Affine vector space partitions

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Abstract

An affine vector space partition of $\text{AG}(n, q)$ is a set of proper affine subspaces that partitions the set of points. Here we determine minimum sizes and enumerate equivalence classes of affine vector space partitions for small parameters. We also give parametric constructions for arbitrary field sizes.

Keywords — finite geometry, vector space partitions, spreads, Klein quadric, Fano plane, hitting formulas

1 Introduction

A vector space partition $\mathcal{P}$ of the projective space $\text{PG}(n - 1, q)$ is a set of subspaces in $\text{PG}(n - 1, q)$ which partitions the set of points. For a survey on known results we refer to [Hed12]. We say that a vector space partition $\mathcal{P}$ has type $(n - 1)^{m_{n-1}} \ldots 2^{m_2} 1^{m_1}$ if precisely $m_i$ of its elements have dimension $i$, where $1 \leq i \leq n$. The classification of the possible types of a vector space partition, given the parameters $n$ and $q$, is an important and difficult problem. Based on [Hed86], the classification for the binary case $q = 2$ was completed for $n \leq 7$ in [EZSS+09]. Under the assumption $m_1 = 0$ the case $(q, n) = (2, 8)$ has been treated in [EZHS+10]. It seems quite natural to define a vector space partition $\mathcal{A}$ of the affine space $\text{AG}(n, q)$ as a set of subspaces in $\text{AG}(n, q)$ that partitions the set of points. However, it turns out that those partitions exist for all types which satisfy a very natural numerical condition. If we impose the additional condition of tightness, that is that the projective closures of the elements of $\mathcal{A}$ have an empty intersection, then the classification problem becomes interesting and challenging. This condition is natural in the context of hitting formulas as introduced in [Iwa89], that is for logical formulas in full disjunctive normal form (DNF) such that each truth assignment to the underlying variables satisfies precisely one term. For a more recent treatment and applications we refer to [PS22]. Here we consider the geometrical and the combinatorial point of view.

Variants of vector space partitions of $\text{PG}(n - 1, q)$ have been studied in the literature. In [EZSS+11] the authors study (multi-)sets of subspaces covering each point exactly $\lambda$ times. The problem of covering each $k$-space exactly once is considered in [HHKK19]. A more general partition problem for groups is studied in [Hed86]. However, we are not aware of any publication directly treating the introduced affine vector space partitions.

The paper is organized as follows. In Section\textsuperscript{2} we formally introduce affine vector space partitions, state the preliminaries, and develop the first necessary existence conditions. Here we are guided by the published necessary conditions for vector space partitions. We also argue why tightness (see above) and
irreducibility, that is there exists no proper subset $A' \subseteq A$ such that the union of all elements of $A'$ is a subspace of $AG(n, q)$, are necessary to obtain an interesting existence question. In Section 3 we classify affine vector space partitions for arbitrary field sizes but small dimensions. Section 3 is concerned with the binary case. We completely determine the possible dimension distributions of tight irreducible affine vector space partitions of $PG(n - 1, 2)$ for all $n \leq 7$. In a few cases we give theoretical or computational classifications of the corresponding equivalence classes of tight irreducible vector space partitions. A very nice example consists of eight solids in $PG(6, 2)$ whose parts at infinity live on the Klein quadric $Q^+(5, 2)$.

A generalization to arbitrary finite fields of characteristic 2 is given in Subsection 5.2. Parametric constructions of tight irreducible affine vector space partitions using spreads or hitting formulas complete Section 5. In Section 6 we determine the smallest possible size of an irreducible tight affine vector space partition of $PG(7, 2)$ and give a parametric upper bound for $PG(n - 1, 2)$ of size roughly $\frac{2n(q+1)}{2}$, which is significantly smaller than the conjectured smallest size of an irreducible hitting formula mentioning all variables. We close with a conclusion and a list of open problems in Section 7. To keep the paper self-contained we present some additional material in an appendix. Section A contains details on integer linear programming formulations that we have utilized to obtain some computational results. Section B contains a few technical results that might be left to the reader or collected from the literature. Lists of hitting formulas that can be used to construct tight irreducible affine vector space partitions of the minimum possible size are given in Section 8.

2 Preliminaries and necessary conditions

Definition 1. An affine vector space partition $A$ of $AG(n, q)$ is a set $\{A_1, \ldots, A_r\}$ of subspaces of $AG(n, q)$ such that $0 \leq \dim(A_i) \leq n - 1$ for all $1 \leq i \leq r$ and every point (element of $F_q^n \setminus \{0\}$) is contained in exactly one element $A_i$. The integer $r$ is called the size of the affine vector space partition.

We write $\#A$ for the size of $A$. For each affine subspace $A \in AG(n, q)$ we write $\overline{A}$ for its projective closure. With this $A := \{\overline{A} : A \in A\}$ is the natural embedding of an affine vector space partition of $AG(n, q)$ in $PG(n, q)$. Denoting the hyperplane at infinity by $H_\infty$, we can directly define an affine vector space partition in $PG(n, q)$:

Definition 2. An affine vector space partition $U$ of $PG(n - 1, q)$ is a set $\{U_1, \ldots, U_r\}$ of subspaces of $PG(n - 1, q)$ such that $1 \leq \dim(U_i) \leq n - 1$ for all $1 \leq i \leq r$ and there exists a hyperplane $H_\infty$ such that every point (1-dimensional subspace) outside of $H_\infty$ is contained in exactly one element $U_i$ and $U_i \not\subseteq H_\infty$ for all $1 \leq i \leq r$. The integer $r$ is called the size of the affine vector space partition and also denoted by $\#U$.

Here we use the algebraic dimension for subspaces in $PG(n - 1, q)$, i.e., if $\dim(U) = u$, then $\#U = \left[\frac{n}{u}\right]_q := \frac{q^n - 1}{q^u - 1}$ and we also speak of $u$-spaces. Using the geometric language, we call 1-, 2-, 3-, 4-, and $n-1$-spaces points, lines, planes, solids, and hyperplanes, respectively. For each $1 \leq i \leq r$ the set $U_i \setminus H_\infty$ is an affine space containing $q^{\dim(U_i) - 1}$-points.

In the following we will mostly speak of an affine vector space partition, abbreviated as avsp, and will consider its embedding in $PG(n - 1, q)$. The type of an avsp $U = \{U_1, \ldots, U_r\}$ is given by $(n - 1)^{m_{n-1}} \cdots 2^{m_2}1^{m_1}$, where $m_i = \#\{U_j : 1 \leq j \leq r, \dim(U_j) = i\}$. Counting points outside of $H_\infty$ gives

$$\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}. \tag{1}$$

The analog of Equation 1 for vector space partitions of $PG(n - 1, q)$ is called the packing condition. While the packing condition for vector space partitions of $PG(n - 1, q)$ is just a necessary but not a sufficient condition for the existence with a given type, for avsps Equation 1 is both necessary and sufficient.

Lemma 3. For each type $(n - 1)^{m_{n-1}} \cdots 2^{m_2}1^{m_1}$ that satisfies the packing condition 1 there exists an avsp $U$ of $PG(n - 1, q)$ attaining that type.
Proof. Consider a subspace $K$ of $H_\infty$ with $\dim(K) = n - 2$. By $H_1, \ldots, H_q$ we denote the $q$ hyperplanes containing $K$ that are not equal to $H_\infty$. Clearly, we have $0 \leq m_{n-1} \leq q$ and we can choose $H_1, \ldots, H_{m_{n-1}}$ as the first elements of $\mathcal{U}$. The remaining elements are constructed recursively. For each index $m_{n-1}+1 \leq j \leq q$ we consider an avsp of type $(n-2)^{m_{n-1}^0} \cdots 2^{m_j} \cdot 1^{m_j}$ where the $m_i^{(j)} \in \mathbb{N}_0$ are chosen such that the packing condition is satisfied for $H_j$ and

$$
\sum_{j=m_{n-1}+1}^{q} m_j^{(j)} = m_i
$$

for all $1 \leq i \leq n-2$. Such a decomposition can be easily constructed, see e.g. Algorithm 1 in Section B. □

Definition 4. We call an avsp $\mathcal{U} = \{U_1, \ldots, U_r\}$ reducible if there exists a subspace $U$ and a subset $S \subseteq \{1, \ldots, r\}$ such that $\dim(U) < n$, $\# S > 1$ and $\{U_i:i \in S\}$ is an avsp of $S$. Otherwise $\mathcal{U}$ is called irreducible.

Lemma 5. The smallest size of an irreducible avsp $\mathcal{U}$ of $\text{PG}(n-1, q)$ is given by $\# \mathcal{U} = q$.

Proof. Let $\mathcal{U}$ be an avsp of $\text{PG}(n-1, q)$. Since there are $q^{n-1}$ points to cover and each subspace covers at most $q^{n-2}$ points, we have $\# \mathcal{U} \geq q$. Now consider a hyperplane $K$ of $H_\infty$. By $H_1, \ldots, H_q$ we denote the $q$ hyperplanes containing $K$ and not being equal to $H_\infty$. With this, $\{H_1, \ldots, H_q\}$ is an irreducible avsp of $\text{PG}(n-1, q)$.

For a vector space partition $\mathcal{P}$ of $\text{PG}(n-1, q)$ we have $\dim(A) + \dim(B) \leq n$ for each pair $\{A,B\}$ of different elements of $\mathcal{P}$, which is also called dimension condition. Using this it can be easily shown that $\# \mathcal{P} \geq \frac{q^{n-1}}{q^{n-2} - 1} = q^n/2 + 1$ if $n$ is even and $\# \mathcal{P} \geq q^{(n+1)/2} + 1$ if $n$ is odd. Both bounds can be attained by spreads, i.e., vector space partitions of type $(n/2)q^{n/2+1}$, and lifted MRD codes of maximum possible rank distance, i.e., vector space partitions of type $(n+1)/2)((n-1)/2)^{(n+1)/2}$, respectively. In [NS11] the authors determine the minimum size $\sigma_q(n, t)$ of a vector space partition of $\text{PG}(n, q)$ whose largest subspace has dimension $t$.

Lemma 6. Let $\mathcal{U}$ be an irreducible avsp of $\text{PG}(n-1, q)$ and $U_1, \ldots, U_q \in \mathcal{U}$ be $q$ different elements with $\dim(U_1) = \cdots = \dim(U_q)$ and $\dim(\langle U_1, \ldots, U_q \rangle) = \dim(U_1) + 1$. Then we have $\dim(U_1) = \cdots = \dim(U_q) = n-1$.

Proof. Let $U := \langle U_1, \ldots, U_q \rangle$ and $u := \dim(U_1)$. Since $U \setminus H_\infty$ contains $q^u$ points and $U_1 \setminus H_\infty$ contains $q^{u-1}$ points for each $1 \leq i \leq q$, the set $\mathcal{U} \setminus \{U_1, \ldots, U_q\} \cup \{U\}$ is an avsp unless $\dim(U) = u + 1 = n$. □

Corollary 7. Let $\mathcal{U}$ be an irreducible avsp of $\text{PG}(n-1, 2)$ and $U_1, U_2 \in \mathcal{U}$ be two different elements with $\dim(U_1) = \dim(U_2) = \dim(U_1 \cap U_2) + 1$. Then, we have $\dim(U_1) = \dim(U_2) = n-1$.

As an analog of the dimension condition for vector space partitions in $\text{PG}(n-1, q)$ we have:

Lemma 8. Let $\mathcal{U}$ be an avsp in $\mathcal{U}$. For each $U, U' \in \mathcal{U}$ we have

$$
\dim(U \cup U') = \dim(U \cup U' \cap H_\infty) \geq \dim(U) + \dim(U') - n.
$$

Proof. Since $U \setminus H_\infty, U' \setminus H_\infty$ are disjoint and $U, U' \not\subseteq H_\infty$ we have $\dim(U \cap U') = \dim(U \cap U' \cap H_\infty)$. The inequality follows from $\dim(U_1 \cup U_2) + \dim((U_1, U_2)) = \dim(U_1) + \dim(U_2)$ and $\dim((U_1, U_2)) \leq n$. □

Due to the following general construction for (irreducible) avsps we introduce a further condition.

Lemma 9. Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be an avsp of $\text{PG}(n-1, q) := V$ and $P$ be a point outside of $V$ (embedded in $\text{PG}(n' - 1, q)$ for some larger value of $n'$). Then, $\mathcal{U}' := \{\langle U_1, P \rangle, \ldots, \langle U_r, P \rangle\}$ is an avsp of $\langle V, P \rangle \cong \text{PG}(n+1 - 1, q)$. Reducibility inherits, i.e. $\mathcal{U}'$ is irreducible iff $\mathcal{U}$ is irreducible.

