FOURTH-ORDER BESSEL-TYPE SPECIAL FUNCTIONS:
A SURVEY

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This paper is dedicated to the memory and achievements of
George Neville Watson (1886 to 1965)

Abstract. This survey paper reports on the properties of the fourth-order Bessel-type
linear ordinary differential equation, on the generated self-adjoint differential operators in
two associated Hilbert function spaces, and on the generalisation of the classical Hankel
integral transform.

These results are based upon the properties of the classical Bessel and Laguerre second-
order differential equations, and on the fourth-order Laguerre-type differential equation.
From these differential equations and their solutions, limit processes yield the fourth-order
Bessel-type functions and the associated differential equation.

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Date: 25 August 2005 (File: C:\Swp50\Bessel\munich9.tex).
2000 Mathematics Subject Classification. Primary: 33C10, 34B05, 34L05. Secondary: 33C45, 34B30,
34A25.

Key words and phrases. Bessel functions, Bessel-type functions, linear ordinary and partial
differential equations, self-adjoint ordinary differential operators, Hankel transforms.
1. Introduction

This survey paper is based on joint work with the following named colleagues:

Jyoti Das, University of Calcutta, India
D.B. Hinton, University of Tennessee, USA
H. Kalf, University of Munich, Germany
L.L. Littlejohn, Utah State University, USA
C. Markett, Technical University of Aachen, Germany
M. Plum, University of Karlsruhe, Germany
M. van Hoeij, Florida State University, USA

2. History

We see below that the structured definition of the general-even order Bessel-type special functions is dependent upon the Jacobi and Laguerre classical orthogonal polynomials, and the Jacobi-type and Laguerre-type orthogonal polynomials.

These latter orthogonal polynomials were first defined by H.L. Krall in 1940, see [13] and [14], and later studied in detail by A.M. Krall in 1981, see [12], and by Koornwinder in 1984, see [15]. In this respect see the two survey papers, [7] of 1990 and [6] of 1999.

The Bessel-type special functions of general even-order were introduced by Everitt and Markett in 1994, see [8].

The properties of the fourth-order Bessel-type functions have been studied by the present author and the seven colleagues named in Section 1 above, in the papers [2], [4] and [5].

3. The Fourth-Order Differential Equation

The fourth-order Bessel-type differential equation takes the form

\[(xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \text{ for all } x \in (0, \infty)\]

where \(M \in (0, \infty)\) is a positive parameter and \(\Lambda \in \mathbb{C}\), the complex field, is a spectral parameter. The differential equation (3.1) is derived in the paper [8, Section 1, (1.10a)], by Everitt and Markett.

This linear, ordinary differential equation on the interval \( (0, \infty) \subset \mathbb{R} \), the real field, is written in Lagrange symmetric (formally self-adjoint) form, or equivalently Naimark form, see [18, Chapter V].

The structured Bessel-type functions of all even orders, and their associated linear differential equations, were introduced in the paper [8, Section 1] through linear combinations of, and limit processes applied to, the Laguerre and Laguerre-type orthogonal polynomials, and to the classical Bessel functions. This process is best illustrated through the following diagram, see [8, Section 1, Page 328] (for the first two lines of this table see the earlier work
of Koornwinder [15] and Markett [17]):

\[
\begin{align*}
\text{Jacobi polynomials} & \quad \quad \rightarrow \quad \text{Jacobi-type polynomials} \\
& \quad \quad k(\alpha, \beta)(1-x)^{\alpha}(1+x)^{\beta} \\
& \quad \quad + M\delta(x+1) + N\delta(x-1) \\
\downarrow & \quad \quad \downarrow \\
\text{Laguerre polynomials} & \quad \quad \rightarrow \quad \text{Laguerre-type polynomials} \\
& \quad \quad k(\alpha)x^{\alpha}\exp(-x) \\
& \quad \quad + N\delta(x) \\
\downarrow & \quad \quad \downarrow \\
\text{Bessel functions} & \quad \quad \rightarrow \quad \text{Bessel-type functions} \\
& \quad \quad \kappa(\alpha)x^{2\alpha+1} \\
\end{align*}
\]

The symbol entry (here \(k\) and \(\kappa\) are positive numbers depending only on the parameters \(\alpha\) and \(\beta\) under each special function indicates a non-negative (generalised) “weight”, on the interval \((-1, 1)\) or \((0, \infty)\), involved in:

(a) the orthogonality property of the special functions
(b) the weight coefficient in the associated differential equations.

It is important to note in this diagram that:

(i) a horizontal arrow \(\rightarrow\) indicates a definition process either by a linear combination of special functions of the same type but of different orders, or by a linear-differential combination of special functions of the same type and order (alternatively by an application of the Darboux transform, see [10])

(ii) a vertical arrow \(\downarrow\) indicates a confluent limit process of one special function to give another special function

(iii) the use of the symbol \(M\delta(\cdot)\) is a notational device to indicate that the monotonic function on the real line \(\mathbb{R}\) defining the weight has a jump at an end-point of the interval concerned, of magnitude \(M > 0\)

(iv) the combination of any vertical arrow \(\downarrow\) with a horizontal arrow \(\rightarrow\) must give a consistent single entry.

Information about the Jacobi-type and Laguerre-type orthogonal polynomials, and their associated differential equations, is given in the Everitt and Littlejohn survey paper [7]; see in particular the references in this paper to the introduction of the fourth-order Laguerre-type differential equation by H.L. and A.M. Krall, Koornwinder and by Littlejohn. The general Laguerre-type differential equation is introduced in the paper [11] by Koekoek and Koekoek; the order of this linear differential equation is determined by \(4 + 2\alpha\) with \(\alpha \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\).

It is significant that the general order Bessel-type functions also satisfy a linear differential equation of order \(4 + 2\alpha\) (with \(\alpha \in \mathbb{N}_0\)), being an inheritance from the order of the general Laguerre-type equation.

The purpose of this survey paper is to discuss the properties of the Bessel-type linear differential equation in the special case when \(\alpha = 0\), as given in the bottom right-hand corner of the diagram; this is the fourth-order differential equation \(4.1\) and involves the weight coefficient \(\kappa(0)x\); its solutions should, in some sense, have orthogonality properties.
with respect to the generalised weight function \( \kappa(0)x + M\delta(0) \), where \( M > 0 \) is the parameter appearing in the differential equation (3.1); see [8, Section 1].

Our knowledge of the special function solutions of the Bessel-type differential equation (3.1) is now more complete than at the time the paper [8] was written. However, the results in [8, Section 1, (1.8a)], with \( \alpha = 0 \), show that the function defined by

\[
J_{\lambda}^{0,M}(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x) \quad \text{for all } x \in (0, \infty),
\]

is a solution of the differential equation (3.1), for all \( \lambda \in \mathbb{C} \), and hence for all \( \Lambda \in \mathbb{C} \), and all \( M > 0 \). Here:

(i) the parameter \( M > 0 \)

(ii) the parameter \( \lambda \in \mathbb{C} \)

(iii) the spectral parameter \( \Lambda \) and the parameter \( M \), in the equation (3.1), and the parameters \( M \) and \( \lambda \), in the definition (3.3), are connected by the relationship

\[
\Lambda \equiv \Lambda(\lambda, M) = \lambda^2(\lambda^2 + 8M^{-1}) \quad \text{for all } \lambda \in \mathbb{C} \text{ and all } M > 0
\]

(iv) \( J_0 \) and \( J_1 \) are the classical Bessel functions (of the first kind), see [24, Chapter III].

