New results on $g$-2 calculation

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Abstract. In this paper I summarize the current situation of the theoretical calculation of the electron $g$-2, including the result from the recent calculation of the QED mass-independent 4-loop contribution with a precision of 1100 digits.

A particle of mass $m$ and spin $s$ possesses a magnetic moment $\mu$
\[ \mu = \frac{g e \hbar}{4\pi mc}s, \]  
where $g$ is the gyromagnetic ratio. According to the Dirac’s theory [1], an electron has $g = 2$. This agreed with the experimental measurements until Kusch and Foley [2] measured a value of the anomaly $a_e$ slightly different from zero:
\[ a_e = \frac{g - 2}{2} = 0.001 15(4). \]  
The deviation is due to the interaction of the electron with photons; using Q.E.D. Schwinger [3,4] was able to calculate at the first order that
\[ a_e = \frac{\alpha}{2\pi} = 0.001 161\ldots, \]  
where $\alpha$ is the fine structure constant
\[ \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}. \]  
The current measurements of $a_e$ are based on the Penning trap method, developed by the group at the University of Washington. This trap uses an axial magnetic field and a quadrupole electric field; the anomaly is expressed as the ratio of two frequencies, which can measured to a very high precision. For the development of this technique, the Nobel prize in Physics 1989 was awarded to H. Dehmelt. Their final results were [5]:
\[ a_{e^-}^{exp} = 1 159 652 188.4(4.3) \times 10^{-12} \ \text{(4.3 ppb)}, \]  
\[ a_{e^+}^{exp} = 1 159 652 187.9(4.3) \times 10^{-12} \ \text{(4.3 ppb)}. \]  

1 Invited talk to ACAT 2017, University of Washington, Seattle, 21-25 August 2017
2 since 1 September 2017
The QED contribution can be split up in mass-independent and mass-dependent parts:

The mass-independent coefficients at 1, 2 and 3 loop are known in analytical form \[3, 4, 8–10\]:

\[
A^{(1)} = \frac{1}{2},
\]

\[
A^{(2)} = \frac{197}{144} + \frac{1}{12} \pi^2 - \frac{1}{2} \pi^2 \ln 2 + \frac{3}{4} \zeta(3) = -0.328 478 965 579 \ldots,
\]

\[
A^{(6)} = \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) + \frac{100}{3} \left( a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{1}{24} \pi^2 \ln^2 2
\]

\[
- \frac{239}{1260} \pi^4 + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 - \frac{28259}{5184} = 1.181 241 456 \ldots,
\]

where \( \zeta(n) = \sum_{i=0}^{\infty} i^{-n} \), \( a_n = \sum_{n=0}^{\infty} 2^{-n} i^{-n} \). In table 1 we list some older theoretical evaluations of the two, three and four loop coefficients. In Ref. [11] I have evaluated up to 1100 digits of precision the 4-loop contribution \( A^{(8)} \), finalizing a twenty-year effort [13–19] begun after the completion of the calculation \( A^{(6)} \) [10]. The first digits of the result are

\[
A^{(8)} = -1.912245764926445574152647167439830054060873390658725345171329848\ldots.
\]

The full-precision result is shown in table 3. The result (13) is in excellent agreement (0.9\( \sigma \)) with the numerical value

\[
A^{(8)}(\text{Ref. [29]}) = -1.91298(84),
\]

latest result of a really impressive pluridecennial effort [20–29], and with the independent value

\[
A^{(8)}(\text{Ref. [30]}) = -1.87(12).
\]

| \( A^{(1)} \) | \( A^{(2)} \) | \( A^{(3)} \) |
|---|---|---|
| -2.973 | 1.496 | -1.434 |
| -0.328 478 965 579 | 1.195 | 1.766 |
| (6) | (8) | (10) |

Table 1. Numerical results of the evaluations of \( A^{(1)} \), \( A^{(6)} \), \( A^{(8)} \) and \( A^{(10)} \).
At 5-loop level there is only the numerical evaluation by the Kinoshita’s group

\[ A_1^{(10)} \text{(Ref. [29])} = 6.599(223) \]  

Concerning the mass-dependent part \( A_2(r) \), \( A_2^{(4)}(r) \) is known in analytical form [36], as well as \( A_2^{(6)}(r) \) [37–41]; the first terms of the expansion for small \( r \) of the 4-loop coefficient \( A_2^{(8)}(r) \) are known analytically [42,43]. \( A_2^{(10)}(m_e/m_\mu) \) and \( A_2^{(10)}(m_e/m_\tau) \) have been calculated numerically [29]; the first terms of the expansion for small mass ratios of \( A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) \) and \( A_3^{(8)}(m_e/m_\mu, m_e/m_\tau) \) are known analytically [43]. The hadronic and weak contribution are

\[
\begin{align*}
 a_e(\text{hadronic v.p.}) & = 1.866(11) \times 10^{-12} \quad \text{(see Ref. [44])} , \\
 a_e(\text{hadronic v.p.,NLO}) & = -0.223(1) \times 10^{-12} \quad \text{(see Ref. [45])} , \\
 a_e(\text{hadronic v.p.,NNLO}) & = 0.028(1) \times 10^{-12} \quad \text{(see Ref. [45])} , \\
 a_e(\text{hadronic I-1}) & = 0.035(10) \times 10^{-12} \quad \text{(see Ref. [46])} , \\
 a_e(\text{weak}) & = 0.0297(5) \times 10^{-12} \quad \text{(see Ref. [47])} .
\end{align*}
\]

