Hecke Algebra Actions on the Coinvariant Algebra

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Abstract

Two actions of the Hecke algebra of type A on the corresponding polynomial ring are studied. Both are deformations of the natural action of the symmetric group on polynomials, and keep symmetric functions invariant. We give an explicit description of these actions, and deduce a combinatorial formula for the resulting graded characters on the coinvariant algebra.

1 Introduction

1.1. The symmetric group $S_n$ acts on the polynomial ring $P_n = F[x_1, \ldots, x_n]$ (where $F$ is a field of characteristic zero) by permuting variables. Let $I_n$ be the ideal of $P_n$ generated by the symmetric ($S_n$-invariant) polynomials without a constant term. The coinvariant algebra of type $A$ is the quotient $P_n/I_n$. Schubert polynomials, constructed in the seminal papers [BGG] and [De], form a distinguished basis for the coinvariant algebra. These polynomials correspond to Schubert cells in the corresponding flag variety.

1.2. In this paper we present two deformations of this action. For these deformations we can take $F = C(q)$, the field of rational functions in an indeterminate $q$. Most of the results actually hold when $F$ is replaced by the ring $Z[q]$ of polynomials in $q$ with integer coefficients.

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Let $T_1, \ldots, T_{n-1}$ be the standard generators of the Hecke algebra $\mathcal{H}_n(q)$ of type $A$; for definitions see Section 2.1 below.

The first action $\rho_1 : \mathcal{H}_n(q) \to \text{Hom}_F(P_n, P_n)$ is defined using $q$-commutators:

\begin{equation}
\rho_1(T_i) := \partial_i X_i - qX_i \partial_i \quad (1 \leq i < n),
\end{equation}

where

$$\partial_i := \frac{1}{x_i - x_{i+1}}(1 - s_i)$$

is the divided difference operator (see Section 2.2), and $X_i$ denotes multiplication by $x_i$. This action belongs to a family introduced in [LS] (see Section 7.1 below). For a geometric interpretation see [DKLLST]. In [DKLLST, Section 1] such families of operators are attributed to Hirzebruch [Hr].

The second action is naturally defined on monomials by the formula

\begin{equation}
\rho_2(T_i)(x_i^\alpha x_{i+1}^\beta m) := \begin{cases} \frac{q x_i^\beta x_{i+1}^\alpha m}{(1 - q) x_i^\alpha x_{i+1}^\beta + x_i^\alpha x_{i+1}^\beta m}, & \text{if } \alpha > \beta; \\ x_i^\alpha x_{i+1}^\beta m, & \text{if } \alpha = \beta; \\ (1 - q)x_i^\alpha x_{i+1}^\beta m + x_i^\alpha x_{i+1}^\beta m, & \text{if } \alpha < \beta; \end{cases}
\end{equation}

Here $m$ is a monomial involving neither $x_i$ nor $x_{i+1}$.

For a closely related action (defined in the context of quantum groups) see [Ji].

**Claim:** The ideal $I_n$ is invariant under both actions. The resulting graded characters on the coinvariant algebra have a common combinatorial formula.

This shows, in particular, that $\rho_1$ and $\rho_2$ lead to equivalent representations of $\mathcal{H}_n(q)$ on the coinvariant algebra. For $q = 1$ they both reduce to the natural $S_n$ action.

**1.3.** Since the ideal $I_n$ is invariant under both $\rho_1$ and $\rho_2$, the coinvariant algebra $P_n/I_n$ carries appropriate actions $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Let $\chi_1^k$ and $\chi_2^k$ be the characters of these representations on the $k$-th homogeneous component of $P_n/I_n$. We shall give an explicit formula for these characters, using the following combinatorial function.

For any permutation $w \in S_n$, define

\begin{equation}
m_q(w) := \begin{cases} (-q)^m, & \text{if there exists a unique } 0 \leq m < n \text{ so that } w(1) > \ldots > w(m+1) < w(m+2) < \ldots < w(n); \\ 0, & \text{otherwise}. \end{cases}
\end{equation}
Let \( \mu := (\mu_1, \ldots, \mu_t) \) be a partition of \( n \), and let \( S_{\mu} := S_{\mu_1} \times \cdots \times S_{\mu_t} \) be the corresponding Young subgroup of \( S_n \). For any permutation \( w \in S_n \) write \( w = r \cdot (w_1 \times \cdots \times w_t) \), where \( w_i \in S_{\mu_i} \) (\( 1 \leq i \leq t \)) and \( r \) is a representative of minimal length for the left coset \( wS_{\mu} \) in \( S_n \). Define

\[
\text{weight}_q^\mu(w) := \prod_{i=1}^t m_q(w_i).
\]

**Theorem.** For all \( k \geq 0 \) and \( \mu \vdash n \),

\[
\chi^1_k(T_{\mu}) = \chi^2_k(T_{\mu}) = \sum_{\{w \in S_n : \ell(w) = k\}} \text{weight}_q^\mu(w)
\]

where \( T_{\mu} := T_1T_2\cdots T_{\mu_1-1}T_{\mu_1+1}\cdots T_{\mu_1+\ldots+\mu_t-1} \) is the subproduct of \( T_1T_2\cdots T_{n-1} \) omitting \( T_{\mu_1+\ldots+\mu_i} \) for all \( 1 \leq i < t \).

The proof relies on an explicit description of the action with respect to the Schubert basis of the coinvariant algebra. See Theorem 4.1 and Theorem 6.5 below.

**Remark.** This character formula is a natural \( q \)-anologue of a weight formula for \( S_n \) presented in [Ro2]. A formally similar result appears also in Kazhdan-Lusztig theory. Kazhdan-Lusztig characters may be represented as sums of exactly the same weights, but over different summation sets [Ro1 Corollary 4, Ra2].

1.4. The rest of this paper is organized as follows. Preliminaries and necessary background are given in Section 2. In Section 3 we introduce \( q \)-commutators and study their representation matrices. The character formula for \( q \)-commutators is proved in Section 4. Natural randomized operators are introduced in Section 5. In Section 6 we show that the representations induced by the two different actions are equivalent. Section 7 concludes the paper with remarks regarding related families of operators, connections with Kazhdan-Lusztig theory, and open problems.

