A $q$-analogue of the embedding chain $U(6) \supset G \supset S0(3)$

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Abstract

A $q$-analogue of the embedding chains of the Arima-Iachello model is proposed. The generators of the deformed $U(6)$ subalgebras are written in terms of the generators of $gl_q(6)$, using $q$-bosons.
1 Introduction

Since their introduction [1], [2], the quantum algebras $G_q$ or $U_q(G)$, i.e. the $q$-deformed universal enveloping algebra of a semi-simple Lie algebra $G$ have been a topic of active research both in physics and mathematics. The underlying idea in some of their applications is to use a $q$-deformed algebra instead of a Lie algebra to realize a generalized dynamical symmetry. For a review of the applications and methods of the dynamical or spectrum generating algebras in physics, see [3], and, in nuclear physics, [4]. The key idea of dynamical symmetry scheme is to write the Hamiltonian of a physical system as a sum of invariants, usually second order Casimir $C$, with constants to be determined by experimental data, of the embedding chains of algebras of the type:

$$G \supset L \supset \ldots \supset SO(3)$$  \hfill (1)

$$\mathcal{H} = C(G) + C(L) \ldots + C(SO(3))$$  \hfill (2)

where $SO(3)$ describes the angular momentum and, usually, the Casimir operators are written using Jordan-Schwinger like realization of the algebra $G$ by means of bosonic creation-annihilation operators. The idea of dynamical symmetry has countless applications in molecular, atomic, nuclear, hadronic and chemical physics. The most simple example is the rigid rotator where the Hamiltonian is written as the Casimir operator of $SU(2)$

$$C = k J(J + 1)$$  \hfill (3)

The energy spectrum of eq.(3) is of the form

$$E_j = k j(j + 1)$$  \hfill (4)

The replacement in eq.(3) of the Casimir of $SU(2)$ by the Gasimir operator of $sl_q(2)$ [5] provides the first example of application of deformed algebra as dynamical symmetry (see below for notation)

$$C = K [J]_q [J + 1]_q$$  \hfill (5)

Now the energy spectrum will depend on a parameter $q = \exp i \tau$ and we have

$$E_j^q = K [j]_q (j + 1)_q = K \frac{\sin(|\tau|j) \sin(|\tau(j + 1)|)}{\sin^2(|\tau j|)}$$  \hfill (6)
Eq.(6) fits experimental data for several deformed nuclei, [5], better than eq.(4) ($|\tau| = 0$). The results of this simple model suggests that it may be worthwhile to further investigate the idea of generalized dynamical symmetry based on $q$-algebras. Indeed in this last decade many applications, mainly, but not uniquely, in molecular and nuclear physics have been investigated. For an excellent review of the subject with an exhaustive list of references see [6]. The simplest, not trivial, embedding chain is the so called Elliot model:

$$SU(3) \supset SO(3)$$  \hspace{1cm} (7)

The deformation of the simple embedding chain of eq.(7) is not trivial. Indeed in [7] it has been shown that the generators of $so_q(3)$ can be expressed by means of the generators of $gl_q(3)$, not of $sl_q(3)$, iff the algebraic relations are restricted to the symmetric representations. Moreover the coproduct of $gl_q(3)$ does not induce the standard coproduct on $so_q(3)$. It is useful to emphasize that the definition of the coproduct is essential to define the tensor product of spaces. The $q$-analogue of the chain eq.(7) has been widely studied. In [8] the author has proposed a possible solution, but the problem has been tackled from several points of view, see [8] for references to the different solutions and for physical applications of the $q$-analogue of the embedding chain eq.(7). Therefore it is clear that an essential step to carry forward the program of application of $q$-algebras as generalized dynamical symmetry, beyond the simple models above discussed, is to dispose of a formalism which allows to build up analogous chains of eq.(1) replacing the Lie algebras by the deformed ones. Of course, as we are no more dealing with Lie algebras, the term embedding has to be intended in the loose sense that the generators of the embedded deformed subalgebra are expressed in terms of the generators of the algebra while the Hopf structure can be or inherited from that of the embedding algebra or imposed on the generators of the embedded algebra. The root of the problem, as it has been discussed in [9], lies on the fact that $G_q$ are well defined only in the Cartan-Chevalley basis and this basis is not suitable to discuss embedding of subalgebras except the regular ones. The classification of so called singular subalgebras of Lie algebras has been started by Dynkin and we refer to the clear paper of Gruber and Lorente [10], where the embedding matrices are explicitly computed for low rank algebras. In particular in [9] it has been shown that, in the case where the rank of $L$, maximal singular algebra of $G$, is equal to the rank of $G$ minus one, it is possible, using realization of $G_q$ in terms of $q$-bosons and/or in terms of the so called $q$-fermions, to write the
Cartan-Chevalley generators of $L_q$ in terms of the generators of $G_q$. Let us remark that this result is not at all a priori obvious due to the non linear structure of $G_q$. It has also been discussed what kind of deformed $G$ is obtained if the standard coproduct is imposed on the generators of $L_q$ in the standard way instead of being derived from that of $G_q$. The aim of this paper is to focus on the embedding chains appearing in the Arima-Iachello model [11], [12] and to discuss in which sense one can write analogous chains of $q$-algebras. This very successful model is based on the following three embedding chains

