Complex structure and solutions of classical nonlinear equation with the interaction $u_4^4$

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Abstract

We consider the (real) nonlinear wave equation

$$\Box u + m^2 u + \lambda u^3 = 0, \quad m > 0, \quad \lambda > 0,$$

on four-dimensional Minkowski space. Let $U(t)$, $W$, and $S$ be the (nonlinear) operator of dynamics and, respectively, the (nonlinear) wave and scattering operators for this nonlinear wave equation. We introduce the complex structure and show that the operators $U(t)$, $W$, and $S$ define complex analytic maps on the space of initial Cauchy data with finite energy. In other words, let $R(\varphi, \pi) = \varphi + i\mu^{-1}\pi$ be the map of initial data on the positive frequency part of the solution of the free Klein-Gordon equation with these initial data. The operators $RU(t)R^{-1}$, $RW R^{-1}$, and $RSR^{-1}$ are defined correctly and are complex analytic on the complex Hilbert space $H^1(\mathbb{R}^3, \mathbb{C})$. In particular, for $z(\alpha) = \sum_{1 \leq n \leq N} \alpha_n z_n$, $z_n \in H^1(\mathbb{R}^3, \mathbb{C})$, $\alpha_n \in \mathbb{C}$, $\langle RU(t)R^{-1}z(\alpha), h \rangle$, $\langle RW R^{-1}z(\alpha), h \rangle$, and $\langle RSR^{-1}z(\alpha), h \rangle$ are entire antiholomorphic functions in $\alpha_n$, $1 \leq n \leq N$.  

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1This paper is the first part of the project $\phi_4^4 \cap M$. 
1. Introduction

We consider the classic nonlinear wave equation

$$\Box u + m^2 u + \lambda u^3 = 0, \ m > 0, \ \lambda > 0$$

(1.1)

in four-dimensional Minkowski space-time. We introduce the (global) complex structure and prove that the wave and scattering (nonlinear) operators for this equation are complex analytic maps on the space of finite energy Cauchy data. This means that the wave and scattering operators define the complex analytic maps of positive frequency parts given by initial data with finite energy.

This result allows us to consider the quantum field with the help of its Wick kernel and to construct a bilinear form which is the solution of nonlinear quantum wave equation with the cubic nonlinearity in four-dimensional space-time \([12, 2, 3, 4, 5]\). This bilinear form-solution is defined in the Fock space of the free in-field on the subspace, generated by linear combinations of coherent vectors corresponding to coherent in-states with finite energy.

To consider vacuum averages and the integral of the type of Paneitz, Pedersen, Segal, Zhou \([1]\) we need namely this holomorphity of Wick kernels.

The complex structure for the solutions of nonlinear equations is considered as follows. We introduce the indexation of solutions of free equations with the help of complex variable. This complex variable is the positive frequency part of a free real solution considered at a fixed time (for instance, at time zero). The unique (real) free solution (with finite energy) can be defined uniquely by its positive frequency part \(\varphi^+\), as well as by its initial data, (real) canonical coordinate and (real) canonical momentum.

In this case the map

$$\varphi^+ := \varphi + i \mu^{-1} \pi = R(\varphi, \pi), \quad R^{-1} \varphi^+ = (\text{Re} \ \varphi^+, \mu \text{Im} \ \varphi^+),$$

where \((\varphi, \pi) \in H^1(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3), \ \varphi^+ \in H^1(\mathbb{R}^3, \mathbb{C}), \ \mu = (-\Delta + m^2)^{1/2}\), defines the one-to-one correspondence between these variables. Thus, we have a possibility to index solutions of nonlinear equation with the help of complex variables.

The main consequence of our consideration is the following principal result. The dynamics of the non-linear equation is given by (infinite-dimensional) holomorphic maps. Namely, we construct the complex structure for the nonlinear equation (1.1) and we prove that the nonlinear dynamics, wave and scattering operators generate holomorphic maps of positive frequency parts defined by solutions.

The proof of holomorphity of these maps is closely connected with unitarity of corresponding operators. (these operators appear when we consider corresponding quantum objects). To proof holomorphity of these maps we consider the complex structure for small initial data, i.e. for initial data from a neighborhood of zero in the space \(H^1 \oplus L_2\). It means that we consider solutions with small energy. At this point, it is very important the stability of solutions. This stability of solutions gives (for sufficiently small initial data) the uniqueness of essential unitarizability (of the derivative) of the wave operator and of the scattering operator.

The stability is closely related with the fact that the equation corresponds physically to a massive (scalar) field and for small energies (less than the double mass constant) particles
cannot be created. In addition, this nonlinear equation has no solutions interpreted as bounded states.

Then, the consideration of solutions smoothed with appropriate functions (= functions with compact support in momentum space) allows to extend the complex analyticity on large initial data. Moreover, this complex analyticity due to the real analyticity over (real) initial data gives the values that are equal to the values given by solutions with large initial data.

This global complex structure gives entire holomorphic functions on finite-dimensional subspaces. With the help of uniform convergence we can easily extend this complex structure on larger class of smoothing functions and on the all solutions with initial data with finite energy. The (anti)holomorphity of functions \( \langle RWR^{-1}(z_{\text{in}}(\alpha)), h \rangle \) on finite-dimensional subspaces and integrals over these subspaces (see Paneitz, Pedersen, Segal, Zhou) are very important for the construction of the quantum field. This holomorphity with the help of Wick kernels allows to construct the quantum field and its vacuum averages (see Heifets, Osipov).

Thus, we prove the following theorem.

**Theorem 1.1.** Let \( R(\varphi, \pi) = \varphi + i\mu^{-1}\pi \) be the isomorphism \( (H^{1/2} \oplus H^{-1/2}, J) \) onto \( H^{1/2}(\mathbb{R}^3, \mathbb{C}) \). Let \( W \) be the wave operator for the nonlinear wave equation (1.1). Then the map \( RWR^{-1} \) is defined correctly as a map of \( D(\mu^{1/2}) \) on \( D(\mu^{1/2}) \) and is a complex analytic map of the complex Hilbert space \( D(\mu^{1/2}) \simeq H^1(\mathbb{R}^3, \mathbb{C}) \) onto itself. In particular, for \( z_{\text{in}}(\alpha) = \sum_{j=1}^N \alpha_j z_{\text{in}, j}, z_{\text{in}, j} \in D(\mu^{1/2}), \alpha_j \in \mathbb{C} \) (i.e. \( z_{\text{in}, j} \in H^1(\mathbb{R}^3, \mathbb{C}) \)), \( h \in H^{1/2}(\mathbb{R}^3, \mathbb{C}) \), the functions \( \langle RWR^{-1}z_{\text{in}}(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathbb{C})} \) are entire antiholomorphic functions on \( (\alpha_1, ..., \alpha_N) \in \mathbb{C}^N \).

The same assertion is valid for the map \( RSR^{-1} \), where \( S \) is the scattering operator of the nonlinear wave equation (1.1).

**Remark.** We emphasize that variables \( (\varphi, \pi) \), \( \varphi^+ = \varphi + i\mu^{-1}\pi \) should be considered as vectors from the spaces \( (H^{1/2} \oplus H^{-1/2}, J), H^{1/2}(\mathbb{R}^3, \mathbb{C}) \), respectively. The complex structure corresponding to the symplectic 2-form of the nonlinear wave equation acts in the complex Hilbert space \( (H^{1/2} \oplus H^{-1/2}, J) \), namely. The vector \( \varphi^+ \) from \( H^{1/2}(\mathbb{R}^3, \mathbb{C}) \) defines the (free real) solution \( \exp(i\mu t)\varphi^+ + \exp(-i\mu t)\varphi^- \). In momentum space this vector \( \varphi^+ \) is defined by the function \( \mu(k)(\varphi^+)(k) \). Here \( \cdot(k) \) is the Fourier transformation and the function \( \mu(k)(\varphi^+)(k) \) is a square integrable function over the measure \( \vartheta(k_0)\delta(k^2 - m^2)d^4k \) (in momentum space).

At the same time the solution of the nonlinear equation have the initial data from \( H^1 \oplus L_2 \) and the natural map

\[
R(\varphi, \pi) = \varphi + i\mu^{-1}\pi
\]

is the isomorphism \( H^1 \oplus L_2 \rightarrow H^1(\mathbb{R}^3, \mathbb{C}) \). Note that the space \( H^1(\mathbb{R}^3, \mathbb{C}) \) is the domain \( D(\mu^{1/2}) \) of definition of the operator \( \mu^{1/2} \) in the space \( H^{1/2}(\mathbb{R}^3, \mathbb{C}) \). This natural map appears as the change of initial data on positive frequency part of the same solution (or as the change of a pair of real variables on the unique complex variable).

Therefore, a unique solution is represented uniquely by the (unique) pair of initial data \( (\varphi, \pi) \), by the pair of real variables \( (\varphi, \mu^{-1}\pi) \), by the positive frequency part of the solution \( \varphi^+(t, x) \), by the complex variable

\[
\varphi^+(0, x) = \varphi(x) + i(\mu^{-1}\pi)(x),
\]
and by the square integrable (over the measure $d^3k/\mu(k)$) complex function $\mu\varphi^\sim(k) + i\pi^\sim(k)$. The last complex-valued function is used in the isomorphism $H^1 \oplus L_2 \rightarrow L_2(\mathbb{R}^3, \mathcal{C})$ defined by Baez, Zhou [3, 7]. This isomorphism gives the complex variable, corresponding to the solution (in this case this variable is $\varphi(x) + i\mu^{-1}\pi(x)$ in spatial coordinates and $\mu(k)\varphi^\sim(k) + i\pi^\sim(k)$ in momentum coordinates).

In addition, the transformation

$$(P(\varphi, \pi))^\sim = \mu\varphi^\sim(k) + i\pi^\sim(k)$$

is the projection on the massive hyperboloid by Goodman [8]. That is, by definition

$$\int v^+(t, x) f(t, x) dtdx = \int \mu^{1/2}v^+(0, x)\mu^{1/2}\mu^{-1} \int e^{i\mu t} f(t, x)dtd^3x$$

$$= \int v^+(0, k) e^{i\mu t}(F_3 f)(t, k)dtd^3k = \int v^+(0, k) f^\sim(\mu, k)d^3k$$

$$= \int \mu v^+(0, k) f^\sim(\mu, k)\frac{d^3k}{\mu} = \langle v, Pf\rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})},$$

$$(Pf)(x) = \mu^{-1} \int e^{i\mu t} f(t, x)dt$$

and

$$(Pf)^\sim(k) = \int e^{i\mu t} e^{ikx} f(t, x) dtd^3x.$$ 

Thus, if a solution $u$ has the initial data $(\varphi, \pi)$, then its positive frequency part is $\varphi^+$, $\varphi^+(x) = \varphi(x) + i(\mu^{-1}\pi)(x)$, and the projection on the massive hyperboloid in momentum space is equal to

$$(P(u))^\sim(k) = e^{i\mu(k)t}(\mu(k)\varphi^\sim(k) + i\pi^\sim(k)).$$

In this case the solution itself is equal to

$$u(t, x) = (\cos \mu t \varphi)(x) + \frac{\sin \mu t}{\mu}(\pi)(x)$$

$$= (\exp i\mu t \varphi^+)(x) + (\exp i\mu t \varphi^+)(x)$$

$$= (\cos \mu t \text{ Re}(\varphi^+))(x) + (\sin \mu t \text{ Im}(\varphi^+))(x)$$

$$= \int e^{i\mu t} e^{ikx} P(u)^\sim(k)\frac{d^3k}{2\mu(k)}$$

$$= \int e^{ik\omega t + ikx} P(u)^\sim(k_0, k)\delta(k_0)\delta(k^2 - m^2)d^4k.$$ 

Therefore, the isomorphism introduced by Baez, Zhou [3, 7], and Osipov [5] agrees with the isomorphism introduced in our paper. In this case an isomorphism maps the same solution in the different spaces with the help of which we index solutions: that is the space $H^1 \oplus L_2$ for initial data given by vector $(\varphi(x), \pi(x))$ in coordinate space, or $(\varphi^\sim(k), \pi^\sim(k))$ in momentum space, or the space $H^1(\mathbb{R}^3, \mathcal{C})$ for positive frequency part $\varphi^+$, $\varphi^+ = \varphi + i\mu^{-1}\pi$, of solutions with finite energy, or the Baez-Zhou isomorphism [3, 7], or the projection by Goodman [8] on the massive hyperboloid given by the vector $\mu\varphi^\sim(k) + i\pi^\sim(k)$ in momentum space with the measure $\delta(k_0)\delta(k^2 - m^2)d^4k$ (= a complex valued square integrable function in momentum space with the measure $d^3k/2\mu(k)$).
2. Global complex structure. Ideas of the proof of Theorem 1.1

In this section we outline ideas and principles of the proof of Theorem 1.1. The proof of Theorem 1.1 can be reduced to the proofs of theorems that are given in the next sections. We complete Section 2 by the deduction of Theorem 1.1 from Theorem 5.1.

