Quantum Solitons with Cylindrical Symmetry

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Abstract

Soliton solutions with cylindrical symmetry are investigated within the non-linear $\sigma$-model disregarding the Skyrme-stabilization term. The solitons are stabilized by quantization of collective breathing mode and collapse in the $\hbar \to 0$ limit. It is shown that for such stabilization mechanism the model, apart from solitons with integer topological number $B$, admits the solitons with half-odd $B$. The solitons with integer $B$ have standard spin-isospin classification, while $B = \frac{1}{2}$ solitons are shown to be characterized by spin, isospin and some additional "momentum".

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1 Introduction

Few years ago a new class of spherically-symmetrical solitons in the nonlinear $\sigma$-model, whose stability is provided on the basis the quantization of collective coordinates chosen suitably was shown in [1-3]. Such solitons does not exist when the Planck constant tends to zero and in this sense they are called quantum solitons.

On the other hand, in the framework of the standard Skyrme model [4] (the model, where solitons are stabilized by a fourth-order derivative term added by hand) a spherically-symmetrical hedgehog configuration was shown to have the lowest energy only in a sector with topological number $B = 1$. For example, for the systems with the baryon number 2 an Ansatz with the cylindrical symmetry is more preferable and was studied by many authors [5-7]. The purpose of this paper is to explore the cylindrically-symmetrical Ansatz in the theory of quantum solitons.

We are starting from the action of the standard $SU(2)$ nonlinear $\sigma$-model:

$$W = -\frac{f_\pi^2}{4} \int d^4x Tr(L_\mu L^\mu), \quad L_\mu = U^+ \partial_\mu U,$$

where $x^\mu = (t, \vec{x})$ is the Minkowski space coordinate, $U = U(t; \vec{x})$ is the $(2 \times 2)$ chiral field matrix, $f_\pi$ is the pion decay constant and $L^\mu = \eta^{\mu\nu} L_\nu$ with the metric tensor $\eta^{\mu\nu}$ of Minkowski space; the space signature is taken to be $(+---)$.

According to the Derrick theorem [8] the static soliton configuration does not exist in the model with the action (1). But for a time-dependent configuration the conditions of the Derrick theorem are not fulfilled and therefore we start from the following Ansatz, which is explicitly time-dependent:

$$U(t; r, \varphi, \vartheta) = e^{-i\frac{m}{2} \varphi \tau_3} \exp \left\{ iF_a \left( \frac{r}{\lambda(t)}, \vartheta \right) \tau_a \right\} e^{i\frac{m}{2} \varphi \tau_3},$$

where $r, \varphi, \vartheta$ are the usual spherical coordinates, $\tau_a (a = 1, 2, 3)$ are the isospin Pauli matrices, $m$ is some integer number and $\lambda(t)$ is a configuration size parameter; the latter will be considered in this paper as a collective ("breathing") coordinate.

In Ref.7 the static limit $(\lambda(t) = 1)$ of the Ansatz (2) was applied for the configurations with $m = 2$ and $B = 2$ in the framework of the Skyrme model. From the Ansatz (2) one can also simply obtain the hedgehog configuration by using here the static limit and setting $m = 1$ and $F_1 = F(r) \sin \vartheta$, $F_2 = 0$, $F_3 = F(r) \cos \vartheta$.

The outline of the paper is as follows. Taking into account the noncommutativity of the breathing coordinate $\lambda(t)$ and the velocity $\dot{\lambda} = \frac{d\lambda(t)}{dt}$ we derive an effective Lagrangian for breathing motion in Sec.II. It is shown that due to the uncertainty principle a repulsive term is generated, what prevents a collapse of the chiral field.
To define the chiral field functions $F_a$ for arbitrary $m$ a system of coupled differential
equations in partial derivatives is obtained. We demonstrate that for the cylindrically-
symmetrical configurations there appear solitons with integer as well as half-odd topological charge. In Sec.III the quantization of spatial and isospatial rotations of soliton
is considered. For solitons with integer topological charge the usual quantum numbers
(spin and isospin) are obtained, meanwhile for solitons with $\frac{1}{2}$-topological charge there
appears additional conserved ”momentum” operator, which commutes with spin and
isospin operators (Sec.IV). This momentum $\zeta$ is shown to be integer and is restricted
by $|j−t| ≤ \zeta ≤ j+t$, where $j$ and $t$ are spin and isospin, respectively. Conclusions and
some speculations related to status of the quantum solitons with cylindrical symmetry
are given in the last section.

