Characters of the Positive Energy UIRs of 
D=4 Conformal Supersymmetry

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Abstract

We give character formulae for the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). Using these we also derive decompositions of long superfields as they descend to the unitarity threshold. These results are also applicable to irreps of the complex Lie superalgebras sl(4/N). Our derivations use results from the representation theory of su(2,2/N) developed already in the 80s.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, cf. [1–96] and references therein. Particularly important are those for $D \leq 6$ since in these cases the relevant superconformal algebras satisfy [97] the Haag-Lopuszanski-Sohnius theorem [98]. This makes the classification of the UIRs of these superalgebras very important. Until recently such classification was known only for the $D = 4$ superconformal algebras $su(2,2/1)$ [99] and $su(2,2/N)$ [100–103] (for arbitrary $N$). Recently, the classification for
\[ D = 3 \text{ (for even } N), \quad D = 5, \quad \text{and } D = 6 \text{ (for } N = 1, 2) \] 
was given in [104] (some results being conjectural), and then the \( D = 6 \) case (for arbitrary \( N \)) was finalized in [105].

Once we know the UIRs of a (super-)algebra the next question is to find their characters, since these give the spectrum which is important for the applications. Some results on the spectrum were given in the early papers [106–108,102] but it is necessary to have systematic results for which the character formulae are needed. This is the question we address in this paper for the UIRs of \( D = 4 \) conformal superalgebras \( su(2,2/N) \). From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra \( sl(4/N) \). But for \( su(2,2/N) \) even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. More than that, in the applications the most important role is played by the representations with “quantized” conformal dimensions at the unitarity threshold and at discrete points below. In the quantum field or string theory framework some of these correspond to operators with “protected” scaling dimension and therefore imply “non-renormalization theorems” at the quantum level, cf., e.g., [22],[23].

Thus, we need detailed knowledge about the structure of the UIRs from the representation-theoretical point of view. Fortunately, such information is contained in [100–103]. Following these papers in Section 2 we recall the basic ingredients of the representation theory of the \( D=4 \) superconformal algebras. In particular we recall the structure of Verma modules and UIRs. Using this information we are able to derive character formulae, some of which are very explicit, cf. Section 3. We also pin-point the difference between character formulae for \( sl(4,N) \) and \( su(2,2/N) \) since for the latter we need to introduce and use the notion of counter-terms in the character formulae. The general formulae are valid for arbitrary \( N \). For illustration we give more explicit formulae for \( N = 1, 2 \), but we leave the example \( N = 4 \) for a follow-up paper, since that would take too many pages, and the present paper is long enough. In Section 4 we summarize our results on the decompositions of long superfields as they descend to the unitarity threshold. These results may be applied to the problem of operators with protected dimensions.
2. Representations of D=4 conformal supersymmetry

2.1. The setting

The superconformal algebras in $D = 4$ are $G = su(2,2/N)$. The even subalgebra of $G$ is the algebra $G_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $G$ by the signature:

$$\chi = [d; j_1, j_2; z; r_1, \ldots, r_{N-1}]$$

(2.1)

where $d$ is the conformal weight, $j_1, j_2$ are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, $z$ represents the $u(1)$ subalgebra which is central for $G_0$ (and for $N = 4$ is central for $G$ itself), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or $R$) symmetry algebra $su(N)$.

We recall that the algebraic approach to $D = 4$ conformal supersymmetry developed in [100–103] involves two related constructions - on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of $G$ (and of the corresponding supergroup $G = SU(2,2/N)$) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of $G$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form [109] to the Verma modules [102],[103]. Here we shall need only the second - Verma module - construction.

2.2. Verma modules

To introduce Verma modules one needs the standard triangular decomposition:

$$G^\mathcal{C} = G^+ \oplus H \oplus G^-$$

(2.2)

where $G^\mathcal{C} = sl(4/N)$ is the complexification of $G$, $G^+, G^-$, resp., are the subalgebras corresponding to the positive, negative, roots of $G^\mathcal{C}$, resp., and $H$ denotes the Cartan subalgebra of $G^\mathcal{C}$.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(G^+) \otimes v_0$, where $U(G^+)$ is the universal enveloping algebra of $G^+$, $\Lambda \in \mathcal{H}^*$ is the lowest weight, and $v_0$ is the lowest weight vector $v_0$ such that:

$$Xv_0 = 0, \quad X \in G^-,$$

$$Hv_0 = \Lambda(H)v_0, \quad H \in \mathcal{H}.$$ 

(2.3)

Further, for simplicity we omit the sign $\otimes$, i.e., we write $Pv_0 \in V^\Lambda$ with $P \in U(G^+)$. 

3
The lowest weight \( \Lambda \) is characterized by its values on the Cartan subalgebra \( \mathcal{H} \), or, equivalently, by its products with the simple roots (given explicitly below). In general, these would be \( 3 + N \) complex numbers, however, in order to be useful for the representations of the real form \( \mathcal{G} \) these values would be restricted to be real and furthermore to correspond to the signatures \( \chi \), and we shall write \( \Lambda = \Lambda(\chi) \), or \( \chi = \chi(\Lambda) \). Note, however, that there are Verma modules to which correspond no ERs, cf. [101] and below.

If a Verma module \( V^\Lambda \) is irreducible then it gives the lowest weight irrep \( L_\Lambda \) with the same weight. If a Verma module \( V^\Lambda \) is reducible then it contains a maximal invariant submodule \( I^\Lambda \) and the lowest weight irrep \( L_\Lambda \) with the same weight is given by factorization: \( L_\Lambda = V^\Lambda / I^\Lambda \) [110],[111],[112].

Thus, we need first to know which Verma modules are reducible. The reducibility conditions for highest weight Verma modules over basic classical Lie superalgebra were given by Kac [112]. Translating his conditions to lowest weight Verma modules we have [101]: A lowest weight Verma module \( V^\Lambda \) is reducible only if at least one of the following conditions is true:

(2.4a) \[
(r - \Lambda, \beta) = m(\beta, \beta)/2, \quad \beta \in \Delta^+, \quad (\beta, \beta) \neq 0, \quad m \in \mathbb{N},
\]

(2.4b) \[
(r - \Lambda, \beta) = 0, \quad \beta \in \Delta^+, \quad (\beta, \beta) = 0,
\]

where \( \Delta^+ \) is the positive root system of \( \mathcal{G}^F \), \( \rho \in \mathcal{H}^* \) is the very important in representation theory element given by \( \rho = \rho_0 - \rho_1 \), where \( \rho_0, \rho_1 \) are the half-sums of the even, odd, resp., positive roots, \( (\cdot, \cdot) \) is the standard bilinear product in \( \mathcal{H}^* \).

If a condition from (2.4a) is fulfilled then \( V^\Lambda \) contains a submodule which is a Verma module \( V^{\Lambda'} \) with shifted weight given by the pair \( m, \beta \) : \( \Lambda' = \Lambda + m\beta \). The embedding of \( V^{\Lambda'} \) in \( V^\Lambda \) is provided by mapping the lowest weight vector \( v_0 \) of \( V^{\Lambda'} \) to the singular vector \( v^{m,\beta}_s \) in \( V^\Lambda \) which is completely determined by the conditions:

\[
X v^{m,\beta}_s = 0, \quad X \in \mathcal{G}^-,
\]

\[
H v^{m,\beta}_s = \Lambda'(H) v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + m\beta.
\]

Explicitly, \( v^{m,\beta}_s \) is given by an even polynomial in the positive root generators:

\[
v^{m,\beta}_s = P^{m,\beta} v_0, \quad P^{m,\beta} \in U(\mathcal{G}^+).
\]

Thus, the submodule of \( V^\Lambda \) which is isomorphic to \( V^{\Lambda'} \) is given by \( U(\mathcal{G}^+) P^{m,\beta} v_0 \).

[More on the even case following the same approach may be seen in, e.g., [113],[114].]

If a condition from (2.4b) is fulfilled then \( V^\Lambda \) contains a submodule \( I^\beta \) obtained from the Verma module \( V^{\Lambda'} \) with shifted weight \( \Lambda' = \Lambda + \beta \) as follows. In this situation \( V^\Lambda \) contains a singular vector

\[
X v^\beta_s = 0, \quad X \in \mathcal{G}^-,
\]

\[
H v^\beta_s = \Lambda'(H) v_0, \quad H \in \mathcal{H}, \quad \Lambda' = \Lambda + \beta.
\]

1 Many statements below are true for any basic classical Lie superalgebra, and would require changes only for the superalgebras \( \text{osp}(1/2N) \).
Explicitly, \( v_s^\beta \) is given by an odd polynomial in the positive root generators:

\[
v_s^\beta = P^\beta v_0 , \quad P^\beta \in U(G^+) .
\]  \hspace{1cm} (2.8)

Then we have:

\[
I^\beta = U(G^+) P^\beta v_0
\]  \hspace{1cm} (2.9)

which is smaller than \( V_{\Lambda'} = U(G^+) v_0' \) since this polynomial is Grassmannian:

\[
(P^\beta)^2 = 0 .
\]  \hspace{1cm} (2.10)

To describe this situation we say that \( V_{\Lambda'} \) is **oddly embedded** in \( V_{\Lambda} \).

Note, however, that the above formulae describe also more general situations when the difference \( \Lambda' - \Lambda = \beta \) is not a root, as used in \([101]\), and below.

The weight shifts \( \Lambda' = \Lambda + \beta \), \([101]\) when \( \beta \) is an odd root are generalized **odd reflections**, and for future reference will be denoted as:

\[
\hat{s}_\beta \cdot \Lambda \equiv \Lambda + \beta , \quad (\beta, \beta) = 0, \quad (\Lambda, \beta) = (\rho, \beta) .
\]  \hspace{1cm} (2.11)

Each such odd reflection generates an infinite discrete abelian group:

\[
\tilde{W}_\beta \equiv \{(\hat{s}_\beta)^n | n \in \mathbb{Z}\} ,
\]  \hspace{1cm} (2.12)

where the unit element is obviously obtained for \( n = 0 \), and \((\hat{s}_\beta)^{-n}\) is the inverse of \((\hat{s}_\beta)^n\), and for future use we have also defined the length function \( \ell(\cdot) \) on the elements of \( \tilde{W}_\beta \). This group acts on the weights \( \Lambda \) extending (2.11):

\[
(\hat{s}_\beta)^n \cdot \Lambda = \Lambda + n\beta , \quad n \in \mathbb{Z} , \quad (\beta, \beta) = 0, \quad (\Lambda, \beta) = (\rho, \beta) .
\]  \hspace{1cm} (2.13)

This is related to the fact that there is a doubly-infinite chain of oddly embedded Verma modules whenever a Verma module is reducible w.r.t. an odd root. This is explained in detail and used for the classification of the Verma modules in \([100]\), and shall be used below.

Further, to be more explicit we need to recall the root system of \( G^\mathbb{E} \) - for definiteness - as used in \([101]\). The positive root system \( \Delta^+ \) is comprised from \( \alpha_{ij} \), \( 1 \leq i < j \leq 4+N \). The even positive root system \( \Delta^+_0 \) is comprised from \( \alpha_{ij} \), with \( i, j \leq 4 \) and \( i, j \geq 5 \); the odd positive root system \( \Delta^+_1 \) is comprised from \( \alpha_{ij} \), with \( i \leq 4, j \geq 5 \). The simple roots are chosen as in (2.4) of \([101]\):

\[
\gamma_1 = \alpha_{12} , \quad \gamma_2 = \alpha_{34} , \quad \gamma_3 = \alpha_{25} , \quad \gamma_4 = \alpha_{4,4+N} , \quad \gamma_k = \alpha_{k,k+1} , \quad 5 \leq k \leq 3+N
\]  \hspace{1cm} (2.14)

Thus, the Dynkin diagram is:

\[
\begin{array}{ccccccc}
\circ & \times & \circ & \cdots & \circ & \times & \circ \\
1 & 3 & 5 & 3+N & 4 & 2
\end{array}
\]  \hspace{1cm} (2.15)
This is a non-distinguished simple root system with two odd simple roots (for the various root systems of the basic classical superalgebras we refer to [115]).

Let \( \Lambda = \Lambda(\chi) \). The products of \( \Lambda \) with the simple roots are [101]:

\[
\begin{align*}
(\Lambda, \gamma_a) &= -2j_a, \quad a = 1, 2, \\
(\Lambda, \gamma_3) &= \frac{1}{2}(d + z') + j_1 - m/N + 1, \\
(\Lambda, \gamma_4) &= \frac{1}{2}(d - z') + j_2 - m_1 + m/N + 1, \\
\gamma' &= z(1 - \delta_{N4}) \\
(\Lambda, \gamma_j) &= r_{N+4-j}, \quad 5 \leq j \leq 3 + N
\end{align*}
\]

These formulae give the correspondence between signatures \( \chi \) and lowest weights \( \Lambda(\chi) \).

In the case of even roots \( \beta \in \Delta^+_0 \) there are six roots \( \alpha_{ij}, j \leq 4 \), coming from the \( sl(4) \) factor (which is complexification of \( su(2,2) \)) and \( N(N-1)/2 \) roots \( \alpha_{ij}, 5 \leq i \), coming form the \( sl(N) \) factor (complexification of \( su(N) \)).

The reducibility conditions w.r.t. to the positive roots coming from \( sl(4)(su(2,2)) \) coming from (2.4) (denoting \( m \rightarrow n_{ij} \) for \( \beta \rightarrow \alpha_{ij} \)) are:

\[
\begin{align*}
n_{12} &= 1 + 2j_1 \equiv n_1 \\
n_{23} &= 1 - d - j_1 - j_2 \equiv n_2 \\
n_{34} &= 1 + 2j_2 \equiv n_3 \\
n_{13} &= 2 - d + j_1 - j_2 = n_1 + n_2 \\
n_{24} &= 2 - d - j_1 + j_2 = n_2 + n_3 \\
n_{14} &= 3 - d + j_1 + j_2 = n_1 + n_2 + n_3
\end{align*}
\]

Thus, reducibility conditions (2.17a, c) are fulfilled automatically for \( \Lambda(\chi) \) with \( \chi \) from (2.1) since we always have: \( n_1, n_3 \in \mathbb{N} \). There are no such conditions for the ERs since they are induced from the finite-dimensional irreps of the Lorentz subalgebra (parametrized by \( j_1, j_2 \).) However, to these two conditions correspond differential operators of order \( 1 + 2j_1 \) and \( 1 + 2j_2 \) (as we mentioned above) and these annihilate all functions of the ERs with signature \( \chi \).

The reducibility conditions w.r.t. to the positive roots coming from \( sl(N)(su(N)) \) are all fulfilled for \( \Lambda(\chi) \) with \( \chi \) from (2.1). In particular, for the simple roots from those condition (2.4) is fulfilled with \( \beta \rightarrow \gamma_j, m = 1 + r_{N+4-j} \) for every \( j = 5, 6, \ldots, N + 3 \). There are no such conditions for the ERs since they are induced from the finite-dimensional

\[ \text{For } N = 4 \text{ the factor } u(1) \text{ in } \mathcal{G}_0 \text{ becomes central in } \mathcal{G} \text{ and } \mathcal{G}^C. \text{ Consequently, the representation parameter } z \text{ can not come from the products of } \Lambda \text{ with the simple roots, as indicated in (2.16). In that case the lowest weight is actually given by the sum } \Lambda + \tilde{\Lambda}, \text{ where } \tilde{\Lambda} \text{ carries the representation parameter } z. \text{ This is explained in detail in [101] and further we shall not comment more on it, but the peculiarities for } N = 4 \text{ will be evident in the formulae.} \]
UIRs of $su(N)$. However, to these $N-1$ conditions correspond $N-1$ differential operators of orders $1 + r_k$ (as we mentioned) and the functions of our ERs are annihilated by all these operators [101].

For future use we note also the following decompositions:

$$\Lambda = \sum_{j=1}^{N+3} \lambda_j \alpha_{j,j+1} = \Lambda^s + \Lambda^z + \Lambda^u$$

(2.18a)

$$\Lambda^s \equiv \sum_{j=1}^3 \lambda_j \alpha_{j,j+1} , \quad \Lambda^z \equiv \lambda_4 \alpha_{45} , \quad \Lambda^u \equiv \sum_{j=5}^{N+3} \lambda_j \alpha_{j,j+1}$$

(2.18b)

which actually employ the distinguished root system with one odd root $\alpha_{45}$.

The reducibility conditions for the $4N$ odd positive roots of $G$ are [102],[101]:

$$d = d^1_{Nk} - z\delta_N \quad 4 - 2k + 2j_2 + z + 2m_k - 2m/N$$

(2.19a.k)

$$d^1_{Nk} \equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N$$

(2.19b.k)

$$d = d^2_{Nk} - z\delta_N \quad 2 - 2k - 2j_2 + z + 2m_k - 2m/N$$

(2.19c.k)

$$d^2_{Nk} \equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N$$

(2.19d.k)

$$d = d^3_{Nk} + z\delta_N \quad 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N$$

(2.19c.k)

$$d^3_{Nk} \equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N$$

(2.19d.k)

$$d = d^4_{Nk} + z\delta_N \quad 2k - 2N - 2j_1 - z - 2m_k + 2m/N$$

(2.19d.k)

$$d^4_{Nk} \equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N$$

where in all four cases of (2.19) $k = 1, \ldots, N$, $m_N \equiv 0$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i , \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} kr_k$$

(2.20)

$m_k$ is the number of cells of the $k$-th row of the standard Young tableau, $m$ is the total number of cells. Condition (2.19a.k) corresponds to the root $\alpha_{3,N+5-k}$, (2.19b.k) corresponds to the root $\alpha_{4,N+5-k}$, (2.19c.k) corresponds to the root $\alpha_{1,N+5-k}$, (2.19d.k) corresponds to the root $\alpha_{2,N+5-k}$.

Note that for a fixed module and fixed $i = 1, 2, 3, 4$ only one of the odd $N$ conditions involving $d^i_{Nk}$ may be satisfied. Thus, no more than four (two, for $N = 1$) of the conditions (2.19) may hold for a given Verma module.

---

3 Note that there are actually as many operators as positive roots of $sl(N)$ but all are expressed in terms of the $N-1$ above corresponding to the simple roots [101].
Remark: Note that for \( n_2 \in \mathbb{N} \) (cf. (2.17)) the corresponding irreps of \( su(2,2) \) are finite-dimensional (the necessary and sufficient condition for this is: \( n_1, n_2, n_3 \in \mathbb{N} \)). Then the irreducible LWM \( L_\Lambda \) of \( su(2,2/N) \) are also finite-dimensional (and non-unitary) independently on whether the corresponding Verma module \( V^\Lambda \) is reducible w.r.t. any odd root. If \( V^\Lambda \) is not reducible w.r.t. any odd root, then these finite-dimensional irreps are called 'typical' [112], otherwise, the irreps are called 'atypical' [112]. In our considerations \( n_2 \notin \mathbb{N} \) in all cases, except the trivial 1-dimensional UIR (for which \( n_2 = 1 \), cf. below). ♦

We shall consider quotients of Verma modules factoring out the even submodules for which the reducibility conditions are always fulfilled. Before this we recall the root vectors following [101].

The positive (negative) root vectors corresponding to \( \alpha_{ij}, (-\alpha_{ij}) \), are denoted by \( X^+_{ij}, (X^-_{ij}) \). In the \( su(2,2/N) \) matrix notation the convention of [101], (2.7), is:

\[
X^+_{ij} = \begin{cases} 
  e_{ji} & \text{for } (i,j) = (3,4), (3,j), (4,j), \ 5 \leq j \leq N + 4 \\
  e_{ij} & \text{otherwise}
\end{cases}
\]

\[
X^-_{ij} = t(X^+_{ij})
\]

where \( e_{ij} \) are \((N+4) \times (N+4)\) matrices with all elements zero except the element equal to 1 on the intersection of the i-th row and j-th column. The simple root vectors \( X^+_{i} \) follow the notation of the simple roots \( \gamma_{i} \) (2.14):

\[
X^+_1 \equiv X^+_{12}, \ X^+_2 \equiv X^+_{34}, \ X^+_3 \equiv X^+_{25}, \ X^+_4 \equiv X^+_{4,4+N}, \ X^+_k \equiv X^+_{k,k+1}, \ 5 \leq k \leq 3 + N
\]  

(2.22)

The mentioned submodules are generated by the singular vectors related to the even simple roots \( \gamma_1, \gamma_2, \gamma_5, \ldots, \gamma_{N+3} \) [101]:

\[
v^+_s = (X^+_1)^{1+2j_1} v_0,
\]

\[
v^+_s = (X^+_2)^{1+2j_2} v_0,
\]

\[
v^+_s = (X^+_j)^{1+r_{N+4-j}} v_0, \quad j = 5, \ldots, N + 3
\]

(2.23a) (2.23b) (2.23c)

(for \( N = 1 \) (2.23c) being empty). The corresponding submodules are \( I^\Lambda_k = U(\mathcal{G}^+) v^+_s \), and the invariant submodule to be factored out is:

\[
I^\Lambda_c = \bigcup_k I^\Lambda_k
\]

(2.24)

Thus, instead of \( V^\Lambda \) we shall consider the factor-modules:

\[
\tilde{V}^\Lambda = V^\Lambda / I^\Lambda_c
\]

(2.25)

which are closer to the structure of the ERs. In the factorized modules the singular vectors (2.23) become null conditions, i.e., denoting by \( |\Lambda\rangle \) the lowest weight vector of \( \tilde{V}^\Lambda \), we have:

\[
(X^+_1)^{1+2j_1} |\Lambda\rangle = 0,
\]

\[
(X^+_2)^{1+2j_2} |\Lambda\rangle = 0,
\]

\[
(X^+_j)^{1+r_{N+4-j}} |\Lambda\rangle = 0, \quad j = 5, \ldots, N + 3
\]

(2.26a) (2.26b) (2.26c)

8
2.3. Singular vectors and invariant submodules at the unitary reduction points

We first recall the result of [102] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$
\begin{align*}
\text{(2.27a)} & \quad d \geq d_{\text{max}} = \max(d_{N1}^1, d_{NN}^4) , \\
\text{(2.27b)} & \quad d = d_{NN}^4 \geq d_{N1}^1 , \quad j_1 = 0 , \\
\text{(2.27c)} & \quad d = d_{N1}^2 \geq d_{NN}^3 , \quad j_2 = 0 , \\
\text{(2.27d)} & \quad d = d_{N1}^3 = d_{NN}^2 , \quad j_1 = j_2 = 0 ,
\end{align*}
$$

where $d_{\text{max}}$ is the threshold of the continuous unitary spectrum.\(^4\) Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$ since then the internal symmetry algebra $su(N)$ is trivial and by definition $m_1 = m = 0$ (the resulting irrep is 1-dimensional with $d = z = j_1 = j_2 = 0$). The UIRs for $N=1$ were first given in [99].

Next we note that if $d > d_{\text{max}}$ the factorized Verma modules are irreducible and coincide with the UIRs $L_{\Lambda}$. These UIRs are called long in the modern literature, cf., e.g., [8],[17],[23],[33],[32],[35],[34]. Analogously, we shall use for the cases when $d = d_{\text{max}}$, i.e., (2.27a), the terminology of semi-short UIRs, introduced in [8],[23], while the cases (2.27b,c,d) are also called short UIRs, cf., e.g., [17],[23],[33],[32],[35],[34].

Next consider in more detail the UIRs at the four distinguished reduction points determining the list above:

$$
\begin{align*}
\text{(2.28)} & \quad d_{N1}^1 = 2 + 2j_2 + z + 2m_1 - 2m/N , \\
& \quad d_{N1}^2 = z + 2m_1 - 2m/N , \quad (j_2 = 0) , \\
& \quad d_{NN}^3 = 2 + 2j_1 - z + 2m/N , \\
& \quad d_{NN}^4 = -z + 2m/N , \quad (j_1 = 0) .
\end{align*}
$$

\(^4\) Note that from (2.27a) follows:

$$
\begin{align*}
& \quad d_{\text{max}} \geq 2 + j_1 + j_2 + m_1 ,
\end{align*}
$$

the equality being achieved only when $d_{N1}^1 = d_{NN}^4$, while from (2.27b,c) follows:

$$
\begin{align*}
& \quad d \geq 1 + j_1 + j_2 + m_1 , \quad j_1j_2 = 0 ,
\end{align*}
$$

the equality being achieved only when $d_{NN}^3 = d_{N1}^4$, or $d_{N1}^2 = d_{NN}^3$, for (2.27b), (2.27c), resp. Recalling the unitarity conditions [116] for the conformal algebra $su(2,2)$:

$$
\begin{align*}
& \quad d \geq 2 + j_1 + j_2 , \quad j_1j_2 > 0 , \\
& \quad d \geq 1 + j_1 + j_2 , \quad j_1j_2 = 0 ,
\end{align*}
$$

we see that the superconformal unitarity conditions are more stringent that the conformal ones.
First we recall the singular vectors corresponding to these points. The above reducibilities occur for the following odd roots, resp.:

\[ \alpha_{3,4+N}, \quad \alpha_{4,4+N}, \quad \alpha_{15}, \quad \alpha_{25} . \]  

The second and the fourth are the two odd simple roots:

\[ \gamma_3 = \alpha_{25}, \quad \gamma_4 = \alpha_{4,4+N} \]  

and the other two are simply related to these:

\[ \alpha_{15} = \alpha_{12} + \alpha_{25} = \gamma_1 + \gamma_3, \quad \alpha_{3,4+N} = \alpha_{34} + \alpha_{4,4+N} = \gamma_2 + \gamma_4 . \]  

Thus, the corresponding singular vectors are:

\[ v^1_{\text{odd}} = P_{3,4+N} v_0 = (X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2) v_0 = \]  

\[ = (2j_2 X^+_2 X^+_4 - (2j_2 + 1)X^+_4 X^+_2) v_0 = \]  

\[ = (2j_2 X^+_{3,4+N} - X^+_4 X^+_2) v_0, \quad d = d_{N1} \]  \hspace{1cm} (2.32a)

\[ v^2_{\text{odd}} = X^+_4 v_0, \quad d = d^2_{N1} \]  \hspace{1cm} (2.32b)

\[ v^3_{\text{odd}} = P_{15} v_0 = (X^+_3 X^+_1 (h_1 - 1) - X^+_1 X^+_3 h_1) v_0 = \]  

\[ = (2j_1 X^+_1 X^+_3 - (2j_1 + 1)X^+_3 X^+_1) v_0 = \]  

\[ = (2j_1 X^+_3 X^+_1 - X^+_3 X^+_1) v_0, \quad d = d^3_{N1} \]  \hspace{1cm} (2.32c)

\[ v^4_{\text{odd}} = X^+_3 v_0, \quad d = d^4_{N1} \]  \hspace{1cm} (2.32d)

where \( X^+_{3,4+N} = [X^+_3, X^+_4] \) is the odd root vector corresponding to the root \( \alpha_{3,4+N} \), \( X^+_{15} = [X^+_1, X^+_3] \) is the odd root vector corresponding to the root \( \alpha_{15} \), \( h_1, h_2 \in \mathcal{H} \) are Cartan generators corresponding to the roots \( \gamma_1, \gamma_2 \), (cf. [101]), and passing from the (2.32a), (2.32c)), to the next line we have used the fact that \( h_2 v_0 = -2j_2 v_0 \), \( (h_1 v_0 = -2j_1 v_0) \), consistently with (2.16b), ((2.16a)). These vectors are given in (8.9b),(8.7b),(8.8a),(8.7a), resp., of [101].

These singular vectors carry over for the factorized Verma modules \( \tilde{V}^\Lambda \):

\[ \tilde{v}^1_{\text{odd}} = P_{3,4+N} |\Lambda\rangle = (X^+_4 X^+_2 (h_2 - 1) - X^+_2 X^+_4 h_2) |\Lambda\rangle = \]  

\[ = (2j_2 X^+_{3,4+N} - X^+_4 X^+_2) |\Lambda\rangle, \quad d = d^1_{N1} \]  \hspace{1cm} (2.33a)

\[ \tilde{v}^2_{\text{odd}} = X^+_4 |\Lambda\rangle, \quad d = d^2_{N1} \]  \hspace{1cm} (2.33b)

\[ \tilde{v}^3_{\text{odd}} = P_{15} |\Lambda\rangle = (X^+_3 X^+_1 (h_1 - 1) - X^+_1 X^+_3 h_1) |\Lambda\rangle = \]  

\[ = (2j_1 X^+_1 X^+_3 - X^+_3 X^+_1) |\Lambda\rangle, \quad d = d^3_{N1} \]  \hspace{1cm} (2.33c)

\[ \tilde{v}^4_{\text{odd}} = X^+_3 |\Lambda\rangle, \quad d = d^4_{N1} \]  \hspace{1cm} (2.33d)
For \( j_1 = 0, \ j_2 = 0 \), resp., the vector \( v_{3\text{odd}}^3, v_{1\text{odd}}^1 \), resp., is a descendant of the singular vector \( v_1^3, v_2^2 \), resp., cf. (2.23a), (2.23b), resp. In the same situations the tilde counterparts \( \tilde{v}_1^3, \tilde{v}_2^2 \) are just zero - cf. (2.26a), (2.26b), resp. However, then there is another independent singular vector of \( \tilde{V}^\Lambda \) in both cases. For \( j_1 = 0 \) it corresponds to the sum of two roots: \( \alpha_{15} + \alpha_{25} \) (which sum is not a root!) and is given by formula (D.1) of [101]:

\[
\tilde{v}^{34} = X_3^+ X_1^+ X_3^+ \langle \Lambda \rangle = X_3^+ X_1^+ \langle \Lambda \rangle , \quad d = d_{NN}^3 , \ j_1 = 0
\]  

(2.34)

Checking singularity we see at once that \( X_k^- \tilde{v}^{34} = 0 \) for \( k \neq 3 \). It remains to calculate the action of \( X_3^- \) :

\[
X_3^- \tilde{v}^{34} = h_3 X_1^+ X_3^+ \langle \Lambda \rangle - X_3^+ X_1^+ h_3 \langle \Lambda \rangle =
\]

\[
= X_1^+ X_3^+ (h_3 - 1) \langle \Lambda \rangle - X_3^+ X_1^+ h_3 \langle \Lambda \rangle = 0 ,
\]

\( h_3, h_4 \in \mathcal{H} \) are Cartan generators corresponding to the roots \( \gamma_3, \gamma_4 \), (cf. [101]), the first term is zero since \( \Lambda(h_3) - 1 = \frac{1}{2}(d - d_{NN}^3) = 0 \), while the second term is zero due to (2.26a) for \( j_1 = 0 \).

For \( j_2 = 0 \) there is a singular vector corresponding to the sum of two roots: \( \alpha_{3,4+} + \alpha_{4,4+} \) (which sum is not a root) and is given in [101] (cf. the formula before (D.4) there):

\[
\tilde{v}^{12} = X_4^+ X_2^+ X_4^+ \langle \Lambda \rangle = X_4^+ X_{3,4+} \langle \Lambda \rangle , \quad d = d_{NN}^1 , \ j_2 = 0
\]  

(2.35)

As above, one checks that \( X_k^- \tilde{v}^{12} = 0 \) for \( k \neq 4 \), and then calculates:

\[
X_4^- \tilde{v}^{12} = h_4 X_2^+ X_4^+ \langle \Lambda \rangle - X_4^+ X_2^+ h_4 \langle \Lambda \rangle =
\]

\[
= X_2^+ X_4^+ (h_4 - 1) \langle \Lambda \rangle - X_4^+ X_2^+ h_4 \langle \Lambda \rangle = 0
\]

using \( \Lambda(h_4) - 1 = \frac{1}{2}(d - d_{NN}^1) = 0 \), and (2.26b) for \( j_2 = 0 \).

