ON RIGHT COIDEAL SUBALGEBRAS

V.K. KHARCHENKO

Abstract. Let $H$ be a character Hopf algebra. Every right coideal subalgebra that contains the coradical has a PBW-basis which can be extended up to a PBW-basis of $H$.

1. Introduction

In the present paper we consider right coideal subalgebras in character Hopf algebras, the Hopf algebras generated by skew-primitive semi-invariants. This class includes quantum enveloping algebras of Kac–Moody algebras, all their generalizations (see, G. Benkart, S.-J. Kang, and D. Melville [1]; M. Costantini, and M. Varagnolo [2]; S.-J. Kang [3]), the bosonizations of quantum symmetric algebras related to diagonal braidings, and so on. We prove the following general statement on the structure of the right coideal subalgebras.

Theorem 1.1. Let $H$ be a character Hopf algebra. Every right coideal subalgebra that contains all group-like elements has a PBW-basis which can be extended up to a PBW-basis of $H$.

This theorem is new for Hopf subalgebras too.

2. Preliminaries

2.1. PBW-generators. Let $S$ be an algebra over a field $k$ and $A$ is a subspace of $S$ with a basis $\{a_j \mid j \in J\}$. A linearly ordered subset $V \subseteq S$ is said to be a set of PBW-generators of $S$ over $A$, called the height function, such that the set of all products

\[ a_j v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k}, \]

where $j \in J, \ v_1 < v_2 < \cdots < v_k \in V, \ n_i < h(v_i), 1 \leq i \leq k$ is a basis of $S$. The value $h(v)$ is referred to as the height of $v$ in $V$. If $A = k$ is the ground field, then we shall call $V$ simply as a set of PBW-generators of $S$.

Definition 2.1. Let $V$ be a set of PBW-generators of $S$ over a subalgebra $A$. Suppose that the set of all words in $V$ as a free monoid has its own order $\prec$ (that is, $a \prec b$ implies $cad \prec cdb$ for all words $a, b, c, d$ in $V$).

1) A leading word of $s \in S$ is the maximal word $m = v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k}$ that appears in the decomposition of $s$ in the basis (2.1).

2) A leading term of $s$ is the sum $am$ of all terms $\alpha_i a_i m$ that appear in the decomposition of $s$ in the basis (2.1), where $m$ is the leading word of $s$.

1991 Mathematics Subject Classification. Primary 16W30, 16W35; Secondary 17B37.

Key words and phrases. Hopf algebra, coideal subalgebra, PBW-basis.

The author was supported by PAPIIT IN 108306-3, UNAM, México.
3) The order \( \prec \) is compatible with the PBW-decomposition related to \( V \) if the leading word of each product \( W = w_1 w_2 \cdots w_m, w_j \in V \cup \{a_j\} \) (considered as an element of \( S \)) is less than or equal to the word \( W' \) that appears from \( w_1 w_2 \cdots w_m \) by deletion of all letters \( a_j \).

2.2. Associated graded algebra. Let \( \Gamma \) be a completely ordered additive (commutative) monoid. A \( \Gamma \)-filtration on an algebra \( S \) is a map \( D \) from \( S \) to \( \Gamma \) extended by a symbol \( -\infty \) with the properties

1. \( D(x) = -\infty \iff x = 0 \), \( D(1 \cdot k^*) = 0 \), \( D(x - y) \leq \max\{D(x), D(y)\} \), \( D(xy) \leq D(x) + D(y) \).

The element \( D(x) \) is the degree of \( x \). We denote by \( S_\gamma \) the subspace of all elements of degree \( \leq \gamma \), while by \( S_{\gamma}^- \) the subspace of all elements of degree \( < \gamma \).

Example 2.2. (Filtration by constitution). Suppose that an algebra \( S \) is generated over a subalgebra \( A \) by a set \( X \). That is, there exists a homomorphism \( \xi : A \langle X \rangle \to S \), where \( A \langle X \rangle \) is the free product \( A \ast k \langle X \rangle \). A constitution of a word \( u \) in \( A \cup X \) is a family of non-negative integers \( \{m_x, x \in X\} \) such that \( u \) has \( m_x \) occurrences of \( x \). Certainly almost all \( m_x \) in the constitution are zero. We fix an arbitrary complete order, \( \prec \), on the set \( X \).

Let \( \Gamma \) be the free additive (commutative) monoid generated by \( X \). The monoid \( \Gamma \) is a completely ordered monoid with respect to the following order:

\[
\gamma_1 x_{i_1} + \gamma_2 x_{i_2} + \ldots + \gamma_k x_{i_k} > \gamma'_1 x_{i_1} + \gamma'_2 x_{i_2} + \ldots + \gamma'_k x_{i_k}
\]

if the first from the left nonzero number in \( (\gamma_1 - \gamma'_1, \gamma_2 - \gamma'_2, \ldots, \gamma_k - \gamma'_k) \) is positive, where \( x_{i_1} > x_{i_2} > \ldots > x_{i_k} \) in \( X \). We associate a formal degree \( D(u) = \sum_{x \in X} m_x x \in \Gamma \) to a word \( u \) in \( A \cup X \), where \( \{m_x | x \in X\} \) is the constitution of \( u \). Respectively, if \( f = \sum \alpha_i u_i \in A \langle X \rangle \), \( 0 \neq \alpha_i \in k \) then

\[
D(f) = \max_i \{D(u_i)\}.
\]

On \( S \) we define a \( \Gamma \)-filtration related to \( \xi : A \ast k \langle X \rangle \to S \) as follows:

\[
D(s) = \min \{D(f) | \xi(f) = s\}.
\]

With every \( \Gamma \)-filtered algebra \( S \) a \( \Gamma \)-graded algebra is associated in the obvious way. For each \( \gamma \in \Gamma \), write \( \operatorname{gr}_\gamma S \) for the linear space \( S_\gamma / S^-_\gamma \). Then

\[
\operatorname{gr} S = \bigoplus_{\gamma \in \Gamma} \operatorname{gr}_\gamma S,
\]

with the product defined by

\[
(x + S^-_\gamma)(y + S^-_\delta) = xy + S^-_{\gamma + \delta}.
\]

If the filtration is as in Example 2.2, then the algebra \( \operatorname{gr} S \) is generated by \( \operatorname{gr} X \cong X \) and \( \operatorname{gr} A = A \).