2 Basic Equations

2.1 Effective Lagrangian

Following the approach of Ref.1, which was developed in the theory of spherically-
symmetrical solitons, we are postulating that the generalized coordinate $\lambda$ and velocity
$\dot{\lambda}$ do not commute:

$$[\lambda, \dot{\lambda}] = i f(\lambda), \quad (3)$$

where $f(\lambda)$ is to be determined after the canonical quantization condition is required.

To obtain an effective potential for the breathing motion it is useful to replace the
Minkowski coordinates ($x^\mu$) by a new one ($z^\alpha$) according to

$$(x^\mu) \equiv (t, \vec{x}) = (\tau, \lambda(\tau)\vec{z}), \quad U(t, \vec{x}) = \hat{U}(\vec{z}). \quad (4)$$

Thus one goes from the representation, where the configuration vibrates in Minkowski
space, to a new one, where the configuration is ”static”, but quantum vibrations are
transferred to the geometrical structure of the adopted space.

After simple calculations (see, e.g., Ref.1) and taking into account (3) one obtains
the following effective Lagrangian for the breathing motion:

$$W = \int \Lambda d\tau,$$

$$\Lambda = \sigma_2[F, \Theta] \left( \lambda \dot{\lambda} - ff' - \frac{5}{4} \lambda^{-1} f^2 \right) - \sigma_1[F, \Theta] \lambda, \quad (5)$$
where $f' = \frac{df(\lambda)}{d\lambda}$ and $\sigma_1[F, \Theta]$ and $\sigma_2[F, \Theta]$ are the following functionals:

\begin{align*}
\sigma_1[F, \Theta] &= \frac{f^2}{2} \int d^3z \left[ (\nabla F)^2 + \sin^2 F (\nabla n_a)(\nabla n_a) \right], \quad (6) \\
\sigma_2[F, \Theta] &= \frac{f^2}{2} \int d^3z z_i z_j \left[ (\nabla_i F)(\nabla_j F) + \sin^2 F (\nabla_i n_a)(\nabla_j n_a) \right]. \quad (7)
\end{align*}

Here functions $F = F(z, \vartheta)$, $n_a = n_a(z, \vartheta)$, $a = 1, 2, 3$ parametrize the chiral field $\hat{U}(\vec{z})$ by

\begin{align*}
F_1 &= F \cos \Theta, \quad F_2 = 0, \quad F_3 = F \sin \Theta, \\
n_1 &= \cos m\varphi \cos \Theta, \quad n_2 = \sin m\varphi \cos \Theta, \quad n_3 = \sin \Theta, \quad (8) \\
\end{align*}

where $\Theta = \Theta(z, \vartheta)$, $F = F(z, \vartheta)$, $z = |\vec{z}|$ and $\nabla_i = \frac{\partial}{\partial z_i}$.

When using the transformation $(\lambda, \dot{\lambda}) \rightarrow (\xi, \dot{\xi})$, where

\begin{align*}
\lambda &= \sigma_1[F, \Theta]^{-1} \xi^{2/3}, \quad \dot{\lambda} = \left\{ \dot{\xi}, \frac{\partial \lambda}{\partial \xi} \right\}, \quad (10)
\end{align*}

and $\{a, b\} = \frac{1}{2}(ab + ba)$, the Lagrangian (2.3) is reduced to

\begin{align*}
\Lambda &= \frac{1}{2} M[F, \Theta] \dot{\xi}^2 - v(\xi), \quad v(\xi) = \xi^{2/3} + \frac{1}{8M[F, \Theta]} \xi^{-2}, \\
M[F, \Theta] &= \frac{8 \sigma_2[F, \Theta]}{9 \sigma_1^2[F, \Theta]}.
\end{align*}

To obtain (11) one has to put $f(\lambda) = (2\sigma_2 \lambda)^{-1}$, which is consistent with the canonical quantization condition $[\xi, P_\xi] = i$, where $P_\xi$ is a momentum, conjugative to $\xi$.

