On modules over the mod 2 Steenrod algebra and hit problems

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Abstract  Let us consider the prime field of two elements, \( \mathbb{F}_2 \equiv \mathbb{Z}_2 \). It is well-known that the classical "hit problem" for a module over the mod 2 Steenrod algebra \( \mathcal{A} \) is an interesting and important open problem of Algebraic topology, which asks for a minimal set of generators for the polynomial algebra \( \mathcal{P}_m := \mathbb{F}_2[x_1, x_2, \ldots, x_m] \), regarded as a connected unstable \( \mathcal{A} \)-module on \( m \) variables \( x_1, \ldots, x_m \), each of degree 1. The algebra \( \mathcal{P}_m \) is the \( \mathcal{F}_2 \)-cohomology of the product of \( m \) copies of the Eilenberg-MacLan complex \( K(\mathbb{F}_2, 1) \). Although the hit problem has been thoroughly studied for more than 3 decades, solving it remains a mystery for \( m \geq 5 \). It is our intent in this work is of studying the hit problem of five variables. More precisely, we develop our previous work \[\text{Commun. Korean Math. Soc. 35 (2020), 371-399}\] on the hit problem for \( \mathcal{A} \)-module \( \mathcal{P}_5 \) in a degree of the generic form \( n_t := 5(2^t - 1) + 18.2^t \), for any non-negative integer \( t \). An efficient approach to solve this problem had been presented. Two applications of this study are to determine the dimension of \( \mathcal{P}_6 \) in the generic degree \( 5(2^{t+4} - 1) + n_t.2^{t+4} \) for all \( t > 0 \) and to describe the modular representations of the general linear group of rank 5 over \( \mathbb{F}_2 \). As a corollary, the cohomological "transfer", defined by William Singer \[\text{Math. Z. 202 (1989), 493-523}\], is an isomorphism in bidegree \( (5, n_0) \). Singer’s transfer is one of the relatively efficient tools to approach the structure of mod-2 cohomology of the Steenrod algebra.

Keywords  Adams spectral sequences · Steenrod algebra · Hit problem · Algebraic transfer

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1 Introduction

Let \( O^S(i, \mathbb{F}_2, \mathbb{F}_2) \) denote the set of all stable cohomology operations of degree \( i \), with coefficient in the prime field \( \mathbb{F}_2 \). Then, the \( \mathbb{F}_2 \)-algebra \( \mathcal{A} := \bigoplus_{i \geq 0} O^S(i, \mathbb{F}_2, \mathbb{F}_2) \) is called the mod 2 Steenrod algebra. In other words, the algebra \( \mathcal{A} \) is the algebra of stable operations on the mod 2 cohomology. In [28], Milnor observed that this algebra is also a graded connected cocommutative Hopf algebra over \( \mathbb{F}_2 \). In some cases, the resulting \( \mathcal{A} \)-module structure on \( H^*(X, \mathbb{F}_2) \) provides additional information about CW-complexes \( X \); for instance (see section three for a detailed proof), the CW-complexes \( CP^4/CP^2 \) and \( S^6 \vee S^6 \) have cohomology rings that agree as a graded commutative \( \mathbb{F}_2 \)-algebras, but are different as a module over \( \mathcal{A} \). Afterwards, the Steenrod algebra is widely studied by mathematicians whose interests range from algebraic topology and homotopy theory to manifold theory, combinatorics, representation theory, and more. It is well-known that the \( \mathbb{F}_2 \)-cohomology of the Eilenberg-MacLan complex \( K(\mathbb{F}_2, 1) \) is isomorphic to \( \mathbb{F}_2[x] \), the polynomial ring of degree 1 in one variable. Hence, based upon the Künneth formula for cohomology, we have an isomorphism of \( \mathbb{F}_2 \)-algebras

\[
\mathcal{P}_m := H^*((K(\mathbb{F}_2, 1))^\times m, \mathbb{F}_2) \cong \mathbb{F}_2[x_1] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_2] \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_m] \cong \mathbb{F}_2[x_1, \ldots, x_m],
\]

where \( x_i \in H^1((K(\mathbb{F}_2, 1))^\times m, \mathbb{F}_2) \) for every \( i \). Since \( \mathcal{P}_m \) is the cohomology of a CW-complex, it is equipped with a structure of unstable module over \( \mathcal{A} \). It has been known (see also [47]) that \( \mathcal{A} \) is spanned by the Steenrod squares \( Sq^i \) of degree \( i \).
for \( i \geq 0 \) and that the action of \( \mathcal{A} \) on \( \mathcal{P}_m \) is depicted as follows:

\[
\begin{align*}
S^q(x_i) &= \begin{cases} 
  x_i & \text{if } i = 0, \\
  x_i^2 & \text{if } i = 1, \\
  0 & \text{if } i > 1,
\end{cases} \\
S^q(FG) &= \sum_{0 \leq a \leq \infty} S^q(a)(F)S^q(a)(G), \text{ for all } F, G \in \mathcal{P}_m \text{ (the Cartan formula)}.
\end{align*}
\]

It is to be noted that since \( S^q(\deg(F))(F) = F^2 \) for any \( F \in \mathcal{P}_m \), the polynomial ring \( \mathcal{P}_m \) is also an unstable \( \mathcal{A} \)-algebra. Letting \( GL_m := GL(m, F_2) \) for the general linear group of degree \( m \) over \( F_2 \). This \( GL_m \) when \( m \geq 2 \), which can be generated by two elements (see Waterhouse [55]), acts on \( \mathcal{P}_m \) by matrix substitution. So, in addition to \( \mathcal{A} \)-module structure, \( \mathcal{P}_m \) is also a (right) \( F_2GL_m \)-module. The classical 'hit problem' for the algebra \( \mathcal{A} \), which is concerned with seeking a minimal set of \( \mathcal{A} \)-generators for \( \mathcal{P}_m \), has been initiated in a variety of contexts by Peterson [33], Priddy [42], Singer [45], and Wood [56]. Structure of modules over \( \mathcal{A} \) and hit problems are currently one of the central subjects in Algebraic topology and has a great deal of intensively studied by many authors like Brunetti and collaborators [5, 6], Crabb-Hubbuck [10], Inoue [16, 17], Janfada-Wood [18, 19], Janfada [20, 21], Kameko [22], Mothebe-Uys [29], Mothebe [30], Pengelley-William [32], the present author and N. Sum [34–40, 49–51], Walker-Wood [53, 54], etc. As it is known, \( F_2 \) is an \( \mathcal{A} \)-module concentrated in degree 0, solving the hit problem is to determine an \( F_2 \)-basis for the space of indecomposables, or 'unhit' elements, \( Q^\underline{m} := F_2 \otimes_{\mathcal{A}} \mathcal{P}_m = \mathcal{P}_m/\mathbb{T}_{\mathcal{P}_m} \) where \( \mathbb{T} \) is the positive degree part of \( \mathcal{A} \). It is well-known that the action of \( GL_m \) and the action of \( \mathcal{A} \) on \( \mathcal{P}_m \) commute. So, there is an induced action of \( GL_m \) on \( Q^\underline{m} \). The structure of \( Q^\underline{m} \) has been treated for \( m \leq 4 \) by Peterson [33], Kameko [22] and Sum [49]. The general case is an interesting open problem. Most notably, the study of this space plays a vital role in describing the \( E^2 \)-term of the Adams spectral sequence (Adams SS), \( Ext^m_{\mathcal{A}}(F_2, F_2) \) via the \( m \)-th Singer cohomological 'transfer' [44]. This transfer is a linear map

