Abstract

A new case of a preferential attachment model is considered. The probability that an already existing node in a network acquires a link to a new node is proportional to the product of its intrinsic fitness and its degree. We enrich this already known model by preferential deletion, which removes nodes at random with probability proportional to their fitness to some exponent. Under ‘normal’ conditions, the resulting node degree distribution is an asymptotic power-law (scale-free regime). We derive an exact condition for a phase transition after which one or a few nodes capture a finite fraction of all links in the infinite networks (dragon-king regime). By approximately ‘parametrizing’ the space of fitness distributions through the beta-density, we then show phase-diagrams that separate the two regimes.

In classical preferential attachment [1, 2], the probability of an existing node acquiring a new link is only proportional to the number of links it already has (its degree). Consequently, the older the node, the more links it has, which in turn, makes it even more popular (‘the rich get richer’). In highly competitive environments, this assumption appears oversimplified, and one expects competition coming from new and fitter nodes. This has led to the development of fitness-adjusted preferential attachment [3], in which the probability of an existing node acquiring a new link is proportional to the product of its degree and its intrinsic fitness. The more connected and the more fit a node, the more link it gets (‘the fit get richer’). It has been shown [4–6] that for some distributions of fitnesses in the infinite system, one or several nodes may dominate the system in the sense that they capture a macroscopic fraction of all links.

In a socio-economic context, we call such nodes dragon-kings (DKs) [7, 8], since they are endogenously generated statistical outliers beyond the power law regime that prevails in absence of any such DKs.

In this article, we add an additional ingredient to the picture: preferential removal of nodes. At any instance of time, a node may be removed from the system with some probability, e.g. due to failure, old age, attacks etc. We consider this probability proportional to both the product of its degree and its intrinsic fitness. The fitter and the more connected a node, the more attractive it is. Nodes may also be removed from the system (failure, death, etc.). At each time step, there are then $\ell + t$ nodes $n_1, n_2, \ldots, n_{\ell+t}$, and $mt + e_0$ edges in the network. The special case $\omega = 0$ of

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1. Growing a network with fitness-adjusted preferential attachment and deletion

Consider growing an undirected network by attaching one node per unit time to an already existing network. At time $t = 0$, we start with an initial network consisting of $\ell$ nodes and $e_0$ edges. At each time step $t = 1, 2, 3, \ldots$, a new node is added to the network by attaching it to $m$ randomly chosen vertices. At time $t$, there are then $\ell + t$ nodes $n_1, n_2, \ldots, n_{\ell+t}$, and $mt + e_0$ edges in the network. The fitter and the more connected a node, the more attractive it is. Nodes may also be removed from the system (failure, death, etc.). At each time step, with probability $e$ a randomly selected node is removed from the network, such that, at time $t$, there are on average $N(t) = (1 - e)t$ nodes in the system. It is reasonable to assume that the probability of failure depends on the fitness of the node. Specifically, we assume that the probability that node $n_i$ fails is proportional to $\eta_i^{\omega}$ for some exponent $\omega$. If $\omega > 0$, then the fit nodes are more likely to fail. This may be for instance because fit nodes are subject to targeted attacks. On the other hand, if $\omega < 0$, then weak nodes are likely to fail. This is reasonable for instance if we interpret $\eta$ itself as a measure of robustness with respect to failure. 1 The special case $\omega = 0$ of

