Computing the Scalar Field Couplings in 6D Supergravity

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Abstract

Using non chiral supersymmetry in 6D space time, we compute the explicit expression of the metric the scalar manifold $SO(1,1) \times \frac{SO(4,20)}{SO(4) \times SO(20)}$ of the ten dimensional type IIA superstring on generic K3. We consider as well the scalar field self-couplings in the general case where the non chiral 6D supergravity multiplet is coupled to generic $n$ vector supermultiplets with moduli space $SO(1,1) \times \frac{SO(4,n)}{SO(4) \times SO(n)}$. We also work out a dictionary giving a correspondence between hyperKahler geometry and the Kahler geometry of the Coulomb branch of 10D type IIA on Calabi-Yau threefolds. Others features are also discussed.

Key words: type II superstring compactification, black attractors, Kahler and hyperKahler geometry, harmonic superspace.

1 Introduction

At Planck scale, ten dimensional type IIA superstring compactification on K3 is described by non chiral $\mathcal{N} = 2$ supergravity in six dimensions [1]-[5]. There, the dynamical degrees of freedom come into two kinds of supersymmetric multiplets:

1) the gravity multiplet $\mathcal{G}_{6D}^{\mathcal{N}=2}$ consisting of 32 bosonic and 32 fermionic propagating degrees of freedom. The propagating bosonic fields are:

$$g_{\mu\nu}(x), \quad B_{\mu\nu}(x), \quad A_{\mu}^{a}(x), \quad \sigma(x).$$
They describe respectively the gravity field $g_{\mu\nu}$, the antisymmetric gauge field $B_{\mu\nu}$, four Maxwell type gauge fields $A^a_\mu$, $a = 1, 2, 3, 4$, and the dilaton $\sigma$. 

(2) twenty Maxwell type supermultiplets $\{V_{6D,N=2}^I\}_{1 \leq I \leq 20}$ having $20 \times 8$ bosonic and $20 \times 8$ fermionic propagating degrees of freedom. The 6D bosonic fields content of these supermultiplets consists of

$$A^I_\mu(x) \quad \phi^a_I(x) \ ,$$

that is twenty gauge fields $A^I_\mu$, capturing a local $U^{20}(1)$ gauge invariance,

$$A^I_\mu \rightarrow A^I_\mu + \partial_\mu \phi^I , \quad (1.1)$$

with gauge parameters $\phi^I$; and eighty real scalar $\{\phi^a_I\}$ parameterizing the real eighty dimensional manifold

$$Q_{80} = \frac{SO(4,20)}{SO(4) \times SO(20)} \ , \quad \text{dim} \ Q_{80} = 4 \times 20 .$$

The dynamical scalar fields of the non chiral $6D$ $\mathcal{N} = 2$ supergravity theory that we will deal with are then $\{\sigma, \phi^a_I\}$; they transform in the following particular representations of the $SO(4) \times SO(20)$ isotropy symmetry [6, 7],

$$\sigma \sim (1, 1) \ , \quad \phi^a_I \sim (4, 20) .$$

Generally, instead of $\phi^a_I \sim (4, 20)$, these scalar fields may be thought of as $\phi^a_I \sim (4, n)$ parameterizing, together with $\sigma$, the following typical moduli space family [8] involving a generic number $n$ of Maxwell supermultiplets,

$$M_{6D}^{N=2} = SO(1, 1) \times Q_{4n} .$$

The real dimension of $M_{6D}^{N=2}$ is equal to $(1 + 4n)$; the case of 10D type IIA superstring on K3 corresponds obviously to $n = 20$. For generic $6D$ $\mathcal{N} = 2$ supergravity models, the integer $n$ can however be any positive number; $n \geq 1$. The leading term of the family is particularly remarkable since, as we will show, corresponds to the well known real four dimensional Taub-NUT geometry.

In this paper, we freeze the dilaton $\sigma$ ($d\sigma = 0$) and study the interacting dynamics of the real $4n$ scalars $\phi^a_I$ that parameterize the $4n$ real dimensional scalar manifold

$$Q_{4n} = \frac{SO(4,n)}{SO(4) \times SO(n)} ,$$

with generic integers $n \geq 1$. We use rigid non chiral supersymmetry in 6D space time to determine the explicit expression of the scalar field couplings of the non linear sigma model that governs the dynamics of the scalars fields $\phi^a_I$. Besides the hyperKahler geometry of the underlying non linear sigma model, the knowledge of the scalar fields
self-couplings associated with $Q_{4n}$ is important for the study of the BPS and non BPS attractors in non chiral 6D supergravity [6, 7, 8, 9]. The interacting dynamics of the dilaton $\sigma$ with the scalar field $\phi^{aI}$ is recovered as usual; it will be implemented at the end of this work.

Having introduced the basic ingredients and the main objective of this study; the question that we have to answer is how to get the $\phi^{aI}$ self-interactions. To that purpose, we shall proceed in three main steps as follows:

(a) Introduce a complex representation to deal with $\phi^{aI}$

Instead of working with the $4n$ real coordinates $\phi^{aI}$, we use rather $2n$ complex fields given by the doublets

$$f^{IA}(x) \sim (2, n), \quad \overline{f}^{IA}(x) \sim (\overline{2}, \pi),$$

and transforming in the fundamentals of the group $SU(2) \times U(n)$. In using this complex representation, the above family of real $4n$ manifold $Q_{4n}$ gets replaced by the complex $2n$ manifold family

$$H_{2n} = \frac{U(2,n)}{U(2) \times U(n)}. \quad (1.2)$$

In addition to the power of complex analysis, this representation allows to exhibit manifestly the $U(n)$ (1) gauge symmetry (1.1) by performing phases change in the complex fields $f^{IA}$. The $\phi^{aI}$ are real since they describe matter in adjoint representation of the gauge symmetry (adjoint matter for short).

(b) Supersymmetry as a basic invariance

Besides fermions, the 6D $\mathcal{N} = 2$ Maxwell multiplet $\mathcal{V}_{6D,N=2}$ has, in addition to the gauge field $A_\mu$, the four real scalars $\phi^a$, which now on should be thought of as,

$$\phi^a \equiv (f^I, \overline{f}^I).$$

To study the geometry (scalar fields self-couplings) of the scalar manifold parameterized by these scalars, it is interesting to split the gauge supermultiplet $\mathcal{V}_{6D,N=2}$ in terms of $\mathcal{N} = 1$ supermultiplets as given below

$$\mathcal{V}_{6D,N=2} = \mathcal{V}_{6D,N=1} \oplus \mathcal{H}_{6D,N=1},$$

where $\mathcal{V}_{6D,N=1}$ is the 6D $\mathcal{N} = 1$ Maxwell multiplet and $\mathcal{H}_{6D,N=1}$ is the hypermultiplet.

Notice in passing that the same approach is used in dealing with the Kahler geometry of the Coulomb branch of the 4D $\mathcal{N} = 2$ supergravity theory.

(c) HSS method to get the explicit expression of the metric

In the Harmonic SuperSpace (HSS) method [10]–[20], the $n$ hypermultiplets $\{\mathcal{H}_{6D,N=1}^I\}_{1 \leq I \leq 20}$ are adequately described by the superfields $\Phi^{+A}$ and their conjugate $\Phi^+_A$,

$$\Phi^{+A} = \Phi^{+A}(x, \theta^+, u^\pm), \quad \Phi^+_A = \Phi^+_A(x, \theta^+, u^\pm).$$
where \( x, \theta^{\pm} = u^{\pm}\theta^i \) and \( u^{\pm}_i \) stand for the space time coordinates, the Grassmann variables and harmonic variables respectively. The HSS superfields \( \Phi^{+A} \) and \( \tilde{\Phi}^{+\bar{A}} \) transform in the fundamental representations of the \( U(n) \) isotropy symmetry of (1.2),

\[
\Phi^{+A} \sim \underline{n}, \quad \tilde{\Phi}^{+\bar{A}} \sim \underline{n}.
\]

Notice in passing that the \( \Phi^+ \) and \( \tilde{\Phi}^+ \) description of hypermultiplets as well as their general self-interactions are well established in literature on harmonic superspace [21]; see also [22]-[31] for related matters. The HSS superfield action describing hypermultiplet interactions has the typical form

\[
S = \int d^6x L(x) \quad \text{with}
\]

\[
L(x) = \int_{S^2} du \mathcal{L}(x, u)
\]

\[
\mathcal{L}(x, u) = \int d^4\theta^+ \left[ \Phi^+ D^{++} \Phi^+ - \mathcal{L}_{\text{int}}^{+4} (\Phi^+, \Phi^+, u^{\pm}) \right],
\]

(1.3)

where \( D^{++} \) is the usual harmonic derivative. The first term of the right hand side of the second relation may be thought of as the Kinetic term and \( \mathcal{L}_{\text{int}}^{+4} \) stands for the hypermultiplet self interactions.

Here, we will use known results on HSS method and the prepotential derived in [9] to deal with the interacting dynamics associated with the manifold \( H_{2n} (1.2) \). This dynamics is given by the Lagrangian super-density \( \mathcal{L}^{+4}_n \),

\[
\mathcal{L}_n = \int d^4\theta^+ \left( \sum_{A=1}^n \Phi^{+A} D^{++} \Phi^{+A} + \mathcal{L}_{\text{int}}^{+4}_n \right),
\]

(1.4)

with

\[
\mathcal{L}_{\text{int}}^{+4}_n = \frac{1}{2} \sum_{I,J=1}^n \lambda_{IJ} T^{++I} T^{++J},
\]

\[
T^{++}_I = \frac{1}{i} Tr \left( \tilde{\Phi}^+ H_I \Phi^+ \right).
\]

(1.5)

The real symmetric matrix \( \lambda_{IJ} \) describes the superfield coupling constants and the \( H^I \)'s are the Cartan generators of the \( U(n) \) isotropy group of eq(1.2). Notice that for the particular \( n = 1 \) case, eq(1.4) gets reduced to

\[
\mathcal{L}_1 = \int d^4\theta^+ \left[ \Phi^+ D^{++} \Phi^+ - \frac{\lambda}{2} (\Phi^+ \Phi^+)^2 \right]
\]

(1.6)

which is nothing but the HSS hypermultiplet model that describe the real 4 dimensional Taub-NUT geometry [32]. The successive integration of eq(1.6) first with respect to the Grassmann \( \theta^+ \) and then with respect to the harmonic \( u^\pm \) variables lead to

\[
L_1 \left( f, \bar{f} \right) = \bar{g}_{ij} \partial_{\mu^i} \partial^{\mu} \bar{f}^j + g^{ij} \partial_{\mu} \bar{f}_i \partial_{\mu} \bar{f}_j + 2h^{ij}_l \partial_{\mu^l} \partial^{\mu} \bar{f}_{ij}
\]

(1.7)
with
\[
\bar{g}_{ij} = \frac{\lambda}{2} \left( \frac{2+\lambda f}{2+\lambda f} \right) \bar{T}_i \bar{T}_j,
\]
\[
g^{ij} = \frac{\lambda}{2} \left( \frac{2+\lambda f}{2+\lambda f} \right) f^i f^j,
\]
\[
h^j_i = \delta^j_i (1 + \lambda f) - \frac{\lambda}{2} \left( \frac{2+\lambda f}{2+\lambda f} \right) f^j f^i,
\]
where \( \lambda \) is a real coupling constant. For generic \( n \geq 1 \); the Lagrangian density \((1.7)\) extends as
\[
\mathcal{L}_n (f, \bar{T}) = \bar{g}_{iA\bar{B}} \partial_\mu f^{iA} \partial^\mu \bar{f}^{\bar{B}} + g^{iA\bar{B}} \partial_\mu f^{iA} \partial^\mu \bar{T} + 2 h^j_i \partial_\mu f^{iA} \partial^\mu \bar{T}.
\]

The main purpose of this paper is to first compute explicitly the metric components
\[
\bar{g}_{iA\bar{B}} = \bar{g}_{iA\bar{B}} (f, \bar{T}),
\]
\[
g^{iA\bar{B}} = g^{iA\bar{B}} (f, \bar{T}),
\]
\[
h^j_i = h^j_i (f, \bar{T}).
\]

We also give a dictionary drawing the correspondence between the Kahler geometry of \(10D\) type IIA superstring on Calabi-Yau threefolds and the hyperKahler geometry of \(10D\) type IIA on K3.

The organization of this paper is as follows. In section 2, we describe some basic tools. In section 3, we study the 4D hyperKahler Taub-NUT geometry as it is the leading term of the family \( SO(1,1) \times H_{2n} \). In section 4, we study the quaternionic 2-form and derive the HSS prepotential. In section 5, we consider the real \( 4n \) dimensional generalization of the Taub-NUT supersymmetric model and in section 6 we derive the hyperKahler metric with \( U^n (1) \) abelian symmetry. In section 7, we give the conclusion and make a discussion concerning the correspondence between Kahler and hyperKahler geometries. In sections 8 and 9, we give two appendices A and B where technical computations are presented.

## 2 Basic tools

In this section, we describe the three following points: (1) The mapping from the real fields \( \phi_1^a \) to the complex \( f^{iA} \) and \( \bar{T}_{iA} \). (2) Supersymmetric representations in \( 6D \) \[33\] and reduction down to \( 4D \). (3) Harmonic superspace method (HSS).

### 2.1 From real \( \phi_1^a \) to the complex \( (f^{iA}, \bar{T}_{iA}) \)

With the objective to use HSS method to get eq\((1.10)\), it is interesting to work with the complex field coordinates \( f^{iA} \) and \( \bar{T}_{iA} \) rather than the real fields \( \phi_1^a \). Below we show how
this mapping can be obtained.

First notice that \( \phi_a^I \) is in the \((4, n)\) bi-fundamental representation of \( SO(4) \times SO(n) \) group. Using the property \( SO(4) \sim SU(2) \times SU(2) \) and the usual Pauli \( 2 \times 2 \) matrices \( \sigma_{ij}^a \), we can put \( \phi_a^I \) in the equivalent form \( \phi_a^{ij} \) with \( \phi_a^i = \sum_{i,j=1}^2 \sigma_{ij}^a \phi_{ij}^i \).

Second, using \( \phi_{ij}^i \), the mapping from these real scalars to the complex fields \( f_i^A \) and \( \tilde{T}_{iA} \) is given by the following relation \( [9] \),

\[ \phi_{ik}^I = Tr \left( [\tilde{T}_{kN} H_f^{fi}] \right) , \quad I = 1, \ldots, n . \tag{2.1} \]

where \( Tr \left( [\tilde{T}_{kN} H_f^{fi}] \right) \) stands for

\[ \sum_{A,B=1}^n \tilde{T}_{kA} (H_I)^A_B f_i^B \equiv \tilde{T}_{kN} H_f^{fi} , \quad I = 1, \ldots, n . \tag{2.2} \]

In this relation, the complex field coordinates \( f_i^A \) and \( \tilde{T}_{iA} \) are in the bi-fundamentals of the isotropy group \( SU(2) \times U(n) \) of the moduli space of the Coulomb branch of the supergravity theory. The \( n \times n \) matrices \( \{ H_I \} \) are the Cartan generators of the \( U(n) \) group satisfying the usual properties,

\[ [H^I, H^J] = 0 , \quad [H, H^J] = 0 , \quad (H^I)^\dagger = H^I , \quad (H)^\dagger = H . \tag{2.3} \]

The \( n \times n \) hermitian matrix \( H \) stands for \( H = \sum_{I=1}^n \varphi_I H^I \) with \( \varphi_I \in \mathbb{R} \), where the real functions \( \varphi_I \) are the abelian \( U^n(1) \) group parameters. The reality condition of the adjoint matter, \( \left( \phi_{ik}^J \right) = \phi_{kj}^i \), follows directly from

\[ \left( \tilde{T}_{kN} \right)^A = T_{iA} \quad , \quad (H_I)^\dagger = H_I . \tag{2.4} \]

Third, it is interesting to notice that the change from the real field coordinates \( \phi_a^i \) to the complex \( f_i^A \) and \( \tilde{T}_{iA} \) is not uniquely defined. Indeed under the change

\[ f \longrightarrow q = e^{iH} f , \quad \tilde{T} \longrightarrow \tilde{q} = e^{-iH} \tilde{T} , \tag{2.5} \]

or more explicitly by exhibiting the indices,

\[ \tilde{T}_{kA} \longrightarrow \tilde{q}_{kA} = (e^{iH})^A_B f_{kB} \tilde{T}_{kA} = (e^{-iH})^A_B f_{kB} \tilde{T}_{kA} , \tag{2.6} \]

where \( H \) is as in (2.3), the mapping (2.1) remains invariant

\[ Tr \left( \tilde{T}_{kN} H_f^{fi} \right) = Tr \left( \tilde{T}_{kN} f_i^A \right) . \tag{2.7} \]

Therefore the field change (2.1) has a \( U^n(1) \) gauge symmetry which can be promoted to the local gauge symmetry (1.1) of the Coulomb branch of the non chiral \( 6D \mathcal{N} = 2 \) supergravity theory. Since, we are not interested here by the gauge-hypermultiplet interactions, we then restrict our attention below to global invariance.
2.2 Supersymmetry

The moduli space of non chiral $6D \mathcal{N} = 2$ supergravity multiplet coupled $n$ Maxwell gauge supermultiplets has the form

$$\frac{SO(4, n)}{SO(4) \times SO(n)} \times SO(1, 1), \quad (2.8)$$

where the factor $SO(1, 1)$ is parameterized by $e^{\sigma}$ with $\sigma$ standing for the dilaton. The real $4n$ moduli $\phi^{aI}$ describe the *vevs* of the scalars of the $6D \mathcal{N} = 2$ vector multiplets $V_{6D,N=2}^I$.

