EXTENSION BETWEEN SIMPLE MODULES OF PRO-$p$-IWAHORI
HECKE ALGEBRAS

NORIYUKI ABE

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo,
Hokkaido, 060-0810, Japan
(abenori@math.sci.hokudai.ac.jp)

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Abstract We calculate the extension groups between simple modules of pro-$p$-Iwahori Hecke algebras.

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1. Introduction

Let $F$ be a non-Archimedean local field of residue characteristic $p$ and $G$ a connected reductive group over $F$. Motivated by the modulo $p$ Langlands program, we study the modulo $p$ representation theory of $G$. As in the classical (the representations over the field of complex numbers), Hecke algebras are useful tools for the study of modulo $p$ representations. Especially, a pro-$p$-Iwahori Hecke algebra which is attached to a pro-$p$-Iwahori subgroup $I(1)$ has an important role in the study. (One reason is that any nonzero modulo $p$ representation has a nonzero $I(1)$-fixed vector.) For example, this algebra is one of the most important tools for the proof of the classification theorem [5].

We focus on the representation theory of pro-$p$-Iwahori Hecke algebra. Since the simple modules are classified [3, 14, 18], we study its homological properties. The aim of this paper is to calculate the extension between simple modules. Note that such calculation was used to calculate the extension between irreducible modulo $p$ representations of $G$ when $G = \text{GL}_2(Q_p)$ [16]. As far as the author knows, a calculation of extensions was done only when $G = \text{GL}_2$. Our calculation is in general, namely we do not assume anything about $G$. We also remark a related result in [1]. If $\pi_1, \pi_2$ are modulo $p$ irreducible subquotients of principal series of $G$, by the main theorem of [1], then we have an embedding $\text{Ext}^1_{\mathcal{H}}(\pi_1^{(1)}, \pi_2^{(1)}) \hookrightarrow \text{Ext}^1_G(\pi_1, \pi_2)$. Hence, the calculation in this paper should be helpful to calculate extensions between $\pi_1$ and $\pi_2$. When $\pi_1, \pi_2$ are principal series, some calculations are done by Hauseux [10, 11].

We explain our result. For each standard parabolic subgroup $P$, let $\mathcal{H}_P$ be the pro-$p$-Iwahori Hecke algebra of the Levi subgroup of $P$. Then for a module $\sigma$ of $\mathcal{H}_P$, we
can consider: the parabolic induction $I_P(\sigma)$ which is an $H$-module, a certain parabolic subgroup $P(\sigma)$ containing $P$, a generalized Steinberg module $\text{St}_{Q_1}^P(\sigma)(\sigma)$, where $Q$ is a parabolic subgroup between $P$ and $P(\sigma)$. By \cite{3}, each simple module is constructed by three steps: (1) starting with a supersingular module $\sigma$ of $H_P$, where $P$ is a parabolic subgroup; (2) take a generalized Steinberg module $\text{St}_{Q_1}^P(\sigma)$ (3) and take a parabolic induction $I_P(\sigma)(\text{St}_{Q_1}^P(\sigma))$. (We do not explain the detail of notation here.) Our calculation follows these steps. Let $\pi_1 = I_P(\sigma_1)(\text{St}_{Q_1}^P(\sigma_1)(\sigma_1))$ and $\pi_2 = I_P(\sigma_2)(\text{St}_{Q_2}^P(\sigma_2)(\sigma_2))$ be two simple modules here $\sigma_1$ (resp. $\sigma_2$) is a simple supersingular module of $H_{P_1}$ (resp. $H_P$).

(1) By considering the central characters, the extension $\text{Ext}^i_H(\pi_1,\pi_2)$ is zero if $P_1 \neq P_2$ (Lemma 3.1). Hence, we may assume $P_1 = P_2$. Set $P = P_1$.

(2) We prove

$$\text{Ext}^i_H(I_P(\sigma_1)(\text{St}_{Q_1}^P(\sigma_1)(\sigma_1)),I_P(\sigma_2)(\text{St}_{Q_2}^P(\sigma_2)(\sigma_2))) \simeq \text{Ext}^i_H(\text{St}_{Q_1}^P(\sigma_1),\text{St}_{Q_2}^P(\sigma_2))$$

for some $Q_1$, $Q_2$ and $P$ \cite{4}. For the proof, we use the adjoint functors of parabolic induction and results in \cite{2}. Hence, it is sufficient to calculate the extension groups between generalized Steinberg modules.

(3) We prove

$$\text{Ext}^i_H(\text{St}_{Q_1}(\sigma_1),\text{St}_{Q_2}(\sigma_2)) \simeq \text{Ext}^{i-r}_H(\text{e}(\sigma_1),\text{e}(\sigma_2))$$

for some (explicitly given) $r \in \mathbb{Z}_{\geq 0}$ or 0 (Theorem 3.8) using some involutions on $H$ and results in \cite{2}. Here, $\text{e}(\sigma)$ is the extension of $\sigma$ to $H$ \cite{2}.

(4) We prove

$$\text{Ext}^i_H(\text{e}(\sigma_1),\text{e}(\sigma_2)) \simeq \text{Ext}^i_{H_P/I}(\sigma_1,\sigma_2)$$

for some ideal $I \subset H_P$ which acts on $\sigma_1$ and $\sigma_2$ by zero. We use results of Ollivier–Schneider \cite{15} for the proof. The algebra $H_P/I$ is not a pro-$p$-Iwahori Hecke algebra attached to a connected reductive group but a generic algebra in the sense of Vignéras \cite{20, 4, 3}. Hence, it is sufficient to calculate the extensions between supersingular simple modules of a generic algebra.

(5) Now let $H$ be a generic algebra and $\pi_1, \pi_2$ be simple supersingular modules. The algebra has the following decomposition as vector spaces: $H = H^{\text{aff}} \otimes_{C[Z_\kappa]} C[\Omega(1)]$. Here, $H^{\text{aff}} \subset H$ is an algebra called ‘the affine subalgebra’, $\Omega(1)$ is a certain commutative group acting on $H^{\text{aff}}$, $Z_\kappa$ is a normal subgroup of $\Omega(1)$ and we have an embedding $C[Z_\kappa] \hookrightarrow H^{\text{aff}}$ which is compatible with the action of $\Omega(1)$ on $H^{\text{aff}}$. Set $\Omega = \Omega(1)/Z_\kappa$. By this decomposition and Hochschild–Serre type spectral sequence, we have an exact sequence

$$0 \to H^1(\Omega, \text{Hom}_{H^{\text{aff}}}(\pi_1,\pi_2)) \to \text{Ext}^1_H(\pi_1,\pi_2) \to \text{Ext}^1_{H^{\text{aff}}}(\pi_1,\pi_2)^\Omega.$$ 

We prove that the last map is surjective (Theorem 4.5).

Therefore, it is sufficient to calculate two groups: $H^1(\Omega, \text{Hom}_{H^{\text{aff}}}(\pi_1,\pi_2))$ and $\text{Ext}^1_{H^{\text{aff}}}(\pi_1,\pi_2)^\Omega$. By the classification result of supersingular simple modules \cite{14, 18}, the restriction of $\pi_1, \pi_2$ to $H^{\text{aff}}$ are the direct sum of characters of $H^{\text{aff}}$. Hence, $\text{Hom}_{H^{\text{aff}}}(\pi_1,\pi_2)$ is easily described, and with this description we can calculate $H^1(\Omega, \text{Hom}_{H^{\text{aff}}}(\pi_1,\pi_2))$.
using well-known calculation of group cohomologies. Note that $\Omega$ is commutative. We also calculate $\text{Ext}^1_{\mathcal{H}}(\Xi_1,\Xi_2)$, where $\Xi_1,\Xi_2$ are characters of $\mathcal{H}$ (Proposition 4.1) following the method of Fayers [8]. This is also calculated by Nadimpalli [13]. Using this description, we can calculate $\text{Ext}^1_{\mathcal{H}}(\pi_1,\pi_2)^\Omega$, and this finishes the calculation of extensions between simple $\mathcal{H}$-modules.

In the last two subsections, examples for $\text{GL}_n$ are given.

2. Preliminaries

2.1. Pro-$p$-Iwahori Hecke algebra

Let $\mathcal{H}$ be a pro-$p$-Iwahori Hecke algebra over a commutative ring $C$ [20]. We study modules over $\mathcal{H}$ in this paper. In this paper, a module means a right module. The algebra $\mathcal{H}$ is defined with combinatorial data $(\mathcal{H},S,\Omega,W,W(1),Z)$ and a parameter $(q,c)$.

We recall the definitions. The data satisfy the following:

- $(\mathcal{H},S)$ is a Coxeter system.
- $\Omega$ acts on $(\mathcal{H},S)$.
- $W = \mathcal{H} \rtimes \Omega$.
- $Z$ is a finite commutative group.
- The group $W(1)$ is an extension of $W$ by $Z$, namely we have an exact sequence $1 \to Z \to W(1) \to W \to 1$.

The subgroup $Z$ is normal in $W(1)$. Hence, the conjugate action of $w \in W(1)$ induces an automorphism of $Z$, hence of the group ring $C[Z]$. We denote it by $c \mapsto w \cdot c$.

Let $\text{Ref}(\mathcal{H})$ be the set of reflections in $\mathcal{H}$ and $\text{Ref}(\mathcal{H}(1))$ the inverse image of $\text{Ref}(\mathcal{H})$ in $W(1)$. The parameter $(q,c)$ is maps $q: S \to C$ and $c: \text{Ref}(\mathcal{H}(1)) \to C[Z]$ with the following conditions. (Here, the image of $s$ by $q$ (resp. $c$) is denoted by $q_s$ (resp. $c_s$).)

- For $w \in W$ and $s \in S$, if $wsw^{-1} \in S$, then $q_{wsw^{-1}} = q_s$.
- For $w \in W(1)$ and $s \in \text{Ref}(\mathcal{H}(1))$, $c_{wsw^{-1}} = w \cdot c_s$.
- For $s \in \text{Ref}(\mathcal{H}(1))$ and $t \in Z$, we have $c_{ts} = tc_s$.

Let $S(1)$ be the inverse image of $S$ in $W(1)$. For $s \in S(1)$, we write $q_s$ for $q_s$, where $s \in S$ is the image of $s$. The length function on $\mathcal{H}$ is denoted by $\ell$, and its inflation to $W$ and $W(1)$ is also denoted by $\ell$.

The $C$-algebra $\mathcal{H}$ is a free $C$-module and has a basis $\{T_w\}_{w \in W(1)}$. The multiplication is given by

- (Quadratic relations) $T_s^2 = q_s T_s + c_s T_s$ for $s \in S(1)$.
- (Braid relations) $T_{vw} = T_v T_w$ if $\ell(vw) = \ell(v) + \ell(w)$.

We extend $q: S \to C$ to $q: W \to C$ as follows. For $w \in W$, take $s_1,\ldots,s_l$ and $u \in \Omega$ such that $w = s_1 \cdots s_l u$ and $l = \ell(w)$. Then put $q_w = q_{s_1} \cdots q_{s_l}$. From the definition, we have $q_{w^{-1}} = q_w$. We also put $q_w = q_w$ for $w \in W(1)$ with the image $\overline{w}$ in $W$. 

Extension Between Simple Modules of Pro-$p$-Iwahori Hecke Algebras
2.2. The data from a group

Let $F$ be a non-Archimedean local field, $\kappa$ its residue field, $p$ its residue characteristic and $G$ a connected reductive group over $F$. We can get the data in the previous subsection from $G$ as follows. See [20], especially 3.9 and 4.2 for the details.

Fix a maximal split torus $S$, and denote the centralizer of $S$ in $G$ by $Z$. Let $Z^0$ be the unique parahoric subgroup of $Z$ and $Z(1)$ its pro-$p$ radical. Then the group $W(1)$ (resp. $W$) is defined by $W(1) = N_G(Z)/Z(1)$ (resp. $W = N_G(Z)/Z^0$), where $N_G(Z)$ is the normalizer of $Z$ in $G$. We also let $Z_\kappa = Z^0/Z(1)$. Let $G'$ be the group generated by the unipotent radical of parabolic subgroups $[5,\text{II.1}]$ and $W_{\text{aff}}$ the image of $G' \cap N_G(Z)$ in $W$. Then this is a Coxeter group. Fix a set of simple reflections $S_{\text{aff}}$. The group $W$ has the natural length function, and let $\Omega$ be the set of length zero elements in $W$. Then we get the data $(W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_\kappa)$.

Consider the apartment attached to $S$ and an alcove surrounded by the hyperplanes fixed by $S_{\text{aff}}$. Let $I(1)$ be the pro-$p$-Iwahori subgroup attached to this alcove. Then with $q_s = \#(I(1)\bar{s}I(1)/I(1))$ for $s \in S_{\text{aff}}$ with a lift $\bar{s} \in N_G(Z)$ and suitable $c_s$, the algebra $\mathcal{H}$ is isomorphic to the Hecke algebra attached to $(G, I(1))$ [20, Proposition 4.4].

When the data come from the group $G$, let $W_{\text{aff}}(1)$ be the image of $G' \cap N_G(Z)$ in $W(1)$ and put $\mathcal{H}_{\text{aff}} = \bigoplus_{w \in W_{\text{aff}}(1)} CT_w$. This is a subalgebra of $\mathcal{H}$.

In this paper, except Section 4, we assume that the data come from a connected reductive group.

2.3. The root system and the Weyl groups

Let $W_0 = N_G(Z)/Z$ be the finite Weyl group. Then this is a quotient of $W$. Recall that we have the alcove defining $I(1)$. Fix a special point $x_0$ from the border of this alcove. Then $W_0 \simeq \text{Stab}_W x_0$, and the inclusion $\text{Stab}_W x_0 \hookrightarrow W$ is a splitting of the canonical projection $W \to W_0$. Throughout this paper, we fix this special point and regard $W_0$ as a subgroup of $W$. Set $S_0 = S_{\text{aff}} \cap W_0 \subset W$. This is a set of simple reflections in $W_0$. For each $w \in W_0$, we fix a representative $n_w \in W(1)$ such that $n_{w_1w_2} = n_{w_1}n_{w_2}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

The group $W_0$ is the Weyl group of the root system $\Sigma$ attached to $(G, S)$. Our fixed alcove and special point give a positive system of $\Sigma$, denoted by $\Sigma^+$. The set of simple roots is denoted by $\Delta$. As usual, for $\alpha \in \Delta$, let $s_\alpha \in S_0$ be a simple reflection for $\alpha$.

The kernel of $W(1) \to W_0$ (resp. $W \to W_0$) is denoted by $\Lambda(1)$ (resp. $\Lambda$). Then $Z_\kappa \subset \Lambda(1)$, and we have $\Lambda = \Lambda(1)/Z_\kappa$. The group $\Lambda$ (resp. $\Lambda(1)$) is isomorphic to $Z/Z^0$ (resp. $Z/Z(1)$). Any element in $W(1)$ can be uniquely written as $n_w\lambda$, where $w \in W_0$ and $\lambda \in \Lambda(1)$. We have $W = W_0 \ltimes \Lambda$.

2.4. The map $\nu$

The group $W$ acts on the apartment attached to $S$, and the action of $\Lambda$ is by the translation. Since the group of translations of the apartment is $X_*(S) \otimes \mathbb{Z} \mathbb{R}$, we have a group homomorphism $\nu: \Lambda \to X_*(S) \otimes \mathbb{Z} \mathbb{R}$. The compositions $\Lambda(1) \to \Lambda \to X_*(S) \otimes \mathbb{Z} \mathbb{R}$ and $Z \to \Lambda \to X_*(S) \otimes \mathbb{Z} \mathbb{R}$ are also denoted by $\nu$. The homomorphism $\nu: Z \to X_*(S) \otimes \mathbb{Z} \mathbb{R} \simeq \text{Hom}_\mathbb{Z}(X^*(S), \mathbb{R})$ is characterized by the following: For $t \in S$ and $\chi \in X^*(S)$, we have $\nu(t)(\chi) = -\text{val}(\chi(t))$, where val is the normalized valuation of $F$. 

We call $\lambda \in \Lambda(1)$ dominant (resp. anti-dominant) if $\nu(\lambda)$ is dominant (resp. anti-dominant).

Since the group $W_{aff}$ is a Coxeter system, it has the Bruhat order denoted by $\leq$. For $w_1, w_2 \in W$, we write $w_1 < w_2$ if there exists $u \in \Omega$ such that $w_1 u, w_2 u \in W_{aff}$ and $w_1 u < w_2 u$. Moreover, for $w_1, w_2 \in W(1)$, we write $w_1 < w_2$ if $w_1 \in W_{aff}(1)w_2$ and $\overline{w}_1 < \overline{w}_2$, where $\overline{w}_1, \overline{w}_2$ are the image of $w_1, w_2$ in $W$, respectively. We write $w_1 \leq w_2$ if $w_1 < w_2$ or $w_1 = w_2$.

2.5. Other basis

For $w \in W(1)$, take $s_1, \ldots, s_t \in S_{aff}(1)$ and $u \in W(1)$ such that $l = \ell(w)$, $\ell(u) = 0$ and $w = s_1 \cdots s_t u$. Set $T^+_w = (T_{s_1} - c_{s_1}) \cdots (T_{s_t} - c_{s_t}) T_u$. Then this does not depend on the choice, and $\{T^+_w\}_{w \in W(1)}$ is a basis of $\mathcal{H}$. In $\mathcal{H}[q_{1/2}^\pm]$, we have $T^+_w = q_{w/w_1}^{1/2} q_{w/w_2}^{1/2} E_o(w_1 w_2)$.

