On semi-vector spaces and semi-algebras

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Abstract

It is well-known that the theories of semi-vector spaces and semi-algebras – which were not much studied over time – are utilized/applied in Fuzzy Set Theory in order to obtain extensions of the concept of fuzzy numbers as well as to provide new mathematical tools to investigate properties and new results on fuzzy systems. In this paper we investigate the theory of semi-vector spaces over the semi-field of nonnegative real numbers \( \mathbb{R}_0^+ \). We prove several results concerning semi-vector spaces and semi-linear transformations. Moreover, we introduce in the literature the concept of eigenvalues and eigenvectors of a semi-linear operator, describing in some cases how to compute them. Topological properties of semi-vector spaces such as completeness and separability are also investigated. New families of semi-vector spaces derived from semi-metric, semi-norm, semi-inner product, among others are exhibited. Additionally, some results on semi-algebras are presented.

keywords: semi-vector space; semi-algebras; semi-linear operators

1 Introduction

The concept of semi-vector space was introduced by Prakash and Sertel in [11]. Roughly speaking, semi-vector spaces are “vector spaces” where the scalars are in a semi-field. Although the concept of semi-vector space was investigated over time, there exist few works available in the literature dealing with such spaces [13, 11, 12, 10, 4, 5, 8]. This fact occurs maybe due to the limitations that such concept brings, i.e., the non-existence of (additive) symmetric for some (for all) semi-vector. A textbook in such a topic of research is the book by Kandasamy [6].

Although the seminal paper on semi-vector spaces is [11], the idea of such a concept was implicit in [13], where Radstrom shown that a semi-vector space over the semi-field of nonnegative real numbers can be extended to a real vector space (see [13 Theorem 1-B.]). In [11], Prakash and Sertel investigated the structure of topological semi-vector spaces. The authors were concerned with the study of the existence of fixed points in compact convex sets and also to generate min-max theorems in topological semi-vector...
spaces. In [12], Prakash and Sertel investigated properties of the topological semi-vector space consisting of nonempty compact subsets of a real Hausdorff topological vector space. In [10], Pap investigated and formulated the concept of integrals of functions having, as counter-domain, complete semi-vector spaces. W. Gahler and S. Gahler [4] showed that a (ordered) semi-vector space can be extended to a (ordered) vector space and a (ordered) semi-algebra can be extended to a (ordered) algebra. Moreover, they provided an extension of fuzzy numbers. Janyska et al. [5] developed such theory (of semi-vector space) by proving useful results and defining the semi-tensor product of (semi-free) semi-vector spaces. They were also interested to propose an algebraic model of physical scales. Canarutto [1] explored the concept of semi-vector spaces to express aspects and to exploit nonstandard mathematical notions of basics of quantum particle physics on a curved Lorentzian background. Moreover, he dealt with the case of electroweak interactions. Additionally, in [2], Canarutto provided a suitable formulation of the fundamental mathematical concepts with respect to quantum field theory. Such a paper presents a natural application of the concept of semi-vector spaces and semi-algebras. Recently, Bedregal et al. [8] investigated (ordered) semi-vector spaces over a weak semi-field $K$ (i.e., both $(K,+)$ and $(K, \cdot)$ are monoids) in the context of fuzzy sets and applying the results in multi-criteria group decision-making.

In this paper we extend the theory of semi-vector spaces. The semi-field of scalars considered here is the semi-field of nonnegative real numbers. We prove several results in the context of semi-vector spaces and semi-linear transformations. We introduce the concept of semi-eigenvalues and semi-eigenvectors of an operator and of a matrix, showing how to compute it in specific cases. We investigate topological properties such as completeness, compactness and separability of semi-vector spaces. Additionally, we present interesting new families of semi-vector spaces derived from semi-metric, semi-norm, semi-inner product, metric-preserving functions among others. Furthermore, we show some results concerning semi-algebras. Summarizing, we provide new results on semi-vector spaces and semi-algebras, although such theories are very difficult to be investigated due to the fact that vectors do not even have (additive) symmetrical. These new results can be possibly utilized in the theory of fuzzy sets in order to extend it or in the generation of new results concerning such a theory.

The paper is organized as follows. In Section 2 we recall some concepts on semi-vector spaces which will be utilized in this work. In Section 3 we present and prove several results concerning semi-vector spaces and semi-linear transformations. We introduce naturally the concepts of eigenvalue and eigenvector of a semi-linear operator and of matrices. Additionally, we exhibit and show interesting examples of semi-vector spaces derived from semi-metric, semi-norms, metric-preserving functions among others. Results concerning semi-algebras are also presented. In Section 4 we show relationships between Fuzzy Set Theory and the theory of semi-vector spaces and semi-algebras. Finally, a summary of this paper is presented in Section 5.

2 Preliminaries

In this section we recall important facts on semi-vector spaces necessary for the development of this work. In order to define formally such concept, it is necessary to define the concepts of semi-ring and semi-field.

**Definition 2.1** A semi-ring $(S, +, \cdot)$ is a set $S$ endowed with two binary operations, $+: S \times S \rightarrow S$
(addition), \( \bullet : S \times S \rightarrow S \) (multiplication) such that: (1) \((S,+)\) is a commutative monoid; (2) \((S,\bullet)\) is a semigroup; (3) the multiplication \(\bullet\) is distributive with respect to +: \( \forall \ x,y,z \in S, \ (x+y)\bullet z = x\bullet z + y\bullet z \) and \( x\bullet(y+z) = x\bullet y + x\bullet z \).

We write \( S \) instead of writing \((S,+,\bullet)\) if there is not possibility of confusion. If the multiplication \(\bullet\) is commutative then \( S \) is a commutative semi-ring. If there exists \( 1 \in S \) such that, \( \forall \ x \in S \) one has \( 1 \bullet x = x = x \bullet 1 = x \), then \( S \) is a semi-ring with identity.

**Definition 2.2** [6, Definition 3.1.1] A semi-field is an ordered triple \((K,+,\bullet)\) which is a commutative semi-ring with unit satisfying the following conditions: (1) \( \forall \ x,y \in K, \ (x+y) = 0 \) then \( x = y = 0 \); (2) if \( x,y \in K \) and \( x \bullet y = 0 \) then \( x = 0 \) or \( y = 0 \).

Before proceeding further, it is interesting to observe that in [4] the authors considered the additive cancellation law in the definition of semi-vector space. In [5], the authors did not assume the existence of the zero (null) vector.

In this paper we consider the definition of a semi-vector space in the context of that shown in [4], Sect.3.1.

**Definition 2.3** A semi-vector space over a semi-field \( K \) is a ordered triple \((V,+,\cdot)\), where \( V \) is a set endowed with the operations \( + : V \times V \rightarrow V \) (vector addition) and \( \cdot : K \times V \rightarrow V \) (scalar multiplication) such that:

1. \((V,+)\) is an abelian monoid equipped with the additive cancellation law: \( \forall \ u,v,w \in V, \ if \ u+v = u+w \ then \ v = w; \)
2. \( \forall \ \alpha \in K \ and \ \forall \ u,v \in V, \ \alpha(u+v) = \alpha u + \beta v; \)
3. \( \forall \ \alpha,\beta \in K \ and \ \forall \ v \in V, \ (\alpha + \beta)v = \alpha v + \beta v; \)
4. \( \forall \ \alpha,\beta \in K \ and \ \forall \ v \in V, \ (\alpha \beta)v = \alpha(\beta v); \)
5. \( \forall \ v \in V \ and \ 1 \in K, \ 1v = v. \)

Note that from Item (1) of Definition 2.3, all semi-vector spaces considered in this paper are regular, that it, the additive cancellation law is satisfied. The zero (or null) vector of \( V \), which is unique, will be denoted by \( 0_v \). Let \( v \in V, \ v \neq 0 \). If there exists \( u \in V \) such that \( v + u = 0 \) then \( v \) is said to be symmetrizable. A semi-vector space \( V \) is said to be simple if the unique symmetrizable element is the zero vector \( 0_v \). In other words, \( V \) is simple if it has none nonzero symmetrizable elements.

**Definition 2.4** [5, Definition 1.4] Let \( V \) be a simple semi-vector space over \( \mathbb{R}^+_0 \). A subset \( B \subset V \) is called a semi-basis of \( V \) if every \( v \in V, \ v \neq 0 \), can be written in a unique way as \( v = \sum_{i \in I_v} v^{(i)} b_i, \) where \( v^{(i)} \in \mathbb{R}^+_0 \), \( b_i \in B \) and \( I_v \) is a finite family of indices uniquely determined by \( v \). The finite subset \( B_v \subset B \) defined by \( B_v := \{ b_i \}_{i \in I_v} \) is uniquely determined by \( v \). If a semi-vector space \( V \) admits a semi-basis then it is said to be semi-free.
The concept of semi-dimension can be defined in analogous way to semi-free semi-vector spaces due to the next result.

**Corollary 2.1** [5 Corollary 1.7] Let V be a semi-free semi-vector space. Then all semi-bases of V have the same cardinality.

Therefore, the semi-dimension of a semi-free semi-vector space is the cardinality of a semi-basis (consequently, of all semi-bases) of V. We next present some examples of semi-vector spaces.

**Example 2.1** All real vector spaces are semi-vector spaces, but they are not simple.

**Example 2.2** The set $[\mathbb{R}_0^+]^n = \mathbb{R}_0^+ \times \cdots \times \mathbb{R}_0^+$ endowed with the usual sum of coordinates and scalar multiplication is a semi-vector space over $\mathbb{R}_0^+$.

**Example 2.3** The set $M_{n \times m}(\mathbb{R}_0^+)$ of matrices $n \times m$ whose entries are nonnegative real numbers equipped with the sum of matrices and multiplication of a matrix by a scalar (in $\mathbb{R}_0^+$, of course) is a semi-vector space over $\mathbb{R}_0^+$.

**Example 2.4** The set $P_n[x]$ of polynomials with coefficients from $\mathbb{R}_0^+$ and degree less than or equal to n, equipped with the usual of polynomial sum and scalar multiplication, is a semi-vector space.