Lemma 2.3. If the associated graded algebra \( \operatorname{gr} S \) for a \( \Gamma \)-filtered algebra \( S \) has a set of homogeneous PBW-generators, \( V = \{v_i\} \), over a subspace \( A \subseteq S_0 \), then each set of representatives \( \hat{v}_i, v_i = \hat{v}_i + S^-_{\langle v_i \rangle} \), is a set of PBW-generators of \( S \) over \( A \).

This lemma has been proved by S. Ufer [7, Proposition 46] for the case \( \Gamma = \mathbb{Z}^+ \). In the general case the proof is quite similar.
Lemma 2.4. Let $U$ be a subalgebra of a $\Gamma$-filtered algebra $S$. If $\text{gr} S$ is a free left (right) $\Gamma$-graded module over $\text{gr} U$ then $S$ is a free left (right) module over $U$.

This is part of the folklore.

2.3. Thin elements and replacement of basis. Suppose that an algebra $S$ has a set $V$ of PBW-generators over a subalgebra $A$, and $1 \in \{a_j\}$. Let $<$ be any complete order of the free monoid of words in $V$ which is compatible with the PBW-decomposition related to $V$. An element $c \in S$ is said to be a thin element if its decomposition in the basis (2.1) has the form

$$c = v^m + \sum a_j W_i,$$

where $W_i < v^m$, and either $m$ divides the height of $v$ in $V$, or $h(v) = \infty$.

Let $T \subseteq S$ be a some set of thin elements. Suppose that for each $v \in V$ there exists at most one element $c_v \in T$ with the leading term of the form $v^m$, $m \geq 1$. One may construct a new set of PBW-generators, $P_T$, related to $T$ in the following way.

If in $T$ there does not exist an element with the leading term of the form $v^m$, $m \geq 1$ we include $v$ in $P_T$ and define the height $h_T(v)$ related to $T$ to be equal to $h(v)$.

If there exists an element $c_v \in T$ with the leading term $v^m$ and $m > 1$, we include in $P_T$ both elements: $v$ and $c_v$. In this case we define the height of $v$ in $P_T$ to be equal to $m$, while the height of $c_v$ related to $P_T$ is the quotient $h(v)/m$.

If there exists an element $c_v \in T$ with the leading term $v$ (that is, if $m = 1$) then we include in $P_T$ just $c_v$ (that accidentally may be equal to $v$). In this case the height of $c_v$ with respect to $T$ by definition equals $h(v)$.

We extend on $P_T$ the order $<$ in the natural way: $c_v < w$ if and only if $(v \leq w \& c_v \neq w)$; and $c_v < c_w$ if and only if $v < w$. In particular $c_v < v$, provided that $m > 1$, where $m$ is defined by $c_v$ in (2.5).

Lemma 2.5. The set $P_T$ is a set of PBW-generators of $S$ over $A$.

Proof. We have to prove that the monotonous restricted words $\theta_i^{v_1^{r_1} v_2^{r_2} \ldots v_k^{r_k}}$ with $v_i \in P_T$ are linearly independent in $S$, and they span $S$.

By definition if $\theta_1 < \theta_2 \in P_T$, $\theta_1 = v_1^{m_1} \ldots \cdot \theta_2 = v_2^{m_2} \ldots$, then either $v_1 < v_2$, or $v_1 = v_2 = v$ with $\theta_1 = c_v = v^m + \ldots$, $\theta_2 = v$. Therefore each monotonous word in $P_T$ has a form

$$\theta_v^{v_1^{r_1} v_2^{r_2} \ldots v_k^{r_k}},$$

where $v_1 < v_2 < \ldots < v_k$, $\theta_i = c_{v_i} = v_i^{m_i} + \ldots$. Of course if $m_i = 1$, then $r_i = 0$, while if $m_i > 1$, then $r_i \geq 0$. The word (2.6) is restricted if and only if $n_i < h_T(\theta_i) = h(v_i)$ in the case $m_i = 1$, and $n_i < h_T(\theta_i) = h(v_i)/m_i$, $r_i < m_i$ otherwise. If we replace $\theta_i := v_i^{m_i} + \ldots$ in (2.6) and then develop multiplication, we get a linear combination $W + \sum \alpha_q W_q$ of words in $V \cup \{a_j\}$, where

$$W = v_1^{n_1 m_1 + r_1} v_2^{m_2 + r_2} \ldots v_k^{m_k + r_k}.$$

Let us as above denote by $W^t$ a word in $V$ that appears from $W$ by deletion of all $a_j$. Since $<$ is a monoidal order, we have $W^t_q < W$, for all $q$. If $m_i > 1$ then

$$n_i m_i + r_i \leq \left(\frac{h(v_i)}{m_i} - 1\right) m_i + (m_i - 1) = h(v_i) - 1.$$
If \( m_i = 1 \), again \( n_i m_i + r_i = n_i < h(v_i) \). Hence \((2.7)\) is a monotonous restricted word, provided that so is \((2.6)\). Since the order \( \prec \) is compatible with the PBW-decomposition related to \( V \), all words \( v_1v_2 \cdots v_k \) that appears in the PBW-decomposition \((2.1)\) of \( \sum_q \alpha_q W_q \) are less than \((2.7)\). It is important to note that in this way different monotonous restricted words in \( P_T \) correspond different monotonous restricted words in \( V \).

