MOMENTS OF MARKOVIAN GROWTH–COLLAPSE PROCESSES

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Abstract

We apply general moment identities for Poisson stochastic integrals with random integrands to the computation of the moments of Markovian growth–collapse processes. This extends existing formulas for mean and variance available in the literature to closed-form moment expressions of all orders. In comparison with other methods based on differential equations, our approach yields explicit summations in terms of the time parameter. We also treat the case of the associated embedded chain, and provide recursive codes in Maple and Mathematica for the computation of moments and cumulants of any order with arbitrary cut-off moment sequences and jump size functions.

Keywords: Growth–collapse processes; Poisson shot noise; uniform cut-off rates; stochastic integrals with jumps; moments; cumulants

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1. Introduction

Markovian growth–collapse processes (see [8]) are piecewise-deterministic Markov processes [6] that grow in between random jump times at which they may randomly crash. Growth–collapse processes are used in e.g. earth sciences and physics, and they have also been recently applied to the study of crypto-currencies; see [9].

Let \((N_t)_{t \geq 0}\) denote a standard Poisson process with intensity \(\lambda > 0\) and jump times \((T_k)_{k \geq 1}\), with \(T_0 := 0\). The growth–collapse process \((X_t)_{t \geq 0}\) increases linearly in \(t\) and crashes at times \(T_k\) by the amount \((1 - Z_k)X_{T_k}^{-}\), i.e.,

\[
X_{T_k} = Z_kX_{T_k}^{-}, \quad k \geq 1,
\]

where \(X_{T_k}^{-}\) denotes the left limit of the process at time \(T_k\), and \((Z_k)_{k \geq 1}\) is an independent and identically distributed (i.i.d.) random sequence of cut-off rates on \([0, 1]\), independent of \((N_t)_{t \in \mathbb{R}^+}\).

In other words, \((X_t)_{t \in \mathbb{R}^+}\) solves the jump stochastic differential equation

\[
\frac{dX_t}{dt} = (1 - Z_{N_t})X_t^{-}dN_t, \quad t \geq 0,
\]

with \(X_0 = 0\), or

\[
X_t = t - \int_0^t (1 - Z_{N_s})X_s^{-}dN_s, \quad t \geq 0,
\]

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Moments of Markovian growth–collapse processes

Figure 1. Sample path of a growth–collapse process.

with explicit solution

\[ X_t = t - \sum_{k=1}^{N_t} T_k (1 - Z_k) \prod_{l=k+1}^{N_t} Z_l \]

\[ = t - \int_0^t s (1 - Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l dN_s, \quad t \geq 0. \] (2)

In particular, the process value after the \( n \)th collapse epoch is

\[ X_{T_n} = \sum_{k=1}^{n} T_k \prod_{l=k}^{n} Z_l - \sum_{k=1}^{n-1} T_k \prod_{l=k+1}^{n} Z_l, \]

and its value before the \( n \)th collapse epoch is the left limit

\[ X_{T_n^-} = \sum_{k=1}^{n} T_k \prod_{l=k}^{n-1} Z_l - \sum_{k=1}^{n-1} T_k \prod_{l=k+1}^{n-1} Z_l, \quad n \geq 1; \]

see Figure 1.

The computation of moments of growth–collapse processes has been the object of several approaches; see [3] for the use of conditional distributions for the computation of mean and variance, and [7] for moment expressions of all orders using the solution of differential equations by matrix exponentials.

In this paper, we apply general moment identities written as sums over partitions for Poisson stochastic integrals with random integrands (see [16, 15, 14, 17]) to the computation of the moments of growth–collapse processes. In particular, we obtain closed-form moment expressions in the case of uniformly distributed cut-off rates.

Given \((N_t)_{t \in \mathbb{R}_+}\), a standard Poisson process with intensity \( \lambda > 0 \) on \( \mathbb{R}_+ \), consider a process \((Y_t)_{t \in \mathbb{R}_+}\) of the form

\[ Y_t = \int_0^t h(N_{s-}) dN_s, \quad t \geq 0, \]
where \( h : \mathbb{N} \to \mathbb{R} \) is a deterministic function. As the left limit \( h(N_s^-) \) is predictable with respect to the filtration \((\mathcal{F}_s)_{s \in \mathbb{R}^+}\) generated by \((N_s)_{s \in \mathbb{R}^+}\), the mean of \( Y_t \) can be computed from the smoothing lemma (see e.g. Theorem 9.2.1 in [4]) as

\[
E[Y_t] = \lambda E \left[ \int_0^t h(N_s^-) ds \right] = \lambda \int_0^t E[h(N_s^-)] ds, \quad t \geq 0.
\]

(3)

This calculation does not apply, however, to the process \( h(N_s) \), which is only adapted and not predictable with respect to the filtration \((\mathcal{F}_s)_{s \in \mathbb{R}^+}\). In this case we may apply the Slivnyak–Mecke formula (see [19], [13], or [18, Corollary 3.2.3] and [5, Section 2.3.4]) to obtain

\[
E[Y_t] = \lambda E \left[ \int_0^t \varepsilon_s^+ h(N_s) ds \right] = \lambda E \left[ \int_0^t h(1 + N_s) ds \right], \quad t \geq 0,
\]

where \( \varepsilon_s^+ \) denotes the operator that adds one jump at the location \( s \geq 0 \) to the Poisson process path.

As an example, the first moment of \( X_t \) given by (2) can be computed using the Slivnyak–Mecke formula as

\[
E[X_t] = t - E \left[ \int_0^t s(1 - Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l dN_s \right]
\]

\[
= t - \lambda E \left[ \int_0^t \varepsilon_s^+ \left( s(1 - Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l \right) ds \right]
\]

\[
= t - \lambda E \left[ \int_0^t s - (1 - \mu_1) \mu_1^{N_s-N_{1+s}} ds \right]
\]

\[
= t - \lambda (1 - \mu_1) \int_0^t s e^{-\lambda(1-\mu_1)(t-s)} ds
\]

\[
= \frac{1 - e^{-\lambda(1-\mu_1)t}}{\lambda(1 - \mu_1)}, \quad t \geq 0,
\]

(4)

where \( \mu_1 = E[Z_1] \), which extends [3, Theorem 4] to non-uniform cut-off rates and yields the limiting behavior

\[
\lim_{t \to \infty} E[X_t] = \frac{1}{\lambda(1 - \mu_1)},
\]

provided that \( \mu_1 < 1 \).

In order to compute higher-order moments in a more general setting, we will apply a nonlinear extension of the Slivnyak–Mecke identity (see Proposition 2.1 below), which allows us to express the moments of Poisson stochastic integrals as a sum of multiple integrals with respect to the intensity of the Poisson process over partitions. In Section 3 we consider the computation of moments of jump processes of the form

\[
Y_t = \sum_{k=1}^{N_t} g(T_k, k, N_t) = \int_0^t g(s, N_s, N_t) dN_s,
\]
where \((T_k)_{k \geq 1}\) denotes the sequence of jump times of the Poisson process \((N_t)_{t \in \mathbb{R}_+}\), and \(g(s, k, m)\) is possibly random but independent of \((N_t)_{t \in \mathbb{R}_+}\); see Proposition 3.1 and Corollary 3.1. These identities are then specialized in Section 4 to the case of uniform cut-off distributions, for processes of the form

\[
Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1 - U_k) \prod_{l=k+1}^{N_t} U_l, \quad t \in \mathbb{R}_+,
\]

where \((U_k)_{k \geq 1}\) is an i.i.d. uniform random sequence on [0, 1], independent of the standard Poisson process \((N_t)_{t \in \mathbb{R}_+}\), and \(f_k\) is a sequence of measurable functions on \(\mathbb{R}_+\); see Corollary 4.1.

In particular, in Proposition 5.1 we derive the closed-form summation

\[
\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n} \sum_{k=0}^{n} (-1)^k (k+1)^{n-1} \binom{n}{k} e^{-k\lambda t/(k+1)}, \quad t \in \mathbb{R}_+, \tag{5}
\]

for the moments of all orders \(n \geq 0\) of the growth–collapse process

\[
X_t = t - \sum_{k=1}^{N_t} T_k(1 - U_k) \prod_{l=k+1}^{N_t} U_l, \quad t \in \mathbb{R}_+,
\]

where \((U_k)_{k \geq 1}\) is an i.i.d. uniform random sequence on [0, 1]. This result extends Theorems 4 and 5 as well as Corollary 1 of [3] from mean and variance to higher moments of all orders, and provides a closed-form alternative to Corollary 4 in [7] which uses matrix exponentials.

The expression (5) immediately yields the asymptotic moments

\[
\lim_{t \to \infty} \mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n}, \quad n \geq 1,
\]

which recover the gamma stationary distribution of \((X_t)_{t \in \mathbb{R}_+}\) with shape parameter 2; see Theorem 3 in [3].

