Moduli of holomorphic functions and logarithmically convex radial weights

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Abstract
Let $\mathcal{H}o\ell(D)$ denote the space of holomorphic functions on the unit disk $D$. We characterize those radial weights $w$ on $D$ for which there exist functions $f, g \in \mathcal{H}o\ell(D)$ such that the sum $|f| + |g|$ is equivalent to $w$. Also, we obtain similar results in several complex variables for circular, strictly convex domains with smooth boundary.

1. Introduction
1.1. Growth spaces
Let $\mathcal{H}o\ell(D)$ denote the space of holomorphic functions on the unit disk $D$. Consider a weight function $w$, that is, a non-decreasing, continuous, unbounded function $w : [0, 1) \to (0, +\infty)$. We extend $w$ to a radial weight on $D$ setting $w(z) = w(|z|), z \in \mathbb{D}$. By definition, the growth space $A^w(D)$ consists of $f \in \mathcal{H}o\ell(D)$ such that
\[
|f(z)| \leq Cw(z), \quad z \in \mathbb{D},
\]
for some constant $C > 0$.

Given functions $u, v : \mathbb{D} \to (0, +\infty)$, we write $u \asymp v$ and we say that $u$ and $v$ are equivalent if
\[
C_1u(z) \leq v(z) \leq C_2u(z), \quad z \in \mathbb{D},
\]
for some constants $C_1, C_2 > 0$. The definition of equivalent weight functions is analogous. If $w_1$ and $w_2$ are equivalent radial weights, then the identities
\[
\|f\|_{A^{w_j}(D)} = \sup_{z \in \mathbb{D}} \frac{|f(z)|}{w_j(z)}, \quad j = 1, 2,
\]
define equivalent norms on the Banach space $A^{w_1}(\mathbb{D}) = A^{w_2}(\mathbb{D})$.

1.2. Motivations
In various applications, it is useful to have test functions $f \in A^w(D)$ for which the reverse of estimate (1.1) holds, in a sense. The starting point for the present paper is the following reverse estimate obtained by Ramey and Ullrich [23] with the help of lacunary series.

Theorem 1.1 (cf. [23, Proposition 5.4]). Let $w(t) = 1/(1 - t), \; 0 \leq t < 1$. There exist functions $f_1, f_2 \in A^w(D)$ such that
\[
|f_1(z)| + |f_2(z)| \geq w(|z|)
\]
for all $z \in \mathbb{D}$.
One may consider Theorem 1.1 as a particular solution of the following approximation problem.

Given a radial weight \( w \) on \( D \), find \( f_1, f_2 \in \mathcal{H}(D) \) such that
\[
|f_1| + |f_2| \asymp w.
\]
In other words, we are looking for a holomorphic mapping \( f : D \to \mathbb{C}^2 \) such that \( \|f\| \asymp w \).

The above problem has been solved recently for various explicit radial weights; see, for example, [11–13, 17, 27]. Clearly, the required property still holds if \( w \) is replaced by an equivalent radial weight. To the best of our knowledge, the largest class of weight functions \( w(t) \) is considered in [1], where the direct analog of Theorem 1.1 is proved under assumption that \( w(t) \) has the following doubling property:
\[
w(1 - s/2) < A w(1 - s), \quad 0 < s < 1,
\]
for some constant \( A > 1 \). Essentially, the same result was independently obtained in [19]. As in [23], the arguments in [1, 19] use lacunary series; see [14] for a different proof.

Basically, property (1.2) means that \( w(t) \) grows sufficiently slowly as \( t \to 1 - \). So, it is natural to ask whether a growth restriction is crucial for the corresponding results. In the present paper, we characterize those \( w \) for which the approximation problem under consideration is solvable.

1.3. Main results

Let \( w : [0, 1) \to (0, +\infty) \) be a weight function. Often, \( w \) is called log-convex if \( \log w \) is convex. In this paper, we use a different definition related to Hadamard’s three-circles theorem. Namely, \( w \) is said to be log-convex if \( \log w(t) \) is a convex function of \( \log t \), \( 0 < t < 1 \). If \( \log w(t) \) is a convex function of \( t \), then \( w \) is log-convex, but the converse is not true; see Remark 6.

1.3.1. Unit disk.

**Theorem 1.2.** Let \( w \) be a radial weight on \( D \). Then the following properties are equivalent:
\[
\begin{align*}
&\text{there exist } f_1, f_2 \in \mathcal{H}(D) \text{ such that } |f_1| + |f_2| \asymp w; \quad (1.3) \\
&w(t) \text{ is equivalent to a log-convex weight function on } [0, 1). \quad (1.4)
\end{align*}
\]

In other words, Theorem 1.2 characterizes those \( w \) for which an analog of Theorem 1.1 holds.

It is worth mentioning the ideas behind the proof of Theorem 1.2. The implication (1.3)⇒(1.4) is deduced from Hadamard’s three-circles theorem and properties of log-convex functions. To prove the reverse implication, we use a geometric construction that generates \( f_1 \) and \( f_2 \) as appropriate power series. The construction is based on approximation of convex functions of one real variable by piecewise linear functions. Similar ideas were used in Borichev’s work [7] on Fock-type spaces. Note that the series used in [1, 19, 23] are Hadamard lacunary. In contrast, for rapidly growing weight functions, we obtain weakly lacunary series: the corresponding frequencies grow slower than any geometric progression.