It must be emphasized here that the Lagrangian (11) is invariant under the scale transformation

\begin{align*}
F(z, \vartheta) \rightarrow F_\beta(z, \vartheta) \equiv F(\beta z, \vartheta), \quad \Theta(z, \vartheta) \rightarrow \Theta_\beta(z, \vartheta) \equiv \Theta(\beta z, \vartheta). \quad (12)
\end{align*}

The role of this invariance of an effective Lagrangian in theory of quantum solitons was discussed in Ref.3.
2.2 Variational Principle and Differential Equations

The chiral field for the cylindrically-symmetric configuration is defined by the same variational equation, which was derived for spherically-symmetric chiral field

\[
\frac{3\delta\sigma_1}{\sigma_1} - \frac{\delta\sigma_2}{\sigma_2} = 0,
\]

(13)

where functionals \(\sigma_1\) and \(\sigma_2\) are defined by (3, 7).

According to a prescription of Refs.1,3 we shall introduce a new variable \(\vec{y}\) and functions \(\tilde{F}(\vec{y})\) and \(\tilde{\Theta}(\vec{y})\) instead of the old ones:

\[
\vec{y} = z_0^{-1} \vec{z}, \quad z_0^2 = \frac{3\sigma_2[F, \Theta]}{\sigma_1[F, \Theta]},
\]

(14)

\[
\tilde{F}(\vec{y}) = F(z_0\vec{y}), \quad \tilde{\Theta}(\vec{y}) = \Theta(z_0\vec{y}).
\]

(15)

Calculating in (13) the variations over \(\delta F\) one obtains the first field equation

\[
- \partial_i \partial^i \tilde{F} + \partial_i \left( y^i y^j \partial_j \tilde{F} \right) + \sin \tilde{F} \cos \tilde{F} \left( \partial_i \tilde{n}_a \partial_i \tilde{n}_a - y^i y^j \partial_i \tilde{n}_a \partial_j \tilde{n}_a \right) = 0,
\]

(16)

where \(\partial_k = \frac{\partial}{\partial y_k}\) and \(\tilde{n}_a(\vec{y}) = n_a(z_0\vec{y})\).

The second equation can be derived from Eq.(13) using the standard variation procedure over \(\delta \tilde{n}_a\) with the constraint \(\tilde{n}_a\tilde{n}_a = 1\):

\[
2 \cot \tilde{F} \partial_i \tilde{F} \left( \partial_i \tilde{n}_a - y^i y^j \partial_j \tilde{n}_a \right) + \partial_j \left( \partial_i \tilde{n}_a - y^i y^j \partial_j \tilde{n}_a \right)
\]

\[
+ \left( \partial_i \tilde{n}_b \partial_i \tilde{n}_b - y^i y^k \partial_i \tilde{n}_b \partial_k \tilde{n}_b \right) \tilde{n}_a = 0
\]

(17)

with \(\tilde{n}_a\tilde{n}_a = 1\) and \(\tilde{n}_a\partial_i\tilde{n}_a = 0\).

Apparently, for the hedgehog-like configuration of Refs.1-3 Eq.(17) is satisfied identically, while Eq.(16) is reduced to the field equation of Ref.1.

One can rewrite Eqs.(16, 17) in spherical coordinates \(y = |\vec{y}|, \vartheta\) and \(\varphi\):

\[
\frac{\partial}{\partial y} \left( (y^4 - y^2) \frac{\partial \tilde{F}}{\partial y} \right) - \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left[ \sin \vartheta \frac{\partial \tilde{F}}{\partial \vartheta} \right] + \sin \tilde{F} \cos \tilde{F} \left[ (y^4 - y^2) \left( \frac{\partial \tilde{\Theta}}{\partial y} \right)^2 + \left( \frac{\partial \tilde{\Theta}}{\partial \vartheta} \right)^2 + m^2 \left( \frac{\sin \tilde{\Theta}}{\sin \vartheta} \right)^2 \right] = 0,
\]

(18)
\[
\sin \vartheta \frac{\partial}{\partial y} \left[ (y^4 - y^2) \sin^2 \tilde{F} \frac{\partial \tilde{\Theta}}{\partial y} \right] - \frac{\partial}{\partial y} \left[ \sin \vartheta \sin^2 \tilde{F} \frac{\partial \tilde{\Theta}}{\partial y} \right] + \frac{m^2}{2} \sin^2 \tilde{F} \sin \frac{2\tilde{\Theta}}{\sin \vartheta} = 0. \tag{19}
\]

Both equations have three singular points: \( y = 0, \ y = 1 \) and \( y = \infty \) and for further analysis it is important to know the behavior of the functions \( \tilde{F}(y, \vartheta) \) and \( \tilde{\Theta}(y, \vartheta) \) near these points.

