) which only involve interactions of velocity components with frequencies below \( 2^k \).

The philosophy of assembling the solution one frequency shell at a time is also the guiding philosophy...
because the convergence for $\beta < 1$ is exponential, so in particular Proposition 4.1 implies the $\alpha \leq 1/2$ case of Theorem 1.1. The logarithmic loss in the estimates prevents us from obtaining the endpoint regularity $p \in C^{2\alpha}_{x,t}$, but we remark that the argument of Section 2.3 can be adapted to give some endpoint type regularity in time with a logarithmic loss.

We now begin the proof of Proposition 4.1 by establishing the bounds (68)-(69).
Proof of (68)-(69). We start by expressing
\[ \delta p_{(k)} = \Delta^{-1} \partial_j \partial_l \left[ P_{\leq k+1} (P_{\leq k+1} v^j P_{\leq k+1} v^l) - P_{\leq k} (P_{\leq k} v^j P_{\leq k} v^l) \right] \]
\[ = \Delta^{-1} \partial_j \partial_l P_{k+1} [P_{\leq k+1} v^j P_{\leq k+1} v^l] + \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{\leq k+1} v^j P_{\leq k+1} v^l] + \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{\leq k+1} v^j P_{\leq k+1} v^l] \]
\[ + \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{\leq k+1} v^j P_{\leq k+1} v^l] + \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{\leq k+1} v^j P_{\leq k+1} v^l] \]
\[ = \delta p_{(k),1} + \delta p_{(k),2} + \delta p_{(k),3} \]
\[ \delta p_{(k),1} = \Delta^{-1} \partial_j \partial_l P_{k+1} [P_{\leq k+1} v^j P_{\leq k+1} v^l] \]
\[ \delta p_{(k),2} = \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{k+1} v^j P_{k+1} v^l] \]
\[ \delta p_{(k),3} = \Delta^{-1} \partial_j \partial_l P_{\leq k} [P_{\leq k} v^j P_{\leq k} v^l + P_{k+1} v^j P_{\leq k} v^l] \]

We then use the basic properties of Littlewood-Paley pieces to further decompose \( \delta p_{(k),3} \) into High-High and High-Low interactions.
\[ \delta p_{(k),3} = \Delta^{-1} \partial_j \partial_l P_{\leq k} \left[ P_{\leq k} v^j P_{k+1} v^l + P_{k+1} v^j P_{\leq k} v^l \right] \]
\[ = \delta p_{(k),3HH} + \delta p_{(k),3HL} \]
\[ \delta p_{(k),3HH} = \Delta^{-1} \partial_j \partial_l P_{k-3, \leq k} \left[ P_{k-3} v^j P_{k+1} v^l + P_{k+1} v^j P_{k-3} v^l \right] \]
\[ \delta p_{(k),3HL} = \Delta^{-1} \partial_j \partial_l P_{k-3, \leq k} \left[ P_{\leq k-4} v^j P_{k+1} v^l + P_{k+1} v^j P_{\leq k-4} v^l \right] \]

Here we have taken advantage of the representation of the product as a convolution in frequency space, which ensures a lower bound on the frequency support of the product \( P_{\leq k-4} v^j P_{k+1} v^l \).

Finally, we use the fact that the Littlewood-Paley pieces of \( v^j \) are all divergence free to write
\[ \delta p_{(k),3HL} = \Delta^{-1} \partial_j P_{k-3, \leq k} \left[ \partial_l P_{\leq k-4} v^j P_{k+1} v^l \right] + \Delta^{-1} \partial_l P_{k-3, \leq k} \left[ \partial_j P_{\leq k-4} v^j P_{k+1} v^l \right] \]
\[ \delta p_{(k),3HH} = \Delta^{-1} \partial_j P_{k-3, \leq k} \left[ \partial_l P_{k-3} v^j P_{k+1} v^l \right] + \Delta^{-1} \partial_l P_{k-3, \leq k} \left[ \partial_j P_{k-3} v^j P_{k+1} v^l \right] \]

and
\[ \delta p_{(k),1} = \Delta^{-1} P_{k+1} \left[ \partial_j P_{\leq k+1} v^j \partial_l P_{\leq k+1} v^l \right] \]

The estimate (68) almost follows from the elementary bounds on \( \| P_k v \|_{C^0} \) and \( \| \nabla P_{\leq k} v \|_{C^0} \), and the bound on the operator norm
\[ \| \nabla^D \Delta^{-1} P_k \| \leq C 2^{(D-2)k} \]
just as in the proof of the estimates for \( p_k \) in Proposition (2.2). The only exceptions are the terms \( \delta p_{(k),3HH} \) and \( \delta p_{(k),2} \), which both involve the operator \( \Delta^{-1} \partial_j \partial_l P_{\leq k} \). For these terms we lose a logarithmic factor by estimating the \( L^1 \) norm of the kernel by
\[ \| \Delta^{-1} \partial_j \partial_l P_{\leq k} \| \leq \sum_{l=0}^{k} \| \Delta^{-1} \partial_j \partial_l P_{l} \| \]
\[ \leq C (1 + |k - k_0(T^n)|) \]

It is straightforward to see that all of the above estimates worsen by a factor of \( 2^k \) upon taking a spatial derivative, leading to (69). □
The main task in the proof of (70)-(71) is to compute the commutator of the material derivative \( \partial_t + P_{\leq k} v \cdot \nabla \) with the convolution operators appearing in the expression for \( \delta P(k) \). In general, the commutator of a vector field and a convolution operator can be expressed nicely using the fundamental theorem of calculus:

\[
[P_{\leq k} v \cdot \nabla, K^*] f = - \int_{\mathbb{R}^n} (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \frac{\partial}{\partial x^j} f(x + h) K(h) dh \\
\quad - \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq k} v^i(x + sh) \frac{\partial}{\partial x^a} f(x + h) K(h) h^a dh ds \tag{85}
\]

By observing that \( \frac{\partial}{\partial x^j} f(x + h) = \frac{\partial}{\partial h} f(x + h) \), one can obtain an alternative expression which does not involve the derivative of \( f \) by integrating by parts in the \( h \) variables, giving

\[
[P_{\leq k} v \cdot \nabla, K^*] f = \int_{\mathbb{R}^n} (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) f(x + h) \partial_i K(h) dh \\
\quad + \int_{\mathbb{R}^n} \partial_i P_{\leq k} v^i(x + h) f(x + h) K(h) dh \\
\quad = \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq k} v^i(x + sh) f(x + h) \partial_i K(h) h^a dh ds \\
\quad + \int_{\mathbb{R}^n} \partial_i P_{\leq k} v^i(x + h) f(x + h) K(h) dh \tag{86}
\]

We remark that since the vector fields involved are always divergence free, the latter term in the commutator is actually 0, and we are left with only one of these terms.

\[
[P_{\leq k} v \cdot \nabla, K^*] f = \int_{\mathbb{R}^n} (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) f(x + h) \partial_i K(h) dh \tag{87}
\]

By integrating by parts in the \( h \) variables, giving

\[
[P_{\leq k} v \cdot \nabla, K^*] f = \int_{\mathbb{R}^n} (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) f(x + h) \partial_i K(h) dh \\
\quad = \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq k} v^i(x + sh) f(x + h) \partial_i K(h) h^a dh ds \tag{88}
\]

We now apply the operator \( \partial_t + P_{\leq k} v \cdot \nabla = \partial_t + P_{\leq k} v^i \partial_i \) to the expression (81). This differentiation generates several commutator terms, the most subtle of which is the commutator \( [P_{\leq k} v \cdot \nabla, \Delta^{-1} \partial_j \partial_t P_{\leq k}] \) in (81). The content of the bounds (70)-(71) is simply that this differentiation costs
a factor $2^{(1-\alpha)k} \|v\|_{C^1,\mathcal{C}_2}$ in all the estimates (possibly with a logarithmic loss in some cases). In what follows we will always neglect the difference between $\partial_t + P_{\leq k} v \cdot \nabla$ and $\partial_t + P_{\leq k+1} v \cdot \nabla$ and similar terms which give rise to some harmless factors of the form $P_{k+1} v \cdot \nabla$, since we already know from the bounds on spatial derivatives that the operator $P_{k+1} v \cdot \nabla$ incurs the desired cost of $2^{(1-\alpha)k} \|v\|_{C^1,\mathcal{C}_2}$.

The low frequency terms such as $\delta p_{(k),LL}$ are treated as follows.

$$(\partial_t + P_{\leq k} v \cdot \nabla) \delta p_{(k),LL} = \Delta^{-1} P_{k+1} (\partial_t + P_{\leq k} v \cdot \nabla) f_{(k),LL} + [(\partial_t + P_{\leq k} v \cdot \nabla), \Delta^{-1} P_{k+1}] f_{(k),LL} \tag{95}$$

For the function $f_{(k),LL}$, we have the bounds

$$\|\nabla^D f_{(k),LL}\|_{C^0} \leq C_D 2^{(D+2-2\alpha)k} \|v\|_{C^1,\mathcal{C}_2}^2 \tag{96}$$

$$\|\nabla^D f_{(k),LL}\|_{C^0} \leq C_T (1 + |k - k_0(T^n)|) 2^{(3-3\alpha)k} \|v\|_{C^1,\mathcal{C}_2}^2 \tag{97}$$

which come from Proposition (2.2) and the bound (10) for $\nabla (\partial_t + P_{\leq k} v \cdot \nabla) P_{\leq k} v \|_{C^0}$.

From (90), the commutator can be written in the form

$$(\partial_t + P_{\leq k} v \cdot \nabla) \Delta^{-1} P_{k+1} f_{(k),LL} = \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq k} v^i(x + sh) \frac{\partial}{\partial x^i} f_{(k),LL}(x + h) \Delta^{-1} \eta_{k+1}(h) h^a dh ds \tag{98}$$

which obeys the desired estimate without losing a factor $(1 + |k - k_0(T^n)|)$, as we have the scaling bound

$$\|\Delta^{-1} \eta_{k+1}(h) h\|_{L^1} \leq C 2^{-3k}$$

The term $\delta p_{(k),HH}$ is treated similarly, with only a few differences such as the appearance of high frequency terms such as $(\partial_t + P_{\leq k} v \cdot \nabla) P_{k+1} v$ and a different scaling for the operator.

The term which requires a more subtle analysis is the term $\delta p_{(k),HL}$, which contains the highly nonlocal operator

$$\Delta^{-1} \partial_j \partial_t P_{\leq k} = \sum_{I = k_0(T^n)}^k \Delta^{-1} \partial_j \partial_t P_I \tag{99}$$

In this case, we use the expression (90) to write the commutator term as

$$(\partial_t + P_{\leq k} v \cdot \nabla) \Delta^{-1} \partial_j \partial_t P_{\leq k} f_{(k),HL} = \sum_{I = k_0(T^n)}^k \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq k} v^i(x + sh) \frac{\partial}{\partial x^i} f_{(k),HL}(x + h) \partial_j \partial_t \Delta^{-1} \eta_I(h) h^a dh ds \tag{100}$$

which immediately gives an estimate on the operator norm

$$\sup_t \|[(\partial_t + P_{\leq k} v \cdot \nabla), \Delta^{-1} \partial_j \partial_t P_{\leq k}]\| \leq \sum_{I = k_0(T^n)}^k \|\nabla P_{\leq k} v\|_{C^0} \|h|\nabla^3 \Delta^{-1} \eta_I(h)\|_{L^1_h}$$

$$\leq C \sum_{I = k_0(T^n)}^k \|\nabla P_{\leq k} v\|_{C^0} \cdot 1$$

$$\leq C_T (1 + |k - k_0(T^n)|) 2^{(1-\alpha)k} \|v\|_{C^1,\mathcal{C}_2} \tag{101}$$

This bound is worse than the estimate we used for the operator

$$\|\Delta^{-1} \partial_j \partial_t P_{\leq k}\| \leq C_T (1 + |k - k_0(T^n)|)$$
by exactly the factor
\[ \left| (\partial_t + P_{\leq k} v \cdot \nabla) \right| \leq C 2^{(1-\alpha)k} \| v \|_{C^1 C^2} \]
we desire, and the estimates (70)-(71) follow from differentiating the above formulas in the \( x \) variables at a cost of \( 2^k \) per derivative.

With the expression (100) for the commutator in hand, the bounds (70)-(71) follow quickly from the estimates in Proposition 2.2.

We are now able to prove Theorem 1.4.

**Corollary 4.1.** If \( \alpha > 1/3 \), the distribution \( \partial_t p + \partial_j (pv^j) \) is continuous when \( p \) is normalized to have integral 0 at every time \( t \).

If \( \alpha > 2/3 \), the distribution \( \partial_t \partial_p + \partial_j (v^j \partial_p) \) is continuous.

**Proof.** Set \( p(k) = \sum_{l=k_{(T^\alpha)}}^k \delta p(l) \). Then \( p(k) \to p \) uniformly in space and time, and as a consequence
\[
\partial_t p(k) + \partial_j (p(k) v^j) \to \partial_t p + \partial_j (pv^j) \tag{102}
\]
weakly as distributions from the uniform continuity of \( v \). By regularizing in time, it can be shown that for each \( k \), we have the identity
\[
\partial_t p(k) + \partial_j (p(k) v^j) = \partial_t p(k) + v^j \partial_j p(k) \tag{103}
\]
using the fact that \( \partial_j v^j = 0 \) as a distribution. From this identity we conclude that the convergence of (102) is actually uniform in \((t, x)\) when \( \alpha > 1/3 \), because we have proven the bound
\[
\| (\partial_t + v \cdot \nabla) \delta p(k) \|_{C^0} \leq \| (\partial_t + P_{\leq k} v \cdot \nabla) \delta p(k) \|_{C^0} + \| (v - P_{\leq k} v) \cdot \nabla \delta p(k) \|_{C^0} \\
\leq C_{T^\alpha} (1 + |k - k_{0(T^\alpha)}|) 2^{(1-3\alpha)k} \| v \|_{C^1 C^2}^3
\]
which decays exponentially as \( k \to \infty \) whenever \( \alpha > 1/3 \).

Assuming that \( \alpha > 2/3 \), the continuity of the distribution \( \partial_t \partial_p + \partial_j (v^j \partial_p) \) follows similarly. Namely, we see that
\[
\partial_t \partial_p(k) + \partial_j (v^j \partial_p(k)) \to \partial_t \partial_p + \partial_j (v^j \partial_p) \tag{104}
\]
as distributions, since \( \nabla p(k) \to \nabla p \) uniformly for \( \alpha > 1/2 \). The uniform convergence for \( \alpha > 2/3 \) then follows from the bounds
\[
\| (\partial_t + v \cdot \nabla) \nabla \delta p(k) \|_{C^0} \leq \| (\partial_t + P_{\leq k} v \cdot \nabla) \nabla \delta p(k) \|_{C^0} + \| (v - P_{\leq k} v) \cdot \nabla \nabla \delta p(k) \|_{C^0} \\
\leq \| \nabla (\partial_t + P_{\leq k} v \cdot \nabla) \delta p(k) \|_{C^0} + \| \nabla P_{\leq k} v \|_{C^0} \| \nabla \delta p(k) \|_{C^0} \\
+ \| (v - P_{\leq k} v) \cdot \nabla \nabla \delta p(k) \|_{C^0} \\
\leq C_{T^\alpha} (1 + |k - k_{0(T^\alpha)}|) 2^{(2-3\alpha)k} \| v \|_{C^1 C^2}^3
\]
or alternatively by repeating the commutator estimates above for \( \nabla \delta p(k) \) directly.

The proof above is not the most robust proof of Corollary 4.1, in particular because it does not give any Hölder regularity. As we will see when we have built up the relevant preliminary estimates, it is better to use the approximation
\[
\partial_t p(k) + \partial_j (p(k) P_{\leq k} v^j) \to \partial_t p + \partial_j (pv^j)
\]
which avoids regularizing in time as in (103) and actually converges in the appropriate Hölder spaces.

