Dynamical evolution of rotating dense stellar systems with embedded black holes

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ABSTRACT
The evolution of self-gravitating rotating dense stellar systems (e.g. globular clusters, galactic nuclei) with embedded black holes is investigated. The interaction between the black hole and the stellar component in differentially rotating flattened systems is analysed. The interplay between velocity diffusion resulting from relaxation and black hole star accretion is investigated, together with cluster rotation, using 2D+1 (20 in space and time) Fokker–Planck numerical methods. The models can reproduce the Bahcall–Wolf solution $f \propto E^{1/4}$ $(n \propto r^{-7/4})$ inside the zone of influence of the black hole. Gravo-gyro and gravo-thermal instabilities cause the system to have a faster evolution, leading to shorter collapse times with respect to non-rotating systems. Angular momentum transport and star accretion support the development of central rotation on relaxation time-scales. We explore system dissolution as a result of mass loss in the presence of an external tidal field (e.g. for globular clusters in galaxies).

Key words: black hole physics – gravitation – stellar dynamics – methods: numerical – globular clusters: general – galaxies: nuclei.

1 INTRODUCTION
Observational analysis of globular clusters (GCs) has improved considerably in recent years thanks to the Hubble Space Telescope (HST) (e.g. Piotto et al. 2002; Rich et al. 2005; Beccari et al. 2006; Georgiev et al. 2009). Observations have been used to obtain luminosity functions, mass functions and colour–magnitude diagrams (CMDs) and to perform population and kinematical analyses, leading to a better understanding of the evolutionary processes at work in GCs. At the same time, new questions have been raised owing to the newly discovered complexity of these stellar systems, but analyses of rotation have rarely been undertaken. There is, however, observational evidence for the existence of intermediate-mass black holes (IMBHs) in GCs, although their origin is not as yet very clear. Local core-collapsed GCs are expected to harbour IMBHs owing to their high central densities. Regarding their detection, it has been argued that ‘non-collapsed’ projected density profiles that evolve harbouring IMBHs are fitted well by medium-concentration King models (Baumgardt et al. 2005). Gerssen et al. (2002, 2003) reported a kinematical study (based on HST spectra) on the central part of the collapsed GC M15. They proposed the presence of an IMBH ($M_{\text{BH}} = 3.9 \times 10^3 M_\odot$) in the central region of this GC.

A single or binary IMBH could account for the net rotation observed in the centre of M15 (Gebhardt et al. 2000; Gerssen et al. 2002; Miller & Colbert 2004; Kiselev et al. 2008). Maccarone & Servillat (2008) demonstrated the possible presence of an IMBH in NGC 2808 with a mass of $\sim 2.7 \times 10^4 M_\odot$, and Noyola, Gebhardt & Bergmann (2008) reported a BH in $\omega$ Cen with a mass of $\sim 10^4 M_\odot$. However, Baumgardt et al. (2003) have shown, through self-consistent N-body computations treating stellar evolution with a realistic initial mass function (IMF), that the core-collapse profile of a star cluster with an unseen concentration of neutron stars and heavy-mass white dwarfs can explain the observed central rise of the mass-to-light ratio (see also McNamara et al. 2003). Similarly, a dense concentration of compact remnants might be responsible for the high mass-to-light ratio of the central region of NGC 6752 seen in pulsar timings (Ferraro et al. 2003; Colpi et al. 2003). Outside our own Galaxy, Gebhardt et al. (2002), Gebhardt, Rich & Ho (2005), Zaharijâa (2008) have reported evidence for a $20\,000\,M_\odot$ BH in the M31 globular cluster G1.

Chandra and XMM–Newton observations of ultraluminous X-ray (ULXR) sources also provide evidence for the existence of IMBHs in dense star clusters, which are often associated with young star clusters, and whose high X-ray luminosities in many cases suggest a compact object mass of at least $10^4 M_\odot$ (Ebisuzaki et al. 2001; Miller et al. 2003). Furthermore, some of the ULXR sources detected in other galaxies may be accreting IMBHs (e.g. Miller & Colbert 2004), although the majority are probably stellar-mass BHs (King et al. 2001; Rappaport, Podsadlowski & Pfahl 2005).
The centres of most galaxies, however, embed massive BHs. This has been shown by HST measurements in recent years, and by the fact that theoretical modelling of measured motions requires the presence of a central compact dark object with a mass of $\sim 10^8$ to $10^9 M_\odot$ (Ferrarese et al. 2001; Gebhardt 2002; Pinkney et al 2003; Kormendy 2004). Ground-based infrared (IR) observations of the fast orbital motions of a few stars in the Milky Way have led to the detection of a 3–4 $\times 10^6 M_\odot$ BH at its centre (Schödel et al. 2003; Ghez et al. 2004; Eckart, Genzel & Schodel 2004). Moreover, BH demographics have led to correlations between the BH mass and the luminosity of its host bulge or elliptical galaxy (Kormendy & Richstone 1995), and between BH mass and the velocity dispersion of its host bulge, as $M_{\text{BH}} \propto \sigma^6$ (Ferrarese & Merritt 2000). This leads to strong links between BH formation and the properties of the stellar bulge, such as the formation of density cusps (Bahcall–Wolf solution), which have been investigated by Schödel, Merritt & Eckart (2008).

Dynamical modelling of collisional stellar systems (such as galactic nuclei, rich open clusters and rich galaxy clusters) still holds a considerable challenge for both theory and computational requirements (in terms of hardware and software). On the theoretical side, the validity of certain assumptions used in statistical modelling based on the Fokker–Planck (FP) and other approximations has not been fully investigated. Stochastic noise in a discrete $N$-body system and the impossibility of directly modelling realistic particle numbers with the presently available hardware pose a considerable challenge on the computational side (but see Berczik et al. 2006).

Although all work known to the authors to date concentrates on self-gravitating star clusters, the improvement in our knowledge and methods in the field of rotating dense stellar systems is also extremely important for galactic nuclei in cases where there is a central star-accreting BH (some stationary modelling exists, for example Duncan & Shapiro 1983; Quinlan & Shapiro 1990; Murphy, Cohn & Durisen 1991; Freitag & Benz 2002). The direct integration of orbits ($N$-body method) has been applied to the problem (Gültekin, Miller & Hamilton 2004; Baumgardt et al. 2005). However, $N$-body simulations provide only a very limited number of case studies, owing to the enormous computing time needed even on GRAPE computers. Moreover, recent investigations show that in young dense clusters, supermassive stars may form through the runaway merging of main-sequence stars by means of direct physical collisions, which may in turn collapse to form an IMBH. The collision rate will be greatly enhanced if massive stars have time to reach the core before exploding as supernovae (Portegies Zwart et al. 2004; Gürkan, Fregeau & Rasio 2006; Freitag, Gürkan & Rasio 2006). It is thus very important to develop reliable approximate models of rotating star clusters with a BH, and this is the subject of the present work.

A 2D FP model has been worked out for the case of axisymmetric rotating star clusters (Einsel & Spurzem 1999, hereafter Paper I; Kim et al. 2002, hereafter Paper II). Here, the distribution function is assumed to be a function of energy $E$ and the $z$-component of angular momentum ($J_z$) only; a possible dependence of the distribution function on a third integral is neglected. As in the spherically symmetric case, the neglect of an integral of motion is equivalent to the assumption of isotropy, here between the velocity dispersions in the meridional plane ($\sigma_\sigma$ and $\sigma_z$-directions); anisotropy between velocity dispersion in the meridional plane and that in the equatorial plane ($\sigma_\phi$-direction) is, however, included.

We realize that the evolutionary models provided by us for rotating dense stellar systems are difficult to use for direct comparisons with observations, because they are not easily analytically describable. They are, however, the only ones that fully cope with all observational data now available (full 3D velocity data, including velocity dispersions in the $\sigma_\sigma$- and $\sigma_\phi$-directions, rotational velocity, density, all as full 2D functions of $\sigma$ and $z$; see Fiestas et al. 2006). No other evolutionary model currently exists that is able to provide this information. With the advent of our new post-collapse and multi-mass models (Paper II; Kim et al. 2004, hereafter Paper III) and the inclusion of stellar evolution and binaries (work in progress) we will be able to deliver even more interesting results. Already the existing $N$-body study (Ardi, Spurzem & Mineshige 2005) shows that rotation not only accelerates the collisional evolution (but see Ernst et al. 2007) but also leads to increasing binary activity in the system.