Definition 10. Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be an avsp of $\text{PG}(n-1, q)$. We call $\mathcal{U}$ tight iff the intersection of all $U_i$ does not contain a point, i.e. the intersection is trivial.
We remark that an avsp $A$ of $AG(n, 2)$ is tight iff for any $x \in F_2^n$, there exists an $A \in A$ such that $A$ is not invariant under addition of $x$, that is $A + x \neq A$.

**Lemma 11.** For each integer $n \geq 2$ there exists a tight avsp of $PG(n - 1, q)$ with size $(q - 1) \cdot (n - 1) + 1$.

**Proof.** Apply the following recursive construction. Start with an $(n - 2)$-dimensional subspace $K$ of $H_\infty$ and consider the $q$ hyperplanes $H_1, \ldots, H_q$ containing $K$ but not being equal to $H_\infty$. Choose $q - 1$ out of these and continue the iteration with the remaining hyperplane until it becomes 2-dimensional, i.e. a line. In the final step replace the affine line by $q$ points, so that the resulting avsp is trivially tight. \hfill \square

A classical result in computer science, attributed to Tarsi, states that a minimally unsatisfiable CNF with $m$ clauses mentions at most $m - 1$ variables, see e.g. [DDKB98, Theorem 8]. The proof can be slightly modified to show that for $n \geq 2$ each tight avsp of $PG(n - 1, 2)$ has size at least $n$. We will prove the conjecture that Lemma 14 is tight in a subsequent paper. The determination of the minimum size of an irreducible tight avsp is quite a challenge and we will present our preliminary results in Sections 3 and 4.

Note that tightness and irreducibility can be checked efficiently. In particular, for irreducibility it suffices to calculate the affine closure for all pairs of subspaces in the avsp. We will show efficiency formally and provide detailed algorithms in future work on hitting formulas.

**Lemma 12.** Let $U$, $K$, and $H_\infty$ be subspaces in $PG(n - 1, q)$ with $K \leq H_\infty$, $\dim(K) = n - 2$, $\dim(H_\infty) = n - 1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\subseteq H_\infty$. By $H_1, \ldots, H_q$ we denote the $q$ hyperplanes containing $K$ but not being equal to $H_\infty$. Then the following statements are equivalent:

1. $U \cap H_\infty \leq K$;
2. there exists an index $1 \leq i \leq q$ with $U \subseteq H_i$;
3. there exists an index $1 \leq i \leq q$ with $U \subseteq H_i$ and $U \cap H_j = U \cap H_\infty = U \cap K$ for all $1 \leq j \leq q$ with $j \neq i$;
4. $\dim(U \cap K) = \dim(U) - 1$.

**Lemma 13.** Let $U$, $K$, and $H_\infty$ be subspaces in $PG(n - 1, q)$ with $K \leq H_\infty$, $\dim(K) = n - 2$, $\dim(H_\infty) = n - 1$, and $\dim(U \cap H_\infty) = \dim(U) - 1$, i.e. $U \not\subseteq H_\infty$. By $H_1, \ldots, H_q$ we denote the $q$ hyperplanes containing $K$ but not being equal to $H_\infty$. Then the following statements are equivalent:

1. $U \cap H_\infty \not\leq K$;
2. $\dim(U \cap H_i) = \dim(U)$ for all $1 \leq i \leq q$;
3. there are $q \cdot (\dim(U) - 1)$-spaces in $U$ containing $U \cap K$ and not being contained in $H_\infty$;
4. $\dim(U \cap K) = \dim(U) - 2$.

Assume that $P$ is a vector space partition of $PG(n - 1, q)$ with type $k_1^{m_1} \cdots k_l^{m_l}$, where $k_1 > \cdots > k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. The so-called tail $T$ of $P$ is the set of all $k_i$-spaces in $P$, i.e., the set of all elements with the smallest occurring dimension. In [Hec09] several conditions on $\#T$ have been obtained. In our situation we can also consider the tail $T := \{U \in \mathcal{U} : \dim(U) = k_i\}$ of an avsp of $PG(n - 1, q)$ with type $k_1^{m_1} \cdots k_l^{m_l}$, where $k_1 > \cdots > k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. The packing condition in [Hec09] directly implies that $q^{k_i - 1 - k_j}$ divides $\#T = m_l$ if $l \geq 2$ and that $q$ divides $\#T = m_l$ if $l = 1$. In [Kur18], the results on the tail of a vector space partition of $PG(n - 1, q)$ were refined using the notion of $\Delta$-divisible sets of $k$-spaces.

**Definition 14.** A (multi-)set $S$ of $k$-spaces in $PG(n - 1, q)$ is called $\Delta$-divisible iff $\#S \equiv \#(H \cap S) \pmod{\Delta}$ for every hyperplane $H$, where $H \cap S$ denotes the (multi-)set of elements of $S$ that are contained in $H$.

**Lemma 15.** Let $\mathcal{U}$ be an avsp of $PG(n - 1, q)$ of type $k_1^{m_1} \cdots k_l^{m_l}$, where $k_1 > \cdots > k_l > 1$ and $k_i > 0$ for all $1 \leq i \leq l$. Let $T := \{U \in \mathcal{U} : \dim(U) = k_i\}$ be the tail of $\mathcal{U}$ and $T' := \{T \cap H_\infty : T \in \mathcal{T}\}$. If $l \geq 2$, then $T'$ is $q^{k_i - 1 - k_j}$-divisible and $\#T = \#T' \equiv 0 \pmod{q^{k_i - 1 - k_j}}$. If $l = 1$, then $T'$ is $q$-divisible and $\#T = \#T' \equiv 0 \pmod{q}$. 

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Proof. Clearly we have $\#T = \#T'$. From the packing condition (1) we directly conclude $\#T \equiv 0 \pmod{q^{k_l-1-k_l}}$ if $l \geq 2$ and $\#T \equiv 0 \pmod{q}$ if $l = 1$. Let $K$ be an arbitrary hyperplane of $H_\infty$ and $H_1, \ldots, H_6$ be the $q$ hyperplanes of $\text{PG}(n-1, q)$ not being equal to $H_\infty$. Call the points outside of $H_\infty$ that are contained in some element of $U$ with dimension strictly larger than $k_l$ covered and all others outside of $H_\infty$ uncovered. Since each $k$-space covers either $q^{k_l-1}$ or $q^{k_l-2}$ points of $H_1 \setminus H_\infty$, the number of uncovered points in $H_1 \setminus H_\infty$ is divisible by $q^{k_l-2}$ if $l \geq 2$ and by $q^{k_l-1}$ if $l = 1$, where $1 \leq i \leq q$ is arbitrary. Let $a$ be the number of $k_l$-spaces in $U$ that are completely contained in $H_i$, so that the number of uncovered points in $H_i$ equals

$$x := a \cdot q^{k_l-1} + (\#T - a) \cdot q^{k_l-2}.$$ 

If $l \geq 2$ we have $x \equiv 0 \pmod{q^{k_l-1-2}}$ and $\#T \equiv 0 \pmod{q^{k_l-1-k_l}}$, so that $(q-1)a \equiv 0 \pmod{q^{k_l-1-k_l}}$ and $a \equiv 0 \pmod{q^{k_l-1-k_l}}$. If $l = 1$ we have $x \equiv 0 \pmod{q^{k_l-1}}$ and $\#T \equiv 0 \pmod{q}$, so that $(q-1)a \equiv 0 \pmod{q}$.

$\Delta$-divisible (multi-)sets $S$ of $k$-spaces in $\text{PG}(n-1, q)$ have been studied in [Kur18]. If we replace each $k$-space by its $q^{k_l-1}$ points we obtain a $\Delta q^{k_l-1}$-divisible multiset of $\#S \cdot q^{k_l-1}$ points in $\text{PG}(n-1, q)$. The possible cardinalities, given the divisibility constant and the field size, have been completely characterized in [KK20] Theorem 1]. Here we will use only a few results on the possible structure of the tail (or more precisely of $T'$) which allow more direct proofs.

Lemma 16. Let $U$ be an aswp of $\text{PG}(n-1, q)$ with tail $T$. If $\#T = q$, then either $U$ is reducible or we have $U = T$ and $n = 2$.

Proof. Denote the dimension of the elements of $T$ by $k$. Lemma 15 yields that $T' := \{T \setminus H_\infty : T \in T\}$ is a $q$-divisible multiset of $(k-1)$-spaces. So, each hyperplane of $H_\infty$ contains either all $q$ or zero elements from $T'$, so that $T'$ is a $q$-fold $(k-1)$-space. With this, the stated results follows from Lemma 6.

Corollary 17. Let $U$ be an irreducible aswp of $\text{PG}(n-1, q)$ of type $k_1^{m_1}, \ldots, k_l^{m_l}$, where $k_1 \geq \cdots \geq k_l$ and $k_i > 0$ for all $1 \leq i \leq l$. If $m_l = q$, then we have $l = 1$ and $k_1 = n-1$.

2.1 The structure of the tail for small parameters

If $\#T$ is small, then we can also characterize the tail. To this end, let $S$ denote a set of $k$-spaces in $\text{PG}(n-1, q)$. The corresponding spectrum $(a_i)_{i \in \mathbb{N}_0}$ is given by the numbers $a_i$ of hyperplanes that contain exactly $i$ elements from $S$, so that

$$\sum_{i=0}^{\#S} a_i = \frac{q^n - 1}{q - 1}. \quad (4)$$

The condition that $S$ is spanning, i.e. $\langle S : S \in S \rangle = \text{PG}(n-1, q)$, is equivalent to $a_{\#S} = 0$. Double-counting the $k$-spaces gives

$$\sum_{i=0}^{\#S} i a_i = \#S \cdot \frac{q^{n-k} - 1}{q - 1}. \quad (5)$$

Lemma 18. Let $S$ be a $2$-divisible set of four $k$-spaces in $\text{PG}(n-1, 2)$. Then there exists a $(k-1)$-space $B$, a plane $E$, and a line $L \leq E$ with $\dim(\langle E, B \rangle) = k + 2$, such that $S = \{\langle P, B \rangle : P \in E \setminus L\}$.

Proof. Assume that $P$ is a point that is contained in at least one but not all elements from $S$. Let $x$ denote the number of elements of $S$ that contain $P$. Since all hyperplanes contain an even number of elements from $S$ we have $x \neq 3$. Assume $x = 2$ for a moment and let $S, S' \in S$ be the two elements not containing $P$. There are $2^{q^{n-k-1}}$ hyperplanes that contain $S$ but do not contain $P$, so that all of those hyperplanes contain $S$ and $S'$. The intersection of these hyperplanes has dimension at most $k$ and contains $S$ as well as $S'$, so that $S = S'$, which is a contradiction. Thus, each point $P$ in $\text{PG}(n-1, 2)$ is contained in 0, 1 or 4 elements of $S$. 5
By \((a_i)_{i \in \mathbb{N}_0}\) we denote the spectrum of \(S\). W.l.o.g. we assume that \(S\) is spanning, i.e., we have \(a_4 = 0\). From the equations (4) and (5) we conclude

\[
a_0 = 2^n - 2^{n-k+1} + 1 \quad \text{and} \quad a_2 = 2^{n-k+1} - 2.
\]

If there is no point \(P\) that is contained in all four elements of \(S\), then the elements of \(S\) are pairwise disjoint and double-counting pairs yields

\[
\binom{2}{2} a_2 = \binom{4}{2} \cdot \left(2^{n-2k} - 1\right),
\]

so that

\[
2^{n-k+1} - 2 = 6 \cdot \left(2^{n-2k} - 1\right) \Leftrightarrow 2^{n-k} - 3 \cdot 2^{n-2k} + 2 = 0,
\]

which has the unique solution \(n = 3, k = 1\).

So, by recursively quotienting out points \(P\) that are contained in all elements of \(S\) we conclude the existence of a \((k - 1)\)-space \(B\) that is contained in all four elements of \(S\). Quotienting out \(B\) yields a spanning 2-divisible set of points in \(PG(2,2)\) with \(a_0 = 1\) and \(a_2 = 6\). Choosing \(E\) as the ambient space and \(L\) as the empty hyperplane yields the stated characterization since in \(PG(2,2)\) there are exactly four points outside a hyperplane.

If \(k = 1\), i.e., the \(k\)-spaces are points, the equations (4)-(6) are also known as “standard equations” or the first three MacWilliams equations for the corresponding linear code.

We remark that Lemma 18 is based on the fact that each 2-divisible set of 4 points is an affine plane. For \(q > 2\) there there further possibilities for \(q\)-divisible sets of \(q^2\) points over \(\mathbb{F}_q\), see [DBDMS19, KM21] on the so-called cylinder conjecture.

### 3 Classification of avsps in \(PG(n - 1, q)\) for small parameters

By definition, there is no avsp in \(PG(1-1, q)\). In \(PG(2-1, q)\) there is a unique avsp. It has type \(1^q\) and is irreducible and tight.

