MAGNETIC MONOPOLES, ELECTRIC NEUTRALITY
AND THE STATIC MAXWELL-DIRAC EQUATIONS

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Abstract. We study the full Maxwell-Dirac equations: Dirac field with 
minimally coupled electromagnetic field and Maxwell field with Dirac current 
as source. Our particular interest is the static case in which the Dirac current 
is purely time-like – the “electron” is at rest in some Lorentz frame. In this 
case we prove two theorems under rather general assumptions. Firstly, that if 
the system is also stationary (time independent in some gauge) then the 
system as a whole must have vanishing total charge, i.e. it must be 
electrically neutral. In fact, the theorem only requires that the system be 
*asymptotically* stationary and static. Secondly, we show, in the axially 
symmetric case, that if there are external Coulomb fields then these must 
necessarily be magnetically charged – all Coulomb external sources are 
electrically charged magnetic monopoles.
I. Introduction
The Maxwell-Dirac equations are the classical field (or more “traditionally”, first
quantised) equations for electronic matter. Historically, only the linearised
equations (where the Dirac current is ignored as a source for the Maxwell
equations) have been studied in detail – for a comprehensive survey of the Dirac
equation with various potentials see Thaller [1]. The lack of past interest in the
full Maxwell-Dirac equations is partly due to the very difficult nonlinearities of
the equations. More importantly, the classical problem was swamped by the
extraordinary success of QED.

The difficult nature of these nonlinear equations has meant that the existence
theory has only recently been enunciated – some highlights in this dev elopment
might be Gross [2], Chadam [3], Georgiev [4], Esteban et al [5], and Bournaveas
[6]. This work culminated in a tour de force of nonlinear functional ana lysis, the
global existence proof of Flato, Simon and Taflin [7].

Our aim in studying the Maxwell-Dirac system is to look for possible non-linear
behaviour which would not be apparent in perturbation expansions. The
particular solutions found in [9] and [10] exhibit just this sort of behaviour –
localisation and charge screening. See also Das [8] and the recent wo rk of Finster,
Smoller and Yau [11].

The static Maxwell-Dirac equations were first written down in [9]. In the present
work we use this formulation to prove two theorems. Firstly, that the stationary,
static Maxwell-Dirac system must have vanishing total charge, this is done in
section IV. The second theorem proves that, in the axially symmetr ic case an
external Coulomb field must have an associated magnetic charge – external
Coulomb fields must be electrically charged magnetic monopoles. This theorem is
proved in section VI.

II. The Static Maxwell-Dirac Equations
In standard notation the Dirac-Maxwell equations are
\[
\begin{align*}
\gamma^\alpha (\partial_\alpha - i e A_\alpha) \psi + i m \psi &= 0 \\
F_{\alpha\beta} &= A_{\beta,\alpha} - A_{\alpha,\beta} \\
\partial^\alpha F_{\alpha\beta} &= -4\pi e j_\beta = -4\pi e \psi \gamma_\beta \psi
\end{align*}
\]
In [9] the 2-spinor form of the Dirac equations was employed to solve for the
electromagnetic potential, under the non-degeneracy condition
\[j_\alpha j_\alpha \neq 0.\]

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electromagnetic potential, under the non-degeneracy condition \(j_\alpha j_\alpha \neq 0\).

Requiring \(A^\alpha\) to be a real four-vector then gave a set of partial differential
equations in the Dirac field alone, the reality conditions.

For 2-spinors \(u_A\) and \(v^B\) we have
\[\psi = \left(\begin{array}{c} u_A \\ v^B\end{array}\right), \quad \text{with} \quad u_C v^C \neq 0 \quad (\text{non-degeneracy}).\]

The electromagnetic potential,
\[A^{A\dot{A}} = \frac{i}{e(u^C v_C)} \left\{ v^A \partial^{B\dot{A}} u_B + u^A \partial^{B\dot{A}} v_B + \frac{im}{\sqrt{2}} (u^A \bar{\pi}^\dot{A} + v^A \bar{\pi}^\dot{A}) \right\}.\]

The reality conditions,
\[
\begin{align*}
\partial^{A\dot{A}} (u_A \bar{\pi}_{\dot{A}}) &= -\frac{im}{\sqrt{2}} (u^C v_C - \bar{\pi}^\dot{C} \bar{\pi}_{\dot{C}}) \\
\partial^{A\dot{A}} (v_A \bar{\pi}_{\dot{A}}) &= -\frac{im}{\sqrt{2}} (u^C v_C - \bar{\pi}^\dot{C} \bar{\pi}_{\dot{C}}) \\
u_A \partial^{A\dot{A}} \bar{v}_{\dot{A}} - \bar{\pi}_{\dot{A}} \partial^{A\dot{A}} u_A &= 0
\end{align*}
\]
The Maxwell equations,

$$\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e \sigma^A_{\beta}(u_A \bar{\nu}_A + v_A \bar{\nu}_A).$$

These equations constitute the Maxwell-Dirac system.

We next impose the static condition.

**Definition 1.** The Maxwell-Dirac system is said to be static if there exists a local Lorentz frame in which the Dirac current vector is purely time-like, i.e. $j^\alpha = j^0 \delta^\alpha_0$.

Imposing this condition one quickly finds that

$$v^A = e^{i\chi} \sqrt{2} u^0 \bar{\nu}_A + u^1 \bar{\nu}_A \delta^\alpha_0,$$

with $\chi$ a real function.

The current vector is now

$$j^\alpha = \sqrt{2}(u^0 \bar{u}_0 + u^1 \bar{u}_1) \delta^\alpha_0.$$

As noted in (1) the gauge is fixed by the choice,

$$u^0 = X e^{\frac{i}{2}(\chi + \eta)}$$

$$u^1 = Y e^{\frac{i}{2}(\chi - \eta)},$$

with $X$, $Y$, and $\eta$ real functions on $\mathbb{R}^4$.