A description of the method used in the present work is given in Section 2, and Section 3 describes the initial configurations of the models and numerical tests of the code. Section 4 presents the main results in the isolated case, first reproducing the spherically symmetric model (no rotation), and then giving a description of the rotational behaviour of axisymmetric systems on relaxation time-scales, emphasizing the interplay between the dynamical evolutionary processes. Section 5 explores the system dissolution resulting from the tidal field of a parent galaxy, and Section 6 gives the conclusions and plans for further research.

2 THEORETICAL MODEL

2.1 Equations and assumptions

The pioneering work of Goodman (1983), in his unpublished thesis, and the further development of the FP method given in Papers I, II and III have engendered new interest in the treatment of the case of axisymmetric rotation, which follows the evolution of self-gravitating rotating systems driven by relaxation effects and its consequences for the stellar redistribution and shape of the system.

The evolution of the distribution function $f(r, \phi)$ of stars in phase space $(r, \phi)$ under the influence of the potential $\phi(r)$ is described by the Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \times \nabla f + \frac{F}{m} \times \nabla v = \left( \frac{f}{\partial t} \right)_{\text{coll}},$$

(1)

with space and velocity coordinates $r$ and $v$, respectively. The force $F = -m \nabla \phi$ is applied on stars of mass $m$. The term on the right-hand side of equation (1) takes into account the changes in $f$ resulting from collisions (not real collisions but stellar scatterings, which cause deviations in the orbits). The collision term is given through the (local) FP approximation:

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial u^\mu} \left( f \frac{\Delta u^\mu}{\Delta t} \right) + \frac{1}{2} \frac{\partial^2}{\partial u^\mu \partial u^n} \left( f \frac{\Delta u^\mu \Delta u^n}{\Delta t} \right),$$

(2)

where $\mu = 1, 2, 3$ and $v = 1, 2, 3$ (tensor notation). $u^\mu$ gives the velocity in Cartesian coordinates. The first-order diffusion coefficients $\langle \Delta u^\mu \rangle$ describe the dynamical friction, and the second-order ones $\langle \Delta u^\mu \Delta u^n \rangle$ give the real velocity diffusion.

In order to obtain the solution of the FP equation, the following assumptions are made.

(i) Cluster evolution time-scales obey the following relation:

$$t_{\text{dyn}} \ll t_0 \ll t_3.$$

(3)

1 A proper definition of $f$ corresponding to the initial conditions for this study is given in equation (29).
where \( t_{rh} \) represents the cluster age, \( t_{rh} \) is the half-mass relaxation time, following Spitzer & Hart (1971):

\[
t_{rh} = 0.138 \sqrt{\frac{N r_h}{G M \ln \Lambda}},
\]

where \( N \) is the number of particles (stars), \( G \) is the gravitational constant, \( m \) is the mean stellar mass, \( \ln \Lambda \) is the Coulomb logarithm (\( \Lambda = 0.4N \) is used here) and \( r_h \) is the half-mass radius. The dynamical time is given by

\[
t_{dyn} = 1.58 \left( \frac{r_h}{G M} \right),
\]

where \( M \) is the total mass of the cluster.

The system evolves slowly through diffusion in a sequence of virtual equilibrium states. In a time \( t_{ih} \), information about the initial configuration is lost owing to relaxation. A proof of this statement is given in Section 3.2.

(i) The solution is given for small-angle scatterings (\( \Delta v/v \ll 1 \)), that is, for changes of \( v \) to \( v + \Delta v \).

(ii) There is no correlation between collisions (in contrast to the case for three-body collisions), which could be important for energy generation in the core, which can reverse the collapse.

(iii) Neither binaries nor stellar evolution is considered. Thus, binary heating resulting from three-body encounters is neglected. Note that binary heating can reverse collapse (Hut 1985; McMillan, Hut & Makino 1990; Kim et al. 2002, Paper II).

(vi) The initial BH mass \( (M_{bh}) \) is much smaller than the cluster mass \( (M) \).

(vii) The distribution of stars is represented by an equal-mass particle system, which is initially axisymmetric in space and is able to develop anisotropy in velocity space. No stellar spectrum is included in this model, in order to enable the model to be tested without significant complexity.

The classical isolating integrals of a general axisymmetric potential \( \phi \), in cylindrical coordinates (\( \sigma, z \)), are the energy per unit mass,

\[
E = \frac{1}{2} v^2 + \phi, \quad \phi = \phi_0(\sigma, z) + \phi_{bh}(\sigma, z),
\]

where \( \phi_0(\sigma, z) \) is the potential of the stellar system and \( \phi_{bh}(\sigma, z) = -GM_{bh}/r \) the BH potential (\( r^2 = \sigma^2 + z^2 \)); and the component of angular momentum along the z-axis per unit mass, given by

\[
J_z = \sigma \psi_v \psi_z,
\]

where \( \psi_v \) is the velocity component in the azimuthal direction. \( E \) and \( \phi \) are negative for all particles.

Conservation of \( E \) and \( J_z \) is used in the solution of the FP equation, which becomes a non-linear second-order integro-differential equation (the diffusion coefficients of equation 2 are expressed in terms of integrals over the local field star velocity distribution function). These integrals are given by the Rosenbluth potentials (Rosenbluth, MacDonald & Judd 1957). A derivation of the diffusion coefficients in terms of \( E \) and \( J_z \) can be found in Paper I.

In axisymmetric systems, although \( f \) can be approximately represented as a function of \( E, J_z \) and \( t \) (except for very special forms of the potential), numerical evidence shows that axisymmetric potentials can support orbits that have three integrals of motion: \( E, J_z \), and a third integral commonly designated \( I_3 \). That is, the typical orbit does not spread uniformly over the hypersurface in phase space defined by its \( E \) and \( J_z \) but is confined to a lower-dimensional subset (‘non-ergodic’ orbits) on their \( EJ_z \) surfaces). A solution of the orbit-averaged FP equation in energy–momentum space may represent an artificial case of a true point-mass system, as in the axisymmetric potential a third integral of motion could restrict particle motion in phase space (Goodman 1983). On one hand, the inner parts of the cluster are dominated by relaxation effects and the third integral can be neglected, owing to the efficiency of diffusion in these regions; on the other hand, the outer region of the cluster can be strongly influenced by the third integral, as radially anisotropic anisotropy dominates this region.

In the present study, non-ergodicity of orbits on the hypersurface (given by \( E \) and \( J_z \)) due to a further isolating integral \( I_3 \) is neglected. The potential close to the BH is spherically symmetric (\( \sim 1/r \)), and \( I_3 \) could be reasonably approximated by \( J_z^2 \), as fewer radial orbits are expected in this region (Amaro-Seoane, Freitag & Spurzem 2004; Baumgardt et al. 2004), which are preferentially disrupted by the BH. The angular momentum \( J_z \) is here well represented by its maximum value \( (J_z^{max}) \). However, possible existing meridional circular orbits cannot be distinguished by our model and will be treated as radial orbits (for example for their accretion).

In terms of integrals of motion, the Boltzmann equation (equation (1)) is expressed in the axisymmetric system as

\[
\frac{\partial f}{\partial t} + \frac{\partial \phi}{\partial \sigma} \frac{\partial f}{\partial E} = \left( \frac{\partial f}{\partial t} \right)_{coll}.
\]

The dependence on \( J_z \) is given implicitly by \( \phi \), and the collisional term of equation (2) can be expressed in terms of \( E \) and \( J_z \) as

\[
\left( \frac{\partial f}{\partial t} \right)_{coll} = \frac{1}{V} \left[ -\frac{\partial}{\partial E} \langle \Delta E f \rangle - \frac{\partial}{\partial J_z} \langle \Delta J_z f \rangle \right] + \frac{1}{2} \frac{\partial^2}{\partial E^2} \left( \langle \Delta E^2 f \rangle \right) + \frac{1}{2} \frac{\partial^2}{\partial J_z^2} \left( \langle \Delta J_z^2 f \rangle \right) + \frac{1}{2} \frac{\partial^2}{\partial J_z \partial \sigma} \left( \langle \Delta J_z \Delta \sigma f \rangle \right),
\]

with the volume element in velocity space given by \( V = 2\pi/\sigma \).

