THE GEOMETRY OF THE ELONGATED PHASE IN 4-D SIMPLICIAL QUANTUM GRAVITY

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We discuss the elongated phase of 4D simplicial quantum gravity by exploiting recent analytical results. In particular using Walkup's theorem we prove that the dominating configurations in the elongated phase are tree-like structures called "stacked spheres". Such configurations can be mapped into branched polymers and baby universes arguments are used in order to analyse the critical behaviour of theory in the weak coupling regime.

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I. INTRODUCTION

The key concept of simplicial quantum gravity is to work with a discrete version of space-time. To this purpose it is necessary to consider a simplicial complex $\Sigma$ by gluing four-dimensional simplexes through their three-dimensional faces in such a way that the link of every $p$-dimensional simplex $\sigma$, $p = 0, 1, 2$, $\text{link}(\sigma, \Sigma)$, is homeomorphic to an $(4 - p - 1)$-sphere. Analytically this last condition implies that the following Dehn-Sommerville constraints should hold

$$\sum_{i=0}^{4} (-1)^i N_i(T) = \chi(T) \quad (1.1)$$

$$\sum_{i=2k-1}^{4} (-1)^i \frac{N_i(T)}{(i - 2k + 2)!/(2k - 1)!} = 0 \quad (1.2)$$

with $1 \leq k \leq 2$ and $N_i(T)$ the number of $i$-dimensional simplexes of the simplicial manifold $T$ and where $\chi(T)$ is the Euler-Poincaré characteristic. Equation (1.1) is just the Euler-Poincaré equation, while (1.2) are consequence of the fact that the link of every 2k-simplex is an odd dimensional sphere, and hence has Euler number zero.

Two simplicial manifolds $T$ and $T'$ are said to be PL-equivalent if there exist simplicial subdivisions of them, $\tilde{T}$ and $\tilde{T}'$, such that $\tilde{T}$ and $\tilde{T}'$ are simplicially isomorphic. A simplicial isomorphism is a one to one map $\phi$ between the vertex of the simplicial manifolds such that $(\phi(v_i), \phi(v_j))$ is an edge of $T'$ if and only if $(v_i, v_j)$ is an edge of $T$ and so on for every simplex of any dimension. A simplicial isomorphism is an equivalence relation among simplicial manifolds. An equivalence class of simplicial manifolds respect to the isomorphism above is a PL-Manifold.

A dynamical triangulation $T_a$ is a simplicial manifold, that is a representative of a PL manifold, generated by gluing, along their adjacent faces, $N_4(T_a)$ equilateral simplexes, say $\sigma^4$, with fixed edge-length $a$. A dynamical triangulation gives a metric structure on a PL-manifold (see e.g. [3]).

The numbers $N_i(T)$ $i = 0, \ldots, 4$ can be seen as the component of a five-dimensional vector $f$

$$f \equiv (N_0(T), N_1(T), N_2(T), N_3(T), N_4(T)) \quad (1.3)$$

called $f$-vector of the triangulation $T$. Dehn-Sommerville equations fix three component of $f$, leaving two undetermined. In simplicial gravity $N_2(T_a)$ and $N_4(T_a)$ are, usually, the two undetermined components. Their values are subject to some limitations as will be explained below.

Every two dimensional simplex, in four dimension, is called bone (hinges in Regge calculus). Let us label the bones by an index $\alpha$ and denote by $q(\alpha)$ the number of four simplexes incident on it; the average number of simplexes incident on a bone is

$$b \equiv \frac{1}{N_2} \sum_{\alpha} q(\alpha) = 10 \left( \frac{N_4(T_a)}{N_2(T_a)} \right) \quad (1.4)$$

since each simplex is incident on 10 different bones. $b$ ranges between two kinematical bounds which follows from Dehn-Sommerville and a theorem by Walkup [16]. This latter theorem is also relevant in classifying the "elongated phase" of four-dimensional simplicial quantum gravity as we shall see below.

**Theorem:** If $T$ is a triangulation of a closed, connected four-dimensional manifold then

$$N_1(T) \geq 5N_0(T) - \frac{15}{2} \chi(T) \quad (1.5)$$

Moreover, equality holds if and only if $T \in H^4(1 - \frac{1}{2} \chi(T))$, where the class of triangulations $H^4(n)$ is defined inductively according to: (a) The boundary complex of any abstract five-simplex $(Bd\sigma)$ is a member of $H^4(0)$. (b) If $K$ is in $H^4(0)$ and $\sigma$ is a four-simplex of $K$, then $K'$ is in $H^4(0)$, where $K'$ is any complex obtained from $K$ by deleting $\sigma$ and adding the join of the boundary complex $Bd\sigma$ and a new vertex distinct from the vertices of $K$. (c) If $K$ is in $H^4(n)$, then $K'$ is in $H^4(n+1)$ if there exist two four-simplexes $\sigma_1$ and $\sigma_2$ with no common vertices and a dimension preserving simplicial map $\phi$ from $K - \sigma_1 - \sigma_2$ onto $K'$ which identifies $Bd\sigma_1$ with $Bd\sigma_2$ but otherwise is one to one.