$$ SU(6) \supset \begin{cases} SU(5) \supset SO(5) \supset SO(3) & \text{(vibrational)} \\ SU(3) \supset SO(3) & \text{(rotational)} \\ SO(6) \supset SO(5) \supset SO(3) & \text{(\(\gamma\)-unstable)} \end{cases} \tag{8} $$

A partial answer to the question of finding a $q$-analogue of the above embedding chains has been given in [13], where it has been shown that, using $q$-bosons realization, the deformed maximal $SU(6)$ subalgebras, i.e. $sl_q(3)$, $so_q(6)$, can be written in terms of the generators of $gl_q(6)$ and that this procedure can be extended also to the deformation of $SO(5)$, maximal subalgebra of $SU(5) \subset SU(6)$. It should be mentioned that an attempt of writing a deformed version of the Arima-Iachello model has been carried out in [14], using the notion of complementary subalgebras introduced about thirty years ago by Moshinsky-Quesne [15]. Two subalgebra $L_1$ and $L_2$ of an algebra $G$ are complementary in one definite irrep. of $G$, if there is a correspondence one-to-one between the irreps. of $L_1$ and $L_2$ contained in the considered irrep. of $G$. Using this notion in [14] a hamiltonian has been written in terms of the second order Casimir of $su_q^{sd}(1, 1)$, $su_q^{d}(1, 1)$ and $su_q(2)$, where $s, d$ are boson operator and the second $su(1, 1)$ is contained in the first one. This hamiltonian, in the limit $q \to 1$, for particular value of the coefficients of the Casimir operators, tends to a hamiltonian in the $\gamma$-unstable chain. Although this approach is interesting, it should be pointed out that the content and the embedding of subalgebras is not well defined. Rather than a deformation of the embedding chain eq.(8), it looks like an interesting dynamical model based on $q$-deformed algebras. Our aim is to build up the whole algebraic construction of a $q$-analogue of the chains of eq.(8). From the construction it should be possible to write hamiltonians which in the limit $q \to 1$ tend to hamiltonians of the undeformed model, for any value of the coefficients of the Casimir operators. Indeed we shall show that, replacing $U(6)$ by $gl_q(6)$, we can write the generators of any $U(6)$ subalgebra in terms of the generators of $gl_q(6)$. To make the paper self-contained, in Sec. 2 we briefly recall the formulas we shall
use in the following. In Sec. 3 we write explicitly the $q$-analogue of the embedding chains eq. (8). In Sec. 4 we summarize and discuss our results.

2 Reminder

We recall, also to fix the notation, the definition of deformed Lie algebra $G_q$ in the Cartan-Chevalley basis, of $sl_q(2)$ $q$-tensor operators, the definition of $q$-bosons, which we shall use to write explicit realization of the $q$-algebras, the Curtright-Zachos deforming functional, connecting $SU(2)$ and $sl_q(2)$ and the deforming map between $Sp(4) \equiv SO(5)$ and $sp_q(4) \equiv so_q(5)$.

2.1 Deformed algebras

Let us recall the definition of $G_q$ associated with a simple Lie algebra $G$ of rank $r$ defined by the Cartan matrix $(a_{ij})$ in the Chevalley basis. $G_q$ is generated by $3r$ elements $e_i^\pm$ and $h_i$ which satisfy ($i, j = 1 = 1, \ldots, r$)

$$[e_i^+, e_j^-] = \delta_{ij}[h_i]_{q_i} \quad \quad [h_i, h_j] = 0$$

$$[h_i, e_j^+] = a_{ij}e_j^+ \quad \quad [h_i, e_j^-] = -a_{ij}e_j^-$$

(9)

