Inverse scattering transform analysis of
Stokes-anti-Stokes stimulated Raman scattering

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Abstract

Zakharov-Shabat–Ablowitz-Kaup-Newel-Segur (ZS–AKNS) representation for Stokes-anti-Stokes stimulated Raman scattering (SRS) is proposed. Periodical waves, solitons and self-similarity solutions are derived. Transient and bright threshold solitons are discussed.

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1 Introduction

The equations that describe the propagation in a Raman active medium when Stokes $E_s$, anti-Stokes $E_a$ and pump $E_p$ exists and when there is no frequency mismatch can be written [1, 2, 3, 4, 5, 6] in the form:

\[
\begin{align*}
\frac{\partial E_p}{\partial \zeta} &= \beta_a Q^* E_a - Q E_s, \\
\frac{\partial E_s}{\partial \zeta} &= Q^* E_p, \\
\frac{\partial E_a}{\partial \zeta} &= -\beta_a Q E_p, \\
\frac{\partial Q}{\partial \tau} + \tilde{g} Q &= E_s^* E_p + \beta_a E_p^* E_a,
\end{align*}
\]

(1)

where $Q$ is the normalized effective polarization of the medium, $\zeta = z/L$ and $\tau = t - z/v$ are dimensionless space and retarded time coordinates, respectively, $v$ is the wave group velocity. $T_2$ is the natural damping time of the material excitation and $\tilde{g} = 1/T_2$. By $\beta_a$ we denote the coupling coefficient which determines the number of anti-Stokes photons relative to number of Stokes photons and its magnitude depends on the matrix element that describes the dipole transition [1, 7]. In this paper we consider $\beta_a = 1$ [1], but some results are valid for $\beta_a \neq 1$. For $\beta_a = 0$ i.e. when the anti-Stokes wave $E_a$ is neglected, these equations are the so called transient stimulated Raman scattering (SRS) equations, which possess ZS–AKNS [8, 9] representation when $\tilde{g} = 0$ [10]. The SRS soliton solutions theoretically discovered by Chu and Scott [10] have been experimentally observed first in [11]. The SRS solitons (regarded as transient solitons [12, 13, 14] with a $\pi$ phase jump at the Stokes frequency) have been extensively studied [15]. In later experiments by Duncan et al [16] a careful comparison between theory and experiment showed good agreement. Shortly after this work [16], Hilfer and Menyuk [17] carried out simulations which indicate that in the highly depleted regime the solutions of transient SRS equations always tend toward a self-similar solution. This result has been recovered by Menyuk et al [18, 19] applying the inverse scattering transform method (ISM) to the transient SRS equations, see [12]. Experiment to observe this solution has been proposed in [18]. The Kaup’s theory [13] also indicates that the dissipation, which appears for finite $T_2$ plays a crucial role in soliton formation. The similarity solutions and other group invariant solutions of the SRS equations in the presence of dissipation are studied in [20]. Claude and Leon [21] reformulated the transient SRS equations as an equivalent $\hat{\partial}$–problem and thus were able
to treat the inhomogeneous broadening with initial conditions of Drühl et al [10] experiments. As a consequence they showed that Raman spike observed in the experiment is not a soliton.

For general system (1) with anti-Stokes wave, phase mismatch, $\beta_a \neq 1$ and dissipation, transient $\pi$ solitons have been investigated by M. Scalora et al [11] using numerical methods. They predict the formation of solitonlike pulses at the anti-Stokes frequency. Another type threshold bright $2\pi$ solitons, which have Lorentzian form have been theoretically obtained in [22].

In sect. II we introduce new variables $S_3, S_\pm$ which are quadratic in terms of $E_p, E_s, E_a$. Then the system (1) for $S_3, S_\pm$ derived from (1) with $\beta_a = 1, \tilde{g} = 0$ allows ZS-AKNS representation similar to the one used by Chu and Scott [10] for another physical quantities: difference of normalized Stokes–anti-Stokes local intensities and for normalized (complex) local Rabi frequency. We also introduce “nonlinear time” and renormalized dimensionless variables different from the ones used in [13]. Then we solve the inverse scattering problem (ISP) for the system (14) with the “nonlinear time” $\tau'$ restricted to the finite interval $0 \leq \tau' \leq 1$; i.e. we derive the corresponding Gel’fand–Levitan–Marchenko (GLM) equation.

In sect. III we obtain new periodic, soliton and self–similarity solutions of Stokes–anti-Stokes SRS equations without dissipation are obtained for both formulations (1) and (3). In addition the transient solitons and the bright solitons of Kaplan et al [22] are discussed.

In the last Sect. IV we propose an extension of the Stokes–anti-Stokes SRS equations for $N$ Stokes and $N$ anti-Stokes waves and conjecture that it is also integrable by means of the IST method.

