TOROIDAL COMPACTIFICATIONS OF INTEGRAL MODELS OF
SHIMURA VARIETIES OF HODGE TYPE

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ABSTRACT. We construct smooth projective toroidal compactifications for the integral canonical models of Shimura varieties of Hodge type constructed by Kisin and Vasiu at primes where the level is hyperspecial. This construction is a consequence of the main result of the paper, which shows, without any unramifiedness conditions on the Shimura datum, that the Zariski closure of a Shimura sub-variety of Hodge type in a Chai-Faltings compactification always intersects the boundary in a relative Cartier divisor. This result also provides a new proof of Y. Morita’s conjecture on the everywhere good reduction of abelian varieties (over number fields) whose Mumford-Tate group is anisotropic modulo center. We also construct integral models of the minimal (Satake-Baily-Borel) compactification for Shimura varieties of Hodge type.

INTRODUCTION

Shimura varieties of Hodge type. This paper is concerned with constructing good compactifications for integral canonical models of Shimura varieties of Hodge type at primes where the level is hyperspecial. Given a Shimura datum \((G, X)\) equipped with an embedding \((G, X) \hookrightarrow (GSp(V, \psi), S^\pm)\) into a Siegel Shimura datum, and a suitably small compact open \(K \subset G(A_f)\), the Shimura variety \(\text{Sh}_K(G, X)\) can be viewed as a parameter space for polarized abelian varieties equipped with level structures and additional Hodge tensors.

If we are in the more familiar PEL setting, these additional Hodge tensors can be chosen to consist of endomorphisms and polarizations. One can then define representable PEL type moduli problems over the reflex field \(E = E(G, X)\), and even over a suitable localization of its ring of integers, which recover the moduli interpretation for \(\text{Sh}_K(G, X)\) over \(\mathbb{C}\), and are thus canonical models for \(\text{Sh}_K(G, X)\) over \(E\) or even its ring of integers; cf. [Del71] for the theory over \(E\), and [Kot92] for the integral theory (when the level at \(p\) is hyperspecial). The theory of [Del71] applies more generally to show that Shimura varieties of Hodge type admit canonical models over their reflex field, and Milne has used Deligne’s results on absolute Hodge cycles to give these canonical models a modular interpretation; cf. [Mil94].

Example. An important class of Shimura data of Hodge type arises from quadratic forms over \(\mathbb{Q}\) of signature \((n+, 2-)\). Suppose that we have a vector space \(U\) over \(\mathbb{Q}\) equipped with such a quadratic form. Then the group \(G = \text{GSpin}(U)\) acts naturally on the Clifford algebra \(C\) attached to \(U\). We can equip \(C\) with an appropriate symplectic form such that we have an embedding \(\text{GSpin}(U) \hookrightarrow \text{GSp}(C)\). Moreover, if we take \(X\) to be the space of negative definite oriented 2-planes in \(U\), then \((G, X)\) is a Shimura datum, and we in fact get an embedding \((G, X) \hookrightarrow (\text{GSp}(C), S^\pm)\) of Shimura data. This is the Kuga-Satake construction; cf. [Del72]. It is important, for example, in the study of the moduli of K3 surfaces (when \(n = 19\)). Moreover, the Shimura varieties attached to the Spin group Shimura data play a significant role in S. Kudla’s program (cf. [Kud04]) for relating intersection numbers on Shimura varieties with Fourier coefficients of Eisenstein series. \((G, X)\) is not of PEL type if \(n \geq 5\).

1Unless otherwise specified, we will consider the Shimura variety as a scheme over its reflex field.
2We now know that every Shimura variety admits such a canonical model; cf. [Mil99].
Compactifications in the case of hyperspecial level. Unfortunately, since Hodge cycles are still transcendently defined, there is no natural way to use them to obtain a modular interpretation over the ring of integers of $E$. So, to get a good integral model for $\text{Sh}_K(G, X)$, we have to resort to more ad hoc methods. Suppose that the level at $p$ is hyperspecial, and that $v|p$ is a place of $E$ above $p$. Suppose also that $p > 2$. Then, in [Kis10], Kisin constructed the integral canonical model $\mathcal{I}_K(G, X)_{\mathcal{O}_E,v}$ for $\text{Sh}_K(G, X)$ over the localization of $\mathcal{O}_E$ at $v$.

Since one of the main interests in having good integral models of Shimura varieties is to facilitate the computation of their zeta functions, and hence their cohomology, we are led to consider the question of their compactification. Over $\mathbb{C}$, Mumford and his collaborators (cf. AMRT10) constructed good, toroidal compactifications in the general setting of arithmetic quotients of hermitian symmetric domains. In [Har89] and [Pin90] these compactifications are constructed for Shimura varieties in their natural adelic setting. All these constructions depend on a choice of a certain cone decomposition $\Sigma$, called a smooth complete admissible rppcd (cf. 4.2 for the terminology). Given such a choice they produce a smooth compactification $\text{Sh}^\Sigma_K(G, X)$ of the Shimura variety $\text{Sh}_K(G, X)$. Our main result is the following theorem; cf. [4.6.13] for a more precise statement.

**Theorem 1.** Suppose again that $p > 2$. There is a co-final collection of smooth complete admissible rppcds $\Sigma$ for $(G, X, K)$ such that the integral canonical model $\mathcal{I}_K := \mathcal{I}_K(G, X)_{\mathcal{O}_E,v}$ admits a smooth toroidal compactification $\mathcal{I}_K^\Sigma$ that is a proper integral model of $\text{Sh}^\Sigma_K(G, X)$. In particular, étale locally around any point, the embedding $\mathcal{I}_K \subset \mathcal{I}_K^\Sigma$ is isomorphic to a torus embedding $T \subset \mathcal{T}$, and the boundary $\mathcal{I}_K^\Sigma \setminus \mathcal{I}_K$ is a relative normal crossings divisor over $\mathcal{O}_E,v$ that admits a stratification parameterized by a conical complex that can be described explicitly in terms of the Shimura datum $(G, X, K)$ and the rppcd $\Sigma$.

The original construction of such integral toroidal compactifications is due to Chai and Faltings ([FC90]) in the case of the Siegel Shimura datum. Their methods were amplified and extended to the case of Shimura varieties of PEL type by K.-W. Lan in [Lan08] (cf. also [Rap78] and [Lar92]). Our method of proof takes as input the existence of the Chai-Faltings compactifications, as well as the compactifications in characteristic 0 mentioned above. As such, it makes essential use of the compatibility between arithmetic and analytic compactifications proven in [Lan10a].

In [Kis10], Kisin also constructs integral canonical models for Shimura varieties of abelian type. It should be possible to extend his method to also construct compactifications for these models; for certain special cases, involving orthogonal Shimura varieties, cf. MP13b.

The restriction $p > 2$ is entirely because of a similar restriction in [Kis10]; cf. (4.6.5) for a discussion of this.

**Transversality.** A result along the lines of Theorem 1 in the more general setting of Pink’s mixed Shimura varieties, has been stated in [Hor10]. This, however, is contingent on a crucial conjecture ([Hor10 3.3.2]), which we prove along the way. In fact, it (or, rather, a strengthened form of it) is the key to the whole construction. Let us now describe it. We direct the reader to (4.6.27) for a more precise version of what follows.

We now drop the condition that $G$ is unramified at $p$, and also the condition $p > 2$. Suppose that we have an embedding $(G, X) \hookrightarrow (\text{GSp}(V), S^\pm(V))$ into a Siegel Shimura datum. For any compact open $K' \subset \text{GSp}(\mathbb{A}_f)$ with $K'_p$ hyperspecial, the integral model $\mathcal{I}_{K'}(\text{GSp}, S^\pm)$ over $\mathbb{Z}_p$ admits a toroidal compactification $\mathcal{I}_{K'}^\Sigma(\text{GSp}, S^\pm)$ constructed by Chai-Faltings [FC90], such that the boundary is a relative Cartier divisor over $\mathbb{Z}_p$.

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3We note that a construction due to Vasiu can be found in [Vas99].
Theorem 2. Suppose $K'$ is such that, with $K = G(\mathbb{A}_f) \cap K'$, the induced map of Shimura varieties

$$\text{Sh}_K(G, X) \to \text{Sh}_{K'}(\text{GSp}, S^\pm)_E$$

is a closed embedding. Then the Zariski closure of $\text{Sh}_K(G, X)$ in $\mathcal{S}_{K'}(\text{GSp}, S^\pm)_{\mathcal{O}_{E, (v)}}$ intersects the boundary in a relative Cartier divisor over $\mathcal{O}_{E, (v)}$.

Let us try to explain the main idea behind the proof. Denote the Zariski closure by $\overline{\mathcal{S}}_K$. The result is a local one and can be proven by considering the formal neighborhood of a closed point $x_0$ in $\mathcal{S}_K$. For simplicity, we assume now that $x_0$ is a closed stratum in $\mathcal{S}_{K'}(\text{GSp}, S^\pm)$. In this situation, for any lift $x \in \text{Sh}_K(G, X)(\overline{\mathbb{Q}})_p$ of $x_0$, the fiber $A_x$ of the universal abelian scheme will admit a rigid analytic uniformization $T^\text{an}/t(Y) \cong A_x^\text{an}$, where $T^\text{an} = \text{Hom}(X, G_m)$ is a split rigid analytic torus with character group $X$; $Y$ is a free abelian group of ‘periods’ with $\text{rk} Y = \text{rk} X$; and $t : Y \to T^\text{an}$ is a map of analytic groups.

Set $V_x = H^1_{dR}(A_x, \mathbb{Q}_p)$; then the uniformization endows $V_x$ with a three-step weight filtration $W^0 V_x = W^1 V_x = \text{Hom}(Y, \mathbb{Z}_p)$, and $g_{W^1} V_x = X \otimes \mathbb{Z}_p$.

Let $R$ be the complete local ring of $\mathcal{S}_{K'}(\text{GSp}, S^\pm)$ at $x_0$; it is a completed torus embedding over $W$ for a split torus $\mathcal{E}$, whose co-character group $\mathcal{B}$ is a lattice in $B(Y \otimes \mathbb{Q})$, the space of symmetric bi-linear pairings on $Y \otimes \mathbb{Q}$. Let $U \subset \text{GSp}(V_x)$ be the unipotent sub-group associated with the weight filtration $W^0 V_x$; then $\text{Lie} U \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is identified with $B(Y \otimes \mathbb{Q}_p) = \mathcal{B} \otimes \mathbb{Q}_p$.

Choose tensors $\{s_a\} \subset V^\otimes$ whose point-wise stabilizer is $G$. These give rise to Galois-invariant tensors $\{s_{a, \tilde{e}, x}\} \subset V^\otimes$. Let $G_x \subset \text{GSp}(V_x)$ be the point-wise stabilizer of the tensors $\{s_{a, \tilde{e}, x}\}$, and let $U_G = U \cap G_x$. Let $R_G$ be the quotient of $R$ corresponding to the completed torus embedding for the sub-torus $E_G \subset \mathcal{E}$ with co-character group $\mathcal{B}_G = \mathcal{B} \cap (\text{Lie} U_G \otimes \mathbb{Q}_p)$.

Let $\mathcal{Z}$ be the normalization of the irreducible component of $(\overline{\mathcal{S}}_K)_{x_0}$ passing through $x$. The theorem is proven in this case by showing that $\mathcal{Z}$ can be identified with an appropriate translate of the analytification of $\text{Spf} R_G$. To be precise, this statement holds if we assume that the normalization $\mathcal{S}_K$ of the Zariski closure of $\text{Sh}_K(G, X)$ in $\mathcal{S}_{K'}(\text{GSp}, S^\pm)_{\mathcal{O}_{E, (v)}}$ has reduced special fiber.

One non-obvious thing here is the fact that $R_G$ has the right dimension. This is equivalent to showing that $B_G \otimes \mathbb{Q}$ generates $\text{Lie} U_G \otimes \mathbb{Q}_p$ as a $\mathbb{Q}_p$-vector space; but a priori it is not even clear that $B_G$ is non-zero! Our approach to this problem is as follows: For any other lift $x' \in \text{Sh}_K(G, X)(\overline{\mathbb{Q}})_p$ of $x_0$ lying in the same irreducible component $\mathcal{Z}$, the associated semi-stable abelian variety $A_{x'}$ comes equipped with a monodromy pairing $N_{x'} : Y \times X \to \mathbb{Q}$. We show that this pairing gives rise to an element in $B_G \otimes \mathbb{Q}$. The key here is the existence of a global family of horizontal Hodge tensors over $\text{Sh}_K(G, X)$ arising from $\{s_a\}$, and the compatibility between $N_{x'}$ and the nilpotent operator $N$ on the weakly admissible $(\varphi, N)$-module attached to the $p$-adic Galois representation $H^1(A_{x'}, \mathbb{Q}_p)$. As $x'$ ranges over all lifts of $x_0$, a simple dimension counting argument shows that the monodromic elements $N_{x'}$ generate all of $\text{Lie} U_G \otimes \mathbb{Q}_p$.

The proof in general has the idea above as its essential kernel, but is more technically involved, since it needs to take into account the general case where the abelian part of the reduction is non-trivial, and also needs to account for some torsion phenomena when the special fiber of $\mathcal{S}_K$ is not reduced.

Chai-Faltings compactifications at places of bad reduction. The results above have the following application: Consider the moduli stack $\mathcal{M}_{g, p}$ (over $\mathbb{Z}$) of $g$-dimensional abelian schemes equipped with a polarization of degree $p^2$. Via Zarhin’s trick, it admits a map into the moduli stack $\mathcal{M}_{8g, 1}$ of principally polarized $8g$-dimensional abelian schemes. Theorem 2 shows that the closure of the image of $\mathcal{M}_{g, p}$ in any toroidal compactification of $\mathcal{M}_{8g, 1}$ intersects the boundary transversally. This allows us to prove:
Theorem 3. The toroidal compactifications of Chai-Faltings-Lan \[\text{[FC90, Lan08]}\] of \(M_{g,p}\) over \(\mathbb{Z}[(2p)^{-1}]\) can be extended over \(\mathbb{Z}[2^{-1}]\), and these extensions have the expected properties.

We refer the reader to (4.5.13) for details and precision. We note that there is already a compactification over \(\mathbb{Z}\) of \(M_{g,p}\) (and in fact of all moduli spaces \(M_{g,d}\) for arbitrary \(d\)) available via the work of Alexeev-Nakamura \[\text{[AN99]}\], Alexeev \[\text{[Ale02]}\] and Olsson \[\text{[Ols08]}\]. It is canonical in a very precise sense and has a natural moduli interpretation. However, as observed in the introduction to \[\text{[Lan08]}\], since it is attached to a specific cone decomposition, it seems ill-suited for the study of Hecke actions and other arithmetic information.

In fact, the methods of this paper apply to construct good compactifications of the integral models of many Shimura varieties of PEL type with parahoric level structure, subsuming, for example, the results of \[\text{[Lan08]}\] and \[\text{[Str10]}\]; cf. (4.5.16). Moreover, given \[\text{[PZ12]}\] and work in progress due to Kisin-Pappas, we expect that the methods of this paper will apply to construct compactifications of most Shimura varieties of Hodge type with parahoric level.

Morita’s conjecture. Theorem 2 also has the following pleasant consequence (cf. (4.4.9) in the body of the paper):

\[\text{Theorem 4. Suppose that } A \text{ is an abelian variety defined over a number field } F, \text{ and suppose that its Mumford-Tate group is anisotropic modulo its center. Then, for every finite place } v \text{ of } F, A \text{ has potentially good reduction over } F_v.\]

The hypothesis on the Mumford-Tate group ensures that \(A\) does not ‘degenerate in characteristic 0’. The theorem says that this is enough to keep it from degenerating in finite characteristic as well. This result gives a different proof of Y. Morita’s conjecture (see \[\text{[Mor75]}\]). Related results can be found in \[\text{[Pau04, Vas08]}\] and \[\text{[Lan10c]}\], with a proof of the full conjecture appearing in \[\text{[Lee12]}\]. The first two papers, as part of their hypotheses, impose certain local conditions on \(G\). In \[\text{[Lan10c]}\], Lan also proves the full conjecture as long as \(A\) appears in the family of abelian varieties over a compact Shimura variety of PEL type. Finally, Lee proves the full conjecture in \[\text{[Lee12]}\] using results of \[\text{[Pau04, Vas08]}\]. Our proof is independent of all these efforts, and applies uniformly without any consideration of special cases.

The minimal compactification. The toroidal compactifications of Mumford, et. al. are resolutions of the minimal or Baily-Borel-Satake compactification, which is important from the automorphic perspective, since its \(L^2\) or intersection cohomology is intimately related with the discrete automorphic spectrum of \(G\); cf. \[\text{[Mor10]}\]. Using by now standard techniques (cf. \[\text{[FC90, §V.2, Lan08, §7.2, Cha90]}\]), we can construct the integral model for the minimal compactification via the Proj construction applied to a certain graded ring of automorphic forms on \(\mathcal{S}_K^\mathbb{Z}\). This gives us the following theorem (cf. 4.8.11):

\[\text{Theorem 5. Suppose again that } G \text{ is unramified at } p \text{ with } p > 2 \text{ and that } K_p \text{ is hyperspecial. Then the minimal compactification of } Sh_K(G,X) \text{ admits a proper, normal model } \mathcal{S}_K^\text{min} \text{ over } \mathcal{O}_{E,(v)} \text{ that is stratified by quotients by finite groups of integral canonical models of Shimura varieties of Hodge type. Moreover, the Hecke action of } G(\mathbb{A}_f^p) \text{ on } \mathcal{S}_K \text{ extends naturally to an action on } \mathcal{S}_K^\text{min}. \text{ Given a complete admissible rppcd } \Sigma \text{ as in Theorem 4 there exists a unique map } p : \mathcal{S}_K^\mathbb{Z} \to \mathcal{S}_K^\text{min} \text{ that extends the identity on } \mathcal{S}_K \text{ and is compatible with the stratifications on domain and target.}\]

Tour of contents. We will now briefly describe the contents of the paper.

In \[\text{[2]}\] we review certain relevant results about log 1-motifs, their Dieudonné theory, and their connection to families of degenerating abelian varieties. Although the results of this paper could have been stated without reference to log 1-motifs for the most part, we believe that they are best phrased in this language. Moreover, log 1-motifs provide a more geometric interpretation
of the weakly admissible $(\varphi, N)$-module attached to the first $p$-adic cohomology group of a semi-stable abelian variety. They also allow us to give a new interpretation of the $p$-adic comparison and Hyodo-Kato isomorphisms for such an abelian variety, which is important when relating the Hyodo-Kato isomorphism to parallel transport between the de Rham cohomologies of the fibers of a semi-stable family of abelian varieties; cf. \cite{cho}. In §3 we review the construction by Chai and Faltings \cite{cho} of formal local models at the boundary of a toroidal compactification of the moduli space of polarized abelian schemes. Since we deal with level structures and not-necessarily-principal polarizations, we have elected to work in the more precise notation and setting of the work of K.-W. Lan \cite{lan}. We show that the completions of these local models at any point can be interpreted, in the logarithmic setting, as deformation rings for polarized log 1-motifs. We use this interpretation to write down explicit descriptions of these completions in terms of linear algebraic data, much as was done by Faltings in \cite{fal} for deformation rings of $p$-divisible groups. After this, in (3.3), we find the technical heart of the paper. The key result here is (3.3.29): it essentially describes the local structure at the boundary of a Shimura variety of Hodge type.

§4 is where we have our payoff, and it forms the bulk of this paper. We review some results about Shimura varieties and Hodge cycles on abelian varieties, as well as the characteristic 0 theory of toroidal compactifications, followed by the Chai-Faltings compactifications of the moduli of polarized abelian varieties. In (4.4), we present a proof of a precise version of Theorem 2, and deduce from it Morita’s conjecture. Theorems 3 and 4 also follow easily. We finish in (4.8) with the construction of the minimal compactification.

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1. Conventions

1. All rings and monoids will be commutative, unless otherwise noted.

2. For any prime $p$, $|\cdot|_p$ will denote the standard $p$-adic norm with $|p|_p = p^{-1}$.

3. If $L$ is a discrete valuation field, then $\mathcal{O}_L$ will denote its ring of integers and $m_L \subset \mathcal{O}_L$ its maximal ideal.

4. We will use the geometric notation for change of scalars. If $f : R \to S$ is a map of rings and $M$ is an $R$-module, then we will denote the induced $S$-module $M \otimes_{R,f} S$ by $f^* M$. If the map $f$ is clear from context, then we will also write $M_S$ for the same $S$-module.

5. If $\varphi : R \to R$ is an endomorphism of $R$, then a $\varphi$-module over $R$ is an $R$-module $M$ equipped with a map $\varphi^* M \to M$ of $R$-modules.

6. Suppose that $R$ is a ring and suppose that $C$ is an $R$-linear tensor category that is a faithful tensor sub-category of $\text{Mod}_R$, the category of $R$-modules. Suppose in addition that $C$ is closed under taking duals, symmetric and exterior powers in $\text{Mod}_R$. Then, for any object $D \in \text{Obj}(C)$, we will denote by $D^\oplus$ the direct sum of the tensor, symmetric and exterior powers of $D$ and its dual.

7. We will consistently identify the étale topoi of schemes with the same underlying reduced scheme. In particular, if $k$ is a field and $B$ is a local Artin ring with residue field
$k$, then we will, without comment, consider any étale sheaf over Spec $k$ as a sheaf over Spec $B$. 

Contents

Introduction 1
Tour of contents 4
Acknowledgements 5
1. Conventions 5
2. Preliminaries 6
2.1. Logarithmic preliminaries 6
2.2. Log 1-motifs and log $F$-crystals 8
2.3. Degenerating abelian varieties 11
2.4. $p$-adic Hodge theory 12
3. At the boundary of a Chai-Faltings compactification 16
3.1. Chai-Faltings local models as deformation spaces of log 1-motifs 16
3.2. Explicit co-ordinates for Chai-Faltings local models 22
3.3. Tate tensors 28
4. Compactifications of Shimura varieties of Hodge type 38
4.1. Shimura varieties and absolute Hodge cycles 38
4.2. Compactifications in characteristic 0 40
4.3. Chai-Faltings compactifications 47
4.4. Intersection with the boundary and Morita’s conjecture 49
4.5. Compactifying $M_{V,Z,n,\psi, Z}(p)$ when $p^2 \nmid d$ 53
4.6. Smooth compactifications of Hodge type 57
4.7. Hecke action 61
4.8. The minimal compactification 62
References 68

2. Preliminaries

2.1. Logarithmic preliminaries. We assume that the reader is familiar with the basics of log geometry. References include [Kat89] and [Niz08].

2.1.1. We recall that a log scheme is a pair $(S, M_S)$ consisting of a scheme $S$ and an étale sheaf of commutative monoids $M_S$ over $S$ equipped with a map of monoids $\alpha : M_S \to S$ such that $\alpha^{-1}(O_S^\times) \to O_S^\times$ is an isomorphism. If $S = \text{Spec} A$ is affine, then, abusing terminology, we will refer to $A$ as a log ring or log algebra. If $P$ is an adjective applied to commutative monoids, we will say that $(S, M_S)$ is $P$ if, for all geometric points $s \to S$, the monoid $M_{S,s}/O_{S,s}$ is $P$. Here is a list of such adjectives:

- A monoid $P$ is cancellative if the map into its group envelope $P^{gp}$ is injective.
- It is fine if it is cancellative and finitely generated
- It is saturated if it is cancellative, and if, for any $a \in P^{gp}$, $a^n \in P$, for some $n \in \mathbb{Z}_{>0}$ if and only if $a \in P$.
- It is fs if it is fine and saturated.

Definition 2.1.2. A map of monoids $f : P \to Q$ is continuous if an element $a \in P$ is invertible if and only if $f(a)$ is invertible in $Q$.

A map $f : (S, M_S) \to (T, M_T)$ of log schemes is continuous if, for every geometric point $s : \overline{s} \to S$, the map $M_{T,f(\overline{s})}/O_{T,f(\overline{s})}^\times \to M_{S,s}/O_{S,s}$ is continuous.
If $X$ is an object over $(S, M_S)$ (for a suitable sense of ‘over’), we will allow ourselves to slightly abuse terminology and to refer to $X$ as an object over $S$.

2.1.3. Let $(S, M_S)$ be an fs log scheme. We have the functor $G^\log_m$ on fs log schemes over $(S, M_S)$ given by

$$G^\log_m : (T, M_T) \to \Gamma(T, M_{\text{log}}^m).$$

For the **Kummer log flat topology** on the category of fs log schemes over $(S, M_S)$ (this is a topology refining the fppf topology on $S$; cf. [Niz08, 2.13]), $G^\log_m$ is a sheaf of abelian groups $\Gamma_m$. Let $\mathcal{S}_n^{\log}$ (resp. $\mathcal{S}_n^{\text{cl}}$) be the Kummer log flat (resp. the classical fppf) site over $S$. We have a natural morphism of sites $\epsilon : \mathcal{S}_n^{\log} \to \mathcal{S}_n^{\text{cl}}$. For any fppf sheaf $H$ over $S$, we will denote its pull-back $\epsilon^*H$ over the Kummer log flat site also by $H$.

The étale sheaf of abelian groups $M_{\text{log}}^m$ extends to the fppf sheaf $\epsilon_*G^\log_m$, we will denote this extension also by $M_{\text{log}}^m$. We have a short exact sequence of fppf sheaves

$$1 \to \mathcal{E}_S^\log \to M_{\text{log}}^m \to (M_{\text{log}}^m / \mathcal{E}_S^\log) \to 1. \tag{2.1.3.1}$$

For any $n \geq 1$, let $\mu_n$ be the sheaf over $\mathcal{S}_n^{\text{cl}}$ of $n$th-roots of unity; then we have the **Kummer short exact sequence** of Kummer log flat sheaves:

$$1 \to \mu_n \to G^\log_m \xrightarrow{\text{tr}_n} G^\log_m \to 1. \tag{2.1.3.2}$$

**Proposition 2.1.4.**

1. For any sheaf of abelian groups $G$ over $\mathcal{S}_n^{\text{cl}}$, there exists a natural map

$$\eta_G : \varprojlim_n \text{Hom}(\mu_n, G) \otimes (M_{\text{log}}^m / \mathcal{E}_S^\log) \to R^1\epsilon_*G.$$  

2. Suppose that $S$ is locally Noetherian. If $G$ is representable and is either smooth or finite flat over $S$, then $\eta_G$ is an isomorphism.

3. Suppose that $S = \text{Spec } R$, for a Noetherian local ring $R$. For any $n \in \mathbb{Z}_{>0}$, there is a natural short exact sequence

$$0 \to H^1(\mathcal{S}_n^{\text{cl}}, \mu_n) \to H^1(\mathcal{S}_n^{\log}, \mu_n) \to (\mathbb{Z}/n\mathbb{Z}) \otimes M \to 0,$$

where $M = H^0(\mathcal{S}_n^{\text{cl}}, G^\log_m / G_m)$.

**Proof.** Statements (1) and (2) are from [Niz08, Theorem 3.12] (the results are originally from the unpublished article [Kata]).

For (3), we use the Leray spectral sequence for the functors $\epsilon$ and $H^0(\mathcal{S}_n^{\text{cl}}, \_)$ to get an exact sequence:

$$0 \to H^1(\mathcal{S}_n^{\text{cl}}, \mu_n) \to H^1(\mathcal{S}_n^{\log}, \mu_n) \to H^0(\mathcal{S}_n^{\text{cl}}, R^1\epsilon_*\mu_n) = \text{End}_{\mathcal{S}_n^{\text{cl}}} (\mu_n) \otimes M.$$

All it remains to do is to show that the map

$$H^1(\mathcal{S}_n^{\log}, \mu_n) \to \text{End}_{\mathcal{S}_n^{\text{cl}}} (\mu_n) \otimes M = (\mathbb{Z}/n\mathbb{Z}) \otimes M$$

is surjective. Choose some element $\overline{m}$ in the right hand side, an element $m \in M$ lifting $\overline{m}$, and an element $\overline{m} \in H^0(\mathcal{S}_n^{\log}, G^\log_m)$ lifting it (we can always do this: use the exact sequence (2.1.3.1) and the fact that $H^1(\mathcal{S}_n^{\text{cl}}, \mathcal{E}_S^\log) = 0$). The long exact sequence of cohomology associated with the Kummer sequence (2.1.3.2) gives us a boundary map

$$\partial_n : H^0(\mathcal{S}_n^{\log}, G^\log_m) \to H^1(\mathcal{S}_n^{\log}, \mu_n).$$

One can check that $\partial_n(\overline{m}) \in H^1(\mathcal{S}_n^{\log}, \mu_n)$ maps onto $\overline{m} \in (\mathbb{Z}/n\mathbb{Z}) \otimes M$. \qed
2.2. Log 1-motifs and log $F$-crystals. In this sub-section, we will assume that the reader is familiar with the notion of a bi-extension; cf. [Del74] §10.2, [SGA71] VII (2.1) for details. For the theory of 1-motifs, cf. [Del74] §10 and [ABV05]; for that of log 1-motifs, cf. [KKN08b] and [KT03] §4.6.

For any pair $(H,G)$ of sheaves of groups over a scheme $S$, we will denote by $1_{H \times G}$ the trivial $G_m$-bi-extension of $H \times G$; similarly, $1_{H \times G}^{log}$ will denote the trivial $G_m^{log}$-bi-extension of $H \times G$.

**Definition 2.2.1.** A log 1-motif $Q$ over an fs log scheme $(S,M_S)$ is a tuple $(B,Y,X,c,c',\tau)$, where:

- $B$ is an abelian scheme over $S$, which we will denote $Q^{ab}$.
- $Y$ and $X$ are étale sheaves of free abelian groups of finite rank over $S$, trivialized over a finite étale cover of $S$. We will denote them as $Q^{\acute{e}t}$ and $Q^{mult,C}$, respectively.
- $c : Y \to B$ and $c' : X \to B'$ are maps of sheaves of groups over $S$. We will denote them by $c_Q$ and $c'_Q$, respectively.
- $\tau : 1^{log}_{Y \times X} \xrightarrow{\simeq} (c \times c')^*P_{log}^{B}$ is a trivialization of $G_m^{log}$-bi-extensions of $Y \times X$. We will denote it by $\tau_Q$.

Here, $P_B$ is the Poincaré $G_m$-bi-extension of $B \times B'$, and $P_{log}^{log}$ is the associated $G_m^{log}$-bi-extension.

A map $\varphi : Q_1 \to Q_2$ of log 1-motifs is a tuple $(\varphi^{ab}, \varphi^{\acute{e}t}, \varphi^{mult,C})$, for $? = ab, \acute{e}t, \varphi^Q : Q_1^? \to Q_2^?$, is a map of sheaves over $S$ and $\varphi^{mult,C} : Q_2^{mult,C} \to Q_1^{mult,C}$. The tuple satisfies:

$c_{Q_2}^{\varphi^{\acute{e}t}} = \varphi^{ab}c_{Q_1}^?, c_{Q_1}^{\varphi^{mult,C}} = \varphi^{ab,\varphi^Q,\varphi'}_{? \times Q_2^?}$, and a certain compatibility between $\tau_{Q_1}$ and $\tau_{Q_2}$, for which we direct the reader to [Del74] 10.2.12.

The dual $Q^\vee$ of a log 1-motif $Q$ is the tuple $((Q^{ab})^\vee, Q^{mult,C}, Q^{\acute{e}t}, c, c', \tau^\vee)$, where $\tau^\vee$ is the trivialization of the $G_m^{log}$-bi-extension $(c^\vee_{Q} \times Q)^*P_{log}^{(Q^{ab})^\vee}$ induced from $\tau$ via the symmetricity of the Poincaré bi-extension.

A polarization of a log 1-motif $Q$ is a map $\lambda : Q \to Q^\vee$ such that $\lambda^{ab} : Q^{ab} \to (Q^{ab})^\vee$ is a polarization, and such that $\lambda^{\acute{e}t} : Q^{\acute{e}t} \to Q^{mult,C}$ is injective.

There is a canonical weight filtration $W_iQ$ of a log 1-motif $Q$ with:

$$W_iQ = \begin{cases} 
0, & \text{if } i < -2; \\
(0,0, Q^{mult,C}, 0,0,1), & \text{if } i = -2; \\
(Q^{ab}, 0, Q^{mult,C}, 0, c^\vee_{Q},1), & \text{if } i = -1; \\
Q, & \text{if } i = 0.
\end{cases}$$

A 1-motif $Q$ is a log 1-motif $(B,Y,X,c,c',\tau)$, where the trivialization $\tau$ arises from a trivialization of the $G_m$-bi-extension $(c \times c')^*P_{B}$.

**Remark 2.2.2.** If $Q$ is a 1-motif, then we can think of it, as is done in [Del74] §10, as a two-term complex $[Q^{\acute{e}t} \to Q^{ab}]$, where $Q^{ab}$ is the semi-abelian extension of $Q^{ab}$ classified by $c_Q^{ab}$. Something similar is true for log 1-motifs as well: we can think of a log 1-motif $Q$ as a two term complex $[Q^{\acute{e}t} \to Q^{ab, log}]$ of Kummer log flat sheaves over $(S,M_S)$, where $Q^{ab, log}$ is a logarithmic enhancement of $Q^{ab}$. This is the point of view taken in [KKN08b]; cf. §2.1 of loc. cit. In the case where $Q^{ab}$ is a torus with character group $X$, we set

$$Q^{ab, log} = \text{Hom}(X,G_m^{log}).$$
In the general case, we take $Q^{\text{ab,log}}$ to be the push-out of the diagram

$$
Q^{\text{mult}} \rightarrow Q^{\text{ab}} \simeq \bigcup Q^{\text{log}}_{\text{mult}, \log}
$$

**Definition 2.2.3.** Given a log 1-motif $Q = (B, Y, X, c, c', \tau)$ over $(S, M_S)$, the $\mathbb{G}_{m}^{\text{log}}/\mathbb{G}_{m}$-extension of $Y \times X$ induced from $(c \times c')^* P_{B}$ is canonically trivialized, and so the trivialization $\tau$ determines a pairing $N_\tau : Y \times X \rightarrow \mathbb{G}_{m, S}/\mathbb{G}_{m, S}$. For any geometric point $\overline{s} \rightarrow S$, the **monodromy of $Q$ at $\overline{s}$**, denoted $N_{\tau, \overline{s}}$ is the induced pairing $Y_{\overline{s}} \times X_{\overline{s}} \rightarrow M_{S, \overline{s}}^{\text{gp}}/\mathcal{O}_{S, \overline{s}}^{\times}$.

It is easy to see that a log 1-motif is a classical 1-motif precisely when $N_{\tau, \overline{s}}$ is trivial for all geometric points $\overline{s} \rightarrow S$.

**Definition 2.2.4.** Let $Q$ be a log 1-motif over $(S, M_S)$, thought of as a two-term complex $[Q^{\text{et}} \rightarrow Q^{\text{ab,log}}]$ as in (2.2.2). For any prime $p$, and any $n \in \mathbb{Z}_{>0}$, the $p^n$-torsion $Q[p^n]$ of $Q$ is the Kummer log flat sheaf of groups $H^{-1}(\text{cone}(Q \overset{p^n}{\rightarrow} Q))$.

The $(\log)$ $p$-divisible group $Q[p^n]$ attached to $Q$ is the Kummer log flat sheaf of groups $\bigcup_n Q[p^n]$. When $Q$ is a classical 1-motif, this is a $p$-divisible group over $S$ in the classical sense.

There is a perfect pairing $Q[p^n] \times Q'[p^n] \rightarrow \mu_{p^n}$ identifying $Q'[p^n]$ with the Cartier dual of $Q[p^n]$ defined precisely as in [Del74, 10.2.5].

**Proposition 2.2.5.** There is a contra-variant exact de Rham realization functor $H_{\text{DR}}^1 : (\log 1$-motifs over $S) \rightarrow (\text{filtered locally free } \mathcal{O}_S\text{-modules})$ satisfying the following properties:

1. For every log 1-motif $Q$, the (descending) filtration $\text{Fil}^i H_{\text{DR}}^1(Q)$ is concentrated in degree $1$, with $\text{Fil}^1 H_{\text{DR}}^1(Q) = \text{Lie}(Q^{\text{ab}})^\vee$.
2. The restriction of $H_{\text{DR}}^1$ to the category of 1-motifs is the (dual of the) classical de Rham realization functor defined in [Del74, 10.10].
3. If $Q^{\text{ab}} = 0$, and $\underline{Y} = Q^{\text{et}}, \underline{X} = Q^{\text{mult}, C}$, then $H_{\text{DR}}^1(Q) = \underline{\text{Hom}}(\underline{Y}, \mathcal{O}_S) \oplus \underline{X} \otimes \mathcal{O}_S; \text{Fil}^1 H_{\text{DR}}^1(Q) = \underline{X} \otimes \mathcal{O}_S$.
4. For every log 1-motif $Q$, there is a canonical perfect pairing $H_{\text{DR}}^1(Q) \times H_{\text{DR}}^1(Q') \rightarrow \mathcal{O}_S$.
5. Suppose that we have a map $f : (T, M_T) \rightarrow (S, M_S)$ of fs log schemes, whose underlying map of schemes is quasi-separated. For every log 1-motif $Q$ over $S$, there is a natural isomorphism of $\mathcal{O}_T\text{-modules}$

$$
\left(f^* H_{\text{DR}}^1(Q) \right)^{\vee} \xrightarrow{\simeq} H_{\text{DR}}^1(f^* Q).
$$

**Proof.** In [Del74], the (co-variant) de Rham realization is defined as the Lie algebra of the universal vector extension $E(Q)$ of the two-term complex attached to $Q$. For a general log 1-motif $Q$, viewing $Q/W_{-2}Q$ and $Q$ as two-term complexes, we set

$$
E(Q) = E(Q/W_{-2}Q) \times_{Q/W_{-2}Q} Q.
$$

Then $H_{\text{DR}}^1(Q) = (\text{Lie} E(Q))^{\vee}$, and $\text{Fil}^1 H_{\text{DR}}^1(Q)$ is the image of $\Omega^1_{Q^{\text{ab}}/S}$ in $H_{\text{DR}}^1(Q)$. That this agrees with the definition for 1-motifs follows from [ABV05, 2.4].

The asserted properties are now immediate from their validity in the case of 1-motifs, including (4), for which cf. [Del74, 10.2.7].
2.2.6. Fix a prime $p$. We will assume now that all our log schemes $(S, M_S)$ satisfy one of the following conditions:
- $S = \text{Spec } R$, for $R$ a $p$-adically complete $\mathbb{Z}_p$-algebra; or
- $p$ is nilpotent in $S$.

In this situation, one can define the \textit{log crystalline site} $(\mathcal{O}_{S, \text{cris}})_{\text{log}}$ and can therefore speak of \textit{log crystals} (of locally free $\mathcal{O}_{S, \text{cris}}$-modules) over $(S, M_S)$. Let $\varphi$ be the absolute Frobenius on $(S, M_S) \otimes \mathbb{F}_p$; then we can also speak of \textit{log $F$-crystals} over $(S, M_S)$: these consist of a pair $(\mathcal{M}, \varphi_\mathcal{M})$, where $\mathcal{M}$ is a log crystal over $(S, M_S)$ and $\varphi_\mathcal{M} : \varphi^* \mathcal{M} \to \mathcal{M}$ is a map of log crystals. Here, for any logarithmic crystal $\mathcal{M}$ over $\mathcal{O}_{S, \text{cris}}$, its Frobenius pull-back $\varphi^* \mathcal{M}$ is the logarithmic crystal of $\mathcal{O}_{S, \text{cris}}$-modules and its evaluation any object $(U, M_U) \to (T, M_T)$ that admits a Frobenius lift $\varphi$ is simply given by $\varphi^* \mathcal{M}((T, M_T))$.

\textbf{Definition 2.2.7.} A log Dieudonné $F$-crystal over $S$ is a tuple $(\mathcal{M}, \varphi_\mathcal{M}, V_\mathcal{M}, \text{Fil}^1 \mathcal{M}(S))$, where $(\mathcal{M}, \varphi_\mathcal{M})$ is a log $F$-crystal over $S$, and:
- $V_\mathcal{M} : \mathcal{M} \to \varphi^* \mathcal{M}$ is a map of logarithmic crystals such that $\varphi_\mathcal{M} V_\mathcal{M} = p$.
- $\text{Fil}^1 \mathcal{M}(S) \subset \mathcal{M}(S)$ is a direct summand (called the Hodge filtration) such that
  $$\varphi^* (\text{Fil}^1 \mathcal{M}(S_0)) = \ker (\varphi_{\mathcal{M}(S_0)} : \varphi^* \mathcal{M}(S_0) \to \mathcal{M}(S_0)).$$

Here, $\text{Fil}^1 \mathcal{M}(S_0) = \text{Fil}^1 \mathcal{M}(S) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \subset \mathcal{M}(S_0)$.

We will usually refer to such a tuple simply as the log Dieudonné $F$-crystal $\mathcal{M}$.

A log Dieudonné $F$-crystal is a \textbf{Dieudonné $F$-crystal} if the underlying log $F$-crystal $\mathcal{M}$ arises from a classical $F$-crystal over $S$.

We will write $\mathbf{1}$ for the trivial crystal over $S$; this is naturally a Dieudonné $F$-crystal when equipped with $\varphi_1 = 1$, and $V_\mathbf{1} = p$, with $\text{Fil}^1 \mathbf{1}(S)$ set to the zero summand of $\mathbf{1}(S) = S$. The \textbf{Tate twist} $\mathbf{1}(1)$ is the Dieudonné $F$-crystal whose underlying crystal is still the trivial crystal, but $\varphi_{\mathbf{1}(1)} = p$, $V_{\mathbf{1}(1)} = 1$, and $\text{Fil}^1 \mathbf{1}(1)(S) = \mathbf{1}(1)(S) = S$.

For any log Dieudonné $F$-crystal $(\mathcal{M}, \varphi_\mathcal{M}, V_\mathcal{M}, \text{Fil}^1 \mathcal{M}(S))$ over $S$, its \textbf{Cartier dual} is the log Dieudonné $F$-crystal determined by the tuple

$$(\mathcal{M}^\vee, \varphi_{\mathcal{M}^\vee}, V_{\mathcal{M}^\vee}, \text{Fil}^1 \mathcal{M}^\vee(S)) = (\mathcal{M}^\vee, V_{\mathcal{M}^\vee}, \varphi_{\mathcal{M}^\vee}, (\mathcal{M}(S)/\text{Fil}^1 \mathcal{M}(S))^\vee).$$

There is a natural perfect pairing $\mathcal{M} \times \mathcal{M}^\vee \to \mathbf{1}(1)$ of log Dieudonné $F$-crystals.

\textbf{Proposition 2.2.8.} There is an exact contra-variant functor

$$\mathcal{D} : (\log 1\text{-motifs over }S) \to (\log \text{Dieudonné F-crystals over }S)$$

with the following properties:

1. For every log 1-motif $Q$, there is a canonical identification of filtered $\mathcal{O}_S$-modules $$(\mathcal{D}(Q)(S), \text{Fil}^1 \mathcal{D}(Q)(S)) = (H^1_\text{dR}(Q), \text{Fil}^1 H^1_\text{dR}(Q)).$$
2. If $Q$ is a classical 1-motif, then we have a canonical identification of $F$-crystals $\mathcal{D}(Q) = \mathcal{D}(Q[p^\infty])$, where the right hand side is the classical Dieudonné crystal associated with the $p$-divisible group $Q[p^\infty]$ \cite{Mess72}.
3. For any log 1-motif $Q$ over $S$, there is a natural perfect pairing

$$\mathcal{D}(Q) \times \mathcal{D}(Q^\vee) \to \mathbf{1}(1)$$

identifying $\mathcal{D}(Q^\vee)$ with $\mathcal{D}(Q)^\vee$.
4. If we have a map $f : (T, M_T) \to (S, M_S)$ of fs log schemes, such that the underlying map of schemes is quasi-separated; then, for any log 1-motif $Q$ over $T$, there is a natural isomorphism

$$\mathcal{D}(f^*Q) \cong f^* \mathcal{D}(Q).$$
Definition 2.3.1. A polarized log 1-motif over \((S, M_S)\) is a tuple \((Q, \lambda)\), where \(Q\) is a log 1-motif over \((S, M_S)\) and \(\lambda\) is a polarization of \(Q\).

We can also think of a polarized log 1-motif as an 8-tuple \((B, Y, \underline{X}, c, c^\vee, \lambda^{ab}, \lambda^{\text{et}}, \tau)\), where \(\tau\) is such that \((1 \times \lambda^{\text{et}})^* \tau\) is a symmetric trivialization of the symmetric \(G_\text{m}^{\log}\)-bi-extension \((c \times c^\vee \lambda^{\text{et}})^* \mathcal{P}_B\) of \(Y \times Y\). This gives us a tuple very much like the ones appearing in the category \(\mathcal{D}_\text{pol}\) considered in [F-C90] §III.2.

Definition 2.3.2. Let \((Q, \lambda)\) be a polarized log 1-motif corresponding to a tuple \((B, Y, \underline{X}, c, c^\vee, \lambda^{ab}, \lambda^{\text{et}}, \tau)\).