Suppose that a linear combination, \( \Xi = \sum \alpha_{ij} a_j U_i \), over \( A \) of monotonous restricted words in \( P_T \) equals zero in \( S \). Let us first in \( \Xi \) replace each \( \theta \in P_T \) by its representation, \( \theta := v^m + \cdots \), in terms of PBW-generators \( V \), and next develop the multiplication. We get a linear combination \( \Xi' = \sum \beta_{ij} W_k \) of words in \( P_T \cup \{ a_j | j \in J \} \). Since \( \prec \) is a monoidal order \( (a \prec b \text{ implies } cad \prec cbd) \), the maximal word \( W_0 \) among \( W_k \) appears in the decomposition of summands \( a_j U \) with the only word \( U \). If \( U \) has form \((2.6)\), then \( W_0 \) has form \((2.7)\). Since the order \( \prec \) is compatible with the PBW-decomposition related to \( V \), the maximal word of \( \Xi' \) in the PBW-decomposition appears just in summands \( \alpha_{ij} a_j W_0 \), where the equality \( U_s = U \) defines the index \( s \). Hence \( \Xi' \neq 0 \) in \( S \). A contradiction.

To see that monotonous restricted words in \( P_T \cup \{ a_j \} \) span \( S \) we may use a standard induction on words ordered by \( \prec \). Indeed, for any monotonous restricted word in \( V \),

\[
W = v_1^{s_1} v_2^{s_2} \cdots v_k^{s_k},
\]

we have \( s_i = n_i m_i + r_i, \ r_i < m_i, \ n_i < h_T(\theta_i) \), where either \( \theta_i = c_{\theta_i} = v_i^{m_i} + \cdots \in T \), or the set \( T \) has no elements with the leading word \( v_i^m, \ m \geq 1 \) (and hence \( \theta_i = r_i \in P_T, m_i = 1 \)). Suppose by induction that values of all super-words smaller than \( W \) belong to the linear space span by monotonous restricted words in \( P_T \cup \{ a_j \} \). The difference between \( W \) and \((2.6)\) is a linear combination of words \( U \) in \( P_T \cup \{ a_j \} \) such that \( U' \) are less than \( W \). Due to the comparability of \( \prec \) with the PBW-decomposition related to \( V \), the PBW-decomposition of any such a word has only summands \( \alpha a_j W' \) with \( W' \prec W \). By induction the lemma is proved. \( \square \)

Remark. In the proof we have seen that the leading term of \((2.6)\) in basis \((2.1)\) equals \((2.7)\).

3. PBW-basis of a character Hopf algebra

Recall that a Hopf algebra \( H \) is referred to as a character Hopf algebra if the group \( G \) of all grouplike elements is commutative and \( H \) is generated over \( k[G] \) by skew primitive semi-invariants \( s_i, \ i \in I : \)

\[
\Delta(s_i) = s_i \otimes 1 + g_{s_i} \otimes s_i, \quad g^{-1} s_i g = \chi^{s_i}(g) s_i, \quad g, g_{s_i} \in G,
\]

where \( \chi^{s_i}, \ i \in I \) are characters of the group \( G \). Let us associate a variable \( x_i \) to \( s_i \). For each word \( u \) in \( X = \{ x_i | i \in I \} \) we denote by \( g_u \) an element of \( G \) that appears from \( u \) by replacing each \( x_i \) with \( g_{s_i} \). In the same way we denote by \( \chi^u \) a character that appears from \( u \) by replacing of each \( x_i \) with \( \chi^{s_i} \). We define a bilinear skew commutator by the formula

\[
[u, v] = uv - pvv, \quad p_{uv} = \chi^u(g_v) = p(u, v).
\]

The group \( G \) acts on the free algebra \( k(X) \) by \( g^{-1} u g = \chi^u(g) u \), where \( u \) is an arbitrary monomial in \( X \). The skew group algebra \( G \langle X \rangle \) has the natural Hopf
algebra structure
\[ \Delta(x_i) = x_i \otimes 1 + g_{s_i} \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \quad g \in G. \]

The construction of the PBW-basis given in \[4\] requires \( I \) to be finite. In order to get a PBW-basis in the general case one may slightly modify this construction as follows.

We fix a Hopf algebra homomorphism \( \xi : G(X) \to H, \xi(x_i) = s_i, \xi(g) = g, \quad i \in I, \quad g \in G \), and consider the filtration \( D \) related to \( \xi \) as defined in Example \[2.2\]. This filtration is compatible with the Hopf algebra structure, that is
\[ (3.3) \quad \Delta(H_\gamma) \subseteq \sum_{\delta + \varepsilon = \gamma} H_\delta \otimes H_\varepsilon, \quad \sigma(H_\gamma) \subseteq H_\gamma, \]
where \( \sigma \) is the antipode. Therefore \( \text{gr} \, H \) is also a character Hopf algebra generated by \( k[G] \) and \( \xi(X) \).

By Lemmas \[2.3\] and \[2.4\] in what follows we may suppose that \( H \) is a \( \Gamma \)-graded character Hopf algebra (or, in the other words, it is homogeneous in each of the generators \( s_i \)).

On the set of all words in \( X \) we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself. This order is not monoidal: we have \( x > x^2 \), while \( xy < x^2y \) if \( y < x \). It has neither ACC, nor DCC: \( x > x^2 > \ldots > x^n > \ldots \); \( xy < x^2y < \ldots x^n y < \ldots \). By these reasons we need one more order, the Hall (or deg-lex) one, \( u < v \) if \( D(u) < D(v) \) or \( D(u) = D(v) \) & \( u < v \). Since there exist just a finite number of words in \( X \) of a given constitution, the Hall order indeed is a complete order of the free monoid.

A non-empty word \( u \) is called a standard word (or Lyndon word, or Lyndon-Shirshov word) if \( vw > uv \) for each decomposition \( u = vw \) with non-empty \( v, w \). A nonassociative word is a word where brackets \( [\,] \) somehow arranged to show how multiplication applies. If \([u]\) denotes a nonassociative word then by \( u \) we denote an associative word obtained from \([u]\) by removing the brackets (of course \([u]\) is not uniquely defined by \( u \) in general).

The set of standard nonassociative words is the biggest set \( SL \) that contains all variables \( x_i \), and satisfies the following properties.

1) If \([u] = [[v][w]] \in SL \) then \([v], [w] \in SL \), and \( v > w \) are standard.

2) If \([u] = [vv_1][v_2][w] \in SL \) then \( v_2 \leq w \).

