CANONICAL SYSTEMS AND THEIR LIMITS ON STABLE CURVES

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ABSTRACT. We propose an object called ‘sepcanonical system’ on a stable curve \( X_0 \) which is to serve as limiting object- distinct from other such limits introduced previously- for the canonical system, as a smooth curve degenerates to \( X_0 \). First for curves which cannot be separated by 2 or fewer nodes, the so-called ‘2-inseparable’ curves, the sepcanonical system is just the sections of the dualizing sheaf, which is not very ample iff \( X_0 \) is a limit of smooth hyperelliptic curves (such \( X_0 \) are called 2-inseparable hyperelliptics). For general, 2-separable curves \( X_0 \) this assertion is false, leading us to introduce the sepcanonical system, which is a collection of linear systems on the ‘2-inseparable parts’ of \( X_0 \), each associated to a different twisted limit of the canonical system, where the entire collection varies smoothly with \( X_0 \). To define sepcanonical system, we must endow the curve with extra structure called an ‘azimuthal structure’. We show (Theorem 7.5) that the sepcanonical system is ‘essentially very ample’ unless the curve is a tree-like arrangement of 2-inseparable hyperelliptics. In a subsequent paper [13] we will show that the latter property is equivalent to the curve being a limit of smooth hyperelliptics, and will essentially give defining equation for the closure of the locus of smooth hyperelliptic curves in the moduli space of stable curves.

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INTRODUCTION

A big part of the geometry of a nonsingular curve revolves around its canonical system and canonical map, which is always an embedding except for the well-understood exception of hyperelliptic curves. The purpose of this paper is to identify and study an appropriate ‘limiting object’ of the canonical system as the curve degenerates to a stable one, an issue that is closely related to the extrinsic geometry, especially defining equations, of the closure of the hyperelliptic locus in the moduli space of stable curves.

The default choice for the limiting object is certainly the canonical system itself, i.e. the linear system associated to the dualizing sheaf, on the limiting stable curve. This choice will be analyzed in §1-3 below. What the analysis shows is that the canonical system on a stable curve \( X_0 \) is a good choice of limit as long

Date: January 20, 2013.
1991 Mathematics Subject Classification. 14H10, 14H51.
Key words and phrases. Canonical system, nodal curve, linear system, degeneration of curves.
arXiv.org/math.AG/1104.4747.
The second question is answered by a result in [13] which shows that the various \( |\omega_{X_0}|_Y^{\text{sep}} \) together effectively arise from a single bundle map, the so-called azimuthal modification of the Brill-Noether map.

As to the first question, the choice of twist is dictated by the requirement, motivated as mentioned above by enumerative applications, that \( |\omega_{X_0}|_Y^{\text{sep}} \) should be as large as possible (in particular it should contain \( |\omega_{X_0}|_Y \)), but should never jump as \( X_0 \) deforms keeping \( Y \) intact: in particular the dimension of \( |\omega_{X_0}|_Y^{\text{sep}} \) depends only on the combinatorics. The fact that enlarging \( |\omega_{X_0}|_Y^{\text{sep}} \) must come at the expense of \( Y \)’s neighbors severely limits the possible twist, leading to our particular choice (which, as mentioned above, differs from choices made previously, e.g. in [8] and [14]).

The fact that \( |\omega_{X_0}|_Y^{\text{sep}} \) is '2-inseparable' in the sense that it cannot be disconnected by removing 2 or fewer nodes. We will show (Theorem 3.12) that a 2-inseparable curve is a limit of smooth hyperelliptic curves iff its dualizing sheaf is not very ample. Those curves were classified long ago by Catanese [6].

Going beyond 2-inseparable stable curves \( X_0 \), it becomes less than clear that the canonical system on \( X_0 \) is the correct limiting object: for example it is no longer true that \( X_0 \) is the limit of hyperelliptics whenever its canonical map is not birational; indeed the locus of stable curves of genus \( g \) whose canonical map is not birational has components of fixed codimension \( c \) independent of \( g \) contained in the boundary (in fact, in the boundary component \( \Delta_c \) of the Deligne-Mumford Moduli space \( \mathcal{M}_g \)). One the other hand, the analysis of 2-inseparable curves suggests viewing these as atoms for the purpose of constructing the limiting object for a general curve.

Before proceeding to describe our limiting object, it should be mentioned here that there exist general constructions for limit objects of linear series, such as the one published in an Inventiones paper by Eisenbud and Harris (cf. [8]) and an unpublished one [14]. The Eisenbud-Harris construction is studied in much detail for the case of the canonical series, especially on curves comprised of two components intersecting in generically positioned points, in an Inventiones paper by Esteves-Medeiros [9] (which also references [14]). But this limit linear series is not a good limiting object from our perspective, because it does not vary smoothly, i.e. it jumps, in families (even locally trivial ones), and therefore does not appear to be useful for the sort of enumerative applications that we have in mind (see below).

The limiting object that we propose here is called the sepcanonical system. It is essentially a part of the full limit series associated to the canonical series (for a suitable modification of the family): more specifically, the largest part that never jumps. To define the sepcanonical system the curve first has to be endowed with some additional structure, closely related to Mainò’s notion of enrichment [12], which we call an azimuthal structure, and which consists essentially of a collection of smoothing directions at separating pairs of nodes (so-called ‘bisepts’). The composite object is called an azimuthal curve, and the parameter space for these is a certain blowup of the moduli space, in which the locus of 2-separable curves becomes a divisor. This notion is best behaved for curves of ‘semi-compact type’, meaning that distinct separating pairs of nonseparating nodes (called ‘proper bisepts’) are disjoint, because then the smoothing directions may be chosen independently. Things are more complicated for non-semi compact-type curves, but this turns out not to be much of a problem because we shall see that in a number of specific ways, those curves behave non-hyperelliptically.

Given an azimuthal curve \( X_0 \), the sepcanonical system \( |\omega_{X_0}|^{\text{sep}} \) is defined as a collection of linear systems \( |\omega_{X_0}|_Y^{\text{sep}} \) containing \( |\omega_{X_0}|_Y \), one on each ‘2-component’ \( Y \) of \( X_0 \), i.e. subcurve obtained by blowing up the separating nodes (called ‘seps’) and bisepts of \( X_0 \). The apparently arbitrary definition is justified, in part, by Theorem 7.5 which shows that in a versal deformation of \( X_0 \), each \( |\omega_{X_0}|_Y^{\text{sep}} \) occurs as the restriction on \( Y \) of the limit of a certain twist (by a linear combination of boundary components depending on \( Y \)) of the relative canonical bundle. Still one may ask

- why the particular choice of twist ?
- in what sense do the individual \( |\omega|_Y^{\text{sep}} \) for different \( Y \) form a whole?

As to the first question, the choice of twist is dictated by the requirement, motivated as mentioned above by enumerative applications, that \( |\omega_{X_0}|_Y^{\text{sep}} \) should be as large as possible (in particular it should contain \( |\omega_{X_0}|_Y \)), but should never jump as \( X_0 \) deforms keeping \( Y \) intact: in particular the dimension of \( |\omega_{X_0}|_Y^{\text{sep}} \) depends only on the combinatorics. The fact that enlarging \( |\omega_{X_0}|_Y^{\text{sep}} \) must come at the expense of \( Y \)’s neighbors severely limits the possible twist, leading to our particular choice (which, as mentioned above, differs from choices made previously, e.g. in [8] and [14]).

The second question is answered by a result in [13] which shows that the various \( |\omega_{X_0}|_Y^{\text{sep}} \) together effectively arise from a single bundle map, the so-called azimuthal modification of the Brill-Noether map.
Theorem 7.5 in turn is a consequence of an elementary ‘Residue Lemma’ which for certain twists of the canonical bundle identifies the sections on the special fibre which lift to the general fibre.

The main justification for the sepcanonical system is provided by a result in [13], based on this paper, which extends Theorem 3.12 to the general case and characterizes limits of smooth hyperelliptic curves. As a step towards this result, we prove here a characterization (Theorem 7.5) of stable curves whose sepcanonical system is not ‘essentially’ very ample (e.g. non-birational) on some 2-component, extending Catanese’s result for the canonical system: namely, those curves are trees of hyperelliptic 2-inseparables joined at Weierstrass points and hyperelliptic divisors. These are precisely the curves which are an admissible double cover of a rational tree.

An alternative (enumeratively-qualified) approach to limits of hyperelliptic curves (and more general 1-dimensional linear systems) focuses on the hyperelliptic pencil and its limits (as maps to rational trees), rather than the canonical system. This is based on the theory of admissible covers due to Harris-Mumford [11] and its generalizations such as the notion of relative maps developed by Faber and Pandharipande [10]. As mentioned above, another approach to the limit of the canonical series is due to Esteves-Medeiros [9], based on [8]. Some of the lemmas and propositions in §2-4 are similar in substance, if perhaps not in point of view, to results published previously. Also, our notion of curve with azimuthal structure is related to the notion of ‘enriched curve’ developed in an unpublished Harvard thesis by L. Mainò [12] (which in turn references our unpublished preprint [14]). In general, the notion of enriched curve, in which biseps play no special role, is different from that of (regular) azimuthal curve, in which they do; however, the two notions coincide for semicompact-type curves.

This paper was originally part of [13], but is being published separately due to length considerations. I thank Gwoho Liu for helpful comments, and David Eisenbud for sending a copy of Mainò’s thesis.

1. Canonical Systems

In this paper, all curves are nodal unless otherwise stated.

A semistable curve is a nodal curve whose dualizing sheaf $\omega_X$ has nonnegative degree on each irreducible component; equivalently, the desingularization of each rational component of $X$ contains at least 2 node preimages; a semistable curve is stable if the degree of $\omega_X$ is positive on each irreducible component. A stable pair or stable pointed curve is a pair consisting of a semistable curve $X$ and a smooth point $p$ on it so that $\omega_X(p)$ has positive degree on each irreducible component. Note if $\theta$ is a separating node on a stable curve $X$, then

$$(X, \theta) = (LX, L\theta) \bigcup_{L\theta \to R\theta} (RX, R\theta)$$

with each pair $(sX, s\theta)$ stable (as pair). We call the $(sX, s\theta)$ the left and right parts of $X$ with respect to $\theta$, also denoted $LX(\theta), RX(\theta)$.

2. Inseparables and Base of the Canonical System

The results of this section are not new (see e.g. [6]) and are included merely for completeness and because we have a different viewpoint that will continue later in the paper. We will deal with the base locus of the canonical system. This will turn out to be related to separating nodes. We begin by considering curves without separating nodes.

A semistable curve is said to be separable if it has a separating node, inseparable otherwise. The separation of a nodal curve $X$ is the blowup $X^{\text{sep}}$ of $X$ at all its separating nodes, called seps for short. Clearly the connected components of $X^{\text{sep}}$ are the maximal connected inseparable subcurves of $X$, called its inseparable components. Any irreducible component is contained in a unique inseparable component. The separation tree of a (connected nodal) curve $X$ is the graph, necessarily a tree, whose vertices are the inseparable components of $X$ and whose edges are the separating nodes of $X$. More generally, we may associate a...
separation tree to any collection of separating nodes on $X$. Separability is closely related to freeness of the canonical system, as the following Lemma begins to show:

**Lemma 2.1.** The canonical system of an inseparable connected semistable curve of genus $\geq 0$ is free (of base points).

