HOPF GALOIS STRUCTURES ON PRIMITIVE PURELY INSEPARABLE EXTENSIONS

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ABSTRACT. Let $L/K$ be a primitive purely inseparable extension of fields of characteristic $p$, $[L : K] > p$. It is well known that $L/K$ is Hopf Galois for some Hopf algebra $H$, and it is suspected that $L/K$ is Hopf Galois for numerous choices of $H$. We construct a family of $K$-Hopf algebras $H$ for which $L$ is an $H$-Galois object. For some choices of $K$ we will exhibit an infinite number of such $H$. We provide some explicit examples of the dual, Hopf Galois, structure on $L/K$.

Let $L/K$ be a modular extension. In [Cha76], Chase shows that $L$ is a principal homogenous space for some infinitesimal $K$-group scheme $G$. If $G = \text{Spec } H$ then $H$ is a finite dimensional, commutative, cocommutative $K$-Hopf algebra which is local with local linear dual (hereafter, “local-local”), and $L$ is an $H$-Galois object. Interpreted using duality, this shows that $L/K$ is Hopf Galois for some finite dimensional, commutative, cocommutative, local-local $K$-Hopf algebra.

A natural question arises: for a given extension $L/K$, is it Hopf Galois for a unique choice of $H$? It is well-known that the answer to this question is “no”. Modular extensions are, by their definition, purely inseparable. In the work cited above, Chase writes “[s]crutiny of the simplest examples shows that a modular extension can be a PHS for many different truncated $[K]$-group schemes $G$.” This comment suggests that $L$ is an $H$-Galois object for many choices of $H$.

The question is then modified: for a given extension $L/K$, can we describe all of the Hopf algebras which make it Hopf Galois? The separable analogue, where $L/K$ is a separable extension, is definitively answered in [GPS87], which describes all such $H$ using group theory. The Hopf algebras in the separable case correspond to a certain class of normal, regular subgroups of the symmetric group $S_n$ where $n = [L : K]$. This elegant result shows that the classification of Hopf Galois structures depends not on the fields but on the group. Clearly the number of such Hopf algebras is finite.

In this work, we focus primarily on the simplest class of modular extensions, namely the primitive extensions. It is well-known that if $L$ is an $H$-Galois object then $H$ must be monogenic, that is, $H$ is generated as a $K$-algebra by a single element. We will construct a family of monogenic $K$-Hopf algebras of dimension equal to $[L : K] = p^n$, $n \geq 2$, and show that each makes $L$ into a Hopf Galois object. These Hopf algebras fall into $n - 1$ classes, and the $i^{th}$ class is parameterized by elements of $K^\times / \left( K^{p^{i+1}-1} \right)^\times$. Not only is $L$ an $H$-Galois object for each of our constructed Hopf algebras, the realization of $L$ as an $H$-Galois object can be done in $p^n - p^{n-1}(p - 1)$ different ways. Unlike the separable case, the number of Hopf Galois extensions evidently depends on the fields; in particular, our work will produce

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examples where the extension $L/K$ is Hopf Galois for an infinite number of Hopf algebras.

We will also briefly discuss general modular extensions. The work presented in the simple case extends to modular extensions quite easily, however we will show that there are modular extensions which are $H$-Galois objects for Hopf algebras which cannot be constructed in the manner presented here.

It should be noted that these constructions can also be done geometrically, using the language of group schemes and principal homogenous spaces (or “torsors”). Indeed, the Hopf algebras we construct represent certain subgroups of group schemes of finite length Witt vectors. We have opted to present our results using a purely algebraic approach for three reasons. First, the language in [GP87] is one of Hopf algebras and Hopf-Galois extensions, and as we are investigating an inseparable analogue to the results in that paper it seems natural to try to use the same language as much as possible. Second, we feel the question is stated more naturally using Hopf algebras – “given a field extension $L/K$, for which Hopf algebras is it a Hopf Galois extension” makes the point more directly than “for Spec $K \to$ Spec $L$, $L$ a field, for which group schemes $G$ does Spec $L$ appear as a torsor?” does. Third, in [Koc14] we use these Hopf algebras to describe the ring of integers in the case where $L$ and $K$ are local fields; an explicit description of the Hopf algebra action is necessary in that work.

Throughout, $p$ is a fixed prime and $K$ is an imperfect field containing a perfect field $k$ of characteristic $p$. All unadorned tensors are over $K$. We denote by $K_p^{-\infty}$ the perfect closure of $K$ in some algebraic closure. All rings (and algebras) are assumed to be commutative. All Hopf algebras are assumed to be finite, commutative, cocommutative, of $p$-power rank, and local-local.

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1. Background

We briefly describe the notion of Hopf Galois extensions and Hopf Galois objects. More details can be found, e.g., in [Chi00]. Let $H$ be a $K$-Hopf algebra, comultiplication $\Delta$ and counit $\varepsilon$. We say that $L$ is an $H$-module algebra if $L$ is an $H$-module such that, for all $a, b \in L$ we have

$$h(ab) = \text{mult} \Delta (h)(a \otimes b).$$

If furthermore the $K$-linear map $L \otimes H \to \text{End}_K(L)$, $(a \otimes h) \mapsto (b \mapsto ah(b))$ is an isomorphism, then we say $L/K$ is an $H$-Galois extension, or simply Hopf Galois if the Hopf algebra is understood. This can be seen as a generalization of the usual Galois theory: if $E/F$ is a Galois extension, $\Gamma = \text{Gal}(E/F)$ then $E/F$ is Hopf Galois via the group algebra $F[\Gamma]$.

Loosely, the notion of a Hopf Galois object is dual to that of a Hopf Galois extension. Given a $K$-Hopf algebra $H$, suppose there is a map $\alpha : L \to L \otimes H$ such that

$$(\alpha \otimes 1) \alpha = (1 \otimes \Delta) \alpha : L \to L \otimes H \otimes H$$

$$\text{mult} (1 \otimes \varepsilon) \alpha = \text{id}_L.$$ 

Then $L$ is said to be an $H$-comodule. If furthermore the map $\gamma : L \otimes L \to L \otimes H$ given by $\gamma(a \otimes b) = (a \otimes 1) \alpha (b)$ is an isomorphism, then $L$ is an $H$-Galois object
(or $H$-principal homogeneous space). It can be shown that $L/K$ is $H$-Galois if and only if $L$ is an $H^*$-Galois object, where $H^* = \text{Hom}_K (H, K)$ is the linear dual to $H$.

In the sections that follow, we construct Hopf algebras $H$ for which $L$ is an $H$-Galois object. Dualizing will put $H^*$-Galois structures on $L/K$.

