The group of autoequivalences and the Fourier-Mukai number of a projective manifold

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Abstract

Let $X$ be a smooth projective variety and $\text{Aut}(D(X))$ the group of autoequivalences of the derived category of $X$. In this paper we show that $X$ has no Fourier-Mukai partner other than $X$ when $\text{Aut}(D(X))$ is generated by shifts, automorphisms and tensor products of line bundles.

1 Introduction

In this paper, a projective manifold means a smooth projective variety over the complex number field $\mathbb{C}$. We consider the derived category $D(X)$ of a projective manifold $X$. That is $D(X)$ is the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on $X$. As is well-known since [Muk], for some $X$, there is another projective manifold $Y$ which is not isomorphic to $X$ but the derived category $D(Y)$ is equivalent to $D(X)$ as triangulated categories. We call such a $Y$ a Fourier-Mukai partner of $X$.

Let $\text{FM}(X)$ be the set of isomorphic classes of Fourier-Mukai partners of $X$. It is conjectured in [Kaw] that the set $\text{FM}(X)$ is a finite set. For instance if $X$ is an algebraic surface, the conjecture holds (cf. [BM]). Hence we call the cardinality of $\text{FM}(X)$ the Fourier-Mukai number of $X$. We note that, when $X$ is a projective K3 surface, [HLOY] makes the counting formula of the Fourier-Mukai number of $X$.

Let $\text{Aut}(D(X))$ be the group of autoequivalences of $D(X)$. There are a few cases when the group $\text{Aut}(D(X))$ is exactly determined. For instance $\text{Aut}(D(X))$ is determined by [BO, Theorem 3.1] when the canonical bundle $K_X$ (or $-K_X$) is ample. When $X$ is an abelian variety, $\text{Aut}(D(X))$ is determined by [Orl2].

In this paper we shall study the relation between $\text{Aut}(D(X))$ and $\text{FM}(X)$:
Does \text{Aut}(D(X)) give us informations on \text{FM}(X) ?

We shall give an answer to this question in an easy case. Namely our theorem is the following:

\textbf{Theorem 1.1. (=} \textsc{Theorem [2.1]}\textbf{)} Let \(X\) be a projective manifold. We assume that \text{Aut}(D(X)) is trivially generated (cf. \textsc{Definition 2.1}). Then \text{FM}(X) = \{X\}.

As an application of \textsc{Theorem 1.1} we show the following:

\textbf{Corollary 1.2. (=} \textsc{Corollary [3.5]}\textbf{)} Let \(X\) be a projective manifold such that \(\text{deg} K_X|_C \neq 0\) for any irreducible curve \(C \subset X\). Then \text{FM}(X) = \{X\}.

We found the paper \textsc{Fav} and noticed that \textsc{Theorem 1.1} is a special case of \textsc{Fav, Corollary 4.3} on arXiv after we have finished this paper. However our proof is independent of Favero’s proof and much simpler than his. In addition our motivation is essentially different from his.

\textbf{Acknowledgement.} I thanks Akira Fujiki, Yoshinori Namikawa and Keiji Oguiso who read an earlier draft and suggested several improvements. I’m very grateful to Yukinobu Toda for answering my question and giving me a suggestion for \textsc{Proposition 3.4}.

\section{The group of autoequivalences}

In this section we recall some known results on \text{Aut}(D(X)). First we give the following easy examples of autoequivalences:

1. The shift of complexes \([1] : D(X) \to D(X)\).

2. The (right derived) functor \(\mathbb{R}f_* : D(X) \to D(X)\), where \(f\) is an automorphism of \(X\).

3. The tensor products by \(L \otimes : D(X) \to D(X)\) where \(L \in \text{Pic}(X)\) and \(\text{Pic}(X)\) is the Picard group of \(X\).

