PARABOLIC FREQUENCY ON MANIFOLDS

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Abstract. We prove monotonicity of a parabolic frequency on manifolds. This is a parabolic analog of Almgren’s frequency function. Remarkably we get monotonicity on all manifolds and no curvature assumption is needed. When the manifold is Euclidean space and the drift operator is the Ornstein-Uhlenbeck operator this can been seen to imply Poon’s frequency monotonicity for the ordinary heat equation. Monotonicity of frequency is a parabolic analog of the 19th century Hadamard three circles theorem about log convexity of holomorphic functions on $\mathbb{C}$. From the monotonicity, we get parabolic unique continuation and backward uniqueness.

0. Introduction

Bounds on growth for functions satisfying a PDE give crucial information and have many consequences. One of the oldest bounds of this type is Hadamard’s three circles theorem for holomorphic functions. For elliptic equations, such as the Laplace equation, Almgren proved the monotonicity of a frequency function that measures the rate of growth, $[A]$. Almgren’s frequency played a fundamental role in his regularity results, $[A]$, and other areas; see, e.g., $[GL]$, $[Lo]$. Almgren’s frequency was generalized to the heat equation by Poon, $[P]$, who proved the monotonicity of a parabolic frequency function. The results of Almgren and Poon rely heavily on the scaling structure of $\mathbb{R}^n$ (cf. $[CMI]$) and do not extend globally to general manifolds. Here we prove a very general monotonicity for drift heat equations on any manifold and show that this general monotonicity implies the earlier one. Part of the strength is the simplicity of the argument yet the power of the consequences.

Suppose that $(M,g)$ is a Riemannian manifold. Let $\phi : M \to \mathbb{R}$ be a smooth function and define an operator $\mathcal{L}_\phi$ (drift Laplacian) on vector-valued functions $u : M \to \mathbb{R}^N$ by

$$\mathcal{L}_\phi u = \Delta u - \langle \nabla u, \nabla \phi \rangle = e^\phi \text{div} \left( e^{-\phi} \nabla u \right).$$

(0.1)

These operators play an important role in many parabolic problems; see, e.g., $[CM2]$, $[CM3]$. The prime example of $\mathcal{L}_\phi$ is where $M = \mathbb{R}^n$ with the flat metric, $\phi = \frac{|x|^2}{4}$ and $\mathcal{L}_{\frac{|x|^2}{4}} u = \Delta u - \frac{1}{2} \langle \nabla u, x \rangle$ is the Ornstein-Uhlenbeck operator. We let $L^2_\phi$ and $W^{1,2}_\phi$ be the spaces of square integrable $\mathbb{R}^N$-valued functions and Sobolev functions with respect to the weight $e^{-\phi}$. It follows from (0.1) that $\mathcal{L}_\phi$ is self-adjoint on $W^{1,2}_\phi$ with respect to the weighted volume

$$\int \langle u, \mathcal{L}_\phi v \rangle e^{-\phi} = - \int \langle \nabla u, \nabla v \rangle e^{-\phi}.$$

(0.2)

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Suppose that $u : M \times [a, b] \to \mathbb{R}^N$ is smooth and $u, u_t \in W^{1, 2}_0$ for each $t \in [a, b]$. Set

$$I(t) = \int |u|^2 e^{-\phi},$$

$$D(t) = -\int |\nabla u|^2 e^{-\phi} = \int \langle u, \mathcal{L}_\phi u \rangle e^{-\phi},$$

$$U(t) = D(t).$$

Observe that with our convention $U$ is always non-positive.

The next theorem is a parabolic version of the classical Hadamard’s three circle theorem for holomorphic functions:

**Theorem 0.6.** When $(\partial_t - \mathcal{L}_\phi) u = 0$, then $(\log I)'(t) = 2U(t)$ and $\log I(t)$ is convex so $U' \geq 0$. Moreover, when $U$ is constant, then $u(x, t) = e^{U t} u(x, 0)$ and $u(\cdot, 0)$ is an eigenfunction of $\mathcal{L}_\phi$ with eigenvalue $-U$.

Poon, [P], proved a monotonicity that can be shown (see Section 1) to follow from the special case of Theorem 0.6 when $M = \mathbb{R}^n$, $N = 1$ and $\phi = \frac{|x|^2}{4}$. His monotonicity holds on manifolds with non-negative sectional curvature and parallel Ricci curvature which are exactly the assumptions needed to generalize Hamilton’s work, [H1], [H2], from Euclidean space to manifolds. In contrast our monotonicity holds on any manifold and no curvature assumption is needed.

