One of the main issues in modern network science is the phenomenon of cascading failures of a small number of attacks. Here we define the dimension of a network to be the maximal number of functions or features of nodes of the network. It was shown that there exist linear networks which are provably secure, where a network is linear, if it has dimension one, that the high dimensions of networks are the mechanisms of overlapping communities, that overlapping communities are obstacles for network security, and that there exists an algorithm to reduce high dimensional networks to low dimensional ones which simultaneously preserves all the network properties and significantly amplifies security of networks. Our results explore that dimension is a fundamental measure of networks, that there exist linear networks which are provably secure, that high dimensional networks are insecure, and that security of networks can be amplified by reducing dimensions.

Network security has become a grand challenge in the current science and technology. We proposed a new model of high dimensional networks by natural mechanisms of homophyly, randomness and preferential attachment. We found that low dimensional networks are much more secure than that of the high dimensional networks, and that there exists an algorithm to
reduce dimensions of networks which preserves network properties and significantly amplifies security of the networks. Our model provides a foundation for both theoretical and practical analyses of security of networks, and an approach to amplifying security of networks.

Networks are proven universal topology of complex systems in nature, society and industry (1). One of the main issues of modern network theory is that most networks are vulnerable to a small number of attacks. This poses a fundamental issue of security of networks (2).

The first type of security is the connectivity security against physical attacks of removal of nodes. For this, it has been shown that in scale-free networks of the preferential attachment (PA, for short) model (3), the overall network connectivity measured by the sizes of the giant connected components and the diameters does not change significantly under random removal of a small fraction of nodes, but the overall connectivity of the networks are vulnerable to removal of a small fraction of the high degree nodes (4–6).

The second type of security is the spreading security against cascading failures by a small number of attacks. We notice that cascading failures naturally occur in rumor spreading, disease spreading, voting, and advertising etc (7–9). It has been shown that in scale-free networks of the preferential attachment model even weakly virulent virus can spread (10).

The authors have shown that cascading failures of attacks are much more serious than that of the physical attacks of removal of nodes, that neither randomness in the ER model (11, 12) nor the preferential attachment scheme in the PA model (3) is a mechanism of security of networks, and that homophyly and randomness together resist cascading failures of networks (A. Li, W. Zhang, Y. Pan and X. Li, Homophyly and randomness resist cascading failure in networks). This shows that some community structures play an essential role in network security.

In practice, overlapping communities are omnipresent. This poses a new question: What roles do the overlapping communities play in network security?

In this article, we found that overlapping communities are obstacles for security of networks.
To solve this problem, we propose an algorithm to amplify security of a network by reducing the dimension of the network. The algorithm removes the obstacle of overlapping communities in network security.

**Results**

Let $G = (V, E)$ be a network. Suppose that each node $v \in V$ has a threshold $\phi_v$. Let $S \subset V$ be a subset of vertices of $G$. We define the infection set of $S$ in $G$ recursively as follows: 1) initially we say that every node in $S$ is infected, 2) a node $v \in V$ becomes infected, if $\phi_v$ fraction of $v$’s neighbors are already infected. We use $\inf^G(S)$ to denote the set of all nodes infected by $S$ in $G$.

**Security Model**

We propose a new model of networks, the security model. It proceeds as follows: Given a homophily exponent $a$ and a natural number $d$,

1. Let $G_d$ be an initial $d$-regular graph such that each node has a distinct color and called seed.

   For each step $i > d$, let $G_{i-1}$ be the graph constructed at the end of step $i - 1$, and $p_i = 1/(\log i)^a$.

2. At step $i$, we create a new node, $v$ say.

3. With probability $p_i$, $v$ chooses a new color, in which case,

   a) we call $v$ a seed,

   b) (Preferential attachment) create an edge $(v, u)$ where $u$ is chosen with probability proportional to the degrees of nodes in $G_{i-1}$, and

   c) (Randomness) create $d - 1$ edges $(v, u_j)$, where each $u_j$ is chosen randomly and uniformly among all seed nodes in $G_{i-1}$.
Otherwise, then $v$ chooses an old color, in which case,

(a) (Randomness) $v$ chooses uniformly and randomly an old color as its own color, and

(b) (Homophyly and preferential attachment) create $d$ edges $(v, u_j)$, where $u_j$ is chosen with probability proportional to the degrees of all nodes of the same color as $v$ in $G_{i-1}$.

