GABOR ORTHOGONAL BASES AND CONVEXITY

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ABSTRACT. Let \( g(x) = \chi_B(x) \) be the indicator function of a bounded convex set in \( \mathbb{R}^d \), \( d \geq 2 \), with a smooth boundary and everywhere non-vanishing Gaussian curvature. Using a combinatorial approach we prove that if \( d \not\equiv 1 \mod 4 \), then there does not exist \( S \subset \mathbb{R}^{2d} \) such that \( \{g(x-a)e^{2\pi ix \cdot b}\}_{(a,b) \in S} \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \).

1. Introduction

The basic question we ask in this paper is, which functions \( g \) can serve as window functions for orthogonal Gabor bases for \( L^2(\mathbb{R}^d) \)?

Let \( g \in L^2(\mathbb{R}^d) \) and \( S \subset \mathbb{R}^{2d} \) be a countable subset. The Gabor system associated to \( g \) and \( S \) is defined by

\[ \mathcal{G}(g, S) = \{g(x-a)e^{-2\pi ix \cdot b}\}_{(a,b) \in S}. \]

Definition 1.1. We say that \( \mathcal{G}(g, S) \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \) with the window function \( g \), \( \|g\| = 1 \), and the spectrum \( S \) if \( \mathcal{G}(g, S) \) is a complete basis for \( L^2(\mathbb{R}^d) \) and the basis vectors are mutually orthogonal in the sense that

\[ \int g(x-a)g(x-a')e^{-2\pi ix \cdot (b-b')}dx = 0 \text{ for all } (a,b) \neq (a',b'). \]

The theory of Gabor bases and frames has undergone much development in recent decades. But as Gröchenig points out in his seminar article ([10]), "there has been little progress on the original question of how to determine which windows and lattices generate a Gabor frame". In this paper we take a small step in this direction in the context of general Gabor orthogonal bases, not necessarily lattices. The question of which functions \( g \) can serve as window function for orthogonal Gabor bases is typically studied using the Balian-Low theorem and its variants. See, for example, [2], [4], [5], [8], [22] and the references contained therein. See also [1], [11] and [19] for closely related results. However, in this paper our aim is to rule out a class of window functions which are indicator functions of bounded sets. In such cases, the Fourier transform of the window function is instantly poorly localized and hence Balian-Low type theorems are difficult to utilize.

When \( S = A \times B \) and \( g = \chi_E(x) \), with \( E \) a bounded subset of \( \mathbb{R}^d \), then it is not difficult to see that \( E \) must tile \( \mathbb{R}^d \) by translation with \( A \) serving as a tiling set. Similarly, in this case \( \{e^{-2\pi ix \cdot b}\}_{b \in B} \) would have to be an orthogonal basis for \( L^2(E) \). This largely reduces the orthogonal Gabor basis problem to the tiling and orthogonal exponential basis components. As a result, we can use the

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theory of tiling and orthogonal exponential bases to rule out the possibility that \( \chi_E \), for a given set \( E \), is the window function for an orthogonal Gabor basis.

This allows us to rule out the possibility that \( \chi_E \) is the window function for an orthogonal Gabor basis with \( S = A \times B \) for many classes of sets \( E \). For example, if \( E = B_2 \) is the unit ball, we can rule out \( \chi_{B_2} \) in two different ways. First of all, the unit ball does not tile by translation. It is known that a convex body tiles by translation only if it is a polyhedron of a certain type. See, for example, [21]. Another way to prove that \( \chi_{B_2} \) is not a window function in the case \( S = A \times B \) is by showing that \( L^2(B_2) \) does not possess an orthogonal exponential basis. This was established by the first listed author, Nets Katz and Steen Pedersen in [13].

When \( S \) is not of the form \( A \times B \), the problem of determining the possible window functions becomes considerably more difficult. In this paper we rule the possibility that the window function \( g \) is an indicator function of a symmetric convex set with a smooth boundary and everywhere non-vanishing Gaussian curvature. The approach used by the first listed author, Katz and Pedersen ([13]) or the one employed the first listed author, Katz and Tao in [14] are difficult to apply here due to the fact that the Gabor spectrum is not assumed to be of the form \( A \times B \). However, the geometric approach used in [15] combined with a suitable combinatorial pigeon-holing technique allows us to rule out a large class of window functions. Our main result is the following.