First of all, we note that the isomorphism $R$ maps the space of initial data $H^1 \oplus L_2$ of solutions with finite energy into the space $H^1(\mathbb{R}^3, \mathcal{C})$ of positive frequency parts of these solutions. The complex Hilbert space $H^1(\mathbb{R}^3, \mathcal{C})$ is identified with the domain $D(\mu^{1/2})$ of definition of the operator $\mu^{1/2}$ with topology given by the norm

$$\|\mu^{1/2}(\cdot)\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}.$$ 

$R^{-1}$ is correctly defined on $D(\mu^{1/2})$ and

$$R^{-1}D(\mu^{1/2}) = H^1 \oplus L_2 \simeq L^1_2 \oplus L_2.$$ 

If $H_1, H_2$ are complex Hilbert spaces and $F$ is the map of $H_1$ into $H_2$, then to prove the complex holomorphy of the map $F$ it is sufficient to proof that for every $x, h \in H_1$, $y \in H^*_2$ ($H^*_2$ the (complex) dual to $H_2$, $H^*_2 \simeq H_2$) any function $(y, F(x + \alpha h))_{H_2}$ is holomorphic in $\alpha \in \mathcal{C}$ for sufficiently small $|\alpha|$, see [9, ch. 2.3, p.84-85].

The basic idea of the proof of Theorem 1.1 is the following.

The first step is the proof of holomorphy of maps $RWR^{-1}$ and $RSR^{-1}$ in a neighborhood of small initial data with finite energy (see Section 3). Since a real analyticity is proved by Kumlin [10], see also Appendix and [11, 12, 13, 14, 15], so to prove the complex holomorphy we need to prove commutativity of derivatives of wave operator and of scattering operator with the operator of imaginary unite. To obtain this result, we prove for our purpose the uniqueness of essential unitarizability for the first derivatives of the wave operator and of the scattering operator in some small zero neighborhood of (good) initial data (Theorem 3.1). For this purpose we use Krein and coauthors ideas [16] and results of Paneitz and Segal [17, 18, 19, 20]. These ideas and results give a possibility to use a stability of nonlinear solutions and prove the uniqueness of essential unitarizability for the first derivative of wave equation and for the first derivative of scattering operator. As good initial data we use the following initial $in$–data $u_{in}$. The initial $in$–data is such that $u_{in}$ and $Ju_{in}$ (where $J$ is the operator of imaginary unit, $J = R^{-1}iR$) belong to the intersection (of finite number) of some Hilbert and Banach spaces (and are dense in these spaces). These, in general, various spaces of initial data were considered and used by Morawetz, Strauss [12, 13], Segal [21, 22], Paneitz [17]. The intersection of these spaces contains the Schwartz space $S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3)$. The treatment of finite-dimensional zero neighborhood of this space is sufficient for our constructions.

This result about the unitarizability yields us the commutativity of operator of imaginary unit with the derivative of the wave operator and with the derivative of scattering operator of the nonlinear equation depending on small initial $in$–data.

This commutativity with the operator of imaginary unit allows us to obtain the complex holomorphy on a zero neighborhood of some finite-dimensional subspace of initial $in$-data. This set of finite-dimensional subspace is dense in the mentioned above (finite) intersection of Banach spaces of initial $in$–data. Instead of the mentioned above (finite) intersection of Banach spaces of initial $in$–data we may choose and we choose the Schwartz
space \( S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3) \). The \( C^\infty \) Frechét differentiability of nonlinear solution and the complex holomorphity on finite-dimensional zero neighborhoods allow us to extend the commutativity of imaginary unit on all derivatives of wave operator and scattering operator at zero.

This result and the real analyticity yield us the complex analyticity for initial \( in \)–data in a zero neighborhood in \( H^1(\mathbb{R}^3, \mathcal{C}) \), that is, for \( u_{in}^+ \in H^1(\mathbb{R}^3, \mathcal{C}), \| u_{in}^+ \|_{H^1(\mathbb{R}^3, \mathcal{C})} < \vartheta \) for some \( \vartheta > 0 \) (Theorem 3.2). Here \( u_{in}^+ \) is the positive frequency part of free solution (with corresponding initial data).

In order to extend this holomorphity on large initial data we consider the functions that are the smoothed (positive frequency part of nonlinear) solution. These smoothed solutions are smoothed over temporal and spatial coordinates with appropriate (rapidly decreasing in coordinate space) test functions with compact support in momentum space.

Due to a (strict) positivity of mass constant, real analyticity (in the whole space), and complex holomorphity in a zero neighborhood the smoothed solution coincides with a smoothed polynomial on initial \( in \)–data (Theorem 4.1).

The degree of this polynomial is bounded by the size of test function support (in momentum space) and is less than

\[
\sup\{[p_0/\text{mass constant}] + 1 \mid (p_0, p) \in \text{supp } f^\sim\}.
\]

Thus, we have the smoothed solutions and the smoothed polynomials. The smoothed solutions are real analytic functions and the smoothed polynomial are holomorphic functions on the whole space of initial data with finite energy. These considered smoothed functions coincide on some zero neighborhood. Thus, the uniqueness (Lemma 5.2), the real analyticity of smoothed functions, the complex holomorphity of smoothed polynomials give the coincidence on the whole space \( H^1(\mathbb{R}^3, \mathcal{C}) \). In other words, the smoothed solutions are complex analytic on the whole space of initial data with finite energy.

Uniform continuity allows to extend the holomorphity on more wide class of test functions. This class of test functions contains (smooth) functions with compact support (in space-time coordinates) too (Theorem 5.1). The equation of motion allows also to extend the holomorphity (of smoothed solutions) on test functions of the form \( \delta(t)f(x), f \in \mathcal{S}(\mathbb{R}^3) \). This allows to complete the proof of Theorem 1.1.

Using outlined ideas we give the detailed proof of main statements in Sections 3-5 of the paper. These main statements are Theorems 3.1, 3.2, 4.1, 5.1.

In addition, in Sections 6, 7 we give more detailed consideration of the assertions used by Paneitz and Segal [20, 17] for the proof of a uniqueness of essential unitarizability of operators \( dW(u_{in}) \) and \( dS(u_{in}) \) for small initial data. These assertions are Theorems 6.1-6.3, Corollary 6.4, and Theorem 7.1.

In Appendix we give also the improvement of Kumlin proof [10] of real analyticity for solutions of the nonlinear equation.

To complete this Section we deduce Theorem 1.1 as a consequence of Theorem 5.1.

**Proof of Theorem 1.1.** We derive Theorem 1.1 from Theorem 5.1. Theorem 5.1 implies that the function

\[
\int \langle RWU_0(t)R^{-1}(u_{in}^+(\alpha)), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} \chi(t) dt,
\]

\( h \in \mathcal{S}(\mathbb{R}^3, \mathcal{C}), \chi \in L^1(\mathbb{R}), u_{in}^+(\alpha) = \sum_{j=1}^{N} \alpha_j u_{in,j}^+, u_{in,j}^+ \in H^1(\mathbb{R}^3, \mathcal{C}), \) is an entire anti-holomorphic function in \( \alpha \in \mathcal{C}^N \).
Let $\chi \in \mathcal{S}_{Re}$, $\text{supp} \chi \subset [-1,1]$, $\chi(t) \geq 0$, $\int \chi(t)dt = 1$, $\chi_\sigma(t) = \sigma \chi(\sigma t)$. For $\sigma \to \infty$ $\chi_\sigma(t)$ converges to the $\delta$–function (in the sense of generalized functions). It is clear that for $\sigma \to \infty$

\[
\int \langle R W U_0(t) R^{-1}(u_{in}^+(\alpha)), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} \chi_\sigma(t)dt = \int \langle u^+(t), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} \chi_\sigma(t)dt
\]

converges to $\langle R W R^{-1}(u_{in}^+(\alpha)), h \rangle = \langle u^+(0), h \rangle \equiv \langle u^+, h \rangle$.

To obtain the uniform bound, we use the equality $f(1) - f(0) = \int_0^1 ds \frac{df}{ds}(s)$, and write

\[
\int dt \chi_\sigma(t) \langle u^+(t), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} - \langle u^+, h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} = \int dt \chi_\sigma(t) \langle (u^+(t), h)_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} - \langle u^+, h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} \rangle = \int dt \chi_\sigma(t) t \int_0^1 ds \langle \dot{u}^+(st), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}.\]

The equation of motion (in the sense of generalized functions in spatial coordinates) implies that

\[
\begin{align*}
\dot{u}^+(t) &= u(t) + i \mu^{-1} \dot{u}(t) \\
\ddot{u}^+(t) &= \dot{u}(t) + i \mu u(t) + i \mu^{-1} \lambda u^3(t).
\end{align*}
\]

The Sobolev inequality and the energy conservation imply that

\[
\| \dot{u}^+(t) \|_{L^2(\mathbb{R}^3, \mathcal{L})} \leq c_1 (H(u(t)))^{1/2} + (u(t))^{3/2} = c_2 (H_0(u_{in}(\alpha)))^{1/2} + (u_{in}(\alpha))^{3/2} \leq c(u_{in,1}, ..., u_{in,N})(|\alpha| + |\alpha|^{3/2}).
\]

Here $H(u)$ is the total energy of solution $u$ and $H_0(u_{in})$ is the free energy of the (free) solution $u_{in}$. Thus, we obtain the estimate

\[
| \int dt \chi_\sigma(t) t \int ds \langle \dot{u}^+(st), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} | \leq \sigma^{-1} \sup_{|t| \leq 1/\sigma} | \langle \dot{u}^+(t), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} | \leq \sigma^{-1} c(u_{in,1}, ..., u_{in,N})(|\alpha| + |\alpha|^{3/2}) \| h \|_{H^1(\mathbb{R}^3, \mathcal{L})}
\]

and the uniform convergence

\[
\int dt \chi(t) \langle R W U_0(t) R^{-1} u_{in}^+(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} \to \langle R W R^{-1} u_{in}^+(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}.
\]

Therefore, the function $\langle R W R^{-1} u_{in}^+(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}$, $h \in \mathcal{S}(\mathbb{R}^3, \mathcal{C})$, is antiholomorphic in $\alpha$. If now $h \in H^{1/2}(\mathbb{R}^3, \mathcal{C})$, $h_n \subset \mathcal{S}(\mathbb{R}^3, \mathcal{C})$ and $h_n \to h$ in $H^{1/2}(\mathbb{R}^3, \mathcal{C})$, then

\[
\langle R W R^{-1} u_{in}^+(\alpha), h_n \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}
\]

converges uniformly to

\[
\langle R W R^{-1} u_{in}^+(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}.
\]

This is a consequence of the uniform estimate

\[
| \langle R W R^{-1} u_{in}^+(\alpha), h_n \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} - \langle R W R^{-1} u_{in}^+(\alpha), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} |
\]
Moreover, the following uniform estimate is valid

\[
\|WR^{-1}u^+(\alpha)\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} \|h_n - h\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} \\
\leq c\|u^+_n(\alpha)\|_{H^1(\mathbb{R}^3, \mathcal{C})} \|h_n - h\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} \\
\leq c(u_{in,1}, \ldots, u_{in,N}) |\alpha| \|h_n - h\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}.
\]

This implies that the function \(D_\alpha\) is defined correctly and is a holomorphic function in particular, for \(h = \delta(x_1)h(x_2, x_3)\) also.

Hence, these functions are complex analytic on finite-dimensional subspaces and the map \(WR^{-1}\) is complex holomorphic as a map from \(D(\mu^{1/2})\) into \(D(\mu^{1/2})\). In particular, the function \(\mu^{1/2}WR^{-1}u^+_n(\alpha)\) is holomorphic.