Inserting Eq.s (11-16,17-21), the known \( A_2^{(j)} \) and the measurement of the fine structure constant [48,49]

\[ \alpha^{-1} = 137.035 998 996(85) \quad (0.62 \text{ ppb}) , \]

into Eq.s (8-10) one finds

\[ a_e^{\text{th}} = 1 159 652 182.031(15)(15)(720) \times 10^{-12} , \]

where the first error comes from \( A_1^{(10)} \), the second one from the hadronic and electroweak corrections, the last one from \( \alpha \), respectively. The values of the single contributions to \( a_e \) are listed in table 2. Conversely, assuming the validity of the theory and using the experimental measurement (7) one finds

\[ a_e^{-1}(a_e) = 137.035 999 1500(18)(18)(330)(0.25 \text{ ppb}) , \]

where the errors come from \( A_1^{(10)} \), hadronic and electroweak corrections, and \( a_e \), respectively.

| contribution | value in units of $10^{-12}$ |
|--------------|-----------------------------|
| \( A_1^{(10)}(\alpha/\pi) \) | 1 610 409 783.631(720) |
| \( A_2^{(4)}(\alpha/\pi)^2 \) | -1 772 305.065(3) |
| \( A_3^{(6)}(\alpha/\pi)^3 \) | 14 804.263 |
| \( A_4^{(8)}(\alpha/\pi)^4 \) | -55.667 |
| \( A_5^{(10)}(\alpha/\pi)^5 \) | 0.446(15) |
| \( A_2^{(6)}(m_e/m_\mu)(\alpha/\pi)^2 \) | 2.804 |
| \( A_2^{(8)}(m_e/m_\mu)(\alpha/\pi)^4 \) | -0.992 |
| \( A_2^{(10)}(m_e/m_\mu)(\alpha/\pi)^6 \) | 0.026 |
| \( A_2^{(4)}(m_e/m_\tau)(\alpha/\pi)^2 \) | -0.0002 |
| \( A_2^{(6)}(m_e/m_\tau)(\alpha/\pi)^4 \) | 0.010 |
| \( a_e(\text{hadronic v.p.}) \) | 1.866(11) |
| \( a_e(\text{hadronic v.p.,NLO}) \) | -0.223(1) |
| \( a_e(\text{hadronic v.p.,NNLO}) \) | 0.028(1) |
| \( a_e(\text{hadronic I-1}) \) | 0.035(10) |
| \( a_e(\text{weak}) \) | 0.0297(5) |

Table 2. Contributions to \( a_e \).
Table 3. First 1100 digits of $A_{1}^{(8)}$.

Methods of calculation of $A_{1}^{(8)}$
I sketch here the methods of calculation used in the literature, only for comparison. For further information on the technical aspects of my calculation of the 4-loop coefficient $A_{1}^{(8)}$, see Ref. [51] in these proceedings.

In QED the contributions to $g^{-2}$ at $n$ loops can be expressed as combinations of $n$-loop 4-dimensional Feynman integrals, belonging to a variety of Feynman diagrams.

- In Ref. [29], the $n$-loop 4-dimensional integrals are transformed in $(3n-2)$-dimensional integrals of (huge) rational functions of Feynman parameters. The integrals are computed using the MonteCarlo adaptative routine VEGAS [50]; an enormous amount of computing time is needed to sample adequately the integrands; for more information, see Ref. [20–29].

- My method, used in [11], consists in:
  (i) reduction of contributions from each Feynman diagram to a small number (334 for $A_{1}^{(8)}$) of $n$-loop $D$-dimensional master integrals by using a suitable algorithm [10, 13];
  (ii) determination of systems of difference or differential equations satisfied by the master integrals [13];
  (iii) high precision calculation of these integrals by solving these systems of equations by means of rapidly convergent series expansions [13, 14];

This method allowed to obtain 1100 digits of $A_{1}^{(8)}$ (and up to 9800 digits for some selected important integrals). See Ref. [51] for further details.

- In Ref. [30] the contributions of the various diagrams are reduced to combinations of a small number of master integrals. Most of these master integrals are computed with MonteCarlo methods.

- I note the alternative approach recently introduced by S. Volkov in Ref. [52, 53]. It is also based on MonteCarlo integration. It seems promising at 5-loop level.

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