Layered Tropical Mathematics

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Abstract. Generalizing supertropical algebras, we present a “layered” structure, “sorted” by a semiring which permits varying ghost layers, and indicate how it is more amenable than the “standard” supertropical construction in factorizations of polynomials, description of varieties, properties of the resultant, and for mathematical analysis and calculus, in particular with respect to multiple roots of polynomials. Explicit examples and comparisons are given for various sorting semirings such as the natural numbers and the positive rational numbers, and we see how this theory relates to some recent developments in the tropical literature such as “characteristic 1,” “analytification,” and “hyperfields.”

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1. Introduction

Tropical geometry, a rapidly growing area expounded for example in [Gat, IMS, Lit, MS, SS], has been based on two main approaches. Primarily, tropical curves have been defined as domains of non-differentiability of polynomials over the max-plus algebra, and also in terms of valuation theory applied to curves over Puiseux series. Unfortunately, semirings such as the max-plus algebra possess a limited algebraic structure theory, and also do not reflect the valuation-theoretic properties intrinsic in tropical mathematics (cf. [Pay2], for example), thereby forcing researchers to turn to combinatoric arguments.

To remedy this situation, the first author introduced a modification in [Iz1] of idempotent semirings, which evolved to the supertropical semiring, which we call here the standard supertropical semiring. Its theory is far more compatible with algebraic structure theory and valuation theory than the max-plus algebra, and has been investigated in a sequence of papers including [IR1] and [IR7], focusing on various fundamental
properties involving roots of polynomials, [IR3] – [IR6] and [IKR2] which concern matrices, as well as [IKR1] and [IKR3], which deal directly with supertropical valuation theory. The basic idea is to introduce another “ghost” copy \( R^\nu \) of the max-plus algebra \( \mathbb{R}_{\max,+} \) (graded by \( \{1, \infty\} \)), where by definition \( 1 \cdot 1 = 1 \) and every other sum and product is \( \infty \), which provides a semiring \( \mathcal{R} \) that is a cover of the max-plus algebra \( \mathbb{R}_{\max,+} \) in which we can “resolve” additive idempotents, in the sense that \( a + a = a^\nu \) instead of \( a + a = a \). This modification permits us to detect corner roots of polynomials in terms of the algebraic structure by means of ghosts. Comparing this construction with the “characteristic 1” approach of [CC, Definition 2.7], we have \( 1 + 1 + 1 = 1 + 1 \) rather than \( 1 + 1 = 1 \).

Although the standard supertropical semiring permits one to define tropical varieties algebraically as roots of polynomials (in a certain sense), and is quite successful in working with matrices, it is not so compatible with other basic notions such as multiplicity of roots, and difficulties are encountered in attempting to establish a useful intrinsic differential calculus on the supertropical structure. The standard supertropical theory also has other drawbacks: Unique factorization of polynomials fails, and some of its basic verifications are made via ad hoc arguments.

In this paper we remedy many of these drawbacks by introducing a new structure “sorted” by a (partially) ordered semiring \( L \) that refines the semiring \( \mathcal{R} \) further. This approach introduces different (possibly infinitely many) ghost layers, thereby enabling one intrinsically to handle considerably more mathematical concepts in the tropical environment. One could view \( \mathcal{R} \) as graded by the semiring \( L \), but we prefer the term “layer” or “sort” instead of “grade” because the customary decomposition \( \mathcal{R} = \bigoplus_{\ell \in L} \mathcal{R}_\ell \) is strengthened to the partition \( \mathcal{R} = \bigcup_{\ell \in L} \mathcal{R}_\ell \). Intuitively, the ghost layers now indicate the number of monomials defining a tangible corner root. This paper has four main objectives:

- Introduce the layered structure and develop its basic properties, in analogy with the supertropical theory developed previously. This includes a description of polynomials and their behavior as functions, and the foundations of an intrinsic algebraic geometry in terms of “layering maps.”
- Indicate how the layered structure extends the scope of the supertropical theory, as well as the max-plus theory. For example, we treat multiple roots by means of layers. This enables one to obtain decisive results about resultants. The layered theory gives us a precise description of the resultant in Theorem (although a new intricacy develops with resultants of primary polynomials).
- One of our main earlier supertropical results, concerning the multiplicative properties of the resultant (of two polynomials), cf. [IR7], had been obtained previously only for \( \nu \)-equivalence. The current layered approach gives a considerably stronger result (even for the supertropical case).
- Show how certain supertropical proofs actually become more natural and more accessible in the layered theory.
- Relate these various concepts to notions already existing in the tropical literature. In particular, Definition 4.25 is the layered version of the analytification defined in [Pav2].

One can view the various choices of the sorting set \( L \) as different stages of degeneration of algebraic geometry, where the crudest (for \( L = \{1\} \)) is obtained by passing directly to the max-plus algebra, refined somewhat (for \( L = \{1, \infty\} \)) by the supertropical theory, and further by the layered theory. But taking the sorting set \( L = \mathbb{N} \) (the positive natural numbers) yields better factorization properties for polynomials (although there still are counterexamples to unique factorization), and enables one to work with multiple roots and with derivatives, as seen in Example [ISR]. But in order to have access to integration, we need to take \( L \supset \mathbb{Q}_{>0} \) (the positive rational numbers). Recently, Sheiner has further applied the layered construction to preserve information lost from the original algebraic setting, as outlined in Examples [ISR1,Gus] along the lines of [Par].

Since our goal here is to show how the layered theory enables us to learn much more about the algebraic structure by means of standard techniques in commutative algebra, we do not handle the most general situation, but focus instead on the most important special cases equipped to handle the vast majority of the applications to tropical geometry. A more formal categorical picture is given in [IKR4].

Here is a survey of the main results of this paper. In order not to be forced at the outset to adjoin an extra (minimal) zero element to the max-plus algebra, we find it convenient to work with semirings without zero, which we denote as \textbf{semirings}\footnote{One can think of the ghost elements as uncertainties in classical algebra arising from adding two Puiseux series whose lowest order terms have the same degree.}. (Since semirings lack additive inverses, the zero element loses much
of its significance in semiring theory.) This may seem like a relatively minor matter, since one can always adjoin 0 formally at any stage. However, we shall see that the placement of a zero element involves some basic issues in the theory (including the inclusion of a special 0-layer), which we discuss briefly in §3.3.

In §2 we introduce the basic algebraic structures – ordered monoids and bipotent semirings. These two notions are essentially equivalent, as seen in Corollary 2.22 and Remark 3.1 and each language has its particular advantages. Whereas ordered groups arise as targets of valuations, and have well-studied completions described in Remark 4.4, semirings permit the introduction of notions familiar from classical algebra such as polynomials, matrices, and modules, and provide the framework for our theory.

In §3 we generalize the standard supertropical semiring to a semiring with different layers sorted by elements of an arbitrary (partially) ordered semiring\(^\dagger\) \(L\). In most applications, we build \(R\) by taking copies of an ordered monoid \(\mathcal{G}\). The familiar max-plus algebra is recovered by taking \(L = \{1\}\), whereas the standard supertropical structure is obtained when \(L = \{1, \infty\}\), where \(R_1\) and \(R_\infty\) are two copies of \(\mathcal{G}\), with \(R_1\) identified with the tangible copy of \(\mathcal{G}\) and \(R_\infty\) being the ghost copy. Other useful choices of \(L\) include \(\{1, 2, \infty\}\), \(\mathbb{N}\), \(\mathbb{Q}_{>0}\), \(\mathbb{R}_{>0}\), and the corresponding semirings with 0 adjoined. The 1-layer, which is the part of significance in the “mainstream” tropical theory, is a multiplicative monoid corresponding to the tangible elements in the standard supertropical theory, and the \(\ell\)-layers for \(\ell > 1\) correspond to the ghosts in the standard supertropical theory.

After laying out the basic definitions and the main motivating example (cf. Construction 3.2), we see how this layered structure arises quite naturally in a unified axiomatic framework, focusing on layered domains\(^\dagger\), the analog of supertropical domains \([IR1]\). Although this is the case of interest in most tropical applications, we also present a more general layered version of supertropical semirings, which requires the somewhat more intricate Definition 3.25 and includes Example 3.26. The benefit is that one now has a much broader pool of examples, such as finite tropical structures, given in Example 3.31. We introduce the “surpassing relation” \((5.7)\), generalizing “ghost surpassing”, which plays such a fundamental role in the standard supertropical theory.

In §4.1 we consider the method of “truncating” the sorting semiring\(^\dagger\) \(L\) to pass from a given layered structure to less refined structures (including the standard supertropical structure, and even the max-plus structure). This is a special case of a layered homomorphism of layered domains\(^\dagger\), discussed briefly in §4.2. It turns out that any “natural” layered homomorphism is given in terms of its action on tangible elements, cf. Theorem 4.19.

As in the tropical and “standard” supertropical theories, one then proceeds to the polynomial semiring\(^\dagger\). Polynomials provide a major motivation, since one can define varieties in terms of roots of polynomials (via the layers). Since it is convenient to view polynomials as functions, as well as variants such as Laurent polynomials, we take an excursion in §5.1 to study the function semiring\(^\dagger\) \(\text{Fun}(S, R')\) from an arbitrary set \(S\) to an arbitrary extension \(R'\) of a layered domain\(^\dagger\) \(R\). In Theorem 5.20 we show how one explicit extension, the completion of the 1-divisible closure of the layered 1-semifield\(^\dagger\) of fractions of \(R\), already tests when two polynomials are equal as functions on all extensions of \(R\). Although a self-contained proof is given, this is really a consequence of model theory, and one way of viewing polynomials is as formulae in the appropriate first-order language.

In §6 the layered theory yields a basic Zariski-type correspondence between tropical geometry and ideals of polynomial semirings, by means of layering maps, enabling us to formulate a layered version of Hilbert’s Nullstellensatz in Theorem 6.13. Layering maps also provide a layered Zariski topology, leading to layered varieties, which are expected to play a major role in understanding the underlying algebraic geometry.

The study of polynomials and their corner roots is one area in which the layered theory has a distinct advantage, which we describe in detail in §7 for polynomials in one indeterminate. First of all, we call a monic polynomial \(\alpha\)-primary if \(a\) is the only corner root of \(f\), up to \(\nu\)-equivalence, cf. Lemma 7.4. (In other words, the variety of a primary polynomial is trivial.) As noted by Sheiner \([Sh1]\) and quoted in Theorem 7.23 (under the assumption that the “sorting semiring\(^\dagger\) \(L\) is a semifield), every polynomial can be factored uniquely into primary polynomials that correspond to its corner roots. An explicit computation of layers of evaluations of a polynomial is given in Corollary 7.24. In the traditional algebraic setting over a field, the primary polynomials in one indeterminate are precisely the powers of linear polynomials, i.e., \((\lambda - a)^m\). Over the max-plus algebra, the primary polynomials have the form \(\lambda^m + a^m\).

In the layered theory, primary polynomials are more varied as polynomials, and are the key to the theory of polynomials in one indeterminate, providing counterexamples to unique factorization, but also yielding much information via a transition to the classical polynomial ring, cf. Proposition 7.26.
We then turn to resultants, obtaining some of the main new results of this paper. Surprisingly, despite the negative result in Theorem 4.12, we see in Example 8.8 that the resultant $|R(f,g)|$ is not multiplicative, in the sense that $a$-primary polynomials $f, g$, and $h$ need not satisfy

$$|R(f, gh)| = |R(f, g)||R(f, h)|.$$

But the resultant is multiplicative up to $\nu$-value, cf. Theorem 8.25, and the resultant is multiplicative modulo primary polynomials, cf. Theorem 8.33. These positive results are enough to prove that the resultant is multiplicative in many cases, including the standard supertropical theory, cf. (Theorem 8.37). The positive results for primary polynomials are proved by considering the esoteric behavior of the matrix permanent, which affects the ghost layers of resultants of primary polynomials.

A brief discussion of differentiation and integration is given in [10] including the intriguing fact (Corollary 9.7) that the layered derivative of a separable polynomial is separable. These results open the door to layered discriminants, which are inaccessible in the unlayered theory. The sort of an arbitrary separable polynomial is obtained in Theorem 9.8, thereby enabling us to identify separable polynomials without computing their roots.

In [10] we consider the main examples for the sorting semiring $L$ and discuss their respective advantages and disadvantages in connection with supporting an intrinsic algebraic theory for tropical mathematics. The ability to detect multiple roots requires $R$ to have an extra (finite) ghost layer.

Unique factorization into irreducibles fails in the standard tropical and supertropical theories. Taking $L = \mathbb{N}$ yields enough refinement to permit us to utilize some tools of mathematical analysis, as indicated above. Taking $L = \mathbb{Q}_{>0}$ enables us to factor polynomials in one indeterminate into primary factors, and “almost” restores unique factorization in one indeterminate, as indicated below and explained in detail in [Sh1]. (Unique factorization in several indeterminates still fails in certain situations, but for the geometric reason that certain varieties can be decomposed non-uniquely as unions of hypersurfaces, even when one takes multiple roots into account). One also can integrate polynomials, as observed in Example 10.10.

Although in our applications the sorting semiring $L$ is almost always totally ordered, one can make do with a directed partial order. This more general approach is sketched in Appendix A, where we also formulate and apply the definition of “symmetry” given in [AGG] to the layered structure, by means of a “negation map” on $L$.

The theory works more generally (and perhaps more aesthetically) when the sorting set $L$ is merely a pre-ordered monoid (not necessarily a semiring), as indicated in Appendix B.

Recently, Viro [Vir] has introduced an algebraic approach based on “hyperfields,” which are sets with multi-valued operations. The ghost layers can be viewed in this context, in the following way: Suppose $f$ and $g$ are two Puiseux series with respective value $c$ and $d$. If $c < d$ then $v(f − g) = c$. But if $c = d$, then $f − g$ could have any value $\geq c$, so in our context one could consider $|c| := c + c$ to have the possibility of taking on all values $\geq c$. We examine this connection further in Example 11.6 of Appendix A.

In [CC] a general categorical geometric theory was outlined for “characteristic 1.” In this paper, we intersperse analogs to that paper, to indicate briefly how one may obtain analogous results in the layered theory. Likewise, we indicate how the theory relates to [Par], [Pay1], and [Pay2].

2. Background

Recall that a (multiplicative) monoid $M = (M, \cdot)$ is a semigroup together with a unit element $1 = 1_M$, and a monoid homomorphism $\varphi : M \to M'$ satisfies $\varphi(1) = 1$ and $\varphi(ab) = \varphi(a) \varphi(b)$ for all $a, b \in M$. A monoid ideal $\alpha \triangleleft M$ is a subset $\alpha$ for which $\ell a \in \alpha$ and $a\ell \in \alpha$ for each $a \in \alpha$ and $\ell \in M$.

The monoid $M$ is cancellative if $ab = ac$ implies $b = c$. There is a well-known localization procedure with respect to multiplicatively closed subsets of Abelian monoids, described in [CHWW]; if $M$ is cancellative, then localizing with respect to all of $M$ yields its group of fractions. We say that a monoid $M$ is $\mathbb{N}$-cancellative if $a^n = b^n$ for some $n \in \mathbb{N}$ implies $a = b$.

**Lemma 2.1.** Any ordered, cancellative monoid is $\mathbb{N}$-cancellative.

*Proof.* If $a^n = b^n$ for $a < b$ then $a^n \leq a^{n-1}b \leq b^n = a^n$ implies $a^{n-1}a = a^n = a^{n-1}b$, so $a = b$. \hfill $\square$

**Lemma 2.2.** In any cancellative $\mathbb{N}$-cancellative monoid $L \neq \{1\}$, the powers of any element $a \neq 1$ are all distinct. In particular, $L$ is infinite.

*Proof.* If $a^i = a^j$ for $i < j$ then $a^j - a^i = 1a^i$, implying $a^{j-i} = 1 = 1^{j-i}$, and thus $a = 1$. \hfill $\square$
We say that a monoid $M$ is $\mathbb{N}$-divisible if for each $a \in M$ and $m \in \mathbb{N}$ there is $b \in M$ such that $b^m = a$. For example, $(\mathbb{Q}, +)$ is $\mathbb{N}$-divisible.

**Remark 2.3.** The customary way of embedding an Abelian monoid $M$ into an $\mathbb{N}$-divisible monoid, is to adjoin $\sqrt[n]{a}$ for each $a \in M$ and $m \in \mathbb{N}$, and define

$$\sqrt[n]{a} \sqrt[n]{b} = a^{\sqrt[n]{m}} \cdot b^m.$$  

When $M$ is $\mathbb{N}$-cancellative, one can reduce modulo the equivalence relation

$$a \equiv \sqrt[n]{b} \iff a^m = b^m,$$

thereby yielding an $\mathbb{N}$-cancellative $\mathbb{N}$-divisible monoid $M'$; furthermore, $M'$ is a group if $M$ is a group.

**Remark 2.4.** For any $\mathbb{N}$-divisible, $\mathbb{N}$-cancellative group $(G, \cdot)$, we can uniquely define $n$-th powers for any $n \in \mathbb{N}$, and thus we can uniquely define arbitrary rational powers of elements of $G$. In this way, $G$ becomes a vector space over $\mathbb{Q}$, where we rewrite the operation of $G$ as addition and define $\frac{m}{n} \cdot a$ to be $a^{m/n}$. Indeed,

$$\left(\frac{m_1}{n_1} + \frac{m_2}{n_2}\right) \cdot a = \frac{m_1}{n_1} \cdot a + \frac{m_2}{n_2} \cdot a = \frac{m_1}{n_1} \cdot a + \frac{m_2}{n_2} \cdot a,$$

and the other identifications are analogous. Thus, we may apply linear algebra techniques to the group $G$.

**2.1. Ordered groups and monoids.** The passage to the max-plus algebra in tropical mathematics is done via ordered groups and, more generally, ordered monoids.

**2.1.1. Ordered groups.** The notion of a “pre-ordered group” is quite well known; it satisfies the following property:

$$a \leq b \implies ga \leq gb \text{ and } ag \leq bg,$$

for all elements $a, b, g$.

**Remark 2.5.** Any ordered $\mathbb{N}$-divisible, $\mathbb{N}$-cancellative Abelian group $G$ can be viewed as a metric space (where we define $d(a, b)$ to be $ab^{-1}$), and thus completed in the usual way, as described in [CG] and [Kel] p. 196. This can be done using general model-theoretic methods, [Mar] p. 116 and [Sac] pp. 35,36, since the theory of ordered $\mathbb{N}$-divisible Abelian groups is model complete.

More specifically, let us sketch how one can work directly with Cauchy sequences in the case under consideration here, following ideas given in [Hol]. There is the difficulty of defining convergence to 0, since in general the group $G$ might lack the archimedean property of the real numbers. This is treated in depth in [Hol] §2. As a special case, one can specify a collection $F$ of sets $S$ of subsets of $G$ having the property that $\bigcap_{S \in F} S = \emptyset$ for each $S$ in $F$, and define a sequence $(a_i) := \{a_1, a_2, \ldots\}$ to be $F$-Cauchy if there is some $S \in F$ satisfying the property that for each $S \in S$ there is $m = m(S)$ depending on $S$ such that $a_i a_j^{-1} \in S$ for all $i, j > m$. Intuitively, this means that $a_i a_j^{-1}$ is “small” whenever $i, j$ are “sufficiently large.” An $F$-null sequence is an $F$-Cauchy sequence satisfying the property that for each $S \in S$ there is $m = m(S)$ depending on $S$ such that $a_i \in S$ for all $i > m$.

Here, we define $S_{a, \varepsilon} := \{\alpha a : \alpha < \varepsilon\}$, and take $F$ to be the collection of sets

$$S_a = \{S_{a, \varepsilon} : 0 < \varepsilon \in \mathbb{Q}\},$$

for $a \in G$. If two sequences are $F$-Cauchy with respect to $S_a$ and $S_b$, respectively, then their product is $F$-Cauchy with respect to $S_{ab}$. Thus, as in the general theory of Cauchy sequences, the set of $F$-Cauchy sequences is a pre-ordered group, having a subgroup comprised of the set of $F$-null sequences, and the quotient group is an ordered group $G$ which we call the $F$-completion of $G$.

Now writing the operation additively and viewing $G$ as a $\mathbb{Q}$-vector space as in Remark 2.4, we can view its $F$-completion as a vector space over the completion $\mathbb{R}$ of $\mathbb{Q}$.

**2.1.2. Ordered monoids.** Ordered monoids are trickier than ordered groups, since, for example, $a > b$ in $(\mathbb{R}, \cdot)$ does not imply $-a > -b$, but rather $-a < -b$. The easiest way around this is to require (2.1) to hold anyway; in other words, to declare that all elements are non-negative; we call such a monoid positively ordered.

**Remark 2.6.** If $M$ is a positively ordered monoid, then its divisible closure $M'$ of Remark 2.5 is also positively ordered, by putting $\sqrt[n]{a} \geq \sqrt[n]{b}$ iff $a^m \geq b^m$. 


2.2. Semirings without zero. As with the “standard” supertropical structure, we work in the language of semirings and use \([Gol]\) as a general reference. Unless explicitly stated otherwise, we use algebraic notation, in which \(\mathbb{1}_R\) denotes the multiplicative identity of \(R\). (For our examples, we occasionally use “logarithmic notation,” in which \(\mathbb{1}_R = 0\).)

It is more convenient to consider, slightly more generally, a \emph{semiring without zero}, which we notate as \(R^\dagger\), to be a structure \((R, +, \cdot, 0, \mathbb{1}_R)\) such that \((R, +, \mathbb{1}_R)\) is a monoid and \((R, +)\) is an Abelian semigroup, with distributivity of multiplication over addition on both sides. (In other words, a semiring \(R^\dagger\) does not necessarily have the zero element \(0_R\), but any semiring can also be considered as a semiring \(R^\dagger\).) Ironically, we do assume that every semiring \(R^\dagger\) has the unit element \(\mathbb{1}_R\). A \emph{semifield} \(R^\dagger\) is a semiring \(R^\dagger\) for which \((R, \cdot, \mathbb{1}_R)\) is an Abelian group.

An \emph{ideal} \(A\) of a semiring \(R^\dagger\), denoted \(A \trianglelefteq R\), is defined to be a sub-semigroup of \((R, +)\) which is also a monoid ideal of \((R, \cdot)\). An ideal \(P\) of \(R\) is \emph{prime} if \(ab \in P\) implies \(a \in P\) or \(b \in P\).

**Definition 2.7.** A semiring \(R^\dagger\) has the \emph{infinite element} \(\infty\) if
\[
\infty + a = \infty = \infty \cdot a = a \cdot \infty, \quad \forall a \in R. \tag{2.2}
\]

When \(R\) does have a zero element \(0 := 0_R\), we require instead that
\[
\infty + 0 = \infty \quad \text{but} \quad 0 \cdot \infty = \infty \cdot 0 = 0.
\]
(This is to enable \(0\) to remain the zero element.)

The following observation enables us to pass from a semiring \(R^\dagger\) to a semiring.

**Remark 2.8.**

(i) Given a semiring \(R^\dagger\), one can formally adjoin a zero element \(0_R\) which is multiplicatively “absorbing” in the sense that
\[
0_R \cdot a = a \cdot 0_R = 0_R, \quad \forall a \in R, \tag{2.3}
\]

to obtain a semiring
\[
R \cup \{0_R\}.
\]

(ii) Alternatively, one could formally adjoin an element \(\infty\) to obtain a semiring \(R^\infty := R \cup \{\infty\}\) satisfying Equation (2.2).

(iii) Finally, one could adjoin \(\infty\) and then \(0_R\). Then \(\infty\) satisfies (2.3) for every element \(a \in R\) except \(0_R\).

Although the notion of semiring \(R^\dagger\) is somewhat unusual, it fits our needs like a glove, since the max-plus algebra \((\mathbb{R}, \max, +, 0)\), with \(1_R = 0\), is a semiring \(R^\dagger\) before adjoining the zero element \(0_R := -\infty\) and much of the theory of supertropical domains, cf. [IR1], is stated more concisely when we do not have to consider special cases involving the element \(0_R\).

(An example of how the adjoined element \(0_R = -\infty\) gets in the way: The dual of the max-plus semiring \(R^\dagger\) is the min-plus semiring \(R^\dagger\) where “max” is replaced by “min.” But the dual of the max-plus semiring is not a semiring, since \(-\infty\) no longer performs the role of the zero element!)

A \emph{semiring homomorphism} is a map \(\varphi : R \to R'\) of semirings \(R^\dagger\) satisfying
\[
\varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a) \varphi(b),
\]
for all \(a, b \in R\), and \(\varphi(1_R) = 1_{R'}\).

(When working with semirings, one also requires that \(\varphi(0_R) = 0_{R'}\).)

**Note 2.9.** As with monoids, the class of homomorphisms from a semiring \(R\) to another semiring is far richer than the set of ideals of \(R\). Given a semiring homomorphism \(\varphi : R \to R'\) we can define an equivalence relation \(\equiv_\varphi\) on \(R\) by
\[
a \equiv_\varphi \ b \quad \text{iff} \quad \varphi(a) = \varphi(b).
\]
If \(R\) and \(R'\) are rings, then \(\equiv_\varphi\) is determined by the ideal \(\ker \varphi\), since \(a \equiv_\varphi \ b\) if \(a - b \in \ker \varphi\). But this is no longer true for semirings. “Non-equivalent” onto homomorphisms may have the same kernel. Thus, in contrast to ring theory, the theory of semiring homomorphisms is much richer than the theory of ideals, and we need to cope with this extra complication.
2.3. Ordered semirings†. Since the tropical structure is largely based on orderings, we pause to consider orders on semirings†. In general, this is often handled in the theory of ordered rings (for example, in [VaD], which also treats ordered structures in model theory) by viewing an ordered ring \( R \) as \( R_+ \cup R_- \cup \{0_R\} \), where \( R_+ \) is the set of positive elements and \( R_- \) is the set of negative elements, but in semirings we might lack negative elements altogether. In fact, in the tropical theory our ordered semirings almost never have negative elements.

Here we take a more general approach, which might give a better indication of the idea of “ghost layers.” Given a semiring† \( L \), we designate a sub-semiring† \( L_+ \subseteq L \) of positive elements. By assumption, \( 1 \in L_+ \).

We write \( k \geq \ell \) when \( k = \ell \) or \( k = \ell + p \) for \( p \in L_+ \).

A priori, this relation \((\geq)\) is only a partial pre-order, but we stipulate that \( \ell + p_1 + p_2 = \ell \) for \( p_1, p_2 \in L_+ \) implies \( \ell + p_1 = \ell \). Thus, \((\geq)\) is antisymmetric, and hence a partial order.

For \( \ell \in L \), an \( \ell \)-ghost sort is an element of the form \( \ell + k \), where \( k \in L_+ \). A 1-ghost sort of \( L \) is called a ghost sort.

Lemma 2.10. Suppose \( \ell \geq k \). Then:

(i) \( \ell + m \geq k + m \) for all \( m \in L \).

(ii) \( \ell p \geq kp \) for all \( p \in L_+ \).

Proof. (i): If \( \ell = k + p' \), for some \( p' \in L_+ \), then \( \ell + m = k + m + p' \).

(ii): If \( \ell = k + p' \), for some \( p' \in L_+ \), then \( \ell p = kp + p'p \). \( \square \)

Remark 2.11. To make the exposition simpler, one usually may assume that our partial order on \( L \) is a total order, which implies that for all \( k \neq \ell \in L \) there is \( p \in L_+ \) such that either \( \ell = k + p \) or \( k = p + \ell \). One situation for which only the weaker assumption holds is in \([5,7]\) so we stay with the partial order.

Definition 2.12. The semiring† \( L \) is non-negative, if \( L = L_+ \) or if \( L = L_+ \cup \{0\} \) when \( L \) has a zero element \( 0 := 0_L \).

We define the following non-negative sub-semirings† of \( L \):

\[ L_{\geq 1} := \{\ell \in L : \ell \geq 1\} \]

a sub-semiring† of \( L \), and

\[ L_{> 1} := \{\ell \in L : \ell \text{ is a ghost sort}\} \]

a semiring† ideal of \( L_{\geq 1} \).

Note that \( L_{\geq 1} \) contains the sub-semiring† generated by 1.

In all of our applications except Examples \[10.12\] \[10.13\] and \[10.14\] \( L = L_+ \) or \( L = L_+ \cup \{0\} \). In many major tropical applications we have \( L := L_+ \), and one could assume this throughout. (The theory runs most smoothly in this case.) We permitted the more general situation in the definition, in order to be able to deal with subtle issues regarding factorization such as in Proposition \[7.26\] below, which involve the more esoteric Examples \[10.12\] and Example \[10.13\].

Definition 2.13. An element \( \ell \in L \) is finite if it satisfies the conditions:

\[ \ell + m \neq \ell, \quad \forall m \in L_+ \]

In our examples, \( L \) will have at most one infinite element, often denoted \( \infty \). For example, the semiring† \( L \) could be the following set of finite order \( q \), for any natural number \( q \).

Definition 2.14. The \( q \)-truncated semiring†

\[ L = [1, q] := \{1, 2, \ldots, q\} \]

is given with the obvious total ordering; the sum and product of two elements \( k, \ell \in L \) are taken as usual, if it does not exceed \( q - 1 \), while the sum or product of \( k \) and \( \ell \) in \( L \) is \( q \) otherwise. In other words, \( q \) is the infinite element and could be denoted as \( \infty \).

Remark 2.15. There is a natural semiring† homomorphism \( \mathbb{N} \to [1, q] \) given by sending \( m \to q \) for all \( m \geq q \).

The first examples for infinite sets \( L \) are \( \mathbb{N} \) and \( \mathbb{Q}_{>0} \), with the usual addition, multiplication, and ordering. Note in these two examples that every element is finite as well as positive. In general, nonzero positive elements of \( L \) need not be finite, and we could have several infinite elements (as can be seen easily by means
of ordinals), but we do not deal with such issues in this paper. When \( L \) is ordered we extend the order to \( L \cup \{\infty\} \) by declaring that \( \infty > \ell \) for all \( \ell \in L \).

**Example 2.16.** The semiring \( [1, q] \) can be extended to \( \{ \frac{i}{m} : 1 \leq i \leq q \} \), which is isomorphic as a semiring to \( [1, qm] \) under the map \( \frac{i}{m} \mapsto i \).

We introduce the following notation: \( 1 \ell \) denotes \( \ell \), and inductively, for any integer \( n > 1 \), we define
\[
n\ell = (n-1)\ell + \ell.
\]

**Remark 2.17.** If \( \ell \geq m\ell \geq \ell \) for some \( m \in \mathbb{N} \) and the order \( (\geq) \) on \( L \) is antisymmetric, then \( n\ell = \ell \) for all \( n \). Indeed, \( m\ell = \ell \) by hypothesis, and the assertion then holds for any multiple \( n \) of \( m \). In general, for any \( u \in \mathbb{N} \) with \( um > n \),
\[
\ell \leq n\ell \leq um\ell = \ell,
\]
so again \( \ell = n\ell \).

2.4. The \( \nu \)-pre-order.

**Definition 2.18.** Suppose \( R \) is a semiring \( \dagger \) with a designated equivalence \( \cong_{\nu} \) which respects addition, in the sense that if \( a \cong_{\nu} a' \) and \( b \cong_{\nu} b' \), then \( a + b \cong_{\nu} a' + b' \). We define a transitive binary relation \( \leq_{\nu} \) by
\[
a \leq_{\nu} b \quad \text{iff} \quad a + b \cong_{\nu} b.
\]

**Lemma 2.19.** The relation \( \leq_{\nu} \) is anti-symmetric (and thus is a partial pre-order).

**Proof.** \( a + b = a \) and \( b + a = b \) implies \( a = b \). \( \square \)

**Definition 2.20.** We say that \( b \) is \( \nu \)-greater than \( a \) in a semiring \( \dagger \) \( R \), written \( a <_{\nu} b \), if \( a \leq_{\nu} b \) but \( a \not\cong_{\nu} b \).

A semiring \( \dagger \) \( R \) is \( \nu \)-**bipotent** if the following two conditions hold:

(i) The relation \( \leq_{\nu} \) is a total pre-order on equivalence classes, in the sense that if \( a \not\cong_{\nu} b \), then either 
\[
a <_{\nu} b \quad \text{or} \quad b <_{\nu} a.
\]

(ii) If \( a <_{\nu} b \) and \( a \not\cong_{\nu} b \), then \( a + b = b \).

We write \( a <_{\nu} b \) when \( a \leq_{\nu} b \) but \( a \not\cong_{\nu} b \).

The relation \( \leq_{\nu} \) respects the monoid structure in the following sense:

**Proposition 2.21.** If \( a \leq_{\nu} b \), then \( ac \leq_{\nu} bc \) for all \( c \); furthermore, if \( c \leq_{\nu} d \), then \( ac \leq_{\nu} bd \).

**Proof.** \( ac + bc = (a + b)c \cong_{\nu} bc \). Furthermore, \( ac + bd \cong_{\nu} ac + (a + b)(c + d) = ac + ac + ad + bc + bd = ac + a(c + d) + bc + bd \cong_{\nu} ac + ad + bc + bd = (a + b)(c + d) \cong_{\nu} bd \). \( \square \)

**Corollary 2.22.** If \( R \) is as in Definition 2.18, then \( R/ \cong_{\nu} \) is an ordered monoid.

**Proof.** \( \leq_{\nu} \) obviously induces an order on \( R/ \cong_{\nu} \), which is total in view of \( \nu \)-bipotence. The order is preserved under multiplication, in view of Proposition 2.21. \( \square \)

3. Layered domains without zero

We get to our main algebraic notion.

**Remark 3.1.** Any ordered monoid \( G \) can be viewed as a semiring \( \dagger \) in which addition is given by
\[
a + b := \max\{a, b\},
\]
with respect to the order of \( G \). (See [IKR1] Theorem 1.5 for more details.)

Here is the motivating example for this paper – a mild generalization of [AGG] Proposition 5.1.

**Construction 3.2.** Suppose we are given a cancellative ordered monoid \( G \), viewed as a semiring \( \dagger \) as in Remark 3.1. For any semiring \( \dagger \) \( L \) we define the semiring \( \dagger \) \( R(L, G) \) to be set-theoretically \( L \times G \), where we denote the element \( (\ell, a) \) as \( [\ell]a \) and for \( k, \ell \in L \), \( a, b \in G \), we define multiplication componentwise, i.e.,
\[
[k]\ell a [\ell]b = [k\ell](ab),
\]
as in (3.1)

and addition from the rules:
\[ [k]a + [ℓ]b = \begin{cases} [k]a & \text{if } a > b, \\ [ℓ]b & \text{if } a < b, \\ [k+ℓ]a & \text{if } a = b. \end{cases} \] (3.2)

**Proposition 3.3.** \( R := R(L, G) \) is a semiring\(^1\).

**Proof.** To prove, for example, the distributivity law
\[ x(y + z) = xy + xz, \]
write \( x = [k]a, y = [ℓ]b, \) and \( z = [m]c \). If \( ab > ac \) then clearly \( b > c, \) and
\[ x(y + z) = xy = xy + xz. \]
Thus we are done unless \( ab = ac \), in which case \( b = c \) since \( G \) is cancellative, and
\[ x(y + z) = [k]a([ℓ]b + [m]b) = [k]a([ℓ+m]b) = [kℓ+km](ab) = [kℓ](ab) + [km](ab) = xy + xz. \]

The other verifications of the semiring\(^1\) axioms are straightforward. The unit element is \([1]1_G\). \( \square \)

Conversely, we have:

**Proposition 3.4.** If \( R := R(L, G) \) is a semiring\(^1\), then the monoid \( G \) is cancellative.

**Proof.** If \( b > c \) with \( ab = ac \) then \([1]a([1]b + [1]c) = [1](ab)\)
whereas \([1]a [1]b + [1]a [1]c = [2](ab)\).

