ANTI-HOLOMORPHIC SEMI-INARIANT SUBMERSIONS FROM KÄHLERIAN MANIFOLDS

HAKAN METE TAŞTAN

Abstract. We study anti-holomorphic semi-invariant submersions from Kählerian manifolds onto Riemannian manifolds. We prove that all distributions which are involved in the definition of the submersion are integrable. We also prove that the O'Neill’s tensor $T$ vanishes on the invariant vertical distribution. We give necessary and sufficient conditions for totally geodesicness and harmonicity of this type submersions. Moreover, we investigate the several curvatures of the total manifold and fibers and give a characterization theorem.

1. Introduction

The theory of Riemannian submersions were initiated by O’Neill [9] and Gray [4]. In [16], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions. In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distribution are invariant with respect to the almost complex structure of the total manifold of the submersion. Afterwards, almost Hermitian submersions have been actively studied between different subclasses of almost Hermitian manifolds, for example, see [3]. Most of the studies related to Riemannian or almost Hermitian submersions can be found in the book [2]. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds were initiated by Şahin [12]. In this case, the fibres are anti-invariant with respect to the almost complex structure of the total manifold. He studied this type submersions from a Kählerian manifold onto a Riemannian manifold. Recently, Shahid and Tanveer [11] extended this notion to the case when the total manifold is nearly Kählerian. A Lagrangian submersion is a special case of an anti-invariant Riemannian submersion such that the almost complex structure of the total manifold reverses the vertical and horizontal distributions. In [15], we studied Lagrangian submersions in detail. There are some other recent paper which involve other structures such as almost product [6], almost contact [8], and Sasakian [7]. In any cases, the definition of anti-invariant Riemannian submersion is the same as the above definition. Besides there are many other notions related with that of anti-invariant Riemannian submersion, such as slant submersion [14], semi-invariant submersion [13] and semi-slant submersion [10]. In particular, the notion of semi-invariant is a natural generalization of the notion anti-invariant submersion. In this paper, we consider semi-invariant submersions from a Kählerian manifold onto a Riemannian manifold in a special case.

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2. Riemannian submersions

In this section, we give necessary background for Riemannian submersions.

Let \((M, g)\) and \((N, g_N)\) be Riemannian manifolds, where \(\dim(M) > \dim(N)\). A surjective mapping \(\pi : (M, g) \to (N, g_N)\) is called a Riemannian submersion \([9]\) if:

(S1) \(\pi\) has maximal rank, and

(S2) \(\pi_\ast\), restricted to \((\ker\pi_\ast)^\perp\), is a linear isometry.

In this case, for each \(y \in N\), \(\pi^{-1}(y)\) is a \(k\)-dimensional submanifold of \(M\) and called fiber, where \(k = \dim(M) - \dim(N)\). A vector field on \(M\) is called vertical (resp. horizontal) if it is always tangent (resp. orthogonal) to fibers. A vector field \(X\) on \(M\) is called basic if \(X\) is horizontal and \(\pi\)-related to a vector field \(X_\ast\) on \(N\), i.e., \(\pi_\ast X_x = X_\ast(\pi(x))\) for all \(x \in M\). As usual, we denote by \(V\) and \(H\) the projections on the vertical distribution \(\ker\pi_\ast\) and the horizontal distribution \((\ker\pi_\ast)^\perp\), respectively. The geometry of Riemannian submersions is characterized by O’Neill’s tensors \(T\) and \(A\), defined as follows:

\[
T_E F = \nabla_{V_E} H F + H \nabla_{V_E} V F,
\]

\[
A_E F = \nabla_{H_E} H F + H \nabla_{H_E} V F
\]

for any vector fields \(E\) and \(F\) on \(M\), where \(\nabla\) is the Levi-Civita connection of \(g_\ast\). It is easy to see that \(T\) and \(A\) are skew-symmetric operators on the tangent bundle reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields \(T\) and \(A\). Let \(U, V\) be vertical and \(\xi, \eta\) be horizontal vector fields on \(M\), then we have

\[
T_U V = T_V U,
\]

\[
A_\xi \eta = -A_\eta \xi = \frac{1}{2} \nabla[\xi, \eta],
\]

