Comment on “Probing gravitational wave polarizations with signals from compact binary coalescences”

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In a recent paper “Probing gravitational wave polarizations with signals from compact binary coalescences” [arXiv:1710.03794 [gr-qc]) the authors argue that a single detection of gravitational wave by the LIGO-Virgo network is capable to distinguish between pure tensor and pure vector polarizations of gravitational waves. Here we point out a mistake in the author’s analysis and show that such differentiation is possible only in the unlikely event when gravitational wave propagates in the direction of interferometer zero response for the tensor or vector polarizations. Nevertheless, the LIGO-Virgo network can distinguish between pure tensor and pure vector polarizations by collecting statistics, as we showed in Phys. Scr. 92, 125001 (2017).

Recently, joint detection of gravitational waves by twin LIGO interferometers in the US and Virgo interferometer in Italy became a reality [1,2]. This achievement opens a perspective to measure polarization of gravitational wave and, thus, distinguish between, e.g., tensor [3] or vector [4] nature of gravity.

A possibility of using networks of ground-based detectors to directly measure polarization of gravitational waves from compact binary coalescences has been recently investigated by Isi and Weinstein [5]. In particular, the authors argued that the current Advanced-LIGO-Advanced-Virgo network can already be used to distinguish between pure tensor and pure vector polarizations of gravitational waves even based on a single detection event. Moreover, using such analysis it has been concluded that the first three-detector observation of gravitational waves from a binary black hole coalescence (GW170814) favors the purely tensor polarization of gravitational waves against purely vector [1]. Next we explain why a single detection of gravitational wave by the LIGO-Virgo network is not sufficient to make a decisive conclusion in favor of one of the two polarizations and point out a mistake in Isi and Weinstein’s analysis of Ref. [5].

To be specific, we consider a toy example discussed by Isi and Weinstein in Section III.A of their paper [5]. Namely, we consider an elliptically-polarized gravitational wave composed of two basis vector modes with a waveform described by a simple sine-Gaussian wavepacket, with some characteristic frequency Ω and relaxation time τ.

\[ h_I(t) = \text{Re} \left[ A \left( F_1^I + i e^{i\omega_0} F_2^I \right) e^{i\Omega(t-t_I)} e^{-\left(t-(t_I)^2/\tau^2\right)} \right], \]

where \( F_1^I \) and \( F_2^I \) are the responses of the detector \( I \) to the two basis polarizations, \( A \equiv |A| e^{i\omega_0} \) is a complex-valued amplitude, \( \epsilon \) is an ellipticity parameter controlling the relative amounts of each polarization and \( t_I \) is the arrival time of the wave at the location of the interferometer \( I \) (\( I = H, \ L, \ V \)). Parameters \( F_1^I \) and \( F_2^I \) are determined by the propagation direction of the gravitational wave and orientation of the interferometer arms.

If gravitational wave propagates along the unit vector \( \hat{k} \) then in vector gravity the unit basis polarization vectors \( \hat{e}_1 \) and \( \hat{e}_2 \) are perpendicular to \( \hat{k} \). They can be chosen in any convenient way, e.g., such that \( \hat{e}_1 \times \hat{e}_2 = \hat{k} \).

Assuming that arms of the detector \( I \) point along unit vectors \( \hat{a}_I \) and \( \hat{b}_I \) the detector response functions read

\[ F_1^I = (\hat{a}_I \cdot \hat{k})(\hat{a}_I \cdot \hat{e}_1) - (\hat{b}_I \cdot \hat{k})(\hat{b}_I \cdot \hat{e}_1), \]

\[ F_2^I = (\hat{a}_I \cdot \hat{k})(\hat{a}_I \cdot \hat{e}_2) - (\hat{b}_I \cdot \hat{k})(\hat{b}_I \cdot \hat{e}_2). \]

