Relativistic calculations of $R(D^{(*)})$, $R(D_s^{(*)})$, $R(\eta_c)$ and $R(J/\psi)$

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Abstract: Recently, the deviation of the ratios $R(D)$, $R(D^*)$ and $R(J/\psi)$ have been found between experimental data and the Standard Model predictions, which may be the hint of New Physics. In this work, we calculate these ratios within the Standard Model by using the improved instantaneous Bethe-Salpeter method. The emphasis is paid to the relativistic correction of the form factors. The results are $R(D) = 0.312^{+0.006}_{-0.007}$, $R(D^*) = 0.249^{+0.001}_{-0.002}$, $R(D_s) = 0.320^{+0.009}_{-0.009}$, $R(D_s^*) = 0.251^{+0.002}_{-0.003}$, $R(\eta_c) = 0.384^{+0.032}_{-0.042}$, and $R(J/\psi) = 0.267^{+0.009}_{-0.011}$, which are consistent with predictions of other models and the experimental data. The semileptonic decay rates and corresponding form factors at zero recoil are also given.

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1 Introduction

As we all believe, the Standard Model (SM) is not a perfect theory especially at higher scale, so it is significantly to test SM precisely to search the new physics (NP) beyond SM[1]. Recently, several experiments reported a few anomalous results of $R(D^{(*)})$ and $R(J/\psi)$, which are defined as

$$R(D^{(*)}) = \frac{Br(B \to D^{(*)}\tau\nu)}{Br(B \to D^{(*)}\ell\nu)},$$  \hspace{1cm} (1.1)$$

and

$$R(J/\psi) = \frac{Br(B_c \to J/\psi\tau\nu)}{Br(B_c \to J/\psi\mu\nu)},$$ \hspace{1cm} (1.2)$$

respectively. $R(D^{(*)})$ and $R(J/\psi)$ have become interesting things that people believe can be used to explore NP [1–13]. Unlike the branching fractions, the involved physical observables above are lessly affected by the uncertainties that originates from the Cabibbo-Kobayashi-Maskawa (CKM) matrix and the hadronic transition form factors, which can be reduced in these ratios mostly.

These ratios have been measured by BaBar [14, 15], Belle [16–18], and LHCb [19–21], and the averaging $B$-tagged measurements of $R(D)$ and $R(D^*)$ at the $\Upsilon(4S)$ and the LHCb measurements of $R(D^*)$ yields [22]

$$R(D)^{\text{EX}} = 0.407 \pm 0.039 \pm 0.024,$$

$$R(D^*)^{\text{EX}} = 0.304 \pm 0.013 \pm 0.007. \hspace{1cm} (1.3)$$

Theoretically, there are already many precise SM predictions of these ratios. For example, by fitting the lattice calculations and recent experimental data, Bigi and Gambino obtained [23]

$$R(D)^{\text{SM}} = 0.299 \pm 0.003. \hspace{1cm} (1.4)$$

For $R(D^*)$, by using the heavy quark expansion and combining with the recent measurements of $\bar{B} \to D^*\ell\nu\ell$, Fajfer et al obtained [6]

$$R(D^*)^{\text{SM}} = 0.252 \pm 0.003. \hspace{1cm} (1.5)$$

Flavour Lattice Averaging Group (FLAG) combined recent lattice calculations and gave the averaging value [24]

$$R(D)^{\text{SM}} = 0.300 \pm 0.008. \hspace{1cm} (1.6)$$

We can easily see that the experimental values of $R(D)$ and $R(D^*)$ exceed SM predictions by $2.3\sigma$ and $3.4\sigma$ [22], respectively.

Most recently, LHCb reported the ratio of branching fractions [25]

$$R(J/\psi)^{\text{EX}} = \frac{Br(B_c \to J/\psi\tau\nu)}{Br(B_c \to J/\psi\mu\nu)} = 0.71 \pm 0.17 \pm 0.18. \hspace{1cm} (1.7)$$

The result deviates $2\sigma$ away from the SM predictions which lie in the range $R(J/\psi) \in [0.23, 0.29]$, where both the new physics and the systematic errors were considered to affect the differences[26–28].
The deviations of \( R(D^{(*)}) \) and \( R(J/\psi) \) have motivated lots of theoretical studies on the semi-leptonic decays of \( B_s \) to \( S \)-wave charmed mesons. Besides above mentioned papers, the \( B \rightarrow D(D^*) \) decays have been studied by QCD sum rules \([29–31]\), constituent quark models \([32]\), Lattice QCD in the framework of heavy quark effective theory (HQET) \([33, 34]\), and HQET method with the \( O(\alpha_s, \Lambda_{QCD}/m_b) \) and (part of) the \( O(\Lambda_{QCD}^2/m_c^2) \) corrections \([35]\), etc.

For the \( B_c \rightarrow J/\psi(\eta_c) \) transitions, many other approaches, such as perturbative QCD (PQCD) \([36]\), QCD sum rules (QCDSR) \([37]\), light-cone QCD sum rules (LCSR) \([38, 39]\), nonrelativistic QCD (NRQCD) \([40, 41]\), the covariant light-front quark model (CLFQM) \([42]\), the nonrelativistic quark model (NRQM) \([43]\), the relativistic quark model (RQM) \([44]\), the covariant confined quark model (CCQM) \([27]\) etc, have been used.

To explain the deviations, a lot of new physical models \([4, 7–13]\) have been proposed. However, to make a reliable prediction of the NP, one need more precise detections of these observables \( R(D^{(*)}) \) and \( R(J/\psi) \) and more precise theoretical calculations within the SM. For example, recently, the Belle collaboration presented an updated measurement of \( R(D) \) which is \( 0.307 \pm 0.037 \pm 0.016 \) \([45]\). It is in agreement with the SM prediction within \( 0.2\sigma \).

In this work, we will give a relativistic study of \( R(D^{(*)}) \) and \( R(J/\psi) \) by using the improved instantaneous Bethe-Salpeter (BS) method. One of the essential part of this method is the instantaneous BS wave function (also called Salpeter wave function) of mesons, which is achieved by solving the instantaneous BS equation (also called Salpeter equation). These functions are applied to calculate the hadronic transition matrix element. In our previous work \([46]\), a similar method is used to study the channel \( B \rightarrow D(D^*) \), where the results are not quite consistent with the experimental values. One possible reason is that we made approximations when boosting the wave functions of the final mesons to the initial meson rest frame. This method is improved in our another work \([47]\) to study the rare decays of \( B_c \) meson. Here we will systematically use this improved BS method to calculate the semi-leptonic decays of \( B_q \) and \( B_c \) mesons, and make more reliable predictions of \( R(D^{(*)}) \) and \( R(J/\psi) \). Besides that, we will also give other quantities, including form factors, \( G(1) \), the slope \( \rho^2 \), differential leptonic spectra, branching ratios, etc.

