Coherence and entanglement in a two-qubit system coupled to a finite temperature reservoir: A comparative study

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Abstract

We investigate a system constituted by two interacting qubits in which one of the qubits is isolated and the other is coupled to a thermal reservoir. We analyze the dynamics of the system considering two different models of system-reservoir interaction: i) a “microscopic” model, in which the master equation is derived taking into account the interaction between the two subsystems (qubits); ii) a phenomenological model, in which the master equation consists of a dissipative term simply added to the unitary evolution term. We compare the time evolution of the concurrence (between the two qubits) and the linear entropy (of the isolated qubit) in both models, for different temperatures of the thermal bath. Although both models provide the same results if the reservoir is at \( T = 0 \), we find that there are significant differences if the reservoir is at finite temperature. While the bipartite entanglement is very similar in both cases, we find contrasting results for the (isolated qubit) linear entropy evolution: as, according to the microscopic model the isolated qubit would approach a maximally mixed state faster for higher temperatures, the phenomenological model gives just the opposite behavior, i.e., it may take longer for the qubit state to become maximally mixed for higher temperatures of the reservoir, a kind of “anomalous” behavior.

1 Introduction

The investigation of the coherent interaction between quantum sub-systems is of fundamental importance in the field of quantum information. As quantum systems are normally susceptible to their environment, quantum coherence may be substantially affected by unwanted couplings to their surroundings. It is therefore of relevance to be able to describe the environmental influence as accurately as possible. Several methods have been developed in order to treat such non-ideal quantum systems; for instance, models involving the coupling of a system of interest with a thermal bath (normally modelled by a large number of quantum harmonic oscillators) may account for phenomena such as loss of quantum coherence (decoherence) \[3, 4\]. Besides, the concept of decoherence is of central importance to the field of quantum information \[5\] as well as for the understanding of the emergence of the “classical” world \[6\]. A practical (perturbative) approach is that based on master equations for the reduced density operator \[2, 4\], and it is largely employed to describe the dynamics of quantum systems weakly coupled to reservoirs. Frequently, the system of interest is considered to be a single quantum system, but in studies regarding the dynamics of several coupled quantum systems (such as interacting qubits) under the influence of their environment, very often the primary interaction between the subsystems of interest is not taken into account in the derivation of the corresponding master equation. There are, though, a few works on this particular subject in the literature, such as derivations and applications of master equations which include the subsystems’ interaction \[7, 8\], as well as discussions about aspects of the dynamics of entanglement between two qubits in contact to reservoirs \[9, 11\] and the generation of entangled states \[12\]. We may also cite studies of quantum systems coupled to thermal baths \[13, 14\] in which ad hoc approaches fail to give a proper evolution of the system’s density operator if the thermal bath is at finite temperature. An interesting point is that in the Jaynes-Cummings model with a lossy cavity, a reduced density operator having the correct thermal equilibrium behavior is not obtained, if the system is coupled to a reservoir at finite temperature, unless the interaction between subsystems is included in the formulation of the corresponding master equation, as discussed in \[15\]. More recently, we may cite \[16\] a comprehensive study of a two-qubit system coupled to the electromagnetic vacuum \((T = 0K)\), where the authors compare the dynamics of the system obtained from an ad hoc approach as well as from a more realistic, “microscopic model”. In our opinion, though, there is lack in the literature of studies about the influence of the temperature of the reservoir on the evolution of composite quantum systems when the interaction between subsystems must be taken into account on such non-ideal circumstances. Our purpose in this contribution, is to make a

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direct comparison between the results arising from a particular “microscopic model”, which takes into account the interaction between the subsystems of interest, and a “phenomenological model”, constructed simply by adding the damping terms to the unitary evolution part. Our system of interest will consist of two interacting qubits (XY-type interaction), having one of the qubits isolated (qubit 1) and the other in contact with a thermal reservoir (qubit 2). As a first step we derive a “microscopic” master equation for our two-qubit system, closely following references [18, 9, 17]. One of our aims is to investigate the dynamics of the degree of bipartite entanglement (quantified by the concurrence) in the two-qubit system according to the microscopic model compared to its evolution in the phenomenological model for different temperatures of the reservoir. In addition to that, we are also going to analyse the coherence associated to qubit 1 (quantified by its linear entropy) which may be now considered a system coupled to a “composite bath” constituted by qubit 2 + thermal reservoir. We show that, in the case of having a finite temperature reservoir, the phenomenological and microscopic approaches yield incompatible results.

Our paper is organized as follows: in section II we present the derivation of the master equations as well as their solutions; in section III we discuss the bipartite entanglement between the qubits; in section IV we discuss the coherence associate to qubit 1, and in section V we summarize our conclusions.

