Planar Thirring Model in the U(2N)-symmetric limit

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Abstract

I review the Thirring model in 2+1d dimensions, focussing in particular on possible strongly-interacting UV-stable fixed points of the renormalisation group, corresponding to a continuous phase transition where a U(2N) global symmetry spontaneously breaks to U(N)⊗U(N). Since there is no small parameter in play, a systematic non-perturbative approach such as numerical simulation of lattice field theory is mandated. I compare and contrast various formulations, paying particular attention to models formulated with either staggered or domain wall lattice fermions. Domain wall fermions, which faithfully capture U(2N) symmetry in the limit of wall separation $L_s \rightarrow \infty$, predict a critical flavor number $1 < N_c < 2$.

1 Introduction

Most of our understanding of quantum field theory derives from theories in which interactions are relatively weak, and it makes sense to envisage elementary processes in terms of well-localised particles. A powerful diagrammatic calculus has been developed to treat this situation both quantitatively and conceptually – the shorthand for this being “perturbation theory”. Even in cases such as the strong interaction where perturbation theory fails to capture the essential physics, our calculational techniques, often the numerical simulation of euclidean lattice field theory, rely on there being a well-understood path to a perturbative continuum limit; for QCD this is guaranteed by asymptotic freedom.

In general, neither weak interactions nor well-localised particle degrees of freedom are necessary ingredients for a continuum quantum field theory. Our modern perspective is that QFTs exist as fixed points of the renormalisation group, where the ratio of the physical scale $\mu$ to some UV regulator scale $\Lambda$ can be made arbitrarily small by a suitable tuning of parameters $\beta \rightarrow \beta^*$;
such that \((\mu/\Lambda) \propto (\beta - \beta^*)^\nu\), where \(\nu > 0\) is one of a set of interrelated critical exponents characterising the continuum theory. In this way predictions are essentially independent of the regularisation details. Even within this framework it can be challenging to calculate accurately, especially in the absence of a small dimensionless expansion parameter. Lattice field theory simulation relies on no small parameters; however, there may still be barriers to complete control to be overcome, particularly for theories involving fermions. The lattice provides a natural regularisation for quantities expressible in terms of differential forms, but theories containing a single relativistic fermion species do not fall in this class \[^1\].

2 The Thirring Model

The Thirring model in 2+1\(d\) has Lagrangian density

\[
\mathcal{L} = \bar{\psi}(\partial + m)\psi + \frac{g^2}{2N}(\bar{\psi}\gamma_\mu\psi)^2.
\]

(1)

Here \(\psi\) is a spinor in a reducible representation of the Clifford algebra, meaning that it is acted on by 4 \(\times\) 4 Dirac matrices \(\gamma_\mu\) satisfying \(\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\) in euclidean metric. The index \(i = 1, \ldots N\) runs over \(N\) distinct species of relativistic fermion. The interaction between conserved current densities \(i\bar{\psi}\gamma_\mu\psi\) implies, as in electrodynamics, that opposite charges attract, like charges repel. Some example motivations for a model of this genre are the description of nodal fermions in \(d\)-wave superconductors \[^2\], \[^3\], spin-liquid phases of Heisenberg antiferromagnets \[^4\], \[^5\], certain correlated Chern insulators amenable to quantum simulation with ultra-cold atoms \[^6\], and low-energy electronic excitations in graphene \[^7\], \[^8\].

Our approach to understanding the model \(^1\) at strong coupling is rooted in its symmetries. In 2+1\(d\) the reducible representation of the Dirac algebra contains two elements \(\gamma_3\) and \(\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3\) which anticommute with the kinetic term in \(^1\). For mass \(m \to 0\), the following spinor rotations generate a U\( (2N)\) symmetry \(^2\) with \(\alpha_i\) a Hermitian generator of U\( (N)\):

\[
\psi \mapsto e^{i\alpha_1}\psi; \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\alpha_1}; \quad \psi \mapsto e^{i\alpha_3\gamma_3\gamma_5}\psi; \quad \bar{\psi} \mapsto \bar{\psi}e^{-i\alpha_3\gamma_3\gamma_5};
\]

(2)

\[
\psi \mapsto e^{i\alpha_5\gamma_5}\psi; \quad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha_5\gamma_5}; \quad \psi \mapsto e^{i\alpha_3\gamma_3}\psi; \quad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha_3\gamma_5}.
\]

(3)