Similar arguments to the methods given in [8] show that the function defined by

\[
Y_{\lambda}^{0,M}(x) := [1 + M(\lambda/2)^2]Y_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}Y_1(\lambda x) \quad \text{for all } x \in (0, \infty),
\]

is also a solution of the differential equation (3.1), for all \( \lambda \in \mathbb{C} \), and hence for all \( \Lambda \in \mathbb{C} \) and all \( M > 0 \); here, again, \( Y_0 \) and \( Y_1 \) are classical Bessel functions (of the second kind), see [24, Chapter III].

The earlier studies of the fourth-order differential equation (3.1) failed to find any explicit form of two linearly independent solutions, additional to the solutions \( J_{\lambda}^{0,M} \) and \( Y_{\lambda}^{0,M} \). However, results of van Hoeij, see [22] and [23], using the computer algebra program Maple have yielded the required two additional solutions, here given the notation \( I_{\lambda}^{0,M} \) and \( K_{\lambda}^{0,M} \), with explicit representation in terms of the classical modified Bessel functions \( I_0, K_0 \) and \( I_1, K_1 \). These two additional solutions are defined as follows, where as far as possible we have followed the notation used for the solutions \( J_{\lambda}^{0,M} \) and \( Y_{\lambda}^{0,M} \);

(i) given \( \lambda \in \mathbb{C} \), with \( \arg(\lambda) \in [0, 2\pi) \), \( M \in (0, \infty) \) and using the principal value of \( \sqrt{\cdot} \), define

\[
c \equiv c(\lambda, M) := \sqrt{\lambda^2 + 8M^{-1}} \quad \text{and} \quad d \equiv d(\lambda, M) := 1 + M(\lambda/2)^2
\]

(ii) define the solution \( I_{\lambda}^{0,M} \), for all \( x \in (0, \infty) \),

\[
I_{\lambda}^{0,M}(x) := -dI_0(cx) + \frac{c}{2}cx^{-1}I_1(cx)
\]

(iii) define the solution \( K_{\lambda}^{0,M} \), for all \( x \in (0, \infty) \),

\[
K_{\lambda}^{0,M}(x) := dK_0(cx) + \frac{c}{2}cx^{-1}K_1(cx)
\]

Remark 3.1. We have
(1) The four linearly independent solutions $J_{0}^{0,M}, Y_{0}^{0,M}, I_{0}^{0,M}, K_{0}^{0,M}$ provide a basis for all solutions of the original differential equation (3.1), subject to the $(\Lambda, \lambda)$ connection given in (3.2).

(2) These four solutions are real-valued on their domain $(0, \infty)$ for all $\lambda \in \mathbb{R}$.

(3) The domain $(0, \infty)$ of the solutions $J_{0}^{0,M}$ and $I_{0}^{0,M}$ can be extended to the closed half-line $[0, \infty)$ with the properties

$$J_{0}^{0,M}(0) = I_{0}^{0,M}(0) = 1 \text{ for all } \lambda \in \mathbb{R} \text{ and all } M \in (0, \infty).$$

The classical Bessel differential equation, with order $\alpha = 0$, written in a form comparable to the fourth-order equation (3.1), is best taken from the left-hand bottom corner of the diagram (3.2); from [3 Section 1, (1.2)] with $\alpha = 0$ we obtain

$$-(xy'(x))' = \lambda^2 xy(x) \text{ for all } x \in (0, \infty);$$

here $\lambda \in \mathbb{C}$ is the spectral parameter. It is to be observed that, formally, if the fourth-order Bessel-type equation (3.1) is multiplied by the parameter $M > 0$ and then $M$ tends to zero, we obtain essentially the classical Bessel equation of order zero (3.9), on using the spectral relationship (3.4) between the parameters $\lambda$ and $\Lambda$. This Bessel differential equation (3.9) has solutions $J_0(\lambda x)$ and $Y_0(\lambda x)$ for all $x \in (0, \infty)$ and all $\lambda \in \mathbb{C}$.

For the need to apply the Frobenius series method of solution we also consider the differential equation (3.10) on the complex plane $\mathbb{C}$:

$$w^{(4)}(z) + 2z^{-1}w^{(3)}(z) - (9z^{-2} + 8M^{-1})w''(z) + (9z^{-3} - 8M^{-1}z^{-1})w'(z) - \Lambda w(z) = 0$$

for all $z \in \mathbb{C}$. In this form the equation has a regular singularity at the origin 0, and an irregular singularity at the point at infinity $\infty$ of the complex plane $\mathbb{C}$; all other points of the plane are regular or ordinary points for the differential equation. It should be noted that the classical Bessel differential equation (3.9) has the same classification when considered in the complex plane $\mathbb{C}$.

A calculation shows that the Frobenius indicial roots for the regular singularity of the differential equation (3.10) at the origin 0, are $\{4, 2, 0, -2\}$. The application of the Frobenius series method, using the computer programs [11] and Maple (see [23]), yield four linearly independent series solutions of (3.10), each with infinite radius of convergence in the complex plane $\mathbb{C}$. If these solutions are labelled to hold for the Bessel-type differential equation (3.1) then we have four solutions $\{y_r(\cdot, \Lambda, M) : r = 4, 2, 0, -2\}$, to accord with the indicial roots, to give the theorem, see [2 Section 3]:

**Theorem 3.1.** For all $\Lambda \in \mathbb{C}$ and all $M > 0$, the differential equation (3.1) has four linearly independent solutions $\{y_r(\cdot, \Lambda, M) : r = 4, 2, 0, -2\}$, defined on $(0, \infty) \times \mathbb{C}$, with the following series properties as $x \to 0^+$, where the $O$-terms depend upon the complex spectral parameter $\Lambda$ and the parameter $M$,

$$y_4(x, \Lambda, M) = x^4 + \frac{1}{7}M^{-1}x^6 + O(x^8)$$
$$y_2(x, \Lambda, M) = kx^2 + O(x^4 \ln(x))$$
$$y_0(x, \Lambda, M) = l + O(x^4 \ln(x))$$
$$y_{-2}(x, \Lambda, M) = mx^{-2} + O(|\ln(x)|).$$

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$$y_2(x, \Lambda, M) = kx^2 + O(x^4 \ln(x))$$
$$y_0(x, \Lambda, M) = l + O(x^4 \ln(x))$$
$$y_{-2}(x, \Lambda, M) = mx^{-2} + O(|\ln(x)|).$$
Here the fixed numbers $k, l, m \in \mathbb{R}$ and are independent of the parameters $\Lambda$ and $M$; these numbers are produced by the Frobenius computer program \[1\] and have the explicit values:

$$k = -(27720)^{-1}, \quad l = (174636000)^{-1}, \quad m = -(9779616000)^{-1}.$$  

4. Higher-order differential equations

As mentioned in Section \[3\] above there exist Bessel-type linear differential equations of all even-orders $4 + 2\alpha$, where $\alpha \in \mathbb{N}_0$ is any non-negative integer. The definition and some properties of these differential equations, and the associated Bessel-type functions, are considered in detail in \[8\] Sections 2 and 3.