2 Master equations for the two-qubit system

2.1 Interacting qubits: unitary evolution

Our system of interest consists of two (dipole) coupled qubits, whose dynamics, in the rotating wave approximation is governed by the following Hamiltonian (in units of $\hbar$)

$$H_S = \omega_1 \sigma_+^{(1)} \sigma_-^{(1)} + \omega_2 \sigma_+^{(2)} \sigma_-^{(2)} + \frac{\lambda}{2} \left( \sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)} \right),$$  

(1)

where $\sigma_+^{(i)} = |1^{(i)}\rangle \langle 0^{(i)}|$ and $\sigma_-^{(i)} = |0^{(i)}\rangle \langle 1^{(i)}|$ (with $i = 1, 2$) are the raising and lowering operators for qubit 1 and 2, respectively. Here $\omega_i$ is the frequency of the $i-th$ qubit and $\lambda/2$ is coupling constant between the two qubits. The Hamiltonian above may be diagonalized in the uncoupled basis $\{|0,0\}; |1,0\}; |0,1\}; |1,1\}$, with eigenenergies and eigenstates (dressed states), in the resonant case, $\omega_1 = \omega_2 = \omega$, given by

$$E_a = 0 \quad |a\rangle = |0,0\rangle$$

$$E_b = \omega - \frac{\lambda}{2} \quad |b\rangle = \frac{1}{\sqrt{2}} |1,0\rangle - \frac{1}{\sqrt{2}} |0,1\rangle$$

$$E_c = \omega + \frac{\lambda}{2} \quad |c\rangle = \frac{1}{\sqrt{2}} |1,0\rangle + \frac{1}{\sqrt{2}} |0,1\rangle$$

$$E_d = 2 \omega \quad |d\rangle = |1,1\rangle.$$  

(2)

2.2 Derivation of the microscopic master equation

Now we assume that qubit 1 is isolated from its environment (although it is coupled to qubit 2) and that qubit 2 is in contact with a thermal bath at temperature $T$. The bath is itself modelled as a collection of independent harmonic oscillators with Hamiltonian

$$H_B = \sum_n \omega_n a_n^\dagger a_n.$$  

(3)

We consider the qubit 2-reservoir interaction as being dissipative, with effective interaction Hamiltonian of the form

$$H_{int} = \sigma_+^{(2)} \otimes B,$$  

(4)

where $B$ is the bath operator $B = \sum_n \varepsilon_n (a_n + a_n^\dagger)$, $a_n^\dagger$ and $a_n$ are the creation and annihilation operators of the $n-th$ mode of the bath (frequency $\omega_n$), $\sigma_+^{(2)} = \sigma_-^{(2)} + \sigma_-^{(2)}$ (relative to qubit 2), and $\varepsilon_n$ is the coupling constant of qubit 2 to the $n-th$ mode of the bath. The total Hamiltonian, system of qubits plus bath is then $H = H_S + H_B + H_{int}$.
The master equation for the density operator, $\rho$, of the two qubit system in the Born-Markov and rotating wave approximations is

$$\dot{\rho}(t) = -i [H_S, \rho(t)] + \mathcal{D}(\rho(t)).$$

(5)

The dissipative term may be written as [11]

$$\mathcal{D}(\rho(t)) = \sum_\omega \gamma(\omega) \left(A(\omega) \rho(t) A^\dagger(\omega) - \frac{1}{2} \{A^\dagger(\omega) A(\omega), \rho(t)\}\right).$$

(6)

The rates $\gamma$ are given by $\gamma(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle B^\dagger(\tau) B(0) \rangle$, where $B(\tau)$ is the bath operator in the interaction representation, or $B(\tau) = e^{iH_B \tau} B e^{-iH_B \tau} = \sum_n \varepsilon_n (a_n e^{-i\omega_n \tau} + a_n^\dagger e^{i\omega_n \tau})$, and $\langle B^\dagger(\tau) B(0) \rangle \equiv \text{Tr}_B [B^\dagger(\tau) B(0) \rho_B]$ is the trace over variables of the bath. Here

$$\rho_B = \frac{\exp(-H_B/kT)}{\text{Tr}_B \{\exp(-H_B/kT)\}}$$

(7)

is the thermal state for the bath at temperature $T$. The jump operators $A(\omega)$ are defined as $A(\omega) = \sum_{\varepsilon_1, \ldots, \varepsilon_\omega = \omega} \Pi(\varepsilon_1) A \Pi(\varepsilon_\omega)$, where $\Pi(\varepsilon)$ is the projector acting on the sub-space associated to the energy eigenvalues $\varepsilon$ of the Hamiltonian $H_S$, and the summation is over the eigenstates having fixed energy difference equal to $\omega$ (in unit of $\hbar$). In our case, $A = \sigma_x^{(2)}$.

