QUASI-PERIODIC SOLUTIONS FOR PERTURBED GENERALIZED NONLINEAR VIBRATING STRING EQUATION WITH SINGULARITIES

CHENGMING CAO
School of Mathematical Sciences
Fudan University
Shanghai 200433, China

XIAOPING YUAN
School of Mathematical Sciences
Fudan University
Shanghai 200433, China

(Communicated by Sergei Kuksin)

Abstract. The existence of 2-dimensional KAM tori is proved for the perturbed generalized nonlinear vibrating string equation with singularities $u_{tt} = ((1 - x^2)u_x)_x - mu - u^3$ subject to certain boundary conditions by means of infinite-dimensional KAM theory with the help of partial Birkhoff normal form, the characterization of the singular function space and the estimate of the integrals related to Legendre basis.

1. Introduction. The KAM (Kolmogorov-Arnold-Moser) theory was used to find the quasi-periodic solutions for hamiltonian partial differential equations (PDEs), originally by Kuksin [7, 8, 9] and Wayne [15]. Among those PDEs, the nonlinear wave (NLW) equation

$$u_{tt} - u_{xx} + Vu + f(u) = 0, f(u) = \sum_{k \geq 3} f_k u^k$$  \hspace{1cm} (1)

has been investigated by many authors.

In KAM theory some parameters are needed to overcome resonances arising in the small divisors. Kuksin [7] assumed that the potential $V = V(x; \xi)$ depends on an $n$-dimensional parameter vector $\xi$ and showed that there are many quasi-periodic solutions for NLW for “most” parameters $\xi$’s. See also [2, 13, 4]. See Pöschel [13] for constant-value potential $V(x) \equiv m$ with $m > 0$ and $-1 < m < 0$ and [16] for $V(x) \equiv m \in (-\infty, -1) \setminus \mathbb{Z}$ and [18] for any prescribed nonconstant potential $V \in L^2[0, \pi]$. When $V(x) \equiv 0$ which is called completely resonant, Berti and Procesi [1] proved the existence of 2-dimensional tori and the existence of any dimensional KAM tori was proved in [17].
In the above papers, the potentials $V$ are regular. In physics and mechanics the potentials sometimes contain some kind of singularity. As an example, let us consider the Legendre potential,

$$V_L(x) = -\frac{1}{2} - \frac{1}{4} \tan^2 x, \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \quad (2)$$

Since

$$\lim_{x \to \pm \frac{\pi}{2}} V_L(x) = -\infty,$$

the endpoints $x = \pm \frac{\pi}{2}$ are actually singular.

It is well-known that the singular differential expression

$$\tilde{A} := -\frac{d^2}{dx^2} + V_L(x), \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad (3)$$

is in limit-circle case and is of deficiency index $(2, 2)$. The expression $\tilde{A}$ is a self-adjoint operator in the domain

$$D(\tilde{A}) = \left\{ u(x) \in L^2\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \mid u\left(\pm\frac{\pi}{2}\right) = 0 \right\}.$$  

Introducing the change of variable

$$\begin{cases} 
  y = \sin x, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\
  z = \frac{u}{\sqrt{\cos x}}, 
\end{cases}$$

the operator $\tilde{A}$ with its domain can be written as

$$\tilde{A} = -\frac{d}{dy}(1-y^2)\frac{d}{dy}, \quad y \in [-1, 1] \quad (4)$$

with

$$D(\tilde{A}) = \left\{ z(y) \in L^2[-1, 1] \mid \lim_{y \to \pm 1} z(y)\sqrt{\cos \arcsin y} = 0 \right\}$$

$$= \left\{ z(y) \in L^2[-1, 1] \mid \lim_{y \to \pm 1} z(y)(1-y^2)^{\frac{1}{4}} = 0 \right\}. \quad (5)$$

In convention, we still write $z(y) = u(x), y = x$. The operator $\tilde{A}$ has pure point spectrum $\sigma(\tilde{A}) = \sigma_p(\tilde{A})$. And the property

$$(\tilde{A}u, u) = \int_{-1}^{1} \frac{d}{dx}\left[ (1-x^2)\frac{d}{dx}u(x)\right]u(x)dx = \int_{-1}^{1} (1-x^2)\left| \frac{d}{dx}u(x) \right|^2 dx \geq 0$$

yields

$$\sigma(\tilde{A}) \subset [0, \infty).$$

To ensure the singular differential operator's strict positive definiteness, we use the notation

$$A = \tilde{A} + m \quad (m > 0). \quad (6)$$

Let $\lambda_j^2$ and $\phi_j(j = 1, 2, \ldots)$ be the eigenvalues and eigenfunctions of $A$, respectively. Here $\lambda_j > 0(j = 1, 2, \ldots)$.

Write

$$u(t, x) = \sum_{j \geq 1} \frac{q_j(t)}{\sqrt{\lambda_j}} \phi_j(x) \quad (7)$$
Inserting (7) into the following equation
\[
\begin{align*}
    u_{tt} - ((1 - x^2)u_x)_x + mu + u^3 &= 0, \quad x \in [-1, 1], \\
    \lim_{x \to \pm 1} u(x)(1 - x^2)^{\frac{3}{2}} &= 0
\end{align*}
\]
we have
\[
\ddot{q}_j + \lambda_j^2 q_j + \sqrt{\lambda_j}(u^3, \phi_j) = 0. \tag{8}
\]
This is a hamiltonian system
\[
\begin{align*}
    \dot{q}_j &= \frac{\partial H}{\partial p_j} = \lambda_j p_j \\
    \dot{p}_j &= -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}
\end{align*} \tag{9}
\]
where the hamiltonian $H$ is
\[
H = \Lambda + G = \frac{1}{2} \sum_{j \geq 1} \lambda_j (p_j^2 + q_j^2) + \frac{1}{4} \sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l, \tag{10}
\]
\[
G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx. \tag{11}
\]
Denoting the invariant $2 \times 2$-dimensional linear space by $E$
\[
E = \{(u, v) = (q_1 \phi_1 + q_2 \phi_2, p_1 \phi_1 + p_2 \phi_2) \} = \bigcup_I \mathcal{T}(I),
\]
where $\mathbb{P}^2 = \{I \in \mathbb{R}^2 : I_j > 0 \text{ for } j = 1, 2\}$ is the positive quadrant in $\mathbb{R}^2$,
\[
\mathcal{T}(I) = \{(u, v) : q_j^2 + p_j^2 = I_j \text{ for } j = 1, 2\},
\]
then our main theorem is as follows.

**Main Theorem.** Consider the nonlinear hamiltonian partial differential equation with boundary condition
\[
\begin{align*}
    u_{tt} - ((1 - x^2)u_x)_x + mu + u^3 &= 0, \quad x \in [-1, 1], \\
    \lim_{x \to \pm 1} u(x)(1 - x^2)^{\frac{3}{2}} &= 0
\end{align*} \tag{12}
\]
If $m \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{41}{4})$, then there is a set $\mathcal{C}$ in $\mathbb{P}^2$ with positive lebesgue measure, a family of 2-tori
\[
\mathcal{T}[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}(I) \subset E
\]
over $\mathcal{C}$, as well as a lipschitz continuous embedding into phase space $\mathcal{P}$
\[
\Phi : \mathcal{T}[\mathcal{C}] \hookrightarrow \mathcal{P},
\]
which is a higher order perturbation of the inclusion map $\Phi_0 : E \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}[\mathcal{C}]$, such that the restriction of $\Phi$ to each $\mathcal{T}(I)$ in the family is an embedding of a rotational invariant 2-torus for the nonlinear hamiltonian differential equation (12).

Here are some remarks. We compare our results with those of Pöschel. By and large, the basic idea is the same in reducing the hamiltonian defined by the partial differential equations to a partial Birkhoff normal form such that the KAM theorem (also see [7]) is applicable. However, there are several main differences because of the singularity of the differential operator $A$. In Pöschel
The differential operator $\hat{A} = -\frac{d^2}{dx^2} + m$ with Dirichlet boundary conditions has eigenvalues $\hat{\lambda}_j^2$ and eigenfunction $\hat{\phi}_j$:

$$\hat{\lambda}_j^2 = j^2 + m, \quad \hat{\phi}_j = \sqrt{\frac{2}{\pi}} \sin jx.$$  

In contrast, the singular differential operator $\hat{A}$ has, respectively, the eigenvalues and eigenfunctions

$$\hat{\lambda}_j^2 = 2j(2j - 1) + m, \quad \hat{\phi}_j = \sqrt{\frac{2}{\pi}} P_{2j-1}(x), \quad j = 1, 2, \ldots$$  

where $P_j(x)$ are Legendre polynomials.

On the one hand, under the basis $\{\hat{\phi}_j\}$ the Hamiltonian of $u^3$ can be written as

$$\hat{G}(q) = \sum_{ijkl} \hat{G}_{ijkl} q_i q_j q_k q_l$$

with

$$\hat{G}_{ijkl} = \frac{1}{\sqrt{\hat{\lambda}_i \hat{\lambda}_j \hat{\lambda}_k \hat{\lambda}_l}} \int_0^\pi \hat{\phi}_i \hat{\phi}_j \hat{\phi}_k \hat{\phi}_l dx$$

Since $\hat{\phi}_j$ is a very simple triangle function $\sqrt{\frac{2}{\pi}} \sin x$, it is easy to verify that

$$\hat{G}_{iijj} = \frac{1}{2\pi} \frac{2 + \delta_{ij}}{\hat{\lambda}_i \hat{\lambda}_j},$$

and to fulfill the relationship

$$\hat{G}_{ijkl} = 0 \quad \text{unless} \quad i \pm j \pm k \pm l = 0,$$  

where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. The relationship (15) leads immediately to that the Hamiltonian $\hat{G}(q)$ is the convolution of $q$ and $q$'s, that is,

$$\hat{G}(q) = q \ast q \ast q \ast q,$$

from which the regularity of the vector field $X_{\hat{G}}$ follows. At the same time, since the coefficients $\hat{G}_{iijj}$ can be explicitly calculated in (14), the resonant conditions in both Birkhoff normal form and the KAM theorem can be directly verified.

However, on the other hand, under the Legendre basis $\phi_j$'s, the Hamiltonian of $u^3$ can be written as

$$G(q) = \sum_{ijkl} G_{ijkl} q_i q_j q_k q_l$$

with

$$G_{ijkl} = \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_k \lambda_l}} \int_{-1}^1 \phi_i \phi_j \phi_k \phi_l dx$$

Both the equation (14) and the relationship (15) do not hold true any more in this case. Actually, the calculation of the integral $\int_{-1}^1 \phi_i \phi_j \phi_k \phi_l dx$ is not completely solved even in special function theory. Thus the fulfillment of the regularity of the vector field $X_G$ and those resonant conditions in both Birkhoff normal form and the KAM theorem are not easy. Section 2 will be devoted to verify the regularity of $X_G$. And the loss of (14) accounts for why we choose $m \in (0, \frac{1}{4}) \cup \left(\frac{1}{4}, \frac{1}{2}\right)$ and consider only 2 dimensional KAM tori.
2. Legendre polynomials and algebraic property. In the section, let us introduce some properties about Legendre polynomials $P_n(x)$ first. By using them, we can derive the estimate of $G_{ijkl}$ in next section.

For fixed $n$, the Legendre polynomial $P_n(x)$ is a $n$ order polynomial. It has an usual expression

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^nk!(n-k)!(n-2k)!} x^{n-2k},$$

as well as the Rodrigues’s formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n].$$

At the endpoint $x = \pm 1$, it satisfies

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n,$$

and it has a uniform upper bound

$$|P_n(x)| \leq 1.$$  

The recursion formula is important

$$(n + 1)P_{n+1} - (2n + 1)xP_n + nP_{n-1} = 0.$$  

A routine computation from (20) gives rise to

$$xP'_n - P'_{n-1} = nP_n,$$

and

$$(1 - x^2)P'_n = n(P_{n-1} - xP_n).$$

From the Rodrigues’s formula (17), we get

$$\int_{-1}^{1} x^k P_n(x) dx = \begin{cases} 0 & k < n \\ \frac{2^{n+1}(n!)^2}{(2n+1)!} & k = n \\ 0 & k > n, k - n \in 2\mathbb{Z} + 1 \\ \frac{k!\Gamma\left(\frac{k}{2} - \frac{n}{2} + \frac{1}{2}\right)}{2^k(k-n)!\Gamma\left(\frac{k}{2} + \frac{1}{2} + \frac{1}{2}\right)} & k > n, k - n \in 2\mathbb{Z} \end{cases}$$

A classical formula can express the product of two Legendre polynomials as a sum of such polynomials:

$$P_k(x)P_l(x) = \sum_{m=|k-l|}^{k+l} \frac{\mathbf{A}(s-k)\mathbf{A}(s-l)\mathbf{A}(s-m)2m+1}{\mathbf{A}(s)2s+1} P_m(x),$$

where

$$s = \frac{k + l + m}{2}$$

and

$$\mathbf{A}(n) = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} = \frac{(2n)!}{2^n(n!)^2} = \frac{1}{2^n} \binom{2n}{n}.$$
Theorem 2.1. If

\[
\left( \begin{array}{ccc} k & l & m \\ 0 & 0 & 0 \end{array} \right)^2 := \frac{A(s-k)A(s-l)A(s-m)}{(2s+1)A(s)}.
\]

Thus we could calculate the integral of three Legendre polynomials:

\[
\int_{-1}^{1} P_k(x)P_l(x)P_m(x) \, dx = 2\left( \begin{array}{ccc} k & l & m \\ 0 & 0 & 0 \end{array} \right)^2.
\] (26)

We remark that the result we get in this paper is an extension of the research in special function and refer to [3] for details.

