Exact perturbations for inflation with smooth exit

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Abstract

Toy models for the Hubble rate or the scalar field potential have been used to analyze the amplification of scalar perturbations through a smooth transition from inflation to the radiation era. We use a Hubble rate that arises consistently from a decaying vacuum cosmology, which evolves smoothly from nearly de Sitter inflation to radiation domination. We find exact solutions for super-horizon perturbations (scalar and tensor), and for sub-horizon perturbations in the vacuum- and radiation-dominated eras. The standard conserved quantity for super-horizon scalar perturbations is exactly constant for growing modes, and zero for the decaying modes.

Keywords: cosmology – inflation – cosmological perturbations

1 Introduction

The transition from inflation to radiation-domination involves a number of subtle issues affecting the evolution of perturbations. Recently, the controversy initiated by [1] over whether indeed super-horizon scalar perturbations are strongly amplified in inflation, appears to have been settled, the answer being affirmative (see [2] and the subsequent papers [3, 4, 5]). The transition is often approximated as an instantaneous jump, which is reasonable, since super-horizon modes change on a timescale that is much greater than the transition time. In order to avoid the complications involved in the matching conditions for a jump transition, smooth transitions have also been considered, using toy models for the Hubble rate [3, 5] or the potential of the scalar field [4].

Here we also consider a cosmology with a smooth exit from inflation, but instead of an ad hoc toy model, we use a recently proposed decaying vacuum cosmology [6]. In this model, a simple exact form for the Hubble rate is deduced consistently from simple physical conditions. The model, together with a brief summary of the necessary results from perturbation theory, is discussed in Section 2. Remarkably, the Hubble rate for the model, with the associated total energy density and pressure, leads to an exact solution for super-horizon scalar perturbations, which we present in Section 3.

The toy-model solutions of [3, 4, 5] are given approximately or numerically. The advantage of an exact solution lies in the additional clarity and the definiteness with which certain questions may be answered. As an example, we show that for our solution, the standard conserved quantity for growing modes is exactly constant, and exactly zero for the decaying modes. In line with the toy-model results, our solution also confirms the standard picture of strong amplification of perturbations through the transition from inflation to the radiation era. We extend the toy-model results of [3, 5] in section 3 by finding the exact form of super-horizon tensor perturbations. For completeness, we also give exact expressions for scalar and tensor sub-horizon perturbations in the vacuum- and radiation-dominated eras. In Section 4 we consider the extension of our results to incorporate a second smooth transition from radiation to matter domination.
2 The inflationary model

In a decaying vacuum cosmology (see e.g. [7] and references cited there), the false vacuum, with energy density $\Lambda(t)$, decays into radiation, with energy density $\rho(t)$. The total (conserved) energy density is $\rho_T = \rho + \Lambda$, and the total pressure is $p_T = \frac{4}{3}\rho - \Lambda$. These appear in the field equations for a flat Friedman-Lemaitre-Robertson-Walker (FLRW) universe:

$$\rho_T = 3H^2, \quad p_T = -2\dot{H} - 3H^2,$$

where $H = \dot{a}/a$ is the Hubble rate and $a$ is the scale factor. The decay of the vacuum into radiation is a non-adiabatic process, generating entropy and driving inflation until the vacuum energy falls low enough for inflationary expansion to end, and radiation domination to develop. The transition is inherently smooth. Despite the non-adiabatic interaction between the vacuum and created radiation, the combined vacuum-radiation system behaves like a perfect fluid, and standard adiabatic perturbation results may be applied to analyse metric scalar and tensor fluctuations. This feature also applies to the more general class of ‘warm’ inflationary models [8].

In [6], a decaying vacuum model is subject to the following simple physical conditions:

(a) the initial vacuum for radiation is a regular Minkowski vacuum (i.e. with zero energy density, particle number, entropy, etc.);
(b) the created radiation obeys the first law of thermodynamics for open systems, i.e.

$$d(\rho V) + pdV - \left(\frac{\rho + p}{n}\right)d(nV) = 0,$$

where $p = \frac{1}{3}\rho$ is the radiation pressure, $n$ its number density, and $V$ the comoving volume of the observable (causally connected) universe;
(c) the total number of radiation particles created throughout the expansion of the observable universe is finite.

It is then shown in [6] that these conditions strongly constrain the expansion, and that the simplest evolution consistent with the conditions is given by a Hubble rate

$$H(a) = 2H_e \left(\frac{a^2_e}{a^2 + a^2_e}\right).$$

(2)

Here $a_e$ is the epoch of exit from inflation, defined by $\ddot{a}_e = 0$, and $H_e$ is the Hubble rate at exit. For $a \ll a_e$, equation (2) shows that $H \approx$ constant, so that the initial evolution of the universe is approximately de Sitter inflation. For $a \gg a_e$ the Hubble rate falls off as $a^{-2}$, so that the universe becomes radiation-dominated. In this era, $\Lambda$ is negligible, falling off as $a^{-6}$.