We will say that \((Q, \lambda)\) is \textbf{positively polarized} if, for every geometric point \(\bar{s} \to S\), the pairing

\[
Y \times Y \xrightarrow{1 \times \lambda^{\text{et}}} Y \times X \xrightarrow{N_{\tau,s}} M^{\text{gp}}_{S,\bar{s}} / \mathcal{O}^\times_{S,\bar{s}}
\]

is positive definite. Here, we say that an element in \(M^{\text{gp}}_{S,\bar{s}} / \mathcal{O}^\times_{S,\bar{s}}\) is positive if it lies in \((M_{S,\bar{s}} / \mathcal{O}^\times_{S,\bar{s}}) \setminus \{1\}\).

A log 1-motif \(Q\) is \textbf{positively polarizable} if there exists a polarization \(\lambda\) such that \((Q, \lambda)\) is positively polarized.

2.3.3. Suppose now that \(S = \text{Spec} \, R\), where \(R\) is a complete local normal Noetherian ring, and suppose that the log structure on \(S\) is defined by a divisor \(D \subset S\). Let \(U \subset S\) be the complement of \(D\). Let \(\text{DEG}_{\text{pol}}(S)\) be the category of positively polarized log 1-motifs \((Q, \lambda)\) over \(S\). Let \(\mathcal{D}_{\text{pol}}(S)\) be the category of pairs \((A, \lambda)\), where \(A\) is a semi-abelian scheme whose restriction \(A_{|U}\) to \(U\) is an abelian scheme over \(U\), and \(\lambda\) is a polarization of \(A_{|U}\).

Theorem 2.3.4. With the hypotheses as above, there is a functorial (in \(S\)) exact equivalence of categories

\[
M_{\text{pol}, S}: \mathcal{D}_{\text{pol}}(S) \xrightarrow{\cong} \text{DEG}_{\text{pol}}(S).
\]

Moreover, suppose we have \((Q, \lambda) \in \mathcal{D}_{\text{pol}}(S)\) with \((A, \lambda) = M_{\text{pol}, S}((Q, \lambda))\) in \(\text{DEG}_{\text{pol}}(S)\). Then:

1. For every prime \(\ell\), there is a canonical identification of \(\ell\)-divisible groups \((A_{|U})[\ell^\infty] = Q[\ell^\infty]_{|U}\), compatible with the Weil pairings induced by the polarizations.

2. There is a canonical identification \(H^1_{\text{dR}}(A_{|U}) = H^1_{\text{dR}}(Q_{|U})\) of filtered \(\mathcal{O}_U\)-modules, compatible with the pairings induced by the polarizations.

Proof. The construction/proof can be found in [F-C90] Ch. III]; cf. also [Lan08] 4.4.16]. Note that when \(R = \mathcal{O}_K\) is the ring of integers of a \(p\)-adic field, and \(D = \text{Spec} \, k\) is its special point, then the result is due to Raynaud [Ray71]. \(\square\) follows from [Lan08] 4.5.3.10).

Let us show (2). First, suppose that the divisor \(D \subset S\) does not contain the reduced sub-scheme underlying the special fiber \(S \otimes \mathbb{F}_p\). If \((A, \lambda)\) is a polarized abelian scheme over \(U\) corresponding to the polarized log 1-motif \((Q, \lambda')\) over \(S\), then both \(H^1_{\text{dR}}(A)\) and \(H^1_{\text{dR}}(Q_{|U})\) can be naturally identified with the de Rham cohomology of the universal vector extension \(p\)-divisible group \(A[p^\infty] = (Q_{|U})[p^\infty]\) over \(U\).

In general, after a finite étale base change if necessary, we can assume that \((Q, \lambda')\) corresponds to a tuple \((B, \lambda^{\text{ab}}, Y, X, \lambda^{\text{et}}, c, c^\vee, \tau)\), where \(Y\) and \(X\) are constant. Using [Lan08] 6.4.1.1(6)
one can now show that there exists a complete local ring $R'$ at the boundary of a smooth toroidal compactification of an appropriate moduli space of polarized abelian varieties, equipped with a ‘universal’ tuple $(B', \lambda^{ab}, Y, X, \lambda^{et}, c', e^{c'}, \tau')$, and a map $R' \to R$ that gives rise to $(Q, \lambda')$ over $S$ and the corresponding polarized abelian scheme $(A, \lambda)$ over $U$. Since $R'$ is log formally smooth over $\mathbb{Z}_p$, the boundary divisor in Spec $R'$ does not contain the special fiber of Spec $R'$, and we conclude now from the previous paragraph. □

2.4. $p$-adic Hodge theory.

2.4.1. Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors with coefficients in $k$ equipped with its Frobenius lift $\varphi$, and let $K_0 = W[p^{-1}] \subset K$ be the maximal absolutely unramified sub-field. Fix an algebraic closure $\overline{K}$ for $K$, and let $\overline{K}$ be its completion. Let $G_K = \text{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$.

Fix a uniformizer $\pi \in K$; let $E(u) \in W[u]$ be the associated monic Eisenstein polynomial, so that we can view $\overline{\mathcal{O}}_K$ as the quotient $W[u]/(E(u))$. Let $S$ be the $p$-adic completion of the divided power envelope of the surjection $W[u] \twoheadrightarrow \overline{\mathcal{O}}_K$ carrying $u$ to $\pi$, and set $\text{Fil}^1 S = \ker(S \to \mathcal{O}_K)$: by construction $\text{Fil}^1 S$ is equipped with divided powers compatible with those on $pS$. Concretely, we have (cf. [Bre00, 2.1.1]):

$$S = \left\{ \sum_{i} a_i \frac{u^i}{q(i)!} \in K_0[[u]] : a_i \in W, \lim_{i \to \infty} a_i = 0 \right\}.$$  

For every $i \in \mathbb{Z}_{\geq 0}$, let $\text{Fil}^i S = (\text{Fil}^1 S)^{[i]}$ be the $i^{th}$ divided power of $\text{Fil}^1 S$.

$S$ is equipped with the log structure induced by the divisor $u = 0$, and the natural log structure on $\mathcal{O}_K$ is induced from this one via the surjection $S \twoheadrightarrow \mathcal{O}_K$.

2.4.2. Consider the category of pairs $(A, \alpha)$ consisting of a $p$-adically complete $W$-algebra $A$, and a surjection $\alpha : A \twoheadrightarrow \overline{\mathcal{O}}_K$ whose kernel is equipped with divided powers compatible with the canonical divided power structure on $pR$. By [Fon94a, 2.2.1], this category has an initial object $(A_{\text{cris}}, \theta)$. Put differently, for every $n \in \mathbb{Z}_{>0}$, the PD thickening

$$\text{Spf} \theta_n : \text{Spf} \overline{\mathcal{O}}_K/p \hookrightarrow \text{Spec} A_{\text{cris}}/p^n$$

is the final object in the crystalline site $((\overline{\mathcal{O}}_K/p)/(W/p^n))_{\text{cris}}$.

Concretely, $A_{\text{cris}}$ is constructed as follows. Let

$$R = \lim_{\overset{\longrightarrow}{x \to x^p}} \overline{\mathcal{O}}_K/p$$

be the perfection of $\overline{\mathcal{O}}_K/p$: it consists of coherent sequences of $p$-power roots of elements of $\overline{\mathcal{O}}_K/p$, and the elements of its underlying multiplicative monoid can also be interpreted as coherent sequences of $p$-power roots of elements of $\overline{\mathcal{O}}_K$. Let $W(R)$ be the ring of Witt vectors with coefficients in $R$; there is a natural surjection $\theta : W(R) \twoheadrightarrow \overline{\mathcal{O}}_K$ and $A_{\text{cris}}$ is the $p$-adic completion of the divided power envelope of $\theta$. Note that $G_K$ naturally acts on everything in sight.

By construction, we have

$$A_{\text{cris}} = \lim_{\overset{\longrightarrow}{n}} H^0 \left( ((\overline{\mathcal{O}}_K/p)/(W/p^n))_{\text{cris}}, \mathcal{O}_{\text{cris}} \right),$$

where $\mathcal{O}_{\text{cris}}$ is the crystalline structure sheaf. In particular, $A_{\text{cris}}$ is canonically endowed with a Frobenius lift $\varphi$ that commutes with the $G_K$-action.

Consider the Teichmüller lift $\lfloor \cdot \rfloor : R \setminus \{0\} \to A_{\text{cris}} \setminus \{0\}$: it is a map of multiplicative monoids. We also have a natural injection $\mathbb{Z}_p(1) \hookrightarrow R^\times$, where we view an element $\epsilon \in \mathbb{Z}_p(1)$ as a coherent sequence $\epsilon \in R^\times$ of $p$-power roots of unity. Composing this with the
2.4.5. Suppose that we have a log 1-motif \( T \) over \( \mathcal{O}_K \). For any element \( [\epsilon] \) in its image, the series \( \sum_n (-1)^{n-1}(n-1)!(\epsilon - 1)^n \) converges to an element \( t(\epsilon) \in A_{\text{cris}} \), and the assignment \( \epsilon \mapsto t(\epsilon) \) gives a Galois equivariant embedding \( t : \mathbb{Z}_p(1) \rightarrow A_{\text{cris}} \).

2.4.3. Consider the category of pairs \( (B, \beta) \), where \( B \) is a \( p \)-adically complete log \( S \)-algebra and \( \beta : B \rightarrow \mathcal{O}_K^\wedge \) is a formal log PD thickening. By [Lod07, Theorem 1.1], this category has an initial object \( (B^\dlog_0, \beta_0) \). Here, \( \mathcal{O}_K^\wedge \) is viewed as the direct limit of the fs log \( S \)-algebras \( \mathcal{O}_L \), where \( L \) ranges over all finite extensions of \( K \), and so inherits the direct limit log structure; this log structure is not fs, but it is saturated.

This object can also be described explicitly. Let \( A_{\text{st}} \) be the \( p \)-adic completion of the divided power algebra \( A_{\text{cris}}(X) \) in one variable over \( A_{\text{cris}} \). For any choice \( \pi \in R \) of a coherent sequence of \( p \)-power roots of \( \pi \), we can view \( A_{\text{st}} \) as an \( S \)-algebra via the map \( u \mapsto [\pi](X + 1)^{-1} \). We can also endow it with the log structure associated with the pre-log structure

\[
(R \setminus \{0\}) \oplus \mathbb{N} \rightarrow A_{\text{st}} \quad (r, i) \mapsto [r](\pi)(X + 1)^{-1}.
\]

With this log structure, \( A_{\text{st}} \) can in fact be viewed as a log \( S \)-algebra. We now have a surjection

\[
\theta_\log : A_{\text{st}} \twoheadrightarrow \mathcal{O}_K^\wedge \quad X \mapsto 0
\]

of log \( S \)-algebras extending \( \theta : A_{\text{cris}} \rightarrow \mathcal{O}_K^\wedge \), and \( \ker(\theta_\log) \) has a natural divided power structure that extends that on \( \ker(\theta)A_{\text{st}} \).

**Proposition 2.4.4.** There is a canonical isomorphism

\[
(A_{\text{st}}, \theta_\log) \cong (B^\dlog_0, \beta_0)
\]

of formal log PD thickenings of \( \mathcal{O}_K^\wedge \) over \( S \).

**Proof.** This is due to Kato; a proof can be found in [Lod07, Proposition 1.3]; cf. also [Bre97, §2].

By construction and (2.4.4) above, we have:

\[
A_{\text{st}} = \lim_n H^0 \left( (\mathcal{O}_K/p)/(S/p^n S) \right)_{\log \text{cris}}, \mathcal{O}_{\log \text{cris}} \right),
\]

where \((\mathcal{O}_K/p)/(S/p^n S)\) is the log crystalline site for \( \mathcal{O}_K/p \) over \( S/p^n S \), and \( \mathcal{O}_{\log \text{cris}} \) is its structure sheaf. In particular, \( A_{\text{st}} \) is endowed with a natural \( \mathcal{G}_K \)-action and a commuting Frobenius lift \( \varphi \) that is compatible with the corresponding structures on \( A_{\text{cris}} \). It is also endowed with a logarithmic connection \( A_{\text{st}} \rightarrow A_{\text{st}} \otimes W[u] W[u] \log(u) \), for which \( \varphi \) is parallel. Equivalently, it has an \( S \)-derivation \( \mathcal{N} : A_{\text{st}} \rightarrow A_{\text{st}} \) lying over the derivation \( u \frac{du}{u} \) of \( S \), and satisfying \( \mathcal{N}\varphi = p\varphi \mathcal{N} \). For a concrete description of these structures, cf. [Bre99, 2.2.2]; or [Bre97, §2].

2.4.5. Suppose that we have a log 1-motif \( Q \) over \( \mathcal{O}_K \). Associated with this is the Tate module \( T_p(Q) = \lim_n Q[p^n]\left[\mathcal{O}_K\right] \): this is a continuous \( \mathcal{G}_K \)-representation over \( \mathbb{Z}_p \). Set

\[
\mathcal{M}(Q) = \lim_n \mathcal{D}(Q)(\text{Spec}(\mathcal{O}_K/p\mathcal{O}_K) \rightarrow \text{Spec} S/p^n S).
\]

\( \mathcal{M}(Q) \) is equipped with the structure of a \( \varphi \)-module over \( S \)

\[
\mathcal{M}(Q) : \varphi^*\mathcal{M}(Q) \rightarrow \mathcal{M}(Q),
\]
and a topologically quasi-nilpotent integrable logarithmic connection $\nabla$. This gives rise to the derivation $\mathcal{N} = \nabla(-u \frac{\partial}{\partial u})$ on $\mathcal{M}(Q)$. Moreover, there is a canonical identification
\[\mathcal{M}(Q) \otimes_S \mathcal{O}_K = H_{\text{dR}}^1(Q),\]
This gives us the $S$-sub-module $\text{Fil}^1 \mathcal{M}(Q) \subset \mathcal{M}(Q)$, defined to be the pre-image of the Hodge filtration in $H_{\text{dR}}^1(Q)$: it satisfies $\text{Fil}^1 S \cdot \mathcal{M}(Q) \subset \text{Fil}^1 \mathcal{M}(Q)$, so that $\mathcal{M}(Q)$ has the structure of a filtered module over $S$. Equip $A_{\text{st}}$ with the divided power filtration.

**Proposition 2.4.6.** There exists a natural $\mathcal{G}_K$-equivariant map
\[j_Q : T_p(Q) \to \text{Hom}_{S,\varphi,\text{Fil}^1,N}(\mathcal{M}(Q), A_{\text{st}})\]
This is an injection with finite cokernel in general and an isomorphism when $p > 2$.

**Proof.** We can interpret $T_p(Q)$ as the group
\[\text{Hom}((\mathbb{Q}_p/\mathbb{Z}_p)\varpi, \mathbb{Q}[p^\infty]\varpi).\]
Moreover, we have a natural map
\[T_p(Q) \to \text{Hom}_{\mathbb{Z},V,\text{Fil}^1}(\mathbb{D}(Q\varpi), \mathbb{D}((\mathbb{Q}_p/\mathbb{Z}_p)\varpi)).\]
Evaluating the crystals on the thickening $A_{\text{st}} \to \mathcal{O}_K$ gives us a natural map
\[j_Q : T_p(Q) \to \text{Hom}_{A_{\text{st}},\varphi,\text{Fil}^1,N}(\mathbb{D}(Q\varpi)(A_{\text{st}}), A_{\text{st}}) = \text{Hom}_{S,\varphi,\text{Fil}^1,N}(\mathcal{M}(Q), A_{\text{st}}).\]
If $Q$ is a classical $1$-motif, it follows from [Fon99] Theorem 7] and its argument that $j_Q$ is an injection with finite cokernel and that it is an isomorphism when $p > 2$.

In general, the naturality of $j_Q$ implies that it respects the weight filtration $\mathcal{W}_Q$. Since the associated graded for the weight filtration is a classical $1$-motif, we see that $\mathcal{G}_K W_Q = \mathcal{G}_K Q$, and therefore $j_Q$ itself must be an isomorphism.

2.4.7. We can now define some Fontaine period rings.

- $B^+_{\text{cris}} = A_{\text{cris}}[p^{-1}]$: as observed before, it is endowed with a Frobenius lift $\varphi$ and a $\mathcal{G}_K$-action.
- $B^+_{\text{dR}}$ is the $(\ker \theta)$-adic completion of $B^+_{\text{cris}}$: it is a complete DVR with residue field $\overline{K}$; $B_{\text{dR}} = \text{Fr}(B^+_{\text{dR}})$ is its fraction field. $B_{\text{dR}}$ is filtered by the powers $\text{Fil}^i B_{\text{dR}}$ of the maximal ideal $\text{Fil}^1 B^+_{\text{dR}}$. $B^+_{\text{cris}} \otimes_{K_0} K$ embeds naturally in $B^+_{\text{dR}}$ and inherits its filtration.
- $B^+_{\text{st}}$ is the image of $A_{\text{st}}[p^{-1}]$ in $B^+_{\text{dR}}$ under the map of $A_{\text{cris}}$-algebras $X \to (\frac{1}{x} - 1)$. It is endowed with a Frobenius lift $\varphi$, a $\mathcal{G}_K$-action and a $B^+_{\text{cris}}$-derivation $N$ satisfying $N\varphi = p\varphi N$. Again, $B^+_{\text{st}} \otimes_{K_0} K$ embeds in $B^+_{\text{dR}}$ and also inherits a filtration from it. Note that one can define $B^+_{\text{st}}$ canonically as a $B^+_{\text{cris}}$-algebra, without making the choice of uniformizer $\pi$. The $\mathcal{G}_K$-equivariant embedding $B^+_{\text{st}} \otimes_{K_0} K \to B^+_{\text{dR}}$ of $B^+_{\text{cris}}$-algebras, however, depends on such a choice; cf. [Fon94a] for more details.

2.4.8. Let $Q$ be a log $1$-motif over $\mathcal{O}_K$; let $Q_0$ be the log $1$-motif over $k_N$ obtained by reducing $Q$ along the surjection $\mathcal{O}_K \twoheadrightarrow k_N$. Set
\[H^1_{\text{dR}}(\mathbb{Q}_K/\mathbb{Z}_p) := T_p(Q)^\vee.\]
There is a canonical surjection $W_N \to k_N$ with underlying map of rings $W \to k$ that is obtained as follows: The Teichmüller lift $[\cdot] : k \to W$ is a map of monoids and so we get a canonical map $M_{k_N} \to k \xrightarrow{[\cdot]} W$. This is the pre-log structure to which the log structure
We will give a construction for the isomorphisms \( M_{W_\mathbb{N}} \to W \) is attached. So we obtain a canonical embedding \( M_{k_\mathbb{N}} \hookrightarrow M_{W_\mathbb{N}} \) with quotient \( 1 + pW \). Now the natural inclusion \( 1 + pW \hookrightarrow M_{W_\mathbb{N}} \) gives us a splitting

\[
M_{W_\mathbb{N}} = (1 + pW) \oplus M_{k_\mathbb{N}}.
\]

This gives us the surjection \( M_{W_\mathbb{N}} \to M_{k_\mathbb{N}} \) underlying \( W_\mathbb{N} \to k_\mathbb{N} \). Moreover, it also allows us to define a Frobenius lift on \( W_\mathbb{N} \) that acts as the \( p \)-power map on \( M_{k_\mathbb{N}} \) and restricts to the canonical Frobenius lift on \( 1 + pW \). Set

\[
M_0(Q) := \mathbb{D}(Q_0)(W_\mathbb{N}) = \varprojlim_n \mathbb{D}(Q_0)(\text{Spec} k_N \hookrightarrow \text{Spec}(W/p^n W)_N).
\]

Note that there is a natural surjection \( S \to W_\mathbb{N} \) that provides us with an identification of \( \varphi \)-modules

\[
\mathcal{M}(Q) \otimes_S W = M_0(Q).
\]

Let \( N_0 : M_0(Q) \to M_0(Q) \) be the reduction of \( N \).

**Lemma 2.4.9.** There is a unique isomorphism

\[
(2.4.9.1) \quad \xi : (M_0(Q) \otimes_W S)[p^{-1}] \xrightarrow{\sim} \mathcal{M}(Q)[p^{-1}]
\]

preserving \( \varphi \) and \( N \). Here, we equip \( M_0(Q) \otimes_W S \) with the diagonal \( \varphi \)-module structure and the map \( N_0 \otimes 1 + 1 \otimes N \).

**Proof.** See [Bre97, 6.2.1.1] for the construction of \( \xi \); it is a divided power avatar of Dwork’s trick. \( \square \)

**Theorem 2.4.10.** Fix a uniformizer \( \pi \in \mathcal{O}_K \) as above. Then there exist natural isomorphisms

\[
\beta_{Q, H-K, \pi} : \mathbb{D}(Q_0)(W_\mathbb{N}) \otimes_W K \xrightarrow{\sim} H^1_{\text{dR}}(Q) \otimes_{\mathcal{O}_K} K. \quad \text{(Hyodo-Kato)}
\]

(2.4.10.2) \( \beta_{Q, st, \pi} : H^1_{\text{et}}(Q_{\overline{k}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B^+_{st} \xrightarrow{\sim} \mathbb{D}(Q_0)(W_\mathbb{N}) \otimes_W B^+_{st}. \) \( \text{(p-atic comparison)} \)

They satisfy the following properties:

1. Equip \( \mathbb{D}(Q_0)(W_\mathbb{N}) \otimes_W K \) with the filtration induced from the Hodge filtration on \( H^1_{\text{dR}}(Q)_K \) via the isomorphism \( \beta_{Q, H-K, \pi} \). Then \( \beta_{Q, st, \pi} \) is \( (\varphi, N, \mathcal{G}_K) \)-equivariant, and also respects filtrations once we tensor with \( K \).

2. If \( Q \) is a classical \( 1 \)-motif, both \( \beta_{Q, H-K, \pi} \) and \( \beta_{Q, st, \pi} \) are independent of \( \pi \) and agree with the classical comparison isomorphisms obtained from those for the \( p \)-divisible group \( Q[p^{-\infty}] \) over \( \mathcal{O}_K \) (cf. [Fal99, 6(6)]).

3. Both \( \beta_{Q, H-K, \pi} \) and \( \beta_{Q, st, \pi} \) are compatible with polarization pairings.

**Proof.** We will give a construction for the isomorphisms \( \beta_{Q, H-K, \pi} \) and \( \beta_{Q, st, \pi} \). The enumerated properties will be clear from the construction, which is a direct extension of [Fal99, Theorem 7].

We obtain (2.4.10.1) simply by reducing (2.4.9.1) from \( S \) to \( \mathcal{O}_K \). We also have:

\[
T_p(Q)[p^{-1}] \xrightarrow{\sim} \text{Hom}_{S, \varphi, \text{Fil}^1, \mathcal{A}'}(\mathcal{M}(Q), A_{st})[p^{-1}]
\]

\[
\xrightarrow{\sim} \text{Hom}_{S, \varphi, \text{Fil}^1, \mathcal{A}'}(\mathbb{D}(Q_0)(W_\mathbb{N}) \otimes_S B^+_{st}),
\]

\[
= \text{Hom}_{\varphi, \text{Fil}^1, \mathcal{A}'}(\mathbb{D}(Q_0)(W_\mathbb{N}), B^+_{st}).
\]

Dualizing and tensoring up to \( B^+_{st} \) now gives us (2.4.10.2). \( \square \)

**Remark 2.4.11.** The construction of the isomorphism \( \beta_{Q, H-K, \pi} \) above is along the lines of Deligne’s original construction for abelian schemes over \( \mathcal{O}_K \); cf. [BSS2, 2.9].

We have the following corollary, which we will use without comment in (3.3):
Corollary 2.4.12. \( H^1_{\text{ét}}(Q^\varphi, \mathbb{Z}_p) \) is a semi-stable Galois representation with weights in \( \{0, 1\} \). 

Let 
\[
D_{\text{st}}(Q) = \text{Hom}_{\mathbb{S}^1}(Q_p, B_{\text{st}} \otimes_{\mathbb{Z}_p} H^1_{\text{ét}}(Q^\varphi, \mathbb{Z}_p))
\]

be the attached weakly admissible filtered \((\varphi, N)\)-module. Then there exists a natural isomorphism of filtered \((\varphi, N)\)-modules
\[
D_{\text{st}}(Q) \cong \mathbb{D}(Q_0)(W_N)[p^{-1}]
\]

Moreover, the map \( N : D_{\text{st}}(Q) \to D_{\text{st}}(Q) \) factors as:
\[
(2.4.12.1) \quad \quad D_{\text{st}}(Q) \to D_{\text{st}}(Q^{\text{mult}}) \to \text{Hom}(Q^{\varphi, C}, K_0) \to \text{Hom}(Q^{\varphi, C}, \mathbb{Z})
\]
Here \( N_Q : Q^{\text{mult}, C} \to \text{Hom}(Q^{\varphi, C}, \mathbb{Z}) \) is the map induced by the monodromy pairing of \( Q \) \( (2.2.3) \).

Proof. Since both \( N_0 \) and \( N_Q \) are additive in \( Q \) and are trivial when \( Q \) is a 1-motif, it is actually enough to show that they are equal when \( Q \) is the log 1-motif \((0, \mathbb{Z}, \mathbb{Z}, 0, 0, \tau_\pi) \), where \( \tau_\pi : \mathbb{Z} \times \mathbb{Z} \to K^\times \) is the pairing such that \( \tau(1, 1) = \pi \). In this case, \( D_{\text{st}}(Q) \) is simply the weakly admissible filtered \((\varphi, N)\)-module attached to a Tate curve with parameter \( \pi \), and here the result is well-known; cf. [Ber04 §II.4]. \( \square \)

Remark 2.4.13. If \( Q \) is a positively polarized log 1-motif, then, by \( (2.3.3) \), it corresponds to a semi-stable abelian variety \( A \) over \( K \) such that \( T_p(A) = T_p(Q) \) and \( H^1_{\text{dR}}(A) = H^1_{\text{dR}}(Q)[p^{-1}] \). So the results above can be rephrased appropriately in terms of the étale and de Rham cohomology of \( A \). In this case, \( (2.4.12.1) \) is due to Coleman-Iovita [CI99 §II.4].

3. At the boundary of a Chai-Faltings compactification

3.1. Chai-Faltings local models as deformation spaces of log 1-motifs.

3.1.1. We begin with a pair \((V, \psi)\), where \( V \) is a \( \mathbb{Q} \)-vector space of dimension \( 2g \) and \( \psi \) is a symplectic form on \( V \). We will fix a \( \mathbb{Z} \)-lattice \( V_\mathbb{Z} \subset V \) such that \( \psi \) restricts to an alternating form
\[
\psi : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}.
\]
We will call such a lattice a polarized lattice for \( V \).

For any ring \( R \), set \( V_R = V_\mathbb{Z} \otimes \mathbb{Z}_p \); we will also denote by \( \psi \) the induced pairing on \( V_R \). Let \( V^*_R \subset V \) be the dual lattice, so that \( \psi \) induces a perfect pairing on \( V_\mathbb{Z} \times V^*_\mathbb{Z} \), and let \( d \in \mathbb{Z}_{>0} \) be such that the order of the finite group \( V_\mathbb{Z} / V^*_\mathbb{Z} \) is \( d^2 \). We will call \( d \) the discriminant of the polarized lattice \( V_\mathbb{Z} \). Then \( \text{GSp}(V_{[1/d]}^\vee, \psi) \) is a reductive sub-group of \( \text{GL}(V_{[1/d]}^\vee) \).

Let \( n \) be an integer in \( \mathbb{Z}_{>0} \) such that \( (n, d) = 1 \). Associated with \((V_\mathbb{Z}/n\mathbb{Z}, \psi)\), we have the moduli stack \( \mathcal{M}_{V_\mathbb{Z}, n, \psi} \) over \( \mathbb{Z}[n^{-1}] \); for any \( \mathbb{Z}[n^{-1}] \)-scheme \( S \), \( \mathcal{M}_{V_\mathbb{Z}, n, \psi}(S) \) parameterizes isomorphism classes of tuples \((A, \lambda, \nu, \alpha)\), where:
- \((A, \lambda)\) is a polarized abelian scheme over \( S \);
- \( \nu : \mathbb{Z}/n\mathbb{Z}_S \xrightarrow{\cong} \mu_{n,S} \) is an isomorphism of étale sheaves of groups over \( S \);
- \( \alpha : \mathbb{V}_{\mathbb{Z}/n\mathbb{Z}, S} \xrightarrow{\cong} A[n] \) is an isomorphism of étale sheaves of groups over \( S \) that carries \( \nu \circ \psi \) to the Weil pairing on \( A[n] \) induced by \( \lambda \), and is symplectic liftable in the sense of [Lan08 1.3.6.2].

In particular, the symplectic liftability condition insures that the prime-to-\( d \) part of \( \ker \lambda \) is isomorphic to the prime-to-\( d \) part of the constant sheaf \( V_\mathbb{Z}^\vee / V_\mathbb{Z} \). It is known that the restriction of \( \mathcal{M}_{V_\mathbb{Z}, n, \psi} \) over \( \mathbb{Z}[(nd)^{-1}] \) is smooth.

Remark 3.1.2. We can and will make sense of this moduli space even when \( g = 0 \). It will be the \( \mathbb{Z}[n^{-1}] \)-scheme \( \mathcal{M}_{0, n} \) classifying isomorphisms \( \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cong} \mu_n \). In other words, it is \( \text{Spec} \mathbb{Z}[n^{-1}][\zeta_n] \), where \( \zeta_n \) is a primitive \( n^{\text{th}} \)-root of unity.
Definition 3.1.3. Let $R$ be a $\mathbb{Z}[d^{-1}]$-algebra. We will call a filtration $W_\ast V_R \subset V_R$ admissible if it is $\text{GSp}(V_R, \psi)$-split (cf. [Kis10, 1.1.2]). Concretely, this means that it is of the form

$$0 = W_{-3} V_R \subset W_{-2} V_R \subset W_{-1} V_R = (W_{-2} V_R)^\perp \subset W_0 V_R = V_R,$$

where $W_{-2} V_R \subset V_R$ is an isotropic direct summand. We will call the filtration proper if $W_{-2} V_R \neq 0$.

Definition 3.1.4. Let $W_\ast V_{\mathbb{Z}/n\mathbb{Z}}$ be an admissible filtration, and let $r = \text{rank}_{\mathbb{Z}/n\mathbb{Z}} W_2 V_{\mathbb{Z}/n\mathbb{Z}}$.

A torus argument $\Phi$ for $(V_Z, \psi, W_\ast V_{\mathbb{Z}/n\mathbb{Z}})$ (cf. [Lan08, 5.4.1]) is a tuple

$$(Y, X, \lambda_{\text{et}}, \varphi_{n}, \varphi_{n}^{\text{mult}}),$$

where:

1. $Y$ and $X$ are free $\mathbb{Z}$-modules of rank $r$ and $\lambda_{\text{et}} : Y \to X$ is an injective map of groups.
2. $\varphi_{n}^{\text{et}} : \text{gr}_0^W V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\sim} Y/\text{NY}$;
3. $\varphi_{n}^{\text{mult}} : W_2 V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\sim} \text{Hom}(X, \mathbb{Z}/n\mathbb{Z})$

are isomorphisms of groups such that the pairing

$$\text{gr}_0^W V_{\mathbb{Z}/n\mathbb{Z}} \times W_2 V_{\mathbb{Z}/n\mathbb{Z}} \xrightarrow{\varphi_{n}^{\text{et}} \times \varphi_{n}^{\text{mult}}} Y/\text{NY} \times \text{Hom}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{(\lambda_{\text{et}}(\cdot), \cdot)} \mathbb{Z}/n\mathbb{Z}$$

is equal to the perfect pairing induced from $\psi$.

Definition 3.1.5. An isomorphism of torus arguments $\Phi = (Y, X, \lambda_{\text{et}}, \varphi_{n}, \varphi_{n}^{\text{mult}})$ and $\Phi' = (Y', X', \lambda'_{\text{et}}, \varphi'_{n}, \varphi'_{n}^{\text{mult}})$ for $(V_Z, \psi, W_\ast V_{\mathbb{Z}/n\mathbb{Z}})$ is a pair of isomorphisms $\gamma_X : X' \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y'$, such that:

- $\gamma'_{\text{et}} = \gamma_X \lambda_{\text{et}} \gamma_Y$;
- $\varphi'_{n} = \gamma_Y \circ \varphi_{n}$;
- $\varphi'_{n}^{\text{mult}} = \gamma_X \circ \varphi_{n}^{\text{mult}}$.

3.1.6. We will be considering tuples $(W_\ast V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$, where:

- $W_\ast V_{\mathbb{Z}/n\mathbb{Z}}$ is an admissible $\text{GSp}(V_{\mathbb{Z}/n\mathbb{Z}}, \psi)$-split filtration of $V_{\mathbb{Z}/n\mathbb{Z}}$;
- A torus argument $\Phi = (Y, X, \lambda_{\text{et}}, \varphi_{n}, \varphi_{n}^{\text{mult}})$, for $(V_Z, \psi, W_\ast V_{\mathbb{Z}/n\mathbb{Z}})$; and
- A symplectic splitting $\delta$ of the filtration $W_\ast V_{\mathbb{Z}/n\mathbb{Z}}$.

Definition 3.1.7. Two tuples $(W_\ast V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$ and $(W'_\ast V_{\mathbb{Z}/n\mathbb{Z}}, \Phi', \delta')$ are equivalent if $W_\ast V_{\mathbb{Z}/n\mathbb{Z}} = W'_\ast V_{\mathbb{Z}/n\mathbb{Z}}$, and $\Phi$ and $\Phi'$ are isomorphic torus arguments. A cusp label for $(V_Z, \psi)$ at level $n$ is an equivalence class of tuples of the form $(W_\ast V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$.

3.1.8. Suppose that we have a cusp label for $(V_Z, \psi)$ with representative $(W_\ast V_{\mathbb{Z}/n\mathbb{Z}}, \Phi, \delta)$. We will use this representative to construct certain spaces associated with the cusp label, and interpret their complete local rings as deformation rings for log 1-motifs. One can easily check that our constructions are independent (up to isomorphism) of the choice of representative.

Fix a prime $p > 0$ such that $(p, n) = 1$, let $k$ be a perfect field of characteristic $p > 0$ and let $W = W(k)$ be its ring of Witt vectors. Fix a lift $W_\ast V_Z$ of $W_\ast V_{\mathbb{Z}/n\mathbb{Z}}$ such that the induced filtration $W_\ast V$ is a $\text{GSp}(V)$-split. We begin with the moduli stack $M_{W_\ast} = M_{\text{gr}_0^W V_Z, \text{gr}_1^W \psi}$ over $\mathbb{Z}[[n d^{-1}]]$; here $\text{gr}_1^W \psi$ is the alternating pairing on $\text{gr}_1^W V_Z$ obtained from $\psi$. As explained in [Lan08, 5.2.7.5], the space $M_{W_\ast}$ is determined up to isomorphism by $W_\ast V_{\mathbb{Z}/n\mathbb{Z}}$, and so is independent of the lift $W_\ast V_Z$.

Let $(B_0, \lambda^{0b})$ be the universal polarized abelian scheme over $M_{W_\ast}$. Suppose that we have an object $(B_0, \lambda^{0b}, \nu_0, \alpha_0)$ of $M_{W_\ast}(k)$ corresponding to a map of stacks $x_0^{0b} : \text{Spec} k \to M_{W_\ast}$. 
Proposition 3.1.9. The complete local ring \( R_{\Phi,x_0}^{ab} = \hat{O}_{M_{W^*},x_0} \) equipped with the polarized abelian scheme \((B_{R_{\Phi,x_0}^{ab}}, \lambda_{R_{\Phi,x_0}^{ab}})\) induced from \((B, \lambda^{ab})\) is the universal deformation ring (over \( W \)) for the polarized abelian variety \((B_0, \lambda_0^{ab})\). More precisely, \( R_{\Phi,x_0}^{ab} \) pro-represents the deformation groupoid \( \text{Def}_{(B_0, \lambda_0^{ab})} \) that assigns to every artin \( W \)-algebra \( C \) with residue field \( k(C) \) the category

\[
\text{Def}_{(B_0, \lambda_0^{ab})}(C) = \left\{ \begin{array}{ll}
\text{Tuples } ((B_C, \lambda_C^{ab}), i_C) \text{ where } (B_C, \lambda_C^{ab}) \text{ is a polarized abelian scheme over } C; & \\
\text{and } & \\
i_C : B_C \otimes_C k(C) \xrightarrow{\cong} B_0 \otimes_k k(C) \text{ is an isomorphism of abelian varieties over } k(C). \end{array} \right.
\]

Proof. This is well-known. \( \Box \)

3.1.10. Now consider the \( M_{W^*} \)-scheme:

\[
\tilde{C}_\Phi = \text{Hom}(\frac{1}{n} Y, B) \times_{\text{Hom}(Y, B^\vee)} \text{Hom}(\frac{1}{n} X, B^\vee).
\]

This is the fiber product of the diagram:

\[
\begin{align*}
\text{Hom}(\frac{1}{n} Y, B) \\
\downarrow \\
\text{Hom}(\frac{1}{n} X, B^\vee) \rightarrow \text{Hom}(Y, B^\vee),
\end{align*}
\]

where the vertical arrow is restriction followed by post-composition with \( \lambda^{ab} \), and the horizontal arrow is pull-back along the map \( Y \xrightarrow{\lambda^{et}} X \leftarrow \frac{1}{n} X \). \( \tilde{C}_\Phi \) is a smooth, proper group scheme over \( M_{W^*} \).

It is shown in [Lan08, 6.2.3.4] that there is a natural map of group schemes over \( M_{W^*} \)

\[
\partial : \tilde{C}_\Phi \rightarrow \text{Hom}(\frac{1}{n} Y/Y, B[n])
\]

\[
(c_n, c_n^\vee) \mapsto c_n^\vee \lambda^{et} - \lambda^{ab} c_n,
\]

whose fibers are abelian schemes over \( M_{W^*} \). The splitting \( \delta \) gives us a distinguished element \( b_{\Phi, \delta} \) in the image of \( \partial \) (cf. [Lan08, 6.2.3.1]). Let \( C_{\Phi, \delta} \) be the fiber of \( \partial \) over \( b_{\Phi, \delta} \): this is an abelian scheme over \( M_{W^*} \).

Over \( C_{\Phi, \delta} \), we have the tautological maps

\[
c_{\Phi, \delta} : \frac{1}{n} Y \rightarrow B; \quad c_{\Phi, \delta}^\vee : \frac{1}{n} X \rightarrow B^\vee.
\]

Set \( c_\Phi = c_{\Phi, \delta}|_Y \) and \( c_\Phi^\vee = c_{\Phi, \delta}^\vee|_X \).

Suppose that we have a map \( x_0^{ab} : \text{Spec } k \rightarrow C_{\Phi, \delta} \) corresponding to an object \((B_0, \lambda_0^{ab}, \alpha_0, c_{0,0}, c_{0,0}^\vee)\) over \( k \), where \((B_0, \lambda_0^{ab}, \alpha_0)\) is an object of \( M_{W^*}(k) \) and \( c_{0,0} : \frac{1}{n} Y \rightarrow B_0 \) and \( c_{0,0}^\vee : \frac{1}{n} X \rightarrow B_0^\vee \) are maps such that \( c_{0,0}^\vee \lambda^{et}_0 = \lambda^{ab} c_0 \); here, \( c_0^\vee = c_{0,0}^\vee|_X \) and \( c_0 = c_{0,0}|_Y \). Consider the complete local ring \( R_{\Phi,x_0}^{ab} = \hat{O}_{C_{\Phi, \delta},x_0^{ab}} \): it is naturally an algebra over the deformation ring \( R_{\Phi,x_0}^{ab} \) considered in 3.1.9 above. Over it we have the pair \((c_{\Phi, \delta}, R_{\Phi,x_0}^{ab}, c_{\Phi, \delta}^\vee, R_{\Phi,x_0}^{ab}, c_{\Phi, \delta}^\vee)\) inherited from the universal pair over \( C_{\Phi, \delta} \).

Lemma 3.1.11. Let \( C \) be an Artin local \( W \)-algebra with residue field \( k(C) \). For any fppf sheaf of groups \( H \) over \( C \), \( H_0 \) will denote its reduction to \( k(C) \). For any prime-to-\( p \) isogeny
\( \phi : A' \to A \) of abelian schemes over \( C \), the following square of fpff sheaves over \( C \) is cartesian

\[
\begin{array}{cc}
A' & A_0' \\
\downarrow \phi & \downarrow \phi_0 \\
A & A_0
\end{array}
\]

In particular, given a map \( f'_0 : H \to A'_0 \) of fpff sheaves of groups over \( C \), lifting \( f'_0 \) to a map \( f' : H \to A' \) is equivalent to lifting \( \phi_0 f'_0 : H \to A_0 \) to a map \( f : H \to A \).

**Proof.** This is clear once we note that \( \ker (A \to A_0) \) is \( p \)-power torsion (cf. [Kat81, 1.1.1]). \( \square \)

Let \( \text{Art}_{R_{\Phi,x_0}} \) be the category of Artin local \( R_{\Phi,x_0} \)-algebras. Note that every ring \( C \) in \( \text{Art}_{R_{\Phi,x_0}} \) comes equipped with a polarized abelian scheme \((B_C, \lambda_C^{ab})\) lifting \((B_0, \lambda_0^{ab})\). Consider the following two deformation functors on \( \text{Art}_{R_{\Phi,x_0}} \):

\[
\text{Def}_{(c_{n,0}, c_{n,0}')} (C) = \begin{cases} \\
\text{Pairs } (c_{n,C}, c_{n,C}') \text{ of maps } \begin{array}{c} c_{n,C} : \frac{1}{n} Y \to B_C \ ; \ c_{n,C}' : \frac{1}{n} X \to B_C' \\
\text{lifting } (c_{n,0}, c_{n,0}') \end{array} \\
\end{cases}
\]

\[
\text{Def}_{c_{n,0}'} (C) = \begin{cases} \text{Lifts } c_{C'}' : \frac{1}{n} X \to B_C' \text{ of } c_{n,0}' \end{cases}
\]

**Proposition 3.1.12.** The two deformation functors are naturally isomorphic and are relatively pro-represented over \( R_{\Phi,x_0} \) by \( R_{\Phi,x_0} \).

**Proof.** That \( R_{\Phi,x_0} \) pro-represents \( \text{Def}_{(c_{n,0}, c_{n,0}')} \) is essentially tautological, so we only have to prove the isomorphism of the two functors. There is clearly a natural map \( \text{Def}_{(c_{n,0}, c_{n,0}')} \to \text{Def}_{c_{n,0}'} \) obtained by restricting any lift of \( c_{n,0}' \) to \( X \). We claim that this is an isomorphism. Indeed, by [3.1.11], giving a lift of \( c_{n,0}' \) is equivalent to giving a lift of \( c_{n,0}' \) (apply the lemma to the isogeny \( [n] : B_C' \to B_C' \)). Once we have the lift \( c_{n,C}' \) of \( c_{n,0}' \), consider the map \( c_{n,C}' \lambda_{et} - b_{\Phi,\delta} : \frac{1}{n} Y \to B_C' \); this is a lift of \( \lambda_0^{ab} c_{n,0}' \). Again, from [3.1.11], this time applied to \( \lambda_C^{ab} \), we obtain a lift \( c_{n,C}' \) of \( c_{n,0}' \).

\( \square \)

3.1.13. Set

\[
\Psi_n = (c_{n,\Phi} \times c_{\Phi}')^* \mathcal{P}_B^{-1}; \quad \Psi = (c_{\Phi} \times c_{\Phi}')^* \mathcal{P}_B^{-1}.
\]

Then \( \Psi_n \) is a \( \mathbb{G}_m \)-bi-extension of \( \frac{1}{n} Y \times X \) over \( C_{\Phi,\delta} \), and \( \Psi \) is a \( \mathbb{G}_m \)-bi-extension of \( Y \times X \) over \( C_{\Phi,\delta} \) such that \((1 \times \lambda_{et})^* \Psi \) is a symmetric \( \mathbb{G}_m \)-bi-extension of \( Y \times Y \).

Set

\[
\mathcal{B}_{\lambda^{et}} = \{ \text{Pairings } (, , ) : Y \times X \to \mathbb{Z} : \text{ such that } \langle y, \lambda^{et}(y') \rangle = \langle y, \lambda^{et}(y') \rangle, \text{ for all } y, y' \in Y \}.
\]

Let \( \mathcal{S}_{\lambda^{et}} = \mathcal{B}_{\lambda^{et}}^\dual \) be its dual abelian group. Also set

\[
\mathcal{B}_\Phi = \frac{1}{n} \mathcal{B}_{\lambda^{et}}; \quad \mathcal{S}_\Phi = \frac{1}{n} \mathcal{S}_{\lambda^{et}}.
\]

Set

\[
\mathcal{S}_\Phi = \frac{1}{n} Y \otimes X \langle y \times \lambda^{et}(y') - y' \times \lambda^{et}(y), y \in Y \rangle.
\]

Then \( \mathcal{S}_\Phi \) is just the maximal torsion-free quotient of \( \mathcal{S}_\Phi \). Let \( \mathcal{S}_\Phi^{tor} \) be the maximal torsion sub-group of \( \mathcal{S}_\Phi \).
Consider the sheaf $\Xi_{\Phi,\delta}$ over $C_{\Phi,\delta}$, whose points valued in any $C_{\Phi,\delta}$-scheme $S$ are given by:

$$\Xi_{\Phi,\delta}(S) = \left\{ \begin{array}{lr}
\text{Trivializations } \tau_n : \frac{1}{n} Y \times X \to \Psi_{n,S} \text{ over } (S, M_S) \text{ of } \mathbb{G}_m, \text{ inducing a symmetric trivialization of } \\
\frac{1}{n} Y \times X \text{ inducing a symmetric trivialization of the symmetric } \mathbb{G}_m, \text{ over } S \times Y.
\end{array} \right. $$

This is a torsor under the $C_{\Phi,\delta}$-group $E_{\Phi}$ of multiplicative type with character group $S_{\Phi}$.

