Unique Continuation for Stochastic Parabolic Equations

Xu Zhang

Abstract

This paper is devoted to a study of the unique continuation property for stochastic parabolic equations. Due to the adapted nature of solutions in the stochastic situation, classical approaches to treat the unique continuation problem for deterministic equations do not work. Our method is based on a suitable partial Holmgren coordinate transform and a stochastic version of Carleman-type estimate.

2000 Mathematics Subject Classification. Primary: 60H15; Secondary: 34A12.

Key Words. Unique Continuation, stochastic parabolic equations, Carleman-type estimate, partial Holmgren coordinate transform.
1 Introduction and main result

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a $C^2$ boundary $\partial G$, and $G_0 \neq G$ be a given subdomain of $G$. Put $Q \triangleq (0, T) \times G$ and $Q_0 \triangleq (0, T) \times G_0$. Throughout this paper, we assume that $a^{ij} \in W^{1,\infty}(0, T; W^{2,\infty}_{loc}(G))$ satisfy $a^{ij} = a^{ji}$ ($i, j = 1, 2, \ldots, n$) and for any open subset $G_1$ of $G$, there is a constant $s_0 = s_0(G_1) > 0$ so that

$$\sum_{i,j} a^{ij} \xi^i \xi^j \geq s_0 |\xi|^2, \quad \forall (t, x, \xi) \equiv (t, x, \xi^1, \ldots, \xi^n) \in (0, T) \times G_1 \times \mathbb{R}^n. \quad (1)$$

Here, we denote $\sum_{i,j=1}^n$ simply by $\sum_{i,j}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a 1 dimensional standard Brownian motion $\{w(t)\}_{t \geq 0}$ is defined. Let $H$ be a Fréchet space. We denote by $L^2_F(0, T; H)$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical quasi-norm; by $L^\infty_F(0, T; H)$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded processes, with the canonical quasi-norm; and by $L^2_F(\Omega; C([0, T]; H))$ the Fréchet space consisting of all $H$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$, with the canonical quasi-norm.

Let us consider the following stochastic parabolic equation:

$$\mathcal{F}z \equiv dz - \sum_{i,j} (a^{ij} z_i) dt = [\langle a, \nabla z \rangle + bz] dt + c z dw(t) \quad \text{in} \ Q. \quad (2)$$

Here $a$, $b$ and $c$ are suitable coefficients. For simplicity, we use the notation $z_i \equiv z_i(x) = \partial z(x)/\partial x_i$, where $x_i$ is the $i$-th coordinate of a generic point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. In a similar manner, in the sequel we use the notation $u_i$, $v_i$, etc. for the partial derivatives of $u$ and $v$ with respect to $x_i$. Also, we denote the scalar product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$.

The main result of this paper is stated as follows:

**Theorem 1.** Let $a \in L^\infty_F(0, T; L^\infty_{loc}(G; \mathbb{R}^n))$, $b \in L^\infty_F(0, T; L^\infty_{loc}(G))$, and $c \in L^\infty_F(0, T; W^{1,\infty}_{loc}(G))$. Then any solution $z \in L^2_F(\Omega; C([0, T]; L^2_{loc}(G))) \cap L^2_F(0, T; H^1_{loc}(G))$ of (2) vanishes identically in $Q \times \Omega$, a.s. $dP$ provided that $z = 0$ in $Q_0 \times \Omega$, a.s. $dP$.

The above result is a unique continuation theorem for stochastic parabolic equations. There are numerous references on the unique continuation for deterministic parabolic equations (see, for example, [2, 5, 6, 9] and so on). However, to the author’s best acknowledge, nothing is known for its stochastic counterpart.