In other words $H^4(0)$ is built up by gluing together five-dimensional simplexes through their four dimensional faces and considering only the boundary of this resulting complex. $H^4(n)$ differs from $H^4(0)$ by the fact that it has $n$ handles. This way of constructing a triangulation of a four-sphere has a natural connection with the definition of a baby universe. A baby universe is associated with a triangulation in which we can distinguish two pieces.
A piece that contains the majority of the simplices of the triangulation that is called the "mother", and a small part called the "baby". In the "minibus" (minimum neck baby universes) the two parts are glued together along the boundary of a four dimensional simplex (in four dimension) that is the "neck" of the baby universe. Thus the "stacked spheres" can be considered as a network of minibus in which the mother is disappeared and the babies universes have a minimal volume, that is the boundary of a five simplices minus the simplices of the necks through which they are glued to the others. We will exploit this parallel in section VI to give an estimate of the number of distinct stacked spheres.

Established in [3], Walkup theorem fixes a bound on the values of b, in fact formula (1.3) together with Dehn-Sommerville equations leads to $b \geq 4 - 10\frac{N_2}{N_0}$. From the Dehn-Sommerville equations and the fact that $N_0 \geq 5$, it follows that $b \leq 5 + \frac{10N_2-50}{N_0}$. Thus in the limit of large $N$, one obtains $4 \leq b \leq 5$.

As in Regge calculus the curvature of a dynamical triangulation is concentrated on the bones and it depends on the number $q(B)$ of four-dimensional simplices incident. More precisely, the curvature $K(B)$ at a bone $B$ is

$$K(B) = \frac{48\sqrt{15}}{a^2} \left[ 2\pi - q(B) \cos^{-1} \frac{1}{4} \right]. \quad (1.6)$$

The Regge version of Einstein-Hilbert action with cosmological constant for a dynamical triangulation $T_a$ is

$$S(T_a) = k_4 N_4(T_a) - k_2 N_2(T_a) \quad (1.7)$$

where $k_4$ is a constant depending linearly on the inverse of the gravitational constant and on the cosmological constant, whereas $k_2$ is proportional to the inverse of the gravitational constant; these constants depend also on the edge-length $a$; typically to simplify calculations one sets $a = 1$.

The partition function for 4-D dynamical triangulations is defined as

$$Z(k_2, k_4) = \sum_{\text{Top}(M)} \sum_{N_4(T_a)} \sum_{N_2(T_a)} W_{\text{Top}}(N_2, N_4) e^{-k_4 N_4 + k_2 N_2} \quad (1.8)$$

where $W_{\text{Top}}(N_4, N_2)$ is the number of distinct à la Tutte dynamical triangulations with a fixed number of four simplices $N_4$ and of bones $N_2$, and a fixed topology. Since in three and in four dimensions there is not yet a mathematical status for the sum over topologies we restrict ourselves to the simply connected topologies:

$$Z_{\text{s.c.}}(k_2, k_4) = \sum_{N_4(T_a)} \sum_{N_2(T_a)} W_{\text{s.c.}}(N_2, N_4) e^{-k_4 N_4 + k_2 N_2} \quad (1.9)$$

Equation (1.9) has the structure of a grand canonical partition function:

$$Z_{\text{G.C.}} = \sum_N \sum_{\sigma_N} e^{-\beta H(\sigma_N)} z^N \quad (1.10)$$

with the number of simplexes that is playing the role of the number of particles. Following this analogy we refer to $W(N_2, N_4)$ as the microcanonical partition function and to

$$Z(k_2, N_4) = \sum_{N_2} W(N_2, N_4) e^{k_2 N_2} \quad (1.11)$$

as the canonical partition function.