It admits a natural surjection onto the $C_{\Phi,\delta}$-group $E^\text{tor}_{\Phi}$ (again of multiplicative type) with character group $S^\text{tor}_{\Phi}$. The splitting $\delta$ allows us to naturally pick out a certain fiber $\Xi_{\Phi,\delta}$ of this surjection; then $\Xi_{\Phi,\delta}$ is a torsor under the $C_{\Phi,\delta}$-torus $E_{\Phi}$ with character group $S_{\Phi}$. Note that $\Xi_{\Phi,\delta}$ is open and closed in $\Xi_{\Phi,\delta}$. For all this, cf. [Lan08 §6.2.3].

Suppose that we have a rational, polyhedral, non-degenerate cone $\sigma \subset B_{\Phi} \otimes \mathbb{R}$; we can then form the torus embedding $E_{\Phi} \hookrightarrow E_{\Phi}(\sigma)$ (cf. [KKMSD73 §I.1]). More precisely, we can consider the monoid

$$S_{\Phi,\sigma} = \sigma^\vee \cap S_{\Phi},$$

where

$$\sigma^\vee = \{ n \in S_{\Phi} \otimes \mathbb{R} : \langle n, s \rangle \geq 0, \text{ for all } s \in \sigma \},$$

and we set $E_{\Phi}(\sigma) = \text{Spec} \mathcal{O}_{C_{\Phi,\delta}}[S_{\Phi,\sigma}]$. We will consider the contraction product

$$\Xi_{\Phi,\delta}(S) = \Xi_{\Phi,\delta} \times_{E_{\Phi}} E_{\Phi}(\sigma).$$

This is an fs log scheme over $C_{\Phi,\delta}$ in the evident way with the log structure induced by the divisor that is the complement of $\Xi_{\Phi,\delta}$. Over any fs log scheme $(S, M_S)$ over $C_{\Phi,\delta}$, we have the $\mathbb{G}_m$-bi-extension $\Psi_{n,S}^{\log}$ of $\frac{1}{n} Y \times X$ induced from the $\mathbb{G}_m$-extension $\Psi_{n,S}$. For any such $(S, M_S)$, we have:

$$\Xi_{\Phi,\delta}(S)((S, M_S)) = \left\{ \begin{array}{lr}
\text{Trivializations } \tau_n : \frac{1}{n} Y \times X \to \Psi_{n,(S,M_S)} \text{ over } (S, M_S) \text{ of } \mathbb{G}_m, \text{ inducing a symmetric trivialization of } \\
(1 \times \lambda^{\text{et}})^* \Psi_{n,(S,M_S)}; \\
\text{and such that, for any geometric point } \bar{s} \to S, \\
\text{the pairing } Y \times X \to M_{S_\bar{s}}^\text{tor} / \Phi_{S_\bar{s},\delta} \otimes \mathbb{Z} \text{ lies in } \sigma \subset B_{\Phi} \otimes \mathbb{R}
\end{array} \right. $$

Here, $N_{\tau,\delta}$ is the monodromy pairing at $\bar{s}$ associated with $\tau$ (cf. [2.2.3]).

Let $\Xi_{\Phi,\delta}(\sigma)$ be the closure of $\Xi_{\Phi,\delta}$ in $\Xi_{\Phi,\delta}(S)$: this is an open and closed strict log subscheme of $\Xi_{\Phi,\delta}(\sigma)$. Over $E_{\Phi,\delta}$, we have a tautological trivialization $\tau_n, \Phi$ of the $\mathbb{G}_m$-bi-extension $\Psi_{\Phi,\delta}$ of $\frac{1}{n} Y \times X$ restricting to a symmetric trivialization of the symmetric $\mathbb{G}_m$-bi-extension $(1 \times \lambda^{\text{et}})^* \Psi_{\Phi,\delta}$ of $Y \times X$. Over $\Xi_{\Phi,\delta}(\sigma)$, this induces a trivialization $\tau_{n,\Phi,\sigma}$ of the $\mathbb{G}_m$-bi-extension $\Psi_{\Phi,\delta}$ of $\frac{1}{n} Y \times X$, which restricts to a symmetric trivialization of the symmetric $\mathbb{G}_m$-bi-extension $(1 \times \lambda^{\text{et}})^* \Psi_{\Phi,\delta}$ of $Y \times X$.

The stratification of $E_{\Phi}(\sigma)$ by the orbits of $E_{\Phi}$ gives rise to a stratification on $\Xi_{\Phi,\delta}(\sigma)$ as well. There is a unique closed stratum $Z_{\Phi,\delta}(\sigma)$. Suppose now that we have a map $x_0 : \text{Spec } k \to E_{\Phi,\delta}(\sigma)$ landing inside the closed stratum. Equip Spec $k$ with the log structure induced from that of $\Xi_{\Phi,\delta}(\sigma)$; then we have the tuple $(B_0, \lambda^{ab}_0, a_0, c_{n,0}, c_{n,0}', c_{n,0}'', \tau_n, 0)$ over Spec $k$ obtained by pull-back from the tautological tuple over $\Xi_{\Phi,\delta}(\sigma)$. Let $\tau_0$ be the trivialization of the $\mathbb{G}_m$-bi-extension $\Psi_{x_0}$ of $Y \times X$ induced from $\tau_n, 0$. The tuple $(B_0, Y, X, \lambda^{ab}_0, \lambda^{\text{et}}, c_0, c_0', \tau_0)$ then gives us a polarized log 1-motif $(Q_0, \lambda_0)$ over $k$. Let $R_{\Phi,\sigma, x_0}$ be the complete local ring of $\Xi_{\Phi,\delta}(\sigma)$ at $x_0$.

**Lemma 3.1.4.** Suppose that we have an Artin local fs log $W$-algebra $(C, M_C)$, and a $\mathbb{G}_m$-torsor $E^\log_{C}$ equipped with a trivialization $\beta_0$ of the induced $\mathbb{G}_m$-torsor $E^\log_{k(C)}$ over $k(C)$. Then
the map $\beta_C \mapsto \beta_{C}^{\otimes n}$ sets up a bijection:

\[
\left(\text{Trivializations } \beta_C \text{ of } E_C^{\log} \right) \cong \left(\text{Trivializations } \beta_C^{\otimes n} \text{ of } (E_C^{\log})^{\otimes n} \right)
\]

**Proof.** For any $\beta'_C$ on the right hand side, its ‘$n^{th}$-roots’ in $E_C^{\log}$ form a $\mu_n$-torsor over $(C, M_C)$. So we reduce to:

**Claim 3.1.15.** Suppose that we have a $\mu_n$-torsor $\eta_C$ over $(C, M_C)$. Then there is a natural bijection:

\[
(\text{Trivializations of } \eta_C) \cong (\text{Trivializations of } \eta_{k(C)})
\]

This last claim in turn follows from the following facts:

- The map $H^1(C^{\log}, \mu_n) \to H^1(k(C)^{\log}, \mu_n)$ is a bijection.
- The map $\mu_n(C) \to \mu_n(k(C))$ is a bijection.

The first fact is a consequence of the analogous fact for classical étale $\mu_n$-torsors and (2.1.4)\(^3\). The second follows because $(n, p) = 1$ and because $\ker(C^\times \to k(C)^\times)$ is a $p$-primary group.  \(\square\)

Let $\mathcal{R}_{\Phi,x_0}^{\text{ab}}$ be as in \(\S 3.1.12\), and consider the category of fs log Artin local $\mathcal{R}_{\Phi,x_0}^{\text{ab}}$-algebras: This will be the category $\text{Art}^{\log}_{\mathcal{R}_{\Phi,x_0}^{\text{ab}}}$ whose objects are pairs $((C, M_C), j_C)$, where $(C, M_C)$ is an Artin local algebra equipped with an fs log structure, and $j_C: k \to k(C)$ is a continuous map of fs log algebras extending the natural map of underlying rings. Here, we equip $k(C)$ with the log structure induced from that of $C$. Note that for every such object, we have the $\mathcal{G}_m$-bi-extensions $\Psi_{n,C}$ of $\mathcal{G}_m$-torsors over $Y \times X$ and $\Psi_C$ of $Y \times X$, and also the associated $\mathcal{G}_m$-bi-extensions $\Psi_{n,C}^{\log}$ and $\Psi_C^{\log}$. Furthermore, via the map $j_C$, we can view $\tau_{n,0}$ (resp. $\tau_0$) as a trivialization of $\Psi_{n,k(C)}$ (resp. $\Psi_{k(C)}^{\log}$).

The category $\text{Art}^{\log}_W$ is defined analogously.

Consider two deformation functors on $\text{Art}^{\log}_{\mathcal{R}_{\Phi,x_0}^{\text{ab}}}$:

\[
\text{Def}_{\tau_{n,0}}((C, M_C), j_C) = \left(\text{Trivializations } \tau_{n,C} \text{ lifting } \tau_{n,0} \text{ and } \eta_{C} \text{ inducing symmetric trivializations of } (1 \times \lambda^{\mu})^{\ast} \Psi_C \right)
\]

\[
\text{Def}_{\tau_0}((C, M_C), j_C) = \left(\text{Trivializations } \tau_C \text{ lifting } \tau_0 \text{ and } \eta_{C} \text{ inducing symmetric trivializations of } (1 \times \lambda^{\mu})^{\ast} \Psi_C \right)
\]

**Proposition 3.1.16.**

(1) The two deformation functors above are naturally isomorphic and are relatively pro-represented over $\mathcal{R}_{\Phi,x_0}^{\text{ab}}$ by $\mathcal{R}_{\Phi,\sigma,x_0}$.

(2) $\mathcal{R}_{\Phi,\sigma,x_0}$ pro-represent the deformation groupoid $\text{Def}_{(Q_0, \lambda_0)}$ over $\text{Art}^{\log}_W$ parameterizing deformations of the polarized log 1-motif $(Q_0, \lambda_0)$.

**Proof.** It is clear that $\mathcal{R}_{\Phi,\sigma,x_0}$ relatively pro-represents $\text{Def}_{\tau_{n,0}}$. We need to show that the two deformation functors are isomorphic. It is, however, immediate from \(\S 3.1.14\), that, for any $((C, M_C), j_C)$ in $\text{Art}^{\log}_{\mathcal{R}_{\Phi,x_0}^{\text{ab}}}$, the natural map from $\text{Def}_{\tau_{n,0}}((C, M_C), j_C)$ to $\text{Def}_{\tau_0}((C, M_C), j_C)$ is a bijection.

For the second assertion, we simply have to put the first assertion together with \(\S 3.1.9\) and \(\S 3.1.12\), and note that deforming $(Q_0, \lambda_0)$ is equivalent to deforming the tuple $(B_0, \lambda_0, c_0, c_0', \tau_0)$.  \(\square\)
3.2. Explicit co-ordinates for Chai-Faltings local models. Our goal in this section is to use the deformation theoretic description of $R_{\Phi,\sigma,x_0}$ to write down explicit co-ordinates for it, under the assumption that $(p, nd) = 1$. Let $(B_0, Y, X, \lambda^{ab}_0, \lambda^{\text{et}}_0, \sigma_0, c_{\Phi}^0, \tau_0)$ be the tuple corresponding to the polarized log 1-motif $(Q_0, \lambda_0)$. We will suppress the sub-scripts from now on and refer to the rings $R^{ab}_{\Phi} \times \prod_{\lambda, x, c, \tau}$ from (3.1.3), $R^{ab}_{\Phi}$ from (3.1.7), and $(R^{ab}_{\Phi,\sigma,x_0})$ from (3.1.10) simply as $R^{ab}, R^{ab}$ and $R$, respectively.

3.2.1. Set $k = k(x_0)$: this is a finite field of characteristic $p$. Set $P_{\Phi,\sigma} = S_{\Phi,\sigma}/S^{\text{sp}}_{\Phi,\sigma}$: this is a sharp, fs monoid. By construction, $k(x_0)$ with its induced log structure is isomorphic to $kP_{\Phi,\sigma}$. If there is no likelihood of confusion, we will refer to this log ring simply as $k$; if we need to emphasize the log structure, we will write $k_{\Phi,\sigma}$ instead.

Let $W = W(k)$, and let $W_{P_{\Phi,\sigma}}$ be the log scheme associated with the pre-log structure

$$
M_{k_{\Phi,\sigma}} \to k \to W,
$$

where the first map is the log structure on $k$ and the second map is the Teichmüller lift. Again, if there is no danger of confusion, we will refer to this log ring simply as $W$.

In particular, the construction gives us a natural splitting

$$
M_{W_{P_{\Phi,\sigma}}} = M_{k_{\Phi,\sigma}} \oplus (1 + pW).
$$

We can use this splitting to define a Frobenius lift on $W_{\Phi,\sigma} := W_{P_{\Phi,\sigma}}$: it will be the $p$-power map on $M_{k_{\Phi,\sigma}}$ and the usual Frobenius automorphism on $1 + pW$. This means that any log $F$-crystal over $k$ can be evaluated on $W_{\Phi,\sigma}$ to give a $\varphi$-module over $W$.

Let $\mathbb{D}(Q_0)$ be the log $F$-crystal over $k$ associated with $Q_0$: by the functoriality of the Dieudonné functor, it has a weight filtration

$$
0 = W_{-1}\mathbb{D}(Q_0) \subset W_0\mathbb{D}(Q_0) \subset W_1\mathbb{D}(Q_0) \subset W_2\mathbb{D}(Q_0) = \mathbb{D}(Q_0),
$$

induced by the weight filtration on $Q_0$. It satisfies $W_i\mathbb{D}(Q_0) = \mathbb{D}(Q_0/W_{-i+1}Q_0)$. Note that $W_{-2}Q_0 = \text{Hom}(X, \mathbb{C}_m^{\text{log}})$ and that $gr^W_0 Q_0 = Y[1]$ (where $X, Y$ are as in the cusp label $\Phi$ above): this gives us natural identifications of $F$-crystals $W_0\mathbb{D}(Q_0) = \text{Hom}(Y, 1)$ and $gr^W_2 \mathbb{D}(Q_0) = X \otimes 1(1)$.

The polarization $\lambda_0$ gives us a perfect pairing $\psi_0 : \mathbb{D}(Q_0) \times \mathbb{D}(Q_0) \to 1(1)$. For every $i$, this induces a perfect pairing $gr^W_i \mathbb{D}(Q_0) \times gr^W_{-i-1} \mathbb{D}(Q_0) \to 1(1)$. In particular, the perfect pairing on $W_0\mathbb{D}(Q_0) \times gr^W_2 \mathbb{D}(Q_0)$ is given by the formula:

$$
W_0\mathbb{D}(Q_0) \times gr^W_2 \mathbb{D}(Q_0) = \text{Hom}(Y, 1) \times (X \otimes 1(1)) \to 1(1)
$$

$$
(f, (\lambda^{\text{et}} \otimes 1)(y \otimes 1)) \mapsto f(y).
$$

Note that $\lambda^{\text{et}} \otimes 1 : Y \otimes 1(1) \to X \otimes 1(1)$ is an isomorphism.

Evaluating the polarized log $F$-crystal $(\mathbb{D}(Q_0), \psi_0)$ on $W_{\Phi,\sigma}$ gives us a $\varphi$-module $M_0$ equipped with a weight filtration $W_\bullet M_0$ and a symplectic pairing $\psi_0 : M_0 \otimes M_0 \to W(1)$ of $\varphi$-modules.

By construction, the weight filtration is $\text{GSp}(M_0, \psi_0)$-split: that is, $W_0M_0$ is isotropic for $\psi_0$ and $W_1M_0$ is its annihilator. Let $P_{\text{wt}} \subset \text{GSp}(M_0, \psi_0)$ be the parabolic sub-group stabilizing $W_0M_0$: let $U_{\text{wt}}$ be its unipotent radical, and let $U_{\text{wt}}^{-2}$ be the center of $U_{\text{wt}}$. It is easy to see that $U_{\text{wt}}^{-2}$ is simply the largest sub-group of $U_{\text{wt}}$ that acts trivially on $W_1M_0 \oplus (M_0/W_0M_0)$. In particular, it is commutative, every $N \in \text{End}(M_0)$, and we have an isomorphism of group schemes

$$
\text{Lie} \to U_{\text{wt}}^{-2} \xrightarrow{\sim} U_{\text{wt}}^{-2}
$$

$$
N \mapsto 1 + N.
$$
The natural identifications Hom(Y, W) = W_0M_0 and X \otimes W(1) = \text{gr}^W_2 M_0 combined with the description of the pairing in (32.1.3) give us further canonical identifications:

\begin{equation}
(3.2.1.3) \quad \text{Lie}U_{\text{wt}}^{-2} = \left\{ \begin{array}{l}
\text{Pairings } N : Y \times X \to W \text{ such that } \\
N(y, \lambda^i(y')) = N(y', \lambda^i(y)), \text{ for all } y, y' \in Y
\end{array} \right\} = \text{B}_\Phi \otimes Z W.
\end{equation}

3.2.2. We will now construct explicit models for \( R_{\text{ab}} \) and \( R_{\text{sab}} \), following [Fal99] §7 and [Moo98] §4. Let \( \mu_0 : G_m \otimes k \to \text{GSp}(M_0, \psi_0) \otimes k \) be a co-character splitting the Hodge filtration \( \text{Fil}^1(M_0 \otimes k) \) of \( M_0 \otimes k \). Concretely, this means that we are choosing a Lagrangian decomposition

\[ M_0 \otimes k = \text{Fil}^1(M_0 \otimes k) \oplus (M_0 \otimes k)' \]

By [DOR10] 4.2.17, we can actually choose a splitting co-character \( \mu \) that factors through \( P_{\text{wt}} \otimes k \). We can lift this co-character to a co-character \( \mu : G_m \to P_{\text{wt}} \) giving a Lagrangian decomposition

\[ M_0 = \text{Fil}^1 M_0 \oplus M_0' \]

lifting the decomposition for \( M_0 \otimes k \).

Let \( U^{\text{op}} \subset \text{GSp}(M_0, \psi_0) \) be the opposite unipotent sub-group associated with \( \mu \). Just as for \( U_{\text{wt}}^{-2} \), every section \( N \in \text{Lie} U^{\text{op}} \) satisfies \( N^2 = 0 \) in \( \text{End}(M_0) \), and so \( U^{\text{op}} \) is canonically isomorphic to \( \text{Lie} U^{\text{op}} \) as a group scheme over \( W \). Since the co-character actually factors through \( P_{\text{wt}} \), for every \( i \), we get a decomposition

\[ W_i M_0 = (W_i M_0 \cap \text{Fil}^1 M_0) \oplus (W_i M_0 \cap M_0'). \]

This shows that \( W_0 M_0 \subset M_0' \subset W_1 M_0 \) and in particular implies that \( U_{\text{wt}}^{-2} \subset U^{\text{op}} \).

Let \( U^{\text{ab}} = U^{\text{op}} / U_{\text{wt}}^{-2} \), and let \( U^{\text{sab}} = U^{\text{op}} / (U^{\text{op}} \cap U_{\text{wt}}) \). \( U^{\text{sab}} \) (resp. \( U^{\text{ab}} \)) acts faithfully on \( M_0^{\text{ab}} = M_0 / W_0 M_0 \) (resp. \( M_0^{\text{ab}} = \text{gr}^W_1 M_0 \)). Note that we have a canonical identifications \( M_0^{\text{ab}} = D(J_0)(W) \) and \( M_0^{\text{ab}} = D(B_0)(W) \) of \( \varphi \)-modules over \( W \). Here, \( J_0 \) denotes the 1-motif \( W_{-1}Q_0 \); we can also think of it as the semi-abelian extension of \( B_0 \) classified by \( c_0' \).

In what follows, \( \square \) can be read as either \( \text{ab} \) or \( \text{ab} \). Let \( \bar{U}^\square \) be the completion of \( U^\square \) along the identity section, and let \( A^\square \) be the formally smooth \( W \)-algebra such that \( \text{Spf} A^\square = \bar{U}^\square \).

The identity section gives us an augmentation ideal \( \bar{I}^\square \subset R^\square \) such that \( R^\square / \bar{I}^\square = W \). Fix a basis \( \{ e_i \} \) for \( \text{Lie} U^\square \), and if \( (x_i) \) are the corresponding co-ordinates on \( U^\square \), let \( \varphi : A^\square \to A^\square \) be the Frobenius lift carrying \( x_i \) to \( x_i^p \).

Let \( \varphi^\square \in U^\square(A^\square) \) be the universal element of \( U^\square \). Consider the 3-tuple \( M^\square = (M^\square, \varphi^\square, \text{Fil}^1 M^\square) \), where:

\[ M^\square = M_0^\square \otimes_W A^\square; \]
\[ \varphi^\square : \varphi^* M^\square = \varphi^* M_0^\square \otimes_W A^\square \xrightarrow{\varphi \otimes 1} M_0^\square \otimes_W A^\square = M^\square \varphi^\square \to M^\square; \]
\[ \text{Fil}^1 M^\square = \text{Fil}^1 M_0^\square \otimes_W A^\square. \]

By [Moo98] 4.4, \( M^\square \) can be endowed with a unique, topologically quasi-nilpotent connection \( \nabla^\square \), for which \( \varphi^\square \) is parallel, giving us an object \((M^\square, \nabla^\square)\) in the category \( MF^\square_{[0,1]}(A^\square) \) considered in loc. cit.. Moreover, this latter category is equivalent to the category of \( p \)-divisible groups over \( A^\square \). So we obtain \( p \)-divisible groups \( \varphi^{\text{sab}} \) (resp. \( \varphi^{\text{ab}} \)) over \( A^{\text{sab}} \) (resp. \( A^{\text{ab}} \)) deforming \( J_0[p^\infty] \) (resp. \( B_0[p^\infty] \)). By Serre-Tate theory for 1-motifs [MS11] 1.1.3.1], this gives us a deformation \( J^{\text{sab}} \) (resp. \( B^{\text{ab}} \)) of \( J_0 \) (resp. \( B_0 \)) over \( A^{\text{sab}} \) (resp. \( A^{\text{ab}} \)). Strictly speaking, the cited result only gives us formal deformations, but the polarization \( \lambda^{\text{ab}}_{0} \) on \( B_0 \) lifts to \( B^{\text{ab}} \), and allows us to algebraize both it and \( J^{\text{sab}} \).

By construction, \( M^{\text{sab}} \) has a weight filtration

\[ 0 = W_0 M^{\text{sab}} \subset W_1 M^{\text{sab}} \subset W_2 M^{\text{sab}} = M^{\text{sab}} \]
in $\text{MF}_{[0,1]}^\text{log}(\mathbb{A}^\square)$. This filtration arises from the corresponding weight filtration on $M_0^\text{ab}$. In particular, the polarization $\psi_0$ of $M_0$ induces a polarization $\psi_0^\text{ab}$ of the first graded component $\text{gr}^1 M_0^\text{ab} = M_0^\text{ab} \otimes_{R_\text{ab}} R_\text{ab}$.

Since $R_\text{ab}$ is the universal deformation ring for $(B_0, \lambda_0^\text{ab})$, there exists a unique continuous map $f_\text{ab} : R_\text{ab} \to A_\text{ab}$ inducing $(B_\text{ab}, \lambda_\text{ab})$. Similarly, the map $f_\text{ab}$ extends, by the universal property of $R_\text{ab}$ to a map $f_\text{ab} : R_\text{ab} \to A_\text{ab}$ inducing $J_\text{ab}$.

The following result can be shown as in [Mom98 4.5] via a simple Kodaira-Spencer calculation.

**Proposition 3.2.3.** $f_\text{ab}$ and $f_\text{ab}$ are isomorphisms.

3.2.4. Thanks to (3.2.3) above, we now have a rather explicit description of both $R_\text{ab}$ and the Dieudonné $F$-crystal attached to the universal deformation of $(J_0, \lambda_0^\text{ab})$ over $R_\text{ab}$. We will now build upon this to get a similarly explicit description for the full log deformation ring $R$, and the polarized log Dieudonné $F$-crystal attached to the universal deformation of $(Q_0, \lambda_0)$ over $R$. As expected, we will exhibit $R$ as a completed toric embedding over $R_\text{ab}$.

Let $E_\Phi$ be the torus over $W$ with character group $S_\Phi$, and let $E_\Phi \to E_{\Phi, \sigma}$ be the torus embedding corresponding to $\sigma \subset B_\Phi \otimes \mathbb{R}$. Fix a $k$-valued point $\beta$ in the closed orbit of $E_{\Phi}(\sigma)$; we will think of it as a map of monoids $\beta : S_{\Phi, \sigma} \to k$ satisfying $\beta^{-1}(k^\times) = S_{\phi, \sigma}^\times$. Let $R_{\Phi, \sigma}$ be the complete local ring of $E_{\Phi, \sigma}$ at $\beta$.

The natural map of monoids $S_{\phi, \sigma} \to R_{\phi, \sigma}^\beta$ endows $R_{\phi, \sigma}^\beta$ with the structure of a log smooth $W$-algebra, and we have a surjection $R_{\phi, \sigma}^\beta \setminus k_{\phi, \sigma}$ of log $W$-algebras. In particular, we have a map of groups $\tau_{\beta, n} : S_{\Phi} \to M_{R_{\phi, \sigma}^\beta}$, which we can restrict to $S_{\lambda^m} \subset S_{\Phi}$ to get a map $\tau_\beta : S_{\lambda^m} \to M_{R_{\phi, \sigma}^\beta}$. Let $\tau_{\beta, n, 0} : S_{\Phi} \to M_{k_{\phi, \sigma}^\beta}$ and $\tau_{\beta, 0} : S_{\lambda^m} \to M_{k_{\phi, \sigma}^\beta}$ be the induced maps.

We will think of $\tau_{\beta, n, 0}$ (resp. $\tau_{\beta, 0}$) as a trivialization of the trivial $\mathbb{G}_m$-bi-extension $1^\text{log}_{\mathbb{G}_m, \beta_{, \text{ab}}}$-extension $1_{Y \times X}$ (resp. $1^\text{log}_{\mathbb{G}_m, \beta_{, \text{ab}}}$) that induces symmetric trivializations of $1_{Y \times Y}^\text{log}$ when pulled back along $1 \times \lambda^\text{ab}$.

Let us return to the category $\text{Art}^\text{log}_{Y,Y}$ of pairs $((C, M_C), jC)$. We can use $jC : k \to k(C)$ to think of $\tau_{\beta, n, 0}$ as a trivialization of $1^\text{log}_{\mathbb{G}_m, \beta_{, \text{ab}}}$ over $k(C)$, and similarly for $\tau_{\beta, 0}$.

Consider the deformation functors

$$
\text{Def}_{\tau_{\beta, n, 0}}((C, M_C), jC) = \left\{ \text{Trivializations } \tau_{\beta, n, C} \text{ of } 1^\text{log}_{\mathbb{G}_m, \beta_{, \text{ab}}} \text{ over } C \text{ lifting } \tau_{\beta, n, 0} \text{ and inducing symmetric trivializations of } 1_{Y \times Y}^\text{log} \right\};
$$

$$
\text{Def}_{\tau_{\beta, 0}}((C, M_C), jC) = \left\{ \text{Trivializations } \tau_{\beta, C} \text{ of } 1_{Y \times X} \text{ over } C \text{ lifting } \tau_{\beta, 0} \text{ and inducing symmetric trivializations of } 1_{Y \times Y} \right\}.
$$

**Proposition 3.2.5.** The two deformation functors above are isomorphic and $R_{\Phi, \sigma}$ pro-represents them over $W$.

**Proof.** This is a special case of (3.1.10) [1].

3.2.6. Recall from (3.1.13) that we have canonical $\mathbb{G}_m$-bi-extensions $\Psi_{n, k}$ of $\mathbb{G}_m$-extensions of $Y \times X$ over $R_\text{ab}$. Set $\tau_{n, 0} = \tau_{n, 0, \beta_{, \text{ab}}}$ and $\tau_{0} = \tau_{0, \beta_{, \text{ab}}}$. By construction, $\tau_{n, 0}$ is a trivialization of the $\mathbb{G}_m$-bi-extension $\Psi_{n, 0}$ and $\tau_{0}$ is a trivialization of $\Psi_0$. The tuple $(B_0, Y, X, c_0, c_0^\vee, \lambda_0^\text{ab}, \lambda^\text{et}, \tau_{0})$ corresponds to a polarized 1-motif $(Q_0, \lambda_0^\text{ab})$ over $k$.

**Proposition 3.2.7.** Given a co-character $w : \mathbb{G}_m \to P_{wt}$ splitting the weight filtration $W_n M_0$ and commuting with the Hodge co-character $\mu : \mathbb{G}_m \to P_{wt}$, we can find a canonical trivialization $\tau_{w, n}^\beta$ of $\Psi_{n, 0}$ over $R_\text{ab}$ lifting $\tau_{n, 0}^\beta$ and inducing a symmetric trivialization of $(1 \times \lambda^\text{et})^* \Psi$. 

Proof. Note that such a co-character always exists; indeed, finding one is equivalent to finding a Levi sub-group \( L_{\text{wt}} \subset P_{\text{wt}} \) containing \( \mu(G_m) \).

Fix a co-character \( w \) as above. From \( (3.1.11) \), it follows that it is enough to find a trivialization \( \tau^\beta_w \) of \( \Psi \) lifting \( \tau^\beta_0 \) and inducing a symmetric trivialization of \((1 \times \Lambda^2)^* \Psi\). This is of course equivalent to deforming the polarized 1-motif \((Q_\beta^0, \lambda^0_\beta)\) over \( R^{\text{sab}} \).

We first observe that there is a natural identification of polarized \( \varphi \)-modules \( \mathbb{D}(Q_\beta^0)(W) = M_0 \). Now \( \Def U^{\text{op}} \), being the \(-1\) eigenspace for \( \mu \), is stable under the action of \( w(G_m) \). \( \Def U^{\text{wt}}_w \) is by definition the \(-2\) eigenspace for \( w \), so we get a splitting

\[
\Def U^{\text{op}} = \Def U^{\text{wt}}_{w, \lambda} \oplus \Def U^{\text{sab}},
\]

where \( \Def U^{\text{sab}} \) maps isomorphically onto the sum of the 0 and \(-1\) eigenspaces for \( w \) within \( \Def U^{\text{op}} \). Since \( \Def U^{\text{op}} \) is isomorphic to \( \Def U^{\text{op}} \), this gives us a splitting of group schemes \( \Def U^{\text{op}} = \Def U^{\text{wt}}_{w, \lambda} \times \Def U^{\text{sab}} \).

In particular, we can now view every section of \( \Def U^{\text{sab}} \) as an automorphism of \( M_0 \).

We will use this to define an object \( \underline{M}^{\text{cl}} \) in \( \MF_{[0,1]}^{\text{R^{sab}}} = \MF_{[0,1]}^{\text{A^{sab}}} \). This will be a tuple \((M^{\text{cl}}, \varphi^{\text{cl}}, \Fil^1 M^{\text{cl}})\), where:

\[
M^{\text{cl}} = M_0 \otimes W A^{\text{sab}}, \quad \Fil^1 M^{\text{cl}} = \Fil^1 M_0 \otimes W A^{\text{sab}},
\]

\[
\varphi^{\text{cl}} : \varphi^* M^{\text{cl}} = \varphi^* M_0 \otimes W A^{\text{sab}} \xrightarrow{\varphi_0 \otimes 1} M_0 \otimes W A^{\text{sab}} = M^{\text{cl}} \xrightarrow{\varphi^{\text{cl}}} M^{\text{cl}}.
\]

Here, \( g^{\text{sab}}_0 \in \hat{U}^{\text{sab}}(A^{\text{sab}}) = \hat{U}^{\text{sab}}(R^{\text{sab}}) \) is the tautological element, viewed as an automorphism of \( M^{\text{cl}} = M_0 \otimes_R A^{\text{sab}} \). We can endow \( M^{\text{cl}} \) with the constant polarization \( \psi_0 \otimes 1 \). It evidently has a weight filtration \( W_\bullet M^{\text{cl}} \) with \( M^{\text{cl}} = M^{\text{cl}}_0 \). By \( \text{MST1} \ 1.1.3.1 \) again, we now obtain the polarized 1-motif \((Q_\beta^0, \lambda^0_\beta)\) over \( R^{\text{sab}} \) reducing to \((Q_0^0, \lambda^0_0)\) over \( k \).

Fix a co-character \( w \) as above. Consider the deformation functors \( \Def \tau_\mu \) from \( (3.1.16) \) and \( \Def \tau_{\beta, n, \mu} \) from \( (3.2.3) \). We can consider both as functors on \( \text{Art}^{\text{log}}_{R^{\text{sab}}} \).

**Proposition 3.2.8.** The functor \( \Def \tau_\mu \to \Def \tau_{\beta, n, \mu} \) carrying \( \tau_{n, \beta} \) to \((\tau^\beta_{w,n})^{-1} \tau_{n,C} \) is an isomorphism of deformation functors over \( \text{Art}^{\text{log}}_{R^{\text{sab}}} \). In particular, there is an isomorphism \( R \xrightarrow{\sim} R^{\text{sab}} \otimes R^{\text{g, s}} \) of \( fs \) log \( R^{\text{sab}} \)-algebras. Under this isomorphism, the universal \( \tau_n \) over \( R \) is mapped to \( \tau^\beta_{w,n} \).

**Proof.** This is clear. \( \square \)

3.2.9. For future reference, we summarize all the choices made in the process of constructing the above explicit model for \( R \):

- In \((3.2.2)\), we chose a co-character \( \mu_0 : G_m \to P_{\text{wt}} \otimes k \) splitting the Hodge filtration, and a lift \( \mu : G_m \to P_{\text{wt}} \) of \( \mu_0 \).
- In \((3.2.4)\), we chose a \( k \)-valued point in the closed orbit of \( \mathbf{E}_\Phi(\sigma) \). Equivalently, we chose a map of monoids \( \beta : S_{\Phi, \sigma} \to k \) such that \( \beta^{-1}(k^\times) = S_{\Phi, \sigma}^0 \).
- Finally, in \((3.2.6)\), we chose a co-character \( w : G_m \to P_{\text{wt}} \) splitting the weight filtration \( W_\bullet M_0 \) and commuting with our choice of \( \mu \). Equivalently, we chose a Levi sub-group \( L_{\text{wt}} \subset P_{\text{wt}} \) containing the image of \( \mu \).

3.2.10. Fix some choices as in \((3.2.9)\), and use them to give explicit co-ordinates for \( R \) as in \((3.2.10)\). We can now give an explicit description of the polarized log \( F \)-crystal \( \mathbb{D}(Q) \) over \( R \) that is attached to the universal deformation \((Q, \lambda)\) of \((Q_0, \lambda_0)\) over \( R \).

Since \( R \) is log smooth over \( W \), giving a log \( F \)-crystal over \( R \) amounts to giving a triple \((M, \varphi_M, \nabla_M)\), where \((M, \varphi_M)\) is a finite free \( \varphi \)-module over \( R \), and \( \nabla_M \) is a topologically
quasi-nilpotent logarithmic connection on $M$ for which $\varphi_M$ is parallel. Giving a Dieudonné log $F$-crystal over $R$ amounts to giving a tuple $(M, \varphi_M, \Fil^1 M, \nabla_M)$, where $(M, \varphi_M, \nabla_M)$ corresponds to a log $F$-crystal over $R$ and $\Fil^1 M \subset M$ is a direct such that $(M, \varphi_M, \Fil^1 M)$ is an object in $\BT^F_R$.

We take $M = M_0 \otimes_W R$ and $\Fil^1 M = \Fil^1 M_0 \otimes_W R$. Equip $M$ with the constant polarization $\psi_M = \psi_0 \otimes 1$. Define $\varphi_M$ as follows: write $M$ as $M^{cl} \otimes_{R^{an}} R$, where $M^{cl}$ is as in the proof of (3.2.7), and set $\varphi_M = \varphi^{cl} \otimes 1$.

Note that $M^{cl}$ is equipped with a natural connection $\nabla^{cl}$, for which $\varphi^{cl}$ is parallel; for example, this follows from the fact that $M^{cl}$ is the evaluation of the Dieudonné crystal of $Q_\psi^R$ at $R^{an}$. To finish we will describe $\nabla_M$ in terms of its connection matrix $\theta \in \End(M_0) \otimes_W \hat{\Omega}^{1, log}_{R/W}$. In fact, $\theta$ will lie in $\Lie U^{op} \otimes_W \hat{\Omega}^{1, log}_{R/W}$: its component in $\Lie U^{ab} \otimes_W \hat{\Omega}^{1, log}_{R/W}$ will be the image of the connection matrix of $\nabla^{cl}$, and its component in $\Lie U^{-2}_{wt} \otimes_W \hat{\Omega}^{1, log}_{R/W} = B_{\Phi} \otimes_W \hat{\Omega}^{1, log}_{R/W}$ will correspond to the natural map $S_{\Phi} \xrightarrow{\dlog} \hat{\Omega}^{1, log}_{R^{ab}, W} \rightarrow \hat{\Omega}^{1, log}_{R/W}$. Here, we are using the splitting $\Lie U^{op} = \Lie U^{ab} \oplus \Lie U^{-2}_{wt}$ afforded by the choice of $w$ above.

From the description of $Q$ in (3.2.8) in terms of $Q^\beta_{\psi}$ (equivalently $\tau_{w,n}$) and $\tau_{\beta,n}$, we obtain the following:

**Proposition 3.2.11.** The tuple $(M, \varphi_M, \Fil^1 M, \nabla_M, \psi_M)$ is the one associated with the polarized log $F$-crystal $\mathbb{D}(Q)$ over $R$ by the correspondence described above.

\[ \square \]

3.2.12. Let $R^{an}$ be the global ring of functions over the rigid analytic space $\hat{U}^{an}$ attached to the formal scheme $\hat{U}^{\mathbb{A}}$ and let $R^{an}[\ell_a : a \in M_R]$ be the $R$-algebra freely generated by the variables $\ell_a$ indexed by $M_R$. We set

\[ R^{an, \log} = \frac{R^{an}[\ell_a : a \in M_R]}{\ell_{ab} - \ell_a - \ell_b, \text{ for all } a, b \in M_R; \ell_r - \log(r) : \text{ for all } r \in R^{an} \text{ such that } |1 - r(x)|, \text{ for all } x \in \hat{U}(\mathcal{O}_{\mathbb{A}^n})} \]

The ring $R^{an, \log}$ is endowed with a natural logarithmic connection

\[ \nabla : R^{an, \log} \rightarrow R^{an, \log} \otimes_R \hat{\Omega}^{1, log}_{R/W} \]

\[ \ell(a) \mapsto \dlog(a) \]

We can also equip it with a natural continuous extension of the endomorphism $\varphi$ of $R$ that carries $\ell_a$ to $p\ell_a$, for all $a \in M_R^{op}$. We will extend the augmentation map $R \rightarrow W$ to the unique map $R^{an, \log} \rightarrow W$ carrying $\ell_a$ to 0, for all $a \in S_{\Phi} \setminus S_{\Phi, 0}$. Set $M^{an, \log} = M \otimes_R R^{an, \log}$, so that we have a $\varphi$-equivariant identification $M^{an, \log} \otimes_{R^{an, \log}} W = M_0[p^{-1}]$. Equip $M^{an, \log}$ with the diagonal logarithmic connection.

**Proposition 3.2.13.** There exists a unique $\varphi$-equivariant, parallel section

\[ \xi : M_0^{\otimes}[p^{-1}] \rightarrow M^{an, \log, \otimes} \]

**Proof.** This is essentially [Vol03, Theorem 9]. However, for later use, we will need an explicit version of this isomorphism, which we now present. First, let $R^{ab, an}$ be the global ring of functions over the analytic space $\hat{U}^{ab, an}$, and let $M^{cl}$ be as in (3.2.7). With this, we can

\[ \text{[\footnote{We will be using the analytification functor of Berthelot, exposed in detail in [dJ95a, §7].}}] \]
associate the $R^\text{sab,an}$-module $M^{\text{cl,an}} = M^{\text{cl}} \otimes_{R^\text{sab}} R^\text{sab,an}$. We will first define the $\varphi$-equivariant (and necessarily parallel) section

$$\xi^{\text{cl}} : M_0 \rightarrow M^{\text{cl,an}}$$

$$m \mapsto \lim_n \varphi^n(m).$$

This is simply Dwork’s trick. Note that its definition is clearly compatible with tensor operations on both sides.

Next, we consider the map

$$S_\varphi \rightarrow M^{\text{EP}}_R \xrightarrow{\text{an-log}} R^{\text{an,log}}.$$

This can be viewed as an element

$$A^{\text{log}} \in B_\varphi \otimes R^{\text{an,log}} = \text{Lie} U^{-2}_{\text{wt}} \otimes_W R^{\text{an,log}} \subset \text{End}(M^{\text{an,log}}).$$

Set $\xi^{\text{log}} = \exp(A^{\text{log}}) \in U^{-2}_{\text{wt}}(R^{\text{an,log}})$. One can now easily check that $\xi^{\text{log}}$ commutes with $\xi^{\text{cl}}$ and that $\xi = \xi^{\text{cl}}\xi^{\text{log}}$ is the unique section whose existence had to be shown.

3.2.14. An important consequence of (3.2.13), exhibited in [Vol03, Theorem 6], is the following: Let $L/K$ be a finite extension, let $L_0 \subset L$ be the maximal unramified sub-extension, and let $\pi \in \mathcal{O}_L$ be a uniformizer, allowing us to fix a branch log of the $p$-adic logarithm satisfying $\log_{\pi}(\pi) = 0$. For any map $x : R \rightarrow \mathcal{O}_L$ of log rings, the choice of logarithm gives us a unique extension $x^{\text{log}} : R^{\text{an,log}} \rightarrow L$ carrying $\ell_\pi$ to $\log_{\pi}(x^{2}(a))$. Evaluating the isomorphism of (3.2.13) along this map gives us an isomorphism of $L$-vector spaces:

$$M_0 \otimes_W L \xrightarrow{\simeq} H^1_{\text{dR}}(Q_x)[p^{-1}].$$

Let $l = \mathcal{O}_L/(\pi)$ equipped with its induced log structure, and let $x_0 : R \rightarrow l$ be the reduction of $x$. We can identify $M_0$ with $\mathbb{D}(Q_{x_0})(W(l)_N)$ as a $\varphi$-module over $W$, giving us:

$$\mathbb{D}(Q_{x_0})(W(l)_N) \otimes_{W(l)} L \xrightarrow{\simeq} H^1_{\text{dR}}(Q_x)[p^{-1}].$$

The following result is an immediate consequence of the log smoothness of $R$ and the construction of the Hyodo-Kato isomorphism in (2.4.10), which is also accomplished via parallel transport.

**Proposition 3.2.15.** The isomorphism in (3.2.14.2) agrees with the Hyodo-Kato isomorphism from (2.4.10). \hfill \Box

3.2.16. Let $\varphi_0 \otimes 1 : \varphi^*M \rightarrow M$ be the scalar extension of $\varphi_0 : \varphi^*M_0 \rightarrow M_0$. For any $h \in \text{End}(M[p^{-1}])$, set

$$\varphi(h) = \varphi_0(\varphi^*h)\varphi^{-1}_0 \in \text{End}

(M[p^{-1}]).$$

Let $g^{\text{sab}} \in U^{\text{sab}}(R)$ be the image of the universal element of $U^{\text{sab}}(R^{\text{sab}})$, and let $\Phi(g^{\text{sab}})$ be the convergent product

$$\Phi(g^{\text{sab}}) = g^{\text{sab}} \varphi(g^{\text{sab}})\varphi^2(g^{\text{sab}})\varphi^3(g^{\text{sab}})\cdots \in \text{GL}(M^{\text{an}}).$$

The following corollary is immediate from the description of $\xi$ in the proof of (3.2.13).

**Corollary 3.2.17.** For any $\varphi$-invariant element $s_0 \in M_{\varphi}^{\varphi}[p^{-1}]$, we have:

$$\xi(s_0) = \xi^{\text{log}}\Phi(g^{\text{sab}})(s_0 \otimes 1).$$

\hfill \Box
3.3. Tate tensors.

Definition 3.3.1. A collection of tensors \( \{ s_{\alpha,0} \}_{\alpha \in I} \subset M_0^{\otimes}[p^{-1}] \) is a collection of Tate tensors for \( M_0 \) if:

1. Each \( s_{\alpha} \), for \( \alpha \in I \), is \( \varphi \)-invariant.
2. The point-wise stabilizer of the collection is a reductive sub-group \( G_{K_0} \subset \text{GSp}(M_0, \psi_0)_{K_0} \).