Remark 2.1. The term $q_{w_1 w_2}^{1/2} q_{w_1}^{1/2}$ does not make sense in a usual way. See [4, Remark 2.2].

2.6. Parabolic induction

Since we have a positive system $\Sigma^+$, we have the minimal parabolic subgroup $B$ with a Levi part $Z$. In this paper, parabolic subgroups are always standard, namely containing $B$. Note that such parabolic subgroups correspond to subsets of $\Delta$.

Let $P$ be a parabolic subgroup. Attached to the Levi part of $P$ containing $Z$, we have the data $(W_{aff,P}, S_{aff,P}, \Omega_P, W_P, W_P(1), Z_P)$ and the parameters $(q_P, c_P)$. Hence, we have the algebra $\mathcal{H}_P$. The parameter $c_P$ is given by the restriction of $c$; hence, we denote it just by $c$. The parameter $q_P$ is defined as in [3, 4.1].

For the objects attached to this data, we add the suffix $P$. We have the set of simple roots $\Delta_P$, the root system $\Sigma_P$ and its positive system $\Sigma^+_P$, the finite Weyl group $W_{0,P}$, the set of simple reflections $S_{0,P} \subset W_{0,P}$, the length function $\ell_P$ and the base $\{T^+_w\}_{w \in W_P(1)}$, $\{T^+_w\}_{w \in W_P(1)}$ and $\{E^+_o(w)\}_{w \in W_P(1)}$ of $\mathcal{H}_P$. Note that we have no $\Lambda_P$, $\Lambda_P(1)$ and $Z_{\kappa_P}$ since they are equal to $\Lambda$, $\Lambda(1)$ and $Z_{\kappa}$.

An element $n_w \lambda \in W_P(1)$, where $w \in W_P, 0$ and $\lambda \in \Lambda(1)$ is called $P$-positive (resp. $P$-negative) if for any $\alpha \in \Sigma^+ \setminus \Sigma^+_P$ we have $\langle \alpha, \nu(\lambda) \rangle \leq 0$ (resp. $\langle \alpha, \nu(\lambda) \rangle \geq 0$). Set $\mathcal{H}^+_P = \bigoplus_w CT^+_w$, where $w \in W_P(1)$ runs $P$-positive elements, and define $\mathcal{H}^-_P$ by the similar way. Then these are subalgebras of $\mathcal{H}_P$. The linear maps $j^+_P: \mathcal{H}^+_P \to \mathcal{H}$ and $j^-_P: \mathcal{H}^-_P \to \mathcal{H}$ defined by $j^+_P(T^+_w) = T_w$ and $j^-_P(T^+_w) = T_w$ are algebra homomorphisms.

Proposition 2.2 ([19, Theorem 1.4]). Let $\lambda^+_P$ (resp. $\lambda^-_P$) be in the center of $W_P(1)$ such that $\langle \alpha, \nu(\lambda^+_P) \rangle < 0$ (resp. $\langle \alpha, \nu(\lambda^-_P) \rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma^+_P$. Then $T^+_\lambda^-_P = T^+_\lambda^+_P = E^+_a \lambda^-_P$ (resp. $T^+_\lambda^-_P = T^+_\lambda^+_P = E^+_a \lambda^-_P$) is in the center of $\mathcal{H}_P$, and we have $\mathcal{H}_P = \mathcal{H}^+_P E^+_a \lambda^-_P \lambda^-_P^{-1}$ (resp. $\mathcal{H}_P = \mathcal{H}^+_P E^+_a \lambda^-_P \lambda^-_P^{-1}$).
Now, for an $\mathcal{H}_P$-module $\sigma$, we define the parabolically induced module $I_P(\sigma)$ by

$$I_P(\sigma) = \text{Hom}_{(\mathcal{H}_P, j_P^*)}(\mathcal{H}, \sigma).$$

This satisfies:

- $I_P$ is an exact functor.
- $I_P$ has the left adjoint functor $L_P$. The functor $L_P$ is exact.
- $I_P$ has the right adjoint functor $R_P$.

For the existence and explicit descriptions of adjoint functors $L_P, R_P$, see [4, 5.1].

For parabolic subgroups $P \subset Q$, we also defines $\mathcal{H}_P^{Q \pm} \subset \mathcal{H}_P$ and $j_P^{Q \pm} : \mathcal{H}_P^{Q \pm} \to \mathcal{H}_Q$ and $j_P^{Q \pm,*} : \mathcal{H}_P^{Q \pm} \to \mathcal{H}_Q$. This defines the parabolic induction $I_P^Q$ from the category of $\mathcal{H}_P$-modules to the category of $\mathcal{H}_Q$-modules.

2.7. Twist by $n_{w_G w_P}$

For a parabolic subgroup $P$, let $w_P$ be the longest element in $W_{0,P}$. In particular, $w_G$ is the longest element in $W_0$. Let $P'$ be a parabolic subgroup corresponding to $-w_G(\Delta_P)$; in other words, $P' = n_{w_G w_P} P^\text{op} n_{w_G w_P}^{-1}$, where $P^\text{op}$ is the opposite parabolic subgroup of $P$ with respect to the Levi part of $P$ containing $Z$. Set $n = n_{w_G w_P}$. Then the map $P^\text{op} \to P'$ defined by $p \mapsto npn^{-1}$ is an isomorphism which preserves the data used to define the pro-$p$-Iwahori Hecke algebras. Hence, $T^P_w \mapsto T^P_{nwn^{-1}}$ gives an isomorphism $\mathcal{H}_P \to \mathcal{H}_{P'}$. This sends $T^{P*}_w$ to $T^{P'*}_{nwn^{-1}}$ and $E^{P}_{a_+, P^*(w)}$ to $E^{P'}_{a_+, P^*(w)}$, where $v \in W_{0,P}$.

Let $\sigma$ be an $\mathcal{H}_P$-module. Then we define an $\mathcal{H}_{P'}$-module $n_{w_G w_P} \sigma$ via the pull-back of the above isomorphism: $(n_{w_G w_P} \sigma)(T^P_w) = \sigma(T^P_{nwn^{-1}})$. For an $\mathcal{H}_{P'}$-module $\sigma'$, we define $n_{w_G w_P}^{-1} \sigma'$ by $(n_{w_G w_P}^{-1} \sigma')(T^P_w) = \sigma'(T^P_{nwn^{-1}})$.

2.8. The extension and the generalized Steinberg modules

Let $P$ be the parabolic subgroup and $\sigma$ an $\mathcal{H}_P$-module. For $\alpha \in \Delta$, let $P_\alpha$ be a parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Then we define $\Delta(\sigma) \subset \Delta$ by

$$\Delta(\sigma) =\{ \alpha \in \Delta \mid \langle \Delta_P, \alpha \rangle = 0, \sigma(T^P_\alpha) = 1 \text{ for any } \lambda \in W_{\text{aff}, P_\alpha}(1) \cap \Delta(1) \} \cup \Delta_P.$$

Let $P(\sigma)$ be the parabolic subgroup corresponding to $\Delta(\sigma)$.

Proposition 2.3 ([6, Theorem 3.6]). Let $\sigma$ be an $\mathcal{H}_P$-module and $Q$ a parabolic subgroup between $P$ and $P(\sigma)$. Denote the parabolic subgroup corresponding to $\Delta_Q \setminus \Delta_P$ by $P_2$. Then there exists a unique $\mathcal{H}_Q$-module $e_Q(\sigma)$ acting on the same space as $\sigma$ such that

- $e_Q(\sigma)(T^Q_w) = \sigma(T^P_w)$ for any $w \in W_P(1)$.
- $e_Q(\sigma)(T^Q_w) = 1$ for any $w \in W_{\text{aff}, P_2}(1)$.

Definition 2.4. We call $e_Q(\sigma)$ the extension of $\sigma$ to $\mathcal{H}_Q$.

A typical example of the extension is the trivial representation $1 = 1_G$. This is a one-dimensional $\mathcal{H}$-module defined by $1(T_w) = q_w$, or equivalently $1(T^*_w) = 1$. We have $\Delta(1_P) = \{ \alpha \in \Delta \mid \langle \Delta_P, \alpha \rangle = 0 \} \cup \Delta_P$, and if $Q$ is a parabolic subgroup between $P$ and $P(1_P)$, we have $e_Q(1_P) = 1_Q$. 
Let $P(\sigma) \supset P_0 \supset Q_1 \supset Q \supset P$. Then as in [3, 4.5], we have $I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \subset I_{Q}^{P_0}(e_{Q}(\sigma))$. Define

$$\text{St}_{Q}^{P_0}(\sigma) = \text{Coker} \left( \bigoplus_{Q_1 \supset Q} I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \to I_{Q}^{P_0}(e_{Q}(\sigma)) \right).$$

When $P_0 = G$, we write $\text{St}_Q(\sigma)$ and call it generalized Steinberg modules.

### 2.9. Supersingular modules

In this subsection, we assume that $C$ is a field of characteristic $p$. Let $O$ be a conjugacy class in $W(1)$ which is contained in $\Lambda(1)$. For a spherical orientation $o$, set $z_O = \sum_{\lambda \in O} E_o(\lambda)$. Then this does not depend on $o$ and $z_O \in Z$, where $Z$ is the center of $H[18, \text{Theorem 5.1}]$. The length of $\lambda \in O$ does not depend on $\lambda$. We denote it by $\ell(O)$. For $\lambda \in \Lambda(1)$ and $w \in W(1)$, we put $w \cdot \lambda = w\lambda w^{-1}$.

**Definition 2.5.** Let $\pi$ be an $H$-module. We call $\pi$ supersingular if there exists $n \in \mathbb{Z}_{>0}$ such that $\pi z_O^n = 0$ for any $O$ such that $\ell(O) > 0$.

**Remark 2.6.** Since $\pi z_O \subset \pi$ is a submodule, if $\pi$ is simple, then $\pi$ is supersingular if and only if $\pi z_O = 0$ for any $O$ such that $\ell(O) > 0$. Let $\lambda \in \Lambda(1)$. Then $\ell(\lambda) \neq 0$ if and only if $\langle \alpha, \nu(\lambda) \rangle \neq 0$ for some $\alpha \in \Sigma$ [4, Lemma 2.12]. Hence, a simple $H$-module $\pi$ is supersingular if and only if $\pi(z_{W(1)} \cdot \lambda) = 0$ for any $\lambda$ such that $\langle \alpha, \nu(\lambda) \rangle \neq 0$ for some $\alpha \in \Sigma$.

The simple supersingular $H$-modules are classified in [14, 18]. We recall their results. Let $W^{\text{aff}}(1)$ be the inverse image of $W^{\text{aff}}$ in $W(1)$.

**Remark 2.7.** When we do not assume that the data come from a group, we have no $W^{\text{aff}}(1)$ but we have $W^{\text{aff}}(1)$. Even though the data come from a group, $W^{\text{aff}}(1)$ is not equal to $W^{\text{aff}}(1)$. We have $Z_{\kappa} \subset W^{\text{aff}}(1)$; however, $Z_{\kappa} \not\subset W^{\text{aff}}(1)$ in general. Since we will not assume that the data come from a group, we do not use $W^{\text{aff}}(1)$ here.

Put $H^{\text{aff}} = \bigoplus_{w \in W^{\text{aff}}(1)} CT_w$. Let $\chi$ be a character of $Z_{\kappa}$ and put $S_{\text{aff}, \chi} = \{ s \in S_{\text{aff}} \mid \chi(c_{\tilde{s}}) \neq 0 \}$, where $\tilde{s} \in W(1)$ is a lift of $s \in S_{\text{aff}}$. Note that, if $\tilde{s}'$ is another lift, then $\tilde{s}' = t \tilde{s}$ for some $t \in Z_{\kappa}$. Hence, $\chi(c_{\tilde{s}'}) = \chi(t) \chi(c_{\tilde{s}})$. Therefore, the condition does not depend on a choice of a lift. Let $J \subset S_{\text{aff}, \chi}$. Then the character $\Xi = \Xi_{J, \chi}$ of $H^{\text{aff}}$ is defined by

$$\Xi_{J, \chi}(T_t) = \chi(t) \quad (t \in Z_{\kappa}),$$

$$\Xi_{J, \chi}(\tilde{s}) = \begin{cases} \chi(c_{\tilde{s}}) & (s \in S_{\text{aff}, \chi} \setminus J), \\ 0 & (s \not\in S_{\text{aff}, \chi} \setminus J), \end{cases}$$

where $\tilde{s} \in W^{\text{aff}}(1)$ is a lift of $s$ and the last equality easily follows from the definition of $S_{\text{aff}, \chi}$. Let $\Omega(1)_\Xi$ be the stabilizer of $\Xi$ and $V$ a simple $C[\Omega(1)_\Xi]$-module such that $V|Z_{\kappa}$ is a direct sum of $\chi$. Put $H_\Xi = H^{\text{aff}} C[\Omega(1)_\Xi]$. This is a subalgebra of $H$. For $X \in H^{\text{aff}}$ and $Y \in C[\Omega(1)_\Xi]$, we define the action of $XY$ on $\Xi \otimes V$ by $x \otimes y \mapsto xX \otimes yY$. Then this defines a well-defined action of $H_\Xi$ on $\Xi \otimes V$. Set $\pi_{X, J, V} = (\Xi \otimes V) \otimes_{H_\Xi} H$. 
Proposition 2.8 ([18, Theorem 1.6]). The module $\pi_{\chi,J,V}$ is simple, and it is supersingular if and only if the groups generated by $J$ and generated by $S_{\text{aff, } \chi} \setminus J$ are both finite. If $C$ is an algebraically closed field, then any simple supersingular modules are given in this way.

The construction of $\pi_{\chi,J,V}$ is still valid even if we do not assume that the data come from a group. In Section 4, we do not assume it, and we calculate the extension between the modules constructed as above.

2.10. Simple modules

Definition 2.9. We consider a triple $(P,\sigma,Q)$ which satisfies the following:

- $P$ is a parabolic subgroup of $G$.
- $\sigma$ is a supersingular finite-dimensional $\mathcal{H}_P$-module.
- $Q$ is a parabolic subgroup contained in $P(\sigma)$.

Then we define an $\mathcal{H}$-module $I(P,\sigma,Q)$ by

$$I(P,\sigma,Q) = I_P(\sigma)(\text{St}_Q^P(\sigma)).$$

Theorem 2.10 ([3, Theorem 1.1]). Assume that $C$ is an algebraically closed field of characteristic $p$. The module $I(P,\sigma,Q)$ is simple, and any simple module has this form. Moreover, $(P,\sigma,Q)$ is unique up to isomorphism.

3. Reduction to supersingular representations

In the rest of this paper, we assume that $C$ is a field of characteristic $p$. Let $(P_1,\sigma_1,Q_1),(P_2,\sigma_2,Q_2)$ be triples as in Definition 2.9. We calculate the extension group $\text{Ext}^1_{\mathcal{H}}(I(P_1,\sigma_1,Q_1),I(P_2,\sigma_2,Q_2))$.

3.1. Central character

We prove the following lemma.

Lemma 3.1. If $\text{Ext}^i_{\mathcal{H}}(I(P_1,\sigma_1,Q_1),I(P_2,\sigma_2,Q_2)) \neq 0$ for some $i \in \mathbb{Z}_{\geq 0}$, then $P_1 = P_2$.

To prove this lemma, we calculate the action of the center $Z$ on simple modules. To do it, we need to calculate the action of $Z$ on a parabolic induction.

Lemma 3.2. Let $P$ be a parabolic subgroup, $\sigma$ a right $\mathcal{H}_P$-module. For $W(1)$-orbit $O$ in $\Lambda(1)$, set $O_P = \{ \lambda \in O \mid \lambda$ is $P$-negative$\}$. Then we have the following:

1. The subset $O_P \subseteq \Lambda(1)$ is $W_P(1)$-stable.
2. Let $O = O_1 \cup \cdots \cup O_r$ be the decomposition into $W_P(1)$-orbits. The action of $z_O \in Z$ on $I_P(\sigma)$ is induced by the action of $\sum_i z_{O_i}^P$ on $\sigma$.

Proof. Since $\Sigma^+ \setminus \Sigma^+_P$ is stable under the action of $W_{0,P}$, (1) follows from the definition of $P$-negative.
Let $\varphi \in I_P(\sigma) = \text{Hom}(\mathcal{H}_{P^{-1}}(\mathcal{H}, \sigma))$. Then for $X \in \mathcal{H}$, we have

$$(\varphi z)(X) = \varphi(z X) = \varphi(X z)$$

since $z$ is in the center of $\mathcal{H}$. Hence, by the definition of $z$, we have

$$(\varphi z)(X) = \sum_{\lambda \in \mathcal{O}} \varphi(X E(\lambda)) = \sum_{\lambda \in \mathcal{O}} \varphi(X E(\lambda)) + \sum_{\lambda \in \mathcal{O}, \text{not } P\text{-negative}} \varphi(X E(\lambda))$$

We prove the vanishing of the second term.

