Gravitational Corrections to the Energy-Levels of a Hydrogen Atom

Zhao Zhen-Hua 1,1, Liu Yu-Xiao 2, Li Xi-Guo 3

1 Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China
2 Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China
3 Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China

The first order perturbations of the energy levels of a hydrogen atom in central internal gravitational field are investigated. The internal gravitational field is produced by the mass of the atomic nucleus. The energy shifts are calculated for the relativistic 1S, 2S, 2P, 3S, 3P, 3D, 4S and 4P levels with Schwarzschild metric. The calculated results show that the gravitational corrections are sensitive to the total angular momentum quantum number.

PACS numbers: 04.90.+e., 31.10.+z.
Key words: Hydrogen atom; Gravitational perturbation; Generally covariant Dirac equation.

I. INTRODUCTION

The study of gravitational fields interacting with spinor fields constitutes an important element in constructing a theory that combines quantum physics and gravity. For this reason, the investigation of the behavior of relativistic particles in this context is of considerable interest. It has been known that the energy levels of an atom placed in an external gravitational field will be shifted as a result of the interaction of the atom with space-time curvature see Refs. [1, 2, 3, 4] for examples. And the geometric and topological effects lead to shifts in the energy levels of a hydrogen atom are considered in Ref. [5].

Recently, there has been a dramatic increase in the accuracy of experiments that measure the transition frequencies in hydrogen. The most accurately measured transition is the 1S – 2S frequency in hydrogen; it has been measured with a relative uncertainty of 25 Hz ($\Delta f/f_0 = 1.0 \times 10^{-14}, f_0 = 2466 \text{THz}$) [6, 7], an order of magnitude larger than the natural linewidth of 1.3 Hz natural width of the 2S level [8, 9]. Indeed, it is likely that transitions in hydrogen will eventually be measured with an uncertainty below 1 Hz [10]. Though that accuracy can not explore the gravitational effect produced by the hydrogen atom nucleus, with the progress of experiments we can detect the gravitational effect.

In this paper we investigate another previously neglected gravitational effect of the energy-level shifts of a hydrogen atom. This is to give some explicit values for energy-level shifts of a hydrogen atom by the general relativistic effect with Schwarzschild metric. And the difference with Refs. [1, 2, 3, 4, 5] is that the gravitational field in this paper is not a external field but produced by the mass of hydrogen atom nucleus. To our knowledge no one has given explicit values for energy-level shifts of a hydrogen atom with gravitational corrections. Although the effect is very small, but it also has the physical significance as a test of general relativity at the quantum level.

* This work was supported by National Natural Science Foundation of China 10435080 and 10575123, Chinese Academy of Sciences Knowledge Innovation Project under Grant No. KJCX-SYW-N2 and KJCX2-SW-N16.
† Corresponding author: zhaozhenhua@impcas.ac.cn
This paper is organized as follows: In Sec. II we review the formalism of the generally covariant Dirac equation in curved space-time. In Sec. III we give the tetrad and spinor connections with Schwarzschild metric. The gravitational perturbation of relativistic 1S level is calculated in Sec. IV. The summary and discussion are given in Sec. V.

II. GENERALLY COVARIANT DIRAC EQUATION IN CURVED SPACE-TIME

To write the generally covariant Dirac equation in curved space-time with metric $g_{\mu\nu}$, one first introduces the spinor affine connections $\omega_{\mu} = \frac{1}{2} \omega_{\mu}^{ab} I_{ab}$, where $I_{ab}$ are the generators of $SO(4)$ group, whose spinor representation is

$$I_{ab} = \frac{1}{4} (\gamma_a \gamma_b - \gamma_b \gamma_a).$$

Here $\gamma_a$ are the Dirac-Pauli matrices with the following relation

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab},$$

and

$$\gamma_0 = -\gamma_0, \quad \gamma_i = \gamma_i \quad (i = 1, 2, 3),$$

$$\gamma_0 = i\beta, \quad \gamma_i = -i\beta \alpha_i,$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $I$ is the $2 \times 2$ identity matrix, $\eta^{ab} = \eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor, and $\sigma_i$ are the standard Pauli matrices