Next, let us verify the algebraic property of the function space given below. Employing the result, we can get the regularity of vectorfield in next section.

Let

\[ \mathcal{D}(A^s) = \left\{ u \in L^2[-1, 1] \mid A^{s/2}u \in L^2[-1, 1], \lim_{x \to \pm 1} u(x)(1 - x^2)^{s/2} = 0 \right\}, \] (27)

and

\[ \ell^2_s = \left\{ u = \sum_{j \geq 1} u_j \phi_j \in L^2[-1, 1] \mid \sum_{j \geq 1} j^{2s}\|u_j\|^2 < \infty \right\}, \] (28)

where \(\|A^{s/2}u\|_{L^2[-1, 1]} = \langle A^{s/2}u, A^{s/2}u \rangle^{1/2} = \langle A^{s}u, u \rangle^{1/2},\) and \(\langle \cdot, \cdot \rangle\) denotes the usual scalar product in \(L^2[-1, 1].\)

The property

\[ \mathcal{D}(A^{s+1}) \subseteq \mathcal{D}(A^s), \] (29)

is also necessary.

If the norm of \(\ell^2_s\) is defined by \(\|u\|_s = (\sum_{j \geq 1} j^{2s}\|u_j\|^2)^{1/2},\) then the following norms are equivalent

\[ \|A^s u\|_{L^2[-1, 1]} \sim \|u\|_s. \] (30)

Remark 1. On one hand,

\[ \|A^s \sum_{j=1}^n u_j \lambda_j \phi_j \|_{L^2[-1, 1]} \leq \| \sum_{j=1}^n |u_j| j^{2s} \|_{L^2[-1, 1]} \]

\[ \lesssim \sum_{j=1}^n |u_j| j^{2s} \lesssim \|u\|_s^2 \quad n \to \infty. \]

On the other hand,

\[ \sum_{j=1}^n |u_j|^2 j^{2s} = \int_{-1}^{1} |\sum_{j=1}^n u_j j^s \phi_j|^2 \, dx \lesssim \sum_{j=1}^n j^{2s} |u_j|^2 \, dx \]

\[ \lesssim \int_{-1}^{1} |A^s \sum_{j=1}^n u_j \phi_j|^2 \, dx \lesssim \|A^{s} u\|_{L^2[-1, 1]}^2 \quad n \to \infty. \]

Our theorem is as follows

Theorem 2.1. If \(u(-x) = -u(x), \) \(u \in \mathcal{D}(A^s),\) then

\[ \|A^s u\|_{L^2[-1, 1]} \lesssim \|A^s u\|_{L^2[-1, 1]}^2. \] (31)

Proof. First, we claim that

\[ |\partial_x \phi_j(x)| \leq \sqrt{2j - 1} j^{2j - 1} \lesssim j^{s/2}. \] (32)

Let \(f(x) = j(1 - x^2) - \left[ P_{2j-2}(x) - x P_{2j-1}(x) \right],\) according to [21], we have

\[ f'(x) = 2j \left( P_{2j-1}(x) - x \right). \]
At the critical points $x_0$ such that $P_{2j-1}(x_0) = x_0$, and the endpoint $x = \pm 1$, we find
\[
f(x_0) = j - P_{2j-2}(x_0) - (j - 1)x_0^2 \geq 0
\]
and
\[
f(-1) = f(1) = 0.
\]
Here we use the property (18) and (19). This derives the relationship
\[
j(1 - x^2) \geq P_{2j-2}(x) - xP_{2j-1}(x).
\]
Using (18) and (21) again, we obtain
\[
\lim_{x \to -1} \frac{P_{2j-2}(x) - xP_{2j-1}(x)}{1 - x^2} = \lim_{x \to -1} \frac{P'_{2j-2}(x) - P_{2j-1}(x) - xP'_{2j-1}(x)}{-2x} = \lim_{x \to -1} \frac{-2jP_{2j-1}(x)}{-2x} = j.
\]
The same method gives
\[
\lim_{x \to 1} \frac{P_{2j-2}(x) - xP_{2j-1}(x)}{1 - x^2} = j.
\]
Combining (33), (34), (35), and the property (22), we see that
\[
|P'_{2j-1}(x)| \leq j(2j - 1).
\]
This leads to (32) because of $\phi_j = \sqrt{2j - 1}P_{2j-1}$.

Due to the fact that $u = \sum_{j \geq 1} \frac{q_j(t)}{\sqrt{\lambda_j}} \phi_j \in \mathcal{D}(A^2)$, we have $u \in L^2$. Using (32), we get
\[
\sum_{j \geq 1} \left| \frac{q_j(t)}{\sqrt{\lambda_j}} \partial_x \phi_j \right| \leq \sum_{j \geq 1} \left| \frac{q_j(t)}{\sqrt{\lambda_j}} \sqrt{2j - 1} \left( j - j(2j - 1) \right) \right|
\leq \left( \sum_{j \geq 1} \left| \frac{q_j(t)}{\sqrt{\lambda_j}} \right|^2 \right)^{1/2}
\leq \|u\|_4 \sim \|A^2u\|_{L^2[-1,1]}.
\]
Then the sum $\sum_{j \geq 1} \frac{q_j(t)}{\sqrt{\lambda_j}} \partial_x \phi_j(x)$ is absolutely convergent in $\mathbb{R} \times [-1,1]$, where $(t, x) \in \mathbb{R} \times [-1,1]$. It follows the estimate
\[
|\partial_x u| \lesssim \|A^2u\|_{L^2[-1,1]},
\]
and
\[
\lim_{x \to \pm 1} (1 - x^2) \partial_x u = 0. \quad (r > 0 \text{ in particular } r = 1 \text{ and } r = \frac{1}{2})
\]
Since $u(-x) = -u(x)$, we have
\[
\int_{-1}^{1} u(x) dx = 0, \quad \int_{-1}^{1} u^3(x) dx = 0.
\]
In view of (37), (38), (39) and Poincaré inequality, we get
\[
\|u\|_{L^2[-1,1]} \lesssim \|\partial_x u\|_{L^2[-1,1]} \lesssim \|A^2u\|_{L^2[-1,1]}.
\]
Using Theorem 7.2 (the Gagliardo-Nirenberg Inequality (130)) in Appendix A, for $f(x) = u(x)$,

$$k = 0, p = \infty, q = 2, r = 2, m = 1, a = \frac{1}{2},$$

we obtain

$$\|u\|_{L^\infty[-1,1]} \lesssim \|\partial_x u\|_{L^2[-1,1]}^{\frac{1}{2}} \|u\|_{L^2[-1,1]}^{\frac{1}{2}} \lesssim \|A^2 u\|_{L^2[-1,1]}.$$

(41)

Then we can conclude

$$\|\mathcal{A} u\|_{L^2[-1,1]} \lesssim \|A^2 u\|_{L^2[-1,1]}, \quad \|\mathcal{A}^2 u\|_{L^2[-1,1]} \lesssim \|A^2 u\|_{L^2[-1,1]}.$$

(42)

Noting (38) (39) (40) and using Poincaré inequality, we have the following inequality,

$$\|Au^3\|_{L^2[-1,1]} = \|\mathcal{A} u^3 + mu^3\|_{L^2[-1,1]} \leq \|\mathcal{A} u^3\|_{L^2[-1,1]} + m\|u^3\|_{L^2[-1,1]} \leq 3\|u^3\|_{H^1[-1,1]} \leq \|\mathcal{A} u\|_{L^2[-1,1]} + m\|u\|_{H^1[-1,1]} \leq \frac{1}{3} \|\mathcal{A} u\|_{L^2[-1,1]} + \|A u\|_{L^2[-1,1]} \lesssim \|A^2 u\|_{L^2[-1,1]}.$$

Here, $\|f\|_{H^1[-1,1]} = \left(\|f\|_{L^2[-1,1]} + \|\partial_x f\|_{L^2[-1,1]}\right)^{\frac{1}{2}}$.

In 1959, Nirenberg [12] observed a connection between $L^p$- norms and the Hölder seminorms $|\cdot|_\alpha$. Define

$$\{f\}_\alpha = \begin{cases} [f]_\alpha & \text{when } 0 < \alpha < 1, \\ |f|_{L^\alpha[-1,1]} & \text{when } \alpha = -\frac{1}{p} \leq 0. \end{cases}$$

(44)

By Theorem 7.1 (the General Nirenberg Inequality (128)) in Appendix A, for $f(x) = \phi_j(x)$,

$$j = 0 \quad \beta = \frac{495}{1364} \quad p_\beta = \infty$$
$$k = 1 \quad \alpha = -\frac{1}{20} \quad p_\alpha = 20$$
$$\theta = \frac{5885}{9889} \quad \gamma = -\frac{1}{2} \quad p_\gamma = 2$$

we have

$$\{\phi_j\}^{\frac{495}{1364}} \lesssim \{\partial_x \phi_j\}^{\frac{5885}{9889}} \{\phi_j\}^{\frac{4004}{9889}}.$$

(45)

Using (42), we obtain

$$\frac{495}{1364} \lesssim \frac{5885}{9889} \frac{4004}{9889} \frac{4004}{9889} \frac{4004}{9889} \frac{4004}{9889}.$$

(46)

It follows that
we can deduce that

\[ \mathcal{A} u \mid_{\mathcal{H}^n} = \left[ \mathcal{A} \sum_{j \geq 1} u_j \phi_j \right]_{\mathcal{H}^n} \lesssim \sum_{j \geq 1} |u_j| j^2 \phi_j_{\mathcal{H}^n} \]

and

\[ \lesssim \sum_{j \geq 1} \frac{1}{j^{\frac{4}{1364}}} \left( \sum_{j \geq 1} |u_j|^{2j^8} \right)^{\frac{1}{j}} \sim \| A^2 u \|_{L^2[-1,1]} \] (47)

With the help of Theorem 7.1 (the General Nirenberg Inequality \[128\]) in Appendix A again, for \( f(x) = (1 - x^2) \partial_x u \),

\[
\begin{align*}
  j &= 1, \quad \beta = -\frac{1}{4}, \quad p_\beta = 4 \\
  k &= 1, \quad \alpha = \frac{495}{1364}, \quad p_\alpha = \infty \\
  \theta &= \frac{31}{42}, \quad \gamma = -\frac{43}{44}, \quad p_\gamma = \frac{44}{43}
\end{align*}
\]

we have

\[ \{ \partial_x [(1 - x^2) \partial_x u] \}_{-\frac{1}{4}} \lesssim \{ \partial_x [(1 - x^2) \partial_x u] \}_{\mathcal{H}^\frac{31}{1364}} \{ (1 - x^2) \partial_x u \}_{\mathcal{H}^\frac{44}{43}}. \] (48)

From (38) (47), the Sobolev Imbedding theorem \( H^1[-1,1] \rightarrow L^{\frac{11}{4}} [-1,1] \) and Poincaré Inequality, it follows that

\[ \| \mathcal{A} u \|_{L^4[-1,1]} \lesssim \| \mathcal{A} u \|_{\mathcal{H}^\frac{31}{1364}} \| (1 - x^2) \partial_x u \|_{L^{\frac{44}{43}}[-1,1]} \lesssim \| A^2 u \|_{L^2[-1,1]} \| (1 - x^2) \partial_x u \|_{H^1[-1,1]} \] (49)

\[ \lesssim \| A^2 u \|_{L^2[-1,1]} \| \mathcal{A} u \|_{L^2[-1,1]} \lesssim \| A^2 u \|_{L^2[-1,1]} \]

Then, using (40) (41) and

\[ (1 - x^2) \partial_x^2 u = 2x \partial_x u - \mathcal{A} u \] (50)

we can deduce that

\[ \| A^2 u^3 \|_{L^2[-1,1]} = \| \mathcal{A} (Au^3) + mAu^3 \|_{L^2[-1,1]} \lesssim \| \mathcal{A}^2 u^3 \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \]

\[ \lesssim \| (1 - x^2) \partial_x^2 \left[ 3u^2 (1 - x^2) \partial_x u \right] \|_{H^1[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \]

\[ \lesssim \| -6 ((1 - x^2) \partial_x u) (1 - x^2) (\partial_x u)^2 \partial_x u - 12 x u (1 - x^2) (\partial_x u)^2 \]

\[ + 18 u (1 - x^2)(\partial_x u) \mathcal{A} u + 3u (1 - x^2) \partial_x (\mathcal{A} u) \mathcal{A} u \|_{H^1[-1,1]} \] (51)
By Poincaré inequality and (38), we obtain

\[
\| (1 - x^2) \partial_x u \|_{H^1[-1,1]} \lesssim \| (1 - x^2) (\partial_x u)^2 \|_{H^1[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \| (1 - x^2) (\partial_x u)^2 \|_{H^1[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \| u (1 - x^2) \partial_x (\mathcal{A} u) \|_{H^1[-1,1]} + \| A^2 u \|_{L^2[-1,1]}^3.
\]