Let $\lambda \in \mathcal{O}$, which is not $P$-negative. Then there exists $\alpha \in \Sigma^+ \setminus \Sigma^+_P$ such that $\langle \alpha, \nu(\lambda) \rangle < 0$. Let $\lambda^{-1}_P$ as in Proposition 2.2. Then $\langle \alpha, \nu(\lambda^{-1}_P) \rangle > 0$. Hence, $\nu(\lambda)$ and $\nu(\lambda^{-1}_P)$ does not belong to the same closed Weyl chamber. Therefore, we have $E(\lambda) E(\lambda^{-1}_P) = 0$ in $\mathcal{H}_C$ by [4, (2.1), Lemma 2.11]. Hence, by [4, Lemma 2.6],

$$\varphi(X E(\lambda)) = \varphi(X E(\lambda)) E_P(\lambda^{-1}_P) E_P(\lambda^{-1}_P)^{-1}$$

$$= \varphi(X E(\lambda)) j_P^*(E_P(\lambda^{-1}_P)) E_P(\lambda^{-1}_P)^{-1}$$

$$= \varphi(X E(\lambda)) E(\lambda^{-1}_P) E_P(\lambda^{-1}_P)^{-1} = 0.$$

If $\lambda \in \mathcal{O}_i$, then $E(\lambda) \in \mathcal{H}_{P^{-1}}$. Hence, we have $E(\lambda) = j_P^*(E_P(\lambda))$ by [4, Lemma 2.6]. Therefore,

$$\sum_{\lambda \in \mathcal{O}_i} \varphi(X E(\lambda)) = \varphi(X) \sum_{\lambda \in \mathcal{O}_i} \sigma(E_P(\lambda)) = \varphi(X) \sigma(z_{\mathcal{O}_i}).$$

We get the lemma. 

\begin{lemma}
Let $(P, \sigma, Q)$ be a triple as in Definition 2.9. Let $R$ be a parabolic subgroup and $\lambda = \lambda_R^{-1}$ as in Proposition 2.2. Then $z_{\mathcal{O}_i} \neq 0$ on $I(P, \sigma, Q)$ if and only if $P \subset R$.
\end{lemma}

\begin{proof}
Set $\mathcal{O} = \mathcal{O}_i$. Since $\Lambda(1) \subset W_R(1)$ and $\lambda$ is in the center of $W_R(1)$, $\lambda$ commutes with $\Lambda(1)$. Hence, $\mathcal{O} = \{n_w, \lambda | w \in W_0\}$.

We prove that $W_P(1)$ acts transitively on $\mathcal{O}_P$. Let $\mu \in \mathcal{O}_P$, and take $w \in W_0$ such that $\mu = n_w, \lambda$. Take $v \in W_{0, P}$ such that $v(\nu(\mu))$ is dominant with respect to $\Sigma^+_P$. Since $v^{-1}(\Sigma^+ \setminus \Sigma^+_P) = \Sigma^+ \setminus \Sigma^+_P$ and $\mu$ is $P$-negative, we have $\langle v(\nu(\mu)), \alpha \rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma^+_P$. Hence, $v(\nu(\mu))$ is dominant. Now $\nu(\lambda)$ and $\nu(\nu(\mu)) \equiv vw(\nu(\lambda))$ is both dominant. Hence, $vw \in \text{Stab}_{W_0}(\nu(\lambda)) = W_{0, R}$. Since $\lambda$ is in the center of $W_R(1)$, we have $(n_w, n_w) \cdot \lambda = \lambda$. Hence, $\mu = n^{-1}_v \cdot \lambda$. Therefore, $W_P(1)$ acts transitively on $\mathcal{O}_P$.

By the definition, $I(P, \sigma, Q)$ is a quotient of $I_{P(\sigma)}(I_{Q(\sigma)}(e_Q(\sigma))) = I_Q(e_Q(\sigma))$. Moreover, by the definition of the extension, we have an embedding $e_Q(\sigma) \hookrightarrow I_Q^G(\sigma)$. Hence, we have $I_Q(e_Q(\sigma)) \hookrightarrow I_Q(I_{P(\sigma)}(e_Q(\sigma))) = I_P(\sigma)$. Let $\chi: Z_P \to C$ be a central character of $\sigma$. By the above lemma and the fact that $\mathcal{O}_P$ is a single $W_P(1)$-orbit, on $I_P(\sigma)$, $z_{\mathcal{O}_i}$ acts by $\chi(z_{\mathcal{O}_i})$. Since $\nu(\lambda)$ is dominant, $\lambda$ is $P$-negative. Hence, $\mathcal{O} \subset \mathcal{O}_P$. By the definition of supersingular representations with Remark 2.6, $\chi(z_{\mathcal{O}_i}) = 0$ if and only if $\langle \alpha, \nu(\lambda) \rangle \neq 0$ for some $\alpha \in \Sigma^+_P$. The condition on $\lambda = \lambda_R^{-1}$ tells that $\langle \alpha, \nu(\lambda) \rangle \neq 0$ if and only if $\alpha \in \Sigma^+ \setminus \Sigma^+_R$. Therefore, $\chi(z_{\mathcal{O}_i}) \neq 0$ if and only if $\Sigma^+_R \cap (\Sigma^+ \setminus \Sigma^+_R) = \emptyset$ which is equivalent to $P \subset R$. \hfill \Box
\end{proof}

\begin{proof}
Assume that $P_1 \neq P_2$. Then we have $P_1 \not\subset P_2$ or $P_1 \not\supset P_2$. Assume $P_1 \not\subset P_2$, and take $\lambda = \lambda_{P_2}^{-1}$ as in Proposition 2.2. Put $\mathcal{O} = \{w, \lambda | w \in W(1)\}$. Then $z_{\mathcal{O}} = 0$

\end{proof}
on $I(P_1,\sigma_1,Q_1)$ and $z_\mathcal{O} \neq 0$ on $I(P_2,\sigma_2,Q_2)$. Hence, the vanishing follows from a standard argument since $z_\mathcal{O} \in \mathcal{H}$ is in the center. The case of $P_1 \not\supset P_2$ is proved by the same way. \hfill $\square$

### 3.2. Reduction to generalized Steinberg modules

By Lemma 3.1, to calculate the extension between $I(P_1,\sigma_1,Q_1)$ and $I(P_2,\sigma_2,Q_2)$, we may assume $P_1 = P_2$. We prove the following proposition.

**Proposition 3.4.** The extension group $\text{Ext}^i_\mathcal{H}(I(P,\sigma_1,Q_1),I(P,\sigma,Q_2))$ is isomorphic to

$$\text{Ext}^i_{\mathcal{H}_{P}(\sigma_1) \cap P(\sigma_2)}(\text{St}^{P(\sigma_1) \cap P(\sigma_2)}_{Q_1 \cap P(\sigma_2)}(\sigma_1),\text{St}^{P(\sigma_1) \cap P(\sigma_2)}_{Q_2}(\sigma_2)).$$

if $Q_2 \subset P(\sigma_1)$ and $\Delta(\sigma_1) \subset \Delta Q_1 \cup \Delta(\sigma_2)$. Otherwise, the extension group is zero.

Hence, for the calculation of the extension, it is sufficient to calculate the extensions between generalized Steinberg modules. For an $\mathcal{H}$-module $\pi$, set $\pi^* = \text{Hom}_C(\pi,C)$. The right $\mathcal{H}$-module structure on $\pi^*$ is given by $(fx)(v) = f(v\xi(X))$ for $f \in \pi^*$, $v \in \pi$ and $X \in \mathcal{H}$. Here, the anti-involution $\xi: \mathcal{H} \to \mathcal{H}$ is defined by $\xi(T_w) = T_w^{-1}$.

**Lemma 3.5.** We have $\text{Ext}^i_\mathcal{H}(\pi_1,\pi_2^*) \simeq \text{Ext}^i_\mathcal{H}(\pi_2,\pi_1^*)^*$. In particular, if $\pi_1$ or $\pi_2$ is finite-dimensional, then $\text{Ext}^i_\mathcal{H}(\pi_1,\pi_2^*) \simeq \text{Ext}^i_\mathcal{H}(\pi_2,\pi_1^*)^*$.

**Proof.** We have the isomorphism for $i = 0$ since both sides are equal to $\{ f: \pi_1 \times \pi_2 \to C \mid f(x_1,X,x_2) = f(x_1,\xi(X),x_2) \ (x_1 \in \pi_1,x_2 \in \pi_2,X \in \mathcal{H}) \}$. Hence, in particular, if $\pi$ is projective, then $\pi^*$ is injective. Let $\cdots \to P_1 \to P_0 \to \pi_2 \to 0$ be a projective resolution. Then $\text{Ext}^i_\mathcal{H}(\pi_2,\pi_1^*)$ is a $i$-th cohomology of the complex $\text{Hom}(P_i,\pi_1^*) \simeq \text{Hom}(\pi_1,P_i^*)$. Since $0 \to \pi_2^* \to P_0^* \to P_1^* \to \cdots$ is an injective resolution of $\pi_2^*$, this is $\text{Ext}^i_\mathcal{H}(\pi_1,\pi_2^*)$.

If $\pi_2$ is finite-dimensional, then $\pi_2 \simeq (\pi_2^*)^*$. Hence, we have $\text{Ext}^i_\mathcal{H}(\pi_1,\pi_2) \simeq \text{Ext}^i_\mathcal{H}(\pi_1,\pi_2^*)^* \simeq \text{Ext}^i_\mathcal{H}(\pi_2^*,\pi_1^*)$. By the same argument, we have $\text{Ext}^i_\mathcal{H}(\pi_1,\pi_2) \simeq \text{Ext}^i_\mathcal{H}(\pi_2^*,\pi_1^*)$ if $\pi_1$ is finite-dimensional. \hfill $\square$

**Proposition 3.6.** Let $P$ be a parabolic subgroup, $\pi$ an $\mathcal{H}$-module and $\sigma$ an $\mathcal{H}_P$-module.

1. We have $\text{Ext}^i_\mathcal{H}(\pi,I_P(\sigma)) \simeq \text{Ext}^i_{\mathcal{H}_P}(L_P(\pi),\sigma)$.
2. We have $\text{Ext}^i_\mathcal{H}(I_P(\pi),\pi^*) \simeq \text{Ext}^i_{\mathcal{H}_P}(\sigma,R_P(\pi^*))$. In particular, if $\pi$ is finite-dimensional, then $\text{Ext}^i_\mathcal{H}(I_P(\pi),\pi^*) \simeq \text{Ext}^i_{\mathcal{H}_P}(\sigma,R_P(\pi))$.

**Proof.** The exactness of $I_P$ and $L_P$ implies (1).

Put $P' = n_{w_c,w_p}P^\text{op}\cdot n_{w_c,w_p}^{-1}$. Define the functor $I'_{P'}$ by

$$I'_{P'}(\sigma') = \text{Hom}_{(\mathcal{H}_P',\mathcal{H}_P')}((\mathcal{H},\sigma'))$$

for an $\mathcal{H}_P'$-module $\sigma'$. Then this has the left adjoint functor $L'_{P'}$, defined by $L'_{P'}(\pi) = \pi \otimes_{(\mathcal{H}_P',\mathcal{H}_P')} \mathcal{H}_P'$. This is exact since $\mathcal{H}_P'$ is a localization of $\mathcal{H}_{P'}$, by Proposition 2.2. Set $\sigma_{\ell-\ell_P}(T_w' \sigma) = (-1)^{\ell(w'_P - w)(w)}\sigma(T_w) \ [4, 4.1]$. Using [2, Proposition 4.2], for an $\mathcal{H}_P$-module $\sigma$, we have

$$\text{Ext}^i_\mathcal{H}(I_P(\sigma),\pi^*) \simeq \text{Ext}^i_\mathcal{H}(\pi,I_P(\sigma)^*) \simeq \text{Ext}^i_\mathcal{H}(\pi,I'_{P'}(n_{w_c,w_P}^{-1}\sigma_{\ell-\ell_P})).$$
Put $i = 0$. Then we get $(n_{w_p}^{-1}L_{p'}(\pi))_{\ell-\ell_p} \simeq R_P(\pi^*)$ by $\text{Hom}_\mathcal{H}(I_P(\sigma),\pi^*) \simeq \text{Hom}_\mathcal{H}(\sigma,R_P(\pi^*))$. Hence, we get (2).

If $\pi$ is finite-dimensional, then $\pi = (\pi^*)^*$. Hence, we get $\text{Ext}_\mathcal{H}^i(I_P(\sigma),\pi) \simeq \text{Ext}_\mathcal{H}^i(\sigma,R_P(\pi))$ applying (3) to $\pi^*$.

**Proof of Proposition 3.4.** Since $I(P,\sigma_2, Q_2)$ is finite-dimensional, we have

$$\text{Ext}_\mathcal{H}^i(I(P,\sigma_1, Q_1), I(P,\sigma_2, Q_2)) = \text{Ext}_\mathcal{H}^i(P^{\sigma_1}(\sigma_1), I_P(\sigma_1)(I(P,\sigma_2, Q_2))).$$

We have $R_{P(\sigma_1)}(I(P,\sigma_2, Q_2)) = 0$ if $Q_2 \nsubseteq P(\sigma_1)$ by [4, Theorem 5.20]. If $Q_2 \subseteq P(\sigma_1)$, then $R_{P(\sigma_1)}(I(P,\sigma_2, Q_2)) = I_P(\sigma_1)(P,\sigma_2, Q_2)$. Hence, the extension group is isomorphic to

$$\text{Ext}_\mathcal{H}^i(P^{\sigma_1}(\sigma_1), I_P(\sigma_1)(P,\sigma_2, Q_2))$$

$$= \text{Ext}_\mathcal{H}^i(P^{\sigma_1}(\sigma_1), I_{P(\sigma_1)}(P_{\sigma_2}) \cap (\text{St}_{Q_1}(\sigma_1)))$$

$$= \text{Ext}_\mathcal{H}^i(P^{\sigma_1}(\sigma_1), I_{P(\sigma_1)}(P_{\sigma_2}) \cap (\text{St}_{Q_1}(\sigma_1))).$$

We have $L_{P(\sigma_2)}(\cap (\text{St}_{Q_1}(\sigma_1))) = 0$ if $\Delta(\sigma_1) \neq \Delta(Q_1) \cup \Delta(P(\sigma_1) \cap P(\sigma_2)$ or $P \nsubseteq P(\sigma_1) \cap P(\sigma_2)$ by [4, Proposition 5.10, Proposition 5.18]. If it is not zero, then the extension group is isomorphic to

$$\text{Ext}_\mathcal{H}^i(P^{\sigma_1}(\sigma_1), I_{P(\sigma_1)}(P_{\sigma_2}) \cap (\text{St}_{Q_1}(\sigma_1) \cap (\text{St}_{Q_2}(\sigma_2))).$$

This holds if $Q_2 \subseteq P(\sigma_1)$, $\Delta(\sigma_1) = \Delta(Q_1) \cup \Delta(P(\sigma_1) \cap P(\sigma_2)$ and $P \subseteq P(\sigma_1) \cap P(\sigma_2)$, and otherwise the extension group is zero. Note that we always have $P \subseteq P(\sigma_1) \cap P(\sigma_2)$ since both $P(\sigma_1)$ and $P(\sigma_2)$ contain $P$. Since $Q_1 \subseteq P(\sigma_1)$, $\Delta(Q_1) \cup \Delta(P(\sigma_1) \cap P(\sigma_2) = (\Delta(\sigma_1) \cup \Delta(Q_1)) \cup (\Delta(\sigma_1) \cap \Delta(\sigma_2)) = \Delta(\sigma_1) \cap (\Delta(Q_1) \cup \Delta(\sigma_2))$. (Recall that $P(\sigma_1)$ is the parabolic subgroup corresponding to $\Delta(\sigma_1)$.) Hence, we have $\Delta(\sigma_1) = \Delta(Q_1) \cup \Delta(P(\sigma_1) \cap P(\sigma_2))$ if and only if $\Delta(\sigma_1) \subseteq \Delta(Q_1) \cup \Delta(\sigma_2)$.

Therefore, to calculate the extension groups, we may assume $P(\sigma_1) = P(\sigma_2) = G$.

### 3.3. Extensions between generalized Steinberg modules

We assume that $P(\sigma_1) = P(\sigma_2) = G$, and we continue the calculation of the extension groups.

**Lemma 3.7.** Let $Q_{11},Q_{12},Q_2$ be parabolic subgroups and $\alpha \in \Delta_{Q_{12}}$ such that $\Delta_{Q_{11}} = \Delta_{Q_{12}} \setminus \{\alpha\}$. Then we have

$$\text{Ext}_\mathcal{H}(\text{St}_{Q_{11}}(\sigma_1), \text{St}_{Q_2}(\sigma_2)) \simeq \begin{cases} \text{Ext}_\mathcal{H}^{i-1}(\text{St}_{Q_{12}}(\sigma_1), \text{St}_{Q_2}(\sigma_2)) & (\alpha \in \Delta_{Q_2}), \\ \text{Ext}_\mathcal{H}^{i+1}(\text{St}_{Q_{12}}(\sigma_1), \text{St}_{Q_2}(\sigma_2)) & (\alpha \notin \Delta_{Q_2}). \end{cases}$$
Proof. Let $P_1$ be a parabolic subgroup corresponding to $\Delta \setminus \{\alpha\}$. First, we prove that there exists an exact sequence

$$0 \to \text{St}_{Q_{12}}(\sigma_1) \to I_{P_1}(\text{St}_{Q_{11}}^P(\sigma_1)) \to \text{St}_{Q_{11}}(\sigma_1) \to 0. \quad (3.1)$$

We start with the following exact sequence.

$$0 \to \sum_{P_1 \supset Q \supset Q_{11}} I_Q^P(e_Q(\sigma_1)) \to I_{Q_{11}}^P(e_{Q_{11}}(\sigma_1)) \to \text{St}_{Q_{11}}^P(\sigma_1) \to 0,$$

Apply $I_{P_1}$ to this exact sequence. Then we have

$$0 \to \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1)) \to I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) \to I_{P_1}(\text{St}_{Q_{11}}^P(\sigma_1)) \to 0.$$

Hence, we get the following commutative diagram with exact columns:

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1)) & \to & \sum_{Q \supset Q_{11}} I_Q(e_Q(\sigma)) \\
\downarrow & & \downarrow \\
I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) & \to & I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) \\
\downarrow & & \downarrow \\
I_{P_1}(\text{St}_{Q_{11}}^P(\sigma_1)) & \to & \text{St}_{Q_{11}}(\sigma_1) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$

Hence, $I_{P_1}(\text{St}_{Q_{11}}^P(\sigma_1)) \to \text{St}_{Q_{11}}(\sigma_1)$ is surjective, and the kernel is isomorphic to

$$\sum_{Q \supset Q_{11}} I_Q(e_Q(\sigma)) \bigg/ \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1))$$

by the snake lemma.