$$\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\omega_{\mu}^{ab}$ is defined by the vanish of the generalized covariant derivative \[11, 12\] of the tetrad (or vierbein) field \[13\] $e^{(a)}_{\mu}(x)$

$$D_{\mu} e^{(a)}_{\nu} = \partial_{\mu} e^{(a)}_{\nu} - \Gamma_{\mu\nu}^{\lambda} e^{(a)}_{\lambda} - \eta_{bc} \omega_{\mu}^{ab} e^{(c)}_{\nu} = \nabla_{\nu} e^{(a)}_{\mu} - \eta_{bc} \omega_{\mu}^{ab} e^{(c)}_{\nu} \equiv 0,$$

where the tetrad field $e^{(a)}_{\mu}(x)$ and its inverse $e_{(a)}^{\mu}(x)$ satisfy the following equations

$$g_{\mu\nu}(x) = \eta_{ab} e^{(a)}_{\mu}(x) e^{(b)}_{\nu}(x),$$

$$e^{(a)}_{\mu}(x) e^{(b)}_{\nu}(x) = \delta_{a}^{b}, \quad (\mu, \nu, a, b = 0, 1, 2, 3)$$

$\mu, \nu$ are the space-time indices lowered with the metric $g_{\mu\nu}$, and $a, b$ are the Lorentz group indices lowered with $\eta_{ab}$.

One also needs to introduce generalized Dirac-Pauli matrices $\Gamma_{\mu}(x) = e^{(a)}_{\mu}(x) \gamma_a$, which satisfy the equation \[2\]

$$\Gamma_{\mu}(x) \Gamma_{\nu}(x) + \Gamma_{\nu}(x) \Gamma_{\mu}(x) = 2g_{\mu\nu}(x).$$

The covariant derivative acting on a spinor field $\psi$ is then

$$D_{\mu} \psi = \partial_{\mu} \psi - \omega_{\mu} \psi,$$

and the generally covariant form of the Dirac equation \[4\] in pure gravitational field is

$$\Gamma^{\mu}(x) D_{\mu} \psi(x) + \frac{mc}{\hbar} \psi(x) = 0,$$
where $\Gamma^\mu(x) = g^{\mu\nu} \Gamma_{\nu}(x)$, $m$ is the mass of spinor particles.

For an electron near the atomic nucleus one needs to consider the effect of the electromagnetic vector potential $A_\mu$, here $A_\mu$ satisfy the Maxwell equations \[2, 14\]

\[g^{\lambda\sigma} \nabla_\lambda A_\sigma - R_\mu{}^\nu A_\nu = -4\pi J_\mu,\]

where $J_\mu$ is the current vector. So the covariant derivative acting on a spinor field should be rewritten as

\[D_\mu \psi = (\partial_\mu - \omega_\mu - iqA_\mu)\psi.\]

Then the generally covariant form of the Dirac equation in gravitational and electromagnetic fields is

\[\Gamma^\mu(\partial_\mu - \omega_\mu - iqA_\mu)\psi(x) + \frac{mc}{\hbar}\psi(x) = 0.\]

### III. SPINOR CONNECTIONS IN THE SCHWARZSCHILD SPACE-TIME

In what follows, we will calculate the spinor connections in a Schwarzschild spacetime. The line element corresponding to the spacetime is given by

\[ds^2 = -g_{\mu\nu} dx^\mu dx^\nu = c^2 \left(1 - \frac{R_s}{r}\right) dt^2 - \frac{1}{1 - \frac{R_s}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,\]

where $R_s = 2GM/r$. With the time gauge conditions \[13, 16\] $e_i^{(0)} = 0$ and $e_i^{(0)} = 0$, the tetrad field $e^{(a)}_\mu$ is given as follows:

\[e^{(a)}_\mu = \begin{pmatrix} \sqrt{1 - \frac{R_s}{r}} & 0 & 0 & 0 \\ 0 & \frac{\sin \theta \cos \phi}{\sqrt{1 - \frac{R_s}{r}}} & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \frac{\sin \theta \sin \phi}{\sqrt{1 - \frac{R_s}{r}}} & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \frac{\cos \theta}{\sqrt{1 - \frac{R_s}{r}}} & -r \sin \theta & 0 \end{pmatrix}.\]

