Resemblance of the power–law scaling behavior of a non–Markovian and nonlinear point processes

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Abstract

We analyze the statistical properties of a temporal point process driven by a confined fractional Brownian motion. The event count distribution and power spectral density of this non–Markovian point process exhibit power–law scaling. We show that a nonlinear Markovian point process can reproduce the same scaling behavior. This result indicates a possible link between nonlinearity and apparent non–Markovian behavior.

1 Introduction

If someone knows the state of every particle in the universe, that someone could determine the future of the universe from this single glimpse at its current state. Therefore, if everything is known, the universe would be Markovian (or even deterministic) to them. Though our human awareness is often limited, the universe will often appear non–Markovian to us [1]. The long–range memory phenomenon seems to imply non–Markovian dynamics, yet we have successfully used nonlinear Markovian models to reproduce it [2]. In this paper, we explore a link between nonlinearity and apparent non–Markovian behavior by introducing and analyzing two point processes: one is driven by a non–Markovian fractional noise and the other by a Markovian white noise.

Many observed temporal phenomena in physical, biological, and social systems are either discrete by their nature or continuous yet representable as a discrete temporal pattern of instantaneous events [3]. This temporal pattern can be encoded by a list of event times or inter–event times. Numerous examples of phenomena described by event times or inter–event times include contraction of infectious diseases [4], order execution within financial markets [5], musical rhythm [6], earthquakes [7], traffic accidents [8], human heartbeat [9] or formation of queues [10]. Such phenomena are driven by various complex mechanisms with a degree of randomness and causality (e.g., earthquakes trigger aftershocks). Such processes can be modeled using the point process formalism [2].

Poisson and Hawkes processes are two well–known examples of the point processes. The Poisson point process assumes either a constant event rate (homogeneous Poisson process) or time dependent event rate (inhomogeneous Poisson process). While in the Hawkes process the system effectively excites itself as newly occurring events temporarily increase the event rate. Unlike in the Poisson process, in the Hawkes process time increments are no longer independent. Having such embedded memory property Hawkes process has found numerous applications in finance [11] and other fields [12,13]. Recently it was shown that a certain class of nonlinear Hawkes processes can reproduce power–law distributions [14], which may appeal to practitioners dealing with complex systems exhibiting power–law distributions, such as financial markets.

Long–range memory and $1/f^\beta$ noise (with $0.5 \lesssim \beta \lesssim 1.5$), as its characteristic feature [15–20], are of particular interest as they are observed across different physical [21–23], biological [9] and social systems [24]. Typically long–range memory is modeled using various non–Markovian models: fractional Brownian motion [25–29], ARCH models [30–35] or non–Markovian mechanisms such as trapping [36,37]. Our approach is different: in earlier works, we have used nonlinear Markovian processes, such as point processes [35,40], stochastic differential equations [41,42] and agent–based models [44,45], to model power–law statistical properties and long–range memory phenomenon (for a recent review see [27]). In this paper, we explore how the nonlinearity of a Markovian point process can help imitate the statistical properties of a simple non–Markovian point process.

Similar inquiry was conducted by [46,47] from the perspective of self–similar stochastic processes. Self–similarity of a stochastic process implies that the process is invariant (in the statistical sense) to proper scaling of time and space.

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axes [48, 46] has explored how the nonlinear transformations of time and space axes affect the dynamics of the noisy voter model. While [47] has shown that there is only one Ito diffusion that is self–similar for any given Hurst exponent $H$. Ito stochastic differential equation analyzed in [47] belongs to the same class of stochastic differential equation as the ones arising from our approach [2]. Unlike [47], our analysis here is numerical and focused directly on the long–range memory properties of the processes. Thus we draw direct comparisons between a nonlinear Markovian process and a non–Markovian process.