To the above two singular vectors in the ER picture correspond second-order superdifferential operators given explicitly in formulae (11a,b) of [102], and in formulae (D3),(D5) of [101].

\[\text{Note that w.r.t. } V^\Lambda \text{ the analogues of the vectors } \tilde{v}^{34} \text{ and } \tilde{v}^{12} \text{ are not singular, but subsingular vectors. Indeed, consider the vector in } V^\Lambda \text{ given by the same } U(\mathcal{G}^+) \text{ monomial as } \tilde{v}^{34} : \]

\( v^{34} = X_3^+ X_1^+ X_3^+ \). Clearly, \( X_k^- v^{34} = 0 \) for \( k \neq 3 \). It remains to calculate the action of \( X_3^- \) :

\[
X_3^- v^{34} = h_3 X_1^+ X_3^+ v_0 - X_3^+ X_1^+ h_3 v_0 =
\]

\[
= X_1^+ X_3^+ (h_3 - 1) v_0 - X_3^+ X_1^+ h_3 v_0 = -X_3^+ X_1^+ v_0
\]

where the first term is zero as above, while the second term is a descendant of the singular vector \( v_1^3 = X_1^+ v_0 \), (cf. (2.23a) for \( j_1 = 0 \)), which fulfills the definition of subsingular vector. Analogously, for the vector \( v^{12} = X_4^+ X_2^+ X_4^+ \) we have \( X_k^- v^{12} = 0 \) for \( k \neq 4 \), and:

\[
X_4^- v^{12} = X_4^- X_2^+ X_4^+ = -X_4^+ X_2^+ v_0 ,
\]

(using \( \Lambda(h_4) - 1 \)), which is a descendant of the singular vector \( v_2^3 = X_2^+ v_0 \), cf. (2.23b) for \( j_2 = 0 \).
From the expressions of the singular vectors follow, using (2.9), the explicit formulae for the corresponding invariant submodules $I^\beta$ of the modules $\tilde{V}^\Lambda$ as follows:

\[ I^1 = U(G^+) P_{3,4+n} \langle \tilde{\Lambda} \rangle = U(G^+) \left( \frac{X_4^+ X_2^+ (h_2 - 1) - X_2^+ X_4^+ h_2}{j_2} \right) \langle \tilde{\Lambda} \rangle , \quad d = d_{N1}^1 , \quad j_2 > 0 , \]  
\[ I^2 = U(G^+) X_4^+ \langle \tilde{\Lambda} \rangle , \quad d = d_{N1}^2 , \]  
\[ I^3 = U(G^+) P_{15} \langle \tilde{\Lambda} \rangle = U(G^+) \left( \frac{X_3^+ X_1^+ (h_1 - 1) - X_1^+ X_3^+ h_1}{j_1} \right) \langle \tilde{\Lambda} \rangle , \quad d = d_{NN}^3 , \quad j_1 > 0 , \]  
\[ I^4 = U(G^+) X_3^+ \langle \tilde{\Lambda} \rangle , \quad d = d_{NN}^4 , \]  
\[ I^{12} = U(G^+) \tilde{\nu}^{12} = X_4^+ X_2^+ X_4^+ \langle \tilde{\Lambda} \rangle , \quad d = d_{N1}^1 , \quad j_2 = 0 , \]  
\[ I^{34} = U(G^+) \tilde{\nu}^{34} = X_3^+ X_1^+ X_3^+ \langle \tilde{\Lambda} \rangle , \quad d = d_{NN}^3 , \quad j_1 = 0 . \]

Sometimes we shall indicate the signature $\chi(\Lambda)$, writing, e.g., $I^1(\chi)$; sometimes we shall indicate also the resulting signature, writing, e.g., $I^1(\chi, \chi')$ - this is a redundancy since it is determined by what is displayed already: $\chi' = \chi(\Lambda + \beta)$, but will be useful to see immediately in the concrete situations without calculation.

The invariant submodules were used in [102] in the construction of the UIRs, as we shall recall below.

### 2.4. Structure of single-reducibility-condition Verma modules and UIRs

We discuss now the reducibility of Verma modules at the four distinguished points (2.28). We note a partial ordering of these four points:

\[ d_{N1}^3 > d_{N1}^2 , \quad d_{NN}^3 > d_{NN}^4 , \]  

or more precisely:

\[ d_{N1}^1 = d_{N1}^2 + 2 , \quad (j_2 = 0) ; \quad d_{NN}^3 = d_{NN}^4 + 2 , \quad (j_1 = 0) . \]

Due to this ordering at most two of these four points may coincide. Thus, we have two possible situations: of Verma modules (or ERs) reducible at one and at two reduction points from (2.28).

In this Subsection we deal with the situations in which no two of the points in (2.28) coincide. According to [102] (Theorem) there are four such situations involving UIRs:

\[ d = d_{\text{max}} = d_{N1}^1 > d_{NN}^3 , \]  
\[ d = d_{N1}^2 > d_{NN}^3 , \quad j_2 = 0 , \]  
\[ d = d_{\text{max}} = d_{NN}^3 > d_{N1}^1 , \]  
\[ d = d_{NN}^4 > d_{N1}^1 , \quad j_1 = 0 . \]
We shall call these cases **single-reducibility-condition (SRC)** Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d = d_{\text{max}}$, i.e., (2.39a,c), the terminology of semi-short UIRs, [8],[23], while the cases (2.39b,d) are also called short UIRs, [17],[23],[33],[32],[35],[34].

As we see the SRC cases have supplementary conditions as specified. And due to the inequalities there are the following additional restrictions which are correspondingly given as:

\[
\begin{align*}
  z &> j_1 - j_2 - m_1 + 2m/N , & (2.39a') \\
  z &> j_1 + 1 - m_1 + 2m/N , & (2.39b') \\
  z &< j_1 - j_2 - m_1 + 2m/N , & (2.39c') \\
  z &< -1 - j_2 - m_1 + 2m/N . & (2.39d')
\end{align*}
\]

Using these inequalities the unitarity conditions (2.39) may be rewritten more explicitly:

\[
\begin{align*}
  d &= d^a_{N_1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > 2 + j_1 + j_2 + m_1 , & (2.39a'') \\
  d &= d^b_{N_1} = z + 2m_1 - 2m/N > j_1 + 1 + m_1 , & j_2 = 0 , & (2.39b'') \\
  d &= d^c_{N_1} = d^c \equiv 2 + 2j_1 - z + 2m/N > 2 + j_1 + j_2 + m_1 , & (2.39c'') \\
  d &= d^d_{N_1} = -z + 2m/N > 1 + j_2 + m_1 , & j_2 = 0 , & (2.39d'')
\end{align*}
\]

where we have introduced notation $d^a, d^c$ to designate two of the SRC cases.

To finalize the structure we should check the even reducibility conditions (2.17b,d,e,f). It is enough to note that the conditions on $d$ in (2.39a'',c''):

\[
  d > 2 + j_1 + j_2 + m_1
\]

and in (2.39b'',d''):

\[
  d > 1 + j_1 + j_2 + m_1 , \quad (j_1,j_2 = 0)
\]

are incompatible with (2.17b,d,e,f), except in two cases. The exceptions are in cases (2.39b'',d'') when $d = 2 + j_1 + j_2 = z$ and $j_1j_2 = 0$. In these cases we have $n_{14} = 1$ in (2.17f) and there exists a Verma submodule $V^{\Lambda + \alpha_{14}}$. However, the $su(2,2)$ signature $\chi_0(\Lambda + \alpha_{14})$ is unphysical: $[j_1 - \frac{1}{2}, -\frac{1}{2}; 3 + j_1]$ for $j_2 = 0$, and $[-\frac{1}{2}, j_2 - \frac{1}{2}; 3 + j_1]$ for $j_1 = 0$. Thus, there is no such submodule of $\tilde{V}^{\Lambda}$.

Thus, the factorized Verma modules $\tilde{V}^{\Lambda}$ with the unitary signatures from (2.39) have only one invariant (odd) submodule which has to be factorized in order to obtain the UIRs. These odd embeddings are given explicitly as:

\[
\tilde{V}^{\Lambda} \rightarrow \tilde{V}^{\Lambda + \beta} \quad (2.40)
\]

where we use the convention [100] that arrows point to the oddly embedded module, and there are the following cases for $\beta$:

\[
\beta = \alpha_{3,4+N} , \quad \text{for (2.39a)}, \quad j_2 > 0, \quad (2.41a)
\]
\[ \beta = \alpha_{4,4+N} , \quad \text{for (2.39b)}, \]
\[ = \alpha_{15} , \quad \text{for (2.39c)}, \quad j_1 > 0, \quad (2.41c) \]
\[ = \alpha_{25} , \quad \text{for (2.39d)}, \]
\[ = \alpha_{3,4+N} + \alpha_{4,4+N} , \quad \text{for (2.39a)}, \quad j_2 = 0, \quad (2.41e) \]
\[ = \alpha_{15} + \alpha_{25} , \quad \text{for (2.39c)}, \quad j_1 = 0 \quad (2.41f) \]

This diagram gives the UIR \( L_\Lambda \) contained in \( \tilde{\nu}^\Lambda \) as follows:

\[ L_\Lambda = \tilde{\nu}^\Lambda / I^\beta, \quad (2.42) \]

where \( I^\beta \) is given by \( I^1, I^2, I^3, I^4, I^{12}, I^{34} \), resp., (cf. (2.36)), in the cases (2.41a, b, c, d, e, f), resp.

It is useful to record the signatures of the shifted lowest weights, i.e., \( \chi' = \chi(\Lambda + \beta) \). In fact, for future use we give the signature changes for arbitrary roots. The explicit formulae are [100],[101]:

\[ \beta = \alpha_{3,N+5-k} : \chi' = [d + \frac{1}{2}; j_1, j_2 - \frac{1}{2}; z + \epsilon_N; r_1, \ldots, r_{k-1} - 1, r_k + 1, \ldots, r_{N-1}], \quad (2.43a) \]
\[ j_2 > 0, \quad r_{k-1} > 0 \quad (2.43a') \]

\[ \beta = \alpha_{4,N+5-k} : \chi' = [d + \frac{1}{2}; j_1, j_2 + \frac{1}{2}; z + \epsilon_N; r_1, \ldots, r_{k-1} - 1, r_k + 1, \ldots, r_{N-1}], \quad (2.43b) \]
\[ r_{k-1} > 0 \quad (2.43b') \]

\[ \beta = \alpha_{1,N+5-k} : \chi' = [d + \frac{1}{2}; j_1 - \frac{1}{2}, j_2; z - \epsilon_N; r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1}], \quad (2.43c) \]
\[ j_1 > 0, \quad r_k > 0 \quad (2.43c') \]

\[ \beta = \alpha_{2,N+5-k} : \chi' = [d + \frac{1}{2}; j_1 + \frac{1}{2}, j_2; z - \epsilon_N; r_1, \ldots, r_{k-1} + 1, r_k - 1, \ldots, r_{N-1}], \quad (2.43d) \]
\[ r_k > 0 \quad (2.43d') \]

\[ k = 1, \ldots, N, \quad \epsilon_N \equiv \frac{z}{N} - \frac{1}{2} \quad (2.44) \]

For each fixed \( \chi \) the lowest weight \( \Lambda(\chi') \) fulfills the same odd reducibility condition as \( \Lambda(\chi') \). We need also the special cases used in (2.41e, f):

\[ \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N} : \chi'_{12} = [d + 1; j_1, 0; z + 2\epsilon_N; r_1 + 2, r_2, \ldots, r_{N-1}], \quad (2.43e) \]
\[ j_2 = 0, \quad d = d_{N1}^1 \]

\[ \beta_{34} = \alpha_{15} + \alpha_{25} : \chi'_{34} = [d + 1; 0, j_2; z - 2\epsilon_N; r_1, \ldots, r_{N-2}, r_{N-1} + 2], \quad (2.43f) \]
\[ j_1 = 0, \quad d = d_{NN}^3 \]

The lowest weight \( \Lambda(\chi'_{12}) \) fulfills (2.39b), while the lowest weight \( \Lambda(\chi'_{34}) \) fulfills (2.39d).
The embedding diagram (2.40) is a piece of a much richer picture [100]. Indeed, notice that if (2.4b) is fulfilled for some odd root $\beta$, then it is fulfilled also for an infinite number of Verma modules $V_\ell = V^{\Lambda + \ell \beta}$ for all $\ell \in \mathbb{Z}$. These modules form an infinite chain complex of oddly embedded modules:

$$\cdots \rightarrow V_{-1} \rightarrow V_0 \rightarrow V_1 \rightarrow \cdots$$  \hspace{1cm} (2.45)

Because of (2.10) this is an exact sequence with one nilpotent operator involved in the whole chain. Of course, once we restrict to the factorized modules $\tilde{V}^\Lambda$ the diagram will be shortened - this is evident from the signature changes (2.43a,b,c,d). In fact, there are only a finite number of factorized modules for $N > 1$, while for $N = 1$ the diagram continues to be infinite to the left. Furthermore, when $\beta = \beta_{12}, \beta_{34}$ from the end of the restricted chain one transmutes - via the embeddings (2.36e,f), resp. - to the chain with $\beta = \alpha_{4,N+4}, \alpha_{25}$, resp. More explicitly, when $\beta = \beta_{12}, \beta_{34}$, then the module $V_1$ plays the role of $V_0$ with $\beta = \alpha_{4,N+4}, \alpha_{25}$. All this is explained in detail in [100].

2.5. Structure of double-reducibility-condition Verma modules and UIRs

We consider now the situations in which two of the points in (2.28) coincide. According to [102] (Theorem) there are four such situations involving UIRs:

- The semi-short SRC UIRs (cf. (2.39a,c)) are obtained by factorizing a Verma submodule $V^{\Lambda + \beta}$ containing either another semi-short SRC UIR of the same type (cf. (2.41a,c)) or containing a short SRC UIR of a different type (cf. (2.41e,f)). In contrast, short SRC UIRs (cf. (2.39b,d)) are obtained by factorizing a Verma submodule $V^{\Lambda + \beta}$ whose irreducible factor-module is not unitary (cf. (2.41b,d)).

We shall call these double-reducibility-condition (DRC) Verma modules or UIRs. As in the previous subsection we shall use for the cases when $d = d_{\text{max}}$, i.e., (2.46a), also the terminology of semi-short UIRs, [8],[23], while the cases (2.46b,c,d) shall also be called short UIRs, [17],[23],[33],[32],[35],[34].

For later use we list more explicitly the values of $d$ and $z$

$$d = d^{ac} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1 ,$$

$$z = j_1 - j_2 + 2m/N - m_1 ;$$

$$d = d^1_{N1} = d^4_{NN} = 1 + j_2 + m_1 , \quad j_1 = 0 .$$  \hspace{1cm} (2.46a')
\[ z = -1 - j_2 + 2m/N - m_1 ; \quad (2.46b') \]
\[ d = d_{N1}^2 = d_{NN}^3 = 1 + j_1 + m_1 , \quad j_2 = 0 , \]
\[ z = 1 + j_1 + 2m/N - m_1 ; \quad (2.46c') \]
\[ d = d_{N1}^2 = d_{NN}^3 = m_1 , \quad j_1 = j_2 = 0 , \]
\[ z = 2m/N - m_1 . \quad (2.46d') \]

We noted already that for \( N = 1 \) the last case, \((2.46d, d')\), is trivial. Note also that for \( N = 2 \) we have: \( 2m/N - m_1 = m - m_1 = 0 \).

To finalize the structure we should check the even reducibility conditions \((2.17b, d, e, f)\). It is enough to note that the values of \( d \) in \((2.46)\) are incompatible with \((2.17b, d, e, f)\), except in a few cases. The exceptions are:

\[ d = d_{N1}^1 = d_{NN}^3 = 2 + j_1 + j_2 , \quad m_1 = 0 \quad (2.47a) \]
\[ d = d_{N1}^1 = d_{NN}^3 = 1 + j_2 + m_1 , \quad j_1 = 0 , \quad m_1 = 0, 1 \quad (2.47b) \]
\[ d = d_{N1}^2 = d_{NN}^3 = 1 + j_1 + m_1 , \quad j_2 = 0 , \quad m_1 = 0, 1 \quad (2.47c) \]
\[ d = d_{N1}^2 = d_{NN}^1 = m_1 , \quad j_1 = j_2 = 0 , \quad m_1 = 0, 1, 2 \quad (2.47d) \]

- In case \((2.47a)\) we have \( n_{14} = 1 \) in \((2.17f)\) and there exists a Verma submodule \( V^{\Lambda + \alpha_{14}} \) with \( su(2, 2) \) signature \( \chi_0(\Lambda + \alpha_{14}) = [j_1 - \frac{1}{2}, j_2 - \frac{1}{2}; 3 + j_1 + j_2] \). As we can see this signature is unphysical for \( j_1 j_2 = 0 \). Thus, there is the even submodule \( \tilde{V}^{\Lambda + \alpha_{14}} \) of \( V^{\Lambda} \) only if \( j_1 j_2 \neq 0 \).
- In case \((2.47b)\) there are three subcases:
  \( m_1 = 0, j_2 = \frac{3}{2} \); then \( d = \frac{3}{2} \), \( n_{24} = 1 \), \( n_{14} = 2 \). The signatures of the embedded submodules of \( V^{\Lambda} \) are: \( \chi_0(\Lambda + \alpha_{24}) = [\frac{1}{2}, 0; \frac{3}{2}] \), \( \chi_0(\Lambda + 2\alpha_{14}) = [-1, -\frac{1}{2}; \frac{3}{2}] \). Thus, there is only the even submodule \( \tilde{V}^{\Lambda + \alpha_{24}} \) of \( \tilde{V} \).
  \( m_1 = 1, j_2 = 0 \); then \( d = 1, n_{13} = 1, n_{24} = 1, n_{14} = 2 \). The signatures of the embedded submodules of \( V^{\Lambda} \) are: \( \chi_0(\Lambda + \alpha_{13}) = [-\frac{1}{2}, \frac{1}{2}; 2] \), \( \chi_0(\Lambda + 2\alpha_{24}) = [\frac{1}{2}, \frac{1}{2}; 2] \), \( \chi_0(\Lambda + 2\alpha_{14}) = [-1, -1; 3] \), and are all unphysical. However, the Verma module \( V^{\Lambda} \) has a subsingular vector of weight \( \alpha_{23} + \alpha_{14} \), cf. \([117]\), and thus, the factorized Verma module \( \tilde{V}^{\Lambda} \) has the submodule \( \tilde{V}^{\Lambda + \alpha_{23} + \alpha_{14}} \).
  \( m_1 = 1 \); then \( n_{14} = 1 \), but as above there is no nontrivial even submodule of \( \tilde{V}^{\Lambda} \).
- The case \((2.47c)\) is dual to \((2.47b)\) so we list shortly the three subcases:
  \( m_1 = 0, j_1 = \frac{1}{2} \); then \( d = \frac{3}{2} \), \( n_{13} = 1, n_{14} = 2 \). There is only the even submodule \( \tilde{V}^{\Lambda + \alpha_{13}} \) of \( \tilde{V} \).
  \( m_1 = 0, j_1 = 0 \); then \( d = 1, n_{13} = 1, n_{24} = 1, n_{14} = 2 \). This subcase coincides with the second subcase of \((2.47b)\).
  \( m_1 = 1 \); then \( n_{14} = 1 \) and as above there is no nontrivial submodule of \( \tilde{V}^{\Lambda} \).
- In case \((2.47d)\) there are again three subcases:
  \( m_1 = 0 \); then all quantum numbers in the signature are zero and the UIR is the one-dimensional trivial irrep.
  \( m_1 = 1 \); then \( d = 1, n_{13} = 1, n_{24} = 1, n_{14} = 2 \). Though this subcase has nontrivial isospin from \( su(2, 2) \) point of view it has the same structure as the second subcase of \((2.47b)\) and
the factorized Verma module $\tilde{V}^\Lambda$ has the submodule $\tilde{V}^\Lambda + \alpha_{23} + \alpha_{14}$. 
$m_1 = 2$; then $d = 2$, $n_{14} = 1$ and as above there is no nontrivial even submodule of $\tilde{V}^\Lambda$.

The embedding diagrams for the corresponding modules $\tilde{V}^\Lambda$ when there are no even embeddings are:

$$
\tilde{V}^{\Lambda+\beta'} \\
\uparrow \\
\tilde{V}^\Lambda \rightarrow \tilde{V}^{\Lambda+\beta}
$$

where

$$(\beta, \beta') =$$

$$(\alpha_{15}, \alpha_{3,4+N}), \quad \text{for (2.46a), } m_1 j_1 j_2 > 0 \quad (2.49a)$$

$$(\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.46a), } j_1 > 0, \ j_2 = 0 \quad (2.49b)$$

$$(\alpha_{15} + \alpha_{25}, \alpha_{3,4+N}), \quad \text{for (2.46a), } j_1 = 0, \ j_2 > 0 \quad (2.49c)$$

$$(\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.46a), } j_1 = j_2 = 0 \quad (2.49d)$$

$$(\alpha_{25}, \alpha_{3,4+N}), \quad \text{for (2.46b), } j_2 > 0, \ 2j_2 + m_1 \geq 2 \quad (2.49e)$$

$$(\alpha_{25}, \alpha_{3,4+N} + \alpha_{4,4+N}), \quad \text{for (2.46b), } j_2 = 0, \ m_1 > 0 \quad (2.49f)$$

$$(\alpha_{15}, \alpha_{4,4+N}), \quad \text{for (2.46c), } j_1 > 0, \ 2j_1 + m_1 \geq 2 \quad (2.49g)$$

$$(\alpha_{15} + \alpha_{25}, \alpha_{4,4+N}), \quad \text{for (2.46c), } j_1 = 0, \ m_1 > 0 \quad (2.49h)$$

$$(\alpha_{25}, \alpha_{4,4+N}), \quad \text{for (2.46d), } m_1 \neq 1 \quad (2.49i)$$

This diagram gives the UIR $L^\Lambda$, contained in $\tilde{V}^\Lambda$ as follows:

$$
L^\Lambda = \tilde{V}^\Lambda / I^{\beta, \beta'} \quad I^{\beta, \beta'} = I^\beta \cup I^{\beta'} \quad (2.50)
$$

where $I^\beta, I^{\beta'}$ are given in (2.36), accordingly to the cases in (2.49).

The embedding diagrams for the corresponding modules $\tilde{V}^\Lambda$ when there are even embeddings are:

$$
\tilde{V}^{\Lambda+\beta'} \\
\uparrow \\
\tilde{V}^{\Lambda+\beta_e} \leftarrow \tilde{V}^\Lambda \rightarrow \tilde{V}^{\Lambda+\beta}
$$

where

$$(\beta, \beta', \beta_e) =$$

$$(\alpha_{15}, \alpha_{3,4+N}, \alpha_{14}), \quad \text{for (2.46a), } j_1 j_2 > 0, \ m_1 = 0 \quad (2.52a)$$

$$(\alpha_{25}, \alpha_{3,4+N}, \alpha_{24}), \quad \text{for (2.46b), } j_2 = \frac{1}{2}, \ m_1 = 0 \quad (2.52b)$$

$$(\alpha_{25}, \alpha_{3,4+N} + \alpha_{4,4+N}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.46b), } j_2 = m_1 = 0 \quad (2.52c)$$

$$(\alpha_{15}, \alpha_{4,4+N}, \alpha_{13}), \quad \text{for (2.46c), } j_1 = \frac{1}{2}, \ m_1 = 0 \quad (2.52d)$$

$$(\alpha_{15} + \alpha_{25}, \alpha_{4,4+N}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.46c), } j_1 = m_1 = 0 \quad (2.52e)$$

$$(\alpha_{25}, \alpha_{4,4+N}, \alpha_{23} + \alpha_{14}), \quad \text{for (2.46d), } m_1 = 1 \quad (2.52f)$$
This diagram gives the UIR $L_\Lambda$ contained in $\tilde{V}^\Lambda$ as follows:

$$L_\Lambda = \tilde{V}^\Lambda/I^{\beta,\beta',\beta_e}, \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'} \cup \tilde{V}^{\Lambda+\beta_e}$$  \hspace{1cm} (2.53)

Naturally, the two odd embeddings in (2.48) or (2.51) are the combination of the different cases of (2.40). Similarly, like (2.40) is a piece of the richer picture (2.45), here we have the following analogues of (2.45) \cite{100}[6]

$$\cdots \rightarrow V_{00} \rightarrow V_{10} \rightarrow \cdots$$

$$\uparrow$$

$$\uparrow$$

$$\uparrow$$

$$\uparrow$$

$$\uparrow$$

$$\uparrow$$

where $V_{k\ell} \equiv V^{\Lambda+k\beta+\ell\beta'}$, and $\beta, \beta'$ are the roots appearing in (2.49a, e, g, i), (or \(2.52a, b, d, f\))

$$\cdots \rightarrow V_{10} \rightarrow V_{11} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

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$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

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$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\cdots \rightarrow V_{00} \rightarrow V_{01} \rightarrow \cdots$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

$$\uparrow \uparrow$$

The difference between the cases $N = 1$ and $N > 1$ is due to the fact that if (2.4b) is fulfilled for $V_{00}$ w.r.t. two odd roots $\beta, \beta'$ then for $N > 1$ it is fulfilled also for all

\footnote{These diagrams are essential parts of much richer diagrams (which we do not need since we consider only UIRs-related modules) which are explicitly described for any $N$ in [100], and shown there in Fig. 1 (for N=1) and Fig. 2 (for N=2).}
Verma modules $V_{k\ell}$ again w.r.t. these odd roots $\beta, \beta'$, while for $N = 1$ it is fulfilled only for $V_{k0}$ w.r.t. the odd root $\beta$ and only for $V_{0\ell}$ w.r.t. the odd root $\beta'$.

In the cases (2.49b, c, d, f, h) (or (2.52c, e)) we have the same diagrams though their parametrization is more involved [100] (cf. also what we said about transmutation for the single chains after (2.45)). However, for the modules with $0 \leq k, \ell \leq 1$ (which we use) we have simply as before $V_{k\ell} = V^{\Lambda + k\beta + \ell\beta'}$ for the appropriate $\beta, \beta'$.

The richer structure for $N > 1$ has practical consequences for the calculation of the character formulae, cf. next Section.
3. Character formulae of positive energy UIRs

3.1. Character formulae: generalities

In the beginning of this subsection we follow [110]. Let \( \hat{G} \) be a simple Lie algebra of rank \( \ell \) with Cartan subalgebra \( \hat{\mathcal{H}} \), root system \( \hat{\Delta} \), simple root system \( \hat{\pi} \). Let \( \Gamma \), (resp. \( \Gamma_+ \)), be the set of all integral, (resp. integral dominant), elements of \( \hat{\mathcal{H}}^* \), i.e., \( \lambda \in \hat{\mathcal{H}}^* \) such that \( (\lambda, \alpha_i^\vee) \in \mathbb{Z} \), (resp. \( \mathbb{Z}_+ \)), for all simple roots \( \alpha_i \), \((\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)) \). Let \( V \) be a lowest weight module with lowest weight \( \Lambda \) and lowest weight vector \( v_0 \). It has the following decomposition:

\[
V = \bigoplus_{\mu \in \Gamma_+} V_\mu \quad \text{where } V_\mu = \{u \in V \mid Hu = (\lambda + \mu)(H)u, \quad \forall H \in \mathcal{H}\} \quad (3.1)
\]

(Note that \( V_0 = Gv_0 \).) Let \( E(\mathcal{H}^*) \) be the associative abelian algebra consisting of the series \( \sum_{\mu \in \mathcal{H}^*} c_\mu e(\mu) \), where \( c_\mu \in \mathbb{C} \), \( c_\mu = 0 \) for \( \mu \) outside the union of a finite number of sets of the form \( D(\lambda) = \{\mu \in \mathcal{H}^* \mid \mu \geq \lambda\} \), using some ordering of \( \mathcal{H}^* \), e.g., the lexicographic one; the formal exponents \( e(\mu) \) have the properties: \( e(0) = 1 \), \( e(\mu)e(\nu) = e(\mu + \nu) \).

Then the (formal) character of \( V \) is defined by:

\[
ch_0 V = \sum_{\mu \in \Gamma_+} (\dim V_\mu) e(\Lambda + \mu) = e(\Lambda) \sum_{\mu \in \Gamma_+} (\dim V_\mu) e(\mu) \quad (3.2)
\]

(We shall use subscript ‘0’ for the even case.)

For a Verma module, i.e., \( V = V^\Lambda \) one has \( \dim V_\mu = P(\mu) \), where \( P(\mu) \) is a generalized partition function, \( P(\mu) = \# \) of ways \( \mu \) can be presented as a sum of positive roots \( \beta \), each root taken with its multiplicity \( \dim G_\beta \) (=1 here), \( P(0) \equiv 1 \). Thus, the character formula for Verma modules is:

\[
ch_0 V^\Lambda = e(\Lambda) \sum_{\mu \in \Gamma_+} P(\mu)e(\mu) = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(\alpha))^{-1} \quad (3.3)
\]

Further we recall the standard reflections in \( \hat{\mathcal{H}}^* \):

\[
s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \lambda \in \hat{\mathcal{H}}^*, \quad \alpha \in \hat{\Delta} \quad (3.4)
\]

The Weyl group \( W \) is generated by the simple reflections \( s_i \equiv s_{\alpha_i}, \alpha_i \in \hat{\pi} \). Thus every element \( w \in W \) can be written as the product of simple reflections. It is said that \( w \) is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of \( w \) is called the length of \( w \), denoted by \( \ell(w) \).

The Weyl character formula for the finite-dimensional irreducible LWM \( L_\Lambda \) over \( \hat{G} \), i.e., when \( \Lambda \in -\Gamma_+ \), has the form: \(^7\)

\[
ch_0 L_\Lambda = \sum_{w \in W} (-1)^{\ell(w)} \; ch_0 V^{w \cdot \Lambda}, \quad \Lambda \in -\Gamma_+ \quad (3.5)
\]

\(^7\) A more general character formula involves the Kazhdan–Lusztig polynomials \( P_{y,w}(u), y, w \in W \) [118].
where the dot \( \cdot \) action is defined by \( w \cdot \lambda = w(\lambda - \rho) + \rho \). For future reference we note:

\[
 s_\alpha \cdot \Lambda = \Lambda + n_\alpha \alpha \tag{3.6}
\]

where

\[
 n_\alpha = n_\alpha(\Lambda) = (\rho - \Lambda, \alpha^\vee) = (\rho - \Lambda)(H_\alpha), \quad \alpha \in \Delta^+
\tag{3.7}
\]

In the case of basic classical Lie superalgebras the first character formulae were given by Kac [112],[119]. For all such superalgebras (except \(osp(1/2N)\)) the character formula for Verma modules is [112],[119]:

\[
 ch \ V^\Lambda = e(\Lambda) \left( \prod_{\alpha \in \Delta_0^+} (1 - e(\alpha))^{-1} \right) \left( \prod_{\alpha \in \Delta_1^+} (1 + e(\alpha)) \right)
\tag{3.8}
\]

Note that the factor \( \prod_{\alpha \in \Delta_0^+} (1 - e(\alpha))^{-1} \) represents the states of the even sector:

\[
 V_0^\Lambda \equiv U((\mathcal{G}^\Phi_+)(0))v_0 \quad \text{as above in the even case}, \quad \text{while} \quad \prod_{\alpha \in \Delta_1^+} (1 + e(\alpha)) \text{ represents the states of the odd sector:} \quad \hat{V}^\Lambda \equiv (U(\mathcal{G}^\Phi_+)/U((\mathcal{G}^\Phi_+)(0)))v_0 .
\]

Thus, we may introduce a character for \( \hat{V}^\Lambda \) as follows:

\[
 ch \ \hat{V}^\Lambda \equiv \prod_{\alpha \in \Delta_1^+} (1 + e(\alpha)).
\tag{3.9}
\]

In our case, \( \hat{V}^\Lambda \) may be viewed as the result of all possible applications of the \(4N\) odd generators \(X_\alpha,4+k\) on \(v_0\), i.e., \( \hat{V}^\Lambda \) has \(24N\) states (including the vacuum). Explicitly, the basis of \( \hat{V}^\Lambda \) may be chosen as in [103]:

\[
 \Psi_{\bar{\epsilon}} = \left( \prod_{k=N} \left( X_{1,4+k}^+ \right)^{\bar{\epsilon}_{1,4+k}} \right) \left( \prod_{k=N} \left( X_{2,4+k}^+ \right)^{\bar{\epsilon}_{2,4+k}} \right) \times \\
 \quad \times \left( \prod_{k=1} \left( X_{3,4+k}^+ \right)^{\bar{\epsilon}_{3,4+k}} \right) \left( \prod_{k=1} \left( X_{4,4+k}^+ \right)^{\bar{\epsilon}_{4,4+k}} \right) v_0 ,
\tag{3.10}
\]

\[
 \bar{\epsilon}_{aj} = 0, 1
\]

where \( \bar{\epsilon} \) denotes the set of all \( \epsilon_{ij} \). Thus, the character of \( \hat{V}^\Lambda \) may be written as:

\[
 ch \ \hat{V}^\Lambda = \sum_{\bar{\epsilon}} e(\Psi_{\bar{\epsilon}}) =
\tag{3.11a}
\]

---

8 Kac considers highest weight modules but his results are immediately transferable to lowest weight modules.

9 The order chosen in (3.10) was important in the proof of unitarity in [102],[103] and for that purposes one may choose also an order in which the vectors on the first row are exchanged with the vectors on the second row. For our purposes the order is important as far as to avoid impossible states - this is much of the analysis done in the next subsections.
\[
= \sum_{\varepsilon} \left( \prod_{k=1}^{N} e(\alpha_{1,4+k})^{\varepsilon_{1,4+k}} \right) \left( \prod_{k=1}^{N} e(\alpha_{2,4+k})^{\varepsilon_{2,4+k}} \right) \times \\
\times \left( \prod_{k=1}^{N} e(\alpha_{3,4+k})^{\varepsilon_{3,4+k}} \right) \left( \prod_{k=1}^{N} e(\alpha_{4,4+k})^{\varepsilon_{4,4+k}} \right) = \tag{3.11b}
\]
\[
= \sum_{\varepsilon} e \left( \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k} \alpha_{a,4+k} \right) = \tag{3.11c}
\]

(Note that in the above formula there is no actual dependence from \(\Lambda\).