We prove:

1. $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1)) = \sum_{Q \supset Q_{11}} I_Q(e_Q(\sigma)).$
2. $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) \cap \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1)) = \sum_{Q \supset Q_{11}} I_Q(e_Q(\sigma)).$

We prove (1). Since $Q_{12} \supset Q_{11}$, $I_{Q_{12}}(e_{Q_{12}}(\sigma_1))$ is contained in the right-hand side. Obviously, $\sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1))$ is also contained in the right-hand side. Hence, $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1)) \subseteq \sum_{Q \supset Q_{11}} I_Q(e_Q(\sigma))$. Take $Q \supset Q_{11}$, and we prove that $I_Q(e_Q(\sigma_1)) \subseteq I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supset Q_{11}} I_Q(e_Q(\sigma_1))$. If $P_1 \supset Q$, then it is obvious. We assume that $P_1 \nsubseteq Q$. Since $\Delta_{P_1} = \Delta \setminus \{\alpha\}$, this is equivalent to $\alpha \in \Delta_Q$. Hence, $\Delta_Q \supset \Delta_{Q_{11}} \cup \{\alpha\} = \Delta_{Q_{12}}$. Therefore, we have $Q \supset Q_{12}$. Hence, $I_Q(e_Q(\sigma_1)) \subseteq I_{Q_{12}}(e_{Q_{12}}(\sigma_1))$. 


We prove (2). By [2, Lemma 3.10], the left-hand side is
\[ \sum_{P_1 \supset Q \supset Q_{11}} I_{\langle Q, Q_{12} \rangle} (e_{\langle Q, Q_{12} \rangle}(\sigma_1)), \]
where \( \langle Q, Q_{12} \rangle \) is the subgroup generated by \( Q \) and \( Q_{12} \). We prove
\[ \{ \langle Q, Q_{12} \rangle \mid P_1 \supset Q \supset Q_{11} \} = \{ Q \mid Q \supset Q_{12} \}. \]
If \( Q \) satisfies \( P_1 \supset Q \supset Q_{11} \), then there exists \( \beta \in \Delta_Q \setminus \Delta_{Q_{11}} \). We have \( \beta \in \Delta_Q \subset \Delta_{Q_{11}} \). Therefore, we have \( \beta \neq \alpha \). Hence, \( \beta \notin \Delta_{Q_{11}} \cup \{ \alpha \} = \Delta_{Q_{12}} \). On the other hand, \( \beta \in \Delta_Q \subset \Delta_{Q_{12}} \). Namely, we have \( \beta \in \Delta_{\langle Q, Q_{12} \rangle} \setminus \Delta_{Q_{12}} \). Obviously, \( \langle Q, Q_{12} \rangle \supset Q_{12} \). Therefore, we get \( \langle Q, Q_{12} \rangle \supseteq Q_{12} \).

On the other hand, assume that \( Q \supseteq Q_{12} \). Then \( \alpha \in \Delta_Q \) since \( \alpha \in \Delta_{Q_{12}} \). Let \( Q' \) be the parabolic subgroup corresponding to \( \Delta_Q \setminus \{ \alpha \} \). Then we have \( \Delta_Q' \subset \Delta \setminus \{ \alpha \} = \Delta_{P_1} \) and \( \Delta_{Q'} = \Delta_Q \setminus \{ \alpha \} \supseteq \Delta_{Q_{12}} \setminus \{ \alpha \} = \Delta_{Q_{11}} \). Hence, \( P_1 \supset Q' \supseteq Q_{11} \). We have \( \Delta_{Q', Q_{12}} = \Delta_{Q' \cup Q_{12}} = (\Delta_Q \setminus \{ \alpha \}) \cup \Delta_{Q_{11}} \cup \{ \alpha \} = \Delta_Q \cup \Delta_{Q_{11}} \cup \{ \alpha \} \). This is \( \Delta_Q \) since \( \Delta_Q \supset \Delta_{Q_{12}} = \Delta_{Q_{11}} \cup \{ \alpha \} \). Hence, \( Q = (Q', Q_{12}) \). We get the existence of the exact sequence (3.1).

Assume that \( \alpha \in \Delta_{Q_2} \). Then \( \alpha \in \Delta_{Q_2} \) and \( \alpha \notin \Delta_{Q_2 \cap P_1} \). Hence, \( \Delta_{Q_2} \neq \Delta_{Q_2 \cap P_1} \cup \Delta_{P} \).

Therefore, \( R_{P_1}(St_{Q_2}(\sigma_2)) = 0 \) by [4, Proposition 5.11]. We have an exact sequence
\[ \text{Ext}^j_H(I_{P_1}(St_{Q_{11}}(\sigma_1)),St_{Q_2}(\sigma_2)) \rightarrow \text{Ext}^{j+1}_H(St_{Q_{12}}(\sigma_1),St_{Q_2}(\sigma_2)) \rightarrow \text{Ext}^{j+1}_H(I_{P_1}(St_{Q_{11}}(\sigma_1)),St_{Q_2}(\sigma_2)). \]
Since \( R_{P_1}(St_{Q_2}(\sigma_2)) = 0 \), for any \( j \), we have
\[ \text{Ext}^j_H(I_{P_1}(St_{Q_{12}}(\sigma_1)),St_{Q_2}(\sigma_2)) = \text{Ext}^j_H(St_{Q_{12}}(\sigma_1),R_{P_1}(St_{Q_2}(\sigma_2))) = 0. \]

Therefore, we get
\[ \text{Ext}^i_H(St_{Q_{12}}(\sigma_1),St_{Q_2}(\sigma_2)) \simeq \text{Ext}^{i+1}_H(St_{Q_{11}}(\sigma_1),St_{Q_2}(\sigma_2)). \]

Next assume that \( \alpha \notin \Delta_{Q_2} \). Let \( Q_{11} \) (resp. \( Q_{12}, Q_{52} \)) be the parabolic subgroup corresponding to \( (\Delta \setminus \Delta_{Q_{11}}) \cup \Delta_{P} \) (resp. \( (\Delta \setminus \Delta_{Q_{12}}) \cup \Delta_{P} \)). Let \( T_\iota = \iota: \mathcal{H} \rightarrow \mathcal{H} \) be the involution defined by \( \iota(T_w) = (-1)^\ell(w)T_w^* \), and set \( \pi^* = \pi \circ \iota \) for an \( \mathcal{H} \)-module \( \pi \). Then we have
\[ \text{Ext}^i_H(St_{Q_{11}}(\sigma_1),St_{Q_2}(\sigma_2)) \simeq \text{Ext}_H^i((St_{Q_{11}}(\sigma_1))^\iota,(St_{Q_2}(\sigma_2))^\iota) \]
\[ \simeq \text{Ext}_H^i(St_{Q_{11}}(\sigma^{i\iota}_{1,\ell-\ell_{\iota}}),St_{Q_2}(\sigma^{i\iota}_{2,\ell-\ell_{\iota}})). \]
by [2, Theorem 3.6]. Now we have \( \alpha \in \Delta_{Q_{52}} \). Applying the lemma (where \( Q_{11} = Q_{12} = Q_{51} \)), we have
\[ \text{Ext}^i_H(St_{Q_{11}}(\sigma^{i\iota}_{1,\ell-\ell_{\iota}}),St_{Q_2}(\sigma^{i\iota}_{2,\ell-\ell_{\iota}})) \simeq \text{Ext}^{i-1}_H(St_{Q_{12}}(\sigma^{i\iota}_{1,\ell-\ell_{\iota}}),St_{Q_2}(\sigma^{i\iota}_{2,\ell-\ell_{\iota}})) \]
\[ \simeq \text{Ext}^{i-1}_H((St_{Q_{12}}(\sigma_1))^\iota,(St_{Q_2}(\sigma_2))^\iota) \]
\[ \simeq \text{Ext}^{i-1}_H(St_{Q_{12}}(\sigma_1),St_{Q_2}(\sigma_2)). \]
We get the lemma.

For sets \( X, Y \), let \( X \triangle Y = (X \setminus Y) \cup (Y \setminus X) \) be the symmetric difference.
Theorem 3.8. We have
\[ \text{Ext}^i_\mathcal{H}(\text{St}_{Q_1}(\sigma_1),\text{St}_{Q_2}(\sigma_2)) \simeq \text{Ext}^{i-\#(\Delta_{Q_1} \Delta_{Q_2})}_\mathcal{H}(e_G(\sigma_1),e_G(\sigma_2)). \]

Proof. By applying Lemma 3.7 several times, we have
\[ \text{Ext}^i_\mathcal{H}(\text{St}_{Q_1}(\sigma_1),\text{St}_{Q_2}(\sigma_2)) \simeq \text{Ext}^{i-r_1}_{\mathcal{H}}(e_G(\sigma_1),\text{St}_{Q_2}(\sigma_2)), \]
where \( r_1 = \#\{\alpha \in \Delta \setminus \Delta_{Q_1} \mid \alpha \in \Delta_{Q_2}\} - \#\{\alpha \in \Delta \setminus \Delta_{Q_1} \mid \alpha \notin \Delta_{Q_2}\}. \) Set \( Q'_2 = n_{w_Gw_{Q_2}}Q''_2n_{-1w_Gw_{Q_2}}. \) Then by Lemma 3.5, we get
\[ \text{Ext}^{i-r_1}_{\mathcal{H}}(e_G(\sigma_1),\text{St}_{Q_2}(\sigma_2)) \simeq \text{Ext}^{i-r_1}_{\mathcal{H}}((\text{St}_{Q_2}(\sigma_2))^*,e_G(\sigma_1)^*) \]
\[ \simeq \text{Ext}^{i-r_1}_{\mathcal{H}}(\text{St}_{Q_2}(\sigma_2^*),e_G(\sigma_1^*)). \]

Again using Lemma 3.7, we have
\[ \text{Ext}^{i-r_1}_{\mathcal{H}}(\text{St}_{Q_2}(\sigma_2^*),e_G(\sigma_1^*)) \simeq \text{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_2^*),e_G(\sigma_1^*)), \]
where \( r_2 = \#(\Delta \setminus \Delta_{Q'_2}) = \#(\Delta \setminus \Delta_{Q_2}). \) Applying Lemma 3.5 again, we get
\[ \text{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_2^*),e_G(\sigma_1^*)) \simeq \text{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_1),e_G(\sigma_2)). \]

Since \( r_1 + r_2 = \#(\Delta_{Q_1} \Delta_{Q_2}), \) we get the lemma.

Recall that the trivial module \( 1 \) is defined by \( 1(T_w) = q_w. \) We denote the restriction of \( 1 \) to \( \mathcal{H}_{aff} \) by \( 1_{\mathcal{H}_{aff}}. \)

Corollary 3.9. We have \( \text{Ext}^i_{\mathcal{H}_{aff}}(1_{\mathcal{H}_{aff}},1_{\mathcal{H}_{aff}}) = 0 \) for \( i > 0. \)

Proof. Let \( \mathcal{H}_{aff} \) be the quotient of \( \mathcal{H}_{aff} \) by the ideal generated by \( \{T_t - 1 \mid t \in Z_\kappa \cap W_{aff}(1)\}. \) Then this is the Hecke algebra attached to the Coxeter system \((W_{aff},S_{aff})\). Let \( \pi_2 \) (resp. \( \pi_1 \)) be an \( \mathcal{H}_{aff} \)-module (resp. \( \mathcal{H}_{aff} \)-module). Then we have \( \text{Hom}_{\mathcal{H}_{aff}}(\pi_1,\pi_2) = \text{Hom}_{\mathcal{H}_{aff}}(\pi_1,\pi_2^{Z_\kappa}). \) In particular, \( \pi_2 \mapsto \pi_2^{Z_\kappa} \) sends injective \( \mathcal{H} \)-modules to injective \( \mathcal{H}_{aff} \)-modules. Since the functor \( \pi_2 \mapsto \pi_2^{Z_\kappa} \) is exact, we have \( \text{Ext}^i_{\mathcal{H}_{aff}}(\pi_1,\pi_2) \simeq \text{Ext}^i_{\mathcal{H}_{aff}}(\pi_1,\pi_2^{Z_\kappa}). \)

Therefore, we have \( \text{Ext}^i_{\mathcal{H}_{aff}}(1_{\mathcal{H}_{aff}},1_{\mathcal{H}_{aff}}) \simeq \text{Ext}^i_{\mathcal{H}_{aff}}(1_{\mathcal{H}_{aff}},1_{\mathcal{H}_{aff}}). \) Consider the root system which defines \((W_{aff},S_{aff}),\) and let \( H \) be the split simply-connected semisimple group with this root system. Then the affine Hecke algebra attached to \( H \) is \( \mathcal{H}_{aff}. \) Let \( \mathcal{H}' \) be the pro-p-Iwahori Hecke algebra for \( H. \) Then \( H' = H \) by [5, II.3. Proposition]; hence, \( \mathcal{H}'_{aff} = \mathcal{H}'. \)

Therefore, the above argument implies that \( \text{Ext}^i_{\mathcal{H}}(1_G,1_G) \simeq \text{Ext}^i_{\mathcal{H}}(1_{\mathcal{H}_{aff}},1_{\mathcal{H}_{aff}}). \)

Next, it is sufficient to prove \( \text{Ext}^i_{\mathcal{H}}(1_G,1_G) = 0 \) for \( i > 0 \) assuming \( G \) is a split, simply connected semisimple group. By [15, Proposition 6.20], the projective dimension of \( 1_G \) is equal to the semisimple rank of \( G, \) namely \( \#\Delta. \) Therefore, \( \text{Ext}^{i+\#\Delta}_{\mathcal{H}}(1_G,\text{St}_B(1_B)) = 0 \) for \( i > 0. \) The left-hand side is \( \text{Ext}^i_{\mathcal{H}}(1_G,1_G) \) by Theorem 3.8.

3.4. Extension between extensions

Let \( P \) be a parabolic subgroup and \( \sigma \) an \( \mathcal{H}_P \)-module which has the extension \( e_G(\sigma) \) to \( \mathcal{H}. \)
In particular, \( \Delta_P \) and \( \Delta \setminus \Delta_P \) are orthogonal to each other. Let \( P_2 \) be a parabolic subgroup
corresponding to $\Delta \setminus \Delta_P$. Let $J \subset \mathcal{H}$ be an ideal generated by \{ $T_w - 1 \mid w \in W_{\text{aff}, P_2}(1)$ \}. Then $e_G(\sigma)(J) = 0$. Hence, for any module $\pi$ of $\mathcal{H}$, we have
\[
\text{Hom}_\mathcal{H}(e(\sigma), \pi) = \text{Hom}_{\mathcal{H}/J}(e(\sigma), \{ v \in \pi \mid vJ = 0 \}).
\]
Note that $T^\ell_{P_2} \mapsto T_w$ defines the injection $\mathcal{H}_{\text{aff}, P_2} \to \mathcal{H}$ since the restriction of $\ell$ on $W_{\text{aff}, P_2}(1)$ is $\ell_{P_2}$. Since any generator of $J$ is in $\mathcal{H}_{\text{aff}, P_2}$, we have $\{ v \in \pi \mid vJ = 0 \} = \{ v \in \pi \mid v(J \cap \mathcal{H}_{\text{aff}, P_2}) = 0 \}$. Since the trivial representation $1_{\mathcal{H}_{\text{aff}, P_2}}$ of $\mathcal{H}_{\text{aff}, P_2}$ is isomorphic to $\mathcal{H}_{\text{aff}, P_2}/(J \cap \mathcal{H}_{\text{aff}, P_2})$, we get
\[
\{ v \in \pi \mid v(J \cap \mathcal{H}_{\text{aff}, P_2}) = 0 \} = \text{Hom}_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, \pi).
\]
Hence, we get
\[
\text{Hom}_\mathcal{H}(e_G(\sigma), \pi) = \text{Hom}_{\mathcal{H}/J}(e_G(\sigma), \text{Hom}_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, \pi)).
\]
This isomorphism can be generalized as
\[
\text{Hom}_\mathcal{H}(\pi_1, \pi) = \text{Hom}_{\mathcal{H}/J}(\pi_1, \text{Hom}_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, \pi))
\]
for any $\mathcal{H}/J$-module $\pi_1$. In particular, $\pi \mapsto \text{Hom}_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, \pi)$ from the category of $\mathcal{H}$-modules to the category of $\mathcal{H}/J$-modules preserves injective modules. Hence, we have a spectral sequence
\[
\text{Ext}^i_{\mathcal{H}/J}(e_G(\sigma), \text{Ext}^j_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, \pi)) \Rightarrow \text{Ext}^{i+j}_{\mathcal{H}}(e(\sigma), \pi).
\]
Now let $\sigma_1, \sigma_2$ be $\mathcal{H}_P$-modules such that both have the extensions $e_G(\sigma_1), e_G(\sigma_2)$ to $\mathcal{H}$. Since $e_G(\sigma_2)_{\mathcal{H}_{\text{aff}, P_2}}$ is a direct sum of the trivial representations, we have
\[
\text{Ext}^j_{\mathcal{H}_{\text{aff}, P_2}}(1_{\mathcal{H}_{\text{aff}, P_2}}, e(\sigma_2)) = 0
\]
for $j > 0$ by Corollary 3.9. Hence,
\[
\text{Ext}^i_{\mathcal{H}}(e_G(\sigma_1), e_G(\sigma_2)) \simeq \text{Ext}^i_{\mathcal{H}/J}(e_G(\sigma_1), e_G(\sigma_2)).
\]

**Lemma 3.10** ([6, Proposition 3.5]). Let $I$ be the ideal of $\mathcal{H}_P$ generated by \{ $T^\lambda_P - 1 \mid \lambda \in \Lambda(1) \cap W_{\text{aff}, P_2}(1)$ \}. Then we have $\mathcal{H}/I \simeq \mathcal{H}_P/I$.

