The Reynolds stress in turbulence from a Lagrangian perspective

To cite this article: T-W Lee 2018 J. Phys. Commun. 2 055027

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The Reynolds stress in turbulence from a Lagrangian perspective

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Abstract
We present a unique method for solving for the Reynolds stress in turbulent canonical flows, based on the momentum balance for a control volume moving at the local mean velocity. A differential transform converts this momentum balance to a solvable form. Validations with experimental and computational data in simple geometries show quite good results. An alternate Lagrangian analytical method is offered, leading to a potential closure method for the Reynolds stress in terms of computable turbulence parameters.

Nomenclature

\[ C_1 \] constants
\[ p'_{\text{rms}} \] pressure fluctuation (rms)
\[ s \] fluid particle path
\[ \text{Re}_s \] Reynolds number based on friction velocity
\[ U \] mean velocity in the x direction
\[ U_e \] free-stream velocity
\[ V \] mean velocity in the y direction
\[ u' \] fluctuation velocity in the x direction
\[ u_{\text{rms}}' \] root-mean square of \( u' \)
\[ u'u' \] Reynolds stress
\[ \delta \] boundary layer thickness
\[ \tau \] flow transit time
\[ \nu \] kinematic viscosity
\[ \langle \rangle \] Reynolds-averaged

Subscripts:
\[ a \] location ‘a’
\[ b \] location ‘b’

1. Introduction

Determination of the Reynolds stress in terms of root turbulence parameters has profound implications in fluid physics and engineering. It can be said that finding the Reynolds stress is the central problem in turbulence. Many practical flows are turbulent, and require some method of analysis or computations so that the flow process can be understood, predicted and controlled. This necessity has led to several generations of turbulence models including a genre that models the Reynolds stress components themselves, the Reynolds stress models \[1, 2\]. The conventional approach of writing higher moments for the Reynolds stress, then attempting to model
them in this way, involves ever increasing number of terms and corresponding complexities. For example, in the Reynolds stress budget, there are six groups of source/sink terms \([3]\), which become about three times that number when expanded upon the repeated indices. In addition, one of the six groups of terms is the dissipation rate, which in itself has quite a complex budget \([4]\). Gradient-based turbulence models such as \(k-\varepsilon\) models are very widely used for practical flows, due to their low computational requirements. However, there is a consistent set of data \([1, 2, 5, 6]\) that show inherent inadequacies for gradient-based models.

In spite of the wide range of models and sub-models in all turbulence models, a common denominator is that they use Eulerian transport equations, for turbulent kinetic energy, dissipation and the Reynolds stress itself. The governing equations themselves are quite physically sound, except that the higher-order terms appear in this formulation that need to be modeled with increasing levels of complexity and thus empiricism. On the other hand, Lagrangian analysis can lead to simple and intuitive look at turbulent transport processes. This kind of Lagrangian analysis has been used in studies of statistics of turbulent intermittency, dispersion, and mixing \([7–11]\). Pope \([12]\) shows a Lagrangian method of obtaining statistics of turbulence fluctuations, which has applications toward probability density function modeling of various flows including combustion. Lagrangian approach has also been attempted for sub-grid modeling of large-eddy simulations \([13]\). The work by Bernard and co-workers \([5, 6]\) derived and used a Lagrangian tracking of the Reynolds stress, and showed that the transport of the Reynolds stress is due to a displacement effect and acceleration/deceleration due to the pressure-gradient and viscous shear.

Recently, a theoretical basis for determination of the Reynolds stress in canonical flows has been presented \([14, 15]\). It is based on the turbulence momentum balance for a control volume moving at the local mean flow speed. Therefore, it constitutes a Lagrangian analysis for the momentum transport, which happens to contain the Reynolds stress term. The resulting ‘integral formula’ is shown below, which works quite well in determining the Reynolds stress based on inputs of root turbulence parameters, \(\langle u'^2 \rangle\), the mean velocity and its gradient for canonical flows \([14, 15]\).

\[
\langle u'u' \rangle = -C_2 \left[ U \langle u'^2 \rangle - \epsilon \int_0^1 \frac{dU}{dy} \langle u'^2 \rangle dy \right] + \nu \frac{\partial u_{rms}'}{\partial y} \tag{1}
\]