We denote the security model by $S$. Let $G = (V, E)$ be a network of model $S$. Since every node $v \in V$ has only one color, we define the dimension of $G$ to be 1. In so doing, we call $G$ a linear network.

It has been shown that: for sufficiently large $n$, if $G$ is a network of the security model with $n$ nodes, then almost surely (or with probability $1 - o(1)$), the following properties hold: (Proofs of the results are referred to A. Li, Y. Pan and W. Zhang, Provable security of networks).

1) (Power law) $G$ follows a power law.

2) (Small world property) The diameter of $G$ is $O(\log n)$.

3) (Homophyly) Let $X$ be a homochromatic set of nodes. Then:

(a) The induced subgraph $G_X$ of $X$ in $G$ is connected,

(b) The diameter of $G_X$ is bounded by $(\log \log n)$,

(c) $G_X$ follows a power law with the same power exponent as that of $G$,

(d) The degrees of nodes in $X$ follow a power law of the same exponent as that of $G$,

(e) The size of $X$ is $|X| = O(\log^{\alpha+1} n)$, and

(f) The conductance of $X$ in $G$ is $\Phi(X) = O(\frac{1}{|X|^\beta})$ for some constant $\beta$, where $|X|$ is the size of $X$. 

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4) (Uniform security) There exists an $\phi = o(1)$ such that for threshold $\phi_v = \phi$ for all nodes $v$ in $G$, for any set $S$ of nodes of size within a polynomial of $\log n$, the size of the infection set of $S$ in $G$ is $o(n)$.

5) (Random security) For every node $v$ in $G$, if $v$ defines its threshold randomly and uniformly, i.e., $\phi_v = r/d_v$, where $d_v$ is the degree of $v$, and $r$ is randomly and uniformly chosen from $\{1, 2, \cdots, d_v\}$, then for any set $S$ of size bounded by a polynomial of $\log n$, the size of the infection set of $S$ in $G$ is $o(n)$.

1) - 3) demonstrate that networks of the security model have all the useful properties of networks. 4) - 5) show that the networks of model $S$ are provably secure against cascading failures of attacks.

The proofs of 4) and 5) involved new probabilistic and combinatorial principles, for which we outline the ideas.

Let $G = (V, E)$ be a network constructed from $S$. For a node $v \in V$, we define the length of degrees of $v$ to be the number of colors associated with all the neighbors of $v$, written by $l(v)$. For a $j$, we define the $j$-th degree of $v$ to be the size of the $j$-th largest homochromatic set among all the neighbors of $v$. We use $d_j(v)$ to denote the $j$-th degree of $v$. Then with probability $1 - o(1)$, we have the following degree priority principle:

(i) The length of degrees of $v$ is bounded by $O(\log n)$,

(ii) The first degree of $v$, $d_1(v)$ is the number of neighbors that share the same color as $v$,

(iii) The second degree of $v$ is bounded by a constant $O(1)$, and

(iv) If $v$ is a seed node, then the first degree of $v$ is lower bounded by, or at least, $\Omega(\log^{2/3} n)$.

A community is a homochromatic set $X$ of $V$, or the induced subgraph of a homochromatic set $X$. We say that a community $X$ is strong, if the seed node $x_0$ of $X$ cannot be infected even if all its neighbors with colors different from that of $x_0$ are all infected, unless there are non-seed
nodes in $X$ which have already been infected. Otherwise, we say that the community $X$ is vulnerable. For appropriately chosen homophyly exponent $a$, the properties (i) - (iv) above ensure that almost all communities of $G$ are strong. Therefore, the number of vulnerable communities is negligible.

Let us consider the infection among strong communities. Suppose that $X$, $Y$, and $Z$ are strong communities with seeds $x_0$, $y_0$ and $z_0$ respectively. It is possible that $x_0$ infects a non-seed node $y_1 \in Y$, $y_1$ infects the seed node $y_0 \in Y$, and $y_0$ infects a non-seed node $z_1 \in Z$. In this case, the infection of the seed $x_0$ of $X$ generates a sequence of strong communities $Y$, $Z$ and so on such that each of the communities contains infected nodes. However by the construction of $G$, we have that $x_0$ is created later than $y_1$, that $y_0$ is created later than $z_1$, and that the edges $(x_0, y_1)$, $(y_0, z_1)$ are created by the preferential attachment scheme at the time step at which a seed node is created so that the edges are embedded in a tree $T$. We call the tree $T$ the "infection priority tree" of $G$. The key point is that the infection priority tree $T$ satisfies the following basic principle: With probability $1 - o(1)$, $T$ has height bounded by $O(\log n)$. We call this property the "infection priority tree principle."