We define \( R_ℓ := \{ [ℓ]a \mid a ∈ G, ℓ ∈ L \} \).
Then \( R = R(L, G) \) is the disjoint union of the \( R_ℓ \), and \( R_1 \) is a monoid isomorphic to \( G \), which we call the tangible copy of \( G \). We have 1:1 maps \( ν_ℓ,k : R_k → R_ℓ \) given by \( ν_ℓ,k(k, a) = (ℓ, a) \) for any \( k ≤ ℓ \), and a map \( s : R → L \) given by \( s(ℓ, a) = ℓ \), for any \( a ∈ G, ℓ ∈ L \).

The sub-semiring\(^1\) of \( R \) generated by \( R_1 \) is just \( \bigcup_{ℓ ∈ L} R_ℓ \), and equals \( R \) iff \( L \) is generated as a semiring\(^1\) by its unit element 1.

**Remark 3.5.** We could perform the same construction assuming that \( L \) contains a zero element 0. Then \( R_0 \) is a semiring\(^1\) as well as an ideal of \( R(L, G) \). When \( G \) has a zero element \( 0_G \), then \( R = R(L, G) \) is a semiring, and the corresponding zero element of \( R_0 \) is the zero element of \( R(L, G) \).

Since Construction \[3.2\] lies at the foundation of this paper, let us axiomatize it (in slightly greater generality). The bulk of our applications in this paper are for \( L \) ordered. Note that the construction does not require \( L \) to be ordered. On the other hand, any set \( L \) has the trivial partial pre-order, defined by \( k ≤ ℓ \) for all \( k, ℓ ∈ L \). For this reason, we frame our definition for partially pre-ordered semirings\(^1\).

**Definition 3.6.** Suppose \((L, ≥)\) is a partially pre-ordered semiring\(^1\) (as described in \[2.6\]), with positive elements \( L^+ \). An \( L\)-quasi-layered domain\(^1\)
\[ R := (R, L, (ν_m, ℓ)) \]
is a commutative semiring\(^1\) \( R \), together with a family \( \{R_ℓ : ℓ ∈ L\} \) of disjoint subsets \( R_ℓ ⊂ R \), such that
\[ R := \bigcup_{ℓ ∈ L} R_ℓ, \] (3.3)
and a family
\[ (ν_m, ℓ) := \{ν_m, ℓ : m ≥ ℓ \quad (m, ℓ ∈ L)\} \]
of sort transition maps
\[ ν_m, ℓ : R_ℓ → R_m \]
for each \( m ≥ ℓ \) in \( L^+ \), such that \( ν_ℓ, ℓ = \text{id}_{R_ℓ} \) for every \( ℓ ∈ L \), and
\[ ν_m, ℓ ∘ ν_ℓ,k = ν_m,k \]
whenever \( m ≥ ℓ ≥ k \), satisfying the axioms A1–A4, and B, to be given below.
We define the $\nu$-relation $a \cong \nu b$ for $b \in R_\ell$ if $\nu_{m,k}(a) = \nu_{m,\ell}(b)$ in $R_m$ for some $m \geq k, \ell$. (Note that in this case we also have $\nu_{m',k}(a) = \nu_{m',\ell}(b)$ for every $m' > m$.)

The axioms are as follows, where we assume $k, \ell \in \mathbb{Z}^+$:

A1. $1_R \in R_1$.

A2. If $a \in R_k$ and $b \in R_\ell$, then $ab \in R_{k\ell}$.

A3. The product in $R$ is compatible with sort transition maps: Suppose $a \in R_k, b \in R_\ell$, with $m \geq k$ and $m' \geq \ell$.
   (i) If $ab \in R_{k\ell}$, then $\nu_{m,k}(a) \cdot \nu_{m',\ell}(b) = \nu_{mm',k\ell}(ab)$.
   (ii) If $ab \in R_0$, then $\nu_{m,k}(a) \cdot \nu_{m',\ell}(b) = ab$.

A4. $\nu_{\ell,k}(a) + \nu_{\ell',k}(a) = \nu_{\ell+\ell',k}(a)$ for all $a \in R_k$ and all $\ell, \ell' \geq k$.

B. (Supertropicality) Suppose $a \in R_k, b \in R_\ell$, and $a \cong \nu b$. Then $a + b \in R_{k+\ell}$ with $a + b \cong \nu a$.

If moreover $k \in \mathbb{Z}^+$ is infinite, then $a + b = a$.

We say that any element $a$ of $R_k$ has layer $k$ ($k \in \mathbb{L}$), and call $L$ the sorting semiring of the quasi-layered domain $R = \bigcup_{k \in \mathbb{L}} R_k$.

An $L$-quasi-layered domain $R := (R, L, (\nu_m))$ is called uniform when the sorting semiring $L$ is totally ordered and the sort transition maps $\nu_{m,k}$ are all bijective.

An $L$-layered domain is a $\nu$-bipotent $L$-quasi-layered domain.

We write $1 := 1_L$, for the unit element of $L$. Thus, the relation ($\geq$) on $L$ satisfies

$$\ell \geq k \quad \text{when} \quad \begin{cases} \ell = k \\ \ell \text{ is a } k\text{-ghost sort.} \end{cases}$$

Note that infinite elements of $L$ are “self-ghost sorts,” in the sense that $\ell + m = \ell$ implies that $\ell$ is an $\ell$-ghost sort.

We usually require that the relation ($\geq$) on $L$ is directed in the sense that for any $k, \ell \in L$ there are positive elements $p_1, p_2 \in \mathbb{L}$ such that

$$k + p_1 = \ell + p_2. \quad (3.4)$$

By this condition, any two elements $k$ and $\ell$ in $L$ have an upper bound; i.e., there exists $m \in \mathbb{L}$ such that $m \geq k, \ell$. The existence of “upper bounds” enables us to define direct limits over $L$, which is needed for Remark 3.21 below.

Remark 3.7. Axiom B implies that if both $a, b \in R_k$ with $a \cong \nu b$ and $k \in \mathbb{L}$ is infinite, then $a = a + b = b$.

Remark 3.8. The $L$-quasi-layered domain $R$ has the special layer $R_1$, which is a multiplicative monoid, called the monoid of tangible elements, and acts with the obvious monoid action (given by multiplication) on each layer $R_k$ of $R$.

Definition 3.9. An $L$-layered domain is called an $L$-layered 1-semifield if $(R_1, \cdot)$ is an Abelian group.

In this case, the action of Remark tang1 is simply transitive, in the sense that for any $a, b \in R_\ell$ there is a unique element $R \in R_1$ for which $ar = b$.

Note that according to this definition, an $L$-layered 1-semifield need not be a semifield; cf. Proposition 3.17 below.

Example 3.10. Taking $L = \{1\}$ with $1 + 1 = 1$, then $R = R_1$ gives us the usual (idempotent) max-plus algebra.

Example 3.11.
Remark 3.12. Several initial observations are in order.

(i) Taking $L = \{1, \infty\}$ with $1 < \infty$, $1 \cdot 1 = 1$, and all other sums and products $= \infty$, we recover the (standard) supertropical domains\(^1\) as defined and studied in \[\text{[IR3].}\] (The tangibles are of layer 1, and the ghosts are comprised of layer $\infty$.)

(ii) Another major example, to be considered below, is $L = \mathbb{N}$.

(iii) Given an arbitrary sorting semiring\(^1\) $L$, one could define $L_{\mathbb{N}}$ to be the sub-semiring\(^1\) generated by $1_L$, i.e., $\mathbb{N} \cdot 1_L$. Then one could replace an $L$-quasi-layered domain\(^1\) $R$ by its sub-quasi-layered domain\(^1\) $\sum_{\ell \in L_{\mathbb{N}}} R_\ell$, and thereby assume that $L = L_{\mathbb{N}}$.

Remark 3.13. Several initial observations are in order.

(i) The layered structure resembles that of a graded algebra, with two major differences: On the one hand, the condition that $R$ is the disjoint union of its components is considerably stronger than the usual condition that $R$ is the direct sum of its components; on the other hand, Axioms $A_4$ and $B$ show that the components are not quite closed under addition.

(ii) This paper is mostly about $L$-layered domains\(^1\), and we require $\nu$-bipotence in many results proved here. However, since $\nu$-bipotence does not hold for polynomials, we need the more general $L$-quasi-layered domains\(^1\) when studying polynomial semirings\(^3\).

(iii) For each $\ell \in L$ we introduce the sets

\[
R_{\geq \ell} := \bigcup_{m \geq \ell} R_m \quad \text{and} \quad R_{> \ell} := \bigcup_{m > \ell} R_m.
\]

In many of our current examples, $L = L_{\geq 1}$ and thus $R = R_{\geq 1}$. When $R$ is an $L$-layered domain\(^1\), we claim that $R_{\geq 1}$ is an $L_{\geq 1}$-layered sub-domain\(^1\) of $R$, and $R_{\geq k}$ and $R_{> k}$ are semiring\(^1\) ideals of $R_{\geq 1}$, for each $k \in L_{\geq 1}$. Indeed, this is an easy verification of the axioms, mostly from Axiom $A_2$.

(iv) Given any $L$-layered domain\(^1\) $R$ and any multiplicative submonoid $M$ of $R_{\geq 1}$, we want to define the $L$-layered sub-domain\(^1\) of $R$ generated by $M$. First we take

\[
M' := \{ \nu_{\ell,k}(a) : k, \ell \in L, \ell \geq k, a \in M \cap R_k \},
\]

which is a submonoid closed under the transition maps. Then we take

\[
M'' := M' \cup \{ a + b : a, b \in M' \text{ with } a \cong_\nu b \}.
\]

This is closed under all the relevant operations, so is the desired $L$-layered domain\(^1\). Note that the second stage is unnecessary for $a = b$ (and similarly for uniform $L$-layered domains\(^1\)), in view of Axiom $A_4$.

(v) Although ubiquitous in the definition, the sort transition maps get in the way of computations, and it is convenient to define the elements

\[
e_\ell := \nu_{\ell,1}(1_R) \quad (\ell \geq 1).
\]

If $a \in R_k$, $\ell \in L$, and $\ell \geq 1$, we conclude by Axiom $A_3$ that

\[
\nu_{\ell,k}(a) = \nu_{\ell,k,1}(a \cdot 1_R) = \nu_{\ell,1}(1_R) \cdot \nu_{k,1}(a) = \nu_{\ell,1}(1_R) \cdot a = e_\ell a.
\]

Thus the sort transition map $\nu_{\ell,k}$ means multiplication by $e_\ell$.

Let us introduce the sorting map

\[
s := s_R : R \to L,
\]

which sends every element $a \in R_\ell$ to its sort index $\ell$, and we view the semiring\(^1\) $R$ as an object fibred by $s$ over the sorting semiring\(^1\) $L$.

Remark 3.13. Axioms $A_1$ and $A_2$ yield the conditions

\[
s(1_R) = 1, \quad s(ab) = s(a)s(b), \quad \forall a, b \in R. \quad (3.6)
\]

Also, Axiom $A_4$ yields $s(a + a) = s(a) + s(a) = 2s(a)$, thereby motivating us to view addition of an element with itself as doubling the layer. Applying $\nu$-bipotence to Axiom $B$ shows that

\[
s(a + b) \in \{ s(a), s(b), s(a) + s(b) \}.
\]
To emphasize the sorting map, as well as the order on \( L \), we sometimes write \((R, L, L_, s, (\nu_{m,\ell}))\) for a given \( L \)-layered domain \( R \) with sort transition maps \((\nu_{m,\ell} : m \geq \ell)\) and their accompanying sorting map \( s : R \to L \), where, as usual, \( L_+ \) is the set of positive elements of \( L \).

There are two main examples.

(a) Let \( R = R(L, G) \) (corresponding to the “naive” tropical geometry). By Construction \(3.2\) \( \nu_{m,\ell} \) are all bijective.

(b) (Corresponding to the supertropical structure given in \( IR1 \).) Notation as in \( IR1 \) Theorem 1, we incorporate \( K \) into the structure of \( R \), by putting \( R_\ell \) to be a copy of \( K \) for \( \ell \leq 1 \) and \( R_\ell \) to be a copy of \( G \) for \( \ell > 1 \). We take the \( \nu_{m,\ell} \) to be the valuation \( \nu \) whenever \( m > 1 \geq \ell \), and the identity otherwise.

We focus on the first case (after some additional general observations), since one can reduce to it anyway via the equivalence given below in Definition \(3.14\) (which takes us from the usual algebraic world to the tropical world).

3.0.1. \textbf{Properties of tangible elements.} We say that a monoid \((M, \cdot)\) is \(\nu\)-\textbf{cancellative} if \(ac \equiv_\nu bc\) implies \(a \equiv_\nu b\) for \(a, b, c \in M\). Likewise, \(M\) is \(\nu\)-\textbf{\(n\)}-\textbf{cancellative} if \(a^m \equiv_\nu b^m\) implies \(a \equiv_\nu b\) for \(a, b \in M\).

\textbf{Proposition 3.14.} If \(ac \equiv_\nu bc\) for \(a, b, c\) in an \(L\)-layered domain \(L\) with \(s(ac) \in L_+\) finite, then \(a \equiv_\nu b\). In particular, \(R_1\) is \(\nu\)-cancellative unless \(1 + 1 = 1\) in \(L\).

\textit{Proof.} Otherwise, we may assume \(a >_\nu b\), in which case \(ac \geq_\nu bc\) by Proposition \(2.21\). If \(ac \equiv_\nu bc\), then \(s(ac) = s((a + b)c) = s(ac + bc) > s(ac)\), a contradiction. \(\square\)

\textbf{Corollary 3.15.} If \(R\) is \(\nu\)-bipotent and \(R_1\) has only finitely many elements, then either \(1 \in L\) is infinite (i.e., \(1 + 1 = 1\)), or the \(\nu\)-relation is trivial on \(R_1\) in the sense that \(a \equiv_\nu 1_R\) for all \(a \in R_1\).

\textit{Proof.} Otherwise, \(R_1\) has an element \(a\) with a different \(\nu\)-value from \(0_R\) or \(1_R\), and clearly \(a^i = a^j\) for some \(i > j\). Then \(a^i + a^j\) is not tangible. But \(a^i + a^j = a^j(a^{i-j} + 1_R)\), which is tangible unless \(1_R \equiv_\nu a^{i-j}\), which is false by hypothesis. \(\square\)

In brief, the only finite layered domains\(\dagger\) are \(\nu\)-trivial, and thus are not uniform (for \(L \neq \{1\}\)). (One can get finite structures by means of a 0 layer, as indicated in Example \(3.31\) below.)

Here is an indication of the importance of tangible elements.

\textbf{Lemma 3.16.} If \(L = L_{\geq 1}\) (for example, if \(L = \mathbb{N}\)), then every invertible element \(a\) of \(R\) is tangible.

\textit{Proof.} \(1 = s(aa^{-1}) = s(a)s(a^{-1})\), implying \(s(a^{-1}) = s(a)^{-1}\). But \(s(a^{-1}) \geq 1\), so

\[1 \leq s(a)s(a^{-1}) \leq s(1_R) = 1,\]

implying \(s(a) = s(a^{-1}) = 1\). \(\square\)

3.0.2. The case when the multiplicative monoid of \(L\) is a group. On the other hand, it often is convenient to make the different assumption that \(L\) is a group. In such a situation, Lemma \(3.16\) fails, and the flavor of the layered theory differs considerably from the standard supertropical theory. But here we can describe the structure of \(R\) in terms of \(R_1\), its set of tangible elements.

\textbf{Proposition 3.17.} If an \(L\)-quasi-layered domain \(R\) is a semifield \(\dagger\) (i.e., is a group under multiplication), then \(L\) is a multiplicative group (and thus also a semifield \(\dagger\)) and also \(R\) is an \(L\)-quasi-layered 1-semifield \(\dagger\).

\textit{Proof.} If \(R\) is a semifield \(\dagger\), we must have \(s(a^{-1}) = s(a)^{-1}\), and for \(a \in R_1\) we have \(a^{-1} \in R_1\). \(\square\)

\textbf{Proposition 3.18.} If \(L\) is a multiplicative group, then one can define new sort transition maps \(\nu'_{\ell,k}\) by \(\nu'_{\ell,k}(a_k) = e_m a_k\) where \(\ell = mk\).

\textit{Proof.} If \(\ell = k\) there is nothing to prove, so we assume \(\ell > k\) are non-negative, and write \(\ell = k + p\) for \(p \in L_+\). Then \(\ell = k(1 + pk^{-1})\), and \(m = 1 + pk^{-1}\). Now \(e_m a_k \in R_\ell\), and

\[\nu'_{\ell,k}(a_k) = e_m a_k \equiv_\nu a_k \equiv_\nu \nu_{\ell,k}(a_k).\]

Thus the sort transition maps can be replaced by multiplication by the \(e_m\), and we can remove the sort transition maps from the definition.
3.1. Ghost layers. Next, we want to extend the notion of ghosts from the standard supertropical situation to the context of the layered structure.

**Definition 3.19.** An element \( b \in R \) is an \( \ell \)-ghost (for given \( \ell \in L \)) if \( s(b) \in L \) is an \( \ell \)-ghost sort, i.e., if \( s(b) = \ell + k \) for some \( k \in L_+ \). A ghost element of \( R \) is a 1-ghost.

In other words, for any \( \ell \)-ghost \( b \), either \( s(b) > \ell \) for \( \ell \) finite, or \( s(b) = \ell \) is infinite. This explains the difference between tangible and ghost sorts. Namely, when \( b \in R_\ell \), \( b \) is an \( \ell \)-ghost iff \( \ell \) is infinite.

**Definition 3.20.** We recall \( R_{\geq} \) from Remark 3.11(iii). In analogy to valuation theory, we call \( R_{\geq} \) the layered valuation semiring\(^1\), and we call \( R_{\geq} \) the ghost ideal.

When \( L = L_{\geq} \), the ghost valuation ideal replaces the ghost ideal of the standard supertropical theory, so the different ghost layers are to be thought of as a refinement of the ghost ideal.

On the other hand, when \( L \) contains the infinite element \( \infty \), then \( R_\infty \) is an idempotent semiring\(^1\) and there is a semiring homomorphism \( \nu : R \rightarrow R_\infty \) sending \( a \mapsto \nu_\infty(a) \) where \( s(a) = \ell \), and is the identity map on \( R_\infty \). In this way, \( R \) covers the idempotent semiring\(^1\) \( R_\infty \), which can be viewed as the “absolute” ghost layer, as described above in Remark 3.21.

Likewise, when \( L = L_{\geq} \), there is a semiring homomorphism \( \nu : R \rightarrow R_1 \cup R_\infty \) sending \( a \mapsto \nu_1,\ell(a) \) where \( s(a) = \ell > 1 \), and \( R \) also covers the standard supertropical semiring\(^1\) \( R_1 \cup R_\infty \).

These roles complement each other. In general, the finite elements \( \ell \in L \) greater than 1 can be viewed as “sorting” different “layers” \( R_\ell \) of ghosts.

Note that two partial pre-orders are at play – the given partial pre-order on the sorting set \( L \), and the \( \nu \)-pre-order on \( R \). There is a subtle interplay between the two. It turns out that we could develop the theory under the weaker condition that \( L \) is a partially pre-ordered multiplicative monoid, and we sketch the appropriate changes in Appendix B.

3.2. Adjoining the absolute ghost layer, and the passage to standard supertropical domains\(^1\).

Even when \( L \) originally does not contain an infinite element a priori, \( L \)-layered domains\(^1\) tie in directly with the (standard) supertropical theory, via a ghost layer introduced at a new element \( \infty \) which we adjoin. (This works even when \( (\geq) \) is merely a partial order on \( L \), although it is easier when \( (\geq) \) is a total order.)

**Remark 3.21.** The system \( (R, L, (\nu_{m,\ell})) \) is a directed system with respect to the set \( L \), as described in [Jac, p. 71]. Hence, by [Jac, Theorem 2.8], the layers \( R_k \) have a direct limit which we denote \( R_\infty \), and maps

\[
\nu_{\infty, k} : R_k \rightarrow R_\infty
\]

such that \( \nu_{\infty, k} = \nu_\infty,\ell \circ \nu_{\ell, k} \) for each \( a \in R_k \) and all \( k < \ell \). Since \( R = \bigcup_k R_k \), we can piece together these maps \( \nu_{\infty, k} \) to a map \( \nu : R \rightarrow R_\infty \). We define

\[
e = e_\infty := \nu(1_R),
\]

(3.7)

easily seen to be the unit element of \( R_\infty \).

We write \( a^\nu \) for \( \nu(a) \in R_\infty \). Thus \( a^\nu = b^\nu \iff a \cong b \) in our previous notation.

We call \( R_\infty \) the absolute ghost layer and \( \nu \) the (absolute) ghost map of \( R \). Note that in the uniform case, \( R_\infty \) is just another copy of \( R_1 \), so everything is much simpler.

**Theorem 3.22.** Suppose \( R = (R, L, (\nu_{m,\ell})) \) is an \( L \)-layered domain\(^1\). Then the absolute ghost layer \( R_\infty \) is a bipotent semiring\(^1\). The ghost map \( \nu : R \rightarrow R_\infty \) is a semiring\(^1\) homomorphism. Define

\[
U = U(R) := R \cup R_\infty.
\]

Then \( U \) is a semiring\(^1\) under the given operations of \( R \) and \( R_\infty \), together with

\[
a \cdot b' := (ab)',
\]

\[
a + b' := \begin{cases} a & \text{if } ea > eb, \\ b' & \text{if } ea \leq eb. \end{cases}
\]

Also, extend \( \nu \) to a map \( \nu_U : U \rightarrow R_\infty \) by taking \( \nu_U \) to be the identity on \( R_\infty \). Then \( U \) has ghost ideal \( G = G(U) := R_\infty \), in the sense of [K1], and \( \nu_U(a) = ea \) for every \( a \) in \( U \).

Then \( U \) can be modified to a supertropical semiring\(^1\)

\[
R_{1, \infty} := R_1 \cup G,
\]
retaining the given multiplication \( \cdot \) of \( U \), but with new addition \( \oplus \) given by the rules

\[
a \oplus b := \begin{cases} 
a & \text{if } ea > eb, \\b & \text{if } ea < eb, \\ea & \text{if } ea = eb.
\end{cases}
\]

(3.8)

\[\nu_m(a \cdot b) = \nu_m(a) \cdot \nu_m(b)\]

for any \( a \in R_k \) and \( b \in R_\ell \); taking limits yields

\[\nu(a \cdot b) = \nu(a) \cdot \nu(b)\]

Likewise, Axiom B tells us that

\[\nu(a + b) = \nu(a) + \nu(b)\]

The other verifications are also easy. By (3.7) we have

\[G \sim R\]

for which proving \( \Phi \) is onto, and thus an isomorphism.

\[\square\]

\begin{proof}

Axiom A3 tells us that \( \nu_{m',k'l}(a \cdot b) = \nu_{m,k}(a) \cdot \nu_{m',l}(b) \)
for any \( a \in R_k \) and \( b \in R_\ell \); taking limits yields

\[\nu(a \cdot b) = \nu(a) \cdot \nu(b)\]

Thus \( \nu \circ \nu = \nu \), and also \( \nu : R \to G \) is a semiring\(^{\dagger}\) homomorphism from \( R \) onto \( G = G(U) \).

We extend the \( \nu \)-equivalence relation from \( R \) to \( U \) by decreeing that \( a \equiv_U b \) iff \( a \) and \( b \) have the same value under \( \nu \).

We turn to the last assertion. Due to (3.8) we have

\[a \oplus b = a + b \quad \text{if} \quad a \not\equiv \nu \, b.\]

On the other hand,

\[a \oplus b = e(a + b) \quad \text{if} \quad a \equiv \nu \, b.\]

Note that

\[a \oplus b \equiv \nu \, a + b\]

in all cases. Also, \( G(U) := R_\infty = G(\mathcal R_{1,\infty}) \).

We may regard \( \mathcal R_{1,\infty} := (\mathcal R_{1,\infty}, \oplus, \cdot) \) as a degeneration of the semiring \( U = U(R) \), where all the ghost layers have been coalesced to \( R_\infty \). When \( L = L_{\geq 1} \), then there is a semiring\(^{\dagger}\) homomorphism \( U \to \mathcal R_{1,\infty} \) given by

\[a \mapsto \begin{cases} a & \text{for } a \in R_1 \cup R_\infty, \\\nu(a) & \text{otherwise.}\end{cases}\]

(This map is a special case of truncation, to be discussed in \( \text{[11]} \))

We are now in a position to see why Construction \( \text{[3.2]} \) of uniform \( L \)-layered domain\(^{\dagger}\) is generic. We recall

\[R(L, G) := \left\{ [\ell]^a \mid a \in G, \ \ell \in L \right\}.\]

**Theorem 3.23.** Suppose \( R' \) is an \( L \)-layered domain\(^{\dagger}\) and \( G = R'_\infty \). There is a semiring\(^{\dagger}\) homomorphism

\[\Phi : R' \to R(L, G)\]

given by \( a \mapsto [s(a)](\nu(a)) \). If \( R' \) is a uniform \( L \)-layered domain\(^{\dagger}\), then \( G \cong R_1 \) and \( \Phi \) is an isomorphism.

**Proof.** Clearly \( 1_R(L, G) = [1]^1_G \),

\[\nu_{\ell,k}( [\ell]^a) = [\ell]^a \quad \text{if} \quad k \leq \ell,\]

and \( [\ell]^a = [k]^b \) iff \( a \equiv \nu \, b \) and \( k = \ell \).

We read off from the definition that \( \Phi \) preserves multiplication and addition.

In case the sort transition maps are bijective, the direct limit construction degenerates to the identity, implying \( G \cong R_1 \). The given map \( \Phi \) is clearly 1:1, since \( \Phi(a) = \Phi(b) \) would imply \( a \equiv \nu \, b \) and \( s(a) = s(b) \), implying \( a = b \). But given \( [\ell]^a \in R(L, G) \), there is \( x \in R' \) with \( \nu(x) = a \); taking \( k = s(x) \) we have some \( m > k, \ell \in L \). Write \( x_\ell = \nu_{m,\ell}^{-1}(\nu_{m,k}(x)) \). Then

\[\Phi(x_\ell) = [\ell]^a,\]

proving \( \Phi \) is onto, and thus an isomorphism. \[\square\]

In other words, up to semiring\(^{\dagger}\) isomorphism, \( R := R(L, G) \) is the unique uniform \( L \)-layered domain\(^{\dagger}\) \( R \) for which \( R_\infty = G \).
3.3. **The case where** \( L \) **is a semiring, and the role of the 0-layer.** To avoid dealing with exceptional cases in the sorting set \( L \), it has been more convenient to work in the language of semirings\(\dagger\). In this subsection we deal with a zero layer, i.e., assume \( 0 \in L \). At the same time, we take the opportunity to fit the zero element of \( R \) (if it exists) into the theory. Our treatment here is brief, with our main intent being to introduce Example 3.31. We study this situation in considerably greater detail in \[\text{IKR4}\].

New intricacies arise when \( 0 \in L \) and \( R_0 \neq \emptyset \). One tricky question involving \( 0_R \) (if it exists) is: Where does it reside? In particular, is it tangible or ghost?

**Lemma 3.24.** The layer \( R_0 \) is also an ideal of \( R \). If furthermore \( R \) is a semiring, then \( 0_R \in R_0 \).

**Proof.** The first assertion is clear. Suppose \( 0_R \in R_k \). Then for any \( a \in R_0 \) we have \[0_R = 0_R \cdot a \in R_{k,0} = R_0.\]

On the other hand, \( 0_R \) also acts like a ghost, since it absorbs all elements. Let us formalize this observation.

**Definition 3.25.** Suppose \((L,\geq)\) is a partially pre-ordered semiring (with zero element \( 0 \)). An **\( L \)-quasi-layered semiring**\(\dagger\) is a semiring \( R \), together with a family \( \{R_{\ell} : \ell \in L\} \) of disjoint subsets \( R_{\ell} \subset R \), satisfying precisely the same conditions as in Definition 3.26 with the following exceptions, to take into account the special role of the \( 0 \) layer:

(a) There are sort transition maps \( \nu_{0,\ell} \) for every \( \ell \in L \).

(b) Axiom A2 is replaced by the following axiom:

\[
\text{A2}_0 \quad \text{If } a \in R_k \text{ and } b \in R_{\ell}, \text{ then } ab \in R_{k\ell} \cup R_0.
\]

(c) In Axiom A3, the product in \( R \) is also compatible with the sort transition maps \( \nu_{0,\ell} \):

If \( a \in R_k \), \( b \in R_{\ell} \) with \( m \geq k \), then \( \nu_{0,k\ell}(ab) = \nu_{m,k}(a) \cdot \nu_{0,\ell}(b) \).

**Remark 3.26.** Since \( R_1 \) no longer turns out to be a monoid, we must often consider the fundamental submonoid \( R_0 \cup R_1 \), whereas \( R_0 \cup R_\infty \) is an ideal of \( R \). This formalizes the notion that the “absorbing layer” \( R_0 \) is both tangible and ghost.

Since \( 1_R \in R_1 \), every invertible element of the fundamental submonoid must lie in \( R_1 \). In particular, if (excluding the element \( 0 \)) the fundamental submonoid is a multiplicative group, then it must be \( R_1 \). This reduces us to the layered 1-semifield\(\dagger\) case described earlier.

Here is our main example.

**Example 3.27.** Given any ideal \( a \) of an \( L \)-layered domain\(\dagger\) \( R \), we formally define \( R_a \) to be \( R \) with the same semiring\(\dagger\) operations, and to have the same sort function as \( R \), except that now \( s(a) = 0 \) for every \( a \in a \). In other words,

\[
(R_a)_0 := R_0 \cup a; \quad (R_a)_\ell := R_\ell \setminus (a \cap R_\ell).
\]

**Proposition 3.28.** \( R_a \) is an \( L \)-layered semiring\(\dagger\).

**Proof.** We need to check associativity and distributivity. But this is clear unless we are using elements of \( a \), and then associativity holds because all products have layer 0. Likewise, to see that \( a(b + c) \) and \( ab + ac \) have the same layer, note this is clear if \( s(a) = 0 \) or if \( s(b + c) \neq 0 \). Thus we may assume that \( s(b + c) = 0 \), and again we are done if \( s(b) = s(c) = 0 \), so we may assume that \( s(b) = 0 \) and \( s(c) \neq 0 \) with \( b \succ_\nu c \) but \( ab \cong_\nu ac \). But then

\[
s(a(b + c)) = s(ab) + s(ac) = s(ab),
\]

so \( a(b + c) = ab = ab + ac \).

**Remark 3.29.**

(i) Taking the degenerate case \( a = 0 \) and \( R_0 = \emptyset \) reduces to the original quasi-layered domain\(\dagger\) (Definition 3.20).

(ii) More generally, if \( R \) is an **\( L \)-quasi-layered semiring**\(\dagger\) and \( R_0 \) is a prime ideal of \( R \), then \( R \setminus R_0 \) is an \( L \setminus \{0\}\)-quasi-layered domain\(\dagger\).

In this way, we see that Definition 3.26 encompasses Definition 3.20.
Remark 3.30. If \( R \setminus a \) is finite, then \((R_a)_1\) is a finite set. Thus, we have a way of “shrinking” the tangible component to a finite set.

One instance of arithmetic significance is when \( R = R(L, N \cup \{0\}) \) where \( L \) is finite, and \( a = \{ [i] \colon n > q, \ell \in L \} \) for some \( q \in \mathbb{N} \). In this case, we can “contract” \( a \) to a single element in \( R_0 \).

Example 3.31. (The layered truncated semiring\(^1\)). Take \( L = \mathbb{N} \), \( R = R(L, \mathbb{R}) \), and fixing \( q > 0 \) in \( L \), define \( a = L \times \{ n : n \geq q \} \preceq R \). Then \( R_a \) contracts to the L-layered semiring\(^1\)

\[
\{ [k]a : k \in L, a \in \{1, \ldots, q - 1\} \} \cup \{ [0]q \},
\]

where addition is defined as in Construction \[3.2\] and multiplication \([k]a \cdot [l]b\) is given as in Equation \[3.1\] except for \( ab \geq q \), in which case \([k]a \cdot [l]b = [0]q\) for any \( k, l \in L \). Addition is given by

\[ [k]a + [0]q = [0]q. \]

The sort transition maps are as in Construction \[3.2\] except that we define \( \nu_{0,k}([k]a) = [0]q \) for all \([k]a\). Thus, \([0]q\) is the special infinite element, and the sort transition map \( \nu_{0,k} \) is not 1:1. In this example, \( R_1 \cup \{ [0]q \} \) should be viewed as the fundamental submonoid.

When instead the layering semiring\(^1\) \( L \) is finite, we see that \( R_1 \cup \{ [0]q \} \) is a finite set, which merits further study using arithmetic techniques.

3.3.1 Adjoining the 0-layer. Starting with an \( L \)-quasi-layered domain\(^1\) \( R \) with respect to a semiring\(^1\) \( L \), we can adjoin a zero layer \( R_0 \) formally in several ways. The first way is simply by adjoining a zero element to \( R \).

Remark 3.32. For any quasi-layered domain\(^1\) \( R \) with respect to a semiring\(^1\) \( L \), the semiring

\[ R \cup \{ 0_R \} \]

can be layered with respect to the semiring

\[ L^0 := L \cup \{ 0 \}, \]

where we take \( R_0 := \{ 0_R \} \), putting it in the zero layer as seen by applying the argument of Proposition \[3.25\].

We take the sort transition maps \( \nu_{0,\ell}(a) := 0_R \) for all \( \ell \neq 0 \) and \( a \in R \).

However, this is not the only possibility for the zero layer, as we saw in Remark \[3.3\].

Proposition 3.33. If \( R \) is a uniform \( L \)-layered domain\(^1\), where \( L \) is a semiring\(^1\), then adjoining \( \{ 0 \} \) formally to \( L \) as the unique minimal element, we can form a uniform \( L \)-layered domain\(^1\) \( R \cup R_0 \), where \( R_0 := e_0 R_1 \) is another copy of \( R_1 \), under the same rules of addition and multiplication given by Proposition \[3.34\] and the sorting maps \( \nu_{0,k} \) are the natural identifications.