This paper is organized as follows. A non–Markovian point process based on the fractional Brownian motion is introduced and analyzed from the numerical perspective in Section 2. In Section 3, we derive a simple Markovian point process generating the same distribution of inter–event times. Then in Section 4 nonlinearity is introduced into the Markovian point process. With this correction, we are able to replicate the power–law scaling behavior of the analyzed statistical properties of the fractional point process. Obtained numerical results and their implications are discussed in Section 5. The code used to obtain the reported results is available via the GitHub repository [1].

2 Non–Markovian fractional point process

The intensity of a process described by event times $t_k$ can be obtained by summing over their profiles:

\[ I(t) = \sum_k A_k (t - t_k). \]  

(1)

We can often assume that events happen almost instantaneously, $A_k(x) = a_k \delta(x)$, as the profiles are usually narrow in comparison to the length of the time series. Often modeled systems are driven by the flow of identical or very similar elementary objects, such as electrons, TCP/IP packets, or trades. Therefore we can assume that the modeled events are unit events, $a_k = 1$.

When dealing with unit point events, it is convenient to introduce event count series as an alternative to intensity series:

\[ N(t) = \sum_k 1_{t \leq t_k < t + \Delta t}. \]  

(2)

In the above $1_c$ is the indicator function, which attains a value of 1 if the condition $c$ is satisfied and is zero otherwise. While $\Delta t$ is the time window within which events are counted. Here, let us consider a unit time window, $\Delta t = 1$.

Typically describing a temporal pattern by a set of event times $\{t_k\}$ is equivalent to the description by a set of inter–event times $\{\tau_k = t_{k+1} - t_k\}$ as the dynamics of many processes do not depend on the absolute time. Inter–event times are far more convenient from the statistical analysis and modeling point of view.

Let us consider a point process whose inter–event time $\tau_k$ evolves according to the iterative equation:

\[ \tau_{k+1} = \tau_k + \sigma \varepsilon_k^{(H)}. \]  

(3)

In the iterative equation above $\sigma$ controls the rate of change of the inter–event times, while $\varepsilon_k^{(H)}$ are the samples of the fractional Gaussian noise with Hurst index $H$, zero mean, and unit variance. We obtain samples of the fractional Gaussian noise by using the approximate circulant method [49].

It is important to note that Eq. (3) is a generalization of a nonlinear Markovian point process [38–40]. To simplify the analysis of this fractional point process we have removed the drift term (effectively set $\gamma = 0$ in the original model) and nonlinearity (set $\mu = 0$) with the hope that the noise term with $H \neq \frac{1}{2}$ would cause drift and nonlinearity effects on its own.

By definition, inter–event time can’t be negative. Therefore diffusion of inter–event times has to be restricted to the positive values. Furthermore, the process also needs to be stationary. Thus another boundary condition has to be introduced from the side of the larger values. To do this, we use two inelastic reflective boundary conditions, which limit possible values of $\tau$ to $[\tau_{\text{min}}, \tau_{\text{max}}]$:

\[ \tau_{k+1} = \begin{cases} 
\tau_{\text{min}} & \text{if } \tau_k + \sigma \varepsilon_k^{(H)} < \tau_{\text{min}}, \\
\tau_{\text{max}} & \text{if } \tau_k + \sigma \varepsilon_k^{(H)} > \tau_{\text{max}}, \\
\tau_k + \sigma \varepsilon_k^{(H)} & \text{otherwise.}
\end{cases} \]  

(4)

[1] https://github.com/akononovicius/fractional-point-process
Figure 1: Statistical properties of the fractional point process: (a) inter–event time PDF, (b) event count per unit time window PDF and (c) PSD of the event count time series. Model parameters: $H = 0.35$ and $\sigma = 10^{-3}$ (red dotted line), $H = 0.5$ and $\sigma = 10^{-4}$ (green solid line), $H = 0.8$ and $\sigma = 10^{-4}$ (blue dashed line).

Without loss of generality, let us set $\tau_{\text{min}} = b$ and $\tau_{\text{max}} = 1 - b$ (here $b$ is some small positive real number). In our numerical simulations we have used $b = \sigma$.