Here \(U\) is the mean velocity, \(u'\) turbulence fluctuation, and \(u_{rms}' = \sqrt{\langle u'^2 \rangle}\). It shows that the momentum equation becomes quite simple in this moving coordinate frame, and that the Reynolds stress can be written in a closed-form equation (equation (1)). Bernard and co-workers \([5, 6]\) also showed that the Reynolds stress can be written in terms of its root contributing factors, by integrating the Navier–Stokes equation along a fluid trajectory, as below.

\[
\langle u'u' \rangle = \langle u'_b v'_a \rangle + \langle v'_a (U_b - U_a) \rangle + \int_{-\tau}^{0} \left\{ v'(0) \frac{\partial p}{\partial x}(s) \right\} ds + \int_{-\tau}^{0} v \langle v'(0) \nabla^2 U(s) \rangle ds \tag{2}
\]

This is a Lagrangian analysis \([5, 6]\) where line integration is performed on the Navier–Stokes equation, and turbulence data are also integrated accordingly, to ascertain the contributing factors to the Reynolds stress. Bernard \(et al\) \([5]\) notes that the Reynolds stress is the result of displacement of momentum, the second term on the right-hand side (RHS) of equation (2) and acceleration/deceleration due to the pressure and viscous terms. This was essentially the same conclusion that was arrived at from a different (but also Lagrangian) approach in our previous work \([14, 15]\). Since the velocity gradient term does not appear explicitly in equation (2), the mechanics does not easily follow the gradient law used in \(k-\varepsilon\) class of turbulence models \([5]\).

In this work, we apply a Lagrangian logic by performing a momentum balance for a control volume moving at the local mean velocity. This results in a Lagrangian transport equation for turbulence momentum so that a closure for the Reynolds stress becomes achievable, in quite a simple manner that is unexpected and new. This Lagrangian transport equation is validated using experimental and DNS data.

2. An alternate perspective on momentum transport in turbulent flows

The mechanics of the turbulence momentum transport can be examined from an alternate perspective. The mean velocity in channel flows exhibits the familiar, steep profile near the wall, and then tending to the centerline value as one moves away from the wall. As shown in figure 1(a), \(\langle u'^2 \rangle\) profiles have a peak near the wall, then gradually approach the centerline value. Finding the right combination of \(U\), \(\langle u'^2 \rangle\), and other parameters for the Reynolds stress through artificial means spawned an entire library of turbulence models. Figure 1(a) shows a very basic comparison of the Reynolds stress and \(\langle u'^2 \rangle\) profiles; \(\langle u'^2 \rangle\) has been multiplied by a negative number to look at the two parameters side by side. The data are from DNS results for turbulent channel flows at \(Re = 650\) and 400 \([16]\). We can see that there are some hints that these profiles exhibit similarities: both the Reynolds stress and \(\langle u'^2 \rangle\) profiles have a sharp peak near the wall. The peaks are slightly removed from one
another, and the Reynolds stress goes to zero at the centerline while \( \langle u'^2 \rangle \) reaches a finite value in the DNS data in figure 1(a). However, to satisfy the centerline symmetry boundary conditions, the slopes of both of these profiles become zero at the centerline.

