Generation of helical modes from a topological defect

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The propagation of an electromagnetic wave in a medium with a screw dislocation is studied. Adopting the formalism of differential forms, it is shown that torsion is responsible for quantized modes. Moreover, it is demonstrated that the modes thus obtained have well defined orbital angular momentum, opening the possibility to design liquid-crystal-based optical tweezers.

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I. INTRODUCTION

During the last decades, the interaction of the orbital angular momentum of light with matter has become a very active research field\textsuperscript{[1–4]} due to its large number of potential applications such as optical tweezers (for the manipulation of living cells and nanoobjects), micromachines (molecular engines) or quantum cryptography devices. In electrodynamics, it is indeed well known that light carries an angular momentum. This latter can be divided into two parts: a spin contribution associated to the polarization of the wave and an orbital contribution. A possible way of controlling the orbital angular momentum state of light beams is to use q-plates built out of liquid crystals\textsuperscript{[5]}. These q-plates coincide with the cross sections of topological line defects\textsuperscript{[6]} and therefore, understanding how the angular momentum of light interacts with a line defect is of prime interest.

From this point of view, the most relevant line defects are probably screw dislocations, because they locally induce torsion. A screw dislocation is a line defect that may occur in smectic C* liquid crystals\textsuperscript{[7]}, in ordinary crystals\textsuperscript{[8]} and even in spacetime\textsuperscript{[9]}. The generation of the defected topology is achieved through a “cut and glue” Volterra process, based on ideas of the homology theory\textsuperscript{[10]}: basically, the screw dislocation is generated by cutting the medium along a half-plane, moving the part located over the cut by a vector $\mathbf{b}$ (named Burgers vector) parallelwise to the edge of the cutting plane, and finally gluing the upper and lower sides. Thus, a screw dislocation is associated with a breaking of translational symmetry and it also exhibits an explicit helicity which, as we show below, has a profound influence on the angular momentum of a propagating electromagnetic field. Fig. 1 depicts a screw dislocation in a generic continuous medium. Assuming cylindrical coordinates and taking the axis of the defect to be the $z$-axis, it is clear that the screw dislocation mixes the $r$ and $\varphi$ degrees of freedom. In other words, by going clockwise a complete turn around the axis, one moves up by one unit of Burgers vector $b$.

\begin{center}
FIG. 1: Screw dislocation
\end{center}

An elegant way of taking into account the boundary condition
\[ \varphi \rightarrow \varphi + 2\pi \; \text{implies} \; z \rightarrow z + b \]  \hspace{1cm} (1)
is to use an Einstein-Cartan background \[11\]. This approach has also been used to describe elastic continuous media in analogy with gravity \[12\]. In this work, we look for a simple solution for an electromagnetic wave propagating along the axis of a screw dislocation. Since the main purpose of this article is to demonstrate the acquisition of orbital angular momentum by the propagating fields, this work may be relevant for applications both in condensed matter physics (particularly smectic C\* liquid crystals) and also in cosmology. For example, searching the cosmic microwave background for orbital angular momentum beams could give some clues on the existence of cosmic screw dislocations in the early universe. For simplicity we consider \(c = \varepsilon = \mu = 1\). Even though the problem at hand involves non-relativistic systems, for convenience, we work in a four-dimensional spacetime as it provides a framework in which Maxwell’s equations are naturally covariant. In cylindrical coordinates, the background geometry induced by the screw dislocation is given by the line element \[9, 11\]
\[
 ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + (dz + \beta d\varphi)^2,
\]
where \(\beta = b/2\pi\). Explicitly, the metric tensor is therefore
\[
g_{\mu\nu} = \begin{pmatrix}
 -1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & r^2 + \beta^2 & \beta \\
 0 & 0 & \beta & 1
\end{pmatrix}.
\]
(3)