Defining the null vector $L$,

$$L = (\sigma^\alpha_{AA} u^A \bar{u}^A) = (L^0, \frac{1}{\sqrt{2}} V),$$

with $L^0 = \frac{1}{\sqrt{2}}(X^2 + Y^2)$ and

$$V = (2XY \cos \eta, 2XY \sin \eta, X^2 - Y^2),$$

our equations become,

$$\frac{\partial}{\partial t}(X^2 + Y^2) = 0$$

$$\nabla V = -2m(X^2 + Y^2) \sin \chi$$

$$\frac{\partial V}{\partial t} + (\nabla \chi) \times V = 0.$$

With electromagnetic potential

$$A^0 = \frac{m}{e} \cos \chi + \frac{(X^2 - Y^2)}{2e(X^2 + Y^2)} \frac{\partial \eta}{\partial t} + \frac{(\nabla \chi) \cdot V}{2e(X^2 + Y^2)}$$

$$A = \frac{1}{2e(X^2 + Y^2)} \left[ \frac{\partial \chi}{\partial t} V + (X^2 - Y^2) \nabla \eta - \nabla \times V \right],$$

where $A = (A^1, A^2, A^3)$.

The full system is given by the above two sets of equations and the Maxwell equations.

One further condition we want to impose is that of stationarity.

**Definition 2.** The Maxwell-Dirac system will be called stationary if there exists a gauge in which $\psi = e^{i\omega t} \phi$, with the bi-spinor $\phi$ independent of $t$. Such a gauge will be referred to as a stationary gauge.

We now have the following simple lemma.

**Lemma 1.** The static Maxwell-Dirac system is stationary if and only if, in the gauge given in (1),

$$\frac{\partial \eta}{\partial t} = 0 \quad \text{and} \quad \frac{\partial X}{\partial t} = 0.$$

In the stationary case we also have,

$$\frac{\partial \chi}{\partial t} = 0 \quad \text{and} \quad \frac{\partial V}{\partial t} = 0.$$
Proof. If the system is stationary there exists a gauge transformation such that 
\[ u^A \rightarrow e^{i\xi}u^A = e^{i\xi t}e^{-i\omega t} \], with \( \zeta^A \) independent of \( t \). Consequently,
\[
X e^{i\frac{1}{2}(\chi + \eta)} = e^{i(\omega t - \xi)} \zeta^0 \\
Y e^{i\frac{1}{2}(\chi - \eta)} = e^{i(\omega t - \xi)} \zeta^1.
\]
So, \( |X| = |\zeta^0| \) and \( |Y| = |\zeta^1| \), both independent of \( t \). We also have,
\[
\frac{\partial \chi}{\partial t} + \frac{\partial \eta}{\partial t} = 2(\omega - \frac{\partial \xi}{\partial t}) \\
\frac{\partial \chi}{\partial t} - \frac{\partial \eta}{\partial t} = 2(\omega - \frac{\partial \xi}{\partial t}).
\]
So that, \( \frac{\partial \eta}{\partial t} = 0 \). The argument is easily reversed to get the converse statement. \( \square \)

III. Isolated Systems
An isolated system is one for which all sources are contained in some ball \( B_\rho(\rho < \infty) \) and for which the fields die off as \( |x| = r \rightarrow \infty \).
In what follows we will be considering stationary Maxwell-Dirac systems. For such systems we would expect, in an appropriate stationary gauge, that \( A^\alpha \) should be \( O(1/r) \) as \( r \rightarrow \infty \). We will also need to impose some decay conditions on \( \psi \) as \( r \rightarrow \infty \) in order to appropriately define an isolated Maxwell-Dirac system.
The best language for the discussion of such decay conditions and other regularity issues is the language of weighted function spaces; specifically weighted classical and Sobolev spaces.
We will use the definitions of \[12\], other accounts of the theory may be found in \[13\], \[14\] and \[15\].

Definition 3. Weighted Sobolev spaces can be defined via the weighted Lebesgue spaces \( L^p_\delta \), \( 1 \leq p \leq \infty \) which are spaces of locally measurable functions for which the norms
\[
\|f\|_{p,\delta} = \left\{ \begin{array}{ll}
\left( \int_{\mathbb{R}^n} |f|^{p} \sigma^{-p\delta-n} \, dx \right)^{\frac{1}{p}}, & p < \infty \\
\text{ess sup}_{\mathbb{R}^n} \left( \sigma^{-\delta}|f| \right), & p = \infty,
\end{array} \right.
\]
are finite. We can replace \( \mathbb{R}^n \) with subsets \( \Omega \) of \( \mathbb{R}^n \) in these definitions. The weight \( \sigma \) is usually taken to be \( \sigma = (1+1/r^2)^{\frac{1}{2}} \) or \( \sigma = r \) on subsets excluding \( \{0\} \).
The weighted Sobolev spaces are now defined as consisting of functions with weak derivatives up to order \( k \) for which the following norm is finite
\[
\|f\|_{k,p,\delta} = \sum_{j=0}^{k} \|D^j f\|_{p,\delta-j}.
\]
For \( p < \infty \) we denote the weighted Sobolev space by \( W^{k+p}_\delta \). For \( p = \infty \) we denote the classical weighted function space by \( C^k_\delta \).
We will also require the Sobolev inequality, given here in the form presented in \[12\].

Sobolev Inequality
If \( F \in W^{k+p}_\delta \) then
\[
(i) \quad \|f\|_{k,q,\delta} \leq C \|f\|_{k,p,\delta}, \quad \text{if} \quad n - kp > 0 \quad \text{and} \quad p \leq q \leq \frac{np}{(n-p)} \\
(ii) \quad \|f\|_{\infty,\delta} \leq C \|f\|_{k,p,\delta}, \quad \text{if} \quad n - kp < 0, \quad \text{and} \quad |f(x)| = o(r^\delta) \quad \text{as} \quad r \rightarrow \infty.
\]
Another tool we will require is the “multiplication lemma”. In what follows we will be mainly using Sobolev spaces with $p = 2$, and it is for this case that we give the multiplication lemma (adapted from [13]).

**Multiplication Lemma**

Pointwise multiplication on $\mathbb{R}^n$ is a continuous bilinear mapping

$$W^{k_1,2}_{\delta_1} \times W^{k_2,2}_{\delta_2} \to W^{k,2}_{\delta},$$

if $k_1, k_2 \geq k$, $k < k_1 + k_2 - n/2$, and $\delta > \delta_1 + \delta_2$.