Proof. If \( a = e_0 a_1 \), \( b = e_k b_1 \), and \( c = e_c c_1 \) for \( a_1, b_1, c_1 \in R_1 \), then

\[ (ab)c = e_0 e_{k \epsilon} e_{c \epsilon} (a_1 b_1 c_1) = e_0 e_{k \epsilon} e_{c \epsilon} a_1 (b_1 c_1) = e_0 a_1 (b_1 c_1) = e_0 a_1 (b_1 c_1) = ab(c), \]

yielding associativity of multiplication. To see distributivity, we note that \( e_k b_1 + e_{k \epsilon} c_1 = e_m (b_1 + c_1) \) where \( m \in \{ k, \ell, k + \ell \} \), so

\[ a(b + c) = e_0 e_{m \epsilon} a_1 (b_1 + c_1) = e_0 a_1 (b_1 + c_1) = e_0 a_1 b_1 + e_0 a_1 c_1 = e_0 e_{k \epsilon} a_1 b_1 + e_0 e_{k \epsilon} a_1 c_1 = ab + ac. \]

Associativity of addition is similar. Finally, if \( a = 0_R \in R_0 \) and \( b \in R_\ell \), then \( ab \in R_0 \ell = R_0 \).

To indicate the richness of this approach, let us see how we can remove the restriction in Construction \[3.2\] that the monoid \( \mathcal{G} \) is cancellative, using the idea of Example \[3.27\].

Construction 3.34. Suppose \( L \) is a semiring, and we are given an ordered monoid \( \mathcal{G} \), viewed as a semiring\(^1\) as in Remark \[3.1\]. We say an element in \( \mathcal{G} \) is non-cancellative if it has the form \( ab = ac \) where \( b \neq c \). The set of non-cancellative elements comprises a monoid ideal \( a \) of \( \mathcal{G} \), since if \( ab = ac \in a \) then, for any \( d \in \mathcal{G} \), \( abd = acd \), implying \( abd \in a \). Define the semiring\(^1\) \( R' \) to be set-theoretically \((L \setminus \{ 0 \}) \times (\mathcal{G} \setminus a) \cup \{ 0 \} \times a \), where again we denote the element \((\ell, a)\) as \([\ell] a\) and define multiplication and addition componentwise, according to the rules of Construction \[3.2\] but with

\[ [k]a \cdot [\ell] b = [0]ab \quad (3.9) \]

whenever \( ab \in a \).
The same idea is applicable when we adjoin the 0-layer. Suppose \( L \) is a semiring\(^\dagger\). Adjoin 0 to \( L \) to get
\[
L^0 = L \cup \{0\}.
\]
Again we mimic Construction \[3.32\] and define the semiring\(^\dagger\) \( R(L', \mathcal{G}) \) to be set-theoretically
\[
(L \times (\mathcal{G} \setminus \{a\})) \cup (\{0\} \times a),
\]
where again we denote the element \((\ell, a)\) as \([\ell]a\) and define multiplication and addition componentwise, according to the rules of the previous paragraph, including \( [3.34] \).

**Proposition 3.35.** The layered semiring\(^\dagger\) of Construction \[3.32\] is a semiring\(^\dagger\). If \( \mathcal{G} \) has an absorbing element 0, then \([0]0\) is the zero element of \( R \).

**Proof.** Exactly as in the proof of Proposition \[3.28\] since we need only check the zero layer. Note that 0 is a non-cancellable element of \( \mathcal{G} \), implying \([0]0\) \( \in R \), and it is clearly the zero element. □

Since we have several ways of adjoining a zero layer, the following observation is useful.

**Proposition 3.36.** For any semiring \( R \) layered with respect to a semiring\(^\dagger\) \( L, R \cup \{0_R\} \) is an \( L^0 \)-layered sub-semiring of \( R \cup R_0 \).

More generally, for any ideal \( a \) of \( R \), writing \( a_0 \) for \( a \cap R_0 \), we have \((\bigcup_{\ell \neq 0} R_\ell) \cup a_0 \) is an \( L^0 \)-layered sub-semiring of \( R \cup R_0 \).

**Proof.** If \( a \in R \) and \( b \in R_\ell \), then \( ab \in R_0 \cdot R_\ell = R_0 \), implying \( ab \in a_0 \). □

This gives rise to the question of whether we should adjoin the entire 0-layer, or just \([0]0\)? Although one’s experience from classical algebra might lead one to adjoin only \([0]0\), there are situations in which one might need other elements in \( R_0 \) in order to distinguish polynomials, as illustrated below in Example \[6.35\].

3.3.2. The 0-layer versus the \( \infty \)-layer. So far we have discussed two layers that in our present context could be extraneous, the 0-layer and the \( \infty \)-layer. These two layers act similarly, since both 0 and \( \infty \) are absorbing elements of \( L \), except that 0 also absorbs \( \infty \) in the sense that \( 0 \cdot \infty = 0 \). (Of course, the difference in the tropical theory is that \( 0 + a = a \) whereas \( \infty + a = \infty \), but often their multiplicative properties are more significant.) Thus, in case \( \infty \in L \) but \( 0 \notin L \), \( R_\infty \) is an ideal of \( R \) that can often be used to replace \( R_0 \) in the above discussion. One instance is given in Appendix B.

3.4. The case of onto sort transition maps. Remark \[3.12\] leads us to a key simplification for layered domains\(^1\) when the sort transition maps are onto, which enables us to reduce many results to the tangible case:

**Lemma 3.37.** If \( R \) is an \( L \)-layered domain\(^1\) and \( a \in R_\ell \) with \( \nu_{\ell,1} \) onto, then \( a = e_\ell a_1 \) for some \( a_1 \in R_1 \).

**Proof.** Taking \( a_1 \in R_1 \) for which \( \nu_{\ell,1}(a_1) = a \), we have \( a = \nu_{\ell,1}(a_1) = e_\ell a_1 \) by Remark \[3.12\]. □

**Note 3.38.** Lemma \[3.37\] enables us to simplify the theory for any layer \( \ell > 1 \) for which \( \nu_{\ell,1} \) is onto. When \( \ell < 1 \) we could go in the opposite direction, and define \( e_\ell \) such that \( \nu_{1,\ell}(e_\ell) = 1_R \). This will be well-defined when \( \nu_{1,\ell} \) is 1:1 since, writing \( \ell = \frac{m}{n} \) for any \( a \in R_\ell \) with \( \nu_{1,\ell}(a) = 1_R \), we have
\[
ne_\ell = ne_{m/n} = e_m = \nu_{m,\ell}(a) = na,
\]
implicating \( a = e_\ell \).

Although our examples often have \( L = L_{\geq 1} \), we use this procedure when the occasion arises.

3.5. Uniform \( L \)-Layered domains\(^1\). Let us return to the most important case, that of uniform \( L \)-Layered domains\(^1\), the main example being \( R(L', \mathcal{G}) \) of Construction \[3.2\]. Let us see how the layered theory simplifies for uniform \( L \)-layered domains\(^1\), enabling us to eliminate the sort transition maps \( \nu_{k,\ell} \) from the picture.

**Remark 3.39.** In a uniform \( L \)-layered domain\(^1\), we can define \( \nu_{k,\ell} \) for \( k < \ell \) to be \( \nu_{\ell,k}^{-1} \). Thus, \( \nu_{k,\ell} \) is defined for all \( k, \ell \in L \).

**Lemma 3.40.** Any element \( a \in R_\ell \) can be written uniquely as \( e_\ell a_1 = \nu_{\ell,1}(a_1) \) for \( a_1 \in R_1 \).

**Proof.** The existence of \( a_1 \) follows from Lemma \[3.37\] and uniqueness is clear since \( \nu_{\ell,1} \) is presumed to be 1:1. The last assertion follows from Axiom A3. □
Proposition 3.41. Axiom A2 can be replaced by the following axiom:

\begin{equation*}
A2'. \text{ If } a = e_k a_1 \in R_k \text{ and } b = e_\ell b_1 \in R_\ell, \text{ for } a_1, b_1 \in R_1, \text{ then } ab = (a_1 b_1) e_{k \ell}.
\end{equation*}

Proposition 3.42. In a uniform L-layered domain\(^1\), if \(a \equiv_\nu b\) for \(a \in R_k\) and \(b \in R_\ell\) then \(b = \nu_{\ell,k}(a)\). In particular, if \(a \equiv_\nu b\) for \(a, b \in R_\ell\), then \(a = b\).

Proof. An immediate application of Lemma 3.40. \(\square\)

Corollary 3.43. In a uniform L-layered domain\(^1\), the transition map \(\nu_{m,\ell}\) is given by \(e_\ell a_1 \mapsto e_m a_1\).

Now we can remove the sort transition maps from the definition, when we write \(R = \bigcup_{\ell \in L} e_\ell R_1\).

Corollary 3.44. Suppose \(R\) is a uniform L-layered domain\(^1\). Defining \(\nu_{m,\ell}\) as in Corollary 3.43, we see that Axiom A3 is equivalent to the following axiom:

\begin{equation*}
A3'. \text{ } e_\ell e_k = e_{k\ell} \text{ for all } k, \ell \in L.
\end{equation*}

Furthermore, Axiom A4 now is equivalent to Axiom B, which we can reformulate as:

\begin{equation*}
B'. \text{ If } a = e_k a_1 \text{ and } b = e_\ell a_1 \text{ (so that } a \equiv_\nu b\text{), then } a + b = e_{k+\ell} a_1.
\end{equation*}

Proof. The first assertion follows from the observation that \(e_\ell a_1 e_k b_1 = e_{k\ell}(a_1 b_1)\); when \(a_1, b_1 \in R_1\) then \(a_1 b_1 \in R_1\).

For the last assertion, apply Lemma 3.39 to Proposition 3.42. \(\square\)

Note that \(\nu\)-bipotence and Axiom B’ could then be used as the definition for addition in \(R\), and we summarize our reductions:

Proposition 3.45. A uniform L-layered domain\(^1\) can be described as the semiring\(^6\)

\begin{equation*}
R := \bigcup_{\ell \in L} R_\ell,
\end{equation*}

where each \(R_\ell = e_\ell R_1\), \((R_1, \cdot)\) is a monoid, there is a 1:1 correspondence \(R_1 \to R_\ell\) given by \(a \mapsto e_\ell a\) for each \(a \in R_1\); and operations are given by Axioms A2’, A3’, B’, and \(\nu\)-bipotence.

We have effectively identified any arbitrary uniform L-layered domain\(^1\) with Construction 3.2, as will be seen more precisely in Theorem 3.23 and Proposition 3.20.

3.6. Reduction to the uniform case. In one sense, we can reduce the general case of an L-layered domain\(^1\) \(R\) to the uniform case. First we cut down on superfluous elements. Note that if \(\nu_{k,1}\) are onto for all \(k \geq 1\), then all the \(\nu_{\ell,k}\) are onto for all \(\ell \geq k\). Indeed, if \(a \in R_\ell\) then writing \(a = \nu_{\ell,1}(a_1)\) we have

\[ a = \nu_{\ell,k}(\nu_{k,1}(a_1)).\]

Remark 3.46. Suppose \(L = L_{\geq 1}\). For any L-layered domain\(^1\) \(R := (R, L, (\nu_{m,\ell}))\), if we replace \(R_\ell\) by \(\nu_{\ell,1}(R_1)\) for each \(\ell \in L\), we get an L-layered domain\(^1\) for which all the \(\nu_{m,\ell}\) are onto.

Having reduced many situations to the case for which all the \(\nu_{m,\ell}\) are onto, we can get a uniform L-layered domain\(^1\) by specifying when two elements are “interchangeable” in the algebraic structure.

Definition 3.47. Define the equivalence relation

\[ a \equiv b \text{ when } s(a) = s(b) \text{ and } a \equiv_\nu b.\]

In view of Proposition 3.42 this relation is trivial in case \(R\) is a uniform L-layered domain\(^1\).

Proposition 3.48. The binary relation \(<_\nu\) on an L-layered domain\(^1\) \(R\) induces a pre-order on the equivalence classes \(R/\equiv\). Furthermore, if \(a \equiv b\), then \(ac \equiv bc\) and \(a + c \equiv b + c\) for all \(c \in R\).
Proof. The first assertion is immediate. For the second assertion, \( s(ac) = s(a)s(c) = s(b)s(c) = s(bc) \) and \( ac \cong bc \) proving \( ac \equiv bc \).

Next, we consider addition. If \( a >_\nu c \), then
\[
a + c = a \equiv b = b + c.
\]
If \( a < _\nu c \), then \( a + c = c = b + c \). If \( a \equiv _\nu c \), then
\[
s(a + c) = s(a) + s(c) = s(b) + s(c) = s(b + c),
\]
and \( a + c \cong_\nu a \cong_\nu b \cong_\nu b + c \). \( \Box \)

Remark 3.49. When the transition maps \( \nu_{\ell,k} \) are onto, one can reduce to uniform \( \ell \)-layered domains, by means of the equivalence relation \( \equiv \) of Definition 3.47 since any \( \nu \)-equivalent elements having the same sort are identified. Then Proposition 3.48 shows that \( R/\equiv \) is an \( \ell \)-layered domain, under the natural induced layering, and the transition maps on \( R/\equiv \) clearly are bijective.

3.7. The \( \ell \)-surpassing relation. One of the key features of the supertropical theory is the use of the antisymmetric “ghost surpassing relation,” given by
\[
a \trianglerighteq b \text{ when } \begin{cases} a = b \\ \text{or} \\ a = b \text{ + ghost,} \end{cases}
\]
which replace equality and provides analogs of identities of classical algebra. We need to extend that relation to \( \ell \)-layered domains, usually with respect to a certain finite positive layer \( \ell \). (In the standard supertropical theory, \( \ell = 1 \).) There are two ways of extending ghost layers to an arbitrary sorting semiring \( L \).

Lemma 3.50.

(i) If \( b_1, b_2 \) are \( \ell \)-ghost, then so is \( b_1 + b_2 \).

(ii) If \( b \) is \( k \)-ghost and \( c \in R_\ell \) with \( \ell \in L_+ \), then \( bc \) is \( k\ell \)-ghost.

Proof. (i): The assertion is clear unless \( b_1 \equiv_\nu b_2 \) with \( s(b_1) = s(b_2) = \ell \), and with \( \ell + \ell = \ell \). But then \( \ell \) is infinite and \( s(b_1 + b_2) = 2\ell \).

(ii): If \( s(b) > k \), then \( s(bc) = s(b)c > k\ell \), so we are done unless \( s(b) = k \) with \( k + p = k \) for some positive \( p \in L_+ \). Then \( k\ell + p\ell = k\ell \), and \( p\ell \in L_+ \), implying \( k\ell \) is infinite, so again \( bc \) is \( k\ell \)-ghost. \( \Box \)

Definition 3.51. The \( \ell \)-surpassing relation \( \trianglerighteq_\ell \) is given by
\[
a \trianglerighteq_\ell b \text{ iff either } \begin{cases} a = b + c \text{ with } a \text{ \( \ell \)-ghost,} \\ a = b, \\ a \equiv_\nu b \text{ with } a \text{ \( \ell \)-ghost.} \end{cases}
\]

Proposition 3.52. If \( a \trianglerighteq_\ell b \) with \( s(a) \leq \ell \) and \( \ell \) finite, then \( a = b \).

Proof. We may assume \( a \neq b \), so the only condition of Definition 3.51 that can hold is \( a = b + c \) with \( c \) an \( \ell \)-ghost and \( c \equiv_\nu a \) (since otherwise \( a = b \) and we are done). Thus, \( s(a) \geq s(c) \) is an \( \ell \)-ghost sort, implying \( s(a) = \ell \) which being also an \( \ell \)-ghost sort is infinite, a contradiction. \( \Box \)

But the relation \( \trianglerighteq_\ell \) is only of marginal interest; our main focus is on the next relation.

Definition 3.53. The \( L \)-surpassing relation \( \trianglerighteq_\ell \) is given by \( a \trianglerighteq_\ell b \) if \( a \trianglerighteq_\ell b \) where \( \ell = s(b) \).

The relation \( \trianglerighteq_\ell \) generalizes equality in the following sense:

Lemma 3.54.

(i) If \( a \trianglerighteq_\ell b \) with \( s(a) = s(b) \) finite, then \( a = b \).

(ii) If \( a \trianglerighteq_\ell b \) and \( b \trianglerighteq_\ell a \), then \( a = b \).
Proof. (i) By Proposition 3.52.

(ii) Since \( s(b) \leq s(a) \) and \( s(a) \leq s(b) \), we get \( s(a) = s(b) \) and conclude by using (i) and Remark 3.57. □

**Example 3.55.** When \( L \) is non-negative,

\[
(a + b)^n \models_L a^n + b^n. \tag{3.12}
\]

(Equality holds unless \( a \equiv_{\nu} b \), in which case we are done by Axiom B.)

**Proposition 3.56.** In the standard supertropical case (\( L = \{1, \infty\} \)), \( a \models_L b \) iff \( a \models_{gs} b \).

Proof. We may as well assume that \( a \neq b \). If \( a \equiv_{\nu} b \) with a \( \ell \)-ghost, then clearly \( a \models_{gs} b \). Hence, we may assume that \( a \not\equiv_{\nu} b \) but \( a = b + c \) for a \( s(b) \)-ghost. But this implies \( a >_{\nu} b \), so \( a = c \), which shows that \( c \) is a \( s \)-ghost and \( a = c + b \), again yielding \( a \models b \). □

Nevertheless, the flavor of \( \models_L \) differs from the standard supertropical theory, as we see in Remark 5.13 below.

### 3.8. The layered sub-domain\(^{†}\) generated by \( 1_R \)

Our objective here is to provide the layered analog of \( F_1 \) of [CC].

**Lemma 3.57.** If \( R \) is an \( L \)-layered domain\(^{†} \), then

\[ \varepsilon_L(R) := \{ e_\ell : 1 \leq \ell \in L \} \]

is an \( L \)-layered sub-domain\(^{†} \) of \( R \), and is also has the natural \( \mathbb{N} \)-action given by \( n \cdot e_\ell = e_{n \ell} \). There is a natural semiring\(^{†} \) isomorphism \( L \to \varepsilon_L \) given by \( \ell \mapsto e_\ell \).

Proof. \( e_k + e_\ell = \nu_{k,1}(1_R) + \nu_{\ell,1}(1_R) = \nu_{k+\ell,1}(1_R) = e_{k+\ell} \)

by Axiom A4, and likewise

\[ e_k \cdot e_\ell = \nu_{k,1}(1_R) \cdot \nu_{\ell,1}(1_R) = \nu_{k+\ell,1}(1_R) = e_{k\ell} \]

by Axiom A3; finally, \( \nu_{1,1}(1_R) = 1_R \), and \( e_\ell + \cdots + e_\ell = e_{n \ell} \) for any \( n \in \mathbb{N} \). □

#### 3.8.1. Comparison with the idempotent (characteristic 1) theory

Let us now see how this ties in with the theory espoused in [CC], which we recall takes a monoid \( G \) and makes it into an idempotent semiring by means of an operation which they call \( s \), and which we temporarily denote as \( \tilde{s} \) to avoid confusion with our sorting map \( s \). Intuitively, \( \tilde{s} \) means the operation \( \cdot + 1^{\circ} \). Thus, in the max-plus algebra one would have \( \tilde{s}(a) \in \{a, 1\} \), and \( \tilde{s}^2 = \tilde{s} \).

**Remark 3.58.** Suppose \( R \) is \( \nu \)-bipotent. Then \( \tilde{s}(a) \in \{a, 1_R\} \) whenever \( a \not\equiv_{\nu} 1_R \). Thus, \( \tilde{s}^2(a) = \tilde{s}(a) \) unless \( a \equiv_{\nu} 1_R \). This observation yields an interesting parallel to [CC] given in Proposition 10.7 below.

**Remark 3.59.** \( \varepsilon_L \) of Lemma 3.57 plays the analogous role of \( F_1 \) of [CC].

### 4. Truncation and layered homomorphisms

In this section, we discuss a fundamental way to cut down the sorting semiring\(^{†} \) \( L \) to a more manageable one, usually finite. (The significance of such a situation is discussed in Example 4.7.) This is put in the context of homomorphisms between layered domains\(^{†} \), which also could serve as a prelude to a general discussion of morphisms, for which we give a foretaste. We continue our basic set-up: \( L \) is a non-negative semiring\(^{†} \) equipped with the order \( \geq \), and \( R \) is an \( L \)-quasi-layered domain\(^{†} \).
4.1. **Truncation of the layering semiring**. Let $Q$ be an upper ideal in $L$; i.e., $Q$ is a non-empty ideal of $L$ having the following property:

$$
\text{If } \ell \in Q, \text{ then } m \in Q \text{ for all } m \geq \ell. \quad (4.1)
$$

**Remark 4.1.** Since $L$ is non-negative, the condition $(4.1)$ means $\ell + m \in Q$ for all $m \in L$. Conversely, when $Q$ satisfies $(4.1)$ and $L = L_{\geq 1}$, then $Q$ is automatically an upper ideal, since $k\ell \geq \ell$ for all $k \in L$.

There are two cases of particular interest, where we fix $q \in L$:

(i) $Q = \{ \ell \in L : \ell \geq q \}$;

(ii) $Q = \{ \ell \in L : \ell > q \}$.

Our objective is to “mod out” $Q$ in order to make $q$ the unique largest element which takes on the role of the infinite element $\infty$ with respect to $L \setminus Q$.

Towards this end, we recall the Rees quotient monoid $L/Q$: We first define an equivalence relation $E_L(Q)$ on $L$ as follows, writing $\sim_Q$ for $\sim_{E_L(Q)}$:

For $k, \ell \in L$, we decree:

- $k \sim_Q \ell \iff k \in Q$.
- $k \sim_Q \ell \iff k = \ell$.

This is well-known to be an equivalence relation on $L$ compatible with multiplication and addition, since

$$
\ell \sim_Q \ell \Rightarrow k \cdot m \sim_Q \ell \cdot m, \quad k + m \sim_Q \ell + m,
$$

for any $k, \ell, m \in L$. The equivalence class $[\ell]_Q$ of any $\ell \in L$ is $\{ \ell \}$ if $\ell \notin Q$, but is $Q$ if $\ell \in Q$, and the set $L/Q := L/E_L(Q)$ is a semiring\(^1\) under the rules

$$
[\ell]_Q \cdot [m]_Q := [\ell m]_Q, \quad [\ell]_Q + [m]_Q := [\ell + m]_Q. \quad (4.2)
$$

The equivalence class of $Q$ in $L/Q$ is a single element, which we write as $q$ in the case of Remark 4.1(i), and we identify a class $[\ell]_Q = \{ \ell \}$ with the element $\ell$ of $L$ if $\ell \in L \setminus Q$. The original order induces an order on our semiring\(^1\) $L/Q$, where the class $q$ is larger than every other class (and thus plays the role of $\infty$ in $L/Q$); the natural map

$$
\pi_Q : L \to L/Q, \quad \ell \mapsto [\ell]_Q
$$

becomes order preserving (in the weak sense: $\ell \leq m \Rightarrow [\ell]_Q \leq [m]_Q$).

We then have

$$
L/Q := \{ [\ell]_Q \mid \ell \in L \} = (L \setminus Q) \cup \{ q \}. \quad (4.3)
$$

In short, $L/Q$ arises from $L$ by identifying all $\ell \in Q$ with the single element $q$.

(In the case of Remark 4.1(ii)) the equivalence class of $Q$ in $L/Q$ is a new element which we could denote as $q^+$.)

We also define an equivalence relation $E_R(Q)$ on $R$ as follows, writing again $\sim_Q$ instead of $\sim_{E_R(Q)}$. Let $x, y \in R$.

$$
x \sim_Q y \text{ if: } \begin{cases} 
s(x) \in Q, s(y) \in Q \text{ with } x \equiv_y y \\
or \\
s(x) \notin Q, \text{ with } x = y.
\end{cases} \quad (4.4)
$$

**Proposition 4.2.** $E_R(Q)$ is compatible with addition and multiplication, i.e., for all $x, y, z \in R$:

$$
x \sim_Q y \Rightarrow x + z \sim_Q y + z, \quad xz \sim_Q yz. \quad (4.5)
$$

**Proof.** This is clear if $x = y$, so we may assume that $s(x) \in Q$ and thus $s(y) \in Q$, and we check addition case by case.

- If $x \equiv_y z$, then $x + z = x \equiv_y y + z$.
- If $x <_y z$, then $x + z = z = y + z$.
- If $x \equiv_y z$, then $x + z \equiv_y z \equiv_y y + z$, implying $s(x + z) \in Q, \quad s(y + z) \in Q$

by Axiom B, in view of $(4.1)$. 

In each case, \( x + z \sim_Q y + z \). One verifies easily that \( x \cdot z \sim_Q y \cdot z \).

We write \( R/Q \) as shorthand for \( R/E_R(Q) \). It follows that \( R/Q \) carries the structure of a semiring such that the natural map

\[
\pi^R_Q : R \to R/Q,
\]

which sends every \( x \in R \) to its equivalence class, denoted by \([x]_Q\), is a semiring\(^1\) homomorphism. Moreover, we have a unique map \( \bar{s} : R/Q \to L/Q \), given by

\[
\bar{s}([x]_Q) = [s(x)]_Q.
\]

such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{s} & L \\
\downarrow{\pi^R_Q} & & \downarrow{\pi_Q} \\
R/Q & \xrightarrow{\bar{s}} & L(Q)
\end{array}
\]

commutes, and this map \( \bar{s} \) is a semiring\(^1\) homomorphism. Explicitly, we have for \( x, y \in R \):

\[
[x]_Q + [y]_Q = [x + y]_Q, \quad [x]_Q \cdot [y]_Q = [xy]_Q.
\]

We write more simply

\[
\bar{L} := L/Q, \quad \bar{R} := R/Q,
\]

and \( \bar{\ell} := [\ell]_Q, \bar{x} := [x]_Q \) for \( \ell \in L, x \in R \). We want to turn the semiring\(^1\) \( \bar{R} \) into an \( \bar{L} \)-layered domain\(^1\). Now the surjective map \( \bar{s} : \bar{R} \to \bar{L} \) already partitions \( \bar{R} \) into subsets

\[
\bar{R}_\lambda := \bar{s}^{-1}(\lambda),
\]

with \( \lambda \) running through \( \bar{L} \). It remains to define the sort transition maps

\[
\nu_{\mu, \lambda} : \bar{R}_\lambda \to \bar{R}_\mu
\]

for \( \lambda, \mu \in \bar{L}, \mu \geq \lambda \).

Before we do this, we switch to a notation which we describe in detail for the case of Remark 4.1(i), that better conveys the idea of “truncation.” For any \( \ell \in L \setminus Q \) we identify \( \ell \) with the element \( \ell. \) \{Recall that \( \ell := [\ell]_Q = \{\ell\}. \}

Now \( \ell = q \) for any \( \ell \in Q \). In this way we view \( \bar{L} \) as a subset of \( L \cup \{q\} \):

\[
\bar{L} = (L \setminus Q) \cup \{q\},
\]

and then have \( \bar{\ell} = \ell \) for \( \ell \in (L \setminus Q) \). We further identify an element \( a \) of \( R_{\ell}, \ell \in \bar{L} \), with its image \( \bar{a} \) in \( \bar{R}_\ell \subset \bar{R} \). This makes sense since \( [a]_Q = \{a\} \) if \( \ell \in L \setminus Q \) while, for \( \ell \in Q \), \( a \) is identified with the set

\[
[a]_Q = \{b \in R \mid s(b) \in Q, a \equiv_\nu b\}.
\]

In this notation

\[
\bar{R} = \bigcup_{\ell \in \bar{L}} \bar{R}_\ell.
\]

Since now \( \bar{L} \) and \( \bar{R} \) are compared to subsets of \( L \) and \( R \), respectively, we need to distinguish addition and multiplication in these semirings from the given addition and multiplication \((+,-,\cdot)\) in \( L \) and \( R \).

We indicate these operations in \( \bar{L} \) and \( \bar{R} \) by the subscript “\(q\)” (alluding to “truncation”). Translating our rules (4.2) and (4.7) for addition and multiplication in \( L \) and \( R \) to the new notation, we obtain the following:

If \( k, \ell \in L \setminus Q, a \in R_k \) and \( b \in R_\ell \), then

\[
\begin{align*}
k +_q \ell &= \begin{cases} 
q & \text{if } k + \ell \notin Q, \\
\bar{k} + \ell & \text{if } k + \ell \in Q,
\end{cases} \\
\bar{a} +_q b &= \begin{cases} 
\bar{a} + b & \text{if } k + \ell \notin Q, \\
\bar{a} + \bar{b} & \text{if } k + \ell \in Q,
\end{cases}
\end{align*}
\]

Furthermore, \( \ell \cdot_q q = q = q \cdot_\ell q = q +_q \ell = \ell +_q q \) for any \( \ell \in \bar{L} \).

We now decree that the sort transition maps \( \bar{\nu}_{m, \ell} \) are just the transition maps \( \nu_{m, \ell} \) given in \( R \):

\[
\bar{\nu}_{m, \ell} = \nu_{m, \ell} : R_\ell \to R_m.
\]
if $\ell, m \in \bar{L}$, $m \geq \ell$.

It is easy to check that the semiring $R$, together with the partition $(\mathcal{R}_{\ell} | \ell \in \bar{L})$ and the sort transition maps $\nu_{m, \ell}$ for $\ell, m \in \bar{L}$, $\ell \leq m$, satisfies the axioms A1–A4, and B, cf. Definition 3.6. Thus $\mathcal{R}$ is an $L$-quasi-layered domain.$^1$

**Definition 4.3.** We call this $\bar{L}$-quasi-layered domain $\mathcal{R}$ the *truncation of $R$ at $Q$.*

**Remark 4.4.** In the special case that $Q = \{\ell \in L | \ell > q\}$ for some $q \in L$, $q \neq \infty$, we also say that $\mathcal{R}$ is the truncation of $R$ at $q$. Furthermore, $\mathcal{R}$ is $\nu$-bipotent if $R$ is $\nu$-bipotent; $\mathcal{R}$ is uniform if $R$ is uniform.

The truncation $R/Q$ has the map

$$\nu_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{G},$$

obtained from $\nu_{\mathcal{R}}$ by restriction. Also, the elements $e_\ell = e_{\ell, \mathcal{R}}$ in $\mathcal{R}_{\geq 1}$ are the elements $e_\ell = e_{\ell, R}$ in $R_{\geq 1}$ for $\ell \in L \setminus Q$, $\ell \geq 1$, whereas $e_q = [e_k]_Q$ for $k \in Q$.

**Example 4.5.** In the example $R = R(L, G)$ of a uniform layered domain$^1$, from Example 3.2, we obtain

$$R(L, G)/Q = R(L/Q, \mathcal{G}).$$

(4.8)

**Remark 4.6.** If $a \in R(L, G)$ with $s(a) \in Q$, then $\bar{a} + \bar{a} = \bar{a}$, since $Q$ is an upper ideal. Thus, the “top” layer of $R(L, G)/Q$ is ghost.

**Example 4.7.** For $L = \{1, \infty\}$ and $Q = \{1\}$, we have the truncation from the standard supertropical semiring$^1$ to the (idempotent) max-plus semiring$^1$. This shows how the standard supertropical theory “covers” the tropical theory.

For $L = \mathbb{N}$ and $Q = \{k : k \geq n\} \subset \mathbb{N}$, the truncation of $R(\mathbb{N}, G)$ at $Q$ is layered by $\{1, \ldots, n\}$. We can continue by taking $L = \{k : k \leq n\}$ and $Q = \{k : m \leq k \leq n\}$ for some given $m < n$. In this way, we get an infinite sequence of successive covers of the max-plus algebra, each of which provides more tropical information as $n$ increases.

### 4.2. Layered homomorphisms.

Truncation can be understood in terms of universal algebra. We assume that $L$ is non-negative.

**Definition 4.8.** A *layered homomorphism*

$$(\varphi, \rho) : (R, L, s, (\nu_{m, \ell})) \rightarrow (R', L', s', (\nu'_{m, \ell}))$$

in the category of $L$-quasi-layered domains$^1$ is a semiring$^1$ homomorphism $\rho : L \rightarrow L'$ preserving the given partial orders, i.e., satisfying the condition:

M1. $k \leq \ell$ implies $\rho(k) \leq \rho(\ell)$.

Together with a semiring$^1$ homomorphism $\varphi : R \rightarrow R'$ such that

M2. $s'(\varphi(a)) \geq \rho(s(a)), \forall a \in R$.

The definition becomes more complicated when $0 \in L$; then we need to modify Axiom M2 to:

M2'. $s'(\varphi(a)) = \ell$, where $\ell = 0$ or $\ell \geq \rho(s(a)), \forall a \in R$.

We always write $\Phi := (\varphi, \rho)$. From now on, we assume that the $L$-quasi-layered domain$^1$ $R$ is uniform. Furthermore, we often assume $L = L'$ and $\rho = \text{id}_L$; we call $\Phi$ a natural homomorphism in this situation. If $\Phi : (R, L, s, (\nu_{m, \ell})) \rightarrow (R', L', s', (\nu'_{m, \ell}))$ is a natural homomorphism such that $\varphi$ is 1:1, we say that $R'$ is an extension of $R$.

**Digression 4.9.** When defining layered homomorphisms over $L$-quasi-layered domains$^1$ which are not necessarily uniform, in order to preserve all the given structure, we should also require the condition:

$$\varphi(\nu_{e, \ell}(a)) = \nu'_{e, \ell}*(\varphi(a)), \forall k, \ell \in L, \forall a \in R_k,$$

(4.9)

with $k' = s'(\varphi(a)) \leq \ell' = s'(\varphi(\nu_{e, \ell}(a)))$.

On the other hand, this condition is rather technical, even when $\Phi$ is natural, which is why we focus on uniform $L$-layered domains$^1$. 

Lemma 4.10. Write $e_{\ell,R}$ for $e_{\ell}$ in $R$. Then $\varphi(e_{\ell,R}) = e_{\ell,R'}$ for each $\ell$ in the sub-semiring\(^1\) of $L$ (resp. $L'$) generated by 1.

Proof. Then $\varphi(e_{1,R}) = \varphi(1_R) = 1_{R'} = e_{1,R'}$. Thus, for each $n \in \mathbb{N}$, we have

$$\varphi(e_{n,R}) = \varphi(e_{1,R} + \cdots + e_{1,R}) = \varphi(e_{1,R}) + \cdots + \varphi(e_{1,R}) = e_{1,R'} + \cdots + e_{1,R'} = e_{n,R'}.$$ 

It follows at once that $\varphi$ is given by its action on $R_1$.

Proposition 4.11. If $a = e_{\ell}a_1$ as in Lemma 3.37, then

$$\varphi(a) = \varphi(e_{\ell,R})\varphi(a_1) = e_{\ell,R'}\varphi(a_1). \quad (4.10)$$

Proof. $\varphi(a) = \varphi(e_{\ell,R})\varphi(a_1) = e_{\ell,R'}\varphi(a_1)$.

Corollary 4.12. Equation (4.10) holds automatically whenever $R$ is uniform $L$-layered.

Proof. Lemma 3.37 is applicable.

4.2.1. Examples of layered homomorphisms.

Example 4.13 (Truncation). The commutative square (1.6) says that the map

$$r^R_Q : R \to R/Q$$

is a layered homomorphism from an $L$-layered domain\(^1\) to an $L/Q$-layered domain\(^4\), where $\rho$ is the natural map sending $m \to q$ for all $m \in Q$.

Example 4.14 (1-localization). If $R$ is an $L$-layered domain\(^1\), then taking any multiplicative submonoid $S$ of $R_1$, we can form the localization $S^{-1}R$ as a monoid, and define addition via

$$\frac{a}{u} + \frac{b}{v} = \frac{av + bu}{uv}$$

for $a, b \in R$, $u, v \in S$. $S^{-1}R$ becomes an $L$-layered domain\(^1\) when we define $s\left(\frac{a}{u}\right) = s(a)$. There is a natural layered homomorphism $R \to S^{-1}R$ given by $a \mapsto \frac{a}{S^1}$, which is injective since $R_1$ is cancellative.