Finally, in Section 6 we show that our approach can also be applied to discrete-time embedded processes of the form

\[
Y(m) = \sum_{k=1}^{m} g(T_k, k, m) = \int_0^{T_m} g(s, N_s, m) dN_s, \quad m \geq 1
\]

(see Corollaries 6.1–6.2), and to the embedded growth–collapse chain

\[
X(m) = T_m - \sum_{k=1}^{m} T_k(1 - U_k) \prod_{l=k+1}^{m} U_l, \quad m \geq 1,
\]

where \((U_k)_{k \geq 1}\) is an i.i.d. uniform random sequence on [0, 1] (see Corollaries 6.3–6.4). This recovers Theorem 7, stated for mean and variance, in [3], and provides moment expressions of all orders.

We proceed as follows. In Section 2 we review the derivation of moment identities for stochastic integrals using sums over partitions, and in Section 3 we apply them to the moments of jump processes driven by a Poisson process. These expressions are then specialized as closed-form identities in Section 4 in the case of uniform cut-off distributions. The moments of growth–collapse processes are considered in Section 5, and the case of embedded chains is treated in Section 6.
2. Moment identities for Poisson stochastic integrals

In this section we review the computation of moments of Poisson stochastic integrals with random integrands using sums over partitions; see Proposition 3.1 in [16]. Consider a Poisson process \((N_t)_{t \in \mathbb{R}_+}\) constructed as \(N_t = \omega([0, t])\), where \(\omega(ds)\) is a Poisson random measure of intensity \(\lambda(ds)\), with sequence \((T_k)_{k \geq 1}\) of jump times. For any \(s_1, \ldots, s_k \in \mathbb{R}_+\), we let \(\epsilon^{+}_{s_1} \cdots \epsilon^{+}_{s_k}\) denote the operator

\[
(e^{+}_{s_1} \cdots e^{+}_{s_k} F)(\omega) = F(\omega \cup \{s_1, \ldots, s_k\})
\]

acting on random variables \(F\) by addition of points at locations \(s_1, \ldots, s_k\) to the point process \(\omega(dx)\). For example, if \(F\) takes the form \(F = f(N_t, \ldots, N_{t_n})\), then we have

\[
\epsilon^{+}_{s_1} \cdots \epsilon^{+}_{s_k} F = f(N_{t_1} + \# \{k : s_k \leq t_1\}, \ldots, N_{t_n} + \# \{k : s_k \leq t_n\})
\]

The following moment identity (see [16, Proposition 3.1] and [17, Theorem 1]) uses sums over partitions \(\{\pi_1, \ldots, \pi_k\}\) of \(\{1, \ldots, n\}\), and applies to random integrands \(u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}\).

**Proposition 2.1.** Let \((u_s(\omega))_{s \in [0, t]}\) denote a stochastic process indexed by \(s \in [0, t]\). For any \(n \geq 1\), we have

\[
\mathbb{E} \left[ \left( \int_0^t u_s dN_s \right)^n \right] = \sum_{k=1}^n \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}} \mathbb{E} \left[ \int_0^t \cdots \int_0^t \epsilon^{+}_{s_1} \cdots \epsilon^{+}_{s_k} (u_{|\pi_1|}(s_1, \omega) \cdots u_{|\pi_k|}(s_k, \omega)) \lambda(ds_1) \cdots \lambda(ds_k) \right],
\]

where the power \(|\pi_i|\) denotes the cardinality of the subset \(\pi_i\) and the above sum runs over all partitions \(\pi_1, \ldots, \pi_k\) of \(\{1, \ldots, n\}\).

In the sequel we will frequently use the equivalent combinatorial expressions

\[
\sum_{k=1}^n \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}} f_k(|\pi_1|, \ldots, |\pi_k|) = \sum_{k=1}^n \frac{n!}{k!} \sum_{p_1 + \cdots + p_k = n, p_1, \ldots, p_k \geq 1} \frac{f_k(p_1, \ldots, p_k)}{p_1! \cdots p_k!} \tag{6}
\]

for \(f_k\) a function on \(\mathbb{N}^k, k = 1, \ldots, n\). In particular, for \(x_1, \ldots, x_n \in \mathbb{R}\) and \(f_k(p_1, \ldots, p_k) = x_{p_1} \cdots x_{p_k}\) this yields the Bell polynomial of order \(n \geq 1\) as

\[
B_n(x_1, \ldots, x_n) = \sum_{k=1}^n \frac{n!}{k!} \sum_{p_1 + \cdots + p_k = n, p_1, \ldots, p_k \geq 1} \frac{x_{p_1} \cdots x_{p_k}}{p_1! \cdots p_k!}
\]

\[
= \sum_{k=1}^n \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1, \ldots, n\}} x_{|\pi_1|} \cdots x_{|\pi_k|} \tag{7}
\]

\[
= \sum_{k=1}^n B_{k,n}(x_1, \ldots, x_{n-k+1}),
\]
where $B_{k,n}$ is the partial Bell polynomial of order $(k,n)$. We will also use the relation $\mathbb{E}[X^n] = B_n(\kappa_X^{(1)}, \ldots, \kappa_X^{(n)})$ between the moments $\mathbb{E}[X^n]$ and the cumulants $\kappa_X^{(n)}$ of a random variable $X$, and the inversion relation

$$\kappa_X^n = \sum_{k=1}^{n} (-1)^{(k-1)} \cdot \sum_{\pi_1 \cup \cdots \cup \pi_k = [1, \ldots, n]} \mathbb{E}[X^{\pi_1}] \cdots \mathbb{E}[X^{\pi_k}]$$

(8)

$$= \sum_{k=1}^{n} (-1)^{(k-1)} \cdot (k-1)! \cdot B_{k,n}(\mathbb{E}[X], \ldots, \mathbb{E}[X^{n-k+1}]), \quad n \geq 1;$$

see [12, Theorem 1] or [11].

The case of shot noise processes. Before moving to the setting of Markovian growth–collapse processes, we use the case of Poisson shot noise processes as an illustration for the result of Proposition 2.1. Consider a shot noise process $(S_t)_{t \in \mathbb{R}^+}$ of the form

$$S_t = \sum_{k=1}^{N_t} h(T_k, t)J_k = \sum_{k=1}^{N_t} h(T_k, t)J_{N_t} = \int_{0}^{t} h(s, t)J_{N_s} dN_s, \quad t \in \mathbb{R}^+,$$

where $(J_k)_{k \geq 0}$ is a sequence of i.i.d. random variables admitting moments of all orders, and $h(\cdot, \cdot)$ is a sufficiently integrable deterministic function. The next proposition provides a closed-form expression for the moments of shot noise processes using standard Bell polynomials; see also [7, Corollary 2] for another expression using matrix exponentials in case $\lambda(ds) = \lambda ds$ for some rate $\lambda > 0$, and $h(s, t) = e^{-\beta(t-s)}$ for some $\beta > 0$.

Proposition 2.2. For any $n \geq 1$, we have

$$\mathbb{E}[S_t^n] = B_n\left(\mathbb{E}[J_1], \int_{0}^{t} h(s, t)ds, \ldots, \int_{0}^{t} h^n(s, t)ds\right),$$

(9)

where $B_n$ is the Bell polynomial of order $n \geq 1$.

Proof. Taking $u_s(\omega) = J_{N_s} h(s, t)$, by Proposition 2.1 we have

$$\mathbb{E}[S_t^n] = \mathbb{E}\left[\left(\sum_{k=1}^{N_t} J_{N_t} h(T_k, t)\right)^n\right] = \sum_{l=1}^{n} \frac{n!}{l!} \sum_{\pi_1 \cup \cdots \cup \pi_l = [1, \ldots, n]} \int_{0}^{t} \cdots \int_{0}^{t} h^{\pi_1}(s_1, t) \cdots h^{\pi_l}(s_l, t) \mathbb{E}\left[\epsilon_{s_1}^{+} \cdots \epsilon_{s_l}^{+} (J_{N_{s_1}}^{\pi_1} \cdots J_{N_{s_l}}^{\pi_l}) \lambda(ds_1) \cdots \lambda(ds_l)\right].$$

Next, we note that for $0 < s_1 < \cdots < s_l < t$ we have

$$\epsilon_{s_1}^{+} \cdots \epsilon_{s_l}^{+} N_{s_l} = i + N_{s_l}, \quad 1 \leq i \leq l \leq n;$$

(10)

hence

$$\mathbb{E}\left[\epsilon_{s_1}^{+} \cdots \epsilon_{s_l}^{+} (J_{N_{s_1}}^{\pi_1} \cdots J_{N_{s_l}}^{\pi_l})\right] = \mathbb{E}\left[J_{1+N_{s_1}}^{\pi_1} J_{2+N_{s_2}}^{\pi_2} \cdots J_{l+N_{s_l}}^{\pi_l}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[J_{1+N_{s_1}}^{\pi_1} \cdots J_{l+N_{s_l}}^{\pi_l} \left| N_{s_1}, \ldots, N_{s_l}\right.\right]\right] \times \cdots \times \mathbb{E}\left[\mathbb{E}\left[J_{1+N_{s_1}}^{\pi_1} \cdots J_{l+N_{s_l}}^{\pi_l} \left| N_{s_1}, \ldots, N_{s_l}\right.\right]\right]$$
and therefore
\[
\mathbb{E}[S^n_t] = \sum_{l=1}^{n-1} \sum_{\pi_1 \cup \ldots \cup \pi_l = \{1, \ldots, n\}} \mathbb{E}[J_{\pi_1}^1] \ldots \mathbb{E}[J_{\pi_l}^1] \int_0^t \int_0^t \ldots \int_0^t h_{\pi_1}(s_1, t) \ldots h_{\pi_l}(s_l, t) \lambda(ds_1) \ldots \lambda(ds_l),
\]
which yields (9) from (7).