Using Poincaré inequality and (38) again with the help of (37) and (50), we have

\[
\| (1 - x^2) (\partial_x u)^2 \|_{H^1[-1,1]} \lesssim \| \partial_x [((1 - x^2)(\partial_x u)) (\partial_x u)] \|_{L^2[-1,1]} + \| \partial_x u (1 - x^2) \partial_x^2 u \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \| \mathcal{A} u \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]}^2.
\]

From (29), we know for \( u \in \mathcal{D}(A^2), A u \in \mathcal{D}(A), \) then \( \lim_{x \to \pm 1} (1 - x^2) A u = 0. \) So we have

\[
\lim_{x \to \pm 1} (1 - x^2) \mathcal{A} u = 0
\]

Using (38), Cauchy Inequality and Poincaré Inequality, we get

\[
\left| \int_{-1}^{1} (1 - x^2) \partial_x (\mathcal{A} u) \, dx \right| = \left| \int_{-1}^{1} \partial_x [(1 - x^2) \mathcal{A} u] \, dx - \int_{-1}^{1} [\partial_x (1 - x^2)] \mathcal{A} u \, dx \right| = 2 \left| \int_{-1}^{1} x [2 x \partial_x u - (1 - x^2) \partial_x^2 u] \, dx \right| = 2 \left| \int_{-1}^{1} 2 x^2 \partial_x u \, dx - \int_{-1}^{1} x (1 - x^2) \partial_x^2 u \, dx \right| = 2 \left| \int_{-1}^{1} 2 x^2 \partial_x u \, dx - \int_{-1}^{1} \partial_x (x (1 - x^2) \partial_x u) \, dx + \int_{-1}^{1} \partial_x (x (1 - x^2)) \partial_x u \, dx \right| = 2 \left| \int_{-1}^{1} (1 - x^2) \partial_x u \, dx \right| \lesssim \| (1 - x^2) \partial_x u \|_{L^2[-1,1]} \lesssim \| \mathcal{A} u \|_{L^2[-1,1]}.
\]
Then it follows from (55)
\[
\| (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} \\
\leq \| \partial_x (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} + \left| \int_{-1}^1 (1 - x^2) \partial_x (\mathcal{A} u) \, dx \right| \\
\leq \| \mathcal{A}^2 u \|_{L^2[-1,1]} + \| \mathcal{A} u \|_{L^2[-1,1]} \\
\leq \| A^2 u \|_{L^2[-1,1]}.
\]
(56)
By Poincaré Inequality with the help of (37) (38) (49) and (56), we have
\[
\| (1 - x^2) (\partial_x u) \mathcal{A} u \|_{H^1[-1,1]} \\
\leq \| \partial_x (1 - x^2) (\partial_x u) \mathcal{A} u \|_{L^2[-1,1]} \\
\leq \| (\partial_x (1 - x^2) (\partial_x u) \mathcal{A} u) \|_{L^2[-1,1]} + \| (\partial_x u) (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} \\
\leq \| (\mathcal{A}^2 u) \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \| (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} \\
\leq \| \mathcal{A} u \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \\
\leq \| A^2 u \|_{L^2[-1,1]}.
\]
(57)
Then we obtain the following inequality by (37) (41) (56) and Poincaré Inequality
\[
\| u (1 - x^2) \partial_x (\mathcal{A} u) \|_{H^1[-1,1]} \\
\leq \| \partial_x [u (1 - x^2) \partial_x (\mathcal{A} u)] \|_{L^2[-1,1]} \\
\leq \| u \|_{L^2[-1,1]} \| (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} + \| (\partial_x u) (1 - x^2) \partial_x (\mathcal{A} u) \|_{L^2[-1,1]} \\
\leq \| \mathcal{A}^2 u \|_{L^2[-1,1]} + \| A^2 u \|_{L^2[-1,1]} \\
\leq \| A^2 u \|_{L^2[-1,1]}.
\]
(58)
In view of (52) (53) (57) (58), we proof the inequality (31).

3. The Hamiltonian.

3.1. The regularity of vectorfield. From introduction, we have already obtain the hamiltonian (10)
\[
H = \Lambda + G = \frac{1}{2} \sum_{j \geq 1} \lambda_j (p_j^2 + q_j^2) + \frac{1}{4} \sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l,
\]
with equations of motions (9)
\[
q'_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \quad p'_j = \frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j},
\]
in some neighbourhood of the origin in the Hilbert space $\ell^2_2 \times \ell^2_2$ with standard symplectic structure $\sum_j dq_j \wedge dp_j$. Then we have the following lemma

Lemma 3.1. For $u = (\sum_{j \geq 1} \sqrt{\lambda_j} \phi_j) \in \mathcal{D}(A^2)$ or $q = (q_j)_{j \geq 1} \in \ell^2_2$, the gradient
\[
G_q = (\frac{\partial G}{\partial q_j})_{j \geq 1}
\]
is real analytic as a map from some neighbourhood of the origin in $\ell^2_2$ into $\ell^2_2$, with
\[
\| G_q \|_2 = O(\| q \|^3_2).
\]
(59)
3.2. The Legendre sequences. It is necessary to make clear the coefficient $G_{ijkl}$ in hamiltonian $H$. In particular $G_{ijij}$. Then we acquire the property of Legendre sequences denoted by $\mathbf{P}(m, n) = \int_{-1}^{1} P_m P_n P_n dx, m, n \in \mathbb{N}$ below.

**Theorem 3.2. (Legendre sequences)** The Legendre sequences $\mathbf{P}(m, n)$ satisfy the following recursion formula

$$
\begin{align*}
\mathbf{P}(m + 1, n) &= \alpha_{n-1}^m \mathbf{P}(m, n - 1) + \alpha_n^m \mathbf{P}(m, n) + \alpha_{n+1}^m \mathbf{P}(m, n + 1) \\
&- \alpha_{m-1}^n \mathbf{P}(m - 1, n - 1) - \alpha_n^m \mathbf{P}(m - 1, n) - \alpha_{m+1}^n \mathbf{P}(m - 1, n + 1) \\
&+ \alpha_{m-2}^n \mathbf{P}(m - 2, n),
\end{align*}
$$

where

$$
\begin{align*}
\alpha_{n-1}^m &= \left[ \frac{(2m + 1)n}{(m + 1)(2n + 1)} \right]^2, \\
\alpha_n^m &= \frac{m(2m + 1)(n^2 + n - m^2 - m)}{(m + 1)^3(2m - 1)} \left[ \frac{(m - 1)m}{n^2 + n - m^2 + m} + \frac{2}{(2n + 1)^2} \right], \\
\alpha_{n+1}^m &= \left[ \frac{(2m + 1)(n + 1)}{(m + 1)(2n + 1)} \right]^2, \\
\alpha_{m-1}^n &= \frac{(m - 1)(2m - 1)(2m + 1)(n^2 + n - m^2 - m)n^2}{(m + 1)^3(n^2 + n - m^2 + m)(2n + 1)^2}, \\
\alpha_{m}^{n-1} &= \frac{2m(n^2 + n - m^2 - m)}{(m + 1)^3(2n + 1)^2} + \left( \frac{m}{m + 1} \right)^2, \\
\alpha_{m}^{n+1} &= \frac{(m - 1)(2m - 1)(2m + 1)(n^2 + n - m^2 - m)(n + 1)^2}{(m + 1)^3(n^2 + n - m^2 + m)(2n + 1)^2}, \\
\alpha_{m-2}^n &= \frac{(m - 1)^3(2m + 1)(n^2 + n - m^2 - m)}{(m + 1)^3(2m - 1)(n^2 + n - m^2 + m)},
\end{align*}
$$

satisfying

$$
\alpha_{n-1}^m + \alpha_n^m + \alpha_{n+1}^m - \alpha_{m-1}^n - \alpha_n^m - \alpha_{m+1}^n + \alpha_{m-2}^n = 1,
$$

and

$$
\begin{align*}
\frac{1}{\sqrt{\lambda_j}} \langle u^3, \phi_j \rangle, \\
\|G_q\|_2 &\lesssim \|\langle (u^3, \phi_j) \rangle_{j \geq 1} \|_4 \\
&\sim \|A^2 u^3\|_{L^2[-1,1]} \\
&\lesssim \|A^2 u\|_4^3 \|_{L^2[-1,1]} \\
&\lesssim \|\langle (u, \phi_j) \rangle_{j \geq 1} \|_4^3 \\
&\lesssim \|q\|_3^3.
\end{align*}
$$

Since $G$ is independent of $p$, the associated hamiltonian vectorfield,

$$
X_G = \sum_{j \geq 1} \left( \frac{\partial G}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial G}{\partial q_j} \frac{\partial}{\partial p_j} \right),
$$

is smoothing of order 1. By contrast, $X_A$ is unbounded of order 1. \qed

The Legendre sequences $(11)$ in hamiltonian $H$.
with
\[ P(0, n) = \frac{2}{2n + 1}, \]  
\[ P(1, n) = \frac{2(2n^2 + 2n - 1)}{(2n - 1)(2n + 1)(2n + 3)}, \]  
\[ P(2, n) = \frac{11n^4 + 22n^3 - 31n^2 - 42n + 18}{(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)}, \]  
\[ P(3, n) = \frac{34n^8 + 102n^7 - 305n^6 - 780n^5 + 703n^4 + 1110n^3 - 450}{(2n - 5)(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)(2n + 7)}. \]

Moreover, if \( \int_{-1}^{1} P_i(x)P_j(x)P_k(x)P_l(x)dx \neq 0 \), then we obtain the estimate of the following integral,
\[ \int_{-1}^{1} P_i(x)P_j(x)P_k(x)P_l(x)dx \lesssim \frac{1}{\sqrt{i + \frac{1}{2}}\sqrt{j + \frac{1}{2}}\sqrt{k + \frac{1}{2}}\sqrt{l + \frac{1}{2}}}, \]  
or in short
\[ P(m, n) \lesssim \frac{1}{mn}. \]

In particular, there exist an absolute constant \( C > 0 \) such that
\[ 0 < \int_{-1}^{1} \phi_i\phi_j\phi_k\phi_l dx \leq C \]

The proof is left in section 6.

Using the property of the Legendre polynomials [23], we can obtain the property about \( G_{ijkl} \) which we need in the next section.

**Lemma 3.3.** Assume \( 0 < i \leq j \leq k \leq l \) and \( i + j + k \geq l \), the coefficients \( G_{ijkl} = 0 \) unless \( i \pm j \pm k \pm l \in 2\mathbb{Z} \).

**Proof.** From the definition of \( G_{ijkl} \) [11] and \( \phi_i \) [13]-[16], we know the product of \( \phi_i\phi_j\phi_k \) is a polynomial like \( f(x) = a_1x^{i+j+k} + a_2x^{i+j+k-2} + \ldots \). Then due to the property [23], under the assumption \( 0 < i \leq j \leq k \leq l \) and \( i + j + k \geq l \), we can check
\[
G_{ijkl} = \frac{1}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \int_{-1}^{1} f(x)\phi_l dx
\]
\[ = \frac{a_1}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \int_{-1}^{1} x^{i+j+k} P_l dx + \frac{a_2}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \int_{-1}^{1} x^{i+j+k-2} P_l dx + \ldots
\]
\[ = 0
\]
unless
\[ i \pm j \pm k \pm l \in 2\mathbb{Z}. \]

\[ \blacksquare \]
4. Partial Birkhoff normal form. Next we introduce complex coordinates
\[
    z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - ip_j).
\]
Then we obtain a real analytic hamiltonian \( H = \sum_j \lambda_j |z_j|^2 + \ldots \) on the complex Hilbert space \( \ell^2 \) with symplectic structure \( i \sum_j d\bar{z}_j \wedge dz_j \).

In the following, \( A(\ell^2, \ell^2 + 1) \) denotes the class of all real analytic maps from some neighbourhood of the origin in \( \ell^2 \) into \( \ell^2 + 1 \). Thus we can also obtain the main proposition like Pöschel [13] but the handling of the small denominator is more complex.

**Proposition 1.** For the 2-dimension of the invariant tori and \( m \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{4}{7}) \), there exists a real analytic, symplectic change of coordinates \( \Gamma \) in some neighbourhood of the origin in \( \ell^2 \) that takes the hamiltonian \( H = \Lambda + G \) with nonlinearity into
\[
    H \circ \Gamma = \Lambda + \bar{G} + \bar{\tilde{G}} + K,
\]
where \( X_G, X_{\bar{G}}, X_K \in A(\ell^2, \ell^2 + 2) \),
\[
    \bar{G} = \frac{1}{2} \sum_{\min(i,j) \leq 2} \overline{G}_{ij} |z_i|^2 |z_j|^2
\]
with uniquely determined coefficient, and
\[
    |\bar{G}| = O(\|\bar{z}\|^4), \quad |K| = O(\|\bar{z}\|^6), \quad \bar{z} = (z_3, z_4, \ldots).
\]
Moreover, the neighbourhood can be chosen uniformly for every compact \( m \)–interval in \( (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{4}{7}) \), and the dependence of \( \Gamma \) on \( m \) is real analytic.