There are two classical tools in the study of the unique continuation for deterministic partial differential equations. One is Holmgren-type uniqueness theorem, another is Carleman-type estimate. Note however that the solution of a stochastic equation is generally non-analytic in time even if the coefficients of the equation are constants. Therefore, one
cannot expect a Holmgren-type uniqueness theorem for the unique continuation for stochastic equations except some very special cases. On the other hand, the usual approach to employ Carleman-type estimate for the unique continuation needs to localize the problem. The difficulty of our present stochastic problem consists in the fact that one cannot simply localize the problem as usual because the classical localization technique may change the adaptedness of solutions, which is a key feature in the stochastic setting. In our equation (2), for the space variable \( x \), we may proceed as in the classical argument. However, for the time variable \( t \), due to the adaptedness requirement, we will have to treat it in a deliberate way. For this purpose, we shall introduce a suitable “partial Holmgren coordinate transform” (see (17)) and deduce a key stochastic version of Carleman-type estimate (see Theorem 2 in the next section).

It is well-known that, unique continuation is an important problem not only in partial differential equations itself, but also in some application problems such as controllability ([10]), inverse problems ([3]), optimal control ([4]) and so on. Numerous studies on unique continuation for deterministic partial differential equations can be found in [1, 11] and the rich references cited therein. It would be quite interesting to extend the deterministic unique continuation theorems to the stochastic ones, but there are many things which remain to be done, and some of which seem to be challenging. In this paper, in order to present the key idea in the simplest way, we do not pursue the full technical generality.

The rest of this paper is organized as follows. In Section 2, as a key preliminary, we show a Carleman-type estimate for stochastic parabolic operators. Section 3 is devoted to the proof of Theorem 1.

2 Carleman-type estimate for stochastic parabolic operators

For any nonnegative and nonzero function \( \psi \in C^{3}(G) \), any \( k \geq 2 \), and any (large) parameters \( \lambda > 1 \) and \( \mu > 1 \), put

\[
\ell = \lambda \alpha, \quad \alpha(t, x) = \frac{e^{\mu \psi(x)} - e^{2\mu |\psi|_{C(G)}}}{t^{k}(T-t)^{k}}, \quad \varphi(t, x) = \frac{e^{\mu \psi(x)}}{t^{k}(T-t)^{k}}.
\]

In the sequel, we will use \( C \) to denote a generic positive constant depending only on \( T \), \( G \), \( G_0 \) and \((a^{ij})_{n \times n}\), which may change from line to line. Also, for \( r \in \mathbb{N} \), we denote by \( O(\mu^{r}) \) a function of order \( \mu^{r} \) for large \( \mu \) (which is independent of \( \lambda \)); by \( O_{\mu}(\lambda^{r}) \) a function of order \( \lambda^{r} \) for fixed \( \mu \) and for large \( \lambda \). We recall the following known result.

Lemma 1. ([7, 8]) Let \( b^{ij} \in C^{1,2}(\overline{Q}) \) satisfying \( b^{ij} = b^{ji} \). Assume that either \((b^{ij})_{n \times n} \) or \(- (b^{ij})_{n \times n} \) is a uniformly positive definite matrix, and \( s_{0}(>0) \) is its smallest eigenvalue. Let \( u \) be a \( C^{2}(\overline{G}) \)-valued semimartingale. Set

\[
\theta = e^{\ell}, \quad v = \theta u, \quad \Psi = 2 \sum_{i,j} b^{ij} \ell_{ij}.
\]
Then for any $x \in G$ and $\omega \in \Omega$ (a.s. $dP$),

\begin{align*}
2 \int_0^T \theta \left[ - \sum_{i,j} (b^{ij}_t v_i)_j + Av \right] [du - \sum_{i,j} (b^{ij}_t u_i)_j dt] + 2 \int_0^T \sum_{i,j} (b^{ij}_t v_i dv)_j \\
+ 2 \int_0^T \sum_{i,j} \left[ \sum_{i',j'} \left( 2b^{ij}_t b^{i'j'}_t \ell_{i',j'} v_{i',j'} + b^{ij}_t b^{i'j'}_t \ell_{i',j'} v_{i',j'} \right) + \Psi b^{ij}_t v_i \right] dt \\
\geq 2 \sum_{i,j} \int_0^T c^{ij}_t v_i v_j dt + \int_0^T Bv^2 dt + \int_0^T \left| - \sum_{i,j} (b^{ij}_t v_i)_j + Av \right|^2 dt \\
- \int_0^T \theta^2 \sum_{i,j} b^{ij}_t du_i v_j - \int_0^T \theta^2 \left[ A - \sum_{i,j} \left( b^{ij}_t \ell_i \ell_j + (b^{ij}_t)_j \right) \right] (dv)^2,
\end{align*}