The first Bohr frequency is

$$\omega_I = \omega - \frac{\lambda}{2}$$

for the transitions $|b\rangle \rightarrow |a\rangle$ and $|d\rangle \rightarrow |c\rangle$ and is related to the jump operator

$$\sigma_x^{(2)}(\omega_I) = \langle a| \sigma_x^{(2)} |b\rangle \langle b| + \langle c| \sigma_x^{(2)} |d\rangle \langle d| ,$$

(8)

while the second Bohr frequency

$$\omega_{II} = \omega + \frac{\lambda}{2}$$

for the transitions $|c\rangle \rightarrow |a\rangle$ e $|d\rangle \rightarrow |b\rangle$ and is related to

$$\sigma_x^{(2)}(\omega_{II}) = \langle a| \sigma_x^{(2)} |c\rangle \langle c| + \langle b| \sigma_x^{(2)} |d\rangle \langle d| .$$

(9)

After identifying each term in Eq. (6), we may rewrite it as

$$\mathcal{D}(\rho(t)) = \sum_{i=1}^I \gamma_i \left(\sigma_x^{(2)}(\omega_i) \rho(t) \sigma_x^{(2)\dagger}(\omega_i) - \frac{1}{2} \{\sigma_x^{(2)\dagger}(\omega_i) \sigma_x^{(2)}(\omega_i), \rho(t)\}\right)$$

$$+ \sum_{i=1}^I \mathcal{R}(\omega_i) \left(\sigma_x^{(2)\dagger}(\omega_i) \rho(t) \sigma_x^{(2)}(\omega_i) - \frac{1}{2} \{\sigma_x^{(2)}(\omega_i) \sigma_x^{(2)\dagger}(\omega_i), \rho(t)\}\right),$$

(10)

with the Kubo-Martin-Schwinger relation [18]

$$\mathcal{R}(\omega_i) = \exp(-\hbar \omega_i/kT) \gamma(\omega_i)$$

(11)

and $\sigma_x^{(2)\dagger}(\omega_i) = \sigma_x^{(2)}(-\omega_i)$.

Now, working out the equation above using the expressions [2] for the eigenstates of $H_S$ as well as the jump operators [8] and [9], the microscopic master equation will read

$$\dot{\rho}(t) = -i [H_S, \rho(t)] + c_I \left(\langle a| \langle b| \rho \ |b\rangle \langle a| - \frac{1}{2} \{|b\rangle \langle b|, \rho\}\right) + c_{II} \left(\langle a| \langle c| \rho \ |c\rangle \langle a| - \frac{1}{2} \{|c\rangle \langle c|, \rho\}\right)$$

$$+ c_{II} \left(\langle b| \langle d| \rho \ |d\rangle \langle b| - \frac{1}{2} \{|d\rangle \langle d|, \rho\}\right)$$

$$+ \mathcal{C}_I \left(\langle b| \langle a| \rho \ |a\rangle \langle b| - \frac{1}{2} \{|a\rangle \langle a|, \rho\}\right) + \mathcal{C}_I \left(\langle d| \langle c| \rho \ |c\rangle \langle d| - \frac{1}{2} \{|c\rangle \langle c|, \rho\}\right)$$

$$+ \mathcal{C}_I \left(\langle c| \langle b| \rho \ |b\rangle \langle c| - \frac{1}{2} \{|b\rangle \langle b|, \rho\}\right) + \mathcal{C}_I \left(\langle d| \langle a| \rho \ |a\rangle \langle d| - \frac{1}{2} \{|a\rangle \langle a|, \rho\}\right).$$
+ \tau_{II} \left( |d\rangle \langle b| \rho |b\rangle \langle d| - \frac{1}{2} \{ |b\rangle \langle b|, \rho \} \right) + \tau_{II} \left( |c\rangle \langle a| \rho |a\rangle \langle c| - \frac{1}{2} \{ |a\rangle \langle a|, \rho \} \right) \\
- c_I (|a\rangle \langle b| \rho |b\rangle \langle a| + |c\rangle \langle d| \rho |d\rangle \langle c|) + c_{II} (|a\rangle \langle c| \rho |d\rangle \langle b| + |b\rangle \langle d| \rho |c| \langle a| \\
- \tau_{I} (|b\rangle \langle a| \rho |c| \langle d| + |d\rangle \langle c| \rho |a| \langle b|) + \tau_{II} (|d\rangle \langle b| \rho |a| \langle c| + |c\rangle \langle a| \rho |b| \langle d|) ,
\end{aligned}
\tag{12}
\end{equation}

with the coefficients
\begin{align*}
c_I &= \frac{1}{2} \gamma (\omega_I) \\
c_{II} &= \frac{1}{2} \gamma (\omega_{II}) \\
\tau_I &= \frac{1}{2} \gamma (\omega_I) \\
\tau_{II} &= \frac{1}{2} \gamma (\omega_{II}) .
\end{align*}
\tag{13}

