RMR-Efficient Randomized Abortable Mutual Exclusion*

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Abstract

Recent research on mutual exclusion for shared-memory systems has focused on local spin algorithms. Performance is measured using the remote memory references (RMRs) metric. As common in recent literature, we consider a standard asynchronous shared memory model with \( N \) processes, which allows atomic read, write and compare-and-swap (short: CAS) operations.

In such a model, the asymptotically tight upper and lower bounds on the number of RMRs per passage through the Critical Section is \( \Theta(\log N) \) for the optimal deterministic algorithms [27, 7]. Recently, several randomized algorithms have been devised that break the \( \Omega(\log N) \) barrier and need only \( o(\log N) \) RMRs per passage in expectation [16, 17, 8]. In this paper we present the first randomized abortable mutual exclusion algorithm that achieves a sub-logarithmic expected RMR complexity. More precisely, against a weak adversary (which can make scheduling decisions based on the entire past history, but not the latest coin-flips of each process) every process needs an expected number of \( O(\log N / \log \log N) \) RMRs to enter and exit the critical section. If a process receives an abort-signal, it can abort an attempt to enter the critical section within a finite number of its own steps and by incurring \( O(\log N / \log \log N) \) RMRs.

1 Introduction

Mutual exclusion, introduced by Dijkstra [11], is a fundamental and well studied problem. A mutual exclusion object (or lock) allows processes to synchronize access to a shared resource. Each process obtains a lock through a capture protocol but at any time, at most one process can own the lock. A process is said to own a lock if it participates in a “capture” protocol designed for the object, and completes it. The owner of the lock can access the shared resource, while all other processes wait in their capture protocol for the owner to “release” the lock. The owner of a lock can execute a release protocol which frees up the lock. The capture protocol and release protocol are often denoted entry and exit section, and a process that owns the lock is in the critical section.

In this paper, we consider the standard cache-coherent (CC) shared model with \( N \) processes that supports atomic read, write, and compare-and-swap (short: CAS) operations. In this model, all shared registers are stored in globally accessible shared memory. In addition, each process has a local cache and a cache protocol ensures coherency. A Remote Memory Reference (short: RMR) is a shared memory access of a register that cannot be resolved locally (i.e., a cache miss). Mutual exclusion algorithms require processes to busy-wait, so the traditional step complexity measure, which counts the number of shared memory accesses, is not useful.

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Early mutual exclusion locks were designed for uniprocessor systems that supported multitasking and time-sharing. A comprehensive survey of these locking algorithms is presented in [25]. One of the biggest shortcomings of these early locking algorithms is that they did not take into account an important hardware technology trend – the steadily growing gap between high processor speeds and the low speed/bandwidth of the processor-memory interconnect [9]. A memory access that traverses the processor-to-memory interconnect, called a remote memory reference, takes much more time than a local memory access.

Recent research [5, 24, 2, 23, 8, 7, 10, 21, 22] on mutual exclusion algorithms therefore focuses on minimizing the number of remote memory references (RMR). The maximum number of RMRs that any process requires (in any execution) to capture and release a lock is called the RMR complexity of the mutual exclusion algorithm. RMR complexity is the metric used to analyze the efficiency of mutual exclusion algorithms, as opposed to the traditional metric of counting steps taken by a process (step complexity). Step complexity is problematic, since for mutual exclusion algorithms, a process may perform an unbounded number of memory accesses (each considered a step) while busy-waiting for another process to release the lock [1].

Algorithms that perform all busy-waiting by repeatedly reading locally accessible shared variables, achieve bounded RMR complexity and have practical performance benefits [5]. Such algorithms are termed local spin algorithms. A comprehensive survey of these algorithms is presented in [4]. Yang and Anderson presented the first $O(\log N)$ RMRs mutual exclusion algorithm [27] using only reads and writes. Anderson and Kim [2] conjectured that this was optimal, and the conjecture was proved by Attiya, Hendler, and Woelfel [7].

Local spin mutual exclusion locks do not meet a critical demand of many systems [26]. Specifically, the locks employed in database systems and in real time systems must support a “timeout” capability which allows a process that waits “too long” to abort its attempt to acquire the lock. The ability of a thread to abort its lock attempt is crucial in database systems; for instance in Oracle’s Parallel Server and IBM’s DB2, this ability serves the dual purpose of recovering from transaction deadlock and tolerating preemption of the thread that holds the lock [26]. In real time systems, the abort capability can be used to avoid overshooting a deadline. Locks that allow a process to abort its attempt to acquire the lock are called abortable locks. Jayanti presented an efficient deterministic abortable lock [21] with worst-case $O(\log N)$ RMR complexity, which is optimal for deterministic algorithms.

In this paper we present the first randomized abortable mutual exclusion algorithm that achieves a sub-logarithmic RMR complexity. Due to the inherent asynchrony in the system, the RMRs incurred by a process during a lock capture and release depend on how the steps of all the processes in the system were scheduled one after the other. Therefore, the maximum RMRs incurred by any process during any lock attempt are determined by the “worst” schedule that makes some process incur a large number of RMRs. To analyze the RMR complexity of lock algorithms, an adversarial scheduler called the adversary is defined. The lower bound of $\Omega(\log N)$ in [7] for mutual exclusion algorithms that use only reads and writes holds for deterministic algorithms where the adversary knows all processes’ future steps. The lower bound does not hold for randomized algorithms where processes flip coins to determine their next steps. Randomized algorithms limit the power of an adversary since the adversary cannot know the result of future coin flips. Adversaries of varying powers have been defined. The most common ones are the oblivious, the weak, and the adaptive adversary [6]. An oblivious adversary makes all scheduling decisions in advance, before any process flips a coin. This model corresponds to a system, where the coin flips made by processes have no influence on the scheduling. A more realistic model is the weak adversary, who sees the coin flip of a process not before that process has taken a step following that coin flip. The adaptive adversary models the strongest adversary with reasonable powers, and it can see every coin flip as it appears,
and can use that knowledge for any future scheduling decisions. Hendler and Woelfel [16] and later Giakkoupis and Woelfel [12] established a tight bound of $\Theta(\log N / \log \log N)$ expected RMR complexity for randomized mutual exclusion against the adaptive adversary. Recently Bender and Gilbert [8] presented a randomized lock that has amortized $O(\log^2 \log N)$ expected RMR complexity against the oblivious adversary. Unfortunately, this algorithm is not strictly deadlock-free (processes may deadlock with small probability, so deadlock has to be expected in a long execution). Our randomized abortable mutual exclusion algorithm is deadlock-free, works against the weak adversary and achieves the same expected RMR complexity as the algorithm by Hendler and Woelfel, namely $O(\log N / \log \log N)$ expected RMR complexity against the weak adversary.

The randomized algorithm we present uses CAS objects and read-write registers. Golab, Hadzilacos, Hendler, and Woelfel [14] (see also [13]) presented an $O(1)$-RMRs implementation of a CAS object using only read-write registers. Moreover, they proved that one can simulate any deterministic shared memory algorithm that uses reads, writes, and conditional operations (such as CAS operations), with a deterministic algorithm that uses only reads and writes, with only a constant increase in the RMR complexity. Recently in [15], Golab, Higham and Woelfel demonstrated that using linearizable implemented objects in place of atomic objects in randomized algorithms allows the adversary to change the probability distribution of results. Therefore, in order to safely use implemented objects in place of atomic ones in randomized algorithms, it is not enough to simply show that the implemented objects are linearizable. Also in [15], it is proved that there exists no general correctness condition for the weak adversary, and that the weak adversary can gain additional power depending on the linearizable implementation of the object. Therefore, in this paper we assume that CAS operations are atomic.

**Abortable Mutual Exclusion.** We formalize the notion of an abortable lock by specifying two methods, lock() and release(), that processes can use to capture and release the lock, respectively. The model assumes that a process may receive a signal to abort at any time during its lock() call. If that happens, and only then, the process may fail to capture the lock, in which case method lock() returns value ⊥. Otherwise the process captures the lock, and method lock() returns a non-⊥ value, and the lock() call is deemed successful. Note that a lock() call may succeed even if the process receives a signal to abort during a lock() call.

Code executed by a process after a successful lock() method call and before a subsequent release() invocation is defined to be its Critical Section. If a process executes a successful lock() call, then the process’s passage is defined to be the lock() call, and the subsequent Critical Section and release() call, in that order. If a process executes an unsuccessful lock() call, then it does not execute the Critical Section or a release() call, and the process’s passage is just the lock() call. Code executed by a process outside of any passage is defined to be its Remainder Section.

The abort-way is defined to be the steps taken by a process during a passage that begins when the process receives a signal to abort and ends when the process returns to its Remainder Section. Since it makes little sense to have an abort capability where processes have to wait for other processes, the abort-way is required to be bounded wait-free (i.e., processes execute the abort-way in a bounded number of their own steps). This property is known as bounded abort. Other properties are defined as follows. **Mutual Exclusion:** At any time there is at most one process in the Critical Section; **Deadlock Freedom:** If all processes in the system take enough steps, then at least one of them will return from its lock() call; **Starvation Freedom:** If all processes in the system take enough steps, then every process will return from its lock() call. The abortable mutual exclusion problem is to implement an object that provides methods lock() and release() such that it that satisfies mutual exclusion, deadlock freedom, and bounded abort.
1.1 Model

Our model of computation, the asynchronous shared-memory model [20] with \( N \) processes which communicate by executing operations on shared objects. Every process executes its program by taking steps, and does not fail. A step is defined to be the execution of all local computations followed by an operation on a shared object. We consider a system that supports atomic read-write registers and \( \text{CAS}() \) objects.

A read-write register \( R \) stores a value from some set and supports two atomic operations \( R.\text{Read()} \) and \( R.\text{Write()} \). Operation \( R.\text{Read()} \) returns the value of the register and leaves its content unchanged, and operation \( R.\text{Write}(v) \) writes the value \( v \) into the register and returns nothing. A \( \text{CAS} \) object \( O \) stores a value from some set and supports two atomic operations \( O.\text{CAS()} \) and \( O.\text{Read()} \). Operation \( O.\text{Read()} \) returns the value stored in \( O \). Operation \( O.\text{CAS}(\text{exp}, \text{new}) \) takes two arguments \( \text{exp} \) and \( \text{new} \) and attempts to change the value of \( O \) from \( \text{exp} \) to \( \text{new} \). If the value of \( O \) equals \( \text{exp} \) then the operation \( O.\text{CAS}(\text{exp}, \text{new}) \) succeeds, and the value of \( O \) is changed from \( \text{exp} \) to \( \text{new} \), and \( \text{true} \) is returned. Otherwise, the operation fails, and the value of \( O \) remains unchanged and \( \text{false} \) is returned.

In addition, a process can execute local coin flip operations that returns an integer value distributed uniformly at random from an arbitrary finite set of integers. The scheduling, generated by the adversary, can depend on the random values generated by the processes. We assume the weak adversary model (see for example [6]) that decides at each point in time the process that takes the next step. In order to make this decision, it can take all preceding events into account, except the results of the most recent coin flips by processes that are yet to execute a shared memory operation after the coin flip.

As mentioned earlier, we consider the cache-coherent (CC) model where each processor has a private cache in which it maintains local copies of shared objects that it accesses. The private cache is logically situated “closer” to the processor than the shared memory, and therefore it can be accessed for free. The shared memory is an external memory accessible to all processors, and is considered remote to all processors. We assume that a hardware protocol ensures cache consistency (i.e., that all copies of the same object in different caches are valid and consistent). A memory access to a shared object that requires access to remote memory is called a remote memory reference (RMR). The RMR complexity of an algorithm is the maximum number of RMRs that a process can incur during any execution of the algorithm.

1.2 Results

We present several building blocks for our algorithm in Section 2. In Sections 3 and 4 we give an overview of the randomized mutual exclusion algorithm. Our results are summarized by the following theorem.

Theorem 1.1. There exists a starvation-free randomized abortable \( N \) process lock against the weak adversary, where a process incurs \( O(\log N/\log \log N) \) RMRs in expectation per passage. The lock requires \( O(N) \) \( \text{CAS} \) objects and read-write registers.

2 Building Blocks

A Randomized \( \text{CAS} \) Counter. A \( \text{CAS} \) counter object with parameter \( k \in \mathbb{Z}^+ \) complements a \( \text{CAS} \) object by supporting an additional \( \text{inc()} \) operation (apart from \( \text{CAS()} \) and \( \text{Read()} \) operations) that increments the object’s value. The object takes values in \( \{0, \ldots, k\} \), and initially the object’s value is 0. Operation \( \text{inc()} \) takes no arguments, and if the value of the object is in \( \{0, \ldots, k-1\} \),
then the operation increments the value and returns the previous value. Otherwise, the value of
the object is unchanged and the integer \( k \) is returned. We will use such an object for \( k = 2 \) to
assign three distinct roles to processes.

Our implementation of the \( \text{inc}() \) operation needs only \( O(1) \) RMRs in expectation. A determin-
istic implementation of a \( \text{CAS} \) counter for \( k = 2 \) and constant worst-case RMR complexity does not
exist: Replacing our randomized \( \text{CAS} \) counter with a deterministic one that has worst-case RMR
complexity \( T \) yields a deterministic abortable mutual exclusion algorithm with worst-case RMR
complexity \( O(T \cdot \log N / \log \log N) \). From the lower bound for deterministic mutual exclusion by
Attiya et al. [7], such an algorithm does not exist, unless \( T = \Omega((\log \log N)^2) \).

In Appendix A we describe a randomized \( \text{CAS} \) counter, called \( \text{RCAScounter}_k \), where the \( \text{inc}() \)
method is allowed to fail. The idea is, that to increase the value of the object, a process randomly
guesses its current value, \( v \), and then executes a \( \text{CAS}(v, v + 1) \) operation. An adaptive adversary
could intervene between the steps involving the random guess and the subsequent \( \text{CAS} \) operation,
thereby affecting the failure probability of an \( \text{inc}() \) method call, but a weak adversary cannot do
so.

**Lemma 2.1.** Object \( \text{RCAScounter}_k \) is a randomized wait-free linearizable \( \text{CAS} \) Counter, where the
probability that an \( \text{inc}() \) method call fails is \( \frac{1}{k+1} \) against the weak adversary. Each of the methods
of \( \text{RCAScounter}_k \) has \( O(1) \) step complexity.

**A Single-Fast-Multi-Slow Universal Construction.** A universal construction object pro-
vides a linearizable concurrent implementation of any object with a sequential specification that
can be given by deterministic code. In Appendix B we devise a universal construction object \( \text{SFMSUnivConst}(T) \)
for \( N \) processes \(^1\) which provides two methods, \( \text{doFast}(op) \) and \( \text{doSlow}(op) \), to perform an operation \( op \) on an object of type \( T \). The idea is that \( \text{doFast}() \) methods cannot be
called concurrently, but are executed very fast, i.e., they have \( O(1) \) step complexity. On the other
hand, \( \text{doSlow}() \) methods need \( O(N) \) steps. The algorithm is based on a helping mechanism in
which \( \text{doSlow}() \) methods help a process that wants to execute a \( \text{doFast}() \) method.

**Lemma 2.2.** Object \( \text{SFMSUnivConst}(T) \) is a wait-free universal construction that implements an
object \( O \) of type \( T \), for \( N \) processes, and an operation \( op \) on object \( O \) is performed by executing either
method \( \text{doFast}(op) \) or \( \text{doSlow}(op) \), and no two processes execute method \( \text{doFast}() \) concurrently.
Methods \( \text{doFast}() \) and \( \text{doSlow}() \) have \( O(1) \) and \( O(N) \) step complexity respectively.

**The Abortable Promotion Array.** An object \( O \) of type \( \text{AbortableProArray}_k \) stores a vector
of \( k \) integer pairs. It provides some specialized operations on the vector, such as conditionally
adding/removing elements, and earmarking a process (associated with an element of the vector) for
some future activity. Initially the value of \( O = (O[0], O[1], \ldots, O[k-1]) \) is \((0, \bot), \ldots, (0, \bot)) \).

The object supports operations \( \text{collect}(), \text{abort}(), \text{promote}(), \text{remove}() \) and \( \text{reset}() \) (see Figure 5
in the appendix). Operation \( \text{collect}(X) \) takes as argument an array \( X[0 \ldots k-1] \) of integers, and
is used to “register” processes into the array. The operation changes \( O[i] \), for all \( i \in \{0, \ldots, k-1\} \),
to value \( \langle \text{REG}, X[i] \rangle \) except if \( O[i] \) is \( \langle \text{ABORT}, s \rangle \), for some \( s \in Z \). In the latter case the value of
\( O[i] \) is unchanged. Process \( i \) is said to be registered in the array if a \( \text{collect}() \) operation changes
\( O[i] \) to value \( \langle \text{REG}, s \rangle \), for some \( s \in Z \). The object also allows processes to “abort” themselves from

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\(^1\) For the DSM model, this also follows from a result by Golab, Hadzilacos, Hendler, and Woelfel [14]. They
established a super-constant lower bound on the RMR complexity of a deterministic bounded counter that can count
up to two, and also supports a reset operation.

\(^2\) We use the universal construction object for smaller sets of processes, specifically for sets of size
\( O(\log N / \log \log N) \).
the array using the operation \texttt{abort()}. Operation \texttt{abort}(i, s) takes as argument the integers \(i\) and \(s\), where \(i \in \{0, \ldots, k - 1\}\) and \(s \in \mathbb{Z}\). The operation changes \(O[i]\) to value \(\langle \text{ABORT}, s \rangle\) and returns \texttt{true}, only if \(O[i]\) is not equal to \(\langle \text{PRO}, s' \rangle\), for some \(s' \in \mathbb{Z}\). Otherwise the operation returns \texttt{false}. Process \(i\) \textit{aborts} from the array if it executes an \texttt{abort}(i, s) operation that returns \texttt{true}. A registered process in the array that has not aborted can be “promoted” using the \texttt{promote()} operation. Operation \texttt{promote()} takes no arguments, and changes the value of the element in \(O\) with the smallest index and that has value \(\langle \text{REG}, s \rangle\), for some \(s \in \mathbb{Z}\), to value \(\langle \text{PRO}, s \rangle\), and returns \(\langle i, s \rangle\), where \(i\) is the index of that element. If there exists no element in \(O\) with value \(\langle \text{REG}, s \rangle\), for some \(s \in \mathbb{Z}\), then \(O\) is unchanged and the value \(\langle \perp, \perp \rangle\) is returned. Process \(i\) is \textit{promoted} if a \texttt{promote()} operation returns \(\langle i, s \rangle\), for some \(s \in \mathbb{Z}\). Operation \texttt{reset()} resets the entire array to its initial state.

Note that an aborted process in the array, cannot be registered into the array or be promoted, until the array is reset. If a process tries to abort itself from the array but finds that it has already been promoted, then the abort fails. This ensures that a promoted process takes responsibility for some activity that other processes expect of it.

In the context of our abortable lock, the \(i\)-th element of the array stores the current state of process with ID \(i\), and a sequence number associated with the state. Operation \texttt{collect()} is used to register a set of participating processes into the array. Operation \texttt{abort}(i, s) is executed only by process \(i\), to abort from the array. Operation \texttt{promote()} is used to promote an unaborted registered process from the array, so that the promoted process can fulfill some future obligation.

In our abortable lock of Section 3 we need a wait-free linearizable implementation of type \texttt{AbortableProArray_{\Delta}}, where \(\Delta\) is the maximum number of processes that can access the object concurrently, and we achieve this by using object \texttt{SFMSUnivConst(AabortableProArray_{\Delta})}. We ensure that no two processes execute operations \texttt{collect()}, \texttt{promote()}, \texttt{reset()} or \texttt{remove()} concurrently, and therefore by we get \(O(1)\) step complexity for these operations by using method \texttt{doFast()}. Operation \texttt{abort()} has \(O(\Delta)\) step complexity since it is performed using method \texttt{doSlow()}, which allows processes to call \texttt{abort()} concurrently.

### 3 The Tree Based Abortable Lock

Our abortable lock algorithm is based on an arbitration tree with branching factor approximately \(\Theta(\log N/\log \log N)\). For convenience we assume (w.l.o.g.) that \(N = \Delta^{\Delta - 1}\) for some positive integer \(\Delta\), where \(N\) is the maximum number of processes in the system. Then it follows that \(\Delta = \Theta(\log N/\log \log N)\).

As in the algorithm by Hendler and Woelfel [16], we consider a tree with \(N\) leaves and where each non-leaf node has \(\Delta\) children. Every non-leaf node is associated with a lock. Each process is assigned a unique leaf in the tree and climbs up the tree by capturing the locks on nodes on its path until it has captured the lock at the root. Once a process locks the root, it can enter the Critical Section.

The main difficulty is that of designing the locks associated with the nodes of the tree. A simple \texttt{CAS} object together with an “announce array” as used in [16] does not work. Suppose a process \(p\) captures locks of several nodes on its path up to the root and aborts before capturing the root lock. Then it must release all captured node locks and therefore these lock releases cause other processes, which are busy-waiting on these nodes, to incur RMRs. So we need a mechanism to guarantee some progress to these processes, while we also need a mechanism that allows busy-waiting processes to abort their attempts to capture node locks. In [16] progress is achieved as follows: A process \(p\), before releasing a lock on its path, searches(with a random procedure) for other processes that are
busy-waiting for the node lock to become free. If \( p \) finds such a process, it promotes it into the critical section. This is possible, because at the time of the promotion \( p \) owns the root lock and can hand it over to a promoted process. Unfortunately, this promotion mechanism fails for abortable mutual exclusion: When \( p \) aborts its own attempt to enter the Critical Section, it may have to release node locks at a time when it doesn’t own the root lock. Another problem is that if \( p \) finds a process \( q \) that is waiting for \( p \) to release a node-lock, then \( q \) may have already decided to abort. We use a carefully designed synchronization mechanism to deal with such cases.