Once \(m \neq 0\), only \(^2\) remain valid; in a gapped theory the symmetry is therefore broken to U\( (N)\)\( \otimes\) U\( (N)\). A second important symmetry frequently motivated by condensed-matter applications is symmetry under inversion

\[^1\]To demonstrate this using infinitesimal rotations it is convenient to define \(\tilde{\psi} = \bar{\psi}\gamma_3\gamma_5\)
of spatial axes – in a particle physics context this is equivalent to parity, arising from inversion of an odd number of euclidean axes; our convention is \( x_\mu \mapsto -x_\mu, \mu = 0, 1, 2 \). For reducible spinors there are two inequivalent parity flips:

\[
P_3 : \psi(x) \mapsto \gamma_3 \psi(-x) \quad \tilde{\psi}(x) \mapsto \tilde{\psi}(-x) \gamma_3
\]

(4)

\[
P_5 : \psi(x) \mapsto \gamma_5 \psi(-x) \quad \tilde{\psi}(x) \mapsto \tilde{\psi}(-x) \gamma_5.
\]

(5)

We thus identify the full global symmetry of (1) with \( m = 0 \) as \( U(2^N) \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \).

Three mass terms break \( U(2^N) \to U(N) \otimes U(N) \):

\[
++ m_h \bar{\psi} \psi; \quad + - im_3 \bar{\psi} \gamma_3 \psi; \quad - + im_5 \bar{\psi} \gamma_5 \psi,
\]

(6)

where the \( \pm \) indicate parity under \( P_{3,5} \) flips. In graphene, the ++ mass corresponds to a charge density wave in which electrons preferentially sit on one of two sub-lattices on the honeycomb [9], whereas a linear combination of +− and −+ yields a bond density wave in which electrons are distributed on both sublattices in a Kekulé texture [10]. Spontaneous breaking of the continuous symmetry is accompanied by \( 2N^2 \) Goldstones, \( N^2 \) with \( J^P = 0^- \) and \( N^2 \) \( J^P = 0^+ \), with \( P \) the unbroken parity. Finally, note that there is another possible mass term:

\[
- - m_H \bar{\psi} \gamma_3 \gamma_5 \psi,
\]

(7)

not related to (6) by any \( U(2N) \) rotation. This is the non-time-reversal invariant Haldane mass realised in graphene-like systems by alternately circulating currents in adjacent half-unit cells [11]; we will not consider it further.

The Thirring model is often analysed by introducing a vectorlike auxiliary boson field \( A_\mu \), and (1) replaced by the equivalent form

\[
\mathcal{L}' = \bar{\psi}_i (\partial^\mu + ig \sqrt{N} A^\mu + m) \psi_i + \frac{1}{2} A_\mu A_\mu.
\]

(8)

In an expansion in powers of \( N^{-1} \), the leading order quantum correction arises in the auxiliary two-point function \( D_{\mu\nu} \), due to a diagram analogous to vacuum polarisation in electrodynamics. The result is UV-finite if the regularisation respects current conservation [12, 13]:

\[
D_{\mu\nu}(k) = \frac{\mathcal{P}_{\mu\nu}(k)}{1 - \Pi(k^2)} + \frac{k_\mu k_\nu}{k^2},
\]

(9)

where \( \mathcal{P}_{\mu\nu}(k) = \delta_{\mu\nu} - k_\mu k_\nu/k^2 \) is the transverse projector, and in \( d \) spacetime dimensions

\[
\Pi(k^2) = -g^2 \frac{4\Gamma(2 - \frac{d}{2})}{3(4\pi)^{\frac{d}{2}} m^{4-d}} k^2 F(2; 2 - \frac{d}{2}; \frac{5}{2}; -\frac{k^2}{4m^2}).
\]

(10)
Asymptotically,
\[ \lim_{k^2 \to \infty} \Pi(k^2) = -g^2 \left( k^2 \right)^{d/2-1} \frac{P (k^2)}{A_d} \]  
(11)
with \( A_d \) a numerical constant taking the value 8 for \( d = 3 \). Now, if we restrict attention to \( A \bar{\psi} \psi \) vertices involving a conserved fermion current, as in a gauge theory, then the longitudinal component of (9) is physically irrelevant, and hence the UV asymptotics of diagrams contributing to higher order corrections, following (11), is controlled by \( D_{\mu \nu} \propto P_{\mu \nu}/g^2 k^{d-2} \). The outcome is that the Thirring model is power-counting renormalisable in a continuous range of dimensions \( d \in (2, 4) \), with only logarithmic divergences which are absorbed in fermion wavefunction and mass renormalisations. The coupling \( g^2 \) requires no further renormalisation, the model being governed by the dimensionless combination \( m^{d-2} g^2 \). A key factor in assessing the model’s dynamics is the ratio \( M_{V}/m \) where the mass of the vector bound state \( M_{V} \) is given by the physical pole of (9) \[ \langle \bar{\psi} \psi \rangle \neq 0 \] and the consequent dynamical generation of a massgap \( \Sigma \), essentially because all Feynman diagrams contributing to the signal vanish due to Furry’s theorem, just as in QED. \[ G \] However, non-perturbative approaches suggest bilinear condensation may occur for sufficiently large \( g^2 \) and/or small \( N \). For instance self-consistent solution of the Schwinger-Dyson (SD) equation, using the bifurcation method and approximating the full vector propagator by (9), finds \( \Sigma \neq 0 \) in the strong-coupling limit \( g^2 \to \infty \) for \( N < N_c = 128/3\pi^2 \approx 4.32 \) \[ 15 \]. The critical line extends into the plane with \( g^2_c (N < N_c) \) a monotonically decreasing function \[ 16 \]. Alternative approximations to the exact SD equations find \( N_c = 32/\pi^2 \approx 3.24 \) \[ 12 \], \( N_c = 2 \) \[ 17 \], or even \( N_c = \infty \) \[ 18 \].