**Lemma 19.** Let \(\mathcal{U}\) be an avsp of \(PG(n - 1, q)\), where \(n \geq 3\). If there exist pairwise different hyperplanes \(U_1, \ldots, U_l \in \mathcal{U}\), then there exists an \((n - 2)\)-space \(K \leq H_\infty\) such that \(K \leq U_i\) for all \(1 \leq i \leq l\).

**Proof.** The statement is trivial for \(l \leq 1\), so that we assume \(l \geq 2\). Due to the dimensions we have \(\dim(U_i \cap U_j) = n - 2\) for all \(1 \leq i < j \leq l\). Since the sets of points \(U_i \setminus H_\infty\) and \(U_j \setminus H_\infty\) are disjoint, we have \(U_i \setminus U_j \leq H_\infty\) and \(U_i \cap U_j = U_i \cap H_\infty = U_j \cap H_\infty\). So, we set \(K = U_i \cap H_\infty\).

**Proposition 20.** Let \(\mathcal{U}\) be an irreducible avsp of \(PG(n - 1, q)\), where \(n \geq 3\). If \(\mathcal{U}\) is of type \(n - 1)^{m_{n-1}} \ldots 2^{m_1}\), then we have \(m_{n-1} \leq q - 2\) or \(m_{n-1} = q\). In the latter case \(\mathcal{U}\) is not tight.

**Proof.** We assume \(m_{n-1} = q - 1 \geq 1\) and let \(K \leq H_\infty\) as in Lemma 19. With this, let \(H \neq H_\infty\) be the unique hyperplane with \(K \leq H\) that is not contained as an element in \(\mathcal{U}\) and \(\mathcal{U}'\) arise from \(\mathcal{U}\) by removing the \(q - 1\) elements. Thus, \(\mathcal{U}'\) is an avsp of \(H\), i.e., \(\mathcal{U}'\) is reducible.

If \(m_{n-1} = q\), then the \((n - 2)\)-space \(K \leq H_\infty\) (as in Lemma 19) is contained in all elements of \(\mathcal{U}\), i.e., \(\mathcal{U}\) is not tight.

**Corollary 21.** Let \(\mathcal{U}\) be an irreducible tight avsp of \(PG(n - 1, 2)\) of type \(n - 1)^{m_{n-1}} \ldots 1^{m_1}\), where \(n \geq 3\). Then we have \(m_{n-1} = 0\).

Let \(\mathcal{U} = \{U_1, \ldots, U_r\}\) be an avsp of \(PG(n - 1, q)\), \(I \subseteq \{1, \ldots, r\}\), and \(V\) be a proper subspace with \(V \not\subseteq H_\infty\). If \(#I \geq 2\) and \(\{U_i : i \in I\}\) is an avsp of \(V\), then we say that the spaces \(U_i\) with \(i \in I\) can be joined to \(V\). Note that this is exactly the situation when \(\mathcal{U}\) is reducible. In \(PG(n - 1, 2)\) any two points outside of \(H_\infty\) can be joined to a line, so that:
Lemma 22. Let $U$ be an irreducible tight avsp of $PG(n-1, 2)$ of type $(n-1)^{m_{n-1}} \ldots 1^{m_1}$, where $n \geq 3$. Then, we have $m_1 = 0$.

Theorem 23. Let $U$ be an avsp of $PG(3-1, q)$ with type $2^{m_2}1^{m_1}$. Then, we have $0 \leq m_2 \leq q$, $m_1 = q \cdot (q - m_2)$, all lines in $U$ contain a common point $P \leq H_{\infty}$, and the 1-dimensional elements can be grouped into pairwise disjoint sets of size $q$ that can be joint to a line each.

Proof. The existence of $P$ follows from Lemma 19 and the parameterization of $m_2, m_1$ follows from the packing condition (1). If $m_2 = 0$ then choose an arbitrary point $P \leq H_{\infty}$. By $L_1, \ldots, L_q$ we denote the $q$ lines containing $P$ that are not equal to $H_{\infty}$. For each line $L_i$ that is not an element of $U$ there exist $q$ points in $U$ that can be joined to $L_i$. (Note that $L_i \cap L_j = P$ for all $1 \leq i < j \leq q$.)

We remark that all possibilities for $0 \leq m_2 \leq q$ can indeed by attained. In general there exist several non-isomorphic examples.

Corollary 24.

(1) Let $U$ be an irreducible avsp of $PG(3-1, q)$. Then $U$ is of type $2^q$ and non-tight.

(2) Let $U$ be an avsp of $PG(3-1, q)$ with type $2^{m_2}1^{m_1}$. Then, $U$ is tight iff $m_1 \geq 1$. In that case $U$ is reducible.

(3) No irreducible tight avsp of $PG(3-1, q)$ exists.

Let $U$ be an avsp of $PG(n-1, q)$, where $n \geq 3$, and $K \leq H_{\infty}$ be an arbitrary $(n-2)$-space. We say that $U^{(1)}, \ldots, U^{(q)}$ is a $K$-decomposition of $U$ if the $q$ hyperplanes containing $K$ and not being equal to $H_{\infty}$ can be labeled as $H_1, \ldots, H_q$ such that

$$U^{(i)} = \{U \cap H_i : U \in U, U \cap H_i \not\leq H_{\infty}\}$$

for all $1 \leq i \leq q$. Note that $U^{(i)}$ is an avsp of $H_i$ for each $1 \leq i \leq q$ (including the case $U^{(i)} = \{H_i\}$). Moreover, any labeling of the $q$ hyperplanes $H_i$ induces a $K$-decomposition. Observe that for a fixed $(n-2)$-space $K \leq H_{\infty}$ each pair of $K$-decompositions arises just by relabeling, so that we also speak of the $K$-decomposition of $U$ since the actual labeling will not matter in our context.

Proposition 25. Let $U$ be an avsp of $PG(n-1, q)$, where $n \geq 3$, with type $(n-1)^{m_{n-1}} \ldots 2^{m_2}1^{m_1}$. If $1 \leq m_{n-1} \leq q$, then there exists an $(n-2)$-space $K \leq H_{\infty}$ such that the $K$-decomposition $U^{(1)}, \ldots, U^{(q)}$ partitions $U$, i.e.,

$$\bigcup_{1 \leq i \leq q} U^{(i)} = U.$$ 

Moreover, if $m_{n-1} \leq q - 1$, then $U$ is reducible. (More precisely, for each index $1 \leq i \leq q$ with $\#U^{(i)} > 1$ the elements in $U^{(i)}$ can be joined to $H_i$.)

Proof. Choose some arbitrary $U \in U$ with $\dim(U) = n-1$, set $K := U \cap H_{\infty}$, and let $U^{(1)}, \ldots, U^{(q)}$ be the $K$-decomposition of $U$ and $H_1, \ldots, H_q$ be the corresponding hyperplanes. Due to Lemma 19 each $U \in U$ with $\dim(U) = n-1$ results in the same $(n-2)$-space $K$ and the same $K$-decomposition $U^{(1)}, \ldots, U^{(q)}$ (up to relabeling). Especially we have that for each $U' \in U$ with $\dim(U') = n-1$ there exists an index $1 \leq i \leq q$ with $U^{(i)} = \{U'\}$. W.l.o.g. we assume $\#U^{(1)} = 1$.

Now consider an element $U \in U$ with $\dim(U) < n-1$. From Lemma 19 we conclude $U \cap H_{\infty} \leq K$ since otherwise $\#U^{(1)} > 1$ (more precisely, $U$ would split into $q$ $(\dim(U) - 1)$-spaces where one of these would be contained in $U^{(1)}$ that also contains an entire hyperplane, which contradicts the packing condition (1)), which would contradict our assumption. Thus, for each $U \in U$ there exists exactly one index $1 \leq i \leq q$ with $U \in U^{(i)}$ and for each index $1 \leq j \leq q$ either $U \in H_j$ or $U \cap H_j \leq K \leq H_{\infty}$. □

Corollary 26. Let $U$ be an avsp of $PG(n-1, q)$ with type $(n-1)^{m_{n-1}} \ldots 2^{m_2}1^{m_1}$, where $n \geq 3$. If $U$ is irreducible, then we have $m_{n-1} = 0$. 

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Affine vector space partitions of $\text{PG}(4 - 1, q)$ that contain at least one hyperplane as an element can be characterized easily.

**Proposition 27.** Let $\mathcal{U}$ be an avsp of $\text{PG}(4 - 1, q)$ of type $3^{m_3}2^{m_2}1^{m_1}$ with $m_3 \geq 1$. Then, we have $1 \leq m_3 \leq q$, $0 \leq m_2 \cdot (q - m_3)$, and $m_1 = q^3 - q^2m_3 - qm_2$. Moreover, there exists an $(n - 2)$-space $K \leq H_\infty$ such that the $K$-decomposition $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(q)}$ partitions $\mathcal{U}$, so that $\mathcal{U}$ especially is reducible if $m_{n-1} \leq q - 1$.

**Proof.** The equation $m_1 = q^3 - q^2m_3 - qm_2$ directly follows from the packing condition (1) and the ranges $0 \leq m_2 \leq q \cdot (q - m_3)$, $0 \leq m_3 \leq q$ follow from the non-negativity of $m_1, m_2, m_3$. Note that we have $m_3 \geq 1$ by assumption. The remaining part follows from Proposition 25. 

4 Classification of tight irreducible avsps in $\text{PG}(n - 1, 2)$ for small dimensions $n$

The cases $n \leq 3$ have already been treated in Section 3 so that we assume $n \geq 4$ in the following. Our aim is to classify all possible types $(n - 1)^{m_{n-1}} \ldots 1^{m_1}$ such that a tight irreducible avsp $\mathcal{U}$ exists in $\text{PG}(n - 1, 2)$. We have $m_{n-1} = 0$ and $m_1 = 0$ due to Corollary 21 and Lemma 22. From Lemma 17, we conclude $m_l \neq 2$ for the smallest index $1 \leq l \leq n - 1$ with $m_l > 0$. The possible vectors $(m_{n-2}, \ldots, m_2) \in N_0^{n-3}$ are quite restricted by the packing condition (1). For $n = 4$ the only remaining possibility is type $2^1$. From Lemma 6 we conclude that the four lines are pairwise disjoint, i.e., they form a partial line spread of cardinality 4. It is well known that each partial line spread of cardinality $q^2$ in $\text{PG}(3, q)$ can be extended to a line spread, which has size $q^2 + 1$. For $q = 2$ there is only the Desarguesian line spread and since it has a transitive automorphism group, there is only one equivalence class. The numbers of line spreads in $\text{PG}(3, q)$ are 1, 2, 3, 21, 1347 for $q = 2, 3, 4, 5, 7$.

In the three subsequent subsections, we will consider tight irreducible avsps in $\text{PG}(n - 1, 2)$ for $n \in \{5, 6, 7\}$. The possible types are completely determined in all cases, where realizations are computed using an integer linear programming (ILP) formulation, see Section 5 in the appendix for the details. If the sizes of the avsps are not too large we were able to also compute all equivalence classes of avsps using a slight modification of an algorithm from [Lin04], see also [KÖ06, Algorithm 4.5]. A GAP implementation, based on the GAP package “FinInG” [BBC18] for computations in finite incidence geometry, can be obtained from the authors upon request. In the theoretical parts, we will also use classification for 2-divisible sets points that can e.g. be found in [HHK+17] or [Kur21]. For the convenience of the reader, we will also give a few selected proofs in Section 6 in the appendix.

4.1 Tight irreducible avsps in $\text{PG}(4, 2)$

We may use Lemma 15 and Lemma 18 to conclude that each avsp of $\text{PG}(n - 1, 2)$ of type $(n - 2)^4$ is non-tight if $n \geq 5$. However, we can further tighten the statement to:

**Proposition 28.** Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be an avsp of $\text{PG}(n - 1, 2)$, where $n \geq 4$, $r \geq 4$, and $\dim(U_i) = n - 2$ for all $1 \leq i \leq 3$. Then, the elements $\{U_4, \ldots, U_r\}$ can be joined to an $(n - 2)$-space $B$ (including the case $r = 4$ and $U_4 = B$) and there exists an $(n - 4)$-space $C$ that is contained in all elements of $\{U_1, U_2, U_3, B\}$.