2 Preliminaries
2.1 The Hecke Algebra of Type A

The symmetric group $S_n$ is generated by $n - 1$ involutions $s_1, s_2, \ldots, s_{n-1}$ satisfying the Moore-Coxeter relations
\[ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad (1 \leq i < n - 1) \]
and
\[ s_is_j = s_js_i \quad \text{if } |i - j| > 1. \]

These involutions are known as the Coxeter generators of $S_n$.

All reduced expressions of a permutation $w \in S_n$ with respect to these generators have the same length, denoted by $\ell(w)$.

The Hecke algebra $H_n(q)$ of type $A$ is the algebra over $F := C(q)$ generated by $n - 1$ generators $T_1, \ldots, T_{n-1}$, satisfying the Moore-Coxeter relations (2.1) and (2.2) as well as the following “deformed involution” relation:
\[ T_i^2 = (1 - q)T_i + q \quad (1 \leq i < n). \]

It should be noted that the last relation is slightly non-standard; this is done in order to get more elegant $q$-analogues. In order to shift to the standard version, one should replace each $T_i$ by $-T_i$.

Let $w$ be a permutation in $S_n$ and let $s_{i_1} \cdots s_{i_{\ell(w)}}$ be a reduced expression for $w$. It follows from the above relations that $T_w := T_{i_1} \cdots T_{i_{\ell(w)}}$ is independent of the choice of reduced expression; the set $\{T_w|w \in S_n\}$ forms a linear basis for $H_n(q)$.

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of $n$. Define $T_\mu \in H_n(q)$ to be the product
\[ T_\mu := T_1T_2\cdots T_{\mu_1-1}T_{\mu_1+1}T_{\mu_1+2}\cdots T_{\mu_1+\mu_2-1}T_{\mu_1+\mu_2+1}\cdots \cdots T_{\mu_1+\cdots+\mu_t}. \]

This is the subproduct of the product $T_1T_2\cdots T_{n-1}$ of all generators (in the usual order), obtained by omitting $T_{\mu_1+\cdots+\mu_i}$ for all $1 \leq i < t$. These elements play an important role in the character theory of $H_n(q)$. For $q = 1$, the elements $T_\mu$ are representatives of all conjugacy classes in $S_n$. It follows that, for $q = 1$, a character is determined by its values at these elements. This is also the case for arbitrary $q$, as the following theorem shows.

**Theorem 2.1.** [Ra1, Theorem 5.1] For each $w \in S_n$ there exists a linear combination
\[ C_w = \sum_\mu a_{w\mu}T_\mu \in H_n(q), \]
with \( a_{w\mu} \in \mathbb{Z}[q] \), such that
\[
\chi(T_w) = \chi(C_w)
\]
for all characters \( \chi \) of the Hecke algebra \( \mathcal{H}_n(q) \).

Let \( \mu = (\mu_1, \ldots, \mu_t) \) be a partition of \( n \). Each permutation \( w \in S_n \) has an associated weight, \( \text{weight}_q^\mu(w) \), as defined in (1.3)–(1.4). The irreducible characters of \( \mathcal{H}_n(q) \) are indexed by the partitions of \( n \). These characters may be represented as weighted sums over Knuth equivalence classes.

**Theorem 2.2.** [Ro1, Corollary 4] Let \( \mathcal{C} \) be a Knuth equivalence class of shape \( \lambda \). Then
\[
\chi^\lambda(T_\mu) = \sum_{w \in \mathcal{C}} \text{weight}_q^\mu(w),
\]
where \( \chi^\lambda \) is the irreducible character of \( \mathcal{H}_n(q) \) corresponding to the shape \( \lambda \).

### 2.2 Schubert Polynomials and the Coinvariant Algebra

#### 2.2.1 Basic Actions on the Polynomial Ring

Let \( x_1, x_2, \ldots, x_n \) be independent variables, and let \( P_n \) be the polynomial ring \( F[x_1, x_2, \ldots, x_n] \). The symmetric group \( S_n \) acts on \( P_n \) by permuting the variables \( x_i \). Let \( \Lambda_n = \Lambda[x_1, x_2, \ldots, x_n] \) be the subring of symmetric functions (i.e., polynomials which are invariant under the action of \( S_n \)). Denote by \( \Lambda_n(i) \) the ring of all polynomials which are invariant under the action of \( s_i \) for a fixed \( i \), \( 1 \leq i < n \). Clearly, \( f \in \Lambda_n(i) \) if and only if \( f \) is symmetric in the variables \( x_i \) and \( x_{i+1} \). We call the polynomials in \( \Lambda_n(i) \), \( i \)-symmetric polynomials.

For \( 1 \leq i < n \) define a divided difference operator \( \partial_i : P_n \to P_n \) by
\[
\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i).
\]
If \( f \in P_n \) is a homogeneous polynomial of degree \( d \), which is not \( i \)-symmetric, then \( \partial_i(f) \) is homogeneous of degree \( d - 1 \). For \( i \)-symmetric polynomials \( \partial_i(f) = 0 \).

The operators \( \partial_i \) satisfy the nil-Coxeter relations [Ma, (2.1)]:
\[
\partial_i^2 = 0 \quad (1 \leq i < n),
\]
\[ \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \quad (1 \leq i < n - 1), \]

(2.6) \[ \partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i - j| > 1. \]

Let \( X_i \) be the operator on \( P_n \) corresponding to multiplication by \( x_i \). Clearly, \( X_i \) increases degree by 1.

The algebra generated by the operators \( \partial_i, 1 \leq i < n \), and \( X_i, 1 \leq i \leq n \) was studied in [De] and [BGG]. The generators satisfy the following commutation relations:

(2.7) \[ \partial_i X_j = X_j \partial_i \quad \text{if } |i - j| > 1, \]

(2.8) \[ \partial_i X_i = 1 + X_{i+1} \partial_i \quad (1 \leq i < n), \]

(2.9) \[ X_i \partial_i = 1 + \partial_i X_{i+1} \quad (1 \leq i < n), \]

2.2.2 Schubert Polynomials

For any sequence \( a = (a_1, \ldots, a_k) \) of positive integers less than \( n \), define \( \partial_a := \partial_{a_1} \cdots \partial_{a_k} \). It follows from the relations (2.5)–(2.6) that if \( a, b \) are two reduced expressions for the same permutation \( w \in S_n \) then \( \partial_a = \partial_b \). We can therefore use the notation \( \partial_w \) for \( w \in S_n \), and in particular \( \partial_{s_i} := \partial_i \) for \( 1 \leq i < n \).