**Theorem 1.2.** Let \( K \) denote a bounded convex set in \( \mathbb{R}^d \), \( d \neq 1 \mod 4 \), symmetric with respect to the origin. Suppose that \( \partial K \) is smooth and has everywhere non-vanishing Gaussian curvature. Then there does not exist an orthonormal Gabor basis for \( L^2(\mathbb{R}^d) \) with the window function \( g(x) = \frac{|K|^{-1/2}}{2} \chi_K(x) \).

**Remark 1.3.** We do not know what happens in the case when \( d = 1 \mod 4 \). It should not be difficult to modify the proof of Theorem 1.2 to handle the more general case when \( \partial K \) is smooth and the boundary contains at least one point where the Gaussian curvature does not vanish. We shall address these issues in the sequel. Other related issues are raised in Section 3 below.

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2. **Proof of Theorem 1.2**

The structure of our argument is the following.

- **i)** First we use the general theory of Gabor frames to prove that \( S \) is well-distributed in the sense that \( \mathbb{R}^{2d} \) can be tiled by cubes of side-length \( C > 0 \) such that every cube contains at least one pair \( (a, b) \in S \). This argument is based purely on the completeness of Gabor frames.

- **ii)** We then use a combinatorial argument to show that there exist at least three points \( (a_j, b_j) \in S, j = 1, 2, 3 \), such that \( |a_1 - a_j| \) is small and \( |b_i - b_j|, i \neq j \), is suitably large.

- **iii)** Finally, we use orthogonality and an asymptotic formula for the Fourier transform of the indicator function of a symmetric convex body with a smooth boundary and everywhere non-vanishing Gaussian curvature to see that the triplet in item ii) cannot exist, obtaining a contradiction.
2.1. Basic structure of orthonormal Gabor bases. We need the following basic structure theorem.

**Lemma 2.1.** Let \( G(g, S) \) be an orthonormal basis for \( L^2(\mathbb{R}^d) \), where \( g \) is in \( L^2(\mathbb{R}^d) \) with norm 1. Then the following hold:

- **i)** There exists \( c > 0 \) such that for any \((a, b), (a', b') \in S, (a, b) \neq (a', b')\),
  \[ |a - a'| + |b - b'| \geq c. \]
- **ii)** There exists \( C > 0 \) such that any cube of side-length \( C \) in \( \mathbb{R}^{2d} \) contains at least one point of \( S \).
- **iii)** \( S \) has uniform density equal to 1: \( D^+(S) = D^-(S) = 1 \).

2.1.1. **Proof of part i).** Observe that if \((a, b) = (a', b')\), then the orthogonality relation takes the form
  \[ \int |g(x)|^2 dx = 1. \]
  The continuity of the integral implies that if \((a, b)\) is sufficiently close to \((a', b')\), then
  \[ \int g(x - a)g(x - a')e^{-2\pi ix \cdot (b - b')} dx \neq 0. \]
  The claim follows.

2.2. **Proof of part ii).** The proof follows from the following result. Define the short time Fourier transform (STFT) (see e.g. [7]) by
  \[ V_g f(t, \nu) = \int f(x)g(x - t)e^{-2\pi i x \cdot \nu} dx, \quad (t, \nu) \in \mathbb{R}^{2d}. \]

**Lemma 2.2.** ([1]) Given \( g \in L^2(\mathbb{R}^d) \) with \( \|g\| = 1 \) and a countable set \( S \subset \mathbb{R}^{2d} \), the system \( G(g, S) \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \) if and only if for any \( f \in L^2(\mathbb{R}^d) \) with \( \|f\| = 1 \),
  \[ \sum_{a \in S} |V_g f(w - a)|^2 = 1 \quad \text{for almost all} \ w \in \mathbb{R}^{2d}. \]
  In other words, \( |V_g f|^2 + S = 1 \) is a tiling, hence \( S \) has asymptotic density 1 and the conclusion of (ii) follows from [16].

**Proof of iii) of Theorem 2.1.** The proof follows from Corollary 4 of [23].

2.3. Combinatorial extraction of an (essentially) linear triple. Having established the fact that \( S \) is well-distributed in the sense of Theorem 2.1, we are going to tile \( \mathbb{R}^{2d} \) by cubes of side-length \( C \). Without loss of generality we may take \( C = 1 \). We may also assume that each cube contains exactly one pair \((a, b)\) from \( S \) because we can always throw the extra pairs out for the purpose of what we are about to do.

We first establish the following general principle. Assume that \( U \) is a countable set of \( n \)-tuples in \( \mathbb{R}^n, n \geq 2 \). Assume that \( U \) has at least one point in any cube of length 1. Given a small number
\( \delta > 0 \) and a big number \( R > 0 \), we are going to find three \( n \)-tuples \( x_i = (x_{i,1}, \cdots, x_{i,n-1}, x_{i,n}), i = 1, 2, 3 \) satisfying the following properties.