We shall use the above to write for the character of \(V^\Lambda\):

\[
ch V^\Lambda = ch \hat{V}^\Lambda \cdot ch_0 V_0^\Lambda = 
\]
\[
= \sum_{\varepsilon} e \left( \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k} \alpha_{a,4+k} \right) \cdot e(\Lambda) \left( \prod_{\alpha \in \Delta_0^+} (1 - e(\alpha))^{-1} \right) = 
\]
\[
= \sum_{\varepsilon} e \left( \Lambda + \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k} \alpha_{a,4+k} \right) \left( \prod_{\alpha \in \Delta_0^+} (1 - e(\alpha))^{-1} \right) = 
\]
\[
= \sum_{\varepsilon} ch_0 V_0^\Lambda + \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k} \alpha_{a,4+k} 
\]

where \(ch_0 V_0^\Lambda\) is the character obtained by restriction of \(V^\Lambda\) to \(V_0^\Lambda\):

\[
ch_0 V_0^\Lambda = e(\Lambda^z) \cdot ch_0 V^\Lambda^z \cdot ch_0 V^\Lambda^u \tag{3.13}
\]

where we use the decomposition \(\Lambda = \Lambda^s + \Lambda^z + \Lambda^u\) from (2.18a), and \(V^\Lambda^z, V^\Lambda^u\), resp., are Verma modules over the complexifications of \(su(2,2), su(N)\), resp., cf. Appendix C.

Analogously, for the factorized Verma modules \(\hat{V}^\Lambda\) the character formula is:

\[
ch \hat{V}^\Lambda = ch \hat{V}^\Lambda \cdot ch_0 \hat{V}_0^\Lambda = 
\]
\[
= \sum_{\varepsilon} ch_0 \hat{V}_0^\Lambda + \sum_{k=1}^{N} \sum_{a=1}^{4} \varepsilon_{a,4+k} \alpha_{a,4+k} 
\]

where \(ch_0 \hat{V}_0^\Lambda\) is the character obtained by restriction of \(\hat{V}^\Lambda\) to \(\hat{V}_0^\Lambda \equiv U((G^F_{\pm})_{(0)}) |\Lambda\rangle\), or more explicitly:

\[
ch_0 \hat{V}_0^\Lambda = e(\Lambda^z) \cdot ch_0 L_{\Lambda^z} \cdot ch_0 L_{\Lambda^u} \tag{3.15}
\]

where we use the decomposition \(\Lambda = \Lambda^s + \Lambda^z + \Lambda^u\) from (2.18a), and character formulae (C.2),(C.3),(C.4) for the irreps of the even subalgebra (from Appendix C).
Formula (3.14) represents the expansion of the corresponding superfield in components, and each component has its own even character. We see that this expansion is given exactly by the expansion of the odd character (3.11).

We have already displayed how the UIRs \( L_\Lambda \) are obtained as factor-modules of the (even-submodules-factorized) Verma modules \( \tilde{V}^A \). Of course, this factorization means that the odd singular vectors of \( \tilde{V}^A \) from (2.33) are becoming null conditions in \( \hat{L}_\Lambda \). However, this is not enough to determine the character formulae even when considering our UIRs as irreps of the complexification \( sl(4/N) \). The latter is a well known feature even in the bosonic case. Here the situation is much more complicated and much more refined analysis is necessary.

The most important aspect of this analysis is the determination of the superfield content. (This analysis was used in [102],[103] but was not explicated enough.) This is given by the positive norm states \( \hat{L}_\Lambda \) among all states in the odd sector \( \hat{V}^A \). Of course, \( \hat{L}_\Lambda \) may have less than \( 2^{4N} \) states.

For future use we introduce notation for the levels of the different chiralities \( \varepsilon_i \) and the overall level \( \varepsilon \)

\[
\varepsilon_i = \sum_{k=1}^{N} \varepsilon_{i,4+k}, \quad i = 1,2,3,4, \quad \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4. \tag{3.16}
\]

The odd null conditions entwine with the even null conditions as we shall see. The even null conditions follow from the even singular vectors in (2.23) (alternatively, one may say that they carry over from the even null conditions (2.26) of \( \tilde{V}^A \)). We write down the even null conditions first since they hold for any positive energy UIR:

\[
\begin{align}
(X_1^+)^{1+2j_1} |\Lambda\rangle &= 0, \\
(X_2^+)^{1+2j_2} |\Lambda\rangle &= 0, \\
(X_j^+)^{1+r_{N+4-j}} |\Lambda\rangle &= 0, \quad j = 5, ..., N + 3
\end{align}
\tag{3.17a-b-c}
\]

((3.17c) being empty for \( N = 1 \)), where by \( |\Lambda\rangle \) we shall denote the lowest weight vector of the UIR \( L_\Lambda \).

3.2. Character formulae for the long UIRs

As we mentioned if \( d > d_{\text{max}} \) there are no further reducibilities, and the UIRs \( L_\Lambda = \tilde{V}^A \) are called long since \( \hat{L}_\Lambda \) may have the maximally possible number of states \( 2^{4N} \) (including the vacuum state).

However, the actual number of states may be less than \( 2^{4N} \) states due to the fact that depending on the values of \( j_a \) and \( r_k \) - not all actions of the odd generators on the vacuum would be allowed. The latter is obvious from formulae (2.43). Using the latter we can give
the resulting signature of the state $\Psi_\bar{\varepsilon}$:
\[
\chi (\Psi_\bar{\varepsilon}) = [d + \frac{1}{2} \varepsilon; j_1 + \frac{1}{2} (\varepsilon_2 - \varepsilon_1), j_2 + \frac{1}{2} (\varepsilon_4 - \varepsilon_3); z + \varepsilon N (\varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_2);
\]
\[
\ldots, r_i + \varepsilon_{1, N+4-i} - \varepsilon_{1, N+5-i} + \varepsilon_{2, N+4-i} - \varepsilon_{2, N+5-i} -
\]
\[
- \varepsilon_{3, N+4-i} + \varepsilon_{3, N+5-i} - \varepsilon_{4, N+4-i} + \varepsilon_{4, N+5-i}, \ldots]
\]

Thus, only if $j_1, j_2 \geq N/2$ and $r_i \geq 4$ (for all $i$) the number of states is $2^{4N}$ [102], and the character formula for the irreducible lowest weight module is (3.14):
\[
ch L_\Lambda = ch \hat{V}^\Lambda = ch \hat{V}^\Lambda \cdot ch_0 \tilde{V}_0^\Lambda, \quad d \geq d_{\text{max}},
\]
\[
j_1, j_2 \geq N/2, \quad r_i \geq 4, \quad i = 1, \ldots, N - 1,
\]

The general formula for $ch L_\Lambda$ shall be written in a similar fashion:
\[
ch L_\Lambda = ch \hat{L}_\Lambda \cdot ch_0 \tilde{V}_0^\Lambda.
\]

Moreover, from now on we shall write only the formulae for $ch \hat{L}_\Lambda$. Thus, formula (3.19) shall be written equivalently as:
\[
ch \hat{L}_\Lambda = ch \hat{V}^\Lambda, \quad j_1, j_2 \geq N/2, \quad r_i \geq 4, \forall i.
\]

As we have noted after (3.14) we do not lose information using this factorized form which has the advantage of brevity.

If the auxiliary conditions (3.19b) are not fulfilled then a careful analysis is necessary. To simplify the exposition we classify the states by the following quantities:
\[
\varepsilon^c_j \equiv \varepsilon_1 - \varepsilon_2,
\]
\[
\varepsilon^a_j \equiv \varepsilon_3 - \varepsilon_4,
\]
\[
\varepsilon^i_r \equiv \varepsilon_{1, 5+i} + \varepsilon_{2, 5+i} + \varepsilon_{3, 4+i} + \varepsilon_{4, 4+i} - \varepsilon_{1, 4+i} - \varepsilon_{2, 4+i} - \varepsilon_{3, 5+i} - \varepsilon_{4, 5+i},
\]
\[
i = 1, \ldots, N - 1.
\]

This gives the following necessary conditions on $\varepsilon_{ij}$ for a state to be allowed:
\[
\varepsilon^c_j \leq 2j_1,
\]
\[
\varepsilon^a_j \leq 2j_2,
\]
\[
\varepsilon^i_r \leq r_{N-i}, \quad i = 1, \ldots, N - 1.
\]

These conditions are also sufficient only for $N = 1$ (when (3.23c) is absent). The exact conditions are:
Criterion: The necessary and sufficient conditions for the state $\Psi_\varepsilon$ of level $\varepsilon$ to be allowed are that conditions (3.23) are fulfilled and that the state is a descendant of an allowed state of level $\varepsilon - 1$. ♦

The second part of the Criterion will take care first of all of chiral (or anti-chiral) states when some $\varepsilon_{a_j}$ contribute to opposing sides of the inequalities in (3.23a) and (3.23c), (or (3.23b) and (3.23c)). This phenomena happens for $j_1 = r_i = 0$, (or $j_2 = r_i = 0$).

We shall give now the most important such occurrences. Take first chiral states, i.e., all $\varepsilon_{3,4+k} = \varepsilon_{4,4+k} = 0$. Fix $i = 1, \ldots, N - 1$. It is easy to see that the following states are not allowed:

$$
\psi_{ij} = \phi_{ij} |\Lambda\rangle = X^+_{1,i+4} X^+_{2,i+5} X^+_{a_1,i+6} \cdots X^+_{a_j-1,i+4+j} |\Lambda\rangle, \quad a_n = 1, 2, \quad (3.24),
$$

$$
j = 1, \ldots, N - i, \quad j_1 = r_{N-i} = \cdots = r_{N-i-j+1} = 0,
$$
in addition, for $N > 2, i > 1$ holds $r_{N-i-1} \neq 0$.

Demonstration: Naturally, this statement is nontrivial only when these states are allowed by condition (3.23a) (i.e., the number of $a_n$ being equal to 2 is not less than the number of $a_n$ being equal to 1), thus we restrict to those. By design these states fulfil also (3.23c), (3.23b) is not relevant), however, they are not descendants of allowed states. First, all states $\psi_{ij} = X^+_{2,i+5} X^+_{a_1,i+6} \cdots X^+_{a_{j-1},i+4+j} |\Lambda\rangle$ violate (3.23c) with $r_{N-i} = 0$. Next, the state $\psi_{11}$ is not allowed since in addition to $\psi_{11}$ also the state $X^+_{1,i+4} |\Lambda\rangle$ is not allowed (it violates (3.23a) with $j_1 = 0$). Due to this the state $\psi_{i2}$ is not descendant of any allowed states, and so on, for all $\psi_{ij}$. Note that the last part of the proof trivializes unless all $a_n = 2$. ■

Remark: The additional condition on the last line of (3.24) is there, since if $r_{N-i+1} = 0$, the states $\psi_{ij} |\Lambda\rangle$ (for $i > 1$) violate (3.23c) with $r_{N-i+1} = 0$ and are excluded without use of the Criterion. ♦

Consider now anti-chiral states, i.e., such that $\varepsilon_{1,4+k} = \varepsilon_{2,4+k} = 0$, for all $k = 1, \ldots, N$. Fix $i = 1, \ldots, N - 1$. Then the following anti-chiral states are not allowed:

$$
\psi'_{ij} = \phi'_{ij} |\Lambda\rangle = X^+_{3,i+5} X^+_{4,i+4} X^+_{b_1,i+3} \cdots X^+_{b_j-1,i+5-j} |\Lambda\rangle, \quad b_n = 3, 4, \quad (3.25)
$$

$$
j = 1, \ldots, i, \quad j_2 = r_{N-i} = \cdots = r_{N-i-j-1} = 0,
$$
in addition, for $N > 2, i > 1$ holds $r_{N-i-1} \neq 0$.

Furthermore, any combinations of $\phi_{ij}$ and $\phi'_{ij}$ are not allowed.

Note that for $N \geq 4$ the states in (3.24),(3.25) do not exhaust the states forbidden by our Criterion. For example, for $N = 4$ there are the following forbidden states:

$$
\psi_4 = \phi_4 |\Lambda\rangle = X^+_{25} X^+_{17} X^+_{16} X^+_{25} |\Lambda\rangle, \quad j_1 = r_1 = r_2 = r_3 = 0 \quad (3.24'),
$$

$$
\psi'_4 = \phi'_4 |\Lambda\rangle = X^+_{45} X^+_{36} X^+_{37} X^+_{48} |\Lambda\rangle, \quad j_2 = r_1 = r_2 = r_3 = 0 \quad (3.25').
$$

Summarizing the discussion so far, the general character formula may be written as follows:

$$
ch \hat{L}_\Lambda = ch \hat{V}^\Lambda - R_{\text{long}}, \quad d > d_{\text{max}}, \quad (3.26)
$$

$$
R = e(\hat{V}_{\text{excl}}^\Lambda) = \sum_{\text{excluded states}} e(\Psi_\varepsilon),
$$

25
where the counter-terms denoted by $\mathcal{R}_{\text{long}}$ are determined by $\hat{V}_{\text{excl}}^\Lambda$ which is the collection of all states (i.e., collection of $\varepsilon_{jk}$) which violate the conditions (3.23), or are impossible in the sense of (3.24) and/or (3.25). Of course, each excluded state is accounted for only once even if it is not allowed for several reasons.\footnote{We should stress that the necessity of the counter-terms above is related to the fact that our representations of $su(2,2/N)$ have physical meaning and the states of $\hat{L}_\Lambda$ represent components of a superfield. There are no counter-terms when we consider these UIRs as irreps of $sl(4/N)$. Thus, formula (3.26) and almost all character formulae derived further in this Section are character formulae of $sl(4/N)$ by just dropping the counter-term $\mathcal{R}$, cf. next Section.}

Finally, we consider two important conjugate special cases.

First, the chiral sector of $R$-symmetry scalars with $j_1 = 0$. Taking into account (3.23a, c) ((3.23b) is trivially satisfied for chiral states) and our Criterion it is easy to see that the appearance of the generators $X_{1,4+k}^+$ is restricted as follows. The generator $X_{15}^+$ may appear only in the state

$$X_{15}^+ X_{25}^+ |\Lambda\rangle$$

and its descendants. The generator $X_{16}^+$ may only appear either in states descendant to the state (3.27) or in the state

$$X_{16}^+ X_{25}^+ |\Lambda\rangle$$

and its descendants including only generators $X_{a,5+\ell}^+$, $a = 1,2$, $\ell > 1$. Further, the restrictions are described recursively, namely, fix $\ell$ such that $1 < \ell \leq N - 1$. The generator $X_{1,5+\ell}^+$ may only appear either in states containing generators $X_{1,5+j}^+$, where $0 \leq j < \ell$, or in the state

$$X_{1,5+\ell}^+ X_{2,4+\ell}^+ X_{2,3+\ell}^+ \cdots X_{2,5}^+ |\Lambda\rangle$$

and its descendants including only generators $X_{a,5+\ell'}^+$, $a = 1,2$, $\ell' > \ell$.

The chiral part of the basis is further restricted. Namely, there are only $N$ chiral states that can be built from the generators $X_{2,4+k}^+$ alone, given as follows:

$$X_{2,4+k}^+ \cdots X_{25}^+ |\Lambda\rangle, \quad k = 1, \ldots, N, \quad j_1 = r_i = 0, \quad \forall i.$$  

(3.30)

This follows from (3.23c) which in this case is reduced to $\varepsilon_{1i} \leq \varepsilon_{1,i+1}$ for $i = 1, \ldots, N-1$.

Second, the anti-chiral sector of $R$-symmetry scalars with $j_2 = 0$. Taking into account (3.23b, c) ((3.23a) is trivially satisfied for anti-chiral states) and our Criterion it is easy to see that the appearance of the generators $X_{3,4+k}^+$ is restricted as follows. The generator $X_{3,4+N}^+$ may appear only in the state

$$X_{3,4+N}^+ X_{4,4+N}^+ |\Lambda\rangle$$

and its descendants. The generator $X_{3,3+N}^+$ may only appear either in states descendant to the state (3.31) or in the state

$$X_{3,3+N}^+ X_{4,4+N}^+ |\Lambda\rangle$$

(3.32)
and its descendants including only generators \( X_{a,4+N-\ell}^+ \), \( a = 3, 4, \ell > 1 \). Further, fix \( \ell \) such that \( 1 < \ell \leq N-1 \). The generator \( X_{3,4+N-\ell}^+ \) may only appear either in states containing generators \( X_{3,4+N-j}^+ \), where \( 0 \leq j < \ell \), or in the state

\[
X_{3,4+N-\ell}^+ X_{4,5+N-\ell}^+ X_{4,6+N-\ell}^+ \cdots X_{4,4+N}^+ |\Lambda\rangle \tag{3.33}
\]

and its descendants including only generators \( X_{a,4+N-\ell'}^+ \), \( a = 3, 4, \ell' > \ell \).

The anti-chiral part of the basis is further restricted. Namely, there are only \( N \) antichiral states that can be built from the generators \( X_{4,4+k}^+ \) alone, given as follows:

\[
X_{4,5+N-k}^+ X_{4,6+N-k}^+ \cdots X_{4,4+N}^+ |\Lambda\rangle , \quad k = 1, \ldots, N , \quad j_2 = r_i = 0, \forall i . \tag{3.34}
\]

This follows from (3.23c) which for such states becomes \( \varepsilon_{4,4+N-i} \leq \varepsilon_{4,5+N-i} \) for \( i = 1, \ldots, N - 1 \).

### 3.3. Character formulae of SRC UIRs

Here we consider the four SRC cases.

- **a** \( d = d_{N1}^1 = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d_{NN}^3 \).
- **b** Let first \( j_2 > 0 \).

In these semi-short SRC cases holds the odd null condition (following from the singular vector (8.9a) of [101], cf. also (2.32a'), (2.33a'), (2.36a)):

\[
P_{3,4+N} |\Lambda\rangle = (X_4^+ X_2^+(h_2 - 1) - X_2^+ X_4^+ h_2) |\Lambda\rangle = (2j_2 X_{3,4+N}^+ - X_4^+ X_2^+) |\Lambda\rangle = 0 \tag{3.35}
\]

where \( X_{3,4+N}^+ = [X_2^+, X_4^+] \). Clearly, condition (3.35) means that the generator \( X_{3,4+N}^+ \) is eliminated from the basis that is built on the lowest weight vector \( |\Lambda\rangle \). Thus, for \( N = 1 \) and if \( r_1 > 0 \) for \( N > 1 \) the character formula is:

\[
ch \hat{L}_\Lambda = \prod_{\alpha \in \Delta_1^+ \atop \alpha \neq \alpha_{3,4+N}^+} (1 + c(\alpha)) - \mathcal{R} , \tag{3.36}
\]

\[
d = d_{\text{max}} = d_{N1}^1 > d_{NN}^3 , \quad j_2 r_1 > 0 .
\]

There are no counter-terms when \( j_1 \geq N/2 , j_2 \geq (N - 1)/2 \) and \( r_i \geq 4 \) (for all \( i \)), and then the number of states is \( 2^{N-1} \). The change of statement (as compared to the long superfields) w.r.t. \( j_2 \) comes because of the elimination of the generator \( X_{3,4+N}^+ \).

**Remark:** For the finite-dimensional irreps of \( \mathfrak{sl}(4/N) \) (in fact, of all basic classical Lie superalgebras) such situations are called 'singly atypical' and the character formulae look exactly as (3.36) with \( \mathcal{R} = 0 \), cf. [120],[121],[122].\(^{11}\)

\(^{11}\) For character formulae of finite-dimensional irreps beyond the singly atypical case cf. [123], [124], [125], [126], and references therein.
When there are no counter-terms (also for the complex \( sl(4/N) \) case) this formula follows easily from (2.42). Indeed, in the case at hand \( I^\beta = I^1, \) (cf. (2.36a)); then from \( L_\Lambda = \tilde{V}^\Lambda /I^1 \) follows:

\[
ch L_\Lambda = ch \tilde{V}^\Lambda - ch I^1, \quad \text{or equivalently,}
\]

\[
ch \hat{L}_\Lambda = ch \tilde{V}^\Lambda - ch \hat{I}^1, \tag{3.37}
\]

where \( \hat{I}^1 \) is the projection of \( I^1 \) to the odd sector. Naively, the character of \( \hat{I}^1 \) should be given by the character of \( \tilde{V}^{\Lambda+\alpha_{3,4+N}} \), however, as discussed in general - cf. (2.9), \( I^1 \) is smaller than \( \tilde{V}^{\Lambda+\alpha_{3,4+N}} \) and its character is given with a prefactor\(^{12}\):

\[
ch \hat{I}^1 = \frac{1}{1 + e(\alpha_{3,4+N})} ch \tilde{V}^{\Lambda+\alpha_{3,4+N}} = \frac{e(\alpha_{3,4+N})}{1 + e(\alpha_{3,4+N})} ch \tilde{V}^\Lambda. \tag{3.38}
\]

Now (3.36) (with \( R = 0 \)) follows from the combination of (3.37) and (3.38).

Formula (3.36) may also be described by using the odd reflection (2.11) with \( \beta = \alpha_{3,4+N} \):

\[
ch \hat{L}_\Lambda = ch \tilde{V}^\Lambda - \frac{1}{1 + e(\alpha_{3,4+N})} ch \tilde{V}^{\hat{s}_{\alpha_{3,4+N}} \Lambda} - R = \tag{3.39a}
\]

\[
= ch \hat{V}^\Lambda - \hat{s}_{\alpha_{3,4+N}} \cdot ch \tilde{V}^\Lambda - R = \tag{3.39b}
\]

\[
= \sum_{\hat{s} \in \hat{W}_{\alpha_{3,4+N}}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{V}^\Lambda - R, \tag{3.39c}
\]

where \( \hat{W}_\beta \equiv \{1, \hat{s}_\beta\} \) is a two-element semi-group restriction of \( \hat{W}_\beta \), and we have formalized further by introducing notation for the action of an odd reflection on characters:

\[
\hat{s}_\beta \cdot ch V^\Lambda = \frac{1}{1 + e(\beta)} ch V^{\hat{s}_\beta \cdot \Lambda} = \frac{1}{1 + e(\beta)} ch V^{\Lambda+\beta} = \frac{e(\beta)}{1 + e(\beta)} ch V^\Lambda. \tag{3.40}
\]

In particular, we shall show that in many cases character formulae (3.36),(3.39) may be written as follows:

\[
ch \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_\beta} (-1)^{\ell(\hat{s})} \hat{s} \cdot (ch \hat{V}^\Lambda - R_{\text{long}}), \tag{3.41}
\]

where \( R_{\text{long}} \) represents the counter-terms for the long superfields for the same values of \( j_1 \) and \( r_i \) as \( \Lambda \), while the value of \( j_2 \) is zero when \( j_2 \) from \( \Lambda \) is zero, otherwise it has to be the generic value \( j_2 \geq N/2 \). (As we know, restriction (3.23b) trivializes for \( j_2 \geq N/2 \) and thus the structure of the irrep is the same for any such generic value.)

---

\(^{12}\) This technique was applied first when deriving the characters of the \( N = 2 \) super-Virasoro algebras, cf. [127].
Writing (3.36) as (3.39) (or (3.41)) may look as a complicated way to describe the cancellation of a factor from the character formula for $\hat{V}^\Lambda$, however, first of all it is related to the structure of $\hat{V}^\Lambda$ given by (2.42), and furthermore may be interpreted - when there are no counter-terms - as the following decomposition:

$$\hat{V}^\Lambda = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta},$$

(3.42)

for $\beta = \alpha_{3,4+N}$. Indeed, for generic signatures $\hat{L}_{\Lambda+\beta}$ is isomorphic to $\hat{L}_\Lambda$ as a vector space (this is due to the fact that $V^{\Lambda+\beta}$ has the same reducibilities as $V^\Lambda$, cf. Section 2), they differ only by the vacuum state. Thus, when there are no counter-terms, both $\hat{L}_\Lambda$ and $\hat{L}_{\Lambda+\beta}$ have the same $2^{4N-1}$ states. If we describe them for shortness as:

$$\Phi_i |\Lambda\rangle, \quad \Phi_i |\Lambda + \beta\rangle,$$

(3.43)

where none of $\Phi_i$ contains $X^+_{3,4+N}$ and recall that the embedding of $\hat{V}^{\Lambda+\beta}$ into $\hat{V}^\Lambda$ is given essentially by the generator $X^+_{3,4+N}$ (cf. (2.36a)) then we see that after the embedding the states in (3.43) restore all $2^{4N}$ states in $\hat{V}^\Lambda$:

$$\Phi_i |\Lambda\rangle, \quad X^+_{3,4+N} \Phi_i |\Lambda\rangle.$$

(3.44)

It is more important that there is a similar decomposition valid for many cases beyond the generic, i.e., we have:

$$\left(\hat{L}_{\text{long}}\right)_{|_{d=d^a}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{3,4+N}}, \quad N = 1 \text{ or } r_1 > 0 \text{ for } N > 1,$$

(3.45)

where $\hat{L}_{\text{long}}$ is a long superfield with the same values of $j_1$ and $r_i$ as $\Lambda$, while the value of $j_2$ has to be specified as above for $R_{\text{long}}$, and equality is as vector spaces.

For $N > 1$ there are possible additional truncations of the basis. To make the exposition easier we need additional notation. Let $i_0$ be an integer such that $0 \leq i_0 \leq N - 1$, and $r_i = 0$ for $i \leq i_0$, and if $i_0 < N - 1$ then $r_{i_0+1} > 0$.\footnote{This is formally valid for $N = 1$ with $i_0 = 0$ since $r_0 \equiv 0$ by convention. This shall be used to make certain statements valid for general $N$.}

Let now $N > 1$ and $i_0 > 0$, then the generators $X^+_{3,4+N-i}, \quad i = 1, \ldots, i_0$, are eliminated from the basis.

Demonstration: First we consider the vector:

$$P_{3,3+N} v_0 = (2j_2 X^+_{3,3+N} - X^+_{4,3+N} X^+_2) v_0 =$$

$$= 2j_2 \left( X^+_{3,4+N} X^+_3 - X^+_{3+N} X^+_3 \right) v_0 -$$

$$- (X^+_{4} X^+_{3+N} - X^+_{3+N} X^+_4) X^+_2 v_0 =$$

$$= \left( P_{3,4+N} X^+_{3+N} - X^+_{3+N} P_{3,4+N} \right) v_0.$$

(3.46)
For $r_1 = 0$ it is descendant of (2.23c) and (2.32) and leads to the null condition:

$$P_{3,3+N}|\Lambda\rangle = \left(2j_2X_{3,3+N}^+ - X_{4,3+N}^+X_2^+\right)|\Lambda\rangle = 0 \quad (3.47)$$

which naturally follows from (3.17c) and (3.35), and which means that the generator $X_{3,3+N}^+$ is eliminated from the basis. Analogously, we define the vectors:

$$P_{3,4+N-i}v_0 = \left(2j_2X_{3,4+N-i}^+ - X_{4,4+N-i}^+X_2^+\right)v_0 \quad (3.48)$$

which are recursively related:

$$P_{3,4+N-i}v_0 = 2j_2\left(X_{3,5+N-i}^+X_{4,4+N-i}^+ - X_{4+N-i}^+X_{3,5+N-i}^+\right)v_0 - \left(X_{4,5+N-i}^+X_{4+N-i}^+ - X_{4+N-i}^+X_{4,5+N-i}^+\right)X_2^+v_0 = (P_{3,5+N-i}X_{4+N-i}^+ - X_{4,N-i}^+P_{3,5+N-i})v_0. \quad (3.49)$$

Thus, in the situation: $r_i = 0$, $i = 1, \ldots, i_0$, there are the following null conditions:

$$P_{3,4+N-i}|\Lambda\rangle = \left(2j_2X_{3,4+N-i}^+ - X_{4,4+N-i}^+X_2^+\right)|\Lambda\rangle = 0, \quad r_j = 0, \quad 1 \leq j \leq i \leq i_0. \quad (3.50)$$

These are recursively descendant null conditions which means that a condition for fixed $i$ is a descendant of the one for $i - 1$ (since $X_{4+N-i}^+|\Lambda\rangle = 0$ due to (3.17c)). Conditions (3.50) mean that the generators $X_{3,4+N-i}^+$, $i = 1, \ldots, i_0$, are eliminated from the basis. ■

From the above follows that for $i_0 > 0$ the decomposition (3.45) can not hold. Indeed, the generators $X_{3,4+N-i}^+$, $i = 1, \ldots, i_0$, are eliminated from the irrep $\hat{L}_\Lambda$ due to the fact that we are at a reducibility point, but there is no reason for them to be eliminated from the long superfield. Certainly, some of these generators are present in the second term $\hat{L}_{\Lambda + \alpha_{3,4+N}}$ in (3.45), but that would be only those which in the long superfield were in states of the kind: $\Phi X_{3,4+N}^+|\Lambda\rangle$, and, certainly, such states do not exhaust the occurrence of the discussed generators in the long superfield. Symbolically, instead of the decomposition (3.45) we shall write:

$$\left(\hat{L}_{\text{long}}\right)_{d = d_a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{3,4+N}} \oplus \hat{L}_A', \quad N > 1, \quad i_0 > 0, \quad (3.51)$$

where we have represented the excess states by the last term with prime. With the prime we stress that this is not a genuine irrep, but just a book-keeping device. Formulae as (3.51) in which not all terms are genuine irreps shall be called \textit{quasi-decompositions}.

The corresponding character formula is:

$$ch \hat{L}_\Lambda = \prod_{\alpha \in \Delta^-_{3,4}} (1 + e(\alpha)) - R = \quad (3.52a)$$
\[ \mathcal{R} = \sum_{\hat{s} \in \hat{W}_{i_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \text{ch} \, \hat{V}^\Lambda - \mathcal{R} = \]
\[ \sum_{\hat{s} \in \hat{W}_{i_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \left( \text{ch} \, \hat{V}^\Lambda - \mathcal{R}_{\text{long}} \right) , \quad (3.52c) \]
\[ \hat{W}_{i_0}^a \equiv \hat{W}_{\alpha_3,N+i_0} \times \hat{W}_{\alpha_3,n_2} \times \cdots \times \hat{W}_{\alpha_3,N+i_0} , \quad (3.52d) \]
\[ d = d_{\text{max}} = d_{N1}^1 > d_{NN}^3 , \quad j_2 > 0 , \quad r_i = 0 , \quad i \leq i_0 . \]

The restrictions (3.23) used to determine the counter-terms are, of course, with \( \varepsilon_{3,5+N-k} = 0 \), \( k = 1, \ldots, 1+i_0 \). Formuлаe (3.36), (3.39), (3.41) are special cases of (3.52a, b, c), resp., for \( i_0 = 0 \). The maximal number of states in \( \hat{L}_\Lambda \) is \( 2^4N - 1 \). The maximal number of states that is obtained from the action of the Weyl group \( \hat{W}_{i_0}^a \) on \( \text{ch} \, \hat{V}^\Lambda \), while the actual counter-term is obtained from the action of the Weyl group on \( \mathcal{R}_{\text{long}} \).