1 Alternatively, we could also assume that the failure rate is proportional to $(k,\eta_i)^{\omega}$, at the cost of making the following calculations much harder to track.
uniform failure rates has been considered in ref [9]. Assuming that
the i-th node is still alive at time t, the rate at which the
degree of node ni increases on average is
\[ \frac{\partial k_i}{\partial t} = m \eta_i k_i S(t) - c k_i \sum_{\text{nbrs } j} \eta_j^{\nu_i} N_i (\eta j)^{\nu_j}, \]
where
\[ S(t) = \int_{1}^{t} d j D(j, t) k_j(t) \eta_j, \tag{2} \]
is a normalization factor and D(j, t) is the probability that the
j-th node is still present in the system at time t. The first term
on the right hand side of equation (1) is a direct consequence of
the attachment rule described above, and m accounts for the m
edges of the new node. The second factor is a product of two
independent probabilities. The probability c that a randomly
chosen node is deleted, and the probability that the node ni is con-
ected to the randomly failed node. The sum in (1) extends over
all neighbors of node ni and we have normalized according to \[ \sum_j \eta_j^\nu_j = N_i N_i^\nu_j \approx N_i (\eta j)^{\nu_j}. \]
Because that sum depends on the specific environment of node ni, we instead replace this expression
by the average node fitness \( \eta \), where the average is over
the fitness distribution \( \rho \). The mean-field version of (1) then becomes
\[ \frac{\partial k_i}{\partial t} = m \eta_i k_i S(t) \left[ 1 - c \frac{\langle \eta \rangle^{\nu_i}}{\sum_j \eta_j^{\nu_j}} \right]. \tag{3} \]
where we have used that \( N(t) \approx (1 - c \eta) t \). We make a multi-
scaling ansatz for the solution of (3), imposing that
\[ k_i = m \left( \frac{1}{i} \right)^{\gamma(\eta_i)}. \tag{4} \]
The scaling exponent \( \gamma \) depends on the fitness \( \eta_i \) of the node.
This is motivated for instance through the well-known result
[1] that in classical preferential attachment (no node removal,
uniform fitnesses) the degree of node ni scales like \( k_i \sim t^{1/2} \).

Next, we derive an explicit expression for D(i, t), the proba-
bility of node ni still being alive at time t. Clearly, it holds iteratively
\[ D(i, t + 1) = D(i, t) \left[ 1 - c \frac{\eta^{\nu_{1-i}}}{\sum_j \eta_j^{\nu_j}} \right], \tag{5} \]
which, in the continuous limit, yields
\[ \frac{\partial D(i, t)}{\partial t} = - \frac{c}{\sum_j \eta_j^{\nu_j}} \eta_i^{\nu_i} \frac{D(i, t)}{t}. \tag{6} \]
Together with the initial condition \( D(i, t) = 1 \) (the node that
is just introduced cannot be deleted in the same time-step), we find
\[ D(i, t) = \left( \frac{1}{i} \right)^{\gamma(\eta_i)}. \tag{7} \]

Next, we want to determine \( S \). Plugging (7) into (2) yields
\[ S(t) = m \int_{1}^{t} d i \left( \frac{1}{i} \right)^{\gamma(\eta_i)} \eta_i \]. If we want to perform this in-
tegration, we have to integrate over the \( \eta_i \), which are stochas-
tically sampled from \( \rho(\eta) \). To overcome this problem, we consider
the average quantity instead,
\[ \langle S(t) \rangle = m \int_{0}^{1} d \eta \rho(\eta) \int_{1}^{t} d i \left( \frac{1}{i} \right)^{\gamma(\eta)} \eta_i. \tag{8a} \]
\[ = m \int_{0}^{1} d \eta \rho(\eta) \frac{t - t^\gamma}{1 - t^\gamma}. \tag{8b} \]
To see that this approximation is justified in the limit of large
\( t \), consider the integration over some interval \([i, i + di] \) with di
small. Since \( (t/j)^{\gamma(\eta)} \) is approximately constant over that
interval, it amounts to considering some \( di \) iid realizations of random
variables \( \eta \) sampled from \( \rho(\eta) \). We then replace these samples by
their average and change the order of integration, leading to (8).
As is explained in Appendix A, it holds \( t^\gamma \in (0, 1) \). Thus, in the
limit \( t \to \infty \), the term \( t^\gamma \) is negligible with respect the term
linear in \( t \), and we end up with \( \langle S(t) \rangle \approx \tilde{A} m t \) where
\[ A = \int_{0}^{1} d \eta \rho(\eta) \frac{t - t^\gamma}{1 - t^\gamma}. \tag{9} \]
Plugging this asymptotic limit into (3) yields
\[ \frac{\partial k(\eta)}{\partial t} = \frac{\eta}{A} - \Omega \frac{k(\eta)}{t}. \tag{10} \]
The solution of this equation is exactly of the imposed form (4),
if we identify
\[ \gamma(\eta) = \frac{\eta}{A} - \Omega. \tag{11} \]
Finally, the constant \( A \) remains to be determined by self-
consistently plugging (11) into (9), leading to the condition
\[ 1 = \int_{0}^{1} d \eta \rho(\eta) \frac{\gamma(\eta)}{A (\frac{\gamma(\eta)}{A} + \frac{c}{\eta^{\nu_i}})} - 1. \tag{12} \]
This ends the general discussion, and we now have to distin-
guish between two different cases: The case where equation
(12) has a solution and the case where it does not.