Here we give the form of the sixth-order and eighth-order differential equations, as given in \[8\] Section 1, (1.10b) and (1.10c)]. (Note that there is a printing error in the display (1.10b); the numerical factor 255 is to be replaced by 225. Also printing errors in the display (1.10c) which are now to be corrected using the form of the differential equation (4.2) below.)

(i) The sixth-order equation derived from the corrected differential expression for \[8\] Section 1, (1.9) and (1.10b) is

$$-(x^3y^{(3)}(x))^{(3)} + (33xy''(x))'' - ((225x^{-1} - 96M^{-1}x^3)y'(x))' = (\lambda^6 + M^{-1}2^4(3!) \lambda^2)x^3y(x) \quad \text{for all} \quad x \in (0, \infty),$$

where, as before, the parameters $M \in (0, \infty)$ and $\lambda \in \mathbb{C}$.

When this equation is considered in the complex plane $\mathbb{C}$ the Frobenius indicial roots for the regular singularity at the origin 0 are $\{6, 4, 2, 0, -2, -4\}$, using the methods provided by \[23\].

(ii) The eighth-order equation derived from the corrected differential expression for \[8\] Section 1, (1.9) and (1.10b) is

$$(x^5y^{(4)}(x))^{(4)} - (78x^3y^{(3)}(x))^{(3)} + (1809xy''(x))'' - ((11025x^{-1} - 2^6(4!)M^{-1}x^5)y'(x))' = (\lambda^8 + M^{-1}2^6(4!) \lambda^2)x^5y(x) \quad \text{for all} \quad x \in (0, \infty),$$

where, as before, the parameters $M \in (0, \infty)$ and $\lambda \in \mathbb{C}$.

When this equation is considered in the complex plane $\mathbb{C}$ the Frobenius indicial roots for the regular singularity at the origin 0 are $\{8, 6, 4, 2, 0, -2, -4, -6\}$, using the methods provided by \[23\].

We note that, formally, if the equations in (4.1) and (4.2) are multiplied by $M > 0$, and then letting $M$ tend to zero we obtain, respectively, the two Sturm-Liouville differential equations, see \[8\] Section 1, (1.2)],

$$-(x^3y'(x))' = \lambda^2x^3y(x) \quad \text{and} \quad -(x^5y'(x))' = \lambda^2x^5y(x) \quad \text{for all} \quad x \in (0, \infty).$$

For the solutions of these equations in classical Bessel functions see \[8\] Section 1, (1.4)].

5. The fourth-order differential expression $L_M$

We define the differential expression $L_M$ with domain $D(L_M)$ as follows:

$$(5.1) \quad D(L_M) := \{f : (0, \infty) \to \mathbb{C} : f^{(r)} \in AC_{loc}(0, \infty) \text{ for } r = 0, 1, 2, 3\},$$
and for all \( f \in D(L_M) \)

\[
L_M[f](x) := (xf''(x))'' - ((9x^{-1} + 8M^{-1}x)f'(x))' \quad (x \in (0, \infty));
\]

it follows that

\[
L_M : D(L_M) \to L^1_{\text{loc}}(0, \infty).
\]

The Green’s formula for \( L_M \) on any compact interval \([\alpha, \beta] \subset (0, +\infty)\) is given by

\[
\int_{\alpha}^{\beta} \left\{ \overline{\mu(x)L_M[f]}(x) - f(x)L_M[g](x) \right\} \, dx = [f, g](x)_{\alpha}^{\beta},
\]

where the symplectic form \([\cdot, \cdot] : D(L_M) \times D(L_M) \times (0, +\infty) \to \mathbb{C}\) is defined by

\[
[f, g](x) := \overline{\mu(x)(xf''(x))'} - (x\overline{\mu''(x)})'f(x)
\]

\[
- x\overline{(\mu'(x)f''(x) - \mu'(x)f'(x))}
\]

\[
- (9x^{-1} + 8M^{-1}x)\overline{(\mu(x)f'(x) - \mu(x)f(x))}.
\]

The Dirichlet formula for \( L_M \) on any compact interval \([\alpha, \beta] \subset (0, +\infty)\) is given by

\[
\int_{\alpha}^{\beta} \{xf''(x)\overline{\mu''(x)} + (9x^{-1} + 8M^{-1}x)f'(x)\overline{\mu'(x)}\} \, dx
\]

\[
= [f, g]_{D}(x)_{\alpha}^{\beta} + \int_{\alpha}^{\beta} L_M[f](x)\overline{\mu(x)} \, dx,
\]

where the Dirichlet form \([\cdot, \cdot]_{D} : D(L_M) \times D_{0}(L_M) \times (0, +\infty) \to \mathbb{C}\) is defined by, for \( f \in D(L_M) \) and \( g \in D_{0}(L_M) \), with

\[
D_{0}(L_M) := \{ g : (0, +\infty) \to \mathbb{C} : g^{(r)} \in AC_{\text{loc}}(0, +\infty) \text{ for } r = 0, 1 \}
\]

and

\[
[f, g]_{D}(x) := -\overline{\mu(x)(xf''(x))'} + \overline{\mu'(x)}xf''(x) + \overline{\mu(x)}(9x^{-1} + 8M^{-1}x)f'(x).
\]

6. Hilbert function spaces

The spectral properties of the fourth-order Bessel differential equation

\[
(xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \Lambda xy(x) \quad \text{for all } x \in (0, \infty),
\]

with \( \Lambda \in \mathbb{C} \) as the spectral parameter, are considered in two Hilbert function spaces:

1. The Lebesgue weighted space

\[
L^2((0, \infty); x) := \left\{ f : (0, +\infty) \to \mathbb{C} : \int_{0}^{\infty} x|f(x)|^2 \, dx < +\infty \right\}
\]

with inner-product and norm defined by, for all \( f, g \in L^2((0, \infty); x) \),

\[
(f, g) := \int_{0}^{\infty} xf(x)\overline{y(x)} \, dx \quad \text{and} \quad ||f|| := (f, f)^{1/2}.
\]

This space takes into account the weight function \( x \) on the right-hand side of (6.6).
(2) The Lebesgue-Stieltjes jump space $L^2([0, \infty); m_k)$, as suggested by the results in [8, Section 4].