To ensure that waiting processes make some progress, we desire that \( p \) “collect” busy-waiting processes (if any) at a node into an instance of an object of type \( \text{AbortableProArray}_\Delta, \text{PawnSet} \), using the operation \( \text{collect()} \). Once busy-waiting processes are collected into \( \text{PawnSet} \), \( p \) can identify a busy-waiting process, if present, using the \( \text{PawnSet.promote()} \) operation, while busy-waiting processes themselves can abort using the \( \text{PawnSet.abort()} \) operation. Note that \( p \) may have to read \( O(\Delta) \) registers just to find a single busy-waiting process at a node, where \( \Delta \) is the branching factor of the arbitration tree. This is problematic since our goal is to bound the number of steps during a passage to \( O(\Delta) \) steps, and thus a process cannot collect at more than one node. For this reason we desire that \( p \) transfer all unreleased node locks that it owns to the first busy-waiting process it can find, and then it would be done. And if there are no busy-waiting processes at a node, then \( p \) should somehow be able to release the node lock in \( O(1) \) steps. Since there are at most \( \Delta \) nodes on a path to the root node, \( p \) can continue to release captured node locks where there are no busy-waiting processes, and thus not incur more than \( O(\Delta) \) overall. We use an instance of \( \text{RCAScounter}_2, \text{Ctr} \), to help decide if there are any busy-waiting processes at a node lock. Initially, \( \text{Ctr} \) is 0, and processes attempt to increase \( \text{Ctr} \) using the \( \text{Ctr.inc()} \) operation after having registered at the node. Process \( p \) attempts to release a node lock by first executing a \( \text{Ctr.CAS}(1,0) \) operation. If the operation fails then some process \( q \) must have further increased \( \text{Ctr} \) from 1 to 2, and thus \( p \) can transfer all unreleased locks to \( q \), if \( q \) has not aborted itself. If \( q \) has aborted, then \( q \) can perform the collect at the node lock for \( p \), since \( q \) can afford to incur an additional one-time expense of \( O(\Delta) \) RMRs. If \( q \) has not aborted then \( p \) can transfer its captured locks to \( q \) in \( O(1) \) steps, and thus making sure some process makes progress towards capturing the root lock. We encapsulate these mechanisms in a randomized abortable lock object, \( \text{ALockArray}_\Delta \).

More generally, we specify an object \( \text{ALockArray}_n \) for an arbitrary parameter \( n < N \). Object \( \text{ALockArray}_n \) provides methods \( \text{lock()} \) and \( \text{release()} \) that can be accessed by at most \( n + 1 \) processes concurrently. The object is an abortable lock, but with an RMR complexity of \( O(n) \) for the abort-way, and constant RMR complexity for \( \text{lock()} \). The \( \text{release()} \) method is special. If it detects contention (i.e., other processes are busy-waiting), then it takes \( O(n) \) RMRs, but helps those other processes to make progress. Otherwise, it takes only \( O(1) \) RMRs. Each non-leaf node \( u \) in our arbitration tree will be associated with a lock \( \text{ALockArray}_\Delta \) and can only be accessed concurrently by the processes owning locks associated with the children of \( u \) and one other process.

Method \( \text{lock()} \) takes a single argument, which we will call pseudo-ID, with value in \( \{0, \ldots, n - 1\} \). We denote a \( \text{lock()} \) method call with argument \( i \) as \( \text{lock}_i() \), but refer to \( \text{lock}_i() \) as \( \text{lock}() \) whenever the context of the discussion is not concerned with the value of \( i \). Method \( \text{lock}() \) returns a non-\( \bot \) value if a process captures the lock, otherwise it returns a \( \bot \) value to indicate a failed \( \text{lock}() \) call. A \( \text{lock}() \) by process \( p \) can fail only if \( p \) aborts during the method call. Method \( \text{release()} \) takes two arguments, a pseudo-ID \( i \in \{0, \ldots, n - 1\} \) and an integer \( j \). Method \( \text{release}_i(j) \) returns \text{true} if and only if there exists a concurrent call to \( \text{lock}() \) that eventually returns \( j \). Otherwise method \( \text{release}_i(j) \) returns \text{false}. The information contained in argument \( j \) determines the transferred node locks. Process pseudo-IDs are passed as arguments to the methods to allow the ability for a process to “transfer” the responsibility of releasing the lock to another process. Specifically, we desire that if a process \( p \) executes a successful \( \text{lock}_i() \) call and becomes
the owner of the lock, then \( p \) does not have to release the lock itself, if it can find some process \( q \) to call \( \text{release}_q() \) on its behalf. In Section 4 we implement object \( \text{ALockArray}_n \), and prove its properties in Appendix D.2, and thus we get the following lemma.

**Lemma 3.1.** Object \( \text{ALockArray}_n \) can be implemented against the weak adversary for the CC model with the following properties using only \( O(n) \) CAS objects and read-write registers.

(a) Mutual exclusion, starvation freedom, bounded exit, and bounded abort.

(b) The abort-way has \( O(n) \) RMR complexity.

(c) If a process does not abort during a \( \text{lock}() \) call, then it incurs \( O(1) \) RMRs in expectation during the call, otherwise it incurs \( O(n) \) RMRs in expectation during the call.

(d) If a process’ call to \( \text{release}(j) \) returns \text{false}, then it incurs \( O(1) \) RMRs during the call, otherwise it incurs \( O(n) \) RMRs during the call.

High Level Description of the Abortable Lock. We use a complete \( \Delta \)-ary tree \( T \) of height \( \Delta \) with \( N \) leaves, called the arbitration tree. The root has height \( \Delta \) and the leaves of the tree have height 0. The \( N \) processes in the system line up as \( \text{ALockArray}_n \). Process \( p \) attempts to capture a node \( u \) on its path \( \text{path}_p \) that it does not own, as long as \( p \) has not received a signal to abort. Process \( p \) attempts to capture a node \( u \) by executing a call to \( u.\text{lock}() \). If \( p \)'s \( u.\text{lock}() \) call returns \( \infty \) then \( p \) is said to have captured \( u \), and if the call returns an integer \( j \), then \( p \) is said to have been handed over all nodes from \( u \) to \( v \) on \( \text{path}_p \), where \( h_v = j \). We ensure that \( j \geq h_u \). Process \( p \) starts to own node \( u \) when \( p \) captures \( u.L \) or when \( p \) is handed over node \( u \) from the previous owner of node \( u \). Process \( p \) can enter its Critical Section when it owns the root node of \( T \). Process \( p \) may receive a signal to abort during a call to \( u.L.\text{lock}() \) as a result of which \( p \)'s call to \( u.L.\text{lock}() \) returns either \( \bot \) or a non-\( \bot \) value. In either case, \( p \) then calls \( \text{release}_p() \) to release all locks of nodes that \( p \) has captured in its passage, and then returns from its \( \text{lock}_p() \) call with value \( \bot \).

Lock capture protocol - \( \text{lock}_p() \). During \( \text{lock}_p() \) a process \( p \) attempts to capture every node on its path \( \text{path}_p \) that it does not own, as long as \( p \) has not received a signal to abort. Process \( p \) attempts to capture a node \( u \) by executing a call to \( u.\text{lock}() \). If \( p \)'s \( u.\text{lock}() \) call returns \( \infty \) then \( p \) is said to have captured \( u \), and if the call returns an integer \( j \), then \( p \) is said to have been handed over all nodes from \( u \) to \( v \) on \( \text{path}_p \), where \( h_v = j \). We ensure that \( j \geq h_u \). Process \( p \) starts to own node \( u \) when \( p \) captures \( u.L \) or when \( p \) is handed over node \( u \) from the previous owner of node \( u \). Process \( p \) can enter its Critical Section when it owns the root node of \( T \). Process \( p \) may receive a signal to abort during a call to \( u.L.\text{lock}() \) as a result of which \( p \)'s call to \( u.L.\text{lock}() \) returns either \( \bot \) or a non-\( \bot \) value. In either case, \( p \) then calls \( \text{release}_p() \) to release all locks of nodes that \( p \) has captured in its passage, and then returns from its \( \text{lock}_p() \) call with value \( \bot \).

Lock release protocol - \( \text{release}_p() \). An exiting process \( p \) releases all nodes that it owns during \( \text{release}_p() \). Process \( p \) is said to release node \( u \) if \( p \) releases \( u.L \) (by executing \( u.L.\text{release}() \) call), or if \( p \) hands over node \( u \) to some other process. Recall that \( p \) hands over node \( u \) if \( p \) executes a \( v.L.\text{release}(j) \) call that returns \text{true} \) where \( h_v \leq h_u \leq j \). Let \( s \) be the height of the highest node \( p \) owns. During \( \text{release}_p() \), \( p \) climbs up \( T \) and calls \( u.L.\text{release}_p(s) \) at every node \( u \) that it owns, until a call returns \text{true}. If a \( u.L.\text{release}_p(s) \) call returns \text{false} \) (process \( p \) incurs \( O(1) \) steps), then \( p \) is said to have released lock \( u.L \) (and therefore released node \( u \)), and thus \( p \) continues on its path. If a \( u.L.\text{release}_p(s) \) call returns \text{true} \) (process \( p \) incurs \( O(\Delta) \) steps), then \( p \) has handed over all remaining nodes that it owns to some process that is executing a concurrent \( u.L.\text{lock}() \) call at node \( u \), and thus \( p \) does not release any more nodes.

Notice that our strategy to release node locks is to climb up the tree until all node locks are released or a hand over of remaining locks is made. Climbing up the tree is necessary (as opposed to climbing down) in order to hand over node locks to a process, say \( q \), such that the handed over nodes lie on \( \text{path}_q \).
4 The Array Based Abortable Lock

We specified object $\text{ALockArray}_n$ in Section 3 and now we describe and implement it (see Figures 1 and 2). Let $L$ be an instance of object $\text{ALockArray}_n$.

**Registering and Roles at lock $L$.** At the beginning of a $\text{lock}()$ call processes register themselves in the $\text{apply}$ array by swapping the value $\text{REG}$ atomically into their designated slots ($\text{apply}[i]$ for process with pseudo-ID $i$) using a $\text{CAS}$ operation. The array $\text{apply}$ of $n$ $\text{CAS}$ objects is used by processes to register and “deregister” themselves from lock $L$, and to notify each other of certain events at lock $L$.

On registering in the $\text{apply}$ array, processes attempt to increase $\text{Ctr}$, an instance of $\text{RCAScounter}_2$, using operation $\text{Ctr}.\text{inc}()$. Recall that $\text{RCAScounter}_2$ is a bounded counter, initially 0, and returns values in $\{0, 1, 2\}$ (see Section 2). Each of these values corresponds to a role at lock $L$. There are four roles that a process can assume during its passage of lock $L$, namely king, queen, pawn and promoted pawn, and a role defines the protocol a process follows during a passage. During an execution, $\text{Ctr}$ cycles from its initial value 0 to non-0 values and then back to 0, multiple times, and we refer to each such cycle as a $\text{Ctr}$-cycle. The process that increases $\text{Ctr}$ from 0 to 1 becomes the king. The process that increases $\text{Ctr}$ from 1 to 2 becomes the queen. All processes that attempt to increase $\text{Ctr}$ any further, are returned value 2 (by specification of object $\text{RCAScounter}_2$), and they assume the role of a pawn process. A pawn process busy-waits until it gets “promoted” at lock $L$ (a process is said to be promoted at lock $L$ if it is promoted in $\text{PawnSet}$), or until it sees the $\text{Ctr}$ value decrease, so that it can attempt to increase $\text{Ctr}$ again. We ensure that a pawn process repeats an attempt to increase $\text{Ctr}$ at most once, before getting promoted. We ensure that at any point in time during the execution, the number of processes that have assumed the role of a king, queen and promoted pawn at lock $L$, respectively, is at most one, and thus we refer to them as $\text{king}_L$, $\text{queen}_L$ and $\text{ppawn}_L$, respectively. We describe the protocol associated with each of the roles in more detail shortly. An array Role of $n$ read-write registers is used by processes to record their role at lock $L$.

**Busy-waiting in lock $L$.** The king process, $\text{king}_L$, becomes the first owner of lock $L$ during the current $\text{Ctr}$-cycle, and can proceed to enter its Critical Section, and thus it does not busy-wait during $\text{lock}()$. The queen process, $\text{queen}_L$, must wait for $\text{king}_L$ for a notification of its turn to own lock $L$. Then $\text{queen}_L$ spins on $\text{CAS}$ object $\text{Sync1}$, waiting for $\text{king}_L$ to $\text{CAS}$ some integer value into $\text{Sync1}$. Process $\text{king}_L$ attempts to $\text{CAS}$ an integer $j$ into $\text{Sync1}$ only during its call to $\text{release}(j)$, after it has executed its Critical Section. The pawn processes wait on their individual slots of the $\text{apply}$ array for a notification of their promotion.

**A collect action at lock $L$.** A collect action is conducted by either $\text{king}_L$ during a call to $\text{release}()$, or by $\text{queen}_L$ during a call to $\text{abort}()$. A collect action is defined as the sequence of steps executed by a process during a call to $\text{doCollect}()$. During a call to $\text{doCollect}()$, the collecting process (say $q$) iterates over the array $\text{apply}$ reading every slot, and then creates a local array $A$ from the values read and stores the contents of $A$ in the $\text{PawnSet}$ object in using the operation $\text{PawnSet}.\text{collect}(A)$. A key point to note is that operation $\text{PawnSet}.\text{collect}(A)$ does not overwrite an aborted process’s value in $\text{PawnSet}$ (a process aborts itself in $\text{PawnSet}$ by executing a successful $\text{PawnSet}.\text{abort}()$ operation).

**A promote action at lock $L$.** Operation $\text{PawnSet}.\text{promote}()$ during a call to method $\text{doPromote}()$ is defined as a promote action. The operation returns the pseudo-ID of a process that was collected during a collect action, and has not yet aborted from $\text{PawnSet}$. A promote action is conducted at lock $L$ either by $\text{king}_L$, $\text{queen}_L$ or $\text{ppawn}_L$.

**Lock handover from $\text{king}_L$ to $\text{queen}_L$.** As mentioned, process $\text{queen}_L$ waits for $\text{king}_L$ to finish its Critical Section and then call $\text{release}(j)$. During $\text{king}_L$’s $\text{release}(j)$ call, $\text{king}_L$ attempts to swap integer $j$ into $\text{CAS}$ object $\text{Sync1}$, that only $\text{king}_L$ and $\text{queen}_L$ access. If $\text{queen}_L$ has not
Object ALockArrayₙ

shared:
- Ctr: RCAScounter, init 0;
- PawnSet: Object of type AbortableProArrayₙ, init ∅;
- apply: array [0...n − 1] of int pairs, init all (⊥, ⊥);
- Role: array [0...n − 1] of int, init ⊥;
- Sync1, Sync2: int, init ⊥;
- KING, QUEEN, PAWN, PAWN_P, REG, PRO: const int 0, 1, 2, 3, 4, 5 respectively;
- getSequenceNo(): returns integer k on being called for the k-th time from a call to lockᵢ(). (Since calls to lockᵢ() are executed sequentially, a sequential shared counter suffices to implement method getSequenceNo().)

local:
- s, val, seq, dummy: int, init ⊥;
- flag, r: boolean, init false;
- A: array [0...n − 1] of int, init ⊥;

// If process 𝑖 satisfies the loop condition in line 2, 7, or 14, and 𝑖 has received a signal to abort, then 𝑖 calls abortᵢ()

Method lockᵢ()

1. \( s ← \text{getSequenceNo}() \)
2. \( \text{await}(\text{apply}[i].\text{CAS}((⊥, ⊥), ⟨\text{REG}, s⟩)) \)
3. flag ← true
4. repeat
5.   Role[𝑖] ← Ctr.inc()
6.   if (Role[𝑖] = PAWN) then
7.     await \( (\text{apply}[i] = ⟨\text{PRO}, s⟩) ∨ \text{Ctr.Read}() ≠ 2 \)
8.     if (apply[𝑖] = ⟨PRO, s⟩) then
9.       Role[𝑖] ← PAWN_P
10.  end
11. until (Role[𝑖] ∈ {KING, QUEEN, PAWN_P})
12. if (Role[𝑖] = QUEEN) then
13.   await (Sync1 ≠ ⊥)
14. end
15. apply[i].CAS(⟨REG, s⟩, ⟨PRO, s⟩)
16. if (Role[i] = QUEEN) then return Sync1 else return ∞

Method abortᵢ()

18. if ¬flag then return ⊥
19. apply[i].CAS(⟨REG, s⟩, ⟨PRO, s⟩)
20. if Role[i] = PAWN then
21.   if ¬PawnSet.abort(i, s) then
22.     Role[i] ← PAWN_P
23.     return ∞
24. end
25. else
26.   if ¬Sync1.CAS(⊥, ∞) then
27.     return Sync1
28. end
29. doCollectᵢ()
30. helpReleaseᵢ()
31. end
32. apply[i].CAS(⟨PRO, s⟩, ⟨⊥, ⊥⟩)
33. return ⊥

Method doCollectᵢ()

51. for \( k ← 0 \) to \( n − 1 \) do
52.   ⟨val, seq⟩ ← apply[k]
53.   if val = REG then \( A[k] ← \text{seq} \) else \( A[k] ← ⊥ \)
54. end
55. PawnSet.collect(A)

Figure 1: Implementation of Object ALockArrayₙ

10
now takes on the responsibility of collecting all registered processes in locking to prevent into the PawnSet. In this case, king aborted, and thus its by king into Sync1 into king to apply that Section. from the apply ⟨⊥⟩ = helpRelease ⟨dummy, s⟩ ← apply[i] apply[i].CAS(⟨PRO, s⟩, ⟨⊥, ⊥⟩) return r

Method release_i(int j)

“aborted”, then king_L successfully swaps j into Sync1, and this serves as a notification to queen_L that king_L has completed its Critical Section, and that queen_L may now proceed to enter its Critical Section.

**Aborting an attempt at lock L by queen_L.** On receiving a signal to abort, queen_L abandons its lock() call and executes a call to abort() instead. queen_L first changes the value of its slot in the apply array from REG to PRO, to prevent itself from getting collected in future collects. Since king_L and queen_L are the first two processes at L, king_L will eventually try to handover L to queen_L. To prevent king_L from handing over lock L to queen_L, queen_L attempts to swap a special value ∞ into Sync1 in one atomic step. If queen_L fails then this implies that king_L has already handed over L to queen_L, and thus queen_L returns from its call to abort() with the value written to Sync1 by king_L, and becomes the owner of L. If queen_L succeeds then queen_L is said to have successfully aborted, and thus king_L will eventually fail to hand over lock L. Since queen_L has aborted, queen_L now takes on the responsibility of collecting all registered processes in lock L, and storing them into the PawnSet object. After performing a collect, queen_L then synchronizes with king_L again, to perform a promote, where one of the collected processes is promoted. After that, queen_L deregisters from the apply array by resetting its slot to the initial value ⟨⊥, ⊥⟩.