Every standard word has the only alignment of brackets such that the appeared nonassociative word is standard (the Shirshov theorem). In order to find this alignment one may use the following procedure: The factors \( v, w \) of the nonassociative decomposition \([u] = [v][w]\) are the standard words such that \( u = vw \) and \( v \) has the minimal length (\[3\], see also \[5\]).

**Definition 3.1.** A super-letter is a polynomial that equals a nonassociative standard word where the brackets mean \([\underline{3.2}]\). A super-word is a word in super-letters.

By Shirshov theorem every standard word \( u \) defines the only super-letter, in what follows we will denote it by \([u]\). The order on the super-letters is defined in the natural way: \([u] > [v] \iff u > v \). We should stress that this order is not complete: If \( x > y \), there are infinite chains of super-letters
\[ [xy] > [xy^2] > \ldots > [xy^n] > \ldots ; [xy] < [x^2y] < \ldots < [x^n y] < \ldots. \]
Nevertheless the Hall order on super-words is still a complete monoidal order.

**Definition 3.2.** A super-letter \([u]\) is called hard in \(H\) provided that its value in \(H\) is not a linear combination of values of super-words in smaller than \([u]\) super-letters.

**Definition 3.3.** We say that a height of a hard in \(H\) super-letter \([u]\) equals \(h = h([u])\) if \(h\) is the smallest number such that: first, \(p_u\) is a primitive \(t\)-th root of unity and either \(h = t\) or \(h = tl^r\), where \(l = \text{char}(k)\); and then the value in \(H\) of \([u]^h\) is a linear combination of super-words in less than \([u]\) super-letters. If there exists no such number then the height equals infinity.

**Theorem 3.4.** Values of all hard in \(H\) super-letters form a set of PBW-generators of \(H\) over \(k[G]\).

**Proof.** If \(X\) is a finite set, this statement has been proved in [4, Theorem 2], see a footnote on page 267. In the general case the algebra \(G(X)\) has a covering by \(G(X_\alpha)\), where \(X_\alpha\) runs through all finite subsets of \(X\). We shall consider each of \(X_\alpha\) as an ordered subset of \(X\), and \(G(X_\alpha)\) as a \(\Gamma\)-graded subalgebra of \(G(X)\). Respectively \(H\) has a covering by the finitely generated \(\Gamma\)-graded character Hopf subalgebras \(H_\alpha \overset{\text{def}}{=} \xi(G(X_\alpha))\).

By Definition 3.2 every hard in \(H\) super-letter \([u]\) is hard in \(H_\alpha\) as soon as \(u \in G(X_\alpha)\). If a super-letter \([v]\) is not hard in \(H\) then by definition \(\xi([v]) = \xi(A)\), where \(A\) is a linear combination of super-words in less than \([v]\) super-letters. Since \(H\) is \(\Gamma\)-graded, all super-words of the sum \(A\) have the same constitution as the word \(v\) does. In particular all super-words of \(A\) belong to \(G(X_\alpha)\). Hence \([v]\) is not hard in \(H_\alpha\).

In the same way the height function is independent of \(\alpha\). \(\square\)

The following statements provide important properties of PBW-basis defined by the hard super-letters.

**Lemma 3.5.** ([4 Lemma 8]). The coproduct of a super-letter \([w]\) has a representation

\[
\Delta([w]) = [w] \otimes 1 + g_w \otimes [w] + \sum \alpha_i g(W''_i)W'_i \otimes W''_i,
\]

where \(W'_i\) are non-empty words in less than \([w]\) super-letters.

**Lemma 3.6.** The Hall order of the super-words is compatible with the PBW-decomposition related to the hard super-letters (see Definition 2.4).

**Proof.** Let \(W\) be a super-word. There exists the following natural diminishing process of the decomposition. First, in \(k[X]\) according to [4 Lemma 7] we decompose the super-word \(W\) in a linear combination of smaller monotonous super-words, then, we replace each non hard super-letter with the decomposition of its value in \(H\) that exists by Definition 3.2 and again decompose the appeared super-words in linear combinations of smaller monotonous super-words, and so on, until we get a linear combination of monotonous super-words in hard super-letters. If they are not restricted, we may apply Definition 3.3 and repeat the process until we get only monotonous restricted words in hard super-letters. \(\square\)
Lemma 3.7. If $[w]$ is a super-letter then
\[
\Delta(\xi([w]^m)) = \sum_{j=0}^{m} \binom{m}{j}_q g_{w}^{m-j} \xi([w])^j \otimes \xi([w])^{m-j} + \sum_{i} \alpha_i g(V_i) \xi(U_i) \otimes \xi(V_i),
\]
where $\binom{m}{j}_q$ are the $q$-binomial coefficients with $q = p(w, w)$, and $U_i$ are basis super-words such that the first from the left super-letter is less than $[w]$.

**Proof.** The $m$-th power of the right hand side of (3.4) after developing of the product takes the form (3.5), where each of $U_i$ is a product of $m$ super-words some of whom equal to $[w]$ (but not all of them!) and others equal to $W_i$'s. Hence $U_i$ as a super-word is less than $[w]^m$ with respect to the lexicographical ordering of words in super-letters. Let us decompose $\xi(U_i)$ in the PBW-basis defined by the hard super-letters. According to Lemma 3.6 we turn to the formula (3.5) where still $U_i < [w]^m$. This implies the required property since the first super-letter in a basis super-word is always the minimal one. \(\square\)

4. Right coideal subalgebras

**Theorem 4.1.** Let $H$ be a character Hopf algebra. Every right coideal subalgebra $U \supseteq k[G]$ has a PBW-basis that may be extended up to a PBW-basis of $H$.

By Lemma 2.3 it suffices to show that $\text{gr}\mathcal{U}$ as a subalgebra of gr$H$ has a required basis. Since $\text{gr}\mathcal{U}$ is a right coideal subalgebra of gr$H$, we still may suppose that $H$ is a $\Gamma$-graded character Hopf algebra and $\mathcal{U}$ is a $\Gamma$-graded coideal subalgebra.

**Lemma 4.2.** Let $w \in \mathcal{U}$. If $\pi, \mu : H \rightarrow H$ are linear maps such that $(\pi \otimes \mu)\Delta(w) = a \otimes b$ with $b \neq 0$, then $a = \pi(c)$ for some $c \in \mathcal{U}$.

