Membrane Paradigm from Near Horizon Soft Hair

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Abstract

The membrane paradigm posits that black hole microstates are dynamical degrees of freedom associated with a physical membrane vanishingly close to the black hole’s event horizon. The soft hair paradigm postulates that black holes can be equipped with zero-energy charges associated with residual diffeomorphisms that label near horizon degrees of freedom. In this essay we argue that the latter paradigm implies the former. More specifically, we exploit suitable near horizon boundary conditions that lead to an algebra of “soft hair charges” containing infinite copies of the Heisenberg algebra, associated with area-preserving shear deformations of black hole horizons. We employ the near horizon soft hair and its Heisenberg algebra to provide a formulation of the membrane paradigm and show how it accounts for black hole entropy.

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Black holes are spacetimes characterized by two causally disconnected regions, separated by an event horizon. For stationary generic black holes, on which we mainly focus in this essay, the horizon is a bifurcate null surface, with past and future horizons intersecting on the bifurcation surface, a compact spacelike surface of co-dimension two. Therefore, constant time slices of black hole spacetimes are divided into inside and outside regions. The horizon is classically a semi-permeable membrane through which things fall in but do not come out.

Black holes behave like thermal states with a large entropy, as pioneered by Bekenstein [1], at a certain temperature, first calculated by Hawking [2]. According to Einstein’s equivalence principle an outside observer should be able to give a complete description of physics with the information available to her, without the need to know the inside horizon information. Therefore, one expects to be able to associate black hole thermodynamics to a region outside the horizon, infinitesimally close to it, the so-called stretched horizon [3].

The membrane paradigm [4] was originally proposed to give a simple classical picture for black holes and their (thermo)dynamics, particularly from the outside observer’s viewpoint, without re-crossing into intricate and subtle points of quantum field theory on curved backgrounds containing causally disconnected regions. The paradigm is based on replacing the region inside the stretched horizon by a thin, classically radiating, membrane that embodies thermodynamical properties of the black hole. This membrane has a tension, can be electrically charged and conducting, has finite entropy and temperature, but cannot conduct heat. The interaction of the stretched horizon with the external universe is described in terms of the Navier–Stokes equation associated with the membrane, Ohm’s law, a tidal-force equation, and the first and second laws of thermodynamics.

Studying quantum field theory on a black hole background yields Hawking radiation. This confirms the thermodynamical picture discussed above and shows that stationary black holes, like any thermodynamical system in equilibrium, emit black-body radiation (at the Hawking temperature). It is hence natural to pose the following question: can the membrane paradigm be extended semi-classically?

To answer this question one needs to convert the membrane paradigm into a specific membrane model, which particularly accounts for the black hole entropy and its associated microstates. Different aspects of this issue were addressed in light of black hole complementarity [3], which was challenged a few years ago through the firewall controversy [5]. While there are other approaches for formulating the membrane paradigm, e.g. [6], in this essay we revisit and motivate it from a novel viewpoint. We start from the concept of near-horizon soft hair and the associated near horizon symmetries, which we shall spell out explicitly for generic black holes in four spacetime dimensions. As we shall see, our soft hair analysis unexpectedly leads us back to the membrane paradigm. Among other things, our approach provides us with a statistical mechanical setup to account for black hole entropy within the membrane picture.
Near-horizon soft hair. Hawking, Perry and Strominger (HPS) [7] introduced about two years ago the notion of soft hair, alluding to the famous statement by John Wheeler “black holes have no hair”. While classical black holes have no hair in Wheeler’s sense (meaning that they are uniquely specified by a handful of parameters like mass, angular momentum and electric charge), HPS argued that black holes can be dressed by zero energy (=soft) excitations.