This unexpected renormalisability may not be the end of the story – the \( 1/N \) expansion has nothing to say about possible ground states in which U(2\( N \)) is spontaneously broken by the generation of a bilinear condensate such as \( \langle \bar{\psi} \psi \rangle \neq 0 \) and the consequent dynamical generation of a massgap \( \Sigma \), essentially because all Feynman diagrams contributing to the signal vanish due to Furry’s theorem, just as in QED. \[ G \] However, non-perturbative approaches suggest bilinear condensation may occur for sufficiently large \( g^2 \) and/or small \( N \). For instance self-consistent solution of the Schwinger-Dyson (SD) equation, using the bifurcation method and approximating the full vector propagator by (9), finds \( \Sigma \neq 0 \) in the strong-coupling limit \( g^2 \to \infty \) for \( N < N_c = 128/3\pi^2 \approx 4.32 \) \[ 15 \]. The critical line extends into the plane with \( g^2_c (N < N_c) \) a monotonically decreasing function \[ 16 \]. Alternative approximations to the exact SD equations find \( N_c = 32/\pi^2 \approx 3.24 \) \[ 12 \], \( N_c = 2 \) \[ 17 \], or even \( N_c = \infty \) \[ 18 \].

In the context of graphene, the existence of a symmetry-breaking phase transition separating a conducting semi-metal from a Mott insulator has
technological significance; if undoped graphene were an insulator the fabrication of fast graphene-based switches with a stable “off” state would be possible, with the potential for a new generation of fast processors. For monolayer graphene the physical value \( N = 2 \) 4-spinors, corresponding to 2 atoms/unit cell on the honeycomb lattice × 2 distinct “Dirac points” (where the gap closes linearly) in the first Brillouin Zone × 2 electron spin states. Unfortunately simulations of graphene using a realistic Hamiltonian appear to preclude this possibility [19]. However, the question also has intrinsic theoretical interest. If the transition is second-order as suggested by the self-consistent approaches, it suggests a correlation length \( \xi \sim \Lambda/\Sigma \) diverging as \( g^2 \to g_c^2(N) \) and the possibility of an interacting theory in the continuum limit, described by a UV-stable RG fixed point as set out above. We could also describe this as a Quantum Critical Point, with a distinct QCP for every integer \( N < N_c \). Since there are no small dimensionless parameters in play, elucidation of the QCP properties, and even determining the value of \( N_c \), requires essentially non-perturbative calculational techniques. In addition to SD, the Functional Renormalisation Group (FRG) has also been applied to this question [20]. In the remainder of this article, however, I will focus on lattice field theory.

3 The Staggered Thirring Model

In any question of dynamical fermion mass generation, the natural starting point on the lattice is the staggered formulation, involving single-component Grassmann fields \( \chi, \bar{\chi} \) living on the sites \( x \) of a cubic lattice. These are technically straightforward to implement and have a \( U(N) \otimes U(N) \) global symmetry protecting massless fermions from acquiring a gap in perturbation theory. The version of the Thirring model which has been most studied has action [21]

\[
S_{stagg} = \frac{1}{2} \sum_{x, \mu} \bar{\chi}^i_x \eta_{\mu x} (1 + i A_{\mu x}) \chi^i_{x+\hat{\mu}} - \bar{\chi}^i_x \eta_{\mu x} (1 - i A_{\mu x-\hat{\mu}}) \chi^i_{x-\hat{\mu}} \\
+ m \sum_x \bar{\chi}^i_x \chi^i_x + \frac{N}{4g^2} \sum_{x, \mu} A_{\mu x}^2. 
\]

(14)

This is recognisably of the same form as [8] with Dirac matrices replaced by Kawamoto-Smit phases \( \eta_{\mu x} \equiv (-1)^{x_0+\cdots+x_{\mu-1}} \). In the absence of interactions staggered fermions recover the continuum action of \( N_f \) reducible flavors in the long-wavelength limit, with \( N_f = 2N \) [22]. The vector auxiliary field \( A_{\mu} \) is defined on the links joining the sites, just like a lattice gauge field,
with the important distinction that the resulting link field is unbounded in magnitude and accordingly not unitary. Lattice formulations with compact auxiliary fields have also been explored \cite{23,24,25}. The formulation \cite{14} has the virtue that integration over $A_\mu$ recovers a fermion interaction term containing no higher than four-point couplings.