In discussing the decay conditions on $\psi$ it will be useful to have some new notation. Firstly, we introduce two new 2-spinors $o_A$ and $\iota_A$ by,

$$u_A = \sqrt{Re}^{\pm i} o_A$$
$$v_A = \sqrt{Re}^{\pm i} \iota_A,$$

with $\iota^C o_C = 1$.

In discussing decay conditions on $A_\alpha$ we need, of course to be aware that $A_\alpha$ is defined only up to a gauge transformation. This problem is usually resolved (to some extent) by imposing gauge conditions, such as the Lorenz gauge. We restrict our attention to stationary gauges and demand that $A_0$ be $O(1/r)$ as $r \to \infty$ in some gauge. If we demand that $A$ be $O(1/r)$ then the question is, can we find a gauge transformation which takes us to the Lorenz gauge with the new $A$ still satisfying the appropriate $O(1/r)$ decay? The answer is yes provided we choose the original $A$ in the correct function space.
We need $A^\alpha$ to be at least twice (weakly) differentiable to make sense of the Maxwell equations. So to get the appropriate differentiability and decay we will take (see definition, below)

$$A^\alpha \in W^{2,p}_{-1+\epsilon}(E_\rho),$$

for some $p > 3/2$ and all $\epsilon > 0$. Here, $E_\rho = \mathbb{R}^3 \setminus B_\rho$, with $\rho < \infty$ large enough so that $B_\rho$ encloses all external sources. This means we can now solve the gauge equation for $\phi \in W^{3,p}_\epsilon(E_\rho)$, with $A' = A + \nabla \phi \in W^{2,1+\epsilon}_\rho(E_\rho)$, in fact, for $0 < \epsilon < 1$ the Laplacian gives an isomorphism between the function spaces $W^{3,p}_\epsilon$ and $W^{1,2}_{3+\epsilon}$, see [12].

The Maxwell equations imply that $j^\alpha \in W^{0,p}_{3+\epsilon}(E_\rho)$. In the vector basis co-moving with $j^\alpha$ (induced from the co-moving dyad) the charge density is $\sqrt{2}R$, we would expect therefore that $R$ is $O(1/r^3)$ as $r \to \infty$. We also require at least three derivatives for $R$ to define the Maxwell equations when the $A^\alpha$ are written in terms of the components of $\psi$. This suggests we should take $R \in W^{3,p}_{3+\epsilon}(E_\rho)$. The $o_A$ and $\iota_A$ must also have at least three (weak) derivatives and we also need to ensure that $j^\alpha \in W^{0,p}_{3+\epsilon}(E_\rho)$. We will require $o_A, \iota_A \in W^{3,p}_\epsilon(E_\rho)$, for any $\epsilon > 0$.

This leaves the differentiability and decay of $\chi$ to be determined. Again we will require at least three derivatives of $\chi$. The decay rate, however must be determined from the equations.

We can now give our definition of an isolated Maxwell-Dirac system. For concreteness and ease of manipulation we will restrict our attention to Sobolev spaces $W^{k,2}_\delta(E_\rho)$.

**Definition 4.** A stationary Maxwell-Dirac system will be said to be isolated if, in some stationary gauge, we have

$$\psi = e^{iEt}R \left( \begin{array}{c} e^{\frac{i}{2} o_A} \\ e^{-\frac{i}{2} \iota_A} \end{array} \right),$$

with $E$ constant and

$$R \in W^{3,2}_{3+\epsilon}(E_\rho); \quad o_A, \iota_A \in W^{3,2}_\epsilon(E_\rho) \text{ and } A^\alpha \in W^{2,2}_{-1+\epsilon}(E_\rho), \quad \text{for some } \rho > 0 \text{ and any } \epsilon > 0.$$

**Remark**

This definition ensures, after use of the Sobolev inequality and the multiplication lemma, that $\psi = o(r^{-\frac{3}{2}+\epsilon})$ and $A^\alpha = o(r^{-1+\epsilon})$.

We are now in a position to prove a lemma which will be used in the proof of theorem 1 in the next section. But we first need some new notation.

We introduce the complex null tetrad vectors

$$l^\alpha = \sigma^\alpha_{\dot{A}} \sigma^A_{\dot{\bar{A}}} \ , \quad n^\alpha = \sigma^\alpha_{\dot{A}} \iota^A_{\dot{\bar{A}}}$$
$$m^\alpha = \sigma^\alpha_{\dot{A}} \sigma^A_{\dot{\bar{B}}} \ , \quad \bar{m}^\alpha = \sigma^\alpha_{\dot{A}} \iota^A_{\dot{\bar{B}}}.$$  

This null tetrad can now be used to define the following (Newman-Penrose) intrinsic derivatives

$$D = l^\alpha \frac{\partial}{\partial x^\alpha}, \quad \triangle = n^\alpha \frac{\partial}{\partial x^\alpha}$$
$$\delta = m^\alpha \frac{\partial}{\partial x^\alpha}, \quad \bar{\delta} = \bar{m}^\alpha \frac{\partial}{\partial x^\alpha}$$
With this notation and the expression for $A^\alpha$ of section II we find that the (real) potential $A^\alpha$ has to have the following components with respect to the null tetrad

$$A_l = l_\alpha A^\alpha = \frac{1}{2e} \left[ \sqrt{2} m \cos \chi - \Delta \chi + i(\mu - \bar{\mu} + \gamma - \bar{\gamma}) \right]$$

$$A_n = n_\alpha A^\alpha = \frac{1}{2e} \left[ \sqrt{2} m \cos \chi + D \chi + i(\rho - \bar{\rho} + \bar{\varepsilon} - \varepsilon) \right]$$

$$A_m = m_\alpha A^\alpha = \frac{i}{2e} \left[ -\frac{\delta R}{R} + \bar{\alpha} - \beta + \tau - \bar{\pi} \right]$$

$$A_{\bar{m}} = \bar{m}_\alpha A^\alpha = \frac{i}{2e} \left[ \frac{\delta R}{R} - \alpha + \bar{\beta} - \bar{\tau} + \pi \right]$$

Here $\alpha, \beta, \gamma, \tau, \mu, \rho, \bar{\varepsilon}$ are the NP spin coefficients (Ricci rotation coefficients for the non-holonomic NP tetrad), see [16]. Their exact form is not important here, what we do need to know is that they are all of the form $o\partial o, i\partial i, o\partial i$ or $i\partial o$, where $\partial$ is any one of the NP intrinsic derivatives, $D, \Delta, \delta$ or $\bar{\delta}$. Notice that here we are using a gauge which has the factor $e^{iEt}$ of definition 4 removed – for the stationary case this means all variables are independent of $t$ and $A^0 \to E/e$ as $r \to \infty$.