In the extreme case of \( R \)-symmetry scalars: \( i_0 = N - 1 \), i.e., \( r_i = 0 \), \( i = 1, \ldots, N - 1 \), or, equivalently, \( m_i = 0 = m \), all the \( N \) generators \( X^+_{3,4+k} \) are eliminated. The character formula is again (3.52) taken with \( i_0 = N - 1 \).

- Let now \( j_2 = 0 \).

Then all null conditions above (valid for \( j_2 > 0 \)) follow from (2.26b), so these conditions do not mean elimination of the mentioned vectors. As we know in this situation we have the singular vector (2.35) which leads to the following null condition:

\[ X^+_{3,4+N} X^+_{4,4+N} \langle \Lambda \rangle = X^+_{4} X^+_{2} X^+_{4} \langle \Lambda \rangle = 0 . \quad (3.53) \]

The state in (3.53) and all of its \( 2^4N - 2 \) descendants are zero for any \( N \). Thus, the character formula is similar to (3.39), but with \( \alpha_{3,4+N} \) replaced by \( \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N} \), (cf. (2.43e)):

\[ \text{ch} \, \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \text{ch} \, \hat{V}^\Lambda - \mathcal{R} = \]
\[ = \sum_{\hat{s} \in \hat{W}_{\beta_{12}}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \left( \text{ch} \, \hat{V}^\Lambda - \mathcal{R}_{\text{long}} \right) , \quad (3.54b) \]

where \( \hat{W}_{\beta_{12}} \equiv \{ 1, \beta_{12} \} \).

Note that for \( N = 1 \) formula (3.54) is equivalent to (3.36) since due to (3.23b) the generator \( X^+_{3,4+N} \) could appear only together with \( X^+_{4,4+N} \) but the resulting state (3.53) is zero.

Here holds a decomposition similar to (3.45):

\[ \left( \hat{L}_{\text{long}} \right)_{d=d^a} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta_{12}} , \quad N = 1 \text{ or } r_1 > 0 \text{ for } N > 1 , \quad (3.55) \]
where $\hat{\mathcal{L}}_{\text{long}}$ is with the same values of $j_1, j_2(=0), r_i$ as $\Lambda$. Note, however, that the UIR $\hat{\mathcal{L}}_{\Lambda+\beta_{12}}$ belongs to type $b$ below.

There are more eliminations for $N > 1$ when $i_0 > 0$. For instance we can show that all states as in (3.33) considered for $\ell = 1, \ldots, i_0$ are not allowed.

**Demonstration:** We show this by induction. Consider first the case $\ell = 1$:

\[
X_{3,3+N}^+ X_{4,4+N}^+ |\Lambda\rangle = \left(X_{3+N}^+ X_{3,4+N}^+ - X_{3,4+N}^+ X_{3+N}^+ \right) X_{4,4+N}^+ |\Lambda\rangle =
\]

\[
= -X_{3,4+N}^+ X_{3+N}^+ X_{4,4+N}^+ |\Lambda\rangle = -X_{3,4+N}^+ X_{4,3+N}^+ |\Lambda\rangle,
\]

where the first term is zero due to (3.53), and the second term is transformed by pulling $X_{3+N}^+$ to the right, where it annihilates the vacuum (due to (3.17c) with $j = N + 3$ for $r_1 = 0$), and the resulting state is the forbidden $\psi_{N-1,1}^N$ from (3.25). Thus, the above state is not allowed.

Now fix $k$ such that $1 < k \leq i_0$ and suppose that we have already shown that all states in (3.33) for $\ell < k$ are not allowed, and we shall show this for $\ell = k$. Indeed, this state is not allowed:

\[
X_{3,4+N}^+ X_{4,4+N}^+ - X_{3,4+N}^+ X_{4,4+N}^+ |\Lambda\rangle =
\]

\[
= -X_{4,5+N-k}^+ X_{3,4+N-k}^+ X_{4,6+N-k}^+ \cdots X_{4,4+N}^+ |\Lambda\rangle =
\]

\[
= -X_{4,5+N-k}^+ \left(X_{4+N-k}^+ X_{3,5+N-k}^+ - X_{3,5+N-k}^+ X_{4+N-k}^+ \right) X_{4,6+N-k}^+ \cdots X_{4,4+N}^+ |\Lambda\rangle,
\]

where the first term on the last line is a state descendant of (3.33) with $k \to k - 1$, which is not allowed by the induction hypothesis and the second term is zero due to pulling $X_{4+N-k}^+$ to the right, where it annihilates the vacuum (due to (3.17c) with $j = N + 4 - k$ for $r_k = 0$).

From the above follows that if $i_0 > 0$ the decomposition (3.55) does not hold. Instead, there is a quasi-decomposition similar to (3.51).

We can be more explicit in the case when all $r_i = 0$. In that case all the vectors $X_{3,5+N-k}^+$ are eliminated from all anti-chiral states.

**Demonstration:** We show this by induction in $k$ starting with $k = 1, 2$. Take first the generator $X_{3,4+N}^+$. As we know when $j_2 = r_i = 0, \forall i$, the only anti-chiral state containing it in a long superfield is the state (3.31) and its descendants. However, here all these possible states are zero due to (3.53). Thus, there are no anti-chiral states containing $X_{3,4+N}^+$. Take next the vector $X_{3,3+N}^+$. As we know the only anti-chiral states containing it in a long superfield are the states (3.31),(3.32), and their descendants. The first is zero, while the second is not allowed as we showed above. Thus, the vector $X_{3,3+N}^+$ is eliminated from all anti-chiral states.

Now fix $\ell$ such that $1 < \ell \leq N - 1$ and suppose that we have already shown elimination of $X_{3,5+N-k}^+$ for $k = 1, \ldots, \ell$, from all anti-chiral states. We want to show elimination for $k = \ell + 1$, i.e., of the generator $X_{3,4+N-\ell}^+$. As we know from the similar consideration of long superfields all anti-chiral states including $X_{3,4+N-\ell}^+$ and which are not yet excluded may be written as the state (3.33) and its descendants including only generators $X_{a,4+N-\ell'}^+$, $a = 3, 4$, $\ell' > \ell$. However, above we have shown that this state is not allowed. Thus, all generators $X_{3,4+k}^+$ for $k = 1, \ldots, N$ are eliminated from the anti-chiral part of the basis. □
The anti-chiral part of the basis is further restricted. As we know, when \( j_2 = r_i = 0, \forall i \), there are only \( N \) anti-chiral states that can be built from the generators \( X_{4,4+k}^+ \) alone, given in (3.34). Thus the corresponding character formula is:

\[
ch \hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\alpha \subseteq \Delta^+_1, \epsilon_1+\epsilon_2 > 0} (1 + e(\alpha)) - \mathcal{R},
\]

(3.58)

\[
d = d_{\text{max}} = d_{NN}^1 > d_{NN}^3, \quad j_2 = 0, \quad r_i = 0, \forall i.
\]

**b** \( d = d_{N1}^2 = z + 2m_1 - 2m/N > d_{NN}^3, \quad j_2 = 0 \).

In these short single-reducibility-condition cases holds the odd null condition (following from the singular vector (2.32b) or (2.33b))

\[
X_{4}^+ |\Lambda\rangle = X_{4,4+N}^+ |\Lambda\rangle = 0.
\]

(3.59)

Since \( j_2 = 0 \) from (2.26b) and (3.59) follows the additional null condition:

\[
X_{3,4+N}^+ |\Lambda\rangle = [X_{2}^+, X_{4}^+] |\Lambda\rangle = 0
\]

(3.60)

For \( N > 1 \) and \( r_1 > 2 \) each of these UIRs enters as the second term in decomposition (3.55), when the first term is an UIR of type \( a \) with \( j_2 = 0 \), as explained above.

Further, for \( N > 1 \) there are additional null conditions if \( r_i = 0, i \leq i_0 \). Indeed, let \( r_1 = 0 \); then from (2.26c) and (3.60) follow the additional null conditions:

\[
X_{4,3+N}^+ |\Lambda\rangle = [X_{4,4+N}^+, X_{3+N}^+] |\Lambda\rangle = 0, \quad r_1 = 0,
\]

(3.61a)

\[
X_{3,3+N}^+ |\Lambda\rangle = [X_{3,4+N}^+, X_{3+N}^+] |\Lambda\rangle = 0, \quad r_1 = 0
\]

(3.61b)

Analogously, in the situation: \( r_i = 0, i = 1, \ldots, i_0 \), there are recursive null conditions:

\[
X_{3,4+N-j}^+ |\Lambda\rangle = [X_{3,5+N-j}^+, X_{4+N-j}^+] |\Lambda\rangle = 0, \quad r_j = 0, \quad 1 \leq j \leq i \leq i_0 \quad (3.62a)
\]

\[
X_{3,4+N-j}^+ |\Lambda\rangle = [X_{3,5+N-j}^+, X_{4+N-j}^+] |\Lambda\rangle = 0, \quad r_j = 0, \quad 1 \leq j \leq i \leq i_0 \quad (3.62b)
\]

Thus, \( 2(1+i_0) \) generators \( X_{3,5+N-k}^+, X_{4,5+N-k}^+ \), \( k = 1, \ldots, 1+i_0 \), are eliminated. The maximal number of states in \( \hat{L}_\Lambda \) is \( 2^{4N-2+i_0} \).

The corresponding character formula is:

\[
ch \hat{L}_\Lambda = \prod_{\alpha \subseteq \Delta^+_1, \epsilon_1+\epsilon_2 > 0} (1 + e(\alpha)) - \mathcal{R}
\]

(3.63a)

\[
= \sum_{\hat{s} \in \hat{W}_{i_0}^b} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{V}^\Lambda - \mathcal{R},
\]

(3.63b)

\[
\hat{W}_{i_0}^b \equiv \hat{W}_{\alpha_{3,N+4}} \times \hat{W}_{\alpha_{3,N+4}} \times \cdots \times \hat{W}_{\alpha_{3,N+4-i_0}} \times \hat{W}_{\alpha_{4,N+4}} \times \hat{W}_{\alpha_{4,N+4}} \times \cdots \times \hat{W}_{\alpha_{4,N+4-i_0}}
\]

(3.63c)

\[
d = d_{N1}^2 > d_{NN}^3, \quad j_2 = 0, \quad r_i = 0, \quad i \leq i_0,
\]

33
where determining the counter-terms we use \( \varepsilon_{j,5+N-k} = 0, \ j = 3, 4, \ k = 1, \ldots, 1+i_0 \).

In the case of \( R \)-symmetry scalars \( (i_0 = N - 1) \) we have:

\[
X_{3,4+k}^+ | \Lambda \rangle = 0, \quad X_{4,4+k}^+ | \Lambda \rangle = 0, \quad k = 1, \ldots, N, \quad r_i = 0, \forall i. \tag{3.64}
\]

The character formula is (3.63) taken with \( 1 + i_0 = N \). These UIRs should be called chiral since all anti-chiral generators are eliminated.

The next two cases are conjugates of the first two and the exposition will be compact.

1. \( c \quad d = d_N^3 N = d^c \equiv 2 + 2j_1 - z + 2m/N > d_N^1 \).
2. Let first \( j_1 > 0 \).

In these semi-short SRC cases holds the odd null condition (following from the singular vector (8.8a) of [101]), here cf. (2.32c') or (2.33c')):

\[
P_{15} | \Lambda \rangle = (2j_1 X_{15}^+ - X_{3}^+ X_{1}^+ ) | \Lambda \rangle = 0 \tag{3.65}
\]

where \( X_{15}^+ = [X_{1}^+, X_{3}^+] \). Clearly, condition (3.65) means that the generator \( X_{15}^+ \) is eliminated from the basis.

Let now \( i'_0 \) be an integer such that \( 0 \leq i'_0 \leq N - 1 \), and \( r_{N-i} = 0 \) for \( i \leq i'_0 \), and if \( i'_0 < N - 1 \) then \( r_{N-1-i'_0} > 0 \).\(^{14}\) For \( N > 1 \) and \( i'_0 > 0 \) there are additional truncations due to the vectors (cf. (C.7) of [101]):

\[
P_{1,5+i} v_0 = (2j_1 X_{1,5+i}^+ - X_{2,5+i}^+ ) X_{1}^+ v_0 =
= 2j_1 (X_{1,4+i}^+ X_{4+1}^+ - X_{4+i}^+ X_{1,4+i}^+ ) v_0 -
- (X_{2,4+i}^+ X_{4+i}^+ - X_{4+i}^+ X_{2,4+i}^+ ) X_{1}^+ v_0 =
= (P_{1,4+i} X_{4+i}^+ - X_{4+i}^+ P_{1,4+i} ) v_0
\]

which produced recursive null conditions:

\[
P_{1,5+i} | \Lambda \rangle = (2j_1 X_{1,5+i}^+ - X_{2,5+i}^+ X_{1}^+ ) | \Lambda \rangle = 0, \quad r_{N-j} = 0, \quad 1 \leq j \leq i \leq i'_0, \tag{3.67}
\]

which means that the generators \( X_{1,5+i}^+ \) are eliminated from the basis.

The corresponding character formula is:

\[
ch \hat{L}_\Lambda = \prod_{\alpha \in \Delta^+_1 \atop \alpha \neq \alpha_{1,4+k} \atop k = 1, \ldots, 1+i_0'} (1 + e(\alpha)) - R \tag{3.68a}
\]

\(^{14}\) This is formally valid for \( N = 1 \) with \( i'_0 = 0 \) since \( r_N = 0 \) by convention.
\[\sum_{s \in \hat{W}_{i_0}^c} (-1)^{\ell(s)} \hat{s} \cdot \text{ch} \, \hat{\nu}^A = \mathcal{R} = \quad (3.68b)\]

\[\sum_{s \in \hat{W}_{i_0}^c} (-1)^{\ell(s)} \hat{s} \cdot (\text{ch} \, \hat{\nu}^A - \mathcal{R}_{\text{long}}), \quad (3.68c)\]

\[\hat{W}_{i_0}^c \equiv \hat{W}_{\alpha_{15}} \times \hat{W}_{\alpha_{16}} \times \cdots \times \hat{W}_{\alpha_{1,5+i_0}}, \quad (3.68d)\]

\[d = d_{\text{max}} = d_{NN}^3 > d_{N1}^1, \quad j_1 > 0, \quad r_{N-i} = 0, \quad i \leq i_0 \leq N-1.\]

This formula is valid also for \(N = 1\) or when \(r_{N-1} > 0\) by setting \(i_0' = 0\). The maximal number of states in \(\hat{L}_A\) is \(2^{4N-1-i_0'}\). The restrictions (3.23) used for the counter-terms are with \(\varepsilon_{1,N-i} = 0, \quad i = 0,1, \ldots, i_0'\).

When \(i_0' = 0\) holds decomposition similar to (3.45):

\[(\hat{L}_{\text{long}})_{d=d_{\text{max}}} = \hat{L}_A \oplus \hat{L}_{A+\alpha_{15}}, \quad N = 1 \text{ or } r_{N-1} > 0 \text{ for } N > 1, \quad (3.69)\]

where \(\hat{L}_{\text{long}}\) is a long superfield with the same values of \(j_2\) and \(r_i\) as \(\Lambda\), while the value of \(j_1\) is zero when \(j_1\) from \(\Lambda\) is zero, otherwise it has to be the generic value \(j_1 \geq N/2\). From the above follows also that when \(i_0' > 0\) the decomposition (3.69) does not hold.

In the case of \(R\)-symmetry scalars \((i_0' = N-1)\) all the \(N\) generators \(X_{1,4+k}^+\) \((k = 1, \ldots, N)\) are eliminated. The maximal number of states in \(\hat{L}_A\) is \(2^{3N}\).

**Let now \(j_1 = 0.\)**

Then the null conditions above all follow from (2.26a) so these conditions do not mean elimination of the mentioned vectors. In this situation we have the singular vector (2.34) which leads to the following null condition:

\[X_{15}^+ X_{25}^+ |\Lambda\rangle = X_3^+ X_1^+ X_3^+ |\Lambda\rangle = 0. \quad (3.70)\]

The state in (3.70) and all of its \(2^{4N-2}\) descendants are zero for any \(N\). Thus, for \(N = 1\) or if \(r_{N-1} > 0\), the character formula is as (3.68) for \(i_0' = 0\), but with \(\alpha_{15}\) replaced by \(\beta_{34} = \alpha_{25} + \alpha_{25}\), (cf. (2.43f)):

\[
\text{ch} \, \hat{L}_A = \sum_{s \in \hat{W}_{\beta_{34}}} (-1)^{\ell(s)} \hat{s} \cdot \text{ch} \, \hat{\nu}^A - \mathcal{R} = \quad (3.71a)
\]

\[
= \sum_{s \in \hat{W}_{\beta_{34}}} (-1)^{\ell(s)} \hat{s} \cdot (\text{ch} \, \hat{\nu}^A - \mathcal{R}_{\text{long}}), \quad N = 1 \text{ or } r_{N-1} > 0, \quad (3.71b)
\]

where \(\hat{W}_{\beta_{34}} \equiv \{1, \beta_{34}\}\).

For \(N = 1\) formula (3.71) is equivalent to (3.68) for \(i_0' = 0\) since due to (3.23a) the generator \(X_{15}^+\) could appear only together with \(X_{25}^+\) but the resulting state (3.70) is zero.
Demonstration: We show this by induction. Consider first the case all states as in (3.29) considered for \( \ell \)
where \( \hat{\Lambda} \) induction similarly to the conjugate case, cf. (3.30). Thus, the corresponding character formula is:
\[
N \quad \text{there are only}
\]
where \( \hat{\Lambda} \) belongs to type \( d \) below.

There are more eliminations for \( N > 1 \) when \( i'_0 > 0 \). For instance we can show that all states as in (3.29) considered for \( \ell = 1, \ldots, i'_0 \) are not allowed.

**Demonstration:** We show this by induction. Consider first the case \( \ell = 1 \):
\[
X^+_1 X^+_2 |\Lambda\rangle = (X^+_1 X^+_x - X^+_5 X^+_5) X^+_2 |\Lambda\rangle =
\]
where the second term is zero due to (3.70), and the first term is transformed by pulling \( X^+_5 \) to the right, where it annihilates the vacuum (due to (3.17c) with \( j = 1 \) for \( r_{N-1} = 0 \), and the resulting state is the forbidden \( \psi_{11} \). Thus, the above state is not allowed. Further, we proceed by induction similarly to the conjugate case, cf. (3.57).

From the above follows that when \( i'_0 > 0 \) the decomposition (3.72) does not hold.

We can be more explicit in the case when all \( r_i = 0 \). In that case all the generators \( X^+_1, X^+_2, \ldots, X^+_N \), are eliminated from all chiral states.

**Demonstration:** Take first the vector \( X^+_1, X^+_2, \ldots, X^+_N \). As we know when \( j_1 = r_i = 0, \forall i \), the only chiral state containing it in a long superfield is the state (3.27) and its descendants. However, here all these possible states are zero due to (3.70). Thus, there are no chiral states containing \( X^+_1, X^+_2, \ldots, X^+_N \), are eliminated as we showed above. Hence, the vector \( X^+_1, X^+_2, \ldots, X^+_N \) is eliminated from all chiral states.

Now fix \( \ell \) such that \( 1 < \ell \leq N - 1 \) and suppose that we have already shown elimination of \( X^+_1, X^+_2, \ldots, X^+_N \) for \( k = \ell + 1 \). We want to show elimination of \( X^+_1, X^+_2, \ldots, X^+_N \) for \( k = \ell + 1 \). As we know from the similar consideration of long superfields all chiral states including \( X^+_1, X^+_2, \ldots, X^+_N \) which are not yet excluded may be written as the state (3.29) and its descendants including only generators \( X^+_a, a = 1, 2, \ell' > \ell \). Then it is shown (analogously to (3.33)) that this state is also not allowed. Thus, all generators \( X^+_1, X^+_2, \ldots, X^+_N \) are eliminated from the chiral part of the basis.

The chiral part of the basis is further restricted. As we know, when \( j_1 = r_i = 0, \forall i \), there are only \( N \) chiral states that can be built from the generators \( X^+_1, X^+_2, \ldots, X^+_N \) alone, given in (3.30). Thus, the corresponding character formula is:
\[
ch \hat{\Lambda} = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \prod_{\alpha \in \Delta^+_+} (1 + e(\alpha)) - \mathcal{R},
\]
\[
d = d_{\max} = d_{NN}^{3} > d_{N1}^{1}, \quad j_1 = 0, r_i = 0, \forall i.
\]
\( d = d_{NN}^N = -z + 2m/N > d_{N1}^1, \ j_1 = 0. \)

In these short single-reducibility-condition cases holds the odd null condition (following from the singular vector \((2.32d)\) or \((2.33d)\)):

\[
X_{3\epsilon}^+ \langle \Lambda \rangle = X_{25}^+ \langle \Lambda \rangle = 0. \tag{3.75}
\]

Since \(j_1 = 0\) from \((2.26a)\) and \((3.75)\) follows the additional null condition:

\[
X_{15}^+ \langle \Lambda \rangle = [X_1^+, X_{3\epsilon}^+] \langle \Lambda \rangle = 0 \tag{3.76}
\]

For \(N > 1\) and \(r_{N-1} > 2\) each of these UIRs enters as the second term in decomposition \((3.72)\), when the first term is an UIR of type \(c\) maximal number of states in \(\hat{L}\). Further, for \(N > 1\) there are additional null conditions if \(r_{N-i} = 0, i \leq i'_0\). These are recursive null conditions:

\[
X_{1,5+i}^+ \langle \Lambda \rangle = [X_{1,4+i}, X_{4+i}] \langle \Lambda \rangle = 0, \quad r_{N-j} = 0, \quad 1 \leq j \leq i \leq i'_0, \tag{3.77a}
\]

\[
X_{2,5+i}^+ \langle \Lambda \rangle = [X_{2,4+i}, X_{4+i}] \langle \Lambda \rangle = 0, \quad r_{N-j} = 0, \quad 1 \leq j \leq i \leq i'_0 \tag{3.77b}
\]

Thus, \(2(1 + i'_0)\) generators \(X_{1,4+k}^+, X_{2,4+k}^+, k = 1, \ldots, 1 + i'_0\), are eliminated. The maximal number of states in \(\hat{L}_\Lambda\) is \(2^{4N-2-2i_0} \).

The corresponding character formula is:

\[
ch \ \hat{L}_\Lambda = \prod_{\alpha \in \Delta_1^+ \atop \alpha \neq \epsilon_{j,4+k} \atop j = 1, 2, k = 1, \ldots, 1 + i'_0} (1 + e(\alpha)) - \mathcal{R} = \tag{3.78a}
\]

\[
= \sum_{\hat{s} \in \hat{W}^{d}_{i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \ V^\Lambda - \mathcal{R}, \tag{3.78b}
\]

\[
\hat{W}^{d}_{i'_0} \equiv \hat{W}_{\alpha_{15}} \times \hat{W}_{\alpha_{16}} \times \cdots \times \hat{W}_{\alpha_{1,5+i_0}} \times \hat{W}_{\alpha_{25}} \times \hat{W}_{\alpha_{26}} \times \cdots \times \hat{W}_{\alpha_{2,5+i_0}}, \tag{3.78c}
\]

\[
d = d_{NN}^N > d_{N1}^1, \quad j_1 = 0, \quad r_{N-i} = 0, i \leq i'_0 \leq N - 1, \quad r_{N-1-i'_0} > 0,
\]

where \(\mathcal{R}\) designates the counter-terms due to our Criterion, in particular, due to \((3.23)\) taken with \(\epsilon_{j,4+k} = 0, \ j = 1, 2, \ k = 1, \ldots, 1 + i'_0\).

In the case of \(R\)-symmetry scalars we have:

\[
X_{1,4+k}^+ \langle \Lambda \rangle = 0, \quad X_{2,4+k}^+ \langle \Lambda \rangle = 0, \quad k = 1, \ldots, N, \quad r_i = 0, \forall i \tag{3.79}
\]

The character formula is \((3.78)\) taken with \(1 + i'_0 = N\). These are chiral UIRs conjugate to the anti-chiral ones in \((3.64)\).
3.4. Character formulae of DRC UIRs

Each of the DRC cases is the obvious combination of two SRC cases and some results follow from this. In fact, in the generic cases, we can give a general character formula which follows directly from embedding diagram (2.55).

So let first $N > 1$ and $r_1 r_{N-1} > 0$, (i.e., $i_0 = i_0' = 0$). Then holds the following character formula:

\[
\begin{align*}
ch \hat{L}_\Lambda &= \sum_{\hat{s} \in \hat{W}_{\beta,\beta'}} (-1)^{(\hat{s})} \hat{s} \cdot ch \hat{X}^\Lambda - R = \\
&= ch \hat{X} - \frac{1}{1 + e(\beta)} ch \hat{X}^{\Lambda + \beta} - \frac{1}{1 + e(\beta')} ch \hat{X}^{\Lambda + \beta'} + \\
&+ \frac{1}{(1 + e(\beta))(1 + e(\beta'))} ch \hat{X}^{\Lambda + \beta + \beta'} - R , \quad (3.80b)
\end{align*}
\]

\[
\hat{W}_{\beta,\beta'} \equiv \hat{W}_\beta \times \hat{W}_{\beta'}
\]

The above formula is proved similarly to what we had in the SRC cases. It reflects the contribution of the modules on embedding diagram (2.55). In fact, the two terms with minus sign on the first line of (3.80b) take into account the factorization of the oddly embedded submodules $I^\beta, I^{\beta'}$, cf. (2.50), coming from the modules $V_{10}, V_{01}$, resp. There can be no contribution of the modules along the same lines of embeddings $V_{k\ell}$, $k, \ell > 1$, due to the Grassmannian nature of the odd embeddings involved. Consequently, all modules $V_{k\ell}$ for $k, \ell > 1$ cannot contribute to the character formula of UIR in $V_{00}$. Only the module $V_{11}$ can contribute since it is also a non-zero submodule of $V_{00}$. However, since it is oddly embedded in $V_{00}$ via both submodules $V_{10}, V_{01}$, its contribution is taken out two times - once with $I^\beta$, and a second time with $I^{\beta'}$. Thus, we need the term with plus sign on the second line of (3.80b) to restore its contribution once.\(^{15}\) We can not apply the same kind of arguments for $N = 1$, nevertheless, formula (3.80) holds also then for the case (2.49a), cf. Appendix A.1.

In accord with (3.80) for $N > 1$ and $d = d^{ac}$ holds the following decomposition:

\[
\left( \hat{L}_{\text{long}} \right)_{d = d^{ac}} = \hat{L}_\Lambda + \hat{L}_{\Lambda + \beta} + \hat{L}_{\Lambda + \beta'} + \hat{L}_{\Lambda + \beta + \beta'} , \quad r_1 r_{N-1} > 0 , \quad (3.81)
\]

where $\hat{L}_{\text{long}}$ is a long superfield with the same values of $r_i$ as $\Lambda$, while the value of $j_1$, (resp. $j_2$), is zero when $j_1$, (resp. $j_2$), from $\Lambda$ is zero, otherwise it has to be the generic value $j_1 \geq N/2$, (resp. $j_2 \geq N/2$).

Next we consider the four DRC cases separately.

- \textbf{ac}$\quad d = d_{\text{max}} = d_{N1}^{1} = d_{NN}^{3} = d^{ac} \equiv 2+j_1+j_2+m_1 , \quad z = j_1-j_2+2m/N-m_1$

\(^{15}\) For more complicated application of similar arguments we refer to [127].
In these semi-short DRC cases hold the two null conditions (3.35) and (3.65). In addition, for \( N > 1 \) if \( r_i = 0, i = 1, \ldots, i_0 \), hold (3.50) and if \( r_{N-i} = 0, i = 1, \ldots, i'_0 \), hold (3.67).

There are two basic situations. The first is when \( i_0 + i'_0 \leq N - 2 \). (This situation is not applicable for \( N = 1 \).) This means that not all \( r_i \) are zero and all eliminations are as described separately for cases \( \bullet a \) and \( \bullet c \). These semi-short UIRs may be called Grassmann-analytic following [23], since odd generators from different chiralities are eliminated. The maximal number of states in \( \hat{L}_\Lambda \) is \( 2^{4N-2-i_0-i'_0} \).

The second is when \( i_0 + i'_0 \leq N - 2 \) does not hold which means that all \( r_i \) are zero, (\( R \)-symmetry scalars, \( m_1 = 0 = m \)), and in fact we have \( i_0 = i'_0 = N - 1 \) and all generators \( X^{+}_{1,4+k} \) and \( X^{+}_{3,4+k} \) are eliminated. The maximal number of states in \( \hat{L}_\Lambda \) is \( 2^{2N} \).

Note that below only one case is applicable for \( N = 1 \).

- For \( j_1j_2 > 0 \) the corresponding character formulae are combinations of (3.52) and (3.68):

\[
\begin{align*}
ch \hat{L}_\Lambda &= \prod_{\alpha \in \Delta^+_I^1, \alpha \neq \alpha_{1,4+k},\ k=1, \ldots, 1+i'_0, \ \alpha \neq \alpha_{3,4+N-j}, \ j=1, \ldots, 1+i_0} (1 + \epsilon(\alpha)) - \mathcal{R} = \\
&= \sum_{\hat{s} \in \hat{W}^{ac}_{i_0, i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{V}^\Lambda - \mathcal{R} , \quad \text{(3.82a)}
\end{align*}
\]

\[
\begin{align*}
&= \sum_{\hat{s} \in \hat{W}^{ac}_{i_0, i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \left( ch \hat{V}^\Lambda - \mathcal{R}_{\text{long}} \right) , \quad \text{(3.82b)}
\end{align*}
\]

\[
\hat{W}^{ac}_{i_0, i'_0} \equiv \hat{W}^a_{i_0} \times \hat{W}^c_{i'_0} , \quad \text{(3.82c)}
\]

\[
d = d_{\text{max}} = d_{N1}^1 = d_{NN}^3 = 2 + j_1 + j_2 + m_1 , \quad j_1j_2 > 0 ,
\]

either \( i_0 + i'_0 \leq N - 2 \),

\[
\begin{align*}
&= \hat{L}_\Lambda \oplus \hat{L}_\Lambda^{+15} \oplus \hat{L}_\Lambda^{+3,4+N} \oplus \hat{L}_\Lambda^{+15+3,4+N} , \quad r_1r_{N-1} > 0 , \quad \text{(3.83)}
\end{align*}
\]

The last subcase is of \( R \)-symmetry scalars. It is also the only formula in the case under consideration - \( ac \) - valid for \( N = 1 \) (where there are no counterterms since (3.23a,b) bring no restrictions, cf. also Appendix A.1).

For \( N > 1 \) and \( i_0 = i'_0 = 0 \) formula (3.82) is equivalent to (3.80) with \( \beta = \alpha_{15}, \beta' = \alpha_{3,4+N} \). Also (3.81) holds with these \( \beta, \beta' \):

\[
\left( \hat{L}_{\text{long}} \right)_{d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_\Lambda^{+15} \oplus \hat{L}_\Lambda^{+3,4+N} \oplus \hat{L}_\Lambda^{+15+3,4+N} ,
\]

39
and with $\hat{L}_{long}$ being a long superfield with the same values of $r_i$ as $\Lambda$ and with $j_1, j_2 \geq N/2$.

All formulae below to the end of case ac are for $N > 1$.

• For $j_1 > 0, j_2 = 0$ the corresponding character formulae are combinations of (3.54) and (3.68):

$$ch \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_{0i}^{ac}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{V}^\Lambda - \mathcal{R}$$

$$= \sum_{\hat{s} \in \hat{W}_{0i}^{ac}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \left( ch \hat{V}^\Lambda - \mathcal{R}_{\text{long}} \right) ,$$

$$\hat{W}_{0i}^{ac} \equiv \hat{W}_{\beta_{12}} \times \hat{W}_{0i}^{c}, \quad \beta_{12} = \alpha_{3,4+N} + \alpha_{4,4+N} ,$$

$$d = d_{\text{max}} = d_{N1}^4 = d_{NN}^3 = 2 + j_1 + m_1 ,$$

$$j_1 > 0, j_2 = 0 , \quad r_1 > 0 .$$

For $i_0 = i'_0 = 0$ holds decomposition (3.81):

$$\left( \hat{L}_{\text{long}} \right)_{d = d_{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{15}} \oplus \hat{L}_{\Lambda + \alpha_{12}} \oplus \hat{L}_{\Lambda + \alpha_{15} + \beta_{12}} , \quad r_1 r_{N-1} > 0 ,$$

where $\hat{L}_{\text{long}}$ is a long superfield with the same values of $j_2(= 0), r_i$ as $\Lambda$ and with $j_1 \geq N/2$. Note that the UIR $\hat{L}_{\Lambda + \alpha_{15}}$ is also of the type ac under consideration, while the last two UIRs are short from type bc considered below.

