Bond-disordered spin systems: Theory and application to doped high-$T_c$ compounds

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We examine the stability of magnetic order in a classical Heisenberg model with quenched random exchange couplings. This system represents the spin degrees of freedom in high-$T_c$ compounds with immobile dopants. Starting from a replica representation of the nonlinear $\sigma$-model, we perform a renormalization-group analysis. The importance of cumulants of the disorder distribution to arbitrarily high orders necessitates a functional renormalization scheme. From the renormalization flow equations we determine the magnetic correlation length numerically as a function of the impurity concentration and of temperature. From our analysis follows that two-dimensional layers can be magnetically ordered for arbitrarily strong but sufficiently diluted defects. We further consider the dimensional crossover in a stack of weakly coupled layers. The resulting phase diagram is compared with experimental data for La$_{2-x}$Sr$_x$CuO$_4$.

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I. INTRODUCTION

Although the interest in disordered spin systems reaches back several decades (for a review, see e.g. Ref. [8]), interest has strongly revived in recent years by the recognition that high-$T_c$ compounds exhibit phases with antiferromagnetic (AFM) order, spin-glass order or stripe order in certain ranges of temperature and doping. A prominent example for such materials is La$_{2-x}$Sr$_x$CuO$_4$, in which every Cu atom carries a spin $\frac{1}{2}$ (for an overview on this material, see e.g. Ref. [2]).

In these materials, the spins are located in weakly coupled layers. Within each layer, the spins constitute a square lattice with antiferromagnetic exchange coupling between nearest neighbors. In the undoped system, these spins can be represented to a good approximation by a classical two-dimensional model since quantum fluctuations lead to a merely quantitative renormalization of the classical parameters. Although a weak coupling between the layers and a weak easy-plane anisotropy are present in these materials (both are about five orders of magnitude smaller than the isotropic intra-plane exchange), they become relevant only on relatively large scales. Thus, on the finite length scales of experimental relevance, the spins can be described in a first approximation by a classical Heisenberg antiferromagnet in two dimensions.

Doping induces holes in the layers, which can lead to quenched frustration of the exchange couplings. Frustration can occur when the holes are localized individually, as well as when they condense into a topologically defective array of stripes which act as antiphase domain walls for the AFM order. Both cases can be represented by an effective bond-disordered Heisenberg model. Such a system is expected to display generic spin-glass behavior, as is observed in experiments beyond a critical doping.

On the theoretical side, Heisenberg spin glasses are much less understood than Ising spin glasses. While it is well established for the pure two-dimensional system that the magnetic correlation length $\xi$ decays exponentially with increasing temperature, the dependence of $\xi$ on disorder is controversial. At $T = 0$ it is not clear whether $\xi$ is finite for every small but finite concentration of defect bonds or whether $\xi$ is infinite up to a critical concentration even for strong defects. Concerning the temperature dependence of $\xi$, there is no consensus as to whether it is reentrant or not.

II. MODEL

In a system with sparse substitutional doping, the dopants can be considered as randomly distributed with negligible correlations and as quenched over a wide temperature range. In doped Mott insulators, the effect of the dopants is twofold: they induce holes in the cuprate planes and at the same time they provide a random potential that localizes these holes. Therefore, they are commonly assumed to be localized on the oxygen atoms between neighboring copper atoms.

A simple argument suggests that such a hole trans-
mutes the antiferromagnetic superexchange coupling between the neighboring copper spins into a ferromagnetic one. Thereby frustration is induced in the spin system. The irrelevance of quantum fluctuations of the spins in the pure system (in the sense that they lead only to a quantitative renormalization of classical parameters such as the spin stiffness) suggests that they may be neglected in a good approximation also for small doping.

A. Definition

Starting from a classical description of the AFM, we may examine an equivalent model where all spins of one bipartite sublattice are flipped and the sign of all exchange couplings is reversed, i.e., we now consider the coupling of the undoped system to be ferromagnetic and the defect couplings to be antiferromagnetic. We base our analysis on the Hamiltonian

$$H = \frac{1}{2} \sum_{r,i} J_i(r) |\nabla_i \vec{S}(r)|^2$$  \hspace{1cm} (1)

for spins on a (hyper)cubic lattice in $d$ dimensions (a single layer is described by $d = 2$). Spins are treated as classical $N$-component vectors of unit length, $\vec{S}(r) = 1$. A spin at site $r$ is coupled to its nearest neighbor in direction $i = 1, \ldots, d$ via the exchange coupling $J_i(r)$. We define $\nabla_i \vec{S}(r) := \vec{S}(r + a_i) - \vec{S}(r)$ for a basis vector $a_i$. The global symmetry $O(N)$ of spin rotations is preserved for arbitrarily disordered exchange couplings.

It is convenient to rewrite the exchange coupling as

$$J_i(r) = |1 - \Delta_i(r)| J$$  \hspace{1cm} (2)

with the value $J > 0$ of the pure system and the quenched random variable $\Delta_i(r)$. Frustration effects emerge for $\Delta_i(r) > 1$, when a ferromagnetic bond becomes antiferromagnetic. To further specify the nature of disorder, we assume a bimodal distribution of the exchange couplings,

$$\Delta_i(r) = \begin{cases} \Delta & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$  \hspace{1cm} (3)

For square lattices as in La$_{2-x}$Sr$_x$CuO$_4$, the concentration $x$ is related to this probability by $x = 2p$. Correlations between different bonds are assumed to be absent.

B. Single defects

Before we address the magnetic order in the presence of a finite concentration of defects, it is instructive to recall briefly the physics of a single defect for $N \geq 2$. While the ground state of the pure system is collinear (all spins are strictly parallel), it is canted (i.e., no longer collinear) for a single defect beyond a certain critical strength $\Delta_{\text{single}}$. The threshold $\Delta_{\text{single}}$ d value can be obtained from a spin-wave calculation (see also appendix B). The ground state for a single defect with $\Delta > \Delta_{\text{single}}$ remains planar for all $N \geq 2$, i.e., apart from a global rotation, the defect texture for Heisenberg spins ($N = 3$) is identical to the texture for XY spins ($N = 2$). Far away from the defect bond, the texture is described by the solution of the Laplace equation,

$$\nabla^2 \vec{S}(r) = 0$$  \hspace{1cm} (4)

and the spins approach a collinear configuration, $\vec{S}(r) \rightarrow \vec{S}_0$ for $r \rightarrow \infty$. To be specific, we discuss a defect bond located at $r = 0$ and oriented in direction $a$. It acts as a source of a dipolar distortion with a moment $\mu$ perpendicular to $\vec{S}_0$. For $d = 2$, in particular,

$$\vec{S}(r) - \vec{S}_0 \approx \frac{\mu}{2\pi} \frac{a \cdot r}{r^2}.$$  \hspace{1cm} (5)

The amplitude of the dipole moment is determined by the nonlinearity of the model, which may be considered as interaction between spin waves. The magnetic textures of several defect bonds are subject to a dipolar interaction at large distances. The effects of a finite density of defects are nontrivial because of the frustrated long-range interaction between the dipoles. In the presence of more than two defects, the interaction among the dipoles typically becomes frustrated. Then the ground state is no longer planar for $N > 2$.