Then, what about the gradient of the Reynolds stress and \( \langle u'^2 \rangle \)? These are plotted in figure 1(b), which mostly reflects what we saw in figure 1(a). The slopes of these profiles become zero at the centerline, and even though the location for the minimum in the gradient is not quite aligned the overall shape of the profiles is quite similar. We can examine if such similarity is just a matter of coincidence or somehow related to the momentum transport by making a slight adjustment to \( \frac{\partial \langle u'v' \rangle}{\partial y} \) and \( \frac{\partial \langle u'^2 \rangle}{\partial y} \) by \( C_1 U \); we multiply \( \frac{\partial \langle u'^2 \rangle}{\partial y} \) by \( C_1 U(y) \), where \( U \) is the mean velocity, and this is again compared with \( \frac{\partial \langle u'v' \rangle}{\partial y} \) in figure 1(c). Multiplying by \( C_1 U(y) \) magically aligns the peaks, and also the shape of the profiles become quite self-similar. As shown later, this is due to the transport of \( \langle u'^2 \rangle \) by the local mean velocity \( U(y) \). To anticipate the mathematical formulation in the next section, we can also compare the gradient of the Reynolds stress with a viscous term, \( \frac{\partial \langle u'^2 \rangle}{\partial y} + \nu \frac{\partial^2 u_{\text{rms}}}{\partial y^2} \), where \( \nu \) is taken as the square root of \( \langle u'^2 \rangle \). This is plotted in figure 1(d), and again compared with \( \frac{\partial \langle u'v' \rangle}{\partial y} \). Since the \( \langle u'^2 \rangle \) profiles reach a nearly constant slope in figure 1(a), we can deduce that the second derivative would go to zero away from the wall, as verified in figure 1(d). Moreover, they have pronounced negative dips near the wall, in the so-called viscous sub-layer. Adding the two profiles in figures 1(c) and (d) results in a very close agreement with the Reynolds stress gradient, as shown in figure 1(e). The match is almost exact. From this analysis, it is seen that the contributions to the Reynolds stress

Figure 1. (a)–(e). Examination of the relationship between the Reynolds stress and \( \langle u'^2 \rangle \) in various forms, for \( \text{Re}_\tau = 650 \) (circle symbols and 400 (diamond): (a) \( \langle u'v' \rangle \) (symbols) and – K(\( u'^2 \)) (lines), \( K = 0.015 \); (b) \( \frac{\partial \langle u'v' \rangle}{\partial y} \) and \( \frac{\partial \langle u'^2 \rangle}{\partial y} \); (c) \( \frac{\partial \langle u'v' \rangle}{\partial y} \) and \( -C_1 U \frac{\partial \langle u'^2 \rangle}{\partial y} \); (d) \( \frac{\partial \langle u'^2 \rangle}{\partial y} \) and \( \nu \frac{\partial^2 u_{\text{rms}}}{\partial y^2} \); and (e) \( \frac{\partial \langle u'v' \rangle}{\partial y} \) and \( \nu \frac{\partial^2 u_{\text{rms}}}{\partial y^2} \). Symbols are from DNS data for \( \langle u'v' \rangle \) \[16\], while lines represent various terms computed using \( \langle u'^2 \rangle \) and \( U \), also from the same DNS data set.
are two-fold: from both $\langle u'^2 \rangle$ and the viscous shear stress due to mean turbulent fluctuations near the wall, and only the $\langle u'^2 \rangle$ effect is significant away from the wall. In channel flows, the sequence of figures 1(a)–(e) shows that the Reynolds stress can be prescribed by $U$ and $\langle u'^2 \rangle$, and no other variables or complex permutations of turbulence parameters seem necessary. This brings us to the main element of this paper: an alternative derivation of the Reynolds stress.
3. Development of the lagrangian momentum transport

The above examination of the Reynolds stress and the longitudinal momentum terms, \( \langle u'^2 \rangle \) and \( U \), point to a relationship between these variables. We start by considering a control volume moving at the local mean speed, as shown in figure 2, for which the x-momentum balance is due to turbulence fluctuations since the mean speed is cancelled out by the motion of the control volume itself. This kind of coordinate transform is used in analysis of normal shocks and premixed flames which all move at the respective wave speeds, and placing a control volume on the wave front leads to much simplifications. Following this logic, the momentum balance is further illustrated in figure 2, where on the x-interfaces the net momentum flux is given by \( u \partial u / \partial x \) while on the y-interfaces the net flux is \( \partial (u'v') / \partial y \). This is exactly analogous to the mean momentum balance in the Eulerian control volume, where the respective momentum fluxes will be \( uU / \partial y \) and \( \partial uV / \partial y \). The resulting Lagrangian turbulent momentum balance is shown below, Reynolds-averaging is denoted by \( \langle \cdots \rangle \), and it naturally contains the Reynolds stress term from the y-interface momentum flux:

\[
\frac{\partial \langle u'v' \rangle}{\partial y} = - \frac{\partial \langle u'^2 \rangle}{\partial x} - \frac{1}{\rho} \frac{\partial \rho' u_{ms}'}{\partial x} + \nu \frac{\partial^2 u_{ms}'}{\partial y^2} \tag{3}
\]

