Open and Closed Cosmological Solutions of Hořava-Witten Theory

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Abstract

The cosmological solutions of Hořava-Witten theory discovered by Lukas, Ovrut and Waldram are generalized to allow non vanishing spatial curvature. The solution with closed spatial sections has initial and final five dimensional curvature singularities. We find two solutions with open spatial sections, both of which evolve from an initial curvature singularity to the supersymmetric domain wall solution at late times. We also present a solution with open spatial sections and a non-zero Ramond-Ramond scalar. The behaviour of the solutions in eleven dimensions is discussed.
I. INTRODUCTION

The strongly coupled limit of the $E_8 \times E_8$ heterotic superstring theory has been identified by Hořava and Witten with $M$−theory compactified on a $S^1/Z_2$ orbifold with $E_8$ gauge fields on each orbifold fixed plane \[1,2\]. This can be further compactified to four dimensions using a Calabi-Yau manifold \[3\]. Matching the predicted values for the four dimensional gravitational and GUT couplings leads one to the conclusion that the orbifold is an order of magnitude larger than the Calabi-Yau space \[3,4\]. This has the interesting consequence that the early universe may have been effectively five dimensional.

Cosmological solutions of Hořava-Witten theory have been constructed from brane solutions in \[5,6\]. These have non-trivial gauge field configurations on the orbifold fixed planes. An alternative approach has been used by Lukas et al who have constructed an effective five dimensional version of Hořava-Witten theory compactified on a Calabi-Yau space and shown that the theory admits a supersymmetric solution in which our spacetime is identified with a four dimensional domain wall \[7\]. They generalized this solutions to allow a cosmological time dependence by seeking separable solutions of the equations of motion of their effective theory \[8\]. These solutions correspond to making the moduli of the domain wall solution time dependent and either evolve from or to a five dimensional curvature singularity.

In this paper separable solutions with non-vanishing spatial curvature are presented. We find a solution with closed spatial sections and two solutions with open spatial sections. The latter solutions are of particular interest because they approach the static domain wall solution at late times. All of the solutions have an initial five dimensional curvature singularity and the closed solution has a final curvature singularity. These singularities are not resolved in the eleven dimensional low energy theory.

Solutions with a non vanishing ‘Ramond-Ramond’ scalar (so called because it would be a type II RR scalar if the orbifold of Hořava-Witten theory were replaced by a circle) were found in \[8\]. We have found such a solution with open spatial sections. However it is of rather a special type and more general solutions can be expected to exist.

The first part of this paper consists of a short review of the five dimensional effective theory constructed in \[7\]. The second section together with the appendix gives an exhaustive discussion of the various cases arising in the separation of variables. We present it here because the details were omitted in \[8\]. In particular we show that all separation constants must vanish, which was assumed in \[8\]. The general solution of the separated equations for the case of vanishing RR scalar but non-vanishing spatial curvature is derived and a special solution with non-vanishing RR scalar and open spatial sections is presented. The final section discusses the behaviour of these solutions in five and eleven dimensions and how our work relates to previous work on cosmological solutions with varying moduli.

II. FIVE DIMENSIONAL EFFECTIVE ACTION

Lukas et al reduce Hořava-Witten theory to five dimensions using the metric ansatz

$$ds^2 = V^{-\frac{2}{3}}g_{\mu\nu}dx^\mu dx^\nu + V^{\frac{2}{3}}\Omega_{mn}dy^m dy^n. \quad (2.1)$$

Here $x^\mu$ ($0 \leq \mu \leq 4$) are coordinate on the five dimensional spacetime (which includes the orbifold direction) with metric $g_{\mu\nu}(x)$ and $y^m$ are coordinate on the Calabi-Yau space with
metric $\Omega_{mn}(y)$. $V(x)$ is a scalar field measuring the deformation of the Calabi-Yau space. In performing the reduction it is necessary to retain a non-zero mode of the three form potential of eleven dimensional supergravity. See [7] for the details. For us the relevant part of the reduced action is

$$S = \int_{M_5} \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 - e^{-\sqrt{\phi}} \partial \xi \cdot \partial \xi - \frac{1}{6} \alpha^2 e^{-2\sqrt{\phi}} \right) +$$

$$+ \sqrt{2} \int_{M_4^{(i)}} \sqrt{-\tilde{g}} \alpha e^{-\sqrt{\phi}} - \frac{1}{2} \int_{M_4^{(j)}} \sqrt{-\tilde{g}} \alpha e^{-\sqrt{\phi}}, \quad (2.2)$$