C. Finite defect concentration

Depending on the relative position of several defect bonds in a cluster, the generated texture may have dipolar or higher-order multipolar structure. In contrast to electrodynamics, the multipolar moments of defect textures are not additive since the spins act as a nonlinear medium for the dipoles, giving rise to many-body interactions. In particular, clusters of defects can lead to canting already for defect-bond strengths below $\Delta_{\text{single}}$, i.e., in clusters of defects the threshold strength of the defect bonds is reduced. The threshold values of some specific bond configurations in the XY model are given in Refs. 12, 21, and 22 (cf. in particular Table I in the latter one). In general, threshold values in the XY model are upper bounds for the thresholds in a Heisenberg system since with increasing $N$ the spins have a larger space of canted states which they may explore to minimize energy. Thus, as soon as the defects are antiferromagnetic ($\Delta > 1$) they can induce canted textures due to the presence of arbitrarily large clusters. Hence the ground state will be canting for every $p > 0$ and $\Delta > 1$ (assuming that the defect distribution has no pathological spatial correlations). Therefore one may expect a system with a certain density of weak defects $1 < \Delta < \Delta_{\text{single}}$ to be qualitatively equivalent to a system with a (possibly much) smaller concentration of strong defects $\Delta > \Delta_{\text{single}}$. 


The presence of canting certainly implies a reduction of magnetization. However, one cannot draw direct conclusions about the range of magnetic order. For a sufficiently high density of strong defects, one certainly has short-range order. For defects of a weaker strength or a lower concentration, one can have also quasi-long-range order (as in a XY model with quenched and uncorrelated dipoles). A phase with true long-range order is also possible but unlikely since it would require a highly ordered dipole configuration.

It is instructive to recall a “duality” relation in the $(p, \Delta)$ parameter space, since in a mixture of ferromagnetic and antiferromagnetic bonds there are two ways to declare one type as “regular” bond and the other type as “defect” bond (see appendix A). This duality implies a relation

$$
\Delta_{AFM}(p) = \frac{\Delta_{FM}(1 - p)}{\Delta_{FM}(1 - p) - 1}
$$

between boundary lines $\Delta_{FM}(p)$ and $\Delta_{AFM}(p)$ limiting regions with ferromagnetic order (for $\Delta < \Delta_{FM}$) and antiferromagnetic order (for $\Delta > \Delta_{AFM}$) in the $(p, \Delta)$ plane. Since the duality relation maps the region $p\Delta > 1$ onto $p\Delta < 1$ it is sufficient to examine the latter region.

To perform first qualitative estimates, one may consider the disorder-averaged exchange coupling, which is

$$
\overline{J_i(r)} = (1 - p\Delta)J
$$

for the bimodal distribution (3). Thus, there is a tendency towards the formation of ferromagnetic order for $p\Delta < 1$ (where $\overline{J} > 0$) and a tendency towards antiferromagnetic order for $p\Delta > 1$ (where $\overline{J} < 0$). Certainly, the presence of order requires more than such a tendency. A further minimum requirement should be that the relative fluctuations

$$
\sigma^2 := \frac{\overline{J_i^2(r)} - \overline{J_i(r)}^2}{\overline{J_i(r)}^2} = \frac{p(1 - p)\Delta^2}{(1 - p\Delta)^2}
$$

of the exchange couplings must be small. Since $\Delta > 1$ and $p\Delta < 1$ in the range of interest, $\sigma^2 \ll 1$ requires $p\Delta^2 \ll 1$. A very crude estimate of the boundary from $\sigma^2 = 1$ suggests

$$
\Delta_{FM}(p) \approx p^{-1/2}.
$$

For the special case $\Delta = 1$, the defect bonds have a vanishing exchange coupling and the system is bond-diluted. The presence of magnetic order (in the sense of a finite magnetization) then requires that a finite fraction of spins is connected by regular bonds. This is the case below the percolation transition, i.e., for $p < \frac{1}{4}$ in $d = 2$. Therefore one expects $\Delta_{FM}(p) = 1$ for $p \geq \frac{1}{2}$.

### D. Previous work

There are only a few approaches in the literature aiming at a more sophisticated analysis, which we briefly summarize in order to highlight our motivation to reconsider this problem in an alternative way. First, the coherent-potential approximation (CPA) provides a simple self-consistent approach to determine an effective spin stiffness for the random system. For $N = 2$, the CPA yields a transition line $\Delta_{FM}(p)$ smoothly interpolating between $\Delta_{FM}(0) = d$ (reflecting the canting threshold for individual defect bonds) and $\Delta_{FM}(p) = 1$ for $p \geq \frac{1}{2}$ in $d = 2$ and for $p \geq \frac{1}{2}$ in $d = 3$. In $d = 2$, the location of the percolation transition is captured exactly. This has to be considered as a fortunate coincidence which is absent in $d \geq 3$, where the location is found only approximately. In the limit $N \to \infty$ the CPA yields that the stability of order requires $\sigma^2 < 1$, which results in $\Delta_{FM}(p) \approx p^{-1/2}$ for small $p$ in agreement with the naive estimate (1).

Unfortunately, the interpretation of the CPA in somewhat ambiguous. According to its construction, the CPA replaces the disordered system by a ferromagnetic one with a homogeneous effective exchange constant. The strength of this effective coupling is determined from a self-consistency condition that considers the fluctuations of a single bond. Thus, the CPA misses effects of clusters of defect bonds. The putative transition line is defined as the border line up to which selfconsistent solutions exist. A priori, this border line can be interpreted in two ways: either as the onset of the canting instability or as the onset of short-range magnetic order. Therefore, the CPA can be only of qualitative use.

To gain insight into the nature of the ground state, Gawiec and Grempe (GG) performed numerical studies of the XY model in $d = 2$. Their data suggest a transition line $\Delta_{FM}(p)$ between a phase with quasi-long-range order and short-range order. It starts from at the canting instability $\Delta_{FM}(0) = \Delta_{single} = 2$ and follows $\Delta_{FM}(p) = 1$ beyond the percolation transition. Thus, the data suggest short-range order for an arbitrarily small concentration of defects of a strength exceeding the canting instability of single defects in agreement with CPA. However, as stated in Ref. 23, the question of whether one can have order for $\Delta > \Delta_{single}$ at sufficiently small $p$ remains far from settled due to finite-size effects.