The function $\gamma (\omega_i)$ is related (see, for instance [18]) to the spectral density $J (\omega_i)$ through
\begin{equation}
\gamma (\omega_i) = J (\omega_i) \left[ 1 + \pi (\omega_i) \right] ,
\end{equation}
\tag{14}

where the index $i = I, II$ corresponds to the Bohr frequencies of the model. Here $\pi (\omega_i)$ is the mean photon number associated to the mode of frequency $\omega_i$ of a thermal state at temperature $T$,
\begin{equation}
\pi (\omega_i) = \frac{1}{e^{\hbar \omega_i/kT} - 1} .
\end{equation}
\tag{15}

We have chosen a Lorentzian function for the spectral density, or
\begin{equation}
J (\omega_i) = \frac{\gamma_0 \Gamma^2}{(\omega_i - \omega_0)^2 + \Gamma^2} ,
\end{equation}
\tag{16}

where $\gamma_0$ is the single qubit decay rate, $\omega_0$ is the central frequency, and $\Gamma$ is the half-width of the distribution.

### 2.2.1 Differential equations for the matrix elements: microscopic case

From the master equation [12] we may obtain a set of coupled differential equations for the dressed state populations
\begin{align*}
\dot{\rho}_{aa}(t) &= -(\bar{c}_I + \bar{c}_{II}) \rho_{aa}(t) + c_I \rho_{bb}(t) + c_{II} \rho_{cc}(t) , \\
\dot{\rho}_{bb}(t) &= \bar{c}_I \rho_{aa}(t) - (c_I + \bar{c}_{II}) \rho_{bb}(t) + c_{II} \rho_{dd}(t) , \\
\dot{\rho}_{cc}(t) &= \bar{c}_{II} \rho_{aa}(t) - (c_I + \bar{c}_{II}) \rho_{cc}(t) + c_I \rho_{dd}(t) , \\
\dot{\rho}_{dd}(t) &= \bar{c}_{II} \rho_{bb}(t) + \bar{c}_I \rho_{cc}(t) - (c_I + \bar{c}_{II}) \rho_{dd}(t) ,
\end{align*}
\tag{17}

as well as for the coherences
\begin{align*}
\dot{\rho}_{ab}(t) &= \left[ i \left( \omega - \frac{\lambda}{2} \right) - \frac{(c_I + \bar{c}_I + 2\bar{c}_{II})}{2} \right] \rho_{ab}(t) + c_{II} \rho_{cd}(t) , \\
\dot{\rho}_{ac}(t) &= \left[ i \left( \omega + \frac{\lambda}{2} \right) - \frac{(2\bar{c}_I + c_{II} + \bar{c}_{II})}{2} \right] \rho_{ac}(t) - c_I \rho_{bd}(t) , \\
\dot{\rho}_{ad}(t) &= \left[ 2i\omega - \frac{(c_I + c_{II} + \bar{c}_I + \bar{c}_{II})}{2} \right] \rho_{ad}(t) ,
\end{align*}
\tag{18}
\[ \dot{\rho}_{bc}(t) = \left[ i\lambda - \frac{(c_I + c_{II} + \bar{c}_I + \bar{c}_{II})}{2} \right] \rho_{bc}(t), \]

\[ \dot{\rho}_{bd}(t) = \left[ i \left( \omega + \frac{\lambda}{2} \right) - \frac{(2c_I + c_{II} + \bar{c}_I + \bar{c}_{II})}{2} \right] \rho_{bd}(t) - \bar{c}_I \rho_{ac}(t), \]

\[ \dot{\rho}_{cd}(t) = \left[ i \left( \omega - \frac{\lambda}{2} \right) - \frac{(c_I + 2c_{II} + \bar{c}_I)}{2} \right] \rho_{cd}(t) + \bar{c}_{II} \rho_{ab}(t). \]

The differential equations above may be solved exactly, and for completeness their solutions are presented in the Appendix.

### 2.3 Phenomenological master equation

The phenomenological master equation is obtained simply by adding a dissipative term to the unitary evolution term, or

\[ \dot{\rho}(t) = -i [H_S, \rho(t)] + \gamma \left( \sigma_+^2 \rho(t) \sigma_-^2 - \frac{1}{2} \left\{ \sigma_+^2 \sigma_-^2, \rho(t) \right\} \right) \]

\[ + \bar{\gamma} \left( \sigma_+^2 \rho(t) \sigma_-^2 - \frac{1}{2} \left\{ \sigma_+^2 \sigma_-^2, \rho(t) \right\} \right). \tag{19} \]

Here, analogously to the microscopic case, the function \( \gamma \equiv \gamma(\omega) \) and the spectral density \( J(\omega) \) are related through \( \gamma(\omega) = J(\omega) [1 + \Pi(\omega)] \). The quantities \( \bar{\gamma} \equiv \bar{\gamma}(\omega), \Pi(\omega) \) and \( J(\omega) \) are just the same expressions as in Eqs. (11), (15) and (16) respectively, but having \( \omega \) as argument, instead.