3.3.2. Let \( \{ s_{\alpha,0} \}_{\alpha \in I} \subset M_0^{\otimes}[p^{-1}] \) be a collection of Tate tensors. In this case, the weight filtration \( W_\bullet M_0[p^{-1}] \), being a \( \varphi \)-stable filtration, can be split by a co-character \( \bar{w} : G_m \to G_{K_0} \).

In particular, the intersection \( P_{w_0,G,K_0} = P_{w_0,K_0} \cap G_{K_0} \) is a parabolic sub-group of \( G_{K_0} \). Let \( U_{w_0,G,K_0}^{-2} = U_{w_0,K_0}^{-2} \cap G_{K_0} \); then we have

\[
\text{Lie} U_{w_0,G,K_0}^{-2} \subset \text{Lie} U_{w_0,K_0}^{-2} = B_\varphi \otimes K_0.
\]

Set \( B_{\Phi,G} = B_\varphi \cap \text{Lie} U_{w_0,G,K_0}^{-2} \); this is a direct summand of \( B_\varphi \). Set \( S_{\Phi,G} = B_{\Phi,G}^\times \); this is a quotient of \( S_\Phi \), and is again a free abelian group. Let \( E_{\Phi,G} \) be the torus over \( W \) with character group \( S_{\Phi,G} \); it is a sub-torus of \( E \). Let \( \sigma_G = \sigma \cap (B_{\Phi,G} \otimes \mathbb{R}) \), and let \( S_{\Phi,G,\sigma_G} \subset S_{\Phi,G} \) be the \( \mathfrak{s} \mathfrak{f} \) monoid associated with the non-degenerate, rational, polyhedral cone \( \sigma_G \). Attaching to this, we have the torus embedding

\[
E_{\Phi,G} \hookrightarrow E_{\Phi,G}(\sigma_G).
\]

Finally, set \( P_{\Phi,G,\sigma_G} = S_{\Phi,G,\sigma_G}/S_{\Phi,G,\sigma_G}^\times \).

Definition 3.3.3. A collection of Tate tensors \( \{ s_{\alpha,0} \} \subset M_0^{\otimes}[p^{-1}] \) satisfies the continuity property if the natural map of monoids \( S_{\Phi,\sigma} \to S_{\Phi,G,\sigma_G} \) is continuous; that is, if only invertible elements in \( S_{\Phi,\sigma} \) are mapped to invertible elements in \( S_{\Phi,G,\sigma_G} \).

If \( \{ s_{\alpha,0} \} \) satisfies the continuity property, in making our choices as in (3.3.2), we can find \( \beta : S_{\Phi,\sigma} \to k \) such that \( \beta \) factors through \( S_{\Phi,G,\sigma_G} \). Let \( R_{\Phi,G,\sigma_G}^3 \) be the complete local ring of \( E_{\Phi,G}(\sigma_G) \) at \( \beta \). This is the normalization of a quotient ring \( R_{\Phi,G}^3 \) of \( R_{\Phi,\sigma}^3 \); cf. [Har89, 3.1]. We will assume that we have made such a choice of \( \beta \) for the rest of this section.

3.3.4. Here is one way to obtain Tate tensors for \( M_0 \) satisfying the continuity property: Let \( \mathcal{O}_K \) be the ring of integers in a finite extension \( K/\mathbb{Q}_p \) with residue field \( k \) and maximal ideal \( m_K \subset \mathcal{O}_K \). Equip it with its canonical log structure. Suppose that we have:

1. A polarized log 1-motif \((Q_\lambda, \Lambda_\lambda)\) over \( \mathcal{O}_K \);
2. A continuous map of log rings \( j_\lambda : k_{\Phi,\sigma} \to \mathcal{O}_K/m_K \), where we have equipped the right hand side with the log structure induced from \( \mathcal{O}_K \); and
3. An identification \( j_\lambda : (Q_\lambda, \lambda_0) = (Q_{x,0}, \lambda_0) \) := \( (Q_x, \lambda_0) \otimes_{\mathcal{O}_K} \mathcal{O}_K/m_K \).

In the language of (3.3.1) we have an object of Def\((Q_{x,0}, \lambda_0)(\mathcal{O}_K)\); this is of course equivalent to giving a local map \( x : R \to \mathcal{O}_K \) of \( \mathfrak{s} \mathfrak{f} \) log algebras.

Let \( \Lambda_x = T_p(Q_x) \), and suppose, in addition, that we have a collection \( \{ s_{\alpha,x,\sigma} \} \subset \Lambda_x^{\otimes}[p^{-1}] \) of Galois-invariant tensors over \( L_x \) defining a reductive sub-group \( G_{Q_\varphi} \subset \text{GSp}(\Lambda_x, \psi_x)_{Q_{\varphi}} \), where \( \psi_x \) is the \( \mathbb{Z}_p(1) \)-valued symplectic form on \( \Lambda_x \) induced from \( \lambda_\pi \).

Via the p-adic comparison isomorphism (2.4.10.2), we now obtain \( \varphi \)-invariant tensors \( \{ s_{\alpha,x,\sigma} \} \subset \mathbb{D}(Q_{x,0})(W_{\mathbb{N}})^{\otimes}[p^{-1}] \) satisfying \( N(s_{\alpha,x,\sigma}) = 0 \), for all \( \alpha \). Note that the condition \( N(s_{\alpha,x,\sigma}) = 0 \) ensures that the \( \varphi \)-invariance of \( s_{\alpha} \) is independent of the choice of Frobenius lift on \( W_{\mathbb{N}} \). Choose any Frobenius lift on \( W_{\mathbb{N}} \); this amounts to a choice of a splitting \( M_{W_{\mathbb{N}}} = M_{\sigma_k} / m_K \otimes (1 + pW) \). Since we have already chosen a splitting \( M_{W_{\varphi,\sigma}} = M_{\sigma_k} / m_K \otimes (1 + pW) \), there now exists a unique map \( \tilde{j}_x : W_{\Phi,\sigma} \to W_{\mathbb{N}} \) lifting \( j_\lambda \) and respecting the chosen splittings. In particular, \( \tilde{j}_x \) is \( \varphi \)-equivariant, and so we have an equality of \( \varphi \)-modules \( M_0 = \mathbb{D}(Q_{x,0})(W_{\mathbb{N}}) \), giving us \( \varphi \)-invariant tensors \( \{ s_{\alpha,0} \} = \{ s_{\alpha,x,\sigma} \} \subset M_0^{\otimes}[p^{-1}] \).

We have the following:
Lemma 3.3.5. \( \{ s_{\alpha,0} \} \) is a collection of Tate tensors satisfying the continuity property.

\[ \square \]

3.3.6. Fix a collection of Tate tensors \( \{ s_{\alpha,0} \} \subseteq M_0^\otimes [p^{-1}] \) satisfying the continuity property. Let \( R^\text{an,log} \) and \( M^\text{an,log} \) be as in (3.2.12). The Tate tensors \( \{ s_{\alpha,0} \} \) give rise to a parallel \( \varphi \)-invariant tensors

\[ \{ s_{\alpha} \} = \{ \xi(s_{\alpha,0} \otimes 1) \} \subseteq M^\text{an,log,\otimes}, \]

where \( \xi \) is as in (3.2.13).

Suppose that \( L/K_0 \) is a finite extension and that \( x : R \to \mathcal{O}_L \) is a local map of log \( W \)-algebras. Then, for each choice of uniformizer \( \pi \in \mathcal{O}_L \), the Hyodo-Kato isomorphism (2.1.10.1) carries the Tate tensors \( \{ s_{\alpha,0} \} \) to tensors \( \{ s_{\alpha,\pi,x} \} \subseteq H^1_{\text{dR}}(Q_x)[p^{-1}] \). According to (3.2.15) these tensors are exactly the evaluation of \( \{ s_{\alpha} \} \) under the map \( x^{\text{log}} : R^\text{an,log} \to L \) induced by the branch of logarithm \( \log_\pi \) attached to \( \pi \). Given a different choice of uniformizer \( \pi' \in \mathcal{O}_L \), we have:

\[ s_{\alpha,\pi',x} = \exp(\log(\pi' \pi^{-1})N_x)(s_{\alpha,\pi,x}), \]

where \( N_x \) is the monodromy map on \( M_x := H^1_{\text{dR}}(Q_x)[p^{-1}] \). This follows, for example, from the explicit description of \( \xi \) in (3.2.13).

Lemma 3.3.7. Suppose that \( s_{\alpha,\pi,x} \) belongs to \( \text{Fil}^0 M_x^\otimes \); then it is invariant under monodromy, and is therefore determined independently of the choice of \( \pi \).

Proof. Let \( L_0 \subseteq L \) be the maximal unramified sub-extension. Set \( M_{x,0} = M_0 \otimes_W L_0 \); then \( M_{x,0} \) is equipped with the isomorphism

\[ \xi_{x,\pi} : M_{x,0} \otimes L_0 \xrightarrow{\sim} M_x. \]

Along this isomorphism, the monodromy \( N_x \) descends to a map \( N_{x,0} \) on \( M_{x,0} \), and this gives \( M_{x,0} \) the structure of a weakly admissible filtered \((\varphi, N)\)-module with Hodge-Tate weights in \( \{0,1\} \). If \( s_{\alpha,\pi,x} = \xi_{x,\pi}(s_{\alpha,0}) \) belongs to \( \text{Fil}^0 M_{x,0} \), then, since \( N_{x,0} \varphi = p\varphi N_{x,0} \), \( N_{x,0}(s_{\alpha,0}) = N_x(s_{\alpha,\pi,x}) \) belongs to the intersection \( (M_{x,0}^\otimes)^{\varphi=p^{-1}} \cap \text{Fil}^0 M_{x,0}^\otimes \).

But, if we ‘forget’ the monodromy \( N_{x,0} \), the remaining filtered \( \varphi \)-module is still weakly admissible. Therefore, one easily sees that the intersection in question must be zero: each element in \( (M_{x,0}^\otimes)^{\varphi=p^{-1}} \) spans a \( \varphi \)-stable sub-space of slope \(-1\), and so must have a Hodge polygon with negative slopes. In particular, \( s_{\alpha,\pi,x} \) must be invariant under monodromy.

\[ \square \]

Definition 3.3.8. We will say that the collection \( \{ s_{\alpha} \} \) is Hodge at \( x \) if, for some (hence any) choice of uniformizer \( \pi \), we have \( \{ s_{\alpha,\pi,x} \} \subseteq \text{Fil}^0 M_x^\otimes \). In this case, by (3.3.7), the specializations \( \{ s_{\alpha,\pi,x} \} \) are determined independently of \( \pi \), and so we will write them simply as \( \{ s_{\alpha,x} \} \).

Definition 3.3.9. For any finite extension \( L/K_0 \), and any quotient \( T \) of \( R_{\mathcal{O}_L} \), write \( L M(T) \) for the set of continuous maps of log \( W \)-algebras \( x : R \to \mathcal{O}_{K_0} \) that factor through \( T \).

Note that a continuous map \( x : R \to \mathcal{O}_{K_0} \) is a map of log algebras precisely when the associated map of monoids \( x^\sharp : \mathcal{S}_{\varphi,\sigma} \to \mathcal{O}_{K_0} \) takes only non-zero values.

Lemma 3.3.10. Fix a finite extension \( L/\mathbb{Q}_p \) and \( x : R \to \mathcal{O}_L \) in \( L M(R) \). For any \( h \in U_{\text{wt}}^W(L) \), set \( \text{Fil}^1 h M_x = h \cdot \text{Fil}^1 M_x \). Then the tuple \( (M_{x,0}, \text{Fil}^1 h M_x, \varphi, N_{x,0}) \) is still weakly admissible.

Proof. This follows because \( h \) acts trivially on \( W_0 M_{x,0} \), as well as on \( \gamma_2^W M_{x,0} \), and because the category of weakly admissible filtered \((\varphi, N)\)-modules is closed under extensions within the category of filtered \((\varphi, N)\)-modules.

\[ \square \]
Lemma 3.3.11. Let \( L/\mathbb{Q}_p \) be a finite extension, and let \((D, \text{Fil}^* D_L, \varphi, N)\) be a weakly admissible filtered \((\varphi, N)\)-module over \( L \). Suppose that \( H \subset \text{GL}(D) \) is a reductive sub-group that is the point-wise stabilizer of a collection of tensors \[
\{v_\beta\} \subset (D^\otimes)^{x=1,N=0} \bigcap \text{Fil}^0 D_L^\otimes.\]
Then \( \text{Fil}^* D_L \) is split by a co-character \( \mu : \mathbb{G}_{m,L} \to H_L \).

Proof. This follows from the argument in [Kis10, 1.4.5]. □

Proposition 3.3.12. Fix \( x : R \to \mathcal{O}_L \) in \( \text{LM}(R) \). For \( \alpha \in I \), set \( s_{\alpha}^{\text{cl}} = \xi^{\text{cl}}(s_{\alpha,0} \otimes 1) \), and let \( s_{\alpha,x}^{\text{cl}} \) denote its specialization at \( x \). Then the following are equivalent:

1. \( \{s_{\alpha}\} \) is Hodge at \( x \).
2. There exists \( u \in E_\Phi(\mathcal{O}_L) \) such that \( \{1 - \log(u)\} s_{\alpha,x}^{\text{cl}} \subset \text{Fil}^0 M_x^\otimes \), and, for any such \( u \), there exists a torsion point \( e \in E_\Phi(\mathcal{O}_L) \) such that \( eu^x : S_\Phi \to L_x^\times \) factors through \( S_{\Phi_G} \).

Proof. We first note that, given \( u \in E_\Phi(\mathcal{O}_L) = \text{Hom}(S_\Phi, \mathcal{O}_L^\times) \), \( \log(u) \in B_\Phi \otimes L = \text{Lie} U_{\text{wt},L}^{-2} \) is the element attached to the composition:

\[
S_\Phi \xrightarrow{u} \mathcal{O}_L^\times \xrightarrow{\log} L.
\]

Here, the logarithm on \( \mathcal{O}_L^\times \) is defined to be trivial on the Teichmüller lift of \((\mathcal{O}_L/m_L)^\times\) into \( \mathcal{O}_L^\times \), and via the usual power series on \( 1 + m_L \).

Now, fix a uniformizer \( \pi \) in \( L \). Let \( \left( \frac{\log}{x,\pi} \right) \in U_{\text{wt}}^{-2}(L) = 1 + \text{Lie} U_{\text{wt},L}^{-2} \) be the automorphism obtained by viewing the composition

\[
S_\Phi \xrightarrow{x^\pi} L_x^\times \xrightarrow{\log} L
\]
as an element of \( B_\Phi \otimes L = \text{Lie} U_{\text{wt},L}^{-2} \). Then \( \{s_{\alpha}\} \) is Hodge at \( x \) if and only if \( \{\xi^{\text{cl}}(s_{\alpha,x}^{\text{cl}})\} \subset \text{Fil}^0 M_x^\otimes \).

Suppose now that \( (2) \) holds. Then, for each \( \alpha \),

\[
\xi^{\text{cl}}_{x,\pi}(s_{\alpha,x}^{\text{cl}}) \subset (1 - \log(u))(s_{\alpha,x}^{\text{cl}}) \subset \text{Fil}^0 M_x^\otimes.
\]

This shows that \( (2) \Rightarrow (1) \).

For the other implication, assume that \( \{s_{\alpha}\} \) is Hodge at \( x \). Then it follows from the proof of (3.3.7) that the monodromy \( N_x \) satisfies \( N_x(s_{\alpha,0}) = 0 \), for all \( \alpha \). This means that the composition

\[
S_\Phi \xrightarrow{x^\pi} L_x^\times \xrightarrow{\log} L_x^\times / \mathcal{O}_L^\times
\]
factors through \( S_{\Phi_G} \). Therefore, we can find \( u \in E_\Phi(\mathcal{O}_L) \) such that \( ux^\pi \) factors through \( S_{\Phi_G} \). So we find:

\[
(1 - \log(u))(s_{\alpha,x}^{\text{cl}}) = \xi^{\text{cl}}_{x,\pi}(s_{\alpha,x}^{\text{cl}}) = s_{\alpha,x}^{\text{cl}} \in \text{Fil}^0 M_x^\otimes.
\]

To finish the proof, suppose that \( v \in E_\Phi(\mathcal{O}_L) \) is another element such that \( (1 - \log(v))(s_{\alpha,x}^{\text{cl}}) \) lies in \( \text{Fil}^0 M_x^\otimes \) for all \( \alpha \). Then, for all \( \alpha \),

\[
(1 - \log(u))(s_{\alpha,x}^{\text{cl}}) = \xi^{\text{cl}}_{x,\pi}(1 + \log(v))(1 - \log(v))(s_{\alpha,x}^{\text{cl}})) = s_{\alpha,x}^{\text{cl}} \in \text{Fil}^0 M_x^\otimes.
\]

Let \( G_{x,v} \subset \text{GL}(M_x) \) be the point-wise stabilizer of the collection \( \{(1 - \log(v))(s_{\alpha,x}^{\text{cl}})\} \). Then, by (3.3.10) and (3.3.11), the filtration \( \text{Fil}^1 M_x \subset M_x \) is split by a co-character \( \mu : \mathbb{G}_{m,L} \to G_{x,v} \).

Consider the map

\[
\text{End}(M_x) \to \oplus_{\alpha \in I} M_x^\otimes, \quad f \mapsto (f((1 - \log(v))(s_{\alpha,x}^{\text{cl}})))_{\alpha \in I}.
\]
It is easy to see that this map is $G_{x,v}$-equivariant, and, using the fact that $\operatorname{Fil}^1 M_\zeta$ is $G_{x,v}$-split, we see that the pre-image of $\oplus_{\alpha \in I} \operatorname{Fil}^0 M_\zeta^\alpha$ under this map is precisely $\operatorname{Lie} G_{x,v} + \operatorname{Lie} P_\zeta$.

Here, $P \subset \operatorname{GL}(M_0)$ is the parabolic sub-group stabilizing the Hodge filtration on $M_0$ and $P_\zeta \subset \operatorname{GL}(M_\zeta) = \operatorname{GL}(M_0) \otimes_W L$ is the stabilizer of $\operatorname{Fil}^1 M_\zeta$.

Moreover, we have

\[(3.3.12.2)\]
\[
\operatorname{Lie} U_{w,t,L}^{-2} \cap (\operatorname{Lie} G_{x,v} + \operatorname{Lie} P_\zeta) = \operatorname{Lie} U_{w,t,L}^{-2} \cap \operatorname{Lie} G_{x,v} = \operatorname{Lie} U_{w,t,L}^{-2} \cap \operatorname{Lie} G = \operatorname{Lie} U_{w,t,G,L}^{-2}.
\]

The second to last equality can be deduced as follows: By its definition and by \[(3.2.17)\], we have:

\[
G_{x,v} = (1 - \log(v))\Phi(g_{\text{ab}})^{-1} G_{\text{ab}} \Phi(g_{\text{ab}})^+ (1 + \log(v)).
\]

Given this, we only have to observe that $U_{w,t,L}^{-2}$ commutes with both $(1 - \log(v))$ and $\Phi(g_{\text{ab}})^x$ (for the latter, note that $U_{w,t}^{-2}$ is invariant under conjugation by $\phi$ and that it commutes with $g_{\text{ab}}$).

From \[(3.3.12.1)\] and \[(3.3.12.2)\], we find that $\tilde{\xi}_{x,v}^{-1}(1+\log(v))$ belongs to $U_{w,t,G,L}^{-2}$. In particular, since we already know that $\tilde{\xi}_{x,v}^{-1}(1+\log(u))$ belongs to $U_{w,t,G,L}^{-2}$, this shows that $\log(v^{-1})$ belongs to $\operatorname{Lie} U_{w,t,G,L}^{-2}$, implying in turn that $ev_{x,v}^{-1}$ factors through $S_{\Phi_G}$, for some torsion point $\epsilon \in E_\Phi(\mathcal{O}_L)$. Therefore, $ev_{x,v}^{-2} = (ev_{x,v}^{-1})^2$ must factor through $S_{\Phi_G}$. \[\square\]

3.3.13. Let $\operatorname{Gr}$ (resp. $\operatorname{Gr}^{\text{ab}}$) be the Grassmannian over $K_0$ that parameterizes direct summands of $M_0[p^{-1}]$ (resp. $M_0^{\text{ab}}[p^{-1}]$) of rank rank $\operatorname{Fil}^1 M_\zeta$. Let $\mathcal{F}_{\operatorname{wt},G,K_\zeta}$ be the image of $\operatorname{Pert}_{\operatorname{wt},G,K_\zeta}$ in $\operatorname{GL}(M_0^{\text{ab}})[p^{-1}]$, and let $L_{\operatorname{wt},G,K_\zeta}$ be its Levi quotient. Let $\operatorname{Gr}_G \subset \operatorname{Gr}$ (resp. $\operatorname{Gr}_G^{\text{ab}} \subset \operatorname{Gr}^{\text{ab}}$) be the sub-scheme consisting of those summands such that the attached two-step filtration of $M_0[p^{-1}]$ (resp. of $M_0^{\text{ab}}[p^{-1}]$) is $G_{K_\zeta}$-split (resp. $\mathcal{F}_{\operatorname{wt},G,K_\zeta}$-split). Here, given a ring $A$, a finite free $A$-module $V$, a filtration $F^\bullet V$ of $V$ by $A$-sub-modules that are local direct summands, and a closed sub-group $H \subset \operatorname{GL}(V)$, we say that $F^\bullet V$ is $H$-split if, étale locally on $\operatorname{Spec} A$, there exists a co-character $\mu : G_m \rightarrow H$ that splits $F^\bullet V$; cf. \cite{SR72} IV.2.2.

Let $U \subset \operatorname{Gr}_G^{\text{ab}}$ be the open sub-scheme consisting of summands $F^1 M_0[p^{-1}] \cap W_0 M_0 = 0$ and $F^1 M_0[p^{-1}] + W_1 M_0[p^{-1}] = M_0[p^{-1}]$. Let $U_G = U \cap \operatorname{Gr}_G$. Let $U^{\text{ab}}$ be the image of $U$ in $\operatorname{Gr}_G^{\text{ab}}$, and set $U_G^{\text{ab}} = U^{\text{ab}} \cap \operatorname{Gr}_G^{\text{ab}}$. According to \cite[DOR10 4.2.17]{}, for any point $y \in \operatorname{Gr}_G(\mathcal{O}_L)$, the attached filtration $F^1_y(M_0 \otimes K_\zeta)$ can be split by a co-character $\mu : G_m \rightarrow P_{\operatorname{wt},G,K_\zeta}$. This shows that the natural map $U \rightarrow \operatorname{Gr}^{\text{ab}}$ gives rise to a map $U_G \rightarrow U_G^{\text{ab}}$. Viewed as a $U_G^{\text{ab}}$-scheme, $U_G$ is equipped with a natural action of $U_{w,t,G,K_\zeta}^{-2}$.

We have:

\[\text{Lemma 3.3.14.}\quad \text{If } U_G \text{ with its natural action is a (trivializable) } U_{w,t,G,K_\zeta}^{-2}\text{-torsor over } U_G^{\text{ab}}. \text{ In particular, for any connected component } Z \subset U_G^{\text{ab}}, \text{ we have:}
\]

\[
\dim Z = d - \dim U_{w,t,G,K_\zeta}^{-2}.
\]

Here, $d = \dim G_{K_\zeta} - \dim P_{G,y}$, where $y \in U(\mathcal{O}_L)$ is any point mapping into $Z$ and $P_{G,y} \subset \operatorname{GL}(M_0) \otimes_W \mathcal{K}_0$ is the parabolic sub-group stabilizing the attached filtration $F^1_y(M_0 \otimes \mathcal{K}_0) \subset M_0 \otimes \mathcal{K}_0$.

\[\square\]

3.3.15. Suppose that $L/K_\zeta$ is a finite extension and that $x : R \rightarrow \mathcal{O}_L$ in $\operatorname{LM}(R)$ is such that $\{s_a\}$ is Hodge at $x$. Let $G_x \subset \operatorname{GL}(M_x)$ be the point-wise stabilizer of $\{s_a,x\}$. Set $P_{G,x} = G_x \cap P_L$, and set $d_x = \dim G_{K_\zeta} - \dim P_{G,x}$. It follows from \[(3.5.11)\] that $\operatorname{Fil}^1 M_\zeta$ is $G_x$-split, so that $P_{G,x}$ is a maximal parabolic sub-group of $G_x$.\[\square\]
Lemma 3.3.16. Suppose that $T$ is a quotient domain of $R_{\Phi_L}$ such that $\text{LM}(T)$ is non-empty, and such that, for every $x \in \text{LM}(T)$, $\{s_{\alpha}\}$ is Hodge at $x$. Let $T^{\text{sub}}$ be the image of $R_{\Phi_L}^{\text{sub}}$ in $T$. Then

\[(3.3.16.1) \quad \dim T^{\text{sub}} \leq d - \dim U_{\text{wt},G,K_0}^{-2} + 1,\]

where $d = d_x$, for some (hence any) $x \in \text{LM}(T)$.

Proof. The isomorphism

\[\xi^\text{cl} : M_0^{\text{sub}} \otimes_W R_{\text{an}}^{\text{sub}} \cong M^{\text{sub},\text{an}}\]

over $\hat{U}^{\text{sub},\text{an}}$ along with the filtration $(\xi^\text{cl})^{-1}(\text{Fil}^1 M^{\text{sub},\text{an}})$ defines a map of analytic spaces $f : \hat{U}^{\text{sub},\text{an}} \to U^{\text{sub},\text{an}}$. One can check that this map is unramified; that is, it induces injections on tangent spaces. This essentially follows from the universality of $R^{\text{sub}}$. In fact, it is easily seen that $f$ factors through an étale map $\hat{U}^{\text{sub},\text{an}} \to U^{\text{sub},\text{an}}$, which we will also denote by $f$.

Here, $U_G^{\text{sub}} \subset U^{\text{sub}}$ is the closed sub-scheme consisting of those direct summands of $M_0^{\text{sub}}[p^{-1}]$ whose images in $M_0[p^{-1}]$ are Lagrangian sub-spaces.

By our hypothesis, the composition $(\text{Spf} T)_{\text{an}} \to (\text{Spf} T^{\text{sub}})_{\text{an}} \to U^{\text{sub},\text{an}}$ is carried under $f$ into $U_G^{\text{sub}}$, and since $(\text{Spf} T)_{\text{an}} \to (\text{Spf} T^{\text{sub}})_{\text{an}}$ is dominant, we conclude that $f$ carries $(\text{Spf} T^{\text{sub}})_{\text{an}}$ into $U_G^{\text{sub}}$. The result now follows from (3.3.14). \hfill $\Box$

Proposition 3.3.17. Suppose that there exists a quotient domain $T$ of $R_{\Phi_L}$ that enjoys the following properties:

1. $\text{LM}(T)$ is non-empty, and for every $x \in \text{LM}(T)$, $\{s_{\alpha}\}$ is Hodge at $x$.
2. $\dim T = d + 1$, where $d = \dim G_{K_0} - \dim P_{G,x}$, for one (hence any) $x \in \text{LM}(T)$.

Let $T^{\text{sub}}$ be the image of $R^{\text{sub}}$ in $T$. Then:

1. $\text{rank} B_{\Phi_G} = \dim U_{\text{wt},G,K_0}^{-2}$.
2. Suppose that $x : T^{\text{sub}} \to \mathcal{O}_{L'}$ admits a lift $y : T \to \mathcal{O}_{L'}$ in $\text{LM}(T)$. Then there exist $u(x) \in \hat{E}_{\Phi}(\mathcal{O}_{L'})$ and a finite set of torsion points $\{e_i(x)\}_{1 \leq i \leq s} \subset \hat{E}_{\Phi}(\mathcal{O}_{K_0})$ such that the set of irreducible components of $\text{Spec}(T \otimes_{T^{\text{sub}},x} \mathcal{O}_{K_0})$ that are flat over $\mathcal{O}_{K_0}$ and not contained in the boundary divisor is $\{e_i(x)u(x) \cdot (\text{Spec} \hat{R}_i^{\text{cl}}(\mathcal{O}_{G,K_0},\mathcal{O}_{K_0}))\}_{1 \leq i \leq s}$.

Proof. Note that $E_{\Phi}$ acts naturally on $\text{Spec} \hat{R}_i^{\text{cl}}$ via translation. It is this translation action that appears in (2).

Fix $x : T^{\text{sub}} \to \mathcal{O}_{L'}$ and let $\mathfrak{p}_x \subset T^{\text{sub}}$ be its kernel. Let $\mathfrak{p} \subset T$ be any prime minimal over $\mathfrak{p}_x T$ such that $\mathfrak{p} \cap T^{\text{sub}} = \mathfrak{p}_x$ (this condition is equivalent to requiring that $p \notin \mathfrak{p}$). Then, by [Mat93, Theorem 15.1], we have:

\[(3.3.17.1) \quad \dim T_{\mathfrak{p}} \leq \dim T^{\text{sub}} \leq d - \dim U_{\text{wt},G,K_0}^{-2},\]

Here, we are using the bound (3.3.16.1) on the dimension of $T^{\text{sub}}$. This implies, using the fact that every complete local Noetherian ring is catenary [Mat93, Theorem 29.4], that:

\[(3.3.17.2) \quad \dim T/\mathfrak{p} = \dim T - \dim T_{\mathfrak{p}} \geq \dim U_{\text{wt},G,K_0}^{-2} + 1.\]

We will now treat $T/\mathfrak{p}_x T$ as a quotient of $\mathcal{O}_{L'} \otimes_{R^{\text{sub}},x} R = R^{\beta}_{\Phi,G,\mathcal{O}_{L'}}$. Fix $u(x) \in \hat{E}_{\Phi}(\mathcal{O}_{L'})$ such that

\[\{(1 + \log(u(x))(s_{\alpha,x}))\} \subset \text{Fil}^0 M_0^\beta.\]

Since, by hypothesis, $x$ lifts to $y : T \to \mathcal{O}_{L'}$ in $\text{LM}(T)$, such an element $u(x)$ can be found by (3.3.12). Moreover, (2) of loc. cit. shows that $\text{LM}(T/\mathfrak{p}_x T)$ lies within $\bigcup e \text{eu}(x) \cdot \text{LM}^{\beta}_{\Phi,G,\mathcal{O}_{K_0},\mathcal{O}_{K_0}}$, where $e$ ranges over the torsion points of $E_{\Phi}(\mathcal{O}_{K_0})$. Here, we are viewing elements of $\text{LM}(R^{\beta}_{\Phi,G})$...
as continuous monoid homomorphisms $S_{\Phi,\sigma} \to \mathcal{O}_{\tau_0} \setminus \{0\}$. In particular, for any lift $y \in \text{LM}(T)$ of $x$, if $y$ factors through $T/\mathfrak{p}$ with $\mathfrak{p}$ minimal over $p_x T$, we find that:

$$\dim U_{\text{wt},G,K_0}^{-2} \geq \dim R_{\Phi,G,\sigma_G}^\beta [p^{-1}] \geq \dim (T/\mathfrak{p})[p^{-1}] \geq \dim U_{\text{wt},G,K_0}^{-2}$$

This shows that $\text{rank } B_{\Phi,G} = \dim R_{\Phi,G,\sigma_G}^\beta - 1 = \dim U_{\text{wt},G,K_0}^{-2}$, thus proving (1).

It also proves (2). We only have to make the additional observation that $\text{Spec}(T \otimes_{T^{\text{an},x},\tau_0})$ has finitely many irreducible components. This follows from [EGAIV2, 4.5.10] and the fact that $T$ is finite over a power series ring in finitely many variables over $\mathcal{O}_L$ [MatS9, 29.4(iii)]. □

**Definition 3.3.18.** We will say that a quotient domain $T$ that satisfies properties (1) and (2) in (3.3.17) above is **adapted to** $\{s_{a,0}\}$.

**Corollary 3.3.19.** If $T$ is a quotient domain of $R_{\Theta L}$ adapted to $\{s_{a,0}\}$, we have $\dim T^{\text{an}} = d - \dim U_{\text{wt},G,K_0}^{-2} + 1$ and the map

$$f : (\text{Spf } T^{\text{an}})^{\text{an}} \to U_G^{\text{an}}$$

from (3.3.16) is an étale map of rigid spaces.

**Proof.** It follows from the proof of (3.3.16) that the closed sub-space $f^{-1}(U_G^{\text{an}})$ of $\hat{U}_G^{\text{an}}$ is étale over $U_G^{\text{an}}$. Therefore, it is enough to prove the dimension count. But this follows from (3.3.17.1), (3.3.17.2) and (3.3.17.3). □

3.3.20. Recall that, by construction (cf. 3.1.13), we have over $R_{\text{an}}$ the $G_m$-bi-extension $\Psi_n$ of $\frac{1}{n}Y \times X$. The trivializations of this $G_m$-bi-extension inducing a symmetric trivialization of $(1 \times \lambda^\ell)^* \Psi$ form an $E_\Phi$-torsor $\Xi_{\Phi,R_{\text{an}}}$ over $\text{Spec } R_{\text{an}}$. Let $\Xi_{\Phi,R_{\text{an}}}^G$ be the induced torsor over $E_{\Phi}^G := E_\Phi/E_{\Phi,G}$. For any $y \in Z_{\geq 0}$, write $\Xi_{\Phi,R_{\text{an}}}^{G(r)}$ (resp. $\Xi_{\Phi,R_{\text{an}}}^{G(r)}$) for the $E_\Phi$-torsor (resp. $E_{\Phi}^G$-torsor) obtained via push-forward along the multiplication-by-$n$ endomorphism.

By definition, $\Xi_{\Phi,R_{\text{an}}}^G$ is nothing but $\text{Spec}(\bigoplus_{i \in S_\Phi} \Psi_n(l)^{-1})$. Here, if $l = \sum_i [y_i \otimes x_i] \in S_\Phi$, for $y_i \in \frac{1}{n}Y$ and $x_i \in X$, we set $\Psi_n = \bigotimes_i (y_i \otimes x_i)^* \Psi_n$. Similarly, if $S_{\Phi,G} = \text{ker}(S_\Phi \to S_{\Phi,G})$, then $\Xi_{\Phi,R_{\text{an}}}^{G(r)} = \text{Spec}(\bigoplus_{i \in S_{\Phi,G}} \Psi_n(l)^{-1})^{-1})$.

Over $\Xi_{\Phi,R_{\text{an}}}^G$, we have the tautological trivialization $\tau$ of $\Xi_{\Phi,R_{\text{an}}}^G$. Let $a \in \hat{R}_{\Phi,\sigma}^\beta \subset R$ be an equation for the boundary divisor in $\text{Spec } R$; for example, we can take $a$ to be the product of the lifts to $S_{\Phi,\sigma}$ of a set of generators for $S_{\Phi,\sigma}/S_{\Phi,\sigma}^G$. Then we have a canonical map of $R_{\text{an}}$-schemes

$$\text{Spec } R[a^{-1}] \to \Xi_{\Phi,R_{\text{an}}}^G.$$

In particular, we have a canonical trivialization of $\Xi_{\Phi,R[a^{-1}]}$, and this induces a canonical trivialization of $\Xi_{\Phi,R[a_{\text{an}},-1]}$, for all $r \in Z_{\geq 1}$.

3.3.21. Let $T$ be a quotient domain of $R_{\Theta L}$ adapted to $\{s_{a,0}\}$. Let $T_n$ be its normalization, and let $T_{\text{an}}^{\text{an}}$ be the normalization of $T_{\text{an}}$. Note that both $T_n$ and $T_{\text{an}}^{\text{an}}$ are complete local domains by a theorem of Nagata [EGAIVII, 0, 23.1.5], and the Henselian property of complete local rings.

Over $R$, the tautological trivialization $\tau$ gives rise to the universal deformation $\tau_n$ of $\tau_{n,0}$. We saw in (3.3.16) that, given our choice of co-character $w : G_m \to P_{\text{wt}}$ and the map $\beta : S_{\Phi,G,\sigma_G} \to r$, we have $\tau_n = \tau^\beta_{w,n} \tau_{\beta,n}$, where $\tau^\beta_{w,n}$ is a trivialization of $\Psi_n$ over $R_{\text{an}}$, and

$$\tau_{\beta,n} : S_\Phi \to \text{M}_{R_{\Phi,\sigma}}^{E_\Phi} \subset R[a^{-1}]^\times$$

is the natural map, viewed as a trivialization of the trivial $G_m^{\text{bi-extension}}$ of $\frac{1}{n}Y \times X$. For $r \in Z_{\geq 1}$, we will consider the induced map

$$\tau_r : S_{\Phi}^G \leftarrow S_\Phi \xrightarrow{\tau_n} R[a^{-1}]^\times \to T_n[a^{-1}]^\times.$$
Let \( X = \mathrm{Spf} \ T_n^{\text{an}} \) (resp. \( Y = \mathrm{Spf} \ T_n^{\text{an}} \)) be the analytic space over \( L \) attached to \( T_n^{\text{an}} \) (resp. \( T_n \)). Then \( X \) and \( Y \) are normal and connected \[4.1.1.23\] 7.2.4, 7.3.5, and there is a natural dominant map \( \pi : X \to Y \). For any analytic space \( Z \) over \( L \), let \( \mathcal{O}(Z) \) be the ring of global analytic functions on \( Z \), and let \( \mathcal{O}(Z)^0 \subset \mathcal{O}(Z) \) be the sub-ring of functions \( f \) such that, for all points \( x \in Z(\overline{K})_0 \), \( |f(x)| \leq 1 \). It is shown in \[4.1.1.23\] 7.3.6] that we have \( T_n^{\text{an}} = \mathcal{O}(X)^0 \) (resp. \( T_n = \mathcal{O}(Y)^0 \)).

We thank J. Rabinoff for helpful suggestions regarding the next lemma.

**Lemma 3.3.22.** For every \( l \in S_0^G \), \( \tau_l(l) \in T_n^\times \). In particular, the canonical trivialization of \( \mathcal{B}_l^G_{T_n[a^{-1}]} \) extends to a trivialization of \( \mathcal{B}_{l}^G_{T_n} \).

**Proof.** We can write \( \tau_l(l) = \frac{a^r}{r} \), for some \( f \in T_n \) and some \( r \in \mathbb{Z}_{>0} \). Set \( g = a^r \); to prove the lemma, we have to show that \( f = ug \), where \( u \in T_n^\times \).

By \[3.3.17(2)\], we find that \( |f(y)| = |g(y)| \) for any \( y \in Y(\overline{K})_0 \) such that \( a(y) \neq 0 \). By continuity, this implies that, for any \( y \in Y(\overline{K})_0 \), if \( f/g \) is defined in a neighborhood of \( y \), then \( |(f/g)(y)| = 1 \).

Suppose now that \( p \subset T_n \) is a height 1 prime not containing \( p \) such that \( \text{ord}_p(f) \geq \text{ord}_p(g) \). Then there exists a Zariski open neighborhood \( U \subset Y \) such \( f/g \) is defined on \( U \) and such that \( U \cap V(p) \neq \emptyset \). But then \( |f/g| = 1 \) on all of \( U \), implying that \( g/f \) is also defined on all of \( U \). Similarly, if \( \text{ord}_p(g) \geq \text{ord}_p(f) \), we again find that both \( f/g \) and \( g/f \) are defined on a non-empty open sub-space of \( V(p) \). We therefore find that the complement of the locus where \( f/g \) is defined has codimension at least 2 in \( Y \). Since \( Y \) is normal, \( u = f/g \) must be defined on all of \( Y \), and must also have absolute value 1 everywhere. This implies that \( u \) belongs to \( \mathcal{O}(Y)^{\text{an}} = T_n^\times \).

**Proposition 3.3.23.** There exists \( r \in \mathbb{Z}_{\geq 1} \) such that the image of \( \tau_r \) lies in \( (T_n^{\text{an}}[p^{-1}])^\times \cap T_n^\times \).

**Proof.** Since \( S_0^G \) is a finitely generated group, it is enough to show that, for every \( l \in S_0^G \), there exists \( r \) such that \( \tau_r(l) \in T_n^{\text{an}}[p^{-1}] \).

Now, since \( T_n \) has finite residue field, for \( r_1 \in \mathbb{Z}_{\geq 1} \) large enough, \( \tau_{r_1}(l) \) has residue class 1 in the residue field of \( T_n \), for every \( l \in S_0^G \). Therefore, viewed as an element of \( \mathcal{O}(Y) \), \( \tau_{r_1}(l) \) satisfies \( |\tau_{r_1}(l)(y) - 1| < 1 \), at all points \( y \in Y(\overline{K})_0 \). In particular, \( \log(\tau_{r_1}(l)) \), defined via the usual power series, converges to an analytic function on \( Y \). We obtain a group homomorphism:

\[
\log \circ \tau_{r_1} : S_0^G \to \mathcal{O}(Y).
\]

We can view this as a map \( \log \circ \tau_{r_1} : Y \to \text{Lie}(H) \), where \( H \) is the analytic \( L \)-group attached to the algebraic vector group \( U_{2,L}^2/U_{2,G,L}^2 \).

On the other hand, we have a canonical map \( \ell_u : X \to \text{Lie}(H) \). It attaches to each \( x \in X(\overline{K})_0 \), the unique element \( \ell_u(x) \in \text{Lie} H \otimes \overline{K}_0 \) such that, for all \( \alpha \in I \),

\[
(1 + \ell_u(x))(s_{\alpha,x}^1) \in \mathbb{F}_l^0 M_x^\oplus.
\]

Here, the notation is as in \[3.3.12\]. We are using the fact that \( U_{2,G,K_0}^2 \) acts trivially on the tensors \( s_{\alpha,x}^1 \), so that the action of \( U_{2,G,K_0}^2 \) on them factors through \( H \).

One sees from \[3.3.17\] \( (2) \) that we have \( \log(\tau_{r_1}) = \tau_{r_1}(\ell_u \circ \pi) \), where \( \pi : Y \to X \) is the natural map. Fix \( l \in S_0^G \), and set \( t = \tau_{r_1}(l) \in T_n^\times \). We find that \( \log(t) \in \mathcal{O}(Y) \) actually lies in \( \mathcal{O}(X) \).

Write \( B \subset T_n \) be the \( T_n^{\text{an}} \)-sub-algebra generated by \( t \). Then \( B \) is a quotient of the polynomial algebra \( T_n[X] \). Since \( T_n^{\text{an}} \subset B \), we must have \( B = T_n^{\text{an}}[X]/I \), where \( I \subset T_n^{\text{an}}[X] \) is a prime ideal such that \( I \cap T_n^{\text{an}} = (0) \).

Given any affinoid open \( U \subset X \), there exists \( C \in \mathbb{Q}_{\geq 0} \) such that \( |\log(t)| \) is bounded above by \( C \) on \( U \). In particular, for \( s \in \mathbb{Z}_{\geq 1} \) large enough, \( |p^s \log(t)| \) is bounded above by \( p^{s-1} \).
everywhere on $U$, and so the power series
\[
\exp(p^n \log(t)) = \sum_{i=0}^{\infty} \frac{(p^n \log(t))^i}{i!}
\]
converges to an analytic function on $U$ satisfying $\log(\exp(p^n \log(t))) = p^n \log(t)$. Therefore, there exists $r_2 \in \mathbb{Z}_{\geq 1}$ such that $t^{r_2}$ belongs to $O(U)$. In particular, this implies that $O(U) \otimes_{\mathcal{O}^{\text{ab}}} B$ is a quotient of the finite étale $O(U)$-algebra $O(U)[x]/(x^{r_2} - t^{r_2})$.

Since $U$ was an arbitrary affinoid open, we can conclude by faithfully flat descent and \cite{Dj95a} 7.1.9 that $B[p^{-1}]$ is a finite algebra over $\mathcal{O}^{\text{ab}}[p^{-1}]$. Since $\mathcal{O}^{\text{ab}}$ is normal and $B[p^{-1}]$ is monogenic over $\mathcal{O}^{\text{ab}}[p^{-1}]$, this in turn means that we have
\[
B[p^{-1}] = \mathcal{O}^{\text{ab}}[p^{-1}][x]/(q(x)),
\]
for some monic irreducible polynomial $q(x) \in \mathcal{O}^{\text{ab}}[p^{-1}][x]$. In particular, $B[p^{-1}]$ is finite free over $\mathcal{O}^{\text{ab}}[p^{-1}]$, and, if $s := \deg(q(x))$, $1, t, \ldots, t^{s-1}$ form a basis for $B[p^{-1}]$ over $\mathcal{O}^{\text{ab}}[p^{-1}]$. Given any $r \in \mathbb{Z}_{\geq 1}$, we can write $t^r$ in this basis: $t^r = \sum_{i=0}^{s-1} a_i t^i$. The element $t^r$ belongs to $\mathcal{O}^{\text{ab}}[p^{-1}]$ if and only if the only non-zero co-ordinate is $a_0$. We can check that an element of $\mathcal{O}^{\text{ab}}[p^{-1}]$ is non-zero after completing at any maximal ideal, so the result follows from the previous paragraph.

\begin{proof}
3.3.24. We will put ourselves in the following situation: $B$ will be a complete local normal domain, essentially of finite type and faithfully flat over $\mathcal{O}$. We will assume that $B$ has residue field $k$. For any $r \in \mathbb{Z}_{\geq 1}$, let $E_{\Phi,(r)} \subset E_{\Phi}$ be the kernel of the composition
\[
E_{\Phi} \overset{[r]}{\longrightarrow} E_{\Phi} \to E_{\Phi}^G.
\]
This is a multiplicative group over $\mathbb{Z}$ that is an extension
\[
1 \to E_{\Phi,G} \to E_{\Phi,(r)} \to E_{\Phi}^G[r] \to 1.
\]
Here, $E_{\Phi}^G[r]$ is the $r$-torsion in $E_{\Phi}^G$. In particular, $E_{\Phi,(r)}$ is a torus if and only if $r = 1$, in which case it is identified with $E_{\Phi,G}$.