The paper is organized as follows. In Section 2, we present the definitions of form factors for different decay channels and the differential decay width. In Section 3, we use the improved BS method to calculated the form factors. In Section 4, we give the numerical results, including the form factors, differential leptonic spectra, decay widths, and the ratio of branching fractions. A conclusion is given finally.

## 2 Formalism of semi-leptonic decays

In this section, we will present the formula of a \( B_q \) \((q = u, d, s, c)\) meson semi-leptonic decays to a charmed meson with the improved BS method. Fig.1 is the Feynmann diagram responsible for the semileptonic decay \( B \rightarrow D_q \ell \bar{\nu} \), whose amplitude is written as

\[
T = \frac{G_F}{\sqrt{2}} V_{ke} \bar{u}_\ell \gamma^\mu (1 - \gamma_5) v_{\bar{\nu}_\ell} \langle D_q | J_\mu | B_q \rangle ,
\]  

(2.1)
where $G_F$ is the Fermi coupling constant, $V_{bc}$ is the CKM matrix element, $J_\mu = V_\mu - A_\mu$ is the charged electroweak current.

The hadronic matrix element $\langle D_q^{(*)}|(V_\mu - A_\mu)|B_q^-\rangle$ can be characterized by the corresponding form factors. If the final meson is a pseudoscalar state, the matrix element can be written as

$$\langle D_q^{(*)}|(V_\mu - A_\mu)|B_q^-\rangle = \begin{pmatrix} f_1(Q^2)(P^\mu + P_f^\mu) + f_2(Q^2)(P^\mu - P_f^\mu) \\ = f_1(Q^2)(P^\mu + P_f^\mu) + f_2(Q^2)Q^\mu \\ = f_1(Q^2)(P^\mu + P_f^\mu) - \frac{M^2 - M_f^2}{Q^2}Q^\mu + f_0(Q^2)\frac{M^2 - M_f^2}{Q^2}Q^\mu \end{pmatrix}, \quad (2.2)$$

where $P$ and $P_f$ are the momenta of the initial and final mesons with masses $M$ and $M_f$, respectively; the definition $Q^\mu = P^\mu - P_f^\mu$ is used; $f_1(Q^2)$, $f_0(Q^2)$ are the form factors which are related to the functions $F_+(Q^2)$ and $F_-(Q^2)$ by

$$f_1(Q^2) = F_+(Q^2), \quad f_0(Q^2) = F_+(Q^2) + \frac{Q^2}{M^2 - M_f^2}F_-(Q^2). \quad (2.3)$$

If the final meson is a vector state, the matrix element is written as

$$\langle D_q^{(*)}|V_\mu|B_q\rangle = ig(Q^2)\epsilon^{\mu\alpha\beta\gamma}P_\alpha P_\beta = \frac{2iV(Q^2)}{M + M_f}\epsilon^{\mu\alpha\beta\gamma}P_\alpha P_\beta,$$

$$\langle D_q^{(*)}|A_\mu|B_q\rangle = f(Q^2)\epsilon^\mu + a_+(Q^2)(\epsilon \cdot q)(P^\mu + P_f^\mu) + a_-(Q^2)(\epsilon \cdot q)Q^\mu$$

$$= 2M_fA_0(Q^2)\frac{\epsilon \cdot q}{Q^2}Q^\mu + (M + M_f)A_1(Q^2)[\epsilon^\mu - \frac{\epsilon \cdot q}{Q^2}Q^\mu] \quad (2.4)$$

$$- A_2(Q^2)\frac{\epsilon \cdot q}{M + M_f}[P^\mu + P_f^\mu - \frac{M^2 - M_f^2}{Q^2}Q^\mu].$$

**Figure 1**: Feynman diagram of the semileptonic $B_q$ decays to a charmed $D_q^{(*)}$ ($q = u, d, s$).
where $\epsilon^\mu$ is the polarization vector of final meson $D^*_q$, $\epsilon^{\mu\nu\alpha\tau}$ is the totally antisymmetric Levi-Civita tensor; $V(Q^2)$, $A_0(Q^2)$, $A_1(Q^2)$, $A_2(Q^2)$ are the form factors which are related to the functions $f(Q^2)$, $a_+(Q^2)$, $a_-(Q^2)$, $v(Q^2)$ by

\[
V(Q^2) = \frac{M + M_f}{2} g(Q^2), \quad A_1(Q^2) = -\frac{f(Q^2)}{M + M_f}, \quad A_2(Q^2) = (M + M_f) a_+(Q^2)
\]

\[
A_0(Q^2) = -\frac{1}{2M_f} (Q^2 a_-(Q^2) + f(Q^2) + (M^2 - M_f^2) a_+(Q^2)).
\]

The square of the transition amplitude can be written as

\[
\sum |T|^2 = \frac{G_F^2}{2} |V_{bc}|^2 L_{\mu\nu} H^{\mu\nu},
\]

where we have summed up the possible polarization of final state. $L_{\mu\nu}$ is the leptonic tensor, which has the form

\[
L_{\mu\nu} \equiv \bar{\nu}_\ell \gamma_\mu (1 - \gamma_5) \nu_\ell \bar{\nu}_\nu (1 + \gamma_5) \gamma_\nu u_\ell
\]

\[
= 8 \left( P_\mu P_\nu + P_\nu P_\mu - g_{\mu\nu} P_1 \cdot P_2 - i\epsilon_{\mu\nu\rho\sigma} P_\rho P_\sigma \right),
\]

where $P_1$ and $P_2$ are the momenta of $l$ and $\bar{\nu}$, respectively. The hadronic tensor $H^{\mu\nu}$ can be written as

\[
H^{\mu\nu} \equiv \sum \langle D_q | J^\mu | B_q \rangle \langle B_q | J^\nu | D_q \rangle
\]

\[
= -\alpha g^{\mu\nu} + \beta_{++} (P + P_f)^\mu (P + P_f)^\nu + \beta_{+-} (P + P_f)^\mu (P - P_f)^\nu
\]

\[
+ \beta_{-+} (P - P_f)^\mu (P + P_f)^\nu + \beta_{--} (P - P_f)^\mu (P - P_f)^\nu
\]

\[
+ i\gamma\epsilon^{\mu\nu\alpha\tau} (P + P_f)_\mu (P - P_f)_\tau,
\]

where the functions $\alpha$, $\beta_{++}$, $\beta_{+-}$, $\beta_{-+}$ and $\gamma$ directly relate to the form factors. For the decays when the final state is a $0^-$ meson, we have

\[
\alpha = \gamma = 0,
\]

\[
\beta_{++} = F_{+}^2, \quad \beta_{+-} = F_{-},
\]

\[
\beta_{-+} = F_+ F_{-}, \quad \beta_{--} = F_- F_+.
\]