where $\phi$ is defined by $V = e^{\sqrt{\phi}}$, $\alpha$ is a constant, $M_5$ is the five dimensional spacetime bounded by the orbifold fixed planes $M_5^{(i)}$ and $\tilde{g}_{ij}$ ($0 \leq i, j \leq 3$) denotes the pull-back of the metric on $M_5$ onto $M_4^{(i)}$. $\xi$ is a scalar field related to the components of the three form on the Calabi-Yau space. We have omitted several terms from the action; this is legitimate provided the current $j_\mu \equiv \iota(\xi \partial_\mu \xi - \xi \partial_\mu \xi)$ vanishes. Let $\xi = e^{\rho + i\theta}$. Then $j_\mu$ vanishes if, and only if, $\theta$ is constant. This is what we shall assume in the following.

Let $y \equiv x^4$ be a coordinate in the orbifold direction with $y \in [-\pi \lambda, \pi \lambda]$ and $Z_2$ acting on $S^1$ by $y \to -y$. The orbifold fixed planes are at $y = 0, \pi \lambda$. Then the equations of motion following from the action are

$$G_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + 2e^{2\rho - \sqrt{\phi}} \partial_\mu \rho \partial_\nu \rho - g_{\mu\nu} \left( \frac{1}{2} (\partial \phi)^2 + e^{2\rho - \sqrt{\phi}} (\partial \rho)^2 + \frac{1}{6} \alpha^2 e^{-2\sqrt{\phi}} \right) +$$

$$+ \sqrt{2} \alpha \sqrt{\frac{g}{\bar{g}}} g^{ij} g_{i\mu} g_{j\nu} e^{\sqrt{\phi}} (\delta(y) - \delta(y - \pi \lambda)), \quad (2.3)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) = -\sqrt{2} e^{2\rho - \sqrt{\phi}} (\partial \rho)^2 - \frac{\sqrt{2}}{3} \alpha^2 e^{-2\sqrt{\phi}} + 2 \alpha \sqrt{\frac{g}{\bar{g}}} e^{-\sqrt{\phi}} (\delta(y) - \delta(y - \pi \lambda)) \quad (2.4)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^{\rho - \sqrt{\phi}} \partial^\mu \rho) = 0, \quad (2.5)$$

where $G_{\mu\nu}$ is the five dimensional Einstein tensor and $\bar{g}^{ij}$ denotes the inverse of $\bar{g}_{ij}$.

**III. SEPARABLE COSMOLOGICAL SOLUTIONS**

Following Lukas et al we seek solutions of the equations of motion of the form

$$ds^2 = -e^{2U(t,y)} dt^2 + e^{2A(t,y)} ds_3^2 + e^{2B(t,y)} dy^2, \quad \phi = \phi(t,y), \quad \rho = \rho(t,y), \quad (3.1)$$

where $ds_3^2$ is the line element on a three dimensional space of constant curvature of sign $k \in \{-1,0,1\}$. Only the case $k = 0$ was considered in [8]. Substituting into the equations of motion gives six partial differential equations. We shall solve these by separation of variables. Set $U(t,y) = U_1(t) + U_2(y)$ etc. Then the $tt$ component of the Einstein equation is
\[ e^{2(U_2-B_2)}(3A_2'' + 6A_2' - 3A_2'B_2' + \frac{1}{2} \phi_1'' + e^{2\rho_1-\sqrt{2}\phi_1} e^{2\rho_2-\sqrt{2}\phi_2} \rho_1'^2) + \]
\[ + \frac{1}{6} \alpha^2 e^{2(B_1-\sqrt{2}\phi_1)} e^{2(U_2-\sqrt{2}\phi_2)} - \sqrt{2} \alpha e B_1-\sqrt{2}\phi_1 e^{2U_2-B_2-\sqrt{2}\phi_2} (\delta(y) - \delta(y - \pi \lambda)) \]
\[ = e^{-2(U_1-B_1)} \left[ 3\dot{A}_1 \dot{B}_1 - \frac{1}{2} \phi_1'' - e^{2\rho_1-\sqrt{2}\phi_1} e^{2\rho_2-\sqrt{2}\phi_2} \dot{\rho}_1^2 \right] + \]
\[ + 3k e^{-2(A_1-B_1)} e^{-2(A_2-U_2)}. \quad (3.2) \]