For $R$-symmetry scalars we combine (3.58) and (3.68a):

$$ch \hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\alpha \in \Delta^+_1}^{\alpha \neq \alpha_{1,4+k}, k=1,...,N} (1 + e(\alpha)) - \mathcal{R} ,$$

$$d = d_{\text{max}} = d_{N1}^4 = d_{NN}^3 = 2 + j_1 ,$$

$$j_1 > 0, j_2 = 0 , \quad r_i = 0 , \forall i .$$

• For $j_1 = 0, j_2 > 0$ the corresponding character formulae are combinations of (3.71) and (3.52):

$$ch \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_{0i}^{ac}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{V}^\Lambda - \mathcal{R}$$

40
\[
\hat{\chi} \mid_{d=d_a} = \hat{\chi} \mid_{d=d_a} \times \hat{\chi} \mid_{d=d_a}, \quad \beta_{34} = \alpha_{15} + \alpha_{25}, \quad d = d_{\max} = d_{N1}^1 = d_{NN}^3 = 2 + j_2 + m_1, \quad j_1 = 0, \quad j_2 > 0, \quad r_{N-1} > 0.
\]

For \( i_0 = i'_0 = 0 \) holds decomposition (3.81):
\[
(\hat{L}_{\text{long}})_{d=d_{a,c}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{3,4} + N} \oplus \hat{L}_{\Lambda + \beta_{34}} \oplus \hat{L}_{\Lambda + \alpha_{3,4,4} + \beta_{34}}, \quad r_1 r_{N-1} > 0, \quad (3.88)
\]
where \( \hat{L}_{\text{long}} \) is a long superfield with the same values of \( j_1(=0), r_i \) as \( \Lambda \) and with \( j_2 \geq N/2 \). Note that the UIR \( \hat{L}_{\Lambda + \alpha_{3,4} + N} \) is again of the type \( \text{ac} \) under consideration, while the last two UIRs are actually from type \( \text{ad} \) considered below.

For \( R \)-symmetry scalars we combine (3.52a) and (3.74):
\[
ch \hat{L}_\Lambda = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \prod_{\substack{\alpha \in \Delta_1^+ \\ \alpha \neq \alpha_{3,4+k} \in \Delta_1 \leq N \leq N}} (1 + e(\alpha)) - R,
\]
\[
d = d_{\max} = d_{N1}^1 = d_{NN}^3 = 2 + j_2,
\]
\[
j_1 = 0, \quad j_2 > 0, \quad r_i = 0, \forall i.
\]

- For \( j_1 = j_2 = 0 \) the corresponding character formulae are combinations of (3.54) and (3.71):
\[
ch \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_{\alpha' \epsilon' c'}} (-1)^{\ell(\hat{s})} \hat{s} \cdot ch \hat{\nu}^\Lambda - R = \quad (3.90a)
\]
\[
\sum_{\hat{s} \in \hat{W}_{\alpha' \epsilon' c'}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \left( ch \hat{\nu}^\Lambda - R_{\text{long}} \right), \quad (3.90b)
\]
\[
\hat{W}_{\alpha' \epsilon' c'} \equiv \hat{W}_{\beta_{12}} \times \hat{W}_{\beta_{34}}, \quad (3.90c)
\]
\[
d = d_{\max} = d_{N1}^1 = d_{NN}^3 = 2 + m_1, \quad j_1 = j_2 = 0, \quad r_1 r_{N-1} > 0.
\]

For \( i_0 = i'_0 = 0 \) holds decomposition (3.81):
\[
(\hat{L}_{\text{long}})_{d=d_{a,c}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \beta_{12}} \oplus \hat{L}_{\Lambda + \beta_{34}} \oplus \hat{L}_{\Lambda + \beta_{12} + \beta_{34}}, \quad r_1 r_{N-1} > 0, \quad (3.91)
\]

41
where $\hat{L}_{\text{long}}$ is a long superfield with the same values of $j_1(=0), j_2(=0), r_i$ as $\Lambda$. Note that the UIR $\hat{L}_{\Lambda+\beta_{12}}$ is of the type bc, $\hat{L}_{\Lambda+\beta_{34}}$ is of the type ad, $\hat{L}_{\Lambda+\beta_{12}+\beta_{34}}$ is of the type bd, these three being considered below.

For $R$-symmetry scalars we combine (3.58) and (3.74):

$$
\text{ch} \hat{L}_{\Lambda} = \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) + \prod_{\alpha \in \Delta^+ \atop \epsilon_{\alpha} > 0} (1 + e(\alpha)) - \mathcal{R},
$$

(3.92)

$\bullet$ ad \quad $d = d_{N1} = d_{N3} = 1 + j_2 + m_1$, $j_1 = 0$, $z = 2m/N - m_1 - 1 - j_2$.

In these short DRC cases hold the three null conditions (3.35), (3.75) and (3.76). In addition, for $N > 1$ if $r_i = 0$, $i = 1, \ldots, i_0$, hold (3.50) and if $r_{N-i} = 0$, $i = 1, \ldots, i'_0$, hold (3.77).

If $i_0 + i'_0 \leq N - 2$ all eliminations are as described separately for cases $\bullet$ a and $\bullet$ d. All these are Grassmann-analytic UIRs. The maximal number of states in $\hat{L}_{\Lambda}$ is $2^{4N-3-i_0-2i'_0}$. Interesting subcases are the so-called BPS states, cf., [16], [20], [23], [35], [41], [42], [44], [53]. They are characterized by the number $\kappa$ of odd generators which annihilate them - then the corresponding case is called $\frac{\kappa}{4N}$-BPS state. For example consider $N = 4$ and $\frac{1}{4}$-BPS cases with $z = 0 \Rightarrow d = 2m/N$. One such case is obtained for $i_0 = 1, i'_0 = 0, j_2 > 0$, then $d = \frac{1}{2}(2r_2 + 3r_3)$, $r_1 = 0$, $r_2 > 0$, $r_3 = 2(1 + j_2)$.

For $j_2m_1 > 0$ the corresponding character formula is a combination of (3.52) and (3.78):

$$
\text{ch} \hat{L}_{\Lambda} = \prod_{\alpha \in \Delta^+ \atop \alpha \neq \alpha_{3,5+N-k} \atop k=1, \ldots, 1+i_0 \atop \alpha \neq \alpha_{a,4+j} \atop a=1,2, \ j=1, \ldots, 1+i'_{0}} (1 + e(\alpha)) - \mathcal{R} =
$$

(3.93a)

$$
= \sum_{\hat{s} \in \hat{W}^{ad}_{i_0,i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \text{ch} \hat{V}_{\Lambda} - \mathcal{R},
$$

(3.93b)

$$
\hat{W}^{ad}_{i_0,i'_0} \equiv \hat{W}^a_{i_0} \times \hat{W}^d_{i'_0},
$$

(3.93c)
\[ d = d_{N1}^1 = d_{NN}^1 = 1 + j_2 + m_1, \quad j_1 = 0, j_2 > 0, \quad i_0 + i'_0 \leq N - 2, \]
\[ r_i = 0, \ i = 1, 2, \ldots, i_0, N - i'_0, N - i'_0 + 1, \ldots, N - 1, \]
\[ r_i > 0, \ i = i_0 + 1, N - i'_0 - 1. \]

For \( i_0 = i'_0 = 0 \) some of these UIRs appear (up to two times) in the decomposition (3.88). More precisely, those with \( r_i > 2\delta_{i,N-1}, \ i = 1, N - 1 \), appear as the term \( \hat{\Lambda}_{i+\beta_{34}} \), while those with \( r_i > \delta_{i1} + 2\delta_{i,N-1}, \ i = 1, N - 1 \), appear also as the term \( \hat{\Lambda}_{i+\alpha_{3,4}+N+\beta_{34}} \).

For \( j_2 = 0, m_1 > 0 \) the corresponding character formula is a combination of (3.54) and (3.78b):

\[ \chi \hat{\Lambda} = \sum_{s \in \hat{\Lambda}_d} (-1)^{\ell(s)} \hat{s} \cdot \chi \hat{\Lambda} = \mathcal{R}, \quad (3.94a) \]
\[ \hat{\theta} \cdot \hat{\theta} = \hat{\theta}_0 \cdot \hat{\theta} = (3.94b) \]

where \( \beta_{12} = \alpha_{3,4} + \alpha_{4,4} \). For \( i_0 = i'_0 = 0 \) some of these UIRs appear in the decomposition (3.91) or (3.88). More precisely, those with \( r_i > 2\delta_{i,N-1}, \ i = 1, N - 1 \), appear as the term \( \hat{\Lambda}_{i+\beta_{34}} \) of (3.91), while those with \( r_i > \delta_{i1} + 2\delta_{i,N-1}, \ i = 1, N - 1 \), appear as the term \( \hat{\Lambda}_{i+\alpha_{3,4}+N+\beta_{34}} \) of (3.88) but only when \( j_2 = \frac{1}{2} \) in \( \Lambda \) there.

In the case of \( R \)-symmetry scalars we have \( i_0 = i'_0 = N - 1, \ \kappa = 3N \) and all generators \( X_{1,4+k}^+, X_{2,4+k}^+, X_{3,4+k}^+ \) are eliminated. Here holds \( d = -z = 1 + j_2 \). These anti-chiral irreps form one of the three series of \textbf{massless} UIRs; they are denoted \( \chi_s^+ \), \( s = j_2 = 0, \frac{1}{2}, 1, \ldots \), in Section 3 of [102]. Besides the vacuum they contain only \( N \) states in \( \hat{\Lambda} \) given by (3.34) for \( k = 1, \ldots, N \). These should be called ultrashort UIRs. The character formula can be written in the most explicit way:

\[ \chi \hat{\Lambda} = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{4,5+N-i}) \quad (3.95) \]
\[ d = d_{N1}^1 = d_{NN}^1 = 1 + j_2 = -z, \]
\[ j_1 = 0, \ r_i = 0, \ \forall i, \]

and it is valid for any \( j_2 \). In the case under consideration - \textbf{ad} - only the last character formula is valid for \( N = 1 \) (cf. Appendix A.1).

The next case is conjugate to the previous one.

\[ \textbf{bc} \quad d = d_{N1}^2 = d_{NN}^2 = 1 + j_1 + m_1, \quad j_2 = 0, \quad z = 2m/N - m_1 + 1 + j_1. \]

In these short DRC cases hold the three null conditions (3.59), (3.60) and (3.65). In addition, for \( N > 1 \) if \( r_i = 0, \ i = 1, \ldots, i_0 \), hold (3.62) and if \( r_{N-i} = 0, \ i = 1, \ldots, i'_0 \), hold (3.67).
If \( i_0 + i'_0 \leq N - 2 \) all eliminations are as described separately for cases \( \bullet \) and \( \bullet c \). These are also Grassmann-analytic UIRs. The maximal number of states in \( \hat{L}_\Lambda \) is \( 24 N - 3 - 2 i_0 - i'_0 \). Here for \( N = 4 \) one \( \frac{1}{2} \) BPS case is obtained for \( i_0 = 0, i'_0 = 1, j_1 > 0 \), then \( d = \frac{1}{2} (2 r_2 + 3 r_1), r_1 = 2 (1 + j_1), r_2 > 0, r_3 = 0 \).

For \( j_1 m_1 > 0 \) the corresponding character formula is a combination of (3.63) and (3.68):

\[
\text{ch} \hat{L}_\Lambda = \prod_{\alpha \in \Delta^+_t, \alpha \neq \alpha_{1,4+k}, k = 1, \ldots, 1+i_0} (1 + e(\alpha)) - \mathcal{R} = \sum_{\hat{s} \in \hat{W}_{i_0,i'_0}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \text{ch} \hat{V}_\Lambda - \mathcal{R}, \tag{3.96a}
\]

\[
\hat{W}_{i_0,i'_0}^{bc} \equiv \hat{W}_i^b \times \hat{W}_{i'_0}^c \tag{3.96b}
\]

\[
d = d^2_{N1} = d^3_{NN} = 1 + j_1 + m_1, \quad j_1 > 0, j_2 = 0, \quad i_0 + i'_0 \leq N - 2, \quad r_i = 0, \quad i = 1, 2, \ldots, i_0, N - i'_0, N - i'_0 + 1, \ldots, N - 1, \quad r_i > 0, \quad i = i_0 + 1, N - i'_0 - 1. \tag{3.96c}
\]

For \( i_0 = i'_0 = 0 \) some of these UIRs appear in the decomposition (3.85). More precisely, those with \( r_i > 2 \delta_{i1}, i = 1, N - 1, \) appear as the term \( \hat{L}_{\Lambda+\beta_{12}} \), while those with \( r_i > 2 \delta_{i1} + \delta_{i,N-1}, i = 1, N - 1, \) appear as the term \( \hat{L}_{\Lambda+\alpha_{15}+\beta_{12}} \).

For \( j_1 = 0, m_1 > 0 \) the corresponding character formula is a combination of (3.71) and (3.63b):

\[
\text{ch} \hat{L}_\Lambda = \sum_{\hat{s} \in \hat{W}_{i_0}^{bc'}} (-1)^{\ell(\hat{s})} \hat{s} \cdot \text{ch} \hat{V}_\Lambda - \mathcal{R}, \tag{3.97a}
\]

\[
\hat{W}_{i_0}^{bc'} \equiv \hat{W}_{\beta_{34}}^{bc} \times \hat{W}_{i_0}^{b}, \tag{3.97b}
\]

where \( \beta_{34} = \alpha_{15} + \alpha_{25} \). For \( i_0 = i'_0 = 0 \) some of these UIRs appear in the decomposition (3.91) or (3.85). More precisely, those with \( r_i > 2 \delta_{i1}, i = 1, N - 1, \) appear as the term \( \hat{L}_{\Lambda+\beta_{12}} \) of (3.91), while those with \( r_i > 2 \delta_{i1} + \delta_{i,N-1}, i = 1, N - 1, \) appear as the term \( \hat{L}_{\Lambda+\alpha_{15}+\beta_{12}} \) of (3.85) but only when \( j_1 = \frac{1}{2} \) in \( \Lambda \) there.

In the case of \( R \)-symmetry scalars we have \( i_0 = i'_0 = N - 1, \kappa = 3N \) and all generators \( X^+_{1,4+k}, X^+_{3,4+k}, X^+_{4,4+k} \), are eliminated. These chiral irreps form another series of massless UIRs, conjugate to the first above; they are denoted \( \chi_s, s = j_1 = 0, \frac{1}{2}, 1, \ldots, \) in Section 3 of [102]. Besides the vacuum they contain only \( N \) states in \( \hat{L}_\Lambda \) given by (3.30) for \( k = 1, \ldots, N \). These should also be called ultrashort UIRs. The character formula is:

\[
\text{ch} \hat{L}_\Lambda = 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} e(\alpha_{2,4+i}) \tag{3.98b}
\]
\[ d = d_{N_1}^2 = d_{N_N}^3 = 1 + j_1 = z, \]
\[ j_2 = 0, \quad r_i = 0, \forall i, \]

and it is valid for any \( j_1 \). In the case under consideration - bc - only the last character formula is valid for \( N = 1 \) (cf. Appendix A.1).

\begin{itemize}
  \item \( \bullet \text{bd} \quad d = d_{N_1}^2 = d_{N_N}^3 = m_1, \quad j_1 = j_2 = 0, \quad z = 2m/N - m_1. \)

In these short DRC cases hold the four null conditions (3.59), (3.60), (3.75) and (3.76).

For \( N = 1 \) this is the trivial irrep with \( d = z = 0 \). This follows from the fact that since \( d = j_1 = j_2 = 0 \) also holds the even reducibility condition (2.17b) (and consequently (2.17d, e, f)). Thus, we have the null conditions: \( X_k^+ |\Lambda\rangle = 0 \) for all simple root generators (and consequently for all generators) and the irrep consists only of the vacuum \(|\Lambda\rangle\).

For \( N > 1 \) the situation is non-trivial. In addition to the mentioned conditions, and if \( r_i = 0, i = 1, \ldots, i_0, \) hold (3.62) and if \( r_{N-i} = 0, i = 1, \ldots, i'_0, \) hold (3.77).

If \( i_0 + i'_0 \leq N - 2 \) all eliminations are as described separately for cases \( \bullet \text{b} \) and \( \bullet \text{d} \). These are also Grassmann-analytic UIRs. The maximal number of states in \( \hat{L}_\Lambda \) is \( 2^{N-4-2i_0-2i'_0} \). For \( N = 4 \) for the BPS cases we take \( z = (r_3 - r_1) = 0 \Rightarrow d = 2r_1 + r_2 \).

In the \( \frac{1}{2} \)-BPS case we have \( i_0 = i'_0 = 0, r_1 = r_3 > 0 \).

For \( i_0 = i'_0 = 0 \) some of these UIRs appear in the decomposition (3.91). More precisely, those with \( r_i > 2\delta_{i_1} + 2\delta_{i,N-1}, i = 1, N - 1, \) appear as the term \( \hat{L}_\Lambda+\beta_{12}+\beta_{34} \).

Most interesting is the case \( i_0 + i'_0 = N - 2 \), then there is only one non-zero \( r_i \), namely, \( r_{1+i_0} = r_{N-1-i'_0} > 0 \), while the rest \( r_i \) are zero. Thus, the Young tableau parameters are: \( m_1 = r_{1+i_0}, \quad m = (1+i_0)r_{1+i_0}. \)

An important subcase is when \( d = m_1 = 1 \), then \( m = i_0 + 1 = N - 1 - i'_0, \quad r_i = \delta_{mi}, \) and these irreps form the third series of massless UIRs. In Section 3 of [102] they are parametrized by \( n \in \mathbb{N}, \frac{1}{2}N \leq n < N \), and denoted by \( \chi_n, n = m, (z = 2n/N - 1), \chi_n^+ \), \( n = N - m, (z = 1 - 2n/N) \). Note that for even \( N \) there is the coincidence: \( \chi'_n = \chi_n^+ \), where \( n = m = N - m = N/2 \). Here we shall parametrize these UIRs by the parameter \( i_0 = 0,1, \ldots, N - 2 \).

Another interesting subcase here is for even \( N \) with \( z = 0 \Rightarrow d = m_1 = 2m/N \Rightarrow i_0 = i'_0 = N/2 - 1 \Rightarrow m_1 = r_{N/2}, \quad m = \frac{N}{2} r_{N/2}. \) These are \( \frac{1}{2} \)-BPS states for \( m_1 > 1 \) and \( \frac{3}{4} \)-BPS for \( m_1 = 1 \). The latter are the massless self-conjugate case: \( \chi'_n, n = N/2 \) mentioned above. For \( N = 4 \) we have: \( i_0 = i'_0 = 1, r_1 = r_3 = 0, r_2 > 0 \), which is massless for \( r_2 = 1 \).

Finally, in the case of \( R \)-symmetry scalars we have \( i_0 = i'_0 = N - 1 \) and all \( 4N \) odd generators \( X_{1,4+k}^+, X_{2,4+k}^+, X_{3,4+k}^+, X_{4,4+k}^+ \), are eliminated. More than this, all quantum numbers are zero, (cf. (2.46d, d')) , and this is the trivial irrep. The latter follows exactly as explained above for the case \( N = 1 \).
For \( m_1 > 0 \) the corresponding character formula is a combination of (3.63) and (3.78):

\[
ch \, \hat{L}_\Lambda = \prod_{\alpha \in \Delta^+_{i \neq a, j, 5+N-k}} (1 + e(\alpha)) - \mathcal{R} = \tag{3.99a}
\]

\[
\mathcal{R} = \sum_{\bar{s} \in \hat{W}_{i_0,i_0'}^{bd}} (-1)^{\ell(\bar{s})} \bar{s} \cdot ch \, \hat{V}^\Lambda - \mathcal{R}, \tag{3.99b}
\]

\[
\hat{W}_{i_0,i_0'}^{bd} \equiv \hat{W}_{i_0}^b \times \hat{W}_{i_0'}^d, \tag{3.99c}
\]

where \( \mathcal{R} \) designates the counter-terms due to our Criterion, in particular, due to (3.23) taken with \( \bar{\varepsilon}_{j,5+N-k} = 0, \ j = 3, 4, \ k = 1, \ldots, 1+i_0, \bar{\varepsilon}_{j',4+k'} = 0, \ j' = 1, 2, k' = 1, \ldots, 1+i_0'. \)

Also for the third series of massless UIRs we can give a much more explicit character formula without counter-terms. Fix the parameter \( i_0 = 0,1,\ldots,N-2. \) Then there are only the following states in \( \hat{L}_\Lambda \):

\[
X_{2, N+4-j}^+ \cdots X_{2, N+4-i_0}^+ |\Lambda\rangle, \ j = 0,1,\ldots,i_0, \tag{3.100a}
\]
\[
X_{4,4+k}^+ \cdots X_{4, N+3-i_0}^+ |\Lambda\rangle, \ k = 1,\ldots,N-1-i_0, \tag{3.100b}
\]

altogether \( N \) states besides the vacuum.

**Demonstration:** Indeed, besides (3.100) no other states involving generators \( X_{a,4+k}^+ \) for \( a = 2, 4 \) are possible due to the restrictions (3.23). Note that the generators of the latter kind which do not appear in (3.100) are eliminated due to (3.59), (3.62b) and (3.75), (3.77b). We have to discuss the generators \( X_{a,4+k}^+ \) for \( a = 3, 5 \). Part of them are eliminated due to (3.62a) and (3.77a). The rest are: \( X_{1, N+4-j}^+, \ j = 0,1,\ldots,i_0 \) and \( X_{4,4+k}^+, \ k = 1,\ldots,N-1-i_0. \) They can not act on the vacuum, so they can only act on some of the states in (3.100a) or (3.100b), resp. For two of these: \( X_{1, N+4-i_0}^+ \) and \( X_{5, N+3-i_0}^+ \) it is easy to see that they can not act on any state. For the rest: \( X_{1, N+4-j}^+, \ j = 0,1,\ldots,i_0-1 \) and \( X_{4,4+k}^+, \ k = 1,\ldots,N-2-i_0, \) the only possibility for action which can not be excluded in an obvious way, is:

\[
X_{1, N+4-j}^+ X_{2, N+3-j}^+ \cdots X_{2, N+4-i_0}^+ |\Lambda\rangle, \ j = 0,1,\ldots,i_0-1, \tag{3.101a}
\]
\[
X_{3,4+k}^+ X_{4,5+k}^+ \cdots X_{4, N+3-i_0}^+ |\Lambda\rangle, \ k = 1,\ldots,N-2-i_0. \tag{3.101b}
\]

However, all these states are not allowed. This is shown as for the states (3.33) and (3.29). Thus, besides the vacuum, \( \hat{L}_\Lambda \) contains only the \( N \) states given in (3.100). \( \blacksquare \)
The corresponding character formula for the massless UIRs of this series is therefore:

\[
\text{ch } \hat{L}_\Lambda = 1 + \sum_{j=0}^{i_0} \prod_{i=j}^{i_0} e(\alpha_{2,N+4-i}) + \sum_{k=1}^{N-1-i_0} \prod_{i=k}^{N-1-i_0} e(\alpha_{4,4+i}),
\]

\[
d = d_{N1}^2 = d_{NN}^4 = m_1 = 1, \quad i_0 = 0, \ldots, N - 2,
\]

\[
z = 2(i_0 + 1)/N - 1, \quad j_1 = j_2 = 0, \quad r_i = \delta_{i,i_0+1}. \tag{3.102}
\]

Remark: In this paper we use the Verma (factor-)module realization of the UIRs. We give here a short remark on what happens with the ER realization of the UIRs. As we know, cf. [101], the ERs are superfields depending on Minkowski space-time and on \(4N\) Grassmann coordinates \(\theta^i_a, \bar{\theta}^k_b, a, b = 1, 2, i, k = 1, \ldots, N\). 16 There is 1-to-1 correspondence in these dependencies and the odd null conditions. Namely, if the condition \(X^+_{a,4+k} |\Lambda\rangle = 0, a = 1, 2,\) holds, then the superfields of the corresponding ER do not depend on the variable \(\theta^i_a\), while if the condition \(X^+_{a,4+k} |\Lambda\rangle = 0, a = 3, 4,\) holds, then the superfields of the corresponding ER do not depend on the variable \(\bar{\theta}^k_{a-2}\). These statements were used in the proof of unitarity for the ERs picture, cf. [103], but were not explicated. They were analyzed in detail in the papers [16],[17],[20],[23], using the notions of 'harmonic superspace analyticity' and Grassmann analyticity. 

16 A mathematically precise formulation is given in [101], while for the even case we refer to [113],[114].
4. Discussion and Outlook

First we summarize the results on decompositions of long irreps as they descend to the unitarity threshold.

In the SRC cases we have embedding formula (2.40) and UIRs are given by formula (2.42). Starting from this in subsection (3.3) we have established that for \( d = d_{\text{max}} \) there hold the following decompositions:

\[
\left( \hat{L}_{\text{long}} \right)_{|d = d_{\text{max}}} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda + \beta} ,
\]

where there are two possibilities for \( \Lambda \) and four possibilities for \( \beta \) as given in (2.41a, c, e, f), however, for \( N > 1 \) there are additional conditions on \( r_i \). In more detail, \( \Lambda \) and \( \beta \) are specified as follows:

\[
d = d_{\text{max}} = d^a = d_{N1}^1 > d_{NN}^3 , \quad r_1 > 0 ,
\]

\[
\beta = \alpha_{3,4+N} , \quad j_2 > 0 ,
\]

\[
\beta = \alpha_{3,4+N} + \alpha_{4,4+N} , \quad j_2 = 0 ,
\]

\[
d = d_{\text{max}} = d^c = d_{N1}^3 > d_{N1}^1 , \quad r_{N-1} > 0 ,
\]

\[
\beta = \alpha_{15} , \quad j_1 > 0 ,
\]

\[
\beta = \alpha_{15} + \alpha_{25} , \quad j_1 = 0 .
\]

The corresponding four decompositions are given in formulae (3.45), (3.55), (3.69), (3.72), resp., and in each case it is explained how \( \hat{L}_{\text{long}} \) is specified. It is also noted that in cases (4.2a′′, c′′) the UIRs \( \hat{L}_{\Lambda + \beta} \) are short from types given in (2.39b, d), resp., and with \( r_1 > 2, r_{N-1} > 2 \), resp.

In the DRC cases we have embedding formulae (2.48), (2.54), (2.55), and UIRs are given by formula (2.50). Starting from this in subsection (3.4) we have established that for \( N > 1 \) and \( d = d_{\text{max}} = d^{ac} \) hold the following decompositions:

\[
\left( \hat{L}_{\text{long}} \right)_{|d = d^{ac}} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda + \beta} \oplus \hat{L}_{\Lambda + \beta'} \oplus \hat{L}_{\Lambda + \beta + \beta'} , \quad r_1 r_{N-1} > 0 ,
\]

where \( \Lambda \) is the semi-short DRC designated as type \( ac \) and there are four possibilities for \( \beta, \beta' \) as given in (2.49a, b, c, d). The corresponding four decompositions are given in formulae (3.83), (3.85), (3.88), (3.91), resp., and in each case it is explained how \( \hat{L}_{\text{long}} \) is specified. Note that in (3.83) all UIRs are semi-short. In (3.85) the first two UIRs are semi-short, the last two UIRs are short of type \( bc \). From the latter two the first is with \( r_1 > 2, r_{N-1} > 0, (r_1 > 2 \text{ if } N = 2), \) the second is with \( r_1 > 2, r_{N-1} > 1, (r_1 > 3 \text{ if } N = 2) \). In (3.88) the first two UIRs are semi-short, the last two UIRs are short of type \( ad \). From the latter two the first is with \( r_1 > 0, r_{N-1} > 2, (r_1 > 2 \text{ if } N = 2), \) the second is with \( r_1 > 1, r_{N-1} > 2, (r_1 > 3 \text{ if } N = 2) \). In (3.91) the first UIR is the semi-short, the other three UIRs are short of types \( bc, ad, bd \), resp. From the latter three the first is with \( r_1 > 2, r_{N-1} > 0, \) the second is with \( r_1 > 0, r_{N-1} > 2, \) the third is with \( r_1, r_{N-1} > 2, (r_1 > 4 \text{ if } N = 2) \).
Summarizing the above, we note first that for $N = 1$ all SRC cases enter some decomposition (4.1), while no DRC cases enter any decomposition (4.3). For $N > 1$ the situation is more diverse and so we give the list of UIRs that do not enter decompositions (4.1) and (4.3):

**SRC cases:**

- **a** $d = d_{\text{max}} = d^a = d^b_{N1} = 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN}$, $j_1, j_2$ arbitrary, $r_1 = 0$.
- **b** $d = d^2_{N1} = z + 2m_1 - 2m/N > d^3_{NN}$, $j_2 = 0$.
- **c** $d = d_{\text{max}} = d^c = d^b_{NN} = 2 + 2j_1 - z + 2m/N > d^1_{N1}$, $j_1, j_2$ arbitrary, $r_{N-1} = 0$.
- **d** $d = d^1_{NN} = -z + 2m/N > d^1_{N1}$, $j_1 = 0$.

**DRC cases:**

all non-trivial cases for $N = 1$, while for $N > 1$ the list is:

- **ac** $d = d_{\text{max}} = d^{ac} = d^1_{N1} = d^3_{NN} = 2 + j_1 + j_2 + m_1$, $z = j_1 - j_2 + 2m/N - m_1$, $j_1, j_2$ arbitrary, $r_1 r_{N-1} = 0$.
- **ad** $d = d^3_{N1} = d^1_{NN} = 1 + j_2 + m_1$, $j_1 = 0$, $z = -1 - j_2 + 2m/N - m_1$, $j_2$ arbitrary, $r_{N-1} = 2$, $r_1 = 0$ for $N > 2$.
- **bc** $d = d^3_{N1} = d^3_{NN} = 1 + j_1 + m_1$, $j_2 = 0$, $z = 1 + j_1 + 2m/N - m_1$, $j_1$ arbitrary, $r_1 \leq 2$, $r_{N-1} = 0$ for $N > 2$.
- **bd** $d = d^2_{N1} = d^1_{NN} = m_1$, $j_1 = j_2 = 0$, $z = 2m/N - m_1$, $r_1, r_{N-1} \leq 2$ for $N > 2$, $r_1 \leq 4$ for $N = 2$.

We would like to point out possible application of our results to current developments in conformal field theory. Recently, there is interest in superfields with conformal dimensions which are protected from renormalisation in the sense that they cannot develop anomalous dimensions [23],[31],[32],[33],[34],[51]. Initially, the idea was that this happens because the representations under which they transform determine these dimensions uniquely. Later, it was argued that one can tell which operators will be protected in the quantum theory simply by looking at the representations they transform under and whether they can be written in terms of single trace 1/2 BPS operators (chiral primaries or CPOs) on analytic superspace [34]. In [51] it was shown how, at the unitarity threshold, a long multiplet can be decomposed into four semi-short multiplets, and decompositions similar to (3.81), i.e., involving the modules in (2.55) (as given in [100]), were considered for $N = 2, 4$. However, the decompositions of [51] are justified on the dimensions of the finite-dimensional
irreps of the Lorentz and $su(N)$ subalgebras involved in the superfields involved in the decompositions, and in particular, the latter hold also when $r_1 r_{N-1} = 0$.

Independently of the above, we would like to make a mathematical remark. As a by-product of our analysis we have obtained character formulae for the complex Lie super-algebras $sl(4/N)$. The point is that our character formulae have as starting point character formulae of Verma modules and factor-modules over $sl(4/N)$. Thus, almost all character formulae in Section 3, more precisely, formulae (3.26), (3.36), (3.39), (3.41), (3.52), (3.54), (3.63), (3.68), (3.71), (3.78), (3.80), (3.82), (3.84), (3.87), (3.90), (3.93), (3.94), (3.95a), (3.96), (3.97), (3.98a), (3.99), become character formulae for $sl(4/N)$ for the same values of the representation parameters by just discarding the counter-terms $\mathcal{R}$, $\mathcal{R}_{\text{long}}$, resp.

