Rees cones and monomial rings of matroids

Rafael H. Villarreal

Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14–740
07000 Mexico City, D.F.
e-mail: vila@math.cinvestav.mx

Abstract

Using linear algebra methods we study certain algebraic properties of monomial rings and matroids. Let $I$ be a monomial ideal in a polynomial ring over an arbitrary field. If the Rees cone of $I$ is quasi-ideal, we express the normalization of the Rees algebra of $I$ in terms of an Ehrhart ring. We introduce the basis Rees cone of a matroid (or a polymatroid) and study their facets. Some applications to Rees algebras are presented. It is shown that the basis monomial ideal of a matroid (or a polymatroid) is normal.

1 Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ and let $I = (x^{v_1}, \ldots, x^{v_q})$ be a proper monomial ideal of $R$ generated by $F = \{x^{v_1}, \ldots, x^{v_q}\}$. As usual for $a = (a_i)$ in $\mathbb{N}^n$ we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$. The Rees cone of $I$ is the rational polyhedral cone in $\mathbb{R}^{n+1}$, denoted by $\mathbb{R}_+ A'$, consisting of all the non-negative linear combinations of the set

$$A' := \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where $e_i$ is the $i$th unit vector. Rees cones have been used in commutative algebra to study algebraic properties and invariants of blowup algebras of monomial ideals [1, 6, 7]. Notice that $\mathbb{R}_+ A'$ has dimension $n+1$. Hence, by the finite basis theorem, the Rees cone is defined by a finite system of linear inequalities, i.e., there is a unique (up to permutation of rows) integer matrix $B = (b_{ij})$ of order $m \times (n+1)$ with non-zero rows such that

$$\mathbb{R}_+ A' = \{y| By \geq 0\},$$

the non-zero entries of each row of $B$ are relatively prime, and none of the rows of $B$ can be omitted. In this situation it is well known that the facets (faces of maximal dimension) of $\mathbb{R}_+ A'$ are given by

$$E_i = \{y| \sum_j b_{ij} y_j = 0\} \cap \mathbb{R}_+ A'.$$

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for $i = 1, \ldots, m$. Consider the convex polytope $P = \text{conv}(v_1, \ldots, v_q)$, where “conv” stands for “convex hull”. The Ehrhart ring of $P$ and the Rees algebra of $I$ are defined as the $k$-subrings of $k[x_1, \ldots, x_n, t]$ given by

$$A(P) = k[\{x^a t^b | a \in bP \cap \mathbb{Z}^n \}] \quad \text{and} \quad R[I] = k[x_1, \ldots, x_n, x_m t, \ldots, x_n t]$$

respectively, where $t$ is a new variable. The normalization of $R[I]$, denoted by $\overline{R[I]}$, is the integral closure of $R[I]$ in its field of fractions. Notice that the subgroup of $\mathbb{Z}^{n+1}$ generated by $\mathcal{A}'$ is equal to $\mathbb{Z}^{n+1}$. Thus using [15, Theorem 7.2.28] we obtain:

$$\overline{R[I]} = k[\{x^a t^b | (a, b) \in \mathbb{R}_+ \mathcal{A}' \cap \mathbb{Z}^{n+1} \}] \subset R[t]. \quad (1)$$

This formula explains the importance of Rees cones in the study of normalizations of Rees algebras. These algebras are important objects of study in algebra and geometry [3, 14]. Since we have the inclusion $A(P)[x_1, \ldots, x_n] \subset \overline{R[I]}$, it is natural to ask when the equality occurs. We are able to prove equality if $\mathbb{R}_+ \mathcal{A}'$ is quasi-ideal (Theorem 2.1), i.e., if the entries of $B$ are sufficiently nice or more precisely if the first $n$ columns of $B$ have all their entries in $\{0, 1\}$. If $I$ is square-free, in [6] it is shown that $\overline{R[I]}$ is equal to the symbolic Rees algebra of $I$ if and only if $\mathbb{R}_+ \mathcal{A}'$ is ideal, i.e., if and only if $B$ has all its entries in $\{0, \pm 1\}$.

The terms ideal and quasi-ideal refer to the facet structure of $\mathbb{R}_+ \mathcal{A}'$ and not to the algebraic structure of an ideal in the sense of ring theory. This terminology is in harmony with the combinatorial optimization notion of ideal clutter and ideal matrix introduced in [4]. Indeed, let $A$ be the matrix with column vectors $v_1, \ldots, v_q$ and assume that the entries of $A$ are in $\{0, 1\}$, by [8, Theorem 3.2] the Rees cone $\mathbb{R}_+ \mathcal{A}'$ is ideal if and only if the polyhedron $\{ y | yA \geq 1; y \geq 0 \}$ has only integral vertices.