**Definition 2.5** Let $(V,+,\cdot)$ be a semi-vector space over $\mathbb{R}_0^+$. We say that a non-empty subset $W$ of $V$ is a semi-subspace of $V$ if $W$ is closed under both addition and scalar multiplication of $V$, that is,

1. $\forall w_1,w_2 \in W \implies w_1 + w_2 \in W$;
2. $\forall \lambda \in \mathbb{R}_0^+ \text{ and } \forall w \in W \implies \lambda w \in W$.

The uniqueness of the zero vector implies that for each $\lambda \in \mathbb{R}_0^+$ on has $\lambda 0_V = 0_V$. Moreover, if $v \in V$, it follows that $0v = 0v + 0v$; applying the regularity one obtains $0v = 0_V$. Therefore, from Item (2), every semi-subspace contains the zero vector.

**Example 2.5** Let $\mathbb{Q}_0^+$ denote the set of nonnegative rational numbers. The semi-vector space $\mathbb{Q}_0^+$ considered as an $\mathbb{Q}_0^+$ space is a semi-subspace of $\mathbb{R}_0^+$ considered as an $\mathbb{Q}_0^+$ space.

**Example 2.6** For each positive integer $i \leq n$, the subset $\mathcal{P}_{(i)}[x] \cup \{0_p\}$, where $\mathcal{P}_{(i)}[x] = \{p(x); \partial(p(x)) = i\}$ and $0_p$ is the null polynomial, is a semi-subspace of $\mathcal{P}_n[x]$, shown in Example 2.4.

**Example 2.7** The set of diagonal matrices of order $n$ with entries in $\mathbb{R}_0^+$ is a semi-subspace of $\mathcal{M}_n(\mathbb{R}_0^+)$, where the latter is the semi-vector space of square matrices with entries in $\mathbb{R}_0^+$ (according to Example 2.3).

**Definition 2.6** [5 Definition 1.22] Let $V$ and $W$ be two semi-vector spaces and $T : V \rightarrow W$ be a map. We say that $T$ is a semi-linear transformation if: (1) $\forall v_1,v_2 \in V$, $T(v_1 + v_2) = T(v_1) + T(v_2)$; (2) $\forall \lambda \in \mathbb{R}_0^+$ and $\forall v \in V$, $T(\lambda v) = \lambda T(v)$.

If $U$ and $V$ are semi-vector spaces then the set $\text{Hom}(U,V) = \{T : U \rightarrow V; T \text{ is semi-linear}\}$ is also a semi-vector space.
3 The New Results

We start this section with important remarks.

Remark 3.1  (1) Throughout this section we always consider that the semi-field $K$ is the set of nonnegative real numbers, i.e., $K = R^+_0 = R^+ \cup \{0\}$.

(2) In the whole section (except Subsection 3.2) we assume that the semi-vector spaces $V$ are simple, i.e., the unique symmetrizable element is the zero vector $0_V$.

(3) It is well-known that a semi-vector space $(V,+,\cdot)$ can be always extended to a vector space according to the equivalence relation on $V \times V$ defined by $(u_1,v_1) \sim (u_2,v_2)$ if and only if $u_1 + v_2 = v_1 + u_2$ (see [13]; see also [4, Section 3.4]). However, our results are obtained without utilizing such a natural embedding. In other words, if one want to compute, for instance, the eigenvalues of a matrix defined over $R^+_0$ we cannot solve the problem in the associated vector spaces and then discard the negative ones. Put differently, all computations performed here are restricted to nonnegative real numbers and also to the fact that none vector (with exception of $0_V$) has (additive) symmetrical. However, we will show that, even in this case, several results can be obtained.

Proposition 3.1 Let $V$ be a semi-vector space over $R^+_0$. Then the following hold:

1. let $v \in V$, $v \neq 0_V$, and $\lambda \in R^+_0$; if $\lambda v = 0_V$ then $\lambda = 0$;
2. if $\alpha,\beta \in R^+_0$, $v \in V$ and $v \neq 0_V$, then the equality $\alpha v = \beta v$ implies that $\alpha = \beta$.

Proof: (1) If $\lambda \neq 0$ then there exists its multiplicative inverse $\lambda^{-1}$, hence $1v = \lambda^{-1}0_V = 0_V$, i.e., $v = 0_V$, a contradiction.

(2) If $\alpha \neq \beta$, assume w.l.o.g. that $\alpha > \beta$, i.e., there exists a positive real number $c$ such that $\alpha = \beta + c$. Thus, $\alpha v = \beta v$ implies $\beta v + cv = \beta v$. From the cancellation law we have $cv = 0_V$, and from Item (1) it follows that $c = 0$, a contradiction. \qed

We next introduce in the literature the concept of eigenvalue and eigenvector of a semi-linear operator.

Definition 3.1 Let $V$ be a semi-vector space and $T : V \rightarrow V$ be a semi-linear operator. If there exist a non-zero vector $v \in V$ and a nonnegative real number $\lambda$ such that $T(v) = \lambda v$, then $\lambda$ is an eigenvalue of $T$ and $v$ is an eigenvector of $T$ associated with $\lambda$.

As it is natural, the zero vector joined to the set of the eigenvectors associated with a given eigenvalue has a semi-subspace structure.

Proposition 3.2 Let $V$ be a semi-vector space over $R^+_0$ and $T : V \rightarrow V$ be a semi-linear operator. Then the set $V_\lambda = \{v \in V; T(v) = \lambda v\} \cup \{0_V\}$ is a semi-subspace of $V$. 
Proof: From hypotheses, \( V_\lambda \) is non-empty. Let \( u, v \in V_\lambda \), i.e., \( T(u) = \lambda u \) and \( T(v) = \lambda v \). Hence, 
\[
T(u + v) = T(u) + T(v) = \lambda(u + v),
\]
i.e., \( u + v \in V_\lambda \). Further, if \( \alpha \in \mathbb{R}_0^+ \) and \( u \in V \), it follows that 
\[
T(\alpha u) = \alpha T(u) = \lambda(\alpha u),
\]
that is, \( \alpha u \in V_\lambda \). Therefore, \( V_\lambda \) is a semi-subspace of \( V \). \[\square\]

The next natural step would be to introduce the characteristic polynomial of a matrix, according to the standard Linear Algebra. However, how to compute \( \det(A - \lambda I) \) if \( -\lambda \) can be a negative real number? Based on this fact we must be careful to compute the eigenvectors of a matrix. In fact, the main tools to be utilized in computing eigenvalues/eigenvectors of a square matrix whose entries are nonnegative real numbers is the additive cancellation law in \( \mathbb{R}_0^+ \) and also the fact that positive real numbers have multiplicative inverse. However, in much cases, such a tools are not sufficient to solve the problem. Let us see some cases when it is possible to compute eigenvalues/eigenvectors of a matrix.

Example 3.1 Let us see how to obtain (if there exists) an eigenvalue/eigenvector of a diagonal matrix \( A \in \mathcal{M}_2(\mathbb{R}_0^+) \),
\[
A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},
\]
where \( a \neq b \) not both zeros.

Let us assume first that \( a, b > 0 \). Solving the equation \( Av = \lambda v \), that is,
\[
\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix},
\]
we obtain \( \lambda = a \) with associated eigenvector \( x(1,0) \) and \( \lambda = b \) with associated eigenvector \( y(0,1) \).

If \( a \neq 0 \) and \( b = 0 \), then \( \lambda = a \) with eigenvectors \( x(1,0) \).

If \( a = 0 \) and \( b \neq 0 \), then \( \lambda = b \) with eigenvectors \( y(0,1) \).

Example 3.2 Let \( A \in \mathcal{M}_2(\mathbb{R}_0^+) \) be a matrix of the form
\[
A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},
\]
where \( a \neq b \) are positive real numbers. Let us solve the matrix equation:
\[
\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.
\]

If \( y \neq 0 \), \( \lambda = a \); hence \( by = 0 \), which implies \( b = 0 \), a contradiction. If \( y = 0 \), \( x \neq 0 \); hence \( \lambda = a \) with eigenvectors \( (x,0) \).

If \( V \) and \( W \) are semi-free semi-vector spaces then it is possible to define the matrix of a semi-linear transformation \( T : V \rightarrow W \) as in the usual case (vector spaces).
Definition 3.2 Let $T : V \to W$ be a semi-linear transformation between semi-free semi-vector spaces with semi-basis $B_1$ and $B_2$, respectively. Then the matrix $[T]_{B_2}^{B_1}$ is the matrix of the transformation $T$.

Theorem 3.1 Let $V$ be a semi-free semi-vector space over $\mathbb{R}^{\times}$ and let $T : V \to V$ be a semi-linear operator. Then $T$ admits a semi-basis $B = \{v_1, v_2, \ldots, v_n\}$ such that $[T]_B$ is diagonal if and only if $B$ consists of eigenvectors of $T$.

Proof: The proof is analogous to the case of vector spaces. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a semi-basis of $V$ whose elements are eigenvectors of $T$. We then have:

\[
T(v_1) = \lambda_1 v_1 + 0 v_2 + \ldots + 0 v_n,
\]

\[
T(v_2) = 0 v_1 + \lambda_2 v_2 + \ldots + 0 v_n,
\]

\[
\vdots
\]

\[
T(v_n) = 0 v_1 + 0 v_2 + \ldots + \lambda_n v_n,
\]

which implies that $[T]_B$ is of the form

\[
[T]_B = \begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_n
\end{bmatrix}.
\]

On the other hand, let $B^* = \{w_1, w_2, \ldots, w_n\}$ be a semi-basis of $V$ such that $[T]_{B^*}$ is diagonal:

\[
[T]_{B^*} = \begin{bmatrix}
\alpha_1 & 0 & 0 & \ldots & 0 \\
0 & \alpha_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_n
\end{bmatrix};
\]

thus,

\[
T(w_1) = \alpha_1 w_1 + 0 w_2 + \ldots + 0 w_n = \alpha_1 w_1,
\]

\[
T(w_2) = 0 w_1 + \alpha_2 w_2 + \ldots + 0 w_n = \alpha_2 w_2,
\]

\[
\vdots
\]

\[
T(w_n) = 0 w_1 + 0 w_2 + \ldots + \alpha_n w_n = \alpha_2 w_n.
\]

This means that $w_i$ are eigenvectors of $T$ with corresponding eigenvalues $\alpha_i$, for all $i = 1, 2, \ldots, n$. □

Definition 3.3 Let $T : V \to W$ be a semi-linear transformation. The set $\text{Ker}(T) = \{v \in V ; T(v) = 0\}$ is called kernel of $T$. 