On the other hand, from (2.1) and (2.2), we obtain

\[
\nabla_U V = T_U V + \hat{\nabla}_U V,
\]

\[
\nabla_U \xi = T_U \xi + H \nabla_U \xi,
\]

\[
\nabla_\xi U = A_\xi U + V \nabla_\xi U,
\]

\[
\nabla_\xi \eta = H \nabla_\xi \eta + A_\xi \eta,
\]

where \(\hat{\nabla}_U V = V_\xi V + H \nabla_V \xi = A_\xi V\), if \(\xi\) is basic. It is not difficult to observe that \(T\) acts on the fibers as the second fundamental form while \(A\) acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O’Neill’s paper \([9]\) and to the book \([2]\).

Finally, we recall that the notion of the second fundamental form of a map between Riemannian manifolds. Let \((M, g)\) and \((N, g_N)\) be Riemannian manifolds.
and \( \varphi : (M, g) \to (N, g_N) \) a smooth map. Then the second fundamental form of \( \varphi \) is given by

\[
(\nabla \varphi_*) (E, F) = \nabla^\varphi E \cdot \varphi_* F - \varphi_* (E, F)
\]

for \( E, F \in TM \), where \( \nabla^\varphi \) is the pull back connection and we denoted conveniently by \( \nabla \) the Riemannian connections of the metrics \( g \) and \( g_N \) \(^1\).

### 3. Anti-holomorphic semi-invariant submersions

A smooth manifold \( M \) is called almost Hermitian \(^{17}\) if its tangent bundle has an almost complex structure \( J \) and a Riemannian metric \( g \) such that

\[
g(E, F) = g(JE, JF)
\]

for any vector fields \( E, F \in TM \), where \( TM \) is the tangent bundle of \( M \). Before, giving our definition recall that the definition of semi-invariant submersion.

**Definition 3.1.** (\(^{13}\)) Let \( M \) be a \( 2m \)-dimensional almost Hermitian manifold with Hermitian metric \( g \) and almost complex structure \( J \), and \( N \) be a Riemannian manifold with Riemannian metric \( g_N \). A Riemannian submersion \( \pi : (M, g, J) \to (N, g_N) \) is called a semi-invariant submersion if there is a distribution \( \mathcal{D} \subset \ker \pi_* \) such that

\[
(\ker \pi_*) = \mathcal{D} \oplus \mathcal{D}^\perp, \quad J(\mathcal{D}) = \mathcal{D}, \quad J(\mathcal{D}^\perp) \subset (\ker \pi_*)^\perp,
\]

where \( \mathcal{D}^\perp \) is the orthogonal complement of \( \mathcal{D} \) in \( \ker \pi_* \).

In this case, the horizontal distribution \( (\ker \pi_*)^\perp \) is decomposed as

\[
(\ker \pi_*)^\perp = J(\mathcal{D}^\perp) \oplus \mu,
\]

where \( \mu \) is the orthogonal complementary distribution of \( J(\mathcal{D}^\perp) \) in \( (\ker \pi_*)^\perp \) and it is invariant with respect to \( J \). Note that, a semi-invariant Riemannian submersion is a natural generalization of an anti-invariant Riemannian submersion \(^{12}\). For the details, see \(^{12, 13}\).

**Definition 3.2.** Let \( \pi : (M, g, J) \to (N, g_N) \) be a semi-invariant submersion. Then we call \( \pi \) an anti-holomorphic semi-invariant submersion, if \( (\ker \pi_*)^\perp = J(\mathcal{D}^\perp) \), i.e., \( \mu = \{0\} \).

Suppose the dimension of distribution \( \mathcal{D}^\perp \) (resp. \( \mathcal{D}^\perp \)) is \( 2p \) (resp. \( q \)). Then, we have \( \text{dim}(M) = 2p + 2q \) and \( \text{dim}(N) = q \). An anti-holomorphic semi-invariant submersion is called a proper anti-holomorphic semi-invariant submersion if \( p \neq 0 \) and \( q \neq 0 \).

**Example.** Define \( \pi : \mathbb{R}^4 \to \mathbb{R} \) by \( \pi(x_1, x_2, x_3, x_4) = \frac{x_3 - x_4}{\sqrt{2}} \).