The construction of both the comultiplication maps $\Delta$ and coaction maps $\alpha$ that follow rely heavily on Witt vector addition. Much of the background on Witt vectors can be found in [Jac75]. For the convenience of the reader we will briefly recall the construction and illustrate some its properties which will be necessary for the rest of the paper.

For each positive integer $d$, define

$$w_d (Z_0, \ldots, Z_d) = Z_0^{p^d} + pZ_1^{p^{d-1}} + \cdots + p^d Z_d \in \mathbb{Z}[Z_0, \ldots, Z_d].$$

The polynomials $w_d$ are called Witt polynomials. Define $S_d := S_d ((X_0, \ldots, X_d) ; (Y_0, \ldots, Y_d))$ recursively by

$$w_d (S_0, \ldots, S_d) = w_d (X_0, \ldots, X_d) + w_d (Y_0, \ldots, Y_d),$$

i.e.,

$$S_d = \frac{1}{p^d} \left( w_d (X_0, \ldots, X_d) + w_d (Y_0, \ldots, Y_d) - S_0^{p^d} - S_1^{p^{d-1}} - \cdots - S_{d-1}^{p} \right).$$

Clearly $S_d \in \mathbb{Q}[X_0, \ldots, X_d, Y_0, \ldots, Y_d]$; a less obvious, but fundamental, result is that we in fact have $S_d \in \mathbb{Z}[X_0, \ldots, X_d, Y_0, \ldots, Y_d]$. To give two explicit examples of these polynomials,

$$S_0 (X_0; Y_0) = X_0 + Y_0$$

$$S_1 ((X_0, X_1) ; (Y_0, Y_1)) = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{(p-1)!}{i! (p-i)!} X_0^i Y_0^{p-i}.$$ 

Let $W (\mathbb{Z}) = \{(a_0, a_1, \ldots) : a_i \in \mathbb{Z}\}$, and define a binary operation on $W (\mathbb{Z})$ by

$$(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (S_0 (a_0; b_0), S_1 ((a_0, a_1) ; (b_0, b_1)), \ldots).$$

Then $W (\mathbb{Z})$ is a group: the identity is $(0, 0, \ldots)$ and the additive inverse is obtained by negating the components. This is typically proved by observing that the map

$$W (\mathbb{Z}) \to \prod_{i=0}^{\infty} \mathbb{Z}$$

$$(a_0, a_1, \ldots) \mapsto (w_0 (a_0), w_1 (a_0, a_1), \ldots)$$

is a bijection from which $W (\mathbb{Z})$ inherits its structure from the product on the right. As $W (\mathbb{Z})$ is a group, the component operation $S_d$ is associative for all $d$.

By replacing $\mathbb{Z}$ with a $\mathbb{Z}$-algebra $R$, we obtain the group $W (R)$. The polynomials $S_d$ can then be viewed as elements of $R [X_0, \ldots, X_d, Y_0, \ldots, Y_d]$. In fact, for any commutative ring $R$ we may view $W$ as an $R$-group scheme.

We conclude this section by recording two well-known observations which will be needed later. The first equality holds because $S_d$ is a polynomial expression in $x_0, \ldots, x_d, y_0, \ldots, y_d$. The second follows from that fact that $S_d ((X_0, \ldots, X_d) ; (Y_0, \ldots, Y_d))$ is a homogenous polynomial of degree $p^d$, where $X_i$ and $Y_i$ each have weight $p^i$, $0 \leq i \leq d$. 
Lemma 1.1. Let $A$ and $B$ be $K$-algebras. Let $f : A \to B$ be a $K$-algebra map. Then, for $(x_0, x_1, \ldots), (y_0, y_1, \ldots) \in W(A), c \in K$ we have, for all $d$,

$$f(S_d((x_0, \ldots, x_d); (y_0, \ldots, y_d))) = S_d((f(x_0), \ldots, f(x_d)); (f(y_0), \ldots, f(y_d))) \in B$$

$$cS_d((x_0, \ldots, x_d); (y_0, \ldots, y_d)) = S_d\left((e^{\theta_d}x_0, \ldots, cx_d); (e^{\theta_d}y_0, \ldots, cy_d)\right) \in A$$

Remark 1.2. Since $K$ is not perfect, it is possible that $e^{\theta_1} \notin K$. However, we may view these as elements of $K^{\theta_1}$.

2. Monogenic Hopf Algebras

The objective of this section is to introduce a new family of monogenic $K$-Hopf algebras. We will accomplish this by generalizing a classification of monogenic Hopf algebras (recall $k$ is perfect). We do not claim that our adaptation to $K$ yields all monogenic Hopf algebras.

First, we briefly describe the collection of monogenic Hopf algebras over $k$. This classification appears in Dieudonné module form in [Koc01] and explicit Hopf algebra descriptions are given in [Koc03].

Fix a positive integer $n$. By [Wat79, 14.4], all monogenic Hopf algebras of rank $p^n$ share the same $k$-algebra structure, namely $H = k[t]/(t^n)$, so a study of Hopf algebra structures reduces to studying the various comultiplications one can put on this $k$-algebra. The simplest comultiplication can be obtained by letting $\Delta(t) = t \otimes_k 1 + 1 \otimes_k t$ - that is, $t$ is a primitive element. The others are best described using Witt vector addition polynomials.

Let $0 < r < n$. For $\eta \in k^\times$, define a sequence $\{\eta_i : i \in \mathbb{Z}^+\}$ recursively by

$$\eta_1 = \eta$$

$$\eta_i = \eta^{p^{i-1}} \eta_{i-1}^{p^r}.$$ 

Notice that the $\eta_i$ implicitly depend on $r$. Explicitly, we have $\eta_i = \eta^{p^i}$, where

$$e_i = p^{-(i-1)} + p^{r-(i-2)} + p^{2r-(i-3)} + \ldots + p^{(i-1)r} = \sum_{j=0}^{i-1} p^{r-(i-j-1)}.$$ \hfill (1)

Let $H = k[t]/(t^n)$, and let $d = [n/r] - 1$. Define $\Delta : H \to H \otimes_k H$ by

$$\Delta(t) = S_d\left((\eta_d t^{dr} \otimes_k 1, \ldots, \eta_1 t^{dr} \otimes_k 1, t \otimes_k 1); (1 \otimes_k \eta_d t^{dr}, \ldots, 1 \otimes_k \eta_1 t^{dr}, 1 \otimes_k t)\right).$$