It must be emphasized that for solid crystals, this geometrization of matter is actually qualitatively equivalent to determinations of actual properties of screw-dislocated dielectrics based on usual elasticity theory. An elastic defect is indeed expected to modify the dielectric properties in the vicinity of the dislocation as prescribed in eq. (11) of reference \[13\]. This latter has to be compared to the spatial part of the metric (3) rewritten in Cartesian coordinates, that is
\[
g_{ij} = \delta_{ij} + \frac{b^2}{2\pi r^2} \begin{pmatrix}
 0 & 0 & -y \\
 0 & 0 & x \\
 -y & x & 0
\end{pmatrix}.
\]
(4)

up to first order in \(\frac{b^2}{2\pi r^2}\). Therefore, it is clear that the anisotropy introduced in the dielectric medium by the screw dislocation, to a good approximation, can effectively be described by a background space with unit dielectric constant given by metric (4). However, this is done in a qualitative way since the coupling constant \(P_2\), between the strain field and the electromagnetic field, does not appear explicitly in our model. This is due to the fact that the starting point of the geometric approach is the boundary condition (11) and not the elasto-optic effect.

On the other hand, in order to describe electromagnetic waves propagating along the axis of a cosmic screw dislocation, we assume that there is no other source of geometry (gravitational field) in the vicinity of the defect. Also, the dislocation is supposed not to be rotating, which would include a coupling between \(t\) and \(\varphi\) in the metric, just like the one between \(z\) and \(\varphi\) due to the dislocation. Moreover, we consider a fixed background geometry, that is, we assume that the electromagnetic wave energy contribution to the local gravitational field is negligible. In either case of propagating electromagnetic fields along a screw dislocation, be it in condensed matter or in the cosmos, we have a geometrical background given by (2). Once the metric tensor is known, the language of General Relativity provides a powerful tool to determine the equations governing electrodynamics in the distorted background. This is the object of the next section.

II. MAXWELL’S EQUATIONS

In reference \[20\], Maxwell’s equations were found in the geometry induced by the presence of a cosmic dislocation using the differential forms formalism \[21, 22\]. Besides its natural elegance, the main advantage of this formalism is the fact that it provides a coordinate-free formulation of electrodynamics. Coordinates are introduced only when a specification of the field components is required. In what follows, the derivation of Maxwell’s equations in the screw-dislocated background is presented, following the steps of reference \[20\]. From the point of view of spacetime, we consider the approximation \[20\] where the electromagnetic field is taken as a weak perturbation on the spacetime metric. That way, the contribution of the electromagnetic field to the spacetime geometry is neglected and the Einstein-Maxwell equations are decoupled. From the point of view of solid-state physics, this approximation means that the electromagnetic field is supposed not to affect the elastic properties of the material medium.

In language of differential forms, Maxwell’s equations can be concisely expressed as \[22\]
\[
dF = 0
\]
(5)
and

\[ \star d \star F = J. \] (6)

Here, \( d \) denotes the exterior derivative, \( \star \) is the Hodge star operator (see Appendix), \( F \) is the Faraday 2-form defined as:

\[ F \equiv B + E \wedge dt. \] (7)

and \( J \) is the current density 1-form

\[ J = -\rho dt + J_r dr + J_\phi d\phi + J_z dz, \] (8)

The electric field 1-form is written as

\[ E = E_r dr + E_\phi d\phi + E_z dz, \] (9)

and the magnetic field 2-form is given by

\[ B = B_{\phi z} d\phi \wedge dz + B_{zr} dz \wedge dr + B_{r\phi} dr \wedge d\phi. \] (10)

In order to express Maxwell’s equations in terms of the electric and magnetic field components of the usual Euclidean space, it is necessary to find the transformation laws between the components of a differential form and its components in the Euclidean basis. The vector basis of 3-dimensional Euclidean space is the space subset of \( \mathcal{B}_v = \{ \hat{e}_t, \hat{e}_r, \hat{e}_\phi, \hat{e}_z, \} \) such that \( \hat{e}_\mu \cdot \hat{e}_\nu = g_{\mu\nu} \), where \( g_{\mu\nu} \) is the flat Minkowski metric. The above basis is not \( \mathcal{B}_v = \{ \hat{e}_t, \hat{e}_r, \hat{e}_\phi, \hat{e}_z, \} \), the dual basis of \( \mathcal{B}_v \), which is such that \( \hat{e}_\mu \cdot \hat{e}_\nu = g^{\mu\nu} \). The relation between the vectors in the basis \( \mathcal{B}_v \) and \( \mathcal{B}_v \) is therefore:

\[ \hat{e}_\mu = \frac{e_\mu}{\sqrt{|g_{\mu\mu}|}}. \] (11)