2. Generic case: The fit get richer
In case there is an A solving equation (12), the cumulative
degree distribution \( P_k \) for fixed \( \eta \) is derived through
\[ P_k [k(\eta) < k] = P_k \left[ \left( \frac{m}{k} \right)^{\gamma(\eta)} < k \right] \tag{13a} \]
\[ = P_k \left[ i > \left( \frac{m}{k} \right)^{\gamma(\eta)} t \right] \tag{13b} \]
\[ = 1 - P_k \left[ i \leq \left( \frac{m}{k} \right)^{\gamma(\eta)} t \right] \tag{13c} \]
\[ = 1 - \frac{1}{1 - \frac{c}{\eta^{\nu_i}}} \left( \frac{m}{k} \right)^{\gamma(\eta)}. \tag{13d} \]
In the last step we have used that the probability of choosing any node \( n \) at random is uniform, and there are \((1 - c)I\) nodes in the network. The pdf is then obtained by taking the derivative, giving \( p_\eta \propto k^{-1/(1 - \frac{1}{\eta \rho^\eta})} \). The overall pdf is then just
\[
p(k) \propto \int_0^1 d\eta \rho(\eta) k^{-1/(1 - \frac{1}{\eta \rho^\eta})}.
\] (14)

Different fitness distributions give rise to different power law exponents through \( \gamma \), that are asymptotically dominated by \( \max_0 \gamma(\eta) \), as is made intuitive through a saddle-point approximation, see for instance [9, 11]. We call this case the ‘scale-free’ phase, in contrast to the case discussed next, in which some of the fittest nodes completely dominate the system.

3. Special case: Dragon-king nodes

It may happen that no \( A \) satisfying equation (12) can be found. To gain an understanding of what this means, note that \( S(t) \) has the interpretation of the (fitness adjusted) number of nodes in the system at time \( t \), and should hence scale linearly with \( t \). Specifically, if there is no removal of nodes \((c = 0)\) and all fitnesses are equal, \( S(t) \) reduces to the total number of outgoing links in the system, \( S(t) = 2mt \) and \( A = 2 \). If nodes have varying fitnesses and may be removed, \( A \) has to be adjusted. But if (12) cannot be satisfied, this means no such \( A \) exists. Going again through the above derivation, one notices that the only assumption that could have been violated is \( \tilde{\gamma} < 1 \). If \( \tilde{\gamma} = 1 \) for some nodes, they attract links proportional to the number of nodes added to the system. In a system of infinite size, they still connect a macroscopic fraction of all nodes in the system. They are dragon-king nodes. This intuitive reasoning has been confirmed more formally for \( c = 0 \), using a mapping of the network to a Bose-Einstein condensate [4]. In that mapping, the fitness \( \eta \) is mapped to an energy level \( \epsilon = -\log(\eta)/\beta \) and \( \beta \) the inverse temperature. The associated distribution \( g(\epsilon) = Be^{-\beta\epsilon} \rho(\exp(-\beta\epsilon)) \) can then be interpreted as the normalized degeneracy of the energy level \( \epsilon \). While elegant, it is not always straightforward to see how this mapping can be extended to different conditions, or arbitrary fitness distributions. For instance, in the fitness distribution has an integrable singularity at \( \eta \uparrow 1 \) (c.f. the \( \alpha = \beta = 1/2 \) line in figure 1), the degeneracy distribution \( g \) has a divergence at \( \epsilon = 0 \), i.e. an infinitely degenerate ground-state, with an interpretation that is far from trivial. Other derivations relying on master- or rate equations have been developed [5, 6], but to the best of our knowledge not in combination with preferential deletion. Here, we therefore derive a systematic classification of the existence of DK nodes that does not rely on a mapping to Bose-Einstein statistics. Specifically, we derive an exact condition for equation (12) having no solution. To this end, we define