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$$

(10)

and $q_i = q^{d_i}$, $d_i$ being non-zero integers with greatest common divisor equal to one such that $d_i a_{ij} = d_j a_{ji}$. For simple laced algebras $d_i = 1$ while for $so_q(2n+1)$ ($sp_q(2n)$) $d_i = 2 \ (1)$, $i \neq n$, $d_n = 1 \ (2)$. Further the generators have to satisfy the Serre relations:

$$\sum_{0 \leq n \leq 1 - a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right]_{q_i} (e_i^+)^{1-a_{ij}-n} e_j^+ (e_i^+)^n = 0$$

(11)

where

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m-n]_q! [n]_q}$$

(12)

$$[n]_q! = [1]_q [2]_q \ldots [n]_q$$

Analogous equations hold replacing $e_i^+$ by $e_i^-$. In the following we assume $h_i = (h_i)^\dagger$ and the deformation parameter $q$ to be different from the roots of the unity. The algebra $G_q$ is endowed with a Hopf algebra structure, i.e. on $G_q$ the action
of the coproduct $\Delta$, antipode $S$ and co-unit $\varepsilon$ is defined. This extremely relevant aspect will not discussed here. We recall only the definition of the coproduct which we shall briefly refer to in the following.

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i$$

$$\Delta(e_i^\pm) = e_i^\pm \otimes q_i^{h_i/2} + q_i^{-h_i/2} \otimes e_i^\pm$$

As the coproduct in a Hopf algebra satisfies $(g_i, g_j \in G_q)$

$$\Delta(g_i g_j) = \Delta(g_i) \Delta(g_j)$$

it is essential to define which elements $\{g_i\}$ are the “basis” of $G_q$.

Let us recall the definition of $gl(n)_q$, $(i, j = 1, 2, \ldots, n-1, k = 1, 2, \ldots, n-1, n)$:

$$[e_i^+, e_j^-] = \delta_{ij}[n_i - n_{i+1}]_q \quad [n_i, n_k] = 0$$

$$[n_k, e_i^\pm] = \pm(\delta_{k,j} - \delta_{k-1,j})$$

The Serre relations are computed using $a_{k,j} = - (\delta_{k-1,j} + \delta_{k,j-1})$. So $gl(n)_q$ can be considered as $sl(n)_q \oplus n_n$ with $h_i = n_i - n_{i+1}$.

2.2 $q$-Bosons

Let us recall the definition of Biedenharn-MacFarlane $q$-bosons [16], [17] which we denote by $b_i^+, b_i$

$$b_i b_j^+ - q^{\delta_{ij}} b_j^+ b_i = \delta_{ij} q^{-N_i}$$

$$[N_i, b_j^+] = \delta_{ij} b_j^+ \quad [N_i, b_j] = -\delta_{ij} b_j \quad [N_i, N_j] = 0$$

It is useful to keep in mind the following identities:

$$b_i^+ b_i = \frac{q^{N_i} - q^{-N_i}}{q - q^{-1}} \quad b_i b_j^+ = \frac{q^{N_i+1} - q^{-N_i-1}}{q - q^{-1}}$$

$$b_i^+ b_k = [b_i^+ b_j, b_j^+ b_k]_q q^{N_j}$$

where the $q$-commutator is defined as

$$[A, B]_q = AB - qBA$$

The explicit construction of $q$-bosons in terms of non-deformed standard bosonic oscillators ($\tilde{b}_i^+, \tilde{b}_i$) is [18]

$$b_i^+ = \sqrt{\frac{[N_i]_q}{N_i}} \tilde{b}_i^+ \quad b_i = \tilde{b}_i \sqrt{\frac{[N_i]_q}{N_i}}$$
From the above equation, as it is well known that the deformed algebras $su_q(n)$ and $sp_q(2n)$ can be written in terms of bilinears of $q$-bosons, it follows that one can write the following correspondences $su_q(n) \subset gl_q(n) \Longleftrightarrow gl(n) \supset sl(n)$ and $sp_q(2n) \Longleftrightarrow sp(2n)$, if each generator of algebras and $q$-algebras is written as a bilinear of bosons and, respectively, of $q$-bosons. Remark that, if $\tilde{b}_i^+$ is the adjoint of $\tilde{b}_i$, then $\tilde{b}_i^+$ is the adjoint of $\tilde{b}_i$ iff $q$ is real or $q = \exp i\tau$, $\tau$ real.