### 2.3 Behavior at Singular Points

Suggesting the analytical properties of the solutions of Eqs. (18,19) one has to expand the functions \( \tilde{F}(y, \vartheta) \) and \( \tilde{\Theta}(y, \vartheta) \) near the origin \( y = 0 \) as

\[
\tilde{F}(y, \vartheta) = F_0 + y^\mu F_1(\vartheta) + \ldots, \quad \mu > 0, \tag{20}
\]

\[
\tilde{\Theta}(y, \vartheta) = \Theta_0(\vartheta) + y^\nu \Theta_1(\vartheta) + \ldots, \quad \nu > 0, \tag{21}
\]

where \( \mu \) and \( \nu \) have to be determined from the field equations (18,19). Putting (20,21) into Eq. (18) one easily obtains at zero order of \( y \)-expansion:

\[
\sin F_0 \cos F_0 \left[ \left( \frac{\partial \Theta_0}{\partial \vartheta} \right)^2 + m^2 \left( \frac{\sin \Theta_0}{\sin \vartheta} \right)^2 \right] = 0. \tag{22}
\]

To satisfy this equation we should take

\[
F_0 = \frac{\pi}{2} n, \quad n = 0, \pm 1, \pm 2, \ldots \tag{23}
\]

The topological charge for the solitons is defined as usual

\[
B = \int d^3x \, B^0(x), \quad \text{where} \quad B^\mu(x) = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} Tr(L_\nu L_\rho L_\sigma), \tag{24}
\]

while \( L_\nu \) was defined in (1). Simple calculations give \( B = \frac{nm}{2} \) and the field equations (18,19) admit solitons with integer and half-odd topological charge. The latter does
not exist in the standard Skyrme model \(^[4]\). Introducing a variable \(\chi = m \ln \left(\tan \frac{\vartheta}{2}\right)\) one obtains the sine-Gordon equation for \(\Theta_0(\vartheta(\chi))\) at the zero order of \(y\)-expansion of Eq.(19):

\[
\frac{d^2 \Theta_0}{d\chi^2} = \frac{\sin 2\Theta_0}{2},
\]

which has the well known solution

\[
\Theta_0 = 2 \tan^{-1} (e^\chi) = 2 \tan^{-1} \left( \tan \frac{\vartheta}{2} \right)^m.
\]

The next order of the \(y\)-expansion of Eq.(18) gives an equation for \(F_1(\vartheta)\):

\[
\frac{d}{dz} \left[ (1 - z^2) \frac{dF_1}{dz} \right] + \left[ \mu(\mu + 1) - (-1)^n 8m^2 \frac{(1 + z^2)^{-n}}{(1 + z)^m (1 - z)^m} \right] F_1 = 0,
\]

where \(z\) denotes \(\cos \vartheta\), and \(n\) is determined by \(\[23\]\).

At zero order of \((y - 1)\)-expansion of Eqs.(18,19) one gets

\[
\frac{\partial F(1, \vartheta)}{\partial y} = \frac{1}{2 \sin \vartheta} \frac{d}{d\vartheta} \left[ \sin \vartheta \frac{dF(1, \vartheta)}{d\vartheta} \right],
\]

\[
- \frac{\sin 2F(1, \vartheta)}{4} \left[ \left( \frac{d\Theta(1, \vartheta)}{d\vartheta} \right)^2 + m^2 \left( \frac{\sin \Theta(1, \vartheta)}{\sin \vartheta} \right)^2 \right],
\]

\[
\frac{\partial \Theta(1, \vartheta)}{\partial y} = \frac{d}{d\vartheta} \left[ \frac{\sin^2 F(1, \vartheta) \sin \vartheta \frac{d\Theta(1, \vartheta)}{d\vartheta}}{2 \sin \vartheta \sin^2 F(1, \vartheta)} \right] - m^2 \left[ \frac{\sin 2\Theta(1, \vartheta)}{4 \sin^2 \vartheta} \right].
\]