**Remark 2.** Here \( m = \frac{1}{4} \) and \( m = \frac{41}{7} \) play an important role in controlling the small divisor. For \( m = \frac{1}{4} \), the resonance occurs when we check if \( \lambda_i + \lambda_j + \lambda_k + \lambda_l = 0 \). For \( m = \frac{41}{7} \), one can find the answer in (79) of Lemma 4.3.

Thus, the hamiltonian \( \Lambda + \bar{G} \) is integrable with integrals \( |z_j|^2, j = 1, 2 \), while the not-normalized fourth order term \( \bar{\tilde{G}} \) is not integrable, but independent of the first 2 modes.

**Proof of property.** Let us introduce another set of coordinates \((\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots)\) in \( \ell^2 \) by setting \( z_j = w_j, \bar{z}_j = w_{-j} \). The hamiltonian under consideration then reads
\[
    H = \Lambda + G
    = \sum_{j \geq 1} \lambda_j z_j \bar{z}_j + \frac{1}{4} \sum_{i,j,k,l} G_{ijkl}(z_i + \bar{z}_i) \ldots (z_l + \bar{z}_l)
    = \sum_{j \geq 1} \lambda_j w_j w_{-j} + \sum_{i,j,k,l} G_{ijkl} w_i w_j w_k w_l.
\]
The prime indicates that the subscripted indices run through all nonzero integers. The coefficients are defined for arbitrary integers by setting \( G_{ijkl} = G_{i|j|k|l|} \).

Formally, the transformation \( \Gamma \) is obtained as the time-1-map of the flow of a hamiltonian vectorfield \( X_F \) given by a hamiltonian
\[
    F = \sum_{i,j,k,l} \prime F_{ijkl} w_i w_j w_k w_l,
\]
with coefficients
\[ iF_{ijkl} = \begin{cases} G_{ijkl} & \text{for } (i, j, k, l) \in \mathcal{L} \setminus \mathcal{N}, \\ \lambda_i' + \lambda_j' + \lambda_k' + \lambda_l' & \text{otherwise.} \end{cases} \] (73)

Here, \( \lambda_i' = \text{sgn} j \cdot \lambda_{ij} \),
\[ \mathcal{L} = \{(i, j, k, l) \in \mathbb{Z}^4 : 0 \neq \min(|i|, \ldots, |l|) \leq 2 \}, \]
and \( \mathcal{N} \subset \mathcal{L} \) is the subset of all \( (i, j, k, l) \equiv (p, -p, q, -q) \). That is, they are of the form \( (p, -p, q, -q) \) or some permutation of it.

Next, we will estimate the denominator \( \lambda_i' + \lambda_j' + \lambda_k' + \lambda_l' \) to ensure the correction of the definition of (73), the proof of the lemma is left at the end of this section.

**Lemma 4.1.** If \( i, j, k, l \) are non-zero integers, such that \( i \pm j \pm k \pm l \in 2\mathbb{Z} \), but \( (i, j, k, l) \neq (p, -p, q, -q) \), then
\[ |\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l'| \geq \sigma(m, n) = \sigma > 0, \quad n = \min(|i|, \ldots, |l|), \]
Hence, the denominators in (73) are uniformly bounded away from zero on every compact \( m \)–interval in \((0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2}) \). The notation \( \sigma(m, n) \) is defined as
\[ \sigma(m, n) = \min \{W(m, n), V(m, n)\}, \]
where
\[ W(m, n) = \min \begin{cases} \frac{(1 - 4m)(n + m)}{2(\sqrt{n(n + 1)} + m + n)(\sqrt{n(n + 1)} + m + n + 2m)} & m \in (0, \frac{1}{4}) \\ \frac{2 - 4m - 1}{4n(m + 1) + m + 4n + 2} & m \in (\frac{1}{4}, \frac{41}{4}) \end{cases} \]
and
\[ V(m, n) = \min_{m \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})} \left\{ \begin{array}{c} m \frac{m}{\sqrt{n(n + 1) + m}}, \quad \frac{n}{\sqrt{m + 2}}, \quad \frac{4m - 1}{4(n(n + 1) + m)^2} \end{array} \right\}. \]

We continue the proof of the property. Expanding at \( t = 0 \) and using Taylor’s formula we formally obtain
\[ H \circ \Gamma = H \circ X_P'|t=1 = H + \{H, F\} + \int_0^1 (1 - t) \{\{H, F\}, F\} \circ X_P' dt \]
\[ = \Lambda + G + \{\Lambda, F\} + \{G, F\} + \int_0^1 (1 - t) \{\{H, F\}, F\} \circ X_P' dt, \]
where \( \{H, F\} \) denotes the Poisson bracket of \( H \) and \( F \). The last line consists of terms of order six or more in \( w \) and constitutes the higher order term \( K \). In the second to last line,
\[ \{\Lambda, F\} = -i \sum_{i, j, k, l} (\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l') F_{ijkl} w_i w_j w_k w_l, \]
hence
\[ G + \{\Lambda, F\} = \sum_{(i, j, k, l) \in \mathcal{N}} G_{ijkl} w_i w_j w_k w_l = \tilde{G} + \hat{G}. \]
Re-introducing the notations \( z_j, \bar{z}_j \) and counting multiplicities, we obtain that
\[
\bar{G} = \frac{1}{2} \sum_{\min (i,j) \leq 2} \bar{G}_{ij} |z_i|^2 |z_j|^2,
\]
with
\[
\bar{G}_{ij} = \begin{cases} 
\frac{2(4i-1)(4j-1)}{\lambda_i \lambda_j} P(2i-1, 2j-1) & i \neq j, \\
\frac{(4i-1)(4j-1)}{\lambda_i \lambda_j} P(2i-1, 2j-1) & i = j.
\end{cases}
\]
(74)

Thus, we have \( H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K \) as claimed formally.

To prove analyticity and regularity of the preceding transformation, we first show that
\( X_F \in A(\ell^2_2, \ell^2_2) \).

Assume a “threshold function”
\[
\tilde{F} = \int_{-1}^{1} v^4 dx,
\]
where
\[
v = \frac{1}{\sigma^2} \sum_j \tilde{w}_j \phi_j \quad \text{and} \quad \tilde{w}_j = \frac{|w_j| + |w_{-j}|}{\sqrt{|j|}}.
\]

The notation \( \sigma \) comes from the estimate of denominator \( |\lambda'_i + \lambda'_j + \lambda'_k + \lambda'_{-l}| \gtrsim \sigma \).

It is easy to check that, by (71), the integral of \( \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \) is uniformly bounded.

\[
\left| \frac{\partial F}{\partial w_l} \right| \leq \sum_{i,j,k} \left| F_{ijkl} \right| |w_i w_j w_k| \\
\lesssim \frac{1}{\sigma |l|} \sum_{i,j,k} \left| \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \right| \frac{|w_i w_j w_k|}{\sqrt{|ijk|}} \quad (77)
\]

while \( \tilde{F} = \frac{1}{2} \sum_{i,j,k,l} \tilde{w}_i \tilde{w}_j \tilde{w}_k \tilde{w}_l \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \).

Then it follows that
\[
\left| \frac{\partial \tilde{F}}{\partial w_l} \right| = \left| \frac{\partial \tilde{F}}{\partial \tilde{w}_l} \cdot \frac{\partial \tilde{w}_l}{\partial w_l} \right| \\
= \frac{1}{\sigma} \sum_{i,j,k} \tilde{w}_i \tilde{w}_j \tilde{w}_k \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \cdot \frac{1}{\sqrt{|l|}} \left| \tilde{w}_l \right| \\
\gtrsim \frac{1}{\sigma |l|} \sum_{i,j,k} \left| \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \right| \frac{(|w_i| + |w_{-i}|)(|w_j| + |w_{-j}|)(|w_k| + |w_{-k}|)}{\sqrt{|ijk|}} \\
\gtrsim \frac{1}{\sigma |l|} \sum_{i,j,k} \left| \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l dx \right| \frac{|w_i w_j w_k|}{\sqrt{|ijk|}}.
\]
Hence, the second inequality of \(\partial F/\partial w_l\) implies that
\[
\left| \frac{\partial F}{\partial w_l} \right| \lesssim \left| \frac{\partial \tilde{F}}{\partial w_l} \right|,
\]
which means
\[
\| F_w \|_2 \lesssim \| \tilde{F}_w \|_2.
\]
On the other hand,
\[
\left| \frac{\partial \tilde{F}}{\partial w_l} \right| = \left| \int_{-1}^{1} 4v^3 \frac{1}{\sigma^l} \phi_l \cdot \frac{1}{\sqrt{|l|}} \bar{w}_l \, dx \right| \lesssim \frac{1}{\sqrt{|l|}} |(v^3, \phi_l)|,
\]
hence,
\[
\| \tilde{F}_w \|_2 \leq \left( \sum_l l^4 \left\| \frac{1}{\sqrt{|l|}} \langle v^3, \phi_l \rangle \right\|^2 \right)^{1/2} \leq \left( \sum_l l^4 \langle v^3, \phi_l \rangle^2 \right)^{1/2} = \| v^3 \|_4 \sim \| A^2 v^3 \|_{L^2[-1,1]} \leq \| A^2 v \|_{L^2[-1,1]} \leq \| v \|_4^2.
\]
In the end, we obtain
\[
\| F_w \|_2 \lesssim \| v \|_2^3.
\]
The analyticity of \( F_w \) follows from the analyticity of each component function and its local boundedness.

In a sufficiently small neighbourhood of the origin in \(\ell^2_\theta\), the time-1-map \(X^1_H\) is well defined and gives rise to a real analytic symplectic change of coordinates \(\Gamma\) with the estimates
\[
\| \Gamma - id \|_2 = O(\| w \|_{\frac{3}{2}}^3), \quad \| D\Gamma - I \|_{\frac{3}{2}, \frac{3}{2}}^{\text{op}} = O(\| w \|_{\frac{3}{2}}^2),
\]
where the operator norm \(\| \cdot \|_{r,s}^{\text{op}}\) is defined by
\[
\| A \|_{r,s}^{\text{op}} = \sup_{w \neq 0} \frac{\| Aw \|_r}{\| w \|_s}.
\]
Obviously, \(\| D\Gamma - I \|_{\frac{3}{2}, \frac{3}{2}}^{\text{op}} \leq \| D\Gamma - I \|_{\frac{3}{2}, \frac{3}{2}}^{\text{op}}\), whence in a sufficiently small neighbourhood of the origin, \(D\Gamma\) defines an isomorphism of \(\ell^2_{s+1}\). It follows that with \(X_H \in A(\ell^2_2, \ell^2_{\theta})\),
\[
\Gamma^* X_H = D\Gamma^{-1} X_H \circ \Gamma = X_{H \circ \Gamma} \in A(\ell^2_2, \ell^2_{\theta}).
\]
The same holds for the Lie bracket: the boundedness of $\|DX_F\|_{\ell_2}^{op}$ implies that

$$[X_F, X_H] = X_{H,F} \in A(\ell_2^2, \ell_2^2).$$

These two facts show that $X_K \in A(\ell_2^2, \ell_2^2)$. The analogue claims for $X_{\delta}$ and $X_{\lambda}$ are obvious. 

**Proof of Lemma 4.2.** In fact, we want to prove there exists the lower bound of $X_i', X_j', X_k', X_l'$, it does not matter to use the renumbered notation $\lambda^2_j = j(j+1) + m$ instead of $\lambda^2_j = 2j(2j-1) + m$. This also makes it easier to use Lemma 4 in Pöschel [13] in our proof.