7
Proposition 3.3 Let $T : V \rightarrow W$ be a semi-linear transformation. Then the following hold:

1. $\ker(T)$ is a semi-subspace of $V$;
2. if $T$ is injective then $\ker(T) = \{0_V\}$;
3. if $V$ has semi-dimension 1 then $\ker(T) = \{0_V\}$ implies that $T$ is injective.

Proof: (1) We have $T(0_V) = T(0_V) + T(0_V)$. Since $W$ is regular, it follows that $T(0_V) = 0_W$, which implies $\ker(T) \neq \emptyset$. If $u, v \in \ker(T)$ and $\lambda \in \mathbb{R}_0^+$, then $u + v \in \ker(T)$ and $\lambda v \in \ker(T)$, which implies that $\ker(T)$ is a semi-subspace of $V$.

(2) Since $T(0_V) = 0_W$, it follows that $\{0_V\} \subseteq \ker(T)$. On the other hand, let $u \in \ker(T)$, that is, $T(u) = 0_W$. Since $T$ is injective, one has $u = 0_V$. Hence, $\ker(T) = \{0_V\}$.

(3) Let $B = \{v_0\}$ be a semi-basis of $V$. Assume that $T(u) = T(v)$, where $u, v \in V$ are such that $u = \alpha v_0$ and $v = \beta v_0$. Hence, $\alpha T(v_0) = \beta T(v_0)$. Since $\ker(T) = \{0_V\}$ and $v_0 \neq 0$, it follows that $T(v_0) \neq 0$. From Item (2) of Proposition 3.1, one has $\alpha = \beta$, i.e., $u = v$.

Definition 3.4 Let $T : V \rightarrow W$ be a semi-linear transformation. The image of $T$ is the set of all vectors $w \in W$ such that there exists $v \in V$ with $T(v) = w$, that is, $\text{Im}(T) = \{w \in W; \exists v \in V \text{ with } T(v) = w\}$.

Proposition 3.4 Let $T : V \rightarrow W$ be a semi-linear transformation. Then the image of $T$ is a semi-subspace of $W$.

Proof: The set $\text{Im}(T)$ is non-empty because $T(0_V) = 0_W$. It is easy to see that if $w_1, w_2 \in \text{Im}(T)$ and $\lambda \in \mathbb{R}_0^+$, then $w_1 + w_2 \in \text{Im}(T)$ and $\lambda w_1 \in \text{Im}(T)$.

Theorem 3.2 Let $V$ be a $n$-dimensional semi-free semi-vector space over $\mathbb{R}_0^+$. Then $V$ is isomorphic to $(\mathbb{R}_0^+)^n$.

Proof: Let $B = \{v_1, v_2, \ldots, v_n\}$ be a semi-basis of $V$ and consider the canonical semi-basis $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ of $(\mathbb{R}_0^+)^n$, where $i = 1, 2, \ldots, n$. Define the map $T : V \rightarrow (\mathbb{R}_0^+)^n$ as follows: for each $v = \sum_{i=1}^n a_i v_i \in V$, put $T(v) = \sum_{i=1}^n a_i e_i$. It is easy to see that $T$ is bijective semi-linear transformation, i.e., $V$ is isomorphic to $(\mathbb{R}_0^+)^n$, as required.

3.1 Complete Semi-Vector Spaces

We here define and study complete semi-vector spaces, i.e., semi-vector spaces whose norm (inner product) induces a metric under which the space is complete.
Definition 3.5 Let $V$ be a semi-vector space over $\mathbb{R}_0^+$. If there exists a norm $\| \cdot \| : V \to \mathbb{R}_0^+$ on $V$ we say that $V$ is a normed semi-vector space (or normed semi-space, for short). If the norm defines a metric on $V$ under which $V$ is complete then $V$ is said to be Banach semi-vector space.

Definition 3.6 Let $V$ be a semi-vector space over $\mathbb{R}_0^+$. If there exists an inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}_0^+$ on $V$ then $V$ is an inner product semi-vector space (or inner product semi-space). If the inner product defines a metric on $V$ under which $V$ is complete then $V$ is said to be Hilbert semi-vector space.

The well-known norms on $\mathbb{R}^n$ are also norms on $[\mathbb{R}_0^+]^n$, as we show in the next propositions.

Proposition 3.5 Let $V = [\mathbb{R}_0^+]^n$ be the Euclidean semi-vector space (over $\mathbb{R}_0^+$) of semi-dimension $n$. Define the function $\| \cdot \| : V \to \mathbb{R}_0^+$ as follows: if $x = (x_1, x_2, \ldots, x_n) \in V$, put $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. Then $\| \cdot \|$ is a norm on $V$, called the Euclidean norm on $V$.

Proof: It is clear that $\|x\| = 0$ if and only if $x = 0$ and for all $\alpha \in \mathbb{R}_0^+$ and $x \in V$, $\|\alpha x\| = |\alpha|\|x\|$. To show the triangle inequality it is sufficient to apply the Cauchy-Schwarz inequality in $\mathbb{R}_0^+$: if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are semi-vectors in $V$ then $\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \cdot \left( \sum_{i=1}^n y_i^2 \right)^{1/2}$. □

In the next result we show that the Euclidean norm on $[\mathbb{R}_0^+]^n$ generates the Euclidean metric on it.

Proposition 3.6 Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be semi-vectors in $V = [\mathbb{R}_0^+]^n$. Define the function $d : V \times V \to \mathbb{R}_0^+$ as follows: for every fixed $i$, if $x_i = y_i$, put $c_i = 0$; if $x_i \neq y_i$, put $\varphi_i = \psi_i + c_i$, where $\varphi_i = \max\{x_i, y_i\}$ and $\psi_i = \min\{x_i, y_i\}$ (in this case, $c_i > 0$); then consider $d(x, y) = \sqrt{c_1^2 + \ldots + c_n^2}$. The function $d$ is a metric on $V$.

Remark 3.2 Note that in Proposition 3.4 we could have defined $c_i$ simply by the nonnegative real number satisfying $\max\{x_i, y_i\} = \min\{x_i, y_i\} + c_i$. However, we prefer to separate the cases when $c_i = 0$ and $c_i > 0$ in order to improve the readability of this paper.

Proof: It is easy to see that $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$.

We will next prove the triangle inequality. To do this, let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be semi-vectors in $V = [\mathbb{R}_0^+]^n$. We look first at a fixed $i$. If $x_i = y_i = z_i$ or if two of them are equal then $d(x, z_i) \leq d(x, y_i) + d(y, z_i)$. Let us then assume that $x_i$, $y_i$ and $z_i$ are pairwise distinct. We have to analyze the six cases: (1) $x_i < y_i < z_i$; (2) $x_i < z_i < y_i$; (3) $y_i < x_i < z_i$; (4) $y_i < z_i < x_i$; (5) $z_i < x_i < y_i$; (6) $z_i < y_i < x_i$. In order to verify the triangle inequality we will see what occurs in the worst cases. More precisely, we assume that for all $i = 1, 2, \ldots, n$ we have $x_i < y_i < z_i$ or, equivalently, $z_i < y_i < x_i$. Since both cases are analogous we only verify the (first) case $x_i < y_i < z_i$, for all $i$. In such cases there exist positive real numbers $a_i$, $b_i$, for all $i = 1, 2, \ldots, n$, such that $y_i = x_i + a_i$ and $z_i = y_i + b_i$, which implies $z_i = x_i + a_i + b_i$. We need to show that $d(x, z) \leq d(x, y) + d(y, z)$, i.e., $\left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2}$. The last inequality is equivalent to the inequality
\[
\sum_{i=1}^{n} (a_i + b_i)^2 \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}.
\]
Again, the last inequality is equivalent to
\[
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2},
\]
which is the Cauchy-Schwarz inequality in \( \mathbb{R}_0^+ \). Therefore, \( d \) satisfies the triangle inequality, hence it is a metric on \( V \). \( \square \)

Remark 3.3 Note that Proposition 3.6 means that the Euclidean norm on \( [\mathbb{R}_0^+]^n \) (see Proposition 3.5) generates the Euclidean metric on \( [\mathbb{R}_0^+]^n \). This result is analogous to the fact that every norm defined on vector spaces generates a metric on it. Further, a semi-vector space \( V \) is Banach (see Definition 3.5) if the norm generates a metric under which every Cauchy sequence in \( V \) converges to an element of \( V \).

Proposition 3.7 Let \( V = [\mathbb{R}_0^+]^n \) and define the function \( \langle \, , \rangle : V \times V \longrightarrow [\mathbb{R}_0^+] \) as follows: if \( u = (x_1, x_2, \ldots, x_n) \) and \( v = (y_1, y_2, \ldots, y_n) \) are semi-vectors in \( V \), put \( \langle u, v \rangle = \sum_{i=1}^{n} x_i y_i \). Then \( \langle \, , \rangle \) is an inner product on \( V \), called dot product.

Proof: The proof is immediate. \( \square \)

Proposition 3.8 The dot product on \( V = [\mathbb{R}_0^+]^n \) generates the Euclidean norm on \( V \).

Proof: If \( x = (x_1, x_2, \ldots, x_n) \in V \), define the norm of \( x \) by \( \| x \| = \sqrt{\langle x, x \rangle} \). Note that the norm is exactly the Euclidean norm given in Proposition 3.5. \( \square \)

Remark 3.4 We observe that if an inner product on a semi-vector space \( V \) generates a norm \( \| \, \| \) and such a norm generates a metric \( d \) on \( V \), then \( V \) is a Hilbert space (according to Definition 3.6) if every Cauchy sequence in \( V \) converges w.r.t. \( d \) to an element of \( V \).

Proposition 3.9 Let \( V = [\mathbb{R}_0^+]^n \) and define the function \( \| \, \|_1 : V \longrightarrow [\mathbb{R}_0^+] \) as follows: if \( x = (x_1, x_2, \ldots, x_n) \in V \), \( \| x \|_1 = \sum_{i=1}^{n} x_i \). Then \( \| x \|_1 \) is a norm on \( V \).