The fractional point process considered here shouldn’t be confused with so–called fractional Poisson processes [50–54]. In the fractional Poisson process, $\tau_k$ are independent and identically distributed. The apparent non–Markovian behavior emerges not due to inherent correlations (like those accounted for in fractional Brownian motion) but due to Mittag–Leffler heavy–tailed inter–event time distribution. As Mittag–Leffler distribution is a solution of the fractional Kolmogorov–Feller equation, this Poisson process is referred to as the “fractional” Poisson process. For similar reasons apparent non–Markovian behavior emerges in the nonlinear point process we will consider in a later section.

To find the stationary distribution of Eq. (4), let us note that the iterative equation describes the confined fractional Brownian motion. The stationary distribution of this process was already considered from the numerical perspective in [55,56]. In [55,56] it was shown that: for $H = 0.5$ stationary distribution is uniform (the only case which can be derived analytically), while for $H < 0.5$ and $H > 0.5$, probability depletion and accretion zones (respectively) form close to the boundaries. In fact, [55,56] reports power–law scaling behavior of the stationary distribution:

$$ p(w_k) \propto w_k^\alpha, \quad \alpha = \frac{1}{H} - 2. \quad (5) $$

In the above, $w_k$ is the distance to the nearest boundary condition (i.e., $w_k = \min(\tau_k - \tau_{\text{min}}, \tau_{\text{max}} - \tau_k)$). This power–law scaling behavior is observed for $w_k > \sigma$, while for $w_k < \sigma$, the exact form of the stationary distribution depends on the type of the boundary condition used [55,56].

In Fig. 1 (a), we see that $p(\tau)$ exhibits power–law scaling behavior as predicted by [55,56]. The same scaling behavior is present in the Beta distribution, particularly by its symmetric case $\text{Beta}(\alpha + 1, \alpha + 1)$. Beta distribution has finite support, which makes sense with the boundary conditions we use. Thus, further we will assume that the stationary distribution of the fractional point process is well approximated by the symmetric Beta distribution.

Deriving analytical forms for the statistical properties of the event count time series is an even more complicated problem. Analytical approximations of probability density function (abbr. PDF) and power spectral density (abbr. PSD) are only known for the $H = 0.5$ case [38–40], while the $H \neq 0.5$ case is up to numerical exploration. Though, as can be seen in Fig. 1(b) and (c), both PDF and PSD exhibit power–law scaling behavior, which appears to have different exponents than in the $H = 0.5$ case (solid green line). The fluctuations observed in Fig. 1(b) for small $N$ are numerical artifacts arising due to $N$ being a discrete random variable. In the following sections when estimating power–law scaling behavior of $p(N)$ we ignore small $N$ value range and consider only intermediate range, where the power–law scaling behavior is clearly observed.

### 3 Simple Markovian point process

Let us build a Markovian point process as a benchmark model to understand the impact of the memory effects of the fractional Brownian motion of the inter–event times. Our goal is to create a Markovian point process with a similar inter–event time distribution, but driven by a standard uncorrelated Gaussian noise instead of the fractional Gaussian noise. Here and later on, we refer to point processes as Markovian if the stochastic increments of inter–event times are independent. In some of the existing literature, only Poisson point processes are considered to be Markovian [50–54].
Figure 2: Power–law exponent of the PDF of inter–event times obtained by numerically evaluating the fractional point process (red circles) and the Markovian point process (blue squares). Black curve corresponds to (9). Other model parameters: $\sigma = 10^{-3}$ (for the simulations of the fractional point process with $H = 0.35$ and $0.4$) or $10^{-4}$ (in all other cases).