Therefore, we get
\[
\text{Ext}^i_{\mathcal{H}}(e_G(\sigma_1), e_G(\sigma_2)) \simeq \text{Ext}^i_{\mathcal{H}/P/I}(\sigma_1, \sigma_2).
\]

**Proposition 3.11.** Set $W'_{\text{aff}} = W_{\text{aff}, P}$, $S'_{\text{aff}} = S_{\text{aff}, P}$, $W' = W_{P}/(\Lambda \cap W_{\text{aff}, P_2})$, $\Omega' = \Omega_P/(\Lambda \cap W_{\text{aff}, P_2})$, $W'(1) = W_{P}(1)/(\Lambda(1) \cap W_{\text{aff}, P_2}(1))$, $Z'_\kappa = Z_\kappa/(Z_\kappa \cap W_{\text{aff}, P_2}(1))$. Then $\mathcal{W}'_{\text{aff}}, S'_{\text{aff}}, \Omega', W', W'(1), Z'_\kappa$ satisfies the condition of subsection 2.1, and the attached algebra is $\mathcal{H}/I$. Moreover, $\Omega'$ is commutative.

**Proof.** Since $\Delta = \Delta_P \cup \Delta_{P_2}$ is the orthogonal decomposition, we have $W_{\text{aff}} = W_{\text{aff}, P} \times W_{\text{aff}, P_2}$ and $S_{\text{aff}} = S_{\text{aff}, P} \cup S_{\text{aff}, P_2}$. The pair $(W'_{\text{aff}}, S'_{\text{aff}}) = (W_{\text{aff}, P}, S_{\text{aff}, P})$ is a Coxeter system, and $\Omega_{P}$ acts on it. Since $W_{\text{aff}, P_2}$ commutes with $W_{\text{aff}, P}$, this gives the action of $\Omega'$ on $(W'_{\text{aff}}, S'_{\text{aff}})$. We have $W_{\text{aff}}' \subset W_{P}$, and since $W_{\text{aff}, P} \cap W_{\text{aff}, P_2}$ is trivial, we have the embedding $W_{\text{aff}}' \subset W'$. We also have $\Omega_{P} \subset W'$. Since $W_{P} = W_{\text{aff}, P_2} \Omega_{P}$, we
have \( W' = W'_\text{aff} \Omega' \). Since \( W_{\text{aff},P} \cap \Omega_P = \{1\} \), we have \( W'_{\text{aff}} \cap \Omega' = \{1\} \) in \( W' \). Hence, \( W' = W'_{\text{aff}} \times \Omega' \). Since \( Z_\kappa \) is finite and commutative, \( Z'_\kappa \) is also a finite commutative group. The existence of the exact sequence

\[
1 \to Z'_\kappa \to W'(1) \to W' \to 1
\]

is obvious. Note that the length function \( \ell' : W'(1) \to \mathbb{Z}_{\geq 0} \) is given by \( \ell_P : W'_P(1) \to \mathbb{Z}_{\geq 0} \) since \( S'_{\text{aff}} = S_{\text{aff},P} \) and \( \Omega' \) is the image of \( \Omega_P \).

We put \( q'_s = q_s \) for \( s \in S_{\text{aff},P} \). (Note that \( q_s = q_{s,P} \) since \( \Delta = \Delta_P \cup \Delta_{P_2} \) is an orthogonal decomposition.) For \( s \in \text{Ref}(W'(1)) \), take its lift \( \bar{s} \in \text{Ref}(W_P(1)) \) and let \( c'_s \) be the image of \( c_s \) in \( C[Z'_\kappa] \). We prove that this is well-defined. Let \( \bar{s}' \) be another lift, and take \( \lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1) \) such that \( \bar{s}' = \bar{s}\lambda \). The image of \( \bar{s} \) in \( \bar{W} \) is in \( \text{Ref}(W_P) \subset W_{P, \text{aff}} \) since \( S_{P, \text{aff}} \subset W_{P, \text{aff}} \) and \( W_{P, \text{aff}} \) is normal. (Recall that a reflection is an element which is conjugate to a simple reflection.) Let \( \bar{\lambda} \) be the image of \( \lambda \) in \( \bar{\Lambda} \). Since \( \bar{s}, \bar{s}' \in W_{P, \text{aff}}(1) \), we have \( \bar{s} \bar{\lambda} = \bar{s}' \bar{\lambda} \). Hence, \( \lambda \in Z_\kappa \). Since \( \lambda \in W_{\text{aff},P_2}(1) \), we have \( \lambda \in Z_\kappa \cap W_{\text{aff},P_2}(1) \). Hence, the image of \( c_s \) is the same as that of \( c_{s,P} \) in \( C[Z'_\kappa] \).

We get the parameter \( (q', c') \) and let \( \mathcal{H}' = \bigoplus_{w \in W'(1)} T'_w \) be the attached algebra. Consider the linear map \( \Phi : \mathcal{H}_P \to \mathcal{H}' \) defined by \( T'_w \mapsto T'_w \), where \( w \in W_P(1) \) and \( w \in W'(1) \) is the image of \( w \).

First, we prove that the map \( \Phi \) preserves the relations. Let \( s \in W_P(1) \) be a lift of an affine simple reflection in \( S_{\text{aff},P} \). Then we have \( (T'_s)^2 = q_s T'_s + c_s T'_s^2 \). Let \( \tilde{s} \) be the image of \( s \) in \( W'(1) \). Then we have \( (T'_w)^2 = q'_w T'_w + c'_w T'_w^2 \). The definition of \( (q', c') \) says \( q'_w = q_s \) and \( \Phi(c_s) = c'_w \). Hence, \( \Phi \) preserves the quadratic relations. The compatibility between \( \ell_P \) and \( \ell' \) implies that \( \Phi \) preserves the braid relations.

Obviously, \( \Phi \) is surjective. We prove that \( \ker \Phi = I \). Clearly, we have \( I \subset \ker \Phi \). Let \( \sum_{w \in W_P(1)} c_w T_w \in \ker \Phi \), where \( c_w \in C \). Fix a section \( x \) of \( W_P(1) \to W'(1) \). Then we have \( \sum_{w \in W_P(1)} c_w T_w = \sum_{w \in W'(1)} \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)\lambda} \). Hence,

\[
0 = \Phi \left( \sum_{w \in W_P(1)} c_w T_w \right) = \sum_{w \in W'(1)} \left( \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)\lambda} \right).
\]

Therefore, for each \( w \in W'(1) \), we have \( \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda = 0 \). Hence,

\[
\sum_{w \in W_P(1)} c_w T_w = \sum_{w \in W'(1)} \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)\lambda} = \sum_{w \in W'(1)} \left( \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)\lambda} - \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)} \right) = \sum_{w \in W'(1)} \left( \sum_{\lambda \in \Lambda(1) \cap W_{\text{aff},P_2}(1)} c_x(w) \lambda T'_{x(w)} (T'_{x(w)} - 1) \right) \in I.
\]

Finally, \( \Omega' \) is commutative since \( \Omega_P \) is commutative. \( \square \)
Remark 3.12. The data do not come from a reductive group in general.

3.5. Example
Let $G = \operatorname{PGL}_2$. We have $\Lambda(1) \simeq F^\times/(1 + (\varpi)) \simeq \mathbb{Z} \times \kappa^\times$. Consider $\widetilde{G} = \operatorname{SL}_2$. Then $G'$ is the image of $\widetilde{G} \to G$ [5, II.4 Proposition]. By this description, we have $\Lambda(1) \cap W(1) = \{ \lambda^2 \mid \lambda \in \Lambda(1) \}$. Therefore, with the notation in Proposition 3.11, we have $W' = \{ 1 \}$, $\mathcal{S}' = \emptyset$, $W'(1) = \Lambda(1)/\{ \lambda^2 \mid \lambda \in \Lambda(1) \}$, $Z_\kappa = Z_\kappa/\{ t^2 \mid t \in Z_\kappa \}$. We have $\mathcal{H}_B/I = C[W'(1)]$.

Consider the trivial module $1_G$. Then we have $1_G = e_G(1_B)$, and we have
\[ \operatorname{Ext}^i_{\mathcal{H}}(1_G, 1_G) \simeq \operatorname{Ext}^i_{\mathcal{H}_B/I}(1_B, 1_B) \simeq \operatorname{Ext}^i_{C[W'(1)]}(1_B, 1_B) = H^i(W'(1), C). \]
Here, $C$ is the trivial $W'(1)$-module. Since the group $W'(1)$ is a 2-group, this cohomology is zero if the characteristic of $C$ is not 2. However, if the characteristic of $C$ is 2, since $W'(1) \simeq \mathbb{Z}/2\mathbb{Z}$ ($p = 2$) or $(\mathbb{Z}/2\mathbb{Z})^\oplus 2$ ($p \neq 2$), $H^i(W'(1), C) \neq 0$ if $i$ is even. Therefore, we have infinitely many $i$ with $\operatorname{Ext}^i_{\mathcal{H}}(1_G, 1_G) \neq 0$. This recovers Koziol’s example [12, Example 6.2].

3.6. Summary
Now we get a reduction. The $\operatorname{Ext}^1$ between simple modules is equal to $\operatorname{Ext}^{1-r}$ between supersingular simple modules for some $r \geq 0$ or zero. In particular, if $r \geq 2$, then $\operatorname{Ext}^1$ between simple modules is zero. If $r = 1$, then $\operatorname{Ext}^{1-r} = \operatorname{Hom}$, so it is zero or one-dimensional. If $r = 0$, we have to calculate $\operatorname{Ext}^1$ between supersingular simple modules. Therefore, the only remaining task is to calculate $\operatorname{Ext}^1$ between supersingular simple modules.

4. $\operatorname{Ext}^1$ between supersingular modules
In this section, we fix data $(W, S, \Omega, W(1), Z_\kappa)$ and let $\mathcal{H}$ be the algebra attached to this data. We do not assume that these data come from a group. We also assume:

- our parameter $q_s$ is zero.
- $\#Z_\kappa$ is prime to $p$.

As in subsection 2.9, let $W(1)$ be the inverse image of $W$ in $W(1)$, and put $\mathcal{H} = \bigoplus_{w \in W} CT_w$.

For a character $\chi$ of $Z_\kappa$ and $w \in W$, we define $(w\chi)(t) = \chi(\bar{w}^{-1}tw)$ where $\bar{w} \in W(1)$ is a lift of $W$. Since $Z_\kappa$ is commutative, this does not depend on a lift $\bar{w}$ and defines a character $w\chi$ of $Z_\kappa$. For a character $\Xi$ of $\mathcal{H}$ and $\omega \in \Omega(1)$, we write $\Xi \omega$ for the character $T_w \mapsto \Xi(T_{w\omega}^{-1})$ for $w \in W(1)$. Since $\Xi \omega$ only depends on the image $\bar{\omega}$ of $\omega$ in $\bar{\omega}$, we also write $\Xi \bar{\omega}$.

Note that, since $s \cdot c_s = c_s$ for $s \in S$ with a lift $\bar{s}$ by the conditions of the parameter $c$, we have $(s\chi)(c_s) = \chi(c_s)$.

4.1. $\operatorname{Ext}^1$ for $\mathcal{H}$
Let $\chi, \chi'$ be characters of $Z_\kappa$ and $J \subseteq S, \chi, J' \subseteq S, \chi'$ subsets. Then we have characters $\Xi = \Xi_J, \chi, \Xi' = \Xi_{J', \chi'}$ of $\mathcal{H}$. We calculate $\operatorname{Ext}^1_{\mathcal{H}}(\Xi, \Xi')$. 
To express the space of extensions, we need some notation. For each \( s \in S_{\text{aff}} \), let \( C_s \) be the set of functions \( a \) on \( \{ \tilde{s} \in W(1) \mid \tilde{s} \mapsto s \in W \} \) such that \( a(t \tilde{s}) = \chi'(t)a(\tilde{s}), a(\tilde{s}t) = a(\tilde{s})\chi(t) \) for any \( t \in Z_n \). Then \( C_s \neq 0 \) if and only if \( \chi' = s\chi \) and if \( \chi' = s\chi \) then \( \dim_C C_s = 1 \).

Now we define some subsets of \( S_{\text{aff}} \). First, consider the sets

\[
A_1(\Xi,\Xi') = \{ s \in S_{\text{aff}} \mid \Xi(T_{\tilde{s}}) = \Xi'(T_{\tilde{s}}) = 0 \},
\]

\[
A_2(\Xi,\Xi') = \{ s \in S_{\text{aff}} \mid \Xi(T_{\tilde{s}}) \neq 0, \Xi'(T_{\tilde{s}}) = 0 \},
\]

\[
A_3(\Xi,\Xi') = \{ s \in S_{\text{aff}} \mid \Xi(T_{\tilde{s}}) = 0, \Xi'(T_{\tilde{s}}) \neq 0 \},
\]

\[
A_4(\Xi,\Xi') = \{ s \in S_{\text{aff}} \mid \Xi(T_{\tilde{s}}) \neq 0, \Xi'(T_{\tilde{s}}) \neq 0 \},
\]

where \( \tilde{s} \) is a lift of \( s \). We define

\[
S_2(\Xi,\Xi') = A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi'),
\]

\[
S_1(\Xi,\Xi') = \{ s \in A_1(\Xi,\Xi') \setminus S_{\text{aff},\chi} \mid s\chi = \chi', (ss_1)^2 \neq 1 \text{ for any } s_1 \in S_2(\Xi,\Xi') \}.
\]

If \( s \in S_{\text{aff},\chi} \) and \( a \in C_s \), then \( a(\tilde{s})\chi(c_s)^{-1} \in C \) does not depend on a lift \( \tilde{s} \) of \( s \). We denote it by \( a\chi(c_s)^{-1} \). We also have that if \( s \in S_{\text{aff},\chi'} \), then \( a(\tilde{s})\chi'(c_s)^{-1} \) does not depend on a lift \( \tilde{s} \). We denote it by \( a\chi'(c_s)^{-1} \). If \( a \neq 0 \), then \( \chi' = s\chi \). Hence, if \( s \in S_{\text{aff},\chi} \), then \( s \in S_{\text{aff},\chi'} \) and \( a\chi(c_s)^{-1} = a\chi'(c_s)^{-1} \).

For the Hecke algebra attached to a finite Coxeter system, the following proposition is [8, Theorem 5.1], and we use a similar proof.

**Proposition 4.1.** Consider the subspace \( E_2(\Xi,\Xi') \) of \( \bigoplus_{s \in S_2(\Xi,\Xi')} C_s \) consisting \( (a_s) \) such that

- If \( s_1, s_2 \in A_2(\Xi,\Xi') \), then \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \).
- If \( s_1, s_2 \in A_3(\Xi,\Xi') \), then \( a_{s_1}\chi'(c_{s_1})^{-1} = a_{s_2}\chi'(c_{s_2})^{-1} \).
- If \( s_1 \in A_2(\Xi,\Xi') \), \( s_2 \in A_3(\Xi,\Xi') \) and \( (s_1 s_2)^2 = 1 \), then \( a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0 \),

and put \( E_1(\Xi,\Xi') = \bigoplus_{s \in S_1(\Xi,\Xi')} C_s \), \( E(\Xi,\Xi') = E_1(\Xi,\Xi') \oplus E_2(\Xi,\Xi') \). For \( (a_s) \in E(\Xi,\Xi') \), consider the linear map \( \mathcal{H} \to M_2(C) \) defined by

\[
T_{\tilde{s}} \mapsto \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ a_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix},
\]

where \( a_s = 0 \) if \( s \notin S_1(\Xi,\Xi') \cup S_2(\Xi,\Xi') \). Then this gives an extension of \( \Xi \) by \( \Xi' \), and it gives a surjective map \( E(\Xi,\Xi') \to \text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi,\Xi') \). The kernel is

\[
\left\{ (a_s) \in E(\Xi,\Xi') \mid a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0 \quad (s_1 \in A_2(\Xi,\Xi'), s_2 \in A_3(\Xi,\Xi')) \right\}.
\]

**Remark 4.2.** Let \( V_2 \) be a subspace of \( \bigoplus_{s \in A_2(\Xi,\Xi')} C_s \) consisting \( (a_s) \) such that \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \) for any \( s_1, s_2 \in A_2(\Xi,\Xi') \). Then \( \dim V_2 \leq 1 \) and \( V_2 \neq 0 \) if and only if \( C_s \neq 0 \) for any \( s \in A_2(\Xi,\Xi') \). Define \( V_3 \) by the similar way. Then \( \dim V_3 \leq 1 \) and \( V_3 \neq 0 \) if and only if \( \chi = \chi' \) for any \( s \in A_3(\Xi,\Xi') \). If there is no \( s_1 \in A_2(\Xi,\Xi') \) and \( s_2 \in A_3(\Xi,\Xi') \) such that \( (s_1 s_2)^2 = 1 \), then \( E_2(\Xi,\Xi') = V_2 \oplus V_3 \). Otherwise, \( \dim E_2(\Xi,\Xi') = \max\{0, \dim V_2 + \dim V_3 - 1\} \).
Proof. Let $M$ be an extension of $\Xi$ by $\Xi'$. Since $\#Z_\kappa$ is prime to $p$, the representation of $Z_\kappa$ over $C$ is completely reducible. Hence, we can take a basis $e_1, e_2$ such that $T_1 e_1 = \chi(t) e_1$ and $T_1 e_2 = \chi'(t) e_2$. With this basis, the action of $T_{\tilde{s}}$ where $\tilde{s} \in S_{\text{aff}}(1)$ with the image $s \in S_{\text{aff}}$ is described as

$$T_{\tilde{s}} = \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ a_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix},$$

for some $a_s(\tilde{s}) \in C$. The action of $T_t$ where $t \in Z_\kappa$ is given by

$$\begin{pmatrix} \chi(t) & 0 \\ 0 & \chi'(t) \end{pmatrix}.$$

Since $T_t T_{\tilde{s}} = T_{t\tilde{s}}$, we have

$$(\begin{pmatrix} \chi(t) & 0 \\ 0 & \chi'(t) \end{pmatrix}) \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ a_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix} = \begin{pmatrix} \Xi(T_{t\tilde{s}}) & 0 \\ a_s(t\tilde{s}) & \Xi'(T_{t\tilde{s}}) \end{pmatrix}.$$

Hence, $a_s(t\tilde{s}) = \chi'(t)a_s(\tilde{s})$. Similarly, we have $a_s(\tilde{st}) = a_s(\tilde{s})\chi(t)$. Hence, $a_s \in C_s$.