For now we record the following bounds to accompany Proposition 4.1 for use in the later applications of Section 6.
Proposition 4.2. Under the assumptions of Proposition (4.1),

\[ \|\nabla^{2+D}p_{(k)}\|_{C^0} \leq C_{D,T^n}(1 + |k - k_0(T^n)|)2^{(D+2(1-\alpha))k}\|v\|_{C_1C_2}^2 \tag{105} \]

\[ \|\nabla^D(\partial_t + P_{\leq k}v \cdot \nabla)\nabla^2p_{(k)}\|_{C^0} \leq C_{D,T^n}(1 + |k - k_0(T^n)|)2^{(D+3(1-\alpha))k}\|v\|_{C_1C_2}^3 \tag{106} \]

Furthermore, if \( \alpha < 2/3 \), we have

\[ \|\nabla^D(\partial_t + P_{\leq k}v \cdot \nabla)\nabla p_{(k)}\|_{C^0} \leq C_{D,T^n}(1 + |k - k_0(T^n)|)2^{(D+2-3\alpha)k}\|v\|_{C_1C_2}^3 \tag{107} \]

Proof. The bound (105) is immediate from the representation

\[ \nabla^2 p_{(k)} = \nabla^2 \partial_j \partial_t \Delta^{-1} P_{\leq k}(P_{\leq k} v^j P_{\leq k} v^l) \]

\[ = \nabla^2 \Delta^{-1} P_{\leq k} \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \tag{108} \]

and define

\[ \delta_{(k)} \left[ \frac{D_{\leq k}}{\partial t} \nabla p_{(k)} \right] = \frac{D_{\leq k}}{\partial t} \nabla p_{(k-1)} + P_{\leq k} v \cdot \nabla p_{(k-1)} \]

\[ = \frac{D_{\leq k}}{\partial t} \nabla p_{(k)} - \frac{D_{\leq k-1}}{\partial t} \nabla p_{(k-1)} \tag{111} \]

Then

\[ \frac{D_{\leq k}}{\partial t} \nabla p_{(k)} = \sum_{I = k_0(T^n)}^k \delta_{(I)} \left[ \frac{D_{\leq I}}{\partial t} \nabla p_{(I)} \right] \tag{113} \]

so (107) follows from the formula (111) with the bounds (105)-(106) as in

\[ \|(\partial_t + P_{\leq k}v \cdot \nabla)\nabla p_{(k)}\|_{C^0} \leq C_{T^n} \sum_{I = k_0(T^n)}^k (1 + |I - k_0(T^n)|)2^{(2-3\alpha)I}\|v\|_{C_1C_2} \]

\[ \leq C_{T^n}(1 + |k - k_0(T^n)|)2^{(2-3\alpha)k}\|v\|_{C_1C_2}^3 \tag{114} \]

(115)

Here we have used summation by parts in \( I \) and the condition \( \alpha < 2/3 \) to bound the sum. The bound on higher spatial derivatives follows similarly.

\[ \square \]

5 Material derivative estimates for the forcing terms

So far we have established Theorems (1.4) and (1.5) as well as the case \( \beta \leq 1/2 \) of Theorem (1.1) by drawing on the basic estimates on Littlewood Paley pieces of Proposition (2.2). Proving the full strength of Theorem (1.1) requires going beyond the first time derivative of the pressure, so we will be interested in developing further estimates on second material derivatives for \( P_k v \) and \( \nabla P_{\leq k} v \) as a preliminary step in this direction.
5.1 Material derivative estimates on LP pieces for the pressure

As a first step, we will prove the following bounds for coarse scale material derivatives of Littlewood Paley pieces of the pressure.

**Proposition 5.1.** Assume that \( v \in C_t C^\alpha_x \) for some \( 1/3 < \alpha < 1 \) is a solution to incompressible Euler with pressure \( p \). Then for any integer \( D \geq 0 \) we have the bound

\[
\| \nabla^D (\partial_t + P_{\leq k} P \cdot \nabla) P k_p \|_{C^0} \leq C_{D,\alpha} 2^{(D+1-3\alpha)k} \| v \|_{C^3_t C^\alpha_x}^3
\]  

(116)

**Proof.** We first consider the case \( D = 0 \).

In the proof of Proposition (12), we used the incompressibility of \( v \) to obtain a decomposition

\[
\begin{align*}
P_k p &= HH_k p + HL_k p + LL_k p \\
HH_k p &= \partial_j \partial_t \Delta^{-1} P_k [ (v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l) ] \\
HL_k p &= \partial_j \Delta^{-1} P_k [ (v^j - P_{\leq k} v^j) \partial_j P_{\leq k} v^l ] + \partial_j \Delta^{-1} P_k [ \partial_t P_{\leq k} v^j (v^l - P_{\leq k} v^l) ] \\
LL_k p &= \Delta^{-1} P_k [ \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l ]
\end{align*}
\]  

(117)

(118)

(119)

(120)

We would like to estimate \((\partial_t + P_{\leq k} P \cdot \nabla) P k_p\) and its derivatives by commuting the advective derivative \((\partial_t + P_{\leq k} v \cdot \nabla)\) with the various convolution operators appearing on the right hand side of (117). The difficulty which restricts us to \( \alpha > 1/3 \) is that the high frequency components of \((v - P_{\leq k} v)\) do not have a good estimate for the material derivative at the scale \( 2^{-k} \). For the High-Low terms in (117), we can escape this difficulty with the higher frequencies using the bandlimited property of Littlewood-Paley projections to write

\[
HH_k p = 2 \partial_j \Delta^{-1} P_k [ \partial_t P_{\leq k} v^j (v^l - P_{\leq k} v^l) ] \\
= 2 \partial_j \Delta^{-1} P_k [ \partial_t P_{\leq k} v^j P_{|k,k+2|} v^l ] .
\]  

(121)

However, it seems that the best we can do for the High-High interactions is to write

\[
HH_k p = \partial_j \partial_t \Delta^{-1} P_k [ (v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l) ] \\
= \sum_{l=k}^{\infty} \sum_{j=1}^{\infty} \sum_{|l-j| \leq 2} \partial_j \partial_t \Delta^{-1} P_k [ P_{l} v^j P l v^l ]
\]  

(122)

and to bound the material derivative of this term by first writing

\[
(\partial_t + P_{\leq k} v \cdot \nabla) HH_k p = \sum_{l=k}^{\infty} ((\partial_t + P_{\leq l} v \cdot \nabla) - P_{|k,l|} v \cdot \nabla) [ \partial_j \partial_t \Delta^{-1} P_k ] \sum_{|l-j| \leq 2} P_{l} v^j P l v^l \\
= \sum_{l=k}^{\infty} (A(l) - B(l))
\]  

(123)

(124)

We can estimate the latter term by

\[
\| B(l) \|_{C^0} \leq \sum_{|l-j| \leq 2} \| P_{|k,l|} v^j \|_{C^0} \cdot \| \nabla \partial_j \partial_t \Delta^{-1} P_k \| \| P_{l} v^j P l v^l \|_{C^0} \\
\leq C 2^{(1-\alpha)k} 2^{-2\alpha l} \| v \|_{C^3_t C^\alpha_x}^3
\]  

(125)
which is acceptable for \( (116) \) upon summing over \( I \geq k \).

The term \( A(I) \) is more dangerous, and involves a commutator,

\[
A(I) = \sum_{\substack{j \geq k \\ |I-J| \leq 2}} \partial_j \partial_t \Delta^{-1} P_k \left[ (\partial_t + P_{\leq I} v \cdot \nabla) \left[ P_I v^j P_J v^l \right] \right]
\]

\[
+ \sum_{\substack{j > k \\ |I-J| \leq 2}} \left[ (\partial_t + P_{\leq I} v \cdot \nabla), \partial_j \partial_t \Delta^{-1} P_k \right] \left[ P_I v^j P_J v^l \right]
\]

\[
= A(I,1) + A(I,2) \tag{127}
\]

For the term \( A(I,1) \) we use the bounds

\[
\| \partial_j \partial_t \Delta^{-1} P_k \| \leq C \\
\| (\partial_t + P_{\leq I} v \cdot \nabla) \left[ P_I v^j P_J v^l \right] \|_{C^0} \leq C2^{(1-3\alpha)I} \| v \|_{C_1 C_2}^3
\]

For the commutator term \( A(I,2) \), we have an estimate for the operator

\[
\| [ (\partial_t + P_{\leq I} v \cdot \nabla), \partial_j \partial_t \Delta^{-1} P_k ] \| \leq C2^{(1-\alpha)I} \| v \|_{C_1 C_2}
\]

which is proven by the same integration by parts used to bound the commutator in Section 4.

This bound together with the estimate \( \| P_I v^j P_J v^l \|_{C^0} \leq C2^{-2\alpha I} \) gives \( (116) \) for \( D = 0 \) after summing over \( I \).

To obtain the estimate \((116)\) for \( D > 0 \), it is not safe to differentiate the terms in \((123)\) since the sum over \( I \) will diverge. Instead, one can first differentiate \( \nabla^D P_k p \) in space, letting the derivatives fall on the Littlewood Paley projections. The estimate \((116)\) follows by first repeating the proof above to obtain the desired bound for \( \| (\partial_t + P_{\leq k} v \cdot \nabla) \nabla^D P_k p \|_{C^0} \), and then commuting the material and spatial derivatives to obtain \((116)\).

The estimate \((116)\) can be used to give another proof of Theorem 1.4 along the same lines as the proof in Corollary 4.1.

The same method also gives the following estimate

**Proposition 5.2.** Under the assumptions of Proposition 5.1 and \( 0 < \alpha < 1 \),

\[
\| \nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) \nabla^2 P_{\leq k} p \|_{C^0} \leq C_{D,\alpha} 2^{(D+3-3\alpha)k} \| v \|_{C_1 C_2}^3 \tag{128}
\]

Applying the method of proof from Proposition 5.1 leads to an extra loss of \( (1 + |k - k_0(\mathbb{T}^n)|) \) that we prefer to avoid.

**Proof.** We write

\[
(\partial_t + P_{\leq k} v \cdot \nabla) \nabla^2 P_{\leq k} p = \sum_{l=k_0(\mathbb{T}^n)}^k \left[ (\partial_t + P_{\leq l+1} v \cdot \nabla) \nabla^2 P_{\leq l+1} p - (\partial_t + P_{\leq l} v \cdot \nabla) \nabla^2 P_{\leq l+1} p \right] \tag{129}
\]

\[
= \sum_{l=k_0(\mathbb{T}^n)}^k (\partial_t + P_{\leq l+1} v \cdot \nabla) \nabla^2 P_{l+1} p + P_{l+1} v \cdot \nabla (\nabla^2 P_{\leq l+1} p) \tag{130}
\]

The estimate \((128)\) now follows from Propositions 5.1 and (2.3) by differentiating and summing.

\[
\| \nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) \nabla^2 P_{\leq k} p \|_{C^0} \leq C \sum_{l=-\infty}^k 2^{(D+3-3\alpha)l} \| v \|_{C_1 C_2}^3
\]

Since \( \alpha < 1 \), the last term controls the geometric series. \( \square \)
In Section 5.2 below, we establish analogous estimates for the Reynolds stress. The proofs are very similar, but we will improve on the treatment of the High-High terms to give a proof which turns out to be more robust.

5.2 Estimates for the Reynolds stress

Here we collect all the necessary bounds on the Reynolds stress

\[ R_{\leq k}^{jl} = P_{\leq k}v^jP_{\leq k}v^l - P_{\leq k}(v^jv^l) \]  

and its derivatives which are used in the proofs of Theorems 1.1-1.2. In the process, we also illustrate all the main technical ideas necessary for estimating \( R_{\leq k}^{jl} \) that will be used in the remainder of the paper.

We start by giving an alternative proof of the commutator estimate from \( \text{CET94} \) and the generalization in \( \text{CDLS10} \) which includes bounds on spatial derivatives \( \| \nabla^D R_{\leq k} \|_{C^0} \). The proof we give here will be more flexible in that we will be able to obtain the necessary bounds on the material derivative \( (\partial_t + P_{\leq k}v \cdot \nabla)R_{\leq k}^{jl} \) using the same method.

**Proposition 5.3.** If \( 0 < \alpha \leq 1 \), then for any \( D \geq 0 \)

\[ \| \nabla^D R_{\leq k} \|_{C^0} \leq C_D 2^{(D-2\alpha)k} \| v \|_{C^\alpha}^2 \]

The proof given here does not use any special properties of Littlewood-Paley projections and generalizes to other mollifiers in the Schwartz class as well.

**Proof.** We start by observing a “Galilean invariance” of the commutator \([131]\). Namely, the expression

\[ R_{\leq k}^{jl} = \int \int v^j(x + h_1)v^l(x + h_2)\eta_{\leq k}(h_1)\eta_{\leq k}(h_2)dh_1dh_2 - \int v^j(x + h)v^l(x + h)\eta_{\leq k}(h)dh \]

for \( x \in \mathbb{T}^n \) has a schematic form similar to a variance

\[ -R_{\leq k}(t, x) = \mathbb{E}[v^2](t, x) - \mathbb{E}[v]^2(t, x) \]

since \( \int \eta_{\leq k}(h)dh = 1 \). For example, \( R_{\leq k}^{jl} \) would be negative definite if \( \eta_{\leq k} \) were positive definite, although this is not the case for Littlewood-Paley projections.

Just as the variance of a random variable remains invariant under the addition of a constant, we can observe that \([132]\) remains invariant when we subtract from \( v^j \) any vector \( A^j(x) \) at each point

\[ R_{\leq k}^{jl} = \int \int (v^j(x + h_1) - A^j(x))(v^l(x + h_2) - A^l(x))\eta_{\leq k}(h_1)\eta_{\leq k}(h_2)dh_1dh_2 \]

\[ - \int ((v^j(x + h) - A^j(x))(v^l(x + h) - A^l(x))\eta_{\leq k}(h)dh \]

By choosing \( A^j(x) = v^j(x) \) and \( A^l(x) = v^l(x) \) we immediately obtain the \( C^0 \) bound in \([132]\) (this choice leads also to the decomposition in the \( \text{CET94} \) argument). However, it is also natural to choose \( A^j(x) = P_{\leq k}v^j(x) \) and \( A^l(x) = P_{\leq k}v^l(x) \) in analogy with the expression \( \mathbb{E}[(v - \mathbb{E}[v])^2] \) for a variance.

With this latter choice, the first term \([134]\) disappears, leaving the expression

\[ R_{\leq k}^{jl} = \int (v^j(x + h) - P_{\leq k}v^j(x))(v^l(x + h) - P_{\leq k}v^l(x))\eta_{\leq k}(h)dh \]
We now expand (136) by adding and subtracting $P_{\leq k}v(x + h)$ to each term and obtain

\[ R_{\leq k}^i = R_{\leq k,HH}^i + R_{\leq k,HL}^i + R_{\leq k,LL}^i \]  
(137)

\[ R_{\leq k,HH}^i = \int (v^i(x + h) - P_{\leq k}v^i(x + h))(v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh \]  
(138)

\[ = P_{\leq k}[(v^i - P_{\leq k}v^i)(v^i - P_{\leq k}v^i)] \]  
(139)

\[ R_{\leq k,HL}^i = \int (v^i(x + h) - P_{\leq k}v^i(x + h))(P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh \]  
+ \int (P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))(v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh \]  
(140)

\[ R_{\leq k,LL}^i = \int (P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))(P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh \]  
(141)

The bound (132) now follows quickly from the expanded form (137). Namely, it is easy to see that

\[ \|\nabla^D (P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\|_{C^0} \leq C_D \|v\|_{C^0}C_\alpha^a \]  

For example, every low frequency term can be written as

\[ (P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x)) = \int_0^1 \partial_t P_{\leq k}v^i(\eta + sh)h^\alpha ds \]  
(142)

and the factor of $h^\alpha$ can be incorporated into the mollifier $\eta_{\leq k}(h)$, which we estimate in $L^1_1$.