Primes and dots denote derivatives with respect to \( y \) and \( t \) respectively. The delta functions are \( y \)-dependent so the terms involving them must be made independent of \( t \) if this equation is to separate. Thus we need \( B_1 = \sqrt{2}\phi_1 \). The final (curvature) term can be made independent of either \( y \) or \( t \). We shall make the first choice by setting \( A_2 = U_2 \) (the latter choice would give the scale factors of the three space and orbifold the same time dependence; this is clearly not a good description of our universe). Note that any constants arising in these relations can be absorbed into \( B_2 \) or \( A_1 \). Finally we must deal with the \( \rho \) terms. To separate these we must either take \( \rho = \sqrt{2}\phi + \text{constant} \), which is discussed in the appendix, or take

\[ e^{-2(U_1-B_1)} e^{2\rho_1-\sqrt{2}\phi_1} \dot{\rho}_1^2 = a^2, \quad (3.3) \]
\[ e^{2(U_2-B_2)} e^{2\rho_2-\sqrt{2}\phi_2} \dot{\rho}_2^2 = b^2, \quad (3.4) \]
where \( a \) and \( b \) are constants.

These conditions also ensure the separation of the other equations of motion. After separation if one adds the \( y \)-dependent part of the \( yy \) equation to the \( y \)-dependent parts of the \( tt \) and \( ij \) equations then the \( y \)-dependent equations of motion can be written

\[ \frac{d}{dy} \left( e^{4A_2-B_2} A_2' \right) + \frac{1}{9} \alpha^2 e^{4A_2+B_2-2\sqrt{2}\phi_2} = \frac{1}{3} (\lambda_1 + \lambda_3) e^{2A_2+B_2}, \quad (3.5) \]
\[ \frac{d}{dy} \left( e^{4A_2-B_2} A_2' \right) + \frac{1}{9} \alpha^2 e^{4A_2+B_2-2\sqrt{2}\phi_2} - \frac{2}{3} a^2 e^{2A_2+B_2+2\rho_2-\sqrt{2}\phi_2} = \frac{1}{3} (\lambda_2 + \lambda_3) e^{2A_2+B_2}, \quad (3.6) \]
\[ 6A_2'' - \frac{1}{2} \phi_1'' + \frac{1}{6} \alpha^2 e^{2(B_2-\sqrt{2}\phi_2)} = \lambda_3 e^{-2(A_2-B_2)}, \quad (3.7) \]
\[ \frac{d}{dy} \left( e^{4A_2-B_2} \phi_2' \right) + \frac{\sqrt{2}}{3} \alpha^2 e^{4A_2+B_2-2\sqrt{2}\phi_2} - \sqrt{2} a^2 e^{2A_2+B_2+2\rho_2-\sqrt{2}\phi_2} = \lambda_4 e^{2A_2+B_2}. \quad (3.8) \]
The \( \lambda_i \) are the separation constants.

If one adds the \( t \)-dependent part of the \( tt \) equation to the \( t \)-dependent parts of the \( ij \) and \( yy \) equations then the \( t \)-dependent equations of motion can be written

\[ 3\dot{A}_1^2 + 3\dot{A}_1 \dot{B}_1 - \frac{1}{4} \dot{B}_1^2 - b^2 e^{2U_1-3B_1+2\rho_1} + 3k e^{2(U_1-A_1)} = \lambda_1 e^{2(U_1-B_1)}, \quad (3.9) \]
\[
\frac{d}{dt} \left( e^{-U_1+3A_1+B_1}(2\dot{A}_1 + \dot{B}_1) \right) - 2b^2 e^{U_1+3A_1-2B_1+2\rho_1} + 4ke^{U_1+A_1+B_1} = (\lambda_1 + \lambda_2)e^{U_1+3A_1-B_1},
\]
(3.10)

\[
\frac{d}{dt} \left( e^{-U_1+3A_1+B_1} \dot{A}_1 \right) + 2ke^{U_1+A_1+B_1} = \frac{1}{3}(\lambda_1 + \lambda_3)e^{U_1+3A_1-B_1},
\]
(3.11)

\[
\frac{d}{dt} \left( e^{-U_1+3A_1+B_1} \dot{B}_1 \right) - 2b^2 e^{U_1+3A_1-2B_1+2\rho_1} = \sqrt{2}\lambda_4 e^{U_1+3A_1-B_1}.
\]
(3.12)

Clearly the last three equations require \(3\sqrt{2}\lambda_4 = \lambda_1 + 3\lambda_2 - 2\lambda_3\) for consistency.