**Proof.** First we assume that two elements of $\{U_1, U_2, U_3\}$ can be joined to an $(n - 1)$-space $H$. Without loss of generality, we assume that $U_1$ and $U_2$ can be joined to $H$. Let $K := H \cap H_\infty$, so that $\dim(K) = n - 2$. By $H'$ we denote the unique hyperplane containing $K$ that is not equal to $H$ or $H_\infty$. Observe that $\{U_3, \ldots, U_r\}$ is an avsp of $H'$ and $K$ is “the hyperplane at infinity” of $H'$. Next we set $K' := K \cap U_3$, so that $\dim(K') = n - 3$. Let $B$ denote the unique $(n - 2)$-space in $H'$ that contains $K'$ and is not equal to $U_3$ or $K$. With this,

---

1One argumentation is based on the fact that each $qk$-divisible (multi- set) of $\frac{q^k+1}{q+1}$ points forms a $(k + 1)$-space for each positive integer $k$, see e.g. [HHK18].
\{U_4, \ldots, U_r\} \text{ is an avsp of } B \text{ (including the case } r = 4, U_4 = B). \text{ Note that the } (n-3)\text{-space } K' \text{ is contained in all elements of } \{H, U_3, B\}. \text{ Since } \{U_1, U_2\} \text{ forms an avsp of } H \text{ and } \dim(U_1) = \dim(U_2) = n-2, \text{ there exists an } (n-4)\text{-space } C \text{ that is contained in all elements of } \{U_1, U_2, U_3, B\}.

Otherwise, we assume that no two elements of \{U_1, U_2, U_3\} can be joined to an \((n-1)\)-space, so that \(\dim(U_i \cap U_j) = n-4 \) for all \(1 \leq i < j \leq 3\). \text{ We set } \mathcal{E}_i := U_i \cap U_\infty, \text{ so that } \dim(\mathcal{E}_i) = n-3, \text{ for all } 1 \leq i \leq 3 \text{ and } \dim(\mathcal{E}_i \cap \mathcal{E}_j) = n-4 \text{ for all } 1 \leq i < j \leq 3. \text{ Let } \mathcal{K} := \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\} \subseteq U_\infty, \text{ so that } n-2 \leq \dim(K) \leq n-1. \text{ If } \dim(K) = n-2, \text{ then consider the } K\text{-decomposition } \mathcal{U}^{(1)}(U^{(2)} \cap \mathcal{U}) \text{ of } \mathcal{U} \text{ and let } H_1, H_2 \text{ be the corresponding hyperplanes. Since } \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \subseteq K, \text{ we have that either } U_i \subseteq H_1 \text{ or } U_i \subseteq H_2 \text{ for all indices } 1 \leq i \leq 3. \text{ By the pigeonhole principle two of the } (n-2)\text{-spaces in } \mathcal{U} \text{ can be contained in the same hyperplane, so that they can be joined, which is a contradiction. Thus, we have } \dim(K) = n-1, \text{ i.e., } K = U_\infty. \text{ Since } \dim((\mathcal{E}_1, \mathcal{E}_2)) = n-2, \dim(\mathcal{E}_3) = n-3, \text{ and } \dim(K) = n-1, \text{ we have}

\[
\dim((\mathcal{E}_1, \mathcal{E}_2) \cap \mathcal{E}_3) = n-4.
\]

Since \(\dim(\mathcal{E}_1 \cap \mathcal{E}_3) = \dim(\mathcal{E}_2 \cap \mathcal{E}_3) = n-4\), we have \(\dim(C) = n-4\) for \(C := \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3\). \text{ Pick three linearly independent vectors } v_1, v_2, v_3 \text{ such that } \mathcal{E}_1 = (C, v_1), \mathcal{E}_2 = (C, v_2), \mathcal{E}_3 = (C, v_3), \text{ and } U_\infty = (C, v_1, v_2, v_3). \text{ Let } P_1, P_2 \text{ be two different arbitrary points outside of } H_\infty \text{ that or not covered by } U_1, U_2, \text{ or } U_3. \text{ For pairwise different } i, j, h \in \{1, 2, 3\} \text{ consider the } (n-2)\text{-space } K_{i,j,h} := (C, v_i, v_j + v_h) \text{ and let } H_{i,j,h} \text{ be the hyperplane that contains } K_{i,j,h} \text{ and } U_i. \text{ Since all points in } H_{i,j,h} \setminus H_\infty \text{ are covered by } U_1, U_2, U_3 \text{ the points } P_1, P_2 \text{ have to be contained in the other hyperplane containing } K_{i,j,h} \text{ not equal to } H_{i,j,h} \text{ and } H_\infty, \text{ so that } P_i \in (C, v_i, v_j + v_h). \text{ Since}

\[
(C, v_1, v_2 + v_3) \cap (C, v_2, v_1 + v_3) \cap (C, v_3, v_1 + v_2) = (C, v_1 + v_2 + v_3),
\]

the \(2^{n-3}\) points outside of \(H_\infty\) that are not covered by \(U_1, U_2, \text{ or } U_3\) have to form an affine subspace \(B \supseteq C\). \text{ If } \#\mathcal{U} = r = 4, \text{ then } B = U_4. \text{ If } \#\mathcal{U} > 5, \text{ then the elements in } \{U_4, \ldots, U_r\} \text{ form an avsp of } B.

**Corollary 29.** \text{ Let } \mathcal{U} \text{ be an irreducible tight avsp of } PG(n-1, 2) \text{ of type } (n-2)^{m_{n-2}} \cdots 2^{m_2}, \text{ where } n \geq 5. \text{ Then, we have } m_{n-2} \leq 2.

Together with the conditions \(m_{n-1} = m_1 = 0\) and the packing condition (1) we obtain:

**Corollary 30.** \text{ Let } \mathcal{U} \text{ be an irreducible tight avsp of } PG(5-1, 2). \text{ Then the type of } \mathcal{U} \text{ is given by } 3^22^4, 3^12^4, \text{ or } 2^8.

All types can indeed be realized and corresponding numbers of equivalence classes are given by 3, 4, and 2, respectively. I.e., for \(n = 4\) we have 9 non-isomorphic examples in total. Below are representatives:

\begin{align*}
\text{E1, } & 2^8: \{0000, 01000, 01010, 01001, 01100, 00100, 01101, 00110, 01110, 01011\}, \\
\text{E2, } & 2^83^1: \{0000, 01000, 01010, 01001, 01100, 00100, 00110, 01001, 01010, 01111, 01110, 00110\}, \text{ and so on.}
\end{align*}

We remark that the hypothetical type \(3^22^2\) is also excluded by Corollary 17.

Similarly as we have constructed \(T'\) from the tail \(T\) in Lemma 15, we can consider the set \(\mathcal{U}' := \{U \cap H_\infty : U \in \mathcal{U}\}\) for an avsp \(\mathcal{U}\) of \(PG(n-1, q)\). \text{ If } \mathcal{U} \text{ is an irreducible tight avsp of } PG(n-1, q) \text{ of type } 2^{m_2}3^{m_3}, \text{ where } m_2 = q^{m_3-2} - qm_3, \text{ then } \mathcal{U}' \text{ is a configuration of } m_2 \text{ points and } m_3 \text{ lines in } H_\infty \cong PG(n-2, q). \text{ The points are pairwise disjoint, so that Lemma 15 yields that they form a } q\text{-divisible set. Any two lines can}
meet in at most a point. If \( n = 5 \), then any two lines indeed intersect in a point. So, the maximum point multiplicity is at most \( m_3 + 1 \). In the following we will theoretically classify the possibilities for \( \mathcal{U} \) for tight irreducible avsps of \( \text{PG}(4,2) \) of type \( 2^{m_2}3^{m_3} \) up to symmetry. Corollary 29 gives \( m_3 \in \{0,1,2\} \). First we will deduce two general necessary criteria for \( \mathcal{U} \).

**Lemma 31.** Let \( \mathcal{U} \) be an irreducible avsp of \( \text{PG}(n-1,q) \) not of type \((n-1)^9\) and \( \mathcal{U}' := \{ U \cap H_\infty : U \in \mathcal{U} \} \). Then \( \mathcal{U}' \) is spanning, i.e., \( \mathcal{U}' \) spans \( H_\infty \).

**Proof.** Assume that \( K \) is a hyperplane of \( H_\infty \) that contains all elements of \( \mathcal{U}' \). From Lemma 12 we can conclude that the \( K \)-decomposition \( \mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(q)} \), with corresponding hyperplanes \( H_1, \ldots, H_q \), is a partition of \( \mathcal{U} \), i.e., the elements of \( \mathcal{U}^{(i)} \) can be joined to \( H_i \) for all \( 1 \leq i \leq q \). Since we have assumed that \( \mathcal{U} \) is not of type \((n-1)^9\) we obtain a contradiction. \( \square \)

**Lemma 32.** Let \( \mathcal{U} \) be an avsp of \( \text{PG}(n-1,q) \) of type \((n-1)^{m_{n-1}} \ldots 2^{m_2} \) and \( \mathcal{U}' := \{ U \cap H_\infty : U \in \mathcal{U} \} \). For each hyperplane \( K \) of \( H_\infty \) let \( a^K_i \) denote the number of \( i \)-dimensional elements of \( \mathcal{U}' \) that are contained in \( K \) and \( b^K_i = m_{i+1} - a^K_i \) the number of \( i \)-dimensional elements of \( \mathcal{U}' \) that are contained in \( K \), where \( 1 \leq i \leq n-2 \). Then there exist \( c_{i,j}^{K} \in \mathbb{N}_0 \) for all \( 1 \leq j \leq q \), \( 1 \leq i \leq n-2 \), such that

\[
\sum_{j=1}^{q} c_{i,j}^{K} = a_{i}^{K} \quad \forall 1 \leq i \leq n-2, \\
\sum_{i=1}^{n-2} c_{i,j}^{K} \cdot q^i + b_{1}^{K} \cdot q^{i-1} = q^{n-2} \quad \forall 1 \leq j \leq q.
\]

**Proof.** For an arbitrary but fix hyperplane \( K \) of \( H_\infty \) let \( \mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(q)} \) be the \( K \)-decomposition of \( \mathcal{U} \) with corresponding hyperplanes \( H_1, \ldots, H_q \). From Lemma 12 we conclude that for each element \( U \in \mathcal{U} \) with \( U \cap H_\infty \leq K \) there exists an index \( 1 \leq j \leq q \) such that \( U \subseteq H_j \). The \( c_{i,j}^{K} \) just count how many \((i+1)\)-dimensional elements of that type are contained in \( H_j \). Since the hyperplanes \( H_1, \ldots, H_q \) are pairwise disjoint, we obtain Equation (6). From Lemma 13 we conclude that for each element \( U \in \mathcal{U} \) such that \( U \cap H_\infty \not\leq K \) we have \( \#(U \cap H_\infty) = q^{\dim(U)-2} \), so that the packing condition yields Equation (9). \( \square \)

Let us consider the case \( m_4 = 0 \) first. Here the \( 2^3 \) points in \( H_\infty \cong \text{PG}(3,2) \) form a spanning 2-divisible set \( \mathcal{P} \) of points. So, \( \mathcal{P} \) is either an affine solid attained in Example E6 or given by the points of a plane and an intersecting line without the intersection point attained in Example E1, cf. Lemma 52. In the first case the geometrical object of the pairwise disjoint eight lines, that are disjoint from a special plane, in \( \text{PG}(4,2) \) is also known under the name of a lifted MRD code or a vector space partition of \( \text{PG}(4,2) \) of type \( 2^3 3^1 \). The uniqueness up to symmetry is a well known result.

For \( m_3 = 1 \) the six points in \( H_\infty \cong \text{PG}(3,2) \) form a 2-divisible set \( \mathcal{P} \) of points, so that \( \mathcal{P} \) is given by two disjoint lines \( L_1, L_2 \), cf. Lemma 51. Let us denote the unique line in \( H_\infty \) by \( L \). Up to symmetry, the lines \( L_1, L_2, \) and \( L \) can be arranged as follows:

- all three lines are pairwise disjoint, so that there are no multiple points, see Example E5;
- \( L = L_1 \), so that there are three double points forming a line, see Example E2;
- \( L \) intersects both \( L_1 \) and \( L_2 \) in a point, so that we have two double points, see Example E3;
- \( L \) intersects exactly one of the lines \( L_1 \) and \( L_2 \) in a point, so that we have a unique double points, see Example E4.