Finally, let us mention that we explicate our results for $N = 1, 2$ in Appendix A. There we display explicitly all decompositions (4.1), (4.3), and when these do not hold, all quasi-decompositions (like (3.51)) that replace them. We leave similar detailed discussion for $N = 4$ for the follow-up paper.

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Appendix A. Explicit character formulae for \( N=1,2 \)

A.1. \( N=1 \)

For \( N = 1 \) the displayed results are almost explicit, so we can allow telegram style.

- **Long superfields**

If \( j_1 j_2 > 0 \) then \( \hat{L}_\Lambda \) has the maximum possible number of states: 16. The character formula is (3.19).

If \( j_1 = 0, j_2 > 0 \) then the generator \( X_{1,15}^+ \) can appear only together with the generator \( X_{2,25}^+ \), and \( \hat{L}_\Lambda \) has 12 states = 3(chiral)\( \times 4(\text{anti-chiral}) \) states.\(^{17}\) The character formula is (3.26) with:

\[
\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{35}))(1 + e(\alpha_{45})) \quad (A.1)
\]

The next case is conjugate. If \( j_1 > 0, j_2 = 0 \) then the generator \( X_{3,35}^+ \) can appear only together with the generator \( X_{4,45}^+ \), and \( \hat{L}_\Lambda \) has 12 states. The character formula is (3.26) with:

\[
\mathcal{R} = e(\alpha_{35})(1 + e(\alpha_{15}))(1 + e(\alpha_{25})) \quad (A.2)
\]

The next case combines the previous two. If \( j_1 = j_2 = 0 \) then the generator \( X_{1,15}^+ \) can appear only together with the generator \( X_{2,25}^+ \), the generator \( X_{3,35}^+ \) can appear only together with the generator \( X_{4,45}^+ \), and \( \hat{L}_\Lambda \) has 9 states = 3(chiral)\( \times 3(\text{anti-chiral}) \) states. The character formula is (3.26) with:

\[
\mathcal{R} = e(\alpha_{15})(1+e(\alpha_{35}))(1+e(\alpha_{45})) + e(\alpha_{35})(1+e(\alpha_{15}))(1+e(\alpha_{25})) - e(\alpha_{15})e(\alpha_{35}) , \quad (A.3)
\]
i.e., we combine the counter-terms of the previous two cases, but need to subtract a counter-term that is counted twice.

- **SRC cases**

\( a \quad d = d_{\text{max}} = d_{11}^1 = 2 + 2j_2 + z > d_{11}^3 \).

\( j_1 > 0 \). The generator \( X_{3,35}^+ \) is eliminated (though for different reasons for \( j_2 > 0 \) and \( j_2 = 0 \), cf. (3.35), resp. (3.53)) and there are only 8 states.\(^{18}\) Then the character formula is (3.36) (or equivalently (3.39)) without counter-terms:

\[
\text{ch } \hat{L}_\Lambda = \prod_{\alpha \in \Delta_1^+ \atop \alpha \neq \alpha_{35}} (1 + e(\alpha)) , \quad (A.4)
\]

\[
d = d_{\text{max}} = d_{11}^1 > d_{11}^3 , \quad j_1 > 0 .
\]

\(^{17}\) In statements like this each sector includes the vacuum.

\(^{18}\) For brevity, here and often below we shall say ”there are M states” meaning ”there are M states in \( \hat{L}_\Lambda \)”.

51
For \( j_2 > 0 \) the decomposition (3.45) is fulfilled with \( \hat{L}_{\text{long}} \) having 16 states as the maximal long superfield with \( j_1 j_2 > 0 \), while \( \hat{L}_{\Lambda + \alpha_{35}} \) has 8 states (being the same type as \( \hat{L} \)).

For \( j_2 = 0 \) the decomposition (3.55) is fulfilled with \( \hat{L}_{\text{long}} \) having 12 states as the long superfield with \( j_1 > 0, j_2 = 0, \beta = \alpha_{35} + \alpha_{45} \) and \( \hat{L}_{\Lambda + \alpha_{35} + \alpha_{45}} \) having 4 states - it actually belongs to case b below (for \( j_1 > 0, j_2 = 0 \)).

- \( j_1 = 0 \). The generator \( X_{35}^+ \) is eliminated, the generator \( X_{15}^+ \) can appear only together with the generator \( X_{25}^+ \) and there are only 6 states. Then the character formula is (3.36) (equivalently (3.39)):

\[
\text{ch} \hat{L}_{\Lambda} = \prod_{\substack{\alpha \in \Delta^+_i \\ \alpha \neq \alpha_{35}}} \left( 1 + e(\alpha) \right) - \mathcal{R} ,
\]

\[
\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{45}))
\]

\[
d = d_{\text{max}} = d_{11}^1 > d_{11}^3 , \quad j_1 = 0 .
\]

This formula is equivalent also to (3.41), noting:

\[
\mathcal{R} = e(\alpha_{15})(1 + e(\alpha_{45})) = (1 - \hat{s}_{\alpha_{35}}) \cdot \mathcal{R}_{\text{long}} ,
\]

taking \( \mathcal{R}_{\text{long}} \) from (A.1).

For \( j_2 > 0 \) the decomposition (3.45) is fulfilled with \( \hat{L}_{\text{long}} \) having 12 states as the long superfield with \( j_1 = 0, j_2 > 0 \), while \( \hat{L}_{\Lambda + \alpha_{35}} \) has 6 states (being the same type as \( \hat{L} \)).

For \( j_2 = 0 \) the decomposition (3.55) is fulfilled with \( \hat{L}_{\text{long}} \) having 9 states as the long superfield with \( j_1 = j_2 = 0 \), while \( \hat{L}_{\Lambda + \alpha_{35} + \alpha_{45}} \) has 3 states - it actually belongs to the next case b, cf. below (for \( j_1 = j_2 = 0 \)).

- b \( \quad d = d_{11}^2 = z > d_{11}^1 , \quad j_2 = 0 .\)

- \( j_1 > 0 \). The generators \( X_{35}^+ \) and \( X_{45}^+ \) are eliminated and there are only 4 states. Then the character formula is (3.63) (for \( i_0 = 0 \)) without counter-terms:

\[
\text{ch} \hat{L}_{\Lambda} = (1 + e(\alpha_{15}))(1 + e(\alpha_{25})) ,
\]

\[
d = d_{11}^2 > d_{11}^3 , \quad j_1 > 0, j_2 = 0 .
\]

These UIRs and the next subcase enter formula (3.45) together with UIRs of case a as we have shown above.

- \( j_1 = 0 \). The generators \( X_{35}^+ \) and \( X_{45}^+ \) are eliminated, the generator \( X_{15}^+ \) can appear only together with the generator \( X_{25}^+ \), and there are only 3 states. Then the character formula is (3.63) (for \( i_0 = 0 \)) with counter-term \( \mathcal{R} = e(\alpha_{15}) \):

\[
\text{ch} \hat{L}_{\Lambda} = 1 + e(\alpha_{25}) + e(\alpha_{15})e(\alpha_{25}) ,
\]

\[
d = d_{11}^2 > d_{11}^3 , \quad j_1 = j_2 = 0 .
\]

52
Here holds also an analog of (3.41) with \( \hat{W}_\beta \) replaced by \( \hat{W}_0^b \) and \( R_{\text{long}} \) from (A.1).

- **c**  \( d = d_{\text{max}} = d_{11}^3 = 2 + 2j_1 - z > d_{11}^1 \).
- \( j_2 > 0 \). The generator \( X_{15}^+ \) is eliminated (though for different reasons for \( j_1 > 0 \) and \( j_1 = 0 \), cf. (3.65), resp. (3.70)) and there are only 8 states. Then the character formula is (3.68) without counter-terms:

\[
ch \hat{L}_\Lambda = \prod_{\alpha \in \Delta^+_1 \atop \alpha \neq \alpha_{15}} (1 + e(\alpha)) ,
\]

\[
d = d_{\text{max}} = d_{11}^3 > d_{11}^1 , \quad j_2 > 0 .
\]

For \( j_1 > 0 \) the decomposition (3.69) is fulfilled with \( \hat{L}_{\text{long}} \) having 16 states as the maximal long superfield with \( j_1j_2 > 0 \). For \( j_1 = 0 \) the decomposition (3.72) is fulfilled with \( \hat{L}_{\text{long}} \) having 12 states as the long superfield with \( j_1 = 0, j_2 > 0 \), and \( L_{\Lambda + \alpha_{15} + \alpha_{25}} \) having 4 states - it actually belongs to the next case d, cf. below (for \( j_1 = 0, j_2 > 0 \)).

- \( j_2 = 0 \). The generator \( X_{15}^+ \) is eliminated, the generator \( X_{35}^+ \) can appear only together with the generator \( X_{45}^+ \) and there are only 6 states. Then the character formula is (3.68):

\[
ch \hat{L}_\Lambda = \prod_{\alpha \in \Delta^+_1 \atop \alpha \neq \alpha_{15}} (1 + e(\alpha)) - R ,
\]

\[
R = e(\alpha_{35})(1 + e(\alpha_{25})),
\]

\[
d = d_{\text{max}} = d_{11}^3 > d_{11}^1 , \quad j_2 = 0 .
\]

This formula is equivalent also to (3.41) with \( R_{\text{long}} \) from (A.2).

For \( j_1 > 0 \) the decomposition (3.69) is fulfilled with \( \hat{L}_{\text{long}} \) having 12 states as the long superfield with \( j_1 > 0, j_2 = 0 \). For \( j_1 = 0 \) the decomposition (3.72) is fulfilled with \( \hat{L}_{\text{long}} \) having 9 states as the long superfield with \( j_1 = j_2 = 0 \), and \( L_{\Lambda + \alpha_{15} + \alpha_{25}} \) having 3 states - it actually belongs to the next case d, cf. below (for \( j_1 = j_2 = 0 \)).

- **d**  \( d = d_{11}^1 = -z > d_{11}^1 , \quad j_1 = 0 .
\)
- \( j_2 > 0 \). The generators \( X_{15}^+ \) and \( X_{25}^+ \) are eliminated and there are only 4 states. Then the character formula is (3.78) (for \( i_0' = 0 \)) without counter-terms:

\[
ch \hat{L}_\Lambda = (1 + e(\alpha_{35}))(1 + e(\alpha_{45})) ,
\]

\[
d = d_{11}^1 > d_{11}^1 , \quad j_1 = 0, j_2 > 0 .
\]

These UIRs and the next subcase enter formula (3.69) together with UIRs of case c as we have shown above.
• $j_2 = 0$. The generators $X^+_{15}$ and $X^+_{25}$ are eliminated, the generator $X^+_{35}$ can appear only together with the generator $X^+_{45}$, and there are only 3 states. Then the character formula is (3.78) (for $i'_0 = 0$) with counter-term $\mathcal{R} = e(\alpha_{35})$:

\[
\begin{align*}
\text{ch} \hat{L}_\Lambda &= 1 + e(\alpha_{45}) + e(\alpha_{35})e(\alpha_{45}) , \\
d &= d^4_{11} > d^3_{11}, \quad j_1 = j_2 = 0.
\end{align*}
\] (A.12)

Here holds also an analog of (3.41) with $\hat{W}_\beta$ replaced by $\hat{W}_0^d$ and $\mathcal{R}_{\text{long}}$ from (A.2).

• **DRC cases**

• **ac** $d = d_{\text{max}} = d^3_{11} = d^{ac} = 2 + j_1 + j_2$, $z = j_1 - j_2$.

The generators $X^+_{15}$ and $X^+_{35}$ are eliminated (though for different reasons for $j_1 > 0$ and $j_1 = 0$, resp., for $j_2 > 0$ and $j_2 = 0$). There are only 4 states and the character formula is (3.82) (for $i_0 = i'_0 = 0$) without counterterms:

\[
\begin{align*}
\text{ch} \hat{L}_\Lambda &= (1 + e(\alpha_{25})) (1 + e(\alpha_{45})) = \\
&= \text{ch} \hat{V}_\Lambda - \frac{1}{1 + e(\alpha_{15})} \text{ch} \hat{V}_{\Lambda + \alpha_{15}} - \frac{1}{1 + e(\alpha_{35})} \text{ch} \hat{V}_{\Lambda + \alpha_{35}} + \\
&\quad + \frac{1}{(1 + e(\alpha_{15}))(1 + e(\alpha_{35}))} \text{ch} \hat{V}_{\Lambda + \alpha_{15} + \alpha_{35}} ,
\end{align*}
\] (A.13)

where the terms with minus may be interpreted as taking out states, while the last term indicates adding back what was taken two times. This may written also in the form of the following pseudo-decomposition:

\[
\begin{align*}
\left( \hat{L}_{\text{long}} \right)_{d = d^{ac}} &= \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{15}} \oplus \hat{L}_{\Lambda + \alpha_{35}} \oplus \hat{L}_{\Lambda + \alpha_{15} + \alpha_{35}}
\end{align*}
\] (A.14)

where $\hat{L}_{\Lambda + \alpha_{15}}$, $\hat{L}_{\Lambda + \alpha_{35}}$, are SRC UIRs with 8 states each described above in cases c, a, resp. They are embedded in $\hat{V}_\Lambda$ via the generators $X^+_{15}$, $X^+_{35}$, resp. Together with $\hat{L}_\Lambda$ this brings in terms which has to be taken out with the last term in which the representation denoted $\hat{L}_{\Lambda + \alpha_{15} + \alpha_{35}}$ is supposed to have the same 4 states as $\hat{L}_\Lambda$ and these excessive states are de-embedded via the composition of the other two maps, i.e., via the product of generators $X^+_{15} X^+_{35}$.

• **ad** $d = d^4_{11} = d^3_{11} = 1 + j_2 = -z$, $j_1 = 0$.

The generators $X^+_{15}$, $X^+_{25}$ and $X^+_{35}$ are eliminated (for the latter for different reasons for $j_2 > 0$ and $j_2 = 0$). These are the first series of massless UIRs, and everything is already explicit in the general formulae. There are only 2 states and the character formula is (3.95) for $N = 1$:

\[
\begin{align*}
\text{ch} \hat{L}_\Lambda &= 1 + e(\alpha_{45}) .
\end{align*}
\] (A.15)

• **bc** $d = d^3_{11} = d^2_{11} = 1 + j_1 = z$, $j_2 = 0$.  

54
The generators \( X_{15}^+ \), \( X_{35}^+ \) and \( X_{45}^+ \) are eliminated (for the first for different reasons for \( j_1 > 0 \) and \( j_1 = 0 \)). These are the second series of massless UIRs. There are only 2 states and the character formula is (3.98) for \( N = 1 \):
\[
ch \hat{L}_\Lambda = 1 + \epsilon(\alpha_{25}) .
\] (A.16)

- **bd** \( d = d_{11}^2 = d_{11}^4 = j_1 = j_2 = z = 0 \)

As we explained in the detail this is the trivial 1-dimensional irrep consisting of the vacuum.

### A.2. \( N=2 \)

- **Long superfields**

We first write down conditions (3.23) explicitly for \( N = 2 \):
\[
\begin{align*}
\epsilon_{15} + \epsilon_{16} &\leq \epsilon_{25} + \epsilon_{26} + 2j_1 \quad & (A.17a) \\
\epsilon_{35} + \epsilon_{36} &\leq \epsilon_{45} + \epsilon_{46} + 2j_2 \quad & (A.17b) \\
\epsilon_{16} + \epsilon_{26} + \epsilon_{35} + \epsilon_{45} &\leq \epsilon_{15} + \epsilon_{25} + \epsilon_{36} + \epsilon_{46} + r_1 \quad & (A.17c)
\end{align*}
\]

To simplify the exposition we classify the generators by their contribution to (A.17). Namely, the chiral, anti-chiral, operators:
\[
\begin{align*}
\Phi^c &= (X_{16}^+)^{\epsilon_{16}} (X_{15}^+)^{\epsilon_{15}} (X_{25}^+)^{\epsilon_{26}} (X_{26}^+)^{\epsilon_{25}} , \\
\Phi^a &= (X_{36}^+)^{\epsilon_{35}} (X_{35}^+)^{\epsilon_{36}} (X_{45}^+)^{\epsilon_{45}} (X_{46}^+)^{\epsilon_{46}} ,
\end{align*}
\] (A.18)

will be distinguished by the values (cf. also (3.22)):
\[
\begin{align*}
\epsilon_j^c &= \epsilon_{25} + \epsilon_{26} - \epsilon_{15} - \epsilon_{16} , \\
\epsilon_j^a &= \epsilon_{45} + \epsilon_{46} - \epsilon_{35} - \epsilon_{36} , \\
\epsilon_r^c &\equiv \epsilon_{15} + \epsilon_{25} - \epsilon_{16} - \epsilon_{26} , \\
\epsilon_r^a &\equiv \epsilon_{36} + \epsilon_{46} - \epsilon_{35} - \epsilon_{45} , \\
\epsilon_r &\equiv \epsilon_r^c + \epsilon_r^a .
\end{align*}
\] (A.19)

Explicitly, the chiral operators are arranged as follows:
\[
\begin{align*}
X_{15}^+ X_{25}^+ , &\quad \epsilon_r^c = 2 , \quad \epsilon_j^c = 0 , \\
X_{25}^+ , &\quad X_{26}^+ X_{15}^+ X_{25}^+ , \quad \epsilon_r^c = 1 , \quad \epsilon_j^c = 1 , \\
X_{15}^+ , &\quad X_{16}^+ X_{15}^+ X_{25}^+ , \quad \epsilon_r^c = 1 , \quad \epsilon_j^c = -1 , \\
X_{26}^+ X_{25}^+ , &\quad \epsilon_r^c = 0 , \quad \epsilon_j^c = 2 , \\
1 , &\quad X_{16}^+ X_{25}^+ , \quad X_{26}^+ X_{15}^+ , \quad X_{16}^+ X_{26}^+ X_{15}^+ X_{25}^+ , \quad \epsilon_r^c = 0 , \quad \epsilon_j^c = 0 , \\
X_{16}^+ X_{15}^+ , &\quad \epsilon_r^c = 0 , \quad \epsilon_j^c = -2 , \\
X_{26}^+ , &\quad X_{26}^+ X_{16}^+ X_{25}^+ , \quad \epsilon_r^c = -1 , \quad \epsilon_j^c = 1 , \\
X_{16}^+ , &\quad X_{16}^+ X_{15}^+ X_{26}^+ , \quad \epsilon_r^c = -1 , \quad \epsilon_j^c = -1 , \\
X_{16}^+ X_{26}^+ , &\quad \epsilon_r^c = -2 , \quad \epsilon_j^c = 0 , \quad (A.20)
\end{align*}
\]
while the anti-chiral operators are arranged as follows:

\[
X_{36}^+ X_{46}^+ , \quad \varepsilon_r^a = 2 , \varepsilon_j^a = 0 , \\
X_{46}^+ , \quad X_{45}^+ X_{36}^+ X_{46}^+ , \quad \varepsilon_r^a = 1 , \varepsilon_j^a = 1 , \\
X_{36}^+ , \quad X_{35}^+ X_{36}^+ X_{46}^+ , \quad \varepsilon_r^a = 1 , \varepsilon_j^a = -1 , \\
X_{45}^+ X_{46}^+ , \quad \varepsilon_r^a = 0 , \varepsilon_j^a = 2 , \\
1 , \quad X_{35}^+ X_{46}^+ , \quad X_{45}^+ X_{36}^+ X_{46}^+ , \quad \varepsilon_r^a = 0 , \varepsilon_j^a = 0 , \\
X_{35}^+ X_{36}^+ , \quad \varepsilon_r^a = 0 , \varepsilon_j^a = -2 , \\
X_{45}^+ , \quad X_{45}^+ X_{35}^+ X_{46}^+ , \quad \varepsilon_r^a = -1 , \varepsilon_j^a = 1 , \\
X_{35}^+ , \quad X_{35}^+ X_{36}^+ X_{45}^+ , \quad \varepsilon_r^a = -1 , \varepsilon_j^a = -1 , \\
X_{35}^+ X_{45}^+ , \quad \varepsilon_r^a = -2 , \varepsilon_j^a = 0 . \quad (A.21)
\]

The same arrangement applies to the states obtained by applying the operators on the vacuum (for which all these indices naturally have zero value). We have added also the identity operator 1 in order to be able to take into account the vacuum automatically.

The allowed states satisfy: \( \varepsilon_j^a + 2j_1 \geq 0 , \varepsilon_j^a + 2j_2 \geq 0 \), \( \varepsilon_r + r_1 \geq 0 \), cf. (3.23). Now we are ready to classify the allowed states depending on the values of \( j_1, j_2, r_1 \). Actually what we do below amounts to giving explicitly formula (3.26).

- First we give the possible states when \( j_1, j_2 \geq 1 \):

  \[
  \Phi^c \Phi^a |\Lambda\rangle , \quad j_1, j_2 \geq 1 , \quad r_1 \geq 4 , \quad 256 \text{ states} ; \quad (A.22a)
  \]

  \[
  \Phi^c \Phi^a |\Lambda\rangle , \quad j_1, j_2 \geq 1 , \quad r_1 = 3 , \quad 255 \text{ states} , \quad \text{excluding the state} \quad X_{16}^+ X_{26}^+ X_{35}^+ X_{45}^+ , \quad \text{with} \quad \varepsilon_r = -4 ; \quad (A.22b)
  \]

  \[
  \Phi^c \Phi^a |\Lambda\rangle , \quad j_1, j_2 \geq 1 , \quad r_1 = 2 , \quad 247 \text{ states} , \quad \text{excluding the 9 states with} \quad \varepsilon_r \leq -3 ; \quad (A.22c)
  \]

  \[
  \Phi^c \Phi^a |\Lambda\rangle , \quad j_1, j_2 \geq 1 , \quad r_1 = 1 , \quad 219 \text{ states} , \quad \text{excluding the 37 states with} \quad \varepsilon_r \leq -2 ; \quad (A.22d)
  \]

  \[
  \Phi^c \Phi^a |\Lambda\rangle , \quad j_1, j_2 \geq 1 , \quad r_1 = 0 , \quad 163 \text{ states} , \quad \text{excluding the 93 states with} \quad \varepsilon_r \leq -1 . \quad (A.22e)
  \]

Further we classify the states when \( j_1, j_2 \geq 1 \) is not fulfilled using the five cases in (A.22) as reference point.

- \( j_1 \geq 1 , \quad j_2 = \frac{1}{2} \).

With respect to (A.22) we exclude 16 states with \( \varepsilon_j^a = -2 \), (so \( \varepsilon_j^a + 2j_2 = -1 \)):

\[
X_{35}^+ X_{36}^+ \Phi^c |\Lambda\rangle . \quad (A.23)
\]

However, for (A.22d, e) the case when \( \Phi^c = X_{16}^+ X_{26}^+ \) (with \( \varepsilon_r^c = -2 \)) is already taken out, and for (A.22e) the four cases of \( \Phi^c \) with \( \varepsilon_r^c = -1 \) are already taken out. Thus, altogether,
in the five cases corresponding to (A.22a, b, c, d, e) we take out 16, 16, 16, 15, 11 states and so there remain now 240, 239, 231, 204, 152 states.

- $j_1 = \frac{1}{2}, j_2 \geq 1$.
  This is the case conjugate to the previous. With respect to (A.22) we exclude 16 states with $\varepsilon_j = -2$, (so $\varepsilon_j + 2j_1 = -1$):

$$X_{16}^+ X_{15}^+ \Phi^a |\Lambda\rangle.$$  \hspace{1cm} (A.24)

Noting the double-counting for the five cases $\Phi^a$ with $\varepsilon_r = -2, -1$, in the cases corresponding to (A.22a, b, c, d, e) now have 240, 239, 231, 204, 152 states.

- $j_1 = j_2 = \frac{1}{2}$.
  This is a combination of the previous two cases. With respect to (A.22) we exclude the states we excluded in both, which would double the numbers (to 32, 32, 32, 30, 22), however, we have to take into account that the state $X_{16}^+ X_{15}^+ X_{35}^+ X_{36}^+ |\Lambda\rangle$ is counted two times. Thus, altogether, in the five cases corresponding to (A.22a, b, c, d, e) we take out 31, 31, 31, 29, 21 states and so there remain now 225, 224, 216, 190, 142 states.

- $j_1 \geq 1, j_2 = 0$.
  In addition to the states excluded in the case $j_1 \geq 1, j_2 = \frac{1}{2}$, we exclude 64 states with $\varepsilon_j = -1$, (so $\varepsilon_j + 2j_2 = -1$):

$$X_{36}^+ \Phi^c |\Lambda\rangle, \quad X_{35}^+ X_{36}^+ X_{46}^+ \Phi^c |\Lambda\rangle,$$

$$X_{35}^+ \Phi^c |\Lambda\rangle, \quad X_{35}^+ X_{36}^+ X_{45}^+ \Phi^c |\Lambda\rangle.$$ \hspace{1cm} (A.25a)

We have to take into account that certain states were already taken out, namely, the following:

- for (A.22c, d, e) the two cases (A.25b) with $\Phi^c = X_{16}^+ X_{26}^+$, (so that $\varepsilon_r = -3$);
- for (A.22d, e) the eight cases obtained by combining (A.25b) with $\Phi^c$ with $\varepsilon_r = -1$ (so that $\varepsilon_r = -2$);
- for (A.22e) the 12 cases obtained by combining (A.25b) with $\Phi^c$ with $\varepsilon_r = 0$ (so that $\varepsilon_r = -1$);
- for (A.22e) the two cases (A.25a) with $\Phi^c = X_{16}^+ X_{26}^+$, (so that $\varepsilon_r = -1$).

Altogether, for (A.22c, d, e) the overcounting is by 2, 10, 24 states. Thus, the states we actually take out w.r.t. the case $j_1 \geq 1, j_2 = \frac{1}{2}$ are 64, 64, 62, 54, 40. Finally, for (A.22e) we have to take out the impossible state $X_{36}^+ X_{45}^+ |\Lambda\rangle$, cf. (3.25.) Altogether the states remaining in the cases corresponding to (A.22a, b, c, d, e) are 176, 175, 169, 150, 111, resp.

- $j_1 = 0, j_2 \geq 1$.
  This is the case conjugate to the previous. In addition to the states excluded in the case $j_1 = \frac{1}{2}, j_2 \geq 1$, we exclude 64 states with $\varepsilon_j = -1$, (so $\varepsilon_j + 2j_1 = -1$):

$$X_{15}^+ \Phi^a |\Lambda\rangle, \quad X_{16}^+ X_{15}^+ X_{25}^+ \Phi^a |\Lambda\rangle,$$

$$X_{16}^+ \Phi^a |\Lambda\rangle, \quad X_{16}^+ X_{15}^+ X_{26}^+ \Phi^a |\Lambda\rangle.$$ \hspace{1cm} (A.26a)

We have to take into account that certain states were already taken out, namely, the following:
- for (A.22c, d, e) the two cases (A.26b) with \( \Phi^a = X^+_{35} X^+_{45} \), (so that \( \varepsilon_r = -3 \));
- for (A.22d, e) the eight cases obtained by combining (A.26b) with \( \Phi^a \) with \( \varepsilon_r = -1 \) (so that \( \varepsilon_r = -2 \));
- for (A.22e) the 12 cases obtained by combining (A.26b) with \( \Phi^a \) with \( \varepsilon_r = 0 \) (so that \( \varepsilon_r = -1 \));
- for (A.22e) the two cases (A.26a) with \( \Phi^a = X^+_{35} X^+_{45} \), (so that \( \varepsilon_r = -1 \)).

Altogether, excluding also the impossible state \( X^+_{15} X^+_{26} | \Lambda \) (when \( r_1 = 0 \), cf. (3.24)), in the five cases corresponding to (A.22a, b, c, d, e) we now have 176, 175, 169, 150, 111 states.

- \( j_1 = \frac{1}{2} \), \( j_2 = 0 \).

This is a combination of previous cases so w.r.t. (A.22) we exclude the states in (A.23), (A.24), (A.25). Due to overlaps there are five states which are counted twice - those in (A.23), (A.25) when \( \Phi^c = X^+_{15} X^+_{15} \). Thus, w.r.t. the case \( j_1 \geq 1 \), \( j_2 = \frac{1}{2} \) we would take out 11 states. However, from those the state (A.24) with \( \Phi^a = X^+_{35} X^+_{45} \) was taken out in (A.22d, e) and the states (A.24) with \( \Phi^a = X^+_{35} \), \( \Phi^a = X^+_{45} X^+_{35} X^+_{45} \) were taken out in (A.22e). Thus, w.r.t. the case \( j_1 \geq 1 \), \( j_2 = \frac{1}{2} \) we take out 11, 11, 11, 10, 8 states. Thus, in the cases corresponding to (A.22a, b, c, d, e) there are 165, 164, 158, 140, 103 states.

- \( j_1 = 0 \), \( j_2 = \frac{1}{2} \).

This case is conjugate to the previous one and so w.r.t. (A.22) we exclude the states in (A.23), (A.24), (A.26). W.r.t. the case \( j_1 = \frac{1}{2} \), \( j_2 \geq 1 \) we take out 11, 11, 11, 10, 8 states. Thus, in the cases corresponding to (A.22a, b, c, d, e) there are 165, 164, 158, 140, 103.

- \( j_1 = j_2 = 0 \).

This is a combination of previous cases so we exclude the states in (A.23), (A.24), (A.25), (A.26). Due to overlaps of (A.26) with (A.23) and (A.24) w.r.t. to the case \( j_1 = \frac{1}{2} \), \( j_2 = 0 \) we would take out 44 states (instead of 64). However, from those the two states (A.26b) with \( \Phi^a = X^+_{35} X^+_{45} \) were taken out in (A.22c, d, e), the four states obtained from (A.26b) with \( \Phi^a = X^+_{45} \), \( \Phi^a = X^+_{35} X^+_{46} \) were taken out in (A.22d, e), the two states (A.26b) with \( \Phi^a = X^+_{45} X^+_{46} \) were taken out in (A.22e), the eight states obtained from (A.26b) with \( \Phi^a = 1 \), \( \Phi^a = X^+_{35} X^+_{46} \), \( \Phi^a = X^+_{45} X^+_{36} \), \( \Phi^a = X^+_{35} X^+_{45} X^+_{36} X^+_{46} \) were taken out in (A.22e), the two states (A.26a) with \( \Phi^a = X^+_{35} X^+_{45} \) were taken out in (A.22e). Thus, the states we actually take out w.r.t. the case \( j_1 = \frac{1}{2} \), \( j_2 = 0 \) are 44, 44, 42, 38, 26. For (A.22e) we have also to take out two impossible states: (3.24) and its combination with (3.25):

\[
X^+_{15} X^+_{26} X^+_{36} X^+_{45} | \Lambda \rangle. \tag{A.27}
\]

Altogether the states remaining in the cases corresponding to (A.22a, b, c, d, e) are 121, 120, 116, 102, 75 states.

Thus, the smallest \( N = 2 \) long superfield has 75 states in \( \hat{L}_A \). Since above the states we described by exclusion we would like to list these 75 states. First there are 6 chiral states:

\[
X^+_{25} | \Lambda \rangle, \quad X^+_{15} X^+_{25} | \Lambda \rangle, \quad X^+_{26} X^+_{15} X^+_{25} | \Lambda \rangle, \tag{A.28a}
X^+_{26} X^+_{25} | \Lambda \rangle, \quad X^+_{16} X^+_{25} | \Lambda \rangle, \quad X^+_{16} X^+_{26} X^+_{15} X^+_{25} | \Lambda \rangle. \tag{A.28b}
\]

and 6 anti-chiral states:

\[
X^+_{46} | \Lambda \rangle, \quad X^+_{36} X^+_{46} | \Lambda \rangle, \quad X^+_{45} X^+_{36} X^+_{46} | \Lambda \rangle. \tag{A.29a}
\]
Now let $\Phi_c \Lambda \rangle$, $\Phi_a \Lambda \rangle$, denote any of the six states in (A.28), (A.29), resp., $\Phi'_c \Lambda \rangle$, $\Phi'_a \Lambda \rangle$, denote any of the three states in (A.28a), (A.29a), resp. Then, there are the following states:

$$\begin{align*}
|\Lambda \rangle, & \Phi_c \Phi_a |\Lambda \rangle, \\
X_{15}^+ X_{26}^+ \Phi_a |\Lambda \rangle, & \quad (A.30b) \\
X_{36}^+ X_{45}^+ \Phi_c |\Lambda \rangle, & \quad (A.30c) \\
X_{26}^+ \Phi'_a |\Lambda \rangle, & \quad (A.30d) \\
X_{16}^+ X_{26}^+ X_{25}^+ \Phi'_a |\Lambda \rangle, & \\
X_{45}^+ \Phi'_c |\Lambda \rangle, & \quad (A.30e) \\
X_{35}^+ X_{45}^+ X_{15}^+ X_{25}^+ |\Lambda \rangle, & \quad (A.30f)
\end{align*}$$

Obviously, there are 63 states in (A.30) (37 + 6 + 6 + 6 + 6 + 2), and altogether 75 states in (A.29), (A.28) and (A.30).