Taking the approximation $\sqrt{1 - \frac{R_s}{r}} \approx 1 - \frac{R_s}{2r}$, we have

\[e^{(a)}_\mu = \begin{pmatrix} 1 - \frac{R_s}{2r} & 0 & 0 & 0 \\ 0 & \frac{\sin \theta \cos \phi}{1 - \frac{R_s}{2r}} & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \frac{\sin \theta \sin \phi}{1 - \frac{R_s}{2r}} & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \frac{\cos \theta}{1 - \frac{R_s}{2r}} & -r \sin \theta & 0 \end{pmatrix}.\]

From Eq. 14, it follows

\[\omega^{ab} = (\nabla_\mu e^{(a)}_\nu) e^{(b)}_\lambda g^{\lambda\mu},\]

and

\[\omega_\mu = \frac{1}{2} \omega^{ab} I_{ab} = \frac{1}{2} I_{ab} (\nabla_\mu e^{(a)}_\nu) e^{(b)}_\lambda g^{\lambda\mu} \approx \frac{1}{2} I_{ab} (-\Gamma^\rho_{\mu\nu}) e^{(a)}_\rho e^{(b)}_\lambda g^{\lambda\mu}.\]
Thus using Eqs. (16), (18), (19) and (20), we obtain the explicit expressions of the nonzero components of spinor connections

$$\omega_0 = \begin{pmatrix}
    0 & 0 & - \frac{R_s \cos \theta}{4r^2} & 0 \\
    0 & 0 & - \frac{R_s \sin \theta e^{-i\phi}}{4r^2} & 0 \\
    - \frac{R_s \cos \theta}{4r^2} & - \frac{R_s \sin \theta e^{i\phi}}{4r^2} & 0 & 0 \\
    - \frac{R_s \sin \theta e^{i\phi}}{4r^2} & \frac{R_s \cos \theta}{4r^2} & 0 & 0
\end{pmatrix},$$

(21)

$$\omega_2 = \begin{pmatrix}
    0 & 0 & \frac{R_s (r-R_s) e^{-i\phi}}{2r-R_s} & 0 \\
    0 & 0 & 0 & \frac{R_s (r-R_s) e^{-i\phi}}{2r-R_s} \\
    \frac{R_s (r-R_s) e^{-i\phi}}{2r-R_s} & 0 & 0 & 0 \\
    0 & 0 & 0 & \frac{R_s (r-R_s) e^{-i\phi}}{2r-R_s}
\end{pmatrix},$$

(22)

$$\omega_3 = \begin{pmatrix}
    \frac{iC_1}{sr-4R_s} & \frac{R_s \sin 2\theta e^{-i\phi}}{sr-4R_s} & 0 & 0 \\
    \frac{R_s \sin 2\theta e^{i\phi}}{sr-4R_s} & - \frac{iC_1}{sr-4R_s} & 0 & 0 \\
    0 & 0 & \frac{iC_1}{sr-4R_s} & \frac{R_s \sin 2\theta e^{i\phi}}{sr-4R_s} \\
    0 & 0 & \frac{R_s \sin 2\theta e^{i\phi}}{sr-4R_s} & - \frac{iC_1}{sr-4R_s}
\end{pmatrix},$$

(23)

where

$$C_1 = 4r - 3R_s + R_s \cos(2\theta).$$

(24)

IV. GRAVITATIONAL PERTURBATION OF THE RELATIVISTIC HYDROGEN ATOM: THE $1S_{1/2}$ STATES

From Eq. (15) the corresponding Hamiltonian in curved space-time follows

$$H = -i\hbar c \Gamma_0 \Gamma^i (\partial_i - \omega_i - iqA_i) + i\hbar c (\omega_0 + iqA_0) - imc^2\Gamma_0.$$  

(25)

The Dirac Hamiltonian in flat space is

$$H_0 = -i\hbar c \gamma_0 \gamma^i (\partial_i - iqA_i) - \hbar c q A'_0 - imc^2\gamma_0,$$  