For \( m_3 = 2 \) let us denote the two lines by \( L_1, L_2 \), their intersection point by \( P \), and their span by \( E \), which is a plane. The four points in \( \mathcal{P} \) form an affine plane, cf. Lemma 18. I.e., there exist a plane \( E' \) and a line \( L' \leq E' \) with \( \mathcal{P} = E' \setminus L' \). From Lemma 51 we conclude \( E \neq E' \), so that we set \( L^* := E \cap E' \) with \( \dim(L^*) = 2 \). From Lemma 52 we conclude that each hyperplane \( H \) of \( H_\infty \) that contains either the line \( L_1 \) or the line \( L_2 \), but not both, has to intersect \( \mathcal{P} \) in at least two points. (More technically, \( a^K_1 = 1 \) implies \( a^K_1 \geq 2 \).) With this we obtain the following list of possible arrangements of \( L_1, L_2 \), and \( E' \setminus L' \):
Proof. Since \( L \not\subset \langle P, L' \rangle \) for \( i = 1, 2 \), we have that \( |L_i \cap E' \setminus L'| = 1 \) for \( i = 1, 2 \), i.e., we have three double points forming a basis, see Example E7;

\( P \in \mathcal{P} = E' \setminus L' \). Since \( E \neq E' \), one of lines \( L_1, L_2 \) intersects \( E' \) exactly in the point \( P \), so that the other line meets \( \mathcal{P} \) in a second point, i.e., we have one triple and one double point, see Example E9;

\( P \in L' \). W.l.o.g. we assume \( L_1 \cap E' = P \), so that \( L_2 \leq E' \). Since the plane \( \pi := \langle L_1, L' \rangle \) is disjoint to \( \mathcal{P} \), we conclude \( L_2 \in \pi \), so that \( L_2 = L' \). So, \( P \) is the unique double point, see Example E8.

Thus we have completed the classification of possibilities for \( \mathcal{U}' \) that may correspond to tight irreducible avsps \( \mathcal{U} \) of \( \text{PG}(4, 2) \). We call the process of moving from \( \mathcal{U}' \) to \( \mathcal{U} \) the extension problem. An integer linear programming formulation is given in Section A in the appendix. Note that the extension problem comprises additional symmetry given by the pointwise stabilizer of \( H_\infty \) of order \( q^{n-1} \).

Given a set \( \mathcal{U}' \) satisfying all of the necessary conditions mentioned so far it is neither clear that an extension to a corresponding avsp \( \mathcal{U} \) always exists nor that it is, in the case of existence, unique up to symmetry. Indeed, we will give counter examples later on. However, for the nine classified configurations \( \mathcal{U}' \) in \( \text{PG}(3, 2) \) it turns out that there always is an up to symmetry unique extension.

### 4.2 Tight irreducible avsps in PG(5, 2)

**Lemma 33.** For \( n \geq 6 \) no tight irreducible avsp of type \((n - 2)^2(n - 3)^4\) in \( \text{PG}(n - 1, 2) \) exists.

**Proof.** Assume that such an avsp \( \mathcal{U} \) exists and consider the intersections of the elements with the hyperplane \( H_\infty \) at infinity, i.e., \( \mathcal{U}' := \{ U \cap H_\infty : U \in \mathcal{U} \} \). By \( E_1, E_2 \) we denote the two \((n - 3)\)-spaces and by \( L_1, \ldots, L_4 \) the four \((n - 4)\)-spaces. The intersection of \( E_1 \) and \( E_2 \) is an \((n - 4)\)-space \( L' \) and \( \dim(E_1, L_j) \geq n - 5 \) for all \( i = 1, 2 \) and \( j = 1, \ldots, 4 \). From Lemma 15 we conclude that \( \mathcal{T}' = \{ L_1, \ldots, L_4 \} \) is a 2-divisible set of \((n - 4)\)-spaces, so that Lemma 18 implies the existence of a plane \( E \leq H_\infty \), a line \( L \leq E \), and an \((n - 5)\)-space \( B \leq H_\infty \) with \( B \cap E = \emptyset \) and

\[
\{L_1, \ldots, L_4\} = \{ \langle Q, B \rangle : Q \in E \setminus L \}.
\]

Since \( E \) is tight we have \( B \cap L' = \emptyset \). However, \( \dim(E_1, L_j) \geq n - 5 \) implies \( \dim(L', L_j) \geq n - 6 \) for all \( 1 \leq j \leq 4 \). So, we clearly have \( n \leq 7 \).

For \( n = 7 \) we conclude \( E \leq E_1 \), \( E \leq E_2 \), \( \dim(B) = 2 \), \( \dim(E_1) = \dim(E_2) = 4 \), \( \dim(E) = 3 \), and \( \dim(L') = 3 \), so that \( L' \leq E \) and \( \dim(E_i \cap B) \geq 1 \) for \( i = 1, 2 \). Thus, we have \( \langle \mathcal{U}' \rangle \leq \langle E, B \rangle \), i.e., \( \mathcal{U}' \) is not spanning, which is a contradiction.

For \( n = 6 \) we have \( \dim(B) = 1 \), i.e., \( B \) is a point. Since \( E \) is tight we have \( B \not\subset L' \). W.l.o.g. we assume \( B \not\subset E_1 \). Since \( E_1 \) intersects each of the lines \( L_1 \) at least a point, we have \( E_1 = E \). Since \( \mathcal{U} \) is irreducible \( E_2 \) is not contained in the solid \( S := \langle E, B \rangle \). Since \( E_1 \) intersects each of the lines \( L_j \) at least a point, we have that the line \( L' \leq E \) intersects each of the lines \( L_j \) in at least a point. Since \( B \not\subset L' \) this is impossible.

**Proposition 34.** Let \( \mathcal{U} \) be a tight irreducible avsp of \( \text{PG}(5, 2) \), then \( \mathcal{U} \) has one of the following types:

- \( 4^{2i}2^{8-2i} \) for \( i \in \{0, 1, 2\} \);
- \( 4^{i}3^{i}2^{12-2i} \) for \( i \in \{0, \ldots, 6\}\setminus\{5\} \); and
- \( 3^{i}2^{16-2i} \) for \( i \in \{0, \ldots, 8\}\setminus\{7\} \).

All types are realizable.

**Proof.** Let the type of \( \mathcal{U} \) be \( 5^{m_5} \ldots 1^{m_1} \). From Corollary 21 and Lemma 22 we conclude \( m_5 = 0 \) and \( m_1 = 0 \), so that the packing condition 1 gives \( 4m_4 + 2m_3 + m_2 = 16 \). Corollary 24 gives \( m_4 \leq 2 \) and Lemma 33 excludes \( (m_4, m_3, m_2) = (2, 4, 0) \). Moreover, Corollary 17 implies \( m_2 \neq 2 \). All remaining possibilities \( (m_4, m_3, m_2) \in \mathbb{N}_0^3 \) are listed in the statement and for each type we found a realization using ILP computations.
Corollary 35. If \( \mathcal{U} \) is a tight irreducible avsp of \( \text{PG}(5, 2) \) of minimum possible size, then \( \#\mathcal{U} = 7 \) and \( \mathcal{U} \) has type \( 4^13^6 \).

For small sizes we have enumerated the isomorphism types of tight irreducible avsps in \( \text{PG}(5, 2) \), see Table I. The last row concerns the parts \( \mathcal{U}' \) at the hyperplane \( H_\infty \) at infinity w.r.t. the avsps \( \mathcal{U} \) counted up to isomorphy in the second row. So, for e.g. types \( 4^13^42^4 \) and \( 2^22^8 \) there exist configurations \( \mathcal{U}' \) that allow more than one extension up to symmetry.

| type       | \( 4^13^6 \) | \( 4^22^22^4 \) | \( 3^8 \) | \( 4^23^12^6 \) | \( 4^13^24^4 \) | \( 4^22^8 \) |
|------------|--------------|----------------|--------|----------------|----------------|----------|
| \# in \( H_\infty \) | 6            | 38             | 32     | 55             | 827            | 83       |

Table 1: Number of isomorphism types of tight irreducible avsps in \( \text{PG}(5, 2) \).

For the minimum possible size of a tight irreducible avsp in \( \text{PG}(5, 2) \) we can write down all implications of the stated necessary conditions for the part \( \mathcal{U}' \) at infinity. So, for type \( 4^13^6 \) configuration \( \mathcal{U}' \) consists of one plane \( E \) and six lines \( \mathcal{L} = \{ L_1, \ldots, L_6 \} \) satisfying the following conditions:

1. the configuration is spanning, i.e., \( \langle E, L_1, \ldots, L_6 \rangle = \text{PG}(4, 2) \);
2. the configuration is tight, i.e., there does not exist a point \( P \) that is contained in \( E \) and all lines in \( \mathcal{L} \);
3. the lines in \( \mathcal{L} \) form a 2-divisible set of lines, i.e., each hyperplane contains an even number of lines;
4. each line \( L_i \) intersects \( E \) in at least a point;
5. hyperplanes that contain \( E \) also contain at least two lines.

Up to symmetry ten such configurations exist:

\begin{align*}
E_1: & (10000, 01000, 00100), (10000, 01000), (10000, 00100), (10100, 00001), (10100, 01001) \\
E_2: & (10000, 01000, 00100), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_3: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10100, 01000) \\
E_4: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_5: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_6: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_7: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_8: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_9: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
E_{10}: & (10000, 01000, 00001), (10000, 01000), (10000, 00001), (10100, 00001), (10110, 01000) \\
\end{align*}

It turns out that \( E_2 \), \( E_4 \), \( E_7 \), and \( E_9 \) are not extendable to an avsp while the other six cases are. Moreover, the extension is unique up to symmetry in these cases.

4.3 Tight irreducible avsps in \( \text{PG}(6, 2) \)

Lemma 36. In \( \text{PG}(6, 2) \) no tight irreducible avsp of type \( 5^24^23^4 \) or \( 5^14^6 \) exists.

Proof. All two possibilities are excluded using ILP computations, see Section A. They are also excluded using GAP computations. \( \square \)

Proposition 37. Let \( \mathcal{U} \) be a tight irreducible avsp of \( \text{PG}(6, 2) \), then \( \mathcal{U} \) has one of the following types:

- \( 5^24^3i^22^{6-2j-4i} \) for \( i \in \{0, 1, 2\} \) and \( 0 \leq j \leq 8 - 2i \), where \( j + 2i \neq 7 \) and \( (i, j) \neq (2, 4) \);
- \( 5^14^3i^22^{4-2j-4i} \) for \( 0 \leq i \leq 4 \) and \( 0 \leq j \leq 12 - 2i \), where \( j + 2i \neq 11 \);
- \( 4^3i^22^{2-2j-4i} \) for \( 0 \leq i \leq 8 \) and \( 0 \leq j \leq 16 - 2i \), where \( j + 2i \neq 15 \) and \( i \neq 7 \).

All types are realizable.
Proof. Let the type of \( \mathcal{U} \) be \( 6^m \ldots 1^m \). From Corollary 21 and Lemma 23 we conclude \( m_6 = 0 \) and \( m_1 = 0 \), so that the packing condition (11) gives \( 8m_5 + 4m_4 + 2m_3 + m_2 = 32 \). Corollary 29 yields \( m_5 \leq 2 \) and Lemma 33 excludes \( (m_5,m_4,m_3,m_2) = (2,4,0,0) \). Moreover, Corollary 17 implies \( m_5 \neq 2 \) for the smallest index with \( m_j > 0 \). The two hypothetical types \( 5^24^23^4 \) and \( 5^14^6 \) are excluded in Lemma 36. All remaining possibilities \( (m_5,m_4,m_3,m_2) \in \mathbb{N}_4^4 \) are listed in the statement and for each type we found a realization using an ILP formulation, see Section A.

Corollary 38. If \( \mathcal{U} \) is a tight irreducible avsp of \( \text{PG}(6,2) \) of minimum possible size, then \( \# \mathcal{U} = 8 \) and \( \mathcal{U} \) has type \( 4^8 \).

Here we describe all four isomorphism types of homogeneous irreducible tight avsps \( \mathcal{U} \) of \( \text{PG}(6,2) \) of type \( 4^8 \). Geometrically each \( \mathcal{U} \) is given by eight solids \( S_1, \ldots, S_8 \) in \( \text{PG}(6,2) \) intersecting a hyperplane \( H_{\infty} \) in a plane (plus some extra conditions). Here we directly consider the part \( \mathcal{U}^* \) at infinity, i.e. the eight planes \( \pi_1, \ldots, \pi_8 \in H_{\infty} \cong \text{PG}(5,2) \) given by \( \pi_i = S_i \cap H_{\infty} \). The conditions for the pairwise intersections are

\[
1 \leq \dim(\pi_i \cap \pi_j) \leq 2 \quad \forall 1 \leq i < j \leq 8.
\]

(10)

Since the planes form a spanning 2-divisible set we have

\[
\# \{1 \leq i \leq 8 : \pi_i \not\subset H\} \in \{2,4,6,8\}
\]

(11)

for every hyperplane \( H \) of \( H_{\infty} \cong \text{PG}(5,2) \).