**Proof.** We have $(\pi \otimes \mu)\Delta(U) \subseteq \pi(U) \otimes H$. Hence $a \otimes b \in \pi(U) \otimes H$, and $a \in \pi(U)$. \(\square\)

We shall use this evident statement as a basic tool. Note that once we have a PBW-basis of $H$, to define a linear map it suffices to fix its values on the restricted monotonous words in an arbitrary way.

Let $[u]$ be a hard super-letter. Suppose that in $\mathcal{U}$ there exists an element $c$ with the leading super-word $[u]^m$, $m \geq 1$. Since $G \subseteq \mathcal{U}$, we may suppose that the super-word $[u]^m$ appears one time with the trivial coefficient:
\[
c = \xi([u]^m + \sum_{j} \alpha_j g_j[u]^m + \sum_{i} \alpha_i g_i W_i R_i),
\]
where $W_i$ are nonempty basis super-words in less than $[u]$ super-letters, while $R_i$ are basis super-words in greater than or equal to $[u]$ super-letters, $\alpha_i, \alpha_j \in k, g_i, g_j \in G$, $g_j \neq 1$.

Denote by $\iota$ the natural projection $H \rightarrow k$, $\iota(g\xi(W)) = 0$, unless $g = 1$, $W = \emptyset$. Since $\Delta(gW) = (g \otimes g)\Delta(W)$, we have
\[
(id \otimes \iota)(\Delta(c)) = \xi([u]^m + \sum_{g_i=1} \alpha_i W_i R_i) \otimes 1.
\]
Thus, by Lemma 4.2
\[
c' = \xi([u]^m + \sum_{g_i=1} \alpha_i W_i R_i) \in \mathcal{U}.
\]
In what follows we fix the notation \( c_u \) for one of the elements from \( U \) of the form (4.2) that has the minimal possible \( m \).

**Lemma 4.3.** In the representation (4.2) of the chosen element \( c_u \) either \( m = 1 \), or \( r_{\mu_u} \) is a primitive \( t \)-th root of unity and \( m = t \) (in the case of positive characteristic) \( m = t(\text{char } k)^s \). In particular \( c_u \) is a thin element, see (2.7).

**Proof.** If \( m = 1 \) there is nothing to prove. Let \( m > 1 \). For each \( k, 1 \leq k < m \) we consider the following linear map set up on the PBW-basis of super-words.

\[
\pi_k(g\xi(W)) = \begin{cases} 0, & \text{if } W \prec [u]_k, \\ g\xi(W), & \text{otherwise}; \end{cases}
\]

By means of formula (3.3) we have

\[
(\pi_k \otimes \pi_{m-k})\Delta(\xi([u])_m) = \binom{m}{k}_q g_u^{m-k} \xi([u])^k \otimes \xi([u])^{m-k}.
\]

By Lemma 3.5 and Lemma 3.6 the coproduct \( \Delta(\xi(W_i)) \) is the sum of tensors \( g\xi(W_i') \otimes \xi(W_i'') \), where all \( W_i' \) are the basis super-words lexicographically smaller than \([u]_m^n \) with the only exception that equals \( g(W_i) \otimes \xi(W_i) \). Hence \( (\pi_k \otimes \pi_{m-k})\Delta(\xi(W_i)) \) is a sum of tensors of the form \( g\pi_k(\xi(R_i')) \otimes \pi_{m-k}(\xi(W_i R_i'')) \). Again by Lemma 3.6 we have \( \pi_{m-k}(\xi(W_i R_i'')) = 0 \). Hence

\[
(\pi_k \otimes \pi_{m-k})\Delta(\xi(W_i R_i)) = 0.
\]

Thus we may write

\[
(\pi_k \otimes \pi_{m-k})\Delta(c_u) = \binom{m}{k}_q g_u^{m-k} \xi([u])^k \otimes \xi([u])^{m-k}.
\]

If \( \binom{m}{k}_q \neq 0 \), then by Lemma 4.2 we find \( c \in U \) such that \( \pi_k(c) = g_u^{m-k} \xi([u])^k \). By definition of \( \pi_k \) this equality means that the PBW-decomposition of \( g_u^{m-k} c \) has the form (1.2) with \( k \) in place of \( m \). Since by the choice of \( c_u \) the number \( m \) is minimal, we get \( \binom{m}{k}_q = 0, 1 \leq k < m \). This system of equations implies \( q^m = 1 \) and either \( m \) equals the multiplicative order \( t \) of \( q \) or \( m = t(\text{char } k)^s \). \( \square \)

By Lemma 2.7 the set \( T \) of all above defined elements \( c_u \) has an extention up to a set \( P_T \) of PBW-generators of \( H \) over \( k[G] \). Now Theorem 4.1 follows from the proposition below.

**Proposition 4.4.** An element \( c \in H \) belongs to \( U \) if and only if all PBW-generators in the PBW decomposition of \( c \) with respect to \( P_T \) belong to \( T \). In particular \( T \) is a set of PBW-generators of \( U \) over \( k[G] \).

To prove this statement we will need some additional properties of the PBW-basis defined by \( P_T \). We extend the order \( \prec \) already defined on \( P_T \) onto the set of all words in \( P_T \) as the lexicographical order. The order \( \prec \) is the Hall order induced by the degree function \( D \); that is, \( W \prec U \) if and only if either \( D(W) < D(U) \), or \( D(W) = D(U) \) & \( W < U \). We have to stress that the order on the set of letters \( P_T \) differs from the order on the set of one-letter words \( P_T \).

We start with connections between these two PBW-decompositions.