**Lemma 2.** For an isolated, stationary Maxwell-Dirac system the following must hold

$$\delta R/R - \frac{E m_0}{E} \in W^2_{-1+\epsilon}(E_\rho),$$

for any $\epsilon > 0$ and some constant $E$.

**Proof.** A straightforward application of the multiplication lemma.

With $o_A, t_A \in W^{3,2}(E_\rho)$ and $A^0 - E/e, A \in W^{2,2}_{1+\epsilon}(E_\rho)$, for any $\epsilon > 0$, we have

$$A_m - m_0 \frac{E}{e}, A_{\bar{m}} - \bar{m}_0 \frac{E}{e} \in W^{2,2}_{\epsilon_1}(E_\rho), \quad \delta_1 > -1 + 3\epsilon$$

and

$$\alpha, \beta, \gamma, \tau, \mu, \rho, \bar{\varepsilon} \in W^{2,2}_{\delta_2}(E_\rho), \quad \delta_2 > -1 + 4\epsilon.$$

The result then follows from (4). \qed

**IV. Vanishing Total Charge**

We will be working with the stationary, static Maxwell-Dirac equations. From section II they are

$$\nabla \times \nabla \chi = 0$$

$$\nabla \times \nabla \chi = -2m(X^2 + Y^2) \sin \chi$$

$$A^0 = m \frac{e}{e} \cos \chi + \frac{(\nabla \chi) \cdot \nabla}{2e(X^2 + Y^2)}$$

$$A = \frac{1}{2e(X^2 + Y^2)} [(X^2 - Y^2) \nabla \eta - \nabla \times V]$$

$$\nabla^2 A^0 = 4\pi e \omega^0$$

$$\nabla \times (\nabla \times A) = 0.$$

(5)

We can now state and prove our theorem of vanishing total charge.

**Theorem 1.** An isolated, stationary, static Maxwell-Dirac system is electrically neutral.
Proof. A more restricted version of this theorem was proved in [17] by one of us (H.B.).

The stationary gauge of definition 4 is the one for which \( A_0 \rightarrow 0 \) as \( r \rightarrow 0 \), the stationary gauge used in equations (5) has \( \psi \) independent of \( t \). A gauge transformation of the type \( \psi \rightarrow e^{-iEt} \psi \) will bring the gauge of definition 4 into that of equations (5). The \( A_0 \) of these equations then differs by a constant, \(-E/e\), from the \( A_0 \) of definition 4. In fact, we will be able to determine \( E \) in the proof of the theorem, see also corollary 1. As we noted earlier we have

\[
\begin{align*}
o_A &= \begin{pmatrix} \sin \frac{\tau}{2} e^{-i\frac{\eta}{2}} \\ -\cos \frac{\tau}{2} e^{i\frac{\eta}{2}} \end{pmatrix} \\
A^A &= \begin{pmatrix} \sin \frac{\tau}{2} e^{i\frac{\eta}{2}} \\ -\cos \frac{\tau}{2} e^{-i\frac{\eta}{2}} \end{pmatrix}
\end{align*}
\]

in the static case. The system is isolated so using the multiplication lemma we have that

\[
\sin \tau, \cos \tau, \sin \eta, \cos \eta \in W^{3,2}(E_\rho).
\]

Now we have \( m_0 = \sigma_0 A^A o_A t_A = 0 \), so using lemma 2 we have

\[
\frac{\delta R}{R} \text{ and } \frac{\delta R}{\hat{R}} \in W^{2,2}_{-1+\epsilon}(E_\rho).
\]

Which in our static case gives,

\[
\begin{align*}
\cos \tau \left( \cos \eta \frac{\partial R}{\partial x} + \sin \eta \frac{\partial R}{\partial y} \right) - \sin \tau \frac{\partial R}{\partial z} &\in W^{2,2}_{-1+\epsilon}(E_\rho) \\
\text{and} \quad \sin \eta \frac{\partial R}{\partial x} - \cos \eta \frac{\partial R}{\partial y} &\in W^{2,2}_{-1+\epsilon}(E_\rho).
\end{align*}
\]

The second of equations (5) can be written as

\[
-2m \sin \chi = \frac{\nabla \cdot V}{R} = \frac{\hat{V} \cdot \nabla R}{R} + \nabla \cdot \hat{V},
\]

with \( \hat{V} = (\sin \tau \cos \eta, \sin \eta \sin \tau, \cos \tau) \). Using the multiplication lemma we have

\[
\hat{V} \in W^{3,2}_\rho \text{ and } \nabla \cdot \hat{V} \in W^{2,2}_{-1+2\epsilon}(E_\rho),
\]

for any \( \epsilon > 0 \). We also have

\[
\frac{\hat{V} \cdot \nabla R}{R} = \sin \tau \cos \eta \frac{\partial R}{\partial x} + \sin \tau \sin \eta \frac{\partial R}{\partial y} + \cos \tau \frac{\partial R}{\partial z}.
\]

Next we utilise the invariance of our equations under Lorentz transformations. In fact, we know that if \((u_A(x^\alpha), v_A(x^\alpha))\) is a solution to the Maxwell-Dirac equations then \((u_A(\hat{x}^\alpha), v_A(\hat{x}^\alpha))\) is a solution to the original system in the \(x^\alpha\) coordinates; here \(\hat{x}^\alpha\) are the Lorentz transformed Cartesian coordinates. This is, of course, true for any linear Lorentz invariant theory. Consider the rotation

\[
\hat{x} = x \cos \omega + y \sin \omega \\
\hat{y} = -x \sin \omega + y \cos \omega \\
\hat{z} = z, \quad \text{with} \quad \omega = -\frac{\pi}{2}.
\]