The integral formula for the Reynolds stress (equation (1)) is an integrated form of this momentum transport equation for the case of negligible pressure fluctuation gradient. In equation (3), the viscous term is somewhat difficult to visualize because the fluctuation can be in either + or – direction relative to \( U \). However, since we are dealing with the second derivative of \( u_{ms}' \), the sign of the first derivative becomes inconsequential, and we are interested in the effect of mean difference in the turbulence intensity. Bernard et al\[5, 6\] also shows that the Navier–Stokes equation becomes quite simple, for analysis of the Reynolds stress, when integrated along fluid particle trajectory:

\[
\langle u'(0) \rangle = \langle u'(-\tau) \rangle + [U(-\tau) - U(0)] - \int_{-\tau}^{0} \frac{1}{\rho} \frac{\partial \rho'}{\partial s}(s)ds + \nu \int_{-\tau}^{0} \nabla^2 U(s)ds \tag{4}
\]

After a Taylor expansion of \( U \), equation (4) becomes a Lagrangian expression for the Reynolds stress \[5, 6\]. A major difference in their work is that the line integrals are applied for each of the terms in the Navier–Stokes equation so that spatial correlations or integrals can be computed for contributing factors to the Reynolds stress. We have simply applied the Lagrangian logic directly to find the transport of turbulence momentum.

Figure 2. Illustration of the momentum balance for a moving control volume.
The $u'^2$ and pressure fluctuation terms are x-derivatives. In conventional calculations in the Eulerian coordinate frame, the x-derivatives would have been set to zero for fully-developed flows. However, for a boundary-layer flow as an example (figure 3), the boundary layer grows due to the ‘displacement’ effect. The mass is displaced due to the fluid slowing down at the wall, as is the turbulence momentum. The boundary layer thickness grows at a monotonic rate, depending on the Reynolds number. Thus, if one rides with the fluid moving at the mean velocity, one would see a change in all of the turbulence properties, as illustrated in figure 3. This displacement effect may be approximated as:

$$\frac{\partial}{\partial x} = C_1 U \frac{\partial}{\partial y} \quad (5)$$

I.e., the fluid parcel will see a different portion of the boundary layer in the y-direction, and how much difference it will see depends on how fast the fluid is moving along in the boundary layer. Thus, the mean velocity, $U$, appears as a multiplicative factor in equation (5). It may be considered as a type of Blasius transform, where the lateral coordinate scales with the streamwise distance. This is intuitive for boundary layer or jet flows, but it is not expected to work as well for channel flows. Indeed, this transform works quite well near the wall as shown below, but the results start to depart toward the centerline in channel flows, requiring a correction [14, 15]. $C_1$ is a constant that depends on the Reynolds number, and has the unit of inverse of velocity. As will be seen later, this variation is only needed for channel flows, since use of similarity variable and normalization by the friction velocity collapses the Reynolds stress profiles for turbulent jets and flow over a flat plate, respectively. Equation (5) is an approximation or a hypothesis, which is confirmed through validation with DNS or experimental data in various geometries.

Equations (3) and (5) lead to the ‘turbulence momentum transport equation’.

$$\frac{\partial \langle u'v' \rangle}{\partial y} = -C_1 U \left[ \frac{\partial \langle u'^2 \rangle}{\partial y} + \frac{1}{\rho} \frac{\partial \rho \langle u' \rangle}{\partial y} \right] + \nu \frac{\partial^2 \langle u' \rangle}{\partial y^2} \quad (6)$$