**Lemma 4.5.** The leading term of a monotonous restricted word in \( P_T \),

\[
\theta_1^{n_1}[u_1]^{r_1} \theta_2^{n_2}[u_2]^{r_2} \cdots \theta_k^{n_k}[u_k]^{r_k},
\]
under the PBW-decomposition related to the hard super-letters equals
\[ (4.8) \quad [u_1]^{n_1m_1+r_1}[u_2]^{n_2m_2+r_2} \cdots [u_k]^{n_km_k+r_k}, \]
where \( \theta_i = [u_i]^{m_i} + \cdots. \) Conversely if
\[ (4.9) \quad W = [u_1]^{n_1}[u_2]^{n_2} \cdots [u_k]^{n_k} \]
is a monotonous restricted super-word in hard super-letters, then its leading term in the decomposition with respect to \( P_T \) equals \( (4.7) \), where \( n_i = [s_i/m_i], \) \( r_i = s_i - n_im_i. \)

**Proof.** The first part of the lemma has been proved in Lemma 2.5. The second part follows by induction on super-words ordered by the Hall order. Indeed the difference \( E \) between \( W \) and \( (4.7) \) is a linear combination of super-words that are less than \( W. \) By Lemma 3.6 all basis super-words in the PBW-decomposition of \( E \) are less than \( W. \) It remains to note that if \( W' < W \) is another basis super-word, then the word \( (4.7) \) related to \( W' \) is less than that related to \( W. \) \( \square \)

**Lemma 4.6.** The Hall order on the words in \( P_T \) is compatible with the PBW-decomposition related to \( P_T \) (see Definition 2.7).

**Proof.** Let \( W \) be a word in \( P_T. \) The word \( W \) has the form \( (4.7), \) where \( \theta_i = c_{n_i} = \xi([u_i]^{m_i} + \cdots), \) while \( u_i \)'s are not necessary increase from the left to the right. If we replace \( \theta_i := [u_i]^{m_i} + \cdots \) in \( (4.7) \) and then develop the multiplication in \( G \langle X \rangle, \) we get a linear combination \( \Sigma \) over \( k[G] \) of super-words with the leading term
\[ (4.10) \quad [u_1]^{n_1m_1+r_1}[u_2]^{n_2m_2+r_2} \cdots [u_k]^{n_km_k+r_k}. \]
By Lemma 3.6 each super-word \( W_1 \) in the PBW-decomposition of \( \xi(\Sigma) \) is less than or equal to \( (4.10). \) That is, if \( W_1 \) is different from \( (4.10), \) we may write
\[ (4.11) \quad W_1 = [u_1]^{n_1m_1+r_1}[u_2]^{n_2m_2+r_2} \cdots [u_s]^{n_sm_s+r_s}[u_{s+1}]^t[v] \cdots, \]
here \( [v] \) is the first from the left super-letter where \( W_1 \) differ from \( (4.10), \) and \( 0 \leq t < n_{s+1}m_{s+1} + r_{s+1}. \) Since \( W_1 \) is monotonous, we have \( v > u_{s+1}, \) provided that \( t \neq 0. \) Hence in this case \( W_1 \) is greater than \( (4.10). \) Hence \( t = 0, \) and \( u_{s+1} > v \geq u_s \geq u_{s-1} \geq \cdots \leq u_1. \)

If all inequalities among \( u_i \)'s in the above chain are strict, then according to Lemma 4.5 all words of the PBW-decomposition with respect to \( P_T \) of \( \xi(W_1) \) are less than or equal to
\[ (4.12) \quad \theta_1^{n_1}[u_1]^{r_1}\theta_2^{n_2}[u_2]^{r_2} \cdots \theta_s^{n_s}[u_s]^{r_s}\theta_v, \]
where \( \theta_v = c_v, \) or \( \theta_v = [v]. \) Hence they are less than \( W. \)

If not all inequalities are strict, say \( u_1 < u_2 < \cdots < u_p = \cdots = u_q < u_{q+1}, \) \( q \leq s, \) then again by Lemma 4.5 the leading word of \( \xi(W_1) \) starts with
\[ (4.13) \quad \theta_1^{n_1}[u_1]^{r_1}\theta_p^{n_p-1}[u_{p-1}]^{r_{p-1}}\theta^{n_p}([u_p]^{m_p} + r_0), \]
where \( r_p + \cdots + r_q = n_qm_q + r_0, \) \( 0 \leq r_0 < m_p. \) It is still less than \( W \) since by definition \( \theta_p < [u_p], \) provided that \( m_p > 1. \) \( \square \)

**Lemma 4.7.** The coproduct of each \( \theta \in P_T \) has a representation
\[ (4.14) \quad \Delta(\theta) = \theta \otimes 1 + g_\theta \otimes \theta + \sum \alpha_i g_i W'_i \otimes W''_i, \]
where \( W'_i, W''_i \) are restricted monotonous words (products) in \( P_T, \) and for every \( i \) either \( W'_i \) or \( W''_i \) starts with a letter that is less than \( \theta. \)
Proof. Let $\theta = \xi([u]^m + \sum_i W_i R_i)$ be the decomposition of $\theta \in P_T$ with respect to the hard super-letters. By Lemma 3.7 and Lemma 4.3 we have

$$\Delta(\xi([u]^m)) - \xi([u]^m) \otimes 1 - g_w^m \otimes \xi([u]^m) = \sum_i \alpha_i g_i \xi(U_i) \otimes \xi(V_i),$$

where $U_i$ are basis super-words starting with smaller than $[u]$ super-letters. By the second part of Lemma 4.5 each summand of the decomposition of $\xi(U_i)$ in the PBW-basis defined by $P_T$ starts by $c_w$ or $[u]$ with $w < u$.

Let $W_i = [w]U$, where $[w] < [u]$. By Lemma 3.5 we have

$$\Delta(W_i R_i) = \sum_j (V'_ij \otimes V''ij)\Delta(U R_i) + (g_w \otimes [w])\Delta(U R_i),$$

where all $V'_ij$ are non empty super-words in less than $[u]$ super-letters. All left hand sides of the tensors $(V'_ij \otimes V''ij)\Delta(U R_i)$ start with smaller than $[u]$ super-letters. Hence by Lemma 3.6 it remains to use the converse part of Lemma 4.5.