Finally, at \(y \to \infty\) the functions \(F\) and \(\Theta\) behave as

\[
F(y, \vartheta) = y^{-3} f_0(\vartheta) + y^{-5} f_1(\vartheta) + \ldots,
\]

\[
\Theta(y, \vartheta) = \theta_0(\vartheta) + y^{-2} \theta_1(\vartheta) + \ldots,
\]

where the relationships between functions in these expressions are given by

\[
f_1(\vartheta) = 3 \left[ \frac{1}{10} \frac{d}{d\vartheta} \left( \frac{\sin \vartheta \frac{df_0}{d\vartheta}}{10 \sin \vartheta} \right) - f_0 \left( \frac{d\Theta_0}{d\vartheta} \right)^2 + m^2 \left( \frac{\sin \Theta_0}{\sin \vartheta} \right)^2 \right],
\]
\[ \theta_1(\vartheta) = \frac{d}{d\vartheta} \left[ \sin \vartheta \frac{d\Theta_0}{d\vartheta} \right] - \frac{m^2 \sin 2\Theta_0}{20 \sin^2 \vartheta}. \] (33)

3 Spatial and Isospatial Rotations

From the configuration considered in the previous section the infinite set of configurations, all degenerate in energy, can be constructed by spatial and isospatial rotations:

\[ \hat{U}(\vec{z}, A, B) = A U \begin{bmatrix} D(B) \vec{z} \end{bmatrix} A^+, \] (34)

where \( D_j^i(B) = \frac{1}{2} Tr [\tau_i B \tau_j B^+] \) is an orthogonal matrix of spatial rotation given by a \( SU(2) \) matrix \( B \); an isospatial rotation of the chiral field \( U \) is given by a \( SU(2) \) matrix \( A \). The usual way to quantize the rotating degrees of freedom is to promote the matrices \( A \) and \( B \) to collective coordinates \( A(t) \) and \( B(t) \).

Here one comment must be given. Due to the fact that the theory in question gives no stable soliton solution in the classical limit \( \hbar \to 0 \) the traditional semi-classical quantization scheme became nonapplicable here. But an example of a vibrating and rotating quantum soliton for the hedgehog-like configuration \( \hat{U}(\vec{z}) \) [3] shows that the contribution of a rotational energy of the field is less than one quarter of total soliton energy. Therefore we shall assume further that the chiral field matter distribution is that for the nonrotating soliton considered in Sec.II and the total soliton energy is given by

\[ E = E_0 + T, \] (35)

where \( E_0 \) is an energy of the nonrotating soliton, which is an eigenvalue of the Hamiltonian corresponding to the effective Lagrangian \( \Lambda \) for the vibrating motion (Eq.(5)):

\[ \hat{H} \Psi = E_0 \Psi, \quad \hat{H} = -\frac{1}{2M[F, \Theta]} d^2 \frac{d}{d\xi} + v(\xi); \] (36)

\( T \) is the kinetic energy related to rotations. It is determined by the Hamiltonian [7]:

\[ \hat{H}_{rot} = \frac{1}{2} a_i U_{ij} a_j - a_i W_{ij} b_j + \frac{1}{2} b_i V_{ij} b_j, \] (37)
\[ a_j = -i Tr[\tau_j A^+ \dot{A}], \quad b_j = i Tr[\tau_j \dot{B} B^+], \quad \dot{A} \equiv \frac{dA(t)}{dt}, \quad \dot{B} \equiv \frac{dB(t)}{dt} \]

and the inertia tensors are expressed in terms of chiral field \( U \equiv U(\vec{z}) \) by

\[ U_{ij} = \frac{\xi^2}{\sigma_1^2[F, \Theta]} u_{ij}, \quad W_{ij} = \frac{\xi^2}{\sigma_1^2[F, \Theta]} w_{ij}, \quad V_{ij} = \frac{\xi^2}{\sigma_1^2[F, \Theta]} v_{ij}, \quad (38) \]