Here, we include the pressure fluctuation term in the displacement term bracket, but the data show that the pressure fluctuation gradients are about two orders of magnitude smaller than that for $u'^2$ in the experimental and DNS data sets [16, 17] (see figure 4). In such instances, the Reynolds stress gradient (LHS) is balanced by the $u'^2$ transport, or the displacement term, and the viscous term on the RHS of equation (6). This is similar to the conclusion arrived at by Bernard and co-workers [5, 6]. Equation (6) would indicate, if validated, that complex modeling of the Reynolds stress is not necessary, and that we can use the turbulence momentum balance to determine the Reynolds stress from root turbulence parameters. Only $U$ and $\langle u'^2 \rangle$ are needed to find the gradient of $\langle u'v' \rangle$, from which $\langle u'v' \rangle$ itself can be determined by integrating from a known boundary condition (e.g. at the wall). Equation (6) is a far simplification than the conventional Eulerian budget for the Reynolds stress [1–4]. One can state that by employing a Lagrangian coordinate frame, the mean velocity ($U$) is cancelled out in the momentum balance, and only re-appears for the displacement term, and this leaves just a small number of terms to determine the gradient of the Reynolds stress as shown in equation (6). Thus, a conversion of the perspective from Eulerian to Lagrangian transforms the turbulence momentum transport to a simple, intuitive form. At times, a change in perspective (Eulerian to Lagrangian in this case, and from differential to integral analysis in [18]) can lead to unexpected solutions to a seemingly complex fluid mechanics problem.

4. Results and discussion

We can check the validity of the turbulent momentum transport in equation (6), and also of the displacement approximation in equation (5). The gradient of the Reynolds stress (LHS of equation (6)) can be compared with the RHS, by using DNS and experimental data for $\langle u'v' \rangle$, $\langle u'^2 \rangle$ and $U$. We first consider rectangular channel flows. In Iwamoto et al [16, 17], DNS data for mean and fluctuation quantities are tabulated, including the
Reynolds stress $\langle u'v' \rangle$, for $Re_\tau$ up to 650. We can obtain the left-hand side (LHS) of equation (6), the gradient of the Reynolds stress, from $\langle u'v' \rangle$ in the DNS data [16, 17], by computing the derivative $\frac{\partial \langle u'v' \rangle}{\partial y}$ numerically. We can also use $U$ and $\langle u'^2 \rangle$ profiles in the same data set to compute the terms on the RHS of equation (6). $u'_{ rms}$ is equal to the square root of $\langle u'^2 \rangle$.

The comparison is shown in figure 4, where the LHS and RHS of equation (6) match almost exactly. Therefore, the turbulence momentum transport (equation (6)) holds quite well for channel flows. Figure 4 also appears to confirm the appropriateness of neglecting the pressure fluctuation gradient term. Figure 5 quantifies the pressure fluctuation gradient in comparison to the $\langle u'^2 \rangle$ gradient for channel flow data [16, 17], where the former is about two orders of magnitude smaller than the latter. Thus, the comparisons in figures 4 and 5 indicate that the pressure fluctuation gradient term may be neglected in certain types of turbulent flows.

The constant, $C_1$, in equation (6), arises due to the displacement effect as described in equation (5). We expect this displacement effect to be larger for higher Reynolds numbers, so that we also expect $C_1$ to increase with the Reynolds number. For example, in turbulent boundary layers, the boundary layer thickness scales as $\delta \sim 0.16xRe_x^{-1/7}$, where $x$ is the stream-wise distance. $C_1$ ranged from 0.000 536 at $Re_\tau = 110$ to 0.0012 at $Re_\tau = 650$ for channel flows shown in figure 4. Scaling collapses the mean velocity and the Reynolds stress profiles in turbulent jet and boundary layer flows (shown below), so that the variation in this constant is only needed for channel flows. In turbulent boundary layers, the boundary layer thickness scales as $\delta \sim 0.16xRe_x^{-1/7}$ where $x$ is the stream-wise distance.
We can perform a similar verification for planar jets, for which we use the experimental data of Gutmark and Wygnanski [19, 20]. For free jets, equation (6) becomes yet simpler because there are no solid boundaries and sharp gradients are not present in \( \langle u'^2 \rangle \). This causes the viscous term in equation (6) to be negligible except near the peak of \( \langle u'^2 \rangle \) profiles, and the Reynolds stress is mostly determined by the transport term, \(-C_1 U \partial \langle u'^2 \rangle / \partial y\).