### 2.3 $q$-bosons realizations of $sl_q(2)$ and $so_q(3)$

In order to clarify what we mean by $sl_q(2)$ and $so_q(3)$, let us write explicitly the $q$-boson realization of $sl_q(2)$

$$J_+ = b_1^+ b_2 \quad J_- = b_2^+ b_1 \quad 2J_0 = N_1 - N_2$$

the states of the irreducible representation $(j, m)$ in the corresponding Fock space are

$$\psi_{jm} = \frac{(b_1^+)^j m (b_2^+)^{j-m}}{\sqrt{|j + m|_q! |j - m|_q!}} \psi_0$$

and of $so_q(3)$

$$L_+ = q^{N-1} q^{-N_0/2} \sqrt{q^{N_1} + q^{-N_1}} b_0^+ b_1$$

$$L_- = q^N b_1 q^{N_1} q^{-N_0/2} \sqrt{q^{N_1} + q^{-N_1}}$$

$$L_0 = N_1 - N_1$$

the states of the odd dimensional irreducible representation $(L, m = n_1 - n_{-1})$ $(L = \max\{n_1\} = n_1 + n_0 + n_{-1})$ in the corresponding Fock space are linear combinations of

$$\psi_{n_1,n_0,n_{-1}} = \frac{(b_1^+)^{n_1} (b_0^+)^{n_0} (b_1^+)^{n_{-1}}}{\sqrt{|n_1|_q! |n_0|_q! |n_{-1}|_q!}} \psi_0$$

### 2.4 $q$-Tensor operator

From the formula of coproduct eq.(13) Biedenharm and Tarlini [20], see also [21] and [22], have derived the general structure of $q$-tensor operators for $sl_q(2)$.

$$\left[ J_\pm, T^k_m(q) \right] = q^{-J_0} \sqrt{|k + m|_q [k \pm m + 1]_q} T^{k}_{m\pm1}(q)$$

$$\left[ J_0, T^k_m(q) \right] = m T^k_m(q)$$

(26)
The Wigner-Eckart theorem for $sl_q(2)$ reads:

$$<JM|T^k_m(q)|j_1 m_1> = (-1)^{2k} [2J + 1]^{-1/2}_q <JM|km,j_1 m_1>_q$$

$$\times <J||T^k||j_1>$$

(27)

where $<JM|km,j_1 m_1>_q$ is the q-Clebsch-Gordan coefficient, see [21]

2.5 Deforming map between $sl(2)$ and $sl_q(2)$

In [23] an invertible deforming functional $Q_\pm$ has been introduced which allows to relate $sl(2) \leftrightarrow sl_q(2)$. Denoting by small (resp. capital) letter $j_{\pm,0}$ ($J_{\pm,0}$) the generator of $sl(2)$ ($sl_q(2)$) it is possible to write ($q$ real)

$$J_+ = Q_+(j_{\pm},j_0) j_+ \quad J_- = Q_-(j_{\pm},j_0) j_- \quad J_0 = j_0$$

(28)

where

$$Q_+ = \sqrt{\left\{ [J_0 + J][J_0 - J - 1]_q \right\} \left\{ (j_0 + j)(j_0 - j - 1) \right\}}$$

(29)

$$(Q_- = Q_+^\dagger)$$ and the operator $j$ ($J$) is defined by the Casimir operator of $sl(2)$ ($sl_q(2)$)

$$C = j(j + 1) \quad (C_q = [J]_q [J + 1]_q)$$

(30)

2.6 Deforming map between $SO(5)$ and $so_q(5)$

In [24], in the space of the symmetric irreps., invertible deforming maps have been derived which allow to express $sl_q(n)$ and $sp_q(2n)$ respectively in terms of $U(n)$ and $Sp(2n)$. As irreducible representations in the Fock space of bosons or $q$-bosons are symmetric and $SO(5) \equiv Sp(4)$, we report here explicitly the map $sp_q(4) \leftrightarrow Sp(4)$ which we shall use in the following. Denoting by small (capital) letters the generators of deformed (undeformed) algebra, we have

$$e_1^+ = E_1^+ \sqrt{\frac{[H_1 + H_2 + 1]_q [H_2]_q}{(H_1 + H_2 + 1) H_2}}$$

$$e_2^+ = \frac{2}{q+q} E_2^+ \sqrt{\frac{[H_2+1]_q [-H_2-2]_q}{(H_2+1)(-H_2-2)}}$$

$$h_k = H_k \quad k = 1,2$$

(31)

(32)
3  \( q \)-Embedding

In this Section we discuss in detail the meaning of the \( q \)-embedding chain, \( L \) being a maximal singular subalgebra of \( G \)

\[
G_q \supset L_q \supset so_q(3)
\]

(33)

for the case where \( G_q = gl_q(6) \). In the following we denote by small (resp. capital) letter \( e^\pm, h \) (\( E^\pm, H \)) the generators of \( gl_q(6) \) (\( L_q \)) and by \( L_{\pm,0} \) the generators of \( so_q(3) \). With a hat on \( E^\pm, H \) we denote, when required, the generators of a maximal subalgebra of \( L \). Let us recall once again that eq.(33) holds if the generators of \( L_q \) can be expressed in terms of the generators of \( G_q \), at least in a particular realization of \( G_q \), in the present work using a \( q \)-boson realization. In the following we assume \( q \) real, therefore any generator \( X^- \) is the adjoint of \( X^+ \) and the generators of the Cartan subalgebra are self-adjoint. We shall comment in the Conclusions on the more general case.