Let $\mathbb{N} \mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integer coefficients. The Rees algebra of $I$ can be rewritten as

$$R[I] = k[\{x^a t^b | (a, b) \in \mathbb{N} \mathcal{A}' \}]. \quad (2)$$

Hence, by Eqs. (1) and (2), we get that $R[I]$ is a normal domain if and only if the following equality holds:

$$\mathbb{N} \mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+ \mathcal{A}'.$$ 

In geometric terms this equality means that $R[I] = \overline{R[I]}$ if and only if $\mathcal{A}'$ is an integral Hilbert basis. As an application we prove that $R[I]$ is normal if the Rees cone of $I$ is quasi-ideal and $k[F t] = A(P)$ (Corollary 2.3). There is a partial converse of this result [5, 13], namely, if all monomials of $F$ have the same degree and $R[I]$ is normal, then $k[F t] = A(P)$.
Ideals and Rees cones of Matroids  

Let \([n] = \{1, \ldots, n\}\) and let \(M = ([n], \mathcal{I})\) be a matroid on \([n]\), i.e., there is a collection \(\mathcal{I}\) of subsets of \([n]\) containing the empty set and satisfying the following two conditions:

(i) If \(I \in \mathcal{I}\) and \(I' \subset I\), then \(I' \in \mathcal{I}\).

(ii) If \(I_1\) and \(I_2\) are in \(\mathcal{I}\) and \(|I_1| < |I_2|\), then there is an element \(e\) of \(I_2 \setminus I_1\) such that \(I_1 \cup \{e\} \in \mathcal{I}\).

The members of \(\mathcal{I}\) are the independent sets of \(M\). A maximal independent set of \(M\) with respect to inclusion is called a basis. The reader is referred to [12] for the general theory of matroids. It is well known that all bases of \(M\) have the same number of elements, this common number \(d\) is called the rank of \(M\). Let \(\mathcal{B} = \{B_1, \ldots, B_q\}\) be the collection of bases of \(M\). For use below we set \(A' = \{e_1, \ldots, e_n, f_{B_1}, \ldots, f_{B_q}\}\); \(f_{B_i} = e_{n+1} + \sum_{j \in B_i} e_j\).

An important aim here is to study the structure of the facets of \(R_+A'\), the basis Rees cone of \(M\). An earlier result shows that the minimal primes of \(I\) are in one to one correspondence with a certain collection of facets of \(R_+A'\), see [6]. One of our main results shows that the basis Rees cone of \(M\) is quasi-ideal (Theorem 3.5). Let \(F_M\) be the set of all monomials \(x_{i_1} \cdots x_{i_d}\) in \(R\) such that \(\{i_1, \ldots, i_d\}\) is a basis of \(M\), and let \(I = (F_M)\) be the basis monomial ideal of \(M\). By a classical result of White [17], the subring \(k[F_M] \subset R\) is normal. As an application to Rees algebras, using White’s result and the quasi-ideal property of the basis Rees cone, we prove that \(R[I]\) is normal (Corollary 3.8). At the end we introduce the family of bases of a discrete polymatroid and show how to generalize Theorem 3.5 and Corollary 3.8 to discrete polymatroids.

Along the paper we introduce most of the notions that are relevant for our purposes. For unexplained terminology and notation on polyhedral geometry we refer to [2, 16].

2 Quasi-ideal Rees cones

Let \(R = k[x_1, \ldots, x_n]\) be a polynomial ring over a field \(k\) and let \(I = (x^{v_1}, \ldots, x^{v_q})\) be a proper monomial ideal of \(R\) generated by \(F = \{x^{v_1}, \ldots, x^{v_q}\}\). The Rees cone of \(I\) is the polyhedral cone in \(\mathbb{R}^{n+1}\), denoted by \(R_+A'\), generated by the set

\[
A' := \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},
\]

where \(e_i\) is the \(i\)th unit vector. See [6, Section 3] for information about some of the interesting properties of Rees cones of square-free monomial ideals. Consider the index set