\[
I(A) \equiv \int_0^1 d\eta \frac{\rho(\eta)}{A(\frac{4\eta}{\tau} + \frac{\epsilon}{\eta \rho^\eta}) - 1},
\] (15)

and ask if there is an \( A \) that satisfies \( I(A) = 1 \). To answer this question, we look for a lower boundary \( A \) for which we know that that definitely no \( A < A \) with \( I(A) = 1 \) exists. The claim is that DK nodes are present in the system if and only if \( I^* = I(A) < 1 \). To see this, note that \( I(A) \) is continuous and monotonically decreasing from \( I^* \) to zero as a function of \( A \). Assuming that \( I^* \geq 1 \) but finite, we can then continuously increase \( A \) from \( I(A = A^*) \) until \( I(A) = 1 \) is satisfied. Conversely, if \( I^* < 1 \), we cannot find any \( A \) with \( I(A) = 1 \). A case that needs special consideration is when \( I^* \) diverges. However, in that case, it is easy to see that we can always find some \( A_1 > A \) small enough such that \( I(A_1) > 1 \) but finite, because \( \lim_{A \downarrow A_1^*} I(A_1) = +\infty \). On the other hand, we can always choose an \( A_2 \) large enough such that the denominator in (15) is larger than 1 for all \( \eta \in (0, 1) \). Then, \( I(A_2) < \int d\eta \rho = 1 \) by normalization. Therefore, an \( A \in (A_1, A_2) \) exists with \( I(A) = 1 \). This finishes the proof of our claim.

So all we need is this lower boundary \( A \). Note that if the integrand of (15) has a zero value on its integration domain, \( I(A) \) diverges (for general \( \rho \)). and hence cannot satisfy \( I(A) = 1 \). Denote by \( \eta_{\min} \in (0, 1) \] the \( \eta \) that minimizes the expression \((1 + \Omega)/\eta + \kappa/\eta^{1 - \omega} \). Then, we know that the lower bound \( A \) is determined as \( A = \left( \frac{1}{\eta_{\min}} + \frac{\kappa}{\eta_{\min}^{1 - \omega}} \right)^{-1} \), since it will make the denominator in (15) vanish at \( \eta_{\min} \). Basic calculus lets us determine this \( \eta_{\min} \) explicitly as

\[
\eta_{\min} = \begin{cases} 
\frac{1}{\Omega^{1 - \omega}} & \text{if } \langle \eta \rangle^{1 - \omega} + \frac{\kappa}{\eta^{1 - \omega}} < \omega - 1 \\
\text{else} & 
\end{cases}
\] (16)

We then finally conclude that \( A = \frac{\eta_{\min}}{1 + \Omega + \kappa^{1 - \omega}} \). This finishes the calculations. To wrap up, we plug \( A \) into (15) and conclude that there are DK nodes (no solution for \( A \) can be found) if and only if

\[
I^* = \int_0^1 d\eta \frac{\rho(\eta)}{\eta_{\min} + \frac{1}{\Omega^{1 - \omega}} \eta + \frac{\kappa}{\eta^{1 - \omega}}} - 1 < 1,
\] (17)

with \( \eta_{\min} \) given by (16).

For \( \omega = 0 \), condition (17) simplifies to
\[
\int_0^1 d\eta \rho \left( \frac{1}{\eta} - 1 \right) < 1,
\]
which is exactly the condition derived in ref. [4] through the mapping to the Bose-Einstein gas. Note that this holds even in case of node deletion \( c > 0 \). The interpretation is that when nodes are removed uniformly at random, this has no influence on the phase-transition. In contrast, if there is node deletion, the rate at which nodes are removed matters.