\[
Tr_{m}^\mathcal{A} : (F_2 \otimes_{GL_m} F_{\mathcal{A}}((\mathcal{P}_m)^*))_n \to Ext^m_{\mathcal{A}}((F_2, F_2) = H^{m,m+n}(\mathcal{A}, F_2),
\]

from the subspace of all \( \mathbb{T} \)-annihilated elements to the \( E^2 \)-term of the Adams SS. Here \( (\mathcal{P}_m)^* = H_*(K(F_2, 1)^{\times m}, F_2) \) and \( F_2 \otimes_{GL_m} F_{\mathcal{A}}((\mathcal{P}_m)^*) \) are the dual of \( \mathcal{P}_m \) and \( (Q^\underline{m})GL_m \)-respectively, where \( (Q^\underline{m})GL_m \) denotes the space of \( GL_m \)-invariants. A natural question arises: Why do we need to calculate the Adams \( E^2 \)-term? The answer is that it is involved in determining the stable homotopy groups of spheres. These groups are pretty fundamental and interesting. Nevertheless, they are also not fully-understood subjects yet. Therefore, the clarification of these problems is an important task of Algebraic topology. It has been shown (see [2], [44]) that the algebraic transfer is highly nontrivial, more precisely, that \( Tr_{m}^\mathcal{A} \) is an isomorphism for \( 0 < m < 4 \) and that the 'total' transfer \( \bigoplus_{m \geq 0} Tr_{m}^\mathcal{A} : \bigoplus_{m \geq 0}(F_2 \otimes_{GL_m} F_{\mathcal{A}}((\mathcal{P}_m)^*))_n \to \bigoplus_{m \geq 0} Ext^m_{\mathcal{A}}(F_2, F_2) \) is a homomorphism of bigraded algebras with respect to the product by concatenation in the domain and the usual Yoneda product for the Ext group. Minami's works [26, 27] have shown the usefulness of the Singer transfer and the hit problem for surveying the Kervaire invariant one problem. This problem, which is a long standing open topic in Algebraic topology, asks when there are framed manifolds with Kervaire invariant one. (Note that a framing on a closed smooth manifold \( M^n \) is a trivialization of the normal bundle \( \nu(M, i) \) of some smooth embedding \( i : M \hookrightarrow \mathbb{R}^{n+1} \).)

Here \( \nu(M, i) \) is a smooth real vector bundle over \( M^n \). More details, we refer the reader to [46].] Framed manifolds of Kervaire invariant one have been constructed in dimension \( 2^k - 2 \) for \( 2 \leq k \leq 6 \). In 2016, by using mod 8 equivariant homotopy theory, Hill, Hopkins, and Ravenel claimed in their surprising work [13] that the Kervaire invariant is 0 in dimension \( 2^k - 2 \) for \( k \geq 8 \). Up to present, it remains undetermined for \( k = 7 \) (or dimension 126) and this has the status of a hypothesis by Snith [46].

Return to Singer's transfer, in higher homological degrees, the works [4], [11], [14], [31], and [15] determined completely the image of \( Tr_{4}^\mathcal{A} \). The authors show that the image of the fourth transfer contains every element in the four families \( \{d_0t \geq 0\}, \{e_1t \geq 0\}, \{f_i \geq 0\}, \{g_i \geq 0\}, \{D_0t \geq 0\}, \{D_1t \geq 0\}, \{p_i \geq 0\} \), whereas it does not contain any element in the three families \( \{g_{i+1} \geq 0\}, \{D_{0t} \geq 0\}, \{p_{i+1} \geq 0\} \). More explicitly, the result on \( g_{i+1} \geq 0 \) is due to [4]; that on \( D_{0t} \geq 0 \), and \( p_{i+1} \geq 0 \) is proved by [11], while that on \( f_i \geq 0 \) is proved by [31]. Remarkably, the results by [4] and [14] gave a negative answer to Minami’s hypothesis [27] predicting that the localization of \( Tr_{4}^\mathcal{A} \) given by inverting the squaring operation \( S_0^0 \) (see section two) is an isomorphism. In [14], Hung indicated that \( Tr_{4}^\mathcal{A} \) is not an isomorphism in infinitely many degrees. In particular, from preliminary calculations in [44], Singer proposed the following.

**Conjecture 1.1** The transfer homomorphism is a monomorphism in every rank \( m > 0 \).

One has seen above that \( Tr_{4}^\mathcal{A} \) is an isomorphism for \( m < 4 \), and so the conjecture holds in these ranks \( m \). Our recent work [41] has shown that it is also true for \( m = 4 \), but the answer to the general case remains a mystery, even in the case of \( m = 5 \) with the help of a computer algebra. It is known, in ranks \( \leq 4 \), the calculations of Singer [44], Hà [11], and Nam [31]
tell us that the non-zero elements \( h_t \in \text{Ext}_{\mathcal{A}}^{12t}(F_2, F_2), e_t \in \text{Ext}_{\mathcal{A}}^{42t+4+2t^2+2^t}(F_2, F_2), f_t \in \text{Ext}_{\mathcal{A}}^{42t+4+2t^2+2^t+1}(F_2, F_2) \), for all \( t \geq 0 \), are detected by the cohomological transfer. In rank 5, based on invariant theory, Singer [44] gives an explicit element in \( \text{Ext}_{\mathcal{A}}^{5,5+9}(F_2, F_2) \), namely \( P_{11} \), that is not detected by \( Tr_{\mathcal{A}}^5 \). In general, direct calculating the value of \( Tr_{\mathcal{A}}^d_f \) on any non-zero element is difficult. Moreover, there is no general rule for that, and so, each computation is important on its own. By this and the above results, in the present text, we would like to investigate the family \( \tilde{Q}^{\mathcal{O}} \) of degree 5(2^t − 1) + 18.2^t for the cases \( t \geq 1 \). Then, Singer’s conjecture for \( m = 5 \) and these degrees will be discussed at the end of section two. We hope that our results would be necessary to formulate general solutions.