Theorem 0.6 has the following immediate consequences (recall that $U$ is non-positive):

**Corollary 0.7.** If $u : M \times [a, b] \to \mathbb{R}^N$ and $(\partial_t - \mathcal{L}_\phi) u = 0$, then

$$I(b) \geq I(a) e^{2U(a)(b-a)}.$$

In particular, if $u(\cdot, b) = 0$, then $u \equiv 0$.

**Proof.** By Theorem 0.6

$$\log I(b) - \log I(a) = \int_a^b (\log I)'(s) \, ds = 2 \int_a^b U(s) \, ds \geq 2U(a) (b-a).$$

Equation (0.8) can be thought of as a bound for the vanishing order at $\infty$ whereas the second part is a version of backward uniqueness. The first part implies strong unique continuation at $\infty$. That is, if $u$ vanishes to infinite order at $\infty$, then it vanishes. We say that $u : M \times (a, \infty) \to \mathbb{R}$ vanishes to infinite order at $\infty$ if $\lim_{t \to \infty} e^{ct} I(t) = 0$ for all constants $c$.

Suppose more generally $u$ satisfies the equation:

$$\partial_t - \mathcal{L}_\phi - \lambda) u = 0.$$
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Where \( \phi \) is as above and \( \lambda = \lambda(t) \) is a function depending on \( t \) only. By considering
\[
v(x, t) = e^{-\int_0^t \lambda(s) \, ds} u(x, t)
\]
and observing that \( v \) satisfies (0.1). It follows that our results apply to \( v \) and hence we get a monotonicity for \( u \).

Our results holds also for more general operators (cf. [ESS], [W]) where
\[
|(\partial_t - \mathcal{L}_\phi) u| \leq C(t) \left(|u| + |\nabla u|\right),
\]
and \( C(t) \) is allowed to depend on \( t \); see Theorem 2.10 and Corollary 2.15.

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1. Parabolic frequency on manifolds

Proof. (of Theorem 0.6). Calculating and integrating by parts gives
\[
I'(t) = 2 \int \langle u, u_t \rangle e^{-\phi} = 2 \int \langle u, \mathcal{L}_\phi u \rangle e^{-\phi} = -2 \int |\nabla u|^2 e^{-\phi} = 2 D(t).
\]
(1.1)

\[
D'(t) = -2 \int \langle \nabla u, \nabla u_t \rangle e^{-\phi} = -2 \int \langle \nabla u, \nabla \mathcal{L}_\phi u \rangle e^{-\phi} = 2 \int |\mathcal{L}_\phi u|^2 e^{-\phi}.
\]
(1.2)

By (1.1) and the definition of \( U \) we get
\[
(\log I)'(t) = 2 \frac{D(t)}{I(t)} = 2 U(t).
\]
(1.3)

Therefore, using (1.1), (1.2) and (0.4)
\[
D' I - I' D = \left( 2 \int |\mathcal{L}_\phi u|^2 e^{-\phi} \right) \left( \int |u|^2 e^{-\phi} \right) - 2 D^2(t)
\]
(1.4)

Here the inequality follows from the Cauchy-Schwarz inequality. Finally, from this we get
\[
U' = \frac{D' I - I' D}{I^2} \geq 0.
\]
(1.5)

When \( U \) is constant \( U' = 0 \) and we therefore have equality in the Cauchy-Schwarz inequality (1.4). It follows that
\[
\mathcal{L}_\phi u = c(t) u.
\]
(1.6)

Next to evaluate \( c \) we observe that by the second equality in (0.4)
\[
D(t) = c(t) \int |u|^2 e^{-\phi} = c(t) I(t).
\]
(1.7)

It follows that \( c(t) = U \) and \( \mathcal{L}_\phi u = U u \). If we set
\[
v(x, t) = e^{-U t} u(x, t),
\]
(1.8)

then we have that
\[
\partial_t v = e^{-U t} (-U u + \partial_t u) = e^{-U t} (-U u + \mathcal{L}_\phi u) = 0.
\]
(1.9)
There is a natural correspondence on $\mathbb{R}^n$ between solutions of the ordinary heat equation and solutions of the drift heat equation: Given $u : \mathbb{R}^n \times (-\infty, 0) \rightarrow \mathbb{R}$, define $v(x, t) = u(\sqrt{-t} x, t)$, $w(x, s) = v(x, -e^{-s})$ and $t = -e^{-s}$. We have the following:

**Lemma 1.10.** The function $w : \mathbb{R}^n \times (-\infty, 0) \rightarrow \mathbb{R}$ defined as above satisfies

$$
(\partial_s - L \frac{|x|^2}{4}) w(x, s) = e^{-s} \left( u_t - \Delta u \right)(e^{-\frac{s}{2}} x, -e^{-s}).
$$

**Proof.** To prove (1.11), we use the chain rule to get

$$
\partial_t v = -\frac{1}{2 \sqrt{-t}} \langle \nabla u, x \rangle + u_t,
$$

$$
\partial_s w = -\frac{\sqrt{-t}}{2} \langle \nabla u, x \rangle - t u_t,
$$

$$
\nabla w = \sqrt{-t} \nabla u,
$$

$$
\Delta w = -t \Delta u.
$$

Combining (1.13)–(1.15) gives (1.11). □

Poon, [P], considered solutions $u : \mathbb{R}^n \times (-\infty, 0) \rightarrow \mathbb{R}$ to the ordinary heat equation on Euclidean space. He showed a monotonicity that is easily seen to be equivalent to that $s \rightarrow \log H(e^{s^2})$ is convex, where

$$
H(R) = (-4 \pi R^2)^{-\frac{n}{2}} \int u^2(y, -R^2) e^{-\frac{|y|^2}{4R^2}}.
$$

The convexity of $\log H(e^{s^2})$ follows from Theorem 0.6 when $M = \mathbb{R}^n$ and $\phi = \frac{|x|^2}{4}$. To see this suppose $u_t = \Delta u$, so that $(\partial_s - L \frac{|x|^2}{4}) w = 0$ by Lemma 1.10. Using the definition of $I_w(s)$ and making the change of variables $y = e^{-\frac{s}{2}} x$ and $R = e^{-\frac{s}{2}}$ gives

$$
I_w(s) = \int u^2(e^{-\frac{s}{2}} x, -e^{-s}) e^{-\frac{|x|^2}{4}} \, dx = R^{-n} \int u^2(y, -R^2) e^{-\frac{|y|^2}{4R^2}} \, dy = H(e^{s^2}).
$$

From this and Theorem 0.6 the convexity of $\log H(e^{s^2})$ follows.

## 2. More general operators

**Theorem 2.1.** If $u : M \times [a, b] \rightarrow \mathbb{R}^N$ satisfies (0.11), then

$$
U' \geq C^2 \left( U - 1 \right),
$$

$$
C^2 \geq \left[ \log(1 - U) \right]'.
$$

**Proof.** First we rewrite $D$ as follows

$$
D = \int \langle u, L \phi u \rangle e^{-\phi} = \int \langle u, [u_t - \frac{1}{2} (u_t - L \phi u)] \rangle e^{-\phi} - \frac{1}{2} \int \langle u, (u_t - L \phi u) \rangle e^{-\phi}.
$$
Differentiating $I(t)$ and rewriting gives
\[
I'(t) = 2 \int \langle u, u_t \rangle e^{-\phi} = 2 \int \langle u, \mathcal{L}_\phi u \rangle e^{-\phi} + 2 \int \langle u, (u_t - \mathcal{L}_\phi u) \rangle e^{-\phi}
\]
\[(2.5) \quad = 2 \int \langle u, [u_t - \frac{1}{2} (u_t - \mathcal{L}_\phi u)] \rangle e^{-\phi} + \int \langle u, (u_t - \mathcal{L}_\phi u) \rangle e^{-\phi}.
\]
Hence,
\[(2.6) \quad I'(t) D(t) = 2 \left( \int \langle u, [u_t - \frac{1}{2} (u_t - \mathcal{L}_\phi u)] \rangle e^{-\phi} \right)^2 - \frac{1}{2} \left( \int \langle u, (u_t - \mathcal{L}_\phi u) \rangle e^{-\phi} \right)^2.
\]
Differentiating $D(t)$ and integrating by parts gives
\[
D'(t) = -2 \int \langle \nabla u, \nabla u_t \rangle e^{-\phi} = 2 \int \langle u_t, \mathcal{L}_\phi u \rangle e^{-\phi} = 2 \int \langle u_t, (u_t - [u_t - \mathcal{L}_\phi u]) \rangle e^{-\phi}
\]
\[(2.7) \quad = 2 \int \left\{ [u_t - \frac{1}{2} (u_t - \mathcal{L}_\phi u)]^2 - \frac{1}{4} |u_t - \mathcal{L}_\phi u|^2 \right\} e^{-\phi}.
\]
So
\[(2.8) \quad D'(t) I(t) = 2 I(t) \int |u_t - \frac{1}{2} [u_t - \mathcal{L}_\phi u]|^2 e^{-\phi} - \frac{I(t)}{2} \int |u_t - \mathcal{L}_\phi u|^2 e^{-\phi}.
\]
Combining $(2.6)$ and $(2.8)$ and using the Cauchy-Schwarz inequality, $(0.11)$ and the elementary inequality $(a + b)^2 \leq 2 (a^2 + b^2)$ gives
\[
D' I - I' D = 2 \left[ \int |u|^2 e^{-\phi} \int |u_t - \frac{1}{2} [u_t - \mathcal{L}_\phi u]|^2 e^{-\phi} - \left( \int \langle u, [u_t - \frac{1}{2} (u_t - \mathcal{L}_\phi u)] \rangle e^{-\phi} \right)^2 \right]
\]
\[(2.9) \quad - I(t) \left[ \frac{1}{2} \int |u_t - \mathcal{L}_\phi u|^2 e^{-\phi} + \frac{1}{2} \left( \int \langle u, (u_t - \mathcal{L}_\phi u) \rangle e^{-\phi} \right)^2 \right]
\]
\[\geq - \frac{I(t)}{2} \int |u_t - \mathcal{L}_\phi u|^2 e^{-\phi} \geq -C^2 \frac{I(t)}{2} \int (|u| + |\nabla u|)^2 e^{-\phi} \geq -C^2 I(t) (I(t) - D(t)).
\]
Dividing both sides by $I^2(t)$ gives the first claim. The second follows from the first. \[\Box\]