Taking $S = R_1$, we call $S^{-1}R$ the $L$-layered 1-semifield\(^4\) of fractions of $R$; this construction shows that any uniform $L$-layered domain\(^4\) can be embedded into a uniform $L$-layered 1-semifield\(^4\).

Example 4.15 (1-divisible closure). We say that an $L$-layered domain\(^1\) $R$ is 1-divisibly closed if for each $a \in R_1$ and $n \in \mathbb{N}$ there is $b \in R_1$ such that $b^n = a$. As in Example 3.14, any uniform $L$-layered 1-semifield\(^4\) $R$ can be embedded into a 1-divisibly closed uniform $L$-layered 1-semifield\(^4\) $F$. Namely, adjoin $\sqrt[n]{a}$ to $R_1$ for each $a \in R_1$ and $n \in \mathbb{N}$, as in Remark 2.3 and enlarge all the other layers accordingly. We call $F$ the 1-divisible closure of $R$.

Although Example 4.15 is all we need for the applications in this paper, let us put this construction in its proper context for $L$-layered domains\(^4\).

Example 4.16 (Digression: $\nu$-divisible closure). We say that an $L$-layered domain\(^4\) $R$ is $\nu$-divisibly closed if for each $a \in R$ and $n \in \mathbb{N}$ there is $b \in R$ such that $b^n \equiv a$ under the equivalence of Definition 3.47. Note that if $s(a) = \ell$ then $s(b) = \sqrt[\ell]{T}$. This implies that $L$ must be closed under taking $n$-th roots for each $n$.

Assuming that $L$ is a group satisfying this condition, one can construct the $\nu$-divisible closure, sketched as follows:

Step 1: Given $a \in R_\ell$, adjoin a formal element $b \in R_{\sqrt[\ell]{T}}$, and consider all formal sums

$$f(b) := \sum_i \alpha_i b^i : \alpha_i \in R, \quad \alpha_i^\nu \equiv \nu, \alpha_i^{n_i} \equiv \nu^{n_i}, i, i'.$$ 

(\(\sum_i \alpha_i b^i\) is to be considered as the $n$-th root of $\sum_i \alpha_i^{(n)} a^i$.)

Define $R_\nu$ to be the set of all elements of the form (4.11), where any $a \in R$ is identified with $\alpha b^0$. (This could be stated more precisely in terms of evaluations of polynomials; compare with Definition 2.5 below.) We can define the sorting map $s : R_\nu \to L$ via

$$s(f(b)) = \sum_i s(\alpha_i) \sqrt[\ell]{i} \in L.$$
We define \( \equiv_{\nu} \) on \( R_{b} \) (notation as in (4.11)) by saying \( f_{b} \equiv_{\nu} f'_{b} := \sum'_{j=0} \alpha'_{j} b^{j} \) if \( \alpha_{i} \equiv_{\nu} \alpha'_{j} \). In particular, \( f_{b} \equiv_{\nu} c \) for \( c \in R \) if \( \alpha_{i} b^{i} \equiv_{\nu} c^{n} \). Likewise, we write \( f_{b} >_{\nu} f'_{b} := \sum'_{j=0} \alpha'_{j} b^{j} \) if \( \alpha_{i} \equiv_{\nu} \alpha'_{j} n b^{j} \).

Now we can define addition on \( R_{b} \) so as to be \( \nu \)-bipotent, where for \( \nu \)-equivalent elements we define \( f(b) + g(b) \) to be their formal sum (combining coefficients of the same powers of \( b \)); multiplication is then defined in the obvious way, via distributivity over addition. Now \( R_{b} \) is an \( \nu \)-layered domain\(^{1}\), in view of Proposition 3.47.

This construction is unique up to isomorphism, since one could replace \( a \) by any equivalent element in terms of Definition 3.47.

**Step 2:** Using Step 1 as an inductive step, one can construct the \( \nu \)-divisible closure by means of Zorn’s Lemma, analogously to the well-known construction of the algebraic closure, cf. [Row] Theorem 4.88.

**Step 3:** This construction is unique up to isomorphism, again by the same argument known for the algebraic closure.

The last two steps should be viewed in terms of “model completeness,” cf. [Mar] §4.3 or [VdD].

**Example 4.17** (Completion). One can construct the **completion** of any \( \nu \)-layered domain\(^{1}\) \( R \) as follows: First, take the completion of the ordered group \( R/ \equiv_{\nu} \) as described in Remark 2.2. We define \( \nu \)-Cauchy sequences in \( R \) to be those sequences \( (a_{i}) := \{a_{1}, a_{2}, \ldots\} \) which become Cauchy sequences modulo \( \equiv_{\nu} \), but which satisfy the extra property that there exists an \( m \) (depending on the sequence) for which \( s(a_{i}) = s(a_{i+1}) \), \( \forall i \geq m \). This permits us to define the sort of the \( \nu \)-Cauchy sequence to be \( s(a_{m}) \). Then we define the null \( \nu \)-Cauchy sequences in \( R \) to be those sequences \( (a_{i}) := \{a_{1}, a_{2}, \ldots\} \) which become null Cauchy sequences modulo \( \equiv_{\nu} \), and the completion \( \hat{R} \) is the factor group.

We also extend our given pre-order \( \nu \) to \( \nu \)-Cauchy sequences by saying that \( (a_{i}) \equiv_{\nu} (b_{i}) \) if \( (a_{i}b_{i}^{-1}) \) is a null \( \nu \)-Cauchy sequence, and, for \( (a_{i}) \not\equiv_{\nu} (b_{i}) \), we say \( (a_{i}) >_{\nu} (b_{i}) \) when there is \( m \) such that \( a_{i} >_{\nu} b_{i} \) for all \( i > m \). The completion \( \hat{R} \) becomes an \( \nu \)-layered domain\(^{1}\) under the natural operations, i.e., componentwise multiplication of \( \nu \)-Cauchy sequences, and addition given by the usual rule that

\[
(a_{i}) + (b_{i}) = \begin{cases} 
(a_{i}) & \text{if } (a_{i}) >_{\nu} (b_{i}), \\
(b_{i}) & \text{if } (a_{i}) <_{\nu} (b_{i}), \\
\nu s(a_{i}) + s(b_{i}), s(a_{i}+b_{i}) & \text{if } (a_{i}) \equiv_{\nu} (b_{i}).
\end{cases}
\] (4.12)

(In the last line, we arranged for the layers to be added when the \( \nu \)-Cauchy sequences are \( \nu \)-equivalent.) It is easy to verify \( \nu \)-bipotence for \( \hat{R} \).

These constructions are universal, in the following sense:

**Proposition 4.18.** Suppose there is an embedding \( \varphi : R \to F' \) of a uniform \( \nu \)-layered domain\(^{1}\) \( R \) into a 1-divisibly closed, uniform \( \nu \)-layered 1-semifield\(^{1}\) \( F' \), and let \( F \) be the 1-divisible closure of the 1-semifield\(^{1}\) of fractions of \( R \). Then \( F' \) is an extension of \( F \). If \( F' \) is complete with respect to the \( \nu \)-pre-order, then we can take \( F' \) to be an extension of the completion of \( F \).

**Proof.** This is standard, so we just outline the argument. First we embed the \( \nu \)-layered 1-semifield\(^{1}\) of fractions of \( R \) into \( F' \), by sending \( \frac{a_{1}}{c_{1}} \to \frac{\varphi(a)}{\varphi(c_{1})} \). This map is 1:1, since if \( \frac{a_{1}}{c_{1}} = \frac{d}{c_{2}} \), then \( c_{2}b = c_{1}d \), implying \( \varphi(c_{1}b) = \varphi(a_{1}d) \), and thus \( \frac{\varphi(b)}{\varphi(c_{1})} = \frac{\varphi(d)}{\varphi(c_{2})} \). Thus, we may assume that \( R \) is an \( \nu \)-layered 1-semifield\(^{1}\). Now we define the map \( F \to F' \) by sending \( \sqrt[n]{a} \to \sqrt[n]{\varphi(a)} \), for each \( a \in F_{1} \). This is easily checked to be a well-defined, 1:1 layered homomorphism.

In case \( F' \) is complete, then we can embed the completion of \( F \) into \( F' \). (The completion of a 1-divisibly closed 1-semifield\(^{1}\) is 1-divisibly closed, since taking roots of a \( \nu \)-Cauchy sequence in \( F_{1} \) yields a \( \nu \)-Cauchy sequence.)

**Theorem 4.19.** Suppose \( \Phi = (\varphi, \rho) : (R, L, s, (\nu_{m, \ell})) \to (R', L', s', (\nu'_{m, \ell})) \) is a layered homomorphism of layered domains\(^{1}\), where \( L \) is totally ordered. Then the restriction of \( \Phi \) to \( L \) is determined by the action of \( \varphi \) on \( R_{1} \).

**Proof.** By Lemma 3.37, for any \( a \in R_{k} \) we have \( a = e_{\ell, R}a_{1} \), for some \( a_{1} \in R_{1} \), and thus \( \varphi(a) = \varphi(e_{\ell, R})\varphi(a_{1}) = e_{\ell, R'}\varphi(a_{1}) \).
Layered homomorphisms can also be used to understand Theorem \[4.28\]

**Proposition 4.20.** Given any semiring\(^1\) \(L\) and \(L\)-layered domain\(^1\) \(R = (R, L, s, (\nu_m, \ell))\), take \(\mathcal{G}\) to be the direct limit of the \(R_\ell\), as described in Remark \[4.27\]. Then we have the layered homomorphism \(\Phi = (\varphi, \text{id}_{L})\) where \(\varphi : R \to R(L, \mathcal{G})\) is given by
\[
a \mapsto (s(a), a^\nu).
\]
Moreover, if \((R', L', s', (\nu'_m, \ell'))\) is a uniform \(L'\)-layered domain\(^1\), then \(\Phi\) is universal, in the sense that any layered homomorphism
\[
(\psi, \text{id}_{L}) : R \to R'
\]
factors through \(\varphi\), via a layered homomorphism
\[
(\pi, \text{id}_{L}) : R(L, \mathcal{G}) \to R'
\]
such that \(\psi = \pi \circ \varphi\).

Finally, any uniform \(L\)-layered domain\(^1\) \(R\) is isomorphic to \(R(L, \mathcal{G})\), where \(\mathcal{G} = R_1\) (viewed as a monoid).

**Proof.** \(\varphi(ab) = (s(ab), ab^\nu) = (s(a), a^\nu)(s(b), b^\nu)\). Addition is trickier, since we have to handle the case of \(a + b\) where \(a \approx \nu b\). But here
\[
\varphi(a + b) = (s(a + b), (a + b)^\nu) = (s(a) + s(b), a^\nu) = \varphi(a) + \varphi(b).
\]
To prove the next assertion, define \(\pi : R(L, \mathcal{G}) \to R'\) by \(\pi((s(a), a^\nu)) = \psi(a)\), and note that \(\pi\) is a homomorphism in view of Corollary \[6.34\] and is well-defined because the sort transition maps in \(R'\) are bijective.

The last assertion is seen by considering the natural homomorphism \(R(L, \mathcal{G}) \to R\), where the restriction to \(R_1\) is the identity. The \(\ell\) component \(\ell R_1\) then can be identified with \(R_\ell\), in view of Theorem \[1.19\] \(\Box\)

A useful layered isomorphism is given in Remark \[4.3\].

### 4.3. Layered supervaluations and the layered analytification

In case the layered domain\(^1\) is not uniform, we need a more general notion of morphism, treated in \[IKR1\]. To understand what is going on, we need to generalize the notion of “valuation.” Valuations are important in algebraic geometry, and play a key role in tropical theory largely because of the following example.

**Example 4.21.** Suppose \(K\) is the field of Puiseux series \(\{f := \sum_{n \in \mathbb{R}} \alpha_n \lambda^n : f\text{ has well-ordered support}\}\) over a given field \(F\). Then we have the \(m\)-valuation \(v : K \to \mathcal{G}\) taking any Puiseux series \(f\) to the lowest real number \(u\) in its support.

A word about notation: Given a valuation \(v : K \to \mathcal{G}\), one can replace \(v\) by \(-v\) and reverse the customary inequality to get
\[
v(a + b) \leq \max\{v(a), v(b)\},
\]
which is more compatible with the max-plus set-up. In what follows, we define an \(m\)-valuation to be a valuation whose target is a monoid, cf. \[IKR1\] Definition 2.1. In other words, we weaken the assumption that \(\mathcal{G}\) be an ordered group to \(\mathcal{G}\) merely being an ordered monoid, whose operation we write from now on in multiplicative notation. (In other words, \(v(ab) = v(a)v(b)\).) This fits in better with our algebraic notation for semirings\(^1\). Thus, any valuation \(v : K \to \mathcal{G}\) is an \(m\)-valuation, where we just disregard addition in \(K\).

Payne [Pay2] has developed an algebraic version of Berkovich’s theory of analytification, which can be viewed as the limit of tropicalizations. In his theory, a multiplicative seminorm \(| - | : W \to \mathbb{R}\) on a ring \(W\) is a multiplicative map satisfying the triangle inequality
\[
|a + b| \leq |a| + |b|.
\]
The underlying space in Payne [Pay2] is the set of multiplicative seminorms from \(K[\lambda_1, \ldots, \lambda_n]\) to \(\mathbb{R}_{>0}\) extending \(v\), for a given \(m\)-valuation \(v : K \to \mathbb{R}_{>0}\). We generalize this definition by taking an arbitrary ordered semiring\(1\) instead of \(\mathbb{R}_{>0}\).

The supertropical version, the strong supervaluation, is defined in \[IKR1\] Proposition 4.1 and Definition 9.9] as a monoid homomorphism \(\varphi\) satisfying \(\varphi(a) + \varphi(b) \triangleright \varphi(a + b)\), where \(\triangleright\) is the ghost surpassing relation of \[IKR1\] Definition 9.1. In this way, strong supervaluations generalize seminorms.

Here is the layered analog.
Definition 4.22. A layered supervaluation on a ring $W$ is a map $\varphi : W \to R$ from $W$ to an $L$-layered semiring $R$ with the following properties:

\[
\begin{align*}
LV1 : \varphi(1) &= 1_R, \\
LV2 : \forall a, b \in R : \varphi(ab) &= \varphi(a)\varphi(b), \\
LV3 : \forall a, b \in R : \varphi(a + b) &\leq \varphi(a) + \varphi(b), \\nonumber
LV4 : \varphi(0) &= 0_R.
\end{align*}
\]

A $\{0, 1\}$-layered supervaluation on a ring $W$ is a layered supervaluation $\Phi : W^\times \to R$, where $W^\times := W \setminus \{0\}$, such that $\Phi(W) \subseteq R_0 \cup R_1$.

Specifically, in the special case where $0 \notin L$, a tangible layered supervaluation† on an integral domain $W$ is a map $\Phi : W^\times \to R$ from $W \setminus \{0\}$ to an L-layered domain† $R$ with the following properties.

\[
\begin{align*}
LV1^\dagger : \Phi(1) &= 1_R, \\
LV2^\dagger : \forall a, b \in R : \Phi(ab) &= \Phi(a)\Phi(b), \\
LV3^\dagger : \forall a, b \in R : \Phi(a) + \Phi(b) &\leq \nu \Phi(a + b).
\end{align*}
\]

A tangible layered supervaluation† on an integral domain $W$ is a layered supervaluation† such that $\Phi(W) \subseteq R_1$.

Proposition 4.23. Suppose that $R = R(L, G)$ an L-layered domain†. If $\Phi : W \to G$ is a $\{0, 1\}$-layered supervaluation of a ring $W$, then $\Phi(a)$ is tangible for every invertible element $w$ of $W$. (In particular, if $W$ is a field, then $\Phi(W^\times)$ is tangible.)

Proof. $\Phi(w)\Phi(w^{-1}) = \Phi(1) = 1_R$, so $\Phi(w)$ is tangible by Lemma 3.16. \qed

In this situation, the tangible layer determines the layered supervaluation.

Remark 4.24. Under the assumptions of the proposition, it follows that the transmissions treated in [IKR4, Proposition 6.41] arise from layered homomorphisms.

The morphisms in the layered category should then be those maps which transfer one layered supervaluation to another. In the standard supertropical situation, these are the transmissions of [IKR3], which are given in the layered setting in [IKR4]. This paves the way for the following concept, with, notation as in Example 4.21.

Definition 4.25. Let $R = R(L, G)$, and view $v$ as the composite map of monoids

\[K \xrightarrow{v} G \cong R_1 \subseteq R.\]

Then for any affine algebraic variety $X$ over $K$, we define $K^\text{layered-an}$ to be the set of $\{0, 1\}$-layered valuations from $K[\lambda_1, \ldots, \lambda_n]$ to $R$ that extend $v$.

The space $K^\text{layered-an}$ extends $K^\text{an}$ of [Pay2], and its theory invites further study.

5. Layered functions and their roots, with multiplicities

As usual, we assume throughout that $R = (R, L, s, (\nu_m, \ell))$ is an L-layered domain†. We recall the multiplicative sort function $s : R \to L$, and write $[\ell]_a$ when we need to emphasize that $s(a) = \ell$.

5.1. The layered function semiring†. Before constructing the polynomial semiring† and the Laurent polynomial semiring†, we need an umbrella structure in which to develop the theory.

Definition 5.1. For any set $S$, and any semiring† $R$, $\text{Fun}(S, R)$ denotes the set of functions $f : S \to R$, which are $\nu$-compatible, in the sense that if $a \equiv_\nu a'$, then $f(a) \equiv_\nu f(a')$.

$\text{Fun}(S, R)$ also is a semiring†, whose operations are given pointwise:

\[ (fg)(a) = f(a)g(a), \quad (f + g)(a) = f(a) + g(a), \]

for all $a \in S$. The unit element of $\text{Fun}(S, R)$ is the constant function always taking on the value $1_R$. 

Remark 5.2. We write $f \models g$, for $f,g \in \text{Fun}(S,R)$, when $f(a) \models g(a)$, $\forall a \in S$. Likewise, we write $f \equiv g$ when $f(a) \equiv g(a)$, $\forall a \in S$. Now we also have the Frobenius-type properties:

$$(f + g)^k \models f^k + g^k, \quad \forall f,g \in \text{Fun}(S,R),$$

and

$$(f + g)^k \equiv g^{f^k} + g^k, \quad \forall f,g \in \text{Fun}(S,R).$$

Remark 5.3. If $\varphi : R \to R'$ is a semiring homomorphism, then there is a semiring homomorphism $\text{Fun}(S,R) \to \text{Fun}(S,R')$ given by $f \mapsto \varphi \circ f$, where

$$\varphi \circ f : a \mapsto \varphi(f(a)), \quad \forall a \in S.$$  

5.1.1. Polynomials. Given a semiring $R$, we have the polynomial semiring $R[\Lambda]$ in the commuting indeterminates

$${\Lambda} := \{\lambda_1, \ldots, \lambda_n\}.$$  

By convention, $[i]_{\Lambda}$ denotes $[i]_R \lambda$. Thus, any monomial can be written in the form $\alpha_1 \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ where $i = (i_1, \ldots, i_n)$, which has layer $s(\alpha_i)$. For any polynomial $f = \sum_i \alpha_i \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, we may apply the sort transition maps $\nu_{\ell,k}$ to its coefficients. We say a polynomial $f$ is tangible if each of its coefficients is tangible.

Lemma 5.4. Any polynomial is $\nu$-compatible.

Proof. Any monomial obviously is $\nu$-compatible, and a polynomial is a sum of monomials.

Just as in [IR1], we view polynomials in $R[\Lambda]$ as functions, but perhaps taking values in some extension $R'$ of $R$. More precisely, for any subset $S \subseteq R^{(n)}$, there is a natural homomorphism

$$\psi : R[\Lambda] \to \text{Fun}(S,R),$$  

obtained by viewing a polynomial as a function on $S$. In fact, for any subset $S \subseteq R^{(n)}$ and semiring homomorphism $\varphi : R \to R'$, there is a natural homomorphism

$$\psi_{\varphi} : R[\Lambda] \to \text{Fun}(S,R'),$$

which is the composite of the natural map $\bar{\varphi} : R[\Lambda] \to R'[\Lambda]$, with the natural homomorphism

$$\psi' : R'[\Lambda] \to \text{Fun}(S,R').$$

When $R$ is a 1-semifield, the same analysis is applicable to Laurent polynomials $R[\Lambda, \Lambda^{-1}]$, since the homomorphism $\lambda_i \mapsto a_i$ then sends $\lambda_i^{-1} \mapsto a_i^{-1}$. Likewise, we can also define the semiring of rational polynomials $R[\Lambda]_{\text{rat}}$, where the exponents of the indeterminates $\lambda_i$ are taken to be arbitrary rational numbers. Then in view of Remark 5.3 when $R$ is 1-divisibly closed and N-cancellative, the homomorphism $\lambda_i \mapsto a_i$ then sends $\lambda_i^{m/n} \mapsto a_i^{m/n}$.

In general, we work in some natural sub-semiring of $\text{Fun}(S,R')$, which we denote as $\mathcal{R}$, which might be identified with the semiring of polynomials, Laurent polynomials, or rational polynomials.

5.2. Decompositions of functions. We want to decompose functions as sums of “nice” functions, the way that polynomials are sums of monomials. This can be done axiomatically, in a rather natural way. We generalize a notion from [IR1], defined there on $R[\lambda_1, \ldots, \lambda_n]$. Essential summands of a function of $\text{Fun}(S,R)$ can be described more easily in the layered setting than in the standard supertropical setting, because we can utilize the different layers. Throughout, we fix a sub-semiring $\mathcal{R}$ of $\text{Fun}(S,R')$, where $R'$ is a suitable 1-divisibly closed, $L$-layered 1-semifield containing $R$. (One would expect $R'$ to be obtained via Examples 5.1.4 and 5.1.9, but this is a nontrivial issue that needs separate consideration.)

Definition 5.5. A function $f \in \text{Fun}(S,R)$ dominates $g \in \text{Fun}(S,R)$ at $a \in S$ if $f(a) \geq_g g(a)$; $f$ strictly dominates $g$ at $a$ if $f(a) >_g g(a)$.

Write $f = \sum_i h_i$, where $h_i \in \mathcal{R}$. The summand $h_i$ is essential at $a \in S$ if $f(a) = h_i(a)$; $h_i$ is inessential at $a \in S$ if $f(a) = \sum_{i \neq \ell} h_i(a)$. Also, $h_i$ is quasi-essential at $a$ if $h_i$ is neither essential nor inessential at $a$; it follows that $f(a) \geq_h h_i(a)$ but $f(a) \neq h_i(a)$. We say $h_i$ is essential in $f$ if $h_i$ is essential at some $a \in S$; $h_i$ is inessential in $f$ if $h_i$ is inessential at every $a \in S$. Finally, $h_i$ is quasi-essential if $h_i$ is neither essential nor inessential; in other words, $h_i$ is quasi-essential at some points of $S$, but not essential at any point.
A decomposition of $f$ is a sum $f = \sum_i h_i$ where each $h_i$ is essential or quasi-essential. The shell of the decomposition is the sum of those $h_i$ that are essential. The support of the decomposition at a point $a$ is the sum of those $h_i$ that are either essential or quasi-essential at $a$.

5.3. The layering map. $\text{Fun}(S, R)$ plays an extremely important role in our research, so we look for a layered framework with respect to an appropriate sorting semiring\(^3\), which turns out to be $\text{Fun}(S, L)$. We assume throughout this discussion that $L$ is non-negative, since we do not know how to interpret negative layers. (The zero layer itself is problematic enough, since it does not add to the value of a polynomial; for example, if $f = \lambda + [0]1$, then $f([0]1) = [0]1$, for any $\ell \in L$.)

Remark 5.6. When $L$ is a partially pre-ordered semiring\(^4\), $\text{Fun}(S, L)$ is partially pre-ordered by the relation $p \leq q$ if $p(a) \leq q(a)$ for all $a \in S$. This partial pre-order is directed, since $p(a), q(a) \in \text{Fun}(S, L)$ are bounded by $p(a) + q(a)$.

Definition 5.7. The layering map of a function $f \in \text{Fun}(S, R)$ is the map $\vartheta_f : S \to L$ given by

$\vartheta_f(a) := s(f(a)), \quad \forall a \in S.$

Thus, $\vartheta_f \in \text{Fun}(S, L)$. If $R$ is $L$-layered, then $\text{Fun}(S, R)$ inherits a layered structure from $R$ pointwise with respect to $\text{Fun}(S, L)$, in the following sense: We can define a sorting map $s : \text{Fun}(S, R) \to \text{Fun}(S, L)$ by sending $f \mapsto \vartheta_f$.

Remark 5.8. The sorting map $s$ plays the analogous role, with respect to functions, as the original sorting map $s : R \to L$. The semiring\(^1\) $\text{Fun}(S, R)$ satisfies Axioms A1–A3 and B with respect to the sorting semiring\(^1\) $\text{Fun}(S, L)$, all verified pointwise, but $\text{Fun}(S, R)$ is not $\nu$-bipotent, since some of the evaluations of $f + g$ might come from $f$ and others from $g$.

The layering map of a function is a mixture of variety. The geometry is contained in the information it provides, as indicated in Remark 6.23. In the standard supertropical theory, $\vartheta_f(a) = \infty$ iff $a$ is a root of $f$, and $\vartheta_f^{-1}(1)$ is the complement set of the root set of $f$. We return to this idea in \([6,1]\).

Since the layering map $\vartheta_f$ is so important, one is led to ask how far $\vartheta_f$ can be from a constant. In one sense, this is easy. View $f$ in $\text{Fun}(R_1^{(n)}, R')$ if a polynomial $f$ is a sum of $m$ monomials with tangible coefficients, then clearly for any $a \in R_1^{(n)}$, $s(f(a)) \leq m$, so $\vartheta_f$ is bounded by $m$. Such considerations lead to a connection between $\vartheta_f$ and simplicial theory.

Example 5.9. Take $R = R(\mathbb{N}, R)$, written in logarithmic notation.

(i) Figure\(^7\) shows the values of the layering map of the (tangible) linear polynomial $f = \alpha \lambda_1 + \beta \lambda_2 + \gamma$, where $\alpha, \beta, \gamma \in R_1$.

(ii) The values of the layering map of the (non-tangible) quadratic polynomial $f = \lambda_1 \lambda_2 + [k]1 \lambda_1 + 1 \lambda_2 + 0$ are presented in Figure\(^8\).

Remark 5.10. For any $a \in S$, the substitution map

$\Phi_a : \text{Fun}(S, R) \to R,$

given by $f \mapsto f(a)$ defines a homomorphism of semirings\(^1\). Thus, it makes sense to study evaluations of functions in this general situation.

More generally, if $S \supseteq S'$ there is a natural homomorphism $\text{Fun}(S, R) \to \text{Fun}(S', R)$ given by restricting the domain of a function from $S$ to $S'$. (We get the previous paragraph by taking $S' = \{a\}$.)

Remark 5.11. Taking $E_L$ as in Remark 3.55\(^5\), we have the semiring\(^1\) $\text{Fun}(S, E_L)$ which plays the parallel role to \([CC]\) Example 3.24 in enabling us analogously to define representable functors and schemes.

Definition 5.12. Given two functions $f, g \in \text{Fun}(S, R)$, define $f \parallel g$ when $f(a) \parallel g(a)$ for all $a \in S$. We write $f \equiv g$ when $f(a) \equiv g(a)$ for each $a \in S$.

Remark 5.13. Definition 5.12 differs considerably in the general layered setting from the standard supertropical setting. In the standard supertropical theory, when $f, g$ are rational polynomials in essential form, $f \parallel g$ means that each coefficient of $f$ surpasses the corresponding coefficient of $g$. For example, $\lambda + 3'' \parallel \lambda + 2$. 

But in the general layered setting, this is no longer true: \( f = \lambda + \lbrack 3 \rbrack \) does not surpass \( g = \lambda + \lbrack 2 \rbrack \), as seen by specializing \( \lambda \) to \( a = \lbrack 2 \rbrack \), since now \( f(a) = \lbrack 3 \rbrack \) whereas \( g(a) = \lbrack 2 \rbrack \). This phenomenon affects roots and varieties.

This rather general framework encompasses some very useful concepts, when we work with some other construction such as polynomials, to be considered shortly.
We may take $S \subseteq R_r^{(n)}$ when we want to restrict our attention to functions evaluated on tangible elements. To obtain a richer but more complicated structure, we could take $S = R_r^{(n)}$ or $S = (R \cup \{0_R\})^{(n)}$, and work with $\text{Fun}(S, R')$ for a suitable extension $R'$ of $R$.

Later, when we define layered varieties, we can take $S$ to be a given layered variety.

5.3.1. Roots of polynomials. The notion of root is crucial in geometry. In the standard supertropical theory recall that an element $a \in S$ is a (ghost) root of a function $f \in R$ iff $f(a)$ is ghost, which can occur in two ways, either by an essential monomial (at $a$) with ghost coefficient (cluster root), or by a pair (or more) of quasi-essential monomials (corner root). The situation in the general layered theory is analogous, although we need to deal with different ghost layers.

**Definition 5.14.** An element $a \in S$ is an $\ell$-root of $f \in R$ if $f(a)$ is an $\ell$-ghost.

**Remark 5.15.** $\bar{\partial}_f^{-1}(L_{\geq 1})$ is just the set of 1-roots of $f$ in $S$.

More important for us is the notion of “corner root.” We write a (rational) polynomial $f = \sum h_i$, where the $h_i$ are (rational) monomials, and call the $h_i$ the monomials of $f$.

**Definition 5.16.** An element $a \in S$ is a corner root of a (rational) polynomial $f$ if $f(a) \neq h(a)$ for each (rational) monomial $h$ of $f$.

**Example 5.17.** Take the uniform $\mathbb{N}$-layered domain $R = R(\mathbb{N}, R)$, written in logarithmic notation, and let $f = [b] \lambda + [1]^2$ and $g = [b] \lambda + [1]^3$. For any $a \in R_\ell$, we have the following values of $\{s(f(a)), s(g(a))\}$:

$$
\begin{align*}
\{j\ell, k\ell\} & \quad \text{for } a > 3; \\
\{j\ell, k\ell+1\} & \quad \text{for } a = 3; \\
\{j\ell, 1\} & \quad \text{for } 2 < a < 3; \\
\{j\ell+1, 1\} & \quad \text{for } a = 2; \\
\{1\} & \quad \text{for } a < 2.
\end{align*}
$$

In this way, we see that the layering map of a polynomial $f$ contains much information about the layers of the coefficients of $f$, and we can distinguish sets of polynomials via their roots, much more effectively than in the non-layered case. But of course we might have to go to some extension of $R$ to find the roots of $f$; for example if we take $f = \lambda^2 + 3$ with $R = R(\mathbb{N}, \mathbb{Z})$.

**Lemma 5.18.** The shell of any (rational) polynomial $f$ has at least two monomials. (In fact, those monomials in the support having maximal and minimal degree under the lexicographic order are essential.)

**Proof.** Increasing (resp. decreasing) the indeterminate of highest degree by a small amount yields a point at which the monomial is essential. \hfill $\Box$

5.3.2. Properties of monomials. Our usage of the word “monomial” is in the appropriate context, either for $R[\Lambda]$, $R[\Lambda, \Lambda^{-1}]$, or $R[\Lambda]_{\text{rat}}$. (Rational) monomials have several nice properties.

**Remark 5.19.** We write $\alpha \Lambda^i$, with $i = (i_1, \ldots, i_n)$, for the monomial $\alpha \Lambda^{i_1} \cdots \Lambda^{i_n}$.

(i) A rational polynomial is a monomial iff its shell has no more than one (essential) summand. (Indeed, a monomial $h$ has a decomposition $h = h_1 + h_2$ iff $h_1 = \alpha \Lambda^i$ and $h_2 = \beta \Lambda^j$ for suitable $\alpha, \beta$, implying $h = (\alpha + \beta) \Lambda^i$, so $h_1$ and $h_2$ are quasi-essential.)

(ii) The only rational polynomials which are tangible are the monomials with tangible coefficients.

**Remark 5.20.** For $R$ is 1-divisibly closed, define the path $P(a, b)$ joining elements $a$ and $b$ of $R^{(n)}$ to be $\{a^t b^{1-t} : t \in \mathbb{Q}, 0 < t < 1\}$. Paths contain elements “close” to $a$. Monomials preserve paths, in the sense that for every monomial $h(\lambda_1, \ldots, \lambda_n)$, all $a, b \in R^{(n)}$, and all $t \in \mathbb{Q}$,

$$
h(a^t b^{1-t}) = h(a)^t h(b)^{1-t},$$

so $h(P(a, b)) = P(h(a), h(b))$.

**Remark 5.21.** As in [IR1] Lemma 5.20, one sees the following, for any monomials $h_1$ and $h_2$ and all $c \neq a, b$ in the path $P(a, b)$:

(i) If $h_1(a) \succcurlyeq c, h_2(a)$ and $h_1(b) \succcurlyeq c, h_2(b)$, then $h_1(c) \succcurlyeq c, h_2(c)$;
(ii) If \( h_1(a) >_\nu h_2(a) \) and \( h_1(b) \geq_\nu h_2(b) \), then \( h_1(c) >_\nu h_2(c) \);
(iii) If \( h_1(a) \geq_\nu h_2(a) \) and \( h_1(b) \geq_\nu h_2(b) \), then \( h_1(c) \geq_\nu h_2(c) \).

5.4. The test algebra. It could be that two polynomials \( f \) and \( g \) agree on all values on \( R^{(n)} \) but not on all values on \( R' \), for an extension \( R' \) of \( R \). In the extreme case, if \( R = \{ 1 \} \) and \( L = \{ 1 \} \), then all polynomials agree on \( R \), but not necessarily on extensions of \( R \).

Example 5.22. The monomials \( \lambda^0 \) and \( \lambda^{n+1} \) agree on the semiring \([1, 2^n]\) but not on its extension \([1, 2^{n+1}]\), cf. Example 2.10.

We want to choose a specific extension \( \tilde{R} \) that will check this for all extensions. For example, in classical algebraic geometry, for any integral domain \( R \) one would take \( \tilde{R} \) to be the algebraic closure of \( R \).

Remark 5.23. For the purposes of this remark only, we define an equivalence \( \equiv_{R'} \) of (rational) polynomials by saying two (rational) polynomials \( f, g \) satisfy \( f \equiv_{R'} g \) iff \( \psi(f) = \psi(g) \) (cf. (5.2)) for every possible homomorphism \( \varphi : R \to R' \). The question arises as to how we choose \( R' \). Presumably, we could have \( f \equiv_{R_1} g \) but not \( f \equiv_{R_2} g \) for extensions \( R_1 \) and \( R_2 \) of \( R \). Let us call a layered extension \( \tilde{R} \) a test algebra for a class of (rational) polynomials \( C \) over \( R \) if, whenever \( f \equiv_{\tilde{R}} g \), then \( f(a) = g(a) \) for all \( a \in R' \), for any extension \( R' \) of \( R \).