Near horizon soft hair emerges from the imposition of suitable near horizon boundary conditions, like the ones discovered in [8]. The near horizon boundary conditions proposed in [9] led to infinite copies of the Heisenberg algebra as near horizon symmetries, albeit only in three spacetime dimensions, and to a specific semi-classical proposal for black hole microstates [10]. Building on ongoing work with Perez and Troncoso [11] we generalize now the boundary conditions of [9] to four dimensions with the aim to exhibit again infinite copies of the Heisenberg algebra as near horizon symmetries. Expanding the metric around the horizon at $\rho = 0$ yields

$$\text{d}s^2 = -\kappa^2 \rho^2 \text{d}t^2 + \text{d}\rho^2 + 2\rho N_a \text{d}\rho \text{d}x^a + \Omega_{ab} \text{d}x^a \text{d}x^b + \mathcal{O}(\rho^2) \quad a, b = 1, 2, \quad (1)$$

where $\kappa$ is (constant and fixed) surface gravity and $\Omega_{ab}$ is the metric on the bifurcate horizon $\mathcal{H}$. Metric fluctuations of order

$$\delta g_{\rho a} = \mathcal{O}(\rho), \quad \delta g_{ab} = \mathcal{O}(1) \quad \Rightarrow \quad N_a = \mathcal{O}(1) = \delta \Omega_{ab}, \quad (2)$$

turn out to generate near horizon soft hair, while all other fluctuations either are determined from these soft hair excitations or correspond to small diffeomorphisms.

The full charge analysis will appear elsewhere [11]; here we just quote its main results: there are three sets of non-trivial diffeomorphisms and associated near horizon charges: 1. near horizon supertranslations, 2. area preserving shear deformations (APSDs), and 3. area preserving twist deformations. Since we are ultimately interested in black hole microstates we freeze the latter, as twist deformations are not expected to affect the microstates and their number. The supertranslations are generated by angle-dependent time-translations, $\xi^t(\theta, \varphi)$, and the APSDs by the gradient part of diffeomorphisms of the 2-sphere, $\xi^a_{\text{grad}}(\theta, \varphi)$. (The twist deformations would correspond to the curl part.) The associated near horizon charges

$$Q[\xi^t, \xi^a_{\text{grad}}] = \int_{\mathcal{H}} \text{d}^2 x \left[ \xi^t P + \xi^a_{\text{grad}} \partial_a \Phi \right] \quad (3)$$

contain the state-dependent functions ($\Omega := \text{det} \Omega_{ab}$)

$$P = \frac{\sqrt{\Omega}}{8\pi G}, \quad \partial_a \Phi = \frac{\Omega_{ab} \pi^{\rho b}}{8\pi G \sqrt{\Omega_{\text{grad}}}}, \quad (4)$$

where $G$ is Newton’s constant, $\pi^{\rho b}$ is the canonical momentum associated with the metric component $g_{\rho b}$ and the subscript ‘grad’ means that we take only the gradient contribution (and not the curl part) on the right hand side of the right equality (4). In contrast to the boundary conditions of [8] we use densitized superrotation parameters $\xi^a_{\text{grad}}$, which is a small but significant change reminiscent of the transformation between standard and Ashtekar variables [12].
Near horizon Heisenberg algebra and entropy. A straightforward canonical analysis reveals the following Poisson brackets between the supertranslation charges $P$ and APSD-charges $\Phi$

\[
\{\Phi(x), P(y)\} = \frac{1}{8\pi G} \delta^{(2)}(x - y) \tag{5}
\]

\[
\{P(x), P(y)\} = 0, \quad \{\Phi(x), \Phi(y)\} = B, \quad (6)
\]

where the explicit form of $B$ is not needed here. Note that $B$ vanishes trivially for non-rotating black holes (like Schwarzschild) and for black holes with flat horizons (like toroidal Kerr-anti-de Sitter) [11].

The commutation relation (5) is a key result and means that the charge associated with supertranslations is the Heisenberg conjugate of the charge associated with APSDs. This result has profound physical consequences and will lead to a formulation of the membrane paradigm, accounting also for the black hole entropy. To discuss some of the physical consequences note that the algebra (5) is nothing but the Heisenberg algebra, where the role of Planck’s constant $h$ is played by a quarter of the inverse of Newton’s constant.

\[
h \simeq \frac{1}{4G} \tag{7}
\]

We stress that the identification (7) is not an input but rather a result emerging from our near horizon analysis.