(26)

where $A'_\mu$ are the electromagnetic vector potentials in flat spacetime. Here we can take the approximation $A_i \equiv A'_i = 0$ and $A_0 \equiv A'_0 = -er^{-1}$, the detailed discussions of this problem is contained in Ref. [2]. So the Hamiltonian of the gravitational perturbation is given by

$$H_I = H - H_0$$

$$= -i\hbar c \Gamma_0 \Gamma^i (\partial_i - \omega_i) + i\hbar c \omega_0 - imc^2\Gamma_0$$

$$+ i\hbar c \gamma_0 \gamma^i \partial_i + imc^2\gamma_0.$$  

(27)

The exact solutions of the Dirac equation for a hydrogen atom in flat space-time serve as the basis for perturbation theory. The energy eigenvalues of a hydrogen atom are

$$E_{n\kappa} = m_e c^2 \sqrt{1 + \left( \frac{\zeta}{n - |\kappa| + s} \right)^2},$$  

(28)
where \( \zeta = Ze^2 \), \( s = \sqrt{\kappa^2 - \zeta^2} \), \( n = 1, 2, \cdots \) is the principal quantum number.

The bound state functions of a hydrogen atom can be written in standard representation \([17, 18]\) as

\[
\psi = \psi_{\kappa}^M = \begin{pmatrix}
g(r)\chi_{\kappa}^M \\
i f(r)\chi_{-\kappa}^M
\end{pmatrix},
\]

(29)

here \( M \) is the eigenvalue of \( J_z \), \( \kappa \) is the eigenvalue of \( K = \beta(\hat{\sigma} \cdot \hat{L} + I) \), the functions \( f(r) \), \( g(r) \) and spinors \( \chi_{\kappa}^M \), \( \chi_{-\kappa}^M \) are given by

\[
f(r) = \frac{2s-\frac{1}{2}\lambda^s+\frac{3}{2}}{\Gamma(2s+1)} \frac{\Gamma(2s+n_r+1)}{n_r\zeta K_c(\zeta K_c - \lambda \kappa)} \sqrt{1 - \frac{W_c r^{s-1} e^{-\lambda r}}{K_c}}
\left(\begin{array}{c}
\kappa - \frac{\zeta K_c}{\lambda} \\
F(-n_r, 2s+1, 2\lambda r) - n_r F(-n_r + 1, 2s+1, 2\lambda r)
\end{array}\right),
\]

(30)

\[
g(r) = \frac{2s-\frac{1}{2}\lambda^s+\frac{3}{2}}{\Gamma(2s+1)} \frac{\Gamma(2s+n_r+1)}{n_r\zeta K_c(\zeta K_c - \lambda \kappa)} \sqrt{1 - \frac{W_c r^{s-1} e^{-\lambda r}}{K_c}}
\left(\begin{array}{c}
\kappa - \frac{\zeta K_c}{\lambda} \\
F(-n_r, 2s+1, 2\lambda r) + n_r F(-n_r + 1, 2s+1, 2\lambda r)
\end{array}\right),
\]

(31)

\[
\chi_{\kappa}^M = C_{1/2} Y_l^{M-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_{-1/2} Y_l^{M+1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(32)

\[
\chi_{-\kappa}^M = -C_{1/2} Y_l^{M-1/2} \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix} - C_{-1/2} Y_l^{M+1/2} \begin{pmatrix} e^{-i\phi} \sin \theta & -\cos \theta \\ \cos \theta & e^{i\phi} \sin \theta \end{pmatrix},
\]

(33)

where \( W_c = E_{n\kappa}/m_e e^2 \), \( K_c = m_e c^2/\hbar c \), \( \lambda = \sqrt{m_e^2 c^4 - E_{n\kappa}^2}/\hbar c \), \( C_{1/2} \) and \( C_{-1/2} \) are the C-G coefficients.

For a hydrogen atom there are two \( 1S_{1/2} (n = 1, l = 0, J = 1/2, \kappa = -1) \) states, which correspond to \( M = \pm 1/2 \).