\textbf{Definition 2.1.} We define the subgroup \text{Tri}(X) of \text{Aut}(D(X)) by the following condition: \text{Tri}(X) is the subgroup generated by shifts, automorphisms and tensor products with line bundles. \text{Tri}(X) is called the \textit{trivial group generated by} \(X\). If \text{Aut}(D(X)) = \text{Tri}(X), \text{Aut}(D(X))\) is said to be \textit{trivially generated}. 

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Remark 2.2. The trivial group Tri(X) is written by the following:

$$\text{Tri}(X) = (\text{Aut}(X) \times \text{Pic}(X)) \times \mathbb{Z}[1],$$

where Aut(X) is the group of automorphisms of X. Namely for $f \in \text{Aut}(X)$ and $L \in \text{Pic}(X)$, we have

$$f_* \circ L \otimes f_*^{-1}(?) = f_*(L) \otimes (?).$$

For instance, when $K_X$ or $-K_X$ is ample, Aut(D(X)) is trivially generated by [BO, Theorem 3.1]. The first nontrivial example of an autoequivalence was found by Mukai. Let us recall his example.

Let A be an abelian variety, $\hat{A}$ the dual abelian variety of A and $P$ the Poincaré line bundle on $A \times \hat{A}$. We define the functor $\Phi : D(\hat{A}) \rightarrow D(A)$ by the following way:

$$\Phi : D(\hat{A}) \rightarrow D(A), \quad \Phi(?) := R\pi_A^*(P \otimes \pi^{*}_{\hat{A}}(?)), \quad (2.1)$$

where $\pi_A : A \times \hat{A} \rightarrow A$ and $\pi_{\hat{A}} : A \times \hat{A} \rightarrow \hat{A}$ are the natural projections. Then $\Phi$ is an equivalence between $D(\hat{A})$ and $D(A)$ by [Muk] Theorem 2.2. The definition (2.1) seems special, but the following theorem claims that it is sufficiently general.

Theorem 2.3. ([Orl, Theorem 2.18]) Let X be a projective manifold and Y a Fourier-Mukai partner of X. Then, for any equivalence $\Phi : D(X) \rightarrow D(Y)$, there is an object $P^\bullet \in D(X \times Y)$ such that

$$\Phi(?) = R\pi_Y^*(P^\bullet \otimes \pi_X^* (?)), \quad (2.2)$$

where $\pi_X$ (resp. $\pi_Y$) is the natural projection from $X \times Y$ to X (resp. Y). Moreover $P^\bullet$ is unique up to isomorphism.

Thus we obtain the following useful corollary:

Corollary 2.4. Let $x_0$ be a closed point of X and $O_{x_0}$ the skyscraper sheaf of $x_0$. If $\Phi(O_{x_0}) \simeq O_{y_0}$ for a closed point $y_0 \in Y$, then there is a Zariski open subset $U$ of X such that

$$x \in U \Rightarrow \exists y \in Y \text{ such that } \Phi(O_x) \simeq O_y.$$ 

In addition, assume that for all $x \in X$ there is a closed point $y \in Y$ such that $\Phi(O_x) \simeq O_y$. Then there is an isomorphism $f : X \rightarrow Y$ and $L \in \text{Pic}(Y)$ such that $\Phi(?) \simeq L \otimes (f_*(?))$.

Proof. See [Huy] Corollary 5.23 and Corollary 6.14.]
3 Proof of Theorem 1.1

In this section we shall prove our main theorem. We first cite a key lemma of the proof essentially due to [BO]. We define the support $\text{Supp}(E)$ of $E \in D(X)$ by

$$\text{Supp}(E) = \bigcup_i \text{Supp}(H^i(E)),$$

where $H^i(E)$ is the $i$-th cohomology with respect to the t-structure $\text{Coh}(X)$.

**Lemma 3.1.** ([BO, Proposition 2.2] or [Huy, Lemma 4.5]) Let $X$ be a projective manifold and $E \in D(X)$. Assume that $\dim \text{Supp}(E) = 0$ and $\text{Hom}_{D(X)}(E, E[i]) = \begin{cases} 0 & (i < 0) \\ \mathbb{C} & (i = 0). \end{cases}$

Then $E$ is isomorphic to $O_x[n]$ for some $x \in X$ and $n \in \mathbb{Z}$.