For high frequency terms, it is always possible to integrate by parts to estimate the derivatives in (132). For example, we have

\[ \frac{\partial}{\partial x^i} R_{\leq k,HH}^i (x) = - \int (v^i(x + h) - P_{\leq k}v^i(x + h))(v^i(x + h) - P_{\leq k}v^i(x)) \frac{\partial}{\partial t}[\eta_{\leq k}(h)] dh \]  
(143)

and one can similarly integrate by parts when the derivative hits the high-frequency factor in the High-Low terms

\[ \int \frac{\partial}{\partial x^i}(v^i(x + h) - P_{\leq k}v^i(x + h))(P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh = \]  
(144)

\[ = \int \frac{\partial}{\partial h^i}(v^i(x + h) - P_{\leq k}v^i(x + h))(P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)dh \]  
(145)

\[ = - \int (v^i(x + h) - P_{\leq k}v^i(x + h)) \frac{\partial}{\partial h^i} [(P_{\leq k}v^i(x + h) - P_{\leq k}v^i(x))\eta_{\leq k}(h)] dh \]  
(146)

In every case, each spatial derivative in the $x$ variable costs a factor $2^k$ in the estimate, so combining these observations gives (132).

The proof above allows us to estimate the material derivative of the Reynolds stress arising from mollifying a solution to the Euler equations provided we assume sufficient regularity on $v$.

**Proposition 5.4.** As in Section (5.1), assume that $v \in C^1C_\alpha^a$ for some $1/3 < \alpha < 1$ is a solution to incompressible Euler with pressure $p$. Then for any integer $D \geq 0$ we have the bound

\[ \|\nabla^D (\partial_t + P_{\leq k}v \cdot \nabla) R_{\leq k}\|_{C^0} \leq C_{D,\alpha} 2^{(D+1-3\alpha)k}\|v\|_{C^0}^3 \]  
(147)
We start the proof by considering the case $D = 0$. The case $D > 0$ can be deduced by establishing the same bound for $\|(\partial_t + P_{\leq k} v \cdot \nabla) \nabla^0 R_{\leq k}\|_{C^0}$, and this latter estimate can be obtained by modifying the argument below.

**Proof.** The proof closely mirrors that of Proposition (5.1), but involves different types of commutator terms. As in the proof of Proposition (5.1), the main difficulty arises from taking material derivatives of high frequency terms. As in the proof of Proposition (5.1), we can also see that $R_{\leq k, HH}$ is the only term to which frequencies much larger than $2^k$ contribute. Namely, due to the bandlimited property of Littlewood-Paley projections, we have that the High-Low term can be expressed as

\[
R_{\leq k, HL}^{ij} = \int (v^j(x + h) - P_{\leq k} v^j(x + h)) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
+ \int (P_{\leq k} v^j(x + h) - P_{\leq k} v^j(x)) (v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
= \int (P_{\leq k+2} v^j(x + h) - P_{\leq k} v^j(x + h)) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
+ \int (P_{\leq k} v^j(x + h) - P_{\leq k} v^j(x)) (P_{\leq k+2} v^i(x + h) - P_{\leq k} v^i(x + h)) \eta_{\leq k}(h) dh
\]

(148)

\[
R_{\leq k, HL}^{ij} = \int P_{[k, k+2]} v^j(x + h) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
+ \int (P_{\leq k} v^j(x + h) - P_{\leq k} v^j(x)) P_{[k, k+2]} v^i(x + h) \eta_{\leq k}(h) dh
\]

(150)

\[
= R_{\leq k, HL1}^{ij} + R_{\leq k, HL2}^{ij}
\]

(151)

We now wish to compute and estimate the material derivatives of these terms, so to begin we compute

\[
(\partial_t + P_{\leq k} v^i(x) \frac{\partial}{\partial x^i}) R_{\leq k, HL1}^{ij}(x) =
\]

\[
= \int (\partial_t + P_{\leq k} v^i(x) \frac{\partial}{\partial x^i}) P_{[k, k+2]} v^i(x + h) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
+ \int P_{[k, k+2]} v^i(x + h) (\partial_t + P_{\leq k} v^i(x) \frac{\partial}{\partial x^i}) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

(152)

\[
= A_{HL, I}^{ij} + A_{HL, IT}^{ij}
\]

(154)

The first of these terms can be expressed as

\[
A_{HL, I}^{ij} = \int [(\partial_t + P_{\leq k} v \cdot \nabla) P_{[k, k+2]}] v^i(x + h) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
- \int (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \partial_t P_{[k, k+2]} v^i(x + h) (P_{\leq k} v^i(x + h) - P_{\leq k} v^i(x)) \eta_{\leq k}(h) dh
\]

\[
+ \int \int [(\partial_t + P_{\leq k} v \cdot \nabla) P_{[k, k+2]}] v^i(x + h) \partial_s P_{\leq k} v^j(x + sh) h^a \eta_{\leq k}(h) dh
\]

\[
- \int \int \int \partial_s P_{\leq k} v^i(x + s_1 h) \partial_t P_{[k, k+2]} v^j(x + s_1 h) \partial_s P_{\leq k} v^j(x + s_2 h) h^{a_1} h^{a_2} \eta_{\leq k}(h) dh ds_1 ds_2
\]

(155)
so that the bound (147) is visible for any $0 < \alpha \leq 1$. The second term can likewise be written as

\[
A_{HL,II}^I = \int_0^1 \int P_{[k,k+2]}v^j(x+h)\partial_t[\partial_t + P_{\leq k}v^j(x)] \frac{\partial}{\partial x^i} \partial h P_{\leq k}v^j(x+sh) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds (157)
\]

\[
= \int_0^1 \int P_{[k,k+2]}v^j(x+h)\partial_t[\partial_t + P_{\leq k}v^j(x)] \partial h P_{\leq k}v^j(x+sh) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds
\]

\[
- \int_0^1 \int P_{[k,k+2]}v^j(x+h)[P_{\leq k}v^j(x+sh) - P_{\leq k}v^j(x)] \partial_t \partial h P_{\leq k}v^j(x+sh) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds (158)
\]

\[
= \int_0^1 \int P_{[k,k+2]}v^j(x+h)\partial_t[\partial_t + P_{\leq k}v^j(x)] \partial h P_{\leq k}v^j(x+sh) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds
\]

\[
- \int_0^1 \int_0^1 \int P_{[k,k+2]}v^j(x+h) \partial_t \partial h P_{\leq k}v^j(x+s_1 sh) \partial h P_{\leq k}v^j(x+sh)(sh^a) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds \, ds_s (159)
\]

and can quickly be seen to obey (147) as well thanks to Proposition (2.3).

The Low-Low term $R_{\leq k,LL}^I$ can be treated similarly after it has been represented in the form

\[
R_{\leq k,LL}^I = \int (P_{\leq k}v^j(x+h) - P_{\leq k}v^j(x))(P_{\leq k}v^j(x+h) - P_{\leq k}v^j(x)) \eta_{\leq k}(h) dh (160)
\]

\[
= \int_0^1 \int_0^1 \partial_t \partial h P_{\leq k}v^j(x+s_1 sh) \partial h P_{\leq k}v^j(x+sh)(sh^a) h^{b}_{\leq k} \eta_{\leq k}(h) dh \, ds \, ds_s (161)
\]

The only term remaining for the proof of (147) is the term

\[
R_{\leq k,HH}^I = P_{\leq k}[v^j - P_{\leq k}v^j][v^j - P_{\leq k}v^j] (162)
\]

which also limits the present method to $\alpha > 1/3$. This term can be handled by the exact same technique as in the proof of Proposition (5.1), namely by expanding in Littlewood-Paley pieces as in (122).

Here we improve on the approach in the proof of Proposition (5.1) to obtain the bound for the derivatives $D > 0$ in (147). Namely, following the proof of Proposition (5.1) we could first differentiate in space to

\[
\nabla D R_{\leq k,HH}^I = \nabla D P_{\leq k}[(v^j - P_{\leq k}v^j)(v^j - P_{\leq k}v^j)] (163)
\]

and then expand the nonlinearity into pieces $P_{[k,k+2]}v^j$ and commute with $(\partial_t + P_{\leq k}v \cdot \nabla)$ to obtain (147).

Instead, we proceed directly, by observing that we can write the operator $(\partial_t + P_{\leq k}v \cdot \nabla)P_{\leq k}$ appearing in $(\partial_t + P_{\leq k}v \cdot \nabla)R_{\leq k,HH}^I$ as

\[
(\partial_t + P_{\leq k}v \cdot \nabla)P_{\leq k} = P_{\leq k+2}(\partial_t + P_{\leq k}v \cdot \nabla)P_{\leq k} (164)
\]

for $I \geq k$ using the bandlimited property of Littlewood-Paley projections. With this representation the bounds for the derivatives $D > 0$ in (147) follow from the $D = 0$ case as the spatial derivatives fall on the operator $P_{\leq k+2}$.

In fact, as we will see later, it is helpful to observe some further cancellation using the bandlimited property. Namely, following (123) we expand

\[
(\partial_t + P_{\leq k}v \cdot \nabla)R_{\leq k,HH}^I = P_{\leq k+2}(\partial_t + P_{\leq k}v \cdot \nabla)P_{\leq k}[(v^j - P_{\leq k}v^j)(v^j - P_{\leq k}v^j)] (165)
\]

\[
= \sum_{I \geq k} P_{\leq k+2}[(\partial_t + P_{\leq k}v \cdot \nabla) - P_{[k,I]}v \cdot \nabla]P_{\leq k}(P_{I}v^j P_{\leq k}v^j) (166)
\]

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The bandlimited property of the Littlewood-Paley projections allows us to throw out the high frequency components in the term \( P_{[k,t]} v \cdot \nabla \) and obtain

\[
(\partial_t + P_{\leq k} v \cdot \nabla) R_{[k,t],HH}^I = P_{\leq k+2} ((\partial_t + P_{\leq k} v \cdot \nabla) P_{\leq k} [(v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l)]) = \sum_{I \geq k} P_{\leq k+2} [(\partial_t + P_{\leq I} v \cdot \nabla) - P_{[k,k+1]} v \cdot \nabla)] P_{\leq k} (P_I v^j P_{\leq I} v^l)
\]

The technique we have employed here to eliminate the high frequency components in the term \( P_{[k,t]} v \cdot \nabla \) is fairly subtle, but this cancellation will be crucial when estimate further advective derivatives of this term later on in Section 7.9.

The bounds in (147) are now immediate from the form (108). As in the proof of Proposition (5.1), the most dangerous terms come from the operator \((\partial_t + P_{\leq I} v \cdot \nabla)\), which always costs a factor of \(2^{(1-\alpha)I}\|v\|_{C^1_t C^2_x}\) in the estimate regardless of whether it falls on \(P_{\leq I} v\), or whether it falls on the operator \(P_{\leq k}\), giving rise to a commutator \([(\partial_t + P_{\leq I} v \cdot \nabla), P_{\leq k}]\) of norm

\[
\sup_t \|[(\partial_t + P_{\leq I} v \cdot \nabla), P_{\leq k}]\| \leq C 2^{(1-\alpha)I}\|v\|_{C^1_t C^2_x}.
\]

The main advantage of the form (108) is that this form facilitates commutator estimates for taking higher order material derivatives.

As an immediate corollary to the estimates (116), (128) and (147), we are able to estimate second order material derivatives of Littlewood-Paley projections of the velocity.

**Proposition 5.5.** If \(1/3 < \alpha < 1\) and \((v,p)\) solve the incompressible Euler equations, then

\[
\| \nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) P_{k+1} v \|_{C^0} \leq C_D 2^{D+(1-\alpha)k} \|v\|_{C^1_t C^2_x}^2 \tag{169}
\]

\[
\| \nabla^D (\partial_t + P_{\leq k} v \cdot \nabla)^2 P_{k+1} v \|_{C^0} \leq C_D 2^{D+2(1-\alpha)k} \|v\|_{C^1_t C^2_x}^3 \tag{170}
\]

\[
\| \nabla^D (\partial_t + P_{\leq k} v \cdot \nabla) \nabla P_{\leq k} v \|_{C^0} \leq C_{D,T} 2^{D+3(1-\alpha)k} \|v\|_{C^1_t C^2_x}^3 \tag{171}
\]

\[
\| \nabla^{D+1} (\partial_t + P_{\leq k} v \cdot \nabla) P_{\leq k} v \|_{C^0} \leq C_{D,T} 2^{D+2(1-\alpha)k} \|v\|_{C^1_t C^2_x}^2 \tag{172}
\]

\[
\| \nabla^{D+1} (\partial_t + P_{\leq k} v \cdot \nabla)^2 P_{\leq k} v \|_{C^0} \leq C_{D,T} 2^{D+3(1-\alpha)k} \|v\|_{C^1_t C^2_x}^3 \tag{173}
\]

**Proof.** The bounds (169), (171) and (173) were already proven with no restriction on \(\alpha\) by applying the basic estimates for Littlewood-Paley pieces of the pressure and the Reynolds stress to the equations

\[
(\partial_t + P_{\leq k} v^j \partial_j) [P_{\leq k} v^l] + \partial^l P_{\leq k} \rho = \partial_t R_{\leq k}^{jl} \tag{175}
\]

\[
(\partial_t + P_{\leq k} v^j \partial_j) [\partial_a P_{\leq k} v^l] + \partial^l \partial_a P_{\leq k} v^j + \partial^l \partial^j P_{\leq k} \rho = \partial_a \partial_j R_{\leq k}^{jl} \tag{176}
\]

\[
(\partial_t + P_{\leq k} v^j \partial_j) [P_{k+1} v^l] + P_{k+1} v^j \partial_j P_{\leq k+1} v^l + \partial^l P_{k+1} v^j = \partial_j (R_{\leq k+1}^{jl} - R_{\leq k}^{jl}) \tag{177}
\]

The remaining estimates (170) and (172) and the equivalent bounds (172) and (174) follow by applying the operators \((\partial_t + P_{\leq k} v^j \partial_j)\) and \(\nabla^D\) to equations (175) - (177). The bounds (110), (128) and (147) together with the basic estimates of Proposition (2.2) imply that the derivative \((\partial_t + P_{\leq k} v^j \partial_j)\) always costs a factor

\[
|[(\partial_t + P_{\leq k} v \cdot \nabla)]| \leq C 2^{(1-\alpha)k}\|v\|_{C^1_t C^2_x}.
\]

in the estimates provided \(\alpha > 1/3\).  

Later on we will generalize the estimates (116), (128), (147) and Proposition (5.5) to higher order material derivatives, but first we will study how the estimates obtained so far can be used to prove some higher order regularity in time for the pressure and the velocity.
6 Second material derivatives of the pressure increments and applications

Here we show that the estimates of Section 5 can be applied to give higher regularity in time for the pressure and the pressure gradient. Our focus will be on proving the following theorems

Theorem 6.1. For $1/2 < \alpha < 1$, we have that

$$(\partial_t p, \nabla p) \in C^\beta_{t,x}$$

for all $\beta < 2\alpha - 1$.

If $2/3 < \alpha < 1$, then $D_{t} \nabla p = \partial_t \nabla p + \text{div} (v \otimes \nabla p)$, which is well-defined as a distribution by (178), is also Hölder continuous, with

$$D_t \nabla p \in C^\beta_{t,x}$$

for all $\beta < 3\alpha - 2$. If $3/4 < \alpha < 1$, we have furthermore that

$$D_{t}^2 \nabla p = \partial_t D_t \nabla p + \partial_j (v^j D_t \nabla p) \in C^0$$

We will also prove a regularity theorem for $D_t^2 p$ and $D_{t}^2 \nabla p$ as well.

The proof of Theorem (6.1), which is contained in Section 6.2 below, proceeds by estimating first and second material derivatives of the pressure increments defined in Section 4. In order to establish the necessary estimates, we will start by proving some preliminary bounds for higher order commutators between material derivatives and the relevant convolution operators.

At this point almost all of the constants depend on the torus $\mathbb{T}^n$, so for simplicity we will no longer keep track of this dependence.