**II. CANONICAL PARTITION FUNCTION**

Recently it has been analytically shown [3] that the dynamical triangulations in four dimensions are characterized by two phases: a strong coupling phase in the region $k_2^{\text{crit}} = \log \theta / 8 < k_2 < k_2^{\text{crit}}$ and a weak coupling phase for $k_2 > k_2^{\text{crit}}$. $k_2^{\text{crit}}$ is the value of $k_2$ for which in the infinite volume limit the theory has a phase transition from the strong to the weak phase (for a detailed analysis see [3]). The transition between these two phases is characterized by the fact that the subdominant asymptotics of the number of distinct triangulations passes from an exponential to
In the strong coupling phase the leading term of the asymptotic expansion of the canonical partition function is (c.f. volume limit the average value of $b$ and it is constantly equal to 4 in the weak coupling regime, that is to say $\tau$ in which $\eta^*(k)$.

From this form of the canonical partition function it follows that in the case of the sphere polynomial behaviour (c.f. $[2]$). Presently the precise value of $k^c$ has not been established yet, it is just known that it is close but distinct from $k^c_{\text{max}} = \log 4$, recent numerical simulations suggest that $k^c = 1.24$.

In the strong coupling phase the leading term of the asymptotic expansion of the canonical partition function is (c.f. $\eta$, the Hausdorff dimension, are at present unknown.

The weak phase $k^c > k^c_{\text{crit}}$ the number of distinct dynamical triangulations with equal curvature assignment have a power law behaviour in $N$. The phase is characterized by two distinct asymptotic regimes of the leading term of the canonical partition function: the critical coupling regime and the weak coupling regime. In the critical coupling regime $k^c < k < k^c_{\text{max}}$ we have

$$Z(N_4, k_2) = c_4 \left( \frac{A(k_2) + 2}{3A(k_2)} \right)^{-4} N_4^{\tau(\eta^*) - 5} \exp \left[ 10 \log \frac{A(k_2) + 2}{3} \right] N_4$$ (2.1)

where for notational convenience we have set

$$A(k_2) = \left[ \frac{27}{2} e^{k_2} + 1 + \sqrt{\left( \frac{27}{2} e^{k_2} + 1 \right)^2 - 1} \right]^{1/3} + \left[ \frac{27}{2} e^{k_2} + 1 - \sqrt{\left( \frac{27}{2} e^{k_2} + 1 \right)^2 - 1} \right]^{1/3} - 1$$ (2.2)

and

$$\eta^*(k_2) = \frac{1}{3} (1 - \frac{1}{A(k_2)})$$ (2.3)

The explicit form of $m(\eta^*(k_2))$ and $n_H$, the Hausdorff dimension, are at present unknown.

In the weak phase $k^c > k^c_{\text{crit}}$ the number of distinct dynamical triangulations with equal curvature assignment have a power law behaviour in $N$. The phase is characterized by two distinct asymptotic regimes of the leading term of the canonical partition function: the critical coupling regime and the weak coupling regime. In the critical coupling regime $k^c < k < k^c_{\text{max}}$ we have

$$Z(N_4, k_2) = \frac{c_4 e^{-1/2} \eta_{\text{max}}^{-1} (1 - 2\eta_{\text{max}})^{-4}}{\sqrt{2\pi} \sqrt{(1 - 3\eta_{\text{max}})/(1 - 2\eta_{\text{max}})}} \left( N + 1 \right)^{\tau - 1/2} \sqrt{e^{(N+1)f(\eta_{\text{max}}, k_2)}}$$ (2.5)

in which $\eta_{\text{max}} = \frac{1}{4}$ for the sphere $S^4$ and

$$f(\eta, k_2) = -\eta \log \eta + (1 - 2\eta) \log(1 - 2\eta) - (1 - 3\eta) \log(1 - 3\eta) + k_2 \eta$$ (2.6)

From this form of the canonical partition function it follows that in the case of the sphere $S^4$ and in the infinite volume limit the average value of $b$, see equation (1.4), is a decreasing function of $k_2$ in the critical coupling regime and it is constantly equal to 4 in the weak coupling regime, that is to say

$$\lim_{N_4 \to \infty} < b >_{N_4} = \frac{1}{\eta(k_2)}, \quad k^c \leq k_2 \leq k^c_{\text{max}}$$

$$\lim_{N_4 \to \infty} < b >_{N_4} = 4, \quad k_2 \geq k^c_{\text{max}}$$ (2.7)

If we look at the average curvature we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_B K(B) vol_4(B) = \pi a^2 \sqrt{3} \left( 10\eta^*(k_2) - \frac{5}{\cos^{-1} \frac{1}{4}} \right)$$ (2.8)
and
\[
\lim_{N \rightarrow \infty} \frac{1}{N} < \sum_B K(B) \text{vol}_4(B) > = \pi a^2 \sqrt{3} \left( \frac{5}{2} - \frac{5}{\pi} \cos^{-1} \frac{1}{4} \right) \quad k_2 > k_2^{\text{max}},
\]

in which \( \text{vol}_4(B) \) is the volume of the four simplexes incident on the bone \( B \), and it is \( a^2 \sqrt{3} q(B) \). As we have already said the value of the average curvature is saturated.