Lemma 3.1 is essentially due to [BO]. The above formulation of the lemma is due to [Huy].

Now let $X$ and $Y$ be projective manifolds and $\Phi : D(X) \to D(Y)$ an equivalence. Then we remark that $\Phi$ induces the natural group isomorphism $\Phi_* : \text{Aut}(D(X)) \to \text{Aut}(D(Y))$ by the following way:

$$\Phi_* : \text{Aut}(D(X)) \to \text{Aut}(D(Y)), \quad \Phi_*(F) := \Phi \circ F \circ \Phi^{-1}.$$

**Theorem 3.2.** (=Theorem 1.1) Let $X$ be a projective manifold. Assume that $\text{Aut}(D(X))$ is trivially generated. Then $\text{FM}(X) = \{X\}$.

**Proof.** Let $Y$ be an arbitrary Fourier-Mukai partner of $X$ and $\Phi : D(X) \to D(Y)$ an equivalence. We fix $\Phi$. We would like to show that $Y$ is isomorphic to $X$.

Choose a very ample line bundle $L_Y$ on $Y$ and fix it. Since the induced morphism $\Phi_* : \text{Aut}(D(X)) \to \text{Aut}(D(Y))$ is an isomorphism, we have

$$\exists F \in \text{Aut}(D(X)) \text{ s.t. } \Phi_*(F) = L_Y \otimes (-).$$

We remark that the following diagram commutes:

$$\begin{align*}
D(X) & \xrightarrow{\Phi} D(Y) \\
F \downarrow & \quad \downarrow L_Y \otimes (-) \\
D(X) & \xrightarrow{\Phi} D(Y).
\end{align*}$$
Since \( kL_Y \) has a global section for any positive integer \( k \in \mathbb{Z}_{>0} \), we can make a morphism \( E \to E \otimes kL_Y \) for any \( E \in D(Y) \). Thus, for any \( k \in \mathbb{Z}_{>0} \) and \( E \in D(Y) \), we have \( \text{Hom}_{D(Y)}(E, E \otimes kL_Y) \neq 0 \). Thus we have

\[
\text{Hom}_{D(X)}(O_x, F^k(O_x)) \cong \text{Hom}_{D(Y)}(\Phi(O_x), \Phi(O_x) \otimes kL_Y) \neq 0.
\]

As \( \text{Aut}(D(X)) \) is trivially generated, \( F \) should be written by

\[
F(?) = L_X \otimes f_*(?)[n],
\]

where \( f \in \text{Aut}(X) \), \( L_X \in \text{Pic}(X) \) and \( n \in \mathbb{Z} \). We shall prove that \( n = 0 \) and \( f = \text{id}_X \).

Suppose to the contrary that \( n \neq 0 \). Since \( n \neq 0 \), for sufficiently large \( \ell \in \mathbb{Z}_{>0} \), we have \( \text{Hom}_{D(X)}(O_x, F^\ell(O_x)) = 0 \) where \( F^\ell \) is the \( \ell \) times composition of \( F \). This is contradiction. Hence \( n \) should be 0.

We assume that \( f \neq \text{id}_X \). Then there is a closed point \( x \in X \) such that \( f(x) \neq x \). Since \( F(O_x) = O_{f(x)} \), we have

\[
\text{Hom}_{D(X)}(O_x, F(O_x)) = 0.
\]

This is contradiction.