**Proof.** We may assume $X$ is singular, and let $X' \to X$ be the blowup of a (nonseparating) node $\theta$, with node preimages $p_1, p_2$. Then $X'$ is a connected nodal curve of genus $g - 1$ and $h^0(\omega_X(p_1 + p_2)) = g > h^0(\omega_{X'})$. From the definition of dualizing sheaf it follows that $\omega_{X'}(p_1 + p_2)$ has a section nonvanishing at both $p_1, p_2$ hence $\omega_X$ is free at $\theta$. Thus $\omega_X$ is free at all (automatically nonseparating) nodes. It follows easily that any base point of $\omega_X$ must occur on a smooth rational components $C$, each of which must contain $r \geq 2$ nodes. Consider the restriction map

$$\rho : H^0(\omega_X) \to H^0(\omega_X|_C) = H^0(\omega_C(r)).$$

Its kernel may be identified with the differentials on the complementary subcurve to $C$, which has genus $g - r + 1$. Therefore $\rho$ has rank $r - 1$, hence is surjective, so $\omega_X$ has no base points on $C$. Therefore $\omega_X$ is free. □

A smooth rational inseparable component of a nodal curve $X$ is called a a separating line of $X$ (see [3], Definition 4.7) and $X$ is said to be a comb if it has a separating line. The following result is due to Catanese ([6], Thm. D), who uses a different terminology.

**Corollary 2.2 (Catanese).** The base locus of the canonical system on a semistable curve is the union of the separating lines and the separating nodes.

**Proof.** To begin with, it is elementary from Riemann-Roch that the base locus contains the separating lines and the separating nodes. For the converse, we use induction on the number of irreducible components of the curve $X$, assumed connected and of genus $\geq 0$. If $X$ has a separating line, we can conclude easily by applying induction to the remaining inseparable components. Else, let $Y$ be an inseparable component that constitutes an end vertex of the separation tree of $X$. Then $X = Y \cup_p Z$ with $Z$ connected and semistable and without separating lines, and $H^0(\omega_Y) = H^0(\omega_Y(p))$ again by Riemann-Roch. By Lemma 2.1, the base locus of $\omega_X$ on $Y$ is exactly $p$. As $H^0(\omega_X) = H^0(\omega_Y) \oplus H^0(\omega_Z)$, the proof is completed by applying induction to $Z$. □

A pair of smooth points on a nodal curve is said to be separated if they lie on different inseparable components.

**Corollary 2.3.** A separated pair of smooth points, not on any separating lines, of a connected stable curve, impose independent conditions on the canonical system.

**Proof.** If $p$ is a smooth point on a non-separating-line inseparable component $C$ of $X$, there is a $\omega_C$ differential on $C$ nonzero at $p$, and using the natural map $H^0(\omega_C) \to H^0(\omega_X)$, this may be extended to a dualizing differential on $X$ that is zero on $X - C$. □

An inseparable semistable curve $X$ is said to be PHEL (acronym for pseudo-hyperelliptic), if the canonical system $|\omega_X|$ fails to define an embedding. This includes all semistable curves or arithmetic genus 0 or 1. A stable pair $(X, p)$ with $X$ inseparable is said to be PHEL if $|\omega_X|$ is ramified at $p$. This terminology is temporary because we will see later that PHEL is equivalent to another notion, to be defined below even for possibly separated curves, that we will call 'hyperelliptic'.

**Lemma 2.4.** Suppose $(X, p)$ is PHEL, inseparable, and singular.

(i) If $X$ is irreducible of arithmetic genus $> 1$, the canonical system of $X$ defines a 2:1 map to a smooth rational curve, ramified on $p$.

(ii) If $X$ is reducible and $p$ lies on an irreducible component $C$ of positive arithmetic genus, then $C$ meets the rest of $X$ in precisely 2 points $p_1, p_2$; if $C$ has genus $> 1$ then $C$ is PHEL and its $g_2^1$ contains $2p$ and $p_1 + p_2$. 4
Proof. (i) is a special case of Proposition 3.11 (ii) follows from the fact that for any Cartier divisor of degree $> 2$, $\omega_C(D)$ is very ample. Of course, $C$ cannot meet the rest of $X$ in $< 2$ points. □

3. 2-INSEPARABLES AND SEPARATION BY THE CANONICAL SYSTEM

The purpose of this section is to develop some lemmas needed in the proof of the main result. These lemmas concern point separation (very ampleness) properties of the canonical system and some of its twists, which are closely related to separating binodes. They also yield short proofs of results of Catanese and others classifying the 2-inseparable nodal (more generally, Gorenstein) curves whose dualizing sheaf is not very ample, and characterization of the limits of hyperelliptic curves (Proposition 3.12).

First some definitions and remarks.

- A pair of nodes on a nodal curve $X$ is called a binode. A binode $(\theta_1, \theta_2)$ is properly separating or a bisep, for short, if each $\theta_i$ is nonseparating but $X - \{\theta_1, \theta_2\}$ is disconnected, hence necessarily has exactly 2 connected components whose closures are, at this point, arbitrarily-designated the left and right parts of $(\theta_1, \theta_2)$, denoted $\sl{X}(\theta_1, \theta_2), \sr{X}(\theta_1, \theta_2)$; in particular, $\theta = (\theta_1, \theta_2)$ itself will have a left and right preimages denoted $\sl{\theta}, \sr{\theta}$. A subcurve of $X$ of the form $\sl{X}(\theta), \sr{X}(\theta)$ for a sep or bisep $\theta$ is simply called a side of $X$. We will call a subset $\theta$ of $X$ a $^*$-sep if it is either a sep or bisep.

- A nodal curve is 2-inseparable if it is not disconnected by removal of any 2 or fewer nodes.

- The 2-separation of $X$ is the blowing-up of all seps and biseps. The connected components of the 2-separation are called 2-components of $X$.

**Lemma 3.1.** A 2-component is always inseparable.

**Proof.** Suppose a 2-component $Y$ is separable and let $\theta$ be a sep of $Y$, necessarily not a sep of $X$. If $\theta = (\theta_1, \theta_2)$ is a bisep of $X$ contained in $Y$ and $\theta \subset \sl{X}(\theta)$ or $\theta \subset \sr{X}(\theta)$, $\theta$ is already a sep on $X$. Otherwise, $(\theta_1, \theta)$ is a bisep on $X$, contradiction. A sep of $X$ contained in $Y$ leads to a similar contradiction. □

**Lemma 3.2.** Let $X$ be a 2-inseparable nodal curve and $a$ an effective Cartier divisor of degree $\geq 3$ on it. Then $\omega_X(a)$ is very ample.

**Proof.** Set $a = \deg(a), g = p_a(X)$. As $h^0(a) = g + a - 1$, we must prove

$$h^0(\omega_X(a - b)) = g + a - 3$$

for every length-2 subscheme $b$ of $X$. Assume first that $b$ is regular. Set $L = O(b - a)$. This has degree $< 0$ overall and $\leq 2$ on each subcurve. By Riemann Roch and Serre duality, what has to be shown is $h^0(L) = 0$. If not, let $s \in H^0(L)$ and $C$ the union of all irreducible components of $X$ on which $s$ is not identically zero. Because $\deg(L) < 0$ while $\deg(L|_C) \geq 0$, there must be a connected component $Y$ of the complementary curve $X - C$ such that $\deg(L|_Y) < 0$, hence $s$ vanishes identically on $Y$. Because $|Y \cap C| \geq 3$, $s|_C$ is a regular section of $L|_C$ with $\geq 3$ zeros, which contradicts $\deg(L|_C) \leq 2$.

This leaves only the case where $b$ is tangent to a branch at a node $\theta$ of $X$. Let $b : X' \rightarrow X$ be the blowing up of $\theta$ and $a' = b^*(a)$, of degree $a$. Then $X'$ is inseparable of arithmetic genus $g - 1$, and

$$h^0(\omega_X(a - b)) = h^0(\omega_{X'}(a' - p))$$

with $p$ the preimage of $\theta$ corresponding to $b$. What must be proven here is

$$H^0(O_{X'}(p - a')) = 0.$$ This proof is similar to the above, using inseparability of $X'$.

This result admits a useful partial extension to the inseparable case. First a definition.
**Definition 3.3.** A nodal curve $X$ is said to be 2-inseparable (resp. inseparable) relative to a subset (or divisor or subscheme...) $\theta$ if given a any bisep (resp. sep) $\theta$ of $X$, (the support of) $\theta$ meets both $L_X(\theta)$ and $g_X(\theta)$. Equivalent terminology is that $(X, \theta)$ is 2-inseparable or inseparable.

A useful remark about this notion is the following.

**Lemma 3.4.** Let $Y$ be a 2-component of $X$ and $\theta$ a bisep of $X$ contained in $Y$. Then $Y$ is inseparable and 2-inseparable relative to $\theta$.

**Proof.** We have seen that $Y$ is inseparable. Let $\theta'$ be a 2-sep of $Y$. If $\theta \subset L_Y(\theta')$ or $\theta' \subset R_Y(\theta')$, then $\theta'$ would be a bisep of $X$ itself, against our hypothesis that $Y$ is a 2-component.

□

The following Lemma on nonpositive bundles will prove useful.

**Lemma 3.5.** Let $L$ be a nontrivial line bundle of degree $\leq 0$ on an inseparable curve $X$, such $L$ has degree $\leq 2$ on any subcurve of $X$ and degree $\leq 1$ on either side of any bisep. Then $H^0(L) = 0$.

**Proof.** If $s$ is a nonzero section vanishing somewhere, let $Y_1, Y_2 \subset X$ be the union of all irreducible components on which $s$ is (resp. is not) identically zero. Since $s$ is a regular section of $L$ over $Y_2$, it cannot have more than 3 zeros there. Also, $Y_1 \neq \emptyset$. Therefore $Y_1$ and $Y_2$ meet in at most, hence exactly, 2 points which constitute a bisep. By the assumption on biseps, $L$ has degree at most 1 on $Y_2$, so again we have a contradiction.

□

Now we can prove the desired partial extension of Lemma 3.2.

**Lemma 3.6.** Let $X$ be an inseparable nodal curve and $a$ an effective smoothly supported divisor of degree $\geq 3$. Assume $X$ is 2-inseparable relative to $a$. Then $\omega_X(a)$ is very ample.

**Proof.** We follow the outline and notations of the proof of Lemma 3.2. We may assume $X$ is 2-separable. Consider first the case where $b$ is regular. Then $L = \mathcal{O}(b - a)$ satisfies the hypotheses of Lemma 3.5, hence $H^0(L) = 0$, as desired. Now to complete the proof, following the argument of the Lemma 3.2 it will suffice to consider the case where $b$ is tangent to the left side at a node $\theta_1$ belonging to a bisep $(\theta_1, \theta_2)$. Using the above notations, we need to show $H^0(\mathcal{O}_X'(p - a')) = 0$ where $p \in L_X'(\theta_2)$. Now $X'$ is inseparable, and clearly $L = \mathcal{O}_X'(p - a')$ satisfies the hypotheses of Lemma 3.5, so we are done.

□

According to Caporaso, an $r$-vine is a nodal curve of genus $g = r - 1$ of the form

$$(\mathbb{P}^1, p_1, \ldots, p_r) \bigcup (\mathbb{P}^1, q_1, \ldots, q_r).$$

$p_i \leftrightarrow q_i$

$i = 1, \ldots, r$

An $r$-vine is said to be an *interlace* if the pair of pointed $\mathbb{P}^1$’s are isomorphic (this notion, but not the term, already occur in Catanese’s article [5]). Note that an interlace $X$ admits a 2:1 morphism $\eta$ to $\mathbb{P}^1$, ramified precisely over the nodes and by Riemann-Hurwitz we have

$$\omega_X \sim (g - 1)\eta^{-1}(pt).$$

Because we can always assume $(p_1, p_2, p_3) = (q_1, q_2, q_3) = (0, 1, \infty)$, the parameter space for $r$-vines is an open subset of $(\mathbb{C}^* - \{0\})^2$ and that of $r$-interlaces is the diagonal.