As a consequence, the transformation law between the Euclidean components of a vector \( \vec{v} \) and its contravariant components are related by \( [24] \)

\[ v^\mu = \frac{\hat{v}^\mu}{\sqrt{|g_{\mu\mu}|}}. \] (12)

(Notice that, in the last two equations, the sum convention for repeated indices should not be used.) Using the metric to obtain the dual vectors of the 1-form (that is the contravariant vectors) and \( [12] \), one finally gets the transformation laws between the components of a 1-form \( A \) and its components in the Euclidean basis:

\[ A_r = A^r, \quad A_\phi = \sqrt{\alpha^2 r^2 + \beta^2} A^\phi + \beta A^z \quad e \quad A_z = A^z + \frac{\beta}{\sqrt{\alpha^2 r^2 + \beta^2}} A^\phi. \] (13)

After some algebra, one finally obtains Maxwell’s equation in the presence of the screw dislocation as \( [20] \):

\[ \frac{1}{r} \frac{\partial}{\partial r} (r E^r) + \frac{1}{\sqrt{\alpha^2 r^2 + \beta^2}} \frac{\partial E^\phi}{\partial \phi} + \frac{\partial E^z}{\partial z} = \rho, \] (14)

for Gauss’ law, whereas for the three components of Ampère-Maxwell’s law, it comes:

\[ \frac{1}{r} \left[ \frac{\beta}{\sqrt{\alpha^2 r^2 + \beta^2}} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial z} \right] B^\phi + \left( \frac{\partial}{\partial \phi} - \beta \frac{\partial}{\partial z} \right) B^z = J^r + \frac{\partial E^\phi}{\partial t}, \] (15)

\[ \frac{\sqrt{\alpha^2 r^2 + \beta^2}}{r} \left[ \frac{\partial B^\phi}{\partial z} - \frac{\partial}{\partial r} \left( B^z + \frac{\beta}{\sqrt{\alpha^2 r^2 + \beta^2}} B^\phi \right) \right] = J^\phi + \frac{\partial E^\phi}{\partial t}, \] (16)
\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \sqrt{r^2 + \beta^2} \ B^\phi + \beta B^z \right) - \frac{\partial B^\phi}{\partial \phi} \right] = J^z + \frac{\partial E^z}{\partial t}. \tag{17}
\]

The equations describing the absence of magnetic monopoles and Faraday’s law are both obtained by making

\[
\rho = 0, \quad J^\phi = J^r = J^z = 0, \quad B^i \rightarrow E^i \quad \text{e} \quad E^i \rightarrow -B^i,
\]

where \( i \) corresponds to the indices \( r, \phi \) and \( z \). The absence of magnetic monopoles leads to

\[
\frac{1}{r} \partial_r \left( r B^r \right) + \frac{1}{\sqrt{r^2 + \beta^2}} \frac{\partial B^\phi}{\partial \phi} + \frac{\partial B^z}{\partial z} = 0, \tag{18}
\]

and the three components of Faraday’s law are given by:

\[
\frac{1}{r} \left( \frac{\beta}{\sqrt{r^2 + \beta^2}} \frac{\partial}{\partial \phi} - \frac{\sqrt{r^2 + \beta^2}}{\partial z} \right) E^\phi + \frac{1}{r} \left( \frac{\partial}{\partial \phi} - \beta \frac{\partial}{\partial z} \right) E^z = -\frac{\partial B^\phi}{\partial t}, \tag{19}
\]

\[
\frac{\sqrt{r^2 + \beta^2}}{r} \left[ \frac{\partial E^\phi}{\partial z} - \frac{\partial}{\partial r} \left( E^z + \frac{\beta}{\sqrt{r^2 + \beta^2}} E^\phi \right) \right] = -\frac{\partial B^\phi}{\partial t}, \tag{20}
\]