## 2 ZS–AKNS Representation and GLM Equation

### 2.1 ZS–AKNS Representation

Let us introduce the following variables

$$S_3 = \frac{1}{2}(|E_s|^2 - |E_a|^2), \quad S_+ = \frac{i}{2}(E_s^*E_p + E_p^*E_a), \quad S_- = S^*_+.$$  \hspace{1cm} (2)
In terms of the new quadratic variables (2) the initial system with $\tilde{g} = 0, \beta_a = 1$ is rewritten as

$$\frac{\partial S_3}{\partial \zeta} = -iQ^*S_+ + iQS_-, \quad \frac{\partial S_+}{\partial \zeta} = -iQS_3, \quad \frac{\partial Q}{\partial \tau} = -2iS_+.$$  \hfill (3)

Then the eq. (3) can be written down as the compatibility condition

$$\partial_\tau U - \partial_\zeta V + [U, V] = 0,$$  \hfill (4)

of the following linear systems:

$$L(\lambda)F(\zeta, \tau, \lambda) \equiv \frac{\partial F}{\partial \zeta} - U(\zeta, \tau, \lambda)F(\zeta, \tau, \lambda) = 0,$$  \hfill (5)

$$M(\lambda)F(\zeta, \tau, \lambda) \equiv \frac{\partial F}{\partial \tau} - V(\zeta, \tau, \lambda)F(\zeta, \tau, \lambda) = F(\zeta, \tau, \lambda)C(\lambda),$$  \hfill (6)

where

$$U(\zeta, \tau, \lambda) = -\frac{i}{\lambda}\sigma_3 + \frac{1}{\sqrt{2}}q(\zeta, \tau), \quad V(\zeta, \tau, \lambda) = \frac{\lambda}{2i}S(\zeta, \tau),$$  \hfill (7)

$$q(\zeta, \tau) = \begin{pmatrix} 0 & Q \\ -Q^* & 0 \end{pmatrix}, \quad S(\zeta, \tau) = \begin{pmatrix} S_3 & -i\sqrt{2}S_+ \\ i\sqrt{2}S_- & -S_3 \end{pmatrix}.$$  \hfill (8)

and $C(\lambda)$ will be fixed up below.

From physical point of view [10, 11] the initial value problem associated to the system (3) is the following

$$Q(\zeta, 0) = 0, \quad E_p(0, \tau) = E_{p0}(\tau),$$  \hfill (9)

$$E_s(0, \tau) = E_{s0}(\tau), \quad E_a(0, \tau) = E_{a0}(\tau)$$

and the problem is to determine the output quantities $E_p(L, \tau), E_s(L, \tau), E_a(L, \tau)$, where $L$ is the total length of the beam path in the Raman cell. Analogically the initial value problem for system (3) is

$$Q(\zeta, 0) = 0, \quad S_3(0, \tau) = S_{30}(\tau), \quad S_+(0, \tau) = S_{+0}(\tau).$$  \hfill (10)

We follow the main idea of [12, 20], namely that as a Lax operator one should consider the operator $M(\lambda)$ in (3) and solve the inverse scattering
problem for it. Then we will use the second operator $L(\lambda)$ in (5) and determine the $\zeta$–dependence of the corresponding scattering data. However there will be substantial differences in the details.

First of all we will approach the inverse scattering problem for the $M(\lambda)$ operator directly rather than via its gauge equivalence to a ZS–AKNS type system. Indeed, this equivalence is realized with $F(\zeta, \tau, \lambda)$ evaluated at $\lambda = 0$. However, in our case the other linear problem has a pole singularity at $\lambda = 0$. Due to this fact makes one cannot evaluate the $\zeta$–dependence of the gauge function, which makes impossible the comparison with the results in [12, 20].

It is well known how to solve the ISP for the system (6) considered on the whole $\tau$–line $-\infty \leq \tau \leq \infty$ and with boundary conditions of ferromagnetic type, i.e. $\lim_{\tau \to \pm \infty} S(\zeta, \tau) = \sigma_3$, see [9]. We will make use of these ideas adopting them to our case. First we have to take into account that the eigenvalues of our $S(\zeta, \tau)$ differ from $\pm 1$ and are generically $\tau$–dependent. In order to calculate them it is enough to know, that $\text{tr} S(\zeta, \tau) = 0$ and

$$-\det S(\zeta, \tau) = S_3^2 + 2S_+S_- = \frac{1}{4} \left(|E_a|^2 - |E_s|^2\right)^2 + \frac{1}{2} |E_p E_a + E_p E_s|^2 = K^4(\tau). \tag{11}$$

Using the evolution equations (1) we check that

$$\frac{dK}{d\zeta} = 0. \tag{12}$$

Besides from (11) we conclude that $K(\tau)$ is real–valued function. In order to proceed further we require in addition that $K^2(\tau)$ is monotonic function of $\tau$. Then we can introduce a new “nonlinear time” $\tau'$ by

$$d\tau' = K^2(\tau)d\tau \tag{13}$$

and the following dimensionless variables:

$$\tau' = \int_0^\tau K^2(\tau'')d\tau''/T_\infty, \quad T_\infty = \int_0^\infty K^2(\tau)d\tau, \quad \zeta' = \zeta T_\infty \tag{14}$$

$$E_p' = \frac{E_p}{K(\tau)}, \quad E_s' = \frac{E_s}{K(\tau)}, \quad E_a' = \frac{E_a}{K(\tau)}, \quad Q' = \frac{Q}{T_\infty}.$$

The primed variables introduced above satisfy the same system (1) of NLEE provided $\tilde{g}' = \tilde{g} T_\infty/K^2$; in what follows below we put $\tilde{g} = 0$. Note, that the
transformation \( \{E_p, E_s, E_a, Q\} \rightarrow \{E'_p, E'_s, E'_a, Q'; K(\tau)\} \) is one-to-one and invertible. In order to obtain the evolution of \( \{E_p, E_s, E_a, Q\} \), one must first determine the evolution of \( \{E'_p, E'_s, E'_a, Q'\} \) and then use the given function \( K(\tau) \) to return to the original variable set.