An irreducible nodal curve is said to be hyperelliptic if it is obtained from a smooth hyperelliptic curve of genus $> 1$ by identifying some pairs belonging to the $g_2^1$, or obtained from a irreducible nodal or smooth curve of arithmetic genus 1 by identifying some pairs in a single $g_2^1$. The following is an initial characterization of 2-inseparable hyperelliptics; the definitive result is Proposition 3.11 below.
Lemma 3.7. Let $p, q$ be smooth, not necessarily distinct points on a 2-inseparable curve $X$. Then $p + q$ imposes independent conditions on $\omega_X$ unless either

(i) $X$ is irreducible hyperelliptic with $p + q \in g^1_2$; or

(ii) $X$ is an interlace and $p + q = \eta^{-1}(pt)$.

Proof. By Riemann-Roch, $h^0(\mathcal{O}(p + q)) = 2$. Assume $p, q$ are in a single irreducible component $C$. Suppose $X$ is reducible and let $Y_1$ be a connected component of the complementary curve $Y = X - C$. The $Y_1$ meets $C$ in at least 3 points, and we have an exact sequence

$$0 \to \omega_{Y_1} \to \omega_{C \cup Y_1} \to \omega_C(a) \to 0$$

As $H^1(\omega_{Y_1}) \cong H^1(\omega_{C \cup Y_1})$ (both curves being connected), the map $H^0(\omega_{C \cup Y_1}) \to H^0(\omega_C(a))$ is surjective. But by the Lemma above, $p + q$ clearly imposes 2 conditions on $\omega_C(a)$, therefore also on $\omega_X$, contradiction. Therefore $Y_1$ cannot exist and $X$ is irreducible. As $|p + q|$ induces a degree-2 pencil on the normalization of $X$, or on a genus-1 partial normalization if $X$ is rational, it is easy to see that $X$ is irreducible hyperelliptic.

If $p \in C, q \in D$ are in different irreducible components, connectedness ensures injectivity of the restriction maps

$$H^0(\mathcal{O}_X(p + q)) \to H^0(\mathcal{O}_C(p)), H^0(\mathcal{O}_X(p + q)) \to H^0(\mathcal{O}_D(q)).$$

Therefore

$$h^0(\mathcal{O}_C(p)), h^0(\mathcal{O}_D(q)) \geq 2,$$

so $C, D$ are smooth rational and there is a 2:1 map $C \cup D \to \mathbb{P}^1$. We claim $C \cup D = X$. If not, let $Y$ be a connected component of the complementary curve. Because $Y$ meets $C \cup D$ in at least 3 points, we may assume $Y$ meets $C$ in at least 2 points, therefore $\omega_C(Y \cap C)$ has nonnegative degree, hence is free. Then from surjectivity of

$$H^0(\omega_{C \cup Y}) \to H^0(\omega_C(C \cap Y))$$

we conclude there is a differential on $C \cup Y$ not vanishing at $p$, hence a section of $\omega_X$ zero on $D$ and nonvanishing at $p$. On the other hand because $X$ is inseparable, $\omega_X$ is free so there is a section of $\omega_X$ nonvanishing at $q$. Therefore $p, q$ impose independent conditions on $\omega_X$, contradiction.

Therefore $X = C \cup D$ is an interlace.

The above argument proves more generally the following

Lemma 3.8. Let $X, Y$ be 2-inseparable nodal curves with $X = C \cup Y$. Then

(i) the restriction map $H^0(\omega_X) \to H^0(\omega_C(C \cap Y))$ is surjective;

(ii) if $|C \cap Y| \geq 3$, then $|\omega_X|$ is very ample on the interior of $C$, i.e. $C \setminus C \cap Y$.

Next we extend the result of Lemma 3.2 to the case of twisting by a degree-2 divisor:

Lemma 3.9. Let $X$ be a 2-inseparable nodal curve and $a$ an effective Cartier divisor of degree 2 on it. Then $\omega_X(a)$ is very ample off $a$, i.e. separates every length-2 scheme except $a$, unless $(X, a)$ is hyperelliptic.

Proof. Actually, Proposition 3.11 below- which is proved independently of Lemma 3.9- already implies $X$ is hyperelliptic and this could be used to show $a$ is a hyperelliptic divisor. But we take a different tack instead, and follow the outline and notations of the proof of Lemma 3.2. Consider first $b$, a regular length-2 scheme other than $a$ and $L = \mathcal{O}(b - a)$, this time of degree 0. If $h^0(L) = 0$, $\omega_X(a)$ separates $b$. If $s \neq 0 \in H^0(L)$ vanishes nowhere, then $\mathcal{O}(a) = \mathcal{O}(b)$ and $a$ is a hyperelliptic divisor by Lemma 3.7. If $s$ vanishes somewhere, it is identically zero on some component, and an argument as in the proof of Lemma 3.2 applies. This settles the case $b$ regular. The case $b$ irregular is identical to that of Lemma 3.7.

We can partly extend this to the inseparable case:

\[ \]
Lemma 3.10. Let $Y$ be an inseparable curve, $p \neq q$ smooth points such that $Y$ is 2-inseparable relative to $p + q$. Then $\omega_Y(p + q)$ is very ample off $p + q$, unless $Y$ is 2-inseparable and $(Y, p + q)$ is hyperelliptic.

Proof. If $Y$ is 2-inseparable, the above Lemma applies. Else, $p, q$ belong to different 2-components $Y_1, Y_2$. It will suffice prove that $|\omega(p + q)|$, separates every length-2 scheme $b$ except at $p + q$. We will focus on the $b$ regular case, as the irregular one is handled as in Lemma 3.6. If $\mathcal{O}(b)$, hence $L = \mathcal{O}(b - p - q)$ has degree $\leq 1$ on each component, Lemma 3.7 applies, so we may assume $\mathcal{O}(b)$ has degree 2 on a unique 2-component $Z$. We may assume $p \not\in Z$. Let $\theta$ be a bisep on $Z$ with $p \in _1Y(\theta), Z \subset _2Y(\theta)$. Then clearly any section $s \in H^0(L)$ vanishes on $_1Y(\theta)$, hence on $_2\theta \subset Z$. If $q \in Z$, then $L$ has degree 1 on $Z$ but $s$ has 2 zeros there, so $s$ vanishes on $Z$, hence on all of $Y$. If $q \not\in Y$, there exists another bisep $\theta' \neq \theta$ with $q \in _1Y(\theta'), Z \subset _2Y(\theta')$. Then $s$ has 3 zeros on $Y$, hence vanishes $s$ above.

The following result due to Catanese [6] classifies the hyperelliptic 2-inseparable curves. Given the foregoing Lemmas, the proof is short and we will include it.

Proposition 3.11 (Catanese). Let $X$ be a 2-inseparable nodal curve of arithmetic genus $g > 1$, such that $\omega_X$ is not very ample; then $X$ is either

(i) irreducible hyperelliptic, or

(ii) an interlace.

In either case, the canonical system $|\omega_X|$ yields a 2:1 morphism onto a rational normal curve in $\mathbb{P}^{g-1}$.

Proof. Assume $X$ is neither irreducible hyperelliptic nor an interlace. We need to show every length-2 scheme imposes independent conditions, i.e. is embedded by $|\omega_X|$. The case of a scheme supported at smooth points was considered above. The case of a scheme $p + q$ where $p$ is a node and $q \neq p$ is elementary. Next, consider the case of a length-2 scheme $\zeta$ supported at a node $p$. If $X'$ is the blowup of $p$ with node preimages $p_1, p_2$ then $\zeta$ dependent for $\omega_X$ means $p_1 + p_2$ dependent for $\omega_{X'}$ (this is true whether $\zeta$ is regular or not). Note $X'$ is inseparable. If $X'$ is 2-inseparable then Lemma 3.7 applies. Else, let $\bar{g}$ be a separating binode of $X'$. If $\bar{g}$ is properly contained in a polyseparator $\Theta$ (see §4 below), then some connected component of $(X')^{\Theta}$ contains neither $p_1$ nor $p_2$, which makes $X$ 2-separable. Therefore $\bar{g}$ is already a maximal polyseparator, hence by Lemma 4.3 its sides $L_{1}X', R_{1}X'$ are both inseparable and again by 2-inseparability of $X$, each side contains precisely one of $p_1, p_2$. Because $L_{1}X', R_{1}X'$ are both free and $H^0(\omega_{1X'}), H^0(\omega_{1X'}) \subset H^0(\omega_{X'})$, this implies $p_1, p_2$ impose 2 conditions on $\omega_{X'}$.

The following characterization of the limits of smooth hyperelliptic curves among 2-inseparables is probably not new (compare for instance [2], §5.2) but is included because it follows easily from the foregoing discussion.

Proposition 3.12. Let $X$ be a 2-inseparable nodal curve of arithmetic genus $g > 1$. Then the following are equivalent:

(i) $X$ is a limit of smooth hyperelliptic curves.

(ii) The dualizing sheaf $\omega_X$ is not very ample.

(iii) $X$ is irreducible hyperelliptic or an interlace.

Proof. The equivalence of (ii) and (iii) is just Catanese’s result (Proposition 3.11). That (i) implies (ii) is trivial: if a smooth hyperelliptic curve $X$ specializes to a stable curve $X_0$, then a general point $p \in X_0$ will be contained in a length-2 scheme imposing 1 condition on $\omega_{X_0}$, so $\omega_{X_0}$ is not even birationally very ample. It remains to prove that (iii) implies (i). This is basically a folklore dimension counting argument that we will give in the case of an interlace, as that of an irreducible curve is similar. Thus let $X_0$ be an interlace, which we may assume corresponds to 2 identical $(g + 1)$-pointed $\mathbb{P}^1$’s of the form

$$(\mathbb{P}^1_1, 0, 1, \infty, p_1, ..., p_{g-2}) \cong (\mathbb{P}^1_2, 0, 1, \infty, q_1, ..., q_{g-2}).$$
Let $X/B$ be a versal deformation of $X_0$. Let $B_1 \subset B$ denote the locus of $(g+1)$-nodal curves, which may be identified locally with $(C_{g-2})^2 \times T$ for a polydisk $T$, with the $2(g-2)$ parameters corresponding to $p_1, ..., q_{g-2}$. On $B, B_1$ is locally defined by $(g+1)$ equations corresponding to the nodes, say $u_1, ..., u_{g+1}$. Let $p$ be a general point of $\mathbb{P}^1$, and let $p' \in \mathbb{P}^1_X, p'' \in \mathbb{P}^1_2 \subset X_0$ be the points corresponding to $p$. Then in a neighborhood of $(p', p'')$ we may identify $X^2_{B_1}$ with $(C_{g-1})^2 \times T$.