The vast dynamical parameter range of relaxed and unrelaxed cluster systems (as presented in Fiestas et al. 2006) was treated, applying appropriate boundary conditions at the inner potential cusp of the BH and the outer cluster tidal boundary (in the presence of a parent galaxy). A double-logarithmic \( (\sigma, z) \) space grid is used, and the FP equation is written in a dimensionless flux form by introducing the dimensionless energy

\[
X(E) \equiv \ln \left( \frac{E}{2\phi - E_0 - E} \right),
\]

where \( E_0 \) is a characteristic energy, which allows a higher resolution at higher energy (and low angular momentum) levels, as well as in the outer parts of the system (halo), where the logarithmic scale increases the spacing of the radii of circular orbits with given energies in the direction of the tidal boundary. The dimensionless angular momentum is given by

\[
Y(J_z, E) \equiv \frac{J_z}{J_z^{max}}.
\]

At each time-step, \( r_{col}(E) \) and \( J_z^{max}(E) \) are determined in the equatorial plane from the evolving potential by a simple
Newton–Raphson scheme, using the relationship

\[ E - \phi(\sigma_{\text{circ}}, z = 0) = \frac{1}{2} \sigma_{\text{circ}} \frac{\partial \phi}{\partial \sigma} \]  

(12)
in order to obtain \( \sigma_{\text{circ}} \) (or \( r_{\text{circ}} \) at \( z = 0 \)), and computing \( J^\text{max}_x(E) \), using

\[ (J^\text{max}_x(E))^2 = \frac{1}{Z^\text{circ}} \frac{\partial \phi}{\partial \sigma} \]  

(13)

### 2.2 Diffusion and loss-cone accretion

Given the values of \( E \) and \( J_x \), the orbit average of the FP equation in the form of equation (8) is obtained by integrating it over an area \( P(E, J_x, t) \) of the hypersurface in phase space, given by

\[ P(E, J_x, t) = 4\pi^2 \int \int_{AE, J_x} d\sigma \, dz. \]  

(14)

This weighting factor also gives the number of stars in the system taking part in the diffusion, as

\[ N(E, J_x, t) = P(E, J_x, t) f(E, J_x, t). \]  

(15)

\( A(E, J_x) \) is given by the intersection of the hypersurface with the \( \sigma z \)-plane, where the sum of the squares of the velocity components is non-negative:

\[ A(E, J_x) = \left\{ \left( \sigma z \right) \left| 2v^2_{\sigma} + v^2_{z} = E - \phi - \frac{J^2_x}{2\sigma^2} \geq 0 \right. \right\}. \]  

(16)

The condition (16) is rastered numerically in the code by given \( E \) and \( J_x \).

In a general axisymmetric potential, almost none of the orbits are closed, so that the orbital period is not well defined. There exist two distinct epicyclic periods, one for each of the oscillations in the \( \sigma \) - and \( z \)-directions. The orbit average is taken over a time that is larger than both and is the required period for the orbit to spread uniformly over the area \( A(E, J_x) \) only because of encounters, that is, on a relaxation time-scale (if the third integral is well conserved).

The FP equation is solved numerically in flux conservation form as

\[ \frac{df}{dt} = \frac{1}{p} \left( -\frac{\partial F_x}{\partial X} - \frac{\partial F_y}{\partial Y} \right), \]  

(17)

where \( p \) is the phase volume per unit \( X \) and \( Y \), with particle flux components in the \( X \) - and \( Y \)-directions of

\[ F_x = -D_{x\sigma} \frac{\partial f}{\partial X} - D_{x\phi} \frac{\partial f}{\partial \phi} - D_x f, \]  

(18)

\[ F_y = -D_{y\sigma} \frac{\partial f}{\partial Y} - D_{y\phi} \frac{\partial f}{\partial \phi} - D_y f. \]  

(19)

The orbit-averaged flux coefficients \( D_x \) are derived from the local diffusion coefficients and transformed to dimensionless variables \( D_{X, \sigma}, D_{X, \phi}, D_{Y, \sigma}, D_{Y, \phi} \) (Paper I).

The loss-cone limit is defined by the minimum angular momentum for an orbit of energy \( E \):

\[ J^\text{min}_x(E) = r_3 \sqrt{2(E - GM_{\text{BH}}/r)}, \]  

(20)

where \( r_3 \) is the disruption radius of the BH, calculated following Frank & Rees (1976):

\[ r_3 \propto r_s(M_{\text{BH}}/m_*)^{1/3}, \]  

(21)

where \( r_s \) and \( m_* \) are the stellar radius and mass, respectively. Their adopted values are given in Section 3.1.

The central potential cusp of an embedded massive BH disturbs the redistribution of stars through collisional interactions. Thus, the following assumptions are made in order to obtain the structural parameters of the cluster.

(i) A seed initial BH mass, which is much larger than a stellar mass, is calculated numerically using a first perturbation of the potential in the initial models (see also Section 3.1).

(ii) Accretion is driven by angular momentum diffusion. A star is completely accreted if its \( z \)-component of angular momentum is less than \( J^\text{min}_x \), which defines the loss-cone boundary. Energy diffusion for accretion is neglected because the changes in \( E \) as a result of collisions are considered small in comparison to the changes in angular momentum (Cohn & Kulsrud 1978).

(iii) The distribution function vanishes for \( J_z > J^\text{max}_z \) and \( J_z < -J^\text{max}_z \).

(iv) The central BH grows slowly, through the accretion of stars, leading to a new distribution \( f(E, J_z) \) and a new \( \phi(\sigma, z) \).

(v) The unit of time is proportional to the relaxation time at the radius of influence of the BH, \( r_s \), defined as the radius at which the mass of the cluster equals \( M_{\text{BH}} \). The time-step is given by \( \Delta t = \xi(t) r_3 \), where

\[ \tau_3 = \frac{0.338 \sigma^3}{n_3 (GM_*)^2 \ln \Lambda} \]  

(22)

(Spitzer & Hart 1971). \( \sigma_x \) and \( n_3 \) are the velocity dispersion and density evaluated at \( r_s \), \( \xi(0) \) depends on the initial model and is increased every time-step by a factor of 4/3 in order to have a fractional increase of central density of between 2 and 4 per cent per time-step. In analogy with the computations of Cohn (1979), one Vlasov step (i.e. one recomputation of the potential) follows every FP (diffusion) step.

A schematic diagram of the numerical \( XY \)-grid, as used for the solution method of the discretized FP equation, is shown in Fig. 1. Energy limits are the central potential \( (X(\phi)) \) and the tidal energy \( (X(E_{\text{tid}})) \); angular momentum limits are the maximum values of \( Y \) in both directions \( (Y = \pm 1) \). For the purpose of this illustration, only the upper half of the grid is shown \( (0 \leq Y < +1) \), as the lower half \( (-1 < Y < 0) \) is symmetric with respect to the axis \( Y = 0 \). Angular momentum diffusion of stars into and out of neighbouring cells is illustrated in the right-hand part of the figure. The distribution

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**Figure 1.** Schematic diagram of the numerical \( XY \)-grid and definition of the loss-cone. Half of the grid is shown on the left-hand side of the figure. The other lower half corresponds to negative values of \( Y \) and is symmetric with respect to the axis \( Y = 0 \). The dark shaded area represents the loss-cone in \( X-Y \)-space, and is limited by \( Y_{\text{min}} \). Stars are able to go into and out of it through angular momentum diffusion, as shown on the right-hand side of the figure for one grid cell.
function is defined in the centre of each cell, whereas the diffusion terms are computed at the cell boundaries. As shown in Fig. 1, the fluxes at the $Y = 1$ boundaries are set to zero. Open boundaries are the loss-cone region, and $E = E_{o1}$ in the tidally limited models. The loss-cone is limited by $Y_{min}(E)$ and the limit of maximum energy ($E(\phi)$). The first derivative of $f$ with respect to $E$ is non-zero at the boundary of each grid cell and is evaluated just inside the boundary in order to obtain accurate escape fluxes. During the solution, the whole grid is rastered and the angular momentum fluxes are saved at each time-step. In the case of isolated models, $X(E_{al})$ is set to 1/10 of the potential at the tidal radius of the corresponding King model. Evaporation of stars in isolated systems is neglected. In tidally limited systems, stars are able to escape from the system through the tidal boundary at $X(E_{al})$, influenced by the potential field of the parent galaxy.