where

\[ u_{ij} = \frac{1}{4f_H} \int d^3z Tr \left( U^+ \left[ \frac{1}{2} \tau_i, U \right] U^+ \left[ \frac{1}{2} \tau_j, U \right] \right), \quad (41) \]

\[ w_{ij} = -u_{ij} \left\{ \left[ \frac{1}{2} \tau_j, U \right] \rightarrow -i(\vec{r} \times \vec{\nabla})_j U \right\}, \quad (42) \]

\[ v_{ij} = -w_{ij} \left\{ \left[ \frac{1}{2} \tau_i, U \right] \rightarrow -i(\vec{r} \times \vec{\nabla})_i U \right\}. \quad (43) \]

Note that the inertia tensors \( U_{ij}, W_{ij}, V_{ij} \) as well as the "breathing mass" \( M \) are invariant under a scale transformation

\[ F(z, \vartheta) \rightarrow F_\beta(z, \vartheta), \quad \Theta(z, \vartheta) \rightarrow \Theta_\beta(z, \vartheta), \quad (44) \]

Using the Ansatz (2) with (3, 4) one obtains the following expressions for nonzero components of the inertia tensors (41-43):

\[ u_{11} = \frac{\pi}{2f_H} \int_0^\infty dz z^2 \int_0^\pi d\vartheta \sin \vartheta (1 + \cos^2 \Theta) \sin^2 F, \quad (45) \]

\[ u_{33} = \frac{\pi}{f_H} \int_0^\infty dz z^2 \int_0^\pi d\vartheta \sin \vartheta \sin^2 \Theta \sin^2 F, \quad (46) \]

\[ w_{11} = \delta_{1m} \frac{\pi}{2f_H} \int_0^\infty dz z^2 \int_0^\pi d\vartheta \sin \vartheta \left[ \frac{\partial \Theta}{\partial \vartheta} + \cot \vartheta \cos \Theta \sin \Theta \right] \sin^2 F, \quad (47) \]

\[ v_{11} = \frac{\pi}{2f_H} \int_0^\infty dz z^2 \int_0^\pi d\vartheta \sin \vartheta \left\{ \left[ \frac{d\Theta}{d\vartheta} \right]^2 + m^2 \cot^2 \vartheta \sin^2 \Theta \right\} \sin^2 F + \left( \frac{\partial F}{\partial \vartheta} \right)^2, \quad (48) \]
\[ u_{22} = u_{11}, \ w_{22} = w_{11}, \ v_{22} = v_{11}, \]  
\[ v_{33} = mw_{33} = m^2u_{33}, \]  
\[ u_{ij} = w_{ij} = v_{ij} = 0 \text{ if } i \neq j. \]  
(49)  

Due to the relations (50) the third components of the canonically-conjugate momenta

\[ K_j \equiv U_{ij}a_i - W_{ij}b_i, \]  
\[ L_j \equiv -W_{ij}a_i + V_{ij}b_i, \]  
(52)  

are dependent:

\[ mK_3 = -L_3, \]  
(54)

which is a result of the cylindrical symmetry. Now the rotational part of the Hamiltonian becomes:

\[
H_{\text{rot}} = \frac{V_{11} - W_{11}}{2D} \vec{K}^2 + \frac{U_{11} - W_{11}}{2D} \vec{L}^2 + \frac{W_{11}}{2D} (\vec{K} + \vec{L})^2 + \frac{1}{2} \left[ \frac{1}{U_{33}} - \frac{V_{11} + m^2U_{11} - 2mW_{11}}{D} \right] K_3^2,
\]
(55)

where \( D \equiv \begin{vmatrix} U_{11} & W_{11} \\ W_{11} & V_{11} \end{vmatrix} \). The canonical angular momenta \( \vec{K} \) and \( \vec{L} \) are the so-called bodyfixed angular momenta in isospace and in usual space, respectively; the conventional angular momenta \( \vec{T} \) and \( \vec{J} \) are defined by the following rotations of this angular momenta \( \vec{K} \) and \( \vec{L} \), respectively:

\[
T_i = -D_{ij}(A)K_j, \quad J_i = -D_{ij}(B)L_j.
\]  
(56)