Now let $E$ be the Hodge bundle $\pi_*(\omega_{X/B})$ pulled back to $X^2_B$ and consider the evaluation map
\[ \phi : E \rightarrow p_1^*(\omega_{X/B}) \oplus p_2^*(\omega_{X/B}). \]

At $(p', p'')$, $\phi$ has rank 1 hence in a neighborhood of $(p', p'')$ on $X^2_B$, the degeneracy locus $D$ of $\phi$ is defined by $g-1$ equations, say $f_1, ..., f_{g-1}$, not necessarily forming a regular sequence. On the other hand, consider the restriction of $\phi$ over $X^2_{B_1}$, identified with $(C_{g-1})^2 \times T$. There, the degeneracy locus of $\phi$ corresponds to hyperelliptic $(g+1)$-nodal curves plus member of the $g_2$ and by Proposition 3.11 this can be identified with $C_{g-1} \times T$ sitting diagonally in $(C_{g-1})^2 \times T$, hence is still defined by a regular sequence of length $g-1$. It follows that $f_1, ..., f_{g-1}$ forms a regular sequence mod $(u_1, ..., u_{g+1})$, i.e. $f_1, ..., f_{g-1}, u_1, ..., u_{g+1}$ together form a regular sequence, hence no $u_i$ is a zero-divisor modulo $f_1, ..., f_{g-1}$ i.e. $D$ has no component contained in any boundary divisor corresponding to a node of $X_0$. Therefore $(p', p'')$ is a limit of hyperelliptic divisors on a smooth hyperelliptic curve.

**Remark 3.13.** Via the theory of admissible covers [11] and its generalizations as in [8] and [10], the limits of hyperelliptics can also be characterized among the stable 2-inseparables as those admitting a 2:1 map to $\mathbb{P}^1$. A drawback of this characterization is that it does not readily yield defining equations for the hyperelliptic locus. By contrast, the above result readily yields such equations, more precisely, equations for the locus of pairs (hyperelliptic curve, fibre of hyperelliptic map) as a degeneracy locus (always among 2-inseparables).

Now as soon as one leaves the realm of 2-inseparable curves, birational non-very ampleness of the canonical system is no longer a good notion of hyperellipticity: as the example below shows, the canonical map can have different degrees (1 or 2) over different components of its image, and such curves cannot be limits of smooth hyperelliptics. This is precisely the motivation for developing the notion of the ‘sepcanonical system’ the will occupy us in subsequent sections. One of the main results of the paper [13] will show that the limits of hyperelliptics are precisely the curves whose sepcanonical system is not birationally very ample.

**Example 3.14.** Let $\mathcal{H}$ be a separating binode on a nodal curve $X$, so that
\[ (1, X(\mathcal{H}), 1, \theta_1 + 1, \theta_2) \]

is a smooth hyperelliptic pair while $\mathbb{R}X$ is non-hyperelliptic. Then $\omega_X$ is not essentially very ample but $X$ cannot be a fibre of a family with smooth total space and general fibre a smooth hyperelliptic curve $X'$. This will follow easily from the results of the later sections but in an elementary fashion can be seen from the fact that the locus of hyperelliptic divisors (pairs not separated by the canonical system) is a divisor on $(X')^{(2)}$, which would extend to a divisor on the relative Hilbert scheme of the family (which can be assumed smooth); however, this locus meets the component $(\mathbb{R}X)^{(2)}$ of the Hilbert scheme $X^{[2]}$ in the isolated point $\theta_1 + \theta_2$.

A similar can be made with $\mathcal{H}$ replaced by a separating node, hyperelliptic only on one side.

### 4. Polyseparators

This section is essentially trivial in nature. Its purpose of this section is to collect for future reference some elementary, and probably well-known (compare [4], [5]), combinatorial remarks about nondisjoint collections of separating binodes on a curve. We begin with a few definitions.

**Definition 4.1.** Fix an inseparable nodal curve $X$. 


Lemma 4.4. (i) A polyseparator of degree $n$ on $X$ is a collection of nodes $(\theta_1, ..., \theta_n)$, $n \geq 2$ with the property that any distinct pair of nodes $\theta, \theta' \in \Theta$ is a separating binode.

(ii) $\Theta$ is a proper polyseparator if $n \geq 3$.

(iii) A binode or polyseparator is maximal if it is not properly contained in a polyseparator.

(iv) We associate to a polyseparator $\Theta$ a graph $G(\Theta)$ having as vertices the connected components of the separation or blowup $X^\Theta = \text{bl}_\Theta X$ and as edge-set $\Theta$.

**Definition 4.2.** (i) On a general nodal curve $X$, we define a polyseparator as a polyseparator on one of its inseparable components.

(ii) $X$ is said to be of semicompact type (or 'polyunseparated') if it has no proper polyseparators.

**Lemma 4.3.** Let $\Theta$ be a polyseparator of degree $n$ on an inseparable curve $X$. Then (i) $G(\Theta)$ is a simple $n$-gon and, with suitable notation,

$$X = \bigcup_{i=1}^{n} RX(\theta_i, \theta_{i+1}), \theta_{n+1} := \theta_1;$$

(ii) given a node $\theta \not\in \Theta, \Theta \cup \{\theta\}$ is a polyseparator iff $\theta$ is a separating node on the unique $RX(\theta_i, \theta_{i+1})$ containing it.

**Proof.** Induction on $n \geq 2$. With (i) being obvious for $n = 2$, we first prove (i) implies (ii) for given $n$. Notations as above, suppose $\theta$ is a node on $RX(\theta_1, \theta_2)$ separating it in 2 connected components $L, R$. Because $X$ is inseparable, precisely one of $\theta_1, \theta_2$ is on each of $L, R$ and we may assume $\theta_1 \in L, \theta_2 \in R$. But then clearly the separation

$$X^{\theta, \theta_1} = [R \cup \bigcup_{j=1}^{i-1} RX(\theta_j, \theta_{j+1})] \bigcup \bigcup_{j=i}^{n} RX(\theta_j, \theta_{j+1}) \cup L$$

so that $(\theta, \theta_1)$ is a separating binode. Conversely, suppose $\theta$ is a nonseparating node on $RX(\theta_1, \theta_2)$. Then $X^{\theta, \theta_1, \theta_2}$ has just 2 connected components and becomes connected when the two preimages of $\theta_2$ are identified, i.e. $X^{\theta, \theta_1}$ is connected, so $(\theta, \theta_1)$ is nonseparating and $\Theta \cup \{\theta\}$ is not a polyseparator.

To complete the proof it suffices to show that (i) and (ii) for given $n$ implies (i) for $n + 1$. So suppose $\Theta = (\theta_1, ..., \theta_{n+1})$ is a polyseparator of degree $n + 1$. Then $(\theta_1, ..., \theta_n)$ is a polyseparator and we may assume the above notations apply. Using (ii) and shifting cyclically, we may assume $\theta_{n+1}$ is a separating node on $RX(\theta_n, \theta_1)$ and inseparability of $X$ again implies that $\theta_{n+1}$ has exactly one of $\theta_n, \theta_1$ or either of its sides in $RX(\theta_n, \theta_1)$. This shows $G(\Theta)$ is an $(n + 1)$-gon.

A notation as above is called a cyclic arrangement for the polyseparator $\Theta$.

**Lemma 4.4.** Any polyseparator on an inseparable curve is contained in a unique maximal polyseparator.

**Proof.** Let $\Theta$ be a polyseparator and use a cyclic arrangement as in the previous lemma. Let $M(\Theta)$ be the union of $\Theta$ and the separating nodes on $RX(\theta_i, \theta_{i+1})$. By the Lemma, any polyseparator containing $\Theta$ is contained in $M(\Theta)$. Therefore it suffices to prove $M(\Theta)$ is a polyseparator. The proof is by induction on $m(\Theta) := |M(\Theta) \setminus \Theta|$. If $m(\Theta) > 0$, pick any $\theta \in M(\Theta) \setminus \Theta$ and let $\Theta' = \Theta \cup \{\theta\}$. To finish the induction it suffices to prove $M(\Theta') = M(\Theta)$. With no loss of generality, we may assume $\Theta' = (\theta_1, ..., \theta_{n+1})$ is a cyclic arrangement as above. If $\alpha \in M(\Theta')$, we may assume $\alpha$ is a separating node on $RX(\theta_n, \theta_{n+1})$. By inseparability of $X$ again, $\alpha$ has exactly one of $\theta_n, \theta_{n+1}$ on either side in $RX(\theta_n, \theta_{n+1})$, which makes it a separating node on $RX(\theta_n, \theta_1)$, so $\alpha \in M(\Theta)$. Conversely, if $\alpha \in M(\Theta)$, we may assume $\alpha$ is a separating node on $Y = RX(\theta_n, \theta_1)$ with $\theta_n \in LY(\alpha), \theta_1 \in RX(\alpha)$. But then if $\theta_{n+1} \in LY(\alpha)$, then $\alpha$ is a separating node on $RX(\theta_{n+1}, \theta_1)$, while if $\theta_n \in RX(\alpha)$, then $\alpha$ is a separating node on $RX(\theta_n, \theta_{n+1})$. Therefore in either case $\alpha \in M(\Theta')$.

**Lemma 4.5.** Any two maximal polyseparators on an inseparable nodal curve are disjoint.
Proof. Let \( \Theta = (\theta_1, ..., \theta_n) \) be a maximal polyseparator, cyclically arranged, and \( \theta \) another node, say \( \theta \in gX(\theta_j, \theta_{j+1}) \). If \( \theta \) separates \( gX(\theta_j, \theta_{j+1}) \), then \( \theta \in \Theta \) by maximality. If not, then clearly \( X^\theta/\theta \) is connected for all \( j \) (we can go around the circle starting from \( \theta_j \) and avoiding \( \theta \)). Therefore, \( \theta_j \) cannot be in any polyseparator not contained in \( \Theta \), so cannot be in any other maximal polyseparator. This proves the required disjointness. \( \square \)

We summarize the above results as follows.

**Proposition 4.6.** Let \( X \) be an inseparable nodal curve. Then there is a collection of disjoint sets of nodes on \( X \) called maximal polyseparators, with the property that a given node \( \theta \) is properly separating iff \( \theta \) is contained in some maximal polyseparator.

A (oriented) bisep \( \theta \) is said to be adjacent (or right-adjacent) on \( X \) if \( R_X(\theta) \) is inseparable. This terminology is justified by the following result, which follows easily from the above discussion:

**Corollary 4.7.** \( (\theta_1, \theta_2) \) is adjacent on \( X \) iff the unique maximal polyseparator containing it has the form, in cyclic arrangement, \( \Theta = (\theta_1, \theta_2, ..., \theta_m) \), \( m \geq 2 \).

**Lemma 4.8.** In a curve of semi-compact type, any two separating binodes lie entirely to one side of each other.

Proof. Let \( \theta \neq \theta' \) be separating binodes of \( X \), and assume for contradiction that \( \theta'_1 \in L_X(\theta), \theta'_2 \in R_X(\theta) \). Then \( b_{\theta'_1}L_X(\theta), b_{\theta'_2}R_X(\theta) \) are connected and their images in \( b_\theta X \) clearly meet, so \( b_\theta X \) is connected, contradiction. \( \square \)

The following two results collect some elementary properties of graphs of curves and of curves built up from 2-inseparable hyperelliptics, respectively. All the required proofs are easy enough to warrant omission.

**Proposition-definition 4.9.** Let \( X \) be a nodal curve.

(i) The 2-separation tree \( G_2(X) \) is the dual graph having as edges the separating nodes and all maximal (i.e. not contained in a proper polyseparator) properly separating binodes, and as vertices the connected components of the blowup of all the latter nodes and binodes.

(ii) This graph is a tree.

For a nodal curve \( X \), we will denote by \( S(X) \) the set of all its seps and biseps. Recall from the beginning of the previous section that a A 2-separation component (or 2-component) of \( X \) is by definition a connected component of the blowup of \( X \) in \( S(X) \), and that a 2-component is always inseparable.