For the Heisenberg model with the bimodal bond distribution, there are only very few numerical studies. For the special case of two dimensions and $\Delta = \Delta_{single} = 2$, Nonomura and Ozeki postulate order for $p \lesssim 0.11$ from an exact-diagonalization method for $S = 1/2$ quantum spins. This implies an even larger stability of the classical model, and in particular $\Delta_{FM}(0) > 2$. However, also their conclusion has to be considered with care because of finite-size effects.

Previous analytic approaches are implicitly restricted to defect bond strengths exceeding the canting threshold, $\Delta \gg \Delta_{single}$. The spin system with random bonds is replaced by a spin system with homogeneous bonds coupled to an additional canting field that also generates dipolar spin textures. In order to shed some light on the quality of this replacement, we perform...
a Hubbard–Stratonovich transformation introducing an auxiliary bivectorial field \( \tilde{f}_i \) via

\[
\exp(-H[\tilde{S}] / T) = \int D\{\tilde{f}\} \exp[-\tilde{H}[\tilde{S}, \tilde{f}] / T].
\]

with

\[
\tilde{H}[\tilde{S}, \tilde{f}] = \sum_{r,i} \left\{ \frac{1}{2} |\nabla_r \tilde{S}(r)|^2 - \tilde{f}_i(r) \cdot \nabla_r \tilde{S}(r) \right. \\
+ \left. \frac{1}{2\Delta_i(r)} \tilde{f}_i^2(r) \right\}.
\]

In this representation, \( \tilde{S} \) and \( \tilde{f}_i \) are thermally fluctuating variables, \( \tilde{S} \) with spherical constraint, \( \tilde{f}_i \) unconstrained. In the Hamiltonian \([10]\) of the transformed system, the spins interact directly via the homogeneous exchange coupling \( J \). In addition, they couple also to the canting field \( \tilde{f}_i \). It is precisely such a coupling that was considered in Refs. [3][4].

The Hubbard–Stratonovich transformation – which yields an exact representation of the original model \([11]\) – shows that the field \( \tilde{f}_i \) has a local selfinteraction potential

\[
V_{ij}(r, r') = \delta_{ij} \delta_{r,r'} J / \Delta_i(r).
\]

On regular bonds with \( \Delta_i(r) = 0, \tilde{f}_i(r) \) is suppressed. On defect bonds with \( \Delta_i(r) > 0, \tilde{f}_i(r) \) has finite fluctuations. For \( \nabla_r \tilde{S}(r) = 0 \) one would have

\[
\langle \tilde{f}_i^a(r) \tilde{f}_j^b(r') \rangle = \delta_{ij} \delta_{r,r'} T / J \Delta_i(r).
\]

However, it is crucial to retain the full correlations between the fluctuations of \( \tilde{S} \) and \( \tilde{f}_i \). The spin-wave saddlepoint equation

\[
\nabla_r \tilde{S}(r) = \tilde{f}_i(r)
\]

immediately shows that \( \tilde{f}_i \) induces a canting of the spin field. If \( \tilde{f}_i(r) \) is considered as fixed and nonvanishing only on a single bond, it induces a dipolar spin texture. It is important to notice that the original spin rotation symmetry of Hamiltonian \([11]\) is preserved in the transformed model \([10]\) only if \( \tilde{f}_i \) is rotated simultaneously with \( \tilde{S} \).

Instead of solving the full problem of two fluctuating fields, Refs. [3][4] proceed with additional assumptions about the nature of \( \tilde{f}_i \). Glazman and Ioselevich \([3]\) (GI) consider a Hamiltonian, where \( \tilde{f}_i(r) \) has a fixed length on the defect bonds. Only the orientation of \( \tilde{f}_i(r) \) is considered as a thermal degree of freedom. Thereby the rotation symmetry is preserved. However, fixing the length of \( \tilde{f}_i(r) \) contradicts to Eq. \([12]\) which shows that also the magnitude of \( \tilde{f}_i(r) \) is a thermally fluctuating quantity that vanishes in the limit of zero temperature. This means that in the approach of GI the strength of disorder is overestimated at low temperatures. This may be a reason for which GI find a reentrant temperature dependence of \( \xi \).

On the other hand, Cherepanov et al. \([4]\) consider \( \tilde{f}_i \) as a quenched field with Gaussian correlations. This is in contradiction to the annealed nature of \( \tilde{f}_i \) as is revealed by the transformed model \([10]\). In addition, spin rotation symmetry is explicitly broken. This treatment is based on the assumption that the spin textures freeze at low temperatures. However, a spontaneous breaking of this symmetry in \( d \leq 2 \) is ruled out by the Mermin–Wagner theorem \([3] \). Thus, one may worry that disorder effects may be overestimated also in this approach. Possibly, the artificial symmetry breaking is related to the spurious generation of random fields in a replica treatment as found by Cherepanov et al. \([4]\).

## III. RG ANALYSIS

Although the Hubbard–Stratonovich transformation was useful to relate previous work to the original model, the introduction of \( f \) is accompanied with additional difficulties. Therefore we choose to continue to work with the original model where the field \( f \) is integrated out. Before we perform a renormalization-group (RG) analysis, we use the standard replica trick to treat disorder.

### A. Replica representation

After \( n \)-fold replication the Hamiltonian reads

\[
H^{(n)} = \frac{1}{2} \sum_{r,i} |1 - \Delta_i(r)| J \psi_i(r)
\]

with the abbreviation

\[
\psi_i(r) := \sum_{\alpha=1}^n |\nabla_r \tilde{S}^\alpha(r)|^2.
\]

(Upper Greek indices label replicas.) Since we assume that the probability distribution of \( \Delta_i(r) \) is uncorrelated and identical for all bonds, disorder averaging leads to the local and translation symmetric Hamiltonian \( \mathcal{H} \) includes the factor \( 1 / T \)

\[
\mathcal{H} = \sum_{r,i} \left\{ \frac{1}{2} K \psi_i(r) - R(\psi_i(r)) \right\}.
\]

The cumulant function \( R \) is specified by

\[
R(\psi_i(r)) := \ln \exp[\Delta_i(r) K \psi_i(r) / 2]
\]

and depends implicitly on \( K := J / T \). For the special case of bimodal disorder \([3]\)

\[
R(\psi) = \ln \left[ 1 - p + p e^{\Delta K \psi / 2} \right]
\]

\[
\approx \begin{cases} 
\frac{1}{2} p \Delta K \psi & \text{for } \Delta K \psi \to 0, \\
\frac{1}{2} \Delta K \psi + \ln p & \text{for } \Delta K \psi \to \infty.
\end{cases}
\]
Note that naturally $R(\psi) = 0$ in the absence of disorder (for $p = 0$ or $\Delta = 0$) and that $R(\psi) = \frac{1}{2} \Delta K \psi$ for $p = 1$, which amounts to an unfrustrated dual model with a corresponding stiffness $(1 - \Delta)J$. In the general case, $R(\psi)$ is a nonlinear function which has a linear asymptotics for small and large arguments, cf. Rq. (17).