This gives $H$ the structure of a $k$-Hopf algebra with counit $\varepsilon(t) = 0$ and antipode $\lambda(t) = -t$. We will denote this Hopf algebra by $H_{n,r,\eta}$. To see that the Hopf algebra axioms are satisfied, first notice that coassociativity follows from the associativity of $S_d$. Also we use Lemma 1.1 to obtain

$$\text{mult}(1 \otimes \varepsilon) \Delta(t) = S_d\left((\eta_d t^{dr} \varepsilon(1), \ldots, t \varepsilon(1)); (\eta_d \varepsilon(t) t^{dr}, \ldots, \varepsilon(t))\right) = 0$$

$$\text{mult}(1 \otimes \lambda) \Delta(t) = S_d\left((\eta_d t^{dr} \lambda(1), \ldots, t \lambda(1)); (\eta_d \lambda(t) t^{dr}, \ldots, \lambda(t))\right) = 0.$$ \hfill (2)

It is shown in [Koc01] that all of the monogenic local-local Hopf algebras of dimension $p^n$ are of the form $H_{n,r,\eta}$. Furthermore, $H_{n,r,\eta'} \cong H_{n,r,\eta}$ if and only if $r = r'$ and $\eta'/\eta = \beta p^{r-n-1}$ for some $\beta \in k$. In the case where $k$ is finite, this allows
us to count the number of monogenic Hopf algebras \[ \text{Koc01 Cor. 3.2}. \] On the other hand, if \( k \) is algebraically closed, there are exactly \( n \) monogenic Hopf algebras of rank \( p^n \).

Now, we adapt the classification in the perfect field case to the case where \( K \) contains \( k \) and is imperfect. Certainly, \( H_{n,r,n} \otimes_k K \) is a \( K \)-Hopf algebra of dimension \( p^n \). However, a careful reading of the results above reveals that \( \eta \) can be replaced by a more general element of \( K \).

Pick \( 0 < r < n \). Let \( g \in \left( K^{p^{-\infty}} \right)^{\times} \) and define
\[
g_i = g, \ g_i = g^{p^{i-1}}, g_i^{p^r}, \ i > 1.
\]
Note that \( g_i \notin K \) in general, even if \( g \in K \). However, it can easily be shown that if \( g^p \in K \) then \( g_i \in K \).

Our strategy will be to construct a comultiplication \( \Delta \) on \( H = K [t] / (t^p) \) such that
\[
\Delta (t) = S_d \left( \left( g_d t^p \otimes 1, \ldots, g_1 t^p \otimes 1, t \otimes 1 \right) ; \left( 1 \otimes g_d t^p, \ldots, 1 \otimes g_1 t^p, 1 \otimes t \right) \right)
\]
for \( g \in K^{1/p} \) (although the final form will differ slightly from this). In order to do so, we need to prove that the expression above is an element of \( H \otimes H \). The following result accomplishes this.

**Lemma 2.1.** Let \( g \in K^{1/p} \), and let \( \{ g_i \} \) be defined as above. Then for all \( d \geq 0 \),
\[
S_d \left( \left( g_d t^p, \ldots, g_1 t^p, u \right) ; \left( g_d t^p, \ldots, g_1 t^p, v \right) \right) \in K [u, v].
\]

**Proof.** In a manner similar to eq. \[ \] we have
\[
g_i = g^{e_i}, e_i = \sum_{j=0}^{i} p^{j(r-(i-j)-1)} = p^{-i+1} \sum_{j=0}^{i} p^{j(r+1)}
\]
for all \( 0 \leq i \leq d \). Thus,
\[
g_i = g^{p^{-i+1} \sum_{j=0}^{i} p^{j(r+1)}},
\]
and the second factor, which we will denote by \( g_i' \), is in \( K \). Then \( g_i = (g^p)^{p^{-i}} g_i' \), and by Lemma \[ \] we can factor \( g^p \) out of \( S_d \left( \left( g_d t^p, \ldots, g_1 t^p, u \right) ; \left( g_d t^p, \ldots, g_1 t^p, v \right) \right) \) and obtain
\[
S_d \left( \left( g_d t^p, \ldots, g_1 t^p, u \right) ; \left( g_d t^p, \ldots, g_1 t^p, v \right) \right) = S_d \left( \left( g_d t^p, \ldots, g_1 t^p, u \right) ; \left( g_d t^p, \ldots, g_1 t^p, v \right) \right) + u + v
\]
where \( S_d ((X_0, \ldots, X_{d-1}) ; (Y_0, \ldots, Y_{d-1})) \) is a homogeneous polynomial (where \( X_i \) and \( Y_i \) each have weight \( p^i \), as before). Thus, since \( g^p \in K \) and \( g_i' \in K \) for all \( i \),
\[
S_d \left( \left( g_d t^p, \ldots, g_1 t^p, u \right) ; \left( g_d t^p, \ldots, g_1 t^p, v \right) \right) \in K [u, v].
\]

By picking \( g \in K^{1/p} \) we get a well-defined algebra map on \( H \) using the lemma above with \( u = t \otimes 1, v = 1 \otimes t \). However, we obtain a nicer parameterization of these maps by letting \( f = g^p \in K^{\times} \). Let \( f_1 = f^{1/p}, f_i = f_1^{p^{i-1}}, f_i^{p^r}, f_i^{p^{r-1}} = f_1^{p^{r-1}} f_i^{p^r}.\)
Proposition 2.2. Let $0 < r < n$ be integers. Let $d = \lceil n/r \rceil - 1$. Let $f \in K^\times$. Let $f_1, f_2, \ldots, f_d$ be the sequence given recursively by

$$f_1 = f^{1/p}, \quad f_i = f^{p-1} f_{i-1}^{p^r}, \quad i \geq 2$$

as above. Let $H_{n,r,f}$ be the $K$-algebra $K[t]/(t^p)$, and let

$$\Delta (t) = S_d \left( f_{d+1}^{p^{d+1}} \otimes 1, \ldots, f_1 t^{p^r} \otimes 1, t \otimes 1 \right) \left( 1 \otimes f_{d+1}^{p^{d+1}} \otimes \ldots, 1 \otimes f_1 t^{p^r} \otimes 1, 1 \otimes t \right)$$

$$\varepsilon (t) = 0$$

$$\lambda (t) = -t$$

Then these maps endow $H_{n,r,f}$ with the structure of a $K$-Hopf algebra.