The nonlinear time \( \tau' \) is introduced in analogy to the one in [23]; the difference is that now \( K^2(\tau) \) can not be interpreted as the total energy density:

\[
E(\tau) = |E_p(\zeta, \tau)|^2 + |E_s(\zeta, \tau)|^2 + |E_a(\zeta, \tau)|^2, \tag{15}
\]

which is constant at every point \( \tau \) as function of \( \zeta \). Since all physical solutions possess finite energy we conclude, that each of the terms in (15) must be integrable functions of \( \tau \). In particular, each of these functions must vanish for \( \tau \rightarrow \infty \). As a consequence of this fact we conclude that \( K^2(\tau) \) must have the same properties. Therefore for this class of solutions we have \( T_\infty < \infty \) and in terms of \( \tau' \) we get the system:

\[
M'(\lambda)F(\zeta, \tau', \lambda) \equiv \frac{\partial F}{\partial \tau'} - \lambda S'(\zeta, \tau')F(\zeta, \tau', \lambda) = 0, \quad S'(\zeta, \tau') = \frac{S(\zeta, \tau)}{K^2(\tau)} \tag{16}
\]

where \( S'(\zeta, \tau') \) satisfies \( \text{tr} S' = 0 \) and \( \det S' = -1 \). As a result the eigenvalues of \( S'(\zeta, \tau') \) become equal to \( \pm 1 \), i.e. we can write down

\[
S'(\zeta, \tau') = g(\zeta, \tau')\sigma_3g^{-1}(\zeta, \tau') \tag{17}
\]

Note that the constancy of \( E(\tau) \) corresponds to pointwise conservation of the photon intensity. From Eq. (1) it is easy to derive also the following important relation:

\[
\frac{1}{2} \frac{\partial}{\partial \tau}|Q|^2 + \tilde{g}|Q|^2 + \frac{\partial}{\partial \zeta}S_3 = 0. \tag{18}
\]

from which for \( \tilde{g} = 0 \) we find that \( \int |Q|^2d\zeta \) is an integral of motion if \( S_3(0, \tau) - S_3(L, \tau) = 0 \). This will be fulfilled if \( E_a \) and \( E_s \) satisfy quasiperiodic boundary conditions, i.e., if \( E_{a,s}(\tau' = 0) = e^{i\phi_{a,s}}E_{a,s}(\tau' = 1) \) with any \( \phi_{a,s} \).

In the next subsection we will use only renormalized quantities and the “nonlinear time” \( \tau' \) and for the simplicity of the notations will drop all primes.
2.2 The GLM equation

We briefly sketch the derivation of the GLM equation related to the left end \( \tau = 0 \) of the interval. Of course we have to introduce also slight modifications in order to take into account the fact that \( S(\tau = 0, \zeta) = S_0(\zeta) \neq \sigma_3 \). The operator \( M(\lambda) \) on finite interval generically possesses purely discrete spectrum with an infinite number of simple discrete eigenvalues. As a consequence, the kernel of the GLM equation contains only a sum over the discrete spectrum. Skipping the details we write down the results.

Let the Jost solution of (16) normalized to the left end \( \tau = 0 \) of the interval, be fixed up by:

\[
\phi_0(\tau, \zeta, \lambda) = g_0 e^{\frac{\lambda \tau \sigma_3}{2} i}, \quad \lim_{\tau \to 0} \phi(\tau, \zeta, \lambda) = \lim_{\tau \to 0} \phi_0(\zeta, \lambda) = 1, \quad (19)
\]

\[
\lim_{\tau \to 0} \phi(\tau, \zeta, \lambda) = \lim_{\tau \to 0} g_1(\zeta) e^{\frac{\lambda \tau \sigma_3}{2} i} T(\zeta, \lambda), \quad (20)
\]

\[
g_0(\zeta) = g(\zeta, \tau = 0), \quad g_1(\zeta) = g(\zeta, \tau = 1), \quad (21)
\]

Then its behaviour at \( \tau \simeq 1 \) determines the scattering matrix \( T(\zeta, \lambda) \) according to (20). It remains now to evaluate the “evolution” of \( T \) in \( \zeta \). In order to do this we have to calculate first \( C(\lambda) \) in (14) by taking the limit of (19) for \( \tau \to 0 \) with the result \( C(\zeta, \lambda) = (i/\lambda) g_0^{-1} \sigma_3 g_0(\zeta) \). Then we take the limit of (19) for \( \tau \to 1 \) which gives the following result for the evolution of \( T(\zeta, \lambda) \):

\[
\frac{dT}{d\zeta} = i \frac{\lambda}{\lambda} \left( T(\zeta, \lambda) g_0^{-1} \sigma_3 g_0 - g_1^{-1} \sigma_3 g_1 T(\zeta, \lambda) \right) \quad (22)
\]

As we already noted, in order to apply the ISM for the solution of our problem we will need first to calculate not only \( g_0(\zeta) \), which is determined from the initial conditions, but also \( g_1(\zeta) \). The situation is greatly simplified if we impose a quasiperiodic boundary conditions on the fields \( E_{a,s,p}(\zeta, \tau) \) in such a way, that \( S_0(\zeta) = S_1(\zeta) \). Then we get \( g_0(\zeta) = g_1(\zeta) \) and the r.h. side of (22) becomes proportional to the commutator \( [T, g_0^{-1}(\zeta) \sigma_3 g_0(\zeta)] \). The importance of this imposition can be seen from the fact, that it immediately provides us with the hierarchy of conservation laws. The generating function of this hierarchy is \( \text{tr} \ T(\lambda) \), which is now \( \zeta \)-independent.