**Aborting an attempt at lock L by a pawn process.** On receiving a signal to abort a pawn process (say p) busy-waiting in lock L, abandons its lock() call and executes a call to abort() instead. Process p first changes the value of its slot in the apply array from REG to PRO, to prevent itself from getting collected in future collects. It then attempts to abort itself in PawnSet by executing the operation PawnSet.abort(p)). If p’s attempt is unsuccessful then it implies that p has already been promoted in PawnSet, and thus p can assume the role of a promoted pawn, and become the owner of L. In this case, p returns from its abort() call with value ∞ and becomes the owner of L. If p’s attempt is successful then p cannot be collected or promoted in future collects and promotion events. In this case, p deregisters from the apply array by resetting its slot to the

```
Method release_i(int j)
34  r ← false
35  if Role[i] = KING then
36      if ¬Ctr.CAS(1, 0) then
37          r ← Sync1.CAS(⊥, j)
38          if r then doCollect();
39          helpRelease_i();
40      end
41  end
42  if Role[i] = QUEEN then
43      helpRelease_i();
44  end
45  if Role[i] = PAWN_P then
46      doPromote_i();
47  end
48  (dummy, s) ← apply[i]
49  apply[i].CAS(⟨PRO, s⟩, ⟨⊥, ⊥⟩)
50  return r

Method helpRelease_i()
56  if ¬Sync2.CAS(⊥, i) then
57      j ← Sync1.Read()
58      Sync1.CAS(j, ⊥)
59      j ← Sync2.Read()
60      Sync2.CAS(j, ⊥)
61      PawnSet.remove(j)
62      doPromote_i()
63 end

Method doPromote_i()
64      PawnSet.remove(i)
65      (j, seq) ← PawnSet.promote()
66  if j = ⊥ then
67      PawnSet.reset()
68      Ctr.CAS(2, 0)
69  else
70      apply[j].CAS(⟨REG, seq⟩, ⟨PRO, seq⟩)
71 end

Figure 2: Implementation of Object ALockArray_n (continued)
initial value \(\langle \perp, \perp \rangle\), and returns \(\perp\) from its call to \texttt{abort()}.  

**Releasing lock \(L\).** Releasing lock \(L\) can be thought of as a group effort between the \texttt{king\(_L\)}, \texttt{queen\(_L\)} (if present at all), and the promoted pawns (if present at all). To completely release lock \(L\), the owner of \(L\) needs to reset \(\texttt{Ctr}\) back to 0 for the next \(\texttt{Ctr}\)-cycle to begin. However, the owner also has an obligation to hand over lock \(L\) to the next process waiting in line for lock \(L\). We now discuss the individual strategies of releasing lock \(L\), by \texttt{king\(_L\)}, \texttt{queen\(_L\)} and the promoted processes. To release lock \(L\), the owner of \(L\) executes a call to \texttt{release}(j), for some integer \(j\).

**Synchronizing the release of lock \(L\) by \texttt{king\(_L\)} and \texttt{queen\(_L\)}.** Process \texttt{king\(_L\)} first attempts to decrease \(\texttt{Ctr}\) from 1 to 0 using a CAS operation. If it is successful, then \texttt{king\(_L\)} was able to end the \(\texttt{Ctr}\)-cycle before any process could increase \(\texttt{Ctr}\) from 1 to 2. Thus, there was no \texttt{queen\(_L\)} process or pawn processes waiting for their turn to own lock \(L\), during that \(\texttt{Ctr}\)-cycle. Then \texttt{king\(_L\)} is said to have released lock \(L\).

If \texttt{king\(_L\)}’s attempt to decrease \(\texttt{Ctr}\) from 1 to 0 fails, then \texttt{king\(_L\)} knows that there exists a \texttt{queen\(_L\)} process that increased \(\texttt{Ctr}\) from 1 to 2. Since \texttt{queen\(_L\)} is allowed to abort, releasing lock \(L\) is not as straightforward as raising a flag to be read by \texttt{queen\(_L\)}. Therefore, \texttt{king\(_L\)} attempts to synchronize with \texttt{queen\(_L\)} by swapping the integer \(j\) into the object \texttt{Sync1} using a \texttt{Sync1.CAS}(\(\perp, j\)) operation. Recall that \texttt{queen\(_L\)} also attempts to swap a special value \(\infty\) into object \texttt{Sync1} using a \texttt{Sync1.CAS}(\(\perp, j\)) operation, in order to abort its attempt. Clearly only one of them can succeed. If \texttt{king\(_L\)} succeeds, then \texttt{king\(_L\)} is said to have successfully handed over lock \(L\) to \texttt{queen\(_L\)}. If \texttt{king\(_L\)} fails, then \texttt{king\(_L\)} knows that \texttt{queen\(_L\)} has aborted and thus \texttt{king\(_L\)} then tries to hand over its lock to one of the waiting pawn processes. The procedure to hand over lock \(L\) to one of the waiting pawn processes is to execute a collect action followed by a promote action.

The collect action needs to be executed only once during a \(\texttt{Ctr}\)-cycle, and thus we let the process (among \texttt{king\(_L\)} or \texttt{queen\(_L\)}) that successfully swaps a value into \texttt{Sync1}, execute the collect action.

If \texttt{king\(_L\)} successfully handed over \(L\) to \texttt{queen\(_L\)}, it collects the waiting pawn processes, so that eventually when \texttt{queen\(_L\)} is ready to release lock \(L\), \texttt{queen\(_L\)} can simply execute a promote action. Since there is no guarantee that \texttt{king\(_L\)} will finish collecting before \texttt{queen\(_L\)} desires to execute a promote action, the processes synchronize among themselves again, to execute the first promote action of the current \(\texttt{Ctr}\)-cycle. They both attempt to swap their pseudo-IDs into an empty CAS object \texttt{Sync2}, and therefore only one can succeed. The process that is unsuccessful, resets \texttt{Sync1} and \texttt{Sync2} to their initial value \(\perp\), and then executes the promote action, where a waiting pawn process is promoted and handed over lock \(L\). If no process were collected during the \(\texttt{Ctr}\)-cycle, or all collected pawn processes have successfully aborted before the promote action, then the promote action fails, and thus the owner process resets the \texttt{PawnSet} object, and then resets \(\texttt{Ctr}\) from 2 to 0 in one atomic step, thus releasing lock \(L\), and resetting the \(\texttt{Ctr}\)-cycle.

**The release of lock \(L\) by \texttt{ppawn\(_L\)}.** If a process was promoted by \texttt{king\(_L\)} or \texttt{queen\(_L\)} as described above, then the promoted process is said to be handed over the ownership of \(L\), and becomes the first promoted pawn of the \(\texttt{Ctr}\)-cycle. Since a collect for this \(\texttt{Ctr}\)-cycle has already been executed, process \texttt{ppawn\(_L\)} does not execute any more collects, but simply attempts to hand over lock \(L\) to the next collected process by executing a promote action. This sort of promotion and handing over of lock \(L\) continues until there are no more collected processes to promote, at which point the last promoted pawn resets the \texttt{PawnSet} object, and then resets \(\texttt{Ctr}\) from 2 to 0 in one atomic step, thus releasing lock \(L\), and resetting the \(\texttt{Ctr}\)-cycle.

All owner processes also \texttt{deregister} themselves from lock \(L\), by resetting their slot in the \texttt{apply} array to the initial value \(\langle \perp, \perp \rangle\). This step is the last step of their \texttt{release}(j) calls, and processes return a boolean to indicate whether they successfully wrote integer \(j\) into \texttt{Sync1} during their
release\((j)\) call. Note that only king\(_L\) could possibly return true since it is the only process that attempts to do so, during its release\((j)\) call.

5 Conclusion

We presented the first randomized abortable lock that achieves sub-logarithmic expected RMR complexity. While the speed-up is only a modest \(O(\log \log n)\) factor over the most efficient deterministic abortable mutual exclusion algorithm, our result shows that randomization can help in principle, to improve the efficiency of abortable locks. Unfortunately, our algorithm is quite complicated; it would be nice to find a simpler one. It would also be interesting to find an algorithm with sub-logarithmic RMR complexity that works against the stronger adversary. In the weak adversary model, no non-trivial lower bounds for mutual exclusion are known, but it seems hard to improve upon \(O(\log n / \log \log n)\) RMR complexity, even without the abortability property.

As shown by Bender and Gilbert, \[8\], the picture looks different in the oblivious adversary model. However, their algorithm is only lock-free with high probability. It would be interesting to find a mutual exclusion algorithm with \(o(\log n / \log \log n)\) RMR complexity against the oblivious adversary that is lock-free with probability one. It would also be interesting to know whether such an algorithm can be made abortable.

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Appendix

A Implementation of Object RCAScounter\(k\)

The sequential specification of the CAS Counter object is presented in Figure 3 in the form of type CAScounter\(k\). The implementation of our randomized CAS counter object, RCAScounter\(k\) of type CAScounter\(k\) is presented in Figure 4. A shared CAS object Count is used to store the value of the counter object, and is initialized to 0. The object provides methods inc(), CAS() and Read(), where the inc() method is allowed to fail, in which case the operation does not change the object state, and returns \(\perp\) to indicate the failure.

### Figure 3: Sequential Specification of Type CAScounter\(k\)

| Type CAScounter\(k\) | Operation inc() | Operation CAS(old, new) | Operation Read() |
|----------------------|------------------|------------------------|------------------|
| \(x: \text{int init } 0\) | 1 if \(x = k\) then return \(x\) | 4 if \(x \neq \text{old} \lor \text{new} \notin \{0, \ldots, k\}\) then return false | 7 return \(x\) |
| | 2 \(x \leftarrow x + 1\) | 5 \(x \leftarrow \text{new}\) | |
| | 3 return \(x - 1\) | 6 return true | |

### Figure 4: Implementation of Object RCAScounter\(k\)

| Object RCAScounter\(k\) | Method inc( ) | Method CAS(old, new) | Method Read( ) |
|--------------------------|---------------|---------------------|----------------|
| shared: Count: int init 0 | 8 \(\beta \leftarrow \text{random}(0, 1, \ldots, k)\) | 15 if \(\text{new} \notin \{0, \ldots, k\}\) then return false | 17 return Count.Read() |
| local: \(\beta: \text{int init } 0\) | 9 if \(\beta = k\) then | 16 return Count.CAS(old, new) | |
| | 10 | if (Count.Read() = k) then return \(k\) | |
| | 11 else | 12 if Count.CAS(\(\beta, \beta + 1\)) then return \(\beta\) | |
| | 13 end | 14 return \(\perp\) | |

During the inc() method, a process \(p\) first makes a guess at the counter’s current value by rolling a \((k + 1)\)-sided dice (in line 8) that returns a value in \(\{0, \ldots, k\}\) uniformly at random, and stores the value in local variable \(\beta\). If \(\beta = k\), then \(p\) performs a Read() on Count(in line 10) to verify the correctness of its guess. If \(p\)’s guess is correct, then it returns \(k\), otherwise it returns \(\perp\) (in line 14) to indicate a failed inc() method call. If \(\beta \in \{0, \ldots, k - 1\}\), then \(p\) performs a Count.CAS(\(\beta, \beta + 1\)) operation (in line 10) in order to verify the correctness of its guess and to
increment \( \text{Count} \) in one atomic step. If \( p \)'s guess is correct, then the \( \text{CAS} \) operation succeeds and the \( \text{inc()} \) method returns the previous value. Otherwise the \( \text{inc()} \) method returns \( \bot \) (in line 14) to indicate a failed \( \text{inc()} \) method call.

Method \( \text{Read()} \) simply reads the current value of \( \text{Count} \) using a \( \text{Count.Read()} \) operation (line 17) and returns the result of the operation. Method \( \text{CAS()} \) takes two integer parameters \( old, new \), and in line 15 performs a safety check, where it checks whether the value of \( new \) is in \( \{0, \ldots, k\} \). If the safety check fails, then the method simply returns \( \text{false} \). Otherwise, it attempts to change the value of \( \text{Count} \) from \( old \) to \( new \) using the \( \text{Count.CAS(old,new)} \) operation (in line 16) and returns the result of the operation.

### A.1 Analysis and Properties of Object \( \text{RCAScounter}_k \)

Consider an instance of the \( \text{RCAScounter}_k \) object. Let \( H \) be an arbitrary history that consists of all method calls on the instance, except failed \( \text{inc()} \) calls and pending calls that are yet to execute line 10 (\( \text{Read} \) operation), line 12 (\( \text{CAS} \) operation), line 16 (\( \text{CAS} \) operation) or line 17 (\( \text{Read} \) operation). If a failed \( \text{inc()} \) is in the history, it can be linearized at an arbitrary point between its invocation and response, as it does not affect the validity of any other operations. Therefore, it suffices to prove that the history without failed \( \text{inc()} \) operations is linearizable, and then linearizability of the original history follows. The same argument applies to omitting the selected pending method calls. Since the selected pending method calls do not change any shared object, they cannot affect the validity of any other operations.

We define a point \( pt(u) \) for every method \( u \) in \( H \). Let \( I(u) \) be the interval between \( u \)'s invocation and response. Let \( S \) be the sequential history obtained by ordering the method calls in \( H \) according to the points \( pt(u) \). To show that \( \text{RCAScounter}_k \) is a randomized linearizable implementation of the type \( \text{CAScounter}_k \), we need to show that the sequential history \( S \) is valid, i.e., \( S \) lies in the specification of type \( \text{CAScounter}_k \) object, and that \( pt(u) \) lies in \( I(u) \). Let \( C \) be an object of type \( \text{CAScounter}_k \), and let \( S_v \) be the sequential history obtained when the operations of \( S \) are executed sequentially on object \( C \) in the order as given in \( S \). Clearly, \( S_v \) is a valid sequential history in the specification of type \( \text{CAScounter}_k \) by construction. Then to show that \( S \) is valid, we show that \( S = S_v \).

**Lemma A.1.** Object \( \text{RCAScounter}_k \) is a randomized linearizable implementation of type \( \text{CAScounter}_k \).

**Proof.** Let \( A \) be an instance of the \( \text{RCAScounter}_k \) object. Consider an arbitrary history \( H \) that consists of all completed method calls on \( A \), except failed \( \text{inc()} \) calls, and all pending method calls on \( A \) that have executed a successful \( \text{CAS} \) operation. We now define point \( pt(u) \) for every method \( u \) in \( H \).

If \( u \) is a \( \text{Read()} \) method call then define \( pt(u) \) to be the point in time when the \( \text{Read} \) operation in line 17 is executed.

If \( u \) is an \( \text{inc()} \) method call that returns from line 10 then \( pt(u) \) is the point in time of the \( \text{Read} \) operation in line 10 and if \( u \)'s \( \text{CAS} \) operation in line 12 succeeds then \( pt(u) \) is the point in time of the \( \text{CAS} \) operation in line 12. By construction, a \( \text{Read} \) or \( \text{CAS} \) operation has been executed during every \( \text{inc()} \) call in \( H \), and no failed \( \text{inc()} \) calls are in \( H \). Then it follows that we have defined \( pt(u) \) for every \( \text{inc()} \) call \( u \) in \( H \).

If \( u \) is a \( \text{CAS()} \) method call that returns from line 15 then \( pt(u) \) is any arbitrary point during \( I(u) \), and if \( u \) returns from line 16 then \( pt(u) \) is the point in time of the \( \text{CAS} \) operation in line 16. Clearly \( pt(u) \in I(u) \) for every method \( u \) in \( H \).
Let $u_i$ be the $i$-th operation in $S$ and $v_i$ be the $i$-th operation in $S_v$. Let $\text{Count}(u_i)^+$ denote the value of object $\text{Count}$ immediately after $pt(u_i)$, and let $\beta(v_i)^+$ denote the value of object $\beta$ after operation $v_i$ in $S_v$. We assume that $u_0$ is a method call that does not change the state of any shared object of instance $\text{A}$ (such as a $\text{Read}(\cdot)$ method) and returns the initial value of the object. This assumption can be made without loss of generality, because the removal of a method call that does not change the state of the object from a linearizable history always leaves a history that is also linearizable. The purpose of the assumption is to simplify the base case of our induction hypothesis.

We now prove by induction on integer $i$, that $\text{Count}(u_i)^+ = \beta(v_i)^+$, and that the return value of $u_i$ matches the value returned by $v_i$, thereby proving $S = S_v$.

**Basis** ($i = 0$) Since initially the value of object $\text{Count}$ and the value of the atomic $\text{CAScounter}_k$ object is 0, it follows from the definition of the method call $u_0$, that $\text{Count}(u_0)^+ = \beta(v_0)^+ = 0$, and the return value of $u_0$ matches that of $v_0$.

**Induction Step** ($i > 0$) From the induction hypothesis, $\text{Count}(u_{i-1})^+ = \beta(v_{i-1})^+$.

**Case a** - $u_i$ is an $\text{inc}()$ method call that executes a successful $\text{CAS}()$ operation in line 12. Then $pt(u_i)$ is when object $\text{Count}$ is incremented from $\beta$ to $\beta+1$ by a successful $\text{Count}.$CAS ($\beta, \beta+1$) operation in line 12 and thus $\text{Count}(u_{i-1})^+ = \beta$ holds. Also, $u_i$ returns $\beta = \text{Count}(u_{i-1})^+$. Since $u_i$ fails the if-condition of line 9 $\beta \neq k$ and therefore $\text{Count}(u_{i-1})^+ = \beta \neq k$ holds. Now consider operation $v_i$ in $S_v$. Since $\beta(v_{i-1})^+ = \text{Count}(u_{i-1})^+ \neq k$, the if-condition of line 1 fails, and the value of the atomic $\text{CAScounter}_k$ is incremented in line 2 and $\beta(v_{i-1})^+$ returned in line 3. Hence $\text{Count}(u_i)^+ = \beta(v_i)^+$ and the return values match.

**Case b** - $u_i$ is an $\text{inc}()$ method call that returns from line 10. Then $pt(u_i)$ is when the $\text{Read}()$ operation on the object $\text{Count}$ is executed in line 10. Clearly, the value returned by the $\text{Read}()$ operation on the object $\text{Count}$ at $pt(u_i)$ is $\text{Count}(u_{i-1})^+$. Since the if-condition of line 10 is satisfied, $\text{Count}(u_{i-1})^+ = k$ and $u_i$ returns integer $k$ without changing object $\text{Count}$. Now consider operation $v_i$ in $S_v$. Since $\beta(v_{i-1})^+ = \text{Count}(u_{i-1})^+ \neq k$, the if-condition of line 1 fails, and the value of the atomic $\text{CAScounter}_k$ is returned in line 3. Hence $\text{Count}(u_i)^+ = \beta(v_i)^+$ and the return values match.

**Case c** - $u_i$ is a $\text{CAS}()$ method call that returns from line 15. Then the if-condition of line 15 is satisfied and thus new $\notin \{0, 1, \ldots, k\}$ and $u_i$ returns $\text{false}$ without changing $\text{Count}$. Now consider operation $v_i$ in $S_v$. Since new $\notin \{0, 1, \ldots, k\}$, the if-condition of line 4 will be satisfied and the Boolean value $\text{false}$ is returned without changing the value of object $\beta$. Hence $\text{Count}(u_i)^+ = \beta(v_i)^+$ and the return values match.

**Case d** - $u_i$ is a $\text{CAS}()$ method call that returns from line 16. Then $pt(u_i)$ is when the $\text{CAS}$ operation on the object $\text{Count}$ is executed in line 16 and $u_i$ returns the result of this $\text{CAS}$ operation. The $\text{CAS}$ operation attempts to change the value of $\text{Count}$ from old to new, therefore if $\text{Count}(u_{i-1})^+ = \text{old}$ then $\text{Count}(u_i)^+ = \text{new}$ and $u_i$ returns $\text{true}$, or else $\text{Count}$ remains unchanged and $u_i$ returns $\text{false}$. Now consider operation $v_i$ in $S_v$. From the code structure, if $\beta(v_{i-1})^+ = \text{old}$ then $\text{Count}(u_i)^+ = \text{new}$ and the Boolean value $\text{true}$ is returned. And if $\beta(v_{i-1})^+ \neq \text{old}$ then the value of object $\beta$ remains unchanged and the Boolean value $\text{false}$ is returned. Hence $\text{Count}(u_i)^+ = \beta(v_i)^+$ and the return values match.

**Lemma A.2.** The probability that an $\text{inc}()$ method call returns $\bot$ is $k/(k+1)$ against the weak adversary.

**Proof.** Let the process calling the $\text{inc}()$ method call (say $u$) be $p$ and let the value of the object $\text{Count}$ immediately before $p$ executes line 8 be $z$. Since the adversary is weak, no other process executes a shared memory operation after $p$ chooses $\beta$ in line 8 and before $p$ finishes executing its
next shared memory operation. From the code structure, \( p \) returns \( \bot \) during \( u \) (in line 14) if and only if \( z \neq \beta \). Since

\[
\text{Prob}(z \neq \beta) = 1 - \text{Prob}(z = \beta) = 1 - \frac{1}{k+1} = \frac{k}{k+1},
\]

the claim follows.

\[\square\]

The following claim follows immediately from an inspection of the code.

**Lemma A.3.** Each of the methods of RCAScounter\(_k\) has step complexity \( \mathcal{O}(1) \), and is wait-free.

Lemma 2.1 follows from Lemmas A.1, A.2 and A.3.