**Case 1.** Assume $0 < m < \frac{1}{4}$, using a convenient mark $\sigma_h = \text{sgn} h$, we can write

$$\delta = \lambda_i' + \lambda_j' + \lambda_k' + \lambda_l' = \sigma_i \lambda_i + \sigma_j \lambda_j + \sigma_k \lambda_k + \sigma_l \lambda_l.$$

The skill in [3] will be used. If $\sigma_i |i| + \sigma_j |j| + \sigma_k |k| + \sigma_l |l| \neq 0$, then $|\sigma_i |i| + \sigma_j |j| + \sigma_k |k| + \sigma_l |l| | \geq 2$. Since

$$\lambda_{|i|} = |h| + \sqrt{|h|(|h|+1) + m} - |h|$$

and $f(x) = \frac{x + m}{\sqrt{x(1+x) + m + x}} < \frac{1}{2}$, $(x \geq 0, 0 < m < \frac{1}{4})$, we have

$$|\delta| = |\lambda_i' + \lambda_j' + \lambda_k' + \lambda_l'|$$

$$= |\sigma_i \lambda_i + \sigma_j \lambda_j + \sigma_k \lambda_k + \sigma_l \lambda_l|$$

$$\geq |\sigma_i |i| + \sigma_j |j| + \sigma_k |k| + \sigma_l |l|$$

$$- \left( \frac{|i| + m}{\sqrt{|i|(|i| + 1) + m} + |i|} + \frac{|j| + m}{\sqrt{|j|(|j| + 1) + m} + |j|} \right.$$

$$\left. + \frac{|k| + m}{\sqrt{|k|(|k| + 1) + m} + |k|} + \frac{|l| + m}{\sqrt{|l|(|l| + 1) + m} + |l|} \right)$$

$$\geq \frac{1}{2} - \frac{|i| + m}{\sqrt{|i|(|i| + 1) + m} + |i|}$$

$$= \frac{1}{2(\sqrt{|i|(|i| + 1) + |i|}(|i|(|i| + 1) + m + |i| + 2m)).}$$

If $\sigma_i |i| + \sigma_j |j| + \sigma_k |k| + \sigma_l |l| = 0$, then using Lemma 4 of Pöschel’s article, we get

$$|\delta| \geq \min_{m \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \infty)} \left\{ \frac{m}{\sqrt{n(n+1) + m}}, \frac{n}{\sqrt{m+2}}, \frac{4m - 1}{4(n(n+1) + m)^2} \right\}.$$

**Case 2.** Assume $\frac{1}{4} < m < \frac{41}{4}$, consider the following two cases

$$\begin{cases} i \pm j \pm k \pm l = 2\alpha, \quad \alpha \in \mathbb{Z}_+, \\ i \pm j \pm k \pm l = 2\beta, \quad \beta \in \mathbb{Z}_-, \end{cases}$$

where $0 < i \leq j \leq k \leq l$. The case “$i \pm j \pm k \pm l = 0$ ” is discussed in Lemma 4 of Pöschel’s article.

Before proving Lemma 4.2 in Case 2., we have to show another lemma and state a basic property of Legendre polynomials.
Lemma 4.2. Assume $\frac{1}{4} < m < \frac{41}{4}$, $j - i > l - k$, $0 < i \leq j \leq k \leq l$, and $i \pm j \pm k \pm l$ is even, if $\delta = (\lambda_j - \lambda_i) - (\lambda_l - \lambda_k)$, then $|\delta| \geq 2 - \frac{4m-1}{4\sqrt{(i+1)^2+m+4i+2}} > 0$.

Proof. It is easy to get $j - i \geq l - k + 2$, hence we have

$|\delta| = |(\lambda_j - \lambda_i) - (\lambda_l - \lambda_k)|$

$= |[(\lambda_j - \lambda_i) - (l - k)] - [(\lambda_l - \lambda_k) - (l - k)]|$

$\geq |(\lambda_j - \lambda_i) - (l - k)| - |(\lambda_l - \lambda_k) - (l - k)|$

$\geq |\lambda_j - \lambda_i| - |l - k| + |(l - k) - (\lambda_l - \lambda_k)|$

$\geq 2 - [(j - i) - (\lambda_j - \lambda_i)] + [(l - k) - (\lambda_l - \lambda_k)]$

$\geq 2 - [(j - i) - (\lambda_j - \lambda_i)].$

It is clear that

$$\lambda_j - \lambda_i = \sqrt{j(j + 1) + m - \sqrt{i(i + 1) + m}}$$

$$= \frac{(j + i + 1)(j - i)}{\sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}} < j - i,$$

when $\frac{1}{4} < m < \frac{41}{4}$. Then

$$(j - i) - (\lambda_j - \lambda_i)$$

$$= \left(1 - \frac{(j + i + 1)}{\sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}}\right)(j - i)$$

$$= \frac{\sqrt{j(j + 1) + m} - (j + \frac{1}{2})}{\sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}} \cdot (j - i)$$

$$+ \frac{\sqrt{i(i + 1) + m} - (i + \frac{1}{2})}{\sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}} \cdot (j - i)$$

$$= \frac{(m - \frac{1}{2})(j - i)}{\sqrt{j(j + 1) + m + (j + \frac{1}{2})} \sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}}$$

$$+ \frac{(m - \frac{1}{2})(j - i)}{\sqrt{i(i + 1) + m + (i + \frac{1}{2})} \sqrt{j(j + 1) + m + \sqrt{i(i + 1) + m}}}.$$ (78)

If $j - i = h \geq 0$, then the function with respect to $h$ satisfies the following property,

$$f(h) = (j - i) - (\lambda_j - \lambda_i)$$

$$= h - \left[\sqrt{(i + h)(i + h + 1) + m} - \sqrt{(i)(i + 1) + m}\right]$$

is monotone increasing, since

$$f'(h) = 1 - \frac{2h + 2i + 1}{2\sqrt{(i + h)(i + h + 1) + m}} > 0,$$

when $\frac{1}{4} < m < \frac{41}{4}$. 
Hence, we completes Lemma 4.3. by letting $h$ goes to infinity,

$$|\delta| \geq 2 - f(h) \geq 2 - \lim_{h \to \infty} f(h) = 2 - \frac{m - \frac{1}{4}}{\sqrt{i(i + 1) + m + i + \frac{1}{2}}} > 0.$$ \hspace{1cm} (79)

In fact, $\frac{1}{4} < m < \frac{41}{4}$ comes from the last inequality of (79) if $i = 1$. \hfill \Box

In order to prove Case 2., we need to divide it into the following 9 subcases:

**Subcase 2.1.** $i + j + k + l = 2\alpha$. Since $i + j + k + l \geq i + 3j$, we have $\alpha > j$. Then, we get $i - j + k + l = 2(\alpha - j)$, which convert to Subcase 2.4 below.

**Subcase 2.2.** $i + j + k + l = 2\alpha$. If $1 \leq \alpha < i$, then we have $l - k = (i - \alpha) + (j - \alpha) > (j - \alpha) - (i - \alpha) = j - i$. Using the idea of Lemma 4 in Pöschel [13], one can obtain

$$|\delta| \geq 2(i - \alpha)f'(j) \geq \frac{2j + 1}{\sqrt{j(j + 1) + m}}.$$ 

On the other hand, it is easy to obtain $\alpha \leq \frac{i + j}{2}$. If $\alpha = i$ or $\alpha = \frac{i + j}{2}$, then it converts to Pöschel’s case. So it suffices to consider the case $i < \alpha < \frac{i + j}{2}$, which means $l - k < j - i$. This can be solved by using Lemma 4.3.

**Subcase 2.3.** $i + j - k + l = 2\alpha$. Using the basic assumption, we can get $l - k \leq \alpha \leq i + j$. The case $\alpha = l - k$ or $\alpha = i + j$ can be solved by using Lemma 4 in Pöschel [13]. If $\alpha > j$, then $i - j - k + l = 2(\alpha - j)$, which converts to Subcase 2.5 below. So it suffices to consider the case $l - k < \alpha \leq j$, which means $l - k \leq j - i$. Use Lemma 4 in Pöschel [13] when the equality holds, while use Lemma 4.3 when equality does not hold.

**Subcase 2.4.** $i + j - k + l = 2\alpha$. Using the basic assumption, we get $\alpha \geq l - j$. If $\alpha > k$, then $i - j - k + l = 2(\alpha - k)$, which converts to Subcase 2.5 below. Otherwise $j - i = k + l - 2\alpha \geq k + l - 2k = l - k$, then use the same skill like Subcase 2.3.

**Subcase 2.5.** $i + j - k + l = 2\alpha$. It is easy to be solved when we observe that $l - k = j - i + 2\alpha$ by using the idea of Lemma 4 in Pöschel [13].

**Subcase 2.6.** $i + j - k + l = 2\beta$. If $\beta + l \geq 1$, then it converts to Subcase 2.5. Next, using the basic assumption, we get $-j - k \leq \beta \leq -l$, which means $i - j - k - l = 2\beta \leq -2l$. This concludes that $l - k \leq j - i$, then we can use the same skill above.

**Subcase 2.7.** $i + j - k + l = 2\beta$. Observe that $l - k = j - i - 2|\beta|$.

**Subcase 2.8.** $i + j + k + l = 2\beta$. Using the basic assumption, we get $|\beta| \leq k$. If $\beta + l < 1$, then $i + j - k - l = 2\beta \leq -2l$, i.e., $i + j + l - k \leq 0$, which is a contradiction. So we obtain $i + j - k + l = 2(\beta + l)$, which can converts to Subcase 2.3.
Subcase 2.9. \( i - j + k - l = 2\beta \). Observe that \( l - k + j - i = 2|\beta| \), then \( l - k \leq j - i + 2 \) or \( l - k \geq j - i \), which can use the same skill as above.

Hence, we finish the proof of Lemma 4.2.

5. The Cantor Manifold Theorem. In this section, we will state Cantor manifold theorem in Pöschel’s article [13] which is proven in [10] using the KAM-theorem for partial differential equations from [14]. The difficulty here is to check the non-degeneracy condition (86) for Cantor manifold theorem.

In a neighbourhood of the origin in \( \ell^2_s \), we now consider more generally hamiltonian of the form

\[
H = \Lambda + Q + R,
\]

where \( \Lambda + Q \) is integrable and in normal form and \( R \) is a perturbation term. Letting \( z = (\tilde{z}, \hat{z}) \) with \( \tilde{z} = (z_1, z_2) \) and \( \hat{z} = (z_3, z_4, \ldots) \), as well as

\[
I = \frac{1}{2}(|z_1|^2, |z_2|^2), \quad Z = \frac{1}{2}(|z_3|^2, |z_4|^2, \ldots),
\]

we assume that

\[
\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \quad Q = \langle A I, I \rangle + \langle B I, Z \rangle,
\]

with constant vectors \( \alpha, \beta \) and constant matrices \( A, B \),

\[
\alpha_i = \lambda_i (i = 1, 2), \quad \beta_j = \lambda_j (j \geq 3),
\]

\[
A = \begin{pmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{pmatrix}, \quad B_{jk} = \bar{G}_{jk} (j \geq 3, k = 1, 2),
\]

In the Birkhoff normal form lemma, \( \Lambda + \bar{G} \) is of that form.

The equations of motion of the hamiltonian \( \Lambda + Q \) are

\[
\begin{align*}
\dot{\tilde{z}} &= i(\alpha + AI + B^T z)\tilde{z}, \\
\dot{\hat{z}} &= i(\beta + BI)\hat{z},
\end{align*}
\]

Thus, the complex 2-dimensional manifold \( E = \hat{z} = 0 \) is invariant, and it is completely filled up to the origin by the invariant tori

\[
\mathcal{T}(I) = \{ \tilde{z} : |\tilde{z}_j|^2 = 2I_j, 1 \leq j \leq 2 \}, \quad I \in \mathbb{P}^2.
\]

On \( \mathcal{T}(I) \) the flow is given by the equations

\[
\dot{\tilde{z}}_j = i\omega(I)\tilde{z}_j, \quad \omega(I) = \alpha + AI,
\]

and in its normal space by

\[
\dot{\hat{z}}_j = i\Omega(I)\hat{z}_j, \quad \Omega(I) = \beta + BI.
\]

They are linear and in diagonal form. In particular, since \( \Omega(I) \) is real, \( \hat{z} = 0 \) is an elliptic fixed point, all the tori are linearly stable, and their orbits have zero Lyapunov exponents. The Cantor manifold theorem proves the persistence of a large portion of \( E \) forming an invariant Cantor manifold \( \mathcal{E} \) for the hamiltonian

\[
H = \Lambda + Q + R.
\]

For the existence of \( \mathcal{E} \), the following assumptions are made.

A. Nondegeneracy. The normal form \( \Lambda + Q \) is nondegenerate in the sense that

\[
\begin{align*}
(A_1) & \quad \det A \neq 0, \\
(A_2) & \quad \langle l, \beta \rangle \neq 0, \\
(A_3) & \quad \langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \neq 0,
\end{align*}
\]

for all \( (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^\infty \) with \( 1 \leq |l| \leq 2 \).
B. Spectral asymptotics. There exists $d \geq 1$ and $\delta < 1$ such that
\[
\beta_j = j^d + \ldots + O(j^\delta),
\]
where the dots stand for terms of order less than $d$ in $j$. Note that the normalization of the coefficient of $j^d$ can always be achieved by a scaling of time.

C. Regularity.
\[
X_Q, X_R \in A(\ell_2^2, \ell_2^2), \quad \left\{ \begin{array}{l}
\tilde{s} \geq s \quad \text{for} \quad d > 1, \\
\tilde{s} > s \quad \text{for} \quad d = 1.
\end{array} \right.
\]

By the regularity assumption, the coefficients of $B = (B_{ij})_{1 \leq j \leq 2i}$ satisfy the estimate $B_{ij} = O(i^{s-\delta})$ uniformly in $1 \leq j \leq 2$. Consequently, for $d = 1$ there exists a positive constant $\kappa$ such that
\[
\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}), \quad i > j,
\]
uniformly for bounded $I$. For $d > 1$, we set $\kappa = \infty$.

The following theorem is in Pöschel [13].