6D $\mathcal{N} = 1$ formalism

In the language of $6D \mathcal{N} = 1$ supersymmetric representations, the Maxwell supermultiplet $V_{6D,N=2} = (\begin{pmatrix} 1 & \frac{1}{2} & 0^4 \end{pmatrix})_{6D}$, split into a vector $V_{6D,N=1}$ and a hypermultiplet $H_{6D,N=1}$. We have

$$V_{6D,N=2}^I = V_{6D,N=1}^I \oplus H_{6D,N=1}^I, \quad I = 1, \ldots, n, \quad (2.9)$$

with the following fields content

$$V_{6D,N=2} = (1, \frac{1}{2})_{6D}, \quad H_{6D,N=1} = (\frac{1}{2}, 0^4)_{6D}. \quad (2.10)$$

As we see, the vector supermultiplets $V_{6D,N=2}^I$ have no scalars. The real $4n$ scalar fields are all of them in the hypermultiplets $H_{6D,N=1}^I, I = 1, \ldots, n$.

4D $\mathcal{N} = 2$ formalism

A more convenient way to deal with $6D \mathcal{N} = 1$ hypermultiplets is to use $4D \mathcal{N} = 2$ superspace. In this $4D \mathcal{N} = 2$ language, the hypermultiplet fields content decomposed as follows,

$$H_{6D,N=1} = (\frac{1}{2}, 0^4)_{6D} \rightarrow H_{4D,N=2} = (\frac{1}{2}, 0^4)_{4D}. \quad (2.11)$$

A similar relation is valid for $V_{6D,N=1}$ which decomposes like

$$V_{6D,N=1} = (1, \frac{1}{2})_{6D} \rightarrow V_{4D,N=2} = (1, \frac{1}{2}, 0^2)_{4D}. \quad (2.12)$$

This reduction is obtained by decomposing $6D$ vectors as $4D$ vectors plus 2 scalars. The $6D$ spinors; say $\theta^{\hat{a}i}$, split equality into a $4D$ Weyl spinor $\theta^{ai}$ and its complex conjugate $\overline{\theta}^{\dot{a}i}$ like,

$$\left( \theta^{\hat{a}i} \right)_{1 \leq \hat{a} \leq 4} \rightarrow \left( \theta^{ai} \overline{\theta}^{\dot{a}i} \right)_{a=1,2}. \quad (2.13)$$

Hypermultiplets couplings in $6D \mathcal{N} = 1$ supersymmetric gauge theory can be then conveniently studied in the framework of the $4D \mathcal{N} = 2$ HSS formalism \[10\] where several results have been obtained. Below, we give a brief description of the $4D \mathcal{N} = 2$ HSS and make comments regarding our purposes.
2.3 General on HSS in 4D

In the HSS formulation of 4D $\mathcal{N} = 2$ hypermultiplet theory, the ordinary superspace with $SU_R(2)$ R-symmetry,

$$z^M = \left( x^\mu, \theta^i_a, \bar{\theta}^\dot{a}_a \right), \quad i = 1, 2, \quad a, \dot{a} = 1, 2,$$

(2.14)

gets mapped into the harmonic superspace $z^M = \left( Y^m, \theta^+_a, \bar{\theta}^\dot{a}_\dot{a}, u^+_i \right)$, with an analytic sub-superspace parameterized by the super-coordinates

$$Y^m = \left( y^\mu, \theta^+_a, \bar{\theta}^\dot{a}_\dot{a} \right),$$

(2.15)

and

$$y^\mu = x^\mu + i \left( \theta^+_a \sigma^\mu \bar{\theta}^a + \theta^- a \sigma^\mu \bar{\theta}^a \right),$$

$$\theta^+_a = u^+_a \theta^a, \quad a = 1, 2,$$

$$\bar{\theta}^\dot{a}_\dot{a} = u^+_\dot{a} \bar{\theta}^\dot{a}_\dot{a}, \quad \dot{a} = 1, 2,$$

(2.16)

where $u^+_i$ are the harmonic variables satisfying the relations $u^+_i u^-_i = 1$ and $u^+_i u^+_j = 0$.

The hypermultiplets are described by an analytic HSS function $\Phi^+ = \Phi^+(Y, u),$

$$D^+_a \Phi^+ = 0, \quad \overline{D}^\dot{a} \Phi^+ = 0,$$

(2.17)

with covariant spinor derivatives as $D^+_a = u^+_a D^a_i, \quad D^+_a = \frac{\partial}{\partial \theta^a} \quad \text{and} \quad \overline{D}^\dot{a} = \frac{\partial}{\partial \bar{\theta}^\dot{a}}$. The superfield $\Phi^+$ satisfy as well the property

$$[D^0, \Phi^+] = \Phi^+.$$

(2.18)

In this relation $D^0$ is a $U(1)$ charge operator given by

$$D^0 = \partial^0 + \left( \theta^+_a \frac{\partial}{\partial \theta^a} + \bar{\theta}^\dot{a}_\dot{a} \frac{\partial}{\partial \bar{\theta}^\dot{a}_\dot{a}} \right) - \left( \theta^- a \frac{\partial}{\partial \theta^a} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^a} \right).$$

(2.19)

It generates, together with the two following operators

$$D^{++} = u^+_i \frac{\partial}{\partial u^+_i} - 2i \theta^+ \sigma^\mu \bar{\theta}^a \partial_\mu - 2i \theta^+ \frac{\partial}{\partial \theta^a} - 2i \bar{\theta}^- \frac{\partial}{\partial \theta^\dot{a}_\dot{a}} - 2i \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^\dot{a}_\dot{a}},$$

$$D^{--} = u^- i \frac{\partial}{\partial u^- i} - 2i \theta^- \sigma^\mu \bar{\theta}^a \partial_\mu - 2i \theta^- \frac{\partial}{\partial \theta^a} - 2i \bar{\theta}^+ \frac{\partial}{\partial \theta^\dot{a}_\dot{a}} - 2i \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^\dot{a}_\dot{a}},$$

(2.20)

where we have set $\tau = x^4 + i x^5$, the SU$_R(2)$ symmetry. In particular, we have the usual commutation relations of the $su(2)$ algebra,

$$[D^0, D^{++}] = +2 D^{++},$$

$$[D^0, D^{--}] = -2 D^{--},$$

$$[D^{++}, D^{--}] = D^0,$$

(2.21)
These operators play a crucial role in the HSS formulation of $4D N = 2$ supersymmetric field theory and obey the twild reality property

$$\tilde{D}^{++} = D^{++}, \quad \tilde{D}^0 = D^0, \quad \tilde{D}^{--} = D^{--}. \quad (2.22)$$

The $\theta^+$- expansion of the HSS hypermultiplet superfield $\Phi^+$ reads as

$$\Phi^+ (Y, u) = q^+ + \theta^+ F^- + \tilde{\theta}^{+2} G^- + i\theta^+ a \tilde{\theta}^{+\dot{a}} B^-_{a\dot{a}} + \theta^+ \tilde{\theta}^{+2} \Delta^{---}, \quad (2.23)$$

where we have ignored fermions for simplicity. Notice that the components,

$$F^- = F^- (x,u), \quad G^- = G^- (x,u), \quad (2.24)$$

are auxiliary fields scaling as a mass squared; i.e $(mass)^2$. The extra remaining one,

$$\Delta^{---} = \Delta^{---} (x,u), \quad (2.25)$$

is also an auxiliary field; but scaling as $(mass)^3$. All these auxiliary fields are needed to have off shell supersymmetry; in particular for the computation of eqs [1.10]. We also have

$$\tilde{\Phi}^+ (Y, u) = \tilde{q}^+ + \theta^{+2} \tilde{G}^- + \tilde{\theta}^{+2} \tilde{F}^- + i\theta^+ a \tilde{\theta}^{+\dot{a}} \tilde{B}^-_{a\dot{a}} + \theta^+ \tilde{\theta}^{+2} \tilde{\Delta}^{---}, \quad (2.26)$$

where $(\sim) = (\bar{\tau})$ stands for the twild conjugation preserving the harmonic analiticity [10]. Moreover, the component fields $F^q = F^q (x,u)$, with Cartan charge $q$, can be also expanded in a harmonic series as follows:

$$F^q (y,u) = \sum_{n=0}^{\infty} u_{t_1}^{+n+q} u_{j_1}^{-n} \cdots j_n \mathcal{F}^{(i_1 \cdots i_{n+q} j_1 \cdots j_n)} (y), \quad (2.27)$$

where we have taken $q \geq 0$ and set for convenience

$$u_{t_1}^{+n+q} u_{j_1}^{-n} \cdots j_n = u_{i_1}^{+} u_{i_2}^{+} \cdots u_{i_{n+q}}^{+} u_{j_1}^{-} \cdots j_n. \quad (2.28)$$

### HSS hypermultiplet action

Following [10], the HSS action $S$ describing the dynamics of interacting hypermultiplets $\Phi^{+A}$ and their conjugate $\Phi_{A}^+$ has the form

$$S_n = \int d^4x \left( \int_{S^2} d^4u \left[ \int d^4\theta^+ L^+_n \left( \Phi^+, \tilde{\Phi}^+, u^\pm \right) \right] \right), \quad (2.29)$$

---

1 The twild $(\sim)$ is an automorphism combining the usual complex conjugation $(-)$ and the conjugation $(\ast)$ of the charge of the $U(1)$ Cartan sub-symmetry of $SU_R (2)$.  

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where \( d^4 \theta^+ = d^2 \theta^+ d^2 \bar{\theta}^+ \) should be understood in the usual way; that is as the derivatives 
\( d^4 \theta^+ \sim (D^{-a} D_a) \left( \overline{D}_a D^{-a} \right) \). This integral measure captures four negative charges. As such, the \( SU(2) \) invariance of \( S \) requires the Lagrangian super-density to carry four positive Cartan charges and reads as

\[
\mathcal{L}^{4+} = \Phi^+ A^{++} \Phi^+ A + \mathcal{L}_{\text{int}}^{4+},
\]

with hypermultiplet self interactions \( \mathcal{L}_{\text{int}}^{4+} \) given by

\[
\mathcal{L}_{\text{int}}^{4+} = -\frac{\lambda}{2} \left[ \text{Tr} \left( \Phi^+ H I \Phi^+ \right) \right] g_{IJ} \left[ \text{Tr} \left( \Phi^+ H^I \Phi^+ \right) \right],
\]

where \( H^I \) as in eqs(2.3). The coupling constant matrix \( \lambda_{IJ} \) has been factorized as \( \lambda g_{IJ} \). The scale \( \lambda \) can be interpreted in terms of the black hole horizon radius. For \( n = 20 \), the matrix \( g_{IJ} \) can be interpreted as the intersection matrix of the 2-cycles of the real second homology of K3.

### 3 \( U(1) \) supersymmetric model

In this section, we study the scalar field self-couplings for the simplest case of 6D \( \mathcal{N} = 2 \) supergravity with one \( (n=1) \) Maxwell supermultiplet \( \mathcal{V}_{6D,\mathcal{N}=2} \). This study has been first considered in [32]; but here it will be used as a first step towards the derivation the \( U^0(1) \) extension of the Taub-NUT geometry. We also take this opportunity to give a geometric interpretation of the harmonic superspace prepotential

\[
\mathcal{L}_{\text{taub-NUT}}^{4+} = -\frac{\lambda}{2} \left( \Phi^+ \Phi^+ \right)^2,
\]

in the framework of 10D type IIA superstring compactification on complex surfaces.

As noted earlier, the vector supermultiplet \( \mathcal{V}_{6D,\mathcal{N}=2} \) has, besides fermions, the following bosonic fields:

1. A gauge field \( A_\mu \) with the abelian gauge symmetry \( (1,1) \),
2. Four real scalars \( \phi^a \) parameterizing the real scalar manifold \( Q_4 = \frac{SO(4)}{SU(4)} \).

In the complex coordinates \((t^i, \bar{t}^i)\), the manifold \( Q_4 \) gets mapped to the complex surface

\[
H_2 = \frac{SU(2,1)}{SU(2) \times U(1)}.
\]

To deal with the underlying geometry of \( H_2 \), it is useful to freeze the dynamics of the gauge field \( A_\mu \) and use the 6D \( \mathcal{N} = 1 \) supersymmetric formalism. There, the real four scalars are all of them in the hypermultiplet \( \mathcal{H}_{6D,\mathcal{N}=1} \) which in turn is conveniently described in the 4D \( \mathcal{N} = 2 \) harmonic superspace formalism where several results have
been obtained. In HSS, the hypermultiplet is represented by the superfield \( \Phi^+ \) \(^{223-226}\); and its self-coupling is described by the action

\[
S_1 = \int d^4x \left( \int_{S^2} du \left[ \int d^4\theta^+ \mathcal{L}_1^{++} \left( \Phi^+, \tilde{\Phi}^+ \right) \right] \right) . \tag{3.3}
\]

The Lagrangian super-density \( \mathcal{L}_1^{++} \) is given by the supersymmetric Taub-NUT model

\[
\mathcal{L}_1^{++} = \tilde{\Phi}^+ D^{++} \Phi^+ - \frac{\lambda}{\ell} \left( \tilde{\Phi}^+ \Phi^+ \right)^2 , \tag{3.4}
\]

where \( \lambda \) is a coupling constant to be interpreted later in terms of the mass \( M \) of the Taub-NUT black hole (\( \lambda \sim M^{-2} \)). The HSS Lagrangian density \( \mathcal{L}_1^{++} \) is invariant under the abelian global \( U(1) \) symmetry \(^{225}\)

\[
\Phi^{'+} = e^{i\Lambda} \Phi^+ , \quad \tilde{\Phi}^{'+} = e^{-i\Lambda} \tilde{\Phi}^+ , \tag{3.5}
\]

with super-parameter \( \Lambda \) constrained as \( D^{++} \Lambda = 0 \). As noted before, this symmetry can be promoted to a local gauge invariance

\[
D^{++} \Lambda \neq 0 \tag{3.6}
\]

by coupling the hypermultiplet \( \Phi^+ \) to a \( 4D \) \( \mathcal{N} = 2 \) Maxwell gauge superfield \( V^{++} \) with the abelian gauge symmetry

\[
V^{'+} = V^{++} - D^{++} \Lambda . \tag{3.7}
\]

Below, we shall not develop this issue; and focus just on the \( U(1) \) global gauge invariance of \( \mathcal{L}_1^{++} \left( \Phi^+, \tilde{\Phi}^+ \right) \).

To get the explicit component field expression of the action, we have to integrate eq\(^{3.3}\) with respect to the Grassmann variables \( \theta^+ \), then eliminate the auxiliary fields through their eqs of motion and finally integrate with respect to the harmonic variables. These technical steps are a little bit cumbersome; they are collected in appendix A.