- \( |(x_{i,1}, \cdots, x_{i,n-1}) - (x_{j,1}, \cdots, x_{j,n-1})| \leq \sqrt{n - 1} \delta, i, j = 1, 2, 3 \)
- \( |x_{i,n} - x_{j,n}| \geq R, i, j = 1, 2, 3 \).

To this end, we are going to do the following. Consider a stack of vertical cubes of length 1, where one of the cubes is \([0, 1]^n\). For each cube, consider its projection onto the first \( n - 1 \) coordinates, which is just a \( n - 1 \)-dimensional cube of length 1. We denote this cube by \( Q_{n-1} \). We tile this cube by sub-cubes of length \( \delta \) of the form \([0, \delta]^{n-1} + \delta(m_1, \cdots, m_{n-1})\). We can do the sub-tiling with length of approximately \( \delta = N^{-1} \) for some large \( N \). Thus, \( 0 \leq m_i \leq N \approx \delta^{-1} \). Recall that by the well distributivity of the set \( S \), every cube of length 1 contains at least one point of \( S \). We can throw all the points and keep only one. This point is of the form \((x_1, \cdots, x_{n-1}, x_n)\). The projection of \((x_1, \cdots, x_{n-1}, x_n)\) onto the first \( n - 1 \) coordinate has the form of \((x_1, \cdots, x_{n-1})\).

Since this point belongs to one of the sub-cubes of side length \( \delta \), assume that

\[
(x_1, \cdots, x_{n-1}) \in [0, \delta]^{n-1} + \delta(m_1, \cdots, m_{n-1}).
\]

We let \((m_1, \cdots, m_{n-1})\) denote the address of \((x_1, \cdots, x_{n-1})\) in the unit cube \( Q_{n-1} \).

Further, we are going to choose many cubes in our vertical stake with large enough gap between them. For example, start from \( Q_n := [0, 1]^n \) and go up by \( 10R \) and choose another cube. We repeat this procedure big number times, e.g. \( 10^{10} \delta^{-n+1} \). Recall that each cube has exactly one point from \( S \). Thus we have collected \( 10^{10} \delta^{-n+1} \) points in our stake where their \( n \)-th coordinates have difference of larger than \( R \).

It is clear that the number of selected cubes in \( \mathbb{R}^n \), thus the number of the points in \( S \) contained in the collection of the \( 10^{10} \delta^{-n+1} \)-cubes, exceed the number of sub-cubes of length \( \delta \) in \( Q_{n-1} \) in \( \mathbb{R}^{n-1} \). Indeed, we have \( 10^{10} \delta^{-n+1} \) points while we have \( \delta^{-n} \) sub-cubes of side length \( \delta \). Thus, the number of the points exceed the number of addresses by a factor larger than 2. By the pigeon-hole principle we conclude that there are at least 3 points in our collections with the same address. In other words, there are three \( n \)-tuples \( x_i = (x_{i,1}, \cdots, x_{i,n-1}, x_{i,n}), 1 \leq i \leq 3 \) such that

- \( |(x_{i,1}, \cdots, x_{i,n-1}) - (x_{j,1}, \cdots, x_{j,n-1})| \leq \sqrt{n} \delta, i, j = 1, 2, 3 \)

Notice our points in the stake of vertical cubes are far from each other at least by \( 10R \). This forces that \( |x_{i,n} - x_{j,n}| \geq R, i, j = 1, 2, 3 \), as claimed.

We now apply the principle we just established. Given \((a, b) \in S\), let \( P(d) \) denote the first \( d - 1 \) coordinates of \( b \). Let \( P_d(b) \) denote the projection of \( b \) onto the \( d \)-th coordinate. By running the procedure above with \( n = 2d \) and \( U = \{(a, P(d), P_d(b)) : (a, b) \in S\} \), we produce three elements of \( S \), \((a_1, b_1), (a_2, b_2), (a_3, b_3)\) satisfying the following properties:

- i) \( |a_1 - a_2|, |a_1 - a_3|, |a_2 - a_3| \leq \sqrt{2d} \delta \).
- ii) \( P_d(b_1) < P_d(b_2) < P_d(b_3) \).
- iii) \( |P(b_2) - P(b_1)|, |P(b_3) - P(b_2)|, |P(b_3) - P(b_1)| \leq \sqrt{2d} \delta \).
- iv) \( |P_d(b_3) - P_d(b_2)|, |P_d(b_2) - P_d(b_1)|, |P_d(b_3) - P_d(b_1)| \geq R \).