The relation \( \partial_i^2 = 0 \) implies that for any \( w \in S_n \) and any \( 1 \leq i < n \)

(2.10) \[ \partial_i \partial_w = \begin{cases} \partial_{s_i w}, & \text{if } \ell(s_i w) > \ell(w); \\ 0, & \text{if } \ell(s_i w) < \ell(w). \end{cases} \]

For each \( w \in S_n \) we define the Schubert polynomial \( S_w \) by

\[ S_w := \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1}) \]

where \( w_0 \) is the longest element in \( S_n \).

By definition, \( S_w \) is a homogeneous polynomial of degree \( \ell(w) \).

It follows from (2.10) that

(2.11) \[ \partial_i(S_w) = \begin{cases} S_{ws_i}, & \text{if } \ell(ws_i) < \ell(w); \\ 0, & \text{if } \ell(ws_i) > \ell(w). \end{cases} \]
Denote $S_i$ by $S_i$. For any $1 \leq i < n$,
\begin{equation}
S_i = x_1 + \cdots + x_i.
\end{equation}
See [Ma, (4.4)]. The following is an important variant of Monk’s formula.

**Monk’s Formula.** [Ma, (4.11)] Let $1 \leq i < n$ and $w \in S_n$. Then
\begin{equation}
S_i S_w = \sum_t S_{wt}
\end{equation}
where the sum extends over all transpositions $t = t_{jk}$ interchanging $j$ and $k$, with $1 \leq j \leq i < k \leq n$ and $\ell(wt) = \ell(w) + 1$.

The description of the action of the operator $X_i$ on Schubert polynomials follows from Monk’s formula and (2.12):
\begin{equation}
X_i(S_w) = (S_i - S_{i-1})S_w = \sum_{j \leq i < k} S_{wt_{jk}} - \sum_{j < i \leq k} S_{wt_{jk}}
\end{equation}
where all summations are over the transpositions $t = t_{jk}$ satisfying $\ell(wt) = \ell(w) + 1$, with $j$ and $k$ in the indicated ranges.

### 2.2.3 The Coinvariant Algebra

Recall that $\Lambda_n = \Lambda[x_1, \ldots, x_n]$ is the subring of $P_n$ consisting of symmetric functions, and let $I_n$ be the ideal of $P_n$ generated by symmetric functions without a constant term. The quotient $P_n/I_n$ is called the **coinvariant algebra** of $S_n$. $S_n$ acts naturally on this algebra. The resulting representation is isomorphic to the regular representation of the symmetric group. See, e.g., [Hu, §3.6] and [Hi, §II.3].

Let $R^k$ ($0 \leq k \leq \binom{n}{2}$) be the $k$-th homogeneous component of the coinvariant algebra: $P_n/I_n = \bigoplus_{k=0}^{\binom{n}{2}} R^k$. Each $R^k$ is an $F[S_n]$-module; let $\chi^k$ be the corresponding character. The set $\{S_w | w \in S_n\}$ of Schubert polynomials forms a basis for $P_n/I_n$, and the set $\{S_w | \ell(w) = k\}$ forms a basis for $R^k$.

The action of the simple reflections on Schubert polynomials is described by the following proposition, which is a reformulation of [BGG, Theorem 3.14 (iii)].
Proposition 2.3. For any simple reflection $s_i$ and any $w \in S_n$,

$$s_i(G_w) = \begin{cases} 
G_w, & \text{if } \ell(ws_i) > \ell(w); \\
-G_w + \sum_{k<i} G_w(k,i+1,i) - \sum_{k<i} G_w(k,i,i+1) + \\
+ \sum_{k>i+1} G_w(k,i,i+1) - \sum_{k>i+1} G_w(k,i+1,i), & \text{if } \ell(ws_i) < \ell(w), 
\end{cases}$$

where $(k, i, i + 1), (k, i + 1, i)$ are cycles of length 3, and the sums extend over those values of $k$ (in the prescribed ranges) for which $w(k,i,i + 1)$ (respectively, $w(k,i + 1,i)$ ) has the same length as $w$.

Note that the signs in this proposition may depend on notational conventions.

Let $\mu$ be a partition of $n$, and let $\chi^k$ be the $S_n$-character on $R^k$ as above. The following character formula is analogous to Theorem 2.2.

Theorem 2.4. [Ro2, Theorem 2] With the notations of Theorem 2.2,

$$\chi^k(w_{\mu}) = \sum_{\ell(w) = k} \text{weight}_{1}^{\mu}(w)$$

where $\text{weight}_{1}^{\mu}(w)$ is the weight (1.4) with $q = 1$, and $w_{\mu}$ is any permutation of cycle-type $\mu$.

The goal of this paper is to define a Hecke algebra action on the polynomial ring $P_n$ which produces a $q$-analogue of Theorem 2.4.

3 $q$-Commutators

For $1 \leq i < n$ define the $q$-commutator $[\partial_i, X_i]_q$ as follows:

$$[\partial_i, X_i]_q := \partial_i X_i - qX_i \partial_i.$$

It should be noted that for $q = 1$, $[\partial_i, X_i]_1 = s_i$. Let $A_i := [\partial_i, X_i]_q$.

Claim 3.1. The operators $A_i$, $1 \leq i < n$, satisfy the Hecke algebra relations (2.1)–(2.3).

