Accuracy of the box-counting algorithm for noisy fractals

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Abstract

The box-counting (BC) algorithm is applied to calculate fractal dimensions of four fractal sets. The sets are contaminated with an additive noise with amplitude $\gamma = 10^{-5} \div 10^{-1}$. The accuracy of calculated numerical values of the fractal dimensions is analyzed as a function of $\gamma$ for different sizes of the data sample ($n_{\text{tot}}$). In particular, it has been found that a tiny ($10^{-5}$) addition of noise generates much larger (three orders of magnitude) error of the calculated fractal exponents. Natural saturation of the error for larger noise values prohibits the power-like scaling. Moreover, the noise effect cannot be cured by taking larger data samples.

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1 Introduction

In last decades computations of fractal dimensions (exponents) have become very popular. A search through the Web of Science reveals about 10,000 journal articles with the term \textit{fractal dimension} in title, keywords, or abstract [1]. The fractal structures have been found in a wide spectrum of problems, ranging from high energy physics [2] to cosmology [3] and from bio-medical sciences [4] to econophysics [5].

Although the concept of fractal exponents has been popularized long time ago [6, 7], the accuracy of obtained results is usually either not discussed or overestimated. Moreover, it has been found that in quite a few papers misleading numerical results and conclusions have been published (see \textit{e.g.}: [4, 8, 9, 10]). In many cases, especially for the time series analysis, the fractal exponents are computed indirectly, from the Hurst exponents. However, a simple relation between the two exponents holds only in some special cases [11]. What is more important, the computational algorithms of fractal exponents have systematic errors much larger than the corresponding statistical errors. This is mainly due to the finiteness of the data sample (of size denoted by \(n_{tot}\)). This problem is independent on the computational algorithm applied. For the box-counting (BC) algorithm, it has been shown that the error (roughly) scales according to the power law, \(\sim 1/n_{tot}^\alpha\) [12], for pure fractals, without noise.

The aim of this paper is to perform an analysis of the fractal structures contaminated with an additive external noise and to estimate the accuracy of the BC algorithm in these cases. A similar analysis for the influence
of noise on the Hurst exponents, which were computed by the MF-DFA method, has been performed recently [13]. It should be stressed that the type of algorithm is not so important, as the accuracy depends more on the choice of a particular fractal set than on the type of the algorithm applied. Algorithms that work better with some classes of the fractal sets are not so efficient for other classes (see e.g. Ref. [14]).

We generate the mathematical fractal sets of different size \( n_{\text{tot}} \), namely, the standard \((1/3)\) Cantor set (CS), \(1/2\)-Cantor set (in 1-dimensional embedding space), Sierpiński triangle and Sierpiński carpet (in 2-dimensional embedding space). The data set \( \{x_i\} \) is contaminated by the external additive noise, \( \xi_i \) (\(|\xi_i| \leq 1\)), with different amplitudes \( \gamma \)

\[
x_i \rightarrow x_i + \gamma \xi_i .
\]  

For \( \gamma = 0 \), no external noise is present. The values of \( \gamma \) that will be considered are in the range \(10^{-5} \div 10^{-1}\) (four orders of magnitude). Also, the size of the data sample, \( n_{\text{tot}} \), has been varied in the range \(2^6 \div 2^{20}\) (about four orders of magnitude). The fractal exponent has been calculated using the BC algorithm, according to the standard formula [9]

\[
d(q) = \frac{1}{1 - q} \lim_{N \to \infty} \log \sum_i p_i^q(N) \log N ,
\]

where \( N \) denotes the number of divisions, \( p_i(N) \) is the measure of the subset in the \( i \)th box for a given division \( N \) and the box size \( \epsilon = 1/N \). In practical calculations the numerical data sets are always finite. Hence, the scaling
range (the limit \( N \to \infty \)) is chosen as the interval within which the log-log plot is closest to the linear function. In fact, the problems with choosing the proper scaling range are the main sources of the systematic error. Furthermore, for monofractal sets one has \( d(q) = d_0 = \text{const.} \) All cases discussed here are monofractals and one can set \( q = 0 \).