**Remark 1.** Condition (1.4) is not a growth restriction: given a weight function \( v \), there exists a log-convex weight function \( w \) such that \( w(t) \asymp v(t), \ 0 \leq t < 1 \). As mentioned above, (1.4) is a natural regularity condition.

**Remark 2.** Theorem 1.2 guarantees that property (1.4) also characterizes those radial weights that are equivalent to \( \max(|f_1|, |f_2|) \) for some \( f_1, f_2 \in \mathcal{H}(D) \).
Remark 3. Given an integer $M$, one may consider the following problem: Characterize those radial weights $w$ for which there exist $f_1, \ldots, f_M \in \text{Hol}(D)$ such that
\[
|f_1| + \cdots + |f_M| \asymp w.
\]
The problem degenerates for $M = 1$: by the maximum modulus principle, no radial weight is equivalent to the modulus of a holomorphic function. For $M = 2, 3, \ldots$, the solution of the problem does not depend on $M$ and it is given by (1.4); see Remark 5.

1.3.2. Circular domains. We extend Theorem 1.2 to certain circular domains in $\mathbb{C}^d, d \in \mathbb{N}$. In particular, an analog of Theorem 1.2 holds for the unit ball of $\mathbb{C}^d$.

Assume that $D \subset \mathbb{C}^d$ is a bounded, circular, strictly convex domain with $C^2$-boundary. Let $r_D(z)$ denote the Minkowski functional on $D$, that is,
\[
r_D(z) = \inf\{\rho > 0 : z \in \rho D\}.
\]
If $D$ is the unit ball, then $r_D(z) = |z|$.

Let $\text{Hol}(D)$ denote the space of holomorphic functions on $D$. Given a weight function $w : [0, 1) \to (0, +\infty)$, the growth space $A^w(D)$ consists of $f \in \text{Hol}(D)$ such that
\[
\|f\|_{A^w(D)} = \sup_{z \in D} \frac{|f(z)|}{w(r_D(z))} < \infty.
\]
We obtain the following extension of Theorem 1.2 to several complex variables:

**Theorem 1.3.** Let $D \subset \mathbb{C}^d$ be a bounded, circular, strictly convex domain with $C^2$-boundary. If $w : [0, 1) \to (0, +\infty)$ is a log-convex weight function, then there exist functions $f_m \in A^w(D), 1 \leq m \leq M = M(D)$ such that
\[
\sum_{m=1}^{M} |f_m(z)| \geq w(r_D(z)), \quad z \in D.
\]
Conversely, if (1.5) holds for some $M \in \mathbb{N}$ and some $f_1, \ldots, f_M \in \text{Hol}(D)$, then $w(t)$ is equivalent to a log-convex weight function.

Remark 4. Recall that, by Theorem 1.2 and Remark 3, the optimal value of $M$ for $D$ is equal to 2. It would be interesting to find the optimal value of $M(B_d)$ for the unit ball $B_d$ of $\mathbb{C}^d, d \geq 2$.

1.4. Applications

We consider composition and multiplication operators, Volterra-type operators and extended Cesàro operators on $A^w(D)$.

1.5. Organization of the paper

Section 2 is devoted to the proof of Theorem 1.2. Circular domains in $\mathbb{C}^d$ are considered in Section 3. Applications are discussed in the final Section 4; main applications are related to the unit ball of $\mathbb{C}^d, d \geq 1$.

2. $D$ is the unit disk

In this section, we prove Theorem 1.2.
2.1. Equivalence to a log-convex weight function is necessary

Given a continuous function \( u : D \to (0, +\infty) \), put
\[
M_u(r) = \max_{|z|=r} u(z), \quad 0 \leq r < 1.
\]

Now, suppose that \( w \) is a radial weight, \( f_1, f_2 \in \mathcal{H}ol(D) \) and \( |f_1| + |f_2| \asymp w \). Without loss of generality, we also assume that \( f_j(0) \neq 0 \) for \( j = 1, 2 \). By Hadamard’s three-circles theorem, \( M_{|f_1|} \) and \( M_{|f_2|} \) are log-convex functions on the interval \([0,1)\); see, for example, [8, Chapter 6, Subsection 3.13]. Hence, the sum \( M_{|f_1|} + M_{|f_2|} \) is also log-convex (see, for example, [18, p. 51]).

Observe that
\[
M_{|f_j|} \leq M_{|f_1|+|f_2|} \leq M_{|f_1|} + M_{|f_2|}, \quad j = 1, 2.
\]

Thus, we have the following equivalences on \([0,1)\):
\[
w \asymp M_{|f_1|+|f_2|} \asymp M_{|f_1|} + M_{|f_2|}.
\]

Since the function \( M_{|f_1|} + M_{|f_2|} \) is log-convex, the implication (1.3)\( \Rightarrow \) (1.4) is proved.

**Remark 5.** Let \( w \) be a radial weight such that
\[
w \asymp |f_1| + \cdots + |f_K|
\]
for some \( f_1, \ldots, f_K \in \mathcal{H}ol(D), \ K \in \mathbb{N}. \) Repeating the arguments used above for \( K = 2 \), we deduce that \( w(t) \) is equivalent to a log-convex weight function.

The rest of the section is devoted to a constructive proof of the implication (1.4)\( \Rightarrow \) (1.3).