3. Moments of jump processes

From now on we assume that \((N_t)_{t \in \mathbb{R}_+}\) is a standard Poisson process with intensity \(\lambda > 0\), and in this section we consider jump processes built as the anticipating Poisson integrals
\[
Y_t = \sum_{k=1}^{N_t} g(T_k, k, N_t) = \int_0^t g(s, N_s, N_t) dN_s, \quad t \in \mathbb{R}_+.
\]  

Proposition 3.1. Let \((Y_t)_{t \in \mathbb{R}_+}\) be defined as in (11). For all \(n \geq 1\), we have
\[
\mathbb{E}[(Y_t)^n] = \sum_{k=1}^{n} \lambda^k \sum_{\pi_1 \cup \ldots \cup \pi_k = \{1, \ldots, n\}} \int_0^t \int_0^t \ldots \int_0^t \mathbb{E} \left[ \prod_{l=1}^{k} g_{\pi_l}(s_l, l + N_{s_l}, k + N_t) \right] ds_1 \ldots ds_n,
\]

where the sum runs over all partitions \(\pi_1, \ldots, \pi_k\) of \(\{1, \ldots, n\}\).
Moments of Markovian growth–collapse processes

Proof. Taking \( u_s(\omega) := g(s, N_s, N_t), 0 \leq s \leq t \), by Proposition 2.1 we have
\[
\mathbb{E}[(Y_t)^n] = \mathbb{E} \left[ \left( \int_0^t g(s, N_s, N_t) dN_s \right)^n \right]
\]
\[
= \sum_{k=1}^n \lambda^k \sum_{\pi_1 \cup \ldots \cup \pi_k = \{1, \ldots, n\}} \int_0^t \cdots \int_0^t \mathbb{E} \left[ \epsilon_{s_1}^+ \cdots \epsilon_{s_k}^+ \prod_{l=1}^k g^{|\pi_l|}(s_l, N_{s_l}, N_t) \right] ds_1 \cdots ds_n
\]
\[
= \sum_{k=1}^n \lambda^k \sum_{\pi_1 \cup \ldots \cup \pi_k = \{1, \ldots, n\}} \int_0^t \cdots \int_0^t \mathbb{E} \left[ \prod_{l=1}^k g^{|\pi_l|}(s_l, l + N_{s_l}, k + N_t) \right] ds_1 \cdots ds_n,
\]
where we applied (10).

Next, we specialize Proposition 3.1 to the case where the process \( g(s, k, n) \) takes the form
\[
g(s, k, n) = f_k(s)(1 - Z_k) \prod_{l=k+1}^n Z_l,
\]
where \( (Z_k)_{k \geq 1} \) is an i.i.d. random sequence independent of the Poisson process \((N_t)_{t \in \mathbb{R}_+}\), i.e., we have
\[
Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1 - Z_k) \prod_{l=k+1}^n Z_l, \quad t \in \mathbb{R}_+.
\]
The corresponding growth–collapse process can be written as
\[
X_t = f_{N_t}(t) - Y_t
\]
\[
= f_{N_t}(t) - f_{N_t}(T_{N_t}) + \sum_{k=1}^{N_t} (f_k(T_k) - f_{k-1}(T_{k-1})) \prod_{l=k}^{N_t} Z_l,
\]
which corresponds to the process in [10, Equation (1)] if we take \( I_n(u) := f_n(u + T_{n-1}) - f_{n-1}(T_{n-1}) \) therein, for \( u \in [0, T_n) \), \( n \geq 1 \), with \( f_0 := 0 \).

Corollary 3.1. Let \((Y_t)_{t \in \mathbb{R}_+}\) be defined as in (12) with \( f_k(s) = f(s) \) independent of \( k \geq 1 \). For all \( n \geq 1 \), we have
\[
\mathbb{E}[(Y_t)^n] = n! \lambda^{n(n-1)} \sum_{k=1}^n \lambda^k \sum_{q_0=0, q_1 < \ldots < q_k = n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_k} \prod_{l=1}^k \left( \frac{f_{q_l-q_{l-1}}(s_l)}{(q_l - q_{l-1})!} C_{q_l-1, q_l-1} e^{\lambda_{q_l-1} - \mu_{q_l-1}} \right) ds_1 \cdots ds_k,
\]
\( t \in \mathbb{R}_+ \), where
\[
C_{p, q} := \mathbb{E}[Z^p(1 - Z)^q] = \sum_{k=0}^p \binom{p}{k} (-1)^k \mu_{q+k} \quad \text{and} \quad \mu_p := C_{p, 0} = \mathbb{E}[Z^p], \quad p, q \geq 0.
\]

Proof. By Proposition 3.1, letting
\[
W_{k, n} := (1 - Z_k) \prod_{l=k+1}^n Z_l, \quad 1 \leq k \leq n,
\]
for all \( n \geq 1 \) we have

\[
\mathbb{E}[(Y_t)^n] = n! \sum_{k=1}^{n} \lambda^k \sum_{p_1 + \cdots + p_k = n \atop p_1, \ldots, p_k \geq 1} \int_0^t \int_0^{s_k} \cdots \int_0^{s_1} \frac{f^{p_1}(s_1) \cdots f^{p_k}(s_k)}{p_1! \cdots p_k!} ds_1 \cdots ds_k.
\]  

(15)

Next, when \( p_1 + \cdots + p_k = n \) and \( 0 \leq s_1 < \cdots < s_k \leq t \), we have

\[
\mathbb{E}\left[\epsilon_{s_1}^+ \cdots \epsilon_{s_k}^+ \left( (W_{N_{s_1}, N_{s_2}})^{p_1} \cdots (W_{N_{N_{s_k}}, N_{N_{s_k}}})^{p_k} \right) \right] = \mathbb{E} \left[ (W_{1+N_{s_1}, k+N_{s_2}})^{p_1} \cdots (W_{k+N_{N_{s_k}}, k+N_{N_{s_k}}})^{p_k} \right]
\]

\[
= \mathbb{E} \left[ \left( (1 - Z_{l+1+N_{s_1}}) \prod_{l=2+N_{s_1}}^{k+N_{s_1}} Z_l \right)^{p_1} \cdots \left( (1 - Z_{k+N_{s_k}}) \prod_{l=k+1+N_{s_k}}^{k+N_{s_k}} Z_l \right)^{p_k} \right]
\]

\[
= \mathbb{E} \left[ \prod_{l=1}^{k} \left( (1 - Z_{l+1+N_{s_l}})^{p_l} (Z_{l+1+N_{s_l}})^{N_{s_{l+1}} - N_{s_l}} \right)^{p_l} \right]
\]

\[
= \prod_{l=1}^{k} \left( C_{p_1 + \cdots + p_{l-1}, p_l} (\mu_{p_1 + \cdots + p_l})^{N_{s_{l+1}} - N_{s_l}} \right)
\]

\[
= \prod_{l=1}^{k} \left( C_{p_1 + \cdots + p_{l-1}, p_l} e^{\lambda(1-s)\mu_{p_1 + \cdots + p_l}} \right)
\]

\[
= e^{\lambda(n-1)\mu_{n-1}} \prod_{l=1}^{k} \left( C_{p_1 + \cdots + p_{l-1}, p_l} e^{\lambda s_l\mu_{p_1 + \cdots + p_l}} \right),
\]

where we used (14) and the independence of the sequence \( (Z_k)_{k \geq 1} \), which leads to (13) from (15).