\[
J = \{i \mid 1 \leq i \leq n \text{ and } \langle e_i, v_j \rangle = 0 \text{ for some } j\} \cup \{n + 1\},
\]
where $\langle \cdot , \cdot \rangle$ denotes the standard inner product. Notice that the cone $\mathbb{R}_+A'$ has dimension $n + 1$, i.e., the linear space generated by $A'$ is equal to $\mathbb{R}^{n+1}$. Hence using [16, Theorem 3.2.1] it is seen that the Rees cone has a unique irreducible representation:

$$\mathbb{R}_+A' = \left( \bigcap_{i \in J} H_{e_i}^+ \right) \cap \left( \bigcap_{i=1}^r H_{\ell_i}^+ \right)$$

such that none of the closed halfspaces can be omitted from Eq. (3). $0 \neq \ell_i \in \mathbb{Z}^{n+1}$ for all $i$, and the non-zero entries of each $\ell_i$ are relatively prime. Here $H_{a}^+$ denotes the closed halfspace $H_{a}^+ = \{ y \mid \langle y, a \rangle \geq 0 \}$ and $H_a$ is the hyperplane through the origin with normal vector $a$. It is easy to see that the first $n$ entries of each $\ell_i$ are non-negative and that the last entry of each $\ell_i$ is negative, the second assertion follows from the irreducibility of Eq. (3).

To avoid repetitions in this section we shall use the notation introduced in Section 1. Thus $A(P)$ denotes the Ehrhart ring of the convex polytope $P$, $R[It]$ denotes the Rees algebra of $I$, and $R[It]$ denotes the normalization of $R[It]$.

**Notation** For use below we set $[n] = \{1, \ldots, n\}$.

**Theorem 2.1** If each vector $\ell_k$ has the form $\ell_k = -d_k e_{n+1} + \sum_{i \in A_k} e_i$ for some $A_k \subset [n]$ and some $d_k \in \mathbb{N}$, then $A(P)[x_1, \ldots, x_n] = R[It]$.

**Proof.** Clearly the left hand side is contained in the right hand side because $A(P) \subset R[It]$. To prove the reverse inclusion let $x^a t^b = x_1^{a_1} \cdots x_n^{a_n} t^b \in R[It]$ be a minimal generator, that is, $(a,b)$ cannot be written as a sum of two non-zero integral vectors in the Rees cone $\mathbb{R}_+A'$. If $b = 0$, then $x^a t^b = x_i$ for some $i \in [n]$. Thus we may assume that $a_i \geq 1$ for $i \leq m$, $a_i = 0$ for $i > m$, and $b \geq 1$.

Case (I): $\langle (a,b), \ell_i \rangle > 0$ for all $i$. The vector $\gamma = (a,b) - e_1$ satisfies $\langle \gamma, \ell_i \rangle \geq 0$ for all $i$, that is, $\gamma \in \mathbb{R}_+A'$. Hence, since $(a,b) = e_1 + \gamma$ and $\gamma \neq 0$, we derive a contradiction.

Case (II): $\langle (a,b), \ell_i \rangle = 0$ for some $i$. We may assume that

$$\{ \ell_j \mid \langle (a,b), \ell_j \rangle = 0 \} = \{ \ell_1, \ldots, \ell_p \}.$$

Subcase (II.a): $e_i \in H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for some $i \in [m] = \{1, \ldots, m\}$. It is not hard to verify that the vector $\gamma = (a,b) - e_i$ satisfies $\langle \gamma, \ell_k \rangle \geq 0$ for all $\ell_k$. Thus $\gamma \in \mathbb{R}_+A'$, a contradiction because $(a,b) = e_i + \gamma$ and $\gamma \neq 0$.

Subcase (II.b): $e_i \notin H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for all $i \in [m]$. Since the vector $(a,b)$ belongs to the Rees cone it follows that we can write

$$(a,b) = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1) \quad (\lambda_i \geq 0).$$

Hence $a/b \in P$ and $a \in bP \cap \mathbb{Z}^n$, i.e., $x^a t^b \in A(P)$. \qed
Definition 2.2 If $\ell_1, \ldots, \ell_r$ satisfy the condition of Theorem 2.1 (resp. $d_k = 1$ for all $k$), we say that the Rees cone of $I$ is quasi-ideal (resp. ideal).

As pointed out in the introduction the notions of ideal and quasi-ideal Rees cones refer to the facet structure of the polyhedral cone $\mathbb{R}_+A'$, i.e., they refer to the irreducible representation of the Rees cone of $I$.