6.1 Second order commutator estimates for material derivatives and convolution operators

In this Section, we show how our bounds from the Euler equations can be used to bound the operator norms of second and third order commutators between coarse scale material derivatives and convolution operators.

We start by introducing some further notation. We will continue to use the notation

$$D_{t}^k = (\partial_t + P_{<k} v \cdot \nabla)$$

and

$$D_{t}^2 = (\partial_t + P_{<k} v \cdot \nabla)^2.$$ 

The expression $[X,]_r T$ will denote operator obtained by commuting $T$ with $X$ repeatedly $r$ times (e.g. $[X,]_2 T = [X, [X, T]]$).

The following proposition describes the general estimate one has available for commutators of coarse scale material derivatives with operators of convolution form. A main example to keep in mind is the operator $T = \Delta^{-1} \nabla^2 P_{<k}$ which appears in the definition of the pressure increments introduced in Section 4.
Proposition 6.1 (Commuting material derivatives and smoothing operators). Suppose that \((v,p)\) solve the incompressible Euler equations, \(0 < \alpha < 1\). Suppose that \(T\) takes the form
\[
Tf = \int_{\mathbb{R}^n} f(x + h)K(h)dh
\]
and that the kernel \(K\) satisfies the estimates
\[
\|\nabla^D K\|_{L^1_h} + \|h|\nabla^{1+D} K\|_{L^1_h} + \|h^2 \nabla^{2+D} K\|_{L^1_h} \leq 2^{Dk}
\] (181)
for all \(0 \leq D \leq M\). Then we have the estimates
\[
\sup_t \|\nabla^D \left[ \frac{D_{\leq I}}{\partial t} K^* \right] T \| \leq C_D 2^{D \max(k,l)} . 2^{r(1-\alpha)l} \|v\|_{C^0_{x,t}}^r
\] (182)
for all \(0 \leq D \leq M\) and all \(0 \leq r \leq 2\).

If \(\alpha > 1/3\) and we also assume
\[
\|h^3 \nabla^{3+D} K\|_{L^1_h} \leq 2^{Dk}
\] (183)
then (182) holds as well for \(r = 3\).

We remark that there is actually no need to invoke the Euler equations for the cases \(r = 0,1\) in Proposition 6.1.

Proof. First recall the basic commutator formulas
\[
\left[ \frac{D_{\leq I}}{\partial t}, K^* \right] f(x) = \int_{\mathbb{R}^n} (P_{\leq I} v^i(x + h) - P_{\leq I} v^i(x)) f(x + h) \partial_i K(h) dh
\] (184)
\[
= \int_0^1 \int_{\mathbb{R}^n} \partial_a P_{\leq I} v^i(x + sh) f(x + h) \partial_i K(h) h^a dh ds
\] (185)
from (90) obtained using integration by parts. Also recall that these formulas have been simplified using the fact that \(P_{\leq I} v\) is divergence free, but the extra term which would arise otherwise would in any case have a similar form and obey the same bounds.

The bounds (217)-(219) for \(r = 0,1\) are then immediate from the form (180) without using the Euler equations. When estimating \(\|\nabla^D \left[ \frac{D_{\leq I}}{\partial t}, K^* \right] T\|\) we do not let the derivatives hit the function \(f\), but rather integrate by parts using the form (185).

We will now commute again with the operator \(\frac{D_{\leq I}}{\partial t} = (\partial_t + P_{\leq I} v \cdot \nabla)\). Here it is important to use the form (184) rather than (185), since we do not always have the control we would require over \(\|h^2 \nabla K\|_{L^1}\) when \(K\) is a long range kernel such as \(\Delta^{-1} \nabla^2 \eta_{\leq k}\). On the other hand, it is safe to absorb extra powers of \(h\) onto the kernel when \(K\) is short range, such as \(\Delta^{-1} \nabla^2 \eta_k\).

We express the second commutator with \(\frac{D_{\leq I}}{\partial t} = (\partial_t + P_{\leq I} v \cdot \nabla)\) as follows
\[
\left[ \frac{D_{\leq I}}{\partial t}, g \right] K^* [f](x) = (\partial_t + P_{\leq I} v^i(t, x) \partial_{x^i}) \int_{\mathbb{R}^n} (P_{\leq I} v^{i_1}(x + h) - P_{\leq I} v^{i_1}(x)) f(t, x + h) \partial_{i_1} K(h) dh
\] (186)
\[
- \int_{\mathbb{R}^n} (P_{\leq I} v^{i_1}(x + h) - P_{\leq I} v^{i_1}(x)) \left( \partial_t + P_{\leq I} v^{i_2}(t, x + h) \partial_{x^{i_2}} \right) f(t, x + h) \partial_{i_1} K(h) dh
\] (187)

\footnote{However, this remark does allow us to remove the term \(\|\nabla^D K\|_{L^1_h}\) in the assumption (181) when \(r \geq 1\).}

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The resulting operator acts on \( f \) only in the spatial variables, and only on a fixed time slice. A full expansion of (180)-(187) gives

\[
\left[ \frac{D_{\leq I}}{\partial t} \right]^2 K * [f](x) = T_{(I)}[f] - T_{(II)}[f] - T_{(III)}[f]
\]

\[
T_{(I)}[f] = \int_{\mathbb{R}^n} \left( \frac{D_{\leq I}}{\partial t} P_{\leq I} v^i(t, x + h) P_{\leq I} v^i(t, x) \right) f(t, x + h) \partial_i K(h)dh
\]

\[
T_{(II)}[f] = \int_{\mathbb{R}^n} \left( P_{\leq I} v^i(t, x + h) - P_{\leq I} v^i(t, x) \right) \partial_i P_{\leq I} v^i(t, x + h) f(t, x + h) \partial_i K(h)dh
\]

\[
T_{(III)}[f] = \int_{\mathbb{R}^n} \left( P_{\leq I} v^i(x + h) - P_{\leq I} v^i(x) \right) \left( P_{\leq I} v^i(t, x + h) - P_{\leq I} v^i(t, x) \right) \partial_i f(t, x + h) \partial_i K(h)dh
\]

(188)

(189)

(190)

(191)

As we have seen before, the main observation here which confirms that the commutator is indeed a smoothing operator is that we can integrate by parts in the \( h \) variables when the derivative hits the \( f \) in (191) by noticing that \( \partial_i f(t, x + h) = \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial h^i} \). The resulting expression can then be simplified by observing that

\[
\frac{\partial}{\partial h^i} \left( P_{\leq I} v^i(t, x + h) - P_{\leq I} v^i(t, x) \right) = 0
\]

from the fact that \( v \) is divergence free, but this observation is not important for the estimates, since a nonzero term of the same type will appear. Performing this integration by parts gives

\[
T_{(III)}[f] = -T_{(III,1)}[f] - T_{(III,2)}[f]
\]

(192)

\[
T_{(III,1)}[f] = \int_{\mathbb{R}^n} \partial_i P_{\leq I} v^i(t, x + h) \left( P_{\leq I} v^i(t, x + h) - P_{\leq I} v^i(t, x) \right) f(t, x + h) \partial_i K(h)dh
\]

(193)

\[
T_{(III,2)}[f] = \int_{\mathbb{R}^n} \left( P_{\leq I} v^i(x + h) - P_{\leq I} v^i(x) \right) \left( P_{\leq I} v^i(t, x + h) - P_{\leq I} v^i(t, x) \right) f(t, x + h) \partial_i \partial_i K(h)dh
\]

(194)

We are now able to read off the bound (182) using the estimates of Proposition (5.3). The main step here is to apply the fundamental theorem of calculus to every term which has the difference form

\[
\delta_h F(x) = F(x + h) - F(x)
\]

(195)

and then absorb the factor of \( h \) into the kernel. For example, we have

\[
T_{(I)}[f] = \int_{\mathbb{R}^n} \delta_h \frac{D_{\leq I}}{\partial t} P_{\leq I} v^i(t, x) f(t, x + h) \partial_i K(h)dh
\]

(196)

\[
= \int_0^1 \int_{\mathbb{R}^n} \partial_i \delta_h \frac{D_{\leq I}}{\partial t} P_{\leq I} v^i(t, x + sh) f(t, x + h) \partial_i K(h)h^a \partial_i dhds
\]

(197)

and

\[
T_{(III,2)}[f] = \int_{\mathbb{R}^n} \delta_h P_{\leq I} v^i(t, x) \delta_h P_{\leq I} v^i(t, x) f(t, x + h) \partial_i \partial_i K(h)dh
\]

(198)

\[
= \int_0^1 \int_0^1 \partial_i \delta_h P_{\leq I} v^i(t, x + sh) \partial_i \partial_i P_{\leq I} v^i(t, x + sh) f(t, x + h) \partial_i \partial_i K(h)h^a h^b dhds
\]

(199)
The bounds (182) now follow from Proposition (155) and the bounds (181) assumed for K. As one would expect, to bound derivatives \( \nabla^D \left[ \frac{D_{\leq t}}{dt} \right]^2 K * [f] \) one must always integrate by parts in the \( h \) variable when the derivative hits the function \( f(t, x + h) \).

Now that we have obtained a good expansion

\[
\left[ \frac{D_{\leq t}}{dt} \right]^2 K * [f](x) = T_{(I)} - T_{(II)} + T_{(III,1)} + T_{(III,2)}
\]

(200)

from (189), (190) and (192), it is worthwhile to observe for future applications that the structure of the commutator survives to allow estimates for higher order commutators after introducing one more trick.

In what follows, we will often suppress the dependence in \( t \) of all the terms; however, every tensor field that appears besides the kernel \( K \) depends on time.

Most of the terms which arise in the expansion of \( \left[ \frac{D_{\leq t}}{dt} \right] K * [f](x) \) are estimated by techniques we have already used for the first two commutators. We will focus on the term

\[
T_{(II)} = \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x) \partial_{i_3} f(t, x + h) \partial_{i_3} K(h) dh
\]

from (190) since this term requires one additional trick to estimate, while the other terms are treated similarly.

To begin, we expand

\[
\left[ \frac{D_{\leq t}}{dt} \right] T_{(I)}[f](x) = \left( \partial_t + P_{\leq 1} t \partial_{x_3} (t, x) \frac{\partial}{\partial x_3} \right) \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) f(t, x + h) \partial_{i_3} K(h) dh
\]

\[
- \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x) \partial_{i_3} f(t, x + h) \frac{D_{\leq t}}{dt} \partial_{i_3} K(h) dh
\]

\[
= T_{(II, A)} + T_{(II, B)}
\]

(201)

\[
T_{(II, A)} = \int_{\mathbb{R}^n} \left( \partial_t + P_{\leq 1} t \partial_{x_3} (t, x) \frac{\partial}{\partial x_3} \right) \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) f(t, x + h) \partial_{i_3} K(h) dh
\]

(202)

\[
T_{(II, B)} = \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x) (\partial_t + P_{\leq 1} t \partial_{x_3} (t, x) \frac{\partial}{\partial x_3}) \left[ \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) f(t, x + h) \right] \partial_{i_3} K(h) dh
\]

\[
- \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) \partial_{i_3} f(t, x + h) \frac{D_{\leq t}}{dt} \partial_{i_3} K(h) dh
\]

(203)

(204)

Whenever we encounter a term of the form \( \delta_h F(x) = F(x + h) - F(x) \) as in (203), we always commute the material derivative and the difference operator \( \delta_h \) as in

\[
(\partial_t + P_{\leq 1} t \partial_t) \delta_h F(x) = \delta_h [(\partial_t + P_{\leq 1} t \nabla) F](x) - \delta_h P_{\leq 1} t \partial_t (x + h) \partial_t F(x + h)
\]

(205)

For term (203), this operation gives rise to two more terms

\[
T_{(II, A)} = T_{(II, A, 1)} - T_{(II, A, 2)}
\]

(206)

\[
T_{(II, A, 1)} = \int_{\mathbb{R}^n} \delta_h \frac{D_{\leq t}}{dt} \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) f(t, x + h) \partial_{i_3} K(h) dh
\]

(207)

\[
T_{(II, A, 2)} = \int_{\mathbb{R}^n} \delta_h P_{\leq 1} t \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) \partial_{i_3} P_{\leq 1} t \partial_{i_3} (t, x + h) f(t, x + h) \partial_{i_3} K(h) dh
\]

(208)

which are controlled by Proposition (155).
For the term $T_{(I,I,B)}$ there are derivatives on the function $f(t, x + h)$, so it will be necessary to integrate by parts in order to control this term. It is important to be careful how this integration by parts is executed, since a naive application of the product rule in (204) will lead to terms such as

$$\int_{\mathbb{R}^n} \delta_h v^{i_1}(t, x) \delta_h v^{i_3}(t, x) \partial_{i_3} \partial_{i_2} \int_{I} P_{\leq I} v^{i_1}(t, x + h) f(t, x + h) \partial_{i_1} K(h) dh$$

which cannot be controlled for long-range kernels, since we have not assumed control of $\|\| \partial_t \nabla L \|_{L^1}$. To avoid seeing such terms, we expand $T_{(I,I,B)}[f]$ in a way that keeps the product term

$$\partial_{i_2} P_{\leq I} v^{i_1}(t, x + h) f(t, x + h)$$
in tact

$$T_{(I,I,B)} = -T_{(I,I,B1)} + T_{(I,I,B2)}$$

$T_{(I,I,B1)} = \int_{\mathbb{R}^n} \delta_h P_{\leq I} v^{i_2}(x) \delta_h P_{\leq I} v^{i_3}(x) \partial_{i_3} \int_{I} P_{\leq I} v^{i_1}(x + h) f(x + h) \partial_{i_1} K(h) dh$

$T_{(I,I,B2)} = \int_{\mathbb{R}^n} \delta_h P_{\leq I} v^{i_2}(x) \left[ \frac{D}{\partial t} \left[ \partial_{i_2} P_{\leq I} v^{i_1}(x + h) f(x + h) \right] - \partial_{i_2} P_{\leq I} v^{i_1}(x + h) \frac{D}{\partial h} f(x + h) \right] K(h) dh$

The term (212) is under control by the bounds of Proposition (6.6). For the term (210), we observe that the derivative $\partial_{i_3} = \frac{\partial}{\partial x_{i_3}} = \frac{\partial}{\partial h}$ can be viewed as a derivative in the $h$ variables, which allows us to integrate by parts in order to control this term. It is important to be careful how this integration by parts is executed, since a naive application of the product rule in (204) will lead to terms such as

$$\int_{\mathbb{R}^n} \delta_h v^{i_1}(t, x) \delta_h v^{i_3}(t, x) \partial_{i_3} \partial_{i_2} \int_{I} P_{\leq I} v^{i_1}(t, x + h) f(t, x + h) \partial_{i_1} K(h) dh$$

The estimates for the terms $\left[ \frac{D}{\partial t}, T_{(I)} \right] [f]$ and $\left[ \frac{D}{\partial h}, T_{(III)} \right] [f]$ from the decomposition (188) involve the same techniques, but also use additional assumptions. The assumption that $\alpha > 1/3$ comes into play in order to estimate the second material derivative in the term

$$\int_{\mathbb{R}^n} \delta_h \frac{D}{\partial t} P_{\leq I} v^{i_1}(x) f(x + h) \partial_{i_1} K(h) dh$$

which arises in the expansion of $\left[ \frac{D}{\partial t}, T_{(I)} \right] [f]$ after commuting the advective derivative with $\delta_h$. The assumption (183) on $\|h^3 \nabla^3 K\|_{L^1}$ comes into play in order to estimate the third derivative of the kernel in the term

$$\int_{\mathbb{R}^n} \delta_h P_{\leq I} v^{i_1}(x) \delta_h P_{\leq I} v^{i_2}(x) \delta_h P_{\leq I} v^{i_3}(x) f(x + h) \partial_{i_3} \partial_{i_2} \partial_{i_1} K(h) dh$$

which arises in the expansion of $\left[ \frac{D}{\partial h}, T_{(III,2)} \right] [f]$ in (199) after integrating by parts. \qed
As a corollary of Proposition 6.1 we have the following commutator estimates:

**Lemma 6.1.** Under the assumptions and notation of Proposition (6.1), we have the estimates

\[
\sup_t \left\| \nabla^D \left[ \frac{D \leq t}{\partial t} \right] (\Delta^{-1} \nabla^A P_k) \right\| \leq C_{D,A} 2^{(D+\tau(1-\alpha))} 2^{(A-2)} k^{2r} \|v\|_{C_tC_r^2} \tag{217}
\]

\[
\sup_t \left\| \nabla^D \left[ \frac{D \leq t}{\partial t} \right] (\Delta^A P_{\leq k}) \right\| \leq C_{D,A} 2^{(D+\tau(1-\alpha))} 2^{A} k^{2} \|v\|_{C_tC_r^2} \tag{218}
\]

for \( r = 0, 1, 2, \) all integers \( k \) and all \( I \geq k \). Also, for \( k \geq k_0(\mathbb{T}^n) \) we have

\[
\sup_t \left\| \nabla^D \left[ \frac{D \leq t}{\partial t} \right] (\Delta^{-1} \nabla^{2+A} P_{\leq k}) \right\| \leq C_{D,A} (1 + |k - k_0(\mathbb{T}^n)|) 2^{(D+\tau(1-\alpha))} 2^{A} k^{2} \|v\|_{C_tC_r^2} \tag{219}
\]

If \( 1/3 < \alpha < 1 \), all the above estimates hold as well for \( r = 3 \). The estimate also holds with a different constant if \( P_{\leq k} \) is replaced by a comparable projection \( P_{\leq k} = P_{\leq k+a} \) provided that \(|a|\) is bounded. Similarly, one can replace \( P_k \) in (217) with any comparable operator

\[
P_{\leq k} = P_{[k,k_2]}
\]

provided \(|k_1 - k|\) and \(|k_2 - k|\) are bounded.

