On A Moving Average With Internal Degrees of Freedom

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Abstract—A new type of moving average is developed. Whereas a regular moving average (e.g. of price) has a built-in internal time scale (time–window, exponential weight, etc.), the moving average developed in this paper has the weight as the product of a polynomial by window factor. The polynomial is the square of a wavefunction obtained from an eigenproblem corresponding to other observable (e.g. execution flow \( I = dV/dt \), the number of shares traded per unit time). This allows to obtain an immediate “switch” without lagging typical for regular moving average. In previous work \[30\], \[31\] the “switch” was obtained by applying an advanced technique of secondary sampling, where a calculated value was used as it were a new observable for sampling. In this work we have implemented “switching” regime using secondary sampling and generalized previous results to be applied to a regular moving average. The main new result is the developments of a new concept — a moving average with internal degrees of freedom.

I. INTRODUCTION

In the modern world, when conducting experiments and measurements, data processing is of critical importance \[1\], \[2\], \[3\], \[4\]. To solve various problems, the data obtained as a result of measurements or calculations are grouped and further processed \[5\], \[6\], \[7\], \[8\]. One of these options is timeseries. Timeserie data is widely available and used. A typical problem in data analysis to study timeserie properties and, possibly, trying to build a predictive model. Built model are typically of in data analysis to study timeserie properties and, possibly, trying to build a predictive model. Built model are typically of in data analysis to study timeserie properties and, possibly, trying to build a predictive model. Built model are typically of

\[
\langle Q_m f \rangle = \int_{-\infty}^{t_{new}} dt \omega(t)Q_m(x(t))f(t)
\]

The \( \omega(t) \) is decaying exponent and \( x(t) \) is either linear or exponential function:

\[
\omega(t) = \exp\left(-\frac{(t-t_{now})}{\tau}\right)
\]

\[
x(t) = \begin{cases} \frac{(t-t_{now})}{\tau} & \text{Laguerre basis} \\ \exp\left(-\frac{(t-t_{now})}{\tau}\right) & \text{shifted Legendre basis} \end{cases}
\]

The moments of \( f \) are usually obtained from direct sampling of all available observations \( l = 1 \ldots M \) in a timeserie:

\[
\langle Q_m, f \rangle = \sum_{l=1}^{M} f(t_l)Q_m(x(t_l))\omega(t_l) [t_l - t_{l-1}]
\]

the moments of a derivative \( df/dt \) can also be obtained from direct sampling:

\[
\langle Q_m \frac{df}{dt} \rangle = \sum_{l=1}^{M} Q_m(x(t_l))\omega(t_l) [f(t_l) - f(t_{l-1})]
\]

Given a good choice of basis polynomials:

\[
Q_m(x) = \begin{cases} L_m(-x) & \text{Laguerre basis} \\ P_m(2x - 1) & \text{shifted Legendre basis} \end{cases}
\]

one can calculate (with double precision arithmetic) the moments to a very high order \( m \lesssim 50 \) (limited by the divergence of \( c_m^h \) multiplication coefficients \[11\]) in Laguerre basis and \( m \lesssim 150 \) (limited by poorly conditioned matrices) in shifted Legendre basis; Chebyshev polynomials \( T_m(2x - 1) \) also
provide very stable calculations in shifted Legendre basis (Chebyshev polynomials have perfectly stable multiplication: all \( c_m^j = 0 \) except \( c_{j-k}^j = c_{j+k}^k = 0.5, j \geq k \). All obtained results are invariant with respect to basis choice, \( Q_m(x) = x^m \) and the ones from \( \psi \) give identical results, but numerical stability can be drastically different\[32, 33\].