Using the results obtained in appendix A, we can put the superfield action \( \mathcal{L}_1^{++} \left( \Phi^+, \tilde{\Phi}^+ \right) \) into the following component field one,

\[
S_1 = \frac{-1}{2} \int d^4x \left( \bar{g}_{ij} \partial_\mu \bar{f}_i \partial^\mu f_j + g^{ij} \partial_\mu \bar{f}_i \partial^\mu f_j + 2h^i_j \partial_\mu \bar{f}_i \partial^\mu \bar{f}_j \right) , \tag{3.8}
\]

with,

\[
\bar{g}_{ij} = \frac{\lambda}{\ell} \frac{(2+\lambda \tilde{f})}{(1+\lambda \tilde{f})} \bar{f}_i \bar{f}_j , \tag{3.9}
\]

\[
g^{ij} = \frac{\lambda}{\ell} \frac{(2+\lambda \tilde{f})}{(1+\lambda \tilde{f})} f^i f^j ,
\]

\[
h^i_j = \delta^i_j (1 + \lambda \tilde{f}) - \frac{\lambda}{\ell} \frac{(2+\lambda \tilde{f})}{(1+\lambda \tilde{f})} f^i \bar{f}_j .
\]
Before proceeding ahead, let us make three comments: (1) Using the following variables change mapping the complex coordinates to the real ones \((r, \theta, \psi, \varphi)\),

\[
\begin{align*}
f^1 &= \rho e^{i(\psi + \varphi)} \cos \frac{\theta}{2}, \\
f^2 &= \rho e^{i(\psi - \varphi)} \sin \frac{\theta}{2},
\end{align*}
\] (3.10)

with

\[
\rho^2 = 2 (r - M) M, \quad r > M = \frac{1}{2\sqrt{\lambda}}
\] (3.11)

the Taub-NUT metric

\[
ds^2 = 2 \h_{ij} d\bar{f}_j + g_{ij} d\bar{f}_i d\bar{f}_j + \bar{g}_{ij} d\bar{f}_i d\bar{f}_j,
\] (3.12)

becomes

\[
ds^2 = \frac{(r + M)}{2 (r - M)} dr^2 + 2 \frac{(r - M)}{(r + M)} (d\psi + \cos \theta d\varphi)^2 \\
+ \frac{(r^2 - M^2)}{2} (d\theta^2 + \sin^2 \theta d\varphi^2).
\] (3.13)

This expression of the metric is precisely the standard form of the Taub-NUT metric where the singularity is manifestly exhibited in real coordinates [34]. Moreover, from (3.11), we learn that the coupling constant \(\lambda\) is proportional to the mass \(M\) of the Taub-NUT black hole with horizon at \(r = M\). Notice that the origin of the conic field variable \(\rho = 0\) corresponds exactly to the singularity \(r = M\). So, the field modulus \(\rho\) can be interpreted as describing fluctuations near the Taub-NUT horizon.

(2) The metric (3.12) can be rewritten in another equivalent form as follows:

\[
ds^2 = G_{i\alpha,j\beta} d\xi^{i\alpha} d\xi^{j\beta},
\] (3.14)

with \(\xi^{i\alpha}\) standing for the \(SU(2) \times SU(2)\) doublet \((\bar{f}^i, \bar{f}_i)\) and where the tensor \(G_{i\alpha,j\beta}\) is given by the \(4 \times 4\) matrix,

\[
G_{i\alpha,j\beta} = \begin{pmatrix}
\bar{g}_{ij} & h^j_i \\
\bar{h}^j_i & g^{ij}
\end{pmatrix}.
\] (3.15)

This way of writing the metric \(G_{i\alpha,j\beta}\) is interesting since it allows to express it in terms of vielbeins \(E^{k\gamma}_{i\alpha}\) as \(G_{i\alpha,j\beta} = E^{k\gamma}_{i\alpha} E^{l\delta}_{j\beta} \varepsilon_{kl} \varepsilon_{\gamma\delta}\) with

\[
E^{k\gamma}_{i\alpha} = \begin{pmatrix}
\delta^k_i \frac{(2 + \lambda \bar{f}^k - \lambda \bar{f}^l \bar{t}^k)}{2\sqrt{1 + \lambda \bar{f}}} & \frac{\lambda \bar{t}^k}{2\sqrt{1 + \lambda \bar{f}}} \\
- \frac{\lambda \bar{t}^k}{2\sqrt{1 + \lambda \bar{f}}} & \delta^k_i \frac{(2 + \lambda \bar{f}^k + \lambda \bar{f}^l \bar{t}^k)}{2\sqrt{1 + \lambda \bar{f}}}
\end{pmatrix}.
\] (3.16)

Following [32], the hyperKahler 2-form \(\Omega^{(kl)}\) reads as,

\[
\Omega^{(kl)} = \frac{1}{1 + \lambda \rho^2} \left( \varepsilon_{\gamma\delta} E^{(k\gamma)}_{i\alpha} E^{(l\delta)}_{j\beta} \right) d\xi^{i\alpha} \wedge d\xi^{j\beta}.
\] (3.17)
As we see, this 2-form is given by the irreducible isotriplet factor of the following reducible quaternionic 2- form

$$\Omega^{kl} = d\xi^{i\alpha} \wedge d\xi^{j\beta} \left( \frac{1}{1 + \lambda^2 \epsilon^{\gamma\delta} E^k E^l E^{i\alpha} E^{j\beta}} \right). \quad (3.18)$$

The extra irreducible term, namely the isosinglet

$$\Omega^0 = \Omega^{kl} \epsilon_{kl},$$

can be also written as $d\xi^{i\alpha} \wedge d\xi^{j\beta} \left( \frac{1}{1 + \lambda^2 \epsilon^{i\alpha j\beta}} \right)$. As we will see in the discussion section, this term may be interpreted in terms of the flux of the NS-NS antisymmetric B-field.

## 4 Quaternionic 2- form

In this section, we want to give a geometric interpretation of the Taub-NUT geometry discussed in the above section in terms of the periods $\phi^a$ of the quaternionic form $\Omega^a$ like,

$$\phi^a = \int_{C_2} \Omega^a,$$  \quad (4.1)

where the real 2-cycle $C_2$ will be specified later on.

Using the homomorphism $SO(4) \simeq SU(2) \times SU(2)$, we can rewrite the field moduli $\phi^a$ and the quaternionic 2- form $\Omega^a$ like,

$$\phi_j^i = \sum_{a=1}^{4} (\sigma^a)^{ij}_j \phi^a, \quad \Omega_j^i = \sum_{a=1}^{4} (\sigma^a)^{ij}_j \Omega^a. \quad (4.2)$$

The analysis to be given in this section can be also viewed as a first step towards the study of the $U^n(1)$ supersymmetric model based on the moduli space (1.2). The non linear $U^n(1)$ supersymmetric sigma model in six dimension and the underlying hyperKahler metric (3.12) will be studied in the next sections.

To that purpose, we first study the quaternionic 2- form by borrowing methods from Kahler geometry and type II superstring compactification on Calabi-Yau threefolds. Then, we consider the derivation of the HSS potential (1.5).

### 4.1 Quaternionifying the hyperKahler form

We begin by describing the complexified Kahler 2- form in type IIA superstring on Calabi-Yau threefolds. Then we study the case of 10D type IIA superstring on K3.
**Complexified Kahler 2- form**

In 10D type IIA superstring on Calabi-Yau threefolds \(X_3\), the usual Kahler form on \(X_3\)

\[
K = \Omega^{(1,1)},
\]

gets complexified by the implementation of the NS-NS B-\ field as follows

\[
J = B_{NS} + iK.
\]

The Kahler moduli \(z^a\) capturing the Kahler deformations of the Calabi-Yau threefold are given by the periods

\[
z^a = \int_{C_2^a} J,
\]

where \(C_2^a\) is a real 2-cycle basis of the second homology of \(X_3\). The holomorphic prepotential \(F(z)\) is given by

\[
F(z) = \int_{X_3} J \wedge J \wedge J,
\]

which, up on using the tri-intersection tensor \(d_{abc}\), gives the well known relation

\[
F(z) = \sum_{a,b,c=1}^{h_{X_3}^{(1,1)}} d_{abc} z^a z^b z^c.
\]

Below, we will show that the analog of this relation in the case of 10D type IIA superstring on K3 is precisely given by eq(1.5).

**From HyperKahler 2- form to quaternionic \(\Omega^a\)**

To begin recall that in the case of the complex surface K3, the complex 2- forms

\[
\Omega^+ = \Omega^{(2,0)} , \quad \Omega^- = \Omega^{(0,2)} ,
\]

and the Kahler 2- form

\[
\Omega^0 = \Omega^{(1,1)} ,
\]

are in the same cohomology class \(H^2(K3)\). This property reflects the fact that K3 has a hyperKahler structure described by the isotriplet

\[
\Omega^{(ij)} \equiv \begin{pmatrix} \Omega^+ \\ \Omega^0 \\ \Omega^- \end{pmatrix}.
\]

In 10D type IIA superstring on K3, the hyperKahler 2- form \(\Omega^{(ij)}\) gets quaternionified as follows

\[
\Omega^{ij} = B_{NS} \varepsilon^{ij} + \Omega^{(ij)},
\]

14
where the singlet $\mathcal{B}_{NS}$ stands for the NS-NS antisymmetric 2-form B-field of the non-chiral $6D$ $\mathcal{N} = 2$ supergravity theory. The above relation can be also put in the equivalent form

$$\Omega^a = \sum_{a=1}^{4} \sigma_{ij}^a \Omega^{ij}, \quad (4.12)$$

where $\sigma_{ij}^a$ are the usual Pauli $2 \times 2$ matrices.

### 4.2 Periods

The scalar fields $\phi^{ai}$ of the hypermultiplets $\mathcal{H}_{6D,N=1}$ (2.9[2.11]) have a geometric interpretation in terms of periods of the above quaternionic 2-form $\Omega^a$. We have

$$\phi^{ai} = \int_{C^I_2} \Omega^a, \quad (4.13)$$

where $C^I_2$ is a generic real 2-cycle of the $H^{(1,1)}(K3)$ Dalbeault homology of K3. For a given 2-cycle $C_2$, the above relation simplifies as

$$\phi^a = \int_{C_2} \Omega^a, \quad (4.14)$$

and is associated with Taub-NUT geometry. Indeed, using eq(4.12), we can rewrite the above relation like,

$$\tilde{f} f^j = \frac{1}{i} \int_{C_2} \Omega^{ij}, \quad (4.15)$$

where we have used the complex coordinates $\phi^{ij} = i \tilde{f} f^j$. Multiplying both sides of this relation by the harmonic variables $u^+_k u^+_l$, we can put the it in the form,

$$\tilde{f}^+ f^+ = \frac{1}{i} \int_{C_2} \Omega^{++}, \quad (4.16)$$

with $w^{++} = \tilde{f}^+ f^+$ and

$$\Omega^{++} = u^+_k u^+_l \Omega^{kl},$$

$$w^{++} = u^+_k u^+_l \tilde{f}^+ f^+,$$

satisfying the obvious identity

$$u^+_i \frac{\partial}{\partial u^+_i} \Omega^{++} = \partial^{++} \Omega^{++} = 0,$$

$$u^+_i \frac{\partial}{\partial u^+_i} w^{++} = \partial^{++} w^{++} = 0,$$  

which should be associated with the conservation law of the HSS current (8.2). Thinking about eq(4.16) as the leading $\theta^+$-component of $\tilde{\Phi}^+ \Phi^+$ (8.3), we can promote it to the following superfield relation

$$T^{++} = \int_{C_2} \mathcal{J}^{++},$$

$$T^{++} = \bar{\Phi}^+ \Phi^+,$$  

\[4.19\]
satisfying
\[ D^{++}T^{++} = 0 \quad \text{and} \quad D^{++}H^{++} = 0 \quad . \] (4.20)

Moreover, denoting by \( \omega \) the real 2-form which is dual to the 2-cycle \( C_2 \) involved in (4.14):
\[ \int_{C_2} \omega = 1 \quad , \] (4.21)
with the normalization
\[ \int_{CY_2} \omega \wedge \omega = 1 \quad , \] (4.22)
then we have
\[ H^{++} = T^{++} \omega = \Phi^+ \Phi^+ \omega \quad . \] (4.23)

Now computing the analog of (1.6), it is not difficult to see that the HSS Lagrangian (1.6) may be defined as
\[ L_1^{++} = \lambda_2 \int_{CY_2} \omega \wedge \omega \quad . \] (4.24)

Substituting \( H^{++} \) by its expression \( T^{++} \omega \) and using the normalization (4.22), we obtain precisely the HSS potential of the Taub-NUT model namely
\[ L_1^{++} = \frac{1}{2} (T^{++})^2 = - \frac{\lambda_2}{2} \left( \Phi^+ \Phi^+ \right)^2 \quad . \] (4.25)

Now we turn to study the generic case.

### 4.3 Deriving the HSS prepotential (1.5)

The above analysis extend naturally to the case of 10D type IIA superstring on K3. There, eq(4.19) and the hypermultiplet coupling \( L_1^{++} \) (4.24-4.25) generalizes as follows:

(i) Instead of one super-current \( T^{++} \), we have twenty HSS conserved currents \( T^{++I} \),
\[ D^{++}T^{++I} = 0 \quad , \quad I = 1, \ldots, 20 \quad , \] (4.26)
given by
\[ T^{++I} = iTr \left( \Phi^+ H^I \Phi^+ \right) \quad . \] (4.27)

They are expressed in terms of the twenty hypermultiplets
\( \Phi^+_A \), \( \Phi^+_A \), \( A = 1, \ldots, 20 \quad , \) (4.28)
and the Cartan generators \( \{ H^I \} \) of the \( U(20) \) isotropy group of the scalar manifold \( S [ U(2) \times U(20) ] \). Similarly as in eq(4.19), we also have
\[ T^{++I} = \int_{C_2} \omega \wedge \omega \quad , \quad I = 1, \ldots, 20 \quad , \] (4.29)
where \( C_2 \) is a generic real 2-cycle of \( \mathcal{H}^{(1,1)}_{(K3)} \). Using the duality relation,
\[
\int_{C_2} \omega_K = \delta^I_K ,
\]
and the intersections,
\[
\int_{K3} \omega_I \wedge \omega_J = g_{IJ} ,
\]
with real intersection matrix \( g_{IJ} = g_{JI} \), we can rewrite the HSS 2-form \( \mathcal{J}^{++} \) like
\[
\mathcal{J}^{++} = \sum_{I=1}^{20} T^{++I} \omega_I = \sum_{I=1}^{20} \omega_I \text{Tr} \left( \Phi^+ H^I \Phi^+ \right) .
\]
Furthermore, using (4.24), we can compute the HSS prepotential
\[
\mathcal{L}^{4+}_{20} = \frac{1}{2} \int_{K3} \mathcal{J}^{++} \wedge \mathcal{J}^{++} .
\]
Substituting \( \mathcal{J}^{++} \) by its expression eq(4.32), we obtain the following HSS Lagrangian density,
\[
\mathcal{L}^{4+}_{20} = \frac{1}{2} \left( \sum_{I,J=1}^{20} g_{KL} T^{++K} T^{++L} \right) ,
\]
or equivalently
\[
\mathcal{L}^{4+}_{20} = -\frac{\lambda}{2} \sum_{I,J=1}^{20} \left[ \text{Tr} \left( \Phi^+ H^K \Phi^+ \right) \right] g_{KL} \left[ \text{Tr} \left( \Phi^+ H^L \Phi^+ \right) \right] .
\]

In the next section, we compute the metric associated with this HSS Lagrangian density. Below, we relax the above hypermultiplets self- coupling (4.35) to generic integers \( n \geq 1 \) dealing with the scalar manifold (1.2). The corresponding HSS prepotential will be denoted as \( \mathcal{L}^{4+}_n \).

5 \( U^n (1) \) supersymmetric model

In the \( \mathcal{N} = 1 \) formalism of the Coulomb branch of the non chiral 6D \( \mathcal{N} = 2 \) supergravity with generic \( n \) Maxwell supermultiplets (2.9), the self- couplings of the hypermultiplets \( \{ \Phi^+ A \} \) is given by the Lagrangian density,
\[
L_n (x) = \int_{S_2} du \ L_n (x, u) ,
\]
with
\[
L_n (x, u) = \int d^4 \theta^+ \left( \sum_{A=1}^{n} \Phi^+_A D^{++} \Phi^+ A + \frac{\lambda}{2} \sum_{K,L=1}^{n} g_{KL} T^{++K} T^{++L} \right) ,
\]
where the \( H^I \)'s are the generators of the \( U(n) \) group and \( T^{++I} = -i \text{Tr} \left( \Phi^+ H^I \Phi^+ \right) \).
5.1 Symmetries

The HSS Lagrangian density (5.2) has the following continuous symmetries:

(1) It has a manifest 4D $\mathcal{N} = 2$ (or equivalently 6D $\mathcal{N} = 1$) supersymmetry captured by the superfield formulation.