In perfect analogy, if $\Delta(U R_i) = \sum_j U'_ij \otimes U''ij$, then $(g_w \otimes [w])\Delta(U R_i) = g_w U'_ij \otimes [w]U''ij R_i$. Therefore the right hand side of each tensor that appears in the $P_T$-decomposition of $(g_w \otimes \xi([w]))\Delta(\xi(U R_i))$ starts with a letter that is less than $\theta$. □

Lemma 4.8. Let $\theta \in P_T$. The coproduct of $\theta^n$ has a decomposition

$$(4.15) \quad \Delta(\theta^n) = \sum_j \left[\begin{array}{c} n \\ j \end{array}\right] g_{\theta}^{n-j} \theta^j \otimes \theta^{n-j} + \sum_i \alpha_i g_i W'_i \otimes W''_i,$$

where $\left[\begin{array}{c} n \\ j \end{array}\right]$ are $q$-binomial coefficients with $q = p(\theta, \theta)$, $W'_i$, $W''_i$ are restricted monotonous words in $P_T$, and for every $i$ either $W'_i$ or $W''_i$ starts with a letter that is less than $\theta$.

Proof. If we develop multiplication in the $n$-th power of the right hand side of (4.14), then we get the first sum of 4.15 and a $k[G]$-linear combination of tensors $\theta^iW' \otimes \theta^jW''$, where $i$ and $j$ may be zero, but either $W'$ or $W''$ starts with a less than $\theta$ letter. Let it be $W'$. By Lemma 4.6 the PBW-decomposition of $\theta^iW'$ has only words that are less than $\theta^iW'$ in lexicographical order. Since these words are monotonous, no one of them may start with a letter that is grater than or equal to $\theta$. The lemma is proved.

Proof of Proposition 4.3. Let in contrary $c \in U$ be an element of the minimal degree whose decomposition in $P_T$-basis has super-letters $[u] \in P_T \setminus T$. Since $U$ is a subalgebra, we may suppose that each term of the decomposition has such a super-letter.

Let $U = \theta_1^n \theta_2 \cdot \cdots \theta_k^n$, $\theta_1 < \theta_2 < \cdots < \theta_k$ be the leading word of $c$ in the $P_T$-basis. Since $G \subseteq U$, we may suppose that $U$ appears one time with the trivial coefficient:

$$(4.16) \quad c = U + \sum_j \alpha_j g_j U + \sum_i \beta_i g_i U_i,$$

where $D(U_i) = D(U)$, $\alpha_j, \beta_i \in k$, $g_i, g_j \in G$, $g_j \neq 1$.

Denote by $t$ the natural projection $H \to k$, $t(gW) = 0$, unless $g = 1, W = \emptyset$. Here $W$ is an arbitrary restricted monotonous word in $P_T$. Since $\Delta(gW) = (g \otimes g)\Delta(W)$,
we have
\[(id \otimes \iota)(\Delta(c)) = (U + \sum_{g_i = 1} \beta_i U_i) \otimes 1,\]
Thus, by Lemma 4.2
(4.17)
\[c' = U + \sum_{g_i = 1} \beta_i U_i \in U\]
where \(U_i < U\). To get a contradiction we consider two cases.

Case 1. Suppose that \(\theta_k \notin T\); that is, \(\theta_k = [u]\) and in \(U\) there does not exist an element with the leading super-word \([u]^m, 1 \leq m \leq n_k\). Let us define the following two linear maps set up on the PBW-basis related to \(P_T\).
(4.18)
\[\pi(gW) = \begin{cases} 0, & \text{if } W \prec \theta_k^n; \\ gW, & \text{otherwise.} \end{cases}\]
(4.19)
\[\nu(gW) = \begin{cases} 0, & \text{if } W \prec \theta_1^n \ldots \theta_{k-1}^{n_k-1}; \\ gW, & \text{otherwise.} \end{cases}\]
Let us show first that
(4.20)
\[(\pi \otimes \nu)(\Delta(U)) = g\theta_k^n \otimes \theta_1^n \ldots \theta_{k-1}^{n_k-1}, \quad g \in G.\]
Since the word \(U\) is monotonous, by Lemma 4.8 we have
(4.21)
\[\Delta(\theta_k^n) = \sum_{j_i} \alpha_{j_i} g_{j_i} W_{j_i} \otimes W_{j_i}, \quad i < k,\]
where for each \(j_i, i \leq k\) either \(W_{j_i}'\) starts by a letter from \(P_T\) that is less than \(\theta_k\), or \(W_{j_i}''\) starts by a letter from \(P_T\) that is less than \(\theta_i\), with the only exception \(W_{j_i}' = \emptyset\), \(W_{j_i}'' = \theta_i^{s_i}\). In the same way
(4.22)
\[\Delta(\theta_k^n) = \sum_{j_k} \alpha_{j_k} g_{j_k} W_{j_k}' \otimes W_{j_k}'',\]
where either \(W_{j_k}'\) or \(W_{j_k}''\) starts by a letter from \(P_T\) that is less than \(\theta_k\), with exceptions of the form \(W_{j_k}' = \theta_k^l, W_{j_k}'' = \theta_k^{n_k-l}\).
\(\Delta(U)\) is a right linear combination over \(k[G]\) of tensors
(4.23)
\[E = W_{j_1}' W_{j_2}' \ldots W_{j_s}' \otimes W_{j_1}'' W_{j_2}'' \ldots W_{j_k}''.\]
Let \(W_{j_s}'\) be the first from the left nonempty factor in \(E\).
If, first, \(s \neq k\), then the tensor get the form
\[E = W_{j_s}' \ldots W_{j_k}' \otimes \theta_1^{n_1} \ldots \theta_{s-1}^{n_{s-1}} W_{j_s}'' \ldots W_{j_k}''.\]
If \(W_{j_s}'\) starts by a less than \(\theta_k\) letter, then the left hand side of \(E\) is less than \(\theta_k^{n_k}\), hence by Lemma 4.6 we get \((\pi \otimes \nu)(E) = 0\).
If \(W_{j_s}''\) starts by a less than \(\theta_s\) letter, then the right hand side is less than \(\theta_1 \ldots \theta_{k-1}\), and again \((\pi \otimes \nu)(E) = 0\).
If, next, \(s = k\), then the tensor has the form
\[E = W_{j_k}' \otimes \theta_1^{n_1} \ldots \theta_{k-1}^{n_{k-1}} W_{j_k}''.\]
Since \(D(W_{j_k}') = n_k D(\theta_k) - D(W_{j_k}') < n_k D(\theta_k), \) unless \(W_{j_k}'' = \emptyset\), we get \(\pi(W_{j_k}') = 0\) with the only exception \(W_{j_k}' = \theta_k^{n_k}\). Thus \(\pi \otimes \nu\) kills all tensors in the PBW-decomposition of \(\Delta(U)\) except one of them. This proves (4.20).
Let us show, further, that

\[(4.24) \quad (\pi \otimes \nu)(U_i) = 0.\]