Remarkably, the energy depends on $\psi$ only through the combination

$$h(\psi) := K \psi - 2R(\psi). \quad (18)$$

_A priori_, it is not clear whether spin fluctuations are governed by the behavior of $h(\psi)$ at large $\psi$ or at small $\psi$.

The stability of the ferromagnetic state with respect to large-scale spin-wave deformations depends on $h(\psi)$ at small $\psi$ and requires $h'(0) > 0$, i.e., $p \Delta < 1$. (For $p \Delta > 1$ the dual antiferromagnetic ground state is locally stable.) This condition is equivalent to the requirement that the average exchange coupling $\bar{J}$ should remain ferromagnetic.

While the effective stiffness is defined from the slope of the function $h(\psi)$ at $\psi = 0$, its behavior $h(\psi) \sim (1 - \Delta)K \psi$ for $\psi \to \infty$ (which is equivalent to $T \to 0$) reflects the presence of frustration. For $\Delta < 1$ — i.e., in the absence of frustration — $h(\psi)$ is positive for all $\psi$ and fluctuations can renormalize the stiffness to a smaller but positive value. For $\Delta > 1$ — i.e., in the presence of frustration — $h(\psi)$ is negative at large $\psi$ and fluctuations can renormalize the stiffness to zero, signaling the destruction of magnetic order.

In the replica representation, the canting instability for a single defect can be retrieved easily. To this end, one has to keep the cumulant function (17) only on a single bond and to switch it off on all other bonds. In the limit $T \to 0$ (considering the statistical weight for fixed $n$ and an fixed but arbitrary spin configuration), the replicas decouple since $R(\psi) \approx \frac{1}{2} \Delta K \psi + \ln p$. The exchange coupling of the defect bond within each replica is $(1 - \Delta)J$ independent of $p$. Thus, canting occurs in the case $p < 1$ for $\Delta > \Delta_{\text{single}}$ as in the case $p = 1$. While the threshold is independent of $p$, the disorder-averaged dipole moment depends on $p$. Since $\psi$ becomes arbitrarily small slightly above the canting threshold, this dipole moment cannot be calculated from the large-$\psi$ limit of $h$.

Therefore it is in general important to retain the _global_ functional form of $h(\psi)$ [or, equivalently, $R(\psi)$]. An approximation of this function by a Taylor series near $\psi = 0$ would amount to an expansion in cumulants of the defect distribution. The relevance of $R(\psi)$ for large $\psi$ shows that one could miss essential physics by dropping high-order cumulants and – in particular – by using a Gaussian distribution for $\Delta$. Even worse: the function $R(\psi)$ is _nonanalytic_, i.e., its behavior at large $\psi$ is outside the radius of convergence of a cumulant expansion.

### B. Flow equations

In order to address the question of magnetic order in the presence of general fluctuations of $\psi$, we generalize the renormalization-group analysis of Polyakov[12] to the replicated model.

Instead of working with the function $h(\psi)$, it is more physical to use an effective bare stiffness and an effective cumulant function defined by

$$K_0 := K - 2R'(0) = (1 - p \Delta)K, \quad (19a)$$

$$R_0(\psi) := R(\psi) - \psi R'(0) = R(\psi) - \frac{1}{2} p \Delta K \psi \quad (19b)$$

as to remove the linear contribution of $R$, which represents a trivial renormalization of the stiffness. The replacement of the original quantities by the effective ones leaves the energy (17) invariant since $K \psi - 2R(\psi) = K_0 \psi - 2R_0(\psi)$.

Following Polyakov’s analysis of the pure system[12] we renormalize the model by a momentum-shell integration. So far, the Hamiltonian (13) was written in a way explicitly retaining the lattice of the spin sites. We choose the lattice spacing as unit of length and go over to a continuum representation by smoothly interpolating the field $\tilde{S}(r)$ between the lattice sites (preserving the normalization $\tilde{S}^2(r) = 1$ everywhere). In this limit, partial spatial derivatives replace the differences in $\psi_i(r) = \sum_{\alpha=1}^n [\partial_r \tilde{S}_\alpha(r)]^2$ and

$$\mathcal{H} = \int d^d r \sum_i \left\{ \frac{1}{2} K \psi_i(r) - R(\psi_i(r)) \right\}. \quad (20)$$

The replacement of differences by partial derivatives and of the sum by an integral should be a reasonably good approximation, i.e., the replacement of $\psi$ should lead to relative errors small compared to unity for an arbitrary spin configuration (even with inhomogeneities on the scale of the lattice spacing). A small relative error in $\psi$ is equivalent to a small relative misrepresentation of $J$ and/or $\Delta$ since these quantities enter the Hamiltonian only as a product with $\psi$. We further use the approximate replacement of the cubic Brillouin zone by a spherical one. Thus, to preserve the volume of the Brillouin zone, its radius $\Lambda$ has to be fixed by $\Lambda^d = d(2\pi)^d/S_d$ with $S_d := 2\pi^{(d/2)} / \Gamma(d/2)$ the surface of the $d$ dimensional unit sphere. In $d = 2$ specifically, $\Lambda^2 = 4\pi$. In order to demonstrate that the fundamental frustration mechanism does not get lost due to the continuum approximation, we briefly rederive the canting threshold in Appendix 3.

According to the scheme of the momentum-shell renormalization group, we now integrate out the spin modes with wave vectors in the shell $\Lambda e^{-d\ell} < k \leq \Lambda$. Thereby the original spin field $\tilde{S}_\alpha$ is mapped onto the slowly varying background field $\bar{s}_\alpha$. They are related by

$$\bar{s}_\alpha = \sqrt{1 - \chi_\alpha \chi_\alpha} s_\alpha + \chi_\alpha \tilde{e}_\alpha \quad (21)$$
with the vector fields $\vec{e}^\alpha_a$ forming a local orthonormal basis \{\(e_1^\alpha, \ldots, e_{N-1}^\alpha, s^\alpha\)\} at each site \(r\) in each replica \(\alpha\). We employ the sum convention for pairs of Latin indices \((a, b = 1, \ldots, N-1)\) only. The field $\chi^\alpha_a$ generates an infinitesimal spin rotation and has contributions only from wave vectors in the momentum shell.

Derivatives of basis vectors can be expanded in the local bases

\[
\begin{align*}
\partial_i s^\alpha_a &= B^\alpha_{ia} e^\alpha_a \\
\partial_i e^\alpha_a &= -B^\alpha_{ia} s^\alpha + A^\alpha_{ia} e^\alpha_b 
\end{align*}
\]  (22a)  (22b)

in terms of potentials \(A\) and \(B\). The arbitrariness of the choice of the vectors $\vec{e}_a$ is reflected in a gauge invariance of the potentials; the gauge transformations are local rotations around $\vec{r}$. One can exploit this gauge symmetry to show that the potential $A$ corresponds to higher order derivatives of the spin field $\vec{\chi}$. We ignore such contributions and therefore omit the potential $A$ from now on.