We would like to remark that as we have taken \( \omega \gg \lambda \), the Bohr frequencies related to the microscopic model turn out to be \( \omega_I \approx \omega_{II} \approx \omega \), which means that \( J(\omega_I) \approx J(\omega) \) and \( \Pi(\omega_I) \approx \Pi(\omega) \); this is surely suitable to make a genuine comparison between the two models.

#### 2.3.1 Differential equations for the matrix elements: phenomenological case

It may be shown that the phenomenological master equation (19) generates the following set of coupled differential equations, for the populations

\[ \dot{\rho}_{11}(t) = -\bar{\gamma} \rho_{11}(t) + \gamma \rho_{22}(t) \]

\[ \dot{\rho}_{22}(t) = \bar{\gamma} \rho_{11}(t) + \gamma \rho_{22}(t) + \frac{i\lambda}{2} \rho_{23}(t) - \frac{i\lambda}{2} \rho_{32}(t) \]

\[ \dot{\rho}_{33}(t) = -\bar{\gamma} \rho_{33}(t) + \gamma \rho_{44}(t) + \frac{i\lambda}{2} \rho_{32}(t) - \frac{i\lambda}{2} \rho_{23}(t) \]

\[ \dot{\rho}_{44}(t) = \bar{\gamma} \rho_{33}(t) - \gamma \rho_{44}(t), \]

and for the coherences

\[ \dot{\rho}_{12}(t) = \left[ i\omega - \frac{(\gamma + \bar{\gamma})}{2} \right] \rho_{12}(t) + \frac{i\lambda}{2} \rho_{13}(t) \]

\[ \dot{\rho}_{13}(t) = \frac{i\lambda}{2} \rho_{12}(t) + (i\omega - \bar{\gamma}) \rho_{13}(t) + \gamma \rho_{24}(t) \]

\[ \dot{\rho}_{14}(t) = \left[ 2i\omega - \frac{(\gamma + \bar{\gamma})}{2} \right] \rho_{14}(t) \]

\[ \dot{\rho}_{23}(t) = \frac{i\lambda}{2} \rho_{22}(t) - \frac{(\gamma + \bar{\gamma})}{2} \rho_{23}(t) - \frac{i\lambda}{2} \rho_{33}(t) \]

\[ \dot{\rho}_{34}(t) = \frac{i\lambda}{2} \rho_{33}(t) - \frac{(\gamma + \bar{\gamma})}{2} \rho_{34}(t) - \frac{i\lambda}{2} \rho_{44}(t) \]
\[
\dot{\rho}_{24}(t) = -\frac{i\lambda}{2}\rho_{34}(t) + (i\omega - \gamma)\rho_{24}(t) + \gamma\rho_{13}(t)
\]
\[
\dot{\rho}_{34}(t) = \left[i\omega - \frac{(\gamma + \gamma)}{2}\right]\rho_{34}(t) - \frac{i\lambda}{2}\rho_{24}(t).
\]

Again, the solutions of the differential equations above consist of relatively large expressions, and we have opted not to include them in this paper.

3 Degree of bipartite entanglement

The concurrence \[19\], an entanglement monotone largely employed to quantify entanglement, is defined as
\[
C(t) = \max[0, \sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)} - \sqrt{\lambda_3(t)} - \sqrt{\lambda_4(t)}],
\]
where \(\lambda_i(t)\) are the eigenvalues of the matrix \(M(t) = \rho(t)\dot{\rho}(t)\) placed in decreasing order in Eq. \[22\], with \(\dot{\rho}(t) = \sigma_y \otimes \sigma_y \rho^*(t) \sigma_y \otimes \sigma_y\) and where \(\sigma_y\) is the usual Pauli matrix. Now we are going to calculate the concurrence as a function of time, for different temperatures of the reservoir in both the microscopic and the phenomenological model.

3.1 Concurrence: microscopic model

According to the microscopic approach, the concurrence is given by
\[
\tilde{C}_m(t) = \max[0, \tilde{C}_m(t)],
\]
with
\[
\tilde{C}_m(t) = 2|\rho_{23}(t)| - 2\sqrt{\rho_{11}(t)\rho_{44}(t)}.
\]

The elements \(\rho_{ij}\) are the solutions of equations \[17\] and \[18\], and \(\text{Im}(\rho_{bc})\) stands for the imaginary part of \(\rho_{bc}\).