Let us also be given \( S_0(\zeta) = S(\zeta, \tau = 0) \) and let it be diagonalizable in the form:

\[
S_0(\zeta) = g_0(\zeta) \sigma_3 g_0^{-1}(\zeta), \quad g_0(\zeta) = g_1(\zeta, \tau = 0). \quad (23)
\]
We introduce the transformation operator which relates the Jost solution \( \phi(\tau, \zeta, \lambda) \) to its asymptotic \( \phi_0(\zeta, \lambda) \) (19):

\[
\phi(\tau, \zeta, \lambda) = \phi_0(\zeta, \lambda) + \int_0^\tau \Gamma_-(\tau, z; \zeta) \phi_0(z, \zeta, \lambda) dz
\] (24)

We have to keep in mind also that all solutions and the scattering matrix of the system (16) are meromorphic functions of \( \lambda \).

Then we obtain that \( \Gamma_-(\tau, y; \zeta) \) must satisfy the following GLM–type equation:

\[
\Gamma_-(\tau, y; \zeta) + S_0(\zeta) K(\tau + y; \zeta) + \int_0^\tau \Gamma_-(\tau, z; \zeta) K'(\tau + z; \zeta) dz = 0, \tag{25}
\]

where the kernel \( K(\tau; \zeta) \) and its derivative \( K' = dK(\tau; \zeta)/d\tau \) are given by:

\[
K(\tau; \zeta) = g_0(\zeta) \begin{pmatrix} 0 & k \\ -k^* & 0 \end{pmatrix} g_0^{-1}(\zeta), \quad k(\tau; \zeta) = -\sum_{\lambda_j \in \mathcal{S}} \frac{m_j(\zeta)}{\lambda_j} e^{i\lambda_j \tau/2}. \tag{26}
\]

Here \( \lambda_j \) are the discrete eigenvalues of \( M(\lambda) \) and \( m_j(\zeta) \) is related to the norm of the corresponding Jost solution of (16); generically \( \lambda_j \) may depend also on \( \zeta \).

The corresponding potential of (16) is recovered from the solution \( \Gamma_-(\tau, y; \zeta) \) of (25) through:

\[
S(\tau, \zeta) = B_-(\tau, \zeta) S_0(\zeta) B^{-1}_-(\tau, \zeta), \quad B_- (\tau, \zeta) = 1 + \Gamma_-(\tau, \tau, \zeta) S_0(\zeta). \tag{27}
\]

The complete solution of the problem requires also the calculation of the \( \zeta \)-dependence of the scattering data, in our case \( m_j(\zeta) \) and \( \lambda_j \).

3 Periodic, Soliton and Similarity Solutions

In this section we generalize the results of [12, 20] to describe the similarity solutions–solitons, periodic and self-similar solutions for the system (2).

We also have not been able to resolve the fundamental problem, inherent to this type of NLEE. We have to solve the ISP for \( M(\lambda) \) on a finite interval, and naturally the \( \zeta \)-dependence of the corresponding scattering data will depend on the boundary values of \( Q \) at both end of this interval. Indeed,
the initial conditions allow us to calculate all the necessary quantities such as $S_3(\zeta, \tau = 0)$, $S_\pm(\zeta, \tau = 0)$ at $\tau = 0$. In order to evaluate them at $\tau = 1$ we have to solve completely the problem.

On the other hand, the initial conditions of such physical system must determine uniquely its evolution. One way out of this problem is to impose certain boundary conditions on the operator $M(\lambda)$ — e.g., (quasi-) periodic. They will relate the values $S_3(\zeta, \tau = 0)$, $S_\pm(\zeta, \tau = 0)$ to $S_3(\zeta, \tau = 1)$, $S_\pm(\zeta, \tau = 1)$.

### 3.1 Periodic (cnoidal) and solitary waves

At first we will study the cnoidal wave similarity solutions which include solitons as a special limit. If we introduce the retarded coordinate $\xi = \zeta - \tau/\alpha$ and the following new variables

\[
B_p(\zeta) = e^{-i\epsilon_p \tau / 2\alpha} E_p(\zeta, \tau), \quad B_s(\zeta) = e^{i\epsilon_s \tau / 2\alpha} E_s(\zeta, \tau), \\
B_a(\zeta) = e^{-i\epsilon_a \tau / 2\alpha} E_a(\zeta, \tau), \quad Y(\zeta) = e^{-i\epsilon \zeta / 2} Q(\zeta, \tau),
\]

where $\epsilon, \epsilon_p, \epsilon_s, \epsilon_a$ are arbitrary real parameters. We find that the reduced variables $B_p$, $B_s$, $B_a$ and $Y$ satisfy:

\[
\frac{dB_p}{d\xi} = \beta_a B_a Y^* e^{-i\epsilon \xi} - e^{i\epsilon \xi} B_s Y, \quad \frac{dB_s}{d\xi} = e^{-i\epsilon \xi} B_p Y^*,
\]

\[
\frac{dB_a}{d\xi} = -\beta_a e^{i\epsilon \xi} B_a Y, \quad \frac{1}{\alpha} \frac{dY}{d\xi} + e^{-i\epsilon \xi} B_p B_s^* + \beta_s e^{-i\epsilon \xi} B_p B_a = 0,
\]