In the quantum theory a scheme of canonical quantization for the chosen rotational coordinates gives rise to the following commutational relations for momenta \( \vec{K}, \vec{L}, \vec{T} \) and \( \vec{J} \):

\[
[K_i, K_j] = i\epsilon^{ijk}K_k, \quad [L_i, L_j] = i\epsilon^{ijk}L_k, \quad [K_i, L_j] = 0, \]
\[
[T_i, T_j] = i\epsilon^{ijk}T_k, \quad [J_i, J_j] = i\epsilon^{ijk}J_k, \quad [T_i, J_j] = 0.
\]  
(57)  
(58)
with $\vec{T}^2 = \vec{K}^2$ and $\vec{J}^2 = \vec{L}^2$.

Now we can consider a rotational Hamiltonian (3.11) as an operator in the Hilbert subspace defined by the physical constraint generated by (3.10):

\[(mK_3 + L_3) |\text{phys} > = 0.\]  

(59)

According to Eq.(47) $W_{11} = 0$ for the solitons with $m \geq 2$ and the rotational Hamiltonian is reduced to

\[H_{\text{rot}} = \frac{1}{2U_{11}}\vec{T}^2 + \frac{1}{2V_{11}}\vec{J}^2 + \frac{1}{2}\left(\frac{1}{U_{33}} - \frac{m^2}{V_{11}} - \frac{1}{U_{11}}\right)K_3^2.\]  

(60)

A special case of this expression with $m = 2$ was considered in Ref.7. The case with $m = 1$ must be analyzed separately.

\section{Solitons with $m=1$. Configurations with half-odd Topological charge}

For $m = 1$ the differential equation (27) is reduced to the Legendre equation

\[\frac{d}{dz}\left[(1 - z^2)\frac{dF_1}{dz}\right] + \left[\mu(\mu + 1) + 2(-1)^{n+1}\right]F_1 = 0.\]  

(61)

From the obvious arguments of finiteness of the solution at $-1 \leq z \leq 1$ one obtains that $F_1(z) = A P_k(z)$, where $A$ is some nonzero constant, $P_k(z)$ stands for the Legendre polynomial of the order of $k$; $\mu$ is imposed by

\[\mu(\mu + 1) + 2(-1)^{n+1} = k(k+1).\]  

(62)

For even $n$ value the lowest-order solution is independent of the azimuth angle: $k = 0$, $F_1 = A$, $\mu = 1$. This means that for such solutions the spherical symmetry is restored and $V_{11} = W_{11} = U_{11} = W_{33}$. Although the determinant $D$ vanishes, the
ratios $\frac{(V_{11} - W_{11})}{D}$ and $\frac{(U_{11} - W_{11})}{D}$ are finite. To compensate the infinitely rising ratio $\frac{W_{11}}{D}$ one has to impose a new constraint on the physical states:

$$\left(\vec{K} + \vec{L}\right) |_{\text{phys}} > 0$$ (63)

instead of the old one (59). So one obtains the standard classification $j = t$ of the rotative excitations for the solitons with the hedgehog-like structure.

Because $\mu$ is real and positive quantity, the solitons for the odd $n$-value exist only for $k \geq 2$ and have a nontrivial dependence on the azimuth angle $\vartheta$. This means that quantum solitons with half-odd topological charge could not be spherically-symmetrical, but have the cylindrical symmetry.

For the half-odd topological charge all measurable quantities such as energy, components of inertia tensor etc. are finite. The topological analysis of these solitons was done by K. Fujii and two of us (N.M.C. and A.P.K.) in Ref.1. The chiral field matrix may be regarded as a point on the sphere $S^3_{\text{int}}$ in the internal space. But the chiral field is not defined uniquely at the origin: $U(r, \vartheta, \varphi) \to i\hat{x}\vec{r}$, $\hat{x} = \vec{r}$, at $r \to 0$. Calculating measurable quantities one considers all points $i\hat{x}\vec{r}$ of the diameter of sphere $S^3_{\text{int}}$ as one point. This means that the internal space is reduced to a bundle of spheres $(S_3 \vee S_3)^{\text{int}}$ (see, e.g., Ref.9). On the other hand all infinite points of the physical space are reflected to one point $U = 1$ of the internal space, and therefore, the physical space is compactified to $S^3_{\text{phys}}$. Thus, the solution for the chiral field is given by the following mapping:

$$S^3_{\text{phys}} \to (S_3 \vee S_3)^{\text{int}}.$$ (64)