This result was already known in numerical simulation [3] and, as we will see in detail in the next section, it is due to the prevalence of stacked spheres in the weak coupling regime.

### III. ELONGATED PHASE

We have seen that in the case of triangulations of the sphere \( S^4 \) for \( k_2 \rightarrow \log 4 \), in the infinite volume limit, \( < b >_{N_4} \rightarrow 4 \). Walkup’s theorem [4] implies that the minimum value of \( b \) is reached on triangulations \( K \) of the sphere \( S^4 \) that belong to \( H^4(0) \) (stacked spheres). Then for \( k_2 > \log 4 \) one has \( < b >_{N_4} = 4 \), that means that in this region of \( k_2 \) the statistical ensemble of quantum gravity is strongly dominated by stacked spheres.

As explicitly shown in [5], the elements of \( H^4(0) \) can be put in correspondence with a tree structure. Let us recall that a \( d \)-dimensional simplicial complex \( T \), \( d \geq 1 \), is called a simple \( d \)-tree if it is the closure of its \( d \)-simplexes \( \sigma_1, ..., \sigma_t \) and these \( d \)-simplexes can be ordered in such a way that:

\[
\text{Cl } \tau_j \cap \left\{ \bigcup_{i=1}^{j-1} \text{Cl } \sigma_i \right\} = \text{Cl } \sigma_j
\]

for some \( (d-1) \)-face \( \tau_j \) of \( \sigma_j \), \( j \geq 2 \), and where the \( \tau_j \) are all distinct. This ordering of the simplexes of \( T \) induces a natural ordering of its vertices in \( v_1, ..., v_{t+d} \), where \( v_{t+d} \) is the vertex of \( \sigma_t \) not in \( \text{Cl } \tau_t \). Note that the interior part of \( T \) contains the simplexes \( \sigma_i \) and faces \( \tau_i \). The boundary of \( T \), \( \partial T \), consists of the boundary of the \( \sigma_i \) minus the \( \tau_i \), and is topologically equivalent to \( S^{d-1} \).

It can be shown [6], and it is very easy to check, that any element of \( K \in H^d(0) \) is the boundary of a simple \( (d+1) \)-tree \( T \), and that \( K \) determines uniquely the simple \( (d+1) \)-tree \( T \) for \( d \geq 2 \).

Note that any stacked sphere \( K \in H^d(0) \) can be mapped into a tree graph. This mapping is defined in the following way, let’s consider the (unique) simple five-simple tree \( T \) associated with \( K \), every five-simple is mapped to a vertex and every four-dimensional face in common with two five simplexes is mapped to an edge which has endpoints at the two vertices which represent the two five simplexes. Since the map between \( K \) and \( T \) is one to one, we have a map from a stacked-sphere into a tree-graph whose number of links at every vertex can be at most 6 (since a five simplex has six faces). This map, from the stacked spheres \( H^d(0) \) to the simple tree-graphs, is not one to one. The mathematical reason is that the previous construction maps every \( T \) to a tree-graph by an application that is the dual map restricted to the five and four dimensional simplexes of \( T \) in its domain and whose image is the tree graphs that are the 1-dimensional skeleton of the dual complex. It is well known that the dual map is a one to one correspondence only if we take in account the simplexes of \( T \) of any dimension. It follows that our map is such that a simple tree-graph may correspond to many stacked spheres. To illustrate better this point we shall use a picture closer to the physical intuition. Let us consider a stacked sphere \( K \) and its simple tree \( T \). Consider on \( T \) one of its ordered simplexes \( \sigma_j \) and let \( \tau_j \) be the face that it shares with one of the simplexes \( \sigma_1, ..., \sigma_t \) introduced previously. Repeating the same arguments used in the calculations of the inequivalent triangulations for baby universes [7], we can cut \( T \) in \( \tau_j \) in two parts, such that each part has two copies of \( \tau_j \) as a part of its boundary. Since \( \tau_j \) is a four dimensional simplex it is easy to admit that we can glue this two different copies in \( 5 \cdot 4 \cdot 3 \) ways to rebuild again a simple five tree \( T' \). In general all the possible gluings will generate distinct triangulations and consequently distinct stacked spheres (if the two parts are highly symmetrical triangulations some of the 60 ways of joining will not be distinct but, since for large \( N_4 \) asymmetrical triangulations will dominate, the number of case in which this will happen will be trascendable). The corresponding tree graph associated with these configurations \( T' \) will always be the same since this operation has not modified the one skeleton of the tree \( T \).