Let \( e_i \) denote the \( i \)-th unit vector, i.e., the vector with a 1 at the \( i \)-th position and zeros everywhere else. If the pairwise intersection of the planes \( \pi_i \) is a line in all cases then they span a solid, which contradicts the condition that not all eight planes can be contained in a hyperplane. W.l.o.g. we assume \( \pi_1 = \langle e_1,e_2,e_3 \rangle \) and \( \pi_2 = \langle e_3,e_4,e_5 \rangle \), i.e., the intersection point between \( \pi_1 \) and \( \pi_2 \) is \( e_3 \). Since the intersection of all eight planes is empty we assume w.l.o.g. that \( \pi_3 \) does not contain \( \pi_1 \cap \pi_2 = e_3 \). Up to symmetry we have the following three cases for \( \pi_3 \):

(a) \( \dim(\pi_1 \cap \pi_3) = 2, \dim(\pi_2 \cap \pi_3) = 1: \pi_3 = \langle e_1,e_2,e_4 \rangle \);

(b) \( \dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1, \dim(\pi_1,\pi_2,\pi_3) = 5: \pi_3 = \langle e_1,e_4,e_2 + e_5 \rangle \); and

(c) \( \dim(\pi_1 \cap \pi_3) = \dim(\pi_2 \cap \pi_3) = 1, \dim(\pi_1,\pi_2,\pi_3) = 6: \pi_3 = \langle e_1,e_4,e_6 \rangle \).

Starting from the three possibilities for \( \pi_1,\pi_2,\pi_3 \) we build up a graph whose vertices consist of the planes that have intersection dimension 1 or 2 with \( \pi_i \) for \( 1 \leq i \leq 3 \), c.f. Condition (10). Two vertices \( \pi \) and \( \pi' \) are connected by an edge if \( 1 \leq \dim(\pi \cap \pi') \leq 2 \), c.f. Condition (10). For these graphs we determine all cliques of size five and check Condition (11) afterwards:

(a) \( 3,014,435,152 \) cliques \( \rightarrow \) 432 cases;

(b) \( 2,198,293,872 \) cliques \( \rightarrow \) 0 cases;

(c) \( 1,218,975,648 \) cliques \( \rightarrow \) 320 cases.

The overall computation took just a few minutes. Note that the constructed 752 cases are just candidates for the extension problem to eight solids. Up to symmetry they decompose into just four non-isomorphic examples. It turns out that they can be distinguished by the maximum number \( \gamma_0 \) of incidences of a point and the eight planes, which has to lie between 2 and 5. In Table 2 we summarize incidence counts.

For \( \gamma_0 = 2 \) we consider an arbitrary plane \( \pi \) contained in the hyperbolic quadric \( Q = Q^+(5,2) \), which form a single orbit under its collineation group \( \text{PGO}^+(6,2) = C_2 \times \text{PGL}(3,2) = S_8 \) of order 40,320. From the 35 points on \( Q \) the points in \( \pi \) have no incidences with the eight planes while all other 28 points on \( Q \) have exactly two incidences. This example is obtained in 16 cases. The symmetry group of the eight planes has order 1344 and type \( C_2^3 : \text{PGL}(3,2) \).
In this section we collect a few general constructions for tight irreducible avsps using different combinatorial methods.

### Constructions of tight irreducible avsps

| $\gamma_0$ | 2 | 3 | 4 | 5 |
|-------------|---|---|---|---|
| point incidences | $2^8$ | $2^{21}2^33^4$ | $1^{16}2^{12}4^4$ | $1^{20}2^63^44^25^4$ |
| line incidences: | $1^{56}$ | $1^{56}$ | $1^{48}2^4$ | $1^{46}2^33^2$ |
| solid incidences: | $1^{56}$ | $1^{56}$ | $1^{48}2^4$ | $1^{40}2^8$ |
| hyperplane incidences: | $2^{28}$ | $2^{28}$ | $2^{24}4^2$ | $2^{23}4^6$ |
| case triples: | $c_{168}$ | $c_{168}$ | $a_{48}c_{96}$ | $a_{72}c_{48}$ |

Table 2: Irreducible tight avsps of $\PG(6,2)$ of type $4^8$.

For $\gamma_0 = 3$ choose a projective base of $\PG(5,2)$, i.e., put $f_i = e_i$ for $1 \leq i \leq 6$ and $f_7 = \sum_{i=1}^6 e_i$.

Consider a Fano plane on the set $\{1, 2, 3, 4, 5, 6, 7\}$:

\[
\ell_1 = \{1, 2, 3\}, \quad \ell_2 = \{1, 4, 5\}, \quad \ell_3 = \{1, 6, 7\}, \quad \ell_4 = \{2, 4, 6\}, \\
\ell_5 = \{3, 4, 7\}, \quad \ell_6 = \{2, 5, 7\}, \quad \ell_7 = \{3, 5, 6\}.
\]

Choose seven planes $\pi_i := \langle f_j : j \in \ell_i \rangle$ for $1 \leq i \leq 7$ and an eight plane. $\pi_8 = K := \langle \sum_{j \in \ell_i} f_j : 1 \leq j \leq 7 \rangle$. Note that $K$ itself is also a Fano plane (of course with a different embedding). The points with three incidences with the eight planes are the $f_i$ for $1 \leq i \leq 7$ and the points with two incidences with the eight planes are the points of $K$. This example is obtained in 112 cases. The symmetry group of the eight planes has order 168 and type $\PGL(3,2)$.

For $\gamma_0 = 4$ let $\{Q_1, Q_2, Q_3, Q_4, R_1, R_2\}$ be a basis of $H_\infty$. With this, we construct the eight planes as

\[
\langle Q_{i+j}, Q_{i+j+1}, R_i \rangle \text{ for } i \in \{1, 2\}, j \in \{0, 2\}, \\
\langle Q_{i+j}, Q_{i+j+1}, R_i + A \rangle \text{ for } i \in \{1, 2\}, j \in \{0, 2\},
\]

where $A = Q_1 + Q_2 + Q_3 + Q_4$ and $Q_5 = Q_1$. The points with four incidences with the eight planes are $Q_1, \ldots, Q_4$. The lines with two incidences with the eight planes are $\langle Q_i, Q_{i+1} \rangle$ for $1 \leq i \leq 4$ (again setting $Q_5 = Q_1$; so this is some kind of a cyclic construction). This example is obtained in 192 cases. The symmetry group of the eight planes has order 128 and type $D_4^2 : C_2$.

For $\gamma_0 = 5$ let $\{Q_1, Q_2, R_1, R_2, S_1, T_1\}$ be a basis of $H_\infty$. With this, we set $S_2 := S_1 + Q_1 + Q_2$, $T_2 := T_1 + Q_1 + Q_2 + R_1 + R_2$ and construct the eight planes as

\[
\langle Q_1, Q_2, R_i \rangle \text{ for } i \in \{1, 2\}, \\
\langle R_1, R_2, S_i \rangle \text{ for } i \in \{1, 2\}, \text{ and} \\
\langle Q_i, R_i, T_j \rangle \text{ for } i, j \in \{1, 2\},
\]

which also reflects the three orbits of the eight planes w.r.t. the action of their automorphism group. The points with five incidences with the eight planes are $R_1$ and $R_2$. The points with four incidences with the eight planes are $Q_1$ and $Q_2$. The points with three incidences with the eight planes are $R_1 + Q_1$ and $R_2 + Q_2$. The lines with three incidences with the eight planes are $\langle R_1, Q_1 \rangle$ and $\langle R_2, Q_2 \rangle$. The lines with two incidences with the eight planes are $\langle R_1, R_2 \rangle$ and $\langle Q_1, Q_2 \rangle$. This example is obtained in 432 cases. The symmetry group of the eight planes has order 1024.

### 5 Constructions of tight irreducible avsps

In this section we collect a few general constructions for tight irreducible avsps using different combinatorial objects. We use spreads, the Klein quadric, and hitting formulas in sections 5.1, 5.2 and 5.3 respectively.
5.1 Constructions from projective spreads

A $k$-spread in PG($n - 1, q$) is a disjoint set of $k$-spaces that partitions PG($n - 1, q$). It is well known that $k$-spreads exist if and only if $k$ divides $n$.

**Proposition 39.** For each positive even integer $n$ there exists a tight irreducible avsp $U$ of PG($n - 1, q$) of type $(n/2)^m$, where $m = q^{n/2}$.

*Proof.* Let $k = n/2$ and $P$ be a $k$-spread of PG($n - 1, q$), which has size $q^k + 1$. Now choose an arbitrary element $K \in P$ and an arbitrary hyperplane $H$ containing $K$. With this we set $U = P \setminus \{K\}$ where we choose $H$ as the hyperplane at infinity. By construction $U$ is an avsp of PG($n - 1, q$). Since all elements are pairwise disjoint $U$ is tight and since any two elements span PG($n - 1, q$) $U$ is irreducible.

We have seen that in PG(5, 2) there exist tight irreducible avsps of types $3^8$ and $2^{16}$. Starting from a 2-spread of PG(5, q) we can clearly obtain a tight avsp $U$ by removing all lines that are completely contained in an arbitrarily chosen hyperplane $H$. However, it may happen that $U$ is reducible. This is indeed the case if we start with the Desarguesian line spread. In PG(5, 2) there exist 131,044 non-isomorphic line spreads [MT09].

**Conjecture 40.** For each integer $1 < k < n$ that divides $n$ there exists a tight irreducible avsp $U$ of PG($n - 1, q$) of type $k^m$, where $m = q^{n-k}$.

If $n$ is odd no $(n-1)/2$-spread exists, but we can construct tight irreducible avsps from some special large partial spreads.

**Proposition 41.** For each odd integer $n \geq 5$ there exists a tight irreducible avsp $U$ of PG($n - 1, q$) of type $(n-1)/2)^m$, where $m = q^{(n+1)/2}$.

*Proof.* Let $k = (n-1)/2$ and $P$ be a vector space partition of PG($n - 1, q$) of type $(k+1)^2k^m$, where $m = q^{k+1}$. Now choose an arbitrary hyperplane $H$ containing the unique $(k+1)$-dimensional element $K$ of $P$. With this we set $U = P \setminus \{K\}$ where we choose $H$ as the hyperplane at infinity. By construction $U$ is an avsp of PG($n - 1, q$). Since all elements are pairwise disjoint $U$ is tight. Any two elements of $U$ span a hyperplane of PG($n - 1, q$). Since the elements of $U' := \{U \cap H_{\infty} : U \in U\}$ span $H_{\infty}$, not all elements of $U'$ can be contained in a hyperplane of $H_{\infty}$ and $U$ is irreducible.

Vector space partitions of the used type can be obtained from lifted MRD codes, see e.g. [SSW19] for a survey on MRD codes. They also occur as extendible partial $k$-spreads, where $k = (n-1)/2$, of the second largest size $q^{k+1}$ and are the main building block in the construction of partial $k$-spreads of size $q^{k+1} + 1$ as described by Beutelspacher [Beu75]. For more details on the relations between these different geometrical objects we refer e.g. to [HKK19].

For each $n \geq 5$ there also exist a vector space partition $P$ of PG($n - 1, q$) of type $(n-2)^22^m$, where $m = q^{n-2}$. Choosing a hyperplane that contains the unique $(n-2)$-space as the hyperplane at infinity we can obtain a tight avsp $U$ of PG($n - 1, q$) of type $2^{q^{n-2}}$. The remaining question is whether we can choose $P$ in such a way that $U$ becomes irreducible.

5.2 Constructions from the Klein quadric

It seems very likely that the avsp of PG(6, 2) of type $4^6$ with maximum point multiplicity 2, see Subsection 1.3, can be generalized to arbitrary field sizes.

**Theorem 42.** There exists a tight irreducible avsp of type $4^9$ in AG(6, q) for $q$ even.
Proof: We will use the following finite field model of AG(6, q). Let \( V = \mathbb{F}_q^3 \times \mathbb{F}_q^3 \times \mathbb{F}_q \) and let \( H_\infty \) be the hyperplane \( X_3 = 0 \). So we identify AG(6, q) with the elements of \( V \) of the form \((a, b, c)\), where \( c \neq 0 \).

Consider the following quadratic form on \( H_\infty \):

\[
Q(x, y, 0) := Tr_{q^3/q}(xy).
\]

Then \( Q \) defines the points of a hyperbolic quadric \( Q \). Next, let \( \pi \) be the plane \( \{(0, y, 0) : y \in \mathbb{F}_q^* \} \). Then \( \pi \) is totally singular with respect to \( Q \). Let \( S_0 := \{(x, 0, 1) : x \in \mathbb{F}_q^3 \} \) and \( S_1 := \{(y, y^q + y^q + 1, 1) : y \in \mathbb{F}_q^3 \} \).

Let \( \alpha \) be a primitive element of \( \mathbb{F}_q^3 \), let \( \sigma \) be the map

\[
\sigma : (x, y, z) \mapsto (\alpha^{-1}x, \alpha y, z),
\]

and let \( G := \langle \sigma \rangle \). We will show that \( \mathcal{S} := \{S_0\} \cup S_0^\sigma \) is a tight irreducible avsp of size \( q^3 \) in AG(6, q).