When \( n = 1 \), Corollary 3.1 yields

\[
\mathbb{E}[Y_t] = \lambda(1 - \mu_1) \int_0^t f(s) e^{-\lambda(1-\mu_1)(t-s)} ds, \quad t \in \mathbb{R}_+,
\]

which recovers (4) when \( f(s) = s, s \in \mathbb{R}_+ \). More generally, \( \mathbb{E}[(Y_t)^n] \) can be computed for any \( n \geq 1 \) and any choice of integrable function \( f(s) \) and moment sequence \( \mu_n = \mathbb{E}[Z^n], n \geq 0 \), using the Maple and Mathematica codes in the online appendix.

### 4. Uniform cut-off rates

In this section we assume that \( (Y_t)_{t \in \mathbb{R}_+} \) takes the form

\[
Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1 - Z_k) \prod_{l=k+1}^{N_t} Z_l, \quad t \in \mathbb{R}_+,
\]

(16)
where \((Z_k)_{k \geq 1}\) is an i.i.d. uniform random sequence on \([0, 1]\). In this case we have 
\[ \mu_n = 1/(n+1), \] and \(C_{p,q}\) is given by the beta function as 
\[ C_{p,q} = \frac{\Gamma(p+q)!}{\Gamma(p+1)! \Gamma(q)!}, \] hence we have 
\[ \prod_{l=1}^{k} C_{p_l + \cdots + p_{l-1}, p_l} = \prod_{l=1}^{k} \frac{(p_1 + \cdots + p_{l-1})!p_l!}{(p_1 + \cdots + p_l + 1)!} = \frac{1}{n!} \sum_{l=1}^{k} \frac{p_l!}{p_1 + \cdots + p_l + 1} \] under the condition \(p_1 + \cdots + p_k = n\), which yields the next consequence of Corollary 3.1.

**Corollary 4.1.** Let \((Y_t)_{t \in \mathbb{R}_+}\) be defined as in (16), where \((Z_k)_{k \geq 1}\) is an i.i.d. uniform random sequence on \([0, 1]\). For any \(n \geq 1\), we have 
\[ \mathbb{E}[Y_t^n] = e^{-n\lambda t/(n+1)} \sum_{k=1}^{n} \lambda^k \sum_{q_0=0}^{n-1} \cdots \sum_{q_k=n} \left( \frac{q_1 \cdots q_k}{1+q_l} \right) e^{s_1(1/(1+q_{l-1})-1/(1+q_l))} ds_1 \cdots ds_k, \] \(t \in \mathbb{R}_+\).

When \(f(s) = s\), \(s \in \mathbb{R}_+\), the relation 
\[ \int_0^{s} s^2 e^{as_1} ds_1 = 1 - \frac{2}{a} \frac{(as_2)^k}{k!}, \] \(s_2 \in \mathbb{R}_+, n \geq 0, \) can be used to compute the integrals in (4.1) by induction using the Maple command \(MY(\lambda, \mu, f, n)\) (resp. the Mathematica command \(MY[\lambda, \mu, f, n]\)) in the online appendix, by taking \(f := s \rightarrow s\) and \(\mu := n \rightarrow 1/(n+1)\) (resp. \(f[s_] := s\) and \(\mu[n_] := 1/(n+1)\)).

**First moment of \(Y_t\) using \(MY(\lambda, \mu, f, 1)\). For \(n = 1\), we have** 
\[ \mathbb{E}[Y_t] = \frac{\lambda}{2} e^{-\lambda t/2} \int_0^t s_1 e^{\lambda s_1/2} ds_1 = t - \frac{2}{\lambda} (1 - e^{-\lambda t/2}), \] which is consistent with Theorem 4 in [3], with a shorter proof; see Figure 2.

**Second moment of \(Y_t\) using \(MY(\lambda, \mu, f, 2)\). For \(n = 2\), we have** 
\[ \mathbb{E}[(Y_t)^2] = \frac{\lambda}{3} e^{-2\lambda t/3} \int_0^t s_1^2 e^{2\lambda s_1/3} ds_1 + \frac{\lambda^2}{6} e^{-2\lambda t/3} \int_0^t s_2^2 e^{2\lambda s_2/6} \int_0^{s_2} s_1 e^{\lambda s_1/2} ds_1 ds_2 \]
\[ = \frac{18}{\lambda^2} e^{-2\lambda t/3} + 4 \frac{e^{-\lambda t/2} \lambda}{\lambda^2} + 2 \frac{6}{\lambda^2} t^2 + \frac{6}{\lambda^2} t^2 + \frac{4}{\lambda^2} t^2, \] hence 
\[ \kappa(2)(t) = \text{Var} [Y_t] = \mathbb{E}[Y_t^2] - (\mathbb{E}[Y_t])^2 = \frac{2}{\lambda^2} (9e^{-2\lambda t/3} - 2e^{-\lambda t} - 8e^{-\lambda t/2} + 1), \] which recovers Theorem 5 in [3] with a shorter proof; see Figure 2. Figures 2–4 are plotted with 10 million Monte Carlo samples and \(\lambda = 2\).

The subsequent integrals for higher-order moments can be evaluated using Mathematica or based on the recurrence relation (19).
Third moment of $Y_t$ using $MY(t, \lambda, \mu, f, 3)$. For $n = 3$, we have

$$\mathbb{E}[(Y_t)^3] = \frac{\lambda}{4} e^{-3\lambda t/4} \int_0^t \int_0^t \int_0^t s_1 e^{3\lambda s_1/4} d1 ds2$$

$$+ \frac{\lambda^2}{8} e^{-3\lambda t/4} \int_0^t \int_0^t \int_0^t s_2 e^{3\lambda s_2/4} ds2 ds3$$

$$+ \frac{\lambda^3}{24} e^{-3\lambda t/4} \int_0^t \int_0^t \int_0^t s_3 e^{3\lambda s_3/4} ds2 ds3$$

$$= \frac{\lambda^3}{e^{-3\lambda t/4}} \left( 384 + 54 e^{\lambda t/12} (\lambda t - 12) + 6 e^{\lambda t/4} (48 + \lambda t (\lambda t - 12)) + e^{3\lambda t/4} (\lambda t (18 + \lambda t (\lambda t - 6)) - 24) \right). \tag{22}$$

see Figure 3 below.
Moments of Markovian growth–collapse processes

Fourth cumulant $\kappa^{(4)}(t)$. Excess kurtosis $\kappa^{(4)}(t)/(\kappa^{(2)}(t))^2$ of $Y_t$.

**FIGURE 4.** Fourth cumulant (30) and excess kurtosis.

*Fourth moment of $Y_t$ using $\text{MY}(t, \lambda, \mu, f, 4)$. For $n = 4$, we have*

$$
E[(Y_t)^4] = \frac{\lambda}{5} e^{-4\lambda t/5} \int_0^t s_1^4 e^{4\lambda s_1/5} ds_1 + \frac{\lambda^2}{10} e^{-4\lambda t/5} \int_0^t s_2^3 e^{3\lambda s_2/10} ds_2 \\
+ \frac{\lambda^2}{20} e^{-4\lambda t/5} \int_0^t s_3^2 e^{2\lambda s_3/20} ds_3 \\
+ \frac{\lambda^3}{15} e^{-4\lambda t/5} \int_0^t s_2^2 e^{\lambda s_2/15} ds_2 \\
+ \frac{\lambda^3}{30} e^{-4\lambda t/5} \int_0^t s_3 e^{\lambda s_3/30} ds_3 \\
+ \frac{\lambda^3}{60} e^{-4\lambda t/5} \int_0^t s_2 e^{\lambda s_2/60} ds_2 \\
+ \frac{\lambda^4}{120} e^{-4\lambda t/5} \int_0^t s_4 e^{\lambda s_4/120} ds_4 \\
= \frac{e^{-4\lambda t/5}}{\lambda^4} \left( 15000 + 1536 e^{4\lambda t/20}(\lambda t - 20) + 108 e^{2\lambda t/15}(180 + \lambda t(\lambda t - 24)) \\
+ 8 e^{3\lambda t/10}(\lambda t(144 + \lambda t(\lambda t - 18)) - 480) \\
+ e^{4\lambda t/5}(120 + \lambda t(\lambda t(36 + \lambda t(\lambda t - 8)) - 96)) \right) \\
$$

see Figure 4 below.
5. Moments of growth–collapse processes

The moments of the growth–collapse process $X_t$ defined as $X_t := f(t) - Y_t$ can be recovered from (18) with $f(s) = f(s)$, $l \geq 1$, and the binomial recursion

$$
\mathbb{E}[(X_t)^n] = \mathbb{E}[(f(t) - Y_t)^n] = (-1)^n \left( \mathbb{E}[(Y_t)^n] - \sum_{k=0}^{n-1} \binom{n}{k} (f(t))^{n-k} (-1)^k \mathbb{E}[(X_t)^k] \right),
$$

(24)

which is implemented in the Maple and Mathematica codes in the online appendix.