When final state is $1^-$ meson, the relations are

\[
\alpha = f^2 + 4M^2 g^2 |\vec{P}_f|^2,
\]

\[
\beta_{++} = \frac{f^2}{4M_f^2} - M^2 g^2 y + \frac{1}{2} \left( \frac{M^2}{M_f^2} (1 - y) - 1 \right) f a_+ + \frac{M^2 |\vec{P}_f|^2}{M_f^2} a_+^2,
\]

\[
\beta_{+-} = \beta_{-+} = g^2 (M^2 - M_f^2) - \frac{f^2}{4M_f^2} - \frac{1}{2} f (a_+ + a_-) - \frac{1}{2} (a_+ - a_-) \frac{M E_f}{M_f^2} + a_+ a_- \frac{M^2 |\vec{P}_f|^2}{M_f^2},
\]

\[
\beta_{--} = -g^2 (M^2 + 2M E_f + M_f^2) + \frac{f^2}{4M_f^2} - f a_- \left( \frac{M E_f}{M_f^2} \right) + a_- \frac{M^2 |\vec{P}_f|^2}{M_f^2},
\]

\[
\gamma = 2g f.
\]
Finally, the decay width $\Gamma$ is read as

$$
\Gamma = \frac{1}{2 M (2\pi)^9} \int \frac{d^3 \vec{P}_f}{2 E_f} \frac{d^3 \vec{P}_\ell}{2 E_\ell} \frac{d^3 \vec{P}_\nu}{2 E_\nu} (2\pi)^4 \delta^4 (P - P_f - P_\ell - P_\nu) \sum |T|^2, $$

(2.11)

where $E_f$, $E_\ell$ and $E_\nu$ the energies of $D_q^\ast$, $\ell$ and $\bar{\nu}_\ell$, respectively. By introducing the symbols $x \equiv E_\ell / M$, $y \equiv (P - P_f)^2 / M^2$, the differential decay width can be written as

$$
\frac{d^2 \Gamma}{dxdy} = |V_{cs}|^2 \frac{G_F^2 M^5}{64 \pi^3} \left\{ \begin{array}{c}
2 \alpha \left( -y + \frac{m^2_f}{M^2} - 3y - \frac{m^2_f}{M^2} \right) \\
+ \beta_+ \left[ 4 \left( 2x \left( 1 - \frac{M^2_f}{M^2} + y \right) - 4x^2 - y \right) + \frac{m^2_f}{M^2} \left( 8x + 4 \frac{M^2_f}{M^2} - 3y - \frac{m^2_f}{M^2} \right) \right] \\
+ (\beta_+ + \beta_-) \frac{m^2_f}{M^2} \left( 2 - 4x + y - 2 \frac{M^2_f}{M^2} + \frac{m^2_f}{M^2} + \beta_- \frac{m^2_f}{M^2} \left( y - \frac{m^2_f}{M^2} \right) \right) \\
- 2\gamma y \left( 1 - \frac{M^2_f}{M^2} - 4x + y + \frac{m^2_\ell}{M^2} \right) + 2\gamma \frac{m^2_\ell}{M^2} \left( 1 - \frac{M^2_f}{M^2} \right) \right\}. 
\right. $$

(2.12)

And the decay width is

$$
\Gamma = \int dxdy \frac{d^2 \Gamma}{dxdy}. $$

(2.13)

3 The improved BS method

The matrix element $\langle D_q^\ast | J_\mu | B^\ast \rangle$ will be calculated by the improved BS method. Within Mandelstam formalism, it can be written as

$$
\langle P_f | J_\mu | P \rangle = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 q_f}{(2\pi)^4} \mbox{Tr} \left[ \chi(P_f, q_f) \Gamma^\mu \chi(P, q) S_2^{-1} (-p_2) \right] (2\pi)^4 \delta^4 (p_2 - p_{2f})
$$

$$
= \int \frac{d^4 q}{(2\pi)^4} \mbox{Tr} \left[ \chi(P_f, q_f) \Gamma^\mu \chi(P, q) S_2^{-1} (-p_2) \right] \bigg|_{q_f=q+\alpha_2 P_f-\alpha_2 P}
$$

(3.1)

where $\chi(P, q)$ and $\bar{\chi}(P, q)$ are the BS wave function of the initial meson and final meson, respectively, and the latter one has the form $\bar{\chi}(P_f, q_f) = \gamma_0 \chi(P_f, q_f) \gamma_0$ in its rest frame; the vertex is $\Gamma^\mu = \gamma^\mu (1 - \gamma^5)$; $S_1$ and $S_2$ are propagators of the quark and anti-quark, respectively. $q$ and is the relative momenta of the quark and antiquark within the initial meson. $p_1$, $p_2$ are respectively the momenta of the quark and anti-quark within the initial meson, which are related to $P$ and $q$ by

$$
p_i = \alpha_i P + J q_i, \quad \alpha_i \equiv \frac{m_i}{m_1 + m_2}, $$

(3.2)

where $m_1$, $m_2$ are the masses of the quark and anti-quark, respectively; $J = 1$ and $-1$ for the cases $i = 1$ and 2, respectively. For the final meson, we define similar relations

$$
p_{1f} = \alpha_{1f} P_f + J q_{1f}, \quad \alpha_{1f} \equiv \frac{m_{1f}}{m_{1f} + m_{2f}}. $$

(3.3)
The BS wave functions fulfill the BS equation which has the form

\[
(p_1 - m_1)\chi(P, q)(p_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(P, k, q)\chi(P, k),
\]

(3.4)

where \(V(P, k, q)\) is the interaction kernel. If we take the instantaneous approximation, the kernel can be reduced to \(V(|\vec{q} - \vec{k}|)\). Now we can introduce two 3-dimensional quantities

\[
\varphi(q_\perp) \equiv i \int \frac{dq_\perp}{2\pi} \chi(P, q), \quad \eta(P, q_\perp) \equiv \int \frac{dk_\perp}{(2\pi)^3} V(k_\perp, q_\perp)\varphi(k_\perp).
\]

(3.5)

where we have used the definitions

\[
q_\perp = \frac{P \cdot q}{M}, \quad q_{\perp}^\mu = q^\mu - q_P P^\mu.
\]