For processes driven by stochastic differential equations, stationary distributions can be easily obtained [57]. Stationary distribution of stochastic differential equation:

$$d\tau_k = \sigma^2 f (\tau_k) \, dk + \sigma dW_k,$$

where $W_k$ is the standard Wiener process, is given by:

$$p (\tau_k) = \exp \left[ 2 \int_{\tau_k}^{\infty} f (u) \, du \right].$$

Let us rearrange [7] to obtain an expression that would allow us to derive the drift function $f (\tau_k)$ based on the desired stationary distribution:

$$f (\tau_k) = \frac{1}{2p (\tau_k)} \frac{d}{d\tau_k} p (\tau_k).$$

If $p (\tau_k)$ corresponds to PDF of the symmetric Beta distribution, then the drift function:

$$f (\tau_k) = \frac{\alpha}{2} \left( 1 - \frac{1}{\tau_k - 1} \right) \left( 1 - \frac{1}{1 - \tau_k} \right).$$

The obtained stochastic differential equation can be solved numerically by iterating the following iterative equation:

$$\tau_{k+1} = \tau_k + \sigma^2 \left( \frac{1}{2H} - 1 \right) \left( 1 - \frac{1}{\tau_k} \right) + \sigma \xi_k.$$

In the above, $\xi_k$ are uncorrelated samples from standard Gaussian distribution (i.e, $\xi_k \sim \mathcal{N} (0, 0.5)$). The smaller $\sigma$, the better the approximation of (6) by (10) will be. Thus as $\sigma$ gets smaller stationary distribution of (10) will become more similar to the stationary distribution of (3).

By comparing these models, we can understand the impact of the fractional diffusion itself, eliminating the effect of different inter–event time distributions. Indeed (see Fig. 2), similar exponents $\alpha$ are obtained from both models for $H \in [0.35, 0.95]$. It means that the inter–event time distributions of both models are similar. Furthermore, the obtained exponents $\alpha$ follow relation (5) reasonably well.

For the Markovian point process, unlike for the fractional model, it is possible to derive analytical approximations of the statistical properties of the event count time series. In this task, particularly useful is the similarity of (10) to the iterative equation of the point process considered in [38–40]:

$$\tau_{k+1} = \tau_k + \sigma^2 \left( \frac{1}{2H} - 1 \right) \left( 1 - \frac{1}{\tau_k} \right) \frac{1}{\tau_k} + \sigma \xi_k.$$

In the limit of short inter–event times ($\tau \to \tau_{\text{min}}$ or $\tau \to 0$) with $\mu = 0$ and $\gamma = \frac{1}{2H} - 1$, the iterative equations match:

$$\tau_{k+1} = \tau_k + \sigma^2 \left( \frac{1}{2H} - 1 \right) \frac{1}{\tau_k} + \sigma \xi_k.$$

For (11), it is known that event count PDF exhibits power–law scaling behavior [58]:

$$p (N) \propto N^{-\lambda}, \quad \lambda = 2 (\gamma - \mu) + 3 = \frac{1}{H} + 1.$$
Figure 3: Power–law exponents of event count PDF (a) and PSD (b) obtained by numerically evaluating the fractional point process (red circles) and the Markovian point process (blue squares). Black curves follow (13) and (14) respectively. Black dashed curve follows (15). Model parameters are the same as in Fig. 2.

Similarly, PSD of the event count time series is known to have a range of frequencies in which it is well approximated by a power–law function [58]:

\[ S(f) \propto f^{-\beta}, \quad \beta = 1 + \frac{2(\gamma - \mu)}{3 - 2\mu} = \frac{1}{3} \left( \frac{1}{H} + 1 \right). \]

(14)

These approximations should hold well for small \( \sigma \) and uncorrelated \( \xi_k \) [40,58,59]. As expected, (13) and (14) predict the power–law exponents of the Markovian point process rather well (see Fig. 3). For the fractional point process, exponent \( \lambda \) behaves as expected for the Markovian point process, while exponent \( \beta \) attains notably different values (with the difference getting bigger further away from \( H = 0.5 \)). It appears that exponent \( \beta \) of the fractional point process changes linearly with \( H \):

\[ \beta \approx \frac{3}{2} - H. \]

(15)

4 Nonlinear Markovian point process

Using a simple Markovian point process driven by iterative equation (10) we were able to mimic the influence of \( H \) on exponents \( \alpha \) and \( \lambda \) of the fractional point process driven by iterative equation (3). Let us introduce a correction to the Markovian point process that would allow us to reproduce \( \beta \) values observed for the fractional point process. Let us follow in the footsteps of [38–40] and add nonlinearity into the Markovian point process.