\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \sqrt{r^2 + \beta^2} \ E^\phi + \beta E^z \right) - \frac{\partial E^\phi}{\partial \phi} \right] = -\frac{\partial B^\phi}{\partial t}. \tag{21}
\]

III. AN HEURISTIC SOLUTION

Except for the plane wave, most of the propagating solutions of Maxwell’s equations are of quite complicated form. They are usually described as superpositions of plane waves or, depending on the coordinate system, special functions or polynomial expansions. Since we are interested in orbital angular momentum, it would appear natural to look for solutions of the Laguerre-Gaussian beam type \[18\], for example. Nevertheless, the aim of this article, being pedagogical, is to find a simple propagating solution that contains the essential physics of the system.

We consider a wave propagating along the Burgers vector of the defect. Therefore, it is licit to assume that variables can be separated according:

\[
\vec{E} = \vec{E}_0(r) u(\phi) \exp [ikz - i\omega t], \tag{22}
\]

\[
\vec{B} = \vec{B}_0(r) u(\phi) \exp [ikz - i\omega t],
\]

where \( u(\phi) \) a complex-valued function. That way, after some calculations, \((14)\) writes as the sum of two terms:

\[
\frac{\sqrt{r^2 + \beta^2}}{E_0^\phi} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r E_0^\phi \right) + ik E_0^\phi \right] + \frac{1}{u(\phi)} \frac{du}{d\phi} = 0, \tag{23}
\]

The first term depends only on \( r \) whereas the second depends only on \( \phi \). As a consequence, it is mandatory that:

\[
\frac{1}{u(\phi)} \frac{du}{d\phi} = C, \tag{24}
\]

with \( C \) being a constant complex number. Moreover, for symmetry reasons, it is required that under the transformation \( \phi \rightarrow \phi + 2\pi, z \rightarrow z + b \), the field remains unchanged so that finally:

\[
u(\phi) = \exp [id\phi], \tag{25}\]
with \( d \) a real number such that \( \nu = id \). As a consequence, the field is expected to have the following form:

\[
\vec{E} = \vec{E}_0(r) \exp \left[ ikz + id\varphi - i\omega t \right], \\
\vec{B} = \vec{B}_0(r) \exp \left[ ikz + id\varphi - i\omega t \right],
\]

(26)

where \( k \) is the wavevector along the \( z \) direction, \( d \) a real number, and \( \omega \) the angular frequency.

By direct substitution of (26) into the Maxwell’s equations (14)-(21), separating the real and imaginary parts of each equation, and solving the resulting system of equations, it comes that:

\[
\vec{E}_0(r) = \frac{a}{r} \hat{r} - \frac{a}{r^2} \sqrt{r^2 + \frac{\beta^2}{r^2}} \hat{\varphi} + \frac{\beta a}{r^2} \hat{z},
\]

(27)

and

\[
\vec{B}_0(r) = \frac{a}{r} \hat{r} + \frac{a}{r^2} \sqrt{r^2 + \frac{\beta^2}{r^2}} \hat{\varphi} - \frac{\beta a}{r^2} \hat{z},
\]

(28)

where the parameter \( a \) sets the intensity of the fields. We also get the dispersion relation

\[
k = \pm \omega
\]

(29)

but more importantly that the integer \( m \) is related to the Burgers vector and to the wavevector by

\[
m = \beta k.
\]

(30)

(Notice that the solution (26) satisfies the boundary condition (1) since equation (30) holds). This implies that solutions of the type (26) are quantized, that is only modes with definite wavevector \( k_m = m/\beta \) are allowed. This brings about interesting applications such as using the medium endowed with a defect as a filter for specific frequencies.