Now we check the conditions that the map defines an action of $\mathcal{H}_{\text{aff}}$. Since we have

$$\begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ a_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix}^2 = \begin{pmatrix} \Xi(T_{\tilde{s}})^2 & 0 \\ (a_s(\tilde{s})(\Xi(T_{\tilde{s}}) + \Xi'(T_{\tilde{s}})) & \Xi'(T_{\tilde{s}})^2 \end{pmatrix},$$

this satisfies the quadratic relation $T_{\tilde{s}}^2 = T_{\tilde{s}} c_{\tilde{s}}$ if and only if

$$a_s(\tilde{s})(\Xi(T_{\tilde{s}}) + \Xi'(T_{\tilde{s}})) = a_s(\tilde{s})\chi(c_{\tilde{s}}).$$

(4.2)

If $s \in A_1(\Xi, \Xi')$, then $a_s(\tilde{s}) = 0$ or $\chi(c_{\tilde{s}}) = 0$, namely $a_s = 0$ or $s \notin S_{\text{aff,}\chi}$.

If $s \in A_2(\Xi, \Xi')$, then $a_s = 0$ or $\Xi(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$. Since $\Xi(T_{\tilde{s}}) \neq 0$, we always have $\Xi(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$. Hence, equation (4.2) is always satisfied.

If $s \in A_3(\Xi, \Xi')$, then $a_s = 0$ or $\Xi'(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$. Note that if $a_s \neq 0$, then $s\chi = \chi'$; hence, $S_{\text{aff,}\chi} = S_{\text{aff,}\chi'}$ and $\chi(c_{\tilde{s}}) = \chi'(c_{\tilde{s}})$. Therefore, under $a_s \neq 0$, we have $\Xi'(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$ if and only if $\Xi'(T_{\tilde{s}}) = \chi'(c_{\tilde{s}})$. This always hold since $\Xi'(T_{\tilde{s}}) \neq 0$. Hence, equation (4.2) is always satisfied.

If $s \in A_4(\Xi, \Xi')$, then we have $\Xi(T_{\tilde{s}}) = \chi(c_{\tilde{s}})$. Hence, we have $a_s(\tilde{s})\Xi'(T_{\tilde{s}}) = 0$. Therefore, we have $a_s = 0$ since $\Xi'(T_{\tilde{s}}) \neq 0$.

Consequently, the quadratic relation holds if and only if $a_s = 0$ or $s \in (A_1(\Xi, \Xi') \setminus S_{\text{aff,}\chi}) \cup A_2(\Xi, \Xi') \cup A_3(\Xi, \Xi')$. The action of $T_{\tilde{s}}$ is given by one of the following matrix:

$$\begin{pmatrix} 0 & 0 \\ a_s(\tilde{s}) & 0 \end{pmatrix} \begin{pmatrix} \chi(c_{\tilde{s}}) & 0 \\ 0 & \chi'(c_{\tilde{s}}) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a_s(\tilde{s}) & 0 \end{pmatrix} \begin{pmatrix} 0 & \chi(c_{\tilde{s}}) \\ 0 & \chi'(c_{\tilde{s}}) \end{pmatrix}.$$

(4.3)

Here, each $\chi(c_{\tilde{s}})$ and $\chi'(c_{\tilde{s}})$ is not zero, and in the first matrix, we assume that $s \notin S_{\text{aff,}\chi}$ if $a_s \neq 0$.

Now we check the braid relations. Let $s_1, s_2 \in S_{\text{aff}}$ and $\tilde{s}_1, \tilde{s}_2$ their lifts. We consider a braid relation $s_1 s_2 \cdots = s_2 s_1 \cdots$. It is easy to see that the action satisfies the braid relation for some lifts $\tilde{s}_1, \tilde{s}_2$ if and only if it is satisfied for any lifts $\tilde{s}_1, \tilde{s}_2$. Take $\tilde{s}_1, \tilde{s}_2$ such that $\tilde{s}_1 \tilde{s}_2 \cdots = \tilde{s}_2 \tilde{s}_1 \cdots$. It is easy to see that if $s_1 \in A_4(\Xi, \Xi')$ or $s_2 \in A_4(\Xi, \Xi')$, then the braid relations hold automatically. So we assume that $s_1, s_2 \in A_1(\Xi, \Xi') \cup A_2(\Xi, \Xi') \cup A_3(\Xi, \Xi')$. 

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Assume that \( s_1 \in A_1(\Xi, \Xi') \). We have

\[
\begin{pmatrix}
0 & 0 \\
a_{s_1}(\bar{s}_1) & a_{s_2}(\bar{s}_2)
\end{pmatrix}
\begin{pmatrix}
\Xi(T_{\bar{s}_2}) \\
\Xi'(T_{\bar{s}_2})
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
a_{s_1}(\bar{s}_1) & a_{s_2}(\bar{s}_2)
\end{pmatrix}
\begin{pmatrix}
\Xi(T_{\bar{s}_2}) \\
\Xi'(T_{\bar{s}_2})
\end{pmatrix}.
\]

Hence, the braid relation is satisfied if and only if

- \( a_{s_1} = 0 \)
- or \( \Xi(T_{\bar{s}_2}) = \Xi'(T_{\bar{s}_2}) \) and the order of \( s_1 s_2 \) is 2
- or \( \Xi(T_{\bar{s}_2}) \Xi'(T_{\bar{s}_2}) = 0 \) and the order of \( s_1 s_2 \) is 3
- or the order of \( s_1 s_2 \) is greater than 3.

If \( s_2 \in A_1(\Xi, \Xi') \), then the condition always holds. If \( s_2 \in A_2(\Xi, \Xi') \cup A_3(\Xi, \Xi') \), then the condition holds if and only if \( a_{s_1} = 0 \) or the order of \( s_1 s_2 \) is not 2, namely \( (s_1 s_2)^2 \neq 1 \).

Replacing \( s_1 \) with \( s_2 \), if \( s_2 \in A_1(\Xi, \Xi') \), we have the similar condition.

Assume that \( s_1, s_2 \in A_2(\Xi, \Xi') \). We have

\[
\begin{pmatrix}
\chi(c_{s_1}) \\
a_{s_1}(\bar{s}_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
a_{s_2}(\bar{s}_2)
\end{pmatrix}
= 
\begin{pmatrix}
\chi(c_{s_1}) \\
a_{s_1}(\bar{s}_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
a_{s_2}(\bar{s}_2)
\end{pmatrix}.
\]

By this calculation, the braid relation is satisfied if and only if \( a_{s_1}(\bar{s}_1) \chi(c_{s_2}) \cdots = a_{s_2}(\bar{s}_2) \chi(c_{s_1}) \cdots \). By [20, Proposition 4.13 (6)], we have \( c_{s_1} c_{s_2} \cdots = c_{s_2} c_{s_1} \cdots \). Hence, the braid relation is satisfied if and only if \( a_{s_1}(\bar{s}_1) \chi(c_{s_1})^{-1} = a_{s_2}(\bar{s}_2) \chi(c_{s_2})^{-1} \), namely \( a_{s_1} \chi(c_{s_1})^{-1} = a_{s_2} \chi(c_{s_2})^{-1} \). By a similar calculation, if \( s_1, s_2 \in A_3(\Xi, \Xi') \), then the braid relation is satisfied if and only if \( a_{s_1} \chi'(c_{s_1})^{-1} = a_{s_2} \chi'(c_{s_2})^{-1} \).

Finally, we assume that \( s_1 \in A_2(\Xi, \Xi') \) and \( s_2 \in A_3(\Xi, \Xi') \). We have

\[
\begin{pmatrix}
\chi(c_{s_1}) \\
a_{s_1}(\bar{s}_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
a_{s_2}(\bar{s}_2)
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
a_{s_1}(\bar{s}_1)
\end{pmatrix}
\begin{pmatrix}
0 \\
a_{s_2}(\bar{s}_2)
\end{pmatrix}.
\]

Hence, the braid relation is satisfied if and only if

- \( a_{s_2}(\bar{s}_2) \chi(c_{s_1}) + a_{s_1}(\bar{s}_1) \chi'(c_{s_2}) = 0 \) and the order of \( s_1 s_2 \) is 2.
- the order of \( s_1 s_2 \) is greater than 2.
We notice that 
\[ a_{s_2}(s_2)\chi(c_{s_1}) + a_{s_1}(s_1)\chi'(c_{s_2}) = 0 \]
if and only if 
\[ a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0. \]

We get the following table which shows the condition for the braid relation:

|    | \( s_1 \) | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) |
|----|-----------|-----------|-----------|-----------|-----------|
| \( s_2 \) | always   | \( a_{s_2} = 0 \) or \( (s_1 s_2)^2 \neq 1 \) | always     | always     | always     |
| \( A_1 \) | \( s_2 = a_{s_1} = 0 \) | \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \) | \( a_{s_1}\chi'(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0 \) | always     | always     |
| \( A_2 \) | \( s_2 = a_{s_1} = 0 \) | \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \) | \( a_{s_1}\chi'(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0 \) | always     | always     |
| \( A_3 \) | \( s_2 = a_{s_1} = 0 \) | \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \) | \( a_{s_1}\chi'(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0 \) | always     | always     |
| \( A_4 \) | always     | always     | always     | always     | always     |

Now we assume that \( (a_s) \in \bigoplus_{s \in S_{aff}} C_s \) defines an action of \( \mathcal{H} \). First, recall that \( C_s \neq 0 \) if and only if \( s\chi = \chi' \). Hence, \( a_s \neq 0 \) implies \( s\chi = \chi' \). Since the quadratic relations hold, if \( a_s \neq 0 \), then \( s \in (A_1(\Xi,\Xi') \setminus S_{aff,\chi}) \cup S_2(\Xi,\Xi') \). If \( s \in A_1(\Xi,\Xi') \setminus S_{aff,\chi} \) and \( (s s_1)^2 = 1 \) for some \( s_1 \in S_2(\Xi,\Xi') \), then the table says that \( a_s = 0 \). Therefore, if \( a_s \neq 0 \), then \( s \in S_1(\Xi,\Xi') \cup S_2(\Xi,\Xi') \). Hence, again by the table, \( (a_s) \) belongs to \( E(\Xi,\Xi') \).

Conversely, if \( (a_s) \in E(\Xi,\Xi') \), then \( a_s \neq 0 \) implies \( s \in S_1(\Xi,\Xi') \cup S_2(\Xi,\Xi') \). Hence, each \( T_s \) satisfies the quadratic relation. Let \( s_1, s_2 \in S_{aff} \). If \( s_1 \in A_1(\Xi,\Xi') \) and \( s_2 \in A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi') \), then the definition of \( S_1(\Xi,\Xi') \) says that \( (s_1 s_2)^2 \neq 1 \) or \( a_{s_1} = 0 \). Then by the table, the braid relation for \( s_1, s_2 \) holds. For other cases, the condition on \( E_2(\Xi,\Xi') \) and the table imply that the braid relation holds too.

Therefore, the map \( E(\Xi,\Xi') \to \text{Ext}_1^\mathcal{H}_{aff}(\Xi,\Xi') \) is well-defined and surjective.

Assume that the extension given by \( (a_s) \) splits, namely each matrix in equation (4.3) is simultaneous diagonalizable. If \( s \in A_1(\Xi,\Xi') \) and \( s \) is diagonalizable if and only if \( a_s = 0 \). Let \( s_1, s_2 \in A_2(\Xi,\Xi') \). Then by \( a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1} \), the matrices corresponding to \( s_1, s_2 \) commute with each other. Hence, these matrices are simultaneous diagonalizable. Similarly, matrices corresponding to \( A_3(\Xi,\Xi') \) are simultaneous diagonalizable.

If \( s_1 \in A_2(\Xi,\Xi') \) and \( s_2 \in A_3(\Xi,\Xi') \), then the corresponding matrices are

\[
\begin{pmatrix}
\chi(c_{s_1}) & 0 \\
a_s(s_1) & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
a_s(s_2) & \chi'(c_{s_2})
\end{pmatrix},
\]

and these commute with each other if and only if \( a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi(c_{s_2})^{-1} = 0 \). Hence, the kernel is equation (4.1). \( \square \)

**Remark 4.3.** Let \( \omega \in \Omega(1)_{\Xi} \cap \Omega(1)_{\Xi'} \). Then \( \omega \) acts on \( \text{Ext}_1^\mathcal{H}_{aff}(\Xi,\Xi') \), and by this action, \( \text{Ext}_1^\mathcal{H}_{aff}(\Xi,\Xi') \) is a right \( \Omega(1)_{\Xi} \cap \Omega(1)_{\Xi'} \)-module. We also have the action of \( \omega \) on \( E(\Xi,\Xi') \) as follows: Let \( s \in S_{aff} \), and put \( s_1 = \omega^{-1} s \omega \in S_{aff} \). Then \( a \mapsto (s_1 \mapsto a(\omega s_1 \omega^{-1})) \) gives an
isomorphism $C_s \simeq C_{s_1}$. We denote this map by $a \mapsto a \cdot \omega$. Then the action is given by $(a_s) \mapsto (a_{\omega^{-1}s_w} \cdot \omega)$. This action commutes with the action of $\omega$ on $\text{Ext}_{K^{(a)}}^i(\Xi, \Xi')$.

4.2. Semidirect product

The argument in this subsection is general. Let $A$ be a $C$-algebra and $\Gamma$ a group acting on $A$. We assume that a finite commutative normal subgroup $\Gamma' \subset \Gamma$ and an embedding $C[\Gamma'] \hookrightarrow A$ are given. Here, we assume that for $\gamma' \in \Gamma'$, the action of $\gamma'$ on $A$ as an element in $\Gamma$ is given by $a \mapsto \gamma' a (\gamma')^{-1}$. We put $B = C[\Gamma] \otimes_{C[\Gamma']} A$ and define a multiplication by $((\gamma_1 \otimes a_1)(\gamma_2 \otimes a_2)) = \gamma_1 \gamma_2 \otimes (\gamma_2^{-1} \cdot a_1) a_2$ for $a_1, a_2 \in A$ and $\gamma_1, \gamma_2 \in \Gamma$. Of course, the example in our mind is $A = \mathcal{H}^{\text{aff}}$, $\Gamma = \Omega(1)$ and $\Gamma' = Z_\kappa$. We have $B = \mathcal{H}$.