The \(ty\) equation of motion is
\[
\left[ e^{-U_1+\frac{3}{2}B_1-\rho_1} \dot{B}_1 \left[ e^{A_2-B_2+\frac{1}{\sqrt{2}}\phi_2-\rho_2} (3A_2' - \frac{1}{\sqrt{2}}\phi_2') \right] = 2ab. \right. (3.13)
\]

The \(\xi\) equation of motion separates to give
\[
be^{A_2-B_2+\frac{1}{\sqrt{2}}\phi_2-\rho_2} (3A_2' - \frac{1}{\sqrt{2}}\phi_2') = \lambda_5, \quad (3.14)
\]

\[
2e^{-U_1+\frac{3}{2}B_1-\rho_1} (3\dot{A}_1 - \frac{1}{2}\dot{B}_1) = \lambda_5. \quad (3.15)
\]

We have not included the delta function terms in the above equations so they are only valid in \(0 < |y| < \pi\lambda\). The delta functions impose the following conditions on the discontinuities of the derivatives of \(A_2\) and \(\phi_2\) at the orbifold fixed planes
\[
[A_2'] = \frac{\pm \sqrt{2}}{3} \alpha e^{B_2-\sqrt{2}\phi_2}, \quad (3.16)
\]
\[
[\phi_2'] = \pm 2\alpha e^{B_2-\sqrt{2}\phi_2}, \quad (3.17)
\]
the upper sign refers to \(y = 0\) and the lower to \(y = \pi\lambda\). The right hand side is evaluated at the fixed plane. If one combines these with the orbifold conditions \(A(t, y) = A(t, -y)\) and \(\phi(t, y) = \phi(t, -y)\) then one obtains the following boundary conditions at \(y = 0\)
\[
A_2'(\pm 0) = \pm \frac{\sqrt{2}}{6} \alpha e^{B_2-\sqrt{2}\phi_2}, \quad (3.18)
\]
\[
\phi_2'(\pm 0) = \pm \alpha e^{B_2-\sqrt{2}\phi_2}, \quad (3.19)
\]
with similar conditions (but opposite signs) at \(y = \pi\lambda\). Substituting these boundary conditions into (3.7) and (3.14) gives \(\lambda_3 = \lambda_5 = 0\).

If one subtracts (3.6) from (3.5) then one obtains
Equations 3.13, 3.14, 3.15 and 3.20 can now be satisfied by choosing \( a = 0 \) and \( \phi'_2 = 3\sqrt{2}A'_2 \). (There are other ways of satisfying these equations but they lead to no new solutions. See the appendix for details.) Equation 3.20 implies \( \lambda_1 = \lambda_2 = 0 \) hence all separation constants vanish. The vanishing of the separation constants was assumed in [8]; we have shown that it is necessary for the consistency of the equations and boundary conditions. We are free to choose the gauge \( B_2 = 4A_2 \) by redefining \( y \to \tilde{y}(y) \). The \( y \)-equations reduce to

\[
A''_2 + \frac{1}{9} \alpha^2 e^{-4A_2} = 0, \tag{3.21}
\]

\[
A'^2_2 - \frac{1}{18} \alpha^2 e^{-4A_2} = 0. \tag{3.22}
\]

It is then straightforward to solve 3.22. The result also satisfies 3.21 (which can be obtained by differentiating 3.22). The solution of the \( y \)-equations is the one found in [7,8], namely

\[
e^{A_2} = e^{U_2} = a_0 H^{\frac{1}{2}}, \quad e^{B_2} = b_0 H^2, \tag{3.23}
\]

where \( H(y) = \frac{\sqrt{2}}{3} \alpha |y| + c_0 \) and \( a_0, b_0 \) and \( c_0 \) are constants.

We are taking \( a = 0 \) so 3.3 implies \( \rho_1 \) is a constant that can be absorbed into \( b \). The solution for \( \xi \) is easily obtained from 3.4:

\[
\xi = e^{i\theta}(d_0 H^4 + \xi_0) \tag{3.24}
\]

where \( \theta, d_0 \) and \( \xi_0 \) are real constants. This solution of the \( y \)-equations is the one found in [8].