From equations (27) and (28), we obtain the Poynting vector

\[
\vec{S} = \frac{1}{2} \vec{E} \times \vec{B}^* = \frac{\beta a^2}{r^3} \hat{\varphi} + \frac{a^2 \sqrt{r^2 + \frac{\beta^2}{r^2}}}{r^3} \hat{z}.
\]

(31)

Moreover, with only \( \varphi \) and \( z \) components, it appears that the Poynting vector spirals along the direction of propagation. To verify this, we identify its components with the components of a tangent vector to a yet unknown space curve given in parametric form by \( r(t), \varphi(t), z(t) \). That is

\[
\dot{r}(t) = 0 \\
r(t) \varphi(t) = \frac{\beta a^2}{r^3} \\
\dot{z}(t) = \frac{a^2 \sqrt{r^2 + \frac{\beta^2}{r^2}}}{r^3}
\]

(32)

After straightforward calculations, the solutions of (32) are obtained as:

\[
r(t) = r_o \\
\varphi(t) = \frac{\beta a^2}{r_o^3} t + \varphi_o \\
z(t) = \frac{a^2 \sqrt{r_o^2 + \frac{\beta^2}{r_o^2}}}{r_o^3} t + z_o,
\]

(33)

where \( r_o, \varphi_o \) and \( z_o \) are integration constants. It is clear that the set of equations (33) describes a helix of radius \( r_o \) and pitch \( 2\pi r_o \sqrt{\frac{r_o^2}{r_o^2} + \frac{\beta^2}{r_o^2}} \). Notice that, in the absence of the defect, \( r = r_o, \varphi = \varphi_o \) and \( z = z_o + \text{const} \cdot t \), which represents a straight line along the \( z \)-axis.

Now, we turn our attention to vector potential \( \vec{A} \) and the scalar potential \( V \). These latter can be obtained from the electric and magnetic fields given by (26) assorted by an appropriate gauge condition. For convenience, the Coulomb gauge is used in all that follows, so that potentials are going to be obtained from:

\[
\vec{B} = \vec{\nabla} \wedge \vec{A} \\
\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V
\]

(34)

(35)

\[
\vec{\nabla} \cdot \vec{A} = 0 \\
\Delta V = 0
\]

(36)

(37)
Using the amplitudes of the fields as prescribed by (27) and (28), the previous set of equations gives for (34):

\[
\frac{1}{r} \frac{\partial A_z}{\partial r} - \frac{\partial A_\phi}{\partial z} = \frac{a}{r} e^{i(m\phi + kz - \omega t)} \tag{38}
\]
\[
\frac{\partial A_r}{\partial z} - \frac{\partial A_\phi}{\partial r} = \frac{a}{r} \sqrt{r^2 + \beta^2} e^{i(m\phi + kz - \omega t)} \tag{39}
\]
\[
\frac{\partial}{\partial r}(r A_\phi) - \frac{\partial A_r}{\partial \phi} = -\frac{\beta a}{r} e^{i(m\phi + kz - \omega t)} \tag{40}
\]

This system suggests that each of the unknown functions \(A_r, A_\phi\), and \(A_z\) are linear with respect to the factor \(e^{i(m\phi + kz - \omega t)}\). In particular, this implies that:

\[
\frac{\partial \bar{A}}{\partial t} = -i\omega \bar{A} \tag{41}
\]

Therefore, (35) gives the system:

\[
i\omega A_r - \frac{\partial V}{\partial r} = \frac{a}{r} e^{i(m\phi + kz - \omega t)} \tag{42}
\]
\[
i\omega A_\phi - \frac{1}{r} \frac{\partial V}{\partial \phi} = -\frac{a}{r} \sqrt{r^2 + \beta^2} e^{i(m\phi + kz - \omega t)} \tag{43}
\]
\[
i\omega A_z + \frac{\partial V}{\partial z} = \frac{\beta a}{r^2} e^{i(m\phi + kz - \omega t)} \tag{44}
\]