2 Statement of results

Some notes. Throughout this paper, let us write

\[
\begin{align*}
(\mathcal{P}_m)_n := \{ f \in \mathcal{P}_m | f \text{ is a homogeneous polynomial of degree } n \}, \\
Q_n^{\otimes m} := \{ [f] \in Q^{\otimes m} | f \in (\mathcal{P}_m)_n \},
\end{align*}
\]

which are \( F_2GL_m \)-submodules of \( \mathcal{P}_m \) and \( Q^{\otimes m} \), respectively. So \( \mathcal{P}_m = \bigoplus_{n \geq 0} (\mathcal{P}_m)_n \) and \( Q^{\otimes m} = \bigoplus_{n \geq 0} Q^{\otimes m}_n \). Recall that to solve the hit problem of three variables, Kameko [22] constructed a \( F_2GL_m \)-modules epimorphism:

\[
(\tilde{Sq}_0^\mathcal{O})(m, m+2n) : Q_n^{\otimes m+2n} \to Q_n^{\otimes m},
\]

which induces the homomorphism \( \tilde{Sq}_0^\mathcal{O} : Q_n^{\otimes m+2n}(GL_m) \to (Q_n^{\otimes m})^{GL_m} \). Since \( \mathcal{A} \) is a cocommutative Hopf algebra, there exists the squaring operations \( Sq^d : \text{Ext}_{\mathcal{A}}^{m,n}(F_2, F_2) \to \text{Ext}_{\mathcal{A}}^{m+1,2m+2n}(F_2, F_2) \), which share most of the properties with \( Sq^d \) on the cohomology of spaces (see [25]), but the classical \( Sq^0 \) is not the identity in general. Remarkably, this \( Sq^0 \) commutes with the dual of \( \tilde{Sq}_0^\mathcal{O} \) through the Singer transfer (see [2], [27]). The reader who is familiar with Kameko’s \( (\tilde{Sq}_0^\mathcal{O})(m, m+2n) \) will probably agree that this map is very useful in solving the hit problem. Indeed, Kameko [22] showed that if \( m = \xi(n) = \min(\gamma \in \mathbb{N} : n = \sum_{1 \leq i \leq \gamma}(2^{d_i} - 1), d_i > 0, \forall i, 1 \leq i \leq \gamma) \), then \( (\tilde{Sq}_0^\mathcal{O})(m, m+2n) \) is an isomorphism of \( F_2GL_m \)-modules. This statement and Wood’s work [56] together are sufficient to determine \( Q_n^{\otimes m} \) in each degree \( n \) of the special “generic” form \( n = r(2^t - 1) + d.2^t \), whenever \( 0 < \xi(d) < r < m \), and \( t \geq 0 \) (see also [39]).

As we mentioned at the beginning, the hit problem was completely solved for \( m = 4 \). Very little information is known for \( m = 5 \) and degrees \( n \) given above. At least, it is surveyed by the present writer [39] for \( (r, d, t) \in \{(5, 18, 0), (5, 8, t) \} \). We now extend for the case \( (r, d, t) = (5, 18, t) \), in which \( t \) an arbitrary non-negative integer. We start with a useful remark.

Remark 2.1 It can be easily seen that \( 5(2^t - 1) + 18.2^t = 2^{t+4}+2^{t+2}+2^{t+1}+2^{t-1}+2^t - 5 \), and so \( \xi(5(2^t - 1) + 18.2^t) = 5 \) for any \( t > 1 \). This implies that the iterated Kameko map

\[
((\tilde{Sq}_0^\mathcal{O})(5, 5(2^t-1)+18.2^t))^{t-1} : Q_n^{\otimes 5(2^t-1)+18.2^t} \to Q_n^{\otimes 5(2^t-1)+18.2^t}
\]

is an isomorphism, for all \( t \geq 1 \), and therefore, it is enough to determine \( Q_n^{\otimes 5(2^t-1)+18.2^t} \) for \( t \in \{0, 1\} \). The case \( t = 0 \) has explicitly been computed by us in [38]. When \( t = 1 \), because Kameko’s homomorphism

\[
(\tilde{Sq}_0^\mathcal{O})(5, 5(2^2-1)+18.2^1) : Q_n^{\otimes 5(2^2-1)+18.2^1} \to Q_n^{\otimes 5(2^2-1)+18.2^0}
\]

is an epimorphism, we have an isomorphism

\[
Q_n^{\otimes 5(2^2-1)+18.2^1} \cong \text{Ker}((\tilde{Sq}_0^\mathcal{O})(5, 5(2^2-1)+18.2^1)) \bigoplus Q_n^{\otimes 5(2^2-1)+18.2^0}.
\]

The space \( Q_n^{\otimes 5(2^2-1)+18.2^0} \) is known by our previous work [38]. Thus, we need compute the kernel of \( (\tilde{Sq}_0^\mathcal{O})(5, 5(2^2-1)+18.2^1) \). For this, our approach can be summarized as follows:
(i) A monomial in \( P \) is assigned a weight vector \( \omega \) of degree \( 5(2^i - 1) + 18.2^l \), which stems from the binary expansion of the exponents of the monomial. The space of indecomposable elements \( \text{Ker}((Sq^5)^{(5,5(2^i-1)+18.2^l)}) \) is then decomposed into a direct sum of \( (Q^5_{(5,5(2^i-1)+18.2^l)})^0 \) and the subspaces \( (Q^5_{(5,5(2^i-1)+18.2^l)})^\omega \) indexed by the weight vectors \( \omega \). Here \( [F]_\omega = [G]_\omega \) in \( (Q^5_{(5,5(2^i-1)+18.2^l)})^\omega \) if the polynomial \( F = G \) is hit, modulo a sum of monomials of weight vectors less than \( \omega \). Basing the previous results by Peterson [33], Kameko [22], Sum [49], and by us [39], one can easily determine \( (Q^5_{(5,5(2^i-1)+18.2^l)})^0 \).

(ii) The monomials in a given degree are lexicographically ordered first by weight vectors and then by exponent vectors. This leads to the concept of admissible monomial; more explicitly, a monomial is admissible if, modulo hit elements, it is not equal to a sum of monomials of smaller orders. The space \( (Q^5_{(5,5(2^i-1)+18.2^l)})^\omega \) above is easily seen to be isomorphic to the space generated by admissible monomials of the weight vector \( \omega \).

(iii) In a given (small) degree, we first list all possible weight vectors of an admissible monomial. This is done by first using a criterion of Singer [44] on the hit monomials, and then combining with the results by Kameko [22] and Sum [49] (see Theorems 4.2, and 4.3 in section four) of the form \( *XZ^2 \) (or \( ZY^2 \)) admissible implying \( Z \) admissible, under some mild conditions∗.

(iv) In a given weight vector, we claim the (strict) inadmissibility of some explicit monomials. The proof is given for a typical monomial in each case by explicit computations. Finally, a direct calculation using Theorems 3.2, 3.3, and some homomorphisms in section three, we obtain a basis of \( (Q^5_{(5,5(2^i-1)+18.2^l)})^\omega \). This approach is much less computational and it can be applied for all certain degrees and all variables \( m \). Moreover, the MAGMA computer algebra [24] has been used for verifying the results.