**Proposition-definition 4.10.** Let \( X \) be a nodal curve.

(i) Any 2-component \( Y \) is endowed with a collection of smooth singletons or ‘unimarks’ coming from separating nodes of \( X \), and smooth pairs or ‘bimarks’ coming from separating binodes. Unless otherwise mentioned, these are always oriented with \( Y \) on the left.

(ii) A sep \( \theta \) is said to be locally left-hyperelliptic if for the 2-component \( Y \) to the left of \( \theta \), \( (Y, L_\theta) \) is hyperelliptic; ditto for locally right- and bilaterally (left + right) hyperelliptic; ditto for bisep in place of sep.

(iii) a locally left or right-hyperelliptic bisep is automatically maximal.

(iv) A stable curve \( X \) is said to be hyperelliptic if it is of semicompact type and every edge of \( G_2(X) \) is locally bilaterally hyperelliptic; given \( X \), it is is hyperelliptic iff every sep and bisep is locally bilaterally hyperelliptic.

(v) For a subset \( \Theta \subset S(X) \), \( X \) is said to be hyperelliptic relative to \( \Theta \) if every \( \theta \in \Theta \) is locally bilaterally hyperelliptic.

(vi) A sep \( \theta \) is said to be left hyperelliptic if \( (L_X(\theta), L_\theta) \) is hyperelliptic; ditto for right and bilateral; ditto for bisep;
Moreover the set of sections satisfying the residue conditions

\[
\text{condition is informally, consider a degeneration of curves}
\]

Lemma about variations of twists of the canonical system in a family. To introduce the Lemma somewhat

\[
\alpha
\]

\[
\text{that a section}
\]

Lemma below says that under some mild hypotheses, this condition is also sufficient.

Then:

\[
\text{which case an étale multisection is just a reduced divisor disjoint from the singular locus. For such}
\]

\[
\text{an effective divisor}
\]

\[
\text{Suppose given }
\]

\[
\text{Composing with the trace map, we get a 'sum of residues' map}
\]

\[
\text{Res : } \pi_\ast(\omega_{X/B}(A + n\sigma)) \to \pi_\ast(\mathcal{O}_\sigma).
\]

Composing with the trace map, we get a ‘sum of residues’ map

\[
\sum_\sigma \text{Res} : \pi_\ast(\omega_{X/B}(A + n\sigma)) \to \mathcal{O}_B.
\]

When $B$ is local, we will identify $\pi_\ast$ with $H^0$.

**Lemma 5.1 (Residue Lemma).** Suppose given

- $\pi : X \to B$, a proper flat family of nodal curves of arithmetic genus $g$, with irreducible generic fibre, over an integral local scheme;
- $B_0 \subset B$, a closed subscheme (the 'boundary') with $\pi_0 : X_0 = X \times_B \overline{B_0} \to \overline{B_0}$ the corresponding subfamily;
- pairwise disjoint étale multisections $\sigma_i, i = 1, \ldots, m$ of $X/B$, identified with the associated relative divisors;
- an effective Cartier divisor $D \subset X$, disjoint from $\sum \sigma_i$ and from the generic fibre, such that, setting $\mathcal{O}_{X_0}(D_0) = \mathcal{O}_X(D) \otimes \mathcal{O}_{X_0}$, we have that the direct image sheaf $\pi_{0\ast}(\omega_{X_0/B_0}(D_0))$ is free of rank $g$ over $B_0$ ('constant sections hypothesis');
- where $\sigma_{i,0} = \sigma_i \cap X_0$, an element $\alpha_0 \in H^0(X_0, \omega_{X_0/B_0}(D_0 + \sum n_i \sigma_{i,0})), n_i \geq 1$.

Then:

1. $\alpha_0$ is a sum $\sum_{i=0}^m \alpha_{0,i}$ of sections

\[
\alpha_{0,i} \in H^0(X_0, \omega_{X_0/B_0}(D + n_i \sigma_{i,0})) \subset H^0(X_0, \omega_{X_0/B_0}(D + \sum_{j} n_j \sigma_{j,0}))
\]

iff $\alpha_0$ satisfies the residue conditions:

\[
\sum_{i,j} \text{Res}(\alpha_0) = 0 \in \mathcal{O}_{B_0}, i = 1, \ldots, m.
\]

Moreover the set of sections satisfying the residue conditions is a free $\mathcal{O}_{B_0}$-module of rank $g + \sum_{i=1}^m n_i \deg(\sigma_{i,0}) - m$. 

5. **Residue Lemma**

The definition of the sepcanonical system to be given in \[\text{is motivated by the following elementary Lemma about variations of twists of the canonical system in a family. To introduce the Lemma somewhat informally, consider a degeneration of curves } X/B \text{ with smooth total space over the disk, and let } D \text{ be a sum of components of the special fibre, and let } s_i : B \to X \text{ be sections disjoint from } D, i = 1, \ldots, n. \text{ Then the bundle } \omega_{X_0} \otimes (\mathcal{O}(D) \otimes \mathcal{O}_{X_0})(\sum s_i(0)) \text{ is a limit of } \omega_{X_0}(\sum s_i(b)) \text{ on the general fibre. What is the condition that a section } \alpha_0 \text{ of the former bundle lift to a section } \alpha_b \text{ of the latter bundle? \text{ An obvious necessary condition is } } \sum \text{Res}_{s_i(0)}(\alpha_0) = 0 \text{ (because the analogous sum for } \alpha_b \text{ vanishes by the Residue Theorem). The Lemma below says that under some mild hypotheses, this condition is also sufficient.}

First some terminology. For a family of curves $\pi : X \to B$, an étale multisection is an effective divisor $\sigma \subset X$ that is finite flat and étale over $B$ and disjoint from the singular locus of $X/B$. $B$ may be a point, in which case an étale multisection is just a reduced divisor disjoint from the singular locus. For such $\sigma$ and an effective divisor $A$ disjoint from $\sigma$ we have the residue map

\[
\text{Res} : \pi_\ast(\omega_{X/B}(A + n\sigma)) \to \pi_\ast(\mathcal{O}_\sigma).
\]

Composing with the trace map, we get a ‘sum of residues’ map

\[
\sum_\sigma \text{Res} : \pi_\ast(\omega_{X/B}(A + n\sigma)) \to \mathcal{O}_B.
\]

When $B$ is local, we will identify $\pi_\ast$ with $H^0$.

**Lemma 5.1 (Residue Lemma).** Suppose given

- $\pi : X \to B$, a proper flat family of nodal curves of arithmetic genus $g$, with irreducible generic fibre, over an integral local scheme;
- $B_0 \subset B$, a closed subscheme (the 'boundary') with $\pi_0 : X_0 = X \times_B \overline{B_0} \to \overline{B_0}$ the corresponding subfamily;
- pairwise disjoint étale multisections $\sigma_i, i = 1, \ldots, m$ of $X/B$, identified with the associated relative divisors;
- an effective Cartier divisor $D \subset X$, disjoint from $\sum \sigma_i$ and from the generic fibre, such that, setting $\mathcal{O}_{X_0}(D_0) = \mathcal{O}_X(D) \otimes \mathcal{O}_{X_0}$, we have that the direct image sheaf $\pi_{0\ast}(\omega_{X_0/B_0}(D_0))$ is free of rank $g$ over $B_0$ ('constant sections hypothesis');
- where $\sigma_{i,0} = \sigma_i \cap X_0$, an element $\alpha_0 \in H^0(X_0, \omega_{X_0/B_0}(D_0 + \sum n_i \sigma_{i,0})), n_i \geq 1$.

Then:

1. $\alpha_0$ is a sum $\sum_{i=0}^m \alpha_{0,i}$ of sections

\[
\alpha_{0,i} \in H^0(X_0, \omega_{X_0/B_0}(D + n_i \sigma_{i,0})) \subset H^0(X_0, \omega_{X_0/B_0}(D + \sum_{j} n_j \sigma_{j,0}))
\]

iff $\alpha_0$ satisfies the residue conditions:

\[
\sum_{i,j} \text{Res}(\alpha_0) = 0 \in \mathcal{O}_{B_0}, i = 1, \ldots, m.
\]

Moreover the set of sections satisfying the residue conditions is a free $\mathcal{O}_{B_0}$-module of rank $g + \sum_{i=1}^m n_i \deg(\sigma_{i,0}) - m$. 

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(ii) If the residue conditions (5.1) are satisfied, then \(a_0\) lifts to an element \(\alpha\) of 
\[H^0(\omega_{X/B}(D + \sum n_i\sigma_i))\] 
satisfying 
\[\sum_{\sigma_i} \text{Res}(\alpha) = 0 \in O_B, i = 1, \ldots, n.\]

Remarks 5.2. (i) Because \(\sum \sigma_i\) and \(D\) are disjoint, \(X\) admits an open affine cover 
\[(U_0 := X \setminus \sum \sigma_i U_1),\]
where \(U_1\) is an affine neighborhood of \(\sum \sigma_i\) contained in \(X \setminus D\). We can use this cover to compute cohomology and over \(U_0 \cap U_1\), naturally identify \(\omega_{X/B}\) and \(\omega_{X/B}(D)\) and compute residues for the latter sheaf using this identification.

(ii) In applications, \(D\) will usually be a sum of boundary components, i.e. components of \(X_{B_0}\).

Proof of Lemma. (i): By the constant sections hypothesis, \(\pi_*(\omega_{X/B}(D))\) is free of rank \(g\) and compatible with base-change. Hence \(R^1\pi_*(\omega_{X/B}(D))\) is free of rank 1 and compatible with base-change, more precisely the map \(R^1\pi_*(\omega_{X/B}) \to R^1\pi_*(\omega_{X/B}(D))\) is an isomorphism and remains one after any base-change. Let \(A\) be the set of sections \(a_0\) expressible as a sum \(\sum a_0, i\) as above, and let \(B\) be the set of sections \(a_0\) satisfying the residue conditions (5.1). Now for each \(i\), consider the long cohomology sequence of 
\[0 \to \omega_{X_0/B_0}(D_0) \to \omega_{X_0/B_0}(D_0 + n_i\sigma_i,0) \to \omega_{X_0/B_0}(D_0) \otimes O_{n_i\sigma_i,0}(n_i\sigma_i,0) \to 0.\]
The \(D_0\) twist is trivial on the third term because \(D\) is disjoint from \(\sigma_i\). This shows that \(H^0(\omega_{X_0/B_0}(D_0 + n_i\sigma_i,0))\) maps surjectively to the set of ‘polar parts’ in \(H^0(\omega_{X_0/B_0}(D_0) \otimes O_{n_i\sigma_i,0}(n_i\sigma_i,0))\) satisfying the \(i\)th residue condition in (5.1). This implies firstly that \(A \subseteq B\), i.e. ‘only if’ holds. Then, considering the maps 
\[A \subseteq B \subseteq H^0(\omega_{X_0/B_0}(D_0 + \sum n_i\sigma_i,0)) \to \bigoplus_i H^0(\omega_{X_0/B_0}(n_i\sigma_i,0))\]
it follows that \(A\), hence \(B\), map onto the submodule \(C\) of \(H^0(\omega_{X_0/B_0}(\sum n_i\sigma_i,0))\) consisting of polar parts satisfying all the residue conditions (5.1) for \(i = 1, \ldots, m\), which is obviously free and cofree of corank \(m\). Because we have an exact sequence 
\[0 \to H^0(\omega_{X_0/B_0}(D_0)) \to A \to C \to 0\]
and likewise for \(B\) in place of \(A\), it follows that \(A = B\) is free of the required dimension.