Let $M_1, M_2$ be right $B$-modules. Then $\text{Hom}_A(M_1, M_2)$ has the structure of a $\Gamma$-module defined by $(f \gamma)(m) = f(m \gamma^{-1}) \gamma$. This action factors through $\Gamma \to \Gamma/\Gamma'$, and we have $\text{Hom}_B(M_1, M_2) = \text{Hom}_A(M_1, M_2)^{\Gamma/\Gamma'}$. Let $N$ be a $\Gamma/\Gamma'$-module and $\varphi \in \text{Hom}_{\Gamma/\Gamma'}(N, \text{Hom}_A(M_1, M_2))$. Set $f : N \otimes M_1 \to M_2$ by $f(n \otimes m) = \varphi(n)(m)$ for $n \in N$ and $m \in M_1$. Then for $\gamma \in \Gamma$, we have $f(n \gamma \otimes m \gamma) = \varphi(n \gamma)(m \gamma) = \varphi(n)(m) \gamma = f(n \otimes m) \gamma$. Namely, $f$ is $\Gamma$-equivariant. We define an action of $\gamma \in A$ on $N \otimes M_1$ by $(n \otimes m) \gamma = n \otimes ma$. Then it coincides with the action of $\gamma$ on $C[\Gamma']$, and it gives an action of $B$. This correspondence gives an isomorphism

$$\text{Hom}_{\Gamma/\Gamma'}(N, \text{Hom}_A(M_1, M_2)) \simeq \text{Hom}_B(N \otimes M_1, M_2).$$

In particular, if $M_2$ is an injective $B$-module, then $\text{Hom}_A(M_1, M_2)$ is an injective $\Gamma/\Gamma'$-module. Therefore, from $\text{Hom}_B(M_1, M_2) = \text{Hom}_A(M_1, M_2)^{\Gamma/\Gamma'}$, we get a spectral sequence

$$E_2^{ij} = H^i(\Gamma/\Gamma', \text{Ext}^j_A(M_1, M_2)) \Rightarrow \text{Ext}_B^{i+j}(M_1, M_2).$$

In particular, we have an exact sequence

$$0 \to H^1(\Gamma/\Gamma', \text{Hom}_A(M_1, M_2)) \to \text{Ext}^1_B(M_1, M_2) \to \text{Ext}^1_A(M_1, M_2)^{\Gamma/\Gamma'} \quad (4.4)$$

Moreover, we assume the following situation. Let $\Gamma_1$ be a finite index subgroup of $\Gamma$ which contains $\Gamma'$, and put $B_1 = A \otimes_{C[\Gamma']} C[\Gamma_1]$. Then this is a subalgebra of $B$, and $B$ is a free left $B_1$-module with a basis given by a complete representative of $\Gamma \setminus \Gamma$. Assume that $M_1$ has a form $L_1 \otimes_{B_1} B$ for some $B_1$-module $L_1$. We have $M_1 = \bigoplus_{\gamma \in \Gamma \setminus \Gamma} L_1 \otimes \gamma$. Since $B$ is flat over $B_1$, we have

$$\text{Ext}_B^1(M_1, M_2) \simeq \text{Ext}_{B_1}^1(L_1, M_2).$$

We have a $B_1$-module embedding $L_1 \hookrightarrow M_1$. This is in particular an $A$-homomorphism, and we get

$$\text{Ext}_A^i(M_1, M_2) \to \text{Ext}_{B_1}^i(L_1, M_2).$$

Since $L_1 \hookrightarrow M_1$ is a $B_1$-homomorphism, this is a $\Gamma_1$-homomorphism. Hence, this induces

$$\text{Ext}_A^i(M_1, M_2) \to \text{Ind}_{\Gamma_1}^{\Gamma} \left( \text{Ext}_A^i(L_1, M_2) \right).$$
The decomposition $M_1 = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} L_1 \otimes \gamma$ respects the $A$-action. Hence,

$$\text{Ext}^i_A(M_1, M_2) = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} \text{Ext}^i_A(L_1 \otimes \gamma, M_2) = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} \text{Ext}^i_A(L_1, M_2) \gamma.$$ 

Therefore, the above homomorphism is an isomorphism

$$\text{Ext}^i_A(M_1, M_2) \simeq \text{Ind}^\Gamma_{\Gamma_1}(\text{Ext}^i_A(L_1, M_2)).$$

This implies

$$H^1(\Gamma/\Gamma', \text{Hom}_A(M_1, M_2)) \simeq H^1(\Gamma_1/\Gamma', \text{Hom}_A(L_1, M_2)),$$

$$\text{Ext}^1_A(M_1, M_2)^{\Gamma/\Gamma'} \simeq \text{Ext}^1_A(L_1, M_2)^{\Gamma_1/\Gamma'}$$

and a commutative diagram

$$\begin{array}{ccc}
0 & \longrightarrow & H^1(\Gamma/\Gamma', \text{Hom}_A(M_1, M_2)) \\
& & \downarrow \text{id} \\
0 & \longrightarrow & H^1(\Gamma_1/\Gamma', \text{Hom}_A(L_1, M_2))
\end{array}$$

We also assume that there exists a finite index subgroup $\Gamma_2$ of $\Gamma$ which contains $\Gamma'$ and $M_2 = L_2 \otimes_{B_2} B$, where $B_2 = A \otimes_{C[\Gamma]} C[\Gamma_2]$. Let $\{\gamma_1, \ldots, \gamma_r\}$ be a set of complete representatives of $\Gamma_2 \backslash \Gamma_1$. Then the decomposition $M_2 = \bigoplus_{\gamma \in \Gamma_2 \backslash \Gamma} L_2 \otimes \gamma = \bigoplus \bigoplus_{\gamma \in (\Gamma_2 \cap \gamma^{-1}_{i} \Gamma_2 \gamma_i), i} L_2 \otimes \gamma_i \gamma$ gives

$$M_2|_{B_1} = \bigoplus_i L_2 \gamma_i \otimes_{B_1 \cap \gamma^{-1}_{i} B_2 \gamma_i} B_1,$$

where $L_2 \gamma_i$ is a $\gamma_i^{-1}B_2 \gamma_i$-module defined by: $L_2 \gamma_i = L_2$ as a vector space and the action is given by $l(\gamma_i^{-1} \gamma_i) = l \cdot b$ for $l \in L_2$ and $b \in B_2$, here $\cdot$ is the original action of $b \in B_2$ on $L_2$. From this isomorphism, we get

$$H^1(\Gamma_1/\Gamma', \text{Ext}^i_A(L_1, M_2)) \simeq \bigoplus_i H^1((\Gamma_1 \cap \gamma_i^{-1} \Gamma_2 \gamma_i)/\Gamma', \text{Ext}^i_A(L_1, L_2 \gamma_i))$$

and

$$\text{Ext}^i_{B_1}(L_1, M_2) = \bigoplus_i \text{Ext}^i_{B_1 \cap \gamma_i^{-1} B_2 \gamma_i}(L_1, L_2 \gamma_i)$$

which is compatible with the exact sequence in equation (4.4).

Set $A = H^\text{aff}, \Gamma = \Omega(1), \Gamma' = \Gamma, \Gamma_1 = \Omega(1)_\Xi$ and $\Gamma_2 = \Omega(1)_{\Xi'}$. Then we get the following lemma. Recall that $\Omega$ is assumed to be commutative. For a character $\Xi$ of $H$, we defined $\Omega(1)_{\Xi}$ as the stabilizer of $\Xi$ in $\Omega(1)$ and $H_{\Xi} = H_{\text{aff}}C[\Omega(1)_{\Xi}]$ in subsection 2.9.

**Lemma 4.4.** Let $\chi, \chi'$ be characters of $\Gamma$ and $J \subset S_{\text{aff}, \chi}, J' \subset S_{\text{aff}, \chi'}$. Put $\Xi = \Xi_{\chi, J}, \Xi' = \Xi_{\chi', J'}$, and let $V, V'$ be irreducible $C[\Omega(1)_{\Xi}], C[\Omega(1)_{\Xi'}]$-modules, respectively. Let $\{\omega_1, \ldots, \omega_r\}$ be a set of complete representatives of $\Omega_{\Xi} \backslash \Omega = \Omega_{\Xi} \Omega_{\Xi'}$, and define $\Xi_i$ by $\Xi_i(X) = \Xi'(\omega_i X \omega_i^{-1})$. Consider the representation of $C[\Omega(1)_{\Xi}] = \omega_i^{-1}C[\Omega(1)_{\Xi'}] \omega_i$
twisting \( V' \) by \( \omega_i \), and we denote it by \( V'_i \). Put \( \Omega \subset \Omega \cap \Omega' \) and \( \mathcal{H} \subset \mathcal{H} \cap \mathcal{H}' \). Then we have a commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
H^1(\bigoplus \mathcal{H} \Omega, \text{Hom}(\pi_{J,\chi}, \pi_{J',\chi'})) & \rightarrow & \bigoplus H^1(\Omega \subset \bigoplus \mathcal{H} \Omega, \text{Hom}(\Xi \subset \Xi' \otimes V'_i)) \\
\downarrow & & \downarrow \\
\bigoplus \mathcal{H} \Omega(\pi_{\chi, J, V, \pi_{\chi', J', V'}}) & \rightarrow & \bigoplus \mathcal{H} \Omega(\Xi \subset \Xi' \otimes V'_i) \\
\downarrow & & \downarrow \\
\bigoplus \mathcal{H} \Omega(\pi_{\chi, J, V, \pi_{\chi', J', V'}}) & \rightarrow & \bigoplus \mathcal{H} \Omega(\Xi \subset \Xi' \otimes V'_i) \\
\end{array}
\]

The following theorem will be proved in subsection 4.4.

**Theorem 4.5.** The map \( \text{Ext}^1(\Xi \subset \Xi' \otimes V') \rightarrow \text{Ext}^1(\Xi \subset \Xi' \otimes V') \) is surjective.

Combining with Lemma 4.4, we get the surjectivity of \( \text{Ext}^1(\pi_{\chi, J, V, \pi_{\chi', J', V'}}) \rightarrow \text{Ext}^1(\pi_{\chi, J, V, \pi_{\chi', J', V'}}) \) stated in the introduction of this paper.

**4.3.** \( \text{Ext}^1(\Xi \subset \Xi' \otimes V') \)

To prove Theorem 4.5, we analyze \( \text{Ext}^1(\Xi \subset \Xi' \otimes V') \). First, we have

\[ \text{Ext}^1(\Xi \subset \Xi' \otimes V') \simeq \text{Ext}^1(\Xi \subset \Xi') \otimes \text{Hom}(V, V') \]

and the surjective homomorphism \( E(\Xi, \Xi') \rightarrow \text{Ext}^1(\Xi \subset \Xi') \). We have the decomposition \( E(\Xi, \Xi') = E_1(\Xi, \Xi') + E_2(\Xi, \Xi') \). Let \( E_1(\Xi, \Xi') \) (resp. \( E_2(\Xi, \Xi') \)) be the image of \( E_1(\Xi, \Xi') \) (resp. \( E_2(\Xi, \Xi') \)). By the description of the kernel (4.1), we have:

- \( \text{Ext}^1(\Xi, \Xi') = E_1(\Xi, \Xi') + E_2(\Xi, \Xi') \).
- \( E_1(\Xi, \Xi') \rightarrow E_2(\Xi, \Xi') \).
- The dimension of the kernel of \( E_2(\Xi, \Xi') \) is at most 1.

Define \( E_i \) by \( E_i = E_2(\Xi, \Xi') \otimes A_i(\Xi, \Xi') \). Then \( \dim E_2 \leq 1 \) and \( E_i \neq 0 \) if and only if for any \( s \in A_i(\Xi, \Xi') \) we have \( s \chi = \chi' \).

Assume that \( s \chi = \chi' \). Fix \( s \in A_2(\Xi, \Xi') \). Then \( a = (a_s) \rightarrow a_{s_0} \chi(c_{s_0})^{-1} \) gives an isomorphism \( E_2 \simeq C \). Let \( \omega \in \Omega(\Xi, \Xi') \). Then

\[ (a \omega)_{s_0} \chi(c_{s_0})^{-1} = a_{s_0} \omega^{-1} (\omega s_0 \omega^{-1}) \chi(c_{s_0})^{-1}. \]
Since \( \omega \) stabilizes \( \chi \), we have \( \chi(c_{\omega_0}) = (\omega^{-1}\chi)(c_{\omega_0}) = \chi(\omega \cdot c_{\omega_0}) = \chi(c_{\omega_0 \omega^{-1}}) \). Therefore, we have
\[
(\omega)_0 \chi(c_{\omega_0})^{-1} = a_{\omega, \omega^{-1}} (\omega \tilde{s}_0 \omega^{-1}) \chi(c_{\omega_0 \omega^{-1}})^{-1} \\
= a_{\omega, \omega^{-1}} \chi(c_{\omega_0 \omega^{-1}})^{-1} \\
= a_s \chi(c_s)^{-1}.
\]

Here, the last equality follows from the definition of \( E'_2(\Xi, \Xi') \). Namely, \( \Omega_{\Xi, \Xi'} \) acts trivially on \( E_2 \). By the same argument, \( \Omega_{\Xi, \Xi'} \) also acts trivially on \( E_3 \). Therefore, it also acts trivially on \( E'_2(\Xi, \Xi') \), hence on \( E'_2(\Xi, \Xi') \). Hence,
\[
(E'_2(\Xi, \Xi') \otimes \text{Hom}_C(V, V'))^{\Omega_{\Xi, \Xi'}} = E'_2(\Xi, \Xi') \otimes \text{Hom}_{\Omega_{\Xi, \Xi'}}(V, V').
\]

### 4.4. Proof of Theorem 4.5

Now we prove Theorem 4.5. Take \( e \in \text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V')^{\Omega_{\Xi, \Xi'}} \), and first, assume that \( e \in E'_1(\Xi, \Xi') \). Therefore, \( e \) gives \( f_s \in C_s \otimes \text{Hom}_C(V, V') \). The space \( C_s \otimes \text{Hom}_C(V, V') \) is the space of functions \( f_s \) on \( \{ \tilde{s} \mid \tilde{s} \text{ is a lift of } s \} \) with values in \( \text{Hom}_C(V, V') \) such that \( f(t_1 \tilde{s} t_2) = \chi(t_1) f(\tilde{s}) \chi(t_2) \) for \( t_1, t_2 \in \mathbb{Z}_\kappa \). Using this \( f_s \), we define an \( \mathcal{H} \)-module structure on \( V \oplus V' \) by
\[
T_{\tilde{s}} \mapsto \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ f_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix}, \quad T_{\omega} \mapsto \begin{pmatrix} V(\omega) & 0 \\ 0 & V'(\omega) \end{pmatrix},
\]
where \( \tilde{s} \in S_{\text{aff}} \), \( s \) its image in \( S_{\text{aff}} \) and \( f_s = 0 \) if \( s \notin S_1(\Xi, \Xi') \). Since \( e \) is \( \Omega_{\Xi, \Xi'} \)-invariant and \( E_1(\Xi, \Xi') \to E'_1(\Xi, \Xi') \) is injective, we have \( V'(\omega)f_s(\tilde{s})V(\omega^{-1}) = f_{\omega, \omega^{-1}} (\omega \tilde{s} \omega^{-1}) \). Hence,
\[
\begin{pmatrix} V(\omega) & 0 \\ 0 & V'(\omega) \end{pmatrix} \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ f_s(\tilde{s}) & \Xi'(T_{\tilde{s}}) \end{pmatrix} \begin{pmatrix} V(\omega) & 0 \\ 0 & V'(\omega) \end{pmatrix}^{-1} = \begin{pmatrix} \Xi(T_{\omega \omega^{-1}}) & 0 \\ f_s(\omega \tilde{s} \omega^{-1}) & \Xi'(T_{\omega \omega^{-1}}) \end{pmatrix}.
\]

Namely, the above action gives an action of \( \mathcal{H}_{\text{aff}} C[\Omega_{\Xi, \Xi'}] \). Hence, this gives an extension class in \( \text{Ext}^1_{\mathcal{H}_{\Xi, \Xi'}}(\Xi \otimes V, \Xi' \otimes V') \), and its image in \( \text{Ext}^1_{\mathcal{H}_{\Xi, \Xi'}}(\Xi \otimes V, \Xi' \otimes V') \) corresponds to \( e \).

Next, we assume that \( e \) comes from \( E_2 \)-part. Then we may assume that there exist \( \varphi \in \text{Hom}_{\Omega(1)_{\Xi, \Xi'}}(V, V') \) and \( c_0 \in E'_2(\Xi, \Xi') \) such that \( e \) is given by \( c_0 \otimes \varphi \). Take a lift \( (a_s) \) of \( c_0 \) in \( E_2(\Xi, \Xi') \), and consider the action of \( \mathcal{H}_{\text{aff}} C[\Omega(1)_{\Xi, \Xi'}] \) on \( V \oplus V' \) defined by
\[
T_{\tilde{s}} \mapsto \begin{pmatrix} \Xi(T_{\tilde{s}}) & 0 \\ a_s(\tilde{s}) \varphi & \Xi'(T_{\tilde{s}}) \end{pmatrix}, \quad T_{\omega} \mapsto \begin{pmatrix} V(\omega) & 0 \\ 0 & V'(\omega) \end{pmatrix},
\]
where \( \tilde{s} \in S_{\text{aff}}(1) \), \( s \) its image in \( S_{\text{aff}} \) and \( a_s = 0 \) if \( s \notin S_2(\Xi, \Xi') \). Recall that \( \Omega(1)_{\Xi, \Xi'} \) acts trivially on \( (a_s) \). Since \( \varphi \) is \( \Omega(1)_{\Xi, \Xi'} \)-equivariant, the calculation as above shows that this gives an action of \( \mathcal{H}_{\Xi, \Xi'} \).
4.5. Calculation of the extensions

We have

$$\text{Ext}^1_H(\pi_X, J, V, \pi_X', J', V') \simeq \bigoplus_i \text{Ext}^1_{\mathcal{H}_\Xi, \Xi'_i}(\Xi \otimes V, \Xi'_i \otimes V').$$

Hence, it is sufficient to calculate \(\text{Ext}^1_{\mathcal{H}_\Xi, \Xi'_i}(\Xi \otimes V, \Xi'_i \otimes V').\) Now replacing \((\Xi', V'_i)\) with \((\Xi, V)\), we explain how to calculate \(\text{Ext}^1_{\mathcal{H}_\Xi, \Xi}(\Xi \otimes V, \Xi \otimes V').\)

Theorem 4.5 implies

$$\dim \text{Ext}^1_{\mathcal{H}_\Xi, \Xi'}(\Xi \otimes V, \Xi' \otimes V') = \dim H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V')) + \dim \text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V')^{\Omega_{\Xi, \Xi'}}.$$

Since \(\mathcal{H}_{\text{aff}}\) acts trivially on \(V\) and \(V\)'s, we have

$$\text{Hom}_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V') = \text{Hom}_{\mathcal{H}_{\text{aff}}}(\Xi, \Xi') \otimes \text{Hom}_{\mathcal{C}}(V, V'),$$

and it is zero if \(\Xi \neq \Xi'\). If \(\Xi = \Xi'\), then

$$\text{Hom}_{\mathcal{H}_{\text{aff}}}(\Xi, \Xi') \otimes \text{Hom}_{\mathcal{C}}(V, V') = \text{Hom}_{\mathcal{C}}(V, V'),$$

and hence,

$$H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V')) \simeq H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\mathcal{C}}(V, V')).$$

This is a group cohomology of an abelian group.