Before going into detail and proceeding to the main results, let us provide some basic concepts. Of course, we assume that the reader is not familiar with the basics of hit problems.

**Weight vector and exponent vector.** Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_t, \ldots) \) be a sequence of non-negative integers. We say that \( \omega \) is a weight vector, if \( \omega_t = 0 \) for \( t \gg 0 \). Then, we also define \( \text{deg}(\omega) = \sum_{i \geq 2} 2^i - 1 \omega_i \). Let \( X = x_1^{u_1} x_2^{u_2} \ldots x_m^{u_m} \) be a monomial in \( P \), define two sequences associated with \( X \) by

\[
\omega(X) := (\omega_1(X), \omega_2(X), \ldots, \omega_t(X), \ldots), \quad u(X) := (u_1, u_2, \ldots, u_m),
\]

where \( \omega_t(X) = \sum_{1 \leq j \leq m} \omega_t(u_j) \) in which \( \alpha_t(n) \) denotes the \( t \)-th coefficients in dyadic expansion of a positive integer \( n \). They are called the weight vector and the exponent vector of \( X \), respectively. We use the convention that the sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

**Linear order on \( P \).** Assume that \( X = x_1^{m_1} x_2^{m_2} \ldots x_m^{m_m} \) and \( Y = x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m} \) are the monomials of the same degree in \( P \). We say that \( X < Y \) if and only if one of the following holds:

(i) \( \omega(X) < \omega(Y) \);

(ii) \( \omega(X) = \omega(Y) \) and \( u(X) < u(Y) \).

**Equivalence relations on \( P \).** For a weight vector \( \omega \), we denote two subspaces associated with \( \omega \) by

\[
P^{\leq \omega}_m = \{X \in P_m | \text{deg}(X) = \text{deg}(\omega), \omega(X) \leq \omega \},
\]

\[
P^{< \omega}_m = \{X \in P_m | \text{deg}(X) = \text{deg}(\omega), \omega(X) < \omega \}.
\]

Let \( F \) and \( G \) be the homogeneous polynomials in \( P \) such that \( \text{deg}(F) = \text{deg}(G) \). We say that

(i) \( F \equiv G \) if and only if \( (F - G) \in \mathfrak{S}P_m = \sum_{i \geq 0} \text{Im}(Sq^2)^i \). Specifically, if \( F \equiv 0 \), then \( F \) is hit (or \( \mathfrak{S} \)-decomposable), i.e., \( F \) can be written in the form \( \sum_{i \geq 0} Sq^i(F_i) \) for some \( F_i \in P_m \);

(ii) \( F \equiv_\omega G \) if and only if \( F, G \in P^{\leq \omega}_m \) and \( (F - G) \in (\mathfrak{S}P_m \cap P^{\leq \omega}_m) + P^{< \omega}_m \).

It is not difficult to show that the binary relations \( \equiv \) and \( \equiv_\omega \) are equivalence ones. So, one defines the quotient space

\[
(Q^{\otimes m})^\omega = P^{\leq \omega}_m / (\mathfrak{S}P_m \cap P^{\leq \omega}_m) + P^{< \omega}_m.
\]

Moreover, due to Sum [51], \( (Q^{\otimes m})^\omega \) is also an \( \mathbb{F}_2 \)GL\(_m\)-module.

**Admissible monomial and inadmissible monomial.** A monomial \( X \in P_m \) is said to be inadmissible if there exist monomials \( Y_1, Y_2, \ldots, Y_k \) such that \( Y_j < X \) for \( 1 \leq j \leq k \) and \( X \equiv \sum_{1 \leq j \leq k} Y_j \). Then, \( X \) is said to be admissible if it is not inadmissible.

Thus, with the above definitions in hand, it is straightforward to see that the set of all the admissible monomials of degree \( n \) in \( P_m \) is a minimal set of \( \mathfrak{S} \)-generators for \( P_m \) in degree \( n \). So, \( Q^{\otimes m}_n \) is a \( \mathbb{F}_2 \)-vector space with a basis consisting
of all the classes represent by the admissible monomials of degree $n$ in $\mathcal{P}_m$. Further, as stated in [37], the dimension of $Q_{5}^{\otimes m}$ can be represented as the sum of the dimensions $(Q_{5}^{\otimes m})^{\omega}$ such that $\deg(\omega) = n$. For later convenience, we need to set some notation. Let $\mathcal{P}_m^0$ and $\mathcal{P}_m^\omega$ denote the submodules of $\mathcal{P}_m$ spanned all the monomials $\prod_{1 \leq j \leq m} x_j^{t_j}$ such that $\prod_{1 \leq j \leq m} x_j^{t_j} = 0$, and $\prod_{1 \leq j \leq m} x_j^{t_j} > 0$, respectively. Let us write $(Q_{5}^{\otimes m})^{\omega} := F_3 \otimes_{\mathbb{F}_2} \mathcal{P}_m^0$ and $(Q_{5}^{\otimes m})^{\omega > 0} := \tilde{F}_2 \otimes_{\mathbb{F}_2} \mathcal{P}_m^\omega$, from which one has that $Q_{5}^{\otimes m} = (Q_{5}^{\otimes m})^{\omega} \oplus (Q_{5}^{\otimes m})^{\omega > 0}$. For a polynomial $F \in \mathcal{P}_m$, we denote by $[F]$ the classes in $(Q_{5}^{\otimes m})^{\omega}$ represented by $F$. If $\omega$ is a weight vector and $F \in \mathcal{P}_m^\omega$, then denote by $[F]_{\omega}$ the classes in $(Q_{5}^{\otimes m})^{\omega}$ represented by $F$.

For a subset $C \subset \mathcal{P}_m$, we also write $|C|$ for the cardinal of $C$ and put $|C| = \{|F| : F \in C\}$. If $C \subset \mathcal{P}_m^\omega$, then put $[C]_{\omega} = \{|F|_\omega : F \in C\}$. Let us denote by $\mathcal{P}_m^{\omega}$ the set of all admissible monomials of degree $n$ in $\mathcal{P}_m$, and let $\omega$ be a weight vector of degree $n$. By setting

\[
(\mathcal{P}_m^{\omega})^{\omega} := \mathcal{P}_m^{\omega} \cap \mathcal{P}_m^\omega, \quad (\mathcal{P}_m^{\omega})^{\omega > 0} := (\mathcal{P}_m^{\omega})^{\omega} \cap (\mathcal{P}_m^{\omega})^{\omega > 0},
\]

then the sets $[\mathcal{P}_m^{\omega})^{\omega}]_\omega$, $[(\mathcal{P}_m^{\omega})^{\omega > 0}]_\omega$ and $[(\mathcal{P}_m^{\omega})^{\omega > 3}]_\omega$ are the bases of the $\mathbb{F}_2$-vector spaces $(\mathcal{P}_m^{\omega})^{\omega}$, $(\mathcal{P}_m^{\omega})^{\omega > 0}$ and $(\mathcal{P}_m^{\omega})^{\omega > 3}$, respectively.

