WHEN IS THE HAWKING MASS MONOTONE UNDER GEOMETRIC FLOWS

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Abstract. In this paper, we study the relation of the monotonicity of Hawking Mass and geometric flow problems. We show that along the Hamilton-DeTurck flow with bounded curvature coupled with the modified mean curvature flow, the Hawking mass of the hypersphere with a sufficiently large radius in Schwarzschild spaces is monotone non-decreasing.

1. Introduction

In this work we plan to study the monotonicity of Hawking mass under the Ricci flow coupled with the geometric flow for hypersurfaces in a slice of the Ricci flow. Actually, H.Bray [4] asked if there exists a flow which makes the quasi-local mass functional nondecreasing.

Given a time interval $I = (a,b)$. The Ricci flow on a manifold $M^{n+1}$ is defined by a one parameter family of metrics $\{g(t); t \in I\}$ satisfying

\begin{equation}
\partial_t g(t) = -2Rc(g(t)), \quad t \in I
\end{equation}

where $Rc(g(t))$ is the Ricci tensor of the metric $g(t)$. This flow was introduced by R.Hamilton in 1982 ([8], [11]) and was used by G.Perelman ([14]) in 2002-3 to give an outline of the proof of Poincare conjecture in dimension three. Ricci flow is a fully nonlinear equation with beautiful geometrical applications (see [8], [14], also [6]). The local existence result on a compact manifold was obtained by R.Hamilton in the famous paper in 1982. In the paper [7], we have proved that the asymptotically locally Euclidean (ALE) property is preserved by the Ricci flow (and the related Hamilton-Ricci flow, which is equivalent to the Ricci flow after a change of gauge). An interesting convergence result is obtained in [13]. The Ricci flow was also been discussed in Physical literature, such as in quantum filed theory, where it was considered as an approximation to the renormalization group (RG) flow for the 2-dimensional nonlinear sigma model.

When interpreted as the statistical physics analogy to Einstein filed equation, Ricci flow can be considered as the gradient flow of the W-functional, which is a non-decreasing entropy functional along the Ricci flow coupling
a heat equation. For a precise definitions of W-functional, one may look at G.Perelman’s paper [14].

We want to find such a flow (which will be the modified mean curvature flow, see [3,5] in section [3] for the definition and see [9] for related flow) so that the Hawking mass ([10], [16], [20], and [21]) is monotone along it. Given a Riemannian manifold \((M^{n+1}, g)\). Assume that \(n = 2\). Let’s recall the definition of Hawking mass for a hypersurface \(F : \Sigma \rightarrow M\). Let \(\nu\) be the outward unit normal vector to \(\Sigma\). Let \(H = H(\Sigma)\) and \(V = V(\Sigma)\) be the mean curvature of \(\Sigma\) and the area of \(\Sigma\) in \(M\) respectively. Then the Hawking mass of the hypersurface \(\Sigma\) is defined by

\[
m_H = m(\Sigma) = \sqrt{\frac{V}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_\Sigma |H|^2 \right).
\]

Given a family of hypersurfaces \(F(\cdot, t)\). We want to find a real valued smooth function \(p(F)\) such that along the geometric flow of the form

\[
\partial_t F = p(F)\nu(F),
\]

the Hawking mass is monotone. This is a natural question. Usually, one should have a lot such kind of flows. We just pick up a natural one. Our main result is Theorem [11] in section [3]. Roughly speaking, the result says that along the Hamilton-DeTurck flow with bounded curvature coupled with the modified mean curvature flow, the Hawking mass of the hypersphere with a sufficiently large radius in Schwarzschild space is monotone non-decreasing. We should say that our result may be extended to non-symmetric cases, however, our aim here is to introduce the flow. This result is interesting in general relativity.

The plan of this paper is the following. In the next section, we give useful geometric formulae and properties of the Ricci flow for warped product spaces. In section [3] we prove the main result stated in Theorem [11].

2. The flow on warped product spaces and related

We denote by \(R(t)\) the scalar curvature of the metric \(g(t)\). Then the function \(R(t)\) satisfies the following evolution equation

\[
\partial_t R = \Delta R + 2|Rc|^2 \geq \Delta R + \frac{2}{n+1}R^2.
\]

This is an important differential inequality which give us the blow up in finite time in compact manifold with initial positive scalar curvature. In particular, we have that the positivity of scalar curvature is preserved along the Ricci flow. However, we assume that the Ricci flow has uniformly bounded curvature (see [7] and [19]).