2.2. Log-convex weight functions: preliminaries

Let \( w : [0,1) \to (0, +\infty) \) be a log-convex weight function. Recall that, by definition, \( \log w(t) \) is a convex function of \( \log t \), that is,
\[
\Phi(x) = \Phi_w(x) = \log w(e^x), \quad x \in (-\infty, 0),
\]
is a convex function.

**Remark 6.** It is natural to compare the above property and the following one: \( \log w(t) \) is a convex function of \( t \). The latter property implies that \( \Phi_w \) is convex, as the composition of two increasing convex functions. The reverse implication does not hold. Moreover, there exist log-convex weight functions that are not even equivalent to the exponent of a convex function.

In what follows, we argue in terms of the function \( \Phi \). Observe that \( w(t) = \exp(\Phi(\log t)) \), \( 0 < t < 1 \).

To prove the implication (1.4)\( \Rightarrow \) (1.3), we may replace \( w \) by an equivalent weight function. So, without loss of generality, we assume that \( \Phi \) is a strictly convex \( C^2 \)-function. In particular, the tangent to the graph of \( \Phi \) is unique at each point \((x, \Phi(x))\), \( x < 0 \). Also, below we repeatedly use the following property without explicit reference: the slope of the tangent to the graph of \( \Phi \) at \((x, \Phi(x))\) is a strictly increasing function of \( x \in (-\infty, 0) \).
Finally, note that \( \log v(e^x) = \log a + \beta x \) for \( v(t) = at^\beta, a > 0, \beta > 0 \). This observation allows one to reduce the proof of the implication (1.4) \( \Rightarrow \) (1.3) to certain manipulations with linear functions.

2.3. Basic induction construction

Fix a number \( x_0 \in (-\infty, 0) \) and a parameter \( h > 0 \). By induction, we construct linear functions \( \ell_k(x) \) and numbers \( x_k \in (x_0, 0), \) \( k = 1, 2, \ldots, \) such that

(i) \( \ell_k(x) \) is a tangent to the graph of \( \Phi(x) \);

(ii) \( \ell_k(x) \) intersects the graph of \( \Phi(x) - h \) at the points whose \( x \)-coordinates are \( x_{k-1} \) and \( x_k, x_{k-1} < x_k \).

Note that \( \ell_k(x) \) and \( x_k \) are uniquely defined by the above properties. So, the induction construction proceeds; see Figure 1.

Given \( \ell_k(x) \), we define parameters \( a_k > 0 \) and \( \beta_k > 0 \) by the following identity:

\[
\ell_k(x) = \log a_k + \beta_k x, \quad x \in \mathbb{R}.
\]

Note that the sequences \( \{x_k\}_{k=0}^\infty \) and \( \{\beta_k\}_{k=1}^\infty \) monotonically increase, \( x_k \to 0 \) and \( \beta_k \to \infty \) as \( k \to \infty \). Also, we use the following brief notation: \( t_k = \exp(x_k), k = 0, 1, \ldots \). Hence, the positive numbers \( t_k \) monotonically increase to 1 as \( k \to \infty \).

Formally, the above construction works for any \( h > 0 \). In applications, we use a sufficiently large parameter \( h \), say, \( h = 2, \) in the case of the unit disk.

2.4. Auxiliary estimates

**Lemma 2.1.** Let the numbers \( x_k, k = 0, 1, \ldots, \) and the linear functions \( \ell_k, k = 1, 2, \ldots, \) be those introduced in Subsection 2.3. Then

\[
\ell_k(x) \geq \ell_{k+1}(x) + h \quad \text{for all } x, x_0 \leq x \leq x_{k-1}, \quad k \geq 1; \tag{2.1}
\]

\[
\ell_{k+1}(x) \geq \ell_k(x) + h \quad \text{for all } x, x_{k+1} \leq x < 0, \quad k \geq 1. \tag{2.2}
\]

**Proof.** We verify property (2.1). The proof of (2.2) is analogous, so we omit it.

For \( k = 1, 2, \ldots, \), let \( x'_{k-1} \) denote the \( x \)-coordinate of the point at which \( \ell_k \) is tangent to the graph of \( \Phi \). Remark that \( x_{k-1} < x'_{k-1} < x_k \) by the definition of \( \ell_k \).
We have \( \ell_{k+1}(x_k) = \ell_k(x_k) + \beta_{k+1} > \beta_k \) and \( x_{k-1} < x_k' \), thus,

\[
\ell_k(x) - \ell_{k+1}(x) \geq \ell_k(x_{k-1}') - \ell_{k+1}(x_{k-1}')
\]

for all \( x, x_0 \leq x \leq x_{k-1} \). Therefore, (2.1) follows from the estimate \( \ell_k(x_{k-1}') - \ell_{k+1}(x_{k-1}') \geq \Phi(x_{k-1}') - (\Phi(x_{k-1}') - h) = h \).

**Lemma 2.2.** Let the numbers \( x_k, t_k, k = 0, 1, \ldots, \) and the linear functions \( \ell_k, k = 1, 2, \ldots, \) be those introduced in Subsection 2.3. Assume that \( h \geq 2 \). Then, for \( k = 1, 2, \ldots, \)

(i) \( a_k t^\beta_k \leq w(t), \quad t \in [t_0, 1]; \)

(ii) \( e^{-h} w(t) \leq a_k t^\beta_k, \quad t \in [t_{k-1}, t_k]; \)

(iii) \( \sum_{m=1}^{\infty} 2^m a_m t^\beta_m < \frac{1}{2} a_k t^\beta_k, \quad t \in [t_{k-1}, t_k]. \)

**Proof.** Let \( k \in \mathbb{N} \). We have \( \ell_k(x) \leq \Phi(x) \) for \( x_0 \leq x < 0 \). Hence, taking the exponentials, we obtain (i) by the definitions of \( \Phi, \ell_k, a_k \) and \( \beta_k \). Analogously, the inequality \( \Phi(x) - h \leq \ell_k(x), \) \( x \in [x_{k-1}, x_k], \) implies (ii).