Since \(U_i < U\), we have \(U_i = \theta_1^{n_1} \cdots \theta_s^{n_s} \theta_{s+1}^{\eta} \cdots\), where \(0 \leq r < n_{s+1}, \eta < \theta_{s+1}\). If \(r \neq 0\), then \(\theta_{s+1} < \eta\) since the word \(U_i\) is monotonous. Thus \(r = 0\); that is,

\[U_i = \theta_1^{n_1} \cdots \theta_s^{n_s} \eta \cdots, \quad \eta < \theta_{s+1}.
\]

By means of Lemma 4.7 we have

\[(4.25) \quad \Delta(\eta) = \sum_j \alpha_j V_j' \otimes V_j'',
\]

where for each \(j\), either \(V_j'\) or \(V_j''\) starts by a letter from \(P_T\) which is less than \(\theta_{s+1}\). Thus \(\Delta(U_i)\) is a right linear combination over \(k[G]\) of tensors

\[(4.26) \quad E = W_1' \cdots W_j' V_j' \cdots \otimes W_s' V_s' \cdots.
\]

If \(W_j' = \cdots = W_j' = V_j' = 0\), then the right hand side takes up the form \(\theta_1^{n_1} \cdots \theta_s^{n_s} \eta \cdots\), which is less than \(\theta_1^{n_1} \cdots \theta_{k-1}^{n_{k-1}}\), hence \((\pi \otimes \nu)(E) = 0\).

Let \(W_j'\), \(r \leq s\) be the first from the left nonempty word in \(E\). The tensor takes up the form

\[E = W_1' \cdots W_j' V_j' \cdots \otimes \theta_1^{n_1} \cdots \theta_s^{n_s} \cdots W_s' V_s' \cdots.
\]

Since either \(W_j'\) starts with a less than \(\theta_k\) letter, or \(W_s'\) starts with a less than \(\theta_l\) letter, again \((\pi \otimes \nu)(E) = 0\).

If the first from the left nonempty factor is \(V_j'\), then in the same way either \(V_j'\) starts with a letter \(\leq \eta < \theta_{s+1} \leq \theta_k\), or \(V_j''\) does. In both cases \((\pi \otimes \nu)(E) = 0\).

Thus \((\pi \otimes \nu)(\Delta(U_i)) = 0\). Now taking into account (4.17), we get

\[(4.27) \quad (\pi \otimes \nu)(\Delta(c)) = g \theta_1^{n_k} \cdots \theta_1^{n_{k-1}} \cdot g \in G.
\]

Lemma 4.2 implies that in \(U\) there exists an element \(c''\) such that \(\pi(c'') = g \theta_k^{n_k}\). Definition (4.18) of \(\pi\) shows that \(\theta_1^{n_k} = [u]^{n_k}\) is the leading word of \(c''\) in the \(P_T\)-decomposition. By Lemma 4.5 the leading super-word of \(c''\) equals \([u]^{n_k}\). According to the construction of \(T\), there exists an element \(c_u \in T\) of the form \(c_u = [u]^{m+n} \cdots\), \(m \leq n_k\). This contradicts the conditions of the first case \((\pi \otimes \nu)(E) = 0\).

**Case 2.** Suppose that \(\theta_k \in T\). In this case \(\theta_k \in U\), therefore one of the letters \(\theta_1, \ldots, \theta_{k-1}\) does not belong to \(T\). By the inductive supposition no one element with the leading word \(\theta_1^{n_1} \cdots \theta_{k-1}^{n_{k-1}}\) belongs to \(U\). At the same time, in perfect analogy with the first case (up to left-right symmetry in the consideration of tensors), we have

\[(\nu \otimes \pi)(\Delta(c')) = g \theta_1^{n_1} \cdots \theta_1^{n_{k-1}} \otimes \theta_k^{n_k}.
\]

Hence by Lemma 4.2 there exists \(c'\) in \(U\) such that \(\nu(c') = g \theta_1^{n_1} \cdots \theta_1^{n_{k-1}}\). Definition (4.19) of \(\nu\) implies that the leading word of \(c''\) indeed equals \(\theta_1^{n_1} \cdots \theta_1^{n_{k-1}}\). A contradiction. Proposition 4.3 and Theorem 4.1 are completely proved.

**References**

1. G. Benkart, S.-J. Kang, and D. Melville, *Quantized enveloping algebras for Borcherds superalgebras*, TAMS, 350, N8(1998), 3297–3319.
2. M. Costantini, M. Varagnolo, *Multiparameter quantum function algebra at roots of 1*, Math. Ann. 306, N4(1996), 759–780.
3. S.-J. Kang, *Quantum deformations of generalized Kac-Moody algebras and their modules*, Journal of Algebra, 175(1995), 1041–1066.
4. V.K. Kharchenko, *A quantum analog of the Poincaré-Birkhoff-Witt theorem*, Algebra i Logika, 38, N4(1999), 476–507. English translation: Algebra and Logic, 38, N4(1999), 259–276 (QA/0005101).
5. M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Publ. Co., 1983.
6. A.I. Shirshov, *Some algorithmic problems for Lie algebras*, Sibirskii Math. J., 3(2)(1962), 292–296.
7. S. Ufer, *PBW bases for a class of braided Hopf algebras*, Journal of Algebra, 280, N1(2004), 84–119.

Universidad Nacional Autónoma de México, Facultad de Estudios Superiores Cuautitlán, Primero de Mayo s/n, Campo 1, CIT, Cuautitlán Izcalli, Edtado de México, 54768, México

E-mail address: vlad@servidor.unam.mx