The phase diagram obtained using the staggered lattice action \cite{14} showing the boundary between semimetal and insulator phases, is shown in Fig. 1. Non-integer values of $N$ were simulated using a hybrid R algorithm which employs a version of rooting. The figure also contains a point obtained in the strong-coupling limit \cite{26} corresponding to a critical flavor number

$$N_{fc} = 6.6(1)$$  \hspace{1cm} (15)

which is at least of the same order as the SD estimates. In contrast to SD predictions however, the critical properties of the QCPs are very sensitive to $N$; in particular the exponent $\delta$ characterising the order parameter response to a small bare mass at criticality varies from $\delta = 2.75(9)$ at $N_f = 2$ \cite{21}, to $\delta = 6.90(3)$ at $N_f = N_{fc}$ \cite{26}. The vertical dashed line marks the estimated location of the strong coupling limit, which does not coincide with $g^{-2} = 0$. Empirically it corresponds to the location of a maximum in the order parameter data $\langle \bar{\psi}\psi(g^2) \rangle$, which is unexpectedly non-monotonic. A possible origin \cite{21} is that the lattice action \cite{14} does not have an exactly conserved fermion current, as a consequence of the non-compactness of the link variable. In the $N^{-1}$ expansion this results in a divergent contribution $\sim g^2 a^{-1}\delta_{\mu\nu}$ to the vacuum polarisation, which must be removed by an additive renormalisation of $g^{-2}$. The unphysical region to the left with $g_{R}^{-2} < 0$ has $D_{\mu\nu}$ negative, corresponding to a violation of positivity.

Overall, the strong $N$-dependence of the staggered Thirring model is striking, and the qualitative resemblance of Fig. 1 to SD predictions \cite{16}
is encouraging. However, the case is not yet closed. The minimal model (14) with \( N = 1 \) (corresponding to \( N_f = 2 \) in Fig. 1) has been studied using a fermion bag algorithm [27], which unlike the workhorse HMC algorithm used elsewhere permits Monte Carlo sampling in the massless limit \( m = 0 \), and hence a much cleaner estimate of critical properties. The exponents obtained are compatible with the older HMC values [21]; however they are also indistinguishable from those of the minimal 2+1d Gross-Neveu model with U(1) global symmetry [28]. When the lattice action is expressed purely in terms of fermion fields \( \chi, \bar{\chi} \) this is not surprising: interactions occur among staggered fields distributed around the vertices of an elementary cubic cell, and the two models differ only by a (presumably irrelevant) body-diagonal term. Only once auxiliary bosons are introduced, as in both \( N^{-1} \) expansion and HMC algorithm, do the models look different. Since the GN auxiliary is scalar/pseudoscalar, symmetry breaking corresponds to auxiliary condensation, is captured by the \( N^{-1} \) expansion [29], and occurs for all \( N \). While Fig. 1 suggests the staggered Thirring and GN models for arbitrary \( N \) are distinct, it is a priori hard to understand why the corresponding continuum models should coincide for \( N \) minimal.

The resolution is that the symmetry-breaking pattern of the staggered model (14) is \( U(N) \otimes U(N) \rightarrow U(N) \), whereas that of the continuum model (8) is \( U(2N) \rightarrow U(N) \otimes U(N) \). Only in the weak coupling limit is the continuum form recovered using \( N_f = 2N \) [22]; at a generic QCP we cannot expect the “taste symmetry restoration” anticipated in the continuum limit of lattice QCD, and must conclude that the QCPs of the staggered lattice Thirring model lie in a different universality class to the continuum model (1). The consequences are in principle profound, as the following non-rigorous argument demonstrates. Observe that according to the large-\( N \) prediction (13), the mass of the vector boson vanishes in the strong coupling limit. At this point the Thirring model becomes a theory of conserved currents interacting via exchange of a massless vector, i.e. QED\(_3\), so it is plausible that the critical \( N_c \) required for symmetry breaking coincides with \( N_c \) defining the IR conformal fixed point of QED\(_3\) [30]. Next, there is an old conjecture [31] that symmetry breaking in any theory is constrained by the relation

\[
 f_{IR} \leq f_{UV},
\]