Proof. Combine the nil-Coxeter relations (2.4)–(2.6) for the operators $\partial_i$ with the commutation relations (2.7)–(2.9) for the operators $\partial_i$ and $X_j$.}$
It follows that the mapping $T_i \mapsto A_i$ $(1 \leq i < n)$ may be extended to a representation $\rho_1$ of $H_n(q)$ on $P_n$:

$$\rho_1(T_i) := A_i = [\partial_i, X_i]_q.$$  

**Remark.** The polynomial action of the Coxeter generators of $S_n$ is multiplicative, i.e., for any generator $s_i$ and any two polynomials $f, g \in P_n$:

$$s_i(fg) = s_i(f)s_i(g).$$

Thus each $s_i$ acts on $P_n$ as an algebra automorphism. It follows that if $f$ is $i$-symmetric (see Section 2.2.1) then $s_i(fg) = fs_i(g)$. In contrast to that, the operators $A_i$ are not multiplicative. Actually, (2.3) implies that the eigenvalues of any linear action of a Hecke algebra generator $T_i$ are 1 and $-q$, and taking $f$ to be a $(-q)$-eigenvector of $A_i$, one would get (if $A_i$ were multiplicative) that $f^2$ is a $q^2$-eigenvector, which is impossible for generic $q$.

**Claim 3.2.** For any $1 \leq i < n$, any $i$-symmetric polynomial $f \in \Lambda_n(i)$, and any polynomial $g \in P_n$:

$$(3.2) A_i(f) = f$$

and

$$(3.3) A_i(fg) = fA_i(g)$$

**Proof.** (3.2) is the special case $g = 1$ of (3.3). The latter follows from the fact that for arbitrary polynomials $f, g \in P_n$

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$

Therefore, if $f \in \Lambda_n(i)$ then $\partial_i(fg) = f\partial_i(g)$, so that

$$A_i(fg) = \partial_i(x_i fg) - q x_i \partial_i(fg) = f[\partial_i(x_i g) - q x_i \partial_i(g)] = fA_i(g).$$

It follows that the ideal $I_n$ of $P_n$ is invariant under all the operators $A_i$, giving rise to a representation $\bar{\rho}_1$ of $H_n(q)$ on the quotient $P_n/I_n$, namely: on the coinvariant algebra. Let $\chi_i^k$ be the character of this representation on the $k$-th homogeneous component $R^k$ of $P_n/I_n$ ($0 \leq k \leq \binom{n}{2}$).

Recall from Section 2.2.3 that the set of Schubert polynomials $\{S_w | \ell(w) = k\}$ forms a basis for $R^k$. 

The representation \( \hat{\rho}_1 \) yields a \( q \)-analogue of Proposition 2.3.

**Theorem 3.3.** For any \( 1 \leq i < n \) and \( w \in S_n \),

\[
\hat{\rho}_1(T_i)(\mathcal{S}_w) = \begin{cases} 
\mathcal{S}_w, & \text{if } \ell(ws_i) > \ell(w); \\
-q\mathcal{S}_w + q \sum_{k<i} \mathcal{S}_w(k,i+1,i) - \sum_{k<i} \mathcal{S}_w(k,i,i+1) + \sum_{k>i+1} \mathcal{S}_w(k,i,i+1) - q \sum_{k>i+1} \mathcal{S}_w(k,i,i+1), & \text{if } \ell(ws_i) < \ell(w),
\end{cases}
\]

where \((k,i,i+1),(k,i+1,i)\) are cycles of length 3, and the sums extend over those values of \( k \) (in the prescribed ranges) for which \( w(k,i,i+1) \) (respectively, \( w(k,i+1,i) \)) has the same length as \( w \).

**Proof.** By the commutation relation (2.8),

\[ A_i = 1 + (X_{i+1} - qX_i)\partial_i. \]

Applying (2.11) and (2.13) completes the proof. \( \square \)

## 4 Characters of \( q \)-Commutators

In this section we prove the following \( q \)-analogue of Theorem 2.4.

**Theorem 4.1.** For any partition \( \mu \vdash n \) and \( k \geq 0 \),

\[ \chi^k_1(T_\mu) = \sum_{\ell(w) = k} \text{weight}^\mu_q(w) \]

where \( \text{weight}^\mu_q(w) \) is defined as in (1.4), and the subproduct \( T_\mu \) is defined as in Section 2.1.

First recall that, by Theorem 3.3, for any \( 1 \leq i < n \) and \( w \in S_n \)

\[ \hat{\rho}_1(T_i) = \begin{cases} 
\mathcal{S}_w, & \text{if } \ell(ws_i) > \ell(w); \\
-q\mathcal{S}_w + \sum_{\ell(zs_i) > \ell(z) = \ell(w)} a_{w,z}(q)\mathcal{S}_z, & \text{if } \ell(ws_i) < \ell(w),
\end{cases} \tag{4.1}
\]

where \( \hat{\rho}_1(T_i) \), \( a_{w,z}(q) \in \mathbb{Z}[q] \) and the summation is over all \( z \in S_n \) with \( \ell(zs_i) > \ell(z) = \ell(w) \).

Denote by \( \langle \cdot, \cdot \rangle \) the inner product on \( P_n/I_n \) defined by \( \langle \mathcal{S}_v, \mathcal{S}_w \rangle := \delta_{vw} \), where \( \delta_{vw} \) is the Kronecker delta. In order to prove Theorem 4.1 we need the following lemma.
Lemma 4.2. Let \( w \in S_n \) be a permutation satisfying \( \ell(ws_i) < \ell(w) \). Then, for any \( \pi \in S_n \):

\[
\langle \tilde{A}_1 \tilde{A}_\pi(S_w), S_w \rangle = -q \langle \tilde{A}_\pi(S_w), S_w \rangle.
\]

Proof of Lemma 4.2. It follows from (4.1) that if \( \ell(ws_i) < \ell(w) \) and \( v \in S_n \) then

\[
\langle \tilde{A}_1(S_v), S_w \rangle = \begin{cases} -q, & \text{if } v = w; \\ 0, & \text{if } v \neq w. \end{cases}
\]

Substituting (4.2) into

\[
\langle \tilde{A}_1 \tilde{A}_\pi(S_w), S_w \rangle = \langle \tilde{A}_1(\sum_v \langle \tilde{A}_\pi(S_w), S_v \rangle S_v), S_w \rangle = \sum_v \langle \tilde{A}_\pi(S_w), S_v \rangle \langle \tilde{A}_1(S_v), S_w \rangle
\]

we obtain the desired conclusion.