This means also that if we write the smoothness as an integral (with the dualization in \(L_2(\mathbb{R}^3, \mathcal{C})\), or in \(L_2(\mathbb{R}^4, \mathcal{C})\)), then

\[
\int u^+(t, x)h(x)dx = \int WR^{-1}(u^+_n(\alpha))(x)h(x)dx
\]

is defined correctly and is a holomorphic function in \(\alpha\) for all \(h \in H^{-1}(\mathbb{R}^3, \mathcal{C})\), that is, in particular, for \(h = \delta(x_1)h(x_2, x_3)\) also.

Hence, these functions are complex analytic on finite-dimensional subspaces and the map \(WR^{-1}\) is complex holomorphic as a map from \(D(\mu^{1/2})\) on \(D(\mu^{1/2})\) in \(H^{1/2}(\mathbb{R}^3, \mathcal{C})\).

The scattering operator \(S\) can be considered analogously. **Theorem 1.1 is proved.**

**Remark.** The essential point (the fine tuning of considered spaces!) is that we need to consider the holomorphy in the space \(H^{1/2}(\mathbb{R}^3, \mathcal{C})\) namely. This follows from symplecticity (and Kählerian structure) of the space with respect to the given symplectic 2-form. The symplecticity, the existence (and the uniqueness) of complex structure and of operator of imaginary unit (with requirements of Poincare-invariance) have a consequence, that the symplecticity implies the equalities

\[
U^+ JU = J
\]

and if \((U^+)^{-1} = U\) (unitarity), then

\[
JU = UJ.
\]

3. Complex structure and complex analyticity in a zero neighborhood

In this Section we consider the complex structure and complex analyticity for solutions of the nonlinear equation with small energy, i.e. for the initial Cauchy in–data from a zero neighborhood in \(H^1 \oplus L_2\).

We introduce some notation that we need. Let \(W, S\) be the nonlinear wave operator and the nonlinear scattering operator, respectively, for the nonlinear wave equation 1.1. The nonlinear operators \(W, S\) are \(C^\infty\) Frechét differentiable invertible maps on the (real) Hilbert space \(H^1 \oplus L_2\) and these maps are \(C^\infty\) symplectomorphisms, see, for instance [7] and references therein.

Let \(\mu\) be the operator, \(\mu := (-\Delta + m^2)^{1/2}\), where \(\Delta\) is the Laplacian and \(m, m > 0\), is a mass constant from Eq. (1.1).
Let $R$ be the operator of isomorphism. This operator maps a pair of real-valued functions (on $\mathbb{R}^3$) onto the complex-valued function,

$$R(\varphi, \pi) = \varphi + i\mu^{-1}\pi := \varphi^+, \quad R^{-1}\varphi^+ = (\Re \varphi^+, \mu \Im \varphi^+).$$

Let $J = R^{-1}iR$. Then $J$ is the (orthogonal) operator of imaginary unit,

$$J(\varphi, \pi) = (-\mu^{-1}\pi, \mu \varphi), \quad J^2 = -I.$$

In particular, $R$ defines a real linear isomorphism of the space $H^{1/2} \oplus H^{-1/2}$ on $H^{1/2}(\mathbb{R}^3, \mathcal{C})$ and the complex linear isomorphism $(H^{1/2} \oplus H^{-1/2}, J)$ on $H^{1/2}(\mathbb{R}^3, \mathcal{C})$. The inner product in $(H^{1/2} \oplus H^{-1/2}, J)$ is given by the expression

$$\langle (\varphi_1, \pi_1), (\varphi_2, \pi_2) \rangle_{(H^{1/2} \oplus H^{-1/2}, J)} = \omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) + i\omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)),$$

where $\omega$, the imaginary part of the inner product, is the symplectic 2-form,

$$\omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \int d^3x (\varphi_1(x)\pi_2(x) - \pi_1(x)\varphi_2(x)).$$

This choice corresponds to the choice of the inner product in the form

$$\langle (\varphi_1, \pi_1), (\varphi_2, \pi_2) \rangle_{(H^{1/2} \oplus H^{-1/2}, J)} = \langle \varphi_1^+, \varphi_2^+ \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} = \int d^3x (\mu^{1/2}\varphi_1^+)(x)(\mu^{1/2}\varphi_2^+)(x),$$

i.e. the inner product is antilinear in the first argument.

By $d^nF$ we denote Frechét derivatives of the transformation $F$ of a Banach space into another Banach space.

**Theorem 3.1.** Let $u_{in}(\alpha) = \sum_{j=1}^N \alpha_j u_{in,j}$, $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{C}^N$, $u_{in,j} \in \mathcal{S}_{Re}(\mathbb{R}^3) \oplus \mathcal{S}_{Re}(\mathbb{R}^3)$. Let $u(\alpha)$ be the solution of the nonlinear wave equation (1.1) with the initial in-data $u_{in}(\alpha)$.

There exists a $\vartheta > 0$ (depending on $u_{in,j}$, $j = 1, ..., N$, such that for $|\alpha| < \vartheta$ the operators $dW(u_{in}(\alpha))$, $dS(u_{in}(\alpha))$ are correctly defined on $H^1 \oplus L_2$ and are bounded operators (on $H^1 \oplus L_2$). These bounded operators are uniquely essentially unitarizable in the complex Hilbert space $(H^{1/2} \oplus H^{-1/2}, J)$, and on $D(\mu^{1/2}) := (H^1 \oplus L_2, J) \subset (H^{1/2} \oplus H^{-1/2}, J)$

$$\langle dW(u_{in}(\alpha))v_{in,1}, dW(u_{in}(\alpha))v_{in,2} \rangle_{(H^{1/2} \oplus H^{-1/2}, J)} = \langle v_{in,1}, v_{in,2} \rangle_{(H^{1/2} \oplus H^{-1/2}, J)},$$

$$JdW(u_{in}(\alpha))v_{in} = dW(u_{in}(\alpha))Jv_{in},$$

$$\langle dS(u_{in}(\alpha))v_{in,1}, dS(u_{in}(\alpha))v_{in,2} \rangle_{(H^{1/2} \oplus H^{-1/2}, J)} = \langle v_{in,1}, v_{in,2} \rangle_{(H^{1/2} \oplus H^{-1/2}, J)},$$

$$JdS(u_{in}(\alpha))v_{in} = dS(u_{in}(\alpha))Jv_{in}.$$

By continuity these equalities for linear operators can be extended uniquely (as linear operator on the variable $v_{in}$) on the whole complex space $(H^{1/2} \oplus H^{-1/2}, J)$.

**Remarks.** 1. The assertion of Theorem 3.1 on the argument $u_{in}$ can be extended by continuity on the zero neighborhood of some Banach space of initial data, or on finite intersection of Banach space used by I. Segal [21, 22], Morawetz and Strauss [14, 17],
Heifets [12], Raczka and Strauss [13], Paneitz [17, 18, 19], Kumlin [10] for initial data and initial in-data. In particular, Paneitz [17], Paneitz and Segal [20] used some Banach space of initial data to prove an analogue of Theorem 3.1. It is obvious that the Schwartz space $S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3)$ is contained in and is dense in these Banach spaces, has a stronger topology, and is a nuclear space.

2. We remark that in Theorem 3.1 (and in the next ones) $\vartheta$ denotes a small strictly positive constant. This constant $\vartheta$ gives a choice of (some) neighborhood of the zero solution, this choice depends on the topology that we need take into account in the considered theorem. In general, these constants are various.

3. It is interesting to consider complex structure in many-dimensional space-time. The stability for small initial data (with a non-zero mass) is similar [20], and at the same time the consideration of complex structure opens new possibilities for global solutions with large initial data. But there is reason to think that the usual Schwartz space is not adequate to describe global solution with large initial data.

Proof of Theorem 3.1. It is clear that the Fréchet derivative of operators $W$, $S$ is defined correctly as a linear operator on the space $H^1 \oplus L^2$ and can be extended by continuity as a linear operator (depending on the point $u_{in}(\alpha)$) on more wide (Banach) space.

To prove Theorem 3.1 we apply Corollary 2.3 [20], Theorem 7 [18], Theorem 16.3 [19, Theorem 16.3], Corollary [17, Corollary, p.115], Theorem 6.3 [19] and verify the required conditions.

In the case of the operator $S$ it is convenient to use directly [17, Corollary, p.115], see also [20, Theorem 4], Theorem 1, 2 [17, pp.114-115], Theorem 6.3 [19]. In this case Corollary [17, Corollary, p.115] implies directly the assertion of Theorem 3.1 for the initial in-data from $S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3)$. In this case the conditions

$$(m^2 - \Delta)^{5/4} u_{in}(t, \cdot), \quad (m^2 - \Delta)^{3/4} \dot{u}_{in}(t, \cdot) \in L^1(\mathbb{R}^3)$$

for some $t$ are fulfilled.

The condition

$$\int_{-\infty}^{+\infty} \|u(t, x)\|_2^2 dt < 2m$$

is fulfilled for the choice $u_{in} \in \mathcal{F}, \|u_{in}\|_F < \vartheta$ for sufficiently small positive $\vartheta$, see Theorem 1(b) [14], here $\mathcal{F}$ is the space of initial data, defined by Morawetz and Strauss [14].

Thus, there exists a constant $\vartheta$, such that $u_{in}(\alpha), |\alpha| < \vartheta$, satisfy all conditions of Corollary 2.3 [20], Theorem 7 [18], Theorem 16.3 [19, Theorem 16.3], Corollary [17, Corollary, p.115] and Theorem 4.4B [17, p. 110] and [22, Corollary 4.4B, p. 491].

In the case of the wave operator $dW(u)$ we apply Theorem 7 [18], Theorem 6.3 [19, Theorem 6.3], and Theorems 1, 2 [17] for the interval $(-\infty, 0]$ (or for the interval $[0, \infty)$ in the case of the wave operator into the future, we may also use intervals $(-\infty, T]$ and $[T, \infty)$).

The proof of absence of “bounded states” was given by Paneitz [17, Corollary, p. 115-116], see also [20]. We give this proof in Section 7. Theorem 3.1 is proved.

We now go to the consideration of holomorphism.

Theorem 3.2. There exists a strict positive $\vartheta$ such that for $z \in D(\mu^{1/2})$ (in $H^{1/2}(\mathbb{R}^3, \mathcal{L})$),

$$\|z\|_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} < \vartheta$$
i.e., for
\[ \|\mu^{1/2}z\|_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} < \vartheta, \]
the transformations \( RW R^{-1}(z), RSR^{-1}(z) \) are holomorphic in \( z \) as transformations of open set \( \|z\|_{H^1(\mathbb{R}^3, \mathcal{L})} < \vartheta \) of the space \( H^1(\mathbb{R}^3, \mathcal{L}) \) into the space \( H^1(\mathbb{R}^3, \mathcal{L}) \). In particular,
\[
RW R^{-1}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n RW R^{-1}(0)(z, \ldots, z),
\]
\[
RSR^{-1}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n RSR^{-1}(0)(z, \ldots, z),
\]
the series converge uniformly for \( \|z\|_{H^1(\mathbb{R}^3, \mathcal{L})} < \vartheta \), and the functions
\[
\langle RW R^{-1}(z), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}, \quad \langle RSR^{-1}(z), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})},
\]
h \( \in H^{1/2}(\mathbb{R}^3, \mathcal{L}) \), are antiholomorphic in \( z \) for \( \|z\|_{H^1(\mathbb{R}^3, \mathcal{L})} < \vartheta \).

Remark. The value of \( \vartheta \) is defined by the radius of convergence of the Taylor series at zero in the real Hilbert space \( H^1 \oplus L_2 \) and its topology.

Proof of Theorem 3.2. First of all, we note that it is possible to write the equality
\[
\langle RW R^{-1} u^+_i, h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} = \langle Wu_i, R^{-1} h \rangle_{\mathcal{H}_S}.
\]
Here we denote \( \mathcal{H}_S := (H^{1/2} \oplus H^{-1/2}, J) \), \( u^+_i \) is the positive frequency part, corresponding to the initial data of \( u_i \).