The near-horizon Hamiltonian $H = Q(\xi^t = \kappa, \xi^a = 0) = \kappa \int_{\mathcal{H}} d^2 x P =: \kappa P_0$ has vanishing Poisson brackets with all near horizon symmetries, which proves the softness property. Its variation

\[
\delta H = \kappa \delta P_0 = T \delta S \tag{8}
\]

establishes the near-horizon first law of thermodynamics, thereby recovering the Bekenstein–Hawking entropy (using $T = 2\pi/\kappa$)

\[
S = 2\pi P_0 = \frac{1}{4G} \int_{\mathcal{H}} d^2 x \sqrt{\Omega} = \frac{\text{Area}}{4G}. \tag{9}
\]

The entropy formula (9) is in accordance with previous near horizon results [8,9] and expected from the (Iyer–)Wald analysis [13]. Before addressing how our near horizon results above relate to the membrane paradigm we consider briefly an example.

The Kerr black hole with mass $M = (r_+ + r_-)/2$ and rotation parameter $a = \sqrt{r_+ r_-}$, but without any soft hair excitations, yields the near horizon charges ($\theta$ is the polar angle on a unit $S^2$)

\[
P_{Kerr} = \frac{M(M + \sqrt{M^2 - a^2}) \sin \theta}{4\pi G}, \quad \Phi_{Kerr} = 0. \tag{10}
\]

The condition that the area preserving twist deformations are fixed implies the scaling relations $\delta \text{Area}/\text{Area} = 2\delta r_+/r_+ = 2\delta r_-/r_-$. Thus, if one keeps the area fixed, $\delta \text{Area} = 0$, the absence of twist deformations implies that mass $M$ and Kerr parameter $a$ are fixed as well. The remaining soft hair excitations (supertranslations and APSDs) do not affect mass or Kerr parameter and thus have a chance to label microstates of a given Kerr black hole.
Reloading membrane paradigm with near-horizon soft hair. The dynamics of a membrane is governed by an action invariant under diffeomorphisms that preserve the volume it sweeps in spacetime. For a membrane wrapping the horizon \( \mathcal{H} \) this includes the APSDs. The dynamics of a codimension one membrane is then described by a single degree of freedom \( \mathcal{P} \) and its Heisenberg conjugate \( \Phi \). This perfectly matches with our near-horizon symmetry analysis above, once we identify \( 1/(4G) \) as the membrane tension, as suggested by the identification (7). In other words, (5) may be viewed as the constant time Poisson bracket, and semi-classically as corresponding commutator, of the degrees of freedom of the membrane theory. The simplest membrane action compatible with these considerations is

\[
I_{\text{membrane}} = \frac{1}{4G} \int dt E dx^2 \sqrt{h_{ij}} \tag{11}
\]

where \( h_{ij} \) is the induced metric. Fixing static gauge on the membrane, the action (11) evaluated in the Euclidean section for the membrane wrapping the stretched horizon (integrating Euclidean time \( t_E \) over a thermal circle of unit radius) yields the Bekenstein–Hawking entropy (9). Thus, the membrane naturally accounts for the black hole entropy given by the logarithm of the partition function derived from the membrane on-shell action (11).

Summary and outlook. Even if one did not know about the membrane paradigm, our near-horizon soft hair analysis not only indirectly implies it and provides a venue for its precise formulation but may also enable us to account for black hole microstates from semi-classical first principles, conceptually along the lines of a recent proposal [10]. Here we only presented a semi-classical analysis that accounts for the entropy, without explicitly specifying the microstates. For the latter one needs to carry out quantization of the membrane action, which may be performed along the lines of [14]. Like in three dimensions [10], a key step will be a controlled cutoff on the soft hair spectrum that selects a precise subset of soft hair excitations as black hole microstates. Our hope is that the relation to the membrane paradigm indicated above can provide such a cutoff.

Finally, it would be excellent to establish a connection between our analysis and holographic entanglement entropy [15], shedding further light on black hole complementarity [3] and the resolution of the firewall issue [5]. Indeed, the boundary conditions (1), (2) eliminate singularities on the horizon and the whole soft hair spectrum is compatible with regularity, suggesting the absence of firewalls.

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