These estimates all follow from Proposition (6.1) after multiplying the operators by the appropriate constant. For example, Proposition (6.1) applies to \( T = C^{-1} (1 + |k - k_0(\mathbb{T}^n)|)^{-1} \Delta^{-1} \nabla^2 P_{\leq k} \) and \( T = C^{-1} 2^{(2-A)k} \Delta^{-1} \nabla^A P_k \) if \( C \) is sufficiently large.

### 6.2 Second order material derivatives of pressure increments and regularity in time for the pressure gradient

With the necessary commutator estimates in hand, we now begin the proof of Theorem (6.1). As in the proof of Corollary (4.1), the proof will proceed by considering the pressure increments defined in Section 4. From this point onward we will no longer record the dependence of the constants on \( \mathbb{T}^n \) and the fixed \( \alpha < 1 \).

The main Lemma which enables us to access higher order advective derivatives despite being unable to differentiate the velocity field itself is the following fact.

**Lemma 6.2.** Suppose \( T \) is a continuous tensor field on \( I \times \mathbb{T}^n \) and \( v \) is a continuous vector field on \( I \times \mathbb{T}^n \). Then if \( v_{(k)} \rightarrow v \) and \( T_{(k)} \rightarrow T \) uniformly as \( k \rightarrow \infty \), then we have weak convergence

\[
\partial_t T_{(k)} + \partial_j (v^{(k)}T_{(k)}) \rightarrow \partial_t T + \partial_j (v^{j}T) \tag{220}
\]

in the sense of distributions.

Our strategy for proving regularity for higher order material derivatives such as \( \partial_t \nabla p + \partial_j (v^j \nabla p) = \frac{D^2 v}{\partial t^2} \) will be to first apply Lemma (6.2) with \( v_{(k)} = P_{\leq k}v \) and, say, \( T_{(k)} = \nabla p_{(k)} \rightarrow \nabla p \), and then to upgrade the weak convergence in (220) to convergence in Hölder spaces.

We start with some preliminary estimates for the pressure increments.

**Proposition 6.2.** Let \( \delta p_{(k)} \) and \( p_{(k)} = \sum_{i=k_0(\mathbb{T}^n)}^{k} \delta p_{(i)} \) be defined as in Section 4. If \( 1/3 < \alpha < 1 \) and \( 0 \leq r \leq 2 \), the following bounds hold

\[
\left\| \nabla^D \frac{D^{\leq k}}{\partial t^r} \delta p_{(k)} \right\|_{C^0} \leq C_{D} (1 + |k - k_0(\mathbb{T}^n)|) 2^{(D+r(1-\alpha)-2\alpha)k} \|v\|_{C_tC_r^{2+r}} \tag{221}
\]

\[
\left\| \nabla^D \frac{D^{\leq k}}{\partial t^r} \nabla^2 p_{(k)} \right\|_{C^0} \leq C_{D} (1 + |k - k_0(\mathbb{T}^n)|) 2^{(D+2r(1-\alpha)-2\alpha)k} \|v\|_{C_tC_r^{2+r}} \tag{222}
\]

\[
\left\| \nabla^{D+1} \frac{D^{\leq k}}{\partial t^r} \nabla p_{(k)} \right\|_{C^0} \leq C_{D} (1 + |k - k_0(\mathbb{T}^n)|) 2^{(D+2r(1-\alpha)-2\alpha)k} \|v\|_{C_tC_r^{2+r}} \tag{223}
\]

\[\text{34}\]
Proof. The bounds (221) and (222) stated in Proposition (6.2) for \( r = 0, 1 \) were already established in Section (6.1) without any assumptions on \( \alpha \). The estimate (223) equivalent to (222) when \( r = 0 \) and follows from (222) and (223) by induction from the cases \( r = 0, 1 \) after commuting the spatial and material derivatives. It therefore suffices to prove (221) and (222) for \( r = 2 \).

We start with (222) since this quantity involves the least number of terms and suffices to illustrate all the main ideas. We begin by writing

\[
\nabla^2 p(k) = \Delta^{-1} \nabla^2 P_{\leq k}[\partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l] \tag{224}
\]

\[
\frac{D_{\leq k}}{\partial t} \nabla^2 p(k) = \left[ \frac{D_{\leq k}}{\partial t}, \Delta^{-1} \nabla^2 P_{\leq k} \right] \left[ \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \right] \tag{225}
\]

\[
+ \Delta^{-1} \nabla^2 P_{\leq k} \left[ \frac{D_{\leq k}}{\partial t}, v^j \partial_j P_{\leq k} v^l \right] \tag{226}
\]

One should regard the differentiation above as an application of the “product rule” for three terms, where the commutator term (225) is what arises when \( \frac{D_{\leq k}}{\partial t} \) “hits” the operator \( \Delta^{-1} \nabla^2 P_{\leq k} \). Taking a second material derivative gives a representation

\[
\frac{D_{\leq k}^2}{\partial t^2} \nabla^2 p(k) = \left[ \frac{D_{\leq k}}{\partial t}, \left[ \frac{D_{\leq k}}{\partial t}, \Delta^{-1} \nabla^2 P_{\leq k} \right] \right] \left( \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \right) \tag{227}
\]

\[
+ 2 \left[ \frac{D_{\leq k}}{\partial t}, \Delta^{-1} \nabla^2 P_{\leq k} \right] \left( \frac{D_{\leq k}}{\partial t} \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \right) \tag{228}
\]

\[
+ \Delta^{-1} \nabla^2 P_{\leq k} \left( \frac{D_{\leq k}^2}{\partial t^2} \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \right) \tag{229}
\]

Since we have assumed \( \alpha > 1/3 \), Proposition (5.3) now guarantees that the terms

\[
\left\| \frac{D_{\leq k}^2}{\partial t^2} \partial_t P_{\leq k} v^j \partial_j P_{\leq k} v^l \right\|_{C^0} \leq C_D 2^{(r(1-\alpha)+2(1-\alpha))k} \|v\|^{2+r}_{C^0} \tag{230}
\]

obey the expected bounds for \( r = 0, 1, 2 \). The bound (222) follows from the Lemma (6.1), which guarantees estimates for the smoothing operators

\[
\sup_r \left\| \nabla^D \left[ \frac{D_{\leq k}}{\partial t} \right]^r \Delta^{-1} \nabla^2 P_{\leq k} \right\| \leq C_D (1 + |k-k_0(T^n)|) 2^{(D+r(1-\alpha))k} \|v\|^{r}_{C^0} \tag{231}
\]

for \( r = 0, 1, 2 \). Here we have used the notation \( [X,]^r T \) from Section (6.1) to denote operator obtained by commuting \( T \) with \( X \) repeatedly \( r \) times.

The proof of estimate (221) is essentially the same, drawing from the decomposition

\[
\delta p(k) = \delta p(k)_{LL} + \delta p(k)_{HL} + \delta p(k)_{HH} \tag{232}
\]

from (93)–(94), but with two new features. First, there are high frequency terms similar to \( \frac{D_{\leq k}}{\partial t} P_{k+1} v \), which are bounded using Proposition (5.3). There are also operators which project to high frequencies of the type

\[
\nabla^D \left[ \frac{D_{\leq k}}{\partial t} \right]^r \left( \Delta^{-1} \nabla^A P_{\leq k} \right) \tag{233}
\]

with \( r = 0, 1, 2 \) and \( A = 0, 1 \), which are bounded by

\[
\sup_r \left\| \nabla^D \left[ \frac{D_{\leq k}}{\partial t} \right]^r \left( \Delta^{-1} \nabla^A P_{\leq k} \right) \right\| \leq C_D 2^{(D+r(1-\alpha)+A)k} \|v\|^{r}_{C^0} \tag{234}
\]

according to Lemma (6.1). \( \square \)
We now define the frequency increments for the material derivative of the pressure gradient.

\[
\delta(k) \frac{D^{\leq k}}{dt} \nabla p(k) = \frac{D^{\leq k+1}}{dt} \nabla p(k+1) - \frac{D^{\leq k}}{dt} \nabla p(k) = P_{k+1}v \cdot \nabla \nabla p(k+1) + \frac{D^{\leq k}}{dt} \nabla \delta p(k) \quad (233)
\]

According to Lemma (6.2), we have

\[
\partial_t \nabla p + \partial_j (v^j \nabla p) = \sum_{k=k_0(T^n)}^\infty \delta(k) \frac{D^{\leq k}}{dt} \nabla p(k) \quad (235)
\]
as distributions whenever \( \alpha > 1/2 \). Our aim is to prove that the summation converges in the appropriate Hölder norms when \( \alpha > 2/3 \). This convergence will follow from the following estimates, which are an immediate consequence of Proposition (6.2) and the formula (234). The important point to observe is that the low frequency parts of \( p(k) \) always appear with at least two derivatives so that the bounds of Proposition (6.2) apply.

**Corollary 6.1 (Bounds for frequency increments of \( \nabla p \) and \( \frac{D}{dt} \nabla p \)).** If \( 1/3 < \alpha < 1 \), then

\[
\| D^{\leq k} \nabla \delta p(k) \|_{C^0} \leq C_D (1 + |k - k_0(T^n)|) 2^{(D + (1 - \alpha) + (1 - 2\alpha)k)} \| v \|_{C_t^3 C_x^\beta}^3 \quad (236)
\]

\[
\| D^{\leq k} \partial_t \nabla p(k) \|_{C^0} \leq C_D (1 + |k - k_0(T^n)|) 2^{(D + (1 - \alpha) + (1 - 2\alpha)k)} \| v \|_{C_t^3 C_x^\beta}^3 \quad (237)
\]

\[
\| D^{\leq k} \partial_t \nabla p(k) \|_{C^0} \leq C_D (1 + |k - k_0(T^n)|) 2^{(D + (2 - 2\alpha) + (1 - 2\alpha)k)} \| v \|_{C_t^3 C_x^\beta}^4 \quad (238)
\]

We can now prove Theorem (6.1).

**Proof of Theorem (6.1).** From Corollary (6.1) we can interpolate with the estimate

\[
\partial_t \nabla p(k) = (\partial_t + P_{\leq k} v \cdot \nabla) \nabla \delta p(k) - P_{\leq k} v \cdot \nabla \nabla \delta p(k) \quad (239)
\]

\[
\Rightarrow \| \partial_t \nabla p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|) 2^{(1 - \alpha)k} \| v \|_{C_t^3 C_x^\beta} \| v \|_{C_t^3 C_x^\beta}^2 \quad (240)
\]
to obtain

\[
\| \nabla \delta p(k) \|_{C_t^\alpha C_x^\beta} \leq C(1 + |k - k_0(T^n)|) 2^{\beta k} (1 + \| v \|_{C_t^3 C_x^\beta})^\beta 2^{(1 - 2\alpha)k} \| v \|_{C_t^3 C_x^\beta}^2 \quad (241)
\]

which implies that \( \nabla p \in C_t^{\beta} C_x^\beta \) for \( \beta < 2\alpha - 1 \) whenever \( 1/2 < \alpha < 1 \).

Similarly, interopolating the bounds in Corollary (6.1) for \( \alpha > 2/3 \) also yields

\[
\| D^{\leq k} \nabla p(k) \|_{C_t^{\beta} C_x^\beta} \leq C(1 + |k - k_0(T^n)|) 2^{\beta k} (1 + \| v \|_{C_t^3 C_x^\beta})^\beta 2^{(2 - 3\alpha)k} \| v \|_{C_t^3 C_x^\beta} \quad (242)
\]

from which it follows that \( \partial_t \nabla p + \partial_j (v^j \nabla p) \in C_t^{\beta} C_x^\beta \) for all \( \beta < 3\alpha - 2 \) provided \( \alpha > 2/3 \).

To establish Hölder regularity for the pressure, we can use the formula

\[
\partial_t^2 \delta p(k) = \partial_t [(\partial_t + P_{\leq k} v \cdot \nabla) \delta p(k) - P_{\leq k} v \cdot \nabla \delta p(k)] \quad (243)
\]

\[
= (\partial_t + P_{\leq k} v \cdot \nabla)^2 \delta p(k) - P_{\leq k} v \cdot \nabla (\partial_t + P_{\leq k} v \cdot \nabla) \delta p(k) \quad (244)
\]

\[
- (\partial_t + P_{\leq k} v \cdot \nabla) [P_{\leq k} v \cdot \nabla \delta p(k)] + P_{\leq k} v \cdot \nabla [P_{\leq k} v \cdot \nabla \delta p(k)] \quad (245)
\]

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which implies
\[ \| \partial^2_t \delta p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|) \| v \|^2_{C_{t,x}^\alpha} 2^{(2-2\alpha)k} \| v \|^2_{C_{t,x}^\alpha} \] (246)

Together with the estimate (240), we have
\[ \| \partial_t \delta p(k) \|_{C^\beta_{t,x}} \leq C(1 + |k - k_0(T^n)|)(1 + \| v \|_{C_{t,x}^\alpha})^\beta 2^{k} \cdot 2^{(1-2\alpha)k} \| v \|_{C_{t,x}^\alpha} \] (247)

and hence \( \partial_t p \in C^\beta_{t,x} \) for all \( \beta < 2\alpha - 1 \) when \( \alpha > 1/2 \).

The method above also applies to the frequency increments for the second order material derivative of \( \nabla p \), which are defined as
\[
\delta(k) \frac{D^2_z}{\partial t^2} \nabla p(k) = \frac{D^2_z}{\partial t^2} \nabla p(k+1) - \frac{D^2_z}{\partial t^2} \nabla p(k) = \frac{D^2_z}{\partial t^2} \nabla p(k) + \left( \frac{D^2_z}{\partial t^2} - \frac{D^2_z}{\partial t^2} \right) \nabla p(k) + \frac{D^2_z}{\partial t^2} \nabla p(k+1) + \frac{D^2_z}{\partial t^2} \nabla p(k) + \frac{D^2_z}{\partial t^2} \nabla p(k+1)
\] (248)

From the bounds
\[ \| \delta(k) \frac{D^2_z}{\partial t^2} \nabla p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|) \| v \|^2_{C_{t,x}^\alpha} \] (249)

we see that \( \frac{D^2_z}{\partial t^2} \nabla p \in C^0 \) if \( 3/4 < \alpha \). These observations together suffice for the proof of Theorem (6.1).