**Main results and applications.** Let us now return to our study of the kernel of the Kameko homomorphism $(\tilde{S}_{5}^{\otimes})((5,5(2^1-1)+18,2^1))$ and state our main results in greater detail. Firstly, by direct calculations using the results by Kameko [22], Singer [44], Sun [49], and Tin [52], we obtain the following, which is one of our main results and is crucial for an application on the dimension of $Q_{5}^{\otimes 6}$.

**Theorem 2.2** We have an isomorphism

\[
\text{Ker}(\tilde{S}_{5}^{\otimes})((5,5(2^1-1)+18,2^1)) \cong (Q_{5}^{\otimes 5}((5(2^1-1)+18,2^1)))^{0} \oplus (Q_{5}^{\otimes 5}((5(2^1-1)+18,2^1)))^{\omega > 0},
\]

where $\tilde{\omega} = (3,3,2,1,1)$ is the weight vector of the degree $5(2^1-1)+18,2^1$.

**Remark 2.3** We are given in [39] that $(Q_{5}^{\otimes 5})^{0} \cong \bigoplus_{1 \leq s \leq 4} \bigoplus_{|\mathcal{J}| = s} (Q_{5}^{\otimes 3})^{\omega > 0}$, where

\[
Q_{5}^{\otimes 3} = \{|x_{j_1} x_{j_2} \ldots x_{j_s}| : t_i \in \mathbb{N}, i = 1, 2, \ldots, s\} \subset Q_{5}^{\otimes 5}
\]

with $\mathcal{J} = (j_1, j_2, \ldots, j_s)$, $1 \leq j_1 < \ldots < j_s \leq 5$, $1 \leq s \leq 4$, and $|\mathcal{J}| := s$ denotes the length of $\mathcal{J}$. This implies that $\dim((Q_{5}^{\otimes 5})^{0}) = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim((Q_{5}^{\otimes 5})^{s})$, for all $n \geq 0$. On the other side, since $\xi(5(2^1-1)+18,2^1) = 3$, by Peterson [33] and Wood [56], the spaces $Q_{5}^{\otimes 1}(5(2^1-1)+18,2^1)$ and $Q_{5}^{\otimes 2}(5(2^1-1)+18,2^1)$ are trivial. Moreover, following Kameko [22] and Sun [49], we have seen that $(Q_{5}^{\otimes 3}(5(2^1-1)+18,2^1))^{0}$ is 15-dimensional and that $(Q_{5}^{\otimes 4}(5(2^1-1)+18,2^1))^{0}$ is 165-dimensional. Therefore, we may conclude that

\[
\dim((Q_{5}^{\otimes 5}(5(2^1-1)+18,2^1))^{0} = 15 \cdot 5 \cdot 3 + 165 \cdot 5 \cdot 4 = 975.
\]

Next, due to Remarks 2.1, 2.2, and to Theorem 2.2, the space $Q_{5}^{\otimes 5}(5(2^1-1)+18,2^1)$ will be determined by computing $(Q_{5}^{\otimes 5}(5(2^1-1)+18,2^1))^{\omega > 0}$.

To achieve this, we use the method described above to explicitly indicate all the admissible monomials in the set $(\mathcal{P}_m^{\omega})^{\omega > 0}$. As a result, it reads as follows.

**Theorem 2.4** There exist exactly 925 admissible monomials of degree $5(2^1-1)+18,2^1$ in $\mathcal{P}_m^{\omega > 0}$ such that their weight vectors are $\tilde{\omega}$. Consequently, $(Q_{5}^{\otimes 5}(5(2^1-1)+18,2^1))^{\omega > 0}$ has dimension 925.

**Corollary 2.5** The space $Q_{5}^{\otimes 5}(5(2^1-1)+18,2^1)$ is 730-dimensional if $t = 0$, and is 2630-dimensional if $t \geq 1$.

As applications, one would also be interested in applying results and techniques of hit problems into the cases of higher ranks $m$ of $Q_{5}^{\otimes m}$ and the modular representations of the general linear groups (see also the relevant discussions in literatures [2], [26, 27], [31], [53, 54]). Two applications below of the contributions of this paper are also not beyond this target.

**First application: the dimension of $Q_{5}^{\otimes 6}$.** The hit problem of six variables has been not yet known. Using Corollary 2.5 for the case $t \geq 1$ and a result in Sun [49], we state that
Theorem 2.6 With the generic degree \(5(2t+4) - 1\) + \(41.2t+4\), where \(t\) is an arbitrary positive integer, then the \(F_2\)-vector space \(Q^\omega_n\) has dimension 165690 in this degree.

Observing from Corollary 2.5 and Theorem 2.6, the readers can notice that the dimensions of \(Q^\omega_5\) and \(Q^\omega_6\) in degrees given are very large. So, a general approach to hit problems, other than providing a monomial basis of the vector space \(Q^\omega_n\), is to find upper/lower bounds on the dimension of this space. However, in this work, we have not studied this side of the problem and it is our concern the next time. It is remarkable that, we have Kameko’s conjecture [22] on an upper bound for the dimension of \(Q^\omega_n\), but unfortunately, it was refuted for \(n \geq 5\) by the brilliant work of Sum [48].

Second application: the behavior of the fifth Singer transfer. We adopt Corollary 2.5 for \(t = 0\), together with a fact of the Adams \(E^2\)-term, \(\text{Ext}^5_{L(G)}(F_2, F_2)\), to obtain information about the behavior of Singer’s cohomological transfer in the bidegree \((5, 5 + (5(2^t - 1) + 18.2^t))\). More precisely, it is known, the calculations of Lin [23], and Chen [8] imply that \(\text{Ext}^5_{L(G)}(F_2, F_2) = \langle h_f, f_t \rangle\) and \(h_t f_t = h_{t+1} f_t \neq 0\) for all \(t \geq 0\). So, to determine the transfer map in the above bidegree, we shall compute the dimension of (the domain of the fifth transfer) \((F_2 \otimes GL_n \text{P}_{\omega}((\mathbb{P}_m)^*))_{5(2^t - 1) + 18.2^t}\) by using a monomial basis of \(Q^\omega_{5(2^t - 1)+18.2^t}\). (We emphasize that computing the domain of \(Tr_m\) in each degree \(n\) is very difficult, particularly for values of \(m\) large as \(m = 5\). The understanding of special cases should be a helpful step toward the solution of the general problem. Moreover, we believe, in principle, that our method could lead to a full analysis of \(F_2 \otimes GL_n \text{P}_{\omega}((\mathbb{P}_m)^*)\) in each \(m\) and degree \(n > 0\), as long as nice decompositions of the space of \(GL_m\)-invariants of \(Q^\omega_n\) in degrees given. However, the difficulty of such a task must be monumental, as \(Q^\omega_n\) becomes much larger and harder to understand with increasing \(m\).) Details for this application are as follows. It may need to be recalled that by the previous discussions [38], we get the technical proposition below.