Now consider the \( t \)-dependent equations. In the \( k = 0 \) case considered in [8] these were integrated by choosing a gauge so that the ‘potential’ (i.e. non-derivative) terms in 3.10, 3.11 and 3.12 become constant i.e. the gauge \( U_1 = -3A_1 + 2B_1 \) was used. This procedure does not work if both \( b \) and \( k \) are non-zero because not all such terms can be made constant simultaneously. However the equations can be cast in a form more amenable to qualitative analysis by working in the four dimensional Einstein frame with metric \( \bar{g}_{ij} = e^{B_1}g_{ij} \) i.e. the line element is

\[
ds^2_4 = e^{2A_2+B_2}(-e^{2U_1+B_1}dt^2 + e^{2A_1+B_1}ds^2_3). \tag{3.25}
\]

Note that there is an overall \( y \)-dependent conformal factor so this cannot really be interpreted as the metric of a four dimensional spacetime. However this is irrelevant in what follows. Define a scale factor \( R(t) = e^{A_1+\frac{1}{2}B_1} \). Then in the ‘comoving’ gauge given by \( U_1 = -\frac{1}{2}B_1 \) the \( t \)-dependent equations reduce to

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{\kappa^2}{3} \left( \frac{1}{2}B^2_1 + V \right) - \frac{k}{R^2}, \tag{3.26}
\]

\[
\frac{1}{R^3} \frac{d}{dt} \left( R^3 \dot{B}_1 \right) = -\frac{dV}{dB_1}. \tag{3.27}
\]
where \( V = \frac{1}{2} b^2 e^{-2\sqrt{2} \kappa B_1} \) and \( \kappa^2 = 8\pi G \) is the four dimensional Planck scale (which took the value 2 in the units used previously). The other two \( t \)–equations are implied by these two. These are the equations governing the homogeneous and isotropic cosmology of a single scalar field \( B_1 \) with potential \( V \). Note that \( V \) is of the exponential type considered in \([9]\) and that the factor of \( 2\sqrt{2} \) occurring in the exponent exceed the critical value of \( \sqrt{2} \) required for a significant inflationary period. Exact analytic solutions of \( 3.26 \) and \( 3.27 \) are only available in the case \( k = 0 \) considered in \([8]\). However special solutions can be found by assuming that the curvature and potential terms are proportional. Consider the slightly more general potential \( V_0 e^{\alpha \kappa B_1} \). If one seeks a solution with \( R(t) = R_0 e^{-\frac{1}{2} \alpha \kappa B_1} \) then one finds \( R \propto t \) and \( e^{\alpha \kappa B_1} \propto \frac{1}{t^2} \) subject to the restriction

\[
\frac{1}{4} \kappa^2 V_0 (2 - \alpha^2) = \frac{k}{R_0^2},
\]

so the geometry of the spatial sections in this solution is dictated by the sign of \( 2 - \alpha^2 \). In our case we have \( \alpha = -2\sqrt{2} \) and hence \( \kappa = -1 \). The scale factor is \( R = \frac{2\sqrt{3}}{t^2} \) which is non-inflationary, as expected.

Lukas et al derived the general solution to the \( t \)–equations in the \( k = 0, b \neq 0 \) case. We shall now do the same for the \( b = 0, k \neq 0 \) case. The equations can be integrated in the gauge given by \( U_1 = A_1 \) i.e. using conformal time. Let \( Y = A_1 + \frac{1}{2} B_1 \). The \( t \)-equations are (using \( \kappa^2 = 2 \) again)

\[
\dot{Y}^2 - \frac{1}{4} \dot{B}_1^2 - \frac{1}{6} \dot{\phi}_1^2 + k = 0,
\]

\[
\frac{d^2}{dt^2} e^{2Y} + 4k e^{2Y} = 0,
\]

\[
\frac{d}{dt} (e^{2Y} \dot{B}_1) = 0,
\]

\[
\frac{d}{dt} (e^{2Y} \dot{\phi}_1) = 0.
\]

Note that we have reintroduced \( \phi_1 \) as an independent field. This is so that we can compare our solutions of Hořava-Witten theory with solutions of eleven dimensional supergravity compactified on a circle rather than an orbifold. If one compactifies from eleven to five dimensions on a Calabi-Yau space and does not include a non-zero mode of the three form on the internal space then the five dimensional effective theory simply consists of gravity and a massless scalar field \( \phi_1 \). One can seek cosmological solutions of this theory in which the fifth dimension is assumed to be a circle, rather than the orbifold considered above (in other words we are considering cosmological solutions of type IIA supergravity compactified on a Calabi-Yau space with \( B_1 \) the dilaton). There is then no need for the metric to depend upon the circle direction so one does not have to separate variables. The equations of motion are then the ones that we have just written down. Of course to recover the solutions
of Hořava-Witten theory, we must set $\sqrt{2}\phi_1 = B_1$ and include the $y-$dependence found above.