Then the map \( \pi \) is a proper anti-holomorphic semi-invariant submersion such that

\[
\ker \pi_* = \mathcal{D} \oplus \mathcal{D}^\perp, \quad \text{where} \quad \mathcal{D} = \text{span}\{\partial_1, \partial_2\}, \quad \mathcal{D}^\perp = \text{span}\{\partial_3 + \partial_4\},
\]

and \( \ker \pi^\perp = \text{span}\{\partial_4 - \partial_3\}, \partial_i = \frac{\partial}{\partial x_i} \).
4. Anti-holomorphic semi-invariant submersions from Kählerian manifolds

In this section, we start to study anti-holomorphic semi-invariant submersions from Kählerian manifolds. An almost Hermitian manifold \((M, g, J)\) is called a Kählerian manifold if

\[(\nabla_E J) F = 0\]

for all \(E, F \in TM\). Let \((M, g, J)\) be a Kählerian manifold and \((N, g_N)\) be a Riemannian manifold. Now we examine how the Kählerian structure on \(M\) places restrictions on the tensor fields \(T\) and \(A\) of an anti-holomorphic semi-invariant submersion \(\pi : (M, g, J) \to (N, g_N)\). In [15], we proved that the following lemma for Lagrangian submersions. For the details of Lagrangian submersions, see [12, 15].

Lemma 4.1. Let \(\pi\) be a Lagrangian submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then we have

a) \(T V J E = J T V E\)

b) \(A_\xi J E = J A_\xi E\),

where \(V\) is a vertical vector field, \(\xi\) is a horizontal vector field, and \(E\) is a vector field on \(M\).

It is easy to show that this lemma also holds for anti-holomorphic semi-invariant submersions.

5. Integrability and Totally Geodesicness

In this section, we shall study the integrability and totally geodesicness of the distributions which are involved in the definition of an anti-holomorphic semi-invariant submersion.

In [13], Sahin proved that the following.

Lemma 5.1. Let \(\pi\) be a semi-invariant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then

a) The anti-invariant distribution \(D^\perp\) is always integrable.

b) The invariant distribution \(D\) is integrable if and only if

\[g(T_Z J W - T_W J Z, J X) = 0\]

for \(Z, W \in D\) and \(X \in D^\perp\).

Thus, using Lemma 4.1 and (2.3), we easily conclude that the following result from Lemma 5.1.

Lemma 5.2. Let \(\pi\) be an anti-holomorphic semi-invariant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then

a) The anti-invariant distribution \(D^\perp\) is always integrable.

b) The invariant distribution \(D\) is always integrable.

Now, we state one of the main results.

Theorem 5.3. Let \(\pi\) be an anti-holomorphic semi-invariant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\). Then horizontal distribution \((\ker \pi^*)^\perp\) is integrable and totally geodesic, i.e., \(A \equiv 0\).

Proof. It is very similar to the proof of Theorem 4.5([15]), so we omit it. \(\Box\)
We remark that the vertical distribution $\text{ker}\pi_*$ is always integrable.

**Lemma 5.4.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation in the fibers $\pi^{-1}(y)$, $y \in N$.

**Proof.** Let $X, Y \in D^\perp$ and $Z \in D$. Then using (2.5) and Lemma 4.1, we get
\[
g(\hat{\nabla}_X Y, Z) = g(\nabla_X JY, Z) = g(-J\nabla_X JY, JZ) = g(J\nabla_X JY, JZ) = g(\nabla_X JY, JZ) = 0.
\]
This completes the proof. □

In a similar way, we have the following result.

**Lemma 5.5.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then the invariant distribution $D$ defines a totally geodesic foliation in the fibers $\pi^{-1}(y)$, $y \in N$.

By Lemma 5.4 and 5.5, we have that:

**Theorem 5.6.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then the fibers of $\pi$ are locally product Riemannian manifolds.

