Parameters of Core Collapse

Holger Baumgardt,1 Douglas C. Heggie,2 Piet Hut,3 Junichiro Makino1

1Department of Astronomy, School of Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
2School of Mathematics, University of Edinburgh, King’s Buildings, Edinburgh EH9 3JZ, UK
3Institute for Advanced Study, Princeton, NJ 08540, USA

Accepted . Received ; in original form

ABSTRACT

This paper considers the phenomenon of deep core collapse in collisional stellar systems, with stars of equal mass. The collapse takes place on some multiple, $\xi^{-1}$, of the central relaxation time, and produces a density profile in which $\rho \propto r^{-\alpha}$, where $\alpha$ is a constant. The parameters $\alpha$ and $\xi$ have usually been determined from simplified models, such as gas and Fokker-Planck models, often with the simplification of isotropy. Here we determine the parameters directly from N-body simulations carried out using the newly completed GRAPE-6.

Key words: celestial mechanics, stellar dynamics - star clusters: core-collapse

1 INTRODUCTION

Consider a spherical non-rotating stellar system in dynamic equilibrium. Two-body encounters drive a slow evolution of the system. In response to the well known gravothermal instability of such systems (Antonov 1962, Lynden-Bell & Wood 1968, Hachisu & Sugimoto 1978) the core contracts. Eventually the central parts of the system evolve in a self-similar manner, unaffected by boundary conditions (Lynden-Bell & Eggleton 1980).

In this self-similar regime all central parameters evolve as powers in $\tau$, where $\tau$ is the time remaining until collapse ends. Here we neglect variations in the Coulomb logarithm in the expression for the relaxation time. If $t_{rec}$ is the central relaxation time (defined as in Spitzer 1987, eq. 2–62), and $\rho_c$ is the central density, it follows that $\frac{\rho_c}{\rho_c}t_{rec} = \xi$, where $\xi$ is constant. At the same time the density profile approaches a power law $\rho \propto r^{-\alpha}$, where $\alpha$ is another constant.

Very little can be said about $\alpha$ and $\xi$ on general grounds. Since the core is nearly isothermal we may expect that the evolution timescale is much larger than $t_{rec}$, and so $\xi << 1$. Lynden-Bell & Eggleton showed on physical grounds that $2 < \alpha < 2.5$. It has been claimed (Lancellotti & Kiessling 2001) that $\alpha = 3$, on the basis of the scale invariance of the Fokker-Planck equation. It is shown in appendix A that this is too restrictive a condition.

Precise determination of the parameters $\alpha$ and $\xi$ have been obtained by a variety of methods (Table 1). Lynden-Bell & Eggleton themselves determined parameters equivalent to $\alpha$ and $\xi$, using an isotropic gaseous model of a stellar system, by determining the self-similar solution directly.

This is an eigenvalue problem in which their two parameters are eigenvalues. In common with all gaseous models, the result for $\xi$ depends on a constant which is usually determined by comparison with results of some other method, and so the value is not given in the Table. Much earlier, Larson (1970) determined equivalent parameters by analysing a time-dependent solution of an anisotropic model based on moments of the Fokker-Planck equation.

These two methods (eigenvalue problems and analysis of time-dependent solutions) have been applied by a number of authors using various models and are listed in Table 1. Where necessary their results have been converted to yield values of $\alpha$ and $\xi$ using relations among core parameters developed by Lynden-Bell & Eggleton, except that we uniformly use Spitzer’s relaxation time. For the theoretical models, the value of the Coulomb logarithm used in the model is assumed to exactly cancel that in Spitzer’s definition. In several cases no values were given by the authors themselves, and so we have added notes to indicate our source for the values given. Only systems with stars of equal mass are considered here.

While earlier discussions of this topic (e.g. Spitzer 1987, Louis 1990, Louis & Spurzem 1991) were restricted to comparison of results from simplified models, our main aim in this paper is to add data from new N-body simulations. These and earlier N-body results are identified in col. 4 of Table 1 by the value of $N$ used, or the range of $N$.