In the following section, the numerical results will be presented and discussed. The final section is devoted to summary and conclusions.

2 Accuracy analysis

For our analysis, the well-known fractals embedded in the one- and two-dimensional embedding space are taken, for which the precise theoretical values of the fractal exponents \( (d^{th}) \) are well known. Approximate values of these exponents \( (d) \) are calculated with the BC algorithm, according to Eq. (2), and the corresponding error is determined as follows

\[
\Delta d = |d^{th} - d| .
\]

These computation is repeated for different values of the noise amplitude \( \gamma \) and for different values of the sample size \( n_{\text{tot}} \). In effect, one can find a dependence of the error on the noise strength and the sample size. Because \( \Delta d \leq E \), from Eq. (3), it is clear that the error \( \Delta d \) has the following upper bound

\[
\Delta d \leq E - d^{th} ,
\]
Figure 1: Error $\Delta d$ as a function of the noise amplitude ($\gamma$) for the standard Cantor set (A) and for the 1/2-Cantor set (B) with the different sizes of the data sample, $n_{tot} = 2^6, 2^8, 2^{10}, 2^{12}, 2^{14}, 2^{16}$. The errors are denoted by the triangle-up, triangle-down, diamond, circle, star and cross symbols, respectively. In the last cases, the power-like fits are also added (solid lines).
where \( E \) is the embedding dimension. Hence, the error can be at most of order of 1. The question remains, how fast the bound (4) is reached. This will be studied below.

First, the standard Cantor set and the 1/2-Cantor set were analyzed. The noise dependence of the errors was shown in Figs. 1-A and 1-B, respectively. Here, the different sample sizes are represented by the triangles-up, triangles-down, diamonds, circles, stars and crosses, respectively. For larger data sets, the power fits were included for a comparison.

From Fig. 1, it is clear that there is no power-like scaling of the error \( \Delta d \) as a function of the noise amplitude (\( \gamma \)), especially for smaller data samples (\( \lesssim 10^3 \)). Also, the value of the error for \( \gamma = 0.01 \) (\( i.e. \), the relative noise as small as about 1%) is rather large, of order 0.1 and the relative value of the error is close to 20%. A surprisingly large error induced by a small additive noise was also found for the fractal stochastic point processes (FSPP). For \( 10^6 \) data points the fractal dimension was estimated within \( \pm 0.1 \) accuracy, \( i.e. \), with the error higher than 10\% \( [15] \). Hence, the noise effect on the computational error is enhanced by at least one order of magnitude.

In Fig 2., the error value is shown versus the sample size (\( n_{\text{tot}} \)). The different noise amplitudes, \( \gamma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 0 \), are denoted by the triangle-up, triangle-down, diamond, circle, star, and cross symbols. The power fits for the smallest-noise amplitudes were also given for comparison. For the both fractals, the error value is almost independent of the sample size — it is dominated by the noise amplitude. Hence, by increasing the sample size one cannot improve numerical results. The triangle and diamond symbols are arranged almost horizontally. For a quite moderate noise (already below
Figure 2: Error $\Delta d$ as a function of the sample size ($n_{tot}$) for the standard Cantor set (A) and for the $1/2$-Cantor set (B) with the different noise amplitudes, $\gamma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 0$. The errors are denoted by the triangle-up triangle-down, diamond, circle, star and cross symbols, respectively. In the last cases, the power-like fits are added (solid lines).
Figure 3: Error $\Delta d$ as a function of the noise amplitude ($\gamma$) for the Sierpiński triangle (A) and for the Sierpiński carpet (B) with the different sizes of the data sample, $n_{tot} = 2^{10}, 2^{12}, 2^{14}, 2^{16}, 2^{18}, 2^{20}$. The errors are denoted by the triangle-up triangle-down, diamond, circle, star and cross symbols, respectively. In the last cases, the power-like fits are added (solid lines).
1%, $\gamma = 0.01 \div 0.001$ — triangles and diamonds), the error approaches the bound (4). Moreover, for a very small noise ($\gamma = 10^{-5}$), the errors are up to three orders of magnitude bigger than the noise itself.