We denote the resulting triple of elements of \( S \) a \((\delta, R)\)-triple.

2.4. Orthogonality and asymptotic expansions of the Fourier transform. Let \( K \) be a symmetric convex body with a smooth boundary and everywhere non-vanishing Gaussian curvature.
Let $K$ be as above written in the form

$$K = \{x : \rho(x) \leq 1\},$$

where $\rho := \rho_K$ is the norm that defines $K$, often called the Minkowski functional (gauge). Define the dual functional by the relation

$$\rho_*(\xi) = \sup_{x \in \partial K} x \cdot \xi,$$

where $\partial K$ denotes the boundary of $K$. We shall need the following result which can be found in Lemma 1.4 from [15]. See [12] and [9] for the original formulation.

**Lemma 2.3.** Let $K$ and $\rho_*$ be as above. Then for any $N$,

$$\hat{\chi}_K(\xi) = \sum_{\alpha=0}^{N} C_\alpha \left( \frac{\xi}{|\xi|} \right) J_{\frac{d}{4} + \alpha}(2\pi \rho_*(\xi))|\xi|^{-\frac{d}{2} - \alpha} + O(|\xi|^{-\frac{d+3}{2} + N})$$

$$= \hat{C}_\alpha \left( \frac{\xi}{|\xi|} \right) \sin \left( 2\pi \rho_*(\xi) - \frac{d}{4} \right) |\xi|^{-\frac{d+1}{4}} + O(|\xi|^{-\frac{d+3}{2}}).$$

For $\alpha = 0, 1, \cdots, N$, the functions $C_\alpha, \hat{C}_0$ are smooth functions of $K$ alone, with $C_0, \hat{C}_0$ being strictly positive.

From this lemma we deduce that if $\hat{\chi}_K(\eta) = 0$, then

$$(2.2) \quad \rho_*(\eta) = \frac{k}{2} + \frac{d-1}{8} + O(|\eta|^{-1}).$$

**2.5. Conclusion of the proof of Theorem 1.2.** We are now in position to put together all the ingredients we have chopped up and seasoned.

Going back to (1.1), we have

$$(2.3) \quad \int \chi_K(x - a_i)\chi_K(x - a_j)e^{2\pi i x \cdot (b_j - b_i)}dx = 0,$$

where $1 \leq i < j \leq 3$ and the pairs $(a_i, b_i)$ form a $(\delta, R)$-triple with $\delta, R$ to be determined in a moment.

Note that

$$\chi_K(x - a_i)\chi_K(x - a_j) = \chi_{K+a_i \cap K+a_j}(x).$$

It follows that the expression in (2.3) equals

$$\hat{\chi}_{K+a_i \cap K+a_j}(b_j - b_i) = \hat{\chi}_{K+a_i}(b_j - b_i) + O(\delta)$$

because $K + a_i$ is the union of $K + a_i \cap K + a_j$ and a set of volume $\leq C\delta$.

It follows that

$$0 = \hat{\chi}_{K+a_i \cap K+a_j}(b_j - b_i) = \hat{\chi}_{K+a_i}(b_j - b_i) + O(\delta),$$

from which we deduce using the argument that yields (2.2) from Lemma 2.3 and trigonometry that

$$\rho_*(b_j - b_i) = \frac{k}{2} + \frac{d-1}{8} + O(|b_j - b_i|^{-1}) + O(\delta |b_j - b_i|^{\frac{d+1}{2}}),$$

since the derivative of cosine is sine and the gradient of $\rho_*$ does not vanish away from the origin by Euler homogeneity relations.
Recall that $|b_j - b_i| \geq R$. Choose $R$ very large and choose $\delta = R^{-\frac{4m+1}{8}}$. It follows that $b_1, b_2, b_3$ live in an $R^{-1}$-neighborhood of a line and hence

$$
|P_d(b_j) - P_d(b_i)| = \frac{k}{2} + \frac{d-1}{8} + O(R^{-1}), \quad 1 \leq i < j \leq 3.
$$

If $R$ is sufficiently large, this yields a contradiction if $d \not\equiv 1 \mod 4$. This completes the proof of Theorem 1.2.

**Remark 2.4.** Note that the contradiction disappears if $d \equiv 1 \mod 4$ because the $\frac{d-1}{8}$ term in (2.4) takes the form $\frac{4m+1}{8} = \frac{2m}{2}$ and thus gets absorbed into the $\frac{2}{3}$ terms that precedes it.