Isi and Weinstein assume that ellipticity parameter \( \epsilon \) in Eq. (1) is a real number and write Eq. (1) in the form

\[ h_I(t) = A_I \cos (\Omega(t-t_I) + \Phi_I) e^{-\left(t-(t_I)^2/\tau^2\right)}, \]

where

\[ A_I = |A| \left| F_1^I + i \epsilon F_2^I \right|, \]

\[ \Phi_I = \phi_0 + \arctan \left( \epsilon F_2^I / F_1^I \right). \]
are the main observables at each detector. They can be obtained, along with \(t_f\), by fitting output of each detector with the waveform [4]. The three timing measurements \(t_f\) are sufficient to find the propagation direction of the gravitational wave \(\hat{k}\). This information, according to Eqs. (2) and (3), will then fix the values of \(F_I^L\) and \(F_I^V\). However, since the two LIGO instruments (in Hanford and Livingston) are almost co-aligned, they have the same response functions: \(F_I^V = F_L^V\), \(F_I^L = F_L^L\) and Eqs. (5) and (6) yield \(A_H = A_L\), \(\Phi_H = \Phi_L\). Thus, we are left with only four independent equations

\[
A_L = |A| |F_1^L + i\epsilon F_2^L|, \tag{7}
\]

\[
A_V = |A| |F_1^V + i\epsilon F_2^V|, \tag{8}
\]

\[
\Phi_L = \phi_0 + \arctan \left(\frac{\epsilon F_2^L}{F_1^L}\right), \tag{9}
\]

\[
\Phi_V = \phi_0 + \arctan \left(\frac{\epsilon F_2^V}{F_1^V}\right), \tag{10}
\]

for the three unknown fitting parameters \(|A|\), \(\phi_0\) and \(\epsilon\). Hence, the system of equations (7)-(10) is overdetermined (there are more equations than unknowns). In general case such a system does not have solutions, which means that it would be impossible to fit the measured waveform unless gravitational wave really has vector polarization. Pure vector polarization of gravitational waves will be compatible with observations only if, subject to the experimental uncertainties, Eqs. (7)-(10) have solutions for \(|A|\), \(\phi_0\) and \(\epsilon\). Based on this analysis, Isi and Weinstein concluded that a single detection event by the LIGO-Virgo network can distinguish between pure tensor and pure vector polarizations of gravitational waves.

Such analysis, however, is erroneous because parameter \(\epsilon\) in Eq. (11) is mistakenly taken as a real number. Instead, it must be taken as a complex number (see Appendix A)

\[
\epsilon = \epsilon_1 + i\epsilon_2, \tag{11}
\]

where

\[
\epsilon_1 = \frac{(\hat{k} \cdot \hat{n})}{1 - (\hat{e}_1 \cdot \hat{n})^2}, \quad \epsilon_2 = \frac{(\hat{e}_1 \cdot \hat{n})(\hat{e}_2 \cdot \hat{n})}{1 - (\hat{e}_1 \cdot \hat{n})^2}. \tag{12}
\]

Here \(\hat{n}\) is a unit vector perpendicular to the orbital plane of the binary system which describes orientation of this plane (see Fig. 1). Direction of \(\hat{n}\) is unknown. It is described by two angles and, thus, \(\epsilon_1\) and \(\epsilon_2\) are independent fitting parameters. As a result, Eqs. (7)-(10) must be replaced with four equations

\[
A_L = |A| |F_1^L - \epsilon_2 F_2^V + i\epsilon_1 F_2^L|, \tag{13}
\]

\[
A_V = |A| |F_1^V - \epsilon_2 F_2^V + i\epsilon_1 F_2^V|, \tag{14}
\]

\[
\Phi_L = \phi_0 + \arctan \left(\frac{\epsilon_1 F_2^L}{F_1^L - \epsilon_2 F_2^V}\right), \tag{15}
\]

\[
\Phi_V = \phi_0 + \arctan \left(\frac{\epsilon_1 F_2^V}{F_1^V - \epsilon_2 F_2^V}\right), \tag{16}
\]

for the four unknown fitting parameters \(|A|\), \(\phi_0\), \(\epsilon_1\) and \(\epsilon_2\). Such system of equations is no longer overdetermined and in most instances has solutions. That is no matter what are the measured values of \(A_I\) and \(\Phi_I\) the measured waveform can be fitted by the gravitational waves of vector polarization. Therefore, in most instances, a single detection event by the LIGO-Virgo network will be compatible with the pure vector and pure tensor polarizations.