2 RESULTS OF N-BODY SIMULATIONS

We have performed a new series of $N$-body simulations of isolated clusters starting from Plummer profiles and contain-
The simulations were carried out on the recently finished GRAPE-6 boards at Tokyo University, using a specially adapted version of the fully collisional N-body code NBODY4 (Aarseth 1999). All runs were performed well into the post-collapse phase. Details of the runs can be found in Table 2.

We first determined the position of the cluster center, using the method of Casertano & Hut (1985). According to Spitzer (1987, p. 149), the average density \( \rho_c \) inside the core radius \( r_c \) is 0.517 times the central density \( \rho(0) \) in an isothermal model. Since also the following relation, connecting \( r_c, \sigma_c \), the 3d-velocity dispersion in the core, and the central density, holds for an isothermal model:

\[
\sigma_c^2 = \frac{4\pi}{3} G \rho(0) r_c^2 ,
\]

(1)

the core radius can be estimated by the following relation:

\[
r_c^2 = 0.517 \frac{3 \sigma_c^2}{4 \pi G \rho_c} .
\]

(2)

Although our clusters initially do not follow isothermal density profiles, their cores approach isothermal profiles as core-collapse proceeds. To calculate \( r_c, \sigma_c \) with sufficient accuracy, after 10 iterations the relative changes of these values between successive iterations were of order 1% or less. Finally, the core collapse rate \( \xi \) was calculated by comparing the core density at two sufficiently separated points in time and calculating the central relaxation time at the midpoint. Fig. 1 shows the evolution of \( \xi \) as a function of the scaled energy \( x_0 = -3\phi(0)/\sigma_c^2 \) for the 64K model. Here \( \phi(0) \) is the central potential, calculated by excluding the star nearest to the cluster centre.

The N-body data is noisy since the time derivative of the central density is used for calculating \( \xi \). Nevertheless, it can clearly be seen that \( \xi \) decreases until about \( x_0 = 10 \), af-

---

Table 1. Determinations of \( \alpha \) and \( \xi \)

| Source                      | \( \alpha \) | \( \xi \)       | Model                      |
|-----------------------------|--------------|-----------------|----------------------------|
| Larson (1970)               | 2.41         | 0.00160         | anisotropic moment         |
| Louis (1990)\(^3\)          | 2.20         | 0.00212         | isotropic moment (eigenvalues) |
| Louis (1990)\(^3\)          | 2.23         | 0.00123         | anisotropic moment (eigenvalues) |
| Lynden-Bell & Eggleton (1980) | 2.208       | –              | isotropic gas (eigenvalue)  |
| Louis & Spurzem (1991)\(^3\) | 2.23         | –              | anisotropic gas (eigenvalues) |
| Cohn (1980)                 | 2.23         | 0.0036         | isotropic Fokker-Planck    |
| Heggie & Stevenson (1988)   | 2.23         | 0.00364        | isotropic Fokker-Planck (eigenvalues) |
| Takahashi (1993)            | 2.23         | 0.003605\(^4\) | isotropic Fokker-Planck (eigenvalues) |
| Cohn (1979)                 | 2.27         | 0.0065         | anisotropic Fokker-Planck  |
| Takahashi (1995)            | 2.23         | 0.0029         | anisotropic Fokker-Planck  |
| Duncan & Shapiro (1982)     | 2.2          | 0.0065         | Monte Carlo anisotropic Fokker Planck |
| Joshi et al. (2000)         | 2.2          | –              | Monte Carlo anisotropic Fokker Planck |
| Giersz & Heggie (1994)      | 2.17\(^6\)   | –              | \( N = 500 \) (average of \( \sim 50 \) cases) |
| Makino (1996)               | 2.36         | –              | \( N = 32k^7 \)            |
| This paper                  | 2.26         | 0.0030         | \( N = 8k - 64k \)         |

Notes
1. From eq.(43), \( \rho_c \)
2. From eq.(44)
3. Several variants are considered in these papers. The values quoted are those highlighted by the authors in the abstract
4. Assumes relation for \( f \) (distribution function), \( \rho_c \), and central velocity dispersion for a Maxwellian.
5. Depends on the assumed value of the Coulomb logarithm; see Spitzer (1987, p.95)