(2) It has a manifest $SU_R(2)$ symmetry captured by the charges of the harmonic variables $u^\pm$. The total charge of $\mathcal{L}_n$ should be zero knowing that the charge of the measure is $Q_{U(1)}(d^4\theta^+) = -4$. This charge is balanced by the charge of the HSS prepotential; i.e.

$$Q_{U(1)}(\mathcal{L}_n^{+4}) = +4 \text{ since } [D^0, \mathcal{L}_n^{+4}] = 4\mathcal{L}_n^{+4}. \quad (5.3)$$

The $SU_R(2)$ invariance will be explicitly exhibited after integration with respect to the harmonic variables $u^\pm$.

(3) It has a manifest $U^n(1)$ global invariance acting by changing the phases of the HSS superfields as follows,

$$\Phi'^+ = e^{iH}\Phi^+ , \quad \tilde{\Phi}'^+ = e^{-iH}\tilde{\Phi}^+ , \quad (5.4)$$

with

$$H = \sum_{I}^n \Lambda_I H^I , \quad [H^I, \Phi_A^+] = q^I_A \Phi_A^+ . \quad (5.5)$$

To be more explicit, we choose the charges $q^I_A$ of the hypermultiplets with respect to the $U(1)$ generators $H^I$ as follows:

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} . \quad (5.6)$$

(4) It has a manifest global $U(n)$ invariance acting by $n \times n$ unitary matrices $U$ and $U^\dagger$ as given below

$$\Phi'^+ = U \Phi^+ , \quad \tilde{\Phi}'^+ = \tilde{\Phi}^+ U^\dagger , \quad (5.7)$$

with $U^\dagger U = I$. Before going ahead notice the two following:

First, the group $U^n(1)$ of eq(5.4) is the maximal abelian subsymmetry of the $U(n)$ group. Its local version ($D^{++}\Lambda_I = 0$) is associated with the Coulomb branch of the 6D $\mathcal{N} = 2$ supergravity theory.

Second, the $U(n)$ invariance allows to extend eq(5.2) to the more general relation

$$\mathcal{L}_n^{+4}_{\text{non abelian}} = -\frac{\lambda}{2} \sum_{K,L} g_{KL} T^{++K}T^{++L} - \frac{\lambda}{2} \sum_{K,\alpha} g_{K\alpha} T^{++K}T^{++\alpha} - \frac{\lambda}{2} \sum_{\alpha,\beta} g_{\alpha\beta} T^{++\alpha}T^{++\beta} . \quad (5.8)$$
where $g_{K\alpha}$ and $g_{\alpha\beta}$ are coupling constants, $T^{++K}$ associated with the Cartan basis as in eq(4.27) and where

$$T^{++\alpha} = T_{r} \left( \Phi^{+} E^{\alpha} \bar{\Phi}^{+} \right). \tag{5.9}$$

In this relation, the $E^{\alpha}$ matrices stand for generic step operators of the $U(n)$ isotropy group. Recall in passing that the set $\{H^{I}, E^{\alpha}\}$ define the $n + (n^{2} - n) = n^{2}$ generators of the $u(n)$ algebra. Invariance under $U(n)$ follows from the trace property

$$Tr \left( U \Phi^{+} E^{\alpha} \bar{\Phi}^{+} U^{\dagger} \right) = Tr \left( \Phi^{+} E^{\alpha} \bar{\Phi}^{+} \right). \tag{5.10}$$

Below, we focus our attention on eq(5.2); i.e $g_{K\alpha} = g_{\alpha\beta} = 0$; but notice that the general case where $g_{K\alpha} \neq 0, g_{\alpha\beta} \neq 0$ is also an interesting issue as it concerns the non abelian extension.

**Equations of motion**

The equations of motion of the hypermultiplets $\Phi^{+A}$ and $\bar{\Phi}_{A}^{+}$ following from the variation of the Lagrangian density (5.2), can be put in the form,

$$(D^{++} - \lambda T^{++}) \Phi^{+} = 0, \quad (D^{++} + \lambda T^{++}) \Phi^{+} = 0, \tag{5.11}$$

with

$$T^{++} = \sum_{I=1}^{n} T^{++}_{I} H^{I} = \sum_{I=1}^{n} T^{++I} H_{I}, \tag{5.12}$$

and where we have set

$$H_{I} = g_{IJ} H^{J}, \quad T^{++}_{I} = g_{IJ} T^{++J}, \quad g_{IJ} g^{JK} = \delta_{I}^{K}. \tag{5.13}$$

Expanding $\Phi^{+}$ and $\bar{\Phi}_{A}^{+}$ as in eqs(2.23,2.26), we can write down the component field eqs of motion. To that purpose, it is interesting to expand $T^{++}$ (5.12) as

$$\frac{1}{l} T^{++} = w^{++} + \theta^{+2} M + \tilde{\theta}^{+2} N + i \theta^{+} \sigma^{+} \tilde{\theta}^{+} A_{\mu} + \theta^{+2} \tilde{\theta}^{+2} P^{-}, \tag{5.14}$$

where, for simplicity, the fermionic contribution have been dropped out and where

$$w^{++} = \sum_{I=1}^{n} w^{++I} H_{I}, \quad A_{\mu} = \sum_{I=1}^{n} A_{I}^{\mu} H_{I}, \tag{5.15}$$

with

$$w^{++I} = \tilde{q}^{+} H^{I} q^{+}, \quad M^{I} = \left( \tilde{q}^{+} H^{I} F^{-} - \tilde{G}^{-} H^{I} q^{+} \right), \quad N^{I} = \left( \tilde{q}^{+} H^{I} G^{-} - \tilde{F}^{-} H^{I} \phi^{+} \right), \quad A_{I}^{\mu} = \left( \tilde{q}^{+} H^{I} B_{\mu}^{-} + \tilde{B}_{\mu}^{-} H^{I} \phi^{+} \right), \tag{5.16}$$
as well as $\partial^{++} w^{++I} = 0$ and $\partial^{++} A^I_\mu = 2i \partial_\mu (w^{++I})$. Then put these relations back into eq(5.11), we obtain the component field eqs of motion: The leading term in $\theta^+$ gives,

$$ (D^{++} - \lambda w^{++}) q^+ = 0 \quad ,$$  

(5.17)

where $w^{++}$, valued in the Cartan subalgebra of the $U(n)$ group, is as in eqs(5.15-5.16). The term in $\theta^+ \sigma^\mu \bar{\theta}^+$ gives,

$$ [\partial^{++} - \lambda w^{++}] B^-_\mu - \lambda A_\mu q^+ = 2 \partial_\mu q^+ \quad ,$$  

(5.18)

where $w^{++}$ and $A_\mu$ are given by eqs(5.15-5.16). The terms in $\theta^+ \theta^+$ and $\bar{\theta}^+ \bar{\theta}^+$ give the equations of motion of the auxiliary fields $F^-$ and $G^-$,

$$ [\partial^{++} - \lambda w^{++}] F^- - \lambda M q^+ = 2 \partial_\tau q^+ \quad ,$$  

$$ [\partial^{++} - \lambda w^{++}] G^- - \lambda N q^+ = 2 \partial_{\bar{\tau}} q^+ \quad ,$$  

(5.19)

with $\tau = (x^5 + ix^6)$. These fields are irrelevant for the determination (1.10). They will be ignored below.

The last relation corresponds to the term $\theta^+ \theta^+$; it gives the space time dynamics of the propagating scalars; this equation is also not needed for the determination of (1.10).

5.2 Solving the constraint eqs(5.17-5.18)

The working of the solution of eqs(5.17) and (5.18) is very technical. For simplicity, we will focus below on the main steps and focus on the results. The details of the computations are presented in the appendix B.

The solution of eq(5.17) expressing $q^+ (x, u)$ in terms of $(f_i (x), \bar{f}_i (x))$ and the harmonics $u_i^\pm$ reads as

$$ q^+ = u_i^+ f_i \exp \left( \frac{\lambda}{2} \sum_{k,l=1}^2 u_k^+ u_l^- \sum_{I=1}^n \left[ Tr \left( \bar{f}_I^k H_I f_l^I \right) \right] H^I \right) \quad ,$$  

$$ \bar{q}^+ = u_i^+ \bar{f}_i \exp \left( \frac{-\lambda}{2} \sum_{k,l=1}^2 u_k^+ u_l^- \sum_{I=1}^n \left[ Tr \left( \bar{f}_I^k H_I f_l^I \right) \right] H^I \right) \quad .$$  

(5.20)

In the limit where the coupling constant $\lambda \rightarrow 0$, we recover the free fields $q^+ = u_i^+ f_i (x)$.

To get the solution of eq(5.18), we need several steps (see appendix B for details): First use the $U^n (1)$ symmetry to make the change

$$ B^-_\mu = e^{\lambda \omega} C^-_\mu \quad , \quad \bar{B}^-_\mu = e^{-\lambda \omega} \bar{C}^-_\mu \quad ,$$  

(5.21)
where $w$ is as in eq(9.2) and where $C_{\mu}^-$ is the new auxiliary field satisfying the differential equation,

$$
\begin{align*}
\partial^{++} C_{\mu}^- - \lambda A_{\mu} f^+ &= 2 \nabla_\mu f^+, \\
\partial^{++} C_{\mu}^- + \lambda A_{\mu} \tilde{f}^+ &= 2 \nabla_\mu \tilde{f}^+, \\
\end{align*}
$$

(5.22)

with

$$
\begin{align*}
\nabla_\mu f^+ &= [\partial_\mu + \lambda (\partial_\mu w)] f^+, \\
\nabla_\mu \tilde{f}^+ &= [\partial_\mu - \lambda (\partial_\mu w)] \tilde{f}^+.
\end{align*}
$$

(5.23)

We also have the decomposition $A_\mu = \sum_{I=1}^n A_I^I H_I$ with

$$
A_\mu^I = \tilde{C}_-^I H^I f^+ + \tilde{f}^+ H^I C_{\mu}^-.
$$

(5.24)

The next step is to use the identity $\lambda A_{\mu} f^+ = \lambda \partial^{++} (\vartheta_{\mu} f^+ - \partial_{\mu} w^- f^+)$ to solve eq(5.22) like,

$$
\begin{align*}
C_{\mu}^- &= 2 \partial_\mu f^- + \lambda \partial_{\mu} f^- + \lambda (\partial_{\mu} w^-) f^+, \\
\tilde{C}_{\mu}^- &= 2 \partial_\mu \tilde{f}^- - \lambda \partial_{\mu} \tilde{f}^- - \lambda (\partial_{\mu} w^-) \tilde{f}^+.
\end{align*}
$$

(5.25)

To determine the quantity $\vartheta_{\mu}$, we have to compute the term $\tilde{f}^+ H^I C_{\mu}^- + \tilde{C}_-^I H^I f^+$ by using eqs(5.25) and derive a constraint equation that allows us to fix $\vartheta_{\mu}$. We have

$$
\begin{align*}
\tilde{f}^+ H^I C_{\mu}^- &= 2 \tilde{f}^+ H^I \partial_{\mu} f^- + \lambda \partial_{\mu} f^- \left( \tilde{f}^+ H^I H^J f^- \right) + 2 \lambda \partial_{\mu} w^- \left( \tilde{f}^+ H^I H^J f^- \right), \\
\tilde{C}_-^I H^I f^+ &= 2 \partial_\mu \tilde{f}^- H^I f^+ - \lambda \partial_{\mu} \tilde{f}^- \left( \tilde{f}^+ H^I H^J f^+ \right) - 2 \lambda \partial_{\mu} w^- \left( \tilde{f}^+ H^I H^J f^+ \right).
\end{align*}
$$

(5.26)

For simplicity of the equations, it is convenient to introduce the following conventional notations:

$$
\begin{align*}
Q_{iI}^A &= u_i^+ Q_i^A \equiv (f^I H_I)^A, \\
\tilde{Q}_B^I &= u_i^+ \tilde{Q}_B^I \equiv (\tilde{f}^I H_I)^I_B.
\end{align*}
$$

(5.27)

with

$$
\begin{align*}
Q_i^A &= (H_I)^A_C f^C, \\
\tilde{Q}_B^I &= \tilde{f}_D (H_I)^D_B, \\
R_B^A &= \tilde{Q}_B^I Q_i^A.
\end{align*}
$$

(5.28)

Using these fields, one can build the following composites

$$
\begin{align*}
\bar{Q}_{iB} Q_i^A, & \quad Q_i^A \bar{Q}_B^J, & \quad Q_i^A Q_j^B, & \quad \bar{Q}_{iB} \bar{Q}_j^D.
\end{align*}
$$

(5.29)

For $n = 1$, the unique Cartan generator reduces to the identity operator, $H_1 = I$, the fields $Q_i^A$ reduce down to $f^i$ and eqs(5.29) to

$$
\begin{align*}
\bar{Q}_{iB} Q_i^A \rightarrow \bar{f}^i f_i^j, & \quad Q_i^A \bar{Q}_B^J \rightarrow f^i \tilde{f}_j, \\
Q_i^A Q_j^B \rightarrow f^i \tilde{f}_k, & \quad \bar{Q}_{iB} \bar{Q}_j^D \rightarrow \tilde{f}_i \tilde{f}_j.
\end{align*}
$$

(5.30)
Using the field moduli $Q^\pm$ and $\tilde{Q}^\pm$, we can rewrite eqs (5.26) like,
\[ \begin{align*}
\tilde{Q}^{+\mu} C_\mu^- &= 2Q^{+\mu} \partial_\mu f^- + \lambda \partial_{\mu J} \left( \tilde{Q}^{+\mu} Q^{-J} \right) + \lambda \partial_\mu \tilde{w}^- (\tilde{Q}^{+\mu} Q^{+J}) , \\
\tilde{C}^-_\mu Q^{+\mu} &= 2\partial_\mu \tilde{f}^- Q^{+\mu} - \lambda \partial_{\mu J} (\tilde{Q}^{-J} Q^{+\mu}) - 2\lambda \partial_\mu \tilde{\varphi}^- (\tilde{Q}^{+\mu} Q^{+J}) .
\end{align*} \] (5.31)

Next, adding the two relations and using eq (5.24), we get
\[ \mathcal{E}_J \partial_\mu^J = v_\mu^- , \] (5.32)
with
\[ \begin{align*}
v_\mu^- &= \left( Q^{iAI} \partial_\mu \tilde{f}_i - Q^{iA} \partial_\mu f^i \right) , \\
\mathcal{E}_J &= [\delta_J + \lambda \tilde{Q}^{iA} Q^{iA}] .
\end{align*} \] (5.33)

Using eq (5.27), these relations can be also put in the equivalent form
\[ \begin{align*}
\tilde{v}_\mu^- &= (f^\mu i H^I \partial_\mu \tilde{f}_i - \tilde{f}_i H^I \partial_\mu f^i) , \\
\tilde{\mathcal{E}}_J &= [\delta_J + \lambda \tilde{Q}^{iA} Q^{iA}] .
\end{align*} \] (5.34)

Then, the solution of $\partial_\mu^J$ reads as,
\[ \begin{align*}
\partial_\mu^J &= \mathcal{F}_J \mathcal{E}_J = \delta_J , \\
\mathcal{F}_J = [1 + \lambda \tilde{f}^i f_i] .
\end{align*} \] (5.35)

Notice that for the leading case $n = 1$, eqs (5.33-5.34) reduce to
\[ \begin{align*}
v_\mu^- \to v_\mu^\prime &= (f^\mu i H^I \partial_\mu \tilde{f}_i - \tilde{f}_i H^I \partial_\mu f^i) , \\
\mathcal{E}_J \to \mathcal{E} &= \left( 1 + \lambda \tilde{f}_i f^i \right) .
\end{align*} \]

and
\[ \mathcal{F}_J \to \mathcal{F} = \left[ 1 + \lambda \tilde{f}^i f_i \right] , \quad \mathcal{F} = 1 . \] (5.36)

Notice moreover that because of the property $\tilde{f}_i H^I H^J f_i = \tilde{f}_i H^I H J f_i$, we have the identity
\[ \tilde{Q}^{iA} Q^{iA} = \tilde{Q}^{iA} Q^{iA} . \]

The solution $C^-_\mu (x, u)$ and $\tilde{C}^-_\mu (x, u)$ read, in terms of $Q^\pm$, as
\[ \begin{align*}
C^-_\mu &= 2\partial_\mu \tilde{f}^- A + \lambda \mathcal{F}_J \mathcal{E}_J = Q^A_- q^{A}_B \left( \partial_\mu \tilde{f}^- \right) + \lambda Q^A_- Q^B_- \left( \partial_\mu \tilde{f}^- \right) , \\
\tilde{C}^-_\mu &= \left( 2\partial_\mu \tilde{f}^- A - \lambda \mathcal{F}_J \mathcal{E}_J = Q^A_- q^{A}_B \left( \partial_\mu \tilde{f}^- \right) + \lambda Q^A_- Q^B_- \left( \partial_\mu \tilde{f}^- \right) .
\end{align*} \] (5.37)

The harmonic dependence of the fields $C^-_\mu = C^-_\mu (x, u)$ and $\tilde{C}^-_\mu = \tilde{C}^-_\mu (x, u)$ is as follows
\[ \begin{align*}
C^-_\mu (x, u) &= u_i C^i_\mu (x) + u_i u_j u_k C^{ijk}_\mu A (x) , \] (5.38)
with
\[
C_{iA}^{\mu} = 2\partial_{\mu}f^{iA} + \lambda F^{ij}v^{i}_{\mu}Q^{jA} \\
+ \frac{3}{2} Q_{jB}^{iA} \left( \partial_{\mu}f^{B} \right) + \frac{\lambda}{3} Q_{iB}^{jA} \left( \partial_{\mu}f^{B} \right) \\
+ \frac{3}{2} Q_{jB}^{iA} Q_{jB}^{iA} \left( \partial_{\mu}f^{B} \right) + \frac{\lambda}{3} Q_{iB}^{jA} Q_{iB}^{jA} \left( \partial_{\mu}f^{B} \right).
\]
\[
(5.39)
\]
Analogous relations for \( C_{\mu iA} \) and \( C_{(ijk)A} \), \( \bar{C}_{(ijk)A} \) are given in appendix B.