where \( f \) is proportional to the negative of the thermodynamic free energy density, and essentially counts the number of light degrees of freedom. For an asymptotically-free theory like QED\(_3\), the count in the UV is the number of spinor degrees of freedom multiplied by a factor \( \frac{3}{4} \) associated with Fermi-Dirac statistics in 2+1d. Assuming a gap-generating symmetry breaking, the IR count is the number of Goldstones, \( N^2 \) for \( U(N) \otimes U(N) \rightarrow U(N) \), and
2N^2 for U(2N) \rightarrow U(N) \otimes U(N). In either limit the count is supplemented by 1 for the photon, which is massless in each phase. Application of (16) then predicts

\[ N^2 \leq \frac{3}{4} \times 2^3 N \Rightarrow N_c \leq 6 \Rightarrow N_{fc} = 12 \text{ staggered;} \quad (17) \]

\[ 2N^2 \leq \frac{3}{4} \times 4N \Rightarrow N_c \leq \frac{3}{2} \text{ continuum.} \quad (18) \]

The big disparity between (17) and (18) hints that capture of the correct symmetry is key to correctly modelling the QCP and predicting \( N_c \).

4 The DWF Thirring Model: Formulation

In recent years, in an attempt to overcome the shortcomings of the staggered model, two alternative lattice fermion formulations have been investigated. One uses the SLAC derivative \[32\], in which global symmetries are manifest while the fermion kinetic bilinear is non-local to suppress the influence of species doublers at the Brillouin Zone edge. The usual arguments that the SLAC derivative leads to non-covariant divergent terms \[33\] do not apply in this case because there is no gauge-invariance requirement forcing a delocalised fermion-boson interaction; rather the auxiliary fields \( A_\mu \) are defined on sites \[34\]. The other approach is motivated by the resemblance of the fermion-boson interaction in (8) to that of a gauge theory, and uses the domain wall fermion (DWF) originally developed to accurately capture flavor symmetries in the chiral limit of lattice QCD.

DWF are formulated on a 2+1+1d lattice in which open boundaries separated by a distance \( L_s \) are imposed in the fictitious third direction \( x_3 \). If we write the Lagrangian density

\[ \mathcal{L}_{\text{DWF}} = \bar{\Psi}(x,s) D_{x,y,s} \Psi(y,s'), \quad (19) \]

with \( s \) the \( x_3 \)-coordinate, then as \( L_s \rightarrow \infty \), near zero-modes of \( D_{\text{DWF}} \) are localised on “domain walls” at \( x_3 = 0, L_s \) as approximate ± eigenmodes of \( \gamma_3 \). Physical fields in the 2+1d target space are then defined by

\[ \psi(x) = P_- \Psi(x,1) + P_+ \Psi(x,L_s); \quad \bar{\psi}(x) = \bar{\Psi}(x,L_s) P_- + \bar{\Psi}(x,1) P_+, \quad (20) \]

with projectors \( P_\pm = \frac{1}{2}(1 \pm \gamma_3) \). As originally shown by Kaplan \[35\], if \( D_{\text{DWF}} \) has the same form as the Wilson lattice derivative, then species doublers are not present in the low-energy spectrum. What about the U(2N) symmetry? While the physical degrees of freedom have readily-identifiable components
which are eigenstates of $\gamma_3$, the full symmetry generated by $(2,3)$ requires that $L_{DWF}$ also has well-controlled behaviour under field rotations generated by $\gamma_5$. The resolution, mirroring the steps originally developed for theories in $3+1d$, is that $D_{DWF}$ is a finite-$L_s$ regularisation of a fermion operator $D$ satisfying the \textit{Ginsparg-Wilson} relations

$$\{\gamma_3, D\} = 2D\gamma_3D; \quad \{\gamma_5, D\} = 2D\gamma_5D; \quad [\gamma_3\gamma_5, D] = 0. \quad (21)$$

Formally the RHS of the GW relations (21) is $O(aD)$, so $U(2N)$ is recovered in the long-wavelength limit provided $D$ is sufficiently local. An overlap operator $D^{ov}$ satisfying (21) but not manifestly local in $2+1d$ can be constructed.

DWF mass terms only involve fields defined on the walls. The equivalents of (6) read:

$$m_h\bar{\Psi}(x, L_s)P_-\Psi(x, 1) + \bar{\Psi}(x, 1)P_+\Psi(x, L_s)$$

$$im_3\bar{\Psi}(x, L_s)\gamma_3P_-\Psi(x, 1) + \bar{\Psi}(x, 1)\gamma_3P_+\Psi(x, L_s)$$

$$im_5\bar{\Psi}(x, 1)\gamma_5P_-\Psi(x, 1) + \bar{\Psi}(x, L_s)\gamma_5P_+\Psi(x, L_s). \quad (22)$$

Note that while $m_{h,3}$ couple fields on opposite walls as in $3+1d$, $m_5$ couples fields on the same wall. Nonetheless, pilot studies with quenched QED$_3$ show that bilinear condensates of the same form as (22) coincide in the limit $L_s \to \infty$ as required by $U(2N)$ symmetry, and moreover the 3 and 5 condensates are numerically indistinguishable. Significantly, finite-$L_s$ corrections are considerably smaller in the 3,5 channels than for $h$; accordingly in subsequent work we focus on $i\langle \bar{\psi}\gamma_3\psi \rangle$ rather than $\langle \bar{\psi}\psi \rangle$. When simulating full fermion dynamics with DWF, the impact of modes propagating in the bulk $0 < x_3 < L_s$ must be decoupled using suitably chosen bosonic Pauli-Villars fields.