(3.6)

Then the BS equation can be rewritten as

\[
\chi(P, q) = S_1(p_1)\eta(P, q_\perp)S_2(-p_2),
\]

(3.7)

Using above equations, we can write Eq. (3.1) as

\[
\langle P_f | J^{\mu} | P \rangle = \int \frac{d^4q}{(2\pi)^4} \text{Tr} [\eta(P_f, q_f \perp)S_1(p_{f1})\Gamma^{\mu}S_1(p_1)\eta(P, q_\perp)S_2(-p_2)]
\]

\[
= \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{p_f}{M_f} (\tilde{\Lambda}_1^+(p_{f1} \perp) + \tilde{\Lambda}_1^-(p_{f1} \perp)) \eta(P_f, q_f \perp)(\tilde{\Lambda}_2^+(p_{f2} \perp) + \tilde{\Lambda}_2^-(p_{f2} \perp)) \right]
\]

\[
+ \tilde{\Lambda}_2^-(p_{f2} \perp)) \frac{p_f}{M_f} S_1(p_{f1})\Gamma^{\mu}S_1(p_1)\eta(P, q_\perp)S_2(-p_2) \right]
\]

\[
\approx \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{p_f}{M_f} \tilde{\Lambda}_1^+(p_{f1} \perp) \eta(P_f, q_f \perp)\tilde{\Lambda}_2^+(p_{f2} \perp) \frac{p_f}{M_f} \right]
\]

\[
\times S_1(p_{f1})\Gamma^{\mu}S_1(p_1)\eta(P, q_\perp)S_2(-p_2) \right].
\]

(3.8)

In the first line of the above equation, we have used Eq. (3.7) with the definition \(q_f \perp \equiv q_f - \frac{p_f \cdot q_f}{M_f} P_f\) for the final meson; in the second line, we have defined the projection operator of the final meson

\[
\tilde{\Lambda}_i^\pm(p_{f1} \perp) = \frac{1}{2\omega_{if}} \left[ \frac{p_f}{M_f} \tilde{\omega}_{if} \pm (Jm_{if} + \varphi_{if \perp}) \right],
\]

(3.9)

with

\[
\tilde{\omega}_{if} \equiv \sqrt{m_{if}^2 - p_{if \perp}^2} = \sqrt{m_{if}^2 - q_{if \perp}^2}.
\]

(3.10)

The relation \(\frac{p_f}{M_f} = \tilde{\Lambda}_1^+(p_{f1} \perp) + \tilde{\Lambda}_1^-(p_{f1} \perp)\) is also applied. In the last equation, we have omitted the contribution of the negative energy part, which is very small compared with that of the positive energy part.

Next, we express the propagators \(S_i(Jp_i)\), and \(S_i(p_{if})\) also in terms of the projection operators,

\[
S_i(Jp_i) = \frac{\Lambda_i^+(p_{i1} \perp)}{p_{ip} - \omega_i + i\epsilon} + \frac{\Lambda_i^-(p_{i1} \perp)}{p_{ip} + \omega_i - i\epsilon},
\]

(3.11)

\[
S_i(p_{if}) = \frac{\Lambda_i^+(p_{if1} \perp)}{p_{ifp} - \omega_{if} + i\epsilon} + \frac{\Lambda_i^-(p_{if1} \perp)}{p_{ifp} + \omega_{if} - i\epsilon},
\]
Then Eq. (3.8) can be written as

\[
\Lambda_i^\pm (p_{1\perp}) = \frac{1}{2\omega_i} \left[ \frac{p_i}{M} \omega_i \pm (Jm_i + p_{1\perp}) \right], \quad \omega_i \equiv \sqrt{m_i^2 - p_{1\perp}^2} = \sqrt{m_i^2 - q_i^2}, \tag{3.12}
\]

\[
\Lambda_i^\pm (p_{1f\perp}) = \frac{1}{2\omega_{if}} \left[ \frac{p_{if}}{M} \omega_{if} \pm (Jm_{if} + p_{1f\perp}) \right], \quad \omega_{if} \equiv \sqrt{m_{if}^2 - p_{1f\perp}^2}.
\]

where

\[
\langle P_f | J^\mu | P \rangle = \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{\bar{P}_f}{M_f} \tilde{\Lambda}_i^+(p_{1f\perp}) \bar{\eta}(P_f, q_{f\perp}) \tilde{\Lambda}_i^+(p_{2f\perp}) \frac{\bar{P}_f}{M_f} \Lambda_i^+(p_{1f\perp}) \Gamma^\mu \Lambda_i^+(p_{1\perp}) \right]
\]

\[
\times \eta(P, q_{\perp}) \Lambda_2^+(p_{2\perp}) \bigg] \frac{1}{(p_{1f\perp} - \omega_{if} + i\epsilon)(p_{1\perp} - \omega_{if} + i\epsilon)(p_{2f\perp} - \omega_2 + i\epsilon)}, \tag{3.13}
\]

where the quantities \( p_{1p}, p_{2p}, \) and \( p_{1f\perp} \) in the denominator are related to \( q_p \) by

\[
p_{1p} = q_p + \alpha_1 M, \\
p_{2p} = -q_p + \alpha_2 M, \\
p_{1f\perp} = q_p + P_{f\perp} - \alpha_2 M. \tag{3.14}
\]

By integrating out \( q_p \) around the upper plane, we get

\[
\langle P_f | J^\mu | P \rangle = -i \int \frac{d^3q}{(2\pi)^3} \text{Tr} \left[ \frac{\bar{P}_f}{M_f} \tilde{\Lambda}_i^+(p_{1f\perp}) \bar{\eta}(P_f, q_{f\perp}) \tilde{\Lambda}_i^+(p_{2f\perp}) \frac{\bar{P}_f}{M_f} \Lambda_i^+(p_{1f\perp}) \Gamma^\mu \right]
\]

\[
\times \Lambda_i^+(p_{1\perp}) \eta(P, q_{\perp}) \Lambda_2^+(p_{2\perp}) \bigg] \frac{1}{(P_{f\perp} - \omega_2 - \omega_{if}) (M - \omega_2 - \omega_{if})}. \tag{3.15}
\]