III. ON AVERAGING OF PAST OBSERVATIONS

Consider familiar demonstration with a moving average. Let \( P^\tau \) be a regular exponential moving average. The average \( \langle \cdot \rangle \) is calculated with the weight \( 2 \):

\[
P^\tau(t_{\text{now}}) = \langle p \rangle = \frac{\langle Q_0 p \rangle}{\langle Q_0 \rangle} = f(t_{\text{now}}) dV(t) p(t) = \int_{-\infty}^{t_{\text{now}}} dV(t) \omega(t) p(t)
\]

The averaging \( d\mu = \omega(t)dt \) takes place between the past and \( t_{\text{now}} \) using exponentially decaying weight \( \omega(t) = \exp(-t_{\text{now}} - t)/\tau \). With \( \tau \) increase, the contributing to integral interval becomes larger and moving average “shifts to the right” (\( \tau \)-proportional time delay, lagging indicator). The \( P \) has no single parameter that can “adjust” the time scale, see Fig. 1 for a demonstration. From this demonstration it is clear that all moving average dependencies are smooth, with the time scale of \( \tau \).

Consider a different approach. Introduce a wavefunction \( \psi(x) \) as a linear combination of basis function \( Q_k(x) \):

\[
\psi(x) = \sum_{k=0}^{n-1} \alpha_k Q_k(x)
\]

Then an observable market–related value \( f \), corresponding to the probability density \( \psi^2(x) \), is calculated by averaging timeseries sample with the weight \( d\mu = \psi^2(x(t))\omega(t)dt \); the expression corresponds to an estimation of Radon–Nikodým derivative \[34\].

\[
f_\psi = \frac{\langle \psi f \psi \rangle}{\langle \psi \psi \rangle}
\]

For averages we use bra–ket notation by Paul Dirac (\( \langle \psi \rangle \) and \( |\psi\rangle \)). The \( \langle \psi \rangle \) is plain ratio of two moving averages, but the weight is not regular decaying exponent \( \omega(t) \) from \( \langle \psi \rangle \), but exponent multiplied by wavefunction squared as \( d\mu = \psi^2(x(t))\omega(t)dt \); the \( \psi^2(x) \) defines how to average a timeseries sample. Any \( \psi(x) \) function is defined by \( n \) coefficients \( \alpha_k \), the value of an observable variable \( f \) in \( \psi(x) \) state is a ratio of two quadratic forms on \( \alpha_k \[10\]. Regular moving average \( \langle \cdot \rangle \) corresponds to \( \psi(x) = \text{const} \). Whereas typical approaches (Fourier, least squares, linear regression, wavelets, etc.) deals with vector of moments \( \langle Q_m f \rangle, m = 0 \ldots n-1 \), the quadratic forms ratio \[10\] operates with matrices \( \langle Q_j | f | Q_k \rangle \) and \( \langle Q_j | Q_k \rangle \). The matrices can be obtained from \( \langle Q_m f \rangle, m = 0 \ldots 2n-2 \), moments using multiplication operator:

\[
Q_j Q_k = \sum_{m=0}^{j+k} c_{j,k}^m Q_m
\]

The main idea of \[33\] is to consider a wavefunction \( \langle \psi \rangle \) then to construct \[10\] quadratic forms ratio. A generalized eigenvalue problem is then considered with the two matrices from \[10\].

We established, that execution flow \( I = dV/dt \) (the number of shares traded per unit time), not trading volume \( V \) (the number of shares traded), is the driving force of the market: asset price is much more sensitive to execution flow \( I \) (dynamic impact), rather than to traded volume \( V \) (regular impact). This corresponds to the matrices \( \langle Q_j | I | Q_k \rangle \) and \( \langle Q_j | Q_k \rangle \). These two matrices are volume- and time- averaged products of two basis functions. Generalized eigenvalue problem for operator \( I = dV/dt \) is then:

\[
\lambda = \sum_{k=0}^{n-1} \langle Q_j | I | Q_k \rangle \alpha_k = \lambda \sum_{k=0}^{n-1} \langle Q_j | Q_k \rangle \alpha_k
\]

Our analysis shows that among the states \( |\psi[i]\rangle \) of the problem \[12\] the state corresponding to the maximal eigenvalue among all \( \lambda[i], \ i = 0 \ldots n - 1 \), is the most important for market dynamics. Consider various observable characteristics in this state \( |\psi[H]\rangle \).