(ii) By (i), it suffices to consider the case \(m = 1\) and we will omit the index. Consider the exact diagram:
\[
\begin{array}{cccccc}
H^0(\omega(D)) & \to & H^0(\omega(D + n\sigma)) & \to & H^0(\omega(D)) & \to & H^1(\omega(D)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(\omega_{X_0/B_0}(D_0)) & \to & H^0(\omega_{X_0/B_0}(D_0 + n\sigma_0)) & \to & H^0(\omega(D)) & \to & H^1(\omega(D)) \\
\end{array}
\]
Here the rightmost horizontal maps are just residue. In fact the entire rightmost square is purely local and it it easy to check that the kernel of the upper right horizontal map– which is a surjection of locally free sheaves– goes surjectively to that to the lower right horizontal map. This proves our assertion. \(\square\)

Remark 5.3. The residue condition is automatically satisfied on an irreducible fibre but not on a reducible one.

Example 5.4. Here is an illustration of how the Residue Lemma is applied. Let \(X_0 = X_0 \cup \theta_1 M X_0 \cup \theta_2 R X_0\) be a 3-component 2-sep nodal curve, varying in a 2-parameter smoothing family \(X/B\), with divisors \(\partial_1, \partial_2\) where \(\theta_1, \theta_2\) persist, respectively. Let \(L, D = L X(\theta_1), \) a divisor on \(X\). Then 
\[\omega_{X/B}(D)|_{M X} = \omega_{M X}(2 R \theta_1 + L \theta_2).\]
We claim that a necessary and sufficient condition for a section \( \alpha \) of the latter sheaf to extend locally to a section of \( \pi_* (\omega_{X/B}(D)) \) is that \( \alpha \) should have zero residue, i.e. trivial pole, at \( _1 \theta_2 \). This of course implies \( \alpha \) has zero residue at \( \theta_1 \) as well.

We will prove sufficiency of the condition as necessity is similar and simpler. To this end, note first that by freeness of \( \omega_{\pi_* X_0} \) at \( _1 \theta_1 \), \( \alpha \) extends to a section of \( \omega_{X/B} |_{\pi_* X_0} \). Then by the Residue Lemma, this extends to a section of

\[
\omega_{\pi_* X_0}(\theta_2) (D + _1 \theta_2) = \omega_{X/B}(D) |_{X(\theta_2)}
\]

over \( \theta_2 \) with zero residue on \( _1 \theta_2 \). Then we glue this to any section of

\[
\omega_{X/B} |_{\pi_* X_0}(\theta_2) = \omega_{X/B}(D) |_{X(\theta_2)}
\]

(which automatically has zero residue on \( \theta_2 \)) and use the Residue Lemma again to get a section of \( \omega_{X/B} \).

\[\square\]

The foregoing example will be generalized in Theorem 7.5 below.

6. SEPCANONICAL SYSTEM

The purpose of this section is first to define intrinsically an object on a reducible curve \( X_0 \), viz. a collection of linear systems on its 2-components, collectively called the sepcanonical system of \( X_0 \). This will later be shown to coincide with a certain collection of limits of the canonical system, i.e. specializations of its twists by certain components of the special fibre. This construction is nontrivial when the special fibre is 2-separable. The sepcanonical system depends on some additional data on the curve called azimuth (essentially, a smoothing direction at each bisep).

**Definition 6.1.**

(i) If \((p_1, p_2)\) is a pair of distinct smooth points on a curve \( X \), the azimuth space at \((p_1, p_2)\) is \( \mathbb{P}(T_{p_1}X \oplus T_{p_2}X) \); an azimuth \( \zeta \) at \((p_1, p_2)\) is an element of the azimuth space. An azimuth is singular if it equals \([1,0]\) or \([0,1]\), regular otherwise. An azimuthal pair \(((p_1, p_2), \zeta)\) consists of a point-pair \((p_1, p_2)\) plus an azimuth \( \zeta \), as above.

(ii) For a bisep (properly separating binode) \( \theta = (\theta_1, \theta_2) \), a left (right) azimuth is an azimuth on its left (right) preimage, i.e. an element \( L\zeta \in \mathbb{P}(L \psi_{\theta_1} \oplus L \psi_{\theta_2}) \) (resp. \( R\zeta \in \mathbb{P}(R \psi_{\theta_1} \oplus R \psi_{\theta_2}) \)), where \( L \psi_{\theta_1} = T^* \psi_{\theta_1} L X(\theta) \) etc.

(iii) For a bisep \( \theta = (\theta_1, \theta_2) \), a middle azimuth is an element

\[
M\zeta \in \mathbb{P}(L \psi_{\theta_1} \oplus R \psi_{\theta_1} \oplus L \psi_{\theta_2} \oplus R \psi_{\theta_2}).
\]

\( M\zeta \) is regular if \( M\zeta = [a, b] \), \( a, b \neq 0 \). An azimuth is a triple \( \zeta_\theta = (L\zeta, M\zeta, R\zeta) \) consisting of a left, middle and right azimuth. \( \zeta_\theta \) is said to be (internally) compatible if

\[
M\zeta = L\zeta R\zeta.
\]

(iv) An azimuthal (resp. middle azimuthal curve) is a nodal curve together with a choice of compatible azimuth (resp. middle azimuth) at each separating binode.

**Definition 6.2.** An azimuthal marking \( \zeta = (p, (q, q')) \) on a curve \( Y \) consists of:

(i) a collection of smooth points (unimarks) \( p_i \in Y \), together with a designation of each as either ‘co-hyperelliptic’, with attached ‘multiplicity’ \( n(\zeta, p) \), or ‘non co-hyperelliptic’, with attached multiplicity \( n(\zeta, p) = 3 \);

(ii) a collection of smooth pairs (bimarks) \( (q_i, q'_i) \), together with a designation of some of them as ‘co-hyperelliptic’, and a choice of regular azimuth \( \zeta(q_i, q'_i) \) (only) at those.

**Remarks 6.3.** (i) The co-hyperelliptic property of a unimark or bimark and the ‘multiplicities’ above are, from \( Y \)’s perspective, completely arbitrary, and in particular is unrelated to being a W-point or hyperelliptic divisor on \( Y \) itself. In applications, there will be another curve attached to \( Y \) at those points, and the data will be related to hyperellipticity of the other curve.
(ii) In an azimuth $\zeta$, any two of $\xi, M\zeta, R\zeta$, if not both singular, uniquely determine a third so that the triple is a compatible azimuth.

(iii) If $p_1 \neq p_2$ and $p_1 + p_2$ belongs to the unique $g_2^1$ on a hyperelliptic curve $X$, there is uniquely determined on $p_1 + p_2$ a hyperelliptic azimuth, automatically regular.

(iv) In particular, given a regular middle azimuth $M\zeta$ on a right-hyperelliptic bisep $\theta$, there is uniquely determined left azimuth on $\theta$, compatible with $M\zeta$ and the hyperelliptic right azimuth.

(v) Given a bisep $\theta = (\theta_1, \theta_2)$ as above on a curve $X_0$ and a smoothing $X/B$ of it, then an element $v$ of the projectivized normal space to $\partial \theta_1 \cap \partial \theta_2$ at $0$ induces a middle azimuth $M\zeta$ at $\theta$. $M\zeta$ is regular iff $v$ corresponds to an infinitesimal smoothing at both $\theta_1$ and $\theta_2$.

Now the sepcanonical system associated to a middle-azimuthal curve $X$, which we are going to define, will consist of

- a sepcanonical bundle $\omega_X^{\text{sep}}$, a line bundle on the 2-separation of $X$, i.e. the disjoint union $\bigcup X_i$ of the 2-components;
- a linear subsystem $|\omega_X^{\text{sep}}| \subset |\omega_X^{\text{sep}}|$, called the sepcanonical system.

More generally, we will define a $\Theta$-sepcanonical system for any collection $\Theta$ of seps and biseps so that the ordinary sepcanonical system corresponds to the case where $\Theta$ consists of all seps and biseps.

**Definition 6.4.** Let $\theta$ be an oriented $*_{\text{sep}}$ (either sep or bisep) on a nodal curve $X$, $\Theta$ a collection of seps and middle-azimuthal biseps.

- $\theta$ is said to be right-hyperelliptic relative to $\Theta$ if
  
  (i) the pair $(R X(\theta), \theta)$ is hyperelliptic;
  
  (ii) every element of $(\Theta \setminus \theta) \cap R X(\theta)$ is hyperelliptic as a sep or middle-azimuthal bisep on $R X(\theta)$.

- if $\theta \in \Theta$ is a right-hyperelliptic bisep relative to $\Theta$, the induced left azimuth on $\theta$ is by definition the one induced by the given middle azimuth at $\theta$ and the hyperelliptic azimuth on $R \theta$.

**Definition 6.5.** Let $(X, \zeta_0)$ be a stable marked curve, $\Theta$ a collection of seps $\theta_i$ and middle-azimuthal biseps $(\bar{\theta}, M\zeta(\bar{\theta}))$, $Y$ a component of the separation $X^\Theta$.

Then the induced azimuthal marking $\zeta(X, \Theta)$ on $Y$ is defined as comprised of the following unimarks and bimarks. Assume all seps and biseps of $\Theta$ are oriented to have $Y$ on their left.

(i) Unimarks: those inherited from $\zeta_0$, plus unimarks $p = 1\theta_i, \theta_i \in \Theta$, designated as co-hyperelliptic iff $\theta_i$ is right-hyperelliptic relative to $\Theta$; the induced multiplicity is denoted $\text{rn}(X, \Theta, p)$ or $\text{rn}(X, \theta_i)$ if $\Theta$ contains all $*_{\text{sep}}$ seps on $X$.

(ii) Bimarks: those inherited from $\zeta_0$, plus bimarks $\{\{q_i, q_i', \zeta\} = \{\theta\}, \theta \in \Theta$, designated as co-hyperelliptic iff $\theta$ is right-hyperelliptic relative to $\Theta$, and in that case endowed with the induced azimuth.

**Definition 6.6.** Let $Y$ be a curve with an azimuthal marking $\zeta = (p = (\theta_i, q_i, \zeta))$. Then define the following items on $Y$:

- The sepcanonical twist:
  
  $$(6.1) \tau(Y, \zeta) = \sum_{\text{marks } p} n(\zeta, p) p + \sum_{\text{binarks } (q, q')} 2(q_i + q_i')$$

- The sepcanonical bundle:
  
  $$(6.2) \omega(Y, \zeta) = \omega_Y(\tau(X, Y))$$

- The sepcanonical system $|\omega(Y, \zeta)| \subset |\omega(Y, \zeta)|$ is the subsystem defined by the following conditions:

  (i) Residue conditions:
  
  $\text{Res}_{p_i}(\alpha) = 0, \forall i$;
  
  $\text{Res}_{q_i} + \text{Res}_{q_i'}(\zeta) = 0, \forall i$. 

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(ii) Azimuthal conditions corresponding to the given azimuth at all co-hyperelliptic biracks.

- When \( \zeta = \zeta(X, \Theta) \) and \( Y \) is a component of \( X^\Theta \), the same objects will be denoted by \( |\omega_X^{|\text{sep}}|_Y \) etc. and the collection of these for different components \( Y \) is called the sepcanonical system of \( X \) associated to \( \Theta \), denoted \( |\omega_X^{|\text{sep}}|_\Theta \).
- If \( a \) is a smoothly supported effective divisor of degree \( \geq 2 \) on \( X \), we define \( \omega_X(a)^{|\text{sep}} \) and \( |\omega_X(a)^{|\text{sep}}| \) analogously, pretending all seps and biseps are non-hyperelliptic and imposing residue conditions only.
- When \( \Theta = \text{Edge}(G_2(X)) \), \( \Theta \) will be omitted from the notation and the corresponding system \( |\omega_X^{|\text{sep}}| \) will be called the (absolute) sepcanonical system of (the middle-azimuthal curve) \( X \).