Let us recall the \( q \)-boson realization of \( sl_q(6) \subset gl_q(6) \) \((i = 1, 2, 3, 4, 5)\)

\[
e_i^+ = b_i^+b_{i+1} \quad h_i = N_i - N_{i+1}
\]

(34)

To get \( gl_q(6) \) one has to add to the previous generators

\[
h_0 = \sum_{i=1}^5 h_i = \sum_{j=1}^6 N_j
\]

(35)

In the following we write explicitly the \( q \)-analogue of the embedding chains of the Arima-Iachello mode. The notation is self-explanatory. We use a vertical arrow in the equations to point out what the embedding chains tend to in the limit \( q \to 0 \). In the following, to save space in some equations we shall denote by \( e_i^\pm \) the generators of \( sl_q(6) \), whose content in \( q \)-bosons is given in eq.(34). Let us remind also that the Cartan generators of the deformed and undeformed algebras are the same.

3.1  \( q \)-analogue of the vibrational embedding chain

\[
gl_q(6) \supset gl_q(5) \supset so_q(5) \supset so_q(3)
\]

(36)

Clearly the generators of \( gl_q(5) \) are obtained from those of \( gl_q(6) \) neglecting \( e_5^\pm, h_5 \) and \( N_6 \) and the \( q \)-boson realization of \( so_q(5) \subset gl_q(5) \subset gl_q(6) \) is \[13\]. Let us remind also that the Cartan generators of the deformed and undeformed algebras are the same.

\[
E_1^\dagger = \left\{ q^{N_1} + q^{-N_1} b_1^+b_2 \sqrt{q^{N_2} + q^{-N_2} q^{-(N_4-N_5)}} \right\}
\]
the generators of $so_q(3)$ can be written in terms of the generators of $gl_q(5)$

$$
\begin{align*}
L_+ &= Q_+ \left\{ 2 \left[ \sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} \right] + \sqrt{6} \left[ \sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} + \sqrt{\frac{N_3}{[N_3]_q}} e_3^+ \sqrt{\frac{N_4}{[N_4]_q}} \right] \right\} \\
L_0 &= 2N_1 + N_2 - N_4 - 2N_5 = 2H_1 + \frac{3}{2}H_2
\end{align*}
$$

where $H_1, H_2$ of Cartan generators of $so_q(5)$ and we have used eqs.(34)-(21) and the deforming map eq.(28).

### 3.2 $q$-analogue of the rotational embedding chain

$$
gl_q(6) \supset gl_q(3) \\
\supseteq^{q+1} so_q(3)
$$

the $q$-boson realization of $sl_q(3) \subset gl_q(6)$,

$$
\begin{align*}
E_1^\dagger &= \left\{ q^{N_1-N_2/2} \sqrt{q^{N_1} + q^{-N_1}} b_1^+ b_2 + b_2^+ b_4 q^{N_1-N_2}/2 \sqrt{q^{N_4} + q^{-N_4}} \right\} q^{-(N_3-N_5)/2} \\
&+ b_3^+ b_5 q^{(N_1-N_4)} \\
E_2^\dagger &= \left\{ q^{N_5-N_6/2} \sqrt{q^{N_4} + q^{-N_4}} b_4^+ b_5 + b_5^+ b_6 q^{N_4-N_5}/2 \sqrt{q^{N_6} + q^{-N_6}} \right\} q^{(N_2-N_3)/2} \\
&+ b_2^+ b_3 q^{(N_4-N_6)} \\
H_1 &= 2N_1 - 2N_4 + N_3 - N_5 \\
H_2 &= N_2 - N_3 + 2N_4 - 2N_6
\end{align*}
$$

the generators of $so_q(3)$ can be written in terms of the generators of $gl_q(6)$

$$
\begin{align*}
L_+ &= Q_+ \left\{ 2 \left[ \sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} \right] + \sqrt{2} \left[ \sqrt{\frac{N_2}{[N_2]_q}} e_2^+ \sqrt{\frac{N_3}{[N_3]_q}} + \sqrt{\frac{N_3}{[N_3]_q}} e_3^+ \sqrt{\frac{N_4}{[N_4]_q}} \right] + 2 \left[ \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} + \sqrt{\frac{N_5}{[N_5]_q}} e_5^+ \sqrt{\frac{N_6}{[N_6]_q}} \right] \right\} \\
L_0 &= 2N_1 + N_2 - N_5 - 2N_6 = H_1 + H_2
\end{align*}
$$