6 Computing the metric

Starting from the superfield relation (5.2) and performing the integration with respect to the Grassmann variables \( \theta^{+} \) and \( \bar{\theta}^{+} \), we obtain the following component field action,
\[
S_{n} = \frac{1}{2} \int d^{4}x \left[ \int_{S^{2}} \frac{1}{2} \sum_{A=1}^{n} \left( B_{\mu}^{-A} \partial^{\mu}q^{+}_{A} - \bar{B}_{\mu}^{-A} \partial^{\mu}q^{+}_{A} \right) \right].
\]
\[
(6.1)
\]
This action still depends on the auxiliary fields \( B_{\mu}^{-A} \) and the harmonic variables. To get the space time field action,
\[
S_{n} = \frac{1}{2} \int d^{4}x \left( 2h^{ijB} \partial_{\mu}f^{iA} \partial^{\mu}f^{jB} + \xi f^{A} \partial_{\mu}f^{jB} + g^{A} \partial_{\mu}f^{jB} \partial^{\mu}f^{jB} \right),
\]
\[
(6.2)
\]
we have to eliminate the \( B_{\mu}^{-A} \)'s and integrate with respect the harmonic variables \( u^{\pm} \). Substituting \( B_{\mu}^{-A} \) and \( q^{+A} \) by their expressions in terms of \( C_{\mu}^{-A} \) and \( f^{+A} \) given by eqs (5.20, 5.21, 5.36, 5.37), we can put \( S_{n} \) as
\[
S = \frac{1}{2} \int d^{4}x \left[ L_{n1} (x) + L_{n2} (x) \right]
\]
\[
(6.3)
\]
with
\[
L_{n1} (x) = \int_{S^{2}} du \left( C_{\mu}^{-A} \partial^{\mu}f^{+}_{A} - \bar{C}_{\mu}^{-A} \partial^{\mu}f^{+A} \right),
\]
\[
L_{n2} (x) = -\lambda \int_{S^{2}} du \left[ \left( Q_{iA}^{+} C_{\mu}^{-A} + \bar{C}_{\mu}^{-A} Q_{iA}^{+A} \right) \left( \partial^{\mu}w^{l} \right) \right],
\]
\[
(6.4)
\]
and \( w^{l} \) as in eq (8.8).

Substituting \( C_{\mu}^{-A} \) and \( \bar{C}_{\mu}^{-A} \) by their expressions in terms of the propagating fields and integrating with respect to the harmonic variables, we find, after some algebra given in appendix B, subsection B2, that \( L_{n1} \) reads as
\[
L_{n1} = -2\partial_{\mu}f^{iA} \partial^{\mu}f^{iA} + \frac{\lambda}{2} N_{kC}^{fID} \partial_{\mu}f^{iA} \partial^{\mu}f^{iA}
\]
\[
- \frac{\lambda}{2} \partial^{\mu}f^{iA} \partial_{\mu}f^{iA} + \frac{\lambda}{2} \partial^{\mu}f^{iA} \partial_{\mu}f^{iA}
\]
\[
(6.5)
\]
with
\[
\mathcal{N}^\mu_{kC} = F_i^\mu Q_i^J P_{kC} + F_i^\mu Q_{kCJ} Q_i^{JD} + \left( \tilde{Q}_{iC} Q_{kJ}^P \right) - \left( \tilde{Q}_{iC} Q_i^P \right) \delta^i_k,
\]
\[
\mathcal{U}_{kC,ID} = F_i^\mu Q_{iD,JK} Q_i^J + \frac{1}{2} \left( \tilde{Q}_{kD} Q_{iCJ} \right) - \frac{1}{2} \left( \tilde{Q}_{iD} Q_{CJ} \right) \varepsilon_{kl},
\]
\[
U^{kC,ID} = F_i^\mu Q_i^{CJ} Q_i^{JD} + 4 \delta \left( Q_i^{CJ} Q_i^{JD} \right) - 4 \delta \left( Q_i^{CJ} Q^P_i \right) \varepsilon_{kl}.
\]
In the particular case where \( n = 1 \), these quantities reduce to
\[
\mathcal{N}^\mu_{k} = \frac{2 \mu T_{I_k}}{1 + \mu T_I} + T_k - \delta^i_k T^i,
\]
\[
\mathcal{U}_{kl} = \frac{T_{I_k}}{1 + \mu T_I} + \frac{1}{2} \frac{T_{I_I}}{T_I},
\]
\[
U^{kl} = \frac{1}{2} \frac{T_{I_k}}{1 + \mu T_I} + \frac{1}{2} \frac{T_{I_I}}{T_I}.
\]
A similar analysis shows that the term \( L_{n2} \) eq\( (9.38) \) has the form
\[
L_{n2} = -\lambda \hat{U}^{kC,ID} \partial_{\mu} \tilde{T}^I_{kC} \partial_{\mu} \tilde{T}^I_{ID} - \lambda \hat{U}_{kC,ID} \partial_{\mu} \tilde{T}^I_{kC} \partial_{\mu} \tilde{T}^I_{ID} + \lambda \hat{U}^{kC,ID} \partial_{\mu} f_{kC} \partial_{\mu} f_{kC} \partial_{\mu} f_{kC},
\]
with
\[
\hat{N}^{kC} = \lambda \left( \tilde{Q}_{C}^I Q^P_{kC} - \tilde{Q}_{iC} Q^{ID} \delta^i_k \right),
\]
\[
\hat{U}_{kC,ID} = \frac{1}{2} \left( \tilde{Q}_{C}^I Q^J_{kC} - \tilde{Q}_{iC} Q_{iD} \varepsilon_{kl} \right),
\]
\[
\hat{U}^{kC,ID} = \frac{1}{2} \left( Q_i^{CJ} Q_i^{JD} - Q_{kC} Q^{ID} \varepsilon_{kl} \right).
\]
In the case \( n = 1 \), these tensors reduce to
\[
\hat{N}^I_k = \lambda \left( T_{kC} - \delta^i_k \right), \quad \hat{U}_{kl} = \frac{1}{2} \frac{T_{I_k}}{T_I}, \quad \hat{U}^{kl} = \frac{1}{2} \frac{T_{I_I}}{T_I}.
\]

The \( U^n (1) \) hyperKähler metric

Adding eqs\( (6.5,6.6) \) and eqs\( (6.8,6.9) \), we get the total Lagrangian density
\[
L_n = + g^{kC,ld} \partial_{\mu} f_{kC} \partial_{\mu} \tilde{T}^I_{ld} + \bar{g}_{kC,ld} \partial_{\mu} f^{ld} \partial_{\mu} f_{kC},
\]
with
\[
2 h_{kC}^{ld} = -2 \delta^l_k \delta^d + \frac{1}{2} \left( N_{kC}^{ID} + \hat{N}_{kC}^{ID} \right),
\]
\[
g_{kC,ID} = -\frac{1}{2} \left( \mathcal{U}_{kC,ID} + \mathcal{U}_{kC,ID} \right),
\]
\[
\bar{g}_{kC,ID} = -\frac{1}{2} \left( \mathcal{U}_{kC,ID} + \mathcal{U}_{kC,ID} \right).
\]
Substituting $\mathcal{N}^{lD}_{kC}$ and $\tilde{\mathcal{N}}^{lD}_{kC}$ by their expressions (6.6-6.9) and using the relation $\mathcal{F}^I J = \delta^K_I$, we get the following explicit field relation of the metric components:

$$h^{lD}_{kC} = -\delta^l_k \left( \delta^D_C + \frac{\lambda}{2} Q_{kC} Q^{lD} \right)$$

$$+ \frac{\lambda}{2} \mathcal{F}^I J \left( \overline{\mathcal{N}}_{kC} Q^{lD}_J + \mathcal{E}^K_J \overline{Q}_C Q^{lD}_k \right) \quad (6.13)$$

and

$$g^{kC,lD} = -\frac{\lambda}{2} \mathcal{F}^I J \left[ Q^k_J Q^{lD}_I + \mathcal{E}^J_K Q^l_I Q^{kD} + \mathcal{E}^J_K Q^{lD}_I Q^{kD} \epsilon^{kl} \right] \quad (6.14)$$

$$\overline{\mathcal{N}}_{kC,lD} = -\frac{\lambda}{2} \mathcal{F}^I J \left( \overline{Q}_{lD,J} \overline{Q}^J_{kC} + \mathcal{E}^J_K \overline{Q}_C Q^{kD} + \mathcal{E}^J_K \overline{Q}_C Q^{kD} \epsilon^{kl} \right).$$

In the special case $n = 1$, these relations reduce to,

$$h^l_k = -2 \delta^l_k \left( 1 + \lambda \overline{f} f \right) + \lambda \left( \frac{1 + (1 + \lambda) \overline{f} f}{1 + \lambda \overline{f} f} \right) f^l \overline{f}_k \quad (6.15)$$

$$\overline{g}_{kl} = -\frac{\lambda}{2} \frac{1 + (1 + \lambda) \overline{f} f}{1 + \lambda \overline{f} f} \overline{f}_k \overline{f}_l \quad (6.16)$$

$$g^{kl} = -\frac{\lambda}{2} \frac{1 + (1 + \lambda) \overline{f} f}{1 + \lambda \overline{f} f} \overline{f}_k \overline{f}_l \quad (6.17)$$

where we have used the identity $\overline{f} f f_k = f^l \overline{f}_k - \delta^l_k (\overline{f} f)$. Comparing this expression with eq(3.3), we recover exactly the Taub-NUT metric.

Furthermore substituting $Q^A_I$ and $\overline{Q}^J_B$ by their expressions in terms of $f^A_I$, $\overline{f}_I A$ and the Cartan matrices $H_I$, we can rewrite the metric components $h^{lD}_{kC}$, $g^{kC,lD}$ and $\overline{g}_{kC,lD}$ as follows:

(i) component $h^{lD}_{kC}$

$$h^{lD}_{kC} = +\delta^l_k \left( \delta^D_C + \frac{\lambda}{2} f^A_I A B g_{IJ} \left( H^I \right)_C \left( H^J \right)_B \right)$$

$$-\frac{\lambda}{2} \mathcal{F}^I J \left( \overline{f} A B g_{IJ} \left( H^I \right)_C \left( H^J \right)_B \right)$$

$$-\frac{\lambda}{2} \mathcal{F}^I J \left( \mathcal{E}^J_K f^A_I A B g_{IJ} \left( H^I \right)_C \left( H^J \right)_B \right) \quad (6.18)$$

where the $n \times n$ matrices $\mathcal{E}^I_J$ and $\mathcal{F}^K_J$ are given by

$$\mathcal{E}^I_J = \left[ \delta^I_J + \lambda \text{Tr} \left( \overline{f} H_I H^I \overline{f} \right) \right] \quad , \quad \mathcal{E}^I_J \mathcal{F}^K_J = \delta^K_I \quad (6.19)$$

(ii) component $g^{kC,lD}$

$$g^{kC,lD} = +\frac{\lambda}{2} \mathcal{F}^I J \left[ f^A_I A B \left( g_{IJ} \left( H^I \right)_A \left( H^J \right)_B \right) \right]$$

$$+\frac{\lambda}{2} \mathcal{F}^I J \left( f^A_I A B \mathcal{E}^K_J g_{IJ} \left( H^I \right)_A \left( H^J \right)_B \right)$$

$$+\frac{\lambda}{2} \mathcal{F}^I J \left( f^A_I A B \left( \mathcal{E}^K_J g_{IJ} \left( H^I \right)_A \left( H^J \right)_B \right) \epsilon^{kl} \right) \quad (6.18)$$
(iii) component $\overline{g}_{kClD}$

\[
\overline{g}_{kClD} = \frac{\lambda}{2} F_I^J \left[ \overline{t}_{IB} \overline{t}_{IA} (H^L)^B_C (H^I)^A_D \right] + \frac{\lambda}{2} F_I^J \left[ \overline{t}_{jA} \overline{t}_{kB} \left( \varepsilon_J^K g_{IL} (H^L)^A_C (H^K)^B_D \right) \right] + \frac{\lambda}{2} F_I^J \left[ \overline{t}_{jA} \overline{t}_{kB} \left( \varepsilon_J^K g_{IL} (H^L)^A_C (H^K)^B_D \right) \varepsilon_{kl} \varepsilon^{ij} \right] ,
\]

which just the hermitian conjugate of $g^{kClD}$. In the limit $\lambda \to 0$, one recover the free field theory and for $n = 1$ the Taub-NUT geometry.

7 Conclusion and discussion

Freezing the dynamics of the dilaton $\sigma$ and using rigid harmonic superspace (HSS) method, we have computed in this paper the explicit expression of the metric of the real eighty dimensional moduli space $M^{N=2}_{6D}$ of the 10D type IIA superstring on K3. This hyperkahler metric has the following form

\[
ds_{80}^2 = \sum_{k,l=1}^{2} \left( \sum_{A,B=1}^{20} 2k_{kB}^A d^{k} B d^{l} A + g^{kBL} d^{k} B d^{l} A + \overline{g}_{kB} \overline{d}^{l} A d^{k} B \right) ,
\]

and should be put in correspondence with the Kahler metric of the Coulomb branch of the moduli space of 10D type IIA superstring on Calabi-Yau threefolds,

\[
ds_{\text{Kahler}}^2 = \sum_{a,b=1}^{n} g_{ab} d^{a} z d^{b} z , \quad g_{ab} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b} .
\]

We have also shown that the metric (7.1) is a particular member of a family of hyperKahler metrics (6.16-6.19) of $4n$ dimensional manifolds with $U^n (1)$ gauge symmetry whose leading term is given by the well known real four dimensional Taub-NUT geometry associated with eqs(1.7-1.8).

The dynamics of the dilaton $\sigma$ can be directly implemented in the supergravity field action along the line given in [8]. In particular we have for the moduli space $SO(1,1) \times U(2,20)$, the following result,

\[
ds_{81}^2 = (d\sigma)^2 - e^{-2\sigma} ds_{80}^2 .
\]

With the analysis given in this study, we have learnt as well that supersymmetry plays a central role in the metric building of the moduli spaces $M^{N=2}_{4D}$ and $M^{N=2}_{6D}$ of the Coulomb branches of the 4D $\mathcal{N} = 2$ and non chiral 6D $\mathcal{N} = 2$ supergravity theories. We also learnt that the constructions are quite similar.

To explicitly exhibit the similarities between the ways to get the two kinds of metrics, we
give below a comment regarding this issue. First, we recall the geometric set up of the moduli spaces $M_{4D}^{N=2}$ and $M_{6D}^{N=2}$ of the two supergravity theories. Then, we describe the way the metrics of the scalar manifolds $M_{4D}^{N=2}$ and $M_{6D}^{N=2}$ can be engineered by combining geometry and supersymmetry in 4D and 6D respectively.