The flux term $F_Y$, per unit energy and unit time across the angular momentum boundary, used to compute the contribution to stellar accretion, is given by the second-order angular momentum diffusion term in equation (19):

$$F_Y = -D_Y \frac{\partial f}{\partial Y}. \tag{23}$$

The dimensionless change in $f(X, Y)$, resulting from diffusion in the inner/outer direction, is then given in a discretized form by

$$\frac{\Delta f}{f} = \frac{\Delta t}{P} \frac{\partial f}{\partial Y} \frac{1}{\partial Y} \tag{24}$$

where $\Delta t$ is the time-step, $P(X, Y)$ is the phase-space volume, and $\Delta F_Y$ is the resultant angular momentum flux in each cell. Energy fluxes are neglected for accretion as they are small in comparison with $F_Y$ (Cohn & Kulsrud 1978).

After the redistribution of orbits, as a result of small-angle collisions, those with $Y \leq Y_{min}$ lie in the loss-cone. Using the timescales of replenishment $t_{in}$ and loss-cone depletions $t_{out}$ (Lightman & Shapiro 1977), the contribution to $f(X, Y)$ of accretion should take into account the ratio

$$q = \frac{t_{out}}{t_{in}} \propto (\frac{Y_{diff}}{Y_{min}})^2. \tag{25}$$

where $Y_{min} = f_{min}/f_{max}$ and we denote the dimensionless angular momentum diffusion term owing to gravitational scattering per time-step, $(Y_{diff})^2$, as $<(\Delta Y)^2>$. If $q < 1$, most loss-cone stars remain inside, and $\Delta f(X, Y)$ is well represented by the flux of equation (24), but, because stars could be scattered out of the loss-cone to orbits of $Y > Y_{min}$ in an orbital time-scale, a correction to the angular momentum flux is necessary. In the classical approximation, the accretion of stars inside $Y_{min}$ leads to an 'empty loss-cone' $(f(X, Y) = 0)$, but this is not a realistic boundary condition. If $q \gg 1$, the angular momentum diffusion term is larger than the loss-cone opening, so that most stars manage to scatter out of it and are not accreted. $q(E_{al}) = 1$ defines a critical energy $(E_{al})$ at a radius $r_{crit}$ in the equatorial plane, which marks the transition between the 'full' and 'empty' loss-cone regimes.

This correction is implemented in the code using a probability of accretion, $P_Y(X, Y, Y_{diff})$, that a star does not escape from the loss-cone as a result of angular momentum change once it is inside, or it can enter the loss-cone from an angular momentum larger than its limit. In the latter case, a probability of accretion $1 - P_Y$ is applied. The probability that an orbit with dimensionless angular momentum $Y$ suffers a change $\Delta Y = |Y - Y_{min}|$, where $|Y - Y_{min}|$ is the distance to the loss-cone boundary, is given by

$$P_Y(Y) = \int_{0}^{Y_{diff}} \frac{2}{\sqrt{\pi}} \exp \left( -\frac{Y^2}{\sigma^2} \right) dY'. \tag{26}$$

That is, centred at each $(X, Y)$ grid cell, a Gaussian distribution of orbits in $Y$ with dispersion $Y_{diff}$ is assumed. Finally, the contribution of $f(X, Y)$ to accretion, at each energy–angular momentum grid cell, is given by

$$\Delta f_a = P_Y(Y)(f_{old} + \Delta f). \tag{27}$$

$f_{old} + \Delta f$ is the result of the redistribution of stars through diffusion processes. In order to obtain the total BH accretion mass, the distribution function $f(E, J)$ is integrated in real and phase space as

$$\Delta m_a = 4\pi \int_{0}^{r_{tid}} \frac{\sigma_1 d\sigma_1}{\sigma} \int_{0}^{\phi_{tid}} \frac{d\phi_1}{\phi_1} \left[ \frac{2\pi}{\sigma} \int_{E_{tid}}^{E_{max}} dE \int_{J_{min}}^{J_{max}} dJ \Delta f_{acc}(E, J) \right], \tag{28}$$

where $\sigma_{tid}$ and $\phi_{tid}$ give the tidal cluster radius in the $\sigma$- and $\phi$-directions respectively. The factor of $2 \times 2\pi = 4\pi$ arises from the consideration of positive and negative zenithal coordinates and because the azimuthal component is symmetric. Moreover, $f_{max} = \sigma \sqrt{2(E - \phi)}$. The accretion mass is added to $M_{bh} = M_{id} + \Delta m_a$, and, furthermore, $M_{id} = M_{tid} - \Delta m_a$.

3 NUMERICAL RESULTS

3.1 Initial conditions

As initial configurations, truncated King models with added bulk motion are used. Their adopted distribution function is

$$f(E, J_z) = \exp \left( -\frac{\Omega_0 J_z}{\sigma_0^2} \right) \exp \left( \frac{E_{al} - E}{\sigma_c^2} \right) - 1, \quad E < E_{al},$$

$$f(E, J_z) = 0, \quad E > E_{al}, \tag{29}$$

where $\sigma_c$ is the central 1D velocity dispersion and $\Omega_0$ is an angular velocity. Fig. 2 shows $f(J_z)$ at constant energy $E$ against $J_z$, for an initially rapidly rotating model ($W_0 = 0.6, w_{0} = 0.9$) (Table 1). $f(J_z)$ covers a wide range of values on a logarithmic scale, and $J_z$ varies from negative to positive values, according to two directions of rotation around the $z$-axis. $J_z = 0$ represents stars on radial orbits, whereas $f_{max}$ represents those on circular orbits. Note that the angular velocity $\Omega_0$ in equation (29) is given by the slope of $f$ in each curve of constant energy (a property of King models). The isoequation sections become shorter, owing to the smaller possible $f_{max}$, at higher absolute values of energy.

The system of units for the initial King models is given by

$$G \equiv M_{bh} = r_{ci} \equiv 1, \tag{30}$$

where $M_{bh}$ is the initial mass of the cluster and $r_{ci}$ is the initial core (King) radius

$$r_{ci} = \sqrt{\frac{9\sigma_c^2}{4\pi G n_c}}, \tag{31}$$

where $n_c$ is the central number density.
by one order of magnitude influence our observables.

The grid dimension is $(200 \times 201)$ in the $(E, J_z)$-space used to construct the models.