where $H_1, H_2$ of Cartan generators of $sl_q(3)$
3.3 \( q \)-analog of the \( \gamma \)-unstable embedding chain

\[ gl_q(6) \supset so_q(6) \supset \hat{gl}_{q+1} so_q(5) \supset \hat{gl}_{q+1} so_q(3) \] (42)

the \( q \)-boson realization of \( so_q(6) \subset sl_q(6) \) \[^{[32]}\]

\[ E_1^\dagger = b_2^+ b_4 q^{(N_3-N_5)/2} + b_3^+ b_5 q^{-(N_2-N_4)/2} \]
\[ E_2^\dagger = b_1^+ b_2 q^{(N_5-N_6)/2} + b_6^+ b_0 q^{-(N_1-N_2)/2} \]
\[ E_3^\dagger = b_2^+ b_3 q^{(N_4-N_5)/2} + b_4^+ b_5 q^{-(N_2-N_3)/2} \]
\[ H_1 = N_2 - N_4 + N_3 - N_5 \]
\[ H_2 = N_1 - N_2 + N_5 - N_6 \]
\[ H_3 = N_2 - N_3 + N_4 - N_5 \] (43)

the \( q \)-boson realization of \( so_q(5) \), deformation of \( SO(5) \) maximal subalgebra of \( SO(6) \subset SU(6) \), using eq. (31), can be written

\[
E_1^\dagger = \left( \sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \right. \sqrt{\frac{N_2}{[N_2]_q}} \left. + \sqrt{\frac{N_3}{[N_3]_q}} e_3^+ \right) \sqrt{\frac{N_4}{[N_4]_q}} \sqrt{\frac{N_5}{[N_5]_q}} \sqrt{\frac{N_6}{[N_6]_q}} \frac{(H_1 + H_2 + 1) [H_2]_q}{(H_1 + H_2 + 1) H_2}
\]
\[
E_2^\dagger = \left( \sqrt{\frac{N_2}{[N_2]_q}} [e_2^+, e_3^+] \sqrt{\frac{N_3}{[N_3]_q}} q^{N_3} + \sqrt{\frac{N_4}{[N_4]_q}} [e_3^+, e_4^+] \sqrt{\frac{N_5}{[N_5]_q}} q^{N_4} \right.
\]
\[
\left. \left. + \sqrt{\frac{N_5}{[N_5]_q}} e_2^+ \sqrt{\frac{N_4}{[N_4]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} \right) \times q^{2+q^{-1}} \sqrt{\frac{(H_2 + 1) [-H_2 - 2]_q}{(H_2 + 1) (-H_2 - 2)}} \right) \] (44)

\[ \hat{H}_1 = N_1 + N_5 - N_2 - N_6 \quad \hat{H}_2 = 2(N_2 - N_5) \] (45)

where \( \hat{H}_1 \), \( \hat{H}_2 \) are the Cartan generators of \( so_q(5) \) and the generators of \( so_q(3) \) can be written in terms of the generators of \( gl_q(6) \)

\[ L_+ = Q_+ \left\{ \sqrt{2} \left[ \sqrt{\frac{N_1}{[N_1]_q}} e_1^+ \sqrt{\frac{N_2}{[N_2]_q}} + \sqrt{\frac{N_3}{[N_3]_q}} e_3^+ \sqrt{\frac{N_4}{[N_4]_q}} \right. \right. \]
\[ \left. \left. + \sqrt{\frac{3}{2}} \left[ \sqrt{\frac{N_2}{[N_2]_q}} [e_2^+, e_3^+] \sqrt{\frac{N_3}{[N_3]_q}} q^{N_3} + \sqrt{\frac{N_4}{[N_4]_q}} [e_3^+, e_4^+] \sqrt{\frac{N_5}{[N_5]_q}} q^{N_4} \right. \right. \]
\[ \left. \left. \left. + \sqrt{\frac{N_5}{[N_5]_q}} e_2^+ \sqrt{\frac{N_4}{[N_4]_q}} + \sqrt{\frac{N_4}{[N_4]_q}} e_4^+ \sqrt{\frac{N_5}{[N_5]_q}} \right) \right\} \]
\[ L_0 = 2N_1 + N_2 - N_5 - 2N_6 = \frac{3}{2}(H_1 + H_3) + 2H_2 \] (46)

where \( H_1, H_2, H_3 \) of Cartan generators of \( so_q(6) \). We recall that

\[ \hat{H}_1 = H_2 \quad \hat{H}_2 = H_1 - H_3 \] (47)
4 Conclusions

Starting from a spectrum generating algebra

\[ U(6) \supset L \supset SO(3) \quad (48) \]