This leads to the following generalization of Corollary $(0.7)$

**Corollary 2.10.** If $u : M \times [a, b] \rightarrow \mathbb{R}^N$ satisfies $(0.11)$ then
\[
I(b) \geq I(a) \exp \left( (b - a) \left( 2 + \sup_{[a,b]} C \right) \left[ \exp \left( \int_a^b C^2(s) \, ds \right) [U(a) - 1] + 1 - \frac{3}{2} \sup_{[a,b]} C \right] \right).
\]
In particular, if $u(\cdot, b) = 0$, then $u \equiv 0$. 


Proof. It follows from (2.5), the Cauchy-Schwarz inequality and the elementary inequality \( a \leq \frac{1}{2} (a^2 + 1) \) applied to \( a = \sqrt{-U} \) that

\[
(\log I)' \geq 2U - \frac{C}{I} \int |u| (|u| + |\nabla u|) e^{-\phi} \geq 2U - C(1 + \sqrt{-U})
\]

(2.11)

\[
\geq \left( 2 + \frac{C}{2} \right) U - \frac{3C}{2}.
\]

From this we get that

\[
\log I(b) - \log I(a) = \int_a^b (\log I)'(s) \, ds
\]

(2.12)

\[
\geq \frac{1}{2} \left( 4 + \sup_{[a,b]} C \right) \int_a^b U(s) \, ds - \frac{3}{2} \sup_{[a,b]} C (b - a).
\]

From (2.3) we get that for \( s \in [a, b] \)

\[
\log(1 - U(s)) \leq \log(1 - U(a)) + \int_a^s C^2(r) \, dr \leq \log(1 - U(a)) + \int_a^b C^2(s) \, ds.
\]

(2.13)

Therefore

\[
U(s) \geq \exp \left( \int_a^s C^2(s) \, ds \right) (U(a) - 1) + 1.
\]

(2.14)

Inserting this lower bound in (2.12) and integrating gives

\[
\log I(b) - \log I(a) \geq (b - a) \left[ \exp \left( \int_a^b C^2(s) \, ds \right) [U(a) - 1] + 1 - \frac{3}{2} \sup_{[a,b]} C \right].
\]

Recall that that \( u : M \times (a, \infty) \to \mathbb{R}^N \) vanishes to infinite order at \( \infty \) if \( \lim_{t \to \infty} e^{ct} I(t) = 0 \) for all constants \( c \). Theorem 2.10 implies the following strong unique continuation at \( \infty \):

**Corollary 2.15.** Suppose that \( \sup C + \int_a^\infty C^2(s) \, ds < \infty \) and \( u : M \times (a, \infty) \to \mathbb{R}^N \) is a solution of (0.11) that vanishes to infinite order at \( \infty \), then \( u \) vanishes.