Let the monotonic non-decreasing function $\hat{m}_k : [0, \infty) \to [0, \infty)$ be defined by, where $k > 0$ is a real parameter,

$$\hat{m}_k(x) = -k \text{ for } x = 0$$

$$= x^2/2 \text{ for all } x \in (0, +\infty).$$

Then $\hat{m}_k$ generates a Baire measure $m_k$ on the $\sigma$-algebra $\mathcal{B}$ of Borel sets on the interval $[0, \infty)$; in turn this measure generates a Lebesgue-Stieltjes integral for Borel measurable functions.

The Hilbert function space $L^2([0, \infty); m_k)$ is defined on all functions with the properties:

(i) $f : [0, \infty) \to \mathbb{C}$ and is Borel measurable on $[0, \infty)$

(ii) $\int_0^\infty x |f(x)|^2 \, dx < +\infty$.

The norm and inner-product in $L^2([0, \infty); m_k)$ are defined by

$$\|f\|^2_k := \int_{[0,\infty)} |f(x)|^2 \, dm_k(x) = k |f(0)|^2 + \int_0^\infty x |f(x)|^2 \, dx$$

and

$$\langle f, g \rangle_k := \int_{[0,\infty)} f(x)\overline{g(x)} \, dm_k(x) = kf(0)\overline{g(0)} + \int_0^\infty xf(x)\overline{g(x)} \, dx.$$

Note that the first integrals in both these definitions are Lebesgue-Stieltjes integrals taken over the set $[0, \infty)$, whilst the second integrals can be taken as Lebesgue integrals.

### 7. Differential operators generated by $L_M$

The Lagrange symmetric differential expression $L_M$ generates self-adjoint operators in both the Hilbert function spaces $L^2((0, \infty); x)$, and in $L^2([0, \infty); m_k)$ for all $k \in (0, \infty)$.

In the space $L^2((0, \infty); x)$ the expression $L_M$ generates a continuum $\{T\}$ of self-adjoint operators, including the significant Friedrichs operator $F$; these properties are developed and considered in Sections 8 to 13 below.

For each $k \in (0, \infty)$ the expression $L_M$ generates a unique self-adjoint operator $S_k$ in the space $L^2([0, \infty); m_k)$; the properties of this operator are considered in Sections 14 and 15.

### 8. Differential operators in $L^2((0, \infty); x)$

The maximal and the minimal differential operators, denoted respectively $T_1$ and $T_0$, as generated by the differential expression $L_M$ in the Hilbert function space $L^2((0, \infty); x)$, are defined as follows, see [18, Chapter V, Section 17]:

(i) $T_1 : D(T_1) \subset L^2((0, \infty); x) \to L^2((0, \infty); x)$ by

$$D(T_1) := \{ f \in D(L_M) : f, x^{-1}L_M(f) \in L^2((0, \infty); x) \}$$
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and

\[ T_1 f := x^{-1}L_M(f) \] for all \( f \in D(T_1) \).

From the Green’s formula (5.4) it follows that the limits

\[ \left[ f, g_1 \right](0^+) := \lim_{x \to 0^+} \left[ f, g \right](x) \quad \text{and} \quad \left[ f, g_1 \right](\infty) := \lim_{x \to \infty} \left[ f, g \right](x) \]

both exist and are finite in \( \mathbb{C} \) for all \( f, g \in D(T_1) \).

\( \text{(ii)} \ T_0 : D(T_0) \subset L^2((0, \infty); x) \to L^2((0, \infty); x) \) by

\[ D(T_0) := \{ f \in D(T_1) : \lim_{x \to 0^+} \left[ f, g \right](x) = 0 \quad \text{and} \quad \lim_{x \to \infty} \left[ f, g \right](x) = 0 \ \forall \ f, g \in D(T_1) \} \]

and

\[ T_0 f := x^{-1}L_M(f) \] for all \( f \in D(T_0) \).

From standard results we have the operator properties, see [18, Chapter V],

\[ T_0 \subseteq T_1, \ T_0^* = T_1 \quad \text{and} \quad T_1^* = T_0, \]

thereby noting that both \( T_0 \) and \( T_1 \) are closed linear operators in \( L^2((0, \infty); x) \).

9. Self-adjoint operators in \( L^2((0, \infty); x) \)

In the weighted space \( L^2((0, \infty); x) \) the Lagrange symmetric (formally self-adjoint) differential expression has the following endpoint classifications at the singular endpoints 0 and \( +\infty \) (for additional details see [2, Section 6]):

\( i) \) At 0+ the singular endpoint is limit-3 in \( L^2((0, \infty); x) \)

\( ii) \) At +\( \infty \) the singular endpoint is Dirichlet and strong limit-2 in \( L^2((0, \infty); x) \).

Based on this information the self-adjoint extensions of the closed symmetric operator \( T_0 \) are determined by the GKN-theorem on singular boundary conditions as given in [18, Chapter V] and [9]. In particular, for the operators \( T_0 \) and \( T_1 \), any self-adjoint operator \( T = T^* \) generated by \( L_M \) in \( L^2((0, \infty); x) \) is a one-dimensional extension of \( T_0 \) or, equivalently, a one-dimensional restriction of \( T_1 \). Let the domain \( D(T) \) as a restriction of the domain \( D(T_1) \) be determined by

\[ D(T) := \{ f \in D(T_1) : [f, \varphi](0) = 0 \} \]

where the function \( \varphi \in D(T_1) \) is a non-null element of the quotient space \( D(T_1)/D(T_0) \) which satisfies the GKN symmetry condition

\[ [\varphi, \varphi](0) = 0. \]

Then the differential operator \( T \) defined by

\[ T f := x^{-1}L_M[f] \] for all \( f \in D(T) \)

satisfies \( T^* = T \), and is self-adjoint in the Hilbert space \( L^2((0, \infty); x) \). All such self-adjoint operators are determined in this way on making an appropriate choice of the boundary condition function \( \varphi \).
10. Boundary properties at $0^+$

The results of the following theorem are essential to obtaining the explicit forms of the
boundary conditions at $0^+$ to determine all self-adjoint extensions of $T_0$.

**Theorem 10.1.** Let $f \in D(T_1)$; then the values of $f, f', f''$ can be defined at the point $0$ so
that the following results hold:

(i) $f \in AC[0, 1]$
(ii) $f' \in AC[0, 1]$ and $f'(0) = 0$
(iii) $f'' \in AC_{loc}(0, 1)$ and $f'' \in C[0, 1]$
(iv) $f^{(3)} \in AC_{loc}(0, 1)$ and $\lim_{x \to 0^+}(xf^{(3)}(x)) = 0$.

For the proof of this theorem see [2, Section 8].

We consider the functions $1, x, x^2$ on the interval $[0, 1]$ but “patched”, see the Naimark
patching lemma [18, Chapter V, Section 17.3, Lemma 2], to zero on $[2, \infty)$ in such a manner
that the patched functions belong to the domain $D(T_1)$; we continue to use the symbols
$1, x, x^2$ for the patched functions.