### 3. Open problems: Spectral sets and Gabor windows

We conclude this paper with a series of questions that arise naturally out of considerations that led to our main result. We have shown that if $K$ is a symmetric convex body in $\mathbb{R}^d$, $d \not\equiv 1 \mod 4$, $d > 1$, where $\partial K$ is smooth and has non-vanishing Gaussian curvature, then the indicator function of $K$ cannot serve as the window function for an orthogonal Gabor basis on $L^2(\mathbb{R}^d)$. It was shown in [14] that under these assumptions, $L^2(K)$ does not possess an orthogonal basis of exponentials. This leads us to the following question.

**Definition 3.1.** We say that $E \subset \mathbb{R}^d$ is spectral if $L^2(\mathbb{R}^d)$ possesses an orthogonal basis of exponentials, namely a basis of the form $\{e^{-2\pi ix \cdot b}\}_{b \in B}$. We shall refer to $B$ as a spectrum.

**Question 3.2.** Suppose that $E$ is a non-spectral subset of $\mathbb{R}^d$. Is it possible that $\mathcal{G}(\chi_E, S) = \{\chi_E(x - a)e^{-2\pi ix \cdot b}\}_{(a, b) \in S}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$?

A natural example to investigate in the context of the Question 3.2 is that of a triangle. It is not spectral which follows, for instance, from a result due to Kolountzakis which proves that non-symmetric convex sets are never spectral ([18]). A similar question can be posed for sets that tile by translation but are not spectral (see [17] and the references contained therein).

Conversely, it is natural to ask whether all spectral sets generate Gabor windows:

**Question 3.3.** Suppose that $E$ is a spectral subset of $\mathbb{R}^d$. Is it true that there exists $S \subset \mathbb{R}^{2d}$ such that $\mathcal{G}(\chi_E, S) = \{\chi_E(x - a)e^{-2\pi ix \cdot b}\}_{(a, b) \in S}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$?

The answer to this question is automatically affirmative if $E$ tiles $\mathbb{R}^d$ by translation. In this case we take $S = A \times B$, where $A$ is a tiling set for $E$ and $B$ is a spectrum. However, examples due Tao ([24]), Kolountzakis, Matolcsi ([17], [20]) and others show that there exist spectral sets which do not tile by translation. It would be natural to investigate the Question 3.3 starting with those constructions.

It is worth recalling in this context that a result due to Fuglede ([6], see also [3]) shows that if $E \subset \mathbb{R}^d$ tiles by a lattice $L$, then $E$ is spectral with a spectrum given by the dual lattice $L^*$. Conversely, if $E$ is spectral with a spectrum $L$ which happens to be a lattice, then $E$ tiles by translation by the dual lattice $L^*$. It follows that if $E$ tiles by a lattice $L$, then $\mathcal{G}(\chi_E, L \times L^*)$ is an orthogonal basis for $L^2(\mathbb{R}^d)$. Similarly, if $E$ is spectral with a lattice spectrum $L$, then $\mathcal{G}(\chi_E, L^* \times L)$ is an orthogonal basis for $L^2(\mathbb{R}^d)$.
A possibly even more basic question arises from the discussion above. As we have seen, the case when $S = A \times B$ is somewhat special in the context of Gabor bases, so it makes sense to try to understand to what extent it is prevalent.

**Question 3.4.** Does there exist a window function $g$ such that $G(g, S) = \{g(x-a)e^{-2\pi i x \cdot b}\}_{(a,b) \in S}$ is not an orthogonal basis for $L^2(\mathbb{R}^d)$ for any $S$ of the form $A \times B$, but is an orthogonal basis for $L^2(\mathbb{R}^d)$ for some $S$ not of that form?

In this general direction, Han and Wang ([11]) proved that if $S = M\mathbb{Z}^{2d}$, where $M$ is a matrix with rational entries and $\det(M) = 1$, then there exists a compactly supported $g \in L^2(\mathbb{R}^d)$ such that $G(g, S) = \{g(x-a)e^{-2\pi i x \cdot b}\}_{(a,b) \in S}$ is an orthogonal basis for $L^2(\mathbb{R}^d)$. However, it is not immediately clear whether $g \in L^2(\mathbb{R}^d)$ can be chosen in this case in such a way that $G(g, S)$ is not an orthogonal basis for $L^2(\mathbb{R}^d)$ for any $S$ of the form $A \times B$.

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