Lemma 5.24. Suppose \( R \) is an \( L \)-layered domain, where \( R_1 \) is a cancellative monoid with \( R \neq \varepsilon_L(R) \) (cf. Lemma 3.57). Let \( F \) be the \( 1 \)-divisible closure of the layered 1-semifield of fractions of \( R \). Then:
(i) \( R \) itself is a test algebra for monomials.
(ii) \( F \) is a test algebra for (rational) polynomials in one indeterminate.

Proof. (i): Suppose we have two monomials \( f \neq g \). Then \( fg^{-1} \) is a monomial \( \neq 1_R \), and thus takes on some value other than \( 1_R \) on \( a \in R^{(n)} \), \( a = (a_1, \ldots, a_n) \) since \( R \) has elements \( a_j \not\equiv_\nu 1_R \), by the argument given in the proof of Corollary 3.15.

(ii): Suppose that \( f \) and \( g \) agree on \( R \). In view of (i), we may assume that \( f \) and \( g \) agree on \( F \). If a monomial \( h \) is essential for \( f \) at some point \( a \) of \( R^{(n)} \), then \( h \) also appears in \( g \) and actually is essential for \( g \) at \( a \), in view of (i) (noting that \( F^{(n)} \) has points “close” to \( a \) since \( F \) is \( 1 \)-divisibly closed). We need to show that any monomial \( h \) of \( f \) which is inessential on \( F \) remains inessential on an extension. But the corner roots of \( f \) all lie in \( F \) (since we only have one indeterminate and \( F \) is divisibly closed), and any corner root has infinitely many other points “closer” to it than the next corner root, seen by taking the path joining them, so the assertion is obvious.

The hypothesis that \( R_1 \) is cancellative is required, in view of Example 5.22.

Remark 5.25. For \( R \) a uniform \( L \)-layered domain, we take \( \tilde{R} \) to be the completion (Example 4.17) of the \( 1 \)-divisible closure (Example 4.15) of the \( L \)-layered 1-semifield of fractions \( F \) of \( R \). If there is a test algebra, there is one that contains \( R \). Indeed, we need to show that any (rational) polynomials \( f, g \) that agree on an extension \( R' \) already agree on \( R \). Replacing \( R' \) by the completion of its \( 1 \)-divisible closure, we may assume that \( R' \) is a complete, \( 1 \)-divisibly closed, \( L \)-layered 1-semifield. But then, in view of Proposition 4.15, there is a layered embedding of \( F \) into \( R' \), so we may replace \( R' \) by \( F \) and assume that \( R = F \). But likewise, we can then embed the \( 1 \)-divisible closure, and then its completion, into \( R' \).

Theorem 5.26. \( \tilde{R} \) of Remark 5.25 is a test algebra for all (rational) polynomials in \( R[A] \).

Proof. Although the easy argument of Lemma 5.24(ii) is not immediately applicable, due to the fact that corner roots are not necessarily defined over \( F \), there is a direct argument which comes from a standard theorem in inequalities. Starting with the same argument as in Lemma 5.24(ii), we may assume that \( f \) and \( g \) have the same essential monomials with respect to evaluations in \( \tilde{R}^{(n)} \), i.e., have the same shells (cf. Definition 5.5), and need to show that if their quasi-essential monomials define the same function on \( \tilde{R}^{(n)} \), then they define the same function on \( R^{(n)} \), for an arbitrary extension \( R' \) of \( R \). In view of Remark 5.26 we may assume that \( R = \tilde{R} \).

First we show that no inessential monomial \( h \) of \( f \) with respect to \( \tilde{R}^{(n)} \) becomes essential with respect to \( R^{(n)} \). Since essentiality is determined according to \( \nu \)-values, we may restrict evaluations to \( R_1 \), so this part of the proof is really a statement about the max-plus algebra. Multiplying through by \( h^{-1} \), we may
assume that \( h \) is the constant \( 1_\mathbb{R} \). Using Remark 2.4, we can view \( \widetilde{R}^{(n)} \) as a vector space over \( \mathbb{Q} \), which becomes a vector space over \( \mathbb{R} \) by means of Remark 2.3. Note that we have switched from multiplicative notation on the monoid \( R_1 \) to additive notation, and the constant \( 1_\mathbb{R} \) becomes the “zero” vector, so the inessentiality of \( h \) translates to the lack of a solution vector \( x = (x_1, \ldots, x_n) \) with
\[
Ax < -b,
\]
where \( b = (\alpha_1, \ldots, \alpha_m) \in R_1^{(m)} \) and \( A = (\alpha_{j,k}) \) is the \( m \times n \) matrix whose \((j,k)\)-entry is the power \((j_k)\) of the monomial \( h_{j} = \alpha_j \lambda_1^{j_1} \cdots \lambda_n^{j_n} \) of \( f \), i.e., obtained by taking the powers of the indeterminates.

Viewing \( R_1 \) as a complete ordered Abelian group, we can define a metric on \( R_1^{(n)} \), which in turn provides a sup metric on \( R_1^{(m)} \). There is a famous theorem from the theory of real inequalities, often called Farkas’ Lemma (cf. [DoSS], [GKT], [BV]), which has already been used in tropical geometry in [DoSS]. Pick some element \( c \in R_1 \) for which, in our original notation, \( c < 1_\mathbb{R} \). (In the vector space notation, \( c < 0 \).) Farkas’ Lemma implies that there is no solution in \( R_1^{(n)} \) to Equation (5.5) iff there is a solution in \( R_1^{(n)} \) for \( y \in \mathbb{R}_{\geq 0}^{(m)} \) to the system
\[
y^t A = (0), \quad y^t b = -c,
\]
(which in the original notation means \( c^{-1} \)), where \( ^t \) indicates the transpose.

The proof of Farkas’ Lemma is topological, cf. [DoSS], and thus works for the metric space \( R_1^{(n)} \) instead of \( \mathbb{R}^{(n)} \). Farkas’ Lemma is actually stated for \( c = 0 \), but one gets this modification by applying Farkas’ Lemma to the matrix inequality
\[
\begin{pmatrix} A & (-c) b \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} < (0),
\]
noting that a solution \((x, d)\), for \( d > 0 \) would yield a solution \( x' = (x_1', \ldots, x_n') \) to Equation (5.5) for \( x_i' = \frac{x_i}{cd_i}. \) But then there cannot be a solution \( x \) to Equation (5.5) in \( \widetilde{R}^{(n)} \), for otherwise
\[
0 = y^t A x < -y^t b = c.
\]

Having disposed of inessential monomials, we only need to concern ourselves with quasi-essential monomials \( h \). By definition, \( h \) is quasi-essential at some corner root \( a = (a_1, \ldots, a_n) \) of \( R^{(n)} \), and by definition, \( \text{csupp}(f) \) has at least two distinct (quasi-essential) monomials at \( a \),
\[
\beta \lambda_1^{i_1} \cdots \lambda_n^{i_n}, \quad \gamma \lambda_1^{j_1} \cdots \lambda_n^{j_n},
\]
where \((i_1, \ldots, i_n) \geq (j_1, \ldots, j_n)\) in the lexicographic order. We claim that \( f(a) = g(a) \). This would yield the theorem, since we already know that the shells of \( f \) and \( g \) are the same.

To prove the claim, we take the smallest \( k \) such that \( i_k > j_k \), we have \( \beta a_k^{i_k} \cdots a_n^{i_n} = \gamma a_k^{j_k} \cdots a_n^{j_n} \), and thus
\[
a_k^{i_k-j_k} \gamma = \beta a_{k+1}^{j_k+1-j_k} \cdots a_n^{j_n-j_k}.
\]
Replacing \( \lambda_k^{i_k-j_k} \) in \( f \) and \( g \) by \( \beta a_{k+1}^{j_k+1-j_k} \cdots a_n^{j_n-j_k} \) throughout, reduces the number of indeterminates by one, and thus by induction we can use this new rational polynomials at \( a \), and thus for \( f \) and \( g \) at \( a \). Since this is true for each possible corner root of \( R^{(n)} \), but the supports are defined over \( \mathbb{R} \), we can apply this argument in turn to all quasi-essential monomials.

\[\square\]

**Corollary 5.27.** The 1-divisible closure of the 1-semifield \( \mathbb{Q} \) of fractions of a uniform \( L \)-layered domain \( \mathbb{R} \) is a test algebra for all (rational) polynomials in \( R[\mathbb{A}] \).

**Proof.** If two polynomials \( f \) and \( g \) differ on \( a = (a_1^{\gamma_1}, \ldots, a_n^{\gamma_n}) \), then say \( f \) has a monomial that dominates \( g \) at \( a \) and thus at \( a' = (a_1^{\gamma_1'}, \ldots, a_n^{\gamma_n'}) \), where \( \gamma_j' \) are rational numbers close enough to the \( \gamma_j \).

A more elegant way of obtaining these results, treated in [KR-I], is to show that \( \mathbb{R} \) is model complete, since the theory of ordered divisible Abelian groups is model complete. [Mar, Corollary 3.17]. See [Mar §4.3] or [VdD] for a general model-theoretic approach, which is being developed in the layered situation by T. Perri.

Many rational polynomials that would be equivalent in the standard supertropical situation now are not equivalent, since quasi-essential monomials could force a root to a different sort; see Example 7.27(iii), (iv), and (v) below. In this setting, we may still discard all monomials strictly dominated by \( f \).
6. Layered topologies

We now get towards one of the main issues – how does one define layered varieties in such a way as to relate to tropical geometry? There are two approaches – the first is related more to a version of the Nullstellensatz, whereas the second is linked more to the simplicial structure of the Newton polytope. To avoid technical complications, we assume throughout this section that the sorting set \( L \) is totally ordered and non-negative.

6.1. The layered component topology. First we look at a relevant topology, assuming \( S \subseteq R^{(n)} \), which reflects further on some of the concepts involving the Nullstellensatz in [IK].

**Definition 6.1.** Write \( f = \sum_i f_i \), a sum of monomials in \( R[\Lambda] \), where \( f_i = \alpha_i \Lambda^i \) for \( i = (i_1, \ldots, i_n) \). Define the components \( D_{f,j} \subseteq S \) of \( f \) to be

\[
D_{f,j} := \{ a \in S : f(a) = f_j(a) \}.
\]

For \( k_1, \ldots, k_n \in L \), the \((k_1, \ldots, k_n)\)-layer of the component \( D_{f,j} \) is

\[
[(k_1, \ldots, k_n)] D_{f,j} := \{ a \in D_{f,j} : s(a_j) = k_j, \ 1 \leq j \leq n, \ \text{where} \ a = (a_1, \ldots, a_n) \}.
\]

**Lemma 6.7.**

**Proof.** \( f g_j \) is one of the monomials \( h_{1+j} \) of the product \( f g \), so \( D_{f,j} \cap D_{g,j} \subseteq D_{f,g,j} \) and \( f g(a) = f_i(a) g_j(a) \)

on \( D_{f,j} \cap D_{g,j} \). Conversely, for any \( a \in S \) for which \( f g(a) = f_i(a) g_j(a) \), we must have

\[
f g(a) \triangleright f_i(a) g_j(a) = f g(a),
\]

implying \( f(a) = f_i(a) \) and \( g(a) = g_j(a) \), and thus \( D_{f,g,1+j} \subseteq D_{f,j} \cap D_{g,j} \).

**Corollary 6.3.** The set of components of polynomials comprises a base for a topology on \( S \).

**Definition 6.4.** We call this topology the **layered component topology**.

Given \( a = (a_1, \ldots, a_n) \) and \( a' = (a'_1, \ldots, a'_n) \), we write \( a \equiv_a a' \) if \( a_i \equiv_a a'_i \) for each \( i = 1, \ldots, n \).

**Lemma 6.5.** Suppose \( R \) is \( L \)-layered domain. If \( a \in D_{f,j} \), then \( a' \in D_{f,j} \) for all \( a' \equiv_a a \) in \( S \).

**Proof.** Assume \( a' \not\in D_{f,j} \), then \( f(a') \equiv_a f_j(a') \) for some \( f_j \neq f_i \). Since \( a \equiv_a a' \), we also have

\[
f_i(a) \equiv_a f_i(a') \equiv_a f_j(a') \equiv_a f_j(a),
\]

a contradiction for \( a \) being in \( D_{f,j} \).

In other words, when \( R \) is \( L \)-layered with each \( \nu_{k,1} \) onto, any component is determined by each \( (k_1, \ldots, k_n) \)-layer, for arbitrary \( k_1, \ldots, k_n \in L \). In particular, when determining the components, we may focus on the \((1, \ldots, 1)\)-layer, i.e., tangible vectors \( a \in R_1^{(n)} \).

**Lemma 6.6.** If \( f = \sum_i f_i \) and \( f' = \sum_i f'_i \) with \( f_i \equiv_a f'_i \) for each \( i \), then \( f \) and \( f' \) have the same sets of components.

**Proof.** Dominance of the monomial \( f_i \) is determined by the \( \nu \)-value, not by the layer. Explicitly, for any \( a \in D_{f,j} \) we have \( f(a) = f_i(a) \equiv_a f'_i(a) \). If \( f'(a) = f'_j(a) \), for some \( j \neq i \), then, since \( f_j \equiv_a f'_j \), we get \( f(a) \equiv_a f_i(a) \equiv_a f_j(a) - a \) contradiction.

**Lemma 6.7.** \( g = (\sum_i f_i) \) and \( h = \sum_i f_i^k \) have the same sets of components.

**Proof.** Derived directly from the Frobenius-type property, \( (\sum_i f_i)^k \equiv_a \sum_i f_i^k \). c.f. Remark 572.

We can understand the layered component topology in terms of the layering maps. First, we recall a well-known fact from algebra, stated in the context of semirings.

**Lemma 6.8.** Suppose \( L \neq \{1\} \) is a cancellative as well as \( \mathbb{N} \)-cancellative monoid. Then for any finite set \( \{f_1, \ldots, f_m\} \subseteq L[\Lambda] \) of monomials with coefficients in \( L \), there exist \( k_1, \ldots, k_n \in L \) such that all the \( f_j(k_1, \ldots, k_n) \) are distinct (since there are finitely many of these).
Proof. A standard induction argument on $n$. For $n = 1$, we note that $ak_i^j = bk_j^l$ for $i \leq j$ iff $a = bk_j^l$, which has a unique solution for $k_1$. But $L$ is infinite, by Lemma 6.11 so almost all elements of $L$ will satisfy the conclusion of the lemma.

In general, write

$$f_i(\lambda_1, \ldots, \lambda_n) = \overline{f_i}(\lambda_1, \ldots, \lambda_{n-1})\lambda_n^{\nu_i},$$

and choose $a_1, \ldots, a_{n-1}$ such that the $\overline{f_i}(a_1, \ldots, a_{n-1})$ are distinct whenever the $\overline{f_i}(\lambda_1, \ldots, \lambda_{n-1})$ are distinct. Then by the previous paragraph almost all choices of $a_n$ will yield the conclusion of the lemma.

\[ \square \]

Theorem 6.9. Suppose $L$ is a cancellative monoid as well as $\mathbb{N}$-cancellative monoid, and $R$ is uniformly $L$-layered. Then the layering map of a polynomial $f \in R[\Lambda]$ determines its components. More precisely, if $D_{f,i}$ are the components of $f = \sum_i f_i$, there exist $k_1, \ldots, k_n \in L$ such that letting $\ell_i = \vartheta_f^\dagger(a)$ where $a = (a_1, \ldots, a_n)$ is in the $(k_1, \ldots, k_n)$-layer of the component $D_{f,i}$, we have

$$[(k_1, \ldots, k_n)]D_{f,i} = \{a \in S \cap (R_{k_1} \times \cdots \times R_{k_n}) : \vartheta_f^\dagger(a) = \ell_i\}.$$ 

Proof. Write $f = \sum_i a_i \lambda_1^{\nu_i} \cdots \lambda_n^{\nu_i}$, and $a = (|k_1| a_1, \ldots, |k_n| a_n)$; then

$$\vartheta_f^\dagger(a) = s(a_1) \prod_{j=1}^n k_j^{\nu_j}.$$ 

In view of Lemma 6.8 there are $k_1, \ldots, k_n \in L$ such that, letting $\ell_i = \vartheta_f^\dagger(a)$, the sums of the $\ell_i$ are distinct. But by definition $\ell_i$ is the sort corresponding to $f$ evaluated on the elements of $[(k_1, \ldots, k_n)]D_{f,i}$, as desired. In other words, the layering map distinguishes among the various components of $f = \sum_i f_i$.

\[ \square \]

6.1.1. The layered Nullstellensatz. The layered component topology is rich enough environment to formulate the Nullstellensatz of \cite{HL}. It is convenient to assume that $R$ is a 1-divisibly closed, L-layered 1-semifield, where $L$ is a cancellative monoid with $L = L_{\geq 1}$. We also assume all the sort transition maps of $R$ are onto.

In analogy to \cite{HL}, we write $f \preceq_{D_{f,i}} g$ if some component $D_{g,j}$ of $g$ contains $D_{f,i}$; we write $f \preceq_{\text{comp}} A$ for $A \subseteq R[\lambda_1, \ldots, \lambda_n]$, if for every essential monomial $f_i$ of $f$ there is some $g \in A$ (depending on $D_{f,i}$) with $f \preceq_{D_{f,i}} g$. Restricting this definition by bringing in the sort map $\vartheta$, we get the following:

Definition 6.10. We write $f \preceq_{\text{comp}} A$ if $f \preceq_{D_{f,i}} g$ and $\vartheta_f^\dagger(a) \geq \vartheta_g^\dagger(a)$ for all $a \in D_{f,i}$, and define $f \preceq_{\text{comp}} A$ for $A \subseteq R[\lambda_1, \ldots, \lambda_n]$, if for every essential monomial $f_i$ of $f$ there is some $g \in A$ with $f \preceq_{\text{comp}} A$ and $f \preceq_{D_{f,i}} g$.

We want to check this property for the tangible part of components. We say that set $S \subseteq R^{(n)}$ is tangible compatible (with respect to $R$) if whenever $a \in S$ we also have $a' \in S$ for all $a' \in R^{(n)}$ such that $a \cong \nu a'$.

Remark 6.11. If the sort transition maps $\nu_{i,1} : R_i \to R_{\ell}$ of $R$ are onto for each $\ell$ in $L$ and $S \subseteq R^{(n)}$ is tangibly compatible, then for each $a \in D_{f,i}$ there exists $\hat{a} \cong \nu a$, where $\hat{a} \in S \cap R_1^{(n)}$.

In this case, the components are determined by the $i$-layer.

Lemma 6.12. Suppose $S \subseteq R^{(n)}$ is tangibly compatible, and $f \preceq_{D_{f,i}} g$, with $f_i = a_i \lambda_1^{\nu_i} \cdots \lambda_n^{\nu_n}$. Then there is some monomial $g_j = \beta_j \lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n}$ of $g$, for which $s(a_1) \geq s(\beta_j)$ and $i_k \geq j_k$ for every $k = 1, \ldots, n$. In fact, $g_j$ can be taken to be the dominant monomial of $g$ at $a$, for any $a \in D_{f,i} \cap R_1^{(n)}$. (Note that $D_{f,i} \cap R_1^{(n)} \neq \emptyset$ by Corollary 6.11.)

Proof. Pick $a \in D_{f,i} \cap R_1^{(n)}$, which exists by Corollary 6.11 then $s(a_1) = \vartheta_f(a) = \vartheta_g(a) = s(\beta_j)$. Assume that $j_k > i_k$ and pick $a = (a_1, \ldots, a_n) \in R^{(n)}$, with $a_k \in R_\ell$ and $\ell = s(a_1)$ then

$$\vartheta_f^\dagger(a) = s(a_1)\ell^{k_1} \cdots \ell^{k_n} = \ell^{\nu_1 + \cdots + \nu_n} \geq s(\beta_j)\ell^{k_1} \cdots \ell^{k_n} = \vartheta_g^\dagger(a)$$

and hence, since $j_k > i_k$, $0 > s(\beta_j)\ell^{k_1} \cdots \ell^{k_{n-1}} \geq s(\beta_j) - a$ contradiction to $L = L_{\geq 1}$. 

\[ \square \]

We say that an ordered monoid is called archimedean when for any $a, b > 1$ there is some $k \in \mathbb{N}$ such that $a^k > b$. For $A \subseteq R[\Lambda]$, define

$$L^\sqrt{\Lambda} = \{f \in R[\Lambda] : f_k \preceq_{D_{f,i}} g \text{ with } f_k \cong g \text{ for some } g \in A\}.$$  \hspace{1cm} (6.1)
Theorem 6.13. (Layered Nullstellensatz) Suppose $L = L_{\geq 1}$ is archimedean, and $R$ is a 1-divisibly closed, $L$-layered 1-semifield whose sort transition maps are all onto, such that $R_1$ is archimedean. Suppose $A \in A[\Lambda]$, and $f \in R[\Lambda]$. Then

$$f \preceq_{\text{comp}} A \iff f \in \text{Lay}\sqrt{A}.$$  

Proof. (⇒) Let $f = \sum_i f_i$, and take $\tilde{f} = \sum_i \tilde{f}_i \in R[\Lambda]$, which exists since the sort transition maps assumed to be onto. Similarly, define $\tilde{g} = \sum_i \tilde{g}_i \in R[\Lambda]$, for every $g \in A$, and let $\tilde{A} := \{ \tilde{g} \mid g \in A \} \subset R_1[\Lambda]$.

In view of Corollary 6.11 restricting the components of $\tilde{f}$ and all $\tilde{g} \in \tilde{A}$ to $S_1 := S \cap R_1^{(m)}$, each $S_1 \cap D_{\tilde{g},j}$ is contained in $S_1 \cap D_{\tilde{g},j}$ for some $\tilde{g} \in \tilde{A}$. Then, since $\tilde{f}$ is tangible, by [IR1] Theorem 7.17], for some $m \in \mathbb{N}$

$$\left(\tilde{f}^m = \left(\sum_i \tilde{f}_i\right)^m = \sum_{\tilde{g} \in \tilde{A}} \hat{\tilde{g}} + \text{ghost}, \quad \tilde{A} \subseteq \tilde{A}, \right.$$  

(6.2)

where $\hat{\tilde{g}} = \sum_k \tilde{h}_k = \tilde{\gamma}_k \Lambda^k$ are polynomials in $R_1[\Lambda]$. Let $\tilde{\Phi} := \sum_{\tilde{g} \in \tilde{A}} \hat{\tilde{g}}$.

Note that by steps 2 and 3 in the proof of [IR1] Theorem 7.17], we know that for every $\tilde{f}^m_i$ in the expansion of $\tilde{f}^m$ we have $\tilde{f}^m_i = \hat{\tilde{h}}_{\tilde{g}_i}$ for distinct $\tilde{g} \in \tilde{A}$. In particular

$$\tilde{f}^m_i(a) = \tilde{f}^m_i(a) = (\hat{\tilde{h}}_{\tilde{g}_i})(a) = (\hat{\tilde{g}})(a), \quad \text{for every } a \in D_{f,i},$$  

(6.3)

and $\tilde{f}^m(a') \geq _{\nu} (\hat{\tilde{h}}_{\tilde{g}_i})(a')$ for any $a' \in S \setminus D_{f,i}$.

We next need to coordinate the layering of the different monomials $f^i_k$ in (6.2). Since $\tilde{f}^m_i = \hat{\tilde{h}}_{\tilde{g}_i}$ we have

$$f^m_i = \alpha^m_i \Lambda_i^j \tilde{\gamma}_k \gamma_j = \tilde{\gamma}_k \Lambda^k \Lambda_j^j = \tilde{\gamma}_k \Lambda^k \Lambda^j \quad \text{with } s(\tilde{\gamma}_k) = 1.$$  

(6.4)

Then, $s(\alpha^m_i) = s(\alpha^m)$, by Lemma 6.12. Let $\ell \in L$ be such that $\ell \cdot s(\beta_j) = s(\alpha^m)$. Take $\gamma_k \equiv_{\\nu} \gamma_k$, where $\gamma_k \in R_\ell$, and define $h = \sum_k h_k$ to be $\hat{h}$ with each $\hat{h}_k$ replaced by $h_k = \gamma_k \Lambda^k$ to get $f^m_i = h_{k,j}$. In conjunction with (6.3), we get $f^m_i = \sum_{\tilde{g} \in \tilde{A}} \hat{\tilde{g}} g$.

(⇐) Taking $g$ as in (6.1), we see that

$$D_{f,i} = D_{f,k,i} = D_{g,i}.$$  

6.1.2. Layering maps. Recall that $\mathcal{R}$ is a sub-semiring of $\text{Fun}(S, R)$.

Definition 6.14. The layering map of a set $I \subset \mathcal{R}$ is the map $\vartheta_I : S \to L$ given by

$$\vartheta_I(a) := \min \{ \vartheta_f(a) : f \in I \},$$

where $\vartheta_f$ is given in Definition 5.7.

This definition carries the implicit assumption that $\min \{ \vartheta_f(a) : f \in I \} \in L$ for every $a \in S$. There are several ways to attain this:

1. $I$ is finite.
2. $L$ satisfies the descending condition (such as $L = \mathbb{N}$).
3. $L$ is complete and bounded from below (such as $\mathbb{R}_{\geq 1}$).

Example 6.15. We take the uniform $\mathbb{N}$-layered domain $R = R(\mathbb{N}, (\mathbb{R}, +))$.

1. $f_k = \lambda_k^\ell + \lambda_2 + 0$ for $k \in \mathbb{N}$. Then for $a = (a_1, a_2)$ tangible we have

$$\vartheta_f(a) = \begin{cases} 3 & \text{for } a_1 = a_2 = 0; \\ 2 & \text{for } a_1 = 0 > a_2 \quad \text{or} \quad a_2 = 0 > a_1 \quad \text{or} \quad a_1^k = a_2 > 0; \\ 1 & \text{otherwise.} \end{cases}$$
(2) \( \mathcal{I} = \{ f_k : k \in \mathbb{N} \} \), \( \mathbf{a} = (a_1, a_2) \) is tangible.

\[
\vartheta_I(\mathbf{a}) = \begin{cases} 
3 & \text{for } a_1 = a_2 = 0; \\
2 & \text{for } a_1 = 0 > a_2 \text{ or } a_2 = 0 > a_1; \\
1 & \text{otherwise.}
\end{cases}
\]

Lemma 6.16. Suppose \( L = L_{\geq 1} \). If \( \mathcal{I} = \sum_{j \in J} R f_j \), then

\[
\vartheta_I(\mathbf{a}) = \inf_j \{ \vartheta_{f_j}(\mathbf{a}) : j \in J \}.
\]

Proof. \((\leq)\) is clear. But for any \( g = \sum_j g_j f_j \in \mathcal{I} \), \( g_j \in R \), we have

\[
\vartheta_I(\mathbf{a}) \geq s(g(\mathbf{a})) \geq \min_{j \in J} \{ s((g_j f_j)(\mathbf{a})) \} = \min_{j \in J} \{ s(g_j(\mathbf{a})) s(f_j(\mathbf{a})) \} \geq \min_{j \in J} \{ s(f_j(\mathbf{a})) \} = \min_{j \in J} \{ \vartheta_{f_j}(\mathbf{a}) \}.
\]

Proposition 6.17. For \( \mathcal{I}_J \subset \text{Fun}(S, R') \),

\[
\vartheta_{\sum_{i \in J} \mathcal{I}_i} = \vartheta_{\bigcup_{i \in J} \mathcal{I}_i} = \inf_j \{ \vartheta_{\mathcal{I}_i} \}; \quad \vartheta_{\mathcal{I}_1 \mathcal{I}_2} = \vartheta_{\mathcal{I}_1} \vartheta_{\mathcal{I}_2}.
\]

Proof. The first assertion is immediate, and the second is clear since the sort map is multiplicative. If \( s(f_i(\mathbf{a})) = \ell_i \) for \( i = 1, 2 \), then \( s(f_1(\mathbf{a})) s(f_2(\mathbf{a})) = \ell_1 \ell_2 = s((f_1 f_2)(\mathbf{a})) \).

In the other direction, we can describe ideals of functions in terms of layering maps.

Definition 6.18. Given a sub-semiring \( R \) of \( \text{Fun}(S, R) \), and \( Z \subseteq S \), define \( R_Z = Z \cap R \). Given any map \( \vartheta : Z \to L \) where \( Z \subseteq S \), define \( \mathcal{I}_\vartheta(Z) \) to be

\[
\mathcal{I}_\vartheta(Z) := \{ f \in R : f(\mathbf{a}) \text{ is } \vartheta(\mathbf{a})\text{-ghost}, \forall \mathbf{a} \in Z \}.
\]

A geometric layered ideal of \( R \) is an ideal of the form \( \mathcal{I}_\vartheta(Z) \) for a suitable map \( \vartheta : Z \to L \). When \( Z \) is understood, we write \( \mathcal{I}_\vartheta \) for \( \mathcal{I}_\vartheta(Z) \).

Strictly speaking, the notation for \( Z \) is redundant, since we can choose \( S \) as we please. But often we start with \( S = \mathbb{R}^{(n)} \), and then take \( Z \) to be a closed subset of \( S \) with respect to the layered component topology, so we utilize the symbol \( Z \) for clarification.

Proposition 6.19. Suppose \( L = L_{\geq 1} \). Then \( \mathcal{I}_\vartheta(Z) \triangleleft R_Z \), and there are 1:1 order-reversing correspondences between the layering maps of \( R \) and the geometric layered ideals of \( R \), given by \( \vartheta \mapsto \mathcal{I}_\vartheta(Z) \) and \( I \mapsto \vartheta_I \).

Proof. Clearly \( \vartheta_I \) is closed under addition, and \( \mathcal{I}_\vartheta(Z) \) is an ideal when \( L = L_{\geq 1} \), since then the layering map increases. For the second assertion, one just follows the standard arguments in the Zariski correspondence. Namely, we need to show that for any layering map \( \vartheta \), defining the geometric layered ideal \( I = I_\vartheta \), that \( \vartheta_I = \vartheta \) and \( I_{\vartheta_I} = I \).

Clearly \( I \supseteq I_{\vartheta_I} \). But if \( f \in I \) then by definition \( f \in \vartheta_I \). Hence \( \vartheta_I = \vartheta \), so \( I_{\vartheta_I} = I_\vartheta = I \).

The hypothesis that \( L = L_{\geq 1} \) is crucial, since otherwise we could multiply by \( e_\ell \) for \( \ell < 0 \) and the definition of \( \vartheta_I \) would become meaningless.

These results indicate that tropical geometry can be understood through a careful study of the algebraic structure of the layering maps, as translated to \( R \).

Definition 6.20. The layering map \( \vartheta_I \) is irreducible if \( \vartheta_I \) cannot be written as the product \( \vartheta_{I_1}, \vartheta_{I_2} \) of two layering maps.

6.2. Layered varieties. We take the standard approach of algebraic geometry, but need to modify it because we do not have negation. Due to space limitations, we give only a rough outline, leaving details for a separate paper. We need a concise algebraic definition of variety, at least in the affine setting.
6.2.1. The corner locus. To introduce layered varieties, we make Definition 5.16 more explicit.

Definition 6.21. Given a rational polynomial \( f = \sum h_i \) written as a sum of rational monomials \( h_i \), for \( i = (i_1, \ldots, i_n) \), define the \textbf{corner support} at \( a \), denoted \( \text{csupp}_a(f) \), to be the set of those \( h_i \) for which \( s(h_i(a)) > 0 \) and \( f(a) \approx_h h_i(a) \).

Definition 6.22. An element \( a \in S \) is a \textbf{corner root} of \( f \) iff \( |\text{csupp}_a(f)| \geq 2 \). We define \( Z_{\text{corn}}(f) \) to be the set of corner roots of \( f \).

Thus, for any corner root \( a \) of \( f \), there are at least two \( h_i \) which are quasi-essential in \( f \) at \( a \), for which \( s(h_i(a)) \in L_+ \) are positive, and \( f(a) \approx_h h_i(a) \)-ghost for these \( h_i \).

Remark 6.23. \( \vartheta^{-1}(L_{\geq 1}) \) is just the set of 1-roots of \( f \) in \( S \).

In classical algebraic geometry, given a polynomial \( f \), one takes its zero locus. Our layered analogy is to take its set of corner roots, which we call the \textbf{corner locus} of \( f \). Note that the complement set of the corner locus is the union of the components of \( f \). This motivates the next definition.

In order to hone in on corner roots of (rational) polynomials, we modify Definition 6.18. For convenience, we take \( R \subseteq R[\Lambda]_{\text{rat}} \).

Digression 6.24. We could mimic Definition 6.7 by defining \( f_a := \sum_i \{ h_i : h_i \in \text{csupp}_a(f) \} \); the same argument as in Proposition 6.2 shows that the sets \( D_{f,a} = \{ b \in S : f(b) = f_a(b) \} \) are a base for a topology that refines the layered component topology of Definition 6.4. This topology better reflects the simplicial nature of tropical geometry.

Definition 6.25. Given \( Z \subseteq S \), define

\[ I_{\text{corn}}(Z) := \{ f \in R : |\text{csupp}_a(f)| \geq 2, \forall a \in Z \} \]

A \textbf{corner layered ideal} of \( R \) is an ideal of the form \( I_{\text{corn}}(Z) \).

The \textbf{corner locus} \( Z_{\text{corn}}(I) \) of a subset \( I \subseteq R \) is the intersection of the corner loci \( Z_{\text{corn}}(f) \) of the functions \( f \) in \( I \). Any such corner locus will also be called an (affine) \textbf{layered variety}.

Remark 6.26. We lose the specific layers used in computing \( Z_{\text{corn}}(I) \). Furthermore, this process leads to unexpected varieties, often arising as degenerate intersections of usual tropical hypersurfaces. For example, if

\[ I_1 = \{ \lambda_1 + \lambda_2 + 2, \lambda_1 + \lambda_2 + (-2) \}, \quad I_2 = \{ \lambda_1 + \lambda_2 + 0, \lambda_1^2 + \lambda_2 + 0 \}, \]

then \( Z_1 := Z_{\text{corn}}(I_1) = \{ (a, a) : a \geq 1 \} \), a ray which is the intersection of two tropical lines not in general position, which is not a customary tropical variety. Furthermore \( Z_2 := Z_{\text{corn}}(I_2) \) is the union of the other two rays in the tropical line \( Z_{\text{corn}}(f) \), where \( f = \lambda_1 + \lambda_2 + 0 \).

In this way, it might seem that we could reduce the tropical line as the union of two layered varieties. This is not desirable, since one would want the tropical line to be an irreducible variety, and its layering map is irreducible, in terms of Definition 6.27. Note that \( \vartheta_{I_1}(0) = 2 \) and \( \vartheta_{I_2}(0) = 3 \) for \( 0 = (0, 0) \), so

\[ \vartheta_{I_1}(0) \vartheta_{I_2}(0) = 6 > \vartheta_f(0) = 3. \]

Lemma 6.27. For monomials \( h_i \) of \( f_i \), we have \( h_i \in \text{csupp}_a(f_i) \) for \( i = 1, 2 \) iff \( h_1 h_2 \in \text{csupp}_a(f_1 f_2) \).

Proof. Each monomial dominates at \( \mathbf{a} \) iff the product dominates at \( \mathbf{a} \). For if \( h'_1 h'_2(\mathbf{a}) >_h h_1 h_2(\mathbf{a}) \) for some other monomial \( h'_1 h'_2 \), then \( h'_1(\mathbf{a}) >_h h_1(\mathbf{a}) \) or \( h'_2(\mathbf{a}) >_h h_2(\mathbf{a}) \).