First note that \( \sigma \) has order \( q^3 - 1 \). Let \( (a, b, 1) \) be a point \( P \) of AG(6, q).

We show that \( P \) lies in a unique element of \( \mathcal{S} \). If \( b = 0 \), then \( P \) lies in \( S_0 \). The condition that \( P \) lies in \( S_0^m \) (where \( 1 \leq m \leq q^3 - 1 \)) can be restated as

\[
a = \alpha^{-m}y, \quad b = \alpha^m(y^{q^2} + y^q + 1).
\]

for some \( y \in \mathbb{F}_q^3 \). We have

\[
ab = y^{q^2+1} + y^{q+1} + y
\]

and the polynomial \( y^{q^2+1} + y^{q+1} + y \) is a permutation on \( \mathbb{F}_q^3 \), by [TZH14, Theorem 4]. Hence, \( y \) and, thus, \( m \) are determined by \( a \) and \( b \). Therefore, \( \mathcal{S} \) is an avsp.

Note that \( \pi_1 := \overline{S_1 \cap H_\infty} = \{(y, y^q + y^q, 0) : y \in \mathbb{F}_q^3 \} \) and \( \pi_0 := \overline{S_0 \cap H_\infty} = \{(x, 0, 0) : x \in \mathbb{F}_q^3 \} \). To compute the image of \( \pi_1 \) under \( \sigma^m \), notice that

\[
(\alpha^{-m}y, \alpha^m(y^{q^2} + y^q))^2 = (\alpha^{-m}y, \alpha^{(q^2+1)m}(\alpha^{-m}y)^q + \alpha^{(q+1)m}(\alpha^{-m}y)^q, 0) = (w, \zeta w^q + \zeta^q w^q, 0)
\]

where \( w = \alpha^{-m}y \) and \( \zeta = \alpha^{(q^2+1)m} \). Therefore, upon application of \( G \),

\[
\mathcal{S}_\infty := \{\overline{S_0 \cap H_\infty} : S \in \mathcal{S} \} = \{\pi_0\} \cup \pi_1^\sigma = \{\pi_\zeta : \zeta \in \mathbb{F}_q^3 \}
\]

where \( \pi_\zeta := \{(y, \zeta y^q + \zeta^q y^q, 0) : y \in \mathbb{F}_q^3 \} \). Note that \( |\mathcal{S}_\infty| = q^3 \) and that \( \mathcal{S} \) consists of totally singular planes of \( Q \) disjoint from \( \pi \). As these are all totally singular planes of \( Q \) disjoint from \( \pi \), these cover the points of \( Q \) uniformly and their common intersection is empty and, thus, \( \mathcal{S} \) is tight. As these pairwise meet in a point, any two elements of \( \mathcal{S} \) span PG(6, q). This shows irreducibility.

Let \( \mathcal{P} \) be the set of planes in the Klein quadric \( Q = Q^+(5, q) \) that is disjoint to an arbitrary but fixed plane \( \pi \) in \( Q \). One can verify that \( \mathcal{P} \) is a spanning \( q \)-divisible set of \( q^3 \) planes in PG(5, q) such that the intersection of a pair of planes is a point, i.e., all known conditions for the part \( U' \) at infinity of a tight irreducible avsp of PG(6, q) of type \( 4q^3 \) are satisfied. The remaining question is whether a solution of the extension problem for \( \mathcal{P} \) exists.

**Conjecture 43.** The extension problem for \( \mathcal{P} \) admits a solution for all prime powers \( q \).

Theorem[12] shows the conjecture for \( q \) even. By computer we showed Conjecture[13] for \( q = 3, 5 \).

### 5.3 Constructions using hitting formulas

A **hitting formula** is a DNF such that each truth assignment to the underlying variables satisfies precisely one term [Iwa89]. For example:

\[
(x \land y \land z) \lor (\bar{x} \land \bar{y} \land \bar{z}) \lor (\bar{x} \land y) \lor (\bar{y} \land z) \lor (\bar{z} \land x).
\]
its stars at the same positions can be joined to an affine subspace. For brevity, we speak of affine subspace, not every affine subspace corresponds to a string. It turns out that any two strings having $U \leq$ implies the upper bound $\#$(...). Interestingly enough, several tight irreducible avsps of $PG(n)$ exist. The minimum possible size of tight irreducible avsps $S$ of subspaces of $PG(3,2)$ such that there exists a configuration as in Lemma 44 and satisfies the conditions of Lemma 31, Lemma 32, and the dimension condition, cf. Lemma 8, admit a point $P$ that is contained in all elements of $U$. The minimum possible size of tight irreducible avsps

6 The minimum possible size of tight irreducible avsps

We have discussed the minimum possible size of a (tight) avsp of $PG(n-1,q)$ in Section 2. Before we consider the minimum possible size $\sigma_q(n)$ of a tight irreducible avsp $U$ of $PG(n-1,q)$ we remark that Lemma 22 implies the upper bound $\#U \leq 2^{n-2}$ for $q = 2$. The constructions mentioned in Section 6 suggest that this upper bound can be attained. In Section 3 and Section 4 we have determined the exact values $\sigma_q(2) = q$, $\sigma_q(3) = \infty$, $\sigma_q(4) = 4$, $\sigma_q(5) = 6$, $\sigma_q(6) = 7$, and $\sigma_q(7) = 8$.

Lemma 44. Let $U'$ be a configuration of a 5-space $K'$, four 4-spaces $S_1',...,S_4'$, and four planes $E_1',...,E_4'$ in $PG(6,2)$ such that there exists a 5-space $A'$ and three 6-spaces $B_1',B_2',B_3'$ satisfying

- $\dim(K' \cap A') = 4$, $S_1',S_2 \leq A'$, $\dim(S_3' \cap A') = \dim(S_4' \cap A') = 3$;
- $K' \not\subseteq B_1'$, $S_3',S_4' \not\subseteq B_1'$, $K',S_3',S_4' \not\subseteq B_2'$, $K' \leq B_3'$, $S_3',S_4' \not\subseteq B_3'$.

Then, then extension problem of $U'$ to a tight irreducible avsp $U$ of $PG(7,2)$ has no solution.

Proof. Assume that $U = \{K,S_1,...,S_4,E_1,...,E_4\}$ is an irreducible avsp of $PG(7,2)$ with $K \cap H_\infty = K'$, $S_i \cap H_\infty = S_i'$, $E_i \cap H_\infty = E_i'$ for all $1 \leq i \leq 4$. Let $A_1,...,A_4$ be the four 6-spaces that contain $A'$ but are not contained in $H_\infty$. From the known intersections with $A'$ we conclude

- $\#(K \setminus H_\infty \cap A_i) \in \{0,2^4\}$;
- $\#(S_j \setminus H_\infty \cap A_i) \in \{0,2^4\}$ for $j = 1,2$;
- $\#(S_j \setminus H_\infty \cap A_i) \in \{0,2^4\}$ for $j = 3,4$;
- $\#(E_j \setminus H_\infty \cap A_i) \in \{0,2^4\}$ for $j = 1,\ldots,4$.

for all $1 \leq i \leq 4$. From the intersections of the $B'_i$, where $1 \leq i \leq 3$, with $S_3'$ and $S_4'$ we conclude $\#(S_i \setminus H_\infty \cap A_j) = \#(S_i \setminus H_\infty \cap A_j)$ for all $1 \leq j \leq 4$. W.l.o.g. we assume $S_1 \setminus H_\infty \cap A_j \neq \emptyset$ for all $i = 1,2$ and $j = 1,2$. From the intersection of $K'$ with $B'_i$ we conclude that either $\#(K \setminus H_\infty \cap A_1) = 2^4$ or $\#(K \setminus H_\infty \cap A_2) = 2^4$. W.l.o.g. we assume $K \setminus H_\infty \cap A_1 = \emptyset$. With this the affine points of the elements of $U$ are either completely contained in $A_4$ or disjoint to $A_4$. This contradicts our assumption that $U$ is irreducible.

Lemma 45. In $PG(6,2)$ every configuration $U'$ of type $5^24^23^4$, $5^24^36$, $5^46$, $5^44^34$, or $4^8$ that does not contain a configuration as in Lemma 44 and satisfies the conditions of Lemma 21, Lemma 22 and the dimension condition, cf. Lemma 8, admits a point $P$ that is contained in all elements of $U'$. 

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Proof. All cases have been excluded by ILP computations.

Corollary 46. In PG(7,2) no tight irreducible avsp of the following types exist: 6^25^24^4, 6^25^44^6, 6^15^6, 6^15^44^4, 5^8.

Corollary 47. The minimum size \( \sigma_2(8) \) of an irreducible tight avsp of PG(7,2) is given by 10.

For attaining examples we refer to Section \( \square \) in the appendix.

Our next aim is a recursive construction which implies an asymptotic upper bound of roughly \( \frac{3n}{2} \) for \( \sigma_2(n) \).

Theorem 48. Let \( \mathcal{U} = \{U_1, \ldots, U_r\} \) be an irreducible tight avsp of PG(\( n-1, 2 \)) with \( \dim(U_1) = n - 2 \) and \( n \geq 3 \). Then, there exists an irreducible tight avsp \( \mathcal{U}' \) of PG(\( n+2-1, 2 \)) of size \#\( \mathcal{U} + 3 = r + 3 \) that contains an element of dimension \( n \).

Proof. Let \( V = PG(n+2-1, 2), H_\infty \) be the hyperplane at infinity, and \( K \leq H_\infty \) be an arbitrary subspace with \( \dim(K) = n \). With this, denote the two hyperplanes containing \( K \) and not being equal to \( H_\infty \) by \( H_1 \) and \( H_2 \). Choose an arbitrary point \( P \leq K \) and a subspace \( K' \leq K \) such that \( \dim(K') = n - 1 \) and \( \langle P, K' \rangle = K \). Now choose an irreducible tight avsp \( \mathcal{U} = \{U_1, \ldots, U_r\} \) of \( H_1/P \) such that \( \dim(U_1) = n - 2 \). We set \( A_i := \langle U_i, P \rangle \) for all \( 1 \leq i \leq r \). Choose an n-space \( B \) with \( B \cap H_1 = A_1 \) and \( B \nsubseteq H_\infty \), so that \( C_1 := B \cap H_2 \) is an \( (n-1) \)-space in \( H_2 \) with \( C_1 \nsubseteq H_\infty \) and \( P \subseteq C_1 \). In \( H_2 \) choose three further \( (n-1) \)-spaces \( C_2, C_3, C_4 \) such that \( \dim(C_i \cap C_j) = \dim(C_1 \cap C_2 \cap C_3 \cap C_4) = n - 3 \) for all \( 1 \leq i < j \leq 4 \), \( C_1 \cap C_2 \cap C_3 \cap C_4 \leq K' \), and that \( \{C_1, C_2, C_3, C_4\} \) forms an avsp of \( H_2 \). (This boils down to an avsp of PG(\( 4-1, 2 \)) of type \( 2^4 \), which is a union of four disjoint lines.) Then,

\[ \mathcal{U}' := \{A_2, \ldots, A_r, B, C_2, C_3, C_4\} \]

is an irreducible tight avsp of \( V \) of size \#\( \mathcal{U} + 3 = r + 3 \). The size follows directly from the construction and \( \dim(B) = n \). Since \( B \cap H_1 \cap H_\infty = B \cap H_2 \cap H_\infty \) we have \( B \cap C_2 \cap C_3 \cap C_4 = C_1 \cap C_2 \cap C_3 \cap C_4 \leq K' \) and \( B \cap A_2 \cap \cdots \cap A_r = A_1 \cap \cdots \cap A_r = P \), so that \( \mathcal{U}' \) is tight. Noting that \( \mathcal{U}'' := \{A_1, \ldots, A_r\} \) is an avsp of \( H_1 \), \( \{C_1, \ldots, C_4\} \) is an avsp of \( H_2 \), and \( \{A_1, C_1\} \) is an avsp of \( B \), we conclude that \( \mathcal{U}' \) is indeed an avsp of \( V \).

It remains to show that \( \mathcal{U}' \) is irreducible. So, assume that there exists a proper subset \( \mathcal{U} \subseteq \mathcal{U}' \) that can be joined to an x-space \( X \). If \( \mathcal{U} \cap \{B, C_2, C_3, C_4\} = \emptyset \), then we have \( \mathcal{U} \subseteq \mathcal{U}'' \) contradicting the fact that \( \mathcal{U}'' \) is irreducible. So, especially we have \( x \in \{n, n+1\} \). Noting that any two elements in \( \{C_1, C_2, C_3, C_4\} \) span \( H_2 \), we conclude \( \#(\mathcal{U} \cap \{B, C_2, C_3, C_4\}) = 1 \).