Solving the final two equations gives

$$\dot{B}_1 = \beta e^{-2Y}, \quad \dot{\phi}_1 = \gamma e^{-2Y},$$ (3.33)

where $\beta$ and $\gamma$ are constants. The other equations can now be solved to give $e^{2Y}$ which can be substituted into 3.33. Integration then yields the solutions

$$e^{A_1} = a_0 |\sin t|^{\frac{1+\delta}{2}}(\cos t)^{\frac{1+\delta}{2}} \quad e^{B_1} = b_0 |\tan t|^\delta \quad \phi_1 = \text{const} + \epsilon \delta \log |\tan t|,$$ (3.34)

$$e^{A_1} = a_0 |t|^{\frac{1+\delta}{2}} \quad e^{B_1} = b_0 |t|^\delta \quad \phi_1 = \text{const} + \epsilon \delta \log |t|,$$ (3.35)

$$e^{A_1} = a_0 \sinh t|^{\frac{1+\delta}{2}}(\cosh t)^{\frac{1+\delta}{2}} \quad e^{B_1} = b_0 |\tanh t|^\delta \quad \phi_1 = \text{const} + \epsilon \delta \log |\tanh t|,$$ (3.36)

in the cases $k = +1, 0, -1$ respectively. The constants $\epsilon$ and $\delta$ are defined by

$$\epsilon = \frac{\gamma}{\beta} \quad \delta = \frac{\beta}{\sqrt{\beta^2 + \frac{2\gamma^2}{3}}}.$$ (3.37)

We see that there is one parameter family of solutions of the type IIA theory. Note that our solutions of Hořava-Witten theory require $\epsilon = \frac{1}{\sqrt{2}}$ so $\delta = \pm \frac{\sqrt{3}}{2}$.

If we define

$$\tau = \begin{cases} \tan t & k = +1 \\ t & k = 0 \\ \tanh t & k = -1 \end{cases}$$ (3.38)

then the solutions can be written in the unified form

$$e^{A_1} = \frac{||\tau|^{\frac{1+\delta}{2}}}{\sqrt{1 + k\tau^2}} \quad e^{B_1} = |\tau|^\delta \quad \phi_1 = \text{const} + \epsilon \delta \log |\tau|.$$ (3.39)

To summarize, in Hořava-Witten theory our separable cosmological solutions have the $y-$dependence of the static domain wall given by 3.23 and time dependence given by the above results with $\delta = \pm \frac{\sqrt{3}}{2}$. Thus, for example, in the $k = -1$ case the metric is

$$ds^2 = a_0^2 |\sinh t|^{1+\frac{\sqrt{3}}{2}}(\cosh t)^{1+\frac{\sqrt{3}}{2}}H(y)(-dt^2 + ds_3^2) + b_0^2 |\tanh t|^{\frac{\sqrt{3}}{2}}H(y)^4 dy^2,$$ (3.40)

and the scalar field is

$$V \equiv e^{\sqrt{2}\phi} = b_0 |\tanh t|^\frac{\sqrt{3}}{2}H(y)^3,$$ (3.41)

where $a_0$ and $b_0$ are positive constants.
IV. DISCUSSION OF SOLUTIONS

We have seen that cosmological solutions of Hořava-Witten theory compactified on a Calabi-Yau space can be obtained by separation of variables. Solutions with curved spatial sections can be obtained and their time dependence is seen to correspond to particular members of a family of type IIA cosmological solutions.

We shall first discuss the solutions with vanishing Ramond-Ramond scalar. The $k = 0$ solutions derived above are the same as those derived in [8]. This can be seen by changing to comoving time, $T$, defined by $dT = e^{A_1} dt$. However our $k = \pm 1$ solutions are new. At early times these solutions behave just like the $k = 0$ ones. This is exactly what happens in conventional (four dimensional) cosmology: spatial curvature is negligible in the early universe. Here we have recovered this result in a five dimensional setting.

The range of $t$ can be divided into two regions, namely $t < 0$ and $t > 0$. In [8] these were referred to as the $(-)$ and $(+)$ branches respectively and are related by time reversal. We shall only consider the $(+)$ solutions (corresponding to a universe that is initially expanding).