Proof: The proof is direct. \( \square \)

Proposition 3.10 Let \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \) be semi-vectors in \( V = [\mathbb{R}_0^+]^n \). Define the function \( d_1 : V \times V \longrightarrow [\mathbb{R}_0^+] \) in the following way. For every fixed \( i \), if \( x_i = y_i \), put \( c_i = 0 \); if \( x_i \neq y_i \), put \( \varphi_i = \psi_i + c_i \), where \( \varphi_i = \max\{x_i, y_i\} \) and \( \psi_i = \min\{x_i, y_i\} \). Let us consider that \( d_1(x, y) = \sum_{i=1}^{n} c_i \). Then the function \( d_1 \) is a metric on \( V \) derived from the norm \( \| \, \|_1 \) shown in Proposition 3.8.
Proof: We only prove the triangle inequality. To avoid stress of notation, we consider the same that was considered in the proof of Proposition 3.6. We then fix $i$ and only investigate the worst case $x_i < y_i < z_i$. In this case, there exist positive real numbers $a_i$, $b_i$ for all $i = 1, 2, \ldots, n$, such that $y_i = x_i + a_i$ and $z_i = y_i + b_i$, which implies $z_i = x_i + a_i + b_i$. Then, for all $i$, $d_1(x_i, z_i) \leq d_1(x_i, y_i) + d_1(y_i, z_i)$; hence, $d_1(x, z) = \sum_{i=1}^{n} d_1(x_i, z_i) = \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} d_1(x_i, y_i) + \sum_{i=1}^{n} d_1(y_i, z_i) = d_1(x, y) + d_1(y, z)$. Therefore, $d_1$ is a metric on $V$.

Proposition 3.11 Let $V = [R_0^+]^n$ be the Euclidean semi-vector space of semi-dimension $n$. Define the function $\parallel \parallel : V \rightarrow R_0^+$ as follows: if $x = (x_1, x_2, \ldots, x_n) \in V$, take $\parallel x \parallel_2 = \max \{ x_i \}$. Then $\parallel x \parallel_2$ is a norm on $V$.

Proposition 3.12 Keeping the notation of Proposition 3.6, define the function $d_2 : V \times V \rightarrow R_0^+$ such that $d_2(x, y) = \max \{ c_i \}$. Then $d_2$ is a metric on $V$. Moreover, $d_2$ is obtained from the norm $\parallel \parallel_2$ exhibited in Proposition 3.11.

Proposition 3.13 The norms $\parallel \parallel$, $\parallel \parallel_1$ and $\parallel \parallel_2$ shown in Propositions 3.6, 3.9 and 3.11 are equivalent.

Proof: It is immediate to see that $\parallel \parallel_2 \leq \parallel \parallel \leq \parallel \parallel_1 \leq n \parallel \parallel_2$.

In a natural way we can define the norm of a bounded semi-linear transformation.

Definition 3.7 Let $V$ and $W$ be two normed semi-vector spaces and let $T : V \rightarrow W$ be a semi-linear transformation. We say that $T$ is bounded if there exists a real number $c > 0$ such that $\parallel T(v) \parallel \leq c \parallel v \parallel$.

If $T : V \rightarrow W$ is bounded and $v \neq 0$ we can consider the quotient $\frac{\parallel T(v) \parallel}{\parallel v \parallel}$. Since such a quotient is upper bounded by $c$, the supremum $\sup_{v \in V, v \neq 0} \frac{\parallel T(v) \parallel}{\parallel v \parallel}$ exists and it is at most $c$. We then define

$$\parallel T \parallel = \sup_{v \in V, v \neq 0} \frac{\parallel T(v) \parallel}{\parallel v \parallel}.$$

Proposition 3.14 Let $T : V \rightarrow W$ be a bounded semi-linear transformation. Then the following hold:

1. $T$ sends bounded sets in bounded sets;
2. $\parallel T \parallel$ is a norm, called norm of $T$;
3. $\parallel T \parallel$ can be written in the form $\parallel T \parallel = \sup_{v \in V, \parallel v \parallel = 1} \parallel T(v) \parallel$.

Proof: Items (1) and (2) are immediate. The proof of Item (3) is analogous to the standard proof but we present it here to guarantee that our mathematical tools are sufficient to perform it. Let $v \neq 0$ be a semi-vector with norm $\parallel v \parallel = a \neq 0$ and set $u = (1/a)v$. Thus, $\parallel u \parallel = 1$ and since $T$ is semi-linear one has

$$\parallel T \parallel = \sup_{v \in V, v \neq 0} \frac{1}{a} \parallel T(v) \parallel = \sup_{v \in V, v \neq 0} \parallel T((1/a)v) \parallel = \sup_{u \in V, \parallel u \parallel = 1} \parallel T(u) \parallel = \sup_{v \in V, \parallel v \parallel = 1} \parallel T(v) \parallel.$$
3.1.1 The Semi-Spaces \( l_+^\infty \), \( l_+^p \) and \( C_+[a,b] \)

In this subsection we investigate topological aspects of some semi-vector spaces over \( \mathbb{R}_0^+ \) such as completeness and separability. We investigate the sequence spaces \( l_+^\infty \), \( l_+^p \), \( C_+[a,b] \), which will be defined in the sequence.

We first study the space \( l_+^\infty \), the set of all bounded sequences of nonnegative real numbers. Before studying such a space, we must define a metric on it, since the metric in \( l_+^\infty \) is defined as \( d(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i| \), where \( x = (x_i) \) and \( y = (y_i) \) are sequences in \( l_+^\infty \), has no meaning to us, because there is no sense in considering \( -y_i \) if \( y_i > 0 \). Based on this fact, we circumvent this problem by utilizing the total order of \( \mathbb{R} \) according to Proposition 3.6.

Let \( x = (\mu_i) \) and \( y = (\nu_i) \) be sequences in \( l_+^\infty \). We then fix \( i \), and define \( c_i \) as was done in Proposition 3.6 if \( \mu_i = \nu_i \) then we put \( c_i = 0 \); if \( \mu_i \neq \nu_i \), let \( \gamma_i = \max\{\mu_i, \nu_i\} \) and \( \psi_i = \min\{\mu_i, \nu_i\} \); then there exists a positive real number \( c_i \) such that \( \gamma_i = \psi_i + c_i \) and, in place of \( |\mu_i - \nu_i| \), we put \( c_i \). Thus, our metric becomes

\[
d(x,y) = \sup_{i \in \mathbb{N}} \{c_i\},
\]

(1)

It is clear that \( d(x,y) \) shown in Eq. (1) defines a metric. However, we must show that the tools that we have are sufficient to prove this fact, once we are working on \( \mathbb{R}_0^+ \).

**Proposition 3.15** The function \( d \) shown in Eq. (1) is a metric on \( l_+^\infty \).

**Proof:** It is clear that \( d(x,y) \geq 0 \) and \( d(x,y) = 0 \iff x = y \). Let \( x = (\mu_i) \) and \( y = (\nu_i) \) be two sequences in \( l_+^\infty \). Then, for every fixed \( i \in \mathbb{N} \), if \( c_i = d(\mu_i, \nu_i) = 0 \) then \( \mu_i = \nu_i \), i.e., \( d(\mu_i, \nu_i) = d(\nu_i, \mu_i) \). If \( c_i > 0 \) then \( c_i = d(\mu_i, \nu_i) \) is computed by \( \gamma_i = \psi_i + c_i \), where \( \gamma_i = \max\{\mu_i, \nu_i\} \) and \( \psi_i = \min\{\mu_i, \nu_i\} \). Hence, \( d(\nu_i, \mu_i) = c_i^* \) is computed by \( \gamma_i^* = \psi_i^* + c_i^* \), where \( \gamma_i^* = \max\{\nu_i, \mu_i\} \) and \( \psi_i^* = \min\{\nu_i, \mu_i\} \), which implies \( d(\mu_i, \nu_i) = d(\nu_i, \mu_i) \). Taking the supremum over all \( i \)'s we have \( d(x,y) = \sup_{i \in \mathbb{N}} \{c_i\} = \sup_{i \in \mathbb{N}} \{c_i^*\} = d(y,x) \).

To show the triangle inequality, let \( x = (\mu_i) \), \( y = (\nu_i) \) and \( z = (\eta_i) \) be sequences in \( l_+^\infty \). For every fixed \( i \), we will prove that \( d(\mu_i, \eta_i) \leq d(\mu_i, \nu_i) + d(\nu_i, \eta_i) \). If \( \nu_i = \mu_i = \eta_i \), the result is trivial. If two of them are equal, the result is also trivial. Assume that \( \mu_i \), \( \nu_i \) and \( \eta_i \) are pairwise distinct. As in the proof of Proposition 3.6 we must investigate the six cases:

1. \( \mu_i < \nu_i < \eta_i \); 2. \( \mu_i < \eta_i < \nu_i \); 3. \( \nu_i < \mu_i < \eta_i \); 4. \( \nu_i < \eta_i < \mu_i \); 5. \( \eta_i < \mu_i < \nu_i \); 6. \( \eta_i < \nu_i < \mu_i \). We only show (1) and (2).

To show (1), note that there exist positive real numbers \( c_i \) and \( c_i' \) such that \( \nu_i = \mu_i + c_i \) and \( \eta_i = \nu_i + c_i' \), which implies \( \eta_i = \mu_i + c_i + c_i' \). Hence, \( d(\mu_i, \eta_i) = c_i + c_i' = d(\mu_i, \nu_i) + d(\nu_i, \eta_i) \).

Let us show (2). There exist positive real numbers \( b_i \) and \( b_i' \) such that \( \eta_i = \mu_i + b_i \) and \( \nu_i = \eta_i + b_i' \), so \( \nu_i = \mu_i + b_i + b_i' \). Therefore, \( d(\mu_i, \nu_i) = b_i < d(\mu_i, \nu_i) + d(\nu_i, \eta_i) = b_i + 2b_i' \).

Taking the supremum over all \( i \)'s we have \( \sup_{i \in \mathbb{N}} \{d(\mu_i, \eta_i)\} \leq \sup_{i \in \mathbb{N}} \{d(\mu_i, \nu_i)\} + \sup_{i \in \mathbb{N}} \{d(\nu_i, \eta_i)\} \), i.e., \( d(x,z) \leq d(x,y) + d(y,z) \). Therefore, \( d \) is a metric on \( l_+^\infty \).