Corollary 2.3 If the Rees cone of $I$ is quasi-ideal and $k[Ft] = A(P)$, then the Rees algebra $R[It]$ is normal.

Proof. By Theorem 2.1 we get

$$R[It] = k[Ft][x_1, \ldots, x_n] = A(P)[x_1, \ldots, x_n] = \overline{R[It]}.$$ 

3 Facets of Rees cones of matroids

It is worth mentioning that the following exchange property allows to define the notion of a basis of a discrete polymatroid [9]. Later we will introduce this notion.

Theorem 3.1 [12, Corollary 1.2.5, p. 18] Let $\mathcal{B}$ be a non-empty family of subsets of $[n]$. Then $\mathcal{B}$ is the collection of bases of a matroid on $[n]$ if and only if the following exchange property is satisfied: If $B_1$ and $B_2$ are members of $\mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there is an element $b_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ is in $\mathcal{B}$.

The family of bases of a matroid satisfies the following symmetric exchange property that will be used below (see [11]):

Theorem 3.2 If $B_1$ and $B_2$ are bases of a matroid $M$ and $b_1 \in B_1 \setminus B_2$, then there is an element $b_2 \in B_2 \setminus B_1$ such that both $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ and $(B_2 \setminus \{b_2\}) \cup \{b_1\}$ are bases of $M$.

Let $M$ be a matroid of rank $d$ on the set $[n] = \{1, \ldots, n\}$ and let $\mathcal{B} = \{B_1, \ldots, B_q\}$ be the collection of bases of $M$. Notice that $|B_i| = d$ for all $i$ and that $\mathcal{B}$ is a clutter on $[n]$, i.e., $B_i \not\subset B_j$ for $i \neq j$. For use below we set

$$A' = \{e_1, \ldots, e_n, f_{B_1}, \ldots, f_{B_q}\}; \quad f_{B_i} = e_{n+1} + \sum_{j \in B_i} e_j.$$ 

We call $\mathbb{R}_+A'$ the basis Rees cone of $M$.

Definition 3.3 An integral matrix $C$ is called totally unimodular if each $i \times i$ minor of $C$ is 0 or $\pm 1$ for all $i \geq 1$. 

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Lemma 3.4 If \( n = q \geq 2 \) and \( B_i = \{i\} \) for \( i \in [n] \), then

\[
\mathbb{R}_+ A' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_b^+,
\]

where \( b = (1, \ldots, 1, -1) \). If \( n = q = 1 \), then \( \mathbb{R}_+ A' = H_{e_1}^+ \cap H^+_{(1, -1)} \). Moreover these are irreducible representations of the Rees cone of \( M \).

**Proof.** It follows readily using that the matrix whose columns are the vectors in \( A' \) is totally unimodular. The lemma also follows at once from the expression for the irreducible representation of a Rees cone given in [8, Theorem 3.2]. \( \square \)

Theorem 3.5 If \( E \) is a facet of \( \mathbb{R}_+ A' \), then \( E \) has the form \( \mathbb{R}_+ A' \cap H_b \), where \( b = e_i \) for some \( i \in [n+1] \) or \( b = (b_i) \) is an integral vector such that \( b_i \in \{0, 1\} \) for \( i \in [n] \) and \( b_{n+1} \geq -d \). In particular the Rees cone of \( M \) is quasi-ideal.

**Proof.** The proof is by induction on \( d \). The case \( d = 1 \) is clear because of Lemma 3.4. Assume \( d \geq 2 \). Let \( E \) be a facet of \( \mathbb{R}_+ A' \). Since the dimension of the Rees cone is \( n+1 \), there is a unique integral vector \( 0 \neq b = (b_i) \) whose non-zero entries are relatively prime such that

(a) \( E = \mathbb{R}_+ A' \cap H_b \neq \mathbb{R}_+ A' \), \( \mathbb{R}_+ A' \subset H_b^+ \), and

(b) there is a linearly independent set \( D \subset H_b \cap A' \) with \( |D| = n \).

If \( D = \{e_1, \ldots, e_n\} \) (resp. \( D \subset \{f_{B_1}, \ldots, f_{B_q}\} \) ), then \( b = e_{n+1} \) (resp. \( b \) is the vector \( (1, \ldots, 1, -d) \) ). The first assertion is clear. To show the second assertion observe that the vectors \( b \) and \( (1, \ldots, 1, -d) \) are in the orthogonal complement of \( \mathbb{R}\{f_{B_1}, \ldots, f_{B_q}\} \), and consequently they must be equal because the non-zero entries of each vector are relatively prime. Notice that \( b_i \geq 0 \) for \( i = 1, \ldots, n \).