\( W(u_i) \) is \( C^\infty \) Frechét differentiable function in \( H^1 \oplus L_2 \). Therefore the equality
\[
\frac{\partial}{\partial \alpha^{(0)}} - i \frac{\partial}{\partial \alpha^{(1)}} (W(u_i + \alpha^{(0)} v_i + J \alpha^{(1)} v_i), R^{-1} h)_{\mathcal{H}_S}
\]
\[
= \langle dW(u_i + \alpha^{(0)} v_i + J \alpha^{(1)} v_i), v_i, R^{-1} h \rangle_{\mathcal{H}_S}
\]
\[
- i \langle dW(u_i + \alpha^{(0)} v_i + J \alpha^{(1)} v_i), J v_i, R^{-1} h \rangle_{\mathcal{H}_S}
\]
\[
= \langle dW(u_i + \alpha^{(0)} v_i + \alpha^{(1)} J v_i), v_i, R^{-1} h \rangle_{\mathcal{H}_S}
\]
\[
+ \langle J dW(u_i + \alpha^{(0)} v_i + \alpha^{(1)} J v_i), J v_i, R^{-1} h \rangle_{\mathcal{H}_S}
\]
is defined correctly for \( u_i, v_i \in H^1 \oplus L_2 \), \( \alpha^{(0)}, \alpha^{(1)} \in \mathbb{R} \). The transformation \( W \) is real analytic in the Hilbert space \( H^1 \oplus L_2 \) and the transformation \( RW R^{-1} \) is real analytic in the Hilbert space \( H^1(\mathbb{R}^3, \mathcal{L}) \) (see \([10]\), and also \([12, 13, 11, 3, 5]\)). Let \( u_i(\alpha) = \sum_{j=1}^{N} (\alpha_j^{(0)} + \alpha_j^{(1)} J) v_{i,j} \) (\( = R^{-1} u^+_i(\alpha) \)), \( v_{i,j} \in S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3) \), \( \alpha_j^{(0)}, \alpha_j^{(1)} \in \mathbb{R} \). Theorem 3.1 implies the existence of \( \vartheta(u_i, N) > 0 \) such that for \( |\alpha| < \vartheta(u_i, N) \)
\[
\langle dW(u_i(\alpha)), J v_i, R^{-1} h \rangle_{\mathcal{H}_S} = \langle J dW(u_i(\alpha)) v_i, R^{-1} h \rangle_{\mathcal{H}_S}
\]
\[
= -i \langle dW(u_i(\alpha)) v_i, R^{-1} h \rangle_{\mathcal{H}_S}.
\]
Therefore,
\[
\left( \frac{\partial}{\partial \alpha^{(0)}} - i \frac{\partial}{\partial \alpha^{(1)}} \right) (W(u_i(\alpha)), R^{-1} h)_{\mathcal{H}_S} = 0.
\]
and the function $\langle W(u_{in}(\alpha)), R^{-1}h \rangle$ is antiholomorphic. The holomorphy and Cauchy-Riemann conditions imply that $(k_j = 0, 1)$

$$
\langle d^n W(u_{in}(\alpha))(J^{k_1}v_{in,1}, ..., J^{k_n}v_{in,n}), R^{-1}h \rangle_{\mathcal{H}_S} = \langle \prod_{j=1}^{n} \frac{\partial^{k_j}}{\partial \alpha_j^{(k_j)}} W(u_{in}(\alpha)), R^{-1}h \rangle_{\mathcal{H}_S} = \langle J^{k_1+...+k_n} d^n W(u_{in}(\alpha))(v_{in,1}, ..., v_{in,n}), R^{-1}h \rangle_{\mathcal{H}_S} = \langle \prod_{j=1}^{n} (-i)^{k_j} \frac{\partial^{k_j}}{\partial \alpha_j^{(k_j)}} W(u_{in}(\alpha)), R^{-1}h \rangle_{\mathcal{H}_S}.
$$

Since $W \in C^\infty(H^1 \oplus L_2, H^1 \oplus L_2)$, so $d^n W(0)$ is a $n$-linear symmetric continuous bounded operator. The density of $S_{Re} \oplus S_{Re}$ in $H^1 \oplus L_2$ allows to extend the equality

$$
d^n W(0)(J^{k_1}v_{in,1}, ..., J^{k_n}v_{in,n}) = J^{k_1+...+k_n} d^n W(0)(v_{in,1}, ..., v_{in,n})
$$

by continuity in $(v_{in,1}, ..., v_{in,n})$ on the whole space $H^1 \oplus L_2$.

Now, if $\|u_{in}\|_{H^1 \oplus L_2} < \vartheta$, where $\vartheta > 0$ and less than the radius of convergence of Taylor series at zero for the real expansion in $H^1 \oplus L_2$, see [10], (and also [12, 13, 11, 7, 3, 5]), for $v_{in} \in H^1 \oplus L_2$ and for sufficiently small $\alpha^{(0)}, \alpha^{(1)} \in \mathbb{R}$, i.e. for

$$
\|u_{in} + \alpha^{(0)} v_{in} + \alpha^{(1)} J v_{in}\|_{H^1 \oplus L_2} < \vartheta,
$$

we have

$$
\langle W(u_{in} + \alpha^{(0)} v_{in} + \alpha^{(1)} J v_{in}), R^{-1}h \rangle_{\mathcal{H}_S} = \sum_{n=1}^{\infty} \frac{1}{n!} \langle d^n W(0)(u_{in} + \alpha^{(0)} v_{in} + \alpha^{(1)} J v_{in}, ..., u_{in} + \alpha^{(0)} v_{in} + \alpha^{(1)} J v_{in}), R^{-1}h \rangle_{\mathcal{H}_S} = \sum_{n_1+n_2+n_3=n} \frac{1}{n_1! n_2! n_3!} \langle d^n W(0)(u_{in}, ..., u_{in}, \alpha^{(0)} v_{in}, ..., \alpha^{(0)} v_{in}, \alpha^{(1)} J v_{in}, ..., \alpha^{(1)} J v_{in}), R^{-1}h \rangle_{\mathcal{H}_S} = \sum_{n_1+n_2+n_3=n} \frac{1}{n_1! n_2! n_3!} \langle \alpha^{(0)} n_2 (\alpha^{(1)} J)^{n_3} d^n W(0)(u_{in}, ..., u_{in}, v_{in}, ..., v_{in}) R^{-1}h \rangle_{\mathcal{H}_S} = \sum_{n_1+n_2=n} \frac{1}{n_1! n_2!} \langle \alpha^{(0)} - i \alpha^{(1)} \rangle^{n_2} \langle d^n W(0)(u_{in}, ..., u_{in}, v_{in}, ..., v_{in}) R^{-1}h \rangle_{\mathcal{H}_S}.
$$

Since the series converges absolutely for $\|u_{in}\|_{H^1 \oplus L_2} < \vartheta$ and $\|u_{in} + (\alpha^{(0)} + \alpha^{(1)} J)v_{in}\| < \vartheta$, i.e. for $|\alpha| < \|v_{in}\|^{-1}(\vartheta - \|u_{in}\|)$, so its sum is antiholomorphic in $\alpha^{(0)} + i \alpha^{(1)}$. Therefore, the function $\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}$ is antiholomorphic in $\alpha$ for sufficiently small $|\alpha|$.

In order to show that $RW R^{-1}$ is holomorphic as a transformation in $H^1(\mathbb{R}^3, \mathcal{C})$, it is sufficient to show (see [3, pp. 84-85]), that for $h \in H^1(\mathbb{R}^3, \mathcal{C})$, $\|u_{in}^+\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} < \vartheta$ (i.e. $\|\mu^{1/2} u_{in}^+\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} < \vartheta$), the function

$$
\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}
$$
is antiholomorphic in $\alpha$. It is evident that for $h \in S(\mathbb{R}^3, \mathcal{C})$ the expressions
\[
\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h \rangle_{H^1(\mathbb{R}^3, \mathcal{C})} = \langle \mu^{1/2} R W R^{-1}(u_{in}^+ + \alpha v_{in}^+), \mu^{1/2} h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}
\]
are correctly defined and antiholomorphic in $\alpha$ (the function $\mu(\cdot)$ is real-valued and the operator $\mu$ commutes with the imaginary unit). If now $h \in H^1(\mathbb{R}^3, \mathcal{C})$ and $h_n \in S(\mathbb{R}^3, \mathcal{C})$, $h_n \to h$ in $H^1(\mathbb{R}^3, \mathcal{C})$, then the uniform in $\alpha$ estimate implies that
\[
|\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h \rangle_{H^1(\mathbb{R}^3, \mathcal{C})} - \langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h_n \rangle_{H^1(\mathbb{R}^3, \mathcal{C})}| \leq \|\mu^{1/2} R W R^{-1}(u_{in}^+ + \alpha v_{in}^+)\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})} \|\mu^{1/2} (h - h_n)\|_{H^{1/2}(\mathbb{R}^3, \mathcal{C})}
\]
This estimate follows from the energy conservation and Equality (3.1). The estimate is uniform in $\alpha$ and implies the uniform convergence of the antiholomorphic function
\[
\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h_n \rangle_{H^1(\mathbb{R}^3, \mathcal{C})}
\]
to the function
\[
\langle RW R^{-1}(u_{in}^+ + \alpha v_{in}^+), h \rangle_{H^1(\mathbb{R}^3, \mathcal{C})}
\]
and its antiholomorphy. Thus, the theorem is proved for the case of the wave operator.

Remark. We note that the following equality for Frechét derivatives is valid,
\[
d^n(RWR^{-1})(0)(z_1, ..., z_n) = Rd^nW(0)(R^{-1}z_1, ..., R^{-1}z_n).
\]
We note also that due to holomorphy (or commutativity with the operator of imaginary unit) Frechét derivatives $d^n(RWR^{-1})(0)$ are $n$-linear forms with respect to the field of complex numbers.

4. Smoothed solution with small initial data

To prove that the maps $W, S$ are holomorphic for large initial data we show that the positive (or negative) frequency part of solution of the nonlinear equation smoothed (in time) with a test function with compact support in momentum space is a polynomial.

Theorem 4.1. Let $\vartheta$ be from Theorem 3.2, $\vartheta > 0$ (i.e. the Taylor expansion at zero of the operators $R W R^{-1}$ and $R S R^{-1}$ converges for $\|z_{in}\|_{H^1(\mathbb{R}^3, \mathcal{C})} < \vartheta$). Let $z_{in} \in H^1(\mathbb{R}^3, \mathcal{C}) \cap S(\mathbb{R}^3, \mathcal{C})$, $\|z_{in}\|_{H^1(\mathbb{R}^3, \mathcal{C})} < \vartheta$, $f \in \mathcal{F}(D(\mathbb{R}))$, then
\[
\int dt f(t)RWU_0(t)R^{-1}z_{in} = \int dt f(t) \sum_{n=1}^{N(f)} \frac{1}{n!} Rd^nW(0)(U_0(t)R^{-1}z_{in}, ..., U_0(t)R^{-1}z_{in}),
\]
where \( N(f) = \max\{[p_0/ \text{ mass constant }] + 1 \mid p_0 \in \text{ supp } f^\sim\} \), \( U_0(t) \) is the free dynamics (i.e. the dynamics defined by the linear Klein-Gordon equation).

Remark. It is clear that \( H^1(\mathbb{R}^3, \mathcal{C}) = D(\mu^{1/2}) \) in \( H^{1/2}(\mathbb{R}^3, \mathcal{C}) \).

Proof of Theorem 4.1. Since \( U_0(t) \) is an orthogonal (and symplectic) transformation, so for \( \| z_{in} \|_{H^1(\mathbb{R}^3, \mathcal{C})} < \vartheta \)
\[
\int dt f(t) R W U_0(t) R^{-1} z_{in}
\]
\[
= \int dt f(t) \sum_{n=1}^\infty \frac{1}{n!} R^m W(0)(U_0(t) R^{-1} z_{in}, ..., U_0(t) R^{-1} z_{in})
\]
\[
= \sum_{n=1}^\infty \int dt f(t) \frac{1}{n!} R^m W(0)(U_0(t) R^{-1} z_{in}, ..., U_0(t) R^{-1} z_{in})
\]
(the series converges in \( H^1(\mathbb{R}^3, \mathcal{C}) \) topology).

The \( n \)-linear form
\[
n!-1^m R W R^{-1}(0)
\]
is the transformation into \( H^1(\mathbb{R}^3, \mathcal{C}) \). The Schwartz theorem implies that there exists a unique generalized function from the space \( S'(\mathbb{R}^{3n}, \mathcal{C}) \) with values in \( H^1(\mathbb{R}^3, \mathcal{C}) \) (i.e. from the space \( S'(\mathbb{R}^{3n}, H^1(\mathbb{R}^3, \mathcal{C})) \)) such that
\[
\frac{1}{n!} R^m W(0)(R^{-1} z_1, ..., R^{-1} z_n) = R_n(z_1 \otimes ... \otimes z_n)
\]
for \( z_1, ..., z_n \in S(\mathbb{R}^3, \mathcal{C}) \).