In statistical mechanics simple-tree structures correspond to self-avoiding "branched-polymers". So we have that for \( k_2 \geq k_2^{\text{max}} \) the dominant configurations for the triangulations of \( S^4 \) are branched polymers. This fact was already observed in numerical simulations in four dimensions. In [8] a network of baby universes of minimum neck (mininb) and of minimum size (blips) without a mother universe was obtained. These have been interpreted as branched polymer like structures.
IV. STACKED SPHERES AND BRANCHED POLYMERS

In this section we will establish a parallel between the standard mean field theory of branched polymers and the stacked spheres, in the sense that we will use the counting techniques of branched polymers to give an upper and lower estimation of the number of inequivalent stacked spheres. As we have stressed in the previous section there exists a map between the stacked spheres and tree graphs and this map is not one to one in the sense that there are more stacked spheres than tree graphs. This means that the number of inequivalent stacked spheres with a fixed number \( N_5 \) of five-simplexes is bounded below by the corresponding number of tree graphs. Now we will study the statistical behaviour of the tree graphs using the measure of simplicial quantum gravity and restricting it to the stacked spheres. In this analysis we will follow [9] in a way adapted to our case. The reader may refer to it for the details. A more mathematical analysis on the enumeration of inequivalent tree graphs is contained in [13].

First of all we notice that the boundary of a five-simplex has six four-simplexes and every edge of a tree graph correspond to a cancellation of two four-simplexes in the corresponding boundary of the stacked sphere. Since in a tree graph with \( N_5 \) vertices there are \( N_5 - 1 \) edges, we have that the boundary of a stacked sphere, whose corresponding tree graph has \( N_5 \) points, is made by a number \( N_4 \) of four-simplexes

\[
N_4 = 4N_5 + 2 \quad .
\]  

(4.1)

From the condition for stacked spheres (4.3), we have that the Einstein-Hilbert action, for dynamical triangulations, restricted to the stacked spheres is

\[
S = N_4(k_4 - \frac{5}{2}k_2) - 5k_2 \quad .
\]  

(4.2)

It follows that the Gibbs factor for the ensemble of tree graphs (branched polymers) corresponding to stacked spheres is

\[
exp \left( -(4N_5 + 2)(k_4 - \frac{5}{2}k_2) + 5k_2 \right) \quad .
\]  

(4.3)

Following the same notation of reference [13], let \( r^6(N_5) \) be the number of inequivalent rooted tree graphs with \( N_5 \) vertices and with at most six incident edges on each vertex, with one rooted vertex with one incident edge. \( \xi^6(N_5) \) is the number of inequivalent tree graphs with \( N_5 \) vertices. By simple geometrical considerations, or from the asymptotic formula [13] for \( r^6(N_5) \) and \( \xi^6(N_5) \), we get the asymptotic formula

\[
\xi^6(N_5) = \frac{1}{N_5} r^6(N_5 + 1)
\]  

(4.4)

Now let \( R^6(k_4,k_2) \) be the partition function for rooted tree graphs with statistical weight (4.3), and \( Z^6(k_4,k_2) \) be the partition function for unrooted tree graphs. We have

\[
R^6(k_4,k_2) = \sum_{N_4=2}^{\infty} e^{-(4(N_5-1)+2)(k_4-\frac{5}{2}k_2)+5k_2} r^6(N_5)
\]  

(4.5)

\[
Z^6(k_4,k_2) = \sum_{N_5=1}^{\infty} e^{-(4N_5+2)(k_4-\frac{5}{2}k_2)+5k_2} \xi^6(N_5)
\]  

(4.6)

Let’s define

\[
R^6(\triangle k_4) \equiv \left( e^{-2(k_4-5)} R^6(k_4,k_2) \right) = \sum_{N_5=2}^{\infty} e^{-4(N_5-1)\triangle k_4} r^6(N_5)
\]  

(4.7)

\[
Z^6(\triangle k_4) \equiv \left( e^{-2(k_4-5)} Z^6(k_4,k_2) \right) = \sum_{N_5=1}^{\infty} e^{-4N_5 \triangle k_4} \xi^6(N_5) \quad ,
\]  

(4.8)

where \( \triangle k_4 \equiv k_4 - \frac{3}{2}k_2 \). It is easy to see, from (4.4), that

\[
R^6(\triangle k_4) = -\frac{1}{4} \frac{d}{d\triangle k_4} Z^6(\triangle k_4) \quad .
\]  

(4.9)
From simplicial quantum gravity it is well known that the susceptibility $\chi(k_4, k_2)$ is given by

$$\chi(k_4, k_2) \propto \frac{\partial^2}{\partial k_4^2} Z(k_4, k_2)$$

(4.10)

By analogous consideration on the graph it follows that

$$\chi(\Delta k_4) \propto \frac{d^2}{d(\Delta k_4)^2} \approx \frac{d}{d(\Delta k_4)} R^6(\Delta k_4)$$

(4.11)

Now we will study the critical behaviour of this system in order to obtain some information about the critical behaviour of stacked spheres.