Proposition 2.7 The following hold:

i) If \(Y \in \mathcal{E}^\omega_{5(2^t-1)+18.2^t}\), then \(\bar{\omega} := \omega(Y)\) is one of the following sequences:

\[
\bar{\omega}_{[1]} := (2, 2, 1, 1), \quad \bar{\omega}_{[2]} := (2, 2, 3), \quad \bar{\omega}_{[3]} := (2, 4, 2),
\]
\[
\bar{\omega}_{[4]} := (4, 1, 1, 1), \quad \bar{\omega}_{[5]} := (4, 1, 3), \quad \bar{\omega}_{[6]} := (4, 3, 2).
\]

ii) \(|\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}| = \begin{cases} 300 & \text{if } k = 1, \\
15 & \text{if } k = 2, 5, \\
10 & \text{if } k = 3, \\
110 & \text{if } k = 4, \\
280 & \text{if } k = 6. 
\end{cases}\]

One should note that \(|\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}| = |\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}|\) for \(k = 2, 3\), and that \(|\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}| = 0 = |\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}|\). Moreover, \(\dim(Q^\omega_{5(2^t-1)+18.2^t}) = \sum_{1 \leq k \leq 6} |\mathcal{E}^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]}| = 730\). Next, applying these results, we explicitly compute the subspaces of \(GL_5\)-invariants \(\langle Q^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]} \rangle_{GL_5}\), for \(1 \leq k \leq 6\), and obtain

Theorem 2.8 The following assertions are true:

i) \(\langle Q^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]} \rangle_{GL_5} = 0\) with \(k \in \{1, 2, 3, 5, 6\}\).

ii) \(\langle Q^\omega_{5(2^t-1)+18.2^t}\bar{\omega}_{[n]} \rangle_{GL_5} = \langle \mathcal{H}_n \rangle_{\mathcal{H}_{[n]}}, \) where

\[
\mathcal{H}_n = x_1 x_2 x_3 x_4 x_5 x_6 + x_1 x_2 x_3 x_4 x_5 x_6 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5.
\]

Now, because \((F_2 \otimes GL_n \text{P}_{\omega}((\mathbb{P}_m)^*))_{5(2^t-1)+18.2^t}\) is isomorphic to \((Q^\omega_{5(2^t-1)+18.2^t})_{GL_5}\), by Theorem 2.8, we have the following estimate:

\[
\dim(F_2 \otimes GL_n \text{P}_{\omega}((\mathbb{P}_m)^*))_{5(2^t-1)+18.2^t} = \dim(Q^\omega_{5(2^t-1)+18.2^t})_{GL_5} \leq \sum_{1 \leq k \leq 6} \dim(Q^\omega_{5(2^t-1)+18.2^t})_{\mathcal{H}_{[n]}} = 1.
\]

On the other side, as shown in section one, \(\{h_t | t \geq 0\} \subset \text{Im}(Tr^m_{\omega})\), and \(\{f_t | t \geq 0\} \subset \text{Im}(Tr^m_{\omega})\). Combining this with the fact that the total transfer \(\mathcal{T}_{m \geq 0}^m Tr^m_{\omega}\) is a homomorphism of algebras, it may be concluded that the non-zero element
Corollary 2.9 The cohomological transfer

\[ T^{55}_{\mathcal{F}} : (\mathcal{F} \otimes_{GL_5} P_{\mathcal{S}}((\mathcal{P}_5)^*))_{5,(2^0-1)+18.2^0} \to \text{Ext}^{55+5(2^0-1)+18.2^0}_{\mathcal{S}}(\mathcal{F} \otimes \mathcal{F}) \]

is an isomorphism. Consequently, Conjecture 1.1 holds in the rank 5 case and the degree 5(2^0 - 1) + 18.2^0.

Comments and open issues. From the above results, it would be interesting to see that \( Q^{\mathcal{S}}_{55} \) is 730-dimensional in degree 5(2^0 - 1) + 18.2^0, but the space of \( GL_5 \)-coinvariants of it in this degree is only one-dimensional. In general, it is quite efficient in using the results of the hit problem of five variables to study \( \mathcal{F} \otimes_{GL_5} P_{\mathcal{S}}((\mathcal{P}_5)^*) \). This provides a valuable method for verifying Singer’s open conjecture on the fifth algebraic transfer. We now close the introduction by discussing about Conjecture 1.1 in the rank 5 case and the internal degree \( n_1 := 5(2^t - 1) + 18.2^t \) for all \( t \geq 1 \). Let us note again that the iterated Kameko homomorphism \((\overline{S}_Q^{\mathcal{S}}(5,n_1))^{t-1} : Q^{\mathcal{S}}_{n_1} \to Q^{\mathcal{S}}_{n_1} \) is an \( \mathcal{F}_2 \otimes GL_5 \)-module isomorphism for all \( t \geq 1 \). So, from a fact of \( \text{Ext}^{\geq n+1}_{\mathcal{F} \otimes \mathcal{F}}(\mathcal{F} \otimes \mathcal{F}) \), to check Singer’s conjecture in the above degree, we need only determine \( GL_5 \)-coinvariants of \( Q_{n_1}^{\mathcal{S}} \) for \( t = 1 \). We must recall that Kameko’s map \((\overline{S}_Q^{\mathcal{S}}(5,n_1))^{t-1} : Q_{n_1}^{\mathcal{S}} \to Q_{n_1}^{\mathcal{S}} \) is an epimorphism of \( GL_5 \)-modules. On the other side, as shown before, the non-zero element \( h_1 f_1 \in \text{Ext}^{5.5+5n_1}_{\mathcal{S}}(\mathcal{F} \otimes \mathcal{F}) \) is detected by the fifth transfer. From these data and Theorem 2.8, one has an estimate

\[ 0 \leq \dim ((\mathcal{F} \otimes GL_5) P_{\mathcal{S}}((\mathcal{P}_5)^*))_{n_1} - 1 \leq \dim (\text{Ker}(\overline{S}_Q^{\mathcal{S}}(5,n_1)))^{GL_5} \]

Moreover, basing the proof of Theorem 2.8 together with a few simple arguments, it follows that the elements in \((\mathcal{F} \otimes GL_5) P_{\mathcal{S}}((\mathcal{P}_5)^*))_{n_1}\) are dual to the classes

\[ \gamma \cdot x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 \]

\[ + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 \]

\[ + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 + x^2 x^2 x^2 x^2 x^2 \]

where \( \gamma \in \mathcal{F}_2 \), and \([\gamma] \in \text{Ker}(\overline{S}_Q^{\mathcal{S}}(5,n_1))\). It could be noticed that calculating explicitly these elements is not easy. However, in view of our previous works [37,39], and motivated by the above computations, we have the following prediction.