Taking into account that \( U_0(t) R^{-1} z = R^{-1}(\exp(-i\mu t) z) \), where \( \mu = (-\Delta + m^2)^{1/2}, \) we obtain
\[
\int dt f(t) \frac{1}{n!} R^m W(0)(U_0(t) R^{-1} z_{in}, ..., U_0(t) R^{-1} z_{in})
\]
\[
= \int dt f(t) R_n(\exp(-i\mu t) z_{in} \otimes ... \otimes \exp(-i\mu t) z_{in})
\]
\[
= R_n(f^\sim(\mu_1 + ... + \mu_n) z_{in} \otimes ... \otimes z_{in}),
\]
here \( \mu_j = (-\Delta_j + m^2)^{1/2} \). The operator \( f^\sim(\mu_1 + ... + \mu_n) \) is a convolution operator, its Fourier transform (in spatial coordinates) is equal to \( f^\sim(\sum_j (p_j^2 + m^2)^{1/2}) \) and \( = 0 \) for
\[
n \geq N(f) = \max\{[p_0/m] + 1 \mid p_0 \in \text{ supp } f^\sim\}.
\]
This implies that
\[
\int dt f(t) R W U_0(t) R^{-1} z_{in}
\]
\[
= \int dt f(t) \sum_{n=1}^{N(f)} \frac{1}{n!} R^m W(0)(R^{-1} \exp(-i\mu t) z_{in}, ..., R^{-1} \exp(-i\mu t) z_{in}).
\] (4.2)

The continuity allows to extend this equality on
\[
z_{in} \in H^1(\mathbb{R}^3, \mathcal{C}), \quad \| z_{in} \|_{H^1(\mathbb{R}^3, \mathcal{C})} < \vartheta.
\]
Indeed, the real analyticity proved by Kumlin \cite{10} and Theorem 26.2.4, ch. XXVI, §1 (or Theorem 26.2.5, ch. XXVI, §1, Theorem 26.2.6, ch. XXVI, §1) by Hille and Phillips, see \cite{14} imply that all derivatives at zero are homogeneous continuous polynomials, and, therefore, all polynomials smoothed with integration over \(f(t)\), are continuous also. These results give the continuity and Equality (4.1) for \(\|z_\in\|_{H^1(\mathbb{R}^3, \mathcal{L})} < \vartheta\). Theorem 4.1 is proved.

5. Holomorphy on finite–dimensional subspaces

In this Section we consider the complex analyticity of functions

\[
\int dt f(t) \langle RWU_0(t)R^{-1}(z_\in(\alpha)), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}, \quad f \in L_1(\mathbb{R}).
\]

Theorem 5.1. Let \(z_\in(\alpha) = \sum_{j=1}^{N} \alpha_j z_{in,j}, \ z_{in,j} \in H^1(\mathbb{R}^3, \mathcal{C}), \ \alpha_j \in \mathcal{C}\). Let \(f \in L_1(\mathbb{R})\), \(h \in H^{1/2}(\mathbb{R}^3, \mathcal{C})\), then

\[
\int dt f(t) \langle RWU_0(t)R^{-1}(z_\in(\alpha)), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})},
\]

is an entire antiholomorphic function in \((\alpha_1, ..., \alpha_N) \in \mathcal{C}^N\).

Proof of Theorem 5.1. For the proof we use Theorem 3.2 about complex analyticity at the zero in \(H^1(\mathbb{R}^3, \mathcal{C}) \simeq D(\mu^{1/2}) \subset H^{1/2}(\mathbb{R}^3, \mathcal{C})\), Equality (4.1) and the real analyticity proved by Kumlin \cite{10}.

Let \(f \in \mathcal{F}(D(\mathbb{R}))\) and

\[
F_1(z_\in(\alpha), f) = \int dt f(t) \langle RWU_0(t)R^{-1}(z_\in(\alpha)), h \rangle_{\mathcal{H}_S},
\]

\[
F_2(z_\in(\alpha), f) = \int dt f(t) \sum_{n=1}^{N(f)} \frac{1}{n!} \langle Rd^nW(0)(U_0(t)R^{-1}z_\in(\alpha), ..., U_0(t)R^{-1}z_\in(\alpha), h \rangle_{\mathcal{H}_S}.
\]

Then \(F_1(z_\in(\alpha), f)\) is a real analytic function for all \(\alpha \in \mathcal{C}^N\) and \(F_2(z_\in(\alpha), f)\) is an entire antiholomorphic function in \(\alpha \in \mathcal{C}^N\). The real analyticity of \(F_1(z_\in(\alpha), f)\) follows from the proof of real analyticity given by Kumlin \cite{10}, from energy conservation and uniform in \(\alpha\) approximation of the integral over time by finite sums. The function \(F_2(z_\in(\alpha), f)\) is an entire antiholomorphic function. This is implied by the facts that this function is a polynomial, generated by Fréchet derivatives at zero and by holomorphy of the transformation \(RWR^{-1}\) in a zero neighborhood in \(H^1(\mathbb{R}^3, \mathcal{C})\), see Theorem 3.2.

Theorem 4.1 implies that there exists \(\vartheta(f) > 0\) (and depending on \(z_{in,1}, ..., z_{in,N}\)) such, that

\[
F_1(z_\in(\alpha), f) = F_2(z_\in(\alpha), f)
\]

for \(|\alpha| < \vartheta(f)\). Then Lemma 5.2 about uniqueness implies the equality

\[
F_1(z_\in(\alpha), f) = F_2(z_\in(\alpha), f)
\]

for all \(\alpha \in \mathcal{C}^N\) and, therefore the function

\[
\int dt f(t) \langle RWU_0(t)R^{-1}(z_\in(\alpha)), h \rangle_{H^{1/2}(\mathbb{R}^3, \mathcal{L})}
\]
is an entire antiholomorphic function in $\alpha$.

If now $f \in L_1(\mathbb{R})$, then there exists a sequence $f_n \in \mathcal{F}(D(\mathbb{R}))$, $f_n \to f$ in $L_1(\mathbb{R})$ and

$$F_1(z_{in}(\alpha), f) = \lim_{n} F_1(z_{in}(\alpha), f_n).$$

Due to the estimate

$$|F_1(z_{in}(\alpha), f - f_n)| \leq \int dt |f(t) - f_n| \|z_{in}(\alpha)\|_{H^1(\mathbb{R}^N)} \|\mu^{-1/2}h\|_{H^1(\mathbb{R}^N)}$$

uniform in $\alpha$ for bounded $\alpha$, the convergence in (5.1) is uniform in $\alpha$ and, thus, $F_1(z_{in}(\alpha), f)$, $f \in L_1(\mathbb{R})$, is an entire antiholomorphic function in $\alpha$. *Theorem 5.1 is proved.*

Now we prove the statement about uniqueness. We prove this statement in the following form.

**Lemma 5.2.** Let $A(\alpha)$ and $B(\alpha)$ be two real analytic functions from $\mathbb{R}^N$ into a Banach space $B$. Let for $|\alpha| < a$ with some $a > 0$ $A(\alpha) = B(\alpha)$. Then $A(\alpha) = B(\alpha)$ for all $\alpha \in \mathbb{R}^N$.

**Remarks.** 1. Real analyticity means a convergence at any point of local Taylor expansion with Frechét derivatives.

2. For us it is sufficient to consider the case $B = C$.

**Proof of Lemma 5.2.** Let $\text{Anal}(\alpha, r)$ be an (open) ball with center at $\alpha$ and radius $r$. Let $r_{A,B}(\alpha) := \min(r_A(\alpha), r_B(\alpha))$, where $r_A(\alpha), r_B(\alpha)$ are radii of convergence of the Taylor expansion with the center at $\alpha$ for the function $A$ and for the function $B$, respectively. By condition of Lemma 5.2 $r_{A,B}(\alpha) > 0$ for all $\alpha \in \mathbb{R}^N$. Let $\mathcal{O}(0, r)$ be an (open) ball with radius $r$ having the properties

$$A(\alpha) = B(\alpha) \quad \text{for} \quad |\alpha| < r.$$

By conditions of Lemma 5.2 $A(\alpha) = B(\alpha)$ in

$$\text{Anal}(0, r_{A,B}(0)) \cap \{\alpha \mid |\alpha| < \vartheta, \alpha \in \mathbb{R}^N\}.$$

Thus, there exists a ball $\mathcal{O}(0, r)$ with radius $r > 0$.

Let

$$r_{\text{max}} = \sup\{r \mid \exists \mathcal{O}(0, r), A(\alpha) = B(\alpha) \forall \alpha \in \mathcal{O}(0, r)\}.$$

We show that $r_{\text{max}} = \infty$.

It is clear that $r_{\text{max}} > 0$. If $r_{\text{max}} < \infty$, then continuity and the real analyticity imply that $A(\alpha) = B(\alpha)$ and $\partial^j A(\alpha) = \partial^j B(\alpha)$ for $\alpha \in \mathcal{O}(0, r_{\text{max}})$ and all $j$. The compact

$$\{\alpha \in \mathbb{R}^N \mid |\alpha| = r_{\text{max}}\} \subset \bigcup_{|\alpha|=r_{\text{max}}} \text{Anal}(\alpha, \frac{1}{4}r_{A,B}(\alpha))$$

and compactness of $\{\alpha \in \mathbb{R}^N \mid |\alpha| = r_{\text{max}}\}$ implies that there exists a finite number of balls $\text{Anal}(\alpha_k, \frac{1}{4}r_{A,B}(\alpha_k))$, $k = 1, ..., K$, such that

$$\{\alpha \in \mathbb{R}^N \mid |\alpha| = r_{\text{max}}\} \subset \bigcup_{1 \leq k \leq K} \text{Anal}(\alpha_k, \frac{1}{4}r_{A,B}(\alpha_k)).$$
Let
\[ \mathcal{O}(K) = \mathcal{O}(0, r_{\text{max}}) \cup \bigcup_{1 \leq k \leq K} \text{Anal}(\alpha_k, \frac{1}{4} r_{A,B}(\alpha_k)). \]

Let
\[ r(K) = \frac{1}{4} \min_{1 \leq k \leq K} r_{A,B}(\alpha_k). \]

Then \( r(K) > 0 \) and \( \mathcal{O}(0, r_{\text{max}} + r(K)) \subset \mathcal{O}(K) \). Really, if \( \alpha \in \mathcal{O}(0, r_{\text{max}} + r(K)), |\alpha| \geq r_{\text{max}}, \) then the vector
\[ \frac{\alpha}{|\alpha|} r_{\text{max}} \in \text{Anal}(\alpha_k, \frac{1}{4} r_{A,B}(\alpha_k)) \cap \mathcal{O}(0, r_{\text{max}}) \]
for some \( k \). Then \( \alpha = \alpha_k + \beta \) and
\[
|\beta| \leq |\alpha - \frac{\alpha}{|\alpha|} r_{\text{max}}| + \frac{1}{|\alpha|} r_{\text{max}} - \alpha_k | \\
\leq |\alpha| - r_{\text{max}} + \frac{1}{|\alpha|} r_{\text{max}} - \alpha_k | \\
\leq r(K) + \frac{1}{4} r_{A,B}(\alpha_k) \\
\leq \frac{1}{2} r_{A,B}(\alpha_k).
\]

Therefore,
\[
A(\alpha) = \sum \frac{1}{j!} \partial^j A(\alpha_k)(\beta - \alpha_k)^j = \sum \frac{1}{j!} \partial^j B(\alpha_k)(\beta - \alpha_k)^j = B(\alpha),
\]
because the series converges in \( \text{Anal}(\alpha_k, r_{A,B}(\alpha)) \) and all derivatives for functions \( A \) and \( B \) at point \( \alpha_k \) coincide. \( \text{Lemma 5.2 is proved.} \)

**Lemma 5.3 (Consequence of Lemma 5.2).** Let \( A(\alpha) \) be a complex holomorphic in \( \mathbb{C}^N \) function with values in a complex Banach space \( \mathbf{B} \) and let \( B(\alpha) \) be a real analytic function in \( \mathbb{C}^N \) with values in the same complex Banach space and \( A(\alpha) = B(\alpha) \) for \( |\alpha| < a \) for some \( a > 0 \). Then \( A(\alpha) = B(\alpha) \) for all \( \alpha \in \mathbb{C}^N \) and \( B(\alpha) \) is a complex holomorphic function.