3.2 Concurrence: phenomenological model

In the phenomenological approach, the concurrence is given by
\[
\tilde{C}_p(t) = \max[0, \tilde{C}_p(t)],
\]
with
\[
\tilde{C}_p(t) = 2|\rho_{23}(t)| - 2\sqrt{\rho_{11}(t)\rho_{44}(t)}.
\]

Here the elements \(\rho_{ij}\) are the solutions of equations \[20\] and \[21\]. Again, due to their large size, we have not explicitly written the expressions for \(\tilde{C}_m\) and \(\tilde{C}_p\), and we will restrict our analysis to a numerical comparison between them.

3.3 Numerical results for the concurrence: a comparison between the microscopic and phenomenological models

In Fig. 1 we have plots of \(\tilde{C}_m\), in Fig. 1a and \(\tilde{C}_p\), in Fig. 1b as a function of time for an initial state for the two qubit system \(|\Psi\rangle_{q1,q2} = |1, 0\rangle\). We note that for the reservoir temperature \(T\) very close to zero, the concurrence curves show an oscillatory pattern as well as decay in both cases. As a matter of fact, despite the differences between the two different approaches, the curves for \(T \approx 0\) virtually coincide. However, as the temperature of the reservoir is raised, we observe that the maxima of the concurrence are lower in the microscopic model compared to the curves obtained in the phenomenological model. Furthermore, a typical pattern of entanglement sudden death (as well as revivals) occurs in both cases, and the concurrence vanishes for long times.

4 Linear entropy of qubit 1: composite reservoir

Now we would like to discuss the simple system considered here from a different perspective. While keeping exactly the same configuration, we consider now that qubit 1 is coupled to a more complex (composite) reservoir, constituted by qubit 2 plus the thermal bath. In other words, we trace over the
Figure 1: (Color online) Concurrence between qubits 1 and 2 as a function of time for an initial two-qubit state $|\Psi_{q1,q2}\rangle = |1, 0\rangle$ according to (a) the microscopic model; (b) the phenomenological model. In both plots, $\omega = 5 \times 10^6 \text{s}^{-1}$, $\gamma_0 = 0.001 \times 5 \times 10^5 \text{s}^{-1}$, $\Gamma = 5 \times 10^5 \text{s}^{-1}$, $\lambda = 4 \times 10^4 \text{s}^{-1}$, and $\omega_0 = 2\omega$. The continuous (blue) curves correspond to a thermal bath at $T = 0.005 \text{ K}$; the dashed (green) curves to $T = 0.05 \text{ K}$ and the dot-dashed (red) curves to $T = 0.15 \text{ K}$.

4.1 Linear entropy: microscopic model

A straightforward calculation leads to the linear entropy for the microscopic model,

$$S(t) = 1 - \left[ \rho_{aa}(t) + \frac{\rho_{bb}(t)}{2} + \frac{\rho_{cc}(t)}{2} - \text{Re} \left[ \rho_{bc}(t) \right] \right]^2 - \left[ \rho_{dd}(t) + \frac{\rho_{bb}(t)}{2} + \frac{\rho_{cc}(t)}{2} + \text{Re} \left[ \rho_{bc}(t) \right] \right]^2,$$

(25)

where the elements $\rho_{ij}$ are the solutions of equations (17) and (18), and $\text{Re} (\rho_{ij})$ stands for the real part of $\rho_{ij}$.

4.2 Linear entropy: phenomenological model

In the phenomenological case, the linear entropy reads

$$S(t) = 1 - \left[ \rho_{11}(t) + \rho_{22}(t) \right]^2 - \left[ \rho_{33}(t) + \rho_{44}(t) \right]^2,$$

(26)

being the elements $\rho_{ij}$ the solutions of equations (20) and (21).

4.3 Numerical results for the qubit 1 linear entropy: a comparison between the microscopic and phenomenological models

Given that the global bipartite state [qubit1] + [qubit 2-thermal bath] state is not pure, the linear entropy and the concurrence are expected to behave differently. In fact, the results according to each model may yield contradictory time evolutions. In Fig. 2, it is shown the linear entropy as a function of time for different temperatures of the reservoir. For a bath at $T \approx 0$, the linear entropy curves are virtually the same in both models: for the set of parameters chosen, they show oscillations and tend to the maximum value of $S_{\text{max}} = 0.5$, which corresponds to a maximally mixed state. Nevertheless, the situation is very different if the reservoir is at finite temperature. Within the microscopic model [see Fig. 2(a)], we notice that linear entropy increases at a faster rate for higher temperatures, meaning that a noisier reservoir has a more destructive effect on the quantum coherence of qubit 1. On the other hand, in the realm of the phenomenological model, the linear entropy is, in a range of times, a decreasing function of temperature, as shown in Fig. 2(b); this is in our opinion a kind of “anomalous behavior”, as a noisier reservoir would induce a more coherent behavior for qubit 1. We note that in both cases qubit 1 is eventually driven to a maximally mixed state, i.e., its linear entropy approaches the (equilibrium) expected asymptotic value of $S_{\text{max}} = 0.5$. 