Before proving Proposition 5.1, we recover the first moments and cumulants of $\lambda$ parameter 2 and scaling parameter $\mu$ from (18) with $s$ see Theorems 4 and 5 of [3].

Proposition 5.1.

The moments of the growth–collapse process

$$
X_t = t - \sum_{k=1}^{N_t} T_k (1 - U_k) \prod_{l=k+1}^{N_t} U_l,
$$

$t \in \mathbb{R}_+$,

with uniform cut-off rates $(U_k)_{k \geq 1}$ on $[0, 1]$, are given by

$$
\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n} \sum_{k=0}^{n} (-1)^k (k+1)^{n-1} \binom{n}{k} e^{-\lambda t/(k+1)} ,
$$

$n \geq 0$, $t \in \mathbb{R}_+$.

As a consequence, the asymptotic moments of $(X_t)_{t \in \mathbb{R}_+}$ are given by

$$
\lim_{t \to \infty} \mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n} ,
$$

$n \geq 1$.

Before proving Proposition 5.1, we recover the first moments and cumulants of $X_t$ from the expressions (20)–(23) and the identity (24). For this we may use the Maple commands $\text{MX}(t, \lambda, \mu, f, n)$, $\text{CX}(t, \lambda, \mu, f, n)$ (resp. the Mathematica commands $\text{MX}[t, \lambda, \mu, f, n]$, $\text{CX}[t, \lambda, \mu, f, n]$) in the online appendix, with $f:= s \to s$ and $\text{mu}:= n \to 1/(n+1)$ (resp. $f[s_\_]:= s$ and $\text{mu}[n_\_]:= 1/(n+1)$).

First and second moments of $X_t$

We find

$$
\mathbb{E}[X_t] = -\mathbb{E}[Y_t] + t = \frac{\lambda}{2} e^{-\lambda t/2} \int_{0}^{t} s_1 e^{\lambda s_1/2} ds_1 = \frac{2}{\lambda} (e^{-\lambda t/2} + 1)
$$

(25)

and

$$
\mathbb{E}[(X_t)^2] = \mathbb{E}[(Y_t)^2] - t^2 + 2t\mathbb{E}[X_t] = \frac{3!}{\lambda^2} \left( 3e^{-2\lambda t/3} - 4e^{-\lambda t/2} + 1 \right);
$$

(26)

see Theorems 4 and 5 of [3].
Third and fourth moments of $X_t$ using $MX(t, \lambda, \mu, f, 3)$ and $MX(t, \lambda, \mu, f, 4)$. Next, from (22) we have
\[
\mathbb{E}(X_t^3) = -\mathbb{E}((Y_t)^3) + t^3 - 3t^2\mathbb{E}[X_t] + 3t\mathbb{E}[X_t^2]
= \frac{4!}{\lambda^3} \left( -16e^{-3\lambda t/4} + 27e^{-2\lambda t/3} - 12e^{-\lambda t/2} + 1 \right),
\] (27)
and the third cumulant of $Y_t$ is given by (8) as
\[
\kappa^{(3)}(t) = \frac{2!}{\lambda^3} \left( -27e^{-7\lambda t/6} + 96e^{-3\lambda t/4} - 135e^{-2\lambda t/3} + 4e^{-\lambda t/2} + 24e^{-\lambda t} + 39e^{-\lambda t/2} - 1 \right);
\] see Figure 3.

From (23) and (25)–(28) we have
\[
\mathbb{E}[(X_t)^4] = \mathbb{E}[(Y_t)^4] - t^4 + 4t^3\mathbb{E}[X_t] - 6t^2\mathbb{E}[(X_t)^2] + 4t\mathbb{E}[X_t^3]
= \frac{5!}{\lambda^4} \left( 125e^{-4\lambda t/5} - 256e^{-3\lambda t/4} + 162e^{-2\lambda t/3} - 32e^{-\lambda t/2} + 1 \right),
\] (29)
and the fourth cumulant of $Y_t$ is given by (8) as
\[
\kappa^{(4)}(t) = \frac{3!}{\lambda^4} \left( -8e^{-2\lambda t} + 504e^{-7\lambda t/6} + 1250e^{-4\lambda t/5} - 256e^{-5\lambda t/4} - 2304e^{-3\lambda t/4}
+ 72e^{-5\lambda t/3} - 81e^{-4\lambda t/3} + 1206e^{-2\lambda t/3} - 64e^{-3\lambda t/2} - 168e^{-\lambda t} - 152e^{-\lambda t/2} + 1 \right);
\] (30)
see Figure 4.

Proof of Proposition 5.1. Applying the Itô formula with jumps to (1), we have
\[
d(X^n_t) = nX^{n-1}_t dt + ((X_t)^n - (X_{t-})^n) dN_t
= nX^{n-1}_t dt + ((X_t - (1 - U_{N_t})X_t - dN_t)^n - (X_{t-})^n) dN_t
= nX^{n-1}_t dt + (X_{t-})^n \sum_{k=1}^{n} \binom{n}{k} (U_{N_t} - 1)^k dN_t
= nX^{n-1}_t dt + (X_{t-})^n((U_{N_t})^n - 1) dN_t,
\]
with $X_0 := 0$; hence
\[
X^n_t = n \int_0^t X^{n-1}_s ds + \int_0^t (X_{s-})^n((U_{N_s})^n - 1) dN_s, \quad t \in \mathbb{R}_+.
\]
Taking expectations on both sides of the above equality and using the smoothing lemma as in (3) yields
\[
\mathbb{E}[X^n_t] = n \int_0^t \mathbb{E}[X^{n-1}_s] ds + \lambda \int_0^t \mathbb{E}[(X_{s-})^n]\mathbb{E}[(U_{N_s})^n] - 1] ds
= n \int_0^t \mathbb{E}[X^{n-1}_s] ds - \frac{\lambda n}{n + 1} \int_0^t \mathbb{E}[(X_{s-})^n] ds, \quad t \in \mathbb{R}_+,
\]
which shows that the moments $\mathbb{E}[(X_t)^n]$ satisfy the differential equation

$$
\frac{d}{dt}\mathbb{E}[(X_t)^n] = n\mathbb{E}[(X_t)^{n-1}] - \frac{kn}{n+1}\mathbb{E}[(X_t)^n], \quad t \in \mathbb{R}_+;
$$

(31)

see also [3, Section 3] and [7, Section 3.5] for proofs using the infinitesimal generator of the Markov process $(X_t)_{t \in \mathbb{R}_+}$. Based on the intuition gained from (25)–(29), we now search for a solution of (31) of the form

$$
\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n} \sum_{k=0}^n a_{k,n} e^{-k\lambda t/(k+1)},
$$

which, by identification of terms, yields the recurrence relation

$$
a_{k,n} = \frac{n(k+1)}{n-k} a_{k,n-1}, \quad 0 \leq k < n,
$$

and hence

$$
a_{k,n} = (k+1)^{n-k} \binom{n}{k} a_{k,k}, \quad 0 \leq k \leq n.
$$

In addition, the initial condition $t = 0$ requires

$$
\sum_{k=0}^n a_{k,n} = 0,
$$

and hence

$$
\sum_{k=0}^n (k+1)^{n-k} \binom{n}{k} a_{k,k} = 0,
$$

which is solved by taking $a_{k,k} = (-1)^k(k+1)^{k-1}$, thanks to the combinatorial relation

$$
S(n, n+1) = \sum_{k=0}^n (-1)^{n-k}(k+1)^n \binom{n+1}{k+1} = (n+1) \sum_{k=0}^n (k+1)^{n-1} \binom{n}{k} (-1)^k = 0,
$$

which follows from the vanishing of the Stirling numbers of the second kind $S(n, n+1)$; see e.g. [1, p. 824].

\[\square\]

6. Embedded growth–collapse chain

In this section we show that Proposition 2.1 can also be used to compute the moments of all orders of the embedded chain

$$
Y(m) = Y_{T_m} = \sum_{k=1}^m g(T_k, k, m) = \int_0^\infty g(s, N_s, m) 1_{[0,T_m]}(s) dN_s, \quad m \geq 1.
$$

(32)

**Proposition 6.1.** Let $(Y(m))_{m \geq 1}$ be of the form (32). For any $n, m \geq 1$, we have

$$
\mathbb{E}[(Y(m))^n] = n! \sum_{k=1}^n \lambda^k \sum_{0=q_0 < q_1 < \cdots < q_{k-1} < q_k = n} \left[ \int_0^{s_1} \cdots \int_0^{s_k} \mathbb{E} \left[ 1_{[N_{s_k} \leq m-k]} \prod_{l=1}^k \frac{(g(s_l, l + N_{s_l}, m))^{q_l-q_{l-1}}}{(q_l-q_{l-1})!} ds_1 \cdots ds_k \right] \right].
$$