It remains to prove (iii). First, fix a \( k \geq 1 \) and assume that \( m \geq k + 2 \). For \( x_0 \leq x \leq x_k \), we have

\[
\ell_k(x) - \ell_m(x) \geq \ell_{k+1}(x) - \ell_m(x) = \sum_{j=k+1}^{m-1} [\ell_j(x) - \ell_{j+1}(x)].
\]

Consider the latter sum. For \( k + 1 \leq j \leq m - 1 \), property (2.1) guarantees that \( \ell_j(x) - \ell_{j+1}(x) \geq h \) for all \( x_0 \leq x \leq x_{j-1} \), hence, for all \( x_0 \leq x \leq x_k \). In sum, we have

\[
\ell_k(x) - \ell_m(x) \geq h(m - k - 1) \quad \text{for } x_0 \leq x \leq x_k.
\]

Taking the exponentials and using the definitions of \( \ell_m \) and \( \ell_k \), we obtain

\[
a_m t^\beta_m \leq a_k t^\beta_k \left( \frac{1}{e^h} + \frac{1}{e^{2h}} + \frac{1}{e^{3h}} + \cdots \right) \quad \text{for } t_0 \leq t \leq t_k.
\]

Thus,

\[
\sum_{m=k+2}^{\infty} a_m t^\beta_m \leq a_k t^\beta_k \left( \frac{1}{e^h} + \frac{1}{e^{2h}} + \frac{1}{e^{3h}} + \cdots \right) \quad \text{for } t_0 \leq t \leq t_k. \tag{2.3}
\]

Secondly, fix \( k \geq 3 \) and assume that \( 1 \leq m \leq k - 2 \). Essentially, we argue as above, replacing (2.1) by (2.2). Namely, for \( 0 > x \geq x_{k-1} \), we have

\[
\ell_k(x) - \ell_m(x) \geq \ell_{k-1}(x) - \ell_m(x) = \sum_{j=m}^{k-2} [\ell_{j+1}(x) - \ell_j(x)].
\]

For \( m \leq j \leq k - 2 \), property (2.2) guarantees that \( \ell_{j+1}(x) - \ell_j(x) \geq h \) for all \( 0 > x \geq x_{k-1} \). In sum, we have

\[
\ell_m(x) \leq \ell_k(x) - h(k - m - 1) \quad \text{for } 0 > x \geq x_{k-1}.
\]

Taking the exponentials and summing over \( m = 1, \ldots, k - 2 \), we obtain

\[
\sum_{m=1}^{k-2} a_m t^\beta_m \leq a_k t^\beta_k \left( \frac{1}{e^h} + \frac{1}{e^{2h}} + \frac{1}{e^{3h}} + \cdots \right) \quad \text{for } 1 > t \geq t_{k-1}. \tag{2.4}
\]

In sum, (iii) holds by (2.3) and (2.4), because \( h \geq 2 \). The proof of the lemma is complete. \( \square \)
2.5. Proof of (1.4) ⇒ (1.3)

Fix a parameter $h \geq 2$ and a number $x_0 \in (-\infty, 0)$ such that $t_0 = \exp(x_0) > \frac{9}{10}$. The basic induction construction provides numbers $x_k$, $t_k$, $\beta_k$ and $a_k$, $k = 1, 2, \ldots$ Put $n_k = \lfloor \beta_k \rfloor + 1$, $k = 1, 2, \ldots$, where $\lfloor \beta_k \rfloor$ denotes the integer part of $\beta_k$. Note that $n_k < n_{k+1}$ for all $k = 1, 2, \ldots$.

Indeed, we have $|x_0| < 1$, thus,

$$\beta_{k+1} - \beta_k \geq \frac{h}{x_k - x'_{k-1}} > \frac{h}{|x_0|} > 1. \quad (2.5)$$

So, put

$$g_1(z) = \sum_{j=0}^{\infty} a_{2j+1}z^{n_{2j+1}};$$
$$g_2(z) = \sum_{j=1}^{\infty} a_{2j}z^{n_{2j}}.$$

The estimates below guarantee that $g_1$ and $g_2$ are well-defined functions of $z \in \mathbb{D}$. In fact, we claim that

$$\frac{2}{5}e^{-h}w(t) < |g_1(z)| + |g_2(z)| < 4w(z), \quad t_0 < |z| < 1. \quad (2.6)$$

To prove the above estimates, put $t = |z|$. For $t \in \left(\frac{9}{10}, 1\right)$, we have

$$1 < \frac{a_k t^{n_k}}{a_k t^\beta_k} \leq \frac{1}{t} \leq \frac{10}{9}.$$

Therefore, properties (i)–(iii) from Lemma 2.2 imply the following inequalities:

(i$'$) $a_k t^{n_k} \leq w(t), \quad t \in [t_0, 1]$;
(ii$'$) $\frac{4}{9}e^{-h}w(t) \leq a_k t^{n_k}, \quad t \in [t_{k-1}, t_k]$;
(iii$'$)

$$\sum_{m \geq 1, \ |m-k| \geq 2} a_m t^{m} < \sum_{m \geq 1, \ |m-k| \geq 2} a_m t^{\beta_m} \leq \frac{1}{2} a_k t^{\beta_k} \leq \frac{5}{9} a_k t^{n_k}, \quad t \in [t_{k-1}, t_k].$$

Now, assume that $t \in [t_{k-1}, t_k]$ for some odd number $k$. On the one hand,

$$\frac{2}{5}e^{-h}w(t) \overset{(ii')}{\leq} \frac{4}{9}a_k t^{n_k} \overset{(iii')}{\leq} a_k t^{n_k} - \sum_{m \geq 1, \ |m-k| \geq 2} a_m t^{m} \leq \sum_{j=0}^{\infty} a_{2j+1}z^{n_{2j+1}} = |g_1(z)| \leq |g_1(z)| + |g_2(z)|.$$

On the other hand, if \( k \geq 3 \), then
\[
|g_1(z)| + |g_2(z)| \leq \sum_{m \geq 1} a_m t^{nm}
\]
\[
= \left( a_k t^{nk} + \sum_{m \geq 1, |m-k| \geq 2} a_m t^{nm} \right) + (a_{k-1} t^{nk-1} + a_{k+1} t^{nk+1})
\]
\[
\leq \left( 1 + \frac{5}{9} \right) a_k t^{nk} + 2w(t)
\]
\[
\leq 4w(t).
\]
For \( k = 1 \), the above arguments are even more simple.

If \( k \) is even, then we have analogous estimates. So, (2.6) holds.

To finish the proof, we modify the functions \( g_1 \) and \( g_2 \). Namely, we obtain functions \( f_1, f_2 \) having the same property as \( g_1, g_2 \) but without common zeros. So, we have \( g_1(z) = z^n G_1(z) \), where \( G_1 \in \mathcal{H}(\mathbb{D}) \) and \( G_1(0) \neq 0 \). Clearly, \( G_1 \in A^w(\mathbb{D}) \) and \( |G_1(z)| \geq |g_1(z)| \) for all \( z \in \mathbb{D} \). Therefore,
\[
\frac{2}{5} e^{-h} w(z) < |G_1(e^{i\theta} z)| + |g_2(z)|, \quad t_0 < |z| < 1,
\]
for all \( \theta \in [0, 2\pi) \). The functions \( G_1 \) and \( g_2 \) have a finite number of zeros in the disk \( \{|z| \leq t_0\} \) and \( G_1(0) \neq 0 \); hence, the required property
\[
|f_1(z)| + |f_2(z)| \asymp w(z), \quad z \in \mathbb{D},
\]
holds for \( f_2(z) = g_2(z) \) and \( f_1(z) = G_1(e^{i\theta} z) \) with an appropriate \( \theta \in [0, 2\pi) \). So, (1.4) implies (1.3). The proof of Theorem 1.2 is finished.

**Remark 7.** As mentioned in the introduction, the sequence \( \{n_k\}_{k=1}^\infty \) is not always lacunary in the classical Hadamard sense. In fact, \( \{n_k\} \) may grow as \( CK^\alpha \) for certain \( C, \alpha > 0 \). We do not use this fact, so we omit its proof. However, the series
\[
\sum_{k=1}^\infty \frac{1}{|\beta_{k+1} - \beta_k|}
\]
converges by (2.5); in other words, the sequence \( \{n_k\}_{k=1}^\infty \) retains certain weak lacunarity for any weight function \( w \) under consideration.

### 3. \( \mathcal{D} \) is a circular domain

We prove Theorem 1.3 in this section, extending the arguments used for the unit disk. So, \( \mathcal{D} \subset \mathbb{C}^d \) is assumed to be bounded and circular. Further restrictions (\( \mathcal{D} \) is a strictly convex domain with \( C^2 \)-boundary) come from Theorem 3.1, which provides building blocks in several complex variables.

**3.1. Equivalence to a log-convex weight function**

This question reduces to the corresponding implication in the unit disk. Indeed, assume that (1.5) holds. Fix a point \( \zeta \in \partial \mathcal{D} \). For \( m = 1, \ldots, M \), consider the slice-functions \( g_m(\lambda) = f_m(\lambda \zeta) \), \( \lambda \in \mathbb{D} \). We have \( g_m \in \mathcal{H}(\mathbb{D}) \), \( m = 1, \ldots, M \), and
\[
|g_1(\lambda)| + \cdots + |g_M(\lambda)| \asymp w(r_{\mathcal{D}}(\lambda \zeta)) = w(|\lambda|), \quad \lambda \in \mathbb{D}.
\]
Hence, as indicated in Remark 5, \( w(t) \) is equivalent to a log-convex weight function.
3.2. Aleksandrov–Ryll–Wojtaszczyk polynomials

The proof of the existence part in Theorem 1.3 will be based on series of special homogeneous holomorphic polynomials with sufficiently sparse degrees (cf. \[2, 24\]).