Proof of Theorem 4.1. In order to prove Theorem 4.1, it suffices to prove that for any partition \( \mu = (\mu_1, \ldots, \mu_t) \) of \( n \)

\[
\langle \tilde{A}_\mu(S_w), S_w \rangle = \text{weight}_q^\mu(w),
\]

where \( \tilde{A}_\mu = \tilde{p}_1(T_\mu) \) is the subproduct of \( \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_{n-1} \) obtained by omitting \( \tilde{A}_{\mu_1 + \cdots + \mu_i} \) for all \( 1 \leq i < t \).

Assume now that there is an index \( i \) such that: \( \tilde{A}_i \) and \( \tilde{A}_{i+1} \) are factors of \( \tilde{A}_\mu \), \( \ell(ws_i) > \ell(w) \), and \( \ell(ws_{i+1}) < \ell(w) \). Then, by Lemma 4.2:

\[
\langle \tilde{A}_{i+1} \tilde{A}_\mu(S_w), S_w \rangle = -q \langle \tilde{A}_\mu(S_w), S_w \rangle.
\]

On the other hand, by the Hecke algebra relations: \( \tilde{A}_{i+1} \tilde{A}_\mu = \tilde{A}_\mu \tilde{A}_i \). Hence

\[
\langle \tilde{A}_{i+1} \tilde{A}_\mu(S_w), S_w \rangle = \langle \tilde{A}_\mu \tilde{A}_i(S_w), S_w \rangle = \langle \tilde{A}_\mu(S_w), S_w \rangle.
\]

The last equality follows from (4.1).

Comparing (4.3) and (4.4) we obtain

\[
-q \langle \tilde{A}_\mu(S_w), S_w \rangle = \langle \tilde{A}_\mu(S_w), S_w \rangle.
\]

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We conclude that, if there is an index \( i \) so that \( \tilde{A}_i \) and \( \tilde{A}_{i+1} \) are factors of \( \tilde{A}_\mu, \ell(ws_i) > \ell(w) \), and \( \ell(ws_{i+1}) < \ell(w) \), then (since \( q \) is indeterminate)
\[
\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \rangle = 0.
\]
Note that in this case \( i, i + 1 \) and \( i + 2 \) belong to the same “block” in the partition \( \mu \), and \( w(i) < w(i + 1) > w(i + 2) \). Thus indeed
\[
\text{weight}^\mu_q(w) = 0.
\]

It remains to check the case in which there is no index \( i \) so that both \( \tilde{A}_i \) and \( \tilde{A}_{i+1} \) appear as factors in the product \( \tilde{A}_\mu \), with \( \ell(ws_i) > \ell(w) \) and \( \ell(ws_{i+1}) < \ell(w) \).

In this case, the relation \( \tilde{A}_i \tilde{A}_j = \tilde{A}_j \tilde{A}_i \) for \( |i - j| > 1 \) gives
\[
\tilde{A}_\mu = \tilde{A}_{i_1} \cdots \tilde{A}_{i_m} \tilde{A}_{i_{m+1}} \cdots \tilde{A}_{i_{\mu_1 + \cdots + \mu_t - 1}},
\]
where \( \ell(ws_{i_j}) < \ell(w) \) for \( j \leq m \), and \( \ell(ws_{i_j}) > \ell(w) \) for \( j > m \). Applying (4.1) and Lemma 4.2 iteratively implies
\[
\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \rangle = (-q)^m = \text{weight}^\mu_q(w),
\]
where \( m = \#\{i \mid \ell(ws_i) < \ell(w) \} \) and \( \tilde{A}_i \) is a factor of \( \tilde{A}_\mu \). \( \square \)

## 5 Randomized Operators

In this section we define a natural “randomized” action of the Coxeter generators on the polynomial ring \( \mathbb{P}_n \), and show that this action satisfies the Hecke algebra relations. This action will be defined initially on monomials, and then extended by linearity to all polynomials in \( \mathbb{P}_n \).

Let \( e_{\alpha,\beta,m} := x_i^\alpha x_{i+1}^\beta m \), where \( m \in \mathbb{P}_n \) is a monomial involving neither \( x_i \) nor \( x_{i+1} \), and \( \alpha, \beta \) are nonnegative integers. Note that the linear subspace \( V_{\alpha,\beta,m} := \text{span}\{e_{\alpha,\beta,m}, e_{\beta,\alpha,m}\} \) is invariant under the action of \( s_i \). In this space \( s_i \) acts as a transposition of the two basis elements (if \( \alpha \neq \beta \)).

A natural randomization of \( e_{\alpha,\beta,m} \) is \( (1 - q)e_{\alpha,\beta,m} + q e_{\beta,\alpha,m} \), where the parameter \( q \) may be interpreted as transition probability \( 0 \leq q \leq 1 \). Motivated by well-known asymmetric physical processes (simulated annealing etc.), we define
\[
R_i^*(e_{\alpha,\beta,m}) := \begin{cases} e_{\beta,\alpha,m}, & \text{if } \alpha \geq \beta; \\
(1 - q)e_{\alpha,\beta,m} + q e_{\beta,\alpha,m}, & \text{if } \alpha < \beta,
\end{cases}
\]
and extend this randomized action to the whole polynomial ring \( P_n \) by linearity. See also [Ji].

**Claim 5.1.** The operators \( R_i^* \), \( 1 \leq i < n \), satisfy the Hecke algebra relations (2.1)–(2.3).

**Proof.** Easily verified by an explicit calculation of the action on the monomials \( e_{\alpha,\beta,m} \). \( \square \)

The operators \( R_i^* \) lead, therefore, to a representation of \( H_n(q) \) on \( P_n \). Unfortunately, the symmetric functions are not invariant under this action. Consider, therefore, the operators whose representation matrices with respect to the basis of monomials are the transposes of those representing \( R_i^* \); i.e., define

\[
R_i(e_{\alpha,\beta,m}) := \begin{cases} 
qe_{\beta,\alpha,m}, & \text{if } \alpha > \beta; \\
(1-q)e_{\alpha,\beta,m} + e_{\beta,\alpha,m}, & \text{if } \alpha < \beta; \\
e_{\alpha,\beta,m}, & \text{if } \alpha = \beta.
\end{cases}
\]

Of course, the operators \( R_i \), \( 1 \leq i < n \), also satisfy the Hecke relations (2.1)–(2.3). It follows that the mapping \( T_i \mapsto R_i \) \((1 \leq i < n)\) may be extended to a representation \( \rho_2 \) of \( H_n(q) \) on \( P_n \):

\[
\rho_2(T_i) := R_i.
\]

The following claim is analogous to Claim 3.2.