The states can be written as

\[
\psi_1 = \begin{pmatrix} 0 \\ f(r) \sin \theta e^{-i\phi} \\ -i g(r) \cos \theta \end{pmatrix},
\]

(34)

and

\[
\psi_2 = \begin{pmatrix} f(r) \\ 0 \\ i g(r) \cos \theta \\ i g(r) \sin \theta e^{i\phi} \end{pmatrix},
\]

(35)

where \( \psi_1 \) corresponds to \( M = 1/2 \) and \( \psi_2 \) to \( M = -1/2 \),

\[
f(r) = \frac{2s-\frac{1}{2}\lambda^s-\frac{3}{2}+s}{K_c \sqrt{\pi \zeta \Gamma(1+2s)}} e^{-r \lambda r} \lambda^{\frac{1}{2}+s} \sqrt{K_c + W_c \sqrt{K_c \zeta + \lambda}},
\]

(36)

\[
g(r) = \frac{2s-\frac{1}{2}\lambda^s+\frac{3}{2}+s}{K_c \sqrt{\pi \zeta \Gamma(1+2s)}} e^{-r \lambda r} \lambda^{\frac{1}{2}+s} \sqrt{K_c - W_c \sqrt{K_c \zeta + \lambda}},
\]

(37)

\( \Gamma(1+2s) \) is the \( \Gamma \) function. The gravitational perturbation matrix elements are

\[
\langle H_I \rangle_{ab} \equiv \langle \psi_a, H_I \psi_b \rangle,
\]

(38)
where the subscripts $a, b$ take on the values 1, 2. Because we take the gravitational field metric as the Schwarzschild metric, so we need to confirm the range of the integration. Here it is taken from $R_n$ to $\infty$, $R_n \approx 1.3 \times 10^{-15}$ m is the atomic nucleus radius. With the computer algebra system Mathematica, we obtain the following results for those perturbation matrix elements

$$\langle H_1 \rangle_{ab} = -\frac{\delta_{ab}}{K_c^2 R_n \zeta \Gamma(1 + 2s)} 2^{-1+2s} c R_n \lambda (R_n \lambda)^{2s} (K_c \zeta + \lambda)$$

$$\left( cm R_n W_c E_{1-2s}(2R_n \lambda) + \sqrt{K_c^2 - W_c^2 h E_{2-2s}(2R_n \lambda)} \right),$$

(39)

where $E_n(z) = \int_1^{\infty} e^{zt}/t^n dt$ is the exponential integral function. Using the equation [2]

$$\text{det}[(\psi_a, H_1 \psi_b) - E_i^{-1} \delta_{ab}] = 0,$$

(40)

from the usual perturbation theory of a degenerate energy eigenvalue, it follows that both of the degenerate $1S_{1/2}$ levels are shifted by the same perturbation:

$$E^1(1S_{1/2}) = -\frac{1}{K_c^2 R_n \zeta \Gamma(1 + 2s)} 2^{-1+2s} c R_n \lambda (R_n \lambda)^{2s} (K_c \zeta + \lambda)$$

$$\left( cm R_n W_c E_{1-2s}(2R_n \lambda) + \sqrt{K_c^2 - W_c^2 h E_{2-2s}(2R_n \lambda)} \right).$$

(41)

Substituting the constant values in Table 1 into Eq. [11], we get

$$E^1(1S_{1/2}) = -1.19956 \times 10^{-38} \text{ ev.}$$

(42)

| Quantity                  | Symbol Value | Units |
|---------------------------|--------------|-------|
| electron charge magnitude | $e$          | $1.60217653 \times 10^{-19}$ C |
| speed of light in vacuum  | $c$          | $2.99792458 \times 10^{-8}$ m s$^{-1}$ |
| electron mass             | $m_e$        | $9.1093826 \times 10^{-31}$ kg |
| Planck constant, reduced  | $\hbar$      | $1.05457168 \times 10^{-34}$ J s |
| permittivity of free space| $\epsilon_0$ | $8.854187817 \times 10^{-12}$ s$^4$ A$^2$ kg$^{-1}$ m$^{-3}$ |
| proton mass               | $M_p$        | $1.6726171 \times 10^{-27}$ kg |
| gravitation constant      | $G$          | $6.6742 \times 10^{-11}$ m$^3$ kg$^{-1}$s$^{-2}$ |