As easily verified the following identity, which is true for rooted tree graphs

$$R^6(N_5) = \sum_{\gamma=0}^{5} \frac{1}{\gamma!} \sum_{n_1, \ldots, n_\gamma \sum_{i=1}^\gamma n_i = n-2} \prod_{i=1}^\gamma r^6(n_i)$$

implies

$$R^6(\Delta k_4) = e^{-4\Delta k_4} \left( \sum_{\gamma=0}^{5} \frac{1}{\gamma!} R^6(\Delta k_4) \right)$$

(4.12)

Differentiating the previous equation we get a differential equation for $R^6(\Delta k_4)$

$$\frac{d}{d\Delta k_4} R^6(\Delta k_4) = -\frac{1}{4} R^6(\Delta k_4) \left[ 1 + \frac{e^{-4\Delta k_4}}{5!} (R^6(\Delta k_4))^5 - R^6(\Delta k_4) \right]^{-1}$$

(4.13)

So $R^6(\Delta k_4)$ shows a singularity when

$$R^6((\Delta k_4)^c) = 1 + \frac{e^{-4\Delta k_4}}{5!} (R^6((\Delta k_4)^c))^5$$

(4.14)

This equation has only one real solution and since $\frac{d}{d\Delta k_4} R^6|_{(\Delta k_4)^c}$ diverges, the inverse function is zero

$$\frac{d(\Delta k_4)(R^6)}{dR^6}|_{(\Delta k_4)^c} = 0$$

(4.15)

By the implicit function theorem $(\Delta k_4)(R^6)$ is analytic function at $R^6((\Delta k_4)^c)$, which implies

$$\Delta k_4(R^6) - \Delta k_4^c = \frac{1}{2} \frac{d^2(\Delta k_4)(R^6)}{d(R^6)^2} \bigg|_{R^6((\Delta k_4)^c)} (R^6(\Delta k_4) - R^6(\Delta k_4^c))^2$$

$$+ o \left( (R^6(\Delta k_4) - R^6(\Delta k_4^c))^2 \right)$$

(4.16)

So near $(\Delta k_4)^c$ we can write

$$R^6(\Delta k_4) \approx R^6((\Delta k_4)^c) + C ((\Delta k_4) - (\Delta k_4)^c)^{\frac{1}{2}}$$

(4.17)

From (4.11) we get

$$\chi^6(\Delta k_4) \propto ((\Delta k_4) - (\Delta k_4)^c)^{-\frac{3}{2}}$$

(4.18)

This means that the critical exponent of the susceptibility for the system of these tree graphs is $\gamma = \frac{1}{2}$.

The partition function that has been studied is a lower estimate of the partition function of the stacked spheres. Consider, now, a stacked sphere $K$ and a face $\tau_j$ through which two five-simplexes are glued together (a link on the corresponding tree graph). We can glue a two four face of a stacked sphere in 5·4·3 different ways, for large number of simplexes this will generate distinct configurations of stacked spheres whose associated tree-graph is always the same. Repeating the same argument for every $j$, $j = 1, \ldots, N_5 - 1$, (that is to say for every link of the dual tree
graph) we obtain a factor, \((5\cdot4\cdot3)^{N_5-1}\), that multiplied by \(r^6(N_5)\) and \(\xi^6(N_5)\), gives an upper bound on the number of , respectively, rooted and unrooted inequivalent stacked spheres. In other words

\[
Z_{tree}(k_4, k_2) \leq Z_{s.s.}(k_4, k_2) \leq \tilde{Z}_{tree}(k_4, k_2)
\]

where \(Z_{tree}(k_4, k_2)\) is the partition function for tree graphs studied above, \(Z_{s.s.}(k_4, k_2)\) is the partition function for stacked spheres and \(\tilde{Z}_{tree}(k_4, k_2)\) is the partition function for tree graphs with the additional weight defined above. A similar analysis as for \(Z_{tree}(k_4, k_2)\) shows that the critical line of \(\tilde{Z}_{tree}(k_4, k_2)\) is, of course, a straight line parallel and above respect to the \(k_4\) axis to \(Z_{tree}(k_4, k_2)\) one, and the susceptibility exponent is again \(\gamma = \frac{1}{2}\).