This in turn implies that the scalar potential \(V\) is linear with respect to the factor \(e^{i(m\phi + kz - \omega t)}\). Bearing in mind there is a similar property for the vector potential, this strongly suggests that for both potentials, variables can be separated in the following way:

\[
A_r = a_r(r) e^{i(m\phi + kz - \omega t)} \tag{45}
\]
\[
A_\phi = a_\phi(r) e^{i(m\phi + kz - \omega t)} \tag{46}
\]
\[
A_z = a_z(r) e^{i(m\phi + kz - \omega t)} \tag{47}
\]
\[
V = v(r) e^{i(m\phi + kz - \omega t)} \tag{48}
\]

Therefore, expressing the Coulomb gauge equations (36)-(37), it comes straightforwardly that:

\[
\frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = 0 \tag{49}
\]
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{50}
\]

Using (48) in (50), it comes that after some algebra that:

\[
\frac{d^2 v}{dX^2} + \frac{1}{r} \frac{dv}{dr} - \left( \frac{m^2}{r^2} + k^2 \right) v = 0 \tag{52}
\]

Performing the change in variable \(X = k r\), we can rearrange the previous expression to get:

\[
X^2 \frac{d^2 v}{dX^2} + X \frac{dv}{dX} - \left( m^2 + X^2 \right) v = 0 \tag{53}
\]

The solutions of this equation are the modified Bessel functions \(I_{\pm m}(X)\) and \(K_m(X)\). As the electric field involves the divergence of the scalar potential, it is natural to retain only the \(K_m(X)\) functions so that the electromagnetic field vanishes at infinity. Therefore, plugging \(v(r) = K_m(k r)\) in eqs (42)-(43) and using the ansatz (45)-(47), we are led to:

\[
a_r(r) = \frac{ik}{2\omega} \left( K_{m+1}(kr) + K_{m-1}(kr) + \frac{2a}{kr} \right) \tag{54}
\]
\[
a_\phi(r) = \frac{i}{\omega} \left( \frac{a}{r^2} \sqrt{r^2 + \beta^2} - im \frac{K_m(kr)}{r} \right) \tag{55}
\]
\[
a_z(r) = \frac{i}{\omega} \left( -\frac{\beta a}{r^2} + ik K_m(kr) \right) \tag{56}
\]
The orbital angular momentum is defined from the vector potential and the electric field by

\[
\vec{L} = \sum_{j=1}^{3} \frac{1}{\mu_0 c^2} \int d^3 x E_j \left( \vec{x} \wedge \vec{\nabla} \right) A_j
\]  

(58)

The volume density of angular momentum, which at the point \( \vec{R} \), is given by

\[
\vec{M} = \vec{R} \times \vec{S}.
\]  

(59)

A straightforward calculation gives

\[
\vec{M} = -z\beta a^2 \frac{\vec{e}_r}{r^2} - a^2 \sqrt{r^2 + \beta^2} \frac{\vec{e}_\phi}{r^2} + \beta a^2 \frac{\vec{e}_z}{r^2}.
\]  

(60)

It is well-known that with a unit system in which \( c = 1 \), the linear momentum density identifies with the Poynting vector. Thus, the ratio between the flux of angular momentum to that of energy across the surface \( z=\text{const} \) is given by:

\[
\frac{L}{P} = \frac{\int_{\delta}^{\infty} dr \int_{\delta}^{2\pi} r d\phi M_z}{\int_{\delta}^{\infty} dr \int_{\delta}^{2\pi} r d\phi S_z} = \beta \frac{\int_{\delta}^{\infty} dx}{\int_{\delta}^{\infty} \sqrt{1 + x^2} dx}
\]

where \( \delta \) is an ultraviolet cut-off corresponding to a core structure. In smectic liquid crystals, \( \delta \) is of the order of the average thickness of a layer [16], whereas in a cosmological context, \( \delta \) is about the inverse of the energy scale at which the symmetry-breaking phase transition occurs [17]. Then after simple manipulations, it comes that

\[
\frac{L}{P} = \beta = \frac{m}{\omega}
\]  

(61)

The solution (26) has therefore a well-defined orbital angular momentum. It has to be emphasized that even if this result is derived from a simple solution of Maxwell’s equations, it corresponds to what is obtained for realistic laser modes (Laguerre-Gaussian beams [18]).