where $\epsilon_p = \epsilon_s + \epsilon, \epsilon_a = \epsilon_p + \epsilon$. The general solution to Eq. \(29\) can be expressed in terms of elliptic integrals. When $\epsilon = 0$, the solutions may be explicitly written in terms of cnoidal functions. Indeed, from \(2\) and \(3\) and using the transformed variable $\xi = \zeta - \tau/\alpha$ we find the system:

\[
\frac{dS_3}{d\xi} = -iQ^* S_+ + iQ S_-, \quad \frac{dS_\pm}{d\xi} = -iQ S_3, \quad \frac{dQ}{d\xi} = 2i S_+,
\]

which has the following first integrals

\[
\frac{1}{2} S_3^2 + S_+ S_- = I_1, \quad S_+ Q^* + S_- Q = I_2, \quad \frac{1}{\alpha} S_3 + \frac{1}{2} |Q|^2 = I_3.
\]
Introducing the new real variables $A_+, \phi_+, \tilde{Q}$ and $\phi$ by:

$$
S_+ = e^{i\phi_+}A_+, \quad S_- = e^{-i\phi_+}A_+, \quad Q = \tilde{Q}e^{i\phi}, \quad \tilde{\phi} = \phi - \phi_+.
$$

(32)

we rewrite (30) and (31) as follows:

$$
\frac{dS_3}{d\xi} = -2\tilde{Q}A_+ \sin \tilde{\phi}, \quad \frac{dA_+}{d\xi} = \tilde{Q}S_3 \sin \tilde{\phi},
$$

$$
\alpha \frac{d\tilde{Q}}{d\xi} = 2A_+ \sin \tilde{\phi}, \quad \frac{d\phi}{d\xi} = -\frac{2A_+}{\alpha} \cos \tilde{\phi},
$$

(33)

$$
I_1 = \frac{1}{2}S_3^2 + A_+^2, \quad I_2 = 2\tilde{Q}A_+ \cos \tilde{\phi}, \quad I_3 = \frac{1}{\alpha}S_3 + \frac{1}{2}\tilde{Q}^2.
$$

(34)

Squaring the equation for $S_3$ and using the expression for $I_k, k = 1, 2, 3$ we obtain

$$
\dot{S}_3^2 = \frac{4}{\alpha}(S_3 - Z_1)(S_3 - Z_2)(S_3 - Z_3),
$$

(35)

where the constants $Z_i$ are related to $I_k$ by:

$$
Z_1 + Z_2 + Z_3 = \alpha I_3, \quad Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = -2I_1
$$

$$
Z_1Z_2Z_3 = \alpha I_3^2/4 - 2\alpha I_1I_3.
$$

(36)

The solutions for $S_3$ may be written explicitly in terms of Jacobian sn function. We have the following periodic (cnoidal) waves:

1. For $\alpha > 0, Z_1 < 0 < Z_2 < Z_3$,

$$
S_3 = Z_1 + (Z_2 - Z_1) \text{sn}^2[p(\xi - \xi_0), k],
$$

$$
p = \sqrt{\frac{Z_3 - Z_1}{\alpha}}, \quad k^2 = \frac{Z_2 - Z_1}{Z_3 - Z_1}.
$$

(37)

2. For $\alpha < 0, Z_1 < Z_2 < 0 < Z_3$,

$$
S_3 = Z_3 + (Z_3 - Z_2) \text{sn}^2[p(\xi - \xi_0), k],
$$

$$
p = \sqrt{\frac{Z_3 - Z_1}{-\alpha}}, \quad k^2 = \frac{Z_3 - Z_2}{Z_3 - Z_1}.
$$

(38)

In the particular case when $k = 1$ we find the corresponding solitary waves
3. For $\alpha > 0$, $Z_1 < 0 < Z_2 = Z_3$,

$$S_3 = Z_2 - \frac{Z_2 - Z_1}{\cosh^2 \sqrt{\frac{Z_2 - Z_1}{\alpha}}(\xi - \xi_0)},$$  \hspace{1cm} (39)

4. For $\alpha < 0$, $Z_1 = Z_2 < 0 < Z_3$,

$$S_3 = Z_2 + \frac{Z_3 - Z_2}{\cosh^2 \sqrt{\frac{Z_3 - Z_2}{-\alpha}}(\xi - \xi_0)}.$$  \hspace{1cm} (40)

Let us concentrate on the most important from the physical point of view soliton solutions, i.e. the third case with additional constraint $Z_1 = Z_2 = -\alpha I_3$. The result of integration is

$$\tilde{Q} = \frac{2\sqrt{T_3}}{\cosh(z)}, \quad S_3 = \alpha I_3(\tanh^2(z) - \text{sech}^2(z)),$$

$$A_+ = \sqrt{2\alpha I_3} \frac{\tanh(z)}{\cosh(z)}, \quad \phi_+ - \phi = \pi/2, \quad z = \sqrt{2I_3}(\xi - \xi_0).$$  \hspace{1cm} (41)

where $\xi_0$ is the arbitrary initial phase. We will return again to this solution in section 3.4.