Obviously the estimates (4.20) are true for the canonical partition functions too,

\[
Z_{tree}(N_4, k_2) \leq Z_{s.s.}(N_4, k_2) \leq \tilde{Z}_{tree}(N_4, k_2) ,
\]

and the previous calculations show that

\[
Z_{tree}(N_4, k_2) \simeq N_4^{-\frac{3}{2}}e^{-N_4(k_4 - \frac{5}{2}k_2 - t_4)}
\]

and

\[
\tilde{Z}_{tree}(N_4, k_2) \simeq N_4^{-\frac{3}{2}}e^{-N_4(k_4 - \frac{5}{2}k_2 + t_4 - \frac{1}{4}\log 60)} ,
\]

where \(t_4\) is a constant that may be calculated by the equation (4.15). More easily from the table in reference [3] we get that \(t_4 \approx \frac{1}{4}\log 0.34\).

The circumstance that in the weak coupling region the partition function of quantum gravity, as found in an analytically way in [2], is strongly dominated by stacked spheres, allows us to write

\[
Z_{s.s.} \simeq N_4^{\gamma_s - 3}e^{k_4^c(k_2)} ,
\]

where \(\gamma_s\) is the susceptibility exponent [3].

Thus the critical line of the stacked spheres is a straight line parallel and between the critical lines of the systems of the two branched polymers. This implies

\[
k_4 - \frac{5}{2}k_2 + t_4 \leq k_4^c(k_2) \leq k_4 - \frac{5}{2}k_2 + t_4 - \frac{1}{4}\log 60
\]

Moreover from the equations (4.22) and (4.23) the one loop green functions [1] of the two model of branched polymers have respectively , near their critical lines , the asymptotic behaviour

\[
G_{tree}(\Delta k_4) \simeq \text{cost}_1 + (\Delta k_4 - \Delta k_4^c)^\gamma \tilde{G}_{tree}(\Delta \tilde{k}_4) \simeq \text{cost}_2 + (\Delta k_4 - \Delta \tilde{k}_4^c)^\gamma
\]

This last equation toghteer the equations (4.21) and (4.24) prove that the one loop function of the stacked spheres \(G_{s.s.}(k_4, k_2)\) near the critical line has the asymptotic behaviour

\[
G_{s.s.}(k_4, k_2) \simeq \text{cost}_3 + (k_4 - k_4^c(k_2))^{1-\gamma_s} ,
\]

with \(\gamma_s < 1\) (the value \(\gamma_s = 1\) is not allowed because in this case the one loop green function of stacked spheres at the critical line will have a behaviour like \(\sum_{N_4=5}^{\infty} 1/N\) that is divergent and then incompatible with the upper bound given by the second equation of (4.26).

Motivated by physical considerations, we can use a well known argument [1] [3] in favour of the fact that the susceptibility exponent of the stacked spheres is \(\gamma_s = \frac{1}{2}\). More precisely we will show that a model of proliferating baby universes, with the measure of quantum gravity restricted to stacked spheres, can be put in correspondence with the statistical system of stacked spheres.

Let us consider four dimensional triangulations that are (boundary of the) stacked spheres in which there can be loops made by two three-dimensional simplexes. This is possible whenever the stacked spheres are pinched on a three simplex creating a bottle neck loop of two three-simplexes. These loops could be either the loops of a Green function or the minimum bottle neck of a baby universes. This class of triangulations, following the notation in literature, is called \(T_2\). The other class of triangulations is the stacked spheres in which the minimum loop length can be made by the boundary of a four-simplex that are five three-simplexes. We call this last class \(T_5\). In the two dimensional theory, the introduction in \(T_2\) of two-link loops (the two dimentional analogous of two three-simplex loop) corresponds in the matrix model \(\phi^4\) to consider Feynman diagrams with self-energy (c.f. [1] [3]).
Since the minima loops of $T_2$ and $T_5$, for which they differ, are of the order of lattice spacing we will expect that the two classes of triangulations, as a statistical mechanics system, coincide in the scaling limit, that is to say they belong to the same universality class.