The 3-dimensional wave functions (Salpeter wave function) of the initial and final mesons fulfill corresponding Salpeter equations

\[
\varphi^{++}(P, q_{\perp}) = \frac{\Lambda_i^+(p_{1\perp}) \eta(P, q_{\perp}) \Lambda_2^+(p_{2\perp})}{M - \omega_{if} - \omega_{if}} ,
\]

\[
\varphi^{++}(P_f, q_{f\perp}) = \frac{\tilde{\Lambda}_i^+(p_{1f\perp}) \eta(P_f, q_{f\perp}) \tilde{\Lambda}_2^+(p_{2f\perp})}{M_f - \omega_{if} - \omega_{if}} , \tag{3.16}
\]

where we have used the definitions

\[
\varphi^{++}(P, q_{\perp}) = \Lambda_i^+(p_{1\perp}) \frac{\bar{P}}{M} \varphi(P, q_{\perp}) \frac{\bar{P}}{M} \Lambda_2^+(p_{2\perp}) \]

\[
\varphi^{++}(P_f, q_{f\perp}) = \tilde{\Lambda}_i^+(p_{1f\perp}) \frac{\bar{P}_f}{M_f} \varphi(P_f, q_{f\perp}) \frac{\bar{P}_f}{M_f} \tilde{\Lambda}_2^+(p_{2f\perp}) \tag{3.17}
\]

whose explicit form can be found in the Appendix. Then the hadronic transition matrix is written as

\[
\langle P_f | J^\mu | P \rangle = -i \int \frac{d^3q}{(2\pi)^3} \text{Tr} \left[ \frac{\bar{P}_f}{M_f} \varphi^{++}(P_f, q_{f\perp}) \frac{\bar{P}_f}{M_f} L_r \Gamma^\mu \varphi^{++}(P, q_{\perp}) \right] , \tag{3.18}
\]

where

\[
L_r = \frac{(M_f - \omega_{if} - \omega_{if})}{(P_{f\perp} - \omega_{if} - \omega_{if})} \Lambda_i^+(p_{1f\perp}). \tag{3.19}
\]
4 Numerical Results and Discussions

In this work, we use the Cornell potential as the interaction kernel [48], which is a linear scalar potential plus a vector interaction potential

$$V(\vec{q}) = V_s(\vec{q}) + V_v(\vec{q}) \gamma_0 \otimes \gamma_0,$$

where the QCD running coupling constant $\alpha_s(\vec{q}) = \frac{12\pi}{33 - 2N_f} \log(a + \vec{q}^2/\Lambda_{QCD}^2)$; the constants $\lambda$, $\alpha$, $a$, $V_0$ and $\Lambda_{QCD}$ are the parameters charactering the potential. In the Eq.(4.1), the symbol $\otimes$ denotes that the Salpeter wave function is sandwiched between the two $\gamma_0$ matrices. Here we use the same parameters as in Ref.[49] which are fixed by fitting mass spectra

$$a = e = 2.7183, \quad \alpha = 0.060 \text{ GeV}, \quad \lambda = 0.210 \text{ GeV}^2,$$

$$m_u = 0.305 \text{ GeV}, \quad m_d = 0.311 \text{ GeV}, \quad m_s = 0.500 \text{ GeV},$$

$$m_c = 1.62 \text{ GeV}, \quad m_b = 4.96 \text{ GeV}, \quad \Lambda_{QCD} = 0.270 \text{ GeV},$$

and the CKM matrix element $|V_{cb}| = 0.0411$ is from PDG [22].

Since we have solved $0^{-}$ and $1^{-}$ meson in previous paper [48, 50], so we will not show the details how to solve the BS equation and directly give the result with numerical wave functions as input. With Eq.(2.2) and Eq.(2.4), we can get the form factors of $\bar{B}^0 \rightarrow D^{+}e\bar{\nu}_e$, $\bar{B}^0 \rightarrow D^{*+}e\bar{\nu}_e$ and $B_c \rightarrow \eta_c(J/\psi)e\bar{\nu}_e$ which are presented in Fig.2, Fig.3 and Fig.4, respectively. In each figure, we plotted two diagrams, the left one is for the case when the final state is a pseudoscalar, and the right one for the vector final state.

![Figure 2](image-url)  

**Figure 2**: The form factors of decays $\bar{B}^0 \rightarrow D^{(*)+}e\bar{\nu}_e$. 

(a) $f_1(Q^2)$, $f_0(Q^2)$ of $0^- \rightarrow 0^-$  
(b) $A_0(Q^2)$, $A_1(Q^2)$, $A_2(Q^2)$, $V(Q^2)$ of $0^- \rightarrow 1^-$
Besides the form factors, there are other experimental observables, which we will give to check our theoretical calculations. According to the paper [3], The differential width can be written as

$$\frac{d\Gamma (B \rightarrow D\ell \nu_{\ell})}{dw} = \frac{G_F^2 m_D^3}{48\pi^3} (m_B + m_D)^2 (w^2 - 1)^{3/2} \eta_{\text{EW}} G^2(w) |V_{cb}|^2 ,$$  

(4.3)

where the factor $\eta_{\text{EW}} = 1 + \alpha/\pi \ln M_Z/m_B \simeq 1.0066$ takes into account the short distance QED corrections. Moreover, the recoil variable $w$ is defined as the product of the 4-velocities
of the B and D mesons, which is related to $Q^2$ by the formula

$$w = v_B \cdot v_D = \frac{m_B^2 + m_D^2 - Q^2}{2m_B m_D}.$$  \hspace{1cm} (4.4)

For the progress $B^0 \rightarrow D^- \ell \nu_\ell$ we can get [51]

$$G(z) = \frac{2\sqrt{r}}{1 + r} f_1(w) = G(1) \left(1 - 8\rho^2 z + (51\rho^2 - 10) z^2 - (252\rho^2 - 84) z^3\right),$$  \hspace{1cm} (4.5)

where

$$z(w) = \frac{\sqrt{w + 1} - \sqrt{2}}{\sqrt{w + 1} + \sqrt{2}}.$$  \hspace{1cm} (4.6)

Here we define $r = m_D/m_B$ and in the limit of negligible lepton masses, the differential decay rate does not depend on $f_0(w)$. Then we can reprint the form factor of channel $\bar{B}^0 \rightarrow D^+ e \bar{\nu}_e$ in the argument of $w$, which is show in Fig.5(a), where in Fig.5(b), we compare our result of $f_1$ with Belle data, to show the uncertainty of the input parameters, we vary all the model parameters simultaneously around their center values within ±10% and take the largest uncertainty as the errors. Within theoretical uncertainties, our results consist with Belle’s data.