3.2 Self–similarity solutions

In order to obtain the self–similarity solutions we introduce the reduced variables

$$E_p(\xi) = e^{-i\epsilon_p \ln \tau} B_p(\zeta, \tau), \quad E_s(\xi) = e^{-i\epsilon_s \ln \tau} B_s(\zeta, \tau),$$

$$E_a(\xi) = e^{-i\epsilon_a \ln \tau} B_a(\zeta, \tau), \quad Q(\zeta) = \frac{1}{\zeta} e^{i\epsilon \ln \zeta} Y(\zeta, \tau),$$  \hspace{1cm} (42)

where $\xi = \zeta \tau$. Then the equations (1) becomes

$$\frac{dB_p}{d\xi} = -\frac{1}{\xi} e^{i\epsilon \ln \zeta} B_s Y + \frac{\beta}{\xi} e^{-i\epsilon \ln \zeta} B_a Y^*, \quad \frac{dB_s}{d\xi} = \frac{1}{\xi} e^{-i\epsilon \ln \zeta} B_p Y^*,$$

$$\frac{dB_a}{d\xi} = -\frac{\beta}{\xi} e^{i\epsilon \ln \zeta} B_p Y, \quad \frac{dY}{d\xi} = e^{-i\epsilon \ln \zeta} (B_p B_s^* + B_a B_p^*),$$  \hspace{1cm} (43)
where \( \epsilon_p = \epsilon_s + \epsilon, \epsilon_a = \epsilon_p + \epsilon \). For \( \epsilon = 0 \) (13) simplifies to:

\[
\frac{d B_p}{d \xi} = -\frac{1}{\xi} B_s Y + \frac{\beta}{\xi} B_a Y^*, \quad \frac{d B_s}{d \xi} = \frac{1}{\xi} B_p Y^*,
\]

\[
\frac{d B_a}{d \xi} = \frac{\beta}{\xi} B_p Y, \quad \frac{d Y}{d \xi} = B_p B_s^* + \beta B_a B_p^*,
\]

(44)

and, while the general solution is singular, nonsingular solutions in terms of series can be obtained by a technique described in [12].

We prefer here to analyze the solutions of (3) with another self–similarity variable \( \xi = 2 \sqrt{2} \zeta \tau \). If we choose

\[
S_3 = \cos[\beta(\xi)], \quad S_+ = \frac{i}{2} \sqrt{2} \sin[\beta(\xi)]
\]

(45)

we find that (3) goes into:

\[
\frac{d^2 \beta(\xi)}{d \xi^2} + \frac{1}{\xi} \frac{d \beta(\xi)}{d \xi} + \sin[\beta(\xi)] = 0,
\]

(46)

Equation (46) was first derived in another context for the transient stimulated Raman scattering by Elgin and O’ Hare [24]. This equation can be reduced to one of the standard forms of the Painlevé (\( P_{III} \)) equation [25]. When \( \xi \gg 1 \), we can use the asymptotic formula given by Novokshenov [25] to obtain

\[
\beta(\xi) = \frac{\tilde{\alpha}}{\xi^{1/2}} \cos \left( \xi + \frac{\tilde{\alpha}^2}{16} \ln \xi + \psi \right),
\]

(47)

where

\[
\tilde{\alpha}^2 = -\frac{16}{\pi} \ln[\cos(\beta_0/2)],
\]

\[
\psi = \frac{2 \ln 2}{\pi} \ln[\cos(\beta_0/2)] + \arg \Gamma \left( \frac{i \tilde{\alpha}^2}{16} \right) - \frac{\pi}{4}.
\]

(48)

Here \( \Gamma(x) \) is the Gamma function with a complex argument, \( \arg[\Gamma(x)] \) indicates its phase and \( \beta_0 = \beta(\xi = 0) \). Similar expressions can be obtained for \( S_3(\xi) \) and \( S_+ \) from (13) and for \( Q \) from (18).
3.3 Discussion

To obtain the bright solitons and to compare our results with the ones of Kaplan et al [22] we slightly generalize equations (1) (see for example [26, 7]). The Raman quantum transition between the lower (ground) and upper (excited) level, i.e. two level atom is described by a $2 \times 2$ hermitian density matrix $\rho$ and the generalized Bloch equations [26, 7]

$$
\frac{\partial Q}{\partial \tau} = \tilde{\Omega}_R^* \Delta, \quad \frac{\partial \Delta}{\partial \tau} = 2 \text{Re}(Q\tilde{\Omega}_R), \quad \Delta = \rho_{11} - \rho_{22},
$$

$$
\tilde{\Omega}_R = \frac{2}{\hbar} \left( \alpha_{s,p} E_s E_p^* + \alpha_{p,a} E_p E_a^* \right), \quad Q = -2i\rho_{12} e^{-i k_0 z + i \omega_0 t}.
$$

Here $\tilde{\Omega}_R$ is the generalized local Rabi frequency [26], $k_0 = k_p - k_s \sim \omega_0/c$ and we have assumed that $\rho_{11} + \rho_{22} = 1$.