(33)
Proof. Taking \( u_s(\omega) := g(s, N_s, m)\mathbf{1}_{[0, T_m]}(s), 0 \leq s \leq t, \) by Proposition 2.1 and the identity \( \{s \leq T_m\} = \{N_s^- < m\}, s > 0, \) we have

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{m} g(T_k, k, m) \right)^n \right] = \mathbb{E} \left[ \left( \int_0^\infty g(s, N_s, m)\mathbf{1}_{[0, T_m]}(s)dN_s \right)^n \right]
\]

\[
= \sum_{k=1}^{n} k! \lambda^k \sum_{\pi_1 \cup \cdots \cup \pi_k = [1, \ldots, n]} \mathbb{E} \left[ \int_0^\infty \int_0^{s_1} \cdots \int_0^{s_k} \epsilon_{s_1}^+ \cdots \epsilon_{s_k}^+ \prod_{l=1}^{k} (g(s_l, N_{s_l}, m)\mathbf{1}_{[N_{s_l}^- < m]} \mid \pi_l) ds_1 \cdots ds_k \right].
\]

Next, since

\[
\{N_{s_1}^- < m\} \subset \{1 + N_{s_2}^- < m\} \subset \cdots \subset \{k - 1 + N_{s_k}^- < m\},
\]

by (10) we have

\[
\epsilon_{s_1}^+ \cdots \epsilon_{s_k}^+ \prod_{l=1}^{k} (g(s_l, N_{s_l}, m)\mathbf{1}_{[N_{s_l}^- < m]} \mid \pi_l) = \prod_{l=1}^{k} (g(s_l, l + N_{s_l}, m)\mathbf{1}_{[l + N_{s_l}^- < m]} \mid \pi_l)
\]

\[
= \mathbf{1}_{[N_{s_k}^- \leq m - k]} \prod_{l=1}^{k} (g(s_l, l + N_{s_l}, m) \mid \pi_l),
\]

which leads to (33). \( \square \)

Next, we specialize Proposition 6.1 to the case where \( g(s, k, m) \) takes the form

\[
g(s, k, m) = f_{k,m}(s)(1 - Z_k) \prod_{l=k+1}^{m} Z_l,
\]

where \( f_{k,m}(s) \) is a deterministic function and \( (Z_k)_{k \geq 1} \) is an i.i.d. random sequence independent of the Poisson process \( (N_t)_{t \in \mathbb{R}_+} \), with moment sequence \( \mu_n = \mathbb{E}[Z^n], n \geq 0 \). The corresponding embedded growth–collapse process can be written as

\[
X(m) = f_{m,m}(T_m) - Y(m) = \sum_{k=1}^{m} (f_{k,m}(T_k) - f_{k-1,m}(T_{k-1})) \prod_{l=k}^{m} Z_l,
\]

which corresponds to the wealth process in [10, Section 2] when \( f_{k,m}(s) = f_k(s) \) is independent of \( m \) and \( Y_k := f_k(T_k) - f_{k-1}(T_{k-1}) \) therein, \( k = 1, \ldots, m \), with \( f_{0,m}(s) := 0 \).

Corollary 6.1. Let \( (Y(m))_{m \geq 1} \) be defined as in (32) from (34), with \( f_{k,m}(s) = f_m(s) \) independent of \( k \geq 1 \). For any \( n, m \geq 1 \), we have

\[
\mathbb{E}[\{Y(m)\}^n] = n! \sum_{k=1}^{n} \sum_{i=0}^{m-k} \frac{\lambda^{k+i}}{i!} \mu_n^{m-i-k} \sum_{0=q_0 < q_1 < \cdots < q_{k-1} < q_k = n}
\]

\[
\prod_{l=1}^{k} C_{q_{l-1}, q_{l} - q_{l-1}} (q_{l} - q_{l-1})! \int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left( \sum_{l=0}^{k-1} \mu_{q_{l}}(s_{l+1} - s_{l}) \right) \prod_{l=1}^{k} f_{m}^{q_{l} - q_{l-1}}(s_l) ds_1 \cdots ds_k,
\]

where we let \( s_0 := 0, \sum_{i=1}^{0} = 0, \) and \( C_{p,q} \) is defined in (14).
Proof. By (6), (34), and Proposition 6.1 we have

\[
\mathbb{E}[Y^2(m)] = \sum_{k=1}^{n} k! \lambda^k \sum_{\pi_1 \cup \ldots \cup \pi_k = [1, \ldots, n]} \mathbb{E}
\left[
\int_0^\infty \int_0^{s_1} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_{m}(s_l)(1 - Z_{l+N_{s_l}}) \prod_{j=l+1+N_{s_l}}^{m} Z_j \right] ds_1 \cdots ds_k
\]

\[
= \sum_{k=1}^{n} k! \lambda^k \sum_{p_1 + \cdots + p_k = n} \frac{n!}{p_1! \cdots p_k!} \prod_{l=1}^{k} f_{m}(s_l)
\int_0^\infty \int_0^{s_1} \cdots \int_0^{s_2} \mathbb{E}
\left[
\prod_{l=1}^{k} (1 - Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1+N_{s_l}}^{m} Z_j
\right] ds_1 \cdots ds_k.
\]

Next, when \( N_{s_k} \leq m - k \) we have

\[
\prod_{l=1}^{k} (1 - Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1+N_{s_l}}^{m} Z_j
\]

\[
= \prod_{l=1}^{k} (1 - Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1}^{m} \prod_{j=l+1+N_{s_l}}^{m} Z_j^{p_l+\cdots+p_k}
\]

\[
= \prod_{l=1}^{k} (1 - Z_{l+N_{s_l}})^{p_l} \left(\prod_{j=l+1}^{m} Z_j^{p_l+\cdots+p_k} \right)
\]

which is a product of three independent random terms, whose expected value given \( N_{s_1}, \ldots, N_{s_k} \) is

\[
\mathbb{E}
\left[
\prod_{l=1}^{k} C_{p_1+\cdots+p_{l-1}, p_l}
\right]
\]

see (14). Therefore, using the fact that the jumps of \((N_{s})_{s \in [0, s_k]}\) are uniformly distributed on \( [0, s_k] \) given that \( N_{s_k} = i \), we have

\[
\mathbb{E}
\left[
\prod_{l=1}^{k} C_{p_1+\cdots+p_{l-1}, p_l}
\right]
\]

\[
= \mathbb{E}
\left[
\sum_{i=1}^{m-k} \mu_{p_1+\cdots+p_k} \prod_{l=1}^{k-1} \mu_{p_1+\cdots+p_l}
\right]
\]
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\[= \left( \prod_{l=1}^{k} C_{p_1+\ldots+p_{l-1},p_l} \right) \sum_{i=0}^{m-k} \mu_{p_1+\ldots+p_l}^{m-k-i} \mathbb{E} \left[ i^{\{N_{s_k} = i\}} \prod_{l=1}^{k-1} \mu_{p_1+\ldots+p_l}^{N_{s_{l+1}}-N_{s_l}} \right] \]

\[= \left( \prod_{l=1}^{k} C_{p_1+\ldots+p_{l-1},p_l} \right) \sum_{i=0}^{m-k} \mathbb{P}(N_{s_k} = i) \mu_n^{m-i-k} \left( \sum_{l=0}^{k-1} \mu_{p_1+\ldots+p_l}(s_{l+1} - s_l) \right)^i \]

since, given \(N_{s_k} = i\), the random vector \((N_{s_1}, N_{s_2} - N_{s_1}, \ldots, N_{s_k} - N_{s_{k-1}})\) is made up of independent binomial random variables with maximum count \(i\) and respective probabilities \(s_1/s_k, (s_2 - s_1)/s_k, \ldots, (s_k - s_{k-1})/s_k\). This leads to

\[
\mathbb{E}[(Y(m))^n] = n! \sum_{k=1}^{n} \sum_{p_1+\ldots+p_k=n} \mu_{p_1+\ldots+p_k}^{m-k-i} \prod_{l=1}^{k} C_{p_1+\ldots+p_{l-1},p_l} \]

\[
\times \int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_m^{p_l}(s_l) \left( \sum_{l=0}^{k-1} \mu_{p_1+\ldots+p_l}(s_{l+1} - s_l) \right)^i ds_1 \cdots ds_k.
\]

The result of Corollary 6.1 can be implemented for any choice of integrable function \(f(s)\) and moment sequence \(\mu_n = \mathbb{E}[Z^n], n \geq 0\), using the Maple and Mathematica codes in the online appendix.