This gives

\[
\begin{align*}
\frac{\sin \eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)} \frac{\partial R(\hat{x}^\alpha)}{\partial x} - \frac{\cos \eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)} \frac{\partial R(\hat{x}^\alpha)}{\partial y} = \frac{\cos \eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)} \frac{\partial R(\hat{x}^\alpha)}{\partial \hat{x}} + \frac{\sin \eta(\hat{x}^\alpha)}{R(\hat{x}^\alpha)} \frac{\partial R(\hat{x}^\alpha)}{\partial \hat{y}}
\end{align*}
\]
Diffeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$ induce an isomorphism of the Sobolev spaces $W^{k,p}_\delta$, see [13]. The rotation above preserves $E_\rho$ and will give an isomorphism of the Sobolev spaces $W^{k,p}_\delta(E_\rho)$. Using our last equation and (3) gives

$$\frac{\cos \eta}{R} \frac{\partial R}{\partial x} + \sin \eta \frac{\partial R}{\partial y} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

This equation and (3) with the multiplication lemma then gives (multiply each equation in turn by $\sin \eta$ and $\cos \eta$ etc.),

$$\frac{1}{R} \frac{\partial R}{\partial x} + \frac{1}{R} \frac{\partial R}{\partial y} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

The rotation

$$\hat{x} = -z, \; \hat{y} = y \; \hat{z} = x$$

gives

$$\frac{\sin \eta(\hat{x}^\alpha)}{R} \frac{\partial (\hat{x}^\alpha)}{\partial x} - \frac{\cos \eta(\hat{x}^\alpha)}{R} \frac{\partial (\hat{x}^\alpha)}{\partial y} = \frac{\sin \eta(\hat{x}^\alpha)}{R} \frac{\partial (\hat{x}^\alpha)}{\partial z} - \frac{\cos \eta(\hat{x}^\alpha)}{R} \frac{\partial (\hat{x}^\alpha)}{\partial y}.$$ Again using the multiplication lemma with (3), we have

$$\frac{\sin \eta}{R} \frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

In the same fashion, using the rotation $\hat{x} = x, \; \hat{y} = z, \; \hat{z} = -y$, we have

$$\frac{\cos \eta}{R} \frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

Another use of the multiplication lemma with the last two equations gives

$$\frac{1}{R} \frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

Altogether we have

$$\frac{1}{R} \frac{\partial R}{\partial x} + \frac{1}{R} \frac{\partial R}{\partial y} + \frac{1}{R} \frac{\partial R}{\partial z} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

A final use of the multiplication lemma with (3) and we have

$$\frac{\hat{V} \cdot \nabla R}{R} \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

We can now conclude from (3) that $\sin \chi \in W^{2,2}_{-1+\epsilon}(E_\rho)$, for any $\epsilon > 0$. By the Sobolev inequality $\sin \chi = o(r^{-1+\epsilon})$ as $r \to \infty$.

Now sine is an invertible $C^\infty$ function on the range of $\chi$ (on $E_\rho$, with $\rho$ large) with $\sin \chi = 0$ for $\chi = 0$. So we can now write

$$\chi = n\pi + \mu, \; \text{with} \; n = 0, \pm 1, \pm 2, \ldots \; \text{and} \;$$

$$\mu \in W^{2,2}_{-1+\epsilon}(E_\rho).$$

Next we use the first of equations (3) to rewrite $A^0$ entirely in terms of $\chi$ and $|\nabla \chi|$. This equation implies we may write $V = \gamma \nabla \chi$, for some function $\gamma$. We also have $|V| = X^2 + Y^2 = |\gamma||\nabla \chi|$, so that

$$A^0 = \frac{m}{e} \cos \chi + \frac{\varepsilon}{2e} |\nabla \chi|$$

with $\varepsilon = \frac{\gamma}{|\gamma|}$.

Which, using (3), may be written as

$$A^0 - \frac{m}{e} \cos(n\pi) = -\frac{2m}{e} \cos(n\pi) \sin^2\left(\frac{\mu}{2}\right) + \frac{\varepsilon}{2e} |\nabla \mu|.$$ Hence,

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{1,2}_{-2+\epsilon}(E_\rho).$$
Note that $|\nabla \chi|$ is bounded on $E_\rho$. So $\gamma$ cannot change sign on $E_\rho$ as $R = X^2 + Y^2 \neq 0$, $\varepsilon$ is fixed.

From the (first) Sobolev inequality applied to $A^0 - \frac{m}{e} \cos n\pi$, with $p = q = 2$, we have that

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{0,6}_{-2+\varepsilon}(E_\rho).$$

Consequently, we also have

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{0,6}(E_\rho).$$

Now, in the static case, $j^0 = \sqrt{2}R$, so $\nabla^2(A^0 - \cos n\pi) \in W^{3,2}_{-3+\varepsilon}$. Hence,

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{5,2}_{-1+\varepsilon}(E_\rho)$$

and from the Sobolev inequality we find that $|\partial_i A^0| < Cr^{-2+\varepsilon}$ (with $\partial_i = \partial/\partial x^i$). So we have,

$$|\partial_i A^0| < Cr^{-2+\varepsilon},$$

for any $0 < \varepsilon < 1/12$. Thus, $\partial_i A^0 \in W^{0,6}_{-9/4}(E_\rho)$. Which finally gives us,

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{3,2}_{-2+\varepsilon}(E_\rho),$$

and the Sobolev inequality now gives

$$A^0 - \cos n\pi = o(r^{-\frac{3}{2}}).$$

From which it is clear that the total electric charge of the system

$$\lim_{\rho \to \infty} \frac{1}{4\pi} \int_{S_\rho} (\nabla A^0) \cdot dS,$$

with $S_\rho$ the sphere of radius $\rho$, must vanish.