A calculation shows that the results given in the next lemma are satisfied:

**Lemma 10.1.** The patched functions $1, x, x^2$ have the following limit properties in respect
of the symplectic form $[\cdot, \cdot]$ and the maximal domain $D(T_1)$:

(i) $1, x^2 \in D(T_1)$ but $x \notin D(T_1)$
(ii) $[1, 1](0^+) = [x, x](0^+) = [x^2, x^2](0^+) = 0$
(iii) $[x, x^2](0^+) = 0$ and $[1, x^2](0^+) = 16$
(iv) $[1, x](0^+)$ does not exist.

The lemmas and corollaries now given below are taken from [2, Section 9], where proofs
are given in detail.

The results of Theorem 10.1 and Lemma 10.1 now provide a basis for the two-dimensional
quotient space $D(T_1)/D(T_0)$:

$$D(T_1)/D(T_0) = \text{span}\{1, x^2\} = \{a + bx^2 : a, b \in \mathbb{C}\}.$$  \hspace{1cm} (10.1)

The linear independence of the functions $\{1, x^2\}$ within the the quotient space follows from
the property $[1, x^2](0^+) = 16 \neq 0$.

A calculation now gives, recall Theorem 10.1

**Lemma 10.2.** Let $f \in D(T_1)$; then the following identities hold:

(i) $[f, 1](0^+) = -8f''(0)$
(ii) $[f, x^2](0^+) = 16f(0)$.

Similarly we have

**Lemma 10.3.** Let $f, g \in D(T_1)$; then

(i) $[f, g](0^+) = 8[f(0)\overline{g''}(0) - f''(0)\overline{g}(0)]$
(ii) $[f, g]_D(0^+) = 8f''(0)\overline{g}(0)$.

We have the corollaries:
Corollary 10.1. The domain of the minimal operator $T_0$ is determined explicitly by

$$D(T_0) = \{ f \in D(T_1) : f(0) = 0 \text{ and } f''(0) = 0 \}.$$  

Corollary 10.2. For all $f \in D(T_1)$

$$\int_0^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx < \infty.$$  

Corollary 10.3. For all $f, g \in D(T_1)$ the Dirichlet formula takes the form

$$\langle T_1 f, g \rangle = 8f''(0)g(0) + \int_0^\infty \left\{ x f''(x)g''(x) + (9x^{-1} + 8M^{-1}x) f'(x)g'(x) \right\} dx.$$  

11. Explicit boundary condition functions at $0^+$

We can now determine all forms of the boundary condition function $\varphi$ satisfying the symmetry condition to determine the domain of all self-adjoint extensions $T$ of the minimal operator $T_0$.

Lemma 11.1. All self-adjoint extensions $T$ of $T_0$ generated by the differential expression $L_M$ in $L^2((0, \infty); x)$ are determined by, using the patched functions $1, x^2$,

$$D(T) := \{ f \in D(T_1) : \langle f, \varphi \rangle(0^+) = 0 \text{ where } \begin{cases} (i) & \varphi(x) = \alpha + \beta x^2 \\ (ii) & \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 \neq 0 \end{cases} \}.$$  

and

$$\langle T f, x \rangle := x^{-1}L_M(f)(x) \text{ for all } x \in (0, \infty) \text{ and all } f \in D(T).$$  

There is an equivalent form of this last result, using the results of Lemma 10.2.

Lemma 11.2. All self-adjoint extensions $T$ of $T_0$ generated by the differential expression $L_M$ in $L^2((0, \infty); x)$ are determined by

$$D(T) := \{ f \in D(T_1) : \begin{cases} (i) & -\alpha f''(0) + 2\beta f(0) = 0 \\ (ii) & \alpha, \beta \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 \neq 0 \end{cases} \}.$$  

and

$$\langle T f, x \rangle := x^{-1}L_M(f)(x) \text{ for all } x \in (0, \infty) \text{ and all } f \in D(T).$$  

Remark 11.1. We note the two special cases:

(i) When $\alpha = 0$ the boundary condition is $f(0) = 0$; this boundary condition plays a special role, and gives an explicit form of the domain of the Friedrichs extension $F$ of $T_0$; see Section 13 below.

(ii) When $\beta = 0$ the boundary condition is $f''(0) = 0$. 


12. Spectral properties of the fourth-order Bessel-type operators

**Theorem 12.1.** The minimal operator $T_0$, defined in (8.4) and (8.5), is bounded below in the space $L^2((0, \infty); x)$ by the null operator $O$, i.e.

$$(T_0 f, f) \geq 0 \text{ for all } f \in D(T_0).$$

**Proof.** Since $T_0$ is a restriction of the maximal operator $T_1$, the result of Corollary 10.3 can be applied to give, using also Corollary 10.1,

$$(T_0 f, f) = 8f''(0)f(0) + \int_0^\infty \left\{ x |f''(x)|^2 + (9x^{-1} + 8M^{-1}x) |f'(x)|^2 \right\} dx$$

for all $f \in D(T_0)$. □

**Theorem 12.2.** (1) Let $T$ be a self-adjoint extension of $T_0$; then:

(i) The essential spectrum $\sigma_{\text{ess}}(T)$ is given by

$$(12.2) \quad \sigma_{\text{ess}}(T) = \sigma_{\text{cont}}(T) = [0, \infty).$$

(ii) There are no embedded eigenvalues of $T$ in the essential spectrum.

(iii) $T$ has at most one eigenvalue; if this eigenvalue is present then it is simple and lies in the interval $(-\infty, 0)$.

(2) Every point $\mu \in (-\infty, 0)$ is the eigenvalue of some unique self-adjoint extension $T$ of $T_0$.

**Proof.** The proof of this theorem is given in detail in [2, Section 13]. □

13. The Friedrichs extension $F$

The closed symmetric operator $T_0$ is bounded below in $L^2((0, \infty); x)$, see Theorem 12.1 and the general theory of such operators implies the existence of a distinguished self-adjoint extension $F$, called the Friedrichs extension of $T_0$.

This Friedrichs operator has the properties:

(i) $T_0 \subset F = F^* \subset T_1$

(ii) $D(F) = \{ f \in D(T_1) : f(0) = 0 \}$

(iii) The essential spectrum $\sigma_{\text{ess}}(F)$ is given by

$$(13.1) \quad \sigma_{\text{ess}}(F) = \sigma_{\text{cont}}(F) = [0, \infty)$$

(iv) $F$ has no eigenvalues.

For a discussion of the definition and properties of this Friedrichs extension see [2, Section 15].