where

\begin{align*}
A & \triangleq - \sum_{i,j} \left[ b^{ij}_t \ell_i \ell_j - (b^{ij}_t)_j \right] - \Psi, \\
B & \triangleq 2 \left[ A\Psi - \sum_{i,j} (Ab^{ij}_t)_j \right] - A_t - \sum_{i,j} (b^{ij}_t \Psi)_j \ell_i^2, \\
c^{ij} & \triangleq \sum_{i',j'} \left[ 2b^{ij}_t (b^{i'j'}_t \ell_{i',j'}) - (b^{ij}_t b^{i'j'}_t \ell_{i',j'}) \right] - \frac{b^{ij}_t}{2} + \Psi b^{ij}_t.
\end{align*}

Moreover, for $\lambda$ and $\mu$ large enough, it holds

\begin{align*}
A &= -\lambda^2 \mu^2 \varphi^2 \sum_{i,j} b^{ij}_t \psi_i \psi_j + \lambda \varphi O(\mu^2), \\
B &\geq 2s_0^2 \lambda^3 \mu^4 \varphi^3 |\nabla \psi|^4 + \lambda^3 \varphi^3 O(\mu^3) + \lambda^2 \varphi^2 O(\mu^4) + \lambda \varphi O(\mu^4) \\
&\quad + \lambda^2 \varphi^{2+2k-1} O(e^{4|\psi|c_{(\mathcal{C})}}) + \lambda^2 \varphi^{2+k-1} O(\mu^2) + \lambda \varphi^{1+k-1} O(\mu^2), \\
\sum_{i,j} c^{ij} v_i v_j &\geq \left[ s_0^2 \lambda \mu^2 \varphi |\nabla \psi|^2 + \lambda \varphi O(\mu) \right] |\nabla v|^2.
\end{align*}

We now show a Carleman-type estimate for stochastic parabolic operators as follows:

**Theorem 2.** Let $b^{ij} \in C^{1,2}(\Omega)$ satisfying $b^{ij} = b^{ji}$. Assume that either $(b^{ij})_{n \times n}$ or $-(b^{ij})_{n \times n}$ is a uniformly positive definite matrix. Let $\psi \in C^3(\overline{G})$ satisfy

\begin{equation}
\min_{x \in G} |\nabla \psi(x)| > 0.
\end{equation}

Then there is some $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, one can find two constants $C = C(\mu) > 0$ and $\lambda_1 = \lambda_1(\mu)$ so that for all $u \in L^2(\Omega; C([0,T]; L^2(G))) \cap L^2_T(0,T; H^0_0(G))$, $f \in L^2_T(0,T; L^2(G))$ and $g \in L^2_T(0,T; H^1(G))$ with

\begin{equation}
du - \sum_{i,j} (b^{ij}_t u_i)_j dt = f dt + gdw(t), \quad \text{in } Q,
\end{equation}

...
and all $\lambda \geq \lambda_1$, it holds

$$
\lambda^3 \mu^4 \mathbb{E} \int_Q \varphi^3 \theta^2 u^2 dx dt + \lambda \mu^2 \mathbb{E} \int_Q \varphi \theta^2 |\nabla u|^2 dx dt
\leq C \left\{ \mathbb{E} \int_Q \theta^2 f^2 dx dt + \mathbb{E} \int_Q \theta^2 \sum_{i,j} b^{ij} g_i g_j dx dt \\
+ \mathbb{E} \int_Q \theta^2 \left[ A - \sum_{i,j} \left( b^{ij} \ell_i \ell_j + (b^{ij} \ell_i)_j \right) \right] g^2 dx dt \right\},
\tag{10}
$$

where

$$
A \triangleq - \sum_{i,j} (b^{ij} \ell_i \ell_j - b^{ij} \ell_i + b^{ij} \ell_{ij}).
\tag{11}
$$