## B Specification of Type AbortableProArray\(_k\)

Type AbortableProArray\(_k\) is presented in Figure 5

| Type AbortableProArray\(_k\) |
|----------------------------------|
| \( A: \) array \([0 \ldots k-1]\) of int pairs \( \text{init} \langle \bot, \bot \rangle \); \( \text{REG, PRO, ABORT: init } 1, 2, 3 \) |

| Operation collect(int[] \( X \)) |
|----------------------------------|
| 1 \( \text{for } i \leftarrow 0 \text{ to } k-1 \text{ do} \) |
| 2 \( \langle v, s \rangle \leftarrow A[i] \) |
| 3 \( \text{if } v \neq \text{ABORT} \land X[i] \neq \bot \text{ then } A[i] \leftarrow \langle \text{REG, X[i]} \rangle \) |
| 4 end |

| Operation abort(int \( i \), int seq) |
|----------------------------------|
| 5 \( \langle v, s \rangle \leftarrow A[i] \) |
| 6 \( \text{if } v = \text{PRO then return false} \) |
| 7 \( A[i] \leftarrow \langle \text{ABORT, seq} \rangle \) |
| 8 return true |

| Operation reset() |
|----------------------------------|
| 9 \( \text{for } i \leftarrow 0 \text{ to } k-1 \text{ do } A[i] \leftarrow \langle 0, \bot \rangle \) |

| Operation promote() |
|----------------------------------|
| 10 \( \text{for } i \leftarrow 0 \text{ to } k-1 \text{ do} \) |
| 11 \( \langle v, s \rangle \leftarrow A[i] \) |
| 12 \( \text{if } v = \text{REG then} \) |
| 13 \( A[i] \leftarrow \langle \text{PRO, s} \rangle \) |
| 14 \( \text{return } \langle i, s \rangle \) |
| 15 end |
| 16 end |
| 17 return \( \langle \bot, \bot \rangle \) |

| Operation remove(int \( i \)) |
|----------------------------------|
| 18 \( \langle v, s \rangle \leftarrow A[i] \) |
| 19 \( A[i] \leftarrow \langle \text{ABORT, s} \rangle \) |

Figure 5: Sequential Specification of Type AbortableProArray\(_k\)

## C The Single-Fast-Multi-Slow Universal Construction

In this section, rather than implementing object SFMSUnivConst\(\langle T \rangle\), we implement a lock-free universal construction object SFMSUnivConstWeak\(\langle T \rangle\), with slightly weaker properties than SFMSUnivConst\(\langle T \rangle\). An object implementation is lock-free, if in any infinite history \( H \) where processes
continue to take steps, and \( H \) contains only operations on that object, some operation finishes. Object \( \text{SFMSUnivConstWeak}(T) \) has the same properties as object \( \text{SFMSUnivConst}(T) \) except method \( \text{doFast}() \) is lock-free with unbounded step-complexity.

There is a standard technique called \textit{operation combining} \cite{18} that can be applied to transform our lock-free object \( \text{SFMSUnivConstWeak}(T) \) to the wait-free object \( \text{SFMSUnivConst}(T) \) with \( O(N) \) step complexity for method \( \text{doFast}() \).

By applying the technique of operation combining we can transform our lock-free universal construction \( \text{SFMSUnivConstWeak}(T) \) into our wait-free object \( \text{SFMSUnivConst}(T) \). We however do not provide a proof of its properties. Doing so would be repeating the same “standard” proof ideas from \cite{18}, and would result in increasing the size of the paper without contributing to the main ideas of this paper. We do provide proofs for our lock-free universal construction \( \text{SFMSUnivConstWeak}(T) \), and the proofs illustrate the main idea from this section, i.e., how to achieve a linearizable concurrent implementation with support for a \( \text{doFast}() \) method of \( O(1) \) step complexity. We now present the implementation of object \( \text{SFMSUnivConstWeak}(T) \) (see Figure 6).

| Object SFMSUnivConstWeak(T) |
|-------------------------------|
| shared: mReg: int init \((s_0, \bot, 0, 0)\); fastOp: int init \((\bot, 0)\); |
| local: state, res, fc, sc, s1, r1, r2, seq: int init 0 |

**Method doFast(op)**

1 \((state, res, fc, sc) \leftarrow mReg.\text{Read}()\)
2 fastOp \(\leftarrow (op, fc + 1)\)
3 if \(\neg \text{helpFast}()\) then
4 \((state, res, fc, sc) \leftarrow mReg.\text{Read}()\)
5 return res

**Method helpFast()**

9 \((s_1, r_1, fc, sc) \leftarrow mReg.\text{Read}()\)
10 \((op, seq) \leftarrow \text{fastOp.\text{Read}()}\)
11 if \(fc \geq seq\) then return true
12 \((s_2, r_2) \leftarrow f(s_1, op)\)
13 return \(mReg.\text{CAS}((s_1, r_1, fc, sc), (s_2, r_2, seq, sc))\)

**Method f(state1, op)**

6 \(state_2 \leftarrow \text{state generated when}\ \ op\ \ \text{is applied to object} \ O\ \ \text{with}\ \ \text{state} \ state_1\)
7 \(res \leftarrow \text{result when}\ \ op\ \ \text{is applied to object} \ O\ \ \text{with}\ \ \text{state} \ state_1\)
8 return \((state_2, res)\)

**Method performSlow(op)**

14 repeat
15 \(\leftarrow mReg.\text{Read}()\)
16 \((s_2, r_2) \leftarrow f(s_1, op)\)
17 if \(s_2 = s_1\) then return \(r_2\)
18 \text{helpFast}()\)
19 until \(mReg.\text{CAS}((s_1, r_1, fc, sc), (s_2, r_1, fc, sc + 1))\)
20 return \(r_2\)

**Figure 6:** Implementation of Object SFMSUnivConstWeak(T).

**Shared Data.** A shared register \( mReg \) stores a 4-tuple \((m_0, m_1, m_2, m_3)\). We use the notation \( mReg[i] \) to refer to the \((i + 1)\)-th tuple element, \( m_i \), stored in register \( mReg \). Element \( mReg[0] \) stores the state of object \( O \). Element \( mReg[1] \) stores the result of the most recent fast operation performed. Elements \( mReg[2] \) and \( mReg[3] \) store counts of the number of fast and slow operations
performed respectively. Initially \( mReg[0] \) stores the initial state of \( O \), \( mReg[1] \) has value \( \bot \) and 
\((mReg[2], mReg[3]) \) is \((0, 0) \).

A shared register \( fastOp \) is used to announce a fast operation to be performed in a pair \((s_0, s_1) \). Element \( fastOp[0] \) stores the complete description of a fast operation to be performed. Element \( fastOp[1] \) stores a sequence number indicating the number of fast operations that have been announced in the past. This sequence number is used by processes to determine whether an announced fast operation is pending execution. Initially \( fastOp \) is \((\bot, 0) \). The methods \( doFast() \) and \( doSlow() \) make use of two private methods \( helpFast() \) and \( f() \) (see Figure 6).

**Description of the \( f() \) method.** Method \( f() \) is implemented using the specification provided by type \( T \). The method takes two arguments \( state_1 \) and \( op \), where \( state_1 \) is a state of object \( O \) and \( op \) is the complete description of an operation to be applied on object \( O \). The method computes the new state \( state_2 \) and the result \( result \), when operation \( op \) is applied on object \( O \) with state \( state_1 \). The method then returns the pair \((state_2, result) \). Since no shared memory operations are executed during the method, the method has 0 step complexity.

**Description of the \( doFast() \) method.** Let \( p \) be a process that executes \( doFast(op) \). In line 1 process \( p \) first copies the 4-tuple read from register \( mReg \) to its local variables \( state, res, fc \) and \( sc \). Then \( p \) announces the operation \( op \) by writing the pair \((op, fc + 1) \) to register \( fastOp \) in line 2. After announcing the operation, process \( p \) helps perform the announced operation by calling the private method \( helpFast() \) in line 3. If the call to \( helpFast() \) returns \( false \), then \( p \) concludes that the announced operation may not have been performed yet. In this case \( p \) makes another call to \( helpFast() \) in line 3 to be sure that the announced operation is performed (we prove later that at most two calls to \( helpFast() \) are required to perform an announced operation). Process \( p \) then reads and returns the result of the performed operation stored in \( mReg[1] \) in line 4 and 5 respectively. Since method \( doFast() \) is not executed concurrently (by assumption), the result of \( p \)'s operation stored in register \( mReg \) is not overwritten before the end of \( p \)'s \( doFast(op) \) call.

**Description of the \( helpFast() \) method.** Let \( q \) be a process that calls and executes \( helpFast() \). In line 9 process \( q \) first copies the 4-tuple read from register \( mReg \) into its local variables \( s1, r1, fc \) and \( sc \). The value read from \( mReg[0] \) constitutes the state of object \( O \), to which \( q \) will attempt to apply the announced operation if required. The value read from \( mReg[1] \) is the result of the last fast operation performed on object \( O \). The value read from \( mReg[2] \) and \( mReg[3] \) is the count of the number of fast and slow operations performed respectively. Process \( q \) then reads \( fastOp \) in line 10 to find out the announced operation \( op \) and the announced sequence number \( seq \). Process \( q \) then determines whether the announced operation has already been performed, by checking whether \( seq \) is less than or equal to \( fc \) in line 11. If so, \( q \) concludes that operation \( op \) has been performed and returns \( true \), otherwise it attempts to perform \( op \) in lines 12 and 13. In line 12 process \( q \) calls the private method \( f() \) to compute the new state \( s2 \) and the result \( r2 \) when operation \( op \) is applied to object \( O \) with state \( s1 \). In line 13 process \( p \) attempts to perform \( op \) by swapping the 4-tuple \((s1, r1, fc, sc) \) with \((s2, r2, fc + 1, sc) \) using a CAS operation on \( mReg \). If the CAS is unsuccessful then no changes are made to \( mReg \). This can happen only if some other process performs an announced fast operation in line 13 or a slow operation in line 19. The result of the CAS operation of line 13 is returned in either case.

**Description of the \( doSlow() \) method.** Let \( p \) be a process that calls and executes \( doSlow(op) \). During the method, \( p \) repeats the while-loop of lines 15-19 until \( p \) is able to successfully apply its operation \( op \). In line 15 process \( p \) first copies the 4-tuple read from register \( mReg \) to its local variables \( s1, r1, fc \) and \( sc \). In line 16 process \( q \) calls the private method \( f() \) to compute the new state \( s2 \) and the result \( r2 \) when operation \( op \) is applied to object \( O \) with state \( s1 \). In the case that operation \( op \) does not cause a state change in object \( O \), i.e., \( s1 = s2 \), then \( p \) returns result \( r2 \) in line 17. Otherwise \( p \) attempts to apply operation \( op \) in line 19 by swapping the 4-tuple
\((s_1,r_1,fc,sc)\) with \((s_2,r_2,fc,sc+1)\) using a CAS operation on register \(mReg\). Before attempting to apply its own operation in line \(19\) \(p\) makes a call to \(\text{helpFast}()\) in line \(18\) to help perform an announced fast operation (if any). On completing the while-loop, \(p\) would have successfully applied its operation \(op\), and thus \(p\) returns the result of the applied operation in line \(20\).

The following lemma (proven in Section \(C.2\)) summarizes the properties of object \(\text{SFMSUnivConstWeak}\langle T \rangle\).

**Lemma C.1.** Object \(\text{SFMSUnivConstWeak}\langle T \rangle\) is a lock-free universal construction object that implements an object \(O\) of type \(T\), for \(n\) processes, where \(n\) is the maximum number of processes that can access object \(\text{SFMSUnivConstWeak}\langle T \rangle\) concurrently and operations on object \(O\) are performed using either method \(\text{doFast}()\) or \(\text{doSlow}()\), and no two processes execute method \(\text{doFast}()\) concurrently. Method \(\text{doFast}()\) has \(O(1)\) step complexity.

### C.1 Operation Combining Technique

In principle the technique works as follows: Processes maintain an \(N\)-element array, say \(\text{announce}\), where process \(i\) “owns” slot \(i\), and processes store in their respective slots the operation that they want to apply. When a process \(p\) wants to apply an operation it first “announces” its operation by writing the operation to the \(p\)-th element of the array. Then \(p\) attempts to help the “next” operation in the \(\text{announce}\) array by attempting to apply that operation if it has not been applied, yet. An index to the “next” operation to be applied is maintained in the same register that stores the state of the concurrent object. Every time an announced operation is applied, the index is also incremented modulo \(N\) in one atomic step. The response of applied operations is stored in another \(N\)-element array, say \(\text{response}\), which can sometimes be combined with the \(\text{announce}\) array. Sequence numbers are used to ensure that an announced operation is not applied more than once. Since the index of the “next” operation cycles the \(\text{announce}\) array, a process needs to help announced operations \(O(N)\) times before its own announced operation is applied, at which point it can stop.

Herlihy \[18\] introduced this technique as a general methodology to transform lock-free universal constructions to wait-free ones. Herlihy presents another example \[19\] that employs the technique of operation combining to transform a lock-free universal construction to a wait-free one, where the step complexity of the method that performs the operation is bounded to \(O(N)\).

On applying the standard technique of operation combining \[18\] to object \(\text{SFMSUnivConstWeak}\langle T \rangle\) we obtain object \(\text{SFMSUnivConst}\langle T \rangle\) and Lemma \(2.2\)

### C.2 Analysis and Proofs of Correctness of Object \(\text{SFMSUnivConstWeak}\langle T \rangle\)

Let a \(\text{helpFast}()\) method call that returns \(\text{true}\) in line \(13\) (on executing a successful CAS operation) be called a successful \(\text{helpFast}()\).

**Claim C.2.** (a) The value of \(\text{fastOp}[1]\) changes only in line \(2\).

(b) The value of \(mReg[3]\) increases by one with every successful CAS operation in line \(19\) and no other operation changes \(mReg[3]\).

(c) The value of \(mReg[2]\) increases with every successful CAS operation in line \(13\) (during a successful \(\text{helpFast}()\)), and no other operation changes \(mReg[2]\).

**Proof.** Part (a) follows immediately from an inspection of the code. Register \(mReg\) is changed only when a process executes a successful CAS operation in lines \(13\) or \(19\). Furthermore, in line \(13\) \(mReg[3]\) is not changed and in line \(19\) \(mReg[2]\) is not changed. Since, in line \(19\) \(mReg[3]\) is incremented
Part (b) follows immediately. Now, for a process to execute line [13] the if-condition of line [11] must fail, hence mReg[2] is increased from its previous value and Part (c) follows.

Consider an arbitrary history \( H \) where processes access an SFMSUnivConstWeak(T) object but no two doFast() method calls are executed concurrently. Since the fast operations are executed sequentially the happens before order on all doFast() method calls in \( H \) is a total order.

Claim C.3. Let \( u_t \) be the \( t \)-th doFast() method call in history \( H \) being executed by process \( p_t \). For \( t \geq 1 \) let \( \alpha_t \) be the point in time when \( p_t \) executes line [2] during \( u_t \) and \( \gamma_t \) be the point when \( p_t \) is poised to execute line [4]. Let \( u_t \)'s helpers be the processes that call helpFast() such that the value read by the processes in line [10] is the value written to register fastOp at \( \alpha_t \). Let \( \beta_t \) be the first point in time when a helper's call to helpFast() succeeds after \( \alpha_t \). Let \( \alpha_0, \beta_0, \gamma_0 \) be the start of execution \( H \). Then the following claims hold for all \( t \geq 0 \):

\[
\begin{align*}
(S_1) & \quad \beta_t \text{ exists and } \beta_t \text{ is in } (\alpha_t, \gamma_t) \\
(S_2) & \quad \text{Throughout } (\alpha_t, \beta_t) : \text{fastOp[1]} = \text{mReg[2]} + 1 = t \\
(S_3) & \quad \text{Throughout } (\beta_t, \alpha_{t+1}) : \text{fastOp[1]} = \text{mReg[2]} = t
\end{align*}
\]

Proof. We prove claims (S1), (S2) and (S3) by induction over \( t \).

Basis: For \( t = 0 \), (S1) and (S2) are trivially true. By assumption the initial value of fastOp[1] and mReg[2] is 0. Consider the interval (\( \beta_0, \alpha_1 \)). From Claim C.2(a) it follows that fastOp is written for the first time at \( \alpha_1 \). The first point when one of the invariants (S3) is destroyed is if a process (say \( p \)) executes a successful CAS operation in line [13] during (\( \beta_0, \alpha_1 \)). Then \( p \) read the value 0 from register fastOp[1] in line [10] since the initial value of fastOp[1] is 0 and fastOp[1] is written to for the first time at \( \alpha_1 \). Since mReg[2] is never decremented (from Claim C.2(b)) and mReg[2] initially has value 0, \( p \) satisfies the if-condition of line [11] and \( p \)'s helpFast() call returns true in line [11]. Therefore, \( p \) does not execute line [13] which is a contradiction.

Induction Step: For \( t \geq 1 \):

Proof of (S1): Consider the interval (\( \alpha_t, \gamma_t \)). To show that (S1) holds for \( t \), we need to show that mReg[2] is changed during (\( \alpha_t, \gamma_t \)). Consider \( p_t \)'s first call to helpFast() in line [8] during \( u_t \). From induction hypothesis (S3) for \( t - 1 \), it follows that fastOp[1] = mReg[2] = \( t - 1 \) during (\( \beta_{t-1}, \alpha_t \)). Then \( p_t \) reads value \( t - 1 \) from mReg[2] in line [11] and writes value \( t \) to fastOp[1] in line [2]. Since fastOp[1] is changed only at \( \alpha_{t+1} \) after \( \alpha_t \), it follows that \( p_t \) reads \( t \) from register fastOp[1] in line [10].

Case a - \( p_t \) returns from line [11]. Then \( p_t \) read a value from mReg[2] in line [9] that is at least \( t \). Since mReg[2] = \( t - 1 \) holds immediately before \( \alpha_t \) some process changed mReg[2] in line [13] during (\( \alpha_t, \gamma_t \)). Hence, (S1) for \( t \) holds.

Case b - \( p_t \) returns true from line [13]. Then \( p_t \) has changed mReg[2] and hence (S1) holds for \( t \).

Case c - \( p_t \) returns false from line [13]. Then some process \( q \) changed register mReg after \( p_t \) read mReg in line [9]. Now, register mReg is written to only in line [13] or line [19] (from an inspection of the code).

Subcase c1 - \( q \) changed mReg by executing line [13]. Then \( q \) has changed mReg[2] and hence (S1) holds for \( t \).

Subcase c2 - \( q \) changed mReg by executing line [19]. Then \( p_t \) executes a second call to helpFast() in line [3]. Let \( m \) be the value of mReg[3] read by \( p_t \) in line [9]. If \( p_t \)'s second helpFast() call satisfies case (a) or (b) then we get that (S1) holds for \( t \).
If $p_t$’s second $\text{helpFast}()$ call returns $\text{false}$ from line 13, then some process changed $mReg$ after $p_t$ read $mReg$ in line 9. If some process changed $mReg$ by executing line 13 then we get that $(S_1)$ holds for $t$. Then some process changed $mReg$ by executing line 19 after $p_t$ read $mReg$ in line 9 and let $r$ be the first process to do so. Therefore, $r$ changes the value of $mReg[3]$ from $m$ to $m+1$ in line 19. Then $r$ executed line 13 after $q$ executed a successful CAS operation in line 19. Then $r$ completed a call to $\text{helpFast}()$ in line 18 after $\alpha_t$. Since $r$ reads $mReg$ after $\alpha_t$, $r$ satisfied the if-condition of line 11 and executed line 13. If $r$ successfully executes the CAS operation in line 13 then we get that $(S_1)$ holds for $t$. Then some process $s$ must have changed $mReg$ after $r$ read $mReg$ in line 9. Since the value of $mReg[3]$ is only incremented (by Claim C.2(b)) and $r$ changes the value of $mReg[3]$ from $m$ to $m+1$, it follows that $s$ changed $mReg$ in line 13 and hence $(S_1)$ holds for $t$.

**Proof of (S2) and (S3):** From $(S_1)$ for $t$ it follows that $\beta_t$ exists and $\alpha_t < \beta_t < \gamma_t < \alpha_{t+1}$. From the induction hypothesis invariants $(S_2)$ and $(S_3)$ are true until $\alpha_t$. Now, one of the invariants $(S_2)$ or $(S_3)$ can be destroyed only if some process executes a successful CAS operation in line 13 and changes $mReg[2]$. By definition of $\beta_t$, $mReg[2]$ is unchanged during $(\alpha_t, \beta_t)$. Then invariants $(S_2)$ and $(S_3)$ continue to hold until $\beta_t$. Therefore, claim $(S_2)$ holds for $t$. It still remains to be shown that claim $(S_3)$ holds for $t$.

Let $p$ be the process that executes a successful CAS operation in line 13 and changes $mReg[2]$ at $\beta_t$. Since $mReg[2] = t-1$ immediately before $\beta_t$ and $p$ executes a successful CAS operation in line 13 at $\beta_t$, $p.fc = t-1$. Then $p$ executed lines 9 and 10 during $(\alpha_t, \beta_t)$ and $p.seq = t$. Therefore, invariant $(S_3)$ is true immediately after $\beta_t$.