Let’s consider the minimum neck one loop function $G(\triangle k_4)$ in $T_2$. In every triangulation of $T_2$ we can cut out the maximal size baby universe of minimum neck and close the two three-simplex loop. We will obtain again a triangulation that belongs to $T_2$. In this way we will obtain all the triangulations of $T_2$ from the triangulations of the stacked spheres $T_5$ considering that for each three-simplex either leave them in their actual form or we can open the triangulation to create a two-three simplex loop and gluing on it a whole one loop universe $G(\triangle k_4)$. We note that in the triangulations of $T_5\,\mathbb{N}_3 = 5/2\mathbb{N}_4$ (Dehn-Sommerville). Calling the one loop function of $T_5\,\,G(\triangle k_4)$, the above considerations lead to the identity

$$G(\triangle k_4) = \sum_{T \in T_5} e^{-\mathcal{N}_4 \triangle k_4} (1 + G(\triangle k_4)) \mathcal{N}_4 = \sum_{T \in T_5} e^{-\mathcal{N}_4 \triangle k_4} = G(\triangle k_4) \ ,$$

in which $\triangle k_4 = k_4 - 5/2k_2$ comes out from restricting the action of quantum gravity to stacked spheres 4.2 and where we have defined

$$\overline{\triangle k_4} = \triangle k_4 - \frac{5}{2} \log (1 + G(\triangle k_4)) \ .$$

By 4.28 we can also write last equation as

$$\triangle k_4 = \overline{\triangle k_4} + \frac{5}{2} \log \left(1 + G(\triangle k_4)\right) \ .$$

By universality and the estimates (4.21) it follows that near the critical point we have that $G(\triangle k_4) \approx \text{cost} + (\triangle k_4 - \overline{\triangle k_4})^{1-\gamma_s}$ with $\gamma_s < 1$. The susceptibility functions of $T_2$ and $T_5$ by 4.28 are

$$\chi(\triangle k_4) \approx -\frac{d}{d\triangle k_4} G(\triangle k_4) \quad \chi(\overline{\triangle k_4}) \approx -\frac{d}{d\overline{\triangle k_4}} G(\overline{\triangle k_4}) \ (4.31)$$

From 4.30 we have

$$\frac{d(\triangle k_4)}{d(\overline{\triangle k_4})} = 1 - \frac{5}{2} \frac{\chi(\overline{\triangle k_4})}{1 + G(\overline{\triangle k_4})} \ . (4.32)$$

If we calculate derivative with respect to $\triangle k_4$ of the one loop function $G(\triangle k_4)$ and use the previous equation we get

$$\chi(\triangle k_4) = \frac{\chi(\overline{\triangle k_4})}{1 - \frac{5}{2} \frac{\chi(\overline{\triangle k_4})}{1 + G(\overline{\triangle k_4})}} \ . (4.33)$$

Now it is clear that $\chi(\triangle k_4) \rightarrow +\infty$ for $\triangle k_4 \rightarrow (\triangle k_4)^c$ and with the same critical exponent $\chi(\overline{\triangle k_4}) \rightarrow +\infty$ for $\overline{\triangle k_4} \rightarrow (\overline{\triangle k_4})^c$. From equation (4.33) when $\chi(\overline{\triangle k_4}) \rightarrow +\infty$ we have that $\frac{\chi(\overline{\triangle k_4})}{1 + G(\overline{\triangle k_4})} \rightarrow \frac{5}{2} \left(1 + G(\overline{\triangle k_4}(\triangle k_4)^c)\right) < +\infty$, then the system $T_5$ is above his critical line, i.e. $\chi(\overline{\triangle k_4}) > \chi(\triangle k_4)^c$. These facts imply that at $\overline{\triangle k_4} = 0$ and around it $\chi(\overline{\triangle k_4})/(1 + G(\overline{\triangle k_4}))$ is a decreasing monotonic function by equation 4.32 because $\chi(\triangle k_4) \rightarrow +\infty$, we can expand equation 4.30 and 4.33 around $\overline{\triangle k_4}(\triangle k_4)^c$

$$\triangle k_4 - \triangle k_4^c = \text{cost} \left(\frac{1}{\triangle k_4 - \overline{\triangle k_4}(\triangle k_4)^c}\right)^2 \ .$$

$$\chi(\triangle k_4) \approx \frac{1}{\triangle k_4 - \overline{\triangle k_4}(\triangle k_4)^c} \approx \frac{1}{\sqrt{\triangle k_4 - \overline{\triangle k_4}}} \ . (4.35)$$

The second asymptotic equality of the last equation implies

$$\gamma_s = \frac{1}{2} \ . (4.36)$$

The dominance of stacked spheres in the weak phase allows us to fix the parameter $\tau$ in the partition function of quantum gravity in the weak coupling regime.

$$\tau - \frac{11}{2} = -\frac{5}{2} \Rightarrow \tau = 3 \ . (4.37)$$
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