For a $2+1d$ theory the key relation is

$$\frac{\det D_{DWF}(m_i)}{\det D_{DWF}(m_h = 1)} = \det D_{L_s}(m_i) \quad \text{with} \quad \lim_{L_s \to \infty} D_{L_s}(m_i) = D^{ov}(m_i). \quad (23)$$

Even after restricting our consideration to formulations with a non-compact auxiliary link field, there are still different plausible choices for the DWF-auxiliary interaction term. The so-called \textit{Surface} formulation confines the interaction to $\Psi, \bar{\Psi}$ defined on the walls, ie.

$$S_{surf} = \frac{i}{2} \sum_{x, \mu} A_{\mu x} [\bar{\Psi}_{x+\mu} P_+ \Psi_{x+\mu} + \bar{\Psi}_{xL_s} \gamma_\mu P_+ \Psi_{x+\mu L_s}] - \text{h.c.} \quad (24)$$

This is by analogy with the treatment of the scalar auxiliary in the Gross-Neveu model formulated with DWF, simulation of which captures a continuous symmetry-breaking phase transition perfectly adequately. A
technical advantage is that the Pauli-Villars determinant in the denominator of (23) does not depend on $A_{\mu}$, so can be ignored in the simulation. By contrast the Bulk formulation treats $1 + iA_{\mu}$ as a gauge connection, albeit non-unitary. Accordingly the auxiliary interacts with all fermion fields including those in the bulk:

$$S_{\text{bulk}} = \frac{i}{2} \sum_{x,\bar{s}} A_{\mu} [\bar{\psi}_{x\bar{s}} (-1 + \gamma_{\mu}) \psi_{x+\bar{\mu}s}] + A_{\mu x - \bar{\mu}} [\bar{\psi}_{x\bar{s}} (1 + \gamma_{\mu}) \psi_{x-\bar{\mu}s}] + \hat{\mu}_{s}. \quad (25)$$

The bulk formulation is much more costly to simulate, in part due to to the non-unitary nature of the link fields, and in part because in the minimal $N = 1$ model there is a slight obstruction to proving positivity of the fermion determinant, so the results to be reviewed below require the RHMC algorithm simulating functional weight $\det(M S_{\text{YM}} M^\dagger)^{\frac{1}{2}}$ [44]. Schematically writing $D_{\text{DWF}} = D_W + D_3$, where $D_W$ resembles an orthodox 2+1d Wilson fermion derivative, then it’s important to note that the property $\gamma_3 D_W \gamma_3 = \gamma_5 D_W \gamma_5 = D_W$ needed in the demonstration [40] of (23) does not require unitary link fields.

5 The DWF Thirring Model: Numerical Results

We now review results of simulations with the Thirring model defined with DWF fermions. Fig. 2 shows the bilinear condensate $\langle \bar{\psi}\psi \rangle$ (actually $\langle \bar{\psi}\gamma_3 \psi \rangle$ as commented above) with $ma = 0.01$ on a $12^3 \times L_s$ as a function of inverse coupling for models with $N = 2$ ($L_s = 16$) [42], and $N = 1$ ($L_s = 8$) [44], using both surface and bulk formulations. The bulk signal is much larger than that for surface, and for $N = 1$ over $N = 2$; indeed for all cases.
examined with $N = 2$, $\langle \bar{\psi} \psi \rangle$ varies linearly with fermion mass $m$ implying U(2$N$) symmetry remains unbroken. The non-monotonic variation of the data suggests strong-coupling artifacts just as found for the staggered model.