4. Parametric classification of the DK regime

In the previous section, we have seen that for certain fitness distributions and deletion rates, DK nodes emerge and completely dominate the system. But can we somehow classify such distributions? Clearly, there is an uncountably infinite number of fitness distributions on the interval \([0, 1] \). Hence, it appears impossible to characterize such distributions systematically. However, consider the beta-distribution, with its density
Figure 1: Beta distribution (18) for different set of parameters $\alpha$ and $\beta$. We assume that this parametric class of distribution approximates the space of fitness distributions sufficiently well for all practical cases. DK nodes emerge when the bulk of the probability mass is concentrated on nodes of low fitness, and there are just a few fit nodes, that then end up dominating the system (e.g. $\alpha = 2, \beta = 5$).

Given by

$$\beta(\eta; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \eta^{\alpha-1} (1-\eta)^{\beta-1} \quad (18)$$

and $B$ the normalizing beta-function. Given the various shapes that are obtained when tuning its parameters (figure 1), we claim that, to first approximation, it captures all practically relevant classes of fitness constellations. In a sense, we thus ‘parametrize’ the space of fitness distributions. This has the advantage that the DK node condition (17) is now a function of four parameters. The phase transition problem now depends on four parameters, $I^* = I^*(\alpha, \beta, \omega, c)$.

In general, the integral in (17) with $\rho$ the beta-density (18) has to be solved numerically. However, the simplified case $\omega = 0$ of uniform node deletion allows for an explicit result:

$$I^* = \begin{cases} \frac{\alpha}{\beta-1}, & \text{if } \beta > 1 \\ \frac{\beta}{\omega-1}, & \text{if } \beta \leq 1. \end{cases} \quad (19)$$

We thus conclude that for $\alpha < \beta - 1$ and $\beta > 1$, it holds $I^* < 1$ and hence there are DK present. The separation between the scale-free and the DK-regime is along the line $\beta = \alpha + 1$. Figure 2 depicts the regime plot for general parameter configurations. 

Comparing with figure 1, we see that the DK regime corresponds to fitness landscapes in which the bulk of the nodes have low fitnesses, such that the few fit nodes end up dominating the system. In line with our intuition, the larger $\omega$, the more fitness distributions fall into the scale-free regime. This effect is enhanced, the larger the value of the deletion rate $c$.

In conclusion, we have extended the calculations of fitness-adjusted preferential attachment by also considering fitness dependent preferential deletion. We have shown under what circumstances the system undergoes a phase transition where some nodes capture a finite fraction of all links in the infinite network. By parametrizing the space of fitness distributions through the beta-density, we could furthermore derive explicit analytical results that characterize the regime change as a function of four parameters.

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Since every node, by construction, will have a fitness $\eta > 0$ and at least $m$ links, it will always have finite probability of attracting more links. Hence, its node degree will always be increasing, implying that $\gamma > 0$. On the other hand, in the most extreme case, node $n_i$ is so strong that it captures a new link every time a new node is added to the system. In that case, the node degree grows linearly, $\gamma = 1$. Since, by construction, we do not have such a scenario under ‘normal circumstances’, we conclude $\gamma < 1$. This constraint breaks down in the case of a phase transition with the emergence of DK nodes, where $\gamma = 1$ for the fittest nodes.

If we allow for node deletion, we must replace the above reasoning for $k_i(t)$ by the expected degree of node $n_i$ at time $t$ [11]. Since, by construction, a node that was removed from the network has zero edges, this quantity is given by $E[k_i] = k_i \cdot D(i, t) \propto t^{\gamma - \kappa \eta} = t^{\tilde{\gamma}}$. With the same reasoning as for the case without node deletion, we conclude that $\tilde{\gamma}$ must be below 1. Note that, somewhat counter-intuitively, the requirement $\tilde{\gamma} < 1$ still allows for $\gamma > 1$. In other words, the individual node may grow super-linearly, but only for a short time, as its lifetime is also limited [9, 11].