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Figure 2: (Color online) Linear entropy of qubit 2 as a function of time for an initial two qubit state $|\Psi_{q_{1},q_{2}}\rangle = |1,0\rangle$ according to (a) the microscopic model; (b) the phenomenological model. In both plots $\omega = 5 \times 10^6 s^{-1}$, $\gamma_0 = 0.01 \times 5 \times 10^5 s^{-1}$, $\Gamma = 5 \times 10^5 s^{-1}$, $\lambda = 4 \times 10^4 s^{-1}$, and $\omega_H = 2\omega$. The continuous (blue) curves correspond to a thermal bath at $T = 0.005 \text{ K}$; the dashed (green) curves to $T = 0.05 \text{ K}$ and the dot-dashed (red) curves to $T = 0.15 \text{ K}$.

5 Conclusions

We have made a comprehensive comparison between two analytically solvable models describing the (weak) coupling of a two-qubit system to a thermal bath. In both formulations we have considered the rotating wave and the Born-Markov approximations, as well as equivalent spectral densities for the bath. The relevant parameters were appropriately chosen in order to accomplish the comparison between the different models. For instance, parameters such as couplings and frequencies compatible with the approximations we have made; in particular, we have chosen the resonant case, i.e., both qubits having the same transition frequency. In this way, we managed to make a straightforward comparison between the models by varying a single parameter, namely, the temperature of the thermal bath. We have found that the microscopic and phenomenological models are virtually equivalent if the reservoir is at $T \approx 0$, which is made evident by the fact that the curves of entanglement between the two qubits as well as the linear entropy of qubit 1 (as a function of time) are coincident in both models in the zero temperature case. We should remark that at $T = 0$, the energy of the system is irreversibly lost to the reservoir via its interaction with qubit 2. However, we have found important differences between the results yielded by each model if the reservoir is at finite temperature. Concerning the entanglement between the two qubits, the differences are small and are more evident for higher temperatures; the microscopic model predicts a more destructive action of the thermal noise compared to the phenomenological construct, as we note in Fig. (1). Differently from the situation at $T = 0$, if the bath is at finite temperature, thermal noise is injected from the reservoir to the two qubit system, and as the interaction between qubits is taken into account within the microscopic model (the reservoir directly induces transitions between the dressed levels of the two qubit system), we expect a more destructive influence of the environmental noise in that case, as we have found.

The differences are even more profound if one considers the linear entropy of qubit 1. Given that the linear entropy is obtained by tracing over the variables of the second sub-system (qubit 2) which is generally in a mixed state, the linear entropy is not an appropriate measure of entanglement, although it may be used to quantify quantum coherence. Even though the curves for the linear entropy are the same for both models if the bath is at $T \approx 0$, in the finite temperature case the phenomenological and microscopic models lead to conflicting results. While according to the microscopic model qubit 1 evolves more rapidly to a mixed state for larger temperatures of the reservoir, the phenomenological model predicts an “anomalous” behavior, according to which the coherence of qubit 1 is degraded more slowly for higher temperatures of the reservoir, as seen in Fig. (2). Our results thus indicate the inadequacy of phenomenological models of this type in the treatment of certain types of quantum composite systems.

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Appendix A: Solution of the master equation: microscopic case

In this appendix we present the explicit solutions of the microscopic master equation [see equations (17) and (18)]. The expressions for dressed state populations are given by

\[
  k \rho_{aa} (t) = c_I c_{II} + e^{-(c_I + \overline{c}_{II}) t} \left\{ \overline{c}_I c_{II} \left[ \rho_{aa} (0) + \rho_{cc} (0) \right] - c_I c_{II} \left[ \rho_{bb} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + \overline{c}_{II}) t} \left\{ c_I \overline{c}_{II} \left[ \rho_{aa} (0) + \rho_{bb} (0) \right] - c_I c_{II} \left[ \rho_{cc} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + \overline{c}_{II} + \overline{c}_{II}) t} \left\{ \overline{c}_I \overline{c}_{II} \rho_{aa} (0) - c_I \overline{c}_{II} \rho_{bb} (0) - \overline{c}_I c_{II} \rho_{cc} (0) + c_I c_{II} \rho_{dd} (0) \right\},
\]