Differentiation between the two polarizations can be made based on a single detection only in the unlikely event when gravitational wave propagates in the direction for which

\[
F_1^I = F_2^I = 0 \tag{17}
\]

for one of the interferometers. Such directions of zero response are different for tensor and vector polarizations [4]. Namely, tensor gravitational wave produces no signal when it propagates parallel to the interferometer plane at 45° angle relative to one of the perpendicular arms (see Fig. 2a). On the other hand, vector gravitational wave yields no signal if it propagates in the direction perpendicular to the interferometer plane (see Fig. 2b), or along one of the interferometer arms. One should mention that propagation direction perpendicular to the interferometer plane is also the direction of zero response for the longitudinal (scalar) gravitational waves which also exist in the vector theory of gravity [4].
Detection of a gravitational wave in the direction of zero response can rule out pure tensor or pure vector polarizations based on a single event. However, wave propagation direction can be obtained accurately only from observing the source electromagnetic counterpart which further reduces chances of making a decisive conclusion based on a single detection. So far the source of gravitational waves was found only for the signal GW170817 [2]. For this event, the gravitational wave propagated about 20° off the nearest direction of zero response for the tensor polarization and about 30° for the vector polarization. Thus, no conclusion in favor of one of the two polarizations can be made.

The LIGO-Virgo network can distinguish between the two polarizations by collecting statistics which should exhibit minima in the distribution function in the vicinities of the directions of the zero response. Tensor and vector polarizations predict that such minima occur at different positions. See Section 16 and Figure 10 in [4] for details.

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Appendix A: Radiation of gravitational waves in vector gravity

For a weak transverse gravitational wave propagating along the x-axis in vector gravity the equivalent metric evolves as [4]

\[ g_{ik} = \eta_{ik} + \begin{pmatrix} 0 & 0 & h_{0y}(t,x) & h_{0z}(t,x) \\ 0 & 0 & 0 & 0 \\ h_{0y}(t,x) & 0 & 0 & 0 \\ h_{0z}(t,x) & 0 & 0 & 0 \end{pmatrix}, \tag{A1} \]

where \( \eta_{ik} \) is Minkowski metric. The three-dimensional polarization vector of this wave is

\[ h = (0, h_{0y}, h_{0z}). \tag{A2} \]

Here we consider a general case when gravitational wave propagates along the unit vector \( \hat{k} \). Then polarization vector \( h \) is perpendicular to \( \hat{k} \) so that \( \hat{k} \cdot h = 0 \).

Signal of the LIGO-like interferometer with perpendicular arms of length \( L \) along the direction of unit vectors \( \hat{a} \) and \( \hat{b} \) is proportional to the relative phase shift \( \Delta \varphi \) of electromagnetic waves traveling a roundtrip distance \( 2L \) along the two arms [4]

\[ \Delta \varphi = \frac{2\omega L}{c} \left[ (\hat{a} \cdot \hat{k})(\hat{a} \cdot h) - (\hat{b} \cdot \hat{k})(\hat{b} \cdot h) \right], \tag{A3} \]

where \( \omega \) is the frequency of electromagnetic wave. Eq. (A3) shows that for any \( h \) the interferometer has zero response if gravitational wave propagates perpendicular to the interferometer plane \( \hat{a} \cdot \hat{k} = \hat{b} \cdot \hat{k} = 0 \) or along one of the interferometer arms. This result is independent of polarization of the transverse gravitational wave and can rule out vector theory of gravity if gravitational wave is detected propagating in the predicted direction of the zero response.

Depending on the inclination angle between \( \hat{k} \) and the orbital plane of binary stars, gravitational wave in vector gravity can be linearly or elliptically polarized in the same way as electromagnetic wave generated by an oscillating quadrupole. Recall that in vector gravity \( h \) is analogous to the vector potential \( \mathbf{A} \) in electrodynamics [4].

To be specific, we consider generation of gravitational waves by two compact stars with equal masses \( m \) moving along circular orbits of diameter \( R \) with angular velocity \( \Omega \) (see Fig. 1). \( \theta(t) = \Omega t + \theta_0 \) is the star azimuthal angle in the orbital plane. Using formulas of Ref. [4] we obtain that far from the binary system

\[ h = -\frac{2Gm\Omega^2r^2}{c^4R} \text{Re} \left[ Pe^{2i\theta(t)} \right], \tag{A4} \]

where \( R \) is the distance to the system, \( P \) is the complex polarization vector

\[ P = (\hat{k} \cdot \hat{n}) \left[ \hat{k} \times [\hat{k} \times \hat{n}] \right] - i[\hat{k} \times \hat{n}], \tag{A5} \]

and \( \hat{n} \) is a unit vector perpendicular to the star orbital plane (see Fig. 1).