Proof. Since the comultiplication is accomplished using Witt vector sums, the computations here are the same as in eq. (2). Alternatively, we could use the facts that $H_{n,r,f} \otimes K^p\cdot \infty$ is a Hopf algebra by [Koc01], and $\Delta (t), \varepsilon (t), \lambda (t)$ are defined over $K$. \hfill \Box

3. ISOMORPHISM QUESTIONS

The Hopf algebras $\{H_{n,r,f} : 0 < r < n, \; f \in K^\times\}$ constructed in Proposition 2.2 are not all unique. While $n$ and $r$ are isomorphism invariants, different choices of $f$ can lead to isomorphic Hopf algebras. Here, we will give a sufficient condition on $f, f'$ for $H_{n,r,f} \cong H_{n,r,f'}$. Additionally, if $r$ is sufficiently large (with respect to $n$) then we will see this condition is necessary as well.

Suppose $H_{n,r,f} = K[t]/(t^p)$. Pick $g \in K^\times$, and let $u = gt$. Then, as a $K$-algebra, $H_{n,r,f} = K[u]/(u^p)$; with the help of Lemma 1.1, we have

$$\Delta (u) = g \Delta (t)$$

$$= g S_d \left( f_{d+1}^{p^{d+1}} \otimes 1, \ldots, f_1 t^{p^r} \otimes 1, t \otimes 1 \right) \left( 1 \otimes f_{d+1}^{p^{d+1}} \otimes \ldots, 1 \otimes f_1 t^{p^r} \otimes 1, 1 \otimes t \right)$$

$$= S_d \left( g^{p^{-d}} f_d t^{p^r} \otimes 1, \ldots, g^{p^{-1}} f_1 t^{p^r} \otimes 1, gt \otimes 1 \right) \left( 1 \otimes g^{p^{-d}} f_d t^{p^r} \otimes 1, \ldots, 1 \otimes g^{p^{-1}} f_1 t^{p^r} \otimes 1, gt \otimes 1 \right)$$

$$= S_d \left( g f_d u t^{p^r} \otimes 1, \ldots, g t f_1 u t^{p^r} \otimes 1, u \otimes 1 \right) \left( 1 \otimes g f_d u t^{p^r} \otimes 1, \ldots, 1 \otimes g t f_1 u t^{p^r} \otimes 1, u \otimes 1 \right)$$

where $g_t = g^{p^{-1} - p^r}$. Now

$$g_t^{p^{i-1}} - g_t^{p^{i-1}} = \left( g^{p^{-i} - p^r} \right)^{p^{i-1}} \left( g^{p - (i-1) - p^{i-1} - r} \right)^{p^r}$$

$$= g^{p^{i-1} - p^{i-1} - p^r + r - r}$$

$$= g^{p^{i-1} - p^r}$$

$$= g_t,$$

and

$$g_t^p = \left( g^{p-1 - p^r} \right)^p = g^{1-p^r+1}.$$

From this it follows that $H_{n,r,f} \cong H_{n,r,(g_1^{-p^r+1})f}$. More generally,
Proposition 3.1. Let \( f, f' \in K^\times \). If \( f/f' \in (K^\times)^{p^{r+1}-1} \) then \( H_{n,r,f} \cong H_{n,r,f'} \).

Proof. The statement that \( H_{n,r,f} \cong H_{n,r,f'} \) whenever \( f/f' \in (K^\times)^{p^{r+1}-1} \) has been proved already. Conversely, suppose \( H_{n,r,f} \cong H_{n,r,f'} \). Then \( H_{n,r,f} \otimes K^{p^{-\infty}} \cong H_{n,r,f'} \otimes K^{p^{-\infty}} \), of course, and, by [Koc01, Sec. 3], \( f/f' \in \left( (K^{p^{-\infty}})^{\times} \right)^{p^{r+1}-1} \).

Thus, the equation

\[ x^{p^{r+1}} + \frac{f}{f'} x = 0 \]

has a solution in \( K^{p^{-\infty}} \). If \( g \) is such a solution, then \( K (g) \) is a separable extension of \( K \) contained in \( K^{p^{-\infty}} \). Since \( K^{p^{-\infty}}/K \) is purely inseparable we have \( g \in K \), hence \( f/f' \in (K^\times)^{p^{r+1}-1} \).

Note the parallel with the result in [Koc01] if we replace \( \beta \) with \( \beta^{p^{-1}} \) of course, if \( f \) and \( f' \) are constants these Hopf algebras are defined over \( k \) and we expect the isomorphism condition above to hold.

4. Hopf Galois Objects

Let \( L = K (x) \), \( x^p = b \in K \). If \( H \) is the monogenic Hopf algebra with primitive generator, then \( L \) is an \( H \)-Galois object: this is the Hopf algebra used in the construction of [Cha76], where he shows that all modular extensions of \( K \) are Hopf Galois objects. The purpose of this section is to show that each of the rank \( p^n \) Hopf algebras constructed above can be used to make \( L \) a Hopf-Galois object.

Let \( H = H_{n,r,f} \) for some choice of \( 0 < r < n \) and \( f \in K \). Define \( \alpha : L \to L \otimes H \) by

\[ \alpha (x) = S_d \left( \left( f_d x^{p^r} \otimes 1, \ldots, f_1 x^r \otimes 1, x \otimes 1 \right) \right). \]

Since exponentiation-by-\( p \) is a \( K \)-algebra map,

\[
\alpha (b) = \alpha (x^n) = S_d \left( \left( f_d^n x^{p^r} \otimes 1, \ldots, f_1^n x^r \otimes 1 \right) \right) = S_d \left( \left( f_d^n x^{p^r} \otimes 1, \ldots, x^n \otimes 1 \right) \right) = x^n \otimes 1 = b \otimes 1,
\]

and so \( \alpha \) is a well-defined \( K \)-algebra map.

Lemma 4.1. The map \( \alpha \) above gives \( L \) the structure of a right \( H \)-comodule.

Proof. We need to show that \( (1 \otimes \Delta) \alpha (x) = (\alpha \otimes 1) \alpha (x) \) and \( \mu (1 \otimes \varepsilon) \alpha (x) = x \). The first follows immediately from the associativity of \( S_d \). The second computation is similar to the one in eq. [2].