Now, assume another process (say $q$) destroys one of the invariants $(S_3)$ or $S_4$ by executing a successful CAS operation in line 13 during $(\beta_t, \alpha_{t+1})$. Then $q$ must have read register $mReg[2]$ and $fastOp[1]$ after $\beta_t$, and therefore $q$ must read the value $t$ from both of them. Then $q$ must have satisfied the if-condition of line 11 and returned $\text{true}$. Hence, $q$ does not execute line 13 which is a contradiction. Therefore, invariant $(S_3)$ is true up to $\alpha_{t+1}$, and thus claim $(S_3)$ holds for $t$.

Let $H'$ be a history that consists of all completed method calls in $H$ and all pending method calls that executed line 2 (Write operation on register fastOp), or which executed a successful CAS operation in line 13 or line 19. We omit all other pending method calls, since during those method calls no operations are executed that changes the state of any shared object, and hence those pending method calls cannot affect the validity of any other operation. Therefore, to prove that history $H$ is linearizable it suffices to prove that history $H'$ is linearizable.

For each method call $u$ in $H'$, we define a point $pt(u)$ and an interval $I(u)$. Let $I(u)$ denote the interval between $u$’s invocation and response. If $u$ is a $\text{doSlow}()$ method call that returns from line 17 then $pt(u)$ is the point in time of the $\text{Read}$ operation in line 15 otherwise, $pt(u)$ is the point in time of the $\text{CAS}$ operation in line 19. If $u$ is a $\text{doFast}()$ method call, let $v$ be a successful $\text{helpFast}()$ method call such that $v$’s line 13 is executed after $u$’s line 2 and before $u$ returns. We define $pt(u)$ to be the point of the successful $\text{CAS}$ operation in $v$’s line 13.

**Claim C.4.** For every method call $u$ in $H$, $pt(u)$ exists and lies in $I(u)$.

**Proof.** There are two types of method calls in $H$, doFast() and doSlow().

**Case a -** $u$ is a doFast() method call.

From Claim C.3 it follows that exactly one of $u$’s helpers (see Claim C.3 for definition) succeeds and the helper performs a successful CAS operation in line 13 at some point in $I(u)$. Therefore, point $pt(u)$ exists and lies in $I(u)$.

**Case b -** $u$ is a doSlow() method call. By definition $pt(u)$ is assigned to a line of $u$’s code, therefore $pt(u)$ exists and lies in $I(u)$. [QED]
Let $S$ be the sequential history obtained by ordering all method calls $u$ in $H'$ according to the points $pt(u)$. To show that $\text{SFMSUnivConstWeak}(T)$ is a linearizable implementation of an object $O$ of type $T$, we need to show that the sequential history $S$ is valid, i.e., $S$ lies in the specification of type $T$, and that $pt(u)$ lies in $I(u)$ (already shown in Claim C.4). Let $S_v$ be the sequential history obtained when the operations of $S$ are executed sequentially on object $O$, as per their order in $S$. Clearly, $S_v$ is a valid sequential history in the specification of type $T$ by construction. Then to show that $S$ is valid, we show that $S = S_v$.

Let $v_t$ be the $t$-th operation in $S_v$ and let $u_t$ be the $t$-th method call in $S$. Let $UC^t_-$ and $UC^t_+$ denote the value of $mReg[0]$ immediately before and after $pt(u_t)$, respectively. Let $O_t^-$ and $O_t^+$ denote the state of object $O$ immediately before and after operation $v_t$, respectively. Let $\alpha_t$ and $\beta_t$ denote the value returned by $u_t$ and $v_t$, respectively. Define $UC^+_0 = UC^-_1$ and $O^+_0 = O^-_1$. Define $\alpha_0 = \beta_0 = \bot$.

**Claim C.5.** Suppose a process calls method $f(x_1, op)$ and the method returns the value pair $(x_2, y)$. If $x_1 = O^-_t$ then $x_2 = O^+_t$ and $y = \beta_t$.

**Proof.** By definition, a call to method $f(x_1, op)$ returns the value pair $(x_2, y)$ such that $x_2$ is the state of $O$ when operation $op$ is applied to $O$ while at state $x_1$ and $y$ is the result of the operation. Then if $x_1 = O^-_t$ then $x_2 = O^+_t$ and $y = \beta_t$. \hfill $\square$

**Claim C.6.** For all $t \geq 1$.

$$(S_1) \quad O^+_t = O^+_{t+1} \quad \text{and} \quad UC^+_t = UC^-_{t+1}$$

$$(S_2) \quad UC^-_t = O^-_t$$

$$(S_3) \quad UC^+_{t-1} = O^+_{t-1} \quad \text{and} \quad \alpha_{t-1} = \beta_{t-1}$$

**Proof.** Proof of $(S_1)$: Since operations in $S_v$ are executed sequentially, it follows that $O^+_t = O^-_{t+1}$. We now show that $UC^+_t = UC^-_{t+1}$. Assume $UC^+_t \neq UC^-_{t+1}$. Then some process $p$ changed the value of $mReg[0]$ by executing a successful $\text{CAS}$ operation in line 13 or line 19 at some point during the interval $(pt(u_t), pt(u_{t+1}))$. By definition, $p$’s successful $\text{CAS}$ operation in line 13 or line 19 is $pt(u_t)$ for some method call $u_t$ where $u_t$ is the $t$-th method call in $H'$. Thus, $t$ is an integer and $t < \ell < t + 1$ holds, which is a contradiction.

Proof of $(S_2)$ and $(S_3)$: We prove $(S_2)$ and $(S_3)$ by induction over $t$.

**Basis** $(t = 1)$ - By assumption, initially, $mReg[0]$ is the initial state of $O$, hence, $UC^-_1 = O^-_1$. Hence, $(S_2)$ is true. $(S_3)$ is true trivially.

**Induction Step** - We assume $(S_2)$ and $(S_3)$ for $t$ are true and prove that $(S_2)$ and $(S_3)$ for $t + 1$ are true. From $(S_1)$ we have, $O^+_{t} = O^-_{t+1}$ and $UC^+_t = UC^-_{t+1}$. From $(S_2)$ for $t$ we have $UC^+_t = O^+_t$. Therefore, it follows that $UC^-_{t+1} = O^-_{t+1}$ and thus $(S_2)$ for $t + 1$ is true.

To show $(S_3)$ for $t + 1$ is true, we need to show $UC^+_t = O^+_t$ and $\alpha_t = \beta_t$. By Claim $(S_2)$ for $t$, $UC^-_t = O^-_t$ holds. Let $p_t$ be the process executing $u_t$.

**Case a** - $u_t$ is a $\text{doSlow}(op)$ method call: Let $x_1$ be the most recent value read by $p_t$ from $mReg[0]$ in line 15 and let $(x_2, y)$ be the value returned when $p_t$ executes line 16. From the code structure, $\alpha_t = y$.

**Subcase (a1)** - $p_t$ returns from line 17. Then $pt(u_t)$ is the point when $p_t$ executes a successful $\text{Read}$ operation on register $mReg$ in line 15. Since $p$ satisfies the if-condition of line 17, $x_2 = x_1$. Thus, $UC^-_t = UC^+_t = x_1$.

**Subcase (a2)** - $p_t$ returns from line 20. Then $pt(u_t)$ is the point when $p_t$ executes a successful $\text{CAS}$ operation on register $mReg$ in line 19. From the definition of a $\text{CAS}$ operation, it follows that $UC^-_t = x_1$ and $UC^+_t = x_2$. 24
For both subcases (a1) and (a2), \( x_1 = UC^-_t = O^-_t \) holds. Then from Claim \( \text{C.5} \) it follows that \( x_2 = O^+_t \) and \( y = \beta_t \). Since \( x_2 = UC^+_t \) and \( y = \alpha_t \), \( O^+_t = UC^+_t \) and \( \alpha_t = \beta_t \).

**Case b -** \( u_t \) is a \( \text{doFast}(op) \) method call: Then \( pt(u_t) \) is the point when a successful \( \text{CAS} \) operation on register \( mReg \) is executed in line \( 13 \) of method call \( w \) where \( w \) is the first successful \( \text{helpFast()} \) method call that begins after \( u_t \)'s line \( 2 \) is executed. Let \( q \) be the process executing \( w \). Let \( x_1 \) be the value read by \( q \) from \( mReg[0] \) in line \( 9 \) and let \( (x_2, y) \) be the value returned when \( q \) executes line \( 12 \). From the definition of a \( \text{CAS} \) operation, it follows that \( UC^-_t = x_1 \) and \( UC^+_t = x_2 \). Since \( x_1 = UC^-_t = O^-_t \), from Claim \( \text{C.5} \) it follows that \( x_2 = O^+_t \) and \( y = \beta_{u_t} \). Since \( x_2 = UC^+_t \), it follows that \( O^+_t = UC^+_t \).

From Claim \( \text{C.3} \) it follows that \( mReg[1] \) is changed exactly once during \( u_t \), specifically at \( pt(u_t) \), where \( q \) writes the value \( y \) to it. Thus, \( p \) reads the value \( y \) from \( mReg[1] \) in line \( 10 \) since \( p \) executes line \( 4 \) after \( pt(u_t) \) (Claim \( \text{C.3} \)). Therefore, it follows that \( \alpha_{u_t} = y = \beta_{u_t} \). \( \square \)

**Lemma C.7.** History \( H' \) has a linearization in the specification of \( T \).

**Proof.** By Claim \( \text{C.4} \) for each method call \( u \) in \( H' \), \( pt(u) \) exists and lies in \( I(u) \). Thus, to show that \( H' \) is linearizable we only need to show that \( S \) lies in the specification of type \( T \). Thus, we need to show that for all \( t \geq 1 \), the value returned by \( v_t \) matches that value returned by \( u_t \). From Claim \( \text{C.6} \) \((S_3)\) it follows that for all \( t \geq 1 \), \( \alpha_t = \beta_t \). \( \square \)

**Lemma C.8.** Object \( \text{SFMSUnivConstWeak}(T) \) is lock-free.

**Proof.** Suppose not. I.e., there exists an infinite history \( H \) during which processes take steps but no method call finishes. It is clear from an inspection of method \( \text{doFast()} \) and private method \( \text{helpFast()} \), that both methods are wait-free. Then if \( H \) contains steps executed by a process that executes a call to \( \text{doFast()} \) then the \( \text{doFast()} \) method call finishes since processes continue to take steps in history \( H \) – a contradiction. Now consider the only other case, where history \( H \) contains steps executed by processes only on \( \text{doFast()} \) method calls. Consider a process \( p \) that takes steps in history \( H \) and fails to complete its \( \text{doSlow()} \) method call. Then during \( p \)'s execution \( p \) reads register \( mReg \) in line \( 13 \) and fails its \( \text{CAS} \) operation in line \( 19 \) during an iteration of the loop of lines \( 14-19 \). Now \( p \)'s \( \text{CAS} \) operation can fail only if some process executes a successful \( \text{CAS} \) operation in line \( 13 \) or line \( 19 \) between \( p \)'s \( \text{Read()} \) and \( \text{CAS} \) operation.

**Case a -** Some process \( q \) executes a successful \( \text{CAS} \) operation in line \( 19 \). Then \( q \) breaks out of the loop of lines \( 14-19 \). Since processes continue to take steps in our infinite history \( H \), \( q \) eventually returns from its \( \text{doSlow()} \) method call – a contradiction.

**Case b -** Some process \( q \) executes a successful \( \text{CAS} \) operation in line \( 13 \). Then \( q \) has performed a successful \( \text{helpFast()} \) method call and incremented \( mReg[2] \). Let the value of \( mReg[2] \) after the increment be \( z \). Now consider the next iteration of the loop by process \( p \), where \( p \)'s \( \text{CAS} \) operation in line \( 19 \) fails again. Since \( \text{Case a} \) leads to a contradiction, some process \( r \) executed a successful \( \text{CAS} \) operation in line \( 13 \). Then \( r \) read incremented \( mReg[2] \) to some value greater than \( z \) in line \( 13 \). From the code structure of the \( \text{helpFast()} \) method, \( r \) failed the if-condition of line \( 11 \) and therefore \( r \) read \( seq = fastOp[1] > z \) in line \( 10 \). Since \( fastOp[1] \) is incremented only in line \( 2 \) during a \( \text{doFast()} \) method call, it follows that a \( \text{doFast()} \) method was called after \( q \) incremented \( mReg[2] \) to \( z \) in line \( 13 \). This is a contradiction to the assumption that processes take steps executing only method \( \text{doSlow()} \) during our history \( H \). \( \square \)

Lemma \( \text{C.1} \) follows from Lemma \( \text{C.7} \) and \( \text{C.8} \)
D The Array Based Randomized Abortable Lock

D.1 Implementation / Low Level Description

We now describe the implementation of our algorithm in detail. (See Figure 1 and 2). We now describe the method calls in detail and illustrate the use of each of the internal objects as and when we require them.

The lock() method. Suppose process $p$ executes a call to lock$_i()$. Process $p$ first receives a sequence number using a call to getSequenceNo() in line 1 and stores it in its local variable $s$. Method getSequenceNo() returns an integer $k$ on being called for the $k$-th time from a call to lock$_i()$. Since calls to lock$_i()$ are executed sequentially, a sequential shared counter suffices to implement method getSequenceNo(). Method getSequenceNo() is used to return unique sequence numbers which helps solve the classic ABA problem. The ABA problem is as follows: If a process reads an object twice and reads the value of the object to be 'A' both times, then it is unable to differentiate this scenario from a scenario where the object was changed to value 'B' in between the two reads of the object. Process $p$ then line 1 and stores it in its local variable $s$. Method getSequenceNo() returns integer $k$ on being called for the $k$-th time from a call to lock$_i()$. Since calls to lock$_i()$ are executed sequentially, a sequential shared counter suffices to implement method getSequenceNo(). Method getSequenceNo() is used to return unique sequence numbers which helps solve the classic ABA problem. The ABA problem is as follows: If a process reads an object twice and reads the value of the object to be 'A' both times, then it is unable to differentiate this scenario from a scenario where the object was changed to value 'B' in between the two reads of the object. Process $p$ then spins on apply$_[i]$ in line 2 until $p$ registers itself by swapping the value $\langle REG, s \rangle$ into apply$_[i]$ using a CAS operation. Processes write the value REG in the apply array to announce their presence at lock L.

Process $p$ then executes the role-loop, lines 4-12, until $p$ either increases the value of $Ctr$ to 1 or 2, or until $p$ is notified of its promotion. Process $p$ begins an iteration of the role-loop by calling the $Ctr$.inc() operation in line 5 and stores the returned value into $Role[i]$. The returned value determines $p$'s current role at lock L. The shared array $Role$ is used by process $p$ to store its role in slot $Role[i]$, which can later be read to determine the actions to perform at lock L. This is important because we want to allow the behavior of transferring locks. Specifically, to enable a process $q$ to call release$_j()$ on behalf of $p$, $q$ needs to determine $p$’s role at lock L, which is possible by reading $Role[i]$.

If the $Ctr$.inc() operation in line 5 fails, i.e., it returns $\perp$, then $p$ repeats the role-loop. Such repeats can happen only a constant number of times in expectation (by Claim A.2). If the value returned in line 5 is 0 or 1, then $p$ has incremented the value of $Ctr$ (from the semantics of a RCAScounter2 object), and it becomes $king_L$ or $queen_L$, respectively, and breaks out of the role-loop in line 12.

If $p$ becomes $king_L$ in line 5 then $p$ fails the if-condition of line 13 and proceeds to execute lines 16-17. In line 16, $p$ changes apply$_[i]$ to the value $\langle PRO, s \rangle$, to prevent itself from getting promoted in future promote actions. In line 17, $p$ returns from its lock() call by returning the special value $\infty$ (a non-$\perp$ value indicating a successful lock() call), since $p$ is $king_L$.

If $p$ becomes $queen_L$ in line 5 then $p$ knows that there exists a king process at lock L, and thus $queen_L$ proceeds to spin on Sync1 in line 14 awaiting a notification from $king_L$. Recall that $king_L$ notifies $queen_L$ of $queen_L$’s turn to own lock L by writing the integer $j$ into Sync1 during a release$_j()$ call. Once $p$ receives $king_L$’s notification (by reading a non-$\perp$ value in Sync1 in line 14), $p$ breaks out of the spin loop of line 14 and proceeds to execute lines 16-17. In line 16, $p$ changes apply$_[i]$ to the value $\langle PRO, s \rangle$, to prevent itself from getting promoted in future promote actions. In line 17, $p$ returns from its lock() call by returning the integer value stored in Sync1 (a non-$\perp$ value indicating a successful lock() call).

If the value returned in line 5 is 2, then $p$ does not become $king_L$ or $queen_L$, and thus $p$ assumes the role of a pawn. Process $p$ then waits for a notification of its own promotion, or, for the $Ctr$ value to decrease from 2, by spinning on apply$_[i]$ and $Ctr$ in line 7. When $p$ breaks out of this spin lock, it determines in line 8 whether it was promoted by checking whether the value of apply$_[i]$ was changed to $\langle PRO, s \rangle$. A process is promoted only by a $king_L$, $queen_L$ or a $pawn_L$ during their
release() call. If \( p \) finds that it was not promoted, then \( p \) is said to have been missed during a Ctr-cycle, and thus \( p \) repeats the role-loop. We later show that a process gets missed during at most one Ctr-cycle.

If \( p \) was promoted, then it writes a constant value \( \text{PAWN}_P = 3 \) into \( \text{Role}[i] \) in line 19 and becomes \( \text{pawn}_L \). Since \( p \) has been promoted, \( p \) knows that both \( \text{king}_L \) and \( \text{queen}_L \) are no longer executing their entry or Critical Section, and thus \( p \) owns lock \( L \) now. Then \( p \) goes on to break out of the role-loop in line 22 and proceeds to return from its \( \text{lock()} \) call by returning the special value \( \infty \) (a non-\( \bot \) value indicating a successful \( \text{lock()} \) call), since \( p \) is \( \text{pawn}_L \).

The \( \text{release()} \) method. Suppose \( p \) executes a call to \( \text{release}_i(j) \) with an integer argument \( j \). We restrict the execution such that a process calls a \( \text{release}_i(j) \) method only after a call to a successful \( \text{lock()} \) has been completed.

In line 34 \( p \) initializes the local variable \( r \) to the boolean value \( \text{false} \). Local variable \( r \) is returned later in line 35 to indicate whether the integer \( j \) was successfully written to \( \text{Sync1} \) during the release method call. In lines 35, 42 and 45 process \( p \) determines its role at the node and the action to perform. In line 49, process \( p \) deregisters itself from lock \( L \) by swapping \( \langle \bot, \bot \rangle \) into \( \text{apply}[i] \). At the end of the method call a boolean is returned in line 50 indicating whether the integer \( j \) was written to \( \text{Sync1} \).

If \( p \) determines that it is \( \text{king}_L \), then it attempts to decrease \( \text{Ctr} \) from 1 to 0 in line 36. This decrement operation will only fail if there exists a queen process at lock \( L \) which increased the \( \text{Ctr} \) to 2 during its \( \text{lock()} \) call. If the decrement operation fails then \( p \) has determined that there exists a queen process at lock \( L \) and it now synchronizes with \( \text{queen}_L \) to perform the collect action. Recall that \( \text{CAS} \) object \( \text{Sync1} \) is used by \( \text{king}_L \) and \( \text{queen}_L \) to determine which process performs a collect. In line 37 \( p \) attempts to swap integer \( j \) into \( \text{Sync1} \) by executing a \( \text{Sync1.CAS}(\bot, j) \) operation and stores the result of the operation in local variable \( r \). If \( p \) is successful then it performs the collect action by executing a call to \( \text{doCollect}_i() \) in line 38. If \( p \) is unsuccessful then it knows that \( \text{queen}_L \) will perform a collect. In line 39 \( p \) calls the \( \text{helpRelease}_i() \) method to synchronize the release of lock \( L \) with \( \text{queen}_L \). We describe the method \( \text{helpRelease}() \) shortly.

If \( p \) determines that it is \( \text{queen}_L \), then it calls the \( \text{helpRelease}_i() \) method call in line 43 to synchronize the release of lock \( L \) with \( \text{king}_L \).

If \( p \) determines that it is a promoted pawn, then it attempts to promote a waiting pawn by making a call to \( \text{doPromote}() \) in line 46.

The \( \text{doCollect}() \) method. Suppose a process \( p \) executing a \( \text{doCollect}_i() \) method call. The collect action consists of reading the \( \text{apply} \) array (left to right), and creating a vector \( A \) of \( n \) values, where the \( k \)-th element is either \( \bot \) (to indicate that the process with pseudo-ID \( k \) is not a candidate for promotion) or an integer sequence number (to indicate that the process with pseudo-ID \( k \) is a candidate for promotion). The vector \( A \) is stored in the \( \text{AbortableProArray}_n \) instance \( \text{PawnSet} \) in line 53 using a \( \text{PawnSet.collect}(A) \) operation. The \( \text{PawnSet.collect}(A) \) operation ensures that if the \( k \)-th element of \( \text{PawnSet} \) has value \( 3 = \text{ABORT} \) (written during a \( \text{PawnSet.abort}(k, \cdot) \) operation), then the \( k \)-th element is not overwritten during the \( \text{PawnSet.collect}(A) \) operation. This is required to ensure that processes that have expressed a desire to abort are not collected and subsequently promoted.