**Proof.** Recalling that $k \geq 2$ and (3), we get

$$
|\lambda^2 \varphi^2 O(\mu^4) + \lambda \varphi O(\mu^4) + \lambda^2 \varphi^{2+2k-1} O(e^{4\varphi|\psi|\sqrt{\theta}}) + \lambda^2 \varphi^{2+k-1} O(\mu^2) + \lambda \varphi^{1+k-1} O(\mu^2)|
\leq \varphi^3 O(\mu^2).
\tag{12}
$$

Integrating (5) (in Lemma 1) on $G$, taking mean value in both sides, and noting (7) (in Lemma 1) and (12), recalling that $u$, and hence $v$, belongs to $L^2_T(0,T; H^2_0(G))$, we conclude that there is a constant $c_0 > 0$ such that

$$
2 \mathbb{E} \int_Q \theta \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] \left[ du - \sum_{i,j} (b^{ij} u_i)_j dt \right] dx
\geq 2 c_0 \mathbb{E} \int_Q [\lambda \mu^2 \varphi |\nabla \psi|^2 + \lambda \varphi O(\mu)] |\nabla v|^2 dx dt
\tag{13}
+ 2 c_0 \mathbb{E} \int_Q \left[ \lambda^3 \mu^4 \varphi^3 |\nabla \psi|^4 + \lambda^3 \varphi^3 O(\mu^3) + \varphi^3 O(\mu^2) \right] v^2 dx dt
\tag{14}
+ \mathbb{E} \int_Q \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right]^2 dx dt
- \mathbb{E} \int_Q \theta^2 \sum_{i,j} b^{ij} du_i du_j dx - \mathbb{E} \int_Q \theta^2 \left[ A - \sum_{i,j} \left( b^{ij} \ell_i \ell_j + (b^{ij} \ell_i)_j \right) \right] (du)^2 dx.
$$

By (9), we have

$$
2 \mathbb{E} \int_Q \theta \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] \left[ du - \sum_{i,j} (b^{ij} u_i)_j dt \right] dx
= 2 \mathbb{E} \int_Q \theta \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] [f dt + gw(t)] dx
\leq 2 \mathbb{E} \int_Q \theta \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right] f dt dx
\leq \mathbb{E} \int_Q \left[ - \sum_{i,j} (b^{ij} v_i)_j + Av \right]^2 dt dx + \mathbb{E} \int_Q \theta^2 f^2 dt dx,
\tag{14}
$$

5
and
\[ E \int_Q \theta^2 \sum_{i,j} b^{ij} du_i du_j dx + E \int_Q \theta^2 \left[ A - \sum_{i,j} \left( b^{ij} \ell_i \ell_j + (b^{ij} \ell_i) \right) \right] (du)^2 dx \]
\[ = E \int_Q \theta^2 \sum_{i,j} b^{ij} g_i g_j dx dt + E \int_Q \theta^2 \left[ A - \sum_{i,j} \left( b^{ij} \ell_i \ell_j + (b^{ij} \ell_i) \right) \right] g^2 dx dt. \] (15)

Combining (13)–(15), we arrive at
\[ 2c_0 E \int_Q \varphi |\lambda \mu^2 | \nabla \psi |^2 + \lambda O(\mu) | \nabla v |^2 dtdx \]
\[ + 2c_0 E \int_Q \varphi^3 \left[ \lambda^3 \mu^4 | \nabla \psi |^4 + \lambda^3 O(\mu^3) + O(\lambda^2) \right] v^2 dtdx \]
\[ \leq E \int_Q \theta^2 f^2 dx dt + E \int_Q \theta^2 \sum_{i,j} b^{ij} g_i g_j dx dt \]
\[ + E \int_Q \theta^2 \left[ A - \sum_{i,j} \left( b^{ij} \ell_i \ell_j + (b^{ij} \ell_i) \right) \right] g^2 dx dt. \] (16)

Finally, combining (16) and (8), and returning v to u, we obtain the desired estimate (10).

3 Proof of Theorem 1

The proof is divided into several steps.