14. Self-adjoint operator $S_k$ in $L^2([0, \infty); m_k)$

In this section, given any $k \in (0, \infty)$, we define the operator $S_k$ generated by the differential expression $L_M$ in the Hilbert function space $L^2([0, \infty); m_k)$, where this space is defined in Section 6 above.
Definition 14.1. Let $k \in (0, \infty)$ be given; then the operator $S_k$

\begin{equation}
S_k : D(S_k) \subset L^2([0, \infty); m_k) \rightarrow L^2([0, \infty); m_k)
\end{equation}

is defined by (see (8.1) and (8.2), and Theorem 10.1 for the definition and properties of the domain $D(T_1) \subset L^2((0, \infty); x)$)

1. $D(S_k) := D(T_1)$
2. for all $f \in D(S_k)$

\begin{equation}
(S_k f) (x) := \begin{cases}
-8k^{-1}f''(0) & \text{for } x = 0 \\
x^{-1}L_M[f](x) & \text{for all } x \in (0, \infty).
\end{cases}
\end{equation}

Theorem 14.1. For all $k \in (0, \infty)$:

1. The linear manifold $D(S_k)$ is dense in $L^2([0, \infty); m_k)$.
2. The operator $S_k$ is hermitian in $L^2([0, \infty); m_k)$.
3. The operator $S_k$ is symmetric in $L^2([0, \infty); m_k)$.
4. The operator $S_k$ is bounded below in $L^2([0, \infty); m_k)$

\begin{equation}
(S_k f, f)_k \geq 0 \text{ for all } f \in D(S_k).
\end{equation}

For the proof of this theorem see [4, Theorem 5.2].

Theorem 14.2. Let $k \in (0, \infty)$ be given; then the symmetric operator $S_k$ on the domain $D(S_k)$ is self-adjoint in the Hilbert function space $L^2([0, \infty); m_k)$.

For the proof of this theorem see [4, Theorem 5.4].

Theorem 14.3. Let $k \in (0, \infty)$ be given; then the operator $S_k$ on the domain $D(S_k)$ is the unique self-adjoint operator generated by the differential expression $L_M$ in the Hilbert function space $L^2([0, \infty); m_k)$.

For the proof of this theorem see [4, Theorem 5.5].

15. Spectral properties of the self-adjoint operator $S_k$

The spectral properties of the self-adjoint operator $S_k$ in $L^2([0, \infty); m_k)$ are given by

Theorem 15.1. For any $k \in (0, \infty)$ let the self-adjoint operator $S_k$ in $L^2([0, \infty); m_k)$ be defined as in Definition 14.1 above; then the spectrum $\sigma(S_k)$ of $S_k$ has the following properties:

1. $S_k$ has no eigenvalues
2. The essential spectrum of $S_k$ is given by

\begin{equation}
\sigma_{ess}(S_k) = \sigma_{cont}(S_k) = [0, \infty).
\end{equation}

For the proof of this theorem see [4, Theorem 6.1].
16. DISTRIBUTIONAL ORTHOGONALITY RELATIONSHIPS

Recall that from the properties of the classical Bessel function $J_0$ we have the result that $J_0(\cdot) \notin L^2((0, \infty); x)$. However from [8, Section 1, (1.7)] we have the following distributional (Schwartzian) orthogonal relationship for the classical Bessel function $J_0$, in the space $\mathcal{D}'$ of distributions,

\[
\lambda \int_0^{\infty} xJ_0(\lambda x)J_0(\mu x) \, dx = \delta(\lambda - \mu) \text{ for all } \lambda, \mu \in (0, \infty);
\]

here $\delta \in \mathcal{D}'$ is the Dirac delta distribution. This is the generalised orthogonality property for the solutions $J_0$ of the classical Bessel differential equation, of order 0, given by (3.9); this result mirrors the spectral properties of this equation, when considered on the half-line $(0, \infty)$, in the space $L^2((0, \infty); x)$; in particular the result that every self-adjoint extension $T$ of the corresponding minimal operator $T_0$ has the property $\sigma_{\text{ess}}(T) = [0, \infty)$.

The distributional proof of (16.1) is discussed in the forthcoming paper [5], where the result is also related to the properties of infinite integrals of Bessel functions as originated by Hankel, see [24, Chapter XIII].

As above for the Bessel function $J_0$ we have, from the explicit representation (3.3), the fourth-order Bessel-type function $J_0^{0, M}$ does not belong to the space $L^2((0, \infty); m_k)$ for all $k, M \in (0, \infty)$.

To obtain a distributional orthogonality for $J_0^{0, M}$, given any $M > 0$, it is necessary to choose a special value of the parameter $k$, i.e. $k = M/2$. Then it is shown in [8, Section 4, Corollary 4.3] that we have the following distributional (Schwartzian) orthogonal relationship for the fourth-order Bessel-type function $J_0^{0, M}$, in the space $\mathcal{D}'$ of distributions,

\[
\lambda [1 + M(\lambda/2)^2]^{-2} \left\{ \int_0^{\infty} xJ_0^{0, M}(x)J_0^{0, M}(x) \, dx + \frac{1}{2} MJ_0^{0, M}(0)J_0^{0, M}(0) \right\} = \delta(\lambda - \mu) \text{ for all } \lambda, \mu \in (0, \infty).
\]

The distributional proof of (16.2) is discussed in the forthcoming paper [5].

As a formal representation it follows that (16.2) may be written as, using the inner-product for the space $L^2\left((0, \infty); m_{M/2}\right)$,

\[
\lambda [1 + M(\lambda/2)^2]^{-2} \left( J_0^{0, M}(\cdot), J_0^{0, M}(\cdot) \right)_{M/2} = \delta(\lambda - \mu) \text{ for all } \lambda, \mu \in (0, \infty).
\]

As another connection between the classical Bessel (3.9) and the fourth-order Bessel-type (3.1) differential equations it is to be noted that, formally, the orthogonality result (16.2) tends to the orthogonality result (16.1), as the parameter $M$ tends to zero.

17. THE GENERALISED HANKEL TRANSFORM

From the general theory of symmetric integrable-square transforms given in [20, Chapter VIII] one form of the classical Hankel transform, for the Bessel function $J_0$ and working in the Hilbert function space $L^2((0, \infty); x)$, is:
(i) Let \( f \in L^2((0, \infty); x) \) then the Hankel transform \( g \in L^2((0, \infty); s) \) is given by, for \( s \in (0, \infty) \),

\[
g(s) = \int_0^\infty \xi J_0(s\xi) f(\xi) \, d\xi
\]

with convergence of the integral in \( L^2((0, \infty); s) \).

(ii) With \( g \in L^2((0, \infty); s) \) the inverse transform, to recover \( f \), is given by, for \( x \in (0, \infty) \),

\[
f(x) = \int_0^\infty s J_0(xs) g(s) \, ds
\]

with convergence of the integral in \( L^2((0, \infty); x) \).

(iii) The Parseval relation holds between \( g \) and \( f \)

\[
\int_0^\infty x |f(x)|^2 \, dx = \int_0^\infty s |g(s)|^2 \, ds.
\]

There is also a direct convergence form of the Hankel transform which is best written as, starting with \( f \in L^1((0, \infty); x) \),

\[
f(x) = \int_0^\infty s J_0(xs) \, ds \int_0^\infty \xi J_0(s\xi) f(\xi) \, d\xi
\]

with \( x \in (0, \infty) \). Here the integrals are Lebesgue or limits of Lebesgue integrals as discussed in \([20\, \text{Chapter VIII}].\)

There is an equivalent generalised Hankel transform involving the fourth-order Bessel-type function \( J^{0,M}_\lambda(\cdot) \) and working now in the Hilbert function space \( L^2([0, \infty); m_{M/2}) \); note again these results require the unique choice of \( k = M/2 \).