Now, we look more closely at the O'Neill’s tensor $T$ of the anti-holomorphic semi-invariant submersion $\pi$. Let $U, V \in \text{ker}\pi_*$ and $\xi \in (\text{ker}\pi_*)^\perp$. Since $(\text{ker}\pi_*)^\perp = J(D^\perp)$, there is a vertical vector field $X \in D^\perp$ such that $\xi = JX$. Then, we have
\[
g(T_U V, \xi) = g(T_U V, JX) = -g(JT_U V, X) = -g(T_U JY, X).
\]
Hence for any $V \in D$, we get
\[
g(T_U V, \xi) = 0.
\]
From (5.1), we deduce that
\[
T_U D = 0
\]
for any $U \in \text{ker}\pi_*$.

Thus, using last equation (5.2), we have the following our main result.

**Theorem 5.7.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then, we have always
\[
a) \quad T_X Z = 0 = T_Z X \quad \text{b) } \quad T_Z W = 0,
\]
where $X \in D^\perp$ and $Z, W \in D$.

At once, from Theorem 5.7, we easily see that $T_Z \xi = 0$ for any $Z \in D$ and $\xi \in (\text{ker}\pi_*)^\perp$. Thus, we have

**Corollary 5.8.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then, we have always $T_Z \equiv 0$ for $Z \in D$.

From the part a) of Theorem 5.7, we have that:
Corollary 5.9. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then the fibers of $\pi$ are always mixed totally geodesic.

From the part b) of Theorem 5.7, we get:

Corollary 5.10. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then the foliations of the invariant distribution $D$ are totally geodesic in the total space $M$.

Also from Theorem 5.7, it follows that:

Corollary 5.11. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then $T \equiv 0$ if and only if $T_{\xi}X = 0$ for all $X, Y \in D^{\perp}$, i.e., $T_{D^{\perp}}D^{\perp} = 0$.

Thus, we obtain the following result.

Corollary 5.12. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then $\ker \pi^*$ defines a totally geodesic foliation if and only if $T_{D^{\perp}}D^{\perp} = 0$.

Since the O'Neill’s tensor $A \equiv 0$, by Corollary 5.12, we have the following.

Theorem 5.13. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then, $M$ is a locally product Riemannian manifold $M_{\ker \pi^*} \times M_{(\ker \pi^*)^{\perp}}$ if and only if $T_{D^{\perp}}D^{\perp} = 0$.

6. Totally Geodesicness and Harmonicity of the anti-holomorphic semi-invariant submersion

In this section, we shall examine the totally geodesicness and harmonicity of an anti-holomorphic semi-invariant submersion. First we give a necessary and sufficient condition for an anti-holomorphic semi-invariant submersion to be a totally geodesic map. Recall that a smooth map $\varphi$ between two Riemannian manifolds is called totally geodesic if $\nabla_{\varphi_*} = 0$.

Theorem 6.1. Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then $\pi$ is a totally geodesic map if and only if $T_{D^{\perp}}D^{\perp} = 0$.

Proof. Since $\pi$ is a Riemannian submersion, we have

\begin{equation}
(\nabla_{\pi_*})(\xi, \eta) = 0
\end{equation}

for all $\xi, \eta \in (\ker \pi_*)^{\perp}$. For any $U, V \in \ker \pi_*$, using (2.5), we get $(\nabla_{\pi_*})(U, V) = -\pi_*(\nabla_U V) = -\pi_*(T_U V + \bar{\nabla}_U V) = -\pi_*(T_U V)$, since $\pi$ is a linear isometry between $(\ker \pi_*)^{\perp}$ and $TN$. Hence, it follows that $(\nabla_{\pi_*})(U, V) = 0$ if and only if $T_U V = 0$, for all $U, V \in \ker \pi_*$, that is:

\begin{equation}
(\nabla_{\pi_*})(U, V) = 0 \iff T \equiv 0.
\end{equation}

In a similar way, for any $U \in \ker \pi_*$ and $\xi \in (\ker \pi_*)^{\perp}$, using (2.7), we get $(\nabla_{\pi_*})(\xi, U) = -\pi_*(\nabla_\xi U) = -\pi_*(A_\xi V + \nabla_\xi U)$. Since $\pi$ is a linear isometry between $(\ker \pi_*)^{\perp}$ and $TN$ and $A \equiv 0$, it follows that

\begin{equation}
(\nabla_{\pi_*})(\xi, U) = 0
\end{equation}
for any $U \in \ker \pi_a$ and $\xi \in (\ker \pi_a)^\perp$. Thus, from (6.1), (6.2) and (6.3), we deduce $
abla \pi_a = 0$ if and only if $T \equiv 0$. But because of Corollary 5.11, this is equivalent to the assertion. \hfill $\square$