We also have

$$\text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi \otimes V, \Xi' \otimes V') = \text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi, \Xi') \otimes \text{Hom}_{\mathcal{C}}(V, V'),$$

and

$$\text{Ext}^1_{\mathcal{H}_{\text{aff}}}(\Xi, \Xi') = E'_1(\Xi, \Xi') \oplus E'_2(\Xi, \Xi').$$

As we saw in subsection 4.3, \(\Omega_{\Xi, \Xi'}\) acts trivially on \(E'_2(\Xi, \Xi').\) Hence,

$$(E'_2(\Xi, \Xi') \otimes \text{Hom}_{\mathcal{C}}(V, V'))^{\Omega_{\Xi, \Xi'}} = E'_2(\Xi, \Xi') \otimes \text{Hom}_{\Omega(1)_{\Xi, \Xi'}}(V, V'),$$

and it is not difficult to calculate this.

Finally, we consider \((E'_1(\Xi, \Xi') \otimes \text{Hom}_{\mathcal{C}}(V, V'))^{\Omega_{\Xi, \Xi'}}.\) By Proposition 4.1, we have

$$E'_1(\Xi, \Xi') \simeq E_1(\Xi, \Xi') = \bigoplus_{s \in S_1(\Xi, \Xi')} C_s.$$ 

Fix \(s_0 \in S_1(\Xi, \Xi')\), and let \(\Omega(1)_{\Xi, \Xi', s_0}\) be the stabilizer of \(s_0\) in \(\Omega(1)_{\Xi, \Xi'}\). Then \(C_{s_0}\) is an \(\Omega(1)_{\Xi, \Xi', s_0}\)-representation. Consider an \(\Omega(1)_{\Xi, \Xi'}\)-orbit \(S \subset S_1(\Xi, \Xi')\). The subspace \(\bigoplus_{s \in S} C_s\) is \(\Omega(1)_{\Xi, \Xi'}\)-stable, and we have an isomorphism

$$\bigoplus_{s \in S} C_s \simeq \text{Ind}_{\Omega(1)_{\Xi, \Xi'}}^{\Omega(1)_{\Xi, \Xi'}} C_{s_0}$$

defined by

$$(a_s) \mapsto (a \mapsto (s_0 \mapsto a_{s_0}(\omega^{-1}s_0 \omega)).$$
Let \( \{s_1, \ldots, s_r\} \) be a complete representative of the \( \Omega(1)_{\Xi, \Xi'} \)-orbits in \( S_1(\Xi, \Xi') \). Then we have
\[
E'_1(\Xi, \Xi') \simeq \bigoplus_i \text{Ind}_{\Omega(1)_{\Xi, \Xi', s_i}}^{\Omega(1)_{\Xi, \Xi'}} C_{s_i}.
\]
Hence,
\[
(E_1(\Xi, \Xi') \otimes \text{Hom}_C(V, V'))^{\Omega(1)_{\Xi, \Xi'}} = \bigoplus_i (C_{s_i} \otimes \text{Hom}_C(V, V'))^{\Omega(1)_{\Xi, s_i}.}
\]

4.6. Example: \( G = \text{GL}_n \)
Assume that the data come from \( \text{GL}_n \). Then the data are as follows (see [17]).
We have \( W_0 = S_n, W = S_n \ltimes (F^\times / O)^n \simeq S_n \ltimes \mathbb{Z}^n, W(1) = S_n \ltimes (F^\times / (1 + (w)))^n = S_n \ltimes (\mathbb{Z} \ltimes \mathbb{X})^n \) and \( W_{\text{aff}} = S_n \ltimes \{(x_i) \in \mathbb{Z}^n \mid \sum x_i = 0\} \). Set
\[
\omega = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} (0, \ldots, 0, 1) \in S_n \ltimes \mathbb{Z}^n \subset W(1),
\]
and denote its image in \( W \) by the same letter \( \omega \). Then \( \Omega \) is generated by \( \omega \) and \( \Omega(1) = \langle \omega \rangle \mathbb{X}^n \). We have \( \omega^n = (1, \ldots, 1) \), and it belongs to the center of \( W(1) \). The element \( c_{s_i} \in C[Z_\kappa] \) is given by \( c_{s_i} = \sum_{t \in \kappa} T_{\nu(t)} \nu(t)\nu(t)\nu(t)^{-1} \), where \( \nu : \kappa^\times \to (\mathbb{X}^n) \) is an embedding to \( i \)-th entry and \( \nu_{\nu(t)} = \nu t \).
Let \( \pi_{\chi, \iota, V} \) and \( \pi_{\chi', \iota', V'} \) be simple supersingular modules, and we assume that the dimension of the modules are both \( n \).

Remark 4.6. An importance of \( n \)-dimensional simple supersingular modules is revealed by a work of Grosse-Klönne [9]. He constructed a correspondence between supersingular \( n \)-dimensional modules of \( \mathcal{H} \) and irreducible modulo \( p \) \( n \)-dimensional representations of \( \text{Gal}(\mathbb{F}/\mathbb{F}) \).

We have \( \dim \pi_{\chi, \iota, V} = (\dim V)[\Omega : \Omega_\Xi] \). Since \( \Omega(1) \) is (hence, \( \Omega(1)_{\Xi} \) is) commutative, we have \( \dim V = 1 \). Therefore, our assumption implies \( [\Omega : \Omega_\Xi] = n \). Since \( \omega^n \) is in the center, \( \langle \omega^n \rangle \subset \Omega_\Xi \). Hence, \( \Omega_\Xi = \langle \omega^n \rangle \) and \( \Omega = \langle \omega^n \rangle \mathbb{X}^n \). Set \( \lambda = V(\omega^n) \). Since \( V|_{\mathbb{X}^n} = \chi, V \) is determined by \( \chi \) and \( \lambda \). We also put \( \lambda = V'(\omega^n) \).

We define \( \chi_j : \kappa^\times \to C^\times \) by \( \chi_j(t_1, \ldots, t_n) = \chi_1(t_1) \cdots \chi_n(t_n) \), and we extend it for any \( j \in \mathbb{Z} \) by \( \chi_{j+n} = \chi_j \). Then
\[
(s_i \chi)_j = \begin{cases} 
\chi_j & (j \neq i, i+1), \\
\chi_{i+1} & (j = i), \\
\chi_i & (j = i+1).
\end{cases}
\]
The description of \( c_{s_i} \) shows \( \chi(c_{s_i}) = 0 \) if and only if \( \chi_i = \chi_{i+1} \) if and only if \( s_i \chi = \chi \). Therefore, \( S_{\text{aff}, \chi} = \{s_i \in S_{\text{aff}} \mid \chi_i = \chi_{i+1}\} \).

We consider \( \text{Ext}_{\mathcal{H}, \text{aff}}(\Xi \otimes V, \Xi' \otimes V') \). By Theorem 4.5, we have the exact sequence
\[
0 \to H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\mathcal{H}, \text{aff}}(\Xi, \Xi') \otimes \text{Hom}_C(V, V')) \to \text{Ext}^1_{\mathcal{H}, \Xi, \Xi'}(\Xi \otimes V, \Xi' \otimes V') \to \text{Ext}^1_{\mathcal{H}, \text{aff}}(\Xi \otimes V, \Xi' \otimes V') \to 0.
\]
The space $\text{Hom}_{\text{aff}}(\Xi, \Xi')$ is $C$ if $\Xi = \Xi'$ and 0 otherwise. We have $\Omega_{\Xi, \Xi'} = \langle \omega^n \rangle \simeq \mathbb{Z}$ and $\omega^n$ acts on $\text{Hom}_C(V, V')$ by $\lambda^{-1} \lambda'$. Therefore, $H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_C(V, V')) = C$ if $\lambda = \lambda'$ and 0 otherwise. Namely, we get

$$\dim H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\text{aff}}(\Xi, \Xi') \otimes \text{Hom}_C(V, V')) = \begin{cases} 1 & \Xi = \Xi', V = V', \\ 0 & \text{otherwise}. \end{cases} \quad (4.5)$$

Note that $\Omega_{\Xi, \Xi'}$ acts on $S_{\text{aff}}$ trivially since $\Omega_{\Xi, \Xi'}$ is in the center of $W$. Hence, the stabilizer of each $s \in S_{\text{aff}}$ in $\Omega_{\Xi, \Xi'}$ is $\Omega_{\Xi, \Xi'}$ itself, and each orbit is a singleton. Therefore, by the previous subsection, we have

$$\text{Ext}^1_{\text{aff}}(\Xi \otimes V, \Xi' \otimes V')_{\Omega_{\Xi, \Xi'}} = \bigoplus_{s \in S_1(\Xi, \Xi')} (C_s \otimes \text{Hom}_C(V, V'))_{\Omega_{\Xi, \Xi'}} \oplus E_2'(\Xi, \Xi') \otimes \text{Hom}_{\Omega_{\Xi, \Xi'}}(V, V').$$

Since $\omega^n \in \Omega_{\Xi, \Xi'}, s = \Omega_{\Xi, \Xi'}$ is in the center of $W(1)$, it acts trivially on $C_s$. Hence, $(C_s \otimes \text{Hom}_C(V, V'))_{\Omega_{\Xi, \Xi'}} = \text{Hom}_{\Omega_{\Xi, \Xi'}}(V, V')$, and it is not zero if and only if $\lambda = \lambda'$. Hence,

$$\text{Ext}^1_{\text{aff}}(\Xi \otimes V, \Xi' \otimes V')_{\Omega_{\Xi, \Xi'}} \simeq \begin{cases} C^{S_1(\Xi, \Xi')} + E_2'(\Xi, \Xi') & \lambda = \lambda', \\ 0 & \text{otherwise}. \end{cases}$$

A complete representative of $\Omega/\Omega_{\Xi, \Xi'}$ is given by $\{1, \omega, \ldots, \omega^{n-1}\}$. Put $\Xi'_i = \Xi' \omega^i$. This is parametrized by $(\chi \omega^i, J_i = \omega^i J \omega^{-i})$. We have $(\chi \omega^i)_j = \chi_{j+i}$ and $\omega^i J \omega^{-i} = \{s_{j+i} \mid s_j \in J\}$. We put $V'_i = V' \omega^i$. Then $V'_i(\omega) = V'(\omega)$ and $V'_i|_{Z_n} = \chi \omega^i$.

The cohomology group $H^1(\Omega_{\Xi, \Xi'}, \text{Hom}_{\text{aff}}(\Xi, \Xi'_i) \otimes \text{Hom}_C(V, V'_i))$ is zero if and only if $(\Xi, V) \neq (\Xi'_i, V'_i)$ by equation (4.5). There exists at most one $i$ such that $(\Xi, \Xi'_i) \neq (V, V'_i)$, and such $i$ exists if and only if $(\Xi, V)$ is $\Omega$-conjugate to $(\Xi', V')$. Hence,

$$\dim \bigoplus_{i=0}^{n-1} H^1(\Omega_{\Xi, \Xi'_i}, \text{Hom}_{\text{aff}}(\Xi, \Xi'_i) \otimes \text{Hom}_C(V, V'_i)) = \begin{cases} 1 & (\Xi, V) \text{ is } \Omega\text{-conjugate to } (\Xi', V'), \\ 0 & \text{otherwise}. \end{cases}$$

We also have

$$\dim \bigoplus_{i=0}^{n-1} \text{Ext}^1_{\text{aff}}(\Xi \otimes V, \Xi'_i \otimes V')_{\Omega_{\Xi, \Xi'}} \simeq \begin{cases} \sum_{i=0}^{n-1} (\#S_1(\Xi, \Xi'_i) + \dim E_2'(\Xi, \Xi'_i)) & \lambda = \lambda', \\ 0 & \text{otherwise}, \end{cases}$$

and each term can be calculated by the description in Proposition 4.1.

**4.7. GL_2**

Now we assume $n = 2$, and we compute $\text{Ext}^1_{\text{aff}}(\pi_{\chi, J, V}, \pi_{\chi', J', V'})$. We continue to use the notation in the previous subsection. Then $\omega$ switches $s_0$ and $s_1$. 
Lemma 4.7. The nonvanishing of $\text{Ext}^1_H(\pi_{\chi,J,V},\pi_{\chi',J',V'})$ implies that $(\chi,J,V)$ is conjugate to $(\chi',J',V')$ by $\Omega$.

Proof. As we have seen in the above, nonvanishing of $\text{Ext}^1$ implies $V(\omega) = V'(\omega) = (V'\omega)(\omega)$. Hence, it is sufficient to prove that $(\chi,J)$ is conjugate to $(\chi',J')$.

If $H^1(\Omega_{\Xi,\Xi'},\text{Hom}_{\mathcal{H}\text{-aff}}(\Xi \otimes \Xi') \otimes \text{Hom}_C(V,V')) \neq 0$, we have $\Xi = \Xi'$ and $V = V'$. Hence, we have the lemma.

If $\text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi' \otimes V')^{\Omega_{\Xi,\Xi'}} \neq 0$, then $C_s \neq 0$, hence $\chi' = s\chi$ for some $s \in S_{\text{aff}}$. Since we assume $G = \text{GL}_2$, $s_0\chi = s_1\chi = \chi\omega$. Since $\pi_{\chi,J,V}$ and $\pi_{\chi',J',V'}$ are both supersingular, the possibility of $(J,J')$ is $(\emptyset,\emptyset)$, $((s_0),\{s_1\})$, $\{(s_0),\{s_1\}\}$, $\{(s_1),\{s_1\}\}$ and except the last two cases, we have $J = \omega J'\omega^{-1}$. If $J = J' = \{s_0\}$, then $s_0 \in S_{\text{aff},\chi}$; hence, $s_0\chi = \chi$. Since $s_0\chi = s_1\chi$, we have $S_{\text{aff},\chi} = S_{\text{aff}}$. Hence, $S_1(\Xi,\Xi') = \emptyset$. We also have $A_2(\Xi,\Xi') = A_3(\Xi,\Xi') = \emptyset$. Therefore, we get $\text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi' \otimes V')^{\Omega_{\Xi,\Xi'}} = 0$. By the same way, if $J = J' = \{s_1\}$, then $\text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi' \otimes V')^{\Omega_{\Xi,\Xi'}} = 0$.

Since $\pi_{\chi,J,V}$ only depends on the $\Omega$-orbit of $(\chi,J,V)$, we may assume $(\chi,J,V) = (\chi',J',V')$. In this case, $H^1(\Omega_{\Xi,\Xi'},\text{Hom}_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi_i \otimes V_i))$ is one-dimensional if $i = 0$ and zero if $i \neq 1$.

1. The case of $\chi_1 = \chi_2$. Then we have $S_{\text{aff},\chi} = S_{\text{aff}}$. By the proof of Lemma 4.7, we have $\text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi \otimes V) = 0$. We have $S_1(\Xi,\Xi_1) = \emptyset$, $A_2(\Xi,\Xi_1) = J_1 = \omega J_2 = J$, and $A_3(\Xi,\Xi_1) = J_0 = J$. Hence, the description in Proposition 4.1 shows that $\dim E'_2(\Xi,\Xi_1) = 1$, and hence, $\dim \text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi \otimes V) = 1$.

2. The case of $\chi_1 \neq \chi_2$. Then we have $S_{\text{aff},\chi} = \emptyset$. Since $\chi \neq s\chi = s\chi_0$ for $s = s_0,s_1$, $C_s = 0$. Therefore, $\text{Ext}^1_{\mathcal{H}\text{-aff}}(\Xi \otimes V,\Xi \otimes V) = 0$. Since $S_{\text{aff},\chi} = \emptyset$, $\Xi(T_s) = \Xi'(T_s) = 0$ for any $s \in S_{\text{aff}}$. We have $A_2(\Xi,\Xi_1) = A_3(\Xi,\Xi_1) = \emptyset$ and $S_1(\Xi,\Xi_1) = S_{\text{aff}}$. Therefore, $E'_1(\Xi,\Xi_1) = 0$ and $\dim E'_2(\Xi,\Xi_1) = \# S_1(\Xi,\Xi_1) = \# S_{\text{aff}} = 2$.

Hence, we have

$$\dim \text{Ext}^1_H(\pi_{\chi,J,V},\pi_{\chi',J',V'}) = \begin{cases} 0 & (\pi_{\chi,J,V} \not\simeq \pi_{\chi',J',V'}), \\ 2 & (\pi_{\chi,J,V} \simeq \pi_{\chi',J',V'}, \chi_1 = \chi_2), \\ 3 & (\pi_{\chi,J,V} \simeq \pi_{\chi',J',V'}, \chi_1 \neq \chi_2). \end{cases}$$

This recovers [7, Corollary 6.7]. (Note that in [7], they calculate the extensions with fixed central character. Since we do not fix the central character here, the dimension calculated here is one greater than the dimension they calculated.)

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