Proposition 4.2. Let \( \gamma : L \otimes L \to L \otimes H \) be given by

\[ \gamma (a \otimes b) = (a \otimes 1) \alpha (b). \]

Then \( \gamma \) is a \( K \)-module isomorphism, hence \( L \) is an \( H \)-Galois object.
Proof. First, notice that since $\alpha$ is a $K$-algebra map we have that $\gamma$ preserves multiplication, i.e., $\gamma((a \otimes b)(c \otimes d)) = \gamma(a \otimes b) \gamma(c \otimes d)$. Also, $\gamma$ is an $L$-module map if we view $L \otimes L$ as an $L$-module via the first factor since $\gamma(a \otimes b) = (a \otimes 1) \gamma(b)$. Since $L \otimes L$ and $L \otimes H$ are both $K$-vector spaces of dimension $p^{2n}$, it suffices to show that $\gamma$ is onto. Now

$$
\gamma(-x \otimes 1 + 1 \otimes x) = -(x \otimes 1) \alpha(1) + (1 \otimes 1) \alpha(x)
$$

$$
= -x \otimes 1 + S_d \left( \left( f_d x^{p^d} \otimes 1, \ldots, x \otimes 1 \right); \left( 1 \otimes f_d x^{p^d}, \ldots, 1 \otimes t \right) \right)
$$

$$
= -x \otimes 1 + x \otimes 1 + 1 \otimes t + t^2 \xi_1
$$

for some $\xi_1 \in L \otimes H$. As $\gamma$ is multiplicative we see that

$$
\gamma \left( (-x \otimes 1 + 1 \otimes x)^i \right) = 1 \otimes t^i + t^{i+1} \xi_i, \, \xi_i \in L \otimes H
$$

for $1 \leq i < p^n$. Thus $\left\{ \gamma \left( (-x \otimes 1 + 1 \otimes x)^i \right) \right\}$ is an $L$-linearly independent set, we have $\dim_L \text{Im} \gamma \geq p^n$. As $[L : K] = p^n$ we have $\dim_K \text{Im} \gamma = L \otimes H$.

Remark 4.3. Proposition [37] can be used to provide examples of finite field extensions $L/K$ with an infinite number of $K$-Hopf algebras which $L$ is an $H$-Galois object. For example, let $K = k(T_1, T_2, \ldots)$ and let $L$ be any purely inseparable extension of degree $p^2$ (or greater). Then $H_{n,r,T_i} \not\cong H_{n,r,T_j}$ unless $i = j$. 

One will notice many parallels between this theory and the Kummer theory of formal groups construction in, e.g., [Chi03, Sec. 39]. This is to be expected since a smooth resolution for $\text{Spec} \, H_{n,r,f}$ can be easily constructed by adapting the resolution in [Koc03, Sec. 4].

Of course, there are many different descriptions for the same field extension $L$. Indeed, pick $g \in K^x$ and let $y = gx$. Then $y^{p^n} = g^{p^n} b \in K$ and so $L = K(y)$. With the coaction above we have

$$
\alpha(y) = \alpha(gx)
$$

$$
= g S_d \left( \left( f_d x^{p^d} \otimes 1, \ldots, x \otimes 1 \right); \left( 1 \otimes f_d x^{p^d}, \ldots, 1 \otimes t \right) \right)
$$

$$
= S_d \left( \left( g f_d x^{p^d} \otimes 1, \ldots, x \otimes 1 \right); \left( 1 \otimes g f_d x^{p^d}, \ldots, 1 \otimes t \right) \right)
$$

$$
= S_d \left( \left( g f_d x^{p^d} \otimes 1, \ldots, g_1 f_1 x \otimes 1, x \otimes 1 \right); \left( 1 \otimes g f_d x^{p^d}, \ldots, g_1 f_1 x^{p^d}, 1 \otimes t \right) \right)
$$

where $g_i = g^{p^{-i-1} - p^i}$ as in the previous section. Thus, since $g_i^n = g^{i - p^{i+1}}$, changing the generator of $L$ in this manner results in the same coaction: $H_{n,r,f}$ acts on $x$ in the same way as $H_{n,r,f} g_{i-1}^{-1} - p^{i+1}$ acts on $y$, and these two Hopf algebras are isomorphic.

On the other hand, let $x_i = x^i, \, 1 \leq i \leq n - 1$, $\gcd(p, i) - 1$. Then $L = K(x_i)$, and defining

$$
\alpha_i(x_i) = S_d \left( \left( f_d x_i^{p^d} \otimes 1, \ldots, f_1 x_i^{p^d} \otimes 1, x_i \otimes 1 \right); \left( 1 \otimes f_d x_i^{p^d}, \ldots, 1 \otimes f_1 x_i^{p^d}, 1 \otimes t \right) \right)
$$

allows for a coaction of $H_{n,r,f}$ on $L$: as $i$ varies each resulting coaction is different. Thus, there are $\phi(p^n) = p^{n-1} (p - 1)$ different ways to view $L$ as an $H_{n,r,f}$-Galois object.
Both Chase’s construction and the Hopf algebras presented here can be considered under one general theory. Indeed, were we to allow \( r = n \) and \( f = 0 \), then \( d = 0 \) and we recover Chase’s Hopf algebra. We have chosen to treat them as separate cases to simplify the question of isomorphic Hopf algebras – clearly, the Hopf algebra “\( H_{n,n,f} \)” does not depend at all on \( f \).

5. Explicit Computations: the Case \( r = n - 1 \)

We shall now explicitly describe the action of \( H := H^*_{n,r,f} \) on \( L \) in the case where \( r = n - 1 \). In this case, \( d = 1 \), and hence the comultiplication on \( H_{n,r,f} \) is

\[
\Delta (t) = t \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} t^{p^\ell} \otimes t^{p^r (p-\ell)}. 
\]

We view this as a restriction on \( r \), not on \( n – \) that is, \( L/K \) can be any extension, but and we only consider the Hopf algebras with \( r = n - 1 \). This will provide a family of explicit Hopf Galois actions on any purely inseparable extension \( L/K \) of degree \( p^n \), \( n \geq 2 \).

As a \( K \)-module, \( H \) has a basis \( \{ z_0 = 1, z_1, \ldots, z_{p^n} \} \) with \( z_i : H \to K \) given by

\[
z_j\left( t^i \right) = \delta_{i,j},
\]

where \( \delta_{i,j} \) is the Kronecker delta. The algebra structure on \( H \) is induced from the coalgebra structure on \( H_{n,r,f} \); explicitly,

\[
z_{j_1} z_{j_2} (h) = \text{mult} (z_{j_1} \otimes z_{j_2}) \Delta (h).
\]

We claim that \( \{ z_p, z_{p^2}, z_{p^3}, \ldots, z_{p^n} \} \) generate \( H \) as a \( K \)-algebra.

We start with a result which will facilitate the study of the algebra structure of \( H \) as well as the action of \( H \) on \( L \).

Lemma 5.1. Let

\[
S_f (u, v) = u + v + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} u^{p^\ell} v^{p^r (p-\ell)}. 
\]

Then, for every positive integer \( i \), \( S_f (u, v)^i \) is a \( K \)-linear combination of elements of the form

\[
u^{i_3 + p^r \ell'} v^{i_2 + p^r \ell''},
\]

where

\[
\ell' = i_{3,1} + 2i_{3,2} + \cdots + (p-1) i_{3,p-1}
\]

\[
\ell'' = (p-1) i_{3,1} + (p-2) i_{3,2} + \cdots + i_{3,p-1},
\]

and \( i_{3,1} + i_{3,2} + \cdots + i_{3,p-1} = i_3 \).