Now, we examine the harmonicity of the submersion. We know that a smooth map $\varphi$ is harmonic if and only if it has minimal fibers $[1]$. Thus the submersion $\pi$ is harmonic if and only if $T \equiv 0$. But because of Corollary 5.11, this is equivalent to the assertion.

Lastly, we investigate several curvatures of the total manifold and fibers and give a characterization theorem for this type submersions. First, we recall that fundamental definitions and notions. Let $(M, g, J)$ be a Kählerian manifold and $\nabla$ is the Levi-Civita connection on $M$. The Riemannian curvature tensor $[17]$ of $(M, g, J)$ is defined by $R(E, F)G = \nabla_{[E, F]}G - \nabla_E (\nabla_F G)$ for vector fields $E, F$ and $G$ on $M$. We put $R(E, F; G, \bar{G}) = g(R(E, F) G, \bar{G})$ where $\bar{G}$ is a vector field on $M$. The sectional curvature $K(E, F)$ of the plane $\sigma$ spanned by the orthogonal unit vector fields $E$ and $F$, is defined by

$$K(E, F) = R(E, F; E, F).$$

The holomorphic bisectional curvature $[5]$ of $M$ is defined for any pair unit vector fields $E$ and $F$ tangent to $M$ by

$$B(E, F) = R(E, J E; F, J F).$$

Then the holomorphic sectional curvature $[5, 17]$ of $M$ is given by

$$H(E) = B(E, E).$$

The manifold $M$ is called a complex space form if it is of constant holomorphic sectional curvature. We denote by $(M, g, J)(c)$ a complex space form of constant holomorphic sectional curvature $c$. Then the Riemannian curvature tensor $R$ of $(M, g, J)(c)$ is given by

$$R(E, F) G = \frac{c}{4} \{ g(F, G) E - g(E, G) F + g(J F, G) J E - g(J E, G) J F + 2 g(E, J F) J G \}.$$
for any vector fields $E, F$ and $G$ on $M$. Hence, we have

\[(7.5) \quad B(E, F) = \frac{1}{2} \{g(E, E)g(F, F) + (g(E, F))^2 + (g(E, JF))^2\}.\]

We note that a Kählerian manifold with vanishing holomorphic sectional curvature is flat \cite{5, 17}.

In view of the O’Neill’s curvature formulas \{0\}, \{1\}, \{2\}, \{2’\} \cite{9}, Lemma 4.1, Theorem 5.7 and Corollary 5.8, from (7.1), we get the following curvature formulas.

**Theorem 7.1.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$ and let $K, \hat{K}$ and $K_*$ be the sectional curvatures of the total space $M$, fibers and the base space $N$, respectively. Then

\[(7.6) \quad K(X, Y) = \hat{K}(X, Y) - g(T_X X, T_Y Y) + \|T_X Y\|^2,\]

\[(7.7) \quad K(X, Z) = \hat{K}(X, Z),\]

\[(7.8) \quad K(Z, W) = \hat{K}(Z, W),\]

\[(7.9) \quad K(X, \xi) = g((\nabla_\xi T)_X X, \xi) - \|T_X \xi\|^2,\]

\[(7.10) \quad K(Z, \xi) = g((\nabla_\xi T)_Z Z, \xi)\]

\[(7.11) \quad K(\xi, \eta) = K_*(\xi, \eta),\]

where $X, Y \in D^\perp$, $Z, W \in D$, $\xi, \eta \in (\ker \pi_\ast)^\perp$, $\xi_\ast = \pi_\ast(\xi), \eta_\ast = \pi_\ast(\eta)$, and all of them are unit vector fields.

From (7.6), (7.7) and (7.8), we have the following result.

**Corollary 7.2.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$. Then any fiber of $\pi^{-1}(y)$ of $\pi$ has constant sectional curvature if and only if $g(T_X X, T_Y Y) = \|T_X Y\|^2$ for all $X, Y \in D^\perp$.