**Uniform cut-off rates.** The next result specializes Corollary 6.1 to the case of embedded growth processes with uniform cut-off rates.

**Corollary 6.2.** Let \((Y(m))_{m \geq 1}\) be defined as in (34), with \(f_{k,m}(s) = f_m(s)\) independent of \(k \geq 1\), where \((Z_l)_{l \geq 1}\) an i.i.d. uniform random sequence on \([0, 1]\). For any \(n, m \geq 1\), we have

\[
\mathbb{E}[(Y(m))^n] = n! \sum_{k=1}^{n} \sum_{i=0}^{m-k} \frac{\lambda^{k+i}}{i!(n+1)^{m-i-k}} \sum_{0=q_0 < q_1 < \cdots < q_{k-1} < q_k = n} \left( \prod_{l=1}^{k} \frac{1}{1+q_l} \right) \int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_m^{p_l}(s_l) \left( \sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1+q_l} \right)^i ds_1 \cdots ds_k,
\]

where we let \(s_0 := 0\) and \(\sum_{l=1}^{0} = 0\).

**Proof.** We rewrite the result of Corollary 6.1 as

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{m} g(T_k, k, m) \right)^n \right] = n! \sum_{k=1}^{n} \sum_{p_1+\ldots+p_k=n} \mu_{p_1+\ldots+p_k}^{m-k-i} \prod_{l=1}^{k} \frac{\lambda^{k+i} \mu_{p_1+\ldots+p_l}^{m-i-k}}{i!} \prod_{l=1}^{k} \frac{(p_1 + \cdots + p_{l-1})!}{(p_1 + \cdots + p_l + 1)!} \]

\[
\times \int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_m^{p_l}(s_l) \left( \sum_{l=0}^{k-1} \mu_{p_1+\ldots+p_l}(s_{l+1} - s_l) \right)^i ds_1 \cdots ds_k
\]
and use the relations $\mu_n = 1/(n + 1)$, $n \geq 0$, and

$$\prod_{l=1}^{k} \frac{(p_1 + \cdots + p_{l-1})!}{(p_1 + \cdots + p_{l})!} = \frac{1}{n!} \prod_{l=1}^{k} \frac{1}{1 + q_l}$$

(see (17)), with $q_l := p_1 + \cdots + p_l$ and $q_k = n$, $1 \leq l \leq k \leq n$.

The first moments of $Y(m)$ can be computed in closed form when $f_m(s) = s$, which corresponds to

$$Y(m) = Y_{\mathcal{T}m} = \sum_{k=1}^{m} T_k (1 - U_k) \prod_{l=k+1}^{m} U_l, \quad m \geq 1,$$

where $(U_k)_{k \geq 1}$ is a uniform random sequence on $[0, 1]$. The next results can be recovered for any integer values of $m \geq 1$ by the Maple command $\text{MYm}(\lambda, m, \mu, f, n)$, (resp. the Mathematica command $\text{MYm}[\lambda, m, \mu, f, n]$) in the online appendix, by setting $f := s \rightarrow s$ and $\mu := n \rightarrow 1/(n + 1)$ (resp. $f[s_] := s$ and $\mu[n_] := 1/(n + 1)$).

First moment of $Y(m)$ using $\text{MYm}(\lambda, m, \mu, f, 1)$. For $n = 1$, Corollary 6.2 yields

$$\mathbb{E}[Y(m)] = \frac{1}{\lambda} \sum_{i=0}^{m-1} \frac{2^{i+1-m}}{i!} \int_0^\infty e^{-s_1} s_1^{i} \frac{s_1}{2} ds_1 = \frac{2^{-m} + m - 1}{\lambda};$$

hence

$$\mathbb{E}[X(m)] = \mathbb{E}[T_m] - \mathbb{E}[Y(m)] = \frac{1 - 2^{-m}}{\lambda}$$

as in Theorem 7 in [3].

Second moment of $Y(m)$ using $\text{MYm}(\lambda, m, \mu, f, 2)$. For $n = 2$, we find

$$\mathbb{E}[(Y(m))^2] = \frac{1}{\lambda^2} \sum_{i=0}^{m-1} \frac{1}{i! 3^{m-i-1}} \int_0^\infty e^{-s_1} s_1^{i} \frac{s_1^2}{3} ds_1$$

$$+ \frac{1}{2 \lambda^2} \sum_{i=0}^{m-2} \frac{1}{i! 3^{m-2-i}} \int_0^\infty e^{-s_1} \int_0^{s_1} \left( \frac{s_1 + s_2}{2} \right)^i s_1^{s_2} ds_1 ds_2$$

$$= \frac{1}{\lambda^2} \left( \frac{2}{3^m} + \frac{m - 1}{2^{m-1}} - m + m^2 \right).$$

Third moment of $Y(m)$ using $\text{MYm}(\lambda, m, \mu, f, 3)$. For $n = 3$, we have

$$\mathbb{E}[Y(m)^3] = \frac{1}{\lambda^3} \sum_{i=0}^{m-1} \frac{1}{i! 4^{m-i-1}} \int_0^\infty e^{-s_1} s_1^{i} \frac{s_1^{i+3}}{4} ds_1$$

$$+ \frac{1}{\lambda^3} \sum_{i=0}^{m-2} \frac{1}{i! 4^{m-2-i}} \int_0^\infty e^{-s_1} \int_0^{s_1} \left( \frac{s_1 + s_2}{2} \right)^i s_1^{s_2} ds_1 ds_2$$

$$+ \frac{1}{\lambda^3} \sum_{i=0}^{m-3} \frac{1}{i! 4^{m-3-i}} \int_0^\infty e^{-s_1} \int_0^{s_1} \int_0^{s_2} \left( \frac{3s_1 + s_2 + 2s_3}{6} \right)^i s_1^{s_2} s_3 ds_1 ds_2 ds_3.$$ (38)
Moments of Markovian growth–collapse processes

(a) (b)

FIGURE 5. Third cumulant and skewness of $Y(m)$.

(a) (b)

FIGURE 6. Fourth cumulant and excess kurtosis of $Y(m)$.

Although the last partial summation (38) does not have a closed-form expression, it can easily be estimated using the Maple and Mathematica codes in the online appendix; see Figure 5, together with Figure 6, where it is plotted with 10 million Monte Carlo samples and $\lambda = 2$.

Fourth moment of $Y(m)$ using $\text{MYm}(\lambda, m, \mu, f, 4)$. For $n = 4$, we find

$$
\mathbb{E}[Y(m)^4] = \frac{1}{\lambda^4} \sum_{i=0}^{m-1} \frac{1}{i!5^{m-1-i}} \int_0^\infty e^{-s_1} \frac{s_1^{i+4}}{5} ds_1 \\
+ \frac{1}{\lambda^4} \sum_{i=0}^{m-2} \frac{1}{i!5^{m-2-i}} \int_0^\infty e^{-s_2} \int_0^{s_2} \left( \left( \frac{s_1 + s_2}{2} \right)^i \frac{s_1^{i+1}}{2} \left( \frac{s_2}{4} \right)^i \frac{s_2^{i+1}}{4} + \left( \frac{2s_1 + s_2}{3} \right)^i \frac{s_1^{i+1}}{3} \left( \frac{s_2}{5} \right)^i \frac{s_2^{i+1}}{5} \right) ds_1 ds_2 \\
+ \frac{1}{\lambda^4} \sum_{i=0}^{m-3} \frac{1}{i!5^{m-3-i}} \int_0^\infty e^{-s_3} \int_0^{s_3} \int_0^{s_3} \left( \frac{s_1^{i+1}}{2} \left( \frac{s_2 + s_3}{5} \right)^i \frac{s_2^{i+1}}{5} \right) ds_1 ds_2 ds_3
$$
where \(Z_k\).

Corollary 6.3. \(g\) of the growth–collapse process (modifying the last term of order \(Cq_k\)) \(0. This process can be obtained from (32) by replacing \(E\) into (39), by changing the last term of order \(i = m - k\) or \(N_{sk} = m - k\) in (35), this amounts to modifying the last term of order \(i = m - k\) or \(N_{sk} = m - k\) in (35), by changing the last term \(C_{q_{k-1} q_k} C_{q_{k-1} q_{k-1}}\) of order \(l = k\) in the product \(\prod_{l=1}^{k} C_{q_{l-1} q_l} C_{q_{l-1} q_{l-1}}\) into \((-1)^{q_k - q_{k-1}} \mu_{q_k} = (-1)^{q_k - q_{k-1}} \mu_{q_k}\).