The extrapolation $L_s \rightarrow \infty$ is crucial in identifying any possible symmetry breaking. Empirically data is well-fitted by an exponential Ansatz:

$$\langle \bar{\psi} \psi \rangle_{L_s \rightarrow \infty} - \langle \bar{\psi} \psi \rangle_{L_s} = C(m, g^2) e^{-\Delta(m, g^2)L_s},$$

which we have tested up to $L_s = 48$, where the decay constant $\Delta \sim O(10^{-2})$ at the strongest couplings. It is also important to test the recovery of U(2$N$) in the same limit, using the dominant residual

$$\delta_h(L_s) = \text{Im} \langle \bar{\Psi}(x, 1) i \gamma_3 \Psi(x, L_s) \rangle = -\text{Im} \langle \bar{\Psi}(x, L_s) i \gamma_3 \Psi(x, 1) \rangle,$$

which empirically matches $\langle \bar{\psi}(1 - i \gamma_3)\psi \rangle$ closely. This is shown as a function of $L_s$ in Fig. 3(a) and of inverse coupling $\beta = ag^{-2}$ at fixed $L_s = 48$ in Fig. 3(b). It appears that $\delta_h$ dwindles away only very slowly as $L_s \rightarrow \infty$;

![Figure 3](image.png)

(a) $\delta_h(\beta, m)$ on $12^3 \times L_s$

(b) $\delta_h(\beta, m)$ on $16^3 \times 48$

Figure 3:

moreover at stronger couplings the residual also grows as $m \rightarrow 0$. The DWF approach to the GW limit defined by \cite{21} will be numerically challenging.

As in any numerical approach involving an auxiliary field on a finite system, we are hampered because the bilinear condensate order parameter vanishes identically for $m \rightarrow 0$, and need to base a search for a critical point on data generated with $m \neq 0$. The approach is to fit $\langle \bar{\psi} \psi(g^2, m) \rangle$ to a renormalisation group-inspired equation of state \cite{21}

$$m = A(\beta - \beta_c) \langle \bar{\psi} \psi \rangle^{\delta - \frac{1}{m}} + B \langle \bar{\psi} \psi \rangle^{\delta},$$

where $\beta_m$ and $\delta$ can be identified with conventional critical exponents:

$$\langle \bar{\psi} \psi \rangle \propto (\beta_c - \beta)^{\beta_m}; \quad \langle \bar{\psi} \psi(\beta_c) \rangle \propto m^{\frac{1}{2}}.$$
Fig. 4 shows such a fit from a $16^3 \times 48$ system. The fitted data has $\beta \in [0.32, 0.52]$ and $m a \in [0.005, 0.05]$, and the resulting critical parameters are $\beta_c = 0.2601(4)$, $\beta_m = 0.413(15)$ and $\delta = 3.44(9)$. These values are compatible with fits to data extrapolated to $L_s \to \infty$ using (26), which are theoretically better-founded, but technically more challenging [45]. A comparison of EoS-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Fits to the Equation of State (28) on $16^3 \times 48$.}
\end{figure}

fitted critical parameters at fixed $L_s$ with results of the $L_s$-extrapolated data is shown for $\beta_c$ in Fig. 5(a) and for the exponents in Fig. 5(b). It is also apparent that finite volume corrections assessed via comparison of system sizes $12^3$ and $16^3$ are rather small as $L_s \to \infty$.

In summary, while Fig. 4 shows data at strong couplings, including from the hypothesised U($2N$)-broken phase, are not well-described by the EoS (28) (we ascribe the mismatch to large finite-$L_s$ corrections in the broken phase, which an ongoing simulation campaign is aiming to control), there is by now rather strong evidence that U($2N$) is broken spontaneously at strong couplings for $N = 1$. The absence of evidence for broken symmetry
for \( N = 2 \) leads to the prediction
\[
1 < N_c < 2.
\] (30)

The contrast with the staggered result (15) fully justifies the investment in DWF. The universality class of the QCP is characterised by critical exponents with estimated values (45) as \( L_s \to \infty \)
\[
\beta_m = 0.31(2); \quad \delta = 4.3(2).
\] (31)

These differ significantly from the values \( \beta_m = 0.57(2), \delta = 2.75(9) \) or \( \beta_m = 0.70(1), \delta = 2.64(3) \) found for the minimal staggered model. So-called hyperscaling relations can be derived on the assumption that dynamics near a QCP is characterised by a single diverging length scale \( \xi \). If this is also applicable to fermion models (29), then we can further deduce values for the exponent \( \nu \) defined by \( \xi \propto |\beta_c - \beta|^{-\nu} \) and \( \eta \) describing critical correlations of the order parameter field \( \langle \psi(0)\overline{\psi}(r) \rangle \propto r^{-\left(d-2+\eta\right)} \). The outcome is
\[
\nu = \frac{1}{d}\beta_m(d+1) = 0.55(3); \quad \eta = \frac{(d+2) - (d-2)\delta}{\delta + 1} = 0.13(4),
\] (32)
again in clear distinction to the minimal staggered Thirring model (27). This is evidence that the DWF model with \( N = 1 \) supports a novel QCP.