Conjecture 2.10 For each \( t \geq 1 \), the space of \( GL_5 \)-invariants elements of \( \text{Ker}(\overline{S}_Q^{\mathcal{S}}(5,n_1)) \) is 1-dimensional.

Since \( h_1 f_1 \in \text{Im}(T^{55}_{\mathcal{F}}) \), for all \( t \geq 0 \), if Conjecture 2.10 is true, then \( T^{55}_{\mathcal{F}} \) is also isomorphism when acting on the coinvariant \((\mathcal{F} \otimes GL_5) P_{\mathcal{S}}((\mathcal{P}_5)^*))_{n_1}\) for \( t \geq 1 \), and so, Conjecture 1.1 holds in bidegree \((5,5+n_1)\). We also wish that our predictions are correct. If not, Singer’s conjecture will be disproved. We leave these issues as future research. At the same time, we also appreciate that some readers may have an interest in solving them.

Overview. Let us give a brief outline of the contents of this paper. Section three contains a brief review of Steenrod squares and some useful linear transformations. The dimensions of the polynomial algebras \( \mathcal{P}_5 \) and \( \mathcal{P}_6 \) in the generic degrees \( n_1 = 5(2^t - 1) + 18.2^t \) and \( 5(2^{t+1} - 1) + n_1 2^{t+4} \) are respectively obtained in section four by proving Theorems 2.2, 2.4, and 2.6. Section five is to present the proof of Theorem 2.8. In the remainder of the text, we give a direct proof of an event claimed above that the non-zero elements \( h_1 f_1 \in \text{Ext}^{5.5+5n_1}_{\mathcal{S}}(\mathcal{F} \otimes \mathcal{F}) \) are detected by \( T^{55}_{\mathcal{F}} \). The proof is based on a representation in the lambda algebra of the fifth Singer transfer. Finally, we describe the set \( (\epsilon^{\mathcal{S}}_{n_1})^{\geq n_0} \) and list some the admissible monomials in \( \epsilon^{\mathcal{S}}_{n_0} \) and the strictly inadmissible monomials in \((\mathcal{P}_5^{n_0})_{n_1}\).

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3 The Necessary Preliminaries

This section begins with a few words on the Steenrod algebra over \( \mathcal{F}_2 \) and ends with a brief sketch of some homomorphisms in [49]. At the same time, we prove some elementary results that will be used in the rest of this text.
3.1 Steenrod squares and their properties

The mod 2 Steenrod algebra $\mathcal{A}$ was defined by Cartan [7] to be the algebra of stable cohomology operations for mod 2 cohomology. This algebra is generated by the Steenrod squares $Sq^i : H^n(X, \mathbb{F}_2) \to H^{n+i}(X, \mathbb{F}_2)$, for $i \geq 0$, where $H^n(X, \mathbb{F}_2)$ denotes the $n$-th singular cohomology group of a topological space $X$ with coefficient over $\mathbb{F}_2$. Steenrod and Epstein [47] showed that these squares are characterized by the following 5 axioms:

(i) $Sq^i$ is an additive homomorphism and is natural with respect to any $f : X \to Y$. So $f^*(Sq^i(x)) = Sq^i(f^*(x))$.

(ii) $Sq^0$ is the identity homomorphism.

(iii) $Sq^i(x) = x \cup x$ for all $x \in H^i(X, \mathbb{F}_2)$ where $\cup$ denotes the cup product in the graded-commutative ring $H^*(X, \mathbb{F}_2)$.

(iv) If $i > \deg(x)$, then $Sq^i(x) = 0$.

(v) Cartan's formula: $Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$.

In addition, Steenrod squares have the following properties:

- $Sq^1$ is the Bockstein homomorphism of the coefficient sequence: $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

- $Sq^i$ commutes with the connecting morphism of the long exact sequence in cohomology. In particular, it commutes with respect to suspension $H^n(X, \mathbb{F}_2) \cong H^{n+1}(\Sigma X, \mathbb{F}_2)$.

- They satisfy the Adem relations: $Sq^i Sq^j = \sum_{0 \leq k \leq i/2} \binom{i-k}{k} Sq^{i-2k} Sq^j$, $0 < i < 2j$, where the binomial coefficients are to be interpreted mod 2. These relations, which were conjectured by Wu [57] and established by Adem [1], allow one to write an arbitrary composition of Steenrod squares as a sum of Serre-Cartan basis elements.

Note that the structure of the cohomology $H^*(X, \mathbb{F}_2)$ is not only as graded commutative $\mathbb{F}_2$-algebra, but also as an $\mathcal{A}$-module. In many cases, the $\mathcal{A}$-module structure on $H^*(X, \mathbb{F}_2)$ provides additional information on $X$.

References

1. J. Adem, The iteration of the Steenrod squares in Algebraic Topology, Proc. Natl. Acad. Sci. USA 38 (1952), 20-726.

2. J.M. Boardman, Modular representations on the homology of real projective space, in Algebraic Topology: Oaxtepec 1991, ed. M. C. Tangor; in Contemp. Math. 146 (1993), 49-70.

3. A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, and J.W. Schlesinger, The mod-p lower central series and the Adams spectral sequence, Topology 5 (1966), 331-342.

4. R.R. Bruner, L.M. Hà, and N.H.V. Hùng, On behavior of the algebraic transfer, Trans. Amer. Math. Soc. 357 (2005), 437-487.

5. M. Brunetti, A. Ciampella, and A.L. Lomonaco, A total Steenrod operation as homomorphism of Steenrod algebra-modules, Ric. Mat. 61 (2012), 1-17.

6. M. Brunetti, and A.L. Lomonaco, A representation of the dual of the Steenrod algebra, Ric. Mat. 63 (2014), 19-24.

7. H. Cartan, Sur l’itération des opérations de Steenrod, Comment. Math. Helv. 29 (1955), 40-58.

8. T.W. Chen, Determination of Ext$^*_F(Z/2, Z/2)$, Topol. Appl. 158 (2011), 669-689.

9. P.H. Chen, and L.M. Hà, Lambda algebra and the Singer transfer, C. R. Math. Acad. Sci. Paris 349 (2009), 1415-1418.

10. M.C. Crabb, and J.R. Hubbuck, Representations of the homology of BV and the Steenrod algebra II, in Algebraic Topology: New trend in localization and periodicity; in Progr. Math. 136 (1996), 143-154.

11. L.M. Hà, Sub-Hopf algebras of the Steenrod algebra and the Singer transfer, Geom. Topol. Monogr. 11 (2007), 101-124.

12. A. Hatcher, Algebraic Topology, Cambridge University Press, 2002, 551 pp.

13. M. A. Hill, M. J. Hopkins, and D. C. Ravenel, On the non-existence of elements of kervaire invariant one, Ann. of Math. (2) 184 (2016), 1-262.

14. N.H.V. Hùng, The cohomology of the Steenrod algebra and representations of the general linear groups, Trans. Amer. Math. Soc. 357 (2005), 4065-4089.