Remarks 6.7. (i) Note the natural inclusion

\[
|\omega_X^{|\text{sep}}|_Y \subset |\omega_X|_Y(2\sum L\theta_i + \sum (q_i + q'_i))
\]

(ii) It follows from Lemma 5.1 that an element of the sepcanonical system on \( Y \) is a sum of sections of \( |\omega_Y|_{Rn(\theta)}(1\theta_i) \) and \( |\omega_Y|_{2(q_j + q'_j)} \) for the various \( i \).

(iii) The definition of \( \omega_X(a)^{|\text{sep}} \) and \( |\omega_X(a)^{|\text{sep}}| \) is certainly not the 'correct' one in general, but it is good enough for the few cases we need.

(iv) The sepcanonical system \( |\omega_X|_Y \) is highly non-local in the sense that its restriction on one 2-component \( Y \) depends on the nature of other 2-components (as well as on the azimuthal structure). For example, we allow a pole of order 3 at \( p \), etc., we allow a pole of order 3 at \( p \) for a sep \( \theta \), when \( p \) is not co-hyperelliptic, i.e. when the other (right) side \( Z \) of \( \theta \) in \( X \) is non-hyperelliptic at \( R \theta \) because \( \omega_Z \) will then have a section with a simple zero at \( R \theta \) to match with a differential on \( Y \) with pole of order 3 at \( p \).

Remark 6.8. We can extend the definition of sepcanonical system to the case where \( X \) is semistable. (This is not essential, because a given family of curves can always be replaced with one with only stable fibres, albeit at the cost of making the total space singular.) Then the stable model \( \bar{X} \) of \( X \) is obtained by contracting a number of 'bridges', i.e. maximal rational chains each contracting to a node \( p \) of \( \bar{X} \). The bridge is said to be separating or not depending on \( p \). We firstly define \( |\omega_X|_{\text{sep}} \) to coincide with the pullback of \( |\omega_X| \) over all components not contained in bridges. Next, we will give a recipe for extending this over the bridges. The recipe may appear arbitrary, but it will be justified later in the discussion of modified Brill-Noether maps. Thus consider a component \( C \) of a bridge \( B \). Then \( C \) is naturally a 2-pointed \( \mathbb{P}^1 \) and may be identified with \( (\mathbb{P}^1, 0, \infty) \). If \( B \) is a separating bridge, with sides \( (L_X, L_P), (R_X, R_P) \), we define \( |(\omega_X)^{|\text{sep}}|_C \) to be the linear system generated by the following differentials (also given in terms of homogeneous polynomials, as subsystem of the appropriate \( \mathcal{O}(n) \), identifying \( \mathcal{O}_{p1} = \mathcal{O}(2\infty) \)):

(i) \( dx, dx/x^2; [X_0^2, X_1^2] \), if \( (L_X, L_P), (R_X, R_P) \) both hyperelliptic;

(ii) \( xdx, dx/x^2; [X_0^3, X_0^2X_1, X_3^2] \), if only \( (L_X, L_P) \) hyperelliptic;

(iii) \( dx, dx/x^2, dx/x^3; [X_0^4, X_0X_1^3, X_3^2] \), if only \( (R_X, R_P) \) hyperelliptic;

(iv) \( xdx, dx/x^2, dx/x^3; [X_0^4, X_0^3X_1, X_0X_3^2, X_4^2] \), if neither side hyperelliptic

(these define respectively a bi-ramified double line, a cusps-flex/flex-cusps plane cubic, and a biflex skew quartic). Note that the linear systems in question are uniquely determined by their degree (as specified by the \( Ln(\theta), Rn(\theta) \) functions) plus the residue condition. Thus, the above apparently arbitrary definition is forced.

If \( C \) is a component of a nonseparating bridge \( B \), with end points \( L_P, R_P \in X \), we define \( |(\omega_X)^{|\text{sep}}|_C \) analogous to case (i) above if \( L_P, R_P \) is a hyperelliptic divisor on the complementary curve \( X' = X - B \); and analogous to case (iv) above, otherwise.

We note that with this definition, it remains that \( |\omega_X|_{\text{sep}} \) defines \( g_2^1 \) on each 2-component, when \( X \) is semistable hyperelliptic.

Remark 6.9. Some inductive arguments may involve the case of a 'shoot' \( C \), i.e. a nonsingular rational curve meeting the rest of \( X \) in a single point \( p \). Such a shoot, if part of a larger semistable curve \( X' \supset X \), will also occur on a bridge on some other subcurve \( X'' \subset X' \), hence dealt with as above.
The following Theorem identifies the sepcanonical system as the limit value of the direct image of a certain twist of the relative canonical bundle. It should be mentioned that for the Eisenbud-Harris limit series associated to the canonical, some analogous results, especially for curves comprised of two components meeting in points in generic position, were obtained by Esteves-Medeiros ([9], §5).

Now the rationale for our particular choice of twist and limit is that it is the 'best of the worst'; namely, its restriction on a given 2-component $Y$ is the 'best behaved'-- from $Y$'s viewpoint-- among those limits of the canonical system which are available in all cases, including the 'worst' case, when components across from $Y$ are hyperelliptic; or, what amounts to the same thing, this limit does not jump as $X$ varies, preserving $Y$. To make this precise, one has to turn the binode boundary loci into divisors, which in general involves the 'azimuthal modification' of a given family which we now define (see [13], §6 for more details).

Let $X/B$ be a family of nodal curves, assumed versal near the special fibre $X_0$. Each bisp $\theta$ of $X_0$ corresponds to a regular codimension-2 subvariety $\partial_2$. Given a collection $\Theta_2$ of disjoint bises, the corresponding loci $\partial_2$ are transverse. Let $B(\Theta_2)$ denote their joint blowup in any order, which is independent of the order. Moreover assume each $\theta \in \Theta_2$ is oriented, so there is a choice of components $L X(\theta)$, $R X(\theta)$ of $X_{\partial_2}$. Then for each $\theta \in \Theta_2$, the inverse image of $L X(\theta)$ on the base-changed family $X_{B(\Theta_2)}$ is a non-Cartier divisor. We let $X_{B(\Theta_2),L}$ be their common blowup, called the left azimuthal modification associated to $\Theta_2$. Let $L X_{B(\Theta_2)}(\theta)$ be the associated Cartier divisor on $X_{B(\Theta_2),L}$.

Given this notion, the result is as follows.

**Theorem 6.10.** Let the following be given:

- $\pi : X \rightarrow B$, a family of nodal or smooth curves with irreducible general fibre, assumed versal near a fibre $X_0$;
- $\Theta_1$, a set of sep's, $\Theta_2$ a set of maximal, regularly middle-azimuthal bises on $X_0$;
- with $\Theta = \Theta_1 \cup \Theta_2$, a connected component $Y$ of the separation $X^0_0$. Assume each $\theta_i \in \Theta_1$ and $\theta_i \in \Theta_2$ is oriented having $Y$ on its left.

Let

$$
\pi' : X' = X_{L,B(\Theta_2)} \rightarrow B' = B(\Theta_2)
$$

be the associated left azimuthal modification and $0' \in B'$ a preimage of 0 corresponding to the given collection of regular middle azimuths $\Theta \in \Theta_2$.

Then: a section

$$
\alpha_0 \in H^0(\omega_Y(3 \sum_{\theta \in Y} L \theta_i + 2 \sum_{\theta \in Y} (L \theta_i)))
$$

extends near $0' \in B'$ to a local section

$$
\alpha \in \pi'_*(\omega_{X'/B'}(-2 \sum_{\theta \in \Theta_1} L X'(\theta) - \sum_{\theta \in \Theta_2} L X'(\theta)))
$$

if and only if

$$
\alpha_0 \in |\omega_{X'/\Theta}|_Y.
$$

**Proof.** The proof is in essence an application of the Residue Lemma 5.1. We first prove sufficiency of (6.4), i.e. 'if'. Denote the line bundle in (6.3) on $X'$ by $\omega'$. Because $L X'(\theta) + R X'(\theta) = \partial_2$ for any $\ast$-sep $\theta$, $\omega'$ is isomorphic locally over $B'$ to

$$
\omega_{X'/B'}(+2 \sum_{\theta \in \Theta_1} R X'(\theta) + \sum_{\theta \in \Theta_2} R X'(\theta))
$$

for some $\ast$-sep $\theta$.
Now the latter sheaf contains an analogous one where $\Theta_1, \Theta_2$ are respectively replaced by $\Theta^0_1, \Theta^0_2$ which refer to the *-seps that meet $Y$. Therefore, locally over $B'$, $\omega'$ contains a subsheaf

$$\omega'_0 \simeq \omega_{X'/B'}(-2 \sum_{\theta \in \Theta^0_1} L X'(\theta) - \sum_{\theta \in \Theta^0_2} L X'(\theta))$$

Moreover, $\omega'_0$ is isomorphic to $\omega'$ near $Y$. Therefore in the proof of sufficiency, we shall henceforth assume all members of $\Theta_1, \Theta_2$ meet $Y$.

Next a remark. Let $(Z,z) = (\mathcal{R} X_0(\theta), \mathcal{R} \theta_i)$ for some $i$, i.e. the side of $X_0$ across $\theta_i$ from $Y$. Then $|\omega_Z|$ is always free at $z$ and $\omega_Z(-z)$ is free at $z$ iff $(Z,z)$ is non-hyperelliptic. Similarly, if $(Z,z_1,z_2) = (\mathcal{R} X_0(\theta), \mathcal{R} \theta_i)$, then the image of $|\omega_Z|$ in $\omega_Z \otimes \mathcal{O}_{X_1,z_2}$ is the hyperelliptic azimuth if $(Z,z_1,z_2)$ is hyperelliptic and the whole 2-dimensional space otherwise. From this it is easy to see that the azimuthal conditions, together with the lowered pole order condition (2 rather than 3) at the hyperelliptic unimarks, cf. Definition 6.4, are necessary and sufficient for $a_0$ to extend to the entire special fibre as a section of $\omega_{X_1}(\sum \mathcal{R} n(\theta_i) \mathcal{R} X_0(\theta_i) + 2 \sum \mathcal{R} X_0(\theta_i))$. Now we use an induction on $m = |\Theta|$. The case $m = 1$ is trivial. Pick any $\theta \in \Theta$ (sep or bisep), let and work over $B(\Theta \setminus \theta)$. Using Lemma 5.1 for $Y \cup L X'(\theta)$, we can deform $a_0$ to a differential $a_1$ on a component $Y_1$ of a nearby fibre $X'_1$ smoothing out exactly $\theta_1$ and keeping $\Theta \setminus \theta$ intact, where $a_1$ satisfies similar residue conditions, $m - 1$ in number, as well as appropriate azimuthal conditions (the fact that the latter can be imposed follows from an argument similar to the *-arbitrary residues summing to zero* argument in the proof of Lemma 5.1). Then by induction $a_1$, hence $a_0$, extends to the general fibre. This completes the proof of sufficiency.

We now prove necessity of (6.4), and accordingly drop the hypothesis that all members of $\Theta$ meet $Y$. We fix a bisep $\theta' \subset Y$ that is right-hyperelliptic relative to $\Theta$ and establish that azimuthal and residue conditions for $a_0$ at $L \theta'$; other cases are similar or simpler.