**SRC cases**

Here we consider the SRC cases similarly to the long superfields taking again the five cases in (A.22) as reference point.

- **a** 
  $d = d_{\text{max}} = d_{21}^1 = 2 + 2j_2 + z + r_1 > d_{22}^3$.
  The maximal number of states is $128 = 16(\text{chiral}) \times 8(\text{anti-chiral})$, achieved for $j_1 \geq 1, r_1 \geq 4$.

- **b** 
  $j_2 > 0$.
  Here hold character formulae (3.36), or equivalently (3.39) or (3.41) when $r_1 > 0$, while for $r_1 = 0$ the character formula is (3.52) (for $i_0 = 1$). We give more detailed description.

  The generator $X_{36}^+$ is eliminated. The eight states in the anti-chiral sector are obtained by applying to the vacuum the following operators:

$$\begin{align*}
X_{46}^+, & \quad \epsilon_r^a = 1, \quad \epsilon_j^a = 1, \\
X_{45}^+ X_{46}^+, & \quad \epsilon_r^a = 0, \quad \epsilon_j^a = 2, \\
1, & \quad X_{35}^+ X_{46}^+, \quad \epsilon_r^a = 0, \quad \epsilon_j^a = 0, \\
X_{45}^+, & \quad X_{45}^+ X_{35}^+ X_{46}^+, \quad \epsilon_r^a = -1, \quad \epsilon_j^a = 1, \\
X_{35}^+, & \quad \epsilon_r^a = -1, \quad \epsilon_j^a = -1, \\
X_{35}^+ X_{45}^+, & \quad \epsilon_r^a = -2, \quad \epsilon_j^a = 0.
\end{align*}$$

The above is equivalent to the anti-chiral part of character formula (3.36):

$$(1 + e(\alpha_{46})) \quad (1 + e(\alpha_{35})) \quad (1 + e(\alpha_{45}))$$

(A.31b)

however, the more detailed description in (A.31a) is necessary to obtain the results on the counter-terms. In particular, for $r_1 = 1$ the last operator does not contribute to
the anti-chiral sector, while for \( r_1 = 0 \) only the first three operators contribute to the anti-chiral sector, and the generator \( X^+_35 \) is also eliminated from the whole basis.

In summary, the results are. When \( j_1 \geq 1 \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 128, 127, 120, 99, 42 states. When \( j_1 = \frac{1}{2} \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 120, 119, 112, 92, 39 states. When \( j_1 = 0 \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 88, 87, 82, 68, 28 states.

When \( r_1 > 0 \) holds formula (3.45) with \( \beta = \alpha_{36} \), where \( \hat{L}_{\text{long}} \) is a long superfield with the same values of \( j_1 \) and \( r_i \) as \( \Lambda \), and with \( j_2 \geq 1 \). Note that when the weight \( \Lambda \) corresponds to cases (A.22a, b, c, d, e) then the weight \( \Lambda + \alpha_{36} \) corresponds to cases (A.22a, a, b, c, d) (since the value of \( r_1 \) is increased by 1). Thus, when \( j_1 \geq 1 \) the UIR \( \hat{L}_{\Lambda + \alpha_{36}} \) has 128, 128, 127, 120, 99 states, when \( j_1 = \frac{1}{2} \) it has 120, 120, 119, 112, 92 states, when \( j_1 = 0 \) it has 88, 88, 87, 82, 68 states. Summed together with the numbers for the UIR \( \hat{L}_{\Lambda} \) from above we obtain the following contributions to \( \hat{L}_{\text{long}} : \) when \( j_1 \geq 1 \) there are 256, 255, 247, 219, 141 states, when \( j_1 = \frac{1}{2} \) there are 240, 239, 231, 204, 131 states, when \( j_1 = 0 \) there are 176, 175, 169, 150, 96 states. Except the last cases (in which \( r_1 = 0 \)) these cases match exactly (not only by numbers) the cases of long superfields for the corresponding values of \( j_1 = 1, \frac{1}{2}, 0 \) and \( j_2 \geq 1 \).

When \( r_1 = 0 \) the long superfields have 163, 152, 111 states i.e., a mis-match of 22, 21, 15 states. All these extra states contain the generator \( X^+_35 \) and do not contain the generator \( X^+_36 \). Explicitly, when \( j_1 \geq 1 \) the 22 states are:

\[
\begin{align*}
X^+_35 \Phi^c_1 |\Lambda\rangle \\
X^+_35 X^+_45 X^+_46 \Phi^c_1 |\Lambda\rangle \\
X^+_35 X^+_46 \Phi^c_2 |\Lambda\rangle \\
X^+_35 X^+_45 X^+_15 X^+_25 |\Lambda\rangle,
\end{align*}
\]

(A.32)

where \( \Phi^c_1 \) denotes the 5 chiral operators of the first three rows of (A.20), \( \Phi^c_2 \) denotes the 11 chiral operators of the first six rows of (A.20). When \( j_1 = \frac{1}{2} \) the 21 states are as in (A.32) except the state \( X^+_35 X^+_46 X^+_16 X^+_15 |\Lambda\rangle \) which is not in the long superfield (since \( \varepsilon^c_j + 2j_1 = -1 \)). When \( j_1 = 0 \) the 16 states are as in (A.32) except the state excluded for \( j_1 = \frac{1}{2} \) and six states which are obtained for \( \Phi^c_1 = \Phi^c_2 = X^+_16, X^+_16 X^+_15 X^+_25 \) (i.e., excluding the third row of (A.20), since for them \( \varepsilon^c_j + 2j_1 = \varepsilon^c_j = -1 \)). Altogether, instead of the decomposition (3.45) we have the quasi-decomposition (3.51):

\[
\left( \hat{L}_{\text{long}} \right)_{|d=d^a} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda + \alpha_{36}} \oplus \hat{L}'_{\Lambda + \alpha_{35}} , \quad r_1 = 0 .
\]

(A.33)

The 28 states of the minimal case are given as follows. There are two anti-chiral states:

\[
X^+_46 |\Lambda\rangle , \quad X^+_45 X^+_46 |\Lambda\rangle ,
\]

(A.34)

and six chiral states (just as in (A.28)):

\[
\begin{align*}
X^+_25 |\Lambda\rangle , \quad X^+_15 X^+_25 |\Lambda\rangle & , \quad X^+_26 X^+_15 X^+_25 |\Lambda\rangle , \\
X^+_26 X^+_25 |\Lambda\rangle , \quad X^+_16 X^+_25 |\Lambda\rangle , \quad X^+_16 X^+_26 X^+_15 X^+_25 |\Lambda\rangle .
\end{align*}
\]

(A.35a)

(A.35b)
Combining the chiral and chiral states would give further 12 states. The rest of the states are obtained by combining these states with impossible states from the opposite chirality, yet obtaining allowed states. Explicitly, the list looks like this. Let \( \Phi_a |\Lambda\rangle \), \( \Phi_c |\Lambda\rangle \), denote any of the states in (A.34), (A.35), resp., \( \Phi'_c |\Lambda\rangle \), denote any of the three states in (A.35a), resp. Thus, there are the following states:

\[
\begin{align*}
|\Lambda\rangle, \quad \Phi_c \Phi_a |\Lambda\rangle, \\
X_{15}^+ X_{26}^+ \Phi_a |\Lambda\rangle, \\
X_{26}^+ X_{46}^+ |\Lambda\rangle, X_{26}^+ X_{16}^+ X_{25}^+ X_{46}^+ |\Lambda\rangle, \\
X_{45}^+ \Phi'_c |\Lambda\rangle.
\end{align*}
\]

(A.36a) (A.36b) (A.36c) (A.36d)

Obviously, there are 20 states in (A.36) \((13 + 2 + 2 + 3)\). Altogether, there are 28 states in (A.34), (A.35) and (A.36). This list amounts to giving explicitly character formula (3.52) (for \( N = 2 \), \( i_0 = N - 1 = 1 \)) without counter-terms. This superfield and its conjugate (considered below) are the shortest semi-short SRC \( N = 2 \) superfields.

* \( j_2 = 0 \)

Here holds character formula (3.54) and for \( r_1 = 0 \) holds also character formula (3.58). A more detailed description follows.

The state \( X_{36}^+ X_{46}^+ |\Lambda\rangle \) and its descendants are eliminated (due to (3.53)). This elimination is described by the second term in character formula (3.54a). The eight states in the anti-chiral sector here come from:

\[
\begin{align*}
X_{46}^+, \quad \varepsilon^a_r = 1, \quad \varepsilon^a_j = 1, \\
X_{45}^+ X_{46}^+, \quad \varepsilon^a_r = 0, \quad \varepsilon^a_j = 2, \\
1, \quad X_{35}^+ X_{46}^+, X_{45}^+ X_{36}^+, \quad \varepsilon^a_r = 0, \quad \varepsilon^a_j = 0, \\
X_{45}^+, \quad X_{45}^+ X_{35}^+ X_{46}^+, \quad \varepsilon^a_r = -1, \quad \varepsilon^a_j = 1, \\
X_{35}^+ X_{45}^+, \quad \varepsilon^a_r = -2, \quad \varepsilon^a_j = 0.
\end{align*}
\]

(A.37)

The above eight differ from (A.31) by one operator: \( X_{35}^+ \) is replaced here by \( X_{45}^+ X_{36}^+ \). For \( r_1 = 1 \) the last operator does not contribute to the anti-chiral sector. Whenever \( r_1 = 0 \) the generators \( X_{35}^+ \) and \( X_{36}^+ \) are eliminated from the anti-chiral part of the basis, which is further restricted due to (3.23c) and there are only two anti-chiral states as given in (A.34).

In summary, when \( j_1 \geq 1 \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 128, 127, 121, 103, 68 states. When \( j_1 = \frac{1}{2} \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 120, 119, 113, 96, 63 states. When \( j_1 = 0 \) correspondingly to the cases in (A.22a, b, c, d, e) we have now 88, 87, 83, 70, 45 states.

We know that when \( r_1 > 0 \) holds formula (3.55) for \( \hat{L}_{\text{long}} \) with the same values of \( j_1, j_2 (= 0), r_1 \) as \( \Lambda \) and with \( \beta = \beta_{12} = \alpha_{36} + \alpha_{46} \). In more detail, when the weight \( \Lambda \) corresponds to cases (A.22a, b, c, d, e) then the weight \( \Lambda + \beta_{12} \) corresponds to cases (A.22a, a, a, b, c) (since the value of \( r_1 \) is increased by 2) and furthermore \( \hat{L}_{\Lambda + \beta_{12}} \) is
actually a SRC of type $b$, see below from where we take the numbers: When $j_1 \geq 1$ the UIR $\hat{L}_{A+\beta_{12}}$ has 48, 48, 48, 47, 42 states, when $j_1 = \frac{1}{2}$ it has 45, 45, 45, 44, 39 states, when $j_1 = 0$ it has 33, 33, 33, 32, 29 states. Summed together with the numbers for the UIR $\hat{L}_A$ from above we obtain the following contributions to $\hat{L}_{\text{long}}$: when $j_1 \geq 1$ there are 176, 175, 169, 150, 110 states, when $j_1 = \frac{1}{2}$ there are 165, 164, 158, 140, 102 states, when $j_1 = 0$ there are 121, 120, 116, 102, 74 states. Except the last cases (when $r_1 = 0$) these cases match exactly the cases of long superfields for the corresponding values of $j_1 = 1, \frac{1}{2}, 0$ and $j_2 = 0$. For completeness one may check that the states of $\hat{L}_{A+\beta_{12}}$ appear in $\hat{L}_{\text{long}}$ being multiplied by $X_{36}^+ X_{46}^+$. In the cases when $r_1 = 0$ there is a mis-match of one state and that extra state is $X_{35}^+ X_{46}^+ (\Lambda)$ which is excluded from $\hat{L}_A$ as explained in general, cf. (3.56). (It is also excluded in case $b$ below.) Thus, instead of (3.55) we have the quasi-decomposition:

$$\left(\hat{L}_{\text{long}}\right)|_{d=d^a} = \hat{L}_A \oplus \hat{L}_{A+\beta_{12}} \oplus \hat{L}_{A+\alpha_{35}+\alpha_{46}}', \quad r_1 = 0,$$

(A.38)

where as in (A.33) we have put a prime on the last term indicating that this is not a genuine irrep.

$\bullet$ $d = d_{21}^2 = z + r_1 > d_{22}^3, \quad j_2 = 0$.

The character formula is (3.63). The generators $X_{36}^+$ and $X_{46}^+$ are eliminated due to (3.59) and (3.60). Due to (3.23b) there are at most two anti-chiral states:

$$X_{45}^+ (\Lambda), \quad X_{35}^+ X_{45}^+ (\Lambda).$$

(A.39)

Thus, the maximal number of states is $48(16 \times 3)$ achieved for $r_1 \geq 4, j_1 \geq 1$. These states are given explicitly as:

$$\Psi_{\xi} = (X_{16}^+)^{\xi_{16}} (X_{15}^+)^{\xi_{15}} (X_{26}^+)^{\xi_{26}} (X_{25}^+)^{\xi_{25}} (X_{35}^+)^{\xi_{35}} (X_{45}^+)^{\xi_{45}} (\Lambda),$$

$$\xi_{aj} = 0, 1; \xi_{35} \leq \xi_{45}; \quad r_1 \geq 4, j_1 \geq 1.$$

(A.40)

In summary, when $j_1 \geq 1$ we have correspondingly to the cases in (A.22a, b, c, d, e) 48, 47, 42, 31, 10 states. When $j_1 = \frac{1}{2}$ we have correspondingly to the cases in (A.22a, b, c, d, e) 45, 44, 39, 29, 9 states. When $j_1 = 0$ we have correspondingly to the cases in (A.22a, b, c, d, e) 33, 32, 29, 23, 7 states. The cases when $r_1 > 2$ were included in decompositions (3.55) in the previous case $a$. (The cases when $r_1 = 2$ were included in quasi-decompositions (3.8) in the previous case $a$.)

The minimal number when $r_1 > 0$ is 23 achieved for $r_1 = 1, j_1 = 0$. Besides the obvious states which include $X_{45}^+ (\Lambda)$, nine chiral states, their combinations and the vacuum, there are the following states:

$$X_{35}^+ X_{45}^+ (\Phi' | \Lambda),$$

$$\Phi' = X_{25}^+, \quad X_{15}^+ X_{25}^+, \quad X_{26}^+ X_{15}^+ X_{25}^+.$$

(A.41)
Whenever $r_1 = 0$ the generators $X_{35}^+$ and $X_{45}^+$ are also eliminated from the basis due to (3.61). Thus, these UIRs are chiral. Due to (3.23c) and excluding the state (3.24) there are 10, 9, 7 states for $j_1 \geq 1$, $\frac{1}{2}$, 0, resp. (as stated above). These states explicitly are:

$$|\Lambda\rangle, \quad X_{25}^+|\Lambda\rangle, \quad X_{15}^+X_{25}^+|\Lambda\rangle, \quad X_{16}^+X_{25}^+|\Lambda\rangle, \quad X_{26}^+X_{25}^+|\Lambda\rangle,$$

$$X_{26}^+X_{15}^+X_{25}^+|\Lambda\rangle, \quad X_{16}^+X_{26}^+X_{15}^+X_{25}^+|\Lambda\rangle, \quad j_1 \geq 0,$$  \hspace{1cm} \text{(A.42a)}

$$X_{15}^+|\Lambda\rangle, \quad X_{16}^+X_{15}^+X_{25}^+|\Lambda\rangle, \quad j_1 \geq \frac{1}{2},$$  \hspace{1cm} \text{(A.42b)}

$$X_{16}^+X_{15}^+|\Lambda\rangle, \quad j_1 \geq 1.$$  \hspace{1cm} \text{(A.42c)}

For $j_1 = 0$ the superfield in (A.42a) and its conjugate (considered below) are the shortest short SRC $N = 2$ superfields.

\[ \bullet \quad d = d_{\max} = d_{22}^2 = 2 + 2j_1 - z + r_1 > d_{21}^2. \]

This case is the conjugate to $\bullet a$ and the maximal number of states is $128 = 8(\text{chiral}) \times 16(\text{anti-chiral})$ achieved for $j_2 \geq 1, r_1 \geq 4$.

\[ \bullet \quad j_1 > 0. \]

The generator $X_{15}^+$ is eliminated. The eight states in the chiral sector are obtained from the following operators:

$$X_{25}^+, \quad \varepsilon_r^c = 1, \quad \varepsilon_j^c = 1,$$

$$X_{26}^+X_{25}^+, \quad \varepsilon_r^c = 0, \quad \varepsilon_j^c = 2,$$

$$1, \quad X_{16}^+X_{25}^+, \quad \varepsilon_r^c = 0, \quad \varepsilon_j^c = 0,$$

$$X_{26}^+, \quad X_{26}^+X_{16}^+X_{25}^+, \quad \varepsilon_r^c = -1, \quad \varepsilon_j^c = 1,$$

$$X_{16}^+, \quad \varepsilon_r^c = -1, \quad \varepsilon_j^c = -1,$$

$$X_{16}^+X_{26}^+, \quad \varepsilon_r^c = -2, \quad \varepsilon_j^c = 0.$$  \hspace{1cm} \text{(A.43)}

In summary, when $j_2 \geq 1$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 128, 127, 120, 99, 42 states. When $j_2 = \frac{1}{2}$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 120, 119, 112, 92, 39 states. When $j_2 = 0$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 88, 87, 82, 68, 28 states. Whenever $r_1 = 0$ the generator $X_{16}^+$ is also eliminated from the basis.

When $r_1 > 0$ holds decomposition (3.69) with $\beta = \alpha_{15}$. When $r_1 = 0$ holds the quasi-decomposition:

$$\left( \hat{L}_{\text{long}} \right)_{|d=de} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{15}} \oplus \hat{L}_{\Lambda + \alpha_{16}}, \quad r_1 = 0,$$  \hspace{1cm} \text{(A.44)}

cf. (A.33). We omit most details since all results and formulae are by conjugation from case $\bullet a$ (when $j_2 \neq 0$).

We still give the 28 states of the minimal case. There are two chiral states:

$$X_{25}^+|\Lambda\rangle, \quad X_{26}^+X_{25}^+|\Lambda\rangle.$$  \hspace{1cm} \text{(A.45)}
and six anti-chiral states (just as in (A.29)):

\begin{align}
X^+_{46} |\Lambda\rangle, \quad X^+_{36} X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{36} X^+_{46} |\Lambda\rangle, \\
X^+_{45} X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{36} X^+_{46} |\Lambda\rangle.
\end{align}

(A.46a) (A.46b)

The rest of the states are obtained as follows. Let \(\hat{\Phi}_a |\Lambda\rangle\), \(\hat{\Phi}_c |\Lambda\rangle\), denote any of the states in (A.45), (A.46), resp., \(\hat{\Phi}_c' |\Lambda\rangle\), denote any of the three states in (A.46a), resp. Thus, there are the following states:

\begin{align}
|\Lambda\rangle, \quad \hat{\Phi}_c \hat{\Phi}_a |\Lambda\rangle, \\
X^+_{36} X^+_{45} \hat{\Phi}_c |\Lambda\rangle, \\
X^+_{45} X^+_{25} |\Lambda\rangle, \quad X^+_{45} X^+_{35} X^+_{46} X^+_{25} |\Lambda\rangle, \\
X^+_{26} \hat{\Phi}_c' |\Lambda\rangle.
\end{align}

(A.47a) (A.47b) (A.47c) (A.47d)

This superfield and its conjugate (considered in \(\bullet a\)) are the shortest semi-short SRC \(N = 2\) superfields.

\(\bullet j_1 = 0.\)

The state \(X^+_{15} X^+_{25} |\Lambda\rangle\) and its descendants are eliminated (due to (3.53)). The eight states in the chiral sector here come from:

\begin{align}
X^+_{25}, \quad \varepsilon^c = 1, \quad \varepsilon^c = 1, \\
X^+_{26} X^+_{25}, \quad \varepsilon^c = 0, \quad \varepsilon^c = 2, \\
1, \quad X^+_{16} X^+_{25}, \quad X^+_{26} X^+_{15}, \quad \varepsilon^c = 0, \quad \varepsilon^c = 0, \\
X^+_{26}, \quad X^+_{26} X^+_{16} X^+_{25}, \quad \varepsilon^c = -1, \quad \varepsilon^c = 1, \\
X^+_{16} X^+_{26}, \quad \varepsilon^c = -2, \quad \varepsilon^c = 0.
\end{align}

(A.48)

The above eight differ from (A.43) by one operator: \(X^+_{16}\) is replaced here by \(X^+_{26} X^+_{15}\). In summary, when \(j_2 \geq 1\) correspondingly to the cases in (A.22a, b, c, d, e) we have now 128, 127, 121, 103, 68 cases. When \(j_2 = \frac{1}{2}\) correspondingly to the cases in (A.22a, b, c, d, e) we have now 120, 119, 113, 96, 63 cases. When \(j_2 = 0\) correspondingly to the cases in (A.22a, b, c, d, e) we have now 88, 87, 83, 70, 45 cases. Whenever \(r_1 = 0\) the generators \(X^+_{16}\) and \(X^+_{15}\) are eliminated from the chiral part of the basis, which is further restricted due to (3.23c) and there are only two chiral states as in (A.45).

When \(r_1 > 0\) holds formula (3.72) for \(\hat{L}_{\text{long}}\) with the same values of \(j_1(= 0), j_2, r_1\) as for \(\Lambda\) and with \(\beta = \beta_{34} = \alpha_{15} + \alpha_{25}\). When \(r_1 = 0\) this decomposition is spoiled by one state \(X^+_{16} X^+_{25} |\Lambda\rangle\) which is excluded from \(\hat{L}_\Lambda\) as explained in general, cf. (3.73), and instead of (3.72) we have the quasi-decomposition:

\[
\left. \left(\hat{L}_{\text{long}}\right) \right|_{d = d_c} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \beta_{34}} \oplus \hat{L}_{\Lambda + \alpha_{15} + \alpha_{25}}, \quad r_1 = 0.
\]

(A.49)

\(\bullet d = d_{22}^+ = -z + r_1 > d_{21}^+, \quad j_1 = 0.\)

This case is the conjugate to \(\bullet b\).

64
The generators $X_{15}^+$ and $X_{25}^+$ are eliminated due to (3.75) and (3.76). Due to (3.23b) there are at most two chiral states depending on the value of $r_1$:

$$
X_{26}^+ |\Lambda\rangle, \quad r_1 \geq 1,
$$

$$
X_{16}^+ X_{26}^+ |\Lambda\rangle, \quad r_1 \geq 2.
$$

(A.50)

The maximal number of states is $48(3 \times 16)$ achieved for $r_1 \geq 4$, $j_2 \geq 1$. These states are given explicitly as:

$$
\Psi_{\varepsilon} = (X_{35}^+)^{\varepsilon_{35}} (X_{36}^+)^{\varepsilon_{36}} (X_{45}^+)^{\varepsilon_{45}} (X_{46}^+)^{\varepsilon_{46}} (X_{16}^+)^{\varepsilon_{16}} (X_{26}^+)^{\varepsilon_{26}} |\Lambda\rangle,
$$

$$
\varepsilon_{aj} = 0, 1; \quad \varepsilon_{35} \leq \varepsilon_{45}; \quad r_1 \geq 2, \quad j_2 \geq 1.
$$

(A.51)

In summary, when $j_2 \geq 1$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 48, 47, 42, 31, 10 states. When $j_2 = \frac{1}{2}$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 45, 44, 39, 29, 9 states. When $j_2 = 0$ correspondingly to the cases in (A.22a, b, c, d, e) we have now 33, 32, 29, 23, 7 states. The cases when $r_1 > 2$ were included in decompositions (3.72) in the previous case c. (The cases when $r_1 = 2$ were included in decompositions (A.49) in the previous case c.)

The minimal number when $r_1 > 0$ is 23 achieved for $r_1 = 1, j_2 = 0$. Besides the obvious states which include $X_{26}^+ |\Lambda\rangle$, nine anti-chiral states, their combinations and the vacuum, there the following states:

$$
X_{16}^+ X_{26}^+ \Phi' |\Lambda\rangle, \quad \Phi' = X_{46}^+, \quad X_{36}^+ X_{46}^+, \quad X_{45}^+ X_{36}^+ X_{46}^+.
$$

(A.52)

Whenever $r_1 = 0$ the generators $X_{16}^+$ and $X_{26}^+$ are also eliminated from the basis due to (3.61). Thus, these UIRs are anti-chiral. Due to (3.23c) and excluding the state (3.25) there are 10, 9, 7 states for $j_2 \geq 1, \frac{1}{2}, 0$, resp. These states explicitly are:

$$
|\Lambda\rangle, \quad X_{46}^+ |\Lambda\rangle, \quad X_{36}^+ X_{46}^+ |\Lambda\rangle, \quad X_{35}^+ X_{46}^+ |\Lambda\rangle, \quad X_{45}^+ X_{46}^+ |\Lambda\rangle,
$$

$$
X_{45}^+ X_{36}^+ X_{46}^+ |\Lambda\rangle, \quad X_{35}^+ X_{45}^+ X_{46}^+ X_{46}^+ |\Lambda\rangle, \quad j_2 \geq 0,
$$

(A.53a)

$$
X_{36}^+ |\Lambda\rangle, \quad X_{35}^+ X_{36}^+ X_{46}^+ |\Lambda\rangle, \quad j_2 \geq 2,
$$

(A.53b)

$$
X_{35}^+ X_{36}^+ |\Lambda\rangle, \quad j_2 \geq 1.
$$

(A.53c)

For $j_2 = 0$ the superfield in (A.53a) and its conjugate (considered above) are the shortest short SRC $N = 2$ superfields.

• DRC cases

Here we consider the DRC cases taking again the five cases of long superfields in (A.22) as reference point.
\[ \bullet \text{ac} \quad d = d_{\text{max}} = d_{21}^2 = d_{22}^2 = 2 + j_1 + j_2 + r_1, \quad z = j_1 - j_2. \]

The maximal number of states is \( 64 = 8(\text{chiral}) \times 8(\text{anti-chiral}) \), achieved for \( r_1 \geq 4 \). The 8 anti-chiral, chiral, states are as described in \( \bullet \text{a,c} \), resp., (differing for \( j_2 > 0 \) and \( j_2 = 0, j_1 > 0 \) and \( j_1 = 0 \), resp.).

\[ \bullet j_1 j_2 > 0. \]

Here hold character formulae (3.82) (without counterterms for \( r_1 \geq 4 \)). The states \( X_{15}^+ |\Lambda \rangle, X_{36}^+ |\Lambda \rangle \) and their descendants are eliminated. Correspondingly to the cases in (A.22a, b, c, d, e) we have now 64, 63, 57, 42, 11 states. In the last case, (where \( r_1 = 0 \)), we eliminate also the generators \( X_{35}^+ \) and \( X_{16}^+ \).

For \( r_1 > 0 \) holds decomposition (3.81) with \( \beta = \alpha_{15}, \beta' = \alpha_{36} \) as stated in the general exposition. We would like demonstrate this and also see how it breaks down for \( r_1 = 0 \), thus, we include for the moment the case \( r_1 = 0 \). Referring to (3.81) we note when the weight \( \Lambda \) corresponds to cases (A.22a, b, c, d, e) then the weights \( \Lambda + \alpha_{15}, \Lambda + \alpha_{36} \), correspond to cases (A.22a, a, b, c, d) (since the value of \( r_1 \) is increased by 1), i.e., the corresponding UIRs have 64, 64, 63, 57, 42 states each. The weight \( \Lambda + \alpha_{15} + \alpha_{36} \), corresponds to cases (A.22a, a, a, b, c) (since the value of \( r_1 \) is increased by 2), i.e., the corresponding UIRs have 64, 64, 64, 63, 57 states. Summed together with the numbers for the UIR \( \hat{L}_\Lambda \) from above we obtain the following contributions to \( \hat{L}_{\text{long}} : 256, 255, 247, 219, 152 \). Except the last case (in which \( r_1 = 0 \)) these cases match exactly the cases of long superfields for the case \( j_1, j_2 \geq 1 \).

When \( r_1 = 0 \) the long superfields for the cases \( j_1, j_2 \geq 1 \) have 163 states, i.e., a mis-match of 11 states.\(^{19}\) These extra states contain either the generator \( X_{16}^+ \) or \( X_{35}^+ \) or both, and they do not contain either \( X_{15}^+ \) or \( X_{36}^+ \). Explicitly, these extra states are:

\[
\begin{align*}
X_{16}^+ X_{46}^+ |\Lambda \rangle, & \quad X_{16}^+ X_{25}^+ |\Lambda \rangle, \quad X_{16}^+ X_{25}^+ X_{46}^+ |\Lambda \rangle, \\
X_{16}^+ X_{26}^+ X_{25}^+ X_{46}^+ |\Lambda \rangle, & \quad X_{16}^+ X_{25}^+ X_{45}^+ X_{46}^+ |\Lambda \rangle, \\
X_{35}^+ X_{46}^+ |\Lambda \rangle, & \quad X_{35}^+ X_{25}^+ X_{46}^+ |\Lambda \rangle, \quad X_{35}^+ X_{25}^+ X_{45}^+ X_{46}^+ |\Lambda \rangle, \\
X_{35}^+ X_{26}^+ X_{25}^+ X_{46}^+ |\Lambda \rangle, & \quad X_{35}^+ X_{25}^+ X_{45}^+ X_{46}^+ |\Lambda \rangle, \\
X_{16}^+ X_{35}^+ X_{25}^+ X_{46}^+ |\Lambda \rangle. &
\end{align*}
\]

Altogether, instead of (3.81) we may write:

\[
\left( \hat{L}_{\text{long}} \right)_{|d = d_{\text{ac}}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda + \alpha_{15}} \oplus \hat{L}_{\Lambda + \alpha_{36}} \oplus \hat{L}_{\Lambda + \alpha_{15} + \alpha_{36}} \oplus \\
\quad \oplus \hat{L}_{\Lambda + \alpha_{16}} \oplus \hat{L}_{\Lambda + \alpha_{35}} \oplus \hat{L}_{\Lambda + \alpha_{16} + \alpha_{35}}, \quad r_1 = 0,
\]

where we have represented the extra states by the last three terms (corresponding to first and second line of (A.54), third and fourth line of (A.54), fifth line of (A.54), resp.), and we have put primes on these since they are not genuine irreps.

\(^{19}\) The reader may wonder whether the long superfield with \( j_1 = \frac{1}{2}, j_2 \geq 1, r_1 = 0 \) may not be used since it has 152 states, however, this is only a coincidence of the total number.
Finally, we give the 11 states of the UIR at $r_1 = 0$:

$$\begin{align*}
|\Lambda\rangle, & \quad X_{25}^{+} X_{46}^{+} |\Lambda\rangle, \quad X_{26}^{+} X_{45}^{+} X_{25}^{+} X_{46}^{+} |\Lambda\rangle, \\
X_{46}^{+} |\Lambda\rangle, & \quad X_{45}^{+} X_{46}^{+} |\Lambda\rangle, \\
X_{25}^{+} |\Lambda\rangle, & \quad X_{26}^{+} X_{25}^{+} |\Lambda\rangle, \\
\Phi_0^{c} X_{46}^{+} |\Lambda\rangle, & \quad \Phi_0^{c} = X_{26}^{+}, \quad X_{26}^{+} X_{25}^{+}, \\
\Phi_0^{a} X_{25}^{+} |\Lambda\rangle, & \quad \Phi_0^{a} = X_{45}^{+}, \quad X_{45}^{+} X_{46}^{+}.
\end{align*}$$

(A.56)

This superfield is the shortest semi-short $N = 2$ superfield.