---

Figure 1. Core collapse rate \( \xi \) as a function of central escape energy \( x_0 \) for the 64K run. Points denote individual values calculated for all times when data was stored. The solid line shows the run of the mean log(\( \xi \)). \( \xi \) decreases steadily as core collapse proceeds, and becomes constant at around \( x_0 = 10 \).
Table 2. Results for the core-collapse time $T_{CC}$, $\alpha$ and $\xi$ from $N$-body simulations

| $N$     | 8192 | 16384 | 32768 | 65536 |
|---------|------|-------|-------|-------|
| $N_{Sim}$ | 8    | 3     | 2     | 1     |
| $< T_{CC} >$ | 1967 | 3640  | 6796  | 12218 |
| $< \alpha >$ | 2.24 | 2.26  | 2.28  | 2.26  |
| $< \xi >$ | 0.0030 | 0.0031 | 0.0030 | 0.0029 |

The other which it becomes nearly constant. This is in good agreement with the behavior found by Cohn (1980) and Takahashi (1995) in their Fokker-Planck simulations. Core-collapse is completed at around $x_0 = 13.0$. Taking the mean $\xi$ from all data points with $x_0 > 11$, we obtain a limiting value of $\xi = 0.0029$ for this run. Similar values are obtained for runs with other $N$ (see Table 2). Taking the mean over all performed runs, we obtain $\xi = 0.0030$, which is in good agreement with what Takahashi (1995) obtained from anisotropic Fokker-Planck calculations (see Table 1).

In order to measure the density gradient at the time of core-collapse, we proceeded in the following way: The time of maximum core contraction was determined from the time when the potential energy at the cluster centre ($\phi(0)$, calculated as above) reached its first minimum. We then calculated the stellar density as a function of distance from the centre and fitted power-law distributions to the density profile inside 0.1 half-mass radii. The slope of the best fitting power-law was determined by a KS-test. Mean values of $\alpha$ can be found in Table 2. In general, we obtain somewhat larger values than the Fokker-Planck calculations, and our results seem to be only marginally compatible with the $\alpha$ preferred by most Fokker-Planck and gas calculations: $\alpha = 2.23$. For the range of particle numbers studied, no clear change of $\alpha$ with $N$ can be seen.

Fig. 2 compares the combined $N$-body data from runs with $N = 8192$ to 65536 stars with various power-law profiles. In order to determine the slope of the density profile, we fitted data up to a maximum radius of $r = 0.1 r_{\text{Half}}$. Fig. 2 shows that outside this radius, the density profile cannot be fitted by a single power-law any more, while inside from there the value of $\alpha$ will not depend on the maximum radius used for the fit. Inside $r = 0.1 r_{\text{Half}}$, we obtain a slope of $\alpha = 2.26$ for the density profile. A value of $\alpha = 2.23$ would give a bad fit to the combined $N$-body data, but might be possible in the $N \to \infty$ limit if $\alpha$ is changing slowly with $N$. A value of $\alpha = 3.0$ is completely ruled out by our $N$-body simulations.

Simplified anisotropic models for star cluster evolution (e.g. Giersz & Spurzem 1994) predict that, at a given radius inside the self-similar regime, the anisotropy approaches a finite value at the end of core collapse. Our results for the evolution of the anisotropy profile are shown in Fig. 3. At the end of core collapse, we obtain an anisotropy profile closely resembling Fig. 2 in Giersz & Spurzem, though the value of the anisotropy parameter $A$ at small radii is a little smaller, around 0.2. The maximum value of about 1.1 (which occurs outside the self-similar regime) is a little larger than theirs. The time evolution of the anisotropy within different Lagrangian shells closely resembles the results from averaged 1000-body models shown in their Fig. 11.
3 CONCLUSIONS

Late core collapse of stellar systems is one of the few regimes where the evolution becomes relatively simple. For systems with particles of equal mass, simple arguments imply that many properties (central density and velocity dispersion, core radius, density profile, etc.) approach simple limiting forms, which can be characterised by just two parameters. A common choice for these is the dimensionless rate of increase of the central density, $\xi$, and the index of the power-law dependence of density on radius outside the core, $-\alpha$.