Step 1. First of all, any neighborhood \( O \) of \( G_0 \) in \( G \) can be covered by a finite number of the images of the following open subset of \( \mathbb{R}^n \)
\[ G' = \left\{ (w_1, \ldots, w_n) \in \mathbb{R}^n \mid 0 < w_n < 1 - \sum_{i=1}^{n-1} w_i^2 \right\} \]
under diffeomorphisms \( x_j = x_j(w_1, \ldots, w_n) \) (1 \( \leq j \leq n \)) of class \( C^2(\overline{G}) \) so that the image of \( \partial G' \cap \{ w_n = 0 \} \) is contained in \( \partial G_0 \). Such diffeomorphisms change the coefficients of the parabolic operator \( \mathcal{F} \) in (2), but do not change its parabolicity and the adoptedness of solutions. Therefore, it suffices to consider \( G = G' \). Note also that those diffeomorphisms do not change the time variable. Hence, to simplify the notations and noting that the original \( z \) vanishes in \( (0, T) \times G_0 \times \Omega \), we may assume the resulting parabolic equation in \( (0, T) \times G' \times \Omega \) reads
\[
\begin{align*}
\mathcal{F} z &\equiv dz - \sum_{i,j} (a^{ij} z_i)_j dt = [(a, \nabla z) + b z] dt + c z dw(t), \quad \text{in } (0, T) \times G' \times \Omega, \\
\text{supp } z &\subset (0, T) \times \{ (x', x_n) \mid x_n \geq 0 \} \times \Omega,
\end{align*}
\]
where \( x' = (x_1, \ldots, x_{n-1}) \) and \( x = (x', x_n) \).
Next, we introduce a “partial Holmgren coordinate transform” \( F : G' \to \mathbb{R}^n \) as follows:
\[
\begin{align*}
\tilde{x}' &= x', \\
\tilde{x}_n &= |x'|^2 + x_n.
\end{align*}
\] (17)

In is easy to see that \( F(G') = \{(\tilde{x}', \tilde{x}_n) \mid |\tilde{x}'|^2 < \tilde{x}_n < 1\} \).

Again, the coordinate transform \( F \) does not change the parabolicity of \( \mathcal{F} \) and the adaptedness of solutions. Hence, to simplify the notations, we may assume the resulting parabolic equation to be the following:
\[
\begin{cases}
\mathcal{F}z \equiv dz - \sum_{i,j} (a^{ij} z_i)_j dt = [\langle a, \nabla z \rangle + bz] dt + czdw(t), & \text{in } (0, T) \times U \times \Omega, \\
\text{supp } z \subset (0, T) \times \{(x', x_n) \mid x_n \geq |x'|^2\} \times \Omega,
\end{cases}
\] (18)

where \( U = \{(x', x_n) \mid |x'|^2 < x_n < 1\} \). It suffices to show that
\[
z \equiv 0, \quad \text{in } (0, T) \times U \times \Omega. \tag{19}
\]

Finally, fix any \( r_0 \) and \( r_1 \) with \( 0 < r_0 < r_1 < 1 \), we choose a function \( \rho \in C^\infty[0, 1] \) so that
\[
\begin{cases}
0 \leq \rho(x_n) \leq 1, & x_n \in [0, 1], \\
\rho(x_n) \equiv 1, & 0 \leq x_n \leq r_0, \\
\rho(x_n) \equiv 0, & r_1 \leq x_n \leq 1.
\end{cases} \tag{20}
\]

Put
\[
u = u(t, x', x_n) \triangleq \rho(x_n) z(t, x', x_n), \quad (t, x) \in (0, T) \times U \times \Omega. \tag{21}
\]

Then, by the first equation in (18), we have
\[
du - \sum_{i,j} (a^{ij} u_i)_j dt = d(\rho z) - \sum_{i,j} (a^{ij} (\rho z)_i)_j dt = \rho \mathcal{F}z - \sum_{i,j} [(a^{ij} \rho_i z)_j + a^{ij} \rho_j z_i] dt = \rho [(\langle a, \nabla z \rangle + bz) - \sum_{i,j} [(a^{ij} \rho_i z)_j + a^{ij} \rho_j z_i]] dt + \rho czdw(t), \quad \text{in } (0, T) \times U \times \Omega; \tag{22}
\]

while, by the second equation in (18) and noting (20), one has
\[
u = 0, \quad \text{on } (0, T) \times \partial U \times \Omega. \tag{23}
\]