In order to ascertain that the GW symmetry (21) actually coincides with \( U(2N) \) it is necessary to establish the locality of the overlap operator \( D^{\text{ov}} \) associated with our DWF implementation. Following a similar analysis for lattice QCD (19), Fig. 6(a) (45) plots \( \max\{||\phi(x)||_2 : ||x - y||_1 = r\} \) for a vector \( \phi \) obtained for various \( \beta \) via \( \phi = D^{\text{ov}} \eta \) with \( \eta \) a delta-function source sited at \( y \). The plot confirms exponential localisation of \( D^{\text{ov}} \) throughout
the critical region, as required. Also shown is the same result obtained using unitary link fields, showing that in this case the localisation length is significantly smaller. Fig. 6(b) from the same work shows the discrepancy \( \delta_{GW} = \| (\gamma_3 D^{ov} + D^{ov} \gamma_3 - 2D^{ov} \gamma_3 D^{ov}) \phi \|_\infty \) as a function of \( L_s \), like Fig. 3(a) it shows the desired symmetry \([21]\) being restored, but only rather slowly, as \( L_s \to \infty \). Results are shown for both the Shamir kernel corresponding to DWF, and the Wilson kernel which performs slightly better. Again, the inset shows that using unitary link fields improves matters markedly.

6 Discussion

In summary, the Thirring model defined using the DWF bulk formulation appears to sustain a QCP where a novel strongly-interacting quantum field theory may be found in the continuum limit. The QCP is essentially defined by dimensionality, count of light fermion degrees of freedom, and pattern of global symmetry breaking. Not even all approaches satisfying these criteria are equivalent: DWF with the surface formulation of the Thirring interaction yield \( 0 < N_c < 1 \) \([44]\), and SLAC fermions predict \( N_c = 0.80(4) \) \([47]\). In such cases a unitary QFT with integer \( N \) cannot exist. We speculate that bulk DWF fall in a different RG basin of attraction due to their promotion of strong dynamics resembling those of abelian gauge theory: similar reasoning underlies the conjectured identification with IR QED \( 3 \) outlined above, and is supported by the large-\( N \) approach. It is clear from the results reviewed above that the 2+1d Thirring model furnishes a challenging new environment in which to stress-test DWF and assess their merit as a non-perturbative specification of massless fermions.

We end with some open questions. The critical exponents \( \beta_m, \delta, \eta, \nu \) studied above all relate to critical fluctuations of the order parameter field, which is a fermion bilinear. Understanding strong dynamics in the symmetric phase also requires the exponent \( \eta_{\psi} \) defined by the critical propagator \( \langle \psi(0) \bar{\psi}(r) \rangle \propto r^{-(2+\eta_{\psi})} \). Unfortunately, exploratory studies of the DWF fermion propagator have proved very noisy \([42]\), and appear to require source smearing, in contrast to the relatively clean results obtained with staggered fermions \([21]\). In the meson sector there is evidence for a spectral separation between Goldstone and non-Goldstone channels at strong coupling \([44]\) but lattices with a longer temporal extent are needed for precision. Finally the ratio \( \langle \psi \psi \rangle / m \chi_\pi \), where \( \chi_\pi \) is the integrated Goldstone correlator (also known as the transverse susceptibility), differs markedly from unity as the coupling strength grows \([42, 44]\) contradicting the axial Ward Identity associated with U(2\( N \)). The problem may lie in unexpectedly strong parameter renormali-
sation, and also possibly in the identification of the physical fields; in a strongly-interacting world it is not clear how firmly tethered the low-energy degrees of freedom are to the walls.

Since the QCP is defined by its symmetry, it also behoves us to consider other possible interactions consistent with $U(2N) \otimes \mathbb{Z}_2$. It has been observed that beyond the current-current interaction of there is another contact interaction, the so-called Haldane interaction $-g_H^2/2N(\bar{\psi} \gamma_3 \gamma_5 \psi)^2$ that shares this symmetry (the sign of the coupling is determined by the requirement for the fermion determinant to be positive definite), and accordingly there is no justification to exclude it from the putative fixed point action; indeed it has been studied using both SLAC fermions and FRG. Fig. 7 shows results of pilot DWF simulations with a Haldane interaction localised on the walls; decreasing $\beta_H \equiv g_H^{-2}$ significantly enhances the bilinear condensate. There is every prospect that the Haldane interaction is present in the fixed-point action, as predicted by FRG.

Finally, what is to be made of the staggered lattice model? In principle it should remain interesting – a model exhibiting a sequence of QCPs defining strongly-interacting QFTs whose universal properties are exquisitely sensitive to the parameter $N$. The real question is what is $N$ counting? Is it possible to formulate a continuum quantum field theory of fermions with symmetry breaking $U(N) \otimes U(N) \to U(N)$? Can this theory be phrased in terms of local fields and interactions? Perhaps the Thirring model still has some surprises in store.
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