Therefore the infection of a seed node $x_0$ generates a path of at most $O(\log n)$ many strong communities each of which contains infected nodes. In addition, each community has size bounded by $O(\log^{a+1} n)$. So even if all the nodes of an infected community are infected, the contribution to the infection set is still negligible.

For any attack $S$ of size bounded by a polynomial of $\log n$, let $k$ be the number of vulnerable communities. Then there are at most $|S| + k$ many seed nodes each of which generates a path of infected strong communities. This allows us to prove that the size of the infection set of $S$ in $G$ is $o(n)$, negligible comparing to $n$.

The arguments above show that the small community phenomenon of $G$ is one of the ingredients in the proofs of security of $G$. We emphasize that the connecting patterns of the small
communities are crucial to the proofs of the security of \( G \). In particular, the degree priority principle ensures that almost all communities are strong, and the infection priority tree principle ensures that any path of infected strong communities has length \( O(\log n) \).

Therefore linear networks from model \( S \) are provably secure. In the proofs of the security result, the community structures play an essential role in security of networks. However it is not the case that every network with a community structure is more secure.

**2-Dimensional Security Model**

We generalize the security model to 2 dimensions such that the networks are rich in overlapping communities. The 2-dimensional security model proceeds as follows: Given a homophyly exponent \( a \) and a natural number \( d \),

1. Let \( G_d \) be an initial \( d \)-regular graph such that each node has a distinct color and called seed.

   For each \( i > d \), let \( G_{i-1} \) be the graph constructed at the end of step \( i-1 \), and \( p_i = 1/(\log i)^a \).

2. At step \( i \), create a new node, \( v \) say.

3. With probability \( p_i \), \( v \) chooses a new color, in which case,

   a. we call \( v \) a seed,

   b. randomly and uniformly chooses an old color, \( c \) say, as the second color of \( v \),

   c. (preferential attachment) create an edge \((v, u)\) where \( u \) is chosen with probability proportional to the degrees of nodes in \( G_{i-1} \),

   d. for each \( j = 1, 2, \ldots, d-1 \):

      - (randomness) with probability \( \frac{1}{2} \), randomly and uniformly chooses a seed node \( u_j \),

      in which case, create an edge \((v, u_j)\).
- (homophyly) otherwise, then chooses a node \( u_j \) with probability proportional to the degrees of nodes among all nodes sharing the second color \( c \) of \( v \), and create an edge \((v, u_j)\).

(4) Otherwise, then \( v \) chooses an old color, in which case,

(a) (randomness) \( v \) chooses uniformly and randomly an old color as its own color, and

(b) (homophyly and preferential attachment) create \( d \) edges \((v, u_j)\), where \( u_j \) is chosen with probability proportional to the degrees of all nodes of the same color as \( v \) in \( G_{i-1} \).

We use \( S^2 \) to denote the 2-dimensional security model. Let \( G = (V, E) \) be a network constructed by \( S^2 \). We define a community of \( G \) is the induced subgraph of a homochromatic set, \( X \) say. In this case, a seed node, \( v \) say, may have two colors \( c_1 \) and \( c_2 \), so that \( v \) is contained in two communities \( G_X \) and \( G_Y \) say. Therefore, \( G \) has an overlapping community structure.

Here we interpret the maximal number of colors (or features) of nodes for all the nodes of the network is the dimension of the network. It is easy to extend the model to \( k \)-dimensional model for arbitrarily given natural number \( k \). We have argue that networks of the security model \( S \) are secure. A natural question is: Are networks of model \( S^2 \) secure?

In Figures 1 and 2, we depict the security curves against a small number of attacks of the top degrees of networks \( G_1 \)'s from the security model \( S \), of networks \( G_2 \) from the 2-dimensional security model \( S^2 \). In both figures, \( G_1 \) and \( G_2 \) have homophyly exponent \( a = 1.5 \), and number of nodes \( n = 10,000 \). The networks \( G_1 \) and \( G_2 \) in Figures 1 and 2 have average numbers of edges \( d = 5 \), and 10 respectively.