The complete discussion of the following results for the generalised Hankel transform are to be found in the forthcoming paper \([5]\).

To state these results the Lebesgue-Stieltjes Hilbert function space \( L^2((0, \infty); n) \) is required. Let the function \( \hat{n} : [0, \infty) \to [0, \infty) \) be defined by

\[
\hat{n}(\lambda) := \frac{1}{2} \lambda^2 \left[ 1 + M(\lambda/2)^2 \right]^{-1} \text{ for all } \lambda \in [0, \infty);
\]

then

\[
\hat{n}'(\lambda) = \lambda \left[ 1 + M(\lambda/2)^2 \right]^{-2} \geq 0 \text{ for all } \lambda \in [0, \infty)
\]

so that \( \hat{n} \) is monotonic increasing on \([0, \infty)\) and generates a Baire measure on the \( \sigma \)-algebra \( \mathcal{B} \) of Borel sets on the interval \([0, \infty)\). The Hilbert space \( L^2((0, \infty); n) \) is then defined as the set of all Borel measurable complex-valued functions \( f \) on \([0, \infty)\) such that

\[
\int_{[0,\infty)} |f(\lambda)|^2 \, dn(\lambda) < +\infty,
\]

with norm and inner-product defined by

\[
\|f\|_n^2 := \int_0^\infty |f(\lambda)|^2 \, dn(\lambda) = \int_0^\infty |f(\lambda)|^2 \lambda \left[ 1 + M(\lambda/2)^2 \right]^{-2} \, d\lambda
\]

\[
(f, g)_n = \int_{[0,\infty)} f(\lambda)\overline{g}(\lambda) \, dn(\lambda) = \int_0^\infty f(\lambda)\overline{g}(\lambda) \lambda \left[ 1 + M(\lambda/2)^2 \right]^{-2} \, d\lambda.
\]
Remark 17.1. This norm $\|\cdot\|_n$ and inner-product $(\cdot, \cdot)_n$ for the space $L^2((0, \infty); n)$ are not to be confused with the norm $\|\cdot\|_k$ and inner-product $(\cdot, \cdot)_k$, introduced in Section 13 for the space $L^2([0, \infty); m_k)$.

We note that the weight function $\lambda \mapsto \lambda [1 + M(\lambda/2)^2]^{-2}$ in the integral in (17.6) is the factor in the distributional orthogonal relationships (16.2) and (16.3).

(1) The $L^2$-theory of the generalised Hankel transform is given by the following results:

**Theorem 17.1.** Let $f \in L^2\left([0, \infty); m_{M/2}\right)$. Then there exists exactly one function $g \in L^2((0, \infty); n)$ with the property that

$$
\int_0^\infty |g(\lambda)|^2 \, dn(\lambda) = \int_{[0,\infty)} |f(x)|^2 \, dm_{M/2}(x);
$$

here $g$ is defined by, for almost all $\lambda \in (0, \infty),$

$$
(F_M f)(\lambda) := g(\lambda) = \int_{[0,\infty)} J^0_M(x) f(x) \, dm_{M/2}(x),
$$

thereby defining also the generalised Hankel operator

$$
F_M : L^2\left([0, \infty); m_{M/2}\right) \to L^2((0, \infty); n).
$$

In addition $g$ satisfies

$$
\int_0^\infty g(\lambda) \, dn(\lambda) = f(0).
$$

**Remark 17.2.** Note that the result (17.8) has to be interpreted as follows, in (i) and (ii):

(i) \[ \int_{[0,X]} J^0_M(x) f(x) \, dm_{M/2}(x) \in L^2((0, \infty); n) \] for all $X \in [0, \infty)$

(ii) \[ \lim_{X \to \infty} \int_0^\infty \left| g(\lambda) - \int_{[0,X]} J^0_M(x) f(x) \, dm_{M/2}(x) \right|^2 \, dn(\lambda) = 0. \]

(iii) Note that the Cauchy-Schwarz inequality shows that if $g \in L^2((0, \infty); n)$ then $g \in L^1((0, \infty); n)$.

**Theorem 17.2.** Let $g \in L^2((0, \infty); n)$. Then there exists exactly one function $f \in L^2\left([0, \infty); m_{M/2}\right)$ with the property that (17.7) is satisfied; here $f$ is defined by

$$
(G_M g)(x) := f(x) := \begin{cases} 
\int_0^\infty g(\lambda) \, dn(\lambda) & \text{if } x = 0 \\
\int_0^\infty J^0_M(x) g(\lambda) \, dn(\lambda) & \text{for } x \in (0, \infty),
\end{cases}
$$

thereby defining also the inverse generalised Hankel operator

$$
G_M : L^2((0, \infty); n) \to L^2\left([0, \infty); m_{M/2}\right).
$$

**Remark 17.3.** Note that the result (17.11) has to be interpreted as follows:
(i) \[ \int_0^\Lambda J_\lambda^0,M(x)g(\lambda)dn(\lambda) \in L^2 ([0, \infty); m_{M/2}) \text{ for all } \Lambda \in (0, \infty) \]

(ii) \lim_{\Lambda \to \infty} \int_0^\infty \left| f(x) - \int_0^\Lambda J_\lambda^0,M(x)g(\lambda)dn(\lambda) \right|^2 xdx = 0.

(2) The direct convergence of the generalised Hankel transform is given by the following results:

**Theorem 17.3.** Let \( \gamma \in (0, \infty) \). If \( f : (0, \infty) \to \mathbb{R} \) has the property
\[ x \mapsto \sqrt{x}f(x) \in L^1(0, \infty) \]
and is of bounded variation in a neighbourhood of \( \gamma \), then
\[ \frac{1}{2}[f(\gamma + 0) + f(\gamma - 0)] = \int_\gamma^\infty J_\lambda^0,M(\gamma) \left( \int_0^\infty J_\lambda^0,M(x)f(x)xdx \right)dn(\lambda). \]

Let \( \mu \in (0, \infty) \). If \( g : (0, \infty) \to \mathbb{R} \) has the property
\[ \lambda \mapsto \sqrt{\lambda}g(\lambda) \in L^1(0, \infty) \]
and is of bounded variation in a neighbourhood of \( \mu \), then
\[ \frac{1}{2}[g(\mu + 0) + g(\mu - 0)] = \int_0^\infty J_\mu^0,M(x) \left( \int_0^\infty J_\lambda^0,M(x)g(\lambda)dn(\lambda) \right)dm_{M/2}(x). \]

**Remark 17.4.** The integrals in Theorem 17.3 are either Lebesgue integrals or limits of such integrals over compact intervals of \((0, \infty)\).