Earlier, we have assumed that the diffusion function of the Markovian point process is equal to unity. If we want to introduce nonlinearity, we will also have to include it in the diffusion function. If the diffusion function is not unity, but some other function \( g(\tau_k) \), then the corresponding drift function [57]:

\[ f(\tau_k) = \frac{g^2(\tau_k)}{2p(\tau_k)} \frac{\mathrm{d}}{\mathrm{d}\tau_k} p(\tau_k) + g(\tau_k) \frac{\mathrm{d}}{\mathrm{d}\tau_k} g(\tau_k), \]

(16)

here \( p(\tau_k) \) would describe the PDF of the desired stationary distribution. In our particular case, \( p(\tau_k) \) is the PDF for the symmetric Beta distribution.

For \( g(\tau_k) \), infinitely many different choices would be valid, though they are suitable for our purposes only as long as \( g(\tau_k) \propto \tau_k^\mu \) in the limit of short inter–event times. Consider the simplest possible alternative:

\[ g(\tau_k) = \tau_k^\mu. \]

(17)

Power–law functions were also considered in [17] with a goal to produce Markovian process with self–similarity property. Putting Eq. (17) and the PDF for the symmetric Beta distribution into Eq. (16) yields:

\[ f(\tau_k) = \frac{1}{2} \left( \frac{\alpha + 2\mu}{\tau_k} - \frac{\alpha}{1 - \tau_k} \right) \tau_k^{2\mu}. \]

(18)

The nonlinear Markovian point process,

\[ \tau_{i+1} = \tau_i + \frac{\sigma^2}{2} \left( \frac{1}{\tau_i} - \frac{2 + 2\mu}{\tau_i} - \frac{1 - 2}{1 - \tau_i} \right) \tau_i^{2\mu} + \sigma^2 \xi_i, \]

(19)
should reproduce all three consider exponents of the fractional point process. Note that this process also retains the similarity to the nonlinear point process considered in [38–40]. In the limit of short inter–event times ($\tau \to \tau_{\text{min}}$) and with $\gamma = \frac{1}{2H} - 1 + \mu$ this iterative equation reduces to (11). Therefore we can reasonably expect that for (19), the relationships between model parameters and the power–law exponents would still hold [38–40]:

$$\alpha = 2 (\gamma - \mu), \quad \lambda = \alpha + 3, \quad \beta = 1 + \frac{\alpha}{3 - 2\mu}.$$  \hspace{1cm} (20)

To reproduce $\beta_F(H)$ observed for the fractional point process (hence the index $F$) we need to figure out how power–law exponent $\mu(H)$ of the nonlinear Markovian point process depends on the Hurst parameter. This is done by solving:

$$\beta_F(H) = 1 + \frac{\alpha(H)}{3 - 2\mu(H)}.$$  \hspace{1cm} (21)

For $\beta_F(H) \neq 1$ the solution of Eq. (21) is given by:

$$\mu(H) = \frac{3}{2} + \frac{\alpha(H)}{2 (1 - \beta_F(H))} = \frac{3}{2} + \frac{\frac{1}{2H} - 2}{2 (1 - \beta_F(H))}.$$  \hspace{1cm} (22)

Unfortunately, with Eq. (13) as $\beta_F(H)$ the value of $\mu$ in the vicinity of $H = \frac{1}{2}$ is close to $-\frac{1}{2}$, while from [3] it is evident that it should be 0. To obtain the proper value of $\mu$ in the vicinity of $H = \frac{1}{2}$ we would need to change the slope of dependence on $H$:

$$\beta_F(H) = 1 - \frac{4}{3} \left( H - \frac{1}{2} \right).$$  \hspace{1cm} (23)