We denote real unit polarization vectors as \( \hat{e}_1 \) and \( \hat{e}_2 \). They are perpendicular to \( \hat{k} \) and can be chosen such that \( \hat{e}_1 \times \hat{e}_2 = \hat{k} \). Using Eq. (A4) one can write \( h \) as

\[ h = -\frac{2Gm\Omega^2r^2}{c^4R} \text{Re} \left[ ((P \cdot \hat{e}_1)\hat{e}_1 + (P \cdot \hat{e}_2)\hat{e}_2) e^{2i\theta(t)} \right], \tag{A6} \]

Then Eq. (A6) yields the following expression for the interferometer response

\[ \Delta \varphi = -\frac{4G\omega Lm\Omega^2r^2}{c^5R} \text{Re} \left[ ((P \cdot \hat{e}_1)F_1 + (P \cdot \hat{e}_2)F_2) e^{2i\theta(t)} \right], \tag{A7} \]

where \( F_1 \) and \( F_2 \) are the detector response functions to the polarizations \( \hat{e}_1 \) and \( \hat{e}_2 \)

\[ F_{1,2} = (\hat{a} \cdot \hat{k})(\hat{a} \cdot \hat{e}_{1,2}) - (\hat{b} \cdot \hat{k})(\hat{b} \cdot \hat{e}_{1,2}) \tag{A8} \]

and

\[ (P \cdot \hat{e}_1) = -(\hat{k} \cdot \hat{n})(\hat{e}_1 \cdot \hat{n}) + i(\hat{e}_2 \cdot \hat{n}), \tag{A9} \]

\[ (P \cdot \hat{e}_2) = -(\hat{k} \cdot \hat{n})(\hat{e}_2 \cdot \hat{n}) - i(\hat{e}_1 \cdot \hat{n}). \tag{A10} \]

Comparing Eq. (A7) with Eq. (11) we find that parameter \( \epsilon \) is given by

\[ i\epsilon = \frac{(P \cdot \hat{e}_2)}{(P \cdot \hat{e}_1)} \tag{A11} \]

or

\[ \epsilon = \frac{(\hat{k} \cdot \hat{n}) + i(\hat{e}_1 \cdot \hat{n})(\hat{e}_2 \cdot \hat{n})}{1 - (\hat{e}_1 \cdot \hat{n})^2}. \tag{A12} \]
When $\hat{k}$ is parallel to the orbital plane ($\hat{k} \cdot \hat{n} = 0$) the gravitational wave is linearly polarized and polarization vector $\mathbf{h}$ is parallel to the orbital plane. In this case $\epsilon$ is pure imaginary and, according to Eqs. (15) and (16), $\Phi_L = \Phi_V$. That is phase shift of the detected signal between different interferometers is determined only by the time delay. Since orientation of the orbital plane is unknown, the direction of $\mathbf{h}$ is a free parameter. Using vector identities one can obtain from Eq. (A3) that for the plane gravitational wave the interferometer signal is equal to zero if $\mathbf{h}$ is parallel to the vector

$$\mathbf{h} \parallel \hat{k} \times \left[(\hat{a} \cdot \hat{k})\hat{a} - (\hat{b} \cdot \hat{k})\hat{b}\right]. \quad (A13)$$

So, if for fixed $\hat{k}$ the wave polarization $\mathbf{h}$ satisfies Eq. (A13) for the orientation of the Virgo (LIGO) interferometer arms then Virgo (LIGO) signal will be zero. At the same time, orientation of the LIGO (Virgo) arms will not satisfy Eq. (A13) and the signal can be nonzero. Thus, the ratio of the LIGO/Virgo signal amplitudes depends on the unknown polarization direction $\mathbf{h}$. For fixed $\hat{k}$, depending on the direction of $\mathbf{h}$, the LIGO/Virgo signal ratio can be any number between 0 and $\infty$.

If inclination angle is nonzero, then gravitational wave will be elliptically polarized. In this case $\epsilon$ has both real and imaginary parts and $\Phi_L \neq \Phi_V$. Degree of ellipticity determines the extra phase shift between LIGO and Virgo signals which is not caused by the difference in the arrival times of the wave at the interferometer locations. The inclination angle is also a free parameter. In total, the orientation of the orbital plane of the binary system is described by two free parameters (angles of $\hat{n}$) which determine the ratio of the LIGO/Virgo signal amplitudes and the extra phase shift between the measured waveforms.

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