We will assume that we are given a section $\tau \in E_{\Phi}^G(B[p^{-1}])$: we can view this as a map of groups $\tau : E_{\Phi}^G \to B[p^{-1}]^\times$. For every $r \in \mathbb{Z}_{\geq 1}$, we obtain an $E_{\Phi,(r)}$-torsor $E_{\tau,(r)}, B[p^{-1}] \subset E_{\Phi,B[p^{-1}]}$ over $B[p^{-1}]$: this is just the pre-image of $\tau$ under the map in (3.3.24.1). Let $\tilde{E}_{\tau,(r)}(\sigma_G)$ be the Zariski closure of $E_{\tau,(r)}, B[p^{-1}]$ in the torus embedding $E_{\Phi,B}(\sigma)$.

We will now assume that $\sigma$ has maximal dimension: this is equivalent to requiring that $\sigma$ be generated by a basis for $B_{\Phi} \otimes \mathbb{Q}$. In this case, $E_{\Phi,B}(\sigma)$ has a unique $k$-valued point $x_0$ in its closed orbit. The complete local ring of $E_{\Phi,B}(\sigma)$ at this point can be identified with the completed monoid ring $B[[S_{\Phi,\sigma}]]$, obtained by completing $B[S_{\Phi,\sigma}]$ along the ideal generated by the non-trivial elements of $S_{\Phi,\sigma}$.

Since the map $S_{\Phi,\sigma} \to S_{\Phi,\sigma,G}$ is continuous, $x_0$ is contained within $\tilde{E}_{\tau,(r)}(\sigma_G)$, for all $r \in \mathbb{Z}_{\geq 1}$. Let $A^r_{\tau}$ be the complete local ring of $E_{\tau,(r)}(\sigma_G)$ at $x_0$. Write $A^r$ for $A^r_{(1)}$.

\begin{lemma}
3.3.25. The following statements are equivalent:
\begin{enumerate}
\item There exists $r \in \mathbb{Z}_{\geq 1}$ and a minimal prime $p \subset A^r_{\tau}$ such that $\tau : S_{\Phi}^G \to B[p^{-1}]^\times$ consists of $t^e$ maps into the group of units of the normalization of $A^r_{\tau}/p$.
\item $\tau : S_{\Phi}^G \to B[p^{-1}]^\times \subset A^r[p^{-1}]^\times$ maps into $A^\times_{\tau}$. \(\quad\)
\item $\tau : S_{\Phi}^G \to B[p^{-1}]^\times$ maps into $B^\times$.
\end{enumerate}
\end{lemma}

\begin{proof}
Let $\left[r\right] : B[[S_{\Phi,\sigma}]] \to B[[S_{\Phi,\sigma}]]$ be the finite flat map induced by the $r$-power map $m \mapsto m^r$ on $S_{\Phi,\sigma}$. Then we have:
\[
A^r_{\tau} = A^r \otimes B[[S_{\Phi,\sigma}]], [r] B[[S_{\Phi,\sigma}]].
\]

In particular, there is a finite flat map of $B$-algebras $A^r \to A^7_{(r)}$. The equivalence of (1) and (2) is now clear, once we observe that $\text{Spec } A^r[p^{-1}]$, and hence $\text{Spec } A^r$, is irreducible.

Clearly, (3) implies the other two statements. It remains to show that it is implied by (2). It is enough to show that every continuous map $y : B \to \mathcal{O}_{\mathbb{K}_0}$ lifts to a continuous map $A^7_{(r)} \to \mathcal{O}_{\mathbb{K}_0}$. Indeed, this will mean that, for any $l \in S^\ell_{\mathbb{O}}$, $y(t(l)) \in \mathbb{K}^r$ in fact belongs to $\mathcal{O}_{\mathbb{K}_0}$. Therefore, $t(l)$ must belong to $B^\times$.

Fix $y : B \to \mathcal{O}_{\mathbb{K}_0}$; we can assume that $y$ factors through $\mathcal{O}_{L}$, for some finite extension $L/K$. To show that it admits a continuous lift to $A^7_{(r)}$, we must show that there exists a homomorphism $y^\circ : S^\ell_{\mathbb{O}} \to L^\times$, whose restriction to $S^\ell_{\mathbb{O},\sigma}$ is a continuous map of monoids with values in $\mathcal{O}_{L}$. Indeed, this is equivalent to finding $g \in B_{\mathbb{O},\sigma}$ such that $\nu \circ y^\circ - g$ lies in the interior of $\sigma$. We can take $g$ to be any sufficiently large multiple of $f$.

Let $y^\circ_1 : S_{\mathbb{O}} \to L^\times$ be the map carrying $l$ to $\pi^g(l)$. Then $y^\circ_1 = y_0 y_1^\circ$ does the job for us.

3.3.26. We will return to the notation of (3.3.24). Suppose that $y : T_n \to \mathcal{O}_{\mathbb{K}_0}$ lies in $\text{LM}(T)$. Then we obtain Hodge tensors $\{s_{\alpha,y}\} \subset \text{Fil}^0 M^\sigma_n$ that are killed by the monodromy $N_y$; cf. (3.3.7). One can now reverse the process described in (3.3.4), and obtain Galois-invariant tensors $\{s_{\alpha,\et,y}\} \subset \text{Fil}^0 \mathbb{O}_K[p^{-1}]^\sigma$ that map to $\{s_{\alpha,y}\}$ under the de Rham comparison isomorphism.

Choose a non-degenerate rational polyhedral cone $\hat{\sigma} \subset B_{\mathbb{O}} \otimes \mathbb{R}$ of maximal dimension such that the monodromy element $N_y$ is contained in $\hat{\sigma}$, but not within any proper face of it. Then the map $y^\circ : S_{\mathbb{O}} \to \mathcal{O}_{\mathbb{K}_0}$ induces a continuous map of monoids $\hat{y}^\circ : S_{\mathbb{O},\hat{\sigma}} \to \mathcal{O}_{\mathbb{K}_0} \setminus \{0\}$.

Let $\hat{R} = R_{\mathbb{O}} \otimes \hat{R}_{\mathbb{O},\hat{\sigma}}^\circ$, where $\hat{R}_{\mathbb{O},\hat{\sigma}} \to k$ is the unique map of monoids carrying all non-trivial elements to $0$. Then there exists a unique continuous map $\hat{y} : \hat{R} \to \mathcal{O}_{\mathbb{K}_0}$ whose restriction to $R_{\mathbb{O}}$ agrees with that of $y$, and whose restriction to $\hat{R}_{\mathbb{O},\hat{\sigma}}^\circ$ agrees with the map induced by $\hat{y}^\circ$. Moreover, we have a polarized log-1-motif $(\hat{Q}_0, \hat{\lambda}_0)$ over the residue field of $\hat{R}$ with its induced log structure, and the process in (3.3.24) associates with $\{s_{\alpha,\et,y}\}$ a natural collection of Tate tensors $\{\hat{s}_{\alpha,0}\} \subset \mathcal{D}((\hat{Q}_0)(W))^\circ$.

**Proposition 3.3.27.** Suppose that $\hat{R}_{\mathbb{O}}$ admits a quotient domain $\hat{T}$ adapted to $\{\hat{s}_{\alpha,0}\}$ through which $\hat{y}$ factors. Then $\hat{\tau}_r$ has its image in $T_{\mathbb{O},\bigtimes}^n$.

**Proof.** We first claim that the image of $R_{\mathbb{O}}$ in $\hat{T}$ can be identified with $T_{\mathbb{O}}$. Indeed, denote this image by $\hat{T}_{\mathbb{O}}$. It follows from (3.3.19) that both $(\text{Spf } T_{\mathbb{O}})^{\text{an}}$ and $(\text{Spf } T_{\mathbb{O}})^{\text{an}}$ are irreducible sub-spaces of $\hat{U}_{\mathbb{O}}^{\text{an}}$ of dimension $d - \dim U^{\text{et}}_{\mathbb{O},G,K_0}$ that are étale over $U_{\mathbb{O},G,K_0}^\text{an}$ under the map $f$ in (3.3.14). In particular, they both coincide with the smooth analytic space $f^{-1}(U_{\mathbb{O},G,K_0}^\text{an})$. However, they both contain the point corresponding to $y_{\mathbb{O}}$, and therefore must be equal.

Now, use the trivialization $\tau_{\mathbb{O}}^\beta$ to identify $\Xi_{\Phi,T_{\mathbb{O}}}$ with the trivial torsor $E_{\Phi,T_{\mathbb{O}}}$. In the notation of (3.3.24), take $B = T_{\mathbb{O}}$, $\tau = \tau_r$, $\sigma = \hat{\sigma}$. The proof of (3.3.23) shows that the natural map

$$\text{Spec } \hat{T}_{\mathbb{O}}[(ap)^{-1}] \to \Xi_{\Phi,T_{\mathbb{O}}} = E_{\Phi,T_{\mathbb{O}}}$$
Now, the irreducible components of Spec $R$.

3.3.28. We will assume from now on that $\Xi$ fibers over their images in $\text{Spec } T$. A canonical reduction of structure group to an $E$-torsor over $\text{Spec } T$ has its image in $T^n_{\text{sab}, \times}$. Then $E_{\Phi, T^n_{\text{sab}}}$ has a canonical reduction of structure group to an $E$-torsor $E_{\Phi, T^n_{\text{sab}}}$ over $T^n_{\text{sab}}$. In the notation of (3.3.24), this is the torsor $E_{\tau_r, (r), T^n_{\text{sab}}}$.

If $E_{\Phi, T^n_{\text{sab}}}[r]$ is the induced $E_{\Phi, T^n_{\text{sab}}}[r]$-torsor, we can view $E_{\Phi, T^n_{\text{sab}}}$ as an $E_{\Phi, T^n_{\text{sab}}}[r]$-torsor over $E_{\Phi, T^n_{\text{sab}}}[r]$. As such, we can attach to the rational polyhedral cone $\sigma_G \subset B_{\Phi} \otimes \mathbb{R}$, the twisted torus embedding over $E_{\Phi, T^n_{\text{sab}}}[r]$:

$$E_{\Phi, T^n_{\text{sab}}}[r] \hookrightarrow E_{\Phi, T^n_{\text{sab}}}(\sigma_G).$$

$E_{\Phi, T^n_{\text{sab}}}(\sigma_G)$ admits a natural finite map to $E_{\Phi, T^n_{\text{sab}}}(\sigma_G)$, whose image contains $x_0$. Let $R_{G, (r)}$ be the complete local ring of $E_{\Phi, T^n_{\text{sab}}}(\sigma_G)$ at $x_0$; it is a finite algebra over $R_{\ell, L}$. By (3.3.17), dim $R_{G, (r)} = \dim T^n$.

The following proposition, which is the main technical result of this paper, can be viewed as a (generalized) analogue at the boundary of a result of Noot for the ordinary locus [Noo96, 2.8].

**Proposition 3.3.29.** Let $r$ be the smallest integer such that $\tau_r$ has its image in $T^n_{\text{sab}, \times}$.

1. The map $R_{\ell, L} \to T^n$ factors through $R_{G, (r)}$ and identifies Spec $T^n$ with the normalization of an irreducible component of Spec $R_{G, (r)}$. In particular, the intersection of Spec $T$ with the boundary divisor in Spec $R_{\ell, L}$ is a relative Cartier divisor over $\ell L$.

2. The special fiber Spec $T^n[a^{-1}]$ is reduced if and only if the following conditions hold:
   - $p \nmid r$;
   - $\text{Spec } T^n[a^{-1}]$ has reduced special fiber.

In this case, there exists a finite étale $T^n_{\text{sab}}$-scheme $T' \subset T^n$, such that the $E_{\Phi, T^n_{\text{sab}}}$-torsor $E_{\Phi, T^n_{\text{sab}}}$ canonically descends to an $E_{\Phi, T^n_{\text{sab}}}$-torsor $E_{\Phi, T'_{\text{sab}}}$ over $T'$, and $T^n$ is identified with the complete local ring of $E_{\Phi, T'}(\sigma_G)$ at a point in its closed stratum.

**Proof.** It follows as in the proof of (3.3.27) that Spec $T^n$ can be identified with the normalization of an irreducible component of Spec $R_{G, (r)}$, and so intersects the boundary in a relative Cartier divisor over $\ell L$. This shows (1).

Fix a basis $e_1, \ldots, e_m$ for $S^n_{\Phi}$, and let $u_i = \tau_r(e_i)$, for $1 \leq i \leq m$. Write $r = p^s r'$, where $p \nmid r'$. We have an isomorphism of $T^n_{\text{sab}}$-schemes:

$$E_{\Phi, T^n_{\text{sab}}}[r] = \text{Spec } T^n_{\text{sab}}[x_1, \ldots, x_m]/(x_i - u_i : 1 \leq i \leq m).$$

Now, the irreducible components of Spec $R_{G, (r)}[a^{-1}]$ are smooth with geometrically connected fibers over their images in $E_{\Phi, T^n_{\text{sab}}}[r]$. It follows that Spec $R_{G, (r)}[a^{-1}]$ admits an irreducible component Spec $T^n[a^{-1}]$ with reduced special fiber precisely when $E_{\Phi, T^n_{\text{sab}}}[r]$ admits an irreducible component with reduced special fiber. For this, two conditions must hold: First, Spec $T^n_{\text{sab}}$ has reduced special fiber; Second, for $1 \leq i \leq m$, $u_i$ admits a $p^s$-power root $v_i \in T^n_{\text{sab}}$ such that the associated irreducible component Spec $T_n$ of Spec $R_{G, (r)}$ is identified with the complete local ring of a twisted torus embedding over an irreducible component of the étale $T^n_{\text{sab}}$-scheme:

$$\text{Spec } T^n_{\text{sab}}[x_1, \ldots, x_m]/(x_i^s - v_i : 1 \leq i \leq m).$$
Since the images of $c_i$ in $T_n$ are identified with those of $x_i$, and since $r$ was chosen to be minimal with respect to the condition that $\tau_r$ has its image in $T_n^{ab}$, we find that $r = r'$. From this, follows.

\section{Compactifications of Shimura Varieties of Hodge Type}

\subsection{Shimura varieties and absolute Hodge cycles}

This is essentially a resumé of the first part of \cite{Kis10} §2, but we will be using Pink’s slightly more general definition of Shimura data from \cite{Pin90}, rather than Deligne’s original definition from \cite{Del71}.

\begin{defn}
A Shimura datum is a triple $(G, X, h)$, where $G$ is a connected reductive group over $\mathbb{Q}$ and $X$ is a $G(\mathbb{R})$-homogeneous space, and $h : X \to \text{Hom}(S, G_\mathbb{R})$ is a $G(\mathbb{R})$-equivariant map (here, $S := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Deligne torus) such that:

1. For any $x \in X$, the composite
\[ S \xrightarrow{h_x} G_\mathbb{R} \xrightarrow{Ad} \text{GL}(\text{Lie}(G)) \]
defines a Hodge structure of type $(-1, 1), (0, 0), (1, -1)$ on $\text{Lie}(G)$;

2. For any $x \in X$, $h_x(i)$ is a Cartan involution of $G_\mathbb{R}$;

3. $G^{ad}$ has no $\mathbb{Q}$-simple factors whose $\mathbb{R}$-points form a compact group.

Usually $h$ will be clear from context, and we will use the pair $(G, X)$ to refer to the Shimura datum.

A map $\iota : (G_1, X_1) \to (G_2, X_2)$ of Shimura data consists of a pair $(\iota_1, \iota_2)$, where $\iota_1 : G_1 \to G_2$ is a map of $\mathbb{Q}$-groups, and $\iota_2 : X_1 \to X_2$ is a $G_1(\mathbb{R})$-equivariant map compatible with $h_1$ and $h_2$ in the obvious sense. It is an embedding if $\iota_1$ is a closed embedding.

\begin{defn}
The weight co-character $w_0 : G_{m, \mathbb{R}} \to G_\mathbb{R}$ is the composition $G_{m, \mathbb{R}} \hookrightarrow S \xrightarrow{h} G_\mathbb{R}$, for $x \in X$. Here, the first map is the natural inclusion. The definition of a Shimura datum ensures that $w_0$ maps into the center of $G_\mathbb{R}$ and is independent of the choice of $x$.

\end{defn}

\begin{ass}
We will assume from now on that $w_0$ is defined over $\mathbb{Q}$.

\end{ass}

\begin{defn}
For any $x \in X$, let $\mu_x : G_{m, \mathbb{C}} \to G_\mathbb{C}$ be the co-character
\[ \mathbb{G}_m, \mathbb{C} \xrightarrow{z} (\mathbb{G}_m, \mathbb{C}) \times (\mathbb{G}_m, \mathbb{C}) \xrightarrow{i} S_\mathbb{C} \xrightarrow{h_x} G_\mathbb{C} \]
The reflex field $E(G, X) \subset \mathbb{C}$ of $(G, X)$ is the field of definition of the conjugacy class of $\mu_h$ in $G$. It is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$.

Let $\mathbb{A}_f$ be the ring of finite adèles, let $K \subset G(\mathbb{A}_f)$ be a compact open sub-group of the adèlic points of $G$. We will write $K = K^f K_p$, where $K_p \subset G(\mathbb{Q}_p)$ and $K^f \subset G(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p \subset \mathbb{A}_f$ denotes the sub-ring of adèles with trivial $p$-component.

By results of Baily-Borel, Shimura, Deligne, Milne, Borovoi and others (see \cite{Mil90} §4.5]), the double coset space
\[ \text{Sh}_K(G, X)_\mathbb{C} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \]
has the natural structure of an algebraic variety over $\mathbb{C}$ with a canonical model $\text{Sh}_K(G, X)$ over the reflex field $E(G, X)$.

\begin{lem}
Let $\iota : (G_1, X_1) \hookrightarrow (G_2, X_2)$ be an embedding of Shimura data, let $K_{2,p} \subset G_1(\mathbb{Q}_p)$ be a compact open sub-group, and let $K_{1,p} = K_{2,p} \cap G_2(\mathbb{Q}_p)$.

1. For any compact open sub-group $K_{1,p}^p \subset G_1(\mathbb{A}_f^p)$, we can find a compact open sub-group $K_{2,p}^p \subset G_2(\mathbb{A}_f^p)$ containing $K_{2,p}^p$ such that $\iota$ induces a closed embedding
\[ i : \text{Sh}_{K_{1,p}^p K_{1,p}}(G_1, X_1) \hookrightarrow \text{Sh}_{K_{2,p}^p K_{2,p}}(G_2, X_2) \]

\end{lem}
defined over $E(G_1, X_1)$.

(2) For $K_1 = K^0_1 K_{1,p}$ sufficiently small, we can choose $K_2 = K^0_2 K_{2,p}$ such that, for any compact open sub-group $K'_2 \subset K_2$, the map

$$i' : \text{Sh}_{K'_2}(G_1, X_1) \to \text{Sh}_{K'_2}(G_2, X_2)$$

is again a closed embedding; here $K'_1 = K'_2 \cap G_1(\mathbb{A}_f)$.

**Proof.** The first assertion is [Kis10, 2.1.2].

For the second, we choose $K_1$ small enough so that $K_2$ can be chosen to be neat (cf. [Lan08, 1.4.1.8]). In this case, for any $K'_2 \subset K_2$ and $r = 1, 2$, the map

$$\text{Sh}_{K'_r}(G_r, X_r) \to \text{Sh}_{K'_r}(G_r, X_r)$$

is finite étale. In particular, the map $i'$ is finite and unramified, and so, to check that it is a closed immersion, it is enough to show that it is injective on $\mathbb{C}$-valued points. Suppose that we have two points $(x, g)$ and $(y, h)$ in $X_1 \times G_1(\mathbb{A}_f)$ mapping to the same point in $\text{Sh}_{K'_2}(G_2, X_2)$. This means that they map to the same point in $\text{Sh}_{K_1}(G_1, X_1)$ as well. So we can find $\gamma_r \in G_r(\mathbb{Q})$, for $r = 1, 2$, $k_1 \in K_1$ and $k'_2 \in K_2$ such that

$$(y, h) = (\gamma_1 x, \gamma_1 g k_1) = (\gamma_2 x, \gamma_2 g k'_2).$$

This implies $\gamma_2^{-1} \gamma_1 \in \text{Stab}_{G_1(\mathbb{Q})}(x) \cap g K_2 g^{-1}$. Since $K'_2$ is neat, this last intersection is trivial, which means that $\gamma_2 = \gamma_1 \in G_1(\mathbb{Q})$ and $k'_2 = k_1 \in K'_1$. $\square$

**Definition 4.1.6.** Let $V$ be a $\mathbb{Q}$-vector-space equipped with a symplectic form $\psi$. The **Siegel Shimura datum** associated to $(V, \psi)$ is the pair $(\text{GSp}(V, \psi), S^\pm(V, \psi))$, where $S^\pm(V, \psi)$ is the $\text{GSp}(V, \psi)(\mathbb{R})$-conjugacy class of maps $h : S \to \text{GSp}(V, \psi)_{\mathbb{R}}$ such that:

1. $h$ induces a Hodge structure of type $(1, 0), (0, 1)$ on $V$, so that we have a corresponding decomposition

$$V_{\mathbb{C}} = V_h^{1,0} \oplus V_h^{0,1};$$

2. The symmetric form $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

The reflex field of a Siegel Shimura datum is $\mathbb{Q}$.

Following [Pit90, 2.6], we can also make sense of a Siegel Shimura datum when $V = 0$. We set $\text{GSp}(0) := \mathbb{G}_m$, and $S^\pm(0)$ to be the set of square roots of $-1$ in $\mathbb{C}$ with the obvious action of $\mathbb{R}^\times$. We equip $S^\pm(0)$ with the constant map $h : S^\pm(0) \to \text{Hom}(\mathbb{Z}, \mathbb{G}_m, \mathbb{R})$ carrying either square root to the norm map $z \mapsto z^2$. We denote this Shimura datum by $(\text{GSp}(0), S^\pm(0))$.

4.1.7. Let $(\text{GSp}, S^\pm)$ be the Siegel Shimura datum associated to $(V, \psi)$ (we assume for now that $V \neq 0$), and let $K = K^p K_{p}$ be a compact open sub-group. For $K^p$ sufficiently small, $\text{Sh}_K(\text{GSp}, S^\pm)$ can be interpreted as the fine moduli space of polarized abelian varieties with level structure. To be more precise, we fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that $\psi$ restricts to a bilinear form on $V_{\mathbb{Z}}$ and such that $V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ is stable under $K$. Let $V^*_{\mathbb{Z}} \subset V$ be the dual lattice with respect to $\psi$, and let $d = 2(V^*_{\mathbb{Z}}/V_{\mathbb{Z}})$.

For an abelian scheme $A$ over a $\mathbb{Q}$-scheme $S$, and for a rational prime $\ell$, let $T_\ell(A)$ be the Tate module of $A$: this is an $\ell$-adic sheaf over $S$. Set $T_{\mathbb{Z}}(A) = \prod_{\text{prime } \ell} T_\ell(A)$. Then, for any $\mathbb{Q}$-scheme $S$, $\text{Sh}_K(\text{GSp}, S^\pm)(S)$ parameterizes isomorphism classes of tuples $(A, \lambda, \eta)$, where

- $A$ is an abelian scheme over $S$;
- $\lambda$ is a polarization of $A$ of degree $d$;
- $\eta$ is a section of the étale sheaf

$$\text{Isom}(\mathbb{L} \otimes_{\mathbb{Z}} \psi), (T_{\mathbb{Z}}(A), [\psi_\lambda])) / K.$$
Here, $[x]$ denotes the line spanned by $x$, and $\psi_\lambda$ is the Weil pairing on $T_{\lambda}(A)$ induced by the polarization $\lambda$. The group $K$ acts on the sheaf of isomorphisms via pre-composition. For more details, see [Del71 §4] or [Kot02 §5]. We see in particular that, for $K_p$ sufficiently small, there exists a universal abelian scheme $A$ over $\text{Sh}_K(GSp, S^\pm)$.

If $V = 0$, then $\text{Sh}_K(GSp(0), S^\pm)$ is the finite étale $\mathbb{Q}$-scheme attached to the $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$-set $\mathbb{A}_f^\times/\mathbb{Q}^{\geq 0}K$.

**Definition 4.1.8.** A Shimura datum $(G, X)$ is of **Hodge type** if it admits an embedding

$$(G, X) \hookrightarrow (\text{GSp}(V, \psi), S^\pm(V, \psi))$$

into a Siegel Shimura datum.

**Remark 4.1.9.** Note that, unless $V = 0$, any such Shimura datum will be a Shimura datum in the sense of Deligne. From now on, unless otherwise specified, we will assume $V \neq 0$.

4.1.10. Let $(G, X)$ be a Shimura datum of Hodge type with reflex field $E = E(G, X)$ equipped with an embedding

$$(G, X) \hookrightarrow (\text{GSp}(V, \psi), S^\pm).$$

Let $K = K_pK_p \subset G(\mathbb{A}_f)$ be a compact open subgroup. By (4.1.5), we can find $K' \subset \text{GSp}(\mathbb{A}_f)$ containing $K$ such that the map $\text{Sh}_K(G, X) \to \text{Sh}_{K'}(\text{GSp}, S^\pm)$ is an embedding defined over $E = E(G, X)$. Moreover, we can ensure that $K_p$ and $K'_p$ are sufficiently small, and fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ as above, so that $\text{Sh}_{K'}(\text{GSp}, S^\pm)$ admits an interpretation as a fine moduli space of polarized abelian schemes with level structure. Let $h : A \to \text{Sh}_K(G, X)$ be the pull-back of the universal family of abelian varieties over $\text{Sh}_{K'}(\text{GSp}, S^\pm)$.

Suppose that we have a finite collection of tensors $\{s_{\alpha,B}\} \subset V^\otimes$ whose pointwise stabilizer in $\text{GSp}$ is $G$. Let $V_{\text{dr}, E} = H^1_{\text{dr}}(A/\text{Sh}_K(G, X))$ be the first relative de Rham cohomology of $A$ over $\text{Sh}_K(G, X)$: this is a vector bundle with integrable connection over $\text{Sh}_K(G, X)$. From [Kis10 §2.2], we see that the tensors $\{s_{\alpha,B}\}$, via the de Rham isomorphism, give rise to parallel tensors

$$\{s_{\alpha,\text{dr}}\} \subset H^0(\text{Sh}_K(G, X), F^0V_{\text{dr}, E})^\wedge = 0.$$

Moreover, for any $\alpha$, any field extension $\kappa$ of $E$, any point $x \in \text{Sh}_K(G, X)(\kappa)$, and any choice of algebraic closure $\overline{\kappa}$ of $\kappa$, we get a $\text{Gal}(\overline{\kappa}/\kappa)$-invariant tensor $s_{\alpha,\text{ét},x} \in H^1_{\text{ét}}(A_{x,\overline{\kappa}}, \mathbb{Q}_p)^\otimes$. Given any choice of embeddings $\sigma : \kappa \hookrightarrow \mathbb{C}$ and $i : \mathbb{Q}_p \hookrightarrow \mathbb{C}$, under the isomorphisms

$$H^1_{\text{dr}}(A_{x}) \otimes_{\kappa,\mathbb{C}} \mathbb{C} \xrightarrow{\sim} H^1(\mathbb{A}_{x,\sigma}(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^1_{\text{ét}}(A_{x,\overline{\kappa},\sigma}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p,\mathbb{C}} \mathbb{C},$$

$s_{\alpha,\text{dr},x}$ is carried to $s_{\alpha,\text{ét},x}$. All of these results are easy consequences of the main result of [DMOS02 Ch. 1]: ‘Hodge implies absolutely Hodge for abelian varieties over $\mathbb{C}$’.

We also have one additional piece of compatibility between $s_{\alpha,\text{dr},x}$ and $s_{\alpha,\text{ét},x}$. For this, consider the case where $\kappa$ is a finite extension of $E_v$, the completion at $v$ for some place $v | p$ of $E$. Then we also have the $p$-adic comparison isomorphism

$$H^1_{\text{dr}}(A_{x}) \otimes_{\kappa} B_{\text{dr}} \xrightarrow[\sim]{\sim} H^1_{\text{ét}}(A_{x,\overline{\kappa}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p,\mathbb{C}} B_{\text{dr}}.$$

**Proposition 4.1.11.** Under the $p$-adic comparison isomorphism above, $s_{\alpha,\text{dr},x}$ is carried to $s_{\alpha,\text{ét},x}$.

**Proof.** This is the main result of [Bla94], which applies directly when $A_x$ is in fact defined over a number field. For the generality we need, as pointed out in [Moo98 5.6.3], we can either appeal to a trick of Lieberman as in [Vas99 5.2.16], or we can directly use the fact that $A_x$ arises from the family $A$ defined over the number field $E$. $$\square$$

4.2. Compactifications in characteristic 0.
4.2.1. Let \((G, X)\) be a Shimura datum, and suppose \(G^{\text{ad}} = G_1 \times G_2 \times \cdots \times G_r\), where, for each \(i = 1, 2, \ldots, r\), \(G_i\) is a \(\mathbb{Q}\)-simple group.

**Definition 4.2.2.** An **admissible parabolic sub-group** \(P \subset G^{\text{ad}}\) is one of the form \(P_1 \times P_2 \times \cdots \times P_r\), where, for each \(i\), \(P_i \subset G_i\) is a parabolic sub-group. Furthermore, we require:

- For each \(i\), either \(P_i = G_i\) or \(P_i\) is a maximal proper parabolic sub-group.
- There is at most one \(i\) such that \(P_i \neq G_i\).

In particular, \(G^{\text{ad}}\) is an admissible parabolic sub-group of itself.

In general, an **admissible parabolic sub-group** of \(G\) is the pre-image of an admissible parabolic sub-group of \(G^{\text{ad}}\).

**Remark 4.2.3.** An admissible parabolic corresponds to a **rational boundary component** \(F\) in the terminology of [AMRT10, §III].

Let \(P \subset G\) be an admissible parabolic sub-group and let \(U_P \subset P\) be its unipotent radical. Let

\[
\cdots \supset (\text{Lie } G)_1 \supset (\text{Lie } G)_0 = \text{Lie } P \supset (\text{Lie } G)_{-1} \supset (\text{Lie } G)_{-2} \supset \cdots
\]

be the natural increasing filtration stabilized by \(P\). Then \(\text{Lie } U_P = (\text{Lie } G)_{-1}\) and \(\text{Lie } U_P^2 = (\text{Lie } G)_{-2}\), where \(U_P^2\) is the center of \(U_P\). Choose a co-character \(w : \mathbb{G}_m \to P\) splitting this natural filtration, and satisfying \(w^\prime w_0^{-1}(G_m) \subset G^{\text{der}}\). Here, \(w_0\) is the weight co-character of the Shimura datum \((G, X)\) (cf. [4.1.2]), and \(G^{\text{der}}\) is the derived sub-group of \(G\). Note that \(w\) endows every representation \(V\) of \(P\) with an increasing filtration \(W_V\).

**Proposition 4.2.4.**

1. Given any \(x \in X\), and any representation \(V\) of \(G\), the filtration \(F_x^\ast V_{\mathbb{C}}\) induced by the map \(\mu_x : \mathbb{G}_{m, \mathbb{C}} \to G_{\mathbb{C}}\) (cf. [4.1.4]) determines a rational mixed Hodge structure on \(V\), for which \(W_{\mathbb{C}}\) is the weight filtration. In particular, \(F_x^\ast\) \text{Lie } \text{P} endows \text{Lie } \text{P} with a polarized mixed Hodge structure of weights \((-1, 1), (0, 0), (1, -1), (0, -1), (-1, 0), (-1, -1)\).

2. For every \(x \in X\), there is a canonically associated homomorphism

\[
\varphi : S_C = \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}} \to P_C
\]

splitting the mixed Hodge structure in (4), and whose restriction to the diagonal embedding of \(\mathbb{G}_{m, \mathbb{C}}\) in \(S_C\) is conjugate under \(P(\mathbb{C})\) to the co-character \(w\).

3. Let \(Q_P \subset P\) be the smallest normal sub-group such that the maps \(\varphi_x\), as \(x\) ranges over \(X\), factor through \(Q_{P,R}\). Let \(G_{P,h}\) be the image of \(Q_P\) in \(L_P\). If \(x\) and \(x'\) are in the same connected component of \(X\), then \(\varphi_x\) and \(\varphi_{x'}\) are conjugate under \(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})\). In particular, the assignment \(x \mapsto \varphi_x\) maps every connected component \(X^+\) of \(X\) into a \(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})\)-orbit of co-characters \(S_C \to Q_{P,C}\). This conjugacy class depends only on the \(Q_P(\mathbb{R})\)-orbit of \(X^+\) within \(\pi_0(X)\), and has a natural holomorphic structure, for which the assignment \(x \mapsto \varphi_x\) is holomorphic and \(P(\mathbb{R})\)-equivariant.

4. Let \(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})\) act on \(\pi_0(X)\) via the maps

\[
\pi_0(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})) = \pi_0(Q_P(\mathbb{R})) \to \pi_0(G(\mathbb{R})).
\]

Let

\[
F_{P,X}^{(2)} \subset \pi_0(X) \times \text{Hom}(S_C, Q_P)\]

be the \(Q_P(\mathbb{R})U_P^{-2}(\mathbb{C})\)-orbit of \(\{X^+\} \times \varphi_x\), for any \(x \in X^+\) (this does not depend on the choice of \(x\) by (3)). Consider the map

\[
\varphi : X \to \pi_0(X) \times \text{Hom}(S_C, Q_C)
\]

\[
x \mapsto ([x], \varphi_x),
\]
where $[x]$ denotes the connected component containing $X$. Then the map $\varphi : \varphi^{-1}(F^{(2)}_{P,X^+}) \to F^{(2)}_{P,X^+}$ is an open immersion such that $\pi_0(\varphi)$ is an isomorphism.

(5) Set $F^{(1)}_{P,X^+} = U_P^{-2}(\mathbb{C})F^{(2)}_{P,X^+}$ and $F_{P,X^+} = U_P(\mathbb{C})F^{(2)}_{P,X^+}$. Then $(G_{P,h}, F_{P,X^+})$ is a Shimura datum with reflex field $E(G, X)$.

Proof. (1) follows from [Bry83, 4.1.5]; cf. also [Pin90, §4] and [AMRT10, §III.4]. In (2) $\varpi_x$ is the map denoted $\omega_x \circ h_{\infty}$ in [Pin90, 4.6].

For (3) and (4), cf. [Pin90, 4.11]. That $(G_{P,h}, F_{P,X^+})$ is a Shimura datum follows from the description of the mixed Hodge structure on $\text{Lie} P$ in (1), and the assertion about the reflex field can be found in [Pin90, 12.1].

Remark 4.2.5. The pairs $(Q_{P,X^+}, F^{(2)}_{P,X^+})$ and $(Q_{P,X^+}/U^{-2}_{P,X^+}, F^{(1)}_{P,X^+})$ are mixed Shimura data in the terminology of [Pin90, Ch. 2]. The first of these is a 

4.2.6. The exponential map gives us an isomorphism of group schemes $\text{Lie}(U^{-2}_P) \cong U^{-2}_P$. From now on, for any $\mathbb{Q}$-algebra $R$, we will use this isomorphism to identify $U^{-2}_P(R)$ with the free $R$-module $\text{Lie} U^{-2}_P \otimes \mathbb{Q}$; in particular, $U^{-2}_P(\mathbb{Q})$ will be a rational pure Hodge structure of weight $(-1, -1)$. We will denote by $U^{-2}_P(\mathbb{Q})(-1)$ the twist of $U^{-2}_P(\mathbb{Q})$ that has weight $(0, 0)$.

Recall that, for any $x \in X$, we obtain a Cartan involution $\sigma_x = h_x(i)$ of $G_R$. Let $B$ be the Killing form on $\text{Lie} G$. It follows from [Bry83, 4.1.2] that the pairing $(\omega_{\sigma_x}) : (v, w) \mapsto B(v, \sigma_x(w))$ induces a positive definite symmetric pairing on $U^{-2}_P(\mathbb{R})$, and hence on $U^{-2}_P(\mathbb{R})(-1)$.

Given a connected component $X^+ \subset X$, we have a canonical continuous map $F^{(2)}_{P,X^+} \to U^{-2}_P(\mathbb{R})(-1)$ defined as follows: To every $([x], \varpi) \in F^{(2)}_{P,X^+}$ it attaches the unique element $w \in U^{-2}_P(\mathbb{R})(-1)$ such that $w\varpi w^{-1}$ is defined over $\mathbb{R}$.

Lemma 4.2.7. There is a canonical open homogeneous self-adjoint (with respect to the pairing $(\omega_{\sigma_x})$, for any choice of $x \in X$) convex cone $\mathcal{H}_{P,X^+} \subset U^{-2}_P(\mathbb{R})(-1)$ such that $X^+ \subset F^{(2)}_{P,X^+}$ is the pre-image of $\mathcal{H}_{P,X^+}$ under the map described above.

Proof. This follows from [Pin90, 4.15].

Lemma 4.2.8. Let $P_1, P_2 \subset G$ be two admissible parabolics, and fix $\gamma \in G(\mathbb{Q})$. Then the following statements are equivalent:

(1) $\gamma U^{-2}_{P_1} \gamma^{-1} \subset U^{-2}_{P_2}$.

(2) $\gamma Q_{P_1} \gamma^{-1} \subset Q_{P_2}$.

Proof. See [AMRT10, §III.4.8].

When $P_1, P_2, \gamma$ satisfy the equivalent conditions of the lemma above, we denote this situation by $P_1 \xrightarrow{\gamma} P_2$. If, in addition, $X^+_1$ and $X^+_2$ are two connected components of $X$ such that $\gamma \cdot X^+_1 = X^+_2$, we will write $(P_1, X^+_1) \xrightarrow{\gamma} (P_2, X^+_2)$. For any admissible parabolic $P \subset G$ and any connected component $X^+$ of $X$, let $\mathcal{H}_{P,X^+}$ be the union of the cones $\gamma^{-1}\mathcal{H}_{P_2,X^+_2} \gamma \subset U^{-2}_P(\mathbb{R})(-1)$, for all $\gamma, P_2, X^+_2$ such that $(P, X^+) \xrightarrow{\gamma} (P_2, X^+_2)$.

4.2.9. Given a Shimura datum $(G, X)$ and a compact open $K \subset G(\mathbb{A}_f)$, we denote by CLR$^K(G, X)$ the category whose objects are cusp label representatives (CLRs for short) for $(G, X)$ of level $K$: these are triples $(P, X^+, g)$, where $P \subset G$ is an admissible parabolic, $X^+$ is a connected component of $X$, and $g \in G(\mathbb{A}_f)$. A map $(P_1, X^+_1, g_1) \xrightarrow{\gamma} (P_2, X^+_2, g_2)$ is an element $\gamma \in (G(\mathbb{Q}))$ such that:

• $(P_1, X^+_1) \xrightarrow{\gamma} (P_2, X^+_2)$; cf. the discussion after (4.2.8).
Proof. This is straightforward. The main point is that, for any admissible parabolic sub-group $P \subset G$, there is a unique admissible parabolic $P' \subset G'$ such that $P' \cap G = P$; cf. [Pin90, 4.16].
Given this, the functor \((\iota, g)_*\) sends \((P, X^+, h)\) to \((P', X'^+, \iota(h)g)\), where \(X'^+\) is the connected component of \(X^+\) containing \(X^+\). The properties in the second assertion are now easily checked.

For (3) the only thing to observe is that, if \(\Phi = (P_\Phi, X^+_\Phi, h)\) and \((\iota, g)_*\Phi = (P_\Psi, X^+_\Psi, h)\), then \(P_\Phi = P_\Psi \cap G\).

Finally, (4) is a consequence of [Pin90, 11.10,11.18]. □

**Definition 4.2.12.** A cusp label for \((G, X, K)\) is an isomorphism class of objects in \(\text{CLR}_K(G, X)\). We will denote the set of cusp labels for \((G, X, K)\) by \(\text{Cusp}_K(G, X)\).

**Remark 4.2.13.** If, for every CLR \(\Phi\), \(Q\Phi(\mathbb{Q})\) acts transitively on the set of connected components of \(X\), then one can easily check that the following definition is equivalent to the one above:

A cusp label for \((G, X, K)\) is an equivalence class of pairs \((P, g)\), where \(P\) is an admissible parabolic sub-group of \(G\), and \(g \in G(k_f)\), where the equivalence relation is as follows: \((P, g) \sim (P', g')\) if there exists \(\gamma \in G(\mathbb{Q})\) such that \(\gamma P \gamma^{-1} = P'\) and \(\gamma g \in QP'(k_f)g'K\).

**Definition 4.2.14.** We will now give a long sequence of definitions that have to do with cone decompositions. See [KKMSD73] §1.2 or [Pin90] §5.1 for further details and any unexplained terminology.

1. Given \(\Phi \in \text{CLR}_K(G, X)\), a **rational polyhedral cone** \(\sigma \subset H^*_\Phi\) is a convex polyhedral cone generated by finitely many elements in \(U^{-2}_\Phi(\mathbb{Q})\). We say that \(\sigma\) is **non-degenerate** if it does not contain any lines. We say that \(\sigma\) is **smooth** if it is generated by part of a basis for \(B_\Phi\). Our convention is that all polyhedral cones are closed.

2. A **rational partial polyhedral cone decomposition** or rppcd \(\Sigma_\Phi\) for \(H^*_\Phi\) is a collection of rational, non-degenerate, polyhedral cones of \(H^*_\Phi\) such that:
   
   (a) Any face of a cone in \(\Sigma_\Phi\) is again a cone in \(\Sigma_\Phi\).
   
   (b) The intersection of any two cones in \(\Sigma_\Phi\) is a face of both of them.

   Given an rppcd \(\Sigma_\Phi\) for \(H^*_\Phi\), let \(\Sigma^*_\Phi \subset \Sigma_\Phi\) be the collection of cones \(\sigma\), whose interior \(\sigma^0\) lies in \(H^*_\Phi\).

3. An rppcd \(\Sigma_\Phi\) for \(H^*_\Phi\) is **smooth** if every cone in \(\Sigma_\Phi\) is smooth. It is **complete** if the union of cones in \(\Sigma_\Phi\) is all of \(H^*_\Phi\).

4. An rppcd \(\Sigma_\Phi\) for \(H^*_\Phi\) is a **refinement** of another decomposition \(\Sigma'_\Phi\) if every cone \(\sigma \in \Sigma'_\Phi\) is the union of cones in \(\Sigma_\Phi\) that are contained in \(\sigma\).

5. A **compatible rppcd** \(\Sigma\) for \((G, X, K)\) is a functorial assignment of an rppcd \(\Sigma_\Phi\) to every \(\Phi \in \text{CLR}_K(G, X)\). By this, we mean that, for every map \(\Phi \twoheadrightarrow \Phi'\),

   \[
   \Sigma_{\Phi'} = (\gamma^*)^{-1}\Sigma_{\Phi} := \{ (\gamma^*)^{-1}(\sigma) : \sigma \in \Sigma_{\Phi}\},
   \]

   where \(\gamma^* : H^*_{\Phi'} \rightarrow H^*_{\Phi}\) is the induced embedding of cones. We say that \(\Sigma\) is **smooth** (resp. **complete**) if every \(\Sigma_{\Phi}\) is smooth (resp. complete).

6. A compatible rppcd \(\Sigma\) for \((G, X, K)\) is a **refinement** of another compatible rppcd \(\Sigma'\) if, for every \(\Phi \in \text{CLR}_K(G, X)\), \(\Sigma_\Phi\) is a refinement of \(\Sigma'_\Phi\).

7. The disjoint union \(\bigsqcup_{\Phi \in \text{CLR}_K(G, X)} \Sigma_\Phi\) has a natural left action by \(G(\mathbb{Q})\) and a natural right action by \(K\) over the corresponding actions on \(\text{CLR}_K(G, X)\). We say that \(\Sigma\) is **admissible** if the double coset space

   \[
   G(\mathbb{Q}) \backslash \bigsqcup_{\Phi \in \text{CLR}_K(G, X)} \Sigma_\Phi / K
   \]

   is **finite**. This is equivalent to requiring that, for each \(\Phi\), the number of \(\text{Aut}(\Phi)\)-orbits in \(\Sigma_\Phi\) is finite.

8. Given an admissible rppcd \(\Sigma\) for \((G, X, K)\), let \(\text{Cusp}_{\Sigma}^K(G, X)\) be the set of equivalence classes of pairs \((\Phi, \sigma)\), where \(\Phi \in \text{CLR}_K(G, X)\) and \(\sigma \in \Sigma_\Phi\). Here, we say that two
such pairs \((\Phi, \sigma)\) and \((\Phi', \sigma')\) are equivalent if there is an isomorphism \(\Phi \cong \Phi'\) such that \((\gamma^*)^{-1} \sigma = \sigma'\).