For the further calculations, $\chi$ can be considered as small since we consider an infinitesimal momentum shell. In addition, we treat temperature and disorder, which drive the fluctuations of $\chi$, as small. In order to expand the Hamiltonian in $\chi$, we first consider

\[
\begin{align*}
\partial_i \vec{\chi}^\alpha &= B^\alpha_{ia} e^\alpha_i \\
&- \chi^\alpha_i B^\alpha_{ia} s^\alpha + \partial_i \chi^\alpha_a e^\alpha_a \\
&- \chi^\alpha_i B^\alpha_{ia} s^\alpha - \frac{1}{2} \chi^\beta_i \chi^\alpha_i B^\alpha_{ia} e^\alpha_a + O(\chi^4) 
\end{align*}
\]  (23)

and, ordered by powers of $\chi$,

\[
\begin{align*}
\psi_i &= \psi_i^{(0)} + \psi_i^{(1)} + \psi_i^{(2)} + \psi_i^{(2')} + O(\chi^3), \\
\psi_i^{(0)} &= \sum_\alpha B^\alpha_{ia} B^\alpha_{ia} = \sum_\alpha (\partial_i s^\alpha)^2, \\
\psi_i^{(1)} &= \sum_\alpha 2\partial_i \chi^\alpha_a B^\alpha_{ia}, \\
\psi_i^{(2)} &= \sum_\alpha \partial_i \chi^\alpha_a \partial_i \chi^\alpha_a, \\
\psi_i^{(2')} &= \sum_\alpha \left\{ \chi^\alpha_i \chi^\beta_i B^\alpha_{ia} B^\beta_{ib} - \chi^\beta_i \chi^\alpha_i B^\beta_{ib} B^\alpha_{ia} \right\}. 
\end{align*}
\]  (24a)  (24b)  (24c)  (24d)

An expansion of the energy density for small $\chi$ gives

\[
h(\psi_i) = h(\psi_i^{(0)}) + h'(\psi_i^{(0)})[\psi_i^{(1)} + \psi_i^{(2)} + \psi_i^{(2')}] \\
+ \frac{1}{2} h''(\psi_i^{(0)})[\psi_i^{(1)}]^2 + O(\chi^3). 
\]  (25)

Substituting this expression back into Eq. (20), we rewrite

\[
\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}
\]

\[
\mathcal{H}_{\text{free}} = \int d^d r \sum_i \left\{ \frac{1}{2} K_i \psi_i^{(0)} + \psi_i^{(2')} \right\} - R_i(\psi_i^{(0)}) \\
\mathcal{H}_{\text{int}} = \int d^d r \sum_i \left\{ \frac{1}{2} K_i \psi_i^{(1)} + \psi_i^{(2')} \right\} - R_i^\prime(\psi_i^{(0)})[\psi_i^{(1)}]^2 + O(\chi^3). 
\]  (26a)  (26b)  (26c)

Hereby we have separated the “free” and “interaction” contributions in a way such that $\mathcal{H}_{\text{int}}$ vanishes for $\chi = 0$ and that $\mathcal{H}_{\text{free}}$ contains the bilinear selfinteraction of $\chi$.

Due to the energy contribution $\mathcal{H}_{\text{int}}$ the fluctuations cannot be integrated out exactly. In analogy to the treatment of the pure system, we apply standard perturbation theory to $\mathcal{H}_{\text{int}}$. In principle, this can be done in a systematic way at low temperatures, where $\chi \sim T^{1/2}$. Aiming at the analysis of the stability of magnetic order for low temperature and weak disorder, we retain only the renormalization effects to first order in $\mathcal{H}_{\text{int}}$. Integration over $\chi$ leads to an infinitesimal renormalization of the Hamiltonian

\[
d\mathcal{H} = \langle \mathcal{H}_{\text{int}} \rangle + O(\mathcal{H}_{\text{int}}^2) 
\]  (27)

where the average is over the fluctuations of $\chi$ weighted by $\mathcal{H}_{\text{free}}$ only.

Using the averages

\[
\langle \partial_i \chi^\alpha_a \partial_j \chi^\beta_b \rangle = \frac{1}{K_\ell} \delta_{ij} \delta_{ab} \delta^\alpha\beta d \ell, \\
\langle \chi^\alpha_a \chi^\beta_b \rangle = \frac{d}{K_\ell A^2} \delta_{ab} \delta^\alpha\beta d \ell, 
\]  (28a)  (28b)

one finds

\[
\langle \psi_i^{(1)} \rangle = 0, \\
\langle \psi_i^{(2')} \rangle = O(n), \\
\langle \psi_i^{(2'')} \rangle = -(N-2) \frac{d}{K_\ell A^2} \psi_i^{(0)} d \ell, \\
\langle \psi_i^{(1)} \rangle^2 = \frac{4}{K_\ell} \psi_i^{(0)} d \ell. 
\]  (29a)  (29b)  (29c)  (29d)

Separating the flow of $K_\ell$ and of $R_\ell$ by the requirement $R_\ell'(0) = 0$ as in Eqs. (3) for the bare quantities, we finally obtain the flow equations

\[
\frac{d}{d\ell} K_\ell = (d-2)K_\ell - \frac{N-2}{A^2/d} - \frac{4}{K_\ell} R_\ell'(0), \\
\frac{d}{d\ell} R_\ell(\psi) = dR_\ell(\psi) - (2 + \frac{N-2}{K_\ell A^2/d}) \psi R_\ell(\psi) \\
+ \frac{2}{K_\ell} \psi[R_\ell''(\psi) - R_\ell''(0)]. 
\]  (30a)  (30b)
Terms explicitly proportional to the number of replicas \( n \) have been dropped. A rescaling \( r \to e^{4t}r \) of lengths has been included in order to keep the value of the cutoff fixed.