Lemma 6.28. If \( |\text{csupp}_a(f)| \geq 2 \), then \( |\text{csupp}_a(f g)| \geq 2 \) for all \( g \in R[\Lambda]_{\text{rat}} \).

Proof. Suppose \( h_1 \neq h_2 \) are the rational monomials in \( \text{csupp}_a(f) \) of respective lowest and highest degree (under the lexicographic order), and \( h'_1 \) and \( h'_2 \) are the rational monomials in \( \text{csupp}_a(g) \) of lowest and highest degree. Then \( h_1 h'_1 \) and \( h_2 h'_2 \) differ and are the rational monomials in \( \text{csupp}_a(f g) \) of respective lowest and highest degree, implying \( |\text{csupp}_a(f g)| \geq 2 \).

Lemma 6.29. \( |\text{csupp}_a(f + g)| \geq \max\{|\text{csupp}_a(f)|, |\text{csupp}_a(g)|\} \).

Proof. All monomials remain in the support of the sum.

Proposition 6.30. \( I_{\text{corn}}(Z) \subset R \), for every \( Z \subseteq S \).

Proof. Combine Lemmas 6.28 and 6.29.
Proposition 6.31. There are 1:1 order-reversing correspondences between the corner loci and the corner layered ideals, given by $Z \mapsto I_{\text{corn}}(Z), Z \subset S$, and $I \mapsto Z_{\text{corn}}(I)$.

Proof. Clearly $I_{\text{corn}}(Z)$ is closed under addition. For the second assertion, one just follows the standard arguments in the Zariski correspondence, paralleling the argument given in the proof of Proposition 6.19. □

6.2.2. The layered Zariski topology.

Theorem 6.32. The sets of the form $Z_{\text{corn}}(I)$ for $I \triangleleft R$ comprise the closed sets of a coarser topology than the layered component topology of Definition 6.4, in which every open set is dense.

Proof. If $I$ is generated by a single element, then $Z_{\text{corn}}(I) = Z_{\text{corn}}(\{f\})$, since any corner root of $f$ is a corner root of a multiple of $f$. Thus, we can reduce to generators of ideals, and

$$\bigcap_j Z_{\text{corn}}(I_j) = Z_{\text{corn}}\left(\bigcup_j I_j\right) = Z_{\text{corn}}\left(\sum_j I_j\right).$$

Moreover,

$$Z_{\text{corn}}(I_1) \cup Z_{\text{corn}}(I_2) = Z_{\text{corn}}(I_1I_2),$$

in view of Lemma 6.27. Thus these sets $Z_{\text{corn}}(I)$ comprise a topology, and all of them are closed in the topology of Definition 6.4.

In view of Proposition 6.2, the intersection of any two non-empty open sets contains some non-empty component and thus is non-empty. Hence every open set is dense. □

Definition 6.33. We call the topology of Theorem 6.32 the layered Zariski topology.

Propositions 6.19 and 6.31 enable us to transfer some tropical geometry to the algebraic theory of $R$. The corner loci are the natural candidates for tropical varieties; the ones corresponding to prime ideals of $R$ (and thus the irreducible layering maps).

(One should note that this correspondence is somewhat weaker than the usual Zariski correspondence in algebraic geometry, since we have not determined which ideals of $R$ have the form $I_{\text{corn}}(Z)$ or $I_{\varnothing}(Z)$.)

Indeed, we should consider all of the ideals obtained from the $\ell$-varieties, but this is outside of the scope of the present paper.

The layered Zariski topology plays a key role in studying tropical dimension.

Remark 6.34. The translation to the algebraic structure of $R$ should be helpful in many basic tasks, such as defining dimension. In ring theory there are three basic definitions of dimension for affine algebras:

(a) The classical Krull dimension (measured by maximal lengths of prime ideals);
(b) The more general module-theoretic version of Krull dimension, considered by Gabriel and studied in depth by Gordon and Robson in \cite{GorR};
(c) The Gelfand-Kirillov dimension \cite{KrL}.

These all coincide for commutative affine algebras, and are defined for semirings as well as rings, so could be used to define the dimension of the coordinate semiring $\uparrow R \cap \text{Fun}(Z, R)$ and thus of the (tropical) layered variety.

We present one interesting example which indicates the direction one might take.

Example 6.35. Take $R = R(\mathbb{N}, R)$, and consider the polynomials $f_1 = 2\lambda_1 + 3\lambda_2 + 5$ and $f_2 = 3\lambda_1 + 2\lambda_2 + 5$. Write $a = (\{8\}a_1, \{6\}a_2) \in R^{(2)}$ in logarithmic notation. The layering maps are

$$\vartheta_{f_1}(a) = \begin{cases} 
  k + \ell + 1 & \text{for } a_1 = 3, \ a_2 = 2; \\
  k + \ell & \text{for } a_1 = 1 + a_2 > 3; \\
  k + 1 & \text{for } a_1 = 3 > 1 + a_2; \\
  \ell + 1 & \text{for } a_2 = 2 > a_1 - 1; \\
  1 & \text{otherwise.}
\end{cases}$$

$$\vartheta_{f_2}(a) = \begin{cases} 
  k + \ell + 1 & \text{for } a_1 = 3, \ a_2 = 2; \\
  k + \ell & \text{for } a_1 = 1 + a_2 > 3; \\
  k + 1 & \text{for } a_1 = 3 > 1 + a_2; \\
  \ell + 1 & \text{for } a_2 = 2 > a_1 - 1; \\
  1 & \text{otherwise.}
\end{cases}$$
Thus,\
\[
\vartheta_{f_1, f_2}(a) = \begin{cases} 
\min\{k, \ell\} + 1 & \text{for } a_1 = a_2 = 2; \\
1 & \text{otherwise.}
\end{cases}
\]

When $L = L_{\geq 1}$, this is dominated by $\vartheta_g$ where $g = \lambda_1 + \lambda_2$, since
\[
\vartheta_g(a) = \begin{cases} 
k + \ell & \text{for } a_1 = a_2; \\
1 & \text{otherwise.}
\end{cases}
\]

But $g$ is not generated by $f_1, f_2$, which would run counter to our intuition. This example shows that the layering map does not suffice to determine the geometry in the standard supertropical theory, which is what led us to a different version of the Nullstellensatz in [IR1].

On the other hand, if we permit $L$ to have a zero element, then taking $a = (0, 0)$, we have $\vartheta_{f_1, f_2}(a) = 1 > \vartheta_g(a)$. Thus, the zero layer could be useful in the theory.

Likewise, we can resolve the difficulty for $L = \mathbb{Q}_{>0}$, since then we could take $k = \ell = \frac{1}{2}$.

Example 6.36. Take $R = R(\mathbb{N}, R), I = (f_1, f_2)$, where $f_1 = \lambda_1 + 1$ and $f_2 = \lambda_1 + 2$.

(1) For $S = R_1^{(n)}, a = (a_1, \ldots, a_n)$ we have
\[
\vartheta_{f_1}(a) = \begin{cases} 
2 & \text{for } a_1 = 1; \\
1 & \text{otherwise.}
\end{cases}
\]
\[
\vartheta_{f_2}(a) = \begin{cases} 
2 & \text{for } a_1 = 2; \\
1 & \text{otherwise.}
\end{cases}
\]

Thus, $\vartheta_I$ is 1 identically on $S$, which is the same as the layering map of a constant, and thus yields the same layered variety (namely all of $S$).

(2) For $n = 1, S = R$, and $L = \mathbb{Q}_{>0}$, we can recover the tangible part of components of $I$ by means of the layering map. Indeed, suppose $a \in R_\ell$.
\[
\vartheta_{f_1}(a) = \begin{cases} 
\ell & \text{for } a > \nu 1; \\
\ell + 1 & \text{for } a \equiv \nu 1; \\
1 & \text{for } a < \nu 1.
\end{cases}
\]
\[
\vartheta_{f_2}(a) = \begin{cases} 
\ell & \text{for } a > \nu 2; \\
\ell + 1 & \text{for } a \equiv \nu 2; \\
1 & \text{for } a < \nu 2.
\end{cases}
\]

Thus, if $\ell > 1$, then
\[
\vartheta_I(a) = \begin{cases} 
\ell & \text{for } a \geq \nu 2; \\
1 & \text{otherwise,}
\end{cases}
\]

thereby yielding one component of $\lambda + 2$ and the closure of the other component, and for $\ell < 1$,
\[
\vartheta_I(a) = \begin{cases} 
1 & \text{for } a \leq \nu 1; \\
\ell & \text{otherwise.}
\end{cases}
\]

In particular, the use of these extra layers enables us to distinguish between $\vartheta_I$ and the layering map of a constant.
Alternatively, using $\ell$-roots, one could define the $\ell$-variety of $I$ to be
\[
Z_\ell(I) \ := \ \{ a \in S : f(a) \text{ is an } \ell\text{-root}, \forall f \in I \} = \{ a \in S : \partial I(a) \text{ is an } \ell\text{-ghost sort} \}.
\]

This notion of variety also takes into account the multiplicity of the root, by applying $\partial^{-1}_I$ to the set of $\ell$-ghost sorts. Of particular interest is the case $\ell = 1$, since this provides the set of elements whose values (under functions in $I$) are ghosts.

7. Polynomials in one indeterminate

In this section, we consider the case of polynomials in one indeterminate, quoting [SII] extensively, in order to understand factorization and multiple corner roots. This is followed up in the next two sections, where we study resultants and then derivatives (and anti-derivatives). In order to obtain decisive results, we assume that $R$ is a uniform $L$-layered 1-semifield$^\dagger$.

Factorization of polynomials behaves much better than in the standard supertropical theory.

We say that a polynomial $f = \sum_{i=0}^{t} [\ell]_i \alpha_i \lambda^i$ is monic if $\ell_0 \alpha_0 \equiv \nu \ 1_R$.

**Remark 7.1.** When studying the $\ell$-roots of a polynomial $f = \sum_{i=0}^{t} [\ell]_i \alpha_i \lambda^i$, we may divide out by $\lambda^u$ and may assume that $u = 0$ without affecting the roots (other than $0_R$). Thus, we may assume throughout that our polynomials are not divisible by $\lambda$.

Furthermore, since the uniform $L$-layered domain$^1$ $R$ is a 1-semifield$^\dagger$, we can always replace $f$ by $\alpha^{-1}_t f$ and thereby assume that $f$ is monic. We often assume that $\ell_t = 1$, since this does not affect the $\nu$-values of the roots.

We write each polynomial as a decomposition into a sum of essential monomials; i.e., deleting any monomials in $\text{csupp}_a(f)$ the dominant monomials of $f$ at $a$. This means that the graph of $f$ is concave up. Let us call $i_{j+1}$ the essential exponent following $i_j$.

**Remark 7.2.** Suppose $a \in R_t$. If $f = \lambda^t + \sum_{i=0}^{t-1} [\ell]_i \alpha_i \lambda^i$ is in essential form, then $f(a) = a^t$ when $a^\nu$ is “large enough,” so in this case $s(f(a)) = \ell^t$. Similarly, when $[\ell_0] \alpha_0 \not\equiv 0_R$, $f(a) = [\ell_0] \alpha_0$ when $a^\nu$ is “small enough,” so in this case $s(f(a)) = \ell_0$.

7.1. Homogeneous parts of a polynomial. Let us give a more explicit version of these considerations.

**Lemma 7.3.** If $f = \sum_i \alpha_i \lambda^i$ is in essential form, then $\alpha_i \alpha_j >_\nu \alpha_{i-1} \alpha_{j+1}$ for all $i > j$.

**Proof.** It is well-known that the slope $\frac{\alpha_{i+1}}{\alpha_{i-1}}$ must be $\nu$-greater than the slope $\frac{\alpha_{i+1}}{\alpha_i}$ in order for the monomials $\alpha_j \lambda^j$ and $\alpha_i \lambda^i$ to be essential. \hfill $\square$

For any tangible $a$, we call those monomials in $\text{csupp}_a(f)$ the dominant monomials of $f$ at $a$.

**Remark 7.4.** Any polynomial $f = \sum_i \alpha_i \lambda^i$ in essential form satisfies the following convexity condition:
\[
\left( \frac{\alpha_{i+1}}{\alpha_i} \right)^{i_{j+1}-i_j} \geq_\nu \left( \frac{\alpha_j}{\alpha_{i_j}} \right)^{i_{j+1}-i_j}.
\]

for each $i_j \leq i \leq i_{j+1}$.

Now, for any $i$ between $i_j$ and $i_{j+1}$, taking
\[
d_j = i_{j+1} - i_j,
\]
we formally replace $\alpha_i$ by $[0]_0 \alpha_i \left( \frac{\alpha_{i+1}}{\alpha_i} \right)^{(i-i_j)/d_j}$. (This is the $\nu$-largest coefficient that can be attached to $\lambda^i$ without affecting the $\nu$-values of $f$ as a function, since it is in the 0-layer.) We call this new polynomial the full form of $f$. Thus, the essential form of $f$ is the “minimal” polynomial equal to $f$ as a function, whereas the full form of $f$ is the “maximal” polynomial equal to $f$ as a function.
Having adjoined these new 0-layer monomials, we can now write \( f = \sum_{i=0}^{t} \alpha_i \lambda^i \), define the slopes
\[
m_i = \frac{\alpha_{i+1}}{\alpha_i},
\]
and note that \( m_i \geq m_{i-1} \) for each \( i \). (These can also be identified with the slopes for the essential form.)

When \( m_{i'} < m_{i''} = \cdots = m_{i''-1} < m_{i'} \)

for suitable \( i' \) and \( i'' \), we call \( \sum_{i''}^{i'} \alpha_i \lambda_i \) the \( m_{i''} \)-homogeneous part of \( f \). (Note that \( i'' \) is the essential exponent following \( i' \).) We write \( \text{dom}(m_{i''}, f) := \{i', \ldots, i''\} \).

We also define homogeneous parts for the extreme cases — When \( f \) has degree \( t \), the top part of \( f \) is that homogeneous part such that \( i'' = t + 1 \), and the bottom part of \( f \) is the \( m_0 \)-homogeneous part, i.e., with \( i' = 0 \).

We also have a semiring homomorphism paralleling the homomorphism \( \Psi_1 \) of [IR3] Remark 3.3].

**Remark 7.5.** Suppose that \( S \) is a multiplicative group. Define the map \( \text{Fun}(S, R) \to \text{Fun}(S, R) \) given by \( f \mapsto \overline{f(a)} = f(a^{-1}) \). This is clearly an isomorphism of order 2, since
\[
\overline{f + g(a)} = (f + g)(a^{-1}) - f(a^{-1}) + g(a) = (\overline{f} + \overline{g})(a),
\]
and likewise \( \overline{fg(a)} = \overline{f(a)} \overline{g(a)} \).

If \( f = \sum h_i \), then \( \overline{f} = \sum \overline{h_i} \). But \( h_i \) is essential at \( a \) iff \( \overline{h_i} \) is essential at \( a^{-1} \).

The map \( f \mapsto \overline{f} \) reverses the order of the corner roots, and thus switches the top part of \( f \) with the bottom part. We make use of this duality to shorten some proofs.

For example, in the uniform case, if \( f = \lambda^4 + 5^2 \lambda^3 + 5^2 \lambda^2 + 5^5 \lambda + 5 \), then the 2-homogeneous part of \( f \) is \( \lambda^4 + 5^2 \lambda^3 + 5^2 \lambda^2 \) and the 1-homogeneous part is \( 5^2 \lambda^2 + 5^5 \lambda + 5 \). Intuitively, the slope is constant on the \( m_i \)-homogeneous parts of the polynomial \( f \). In general, \( m_{\nu} \) is the unique corner root of the \( m_i \)-homogeneous part of \( f \). By definition, \( \text{dom}(m_{\nu}, f) = \text{csup}(m_{\nu})(f) \).

### 7.2. Separable polynomials.

**Definition 7.6.** A corner root \( a \) of a polynomial \( f(\lambda) \) is **simple** if \( f = (\lambda + a)g \), where \( a \) is not a corner root of \( g \). A polynomial \( f \) is **separable** if \( f \) is the product of a constant together with linear factors having \( \nu \)-inequivalent corner roots.

**Remark 7.7.** A polynomial is separable iff each corner root is simple.

The following observation is due to Sheiner [Sh1] Lemma 3.10].

**Proposition 7.8.** If \( R \) is an \( L \)-layered semifield, and if a polynomial \( f = \sum_{i=0}^{t} \alpha_i \lambda^i \) is in essential form, where \( s(\alpha_i) = \ell_i \), then
\[
f = \prod_{i=0}^{t-1} (\lambda + \frac{\alpha_i}{\alpha_{i+1}}),
\]
where \( k_i = \frac{\alpha_i}{\ell_i} \); and \( \beta_i = \frac{\alpha_i}{\ell_i} \); i.e., \( f \) is separable.

**Proof.** Dividing out by \( \prod_{i=0}^{t-1} \), we may assume that \( \prod_{i=0}^{t-1} = 1 \). In analogy to [IR1] Lemma 8.28] (seen by repeated applications of Lemma [23]),
\[
f = (\lambda + \frac{\alpha_{t-1}}{\alpha_t}) \left( \lambda^{t-1} + \sum_{i=0}^{t-2} \frac{\alpha_i}{\alpha_{t-1}} \lambda^i \right),
\]
and we continue by induction. \( \square \)
7.3. Primary polynomials. Unfortunately, not every polynomial in $R[\lambda]$ is separable. Sheiner [Sh1 Lemma 3.10] handles the general situation by treating uniform layered domains with $0 \in L$, but this theory is considerably more technical, and factorization loses uniqueness. In order to treat the general situation, one needs a more technical approach. Our next definition is in opposition to separability.

**Definition 7.9.** A monic polynomial $f$ of degree $t$ is called a-**primary** if

$$ f = \lambda^t + \sum_{j=0}^{t-1} \alpha_j \lambda^j $$

where $\alpha_j \equiv_\nu a^{t-j}$ for all $i$.

**Lemma 7.10.** When $R$ is $\nu$-cancellable and $\nu$-N-cancellable, the monic polynomial $f$ is a-primary iff every corner root of $f$ is $\nu$-equivalent to $a$.

**Proof.** Any corner root $b$ satisfies

$$ a^{t-j} b^j \equiv_\nu a_j \lambda^j b^j \equiv_\nu a_j b^j \equiv_\nu a^{t-j} b^j $$

for some $j, j'$, which implies by $\nu$-cancellation and $\nu$-N-cancellation that $a \equiv_\nu b$. The reverse implication is obtained by reversing this argument. 

From this point of view, the primary polynomials are the ones with the simplest root locus, namely the $\nu$-equivalence class of a single point.

**Proposition 7.11.** If $f = \sum_{i=1}^m h_i$ is in essential form, where $\deg h_i = i$, then

$$ h_j f = \left( \sum_{i=j}^m h_i \right) \left( \sum_{k=1}^j h_k \right). \quad (7.3) $$

**Proof.** For $i > k$, we have $h_i h_k(a) \leq_\nu h_{i-1} h_{k+1}(a)$ for all $a$, with strict inequality unless these monomials are all in the same homogeneous component. It follows that each term in the left side also appears in the right side, and the other terms $h_i h_k$ on the right side (for $i > k$) are dominated by $h_{i-1} h_{k+1}$ and, by induction descending to $j$, are dominated by a term on the left side, with the domination strict at some step.

**Proposition 7.12.** If $f = \sum_{i=0}^m \alpha_i \lambda^i$ has bottom part $\sum_{i=0}^j \alpha_i \lambda^i$, which is $\alpha_j$ times some $a$-primary polynomial $f_a$, then $f \equiv_\nu \lambda^j f_a$, where

$$ \lambda^j f_a = \alpha_j^{-1} \left( \sum_{i=j}^m \alpha_i \lambda^i \right). $$

**Proof.** By definition of bottom part, the slopes change at $\lambda^j$, so we have the same convexity argument as in Proposition [7.11].

Thus, we can factor polynomials at their bottom part. By duality, one could also factor out the top part first. But at any rate, iterating this procedure, we can write any monic polynomial $f$ as a product of primary polynomials, i.e.,

$$ f = \prod_a f_a, $$

where each $f_a$ is a-primary. We call this the **primary decomposition** of the polynomial $f$. This motivates us to study primary polynomials. In the customary theory of polynomials over a field, the only primary polynomials would be powers of linear polynomials. The situation here is considerably more subtle.

**Definition 7.13.** $\mathcal{P}_a \subset R[\lambda], \ a \in R$, denotes the set of a-primary polynomials.

**Example 7.14.** If $R$ is any $L$-layered domain$^1$, and $a \in R$, we define

$$ \langle a \rangle_\nu := \{ b \in R : b \equiv_\nu a \ \text{for some j in N} \}, \quad (7.4) $$

and in particular, taking $a = \mathbb{1}_R$,

$$ \langle \mathbb{1}_R \rangle_\nu = \{ b \in R : b \equiv_\nu \mathbb{1}_R \}. $$

These clearly are $L$-layered sub-semirings$^1$ of $R$.

Although its structure is rather trivial, $\langle a \rangle_\nu$ plays a key role in the factorization theory.
Proposition 7.15. \( \mathcal{P}_a \) is a sub-monoid of \( \langle a \rangle_\nu [\lambda] \), and is also closed under addition of polynomials of the same degree.

Proof. By definition, each coefficient of an \( a \)-primary polynomial belongs to \( \langle a \rangle_\nu \). Also,

\[
\sum_{i=0}^{t} [k_i \alpha_i \lambda^i] + \sum_{i=0}^{t} [\ell_i \beta_i \lambda^i] = \sum_{i=0}^{t} [k_i + \ell_i] (\alpha_i + \beta_i) \lambda^i,
\]

whereas \( [k_i \alpha_i + \ell_i \beta_i] \equiv \nu a^{t-i} + a^{t-i} \equiv \nu a^{t-i} \). Thus it remains to show that the product of \( a \)-primary polynomials is \( a \)-primary.

\[
\left( \sum_{i=0}^{t} [k_i \alpha_i \lambda^i] \right) \left( \sum_{j=0}^{t'} [\ell_j \beta_j \lambda^j] \right)
\]

is a sum of monomials

\[
[k_i \alpha_i \lambda^i] [\ell_j \beta_j \lambda^j] = [k_i \ell_j] (\alpha_i \beta_j) \lambda^{i+j},
\]

and \( [k_i \alpha_i \ell_j \beta_i] \equiv \nu a^{t-i}a^{t'-j} = a^{t+i'+j} \), so the product is indeed \( a \)-primary. \( \square \)

Proposition 7.16. If \( f = \sum_{i=0}^{m} [\ell_i \alpha_i \lambda^i] \) is \( a \)-primary, then for \( b \in R_k \),

\[
f(b) = \begin{cases} [k^m]b^m & \text{if } b > \nu a; \\ [\sum_{i=0}^{m} \ell_i \alpha_i b^i] & \text{if } b \equiv \nu a; \\ [\ell_0]a_0 & \text{if } b < \nu a. \end{cases}
\]

Proof. Clearly the monomial \( \lambda^m \) strictly dominates \( f \) at \( b \) when \( b > \nu a \), and \( [\ell_0]a_0 \) strictly dominates when \( b < \nu a \), whereas when \( b = a \) all the terms have the same \( \nu \)-value, and one just combines the ghost layers. \( \square \)

Corollary 7.17. Suppose \( f = \sum_{i=0}^{t} [\ell_i \alpha_i \lambda^i] \) is \( a \)-primary, and \( b \in R_k \). Then

\[
s(f(b)) = \begin{cases} k_t & \text{if } b > \nu a; \\ \sum_{i=0}^{t} \ell_i k^i & \text{if } b \equiv \nu a; \\ \ell_0 & \text{if } b < \nu a. \end{cases}
\]

We can generalize Proposition 7.16.

Proposition 7.18. Notation as above, in any homogeneous part \( \sum_{i=0}^{m''} [\ell_i \alpha_i \lambda^i] \) of a polynomial \( f \), we have \( f(b) = [k'' \ell'' \alpha_i \lambda^i] b'' \) whenever \( m'' \nu < b < \nu m'' \nu \), where \( i'' \) is the essential exponent following \( i' \) and \( \ell = s(b) \).

Proof. This is the strictly dominant term in the \( m'' \)-homogeneous part, which we claim dominates all other terms. Indeed,

\[
[k'' \ell''] \alpha_i b'' \equiv \nu \alpha_i b'' \nu^{-i} b'' = \alpha_i' m''^{-i} b',
\]

which is greatest when \( i = i'' \). \( \square \)

For the bottom part, \( f(b) = [\ell_0]a_0 \) when \( b < \nu m_0 \).

Proposition 7.19. For any \( a \)-primary polynomial \( f \) in \( R[\lambda] \), we have \( f(b) \equiv \nu (a + b)^{deg f} \) for all \( b \in R \).

Proof. One checks each of the three cases in Proposition 7.16. \( \square \)

Corollary 7.20. For any product \( f = \prod f_a \) of \( a \)-primary polynomials \( f_a \) and any \( b \)-primary polynomial \( g_b \), we have

\[
\prod f_a(b)^{deg g_a} \equiv \nu \prod (a + b)^{deg f_a} \equiv \nu \prod g_b(a)^{deg f_a}.
\]

Proof. Apply Proposition 7.19 twice, for \( f_a \) and for \( g_b \). \( \square \)

Corollary 7.21. For any products \( f = \prod f_a \) and \( g = \prod g_b \) of \( a \)-primary polynomials and \( b \)-primary polynomials respectively, we have

\[
\prod_{a,b} f_a(b)^{deg g_a} \equiv \nu \prod_{a,b} (a + b)^{deg f_a} \equiv \nu \prod_{a,b} g_b(a)^{deg f_a}.
\]
Remark 7.22. Recall from \[\text{Sh1}\] Lemma 3.13 that there is a monoid homomorphism \(\psi_a : \mathcal{P}_a \to L[\lambda]\) given by
\[
\psi_a \left( \sum_{i=0}^{t} [\ell_i]_{\alpha_i} \lambda^i \right) = \sum_{i=0}^{t} \ell_i \lambda^i, 
\]
which also is additive on \(a\)-primary polynomials of the same degree. (This is an easy consequence of Axioms A3 and B. Furthermore, \(\psi_a\) is an isomorphism in the important case that \(R = R(L, \mathcal{G})\) of Construction 3.2. The \(0\)-layer terms obtained in multiplying together \(a\)-primary polynomials are inessential, and thus can be excluded. For example,
\[
(\lambda + [\ell]_{\alpha})(\lambda + [\ell]_{\beta}) = \lambda^2 + [0]_{\alpha} \lambda + [\ell]_{\beta}^2 = \lambda^2 + [\ell]_{\beta}^2 \lambda^2
\]
since the monomial \(0\)-layer is inessential. (Otherwise, the homomorphism would break down, which happens in other situations, as pointed out by Sheiner \[\text{Sh1}\].)

Sheiner \[\text{Sh1}\] obtained the following uniqueness result:

Theorem 7.23. \[\text{Sh1}\] Lemma 3.10, Theorem 3.12 Any polynomial \(f \in R[\lambda]\) over a uniform \(L\)-layered semifield \(R\) can be factored in the form
\[
f = \alpha f_{a_1} \cdots f_{a_d}
\]
where \(\alpha \in R\), \(a_1 >_\nu a_2 >_\nu \cdots >_\nu a_d\) are the corner roots of \(f\), and each \(f_{a_j}\) is \(a_j\)-primary. This factorization is unique with respect to the \(a_j\).

Existence is given in Proposition 7.11. Uniqueness is obtained by noting that for any \(b \in R\),
\[
s(f(b)) = s(\alpha)s(f_{a_1}(b)) \cdots s(f_{a_d}(b))
\]
so any other factorization must have the same corner roots, in view of Remark 7.22 and Corollary 7.17, which show that different corner roots in the factorizations would give different sorts. One concludes by observing that any two distinct \(a\)-primary polynomials are different as functions in view of Remark 7.22 (as is seen by substituting \([\ell]_a\) for different values of \(\ell\).)

This factorization into primary polynomials is sufficient for many applications, such as:

Corollary 7.24. Suppose \(f = \alpha f_{a_1} \cdots f_{a_d}\) as in (7.6), where \(a_1 >_\nu a_2 >_\nu \cdots >_\nu a_d\), and suppose \(b \in R_{\ell}\). Also, write
\[
f_{a_j} = \lambda^{\ell_j} + \sum_{i=1}^{l_j} [\ell_{i,j}]_{\alpha_{i,j}} \lambda^{l_{i,j}}
\]
for \(1 \leq j \leq d\). If \(b \equiv_{\nu} a_j\) for some \(j\), then
\[
s(f(b)) = s(\alpha) \left( \ell^{\ell_j} + \sum_{i=1}^{l_j} \ell_{i,j} \lambda^{l_{i,j}} \right) \prod_{i=j+1}^{d} \ell_{i,j} \prod_{i=1}^{j-1} \ell_{0,i}.
\]
If \(a_j <_\nu b < a_{j+1}\), then
\[
s(f(b)) = \prod_{i=j+1}^{d} \ell_{i,j} \prod_{i=1}^{j} \ell_{0,i}.
\]

This enables us to compute in some sense how much the ghost layer is raised by evaluating at the corner root \(a\). Unfortunately, a certain ambiguity remains — sometimes an \(a\)-primary polynomial could be factored into \(a\)-primary polynomials of smaller degree, and this need not be unique, as seen for example in \[\text{Sh1}\] Corollary 3.14. Ironically, this failure can be viewed in a positive light, by a connection to “classical” algebra.

Remark 7.25. Let \(\mathcal{L}\) denote the “classical” polynomial semiring \(L[\lambda]\). In view of Remark 7.22, any factorization of an \(a\)-primary polynomial can be transferred to a classical factorization in \(\mathcal{L}\). Thus, we obtain information about factorization of \(a\)-primary polynomials in terms of factorizations in \(\mathcal{L}\). But when we take \(L\) to be positive, the classical factorization is modified somewhat, which leads to various difficulties. For instance, taking \(L = \mathbb{Q}_{>0}\) leads us to classical factorization of polynomials into polynomials having positive coefficients, which is not necessarily unique although factorization of polynomials over \(\mathbb{Q}\) is unique.

These considerations lead one towards considering a larger sorting semiring \(L\) which is a field, and in particular would have negative elements, so that we could factor polynomials into linear binomials. But once 0
is adjoined to $L$, the hypothesis of Theorem 7.23 is no longer valid. As mentioned in Remark 7.22, Sh1 explains how the 0 layer can ruin factorization into primary polynomials. Thus, we must tread a narrow path. First we take the primary decomposition, and then study each $\alpha$-primary polynomial in turn by means of the map $\psi_\alpha$.

7.4. Multiple roots. The layered theory enables us to study multiplicities of corner roots.

We have rather precise information, but at the cost of taking $L = \mathbb{Q}$, i.e., considering negative ghost layers.

**Proposition 7.26.** Suppose $f$ is a-primary of degree $t$, with $s(a) = \ell$. Write $f_\psi$ for $\psi_a(f)$ from Remark 7.22.

(i) $f(a) = [f_\psi(\ell)]a^\ell$; in particular, $s(f(a)) = f_\psi(\ell)$.

(ii) $(\lambda + a)$ divides $f$ iff $f_\psi(-\ell) = 0$, iff $f([-\ell]a) \in R_0$.

(iii) $(\lambda + a)^m$ divides $f$, iff $-\ell$ is a root of $f_\psi$ of multiplicity at least $m$.

**Proof.** (i) Write $f = \sum_{i=0}^{m} [\ell_i] \alpha_i \lambda^i$. Since $f(a)$ is a sum of terms each $\nu$-equivalent to $a^\nu$, $f = \sum_{i=0}^{m} [\ell_i] \alpha_i \lambda^i$, we have $s(f(a)) = \sum \ell_i \ell^i = f_\psi(\ell)$.

(ii) Write $f_\psi = (\lambda + \ell)g_\psi + \ell'$, according to the classical Euclidean algorithm, and take $g$ such that $g_\psi = \psi_a(g)$. Then $\ell' = 0$ iff $f_\psi(-\ell) = 0$, iff $f([-\ell]a) \in R_0$ in view of (i). Note that if $f_\psi = (\lambda + \ell)g_\psi$, then $f = (\lambda + a)g$ in view of Remark 7.22.

(iii) Apply induction to (ii). \qed

We may like to express the multiplicity of a corner root $a$ (cf. Definition 5.10) directly in terms of the ghost layer $s(f(a))$, especially in the case where $f$ is $\alpha$-primary. The first guess might be that $a \in R_\ell$ has multiplicity $\geq m$ if $s(f(a)) \geq 2^m s(f(b))$, for suitable $b \in R_\ell$ which is not an $\ell$-root of $f$. There are several difficulties with this approach. The constant term could have a very large ghost layer in comparison with the intermediate terms, which distorts the factorization. Also, non-roots could yield different ghost layers. Nevertheless, here are some examples to aid intuition.

**Example 7.27.** Suppose $f \in R$, and $a \in R_1$.

(i) If $f(a) \in R_1$, then $s(f(a)) = 1 = 2^0$.

(ii) The tangible corner root $a$ of $f = \lambda + a$ satisfies $f(a) = a + a \in R_2$, so $s(f(a)) = 2 = 2^1$.

(iii) The tangible corner root $a$ of $f = (\lambda + a)^m$ satisfies $f(a) = (a + a)^m \in R_{2m}$, so $s(f(a)) = 2^m$.

(iv) The tangible corner root $a$ of $f = \lambda^2 + a^2$ satisfies $f(a) = a^2 + a^2 \in R_2$, so $s(f(a)) = 2 = 2^1$.

(v) The tangible corner root $a$ of $f = \lambda^2 + a\lambda + a^2$ satisfies $f(a) = a^2 + a^2 + a^2 \in R_3$, so $s(f(a)) = 3$.

(vi) The tangible corner root $a$ of $f = \lambda^2 + [\ell] a \lambda + a^2$ satisfies $s(f(a)) = 4$.

(vii) The tangible corner root $a \in R_\ell$ of $f = ([\ell] \lambda + a)^m$ satisfies

$$s(f(a)) = \sum \left( \begin{array}{c} m \\ j \end{array} \right) \ell^j \ell^{m-j} = \ell^m \sum \left( \begin{array}{c} m \\ j \end{array} \right) = 2^m \ell^m,$$

whereas for $b \not\equiv \psi a$, $s(f(b)) = \ell^m$.

8. Layered resultants

Resultants are an attractive tool since they provide a link between linear algebra and geometry, and also provide a criterion for when polynomials are relatively prime. The supertropical resultant was studied in the standard supertropical case in [IR7], and the same definition works more generally in the layered theory.

8.1. The layered resultant. We assume throughout this section that $f, g \in R[\lambda]$ have respective degrees $m, n$ over the $L$-layered domain $R$ and, for convenience, we write $f = \sum_{i=0}^{m} [\ell] \alpha_i \lambda^{m-i}$ in full form, where the inessential coefficients have ghost layer 0, as in Remark 7.3. We also adjoin $[\ell]0_R$ formally to $R$ for each layer $\ell$, in order to be able to deal more easily with matrices. (For example, the off-diagonal entries of the identity matrix are $0 \in R[\ell] := [\ell]0_R$.)
Throughout this example, we take $\alpha := [k] \alpha \in R_k$, $\alpha_i := [k] \alpha \in R_{k_i}$, $\beta := [\ell] \beta$, and $\beta_j := [\ell] \beta \in R_{\ell_j}$. We specify the layer only when it differs from this notation.