(9) Given an admissible \(\Sigma\) and \([\Phi, \sigma]) \in \text{Cusp}^\Sigma_K(G, X)\), a face of \([\Phi, \sigma])\) is an equivalence class in \(\text{Cusp}^\Sigma_K(G, X)\) of the form \([\Phi', \sigma']\), where, for some \(\gamma \in G(\mathbb{Q})\), \(\Phi' \sim \Phi\) and \(\sigma'\) is a face of \((\gamma^*)^{-1} \sigma\).

(10) Unless otherwise indicated, we will also impose the following condition on admissible rppcds \(\Sigma\) (cf. \([\text{Lan08 \ 6.5.2.25}]\); \([\text{Pin90 \ 7.12}]\): Given \(\Phi \in \text{CLR}_K(G, X)\) and \(\sigma \in \Sigma_\Phi\), let \(\Phi \sim \Phi'\) be a map such that \(\sigma\) is in the image of \(\gamma^* : H_{\Phi'} \subset H_{\Phi}\). Then we require any automorphism \(\eta \in \Gamma_\Phi\) with \(\eta \cdot \sigma \cap \sigma \neq \emptyset\) to act trivially on \(\gamma^*(H_{\Phi})\).

**Remark 4.2.15.** Our definition of an admissible rppcd is stricter than the ones found in \([\text{Har89}]\) and \([\text{Pin90}]\) (which are themselves slightly different from each other). In particular, our ‘admissible’ is Pink’s ‘finite admissible’ in \([\text{Pin90}]\). Our definition, however, agrees with the one found in \([\text{Lan08 \ 7.13}]\) in the PEL case.

**Definition 4.2.17.** We will say that a CLR \(\Phi\) for \((G, X, K)\) is improper if \(P_\Phi = G\). A cusp label \([\Phi]\) is improper if it is the class of an improper CLR. For an improper CLR \(\Phi\), the unipotent radical \(U_\Phi\), and hence the objects \(U_{\Phi}^{-2}\) and \(H_\Phi\), are trivial. Moreover, \(H := Q_\Phi \subset G\) is the smallest normal \(\mathbb{Q}\)-rational sub-group generated by \(X\), and is reductive. The improper cusp labels are in bijection with the double coset space \(G(\mathbb{Q})H(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K\), and given such a cusp label \([\Phi]\), and any admissible rppcd \(\Sigma\), we will also denote by \([\Phi]\) the unique class in \(\text{Cusp}^\Sigma_K(G, X)\) that it gives rise to.

4.2.18. Let \(\Sigma\) be an admissible rppcd for \((G, X, K)\), and let \(\text{Sh}^\Sigma_K(G, X)\) be the associated partial toroidal compactification of \(\text{Sh}_K(G, X)\). Suppose \(\Phi\) and \(\Phi'\) are representatives of the same class in \(\text{Cusp}_K(G, X)\), and let \(\gamma \in G(\mathbb{Q})\) be an element such that \(\Phi \sim \Phi'\). Conjugation by \(\gamma\) gives a morphism \(\int(\gamma) : (Q_\Phi, F_\Phi(2)) \to (Q_{\Phi'}, F_{\Phi'}(2))\) of mixed Shimura data. Suppose that \(q \in Q_\Phi(\mathbb{A}_f)\) is such that \(\gamma q \Phi \subset q \Phi; K\); then \(\int(\gamma)(K_\Phi) = K_\Phi(q^{-1})\). We therefore get a map \([\gamma, q] : \xi_\Phi \to \xi_{\Phi'}\) of mixed Shimura varieties. On the level of complex points, for \((\omega, g) \in F_\Phi(2) \times Q_\Phi(\mathbb{A}_f)\), we have \([\gamma, q]([\omega, g]) = ([\gamma \cdot \omega, \int(\gamma)(g)q])\); that this map descends to a map of varieties over \(E(G, X)\) follows from \([\text{Pin90 \ 11.10}]\). It is easy to check from its explicit description over \(\mathbb{C}\) that \([\gamma, q]\) depends only on \(\gamma\) and not on the choice of \(q\); we will therefore denote it simply by \([\gamma]\).

**Theorem 4.2.19** (Ash-Mumford-Rapoport, Pink). Assume now that \(K\) is neat. Given any admissible rppcd \(\Sigma\) for \((G, X, K)\), there exists an algebraic space \(\text{Sh}^\Sigma_K(G, X)\) over \(E(G, X)\) containing \(\text{Sh}_K(G, X)\) as an open dense sub-variety and satisfying the following properties:

1. The complement \(D^\Sigma_K\) of \(\text{Sh}^\Sigma_K(G, X)\) in \(\text{Sh}^\Sigma_K(G, X)\) is an effective Cartier divisor, along which \(\text{Sh}^\Sigma_K(G, X)\) has no toroidal singularities.
2. If \(\Sigma\) is complete, then \(\text{Sh}^\Sigma_K(G, X)\) is proper; if \(\Sigma\) is smooth, then \(\text{Sh}^\Sigma_K(G, X)\) is smooth.
3. There is a stratification by smooth locally closed sub-varieties

\[
\text{Sh}^\Sigma_K(G, X) = \bigsqcup_{[\Phi, \sigma]} Z_{[\Phi, \sigma]},
\]

where \([\Phi, \sigma]\) ranges over \(\text{Cusp}^\Sigma_K(G, X)\). In this stratification, \(Z_{[\Phi, \sigma]}\) is in the closure of \(Z_{[\Phi', \sigma']}\) if and only if \([\Phi', \sigma']\) is a face of \([\Phi, \sigma]\). In particular, the strata of the form \(Z_{[\Phi]}\), for \([\Phi]\) improper are open and closed in \(\text{Sh}^\Sigma_K(G, X)\).
(4) For every \([\Phi, \sigma] \in \text{Cusp}^*_{[\Phi]}(G, X)\) with representative \((\Phi, \sigma)\), \(Z_{[\Phi, \sigma]}(\sigma)\) is canonically isomorphic to the closed stratum \(\xi_{\Phi}(\sigma)\) in the twisted torus embedding \(\xi_{\Phi}(\sigma) = \xi_{\Phi} \times E_{\Phi} \). In fact, the completion of \(\text{Sh}^*_{[\Phi]}(G, X)\) along \(Z_{[\Phi, \sigma]}(\sigma)\) is canonically isomorphic to the completion of \(\xi_{\Phi}(\sigma)\) along \(\xi_{\Phi}(\sigma)\). In particular, \(\text{Sh}^*_{[\Phi]}(G, X)\) itself is the union of the strata indexed by the improper cusp labels.

(5) Suppose that \((\Phi, \sigma)\) and \((\Phi', \sigma')\) are two representatives of a class in \(\text{Cusp}^*_{[\Phi]}(G, X)\), and let \(\gamma \in G(\mathbb{Q})\) be such that \(\Phi \overset{\gamma}{\Rightarrow} \Phi'\) and \(\gamma^*\sigma' = \sigma\). Then the following diagram commutes:

\[
\begin{array}{ccc}
Z_{[\Phi, \sigma]}(\sigma) & \overset{\gamma}{\sim} & Z_{\Phi}(\sigma') \\
\downarrow \sim & & \downarrow \sim \\
\xi_{\Phi}(\sigma) & \overset{\gamma}{\sim} & \xi_{\Phi'}(\sigma').
\end{array}
\]

Here the isomorphism \([\gamma]\) in the bottom row is the one induced from the isomorphism \([\gamma]: \xi_{\Phi} \rightarrow \xi_{\Phi'}\) discussed in (4.2.18), and the diagonal maps are the isomorphisms from (4).

Proof. This follows from (Pin90) 12.4.

Lemma 4.2.20. Suppose that we are given an embedding \(\iota: (G, X) \hookrightarrow (G', X')\) of Shimura data. Let \(K, K', g\) be as in (4.2.11), so that there is a functor

\[(\iota, g)_*: \text{CLR}_K(G, X) \rightarrow \text{CLR}_{K'}(G', X').\]

(1) Every compatible rppcd \(\Sigma\) for \((G', X', K')\) naturally gives rise to a compatible rppcd \(\Sigma = (\iota, g)^*\Sigma\) for \((G, X, K)\).

(2) If \(\Sigma'\) is admissible (resp. complete), then \(\Sigma\) is also admissible (resp. complete).

(3) If \(\Sigma\) is any admissible rppcd for \((G, X, K)\), then we can find an admissible rppcd \(\Sigma'\) for \((G', X', K')\) such that \((\iota, g)^*\Sigma'\) is a refinement of \(\Sigma\).

(4) In (3), we can choose \(\Sigma'\) such that both \(\Sigma'\) and \((\iota, g)^*\Sigma'\) are smooth.

Proof. For \(\Phi \in \text{CLR}_K(G, X)\) with \(\Phi' = (\iota, g)_* \Phi \in \text{CLR}_{K'}(G', X')\), there is a natural embedding of groups \(U_{\Phi}^{-1} \subset U_{\Phi'}^{-1}\) inducing an embedding cones \(H_{\Phi}^* \subset H_{\Phi'}^*\). So the rational cone decomposition \(\Sigma_{\Phi'}\) will determine a rational cone decomposition \(\Sigma_{\Phi}\) for \(H_{\Phi}^*\). This gives us the induced compatible rppcd \(\Sigma = (\iota, g)^*\Sigma\) in (1).

As for (2), the inheritance of completeness by \(\Sigma\) is clear. The inheritance of admissibility follows from (Hars93) 3.3.

We will only indicate the idea of the proofs of (3) and (4); cf. also (Hars10) 2.4.12. For (3), we proceed as follows: For each \(\Phi' \in \text{CLR}_K(G, X)\), we will define a new cone decomposition \(\Sigma_{\Phi'}\) for \(H_{\Phi}^*\) as follows: it will consist of the intersection of the cones in \(\Sigma_{\Phi}\) with the cones in \(\Sigma_{\Phi}\) along the inclusions \(H_{\Phi} \hookrightarrow H_{\Phi'}\) associated with any \(\Phi \in \text{CLR}_K(G, X)\) such that \(\Phi = (\iota, g)_* \Phi\). One can check now that the \(\Sigma_{\Phi'}\) defined in this way patch together to give an admissible rppcd for \((G', X', K')\); clearly, \((\iota, g)^*\Sigma'\) refines \(\Sigma\). Finally, for (4), take any smooth refinement \(\Sigma''\) of the just constructed \(\Sigma'\). Then, by construction, \((\iota, g)^*\Sigma''\) will also be smooth.

Proof. For (4), we proceed as follows: For each \(\Phi' \in \text{CLR}_K(G, X)\), we will define a new cone decomposition \(\Sigma_{\Phi'}\) for \(H_{\Phi}^*\) as follows: it will consist of the intersection of the cones in \(\Sigma_{\Phi}\) with the cones in \(\Sigma_{\Phi}\) along the inclusions \(H_{\Phi} \hookrightarrow H_{\Phi'}\) associated with any \(\Phi \in \text{CLR}_K(G, X)\) such that \(\Phi = (\iota, g)_* \Phi\). One can check now that the \(\Sigma_{\Phi'}\) defined in this way patch together to give an admissible rppcd for \((G', X', K')\); clearly, \((\iota, g)^*\Sigma'\) refines \(\Sigma\). Finally, for (4), take any smooth refinement \(\Sigma''\) of the just constructed \(\Sigma'\). Then, by construction, \((\iota, g)^*\Sigma''\) will also be smooth.

Proof. For (4), we proceed as follows: For each \(\Phi' \in \text{CLR}_K(G, X)\), we will define a new cone decomposition \(\Sigma_{\Phi'}\) for \(H_{\Phi}^*\) as follows: it will consist of the intersection of the cones in \(\Sigma_{\Phi}\) with the cones in \(\Sigma_{\Phi}\) along the inclusions \(H_{\Phi} \hookrightarrow H_{\Phi'}\) associated with any \(\Phi \in \text{CLR}_K(G, X)\) such that \(\Phi = (\iota, g)_* \Phi\). One can check now that the \(\Sigma_{\Phi'}\) defined in this way patch together to give an admissible rppcd for \((G', X', K')\); clearly, \((\iota, g)^*\Sigma'\) refines \(\Sigma\). Finally, for (4), take any smooth refinement \(\Sigma''\) of the just constructed \(\Sigma'\). Then, by construction, \((\iota, g)^*\Sigma''\) will also be smooth.

Proposition 4.2.21. Let \(\iota: (G, X) \hookrightarrow (G', X')\) be an embedding of Shimura data, and let \(K, K', g\) be as in (4.2.20). Suppose that both \(K\) and \(K'\) are neat. Let \(\Sigma'\) be an admissible rppcd for \((G', X', K')\) and \(\Sigma\) be an admissible rppcd for \((G, X, K)\) refining the one induced from \(\Sigma'\) via (4.2.20) (4).

(1) There exists a natural map \(g|_{\text{Sh}^*_{K}}: \text{Sh}_{K}(G, X) \rightarrow \text{Sh}_{K'}^{\Sigma'}(G', X')\) extending the map \(g|_{K, K'}: \text{Sh}_{K}(G, X) \hookrightarrow \text{Sh}_{K'}^{\Sigma'}(G', X')\) induced by the map \((\iota, g)\) of Shimura data.
(2) Under the map in (1), for any \([\Phi, \sigma] \in \text{Cusp}_K^X(G, X)\), the stratum \(Z_{[(\Phi, \sigma)]}\) maps into the stratum \(Z_{[(\Phi', \sigma')]}\), where \([(\Phi', \sigma')] \in \text{Cusp}_K^{X'}(G', X')\) is determined in the following way: \(\Phi' = (\iota, g), \Phi \in \Sigma_{\Phi'}\) is the minimal cone that contains \(\sigma\).

(3) For \((\Phi, \sigma)\) and \((\Phi', \sigma')\) as in (2), the map \([g]_{[(\Phi, \sigma)]} : Z_{[(\Phi, \sigma)]} \to Z_{[(\Phi', \sigma')]}\) can be described as follows: The natural map of mixed Shimura varieties \(\xi_\Phi \to \xi_{\Phi'}\) is \(\mathbf{E}_\Phi\)-equivariant map over the natural map \(C_\Phi \to C_{\Phi'}\). Note, \([g]_{[(\Phi, \sigma)]}\) is isomorphic to the map between the closed stratum \(Z_\Phi(\sigma) \subset \xi_\Phi\) into the closed stratum of \(Z_{\Phi'}(\sigma') \subset \xi_{\Phi'}(\sigma')\). Similarly, the induced map \([g]_{[(\Phi, \sigma)]}\) on the completion of \(\text{Sh}_K^2(G, X)\) along \(Z_{[(\Phi, \sigma)]}\) is isomorphic to the natural map between the completions of \(\xi_\Phi(\sigma)\) and \(\xi_{\Phi'}(\sigma')\) along their closed strata.

**Proof.** See [Pin90, 6.25,12.4]. □

### 4.3. Chai-Faltings compactifications

#### 4.3.1. Suppose \((G, X) = (\text{GSp}, S^\perp)\) is the Siegel Shimura datum associated with a symplectic space \((V, \psi)\) over \(\mathbb{Q}\). In this case, any maximal parabolic \(P \subset G\) is the stabilizer of an isotropic sub-space \(W \subset V\); or, equivalently, of the filtration

\[
0 = W_{-3}V \subset W_{-2}V = W \subset W_{-1}V = W_\perp \subset W_0V = V.
\]

\(X\) is the union of two connected components, and the choice of connected component can be seen as the choice of an isomorphism of vector spaces \(\mathbb{Q} \overset{\sim}{\to} (\mathbb{Q}(1));\) in other words, as a choice of orientation for \(\mathbb{C}\). In particular, choosing a connected component \(X^+\) of \(X\) gives us an isomorphism \(U_p^{-2}(\mathbb{Q})(-1) \overset{\sim}{\to} U_p^{-2}(\mathbb{Q})\).

We now see (cf. [Mor08, 1.2]):

- \(U_p^{-2} \subset P\) is the sub-group of elements acting trivially on \(W_1\) and can be identified with the group of homomorphisms \(V/W_1V \to W_\perp\), which are symmetric with respect to the identification \(W_\perp = (V/W_1V)^\vee\) induced by \(\psi\). In particular, we can identify \(U_p^{-2}(\mathbb{Q})\) with the group of symmetric bilinear pairings on \(V/W_1V\).
- \(L_p\) is identified with the sub-group of \(\text{GL}(W_{-2}V) \times \text{GSp}(\text{gr}_{W_1}^W V) \times \text{GL}(\text{gr}_{W_0}^W V)\) consisting of elements \((g_1, g_2, g_3)\), where \(g_1 = g_3\) under the identification \(W_\perp = (V/W_1V)^\vee\).
- \(Q_p \subset P\) is the normal sub-group of elements acting trivially on \(\text{gr}_{W_1}^W V\) and acts transitively on the connected components of \(S^\perp\).
- \(G_{p,h} = \text{GSp}(\text{gr}_{W_1}^W V)\), and the Shimura datum \((G_{p,h}, X_p)\) is simply the Siegel Shimura datum associated with the symplectic space \((\text{gr}_{W_1}^W V, \text{gr}_{W_1}^W \psi)\) (with the agreed upon meaning when \(\text{gr}_{W_1}^W V = 0\); cf. 4.1.6).
- Given a connected component \(X^+\), the cone \(H_{p,X^+} \subset U_p^{-2}(\mathbb{R})(-1)\) is the pre-image of the cone of positive definite symmetric pairings on \(V/W_1V\) under the isomorphism \(U_p^{-2}(\mathbb{Q})(-1) \overset{\sim}{\to} U_p^{-2}(\mathbb{Q})\) afforded by the choice of \(X^+\). \(H_{p,X^+}\) is the pre-image of the cone of positive semi-definite symmetric pairings on \(V/W_1V\).

#### 4.3.2. Let \(V_\mathbb{Z} \subset V\) be a polarized \(\mathbb{Z}\)-lattice with discriminant \(d^2\). We would like to reconcile the present notion of cusp labels with the one introduced in 3.1.7. Fix \(n \in \mathbb{Z}_{>0}\), and set

\[
K(n) = \text{GSp}(A_f) \bigcap \ker \left( \text{GL}(V_\mathbb{Z} \otimes \widehat{\mathbb{Z}}) \to \text{GL}(V_\mathbb{Z} \otimes (\mathbb{Z}/n\mathbb{Z})) \right) \subset \text{GSp}(A_f).
\]

This is the *level n sub-group associated with* \(V_\mathbb{Z}\). Let \(M_{V_\mathbb{Z}, n, \psi}\) be the moduli space from 3.1.1; then its fiber over \(\mathbb{Q}\) is canonically identified with \(\text{Sh}_{K(n)}(\text{GSp}(V), S^\perp)\). From now on, we will write \(\mathcal{H}_{K(n)}\) for the fiber of \(M_{V_\mathbb{Z}, n, \psi}\) over \(\mathbb{Z}[(nd)^{-1}]\).

**Lemma 4.3.3.**
(1) There is a bijection between the set of cusp labels for $({\mathrm{GSp}, S^\pm, K(n)})$ defined in (3.2.12) and the set of cusps labels for $(V_\mathbb{Z}, \psi)$ at level $n$ defined in (3.1.7).

(2) Let $[\Phi]$ be a cusp label for $({\mathrm{GSp}, S^\pm, K(n)})$, and let $[[W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi', \delta]]$ be the corresponding cusp label for $(V_{\mathbb{Z}}, \psi)$ at level $n$. Then the free abelian groups $B_\Phi$ from (4.2.9) and $B_{\Phi'}$ from (3.1.13) are naturally identified. In particular, the tori $E_\Phi$ and $E_{\Phi'}$ are naturally identified.

(3) Let $[\Phi]$ and $[[W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi', \delta]]$ be as above. Then the fiber over $\mathbb{Q}$ of the tower

$$\Xi_{\Phi', \delta} \to C_{\Phi', \delta} \to M_{W_\bullet}$$

of an $E_\Phi$-torsor over an abelian scheme over $M_{W_\bullet}$ considered in §47 is naturally isomorphic to the analogous tower $\xi_\Phi \to C_\Phi \to \text{Sh}_{K(n)}$ from (4.2.19) [4].

**Proof.** By (3.1.1) and (4.2.13), a cusp label for $(G, X, K(n))$ is an equivalence class of pairs $(W_\bullet V, g)$, where $W_\bullet V$ is a three step filtration of $V$:

$$0 = W_{-3}V \subset W_{-2}V \subset W_{-1}V = (W_{-2}V)^\perp \subset W_0V = V;$$

and $g \in \text{GSp}(\mathbb{A}_f)$. Under this equivalence relation, $(W_\bullet V, g)$ and $(W'_\bullet V, g')$ are equivalent if there exists $\gamma \in \text{GSp}(\mathbb{Q})$ such that $\gamma (W_\bullet V) = W'_\bullet V$, and if $\gamma g \in Q_{p'}(\mathbb{A}_f)g'K(n)$, where $P' \subseteq \text{GSp}$ is the parabolic sub-group stabilizing $W'_\bullet V$ and $Q_{p'} \subset P'$ is as in (3.3.1).

Given a pair $(W_\bullet V, g)$, we can define an associated torus argument $\Phi'$ as in [3.1.1] for the induced filtration $W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$ as follows (cf. also [10a §3.1]): Set $V_\mathbb{Z}(g) = g \cdot V_{\mathbb{Z}} \subset V_{\mathbb{A}_f}$. We take $Y = \text{gr}^W_1 V_\mathbb{Z}(g)$ and $X = \text{Hom}(W_2V_\mathbb{Z}(g), \mathbb{Z})$; $\lambda_{\Phi'} : Y \to X$ will be the map induced by the pairing between $\text{gr}^W_1 V$ and $W_2V$; the maps $\varphi^\natural_n$ and $\varphi^{\text{mult}}_n$ will just be the reduction mod-$n$ of the isomorphisms $\text{gr}^W_1 V_\mathbb{Z} \cong \text{gr}^W_1 V_\mathbb{Z}(g)$ induced by multiplication by $g$. The cusp label (in the sense of (3.1.7)) associated with the pair $(W_\bullet V, g)$ will now be represented by $(W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}, \Phi', \delta)$, where $\delta$ is any splitting of $W_\bullet V_{\mathbb{Z}/n\mathbb{Z}}$. It is now easy to check that this assignment defines a canonical morphism as in (1), and that it satisfies (2).

Finally, (3) is a consequence of [10a 3.6.11]. Note that $M_{W_\bullet}$ has the meaning explained in (3.1.2) when $\text{gr}^W_1 V_{\mathbb{Z}/n\mathbb{Z}} = 0$.

**Remark 4.3.4.** In the situation of (3) above, we will write the tower $\Xi_{\Phi', \delta} \to C_{\Phi', \delta} \to M_{W_\bullet}$ as $\Xi_{\Phi} \to C_{\Phi} \to \mathcal{K}_{K(n)}$. Implicit in this notation is the fact that the tower only depends on the cusps label $[\Phi]$ for $({\mathrm{GSp}, S^\pm, K(n)})$.

4.3.5. Suppose that we have $\Phi \in \text{CLR}_{K(n)}(\text{GSp}, S^\pm)$ and $\sigma \in \text{H}_\Phi$. The description in (3.1.13) shows that over $\Xi_{\Phi}(\sigma)$ there is a canonical polarized log 1-motif $(Q_{(\Phi, \sigma)}, \lambda_{(\Phi, \sigma)})$ attached to the tautological tuple

$$(B, Y, X, c_\Phi, c'_\Phi, \lambda_{ab}, \lambda_{\Phi'}, \tau_{(\Phi, \sigma)}).$$

The fact that $\sigma$ lies within $\text{H}_\Phi$ implies that the polarization is in fact positive.

Given an affine open Spec $R \subset \Xi_{\Phi}(\sigma)$, let $\text{Spf} \hat{R}$ be its completion along the closed stratum of $\Xi_{\Phi}(\sigma)$. Equip Spec $\hat{R}$ with the induced log structure, and let $U \subset \text{Spec} \hat{R}$ be the complement of the boundary divisor; then by (2.3.4), the positively polarized log 1-motif $(Q_{(\Phi, \sigma)}, \lambda_{(\Phi, \sigma)})$ gives rise to a polarized abelian scheme $(A, \lambda)$ over $U$ that extends to a semi-abelian scheme over Spec $\hat{R}$.

Moreover, there exist: a tautological full level-$n$ structure on $B$, and liftings $c_{\Phi, n}$, $c'_{\Phi, n}$ and $\tau_{(\Phi, \sigma), n}$ of $c_\Phi$, $c'_\Phi$ and $\tau_{(\Phi, \sigma)}$, respectively. These combine to equip $(A, \lambda)$ with a full level-$n$ structure and determine a canonical morphism $U \to \mathcal{K}_{K(n)}$.

**Theorem 4.3.6** (Chai-Faltings,Lan). Fix $n \geq 3$ and let $K = K(n)$. Given a smooth admissible rpped $\Sigma$ for $({\mathrm{GSp}, S^\pm, K})$, there exists a smooth algebraic space $\mathcal{K}_\Sigma = \mathcal{K}_{\Sigma}(\text{GSp}, S^\pm)$ over $\mathbb{Z}[[nd]^{-1}]$ containing $\mathcal{K}$ as an open dense sub-scheme and satisfying the following properties:
(1) The boundary $D_K^\Sigma = \mathcal{J}_K^\Sigma \setminus \mathcal{J}_K$ is a relative effective Cartier divisor with normal crossings, and $\mathcal{J}_K^\Sigma$, equipped with the associated log structure, is log smooth over $\mathbb{Z}[(nd)^{-1}]$.

(2) If $\Sigma$ is complete, then $\mathcal{J}_K^\Sigma$ is proper over $\mathbb{Z}[(nd)^{-1}]$.

(3) The generic fiber $\mathcal{J}_K^\Sigma \otimes \mathbb{Q}$ is naturally isomorphic to the partial toroidal compactification $\mathcal{S}_K^\Sigma(G\!\mathfrak{sp}, S^\pm)$ of $\mathcal{S}_K(G\!\mathfrak{sp}, S^\pm)$ defined in (4.2.19).

(4) $\mathcal{J}_K^\Sigma$ admits a stratification by smooth sub-schemes:

$$\mathcal{J}_K^\Sigma = \bigsqcup_{[(\Phi, \sigma)]} Z_{[(\Phi, \sigma)]},$$

where $[(\Phi, \sigma)]$ ranges over $\text{Cusp}_K^\Sigma(G\!\mathfrak{sp}, S^\pm)$. This is compatible with the stratification of its generic fiber in (4.2.19).

(5) For every $[(\Phi, \sigma)] \in \text{Cusp}_K^\Sigma(G\!\mathfrak{sp}, S^\pm)$ with representative $(\Phi, \sigma)$, $Z_{[(\Phi, \sigma)]}$ is canonically isomorphic to the closed stratum $Z_{\Phi}(\sigma)$ in the twisted torus embedding $\Xi_{\Phi}(\sigma)$. In fact, the completion $\mathfrak{X}_{[(\Phi, \sigma)]}$ of $\mathcal{J}_K^\Sigma$ along $Z_{[(\Phi, \sigma)]}$ is canonically isomorphic to the completion of $Z_{\Phi}(\sigma)$ along $Z_{\Phi}(\sigma)$.

(6) Every complete $\Sigma$ admits a refinement $\Sigma'$ such that $\mathcal{J}_K^\Sigma$ is projective.

Proof.

Assertions (1), (2), (3) and (5) follow from [Lan08, 6.4.1.1]; cf. also [FC90, §V.2]. Let $\mathfrak{X}_{\Phi}(\sigma)$ be the completion of $\Xi_{\Phi}(\sigma)$ along its closed stratum. The canonical strata preserving map

$$j : \mathfrak{X}_{\Phi}(\sigma) \to \mathfrak{X}_{[(\Phi, \sigma)]}$$

implicit in the statement of (5) is characterized by the following property: As in (4.3.5), suppose that we are given an affine open $\text{Spec} \mathbb{R} \subset \Xi_{\Phi}(\sigma)$ with completion $\text{Spf} \mathbb{R}$ along the closed stratum. Let $U \subset \text{Spec} \mathbb{R}$ be the complement of the boundary divisor. Then the restriction of $j$ to $\text{Spf} \mathbb{R}$ is the unique map arising from the canonical map $U \to \mathcal{J}_K(n)$ described in loc. cit.

As for (3), this follows from [Lan10a, 4.1.1], though some care must be taken to descend the cited assertion from $\mathbb{C}$ down to the reflex field. To do this, we take the Zariski closure $\Delta^{\text{tor}}$ of the diagonal $\Delta \subset \mathcal{S}_K(G\!\mathfrak{sp}, S^\pm) \times \mathcal{S}_K(G\!\mathfrak{sp}, S^\pm)$ in $\mathcal{S}_K(G\!\mathfrak{sp}, S^\pm) \times (\mathcal{J}_K^\Sigma \otimes \mathbb{Q})$. We have to check that $\Delta^{\text{tor}}$ is the graph of an isomorphism; that is, it maps isomorphically onto both factors. This can be done over $\mathbb{C}$, which is precisely what is accomplished in the proof of loc. cit.

Note that in [FC90, Lan08, Lan10a], all admissible rppcds $\Sigma$ are assumed to be complete, but the construction, and its comparison with the analytic construction, go through for any smooth admissible rppcd.

Finally, (6) follows from [FC90, V.5.8] (cf. also [Lan08, 7.3.3.4]).

4.4. Intersection with the boundary and Morita’s conjecture.

4.4.1. Fix a Shimura datum $(G, X)$ and an embedding $\iota : (G, X) \hookrightarrow (G\!\mathfrak{sp}, S^\pm)$. Let $(V, \psi)$ be the symplectic space to which $G\!\mathfrak{sp}$ is attached. Fix a polarized $\mathbb{Z}$-lattice $V_\mathbb{Z} \subset V$, and let $d = z(V_\mathbb{Z}^\vee / V_\mathbb{Z})$ be its discriminant: we will assume that $(p, d) = 1$. For every integer $n$ such that $(n, pd) = 1$, let $K(n) \subset G\!(\mathbb{A}_f)$ be the sub-group of level $n$ associated with $V_\mathbb{Z}$, and let $K(n) = K(n) \cap G\!(\mathbb{A}_f)$.

For $n \geq 3$ large enough, it follows from (4.1.3) that there is a closed embedding of Shimura varieties $\mathcal{S}_K(n)(G, X) \hookrightarrow \mathcal{S}_{K(n)}(G\!\mathfrak{sp}, S^\pm)_E$ over the reflex field $E = E(G, X)$. Note that $\mathcal{S}_{K(n)}(G\!\mathfrak{sp}, S^\pm)$ is just the fine moduli space $M_{V_\mathbb{Z}/n, \psi} \otimes \mathbb{Z}[1/n] \otimes \mathbb{Q}$ introduced in (3.1.1), and as such has the canonical integral model $\mathcal{J}_{K(n)}(G\!\mathfrak{sp}, S^\pm) = M_{V_\mathbb{Z}/n, \psi} \otimes \mathbb{Z}[1/n] \otimes \mathbb{Q}$. Let $\mathfrak{p}$ be a prime of $E$ lying above $p$; let $E_\mathfrak{p}$ be the completion of $E$ along $\mathfrak{p}$, and let $\mathfrak{O}_{E_\mathfrak{p}}$ be its ring of integers. Let $\mathcal{J}_{K(n)}(G, X)$ be the Zariski closure of $\mathcal{S}_{K(n)}(G, X)$ in $\mathcal{J}_{K(n)}(G\!\mathfrak{sp}, S^\pm)_{E, \mathfrak{p}}$. To
keep notation light, we will write $K'$ for $K'(n)$, $K$ for $K(n)$, and we will contract the notation for the $\mathcal{O}_{E,(\nu)}$-schemes $\mathcal{A}_K'(n)(GSp, S^\pm)$, $\mathcal{A}_K(n)(G, X)$ to $\mathcal{A}_K'$ and $\mathcal{A}_K$, respectively. Similarly, we will denote $\text{Sh}_{K'}'(n)(GSp, S^\pm)_{\mathbb{E}}$ and $\text{Sh}_{K}(n)(G, X)$ by $\text{Sh}_{K'}$ and $\text{Sh}_{K}$, respectively.

Fix $\Phi \in \text{CLR}_{K'}'(GSp, S^\pm)$, a rational polyhedral cone $\sigma \subseteq H_\Phi$, and suppose that $x_0$ is a closed point in the closed stratum $\mathbb{Z}_\Phi(\sigma)_{\mathbb{E},(\nu)} \subseteq \Xi_{\Phi}(\sigma)_{E,(\nu)}$. We will assume that $x_0$ is valued in a finite field $k$ of characteristic $p$.

Let $R_{\Phi, x_0}$ be the complete local ring of $\Xi_{\Phi}(\sigma)$ at $x_0$, and let $R_{\Phi, x_0,v}$ be its base change over $\mathcal{O}_{E, v}$. By the discussion in (4.3.3), the complement $U_{\Phi, x_0,v}$ of the boundary divisor in $\text{Spec} R_{\Phi, x_0,v}$ admits a canonical map to $\mathcal{A}_K'$.

Abusing notation, we will write $\mathcal{A}_K \cap U_{\Phi, x_0,v}$ for the pull-back of $\mathcal{A}_K$ over $U_{\Phi, x_0,v}$.

**Definition 4.4.2.** We will say that $\text{Spec} R_{\Phi, x_0,v}$ intersects $\mathcal{A}_K'$ at $x_0$ if the Zariski closure of the image of $\mathcal{A}_K \cap U_{\Phi, x_0,v}$ in $\text{Spec} R_{\Phi, x_0,v}$ contains the closed point.

In this case, we will write $T_{\Phi, x_0,v}$ for the quotient of $R_{\Phi, x_0,v}$ attached to this Zariski closure.

4.4.3. Let $\Phi, \sigma$ and $x_0$ be as above, and suppose that $k(x_0)$ is a finite field of characteristic $p$. As in (3.2), denote by $(Q_0, \lambda_0)$ the induced polarized log 1-motif over $k(x_0)$. Let $L/W(k)_Q$ be a finite extension, and let $x : \text{Spec} \mathcal{O}_L \to \Xi_{\Phi}(\sigma)$ be a lift of $x_0$ carrying the generic point into the complement of $\mathbb{Z}_\Phi(\sigma)$. Then we have the induced polarized log 1-motif $(Q_x, \lambda_x)$ over $\mathcal{O}_L$. Also, by the above discussion, $x|_{\text{Spec} L}$ can be canonically viewed as a point of $\mathcal{A}_K$, and if $A_x$ is the attached abelian variety over $L$, it follows from (2.3.3) that there exists a canonical isomorphism of filtered $L$-vector spaces:

$$H^1_{\text{dR}}(A_x/L) \cong H^1_{\text{dR}}(Q_x) \otimes_{\mathcal{O}_L} L.$$  

Similarly, if we fix an algebraic closure $\overline{L}/L$, there exists a canonical isomorphism of $\text{Gal}(\overline{L}/L)$-modules:

$$H^1_{\text{et}}(A_x, \overline{\mathbb{Q}}_p, \mathbb{Z}_p) \cong H^1_{\text{et}}(Q_x, \overline{\mathbb{Q}}_p, \mathbb{Z}_p).$$

Fix a uniformizer $\pi \in \mathcal{O}_L$, and let $W = W(k)$; then (2.4.10) shows that we have natural comparison isomorphisms:

(4.4.3.1) $\mathbb{D}(Q_0)(W_{\Phi, x}) \otimes_W L \cong H^1_{\text{dR}}(A_x/L).

(4.4.3.2) $\mathbb{D}(Q_0)(W_{\Phi, x}) \otimes_W B_{\text{st}} \cong H^1_{\text{et}}(A_x, \mathbb{Q}_p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{st}}.$

4.4.4. Choose tensors $\{s_{\alpha}\} \subseteq V^\otimes$ such that $G \subset GSp$ is their pointwise stabilizer. Let $V_{\text{dR}, E}$ be the relative first de Rham cohomology of the universal abelian scheme $A$ over $\text{Sh}_K$. By (4.1.10), we now obtain parallel Hodge tensors over $\text{Sh}_K$:

$$\{s_{\alpha, \text{dR}}\} \subset H^0(\text{Sh}_K, F^0(V_{\text{dR}, E})^{\nabla = 0}).$$

Also, for any field $\kappa/\mathbb{Q}$ with algebraic closure $\overline{\kappa}$, and any point $x \in \text{Sh}_K(\kappa)$, we obtain a canonical collection of $\text{Gal}(\overline{\kappa}/\kappa)$-invariant tensors

$$\{s_{\alpha, \text{et}, x}\} \subset H^1_{\text{et}}(A_x, \overline{\mathbb{Q}}_p, \mathbb{Q}_p)^\otimes.$$  

The point-wise stabilizer of this collection is isomorphic to $G_{\kappa}$, and is therefore reductive.

Now, fix $\Phi, \sigma$ and $x_0 \in \mathbb{Z}_\Phi(\sigma)_{\mathbb{E},(\nu)}(k)$ as above, and assume that $\text{Spec} R_{\Phi, x_0}$ intersects $\mathcal{A}_K$ at $x_0$. Let $L/E_0$ be a finite extension such that there exists an $L$-valued point $x : \text{Spec} L \to \mathcal{A}_K \cap U_{\Phi, x_0}$ specializing to $x_0$. Then we obtain a collection of Tate tensors $\{s_{\alpha, x_0}\} \subseteq \mathbb{D}(Q_0)(W_{\Phi, x})$ via the process described in (4.3.3). This uses the $p$-adic comparison isomorphism (4.4.3.2) (for a given choice of uniformizer $\pi \in \mathcal{O}_L$) and the Galois-invariant collection $\{s_{\alpha, \text{et}, x}\}$ contained in $H^1_{\text{et}}(A_x, \overline{\mathbb{Q}}_p)^\otimes$. 


Proposition 4.4.5. Let $Z$ be the irreducible component of $\text{Spec} T_{\Phi,\sigma,x_0}$ containing $x$, and let $T$ be its ring of functions. Then:

1. As a quotient of $R_{\Phi,\sigma,x_0} \otimes \mathcal{O}_E$, $T$ is adapted to $\{s_{\alpha,x_0}\}$ in the sense of (3.3.18).
2. The collection $\{s_{\alpha,x_0}\}$ is independent of the lift $x$ in $\text{LM}(T)$ and the choice of uniformizer $\pi$, and is therefore canonically attached to $x_0$ and the irreducible component $\text{Spec} T$.

Proof. We will use some notation from (4.3.5). Set $R = R_{\Phi,\sigma,x_0}$, $U^0 = U^0_{\Phi,\sigma,x_0}$, $\tilde{U} = (\text{Spf} R)^{\text{an}}$ and let $\tilde{U}^0 \subset \tilde{U}^{\text{an}}$ be the complement of the boundary divisor. Recall from (3.2.13) that we have a canonical $\varphi$-equivariant horizontal isomorphism

$$\xi : \mathbb{D}(Q_0)(W_{\Phi,\sigma}) \otimes W R^{\text{an},\text{log}} \to \mathbb{D}(Q_0)(\Phi)(R) \otimes R R^{\text{an},\text{log}}$$

Therefore, the tensors $\{s_{\alpha,x_0}\}$ propagate to parallel, $\varphi$-invariant tensors $\{s^{\text{an}}_{\alpha} = \xi(s_{\alpha,x_0})\}$ contained in $\mathbb{D}(Q_0)(\Phi)(R) \otimes R R^{\text{an},\text{log}}$. By construction, (4.1.11) and the compatibility of the Hyodo-Kato isomorphism with $\xi$ (3.2.15), the tensors $\{s^{\text{an}}_{\alpha}\}$ are Hodge at $x$ (cf. 3.3.8); in fact, they specialize to the tensors $\{s_{\alpha,dR,x}\} \subset H^1_{\text{dR}}(A_x)^{\circ}$.

Also, if $(A,\lambda)$ is the polarized abelian variety over $U^0$ attached to $(Q_0,\lambda_{(\Phi,\sigma)})$, then we have (cf. 2.3.4) a natural identification of filtered coherent sheaves

$$\mathbb{D}(Q_0)(\Phi)(U^0_{\text{Zar}}) = H^1_{\text{dR}}(A/U^0)$$

Here, $U^0_{\text{Zar}}$ is the Zariski site of $U^0$. In particular, $\mathbb{D}(Q_0)(\Phi)(U^0_{\text{Zar}})$ is equipped with canonical horizontal Hodge tensors $\{s_{\alpha,dR}\}$ that also specialize to $\{s_{\alpha,dR,x}\}$ at $x$.

Now, $\{s_{\alpha,dR}\}$ and $\{s^{\text{an}}_{\alpha}\}$ are horizontal tensors that agree at the point $x$ of the smooth, irreducible space $Z^{\text{an}} \cap \tilde{U}^0_{E_0}$. So, for any $x' \in \text{LM}(T)$ and any $\alpha$, the specialization of $s^{\text{an}}_{\alpha}$ at $x'$ is equal to $s_{\alpha,dR,x'}$. In particular, the collection $\{s^{\text{an}}_{\alpha}\}$ is Hodge at every $x' \in \text{LM}(T)$. Since the dimensional criterion in (3.3.17) can be verified from the complex analytic uniformization of $\text{Sh}_K(\mathbb{C})$, this finishes the proof of (1).

(2) follows from the argument in the proof of (Kis10) 2.3.5. Suppose that $x' \in \text{LM}(T)$ is another lift of $x_0$ and that $\{s_{\alpha,x'_0}\}$ is the resulting collection of Tate tensors (for some choice of uniformizer $\pi'$). Then, by (4.1.11), $\{s_{\alpha,dR,x'}\}$ is carried to $\{s_{\alpha,x'_0}\}$ under the Hyodo-Kato isomorphism. From the proof of (1), it now follows that $\{s_{\alpha,x'_0}\} = \{s_{\alpha,x_0}\}$. $\square$

4.4.6. Choose any smooth admissible rppc $\Sigma'$ for $(\text{GSp},S^\pm,K')$, and let $\Sigma$ be the admissible rppc for $(G,X,K)$ induced from $\Sigma'$ (cf. 4.2.20). Associated with this is the partial toroidal compactification $\mathcal{F}_K^{\Sigma}$ (cf. 4.3.6) of $\mathcal{F}_K$: This is an integral model over $\mathcal{O}_{E_0}(w)$ of the partial toroidal compactification $\text{Sh}_K^{\Sigma'}$ of $\text{Sh}_K$ from (4.2.19). Let $\mathbb{D}_K^{\Sigma}$, $\mathcal{F}_K^{\Sigma}$ be the boundary divisor in $\mathcal{F}_K^{\Sigma'}$, let $\mathcal{F}_K^{\Sigma}$ be the Zariski closure of $\text{Sh}_K(G,X)$ in $\mathcal{F}_K^{\Sigma'}$.

Given a finite extension $F/E$ and place $w|v$ of $F$, let $\mathcal{F}_K^{\Sigma}(w)$ (resp. $\mathcal{F}_K^{\Sigma}(w)$) be the normalization of $\mathcal{F}_{K_0}(w)$ (resp. $\mathcal{F}_{K_0}(w)$). If $F = E$, we will omit the sub-script $(w)$, and simply write $\mathcal{F}_K$ and $\mathcal{F}_K^{\Sigma}$.

For every pair $(\Phi_G,\sigma_G) \in \text{CLR}_K(G,X)$ with image $(\Phi,\sigma)$ in $\text{CLR}_K(GSp,S^\pm)$, let $Z_{[(\Phi_G,\sigma_G)]}(w)$ be the Zariski closure of $Z_{[(\Phi,\sigma)]}(w)$ in $\mathcal{F}_K^{\Sigma}(w)$. Also, let $Z_{\Phi_G(\sigma)} \subset \xi_{\Phi_G(\sigma)}(\sigma_G)$ (resp. $\mathcal{F}_{K_0}(w)$) be the normalization of $\Xi_{\Phi_G(\sigma)}(w)$ (resp. $Z_{(\Phi_G(\sigma))}(w)$). Let $Z_{\Phi_G(\sigma)}(w)$ be the closed stratum. Let $\mathcal{E}_{\Phi_G(\sigma)}(w)$, $\mathcal{F}_{\Phi_G(\sigma)}(w)$ be the normalization of $\mathcal{E}_{\Phi_G(\sigma)}(w)$ (resp. $Z_{\Phi_G(\sigma)}(w)$). Here, given a flat $\mathcal{O}_F(w)$-scheme $Y$ and a finite map of $F$-schemes $f : X \to Y$, the normalization of $Y$ in $X$ is the normalization in $X$ of the Zariski closure in $Y$ of the image of $f$.

Thus, for every $(\Phi_G,\sigma_G)$, we obtain a tower:

$$\mathcal{E}_{\Phi_G(\sigma)}(w) \to \mathcal{F}_{\Phi_G(\sigma)}(w) \to \mathcal{F}_{\Phi_G(\sigma)}(w).$$
Again, if \( F = E \), we will omit \((w)\) from the sub-scripts.

**Theorem 4.4.7.**

(1) The intersection \( \mathcal{D}_K^\Sigma = \mathcal{D}_K^\Sigma \cap D_K^\Sigma \) is a relative Cartier divisor over \( \mathcal{O}_{E,(w)} \).

(2) \( \mathcal{D}_K \) is proper if and only if \( G \) is anisotropic modulo center.