Table 1. Parameters of initial models used in the simulations. Column 1, model identification name; column 2, King potential; column 3, dimensionless rotation; column 4, concentration; column 5, $\ln(r_h/r_0)$; column 6, dynamical ellipticity; column 7, initial half-mass relaxation time in code units; column 8, ratio of rotational energy to kinetic energy.

| Model | $W_0$ | $o_0$ | $\ln(r_{\mathrm{tid}}/r_0)$ | $\ln(r_h/r_0)$ | $\varepsilon_{\mathrm{dyn}}$ | $t_{\mathrm{rel}}$ | $\varepsilon_{\mathrm{rot}}/\varepsilon_{\mathrm{kin}}$ |
|-------|-------|-------|---------------------------|----------------|-----------------|----------------|----------------------------------|
| M0    | 3.0   | 0.0   | 5.79                      | 1.50           | -0.001          | 29.40         | 0.00                              |
| M1    | 6.0   | 0.0   | 2.92                      | 0.99           | -0.001          | 91.88         | 0.00                              |
| M2    | 6.0   | 0.3   | 2.71                      | 0.96           | 0.105           | 87.73         | 7.00                              |
| M3    | 6.0   | 0.6   | 2.29                      | 0.87           | 0.278           | 76.32         | 19.81                             |
| M4    | 6.0   | 0.9   | 1.92                      | 0.83           | 0.403           | 71.24         | 30.25                             |
| M5    | 6.0   | 1.2   | 1.57                      | 0.82           | 0.500           | 71.28         | 39.85                             |

The initial conditions of each model are given by the pair $(W_0, o_0)$ – see Table 1. Here, $W_0$ is the familiar King parameter

$$W_0 = \frac{\langle \phi(r_{\mathrm{tid}}) \rangle - \phi_\ast}{\sigma_c^2}$$

(King 1966; Spitzer 1987), where $\phi(r_{\mathrm{tid}})$ is the potential at the cluster tidal boundary and $\phi_\ast$ is the central potential.

$$o_0 = \sqrt{9/(4\pi G M_\odot)} W_0$$

is the initial rotational parameter. Radii are given in units of the initial cluster core radius. Table 1 gives the initial parameters of the models. Intermediate models ($W_0 = 6.0$) are expected to reproduce the current evolutionary states of most GCs. Their initial concentrations decrease and their dynamical ellipticities increase, the higher the initial rotation. $\varepsilon_{\mathrm{dyn}} = 1 - h/a$ is calculated$^3$ following Goodman (1983) as defined in Paper I. The tidally limited models presented in Section 5 are denoted by M1T to M5T.

There are two scaling parameters in the simulation: (i) the particle number $N$, which defines the mass of a single star to the total mass of the system, that is, $m = M_{\mathrm{cl}}/N$ (with $M_{\mathrm{cl}} = 1$ in our units); and (ii) the mass of the BH with respect to the total mass, which we specify in terms of the initial seed BH mass $\beta = M_{\mathrm{bh}}/M_{\mathrm{cl}} = 1 \times 10^{-5}$. Because our aim is to study the dependence of the standard losscone accretion model on the new physical scenario of a rotating axisymmetric system surrounding the BH, we fix $\beta$ and make our models scale-invariant to the particle number as long as the pure point-mass interactions are considered. The choice of $\beta$ is somewhat arbitrary, but does not change the physics much, because FP models (e.g. Amaro-Seoane et al. 2004) show that the time evolution of the BH and of the star cluster do not depend sensitively on the initial seed. In order to prove this statement, we performed tests with different values of $\beta$, as shown in Fig. 3. All tests show a common final $M_{\mathrm{BH}}$, and the disruption rates follow the same self-similar evolution. For comparison, we include physical units on the right $Y$-axis and top X-axis. Here we applied our models to a massive GC, using $M_\ast = 5 \times 10^6 M_\odot$. The initial $t_{\mathrm{rel}}$ was calculated following equation (4), after scaling the half-mass radius to the initial King radius using Table 1 ($r_{\mathrm{tid}} = 2.7$ pc). In this equation, we use $N_\ast = 5 \times 10^6$, and thus $m = 1 M_\odot$. We obtain $t_{\mathrm{rel}} \sim 1.4 \times 10^4$ yr, which is a typical value of massive galactic GCs. In our models, the initial seed represents a small perturbation in the central potential of our initial model, which accretes mass corresponding to the losscone accretion onto a fixed BH, scaled down to our system (see below). This unphysical assumption for the initial accretion is used because our goal is to study the long-term evolution of the system (which approaches a self-similar solution) and not the initial growth process.

For stellar disruption we do have another parameter though, namely the stellar radius in our simulation units. This radius defines the disruption radius through equation (21), which grows in time owing to BH growth. Here we use for most simulations a value $r_\ast = r_h/r_\ast = 2 \times 10^{-8}$. In the following we use fiducial values of $(\alpha, \beta) = (2 \times 10^{-8}, 1 \times 10^{-5})$ for most of our models, which (taking $r_h$ as the solar radius) define the parameters for a GC. We have analysed the variation of $\alpha$, and the results are shown in Fig. 4. Changes of $\alpha$ by one order of magnitude influence our observables $M_{\mathrm{BH}}$ and $\mathrm{d}M_{\mathrm{bh}}/\mathrm{d}t$ by a factor of only 2–3. This suggests that our result can be applied by scaling to a wider range of astrophysical systems, including galactic nuclei. Moreover, for a direct application to GCs our initial model is unphysical, because the seed BH is not fixed, and it may grow or be ejected by close three-body encounters, all effects that we are currently not taking into account. However, no matter what the growth mechanism is, if the BH remains in the

$^3$ The axis ratio of an oblate spheroid.

Figure 2. $f(J_z)$, at constant energy $E$, for an initial model ($W_0 = 0.6, o_0 = 0.9$). From top to bottom the curves have smaller $|E|$, listed in the left column. The grid dimension is $(200 \times 201)$ in the $(E, J_z)$-space used to construct the models.

Figure 3. Evolution of Model M1 (6.0, 0.0) for various values of $\beta$ ($\beta = 2 \times 10^{-8}$). (a) Black hole mass against time. The left $Y$-axis shows code units ($M_{\mathrm{BH}}/M_\odot$) and the right $Y$-axis transforms them to units of $M_\odot$ for a $M_{\mathrm{BH}} = 5 \times 10^6 M_\odot$. (b) Evolution of disruption rates. The left $Y$-axis shows code units (d$M_{\mathrm{BH}}$/d$t$/t$_{\mathrm{rel}}$). The right $Y$-axis transforms them to units of $M_\odot$ yr$^{-1}$.
cluster and grows, we think that our scale-invariant solution should be reached.

3.2 Numerical tests

We use a grid size of \( N_X = 200, N_Y = 201, N_{\omega} = N_z = 200 \), obtaining errors of angular momentum, mass and energy as shown in Table 2 for a typical model (our reference model, \( W_0 = 6 \)) by the time the central density has decreased by about 2–3 orders of magnitude, during core expansion. As can be seen from Table 2, a grid convergence study shows that shorter grids are not sufficient to keep the errors small, and in order to achieve and improve the accuracy reported by FP calculations without deep potentials of 1.7, 0.7 and 0.4 per cent in energy, mass and angular momentum respectively (Einsel & Spurzem 1999, Paper I).

There are two main effects that make our numerical problem much more difficult than earlier ones, leading to larger errors. First, the axisymmetry of the system; and second, the deep growing central potential. It makes necessary a better resolution in real and velocity space, especially at the time the region of influence of the BH dominates the core and during expansion. We increased the resolution in the inner parts of the spatial grid \((N_X, N_Y)\) by setting a linear grid for the core and a logarithmic one for the outer regions. The \((N_X, N_Y)\)-grid has a correspondingly higher resolution for absolute energies larger than \( E_{\text{crit}} \), as mentioned in Section 2.1. Here we set \( E_0 = E_{\text{crit}} \) in equation (10). A further test of the code was made by reproducing the spherically symmetric case with the non-rotating initial parameters (as presented in Section 4.1). Comparison with N-body realizations will be presented in a forthcoming paper. See, however, Kim et al. (2008) for a comparative study of non-BH systems.

Table 2.

| Grid      | \( \Delta E/E \) | \( \Delta M/M \) | \( \Delta J_z/J_z \) |
|-----------|------------------|------------------|----------------------|
| 50 \times 50 | \( 7.85 \times 10^{-2} \) | \( 3.17 \times 10^{-2} \) | \( 3.66 \times 10^{-2} \) |
| 100 \times 100 | \( 1.84 \times 10^{-2} \) | \( 8.43 \times 10^{-3} \) | \( 5.88 \times 10^{-3} \) |
| 200 \times 200 | \( 2.08 \times 10^{-3} \) | \( 2.89 \times 10^{-4} \) | \( 8.66 \times 10^{-4} \) |

As is well known, the long-term evolution of relaxed systems does not depend on the details of the initial conditions, as these are erased on a relaxation time-scale. To verify this we also started some of our runs from an initial King profile, with a concentration parameter \( W_0 = 3.0 \) (Model M0). The evolution of the density at the radius of influence is compared with Model M1 (\( W_0 = 6.0 \)) in Fig. 5. The \( W_0 = 3 \) and the \( W_0 = 6 \) models both experience a self-similar expansion, which approximates \( n_{\rho} \sim r^{-2} \) after collapse is prevented. From now on we define the collapse time as the time of maximum density at the radius of influence in the system (see Tables 3 and 5). In Fig. 5 the evolution of the model \( W_0 = 6.0 \) is normalized to its density maximum and to the collapse time. Model \( W_0 = 3.0 \) is overplotted for purposes of comparison.