![Figure 5](attachment:image.png)

(a) $f_1(w)$ and $f_0(w)$ without errors  \hspace{1cm} (b) $f_1(w)$ with errors compared with Belle results [51]

**Figure 5:** The form factors with and without errors of the decay $\bar{B}^0 \rightarrow D^+ e \bar{\nu}_e$.

The normalization $G(1)$ and the slope $\rho^2$ are independent parameters which describe the shape and normalization of the measured decay distributions. Using Eq. (4.5) and the numerical values of form factors as well as $|V_{cb}| = 41.1 \times 10^{-3}$ for the PDG [22], we obtain the values of $G(1)$ and the slope $\rho^2$ which are showed in table 1, where the average of experimental data and LQCD’s results are also given as comparison. For both parameters, as the form factors, we get smaller center values, but considering the uncertainties, the
listed results are still consistent with each other. The normalization parameter $G(1)$ and
the slope $\rho^2$ of the other $B_q$ semileptonic decays to pseudoscalar are shown in table 2.

| parameters | $\eta_{EW}G(1)|V_{cb}|[10^{-3}]$ | $\rho^2$ |
|------------|---------------------------------|---------|
| ours       | $35.6^{+4.9}_{-5.0}$            | $0.97 \pm 0.16$ |
| Averages of EX [3] | $41.57 \pm 0.45_{\text{stat}} \pm 0.89_{\text{syst}}$ | $1.128 \pm 0.024_{\text{stat}} \pm 0.023_{\text{syst}}$ |
| LQCD [33] | $42.81(40)$                     | $1.119(71)$ |

Table 2: The normalization and the slope of $B^0 \rightarrow D^- \ell \nu_\ell$, $B_s$ and $B_c$ decays to a pseudoscalar meson.

| Channel                  | $\eta_{EW}G(1)|V_{cb}|[10^{-3}]$ | $\rho^2$ |
|--------------------------|---------------------------------|---------|
| $B^- \rightarrow D^0 \ell \nu_\ell$ | $35.5^{+4.6}_{-4.8}$           | $0.96 \pm 0.13$ |
| $B_s \rightarrow D_s \ell \nu_\ell$ | $35.9^{+4.7}_{-4.5}$           | $1.13 \pm 0.21$ |
| $B_c \rightarrow \eta_c \ell \nu_\ell$ | $37.4^{+4.8}_{-4.2}$           | $2.64 \pm 0.23$ |

For the progress $\bar{B}^0 \rightarrow D^{++} \ell \nu_\ell$, where the final meson is a vector, we can get a similar formula as the pseudoscalar case [34]

$$
\frac{d\Gamma (\bar{B}^0 \rightarrow D^{++} \ell \nu_\ell)}{dw} = \frac{G^2_f m_{D^+}^3}{48\pi^5} (m_B - m_{D^*})^2 \eta_{EW}^2 \chi(w)F^2(w)|V_{cb}|^2,
$$

where

$$
\chi(w)F^2(w) = h_{A_1}(w)\sqrt{w^2 - 1}(w + 1)^2 \left\{ 2 \left[ \frac{1 - 2w + r^2}{(1 - r)^2} \right] \left[ 1 + R_1^2(w) \frac{w^2 - 1}{w + 1} \right] + \left[ 1 + \frac{1 - R_2(w)}{1 - r} \right]^2 \right\}.
$$

and

$$
h_{A_1}(w) = \frac{2\sqrt{m_Bm_{D^*}}}{m_B + m_{D^*}} \frac{A_1(Q^2)}{G^2} \frac{A_{1}^{(Q^2)}}{(m_B + m_{D^*})^2}.
$$

Here we use the parametrization of these form factor functions introduced by Caprini, Lellouch and Neubert (CLN) [52]

$$
h_{A_1}(w) = h_{A_1}(1) \left[ 1 - 8\rho^2 z + (53\rho^2 - 15) z^2 - (231\rho^2 - 91) z^3 \right],
$$

$$
R_1(w) = \frac{h_V(w)}{h_{A_1}(w)} \approx 1.27 - 0.12(w - 1) + 0.05(w - 1)^2,
$$

$$
R_2(w) = \frac{h_{A_1}(w) + rh_{A_2}(w)}{h_{A_1}(w)} \approx 0.80 + 0.11(w - 1) - 0.06(w - 1)^2.
$$

All these parameters, $h_{A_1}(w)$, $R_1(w)$, $R_2(w)$, etc, are alterations of former form factors, which can be obtained using Eq. (2.12). For example, at zero recoil, where $w = 1$ , $z = 0$
and $Q^2 = Q_{\text{max}}^2 = (m_B + m_D)^2$, we have
\[
\mathcal{F}(1) = h_{A_1}(1) = \frac{m_B + m_D}{2\sqrt{m_Bm_D}}A_1 \left( Q_{\text{max}}^2 \right).
\]

(4.11)

We do not show details of others, only display $\eta_{\text{EW}} \mathcal{F}(1) |V_{cb}|$, $\rho^2$, $R_1(1)$ and $R_2(1)$ in table 3, in this table, LQCD results and averages of experimental data are also listed as comparison. Similarly, in table 4, the normalization $\mathcal{G}(1)$ and the slope $\rho^2$ of the other $0^- \rightarrow 1^-$ channels are shown.

**Table 3:** The normalization and the slope of $B^0 \rightarrow D^* \ell \nu_\ell$.

| Parameters | $\eta_{\text{EW}} \mathcal{F}(1) |V_{cb}|$ [10^{-3}] | $\rho^2$ | $R_1(1)$ | $R_2(1)$ |
|-----------|---------------------------------|-------|---------|---------|
| ours      | 40.06$^{+4.98}_{-4.13}$          | 1.04  | ±0.19   |         |
| Averages of EX [3] | 35.61 ± 0.11 stat ± 0.41 syst | 1.205 ± 0.015 stat ± 0.021 syst |       |
| LQCD [34] | 36.71 ± 0.41 ± 0.84             | 1.29(17) |         |

| Parameters | $R_1(1)$ | $R_2(1)$ |
|-----------|---------|---------|
| ours      | 1.55$^{+0.12}_{-0.11}$          | 0.98$^{+0.12}_{-0.11}$ |
| LQCD [34] | 1.40 ± 0.032                     | 0.854 ± 0.020 |

are shown.