**Theorem 3.1.** Let $D \subset C^d$ be a bounded, circular, strictly convex domain with $C^2$-boundary. There exist $\delta = \delta(D) \in (0, 1)$ and $Q = Q(D) \in \mathbb{N}$ with the following properties: for every $n \in \mathbb{N}$, there exist homogeneous holomorphic polynomials $W_q[n]$ of degree $n$, $1 \leq q \leq Q$, such that

$$||W_q[n]||_{L^\infty(\partial D)} \leq 1;$$

$$\max_{1 \leq q \leq Q} |W_q[n](\zeta)| \geq \delta \quad \text{for all } \zeta \in \partial D.\quad (3.1)$$

As observed in [10], to prove Theorem 3.1, it suffices to repeat mutatis mutandis the arguments used in [16, Theorem 2.6].

3.3. Auxiliary estimates

We need a modification of property (iii) from Lemma 2.2.

**Lemma 3.2.** Let $\delta \in (0, 1)$. Then there exists $h(\delta) \geq 2$ with the following property: Let the numbers $x_k$, $t_k$, $k = 0, 1, \ldots$, and the linear functions $t_k$, $k = 1, 2, \ldots$, be those introduced in Subsection 2.3. Assume that $h \geq h(\delta)$. Then

$$(\text{iii}_2) \sum_{m \geq 1, |m-k| \geq 2} a_m t^{\beta m} < (\delta/2)a_k t^{\beta_k}, \quad t \in [t_{k-1}, t_k],$$

for $k = 1, 2, \ldots$.

**Proof.** The argument coincides with that used in the proof of Lemma 2.2. Indeed, if $h \geq 2$ is sufficiently large, then properties (2.3) and (2.4) imply (iii$_2$).

3.4. Proof of Theorem 1.3: existence part

Let $w : [0, 1) \to (0, +\infty)$ be a log-convex weight function and let $\Phi(x) = \log w(e^x)$, $x \in (-\infty, 0)$.

We modify the arguments used in the case of the unit disk, replacing the monomials $z^n$, $n = 1, 2, \ldots$, by the Aleksandrov–Ryll–Wojtaszczyk polynomials $W_q[n]$, $q = 1, \ldots, Q$, $n = 1, 2, \ldots$.

So, let the constant $\delta \in (0, 1)$ be that provided by Theorem 3.1, and let the constant $h(\delta) \geq 2$ be that provided by Lemma 3.2. Fix a parameter $h \geq h(\delta)$ and a number $x_0 \in (-\infty, 0)$ such that $t_0 = \exp(x_0) > \frac{a}{10}$. The basic induction construction provides numbers $x_k$, $t_k$, $\beta_k$ and $a_k$, $k = 1, 2, \ldots$.

For $s = 0, 1$, put

$$f_{q+sQ}(z) = \sum_{j=0}^{\infty} a_{2j+1+s} W_q[n_{2j+1+s}](z), \quad z \in D, \quad q = 1, \ldots, Q,$$

where $n_k = [\beta_k] + 1, k = 1, 2, \ldots$.

We adapt the argument used to prove (2.6). So, let $t = r_D(z)$. Properties (i) and (ii) from Lemma 2.2 and property (iii$_2$) from Lemma 3.2 imply the following inequalities:

$$(i') \quad a_k t^{\beta_k} \leq w(t), \quad t \in [t_0, 1);$$

$$(ii') \quad \frac{a}{10} e^{-h} w(t) \leq a_k t^{\beta_k}, \quad t \in [t_{k-1}, t_k];$$

$$\max_{1 \leq q \leq Q} |W_q[n](\zeta)| \geq \delta \quad \text{for all } \zeta \in \partial D.\quad (3.2)$$
\[(iii_\delta')\]
\[
\sum_{m \geq 1, |m-k| \geq 2} a_m t^{\nu_m} < \sum_{m \geq 1, |m-k| \geq 2} a_m t^\nu_m
\]
\[
\leq \frac{\delta}{2} a_k t^{\nu_k} \leq \frac{5\delta}{9} a_k t^{\nu_k}, \quad t \in [t_k-1, t_k].
\]

First, remark that \((iii_\delta')\) is more stringent than \((iii')\). So, \(f_{q+sQ}\) are well-defined functions of \(z \in D\). Moreover, combining estimate (3.1) and the arguments from the proof of Theorem 1.2, we obtain \(f_{q+sQ} \in A^w(D), s = 0, 1, q = 1, \ldots, Q\).

Secondly, we claim that
\[
\frac{2}{5} e^{-h} w(t) \left( \sum_{q=1}^{Q} |f_q(z)| + \sum_{q=1}^{Q} |f_q+Q(z)| \right), \quad t_0 < t < 1.
\] (3.3)

To prove the above estimate, assume that \(t \in [t_k-1, t_k]\) for some odd number \(k\). We have \(z = t\zeta\) for some \(\zeta \in \partial D\). Applying (3.2), select \(q(\zeta), 1 \leq q(\zeta) \leq Q\) such that
\[
|W_{q(\zeta)}[n_k](\zeta)| \geq \delta.
\] (3.4)

Recall that \(W_{q(\zeta)}[n]\) is a homogeneous polynomial of degree \(n\). Thus, we obtain
\[
\frac{2\delta}{5} e^{-h} w(t) (iii_\delta') \leq \frac{4\delta}{9} a_k t^{\nu_k}
\]
\[
\leq \delta a_k t^{\nu_k} - \sum_{m \geq 1, |m-k| \geq 2} a_m t^{\nu_m}
\]
\[
\leq \left| \sum_{j=0}^{\infty} a_{2j+1} W_{q(\zeta)}[n_{2j+1}](t\zeta) \right|
\]
\[
= |f_{q(\zeta)}(t\zeta)|
\]
\[
\leq \sum_{q=1}^{Q} |f_q(t\zeta)| + \sum_{q=1}^{Q} |f_q+Q(t\zeta)|.
\]

If \(k\) is even, then we have an analogous estimate. So, (3.3) holds. Putting \(f_{2Q+1} \equiv 1\), we obtain the required estimate
\[
w(r_D(z)) \leq C \sum_{m=1}^{2Q+1} |f_m(z)|
\]
for all \(z \in D\). The proof of Theorem 1.3 is finished.