The bounds also give some H"older regularity in space for \( \frac{D^2_z}{\partial t^2} \nabla p \), but proving convergence in \( C^\beta_{t,x} \) will require higher order estimates.

Before moving on to establish the general higher order estimates for material derivatives, we examine the regularity that can be established already for the pressure itself. The regularity for the pressure appears to be slightly more subtle than the regularity for the velocity field and pressure gradient stated in Theorem (6.1).

## 6.3 Regularity in time for the pressure

Using the methods in Section (6.2) and a few additional bounds, we can also establish the following regularity results for the pressure and its material derivatives.

Here we use the notation
\[ (x)_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \]

**Theorem 6.2.** If \( 1/3 < \alpha < 1 \), then the distribution \( \frac{D}{\partial t} p = \partial_t p + \partial_j (p v^j) \) belongs to
\[ \frac{D}{\partial t} p \in C^\beta_{t,x} \] (252)

for all \( 0 \leq \beta < (1 - \alpha) + (1 - 2\alpha)^+ \).

If \( 1/2 < \alpha < 1 \), we also have \( \frac{\partial_t^2 p}{\partial t^2} = \partial_t \frac{D}{\partial t} p + \partial_j (v^j \frac{D}{\partial t^2} p) \in C^0 \).
Lemma 6.4. Define the frequency increments

\[ \Delta(k) \frac{D}{dt} p(k) = \frac{D}{dt} p(k+1) - \frac{D}{dt} p(k) \]

and, following (257),

\[ \Delta^{2}(k) \frac{D^{2}}{dt^{2}} p(k) = \frac{D^{2}}{dt^{2}} p(k+1) - \frac{D^{2}}{dt^{2}} p(k) \]

\[ = \delta^{2}(k) \frac{D^{2}}{dt^{2}} p(k) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \]

\[ = \delta^{2}(k) \frac{D^{2}}{dt^{2}} p(k) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]

and, following (257),

\[ = \delta^{2}(k) \frac{D^{2}}{dt^{2}} p(k) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]

Observe that here we are unable to show that \( \nabla \frac{D}{dt} p \in C^{0} \) even when \( v \in C_{1}C_{x}^{1} \), whereas Theorem (6.1) guarantees that \( \frac{D^{2}}{dt^{2}} p \in C^{0} \) whenever \( \alpha > 2/3 \). We begin the proof of Theorem (6.2) by recalling a few extra preliminary estimates.

Lemma 6.3. If \( \alpha \neq 2/3 \), then

\[ \| \nabla p(k) \|_{C^{0}} \leq C(1 + |k - k_{0}(T^{n})|) 2^{(1-2\alpha)k} \| v \|_{C^{1}C_{2}^{2}}^{2} \]

\[ \| (\partial_{t} + P_{\leq k} v \cdot \nabla) \nabla p(k) \|_{C^{0}} \leq C(1 + |k - k_{0}(T^{n})|) 2^{(2-3\alpha)k} \| v \|_{C^{1}C_{2}^{2}}^{3} \]

The estimates follow from the arguments of Proposition (4.2). The bound (253) is obtained by summing by parts the bounds for the pressure increments

\[ \| \nabla \Delta p(k) \|_{C^{0}} \leq C(1 + |k - k_{0}(T^{n})|) 2^{(1-2\alpha)k} \| v \|_{C^{1}C_{2}^{2}}^{2} \]

from \( I = k_{0}(T^{n}) \) to \( k \). The new point here is that when \( \alpha > 1/2 \), the most we can say is that \( \nabla p(k) \) is bounded, rather than decaying at the rate of \( 2^{(1-2\alpha)k} \| v \|_{C^{1}C_{2}^{2}}^{2} \) that dimensional analysis would suggest.

The same technique above was used to establish (254) in Proposition (4.2) by summing the bounds for

\[ \delta^{(I)} \frac{D}{dt} \nabla p(I) = (\partial_{t} + P_{\leq I+1} v \cdot \nabla) \nabla p(I+1) - (\partial_{t} + P_{\leq I} v \cdot \nabla) \nabla p(I) \]

from \( I = k_{0}(T^{n}) \) to \( k \). There we used an extra summation by parts in \( I \) when \( \alpha < 2/3 \) – when \( \alpha = 2/3 \) the method leads to an extra factor of \( (1 + |k - k_{0}(T^{n})|) \) in the estimate. When \( \alpha > 2/3 \), we have a decaying geometric series, so the main term is the first term, which is bounded (in particular, the factor \( (1 + |k - k_{0}(T^{n})|) \) does not actually appear in this case). The bound (254) has not been used to establish any of the results proven so far, but we will need it for Theorem (6.2).

With these bounds in hand we can estimate the frequency increments for \( \frac{D}{dt} p \) and \( \frac{D^{2}}{dt^{2}} p \).

Lemma 6.4. Define the frequency increments

\[ \delta^{(k)} \frac{D}{dt} p(k) = \frac{D}{dt} p(k+1) - \frac{D}{dt} p(k) \]

\[ \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k) = \frac{D^{2}}{dt^{2}} p(k+1) - \frac{D^{2}}{dt^{2}} p(k) \]

and, following (257),

\[ \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k) = \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]

and, following (257),

\[ \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k) = \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]

and, following (257),

\[ \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k) = \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]

and, following (257),

\[ \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k) = \delta^{(k)} \frac{D^{2}}{dt^{2}} p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \nabla p(k+1) + \frac{D^{2}}{dt^{2}} p(k+1) \cdot \partial_{j} p(k+1) \]
Then we have the estimates
\[ \| \nabla D \delta(k) \leq \partial_t p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|)2^{(D+2)(1-\alpha)-1+(1-2\alpha)+k} \| v \|_{C^1 C^2} \] (259)
\[ \| \nabla D \delta(k) \leq \partial_t p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|)2^{(D+2)(1-\alpha)-1+(1-2\alpha)+k} \| v \|_{C^1 C^2} \] (260)
\[ \| \nabla D \delta(k) \leq \partial_t p(k) \|_{C^0} \leq C(1 + |k - k_0(T^n)|)2^{(D+2)(1-\alpha)-1+(1-2\alpha)+k} \| v \|_{C^1 C^2} \] (261)

Lemma 6.4 follows by applying the bounds in Proposition 6.2 and Lemma 6.3 to the formulas (256) and (258). In every case, the dominant terms are the ones with pressure gradients that are not differentiated, where we apply the bound
\[ \| \nabla p(k) \|_{C^0} \leq (1 + |k - k_0(T^n)|)2^{(1-2\alpha)-1+(1-2\alpha)+k} \| v \|_{C^1 C^2} \]

Theorem 6.2 now follows from Lemma 6.4 by interpolation as in the arguments of Section 6.2.

Having proven Theorems 6.2 and 6.1 we now move on to the proof of the general Theorems 1.3 and 1.4 which require estimates for higher order material derivatives.

7 Higher order material derivatives

We now begin the proof of Theorems 1.3 and 1.4, which summarize the Hölder regularity of all material derivatives \( D^r \partial_t v \) and \( D^r \partial_t p \) in space and time given that \( v \in C_T C^\alpha_x \). The proof proceeds by generalizing the proof of Theorem 6.1 to allow for higher order advective derivative estimates within a framework that is well-suited for induction.

We start by summarizing the notation we will be using in the rest of the proof (much of which has already been introduced), and by stating some preliminary lemmas.

7.1 Notation and Preliminaries

In this Section we recall some notation that has been introduced during the course of the proof and will be used more heavily in what follows. We also state some algebraic lemmas and conventions that we will follow in the remainder of the proof.

7.2 Algebraic Conventions and Commutator Identities

Let \( X, Y \) and \( Z \) belong to a noncommutative ring of operators. For operators \( Y_1, Y_2, \ldots, Y_n \), we use the notation
\[ \prod_{i=1}^n Y_i = Y_1 Y_2 \cdots Y_{n-1} Y_n \]
to denote the product of the operators taken from left to right. An empty product is equal to 1.

We use the notation
\[ [X, Y] = XY - YX \]
to denote the commutator of \( Y \) with \( X \), and we let
\[ [X,]^r Y \]
denote the operator obtained by commuting \( Y \) with \( X \) repeatedly \( r \) times. For example, \( [X,]^2 Y = [X, [X, Y]] = X(YX - YX) - (XY - YX)X \).

We will often use the following product rule for the commutator
\[ [X,][Y,Z] = [X,[Y,Z]] \]
\[ = (XYZ - YXZ) + (YXZ - YZX) \] (262)
\[ = (XYZ - YXZ) + (YXZ - YZX) \] (263)

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Here and in what follows, we employ the convention that the commutator \([X,\cdot]\) precedes the operator multiplication in the order of operations.

We record here the formula
\[
[X^r, Y] = \sum_{s=1}^{r} \binom{r}{s} [X]^s Y X^{r-s}
\] (264)

Formula (264) can be obtained from (262) by induction on \(r\) using Pascal’s rule and the expression
\[
[X^{r+1}, Y] = -[Y, [XX^r]] = -[Y, X]X^r - X[Y, X^r]
\]
\[
= [X, Y]X^r + [X, [X^r], Y] + [X^r, Y]X.
\]
or by comparing the coefficients of \(t^r\) in the generating function
\[
e^{tX}Y = e^{t[X,\cdot]}Y e^{tX}
\]
The identity \(e^{tX}Y e^{-tX} = e^{t[X,\cdot]}Y\) used above follows from uniqueness of solutions for ODEs.

As a consequence of (264), it is possible to express the power of a sum of noncommutative operators in the form
\[
(X + Y)^n = \sum_{\ell=0}^{n} \sum_{r_1,\ldots,r_\ell,m} C_{r_1,\ldots,r_\ell} \left( \prod_{i=1}^{\ell} [X_i, Y] \right) X^m
\] (265)
for some non-negative integers \(C_{r_1,\ldots,r_\ell}\), where the sum runs over non-negative indices satisfying \(r_1 + \ldots + r_\ell + \ell + m = n\).

In the applications below, we will always take the operator \(X\) in the formulas (264), (265) to be an operator of the form \(D_{\leq k}\) as defined in (267) below.

### 7.3 Coarse scale material derivatives and notation

We denote by \(D_{\leq k}\) the \(r\)-times repeated, coarse scale advective derivative
\[
\frac{D_{\leq k}}{\partial t} = (\partial_t + P_{\leq k}v \cdot \nabla)^r
\] (266)

We denote by \(P_{\leq k}\) any operator of the form
\[
P_{\leq k} = P_{[k_1, k_2]}
\]
for which the differences \(|k_1 - k|\) and \(|k_2 - k|\) are bounded. Thus, operators of the form \(P_{\leq k}\) are supported on a frequency shell \(C^{-1}2^k \leq |\xi| \leq C2^k\), \(\xi \in \mathbb{R}^n\) with \(C\) a constant which will depend only on the number \(\alpha < 1\), which is fixed in the remainder of the proof.

Similarly, we denote by \(P_{\leq k}\) any operator of the form \(P_{\leq k+a}\) where \(|a| \leq C\) for some constant \(C\). Thus, “projections” \(P_{\leq k}\) restrict to frequencies \(|\xi| \leq C2^k\), and the difference between any two such operators has the form
\[
P_{\leq k} - P_{\leq k} = P_{\leq k}
\]

Generalizing (266) we denote by \(D_{\leq k}\) any operator of the form
\[
\frac{D_{\leq k}}{\partial t} = (\partial_t + P_{\leq k}v \cdot \nabla)
\] (267)
7.4 The Main Lemma

The Main Lemma used to establish Theorem \ref{thm1.3} is the following

**Lemma 7.1 (Main Lemma I).** Suppose that \((v, p)\) are solutions to the incompressible Euler equations and fix \(0 < \alpha < 1\). Then for all \(r(1 - \alpha) - 2\alpha < 0\), we have the estimates

\[
\|\nabla^A \frac{D_{\leq k}^s}{\partial t} R_{\leq k} v\|_{C^0} + \|\nabla^A \frac{D_{\leq k}^s}{\partial t} P_{k} p\|_{C^0} \leq C_A 2^{(A + r(1 - \alpha) - 2\alpha)k} \|v\|^{2+r}_{C_t C_x^2} \tag{268}
\]

\[
\|\nabla^A \frac{D_{\leq k}^s}{\partial t} \nabla^2 P_{\leq k} v\|_{C^0} \leq C_A 2^{(A + r(1 - \alpha) + (2 - 2\alpha)k)\|v\|^{2+r}_{C_t C_x^2}} \tag{269}
\]

and for \(0 \leq s \leq r + 1\),

\[
\|\nabla^A \frac{D_{\leq k}^s}{\partial t} P_{\leq k} v\|_{C^0} \leq C_A 2^{(A + s + (1 - \alpha) - \alpha)k} \|v\|^{1+s}_{C_t C_x^2} \tag{270}
\]

\[
\|\nabla^A \frac{D_{\leq k}^s}{\partial t} \nabla P_{\leq k} v\|_{C^0} \leq C_A 2^{(A + s + (1 - \alpha) + (1 - \alpha)k)\|v\|^{1+s}_{C_t C_x^2}} \tag{271}
\]

Furthermore, the vector fields \(Z_s(t, \cdot) : \mathbb{T}^n \to \mathbb{R}^n\) obtained by commuting

\[
\left[ \frac{D_{\leq k}^s}{\partial t}, \right]^s (P_{\leq k} v \cdot \nabla) = Z_s \cdot \nabla \tag{272}
\]

have coefficients \(Z_s(t, x)\) obeying the bounds

\[
\|\nabla^A \frac{D_{\leq k}^s}{\partial t} Z_s\|_{C^0} \leq C_A 2^{(A + (q + s)(1 - \alpha) - \alpha)k} \|v\|^{q+s+1}_{C_t C_x^2} \tag{273}
\]

provided \(q + s \leq r + 1\).

Also, for any operator of the form \(T f(x) = \int f(x + h) K(h) dh\) whose kernel \(K(h)\) satisfies

\[
\|\nabla^A K\|_{L_h^1} + \|h\|^{1 + A} K\|_{L_h^1} + \|h\|^{2r + 2A} K\|_{L_h^1} + \ldots + \|h\|^{r + 2r + 2A} K\|_{L_h^1} \leq 2^{4k} \tag{274}
\]

for \(A = 0, 1, \ldots, M\), we have the commutator estimates

\[
\sup_t \left\| \nabla^A \left[ \frac{D_{\leq k}^s}{\partial t}, \right]^s T \right\| \leq C_A 2^{(A + s + (1 - \alpha) - \alpha)k} \|v\|^{q}_{C_t C_x^2} \tag{275}
\]

for \(0 \leq s \leq r + 2\) and \(A = 0, 1, \ldots, M\).

In the Sections \ref{sec7.5}-\ref{sec7.7} below, we will give the proof of Lemma \ref{lem7.1}. Here we will outline how Lemma \ref{lem7.1} implies Theorems \ref{thm1.3} and \ref{thm1.4} starting with the proof of Theorem \ref{thm1.3}

7.5 Proof of Theorem \ref{thm1.3} on the time regularity of the velocity field

Here we show how Lemma \ref{lem7.1} can be used to establish Theorem \ref{thm1.3}. The key idea is to prove estimates for the following frequency increments for the velocity field and its higher advective derivatives

\[
\delta_{(k)} \frac{D_{\leq k}^s}{\partial t} P_{\leq k} v = \frac{D_{\leq k+1}^s}{\partial t} P_{\leq k+1} v - \frac{D_{\leq k}^s}{\partial t} P_{\leq k} v \tag{276}
\]
These frequency increments are defined so that
\[ \sum_{k=k_0(T^n)}^{\infty} \delta(k) \frac{D^r_{\leq k}}{D^s_{\leq k}} P_{\leq k} v = \frac{D^r}{D^s} v \]  
(277)
when the summation converges uniformly.

Using Lemma 7.1 we obtain the following bounds on the frequency increments defined in (270).