**Proof.** (1) is a local statement. Let \( F/E \) be as above. Let \( T \) be a quotient domain of \( \mathcal{O}_{E,(w)} \) and \( \mathcal{O}_F \) is a finite \( \mathcal{O}_{E,(w)} \)-torsor over \( \mathcal{C}_G \).

Suppose that \( F/E \) is a finite extension and \( w/v \) is a place of \( F \) such that the special fiber of \( \mathcal{D}_K \) is reduced. Then:

(3) \( \mathcal{D}_K \) admits a stratification by normal, \( \mathcal{O}_{F,(w)} \)-flat locally closed sub-schemes:

\[
\mathcal{D}_K = \bigcup_{[\Phi_G, \sigma_G]} \mathcal{D}_{K,(w)},
\]

where \([\Phi_G, \sigma_G]\) ranges over \( \text{Cusp} \Sigma \Sigma \Sigma \).

This is compatible with the stratification of \( \text{Sh}_K^\Sigma(G,X) \) described in (4.2.19) (3). Moreover, the completion \( \mathcal{X}_{\Phi_G, \sigma_G} \) of \( \mathcal{X}_{\Phi_G, \sigma_G} \) along \( \mathcal{D}_{\Phi_G, \sigma_G} \) is canonically isomorphic to the completion of \( \mathcal{X}_{\Phi_G, \sigma_G} \) along \( \mathcal{D}_{\Phi_G, \sigma_G} \).

(4) For every \( \Phi_G \in \text{CLR}_G(G,X) \), \( \mathcal{X}_{\Phi_G, \sigma_G} \) is proper if and only if its generic fiber is.

(5) \( \mathcal{X}_{\Phi_G, \sigma_G} \) is identified with \( \text{Sh}_{\Phi_G, \sigma_G} \). This is compatible with the stratification of \( \text{Sh}_K^\Sigma(G,X) \) described in (4.2.19) (3).

Now, (4.4.5) (1), combined with (3.3.27) and (3.3.29) (1), shows that \( \mathcal{D}_K^\Sigma \cap D_K^\Sigma \) is a finite \( \mathcal{O}_{E,(w)} \)-torsor over \( \mathcal{C}_G \).

Assume now that \( F/E \) and \( w/v \) are such that the special fiber of \( \mathcal{D}_K \) is reduced. Let \( x_0 \) be as above. Let \( T \) be a quotient domain of \( T_{\Phi_G, \sigma_G} \otimes \mathcal{O}_F \) such that \( \text{Spec} T \subset \text{Spec}(\mathcal{O}_{E,(w)} \otimes \mathcal{O}_F) \) is an irreducible component, and let \( a \in T \) be an equation for the boundary divisor. If \( T_n \) is the normalization of \( T \), we claim that \( T_n[a^{-1}] \otimes \mathcal{O}_F \) is reduced. Indeed, \( T_n \) can be identified with an irreducible component of the completion \( (\mathcal{D}_K^\Sigma)_{x_0} \) of \( \text{Sh}_K^\Sigma \).

Now \( \text{EGAIV2} \) 7.8.3(vii) shows that there exists a canonical collection of Tate tensors \( \{s_n, x_0\} \) such that \( T \) is adapted to \( \{s_n, x_0\} \). Let \( T_n \) be the normalization of \( T \).

Now (3.3.29) (2) shows that there exists a finite \( \mathcal{O}_{E,(w)} \)-torsor \( \mathcal{E}_{\Phi_G} \) on \( T_n \), which is a canonical reduction of structure group to an \( \mathcal{E}_{\Phi_G} \)-torsor \( \mathcal{E}_{\Phi_G} \).

Moreover, by loc. cit., \( T_n \) can be identified with the complete local ring at \( x_0 \) of the torus embedding \( \mathcal{E}_{\Phi_G} \).

Viewed as a point \( (\text{Spec} T_n)(k(w), x_0) \) belongs to \( \mathcal{E}_{\Phi_G} \). Moreover, the complete local ring \( \mathcal{E}_{\Phi_G} \) at \( x_0 \) can be identified with that of the closed stratum in \( \mathcal{E}_{\Phi_G} \).

It is now easy to deduce (3), (4) and (5) from the corresponding statements in characteristic 0 in (4.2.19), as well as those for the Chai-Faltings compactifications from (1.3.6). One only needs the following additional fact: The normalization of the Zariski closure of \( \text{Sh}_K \) in \( \text{Sh}_K^\Sigma \) can be identified with \( \text{Sh}_K^\Sigma \). This follows from (Har79) 3.4.

By a result of Borel and Harish-Chandra \( \text{BHCl62} \) 5.6 (cf. also \( \text{Pau01} \) 3.1.5), \( G \) is anisotropic modulo center if and only if, for any level \( K \subset G(A_f) \), the Shimura variety \( \text{Sh}_K(G,X) \) is proper. Therefore, (2) is a consequence of (1). \( \mathcal{D}_K \) is proper precisely when the boundary \( \mathcal{D}_K \) is empty for any \( \text{rppcd} \Sigma \). Since \( \mathcal{D}_K \) is flat over \( \mathcal{O}_{E,(w)} \), it is empty if and only if its generic fiber is. \( \square \)
4.4.8. As stated in the introduction, from \([4.4.7]\), we can now easily deduce Morita’s conjecture. Let us recall the Mumford-Tate group \(\text{MT}_A\) associated with an abelian variety \(A\) over \(\mathbb{C}\). One way to define it is as the Tannaka group of the Tannakian sub-category of the category of polarizable rational Hodge structures generated by the rational Hodge structure \(H^1(A(\mathbb{C}), \mathbb{Q})\) (cf. \([\text{DMOS82}}\) Ch. II)). In particular, it is a connected reductive group and there is a canonical map \(h_A : S \to \text{MT}_A\) that gives rise to the Hodge decomposition of \(H^1(A(\mathbb{C}), \mathbb{C})\). The pair \((\text{MT}_A, X_A)\), where \(X_A\) is the \(\text{MT}_A(\mathbb{R})\)-conjugacy class of \(h_A\), is a Shimura datum of Hodge type.

Suppose now that \(A\) is defined over a number field \(F\). The Mumford-Tate group \(\text{MT}_A\) of \(A\) is \(\text{MT}_{\sigma^*A}\), for any embedding \(\sigma : F \to \mathbb{C}\). The main result of \([\text{DMOS82}}\) Ch. 1] shows that \(\text{MT}_A\) does not depend on the choice of embedding. The following theorem is originally due to Paugam-Vasiu-Lee \([\text{Pau04, Vas08, Lee12}}\].

**Theorem 4.4.9.** Suppose that \(\text{MT}_A\) is anisotropic modulo center. Then \(A\) has potentially good reduction at all finite places of \(F\).

**Proof.** Extending \(F\) if necessary, we can assume that it contains the reflex field \(E = E(\text{MT}_A, X_A)\). Fix \(\sigma : F \to \mathbb{C}\), and set \(V = H^1(\sigma^* A(\mathbb{C}), \mathbb{Q})\) equipped with a pairing attached to some polarization of \(\sigma^* A\). We then have a natural embedding of Shimura data:

\[
(\text{MT}_A, X_A) \hookrightarrow (\text{GSp}(V), S^\pm).
\]

Fix a prime \(p\) and a place \(v|p\) for \(E\). Choose \(V_\mathbb{Z} \subset V\), \(n \geq 3\) and a cone decomposition \(\Sigma'\) as in \([4.4.1]\). Let \(\mathcal{J}_K^\prime\) be the attached Chai-Faltings compactification over \(\mathcal{O}_{E(v)}\) as in \(\text{loc. cit.}\).

With \(K = K' \cap \text{MT}_A(\mathbb{A}_f)\), we see from \([\text{Vas08}}\) Fact 2.6] that, in order to show that \(A\) has potentially good reduction at any place \(w|v\) of \(F\), it is enough to show that the Zariski closure \(\mathcal{J}_K\) of \(\text{Sh}_K(\text{MT}_A, X_A)\) in \(\mathcal{J}_K^\prime\) is proper. We have shown this already in \([4.4.7}\) (\(2\)).

**Remark 4.4.10.** We note that in \([\text{Vas12c}}\) 2.2.6] Vasiu deduces the properness of \(\mathcal{J}_K\) from the validity of Morita’s conjecture. Our arrow of deduction points in the opposite direction.

### 4.5. Toroidal Compactifications

#### 4.5.1. Fix a prime \(p\) such that \(p^2 \nmid d\). For the sake of completeness, we will now briefly describe the local structure of \(\mathcal{J}_{K(n)}(\mathbb{Z}_{(p)})\) for \(p | n\), following \([\text{RZ96}}\) and \([\text{Gör03}}\). We also take the opportunity to show how the powerful results of Vasiu-Zink \([\text{VZ10}}\) can be applied to define integral canonical models of Shimura varieties, and to show their existence, even at primes of bad reduction; cf. \([\text{MP13a}}\) for a similar application to orthogonal Shimura varieties.

Let \(M^\text{loc}_{\mathbb{Z}_{(p)}}\) be the projective \(\mathbb{Z}_{(p)}\)-scheme such that, for every \(\mathbb{Z}_{(p)}\)-algebra \(R\), we have:

\[
M^\text{loc}_{\mathbb{Z}_{(p)}}(R) = \left\{\begin{array}{ll}
\text{Isotropic } R\text{-sub-modules } \text{Fil}^1 V_R & \text{if } V_R \\
\text{that are, locally on Spec } R, \text{ direct summands of rank } g.
\end{array}\right.
\]
Let \( \text{rad}(V_{\overline{F}_p}) \subset V_{\overline{F}_p} \) be the radical for the induced alternating form on \( V_{\overline{F}_p} \): this is trivial if \( p \nmid d \), and is a two-dimensional sub-space if \( p \mid d \). For any point \( x_0 \in \mathcal{M}_{Z(p)}^{\text{loc}}(\overline{F}_p) \) let \( \text{Fil}_{x_0}^1 \subset V_{\overline{F}_p} \) denote the attached isotropic sub-space. The proof of the following proposition can be gleaned from [Gôr03 §5.1]

**Proposition 4.5.2.**

1. If \( \text{Fil}_{x_0}^1 \) does not contain \( \text{rad}(V_{\overline{F}_p}) \), then \( \mathcal{M}_{Z(p)}^{\text{loc}} \) is smooth at \( x_0 \).
2. If \( \text{Fil}_{x_0}^1 \) contains \( \text{rad}(V_{\overline{F}_p}) \), then there exists a neighborhood \( U \) of \( x_0 \) and an isomorphism of \( Z(p) \)-schemes
   \[
   U \xrightarrow{\sim} \mathbb{A}^3_{Z(p)} \times_{\text{Spec} Z(p)} \text{Spec} \left( \frac{[T_1, T_2, T_3, T_4]}{(T_1 T_2 - T_3 T_1 - p)} \right).
   \]

We now recall some definitions from [VZ10].

**Definition 4.5.3.** A \( Z(p) \)-scheme \( X \) is healthy regular if it is regular, faithfully flat over \( Z(p) \); and if, for every open sub-scheme \( U \subset X \) containing \( X_0 \) and all generic points of \( X_{\overline{F}_p} \), every abelian scheme over \( U \) extends uniquely to an abelian scheme over \( X \).

A local \( Z(p) \)-algebra \( R \) with maximal ideal \( m \) is quasi-healthy regular if it is regular, faithfully flat over \( Z(p) \), and if every abelian scheme over \( \text{Spec} R \setminus \{m\} \) extends uniquely to an abelian scheme over \( \text{Spec} R \).

**Theorem 4.5.4** (Vasiu-Zink). Let \( R \) be a regular local \( Z(p) \)-algebra with algebraically closed residue field \( k \), of dimension at least 2. Suppose that it admits a surjection
   \[
   R \to W(k)[[T_1, T_2]]/(p - h),
   \]
   where \( h \notin (p, T_1^p, T_2^p, T_1^{p-1} T_2^{-1}) \). Then \( R \) is quasi-healthy regular. In particular, if \( R \) is a formally smooth \( Z(p) \)-algebra of dimension at least 2, then \( R \) is quasi-healthy regular.

**Proof.** This is [VZ10 Theorem 3].

**Corollary 4.5.5.** Suppose either that \( p > 2 \) or that \( g > 2 \). Then \( \mathcal{M}_{Z(p)}^{\text{loc}} \) is healthy regular.

**Proof.** This follows from the local description of \( \mathcal{M}_{Z(p)}^{\text{loc}} \) in [Gôr02] and the criterion from [Kis10].

**Definition 4.5.6.** A pro-scheme \( X \) over \( Z(p) \) (for some prime \( p \)) satisfies the extension property if, for any healthy regular \( Z(p) \)-scheme \( S \), any map \( S \to X \) extends to a map \( S \to X \).

A pro-scheme \( X \) over \( Z(p) \) is an integral canonical model of its generic fiber \( X_0 \) if it is healthy regular and has the extension property. Clearly, if \( X \) is an integral canonical model of \( X_0 \), then it is uniquely determined by this property.

Consider the inverse system \( \{ \mathcal{I}_{K(n), Z(p)} \}_{(n, p) = 1} \), where the set of integers \( n \) satisfying \( (n, p) = 1 \) is ordered by divisibility. The transition map from \( \mathcal{I}_{K(n), Z(p)} \) to \( \mathcal{I}_{K(m), Z(p)} \) for \( m \mid n \) is the finite étale map that extracts a level-\( m \) structure from a level-\( n \) structure in the obvious way.

**Proposition 4.5.7.** Suppose that \( p > 2 \). Then the pro-\( Z(p) \)-scheme
   \[
   \lim_{(n, p) = 1} \mathcal{I}_{K(n), Z(p)}
   \]
   is the integral canonical model of its generic fiber.

**Proof.** That this pro-scheme has the extension property follows from the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties over local fields, and the definition of healthy regularity; cf. [Kis10 2.3.8]. So we only have to check that it is itself healthy regular. When
if \( p > 2, \) \( M_{Z(p)}^{\text{loc}} \) is a local model for \( \mathcal{S}_{K(n),Z(p)} \) in the sense of [RZ96] [DP94]; cf. [Pap00] Theorem 2.2. This means that there exists an étale covering \( V \to \mathcal{S}_{K(n)} \) equipped with an étale map \( V \to M_{Z(p)}^{\text{loc}}. \) So the proposition follows from (4.5.7).

**Definition 4.5.8.** Mildly abusing terminology, we will also refer to \( \mathcal{S}_{K(n),Z(p)} \) as the integral canonical model of its generic fiber; cf. also (4.6.6) below.

4.5.9. We will now review Zarhin’s trick (cf. [Zar77, §2] or [Zar85 §4]), and formulate it in both moduli and group theoretic terms. To begin, fix a quadruple of integers \( x, y, z, w \) such that \( x^2 + y^2 + z^2 + w^2 = d^2 - 1. \) Set

\[
\beta = \begin{pmatrix}
  x & -y & -z & -w \\
  y & -w & z & x \\
  z & w & x & -y \\
  w & -z & y & x
\end{pmatrix} \in \text{End}(\mathbb{Z}^4).
\]

If \( \beta^T \) is the transpose matrix, then we have \( \beta^T \beta = (d^2 - 1)I_4, \) where \( I_4 \) is the identity matrix.

Set \( V' = V^4 \oplus (V^\vee)^4; \) we will equip it with a symplectic pairing \( \psi' \) defined for \( w_1, w_2 \in V^4 \) and \( f_1, f_2 \in (V^\vee)^4 \) by the following formula:

\[
\psi'((w_1, f_1), (w_2, f_2)) = \psi^4(w_1, w_2) + d^2(\psi^\vee)^4(f_1, f_2) - f_2(\beta^T(w_1)) + f_1(\beta^T(w_2)).
\]

In this formula, \( \psi^\vee \) is the dual symplectic pairing on \( V^\vee \) attached to \( \psi; \) \( \psi^4 \) and \( (\psi^\vee)^4 \) are the induced pairings on \( V^4 \) and \( (V^\vee)^4, \) respectively; and \( \beta^T \) is being viewed as an endomorphism of \( V^4 \) via the identification \( V^4 = V \otimes_{\mathbb{Z}} \mathbb{Z}^4. \) The key point now is that \( \psi' \) restricts to a perfect symplectic pairing on \( V'_Z \) (this is a routine check, using the identity \( \alpha^T \alpha = d^2 - 1). \)

The natural action of \( \text{GSp} := \text{GSp}(V, \psi) \) on \( V' \) embeds it within \( \text{GSp}' := \text{GSp}(V', \psi'), \) and gives rise to an embedding of Shimura data

\[
(\text{GSp}, S^\pm(V)) \to (\text{GSp}', S^\pm(V')).
\]

For any \( n \in \mathbb{Z}_{>0}, \) let \( K(n) \subset \text{GSp}'(\mathbb{A}_f) \) be the level \( n \) sub-group attached to \( V'_Z = V^4 \oplus (V^\vee)^4. \) Then we obtain maps of Shimura varieties:

\[
\exists_{\beta,n} : \text{Sh}_{K(n)}(\text{GSp}, S^\pm(V)) \to \text{Sh}_{K(n)'}(\text{GSp}, S^\pm(V')).
\]

Viewed as a map of moduli spaces

\[
\exists_{\beta,n} : \mathcal{S}_{K(n),\mathbb{Q}} = \mathcal{M}_{V^4, n, \psi, \mathbb{Q}} \to \mathcal{M}_{V'_Z, n, \psi', \mathbb{Q}} = \mathcal{S}_{K(n),\mathbb{Q}},
\]

this can be described as follows: Given a tuple \( (A, \lambda, \nu, \alpha) \) on the left hand side, we map it to the tuple \( (A', \lambda', \nu', \alpha') \), where \( A' = A^4 \times (A^\vee)^4, \lambda' = \alpha^4 \times (\lambda \circ \alpha)^4, \) and \( \lambda' : A' \to (A')^\vee \) is the unique principal polarization making the following diagram commute:

\[
\begin{array}{ccc}
A^4 \times A^4 & \xrightarrow{f} & A^4 \times (A^\vee)^4 \\
\downarrow & & \downarrow \lambda^4 \\
\lambda^4 \times A^4 & \xrightarrow{f^\vee} & (A^\vee)^4 \times A^4 = (A')^\vee.
\end{array}
\]

Here, \([\beta] : A^4 \to A^4 \) is the natural map determined by \( \beta \) via the identification \( A^4 = A \otimes_{\mathbb{Z}} \mathbb{Z}^4. \)

This moduli theoretic description makes sense over any base, and so we obtain an extension

\[
\exists_{\beta,n,\mathbb{Z}[(nr)^{-1}]} : \mathcal{S}_{K(n)} \to \mathcal{S}_{K(n),\mathbb{Z}[(nr)^{-1}]}.
\]
Lemma 4.5.10. \( \mathfrak{J}_{K(n)} \) is a finite map. In particular, \( \mathfrak{J}_{K(n)} \) is the normalization of the Zariski closure of the image (under \( \beta \)) of \( \text{Sh}_{K(n)}(\text{GSp}(S^\pm(V))) \) in \( \mathfrak{J}_{K'(n),\mathbb{Z}[(nr)^{-1}]} \).

Proof. It is easy to see, using the Nerón-Ogg-Shafarevich criterion, that the map is proper. We have to check that it is quasi-finite: for this, it is enough to show that, given a polarized abelian variety \((A,\lambda)\) over an algebraically closed field \(k\), there are, up to isomorphism, only finitely many polarized abelian varieties \((B,\mu)\) over \(k\) with \((A^8,\lambda^8) \simeq (B^8,\mu^8)\). This follows from [Zar77, 4.2.2]. \( \square \)

4.5.11. Choose \(n\) so large that \(\beta_{r,n}\) is a closed immersion. Suppose that we have \(\Phi \in \text{CLR}_{K(n)}(\text{GSp}(S^\pm(V)))\), with associated \(\Phi' \in \text{CLR}_{K'(n)}(\text{GSp'}(S'^\pm(V')))\) (cf. 4.2.11). Taking the normalization of the Zariski closure of the tower \(\xi_\Phi \to C_\Phi \to \text{Sh}_{K_\Phi}\) in the tower \(\Xi_{\Phi'} \to C_{\Phi'} \to \mathfrak{J}_{K_{\Phi'}}\), gives us (cf. 4.4.6):

\[
\Xi_\Phi \to C_\Phi \to \mathfrak{J}_{K_\Phi}.
\]

On the other hand, applying the method of (4.3.3) to \(\Phi\), we can attach to it a tuple \((W_\xi V_{Z/nZ},\Phi_1,\delta)\) (cf. 3.1.7), and construct the associated tower as in (3.1.13) (we consider this again over \(\mathbb{Z}[(nr)^{-1}]\)):

\[
\Xi_{\Phi_1,\delta} \to C_{\Phi_1,\delta} \to M_{W_\xi}.
\]

Lemma 4.5.12. The tower \(\Xi_{\Phi_1,\delta} \to C_{\Phi_1,\delta} \to M_{W_\xi}\) is canonically isomorphic to the tower \(\Xi_{\Phi_1,\delta} \to C_{\Phi_1,\delta} \to M_{W_\xi}\). In particular, \(\mathfrak{J}_{K_\Phi}\) is healthy regular with singularities as described in (4.5.2), \(C_\Phi\) is an abelian scheme over \(\mathfrak{J}_{K_\Phi}\), and \(\Xi_\Phi\) is smooth over \(\mathfrak{J}_{K_\Phi}\).

Proof. As in the proof of (4.3.3), the generic fibers of the two towers are canonically isomorphic. Let \((W_\xi V_{Z/nZ},\Phi'_1,\delta')\) be the tuple attached to \(\Phi'\) by the method of (4.3.3). By definition, the tower \(\Xi_{\Phi'} \to C_{\Phi'} \to \mathfrak{J}_{K_{\Phi'}}\), attached to \(\Phi'\) is identified with the tower

\[
\Xi_{\Phi'_1,\delta'} \to C_{\Phi'_1,\delta'} \to M_{W_\xi V_{Z/nZ}}.
\]

To show the lemma, it is enough to show that there is a natural finite map from the tower in (4.5.11.2) to that in (4.5.12.1) extending that on the generic fibers. This can be checked using (4.4.7) and the fact that \(\mathfrak{J}_{K(n)}\) has reduced special fiber, we can now improve (4.3.6) as follows:

Theorem 4.5.13. Fix \(n \geq 3\) and let \(K = K(n)\). There exists a co-final system \(\{\Sigma\}\) of rppcds for \((\text{GSp}(S^\pm,K))\), such that, for each \(\Sigma\) in this system, there exists a regular algebraic space \(\mathfrak{J}_K^\Sigma\) over \(\mathbb{Z}[(nr)^{-1}]\) containing \(\mathfrak{J}_K\) as a open dense sub-scheme and satisfying the following properties:

1. The fiber of \(\mathfrak{J}_K^\Sigma\) over \(\mathbb{Z}[(nd)^{-1}]\) is identified with the Chai-Faltings compactification from (4.3.6).

2. The boundary \(D_K^\Sigma = \mathfrak{J}_K^\Sigma \backslash \mathfrak{J}_K\) is a relative effective Cartier divisor.

3. If \(\Sigma\) is complete, then \(\mathfrak{J}_K^\Sigma\) is proper over \(\mathbb{Z}[(nr)^{-1}]\).

4. \(\mathfrak{J}_K^\Sigma\) admits a stratification by (healthy) regular sub-schemes:

\[
\mathfrak{J}_K^\Sigma = \bigsqcup_{[(\Phi,\sigma)]} \mathbb{Z}((\Phi,\sigma)),
\]

where \([(\Phi,\sigma)]\) ranges over \(\text{Cusp}_K^\Sigma(\text{GSp}(S^\pm))\). This is compatible with the stratification of its generic fiber in (4.2.19) (5).
Remark 4.5.14. Even though the Siegel Shimura variety is of PEL type, the embedding arising
from Zarhin’s trick realizes $\text{GSp}$ as a sub-group in $\text{GSp}'$ in a somewhat subtle way, involving
tensors that are not just those generated by endomorphisms of $V'$. So, even in this special case,
one really needs to be able to work with arbitrary Hodge tensors to show transversality of the
intersection with the boundary.

Remark 4.5.15. Along the lines of [FC90] IV.5.7(5) or [Lan08] 6.4.1.1(6), we can now check
that the universal polarized abelian scheme over $T$ induces an embedding of the corresponding Shimura data. We can then check that the induced
pairing $\iota$ on $T$ is perfect.

Remark 4.5.16. The same method allows one to construct and describe good compactifications of integral models of many PEL Shimura varieties with parahoric level at $p$. There are essentially
two reducedness of the special fiber, which is known in many cases; cf. [Gör03, [Pap00, PZ12]
and having an analogue of the explicit description (as in (4.5.12)) of the
requirements: Reducedness of the special fiber, which is known in many cases; cf. [Gör03,
induces an embedding of the corresponding Shimura data. We can then check that the induced
pairing $\iota$ on $T$ is perfect.

Lemma 4.6.2. Suppose that $(G, X)$ is a Shimura datum of Hodge type such that $G$ is unramified at $p$ with reductive model $G_{\mathbb{Z}(p)}$.

(1) There exists a $p$-integral embedding $\iota : (G, X) \hookrightarrow (G, X')$ of Shimura data with representative $(\Phi, \sigma)$, $Z_{\iota(\Phi, \sigma)}$ is canonically
isomorphic to the closed stratum $Z_0(\sigma)$ in the twisted torus embedding $\mathcal{F}_0(\sigma)$. In fact,
the completion $\mathcal{X}_{\iota(\Phi, \sigma)}$ of $\mathcal{F}_0(\sigma)$ along $Z_{\iota(\Phi, \sigma)}$ is canonically isomorphic to the completion
of $\mathcal{F}_0(\sigma)$ along $Z_0(\sigma)$.

(6) Every complete $\Sigma$ admits a refinement $\Sigma'$ such that $\mathcal{F}_K^{\Sigma'}$ is projective.

4.6. Smooth compactifications of Hodge type. Let us start with a Shimura datum $(G, X)$,
and a rational prime $p$. Suppose that $G$ is unramified at $p$: this means that $G_{\mathbb{Q}_p}$ is quasi-split
and splits over an unramified extension. This is also equivalent to saying that $G$ has a reductive
model $G_{\mathbb{Z}(p)}$ over $\mathbb{Z}(p)$.

Definition 4.6.1. An embedding $\iota : (G, X) \hookrightarrow (G', X')$ of Shimura data is said to be $p$-
integral if there exist reductive models $G_{\mathbb{Z}(p)}$ of $G$ and $G'_{\mathbb{Z}(p)}$ of $G'$ over $\mathbb{Z}(p)$, and if the
embedding of groups $G \hookrightarrow G'$ underlying $i$ is induced by an embedding $G_{\mathbb{Z}(p)} \hookrightarrow G'_{\mathbb{Z}(p)}$.

Lemma 4.6.2. Suppose that $(G, X)$ is a Shimura datum of Hodge type such that $G$ is unramified at $p$ with reductive model $G_{\mathbb{Z}(p)}$.

(1) There exists a $p$-integral embedding of Shimura datum $\iota : (G, X) \hookrightarrow (G, X')$ into a
Siegel Shimura datum.

(2) Suppose that the embedding $G \hookrightarrow GSp$ arises from an embedding $G_{\mathbb{Z}(p)} \hookrightarrow GSp_{\mathbb{Z}(p)} =
GSp(V_{\mathbb{Z}(p)}, \psi)$, for a symplectic $\mathbb{Z}(p)$-lattice $V_{\mathbb{Z}(p)} \subset V$. Then there exists a collection of
tensors $\{s_\alpha\} \subset V_{\mathbb{Z}(p)}$ such that $G_{\mathbb{Z}(p)} \subset GSp_{\mathbb{Z}(p)}$ is the pointwise stabilizer of $\{s_\alpha\}$.

Proof. For (1), choose any embedding $\iota' : (G, X) \hookrightarrow (G, X')$ of Shimura data. By
[FC90] 2.3.1, there exist a $\mathbb{Z}(p)$-lattice $V_{\mathbb{Z}(p)} \subset V$ and an embedding $G_{\mathbb{Z}(p)} \hookrightarrow GL(V_{\mathbb{Z}(p)})$ that
induces $\iota'$ over $\mathbb{Q}$. The problem is that $\psi$ might not induce a perfect $\mathbb{Z}(p)$-pairing on $V_{\mathbb{Z}(p)}$. To take care of this, we apply Zarhin’s trick (4.5.7), which tells us that there exists a perfect
pairing $\psi'$ on $V'_{\mathbb{Z}(p)} = (V_{\mathbb{Z}(p)} \times V'_{\mathbb{Z}(p)})^4$ and an embedding $GSp(V, \psi) \hookrightarrow GSp(V', \psi')$. This also
induces an embedding of the corresponding Shimura data. We can then check that the induced
embedding \((G, X) \hookrightarrow (\text{GSp}(V', \psi'), S^\pm)\) arises from an embedding \(G_{\mathbb{Z}(p)} \hookrightarrow \text{GSp}(V'_{\mathbb{Z}(p)}, \psi')\) and is thus \(p\)-integral.

We note that, in [Kis10 2.3.1], when \(p = 2\), Kisin restricts attention to groups \(G\) without a factor of type \(B\). In fact, this restriction, which arises from a corresponding restriction in a result of G. Prasad and J.-K. Yu, is unnecessary. To be more precise, consider the following assertion:

- Let \(i : \mathcal{G} \rightarrow \mathcal{H}\) be a map of reductive group schemes over \(\mathbb{Z}_p\), whose fiber over \(\mathbb{Q}_p\) is a closed embedding; then \(i\) is a closed embedding.

According to [PY06 1.3], this assertion is true as long as \(\mathcal{G}_{\mathbb{Z}_p}\) does not admit a normal sub-group isomorphic to \(\text{SO}_{2n+1}\). In [Kis10 2.3.1], this assertion needs to be applied when \(\mathcal{G} = G_{\mathbb{Z}_p}\). But the classification of Shimura data of Hodge type in [Del79 1.3] shows that \(G\) can never have a normal sub-group isomorphic to \(\text{SO}_{2n+1}\). Indeed, any factor of the derived sub-group \(G^{\text{der}}\) of type \(B\) will be simply connected.

(2) follows from [Kis10 1.3.2].

4.6.3. We will now fix an unramified-at-\(p\) Shimura datum \((G, X)\), a reductive model \(G_{\mathbb{Z}(p)}\), for \(G\) over \(\mathbb{Z}_\mathbb{Z}(p)\), and a \(p\)-integral embedding \((G, X) \hookrightarrow (\text{GSp}, S^\pm)\). Let \((V, \psi)\) be the symplectic space to which \(\text{GSp}\) is attached, and let \(V_{\mathbb{Z}(p)} \subset V\) be the \(\mathbb{Z}(p)\)-lattice on which \(\psi\) such that \(G \hookrightarrow \text{GSp}\) is obtained from an embedding of reductive \(\mathbb{Z}(p)\)-groups \(G_{\mathbb{Z}(p)} \hookrightarrow \text{GSp}(V_{\mathbb{Z}(p)})\). Fix a polarized \(\mathbb{Z}\)-lattice \(V \subset V\) such that \(V \otimes \mathbb{Z}(p) = V_{\mathbb{Z}(p)}\), and of discriminant \(\mathfrak{d}\). For every integer \(n\) such that \((n, p) = 1\), let \(K(n) \subset \text{GSp}(\mathbb{A}_f)\) be the sub-group of level \(n\) associated with \(V\), and let \(K(n) = K'(n) \cap G(\mathbb{A}_f)\). Set \(K_p = \text{GSp}(\mathbb{Q}_p)\), \(K_p = G_{\mathbb{Z}(p)}(\mathbb{Z}_p)\).

For \(n\) large enough, it follows from (4.1.3) that we have a closed embedding of Shimura varieties \(\text{Sh}_{K(n)}(G, X) \hookrightarrow \text{Sh}_{K'(n)}(\text{GSp}, S^\pm)_E\) over the reflex field \(E = E(G, X)\). Note that \(\text{Sh}_{K'(n)}(\text{GSp}, S^\pm)\) is just the fine moduli space \(\text{M}_{V_{\mathbb{Z}(p)} \otimes \mathbb{Z}[1/n]}\mathbb{Q}\) introduced in (3.1.1), and as such has the integral canonical model \(\mathcal{S}_{K'(n)}(\text{GSp}, S^\pm) = \text{M}_{V_{\mathbb{Z}(p)} \otimes \mathbb{Z}[1/n]}\mathbb{Q}\) over \(\mathbb{Z}[(nd)^{-1}]\).

The reflex field \(E\) is unramified at \(p\) (cf. [Mil12 4.4.7]). Let \(v \mid p\) be a prime of \(E\) lying above \(p\), and let \(\mathcal{O}_{E,(v)}\) be the localization of \(\mathcal{O}_E\) at \(v\). Let \(\mathcal{S}_{K(n)}(G, X)\) be the normalization of the Zariski closure of \(\text{Sh}_{K(n)}(G, X)\) in \(\mathcal{S}_{K'(n)}(\text{GSp}, S^\pm)\mathcal{O}_{E,(v)}\).

Theorem 4.6.4 (Kisin, Vasiu). Suppose that \(p > 2\). Set

\[
\mathcal{S}_{K_p}(G, X) = \lim_{\substack{n \to \infty}} \mathcal{S}_{K(n)}(G, X).
\]

Then:

1. For each \(n\), \(\mathcal{S}_{K(n)}(G, X)\) is smooth over \(\mathcal{O}_{E,(v)}\), and the transition maps in the inverse limit above are finite étale.

2. \(\mathcal{S}_{K_p}(G, X)\) satisfies the extension property [4.5.7] and is therefore the integral canonical model for \(\text{Sh}_{K_p}(G, X)\).

Proof. This is [Kis10 2.3.8].

Remark 4.6.5. In the case \(p = 2\), work of Vasiu-Zink [VZ10] reduces the problem again to showing the smoothness of \(\mathcal{S}_{K(n)}(G, X)\). It should be possible to prove smoothness using the strategy in [Kis10]. There are three results in loc. cit. whose proofs use the condition \(p > 2\). The first is in [Kis10 2.3.1], where, as we observed in the proof of (4.6.2), the condition is not in fact needed. The second is [Kis10 1.4.2], which has since been extended to the case \(p = 2\) by the work of W. Kim [Kim11], Lau [Lau12], and T. Liu [Liu11]. Finally, the assumption on \(p\) is also used in a deformation theoretic argument in [Kis10 1.5.8]. When the universal abelian scheme over \(\mathcal{S}_{K(n)}(G, X)\) has connected \(p\)-divisible group at every closed point, Zink's
theory of displays allows Kisin to push the argument through even when \( p = 2 \). In general, the condition on \( p \) does seem a little more serious here, and we have been unable to remove it so far.

Nonetheless, Vasiu has made some progress towards integral canonical models even when \( p = 2 \); cf. [Vas12a, Vas12b, Vas12d], and especially [Vas12c], where the case \( p = 2 \) is claimed to have been tackled in a number of cases, including those where the ordinary locus is dense in the special fiber of \( \mathscr{S}_{K(n)}(G, X) \).

**Definition 4.6.6.** For any neat compact open sub-group \( K' \subset G(\mathbb{A}_f) \), we will refer to the quotient \( \mathscr{S}_{K_p,K'}(G, X) := \mathscr{S}_{K_p}(G, X)/K' \) as the integral canonical model for \( \text{Sh}_{K_p,K'}(G, X) \) over \( \mathcal{O}_{E,(v)} \). In particular, for \( n \geq 3 \), the \( \mathcal{O}_{E,(v)} \)-scheme denoted \( \mathscr{S}_{K(n)}(G, X) \) above is the integral canonical model for \( \text{Sh}_{K(n)}(G, X) \).

**Definition 4.6.7.** We will say that a triple \((G, X, K)\) consisting of a Shimura datum \((G, X)\) and a compact open \( K \subset G(\mathbb{A}_f) \) is **unramified at \( p \)** if \( K_p = G_{\mathbb{Z}(p)}(\mathbb{Z}_p) \), for some reductive model \( G_{\mathbb{Z}(p)} \) of \( G \) over \( \mathbb{Z}_p \).

If \( K \subset G(\mathbb{A}_f) \) and \( K' \subset G'(\mathbb{A}_f) \) are compact open subgroups such that \((G, X, K)\) and \((G', X', K')\) are unramified at \( p \), then a **p-integral embedding** \((\iota, g) : (G, X, K) \hookrightarrow (G', X', K')\) consists of a \( p \)-integral embedding \( \iota : (G, X) \hookrightarrow (G', X') \), and an element \( g \in G'(\mathbb{A}_f) \) with \( g_p \in K'_p \), such that \( g^{-1}\iota(K)g \subset K' \). Here \( g_p \) is the \( p \)-primary part of \( g \).

Suppose that we are given an unramified-at-\( p \) triple \((G, X, K)\) corresponding to a reductive model \( G_{\mathbb{Z}(p)} \) for \( G \).

**Definition 4.6.8.** A **\( p \)-integral CLR** for \((G, X, K)\) is a CLR in \( \text{CLR}_K(G, X) \) of the form \((P, X^+, g)\), with \( g_p \in K_p \); here, \( g_p \) is the \( p \)-primary part of \( g \). If \( \Phi, \Phi' \) are two \( p \)-integral CLRs then a map \( \Phi \sim \Phi' \) between them is a map of CLRs with \( \gamma \in G_{\mathbb{Z}(p)}(\mathbb{Z}(p)) \). We will denote the category of \( p \)-integral CLRs by \( \text{CLR}_K^p(G, X) \).

**Lemma 4.6.9.**

1. Every class in \( Cusp_K(G, X) \) has a representative in \( \text{CLR}_K^p(G, X) \).
2. Fix \( \Phi \in \text{CLR}_K^p(G, X) \); then the Shimura datum \((G_{\Phi,h}, P_{\Phi})\) is also unramified at \( p \).
3. If \((G, X)\) is of Hodge type, then so is \((G_{\Phi,h}, P_{\Phi})\). Moreover, for any prime \( v \mid p \) of \( E = E(G, X) \), the Shimura variety \( \text{Sh}_{K_p} \) over \( E(G, X) \) has an integral canonical model \( \mathscr{S}_{K_p} \) over \( \mathcal{O}_{E,(v)} \).

Proof. The main point is that every parabolic sub-group of \( G \) extends uniquely to a parabolic sub-group of \( G_{\mathbb{Z}(p)} \). So every admissible parabolic sub-group \( P \subset G \) extends to a parabolic sub-group \( P_{\mathbb{Z}(p)} \subset G_{\mathbb{Z}(p)} \) with Levi quotient \( L_{\mathbb{Z}(p)} \). Furthermore, it follows from [Hor10 1.6.9] that the normal sub-group \( Q_P \subset P \) extends to a normal sub-group of \( P_{\mathbb{Z}(p)} \) with reductive image in \( L_{\mathbb{Z}(p)} \). From this, both 1 and 2 are clear.

When \((G, X)\) is a Siegel Shimura datum, then so is \((G_{\Phi,h}, P_{\Phi})\) (cf. 1.3.1). The first assertion of 3 is immediate from this. The second now follows from 1 and 2.

**Assumption 4.6.10.** From now on, all CLRs will be assumed to be \( p \)-integral, with respect to a \( p \)-integral structure on \((G, X, K)\) that will be clear from context. All maps considered between such CLRs will also be in the category of \( p \)-integral CLRs.

**Definition 4.6.11.** Let \((G, X)\) be a Shimura datum, and let \( K \subset G(\mathbb{A}_f) \) be a neat compact open sub-group. A **\( p \)-integral Hodge embedding** \( \iota : (G, X, K) \hookrightarrow (\text{GSp}(V), S^\pm, K(n)) \) into a Siegel Shimura datum consists of a \( p \)-integral embedding \( \iota : (G, X) \hookrightarrow (\text{GSp}(V), S^\pm) \), a polarized lattice \( V_\mathbb{Z} \subset V \) of discriminant \( d \), and an \( n \in \mathbb{Z}_{\geq 3} \) prime to \( pd \) such that:

- \( K = K(n) \cap G(\mathbb{A}_f) \), where \( K(n) \) is the level \( n \) sub-group of \( \text{GSp}(V_{\mathbb{A}_f}) \) associated with \( V_\mathbb{Z} \).
The map $\text{Sh}_K(G, X) \to \text{Sh}_{K(n)}(G_{\text{Sp}}, S^\pm)$ is an embedding.

We will say that an admissible rppcd $\Sigma$ for $(G, X, K)$ is associated with the $p$-integral Hodge embedding $\iota : (G, X, K) \to (G_{\text{Sp}}(V), S^\pm, K(n))$ if there exists a smooth rppcd $\Sigma'$ for $(G_{\text{Sp}}, S^\pm, K(n))$ such that $\Sigma$ is induced from $\Sigma'$.

**Assumption 4.6.12.** We will always assume from now on that $K \subset G(\mathbb{A}_f)$ is chosen so that there exists a $p$-integral embedding $\iota : (G, X, K) \to (G_{\text{Sp}}, S^\pm, K(n))$. Furthermore, any admissible rppcd $\Sigma$ for $(G, X, K)$ will be assumed to be associated with a $p$-integral Hodge embedding $(G, X, K) \to (G_{\text{Sp}}, S^\pm, K(n))$. This is not a very strong condition. By (1.1.7) and (4.2.20), we can always arrange this by shrinking $K$ a little away from $p$ and then by refining $\Sigma$.

**Theorem 4.6.13.** Assume that $p > 2$. Let $(G, X)$ be a Shimura variety of Hodge type with reflex field $E$, $K \subset G(\mathbb{A}_f)$ a neat compact open, and $\Sigma$ an admissible rppcd for $(G, X, K)$ (recall our assumptions from 4.6.12). Fix $v|p$, a prime of $E$ over $p$. Then there exists a flat integral model $\mathcal{J}_K^\Sigma$ of $\text{Sh}_K^\Sigma(G, X)$ over $\mathcal{O}_{E, (v)}$ of the toroidal compactification $\text{Sh}_K^\Sigma(G, X)$ from (4.2.19), containing the integral canonical model $\mathcal{J}_K$ of $\text{Sh}_K(G, X)$ as an open dense sub-scheme, and satisfying the following properties:

1. The boundary $D_K^\Sigma = \mathcal{J}_K^\Sigma \setminus \mathcal{J}_K$, equipped with its reduced sub-scheme structure, is a relative effective Cartier divisor over $\mathcal{O}_{E, (v)}$, along which $\mathcal{J}_K^\Sigma$ has at worst toroidal singularities. In particular, $\mathcal{J}_K^\Sigma$, equipped with the log structure associated with $D_K^\Sigma$ is log smooth over $\mathcal{O}_{E, (v)}$.

2. $\Sigma$ can be chosen such that $\mathcal{J}_K^\Sigma$ is smooth, projective, and such that $D_K^\Sigma$ has normal crossings.

3. There is a stratification by smooth $\mathcal{O}_{E, (v)}$-schemes

$$\mathcal{J}_K^\Sigma = \bigsqcup_{[(\Phi, \sigma)]} Z_{[(\Phi, \sigma)]},$$

where $[(\Phi, \sigma)]$ ranges over $\text{Cusp}_K^\Sigma(G, X)$. This stratification is compatible with the stratification of $\text{Sh}_K^\Sigma(G, X)$ described in (4.2.19)3; in particular, $Z_{[(\Phi, \sigma)]}$ lies in the closure of $Z_{[(\Phi', \sigma')]}$ if and only if $[(\Phi', \sigma')]$ is a face of $[(\Phi, \sigma)]$.

4. For each $\Phi \in \text{CLR}_K^\Sigma(G, X)$, $\mathcal{J}_{K_\Phi}$ is the integral canonical model of $\text{Sh}_{K_\Phi}$, and $C_\Phi$ is an abelian scheme over $\mathcal{J}_{K_\Phi}$.

5. For each $[(\Phi, \sigma)]$ with representative $(\Phi, \sigma)$, $Z_{[(\Phi, \sigma)]}$ is canonically isomorphic to the closed stratum in the twisted torus embedding $\Xi_\Phi(\sigma)$ over the $\mathcal{J}_{K_\Phi}$-abelian scheme $C_\Phi$.

In fact, the completion $\mathcal{X}_{[(\Phi, \sigma)]}$ of $\mathcal{J}_K^\Sigma$ along $Z_{[(\Phi, \sigma)]}$ is canonically isomorphic to the completion of $\Xi_\Phi(\sigma)$ along its closed stratum. This description is compatible with the analogous description of the strata found in (4.2.19)4.