In Fig. 2 a demonstration of several observables is presented: the price in \( |\psi[H]\rangle \) state \[15\], maximal eigenvalue \( \lambda[H] \) of \[12\] problem, and minimal eigenvalue \( \lambda[L] \) (for completeness).

\[
P_{[H]} = \frac{\langle \psi[H] | p I | \psi[H] \rangle}{\langle \psi[H] | I | \psi[H] \rangle}
\]

From these observable one can clearly see that singularities in \( I \) cause singularities in price, and that a change in \( |\psi[H]\rangle \)
localization causes an immediate “switch” in an observable. This switch is caused by the presence of $n - 1$ internal degrees of freedom $\alpha_k$ ($n$ coefficients [14], one less due to normalizing $1 = \langle \psi | \psi \rangle$). Such a “switch” is not possible in regular moving average [7] since it has no any internal degree of freedom, hence, all regular moving average dependencies are smooth.

IV. ON LOCALIZATION CHANGES IN $|\psi^{[IH]}\rangle$

Considered above the state $|\psi^{[IH]}\rangle$ maximizes the number of shares traded per unit time on past observations sample; it determines the time scale. Let us consider in this state not the price and execution flow as we studied before, but simply time distance to “now” $T^{[IH]}$:

$$T^{[IH]} = \frac{\langle \psi^{[IH]} | (t_{\text{now}} - t) I | \psi^{[IH]} \rangle}{\langle \psi^{[IH]} | I | \psi^{[IH]} \rangle}$$

(16)

$$T^r = \frac{\langle (t_{\text{now}} - t) I \rangle}{\langle I \rangle}$$

(17)

to compare it with regular moving average $T^r$ (were it an integral over $dt$ the $T^r$ would be a constant; the invervation with $dV$ make it tracking the spikes in $I$). As all the values of time (future and past) are known, the (16) carry information about $|\psi^{[IH]}\rangle$ localization. When the value is small – a large $dV/dt$ spike event happened very recently. When it is large – a large spike happened a substantial time ago, the value is an information when a large spike in $dV/dt$ took place. In Fig. 3 the value of $T^{[IH]}$ (scaled by the factor $10^{-3}$ and shifted up to fit the chart) is presented along with regular moving average $T^r$ for $\tau = 256s$. One can clearly see that there is no smooth transition between the states, the “switch” happens instantly, there is no $\tau$-proportional time delay, what is typical for regular moving averages $T^r$ and the one in Fig. 1.

A linear dependence of $T^{[IH]}$ on time is also observed, this is an indication of stability of $|\psi^{[IH]}\rangle$ state identification. The value of $T^{[IH]}$ is the time scale; typically it is easier to work with the density matrix $\rho_I(\psi^2)$ obtained from $\psi(x) = \psi^{[IH]}(x)$ rather than with the time scale itself.

V. CONCLUSION

A moving average with the weight $d\mu = dV \Phi(x(t))\omega(t)$ is considered.

$$p_{\text{aver}} = \frac{\int_{t_{\text{now}}}^{t_{\text{now}}} dV \Phi(x(t))\omega(t)p(t)}{\int_{t_{\text{now}}}^{t_{\text{now}}} dV \Phi(x(t))\omega(t)}$$

(18)

The $\omega(t)$ is decaying exponent [2]; the polynomial $\Phi(x)$ is obtained solely from observed execution flow $I = dV/dt$ and has the form $\Phi(x) = |\psi^{[IH]}|^2(x)$, it corresponds to the maximal eigenvalue of (12) eigenproblem. Contrary to regular moving average [7] the developed approach has $n - 1$ internal degrees of freedom what adjusts averaging weight according to spikes in other observable (such as execution flow $I = dV/dt$). These internal degrees of freedom allow to obtain an immediate “switch”, what is not possible in regular moving average that always has $\tau$-proportional time delay, lagging indicator. The comparison with regular moving average is most clear in Fig. 3 smooth regular moving average (blue line) vs. “switching” moving average with internal degrees of freedom (green line).

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