We have shown that, using \( q \)-bosons, a deformed analogue of this chain can be obtained replacing \( U(6) \) by \( gl_q(6) \) and writing the generators of \( L_q \) (\( L = SO(6), SO(5), SU(3), SO(5) \subset SO(6) \)) and of \( so_q(3) \) in terms of the generators of \( gl_q(6) \) (or of \( gl_q(5) \) for the vibrational chain) with coefficients taking value in \( gl_q(6) \) (or \( gl_q(5) \)).

In order to write our results we have used the relation between standard bosonic operators and \( q \)-bosons eq.\((21)\), the deforming map between \( su(2) \) and \( su_q(2) \) eq.\((28)\) and between \( sp(4) \equiv so(5) \) and \( sp_q(4) \equiv so_q(5) \) eq.\((31)\). Remark that in the last step of the embedding chain one can keep undeformed \( SO(3) \) and the generators \( l_\pm \) of the undeformed algebra of the angular momentum can be written, e.g., as

\[ l_+ = \sum_k A_k^+ (q, \{ e_i^+, h_i \}) e_k^+ \quad (49) \]

where \( A_k^+ (q, \{ e_i^+, h_i \}) \) are operators taking value in \( gl_q(6) \) (or \( gl_q(5) \)). Let us point out that the problem in a deformation of the subalgebra of a subalgebra in an embedding chain of the type, e.g.,

\[ gl_q(6) \supset L_q \supset so_q(3) \quad (50) \]

is that the generators \( L_\pm \) cannot be written in terms of the generators of \( L_q \), but only in terms of the generators of \( gl_q(6) \). This peculiar feature is common to any deformed algebra when we consider embedding chain with deformed singular subalgebra \( L \subset G \). In conclusion we have shown that it is possible to write the Cartan-Chevalley generators of \( so_q(6) \) and \( sl_q(3) \) in terms of the generators of \( gl_q(6) \) and those of \( so_q(5) \) in terms of \( gl_q(5) \), but that it is not possible to extend further the procedure to obtain, in particular, \( so_q(3) \), as it was argued in \([9]\) and \([13]\). However for the considred chains eq.\((5)\) we can go a step further writing the generators in terms of those of the \textit{grand-mother} \( gl_q(6) \). Even if we have considered a particular case, so our results are not quite general, the used procedure is enough general to be applied successfully to other physically relevant models, even taking into account supersymmetric extensions. In the spirit of the use of spectrum generating algebra, one should write a hamiltonian of a physical system as a sum of invariants of the \( q \)-algebras appearing in the embedding chain of the previous Section. This can be
done as the Casimir operators of $sl_q(n)$ [23], [26] and of $so_q(5)$ [27], [28] are known. For physical application one needs self-adjoint Casimir. For $q$ real this property is guaranteed in the realization we have used. In many applications of $q$-algebras, however, it seems that $q$ being a phase is a favoured value, see [3]. For this value, due to the lack of invariance $q \leftrightarrow q^{-1}$ in our expressions, the product $E^+ E^-$ is no more self-adjoint. In order to restore the self-adjointness of the hamiltonian one has to sum the Casimir operators of $L_q (so_q(3))$ and of $L_{q^{-1}} (so_{q^{-1}}(3))$. As a final comment, if one considers transitions in the physical system induced by $q$-tensor operator under $so_q(3)$ or, if the rotation group in physical space is left undeformed, by tensor operator under $SO(3)$, one has to face the problem of the choice of the coproduct. In order to have the usual structure of $q$-tensor eq.(26) or tensor operators, eq.(28) in the limit $q \to 1$, the coproduct $\Delta$ has to be imposed in $so_q(3)$ or $SO(3)$ and cannot be inherited by that of $G_q$ or $L_q$.