**Remark.** By the space \( \mathbb{C}^N \) we mean the space with standard basis, given, for instance, in the form \((z_1, ..., z_N) \in \mathbb{C}^N, z_j = x_j + iy_j, x_j, y_j \in \mathbb{R} \). The complex analyticity is holomorphy with respect to the variables \((z_1, ..., z_N) \) and the real analyticity is an expansion into a (local) Taylor series with respect variables \((x_1, y_1, ..., x_N, y_N) \).

**6. Unitarizability of operators** \( dW(u_{in}) \) and \( dS(u_{in}) \)

In this Section we complete the proof of the uniqueness of essential unitarizability of operators \( dW(u_{in}) \) and \( dS(u_{in}) \) for small solutions, i.e. Theorem 3.1. For this purpose we give here more detailed proof of required assertions. These assertions are Theorem 6.1, 6.2, 6.3 (Theorem 6.3 is closely connected with Theorem 3.1) and these theorems have been used by Paneitz [17, 18, 19] for the proof of an analogue of Theorem 3.1.
Theorem 6.1 (see [16], ch. III, §1, §2 and also Theorem 6 [18]). Let $t \to A(t)$ be a strongly continuous norm-bounded map from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H})$ (= the algebra of bounded operators on (the real Hilbert space) $\mathcal{H}$) such that

$$N(A) := \int_{-\infty}^{+\infty} \|A(t)\| dt < 2.$$ 

Then

$$U(t; \lambda) = I + \lambda \int_{-\infty}^{t} A(s)U(s; \lambda) ds$$

has a unique continuous solution for all $\lambda$; $\|U(t; \lambda)\| \leq \exp(|\lambda|N(A))$ for all $t$; $W(\lambda) := U(0; \lambda)$, $\lim_{t \to +\infty} U(t; \lambda)$ exists in norms, $S(\lambda) := \lim_{t \to -\infty} U(t; \lambda)$. If $|\lambda| < 2/N(A)$ then $(W(\lambda) + I)^{-1}$ and $(S(\lambda) + I)^{-1}$ exist. $\lambda \to (W(\lambda) + I)^{-1}$ and $\lambda \to (S(\lambda) + I)^{-1}$ are analytic, and

$$\begin{align*}
(W(\lambda) - I)(W(\lambda) + I)^{-1} &= 2 \int_{0}^{\lambda} \int_{-\infty}^{0} (I + W(\rho))^{-1} U(s; \rho) A(s)U(s; \rho)(I + W(\rho))^{-1} ds d\rho, \\
(S(\lambda) - I)(S(\lambda) + I)^{-1} &= 2 \int_{0}^{\lambda} \int_{-\infty}^{+\infty} (I + S(\rho))^{-1} U(s; \rho) A(s)U(s; \rho)(I + S(\rho))^{-1} ds d\rho.
\end{align*}$$

(6.1) (6.2)

Remark. In this chapter notations $W(\lambda)$, $S(\lambda)$ denote linear operators and correspond to the first derivatives of (nonlinear) operators $W(u_0)$, $S(u_0)$ which are wave and scattering operators of the nonlinear equation (1.1).

Proof of Theorem 6.1. The first part is well known, and follows immediately from the usual “time-ordered exponential” form of the solution $U(t; \lambda)$, a norm-convergent power series in $\lambda$, we refer to the Krein proof, see [16].

In the second part we follow to Paneitz [18, Theorem 6], and give the proof for the case of the operator $W(\lambda)$.

We need some notation. Define

$$h(t) = \begin{cases} 
+\frac{1}{2} & \text{for } t \geq 0, \\
-\frac{1}{2} & \text{for } t < 0.
\end{cases}$$

This is the Green function of the linear differential equation of first order in time.

Let $\mathcal{B}$ be the Banach space of continuous functions $f(t)$ from $\mathbb{R}$ to $\mathcal{H}$ having continuous limits $f(\pm\infty)$ as $t \to \pm\infty$, with the sup norm

$$|||f||| = \sup_{t \in \mathbb{R}} ||f(t)||.$$ 

We define an integral operator $K_- : \mathcal{B} \to \mathcal{B}$ by

$$(K_- f)(t) = \lambda \int_{-\infty}^{0} h(t-s) A(s)f(s) ds.$$
If \( N(A) < 2 \) and \(|\lambda| < 2/N(A)\) clearly \( ||K_-|| < 1 \), and then $f$

\[
(I - K_-)g = f
\]  \hfill (6.3)

(for a given constant \( f \in \mathcal{H} \)) has a unique solution \( g \in \mathcal{B} \). This solution \( g \) satisfies the boundary conditions

\[
g(-\infty) + g(0) = 2f
\]  \hfill (6.4)

and the equation

\[
g(t) = g(-\infty) + \lambda \int_{-\infty}^{t} A(s)g(s)ds.
\]  \hfill (6.5)

The boundary condition (6.4) is implied by Equation (6.3) and gives equalities

\[
g(0) - (K_-)(0) = g(0) - \frac{1}{2} \lambda \int_{-\infty}^{0} A(s)g(s)ds = f,
\]

\[
g(-\infty) - \lim_{T \to -\infty} (K_-)(T) = g(-\infty) + \frac{1}{2} \lambda \int_{-\infty}^{0} A(s)g(s)ds = f.
\]

In the continuous case Equation (6.5) is equivalent to the differential equation. Equation (6.5) can be obtained from Equation (6.3) by differentiation in \( t \).

Then by uniqueness \( g(t) = U(t;\lambda)g(-\infty) \), \( g(0) = W(\lambda)g(-\infty) \), and \( 2f = (I + W(\lambda))g(-\infty) \). \( g \), hence \( g(-\infty) \), depends continuously on \( f \), so \( \lambda \to (I + W(\lambda))^{-1} \) is an analytic map into \( \mathcal{L}(\mathcal{H}) \) for \( 0 \leq \lambda \leq 1 \), that is, a power series convergent in norm, this is implied by the inequality \( ||K_-|| < 1 \) for \( |\lambda| < 2/N(A) \). We emphasize that \( \mathcal{H} \) is a real Hilbert space (see [18], p. 316], thus \( \lambda \) is a real constant also.

It is easily checked that

\[
\frac{d}{d\lambda} U(t;\lambda) = U(t;\lambda) \int_{-\infty}^{t} U(s;\lambda)^{-1} A(s)U(s;\lambda)ds
\]

(by solving the first order differential equation which the l.h.s. satisfies),

\[
\frac{dU(t;\lambda)}{d\lambda} = \int_{-\infty}^{t} A(s)U(s;\lambda)ds + \int_{-\infty}^{t} A(s)\frac{dU(s;\lambda)}{d\lambda}ds,
\]

\[
\frac{d^2U(t;\lambda)}{dt d\lambda} = A(t)U(t;\lambda) + \lambda A(t)\frac{dU(s;\lambda)}{d\lambda},
\]

see [16] ch. III, §1, 4, (1.1), (1.10), (1.19)]. This equation corresponds to the linear equation with an external field.

Thus, \( dW(\lambda)/d\lambda = W(\lambda)R_-(\lambda) \), where

\[
R_-(\lambda) = \int_{-\infty}^{0} U(s;\lambda)^{-1} A(s)U(s;\lambda)ds.
\]

Defining \( X_-(\lambda) = (W(\lambda) - I)(W(\lambda) + I)^{-1} \), clearly

\[
\frac{d}{d\lambda} X_-(\lambda) = 2(I + W(\lambda))^{-1} W(\lambda) R_-(\lambda)(I + W(\lambda))^{-1},
\]

\(^2\text{For the case of the wave operator it is sufficient to take the condition } N_- := \int_{-\infty}^{0} ||A(t)||dt < 2.\)
and since $W(0) = I$, $X_-(\lambda)$ has the integral expression (6.1). Theorem 6.1 is proved.

In the case when $A(t) \in sp(\mathcal{H})$ it is well known that then $U(t; \lambda)$, $W(\lambda)$, $S(\lambda) \in Sp(\mathcal{H})$. Here we use Paneitz’s notation [18, pp. 316-319]) and since $W(0) = I$, $X_-(\lambda)$ has the integral expression (6.1). Theorem 6.1 is proved.

Theorem 6.2. (see Theorem 7 [18]). Let $t \rightarrow A(t)$ be a strongly continuously normbounded map from $\mathbb{R}$ into $sp(\mathcal{H})$ taking values in the closed positive cone $\overline{C}_0$ (the notation follows to [18, pp. 316, 319]).

1. If $\int_{-\infty}^{0} \|A(t)\| dt < 2$ then the wave operator $W : X(-\infty) \rightarrow X(0)$ for the equation $dX/dt = A(t)X$ is the Cayley transform of a $Y_- \in \overline{C}_0$.

2. If $\int_{-\infty}^{0} \|A(t)\| dt < 2$ then the scattering operator $S : X(-\infty) \rightarrow X(0)$ for the equation $dX/dt = A(t)X$ is the Cayley transform of a $Y \in \overline{C}_0$.

3. If furthermore $\int_{-\infty}^{0} A(t) dt$ (as a strong-operator topology integral) is in the interior $C_0$, then $Y_- \in C_0$ also, and then $W$ commutes with a unique complex structure, i.e., is uniquely unitarizable.

4. If furthermore $\int_{-\infty}^{0} A(t) dt$ (as a strong-operator topology integral) is in the interior $C_0$, then $Y \in C_0$ also, and then $S$ commutes with a unique complex structure, i.e., is uniquely unitarizable.

Remarks. 1. The conditions $\int_{0}^{\infty} \|A(t)\| dt < 2$ and $\int_{0}^{\infty} \|A(t)\| dt < 2$ are sufficient for the consideration of wave operators. In the quantum case this is connected with an elastic scattering for energies less than $4m$. We intend to consider this question later in connection with the unitarity of the quantum scattering operator.

2. The restriction on the subspace of kernels in the condition of Theorem 6.2 corresponds to the exclusion of “bound states”, i.e. vectors $v \neq 0$ such that $Sv = v$.

Proof of Theorem 6.2. (1W) It follows from the integral formula in Theorem 6.1 for $(W - I)(W + I)^{-1}$, the invariance of $\overline{C}_0$ under $(U_1)^{-1} : T \rightarrow 2(I + W^{-1})^{-1}T(I + W)^{-1}$ (Theorem 5 [18]) and $A_0 W$, for $W \in Sp(\mathcal{H})$, and the fact that $\overline{C}_0$ is a convex cone.

2W) For $\rho = 0$ the integrand in (6.1) is equal to $\frac{1}{2} \int_{-\infty}^{0} A(t) dt$. Thus, if for some $v \in \mathcal{H}$ $\int_{0}^{\infty} a(A(t)v, v) = 0$, then $a(A(t)v, v) = 0$ for all $t \leq 0$, due to continuity of $a$ and the inclusion $A(t) \in C_0$. Since $A(t) \in Sp(\mathcal{H})$, so $a(A(t)v, v) = S_0 (J^{-1} A(t)v, v)$ (in the notation of [18, pp. 316-319]) and $J^{-1} A(t)$ is a symmetrical and positive operator. Therefore, the equality $a(A(t)v, v) = S_0 (J^{-1} A(t)v, v)$ means that $(J^{-1} A(t))^{1/2} v = 0$, and thus, $J^{-1} A(t)v = 0$ and $A(t)v = 0$, that is, $v \in \cap_{0 < s \leq \lambda} \ker A(t) = \{0\}$ and $v = 0$. This means that if $v \neq 0$, then, due to (6.1), $a(A_- v, v) = 0$.

(3W) As noted before, the inner integrand in (6.1) is norm-continuous in $\rho$. Thus the integral is in the interior $C_0$ if $\int_{0}^{\infty} A(t) dt \in C_0$, by the observation in part (2W). Finally, apply Theorem 2 [18, Theorem 2, p. 318] to (Cayley transform)$^{-1}(W)$.

The consideration of (1S), (2S), (3S), i.e. the integral formula (6.2) in Theorem 6.1 for $(S - I)(S + I)^{-1}$ is completely the same as the case of (6.1). Theorem 6.2 is proved.