Figure 6 shows the gradient of the Reynolds stress from Gutmark and Wygnanski [19, 20], compared with the current result using \( U \) and \( \langle u'^2 \rangle \) data. The initial and final slopes of the Reynolds stress are not quite matched, but overall the agreement is good. \( \langle u'^2 \rangle \) increases (positive slope) from the centerline value, and then decreases (negative slope), while the mean velocity gradually decreases from the centerline value to zero [19, 20]. \( C_1 \) is negative since the velocity gradient is negative as measured from the centerline, and the Reynolds stress positive (opposite of wall-bounded flows). This leads to an inverted behavior for the Reynolds stress: positive gradient near the centerline, then converting to negative gradient. We note that when plotted in self-similar variables, the experimental data collapse to a single profile [19, 20], so that only a single value \( C_1 \) is needed for turbulent jets at all Reynolds numbers.

As noted above, the RHS side of equation (6) can be numerically integrated to find the Reynolds stress itself, starting from the boundary of condition of \( \langle u'v' \rangle = 0 \) at the centerline. The result is shown in figure 7, where we can see that the integration is somewhat forgiving of the discrepancy in the initial slope. The agreement for the Reynolds stress is decent in figure 7, although the error does tend to accumulate particularly beyond the peak near \( y / y_{m/2} = 0.75 \). Again, we are taking the derivative of experimental data in equation (3), fitted to a line [20], and considering the potential for numerical errors, the comparisons of figures 6 and 7 are quite reasonable.

Finally, we use the DNS data for flows over a flat plate at zero pressure gradients [20]. For this flow geometry, the Reynolds stresses collapse to a single profile when properly scaled [20]. This is very useful for the current approach, as again we only need a single value of the constant \( C_1 \). \( \langle u'^2 \rangle \) profiles have quite a sharp peak near the wall, and gradually decrease as the distance from the wall increases [20]. Although this is somewhat attenuated near the wall by the mean velocity in the transport term (the first term on the RHS of equation (6)), the resulting contribution of \( \langle u'^2 \rangle \) is still high. However, close to the wall the large gradient in \( \langle u'^2 \rangle \) also increases the magnitude of the viscous term. For experimental data [20], there are data fluctuations due to measurement difficulties near the wall, and this leads to large errors when numerically differentiated. So we apply smooth spline-fit functions for the experimental data, to remove these fluctuations. This is done only to retain the data differentiability, and for high-resolution data (e.g. DNS) this is not critical. Even with the spline fit being used, the differentiation is not clean. Yet, the balance in the turbulence momentum can still be checked, by again numerically integrating the RHS of equation (6) to find the Reynolds stress. Comparison with the experimental data [20] is shown in figure 8. In spite of the sharp gradients in \( \langle u'^2 \rangle \) near the wall, the agreement between the current theory and experimental data is quite good. Similar to the channel flow, the transport term is pervasive while the viscous term adds to the Reynolds stress near the wall. Overall, observations in channel and flat plate flows indicate that the transport for the Reynolds stress is similar in wall-bounded flows, and can be mathematically prescribed using equation (6).
5. Conclusions

Applying the momentum balance to a Lagrangian control volume moving at the local mean velocity yields a closed-form equation for the Reynolds stress gradient. This formulation is confirmed using various data sets, where the gradient of the Reynolds stress is shown to be due to the net momentum balance of the transport and the viscous effects for flows for flow geometries where the gradient in the pressure fluctuations are negligible (figure 4). The gradient in the Reynolds stress can be numerically integrated to find the Reynolds stress itself. Both the gradient and the Reynolds stress obtained through this approach agree quite well with DNS and experimental data in canonical flows.

Solving for the Reynolds stress within the Eulerian framework can be a daunting task, close to being intractable. However, in the Lagrangian perspective, \( u'v' \) is simply a forward transport of \( v' \) momentum, and as such subject to the same Newton’s second law. A shift in the perspective, in this instance and in [18], leads to a great simplification, due to cancellation of the mean velocity with that of the control volume (figure 2). Implications of this work are that the Reynolds stress can be written explicitly in terms of basic turbulence parameters so that it furnishes the \( u'v' \) term in the Reynolds-averaged Navier–Stokes (RANS) equation. The full
closure will of course require the $u'^2$ component. An approach to obtain $u'^2$ using a similar Lagrangian analysis is being investigated in this laboratory.

**Acknowledgments**

This work was inspired after a research visit to Brno University of Technology (VUTB), and a conversation with Zdenka V. on an autumn evening in Brno.

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