(1) **Geometry of scalar spaces**

First recall that moduli space of 10D type II superstring on Calabi-Yau threefolds has two branches: a Kahler branch with scalar manifold $M_{4D}^{N=2}$ and complex a one with a hyperKahler structure [35, 36]. Notice also that in 10D type IIA superstring on K3, the scalar manifold $M_{6D}^{N=2}$ is hyperKahler since complex and Kahler deformations of the metric combine to make a hyperKahler structure. The general picture giving the 4D/6D correspondence is schematized in the following table:

| 4D $\mathcal{N} = 2$ sugra | $\leftrightarrow$ | 6D $\mathcal{N} = 2$ sugra |
|-------------------------------|-----------------|-------------------------------|
| Type II on CY3                | -               | Type IIA on K3                |
| Kahler complex                | -               | hyperKahler $\equiv \begin{cases} \text{Kahler} \\
| Kahler 2- form: $\Omega_2^{(1,1)}$ | -               | complex \end{cases}$          |
| holomorphic form: $\Omega_3^{(3,0)}$ | -               | hyperkahler: $\Omega^{(ij)} = \begin{cases} \Omega_2^{(2,0)} \\
| antiholomorphic: $\Omega_3^{(0,3)}$ | -               | $\Omega_2^{(1,1)}$ \\
|                                |                 | $\Omega_2^{(0,2)}$           |

Notice that in the case of K3 compactification, Kahler and complex 2-form on K3 are in the same cohomology class.

(2) **Kahler $\rightarrow$ hyperKahler and beyond**

The Kahler and the complexified Kahler structures of the Coulomb branch of vacua in 10D type IIA on CY3 can be put in correspondence with the hyperKahler and quaternionified hyperKahler structure of $M_{6D}^{N=2}$ as shown in the following table,
where $\Omega_2^{(1,1)}$ and $\Omega^{(ij)}$ are as in previous tables and where $B_{NS}$ stands for the NS-NS B-field. Notice the remarkable role played by the $B$-field in both cases.

(3) **Metrics building**

As it is well known, the Kahler metric $g_{ab}$ has a nice interpretation in $4D \mathcal{N} = 1$ superspace. Denoting by $\Phi^a$ a generic chiral superfield with leading component scalar field $z^a$; that is $(\Phi^a)_{\theta=0} = z^a$, the metric $g_{ab}$ can be obtained by integrating out the Grassmann variables $\theta$ in the following superspace relation,

$$L(x) = \int d^4\theta K(\Phi, \bar{\Phi}), \quad (7.4)$$

where $K(\Phi, \bar{\Phi})$ is the usual Kahler (super)potential.

HyperKahler metrics are engineered in a quite similar manner; but now by using HSS method. There, the Lagrangian density $L(x)$ is given by

$$L(x) = \int_S du \left[ \int d^4\theta^+ \mathcal{L}^+ \left( \Phi^+, \bar{\Phi}^+ \right) \right], \quad (7.5)$$

where $\Phi^+$ is a hypermultiplet superfield and $\mathcal{L}^+ \left( \Phi^+, \bar{\Phi}^+ \right)$ is the harmonic superspace Lagrangian density as in eq(1.3).

The hyperKahler metric associated with eq(7.5) is obtained as follows:

(i) first integrate out the Grassmann variables $\theta^+$ (7.5) to get the typical relation (6.1).

(ii) Then eliminate the vector auxiliary fields $B_{-\mu}^A$ through their eqs of motion.

(iii) Finally integrate out the harmonic variables $u^\pm$ and end with eq(7.1).

In the case of $4D/6D \mathcal{N} = 2$ supergravity theories embedded in type IIA superstring
compactifications, we have the following correspondence

| 4D \(\mathcal{N} = 2\) sugra | --- | 6D \(\mathcal{N} = 2\) sugra |
|--------------------------|------|--------------------------|
| moduli \(z^a = \int_{C^2} J\) | \(\phi^{ij} = \int_{C^4} J^{ij}\) |
| prepotential \(F(z) = \{ \int_{CY} J \wedge J \wedge J \} = \sum_{a,b,c} d_{abc} z^a z^b z^c\) | \(\mathcal{F}^{ijkl} = \{ \int_{CY} J^{ij} \wedge J^{kl} \} = g_{ij} \phi^{ij} \phi^{kl}\) |

where \(d_{abc}\) are the 3-intersection numbers in the second homology of the Calabi-Yau threefold and \(g_{IJ}\) the 2-intersections of \(H_2(K3,R)\).

Moreover, using the fact that Kahler 2-form can be also expressed as \(K = \frac{1}{2} (J - \bar{J})\), we have

\[
\int_{C^2} K = \frac{1}{2} (z^a - \bar{z}^a) . \tag{7.6}
\]

Then, Kahler potential \(K = \mathcal{K}(\Phi, \bar{\Phi})\) reads in terms of the usual 4D \(\mathcal{N} = 1\) chiral superfields \(\Phi^a\) and \(\bar{\Phi}^a\) as,

\[
e^\mathcal{K} = \frac{i}{8} \sum_{a,b,c} d_{abc} (\Phi^a - \bar{\Phi}^a) (\Phi^b - \bar{\Phi}^b) (\Phi^c - \bar{\Phi}^c) . \tag{7.7}
\]

The analogue of the above 4D \(\mathcal{N} = 1\) superspace (or equivalently 2D \(\mathcal{N} = 2\) relations in six dimensional space time can be also written down in 6D \(\mathcal{N} = 1\) HSS or equivalently 4D \(\mathcal{N} = 2\) HSS. There, the analogue of eq(7.6) is obviously given by the hyperKahler isotriplet moduli

\[
\phi^{(kl)I} = \int_{C^4} \Omega^{(kl)} . \tag{7.8}
\]

In harmonic superspace, this can be achieved by multiplying both sides of \(\phi^{ij} = \int_{C^4} J^{ij}\) by the harmonic variable monomial \(u_i^+ u_j^+\) to end with,

\[
\phi^{++I} = \int_{C^2} \Omega^{++} , \quad B_{NS} e^{ij} u_i^+ u_j^+ = 0 , \tag{7.9}
\]

and then the hyperKahler potential,

\[
\mathcal{L}_{HK}^{++} = \sum_{I,J} g_{IJ} T^{++I} T^{++J} . \tag{7.10}
\]

In the above relation the HSS superfield \(T^{++I}\) is given by the following hypermultiplet composite

\[
T^{++I} = \frac{1}{i} Tr \left( \Phi^+ H^I \Phi^+ \right) , \tag{7.11}
\]

where \(H^I\) are the Cartan generators of the \(U(n)\) isotropy symmetry of the moduli space. The \(T^{++I}\)'s obey the HSS conservation laws

\[
D^{++} T^{++I} = 0 , \tag{7.12}
\]

29
and are interpreted as the Noether super-currents in harmonic superspace. This relation can be also stated as given by the sum of the two following relation,

\[
\begin{align*}
(D^{++} - \lambda \sum_i T_i^{++}H^i) \Phi^+ &= 0 , \\
(D^{++} + \lambda \sum_i T_i^{++}H^i) \tilde{\Phi}^+ &= 0 .
\end{align*}
\tag{7.13}
\]

These eqs are precisely the superfield eqs of motion, from which we can read the HSS superfield action for the hypermultiplets.

We end this study by noting that the analysis given in this paper can be also used to deal with 10D type IIA on ALE spaces.

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8 Appendix A: U(1) model

To bring the superfield action (3.3) to the component fields form (1.10), we start by computing the hypermultiplet equations of motion. They are given by

\[
\begin{align*}
D^{++} \Phi^+ - \lambda (\tilde{\Phi}^+ \Phi^+) \Phi^+ &= 0 , \\
D^{++} \tilde{\Phi} + \lambda (\tilde{\Phi}^+ \Phi^+) \tilde{\Phi}^+ &= 0 .
\end{align*}
\tag{8.1}
\]

Then identify the conserved Noether HSS current \(T^{++}\)

\[
D^{++}T^{++} = 0 ,
\tag{8.2}
\]

associated with the symmetry (3.5). The super-current \(T^{++}\) can obtained by multiplying the first relation of the system (8.1) by \(\tilde{\Phi}^+\), and the second relation by \(\Phi^+\). By adding both relations, we end with the HSS conservation law \(D^{++}(\tilde{\Phi}^+ \Phi^+) = 0\), from which we learn:

\[
T^{++} = i\tilde{\Phi}^+ \Phi^+ , \quad \tilde{T}^{++} = T^{++} .
\tag{8.3}
\]

Moreover, substituting the superfields \(\Phi^+\) and \(\tilde{\Phi}^+\) by their \(\theta^+\)-expansions (2.23, 2.25), we get the following component field relations:

(a) the leading \(\theta^+\) component \((\theta^+ = 0)\) gives the field eqs of motion of \(\Delta^{---}\) and its conjugate \(\Delta^{---}:\)

\[
\begin{align*}
[\partial^{++}q^+ - \lambda (\bar{q}^+q^+)]q^+ &= 0 , \\
[\partial^{++}\bar{q}^+ + \lambda (\bar{q}^+q^+)]\bar{q}^+ &= 0 .
\end{align*}
\tag{8.4}
\]
These are constraint eqs that fix the dependence of the scalar field $q^+$ in the harmonic variables $u^\pm_i$; i.e:

$$q^+ = q^+ (x, u^\pm) \quad (8.5)$$

(b) the $\theta^+ \sigma^\pm \bar{\theta}^+$ component of eq(8.2) gives the field eqs of motions of $B^-_\mu$ and $\bar{B}^-_\mu$:

$$[\partial^{++} - \lambda (\bar{q}^+ q^+) ] B^-_\mu - \lambda \left( \bar{q}^+ B^-_\mu + \bar{B}^-_\mu q^+ \right) = 2 \partial_\mu q^+ ,$$

$$[\partial^{++} + \lambda (q^+ q^+) ] \bar{B}^-_\mu + \lambda \left( q^+ B^-_\mu + B^-_\mu q^+ \right) \bar{q}^+ = 2 \partial_\mu \bar{q}^+ . \quad (8.6)$$

(c) The $\theta^{+2}$ and $\bar{\theta}^{+2}$ components give the eqs of motion of the auxiliary fields $F^-$ and $G^-$. But these relations are irrelevant for the explicit computation the metric (1.10). They are rather needed for the determination of the scalar potential that follows from the compactification from $6D$ down to $4D$.

The solution of eq(8.4) is given by,

$$q^+ (x, u) = u^+_i e^{\lambda w f^i (x)} , \quad (8.7)$$

with $w (\bar{w} = -w)$ given by

$$w = \frac{1}{2} \left( \bar{f}^- f^+ + \bar{f}^+ f^- \right) = \frac{1}{2} u^+_i u^-_j f^{(i} f^{j)} ,$$

$$\partial^{++} w = \bar{f}^+ f^+ , \quad (8.8)$$

where the complex doublets

$$f^\pm = u^+_i f^i \quad , \quad \bar{f}^\pm = u^-_i \bar{f}^i \quad , \quad (8.9)$$

are as in eqs (1.10). The solution of eq(8.6) is given by,

$$B^-_\mu (x, u) = e^{\lambda w} C^-_\mu (x) , \quad (8.10)$$

The field $C^-_\mu$ is given by

$$C^-_\mu = 2 \partial_\mu f^- + \lambda \partial_\mu f^- + \lambda f^+ \partial_\mu \left( \bar{f}^- f^- \right) , \quad (8.11)$$

with

$$\partial_\mu = -\frac{1}{1+\lambda \rho^2} \left( \bar{f}^- \partial_\mu f^+ - f^i \partial_\mu \bar{f}^i \right) ,$$

$$\rho^2 = \bar{f}^- f^- . \quad (8.12)$$

To get this result, we proceed in steps as follows:

(i) compute the $\theta^+ -$ expansion of $\bar{\theta}^+ \Phi^+$ by using (2.23, 2.26). We have,

$$\Phi^+ = w^{++} + \theta^+ M + \bar{\theta}^{+2} N + i \theta^+ \sigma^\mu \bar{\theta}^+ A_\mu + \theta^+ 2 \bar{\theta}^{+2} P^{--} , \quad (8.13)$$
with
\[ w^{++} = \tilde{q}^+ q^+ , \]
\[ M = \left( \tilde{q}^+ F^- - \tilde{G}^- q^+ \right) , \]
\[ \tilde{M} = \left( \tilde{F}^- \phi^+ - \tilde{q}^+ \tilde{G}^- \right) , \]
\[ N = \left( \tilde{q}^+ G^- - \tilde{F}^- q^+ \right) , \]
\[ A_\mu = \left( \tilde{q}^+ B_\mu^- + \tilde{B}_\mu^- q^+ \right) , \] (8.14)

and
\[ P^{--} = \left( \tilde{q}^+ \Delta^3 - \tilde{\Delta}^3 \phi^+ - \tilde{F}^- F^- - \tilde{G}^- G^- - \tilde{B}_\mu^- B^- \mu \right) . \] (8.15)

We also have
\[ \tilde{w}^{++} = -w^{++} , \quad \tilde{A}_\mu = A_\mu , \]
\[ \tilde{M} = -N , \quad \tilde{P}^{--} = -P^{--} . \] (8.16)

(ii) use the HSS conservation law \( D^{++}(\tilde{\Phi}^+ \Phi^+) = 0 \), which leads, at the level of the component fields, to:
\[ \partial^{++}(\tilde{q}^+ q^+) = 0 , \]
\[ \partial^{++}A_\mu = 2\partial_\mu(\tilde{q}^+ q^+) . \] (8.17)

These component fields conservation laws require the factorization (8.7); and allow to bring \( A_\mu \) into the simple form
\[ A_\mu = \tilde{f}^+ C^-_\mu + \tilde{C}^-_\mu f^+ , \]
\[ \partial^{++}A_\mu = 2\partial_\mu(\tilde{f}^+ f^+) . \] (8.18)

The second relation of above eqs shows that \( A_\mu \) can be decomposed like
\[ A_\mu = \vartheta_\mu + 2\partial_\mu w , \]
\[ \partial^{++}\vartheta_\mu = 0 , \] (8.19)

where
\[ w(x, u) = u^{(i)} w^{(ij)}(x) , \] (8.20)

is as in eq[8.8].

The extra term \( \vartheta_\mu = \vartheta_\mu(x) \) is an isosinglet; it is determined as follows:

First perform the change \( B^-_\mu = e^{\lambda u} C^-_\mu \) to first bring eqs[8.6] into the simplest form
\[ \partial^{++}C^-_\mu - \lambda A_\mu f^+ = 2\partial_\mu f^+ , \]
\[ \partial^{++}\tilde{C}^-_\mu + \lambda A_\mu \tilde{f}^+ = 2\partial_\mu \tilde{f}^+ . \] (8.21)

Substituting,
\[ A_\mu f^+ = \partial^{++}(\vartheta_\mu f^- + 2f^+ \partial_\mu w^{--}) , \]
\[ A_\mu \tilde{f}^+ = \partial^{++}(\vartheta_\mu \tilde{f}^- + 2\tilde{f}^+ \partial_\mu w^{--}) , \]
\[ \partial^{++}w^{--} = w , \]
\[ w^{--} = \frac{1}{2}(\tilde{f}^- f^-) , \] (8.22)
we obtain,
\[
\begin{align*}
C^-_\mu & \quad = \quad 2\partial_\mu f^- + \lambda \partial_\mu f^- + 2\lambda f^+ (\partial_\mu \varphi^-) , \\
\bar{C}^-_\mu & \quad = \quad 2\partial_\mu \bar{f}^- - \lambda \partial_\mu \bar{f}^- - 2\bar{f}^+ (\partial_\mu \varphi^-) .
\end{align*}
\] (8.23)

Multiplying the first eq by \( \bar{f}^+ \) and the second by \( f^+ \); we obtain
\[
\begin{align*}
\bar{f}^+ C^-_\mu & \quad = \quad 2\bar{f}^+ \partial_\mu f^- + \lambda \partial_\mu \bar{f}^+ f^- + 2\bar{f}^+ f^+ (\partial_\mu w^-) , \\
f^+ \bar{C}^-_\mu & \quad = \quad 2f^+ \partial_\mu \bar{f}^- - \lambda \partial_\mu f^+ \bar{f}^- - 2f^+ \bar{f}^+ (\partial_\mu w^-) .
\end{align*}
\] (8.24)

Then adding both eqs, we get,
\[
\partial_- \mu + 2\partial_\mu w = 2 \left( \bar{f}^+ \partial_\mu \bar{f}^- + f^+ \partial_\mu f^- \right) - \lambda \partial_\mu \left( \bar{f}^+ f^- \right) ,
\] (8.25)

from which we get the expression of \( \partial_\mu \) (8.12).