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Theorem 6.3 (= Theorem 4.4B [21], [22, p. 491], [17, p. 110]). Let $u_{in}$ be a given finite-energy solution of the free equation
\[ \Box u_{in} + m^2 u_{in} = 0 \] (6.6)
such that grad $u_{in}$ is also of finite energy, and suppose that the $L_\infty$-norms (over space) of $u_{in}(t, \cdot)$ and grad $u_{in}(t, \cdot)$ are bounded by \( \text{const} (1 + |t|)^{-3/2} \). Then if $u_{in}$ in a certain norm is sufficiently small, there exist unique solutions $u$ and $u_{in}$, $u_{out}$ of
\[ \Box u + m^2 u + \lambda u^3 = 0 \]
and (6.6) respectively, such that
\[ \| u - u_{in} \|_{H^{1/4} L_2} \to 0 \quad \text{as} \quad t \to -\infty, \]
\[ \| u - u_{out} \|_{H^{1/4} L_2} \to 0 \quad \text{as} \quad t \to +\infty, \]
and $u$, $u_{in}$, $u_{out} = O(|t|^{-3/2})$ in $L_\infty(\mathbb{R}^3)$.

It may also be discerned from these methods that in these and other similar situations
\[ |t|^{3/2} \| u - u_{in} \|_\infty \to 0 \quad \text{as} \quad t \to -\infty, \]
\[ |t|^{3/2} \| u - u_{out} \|_\infty \to 0 \quad \text{as} \quad t \to +\infty, \]
a fact which we will use later.

Corollary 6.4 (See also Theorem 4.4B [17, p. 110], [22, p. 491], Corollary [17, p. 115]). Take nonvanishing solutions $u$, $u_{in}$, $u_{out}$ as in Theorem 6.3 earlier (which implies in particular that $u$ is uniformly bounded and $\int_\infty^+ \| u(t, \cdot) \|_\infty^2 dt < \infty$), and assume that $\int_\infty^+ \| u(t, \cdot) \|_\infty^2 dt < 2m$ and that for some time $t$
\[ (m^2 - \Delta)^{5/4} u_{in}(t, \cdot) \in L_1(\mathbb{R}^3), \]
(6.8)
then $dW(u_{in}) : (H^{1/2} \oplus H^{-1/2}, J) \to (H^{1/2} \oplus H^{-1/2}, J)$ and $dS(u_{in}) : (H^{1/2} \oplus H^{-1/2}, J) \to (H^{1/2} \oplus H^{-1/2}, J)$ are uniquely essentially unitarizable.

Proof of Corollary 6.4. To prove Corollary we use Theorem 2b) [17, p. 115]) (= Theorem 7(2) [18] = Theorem 6.2 [19], see also Theorem 6.3 [19]), it remains only to show that all
\[ -A(t) = e^{-tQ} \begin{pmatrix} 0 & 0 \\ F'(u) & 0 \end{pmatrix} e^{tQ}, \quad Q = \begin{pmatrix} 0 & 1 \\ -\Delta + m^2 & 0 \end{pmatrix}, \]
vanish on no nonzero vector $(v_1, v_2) \in H^{1/2} \oplus H^{-1/2}$. Now $e^{tQ}(v_1, v_2) = (v(t), \dot{v}(t))$ for $(v_1, v_2)$ with finite $H^{1/2} \oplus H^{-1/2}$ norm satisfying the Klein-Gordon equation, and $F'(u)v := u^2 v = 0$ is equivalent to $uv = 0$. The conclusion $v \equiv 0$ follows from (6.7), the asymptotics of $u_{in}(t, 0)$ in some Lorentz frame [23, Corollary 2] obtainable from (6.8), hyperbolicity, and the vanishing of any solution $v$, $(v(t), \dot{v}(t)) \in H^{1/2} \oplus H^{-1/2}$, which vanishes in a backward time-like cone, see Theorem 7.1 (see also [24, 8], [23, Corollary 1]. Our proof uses also Theorem 6.3 [19, Theorem 6.3], see also Theorem 2 [17, Theorem 2, p. 115].
To consider the case (of the derivative) of wave operator \( dW(u) \) we apply Theorem 7, [18, Theorem 7], for the interval \((-\infty, 0]\) (or \([0, \infty)\) for the forward wave operator) and use Theorem 6.3 [19, Theorem 6.3] and Theorem 2 [17, Theorem 2, p. 115].

In Theorem 7.1 we give the detailed proof of the statement that the condition

\[ u(t, x)v(t, x) = 0 \]

for all \((t, x) \in \mathbb{R}_- \times \mathbb{R}^3\) implies that \(v(t, x) = 0\) for all \((t, x)\) belonging to some backward time-like cone. The last condition implies that \(v = 0\). The ideas of the proof of Theorem 7.1 are analogous to [18, 19, 20, 17] and use the S. Nelson results about asymptotic behavior of free solutions, see [23, Corollary 2, Corollary 1']. Corollary 6.4 is proved.

Remarks. 1. We point out that the paper [20] of Paneitz and Segal formulates a statement, gives some references, but does not contain the proof of this statement. The more detailed exposition of Paneitz [17] formulates the assertion \(v = 0\) in the forward cone (for the case of derivative of the scattering operator) as a condition. This is not quite appropriate because only the condition \(v(t, x) = 0\) on the forward time-like cone appears naturally [20]. The last condition implies that \(v_{in}(t, x) = 0\) on the forward cone and, therefore, \(v_{in} = 0\) [24] and \(v = dS(u_{in})v_{in} = 0\) (and also by Goodman [8] \(v(t, x) (= (dS(u_{in})v_{in})(t, x)\) is equal to zero on the cone). However the Paneitz’s paper [17] uses the absence of solution with initial data from \(H^{1/2} \oplus H^{-1/2}\) and the Morawetz’s paper [24] contains the proof for solution with finite energy only. In this case the use of Theorem 1 [23, Theorem 1] is more appropriate. Theorem 1 [23, Theorem 1] gives the required assertion about the statement that a solution equal to zero in some (backward) light cone is the zero solution.

2. The part of Corollary 6.4 is contained in Theorem 16.3 [19].

3. We remark that to use Corollary [17] we take the space \(R(H^1 \oplus L)\) as the subspace \(D = D(\mu^{1/2})\) in \(H^{1/2}(\mathbb{R}^3, \mathcal{U})\) (see Definitions [17] and also [19]).

4. To apply Theorem 6.3 [17] we use that the condition \(a(Sv, v) > 0\) for all \(v \neq 0\) is implied by Condition 2 of Theorem 7 [18] (i.e. \(\cap_{t \in \mathbb{R}} \{\ker A(t)\} = \{0\}\) and by Statement 2 of Theorem 7 [18] (for the wave operator this condition is \(\cap_{t \in \mathbb{R}, t < 0} \{\ker A(t)\} = \{0\}\) and \(a(Wv, v) > 0\) for all \(v \neq 0\)). Condition 7(2) (= Statement 7(2) [18]) and the simple algebraic identity

\[ a((S - I)(S + I)^{-1}v, v) = 2a(Sw, w), \]

where \(w = (S + I)^{-1}v\), imply positivity of the inverse Cayley transform

\[ Y = (S - I)(S + I)^{-1}, \quad Y_- = (W_- - I)(W_- + I)^{-1}. \]

Here (and in [19]) the symplectic form is denoted by \(a(\cdot, \cdot)\) instead of the notation \(\omega(\cdot, \cdot)\) that is used in the other places.

5. For the consideration of \(d\)-dimensional case we intend to give the detailed (and independent) proof of Theorem 6.3 [18] and Theorem 2 [17] separately.

7. \(\cap \{\ker A(t)\} = \{0\}\). A solution equal zero in an infinite time column is equal zero

\(^{3}\)It is possible to use the interval \((-\infty, T]\), or \([T, \infty)\), also.
To complete the proof of assertions about holomorphity we need the following statement.

Let us consider the system of equations

\[
\begin{align*}
\Box u + m^2 u &+ \lambda v^3 = 0 \\
\Box v + m^2 v + 3\lambda u^2 v &= 0
\end{align*}
\]

(7.1)

(solutions of which are the tangent space of the manifold of solutions of the nonlinear equation).

**Theorem 7.1.** Let \((u, v)\) be solutions of the system of equations (7.1), and let for the initial in-data of the solution \(u\) satisfies hypotheses of Theorem 3.1. Let \(u \not= 0\) and \(u(t, x)v(t, x) = 0\) for all \((t, x) \in \mathbb{R}_- \times \mathbb{R}^3\). Then \(v = 0\).

The analogous statement is valid for the scattering operator and for the out-wave operator.

**Remark.** Here, as in Theorem 3.1, it is sufficient to take initial data with weaker conditions than in Theorem 3.1, see Paneitz \[17, Corollary, p.115-116\], Paneitz, Segal \[20\].

These conditions are required for the consideration of solutions behavior. If the space \(S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3)\) is used as the space of initial in-data these conditions are fulfilled. In any case these conditions are fulfilled for sufficiently small initial in-data.

**Proof of Theorem 7.1** (the idea of the proof is given in \[17\]). In our case, i.e. for Equation (7.1), the operator \(A(t)\) in the statement

\[
\bigcap_{t \in \mathbb{R}, t \leq 0} \{\ker A(t)\} = \{0\}
\]

(see \[17, 18, 19\] and the Paneitz’s notation) corresponds to the operator

\[
U_0(-t) \begin{pmatrix} 0 & 0 \\ -3\lambda u(t)^2 & 0 \end{pmatrix} U_0(t).
\]

This correspondence fulfills in the interaction representation for the second equation in (7.1).

Therefore, the condition \(A(t)v(t) = 0\) is reduced (for the in-wave operator) to the condition

\[ u(t, x)v(t, x) = 0 \text{ for all } (t, x) \in \mathbb{R}_- \times \mathbb{R}^3. \]

Analogously, for the out-wave operator the condition

\[ u(t, x)v(t, x) = 0 \text{ for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \]

appears and for the scattering operator the condition

\[ u(t, x)v(t, x) = 0 \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^3 \]

appears.
The proof of the fulfillment of the condition
\[ \bigcap_{t \in \mathbb{R}} \{ v \mid u(t, x) v(t, x) = 0 \ \forall x \} = \{0\}, \]
or
\[ \bigcap_{t \in \mathbb{R}} \{ v \mid u(t, x) v(t, x) = 0 \ \forall x \} = \{0\}, \]
follows from hyperbolicity, relations
\[ |t|^{3/2} \|u - u_{in}\|_{\infty} \to 0 \quad \text{for} \quad t \to -\infty, \quad (7.2) \]
or
\[ |t|^{3/2} \|u - u_{out}\|_{\infty} \to 0 \quad \text{for} \quad t \to +\infty, \quad (7.3) \]
and the behavior of \( u_{in}(t, x) \), or \( u_{out}(t, x) \), for large times, see \[23, Corollary 2\] and Theorem 7.2.

For this purpose, we prove the existence of some time column in which \( v(t, x) \) are equal zero. To proof this fact we use hyperbolicity and mentioned convergences. This time column, i.e. time-like positive axes with vertices on some space-like base, is directed to the past for the \( in \)-wave operator (and to the future for the \( out \)-wave operator). The base of this column is arranged sufficiently far in the past. Since \( v(t, x) \) for \( t \leq 0 \) satisfies the free equation and the causal envelope of this column is the backward light cone, so the solution is equal to zero in the backward light cone and, thus, it is the zero solution \[23, Corollary 1\].

This column is defined by the solution \( u \) and can be constructed with the help of the solution \( u_{in} \), namely, with the help of (neighborhood of) the point in momentum space at which \( (u_{in})^{-}(k) \neq 0 \) (we note that due to the condition (6.8) \( (u_{in})^{-}(k) \) is continuous in \( k \)). With the help of the S. Nelson’s result \[23, Corollary 2\], which describes the behavior of a free solution for large times and approximating the solution \( u = W(u_{in}) \) by its \( in \)-data we obtain that \( u(t, x) \neq 0 \) for \( (t, x) \) belonging to some set. This set, after the certain Lorentz rotation has the form \( \Sigma(a, r) = \bigcup_{t_n, t_n + a} \{ x \in \mathbb{R}^3 \mid |x| \leq r \} \). Therefore, \( v(t, x) = 0 \) on \( \Sigma(a, r) \). Since for \( t \leq 0 v(t, x) \) satisfies the free equation, hyperbolicity implies that the parameters \( a \) and \( r \) can be taken such that the causal envelope \( \Sigma(a, r) \) contains some (more narrow than \( r \)) column, and, thus, it is the backward light cone \[23, ch. 5, \S 28\].

Remarks. 1. We remark that the time column (to the past) is an infinite time cylinder of the form \( (a_0, a) + \mathbb{R}_{+} \times \{ x \in \mathbb{R}^3 \mid |x| \leq r \} \).