**Corollary 17.1.** (i) If \( \gamma \in (0, \infty) \) is a point of continuity of the function \( f \) then \( \gamma \in (0, \infty) \) \( (17.8, \ 17.11) \) and \( (17.14) \) imply
\[ (\mathcal{G}_M(\mathcal{F}_Mf))(\gamma) = \int_0^\infty J_\lambda^0,M(\gamma)(\mathcal{F}_Mf)(\lambda)dn(\lambda) \]
\[ = (M/2)f(0) \int_0^\infty J_\lambda^0,M(\gamma)dn(\lambda) + \int_0^\infty J_\lambda^0,M(\gamma) \left( \int_0^\infty J_\lambda^0,M(x)f(x)xdx \right)dn(\lambda) \]
\[ = f(\gamma), \]

since
\[ \int_0^\infty J_\lambda^0,M(\eta)dn(\lambda) = 0 \text{ for all } \eta \in (0, \infty). \]

(ii) If \( \gamma = 0 \) then
\[ (\mathcal{G}_M(\mathcal{F}_Mf))(0) = (M/2)f(0) \int_0^\infty dn(\lambda) = f(0) \]
since
\[
\int_{0}^{\infty} J^0_M(0) \left( \int_{0}^{\infty} J^0_M(x)f(x)dx \right) dn(\lambda) = 0,
\]
from (17.18), the use of the Fubini integral theorem and noting that \( J^0_M(0) = 1. \)

(iii) If \( \mu \in (0, \infty) \) is a point of continuity of \( g \) then (17.8), (17.11) and (17.16) imply
\[
(F_M(G_M))(\mu) = g(\mu).
\]

18. The Plum partial differential equation

The Plum equation is a fourth-order linear partial differential in the Euclidean space \( \mathbb{R}^2 \) of two dimensions, derived from a linear partial differential expression which is connected with the fourth-order Bessel-type ordinary differential equation.

If the Laplacian \( \nabla^2 \) partial differential expression is written in polar co-ordinates
\[
(18.1) \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]
then the Plum equation has the form, with \( u = u(r, \theta) \),
\[
(18.2) \quad \nabla^4 u - \gamma \nabla^2 u - \frac{4\gamma}{r^2} u = \Lambda u.
\]
Here \( \gamma > 0 \) is determined by \( \gamma = 8M^{-1} \) where \( M > 0 \) is the parameter in the fourth-order Bessel equation (3.1), and \( \Lambda \in \mathbb{C} \) is a spectral parameter.

Written out the equation (18.2) becomes, see [16] Section 1, (2)],
\[
\frac{\partial^4 u}{\partial r^4} + 2 \frac{\partial^3 u}{\partial r^3} + \left( -\frac{1}{r^2} - \gamma \right) \frac{\partial^2 u}{\partial r^2} + \left( \frac{1}{r^3} - \frac{\gamma}{r} \right) \frac{\partial u}{\partial r} + \frac{1}{r^4} \frac{\partial^4 u}{\partial \theta^4}
\]
\[
+ \frac{2}{r^2} \frac{\partial^2 u}{\partial \theta^2 \partial r} - \frac{2}{r^3} \frac{\partial^3 u}{\partial \theta^2 \partial r} + \left( \frac{4}{r^4} - \frac{\gamma}{r^2} \right) \frac{\partial^2 u}{\partial \theta^2} - \frac{4\gamma}{r^2} u
\]
\[
= \Lambda u. \quad (18.3)
\]

From the results given in [19] and [16] assume that a solution for (18.2) is of the separated form
\[
(18.4) \quad u(r, \theta) = v(r)w(\theta)
\]
where \( w \) is required to be a solution of the second-order Sturm-Liouville differential equation
\[
(18.5) \quad -w''(\theta) = 4w(\theta).
\]
Note that \( w \) is then of the general form
\[
(18.6) \quad w(\theta) = A \cos(2\theta) + B \sin(2\theta)
\]
for, say, \( \theta \in [0, \pi] \) and scalars \( A, B \).

Also note that the factor 4 in the equation (18.5) is critical, and has to be fixed, for the separation method to be effective.
Substitution of \((18.3)\) into \((18.3)\) yields, see \([16, \text{Section } 1]\) and \([19]\),

\[
\left( v^{(4)}(r) + \frac{2}{r} v^{(3)}(r) + \left(-\frac{1}{r^2} - \gamma\right) v''(r) + \left(\frac{1}{r^3} - \frac{\gamma}{r}\right) v'(r) + \frac{16}{r^4} v(r) \right) w(\theta)
\]

\[
+ \left( -\frac{8}{r^2} v''(r) + \frac{8}{r^3} v'(r) + \left(-\frac{16}{r^4} + \frac{4\gamma}{r^2}\right) v(r) - \frac{4\gamma}{r^2} v'(r) \right) w(\theta)
\]

\[
(18.7) = \Lambda v(r) w(\theta) \text{ for all } r \in (0, \infty) \text{ and all } \theta \in [0, \pi].
\]

For \((18.7)\) to hold requires that the function \(v(\cdot)\), on gathering up terms, has to satisfy the ordinary differential equation, see \([16, \text{Section } 1, (4)]\), for all \(r \in (0, \infty)\),

\[
v^{(4)}(r) + \frac{2}{r} v^{(3)}(r) + \left(-\frac{9}{r^2} - \gamma\right) v''(r) + \left(\frac{9}{r^3} - \frac{\gamma}{r}\right) v'(r) - \Lambda v(r) = 0.
\]

This last equation may be written in the Lagrange symmetric form

\[
(\rho v''(r))'' - \left((9r^{-1} + \gamma r) v'(r)\right)' = \Lambda r v(r) \text{ for all } r \in (0, \infty),
\]

which is the Bessel fourth-order differential equation \((3.1)\) when \(\gamma = 8M^{-1}\).

Thus separated solutions of the partial differential equation \((18.2)\) can be written in the form

\[
u(r, \theta) = v(r) w(\theta) \text{ for all } r \in (0, \infty) \text{ and } \theta \in [0, 2\pi],
\]

where \(w(\cdot)\) is any trigonometrical solution \((18.6)\) of \((18.5)\), and \(v(\cdot)\) is any solution of the fourth-order Bessel equation \((18.9)\) for any choice of the spectral parameter \(\Lambda\).

Defining the partial differential expression \(P_\gamma[\cdot]\), for \(\gamma \in (0, \infty)\), by

\[
P_\gamma[u] := \left( \nabla^4 u - \gamma \nabla^2 u - \frac{4\gamma}{r^2} u \right)
\]

it is shown in \([16]\) that \(P_\gamma\) is a formally symmetric linear partial differential expression in \(L^2(E^2)\), using polar co-ordinates \((r, \theta)\).

Some early studies indicate that there may be problems in applied mathematics, for which the partial differential equation \(P_\gamma[u] = \Lambda u\) is involved in one or more of the associated mathematical models.

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