Thus we may assume that \( D \) is the set \( \{e_1, \ldots, e_s, f_{B_1}, \ldots, f_{B_t}\} \), where \( s \in [n-1] \), \( s + t = n \), and \( \{1, \ldots, s\} \) is the set of all \( i \in [n] \) such that \( e_i \in H_b \).

We may assume that \( 1 \in B_k \) for some \( k \in [q] \). Indeed if \( 1 \) is not in \( \cup_{i=1}^q B_i \), then the facets of \( \mathbb{R}_+ A' \) different from \( H_{e_1} \cap \mathbb{R}_+ A' \) are in one to one correspondence with the facets of the Rees cone

\[
\mathbb{R}_+\{e_2, \ldots, e_n, f_{B_1}, \ldots, f_{B_q}\}.
\]

Observe that this argument can be applied replacing \( 1 \) by any other element \( j \) in \([n]\) and \( j \) not in \( \cup_{i=1}^q B_i \). Hence we may as well assume that the set \([n]\) is equal to \( \cup_{i=1}^q B_i \).

Assume that \( 1 \notin B_1 \). Since \( 1 \in B_k \setminus B_1 \) for some \( k \), by the symmetric exchange property of \( B \) (Theorem 3.2), there is \( j \in B_1 \setminus B_k \) such that \( (B_1 \setminus \{j\}) \cup \{1\} = B_i \) for some basis \( B_i \). Hence

\[
f_{B_1} + e_1 = f_{B_i} + e_j \Rightarrow \langle f_{B_1}, b \rangle = -\langle e_j, b \rangle \Rightarrow \langle f_{B_1}, b \rangle = \langle e_j, b \rangle = 0,
\]
i.e., \( f_B \) and \( e_j \) belong to \( H_b \). Thus, as \( 1 \in B_i \), by replacing \( f_B \) by \( f_B \) in \( D \), we may assume from the outset that \( 1 \in B_1 \). Next assume that \( t \geq 2 \) and \( 1 \notin B_2 \). Since \( 1 \in B_1 \setminus B_2 \), by the exchange property there is \( j \in B_2 \setminus B_1 \) such that \( (B_2 \setminus \{j\}) \cup \{1\} = B_i \) for some basis \( B_i \). Hence

\[
f_{B_2} + e_1 = f_{B_1} + e_j \Rightarrow \langle f_{B_1}, b \rangle = \langle e_j, b \rangle = 0,
\]
i.e., \( f_{B_1} \) and \( e_j \) belong to \( H_b \). Notice that \( i \neq 1 \). Indeed if \( i = 1 \), then from the equality above we get that \( \{f_{B_2}, e_1, f_{B_1}, e_j\} \) is linearly dependent, a contradiction because this set is contained in \( D \). Thus, as \( 1 \in B_i \), by replacing \( f_{B_2} \) by \( f_{B_1} \) in \( D \), we may assume from the outset that \( 1 \in B_1 \cap B_2 \). Applying the arguments above repeatedly shows that we may assume that \( 1 \) belongs to \( B_i \) for \( i = 1, \ldots, t \). We may also assume that \( B_1, \ldots, B_t \) is the set of all basis of \( M \) such that \( 1 \in B_i \), where \( t \leq r \). For \( i \in [r] \), we set \( C_i = B_i \setminus \{1\} \). Notice that there is a matroid \( M' \) on \([n]\) of rank \( d - 1 \) whose collection of bases is \( \{C_1, \ldots, C_r\} \). Consider the Rees cone \( \mathbb{R}_+A'' \) generated by

\[
A'' = \{e_1, e_2, \ldots, e_n, f_{C_1}, \ldots, f_{C_r}\}.
\]

Notice that \( \mathbb{R}_+A'' \subset H_b^+ \) and that \( \{e_1, e_2, \ldots, e_s, f_{C_1}, \ldots, f_{C_s}\} \) is a linearly independent set. Thus \( \mathbb{R}_+A'' \cap H_b \) is a facet of the cone \( \mathbb{R}_+A'' \) and the result follows by induction.

The facets of the Rees cone of the ideal generated by all square-free monomials of degree \( d \) of \( R \) were computed in [1, Theorem 3.1], the result above can be seen as a generalization of this result.