Proof. We have

\[
S_f (u, v)^i = \left( u + v + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} u^{p^\ell} v^{p^r (p-\ell)} \right)^i = \sum_{i_1 + i_2 + i_3 = i} \binom{i}{i_{1,1} i_2 i_3} \left( u^{i_1} v^{i_2} \right) \left( f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} u^{p^\ell} v^{p^r (p-\ell)} \right)^{i_3}. 
\]
The last factor in each summand can be expanded as
\[
f_{i_3} \sum_{i_{3,1} + \cdots + i_{3,p-1}} \left( \begin{pmatrix} i_3 \\ i_{3,1}, \ldots, i_{3,p-1} \end{pmatrix} \left( \prod_{j=1}^{p-1} \frac{1}{i_3,j! (p-i_3,j)!} \right) u^{i_1+p/\ell'} v^{i_2+p/\ell''} \right).
\]

The result follows. \hfill \Box

Next, we consider powers of the \(z_{p^r}\)'s.

**Lemma 5.2.** For \(0 \leq s \leq r\), \(1 \leq m \leq p - 1\), \(z_{p^r}^m = m! z_{p^r}\).

**Proof.** Clearly, this holds for \(m = 1\). Suppose \(z_{p^r}^{m-1} = (m-1)! z_{(m-1)p^r}\). By eq. 3 we have
\[
z_{p^r}^m (t^i) = \text{mult} \left( z_{p^r}^{m-1} \otimes z_{p^r} \right) \Delta (t^i)
= \text{mult} \left( z_{p^r}^{m-1} \otimes z_{p^r} \right) S_f (t \otimes 1, 1 \otimes t)
= \text{mult} \left( (m-1)! z_{(m-1)p^r} \otimes z_{p^r} \right) \left( t \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} t^{p/\ell} \otimes t^{p/\ell} \right).
\]

By Lemma 5.1, the tensors are of the form
\[
t^{i_1+p/\ell'} \otimes t^{i_2+p/\ell''},
\]
with \(\ell', \ell''\) as before. Note that \(\ell' + \ell'' = pi_3\). Since \(z_{p^r} \left( t^i \right) = \delta_{p^r,i}\) and \(z_{(m-1)p^r} \left( t^i \right) = \delta_{(m-1)p^r,i}\), the expression \(\left( z_{(m-1)p^r} \otimes z_{p^r} \right) \left( t^{i_1+p/\ell'} \otimes t^{i_2+p/\ell''} \right)\) is nontrivial only if
\[
\begin{align*}
(m-1) p^s &= i_1 + p^r \ell' \\
p^s &= i_2 + p^r \ell''.
\end{align*}
\]

If we add the two equations together we get
\[
mp^s = i_1 + i_2 + p^r + i_3.
\]
From this it is clear that \(i_3 = 0\), which means \(\ell' = \ell'' = 0\) as well. Thus \(i_2 = p^s\) and \(i_1 = (m-1) p^s\), hence \(i = mp^s\) and with the help of Lucas’ Theorem [Fin47] we get
\[
z_{p^r}^m (t^{mp}) = (m-1)! \left( \begin{pmatrix} mp^s \\ (m-1) p^s, p^s, 0 \end{pmatrix} \right) = (m-1)! \left( \begin{pmatrix} mp^s \\ p^s \end{pmatrix} \right) = (m-1)! m = m!.
\]

Therefore, \(z_{p^r}^m = m! z_{mp^s}\). \hfill \Box

**Lemma 5.3.** For \(0 \leq s \leq r - 1\), \(z_{p^r}^s = 0\).

**Proof.** We have
\[
z_{p^r}^s (t^i) = (p-1)! \text{mult} \left( z_{(p-1)p^r} \otimes z_{p^r} \right) \left( \sum_{i_1 + i_2 + i_3 = i} \left( i \begin{pmatrix} i_1, i_2, i_3 \end{pmatrix} \right) \left( t^{i_1} \otimes t^{i_2} \right) \left( \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} t^{p/\ell} \otimes t^{p/\ell} \right)^{i_3} \right).
\]
If \((z_{(p-1)p} \otimes z_p) \left( t^{i_1+p^s \ell'} \otimes t^{i_2+p^s \ell''} \right)\) is nontrivial then
\[
(p-1) p^s = i_1 + p^s \ell' \\
p^s = i_2 + p^s \ell''.
\]
Again, \(i_3 = 0\), so \(i_2 = p^s\) and \(i_1 = (p-1)p^s\), hence \(i = p^{s+1}\). But then
\[
z_{p^{s+1}}^{p^s} \left( t^{p^{s+1}} \right) = (p-1)! \left( \frac{p^{s+1}}{(p-1)p^s, p^s, 0} \right)
= - \left( \frac{p^{s+1}}{p^s} \right)
= 0.
\]
So \(z_{p^{s+1}}^{p} = 0\). □

**Remark 5.4.** While not part of the generating set we are constructing, notice that the above results show \(z_1^{m} = m!z_m\) and \(z_1^0 = 0\).

The behavior is slightly different for \(p^r\).

**Lemma 5.5.** We have \(z_{p^r}^p = f z_1\) and \(z_{p^r}^{p^2} = 0\).

**Proof.** If the expression \((z_{(p-1)p^r} \otimes z_{p^r}) \left( t^{i_1+p^s \ell'} \otimes t^{i_2+p^s \ell''} \right)\) in the expansion of \(z_{p^r}^p (t^i)\) is nontrivial then
\[
(p-1) p^r = i_1 + p^r \ell' \\
p^r = i_2 + p^r \ell''.
\]
If \(i_3 = 0\) then \(i_2 = p^r\), \(i_1 = (p-1)p^r\) and \(i = p^{r+1}\); however, \(i < p^{r+1} = p^n\) so this cannot occur. Thus \(i_3 = 1\), \(\ell' = p-1\), \(\ell'' = 1\) (both of these can occur only by setting \(i_{3,j} = \delta_{j,p-1}\)), \(i_2 = 0\), and \(i_1 = 0\). Hence, \(i = 1\) and
\[
z_{p^r}^p (t) = (p-1)! \left( \frac{1}{0, 0, 1} \right) \frac{1}{(p-1)! (p-(p-1))} = f.
\]
Therefore, \(z_{p^r}^p = f z_1\). That \(z_{p^r}^{p^2} = 0\) follows immediately. □

From the results above, we can deduce that \(\{z_{p^r} : 1 \leq s \leq r\}\) generate \(H\) as a \(K\)-algebra.