**Corollary 1.** In the gauge in which $A^0 \to 0$ as $r \to \infty$ the Dirac bi-spinor of an isolated, stationary, static Maxwell-Dirac system takes the form

$$\psi = e^{\pm imt} \phi, \quad \text{with} \quad \phi \in W^{3,2}_{-\frac{3}{2}+\varepsilon}(E_\rho) \quad (\text{as above}).$$

**Proof.** This result is a simple consequence of the proof of the theorem. We had

$$A^0 - \frac{m}{e} \cos(n\pi) \in W^{3,2}_{-2+\varepsilon}(E_\rho).$$

The constant term, $-\frac{m}{e} \cos n\pi$ is removed by the gauge transformation

$$\psi \to e^{-i \cos(n\pi) mt} \psi.$$
Definition 5. A Maxwell-Dirac system will be called axially symmetric if
\[
\begin{bmatrix} L, \frac{\partial}{\partial \phi} \end{bmatrix} = \begin{bmatrix} N, \frac{\partial}{\partial \phi} \end{bmatrix} = 0,
\]
where \( L = \sigma^\alpha_{\bar{A}A} u^A \frac{\partial}{\partial x^\alpha} \) and \( N = \sigma^\alpha_{\bar{A}A} v^A \frac{\partial}{\partial x^\alpha} \).

For our static systems we require only that \( L \) be invariant under translations in \( \phi \).

In fact, writing \( L \) in cylindrical polar coordinates,
\[
L = L^0 \frac{\partial}{\partial t} + \frac{1}{\sqrt{2}} \left( V^\rho \hat{\rho} + V^\phi \hat{\phi} + V^z \hat{k} \right)
\]
\[
= L^0 \frac{\partial}{\partial t} + \frac{1}{\sqrt{2}} \left( V^\rho \frac{\partial}{\partial \rho} + V^\phi \frac{\partial}{\partial \phi} + V^z \frac{\partial}{\partial z} \right),
\]
we find that
\[
L^0 = \frac{X^2 + Y^2}{\sqrt{2}}, \quad V^\rho = 2XY \cos(\eta - \phi),
\]
\[
V^\phi = 2XY \sin(\eta - \phi) \quad \text{and} \quad V^z = X^2 - Y^2
\]
must all be independent of \( \phi \). This means our static Maxwell-Dirac system is axially symmetric if
\[
\frac{\partial X}{\partial \phi} = \frac{\partial Y}{\partial \phi} = \frac{\partial (\eta - \phi)}{\partial \phi} = 0.
\]

This information lets us characterise stationary, axially symmetric, static Maxwell-Dirac systems as follows.

Lemma 3. A non-trivial static, axially symmetric Maxwell-Dirac system is stationary if and only if \( \eta = \phi \) in the gauge given in (1).

Proof. As \( \mathbf{V} \) is independent of \( \phi \) then so is \( \chi \), as \( \sin \chi = (\nabla \cdot \mathbf{V})/(X^2 + Y^2) \). The reality condition,
\[
\frac{\partial \mathbf{V}}{\partial t} + (\nabla \chi) \times \mathbf{V} = 0
\]
gives,
\[
\frac{\partial V^\rho}{\partial t} - V^\phi \frac{\partial \chi}{\partial z} = 0
\]
\[
\frac{\partial V^\phi}{\partial t} - V^z \frac{\partial \chi}{\partial \rho} + V^\rho \frac{\partial \chi}{\partial \phi} = 0
\]
\[
\frac{\partial V^z}{\partial t} + V^\phi \frac{\partial \chi}{\partial \rho} = 0.
\]
If the system is stationary lemma 1 says \( \mathbf{V} \) is independent of \( t \). So either \( V^\phi = 0 \) or \( \chi \) is constant. Constant \( \chi \) leads to the trivial solution with \( X = Y = 0 \). So we must take \( V^\phi = 2XY \sin(\eta - \phi) = 0 \). We have \( \eta = \phi \mod n\pi \).

On the other hand, if \( \eta = \phi \) we have \( V^\phi = 0 \) and consequently both \( V^\rho \) and \( V^z \) are independent of \( t \). Combining these with \( \partial_t (X^2 + Y^2) = 0 \) gives the result.

From now on we study the axially symmetric, stationary, static Maxwell-Dirac equations.

It will prove convenient in what follows to use spherical polar coordinates \((r, \theta, \phi)\) and to make the following change of variables
\[
X = \sqrt{R} \cos \left( \frac{T}{2} \right) \quad \text{and} \quad Y = \sqrt{R} \sin \left( \frac{T}{2} \right).
\]
All our dependent variables depend only on \((r, \theta)\), the equations are

\[
\begin{align*}
V &= \mathcal{R} \left[ \cos(\tau - \theta) \mathbf{\hat{r}} + \sin(\tau - \theta) \mathbf{\hat{\theta}} \right] \\
(\nabla \chi) \times V &= 0 \\
\nabla \cdot V &= -2mR \sin \chi \\
A^0 &= \frac{m}{e} \cos \chi + \frac{\epsilon}{2e} |\nabla \chi| \\
A &= \frac{1}{2e} \left\{ \frac{\cos \tau}{r \sin \theta} - \frac{1}{rR} \left[ \frac{\partial}{\partial r} (rR \sin(\tau - \theta)) - \frac{\partial}{\partial \theta} (R \cos(\tau - \theta)) \right] \right\} \mathbf{\hat{\phi}},
\end{align*}
\]

(10)

together with the Maxwell equations. Here \(\mathbf{\hat{r}}, \mathbf{\hat{\theta}},\) and \(\mathbf{\hat{\phi}}\) are the unit coordinate vectors.

We note that \(A^\alpha\) automatically satisfies the Lorenz gauge condition.

In the spherically symmetric case \([9]\) we have \(\tau = \theta\) with \(R\) and \(\chi\) functions of \(r\) only. In which case

\[
A = \frac{1}{2e} \cot \theta \frac{1}{r} \mathbf{\hat{\phi}},
\]

the magnetic monopole.

Other tractable cases are those for which \(\tau\) is constant. It is straightforward to show there are really only two cases, see \([7]\).

- the cylindrically symmetric case, \(\tau = \pi/2\), see \([10]\).
- the case \(\tau = 0\), variables depend on \(z\) only, see \([18]\).