Thus we have \( F = L_X \otimes (-) \). Hence for any positive integer \( k \),

\[
\Phi(O_x) \otimes kL_Y = \Phi(O_x \otimes kL_X) = \Phi(O_x).
\]

Thus each Hilbert polynomial of \( H^i(\Phi(O_x)) \) with respect to \( L_Y \) is constant. Since \( L_Y \) is very ample, it follows that \( \dim \text{Supp}(H^i(\Phi(O_x))) = 0 \). Thus \( \dim \text{Supp}(\Phi(O_x)) = 0 \). By Lemma 3.1 we have

\[
\Phi(O_x) = O_{y_x}[n_x],
\]

for some \( y_x \in Y \) and \( n_x \in \mathbb{Z} \). By the first half assertion of Corollary 2.4 \( n_x \) is locally constant. Hence, \( n_x \) is constant. So we put \( n_x = n \). Then \( Y \) is isomorphic to \( X \) by the last half assertion of Corollary 2.4 \( \square \).

**Remark 3.3.** The converse of Theorem 3.2 does not hold. For instance, there are projective K3 surfaces \( X \) with Fourier-Mukai number one (See [HLOY] or [Ogu]). On the other hand, as is well-know by [ST], the spherical twist \( T_S \) by a spherical object \( S \in D(X) \) gives an autoequivalence of \( D(X) \) which does not belong to \( \text{Tri}(X) \). Thus \( \text{Tri}(X) \) is a proper subgroup of \( \text{Aut}(D(X)) \).

\footnote{For example a line bundle on \( X \) is a spherical object.}
Let us consider the following three statements for a projective manifold $X$:

(A) The canonical bundle $K_X$ (or $-K_X$) is ample.
(B) The autoequivalence group $\text{Aut}(D(X))$ is trivially generated.
(C) The Fourier-Mukai number of $X$ is one.

[BÖ] proved that $(A) \Rightarrow (B)$ and $(A) \Rightarrow (C)$. As we wrote in Remark 3.3, the converse does not hold. Our theorem claims that $(B) \Rightarrow (C)$.

Now we would like to show that the proposition $(B) \Rightarrow (A)$ does not hold.

**Proposition 3.4.** Let $X$ be a projective manifold such that $\deg K_X|_C \neq 0$ for any irreducible curve $C \subset X$. Then $\text{Aut}(D(X))$ is trivially generated.

For instance, let $Y$ be a projective manifold such that $K_Y$ is ample and let $X \to Y$ be the blowing up at a point of $Y$. Then $X$ satisfies the assumption.

**Proof.** We choose an arbitrary autoequivalence $F \in \text{Aut}(D(X))$ and fix it. Since the functor $\otimes K_X[\dim X]$ is the Serre functor, the following diagram commutes up to isomorphisms:

$$
\begin{array}{ccc}
D(X) & \xrightarrow{F} & D(X) \\
\otimes K_X & \downarrow & \otimes K_X \\
D(X) & \xrightarrow{F} & D(X).
\end{array}
$$

Thus we have

$$
F(O_x) \simeq F(O_x) \otimes K_X. \quad (3.1)
$$

It suffices to show that $\dim \text{Supp}(F(O_x)) = 0$ by Lemma 3.1 and Corollary 2.4. Suppose to the contrary that $\dim \text{Supp}(F(O_x)) > 0$. Then there is an irreducible curve $C$ contained in $\text{Supp}(F(O_x))$. In particular we assume that $C \subset \text{Supp}(H^i(F(O_x)))$ for some $i \in \mathbb{Z}$. Now we put $\mathcal{F} = H^i(F(O_x))|_C$. Notice that $\text{rank } \mathcal{F} > 0$. Thus we have

$$
\deg \mathcal{F} \otimes K_X|_C - \deg \mathcal{F} = \text{rank } \mathcal{F} \cdot \deg K_X|_C \neq 0
$$

On the other hand $\deg \mathcal{F} \otimes K_X|_C - \deg \mathcal{F}$ should be $0$ by (3.1). This is contradiction.

The next corollary easily follows from Proposition 3.4 and Theorem 3.2.

**Corollary 3.5.** Notations are being as above. Then $\text{FM}(X) = \{X\}$. 

6
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