For arbitrary temperature, the flow of \( K \) can be interpreted as a flow of the spin stiffness at fixed temperature. Using the dimensionless stiffness \( j_\ell := K \ell^2 / K \), the dimensionless temperature \( t := T / J = 1 / K \), the rescaled field \( \phi := \Delta K \psi \) (we recall that \( K \) and \( \Delta \) are unrenormalized quantities), and \( \hat{R}_\ell(\phi) := \hat{R}_\ell(\psi) \), we rewrite the flow equations as

\[
\begin{align*}
\frac{d}{dt} j_\ell &= (d - 2) j_\ell - \frac{N - 2}{\ell^2} t - \frac{4\Delta^2}{j_\ell} \hat{R}_\ell'(0), \quad (31a) \\
\frac{d}{dt} \hat{R}_\ell(\phi) &= d \hat{R}_\ell(\phi) - \left( 2 + \frac{N - 2}{\ell^2} \frac{t}{j_\ell} \right) \phi \hat{R}_\ell'(\phi) + \frac{2\Delta}{j_\ell} \phi [\hat{R}_\ell''(\phi) - \hat{R}_\ell''(0)]. \quad (31b)
\end{align*}
\]

In this form, temperature and disorder strength \( \Delta \) appear as explicit parameters. The flow equations (31) have to be solved with the initial conditions

\[
\begin{align*}
j_0 &= 1 - p \Delta, \quad (32a) \\
\hat{R}_0(\phi) &= \ln \left[ 1 - p + p \ e^{\phi/2} \right] - \frac{1}{2} p \phi. \quad (32b)
\end{align*}
\]

Corresponding to the neglect of higher orders of \( \mathcal{H}_{\text{int}} \) in Eq. (27), these flow equations contain renormalization effects only to the leading order in temperature and disorder. Higher orders of perturbation theory would certainly generate higher order contributions as well as a more complicated functional form of the Hamiltonian.

Anticipating that \( \hat{R}_\ell'(0) \geq 0 \) is preserved under the flow, Eq. (31a) shows that both thermal fluctuations and disorder tend to reduce the effective stiffness.

In the given order, several features of the flow equations are remarkable. The initial function \( \hat{R}_0(\phi) \) depends only on \( p \) (cf. Fig. 3). The crossover from the linear regime at small \( \phi \) to the linear regime at large \( \phi \) occurs at \( \phi^* = 2 \ln \frac{1}{p} \), where the curvature \( \hat{R}_0''(\phi) = 1/16 \cosh^2[(\phi - \phi^*)/4] \) has its maximum \( \hat{R}_0''(\phi^*) = \frac{1}{16} \). The flow equation of the stiffness couples to disorder only through \( \hat{R}_\ell''(0) \). For \( \Delta = 0 \), this coupling vanishes, while \( \hat{R}_\ell(\phi) = 0 \) for \( p = 0 \). The flow equations depend on \( N \) only at finite temperatures (unlike the flow equations of Ref. [4]). Thus, the properties at zero temperature are expected to be independent of \( N \). However, one has to keep in mind that the equations apply only to spins with a continuous rotation symmetry (\( N \geq 2 \)) and that the renormalization scheme ignores the effects of topological defects, which are known to be particularly important for \( N = 2 \).

IV. RESULTS AND DISCUSSION

In order to determine the large-scale properties of the model, we numerically integrate the coupled flow equations (31) for \( j_\ell \) and \( \hat{R}_\ell(\phi) \) from \( \ell = 0 \) to large \( \ell \). If \( j_\ell \) converges for \( \ell \to \infty \) to a finite value much larger than \( t \), the system is in an ordered state with finite renormalized stiffness and infinite correlation length, i.e. \( \langle S(\mathbf{r})S(\mathbf{r}') \rangle \) decays slower than exponentially with the distance \( |\mathbf{r} - \mathbf{r}'| \). If, on the other hand, \( j_\ell \) becomes of order \( t \) on a finite scale \( \ell = \ell^* \), the system is disordered and we identify the correlation length as

\[
\xi = A \ e^{\ell^*} \quad (33)
\]

with some constant \( A \) of the order of the lattice spacing. Following previous references[4,14] we specifically define this scale from

\[
j_\ell^* = \frac{t}{2\pi}. \quad (34)
\]

The critical disorder strength \( \Delta_{\text{FM}} \) separating a ferromagnetically ordered phase from a disordered phase can be identified with the line where \( \xi \) diverges. Unfortunately, this criterion does not allow for a distinction between true long-range order and quasi-long-range order in the ferromagnetic phase.

A. Zero temperature

At \( T = 0 \), we find the transition line \( \Delta_{\text{FM}}(p) \) as shown in Fig. 4. This transition has the following features:

- **Regime of small \( p \)**: For dilute disorder, \( p \to 0 \), we find a slow but unbounded increase of \( \Delta_{\text{FM}}(p) \). In the range \( 10^{-5} \leq p \leq 0.1 \) the line follows roughly the relation (cf. the dashed line in the inset of Fig. 4)

\[
\Delta_{\text{FM}}(p) \approx 0.606 + 0.648 \ln \frac{1}{p}. \quad (35)
\]
However, in the limit $p \to 0$, $\Delta_{FM}(p)$ appears to increase slower than logarithmically.

This finding implies order below a finite defect concentration even for arbitrarily strong defect bonds. This behavior is in disagreement with Refs. 13 and 14, where disorder (with $\Delta > \Delta_{single}$) was suggested to destroy order for infinitesimally small $p$. As already stated in Sec. II D, we believe that the effects of disorder are overestimated in both previous approaches because of special assumptions about the nature of the bivectorial field $\vec{f}_i$.

For the special case $N = 2$, where this disagreement persists, further references can be included in the comparison. (We assume here that XY and Heisenberg systems should have a similar phase diagram at $T = 0$ since our lowest-order flow equations are independent of $N$.

Topological defects are ignored in our work as well as in Ref. 14. Their presence may further reduce $\xi$. The way how disorder is treated by Cherepanov et al. implies that their analysis actually describes an XY model with random phase shifts. Various recent work (see e.g. 24) has provided evidence that quasi-long-range order should exist for weak disorder (even in the presence of vortices). This observation is consistent with our flow equations but contradicting the flow equations of Cherepanov et al.

From their numerical data, Gawiec and Grempel argue for a disordered phase for $\Delta > \Delta_{single}$ and $p > 0$, i.e., for $\Delta_{FM}(0) = 2$. While this conclusion is again in disagreement with our result, the numerical data are not: Gawiec and Grempel present data for $\Delta = 4$ as the only value with $\Delta > 2$ and they demonstrate the absence of order only for $p \gtrsim 0.02$. For $\Delta = 4$, we find order at very small $p \lesssim 0.0062$, which actually is not excluded by the numerical data (cf. Fig. 17 of Ref. 22).

Regime of large $p$. For $p \to 1^-$ we find that $\Delta_{FM}(p) \to 1^+$ in a smooth way. While this is qualitatively correct, a horizontal segment with $\Delta_{FM}(p) = 1$ beyond the percolation transition at $p = \frac{1}{2}$ is absent. However, in this range we find a $\Delta_{FM}(p)$ which is only slightly larger, i.e., this is a quantitative effect which may be attributed to the continuum approximation as argued below Eq. (20). In addition, the regime near $\Delta = 1$ and $p \gtrsim \frac{1}{2}$ is a regime of strong disorder in the sense of Sec. III C since $\sigma^2_1 \gtrsim 1$.