**Definition 3.8** The metric space \( l_+^\infty \) is the set of all bounded sequences of nonnegative real numbers equipped with the metric \( d(x,y) = \sup_{i \in \mathbb{N}} \{c_i\} \) given previously.
We prove that $l_\infty^+$ equipped with the previous metric is complete.

**Theorem 3.3** The space $l_\infty^+$ with the metric $d(x, y) = \sup_{i \in \mathbb{N}} \{c_i\}$ shown above is complete.

**Proof:** The proof follows the same line as the standard proof of completeness of $l_\infty$; however it is necessary to adapt it to the metric (written above) in terms of nonnegative real numbers. Let $(x_n)$ be a Cauchy sequence in $l_\infty^+$, where $x_i = (\eta_1^{(i)}, \eta_2^{(i)}, \ldots)$. We must show that $(x_n)$ converges to an element of $l_\infty^+$. As $(x_n)$ is Cauchy, given $\epsilon > 0$, there exists a positive integer $K$ such that, for all $n, m > K$,

$$d(x_n, x_m) = \sup_{j \in \mathbb{N}} \{c_j^{(n,m)}\} < \epsilon,$$

where $c_j^{(n,m)}$ is a nonnegative real number such that, if $\eta_j^{(n)} = \eta_j^{(m)}$ then $c_j^{(n,m)} = 0$, and if $\eta_j^{(n)} \neq \eta_j^{(m)}$ then $c_j^{(n,m)}$ is given by

$$c_j^{(n,m)} = \max\{\eta_j^{(n)}, \eta_j^{(m)}\} = \min\{\eta_j^{(n)}, \eta_j^{(m)}\} + c_j^{(n,m)}.$$  

This implies that for each fixed $j$ one has

$$c_j^{(n,m)} < \epsilon, \quad (2)$$

where $n, m > K$. Thus, for each fixed $j$, it follows that $(\eta_j^{(1)}, \eta_j^{(2)}, \ldots)$ is a Cauchy sequence in $\mathbb{R}_0^+$. Since $\mathbb{R}_0^+$ is a complete metric space, the sequence $(\eta_j^{(1)}, \eta_j^{(2)}, \ldots)$ converges to an element $\eta_j$ in $\mathbb{R}_0^+$. Hence, for each $j$, we form the sequence $x$ whose coordinates are the limits $\eta_j$, i.e., $x = (\eta_1, \eta_2, \eta_3, \ldots)$. We must show that $x \in l_\infty^+$ and $x_n \to x$.

To show that $x$ is a bounded sequence, let us consider the number $c_j^{(n,\infty)}$ defined as follows: if $\eta_j^{(n)} = \eta_j^{(m)}$ then $c_j^{(n,\infty)} = 0$, and if $\eta_j^{(n)} \neq \eta_j^{(m)}$, define $c_j^{(n,\infty)}$ be the positive real number satisfying

$$c_j^{(n,\infty)} = \min\{\eta_j^{(n)}, \eta_j^{(m)}\} + c_j^{(n,m)}.$$  

From the inequality (2) one has

$$c_j^{(n,\infty)} \leq \epsilon. \quad (3)$$

Because $\eta_j \leq \eta_j^{(n)} + c_j^{(n,\infty)}$ and since $\eta_j^{(n)} \in l_\infty^+$, it follows that $\eta_j$ is a bounded sequence for every $j$. Hence, $x = (\eta_1, \eta_2, \eta_3, \ldots) \in l_\infty^+$. From (3) we have

$$\sup_{j \in \mathbb{N}} \{c_j^{(n,\infty)}\} \leq \epsilon,$$

which implies that $x_n \to x$. Therefore, $l_\infty^+$ is complete. \qed

Although $l_\infty^+$ is a complete metric space, it is not separable.

**Theorem 3.4** The space $l_\infty^+$ with the metric $d(x, y) = \sup_{i \in \mathbb{N}} \{c_i\}$ is not separable.

**Proof:** The proof is the same as shown in [7, 1.3-9], so it is omitted. \qed

Let us define the space analogous to the space $l^p$. 

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13
Definition 3.9 Let $p \geq 1$ be a fixed real number. The set $l^p_+$ consists of all sequences $x = (\eta_1, \eta_2, \eta_3, \ldots)$ of nonnegative real numbers such that \[ \sum_{i=1}^{\infty} |\eta_i|^p < \infty, \] whose metric is defined by \[ d(x, y) = \left[ \sum_{i=1}^{\infty} |c_i|^p \right]^{1/p}, \] where \( y = (\mu_1, \mu_2, \mu_3, \ldots) \) and \( c_i \) is defined as follows: \( c_i = 0 \) if \( \mu_i = \eta_i \), and if \( \mu_i < \eta_i \) (resp. \( \eta_i < \mu_i \)) then \( c_i > 0 \) is such that \( \mu_i = \eta_i + c_i \).

Theorem 3.5 The space $l^p_+$ with the metric $d(x, y) = \left[ \sum_{i=1}^{\infty} |c_i|^p \right]^{1/p}$ exhibited above is complete.

Proof: Recall that given two sequences $(\mu_i)$ and $(\eta_i)$ in $l^p_+$ the Minkowski inequality for sums reads as
\[
\left[ \sum_{i=1}^{\infty} |\mu_i + \eta_i|^p \right]^{1/p} \leq \left[ \sum_{i=1}^{\infty} |\mu_i|^p \right]^{1/p} + \left[ \sum_{k=1}^{\infty} |\eta_k|^p \right]^{1/p}.
\]
Applying the Minkowski inequality as per [7, 1.5-4] with some adaptations, it follows that \( d(x, y) \) is, in fact, a metric. In order to prove the completeness of $l^p_+$, we proceed similarly as in the proof of Theorem 3.3 with some adaptations. The main adaptation is performed according to the proof of completeness of $l^p$ in [7, 1.5-4] replacing the last equality $x = x_m + (x - x_m) \in l^p$ (after Eq. (5)) by two equalities in order to avoid negative real numbers.

(1) If the $i$-th coordinate $x^{(i)} - x^{(i)}_m$ of the sequence $x - x_m$ is positive, then define $c^{(i)}_m = x^{(i)} - x^{(i)}_m$ and write $x^{(i)} = x^{(i)}_m + c^{(i)}_m$. From Minkowski inequality, it follows that the sequence $(x^{(i)})_i$ is in $l^p_+$.

(2) If $x^{(j)} - x^{(j)}_m$ is negative, then define $c^{(j)}_m = x^{(j)}_m - x^{(j)}$ and write $x^{(j)}_m = x^{(j)} + c^{(j)}_m$. Since $x_m \in l^p_+$, from the comparison criterion for positive series it follows that the sequence $(x^{(j)})_j$ is also in $l^p_+$.

\[ \square \]

Theorem 3.6 The space $l^p_+$ is separable.

Proof: The proof follows the same line of [7, 1.3-10].

Definition 3.10 Let $I = [a, b]$ be a closed interval in $\mathbb{R}^+$, where $a \geq 0$ and $a < b$. Then $C_+ [a, b]$ is the set of all continuous nonnegative real valued functions on $I = [a, b]$, whose metric is defined by \[ d(f(t), g(t)) = \max_{t \in I} \{ c(t) \}, \] where $c(t)$ is given by \[ \max \{ f(t), g(t) \} = \min \{ f(t), g(t) \} + c(t). \]

Theorem 3.7 The metric space $(C_+ [a, b], d)$, where $d$ is given in Definition 3.10, is complete.

Proof: The proof follows the same lines as the standard one with some modifications. Let $(f_m)$ be a Cauchy sequence in $C_+ [a, b]$. Given $\epsilon > 0$ there exists a positive integer $N$ such that, for all $m, n > N$, it follows that
\[
d(f_m, f_n) = \max_{t \in I} \{ c_{m,n}(t) \} < \epsilon,
\]

(4)
where \( \max\{f_m(t), f_n(t)\} = \min\{f_m(t), f_n(t)\} + c_{m,n}(t) \). Thus, for any fixed \( t_0 \in I \) we have \( c_{m,n}(t_0) < \epsilon \), for all \( m, n > N \). This means that \( (f_1(t_0), f_2(t_0), \ldots) \) is a Cauchy sequence in \( \mathbb{R}_0^+ \), which converges to \( f(t_0) \) when \( m \to \infty \) since \( \mathbb{R}_0^+ \) is complete. We then define a function \( f : [a, b] \to \mathbb{R}_0^+ \) such that for each \( t \in [a, b] \), we put \( f(t) \). Taking \( n \to \infty \) in \( \text{(3)} \) we obtain \( \max\{c_m(t)\} \leq \epsilon \) for all \( m > N \), where \( \max\{f_m(t), f(t)\} = \min\{f_m(t), f(t)\} + c_m(t) \), which implies \( c_m(t) \leq \epsilon \) for all \( t \in I \). This fact means that \( (f_m(t)) \) converges to \( f(t) \) uniformly on \( I \), i.e., \( f \in C_+[a, b] \) because the functions \( f_m \)'s are continuous on \( I \). Therefore, \( C_+[a, b] \) is complete, as desired. \( \Box \)

3.2 Interesting Semi-Vector Spaces

In this section we exhibit semi-vector spaces over \( K = \mathbb{R}_0^+ \) derived from semi-metrics, semi-metric-preserving functions, semi-norms, semi-inner products and sub-linear functionals.

**Theorem 3.8** Let \( X \) be a semi-metric space and \( M_X = \{d : X \times X \to \mathbb{R} ; d \) is a semi-metric on \( X \} \). Then \( (M_X, +, \cdot) \) is a semi-vector space over \( \mathbb{R}_0^+ \), where \( + \) and \( \cdot \) are the addition and the scalar multiplication (in \( \mathbb{R}_0^+ \)) pointwise, respectively.

**Proof:** We first show that \( M_X \) is closed under addition. Let \( d_1, d_2 \in M_X \) and set \( d := d_1 + d_2 \). It is clear that \( d \) is nonnegative real-valued function. Moreover, for all \( x, y \in X \), \( d(x, y) = d(y, x) \). Let \( x \in X \); \( d(x, x) = d_1(x, x) + d_2(x, x) = 0 \). For all \( x, y, z \in X \), \( d(x, z) = d_1(x, z) + d_2(x, z) \leq |d_1(x, y) + d_2(x, y)| \) + \( [d_1(y, z) + d_2(y, z)] = d(x, y) + d(y, z) \).