Proof. Choose a $p$-integral Hodge embedding $(G, X, K) \to (G_{\text{Sp}}, S^\pm, K(n))$ into a Siegel Shimura datum with which $\Sigma$ is associated. Take $\mathcal{J}_K^\Sigma$ to be the normalization of the Zariski closure of $\text{Sh}_K^\Sigma(G, X)$ in $\mathcal{J}_K^\Sigma[n]$.\footnote{We will see in (4.7.4) below that, when $\Sigma$ is complete, $\mathcal{J}_K^\Sigma$ does not depend on the choice of $p$-integral embedding.}

A good part of the theorem now follows from (4.4.7).\footnote{3 was already shown in (1.1.7)3. Consider 4: that $\mathcal{J}_{K_\Phi}$ is the integral canonical model for $\text{Sh}_{K_\Phi}$ is a consequence of (4.6.4) and (4.6.9). That $C_\Phi$ is an abelian scheme over $\mathcal{J}_{K_\Phi}$ follows from (4.6.11) below. 5 and 6 are now immediate from (4.4.7)5.}

Finally, (2) follows from (4.3.0)6 and (4.2.20).\qed
Lemma 4.6.14. Let X be a healthy regular $\mathbb{Z}_{(p)}$-scheme (for example, X can be formally smooth over $\mathbb{Z}_{(p)}$), and let $A \to X$ be an abelian scheme. Suppose that $B_{\mathbb{Q}} \subset A_{\mathbb{Q}}$ is a finite map of abelian schemes over X. Let $B \to A$ be the normalization of the Zariski closure of the image of $B_{\mathbb{Q}}$ in $A_{\mathbb{Q}}$. Then B is an abelian scheme over X.

Proof. From the usual Néron-Ogg-Shafarevich good reduction criterion, we see that $B_{\mathbb{Q}}$ extends to an abelian scheme over all co-dimension 1 points of X. By healthy regularity, $B_{\mathbb{Q}}$ extends to an abelian scheme $B'$ over X. Let $B \to A$ be the normalization of the Zariski closure of the image of $B_{\mathbb{Q}}$ in $A_{\mathbb{Q}}$. Then B is an abelian scheme over X.

Remark 4.6.15. As is clear from the proof, our methods will also work for $p = 2$ as soon as we know that the construction in (4.6.4) produces integral canonical (i.e. smooth) models.

Corollary 4.6.16. Let $(G, X, K)$, $E$ and $v | p$ be as above. Then every geometric connected component of $\mathcal{I}_K, k(v)$ is the specialization of a unique geometric connected component of $\text{Sh}_K(G, X)$.

Proof. We now know that $\mathcal{I}_K$ admits a smooth compactification over $\mathcal{O}_{E, (v)}$, so the result follows from Zariski’s connectedness theorem; cf. [DM69, 4.17].

Corollary 4.6.17. Suppose that $(G, X, K)$ is an unramified-at-p triple (cf. [4.6.7]) of Hodge type, let $E = E(G, X)$ be its reflex field, and let $v | p$ be a prime of E. Then the integral canonical model $\mathcal{I}_K$ over $\mathcal{O}_{E, (v)}$ is proper if and only if $G / Z(G)$ is anisotropic.

Proof. This is simply a special case of (4.6.7) (2).

4.7. Hecke action. Here, we will show that the Hecke action on the tower of toroidal compactifications can be obtained entirely formally, once we are given integral canonical models and the Hecke action in characteristic 0.

4.7.1. Consider the following categories over $\mathcal{O}_{E, (v)}$: First, we take the category of pairs $(T, D)$, where T is a proper normal flat $\mathcal{O}_{E, (v)}$-algebraic space, and $D \subset T$ is a sub-scheme that is a relative effective Cartier divisor over $\mathcal{O}_{E, (v)}$; the morphisms $(T, D) \to (T', D')$ are simply morphisms $T \to T'$ of $\mathcal{O}_{E, (v)}$-schemes that carry $D$ into $D'$. Next, we consider triples $(T^o, T_E, D_E)$, where $T^o$ is a normal, flat $\mathcal{O}_{E, (v)}$-scheme, $T_E$ is a proper normal algebraic space over $E$ equipped with an open immersion $T^o \otimes E \to T_E$, and $D_E \subset E$ is an effective divisor whose complement is $T^o \otimes E$. A morphism $(T^o, T_E, D_E) \to (T'^o, T'_E, D'_E)$ is a pair of maps $T^o \to T'^o$ and $T_E \to T'_E$ of $\mathcal{O}_{E, (v)}$-schemes that agree on $T^o \otimes E$ and carry $D_E$ into $D'_E$.

Lemma 4.7.2. The natural functor $(T, D) \to (T \setminus D, T \otimes E, D \otimes E)$ is fully faithful.

Proof. Let $k_v$ be the residue field of $\mathcal{O}_{E, (v)}$. Then, the lemma is a consequence of two facts:

(a) Given $(T, D)$ as above, $D \otimes k_v$ has co-dimension 2 in T.

(b) Given a normal algebraic space T over $\mathcal{O}_{E, (v)}$, a closed sub-space $Z \subset T$ of co-dimension at least 2, any map $T \setminus Z \to T'$ to a proper algebraic space $T'$ over $\mathcal{O}_{E, (v)}$ extends uniquely to a map $T \to T'$.

The corresponding assertions for schemes over $\mathcal{O}_{E, (v)}$ are well-known, from which we can easily deduce the above for algebraic spaces as well. 

4.7.3. Suppose that there is a $p$-integral embedding $(t, g) : (G, X, K) \to (G', X', K')$ of unramified-at-$p$ triples, as in (4.6.7) above, and suppose that $(G', X')$ (and hence $(G, X)$) is of Hodge type. Let $E = E(G, X)$ be the reflex field of $(G, X)$, let $v | p$ be a place of $E$ and let $E$ be the completion of $E$ along v. Let $\mathcal{I}_K = \mathcal{I}_K(G, X)$ and $\mathcal{I}_{K'} = \mathcal{I}_{K'}(G', X')$ be the integral canonical models over $\mathcal{O}_{E, (v)}$ of $\text{Sh}_K(G, X)$ and $\text{Sh}_{K'}(G', X')$, respectively.\[\text{By (4.7.1)},

---

\^We are abusing notation here a little, since $\mathcal{I}_K$ has up to now denoted the integral canonical model of $\text{Sh}_{K'}(G', X')$ over the ring of integers of a completion of $E(G', X')$ (as opposed to $E(G, X)$).
with every \( \Phi \in \text{CLR}_K^p(G,X) \), we can associated \( \Phi' = (\iota,g) \in \text{CLR}_K^p(G,X) \) such that \( K_\Phi \subset K_{\Phi'} \). In particular, there is a natural finite map \( K_\Phi \rightarrow K_{\Phi'} \), extending the embedding \( \text{Sh}_{K_\Phi}(G_{\Phi,h},F_{\Phi}) \rightarrow \text{Sh}_{K_{\Phi'}}(G'_{\Phi',h},F_{\Phi'}) \).

For any complete admissible \( \Sigma' \) (resp. \( \Sigma \)) for \( (G',X',K') \) (resp. \( (G,X,K) \)), over \( \text{Sh}_{K'}(1) \) there exists the proper toroidal compactifications \( \mathcal{F}_K^\Sigma \) of \( \mathcal{F}_K' \) and \( \mathcal{F}_K^\Sigma' \) of \( \mathcal{F}_K \) from \( 4.6.13 \).

**Proposition 4.7.4** (Hecke action). Suppose that \( \Sigma \) is a refinement of the complete admissible \( \text{ppcd} \) \( \Sigma' \) for \( (G,X,K) \) induced from \( \Sigma' \) along the embedding \( (\iota,g) \). Then there is a unique proper map \( g: [\Sigma,K,K'] : \mathcal{F}_K^\Sigma \rightarrow \mathcal{F}_K' \) with the following properties:

1. Its restriction to \( \mathcal{F}_K \) agrees with the natural Hecke map \( [g]_{K,K'} : \mathcal{F}_K \rightarrow \mathcal{F}_K' \) restricting to the corresponding Hecke map over \( E \).
2. Over \( E \), \([g]_{K,K'}^\Sigma: \mathcal{F}_K^\Sigma \rightarrow \mathcal{F}_K' \) agrees with the corresponding map defined in \( 4.2.21 \) \([1]\).
3. For any \( ([\Phi,\sigma]) \in \text{Cusp}_K^\Sigma(G,X) \), the stratum \( Z_{E,[\Phi,\sigma]} \) maps into the stratum \( Z_{E,[\Phi',\sigma']}, \) where \( ([\Phi',\sigma']) \in \text{Cusp}_{K'}^\Sigma(G',X') \) is determined in the following way: \( \Phi' = (\iota,g)^* \Phi \) and \( \sigma' \in \Sigma'_{\Phi'} \) is the minimal cone that contains \( \sigma \).
4. For \( ([\Phi,\sigma]) \) and \( ([\Phi',\sigma']) \) as in \( 3 \), the restriction \( [g]_{E,[\Phi,\sigma]} : Z_{E,[\Phi,\sigma]} \rightarrow Z_{E,[\Phi,\sigma'] \} \) of \([g]_{K,K'}^\Sigma \) can be described as follows: There exists a canonical homomorphism \( C_{\Phi} \rightarrow C_{\Phi'} \) lying over the canonical map of integral canonical models \( \mathcal{F}_{K_\Phi} \rightarrow \mathcal{F}_{K_\Phi'}, \) and lifting to an \( E_{K_\Phi} \)-equivariant map \( \Xi_{E,[\Phi,\sigma]} \rightarrow \Xi_{E,[\Phi,\sigma']} \). Now, \([g]_{E,[\Phi,\sigma]} \) is isomorphic to the canonical map between the closed stratum of \( \Xi_{E,[\Phi,\sigma]} \) into the closed stratum of \( \Xi_{E,[\Phi,\sigma']} \). Similarly, the induced map \( [g]_{E,[\Phi,\sigma]} : X_{E,[\Phi,\sigma]} \rightarrow X_{E,[\Phi,\sigma']} \) on the completions along the strata is isomorphic to the canonical map between the completions of \( \Xi_{E,[\Phi,\sigma]} \) and \( \Xi_{E,[\Phi,\sigma']} \) along their closed strata.

In particular, the toroidal compactification \( \mathcal{F}_K^\Sigma \) does not depend on the choice of \( p \)-integral embedding of \( G,X \) into a Siegel Shimura datum.

**Proof.** We first note that the map \([g]_{K,K'} \) mentioned in \( 1 \) is the extension of the corresponding map in characteristic 0 (cf. \( 4.2.21 \)) obtained via the extension property of the integral canonical model \( \mathcal{F}_K' \) (or, rather, of the pro-scheme \( \mathcal{F}_K' \); cf. \( 4.6.3 \)). The existence and uniqueness of the map \([g]_{K,K'}^\Sigma \) with properties \( 1 \) and \( 2 \) now follows from \( 4.7.2 \). Assertions \( 3 \) and \( 4 \) are immediate from their characteristic 0 counterparts \( 4.2.21 \) \( 2 \) and \( 4.2.21 \) \( 3 \). \( \square \)

**Remark 4.7.5.** The arguments above can be used to show that our compactifications agree with those of Lan \( [\text{Lan08}] \) when \( (G,X) \) is of PEL type.

4.8. **The minimal compactification.** With the results of \( 4.6 \) we can now easily construct the minimal or Baily-Borel-Satake compactification of the integral canonical model \( \mathcal{F}_K \) of a Shimura variety \( \text{Sh}_K(G,X) \) of Hodge type at a prime \( p \) where \( (G,X,K) \) is as in \( 4.6.13 \). We will follow the strategy in \( [\text{FC90}] \) \( \S \text{.V.2} \), which is extended to the PEL case in \( \text{Lan08} \) \( \S \text{.7.2} \). Since the method here is not very different from that used in loc. cit., our treatment will be somewhat compressed.

**Definition 4.8.1.** Suppose that there is a \( p \)-integral Hodge embedding \( \iota : (G,X,K) \hookrightarrow (\text{GSp},S^\pm, K(n)) \). The **Hodge bundle** \( \omega(\iota) \) over \( \mathcal{F}_K \) associated with \( \iota \) is the top exterior power of \( V_{\text{dR}}(\iota) \). Then the **Hodge bundle** \( \omega(\iota)^\Sigma \) over \( \mathcal{F}_K^\Sigma \) is the top exterior power of \( V_{\text{dR}}(\iota) \).
**Definition 4.8.2.** Let $S$ be an algebraic space over $\mathcal{O}_{E,(v)}$. Define an equivalence relation on line bundles on $S$ as follows: We say that $\mathcal{L}_1 \sim_{\text{Proj}} \mathcal{L}_2$ for line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $S$ if there exist a map $f : S' \to S$ of $\mathcal{O}_{E,(v)}$-schemes and integers $n, m \in \mathbb{Z}_{>0}$ such that:

- $f_*f^*\mathcal{S} = \mathcal{S}$ and $R^if_*f^*\mathcal{S} = 0$, for all $i > 1$.
- There exists an isomorphism of line bundles $f^*\mathcal{L}_1^\otimes n \simeq f^*\mathcal{L}_2^\otimes m$ over $S'$.

We will denote the set of equivalence classes of line bundles over $S$ under $\sim_{\text{Proj}}$ by $\text{Pic}(S)_{\mathbb{Z}}$.

**Lemma 4.8.3.** Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line bundles over $S$ with $\mathcal{L}_1 \sim_{\text{Proj}} \mathcal{L}_2$. Then the $\mathcal{O}_{E,(v)}$-schemes $\text{Proj}\left(\bigoplus_k H^0(S, \mathcal{L}_1^\otimes k)\right)$ and $\text{Proj}\left(\bigoplus_k H^0(S, \mathcal{L}_2^\otimes k)\right)$ are isomorphic.

**Proof.** Let $f : S' \to S$ and $n, m \in \mathbb{Z}_{>0}$ be as in the definition of $\mathcal{L}_1 \sim_{\mathbb{Z}} \mathcal{L}_2$ above. Then, for $r = 1, 2$, the projection formula shows

$$R^if_*f^*\mathcal{L}_r^\otimes k = \begin{cases} \mathcal{L}_r^\otimes k, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

In particular:

$$\text{Proj}\left(\bigoplus_k H^0(S, \mathcal{L}_1^\otimes k)\right) = \text{Proj}\left(\bigoplus_k H^0(S', f^*\mathcal{L}_r^\otimes k)\right).$$

So we can replace $S$ by $S'$ and assume that $\mathcal{L}_1^\otimes n \simeq \mathcal{L}_2^\otimes m$. Now the result follows from [EGAIII 2.4.7].

**Definition 4.8.4.** For any class $[\mathcal{L}] \in \text{Pic}(S)_{\mathbb{Z}}$, set

$$\mathcal{P}(S, [\mathcal{L}]) = \text{Proj}\left(\bigoplus_k H^0(S, \mathcal{L}^\otimes k)\right).$$

By (4.8.3) above, this $\mathcal{O}_{E,(v)}$-scheme does not depend on the choice of representative $\mathcal{L}$.

**Lemma 4.8.5.** Let $\Sigma$ be a complete admissible rppcd for $(G, X, K)$ associated with two different $p$-integral embeddings $i_k : (G, X, K) \hookrightarrow (\text{GSp}(V_k), S_k^+)$, for $k = 1, 2$.

1. There exist a refinement $\Sigma'$ of $\Sigma$ and integers $r_1, r_2 \in \mathbb{Z}_{>0}$ such that the pull-backs of the line bundles $(\omega(i_1)^r_1)^{\otimes r_1}$ and $(\omega(i_2)^r_2)^{\otimes r_2}$ to $\mathcal{S}_K^{\Sigma'}$ are isomorphic.

2. The class $[\omega^{\Sigma'}] := [\omega(i)^{r_1}] \in \text{Pic}(\mathcal{S}_K^{\Sigma'})$ is independent of the choice of $i$. Moreover, it satisfies $(1_{\Sigma_k'}^{\Sigma_1})^*[\omega^{\Sigma'}] = [\omega^{\Sigma_1'}]$, for any refinement $\Sigma'$ of $\Sigma$.

**Proof.** Let $V_k, Z_{(p)} \subset V_k$, for $k = 1, 2$, be symplectic $\mathbb{Z}_p$-lattices giving rise to the $p$-integral structure on $\text{GSp}(V_k)$. Take $r_1, r_2 \in \mathbb{Z}_{>0}$ such that $V_{1,Z_{(p)}}^{\otimes r_1}$ and $V_{2,Z_{(p)}}^{\otimes r_2}$ are isomorphic as symplectic $\mathbb{Z}_p$-modules. Let $V_{(p)}$ be this common symplectic $\mathbb{Z}_p$-module with associated symplectic $\mathbb{Q}$-space $\tilde{V}$. Then there are natural $p$-integral embeddings of $(\text{GSp}(V_k), S_k^+)$ $(k = 1, 2)$ into the Siegel Shimura datum $(\text{GSp}(V), \tilde{S}^\pm)$.

Choose a polarized lattice $\tilde{V}_{\Sigma} \subset \tilde{V}$ of discriminant $d$. Choose $\tilde{n} \in \mathbb{Z}_{>0}$ prime to $pd$ such that such that both $K(n_1)$ and $K(n_2)$ map into $K(\tilde{n}) \subset \text{GSp}(\tilde{V})(\mathbb{A}_f)$. Refining $\Sigma$, if necessary, we can assume that we have an admissible rppcd $\tilde{\Sigma}$ for $(\text{GSp}(\tilde{V}), \tilde{S}^\pm, K(\tilde{n}))$ inducing admissible rppcds $\Sigma_k$ for $(\text{GSp}(V_k), S_k^+, K(n_k))$ and such that $\Sigma$ refines $\Sigma_k$, for $k = 1, 2$. By (4.7.4), we

\[\text{It is implied by (4.3.11)-(3) below that we can take } \Sigma' = \Sigma.\]
then get a natural diagram

\[
\begin{array}{c}
\mathcal{M}^\Sigma_{K(n_1)} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\mathcal{M}^\Sigma_{K(n_2)}
\end{array}
\]

Let \( \omega_{\Sigma_k} \) (resp. \( \tilde{\omega}_{\Sigma_k} \)) be the Hodge bundle on \( \mathcal{M}^\Sigma_{K(n_k)} \) (resp. \( \mathcal{M}^\Sigma_{\tilde{K}(n)} \)). Then by construction there exist, for \( k = 1, 2 \), natural isomorphisms \( j_k^* \tilde{\omega}_{\Sigma_k} \cong (\omega_{\Sigma_k})^\otimes r_k \). This immediately gives us (1).

Now (2) is immediate from the following

Claim 4.8.6. Write \( f \) for the map \([1]\) \( \mathcal{M}^\Sigma_{\tilde{K}} \rightarrow \mathcal{M}^\Sigma_{K} \); then \( f_\ast \mathcal{O}_{\mathcal{M}^\Sigma_{\tilde{K}}} = \mathcal{O}_{\mathcal{M}^\Sigma_{K}} \) and \( R^i f_\ast \mathcal{O}_{\mathcal{M}^\Sigma_{\tilde{K}}} = 0 \), for \( i > 0 \).

We proceed as in [FC90 V.1.2(b)]: This is a statement that is local on \( \mathcal{M}^\Sigma_{K} \). The description in [4.7.4] of the restriction of \( f \) to the completion of \( \mathcal{M}^\Sigma_{\tilde{K}} \) along its strata further reduces us to the situation where \( f \) is an equivariant blow-up of a torus embedding, where one can write down an explicit Čech complex to compute the cohomology. \( \square \)

4.8.7. Pick \( \Phi \in \text{CLR}^P_K(G,X) \), and set\(^8\)

\[
\Delta_\Phi = \frac{P_{\Phi}(\mathbb{Z}(p))^{+} \cap Q_{\Phi}(\mathbb{A}_f)g_{\Phi}K_{\Phi}^{-1}}{Q_{\Phi}(\mathbb{Z}(p))^{+}}
\]

This is a quotient of the group \( \Gamma_\Phi \subset \text{Aut}(H_\Phi) \) defined in [4.2.9] and is the group denoted \( \Delta_1 \) in [Pin90 6.3]. Moreover, we have a canonical map \( \Delta_\Phi \rightarrow \text{Aut}(\mathcal{M}^\Sigma_{K}) \) given as follows: Each \( \gamma \) in \( P_{\Phi}(\mathbb{Z}(p))^{+} \cap Q_{\Phi}(\mathbb{A}_f)g_{\Phi}K_{\Phi}^{-1} \) determines an automorphism \( \Phi \xrightarrow{\sim} \Phi \). Just as in [4.2.18], this gives us an automorphism \([\gamma]\) of \( \text{Sh}_{K_\Phi} \), which, as one checks easily, depends only on the image of \( \gamma \) in \( \Delta_\Phi \). Since the action of \( \Delta_\Phi \) on \( \text{Sh}_{K_\Phi} \) is defined via prime-to-\( p \) Hecke operators, it extends naturally to an action of \( \Delta_\Phi \) on the integral canonical model \( \mathcal{M}^\Sigma_{K_\Phi} \). As shown in [Pin90 6.3], this action factors through a finite quotient of \( \Delta_\Phi \).

Lemma 4.8.8.

1. The quotient map \( \mathcal{M}^\Sigma_{K_\Phi} \rightarrow \Delta_\Phi \backslash \mathcal{M}^\Sigma_{K_\Phi} \) is a Galois cover. In particular, \( \Delta_\Phi \backslash \mathcal{M}^\Sigma_{K_\Phi} \) is smooth over \( \mathcal{O}_{E,(v)} \).

2. For every admissible \( \Sigma \) for \( (G,X,K) \) and every \([([\Phi_1, \sigma_1]) \in \text{Cusp}^\Sigma_K(G,X) \) with \([\Phi_1] = [\Phi] \), there is a canonical smooth surjective map \( \mathcal{Z}_{([\Phi_1, \sigma_1])} \rightarrow \Delta_\Phi \backslash \mathcal{M}^\Sigma_{K_\Phi} \).

Proof. Choose a \( p \)-integral embedding \( \iota : (G,X,K) \hookrightarrow (G_{\Phi},X_{\Phi},K(n)) \), and let \( \Phi' = \iota_\ast \Phi \), so that we have a natural map \( i_{\Phi, \Phi'} : \mathcal{M}_{K_\Phi} \rightarrow \mathcal{M}_{K(n)_{\Phi'}} \) of integral canonical models over \( \mathcal{O}_{E,(v)} \).

In turn, this gives us a map \( \Delta_\Phi \backslash \mathcal{M}_{K_\Phi} \rightarrow \Delta_{\Phi'} \backslash \mathcal{M}_{K(n)_{\Phi'}} \) of their quotients. It follows from [Mor08 p. 8] that \( \Delta_{\Phi'} \) acts trivially on \( \mathcal{M}_{K(n)_{\Phi'}} \). So we in fact have a factorization

\[
i_{\Phi, \Phi'} : \mathcal{M}_{K_\Phi} \rightarrow \Delta_{\Phi} \backslash \mathcal{M}_{K_\Phi} \rightarrow \mathcal{M}_{K(n)_{\Phi'}}.
\]

But \( i_{\Phi, \Phi'} \) is unramified; indeed, the construction of Kisin [Kis10] identifies the complete local rings of \( \mathcal{M}_{K_\Phi} \) with quotients of complete local rings of \( \mathcal{M}_{K(n)_{\Phi'}} \). So the map \( \mathcal{M}_{K_\Phi} \rightarrow \Delta_{\Phi} \backslash \mathcal{M}_{K_\Phi} \) is also unramified. This gives us (1).

For (2), we note first that \( \Delta_{\Phi} \backslash \mathcal{M}_{K_\Phi} \) is independent of the choice of representative \( \Phi \) in \( [\Phi] = [\Phi_1] \): Indeed, if \( \Phi_2 \) is another representative, then we can choose any map \( \Phi \xrightarrow{\pi} \Phi_2 \), which

\(^8\)The super-script ‘+’ means that we are looking at the sub-group of elements fixing the connected component \( X_\Phi^+ \).
will produce an isomorphism $[\eta]: J_{K_{\phi}} \xrightarrow{\sim} J_{K_{\phi^2}}$. Two such isomorphisms will differ by an automorphism $[\gamma] \in \Delta_{\Phi}$, so $\Delta_{\Phi} \backslash J_{K_{\phi}}$ and $\Delta_{\Phi} \backslash J_{K_{\phi^2}}$ are canonically isomorphic.

If we fix a representative of the form $(\Phi, \sigma)$ for $[(\Phi_1, \sigma_1)]$, then we have a canonical isomorphism $Z_{[(\Phi_1, \sigma_1)]} \xrightarrow{\sim} Z_{\Phi}(\sigma)$, giving us a map $i_{\Phi, \sigma}: Z_{[(\Phi_1, \sigma_1)]} \to J_{K_{\phi}}$. If we have an automorphism $\Phi \to \Phi$ and $\sigma' \subset \Sigma_{\Phi}$ with $\gamma' \sigma' = \sigma$, then, according to \cite[2.11]{Lan08}, $i_{\Phi, \sigma'} = [\gamma] \circ i_{\Phi, \sigma}$. This shows that we have a canonical map $Z_{[(\Phi_1, \sigma_1)]} \to \Delta_{\Phi} \backslash J_{K_{\phi}}$; it is clearly smooth and surjective (use (1)).

\begin{proposition}
Let $\iota: (G, X, K) \to (GSp, S^2, K(n))$ be a $p$-integral Hodge embedding, and suppose that $\Sigma$ is an admissible rppcd for $(G, X, K)$ associated with $\iota$. Let $\omega(\iota)\Sigma$ be the Hodge bundle over $\mathcal{F}_{\Sigma}^n$ induced via $\iota$, and denote by $\mathcal{F}_{\Sigma}^{\min}$ the projective $\Theta_{E,(v)}$-scheme $P(\mathcal{F}_{\Sigma}^n, \omega(\iota)\Sigma)$.

1. A suitable power of $\omega(\iota)\Sigma$ is generated by global sections. The map $f: \mathcal{F}_{\Sigma}^n \to \mathbb{P}^{n}_{\Theta_{E,(v)}}$ into projective space corresponding to the linear system attached to such a power of $\omega(\iota)\Sigma$ has a Stein factorization $\mathcal{F}_{\Sigma}^n \xrightarrow{f^S} \mathcal{F}_{\Sigma}^{\min} \xrightarrow{f^P} \mathbb{P}^{n}_{\Theta_{E,(v)}}$.

2. For any $[(\Phi, \sigma)] \in \text{Cusp}_{\Sigma}^{K\Sigma}(G, X)$ the restriction of $f^\Sigma$ to $Z_{[(\Phi, \sigma)]}$ factors through the canonical smooth surjective map $Z_{[(\Phi, \sigma)]} \to \Delta_{\Phi} \backslash J_{K_{\phi}}$ (cf. \cite{FC90}). The induced map from $\Delta_{\Phi} \backslash J_{K_{\phi}}$ over $\mathcal{F}_{\Sigma}^{\min}$ depends only on the class $[\Phi] \in \text{Cusp}_{\Sigma}^{K\Sigma}(G, X)$.

3. If $[\Phi] \neq [\Phi']$ in $\text{Cusp}_{\Sigma}^{K\Sigma}(G, X)$, then the images of $Z_{[(\Phi, \sigma)]}$ and $Z_{[(\Phi', \sigma')]}$ in $\mathcal{F}_{\Sigma}^{\min}$ are disjoint.

4. The map $Z_{[(\Phi, \sigma)]} \to \Delta_{\Phi} \backslash J_{K_{\phi}}$ is an isomorphism if and only if the unipotent radical $U_{\Phi} \subset P_{\Phi}$ has dimension at most 1.

\end{proposition}

\textit{Proof.} As in \cite[V.2.1]{FC90}, we can use \cite[IX.2.1]{MBS85} to deduce the first part of (1). The second part follows from the argument in \cite[7.2.3]{Lan08}.

2 is shown exactly as for the PEL case; cf. \cite[7.2.3]{Lan08}, esp. the discussion after 7.2.3.5.

We now show (3): Let $\mathcal{F}_{\Sigma}^{\min}$ be a geometric point in the intersection of the images of $Z_{[(\Phi, \sigma)]}$ and $Z_{[(\Phi', \sigma')]},$ and let $C$ be a proper smooth connected curve over $k(\mathcal{F}_{\Sigma}^{\min})$ mapping into $(f^S)^{-1}(\mathcal{F}_{\Sigma}^{\min})$, and whose image intersects both $Z_{[(\Phi, \sigma)]}$ and $Z_{[(\Phi', \sigma')]}. Suppose that the generic point of $C$ maps into the stratum $Z_{[(\Phi, \sigma)]} \subset \mathcal{F}_{\Sigma}^{\min}$; then $C$ maps into the closure of $Z_{[(\Phi, \sigma)]}. From the description of the closure of strata in \cite[4.6.12]{CLR92}, it follows that we have maps $\Phi \xrightarrow{\gamma} \Phi$ and $\Phi' \xrightarrow{\gamma'} \Phi'$ in $\text{CLR}_{\Sigma}^{K\Sigma}(G, X)$. From \cite[V.2.2]{FC90}, we see that the abelian part of the semi-abelian scheme induced over $C$ is iso-trivial. The argument in \cite[7.2.3.6]{Lan08} now shows that $l_\sigma[\Phi] = l_\sigma(\Phi') = l_\sigma(\Phi)$ in $\text{Cusp}^{K\Sigma}(\iota(GSp, S^2)).$ This implies that $[\Phi] = [\Phi'] = [\Phi]';$ cf. \cite[1.2.11]{FC90}.

For the map in (4) to be an isomorphism, we would need $C_{\Phi}$ to be the trivial abelian scheme over $\mathcal{F}_{\Sigma}^{\min}$, which means that $U_{\Phi} = U_{\Phi}^{-2}$, and $E_{\Phi}$ would have to be a torus of rank at most 1, which means that $U_{\Phi}^{-2}$ has dimension at most 1. So we find that $U_{\Phi}$ must have dimension at most 1.

Conversely, if $U_{\Phi}$ has dimension at most 1, then there are two possibilities: Either $U_{\Phi}$ is trivial, in which case we clearly have $\Xi_{\Phi} = \mathcal{F}_{\Sigma}^{\Phi}$. The other possibility is that dim $U_{\Phi} = 1$, in which case $\sigma = H_{\Phi}$, and $\Xi_{\Phi}(\sigma) \to \mathcal{F}_{\Sigma}^{\Phi}$ is isomorphic to the affine line over $\mathcal{F}_{\Sigma}^{\Phi}$, so that $Z_{\Phi}(\sigma) \to \mathcal{F}_{\Sigma}^{\Phi}$ is again an isomorphism.

We now claim that, in both these cases, the conjugation action of $P_{\Phi}(G)$ on $Q_{\Phi}(G)$ is via inner automorphisms. This is clear when $P_{\Phi} = G$. When dim $U_{\Phi} = 1$, it follows from the classification of simple algebraic groups, that $G^\text{ad}$ has a factor isomorphic to $PGL_3$, and that $P_{\Phi}$ is the pre-image in $G$ of a Borel sub-group of this factor. From this description, the claim is easily checked. It now follows that in both cases $\Delta_{\Phi}$ acts trivially on $\mathcal{F}_{\Sigma}^{\Phi}$ and (4) is proved. \qed
4.8.10. Fix \( \Phi \in \text{CLR}_K^p(G, X) \), and consider the tower \( \Xi_\Phi \to C_\Phi \xrightarrow{\pi_\Phi} \mathcal{I}_K \). The action of \( \Delta_\Phi \) on \( \mathcal{I}_K \) extends naturally to an action on the whole tower. This can be seen from the point of view of integral canonical models of mixed Shimura varieties (cf. [Hor10]).

As a \( C_\Phi \)-scheme, we can write

\[
\Xi_\Phi = \text{Spec} \bigoplus_{\ell \in S_\Phi} \Psi_\Phi(\ell),
\]

where, for each \( \ell \), \( \Psi_\Phi(\ell) \) is a line bundle over \( C_\Phi \). Set, for any \( \ell \in S_\Phi \), \( F_{J_\Phi}^{(\ell)} = \pi_{\Phi, \ell} \): this is a coherent sheaf over \( \mathcal{I}_K \). For \( \ell, \ell' \in S_\Phi \), we have a natural ‘multiplication’ map \( F_{J_\Phi}^{(\ell)} \otimes_{\sigma_{E,(v)}} F_{J_\Phi}^{(\ell')} \to F_{J_\Phi}^{(\ell+\ell')} \), and we also have an identification \( F_{J_\Phi}^{(0)} = \sigma_{\mathcal{I}_K} \Phi \). For any geometric point \( \mathfrak{p} \to \mathcal{I}_K \), let \( \widehat{F_{J_\Phi}}^{(\ell)}(\mathfrak{p}) \) be the completion of the stalk of \( F_{J_\Phi}^{(\ell)} \) at \( \mathfrak{p} \). Also, let \( \widehat{\mathcal{I}_K} \) be the maximal ideal of the completion of \( \mathcal{I}_K \) at \( \mathfrak{p} \). Let \( H^S_{\Phi} \subset S_\Phi \) be the collection of elements that pair non-negatively with \( H^S \). Consider \( \prod_\mathfrak{p} H^S_{\Phi} \widehat{F_{J_\Phi}}^{(\ell)}(\mathfrak{p}) \): this is an \( \sigma_{E,(v)} \)-algebra, and it is equipped with a maximal ideal \( \widehat{\mathcal{I}_K} \times \prod_\mathfrak{p} H^S_{\Phi} \widehat{F_{J_\Phi}}^{(\ell)}(\mathfrak{p}) \), along which it is in fact complete. It is also naturally equipped with an action of the discrete automorphism group \( \Delta_\Phi \).

**Theorem 4.8.11.** Let \( (G, X, K) \) be as in [4.6.13]. Then there exists a normal projective \( \sigma_{E,(v)} \)-scheme \( \mathcal{I}_K^{\min} \) in which \( \mathcal{I}_K \) embeds as a dense open sub-scheme, and which enjoys the following properties:

1. For every complete admissible rppcd \( \Sigma \) for \( (G, X, K) \), we have an isomorphism of \( \sigma_{E,(v)} \)-schemes \( \mathbb{F}(\mathcal{I}_K^{\min}, [\omega_{\Sigma}^K]) \cong \mathcal{I}_K^{\min} \).

2. For any \( p \)-integral Hodge embedding \( \iota : (G, X, K) \to (\text{GSp}, S^+, K(n)) \), the Hodge bundle \( \omega(\iota) \) extends to an ample line bundle \( \omega(\iota)^{\min} \) over \( \mathcal{I}_K^{\min} \). The class \( [\omega^{\min}] \) of \( \omega(\iota)^{\min} \) in \( \text{Pic}(\mathcal{I}_K^{\min}) \) is independent of the choice of \( \iota \). In fact, given a different \( p \)-integral embedding \( \iota' \), there exist \( r, s \in \mathbb{Z}_{> 0} \) such that \( (\omega(\iota)^{\min})^\otimes r \) and \( (\omega(\iota')^{\min})^\otimes s \) are isomorphic.

3. For every \( \Sigma \) as in [4], there is a natural proper surjective map \( \hat{f}_\Sigma : \mathcal{I}_K^{\Sigma} \to \mathcal{I}_K^{\min} \) with geometrically connected fibers extending the identity on \( \mathcal{I}_K \), such that \( (\hat{f}_\Sigma)^* [\omega^{\min}] = [\omega^{\Sigma}] \). In fact, if \( \Sigma \) is associated with a \( p \)-integral embedding \( \iota \), then \( (\hat{f}_\Sigma)^* [\omega^{\min}] = [\omega(\iota)^{\Sigma}] \).

4. \( \hat{f}_\Sigma \) is universal in the following sense: if \( f : \mathcal{I}_K^{\Sigma} \to T \) is any other map to an \( \sigma_{E,(v)} \)-scheme \( T \) such that there is an ample line bundle \( \mathcal{L} \) over \( T \) with \( f^* \mathcal{L} \cong (\omega(\iota)^{\Sigma})^\otimes n \), for some \( n \in \mathbb{Z}_{> 0} \), then \( f \) factors through \( \hat{f}_\Sigma \).

5. There is a natural stratification

\[
\mathcal{I}_K^{\min} = \bigcup_{[\Phi] \in \text{Cusp}_K(G, X)} \mathbb{Z}_{[\Phi]}
\]

into locally closed smooth \( \sigma_{E,(v)} \)-sub-schemes. In this stratification, \( \mathbb{Z}_{[\Phi]} \) is in the closure of \( \mathbb{Z}_{[\Phi']} \) if and only if there is a map \( \Phi' \to \Phi \) in \( \text{CLR}_K^p(G, X) \).

6. Fix \( \Phi \) in \( \text{CLR}_K^p(G, X) \). For every geometric point \( \mathfrak{p} \to \mathcal{I}_K^{\min} \) lying in the stratum \( \mathbb{Z}_{[\Phi]} \), we have an isomorphism of complete local \( \sigma_{E,(v)} \)-algebras

\[
\hat{\mathcal{O}}_{\mathcal{I}^{\min}, \mathfrak{p}} \cong \left( \prod_{\ell \in S_\Phi} \widehat{F_{J_\Phi}}^{(\ell)}(\mathfrak{p}) \right)^{\Delta_\Phi}.
\]
In particular, the natural map $\Delta_\Phi \backslash \mathcal{I}_{K_\Phi} \to \mathbb{Z}_{[\Phi]}$ is an isomorphism.

(7) The map $\hat{f}^\Sigma$ is compatible with stratifications in the following sense: For any $[\Phi]$ in $\text{Cusp}_K(G,X)$, the pre-image of $\mathbb{Z}_{[\Phi]}$ under $\hat{f}^\Sigma$ is $\bigsqcup \mathbb{Z}_{[\Phi,\sigma]}$, where the disjoint union ranges over classes in $\text{Cusp}_K(G,X)$ of the form $[(\Phi,\sigma)]$. Moreover, the induced map $\mathbb{Z}_{[\Phi,\sigma]} \to \mathbb{Z}_{[\Phi]}$ is, for any choice of representative $(\Phi,\sigma)$, isomorphic to the natural map from the closed stratum of $\Xi_{\Phi}(\sigma)$ to $\Delta_\Phi \backslash \mathcal{I}_{K_\Phi}$: in particular, it is smooth and surjective.

(8) Let $\mathcal{I}_K \subset \mathcal{I}_K^\Sigma$ be the pre-image of the complement in $\mathcal{I}_K^\min$ of the union of the strata of codimension at least 2. Then $\mathcal{I}_K^\Sigma$ maps isomorphically into $\mathcal{I}_K^\min$. Moreover, for any $p$-integral embedding $\iota : (G,X,K) \to (\text{GSp}, S^\pm, K(n))$ with which $\Sigma$ is associated, and for every $k \in \mathbb{Z}_{\geq 0}$, the natural map

$$H^0\left(\mathcal{I}_K^\Sigma, (\omega(t)^\Sigma)^{\otimes k}\right) \to H^0\left(\mathcal{I}_K^\Sigma, (\omega(t)^\Sigma)^{\otimes k}\right)$$

is an isomorphism.

Proof. As mentioned earlier, everything here follows from arguments in [FC90 §V.2] and [Lan08 §7.2.3], so we allow ourselves to be somewhat terse.

As in (1.8.9), we can take $\mathcal{I}_K^\min = \mathcal{P}(\mathcal{I}_K^\Sigma, [\omega^\Sigma])$, for any admissible rppcd $\Sigma$ for $(G,X,K)$. Then (1.8.5) (2), along with (1.8.5) and (4.8.3), shows that $\mathcal{I}_K^\min$ does not depend on the choice of $\Sigma$. We see from (4.8.9) that $\mathcal{I}_K^\min$ is projective. That it is in fact normal now follows from [Lan08 §7.2.3.1].

Denote by $\mathbb{Z}_{[\Phi]}$ the common image in $\mathcal{I}_K^\min$ under $\hat{f}^\Sigma$ of all the strata of the form $\mathbb{Z}_{[\Phi,\sigma]}$, with $[\Phi] = [\Phi]$. Then (1.8.9) (3) shows that the pre-image of $\mathbb{Z}_{[\Phi]}$ is $\bigsqcup \mathbb{Z}_{[\Phi,\sigma]}$. Assertions (2), (3) and (4) now easily follow from the arguments in [Lan08 §7.2.3.5,7.13]. In particular, we find that $\mathcal{I}_K$ (the union of the strata corresponding to the improper cusps labels) maps isomorphically onto an open dense sub-scheme of $\mathcal{I}_K^\min$.

We still have to show assertions (2), (3) and (4). The first part of (3) is clear: there is at most one map $\hat{f}^\Sigma : \mathcal{I}_K^\Sigma \to \mathcal{I}_K^\min$ extending the identity on $\mathcal{I}_K$, and we have seen in (1.8.9) that there is at least one such map. As for (2), the arguments of [Lan08 §7.2.4.1] show that we can take $\omega^\min(\iota)$ to be the class of $\left(\hat{f}^\Sigma \right)^* \omega(t)^\Sigma$ (the main point is to show that this latter sheaf is a line bundle), for any $\Sigma$ associated with $\iota$. That $\left(\hat{f}^\Sigma \right)^* [\omega^\min] = [\omega^\Sigma]$ is now clear from (1.8.3).

Assertion (4) is clear from the description of $\hat{f}^\Sigma$ as a Stein factorization above.

As for (5), the first part follows from (1.8.9) (4), which shows that $\mathcal{I}_K^\Sigma$ maps bijectively onto its image. It maps isomorphically because the map $\hat{f}^\Sigma$ is proper and because $\mathcal{I}_K^\min$ is normal. The second part about the extension of sections of the Hodge bundle is now deduced in the standard way; cf. [Lan08 §7.2.4.8].

Remark 4.8.12. When $G^{ad}$ does not admit PGL$_{2,\mathbb{Q}}$ as a simple factor, $\mathcal{I}_K^\Sigma = \mathcal{I}_K$, and (4.8.1) (5) gives us Koehler’s principle for sections of powers of the Hodge bundle $\omega(\iota)$.

Remark 4.8.13. It is possible now to repeat the arguments in [Lan08 §7.3] to show that, whenever the rppcd $\Sigma$ admits a polarization function (cf. [Lan08 §7.3.1.1]), $\mathcal{I}_K^\Sigma$ will be the normalization of the blow-up of $\mathcal{I}_K^\min$ along a very explicit sheaf of ideals: in particular, it will be a projective scheme. We do not go into the details here, since our construction already gives projective toroidal compactifications of $\mathcal{I}_K^\Sigma$ for free: We only have to choose a $p$-integral embedding $(G,X,K) \to (\text{GSp}, S^\pm, K(n))$, and then we are free to choose any projective, smooth admissible rppcd $\Sigma'$ for the Siegel Shimura datum. The induced rppcd $\Sigma$ for $(G,X,K)$ will be
automatically projective, and the compactification $\mathcal{H}_K^\Sigma$ will also be projective, since it is finite over the projective compactification $\mathcal{H}_K^{\min}$. 

**Proposition 4.8.14 (Hecke action).** Suppose that we have a $p$-integral embedding $(i,g):(G,X,K) \hookrightarrow (G',X',K')$ of unramified-at-$p$ triples of Hodge type, as in (4.7.4). Let $E = E(G,X)$, and let $v$ be a finite prime of $E$. Then there is a unique map $[g]_{K,K'}^{\min} : \mathcal{I}_K \to \mathcal{I}_{K'}$ of $\mathcal{O}_{E,(v)}$-schemes extending the Hecke map $[g]_{K',K} : \mathcal{I}_K \to \mathcal{I}_{K'}$ and enjoying the following properties:

1. For every $[\Phi] \in \text{Cusp}_K(G,X)$, $[g]^{\min}_{K,K'}$ maps the stratum $Z_{[\Phi]}$ into the stratum $Z_{[\Phi']}$, where $[\Phi'] = (i,g)[\Phi]$.
2. For every $[\Phi] \in \text{Cusp}_K(G,X)$, the restriction of $[g]^{\min}_{K,K'}$ to $Z_{[\Phi]}$ is isomorphic to the natural map $\Delta_{\Phi} \mathcal{I}_{K_{\Phi}} \to \Delta_{\Phi'} \mathcal{I}_{K'_{\Phi}}$ of quotients of integral canonical models, where $\Phi$ is a representative of $[\Phi]$ and $\Phi' = (i,g)\Phi$.
3. Given a $p$-integral embedding $i'_1 : (G',X',K') \hookrightarrow (\text{GSp}(4),\mathbb{Z}(2),K(n))$, we have $\left( [g]_{K,K'}^{\min} \right)^* \omega(i'_1)^{\min} \supseteq \omega(i_1)^{\min}$.

**Proof.** The uniqueness of such an extension is clear. We only have to show its existence. Choose any admissible rppcd $\Sigma'$ for $(G',X',K')$, and let $\Sigma$ be the induced admissible rppcd for $(G,X,K)$. We claim that we have a commuting diagram

$$
\begin{align*}
\mathcal{H}_K^{\Sigma} & \xrightarrow{[g]_{K,K'}^{\Sigma,\Sigma'}} \mathcal{H}_{K'}^{\Sigma'} \\
\mathcal{I}_K^{\min} & \xrightarrow{[g]_{K,K'}^{\min}} \mathcal{I}_{K'}^{\min}
\end{align*}
$$

Here, $[g]_{K,K'}^{\Sigma,\Sigma'}$ is the Hecke map from (4.7.4). This claim is enough, since all the claimed properties of $[g]_{K,K'}^{\min}$ will now follow easily from the corresponding properties of $[g]_{K,K'}^{\Sigma,\Sigma'}$.

To prove the claim, we have to show that the composition $\mathcal{I}_K^{\Sigma'} \circ [g]_{K,K'}^{\Sigma,\Sigma'}$ factors through $\mathcal{I}_K^{\Sigma}$, but this follows from (4.8.11) (3).

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