It is interesting to check whether the 2-D fractals are also so sensitive to an external noise. To this end, a similar analysis was performed for the Sierpiński triangle (Figs. 3A, 4A) and for the Sierpiński carpet (Figs. 3B, 4B). The fractal sets embedded in more dimensions are computationally more demanding. It has been estimated that in order to have a comparable accuracy for the fractals embedded in 2-D space, the number of data points should be squared, in comparison to the 1-D fractals [12]. Hence, the 2-D fractal sample sizes were taken in the range $n_{tot} = 2^{10} \div 2^{20}$.

In Fig. 3, the error dependence on the noise amplitude ($\gamma$) is shown for the Sierpiński triangle (A) and for the Sierpiński carpet (B). The different sample sizes, $n_{tot} = 2^{10}, 2^{12}, 2^{14}, 2^{16}, 2^{18}, 2^{20}$, are represented by the triangle-up, triangle-down, diamond, circle, star and cross symbols, respectively. Again, an approximate power-like scaling is not observed. The total error (i.e. the sum of statistical and systematic errors, $\Delta d$) was found in the range quite similar as for the 1-D fractals (Cantor sets). One can conclude that higher dimensional fractals are not more sensitive to an additive external noise than the 1-D fractals. However, the error remains considerably large, at least about 10% for a small (1% and less) noise and relatively large data samples (of order $10^5$). The error (3) for a small noise ($\gamma \approx 10^{-5}$) is about three–four orders of magnitude larger (approaching $10^{-1}$).

It is worth to mention that, in general, the external noise increases the calculated value of the fractal exponents. This is intuitively clear, as the
Figure 4: Error $\Delta d$ as a function of the sample size ($n_{tot}$) for the Sierpiński triangle (A) and for the Sierpiński carpet (B) with the different noise amplitudes, $\gamma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 0$. The errors are denoted by the triangle-up triangle-down, diamond, circle, star and cross symbols, respectively. In last two cases, the power-like fits are added (solid lines).
white noise embedded in the $E$-dimensional space has the BC dimension equal to $E$. Hence, the obtained exponents for noisy fractals are usually overestimated and for large enough values of $\gamma$ one gets results close to $E$. This effect prohibits the power-like scaling of $\Delta d$ as a function of $\gamma$. In addition, one reaches natural cut-off for larger noise values, $\Delta d \leq E$, where $E$ is the embedding dimension of the fractal.

## 3 Summary and conclusions

In this paper the influence of an external additive noise with the amplitude $\gamma$ on the BC-algorithm calculations of the fractal exponents, for the fractal sets of size $n_{tot}$, was analysed. The numerical results are shown in Figs. 1–4.

It has been found that the noise effect is surprisingly large — a relatively tiny external noise ($\gamma = 10^{-5}$) implies the error values (3) up to three–four orders of magnitude larger. A small noise is usually a part of any real data under investigation. Hence, one should be very careful when drawing conclusions from numerically calculated fractal exponents for experimental data.

As the error value is trivially limited by Eq. (4), for large noise amplitudes, the noise enhancement effect is saturated and a power-like scaling of this effect does not exist. A strong sensitivity to the noise is in agreement with the results obtained earlier for FSPP [15]. It is intuitively clear that the noise increases the computed fractal exponents, with the upper bound $\Delta d \leq E$. Moreover, as can be seen from Figs. 2 and 4, the noise effect cannot be cured by taking much larger data samples. Already for the noise level
approaching 0.1%, the sample size becomes irrelevant.

In conclusion, one should treat very carefully the numerical results of fractal exponents calculations for the data sets containing noise. This is very important since a small addition of noise is usually present in a majority of practical applications.

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