**Claim 5.2.** For any \( 1 \leq i < n \), any \( i \)-symmetric polynomial \( f \in \Lambda_n(i) \), and any polynomial \( g \in P_n \):

\[
R_i(f) = f
\]

and

\[
R_i(fg) = fR_i(g)
\]

**Proof.** Direct calculation. \( \square \)

It follows from the first part of the claim that symmetric functions are pointwise invariant under \( \rho_2(H_n(q)) \). By the second part, the ideal \( I_n \) is also invariant under \( \rho_2(H_n(q)) \). Thus, \( \rho_2 \) gives rise to a representation \( \tilde{\rho}_2 \) of \( H_n(q) \) on the coinvariant algebra \( P_n/I_n \).

The action of \( R_i \) on monomials is transparent. Section 6 is devoted to a better understanding of the action on the coinvariant algebra.
6 Properties of the Randomized Action

The following sequence of assertions concerns the connections between the operators $A_i$ and $R_i$.

Claim 6.1. The operators $A_i$ and $R_i$ have the same invariant vectors:

$$\ker(A_i - 1) = \ker(R_i - 1) = \Lambda_n(i),$$

where $\Lambda_n(i)$ is the set (actually, subalgebra) of all polynomials invariant under $s_i$.

Proof. By the definition of $A_i$ and the commutation relations (2.8),

$$\ker(A_i - 1) = \ker[(X_{i+1} - qX_i)\partial_i] = \ker\partial_i = \Lambda_n(i).$$

As for $R_i - 1$, let $V_{\alpha,\beta,m} := \text{span}\{e_{\alpha,\beta,m}, e_{\beta,\alpha,m}\}$ as in the beginning of Section 5. Note that

$$P_n = \bigoplus_{\{\alpha,\beta,m\} | \alpha \geq \beta} V_{\alpha,\beta,m}$$

is a decomposition of $P_n$ into a direct sum of $R_i$-invariant subspaces.

By (5.2), in $V_{\alpha,\beta,m}$:

$$(R_i - 1)(e_{\alpha,\beta,m}) = \begin{cases} -e_{\alpha,\beta,m} + qe_{\beta,\alpha,m}, & \text{if } \alpha > \beta; \\ -qe_{\alpha,\beta,m} + e_{\beta,\alpha,m}, & \text{if } \alpha < \beta; \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

Thus

$$\ker(R_i - 1) \cap V_{\alpha,\beta,m} = \begin{cases} \text{span}\{e_{\alpha,\beta,m} + e_{\beta,\alpha,m}\}, & \text{if } \alpha \neq \beta; \\ \text{span}\{e_{\alpha,\beta,m}\}, & \text{if } \alpha = \beta, \end{cases}$$

implying

$$\ker(R_i - 1) = \bigoplus_{\{\alpha,\beta,m\} | \alpha \geq \beta} \text{span}\{e_{\alpha,\beta,m} + e_{\beta,\alpha,m}\} = \Lambda_n(i).$$

\[\square\]

Claim 6.2.

(a) For any positive integers $i < n$, $j \leq n$ and nonnegative integer $m$,

$$(6.1) \quad (A_i - R_i)(x_j^m) = (1 - q)\partial_i(x_j^{m+1}).$$
(b) For any polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \), the polynomial \((A_i - R_i)f\) is \(i\)-symmetric and divisible by \(1 - q\) :

\[
(A_i - R_i)f \in \Lambda_n(i) \cap (1 - q) \cdot \mathbb{Z}[x_1, \ldots, x_n].
\]

Proof.
(a) If \( j \not\in \{i, i + 1\} \) or \( m = 0 \) then both sides of (6.1) equal zero. If \( j = i \) and \( m \geq 1 \) then

\[
(A_i - R_i)(x_i^m) = (1 + (x_{i+1} - qx_i) \partial_i - R_i)(x_i^m)
\]

\[
= x_i^m + (x_{i+1} - qx_i) \sum_{t=1}^{m} x_i^{-t} x_{i+1}^{t-1} - qx_i^m
\]

\[
= (1 - q) \sum_{t=1}^{m} x_i^{-t} x_{i+1} = (1 - q) \partial_i(x_i^{m+1}).
\]

If \( j = i + 1 \) and \( m \geq 1 \), by Claim 6.1

\[
(A_i - R_i)(x_i^m) = (A_i - R_i)(x_i^m + x_{i+1}^m - x_i^m) = (A_i - R_i)(-x_i^m)
\]

\[
= -(1 - q) \partial_i(x_i^{m+1}) = (1 - q) \partial_i(x_i^{m+1}).
\]

(b) It suffices to prove this claim for monomials \( x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n} \). Any such monomial has the form \( gx_j^m \), where \( g \in \Lambda_n(i) \), \( j \in \{i, i + 1\} \), and \( m \) is a nonnegative integer. It follows from (3.3), (5.4) and (6.1) that

\[
(6.2)
\]

\[
(A_i - R_i)(gx_j^m) = g(A_i - R_i)(x_j^m) = (1 - q)g \partial_i(x_j^{m+1}),
\]

as claimed.

Lemma 6.3. \( \Lambda_n(i) \) is spanned, as a \( \Lambda_n \)-module, by the Schubert polynomials \( \mathcal{S}_w \) with \( \ell(ws_i) > \ell(w) \). The same holds when the ground field \( F \) is replaced by \( \mathbb{Z} \).

Proof. First of all, if \( \ell(ws_i) > \ell(w) \) then, by Proposition 2.3, \( s_i(\mathcal{S}_w) = \mathcal{S}_w \) and therefore \( \mathcal{S}_w \in \Lambda_n(i) \).