The simple form (2) for $H(a)$ had been introduced in [9], but as an ad hoc toy model, without a consistent physical foundation as given in [6]. The cosmic proper time follows on integrating equation (2), as in [9]:

$$t = t_e + \frac{1}{4H_e} \left[\ln \left(\frac{a}{a_e}\right)^2 + \left(\frac{a}{a_e}\right)^2 - 1\right],$$

(3)

while equation (1) gives

$$\rho_T = 12H_e^2 \left[\frac{a^2_e}{a^2 + a^2_e}\right]^2, \quad p_T = 4H_e^2 \left[\frac{a^2_e(a^2 - 3a^2_e)}{(a^2 + a^2_e)^3}\right].$$

(4)

(5)

It follows that the effective pressure index and adiabatic sound speed are given by

$$w \equiv \frac{p_T}{\rho_T} = \frac{1}{3} \left(\frac{a^2 - 3a^2_e}{a^2 + a^2_e}\right), \quad c_s^2 \equiv \frac{\dot{p_T}}{\dot{\rho_T}} = \frac{1}{3} \left(\frac{a^2 - 5a^2_e}{a^2 + a^2_e}\right).$$

(6)

(7)
A physical length scale $\lambda$, corresponding to a comoving wave number $k$, is given by $\lambda = 2\pi a/k$. This scale crosses the Hubble radius $H^{-1}$ when $a = k/(2\pi H)$. By equation (2), the epoch $a_-$ of leaving and the epoch $a_+$ of re-entering are given exactly by

$$a_+ = \frac{k_e}{k} \left[ 1 \pm \sqrt{1 - \left(\frac{k}{k_e}\right)^2} \right].$$

(8)

where $k_e = 2\pi a_e H_e$ is the comoving wave number of the Hubble radius at exit. Scales with $k \geq k_e$ do not cross the Hubble radius, and remain sub-horizon. Scales which leave the horizon well before the end of inflation have $k \ll k_e$. It follows from (8) that for these super-horizon scales

$$a_+ \approx \left(\frac{2k_e}{k}\right)^\pm 1.$$

The fact that all $k < k_e$ modes re-enter during radiation domination is a consequence of the simplistic nature of the model. A more complete model requires a further transition from radiation to matter domination, and this adjustment will invalidate the expressions for $a_+$.

The evolution of Bardeen’s gauge invariant potential $\Phi$ describing adiabatic scalar perturbations is given in [10] in terms of conformal time. Using the scale factor $a$ as the dynamical variable (as in [9, 3]), we get

$$a^2 \frac{d^2 \Phi}{da^2} + \frac{1}{2a} (7 + 6c_s^2 - 3w) \frac{d\Phi}{da} + \frac{3}{a^2} (c_s^2 - w) \Phi - \frac{1}{a^4 H^2} \nabla^2 \Phi = 0.$$  

(9)

Decomposing $\Phi(a, \vec{x})$ into eigenmodes $\tilde{\Phi}(a, \vec{k})$ of the comoving Laplacian $\nabla^2$, and using equations (6) and (7), this equation becomes

$$a^2 \frac{d^2 \tilde{\Phi}}{da^2} + 4a \left( \frac{a^2}{a^2 + a_e^2} \right) \frac{d\tilde{\Phi}}{da} + \left[ \pi^2 \left( \frac{k}{k_e} \right)^2 \left( \frac{a^2 + a_e^2}{a_e a} \right)^2 - 2 \left( \frac{a_e^2}{a^2 + a_e^2} \right) \right] \tilde{\Phi} = 0.$$  

(10)

The evolution of the modes $\tilde{h}(a, \vec{k})$ of gauge-invariant tensor perturbations is given by

$$\frac{d^2 \tilde{h}}{d\eta^2} + 2a H \frac{d\tilde{h}}{d\eta} + k^2 \tilde{h} = 0.$$  

With $a$ as the dynamical variable, and using equations (1), (6) and (7), this becomes

$$a^2 \frac{d^2 \tilde{h}}{da^2} + 2a \left( \frac{a^2 + a_e^2}{a^2 + a_e^2} \right) \frac{d\tilde{h}}{da} + \pi^2 \left( \frac{k}{k_e} \right)^2 \left( \frac{a^2 + a_e^2}{a_e a} \right)^2 \tilde{h} = 0.$$  

(11)

Super-horizon scales are characterized by $k \ll aH$, so that by equation (3), we can neglect the $k$-term in equation (10). Similarly, the $k$-term in equation (11) may be neglected.