While this now yields the proper value of $\mu$ close to $H = \frac{1}{2}$, predictions of $\beta_F$ for other $H$ noticeably disagree with the results of the numerical simulation. Though the agreement can be improved by including second–order term:

$$\beta_F(H) = 1 - \frac{4}{3} \left( H - \frac{1}{2} \right) + C_2 \left( H - \frac{1}{2} \right)^2.$$  \hspace{1cm} (24)

The best fit to the numerical simulation results is obtained with $C_2 \approx \frac{2}{3}$. Then Eq. (22) becomes:

$$\mu(H) = \frac{3}{2} - \frac{3}{5H} - \frac{6}{5 (5 - 2H)}.$$  \hspace{1cm} (25)

Numerical simulation of the nonlinear Markovian point process, Eq. (19), with $\mu$ determined by Eq. (25), produces power–law exponents similar to the ones obtained by numerically simulating the fractional point process. As seen in Fig. 4 introduction of nonlinearity has enabled reasonable reproduction of the PSD exponent $\beta$. Though the numerical simulation results slightly deviate from the analytical prediction. The agreement between analytical and numerical results of the nonlinear Markovian point process should improve with smaller $\sigma$ [40,58,59], yet decreasing $\sigma$ increases the required length of generated time series, the real simulation time, and required computer memory resources.
5 Conclusions

Here we have introduced a generalization of the point process proposed in [38–40] by replacing the Brownian motion of the inter–event times with fractional Brownian motion. Thus, by construction, our generalization exhibits long–range memory stemming from the persistence or the anti–persistence of inter–event times. We have numerically explored this model from the perspective of its long–range memory properties.

The considered fractional point process can generate time series exhibiting power–law scaling in PDF and PSD. Time series generated with persistent inter–event times (using fractional Brownian motion with $H > 0.5$) exhibit anti–persistent behavior (PSD exponent $\beta < 1$), while time series with anti–persistent inter–event times ($H < 0.5$) appear to exhibit persistent behavior ($\beta > 1$). PDF exponent $\lambda$ also gets smaller as Hurst exponent $H$ of the underlying fractional Brownian motion grows larger. Although, we weren’t able to numerically observe the behavior of the fractional point process for the whole range of $H$. For most cases driven by the anti–persistent fractional Brownian motion the fractional point process doesn’t explore the phase space quick enough, the lower $H$ the longer time series are needed to properly observe the long–term behavior of the process. Cases with $H < 0.35$ were too demanding for the computational resources and took too long to run.

The observed power–law scaling behavior of the generated time series PDF can be replicated by a simple Markovian point process mimicking the inter–event time PDF of the fractional point process. Based on the prior numerical exploration of the confined fractional Brownian motion [55], we have assumed that the inter–event time distribution of the considered fractional point process is the symmetric Beta distribution. From the numerical simulations, we can conclude that this assumption is reasonable as generated inter–event time PDFs appear similar in both models. Event count (time series) PDFs appear similar as well.

Although, the reproduction of the observed power–law scaling behavior in the PSD requires the introduction of nonlinearity into the Markovian point process. We have considered the simplest case, when the diffusion term of the point process is a power–law function of the prior inter–event time. Using the nonlinear Markovian point process with symmetric Beta inter–event time distribution, we have reproduced the power–law scaling behavior in PDF and PSD observed in the fractional point process.

We have introduced two similar point processes – one driven by a fractional noise and the other by a white noise. The latter can mimic the former due to nonlinearity in drift and diffusion terms of process. Having such two similar and comparable models of differing nature should allow building, testing and, tuning indicators attempting to discriminate between the “true” and spurious long–range memory processes. Some of the more recent attempts to discriminate between two different types of memory include [60–67].

Author contributions

Aleksejus Kononovicius: Conceptualization, Methodology, Software, Writing – Original Draft, Writing – Review & Editing, Visualization. Rytis Kazakevičius: Conceptualization, Methodology, Writing – Review & Editing. Bronislovas Kaulakys: Conceptualization, Writing – Review & Editing, Supervision.

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