**Table 4:** The normalization and the slope of $B^-, B_s$ and $B_c$ decay to a vector meson.

| Channel | $\eta_{\text{EW}} \mathcal{F}(1) |V_{cb}|$ [10^{-3}] | $\rho^2$ | $R_1(1)$ | $R_2(1)$ |
|---------|---------------------------------|-------|---------|---------|
| $B^- \rightarrow D^{*0} \ell \nu_\ell$ | 40.1$^{+4.9}_{-4.2}$ | 1.04 ± 0.18 | 1.55$^{+0.11}_{-0.12}$ | 0.98$^{+0.12}_{-0.11}$ |
| $B_s \rightarrow D_s^* \ell \nu_\ell$ | 39.8$^{+4.6}_{-4.4}$ | 1.18 ± 0.15 | 1.34$^{+0.10}_{-0.11}$ | 1.03$^{+0.15}_{-0.14}$ |
| $B_c \rightarrow J/\psi \ell \nu_\ell$ | 38.8$^{+4.8}_{-4.1}$ | 2.67 ± 0.16 |       |         |

The differential branching fraction is another observable. With the numerical values of form factors calculated by the BS method, we straightforwardly obtain differential branching fractions. In Fig.6(a), the spectra of $B \rightarrow D_{\mu \nu}$ and $B \rightarrow D_{\tau \nu}$ are shown, in Fig.6(b), the spectra for $B \rightarrow D^* \mu \nu$ and $B \rightarrow D^* \tau \nu$ are given. Our results of differential branching fraction for the cases of $B \rightarrow D$ agree very well with the previous study [34] which was a lattice calculation. In Fig.7 and Fig.8, differential branching fractions for $B_s$ and $B_c$ decays are given.

Finally, the decay widths and corresponding branching ratios by BS method are shown in table 5 and table 6, where the table 5 are the cases of $B_q$ decays to a pseudoscalar and table 6 the cases of $B_q$ to a vector. The errors are calculated as before by varying all the parameters simultaneously around their center values within ±10%, and the largest uncertainties are taken as the theoretical errors.

In table 5 and table 6, as comparison we also show other theoretical results as well as the experimental data from PDG. In the last column of table 5, we show our results of the ratios $R(D)$, $R(D_s)$ and $R(\eta_c)$, the corresponding vector cases $R(D^*)$, $R(D_s^*)$ and $R(J/\psi)$ are shown in the last column in table 6. From these two tables, we can see that, our results
of branching ratios with errors consist with experimental data, but the center value of the decay $B^- \to D^0 \tau \nu_\tau$ is smaller than that of experimental data, while we get a larger center value of $B^- \to D^0 \tau \nu_\tau$, these result in a larger $R(D)$ value than experimental data.

To compare with each other, we give the table 7, in which we show the ratios of $R(D)$, $R(D^{(*)})$ and $R(J/\psi)$ by this method, other SM predictions and experimental data. We have lots of available SM theoretical results, but we only show very few of them in this paper. We can see that, though we have relative large theoretical uncertainties in the branching ratios, but we get very small uncertainties in the ratios of $R(D)$, $R(D^{(*)})$ and $R(J/\psi)$, most of the uncertainties are cancelled. This also happened in other theoretical prediction. Our
result of $R(D)$ is close to other SM predictions, while larger than experimental data expect the recent Belle data in 2019. For $R(D^{(*)})$, most theoretical results are consistent, but smaller than experimental data. For $R(J/\psi)$, theoretical prediction of SM is much smaller than experimental data.

In conclusion, we give a relativistic calculation of the ratios $R(D)$, $R(D^{*})$ and $R(J/\psi)$ using the instantaneous BS method which has been improved to provide a more covariant formula to calculate the transition matrix element. Within errors, the theoretical branching ratios are consistent with experimental data, while the ratios of $R(D)$, $R(D^{*})$ and $R(J/\psi)$ consist with other SM predictions, but have obvious difference from the experimental data.

Table 5: Decay widths ($10^{-15}$ GeV), branching ratios (%) and $R(D)$.

| Channels | width | Br | Br [32] | Br [30, 31] | Br (PDG [22]) | Ratio ($\tau/\ell$) |
|----------|-------|----|---------|-------------|---------------|-------------------|
| $B^- \rightarrow D^0 \nu_e$ | 7.68$^{+2.67}_{-1.84}$ | 1.91$^{+0.67}_{-0.45}$ | 2.20 - 3.00 | 1.50 - 2.40 | 2.20$^{0.10}_{-0.07}$ | 0.312$^{+0.006}_{-0.007}$ |
| $B^- \rightarrow D^0 \mu \nu$ | 7.65$^{+2.65}_{-1.83}$ | 1.91$^{+0.66}_{-0.45}$ | 2.20$^{0.10}_{-0.07}$ | 0.313$^{+0.006}_{-0.007}$ |
| $B^- \rightarrow D^0 \tau \nu$ | 2.40$^{+0.77}_{-0.54}$ | 0.60$^{+0.20}_{-0.13}$ | 0.77$^{0.25}_{-0.07}$ |
| $B^0 \rightarrow D^+ \nu_e$ | 7.65$^{+2.67}_{-1.83}$ | 1.77$^{+0.64}_{-0.42}$ | 2.20 - 3.00 | 1.30 - 2.20 | 2.20$^{0.10}_{-0.07}$ | 0.315$^{+0.006}_{-0.007}$ |
| $B^0 \rightarrow D^+ \mu \nu$ | 7.64$^{+2.66}_{-1.84}$ | 1.76$^{+0.62}_{-0.42}$ | 2.20$^{0.10}_{-0.07}$ | 0.313$^{+0.006}_{-0.007}$ |
| $B^0 \rightarrow D^+ \tau \nu$ | 2.39$^{+0.77}_{-0.54}$ | 0.55$^{+0.18}_{-0.12}$ | 1.03$^{0.22}_{-0.07}$ |
| $B^0_s \rightarrow D^+ s \nu_e$ | 7.12$^{+2.73}_{-1.84}$ | 1.64$^{+0.65}_{-0.42}$ | 2.80 - 3.80 | 0.320$^{+0.009}_{-0.009}$ |
| $B^0_s \rightarrow D^+ s \mu \nu$ | 7.10$^{+2.73}_{-1.85}$ | 1.63$^{+0.63}_{-0.42}$ | 2.80 - 3.80 | 0.321$^{+0.009}_{-0.009}$ |
| $B^0_s \rightarrow D^+ s \tau \nu$ | 2.28$^{+0.80}_{-0.54}$ | 0.52$^{+0.18}_{-0.12}$ | 1.03$^{0.22}_{-0.07}$ |
| $B^- \rightarrow \eta_c \nu_e$ | 5.25$^{+2.68}_{-1.80}$ | 0.40$^{+0.21}_{-0.14}$ | 0.45$^{0.44}_{0.55}$ | 0.50$^{0.53}_{0.55}$ | 0.384$^{+0.032}_{-0.042}$ |
| $B^- \rightarrow \eta_c \mu \nu$ | 5.25$^{+2.69}_{-1.79}$ | 0.40$^{+0.21}_{-0.14}$ | 0.45$^{0.44}_{0.55}$ | 0.50$^{0.53}_{0.55}$ | 0.384$^{+0.032}_{-0.042}$ |
| $B^- \rightarrow \eta_c \tau \nu$ | 2.02$^{+0.77}_{-0.55}$ | 0.15$^{+0.06}_{-0.04}$ | 0.45$^{0.44}_{0.55}$ | 0.50$^{0.53}_{0.55}$ | 0.384$^{+0.032}_{-0.042}$ |