**Lemma 7.2** (Velocity increment bounds). For all \( r(1-\alpha) - 2\alpha < 0 \) and all \( q + s < r + 1 \), the frequency increments defined in (270) satisfy the bounds
\[ \| \nabla^A \frac{D^r_{\leq k}}{D^s_{\leq k}} \delta(k) \frac{D^s_{\leq k}}{D^r_{\leq k}} P_{\leq k} v \|_{C^0} \leq C A 2^{(A+(q+s)(1-\alpha)-\alpha)k} \| v \|^{q+s+1}_{C^0} C^r \]  
(278)

**Proof.** Letting \( r \) be fixed, we proceed by induction on \( s \). For \( s = 0 \), the frequency increments defined in (270) are simply the Littlewood-Paley projections of \( v \), and the estimate (278) follows for all \( q \leq r + 1 \) from the bound (270).

We assume now by induction that (278) has been established for some \( s \geq 0 \) and consider the bound (278) for \( s + 1 \). Observe that we can write each frequency increment for the \( s + 1 \)st advective derivative in the form
\[ \delta(k) \frac{D^r_{\leq k+1}}{D^s_{\leq k+1}} P_{\leq k} v = \frac{D^r_{\leq k+1}}{D^s_{\leq k+1}} P_{\leq k} v - \frac{D^r_{\leq k}}{D^s_{\leq k}} P_{\leq k} v \]  
(279)
\[ = P_{k+1} v \cdot \nabla \frac{D^r_{\leq k+1}}{D^s_{\leq k+1}} P_{\leq k+1} v + \frac{D^r_{\leq k}}{D^s_{\leq k}} P_{\leq k} v \]  
(280)
The first term in (280) and its first \( q \leq r + 1 - s \) advective derivatives can be estimated using Lemma 7.1. These estimates follow by commuting the spatial derivative in \( P_{k+1} v \cdot \nabla \) onto the term \( P_{\leq k} v \) using the formula (264). All of the terms generated by this commutation obey bounds of the form (278) by (270) + (271).

The second term in (280) obeys an estimate of the form (278) by our induction hypothesis, which concludes the proof of Lemma 7.2. \( \square \)

We can now conclude that the series (277) converges in \( C^0 \) for all \( 0 \leq r < \frac{\alpha}{1-\alpha} \) by induction on \( r \) using Lemma 7.2 and the case \( q = 0 \) of the estimate (278). Theorem 1.3 now follows as in the argument of Section 2.3 by using the cases \( A = 1, q = 0 \) and \( A = 0, q = 1 \) of Lemma 278 to bound the first spatial and temporal derivatives of the frequency increments.

### 7.6 Proof of Theorem 1.4 on the time regularity of the pressure

Our proof of Theorem 1.4 on the regularity in time of the pressure will require an analysis of frequency increments for the pressure which generalizes the analysis of Section 6.2.

The most basic estimates on the pressure increments are provided by Lemma 7.3 below, which is deduced from Lemma 6.1 by generalizing the arguments of Proposition 4.1 and Section 6.2.

**Lemma 7.3** (Pressure Increment bounds). Let \( \delta P_{(k)} \) be defined as in Definition 4.1. Then for all \( r(1-\alpha) - 2\alpha < 0 \) and all \( s \leq r + 1 \)
\[ \| \nabla^A \frac{D^r_{\leq k}}{D^s_{\leq k}} \delta P_{(k)} \|_{C^0} \leq C A (1 + |k - k_0(T^n)|) 2^{(A+s(1-\alpha)-2\alpha)k} \| v \|^{2+s}_{C^0} C^r \]  
(281)
\[ \| \nabla^A \frac{D^r_{\leq k}}{D^s_{\leq k}} \delta P_{(k)} \|_{C^0} \leq C A (1 + |k - k_0(T^n)|) 2^{(A+s(1-\alpha)+2-2\alpha)k} \| v \|^{2+s}_{C^0} C^r \]  
(282)
\[ \| \nabla^A \frac{D^r_{\leq k}}{D^s_{\leq k}} \delta P_{(k)} \|_{C^0} \leq C A (1 + |k - k_0(T^n)|) 2^{(A+s(1-\alpha)+2-2\alpha)k} \| v \|^{2+s}_{C^0} C^r \]  
(283)
For example, (282), which implies (283) by commuting, is obtained from the formula
\[
\frac{D_{x}^{q}k}{\partial t^{s}}\nabla^{2}p_{(k)} = \frac{D_{x}^{q}k}{\partial t^{s}}\Delta^{-1}\nabla^{2}P_{\leq k}[\partial_{j}P_{\leq k}v^{i}\partial_{i}P_{\leq k}v^{j}]
\]
\[
= \sum_{q=0}^{s}C_{q}\left[\frac{D_{x}^{q}k}{\partial t}\right]^{q}\Delta^{-1}\nabla^{2}P_{\leq k}\frac{D_{x}^{q}k}{\partial t^{s-q}}(\partial_{j}P_{\leq k}v^{i}\partial_{i}P_{\leq k}v^{j})
\]
(284)
using the rule (261). Lemma (7.1) guarantees that \(s \leq r + 1\) material derivatives of \(\nabla P_{\leq k}v\) obey the desired estimates. The operator \(\left[\frac{D_{x}^{q}k}{\partial t}\right]^{q}\Delta^{-1}\nabla^{2}P_{\leq k}\) is estimated as in the proof of Lemma 6.1 by applying the commutator estimates in (275) to the operator
\[
C^{-1}(1 + |k - k_{0}(\mathbb{T}^{n})|)^{-1}\Delta^{-1}\nabla^{2}P_{\leq k} = C^{-1}(1 + |k - k_{0}(\mathbb{T}^{n})|)^{-1}\sum_{l=0}^{k}\Delta^{-1}\nabla^{2}P_{l}
\]
(286)
Here \(C\) is a universal constant chosen sufficiently large so that the estimate (274) holds for this operator. Such a choice of \(C\) is possible because, by scaling, each term \(C^{-1}\Delta^{-1}\nabla^{2}P_{l}\) in this decomposition satisfies (274) with \(2^{k}\) replaced by \(2^{l}\) once \(C\) is chosen appropriately.

In order to deduce Theorem (1.4) from Lemma (7.1), we must also supplement the bounds in the Lemma (7.5) with the estimate

Lemma 7.4. For all \(r(1 - \alpha) - 2\alpha < 0\) with \(\alpha \neq 1/2\) and all \(q \leq r + 1\), we have
\[
\|\nabla^{A}\frac{D_{x}^{q}k}{\partial t^{s}}\nabla P_{(k)}\|_{C^{0}} \leq C_{A}(1 + |k - k_{0}(\mathbb{T}^{n})|)^{2(A+q(1-\alpha)+(1-2\alpha)+k)}k\|v\|_{C_{1}}C_{2}^{q+2}
\]
(287)
Proof. We proceed by induction on \(q\). For \(q = 0\), we have
\[
\nabla P_{(k)} = \sum_{l=0}^{k}\nabla\delta P_{(l)}
\]
(288)
The estimate (286) is obtained for \(\alpha < 1/2\) by summing by parts the estimate (281); the largest contribution comes from the last terms in the series. For \(\alpha > 1/2\) and \(A = 0\), the main terms come from the beginning of the series, giving (280). The estimate for larger values of \(A\) is already recorded in Proposition (7.3). Assuming the bound (286) for \(q\), we prove the bound for \(q + 1\) by writing
\[
\delta_{(k)}\frac{D_{x}^{q+1}}{\partial t^{s+1}}\nabla P_{(k)} = P_{k+1}v: \nabla\frac{D_{x}^{q+1}}{\partial t^{s+1}}\nabla P_{(k+1)} + \frac{D_{x}^{q}k}{\partial t}\delta_{(k)}\frac{D_{x}^{q+1}}{\partial t^{s+1}}\nabla P_{(k)}
\]
and applying (283), (286) and our induction hypothesis.

We now define the frequency increments for material derivatives of the pressure
\[
\delta_{(k)}\frac{D_{x}^{s}}{\partial t^{s}}p_{(k)} = \frac{D_{x}^{s}}{\partial t^{s}}p_{(k+1)} - \frac{D_{x}^{s}}{\partial t^{s}}p_{(k)}
\]
(289)
Theorem (1.4) will be deduced from Lemma (7.5) below.

Lemma 7.5. For all \(r(1 - \alpha) - 2\alpha < 0\), \(\alpha \neq 1/2\), and all \(s + q \leq r + 1\) with \(s \geq 1\) we have
\[
\|\nabla^{A}\frac{D_{x}^{q}k}{\partial t^{s}}\delta_{(k)}\frac{D_{x}^{s}}{\partial t^{s}}p_{(k)}\|_{C^{0}} \leq C_{A}(1 + |k - k_{0}(\mathbb{T}^{n})|)^{2(A+(s+q)(1-\alpha)+(1-2\alpha)+k)}k\|v\|_{C_{1}}C_{2}^{s+q+s}
\]
(290)
Proof. For \( s = 1 \), we have

\[
\delta_k \frac{D_{<k}}{\partial t} p(k) = P_{k+1} v \cdot \nabla p_{k+1} + \frac{D_{<k}}{\partial t} \delta p(k)
\]  

(291)

So the bound (290) follows from Lemma (7.3) together with (286). Assuming the bound (290) for \( s \), we write

\[
\delta_k \frac{D_{<k}}{\partial t} p(k) = P_{k+1} v \cdot \nabla p_{k+1} + \frac{D_{<k}}{\partial t} \delta p(k)
\]  

(292)

which gives (290) from the induction hypothesis and estimating the first term with (286).

Applying the case \( q = 0 \) of Lemma 7.5 and applying Lemma 6.2 by induction in \( s \), we conclude that the series

\[
\frac{D_{<k}}{\partial t} p = \sum_{k=k_0(T^n)}^{\infty} \delta_k \frac{D_{<k}}{\partial t} p(k)
\]  

(293)

converges uniformly for all \( s < 2\alpha_1 - \alpha \). As in the arguments of Sections 4 and 6.2, we obtain the Hölder regularity in time and space for \( \frac{D_{<k}}{\partial t} \nabla p \) stated in Theorem 1.3 by interpolating the bounds in Lemma 7.5 for first spatial and temporal derivatives (\( A = 1, q = 0 \) and \( A = 0, q = 1 \)) with the case \( q = 0, A = 0 \) to conclude the proof.

### 7.7 Proof of the Main Lemma, Intro

We now turn to the proof of the Main Lemma (7.1). The proof proceeds by induction on \( r \), so we will assume that Lemma (7.1) has been proven for \( r \leq n \), and we will prove that Lemma (7.1) also holds for \( r = n + 1 \). The base cases \( r = 0, 1 \) have been established in Sections (2.2) through (6), and these proofs contain most of the ideas necessary for the general case of Lemma (7.1).

We start the presentation by showing how the case \( r = n + 1 \) of Lemma (7.1) can be reduced to establishing the bound (269) for \( r = n + 1 \) using the cases \( r \leq n \) of Lemma (7.1) as an inductive hypothesis. The main step which requires a new trick is to prove the estimate (269) for \( r = n + 1 \).

### 7.8 Reducing to the forcing term estimates

In this Section we assume that Lemma (7.1) has been proven for \( r \leq n \) and furthermore that the bound (268) has been established for \( r = n + 1 \). In this section, we show how the rest of the statements in the case \( r = n + 1 \) of Lemma (7.1) excluding (268) follow from these assumptions.

**The estimate (269).** We can obtain the estimate (269) for \( r = n + 1 \) by decomposing into frequency increments

\[
\frac{D_{<k}}{\partial t} \nabla^2 p_{\leq k} p = \sum_{I=k_0(T^n)}^{k-1} \delta_{(I)} \frac{D_{<I}}{\partial t} \nabla^2 p_{\leq I} p
\]  

(294)

\[
\delta_{(I)} \frac{D_{<I}}{\partial t} \nabla^2 p_{\leq I} p = \left( \frac{D_{<I}}{\partial t} \nabla^2 p_{\leq I+1} p - \frac{D_{<I}}{\partial t} \nabla^2 p_{\leq I} p \right).
\]  

(295)

The bound (269) then follows from the bound

\[
\| \nabla^A \delta_{(I)} \frac{D_{<I}}{\partial t} \nabla^2 p_{\leq I} \|_{C^\alpha} \leq 2^{(A+(n+1)(1-\alpha)+(2-2\alpha)I)} \| v \|_{C^\alpha} (296)
\]
for the frequency increments, because the sum grows geometrically, with the main term coming from the last term \( I = k - 1 \).

We now focus our attention on proving the inequality (296).

**Inequality (296)** Inequality (296) can be proven quickly by induction on \( n \) in a manner similar to the proofs of Lemmas 7.2 and 7.5. Here we unwind the induction to give a more direct proof.

Using the “product rule” for \( \delta (I) \), we decompose

\[
\delta (I) \frac{D^{n+1}}{\partial t^{n+1}} \nabla^2 P_{\leq I} = \Gamma_I + \frac{D^{n+1}}{\partial t^{n+1}} \nabla^2 P_{\leq I+1} P
\]

\[
\Gamma_I = (P_{I+1} v \cdot \nabla) \frac{D^n}{\partial t^n} \nabla^2 P_{\leq I+1} P + \frac{D^n}{\partial t^n} (P_{I+1} v \cdot \nabla) \frac{D^{n-1}}{\partial t^{n-1}} \nabla^2 P_{\leq I+1} P
\]

\[
+ \ldots + \frac{D^n}{\partial t^n} (P_{I+1} v \cdot \nabla) \nabla^2 P_{\leq I+1} P
\]

\[
= \sum_{j=0}^{n} \frac{D^j}{\partial t^j} (P_{I+1} v \cdot \nabla) \frac{D^{n-j}}{\partial t^{n-j}} \nabla^2 P_{\leq I+1} P
\]

The \( P_{I+1} P \) term separated from the series in (291) can be estimated by the case \( r = n + 1 \) of (268). Since at most \( n \) material derivatives fall on \( \nabla^2 P_{\leq I} P \), the series (299) can be estimated by the \( r \leq n \) case of Lemma (7.1) once each term has been expanded using the commutator rules (264) and (265).

\[
\frac{D^j}{\partial t^j} (P_{I+1} v \cdot \nabla) \frac{D^{n-j}}{\partial t^{n-j}} \nabla^2 P_{\leq I+1} P = \frac{D^j}{\partial t^j} (P_{I+1} v \cdot \nabla) \left( P_{I+1} v \cdot \nabla + \frac{D^j}{\partial t^j} \right)^{n-j} \nabla^2 P_{\leq I+1} P
\]

\[
= \sum_{1 \leq \ell \leq (n-j)+1} \sum_{r_1, \ldots, r_{\ell}} C_{j, r_1, \ldots, r_{\ell}} \left( \prod_{i=1}^{\ell} \left[ \frac{D^i}{\partial t^i} \right]^{r_i} \right) \frac{D^j}{\partial t^j} \nabla^2 P_{\leq I+1} P
\]

Here the sum only runs over non-negative indices with \( r_1 + \ldots + r_\ell + \ell + m = n + 1 \) and \( \ell \geq 1 \). Therefore, at most \( r_1 + \ldots + r_\ell + m = n + 1 - \ell \leq n \) advective derivatives appear in each term of (301). Applying the \( r \leq n \) case of Lemma (7.1) gives the bound (269).

**Estimates for the commutators.** The estimates (273) follow from (270)–(271) as follows. For \( s = 0 \), the bound (273) is identical to the bound (270). For \( s \leq r + 1 \), the result follows by induction on \( s \) from the recursive formula

\[
Z_{s+1} \cdot \nabla = \left[ \frac{D^{k}}{\partial t^k} \right] Z_s \cdot \nabla
\]

\[
= \left( \frac{D^{k}}{\partial t^k} Z_s - Z_s \cdot \nabla P_{\leq k} v \right) \cdot \nabla
\]

once the bounds (270), (271) are established.