Before closing this section a few remarks on the conservation laws are in order. It is interesting to notice that the Poynting vector (31) obeys the conservation law \( \vec{\nabla} \cdot \vec{S} = 0 \). Furthermore, the transversal part of the Poynting vector is also divergence-free. So, the beam intensity distribution does not change in the plane perpendicular to the direction of propagation. In other words, it is a non-diffracting beam (19). Also, the radial and azimuthal components of the angular momentum density are symmetric about the z-axis. This implies that integration over the beam profile leaves only the \( \vec{e}_z \) component. This is easily seen by writing \( \vec{M} \) in terms of its Cartesian components while keeping the cylindrical coordinates.

IV. CONCLUSION

In this work, we investigated some features of electrodynamics in the neighborhood of a screw dislocation. From the geometric treatment of topological defects, it appears that the torsion induced by the dislocation couples to the electromagnetic field in two ways. First, it is responsible for a quantization of the modes, for which the allowed frequencies depend only on the value of Burgers vector. This may be of prime interest for several potential applications such as defect sounding or X-ray filters or even the design on the heat rectifier devices [26], due to the periodicity of the screw dislocation. Second, the torsion forces the Poynting vector to spiral along the direction of propagation, possibly endowing the electromagnetic field with an orbital angular momentum. Such property is relevant in observational cosmology as a signature of cosmic strings, but it also provides an alternate approach to design optical tweezers from a simple (and tunable) waveguide effect.

One of the main interests of this work is that it can be generalized to other kinds of defects. Indeed, the differential forms formalism provides the general process of dealing with electromagnetism in non-trivial background geometries. Other kinds of line defects (edge dislocations, disclinations) and even distributions of defects can be treated this way, and one may expect strong couplings between the quantized modes in this last case. This will be the object of a next paper.

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APPENDIX: Hodge duality

In cylindrical coordinates, the 1-form basis writes:

\[ B_1 = \{ dt, dr, d\phi, dz \} \]  

In electrodynamics, the components of the Faraday 2-form that accounts for the field write as [21]:

\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \]  

where \( dx^\mu \wedge dx^\nu \) are elements of the 2-form basis.

\[ B_2 = \{ d\phi \wedge dz, dz \wedge dr, dr \wedge d\phi, dr \wedge dt, d\phi \wedge dt, dz \wedge dt \}. \]  

To translate the usual Maxwell’s equations in terms of differential forms, it is convenient to introduce the Hodge star operator \( \star \). This latter acts on a \( p \)-form in \( n \)-dimensional space and turns it into the \( (n - p) \)-form that somehow completes the volume \( n \)-form. Given the product of two \( p \)-forms \( \rho \) and \( \psi \) defined on an oriented \( n \)-manifold described by metric \( g_{\mu\nu} \), then the Hodge star operator is defined as:

\[ \rho \wedge \star \psi = \sqrt{|\text{det}(g_{\mu\nu})|} \langle \rho, \psi \rangle dx^1 \wedge .. dx^{n-p} \]  

Taking into account metric [2], the action of \( \star \) on the 2-forms of \( B_2 \) is then:

\[ \star (d\phi \wedge dz) = -\frac{1}{r} dr \wedge dt, \]  

\[ \star (dz \wedge dr) = -\frac{r^2 + \beta^2}{r} d\phi \wedge dt - \frac{\beta}{r} dz \wedge dt, \]  

\[ \star (dr \wedge d\phi) = -\frac{\beta}{r} d\phi \wedge dt - \frac{1}{r} dz \wedge dt, \]  

\[ \star (dr \wedge dt) = rd\phi \wedge dz, \]  

\[ \star (d\phi \wedge dt) = \frac{1}{r} dz \wedge dr - \frac{\beta}{r} dr \wedge d\phi, \]  

\[ \star (dz \wedge dt) = \frac{r^2 + \beta^2}{r} dr \wedge d\phi - \frac{\beta}{r} dr \wedge dz. \]  