The \( \text{helpRelease}() \) method. Suppose \( \text{king}_L \) calls \( \text{helpRelease}_i() \) and \( \text{queen}_L \) calls \( \text{helpRelease}_q() \). During the course of these method calls, \( \text{king}_L \) and \( \text{queen}_L \) synchronize with each other in order to reset \( \text{CAS} \) objects \( \text{Sync1} \) and \( \text{Sync2} \), remove themselves from \( \text{PawnSet} \), promote a collected process and notify the promoted process. If no process is found in \( \text{PawnSet} \) that can be promoted, then the \( \text{PawnSet} \) object is reset to its initial state and \( \text{Ctr} \) reset to 0. Recall that \( \text{CAS} \) object \( \text{Sync2} \) is used as a synchronization primitive by \( \text{king}_L \) and \( \text{queen}_L \) to determine which process exits last among them, and thus performs all pending release work. In line 56 the process...
which swaps value \( i \) or \( k \) into \texttt{Sync2} by executing a successful \texttt{CAS} operation, exits, and the other process performs the pending release work in lines 57-63. Let us now refer to this other process as the releasing process. In lines 57-58 the releasing process resets \texttt{Sync1} to its initial value \( \perp \). In line 59 the releasing process reads the pseudo-ID written to \texttt{Sync2} by the exited process (process that executed a successful \texttt{CAS} operation on \texttt{Sync1}). The pseudo-ID written to \texttt{Sync2} is required to remove the exited process from getting promoted in a future promote in case it was collected in \texttt{PawnSet}. In line 61 the releasing process removes the exited process from \texttt{PawnSet}. \texttt{CAS} object \texttt{Sync2} is reset to its initial value \( \perp \) in line 60. In line 62 the releasing process calls \texttt{doPromote()} to promote a collected process.

The \texttt{doPromote()} method. Suppose \( p \) executes a call to \texttt{doPromote\textunderscore i()} in line 64. \( p \) removes itself from \texttt{PawnSet} by executing a \texttt{PawnSet.remove}(\( i \)) operation. It does so to prevent itself from getting promoted in case it was collected earlier. In line 65, \( p \) performs a promote action by executing a \texttt{PawnSet.promote()} operation. If a process was collected and the process has not aborted then its corresponding element \((k\text{-th element for a process with pseudo-ID } k)\) in \texttt{PawnSet} will have the value \( \langle \texttt{REG}, \cdot \rangle \). If a process has aborted then its corresponding element in \texttt{PawnSet} will have the value \( \langle \texttt{PRO}, \cdot \rangle \).

If a successful \texttt{promote()} operation is executed then an element in \texttt{ PawnSet} is changed from \( \langle \texttt{REG}, s \rangle \) to \( \langle \texttt{PRO}, s \rangle \), where \( s \in \mathbb{N} \), and the pair \( \langle k, s \rangle \) is returned, where \( k \) is the index of that element in \texttt{PawnSet}. In this case we say that process with pseudo-ID \( k \) was promoted. If an unsuccessful \texttt{promote()} operation is executed, then no element in \texttt{ PawnSet} has the value \( \langle \texttt{REG}, s \rangle \), where \( s \in \mathbb{N} \), and thus the special value \( \langle \perp, \perp \rangle \) is returned. We then say that no process was promoted. The returned pair is stored in local variables \( \langle j, \texttt{seq} \rangle \) in line 65.

If no process was promoted, then \( p \) resets \texttt{PawnSet} to its initial value in line 67 using the \texttt{reset()} operation, and decreases \texttt{Ctr} from 2 to 0 in line 68. If a process was found and promoted in \texttt{PawnSet}, then that process is notified of its promotion, by swapping its corresponding \texttt{apply} array element's value from \texttt{REG} to \texttt{PRO} using a \texttt{CAS} operation in line 70.

Recall that, while executing a \texttt{lock()} method call a process may receive a signal to abort. Suppose a process \( p \) receives a signal to abort while executing a \texttt{lock\textunderscore i()} method call. If process \( p \) is busy-waiting in lines 2-7 or 14 then \( p \) stops executing \texttt{lock\textunderscore i()}, and instead executes a call to \texttt{abort\textunderscore i()}. If \( p \) is poised to execute any line in 10 or 17 then it completes its call to \texttt{lock\textunderscore i()}. If \( p \) is poised to execute any other line then it continues executing \texttt{lock\textunderscore i()} until it begins to busy-wait in lines 2-7 or 14 at which point it stops and calls \texttt{abort\textunderscore i()}. If \( p \) does not begin to busy-wait in lines 2-7 or 14 then it completes its \texttt{lock\textunderscore i()} call.

The \texttt{abort()} method. Suppose \( p \) executes a call to \texttt{abort\textunderscore i()}. Process \( p \) first determines whether it quit \texttt{lock\textunderscore i()} while busy-waiting on \texttt{apply\textunderscore i[]} in line 2 and if so, \( p \) returns \( \perp \) in line 18. If not, then \( p \) changes \texttt{apply\textunderscore i[]} to the value \texttt{PRO} in line 19 to prevent itself from getting collected in future collect actions. In line 20 process \( p \) determines whether it quit \texttt{lock\textunderscore i()} while busy-waiting on \texttt{apply\textunderscore i[]} in line 7 or 14 or while busy-waiting on \texttt{Sync1} in line 14. If \( p \) quit while busy-waiting on \texttt{apply\textunderscore i[]} then clearly it is a pawn process, and if it quit while busy-waiting on \texttt{Sync1} then it is a queen process.

If process \( p \) determines that it is a pawn then it attempts to remove itself from \texttt{ PawnSet} by executing a \texttt{ PawnSet.abort\textunderscore i(\( i, s \))} operation in line 21 where \( s \) was the sequence number returned in line 11. If \( p \) has not been promoted yet, then the operation succeeds and \( p \)'s corresponding element in \texttt{ PawnSet} is changed to a value \( \langle \texttt{ABORT}, s \rangle \), thus making sure that \( p \) can not be collected or promoted anymore. If \( p \) has already been promoted then the operation fails and \( p \) now knows that it is has been promoted, and assumes the role of a promoted pawn, and in line 22, \( p \) writes \texttt{PAWN\textunderscore P} into \texttt{Role\textunderscore i[]} and returns the special value \( \infty \) in line 23.

If process \( p \) determines that it is \texttt{queen\textunderscore i} then it first attempts to swap a special value \( \infty \) into
Sync1 in line 26 by executing a Sync1.CAS(⊥, ∞) operation to indicate its desire to abort. If p is successful then p has determined that it is the first (among kingL and itself) to exit, and therefore p performs the collect action by calling doCollecti() in line 29. Process p then makes a call to helpReleasei() in line 31 to help release lock L by synchronizing with kingL.

If p was unsuccessful at swapping value ∞ into Sync1 then it knows the kingL is executing release(), and kingL will eventually perform the collect action. Then p has determined that it is the current owner of lock L, and returns the integer value stored in Sync1 in line 27.

Process p executes line 32 only if p successfully aborted earlier in its abort() call, and thus it deregisters itself from lock L by swapping ⟨⊥, ⊥⟩ into apply[i]. Finally, in line 33 p returns ⊥ to indicate a successful abort (i.e., a failed lock() call).

### D.2 Analysis and Proofs of Correctness

Let H be an arbitrary history of an algorithm that accesses an instance, L, of object ALockArrayn, where the following safety conditions hold.

**Condition D.1.** (a) No two locki() calls are executed concurrently for the same i, where i ∈ {0, . . . , n − 1}.

(b) If a process p executes a successful locki() call, then some process q eventually executes a releasei() call where the invocation of releasei() happens after the response of locki() (assuming the scheduler is such that q continues to make progress until its releasei() call happens).

(c) For every releasei() call, there must exist a unique successful locki() call that completed before the invocation of the releasei() call.

Then the following claims hold for history H.

**Lemma D.2.** Methods releasei(j), aborti(), helpReleasei(), doCollecti(), doPromotei() are wait-free.

**Proof.** Follows from an inspection of these methods. □

**Claim D.3.** No two releasei() calls where a shared memory step is pending, are executed concurrently for the same i, where i ∈ {0, . . . , n − 1}.

**Proof.** Assume for the purpose of a contradiction that two processes are executing a call to releasei() concurrently for the first time at time t. Then from Condition D.1(b)-(c), it follows that two successful calls to locki() were executed before t. From condition D.1(a) it follows that the two successful locki() calls did not overlap. Consider the first successful locki() call executed by some process p. Since the locki() call returned a non-⊥ value, the method did not return from line 18. Then p did not abort while busy-waiting in line 2, and thus apply[i] was set to a non-⟨⊥, ⊥⟩ value in line 2 during the first locki() call. Let t' be the point in time when apply[i] was set to a non-⟨⊥, ⊥⟩ value in line 2. We now show that the apply[i] ≠ ⟨⊥, ⊥⟩ in the duration between [t', t]. Suppose not, i.e., some process resets apply[i] to ⟨⊥, ⊥⟩ during [t', t]. Now, apply[i] is reset to a ⟨⊥, ⊥⟩ value only in line 32 during aborti() or in line 49 during releasei().

**Case a -** apply[i] reset to ⟨⊥, ⊥⟩ in line 49 during releasei(). Then the last shared memory step of the releasei() has been executed, and the call has ended for the purposes of the claim. Then the two releasei() calls are not concurrent at t, a contradiction.
Case b - apply\([i]\) reset to \(\langle \bot, \bot \rangle\) in line 32 during \(\text{abort}_i()\). Since the two \(\text{lock}_i()\) calls are not concurrent it follows that \(\text{apply}[i] \neq \langle \bot, \bot \rangle\) at the end of the first \(\text{lock}_i()\) call, and thus \(\text{apply}[i]\) is reset to \(\langle \bot, \bot \rangle\) in line 32 during the second successful \(\text{lock}_i()\) call. Now consider the second successful \(\text{lock}_i()\) call executed by some process \(q\). Then \(q\) would repeatedly fail the \(\text{apply}[i].\text{CAS}(\langle \bot, \bot \rangle, \cdot)\) operation of line 2, and the only way \(q\)'s \(\text{lock}_i()\) call could finish, is if \(q\) aborts the busy-wait loop of line 2. In which case \(q\) executes \(\text{abort}_i()\), and satisfies the if-condition of line 18 and return \(\bot\) in line 18. Then the second \(\text{lock}_i()\) does not reset \(\text{apply}[i]\) in line 32 during \(\text{abort}_i()\) - a contradiction.

Since \(\text{apply}[i] \neq \langle \bot, \bot \rangle\) throughout \([t', t]\), it then follows from the same argument of Case b, that the second \(\text{lock}_i()\) call is unsuccessful, and thus a contradiction. \(\square\)

From Claim D.3 and Condition D.1(a) it follows that no two calls to \(\text{lock}_p()\) or \(\text{release}_p()\) are executed concurrently for the same \(p\), where \(p \in \{0, \ldots, n - 1\}\). Then we can label the process executing a \(\text{lock}_p()\) or \(\text{release}_p()\) call, simply \(p\), without loss of generality. We do so to make the rest of the proofs easier to follow.

**Helpful claims based on variable usage.**

**Claim D.4.** (a) \(\text{Role}[p]\) is changed by process \(q\), only if \(q = p\).

(b) \(\text{Role}[p]\) is unchanged during \(\text{release}_p()\).

(c) \(\text{Role}[p]\) can be set to value \text{KING, QUEEN or PAWN} only when \(p\) executes line 5 during \(\text{lock}_p()\).

(d) \(\text{Role}[p]\) is set to value \text{PAWN_P} only when \(p\) executes line 22 during \(\text{lock}_p()\) or when \(p\) executes line 22 during \(\text{abort}_p()\).

**Proof.** All claims follow from an inspection of the code. \(\square\)

**Claim D.5.** (a) The only operations on \(\text{PawnSet}\) are \(\text{collect}(A), \text{promote}(), \text{remove}(i), \text{remove}(j), \text{abort}(k, s)\), and \(\text{reset}()\) (in lines 55, 62, 64, 67, 21 and 27, respectively) where \(A\) is a vector with values in \([\bot] \cup \mathbb{N}\), and \(i, j, k \in \{0, 1, \ldots, n - 1\}\), and \(s \in \mathbb{N}\).

(b) The \(i\)-th entry of \(\text{PawnSet}\) can be changed to \(\langle \text{REG}, s \rangle\), where \(s \in \mathbb{N}\), only when a process executes a \(\text{PawnSet.c}^{\text{ollect}}(A)\) operation in line 55 where \(A[i] = s\).

(c) The \(i\)-th entry of \(\text{PawnSet}\) can be changed to \(\langle \text{PRO}, s \rangle\), where \(s \in \mathbb{N}\), only when a process executes a \(\text{PawnSet.prom}^{\text{ote}}()\) operation in line 64.

(d) The \(i\)-th entry of \(\text{PawnSet}\) can be changed to \(\langle \text{ABORT}, s \rangle\), where \(s \in \mathbb{N}\), only when a process executes a \(\text{PawnSet.remove}(i), \text{PawnSet.remove}(j)\) or \(\text{PawnSet.abort}(k, s)\) operation in lines 62, 61 or 27, respectively.

**Proof.** Part (a) follows from an inspection of the code. Parts (b), (c) and (d) follow from Part (a) and the semantics of type \text{AbortableProArray}_n^{\text{}}. \(\square\)

**Claim D.6.** Let \(s \in \mathbb{N}\).

(a) \(\text{apply}[p]\) is changed from \(\langle \bot, \bot \rangle\) to a non-\(\langle \bot, \bot \rangle\) value only when process \(p\) executes a successful \(\text{apply}[p].\text{CAS}(\langle \bot, \bot \rangle, \langle \text{REG}, s \rangle)\) operation in line 2.

(b) \(\text{apply}[p]\) is changed to value \(\langle \text{REG}, s \rangle\) only when process \(p\) executes a successful \(\text{apply}[p].\text{CAS}(\langle \bot, \bot \rangle, \langle \text{REG}, s \rangle)\) operation in line 2.
(c) \text{apply}[p] \text{ is changed to a } \langle \bot, \bot \rangle \text{ value only when } p \text{ executes a successful } \text{apply}[p].\text{CAS}(\langle \text{PRO}, s \rangle, \langle \bot, \bot \rangle) \text{ operation either in line 32 or line 49.}

\textbf{Proof.} Parts (a), (b) and (c) follow from an inspection of the code. \hfill \square

\textbf{Helpful Notations and Definitions.} We now establish a notion of time for our history \( H \).

Let the \( i \)-th step in \( H \) occur at time \( t_i \). Then every point in time during \( H \) is in \( \mathbb{N} \).

Let \( t^i_p \) denote the point in time immediately after process \( p \) has finished executing line \( i \), and no process has taken a step since \( p \) has executed the last operation of line \( i \) (This operation can be the response of a method call made in line \( i \)). Since some private methods are invoked from more than one place in the code, the point in time \( t^{i_p}_i \), where \( i \) is a line in the method, does not refer to a unique point in time in history \( H \). In those cases we make sure that it is clear from the context of the discussion, which point \( t^i_p \) refers to. Let \( t^{i-}_p \) denote the point in time when \( p \) is poised to execute line \( i \), and no other process takes steps before \( p \) executes line \( i \).

Let \( p \) be an arbitrary process and \( s \) be an arbitrary integer. We say process \( p \) \textit{registers}, when it executes a successful \text{apply}[p].\text{CAS}(\langle \bot, \bot \rangle, \langle \text{REG}, s \rangle) \text{ operation in line 16}. Process \( p \) \textit{captures} and \textit{wins} lock \( L \) when it returns from \text{lock}() \text{ with a non-} \bot \text{ value. Process } p \text{ is said to } \textit{promote} \text{ another process } q \text{ if } p \text{ executes a PawnSet.promote()} \text{ operation in line 65 that returns a value } \langle q, s \rangle, \text{ where } s \in \mathbb{N}. \text{ A process } p \text{ is said to be } \textit{promoted at lock } L, \text{ if some process } q \text{ executes a PawnSet.promote()} \text{ operation that returns value } \langle p, s \rangle, \text{ where } s \in \mathbb{N}.

Process \( p \) is said to \textit{hand over} lock \( L \) to process \( q \) if it executes a successful \text{CAS} operation \text{L.Sync1.CAS}(\langle \bot, j \rangle) \text{ in line 37} \text{ where } j \text{ is the process that last increased } \text{Ctr} \text{ from 1 to 2}. \text{ Process } p \text{ is said to have } \textit{released} \text{ lock } L \text{ by executing a successful } \text{Ctr.CAS}(1, 0) \text{ operation in line 38 or by executing a successful } \text{Ctr.CAS}(2, 0) \text{ operation in line 68. Process } p \text{ either hands over, promotes a process, or releases lock } L \text{ during a call to L.release}_p(j) \text{ where } j \text{ is an arbitrary integer.} \text{ A process } \textit{ceases to own} \text{ a lock either by releasing lock } L \text{ or by promoting another process, or by handing over lock } L \text{ to some other process. Process } p \text{ is } \textit{deregistered} \text{ when } p \text{ executes a successful } \text{apply}[p].\text{CAS}(\langle \text{PRO}, s \rangle, \langle \bot, \bot \rangle) \text{ operation in line 32 or 49}. \text{ A process } p \text{ is said to be } \textit{not registered in PawnSet} \text{ if the } p \text{-th entry of PawnSet is not value } \langle \text{REG}, s \rangle, \text{ where } s \in \mathbb{N}. \text{ The repeat-until loop starting at line 41 and ending at line 12 is called role-loop.}

In some of the proofs we use represent an execution using diagrams, and the legend for the symbols used in the diagrams is given in Figure 7.

\textbf{Releasees of lock and Cease-release events.}

A process \( p \) becomes a \textit{releaser} of lock \( L \) at time \( t \) when

(R1) \( p \) increases \text{Ctr} to 1 (i.e., \text{Ctr.inc}() returns 0 = \text{KING}) \text{ or 2 (i.e., \text{Ctr.inc}() returns 1 = \text{QUEEN}), or when}

(R2) \( p \) is promoted at lock \( L \) by some process \( q \).

\textbf{Claim D.7.} \begin{enumerate} 
\item \( p \) executes a \text{Ctr.CAS}(1, 0) \text{ operation only in line 36 during } \text{release}_p(j). 
\item \( p \) executes a \text{Sync2.CAS}(\langle \bot, p \rangle) \text{ operation only in line 56 during } p \text{'s call to } \text{helpRelease}_p(). 
\item \( p \) executes a PawnSet.promote() \text{ operation only in line 65 during } p \text{'s call to } \text{doPromote}_p(). 
\item \( p \) executes a \text{Ctr.CAS}(2, 0) \text{ operation only in line 68 during } p \text{'s call to } \text{doPromote}_p(). 
\end{enumerate}

\textbf{Proof.} All claims follows from an inspection of the code. \hfill \square

We now define the following \textit{cease-release} events with respect to \( p \):
Figure 7: Legend for Figures 8 to 16

- $\ell$: Atomic shared memory operation executed in line $\ell$.
- $m$: Method call $m$ executed in line $\ell$.
- $\ell_1 \rightarrow \ell_2$: Operation in line $\ell_2$ is executed only after the operation in line $\ell_1$.
- $\mathcal{P}$: Predicate $\mathcal{P}$ is evaluated in line $\ell$.
  - $\ell_1$: The next line executed is $\ell_1$ if and only if $\mathcal{P}$ is true.
  - $\ell_2$: The next line executed is $\ell_2$ if and only if $\mathcal{P}$ is false.
- $\ell_1 \rightarrow \ell_2 \rightarrow \ell_3$: Predicate $\mathcal{P}$ is evaluated in line $\ell$.
  - $\ell_1$: The next line executed is $\ell_1$ if and only if $\mathcal{P}$ is true.
  - $\ell_2$: The next line executed is $\ell_2$ if and only if $\mathcal{P}$ is false and condition $C$ is false.
  - $\ell_3$: The next line executed is $\ell_3$ if and only if $\mathcal{P}$ is false and condition $C$ is true.
- $\ell_1 \rightarrow \ell_2$: Statement $S$ holds after line $\ell_1$ is executed and before line $\ell_2$ is executed.
- $S \rightarrow$: Helpful statement $S$ to improve the readability of the figure.
- $\text{op} \rightarrow \ell$: Operation $\text{op}$ executed in line $\ell$.
\( \phi_p: p \) executes a successful \( \text{Ctr.CAS}(1, 0) \) at \( t_p \) during \( \text{release}_p(j) \).

\( \tau_p: p \) executes a successful \( \text{Sync2.CAS}(\bot, p) \) at \( t_p \) during \( \text{helpRelease}_p() \).

\( \pi_p: p \) promotes some process \( q \) at \( t_p \) during \( \text{doPromote}_p() \).

\( \theta_p: p \) executes an operation \( \text{Ctr.CAS}(2, 0) \) at \( t_p \) during \( \text{doPromote}_p() \).

Process \( p \) ceases to be a releaser of lock \( L \) when one of \( p \)'s cease-release events occurs. We say process \( p \) is a releaser of lock \( L \) at any point after it becomes a releaser and before it ceases to be a releaser.

**Claim D.8.** (a) Method \( \text{doCollect}_p() \) is called only by process \( p \) in lines 29 and 38.

(b) Method \( \text{helpRelease}_p() \) is called only by process \( p \) in lines 39, 43 and 30.

(c) Method \( \text{doPromote}_p() \) is called only by process \( p \) in line 46 and in line 62 (during \( \text{helpRelease}_p() \)).