Corollary 6.3. Let \((X(m))_{m \geq 1}\) be defined as in (39). For any \(n, m \geq 1\), we have

\[
\mathbb{E}\left[(X(m))^n\right] = n!(-1)^n \sum_{k=1}^{n} \sum_{m-k-1}^{m-k-1} \frac{\lambda_{k+i}}{i!} \mu_{n-i-k} \sum_{0=q_0 < q_1 < \ldots < q_{k-1} < q_k = n}^{q_k = n} \int_{0}^{s_k} e^{-\lambda s_k} \prod_{l=0}^{k-1} \frac{\mu_{q_l}(s_{l+1} - s_l)}{(q_l - q_{l-1})!} ds_1 \ldots ds_k
\]

\[
+ \frac{\lambda^m}{(m-k)!} \sum_{k=1}^{\min(n,m)} \sum_{0=q_0 < q_1 < \ldots < q_{k-1} < q_k = n}^{q_k = n} (-1)^{q_k - q_{k-1}} \mu_{n} \int_{0}^{s_k} e^{-\lambda s_k} \prod_{l=0}^{k-1} \frac{\mu_{q_l}(s_{l+1} - s_l)}{(q_l - q_{l-1})!} ds_1 \ldots ds_k
\]

where we let \(s_0 := 0\) and \(\sum_{l=1}^{0} = 0.\)
When \( n = 1 \), Corollary 6.3 yields the first moment

\[
\mathbb{E}[X(m)] = -(1 - \mu_1) \sum_{i=0}^{m-2} \frac{\lambda^{1+i}}{i!} \mu_1^{m-i} \int_0^\infty e^{-\lambda s} f_m(s) ds + \frac{\mu_1 \lambda^m}{(m-1)!} \int_0^\infty e^{-\lambda s} f_m(s) s^{m-1} ds.
\]

Taking \( f_m(s) = s, s \in \mathbb{R}_+ \), this shows that

\[
\mathbb{E}[X(m)] = \frac{\mu_1}{\lambda} \left( \frac{1 - \mu_1^m}{1 - \mu_1} \right),
\]

which extends (37) above and [3, Equation (52)] to non-uniform cut-off rates. In the exponential case with \( \mu_1 \in (-1, 1) \) this recovers the long-range behavior of the first moment (see [10, p. 369], as well as [2, Section 4]) as \( m \) tends to infinity.

The moment \( \mathbb{E}[(X(m))^q] \) can be computed from Corollary 6.2 for specific integer values \( n, m \geq 1 \) and for any choice of function \( f(s) \) and moment sequence \( \mu_n = \mathbb{E}[Z^n], n \geq 0 \), using the Maple and Mathematica codes in the online appendix.

When \( (U_k)_{k \geq 1} \) is an i.i.d. uniform random sequence on \([0, 1]\), computing the moments of

\[
X(m) = T_m - \sum_{k=1}^{m} T_k (1 - U_k) \prod_{l=k+1}^{m} U_l
\]

according to Corollary 6.2 means multiplying the product \( \prod_{l=1}^{k} \frac{1}{1 + q_l} \) for \( i = m - k \) in (36) by

\[
\frac{(-1)^{q_k - q_{k-1}} m_{q_k}}{C_{q_k-1, q_{k-1}}} = \frac{(-1)^{q_k - q_{k-1}} (1 + q_k)!}{(1 + q_k)q_{k-1}!(q_k - q_{k-1})!} = (-1)^{q_k - q_{k-1}} \binom{n}{q_{k-1}},
\]

as done in the next result.

**Corollary 6.4.** Let \((X(m))_{m \geq 1}\) be defined as in (40), with \((U_k)_{k \geq 1}\) an i.i.d. uniform random sequence on \([0, 1]\). For any \( n, m \geq 1 \), we have

\[
\mathbb{E}[(X(m))^q] = (-1)^n \sum_{k=1}^{n} \sum_{i=0}^{m-k-1} \frac{\lambda^{k+i}}{i!(n+1)^{m-i-k}} \sum_{0=q_0 < q_1 < \cdots < q_k < q_k=n} \int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left( \sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1 + q_l} \right) \prod_{l=1}^{k} f_m^{q_l-1}(s_l) ds_1 \cdots ds_k
\]

\[
+ (-1)^n \sum_{k=1}^{m-n} \frac{\lambda^m}{(m-k)!} \sum_{0=q_0 < q_1 < \cdots < q_k < q_k=n} \int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left( \sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1 + q_l} \right) \prod_{l=1}^{k} f_m^{q_l-1}(s_l) ds_1 \cdots ds_k,
\]

where we let \( s_0 := 0 \) and \( \sum_{i=1}^{0} = 0 \).
The following moments and cumulants can be computed from Corollary 6.4 for any specific integer values of \( n, m \geq 1 \) using the Maple commands \( \text{MXm}(\lambda, m, \mu, f, n) \) (resp. the Mathematica commands \( \text{MXm}[\lambda, m, \mu, f, n] \), \( \text{CXm}[\lambda, m, \mu, f, n] \)) in the online appendix, with \( f := s \mapsto s \) and \( \mu := n \mapsto 1/(n+1) \) (resp. \( f[s_] := s \) and \( \mu[n_] := 1/(n+1) \)). When \( f_m(s) = s \), Corollary 6.4 shows that the second moment reads

\[
\mathbb{E}[X(m)^2] = \frac{1}{\lambda^2} \left( 2 - 4 \left( \frac{1}{2} \right)^m + 2 \left( \frac{1}{3} \right)^m \right),
\]

which yields

\[
\text{Var}[X(m)] = \frac{1}{\lambda^2} \left( 1 - 2 \left( \frac{1}{2} \right)^m + 2 \left( \frac{1}{3} \right)^m - \left( \frac{1}{4} \right)^m \right), \quad m \geq 0,
\]
from (37), and recovers Theorem 7 in [3]. Figures 7–8 are plotted with 100 million Monte Carlo samples and $\lambda = 2$.

In particular, the third and fourth cumulants of $X(m)$ can be obtained from Corollary 6.4 as shown in Figures 7–8, along with Monte Carlo simulations used for confirmation.

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**References**

[1] Abramowitz, M., and Stegun, I. (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Vol. 55, 9th edn. Dover, New York.

[2] Boxma, O., Kella, O. and Perry, D. (2011). On some tractable growth–collapse processes with renewal collapse epochs. J. Appl. Prob. 48A, 217–234.

[3] Boxma, O., Perry, D., Stadje, W. and Zacks, S. (2006). A Markovian growth–collapse model. J. Appl. Prob. 38, 221–243.

[4] Brémaud, P. (1999). Markov Chains. Springer, New York.

[5] Chiu, S., Stoyan, D., Kendall, W. and Mecke, J. (2013). Stochastic Geometry and Its Applications, 3rd edn. John Wiley, New York.

[6] Davis, M. (1984). Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models. J. R. Statist. Soc. B [Statist. Methodology] 46, 353–388.

[7] Daw, A. and Pender, J. (2020). Matrix calculations for moments of Markov processes. Preprint. Available at https://arxiv.org/abs/1909.03320.

[8] Eliazar, I. and Klafter, J. (2004). A growth–collapse model: Lévy inflow, geometric crashes, and generalized Ornstein–Uhlenbeck dynamics. Physica A 334, 1–21.

[9] Frolkova, M. and Mandjes, M. (2019). A Bitcoin-inspired infinite-server model with a random fluid limit. Stoch. Models 35, 1–32.

[10] Kella, O. (2009). On growth–collapse processes with stationary structure and their shot-noise counterparts. J. Appl. Prob. 46, 363–371.

[11] Leonov, V. and Shiryaev, A. (1959). On a method of calculation of semi-invariants. Theory Prob. Appl. 4, 319–329.

[12] Lukacs, E. (1955). Applications of Faà di Bruno’s formula in mathematical statistics. Amer. Math. Monthly 62, 340–348.

[13] Mecke, J. (1967). Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. Z. Wahrscheinlichkeitstheor. 9, 36–58.

[14] Privault, N. (2009). Moment identities for Poisson–Skorohod integrals and application to measure invariance. C. R. Acad. Sci. Paris 347, 1071–1074.

[15] Privault, N. (2012). Invariance of Poisson measures under random transformations. Ann. Inst. H. Poincaré Prob. Statist. 48, 947–972.

[16] Privault, N. (2012). Moments of Poisson stochastic integrals with random integrands. Prob. Math. Statist. 32, 227–239.

[17] Privault, N. (2016). Combinatorics of Poisson stochastic integrals with random integrands. In Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener–Itô Chaos Expansions and Stochastic Geometry, eds G. Peccati and M. Reitzner, Springer, Berlin, pp. 37–80.

[18] Schneider, R. and Weil, W. (2008). Stochastic and Integral Geometry. Springer, Berlin.

[19] Slivnyak, I. (1962). Some properties of stationary flows of homogeneous random events. Theory Prob. Appl. 7, 336–341.