Let \(M = \mathbb{R} \times S^n\) with the product metric

\[
g = \phi(x)^2 dx^2 + \psi(x)^2 \hat{g}.
\]

Here \(S^n\) be the standard n-sphere so that \(\hat{g} = d\sigma^2\). We shall write by \(x = r\) as the standard radial variable in the Schwarzschild coordinates.
Introduce the arc-length parameter $s$ and sectional curvatures in the following way. Let

\begin{equation}
(2.3) \quad s = \int_0^x \phi(\tau) d\tau.
\end{equation}

Let

\begin{equation}
K_0 = -\frac{\psi_{ss}}{\psi},
\end{equation}

be the sectional curvature of the 2-planes perpendicular to the spheres $\{x\} \times S^n$ and

\begin{equation}
K_1 = \frac{1 - \psi_s^2}{\psi^2}
\end{equation}

the sectional curvature of the 2-planes tangential to the spheres. Note that

\begin{equation}
\frac{\partial}{\partial s} = \frac{1}{\phi(x)} \frac{\partial}{\partial x}.
\end{equation}

In terms of the arc-length parameter $s$, the metric can be read as

\begin{equation}
g = ds^2 + \psi^2 \hat{g}.
\end{equation}

It is known that the Ricci curvature tensor of the product metric $g$ is given by

\begin{equation}
Rc = n\{-\frac{\psi_{xx}\psi + \psi_x \phi_x}{\phi \psi}\} dx^2 + \{-\frac{\phi \psi_{xx} - (n-1)\phi \psi_x^2 + \psi \phi_x \psi_x}{\phi^3} + n-1\} \hat{g}.
\end{equation}

See [1] and [17].

Let

\begin{equation}
g_s = \psi^2 \hat{g}.
\end{equation}

Then we have the hessian formula for $s$ (see [15]):

\begin{equation}
Hesss = \psi \psi_s \hat{g} = \frac{\psi_s}{\psi} g_s,
\end{equation}

and the Ricci tensor can be written as

\begin{equation}
Rc = nK_0 ds^2 + [K_0 + (n-1)K_1] \psi^2 \hat{g}.
\end{equation}

See [17].

Note that the scalar curvature is given by

\begin{equation}
R = nK_0 + n[K_0 + (n-1)K_1].
\end{equation}

Then for the warped product metrics, the Ricci flow (1.1) becomes the system

\begin{equation}
(2.4) \quad \begin{cases} 
\psi_t = -[K_0 + (n-1)K_1] \psi = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi}, \\
\phi_t = -nK_0 \phi = n\frac{\psi_{xx}}{\psi} \phi.
\end{cases}
\end{equation}

In the xt coordinates, we have

\begin{equation}
(2.5) \quad \begin{cases} 
\psi_t = \frac{1}{\phi(x)} \frac{\partial}{\partial x} \frac{1}{\phi(x)} \frac{\partial}{\partial x} \psi - (n-1)\frac{1}{\psi} \left(\frac{3}{\phi(x)} \frac{\partial}{\partial x} \phi(x)\psi\right)^2, \\
\phi_t = n\frac{\psi_{xx}}{\psi} \phi.
\end{cases}
\end{equation}
We now discuss some related geometric quantities. For a fixed $r > 0$, let
\[ \Sigma := \partial B(r) = \{ p \in M; s(p) = r \} \]
be the sphere of radius $r$ in the warped Riemannian manifold $(M, g)$. Note that the volume of $\partial B(r)$ is
\[ V(r) = |\Sigma| = \int_{\Sigma} d\sigma = \int_{N} \psi(s)^n dv_{\hat{g}} = \psi(r)^n V(N). \]
Here $V(N)$ is the volume of $N = S^n$ in the metric $\hat{g}$. In particular, for $n = 2$, we have $V(N) = 4\pi$.
Let
\[ g_s = \psi^2 \hat{g}. \]
Then we have the hessian formula for $s$ (see [13]):
\[ \text{Hess}(s) = \psi \psi_s \hat{g} = \frac{\psi_s}{\psi} g_s. \]
By the hessian formula for the function $s$, we have the mean curvature of the sphere $\Sigma$ given by
\[ H = \Delta s = n \frac{\psi_s}{\psi} = \frac{V'(s)}{V(s)}. \]
Note that the Hawking mass of $\Sigma$ ($n = 2$) is
\[ m = m(\Sigma) = \sqrt{\frac{V(s)}{16\pi}} (1 - \frac{1}{16\pi} \frac{V'(s)^2}{V(s)}), \]
which can also be written as
\[ m = \frac{\psi(s)}{2} (1 - \psi'^2). \]
In the following, we always assume that $n = 2$.

3. Monotonicity in Schwarzschild spaces

Recall that a Schwarzschild space has the metric of the form
\[ g = \exp(2\lambda(r)) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]
which is assumed to be asymptotically locally Euclidean (in short, ALE).