**Theorem 5.1.** “THE CANTOR MANIFOLD THEOREM. Suppose the hamiltonian $H = \Lambda + Q + R$ satisfies assumptions $A, B$ and $C$, and
\[
|R| = O(\|\tilde{z}\|^2_2) + O(\|z\|^2_2)
\]
with
\[
g > 4 + \frac{4 - \Delta}{\kappa}, \quad \Delta = \min(\tilde{s} - s, 1).
\]
Then there exists a Cantor manifold $\mathcal{E}$ of real analytic, elliptic diophantine $n$-tori given by a Lipschitz continuous embedding $\Psi : \mathcal{T}[\mathcal{E}] \to \mathcal{E}$, where $\mathcal{E}$ has full density at the origin, and $\Psi$ is close to the inclusion map $\Psi_0$:
\[
\|\Psi - \Psi_0\|_{s, B, r, \mathcal{T}[\mathcal{E}]} = O(v^\sigma),
\]
with some $\sigma > 1$. Consequently, $\mathcal{E}$ is tangent to $E$ at the origin.”

We now verify the assumptions of the Cantor Manifold Theorem.

**Proof.** We already known that $X_Q, X_R \in A(\ell_2^2, \ell_2^2)$ with $|R| = O(\|\tilde{z}\|^2_2) + O(\|z\|^2_2)$. On the other hand, we have
\[
\lambda_j = \sqrt{j(j+1)} + m = j + \frac{1}{2} + \frac{m - \frac{1}{2}}{2j} + O(j^{-3}).
\]
So conditions B and C are satisfied with $d = 1, \delta = -1, \tilde{s} = \frac{9}{2}$ and $s = \frac{7}{2}$.

Moreover, since $B_{ij} = \tilde{G}_{ij} = \frac{2(2i+1)(2j+1)}{\lambda_i \lambda_j} P(i, j)$, we have
\[
\Omega_{j-2} = (\beta + BI)_{j-2} = \lambda_j + \frac{\langle v, I \rangle}{j}
\]
with $v = 2(2i+1)(2j+1)P(i, j)(\lambda_i^{-1}, \lambda_j^{-1})$. This gives the asymptotic expansion
\[
\Omega_{j-2} = j + \frac{1}{2} + \frac{m - \frac{1}{2}}{2j} + \frac{\langle v, I \rangle}{j} + O(j^{-3}) = j + \frac{1}{2} + \frac{m_j}{j} + O(j^{-3}),
\]
$m_j = \frac{m - \frac{1}{2}}{2} + \langle v, I \rangle$. Thus, for $i > j$,
\[
\frac{\Omega_i - \Omega_j}{i - j} = 1 - \frac{m_j}{(i+2)(j+2)} + O(j^{-3}) = 1 + O(j^{-2}),
\]
we can obtain
\[
\det A = \det \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\
\triangleq \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \\
\triangleq \frac{1}{\lambda_1^2\lambda_2^2} \det g < 0,
\]
where
\[
g_{11} = \lambda_1^2 G_{11} = 3 \times 3 \times P(1, 1) = \frac{18}{5},
\]
\[
g_{12} = g_{21} = \lambda_1 \lambda_2 G_{12} = 2 \times 3 \times 7 \times P(1, 3) = \frac{92}{15},
\]
\[
g_{22} = \lambda_2^2 G_{22} = 7 \times 7 \times P(3, 3) = 2 \times 7 \times 241 \times 5 \times 11 \times 13,
\]
\[
\det g = g_{11}g_{22} - g_{12}g_{21} = -\frac{663764}{32175}.
\]

The nondegeneracy condition \((A_2)\) is easy to check since \(\lambda_j\) or \(\lambda_i \pm \lambda_j (i \neq j)\) are not equal to zero.

Next, we will check the nondegeneracy condition \((A_3)\). Since the condition \(1 \leq |l| \leq 2\), we only need to consider the following two cases
\[
\langle k, \omega \rangle \pm \Omega_j \neq 0
\]
and
\[
\langle k, \omega \rangle \pm (\Omega_i - \Omega_j) \neq 0
\]
Recall the definition of \(\omega(I)\) \((84)\) and \(\Omega(I)\) \((85)\), we can obtain that \(\Omega = \beta + BA^{-1}(\omega - \alpha)\). Besides, choosing \(\zeta = (\frac{1}{\omega_1}, \frac{1}{\omega_2}) = (\sigma, \zeta_2)\) as another new parameter vector instead of \(\omega = (\omega_1, \omega_2)\), we can obtain the following two expressions \((A_{31})\) and \((A_{32})\) with respect to \(\sigma = \frac{1}{\zeta_1}\) which are equivalent to \((89)\) and \((90)\).

\[
(A_{31}) \quad \left| \frac{df_1(\sigma)}{d\sigma} \right| = \beta_j - \sum_{l=1}^{2} B_{jl}(A^{-1}\alpha)_l > 0,
\]
\[
(A_{32}) \quad \left| \frac{df_2(\sigma)}{d\sigma} \right| = \left| \beta_i - \beta_j - \sum_{l=1}^{2} (B_{il} - B_{jl})(A^{-1}\alpha)_l \right| > 0,
\]
where
\[
\tilde{\omega} = \omega \sigma = (1, \zeta_2),
\]
\[
f_1(\sigma) = \langle k, \tilde{\omega} \rangle + \sigma \beta_j + \sum_{l=1}^{2} B_{jl}[A^{-1}(\tilde{\omega} - \alpha \sigma)]_l,
\]
\[
f_2(\sigma) = \langle k, \tilde{\omega} \rangle \pm \left[ \sigma \beta_i + \sum_{l=1}^{2} B_{il}[A^{-1}(\tilde{\omega} - \alpha \sigma)]_l - (\sigma \beta_j + \sum_{l=1}^{2} B_{jl}[A^{-1}(\tilde{\omega} - \alpha \sigma)]_l) \right].
\]
Here, we denote $g_{j1}$ and $g_{j2}$ according to (75),
\[
g_{j1} = \lambda_1 \lambda_j \bar{G}_{1j} \\
= 2 \times (4 \times 1 - 1) \times (4j - 1) \times P(1, 2j - 1) \\
= \frac{2(8j^2 - 4j - 1)}{(4j - 3)(4j + 1)};
\]
and
\[
g_{j2} = \lambda_2 \lambda_j \bar{G}_{2j} \\
= 2 \times (4 \times 2 - 1) \times (4j - 1) P(3, 2j - 1) \\
= \frac{28 \times (1088 j^6 - 1632 j^5 - 2440 j^4 + 3120 j^3 + 1406 j^2 - 1110j - 225)}{(4j - 7)(4j - 5)(4j - 3)(4j + 1)(4j + 3)(4j + 5)}.
\]

By the basic computation
\[
A^{-1} = \begin{pmatrix}
\bar{G}_{11} & \bar{G}_{12} \\
\bar{G}_{21} & \bar{G}_{22}
\end{pmatrix}^{-1} \\
\frac{\bar{G}_{22} - \bar{G}_{12}}{\det A} \\
\frac{-\bar{G}_{21}}{\det A} \bar{G}_{11},
\]
as well as the definition of $B_{jk}$, $\alpha_i$, $\beta_j$ and $g_{j1}, g_{j2}$ in (80)–(83) and (94) (95), we can obtain the following result.

In (A31), we have
\[
\frac{df_1(\sigma)}{d\sigma} = \begin{vmatrix}
\lambda_j - \bar{G}_{1j} & \bar{G}_{22}\lambda_1 - \bar{G}_{12}\lambda_2 \\
\frac{\det A}{\lambda_1^2 \lambda_2^2 \lambda_j} & -\bar{G}_{21} \bar{G}_{11}\lambda_1 - \bar{G}_{12}\lambda_1
\end{vmatrix}
\]
\[
= \frac{1}{\det A} \begin{vmatrix}
(\det g) \lambda_j^2 - g_{j1}(g_{22}\lambda_1^2 - g_{12}\lambda_2^2) - g_{j2}(g_{11}\lambda_2^2 - g_{12}\lambda_1^2)
\end{vmatrix}
\]
\[
\geq \frac{1}{\det A} \begin{vmatrix}
((\det g) \lambda_j^2) - |(\det g)(\lambda_j^2 - g_{j1}g_{22}\lambda_1^2 + g_{j2}g_{11}\lambda_2^2)|
\end{vmatrix},
\]
where $j \geq 3$ and $m \in \left(0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{41}{4}\right)$. An easy computation gives
\[
g_{22}\lambda_1^2 - g_{12}\lambda_2^2 = -3034 \times \frac{2145}{2145} - 45876 \times \frac{715}{715}
\]
and
\[
g_{11}\lambda_2^2 - g_{12}\lambda_1^2 = -38 \times \frac{15}{15} + 464 \times \frac{15}{15}.
\]
From (88) (96) (99), it is easy to estimate that
\[
|((\det g) \lambda_j^2)| = |((\det g)(2j(2j - 1) + m)| > 618,
\]
and
\[
|g_{j1}(g_{22}\lambda_1^2 - g_{12}\lambda_2^2) + g_{j2}(g_{11}\lambda_2^2 - g_{12}\lambda_1^2)| < 441.
\]
Since then, by substituting (100) and (101) into (97), we obtain
\[
\left| \frac{df_1(\sigma)}{d\sigma} \right| > 0.
\]

In \((A_{32})\), we have
\[
\left| \frac{df_2(\sigma)}{d\sigma} \right| = \left| \beta_i - \beta_j - \sum_l (B_{il} - B_{jl})(A^{-1}A)_l \right|
\]
\[
= \left| (\lambda_i - \lambda_j) \frac{\bar{G}_{i1} - \bar{G}_{j1}}{\det A} - (\bar{G}_{i2} - \bar{G}_{j2}) \frac{\bar{G}_{i1} - \bar{G}_{j1}}{\det A} \right|
\]
\[
= \frac{1}{\lambda_1^2 \lambda_2^2 |\det A|} \left| (\lambda_i - \lambda_j) \det g - \lambda_1^2 \lambda_2^2 \left( \frac{g_{i2}}{\lambda_i \lambda_2} - \frac{g_{j2}}{\lambda_j \lambda_2} \right) \left( \frac{g_{i1}}{\lambda_i^2 \lambda_1} - \frac{g_{j1}}{\lambda_j^2 \lambda_1} \right) \right|
\]
\[
\geq \frac{|i - j|}{\lambda_1^2 \lambda_2^2 |\det A| (\lambda_i + \lambda_j)} \left[ \left| (\det g) \frac{\lambda_i - \lambda_j}{i - j} \right| - \left| (g_{i2} \lambda_1^2 - g_{i2} \lambda_2^2) \frac{\frac{g_{i1}}{\lambda_i^2 \lambda_1} - \frac{g_{j1}}{\lambda_j^2 \lambda_1}}{i - j} \right| \right].
\]

Since
\[
\frac{\lambda_i - \lambda_j}{i - j} = \frac{\sqrt{2i(2i - 1) + m} - \sqrt{2j(2j - 1) + m}}{i - j} = \frac{4(i + j) - 2}{\lambda_i + \lambda_j} > \frac{4(i + j) - 2}{|2(i + j) - 1| + 2 \sqrt{m - \frac{1}{4}}} > \frac{2}{1 + \frac{\sqrt{m}}{13}}, \quad (i, j \geq 3, i \neq j)
\]
we have the following estimate by (88)
\[
\left| (\det g) \frac{\lambda_i - \lambda_j}{i - j} \right| > 27. \quad (103)
\]

Let
\[
v_1(i) = \frac{g_{i1}}{\lambda_i}, \quad (104)
\]
\[
v_2(i) = \frac{g_{i2}}{\lambda_i}. \quad (105)
\]

Then for every fixed \(m \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{41}{4})\), when \(x \in [3, \infty)\)
\[
\partial_x v_1(x) < 0, \quad \partial_x v_2 < 0, \quad (106)
\]
and both monotonically increase with respect to \(x\). Besides,
\[
\frac{g_{i1}}{\lambda_i} - \frac{g_{j1}}{\lambda_j} \in \partial_x v_1([3, \infty)), \quad \frac{g_{i2}}{\lambda_i} - \frac{g_{j1}}{\lambda_j} \in \partial_x v_2([3, \infty)). \quad (107)
\]
From (98) (99) and some computation, we know
\[
\max_{m \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{41}{4})} \left| (g_{22} \lambda_i^2 - g_{12} \lambda_i^2) \frac{g_{11}}{i - j} + (g_{11} \lambda_i^2 - g_{12} \lambda_i^2) \frac{g_{22}}{i - j} \right| < 27. \quad (108)
\]
In the end, by substituting (103) and (108) into (102), we get
\[
\left| \frac{df_2(\sigma)}{d\sigma} \right| > 0.
\]
Thus the main theorem follows.

6. Proof of Theorem 3.2. Before our proof, we need the following four lemmas which will be proved in Appendix B.

The idea is using the recursion formula (20) to obtain lemma 6.2 (with the help of lemma 6.1 also ) and lemma 6.3. Then, we eliminate \( Q(m,n) \) and get the recursion (See lemma 6.4) with respect to \( P(m,n) \) by some technical calculation.

The estimate of \( c(m) = \lim_{n \to \infty} nP(m,n) \) in lemma 6.5 and the symmetry of \( P(m,n) \) give the final result. The idea using another sequence \( \frac{P(m,n)}{m+1} \) similar to \( c(m) \) to obtain a rough upper bound of \( c(m) \) and using the symmetry property to obtain a precise estimate.