15. N.H.V. Hùng and V.T.N. Quyhn, The image of Singer’s fourth transfer, C. R. Math. Acad. Sci. Paris 347 (2009), 1415-1418.

16. M. Inoue, A generators of the cohomology of the steinberg summand $M(n)$, In: D.M. Davis, J. Morava, G. Nishida, W. S. Wilson and N. Yagita (eds.) Recent Progress in Homotopy Theory (Baltimore, MD, 2000). Contemporary Mathematics, vol. 293, pp 125-139. American Mathematical Society, Providence (2002).

17. M. Inoue, Generators of the cohomology of $M(n)$ as a module over the odd primary Steenrod algebra, J. Lond. Math. Soc. (2) 75 (2007), 317-329.

18. A.S. Janfada, and R.M.W. Wood, The hit problem for symmetric polynomials over the Steenrod algebra, Math. Proc. Cambridge Philos. Soc. 133 (2002), 295-303.

19. A.S. Janfada, and R.M.W. Wood, Generating $H^*(BO(3), \mathbb{F}_2)$ as a module over the Steenrod algebra, Math. Proc. Cambridge Philos. Soc. 134 (2003), 239-258.

20. A.S. Janfada, A criterion for a monomial in $P(3)$ to be hit, Math. Proc. Cambridge Philos. Soc. 145 (2008), 587-599.

21. A.S. Janfada, A note on the unstability conditions of the squares on the polynomial algebra, J. Korean Math. Soc. 46 (2009), 907-918.

22. M. Kameko, Products of projective spaces as Steenrod modules, PhD thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990, 29 pages.

23. W.H. Lin, Ext$^*_F(Z/2, Z/2)$ and Ext$^{s+*}_F(Z/2, Z/2)$, Topol. Appl. 155 (2008), 459-496.

24. Magma Computational Algebra System (V2.25-8), the Computational Algebra Group at the University of Sydney, (2020), http://magma.maths.usyd.edu.au/magma/.

25. J.P. May, A General Algebraic Approach to Steenrod Operations, Lect. Notes Math., vol. 168, Springer-Verlag (1970), 153-231.
26. N. Minami, *The Adams spectral sequence and the triple transfer*, Amer. J. Math. 117 (1995), 965-985.
27. N. Minami, *The iterated transfer analogue of the new doomsday conjecture*, Trans. Amer. Math. Soc. 351 (1999), 2325-2351.
28. J.W. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) 67 (1958), 150-171.
29. J.F. Mothebe, and L. Uys, *Some relations between admissible monomials for the polynomial algebra*, Int. J. Math. Math. Sci., Article ID 235806, 2015, 7 pages.
30. M.F. Mothebe, *The admissible monomial basis for the polynomial algebra in degree thirteen*, East-West J. Math. 18 (2016), 151-170.
31. T.N. Nam, *Cohomology operations*, pages.
32. D.J. Pengelley, and F. Williams, *On the generators of the polynomial algebra as a module over the Steenrod algebra*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J, 1962.
33. F.P. Peterson, *Generators of* $H^*(BU(2); F_p)$, *Algebr. Geom. Topol.* 13 (2013), 2061-2085.
34. D.V. Phúc, *On the generators of the polynomial algebra as a module over the Steenrod algebra*, Abstracts Amer. Math. Soc. 833 (1987), 273-288.
35. D.V. Phúc, and N. Sum, *On a minimal set of generators for the polynomial algebra as a module over the Steenrod algebra*, Acta Math. Vietnam. 42 (2017), 149-162.
36. D.V. Phúc, *The hit problem for the polynomial algebra of five variables in degree seventeen and its application*, East-West J. Math. 18 (2016), 27-46.
37. D.V. Phúc, *The 'hit' problem of five variables in the generic degree and its application*, Topol. Appl. 107321 (2020), 34 pages, in press.
38. D.V. Phúc, *A-generators for the polynomial algebra of five variables in degree 5(2^r - 1) + 6.2^r*, Commun. Korean Math. Soc. 35 (2020), 371-399.
39. D.V. Phúc, *On Peterson’s open problem and representations of the general linear groups*, J. Korean Math. Soc. 58 (2021), 643-702.
40. D.V. Phúc, *On the dimension of* $H^*(\mathbb{Z}_2)\otimes \mathbb{Z}_2$ *as a module over Steenrod ring*, Topol. Appl. 303 (2021), 107856.
41. D.V. Phúc, *The answer to Singer’s conjecture on the cohomological transfer of rank 4*, Preprint 2021, available online at https://www.researchgate.net/publication/352284459, submitted for publication.
42. S. Priddy, *On characterizing summands in the classifying space of a group, I*, Amer. Jour. Math. 112 (1990), 737-748.
43. J. Repka, and P. Selick, *On the subalgebra of* $H^*(\mathbb{R}P^\infty)$ *annihilated by Steenrod operations*, J. Pure Appl. Algebra 127 (1998), 273-288.
44. W.M. Singer, *The transfer in homological algebra*, Math. Z. 202 (1989), 493-523.
45. W.M. Singer, *On the action of the Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991), 577-583.
46. V.P. Snaith, *Stable homotopy - around the Arf-Kervaire invariant*, Birhauser Progress on Math. Series vol. 273 (April 2009), 250 pages.
47. N.E. Steenrod, and D.B.A. Epstein, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J, 1962.
48. N. Sum, *The negative answer to Kameko’s conjecture on the hit problem*, Adv. Math. 225 (2010), 2365-2390.
49. N. Sum, *On the Peterson hit problem*, Adv. Math. 274 (2015), 432-489.
50. N. Sum, *On a construction for the generators of the polynomial algebra as a module over the Steenrod algebra*, in Singh M., Song Y., Wu J. (eds), Algebraic Topology and Related Topics. Trends in Mathematics. Birkhäuser/Springer, Singapore (2019), 265-286.
51. N. Sum, *The squaring operation and the Singer algebraic transfer*, Vietnam J. Math. 49 (2021), 1079-1096.
52. N.K. Tín, *The hit problem for the polynomial algebra in five variables and applications*, PhD thesis, Quy Nhon University, 2017.
53. G. Walker, and R.M.W. Wood, *Polynomials and the mod 2 Steenrod Algebra: Volume 1, The Peterson hit problem*, in London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 2018.
54. G. Walker, and R.M.W. Wood, *Polynomials and the mod 2 Steenrod Algebra: Volume 2, Representations of GL(n; F_2)*, in London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 2018.
55. W.C. Waterhouse, *Two generators for the general linear groups over finite fields*, Linear Multilinear Algebra 24 (1989), 227-230.
56. R.M.W. Wood, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambridges Phil. Soc. 105 (1989), 307-309.
57. W. Wu, *Sur les puissances de Steenrod*, Colloque de Topologie de Strasbourg, 1951, no. IX, 9 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.