To keep the specific form of the metrics, we need to use the so-called Hamilton-DeTurck flow. The Hamilton-DeTurck flow is the following
\[ \partial_t g = -2Re(g) + L_X g \]
where $X$ is the vector-field on the manifold $M$ and $L_X g$ is the Lie-derivative of the metric $g(t)$. This flow is important in the sense that we can keep the form of the metrics (3.1) along it by suitable vector field $X$. In fact, we choose
\[ X = \left[ \frac{n-1}{r} (\exp(2\lambda(r,t)) - 1) + \partial_r \lambda \right] \partial_r. \]
Given a manifold \( M \subset \mathbb{R}^n \) with the standard Euclidean polar coordinates \((r, \theta)\). For the Hamilton-DeTurck Ricci flow of the form

\[
g(t) = \exp(2\lambda(r, t))dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]
we have

\[
\partial_t f = 1 + f \partial_r^2 f - \frac{2}{f^3} (f_r)^2 + \left( \frac{n-1}{r} - \frac{1}{rf^2} \right) f_r - \frac{(n-1)}{r^2 f} (f^2 - 1),
\]
for \( f = \exp(\lambda) \) with \( f(0) = 1 \) and \( f(\infty) = 1 \). Using the Maximum principle trick, we have that \( f \) is uniformly bound both in time variable and space variables, which implies the global existence of the flow (see also \cite{13}). Invoking the ALE properties of the Ricci flow \cite{7} and the gradient estimate of Shi \cite{19}, we know that

\[
f \to 1, \quad r \partial_r f \to 0, \quad r^2 \partial_r^2 f \to 0
\]
uniformly for \( r \to \infty \); in particular, the derivative \( f_t \) is also uniformly bounded.

Note that the unit normal \( \nu \) to \( \Sigma \) is

\[
\nu = \frac{1}{\phi} \frac{\partial}{\partial x}.
\]
Hence the geometric flow for hypersurfaces can be written in the following form:

\[
\partial_t s = p(H(s)).
\]
Here we have

\[
H(s) = \frac{2}{rf}.
\]

In the state case when the metric is of the form \((3.2)\), we have

\[
\psi(s) = r(s), \quad \frac{\partial r}{\partial s} = e^{-\lambda} = f^{-1}.
\]
Then we have

\[
K_0 = -\frac{f_r}{rf^3}, \quad K_1 = \frac{1 - f^{-2}}{r^2},
\]
and

\[
R = 4K_0 + 2K_1 = -\frac{4f_r}{rf^3} + 2\frac{1 - f^{-2}}{r^2}.
\]
We also have

\[
m(r) = \frac{r}{2}(1 - f^{-2}),
\]
and

\[
\frac{d}{dr} m(r) = \frac{1 - f^{-2}}{2} - rf^{-3} f_r.
\]
We remark that

\[
m(r) = \frac{r^3 K_1}{2}
\]
for this metric.

In the evolving case, in some sense, we need to understand the evolution of the curvature \( K_1 \). Here, we want to consider in another way and we shall...
determine a geometric nature choice of \( p(r) \) to make the Hawking mass monotone.

Note that along the Hamilton-DeTurck flow coupled with the geometric flow (3.3)

\[
\frac{dm(r,t)}{dt} = \frac{\partial m}{\partial r} \frac{dr}{ds} \frac{ds}{dt} + \partial_t m = \left[ 1 - \frac{f}{2} - rf^{-3}f_r \right]f^{-1}p(H) + \partial_t m.
\]

Note that

\[
\partial_t m = rf^{-3}f_t.
\]

So, putting all these together, we have

\[
\frac{dm(r,t)}{dt} = \left[ 1 - \frac{f}{2} - rf^{-3}f_r \right]f^{-1}p(H) + rf^{-3}f_t.
\]

Then we have

\[
\frac{dm(r,t)}{dt} = \frac{1}{4}r^2Rf^{-1}p(H) + rf^{-3}f_t.
\]

Assume that

\[
(3.5) \quad p(H) = R^{-1}H,
\]

and we shall call such a geometric flow as the negative mean curvature flow. Using \( fH = \frac{2}{7} \), we have

\[
\frac{dm(r,t)}{dt} = \frac{r}{2} + rf^{-3}f_t + O(1).
\]

Note that \( \frac{1}{4H} = \frac{r}{2} + O(1) \) is the leading term in the expression above. Then

\[
\frac{dm(r,t)}{dt} > 0
\]

for large radius \( r > 0 \). Therefore, we get the following result.

**Theorem 1.** Assume that the scalar curvature \( R(g_0) \) is positive for the initial Schwarzschild metric

\[
g_0 = \exp(2\lambda(r))dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

which is assumed to be asymptotically locally Euclidean. Along the Hamilton-DeTurck flow with bounded curvature coupled with the modified Mean curvature flow, there exists a large radius \( r_0 >> 1 \) depending on the bound of curvature of Hamilton-DeTurck flow such that for all \( r > r_0 \), the Hawking mass is monotone non-decreasing.

It is quite clear that we can make the Hawking mass monotone non-increasing if we choose

\[
p(H) = -R^{-1}H.
\]

However, in this case, the geometric flow is a back-ward flow. It is an interesting question to study the same problem along the Cross-curvature flow introduced by Chow-Hamilton [12]. We believe the similar result is also true for general asymptotically flat warped product spaces. The progress is still in the future.
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