\[
  k \rho_{bb} (t) = c_I \overline{c}_I + e^{-(c_I + \overline{c}_{II}) t} \left\{ -\overline{c}_I c_{II} \left[ \rho_{aa} (0) + \rho_{cc} (0) \right] + c_I c_{II} \left[ \rho_{bb} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + \overline{c}_{II}) t} \left\{ -\overline{c}_I \overline{c}_{II} \rho_{aa} (0) + \overline{c}_I c_{II} \rho_{bb} (0) + \overline{c}_I c_{II} \rho_{cc} (0) - c_I c_{II} \rho_{dd} (0) \right\},
\]

\[
  k \rho_{cc} (t) = c_I c_{II} + e^{-(c_I + \overline{c}_{II}) t} \left\{ \overline{c}_I c_{II} \left[ \rho_{aa} (0) + \rho_{cc} (0) \right] - c_I c_{II} \left[ \rho_{bb} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + \overline{c}_{II}) t} \left\{ -c_I \overline{c}_{II} \left[ \rho_{aa} (0) + \rho_{bb} (0) \right] + c_I c_{II} \left[ \rho_{cc} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + c_{II} + \overline{c}_{II}) t} \left\{ -\overline{c}_I c_{II} \rho_{aa} (0) + c_I \overline{c}_{II} \rho_{bb} (0) + \overline{c}_I c_{II} \rho_{cc} (0) - c_I c_{II} \rho_{dd} (0) \right\},
\]

\[
  k \rho_{dd} (t) = c_I \overline{c}_I + e^{-(c_I + \overline{c}_{II}) t} \left\{ -\overline{c}_I c_{II} \left[ \rho_{aa} (0) + \rho_{cc} (0) \right] + c_I c_{II} \left[ \rho_{bb} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + \overline{c}_{II}) t} \left\{ -\overline{c}_I \overline{c}_{II} \left[ \rho_{aa} (0) + \rho_{bb} (0) \right] + \overline{c}_I c_{II} \left[ \rho_{cc} (0) + \rho_{dd} (0) \right] \right\}
  + e^{-(c_I + c_{II} + \overline{c}_{II} + \overline{c}_{II}) t} \left\{ \overline{c}_I \overline{c}_{II} \rho_{aa} (0) - c_I \overline{c}_{II} \rho_{bb} (0) - \overline{c}_I c_{II} \rho_{cc} (0) + c_I c_{II} \rho_{dd} (0) \right\},
\]

where \( k = (c_I + \overline{c}_{II}) (c_{II} + \overline{c}_{II}) \) and the coefficients \( c_i \) are defined in (13). The coherences are given by

\[
  \rho_{ab} (t) = e^{- \frac{i}{2} (\omega - \frac{\frac{\overline{c}_I}{c_I} \overline{c}_{II}}{c_I + \overline{c}_{II}}) t} \left\{ c_{II} + e^{- (c_I + \overline{c}_{II}) t} \overline{c}_{II} \right\} \rho_{ab} (0) + \left[ 1 - e^{- (c_I + \overline{c}_{II}) t} \right] c_{II} \rho_{cd} (0),
\]

\[
  \rho_{ac} (t) = e^{- \frac{i}{2} (\omega - \frac{\frac{c_I}{c_I} c_{II}}{c_I + c_{II}}) t} \left\{ c_I + e^{- (c_I + \overline{c}_{II}) t} \overline{c}_I \right\} \rho_{ac} (0) - \left[ 1 - e^{- (c_I + \overline{c}_{II}) t} \right] c_I \rho_{bd} (0),
\]

\[
  \rho_{ad} (t) = e^{- \frac{i}{2} (\omega - \frac{\frac{c_I}{c_I} c_{II} + \frac{\overline{c}_I}{\overline{c}_{II}} \overline{c}_{II}}{c_I + \overline{c}_{II}}) t} \rho_{ad} (0),
\]
\[
\rho_{bc}(t) = e^{-\left(\frac{c_{II} - c_{I}}{c_{I} + c_{II}}\right)t} \rho_{bc}(0),
\]

\[
\rho_{bd}(t) = e^{\left[i\left(\omega + \frac{c_{II}}{c_{I} + c_{II}}\right)\right]t} \left\{ 1 - e^{-\left(c_{I} + \sigma_{II}\right)t} \right\} c_{I} \rho_{ac}(0) + \left[ c_{I} + e^{-\left(c_{I} + \sigma_{II}\right)t} c_{II} \right] \rho_{bd}(0),
\]

\[
\rho_{cd}(t) = e^{\left[i\left(\omega - \frac{c_{II}}{c_{I} + c_{II}}\right)\right]t} \left\{ 1 - e^{-\left(c_{I} + \sigma_{II}\right)t} \right\} c_{II} \rho_{ab}(0) + \left[ c_{II} + e^{-\left(c_{I} + \sigma_{II}\right)t} c_{II} \right] \rho_{cd}(0).
\]

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