Lemma 6.1. \( \int_{-1}^{1} P_{m}P_{n+1}P_{n-1}dx = \int_{-1}^{1} P_{n}P_{m+1}P_{m-1}dx. \)

If we denote
\[
P(m,n) = \int_{-1}^{1} P_{m}P_{m}P_{n}dx,
\]
\[
Q(m,n) = \int_{-1}^{1} P_{m-1}P_{m+1}P_{n}dx,
\]
then we have

Lemma 6.2.
\[
P(m,n) = \left( \frac{n + 1}{2n + 1} \right)^2 \left( \frac{2m - 1}{m} \right)^2 P(m - 1, n - 1)
+ \frac{2n^2 + 2n - 2m^2 + 2m}{(2n + 1)^2 m^2} Q(m - 1, n) - \left( \frac{m - 1}{m} \right)^2 P(m - 2, n). \quad (109)
\]

Lemma 6.3.
\[
Q(m,n) = \frac{(2m + 1)m}{(m + 1)(2m - 1)} P(m,n)
+ \frac{(2m + 1)(m - 1)}{(2m - 1)(m + 1)} Q(m - 1, n) - \frac{m}{m + 1} P(m - 1, n). \quad (110)
\]

Lemma 6.4.
\[
P(m+1,n)
= \alpha_{m}^{-1}P(m, n - 1) + \alpha_{m}^{n}P(m, n) + \alpha_{m}^{n+1}P(m, n + 1)
- \alpha_{m-1}^{-1}P(m - 1, n - 1) - \alpha_{m-1}^{n}P(m - 1, n) - \alpha_{m-1}^{n+1}P(m - 1, n + 1)
+ \alpha_{m-2}^{n}P(m - 2, n), \quad (111)
\]
with the coefficients $\alpha_i^j$ and $P(k,n), (k=0,1,2,3)$ expressed in Theorem 3.3.

Now we only need to prove
\[
\int_{-1}^{1} P_i(x)P_j(x)P_k(x)P_l(x)dx \lesssim \frac{1}{\sqrt{i + \frac{1}{2}} \sqrt{j + \frac{1}{2}} \sqrt{k + \frac{1}{2}} \sqrt{l + \frac{1}{2}}}
\]

It is obvious to conclude the following proposition by mathematical induction according to the recursion (111) and the expressions of $P(i,n), (i = 0, 1, 2)$ (65) (66) (67).

**Proposition 2.** For every fixed $m$, $P(m,n)$ is a rational fraction with respect to $n$, i.e.
\[
P(m,n) = \frac{\mathcal{R}(n)}{\mathcal{N}(n)}.
\]
The degree of denominator $\text{deg}(\mathcal{R}(n))$ and the degree of numerator $\text{deg}(\mathcal{N}(n))$ satisfy
\[
\text{deg}(\mathcal{N}(n)) - \text{deg}(\mathcal{R}(n)) \geq 1.
\]

**Remark 3.** We conjecture that $\text{deg}(\mathcal{R}(n)) = 2m$ and $\text{deg}(\mathcal{N}(n)) = 2m + 1$.

An immediate consequence of the proposition is that, for every fixed $m$,
\[
\lim_{n \to \infty} n P(m,n) = O(1),
\]
and since the symmetry $P(m,n) = P(n,m)$, we also have that for every fixed $n$,
\[
\lim_{m \to \infty} m P(m,n) = O(1).
\]

In the following, we have to check if,
\[
\lim_{mn \to \infty} mn P(m,n) = O(1),
\]
which seems to be correct intuitively but need to be proved strictly.

If we fix $m$, we can denote
\[
\lim_{n \to \infty} n P(m,n) = c(m),
\]
and it is obvious that
\[
c(0) = 1, \quad c(1) = \frac{1}{2}, \quad c(2) = \frac{11}{32}, \quad c(3) = \frac{17}{64}.
\]
Observing the relationship
\[
\alpha_{m-1}^{n-1} + \alpha_{m}^{n} + \alpha_{m+1}^{n+1} - \alpha_{m-1}^{n} - \alpha_{m}^{n-1} - \alpha_{m-1}^{n+1} + \alpha_{m-2}^{n} \equiv 1,
\]
by letting $n$ goes to infinity, we get
\[
\alpha_{m} - \alpha_{m-1} + \alpha_{m-2} \equiv 1, \quad (114)
\]
where
\[
\alpha_{i} = \lim_{n \to \infty} \sum_{j=n-1}^{n+1} \alpha_{i}^{j}, (i = m-1, m), \quad \alpha_{m-2} = \lim_{n \to \infty} \alpha_{m-2}^{n}, \quad (115)
\]
and
\[
\alpha_{m-1} = \frac{6m^3 - 2m^2 + 1}{2(m+1)^3}, \quad \alpha_{m} = \frac{(2m+1)(6m^3 + 2m^2 - 1)}{2(2m-1)(m+1)^3},
\]
\[
\alpha_{m-2} = \frac{(m-1)^3(2m+1)}{(m+1)^3(2m-1)}, \quad (116)
\]
Then from (111), letting $n$ goes to infinity, we have the following unilateral sequence
\[ c(m + 1) = \alpha_m c(m) - \alpha_{m-1} c(m - 1) + \alpha_{m-2} c(m - 2). \] (117)

**Lemma 6.5.** The sequence $c(m)$ satisfies the following properties:

\[ 0 \leq c(m + 1) < c(m) \leq \frac{1}{2}. \] (118)

and
\[ c(m + 1) \leq \frac{15}{16(m + 2)} + \frac{1}{32}. \] (119)

The proof is given in Appendix B.

From (113) and (119), we know
\[ \sup_{n \geq 1} (nP(m, n)) \lesssim \frac{15}{16(m + 2)} + \frac{1}{32}, \] (120)

On the other hand, since the symmetry (See the definition of $P(m, n)$.

\[ P(m, n) = P(n, m), \] (121)

\[ \lim_{m \to \infty} mP(m, n) = \lim_{m \to \infty} mP(n, m) = c(n), \] (122)

then by (119)
\[ mP(m, n) \lesssim c(n) = \frac{15}{16(n + 1)} + \frac{1}{32}, \] (123)

so
\[ nP(m, n) \lesssim \left( \frac{15n}{16(n + 1)} + \frac{1}{32} \right) \frac{1}{m}. \] (124)

However, from (120) we know
\[ \sup_{n \geq 1} (nP(m, n)) < \infty, \]

so from (124), we obtain
\[ \sup_{n \geq 1} (nP(m, n)) \approx \frac{1}{m}, \]

then it follows that, there exists $\tilde{C} > 0$ such that
\[ P(m, n) < \frac{\tilde{C}}{mn}. \]

By Cauchy Inequality, we have
\[ \int_{-1}^{1} P_i P_j P_k P_l dx \leq \left( \int_{-1}^{1} (P_i P_j)^2 dx \right)^{\frac{1}{2}} \left( \int_{-1}^{1} (P_k P_l)^2 dx \right)^{\frac{1}{2}} \]
\[ \lesssim \frac{1}{\sqrt{i + \frac{1}{2}} \sqrt{j + \frac{1}{2}} \sqrt{k + \frac{1}{2}} \sqrt{l + \frac{1}{2}}}. \]

If $i \leq j \leq k \leq l$, then we already know
\[ \int_{-1}^{1} P_i P_j P_k P_l dx \neq 0 \quad \text{(iff} \quad i + j + k + l = 2\alpha \quad \text{and} \quad i + j + k \geq l) \]

and $P_i P_j P_k P_l$ is an even function, from (25) we know
\[ \int_{-1}^{1} P_i P_j P_k P_l dx > 0, \]
then
\[ 0 < \int_{-1}^{1} \phi_i \phi_j \phi_k \phi_l \, dx \leq C \]
is an easy deduction. \(\square\)

7. **Appendix A.** Recall the definition

\[
\{f\}_\alpha = \begin{cases} 
  \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} & \text{when } 0 < \alpha < 1, \\
  \|f\|_{L^p[-1,1]} & \text{when } \alpha = -\frac{1}{p} \leq 0,
\end{cases}
\]

for any \(f\) defined on \([-1,1]\).

Then we have the following General Nirenberg Inequality.

**Theorem 7.1.** [11] [5] [12] Suppose \(-1 \leq \alpha, \beta, \gamma < 1\) and \(j, k\) are nonnegative integers, \(0 \leq \theta \leq 1\), and

\[
\begin{cases} 
  j + \beta = \theta(k + \alpha) + (1 - \theta)\gamma, \\
  \frac{1}{p_\beta} \leq \frac{\theta}{p_\alpha} + \frac{1 - \theta}{p_\gamma},
\end{cases}
\]

where

\[
p_\delta = \begin{cases} 
  \infty, & \delta \geq 0, \\
  -\frac{1}{\delta}, & \delta < 0.
\end{cases}
\]

When \(k + \alpha\) is an integer \(\geq 1\), and \(-1 < \alpha < 0\) (i.e. \(1 < p_\alpha < \infty\)) we require \(\theta \neq 1\).

Then for any \(f \in C_0^\infty[-1,1]\), we have

\[
\{\partial^j_x f\}_\beta \lesssim \{\partial^k_x f\}_\alpha \{f\}_\gamma^{1 - \theta}.
\]

An immediate consequence of this theorem is the Gagliardo-Nirenberg Inequality.

**Theorem 7.2.** [11] [12] Suppose \(0 \leq \alpha \leq 1, 1 \leq p, q, r \leq \infty, m, k\) are any integers satisfying

\[
\begin{cases} 
  k - \frac{1}{p} = a(m - \frac{1}{q}) + (1 - a)(\frac{1}{r}), \\
  \frac{1}{p} \leq \frac{a}{q} + \frac{1 - a}{r},
\end{cases}
\]

then for any \(f \in C_0^\infty[-1,1]\), we have the following inequality

\[
\|\partial^k_x u\|_{L^p[-1,1]} \lesssim \|\partial^m_x u\|_{L^q[-1,1]}^a \|u\|_{L^r[-1,1]}^{1 - a}
\]

with the following exception: if \(m - \frac{a}{q} = k, 1 < p < \infty\), then \([130]\) holds for \(a \neq 1\).
8. Appendix B.

8.1. Proof of Lemma 6.1

Proof. From [6], we know that

\[
P_{n+1}P_{n-1} = \sum_{k=2}^{2n} \binom{n+1}{k} \binom{n-1}{0} \binom{k}{0} (2k+1)P_k
\]

\[
P_m^2 = \sum_{k=0}^{2m} \binom{m}{k} \binom{m}{0} (2k+1)P_k,
\]

and the same to \(P_{m+1}P_{m-1}\) and \(P_n^2\), then

\[
\int_{-1}^{1} P_{n+1}P_{n-1}P_m^2 dx = 2 \sum_{k=2}^{2\min(m,n)} \binom{n+1}{k} \binom{n-1}{0} \binom{k}{0} (2k+1),
\]

\[
\int_{-1}^{1} P_{m+1}P_{m-1}P_n^2 dx = 2 \sum_{k=2}^{2\min(m,n)} \binom{m+1}{k} \binom{m-1}{0} \binom{k}{0} (2k+1).
\]

It is easy to check that

\[
\binom{n+1}{k} \binom{n-1}{0} \binom{k}{0} = \frac{A(k+1)A(n-k)}{(2n+k+1)A(n+\frac{k}{2})} \frac{A(k)A(n-k)}{(2m+k+1)A(m+\frac{k}{2})}
\]

\[
= \frac{A(k+1)A(n-k)}{(2n+k+1)A(n+\frac{k}{2})} \frac{A(k)A(n-k)}{(2m+k+1)A(m+\frac{k}{2})}
\]

\[
= \binom{m+1}{k} \binom{m-1}{0} \binom{k}{0}.
\]

\[
\square
\]

8.2. Proof of Lemma 6.2

Proof. From the recursion formula [20]

\[(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0,\]

we get

\[
P_m = \frac{(2m-1)xP_{m-1} - (m-1)P_{m-2}}{m}, \quad xP_n = \frac{(n+1)P_{n+1} + nP_{n-1}}{2n+1},
\]
then we can write the demanded integration as follows,

\[
\int_{-1}^{1} P_m P_m P_n P_n \, dx = \int_{-1}^{1} \left[ \frac{(2m-1)x P_{m-1} - (m-1)P_{m-2}}{m} \right]^2 P_n^2 \, dx
\]

\[
= \left( \frac{2m-1}{m} \right)^2 \int_{-1}^{1} P_{m-1}^2 \left[ \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1} \right]^2 \, dx
\]

\[
+ \left( \frac{m-1}{m} \right)^2 \int_{-1}^{1} P_{m-2}^2 P_n^2 \, dx
\]

\[
- \frac{2(2m-1)(m-1)}{m^2} \left[ m P_m + (m-1)P_{m-2} \right] \frac{2m-1}{2m-1} P_{m-2}^2 P_n^2 \, dx
\]

\[
= \left( \frac{n+1}{2n+1} \right)^2 \left( \frac{2m-1}{m} \right)^2 \int_{-1}^{1} P_{m-1}^2 P_{n+1}^2 \, dx
\]

\[
+ \left( \frac{m-1}{m} \right)^2 \left( \frac{n}{2n+1} \right)^2 \int_{-1}^{1} P_{m-1}^2 P_{n-1}^2 \, dx
\]

\[
+ 2 \left( \frac{2m-1}{m} \right)^2 \frac{n+1}{2n+1} \frac{n}{2n+1} \int_{-1}^{1} P_{m-1}^2 P_{n+1} P_{n-1} \, dx
\]

\[
- \left( \frac{m-1}{m} \right)^2 \int_{-1}^{1} P_{m-2}^2 P_n^2 \, dx - \frac{2(m-1)}{m} \int_{-1}^{1} P_m P_{m-2} P_n^2 \, dx.
\]