Typical runs for the evolution of one model up to \( ~50 \times 10^n \) needed about 40 h on a 3-GHz Pentium IV processor (ARI-ZAH, University of Heidelberg). A speed-up through parallel processing would be recommended for multi-mass versions of the present code (work in progress). This performance is not disappointing, taking into account that the number of floating-point operations performed per time-step in our models is \( N_X \times N_Y \times N_{\omega} \times N_z \) and that the time-steps get much shorter close to \( t_{\text{cc}} \). The results presented here concentrate on our standard model \( W_0 = 6 \).

4 ISOLATED SYSTEMS

4.1 Spherical symmetry

In order to test the method, we reproduce the evolution of isolated dense stellar systems in the spherically symmetric case. We realize...
this model by setting the initial rotating parameter $\omega_0$ to zero. $M_{bh}$ starts growing through the accretion of stars in low-$J_z$ orbits. For $\sim 0.3 t_{rh}$ the evolution is unaffected by the presence of the small central BH and is dominated by the contraction of the core. However, the increasing density supports the growth rate of $M_{bh}$ at later times, when the BH potential ($\sim G M_{bh}/r$) dominates the stellar distribution within $r_a$.

The final steady-state, solid curve in Fig. 6, evolves towards a power law of $\lambda = -1.75$, according to $n \propto r^3$. This solution has been extensively studied in the spherical case by Bahcall & Wolf (1976), Lightman & Shapiro (1977), Marchant & Shapiro (1980) and others. It forms inside $r_a$ and is maintained in the post-collapse phase, when the evolution is driven through energy input from the central object, and $r_a$ extends over larger regions. The density profile flattens close to the centre as a result of the effective loss-cone accretion and it remains practically unchanged in the halo, where the loss-cone loses its significance.

As the system evolves, orbits in the region of influence of the BH become Keplerian-bounded. Their velocity dispersion approximates a power law of $-1/2$ within the BH radius of influence $r_a$. Fig. 7 shows the evolution of the total 1D velocity dispersion profile in the same way as in Fig. 6 for the density. The velocity dispersion grows significantly inside $r_a$ and faster when the cluster is close to collapse, owing to the presence of the deep central potential.

Fig. 8 shows the anisotropy profile in the system at various times. Anisotropy is defined as $A \equiv 2[1 - (\sigma_\phi^2/\sigma_r^2)]$, where the total velocity dispersion $\sigma_r^2 = \sigma_\phi^2 + 2 \sigma_\phi^2$ (as $\sigma_\phi = \sigma_z$). The velocity dispersion $\sigma_\phi$, which is the azimuthal velocity dispersion in the direction of rotation, and $\sigma_\phi$ are calculated initially by taking moments of $f$ with respect to $\theta$ and $\phi$ and ensuring conservation of energy and angular momentum per unit volume by encounters between stars (Goodman 1983). The initial profile shows a maximum positive halo anisotropy (radial orbits dominate the halo in the initial configuration) after a very short time (a fraction of $t_{bh}$). The total amount of radial anisotropy, using the rate $K_r/K_\phi$, where $K_r$ is the kinetic energy in the radial degree of freedom and $K_\phi$ is the kinetic energy in the tangential degree of freedom, gives a maximum excess of 13 per cent in $K_r$ present in the system at the time of the final profile shown in Fig. 8. The small excess of $K_r$ can be understood, because $A(r)$ rises only in the outer regions of the system, where the density is low. These results are of the same order as those presented in previous theoretical studies of anisotropy profiles and the evolution of GCs (Louie & Spurzem 1991; Giersz & Spurzem 1994). Moreover, a small negative anisotropy forms slowly inside the BH radius of influence (tangential orbits dominate the centre close to the BH), whereas radial anisotropy remains in the halo (Quinlan,Hernquist & Sigurdsson; Freitag & Benz 2002; Baumgardt et al. 2004).

The evolution of the Lagrangian radii is a good indicator for the contraction and further re-expansion of mass shells. During expansion, core shells increase as $r \propto t^{2/3}$, as expected for a system in which the central object has a small mass and the energy production is confined to a small central volume (Hénon 1965; Shapiro 1977; McMillan, Lightman & Cohn 1981; Goodman 1984). In Fig. 9 the radius of influence is plotted in addition to the Lagrangian radii. $r_{\phi}$ refers to the evaluation of Lagrangian radii at a zenithal angle, where the effects of probable flattening on the mass columns are expected to be less important, that deviations from spherical symmetry are only up to second order in a Legendre expansion, that
is, \( P_2(\cos \theta) = 0 \). This gives \( \theta = 54, 74 \) (Einsel & Spurzem 1999, Paper I).

Figs 3 and 4 show how \( M_{\text{BH}} \) reaches a nearly constant fraction of \( M_{\text{ini}} \) at collapse time (\( t_{\text{cc}} \)), whereas the star accretion rate \( (dM/dt) \) is maximal at \( t_{\text{cc}} \) owing to the higher density of orbits in the core decreasing afterwards very rapidly (Figs 3b and 4b). For a density power law of \( \lambda = -1.75 \), the expected proportionality \( dM/dt \propto r^\alpha \) turns out to be \( \alpha = -1.2 \) (Amaro-Seoane et al. 2004).

During evolution, the core is heated through the consumption of stars in bound, highly energetic orbits in the cusp. Energy flux is achieved by small-angle, two-body encounters, by which some stars lose energy and move closer to the BH, being eventually consumed, while the stars with which they interact gain energy and move outwards from the cusp into the ambient core. Angular momentum transport is initially enhanced by gravo-gyro instabilities and not affected by the BH accretion of stars on orbits of low \( J_a \). Later, when the core density grows, the mass growth rate increases strongly owing to core contraction to higher densities and stronger stellar interaction.

The general behaviour confirms previous studies of spherically symmetric systems. Marchant & Shapiro (1980) follow the evolution of a star cluster containing a central BH (included in their simulations at collapse time). The BH mass stalls after approximately 2 relaxation time-units to a final mass of \( \sim 4000 M_\odot \). In our models, a similar rapid evolution before expansion is observed, and the final masses are comparable (see Tables 4 and 6). 

### 4.2 Axisymmetric isolated systems

The evolution of the density profile of Model M4 is shown in Fig. 10. The extent of the BH gravitational influence is marked on each curve at the position of \( r_\text{cusp} \) (squares). Evolutionary profiles are represented by dashed curves. Note that the limit between the cusp and the core is located at \( \sim r_\text{cusp} \). The central cusp in the density profile grows first very slowly and then faster towards core collapse. It approaches the \( -7/4 \) cusp, as in the spherically symmetric case.

Fig. 11 shows the evolution of density contours in the meridional plane (\( \sigma, z \)). In the regions where BH star accretion dominates (i.e. inside \( r_\text{cusp} \)), the isodensity contours grow stronger owing to the...
presence of the BH (lighter zones in Fig. 11). The cusp forms a strong gradient towards the centre, and it grows faster close to core collapse. Note that the flattened shape of the system remains at later times, during the expansion phase (Fig. 11d). The velocity dispersion is Keplerian within the BH radius of influence $r_\text{rh}$, as Fig. 12 shows. The extension of $r_\text{rh}$ is comparable to that in non-rotating models, but the cusp formation time is shorter the higher the initial rotation parameter (from comparison with Fig. 7), owing to the faster evolution of these models. Moreover, the ratio $K_1/K_2$ is larger the faster the rotation, with maximum values of 1.18, 1.21, 1.23 and 1.26 for Models M2 to M5, respectively.

