A comparison between the performance of different Mindlin’s theory based strain gradient models in local singularities

Resam Makvandi¹ and Daniel Juhre¹,∗

¹ Institute of Mechanics, Faculty of Mechanical Engineering, Otto von Guericke University Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany.

The classical continuum mechanics theories are originally supposed to determine deformations in ranges from millimeter to meter, the so-called macroscopic scale. In fact, these theories are approximations of physical systems neglecting the underlying microstructure. For instance, a Cauchy continuum, i.e. a continuum with an elastic energy determined as a function of the gradients of its macroscopic displacement, can only approximate the behavior of a physical system sufficiently as long as the microstructure has a much smaller length-scale than the macrostructure [1]. Although these models were exploited in studies for large and small scales, experiments have shown that the classical models are not able to properly cover the smaller scales; in particular, problems in micron- and nano-dimensions are frequently observed. Size-effects, which cannot be captured exploiting these theories, seem to be the source of this issue. On top of that, the appearance of local singularities at the crack tips (or more broadly, in the presence of point and line loads) is one of the known limitations of the classical continuum mechanics theories. Generalizing these models by introducing additional kinematic terms to consider the underlying microstructure effects at macroscopic levels is one way of overcoming the already mentioned problems. In this contribution, we will focus on the Mindlin’s theory of elasticity with microstructures [2] and its different forms. Therein, it is shown that for the first strain gradient theory, five additional parameters must be introduced. However, in practice, due to the complexities of measuring the new parameters, various simplified versions of the theory are being explored, among them we name Altan et al. [3] and Reiher et al. [4]. Our aim here is to compare the performance of these simplified theories in removing the local singularities of the conventional continuum mechanics theory.

1 Introduction

Recently, the strain gradient theories have been shown to be applicable and of use in different areas. In strain gradient elasticity, higher-order gradients of the displacement field are deployed to overcome shortcomings of the classical models. In this contribution, we will focus on the stress regularization ability of strain gradient theories for line displacement and/or load boundary conditions. It has been shown earlier that the first strain gradient theory should suffice to regularize the stress fields under these kinds of boundary conditions [4].

2 Strain gradient elasticity

Assuming small strains, we start by defining the symmetric part of the displacement gradient (the linear strain tensor)

\[ \varepsilon := \nabla_{\text{sym}} u = \frac{1}{2} \left( \nabla \otimes u + u \otimes \nabla \right) = \frac{1}{2} (u_{i,j} + u_{j,i}) = \varepsilon_{ij}, \]  

where \( u \) represents the (macroscopic) displacement field. The deformation energy density from Mindlin’s theory of elasticity with microstructures reads (the contractions between tensors are defined as in [5])

\[ W = \frac{1}{2} \varepsilon \cdot C \varepsilon + \frac{1}{2} \gamma \cdot B \gamma + \frac{1}{2} \kappa \cdot A \kappa + \gamma \cdot F \gamma + \varepsilon \cdot G \varepsilon. \]  

In Eq. (2), \( A, B, C, D, F, \) and \( G \) are constitutive tensors where the number in brackets defines the order of the tensor. The third-order tensor \( \kappa \) is the gradient of macroscopic deformation, and \( \gamma = (\nabla u - \varphi) \) is the so-called relative deformation where \( \varphi \) denotes the micro-deformation. This general energy definition consists of 903 independent coefficients (material parameters). Mindlin has formulated simpler versions of his theory to express the deformation energy density only in terms of the macroscopic displacements in three different forms. All these forms, however, produce the same partial differential equations in terms of the macroscopic displacement. In this contribution, we focus on Form I of the theory where

\[ \kappa := \nabla \nabla u = u_{i,j,k} = \kappa_{ijk}. \]

Assuming linear elasticity, Eq. (2) reduces to:

\[ W = \frac{1}{2} \varepsilon \cdot C \varepsilon + \frac{1}{2} \kappa \cdot A \kappa + \varepsilon \cdot F \kappa. \]  

Corresponding author: e-mail daniel.juhre@ovgu.de,

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with 300 independent parameters. We can further reduce the model by assuming centro-symmetric isotropic materials. With the latter assumption, all the odd-order tensors must vanish. In this case, only 7 independent parameters remain ($a_1 \cdots a_7$) \[ W = a_1 (\varepsilon \cdot I)^2 + a_2 \varepsilon \cdot \varepsilon + a_3 (\kappa \cdot I) \cdot (\kappa \cdot I) + a_4 I \cdot \kappa \cdot I + a_5 (I \cdot \kappa) \cdot (I \cdot \kappa) + a_6 \kappa \cdots \kappa + a_7 \kappa \cdots \kappa, \] (4)

where $I$ is the identity tensor. In index notation, Eq. (4) will be

\[ W = \frac{1}{2} \lambda I_{ij} I_{jj} + \mu I_{ij} I_{jj} + b_1 \kappa_{ik} \kappa_{kj} + b_2 \kappa_{ij} \kappa_{kk} + b_3 \kappa_{ik} \kappa_{jk} + b_4 \kappa_{ij} \kappa_{jk} + b_5 \kappa_{ij} \kappa_{kji}, \] (5)

where $\lambda$ and $\mu$ are the Lamé constants and $b_1 \cdots b_5$ are gradient material parameters. Calculating the variation of $\tilde{W}$ gives us the variational formulation of the problem (see, for instance, [6]). Determining the gradient material parameters is by no way a trivial task. To simplify the problem, a few models have been introduced in the literature, among them is the GRADELA model [3] developed by Aifantis and co-workers where the higher-order strains are defined as the gradients of the linear strain tensor and therefore inherit its symmetry. Based on their formulation, the gradient parameters for Form I can be written as

\[ b_1 = 0 \quad b_2 = \frac{1}{2} \lambda l^2 \quad b_3 = 0 \quad b_4 = \frac{1}{2} \mu l^2 \quad b_5 = \frac{1}{2} \mu l^2 \] (6)

where $l$ is the length-scale parameter. Another model which we will consider here is the one introduced by Reiher et al. [4].

\[ b_1 = 0 \quad b_2 = 0 \quad b_3 = 0 \quad b_4 = \frac{1}{2} \lambda_1 \quad b_5 = 0. \] (7)

3 Numerical results

Our aim here is to compare the performance of models introduced in the previous section under the boundary conditions specified in Fig.1a. As for material properties, we use Lamé constants $\lambda = 1.0 \text{ MPa}$ and $\mu = 0.08 \text{ MPa}$ together with $l^2 = 0.04$ for the GRADELA model, and $\lambda_1 = 0.04$ for the model introduced by Reiher et al. It should be noted that these assumptions result in two slightly different materials and therefore comparing the results quantitatively does not give us much insights. However, looking at the distribution of the strain energy density in Figs. 1b and 1c, one can see that these models handle the prescribed boundary conditions differently in terms of the distribution of strain energy density in the cuboid; GRADELA shows some concentrations of the strain energy density in the corners of the cuboid which is undesirable considering the applied boundary conditions.

Fig. 1: a) dimensions and boundary conditions of the problem ($\tilde{u} = 0.05 \text{ mm}$), b) strain energy density for Reiher et al. [4], and c) strain energy density for GRADELA model [3].

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