- $j_1 > 0, j_2 = 0$. Here hold character formulae (3.84) (without counterterms for $r_1 \geq 4$). The states $X_{36}^{+} X_{46}^{+} |\Lambda\rangle, X_{15}^{+} |\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a, b, c, d, e) we have now $64, 63, 58, 45, 16$ states. In the last case, where $r_1 = 0$, we eliminate the generator $X_{16}^{+}$ and exclude the generators $X_{3,4+k}^{+}$ from the anti-chiral sector.

For $r_1 > 0$ holds decomposition (3.85). Note that when the weight $\Lambda$ corresponds to cases (A.22a, b, c, d, e) then the weight $\Lambda + \alpha_{15}$ corresponds to cases (A.22a, a, b, c, d) (since the value of $r_1$ is increased by 1), i.e., the corresponding UIRs have $64, 64, 63, 58, 45$ states. The weight $\Lambda + \beta_{12}$ corresponds to cases (A.22a, a, a, b) (since the value of $r_1$ is increased by 2), but from type bc considered below, i.e., the corresponding UIRs have $24, 24, 24, 23, 19$ states. The weight $\Lambda + \alpha_{15} + \beta_{12}$ corresponds to cases (A.22a, a, a, a, b) (since the value of $r_1$ is increased by 3), also from type bc, i.e., the corresponding UIRs have $24, 24, 24, 24, 23$ states. Summed together with the numbers for the UIR $\hat{L}_{\Lambda}$ from above we obtain the following contributions to $\hat{L}_{\text{long}}$: 176, 175, 169, 150, 103. Except the last case (in which $r_1 = 0$) these cases match exactly the cases of long superfields for the cases when $j_1 \geq 1, j_2 = 0$.

When $r_1 = 0$ the corresponding long superfields have 111 states, i.e., there is a mismatch of 8 states.\(^{20}\) These extra states contain either the generator $X_{16}^{+}$ or $X_{35}^{+}$ or both, and they do not contain $X_{15}^{+}$. Explicitly, they are:

$$\begin{align*}
X_{16}^{+} X_{46}^{+} |\Lambda\rangle, & \quad X_{16}^{+} X_{25}^{+} |\Lambda\rangle, \quad X_{16}^{+} X_{25}^{+} X_{46}^{+} |\Lambda\rangle, \quad X_{16}^{+} X_{26}^{+} X_{25}^{+} X_{46}^{+} |\Lambda\rangle, \\
X_{16}^{+} X_{25}^{+} X_{45}^{+} X_{46}^{+} |\Lambda\rangle, & \quad X_{16}^{+} X_{25}^{+} X_{36}^{+} X_{45}^{+} |\Lambda\rangle, \\
X_{35}^{+} X_{46}^{+} |\Lambda\rangle, & \quad X_{35}^{+} X_{25}^{+} X_{46}^{+} |\Lambda\rangle, \\
X_{16}^{+} X_{35}^{+} X_{25}^{+} X_{46}^{+} |\Lambda\rangle.
\end{align*}$$

(A.57)

Altogether, instead of (3.85) we may write:

$$\left( \hat{L}_{\text{long}} \right)_{d = d} = \hat{L}_{\Lambda} \oplus \hat{L}_{\Lambda + \alpha_{15}} \oplus \hat{L}_{\Lambda + \beta_{12}} \oplus \hat{L}_{\Lambda + \alpha_{15} + \beta_{12}} \oplus \hat{L}_{\Lambda + \alpha_{16}} \oplus \hat{L}_{\Lambda + \alpha_{35}} \oplus \hat{L}_{\Lambda + \alpha_{16} + \alpha_{35}}, \quad r_1 = 0,$$

(A.58)

\(^{20}\) Again the long superfield with correct number of states 103 (with $j_1 = \frac{1}{2}, j_2 = r_1 = 0$) does not fit.
where (as in (A.55)) we have represented the extra states by the last three terms (corresponding to first and second line of (A.57), third line of (A.57), fourth line of (A.57), resp.), and we have put primes on these since they are not genuine irreps.

Finally, we give the 16 states of the UIR at \( r_1 = 0 \):

\[
|\Lambda\rangle, \ X_{25}^+ X_{46}^+ |\Lambda\rangle, \\
X_{46}^+ |\Lambda\rangle, \ X_{45}^+ X_{46}^+ |\Lambda\rangle, \\
X_{25}^+ |\Lambda\rangle, \ X_{26}^+ X_{25}^+ |\Lambda\rangle, \\
\Phi_0^a X_{25}^+ |\Lambda\rangle, \ \Phi^0_a = X_{45}^+ X_{46}^+ , \ X_{35}^+ X_{46}^+ , \ X_{36}^+ X_{45}^+, \\
\Phi^+_{a} X_{25}^+ |\Lambda\rangle, \ \Phi^+_a = X_{45}^+ , \ X_{35}^+ X_{45}^+ X_{46}^+ , \\
\Phi^+_c X_{46}^+ |\Lambda\rangle, \ \Phi^+_c = X_{26}^+ , \ X_{26}^+ X_{25}^+, \\
\Phi^0_a X_{45}^+ X_{25}^+ |\Lambda\rangle.
\]

The states of (A.56) are a subset of (A.59).

The next case is conjugate to the preceding.

\( j_1 = 0, j_2 > 0 \). Here hold character formulae (3.87) (without counterterms for \( r_1 \geq 4 \)). The states \( X_{15}^+ X_{25}^+ |\Lambda\rangle, \ X_{36}^+ |\Lambda\rangle \) and their descendants are eliminated. Correspondingly to the cases in (A.22a, b, c, d, e) we have now 64, 63, 58, 45, 16 states. In the last case, when \( r_1 = 0 \), we eliminate the generator \( X_{35}^+ \) and exclude the generators \( X_{1,4+k}^+ \) from the chiral sector.

For \( r_1 > 0 \) holds decomposition (3.88). When \( r_1 = 0 \) the corresponding long superfields have 111 states, i.e., there is a mis-match of 8 states. These extra states contain either the generator \( X_{16}^+ \) or \( X_{35}^+ \) or both, and they do not contain \( X_{36}^+ \). Explicitly, these extra states are:

\[
X_{35}^+ X_{25}^+ |\Lambda\rangle, \ X_{35}^+ X_{46}^+ |\Lambda\rangle, \ X_{35}^+ X_{46}^+ X_{25}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{46}^+ X_{25}^+ |\Lambda\rangle, \\
X_{35}^+ X_{46}^+ X_{26}^+ X_{25}^+ |\Lambda\rangle, \ X_{35}^+ X_{46}^+ X_{15}^+ X_{26}^+ |\Lambda\rangle, \\
X_{16}^+ X_{25}^+ |\Lambda\rangle, \\
X_{35}^+ X_{16}^+ X_{46}^+ X_{25}^+ |\Lambda\rangle.
\]

Altogether, instead of (3.88) we may write:

\[
\left( \hat{L}_{\text{long}} \right)_{\mid d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\alpha_{36}} \oplus \hat{L}_{\Lambda+\beta_{34}} \oplus \hat{L}_{\Lambda+\alpha_{36}+\beta_{34}} \oplus \\
\oplus \hat{L}_{\Lambda+\alpha_{16}} \oplus \hat{L}_{\Lambda+\alpha_{35}} \oplus \hat{L}_{\Lambda+\alpha_{16}+\alpha_{35}} , \ r_1 = 0 .
\]
Finally, we give the 16 states of the UIR at $r_1 = 0$:

$$
|\Lambda\rangle, \quad X^+_{25} X^+_{46} |\Lambda\rangle,
$$

$$
X^+_{16} |\Lambda\rangle, \quad X^+_{15} X^+_{46} |\Lambda\rangle,
$$

$$
X^+_{22} |\Lambda\rangle, \quad X^+_{26} X^+_{25} |\Lambda\rangle,
$$

$$
\Phi^0_c X^+_{46} |\Lambda\rangle, \quad \Phi^0_c = X^+_c X^+_{25}, \quad \Phi^+_{15} X^+_c X^+_26,
$$

$$
\Phi^+_{16} X^+_{25}, \quad \Phi^+_{16} X^+_c X^+_25,
$$

$$
\Phi^0_c X^+_5 |\Lambda\rangle, \quad \Phi^0_c = X^+_{45} X^+_4 X^+_46,
$$

$$
\Phi^0_c X^+_5 X^+_{46} |\Lambda\rangle.
$$

(A.62)

The states of (A.56) are a subset of (A.62).

- $j_1 = j_2 = 0$. Here hold character formulae (3.90) (without counterterms for $r_1 \geq 4$). The states $X^+_{15} X^+_{25} |\Lambda\rangle, \quad X^+_{36} X^+_{46} |\Lambda\rangle$ and their descendants are eliminated. Correspondingly to the cases in (A.22a,b,c,d,e) we have now 64, 63, 59, 47, 24 states. In the last case, when $r_1 = 0$, we exclude the generators $X^+_{3,4+k}$ from the anti-chiral sector and the generators $X^+_{1,4+k}$ from the chiral sector and also the combination of impossible states (A.27) as explained in the general exposition.

For $r_1 > 0$ holds decomposition (3.91). Note that when the weight $\Lambda$ corresponds to cases (A.22a,b,c,d,e) then the weights $\Lambda + \beta_{12}, \quad \Lambda + \beta_{34}$ corresponds to cases (A.22a,a,a,b,c) (since the value of $r_1$ is increased by 2), but from types $bc, ad$, resp., considered below, i.e., the corresponding UIRs have 24, 24, 24, 20 states each. The weight $\Lambda + \beta_{12} + \beta_{34}$ corresponds to cases (A.22a,a,a,a,a) (since the value of $r_1$ is increased by 4), but from type $bd$, i.e., the corresponding UIRs have 9, 9, 9, 9 states. Summed together with the numbers for the UIR $\hat{L}_\Lambda$ from above we obtain the following contributions to $\hat{L}_{\text{long}}$: 121, 120, 116, 102, 73. Except the last case (in which $r_1 = 0$) these cases match exactly the cases of long superfields for the cases when $j_1 = j_2 = 0$.

When $r_1 = 0$ the corresponding long superfields have 75 states, i.e., there is a mis-match of 2 states. These extra states are:

$$
X^+_{16} X^+_{25} |\Lambda\rangle, \quad X^+_{35} X^+_{46} |\Lambda\rangle.
$$

(A.63)

Altogether, instead of (3.91) we may write:

$$
(\hat{L}_{\text{long}})_{d=c,d=c} = \hat{L}_\Lambda \quad \hat{L}_\Lambda + \beta_{12} \quad \hat{L}_\Lambda + \beta_{34} \quad \hat{L}_\Lambda + \beta_{12} + \beta_{34} \quad \hat{L}_\Lambda + \alpha_{16} \quad \hat{L}_\Lambda + \alpha_{35}, \quad r_1 = 0.
$$

(A.64)

69
Finally, we give the 24 states of the UIR at \( r_1 = 0 \):

\[
|\Lambda\rangle, \quad X^+_{25} X^+_{46} |\Lambda\rangle, \\
X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{46} |\Lambda\rangle, \\
X^+_{25} |\Lambda\rangle, \quad X^+_{26} X^+_{25} |\Lambda\rangle, \\
\Phi^0_c X^+_{46} |\Lambda\rangle, \quad \Phi^0_c = X^+_{26} X^+_{25}, \quad X^+_{16} X^+_{25}, \quad X^+_{15} X^+_{26}, \\
\Phi^- X^+_{46} |\Lambda\rangle, \quad \Phi^- = X^+_{26}, \quad X^+_{16} X^+_{25}, \\
\Phi^0_a X^+_{25} |\Lambda\rangle, \quad \Phi^0_a = X^+_{45} X^+_{46}, \quad X^+_{35} X^+_{46}, \quad X^+_{36} X^+_{45}, \\
\Phi^- X^+_{25} |\Lambda\rangle, \quad \Phi^- = X^+_{45}, \quad X^+_{35} X^+_{46}, \\
\Phi^0_c \Phi^0_a |\Lambda\rangle \\
\]

(A.65)

The states of (A.59),(A.62) are subsets of (A.65).

\[ d = d_{21}^1 = d_{22}^4 = 1 + j_2 + r_1 , \quad j_1 = 0 , \quad z = -1 - j_2 . \]

Here hold character formulae (3.93) when \( j_2 r_1 > 0 \), (3.94) when \( j_2 = 0 , r_1 > 0 \), (both these cases without counterterms for \( r_1 \geq 4 \)), and finally when \( r_1 = 0 \) holds (3.95) independently of the value of \( j_2 \) - these are the anti-chiral massless UIRs.

The generators \( X^+_{15}, X^+_{25} \), and in addition \( X^+_{36} \) for \( j_2 > 0 \) (resp. the state \( X^+_{36}, X^+_{46} |\Lambda\rangle \), and its descendants for \( j_2 = 0 \)) are eliminated. The maximal number of states is 24 = 3(chiral) \times 8(anti-chiral), achieved for \( r_1 \geq 4 \). The chiral sector for \( r_1 > 0 \) consists of the two states in (A.50) and the vacuum, while the anti-chiral sector is given by (A.31) for \( j_2 > 0 \) and by (A.37) for \( j_2 = 0 \).

The 24 states for \( j_2 > 0 \) are given explicitly as:

\[
|\Lambda\rangle, \quad X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{46} |\Lambda\rangle, \quad r_1 \geq 0, \\
X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{26} X^+_{46} |\Lambda\rangle, \quad r_1 \geq 1, \\
X^+_{45} |\Lambda\rangle, \quad X^+_{45} X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{26} X^+_{45} X^+_{46} |\Lambda\rangle, \quad X^+_{35} |\Lambda\rangle, \quad r_1 \geq 1, \\
X^+_{26} |\Lambda\rangle, \quad X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{46} |\Lambda\rangle, \quad r_1 \geq 1, \\
X^+_{26} X^+_{45} |\Lambda\rangle, \quad X^+_{26} X^+_{45} X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{35} X^+_{45} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{45} X^+_{46} |\Lambda\rangle, \quad r_1 \geq 2, \\
X^+_{16} X^+_{26} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{35} X^+_{46} |\Lambda\rangle, \quad X^+_{26} X^+_{35} |\Lambda\rangle, \quad r_1 \geq 2, \\
X^+_{26} X^+_{35} X^+_{45} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{45} |\Lambda\rangle, \quad X^+_{16} X^+_{26} X^+_{35} X^+_{45} |\Lambda\rangle \quad X^+_{16} X^+_{26} X^+_{35} |\Lambda\rangle, \quad r_1 \geq 3, \\
X^+_{16} X^+_{26} X^+_{35} X^+_{45} |\Lambda\rangle, \quad r_1 \geq 4. \\
\]

(A.66)

Thus, correspondingly to the cases in (A.22a, b, c, d, e) we have now 24, 23, 19, 12, 3 states.

The irreps with \( r_1 > 2 \) appear (two times if \( r_1 > 3 \)) in decomposition (3.88) as explained in detail in the main text for type \textbf{ad}. (The irreps with \( r_1 = 2 \) have appeared in quasi-decomposition (A.61).)

The 24 states for \( j_2 = 0 \) are given explicitly as:

\[
|\Lambda\rangle, \quad X^+_{46} |\Lambda\rangle, \quad X^+_{45} X^+_{46} |\Lambda\rangle, \quad r_1 \geq 0 ,
\]

70
\( X_{45}^+ X_{16}^- |\Lambda_+ \), \( X_{26}^+ X_{46}^- |\Lambda_+ \), \( X_{45}^+ X_{36}^- |\Lambda_+ \), \( r_1 \geq 1 \), \\
\( X_{45}^- |\Lambda_+ \), \( X_{45}^+ X_{35}^- X_{46}^- |\Lambda_+ \), \( X_{26} X_{45}^+ X_{46}^- |\Lambda_+ \), \( r_1 \geq 1 \), \\
\( X_{26}^+ |\Lambda_+ \), \( X_{26}^+ X_{35}^- X_{46}^- |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{46}^- |\Lambda_+ \), \( X_{26} X_{45}^+ X_{36}^- |\Lambda_+ \), \( r_1 \geq 1 \), \\
\( X_{36}^+ X_{46}^- |\Lambda_+ \), \( X_{46}^+ X_{45}^- X_{36}^- |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{45}^- |\Lambda_+ \), \( X_{16} X_{26}^+ X_{45}^+ X_{46}^- |\Lambda_+ \), \( r_1 \geq 2 \), \\
\( X_{16}^+ X_{26}^+ |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{35}^- X_{46}^- |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{45}^- X_{36}^- |\Lambda_+ \), \( r_1 \geq 2 \), \\
\( X_{26}^+ X_{35}^+ X_{45}^- |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{45}^- |\Lambda_+ \), \( X_{16}^+ X_{26}^+ X_{45}^+ X_{36}^- |\Lambda_+ \), \( r_1 \geq 3 \), \\
\( X_{16}^+ X_{26}^+ X_{35}^- X_{45}^- |\Lambda_+ \), \( r_1 \geq 4 \).

\( (A.67) \)

Thus, correspondingly to the cases in \( (A.22a, b, c, d, e) \) we have now 24, 23, 20, 13, 3 states.

The irreps with \( r_1 > 2 \) appear as the term \( \tilde{L}_{\Lambda + \beta_{34}} \) of (3.91), while those with \( r_1 > 3 \) appear also as the term \( \tilde{L}_{\Lambda + \alpha_4 + \gamma + \beta_{34}} \) of (3.88) but only when \( j_2 = \frac{1}{2} \) in \( \Lambda \) there. (The irreps with \( r_1 = 2 \) have appeared in quasi-decompositions \( (A.64) \).)

The cases \( (A.66) \) and \( (A.67) \) share 21 states (for \( r_1 \geq 4 \)). The 3 states by which they differ are the last states on the 3rd, 6th, 7th lines of \( (A.66) \) and 2nd, 4th, 6th lines of \( (A.67) \).

\[ \bullet bc \quad d = d_{21}^2 = d_{22}^3 = 1 + j_1 + r_1, \quad j_2 = 0, \quad z = 1 + j_1 . \]

Here hold character formulae (3.96) when \( j_1 r_1 > 0 \), (3.97) when \( j_1 = 0, r_1 > 0 \), (both these cases without counterterms for \( r_1 \geq 4 \)), and finally when \( r_1 = 0 \) holds (3.98) independently of the value of \( j_1 \) - these are the chiral massless UIRs.

The generators \( X_{36}^+, X_{46}^+ \), and in addition \( X_{15}^+ \) for \( j_1 > 0 \) (resp. the state \( X_{15}^+ X_{25}^- |\Lambda_+ \), and its descendants for \( j_1 = 0 \)) are eliminated. The maximal number of states is \( 24 = 8(\text{chiral}) \times 3(\text{anti-chiral}) \), achieved for \( r_1 \geq 4 \). The anti-chiral sector for \( r_1 > 0 \) consists of the two states in \((A.39)\) and the vacuum, while the chiral sector is given by \( (A.43) \) for \( j_1 > 0 \) and by \( (A.48) \) for \( j_1 = 0 \).

The 24 states for \( j_1 > 0 \) are given explicitly as:

\[ |\Lambda_+ \rangle, \ X_{25}^+ |\Lambda_+ \rangle, \ X_{26}^+ X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 0, \]
\[ X_{16}^+ X_{25}^- |\Lambda_+ \rangle, \ X_{45}^+ X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 1, \]
\[ X_{26}^+ |\Lambda_+ \rangle, \ X_{26}^+ X_{16}^- X_{25}^- |\Lambda_+ \rangle, \ X_{45}^+ X_{26}^- X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 1, \]
\[ X_{45}^- |\Lambda_+ \rangle, \ X_{45}^+ X_{16}^- X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 1, \]
\[ X_{45}^+ X_{26}^- |\Lambda_+ \rangle, \ X_{45}^+ X_{26}^- X_{16}^- X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 2, \]
\[ X_{35} X_{45}^- |\Lambda_+ \rangle, \ X_{35}^+ X_{45}^- X_{16}^- X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 2, \]
\[ X_{45}^+ X_{16}^- X_{26}^- |\Lambda_+ \rangle, \ X_{35}^+ X_{45}^- X_{26}^- X_{16}^- X_{25}^- |\Lambda_+ \rangle, \quad r_1 \geq 3, \]
\[ X_{35} X_{45}^- X_{16}^- X_{26}^- |\Lambda_+ \rangle, \quad r_1 \geq 4. \]

\( (A.68) \)

Thus, correspondingly to the cases in \((A.22a, b, c, d, e)\) we have now 24, 23, 19, 12, 3 states.

The irreps with \( r_1 > 2 \) appear (up to two times) in decomposition (3.85) as explained in detail in the main text for type \( bc \). (The irreps with \( r_1 = 2 \) have appeared in quasi-decomposition \((A.58)\).)
The 24 states for $j_1 = 0$ are given explicitly as:

\[
\begin{align*}
&|\Lambda\rangle, \ X_{25}^+ |\Lambda\rangle, \ X_{26}^+ X_{25}^+ |\Lambda\rangle, \quad r_1 \geq 0, \\
&X_{16}^+ X_{25}^+ |\Lambda\rangle, \ X_{45}^+ X_{25}^+ |\Lambda\rangle, \ X_{26}^+ X_{15}^+ |\Lambda\rangle, \quad r_1 \geq 1, \\
&X_{26}^+ |\Lambda\rangle, \ X_{45}^+ X_{26}^+ X_{25}^+ |\Lambda\rangle, \ X_{45}^+ X_{26}^+ X_{25}^+ |\Lambda\rangle, \quad r_1 \geq 1, \\
&X_{45}^+ |\Lambda\rangle, \ X_{45}^+ X_{16}^+ X_{25}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{25}^+ |\Lambda\rangle, \ X_{45}^+ X_{26}^+ X_{15}^+ |\Lambda\rangle, \quad r_1 \geq 1, \\
&X_{45}^+ X_{26}^+ |\Lambda\rangle, \ X_{45}^+ X_{26}^+ X_{16}^+ X_{25}^+ |\Lambda\rangle, \ X_{16}^+ X_{26}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{26}^+ X_{25}^+ |\Lambda\rangle, \quad r_1 \geq 2, \\
&X_{35}^+ X_{45}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{16}^+ X_{25}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{26}^+ X_{15}^+ |\Lambda\rangle, \quad r_1 \geq 2, \\
&X_{45}^+ X_{16}^+ X_{26}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{26}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ X_{26}^+ X_{16}^+ X_{25}^+ |\Lambda\rangle, \quad r_1 \geq 3, \\
&X_{35}^+ X_{45}^+ X_{16}^+ X_{26}^+ |\Lambda\rangle, \quad r_1 \geq 4. \quad (A.69)
\end{align*}
\]

Thus, correspondingly to the cases in (A.22a, b, c, d, e) we have now 24, 23, 20, 13, 3 states.

The irreps with $r_1 > 2$ appear as the term $\hat{L}_{\Lambda+\beta_{12}}$ of (3.91), while those with $r_1 > 3$ appear also as the term $\hat{L}_{\Lambda+\alpha_{15}+\beta_{12}}$ of (3.85) but only when $j_1 = \frac{1}{2}$ in $\Lambda$ there. (The irreps with $r_1 = 2$ have appeared in quasi-decomposition (A.64).)

\[ \text{bd} \quad d = d_{21}^2 + d_{22}^4 = r_1, \quad j_1 = j_2 = 0 = z. \]

The generators $X_{15}^+, X_{25}^+, X_{36}^+, X_{46}^+$ are eliminated. For $r_1 = 1$ also the generators $X_{16}^+, X_{35}^+$ are eliminated. For $r_1 = 0$ the remaining two generators $X_{26}^+, X_{45}^+$ are eliminated and we have the trivial irrep as explained in general.

For $r_1 > 0$ the character formula is (3.99) with $i_0 = i_0' = 0$. The maximal number of states is nine and the list of states together with the conditions when they exist are:

\[
\begin{align*}
&|\Lambda\rangle, \quad r_1 \geq 0, \\
&X_{26}^+ |\Lambda\rangle, \ X_{45}^+ |\Lambda\rangle, \quad r_1 \geq 1, \\
&X_{16}^+ X_{26}^+ |\Lambda\rangle, \ X_{35}^+ X_{45}^+ |\Lambda\rangle, \ X_{26}^+ X_{45}^+ |\Lambda\rangle, \quad r_1 \geq 2, \quad (A.70) \\
&X_{16}^+ X_{26}^+ X_{45}^+ |\Lambda\rangle, \ X_{26}^+ X_{35}^+ X_{45}^+ |\Lambda\rangle, \quad r_1 \geq 3, \\
&X_{16}^+ X_{26}^+ X_{35}^+ X_{45}^+ |\Lambda\rangle, \quad r_1 \geq 4.
\end{align*}
\]

Thus, correspondingly to the cases in (A.22a, b, c, d, e) we have now 9, 8, 6, 3, 1 states. The mixed massless irrep is obtained for $d = r_1 = 1$ and consists of the first three states above - as was shown in general.

The irreps with $r_1 > 4$ have appeared in decomposition (3.91), cf. type ac above. (The irreps with $r_1 = 4$ have appeared in quasi-decomposition (A.64).)
Appendix B. Odd Reflections

We consider two ways to extend the Weyl group $W$ of the even subalgebra $G_{\bar{0}}$ to a larger group by *odd reflections* [101],[128]. One way we introduced in (2.11). The other way is as follows. For $\beta \in \Delta_{\bar{1}}$, $\Lambda \in \chi^*$, we define:

\[
\begin{align*}
\hat{s}_\beta \Lambda &= \Lambda - 2\frac{(\beta,\Lambda)}{(\beta,\beta)} \beta, & (\beta,\beta) \neq 0 \\
\hat{s}_\beta \Lambda &= \Lambda + \beta, & (\beta,\beta) = 0, (\beta,\Lambda) \neq 0 \\
\hat{s}_\beta \Lambda &= \Lambda, & (\beta,\beta) = 0, (\beta,\Lambda) = 0, \beta \neq \Lambda \\
\hat{s}_\beta \beta &= -\beta, 
\end{align*}
\]

where $(\cdot,\cdot)$ is the standard bilinear product in $\chi^*$. As in the even case one has: $\hat{s}_{-\beta}^{-1} = \hat{s}_\beta$, and $\hat{s}_{\beta}^2 = \text{id}_{\chi^*}$.

The two versions of the odd reflections have different uses. The definition (B.1) is useful for the fact that each such odd reflection transforms the root system $\Delta$ into a root system $\Delta'$ so that the corresponding Lie superalgebras $G(\Delta)$ and $G(\Delta')$ are isomorphic [128],[129].

The generalized odd reflections $\hat{s}_\beta$ introduced in [101] and (2.11) are useful for the identification of the generalized Weyl group with the multiplets of reducible Verma modules. For this we used the fact that each $\hat{s}_\beta$ generates an infinite discrete abelian group given in (2.12). In fact, the generalized odd reflections act as translations and do not preserve $\Delta$.

Finally, note that if $\alpha, \beta, \alpha + \beta \in \Delta$ and $(\alpha,\alpha)(\beta,\beta)(\alpha + \beta,\alpha + \beta) = 0$ then $\hat{s}_{\alpha + \beta}$ can not be expressed in terms of $\hat{s}_\alpha$, $\hat{s}_\beta$.
Appendix C. Characters of the even subalgebra

For the characters of the even subalgebra we first recall its structure: $G^G_0 = sl(4) \oplus gl(1) \oplus sl(N)$ of $G^G$. We choose a basis in which the Cartan subalgebra $H$ of $G^G$ is also a Cartan subalgebra of $G^G$. Since the subalgebra $G^G_0$ is reductive the corresponding character formulae will be given by the products of the character formulae of the two simple factors $sl(4)$ and $sl(N)$.

We start with the $sl(4)$ case. We denoted the six positive roots of $sl(4)$ by $\alpha_{ij}$, $1 \leq i < j \leq 4$. For the simplification of the character formulae we use notation for the formal exponents corresponding to the $sl$ characters will be given by the products of the character formulae of the two non-simple roots we have: $e(\alpha_{13}) = t_1 t_2$, $e(\alpha_{24}) = t_2 t_3$, $e(\alpha_{14}) = t_1 t_2 t_3$.

In terms of these the character formula for a Verma module over $sl(4)$ is:

$$ch_0 \ V^{\Lambda^s} = \frac{e(\Lambda^s)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \quad (C.1)$$

where by $\Lambda^s$ we denote the $sl(4)$ lowest weight.

The representations of $sl(4)$ which we consider are infinite-dimensional. When $d > d_{\text{max}}$ then all the numbers: $n_2, n_{13}, n_{24}, n_{14}$ from (2.17) can not be positive integers. Then the only reducibilities of the $sl(4)$ Verma module are related to the complexification of the Lorentz subalgebra of $su(2, 2)$, i.e., with $sl(2) \oplus sl(2)$, and the character formula is given by the product of the two character formulae for finite-dimensional $sl(2)$ irreps. In short, the $sl(4)$ character formula is:

$$ch_0 \ L_{\Lambda^s} = ch_0 \ V^{\Lambda^s} - ch_0 \ V^{\Lambda^s+n_1 \alpha_{12}} - ch_0 \ V^{\Lambda^s+n_3 \alpha_{34}} + ch_0 \ V^{\Lambda^s+n_1 \alpha_{12}+n_3 \alpha_{34}} =$$

$$= \frac{e(\Lambda^s)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_2 t_3)} \quad (C.2)$$

and we have introduced for later use notation $Q_{n_1, n_2}^s$ for the character factorized by $e(\Lambda^s)$. The above formula obviously has the form (3.5) replacing $W \mapsto W_2 \times W_2$, where $W_2$ is the two-element Weyl group of $sl(2)$.

When $d \leq d_{\text{max}}$ there are additional even reducibilities, cf. (2.47), (2.51), (2.52), and the discussion in-between.

Thus, we need additional formulae for $ch_0 \ L_{\Lambda^s}$:

$$ch_0 \ L_{\Lambda^s} =$$

$$= e(\Lambda^s) \ Q_{n_1, n_2}^s \cdot (1 - t_1 t_2 t_3) = \frac{e(\Lambda^s) (1 - t_1^{n_1})(1 - t_3^{n_3})}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_1 t_2)(1 - t_2 t_3)} \quad (C.3a)$$

for (2.52a),

$$d = d_{N1}^1 = d_{NN}^3 = 2 + j_1 + j_2, \quad j_1 j_2 > 0 ;$$

$$= e(\Lambda^s) \ Q_{1, 2}^s \cdot (1 - t_2 t_3) = \frac{e(\Lambda^s) (1 + t_3)}{(1 - t_2)(1 - t_1 t_2)(1 - t_1 t_2 t_3)},$$
for (2.52b), \[ d = d^{1}_{N1} = d^{4}_{NN} = 3/2, \; j_1 = 0, j_2 = \frac{1}{2}; \; (C.3b) \]

\[ = e(\Lambda^s) \cdot Q_{2,1}^s \cdot (1-t_1t_2) = \frac{e(\Lambda^s) (1+t_1)}{(1-t_2)(1-t_2t_3)(1-t_1t_2t_3)}, \]

for (2.52d), \[ d = d^{2}_{N1} = d^{3}_{NN} = 3/2, \; j_1 = \frac{1}{2}, j_2 = 0; \; (C.3c) \]

\[ = e(\Lambda^s) \cdot Q_{1,1}^s \cdot (1-t_1t_2^2t_3) = \frac{e(\Lambda^s) (1-t_1t_2^2t_3)}{(1-t_2)(1-t_1t_2)(1-t_2t_3)(1-t_1t_2t_3)}, \]

for (2.52c, e, f), \[ d = 1, \; j_1 = j_2 = 0. \; (C.3d) \]

In the case of \( sl(N) \) the representations are finite-dimensional since we induce from UIRs of \( su(N) \). The character formula is (3.5), which we repeat in order to introduce the corresponding notation:

\[ ch_0 \; L_{\Lambda^u}(r_1, \ldots, r_{N-1}) = \sum_{w \in W_u} (-1)^{\ell(w)} \; ch_0 \; V^{w \cdot \Lambda^u}, \; \Lambda^u \in -\Gamma^u \; (C.4) \]

The index \( u \) is to distinguish the quantities pertinent to the case.
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