VI. Magnetic Monopoles

The spherically symmetric solution has an external (i.e. not sourced by the Dirac field directly) electrically charged magnetic monopole. In this section we will prove a theorem which shows that this is, to some extent, the generic situation. We will show that the axially symmetric, stationary, static Maxwell-Dirac system can have an external Coulomb point charge only if it is magnetically charged.

First we define what we mean by an external Coulomb field.

**Definition 6.** We will say that a Maxwell-Dirac system has an external Coulomb field if we can choose spherical polar coordinates and a ball \(B_\rho\) centred at \(r = 0\) such that

\[
A^0 = \frac{q}{r} + h, \ \text{in} \ \ B_\rho, \ \rho > 0,
\]

with, \(h\), a bounded function on \(B_\rho\) and \(q\) constant.

**Remarks**

1. \(q/r\) is of course harmonic on \(B_\rho \setminus \{0\}\), so it is not directly sourced by the Dirac field via the Maxwell equation \(\nabla^2 A^0 = 4\pi e j^0\). In this sense the Coulomb field is “external” to the Dirac field.

2. The condition that \(h\) is bounded on \(B_\rho\) is quite weak. In practice it will follow from elliptic regularity of the Poisson equation \(\nabla^2 h = 4\pi e j^0\) \((j^0 = \sqrt{2}(X^2 + Y^2) = \sqrt{2}R\) must be at least \(L^1(B_\rho)\) for the total Dirac charge to be well-defined on \(B_\rho\), see for example \([19]\) and \([20]\). We also require \(R\) to be differentiable at least three times (if only in the weak sense) to satisfy the Maxwell equation for \(A\) — this puts \(R\) in \(W^{3,p}_\delta\) for some \(p \geq 1\). Consequently, \(h\) will be in \(W^{5,p}_{\delta+2}\), which ensures that \(h\) can be included in one of the classical weighted function spaces.

Now for our theorem.

**Theorem 2.** Suppose an axially symmetric, stationary, static Maxwell-Dirac system has an external Coulomb field. Let \(A_{(\rho,\rho_1)} = B_\rho \setminus B_{\rho_1}, \ \text{with} \ \rho > \rho_1 > 0\) and
charge of the magnetic field $B$. We now work on (14) where (15) can be written as

$$\varepsilon \frac{r \partial_r \gamma}{|\nabla \chi|}$$

We will first show that $\sin(\tau - \theta) \to 0$ as $r \to 0$.

Now, $A^0 = \frac{\eta}{r} + h$, so from (10) we have

$$|\nabla \chi| = \frac{q}{r} + g,$$

where $g = h - \frac{\eta}{r}$ and $\cos \chi$ is bounded on $B_{\rho}$. Next, we write

$$\chi = q \ln r + \zeta, \text{ on } B_{\rho \{0\}}.$$

Equation (12) is

$$(\partial_r \zeta)^2 + \frac{2g}{r} \partial_r \zeta + \frac{1}{r^2} \partial_\theta \zeta^2 = \frac{g}{r} + g^2.$$

Note that $g \partial_\theta \zeta \to g$ as $r \to 0$ so that $r \partial_\theta \chi = q + r \partial_r \zeta$ and $\partial_\theta \chi = \partial_\theta \zeta$ are bounded on $A_{\rho \{0\}}$, for $\rho$ small enough. Let $\eta$ be the smallest non-negative number such that $r^\eta \partial_r \zeta \to 0$ as $r \to 0$. We can write (13) as

$$(r \frac{1 + \eta}{r} \partial_r \zeta)^2 + 2r^\eta \partial_r \zeta + (r \frac{1 + \eta}{r} \partial_\theta \zeta)^2 = 2r^\eta g + r^{\eta+1} g^2.$$

It is clear that $r \frac{1 + \eta}{r} \partial_r \zeta \to 0$ and $r \frac{1 + \eta}{r} \partial_\theta \zeta \to 0$. From which we conclude that $\eta \leq 1$ and so $\partial_\theta \chi = \partial_\theta \zeta \to 0$ as $r \to 0$. Consequently,

$$\sin(\tau - \theta) = \frac{\partial_\theta \chi}{r |\nabla \chi|} \to 0, \text{ as } r \to 0.$$

Now $r \partial_r \chi \to q$ as $r \to 0$, so $r \partial_r \chi$ cannot change sign as $\theta$ varies, for $\rho$ small. Writing $\varepsilon = \frac{q}{|	heta|}$ we have, from (11),

$$(14) \begin{aligned}
\quad & \cos \tau(r, \pi) \to -\varepsilon \varepsilon_1 \\
\quad & \cos \tau(r, 0) \to \varepsilon \varepsilon_1
\end{aligned} \quad \text{as } r \to 0.$$

We now work on $A_{\rho \{0\}}$ with $\rho$ small enough that $\cos(\tau - \theta) \neq 0$. The magnetic charge of the magnetic field $B = \nabla \times A$ in $B_{\rho}$, $\rho > r > \rho_1$, is

$$(15) b = \frac{1}{4\pi} \int_{S_r} B \cdot dS = \frac{1}{2} \int_{\theta=0}^{\pi} \partial_\theta (r \sin \theta A) d\theta,$$

where $A = A_0$ is given in (10). After some manipulation the third equation of (10) can be written as

$$\partial_\theta \tau - \frac{r \partial_r R}{R} = \tan(\tau - \theta) \left( r \partial_r \tau - \frac{\partial_\theta R}{R} \right) - 2mr \sin \chi - 3 \cos(\tau - \theta).$$
So that if $r \partial_r \tau - \frac{\partial R}{R}$ is bounded on $A(r, \rho_1)$ then so is $\partial_\theta \tau - 2 + \frac{r \partial_r R}{R}$. From (10) we have

$$r \sin \theta A = \frac{1}{2e} \{ \cos \tau - P \sin \theta \},$$

where $P = \left( r \partial_r \tau - \frac{\partial_R}{R} \right) \cos(\tau - \theta) - \left( \partial_\theta \tau - 2 + \frac{r \partial_R}{R} \right) \sin(\tau - \theta)$. Clearly, under the conditions of the theorem, $P$ is bounded on $A(r, \rho_1)$. From (15) we now have

$$b = \frac{1}{2e} \left[ \cos \tau(r, \pi) - \cos \tau(r, 0) \right] - [P \sin \theta]_{\theta=0}$$