From Figures 1 and 2, we have that networks of the security model are much more secure than that of the 2-dimensional security model. Therefore low dimensional networks are more
secure than that of the high dimensional networks. This experiment shows that dimension is an important measure of networks which plays an essential role in security of networks.

**Algorithm \( \mathcal{R} \): Reducing Dimensions of Networks**

We know that high dimensional networks are less secure than that of low dimensional ones. Can we amplify security of networks by reducing dimensions? For this, we introduce an algorithm to reduce high dimensional networks to low dimensional ones.

Let \( G = (V,E) \) be a network. Suppose that \( \mathcal{X} = \{X_1, X_2, \ldots, X_l\} \) is an overlapping community structure of \( G \). We introduce a graph reduction algorithm to remove the overlapping communities of \( G \) as follows.

1. Let \( X = \bigcup X_j \).

2. For every \( x \in X \), we split \( x \) by the following steps:
   
   (a) suppose that \( Y_1, Y_2, \ldots, Y_k \) are all communities \( X_j \)'s containing \( x \),
   
   (b) For each \( i = 1, 2, \ldots, k \), let \( d_i(x) \) be the number of neighbors of \( x \) that are in \( Y_i \),
   
   (c) Replace \( x \) by a circle of \( k \) nodes \( x_1, x_2, \ldots, x_k \).
   
   (d) For each \( i \in \{1, 2, \ldots, k\} \), all the neighbors of \( x \) that are in \( Y_i \) link to \( x_i \).
   
   (e) For every neighbor \( z \) of \( x \) which is outside of \( Y_i \) for all \( i \), with probability proportional to \( d_i(x) \), we replace the edge \((z, x)\) by \((z, x_i)\).

We use \( \mathcal{R} \) to denote the reduction above. Clearly, \( \mathcal{R} \) splits the overlapping communities of \( G \) into disjoint communities.

**\( \mathcal{R} \) Preserves Network Properties**

In Figure 3, we depict the degree distributions of a network \( G_1 \) from the security model \( S \), a network \( G_2 \) from the 2-dimensional security model \( S^2 \) and the network \( H \) reduced from \( G_2 \)
by $\mathcal{R}$, i.e., $H = \mathcal{R}(G_2)$, where the homophyly exponent $a = 1.5$, the average number of edges $d = 10$ and the number of nodes $n = 10,000$ in the construction of $G_1$ and $G_2$. From the figure, we know that $G_1$, $G_2$ and $H = \mathcal{R}(G)$ all follow a power law of the same power exponent.

In Table I we report the diameters, average distances and clustering coefficients of a network $G_1$ from the security model $S$, a network $G_2$ from the 2-dimensional security model $S^2$, and the reduced network $H = \mathcal{R}(G_2)$, where the homophyly exponent $a = 1.5$, average number of edges $d = 10$, and $n = 10,000$ for both $G_1$ and $G_2$.

From Table I we have the following properties:

1. The diameter and average number of distances of $G_1$ are approximately equal to that of $G_2$ respectively.

2. The diameter and average number of distances of $H = \mathcal{R}(G_2)$ are slightly larger than that of $G_2$ respectively.

3. The clustering coefficient of $G_1$ is significantly larger than that of $G_2$.

Therefore high dimensional networks have less clustering coefficients than that of the low dimensional ones.

4. The clustering coefficient of $H = \mathcal{R}(G_2)$ is approximately equal to that of $G_2$.

We thus have that high dimensions significantly reduce the clustering coefficients, and that high dimensions undermine security of networks. However the high dimensions preserve both the power law and the small world property of low dimensional networks. For the reduction $\mathcal{R}$, we have:

1) It reduces dimensions of networks,

2) It preserves the clustering coefficients of networks,
3) It preserves the power law and small world property, and
4) It slightly increases the diameters and average numbers of distances of networks.

4) is the only disadvantage of $R$, for which we know that the amount of increments of
diameters and average numbers of distances are small, compared to that of the origin networks.

$R$ Amplifies Security of Networks

In Figures 1 and 2, we depict the security curves against a small number of attacks of the top
degrees of networks $G_1$’s from the security model $S$, of networks $G_2$ from the 2-dimensional
security model $S^2$, and the reduced networks from $G_2$’s by $R$, i.e., the networks $H = R(G_2)$. In
both figures, $G_1$ and $G_2$ have homophyly exponent $a = 1.5$, and number of nodes $n = 10,000$.
The networks $G_1$ and $G_2$ in Figures 1 and 2 have average numbers of edges $d = 5$, and 10
respectively.