**Step 2.** The above transforms do not change the adaptedness of \( z \), and hence that of \( u \). We now apply Theorem 2 to \( u \) given by (21), \( Q \) replaced by \( (0, T) \times U \), and
\[
\psi = \psi(x) = 1 - x_n, \quad x \in \overline{U} \tag{24}
\]
By (10) in Theorem 2, and noting (22), we conclude that there is a constant $C > 0$ such that for any sufficiently large $\lambda$ and $\mu$, it holds

$$\lambda^3 \mu^4 E \int_0^T \int_U \phi^3 \theta^2 u^2 \, dx \, dt + \lambda \mu^2 E \int_0^T \int_U \phi \theta^2 |\nabla u|^2 \, dx \, dt$$

$$\leq C \left[ E \int_0^T \int_U \theta^2 \left\{ \rho(\langle a, \nabla z \rangle + b z) - \sum_{i,j} \left[ (a_{ij} \rho_i z)_j + a_{ij} \rho_j z_i \right] \right\}^2 \, dx \, dt \right.$$  

$$+ E \int_0^T \int_U \theta^2 \sum_{i,j} a_{ij} (\rho c z)_i (\rho c z)_j \, dx \, dt$$

$$+ E \int_0^T \int_U \theta^2 \left[ A - \sum_{i,j} \left( b_{ij} \ell_i \ell_j + (b_{ij} \ell_i)_j \right) \right] (\rho c z)^2 \, dx \, dt \left. \right] \quad (25)$$

where $A = \sum_{i,j} (a_{ij} \ell_i \ell_j - a_{ij} \rho_i \ell_i + a_{ij} \rho_j \ell_i)$.  

By the first estimate in (7) and noting our assumptions on $a$, $b$ and $c$, we get

$$E \int_0^T \int_U \theta^2 \left\{ \rho(\langle a, \nabla z \rangle + b z) - \sum_{i,j} \left[ (a_{ij} \rho_i z)_j + a_{ij} \rho_j z_i \right] \right\}^2 \, dx \, dt$$

$$+ E \int_0^T \int_U \theta^2 \sum_{i,j} a_{ij} (\rho c z)_i (\rho c z)_j \, dx \, dt$$

$$+ E \int_0^T \int_U \theta^2 \left[ A - \sum_{i,j} \left( b_{ij} \ell_i \ell_j + (b_{ij} \ell_i)_j \right) \right] (\rho c z)^2 \, dx \, dt$$

$$\leq C E \int_0^T \int_U \theta^2 (\lambda^2 \mu^2 \varphi^2 \rho^2 + |\nabla z|^2) \, dx \, dt. \quad (26)$$

On the other hand, by (20) and (21), one finds

$$\lambda \mu^2 E \int_0^T \int_U \phi \theta^2 |\nabla u|^2 \, dx \, dt + \lambda^3 \mu^4 E \int_0^T \int_U \phi^3 \theta^2 u^2 \, dx \, dt$$

$$\geq E \int_0^T \int_{U \cap \{0 < x < r_0\}} \theta^2 \left[ \lambda^2 \mu^2 \varphi |\nabla z|^2 + \lambda^3 \mu^4 \varphi^3 \rho^2 \right] \, dx \, dt \quad (27)$$

Hence, combining (25)–(27), and choosing $\lambda$ and $\mu$ large enough, we arrive at

$$E \int_0^T \int_{U \cap \{0 < x < r_0\}} \theta^2 \left[ \lambda^2 \mu^2 \varphi |\nabla z|^2 + \lambda^3 \mu^4 \varphi^3 \rho^2 \right] \, dx \, dt$$

$$\leq C E \int_0^T \int_{U \cap \{r_0 < x < 1\}} \theta^2 (\lambda^2 \mu^2 \varphi^2 \rho^2 + |\nabla z|^2) \, dx \, dt. \quad (28)$$