(i) For $f = \lambda + \alpha$, $g = \lambda + \beta$,

$$|\mathcal{R}(f, g)| = \alpha + \beta = \begin{cases} \alpha & \text{for } \alpha >_\nu \beta, \\ [k+\ell] \alpha & \text{for } \alpha \eqsim _\nu \beta, \\ \beta & \text{for } \alpha <_\nu \beta. \end{cases}$$

(ii) For $f = \sum_i \alpha_i \lambda^i$ and $g = \lambda + \beta$,

$$|\mathcal{R}(f, g)| = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{m-1} & \alpha_m \\ \beta & 1 & \cdots \\ \cdots & \cdots & \beta & 1 \end{vmatrix} = \alpha_0 + \alpha_1 \beta + \cdots + \alpha_m \beta^m = f(\beta)$$
which is
\[
\begin{cases}
\alpha_0 & \text{for } \alpha_1 < \nu \alpha_0, \text{ (in other words, } \beta \text{ is smaller than all roots of } f), \\
[\sum_{i \in \text{dom}(\beta, f)} k_i \beta^i]_{\alpha_i \beta^i} & \text{for } \beta \text{ a root of } f, \\
[k_i \beta^i]_{\alpha_i \beta^i} & \text{for } \alpha_i > \nu \alpha_{i-1} \text{(in other words, } \beta \text{ is between roots of } f). 
\end{cases}
\]

In every case this equals \(f(\beta)\). Likewise,
\[
|R(f, \lambda)| = \begin{vmatrix}
\alpha_0 & \alpha_1 & \cdots & \alpha_{m-1} & \alpha_m \\
1 & \ddots & & & \\
& \ddots & \ddots & & \\
& & \ddots & 1 \\
\end{vmatrix} = \alpha_0.
\]

(iii) Suppose \(f = (\lambda + \alpha_1)(\lambda + \alpha_2) = \lambda^2 + (\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2 \) and \(g = \lambda + \beta\), with \(\alpha_1 < \nu \alpha_2\). Then
\[
|R(f, g)| = \begin{vmatrix}
\alpha_1 \alpha_2 & \alpha_1 + \alpha_2 & 1 \\
\beta & 1 & \\
\beta & 1 
\end{vmatrix} = \alpha_1 \alpha_2 + (\alpha_1 + \alpha_2)\beta + \beta^2 = f(\beta),
\]

which is
\[
\begin{cases}
\alpha_1 \alpha_2 & \text{if } \beta < \nu \alpha_1, \\
\alpha_2 [k_1 + \ell] \alpha_1 & \text{if } \beta \approx \nu \alpha_1, \\
\beta \alpha_2 & \text{if } \alpha_1 < \nu \beta < \nu \alpha_2, \\
[\ell(k_2 + \ell)] \alpha_2^2 & \text{if } \beta \approx \nu \alpha_2, \\
\beta^2 & \text{if } \alpha_2 < \nu \beta.
\end{cases}
\]

(iv) When \(f\) is a primary of degree 2, say \(f = \lambda^2 + \alpha_1 \lambda + \alpha_0, \) \(g = \sum_{j=0}^n \beta_j \lambda^j, \)
\[
|R(f, g)| = \begin{vmatrix}
\alpha_0 & \alpha_1 & 1 & \ddots \\
& \alpha_0 & \alpha_1 & \ddots \\
& & \ddots & \ddots \\
& & & \alpha_0 & \alpha_1 & 1 \\
\beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\
\beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n 
\end{vmatrix},
\]

which, in view of Remark 8.3, is the sum of terms of the form
\[
\alpha_{i_1} \cdots \alpha_{i_n} \beta_{j_1} \beta_{j_2},
\]
where \((i_1 + 1, i_2 + 2, \ldots, i_n + n, j_1 + 1, j_2 + 2)\) is a permutation of \((1, \ldots, n + 2)\). Equation 8.2 implies
\[
i_1 + \cdots + i_n = 2n - j_1 - j_2.
\]
Thus each term involving \(\beta_{j_1} \beta_{j_2}\) is \(\nu\)-equivalent to
\[
a^t \beta_{j_1} \beta_{j_2}
\]
where \(t = \sum_{n=1}^n (n - i_n) = n^2 - 2n + j_1 + j_2.\)
Since these \(\nu\)-values are all the same, one sees that the layer of the sum of terms involving \(\beta_{j_1} \beta_{j_2}\) is precisely the permanent of the layer matrix of \(R(f, g)\), with the \(n + 1, n + 2\) rows and \(j_1, j_2\) columns erased. But 8.3 equals
\[
a^{n^2 - 2n} \beta_{j_1} a^{j_1} \beta_{j_2} a^{j_2}.
\]
(8.4)
If \(a\) is not a corner root of \(g\), then \(g(a) = \beta_j a^j\) for some \(j\) (which strictly dominates the other terms of this form), and picking \(j_1 = j_2 = j\) yields
\[
|R(f, g)| \approx \nu a^{n^2 - 2n} g(a)^2.
\]
If \(a\) is a corner root of \(g\), then there are \(j_1 \neq j_2\) for which \(\beta_{j_1} a^{j_1} = \beta_{j_2} a^{j_2}\) dominates 8.4, so we have at least two summands of the form 8.3.
In particular, \(|\Re(f, g)| \approx \nu a^{n^2 - 2n}\beta_0^2\) if the smallest root of \(g\) dominates \(a\). If the smallest root of \(g\) strictly dominates \(a\), then we get a unique dominant term from \(\beta_0^2\), and \(\Re(f, g)| = a^{n^2 - 2n}\beta_0^2\).

Likewise, if \(a\) strictly dominates all the roots of \(g\), then we get the dominant terms in \(\Re(f, g)| by choosing \(\beta_n\) along the lower part of the main diagonal, which means the remaining part in computing the permanent must be \(\alpha_0\) along the upper part of the main diagonal, yielding \(\Re(f, g)| = \alpha_0^2\beta_n^2\).

(v) More generally, when \(g\) is \(b\)-primary and \(b\) strictly dominates every root of \(f\), then the same argument as in (iv) shows that \(\Re(f, g)| = \alpha_n^m\beta_0^m\). Namely, we get the most significant terms when we choose \(\beta_0\) and \(\alpha_n\) in computing the permanent.

(vi) When \(g\) is \(b\)-primary and \(b\) dominates every root of \(f\), then the same argument as in (v) shows that \(\Re(f, g)| \approx \nu a^{m^2}\beta_0^m\), but we could have other terms yielding the same result. Note however that any term contributing to \(\Re(f, g)| must be products of coefficients of the upper part of \(f\) together with coefficients of \(g\). Thus, we would have the same resultant if we replaced \(f\) by its upper part.

**Lemma 8.5.** When \(f\) and \(g\) are both \(a\)-primary, i.e., \(f = \sum_{i=0}^m |\ell_i| a^i \lambda^{m-i}\) and \(g = \sum_{j=0}^n |k_j| a^j \lambda^{n-j}\), then \(\Re(f, g)| = |\ell'| a^{mn}\), where \(|\ell'|\) is the permanent of the matrix of the ghost layers.

**Proof.** Every possible term in the permanent has the same \(\nu\)-value as \(a^{mn}\), and so we add the ghost layers of these terms. \(\square\)

This example turns out to be so instrumental that we introduce some notation.

**Definition 8.6.** The layer matrix of a matrix \(A = ([\ell_{i,j}] a_{i,j})\) is the matrix \((\ell_{i,j}) \in M_n(L)\). When \(L\) is a ring, its (classical) determinant is computed as an element of \(L\).

Given an \(a\)-primary polynomial \(f = \sum_{i=0}^m |k_i| a^i \lambda^{m-i}\) \(\in R[\lambda]\), we let \(\mathfrak{L}_n(f)\) denote the layer matrix of \(A_n(f)\), which is the \(n \times (m + n)\) matrix

\[
\begin{pmatrix}
  k_0 & k_1 & k_2 & \ldots & k_m & \cdots \\
  k_0 & k_1 & k_2 & \ldots & k_m & \cdots \\
  k_0 & k_1 & k_2 & \ldots & k_m & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  k_0 & k_1 & k_2 & \ldots & k_m & \cdots
\end{pmatrix}
\]

where the empty places stand for 0.

Let us write \(|A|_{\text{per}}\) for the permanent of the matrix \(A \in M_n(L)\). For polynomials \(f, g\) of respective degrees \(m, n\), the layer Sylvester matrix \(\mathfrak{L}(f, g)\) of \(f\) and \(g\) is the matrix \(\left(\begin{array}{c|c}
\mathfrak{L}_n(f) & \\
\mathfrak{L}_m(g) & \\
\end{array}\right)\), and the layer permanent of \(f\) and \(g\) is \(|\mathfrak{L}(f, g)|_{\text{per}}\).

We now can restate Lemma 8.5 more succinctly.

**Lemma 8.7.** When \(f\) and \(g\) are \(a\)-primary,

\(\Re(f, g)| = |\mathfrak{L}(f, g)|_{\text{per}} a^{mn}\).

**Example 8.8.** Suppose \(f = \sum_{i=0}^m |k_i| a^i \lambda^i\) and \(g = \lambda + |\ell| a\). The permanent of the layer Sylvester matrix is

\(\mathfrak{L}(f, g)|_{\text{per}} = \begin{vmatrix}
  k_0 & k_1 & \ldots & k_m \\
  \ell & 1 & \cdots & \cdots \\
  \ldots & \ddots & \ddots & \ddots \\
  \ell & 1 & \cdots & \cdots
\end{vmatrix}_{\text{per}} = k_0 + k_1 \ell + k_2 \ell^2 + \cdots = \sum_{i=0}^m k_i \ell^i,
\)

seen by expanding along the first row, and this equals \(f(\ell)\), where \(f = \sum_{i=0}^m k_i \lambda^i\). In particular, if \(\ell \geq 1\) and \(k_0, k_1, k_2 \geq 1\), then \(|\mathfrak{L}(f, g)|_{\text{per}}\) is 2-ghost.

The computations of Example 8.4 might lead us to expect that the resultant is multiplicative, especially in view of [R7] Theorem 4.12. However, Example 8.5 already leads us to a counterexample. To simplify notation, we write \([k] a^i\) to denote \([k](a^i)\), i.e., \(a^i\) given layer \(k\).
Example 8.9. Suppose \( f = \lambda^2 + [k_1]a\lambda + [k_0]a^2 \), \( g = \lambda + [\ell]a \), and \( h = \lambda + [\hat{\ell}]a \). (Thus \( f, g, h \) are all \( a \)-primary.) Let \( |\Sigma(f,g)|_{\per} \) and \( |\Sigma(f,h)|_{\per} \) be the permanents of the layer Sylvester matrices, which are

\[
|\Sigma(f,g)|_{\per} = \left| \begin{array}{ccc}
  k_0 & k_1 & 1 \\
  \ell & 1 & \ell \\
  \ell & 1 & \ell \\
\end{array} \right|_{\per} = \hat{\ell}^2 + k_1\ell + k_0,
\]

\[
|\Sigma(f,h)|_{\per} = \left| \begin{array}{ccc}
  k_0 & k_1 & 1 \\
  \ell & 1 & \ell \\
  \ell & 1 & \ell \\
\end{array} \right|_{\per} = \hat{\ell}^2 + k_1\hat{\ell} + k_0,
\]

Then

\[
|\mathcal{R}(f,g)| = |\Sigma(f,g)|_{\per} a, \quad |\mathcal{R}(f,h)| = |\Sigma(f,h)|_{\per} a,
\]

so their product is

\[
|\Sigma(f,g)|_{\per}|\Sigma(f,h)|_{\per} a = |\nu| a
\]

where

\[
p = (\ell^2 + k_1\ell + k_0)(\hat{\ell}^2 + k_1\hat{\ell} + k_0) = \ell^2(\ell + \hat{\ell}) + \ell(k_1\ell + k_0k_1) + k_0\ell^2 + k_0k_1\hat{\ell} + k_0^2.
\]

On the other hand,

\[
|\Sigma(f,gh)|_{\per} = \left| \begin{array}{ccc}
  k_0 & k_1 & 1 \\
  \ell & 1 & \ell \\
  \ell & 1 & \ell \\
\end{array} \right|_{\per} = \frac{k_0}{k_1}
\]

which clearly is a sum of 13 terms, and turns out to be \( p + 4k_0\ell\hat{\ell} \). (In the determinant computation, these would appear twice with + sign and twice with – sign, and thus cancel.) It follows that when \( k_0\ell\hat{\ell} \neq 0 \) we have

\[
|\mathcal{R}(f,gh)| \neq |\mathcal{R}(f,g)||\mathcal{R}(f,h)|.
\]

It is worth comparing these computations with the fact that in the “classical” world, for the standard determinant, \( |\mathcal{R}(f,gh)| = |\mathcal{R}(f,g)||\mathcal{R}(f,h)| \). Since the terms involved in computing the permanent and the determinant are the same (just with a change of sign), one might be surprised that we had this new term \( k_0\ell\hat{\ell} \). This is clarified when we factor \( f \) into linear factors, i.e., \( f = (\lambda + a_1)(\lambda + a_2) \). Then \( k_0 = s(a_1a_2) \) and \( k_1 = s(a_1 + a_2) \), so the “extra” term \( s(a_1a_2)\ell\hat{\ell} \) now is achieved in

\[
|\Sigma(f,g)|_{\per}|\Sigma(f,h)|_{\per} = (\ell^2 + s(a_1a_2)\ell + s(a_1a_2))(\hat{\ell}^2 + s(a_1 + a_2)\hat{\ell} + s(a_1a_2)),
\]

although with a smaller coefficient.

Example 8.10. For \( f = \lambda^2 + [k_1]a^2\lambda + [k_0]a^3 \), \( g = \lambda^2 + [\ell]a\lambda + [\ell_0]a^2 \), and \( h = \lambda^2 + [\hat{\ell}]a\lambda + [\hat{\ell}_0]a^2 \),

\[
|\Sigma(f,g)|_{\per} = \left| \begin{array}{ccc}
  k_0 & k_1 & 1 \\
  \ell_0 & \ell_1 & 1 \\
\end{array} \right|_{\per} \quad \text{and} \quad |\Sigma(f,h)|_{\per} = \left| \begin{array}{ccc}
  k_0 & k_1 & 1 \\
  \hat{\ell}_0 & \hat{\ell}_1 & 1 \\
\end{array} \right|_{\per},
\]

and we have checked on the computer, using Matematika, that every term in their product is subsumed in \( |\Sigma(f,gh)|_{\per} \).

Example 8.11. For \( f = \lambda^3 + [k_2]a\lambda^2 + [k_1]a^2\lambda + [k_0]a^3 \), and \( g \) and \( h \) as in Example 8.10, again we have checked on the computer that every term in the product is subsumed in \( |\Sigma(f,gh)|_{\per} \).

After the initial shock of these examples, one can find the following conclusions:

(1) \( |\mathcal{R}(f,gh)| \cong_{\nu} |\mathcal{R}(f,g)||\mathcal{R}(f,h)| \),

(2) \( |\mathcal{R}(fg,h)| \cong_{\nu} |\mathcal{R}(f,h)||\mathcal{R}(g,h)| \),
\[ |\Re \left( \prod_a f_a \prod_b g_b \right) | = \prod_{a,b} |\Re(f_a, g_b)|, \]

the products taken over the homogeneous parts of \( f \) and \( g \).

(4) \[ |\Re(f, g)| \leq L |\Re(f, g)| |\Re(f, h)| \]

when \( L \) is small enough (including the standard supertropical case).

We aim for these results. We need a method of factoring out the bottom part of a polynomial. Note that in the computations in Example 8.4 we could disregard all the homogeneous parts of \( g \) except the one in which \( a \) is a root.

Our main objective is to compare \( |\Re(f, g)| \) with \( \prod_{a,b} |\Re(f_a, g_b)| \), where \( f = \prod_a f_a \) is the primary decomposition of \( f \) and \( g = \prod_b g_b \) is the primary decomposition of \( g \).

Already we have the following special case, which was indicated in Example 8.4.

**Lemma 8.12.** Suppose \( f = \sum_{i=0}^m \alpha_i \lambda^i \) is \( a \)-primary and \( g = \sum_{j=0}^n \beta_j \lambda^j \) is \( b \)-primary. Then

\[ |\Re(f, g)| \leq b |\Re(f)\Re(g)| \leq |\Re(f)| |\Re(g)| \leq |f| |g|. \]

If \( a <_\nu b \), then

\[ |\Re(f, g)| = \beta_0^m = g(a)^m \leq b^{mn} = (a + b)^{mn} = f(b)^n. \]

If \( a >_\nu b \), then

\[ |\Re(f, g)| = \alpha_0^n = f(b)^n \leq a^{mn} = (a + b)^{mn} = g(a)^m. \]

**Proof.** By Remark 8.3 \( |\Re(f, g)| \) is a sum of terms of the form

\[ \alpha_{i_1} \cdots \alpha_{i_s} \beta_{j_1} \cdots \beta_{j_m}. \]

When \( b >_\nu a \), noting that

\[ \beta_j a^j <_\nu \beta_0 \leq b^0, \]

we get the \( \nu \)-dominant term when we increase the weight of \( b \) in this term by choosing each \( \beta_j a^j \) to be \( \beta_0 \), i.e., we choose the term in \( |\Re(f, g)| \) involving \( \beta_0^m \). There is only one such nonzero term, and this has each \( \alpha_i = \alpha_m. \) But \( \alpha_m \leq b \) since \( f \) is monic. Thus, the \( \nu \)-dominant term of \( |\Re(f, g)| \) is \( \beta_0^m = b^{mn} \), since all the other terms \( a^j b^{mn-j} \) are dominated by it. The second assertion follows by symmetry, since then \( g(a) \leq a_n = (a + b)^n. \) When \( a \leq b \), we still have the same \( \nu \)-dominant terms, but perhaps have others as well, so we only get \( \nu \)-equality.

To generalize this observation, we turn to an idea from [IR7].

**Definition 8.13.** Given a polynomial \( f = \sum_{i=0}^m \alpha_i \lambda^i \), we define

\[ f_{[u]} = \sum_{i=u}^m \alpha_i \lambda^{i-u}, \quad u = 1, \ldots, m. \]

\( f_{[1]} \) is called the reduction along \( \tilde{a} \), where \( \tilde{a} \) is the corner root \( \frac{\alpha_0}{\alpha_1}. \)

**Lemma 8.14.** Over a layered 1-semifield \( \mathcal{R} \), suppose \( \tilde{a} \) is the root of \( f \) having lowest \( \nu \)-value, and \( \deg f_{\tilde{a}} = u. \) Then \( f = f_{[u]} f_{\tilde{a}}. \)

**Proof.** This is just a restatement of Proposition 7.12 since \( f_{[u]} = \tilde{f} \) and \( f_{\tilde{a}} = \sum_{i=0}^u \alpha_i \lambda^i \).

**Example 8.15.** \( f = \lambda f_{[1]} + \alpha_0. \) For \( \alpha_1 \) invertible, \( f \geq \lambda \geq \alpha_0 (\lambda + \frac{\alpha_0}{\alpha_1}) f_{[1]} \).

Our main tool is the following computation:

**Lemma 8.16.** If \( f = \sum_{i=0}^m \alpha_i \lambda^i \) and \( g = \sum_{j=0}^n \beta_j \lambda^j \), then

\[ |\Re(f, g)| \leq \alpha_0 |\Re(f, g_{[1]})| + \beta_0 |\Re(f_{[1]}, g)|. \]
Proof. We expand the resultant

\[
|\mathbb{R}(f, g)| = \begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots \\
\alpha_1 & \alpha_2 & \ldots \\
\alpha_2 & \ldots & \\
\vdots & \vdots & \ddots & \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_2 & \ldots & \beta_n \\
\vdots & \vdots & \ddots & \\
\beta_n & & & & \\
\end{vmatrix}
\]

along the first column, to get

\[
|\mathbb{R}(f, g)| = \begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots \\
\alpha_1 & \alpha_2 & \ldots \\
\alpha_2 & \ldots & \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_2 & \ldots & \beta_n \\
\vdots & \vdots & \ddots & \\
\beta_n & & & & \\
\end{vmatrix} + \beta_0
\]

In computing the second layered permanent of Equation (8.6) by expanding along the first column, the occurrence of \(\alpha_0\) in the second row must be multiplied by some \(\alpha_i\) in the first row, whereas, switching the first two rows, we also have \(\alpha_1\alpha_{i-1}\). But \(\alpha_1\alpha_{i-1} \geq_{\nu} \alpha_0\alpha_i\), so the term with \(\alpha_0\alpha_i\) is not relevant to the computation of the layered permanent. Thus the occurrence of \(\alpha_0\) in the second row cannot strictly dominate the second layered permanent of (8.6), and we may erase it.

By the same token, each occurrence of \(\beta_0\) does not strictly dominate the first layered permanent of Equation (8.6). Thus, (8.6) is \(\nu\)-equivalent to

\[
|\mathbb{R}(f, g)| = \alpha_0|\mathbb{R}(f, g_{[1]})| + \beta_0|\mathbb{R}(f_{[1]}, g)|.
\]

We want strict equality in (8.6). Towards this end, we have.

**Corollary 8.17.** If \(f = \sum_{i=0}^{m} \alpha_i \lambda^i\) and \(g = \sum_{i=0}^{m} \beta_i \lambda^i\) satisfy \(\alpha_0|\mathbb{R}(f, g_{[1]})| >_{\nu} \beta_0|\mathbb{R}(f_{[1]}, g)|\), then

\(|\mathbb{R}(f, g)| \cong_{\nu} \alpha_0|\mathbb{R}(f, g_{[1]})|\),

and

\[
|\mathbb{R}(f, g)| = \alpha_0 \begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots \\
\alpha_1 & \alpha_2 & \ldots \\
\alpha_2 & \ldots & \\
\vdots & \vdots & \ddots & \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_1 & \beta_2 & \ldots & \beta_n \\
\beta_2 & \ldots & \beta_n \\
\vdots & \vdots & \ddots & \\
\beta_n & & & & \\
\end{vmatrix} \quad \text{(8.7)}
\]

**Corollary 8.18.** If \(f = \sum_{i=0}^{m} \alpha_i \lambda^i\), then

\(|\mathbb{R}(f, \lambda g)| = \alpha_0|\mathbb{R}(f, g)|\).

**Proof.** By the proposition applied to \(\lambda g\), noting that \(g = (\lambda g)_{[1]}\). [\(\square\)
Corollary 8.19. If \( f = \sum_{i=0}^{m} \alpha_i \lambda^i \), then 
\[
|\Re(f, \lambda^t g)| = \alpha_0^t |\Re(f, g)|,
\]
and likewise if \( g = \sum_{i=0}^{n} \beta_i \lambda^i \), then 
\[
|\Re(\lambda^t f, g)| = \beta_0^t |\Re(f, g)|.
\]

Proof. By induction on \( t \), applying Corollary 8.18 repeatedly. \( \square \)

Theorem 8.20. Any monic polynomials \( f \) and \( g \) satisfy:
\[
|\Re(f, g)| \leq \nu \prod_{a,b} |\Re(f_{a}, g_{b})| \leq \nu \prod_{a,b} (a + b)^{m_a n_b},
\]
(8.8)

where \( f = \prod_{a} f_a \) is the primary decomposition of \( f \) and \( g = \prod_{b} g_b \) is the primary decomposition of \( g \), and \( m_a = \deg f_a \) and \( n_b = \deg g_b \).

Furthermore, write \( \bar{a} \) (resp. \( \bar{b} \)) for the root of \( f \) (resp. \( g \)) having smallest \( \nu \)-value. Then in (8.5),
\[
|\Re(f, g)| = \alpha_0 |\Re(f, g_{[1]})|
\]
(8.9)

if \( \bar{a} < \nu \bar{b} \),
\[
|\Re(f, g)| = \beta_0 |\Re(f_{[1]}, g)|
\]
(8.10)

if \( \bar{a} > \nu \bar{b} \), and
\[
|\Re(f, g)| = \alpha_0 |\Re(f, g_{[1]})|^{\nu} = \beta_0 |\Re(f_{[1]}, g)|^{\nu}
\]
(8.11)

if \( \bar{a} \equiv \nu \bar{b} \).

Proof. We prove Theorem 8.20 by double induction on \( \deg f \) and \( \deg g \), and carry this inductive assumption for Equation (8.8) throughout. The base of the induction is Lemma 8.12. Let
\[
P := \prod_{a,b} |\Re(f_{a}, g_{b})|.
\]

We first prove the inequality
\[
|\Re(f, g)| \leq \nu P.
\]
(8.12)

The reverse inequality to (8.12) will follow by considering the leading monomial that is \( \nu \)-equivalent to \( P \).

Since we could only increase the layered resultant by passing to the full forms of \( f \) and \( g \), we may replace \( f \) and \( g \) by their full forms. On the other hand, since we are only interested in \( \nu \)-values at this stage, we may replace all the coefficients by tangible coefficients of the same \( \nu \)-value. Thus, we assume that every power of \( \lambda \) up to \( \deg f \) has a tangible coefficient in \( f \), and likewise for \( g \).

Assume for convenience that \( \bar{a} \leq \nu \bar{b} \). Thus,
\[
\frac{\beta_0}{\beta_1} \leq \nu \frac{\bar{b}}{\bar{a}} = \frac{\alpha_0}{\alpha_1}.
\]

Appealing to (8.5), we want to show that
\[
\alpha_0 |\Re(f, g_{[1]})| \leq P \quad \text{and} \quad \beta_0 |\Re(f_{[1]}, g)| \leq P.
\]

Note that \( g_{[1]} = g_{[1]} \prod_{b \neq \bar{b}} g_b \), so by induction
\[
|\Re(f_{[1]}, g)| \leq \nu \prod_{a} \left( |\Re(f_{a}, g_{b})| \prod_{b \neq \bar{b}} |\Re(f_{a}, g_b)| \right).
\]

But writing \( \bar{m} \) for \( m_{\bar{a}} \) and \( \bar{n} \) for \( n_b \), and \( f_{\bar{a}} = \sum_{i=0}^{\bar{m}} \alpha_i \lambda^i \), we have \( \alpha_{\bar{m}, \bar{n}} \equiv \nu \beta_{\bar{R}} \), and thus
\[
|\Re(f_{\bar{a}, g_{\bar{b}}})| \leq \nu \beta_0^{\bar{m}} = \beta_0^{\bar{m}} \left( \frac{\beta_1}{\beta_0} \right)^{\bar{m}}
\]
by Lemma 8.12, and
\[
\alpha_{0, \bar{a}} \equiv \nu \bar{m} \left( \frac{\alpha_0}{\alpha_1} \right)^{\bar{m}},
\]
yielding
\[
\alpha_{0, \bar{a}} \Re(f_{\bar{a}, g_{\bar{b}}}) \leq \nu \beta_0^{\bar{m}} \left( \frac{\alpha_0}{\alpha_1} \right)^{\bar{m}} \leq \nu \beta_0^{\bar{m}},
\]

with strict inequality when \( \bar{a} <_{\nu} \bar{b} \). Hence, we have
\[
\alpha_0 |R(f, g_1)| \leq |R(f_a, g_b)| \prod_{b \neq b} |R(f_a, g_b)| = P,
\]
with strict inequality when \( \bar{a} <_{\nu} \bar{b} \).

Likewise, by induction,
\[
|R(f_{\bar{u}_1}, g)| \cong_{\nu} \prod_{b} \left( |R(f_{\bar{u}_1}, g)| \prod_{a \neq \bar{u}} |R(f_{a}, g_b)| \right).
\]
But, by induction, \( |R(f_{\bar{u}_1}, g)| \cong_{\nu} \alpha_{m}^m \eta_{0}^{-1} \cong_{\nu} \beta_0^{-1} \) since \( \alpha_m \cong_{\nu} 1 \), so
\[
\beta_0 |R(f_{\bar{u}_1}, g)| \leq \nu |R(f_{\bar{u}_1}, g)|.
\]

Thus, we have proved inductively that \( \alpha_0 |R(f, g_1)| \leq \nu P \) with strict \( \nu \)-inequality if \( \bar{a} <_{\nu} \bar{b} \), and with equality if \( \bar{a} \cong_{\nu} \bar{b} \), and also that \( \beta_0 |R(f_{\bar{u}_1}, g)| \leq \nu P \). This establishes Equation (8.12). On the other hand, as noted in Lemma [8.12] it is easy to find the dominant term \( \alpha_{m}^m \beta_0^{-1} \) in \( R(f_a, g_b) \), so \( R(f_a, g_b) = \alpha_{m}^m \beta_0^{-1} \), and this term also occurs when we use the essential forms of \( f \) and \( g \). Equation (8.12) follows.  

**Corollary 8.21.** If \( f = \sum_{i=0}^{m} \alpha_i^i \) and \( g = \sum_{j=0}^{n} \beta_j^j \) have the property that \( \bar{a} <_{\nu} \bar{b} \), notation as in the theorem, then
\[
|R(f)| \cong_{\nu} \beta_0 |R(f_{\bar{u}_1}, g)|.
\]
if \( \bar{a} <_{\nu} \bar{b} \), then
\[
|R(f)| = \beta_0 |R(f_{\bar{u}_1}, g)|.
\]

**Proof.** Apply Equation (8.13) to Corollary 8.17. 

**Corollary 8.22.** If \( f = \sum_{i=0}^{m} \alpha_i^i \) and \( g = \sum_{j=0}^{n} \beta_j^j \) have the property that \( \bar{a} <_{\nu} \bar{b} \), notation as in the theorem, then
\[
|R(f)| = \beta_0 |R(f_{\bar{u}_1}, g)| = g(\bar{a})^m |R(f_{\bar{u}_1}, g)| = |R(f_{\bar{a}}, g)| |R(f_{\bar{u}_1}, g)|.
\]

**Proof.** Iterate Corollary 8.21.

**Corollary 8.23.** \( |R(f, \prod_b g_b)| \cong_{\nu} \prod_b f(\deg g_b) \), for any product of \( b \)-primary polynomials \( g_b \).

**Proof.** Apply Theorem 8.20 to the primary decompositions.

**Corollary 8.24.** If \( f = \prod_a f_a \) and \( g = \prod_b g_b \) written as products of primary polynomials, then
\[
|R(f, g)| \cong_{\nu} \prod_{a, b} \deg f_a \deg g_b.
\]

**Proof.** Replace \( f \) and \( g \) by their full forms.

**Theorem 8.25.** Any polynomials \( f, g, \) and \( h \) satisfy the relations:
\[
|R(f, gh)| \cong_{\nu} |R(f, g)||R(f, h)| \quad \text{and} \quad |R(gh, h)| \cong_{\nu} |R(g, h)||R(g, h)|.
\]

**Proof.** By symmetry, we need only prove the first equivalence. Taking a decomposition as in Remark 8.24, we write \( f = f_a f_{\bar{u}}, g = g_b g_{\bar{v}}, h = h_\ell h_{\bar{v}}, \) and we may assume that \( b \geq_{\nu} c \). We induct on the degree of \( f \).

In view of Theorem 8.20 factoring \( f \) and \( gh \) into their primary factors via [Sh1] Theorem 3.12 we may assume that \( f \) and \( gh \) are primary. But then we conclude by means of Lemma 8.12.

Note that Theorem 8.25 was proved independently of [IR7], so the layered structure actually gives a stronger result than in [IR7], and with a more direct proof. We can improve Theorem 8.20 in certain cases.

**Proposition 8.26.** For monic polynomials \( f \) and \( g \), assume that the corner roots of \( f \) are distinct from the corner roots of \( g \). Then:
\[
|R(f, g)| = \prod_{a, b} |R(f_a, g_b)|,
\]
where \( f = \prod_a f_a \) is the primary decomposition of \( f \) and \( g = \prod_b g_b \) is the primary decomposition of \( g \).
Proof. Let \( m_a = \deg f_a \) and \( m_b = \deg g_b \), and assume that \( \bar{a} \) (resp. \( \bar{b} \)) is the \( \nu \)-lowest of all the corner roots of the \( f_a \) (resp. of the \( g_b \)). To illustrate the idea, we first assume that \( \bar{a} <_\nu \bar{b} \). We apply Corollary \( \text{8.21} \) repeatedly, alternating from \( f \) to \( g \) when necessary, but at each time having a unique root of lowest \( \nu \)-value to remove.

This also gives us multiplicativity results.

**Corollary 8.27.** In case \( f, g \) and \( h \) have no corner roots in common,
\[
|\Re(f, gh)| = |\Re(f, g)||\Re(f, h)| \quad \text{and} \quad |\Re(fg, h)| = |\Re(f, h)||\Re(g, h)|. \tag{8.16}
\]

**Proof.** Write \( f = \prod_a f_a \) where the \( f_a \) are \( a \)-primary, and likewise \( g = \prod_b g_b \) and \( h = \prod_c h_c \). Then, in view of Proposition \( \text{8.20} \)
\[
|\Re(f, gh)| = \left| \Re\left( \prod_a f_a, \prod_{b,c} g_{b,c} \right) \right| = \prod_{a,b,c} |\Re(f_a, g_{b,c})| |\Re(f_a, h_{b,c})| = |\Re(f, g)| |\Re(f, h)|.
\]
The second assertion is proved analogously. \( \square \)

**Proposition 8.28.** Suppose \( f = \sum_{i=0}^{m} \alpha_i \lambda^i \) and \( g = \sum_{j=0}^{n} \beta_j \lambda^j \), where \( f \) is \( a \)-primary. If \( a \) is strictly dominated by all the corner roots of \( g \), then
\[
|\Re(f, g)| = \beta_0^m.
\]
If \( a \) strictly dominates all the corner roots of \( g \), then
\[
|\Re(f, g)| = \alpha_0^n.
\]

**Proof.** By Proposition \( \text{8.20} \) Letting \( \beta_{0,b} \) denote the constant term of \( g_b \), we have \( \beta_0 = \prod_b \beta_{0,b} \), and for the first assertion Lemma \( \text{8.12} \) then yields
\[
|\Re(f, g)| = \prod_b \beta_{0,b}^m = \beta_0^m.
\]
The second assertion is proved analogously. \( \square \)

**Lemma 8.29.** Suppose \( \bar{a} \) is smaller than any corner root of \( f \) or \( g \). Then
\[
|\Re((\lambda + \bar{a})f(\lambda + \bar{a})g)| = |\alpha_0 \beta_0| |\Re(f, g)|,
\]
where \( \alpha_0, \beta_0 \) are the respective constant terms of \( f \) and \( g \).

**Proof.** Note that the constant terms of \((\lambda + \bar{a})f(\lambda + \bar{a})g\) are respectively \( |\bar{a}| \alpha_0 \) and \( |\bar{a}| \beta_0 \). Thus, applying \( \text{8.12} \) on the left and then \( \text{8.13} \) on the right, taking Example \( \text{8.15} \) into account, we see that the two remaining dominant terms are \( |\bar{a}| \alpha_0 \beta_0 |\Re(f, g)| \) and \( |\bar{a}| \alpha_0 \beta_0 |\Re(f, g)| \), yielding the assertion. \( \square \)

The dual assertion holds when we take the resultant of two polynomials whose leading components are linear with the same \( \nu \)-values.