The proof of Proposition (6.1) explains in detail how to estimate the commutator (275) using the bounds (270), (271) in the cases \( r \leq 1 \) and \( s \leq 3 \). These cases already contain all the necessary ingredients for the general case.

### 7.9 Higher order advective derivatives of forcing terms

To complete the induction, it now remains to show that the estimate (268) holds for \( r = n + 1 \) assuming the cases \( r \leq n \) of Lemma (7.1). We concentrate first on the bound for the Reynolds stress, as the
bounds for Littlewood-Paley projections of the pressure are similar. First we recall the decomposition obtained in \([154\, , \, 150\, , \, 100\, \text{and} \, 108\).
\[
R_{\leq k}^{jl} = R_{\leq k, HH}^{jl} + R_{\leq k, HL}^{jl} + R_{\leq k, LL}^{jl} \tag{304}
\]
\[
R_{\leq k, LL}^{jl} = \int \delta_h P_{\leq k} v^j(x) \delta_h P_{\leq k} v^l(x) \eta_{\leq k}(h) dh \tag{305}
\]
\[
R_{\leq k, HL}^{jl} = \int P_{[k, k+2]} v^j(x + h) \delta_h P_{\leq k} v^l(x) \eta_{\leq k}(h) dh + \int \delta_h P_{\leq k} v^l(x) P_{[k, k+2]} v^j(x + h) \eta_{\leq k}(h) dh \tag{306}
\]
\[
R_{\leq k, HH}^{jl} = \int (v^j(x + h) - P_{\leq k} v^j(x + h))(v^l(x + h) - P_{\leq k} v^l(x + h)) \eta_{\leq k}(h) dh \tag{307}
\]
\[
= P_{\leq k}[(v^j - P_{\leq k} v^j)(v^l - P_{\leq k} v^l)] \tag{308}
\]
\[
= \sum_{l \geq k} P_{\leq k}(P_l v^j P_{\geq l} v^l) \tag{309}
\]

The last decomposition follows from the bandlimited property of Littlewood-Paley projections.

We already saw in the proofs of Propositions \([5.4\, \text{and} \, 6.1\) one way to estimate material derivatives \(\frac{D_{\leq k}}{\partial t^n}\) for terms of the type \((305)\) and \((308)\). It is straightforward to see that we have the desired estimates
\[
\|\nabla^A \frac{D_{\leq k}^n}{\partial t^n} R_{\leq k, LL}^{jl}\|_{C^0} \leq C_2^{2n_1 + 3n_2} e^{|\nabla^A|} \|v\|^{s + 2} \tag{310}
\]
\[
\|\nabla^A \frac{D_{\leq k}^n}{\partial t^n} R_{\leq k, HL}^{jl}\|_{C^0} \leq C_2^{2n_1 + 3n_2} e^{|\nabla^A|} \|v\|^{s + 2} \tag{311}
\]

for \(s \leq n + 1\), since the case \(r = n\) of Lemma \([7.1\) allows us to take up to \(n + 1\) material derivatives of \(P_{\leq k} v\) and \(\nabla P_{\leq k} v\) provided \(n(1 - \alpha) - 2\alpha < 0\).

The restriction \((n + 1)(1 - \alpha) - 2\alpha < 0\) becomes important for summing the estimates in the High-High terms. Here our goal is to estimate
\[
\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} R_{\leq k, HH}^{jl} = \sum_{l \geq k} \frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k}(P_l v^j P_{\geq l} v^l) \tag{312}
\]
\[
= \sum_{l \geq k} \frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} R_{\leq k, HH,l}^{jl} \tag{313}
\]

using our bounds for \(\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k} v\).

The main idea will be to take advantage of the fact that the bandlimited properties of Littlewood Paley projections allow us to express any operator \(\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k} v\) in the form
\[
\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k} v = \sum_{\ell=0}^{n+1} \sum_{r_1, \ldots, r_\ell} C_{r_1, \ldots, r_\ell} \prod_{i=1}^{\ell} \left[ \frac{D_{\leq k}}{\partial t^i} \right]^{r_i} (P_{\leq k} P_{\leq k} v, \nabla P_{\leq k}) \prod_{j=1}^{m} \left( P_{\leq k} \frac{D_{\leq k}}{\partial t^j} P_{\leq k} \right) P_{\leq k} \tag{314}
\]
where the sum runs only over indices \(r_1 + \cdots + r_\ell + \ell + m = n + 1\). The frequencies of the projections \(P_{\leq k} v\) remain bounded by \(C \cdot 2^k\) because the number of factors \(n + 1\) is bounded in terms of the fixed \(\alpha < 1\).
The starting point for the representation (314) is that we can use the bandlimited property of Littlewood-Paley projections to express

\[
P_{\leq k} \frac{D_{\leq k}}{\partial t} P_{\leq k} = \left( P_{\leq k} \frac{D_{\leq k}}{\partial t} P_{\leq k} \right) P_{\leq k} \quad (315)
\]

for some Littlewood-Paley projection \( P_{\leq k} \). The point here is that each material derivative can only increase the overall frequency support by at most a factor of \( C \cdot 2^k \).

Using the bandlimited property again, the operator (315) can then be expressed as

\[
P_{\leq k} \frac{D_{\leq k}}{\partial t} P_{\leq k} = P_{\leq k} \left( \frac{D_{\leq k}}{\partial t} - P_{\approx k} v \cdot \nabla \right) P_{\leq k} \quad (316)
\]

where most of the intermediate frequencies between \( 2^k \) and \( 2^l \) do not contribute to \( P_{\approx k} v \) thanks to the projection operators \( P_{\leq k} \). Our first explicit example of this technique appeared already in the formula (168) above.

The decomposition (314) is achieved by induction on \( n \). First we use (316) to write

\[
\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k} = T_1 - T_2 \quad (317)
\]

\[
T_1 = \frac{D_{\leq k}^n}{\partial t^n} \left( P_{\leq k} \frac{D_{\leq k}}{\partial t} P_{\leq k} \right) P_{\leq k} \quad (318)
\]

\[
T_2 = \frac{D_{\leq k}^n}{\partial t^n} \left( P_{\approx k} P_{\approx k} v \cdot \nabla P_{\leq k} \right) P_{\leq k} \quad (319)
\]

The term (318) has the form (314) after the leftmost factor of \( \frac{D_{\leq k}^n}{\partial t^n} P_{\leq k} \) in (318) has been expressed in the form (314) using the induction hypothesis.

For the term (319), we commute the material derivatives using the rule (264)

\[
T_2 = \sum_{q=0}^{n} \binom{n}{q} \left( \frac{D_{\leq k}^q}{\partial t^q} \right) \left( P_{\approx k} P_{\approx k} v \cdot \nabla P_{\leq k} \right) \frac{D_{\leq k}^{n-q}}{\partial t^{n-q}} P_{\leq k} \quad (320)
\]

These terms all have the form (314) after the factor \( \frac{D_{\leq k}^{n-q}}{\partial t^{n-q}} P_{\leq k} \) has been expressed in the form (314) using the induction hypothesis.

The formula (314) allows us to expand each term in the series (312) as

\[
\frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k,HH,I}^{jl} = \frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k} \left[ P_t v^l P_{\approx I} v^l \right] \quad (321)
\]

\[
= \sum_{\ell=0}^{n+1} \sum_{r_1, \ldots, r_t, m} C_{r_1, \ldots, r_t} \prod_{i=1}^{t} \left( \frac{D_{\leq k}}{\partial t} \right)^{r_i} \left( P_{\approx k} P_{\approx k} v \cdot \nabla P_{\leq k} \right) \prod_{j=1}^{m} \left( P_{\approx k} \frac{D_{\leq k}}{\partial t} P_{\leq k} \right) P_{\leq k} \left[ P_t v^l P_{\approx I} v^l \right] \quad (322)
\]

with \( r_1 + \cdots + r_t + \ell + m = n + 1 \). After fully expanding the commutators using (272) and (264), this decomposition and the cases \( r \leq n \) of Lemma (7.1) give the estimate

\[
\| \nabla A \frac{D_{\leq k}^{n+1}}{\partial t^{n+1}} P_{\leq k,HH,I}^{jl} \|_{C^0} \leq C_A 2^k 2^{((n+1)(1-\alpha)-2\alpha)l} \| v \|^{n+3}_{C_t C_x}, \quad (323)
\]

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where the worst estimate arises from the terms of the form $\frac{D^{n+1}_t}{\partial t^{n+1}} [P_1 v^3 P_{\geq t} v]$ which appears in the case $\ell = 0$ and $m = n + 1$. Note that the leftmost operator on every term in (322) has the form $\left[ \frac{D^{s_k}_t}{\partial t} \right] P_{\leq k}$ for some $s \leq n + 1$, which ensures that spatial derivatives never cost more than $C_0^{s_k}$ by the case $r \leq n$ for the bounds (273) in Lemma (7.1). For $(n + 1)(1 - \alpha) - 2\alpha < 0$, the bound (323) can be summed over $I \geq k$ to give

$$\| \nabla^A \frac{D^{n+1}_t}{\partial t^{n+1}} P_{\leq k} I \|_{C^0} \leq C_2^2(A + (n + 1)(1 - \alpha) - 2\alpha)^k \| v \|^{n+3} C_1 C_2^2,$$  

which concludes the proof of estimate (268) for the Reynolds stress. The bound for the pressure in (268) is proven in essentially the same way using the analogous trichotomy decomposition achieved in Proposition (5.1). From this bound, the Lemma (7.1) for $r = n + 1$ follows from the discussion in Section (7.8), which completes the inductive proof of Lemma (7.1).

## 8 Smoothness of trajectories

Here we show how the results of Section 7 can be used to prove the smoothness of particle trajectories. We consider the setting of Theorem 1.6 and we now assume that the velocity field $v(t, x)$ has borderline regularity $v(t, x) \in C_t^0 C_x^\infty$. Our goal is to construct a smooth particle trajectory $X(t)$ through $x_0$ at time 0; that is, a $C^\infty$ solution to

$$X(t) = x_0 + \int_0^t v(s, X(s)) ds$$  

(325)

For the coarse scale velocity fields $P_{\leq k} v$, we have well-defined particle trajectories $X_{(k)}(t)$ satisfying

$$X_{(k)}(t) = x_0 + \int_0^t P_{\leq k} v(s, X_{(k)}(s)) ds$$  

(326)

From the identity (328) we can see that the curves $X_{(k)}(t)$ are Lipschitz in $t$ uniformly in $k$, and therefore form an equicontinuous family of functions mapping $t \in I$ into $T^n$. Thus, the sequence $X_{(k)}(t)$ has a subsequence converging uniformly on compact sets to some limit as $k \to \infty$. Let $X(t)$ be any such uniform limit. We have that $X(t)$ satisfies (328) by passing to the limit in (328).

Now note that the particle trajectories $X_{(k)}$ are smooth in time by the results of Section 7 with

$$\frac{d^{r+1}}{dt^{r+1}} X_{(k)}(t) = \frac{D^r_{\leq k}}{\partial t^r} P_{\leq k} v(t, X_{(k)}(t))$$  

(327)

In particular, we have Taylor’s formula for all $t_0, t_1 \in I$

$$X_{(k)}(t_1) = X_{(k)}(t_0) + \sum_{r=1}^N \frac{D^{r-1}_{\leq k}}{\partial t^{r-1}} P_{\leq k} v(t, X_{(k)}(t_0)) \frac{(t_1 - t_0)^r}{r!}$$

$$+ \frac{(t_1 - t_0)^{N+1}}{N!} \int_0^1 \frac{D^N_{\leq k}}{\partial t^N} P_{\leq k} v(t_s, X_{(k)}(t_s)) (1 - s)^N ds$$

$$t_s = t_0 + s(t_1 - t_0)$$  

(328)

By the results of Section 7, we have furthermore that $\frac{D^r_{\leq k}}{\partial t^r} P_{\leq k} v$ converges uniformly on $I \times T^n$ to $\frac{D^r}{\partial t^r} v$ for any $r \geq 0$. This observation and the uniform convergence of $X_{(k)}(t) \to X(t)$ allow us to pass to the limit in Taylor’s formula (328) and conclude that $X(t)$ is smooth with $\frac{d^{r+1}}{dt^{r+1}} X(t) = \frac{D^r}{\partial t^r} v(t, X(t)).$ Thus we have proved Theorem 1.6.
9 Concluding Remarks

Several parts of the analysis in this paper give a new point of view on convex integration constructions of Euler flows and the pursuit of Onsager’s conjecture. One point which the analysis clarifies is that some of the special estimates for material derivatives in these constructions are forced by the Euler equations, rather than being artifacts of the constructions. These bounds give another point of view on the constraints one expects for the type of scheme that could be used to approach Onsager’s conjecture. For example, the bounds on material derivatives (inequality (8) in particular) show that the natural time scale associated to frequency \( \lambda \approx 2^k \) is on the order \( \lambda^{-(1-\alpha)/5} \). In particular, any time cutoffs employed in a construction at frequency \( \lambda \) should have a lifespan at least least \( \lambda^{-(1-\frac{1}{5})} \) in order to be compatible with the desired spatial regularity 1/3. The constructions of solutions in [Lsc12, BDLS13] employ time cutoffs with a lifespan of size about \( \lambda^{-(1-\frac{1}{5}+\epsilon)} \) and lead to regularity \( v \in C^{1/5-\epsilon}_{t,x} \). However, Buckmaster has shown [Buc13] that it is possible to obtain \( C^{1/5-\epsilon}_{t,x} \) solutions with \( v(t,\cdot) \in C^{1/3-\epsilon}_x(T^3) \) for almost every \( t \in \mathbb{R} \). His proof is based on the construction in [BDLS13] but modifies the construction while keeping careful track of the time-dependence of the estimates so that the set of “bad times” experiencing consistent contributions from anomalous error terms including sharp time cutoffs has measure 0. This idea has since been upgraded in [BDLS14] to construct solutions in the class \( C^{0}_{t,x} \cap L^1_t C^{1/3-\epsilon}_x \). It remains an important problem to determine if the exponent 1/5 can be improved on in any function space that admits an advective derivative bound as in [8] (for example, in \( L^2 \) based function spaces \( L^1_t H^{1/3+\epsilon} \)).

Theorem 1.5 on the regularity of the total energy suggests some further questions regarding the energy profiles of Euler flows. De Lellis and Székelyhidi have shown [DLS12] that the energy profile of an Euler flow constructed by convex integration can be essentially any smooth, positive function (see also [BDLS13]). Refinements of this result tailored to the initial value problem show that uniqueness for the initial value problem for the Euler equations in Hölder spaces cannot be restored by many natural “entropy criteria” one might propose (see [DLS10, Dan13]). There is a restriction in [DLS12, BDLS13] that the energy profile is bounded below by a positive constant, but this restriction may be purely technical. It is reasonable to suspect that the energy profile can also be made rough as well, and it would be interesting to see whether the regularity in Theorem 1.5 is sharp since the proof of Theorem 1.5 is closely related to the proof of energy conservation in [CET94]. For exponents \( \alpha < 1/5 \), the sharpness of regularity for the energy profile has now been proven in [IO15].

It would be of further interest to show that irregularity of the energy profile is a generic behavior for solutions with regularity strictly below 1/3. That is, for an Euler flow which is generic in a space similar to \( C^{1/5}_t C^\alpha_x \) with \( \alpha < 1/3 \), we expect that the energy profile will not belong to any space with better regularity than \( C^{1/5}_t \), and furthermore should fail to be of bounded variation on every time interval. In particular, a small perturbation of Euler flows in \( C^{1/5}_t C^\alpha_x \) for \( \alpha < 1/3 \) should generically lead to an irregular energy profile which does not consistently decrease or increase on any time interval, in contrast to the discussion of the case \( \alpha = 1/3 \) in Section 3. Such a result would indicate that energy dissipation at regularity below 1/3, while possible, is an unstable phenomenon, so that the 1/3 law [4] would be the only possible law for velocity fluctuations that is compatible with the dissipation of energy in a robust sense. See [IO15] for results in support of this conjecture in the range \( \alpha < 1/5 \).

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