Step 3. From now on, we fix $\mu$. Also, we fix any $\kappa_1 \in (0, r_0)$. Noting the definition of $\psi$
in (24) implies that \( \theta = \theta(t, x_n) \) is decreasing with respect to \( x_n \), from (28), we deduce that

\[
\lambda^3 \mu^4 E \int_0^T \int_{U \cap \{0 < x_n < \kappa_1\}} |\theta(t, \kappa_1)|^2 \varphi^3 z^2 dx dt \\
\leq E \int_0^T \int_{U \cap \{0 < x_n < \kappa_1\}} |\theta(t, \kappa_1)|^2 \left[ \lambda \mu^2 \varphi \|\nabla z\|^2 + \lambda^3 \mu^4 \varphi^3 z^2 \right] dx dt \\
\leq E \int_0^T \int_{U \cap \{0 < x_n < \kappa_1\}} E \left[ \lambda^2 \mu^2 \varphi \|\nabla z\|^2 + \lambda^3 \mu^4 \varphi^3 z^2 \right] dx dt \\
\leq E \int_0^T \int_{U \cap \{0 < x_n < r_0\}} \theta^2 \left[ \lambda^2 \mu^2 \varphi \|\nabla z\|^2 + \lambda^3 \mu^4 \varphi^3 z^2 \right] dx dt \\
\leq C E \int_0^T \int_{U \cap \{r_0 < x_n < \kappa_1\}} \theta^2 \left( \lambda^2 \mu^2 \varphi^2 z^2 + |\nabla z|^2 \right) dx dt
\]  

(29)

for a constant \( C > 0 \), independent of \( \lambda \).

Further, fix any \( \kappa_2 \in (0, 1) \). Noting that \( \theta = \theta(t, x_n) \) is increasing (resp. decreasing) with respect to \( t \) in \([0, T/2]\) (resp. \((T/2, 1)\)), we deduce that

\[
E \int_0^T \int_{U \cap \{0 < x_n < \kappa_1\}} |\theta(t, \kappa_1)|^2 \varphi^3 z^2 dx dt \\
\geq E \int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U \cap \{0 < x_n < \kappa_1\}} |\theta(t, \kappa_1)|^2 \varphi^3 z^2 dx dt \\
\geq |\theta((1 - \kappa_2)T/2, \kappa_1)|^2 E \int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U \cap \{0 < x_n < \kappa_1\}} \varphi^3 z^2 dx dt, \\
\]

and

\[
E \int_0^T \int_{U \cap \{r_0 < x_n < \kappa_1\}} |\theta(t, r_0)|^2 \left( \lambda^2 \mu^2 \varphi^2 z^2 + |\nabla z|^2 \right) dx dt \\
\leq |\theta(T/2, r_0)|^2 E \int_0^T \int_{U \cap \{r_0 < x_n < \kappa_1\}} \left( \lambda^2 \mu^2 \varphi^2 z^2 + |\nabla z|^2 \right) dx dt.
\]  

(30)

(31)

Combining (29)–(31), we end up with

\[
\lambda^3 \mu^4 |\theta((1 - \kappa_2)T/2, \kappa_1)|^2 E \int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U \cap \{0 < x_n < \kappa_1\}} \varphi^3 z^2 dx dt \\
\leq C |\theta(T/2, r_0)|^2 E \int_0^T \int_{U \cap \{r_0 < x_n < \kappa_1\}} \left( \lambda^2 \mu^2 \varphi^2 z^2 + |\nabla z|^2 \right) dx dt.
\]  

(32)
By (4), (3) and (24), we find
\[ |\theta((1 - \kappa_2)T/2, \kappa_1)|^2 = \exp\left\{ \frac{2\lambda[e^{(1-\kappa_2)\mu} - e^{2\mu}]}{(1 - \kappa_2^2)^{kT/2}} \right\}, \tag{33} \]
\[ |\theta(T/2, r_0)|^2 = \exp\left\{ \frac{2\lambda[e^{(1-r_0)\mu} - e^{2\mu}]}{T^{2k}} \right\}. \]