(d) If cease-release event \( \phi_p \) occurs then \( p \) is executing \( \text{release}_p(j) \).

(e) If cease-release event \( \tau_p \) occurs then \( p \) is executing \( \text{helpRelease}_p() \).

(f) If cease-release event \( \pi_p \) or \( \theta_p \) occurs then \( p \) is executing \( \text{helpRelease}_p() \) or \( \text{doPromote}_p() \).

**Proof.** Parts (a), (b) and (c) follow from an inspection of the code. By definition, cease-release event \( \phi_p \) occurs when \( p \) executes a successful \( \text{Ctr.CAS}(1, 0) \) operation in line 36 during \( \text{release}_p(j) \), and thus (d) follows immediately. By definition, cease-release event \( \tau_p \) occurs when \( p \) executes a successful \( \text{Sync2.CAS}(\bot, p) \) in line 56 during \( \text{helpRelease}_p() \), and thus (e) follows immediately.

By definition, cease-release event \( \pi_p \) occurs only when \( p \) executes a \( \text{PawnSet.promote()} \) operation that returns a non-\( \langle \bot, \bot \rangle \) value in line 65 and cease-release event \( \theta_p \) occurs only when \( p \) executes a \( \text{Ctr.CAS}(2, 0) \) operation in line 68. Then if cease-release event \( \pi_p \) or \( \theta_p \) occurs then \( p \) is executing \( \text{doPromote}_p() \). From (e), \( p \) could also call \( \text{doPromote}_p() \) from line 62 during \( \text{helpRelease}_p() \). Then if cease-release event \( \pi_p \) or \( \theta_p \) occurs then \( p \) is executing \( \text{doPromote}_p() \) or \( \text{helpRelease}_p() \). Thus, (f) holds.

**Claim D.9.** Consider \( p \)'s \( k \)-th passage, where \( k \in \mathbb{N} \). Note that \( s = k \). If \( \text{Role}[p] = \text{PAWN}_p \) at some point in time \( t \) during \( p \)'s call to \( \text{lock}_p() \), then some process \( q \) promoted \( p \) at \( t_q < t \) and \( p \) became releaser of \( L \) by condition (R2) at \( t_q < t \).

**Proof.** From Claim D.4(d), \( p \) changes \( \text{Role}_p \) to \( \text{PAWN}_p \) only in line 9 or line 22.

**Case a -** \( p \) changed \( \text{Role}_p \) to \( \text{PAWN}_p \) in line 22. Then \( p \)'s call to \( \text{PawnSet.abort}(p, s) \) returned \( \text{false} \) in line 21. From the semantics of the \( \text{AbortableProArray} \) object, it follows that the \( p \)-th entry of \( \text{PawnSet} \) was set to value \( \langle \text{PRO}, s \rangle = \langle 2, s \rangle \). From Claim D.5(c), the \( p \)-th entry of \( \text{PawnSet} \) is set to value \( \langle \text{PRO}, s \rangle \) only when a \( \text{PawnSet.promote()} \) operation returns \( \langle p, s \rangle \) in line 65.

Then some process \( q \) promoted \( p \) at \( t_q \) and \( p \) became a releaser of \( L \) by condition (R2) at \( t_q < t \).

**Case b -** \( p \) changed \( \text{Role}_p \) to \( \text{PAWN}_p \) in line 9. Then \( p \) broke out of the spin loop of line 4 and thus \( \text{apply}[p] = \langle \text{REG}, s \rangle \neq \langle \text{PRO}, s \rangle \) at \( t_p \). Since \( p \) satisfied the if-condition of line 8, it follows that \( \text{apply}[p] = \langle \text{PRO}, s \rangle \) at \( t_p \). Since \( p \) does not change \( \text{apply}[p] \) to value \( \langle \text{PRO}, s \rangle \) during \( [t_p, t_p] \) it follows that some other process changed \( \text{apply}[p] \) to value \( \langle \text{PRO}, s \rangle \). Now, \( \text{apply}[p] \) is changed to value \( \langle \text{PRO}, s \rangle \) by some other process (say \( q \)) only in line 70 and thus, from the code structure, \( q \) also executed a \( \text{PawnSet.promote()} \) operation that returned \( \langle p, s \rangle \) in line 65. Then \( q \) promoted \( p \) at \( t_q \) and \( p \) became a releaser of \( L \) by condition (R2) at \( t_q < t \).
Claim D.10. Consider p’s k-th passage, where \( k \in \mathbb{N} \). If \( t \in \{ \lceil \frac{18}{p}, \frac{33}{p} \rceil, \lceil \frac{37}{p}, \frac{39}{p} \rceil, \lceil \frac{43}{p}, \frac{43}{p} \rceil, \lceil \frac{46}{p}, \frac{46}{p} \rceil \} \), then cease-release event \( \phi_p \) does not occur before time \( t \).

Proof. By definition, cease-release event \( \phi_p \) occurs when \( p \) executes a successful \( \text{Ctr.CAS}(1,0) \) operation in line 36. From Claim D.8(d) cease-release event \( \phi_p \) occurs only during \( \text{release}_p(j) \).

Case a - \( t \in \{ \frac{18}{p}, \frac{33}{p} \} \). Then \( p \) is executing \( \text{abort}_p() \) and has not yet executed a call to \( \text{release}_p() \). Since cease-release event \( \phi_p \) can occur only during \( \text{release}_p() \), cease-release event \( \phi_p \) did not occur before time \( t \).

Case b - \( t \in \{ \frac{37}{p}, \frac{39}{p} \} \): Then \( p \) must have failed the if-condition of line 35 and thus \( p \) executed an unsuccessful \( \text{Ctr.CAS}(1,0) \) operation in line 36 and cease-release event \( \phi_p \) did not occur before time \( t \).

Case c - \( t \in \{ \frac{43}{p}, \frac{43}{p}, \frac{46}{p}, \frac{46}{p} \} \): From Claim D.11 \( \text{Role}[p] \) \( \in \{ \text{QUEEN}, \text{PAWN}_P \} \) at \( t \). Since \( \text{Role}[p] \) is unchanged during \( \text{release}_p() \) (Claim D.4(b)), it follows that \( \text{Role}[p] \neq \text{KING} \) at \( \frac{35}{p} \). Then \( p \) fails the if-condition of line 35 and does not execute line 36 and thus cease-release event \( \phi_p \) did not occur before time \( t \).

The proof of the following claim has been moved to Appendix F since the proof is long and straightforward.

Claim D.11. The value of \( \text{Role}[p] \) at various points in time during \( p \)’s k-th passage, where \( k \in \mathbb{N} \), is as follows.

| Time       | Value of \( \text{Role}[p] \)       | Time       | Value of \( \text{Role}[p] \)       |
|------------|-------------------------------------|------------|-------------------------------------|
| \( \frac{2}{p} \)       | \{⊥, \text{KING, QUEEN, PAWN} \}               | \( \frac{4}{p} \)       | \{KING, QUEEN, PAWN_P \}               |
| \( \frac{3}{p} \)       | \text{PAWN}                        | \( \frac{21}{p} \)       | \{KING, QUEEN, PAWN_P \}               |
| \( \frac{13}{p} \)      | \text{PAWN}_P                      | \( \frac{14}{p} \)       | \{KING, QUEEN, PAWN_P \}               |
| \( \frac{11}{p} \)      | \text{QUEEN}                      | \( \frac{16}{p} \)       | \{KING, QUEEN, PAWN_P \}               |
| \( \frac{19}{p} \)      | \{KING, QUEEN, PAWN_P \}           | \( \frac{19}{p} \)       | \{KING, QUEEN \}                      |
| \( \frac{23}{p} \)      | \{QUEEN, PAWN \}                  | \( \frac{19}{p} \)       | \{KING, QUEEN \}                      |
| \( \frac{25}{p} \)      | \text{PAWN}                       | \( \frac{19}{p} \)       | \{KING, QUEEN \}                      |
| \( \frac{28}{p} \)      | \text{QUEEN}                      | \( \frac{19}{p} \)       | \{KING, QUEEN, PAWN_P \}               |

Claim D.12. Consider p’s k-th passage, where \( k \in \mathbb{N} \).

(a) If process \( p \) calls \( \text{helpRelease}_p() \) or \( \text{doPromote}_p() \) during \( \text{abort}_p() \) then it does not call \( \text{release}_p(j) \).

(b) Process \( p \) calls \( \text{helpRelease}_p() \) at most once.

(c) Process \( p \) calls \( \text{doPromote}_p() \) at most once.

Proof. Proof of (a): The following observations follow from an inspection of the code. If \( p \) executes \( \text{doPromote}_p() \) during \( \text{abort}_p() \), then it does so during a call to \( \text{helpRelease}_p() \) in line 62. If \( p \) executes \( \text{helpRelease}_p() \) during \( \text{abort}_p() \), then it does so by executing line 30. Then \( p \) calls \( \text{helpRelease}_p() \) or \( \text{doPromote}_p() \) during \( \text{abort}_p() \) in line 30 and goes on to return value \( ⊥ \) in line 33. Then \( p \)’s call to \( \text{lock}_p() \) returns value \( ⊥ \) and \( p \) does not call \( \text{release}_p() \) (follows from conditions 3 and 4).
Proof of (b): From Part (ii), if \( \text{helpRelease}_p() \) is executed during \( \text{abort}_p() \) then \( \text{release}_p(j) \) is not executed. Then to prove our claim we need to show that \( \text{helpRelease}_p() \) is called at most once during \( \text{abort}_p() \) and \( \text{release}_p(j) \), respectively. From Claim D.8(c), method \( \text{helpRelease}_p() \) is called by \( p \) only in lines 39, 43 and 50. Since \( \text{helpRelease}_p() \) is called only once during \( \text{abort}_p() \) (specifically in line 30), it follows immediately that \( p \) executes \( \text{helpRelease}_p() \) at most once during \( \text{abort}_p() \). From Claim D.11 \( \text{Role}[p] \in \{\text{KING, QUEEN, PAWN}_p\} \) at \( t_{p}^3 \). Since \( \text{Role}[p] \) is unchanged during \( \text{release}_p() \) (Claim D.4(b)), it follows that \( p \) satisfies exactly one of the if-conditions of lines 35, 42 and 45, and thus \( p \) does not execute both lines 39 and 43. Then \( p \) executes \( \text{helpRelease}_p() \) at most once during \( \text{release}_p(j) \).

Proof of (c): From Part (iii), if \( \text{doPromote}_p() \) is executed during \( \text{abort}_p() \) then \( \text{release}_p(j) \) is not executed. Then to prove our claim we need to show that \( \text{doPromote}_p() \) is called at most once during \( \text{abort}_p() \) and \( \text{release}_p(j) \), respectively. From Claim D.8(c), method \( \text{doPromote}_p() \) is called by \( p \) only in line 46 and in line 62 (during \( \text{helpRelease}_p() \)).

Case a - \( p \) called \( \text{doPromote}_p() \) in line 46 (during \( \text{helpRelease}_p() \)). Then \( p \) is executing \( \text{helpRelease}_p() \). From Claim D.8(c), method \( \text{helpRelease}_p() \) is called by \( p \) only in lines 39, 43 and 50. Then \( p \) called \( \text{helpRelease}_p() \) either in line 39, 43 or 50.

Case a(i) - \( p \) called \( \text{helpRelease}_p() \) in line 39 or 43 (during \( \text{release}_p(j) \)). Then \( p \) is executing \( \text{release}_p() \), and since \( p \) called \( \text{helpRelease}_p() \) in lines 39 or 43 \( p \) satisfied the if-conditions of lines 35 or 42, and thus \( \text{Role}[p] = \text{KING} \) at \( t_{p}^{35} \) or \( \text{Role}[p] = \text{QUEEN} \) at \( t_{p}^{42} \), respectively. Since \( \text{Role}[p] \) is unchanged during \( \text{release}_p() \) (Claim D.4(b)), it follows that \( \text{Role}[p] \in \{\text{KING, QUEEN}\} \) during \( \text{release}_p() \). Then \( p \) fails the if-condition of line 45 and does not execute \( \text{doPromote}_p() \) in line 46. Hence, \( p \) executes \( \text{doPromote}_p() \) at most once during \( \text{release}_p() \).

Case a(ii) - \( p \) called \( \text{helpRelease}_p() \) in line 50 Then \( p \) is executing \( \text{abort}_p() \) and it goes on to return value \( \bot \) in line 55. Then \( p \)'s call to \( \text{lock}_p() \) returns value \( \bot \) and \( p \) does not call \( \text{release}_p(j) \) (follows from conditions (i) and (ii)). Hence, \( p \) executes \( \text{doPromote}_p() \) at most once during \( \text{abort}_p() \).

Case b - \( p \) called \( \text{doPromote}_p() \) in line 46. Then \( p \) is executing \( \text{helpRelease}_p() \) and \( p \) satisfied the if-condition of lines 35 and 42, and thus \( \text{Role}[p] = \text{PAWN}_p \) at \( t_{p}^{35} \). Since \( \text{Role}[p] \) is unchanged during \( \text{release}_p() \) (Claim D.4(b)), it follows that \( \text{Role}[p] = \text{PAWN}_p \) during \( \text{release}_p() \). Then \( p \) failed the if-condition of lines 35 and 42 and \( p \) did not execute \( \text{helpRelease}_p() \) in lines 39 and 43. Hence, \( p \) executes \( \text{doPromote}_p() \) at most once during \( \text{release}_p() \).

Claim D.13. Consider \( p \)'s \( k \)-th passage, where \( k \in \mathbb{N} \). Let \( t \) be a point in time at which either \( p \) is poised to execute \( \text{release}_p(j) \), or \( t \in \{ t_{p}^{20}, t_{p}^{29}, t_{p}^{37}, t_{p}^{38}, t_{p}^{51}, t_{p}^{55}, t_{p}^{56}, t_{p}^{57}, t_{p}^{62}, t_{p}^{65}, t_{p}^{67}, t_{p}^{68}\} \). Then

(a) none of \( p \)'s cease-release events have occurred before time \( t \), and

(b) \( p \) is a releaser of \( \text{lock}_L \) at time \( t \).

Proof. Proof of (a): First note that if \( t \in \{ t_{p}^{20}, t_{p}^{29}, t_{p}^{37}, t_{p}^{38}\} \) then \( p \) is executing \( \text{doCollect}() \). From Claim D.8(a), \( p \) calls \( \text{doCollect}() \) only in lines 29 and 38. Then if \( t \in \{ t_{p}^{20}, t_{p}^{29}\} \) then \( t \in \{ t_{p}^{29}, t_{p}^{29}\} \) or \( t \in \{ t_{p}^{37}, t_{p}^{38}\} \). Therefore, assume now \( t \in \{ t_{p}^{29}, t_{p}^{29}\} \) or \( t \in \{ t_{p}^{37}, t_{p}^{38}\} \).

Case a - \( t \in \{ t_{p}^{29}, t_{p}^{29}, t_{p}^{37}, t_{p}^{38}\} \): If \( t \in \{ t_{p}^{29}, t_{p}^{29}\} \) then from a code inspection, \( p \) is executing \( \text{abort}_p() \) and \( p \) did not execute a call to \( \text{doPromote}_p() \) or \( \text{helpRelease}_p() \) before time \( t \). If \( t \in \{ t_{p}^{37}, t_{p}^{38}\} \) then \( p \) is executing \( \text{release}_p(j) \) and then from a code inspection and Claim D.12(a) it follows that \( p \) did not execute a call to \( \text{doPromote}_p() \) or \( \text{helpRelease}_p() \) before
time $t$. Then from Claims \([D.8(c)]\) and \([D.8(f)]\) it follows that events $\tau_p$, $\pi_p$ and $\theta_p$ did not occur before time $t$. Since $t \in [\frac{20}{p}, \frac{29}{p})$ or $t \in [\frac{37}{p}, \frac{48}{p}]$, it follows from Claim \([D.10]\) that cease-release event $\phi_p$ did not occur before time $t$. 

**Case b -** $t \in \{\frac{50}{p}, \frac{62}{p}, \frac{68}{p}\}$: Then $p$ is executing $\text{helpRelease}_p()$. Then from Claim \([D.8(b)]\) it follows that $p$ is executing a call to $\text{helpRelease}_p()$ in line $39$, $43$ or $30$. Then from Claim \([D.10]\) it follows that cease-release event $\phi_p$ did not occur before time $t$. From Claim \([D.12(b)]\), it follows that this is $p$'s only call to $\text{helpRelease}_p()$. From Claim \([D.8(c)]\), $p$ calls $\text{doPromote}_p()$ only in line $46$ and in line $62$ (during $\text{helpRelease}_p()$). Since $p$ has not yet executed line $46$ and this is the only call to $\text{helpRelease}_p()$, $p$ has not called $\text{doPromote}_p()$ before time $t$. Then from Claim \([D.8(b)]\) it follows that events $\tau_p$ and $\theta_p$ did not occur before time $t$. By definition, cease-release event $\tau_p$ occurs when $p$ executes a successful $\text{Sync2.CAS}(\bot, p)$ in line $56$. If $t = \frac{50}{p}$, then clearly cease-release event $\tau_p$ did not occur before time $t$. If $t \in [\frac{57}{p}, \frac{62}{p}]$, then $p$ has already been at some point in line $56$, and thus $p$ executed an unsuccessful $\text{Sync2.CAS}(\bot, p)$ operation in line $56$ and thus cease-release event $\tau_p$ did not occur before time $t$. 

**Case c -** $t \in \{\frac{65}{p}, \frac{67}{p}, \frac{68}{p}\}$: Then $p$ is executing $\text{doPromote}_p()$. From Claim \([D.12(c)]\), it follows that this is the only call to $\text{doPromote}_p()$. By definition, cease-release event $\theta_p$ occurs only when $p$ executes a $\text{Ctr.CAS}(2, 0)$ operation in line $68$ of $\text{doPromote}_p()$. Event $\theta_p$ did not occur before time $t$ since $t < \frac{65}{p}$ and this is $p$'s only call to $\text{doPromote}_p()$. By definition, cease-release event $\pi_p$ occurs only when $p$ executes a $\text{PawnSet.promote()}$ operation that returns a non-$\bot$ value in line $65$ of $\text{doPromote}_p()$. If $t = \frac{65}{p}$, then cease-release event $\pi_p$ did not occur before time $t$ since $\frac{65}{p} < \frac{65}{p}$ and thus $p$'s $\text{PawnSet.promote()}$ operation returned value $\bot$. From Claim \([D.4(b)]\), it follows that $\text{PawnSet.promote()}$ operation returned value $\bot$. From Claim \([D.10]\), it follows that cease-release event $\phi_p$ did not occur before time $t$. 

We now show that cease-release event $\tau_p$ did not occur before time $t$ thus completing the proof.

**Subcase c(i) -** $p$ called $\text{doPromote}_p()$ during $\text{helpRelease}_p()$: Then $p$ satisfied the if-condition of line $56$, and thus $p$ executed an unsuccessful $\text{Sync2.CAS}(\bot, p)$ operation in line $56$ and cease-release event $\tau_p$ did not occur before time $t$. 

**Subcase c(ii) -** $p$ called $\text{doPromote}_p()$ in line $46$. From Claim \([D.11]\) $\text{Role}[p] \in \text{PAWN.P}$ at $\frac{46}{p}$. Since $\text{Role}[p]$ is unchanged during $\text{release}_p()$ (Claim \([D.4(b)]\)), it follows that $\text{Role}[p] = \text{PAWN.P}$ at $\frac{65}{p}$ and $\frac{68}{p}$. Then $p$ failed the if-conditions of lines $35$ and $12$ and does not execute a call to $\text{helpRelease}_p()$ before time $t$. Then from Claim \([D.8(c)]\) it follows that cease-release event $\tau_p$ did not occur before time $t$. 

**Proof of (b):** From Part (a), $p$ does not cease to be releaser of L before time $t$. Therefore, to prove our claim we need to show that $p$ becomes a releaser of L at some point $t' < t$. We first show that $\text{Role}[p] \in \{\text{KING, QUEEN, PAWN.P}\}$ at time $t$. Let $t'$ be the point when $p$ is poised to execute $\text{release}_p(j)$. From the inspection of the various points in time chosen for $t$ (including $\frac{43}{p}$, but excluding $t'$) and the table in Claim \([D.11]\) it follows that $\text{Role}[p] \in \{\text{KING, QUEEN, PAWN.P}\}$ at time $t$ (including $\frac{43}{p}$, but excluding $t'$). Clearly $\text{Role}[p]$ is unchanged during $[t', \frac{44}{p}]$. Then the value of $\text{Role}[p]$ at $t'$ is the same as that at $\frac{44}{p}$, i.e., $\text{Role}[p] \in \{\text{KING, QUEEN, PAWN.P}\}$. 

**Case a -** $\text{Role}[p] \in \{\text{KING, QUEEN}\}$ at time $t$: From Claim \([D.4(c)]\), $\text{Role}[p]$ is set to $\text{KING}$ or $\text{QUEEN}$ only when $p$ executes line $5$. Then $p$ changed $\text{Role}[p]$ to $\text{KING}$ or $\text{QUEEN}$ at $\frac{43}{p}$, and thus $p$ became a releaser of lock L by condition (R1) at $\frac{43}{p}$. 

**Case b -** $\text{Role}[p] = \text{PAWN.P}$ at time $t$: From Claim \([D.9]\) it follows that some process $q$
promoted \( p \) at \( \frac{65}{q} \) and \( p \) became a releaser of \( L \) by condition (R2) at \( \frac{65}{q} = t' < t \).  

\[
\text{Claim D.14. Consider } p \text{'s } k \text{-th passage, where } k \in \mathbb{N}. \text{ If any of process } p \text{'s cease-release events occurs at time } t \text{ then } p \text{ ceases to be the releaser of lock } L \text{ at time } t. 
\]

\[
\text{Proof. To prove our claim we need to show that } p \text{ is a releaser of } L \text{ immediately before time } t, \text{ since by definition } p \text{ ceases to be a releaser of } L \text{ when any of } p \text{'s cease-release events occurs. By definition, cease-release event } \phi_p \text{ occurs when } p \text{ executes a successful Ctr.CAS}(1,0) \text{ operation in line } 36 \text{ cease-release event } \tau_p \text{ occurs when } p \text{ executes a successful Sync2.CAS}(\bot, p) \text{ in line } 56 \text{ cease-release event } \pi_p \text{ occurs only when } p \text{ executes a PawnSet.promote()} \text{ operation that returns a non-}\langle \bot, \bot \rangle \text{ value in line } 65 \text{ cease-release event } \theta_p \text{ occurs only when } p \text{ executes a Ctr.CAS}(2,0) \text{ operation in line } 68 \text{.}
\]

From Claim \( \text{D.13(1)} \), \( p \) is a releaser of \( L \) at \( t = \frac{56}{p}, \frac{55}{p}, \frac{65}{p}, \frac{68}{p} \). Hence, the claim follows.  

We say a process has \textit{write-access} to objects Sync1 and Sync2, respectively, if the process can write a value to Sync1 and Sync2, respectively. We say a process has \textit{registration-access} to object PawnSet, if the process can execute an operation on PawnSet that can write values in \( \langle a, b \rangle | a \in \{0, 1, 2\}, b \in \mathbb{N} \rangle \) to some entry of PawnSet. We say a process has \textit{deregistration-access} to object PawnSet, if the process can execute an operation on PawnSet that can write value \( \langle \text{ABORT}, s \rangle = \langle 3, s \rangle \), where \( s \in \mathbb{N} \), to some entry of PawnSet. Object PawnSet is said to be \textit{candidate-empty} if no entry of PawnSet has value \( \langle \text{REG}, \cdot \rangle \) or \( \langle \text{PRO}, \cdot \rangle \).

\[
\text{Claim D.15. Only releasers of } L \text{ have write-access to Sync1, Sync2 and registration-access to PawnSet.}
\]

\[