The generalization of (1) we mentioned above is obtained by replacing the equation for $Q$ in (1) by the system (49). These equations, rewritten for the self–similarity variable $\xi = \zeta - \tau/\alpha$ coincide with equation (6) from Kaplan et al. [7]. The direct comparison of $S_+ + \tilde{\Omega}_R$ shows that the physical interpretation of $S_+$ is the normalized local Rabi frequency. From these equations we obtain also that the quantity

$$
\Delta^2 + |Q|^2 = C(\zeta),
$$

is conserved in $\tau$ and that the following equations

$$
\delta_p \Phi_p + \delta_s \Phi_s + \delta_a \Phi_a = I(\tau), \quad \delta_i = \left( \frac{1}{v_{gi}} - \frac{1}{v_g} \right), \quad i = p, s, a, \quad (51)
$$

$$
\frac{\partial}{\partial \zeta} (\delta_a \Phi_a - \delta_s \Phi_s) - \pi N_0 \frac{\partial}{\partial \tau} \Delta = 0, \quad \Phi = |E_i|^2 \quad (52)
$$

hold. By $N_0$ we have denoted the density of Raman particles. Let us use the following ansatz [22]

$$
\Phi_p = |a_p|^2 \Phi_\Sigma, \quad \Phi_s = |a_s|^2 \Phi_\Sigma, \quad \Phi_a = |a_a|^2 \Phi_\Sigma,
$$

$$
|a_p|^2 = \frac{\gamma_3}{W}, \quad |a_s|^2 = \frac{|w_{s,p}|^2}{\delta_3^2 W}, \quad |a_a|^2 = \frac{|w_{p,a}|^2}{\delta_3^2 W}, \quad (53)
$$

$$
w_{s,p} = \frac{2\pi}{c} \alpha_{s,p} \sqrt{\frac{\omega_s \omega_p}{n_s n_p}}, \quad w_{p,a} = \frac{2\pi}{c} \alpha_{p,a} \sqrt{\frac{\omega_p \omega_a}{n_p n_a}},
$$

13
where

\[ \alpha_{s,p} = \frac{1}{\hbar^2} \sum_m \left[ \frac{(\vec{d}_{1m} \cdot \vec{e}_p)(\vec{d}_{m2} \cdot \vec{e}_s)}{(\omega_m - \omega_p)} + \frac{(\vec{d}_{1m} \cdot \vec{e}_s)(\vec{d}_{m2} \cdot \vec{e}_p)}{(\omega_m + \omega_s)} \right], \quad (54) \]

\[ \alpha_{p,a} = \frac{1}{\hbar^2} \sum_m \left[ \frac{(\vec{d}_{1m} \cdot \vec{e}_a)(\vec{d}_{m2} \cdot \vec{e}_p)}{(\omega_m - \omega_a)} + \frac{(\vec{d}_{1m} \cdot \vec{e}_p)(\vec{d}_{m2} \cdot \vec{e}_a)}{(\omega_m + \omega_p)} \right]. \quad (55) \]

Here \( \vec{d}_{1m}, \vec{d}_{2m} \) are the dipole matrix elements between the Raman quantum levels and the \( m \)-th quantum level, \( m \neq 1, 2 \) and \( n_p, n_s, n_a \) are refractive indexes at frequencies \( \omega_p, \omega_s, \omega_a \) respectively. From (54), (55) we find that if \( \omega_p \gg \omega_a, \omega_s \) and \( \omega_a, \omega_p \ll \omega_m \) then \( \alpha_{s,p} \simeq \alpha_{p,a} \) and \( \omega_{s,p} \simeq \omega_{p,a} \). Using arguments analogous to the ones in [1] we find that one can expect such physical systems to be described by the system (1) with \( \beta_a \simeq 1 \).

Inserting (53) into \( I (51) \), from (52) we obtain:

\[ \gamma_3^2 = - \left( \frac{|w_{s,p}|^2}{\delta_s \delta_p} + \frac{|w_{p,a}|^2}{\delta_a \delta_p} \right). \]

Let us consider the second integral (52) with conserved density \( \delta_a \Phi_a - \delta_s \Phi_s \) and conserved flux \( -\pi N_0 \Delta \). Then

\[ J = 2 (\delta_a \Phi_a - \delta_s \Phi_s) = -\frac{2\pi \Phi_N N_0}{\Phi_0}. \quad (56) \]

Here

\[ \Phi_0 = \pm \frac{\pi N_0 W}{(|w_{s,p}|^2/\delta_s - |w_{p,a}|^2/\delta_a)^{\frac{1}{2}}}; \quad (57) \]

where "-" indicates that the molecules (atoms) are initially at the equilibrium and "+" that the population difference is inversed. Let us also introduce:

\[ \Phi_\Sigma = \Phi_0 S(\xi), \quad \Delta = \pm (1 - S(\xi)), \quad Q(\xi) = -\gamma_3 \xi S(\xi), \quad (58) \]

where \( S(\xi) \) may have different forms (soliton, Lorentzian etc.) [22]. Finally from the normalization condition \( |a_p|^2 + |a_s|^2 + |a_a|^2 = 1 \) we have

\[ W = \frac{|w_{s,p}|^2}{\delta_s \delta_{s,p}} - \frac{|w_{p,a}|^2}{\delta_a \delta_{p,a}}. \quad (59) \]
where
\[
\delta_{s,p} = \frac{1}{\delta_s} - \frac{1}{\delta_p}, \quad \delta_{p,a} = \frac{1}{\delta_p} - \frac{1}{\delta_a}.
\] (60)

Recently these bright solitons, in more general physical situation, cascade SRS [1, 7] have been used to predict generation of subfemtosecond coherent pulses in SRS experiment [22]. From the above analysis it is clear, that bright solitons are obtained in the case of finite group velocity dispersion parameters \(\delta_i\) [1, 22] and non-zero population difference \(\Delta\).

### 3.4 The one soliton solution

Here we will show, that the auxiliary linear problem for the vector NLS equation with some additional reduction is equivalent to the Stokes–anti–Stokes SRS equations without last equation for \(Q\). This formal equivalence allows us to recover \(E_p, E_s, E_a\) from potential \(Q\), already obtained by inverse scattering transform method of Sec. 2 with (18).