### 3 Perturbation solutions

#### 3.1 Super-horizon perturbations

For those modes which leave the Hubble radius (necessarily during inflation), while they remain outside the Hubble radius, we can neglect the $k$ term in equation (10), and use the standard transformation for removing the first derivative (11), i.e. $\phi = \tilde{\Phi} \exp \int 2ada/(a^2 + a_e^2)$. This brings the equation into the remarkably simple form

$$a^2 \frac{d^2 \phi}{da^2} - 2\phi = 0,$$
which has the explicit exact solution
\[ \phi = C_1 a^2 + C_2 a^{-1}, \]
where \( C_1 \) and \( C_2 \) are arbitrary constants. Thus the exact solution for super-horizon scalar perturbations (with \( k \ll aH \)) is
\[ \tilde{\Phi} = A_k \left( \frac{a^2}{a^2 + a_e^2} \right) + B_k \left( \frac{a_e}{a} \right) \left( \frac{a_e^2}{a^2 + a_e^2} \right), \]
(12)
where \( A_k \) and \( B_k \) are arbitrary dimensionless constants, the latter corresponding to the decaying modes.

In the vacuum-dominated era, when \( a \ll a_e \), and assuming that the scales leave the Hubble radius well before exit, it follows that \( |\tilde{\Phi}| \) grows as \( a^3 \). In the radiation-dominated era (\( a \gg a_e \)), \( |\tilde{\Phi}| \) is approximately constant (while the scales are still super-horizon). Therefore we have a consistent model in which the super-horizon scalar perturbations are known exactly via equation (12). They are strongly amplified during inflation and then remain approximately constant after inflation. This is in line with standard results that use an instantaneous transition [10], as well as with the approximate results for toy-model smooth transitions [3, 4, 5]. Although our model is asymptotically de Sitter, \( \Phi \) quickly grows quadratically through the almost-de Sitter era.

For adiabatic super-horizon scalar perturbations, the growing modes have a conserved quantity, given by [12]
\[ \zeta = \tilde{\Phi} + \frac{2}{3(1 + w)} \left[ \tilde{\Phi} + a \frac{d\tilde{\Phi}}{da} \right]. \]
(13)
This is usually used in instantaneous transition models to express the growing perturbations at late times in terms of their early-time forms [10]. In [3, 4, 5], \( \zeta \) is used to estimate the amplification of perturbations with smooth transition. We can use our exact solution for a smooth transition, to show that in our case, \( \zeta \) is exactly constant, even if we include the decaying modes (\( B_k \neq 0 \)). Substituting from (6) and (12) into (13), we find that (to lowest order in \( k \); [5])
\[ \zeta = \frac{3}{2} A_k. \]
(14)
In particular, it follows that for the pure decaying modes (\( A_k = 0 \)), we have \( \zeta \) exactly zero. These results about the decaying modes differ from the standard formulation that \( \zeta \) is only conserved for growing modes, but they are in line with the analysis in [5]. Of course, the fact that \( \zeta = 0 \) for the decaying modes shows that \( \zeta \) is only useful as a conserved quantity for the growing modes.

For tensor perturbations on scales beyond the Hubble radius, so that \( k \ll aH \), we can neglect the \( k \) term in equation (11), leading to the exact solution
\[ \tilde{h} = C_k - D_k \left[ \frac{a_e}{a} + \frac{1}{3} \left( \frac{a_e}{a} \right)^3 \right], \]
(15)
where \( C_k \) and \( D_k \) are arbitrary dimensionless constants, the latter corresponding to the decaying modes and so the amplitude of the gravity waves is approximately constant. During inflation, when \( a \ll a_e \), and assuming that the scales leave the Hubble radius well before exit, equation (15) shows that
\[ \tilde{h} \approx C_k - \frac{1}{3} D_k \left( \frac{a_e}{a} \right)^3. \]
During radiation domination (\( a \gg a_e \)), assuming that the wavelength is still super-horizon, equation (15) leads to
\[ \tilde{h} \approx C_k - D_k \frac{a_e}{a}. \]