The evolution of Lagrangian radii containing the indicated fractions of the initial mass is shown in Fig. 13 in comparison to the non-rotating model. Lagrangian radii also give a qualitative description of the interaction of a growing BH and the cluster mass shells. Initially, the BH mass growth is slow because of the low central density, and Lagrangian radii are dominated by core contraction. Eventually, the collapse is halted and reversed and the mass shells re-expand. This occurs sooner for more rapidly rotating models. Note that the smallest radius contains only 0.01 per cent of the cluster mass, in order to follow the evolution of mass shells closer to the BH. Our single-mass rotating models show some deviations of the self-similar expansion phase, which should be investigated further.

Fig. 14 shows the evolution of density at $r_\text{rh}$ for models with rotation parameters $\omega_0 = 0$ (non-rotating), 0.3, 0.6, 0.9 and 1.2. After a similar initial evolution, collapse is faster the higher the initial rotation. Gravo-thermal and gravo-gyro instabilities drive collapse, and angular momentum is transported out of the core in a progressively more efficient way for the more rapidly rotating models. The two instabilities occur together and support each other. The collapse is reversed as a result of the energy source built by the star-accreting BH, while the central density drops during expansion. Because radial anisotropy dominates first (as shown in Fig. 8), it supports the BH accretion of stars in the core, and these stars are able to interact with the low-$J_z$ (eccentric) orbits in the outer parts.

As seen in Table 3, collapse times for non-BH models are comparable to those in the BH rotating models and are also shorter for higher initial rotation. $t_{\text{cc}}$ varies from 12.20$t_{\text{bh}}$ (non-rotating models) to 4.6$t_{\text{bh}}$ (rapidly rotating models). At $t_{\text{cc}}$, angular momentum diffusion is more effective, owing to the interplay between dynamical instabilities and BH star accretion.

Table 4 shows the final BH mass (in units of $M_\odot$) of each model and the respective maximal accretion rate (in solar masses per year) at the time $t_{\text{bh}}$ of BH stalls and the accretion rate begins to slow down. For a system of $M_\text{cl} = 5 \times 10^6 M_\odot$, $M_{\text{bh}}$ varies between $7.5 \times 10^4 M_\odot$ and $1.5 \times 10^5 M_\odot$, which agrees with the IMBH estimated from theoretical studies and observations of GCs (Gebehart et al. 2000, 2002; Gerssen et al. 2002), as expected according to the initial conditions of our models (Section 3.1). Physical units were derived as described in Section 3.1 using the initial parameters of the corresponding King models (Table 1). The general behaviour exhibits a slightly decreasing $M_{\text{bh}}$, but higher mass growth rates for higher initial rotation. In rotating models, the stalling of $M_{\text{bh}}$ always occurs sooner the higher $\omega_0$ is (Table 3).

It is known that in rotating models without BHs, the total collapse time is shortened by the gravo-gyro effect (Hachisu 1979, 1982), whereby large amounts of initial rotation drive the system into a phase of strong mass loss while it contracts (the core rotates faster, although angular momentum is transported outwards). At the same time, the core is heating, while the source of the so-called ‘gravo-gyro’ catastrophe is consumed and the growth in central rotation levels off after 2–3 $t_{\text{bh}}$ towards core collapse (Paper I).
Simulations of the post-collapse phase, driven by three-body binary heating as in Paper II, exhibit a faster evolution for rotating models. BH rotating models similarly experience the onset of gravo-gyro instabilities, as angular momentum diffuses outwards, leading to an increase in the central rotation (Hachisu 1979, 1982). Moreover, the growth of the BH mass causes the expansion of the system (Figs 9 and 13) and leads to an ordered motion of high-$J_z$ bounded orbits around the central BH (tangential anisotropy), supporting the development of central rotation. However, as the BH reaches its final mass, angular momentum continues to be transported out of the core. Fig. 15 shows snapshots of the evolution of the 2D distribution of $v_{rot}$ in the meridional plane, at representative times; the lighter areas represent contours of more rapid rotation. Note that an important amount of central rotation is still present during the time of expansion (Fig. 15d, cf. Fig. 23).

The ratio of rotational velocity over velocity dispersion represents the importance of ordered motion in comparison to random motion. In Fig. 16 the evolution of the ratio $V_{rot}/\sigma$ at the Lagrangian radii is shown. Up to collapse, there is no considerable influence of the BH, whereas during the expansion $V_{rot}/\sigma$ grows slightly with time, in particular for the inner shells.

The results presented here support the thesis that the formation of a massive central dark object could predict the remaining central rotation in GCs over long evolutionary time-scales. However, the central $V_{rot}/\sigma$ finally decreases owing to the increasing central velocity dispersion, and later falls after $t_{cc}$ together with $V_{rot}$, as angular momentum is carried away from the system. In the outer regions the effect is smaller, with a slower rate of decrease.

### 5 TIDALLY LIMITED MODELS

In these models, mass loss is included, allowing the escape of stars through the energy tidal limit (see Fig. 1). While $M_{\text{th}}$ is growing and the central density is increasing within $r_a$, the system loses mass through the outer tidal boundary as a result of relaxation effects.
Rotating stellar systems with black holes

Figure 18. Evolution of density distribution in the meridional plane for Model M4T in the tidally limited case. Cylindrical coordinates \((\varpi, z)\) are used. Lighter zones represent higher isodensity contours. Note that scales are different in the bottom figures owing to the shrinking of the outer tidal radius.

Figure 19. Equatorial profile \((\zeta = 0)\) of the 1D total velocity dispersion (Model M4T) in the tidally limited case. The dot-dashed line shows the \(-1/2\) slope. Evolutionary profiles are labelled as in Fig. 17. The locations of \(r_a\) are shown as squares.

towards a power law of \(-1/2\) within the BH radius of influence \(r_a\) (Fig. 19).

Fig. 20 shows the anisotropy profile in the system for various times. The initial profile shows radial halo anisotropy (radial orbits dominate the halo in the initial configuration). Tangential anisotropy seems not to form inside the BH radius of influence, as it does in the isolated case. Moreover, at later times, tangential orbits dominate the halo as a consequence of an effective \(J_z\)-transport outwards and the accretion of preferentially radial orbits by the central BH. In general, a faster evolution in more rapidly rotating models leads to a smaller \(M_{\text{stellar}}\) (see Table 6), and thus to a smaller tangential anisotropy (i.e. smaller in Model M4T than in M1T). This is to be expected, as a more massive BH will consume more stars in preferentially radial orbits and the life time of rapidly rotating tidally limited systems is too short to enable the development of higher \(M_{\text{bh}}\). Note that at this evolutionary time \((t \approx 5t_{\text{rhi}}\) for M4T) the cluster has lost more than 50 per cent of its mass (see Table 5) and the densities in the outer parts are much lower than in the isolated models. As a consequence, the measured ratios \(K_r/K_\phi\) range from 0.98 to almost 1.00 for all these models. At the same time, \(r_a\) becomes larger and dominates almost the whole system, which itself is close to dissolution (see Fig. 21).

Fig. 21 shows the evolution of the central density for all models, with rotation parameters \(\omega_0 = 0.0\) (non-rotating), 0.3, 0.6, 0.9 and 1.2 for the tidally limited case. Collapse time is reached more quickly the higher the initial rotation, in comparison to the isolated models. Gravo-thermal and gravo-gyro instabilities support each other and the collapse phase is reversed owing to the energy source built by the star-accreting BH.

The evolution of Lagrangian radii containing the indicated fractions of the initial mass is shown in Fig. 22. Owing to mass loss, the outer mass shells are rapidly truncated, the faster the higher initial rotation. At the same time, the core density grows (higher disruption rates). Later, collapse is halted and reversed (while the accretion rate slows down rapidly) and the mass shells re-expand for a few \(t_{\text{bh}}\).

Collapse times for tidally limited models with same initial conditions can be seen in Table 5, where the collapse parameters of rotating BH and non-BH models are compared. Collapse times are comparable to those for the non-BH models and slightly shorter for...
increasing rotation. The cluster loses, at collapse time, between 40 and 85 per cent of its mass in the non-BH models and between 48 and 87 per cent of the initial cluster mass in the BH models. In both cases the mass loss is higher for more rapid rotation. The time at which the cluster loses half of its mass decreases the higher the initial rotation of the model. The effect is driven by the interplay between relaxation effects (gravo-gyro instabilities), BH star accretion and tidal mass loss. This will also involve a faster dissolution of the cluster in the galactic tidal field.