Let us show that \( M_X \) is closed under scalar multiplication. Let \( d_1 \in M_X \) and define \( d = \lambda d_1 \), where \( \lambda \in \mathbb{R}_0^+ \). It is clear that \( d \) is real-valued nonnegative and for all \( x, y \in X \), \( d(x, y) = d(y, x) \). Moreover, if \( x \in X \), \( d(x, x) = 0 \). For all \( x, y, z \in X \), \( d(x, z) = \lambda d_1(x, z) \leq \lambda[d_1(x, y) + d_1(y, z)] = d(x, y) + d(y, z) \). This means that \( M_X \) is closed under scalar multiplication.

It is easy to see that \( (M_X, +, \cdot) \) satisfies the other conditions of Definition 2.3. \( \Box \)

Let \( (X, d) \) be a metric space. In \( \text{[3]} \), Corazza investigated interesting functions \( f : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) such that the composite of \( f \) with \( d \), i.e., \( X \times X \xrightarrow{d} \mathbb{R}_0^+ \xrightarrow{f} \mathbb{R}_0^+ \) also generates a metric on \( X \). Let us put this concept formally.

**Definition 3.11** Let \( f : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) be a function. We say that \( f \) is metric-preserving if for all metric spaces \( (X, d) \), the composite \( f \circ d \) is a metric.

To our purpose we will consider semi-metric preserving functions as follows.

**Definition 3.12** Let \( f : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) be a function. We say that \( f \) is semi-metric-preserving if for all semi-metric spaces \( (X, d) \), the composite \( f \circ d \) is a semi-metric.

We next show that the set of semi-metric preserving functions has a semi-vector space structure.

**Theorem 3.9** Let \( \mathcal{F}_{\text{pres}} = \{f : \mathbb{R}_0^+ \to \mathbb{R}_0^+ ; f \) is semi-metric preserving\}. Then \( (\mathcal{F}_{\text{pres}}, +, \cdot) \) is a semi-vector space over \( \mathbb{R}_0^+ \), where \( + \) and \( \cdot \) are the addition and the scalar multiplication (in \( \mathbb{R}_0^+ \)) pointwise, respectively.
Proof: We begin by showing that $F_{\text{pres}}$ is closed under addition and scalar multiplication pointwise.

Let $f, g \in F_{\text{pres}}$. Given a semi-metric space $(X, d)$, we must prove that $(f + g) \circ d$ is also semi-metric preserving. We know that $[(f + g) \circ d](x, y) \geq 0$ for all $x, y \in X$. Let $x \in X$; then $[(f + g) \circ d](x, x) = f(d(x, x)) + g(d(x, x)) = 0$. It is clear that $[(f + g) \circ d](x, y) = [(f + g) \circ d](y, x)$. Let $x, y, z \in X$. One has: $[(f + g) \circ d](x, y) = f(d(x, y)) + g(d(y, x)) \leq [(f \circ d)(x, z)] + [(g \circ d)(z, y)] = (f + g)(d(x, y)) + [(f + g)(d(x, z)) + (f + g)(d(z, y))] = (f + g)(d(x, z)) + (f + g)(d(z, y)) = [(f + g) \circ d](x, z) + [(f + g) \circ d](z, y)$.

Here, we show that for each $f \in F_{\text{pres}}$ and $\alpha \in R^+_0$, it follows that $\alpha f \in F_{\text{pres}}$. We show only the triangular inequality since the other conditions are immediate. Let us calculate: $[(\alpha f \circ d)(x, y)] = [(f \circ d)(x, y)] = [f(d(x, y))] + [f(d(y, x))] = (f + g)(d(x, z)) + (f + g)(d(z, y)) = [(f + g) \circ d](x, z) + [(f + g) \circ d](z, y)$.

The null vector is the null function $0_f : R^+_0 \rightarrow R^+_0$. The other conditions are easy to verify. □

**Theorem 3.10** Let $V$ be a semi-normed real vector space and $N_V = \{|| \cdot || : V \rightarrow R; || \cdot || is a semi-norm on V\}$. Then $(N_V, +, \cdot)$ is a semi-vector space over $R^+_0$, where $+$ and $\cdot$ are addition and scalar multiplication (in $R^+_0$) pointwise, respectively.

Proof: From hypotheses, $N_V$ is non-empty. Let $|| \cdot ||_1, || \cdot ||_2 \in N_V$ and set $|| \cdot || := || \cdot ||_1 + || \cdot ||_2$. For all $v \in V$, $||v|| \geq 0$. If $v \in V$ and $\alpha \in R$ then $||\alpha v|| = ||\alpha||||v||$. For every $u, v \in V$, it follows that $||u + v|| := ||u + v||_1 + ||u + v||_2 \leq (||u||_1 + ||u||_2) + (||v||_1 + ||v||_2) = ||u|| + ||v||$. Hence, $N_V$ is closed under addition.

We next show that $N_V$ is closed under scalar multiplication. Let $|| \cdot ||_1 \in N_V$ and define $|| \cdot || := \lambda || \cdot ||_1$, where $\lambda \in R^+_0$. For all $v \in V$, $||v|| \geq 0$. If $\alpha \in R$ and $v \in V$, $||\alpha v|| = ||\alpha||||v||_1 = ||\alpha||||v||$. Let $u, v \in V$. Then $||u + v|| \leq \lambda||u||_1 + \lambda||v||_1 = ||u|| + ||v||$. Therefore, $N_V$ is closed under addition and scalar multiplication over $R^+_0$.

The zero vector is the null function $0 : V \rightarrow R$. The other conditions of Definition 2.3 are straightforward. □

**Remark 3.5** Note that $N_V^0 = \{|| \cdot || : V \rightarrow R; || \cdot || is a norm on V\}$ is also closed under both function addition and scalar multiplication pointwise.

**Lemma 3.1** Let $T : V \rightarrow W$ be a linear transformation.

1. If $|| \cdot || : W \rightarrow R$ is a semi-norm on $W$ then $|| \circ T : V \rightarrow R$ is a semi-norm on $V$.

2. If $T$ is injective linear and $|| \cdot || : W \rightarrow R$ is a norm on $W$ then $|| \circ T$ is a norm on $V$.

Proof: We only show Item (1). It is clear that $|| \circ T||(v) \geq 0$ for all $v \in V$. For all $\alpha \in R$ and $v \in V$, $||\circ T\|(v) = ||\alpha||T(v)|| = ||\alpha||||\circ T||(v)$. Moreover, $\forall v_1, v_2 \in V, ||\circ T||(v_1 + v_2) \leq ||\circ T||(v_1) + ||\circ T||(v_2)$. Therefore, $|| \circ T$ is a semi-norm on $V$. □

**Theorem 3.11** Let $V$ and $W$ be two semi-normed vector spaces and $T : V \rightarrow W$ be a linear transformation. Then

$N_{TV} = \{|| \circ T : V \rightarrow R; || \cdot || is a semi-norm on W\}$
is a semi-subspace of \((N_V, +, \cdot)\).

**Proof:** From hypotheses, it follows that \(N_V\) is non-empty. From Item (1) of Lemma 3.1 it follows that \(\| \| \circ T\) is a semi-norm on \(V\). Let \(f, g \in N_V\), i.e., \(f = \| \|_1 \circ T\) and \(g = \| \|_2 \circ T\), where \(\| \|_1\) and \(\| \|_2\) are semi-norms on \(W\). Then \(f + g = [\| \|_1 + \| \|_2] \circ T \in N_V\). For every nonnegative real number \(\lambda\) and \(f \in N_V\), \(\lambda f = \lambda(\| \|_1 \circ T) = (\lambda \| \|) \circ T \in N_V\). \(\square\)

**Theorem 3.12** Let \(N\) be the class whose members are \(\{N_V\}\), where the \(N_V\) are given in Theorem 3.10. Let \(\text{Hom}(N)\) be the class whose members are the sets

\[
\text{hom}(N_V, N_W) = \{F_T : N_V \to N_W; F_T(\| \|_V) = \| \|_V \circ T\},
\]

where \(T : W \to V\) is a linear transformation and \(\| \|_V\) is a semi-norm on \(V\). Then \((N, \text{Hom}(N), \text{Id}, \circ)\) is a category.

**Proof:** The sets \(\text{hom}(N_V, N_W)\) are pairwise disjoint. For each \(N_V\), there exists \(\text{Id}_{N_V}\) given by \(\text{Id}_{N_V}(\| \|_V) = \| \|_V = \| \|_V \circ \text{Id}_{(V)}\). It is clear that if \(F_T : N_V \to N_W\) then \(F_T \circ \text{Id}_{(N_V)} = F_T\) and \(\text{Id}_{(N_W)} \circ F_T = F_T\).

It is easy to see that for every \(T : W \to V\) linear transformation, the map \(F_T\) is semi-linear, i.e., \(F_T(\| \|_V^{(1)} + \| \|_V^{(2)}) = F_T(\| \|_V^{(1)}) + F_T(\| \|_V^{(2)})\) and \(F_T(\lambda \| \|_V) = \lambda F_T(\| \|_V)\), for every \(\| \|_V\), \(\| \|_V^{(1)}\), \(\| \|_V^{(2)}\) \(\in\) \(N_V\) and \(\lambda \in \mathbb{R}^+\).

Let \(N_U, N_V, N_W, N_X \in N\) and \(F_{T_1} \in \text{hom}(N_U, N_V)\), \(F_{T_2} \in \text{hom}(N_V, N_W)\), \(F_{T_3} \in \text{hom}(N_W, N_X)\), i.e.,

\[
N_U \xrightarrow{F_{T_1}} N_V \xrightarrow{F_{T_2}} N_W \xrightarrow{F_{T_3}} N_X.
\]

The linear transformations are of the forms

\[
X \xrightarrow{T_3} W \xrightarrow{T_2} V \xrightarrow{T_1} U \xrightarrow{\| \|_U} \mathbb{R}.
\]

The associativity \((F_{T_3} \circ F_{T_2}) \circ F_{T_1} = F_{T_3} \circ (F_{T_2} \circ F_{T_1})\) follows from the associativity of composition of maps. Moreover, the map \(F_{T_3} \circ F_{T_2} \circ F_{T_1} \in \text{Hom}(N)\) because \(F_{T_3} \circ F_{T_2} \circ F_{T_1} = (\| \|_U) \circ (T_1 \circ T_2 \circ T_3)\) and \(T_1 \circ T_2 \circ T_3\) is a linear transformation. Therefore, \((N, \text{Hom}(N), \text{Id}, \circ)\) is a category, as required. \(\square\)

**Theorem 3.13** Let \(V\) be a real vector space endowed with a semi-inner product and let \(P_V = \{\langle , \rangle : V \times V \to \mathbb{R}; \langle , \rangle\) is a semi-inner product on \(V\}\). Then \((P_V, +, \cdot)\) is a semi-vector space over \(\mathbb{R}^+_0\), where + and \(\cdot\) are addition and scalar multiplication (in \(\mathbb{R}^+_0\)) pointwise, respectively.