This corollary implies the unique continuation of Poon, [P], who considered functions \( u \) on \( \mathbb{R}^n \) into \( \mathbb{R} \) with

\[
u_i - \Delta u = \langle b(x, t), \nabla u \rangle + c(x, t) u,
\]

where \( |b| + |c| \leq C \) is uniformly bounded (cf. [L]). We will see that the results here apply more generally to functions \( u \) satisfying the differential inequality

\[
|u_t - \Delta u| \leq C (|u| + |\nabla u|).
\]

(2.17)

Applying the transformation in Lemma 1.10 to \( u \), we get a function \( w(y, s) \) with

\[
\left| \left( \partial_s - L_{[\cdot]^{1/2}} \right) w \right| = e^{-s} \left| (\partial_t - \Delta) u \right| \leq C e^{-s} (|u| + |\nabla u|)
\]

(2.18)

\[
\leq C e^{-s}|w| + C e^{-\frac{s}{2}} |\nabla w|.
\]
Since $\int_{0}^{\infty} e^{-s} ds < \infty$, Corollary 2.15 applies. Exponential decay of order $c$, i.e., decay like $e^{-cs}$, corresponds to polynomial decay $t^c$ in the transformed variable $t = -e^{-s}$.

2.1. **Without $\omega$ term.** In this subsection we assume that $u : M \times [a, b] \to \mathbb{R}^N$ satisfies

$$|(\partial_t - L_{\omega}) u| \leq C(t)|\nabla u|. \quad (2.19)$$

In this case we get better estimates when $U(a)$ is small. It follows from (2.9), with obvious simplifications in the second to last inequality from using (2.19) in place of (0.11), that $U' \geq C^2 U$ or, equivalently,

$$[\log(-U)]' \leq \frac{C^2}{2}. \quad (2.20)$$

We therefore get that

$$U(s) \geq U(a) \exp\left(\frac{1}{2} \int_{a}^{s} C^2(\tau) \, d\tau\right). \quad (2.21)$$

With similar simplifications in (2.11) we get that for $s \in [a, b]$

$$[\log I]' \geq 2 U - C \sqrt{-U} \quad (\log I)' \geq 2 U(\omega) \exp\left(\frac{1}{2} \int_{a}^{b} C^2(\tau) \, d\tau\right) - C \sqrt{-U(\omega)} \exp\left(\frac{1}{4} \int_{a}^{b} C^2(\tau) \, d\tau\right). \quad (2.22)$$

Integrating gives

$I(b) \geq I(a) \exp\left[(b - a) \left\{2 U(\omega) \exp\left(\frac{1}{2} \int_{a}^{b} C^2(\tau) \, d\tau\right) - C \sqrt{-U(\omega)} \exp\left(\frac{1}{4} \int_{a}^{b} C^2(\tau) \, d\tau\right)\right\}\right].$

**References**

[A] F. Almgren, Jr., *Q*-valued functions minimizing Dirichlet’s integral and the regularity of area minimizing rectifiable currents up to codimension two*, preprint.

[CM1] T.H. Colding and W.P. Minicozzi II, *Harmonic functions with polynomial growth*, J. Diff. Geom., v. 46, no. 1 (1997) 1–77.

[CM2] T.H. Colding and W.P. Minicozzi II, *Heat equations in analysis, geometry and probability*, to appear.

[CM3] T.H. Colding and W.P. Minicozzi II, *Generic mean curvature flow I; generic singularities*, Annals of Math., 175 (2012), 755–833.

[ESS] L. Escauriaza, G. Seregin and V. Sverak, *Backward uniqueness for parabolic equations*, Archive for Rational Mechanics and Analysis, (2003) volume 169, 147–157.

[GL] N. Garofalo and F.H. Lin, *Monotonicity properties of variational integrals, $A_p$ weights and unique continuation*, Indiana Univ. Math. J. 35 (1986), no. 2, 245–268.

[H1] R.S. Hamilton, *A matrix Harnack estimate for the heat equation*, CAG, 1 (1993) 88–99.

[H2] R.S. Hamilton, *Monotonicity formulas for parabolic flows on manifolds*, CAG, 1 (1993) 100–108.

[L] F.H. Lin, *A uniqueness theorem for parabolic equations*, Comm. Pure Appl. Math. (1988) 42, 125–136.

[Lo] A. Logunov, *Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure*. Ann. of Math. (2) 187 (2018), no. 1, 221–239.

[P] C.C. Poon, *Unique continuation for parabolic equations*. Comm. PDEs (1996) 21, 521–539.

[W] L. Wang, *Uniqueness of self-similar shrinkers with asymptotically conical ends*. J. Amer. Math. Soc. 27 (2014), no. 3, 613–638.

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