2. The causal envelope for the nonlinear wave equation, or for the system of equations (7.1), is the light cone envelope as in the case of the linear wave equation. This form of causal envelope is the consequence of hyperbolicity of wave equations and locality of the interaction, see, for instance, \[23, ch. 5, \S 28\], Reed, Simon \[26, v.2, Theorem X.77\].

To construct the required time column, and/or the corresponding light cone, we formulate as Theorem 7.2 the assertion proved by S. Nelson \[23\].

Theorem 7.2 (see \[23, Corollary 2\], our notations follow the notation in \[23\], see also the required conditions on the considered solutions \[23, 17\]). Let \( u_{in} \) be a free solution
that satisfies the conditions (6.8). Then
\[ \lim_{t \to +\infty} t^{3/2} e^{i\alpha(t,\lambda)} u_{in}^+(t, \frac{\lambda t}{\sqrt{1+\lambda^2}}) = (1 + \lambda^2)^{5/4} (u_{in}^+)\sim(\lambda), \]
where \( \lambda \in \mathbb{R}^3, \)
\[ \alpha(t, \lambda) = \frac{3\pi}{4} + \frac{t}{\sqrt{1+\lambda^2}} \]
and \( u_{in}^+, (u_{in}^+)\sim \) is the positive frequency part of the free (real) solution at the time zero and, correspondingly, its Fourier transform.

Remark. S. Nelson [23] uses the choice of space variables, corresponding to the choice of space variables in the form \((mt, mx)\). In momentum space this choice corresponds to the choice of coordinates in the form \((k^0/m, k/m)\). Here \( m \) is the mass constant from the nonlinear equation.

We continue the proof of Theorem 7.1.

We apply now Theorem 7.2 (see [23, Corollary 2]) to the initial \( in\)-data. For this purpose we construct a column in which \( u(t, x) \neq 0 \). This column is constructed with the help of a point at which \((u_{in}^+)\sim(b) \neq 0 \) and a Lorentz rotation.

We note that we need the assertion about vanishing in the form of Theorem 3.1 about unitarizability. The conditions of Theorem 3.1 imply that \((u_{in}^+)\sim(k) \) is continuous in \( k \). Since \( u_{in}^+ \neq 0 \), so there is a point \( b \) in momentum space such that \((u_{in}^+)\sim(b) \neq 0 \). This fact is equivalent to the existence of a Lorentz transformation such that \((u_{in}^+)\sim(0) \neq 0 \). Here \( u_{in, \Lambda}(t, x) = u_{in}(\Lambda(t, x)) \) and
\[ (u_{in, \Lambda}^+)\sim(k) = (2\pi)^{-3/2} \int e^{ikx} u_{in, \Lambda}(0, x) d^3x. \]
Let \( c(u_{in}) := |(u_{in, \Lambda}^+)\sim(0)| > 0 \). There exists such \( r_1 \), that
\[ \sup_{|k| \leq r_1} |(u_{in, \Lambda}^+)\sim(k) - (u_{in, \Lambda}^+)\sim(0)| \leq \frac{1}{8} c(u_{in}), \]
\[ \sup_{|k| \leq r_1} ((1 + \frac{k^2}{m^2})^{5/4} - 1) \leq \frac{1}{8} c(u_{in}). \]
Let
\[ u_{as, \Lambda}(t, x) = |mt|^{-3/2} (1 + \frac{x^2}{\sqrt{t^2 - x^2}})^{5/4} e^{i\alpha(t, x)} (u_{in, \Lambda}^+)\sim(\frac{x}{\sqrt{t^2 - x^2}}), \]
here
\[ \alpha(t, x) = \frac{3\pi}{4} + m\sqrt{t^2 - x^2} \]
and we have used the transform \( x = \lambda t/\sqrt{m^2 + \lambda^2} \), that is, \( \lambda = mx/\sqrt{t^2 - x^2} \), to introduce some approximation. This approximation allows to use the assertion of Theorem 7.2. We note that \( \text{Re} u_{as, \Lambda}(t, x) \) corresponds to the approximation of the initial solution \( u_{in}(t, x) \). We write
\[ u_{\Lambda}(t, x) = \text{Re} u_{as, \Lambda}(t, x) + u_{in, \Lambda}(t, x) - \text{Re} u_{as, \Lambda}(t, x) + u_{\Lambda}(t, x) - u_{in, \Lambda}(t, x). \]
We take \( r \) (this \( r \) defines the width of the column) and we suppose that \( |x| < r \). Moreover, this value of \( r \) we choose sufficiently large with respect to chinks (of the size \( \leq 2\pi/mass\ constant \)) between the constructed cylinders and such that the causal envelope of constructed cylinders contains the column with the width \( r/2 \).

There exists such (sufficiently large) \( a_1 > 0 \), that for all \( t < -a_1, |x| < r \)

\[
\sup_{|x| \leq r} |mt^{3/2}|u_{in,\Lambda}(t,x) - \text{Re} \ u_{as,\Lambda}^+(t,x)| \leq \frac{1}{8}c(u_{in}), \tag{7.4}
\]

\[
|mt^{3/2}||u_{\Lambda}(t,x) - u_{in,\Lambda}(t,x)||_\infty \leq \frac{1}{8}c(u_{in}), \tag{7.5}
\]

(7.4) is implied by Theorem 7.2, and (7.5) follows from the requirements on norms of initial \( in\)-data \( \mathbb{I} \). For \( t_n \leq t \leq t_n + \pi/2m \), where

\[
t_n = \frac{1}{m}(\frac{\pi}{2} + \pi n) - \arg(u_{in}^+)\sim(0)
\]

and \( m \) is the mass constant, the explicit form of \( u_{as,\Lambda}^+(t,x) \) implies that

\[
|\text{Re} \ u_{as,\Lambda}(t,x)| \geq |mt|^{-3/2}\frac{1}{8}c(u_{in}) > 0.
\]

Here we require that \( a_1 \) is larger than \( 4ma^2/\pi \). Then, we choose \( r \), defined the width of the cylinder, such that it is larger than the chinks between the cylinders and such that the causal envelope of these cylinders contains some (more narrow) column (of nonzero width).

The causal envelope of the cylinder contains the part of some (light) cone from the top to the section of the cone. The section of this cone coincides with the support of the cylinder. This is implied by hyperbolicity, the fact that \( u(t, x) \) for \( t \leq 0 \) satisfies the free wave equation and, therefore, the solution \( v_{\Lambda}(t, x) \), corresponding to the consideration of \( u_{\Lambda}(t, x) \) (i.e. corresponding to the consideration of the solution in some Lorentz frame), satisfies the free wave equation in some domain. This domain is equal to the domain \( \{ (t, x) \in \mathbb{R}^4 \mid t \leq 0 \} \) in this Lorentz frame. This domain with the space-like boundary contains the light cone with the top at zero (in the considered Lorentz frame) and directed into the past. Therefore, the causal envelope of (the set of) cylinders contains the column and, thus, it is the light cone directed into the past.

Therefore, causality and hyperbolicity imply that \( v(t, x) = 0 \) in the backward light cone with the top in the sufficiently large past. Consequently, \( v = 0 \), see \cite{23}, Corollary 1, Corollary 1′, \cite{24}. We note that by S. Nelson \cite{23} Corollary 1, Corollary 1′ Corollary 1, as far as Corollary 1′, implies that

\[
\|v^+\|_{L_2} = \lim_{t \to 0} \|v(t, \cdot)\|_{L_2} = \lim_{\Gamma} \|\chi_{\Gamma} v(t, \cdot)\|_{L_2}.
\]

\footnote{We note that \( t^{3/2} \) convergence is the consequence of conditions that the solution \( u_{in} \) satisfied or the consequence of the same conditions for the Lorentz rotated solution \( u_{in,\Lambda} \). Here the Lorentz rotation is defined by (non-zero) solution \( u_{in} \) and the requirement \( (u_{in,\Lambda}^+)\sim(0) \neq 0 \) (under conditions that are satisfied by the solution \( u_{in} \)). In any case these conditions are fulfilled with our choice of initial data from \( S_{Re}(\mathbb{R}^3) \oplus S_{Re}(\mathbb{R}^3) \) due to the choice of sufficiently small \( \alpha \) in Theorem 3.1.

The choice of the Lorentz rotation (or the choice of the Lorentz frame) and its consideration is equivalent to the fact that instead of the solution \( u(t, x) \) we consider the solution \( u_{\Lambda}(t, x) = u(\Lambda(t, x)) \) with the straight forwarded column, directed into the past.}
Here \( \chi_1 \) is the characteristic function of the cone \( \Gamma \). Since we consider the solutions \( v \) with the initial data in \( H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3) \), it is more simple to use the S. Nelson assertion \[23\], or even the corresponding assertion in the Vladimirov book \[25, \text{ch. V, §29, Theorem 1, Corollary 2}\]. \textbf{Theorem 7.1 is proved.}

\textbf{Appendix}

As pointed out by P. Kumlin \footnote{We are indebted to P. Kumlin for sending us this improvement.} Lemma 3.4 in the proof of real analyticity \[10\] is not correct as it is formulated. That is also the case for Proposition 1.4 in Brenner \[27\], which is the origin of Lemma 3.4. However the conclusion in Step 3 remains true if lines 4 to 25 on page 265 \[10\] are replaced by the argument below.

It remains to show that, for all \( t \in [-T, T] \),

\[
\mathcal{K}(t) := \left\{ \int_0^t K(t - s)(\varphi^2 \chi_R \psi_j(s))ds : j = 1, 2, 3, \ldots \right\}
\]

has a convergent subsequence in \( L_6 \). By the Rellich-Kondrashov theorem it suffices to prove that \( \mathcal{K}(t) \) is bounded in some \( L^p_\epsilon \), \( \epsilon > 0 \), where \( \frac{1}{p} - s' = \frac{1}{6} \) and \( s' > 0 \). Set \( p = \frac{3}{2} \) and \( s = \frac{1}{6} + \frac{s}{3} \) (\( s' = \frac{1}{2} \)). For a fixed \( t \in [-T, T] \), Proposition 3.1 yields

\[
\| \int_0^t K(t - s)(\varphi^2 \chi_R \psi_j(s))ds \|_{L_\epsilon^{s'}} \leq \int_0^t k(t) \| \varphi^2 \chi_R \psi_j(s) \|_{L_\epsilon^p} ds
\]

\[
\leq C \sup_{s \in [0, t]} \| \varphi^2 \chi_R \psi_j(s) \|_{L_\epsilon^p}.
\]

Here we introduce the Besov spaces \( B = B^{s+\epsilon, q}_p \) with norm \((0 < s < 1)\)

\[
\| f \|_B = (\int_0^\infty (t^{-s} \omega_p(f; t))^q \frac{dt}{t})^{1/q}.
\]

\( \omega_p(f; t) \) denote the continuity modulus

\[
\omega_p(f; t) = \sup_{|\nu| \leq t} \| f(\cdot + \nu) - f(\cdot) \|_{L_p}
\]

with the usual modification for \( p = \infty \). The embedding \( L^{s+2\epsilon}_p \subset B^{s+\epsilon, q}_p \subset L^q_p \), \( 1 \leq q < \infty \) is well known. Straight forward calculations give

\[
\omega_p(\varphi^2 \chi_R \psi_j; t) \leq \| \chi_R \|_{L_\infty} \{ \| \varphi \|_{L_\infty}^2 \omega_3(\psi_j; t) + \ldots \} + C \| \varphi \|_{L_\infty} \omega_\infty(\chi_R; t),
\]

where \( \ldots \) denotes cyclic permutation of \( \varphi, \varphi \) and \( \psi_j \). By embedding above and Sobolev’s inequality it follows that

\[
\sup_{x \in \mathbb{R}} \| \varphi^2 \chi_R \psi_j(s) \|_{B^{s+\epsilon, 1}_p} \leq C \| \varphi \|_{L^2}^2 \| \psi_j \|_{L^2}
\]
for $\epsilon$ small enough and we conclude

$$\| \int_0^t K(t-s)(\varphi^2\chi_R\psi_j(s))ds\|_{L^{p'+\epsilon}_{p'}} \leq C \sup_{s \in [0,t]} \|\varphi^2\chi_R\psi_j(s)\|_{B^{p'+\epsilon,1}}$$

$$\leq C\|\varphi\|_{L^2}\|\psi_j\|_Z$$

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