From Figures 1 and 2 we have the following results:

1. For a network $G$ from model $S^2$, $H = R(G)$ is much more secure than $G$.
2. For the same homophyly exponent $a$, average number of edges $d$ and number of nodes

   $n$, let $G_1$ and $G_2$ be the networks constructed from models $S$ and $S^2$ respectively. Then

   $H = R(G_2)$ has approximately the same security as that of $G_1$.

These results demonstrate that algorithm $R$ optimally amplifies the security of networks of
model $S^2$ so that it significantly amplifies security of networks against cascading failures of
attacks. We notice that the only cost of $R$ is that it slightly increases the diameters and average
numbers of distances of networks.

Method Summary

For the cascading failure of attacks, for a network $G = (V, E)$, every node $v \in V$ picks
randomly and uniformly a threshold $\phi_v = r/d_v$, where $r$ is randomly and uniformly chosen
from \{1, 2, \cdots, d_v\}, and \(d_v\) is the degree of \(v\) in \(G\). The curves of cascading failures of all the experiments in Figures 1 and 2 are the maximal ones among 100 times of attacks of the top degree nodes.

**Discussion**

Global cascading failure may occur in many networks by a small number of attacks for which there are various reasons. Clearly the 2-dimensional model can be easily extended to higher dimensional ones. However the 1- and 2-dimensional models are already sufficient for us to investigate the roles of high dimensions in structure and security of networks. We found that high dimensions are the mechanisms of overlapping communities in networks, that high dimensions and their structural characteristics of overlapping communities are fundamental reasons of insecurity of networks against cascading failures of attacks, and that there exists an algorithm to amplify security of networks by reducing the dimensions. Furthermore, our algorithm for reducing dimensions preserve all the network properties of the original networks. The new concepts and discoveries reported here include: 1) dimension is a fundamental notion of networks, 2) there exist linear networks which are provably secure, 3) high dimensions reduce clustering coefficients and undermine security of networks, 4) security of networks can be amplified by reducing dimensions, and 5) algorithms for reducing dimensions preserve all the properties of the original networks. These discoveries may not only point out new directions of network theory, but also have implications and potentials in understanding and analyzing complex systems in general.

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Figure 1: Security curves. We use $G_1$ and $G_2$ to denote the networks constructed from models $S$ and $S^2$ respectively. Let $H = \mathcal{R}(G_2)$. In the construction of $G_1$ and $G_2$, $a = 1.5$, $d = 5$ and $n = 10,000$. The horizon represents the number of attacked top degree nodes, and the vertical line is the number of the largest infected sets among 100 times of attacks. The security of $G_1$, $G_2$ and $H = \mathcal{R}(G_2)$ are colored blue, green and red respectively. In each time of attacks, the threshold of a node is randomly defined.

|               | $G_1$ | $G_2$ | $H = \mathcal{R}(G_2)$ |
|---------------|-------|-------|------------------------|
| Diameter      | 10    | 9     | 12                     |
| Average Distance | 5.68  | 5.19  | 6.33                   |
| Clustering Coefficient | 0.535 | 0.359 | 0.352                  |

Table 1: Network Properties. Diameters, average distances and clustering coefficients of $G_1$, $G_2$ and $H = \mathcal{R}(G_2)$, where $G_1$ and $G_2$ are constructed from the models $S$ and $S^2$ respectively, with $a = 1.5$, $d = 10$ and $n = 10,000$. 
Figure 2: Security curves. The same as Figure 1, the difference is that in $G_1$ and $G_2$, $d = 10$.

Figure 3: Degree distributions of a network $G_1$ from $S$ and a network $G_2$ from $S^2$ and the reduced version of $G_2$ by $R$, that is, $H = R(G_2)$, where the homophy exponent $a = 1.5$, average number of edges $d = 10$, and number of nodes $n = 10,000$ in the construction of $G_1$ and $G_2$. The distributions of $G_1$, $G_2$ and $H$ are colored red, green and blue respectively. From the figure, we know that $G_1$, $G_2$ and $H$ all follow a power law of the same power exponent.
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Additional information

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