There our flow equations are not quantitatively reliable also because of the lowest-order truncation. On the other hand, the transition at small $p$ should be well described since Eq. (3) is satisfied, $p\Delta_{FM}(p) \to 0$ for $p \to 0$.

When the transition line is approached at finite $p$, the correlation length displays a divergence

$$\xi(\Delta, p) \sim [\Delta - \Delta_{FM}(p)]^{-\nu}. \quad (36)$$

From the numerical integration of our flow equations we find with a mean-field like exponent $\nu = 0.500(1)$ [cf. the inset of Fig. 3].

### B. Finite temperatures

Temperature enters the flow equations in two places where it could lead to contrary effects. In Eq. (31b), an increase of temperature leads to a faster renormalization of the spin stiffness to smaller values [ignoring the temperature dependence of $\hat{R}_{\ell}'(0)$]. On the other hand, in Eq. (31a) temperature tends to suppress $\hat{R}$, which might in turn reduce the efficiency of disorder in suppressing the spin stiffness. However, from our flow equations, we
always find that thermal fluctuations reduce stiffness and therefore also $\xi$.

More precisely, $\xi$ decreases monotonously with increasing temperature. Thus, a reentrant temperature dependence as found by GI is absent in the present treatment. We attribute this discrepancy to the fact that GI keep the length of the bivector $\vec{f}_i$ fixed, whereas its typical length should vanish at low $T$ according to Eq. (12). Thereby, with decreasing temperature, the strength of disorder is increasingly overestimated, giving way to an apparent reentrance.

C. Comparison with experiments

We now turn to check the consistency of our theory with measurements on La$_{2-x}$Sr$_x$CuO$_4$. To allow for a comparison of our results with those of Cherepanov et al. we refer to the same experimental data by Keimer et al. for $x = 0.02, 0.03$ and 0.04. For the moment we assume that interlayer couplings and spin anisotropies can be neglected and come back to this issue later on.

The comparison of our results for $\xi$ with experiments involves four parameters, $p$, $\Delta$, $J$, and $A$. As already stated above, our model parameter $p = x/2$ is directly related to the dopant concentration $x$. $A$ is a length scale of the order of the lattice spacing. In principle, this parameter can depend on temperature and on disorder itself (compare the discussion in Ref. 14). Such dependences could modify the function $\xi(x, T)$ in a subdominant way and would involve additional assumptions and parameters. We refrain from including such dependences for the purpose of the subsequent semiquantitative comparison.

At $T = 0$, $J$ does not enter the flow equations since it simply sets the energy scale and $\Delta$ is the only unknown model parameter which enters the numerical calculation of $\ell^\star$. From the consideration of the superexchange across a defect bond one expects $\Delta \gg 1$ i.e., a value clearly above the canting instability. The finiteness of the measured values of $\xi$ for $x \geq 0.02$ implies that the data points lie in the disordered phase, i.e., $\Delta > \Delta_{\text{FM}}(p = 0.01) \approx 3.67$.

We have determined values of the parameters $\Delta$, $A$, and $J$ from the requirement that our theoretical values for $\xi$ should be consistent with the experimental data. In view of the given experimental errors and the approximate nature of our RG calculation, we found a satisfactory agreement for

$$\Delta = 3.7, \quad J = 240 \text{ K}, \quad A = 5.8 a,$$

(37a) (37b) (37c)

where $a = 3.8 \text{Å}$ is the lattice spacing. In Fig. 3 we compare our theory with data for $\xi$ as a function of $x$ at $T = 0$. The inset shows shows for $\Delta = 3.7$ a double-logarithmic plot of $\xi(x)$ which reveals the mean-field-like divergence of $\xi$ according to Eq. (36) near the order-disorder transition.

The temperature dependence of $\xi$ is compared in Fig. 4. Keeping in mind the strong fluctuations of the experimental data, they can be considered as consistent with our analysis. However, the theoretical dependence of $\xi$ on $T$ and $p$ does not quantitatively confirm the empirical formula

$$\xi^{-1}(T, p) = \xi^{-1}(T, 0) + \xi^{-1}(0, p),$$

(38)

which would imply that in Fig. 4 the curves for different $p$ should differ only by a vertical shift. In contrast, we find that thermal fluctuations lead to a stronger increase of $\xi^{-1}$ for smaller $p$.

D. Coupled layers

We now address the effects of a very weak interlayer coupling $J^\perp$. For La$_2$CuO$_4$, a ratio $J^\perp/J = 5 \times 10^{-5}$ was determined from experiments. From a simple scaling analysis one immediately obtains the flow equation

$$\frac{d}{d\ell} J^\perp = 2 J^\perp$$

(39)

which shows the strong relevance of this coupling. From the condition that the interlayer coupling becomes comparable to the intralayer coupling, $J^\perp = J$, one can fix a dimensional crossover scale

$$\ell^\perp = \frac{1}{2} \ln \frac{J}{J^\perp}.$$

(40)

Comparing this scale with the correlation length ($\xi^\perp$) obtained in the absence of the layer coupling, one expects that the coupling actually is irrelevant for $\ell^\star < \ell^\perp$. 

![Graph](image-url)
where 2D fluctuations on small scales renormalize $J^\perp$ to zero. On the other hand, for $\ell^* > \ell^\perp$ fluctuations become three-dimensional on scales larger than $\ell^\perp$. Per definition of $\ell^*$, the exchange coupling on the scale $\ell^\perp$ is still large compared to temperature such that magnetic long-range order should be stable. Therefore, the location of three-dimensional ordering in the $(x, T)$ plane can be determined from the implicit condition

$$\ell^*(x, T) = \frac{1}{2} \ln \frac{J}{J^\perp}. \quad (41)$$

We have evaluated this condition numerically for $J^\perp/J = 5 \times 10^{-5}$ and the parameter set ($\ell^\perp$). The resulting Néel temperature is plotted in Fig. 5 as a function of doping. The transition temperature is normalized by its value $T_N(0) \approx 300$K in the absence of disorder, which essentially reflects the value of $J$. While the precise value of $J$ may vary to some extent with the employed fitting procedure for the parameter set, the shape of normalized transition line is very robust. At $T = 0$, the critical disorder strength $x_{\text{3D}}$ is increased by the interlayer coupling only slightly over $x_{\text{2D}} \approx 0.1906$, $(x_{\text{3D}} - x_{\text{2D}})/x_{\text{2D}} \approx 10^{-2}$, because of the extremely slow divergence of $\xi$ for $x \rightarrow x_{\text{2D}}^+$.