In this paper we explain why $\alpha$ is not determined, as has been argued, by the scale invariance of the Fokker-Planck equation. We review historical determinations of these parameters based on numerical solutions of the Fokker-Planck equation and other simplified models for the evolution of stellar systems. Our main contribution, however, is to present determinations of these parameters directly from new large direct N-body computations. We find that the core collapse rate $\xi (= 0.0030)$ agrees satisfactorily (within the statistical error of the N-body results) with those determined by better simplified models. The density profile index $\alpha$, is slightly steeper, our best value being about 2.26.

ACKNOWLEDGMENTS

We thank the referee for his helpful comments.

APPENDIX A: SCALING OF THE FOKKER-PLANCK EQUATION

It has been shown by Lancellotti & Kiessling (2001) that the Fokker-Planck equation admits a unique scale invariance, and that therefore the self-similar solution requires a limiting density profile $\rho \propto r^{-3}$, i.e. $\alpha = 3$. Here it is shown why the value of $\alpha$ is not determined by the scale invariance of the Fokker-Planck equation.

Consider first the model problem
\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \frac{r}{r^3} \frac{\partial f}{\partial v} = f^2,
\]
which can be interpreted as a Fokker-Planck equation for a distribution of Keplerian oscillators with energy $E = \frac{1}{2} v^2 - 1/r$. The right side of eq. (A1) is a simple collision term, chosen only so that it has the same scaling property as in the Fokker-Planck equation of collisional stellar dynamics. Indeed eq. (A1) admits the unique scaling $f \rightarrow \mu f$, $t \rightarrow \mu^{-1} t$, $r \rightarrow \mu^{-2/3} r$, $v \rightarrow \mu^{1/3} v$. It follows that eq. (A1) admits self-similar solutions of the form
\[
f = t^{-1} F(\mu r^{-2/3}, \mu^{1/3} v),
\]
where $F$ is some function satisfying a certain partial differential equation. Hence the space density is $\rho = t^{-2} \int F(\mu r^{-2/3}, \nu^{1/3}) d\nu$, where $\nu = \nu^{1/3}$. This is stationary at large $r$ only if $\rho \propto r^{-3}$, i.e. $\alpha = 3$.

These are not the only self-similar solutions of eq. (A1), however. There are also solutions of the form
\[
f = t^{-1} F(t^\beta E),
\]
where $\beta$ is any constant and $F$ satisfies a certain ordinary differential equation. Such solutions are certainly self-similar, in the sense that the function $f$ evolves by time-dependent scalings of $f, v$ and $r$. The reason why eq. (A1) admits a wider class of self-similar solutions than those of the form (A2) is that solutions of the form (A3) also satisfy the differential equation
\[
v \cdot \frac{\partial f}{\partial r} = \frac{r}{r^3} \frac{\partial f}{\partial v} = 0
\]
and hence also the simpler Fokker-Planck equation
\[
\frac{\partial f}{\partial t} = f^2.
\]
This pair of differential equations admits a much wider class of scalings $f \rightarrow \mu f$, $t \rightarrow \mu^{-1} t$, $v \rightarrow \nu v$, $r \rightarrow \nu^{-2} r$.

Another interpretation of this situation is to observe that (A4) is the orbit-averaged version of eq. (A1) (cf. Spitzer 1987). In the language of stellar dynamics it is the equation obeyed by solutions which evolve slowly, on a timescale $t_{ev}$ much longer than the orbital or crossing timescale, $t_{cr}$. Indeed the first term on the left of eq. (A1), and the term on the right, are of order $f/t_{ev}$, while the remaining terms of eq. (A1) are of order $f/t_{cr}$. If we restrict attention to self-similar solutions of the form of eq. (A2) we are, in effect, insisting that $t_{ev} \propto t_{cr}$. Indeed the scaling of eq.(A1) is exactly the same as that of the N-body equations
\[
\ddot{r}_i = -G \sum_{j \neq i} \frac{m_j (r_i - r_j)}{|r_i - r_j|^3} t_{ij},
\]
and the analogues of eq.(A2) are then the familiar homothetic solutions in which $r_i \propto t^{2/3}$ (cf. Arnold et al. 1997, p.65).