4. Applications

It is well known that a doubling weight function is equivalent to a log-convex one. Hence, Theorem 1.3 allows one to recover the applications presented in [1, Sections 3–7] for the doubling weight functions \(w\). In this section, we consider several direct applications of Theorem 1.3, assuming that \(w : [0,1) \to (0, +\infty)\) is a log-convex weight function, unless otherwise stated.
4.1. Weighted composition operators

Given $g \in \mathcal{Hol}(D)$ and a holomorphic mapping $\varphi : D \to D$, the weighted composition operator $C^g_\varphi : \mathcal{Hol}(D) \to \mathcal{Hol}(D)$ is defined by the formula

$$(C^g_\varphi f)(z) = g(z)f(\varphi(z)), \quad f \in \mathcal{Hol}(D), \; z \in D.$$  

If $g \equiv 1$, then $C^g_\varphi$ is denoted by $C_\varphi$ and it is called a composition operator. Various properties of $C_\varphi$ are presented in monographs [9, 25].

Consider a linear space $\mathcal{Y}(D)$ that consists of functions $f : D \to \mathbb{C}$. We say that $\mathcal{Y}(D)$ is a lattice if the following implication holds.

Assume that $F \in \mathcal{Y}(D)$, $f : D \to \mathbb{C}$ is a continuous function, and $|f(z)| \leq |F(z)|$ for $z \in D$. Then $f \in \mathcal{Y}(D)$.

Let $D$ be as in Theorem 1.3. The definition of a lattice and Theorem 1.3 imply the following result (cf. [1, Corollary 2]).

**Corollary 4.1.** Suppose that $g \in \mathcal{Hol}(D)$, $\varphi : D \to D$ is a holomorphic mapping and $\mathcal{Y}(D)$ is a lattice. Then the weighted composition operator $C^g_\varphi$ maps $A^w(D)$ into $\mathcal{Y}(D)$ if and only if

$$|g(z)|w(r_D(\varphi(z))) \in \mathcal{Y}(D).$$

4.2. Integral operators

Assume that $D$ is the unit ball $B_d$ of $\mathbb{C}^d$. Given $g \in \mathcal{Hol}(B_d)$ and a holomorphic mapping $\varphi : B_d \to B_d$, the Volterra-type operator $V^g_\varphi : \mathcal{Hol}(B_d) \to \mathcal{Hol}(B_d)$ is defined as

$$(V^g_\varphi f)(z) = \int_0^1 f(\varphi(tz)) \frac{Rg(tz)}{t} \, dt, \quad f \in \mathcal{Hol}(B_d), \; z \in B_d,$$

where

$$Rg(z) = \sum_{j=1}^d z_j \frac{\partial g}{\partial z_j}(z), \quad z \in B_d,$$

is the radial derivative of $g$. If $\varphi(z) \equiv z$, then $V^g_\varphi$ is denoted by $J_g$ and it is called an extended Cesàro operator (see [15]). In fact, if $d = 1$, then we have

$$(J_g f)(z) = \int_0^z f(w)g'(w) \, dw, \quad f \in \mathcal{Hol}(D), \; z \in \mathbb{D}.$$  

The above operator was introduced by Pommerenke [22] as a natural generalization of the classical Cesàro operator. For $d = 1$, various properties of the operator $J_g$ are discussed in surveys [3, 26].

Direct calculations show that

$$RV^g_\varphi f(z) = f(\varphi(z))Rg(z), \quad z \in B_d,$$

for all $f, g \in \mathcal{Hol}(B_d)$ (cf. [15]). Applying (4.1) and Corollary 4.1, we obtain the following fact.

**Corollary 4.2.** Let $\mathcal{Y}(B_d)$ be a lattice on $B_d$. Then the operator $RV^g_\varphi$ maps $A^w(B_d)$ into $\mathcal{Y}(B_d)$ if and only if $|Rg(z)|w(|\varphi(z)|) \in \mathcal{Y}(B_d)$.

Given a space $X \subset \mathcal{Hol}(B_d)$, a typical problem is to characterize $g \in \mathcal{Hol}(B_d)$ such that $J_g$ is a bounded operator on $X$. Often the characterizing property is the following one:

$$\sup_{z \in B_d} |Rg(z)|(1 - |z|) < \infty,$$
that is, $g$ is in the Bloch space $\mathcal{B}(B_d)$ (see, for example, [4, 15]). Hence, it is interesting to find those $X$ for which the answer is different. To give such examples, consider the following exponential weight functions:

$$w_\alpha(r) = \exp\left(\frac{1}{(1-r)^\alpha}\right), \quad 0 \leq r < 1, \quad \alpha > 0.$$ 

Note that the above weight functions are not doubling.