**Proof:** The proof is analogous to that of Theorems 3.8 and 3.10. \(\square\)

**Proposition 3.16** Let \(V, W\) be two vector spaces and \(T_1, T_2 : V \to W\) be two linear transformations. Let us consider the map \(T_1 \times T_2 : V \times V \to W \times W\) given by \(T_1 \times T_2(u, v) = (T_1(u), T_2(v))\). If \(\langle , \rangle\) is a semi-inner product on \(W\) then \(\langle , \rangle \circ T_1 \times T_2\) is a semi-inner product on \(V\).

17
Proof: The proof is immediate, so it is omitted.

Let $V$ be a real vector space. Recall that a sub-linear functional on $V$ is a functional $t : V \rightarrow \mathbb{R}$ which is sub-additive: $\forall u, v \in V$, $t(u + v) \leq t(u) + t(v)$; and positive-homogeneous: $\forall \alpha \in \mathbb{R}_0^+$ and $\forall v \in V$, $t(\alpha v) = \alpha t(v)$.

**Theorem 3.14** Let $V$ be a real vector space. Let us consider $\mathcal{S}_V = \{S : V \rightarrow \mathbb{R}; S$ is sub-linear on $V\}$. Then $(\mathcal{S}_V, +, \cdot)$ is a semi-vector space on $\mathbb{R}_0^+$, where $+$ and $\cdot$ are addition and scalar multiplication (in $\mathbb{R}_0^+$) pointwise, respectively.

Proof: The proof follows the same line of that of Theorems 3.8 and 3.10 and 3.13.

### 3.3 Semi-Algebras

We start this section by recalling the definition of semi-algebra and semi-sub-algebra. For more details the reader can consult [4]. In [9], Olivier and Serrato investigated relation semi-algebras, i.e., a semi-algebra being both a Boolean algebra and an involutive semi-monoid, satisfying some conditions (see page 2 in Ref. [9] for more details). Roy [14] studied the semi-algebras of continuous and monotone functions on compact ordered spaces.

**Definition 3.13** A semi-algebra $A$ over a semi-field $K$ (or a $K$-semi-algebra) is a semi-vector space $A$ over $K$ endowed with a binary operation called multiplication of semi-vectors $\bullet : A \times A \rightarrow A$ such that, $\forall u, v, w \in A$ and $\lambda \in K$:

1. $u \bullet (v + w) = (u \bullet v) + (u \bullet w)$ (left-distributivity);
2. $(u + v) \bullet w = (u \bullet w) + (v \bullet w)$ (right-distributivity);
3. $\lambda(u \bullet v) = (\lambda u) \bullet v = u \bullet (\lambda v)$.

A semi-algebra $A$ is associative if $(u \bullet v) \bullet w = u \bullet (v \bullet w)$ for all $u, v, w \in A$; $A$ is said to be commutative (or abelian) is the multiplication is commutative, that is, $\forall u, v \in A$, $u \bullet v = v \bullet u$; $A$ is called a semi-algebra with identity if there exists an element $1_A \in A$ such that $\forall u \in A$, $1_A \bullet u = u = u \bullet 1_A$; the element $1_A$ is called identity of $A$. The identity element of a semi-algebra $A$ is unique (if exists). If $A$ is a semi-free semi-vector space then the dimension of $A$ is its dimension regarded as a semi-vector space. A semi-algebra is simple if it is simple as a semi-vector space.

**Example 3.3** The set $\mathbb{R}_0^+$ is a commutative semi-algebra with identity $e = 1$.

**Example 3.4** The set of square matrices of order $n$ whose entries are in $\mathbb{R}_0^+$, equipped with the sum of matrices, multiplication of a matrix by a scalar (in $\mathbb{R}_0^+$, of course) and by multiplication of matrices is an associative and non-commutative semi-algebra with identity $e = I_n$ (the identity matrix of order $n$), over $\mathbb{R}_0^+$. 

18
Example 3.5 The set \( P_n[x] \) of polynomials with coefficients from \( \mathbb{R}_0^+ \) and degree less than or equal to \( n \), equipped with the usual of polynomial sum and scalar multiplication is a semi-vector space.

Example 3.6 Let \( V \) be a semi-vector space over a semi-field \( K \). Then the set \( \mathcal{L}(V,V) = \{ T : V \rightarrow V; T \text{ is a semi-linear operator} \} \) is a semi-vector space. If we define a vector multiplication as the composite of semi-linear operators (which is also semi-linear) then we have a semi-algebra over \( K \).

Definition 3.14 Let \( A \) be a semi-algebra over \( K \). We say that a non-empty set \( S \subseteq A \) is a semi-subalgebra if \( S \) is closed under the operations of \( A \), that is,

1. \( \forall u, v \in A, u + v \in A; \)
2. \( \forall u, v \in A, u \cdot v \in A; \)
3. \( \forall \lambda \in K \) and \( \forall u \in A, \lambda u \in A. \)

Definition 3.15 Let \( A \) and \( B \) two semi-algebras over \( K \). We say that a map \( T : A \rightarrow B \) is an \( K \)-semi-algebra homomorphism if, \( \forall u, v \in A \) and \( \lambda \in K \), the following conditions hold:

1. \( T(u + v) = T(u) + T(v); \)
2. \( T(u \cdot v) = T(u) \cdot T(v); \)
3. \( T(\lambda v) = \lambda T(v). \)

Definition 3.15 means that \( T \) is both a semi-ring homomorphism and also semi-linear (as semi-vector space).

Definition 3.16 Let \( A \) and \( B \) be two \( K \)-semi-algebras. A \( K \)-semi-algebra isomorphism \( T : A \rightarrow B \) is a bijective \( K \)-semi-algebra homomorphism. If there exists such an isomorphism, we say that \( A \) is isomorphic to \( B \), written \( A \cong B \).

The following results seems to be new, because semi-algebras over \( \mathbb{R}_0^+ \) are not much investigated in the literature.

Proposition 3.17 Assume that \( A \) and \( B \) are two \( K \)-semi-algebras, where \( K = \mathbb{R}_0^+ \) and \( A \) has identity \( 1_A \). Let \( T : A \rightarrow B \) be a \( K \)-semi-algebra homomorphism. Then the following properties hold:

1. \( T(0_A) = 0_B; \)
2. \( \text{If } u \in A \text{ is invertible then its inverse is unique and } (u^{-1})^{-1} = u; \)
3. \( \text{If } T \text{ is surjective then } T(1_A) = 1_B, \text{ i.e., } B \text{ also has identity; furthermore, } T(u^{-1}) = [T(u)]^{-1}; \)
4. \( \text{If } u, v \in A \text{ are invertible then } (u \cdot v)^{-1} = v^{-1} \cdot u^{-1}; \)
5. \( \text{the composite of } K \text{-semi-algebra homomorphisms is also a } K \text{-semi-algebra homomorphism}; \)
(6) if \( T \) is a \( K \)-semi-algebra isomorphism then also is \( T^{-1} : B \to A \).

(7) the relation \( A \sim B \) if and only if \( A \) is isomorphic to \( B \) is an equivalence relation.

Proof: Note that Item (1) holds because the additive cancelation law holds in the definition of semi-vector spaces (see Definition 2.3). We only show Item (3) since the remaining items are direct. Let \( v \in B \); then there exists \( u \in A \) such that \( T(u) = v \). It then follows that \( v \circ T(1_A) = T(u \circ 1_A) = v \) and \( T(1_A) \circ v = T(1_A \circ u) = v \); which means that \( T(1_A) \) is the identity of \( B \), i.e., \( T(1_A) = 1_B \).

We have: \( T(u) \circ T(u^{-1}) = T(u \circ u^{-1}) = T(1_A) = 1_B \) and \( T(u^{-1}) \circ T(u) = T(u^{-1} \circ u) = T(1_A) = 1_B \), which implies \( T(u^{-1}) = [T(u)]^{-1} \).

\[ \]

Proposition 3.18 If \( A \) is a \( K \)-semi-algebra with identity \( 1_A \) then \( A \) can be embedded in \( L(A, A) \), the semi-algebra of semi-linear operators on \( A \).

Proof: For every fixed \( v \in A \), define \( v^* : A \to A \) as \( v^*(x) = v \circ x \). It is easy to see that \( v^* \) is a semi-linear operator on \( A \). Define \( h : A \to L(A, A) \) by \( h(v) = v^* \). We must show that \( h \) is a injective \( K \)-semi-algebra homomorphism where the product in \( L(A, A) \) is the composite of maps from \( A \) into \( A \). Fixing \( u, v \in A \), we have: \( [h(u + v)](x) = (u + v)^*(x) = (u + v) \circ x = u \circ x + v \circ x = u^*(x) + v^*(x) = [h(u)](x) + [h(v)](x) \), hence \( h(u + v) = h(u) + h(v) \). For \( \lambda \in K \) and \( v \in A \), it follows that \( [h(\lambda v)](x) = (\lambda v)^*(x) = (\lambda v)x = \lambda(vx) = [\lambda h(v)](x) \), i.e., \( h(\lambda v) = \lambda h(v) \). For fixed \( u, v \in A \), \( [h(u \circ v)](x) = (u \circ v)^*(x) = (u \circ v) \circ x = u \circ (v \circ x) = u \circ v^*(x) = u^*(v^*(x)) = [h(u) \circ h(v)](x) \), i.e., \( h(u \circ v) = h(u) \circ h(v) \). Assume that \( h(u) = h(v) \), that is, \( u^* = v^* \); hence, for every \( x \in A \), \( u^*(x) = v^*(x) \), i.e., \( u \circ x = v \circ x \). Taking in particular \( x = 1_A \), it follows that \( u = v \), which implies that \( h \) is injective. Therefore, \( A \) is isomorphic to \( h(A) \), where \( h(A) \subseteq L(A, A) \).

