Students’ Algebraic Thinking Process in Context of Point and Line Properties

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Abstract. Learning of schools algebra is limited to symbols and operating procedures, so students are able to work on problems that only require the ability to operate symbols but unable to generalize a pattern as one of part of algebraic thinking. The purpose of this study is to create a didactic design that facilitates students to do algebraic thinking process through the generalization of patterns, especially in the context of the property of point and line. This study used qualitative method and includes Didactical Design Research (DDR). The result is students are able to make factual, contextual, and symbolic generalization. This happen because the generalization arises based on facts on local terms, then the generalization produced an algebraic formula that was described in the context and perspective of each student. After that, the formula uses the algebraic letter symbol from the symbol that uses the students’ language. It can be concluded that the design has facilitated students to do algebraic thinking process through the generalization of patterns, especially in the context of property of the point and line. The impact of this study is this design can use as one of material teaching alternative in learning of school algebra.

1. Introduction
Learning of schools algebra is only fixed on symbol manipulation and understanding of procedures [1]. Just because of the letter symbols that are operated in accordance with the procedure, then the teachers consider that it is learning about algebra. In fact, Radford stated that letters does not characterize algebra learning [2].

That algebra learning have caused problems that many researchers have highlighted. Walkoe suggested that students in Los Angeles were not successful in algebra classes, as many as 44% of students in Los Angeles who took algebra the first year received a failed score [1]. Research conducted during the 1970s and 80s showed that students faced some difficulties in the subject of algebra [3]. In Indonesia, there are still problems with algebra learning. Students are able to work on problems that only require the ability to operate symbols but they are unable to work on problems that require the ability to generalize a pattern. When students were asked to solve generalizations about the property of point and line as follows

From 2 different points can be made exactly 1 line. From 3 different points and noncollinear, there are at most 3 lines that can be made. From 4 different points and no 3 points are collinear, there are at most 6 lines that can be made.
Illustration:

When given different points that are no 3 points are collinear and are not known in number, then how many lines can be made at most? They don’t make conjecture that the patterns form a regularity. There are students who find terms relationship in patterns but they are difficult to find the generalization of the pattern. For students who have studied the sequence and series, they have idea for using formula of the arithmetic series addition but they are difficult to remember what the formula is. That’s the tradition of mathematics learning that seem to have to solve problems using the formula, making learning meaningless and students forget it easily. Factor that cause student’s difficulties is the lack of students experience to solve the problems that require the ability to generalize the pattern using their creativity. Lack of experience can occur because of learning system that teach algebraic thinking is lack. In fact, as has been explained before, thinking algebra is an important thing to learn. Based on this, the difficulty can be categorized into didactical obstacle. Brousseau stated that didactical obstacle is a learning difficulty that occurs because seem to depend only on a project within an educational system [4]. The existing education system in schools lack of giving the experience of algebraic thinking problems, especially about the generalization of patterns. In fact, in mathematical textbooks published by the government there has been one problem of generalizing patterns in line and corner material, but it is not given as experience to students.

Hitt et al. wrote the results of his research about arithmetic-algebra transition process in seventh grade students [5], Radford introduced algebraic thinking forms through his research [6] and also wrote the results of his longitudinal research on the beginning of algebraic thinking embodiment in elementary school students [7]. Furthermore, Blanton et al. wrote the results of his research on third grade students to develop algebraic thinking [8]. However, based on that researchs, no one who study on algebraic thinking (with the generalization context) linked with the context of geometry about the property of point and line. Learning obstacle test results also show the problems about that. In fact, algebraic thinking is important to mastered by the students and for beginners it still need a spatial structure (which is related to the geometry structure) to understand it.

Because the problems presented are based on the student learning obstacle, the study conducted in the form of Didactical Design Research (DDR). The purpose of this study is to create a didactic design that facilitates students to do algebraic thinking process through the generalization of patterns, especially in the context of geometry about the property of point and line.

1.1. Theoretical background

Point and line are undefined elements that construct geometry [9]. One of postulate about point and line is if two different points are given then there is exactly one line containing them [10]. From that postulate, author related it with generalization in algebraic thinking. For students in Junior High School, that postulate is called “property” by author. The purpose is for simplify students understanding.