By the same proposition, for any \( w \in S_n \)

\[
(1 + s_i)(\mathcal{S}_w) \in \text{span}\{\mathcal{S}_z \mid \ell(zs_i) > \ell(z) = \ell(w)\} \quad \text{(in } P_n/I_n)\)
\]

and therefore, for any \( f \in P_n/I_n : \)

\[
(1 + s_i)(f) \in \text{span}\{\mathcal{S}_z \mid \ell(zs_i) > \ell(z)\} \quad \text{(in } P_n/I_n)\).
\]
If \( f \in \Lambda_n(i)/I_n \) then \((1 + s_i)(f) = 2f\), so that \( \Lambda_n(i)/I_n \) is spanned, as a vector space, by the above Schubert polynomials. Since \( I_n \) is the ideal of \( P_n \) generated by the homogeneous elements in \( \Lambda_n \) of positive degree, a standard argument yields the claimed result for \( \Lambda_n(i) \) as a \( \Lambda_n \)-module.

The result for \( Z \) instead of \( F \) follows from the fact that Schubert polynomials also form a “basis” for polynomials with integer coefficients. \( \square \)

The following proposition provides a description of the action of \( \tilde{\rho}_2(\mathcal{H}_n(q)) \) on Schubert polynomials.

**Proposition 6.4.** For each \( 1 \leq i < n \) and \( w \in S_n \),

\[
\tilde{\rho}_2(T_i)(\mathfrak{S}_w) = \begin{cases} 
\mathfrak{S}_w, & \text{if } \ell(ws_i) > \ell(w); \\
-q\mathfrak{S}_w + \sum_{\ell(zs_i) > \ell(z)} [(1-q)b_{w,z} + c_{w,z}]\mathfrak{S}_z, & \text{if } \ell(ws_i) < \ell(w),
\end{cases}
\]

where \( b_{w,z} \in Z \), \( c_{w,z} \in \{-1,0,1\} \), and the sum extends over all permutations \( z \in S_n \) with \( \ell(zs_i) > \ell(z) = \ell(w) \).

**Proof.** Since \( \mathfrak{S}_w \in \Lambda_n(i) \) for \( w \in S_n \) with \( \ell(ws_i) > \ell(w) \), Claim 6.1 implies that \( \rho_2(T_i)(\mathfrak{S}_w) = \mathfrak{S}_w \) for these \( w \).

Homogeneous components of \( P_n \) are invariant under the action of each \( R_i \), \( 1 \leq i < n \). It follows that the homogeneous components of the coinvariant algebra are invariant under \( \tilde{\rho}_2(\mathcal{H}_n(q)) \), so that each \( \tilde{\rho}_2(T_i)(\mathfrak{S}_w) \) is spanned by Schubert polynomials of degree \( \ell(w) \). Combining this fact with Claim 6.2(b) and Lemma 6.3 shows that for any \( 1 \leq i < n \) and \( w \in S_n \)

\[
(\tilde{\rho}_2(T_i) - \tilde{\rho}_1(T_i))(\mathfrak{S}_w) = (1-q) \sum_{\ell(zs_i) > \ell(z) = \ell(w)} d_{w,z}\mathfrak{S}_z,
\]

where \( d_{w,z} \in Z \), and the sum extends over all permutations \( z \in S_n \) with \( \ell(zs_i) > \ell(z) = \ell(w) \).

Combining (6.3) with (4.1) gives, for any \( w \in S_n \) with \( \ell(ws_i) < \ell(w) \)

\[
\tilde{\rho}_2(T_i)(\mathfrak{S}_w) = \tilde{\rho}_1(T_i)(\mathfrak{S}_w) + (\tilde{\rho}_2(T_i) - \tilde{\rho}_1(T_i))(\mathfrak{S}_w)
= -q\mathfrak{S}_w + \sum_{\ell(zs_i) > \ell(z) = \ell(w)} a_{w,z}\mathfrak{S}_z + (1-q) \sum_{\ell(zs_i) > \ell(z) = \ell(w)} d_{w,z}\mathfrak{S}_z
= -q\mathfrak{S}_w + \sum_{\ell(zs_i) > \ell(z) = \ell(w)} [(1-q)d_{w,z} + a_{w,z}]\mathfrak{S}_z,
\]

where \( a_{w,z} \in \{0, \pm 1, \pm q\} \), \( d_{w,z} \in Z \), and the sum extends over all \( z \in S_n \) with \( \ell(zs_i) > \ell(z) = \ell(w) \),
Substituting
\[ b_{w,z} = \begin{cases} d_{w,z}, & \text{if } a_{w,z} \neq \pm q \\ d_{w,z} - a_{w,z}/q, & \text{if } a_{w,z} = \pm q \end{cases} \]
and
\[ c_{w,z} = \begin{cases} a_{w,z}, & \text{if } a_{w,z} \neq \pm q \\ a_{w,z}/q, & \text{if } a_{w,z} = \pm q \end{cases} \]
completes the proof.

Imitating the proof of Theorem 4.1 we obtain

**Theorem 6.5.**
\[ \chi_k^2(T_\mu) = \sum_{\ell(w)=k} \text{weight}_q^\mu(w) \]
where \( \text{weight}_q^\mu(w) \) are the same weights as in Theorem 4.1.

Combining Theorems 6.5 and 4.1 together with Ram’s result (Theorem 2.1) shows that

**Theorem 6.6.** The representation of \( \mathcal{H}_n(q) \) induced by the \( q \)-commutators \( A_i \) on the homogeneous components of the coinvariant algebra, and the representation induced by the transposed randomized operators \( R_i \) on these components, are equivalent.

**Problem 6.7.** Calculate the coefficients \( b_{w,z} \) in Proposition 6.4.

We conjecture that \( b_{w,z} \in \{-1, 0, 1\} \).

### 7 Final Remarks

#### 7.1 Related Families of Operators

Consider the following family of \( q \)-commutators:
\[ B_i := -[\partial_i, X_{i+1}]_q \quad (1 \leq i < n). \]
This family is closely related to the \( q \)-commutators \( A_i \).

**Fact 7.1.** The operators \( B_i \) satisfy the Hecke algebra relations (2.1)–(2.3).
Proposition 7.2. Let $D_i$ be operators on the polynomial ring $P_n$ of the form
$c_i + P_i(X_i, X_{i+1})\partial_i$, $1 \leq i < n$, where $P_i$ are polynomials of two variables, and $c_i$ are constants. If

(1) $D_i$ satisfy the Hecke algebra relations (2.1)–(2.3) (with $q \neq 0, -1$);
(2) $D_i$ are degree preserving (as operators on $P_n$);
(3) $\Lambda_n$, the subring of symmetric functions, is pointwise invariant under all $D_i$, $1 \leq i < n$;
then, for $n > 2$, either $D_i = A_i$ (\forall i) or $D_i = B_i$ (\forall i).