V. SUMMARY AND DISCUSSION

In a similar calculation as the $1S_{1/2}$ state, we find that all the relativistic $1S$, $2S$, $2P$, $3S$, $3P$, $3D$, $4S$ and $4P$ energy levels are respectively shifted as the same amount listed in Table 2. This means that the first order gravitational perturbations can partly remove the degeneracy of the hydrogen atom states. Although the effect is very small, but form Table 2 we find that the quantity of corrections of the energy levels with same principal quantum number $n$ and total angular momentum quantum number $J$, like $2S_{1/2}$ and $2P_{1/2}$, $3S_{1/2}$ and $3P_{1/2}$, $3P_{3/2}$ and $3D_{3/2}$, are very closely. But for the levels with same principal quantum number and different total angular momentum quantum number, like $3S_{1/2}$ and $3P_{3/2}$, their corrections have obvious difference. Those calculations show that the gravitational corrections are sensitive to the total angular momentum quantum number. It is a very important feature of the interaction between gravitational fields and spinor fields. With this feature we can find the gravitational effect in other system, and make a test of general relativity at the quantum level.
| State | The energy-level shift (Unit: ev) |
|-------|----------------------------------|
| $1S_{1/2}$ | $-1.19956 \times 10^{-38}$ |
| $2S_{1/2}$ | $-8.99637 \times 10^{-39}$ |
| $2P_{1/2}$ | $-8.99562 \times 10^{-39}$ |
| $2P_{3/2}$ | $-2.99862 \times 10^{-39}$ |
| $3S_{1/2}$ | $-6.66389 \times 10^{-39}$ |
| $3P_{1/2}$ | $-6.66353 \times 10^{-39}$ |
| $3P_{3/2}$ | $-2.66544 \times 10^{-39}$ |
| $3D_{3/2}$ | $-2.66538 \times 10^{-39}$ |
| $4S_{1/2}$ | $-5.24777 \times 10^{-39}$ |
| $4P_{1/2}$ | $-5.24756 \times 10^{-39}$ |

[1] L. Parker, Phys. Rev. Lett. **44** (1980) 1559.
[2] L. Parker, Phys. Rev. D **22** (1980) 1922.
[3] L. Parker and L. O. Pimentel, Phys. Rev. D **25** (1982) 3180.
[4] Y. S. Duan, J. Exptl. Theoret. Phys. (U.S.S.R.) **34** (1958) 632; E. Fischbach and B. S. Freeman, Phys. Rev. D **23** (1981) 2157.
[5] G. de A. Marques and V. B. Bezerra, Phys. Rev. D **66** (2002) 105011.
[6] M. Fischer et al., Phys. Rev. Lett. **92** (2004) 230802.
[7] N. Kolachevsky, J. Alnis, S. D. Bergeson, and T. W. Hänsch, Phys. Rev. A **73** (2006) 021801.
[8] C. L. Cesar et al., Phys. Rev. Lett. **77** (1996) 255.
[9] T. C. Killian et al., Phys. Rev. Lett. **81** (1998) 3807.
[10] U. D. Jentschura, P. J. Mohr, and G. Soff, Phys. Rev. Lett. **82** (1998) 53.
[11] T. W. B. Kibble, J. Math. Phys. **2** (1961) 212.
[12] R. Loll, *Discrete Approaches to Quantum Gravity in Four Dimensions,* (http://www.livingreviews.org/lrr-1998-13).
[13] E. Poisson, *The Motion of Point Particles in Curved Spacetime,* (http://www.livingreviews.org/lrr-2004-6).
[14] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation,* Freeman, San Francisco (1973), p.332.
[15] J. Schwinger, Phys. Rev. **130** (1963) 1253.
[16] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toribio, and K. H. Castello-Branco, Phy. Rev. D **65** (2002) 124001.
[17] M. E. Rose, *Relativistic Electron Theory,* Wiley, New York (1961), p.177.
[18] P. Strange, *Relativistic Quantum Mechanics,* Cambridge University Press, Cambridge (1998), p.229.
[19] S. Eidelman, et al., Phys. Lett. B **1** (2004) 592.