The coalgebra structure on \(H\) is induced from the multiplication on \(H_{n,r,f}\) and is much more straightforward. For all \(h \in H\) when we apply the comultiplication \(\Delta\) to it we get a \(K\)-linear map \(H_{n,r,f} \otimes H_{n,r,f} \to K\) given by
\[
\Delta(h) (a \otimes b) = h(ab).
\]
Thus,
\[
\Delta(z_j) \left( t^{i_1} \otimes t^{i_2} \right) = z_j \left( t^{i_1+i_2} \right) = \delta_{j,i_1+i_2}
\]
and so
\[
\Delta(z_j) = \sum_{i=0}^{j} z_{j-i} \otimes z_i.
\]
Note that this is true for all \(j\), not just the powers of \(p\).

We summarize.
Proposition 5.6. The Hopf algebra $H$ above is

$$H = K \left[ z_p, z_{p^2}, \ldots, z_{p^r} \right] / \left( z_{p^r}, z_{p^2}, \ldots, z_{p^r-1}, z_{p^r} \right)$$

$$\Delta (z_{p^r}) = \sum_{i=0}^{p^r-1} z_{p^r-i} \otimes z_i.$$

Of course, it is possible to write $\Delta (z_{p^r})$ solely in terms of $z_p, \ldots, z_{p^s}, z_{p^r}$, but that is not needed for our purposes.

We will now describe the Hopf Galois action of $H$ on $L$. Then the $K$-algebra map $\alpha : L \to L \otimes H_{n,r,f}$ is given by

$$\alpha (x) = x \otimes 1 + 1 \otimes t + \frac{1}{p! (p-\ell)!} x^{p^r \ell} \otimes p^r (p-\ell).$$

This gives $L$ the structure of an $H_{n,r,f}$-comodule – in fact it makes $L$ an $H_{n,r,f}$-Galois object. Here, we compute the induced action of $H$ on $L$ which makes $L/K$ an $H$-Galois extension.

Generally, if $A$ is a $K$-Hopf algebra such that $L$ is an $A$-Galois object, then $A^*$ acts on $L$ by

$$h (y) = \text{mult} \left( 1 \otimes h \right) \alpha (y), \ h \in A^*, \ y \in L. \quad (4)$$

Here, it suffices to compute $z_{p^s} (x^i)$ for $1 \leq s \leq r$, $1 \leq i \leq p^n - 1$, however it will also be useful to compute $z_j (x^i)$ for some choices of $j$ which are not powers of $p$. Notice that we use $z_j$ in two different contexts: one to describe $z_j$ as a map $H_{n,r,f} \to K$, the other to describe how $z_j$ acts on $L$.

The first result handles the case $i = 1$.

Lemma 5.7. We have

$$z_0 (x) = x, \ z_1 (x) = 1, \ z_{p^s} (x) = -fx^{p^r (p-1)}.$$  

For $1 \leq j \leq p^r - 1$, $z_j (x) = 0$.

Proof. Applying eq. (4) to $h = z_j$, $y = x$ gives

$$z_j (x) = xz_j (1) + z_j (t) + \frac{1}{p! (p-\ell)!} x^{p^r \ell} z_j (p^r (p-\ell)),$$

from which the result follows. \qed

The second result handles the cases where $i$ is a nontrivial power of $p$.

Lemma 5.8. For $1 \leq m \leq r$ we have

$$z_0 \left( x^{p^m} \right) = x^{p^m}, \ z_{p^m} \left( x^{p^m} \right) = 1.$$  

For all other choices of $j$, $z_j \left( x^{p^m} \right) = 0$. 

Proof. The computations are facilitated by observing that $\alpha \left( x^{p^m} \right) = x^{P^m} \otimes 1 + 1 \otimes t^{P^m}$. Indeed,

$$\alpha \left( x^{p^m} \right) = \alpha \left( x \right)^{p^m} = \left( x \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{p^\ell} \otimes t^{p^\ell (p-\ell)} \right)^{p^m}$$

$$= x^{P^m} \otimes 1 + 1 \otimes t^{P^m} + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{P^{p^\ell+m}} \otimes t^{P^{p^\ell+m} (p-\ell)}$$

$$= x^{P^m} \otimes 1 + 1 \otimes t^{P^m}$$

since $r + m \geq r + 1 = n$. Now for $0 \leq j \leq p^n - 1$ we have

$$z_j \left( x^{p^m} \right) = x^{P^m} z_j \left( 1 \right) + z_j \left( t^{P^m} \right),$$

from which the result follows.$\square$

Next, we have

**Theorem 5.9.** Let $H = H_{r,f}$ be as in Proposition 5.8, that is,

$$H = K \left[ z_{p^1}, z_{p^2}, \ldots, z_{p^r} \right] \setminus \left( z_{p^1}, z_{p^2}, \ldots, z_{p^r-1}, z_{p^r} \right)$$

$$\Delta \left( z_{p^r} \right) = \sum_{i=0}^{p^r} z_{p^r-i} \otimes z_i.$$

For $0 \leq i \leq p^n - 1$, write

$$i = \sum_{\ell=0}^{r} i(\ell) p^\ell.$$

Then, for $0 \leq s \leq r - 1$ we have

$$z_{p^r} \left( x^i \right) = i(s) x^{i-p^r}.$$

Additionally,

$$z_{p^r} \left( x^i \right) = i(r) x^{i-p^r} - i f x^{p^r (p-1)+i-1}.$$

**Remark 5.10.** Note that if $i < p^r < p^r$ then $z_{p^r} \left( x^i \right) = 0$, and if $i < p^r$ then $z_{p^r} \left( x^i \right) = -i f x^{p^r (p-1)+i-1}$.

*Proof.* We have

$$z_{p^r} \left( x^i \right) = \text{mult} (1 \otimes z_{p^r}) \alpha \left( x^i \right)$$

$$= \text{mult} (1 \otimes z_{p^r}) \left( x \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{p^\ell} \otimes t^{p^\ell (p-\ell)} \right)^i$$

$$= \text{mult} (1 \otimes z_{p^r}) \sum_{i_1+i_2+i_3=i} \left( \binom{i}{i_1, i_2, i_3} \left( x^{i_1} \otimes t^{i_2} \left( f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{p^\ell} \otimes t^{p^\ell (p-\ell)} \right)^{i_3} \right) \right).$$
After expanding, the tensors are of the form $x^{i_1 + p^r l'} \otimes t^{i_2 + p^r l''}$ as before. Applying $1 \otimes z_{p^r}$ to this expression will give 0 unless

$$p^s = i_2 + p^r l''.$$  \hspace{1cm} (5)

Assume first that $s < r$. Since $p^r > p^s$ we see that $i_3 = 0$ and $i_2 = p^s$. Thus $i_1 = i - p^s$ and we get

$$z_{p^s}(x^i) = \left( i - p^s, p^s, 0 \right) x^{i-p^s} z_{p^s}(t^{p^s})$$

$$= \left( p^s \right) x^{i-p^s}$$

$$= i(a)x^{i-p^s},$$

as desired.