Table 6 shows the final BH mass for each model and the respective maximal accretion rates. $M_{\text{bh}}^{\text{final}}$ varies between $2.3 \times 10^3 \, M_\odot$ and $7.7 \times 10^3 \, M_\odot$, which are smaller masses than in the isolated case. The general behaviour exhibits a decreasing $M_{\text{bh}}^{\text{final}}$ but a higher mass growth rate corresponding to a more rapid initial rotation. The stalling of $M_{\text{bh}}$ is always faster the higher $\omega_0$ is.

The cluster mass loss resulting from tidal effects of the parent galaxy is very strong during the re-expansion of the core. The acceleration of mass loss is similar to that observed in Paper II in the post-collapse models driven by binary heating, although the effect in the present BH models is more pronounced, with the consequence of a faster evolution of the cluster towards sphericity and final dissolution.

Tidally limited models experience the onset of gravo-gyro instabilities, as angular momentum diffuses outwards, leading to a large mass loss and a limited increase of central rotation. At the galaxy tidal boundary, mainly circular tidal orbits in the halo are lost, whereas highly eccentric (low $J_z$) orbits can interact with stars in the core.

Fig. 23 shows snapshots of the evolution of the 2D distribution of rotational velocity in the meridional plane, at representative times, with the lighter areas representing contours of more rapid rotation. Rotation is lost more rapidly than in the isolated model. Angular momentum transport and the growing BH potential support the development of ordered motion in the core and, at the same time, trigger mass loss through the tidal boundary.

### 6 CONCLUSIONS AND OUTLOOK

The variety of environments in which dense stellar systems (GCs, galactic nuclei) form and evolve make them the target of studies of fundamental dynamical processes. Observational studies of GCs
suggest the existence of IMBHs in their centres (for example in G1 and M15), and it is well known that some of them show flattening as a result of system rotation. The improvements in observational and theoretical studies of dense stellar systems in recent years has led to a better understanding of central BHs and their environments, but at the same time has raised new questions as a result of their newly discovered complexity. This is the reason why theoretical models are important in elucidating the origin of the observed phenomena, in explaining their formation, evolution and interaction, and in predicting possible evolutionary scenarios, which can be confirmed by observations. The theoretically formulated evolutionary models presented here extend the model complexity of spherically symmetric systems through the implementation of differential rotation and BH star accretion.

Our results can be summarized as follows.

(i) Core collapse is an evolutionary property in self-gravitating systems without a BH (Papers I and II). Gravo-gyro effects are coupled to gravo-thermal instability and drive core contraction. We start with a seed central BH, which grows over relaxation time-scales as a result of stellar accretion. Evolution towards core contraction increases the central density and supports star accretion by the central BH. The BH acts as an energy source, through which energetic stellar orbits are formed, which easily cross the loss-cone limit or interact with stars in the core being able to reverse collapse, and triggering core expansion. A rapid expansion has also been observed in N-body realizations (Baumgardt et al. 2004), which use high-concentration initial models (\(W_0 = 10\)) and higher initial BH masses (∼1–5 per cent \(M_\odot\)) in their simulations. They show accretion rates that agree with the classical approximation (Frank & Rees 1976) applied to a Bahcall–Wolf cusp, as we also show in Figs 3 and 4. In the presence of an external potential (tidal limit), a faster evolution accelerates mass loss and leads to cluster dissolution.

(ii) Final steady-state solutions are found in all isolated models, which approach the −1.75 slope in the density cusp and the −0.5 slope in the velocity dispersion cusp inside the BH radius of influence \(r_s\), corresponding to Keplerian-bounded orbits, independent of the initial rotation.

(iii) The final \(M_{\text{BH}}\) nearly stalls at ∼0.001\(M_\odot\), and grows more slowly during the post-collapse phase, whereas the BH mass accretion rate (\(dM_{\text{BH}}/dt\)) decreases strongly after reaching a maximum before core expansion, owing to the higher density of orbits in the core. In the tidally limited model, mass loss is very strong during the re-expansion of the core. The cluster mass reaches, in a fraction of \(t_{\text{co}}\) after collapse, values at least one order of magnitude smaller than its hosted BH. For a cluster of \(5 \times 10^6 M_\odot\), \(M_{\text{BH}}^\text{final}\) varies between \(7.5 \times 10^3 M_\odot\) and \(1.5 \times 10^4 M_\odot\) for isolated models of different initial rotation, and between \(2.3 \times 10^3 M_\odot\) and \(7.6 \times 10^3 M_\odot\) for tidally limited models. As noted in Section 3.1 these values reproduce the expected mass of IMBHs, as we set our initial parameters according to the physical properties of these objects.

(iv) Rapidly rotating, moderate-concentration models (M4, M5) maintain central rotation at collapse and during the expansion phase in an efficient way, in comparison with models without a BH. They are able to maintain an efficient angular momentum diffusion, and at the same time are concentrated enough to avoid an excessive mass loss. Both effects support the accretion of stars in low-\(J_2\) orbits. These models show a stable evolution of ordered versus random motion \((V_{\text{los}}/\sigma)\) up to the expansion phase.

Because flattening supported by rotation is a well-known phenomenon in GCs and there is observational evidence of the existence of central dark objects in some GCs, with and without rotation, there is motivation for the study of this constraint in the long-term evolution. Although some constraints are still missing, such as a mass spectrum or stellar evolution (work in progress), as well as a more realistic criterion for tidal mass loss (as observations suggest, e.g. Mackey & van den Berg 2005), our models are consistent with theoretical studies on the general evolution of systems embedding BHs and in addition consider the importance of initial differential rotation, which needs to be taken into account for the understanding of GC formation and evolution, especially as it can be high in young clusters (e.g. in the Large Magellanic Cloud, Brocato et al. 2004).

Moreover, galaxy cores are known to harbour supermassive BHs and some of them are ‘collisional’, in the sense that their relaxation times are ∼10^10 yr (such as M32 and the Milky Way). They show cuspy density profiles, which shorten the time of relaxation the smaller the distance to the centre of the nucleus, and might support the formation of a steady-state configuration (Bahcall–Wolf solution) at times of the order of 10 Gyr.

Because the models presented here have only one mass component, the effect of mass segregation in a realistic multi-mass system will shorten times of evolution (Gürkan, Freitag & Rasio 2004), leading to a faster commencement of expansion in the presence of a central BH. The central velocity dispersion arises because of the BH-induced Bahcall–Wolf cusp. This increase is not affected by the presence of a spectrum of stellar masses, but the high-mass stars are expected to show a lower cusp in their central dispersion (as reported in Paper III), leading possibly to a higher or at least more stable \(V_{\text{los}}/\sigma\) for these mass classes. Multi-mass models with a BH are currently being developed, and comparison with N-body models are intended to complement these calculations, using the highest particle number currently permitted \((N \sim 10^9)\) (Berczik et al. 2006).

As shown, the amount of rotation present in the system during its dynamical evolution is strongly influenced by the interplay between angular momentum diffusion (gravo-gyro instability) and the redistribution of high-energy orbits close to the BH (loss-cone refilling). Because a central BH is able to ‘consume’ angular momentum from the system, in the form of stars, it might itself become an angular momentum source, which might be able to rotate (Kerr Black Hole), permitting also a more efficient angular momentum transport outwards, through interaction with core stars, driven by relaxation. A binary BH could in a similar way lead to a more efficient development of rotation in its zone of influence, modifying substantially the final shape of the cluster (Mapelli et al. 2005; Berczik et al. 2006).

The models presented are able to reproduce 2D distributions (in the meridional plane) of density, cluster potential, velocity dispersions, rotational velocity, anisotropy, dynamical ellipticity, among other parameters, at any time of evolution and deep in the stellar cusp surrounding the central BH. They make possible the study of kinematical and structural parameters in time, which can complement and test observational measurements, thus contributing to the understanding of the common evolution of star clusters and galaxies.

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