**Theorem 7.3.** Let $\pi$ be an anti-holomorphic semi-invariant submersion from a Kählerian manifold $(M, g, J)$ onto a Riemannian manifold $(N, g_N)$ and let $B$ and $\hat{B}$ be the holomorphic bisectional curvatures of the total space $M$ and fibers, respectively. Then

\[(7.12) \quad B(X, Y) = g((\nabla_\xi T)_X Y, JY) - g(T_X X, T_Y Y),\]

\[(7.13) \quad B(X, Z) = 0,\]

\[(7.14) \quad B(Z, W) = \hat{B}(Z, W),\]

\[(7.15) \quad B(X, \xi) = -g((\nabla_\xi T)_X J\xi, \xi) + g(T_X JX, T_J \xi),\]

\[(7.16) \quad B(Z, \xi) = g((\nabla_\xi T)_Z J\xi, \xi) - g((\nabla_\xi T)_Z J\xi, \xi),\]

\[(7.17) \quad B(\xi, \eta) = g((\nabla_\xi T)_Z J\xi, \eta) - g(T_J \xi, T_J \eta),\]

where $X, Y \in D^\perp$, $Z, W \in D$, $\xi, \eta \in (\ker \pi_\ast)^\perp$, and all of them are unit vector fields.
Proof. From (7.2), using similar arguments which used in Theorem 7.1, we get all curvature formulas above except (7.13). Next, we prove (7.13). Using the O’Neill’s formula \[ \{9\], we have

\[
B(X, Z) = g((\nabla JX)Z, X) - g((\nabla Z)JX, JX)
\]

for unit vector fields \(X \in D^\perp\) and \(Z \in D\). After some calculation, from (7.18), we get

\[
B(X, Z) = g(\tilde{T}_{\nabla Z}JX - \tilde{T}_{\nabla JX}X, JX).
\]

Because of Lemma 5.5, we know that \(\tilde{T}_{\nabla Z}JX, \tilde{T}_{\nabla JX}X \in D\). Hence, by Theorem 5.7, we find \(\tilde{T}_{\nabla Z}JX = \tilde{T}_{\nabla JX}X = 0\). Thus, (7.19) gives \(B(X, Z) = 0\). □

We have immediately from Theorem 7.3 that:

**Corollary 7.4.** Let \(\pi\) be an anti-holomorphic semi-invariant submersion from a Kählerian manifold \((M, g, J)\) onto a Riemannian manifold \((N, g_N)\) and let \(H\) and \(\tilde{H}\) be the holomorphic sectional curvatures of the total space \(M\) and fibers, respectively. Then

\[
H(X) = g((\nabla JX)X, JX) - \|T_X X\|^2
\]

\[
H(Z) = \tilde{H}(Z),
\]

\[
H(\xi) = g((\nabla Z)JX, JX) - \|T_X \xi\|^2
\]

where \(X \in D^\perp\), \(Z \in D\) and \(\xi \in (\ker \pi^\perp)^\perp\) and all of them are unit vector fields.

With the help of (7.4) and (7.5), from (7.13), we have the following result.

**Theorem 7.5.** Let \(\pi\) be a proper anti-holomorphic semi-invariant submersion from a complex space form \((M, g, J)(c)\) onto a Riemannian manifold \((N, g_N)\), then \(c = 0\). In other word, the total space is flat. In particular, there exists no proper anti-holomorphic semi-invariant submersion from a complex space form \((M, g, J)(c)\) with \(c \neq 0\).

From Theorem 7.5, we deduce that:

**Theorem 7.6.** Let \(\pi\) be an anti-holomorphic semi-invariant submersion from a complex space form \((M, g, J)(c)\) with \(c \neq 0\) onto a Riemannian manifold \((N, g_N)\), then \(\pi\) is either an anti-invariant submersion (Lagrangian case) or an almost Hermitian submersion (Kählerian case).

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İSTANBUL UNIVERSITY, DEPARTMENT OF MATHEMATICS, VEZNECİLER, İSTANBUL, TURKEY
E-mail address: hakmete@istanbul.edu.tr