On the elements of the 3-form basis \( B_3 \), the action of Hodge’s star operator is

\[ \star (dr \wedge d\phi \wedge dt) = -\frac{1}{r} dz - \frac{\beta}{r} d\phi, \]  

\[ \star (dz \wedge dr \wedge dt) = -\frac{r^2 + \beta^2}{r} d\phi - \frac{\beta}{r} dz, \]  

\[ \star (d\phi \wedge dz \wedge dt) = -\frac{1}{r} dr, \]  

\[ \star (dr \wedge d\phi \wedge dz) = -\frac{1}{r} dt. \]  

[1] A. Muthukrishnan and C.R. Stroud, J. Opt. B: Quantum semi-classical Opt. 4, S73 (2002).
[2] M. Babiker, C.R. Bennett, D.L. Andrews, and L.C.D. Romero, Phys. Rev. Lett. 89, 143601 (2002).
[3] A. Alexandrescu, D. Cojoc, and E. Di Fabrizio, Phys. Rev. Lett. 96, 243001 (2006).
[4] S. Thanvanthri, K.T. Kapale and J.P. Dowling, Phys. Rev. A 77, 053825 (2008).
[5] L. Marrucci, C. Manzo and D. Paparo, Phys. Rev. Lett. 96, 163905 (2006).
[6] C. Sátiro and F. Moraes, Eur. Phys. J. E 20, 173 (2006).
[7] M.-F. Achard, M. Kleman, Yu. A. Nastishin and H.-T. Nguyen, Eur. Phys. J. E 16, 37 (2005).
[8] Frank R.N. Nabarro and John P. Hirth (Editors) Dislocations in Solids, Volume 12, Elsevier, Amsterdam (2004).
[9] D. V. Gal’tsov and P. S. Letelier, Phys. Rev. D 47, 4273 (1993).
[10] M. Kleman, Points, Lignes, Parois dans les fluides anisotropes et les solides cristallins, Edition de Physique, Paris (1977).
[11] R. A. Puntigam and H. H. Soleng, Class. Quantum Grav. 14, 1129 (1997).
[12] M. O. Katanaev and I. V. Volovich, Ann. Phys. 216, 1 (1992).
[13] D. Sahoo, A.K. Arora and R. Kesavamoorthy, J. Phys. C: Solid State Phys. 16, 1687 (1983).
[14] C. Furtado, V.B. Bezerra, F. Moraes, Europhys. Lett. 52, 1 (2000).
[15] P.A.M. Dirac, Proc. R. Soc. A 133 60 (1931).
[16] P.G. De Gennes, J. Prost, The physics of liquid crystals (2nd edition), Oxford Science Publication (1995).
[17] B. Allen, A. C. Ottewill, Phys. Rev. D 42, 2669 (1990).
[18] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw and J. P. Woerdman, Phys. Rev. A 45, 8185 (1992).
[19] R. Horak, Z. Bouchal and J. Bajer, Opt. Comm. 133, 315 (1997).
[20] L. Dias and F. Moraes, Braz. J. Phys. 35, 636 (2005).
[21] For a very clear introduction to electromagnetism with differential forms in three space dimensions see: K. F. Warnick, R. Selfridge and D. V. Arnold, IEEE Trans. Ed., 40, 53 (1997).
[22] For a detailed account of the Maxwell’s equations in the differential form approach, in four spacetime dimensions, see: J. Baez and J. P. Muniain, Gauge fields, knots and gravity, World Scientific, Singapore (1994).
[23] J. Petterson, Phys. Rev. D 10, 3166 (1974).
[24] The explanation for that comes from the fact that we have $\vec{v} = v^\mu \vec{e}_\mu = v^\mu \vec{e}_\mu$.
[25] J.D. Jackson, Classical Electrodynamics (3rd edition), John Wiley and Sons (1999) p. 350.
[26] M. Maldovan, Nature 503, 209 (2013).