With the help of Lemma 6.1, we have

\[
\int_{-1}^{1} P_{m-1}^2 P_{n+1} P_{n-1} \, dx = \int_{-1}^{1} P_m P_{m-2} P_n^2 \, dx. \tag{132}
\]

Insert it into equation (131), we complete the proof.

\[\square\]

8.3. Proof of Lemma 6.3.

Proof. From (20) we know

\[
\int_{-1}^{1} [(n+1)P_{n+1}] P_{n+1} P_m^2 \, dx
\]

\[
= \int_{-1}^{1} [(2n+1)x P_n] P_{n+1} P_m^2 \, dx - \int_{-1}^{1} [nP_{n-1}] P_{n+1} P_m^2 \, dx
\]

\[
= (2n+1) \int_{-1}^{1} (x P_{n+1}) P_n P_m^2 \, dx - n \int_{-1}^{1} P_{n+1} P_{n-1} P_m^2 \, dx
\]

\[
= (2n+1) \int_{-1}^{1} P_m (n+2) P_{n+2} + (n+1) P_n P_m^2 \, dx - n \int_{-1}^{1} P_{n-1} P_{n+1} P_m^2
\]

\[\square\]

8.4. Proof of lemma 6.4.

Proof. Using the recursion (20)

\[
(n+1)P_{n+1} - (2n+1)x P_n + nP_{n-1} = 0,
\]

and the expressions of Legendre polynomials

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2},
\]
it is easy to prove
\[ P(1, n) = \frac{2(2n^2 + 2n - 1)}{(2n - 1)(2n + 1)(2n + 3)} \]
and
\[ P(0, n) = \frac{2}{2n + 1} \]
as well as
\[ P(2, n) = \frac{11n^4 + 22n^3 - 31n^2 - 42n + 18}{(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)} \]
Then
\[ Q(1, n) = \int_1^0 P_0 P_1 P_2 dx = \frac{3}{2} P(1, n) - \frac{1}{2} P(0, n) = \frac{2n(n + 1)}{(2n - 1)(2n + 1)(2n + 3)} \]

From (109) and (110), we can find the expressions of \( P(m, n) \) and \( Q(m, n) \), when \( m \) is fixed. In particular, we have
\[ P(3, n) = \frac{34n^6 + 102n^5 - 305n^4 - 780n^3 + 703n^2 + 1011n - 450}{(2n - 5)(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)(2n + 7)} \]

Now, by Lemma 6.3, we have the following recursion with respect to \( \left\{ \frac{m(m+1)}{2m+1} \right\}_{m=1}^{\infty} \)
\[ \frac{m(m+1)}{2m+1} Q(m, n) - \frac{(m-1)(m-1) + 1}{2(m-1) + 1} Q(m-1, n) \]
\[ = \frac{m^2}{2m+1} P(m, n) - \frac{m^2}{2m+1} P(m-1, n) \]
Then it follows that
\[ Q(m, n) = \frac{2m+1}{m(m+1)} \left[ \frac{m^2}{2m+1} P(m, n) + \frac{1}{2m-3}(2m+1) P(m-1, n) \right. \]
\[ + \sum_{i=1}^{m-2} \left( \frac{1}{(2i-1)(2i+3)} P(i, n) - \frac{1}{3} P(0, n) \right) \] \hspace{1cm} (133)
Inserting (133) into (109) gives the following recursion,
\[ P(m+1, n) \]
\[ = \left( \frac{n+1}{2n+1} \right)^2 \left( \frac{2m+1}{m+1} \right)^2 P(m, n + 1) \]
\[ + \left( \frac{n}{2n+1} \right)^2 \left( \frac{2m+1}{m+1} \right)^2 P(m, n - 1) \]
\[ + \frac{2m(2m+1)(n^2 + n - m^2 - m)}{(m+1)^3(2m-1)(2m+1)^2} P(m, n) \]
\[ - \left[ \frac{m}{m+1} \right]^2 \frac{2(n^2 + n - m^2 - m)}{m(m+1)^3(2m-3)(2m+1)^2} P(m-1, n) \] \hspace{1cm} (134)
\[ + \frac{2(n^2 + n - m^2 - m)(2m+1)}{m(m+1)^3(2m+1)^2} \sum_{i=1}^{m-2} \left( \frac{1}{(2i-1)(2i+3)} P(i, n) \right) \]
\[ - \frac{4(2m+1)(n^2 + n - m^2 - m)}{3m(m+1)^3(2n+1)^3} \].
Calculating
\[
\frac{m(m + 1)^3(2n + 1)^2}{2(n^2 + n - m^2 - m)(2m + 1)} \mathbf{P}(m + 1, n) - \frac{(m - 1)m^3(2n + 1)^2}{2(n^2 + n - m^2 + m)(2m - 1)} \mathbf{P}(m, n),
\]
by substituting the expression of recursion (134) into \( \mathbf{P}(m + 1, n) \) and \( \mathbf{P}(m, n) \), we can obtain the recursion formula which the expression and the coefficients are asserted in Theorem 3.3.

Here we would like to mention that the “nonhomogeneous term” (the last term only with respect to \( m \) and \( n \) except \( \mathbf{P}(\cdot, \cdot) \)) is also disappeared with the summation about \( \mathbf{P}(i, n) \)!

So \( \mathbf{P}(m + 1, n) \) is completely finite linear dependent and this is important to get the estimate of integral of (69).

8.5. **Proof of lemma 6.5.**

*Proof.* Using the identity (114), we can fold the unilateral sequence as follows
\[
\mathbf{C}(m) = \beta_{m-1} \mathbf{C}(m - 1) - \beta_{m-2} \mathbf{C}(m - 2),
\]
where
\[
\mathbf{C}(m) = \mathbf{c}(m + 1) - \mathbf{c}(m),
\]
and
\[
\beta_{m-1} = \alpha_m - 1 = \frac{8m^4 - 4m^2 + 1}{2(m + 1)^3(2m - 1)}, \quad \beta_{m-2} = \alpha_{m-2} = \frac{(m - 1)^3(2m + 1)}{(m + 1)^3(2m - 1)}.
\]
and it is easy to calculate that
\[
\mathbf{C}(1) = -\frac{5}{32}, \quad \mathbf{C}(2) = -\frac{5}{64}.
\]

**Remark 4.** It is important to point out that \( \mathbf{c}(m) \) is “homogeneous” (has no terms only with respect to \( m \)) and it belongs to the closure of the set \( \mathcal{C} \),
\[
\mathbf{c}(m) \in \mathcal{C}, \quad \mathcal{C} = \left\{ \mathcal{P}(m) \mid \mathcal{D}(m) \right\}
\]
where \( \mathcal{P}(m) \) and \( \mathcal{D}(m) \) are polynomials with integral coefficients (with respect to \( m \)). Then it excludes the case which has the “nonhomogeneous term”, for example
\[
\mathbf{c}(m) = \frac{m - 1}{m} \mathbf{c}(m - 1) + \frac{1}{m^2}, \quad \mathbf{c}(1) = 1
\]
and we know \( \tilde{\mathbf{c}}(m) = \frac{1 + \frac{1}{m} + \ldots + \frac{1}{m}}{m} \), due to the nonhomogeneous term \( \frac{1}{m^2} \).

If we denote another sequence which can be folded,
\[
\mathbf{D}(m) = -\frac{15}{16(m + 1)(m + 2)} = \frac{15}{16(m + 2)} - \frac{15}{16(m + 1)},
\]
and the error sequence follows
\[
\mathbf{E}(m) = \mathbf{D}(m) - \mathbf{C}(m),
\]
then it is easy to check that, for \( m \geq 3 \),
\[
\mathbf{D}(m) \geq \beta_{m-1} \mathbf{D}(m - 1) - \beta_{m-2} \mathbf{D}(m - 2),
\]
\[
\mathbf{E}(m) \geq \beta_{m-1} \mathbf{E}(m - 1) - \beta_{m-2} \mathbf{E}(m - 2).
\]
It is obvious to see that
\[
\mathbf{E}(1) = \mathbf{E}(2) = 0.
\]
By induction, we obtain that

\[ E(m) \geq \left( \frac{m + \sqrt{2} - 1}{m + 2} \right)^2 E(m - 1). \]  \hfill (138)

Then for any \( m \),

\[ E(m) \geq 0, \]

and it follows that

\[ C(m) \leq D(m) = -\frac{15}{16(m + 1)(m + 2)} < 0, \] \hfill (139)

by the notation \( (136) \), which means

\[ 0 \leq c(m + 1) < c(m) \leq \frac{1}{2}. \] \hfill (140)

and \( c(m) \) has a limitation which we will know it is 0 in the end.

Besides, from \( (139) \), we can deduce that

\[ \sum_{i=1}^{m} C(i) \leq \sum_{i=1}^{m} D(i). \]

**unfolding** the sequence, we obtain

\[ \sum_{i=1}^{m} (c(i + 1) - c(i)) \leq \sum_{i=1}^{m} \frac{15}{16} \left( \frac{1}{m + 2} - \frac{1}{m + 1} \right), \]

\[ c(m + 1) \leq c(1) + \frac{15}{16} \left( \frac{1}{m + 2} - \frac{1}{2} \right). \]

Then we have the upper bound for \( c(m) \),

\[ c(m + 1) \leq \frac{15}{16(m + 2)} + \frac{1}{32}. \] \hfill (141)

\[ \square \]

**Acknowledgments.** The authors would like to thank the referees for their invaluable suggestions and comments. The first author is grateful to Kangkang Zhang for her help, encouragement and patience, without which this paper would not have been written.

**REFERENCES**

[1] M. Berti and M. Procesi, Quasi-periodic solutions of completely resonant forced wave equations, *Comm. Partial Differential Equations*, 31 (2006), 959–985.

[2] J. Bourgain, Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE, *Int. Math. Res. Not.*, 1994 (1994), 475–497.

[3] M. Gao and J. Liu, Quasi-periodic solutions for 1D wave equation with higher order nonlinearity, *J. Differential Equations*, 252 (2012), 1466–1493.

[4] B. Grébert and L. Thomann, KAM for the quantum harmonic oscillator, *Comm. Math. Phys.*, 307 (2011), 383–427.

[5] D. B. Henry, How to remember the Sobolev inequalities, *Differential Equations*, Springer Berlin Heidelberg, 957 (1982), 97–109.

[6] H. Y. Hsu, Certain integrals and infinite series involving ultra-spherical polynomials and Bessel functions, *Duke Math. J.*, 4 (1938), 374–383.

[7] S. B. Kuksin, *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*, Springer-Verlag, 1993.
S. B. Kuksin, Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum, *Funktsional. Anal. i Prilozhen.*, 21 (1987), 22–37.

S. B. Kuksin, Perturbation of quasiperiodic solutions of infinite-dimensional Hamiltonian systems, *Math. USSR Izv.*, 32 (1989), 39–62.

S. B. Kuksin and J. Pöschel, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrodinger equation, *Anal. of Math.*, 143 (1996), 149–179.

L. Nirenberg, An extended interpolation inequality, *Annali Della Scuola Normale Superiore di Pisa-Classe di Scienze*, 20 (1966), 733–737.

L. Nirenberg, On elliptic partial differential equations, *IL Principio Di Minimo E Sue Applicazioni Alle Equazioni Funzionali*, 17 (2011), 1–48.

J. Pöschel, Quasi-periodic solutions for a nonlinear wave equation, *Comment. Math. Helv.*, 71 (1996), 269–296.

J. Pöschel, A KAM-theorem for some nonlinear partial differential equations, *Ann. Sc. Norm. Super. Pisa*, 23 (2000), 119–148.

C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.*, 127 (1990), 479–528.

X. Yuan, Invariant manifold of hyperbolic-elliptic type for nonlinear wave equation, *Int. J. Math. Math. Sci.*, 2003 (2003), 1111–1136.

X. Yuan, Quasi-periodic solutions of completely resonant nonlinear wave equations, *J. Differential Equations*, 230 (2006), 213–274.

X. Yuan, Invariant tori of nonlinear wave equations with a given potential, *Discrete Contin. Dyn. Syst.*, 16 (2006), 615–634.

Received October 2015; revised November 2016.

E-mail address: 12110180001@fudan.edu.cn
E-mail address: xpyuan@fudan.edu.cn