For the Fokker-Planck equation of stellar dynamics the situation is a little more complicated. Its general and orbit-averaged forms are
\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \phi \cdot \frac{\partial f}{\partial v} = \left( \frac{\partial f}{\partial t} \right)_c
\]
and
\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial E} \left( \frac{\partial \phi}{\partial E} \right) = \left( \frac{\partial f}{\partial E} \right)_c,
\]
where $\left( \frac{\partial f}{\partial t} \right)_c$ is the collision term, $\phi$ is the potential, $E = \phi + \frac{1}{2} v^2$ and $\langle \rangle$ denote an orbit average. Eq. (A6) is not obtained from eq. (A5) by assuming that $f = f(E,t)$. Instead, we must assume that a solution of eq. (A5) may be expanded in the form $f = f_0 + \varepsilon f_1 + \ldots$ where $\varepsilon = t_{cr}/t_{ev}$ and $f_0 = f_0(E,t)$. Then $f_0$ obeys eq. (A6), while $f_1$ and higher terms obey appropriate linearised forms of eq. (A5).

For self-similar core collapse, $f_0$ may be taken to be of the form of eq. (A3). There is no reason to suppose that $f_1$ enjoys the same self-similar evolution as $f_0$. Thus the solution of eq. (A5) for core collapse is nearly self-similar (up to terms of order $t_{cr}/t_{ev}$) but is not confined by the highly restrictive scaling properties of eq. (A5) itself. As core collapse comes to an end, with a small number of stars remaining in the core, $t_{ev}$ decreases and becomes nearly comparable with $t_{cr}$. Then new phenomena beyond the Fokker-Planck
equation become important, such as formation of binaries in 3-body encounters.

REFERENCES

Aarseth S., 1999, PASP 111, 1333
Antonov V. A., 1962, Vest. Leningrad Univ. 7, 135; transl. in Dynamics of Star Clusters, eds. J. Goodman, P. Hut, IAU Symposium 113, Reidel, Dordrecht, p.525
Arnold V. I., Kozlov V. V., Neishtadt A. I., 1997, Mathematical Aspects of Classical and Celestial Mechanics, Springer Verlag, Berlin
Casertano S., Hut P., 1985, ApJ 298, 80
Cohn H., 1979, ApJ 234, 1036
Cohn H., 1980, ApJ 242, 765
Duncan M. J., Shapiro S. L., 1982, ApJ 253, 921
Giersz M., Heggie D. C., 1994, MNRAS 270, 298
Giersz M., Spurzem R., 1994, MNRAS 269, 241
Hachisu I., Sugimoto D., 1978, Prog. Theo. Physics 60, 123
Heggie D. C., Stevenson D., 1988, MNRAS 230, 223
Joshi K. J., Rasio F. A., Portegies Zwart S. F., 2000, ApJ 540, 969
Lancellotti C., Kiessling M., 2001, ApJ 549, L93
Larson R. B., 1970, MNRAS 147, 323
Louis P. D., 1990, MNRAS 244, 478
Louis P. D., Spurzem R., 1991, MNRAS 251, 408
Lynden-Bell D., Eggleton P. P., 1980, MNRAS 191, 483
Lynden-Bell D., Wood R., 1968, MNRAS 138, 495
Makino J., 1996, ApJ 471, 796
Spitzer L. Jr., 1987 Dynamical Evolution of Globular Clusters, Princeton University Press, Princeton
Takahashi K., 1993, PASJ 45, 789
Takahashi K., 1995, PASJ 47, 561

This paper has been produced using the Royal Astronomical Society/Blackwell Science \TeX{} style file.