Proposition 7.2 is related to a general theorem of Lascoux and Schützenberger.

LS Theorem. [LS Theorem 1] Let $x_1, x_2, x_3$ be variables, and let $s_i i = 1, 2$ be the simple transpositions as above. Let $D_i$, $i = 1, 2$ be linear operators on the ring of rational functions $\mathbb{C}(x_1, x_2, x_3)$ (considered as a vector space over $\mathbb{C}$) defined by:

$$D_i = P_i + Q_is_i$$

where $P_i, Q_i \in \mathbb{C}(x_i, x_{i+1})$ are rational functions of the corresponding pair of variables.

Assume that:

(1) $D_1D_2D_1 = D_2D_1D_2$;
(2) $D_1$ is invertible and $P_1 \neq 0$.

Then:

$D_1$ and $D_2$ preserve the ring of polynomials $\mathbb{C}[x_1, x_2, x_3]$ if and only if there exist $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{C}$, so that $\Delta := \alpha\delta - \beta\gamma \neq 0$, $\eta \neq 0$, $\eta \neq \Delta$, and

$$P_i(x_i, x_{i+1}) = (x_i - x_{i+1})^{-1}(\alpha x_i + \beta)(\gamma x_{i+1} + \delta)$$
and
$$Q_i = \eta - P_i.$$

Also, in that case, both $D_1$ and $D_2$ satisfy $D_i^2 = \Delta D_i + \eta(\eta - \Delta)$.

Obviously, the initial conditions in this theorem are quite different from those of Proposition 7.2; but the two families $A_i$ and $B_i$ are common solutions of both problems (for the LS theorem in the special case $\Delta = 1 - q$, $\eta = 1$). Note that for $\Delta = 1 - q$, $\eta = -q$ one gets two other families of the $q$-commutator type, for which $\Lambda_n$ is not pointwise invariant.

It should be mentioned that the family $R_i$ of Sections 5 and 6 is not obtainable from the LS theorem (or from Proposition 7.2).
7.2 Connections with Kazhdan-Lusztig Theory

Theorem 3.3 has a remarkable analogue in Kazhdan-Lusztig theory. In their seminal paper [KL] Kazhdan and Lusztig constructed a canonical basis to Hecke algebra representations. A basic theorem in this theory describes the action of the generators $T_s$ on the canonical basis elements $C_w$.

**Theorem 7.3.** [KL (2.3.a)–(2.3.c)] Let $W$ be a Coxeter group, $s$ a Coxeter generator of $W$, $w \in W$, and $C_w$ the corresponding Kazhdan-Lusztig basis element. Then

$$T_s(C_w) = \begin{cases} 
-C_w, & \text{if } \ell(sw) < \ell(w); \\
qC_w + q^\frac{1}{2} \sum_{\ell(z) < \ell(w), \ell(z) = \ell(w)} a_{w,z}C_z, & \text{if } \ell(sw) > \ell(w),
\end{cases}$$

where the coefficients $a_{w,z} \in \mathbb{Z}$ are independent of $q$.

This analogy leads to similar character formulas in the two theories; see Theorems 2.2 and 4.1. This surprising phenomenon seems to warrant further study.

7.3 Probabilistic Aspects

The parameter $q$ in the definition of the Hecke algebra may be interpreted as a *transition probability*. This gives a natural interpretation to the appearance of the coefficients $q$ and $1 - q$ in the basic Hecke relation (2.3). This observation was fundamental to the definition of the randomized action in Section 5. The operators defined there interpolate between two well-studied extreme cases: sorting ($q = 0$) and mixing ($q = 1$) by means of adjacent transpositions. They also form an interesting link between algebra and physics-motivated optimization.

7.4 Other Weyl Groups

Extension of all the above to other Weyl and Coxeter groups is highly desirable. Preliminary computations indicate that this may not be straightforward.
References

[BGG] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Schubert cells and cohomology of Schubert spaces $G/P$. Usp. Mat. Nauk. 28 (1973), 3-26.

[De] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion. Invent. Math. 21 (1973), 287-301.

[DKLLST] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, J.-Y. Thibon, Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras, Publ. Res. Inst. Math. Sci. 31 (1995), 179–201.

[Hi] H. L. Hiller, Geometry of Coxeter Groups. Res. Notes in Math. 54, Pitman, Boston, 1982.

[Hr] F. Hirzebruch, Topological Methods in Algebraic Geometry. Springer, Berlin, 1966.

[Hu] J. E. Humphreys, Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Math. 29, Cambridge Univ. Press, 1992.

[Ji] M. Jimbo, A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys. 10 (1985), 63–69.

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras. Invent. Math. 53 (1979), 165-184.

[LS] A. Lascoux and M. P. Schützenberger, Symmetrization operators on polynomial rings. Functional Anal. Appl. 21 (1987), 324-326 [English]. No. 4, 77-78 [Russian].

[Ma] I. G. Macdonald, Schubert polynomials. Surveys in Combinatorics, 1991 (Guildford, 1991), 73–99, London Math. Soc. Lecture Note Ser., 166, Cambridge Univ. Press, Cambridge, 1991.

[Ra1] A. Ram, A Frobenius formula for the characters of the Hecke algebras. Invent. Math. 106 (1991), 461–488.

[Ra2] A. Ram, An elementary proof of Roichman’s rule for irreducible characters of Iwahori-Hecke algebras of type A. Mathematical Essays in Honor of Gian-Carlo Rota, 335–342, Progr. Math. 161, Birkhäuser, Boston, 1998.

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[Ro1] Y. Roichman, *A recursive rule for Kazhdan-Lusztig characters*. Adv. in Math. (1997), 24-45.

[Ro2] Y. Roichman, *Schubert polynomials, Kazhdan-Lusztig basis and characters*. (with an appendix: *On characters of Weyl groups*. co-authored with R. M. Adin and A. Postnikov). Discrete Math., to appear.