Warren and Cooper argued that the power of mathematics lies in relationships and transformations that give rise to patterns and generalizations [11], so Kaput and Blanton argued that the teaching of mathematics should focus on directives to foster basic skills in generalizing and also expressing and justifying a systematic generalization [11]. This is in line with the statement of Mason et al. that every student starting school have to be given the ability to generalize in certain cases, and this is the root of algebra [12]. Becker and Rivera also stated that the ability to generalize is an important aspect of algebraic thinking and reasoning [12]. Radford revealed that the generalization of the pattern is used as a path to algebra [13].
Thus, algebra learning should not be limited to symbols and operating procedures. Moreover, algebra learning is related to reasoning and ways of thinking. Tunks and Weller suggested that algebraic reasoning is ways of thinking involving semiotic forms that capture the essence of patterns, functions, structures, or modeling situations [14]. Radford argued that algebraic thinking is not enough just to condition the use of the letter (as a symbol) only [7].

Based on the research reviewed by Radford, he stated there are 3 characteristics of algebraic thinking, namely indeterminacy, denotation and analyticity [7]. Indeterminacy is described that the problems in algebraic thinking involve an unknown number, in terms of equations, variables, parameters, and so on. Denotation means that the indeterminate numbers (involved in the problem) must be named or denoted. As for analyticity is described that indeterminate numbers are treated as if they are known, meaning that they are treated with add, subtract, multiply, or divide operation. The generalization of the algebraic pattern according to Radford is depends on the capability of a sequence $S$, recognizing that this similarity applies to the terms of $S$ and can use it to give a direct expression of any term of $S$ [2].

2. Method

This study use qualitative method and because of the purpose of this study is to create a didactic design based on students learning obstacle, this study includes Didactical Design Research (DDR) with three steps of analysis. First, didactical situation analysis before the learning begins and it is realized as Hypothetic of Didactic Design. Second, metapedadidactic analysis, it is analysis of the occurrence of didactic and learning situational changes. Third, retrospective analysis, it is analysis that relates between the results of hypothetic didactical situation analysis and metapedadidactic analysis.

Participants were divided into two groups. The first group of participants is when the authors tested learning obstacles to find students difficulties of learning, they are the eighth grade students in one of Junior High School in Bandung and tenth grade students in one of Senior High School in Bandung. The second group of participants is when the authors do research using didactic designs about the property of the point and line associated with the algebraic thinking process, they are seventh grade students in one of Junior High School in Bandung.

3. Result and Discussion

The didactic design is developed based on students difficulties that have been reviewed previously. Because of the students’ difficulty that was found is the lack of students’ experience to solve problems that require the ability to generalize the patterns, then the designs are made to habituate students in solving the generalization problems of patterns from the following terms in Figure 1.

![Figure 1. Figure of the terms to habituate the generalization of patterns](image-url)

After that, students are given a generalization of the pattern for the property of the point and line (as mentioned in the introduction).

To guide students to generalizations, on the problem of generalization of patterns for the property of point and line, first the students are asked how many number of lines at most in the fourth and fifth terms. The purpose is that students can see patterns locally.

When searching for many lines in the fourth term, some students draw as follows in Figure 2.
Figure 2. Figure of the first type of student’s answer for the fourth term

Figure 2 shows that student drew 5 points and connected every 2 points into a line until she found the number of lines, there are 10 lines. Beside that, there were also students who determined the number of lines as follows in Figure 3.

![Figure 3](image)

Figure 3. Figure of the second type of student’s answer for the fourth term

Figure 3 shows that student determined the number of lines by forming the pattern of numbers that are listed in the picture (in the question) first, then he drew 5 points and connected every 2 points into a line until he found the number of lines. He answered like that because he sees the figure of each terms. The first term that have 1 line is connected with the number in the wanted term so to determine the second term he sees 1 (the number of lines in the first term) and 2 (the number of wanted term, it is the second term), then adds 1 and 2 to produce 3 which is the number of lines in the second term. Likewise the third term, he sees 3 (the number of lines in the second term) and 3 (the number of wanted term, it is the third term), then adds 3 and 3 to produce 6 which is the number of lines in the third term, and so on. So to find the number of lines in the fifth term he answered like that.

There are also student’s answered like in Figure 4.

![Figure 4](image)

Figure 4. Figure of the type of student’s answer for the fifth term

Figure 4 shows that student drew first, after that he found the pattern number by adding the number of lines in the previous term with the number of the wanted term.

To extend a figural sequence, students need to understand a regularity that involves the linkage of two different structures, they are spatial and numerical [7]. In determining the fourth and fifth term, although the students draw the lines first, in finding the patterns locally, they still see the similarity of the numbers in each term, it is means they are stronger in seeing the numeric structure than spatial or geometric structure. What this means is that the attention on the numerical structure somehow leaves in the background the geometric structure [7].