When the function $F(y, \vartheta)$ [Eq.(8)] is varying in the sector $-\frac{\pi}{2} \leq F \leq 0$ the upper sphere of the bundle is covered. The lower sphere is covered when the function $F$ is varying in a sector $-\pi \leq F \leq -\frac{\pi}{2}$ and so on. For the half-odd topological charge the chiral matrix maps the physical space into the spheres from the bundle $(S_3 \vee S_3)^{\text{int}}$ odd times.

Here we do not consider physical applications of the obtained solitons. We would like only to mention that the chiral field, strictly speaking, could be defined only at the hadron periphery; at the soliton origin the field matter is determined by color degrees of freedom (quarks). In this sense there is no problem with the ambiguous behavior of the chiral field at the origin, because this region must be "occupied" by the valence quarks and the ambiguity of the chiral field at the origin is an artifact of an effective theory.

For odd $n$-value $F(y, \vartheta)$ has nontrivial $\vartheta$-dependence and in the expression for the rotative Hamiltonian the denominator $D$ does not vanish. According to (47) $W_{11} \neq 0$
if \( m = 1 \) and, therefore, all terms in (3.11) give nonzero contribution to the rotational energy. Moreover, there appears a new conserved quantity \( \zeta^2 = (\vec{K} + \vec{L})^2 \). It is obvious that the commutation relations between components of the \( \zeta \)-operator are those for angular momentum:

\[
[\zeta_i, \zeta_j] = i\varepsilon_{ijk}\zeta_k. \quad (65)
\]

So, for the soliton with half-odd topological charge the state is given by

\[
|j, t, \zeta, j_3, t_3, k_3>,
\]

where \( j, t \) and \( \zeta \) are soliton spin, isospin and \"new\" momentum, respectively; \( j_3 \) and \( t_3 \) are \( z \)-components of the spin and isospin; \( k_3 = -t, -t+1, \ldots, t \) is an eigenvalue of the operator \( K_3 \). Due to the constraint \((52)\) the state vector is an eigenstate of the \( \zeta_3 \)-operator with zero eigenvalue. Therefore the quantum number \( \zeta \) is integer and is constrained by

\[
|j - t| \leq \zeta \leq j + t, \quad (66)
\]

and the soliton spin and isospin may be either both integer or both half-odd.

5 Conclusions and discussion

In the above consideration we have investigated solitons with the cylindrical symmetry in the nonlinear \( \sigma \)-model. The stabilization of the solitons is provided by quantization of the collective breathing coordinate. The system of two coupled differential equations in partial derivatives for the chiral field is obtained. An analysis of this equations shows that at the origin the chiral angle must be \( \pi/2 \) multiplied by any integer number \( n \) (see Eq.(23)). This gives, additionally to the \"standard\" solitons with integer topological charge, solitons with half-odd topological charge. The latter could not exist in the model, proposed by Skyrme [4], where soliton stabilization is guaranteed by \textit{ad hoc} added term to the Lagrangian density of the nonlinear \( \sigma \)-model [1]. It must be stressed here that the quantum solitons with half-odd topological charge do not appear for spherically-symmetric Ansatz of Refs.1-3. This means that for symmetries, which are more general than the spherical one, the spectrum of the quantum solitons is more rich than that of the static solitons of the model with Skyrme stabilization term.
Quantization of rotative coordinates for spatial and isospacial rotations gives appropriate quantum numbers of the soliton. The soliton with integer topological number is characterized by the standard set of quantum numbers (spin and isospin). At the same time, the solitons with the $\frac{1}{2}$-topological charge are shown to be characterized by spin, isospin and by some additional ”momentum”. The operator for this additional momentum commutes with the soliton Hamiltonian, as well as with its spin and isospin operators.

Between unsolved problems we would like to mention here some physical applications of the obtained solutions, statistics of the solitons with half-odd topological charge and numerical calculations. We hope to discuss some of them in our further publications.

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