**Example 8.30.** In logarithmic notation, we compute \( |\Re(\lambda^2 + 5\lambda + 7, \lambda^2 + 4\lambda + 6)| \) to be
\[
\begin{array}{ccc}
7 & 5 & 0 \\
7 & 5 & 0 \\
6 & 4 & 0 \\
6 & 4 & 0 \\
\end{array}
\]
which is
\[
\begin{array}{ccc|ccc}
7 & 5 & 0 & +6 & 5 & 0 \\
4 & 0 & 0 & 6 & 4 & 0 \\
\end{array} = 7(5 \cdot 4 \cdot 0) + 6(5 \cdot 5 \cdot 0) = |\Re(f, gh)|.
\]

These polynomials factor to \((\lambda + 5)(\lambda + 2)\) and \((\lambda + 4)(\lambda + 2)\), so the lemma yields
\[
|\Re(f, gh)| = |\Re(f, g)||\Re(f, h)|. \tag{8.16}
\]

**Proposition 8.31.** Suppose any common corner root of \( f \) and \( gh \) is a simple root for each of \( f \) and \( gh \). (In particular, this is the case if \( f \) and \( gh \) are separable.) Then
\[
|\Re(f, gh)| = |\Re(f, g)||\Re(f, h)|.
\]
Proof. Factoring to primary polynomials, we may assume that \( f \) is \( a \)-primary and \( gh \) is \( b \)-primary. Write 
\[ f = \sum_{i=0}^{m} \alpha_i \lambda^i, \quad g = \sum_{i=0}^{n} \beta_i \lambda^i, \quad \text{and} \quad h = \sum_{i=0}^{t} \gamma_i \lambda^i. \]

If \( a < b \), then 
\[ |\mathcal{R}(f, gh)| = (|\mathcal{R}(f, g)|)^m = |\mathcal{R}(f, g)||\mathcal{R}(f, h)|. \]

If \( a > b \), then 
\[ |\mathcal{R}(f, gh)| = \alpha_0^{b+b} = |\mathcal{R}(f, g)||\mathcal{R}(f, h)|. \]

Thus we may assume that \( a \cong b \), so \( b \) is a simple root of \( gh \), implying by hypothesis that \( g \) or \( h \) is constant, and again we are done.

Our next obstacle is when \( f \) and \( g \) have \( \nu \)-equivalent corner roots of lowest \( \nu \)-value.

Lemma 8.32. For monic polynomials \( f = \sum_{i=0}^{m} \alpha_i \lambda^i \) and \( g = \sum_{i=0}^{n} \beta_i \lambda^i \), having the same root \( \bar{a} \) of lowest \( \nu \)-value, we have

\[ |\mathcal{R}(f, g)| = |\mathcal{R}(f_a, g_a)| |\mathcal{R}(f_{[m]}, g_{[n]}). \tag{8.17} \]

where \( m = \deg f_a \) and \( n = \deg g_a \).

Proof. By Corollary 8.22 we get \( \cong \nu \). We need to show that all the dominant terms of \( |\mathcal{R}(f, g)| \) come from
\[ |\mathcal{R}(f_a, g_a)| |\mathcal{R}(f_{[m]}, g_{[n]}). \]  
This was done in the simpler case where the roots of lowest \( \nu \)-value differ, in Corollary 8.22 and we want to adapt the proof of Corollary 8.22 to get equality. Although the spirit is exactly the same, the notation seems to be a bit more cumbersome, since Corollary 8.17 shows that we get more terms in the case at hand, which are not as easy to eliminate by the trick of Lemma 8.16.

The straightforward approach would be to apply the expansion along \( m + n \) columns simultaneously, rather than along the first column.

Namely, writing the Sylvester matrix \( \mathcal{R}(f, g) \) as \( \langle c_{i,j} \rangle \), any term in the summation of \( |\mathcal{R}(f, g)| \) has the form
\[ s_{\pi} = \prod_{i,j} c_{1,j_1} c_{2,j_2} \cdots c_{m+n,j_{m+n}}, \]
where \( j_i = \pi(i) \). We say that \( c_{i,j} \) has type 1 if \( i \leq \bar{m} \) and \( j_i \leq \bar{m} \) (in which case \( c_{i,j} \) is a coefficient of \( f_a \)), and \( c_{i,j} \) has type 2 if \( m < i \leq \bar{m} + \bar{n} \) (in which case \( c_{i,j} \) is a coefficient of \( g_a \)). We need that all dominant terms for \( |\mathcal{R}(f, g)| \) and \( b \) have types 1 or 2.

One may see this directly, but the notation is cumbersome. Instead, we take an inductive procedure, expanding Equation 8.7 along the first column, and note that for any term

\[ c = c_{1,j_1} c_{2,j_2} \cdots c_{m+n,j_{m+n}}, \tag{8.18} \]
with \( c_{n+2,j_1} = \beta_0 \) (along the \( n + 2 \) row) we have the corresponding term

\[ c_{1,j_1'} c_{2,j_2'} \cdots c_{m+n,j_{n+m}'} \tag{8.19} \]
where \( j_1' + 1 = 1 \) so that \( c_{1,j_1'} = \beta_1 \), and \( j_2' + 2 = j_2 + 1 \) so that \( c_{1+2,j_{2+1}} = c_{n+2,j_{n+1}} = \beta_{j_{n+1}(n)} \), and all the other \( j'_i = j_i \). Then the 8.19 is the same as 8.18 except that \( \beta_0 \beta_{j_{n+1}(n+1)} \) has been replaced by \( \beta_1 \beta_{j_{n+1}(n+2)} \), for which the \( \nu \)-value could only increase. If \( c \) is a dominant term for \( |\mathcal{R}(f, g)| \), then it has been associated with a dominant term for \( |\mathcal{R}(f, g)| \) coming from \( \beta_0|\mathcal{R}(f_{[1]}, g)| \), and now we can continue the induction to show that we only get dominant terms when we do the reduction along the smallest corner root at each stage, which means the first \( \bar{m} + \bar{n} \) reductions must be \( \bar{m} \) for \( f \) and \( \bar{n} \) for \( g \), as desired.

Theorem 8.33. For monic polynomials \( f \) and \( g \), we have

\[ |\mathcal{R}(f, g)| = \prod_{a,b} |\mathcal{R}(f_a, g_b)|, \tag{8.20} \]

where \( f = \prod_a f_a \) is the primary decomposition of \( f \) and \( g = \prod_b g_b \) is the primary decomposition of \( g \).

Proof. Iterate Lemma 8.32 and Proposition 8.28.

In view of Proposition 8.28, we know \( |\mathcal{R}(f_a, g_b)| \) except when \( a \cong \nu b \). Thus, the determination of \( |\mathcal{R}(f, g)| \) has been reduced to the case where \( f \) and \( g \) are both \( a \)-primary. This case has already been considered in Lemma 8.7, in which it was seen that \( |\mathcal{R}(f, g)| = \prod|\mathcal{R}(f_{[i]}, g)| a \prod \), which however is tricky to compute, as observed in Example 8.9 and the ensuing discussion.

Lemma 8.34. Suppose \( f = \sum_{i=0}^{m} k_i \lambda^{m-i} \) and \( g = (\lambda + [\ell]a)h \), where \( h = \lambda^n + \sum_{i=0}^{n-1} [k_i \lambda^{n-i} \lambda^i, \) with each \( \ell_i \geq 0 \). The permanent of the layer Sylvester matrix satisfies \(|\mathcal{L}(f, g)| \geq f(\ell')|\mathcal{L}(f, h)| \), where \( f = \sum_{i=0}^{m} k_i \lambda^i \in L[\lambda]. \)
Proof. Write $\tilde{h} = \lambda^n + \sum_{i=1}^{n-1} \ell_i \lambda^i$. Then $(\lambda + \ell')\tilde{h} = \lambda^{n+1} + \sum_{j=1}^{n-1} (\ell' \ell_j + \ell_{j-1}) \lambda^j + \ell_0$, so

$$|\mathfrak{L}(f, g)|_{\text{per}} = \begin{vmatrix} k_0 & k_1 & \cdots & k_m \\ k_0 & k_1 & \cdots & k_m \\ \ell' \ell_0 & \ell' \ell_1 + \ell_0 & \cdots & \ell' \ell_0 & \ell' \ell_1 + \ell_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{\text{per}} = k_0 + k_1 \ell' + k_2 \ell'^2 + \cdots.$$ 

Expanding the permanent along the first row shows that we get all the terms of $\sum_{i=0}^{m} k_i \ell'^i |\mathfrak{L}(f, h)|_{\text{per}}$ (and possibly others).

Proposition 8.35. Suppose $L = L_{\geq 0}$, and $f = \sum_{i=0}^{m} [k_i] a^{m-i} \lambda^i$ and $g = \lambda^n + \sum_{i=0}^{n-1} [\ell_i] a^{\ell_i} \lambda^i$. Write $\tilde{g} = \lambda^n + \sum_{j=1}^{n-1} \ell_j \lambda^j$, which as a polynomial over $C$ we factor as

$$\tilde{g} = \prod_{j=1}^{n}(\lambda - u_j).$$

Then the permanent of the layer Sylvester matrix satisfies $|\mathfrak{L}(f, g)|_{\text{per}} \geq \prod_{j=1}^{n-1} f(u_j)$, where $f = \sum_{i=0}^{m} k_i \lambda^i$.

Proof. Iterate Lemma 8.34.

The following question still remains.

Conjecture 8.1.

$$|\Re(f, gh)| = |\Re(f, g)||\Re(f, h)| \quad \text{and} \quad |\Re(f, gh)| \geq |\Re(f, h)||\Re(g, h)|.$$  \hfill (8.21)

Note that by Theorem 8.33 it is enough to show this in the case where $f, g, h$ are all $a$-primary for some $a$, in which case the conjecture reduces to proving that the layer Sylvester matrices satisfy $\mathfrak{L}(f, gh) \geq \mathfrak{L}(f, g)\mathfrak{L}(f, h)$. But the permanent is congruent to the determinant modulo 2. Thus, replacing the layers by indeterminates, we might hope that formally no term in $\mathfrak{L}(f, g)\mathfrak{L}(f, h)$ repeats. This is easy to see for $g$ or $h$ linear, but in general we have the following example.

Example 8.36. Suppose $f = [k_2] \alpha_2 \lambda^2 + [k_1] \alpha_1 \lambda + [k_0] \alpha_0$ and $g = [\ell_2] \beta_2 \lambda^2 + [\ell_1] \beta_1 \lambda + [\ell_0] \beta_0$. Then $|\mathfrak{L}(f, g)|_{\text{per}}$ is the permanent of the matrix

$$\begin{vmatrix} k_0 & k_1 & k_2 \\ k_0 & k_1 & k_2 \\ \ell_0 & \ell_1 & \ell_2 \\ \ell_0 & \ell_1 & \ell_2 \end{vmatrix}_{\text{per}},$$

which has $k_0k_2\ell_0\ell_2$ occurring twice.

We resolve the conjecture in the standard supertropical case:

Theorem 8.37. For $L = L_{\geq 0}$, any polynomials $f, g, h$ satisfy:

$$|\Re(f, gh)| = |\Re(f, g)||\Re(f, h)| \quad \text{and} \quad |\Re(f, gh)| \geq |\Re(f, h)||\Re(g, h)|, \quad \text{for } \ell \in \{1, 2, 3, 4\}. \hfill (8.22)$$

In particular, the conjecture holds when $L$ is truncated to level 4 or less (which includes the supertropical case).

Proof. By Theorem 8.33 we may assume that $f, g, h$ are $a$-primary. Then if $g$ or $h$ is linear we are done by Lemma 8.34 so we may assume that $g$ and $h$ are of degree $\geq 2$, and for deg $f \leq 4$ we are done by Example 8.11.
9. Layered derivatives and the discriminant

In this section we assume that $R$ contains a zero element $0_R$.

**Definition 9.1.** The layered derivative $f'$ of $f$ on $R[\lambda]$ is given by:

$$
\left( \sum_{j=0}^{n} [\ell_j] \alpha_j \lambda^j \right)' := \sum_{j=1}^{n} [\ell_j] \alpha_j \lambda^{j-1}.
$$

(9.1)

In particular, for $\alpha = [1] \in R_1$, 

$$
(\alpha \lambda^j)' := [j] \alpha \lambda^{j-1} \quad (j \geq 2), \quad (\alpha \lambda)' := \alpha, \quad \text{and} \quad \alpha' := 0_R.
$$

We have the familiar formulas:

1. $(f + g)' = f' + g'$;
2. $(fg)' = f'g + fg'$.

**Remark 9.2.** It is clear from (9.1) that if $f$ is essential then so is $f'$, since the coefficients have the same respective $\nu$-values (with the power of $\lambda$ decreased by 1).

**Note 9.3.** Sheiner [Sh2] has noted that the natural homomorphism from the polynomial semiring $\mathbb{P}_n$ in $n$ indeterminates to its image in $\text{Fun}(R(n), R)$ (viewing a polynomial as a function) does not commute with taking the layered derivative, even in the standard supertropical setting. For example, take $f_n = \lambda^2 + a\lambda + 2$. Then the $f_n$ are all equal as functions whenever $a \leq 1$. But $f_n = \lambda^n + a$, which all differ. Thus, when working with $R[\lambda]$, one needs to choose a particular polynomial representative in order for our definition of derivative to be well-defined. Fortunately, the natural candidate, the essential form of the polynomial, works well, by Remark 9.2.

**Example 9.4.** The layered derivative of an $a$-primary polynomial is an $a$-primary polynomial since the leading coefficient is multiplied by $[m] \mathbb{P}_R$. The ghost layer of $f'(a)$ can decrease dramatically when the ghost layer $\ell_0$ of the constant term is large enough.

**Remark 9.5.** Since the layered derivative of $[\ell] \alpha \lambda^m$ is $[m\ell] \alpha \lambda^{m-1}$, we would expect the anti-derivative of $[\ell] \alpha \lambda^m$ to be $[\frac{m-1}{\ell}] \alpha \lambda^{m+1}$. But in general this only makes sense when the sorting semiring $\mathbb{P}_L$ is $\mathbb{N}$-divisible, for example when $L = \mathbb{Q}_{>0}$.

The layered derivative of a separable polynomial can be factored rather easily when $L = \mathbb{Q}_{>0}$.

**Proposition 9.6.** If $f = (\lambda + a_m) \cdots (\lambda + a_1)$ is separable, then 

$$
f' = \left( \frac{m}{m-1} \right) \lambda + a_m \left( \frac{(m-1)/(m-2)}{\lambda + a_{m-1}} \cdots \left( \frac{2}{\lambda + a_2} \right) \right) = \prod_{k=2}^{m} \left( \frac{k}{k-1} \lambda + a_k \right).
$$

**Proof.** Write $f = \sum_{i=0}^{m} [\ell_i] \alpha_i \lambda^i$, and then using Equation (9.1), we write $f' = \sum_{i=1}^{m} [\ell_i] \alpha_i \lambda^{i-1}$ and factor it.

**Corollary 9.7.** The layered derivative of a separable polynomial is separable.

The layered derivative enables one to define the layered discriminant of a polynomial $f$ as the layered resultant $|\mathbb{R}(f, f')|$ of $f$ and $f'$. Although the discriminant may be difficult to compute in general, it is easy for separable polynomials in view of Propositions 8.31 and 9.6, for which ghost layer depends only on the degree of $f$ and not on the particular corner roots. Thus, we can determine whether or not a polynomial is separable by checking the layer of its layered discriminant.

**Theorem 9.8.** The sort of the layered discriminant of a tangible separable polynomial $f(\lambda)$ of degree $m$ is $m^{m-1} \prod_{k=2}^{m} (2k-1)^{(k-1)}$.

**Proof.** Write $f = (\lambda + a_m) \cdots (\lambda + a_1)$; then $f' = \prod_{k=2}^{m} (\frac{k}{k-1} \lambda + a_k)$. Now,

$$
s \left( |\mathbb{R}(\lambda + a_j, \frac{k}{k-1} \lambda + a_k) | \right) = \begin{cases} 
\frac{j}{j-1}, & j > k; \\
\frac{k}{k-1} + 1, & j = k; \\
1, & j < k.
\end{cases}
$$

For each $k$, taking the product as $j$ runs from 1 to $m$ gives $\frac{m}{m} (\frac{k}{k-1} + 1) \cdots 1 = \frac{m}{m} + \frac{m}{m} = \frac{(2k-1)m}{k(k-1)}$. □
When the polynomial is not separable, its multiple root makes the sort higher. Thus, the existence of a multiple corner root of a polynomial $f$ can be recognized via the ghost layer of its layered discriminant, without actually computing the roots of $f$.

10. Major examples of layered domains
d
Having set forth the general layered structure, we pause to indicate how specific examples for $L$ fit into this context. For convenience we usually take the sorting set $L$ to be a semiring with $L = L_+$. One could also take the more general case where $L = L_+ \cup \{0\}$. We consider how the choice of the sorting semiring $L$ affects the mathematical structure of $R$. In all but the last example, $L$ is totally ordered, and usually non-negative.

10.1. Examples of uniform $L$-layered domains for $L$ totally ordered under $\geq$. Since the main goal of this paper is to enrich the ghost structure, we turn first to describe various choices of the $L$-layered domain construction here. We list them and indicate their strengths and weaknesses.

10.1.1. Examples for $L = L_{\geq 1}$. The following examples are relatively easy to describe, and suit most of our purposes. We already considered the max-plus situation in Example 3.10. In all other examples, we assume that $1 + 1 > 1$ in $L$.

Example 10.1. Taking $L = \{1, \infty\}$ yields the (standard) supertropical domains, as noted above in Example 3.11. Although this structure has nice properties obtained in [IR3], [IR4], [IKR4], and [IKR3], especially in connection with linear algebra, it has the deficiency that some basic algebraic properties fail such as unique factorization of polynomials (cf. [IR1] Theorem 8.53), and other properties (such as [IR1] Lemma 2.2) require case-by-case analysis. Furthermore, since roots of polynomials are defined in terms of ghost values and there is only one layer of ghosts, it is difficult to study multiple roots of polynomials.

Differentiation of polynomials is not very useful in this setting, as explained in [4].

Remark 10.2. The standard supertropical case (Example 3.11) does have one advantage over the general $L$-layered structure it satisfies the Frobenius property $(a+b)^m = a^m + b^m$ for all $m$, whereas for the general $L$-layered structure this holds only up to $\nu$-equivalence.

Example 10.3. In Example 3.11, we could take instead $L = \{1, 2\}$ with $1 < 2$, $1 \cdot 1 = 1$, and all other sums and products are 2, i.e., formally replacing the index $\infty$ by 2. This fits in better with our notation of truncation (4.7), as to be explained shortly.

Example 10.4. More generally, we may choose the $q$-truncated semiring

$$L = [1, q] := \{1, 2, \ldots, q\}$$

of Example 2.14. The elements of $R_q$ are all $q$-ghosts. We then obtain an $L$-layered domain with $q$ layers. We recover Example 10.3 when $q = 2$.

Remark 10.5. Note in Example 10.4 that $q$ takes on the role of the infinite element in $L$. Thus, we may relabel $q$ as $\infty$, and call this semiring

$$L = [1, q - 1]^\infty := \{1, 2, \ldots, q - 1, \infty\}.$$

This notation corresponds better with Example 3.11.

When $q > 2$ we can describe when $a \in R^{(n)}$ is a multiple root of a tangible polynomial $f$, in terms of when $f(a)$ is 2-ghost.

Remark 10.6. The Frobenius property $(a+b)^m = a^m + b^m$ does hold in the $q$-truncated domain for all $m \geq q$, since then both $(a+b)^m$ and $a^m + b^m$ have maximal possible layer $q$.

The truncated layered domain generalizes the situation given in [CC], as follows.

Proposition 10.7. (Notation as in Remark 3.53) Suppose $R$ is a $q$-truncated layered domain and also satisfies the property:

$$\nu_{q,k}(a) = e_q \iff a \cong _\nu 1_R, \quad \forall a \in R_k.$$  

(For example, this holds when $R$ is uniform.) Then $s^{q-1}(a) = s^q(a)$ for all $a \cong _\nu 1_R$, implying $s^{q+1} = s^q$. 

Proof. If \( a \cong_\nu 1_R \), then \[ \tilde{s}^{a-1}(a) = e_q = e_{q+1} = \tilde{s}(e_q) = \tilde{s}(\tilde{s}^{a-1}(a)) = \tilde{s}^q(a). \]

But in view of Remark 10.1, if \( a \not\cong_\nu 1_R \), then either \( \tilde{s}(a) = a \), in which case \( \tilde{s}^{a+1}(a) = \tilde{s}^q(a) \), or \( \tilde{s}(a) = 1 \), in which case \( \tilde{s}^{a+1}(a) = \tilde{s}^q(1_R) = \tilde{s}^{a-1}(1_R) = \tilde{s}^q(a) \). \( \Box \)

Example 10.8. Taking \( L = \mathbb{N} \) enables us to deal with arbitrary multiplicities of corner roots, and also deal with layered derivatives, since we can apply the formula \( \tilde{s}^q \). Thus, this situation is useful for studying geometry.

There are difficulties from the algebraic perspective. We still have irreducible non-primary polynomials such as \( [1]0 \lambda^3 + [2]2 \lambda + [1]3 \). Unique factorization of polynomials still fails in one indeterminate, cf. \( \text{Sh1} \).

Also, often one cannot integrate since the antiderivative \( [\pm]1 \lambda^{m+1} \) of \( [1] \lambda^m \) described in Remark 9.4 does not exist unless \( m \) divides \( t \).

10.1.2. Examples for \( L \neq L_{\geq 1} \). We expand \( L \) further, to handle more sophisticated mathematical analysis, such as integration.

Example 10.9. Suppose \( L = \{ \frac{m}{n} : m, n \in \mathbb{N} \} \). Sheiner \( \text{Sh1} \) has pointed out that the polynomial

\[ \lambda^2 + [2]b + [1]ab \quad a < \nu b, \]

which is irreducible over the (standard) supertropical semiring, now has the factorization

\[ \lambda^2 + [2]b + [1]ab = (\lambda + [\pm]a)(\lambda + [2]b), \quad (10.1) \]

which enables one to resolve the different factorizations in \( R[\lambda] \) given in \( \text{IR1} \), Example 8.38(ii). On the other hand, as noted above, unique factorization of primary polynomials still fails.

Note that the ghost elements no longer form an ideal, since we have elements of layer \( < 1 \). The ghost valuation semiring \( \delta \) corresponds to \( L = \{ \frac{m}{n} : m \geq 2^n \} \), which contains the sub-semiring \( \mathbb{N} \).

The situation is even better when \( L \) is a multiplicative group.

Example 10.10. Taking \( L = \mathbb{Q}_{\geq 0} \) enables us to factorize polynomials in one indeterminate uniquely into primary polynomials, as described in Theorem 7.23. Unique factorization of \( \mathbb{Q}_{\geq 0} \)-layered primary polynomials into irreducibles almost always holds, the only exception involving the \( 0 \)-layer, occurring either in the leading monomial or the lowest order monomial, cf. \( \text{Sh1} \).

The ghost valuation semiring \( \delta \) corresponds to \( L = \mathbb{Q}_{\geq 1} \), which contains the sub-semiring \( \mathbb{N} \), and this example should be a useful tool in geometric applications. Also, one can integrate in this setting, since the antiderivative of Remark 9.4 now makes sense.

The situation in several variables is not yet understood completely, because of subtleties in the geometry. Sheiner (cf. \( \text{Sh1} \)) gives an example of a polynomial with multiple factorizations; this corresponds to a tropical hypersurface which can be decomposed into different unions of irreducible supertropical hypersurfaces, even taking layers into account.

Example 10.11. One could also take \( L = \mathbb{R}_{\geq 0} \), which provides better factorizations of some primary polynomials, although we do not yet see much advantage over Example 10.10. In this case, the ghost valuation semiring \( \delta \) corresponds to \( L = \mathbb{R}_{\geq 1} \).

10.2. Non-positive examples. Other relevant examples are more esoteric.

Example 10.12. Taking \( L = \mathbb{R} \) provides unique factorization of primary polynomials into linear and quadratic factors, but polynomials having a \( 0 \) component need not be factorizable into primary polynomials. One could interpret negative layers as “antilayers,” since \( [0]a + [-0]a = [0]a \in R_0 \).

Example 10.13. Taking \( L = \mathbb{C} \) provides unique factorization of primary polynomials into linear factors, at the cost of losing the total order of the reals. (Note that \( \mathbb{C} \) could be pre-ordered via the absolute value.)

We expect this example to be a useful intermediate tool in tropical calculus, since one can pass later to the layered sub-domain \( L_{\geq 1} \).

Example 10.14. More generally, Sheiner \( \text{Sh2} \) has an interesting example taking \( L = F \), notation as in Example 4.27. Let \( R = R(L, \mathcal{G}) \), and define the map \( K \rightarrow R \) by \( p \mapsto [b]_\alpha(p) \) where \( \alpha \) is the coefficient of the lowest monomial of the Puiseux series \( p \). This map, generalizing the Kapranov map, keeps track of the “leading coefficient” of the Puiseux series \( p \) in terms of when the image of \( p \) has layer \( 0 \), and provides a layered version of \( \text{Par} \), as to be explained further in Appendix A.
Example 10.15. If one is willing to forego integration, we could take \( L \) to be a finite field, with the trivial pre-order.

11. Appendix A: Layered domains with symmetry, and patchworking

Akian, Gaubert, and Guterman [AGG, Definition 4.1] introduced an involutory operation on semirings, which they call a symmetry, to unify the supertropical theory with classical ring theory. In this appendix, we put their symmetry in the context of \( L \)-layered domains, where here \( L \) is partially pre-ordered. The main example for our construction is the “patchworking” given in Example 11.6 below.

Definition 11.1. A negation map on a semiring \( L \) is a function \( \tau : L \rightarrow L \) satisfying the properties:

\[
\begin{align*}
\text{N1. } \tau(k\ell) &= \tau(k)\ell = k\tau(\ell); \\
\text{N2. } \tau^2(k) &= k; \\
\text{N3. } \tau(k + \ell) &= \tau(k) + \tau(\ell).
\end{align*}
\]

Suppose the semiring \( L \) has a negation map \( \tau \) of order \( \leq 2 \). We say that an \( L \)-quasi-layered domain \((R, \tau, \sigma)\) is an \((L, \tau)\)-quasi-layered semiring with symmetry \( \sigma \) when \( R \) is a semiring together with a map

\[
\sigma : R \rightarrow R
\]

and a negation map \( \tau \) on \( L \), together with the extra axiom (for all \( a \in R_k, b \in R_\ell \)):

\[
\text{S1. } s(\sigma(a)) = \tau(s(a)).
\]

Note that when \( \sigma \) and \( \tau \) are the identity maps, we are back to \( L \)-layered domains.

Remark 11.2. One big advantage of the symmetry is that it enables one to return to a more classical definition of determinant of a matrix \( A = (a_{ij}) \), defined as

\[
\sum_{\pi \in S_n} \sigma(\text{sgn}(\pi))a_{\pi(1)} \cdots a_{\pi(n)},
\]

(11.1)

Remark 11.3. Concerning truncation in the context of symmetries, we observe briefly that when \( L \) has a given negation map \( \tau \), we should require our upper ideal \( Q \) to be \( \tau \)-invariant; i.e., \( \tau(Q) \subseteq Q \). Then \( \tau \) induces a negation map on \( L \), which can be used to define a natural symmetry on \( \overline{L} \), and a truncation that works in parallel to Definition 4.3.

11.1. Examples of \((L, \tau)\)-layered domains with symmetry.

Example 11.4. Whenever \( L \) is a ring, one can define a negation map by putting \( \tau(\ell) = -\ell \), and then define the layered symmetry via \( \sigma(\ell a) = [-\ell]a \cdot \). Applying this to Example 11.12, Sheiner [S12] has exploited Remark 11.3 to study the linear algebra of this structure via the 0-layer. For example, a matrix is singular iff (11.2) has layer 0.

We conclude this appendix with an example motivated from Viro’s theory of patchworking in tropical geometry, as developed in [IMS, Chapter 2], in which the sorting semiring \( L \) is more intricate, with a partial order which is not total.

Example 11.5. Suppose \( L \) is an ordered semiring. We mimic the construction of \( \mathbb{Z} \) from \( \mathbb{N} \). Define the doubled semiring

\[
D(L) = L_1 \times L_{-1},
\]

the direct product of two copies \( L_1 \) and \( L_{-1} \), where addition is defined componentwise, but multiplication is given by

\[
(k, \ell) \cdot (k', \ell') = (kk' + \ell\ell', k\ell' + \ell k').
\]

In other words, \( D(L) \) is multiplicatively graded by \( \{\pm 1\} \).

\( D(L) \) is endowed with the product partial order, i.e., \((k', \ell') \geq (k, \ell)\) when \( k' \geq k \) and \( \ell' \geq \ell \). To see this, note that if \((k', \ell') \geq (k, \ell)\), then multiplying by \((m, n)\) gives

\[
(k'm + \ell'n, k'n + \ell'm) \geq (km + \ell n, kn + \ell m).
\]
Furthermore, $D(L)$ has the negation map $\tau$ of order 2, given by $\tau(k, \ell) = (\ell, k)$.

In case $L$ is truncated, as in Example 11.3, with maximal element $n$, then $(n, n) \geq (k, \ell)$ for all $k$ and $\ell$, so $(n, n)$ is the unique maximal element of $D(L)$. On the other hand, one could take infinitely many layers, such as $L = \mathbb{N}$ as in Example 11.5.

Here is the $D(L)$-layered domain† with symmetry of greatest interest to us.

**Example 11.6.** Suppose $\mathcal{G}$ is an ordered abelian monoid, viewed as a semiring† as in Construction 3.2. Define the double layered domain†

$$R = R(D(L), \mathcal{G}) = \{((k, \ell), a) : (k, \ell) \neq (0, 0), a \in \mathcal{G}\},$$

but with addition and multiplication given by the following rules:

$$(k, \ell), a + ((k', \ell'), b) = $$

- if $a > b$, then $((k, \ell), a)$
- if $a < b$, then $((k', \ell'), b)$
- if $a = b$, then $((k + k', \ell + \ell'), a)$

$$(k, \ell), a \cdot ((k', \ell'), b) = $$

- if $a > b$, then $((kk' + \ell'k + k\ell'), ab)$

One can check routinely that this is a commutative semiring†. When $L = \{1, \infty\}$, we note that

$$D(L) = \{(1, 1), (1, \infty), (\infty, 1), (\infty, \infty)\},$$

which is applicable to Viro’s theory of patchworking, where the “tangible” part could be viewed as those elements of layer $(1, 1), (1, \infty),$ or $(\infty, 1)$. Explicitly, comparing with Viro’s use of hyperfields in [Vir § 3.5], we can identify these three layers respectively with $0, 1,$ and $-1$ in his terminology, and the element $(\infty, \infty)$ with the set $\{0, 1, -1\}$.

**Remark 11.7.** In the doubled layered domain† $R = R(D(L), \mathcal{G})$, we consider the symmetry $\sigma : R \to R$ given by $\sigma : ((k, \ell), a) \mapsto ((\ell, k), a)$. This symmetry is analogous to the one described in [AGG], and behaves much like the negation.

**Note 11.8.** When the order on $L$ is only partial, $L$ could have several multiplicative idempotents other than 1 and $\infty$, cf. Example 11.6. Thus, one would want to define tangible elements more generally, in terms of these idempotents, and Lemma 3.40 needs to be modified. Otherwise, the theory pretty much follows the same lines given there.

12. **Appendix B: Weakening the structure of $L$ and $R$**

Strictly speaking, we have only generalized the notion of supertropical domain†, not supertropical semiring, since Axiom A2 says that $a, b \in R_1$ implies $ab \in R_1$. We take a brief excursion to consider a slight generalization that covers this case also.

**Note 12.1.** To generalize the notion “supertropical semiring” from the standard supertropical theory, we would weaken Axiom A2 to:

wA2. If $a \in R_k$ and $b \in R_\ell$, then $ab \in R_m$ for some $m \geq k\ell$.

Now we have to modify Axiom A3 to make it compatible; i.e., multiplication commutes with the sort transition maps. Technically, this says:

wA3. If $a \in R_k$ and $a' \in R_{k''}$, with $aa' \in R_{k''}$ and $\nu_{k, k'}(a) \cdot \nu_{k', k''}(a') \in R_{k''}$, and $\nu_{m, \ell}(a) \cdot \nu_{m', \ell'}(a'') \in R_{m''}$, for $m \geq \ell$, $m' \geq \ell'$, and $m'' \geq mm'$, then $\nu_{q, \ell''}(aa') = \nu_{q, \ell''}(\nu_{m, \ell}(a) \cdot \nu_{m', \ell'}(a'))$ for all $q \geq \ell''$, $m''$.

This weakening is of arithmetic interest, since we now have a version of Example 3.31 without requiring a zero layer.
Example 12.2 (The weakly layered truncated semiring). Suppose \( R \) is \( L \)-quasi-layered. Fix \( q > 0 \), and for any semiring \( L \) we formally adjoin an infinite sort \( \infty \), letting \( L_{\infty} = L \cup \infty \). Define
\[
\hat{R}(L_{\infty}, [1, q]) := \{ [k]a : k \in L, \ a \in \{1, \ldots, q - 1\}\} \cup \{ [\infty]q\},
\]
where addition is defined as in Construction 3.2, and the product \([k]a \cdot [\ell]b\) is given as in Equation 3.1 except for \( ab = q \), in which case \([k]a \cdot [\ell]b = [\infty]q\) for any \( k, \ell \in L \). Addition and multiplication by \([\infty]q\) are given by:
\[
[k]a + [\infty]q = [\infty]q = [k]a [\infty]q.
\]
One checks as before that \( \hat{R}(L_{\infty}, [1, q]) \) is indeed a semiring. The sort transition maps are as in Construction 3.2, except that we define \( \nu_{\infty,k}([k]a) = [\infty]q \) for all \((k, a)\). Thus, \([\infty]q\) is the special infinite element.

When we forego \( \nu \)-bipotence, we do not need \( L \) to be a semiring, but merely a directed, partially pre-ordered multiplicative monoid (without addition). Although this material is not needed for our current applications to tropical mathematics, it yields an intriguing parallel between the semiring \( L \) and the sorting set \( L \) (since any ordered monoid becomes a semiring when addition is taken to be the maximum), and may provide guidance for future research.

Remark 12.3. Since \( L \) now is only assumed to be a multiplicative monoid, we need to remove references to addition in \( L \). Thus, we need a formal “doubling function” \( \ell \mapsto 2\ell \) on \( L \), and use strong ghosts, eliminate Axiom A4, and weaken Axiom B to:

\[
\text{wB. (weak supertropicality)} \quad \text{If} \ a \in R_k \text{ and } b \in R_\ell \text{ with } a \parallel b, \text{ then } a + b \in R_m \text{ for some } m \geq k, \ell, \min\{2k, 2\ell\} \text{ with } a + b \equiv_{\nu} b.
\]

It is easy to check that the sorting map \( s : R \to L \) still exists and satisfies Equation 3.4.

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