### 3.2 Sub-horizon perturbations

For perturbations on scales that are inside the Hubble radius, i.e. with \( k/(aH) \) not negligible, equations (11) and (11) can not be solved exactly. However, we can give analytic forms for the solutions in the vacuum-dominated and radiation-dominated eras, since the equations reduce to Bessel form. Using (11) and (11), with \( Z \), denoting a linear combination of the Bessel functions \( J_0 \) and \( Y_0 \), we find the following.
Vacuum-dominated era:
\[
\ddot{\Phi} \approx \left( \frac{a_e}{a} \right)^{1/2} Z_{3/2} \left( -\pi \frac{k a_e}{k_e} a \right) \\
= E_k \left( \frac{a_e}{a} \right) \left[ \frac{k}{k_e} \sin \left( \pi \frac{k a_e}{k_e} a \right) + \frac{a}{a_e} \cos \left( \pi \frac{k a_e}{k_e} a \right) \right] \\
+ F_k \left( \frac{a_e}{a} \right) \sin \left( \pi \frac{k a_e}{k_e} a \right) - \frac{k}{k_e} \cos \left( \pi \frac{k a_e}{k_e} a \right),
\]
(16)
\[
\ddot{h} \approx \left( \frac{a_e}{a} \right)^{3/2} Z_{-3/2} \left( -\pi \frac{k a_e}{k_e} a \right) \\
= G_k \left[ \frac{k a_e}{k_e} a \sin \left( \pi \frac{k a_e}{k_e} a \right) - \cos \left( \pi \frac{k a_e}{k_e} a \right) \right] \\
+ H_k \left[ \frac{k a_e}{k_e} a \cos \left( \pi \frac{k a_e}{k_e} a \right) + \sin \left( \pi \frac{k a_e}{k_e} a \right) \right].
\]
(17)

Radiation-dominated era:
\[
\ddot{\Phi} \approx \left( \frac{a_e}{a} \right)^{3/2} Z_{3/2} \left( \pi \frac{k}{k_e a_e} \right) \\
= I_k \left( \frac{a_e}{a} \right)^2 \frac{a}{a_e} \sin \left( \pi \frac{k a_e}{k_e a_e} a \right) - \frac{k}{k_e} \cos \left( \pi \frac{k a_e}{k_e a_e} a \right) \\
+ J_k \left( \frac{a_e}{a} \right)^2 \left[ \frac{k}{k_e} a \sin \left( \pi \frac{k a_e}{k_e a_e} + a_e \cos \left( \pi \frac{k a_e}{k_e a_e} \right) \right] \\
+ K_k \left( \frac{a_e}{a} \right) \left[ L_k \sin \left( \pi \frac{k a_e}{k_e a_e} \right) - M_k \cos \left( \pi \frac{k a_e}{k_e a_e} \right) \right] \\
(18)
\]
\[
\ddot{h} \approx \left( \frac{a_e}{a} \right)^{1/2} Z_{1/2} \left( \pi \frac{k}{k_e a_e} \right) \\
= \left( \frac{a_e}{a} \right) \left[ L_k \sin \left( \pi \frac{k a_e}{k_e a_e} \right) - M_k \cos \left( \pi \frac{k a_e}{k_e a_e} \right) \right].
\]
(19)

Note that smooth toy models of standard inflation are only applicable for super-horizon modes around the time of transition, since on sub-horizon scales, the dynamics of the reheating era have a significant effect. This is not the case in the decaying vacuum model, which has no reheating era.

4 Concluding remarks

We have shown that a simple decaying vacuum model with a smooth transition from inflation to radiation domination has remarkably simple exact forms for its super-horizon perturbations, and that these forms confirm recent work on amplification and on the conserved quantity \( \zeta \). However, the model remains simplistic in the sense that it does not incorporate the second transition, i.e. from radiation to matter domination. Such an extension is necessary in order to provide quantitative predictions for the degree of amplification in modes which affect the microwave background and structure formation, as well as for the relative contribution of tensor perturbations.

In fact the decaying vacuum model can be extended to incorporate the creation of massive as well as massless particles, on the basis of the same physical requirements as discussed above. In this model, a simple Hubble rate that satisfies the requirements is given by [14]
\[
H(a) = 2H_e \left[ \frac{a_e^2}{a^2 + a_e^2} \right] \left[ \frac{a^2 + a_m^2}{a_m^{3/2} (a_e^2 + a_m^{3/2})} \right],
\]
(20)
where \( a_e \) is approximately the epoch of exit (i.e. \( \dot{a}_e \approx 0 \)), and \( a_m (\gg a_e) \) is approximately the epoch of matter-radiation equality. From [14] we see that
\[
H \sim \text{const} \quad \text{for} \quad a \ll a_e,
\]
\[ H \sim a^{-2} \quad \text{for} \quad a_e \ll a \ll a_m, \]
\[ H \sim a^{-3/2} \quad \text{for} \quad a_m \ll a. \]

Thus (20) describes a smooth evolution from inflation to radiation domination to matter domination, and it can be used to find the properties of super-horizon perturbations that leave the Hubble radius during inflation and re-enter soon after matter-radiation equality. Clearly, it is no longer possible to find exact analytic forms for these perturbations, and numerical integration will be necessary. This is the subject of further work.

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