After that activity, the students were instructed to determine the 117th term in order to see the consistency of the patterns found by the students and also to allow students to find an easier way to determine the number of lines in large numbers of term. In the process of finding this 117th term, they were difficult to find the amount of $1 + 2 + 3 + ... + 117$. Some are counting manually, some are using a calculator. Figure 5 shows a proof of one of student’s answer.
Figure 5. Figure of the first type of student’s answer for the 117th term

Figure 5 shows that the answer indicates that the student is able to make a pattern for a particular term, even if it is not practical. But he has seen the similarity of the patterns of local terms. Because that student’s way was impressed needing a long time, finally there is student who find the following way.

![Figure 5](image1)

Figure 6. Figure of the second type of student’s answer for the 117th term

Figure 6 shows that the student finds the pattern by looking at the number of a term and the number of points on that term. For example, for the fourth term there are 5 points resulting 10 lines, he sees 4, 5, and 10 then thinks that 10 is resulted by multiplying 5 by 4/2. For the fifth term there are 6 points resulting 15 lines, he sees 5, 6, and 15 then thinks that 15 is resulted by multiplying 5 by 6/2. So for the 117th term there are 118 points will result 6.903 lines by multiplying 117 by 118/2. He concludes that in order to find the number of lines in a term that is, if the term number is even, then the number of the wanted term is divided by two then multiplied by the number of points on the term and if the term number is odd, then the number of points in the term is divided by two then multiplied by the number of the wanted term.

The student's answer shows a more practical way than the previous student. Student has been able to make the relationship between the term numbers and the number of lines on that term by looking at the number of points on that term. That is, students are able to make generalization called factual generalization [2], which is the basic level generalization for local term. This generalization arises based on facts on local terms which are then referred to as factual algebraic thinking [6]. In this case, students have begun to show the characteristics of indeterminacy, although still in certain terms. Apparently, with the design of situations to explore students' abilities, students' knowledge is formed as an "optimal" solution to a particular situation (problem) [15].
After that, the students were asked how to count the number of lines in each term in order to see the generalization language that emerged from the students’ minds. There are students who answer as follows in Figure 7.

\[ \text{jumlah garis suku yg sebelumnya} + \text{suku yg dicari} \]

**Figure 7.** Figure of the first type of student’s answer for find any term in patterns

Beside that, there are also students who answered as follows in Figure 8.

\[ 1 + 2 + 3 + y \ldots \text{sampai suku yang ditanyakan} \]

**Figure 8.** Figure of the second type of student’s answer for find any term in patterns

Figure 7 and 8 show that students begin to denote indeterminacy with their own language according to the context of the pattern asked. The denotation is obtained based on the pattern regularity analysis of the first, second, third, and so on. They are able to create a generalized “formula” with their own minds, although the generalization is symbolized by “words”. They realize that the similarity patterns of certain term can be used in each term. This generalization is a deeper level of generalization from factual generalization, called contextual generalization [2]. This generalization produced an algebraic formula that was described in the context and perspective of each student so that Radford named it contextual algebraic thinking [6]. Unfortunately, students who find a more practical pattern do not write the results of their generalizations.

The core of the series of activities that have been done is to make students habit in making a generalization of the pattern until they can denote the pattern. In fact, the activity when students answer as in the figure 7 and 8, it also includes denoting activity, but when students are asked to make a simpler notation, they can denote like in Figure 9.

\[ y + x \]

**Figure 9.** Figure of student’s answer for make a notation in patterns

With a little direction, Figure 9 show that students can create algebraic letter symbols. Initially, the authors thought that this symbolization will take very long time. But it turns out that students are able to use this symbol quickly, although initially it takes time to think long enough. This happens because they have learned about the variables in the previous semester. The formula produced by the students is the result of generalization that uses the algebraic letter symbol from the symbol that uses the students’ language. This generalization is called symbolic generalization [2]. The thinking of students that able to enter this stage can be categorized into standard algebraic thinking [6]. They are able to create a formula expression directly for any term being asked.
4. Conclusion
Based on the discussion that has been done, it can be concluded that the design that has been made has facilitated the students to do algebraic thinking process through the generalization of patterns, especially in the context of property of the point and line. Students are able to generalize patterns and denote patterns using their own language. However, this study is still not completed because in fact students can be facilitated to denote a more practical pattern. Therefore, for further study students should be given direction to denote a more practical pattern than the patterns that have been found. The impact of this study is this design can use as one of material teaching alternative in learning of school algebra.

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