The five dimensional Ricci scalar (computed by taking the trace of 2.3) diverges at $t = 0$ for all of the above solutions, indicating the presence of a five dimensional curvature singularity. However this may be merely a singularity of the effective theory that gets resolved in the full eleven dimensional theory. It is therefore important to examine the solutions from an eleven dimensional perspective. In the $k = 0$ case the metric is

$$ds^2 = a_0^2 t^{1+\frac{1}{2}\sqrt{3}} H(y)^{-1} \eta_{ij} dx^i dx^j + b_0^2 t^{\frac{1}{2}\sqrt{3}} H(y)^2 dy^2 + c_0^2 t^{\pm \frac{1}{6}\sqrt{3}} H(y) ds_6^2,$$

(4.1)

where $a_0$, $b_0$ and $c_0$ are constants. We shall refer to the upper and lower choices of sign as the $(\uparrow)$ and $(\downarrow)$ solutions respectively. The qualitative behaviour of these solutions near $t = 0$ was discussed in [8] in the five dimensional theory. In eleven dimensions the qualitative behaviour of the Calabi-Yau space is the same as that of the orbifold (i.e. collapsing for the $(\uparrow)$ solutions and decompactifying for the $(\downarrow)$ solutions). It is straightforward to compute the eleven dimensional Ricci scalar, which is found to diverge at $t = 0$ for the $(\uparrow)$ solution and vanish there for the $(\downarrow)$ solution. Thus the $(\uparrow)$ solution has a genuine singularity at $t = 0$. For the $(\downarrow)$ solution, the square of the eleven dimensional Ricci tensor vanishes at $t = 0$ but the square of the Riemann tensor is divergent there (the formulae in [10] are useful for computing these quantities). Hence $t = 0$ is also a singularity in this case but of a rather different type than in the $(\uparrow)$ solution. For both types of solution the singularity lies at a finite affine parameter distance in the past on timelike and null geodesics.

After the initial singularity, the induced metrics on the (four dimensional) orbifold fixed planes evolve as non-inflationary FRW universes. For example in the $k = 0$ case [8] the scale factor is $T_{\uparrow,\downarrow}^{\pm \frac{1}{2}\sqrt{3}}$.

Reducing the bulk metric to four dimensions involves taking account of the massive modes arising through integrating out the $y$-dependence. However the time dependence of the resulting four dimensional Einstein frame metric can be read off from 3.25. Interestingly, the $(\uparrow)$ and $(\downarrow)$ solutions both give the same result after this reduction. In the $k = 0$ case (the only case for which a simple analytic expression exists), the scale factor is $T_{\pm}$.

This degeneracy also arises in the reduction of the type IIA solutions discussed above: the four dimensional Einstein frame metric is independent of $\delta$. 

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At late times the behaviour of the solutions is sensitive to the spatial curvature. The $k = 0$ solutions undergo power law expansion or contraction as discussed in [8]. The $k = +1$ solutions have interesting behaviour near $t = \frac{\pi}{2}$. Before analyzing this we should first mention that in the $k = +1$ case there is only one solution because the $(↑)$ and $(↓)$ metrics are related by the coordinate transformation $t \to \frac{\pi}{2} - t$. We shall consider the metric in the $(↑)$ form. In the five dimensional theory the three dimensional spatial sections collapse to zero size at $t = \frac{\pi}{2}$ and the orbifold decompactifies. The five dimensional Ricci scalar diverges. The qualitative behaviour of the spatial section and orbifold is unchanged in the eleven dimensional metric and the behaviour of the Calabi-Yau space is qualitatively the same as that of the orbifold. The eleven dimensional Ricci scalar and Ricci tensor squared both vanish at $t = \frac{\pi}{2}$ (since we know that this is what happens for the $(↓)$ form of the metric at $t = 0$) but the square of the Riemann tensor diverges. Thus this solution expands from an initial curvature singularity to a final one but the nature of the two singularities is different.

The most interesting case is $k = −1$. For large $t$ it has $e^{A_1} \propto e^t$ and $e^{B_1} = \text{constant}$. By changing to comoving time one immediately sees that this is nothing but the static domain wall solution. Thus both the $(↑)$ and $(↓)$ solutions evolve from an initial curvature singularity to a supersymmetric solution appropriate for the reduction to a $N = 1$ supergravity theory in four dimensions [3,7], which has phenomenological appeal.