The set of all monomials \( x_{i_1} \cdots x_{i_d} \in R \) such that \( \{i_1, \ldots, i_d\} \) is a basis of \( M \) will be denoted by \( F_M \) and the subsemigroup generated by \( F_M \) will be denoted by \( \mathbb{M}_M \). The basis monomial ring of \( M \) is the subring \( k[F_M] \subset R \). The square-free monomial ideal \( I(\mathcal{B}) := (F_M) \) is called the basis monomial ideal of \( M \).

**Proposition 3.6 ([17])** If \( x^a \) is a monomial of degree \( \ell d \) for some \( \ell \in \mathbb{N} \) such that \( (x^a)^p \in \mathbb{M}_M \) for some \( p \in \mathbb{N} \setminus \{0\} \), then \( x^a \in \mathbb{M}_M \). In particular \( k[F_M] \) is normal.

The next result is just a reinterpretation of Proposition 3.6 which is adequate to examine the normality of the Rees algebra of \( I(\mathcal{B}) \).

**Corollary 3.7** Let \( P \) be the convex hull of the set of all vectors \( e_{i_1} + \cdots + e_{i_d} \) such that \( \{i_1, \ldots, i_d\} \) is a basis of \( M \). Then \( A(P) = k[F_M t] \).

**Proof.** It suffices to show the inclusion \( A(P) \subset k[F_M t] \). Take \( x^a t^b \in A(P) \), that is, \( a \in \mathbb{Z}^d \cap bP \). Hence \( x^a \) has degree \( bd \) and \( (x^a)^p \in \mathbb{M}_M \) for some positive integer \( p \). By Proposition 3.6 we get \( x^a \in \mathbb{M}_M \). It is rapidly seen that \( x^a t^b \) is in \( k[F_M t] \), as required. \( \square \)

As an application to Rees algebras we obtain the following:
Corollary 3.8 If $I = I(B)$ is the basis monomial ideal of a matroid, then $R[It]$ is normal.

Proof. It follows from Corollaries 2.3 and 3.7 together with Theorem 3.5. □

Polymatroidal sets of monomials

For $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we set $|a| = a_1 + \cdots + a_n$. Let $\mathcal{A} = \{v_1, \ldots, v_q\} \subset \mathbb{N}^n$ be the set of bases of a discrete polymatroid of rank $d$, i.e., $|v_i| = d$ for all $i$ and the following condition is satisfied: given any two $a = (a_i), c = (c_i)$ in $\mathcal{A}$, if $a_i > c_i$ for some index $i$, then there is an index $j$ with $a_j < c_j$ such that $a - e_i + e_j$ is in $\mathcal{A}$. The set $F = \{x^{v_1}, \ldots, x^{v_q}\}$ (resp. the ideal $I = (F) \subset R$) is called a polymatroidal set of monomials (resp. a polymatroidal ideal). Notice that the basis monomial ideal of a matroid is a polymatroidal ideal. We refer to [9, 10] for the theory of discrete polymatroids and polymatroidal ideals. Below we indicate how to generalize Theorem 3.5 and Corollary 3.8 to discrete polymatroids.

Lemma 3.9 The set $F' = \{x^{v_i}/x_1: x_1 \text{ occurs in } x^{v_i}\}$ is also polymatroidal.

Lemma 3.10 Let $G = \{x^{u_1}, \ldots, x^{u_s}\}$ and let $d = \max\{|u_i| : i \in [s]\}$. Suppose that $F = \{x^{u_1}, \ldots, x^{u_t}\}$ is the set of all $x^{u_i}$ of degree $d$. If $Q = \text{conv}(u_1, \ldots, u_s)$ and $k[Gt] = A(Q)$, then $k[Ft] = A(P)$, where $P$ is the convex hull of $u_1, \ldots, u_t$.

Proposition 3.11 If $I = (F)$ is a polymatroidal ideal, then $R[It]$ is normal.

Proof. Let $\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\}$ and let $\mathbb{R}_+\mathcal{A}'$ be the Rees cone of $I$. Using the proof of Theorem 3.5 together with Lemma 3.9 and the fact that $\mathcal{A}$ satisfies the symmetric exchange property [9 Theorem 4.1, p. 241] it is not hard to see that the Rees cone $\mathbb{R}_+\mathcal{A}'$ is quasi-ideal. By [9 Theorem 6.1] and Lemma 3.10 we get $k[Ft] = A(P)$. Thus applying Corollary 2.3 we obtain that $R[It]$ is normal. □

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