In comparison to the theory in Ref. 14, we find that $T_N(x)$ decays less abruptly near $x_{\text{3D}}$. We oppose our results to data from Hall measurements (squares in Fig. 5), susceptibility measurements (triangles), and PAC measurements (crosses). The overall agreement is satisfying, although the experimental data partially suggest that disorder is less effective in reducing the transition temperature. This tendency may be attributed to the fact that the samples were partially oxygen-doped (at higher temperature, oxygen is not quenched and less effective in generating spin frustration) as well as the fact that we have neglected the easy-plane spin anisotropy, which also tends to stabilize the magnetic order.

V. CONCLUSIONS

In this article, we have reexamined a classical model for $N$-component spins ($N \geq 2$) with random exchange couplings. Thereby we have chosen an approach complementary to previous studies by Glazman and Ioselevich and by Cherepanov et al. Special care has been taken to preserve the quenched nature of disorder and the global spin rotation symmetry.

Our analysis involves approximations which we briefly summarize. (i) The originally discrete spin system is represented in a continuum formulation. Thus, features related to the specific lattice structure – such as the location of the percolation transition for $\Delta = 1$ – cannot be captured quantitatively. (ii) As in the pure case, the renormalization scheme accounts only for interactions between spin waves, ignoring the role of topological defects. Thus, the degree of magnetic order may be overestimated (recall that for $N = 2$ the pure system erroneously appears to be ordered at all temperatures if vortices are neglected). (iii) The flow equations are truncated to lowest order in temperature and disorder. In principle, the analysis could be extended to higher orders. In practice, this extension is hampered by a much more complicated functional form of the Hamiltonian which is generated during the flow.

Due to the nonanalytic nature of the cumulant function $R$, we found it necessary to develop a functional renormalization group, i.e., to keep track of arbitrarily high cumulants of the disorder distribution. To the best of our knowledge, functional flow equations for disordered spin systems have been considered previously only for different types of disorder, in particular for random fields and random anisotropy.

The flow equations (31) are the central result of the analytic part of this work. For comparison, the analysis of Cherepanov et al. which employs approximations corresponding to the ones listed above, is restricted to a single disorder parameter that corresponds to the lowest cumulant of the disorder distribution.

As main physical result from our RG analysis, we find that the two-dimensional spin system can be magnetically ordered at $T = 0$ in the presence of a sufficiently small but finite concentration of arbitrarily strong defects ($\Delta > \Delta_{\text{single}}$). At this place it is worthwhile to recall that we have identified magnetic order from the length scale where the spin stiffness is renormalized down to the scale of thermal fluctuations. In the absence of explicit calculations for the spin-spin correlation function it is natural to assume that this scale coincides with the magnetic correlation length $\xi$. Such a calculation would be desirable in order to clarify whether the ordered phase (with $\xi = \infty$) has quasi-long-range order or true long-range order. We have determined $\xi$ by a numerical integration of our flow equations for the strictly two-dimensional system as well as for weakly coupled layers. In the first case our zero-temperature phase diagram is consistent with numerical simulations. In the second case, the calculated depen-
dence of $\xi$ on temperature and on disorder strength is in good agreement with measurements on cuprates. In both cases the comparison was restricted to finite length scales given by the computationally manageable system sizes or by the experimental error bars, respectively.

Nevertheless, concerning the question of whether magnetic order is stable against bond disorder in two dimensions on largest scales, our positive answer, shared by Ref. 13, disagrees with previous negative ones. In the end of Sec. III D we have given specific reasons why in our opinion the previous answers cannot be considered as final. In view of the complementary approaches and approximations involved, this question has to be considered as an open one that calls for additional future research.

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APPENDIX A: DUALITY

For the bimodal distribution 3 with $\Delta > 1$, our system consists of a fraction $p$ of “defective” antiferromagnetic bonds and a fraction $1 - p$ of “regular” ferromagnetic bonds. The relative strength of bonds is $|J_{AFM}/J_{FM}| = 1 - \Delta$. From a dual point of view, one may say that the system consists of a fraction $\tilde{p} := 1 - p$ of “defective” ferromagnetic bonds and a fraction $1 - \tilde{p} = p$ of “regular” antiferromagnetic bonds. Since for classical spins thermodynamic properties are invariant under flipping one sublattice and reversing the sign of the exchange coupling ($J_{AFM} = -J_{FM}$ and $J_{FM} = -J_{AFM}$), the system is equivalent to a system with a fraction $\tilde{p} := 1 - p$ of “defective” antiferromagnetic bonds and a fraction $1 - \tilde{p} = p$ of “regular” ferromagnetic bonds of relative strength $|J_{AFM}/J_{FM}| = 1/(1 - \Delta) =: 1 - \tilde{\Delta}$. Thus duality provides a mapping

$$J \rightarrow \tilde{J} = (1 - \Delta)J,$$
$$p \rightarrow \tilde{p} = 1 - p,$$
$$\Delta \rightarrow \tilde{\Delta} = \frac{\Delta}{\Delta - 1},$$

for all temperatures.

APPENDIX B: CONTINUUM LIMIT

We show that spin frustration mechanism is well captured in the continuum representation of our model. To this end, we rederive the canting instability threshold $\Delta_{\text{single}} = d$ for a single defect bond. For a single defect bond located at $\mathbf{r} = 0$ and oriented in direction $\mathbf{e}_j$ the Hamiltonian reads

$$H = \frac{J}{2} \int d^d \mathbf{r} \sum_i (1 - \delta_{ij} \delta(\mathbf{r})\Delta)|\partial_i \tilde{S}(\mathbf{r})|^2.$$  \hspace{1cm} (B1)

We introduce a canting field $\tilde{\chi}(\mathbf{r})$, $\tilde{\chi}^2(\mathbf{r}) \leq 1$, perpendicular to a collinear ground state $\tilde{S}_0(\mathbf{r}) \equiv \tilde{S}_0$ of the pure system via

$$\tilde{S}(\mathbf{r}) = \sqrt{1 - \tilde{\chi}^2(\mathbf{r})}\tilde{S}_0 + \tilde{\chi}(\mathbf{r}).$$  \hspace{1cm} (B2)

To show the canting instability we insert (B2) into the Hamiltonian and minimize the energy with respect to the canting field $\tilde{\chi}$ after expanding the Hamiltonian up to quadratic order in this field. In Fourier space the saddle-point equation reads

$$\tilde{\chi}_q = \Delta \frac{q_j}{q} \int k_j \tilde{\chi}_k.$$  \hspace{1cm} (B3)

Multiplying equation (B3) with $q_i$ and integrating over $q$ we get

$$\tilde{\mu}_i = \frac{\Delta}{d} \delta_{ij} \tilde{\mu}_j,$$  \hspace{1cm} (B4)

with

$$\tilde{\mu}_i := \int k_i \tilde{\chi}_k.$$  \hspace{1cm} (B5)

This self-consistency condition on $\tilde{\mu}_i$ implies

$$\Delta = \Delta_{\text{single}} = d$$  \hspace{1cm} (B6)

for a non-vanishing solution of the saddle-point equation (B3).

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