We now choose \( \kappa_2 \) to be
\[ \kappa_2 = \sqrt{1 - \frac{e^{2\mu} - e^{(1-\kappa_1)\mu}}{e^{2\mu} - e^{(1-r_0)\mu}}}. \tag{34} \]

Since \( \kappa_1 \in (0, r_0) \), one sees that \( \kappa_2 \in (0, 1) \). Moreover, by (34), we have
\[ \frac{e^{(1-\kappa_1)\mu} - e^{2\mu}}{(1 - \kappa_2^2)^{k}} = e^{(1-r_0)\mu} - e^{2\mu}. \tag{35} \]

Combining (33) and (35), it follows that
\[ |\theta((1 - \kappa_2)T/2, \kappa_1)|^2 = |\theta(T/2, r_0)|^2. \tag{36} \]

Now, by (32) and noting (36), we conclude that
\[ \lambda^3 \mu^4 \mathbb{E} \int_{(1-\kappa_2)T/2}^{(1+\kappa_2)T/2} \int_{U \cap \{0 < x_n < \kappa_1\}} \varphi^3 z^2 dx dt \]
\[ \leq C \mathbb{E} \int_0^T \int_{U \cap \{r_0 < x_n < 1\}} (\lambda^2 \mu^2 \varphi^2 z^2 + |\nabla z|^2) dx dt. \tag{37} \]

Letting \( \lambda \to +\infty \) in (37), we conclude that
\[ z \equiv 0, \quad \text{in } \left( (1 - \kappa_2)T/2, (1 + \kappa_2)T/2 \right) \times (U \cap \{0 < x_n < \kappa_1\}) \times \Omega. \]

Hence,
\[ z(T/2, \cdot) \equiv 0, \quad \text{in } (U \cap \{0 < x_n < \kappa_1\}) \times \Omega. \]

Since \( r_0 \) (resp. \( \kappa_1 \)) can be chosen as close to 1 (resp. \( r_0 \)) as one likes, one concludes that
\[ z(T/2, \cdot) \equiv 0, \quad \text{in } U \times \Omega. \]

Replace \( T \) by any given \( t_0 \in (0, T) \). Then, the above argument yields \( z(t_0/2, \cdot) \equiv 0 \) in \( U \times \Omega \). Hence,
\[ z \equiv 0, \quad \text{in } (0, T/2] \times U \times \Omega. \]

Applying this argument to \( z(\cdot + T/2, \cdot) \), it follows that
\[ z \equiv 0, \quad \text{in } (T/2, 3T/4] \times U \times \Omega. \]

Repeating this procedure, we arrive at (19). This completes the proof of Theorem 1.
References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators (III–IV)*, Springer-Verlag, Berlin, 1985.

[2] V. Isakov, *Carleman type estimates in an anisotropic case and applications*, *J. Differ. Equations*, 105 (1993), 217–238.

[3] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin, 1998.

[4] X. Li, J. Yong, *Optimal control theory for infinite-dimensional systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1995.

[5] J.-C. Saut, B. Scheurer, *Unique continuation for some evolution equations*, *J. Differential Equations*, 66 (1987), 118–139.

[6] C. D. Sogge, *A unique continuation theorem for second order parabolic differential operators*, *Ark. Mat.*, 28 (1990), 159–182.

[7] S. Tang, X. Zhang, *Carleman inequality for backward stochastic parabolic equations with general coefficients*, *C. R. Acad. Sci. Paris Série I*, 339 (2004), 775–780.

[8] S. Tang, X. Zhang, *Null controllability for forward and backward stochastic parabolic equations*, Preprint.

[9] H. Yamabe, *A unique continuation theorem of a diffusion equation*, *Ann. Math.*, 69 (1959), 462–466.

[10] E. Zuazua, *Some problems and results on the controllability of partial differential equations*, in *Progress in Mathematics*, Vol: 169, Birkhäuser Verlag, Basel/Switzerland, 1998, 276–311.

[11] C. Zuily, *Uniqueness and Non-Uniqueness in the Cauchy Problem*, Birkhäuser, Boston, 1983.