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References

[1] M. Jimbo, ”A $q$-difference analogue of $U(q)$ and the Yang-Baxter equation”, Lett. Math. Phys. 10, 63 (1985)

[2] V.G. Drinfeld, ”Quantum Groups”, in Proc.Int.Congr. of Math., MSRI Berkely, California (1986)

[3] Spectrum-Generating Algebras and Dynamic Symmetries in Physics, Proc. of VII Symposium on Symmetry in Physics, Eds. B. Gruber and T. Otsuka, Plenum Press, New York (1993)

[4] P. Van Isacker, ”Dynamical symmetries in the structure of nuclei” Rep.Prog.Phys. 62, 1661 (1999)

[5] P.P. Raychev, R.P. Roussev and Yu.F. Smirnov”The quantum algebra $SU_q(2)$ and rotational spectra of deformed nuclei “, J. Phys. G 16, L137 (1990)
[6] D. Bonatsos and C. Daskaloyannis, “Quantum Groups and Their Applications in Nuclear Physics Prog.Part.Nucl.Phys. 43, 537 (1999)

[7] J. Van der Jeugt, “On the principal subalgebra of quantum enveloping algebras $gl_q(l + 1)$”, J. Phys. A 25, L213 (1992)

[8] A. Sciarrino, “Deformed $U(Gl(3))$” from $SO_q(3)”$

in Proc. Symmetries in Science VII: Spectrum Generating Algebras and Dynamics in Physics, Eds. B. Gruber and T. Otsuka, Plenum Press, New York (1993)

[9] A. Sciarrino, “Deformation of Lie algebras in a non-Chevalley basis and ”embedding” of $q$-algebra”, J. Phys. A 27, 7403 (1994)

[10] M. Lorente and B. Gruber, “Classification of Semisimple Subalgebras of Simple Lie Algebras ”, J. Math. Phys. 13, 1639 (1972)

[11] A. Arima and F. Iachello “Collective Nuclear States as Representations of a $SU(6)$ Group”, PRL 35, 1069 (1975)

[12] F. Iachello and A. Arima “The Interacting Boson Model” , Cambridge University Press, Cambridge (1987)

[13] A. Naddeo and A. Sciarrino, “Deformation of the ”embedding” of $q$-algebras $A_5 \supset G”$, J. Phys. A 30, 4373 (1997)

[14] Yu-Cheng Wang and Ze-Sen Yang ”$q$-Deformed $SU(1, 1) \otimes SO(6)$ Spectra of Interacting s, d Bosons in Nuclei”, Commun.Theor. Phys. 17, 449 (1992)

[15] M. Moshinsky and C. Quesne, ”Noninvariance Groups in the Second-Quantization Picture and Their Applications”, J. Math. Phys. 11, 1631 (1970)

[16] L.C. Biedenharn,”The quantum group $SU_q(2)$ and a $q$-analogue of the boson operators”, J. Phys. A 22, L873 (1989)

[17] A.J. Macfarlane, ”On $q$-analogues of the quantum harmonic oscillator and the quantum group $SU_q(2)$”, J. Phys. A 22, 4581 (1989)

[18] Xing-Chang Song, ”The construction of the $q$-analogues of the harmonic oscillator operators from ordinary oscillator operators”, J. Phys. A 23, L821 (1990)
[19] T. Hayashi, "q-Analogues of Clifford and Weyl algebras - Soinor and Oscillator Representations of Quantum Enveloping Algebras", Commun. Math. Phys. 127, 129 (1990)

[20] L.C. Biedenharn and M. Tarlini, "On q-Tensor Operator for Quantum Groups", Lett. Math. Phys. 20, 272 (1990)

[21] Yu.F. Smirnov, V.N. Tolstoi and Yu.I. Kharitonov, "Method of projection operators and the q analog of the quantum theory of angular momentum, Clebsch-Gordan coefficients and irreducible tensor operators", Sov. J. Nucl. Phys. 53, 593 (1991)

[22] V. Rittenberg and M. Scheunert, "Tensor Operators for quantum groups and application", J. Math. Phys. 33, 436 (1992)

[23] T.L. Curtright and C.K. Zachos, "Deforming maps for quantum algebras", Phys. Lett. B 243, 237 (1990)

[24] A. Sciarrino, "Deforming maps between sl(n), sp(2n) and U_q(sl(n)), U_q(sp(2n))", Preprint DSF-42/01 [math.QA/0112120]

[25] A. Chakrabarti, "SO(5)_q and contraction: Chevalley basis representations for q-generic and root of unity", J. Math. Phys. 35, 4247 (1994)

[26] A.M. Bincer, "Casimir invariants for su_q(n)", J. Phys. A 24, L1133 (1991)

[27] R.B. Zhang, M.D. Gould and A.J. Bracken "Generalized Gelfand invariants and quantum groups", J. Phys. A 24, 937 (1991)

[28] A. Chakrabarti, "SO(5)_q and contraction: Chevalley basis representations for q-generic and root of unity", J. Math. Phys. 35, 4247 (1994)