**Corollary 4.3.** Let $\alpha > 0$ and let $g \in \mathcal{H}(B_d)$. The operator $J_g : \mathcal{A}^{w_\alpha}(B_d) \to \mathcal{A}^{w_\alpha}(B_d)$ is bounded if and only if

$$\sup_{z \in B_d} |g(z)|(1 - |z|)^\alpha < \infty. \quad (4.2)$$

**Proof.** For $h \in \mathcal{H}(\overline{D})$, the norm $\|h\|_{\mathcal{A}^{w_\alpha}(\overline{D})}$ is equivalent to

$$|h(0)| + \|\mathcal{R}h\|_{\mathcal{A}^{w_\alpha}(\overline{D})}$$

by [21, Theorem D]. Given $f \in \mathcal{H}(B_d)$, we apply the above fact to the slice-functions of $J_g f \in \mathcal{H}(B_d)$ and conclude that

$$J_g f \in \mathcal{A}^{w_\alpha}(B_d) \Leftrightarrow \mathcal{R} J_g f \in \mathcal{A}^{w_\alpha}(B_d).$$

By Corollary 4.2, the latter property holds if and only if

$$\sup_{z \in B_d} |\mathcal{R}g(z)| \frac{w_\alpha(|z|)}{w_\alpha(1 - |z|)} < \infty,$$

that is,

$$\sup_{z \in B_d} |\mathcal{R}g(z)|(1 - |z|)^{\alpha + 1} < \infty.$$ 

It remains to observe that the above inequality is equivalent to (4.2). \hfill \Box

**Remark 8.** By definition, the weighted Bergman space $A^p_{w_\alpha}(\overline{D})$, $0 < p < \infty$, consists of $f \in \mathcal{H}(\overline{D})$ such that

$$\int_{\overline{D}} |f(z)|^p \frac{d\nu(z)}{w_\alpha(|z|)} < \infty,$$

where $\nu$ denotes Lebesgue measure on $\mathbb{C}$. Formally, one may consider $A^{w_\alpha}(\overline{D})$ as the weighted Bergman space $A^p_{w_\alpha}(\overline{D})$ with $p = \infty$. So, given $\alpha > 0$, remark that property (4.2) with $d = 1$ also characterizes those $g \in \mathcal{H}(\overline{D})$ for which the operator $J_g : A^p_{w_\alpha}(\overline{D}) \to A^p_{w_\alpha}(\overline{D})$, $0 < p < \infty$, is bounded (see [20]).

### 4.3 Associated weights on the unit ball $B_d$

For $D = B_d$, $d \geq 1$, Theorem 1.3 is also applicable to arbitrary weight functions via the notion of associated weight. Namely, the following definition was formally introduced in [5].

**Definition 1.** Given a radial weight $v$ on $B_d$, $d \geq 1$, the associated weight $\tilde{v}_d$ is defined by

$$\tilde{v}_d(z) = \sup\{|f(z)| : f \in \mathcal{H}(B_d), \quad |f| \leq v \text{ on } B_d\}.$$ 

As observed in [5], $\tilde{v}_1$ is a radial weight, so the associated weight function $\tilde{v}_1 : [0, 1) \to (0, +\infty)$ is correctly defined. Moreover, $\tilde{v}_1$ is known to be log-convex (see [6]).

Now, assume that $d \geq 2$. If $f \in \mathcal{H}(B_d)$ and $|f| \leq v$, then every slice-function $f_\zeta(\lambda) = f(\lambda \zeta)$, $\zeta \in \partial B_d$, $\lambda \in \mathbb{D}$, is in $\mathcal{H}(\mathbb{D})$ and $|f_\zeta(\lambda)| \leq v(|\lambda|)$, $\lambda \in \mathbb{D}$. Thus, on the one hand, $\tilde{v}_d(z) \leq $
\( v_1(|z|), z \in B_d \). On the other hand, if \( f \in \mathcal{H}ol(D) \) and \( |f| \leq v \) on \( D \), then \( F(z_1, \ldots, z_d) := f(z_1) \in \mathcal{H}ol(B_d) \) and \( |F| \leq v \) on \( B_d \); hence, \( \tilde{v}_d(z) \geq \tilde{v}_1(|z|) \). So \( \tilde{v}_d \) is a radial weight and the identity \( \tilde{v}_d = v_1 := \tilde{v} \) holds for the associated weight functions.

Clearly, \( \mathcal{A}^v(B_d) = \mathcal{A}^\phi(B_d), d \geq 1 \), isometrically. Hence, given an arbitrary weight function \( v : [0, 1) \to (0, +\infty) \), the study of \( \mathcal{A}^v(B_d) \) reduces to that of \( \mathcal{A}^\phi(B_d) \), where \( w = \tilde{v} \) is a log-convex weight function. For example, an extension of Corollary 4.1 has the following form.

**Corollary 4.4.** Let \( d \geq 1 \) and let \( v : [0, 1) \to (0, +\infty) \) be an arbitrary weight function. Suppose that \( g \in \mathcal{H}ol(B_d), \phi : B_d \to B_d \) is a holomorphic mapping and \( \mathcal{Y}(B_d) \) is a lattice. Then the weighted composition operator \( C^g_\phi \) maps \( \mathcal{A}^v(B_d) \) into \( \mathcal{Y}(B_d) \) if and only if \( g(z)|\tilde{v}(|\phi(z)|) \in \mathcal{Y}(B_d) \).

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