$$= \frac{1}{2e} \left[ \cos \tau(r, \pi) - \cos \tau(r, 0) \right].$$

Finally, from equations (12) and (13) we obtain the magnetic charge in the limit $r \to 0$

$$b_0 = \frac{-\varepsilon_1}{2e} = \pm \frac{1}{2e}.$$

**Corollary 2.** Suppose we have an axially symmetric, isolated, stationary, static Maxwell-Dirac system with the only external sources being $N$ isolated electrically charged magnetic monopoles. Let the conditions of theorem 2 apply in the $N$ balls $B_\rho_i$, containing the charges. Let the conditions of theorem 1 apply on $E_\rho$. Then, if $N$ is even there are $N/2$ positive charges and $N/2$ negative charges (with corresponding monopoles), with the total magnetic charge of the system being zero. If $N$ is odd there are $(N - 1)/2$ charges with one sign and $(N + 1)/2$ charges with the opposite sign, with the total magnetic charge of the system being $\pm 1/(2e)$.

**Proof.** We have $N$ charged monopoles each in a ball $B_\rho_i$, $i = 1, 2, 3 \ldots N$, there are no other external sources and all the $B_\rho_i$ are properly contained in $B_\rho$. We have $A$ is $O(1/r)$ so that $P$, as defined in the above proof, is bounded on $E_\rho$. Using the results in the proof of theorem 2 and the divergence theorem we find the total magnetic charge of the system.

$$b_{\text{total}} = \lim_{r \to \infty} \frac{1}{4\pi} \int_{S_r} (\nabla \times A) \cdot dS = \frac{1}{2e} \lim_{r \to \infty} \left[ \frac{\cos \tau(r, \pi) - \cos \tau(r, 0)}{2} \right]$$

$$= -\frac{\varepsilon}{2e} \sum_{i=1}^{N} \varepsilon_i.$$

This gives,

$$1 \geq \varepsilon \sum_{i=1}^{N} \varepsilon_i \geq -1,$$

from which the results of the corollary follow.

**VII. Conclusions**

In classical physics one expects stationary or static systems to be the end point of some time evolution. This clearly cannot be the case for a single isolated electron modelled by the Maxwell-Dirac system. To construct such a model we will have to abandon one, or both, of the static and stationary assumptions. The electron must be a dynamic object. The same also applies to other stable objects such as the Hydrogen atom. A Maxwell-Dirac model of such objects must be non-static or non-stationary, or both.
Theorem 1 is remarkable in many ways. No matter what arrangement of external electric and magnetic fields inside the ball \( B_\rho \), no matter what we do to the topology in \( B_\rho \) the total electric charge of the system must be zero. The total charge vanishes purely as a result of the asymptotic decay and regularity conditions.

References

[1] B. Thaller, *The Dirac Equation*, Springer-Verlag Texts and Monographs in Physics, 1992.
[2] L. Gross, *The Cauchy Problem for the Coupled Maxwell-Dirac Equations*, Comm. Pure Appl. Math. 19 (1966), pp. 1-5.
[3] J. Chadam, *Global Solutions of the Cauchy Problem for the (Classical) Coupled Maxwell-Dirac System in One Space Dimension*, J. Funct. Anal. 13 (1973), pp. 495-507.
[4] V. Georgiev, *Small Amplitude Solutions of the Maxwell-Dirac Equations*, Indiana Univ. Math. J. 40(3) (1991), pp. 845-883.
[5] M. Esteban, V. Georgiev, E. Séré, *Stationary Solutions of the Maxwell-Dirac and Klein-Gordon-Dirac Equations*, Calc. Var. 4 (1996), pp. 265-281.
[6] N. Bournaveas, *Local Existence for the Maxwell-Dirac Equations in Three Space Dimensions*, Comm. Part. Diff. Eq. 21(5& 6) (1996), pp. 693-720.
[7] M. Flato, J. C. H. Simon, E. Taffin, *Asymptotic Completeness, Global Existence and the Infrared Problem for the Maxwell-Dirac Equations*, Memoirs of the AMS 127(606) (1997).
[8] A. Das, *General Solutions of the Maxwell-Dirac Equations in 1+1 Dimensional Spacetime and a Spatially Confined Solution*, J. Math. Phys. 34(10) (1993), pp. 3986-3999.
[9] C. J. Radford, *Localised Solutions of the Dirac-Maxwell Equations*, J. Math. Phys. 37(9) (1996), pp. 4418-4433.
[10] H. S. Booth and C. J. Radford, *The Dirac-Maxwell Equations with Cylindrical Symmetry*, J. Math. Phys. 38(3) (1997), pp. 1257-1268.
[11] F. Finster, J. Smoller, and S-T. Yau, *Particle-Like Solutions of the Einstein-Dirac-Maxwell Equations*, preprint gr-qc 9802012 (1998).
[12] R. Bartnik, *The Mass of an Asymptotically Flat Manifold*, Comm. Pure and Appl. Math. 94 (1986), pp. 661-693.
[13] Y. Choquet-Bruhat and D. Christodoulou, *Elliptic Systems in \( H^{s, \delta} \) Spaces on Manifolds which are Euclidean at Infinity*, Acta Math. 146 (1981), pp. 129-150.
[14] J. Chabrowski, *The Dirichlet Problem with \( L^2 \)-Boundary Data for Elliptic Linear Equations*, Springer-Verlag Lecture Notes in Mathematics 1482 (1991).
[15] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press (1993).
[16] R. Penrose and W. Rindler, *Spinors and Space-Time* Vol. 1 and 2, Cambridge Monographs in Mathematical Physics (Cambridge U. P., Cambridge) (1992).
[17] H. S. Booth, *The Static Maxwell-Dirac Equations*, PhD Thesis, University of New England (Australia), (1998).
[18] C. J. Radford, *An ODE Solution of the Static Maxwell-Dirac Equations*, to be published.
[19] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of the Second Order*, Springer-Verlag Grundlehren der mathematischen Wissenschaften 224 (1977).
[20] E. H. Lieb and M. Loss, *Analysis*, AMS Graduate Studies in Mathematics 14 (1997).

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