Notice that using \( 2w^- = \bar{f}^- f^- \), we can split the field \( C^-_\mu \) (8.23) as the sum of two irreducible components as follows
\[
C^-_\mu (x, u) \quad = \quad U^- C^i_\mu (x) + U^- U^- C^{(ijk)} (x) ,
\] (8.26)

where
\[
\begin{align*}
C^i_\mu & \quad = \quad 2\partial_\mu f^i + \lambda \partial_\mu f^i + \frac{\lambda^2}{3} \partial_\mu f^i + \frac{\lambda L^i_x k}{3} \partial_\mu T^i_k + \frac{\lambda i}{3} \partial_\mu \bar{T}^i_k , \\
C^{(ijk)}_\mu & \quad = \quad \lambda f^i g^j \partial_\mu g^k + \lambda f^i \bar{g}^j \partial_\mu \bar{g}^k .
\end{align*}
\] (8.27)

Similar relations can be written down for \( \bar{C}_\mu \) and \( \bar{C}^{(ijk)}_\mu \).

the Taub-NUT metric

By performing the integration of eq (1.8) with respect to the Grassmann variables \( \theta^+ \) and \( \bar{\theta}^+ \), we first get
\[
\mathcal{L}_1 \quad = \quad \frac{1}{2} \int_{S^2} du \left( B^-_\mu \partial^\mu q^+ + \bar{B}^-_\mu \partial^\mu q^+ \right) .
\] (8.28)

Then using the solution (8.7), we can bring the above expression to
\[
\mathcal{L}_1 \quad = \quad \frac{1}{2} \int_{S^2} du \left[ \left( C^-_\mu \partial^\mu f^+ - \bar{C}^-_\mu \partial^\mu f^+ \right) - \lambda A^\mu \partial^\mu w \right] ,
\] (8.29)

where we have used the identity \( A^-_\mu = \bar{f}^+ C^-_\mu + \bar{C}^-_\mu f^+ \).

Next integrating with respect to the harmonic variables \( u^\pm \), we obtain,
\[
\mathcal{L}_1 \quad = \quad \frac{1}{4} \left[ \left( C^i_\mu \partial^\mu T^i_{\bar{i}} - \bar{C}^i_\mu \partial^\mu f^i \right) + 4a \lambda \partial_\mu \left( \bar{f}^i (\bar{T}^i_{\bar{i}}) \right) \partial^\mu \left( \bar{T}^i_{\bar{i}} \right) \right] ,
\] (8.30)

where \( a = \frac{1}{6} \); and where we have used
\[
\begin{align*}
\int_{S^2} du \left[ u^{+i} u^{-j} \right] & \quad = \quad \frac{1}{2} \varepsilon^{ij} , \\
\int_{S^2} du \left[ u^{+(i} u^{-j) u^{-k} u^{-l)} \right] & \quad = \quad \frac{1}{3} \left( \delta^i_k \delta^j_l + \delta^j_k \delta^i_l \right) .
\end{align*}
\] (8.31)

Substituting \( C^i_\mu \) by its expression (8.27), we get,
\[
\mathcal{L}_1 \quad = \quad \frac{1}{4} \left( \bar{g}_{ij} \partial^i_\mu \partial^k_\mu f^j + g^{ij} \partial^i_\mu \bar{T}^j_{\bar{i}} \partial^\mu \bar{T}^j_{\bar{i}} + 2h^i_\mu \partial^i_\mu \partial^\mu \bar{T}^j_{\bar{i}} \right) ,
\]
with \( \bar{g}_{ij} \), \( g^{ij} \) and \( h^i_\mu \) as in eqs (8.30).
9 Appendix B: \textit{U}^n(1) model, \(n > 1\)

In appendix B1, we determine the solution of eqs(5.17-5.18) and in appendix B2, we derive the explicit expression on the component field of the metric of the moduli space \(SO(4,20)/SO(4) \times SO(20)\).

9.1 Solving the constraint eqs(5.17-5.18)

First we consider the solution of eq(5.17). Then we deal with eq(5.18).

\textbf{Solving eq(5.17)}

The solution of \(q^+ \pm \), in terms of the fields doublets \(f^\pm A = u^\pm f^A, \tilde{f}^\pm = u^\pm \tilde{f}^A\) can be obtained by factorizing it as follows

\[
q^+ = u^\pm (e^{\lambda w} f^i), \quad \tilde{q}^+ = u^\pm (\tilde{f}^i e^{-\lambda w}),
\]

where \(w = w(f, \tilde{f})\) is given by

\[
w^I = \sum_{I=1}^{n} w^I H^I, \quad w^I = \frac{1}{2} u^+ h^k u^- Tr\left( H^I f^k + \tilde{f}^I f^k \right).
\]

Notice that \(w^I\) has a particular dependence in the harmonic variables; it captures an \(SU_R(2)\) isotriplet representation,

\[
w_I = \frac{1}{2} Tr\left( H^I f^+ + \tilde{H}^I f^+ \right).
\]

Notice also

\[
\partial^+ w_I = w^+_I = Tr\left( H^I f^+ \right), \quad (\partial^+)^2 w_I = 0, \\
\partial^- w_I = w^-_I = Tr\left( \tilde{H}^I f^- \right), \quad (\partial^-)^2 w_I = 0.
\]

The solution of \(q^+ = q^+(x, u)\) in terms of \((f^I, \tilde{f}^I)\) and the harmonics \(u^I\) reads therefore as

\[
q^+ = \left[ \exp\left( \frac{\lambda}{2} \sum_{k,l=1}^{n} u^+_k u^-_l \sum_{I=1}^{n} \left[ Tr\left( H^I f^I \right) H^I \right] \right) f^I u^+_I \right],
\]

\[
\tilde{q}^+ = \left[ \exp\left( \frac{-\lambda}{2} \sum_{k,l=1}^{n} u^+ u^- \sum_{I=1}^{n} \left[ Tr\left( \tilde{H}^I f^I \right) H^I \right] \right) \tilde{f}^I u^+_I \right].
\]

In the limit \(\lambda \to 0\), we recover the free fields \(q^+ = u^+_I f^I(x)\).

\textbf{Solving eq(5.18)}

To get the solution of eq(5.18), we need several steps:
(i) **Step 1**: We use the $U^n(1)$ symmetry of the Lagrangian density and the equations of motion to make the change of field variables

\[ B^-_\mu = e^{\lambda w} C^-_\mu, \]
\[ \tilde{B}^-_\mu = e^{-\lambda w} \tilde{C}^-_\mu, \]

where $w$ is as in eq(9.2) and where $C^-_\mu$ is the new field to determine. This change of field variables allows us to bring eq(5.18) into the form

\[ \partial^{++} C^-_\mu - \lambda A^I_\mu f^+ = 2\nabla_\mu f^+, \]
\[ \partial^{++} C^-_\mu + \lambda A^I_\mu \tilde{f}^+ = 2\nabla_\mu \tilde{f}^+, \]

where we have set

\[ \nabla_\mu f^+ = [\partial_\mu + \lambda (\partial_\mu w)] f^+, \]
\[ \nabla_\mu \tilde{f}^+ = [\partial_\mu - \lambda (\partial_\mu w)] \tilde{f}^+. \]

We also have

\[ A_\mu = \sum_{I=1}^n A^I_\mu H^I, \]
\[ A^I_\mu = \tilde{C}^-_\mu H^I f^+ + \tilde{C}^-_\mu H^I C^-_\mu, \]

which satisfy

\[ \partial^{++} A^I_\mu = 2\left[ \partial_\mu \left( \tilde{f}^+ H^I f^+ \right) \right], \]
\[ = 2\partial^{++} (\partial_\mu w^I). \]

Eq(9.10) implies in turns $(\partial^{++})^2 A^I_\mu = 0$; and so can be solved as follows,

\[ A^I_\mu = \vartheta^I_\mu + 2\partial_\mu w^I. \]

The $\vartheta^I_\mu$'s are isosinglets,

\[ \partial^{++} \vartheta^I_\mu = 0, \]
\[ \vartheta_\mu = \sum_{I=1}^n \vartheta^I_\mu H^I, \]

they will be determined in terms of the dynamical scalars $f^{iA}$.

(ii) **Step 2**: We use the identity

\[ \lambda A_\mu f^+ = \lambda \partial^{++} (\partial_\mu f^- + \partial_\mu w^- f^+) \]

to solve eq(9.7) like,

\[ C^-_\mu = 2\partial_\mu f^- + \lambda \partial_\mu f^- + \lambda (\partial_\mu w^-) f^+, \]
\[ \tilde{C}^-_\mu = 2\partial_\mu \tilde{f}^- - \lambda \partial_\mu \tilde{f}^- - \lambda (\partial_\mu w^-) \tilde{f}^+. \]

To determine $\vartheta_\mu$, we compute $\left( \tilde{f}^+ H^I C^-_\mu + \tilde{C}^-_\mu H^I f^+ \right)$ by using eqs(9.14) and derive a constraint equation that allows us to fix $\vartheta_\mu$. We have

\[ \tilde{f}^+ H^I C^-_\mu = 2\tilde{f}^+ H^I \partial_\mu \tilde{f}^- + \lambda \vartheta_\mu \left( \tilde{f}^+ H^I H^J f^- \right) + 2\lambda \partial_\mu w^- \left( \tilde{f}^+ H^I H^J f^+ \right), \]
\[ \tilde{C}^-_\mu H^I f^+ = 2\partial_\mu \tilde{f}^- H^I f^+ - \lambda \vartheta_\mu \left( \tilde{f}^- H^I H^J f^+ \right) - 2\lambda \partial_\mu w^- \left( \tilde{f}^+ H^I H^J f^+ \right). \]
Before going ahead, it is convenient to simplify a little bit the above relations by using the following conventional notations:

\[ Q_i^\pm A = u_i^\pm Q_i^A \equiv (f^\pm H_I)^A, \]
\[ \tilde{Q}_i^\pm I = u_i^\pm \tilde{Q}_i^I \equiv \left( \tilde{f}^\pm H_I^I \right)_B, \]

(9.16)

with

\[ Q_i^A = (H_I^I)^A f^C, \]
\[ \tilde{Q}_i^I = \tilde{I}_j^D (H_I^I)_B, \]
\[ R^A_B = \tilde{Q}_i^f Q^A_i \]

(9.17)

From these fields, one can build the following quantities

\[ \tilde{Q}_i^f Q^A_i, \quad Q_i^A \tilde{Q}_j^B, \]
\[ Q_i^A Q^B_B = (H_I^I)^A f^C, \]
\[ Q_i^A Q^B_B = (H_I^I)_B. \]

(9.18)

Notice that for \( n = 1 \), we have \( H_1 = I \), and the fields \( Q_i^A \) reduce to \( f^i \) and eqs(9.18) to

\[ \tilde{Q}_i^f Q^A_i \rightarrow \tilde{I}_i^f, \quad Q_i^A \tilde{Q}_j^B \rightarrow f^i \tilde{T}_j, \]
\[ Q_i^A Q^B_B \rightarrow f^i f^k, \quad \tilde{Q}_i^f Q^B_B \rightarrow \tilde{I}^i \tilde{T}_l. \]

(9.19)

Using these new field moduli, we can rewrite eqs(9.15) like,

\[ \tilde{Q}^+ C^- = 2 \tilde{Q}^+ \tilde{I}^\mu f^- + \lambda \partial_{\mu} \tilde{Q}^+ \left( \tilde{Q}^+ Q^- \right) + \lambda \partial_{\mu} w_j^- \left( \tilde{Q}^+ Q^+ \right), \]
\[ \tilde{C}^- Q^+ = 2 \partial_{\mu} \tilde{I}^- Q^+ - \lambda \partial_{\mu} \tilde{Q}^- \left( \tilde{Q}^+ Q^+ \right) - 2 \lambda \partial_{\mu} \varphi_j^- \left( \tilde{Q}^+ Q^+ \right). \]

(9.20)

Next, adding these two relations and using eq(9.15), we end with

\[ \vartheta^I_\mu = \left( Q_i^A \partial_\mu \tilde{I}^i_A - \tilde{Q}_i^f \partial_\mu f^i \right) - \lambda \vartheta^I_\mu \left( \tilde{Q}_i^f Q^i_A \right), \]

(9.21)

which can be also rewritten as

\[ \mathcal{E}^I_\mu \vartheta^I_\mu = v^I_\mu, \]

(9.22)

with

\[ v^I_\mu = \left( Q_i^A \partial_\mu \tilde{I}^i_A - \tilde{Q}_i^f \partial_\mu f^i \right), \]
\[ \mathcal{E}^I_\mu = \left[ \delta^I_\mu + \lambda \tilde{Q}_i^f Q^i_A \right]. \]

(9.23)

Using eq(9.16), these relations can be also put in the equivalent form

\[ v^I_\mu = (f^I H^I \partial_\mu \tilde{I}^i_i - \tilde{I}^i_i \partial_\mu f^i), \]
\[ \mathcal{E}^I_\mu = \left[ \delta^I_\mu + \lambda \tilde{Q}_i^f Q^i_A \right]. \]

(9.24)

Thus, the solution of \( \vartheta^I_\mu \) reads as,

\[ \vartheta^I_\mu = \mathcal{F}_I^I v^I_\mu, \quad \mathcal{F}_I^I \mathcal{E}_K^I = \delta^I_K. \]

(9.25)
Notice that for \( n = 1 \), eqs (9.23-9.25) reduce to
\[
\begin{align*}
v'_\mu & \rightarrow v_\mu = \left(t_i \partial_\mu \tilde{t}_i - \tilde{t}_i \partial_\mu t_i \right), \\
\mathcal{E}'_I & \rightarrow \mathcal{E} = \left[1 + \lambda \tilde{t}_i t^i \right], \\
\mathcal{F}'_I & \rightarrow \mathcal{F} = \frac{1}{1+\lambda \tilde{t}_i t^i}, \\
\mathcal{E} \mathcal{F} & = 1.
\end{align*}
\]
(9.26)

Notice moreover that because of the property,
\[
\tilde{t}_i H_J H^l t^i = \tilde{t}_i H^l t^i, \quad (9.27)
\]
we have the identity
\[
\overline{Q}_{iAJ} Q^{iA} = \overline{Q}^l_i Q^{iA} .
\]
(9.28)

The solution \( C^{-A}_\mu(x, u) \) and \( \tilde{C}^{-A}_\mu(x, u) \) read, in terms of \( Q^+_J \), as follows:
\[
\begin{align*}
C^{-A}_\mu &= 2 \partial_\mu f^{-A} + \lambda \mathcal{F}'_I v'_\mu Q^{-A}_J + \lambda Q^+_J Q^{-B} (\partial_\mu f^{-B}) \\
&\quad + \lambda Q^+_{BJ} Q^{-J} \left( \partial_\mu \tilde{t}_B \right), \\
\tilde{C}^{-A}_\mu &= 2 \partial_\mu \tilde{t}^{-A} - \lambda \mathcal{F}'_I v'_\mu \tilde{Q}^{-A}_J - \lambda \tilde{Q}^+_J Q^{-BJ} \left( \partial_\mu \tilde{t}_B \right) \\
&\quad - \lambda \tilde{Q}^+_{BJ} \tilde{Q}^{-J} \left( \partial_\mu \tilde{t}_B \right).
\end{align*}
\]
(9.29)

Like in eq (8.26), these fields obey \( (\tilde{\partial}^{-})^2 C^{-A}_\mu = 0 \); they can be then decomposed in quite similar manner like
\[
C^{-A}_\mu(x, u) = u_i C^i_A(x) + u_i u_j u_k C^{(ijk)A}_\mu(x),
\]
(9.31)
with
\[
\begin{align*}
C^i_A &= 2 \partial_\mu f^i + \lambda \mathcal{F}'_I v'_\mu Q^i_J \\
&\quad + \frac{1}{3} Q^j_\mu J B \left( \partial_\mu f^B \right) + \frac{1}{3} Q^J_\mu B \left( \partial_\mu f^B \right) \\
&\quad + \frac{1}{3} Q^j_\mu J B \left( \partial_\mu \tilde{t}_B \right) + \frac{1}{3} Q^B_\mu J \left( \partial_\mu \tilde{t}^B \right), \quad (9.32)
\end{align*}
\]
and
\[
\begin{align*}
\overline{C}^{-i}_A &= 2 \partial_\mu \tilde{t}^{-i} - \lambda \mathcal{F}'_I v'_\mu \overline{Q}^{-i}_J \\
&\quad - \frac{1}{3} \overline{Q}^j_\mu J B \left( \partial_\mu \tilde{t}_B \right) - \frac{1}{3} \overline{Q}^J_\mu B \left( \partial_\mu \tilde{t}^B \right) \\
&\quad - \frac{1}{3} \overline{Q}^j_\mu J B \left( \partial_\mu \tilde{t}^B \right) - \frac{1}{3} \overline{Q}^j_\mu B \left( \partial_\mu \tilde{t}_B \right) .
\end{align*}
\]
(9.33)

Similar relations may be written down for \( C^{(ijk)A}_\mu \) and \( \overline{C}^{(ijk)A}_\mu \).
9.2 Deriving the metric (1.10)

Performing the integration of eq (5.2) with respect to the Grassmann variables $\theta^+$ and $\bar{\theta}^+$, we obtain the following action,

$$S_n = \frac{1}{2} \int d^4x \left[ \int_{S^2} du \left( B_{\mu A}^+ \partial^{\mu} \tilde{q}_A^+ - \tilde{B}_{\mu A}^- \partial^{\mu} q^{+ A} \right) \right]. \quad (9.34)$$

To get the space time field action,

$$S_n = \frac{1}{2} \int d^4x \left( 2 h_{iA}^{B} \partial_{\mu} f_{iA}^{B} \partial^{\mu} \tilde{f}_{jB}^+ + g_{iAjB}^{I} \partial_{\mu} f_{iA}^{I} \partial^{\mu} \tilde{f}_{jB}^I + g_{iAjB}^{I} \partial_{\mu} \tilde{f}_{iA}^I \partial^{\mu} \tilde{f}_{jB}^I \right), \quad (9.35)$$

we have to integrate with respect the harmonic variables $u^\pm$.