Now we consider the case $s = r$. Then $i_3 = 0$, $i_2 = p^r$, $i_1 = i - p^r$ certainly satisfies eq. (5). However, we get an additional solution to this equation, namely $i_3 = 1$, $i_2 = 0$, $i_1 = i - 1$. Thus

$$z_{p^r}(x^i) = \left( i - p^r, p^r, 0 \right) x^{i-p^r} z_{p^r}(t^{p^r}) + \left( i - 1, 0, 1 \right) x^{i-1} f \frac{1}{(p-1)!} \frac{1}{(p-(p-1))!} x^{p^r-p^r} z_{p^r}(t^{p^r})$$

$$= i(r)x^{i-p^r} - i f x^{p^r(p-1)+i-1}.$$

□

The results above do not generalize easily to the case $n > r + 1$. Certainly, if $2r < n$ then the comultiplication on $H_{n,r,f}$ (and its coaction on $L$) becomes much more complicated, making the computations of the algebra structure (and the action) of its dual much more involved as well. If $r+1 < n \leq 2r$, computation of the algebra structure of $H_{n,r,f}$ is somewhat more complex than the case considered here – in particular, $z_{p^r}^p \neq f z_{p^r}$ – but, as a future paper [Koc14] will show, it is possible to show that $H_{n,r,f}$ is generated as a $K$-module by $\left\{ \prod_{s=0}^{n-1} z_{p^r}^{j_s} : 0 \leq j_s \leq p - 1 \right\}$, and much of its action on $L$ can be made explicit.

6. A Note on Modular Extensions

While the focus of this work is primitive purely inseparable extensions, it should be pointed out that the constructions here can be adapted easily to general modular extensions. The following should be clear.

**Proposition 6.1.** Let $L/K$ be modular, $L \cong L_1 \otimes \cdots \otimes L_s$ with $L_i/K$ primitive of degree $p^{n_i}$, $1 \leq i \leq s$. For each $i$, pick $0 < r_i < n_i$ and $f_i \in K^\times$. Set

$$H = H_{n_1, r_1, f_1} \otimes \cdots \otimes H_{n_s, r_s, f_s}.$$  

Then $L$ is an $H$-Galois object.

Thus, the constructions in the previous sections show that any modular extension can be equipped with numerous Hopf Galois structures. However, it is not the case that all (local-local) Hopf Galois structures on modular extensions have been exhibited here, as the following example shows.
Example 6.2. Let $K = \mathbb{F}_p (T_1, T_2)$. Let $L = K (x, y)$ with $x^{p^2} = T_1, y^{p^2} = T_2$. Then $L/K$ is modular. Let $H = K (t, u) / (t^{p^2}, u^{p^2})$, and define $\Delta : H \to H \otimes H$ by
\[
\Delta (t) = S_1 ((u^p \otimes 1, t \otimes 1); (1 \otimes u^p, 1 \otimes t)) \\
\Delta (u) = S_0 (u \otimes 1, 1 \otimes u).
\]
Along with the counit $\varepsilon$ given by $\varepsilon (t) = \varepsilon (u) = 0$ and antipode $\lambda (t) = -t, \lambda (u) = -u$, this gives $H$ the structure of a $K$-Hopf algebra which is not monogenic. While checking that the Hopf algebra axioms hold is straightforward, one can also note that $H \otimes K^p - \infty$ is a special case of a “bigenic” Hopf algebra given in [?, Ex. 6.6]. Define $\alpha : L \to L \otimes H$ by
\[
\alpha (t) = S_1 ((y^p \otimes 1, x \otimes 1); (1 \otimes u^p, 1 \otimes t)) \\
\alpha (u) = S_0 (y \otimes 1, 1 \otimes u).
\]
Then $L$ is an $H$-Galois object.

In fact, this example seems to suggest that the biggest obstacle to a modular extension $L$ being an $H$-Galois object is the algebra structure of $H$; the coalgebra structure seems to naturally give a coaction. If $L$ is an $H$-Galois object, then $L \otimes L \cong L \otimes H$. Also, $L \otimes L$ is a truncated polynomial algebra: in the simple, degree $p^n$ case $L = K (x)$, clearly we have $L (u) / (u^{p^n}) \cong L \otimes L$ via $u \mapsto 1 \otimes x - x \otimes 1$. Thus, for a given modular extension $L/K$, the problem appears to be reduced to finding the Hopf algebra structures on the truncated polynomial algebra $L \otimes L$. We have not found a local-local Hopf algebra of the proper type (as an algebra) which does not give $L$ the structure of an $H$-Galois object.

References

[Cha76] Stephen U. Chase, In infinitesimal group scheme actions on finite field extensions, Amer. J. Math. 98 (1976), no. 2, 441–480.

[Chi00] Lindsay N. Childs, Taming wild extensions: Hopf algebras and local Galois module theory, Mathematical Surveys and Monographs, vol. 80, American Mathematical Society, Providence, RI, 2000.

[Fin47] N. J. Fine, Binomial coefficients modulo a prime, Amer. Math. Monthly 54 (1947), 589–592.

[GP87] Cornelius Greither and Bodo Pareigis, Hopf Galois theory for separable field extensions, J. Algebra 106 (1987), no. 1, 239–258.

[Jac75] Nathan Jacobson, Lectures in abstract algebra. III, Springer-Verlag, New York, 1975, Theory of fields and Galois theory, Second corrected printing, Graduate Texts in Mathematics, No. 32.

[Koc01] Alan Koch, Monogenic bialgebras over finite fields and rings of Witt vectors, J. Pure Appl. Algebra 163 (2001), no. 2, 193–207.

[Koc03] Alan Koch, Monogenic Hopf algebras and local Galois module theory, J. Algebra 264 (2003), no. 2, 408–419.

[Koc14] Alan Koch, Scaffolds and integral Hopf Galois module structure on pi extensions, preprint (2014).

[Wat79] William C. Waterhouse, Introduction to affine group schemes, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York, 1979.