Finally we turn to the special $k = −1$ solution with non-vanishing RR scalar. The scale factor is proportional to the comoving time. In five dimensions the Ricci scalar diverges at $t = 0$. The eleven dimensional metric is

$$ds^2 = a_0^2 H(y)^{-1} \left( -dT^2 + \frac{49}{108} T^2 ds_3^2 \right) + b_0^2 T^2 H(y)^2 dy^2 + c_0 T^4 H(y) ds_6^2,$$

where $T$ is the (eleven dimensional) comoving time. The eleven dimensional Ricci scalar diverges at $T = 0$. Note that the expansion at late times in this solution is slower than in the $k = −1$ solutions with no RR scalar.

We have seen that the cosmological solutions of Lukas et al can be easily generalized to include non-zero spatial curvature. The time dependence is the same as that of particular solutions for a massless scalar field evolving in five dimensions with the fifth dimension a circle rather than an orbifold. Massless scalar fields typically arise when one allows moduli to vary. These were considered in detail by Gibbons and Townsend [11]. Our solutions share the generic properties found by them. This may seem surprising because we have solutions varying in two directions whereas theirs only varied in one. However we have seen that separation of variables forces the potential term into the $y$–equations so the time evolution is just that of a massless scalar field. More general (i.e. non-separable) solutions can be expected to have more interesting time dependence.

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APPENDIX:

This appendix deals with the cases encountered in the analysis of section III that do not lead to new solutions.
We first discuss what would have happened had we taken $\rho = \sqrt{2}\phi + \text{constant}$ in order to separate the field equations instead of 3.3 and 3.4. Define a constant $\beta$ by

$$\beta^2 = 2e^{2\rho - \sqrt{2}\phi}. \quad (A1)$$

Then the separated equations of motion are the as those derived above (with $a = b = 0$) with the exceptions that 3.7 is replaced by

$$6A_2'^2 - \frac{1}{2}(1 + \beta^2)\phi_2'^2 + \frac{1}{6}e^{2(\rho_B - \sqrt{2}\phi_2)} = \lambda_3 e^{-2(A_2 - B_2)}, \quad (A2)$$

3.9 is replaced by

$$3\dot{A}_1^2 + 3A_1\dot{B}_1 - \frac{1}{4}(1 + \beta^2)\dot{B}_1^2 + 3ke^{2(U_1 - A_1)} = \lambda_1 e^{2(U_1 - B_1)} \quad (A3)$$

and 3.13 is replaced by

$$\dot{B}_1 \left(3\sqrt{2}A_2' - (1 + \beta^2)\phi_2' \right) = 0. \quad (A4)$$

The solution $(1 + \beta^2)\phi_2' = 3\sqrt{2}A_2'$ of the final equation is incompatible with the orbifold boundary conditions unless $\beta = 0$ i.e. $\rho = -\infty$ i.e. $\xi = 0$. So this will simply lead to one of the solutions derived above for the case $a = b = 0$. The second possibility is $\dot{B}_1 = 0$. The relations $\lambda_1 = \lambda_2$ and $3\sqrt{2}\lambda_4 = \lambda_1 + 3\lambda_2 - 2\lambda_3$ are easily derived as before and $\dot{B}_1 = 0$ implies $\lambda_4 = 0$ so $\lambda_3 = 2\lambda_1$. If one now subtracts 3.8 from 3.5, chooses the gauge $B_2 = -2A_2$ and integrates then one obtains

$$e^{6A_2}(A_2' - \frac{1}{3\sqrt{2}}\phi_2') = \lambda_1 (y - y_0), \quad (A5)$$

where $y_0$ is a constant. This is clearly incompatible with the boundary conditions unless $\lambda_1 = 0$. Hence all of the separation constants vanish. Now substituting the boundary conditions into 3.13 yields $\beta = 0$ hence $\xi = 0$. Thus this case does not lead to any new solutions either.

Now we turn to the alternative ways of satisfying equations 3.13, 3.14, 3.15 and 3.20. These all require either $i)$ $a = b = 0, \dot{B}_1 = 0$; or $ii)$ $b = 0, \rho_2 = \frac{1}{\sqrt{2}}\phi_2$; Case $i)$ requires $\lambda_4 = 0$ hence (since $\lambda_3 = 0$) $\lambda_2 = -\frac{1}{3}\lambda_1$. However 3.3 and 3.6 require $\lambda_1 = \lambda_2$. Hence all separation constants must vanish. Equation 3.9 can only be solved for $k = 0, -1$. The solution is just the static domain wall. For case $ii)$, equation 3.4 implies that $\phi_2' = 0$. However this is incompatible with the boundary conditions so there are no solutions in this case.
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