Indeed, if we introduce:
\[
\frac{\partial}{\partial \zeta} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix}
\] (61)

and require that:
\[
q_1 = -Q, \quad q_2 = Q^*, \quad \psi_1 = E_p, \quad \psi_2 = E_s, \quad \psi_3 = E_a.
\] (62)

without last equation for \(Q\), which we may obtain from the previous subsection. Using the well known one soliton solution of the vector nonlinear Schrödinger equation under the reduction (62) and the solution of the linear problem (61) with potentials \(Q, Q^*\) we obtain
\[
Q = \sqrt{2}\eta e^{i\phi} \cosh(z), \quad z = \eta \zeta - \frac{1}{\eta} \int_0^\tau K^2(\tau')d\tau'
\] (63)

where \(K(\tau')\) is real, the soliton’s eigenvalue is \(i\eta\) and \(\phi\) is constant real phase. The direct integration of Eq. (61) with reduction (62) is given by
\[
E_p = \sqrt{2}K(\tau) \frac{\tanh(z)}{\cosh(z)} e^{i\phi}, \quad E_s = K(\tau) \tanh^2(z), \quad E_a = \frac{K(\tau)}{\cosh^2(z)} e^{2i\phi}.
\] (64)
These solutions are similar to transient SRS solitons obtained in \cite{12} (see also \cite{23}) and for \(K=1, \phi=0\) coincide with the ones in \cite{1}. If we now calculate the \(S_3\) and \(S_\pm\) using the above expressions for \(E_p, E_s, E_a\) we obtain precisely the soliton solution \(\text{(41)}\).

### 4 \(N\)-component generalizations

In this section we show that the extended model with \(N\)-Stokes and \(N\)-Stokes components is also integrable in the sense means of IST method. The considerations are formal from physical point of view.

Let us consider the following equations

\[
\begin{align*}
\frac{\partial E_p}{\partial \xi} &= \sum_{i=1}^{N}(\beta_a Q^* E_a^{(i)} - Q E_s^{(i)}), & \frac{\partial E^{(i)}}{\partial \xi} &= Q^* E_p, & i = 1, \ldots, N, \\
\frac{\partial E_a^{(i)}}{\partial \xi} &= -\beta_a Q E_p, & \frac{\partial Q}{\partial \tau} + \tilde{g} Q &= \sum_{i=1}^{N}(E_s^{(i)} E_p + \beta_a E_p^* E_a^{(i)}). 
\end{align*}
\]

with \(\beta_a = 1\). The equations for the \(E_i\) can be written down as the same auxiliary linear problem, which solves the \(N\)-component vector NLS equation

\[
\begin{align*}
q_1 &= -Q, & q_2 &= Q^*, & \ldots & q_{2N-1} &= -Q, & q_{2N} &= Q^*, \\
\psi_1 &= E_p, & \psi_2 &= E_s^{(1)}, \\
\psi_3 &= E_a^{(1)}, & \ldots & \psi_{2N} &= E_s^{(N)}, & \psi_{2N+1} &= E_a^{(N)}, \\
\frac{\partial}{\partial \xi} \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\psi}_3 \\
\vdots \\
\tilde{\psi}_{2N} \\
\tilde{\psi}_{2N+1}
\end{pmatrix} &= \begin{pmatrix}
0 & q_1 & q_2 & \ldots & q_{2N-1} & q_{2N} \\
-q_1^* & 0 & 0 & \ldots & 0 & 0 \\
-q_2^* & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-q_{2N-1}^* & 0 & 0 & \ldots & 0 & 0 \\
-q_{2N}^* & 0 & 0 & \ldots & 0 & 0
\end{pmatrix} \begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 \\
\tilde{\psi}_3 \\
\vdots \\
\tilde{\psi}_{2N} \\
\tilde{\psi}_{2N+1}
\end{pmatrix},
\end{align*}
\]

with the spectral parameter \(\lambda = 0\).

We introduce again the quadratic variables

\[
\begin{align*}
S_3 &= \frac{1}{2} \sum_{i=1}^{N}(|E_s|^2 - |E_a|^2), & S_+ &= S_- = \frac{i}{2} \sum_{i=1}^{N}(E_s^* E_p + E_p^* E_a),
\end{align*}
\]

\(16\)
and show that they satisfy the same equation (3). Therefore the ZS–AKNS representation (4) can be used as above for the analysis of the system (6). The procedure of solving Eq. (65) is analogous to the considerations of section IV. Clear physical interpretation and solutions of Eq. (65) will be given elsewhere.

5 Conclusion

In this paper a method of solving Stokes-anti Stokes SRS with $\beta_a = 1$ is presented. New transient solitons (63), (64) are obtained. For bright solitons our results are in agreement with those of Kaplan et al [22]. The traveling and self-similar solutions to Eq. (1) are discussed.

We stress also that for the transient solitons with $\beta_a = 1$ the number of Stokes–anti–Stokes photons are close to each other.

The ISM generically meets with difficulties due to the fact that generically no boundary conditions on the potentials (16) are imposed. In the case of quasiperiodic boundary conditions these difficulties are overcome.

Looking at the Lax operator (16) we recognize the system of equations (3) as belonging to the Heisenberg ferromagnet hierarchy. One can try to apply the expansions over the “squared solutions” in the spirit of [27] and then treat the case $\tilde{g} \neq 0$ as a perturbation.

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