Definition 3.17 Let \( A \) be a semi-vector space over a semi-field \( K \). Then \( A \) is said to be a Lie semi-algebra if \( A \) is equipped with a product \( [\cdot , \cdot] : A \times A \to A \) such that the following conditions hold:

1. \([\cdot , \cdot]\) is semi-bilinear, i.e., fixing the first (second) variable, \([\cdot , \cdot]\) is semi-linear w.r.t. the second (first) one;

2. \([\cdot , \cdot]\) is anti-symmetric, i.e., \([v, v] = 0 \ \forall \ v \in A\);

3. \([\cdot , \cdot]\) satisfies the Jacobi identity: \( \forall u, v, w \in A, [u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0 \)

From Definition 3.17 we can see that a Lie semi-algebra can be non-associative, i.e., the product \([\cdot , \cdot]\) is not always associative.

Let us now consider the semi-algebra \( M_n(\mathbb{R}_+^\alpha) \) of matrices of order \( n \) with entries in \( \mathbb{R}_+^\alpha \) (see Example 3.23). We know that \( M_n(\mathbb{R}_+^\alpha) \) is simple, i.e., with exception of the zero matrix (zero vector), no matrix has (additive) symmetric. Therefore, the product of such matrices can be nonzero. However, in the case of a Lie semi-algebra \( A \), if \( A \) is simple then the unique product \([\cdot,\cdot]\) that can be defined over \( A \) is the zero product, as it is shown in the next result.
Proposition 3.19 If $A$ is a simple Lie semi-algebra over a semi-field $K$ then the semi-algebra is abelian, i.e., \([u,v] = 0\) for all $u, v \in A$.

Proof: Assume that $u, v \in A$ and $[u,v] \neq 0$. From Items (1) and (2) of Definition 3.17 it follows that $[u + v, u + v] = [u, u] + [u, v] + [v, u] + [v, v] = 0$, i.e., $[u, v] + [v, u] = 0$. This means that $[u, v]$ has symmetric $[v, u] \neq 0$, a contradiction. □

Definition 3.18 Let $A$ be a Lie semi-algebra over a semi-field $K$. A Lie semi-subalgebra $B \subseteq A$ is a semi-subspace of $A$ which is closed under $[u,v]$, i.e., for all $u, v \in B$, $[u,v] \in B$.

Corollary 3.1 All semi-subspaces of $A$ are semi-subalgebras of $A$.

Proof: Apply Proposition 3.19. □

4 Fuzzy Set Theory and Semi-Algebras

The theory of semi-vector spaces and semi-algebras is a natural generalization of the corresponding theories of vector spaces and algebras. Since the scalars are in semi-fields (weak semi-fields), some standard properties does not hold in this new context. However, as we have shown in Section 3 even in case of nonexistence of symmetrizable elements, several results are still true. An application of the theory of semi-vector spaces is in the investigation on Fuzzy Set Theory, which was introduced by Lotfali Askar-Zadeh [16]. In fact, such a theory fits in the investigation/extension of results concerning fuzzy sets and their corresponding theory. Let us see an example.

Let $L$ be a linearly ordered complete lattice with distinct smallest and largest elements 0 and 1. Recall that a fuzzy number is a function $x : \mathbb{R} \rightarrow L$ on the field of real numbers satisfying the following items (see [4, Sect. 1.1]): (1) for each $\alpha \in L_0$ the set $x_\alpha = \{ \varphi \in \mathbb{R} | \alpha \leq x(\varphi) \}$ is a closed interval $[x_{al}, x_{ar}]$, where $L_0 = \{ \alpha \in L | \alpha > 0 \}$; (2) $\{ \varphi \in \mathbb{R} | 0 < x(\varphi) \}$ is bounded.

We denote the set $\mathbb{R}_L$ to be the set of all fuzzy numbers; $\mathbb{R}_L$ can be equipped with a partial order in the following manner: $x \leq y$ if and only if $x_{al} \leq y_{al}$ and $x_{ar} \leq y_{ar}$ for all $\alpha \in L_0$. In this scenario, Gahler et al. showed that the concepts of semi-algebras can be utilized to extend the concept of fuzzy numbers, according to the following proposition:

Proposition 4.1 [4, Proposition 19] The set $\mathbb{R}_L$ is an ordered commutative semi-algebra.

Thus, a direct utilization of the investigation of the structures of semi-vector spaces and semi-algebras is the possibility to generate new interesting results on the Fuzzy Set Theory.

Another work relating semi-vector spaces and Fuzzy Set Theory is the paper by Bedregal et al. [8]. In order to study the aggregation functions (geometric mean, weighted average, ordered weighted averaging, among others) w.r.t. an admissible order (a total order $\preceq$ on $L_n([0,1])$ such that for all $x, y \in L_n([0,1])$, $x \preceq^p y \implies x \preceq y$), the authors worked with semi-vector spaces over a weak semi-field.
Let $L_n([0,1]) = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n | x_1 \leq x_2 \leq \ldots \leq x_n\}$ and $U = ([0,1], \oplus, \cdot)$ be a weak semi-field defined as follows: for all $x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$ and $\cdot$ is the usual multiplication. The product order proposed by Shang et al. [15] is given as follows: for all $x = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n\}$ and $y = \{(y_1, y_2, \ldots, y_n) \in [0,1]^n\}$ vectors in $L_n([0,1])$, define $x \leq^p_n y \iff \pi_i(x) \leq \pi_i(y)$ for each $i \in \{1, 2, \ldots, n\}$, where $\pi_i : L_n([0,1]) \rightarrow [0,1]$ is the $i$-th projection $\pi_i(x_1, x_2, \ldots, x_n) = x_i$. With these concepts in mind, the authors showed two important results:

**Theorem 4.1** (see [8, Theorem 1]) $L_n([0,1]) = (L_n([0,1], +, \cdot))$ is a semi-vector space over $U$, where $r \cdot v = (rx_1, \ldots, rx_n)$ and $u + v = (x_1 + y_1, \ldots, x_n + y_n)$. Moreover, $(L_n([0,1]), \leq^p_n)$ is an ordered semi-vector space over $U$, where $\leq^p_n$ is the product order.

**Proposition 4.2** (see [8, Proposition 2]) For any bijection $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$, the pair $(L_n([0,1]), \preceq^f)$ is an ordered semi-vector space over $U$, where $\preceq^f$, defined in [8, Example 1], is an admissible order.

As a consequence of the investigation made, the authors propose an algorithm to perform a multi-criteria and multi-expert decision making method.

Summarizing the ideas: the better the theory of semi-vector spaces is extended and developed, the more applications and more results we will have in the Fuzzy Set Theory. Therefore, it is important to understand deeply which are the algebraic and geometry structures of semi-vector spaces, providing, in this way, support for the development of the own theory as well as other interesting theories as, for example, the Fuzzy Set Theory.

## 5 Summary

In this paper we have extended the theory of semi-vector spaces, where the semi-field of scalars considered here is the nonnegative real numbers. We have proved several results in the context of semi-vector spaces and semi-linear transformations. We introduced the concept of eigenvalues and eigenvectors of a semi-linear operator and of a matrix and shown how to compute it in specific cases. Topological properties of semi-vector spaces such as completeness and separability were also investigated. We have exhibited interesting new families of semi-vector spaces derived from semi-metric, semi-norm, semi-inner product, among others. Additionally, some results concerning semi-algebras were presented. The results presented in this paper can be possibly utilized in the development and/or investigation of new properties of fuzzy systems and also in the study of correlated areas of research.

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## References

[1] D. Canarutto. Positive spaces, generalized semi-densities, and quantum interaction. *Journal of Mathematical Physics*, 53:032302, 2012.

22
[2] D. Canarutto. Special generalized densities and propagators: a geometric account. *International Journal of Geometric Methods in Modern Physics*, 13(01):1530004, 2016.

[3] P. Corazza. Introduction to metric-preserving functions. *American Math. Monthly*, 106(4):309–323, 1999.

[4] W. Gähler, S. Gähler. Contributions to fuzzy analysis. *Fuzzy sets and systems*, 105:201–204, 1999.

[5] J. Janyska, M. Modugno, R. Vitolo. Semi-vector spaces and units of measurement. *Acta. Appl. Math.* 110:1249–1276, 2010.

[6] W.B.V. Kandasamy. *Smarandache semirings, semifields, and semivector spaces*. American Research Press, Rehoboth, NM, 2002.

[7] E. Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley and Sons. Inc., 1978.

[8] T. Milfont, I. Mezzomo, B. Bedregal, E. Mansilla, H. Bustince. Aggregation functions on n-dimensional ordered vectors equipped with an admissible order and an application in multi-criteria group decision-making. *Intern. J. Approx. Reasoning*, 137:34–50, 2021.

[9] J-P. Olivier, D. Serrato. Initial objects, universal objects for squares, equivalences and congruences in relation semi-algebras and algebras. *Math. Log. Quart.*, 41:455–475, 1995.

[10] E. Pap. Integration of functions with values in complete semi-vector space. *Measure theory, Oberwolfach 1979, Lecture Notes in Mathematics*, 794:340–347, 1980.

[11] P. Prakash, M.R. Sertel. Topological semivector spaces, convexity and fixed point theory. *Semi-group Forum*, 9:117–138, 1974.

[12] P. Prakash, M.R. Sertel. Hyperspaces of topological vector spaces: their embedding in topological vector spaces. In *Proceedings of the AMS*, 61(1):163–168, 1976.

[13] H. Radstrom. An embedding theorem for spaces of convex sets. In *Proc. Amer. Math. Soc.* 3:165–169, 1952.

[14] A.K. Roy. Ideals in semi-algebras of continuous, monotone functions on a compact ordered space. *Math. Ann.*, 185:231–246, 1970.

[15] Y. Shang, X. Yuan, E.S. Lee. The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets. *Computers and Mathematics with Applications*, 60:442–463, 2010.

[16] L.A. Zadeh. Fuzzy sets. *Information and Control*, 338–353, 1965.