(i) If \( x = n \), then let \( 2 \leq i \leq 4 \) be the unique index such that \( C_i \in \mathcal{U}' \). Clearly, \( B \notin \mathcal{U}' \). Let \( \hat{C} \) be the other \( (n-1) \)-space in \( X \) not contained in \( H_\infty \) and not equal to \( C_i \) with \( C \cap H_\infty = C_1 \cap H_\infty \), so that the elements of \( \hat{C} \setminus \{C_i\} \) form a vector space partition of \( \hat{C} \). However, since \( P \nsubseteq C_i \) and all elements in \( \mathcal{U}' \setminus \{B, C_1, C_2, C_3\} \) contain \( P \), this is impossible.

(ii) If \( x = n+1 \) and \( \#(\mathcal{U} \cap \{B, C_2, C_3, C_4\}) = 1 \), then we have \( \dim(X') = n \) for \( X' := X \cap H_1 \). If \( B \in \mathcal{U}' \), then \( \mathcal{U}' \setminus \{B\} \cup \{A_1\} \) can be joined to \( X' \) in \( H_1 \), which is a contradiction. If \( C_i \in \mathcal{U}' \), then \( \mathcal{U}' \setminus \{C_i\} \) can be joined to \( X' \) in \( H_1 \), which is also a contradiction.

Thus, \( \mathcal{U}' \) is irreducible.

Corollary 49. For each \( n \geq 4 \) an irreducible tight avsp \( \mathcal{U} \) of PG(\( n-1, 2 \)) of size \( \left\lfloor \frac{3n-3}{2} \right\rfloor \) exists.

Proof. For \( n = 4 \) there exists such an example with type \( 2^4 \) and for \( n = 5 \) there exists such an example with type \( 3^22^4 \). Then, iteratively apply the construction from Theorem 48.

We remark that the constructive upper bound for \( q_2(n) \) is tight for \( n \in \{4, 5, 6, 8\} \).
7 Conclusion

We have introduced the geometrical object of affine vector space partitions. To make their study interesting we need the additional conditions of tightness and irreducibility, which are natural in the context of hitting formulas. A very challenging problem is the determination of the minimum possible size of an irreducible tight avsp of PG($n-1,q$). To this end we have obtained some preliminary results for arbitrary field sizes but small dimensions and for the binary case with medium sized dimensions. We also gave a parametric construction that matches the known exact values in many cases. That irreducible tight avsps are nice geometric objects can be e.g. seen at their sometimes large automorphism groups as well as the mentioned connection to the hyperbolic quadric $Q^+(5,q)$. While we have obtained a few insights, many questions remain open. So, we would like to close with a list of a few open problems:

1. Consider tight irreducible avsps of PG($4,q$) of type $2^m3^m3^3$. What is the largest possible value for $m_3$?
2. Determine a solution of the extension problem for the set $\mathcal{P}$ of $q^3$ planes in PG($5,q$) obtained from the hyperbolic quadric $Q^+(5,q)$ for $q$ odd, cf. Conjecture 43.
3. Determine further constructions for tight irreducible avsps of PG($n-1,q$) with large automorphism groups.
4. Construct a tight irreducible avsp of PG($n-1,q$) of type $2^q^{n-2}$ for all $n \geq 5$.
5. Is it possible that a tight irreducible avsp of PG($n-1,q$) contains 1-dimensional elements if $n \geq 3$ and $q \geq 3$?
6. Determine further exact values of the minimum size $\sigma_q(n)$ of a tight irreducible avsp of PG($n-1,q$).
7. Determine $\lim_{n \to \infty} \sigma_q(n)/n$.
8. Is $\sigma_q(n)$ strictly increasing in $n$?

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A Integer linear programming formulations

Let $\mathcal{U}'$ be an arbitrary set of subspaces of $H_{\infty}$ in PG($n-1,q$). For the question whether $\mathcal{U}'$ can be extended to an avsp $\mathcal{U}$ of PG($n-1,q$) we utilize binary variables $x_C$ for all subspaces $C$ of PG($n-1,q$) such that $C \not\subseteq H_{\infty}$ and $C \cap H_{\infty} \in \mathcal{U}'$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. We denote the set of all of these subspaces by $\mathcal{C}$. For each point $P$ in PG($n-1,q\setminus H_{\infty}$) the equation

$$\sum_{C \in \mathcal{C} : P \leq C} x_C = 1$$

(12)

and for each $U \in \mathcal{U}'$ the equation

$$\sum_{C \in \mathcal{C} : U \leq C} x_C = 1$$

(13)

has to be satisfied. The 0/1 solutions of this equation system are in one-to-one correspondence to extensions of $\mathcal{U}'$ to avsps $\mathcal{U}$ in PG($n-1,q$).
Searching a tight irreducible avsp $\mathcal{U}$ in $\text{PG}(n-1, q)$ directly can be achieved by a similar model. Now let $\mathcal{C}$ be the set of subspaces of $\text{PG}(n-1, q)$ that are not incident with $H_\infty$. Again we use binary variables $x_C$ for all $C \in \mathcal{C}$ with the meaning $x_C = 1$ iff $C \in \mathcal{U}$. Partitioning the affine points is modeled by
\[ \sum_{C \in \mathcal{C} : P \leq C} x_C = 1 \quad (14) \]
for all points $P$ not contained in $H_\infty$. The condition that $\mathcal{U}$ is tight can be written as
\[ \sum_{C \in \mathcal{C} : Q \leq C} x_C + 1 \leq \sum_{C \in \mathcal{C}} x_C \quad (15) \]
for all points $Q \leq H_\infty$. In order to model the condition that $\mathcal{U}$ is irreducible we say that a subspace $A$ escapes a subspace $B$ if $A$ has both points that are contained and points that are not contained in $B$. So, for each $B \in \mathcal{C}$ we require
\[ x_B + \sum_{C \in \mathcal{C} \text{ such that } C \text{ escapes } B} x_C \geq 1. \quad (16) \]
Of course we can fix the type of $\mathcal{U}$ by additional equations. Using a target function we can minimize or maximize $\#\mathcal{U}$ as well as the number of $i$-dimensional elements. We have to mention that this ILP formulation comprises a lot of symmetry, so that it can be solved in reasonable time for small parameters $n$ and $q$ only. However, we can use the inherent symmetry to fix some of the $x_C$ variables. I.e. the symmetry group acts transitively on the set of $a$-spaces that are not contained in $H_\infty$. For pairs of an $a$-space $A$ and a $b$-space $B$ that both are not contained in $H_\infty$, the different orbits under the action of the symmetry group are characterized by the invariant $\dim(A \cap B)$.

B Technical details

In order to keep the paper more readable, we have moved some technical details, that may also be left to the reader, to this section. The proof of Lemma 3 uses the numbers $m_j^{(s)}$ satisfying certain constraints. For completeness we state how those number can be computed in Algorithm 1.

In the three subsequent lemmas we characterize 2-divisible sets in $\text{PG}(3, 2)$ of cardinality $s \in \{3, 6, 8\}$.

Lemma 50. Let $\mathcal{P}$ be a 2-divisible set of three points in $\text{PG}(3, 2)$ then $\mathcal{P}$ forms a line.

Proof. Let $\mathcal{P} = \{P_1, P_2, P_3\}$ and $L := \langle P_1, P_2 \rangle$. Since all hyperplanes containing $L$ have to contain $\mathcal{P}$, we have $P_3 \in L$. □

Lemma 51. Let $\mathcal{P}$ be a 2-divisible set of six points in $\text{PG}(3, 2)$ then $\mathcal{P}$ is the disjoint union of two lines.

Proof. If $H$ is a hyperplane containing all points of $\mathcal{P}$, then there is a unique point $P \leq H$ with $P \notin \mathcal{P}$. Since every hyperplane $H'$ that does not contain $P$ intersects $\mathcal{P}$ in cardinality 3, so that this case cannot occur, i.e., $\mathcal{P}$ is spanning. From the standard equations we compute $a_0 = 0$, $a_2 = 9$, and $a_4 = 6$ for the spectrum. From the MacWilliams transform for the corresponding linear code we conclude the existence of a triple of points $\mathcal{P}'$ forming a line. Since $\mathcal{P}' \setminus \mathcal{P}$ is also 2-divisible the statement follows from Lemma 50. □

We remark that there exists a second 2-divisible set of six points – a projective base of dimension 5, which clearly cannot be embedded in $\text{PG}(3, 2)$.

Lemma 52. Let $\mathcal{P}$ be a 2-divisible set of eight points in $\text{PG}(3, 2)$ then $\mathcal{P}$ is either an affine solid or given by the points of a plane and an intersecting line without the intersection point.
Algorithm 1 Computing $m^{(j)}_i$

Input: $m_{n-1}, \ldots, m_1 \in \mathbb{N}_0$ with $\sum_{i=1}^{n-1} m_i \cdot q^{i-1} = q^{n-1}$

Output: $m^{(j)}_i \in \mathbb{N}_0$ with $\sum_{i=1}^{n-2} m^{(j)}_i \cdot q^{i-1} = q^{n-2}$ for all $n_{n-1} + 1 \leq j \leq q$

and $\sum_{j=m_{n-1}+1}^{q} m^{(j)}_i = m_i$ for all $1 \leq i \leq n-2$

$h \leftarrow n - 2$

for $m_{n-1} + 1 \leq j \leq q$ do

$r \leftarrow q^{n-2}$

while $r > 0$ do

$t \leftarrow \min \{r/q^{h-1}, m_h\}$

$m_h \leftarrow m_h - t$

$r \leftarrow r - t \cdot q^{h-1}$

if $t = 0$ then

$h \leftarrow h - 1$

end if

end while

end for

return $m^{(j)}_i$

Proof. Assume that $\pi$ is a hyperplane, which is a plane in our situation, containing six of the eight points and denote the unique uncovered point of $\pi$ by $P$. Each hyperplane that is incident with $P$ contains either two or six of the points in $\pi$. Thus, the remaining two points form a line $L$ containing $P$. Clearly, there is a unique example up to symmetry. Otherwise each hyperplane contains either 0, 2, or 4 points, so that the standard equations yield that there is a unique empty hyperplane and all other hyperplanes contain exactly four points, i.e., the point set is given by an affine solid.

We remark that both point sets can also be described as unions of two 2-divisible point sets, i.e., the union of two affine planes in the first case and the union of a line and a projective basis of size five in the second case.

C  Tight irreducible affine vector space partitions of minimum size that can be obtained by compression

In Subsection we have shown how avsps of $PG(n - 1, 2)$ can be obtained from hitting formulas by compression. In [PS22] irreducible hitting formulas of minimum possible mentioning all variables where enumerated up to seven variables. Going over their list we obtain the following examples of tight irreducible avsps that can be obtained by compression and that have the minimum possible size $\sigma_2(n)$, see Section []. The pairs of strings that can be compressed to an affine subspace are separated by horizontal lines.
Examples for $n = 5$:

- $00**$
- $1*0*$
- $010*$
- $1*10$
- $*111$
- $0110$
- $1011$

Examples for $n = 6$:

- $00***$
- $100**$
- $*100*$
- $1*10$
- $*1*11$
- $101*1$
- $011*0$
- $*1101$
- $*1010$

For $n = 7$ there is a unique example:

- $000***$
- $10*0**$
- $*1*000$
- $**111*$
- $*10*11$
- $*0110*$
- $*1101*$
- $*1101$
- $*110*1$
- $*10*10$
- $0010**$
- $1001**$

For $n = 8$, there are 26 irreducible hitting formulas of size 13 mentioning all $n - 1 = 7$ variables. Curiously enough, compression was always successful. Moreover, we can also obtain tight irreducible avsps of PG(7, 2) of minimum size $\sigma_2(8) = 10$ by compression starting from an irreducible hitting formulas with strictly more
than 13 terms:

| Term | Term | Term |
|------|------|------|
| **1**1**0** | **10**1**0** | 1*101*1 |
| **110**1**0** | 1101*0** |
| 00**1**0* | 010**0** | 0111**0** |
| 001**0** | 0111**0** | 1101**0** |
| 01**0**0* | 1*011**0** | 0111**0** |
| 1*101**0** | 1101*0** |
| 01111**0** | 1*001**0** | 11111**0** |
| 10111**0** | 00011**0** | 11111**0** |
| 001**0**1 | **001**1**0** | 00111**0** |
| 10111**0** | 11111**0** | 11111**0** |
| 01111**0** | 11011**0** | 11111**0** |
| 01111**0** | 00011**0** | 00111**0** |
| 01111**0** | 11011**0** | 10110**0** |
| 01111**0** | 11011**0** | 10110**0** |
| 01111**0** | 11011**0** | 10110**0** |

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