Figure 8: The differential branching fractions for $B_c$ meson decays.
Table 6: Decay widths (10^{-15} GeV), branching ratios (%) and $R(D^*)$.

| Channels | width (±0.55) | Br [32] | Br [30, 31] | Br (PDG [22]) | Ratio($\tau/\ell$) |
|----------|---------------|---------|-------------|----------------|-------------------|
| $B^- \rightarrow D^0 e^-\bar{\nu}_e$ | 2.62^{+0.75}_{-0.55} 5.41^{+1.88}_{-1.37} | 5.90 – 7.60 | 4.36 – 8.94 | 4.88 ± 0.10 | 0.249^{+0.001}_{-0.002} |
| $B^- \rightarrow D^0 \mu^-\bar{\nu}_\mu$ | 2.61^{+1.75}_{-0.55} 6.51^{+1.88}_{-1.36} | 5.90 – 7.60 | 4.88 ± 0.10 | 0.249^{+0.001}_{-0.002} |
| $B^- \rightarrow D^0 \tau^-\bar{\nu}_\tau$ | 0.65^{+0.18}_{-0.13} 1.65^{+0.48}_{-0.33} | 1.68 ± 0.20 |
| $B_s^0 \rightarrow D^{*+} e^-\bar{\nu}_e$ | 2.61^{+0.75}_{-0.55} 6.03^{+1.74}_{-1.26} | 5.90 – 7.60 | 4.57 – 9.12 | 4.88 ± 0.10 | 0.248^{+0.001}_{-0.002} |
| $B_s^0 \rightarrow D^{*+} \mu^-\bar{\nu}_\mu$ | 2.60^{+1.75}_{-0.54} 6.01^{+1.74}_{-1.26} | 4.88 ± 0.10 | 0.249^{+0.001}_{-0.002} |
| $B_s^0 \rightarrow D^{*+} \tau^-\bar{\nu}_\tau$ | 0.65^{+0.18}_{-0.13} 1.56^{+0.42}_{-0.30} | 1.67 ± 0.13 |
| $B_c^- \rightarrow J/\psi e^-\bar{\nu}_e$ | 2.11^{+0.75}_{-0.56} 1.62^{+0.60}_{-0.43} | 1.36 [44] | 1.67 [53] | 0.267^{+0.001}_{-0.011} |
| $B_c^- \rightarrow J/\psi \mu^-\bar{\nu}_\mu$ | 2.10^{+0.77}_{-0.55} 1.62^{+0.59}_{-0.30} | 0.268^{+0.009}_{-0.012} |
| $B_c^- \rightarrow J/\psi \tau^-\bar{\nu}_\tau$ | 0.56^{+0.18}_{-0.13} 0.43^{+0.14}_{-0.10} |

Table 7: The experiment data and SM prediction of $R(D)$, $R(D^*)$ and $R(J/\psi)$ with the result in this paper.

| experiment | $R(D)$ | $R(D^*)$ | $R(J/\psi)$ |
|------------|--------|----------|-------------|
| ours       | 0.312±0.006 | 0.248±0.001 | 0.267±0.008 |
| Lattice QCD [34] | 0.300±0.008 | | |
| Heavy quark expansion [6] | 0.252±0.003 | | |
| LCSR [29] | 0.260(8) | | |
| CCQM [27] | | | |
| BarBar [3] | 0.440±0.058 | 0.332±0.024 | 0.24 |
| Belle (2017) [3] | 0.375±0.058 | 0.293±0.038 | |
| Belle (2019) [45] | 0.307±0.037 | 0.283±0.018 | |
| LHCb [3, 25] | 0.285±0.019 | 0.71±0.17 | 0.18 |

except the recent Belle data [45].

A The 0− and 1− Salpeter wave function

The relativistic wave function for a 0− meson has the form [48]

$$
\varphi_{0^-}(q_\perp) = M \left[ \frac{P}{M} \varphi_1(q_\perp) + \varphi_2(q_\perp) + \frac{P \cdot \vec{q}}{M^2} \varphi_3(q_\perp) + \frac{P \cdot \vec{q}}{M^2} \varphi_4(q_\perp) \right] \gamma_5, \quad (A.1)
$$

where the radial wave functions $\varphi_1 \sim \varphi_4$ fulfill the constraint conditions

$$
\varphi_3 = \frac{M (\omega_2 - \omega_1)}{m_1 \omega_2 + m_2 \omega_1} \varphi_2, \\
\varphi_4 = -\frac{M (\omega_1 + \omega_2)}{m_1 \omega_2 + m_2 \omega_1} \varphi_1. \quad (A.2)
$$
The numerical values of $\varphi_1$ and $\varphi_2$ can be obtained by solving the Salpeter equation [48].

For the $1^-$ state, the relativistic wave function has the form [50]

$$
\varphi_{1^-}(q_\perp) = (q_\perp \cdot \epsilon) \left[ f_1(q_\perp) + \frac{P}{M} f_2(q_\perp) + \frac{q_\perp}{M} f_3(q_\perp) + \frac{P q_\perp}{M^2} f_4(q_\perp) \right],
$$

$$
+ M \ell \left[ f_5(q_\perp) + \frac{P}{M} f_6(q_\perp) + \frac{q_\perp}{M} f_7(q_\perp) + \frac{P q_\perp}{M^2} f_8(q_\perp) \right],
$$

(A.3)

where the radial wave functions $f_1 \sim f_8$ fulfill the constraint conditions

$$
f_1(q_\perp) = \frac{q_\perp^2 f_3(\omega_1 + \omega_2) + 2M^2 f_5 \omega_2}{M (m_1 \omega_2 + m_2 \omega_1)},
$$

$$
f_2(q_\perp) = \frac{q_\perp^2 f_4(\omega_1 - \omega_2) + 2M^2 f_6 \omega_2}{M (m_1 \omega_2 + m_2 \omega_1)},
$$

$$
f_7(q_\perp) = \frac{M (\omega_1 - \omega_2)}{m_1 \omega_2 + m_2 \omega_1} f_5,
$$

$$
f_8(q_\perp) = \frac{M (\omega_1 + \omega_2)}{m_1 \omega_2 + m_2 \omega_1} f_6.
$$

(A.4)

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