To that purpose, we start by substituting $B_{\mu A}^+$ and $q^{+ A}$ by of their expressions in terms of $C_{\mu A}^-$ and $f^{+ A}$ (9.5-9.6, 9.29). Doing this, we can bring $S_n$ to the form

$$S = \frac{1}{2} \int d^4x \left[ L_{n1} (x) + L_{n2} (x) \right] \quad (9.36)$$

where we have set

$$L_{n1} (x) = \int_{S^2} du \left( C_{\mu A}^- \partial^{\mu} \tilde{f}_{iA}^+ - \tilde{C}_{\mu A}^- \partial^{\mu} f^{+ A} \right), \quad (9.37)$$

$$L_{n2} (x) = -\lambda \int_{S^2} du \left[ \left( \tilde{Q}_{iA}^+ C_{\mu A}^- + \tilde{C}_{\mu A}^- Q_{iA}^+ \right) \left( \partial^{\mu} w^I \right) \right],$$

with $w^I$ as in eq (8.8). Notice that using eq (9.9), we also have

$$L_{n2} = -\lambda \int_{S^2} du \left( \sum_{I=1}^{n} A_{\mu I} \partial^{\mu} w^I \right). \quad (9.38)$$

As the integration with respect to the harmonic variables is technical, let us give details regarding the explicit calculations of $L_{n1}$ and $L_{n2}$.

1. Computing $L_{n1}$

Substituting $C_{\mu A}^-$ and $\tilde{C}_{\mu A}^-$ by their expressions (9.29-9.30) in terms of the dynamical fields $f_{iA}^{\pm}$ and $\tilde{f}_{iA}^{\pm}$, we can determine $L_{n1}$. The calculations are lengthy, we shall then proceed by steps. Setting

$$A_1 = C_{\mu A}^- \partial^{\mu} \tilde{f}_{iA}^+, \quad \tilde{A}_1 = \tilde{C}_{\mu A}^- \partial^{\mu} f^{+ A}, \quad (9.39)$$

we first compute their explicit expression in terms of the dynamical fields $f^{\pm}$ and $\tilde{f}^{\pm}$,

$$A_1 = A_1 \left( f^{\pm}, \tilde{f}^{\pm} \right), \quad \tilde{A}_1 = \tilde{A}_1 \left( f^{\pm}, \tilde{f}^{\pm} \right). \quad (9.40)$$

Then we integrate with respect to the harmonic variables.
(i) Computing $A_1$ and $\tilde{A}_1$
Putting eqs (9.29, 9.30) back into eqs (9.39), we obtain

$$A_1 = 2\tilde{\mu} f^{-A} \partial^A \tilde{T}_{iA} + \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} + \lambda Q^+_J \tilde{\nu}_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA} + \frac{\lambda}{2} Q^+_J Q^{-B} \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA}$$

(9.41)

and

$$\tilde{A}_1 = 2\tilde{\mu} f^{-A} \partial^A \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} - \lambda Q^+_J \tilde{\nu}_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA} - \lambda Q^+_J \tilde{Q}^+_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA}$$

(9.42)

In these relations, $F^I_j$ and $u^I_{\mu}$ are given by eqs (9.22, 9.25) and $\tilde{Q}^+_B$ and $Q^{-B}$ are as in eqs (9.16, 9.18).

(ii) Integration over $S^2$
The integration of the above eqs with respect to the harmonic variables gives

$$\int_{S^2} d\mu A_1 = \mathcal{C}^{iA}_{\mu iA} \partial^\mu \tilde{T}_{iA} , \quad \int_{S^2} d\mu \tilde{A}_1 = \mathcal{C}^{iA}_{\mu iA} \partial^\mu f^{iA}$$

(9.43)

where

$$\mathcal{C}^{iA}_{\mu iA} \partial^\mu \tilde{T}_{iA} = -\tilde{\mu} f^{iA} \partial^\mu \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} + \xi \lambda \tilde{\nu}_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA} + \xi \lambda Q^+_J \tilde{\nu}_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA}$$

(9.44)

and

$$\mathcal{C}^{iA}_{\mu iA} \partial^\mu f^{iA} = -\tilde{\mu} f^{iA} \partial^\mu \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} - \xi \lambda Q^+_J \tilde{\nu}_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA} + \xi \lambda Q^+_J \tilde{Q}^+_B \partial^B \tilde{\mu} f^{+A} \tilde{T}_{iA}$$

For later use it is also interesting to rewrite these relations as follows:

$$\mathcal{C}^{iA}_{\mu iA} \partial^\mu \tilde{T}_{iA} = -\tilde{\mu} f^{iA} \partial^\mu \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} + 2\xi \lambda \left[ \left( Q^{kJ}_{iA} \tilde{Q}^{kJ}_{iA} \right) - \left( \tilde{Q}^{kJ}_{iA} Q^{kJ}_{iA} \right) \right] \partial^\mu \tilde{T}_{iA}$$

(9.45)

with $\xi = \frac{1}{8}$, and

$$\mathcal{C}^{iA}_{\mu iA} \partial^\mu f^{iA} = -\tilde{\mu} f^{iA} \partial^\mu \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} Q^{iA}_{\mu} \partial^A \tilde{T}_{iA} - 2\xi \lambda \left[ \left( Q^{kJ}_{iA} \tilde{Q}^{kJ}_{iA} \right) - \left( \tilde{Q}^{kJ}_{iA} Q^{kJ}_{iA} \right) \right] \partial^\mu \tilde{T}_{iA}$$

(9.46)

Subtracting the two terms as in (9.37), we obtain

$$L_{n1} = -2\tilde{\mu} f^{iA} \partial^\mu \tilde{T}_{iA} - \frac{\lambda}{2} F^I_j u^I_{\mu} \tilde{T}_{iA}$$

(9.47)
Using eqs (9.22-9.25) and the identity
\[ \mathcal{F}^I_{\mu j} v^\mu_j = \mathcal{F}^I_{\mu j} Q^{D^I}_k \partial^I t_{kC} \partial_\mu \bar{t}_{ID} - \mathcal{F}^I_{\mu j} Q^{D^I}_k \partial^I t_{kC} \partial_\mu \bar{t}_{ID} + \mathcal{F}^I_{\mu j} Q^{D^I}_k \partial^I t_{kC} \partial_\mu \bar{t}_{ID} \]
we can put \( L_{1n} \) like
\[ L_{1n} = -2 \partial_\mu F_{iA} \partial^I t_{iA} + \frac{1}{2} \mathcal{N}^{ID}_{kC} \partial^I t_{kC} \partial_\mu \bar{t}_{ID} - \frac{1}{2} \mathcal{U}^{CD}_{kC} \partial^I t_{kC} \partial_\mu \bar{t}_{ID} \]
where we have set
\[ \mathcal{N}^{ID}_{kC} = \mathcal{F}^I_{\mu j} Q^{D^I}_k + \mathcal{F}^I_{\mu j} Q^{D^I}_{kC} \]
\[ \mathcal{U}^{CD}_{kC} = \mathcal{F}^I_{\mu j} Q^{D^I}_k + 8 \xi \left( \mathcal{Q}^I_{kC} Q^{D^I}_j \right) - 8 \xi \left( \mathcal{Q}^{I}_{kC} Q^{D^I}_j \right) \delta^I_k \]
(9.50)
\[ \mathcal{U}^{CD}_{kC} = \mathcal{F}^I_{\mu j} Q^{D^I}_k + 4 \xi \left( \mathcal{Q}^{I}_{kC} Q^{D^I}_j \right) - 4 \xi \left( \mathcal{Q}^{I}_{kC} Q^{D^I}_j \right) \delta^I_k \]
In the particular case where \( n = 1 \), these quantities reduce to
\[ \mathcal{N}^{I}_k = 2 \frac{t_k}{1 + M} + 8 \xi \left( \bar{t}_k - \delta^I_k \left( \bar{t}_I f^I \right) \right) \]
\[ \mathcal{U}^{kl} = 4 \xi \left( \bar{t}_k \bar{t}_l \right) \]
(9.51)
where \( \bar{t}_I f^I \) stands for \( \bar{t}_I f^I \) and \( \xi = \frac{1}{8} \).

(2) Computing \( L_{n2} \) eq (9.38)
Using eqs (9.29, 30), we have
\[ \bar{Q}^+_{AI} C^{-A}_\mu = 2 \bar{Q}^+_{AI} F^I_{\mu j} \bar{Q}^+_{AI} Q^{-A}_j + \lambda \mathcal{F}^I_{\mu j} v^\mu_j \bar{Q}^+_{AI} Q^{-A}_j + \lambda \bar{Q}^+_{AI} Q^+_{AI} Q^{-B} \left( \partial_\mu f^B \right) \]
\[ + \lambda \bar{Q}^+_{AI} Q^+_{AI} Q^{-B} \left( \partial_\mu f^B \right) \]
and
\[ \bar{C}^{-A}_{\mu A} Q^+_{AI} = 2 \partial_\mu \bar{Q}^+_{AI} Q^+_{AI} - \lambda \mathcal{F}^I_{\mu j} v^\mu_j \bar{Q}^+_{AI} Q^+_{AI} - \lambda Q^+_{AI} Q^+_{AI} \left( \partial_\mu \bar{t}_B \right) \]
\[ - \lambda Q^+_{AI} Q^+_{AI} \left( \partial_\mu \bar{t}_B \right) \]
Putting these relations back into \( A_{\mu I} = \left( \bar{Q}^+_{AI} C^{-A}_\mu + \bar{C}^{-A}_{\mu A} Q^+_{AI} \right) \), we obtain
\[ A_{\mu I} = 2 \left( \bar{Q}^+_{AI} F^I_{\mu j} \bar{Q}^+_{AI} Q^{-A}_j \right) + \lambda \mathcal{F}^I_{\mu j} v^\mu_j \left( \bar{Q}^+_{AI} Q^{-A}_j - \bar{Q}^+_{AI} Q^+_{AI} \right) + \lambda \left( \bar{Q}^+_{AI} Q^+_{AI} - \bar{Q}^+_{AI} Q^+_{AI} \right) Q^{-B} \left( \partial_\mu \bar{t}_B \right) \]
\[ + \lambda \left( \bar{Q}^+_{AI} Q^+_{AI} - \bar{Q}^+_{AI} Q^+_{AI} \right) Q^{-B} \left( \partial_\mu \bar{t}_B \right) \]
(9.54)
Moreover, using the following identities
\[
\tilde{Q}_{AI} Q_i^+ A = \tilde{Q}_{AI} Q_j^+ A, \\
\tilde{Q}_{AI} Q_i^+ A = \tilde{Q}_{AI} Q_j^+ A, \\
\tilde{Q}_{AI} Q_i^+ A = \tilde{Q}_{AI} Q_j^+ A,
\]
the above expression of \( A_{\mu I} \) gets reduced to
\[
A_{\mu I} = 2 \left( \tilde{Q}_{AI} \partial_{\mu} f^{-A} + Q_i^+ A \partial_{\mu} \tilde{f}_i^A \right) - \lambda \mathcal{F}_I^I v_\mu \left( \overline{Q}_{iAI} Q_i^A \right). 
\]
(9.56)

Now using eq(9.38), and substituting \( A_{\mu I} \), we have
\[
L_{n2} = -\lambda \int_{S^2} du \left[ \sum_{I=1}^{n} \left( \tilde{Q}_{AI} \partial_{\mu} f^{-A} + Q_i^+ A \partial_{\mu} \tilde{f}_i^A \right) \partial_{\mu} w^{I} \right]. 
\]
(9.57)

The term \( \lambda \mathcal{F}_I^I v_\mu \left( \overline{Q}_{iAI} Q_i^A \right) \partial_{\mu} w^{I} \) drops out because of the property \( \int_{S^2} du \left( \tilde{f}_i^A - f_i^{-A} \right) = 0 \). By integration by parts, we can also put \( L_{n2} \) in the form
\[
L_{n2} = +\lambda \int_{S^2} du \left[ \sum_{I=1}^{n} \left( \tilde{Q}_{AI} \partial_{\mu} f^{-A} + Q_i^+ A \partial_{\mu} \tilde{f}_i^A \right) \partial_{\mu} w^{-I} \right]. 
\]
(9.58)

Substituting \( w^{-I} = \tilde{Q}^{-I} f^{-} = \tilde{f}^{-I} Q^{-I} \), we get the following
\[
L_{n2} = -\frac{2}{\lambda} \left( \widehat{U}_{kl}^{kC,ID} \partial_{\mu} \tilde{f}_k^C \partial_{\mu} \tilde{f}_l^D + \widehat{U}_{kC,ID} \partial_{\mu} f^{ID} \partial_{\mu} f^{kC} \right) \\
+\frac{2}{\lambda} \mathcal{N}_{kC}^{kD} \partial_{\mu} f^{kC} \partial_{\mu} f^{lD}. 
\]
(9.59)

where we have set
\[
\mathcal{N}_{kC}^{kD} = 8 \lambda \xi \left( Q_{kC}^{IC} Q_{kD}^{ID} - Q_{kIC}^{ID} \delta_k^I \right), \\
\widehat{U}_{kC,ID} = 4 \lambda \xi \left( Q_{kC}^{IC} Q_{kD}^{ID} - Q_{kC}^{IC} Q_{kD}^{ID} \delta_k^I \right), \\
\widehat{U}_{kC,ID} = 4 \lambda \xi \left( Q_{kC}^{IC} Q_{kD}^{ID} - Q_{kC}^{IC} Q_{kD}^{ID} \delta_k^I \right).
\]
(9.60)

In the case \( n = 1 \), these tensors reduce to
\[
\mathcal{N}_{k}^{k} = 8 \lambda \xi \left( \tilde{f}_k f_k - \tilde{f} f_k \right), \\
\widehat{U}_{k}^{k} = 4 \lambda \xi \tilde{f}_k f_k, \\
\widehat{U}_{k}^{k} = 4 \lambda \xi \tilde{f}_k f_k.
\]
(9.61)

the \( U^1 \) hyperKahler metric

Adding eqs(9.49,9.50) and eqs(9.59,9.60), we get the total Lagrangian density
\[
L_n = +g_{kC,ID} \partial_{\mu} \tilde{f}_k^C \partial_{\mu} \tilde{f}_l^D + \overline{g}_{kC,ID} \partial_{\mu} f^{ID} \partial_{\mu} f^{kC} \\
+2h_{kC,ID} \partial_{\mu} f^{kC} \partial_{\mu} f^{lD}. 
\]
(9.62)

with \( g_{kC,ID}, \overline{g}_{kC,ID} \) and \( h_{kC,ID} \) as in eqs(6.16,6.19).
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