Consider a *-sep $\theta' \in \Theta$ on $\mathcal{R} X'_0(\theta')$ that is extremal relative to $Y$, i.e. such that there is no element of $\Theta$ strictly right of $\theta'$. Thus $\mathcal{R} X'_0(\theta')$ is a component of $\mathcal{R} X'_0(\Theta)$. Let $X'_1$ be the unique component of $\mathcal{R} X'_0(\Theta)$ containing the left side $L \theta'$. Because our family is versal, there is a nearby fibre $X'_1/B'$ where $\theta'$ deforms to a bisep $\theta'_1$, $\mathcal{R} X'_0(\theta'_1)$ survives intact, while $L X'_0(\theta'_1)$ smooths out. Because $a$ extends to this fibre, an argument as above shows that the residue and azimuthal conditions hold at $L \theta'$. Hence by specialization, they also hold at $L \theta'$. Because this is true for every extremal $\theta'$, it follows that the restrictions of $a$ on $X'_1$ belongs to the linear system $|\omega|^{sep}(X'_1, \bar{\xi})$ where $\bar{\xi}$ is the azimuthal marking on $X'_1$ induced by the collection of the external $\theta'$. Now the latter system is composite with the canonical system of $X'_1$, hence it fails to separate any hyperelliptic pair on $X'_1$ and is ramified at every $W$-point (see Lemma 7.2). Therefore we can continue and *peel off* $X'_1$ just like $\mathcal{R} X'_0(\theta')$; the azimuthal (and residue) conditions are satisfied on the left of every bisep on $X_1$ and similarly for seps. Continuing in this way, we eventually reach $\theta$ on $Y$, so the conditions are satisfied there as well.

\[ \square \]

7. Semistable hyperelliptics: general case

A stable, say semicompact-type, curve $X$ may be viewed as built up from 2-inseparable ‘atoms’; as such, the canonical system of $X$ behaves in an entirely heterogeneous manner: it can be birational on some atoms but not on others. This is not so for the sepcanonical system. Indeed a remarkable property of the sepcanonical system (on a middle-azimuthal stable curve), to be established here, is that its mapping behavior is quite homogeneous: either it is ‘essentially very ample’ (in particular birational on every atom); or else the curve is hyperelliptic in the sense defined above, i.e. comprised of 2-inseparable hyperelliptic atoms in a tree-like arrangement (and in particular is of semicompact type), and the sepcanonical system yields a 2:1 mapping on every atom. In particular, different 2-components influence the behavior of the sepcanonical system on each other. This property is part of the main result of this section (Theorem 7.5).
A stable middle-azimuthal curve is said to be hyperelliptic (as such) if it is hyperelliptic as ordinary curve and if moreover the given middle azimuth $\mathcal{M} \zeta(\theta)$ at every bisep $\theta$ necessarily hyperelliptic, is the hyperelliptic middle azimuth, i.e. the one determined by the hyperelliptic left and right azimuths at $\theta$. A semistable curve is pseudo-hyperelliptic if its stable contraction is. Our purpose here is to extend the characterization of 2-inseparable hyperelliptics in terms of very ampleness of the canonical system to the case of general nodal middle-azimuthal curves, where the canonical system is replaced by the sepcanonical one.

We begin with an elementary Lemma on the effect of azimuthal constraints on very ampleness.

**Lemma 7.1.** Let $Y$ be an inseparable nodal curve, $p \neq q \in Y$ a pair of smooth points such that $Y$ is 2-inseparable relative to $p + q$, and $\zeta$ an azimuth at $(p, q)$. Then the $\zeta$-constrained subsystem of $|\omega_Y(2p + 2q)|$ is essentially very ample unless $Y$ is 2-inseparable hyperelliptic, $p + q$ is a hyperelliptic divisor and $\zeta$ is the hyperelliptic azimuth.

*Proof.* If $Y$ is not 2-inseparable, then $\omega_Y(p + q)$ is already essentially very ample by Lemma 3.10 hence so is the constrained subsystem in question. If $Y$ is 2-inseparable and $p + q$ is not a hyperelliptic divisor (e.g. $Y$ is not hyperelliptic), $\omega_Y(p + q)$ is essentially very ample by Lemma 3.9 and again we are done.

It remains to consider the case where $p + q$ is a hyperelliptic divisor on the 2-inseparable hyperelliptic curve $Y$, where we must determine $\zeta$. Then the linear system $|\omega_Y(p + q)|$ is just $\text{Sym}^g \eta$ where $\eta = |p + q|$ is the $g^1_1$. On the other hand, $|\omega_Y(2p + 2q)|$ is very ample and contains as a hyperplane $\text{Sym}^{g+1} \eta$, which induces the same map to $\mathbb{P}^1$ as $\eta$ (albeit as rational normal curve of degree $g + 1$). This hyperplane corresponds to a point $z_0$ on the secant line $pq \subset \mathbb{P}^{g+2} = \mathbb{P}(H^0(\omega_Y(2p + 2q)))$ which can be identified with the hyperelliptic azimuth. Because projection from $pq$ itself corresponds to $\text{Sym}^g \eta$ (i.e. to the map to a rational normal curve of degree $g$), a projection from any $z \in pq$, can fail to be an embedding off $p + q$ is if the projection as a map on $X$ coincides with the $g^1_1$, coincide with projection from $z_0$; but because $|\omega_Y(2p + 2q)|$ is very ample, this is only possible if $z = z_0$, i.e. the hyperelliptic azimuth.

An azimuthally marked curve $(Y, \zeta)$ is said to be hyperelliptic if all unimarks in $\zeta$ are co-hyperelliptic and hyperelliptic on $Y$ and all bimarks are co-hyperelliptic and hyperelliptic on $Y$ and the associated azimuth in $\zeta$ is the hyperelliptic one. Then combining the above Lemma with the Residue Lemma 5.1 we conclude

**Lemma 7.2.** Let $(Y, \zeta)$ be a 2-inseparable hyperelliptic azimuthally marked curve. Then the sepcanonical system $|\omega|_{\text{sep}}(Y, \zeta)$ is composite with the canonical system on $Y$.

*Proof.* Part (i) of the Residue Lemma implies that we may assume $\zeta$ consists of a single (hyperelliptic) unimark or azimuthal bimark. Then the bimark case follows from the above Lemma, while the unimark case is straightforward.

Our result main result on the sepcanonical system, Theorem 7.5, is that unless a stable curve is hyperelliptic, its sepcanonical system is 'essentially very ample' (see below); only 'essentially' because of some exceptional behaviour at the separators, but this turns out to be good enough for applications. The proof has mostly been given above, in the discussion of 2-inseparables.

**Definition 7.3.** Let $Y$ be a curve with some unimarks and bimarks, all smooth. A linear system $(L, V)$ on $Y$ is said to be essentially very ample if it induces an embedding on every length-2 subscheme, except

(i) if $p$ is a unimark, at least one of

$$(L(-ip), V(-ip)), V(-ip) := V \cap H^0(L(-ip)), i = 0, 1, 2, 3$$

induces an embedding on the subscheme $2p$;

(ii) if $(p, q)$ is a bimark, at least one of $(L(-ip - iq), V(-ip - iq)), i = 0, 1, 2$ induces an embedding on $p + q$.

Another consequence of Lemma 7.1 is
Lemma 7.4. Let θ be a non-hyperelliptic azimuthal *-sep on X. Assume θ is left-extremal, i.e. 1_X(θ) contains no other *-sep of X. Then the sepcanonical system |ω_X|_sep is essentially very ample on the 2-component containing 1_θ.

Proof. Note 1_X(θ) is inseparable and 2-inseparable relative to 1_θ. The cases where θ is non-hyperelliptic as plain *-sep are settled by Lemmas 3.6 and 3.10. The remaining case, where θ is hyperelliptic but the azimuth is not, is settled by Lemma 7.1.

Theorem 7.5. Let X be a semistable middle-azimuthal curve of genus g ≥ 2. Then either
(i) the sepcanonical system |ω_X|_sep is essentially very ample on each 2-component of X; or
(ii) X is hyperelliptic and |ω_X|_sep maps each 2-component of X 2:1 to a rational normal curve.

Proof. First, it is easy to see that no generality is lost by assuming X is stable. By Proposition 3.11 we may assume X is 2-separable. We will assume to begin with that X is of semicompact type.

Assume X is (azimuthally) not hyperelliptic. Consider a 2-component Y of X that is an end-vertex of the separation tree G_2(X). We may assume Y is left-extremal for X. Let θ be the edge into Y. We assume θ is a bisep as the sep case is easier. By definition, θ is not (azimuthally) hyperelliptic. Therefore by Lemma 7.4, |ω_X|_sep is essentially very ample on Y. Now if θ is the unique edge of G_2(X), we are done by applying the same argument to its other vertex. Else, there exists another end-vertex Z with corresponding edge ρ, which we may assume is right-extremal. By the same argument, |ω_X|_sep is essentially very ample on Z. Because θ ⊂ 1_X(ρ), 1_X(ρ) is not hyperelliptic. By induction on the number of components, |ω_1_X(ρ)|_sep is essentially very ample, hence so is |ω_X|_sep|_1_X(ρ). Therefore |ω_X|_sep is essentially very ample.

It remains to consider the case where X is stable and not of semicompact type, hence contains a proper polyseparator Θ. Such X is never hyperelliptic. The claim then is that |ω_X|_sep is always essentially very ample. Suppose first that X contains a sep or maximal 2-sep, i.e. that the separation tree G_2(X) is nontrivial. Let θ be an edge. We may assume Θ ⊂ 1_R X(θ), which implies 1_R X(θ) is non-hyperelliptic. Replacing θ by a terminal edge to its left, we may assume Y = 1_X(θ) is an end-vertex of G_2(X). Because 1_R X(θ) is non-hyperelliptic, |ω_1_R X(θ)|_sep is essentially very ample by induction on the number of components, hence so is |ω_X|_sep|_1_R X(θ). Now if Y is non-hyperelliptic, |ω_X|_sep is similarly essentially very ample on it. Otherwise, Y is hyperelliptic and in particular of semicompact type. Because, as an end-vertex, Y contains no sep or maximal bisep of X, it follows that Y is 2-inseparable relative to 1_θ. Then it follows from Lemma 3.6 that |ω_X|_sep is essentially very ample on Y.

Now we may assume G_2(X) is trivial and X contains a proper polyseparator Θ. It will suffice to prove that |ω_X|_sep is essentially very ample of each connected component Y of X^Θ. We may assume

Y = 1_X(Θ), Θ = (θ_1, θ_2) ⊂ Θ = (θ_1, ..., θ_n), n ≥ 3.

If Y is non-hyperelliptic, then |ω_Y|_sep is already essentially very ample, hence so is the bigger system |ω_X|_sep|_Y. Therefore assume Y is hyperelliptic. Note that Θ is not right- hyperelliptic, so there is no left azimuthal condition attached to it. If Θ is contained in a single 2-component of Y then Y is 2-inseparable, hence |ω_X|_sep is essentially very ample on it by Lemma 5.2. Else, each θ_i, i = 1, 2 is contained in a different 2-component Y_i of X. Then because Y contains no other *-seps of X, it follows that Y must be 2-inseparable relative to 1_θ_1, 1_θ_2, so we can conclude by Lemma 3.6.

Remark 7.6. In [13], we show that the hyperelliptic middle-azimuthal stable curves are precisely the limits of smooth hyperelliptic curves. This is done by constructing a modification of the Hodge bundle on a suitable parameter space (a blowup of the Hilbert scheme), together with a map to the relative canonical bundle, whose degeneracy locus meets the boundary transversely.

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