\text{Proof. The following observations follow from an inspection of the code. A value can be written to Sync1 only in lines } 20, 37, \text{ and } 68 \text{. A value can be written to Sync2 only in lines } 56, 60 \text{. From the semantics of the AbortableProArray}_p \text{ object, only operations collect()}, promote(), and reset() can write values in } \langle a, b \rangle | a \in \{0, \text{REG, PRO}\}, b \in \mathbb{N} \rangle \text{ to PawnSet. From Claim } \text{D.5(b)} \text{, the operations collect()}, promote(), and reset() are executed on PawnSet only in lines } 55, 65 \text{ and } 67 \text{ respectively.}
\]

Suppose an arbitrary process \( p \) writes a value to Sync1 or Sync2, or a value in \( \langle a, b \rangle | a \in \{0, 1, 2\}, b \in \mathbb{N} \rangle \) to an entry of PawnSet. From Claim \( \text{D.13(1)} \), \( p \) is a releaser of \( L \) at \( t = \frac{20}{p}, \frac{37}{p}, \frac{55}{p}, \frac{60}{p}, \frac{65}{p}, \frac{67}{p} \) and \( t = \frac{68}{p} \). Hence, the claim follows.

\[
\text{Claim D.16. The } i\text{-entry of PawnSet can be changed only by process } i \text{ or a releaser of } L.
\]

\[
\text{Proof. The values that can be written to PawnSet are in } \langle a, b \rangle | a \in \{0, 1, 2, 3\}, b \in \mathbb{N} \rangle \text{. A process that can write values in } \langle a, b \rangle | a \in \{0, 1, 2\}, b \in \mathbb{N} \rangle \text{ to any entry of PawnSet is said to have registration-access to PawnSet. From Claim } \text{D.15} \text{ it follows that only a releaser of } L \text{ has registration-access to PawnSet, therefore only a releaser of } L \text{ can write values in } \langle a, b \rangle | a \in \{0, 1, 2\}, b \in \mathbb{N} \rangle \text{ to the } i\text{-th entry of PawnSet. From Claim } \text{D.5(d)} \text{ the value } \langle \text{ABORT}, s \rangle = \langle 3, s \rangle \text{, where } s \in \mathbb{N} \text{, can be written to the } i\text{-th entry of PawnSet only when a process executes a remove}(i), \text{remove}(i) \text{ or PawnSet.abort}(i, s) \text{ operation in line } 64, 61 \text{ or } 21 \text{ respectively. From Claim } \text{D.13(1)} \text{, it follows that a process executing lines } 64 \text{ and } 61 \text{ is a releaser of } L \text{. Since a PawnSet.abort}(i, s) \text{ operation in line } 21 \text{ is executed only by process } i \text{, our claim follows.}
\]

\[
\text{Claim D.17. Sync2 is changed to a non-} \bot \text{ value only by a releaser of } L \text{ (say } r \text{) in line } 76 \text{ which triggers the cease-release event } \tau_r.
\]
Proof. By definition, cease-release event $\tau_p$ occurs when $p$ executes a successful $\text{Sync2.CAS}(\bot, p)$ in line 56. From a code inspection, $\text{Sync2}$ is changed to a non-$\bot$ value only when some process (say $r$) executes a successful $\text{Sync2.CAS}(\bot, r)$ operation in line 56. From Claim D.15 it follows that $\text{Sync2}$ is changed only by a releaser of $L$. Then $r$ is a releaser of $L$ when it changes $\text{Sync2}$ to a non-$\bot$ value in line 56 and doing so triggers the cease-release event $\pi_r$.

Claim D.18. A $\text{PawnSet.promote()}$ operation is executed only by a releaser of $L$ (say $r$), and if the value returned is non-$\langle\bot, \bot\rangle$, the cease-release event $\pi_r$ is triggered.

Proof. By definition, cease-release event $\pi_p$ occurs only when $p$ executes a $\text{PawnSet.promote()}$ operation that returns a non-$\langle\bot, \bot\rangle$ value in line 65. From a code inspection, a $\text{PawnSet.promote()}$ operation is executed only when some process (say $r$) executes line 56. From Claim D.15 it follows that $\text{PawnSet}$ is changed only by a releaser of $L$. Then $r$ is a releaser of $L$ when it executes a $\text{PawnSet.promote()}$ operation, and if the operation returns a non-$\langle\bot, \bot\rangle$ value then the cease-release event $\pi_r$ is triggered.

Claim D.19. During an execution of $\text{doPromote}_p()$ exactly one of the events $\tau_p$ and $\theta_p$ occurs.

Proof. By definition, cease-release event $\tau_p$ occurs when $p$ executes a $\text{PawnSet.promote()}$ operation in line 65 that returns a non-$\langle\bot, \bot\rangle$ value, and cease-release event $\theta_p$ occurs when $p$ executes a $\text{Ctr.CAS}(2, 0)$ operation in line 68 during $\text{doPromote}_p()$.

Case a - the $\text{PawnSet.promote()}$ operation in line 65 returns a non-$\langle\bot, \bot\rangle$ value, and thus cease-release event $\pi_p$ occurs: Then $p$ fails the if-condition of line 66 and line 68 is not executed. Therefore, cease-release event $\theta_p$ does not occur.

Case b - the $\text{PawnSet.promote()}$ operation in line 65 returns $\langle\bot, \bot\rangle$, and thus cease-release event $\pi_p$ does not occur: Then $p$ satisfies the if-condition of line 66 and executes a $\text{Ctr.CAS}(2, 0)$ operation in line 68. Hence, cease-release event $\theta_p$ occurs.

Claim D.20. During an execution of $\text{helpRelease}_p()` exactly one of the events $\tau_p$, $\pi_p$ and $\theta_p$ occurs.

Proof. By Claim D.17 events $\pi_p$ and $\theta_p$ can only occur during $p$’s call to $\text{doPromote}_p()$, and cease-release event $\tau_p$ occurs when $p$ executes a successful $\text{Sync2.CAS}(\bot, p)$ operation in line 56.

Case a - $p$ executes a successful $\text{Sync2.CAS}(\bot, p)$ operation in line 56 and thus cease-release event $\tau_p$ occurs: Then $p$ fails the if-condition of line 56 and returns immediately from its call to $\text{helpRelease}_p()$. Therefore, events $\pi_p$ and $\theta_p$ do not occur.

Case b - $p$ executes an unsuccessful $\text{Sync2.CAS}(\bot, p)$ operation in line 56 and thus cease-release event $\tau_p$ does not occur. Then $p$ satisfies the if-condition of line 56 and calls $\text{doPromote}_p()$ in line 62. From Claim D.19 exactly one of the events $\pi_p$ and $\theta_p$ occurs during $p$’s call to $\text{doPromote}_p()$.

Claim D.21. The value of $\text{Ctr}$ can change only when a $\text{Ctr.inc()}$, $\text{Ctr.CAS}(2, 0)$ or $\text{Ctr.CAS}(1, 0)$ operation is executed in lines 54, 68 or 77.

Proof. From the semantics of the $\text{RCASCounter}_2$ object, if $\text{Ctr}$ is increased to value $i$ by a $\text{Ctr.inc()}$ operation, then its value was $i − 1$ immediately before the operation was executed. Then all claims follow from an inspection of the code.

Claim D.22. If the value of $\text{Ctr}$ changes, it either increases by 1 or decreases to 0. Moreover its values are in $\{0, 1, 2\}$.
Proof. From the semantics of the RCAScounter2 object, a Ctr.inc() operation changes the value of Ctr from \( i \) to \( i + 1 \) only if \( i \in \{0, 1\} \). From Claims [D.21], the value of Ctr can change only when a Ctr.inc(), Ctr.CAS(2, 0) or Ctr.CAS(1, 0) operation is executed (in lines 5, 8 or 30). Then it follows that the values of Ctr are in \{0, 1, 2\}. It also follows that the value of Ctr either changes from 0 to 1 and back to 0, or it changes from 0 to 1 to 2 and back to 0.

Ctr-Cycle Interval \( T \). Let \( T = [t_s, t_e] \) be a time interval where \( t_s \) is a point when Ctr is 0 and \( t_e \) is the next point in time when Ctr is decreased to 0. For \( i \in \{0, 1, 2\} \) let \( I_i = \{ t \in T | Ctr = i \text{ at } t \} \) and let time \( I_i^- = \min(I_i) \) and time \( I_i^+ = \max(I_i) \). From Claim [D.22] it follows immediately that during \( T \) the set \( I_i, i \in \{0, 1, 2\} \), forms an interval \( [I_i^-, I_i^+] \), and \( I_2 = \emptyset \) if and only if Ctr is never increased to 2 during \( T \). Moreover, \( t_s = I_0^- \) and \( I_0 \) is immediately followed by \( I_1 \) (i.e., \( \min(I_1) = \max(I_0) + 1 \)). If \( I_2 \neq \emptyset \) then \( I_2 \) follows immediately after \( I_1 \). The Ctr-cycle interval \( T \) ends either at time \( I_1^+ \) if \( I_2 = \emptyset \), or at time \( I_2^- \) if \( I_2 \neq \emptyset \).

Then it also follows that exactly one process changes Ctr from 0 to 1 during \( T \), and it does so at time \( I_1^- \). Let \( K \) be the process that increases Ctr to 1 at time \( I_1^- \). And if \( I_2 \neq \emptyset \) then exactly one process changes Ctr from 1 to 2 during \( T \), and it does so at time \( I_2^- \). If \( I_2 \neq \emptyset \) let \( Q \) be the process that increases Ctr to 2 at time \( I_2^- \). Let \( R(t) \) denote the set of processes that are the releasers of lock \( L \) at time \( t \in T \).

Claim D.23. If \( R(I_0^-) = \emptyset \) and at \( I_0^- \), Sync1 = Sync2 = \bot and PawnSet is candidate-empty, then the following holds:

(a) \( \forall t \in I_0^- : R(t) = \emptyset \) and throughout \( I_0 \), Sync1 = Sync2 = \bot and PawnSet is candidate-empty.

(b) \( R(I_1^-) = \{K\} \) and at time \( I_1^- \), Sync1 = Sync2 = \bot and PawnSet is candidate-empty.

(c) \( K \) executes lines of code of lock\(_K\)() starting with line 2 as depicted in Figure 8. (A legend for the figure is given in Figure 7.)

\[\text{Apply}[K].\text{CAS}(\bot, 1, \text{REG}, s)\]

\[\text{false} \quad \text{true} \]

5 6 12 13 16 17

\[\text{Role}[K] = \text{KING} = 0\]

Figure 8: \( K \)'s call to lock\(_K\)()

(d) \( K \)'s call to lock\(_K\)() returns \( \infty \) and \( \text{Role}[K] = \text{KING} \) throughout \( [I_1^-, I_2^+] \).

(e) \( K \) executes a Ctr.CAS(1, 0) operation in line 30 during \( T \), and \( K \) does not change Sync1, Sync2 or PawnSet throughout \( [I_1^- , I_2^-] \).

(f) \( \forall t \in I_1^- : R(t) = \{K\} \).

(g) Throughout \( I_1 \), Sync1 = Sync2 = \bot and PawnSet is candidate-empty.

Proof. Proof of (a): Consider the claim \( R(t) = \emptyset \) where \( t \in I_0 \). Since \( R(I_0^-) = \emptyset \) holds by assumption, the claim holds at \( t = I_0^- \). For the purpose of a contradiction assume the claim fails to hold for the first time at some point \( t' \) during \( I_0 \). Then some process \( p \) becomes a releaser of
lock $L$ at time $t'$. Process $p$ cannot become a releaser of $L$ by $p$ increasing $\text{Ctr}$ to 1 or 2 (condition (R1)) at time $t'$, since $\text{Ctr} = 0$ throughout $I_0$. Therefore, assume it becomes a releaser of $L$ when some process $q$ promotes $p$ (condition (R2)) at $t'$. By Claim [D.14] $q$ ceases to be a releaser of lock $L$ at $t'$. This is a contradiction to our assumption that $p$ is the first process during $I_0$ to become a releaser of $L$.

By assumption the variables $\text{Sync1}, \text{Sync2}$ and $\text{PawnSet}$ are at their initial value at $I^+_0$. Since the values of these variables are only changed by a releaser of lock $L$ (by Claim [D.15]) and for all $t \in I_0$, $R(t) = \emptyset$, it follows that the variables are unchanged throughout $I_0$.

Proof of (b): At time $I^-_1$ $\text{Ctr}$ is increased from 0 to 1, and thus the only operation executed is a $\text{Crt.inc()}$ operation by process $K$. Then $K$ becomes a releaser of lock $L$ at time $I^-_1$ by condition (R1). Since for all $t \in I_0$, $R(t) = \emptyset$ (Part (a)), it follows that $R(I^-_1) = \{K\}$. Since $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty throughout $I_0$ (Part (a)), and the only operation at time $I^-_1$ is the $\text{Crt.inc()}$ operation, it follows that $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty at time $I^-_1$.

Proof of (c) and (d): Since $K$ is the process that increased $\text{Ctr}$ from 0 to 1 at time $I^-_1$, and since $K$ can increase $\text{Ctr}$ only by executing a $\text{Crt.inc()}$ operation in line 5 (by Claim [D.21] $\text{Role}[K] = 0 = \text{KING}$ at $t^-_K$). Then from the code structure, $K$ does not execute lines [7,9] and does not repeat the role-loop, and does not busy-wait in the spin loop of line 14 instead $K$ proceeds to execute lines [16,17] and returns value $\infty$ in line 17. Since $K$ does not change $\text{Role}[K]$ during $[t^-_K,t^+_K]$, $\text{Role}[K] = \text{KING}$ throughout $[t^-_K,t^+_K]$.

Proof of (e): Since $K$ is the process that increased $\text{Ctr}$ from 0 to 1 at time $I^-_1$, and since $K$ can increase $\text{Ctr}$ only by executing a $\text{Crt.inc()}$ operation in line 5 (by Claim [D.21] $\text{Role}[K] = 0 = \text{KING}$ at $t^-_K$). From Part (a), $K$ returns from $\text{lockK()}$ with value $\infty$ in line 17 and thus $K$ consequently calls $\text{releaseK(j)}$ (follows from conditions (b) and (d)). Note that $K$ has not executed any operations on $\text{Sync1}, \text{Sync2}$ and $\text{PawnSet}$ in the process. Then $\text{Role}[K] = \text{KING}$ at $t^-_K$ and thus $p$ satisfies the if-condition of line 35 and executes the $\text{Crt.CAS(1,0)}$ operation in line 36 during $T$ without having executed any operations on $\text{Sync1}, \text{Sync2}$ and $\text{PawnSet}$ in the process. Thus $K$ did not change $\text{Sync1}, \text{Sync2}$ or $\text{PawnSet}$ during $[I^-_1,t^+_K]$.

Proof of (f): Since $R(I^-_1) = \{K\}$ (Part (b)), to prove our claim we need to show that during $I_1$ $K$ does not cease to be a releaser and no process becomes a releaser. Suppose not, i.e., the claim $R(t) = \{K\}$ fails to hold for the first time at some point $t'$ in $I_1$.

Case a - Process $K$ ceases to be a releaser of $L$ at $t'$: By definition, cease-release event $\phi_K$ occurs when $K$ executes a successful $\text{Crt.CAS(1,0)}$ operation in line 36. From Part (a) $K$ executes a $\text{Crt.CAS(1,0)}$ operation in line 36. If $K$ executes a successful $\text{Crt.CAS(1,0)}$ operation in line 36 then, by definition, cease-release event $\phi_K$ occurs and by Claim [D.14] $K$ ceases to be the releaser of $L$. Thus, $t' = t^+_K$ and $\text{Ctr}$ changes to value of 0 at $t'$. But since $t' \in I_1$ and $\text{Ctr} = 1$ throughout $I_1$, we have a contradiction. If $K$ executes an unsuccessful $\text{Crt.CAS(1,0)}$ operation in line 36 then $\text{Ctr} \neq 1$ at $t^+_K$. Since $p$ did not cease to a releaser at $t^+_K$, $t^+_K < t'$. Since $I^-_1 = t^-_K < t^+_K < t' < I^+_1$ and $\text{Ctr} = 1$ throughout $I_1$, $\text{Ctr} = 1$ at $t^+_K$, and thus we have a contradiction.

Case b - Some process $q$ becomes a releaser of $L$ at $t'$: Since $\text{Ctr}$ is not increased during $I_1$, it follows from conditions (R1) and (R2) that some process $r$ promoted $q$ at time $t'$. Then by definition, event $\pi_r$ occurs at $t'$, and thus from Claim [D.14] it follows that $r$ is a releaser of $L$ immediately before $t'$. Since $K$ is the only releaser immediately before $t'$, $r = K$. Then cease-release event $\pi_K$ occurred at $t'$ and $K$ ceases to be a releaser at $t'$. As was shown in Case a, this leads to a contradiction.

Proof of (g): At time $I^-_1$ the claim $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty holds by Part (a). Suppose some process $p$ changes $\text{Sync2}$ or $\text{Sync1}$ or $\text{PawnSet}$ for the first time.
at some point \( t' \) during \( I_1 \). From Claim \([D.15]\) it follows that \( p \) is a releaser of lock \( L \) at time \( t' \).

Since for all \( t \in I_1 \), \( R(t) = \{ K \} \) (Part (f)), it follows that \( p = K \). From Part (e), \( K \) does not change any of the variables before the point when it executes a \( \text{Ctr.CAS}(1,0) \) operation in line 36, i.e., \( t' - t < t' \). If \( K \) executes a successful \( \text{Ctr.CAS}(1,0) \) operation in line 36 then the interval \( I_1 \) ends and clearly \( t' \notin I_1 \), hence a contradiction. If \( K \) executes an unsuccessful \( \text{Ctr.CAS}(1,0) \) operation in line 36 then \( \text{Ctr} \neq 1 \) at \( t_36 - K \). Since \( I_1 - 1 = t_5 - K < t_36 - K < t' < I_1 - 1 \) and \( \text{Ctr} = 1 \) throughout \( I_1 \), we have a contradiction.

\( \Box \)

**Claim D.24.** If \( I_2 \neq \emptyset \) and \( R(I_0) = \emptyset \) and at \( I_0^{-1} \), \( \text{Sync1} = \text{Sync2} = \perp \) and \( \text{PawnSet} \) is candidate-empty, then the following claims hold:

(a) \( R(I_2^{-1}) = \{ K, Q \} \) and at time \( I_2^{-1} \), \( \text{Sync1} = \text{Sync2} = \perp \) and \( \text{PawnSet} \) is candidate-empty.

(b) \( K \) and \( Q \) are the first two releasers of \( L \).

(c) During \( (I_2^{-1}, I_2^+) \) a process can become a releaser of \( L \) only if it gets promoted by a releaser of \( L \).

(d) If \( K \) takes enough steps, \( K \) executes lines of code of \( \text{release}_K() \) starting with line 34 as depicted in Figure 9.

(e) \( K \) executes an unsuccessful \( \text{Ctr.CAS}(1,0) \) operation in line 36 and calls \( \text{helpRelease}_K() \) in line 39 such that \( I_2 < I_36 < I_{39} \).

(f) If \( K \) and \( Q \) take enough steps, \( Q \) finishes \( \text{lock}_Q() \) during \( T \).

(g) If \( K \) and \( Q \) take enough steps, \( Q \) executes lines of code of \( \text{lock}_Q() \) starting with line 2 as depicted in Figure 10.

(h) If \( Q \) calls \( \text{release}_Q() \), it executes lines of code of \( \text{release}_Q() \) starting with line 34 as depicted in Figure 11.

(i) \( Q \) calls \( \text{helpRelease}_Q() \) either in line 30 or in line 43 after time \( I_2^{-1} \).

![Figure 9: \( K \)'s call to \( \text{release}_K(j) \)](image)

**Proof.** Proof of (a) and (b): Since \( Q \) is the process that increases \( \text{Ctr} \) from 1 to 2 at time \( I_2^{-1} \), and since \( Q \) can increase \( \text{Ctr} \) only by executing a \( \text{Ctr.inc()} \) operation in line 5 (by Claim \([D.21]\) \( Q \) becomes a releaser of lock \( L \) by condition (R1) at \( I_2^{-1} = I_36 \). Since for all \( t \in I_1 \), \( R(t) = \{ K \} \)
(Claim D.23[1]), it follows that \( R(I^-) = \{K, Q\} \). By claim D.23[g], throughout \( I_1 \), Sync1 = Sync2 = \( \bot \) and PawnSet is candidate-empty, and since the only operation executed at time \( I^+ \) is Ctr.inc(), it follows that at time \( I^+ \), Sync1 = Sync2 = \( \bot \) and PawnSet is candidate-empty. Hence Part (b) holds. Clearly \( K \) and \( Q \) are the first two releasers of \( L \), hence Part (b) holds.

Proof of (c): From conditions (R1) and (R2), a process can become a release \( r \) of \( L \) either by increasing \( Ctr \) to 1 or 2 or by getting promoted. Since \( Ctr \) is not increased during \( (I^-, I^+) \), it follows that during \( (I^- , I^+) \) a process becomes a releaser of \( L \) only if it gets promoted. By definition, a process can be promoted only when a PawnSet.promote() operation is executed in line 65 and from Claim D.18 only a releaser of \( L \) can execute this operation. Then during \( (I^- , I^+) \) a process becomes a releaser of \( L \) only if it gets promoted by a releaser of \( L \).

Proof of (d) and (e): From Claim D.23[e], \( K \) executes the Ctr.CAS(1, 0) operation in line 36 during \( T \). If \( K \)'s Ctr.CAS(1, 0) operation is successful then the value of \( Ctr \) decreases from 1 to 0 and the Ctr-cycle interval \( T \) ends and thus \( I_2 = \emptyset \), which is a contradiction to our assumption that \( I_2 \neq \emptyset \). Then \( K \)'s Ctr.CAS(1, 0) operation is unsuccessful.

Since \( K \) executes an unsuccessful Ctr.CAS(1, 0) operation in line 36, \( K \) satisfies the if-condition of line 36 executes lines 37-38 and calls helpRelease\(_K\)() in line 39 and then executes lines 49-50.

Since \( K \) executes an unsuccessful Ctr.CAS(1, 0) operation in line 36, it follows that \( Ctr \) was changed from 1 to 2 at time \( I^- \) (by definition), and thus \( I^- < I_2^- \). Since \( I_2^- < I_2^+ \), it follows that \( I^- < I_2^- < I_2^+ \).

Proof of (f), (g) and (h): Since \( Q \) is the process that increases \( Ctr \) from 1 to 2 at time \( I^- \), and since \( Q \) can increase \( Ctr \) only by executing a Ctr.inc() operation in line 5 (by Claim D.21) \( Q \) set Role[\( K \)] = QUEEN at \( I^- \). Then from the code structure, \( Q \) does not execute lines 7-9 and does not repeat the role-loop, instead, it proceeds to line 15 and then proceeds to busy-wait in the spin loop of line 14. Then \( Q \) does not finish lock\(_Q\)() only if it spins indefinitely in line 14 and does not receive a signal to abort.

For the purpose of a contradiction assume that \( Q \) does not finish lock\(_Q\)(). Then \( Q \) reads the
value \( \perp \) from \( \text{Sync1} \) in line 14 indefinitely. From Part (c) it follows that \( \mathcal{K} \) executes a \( \text{Sync1.CAS}(\perp, j) \) operation in line 37 during \((I_2^-, I_2^+)\). Since \( \text{Sync1} = \perp \) at time \( I_2^- \) (Part (a)), and only a releaser can change \( \text{Sync1} \) (Claim \( \text{D.15} \)), and \( \mathcal{Q} \) is busy-waiting in line 14, it follows that the only other releaser, \( \mathcal{K} \), executed a successful \( \text{Sync1.CAS}(\perp, j) \) operation in line 37 during \((I_2^-, I_2^+)\) and changed \( \text{Sync1} \) to a non-\( \perp \) value. Then for \( \mathcal{Q} \) to read \( \perp \) from \( \text{Sync1} \) in line 14 indefinitely, some process must reset \( \text{Sync1} \) to \( \perp \) before \( \mathcal{Q} \) reads \( \text{Sync1} \) again.

**Case a -** \( \mathcal{K} \) resets \( \text{Sync1} \) in line 58 before \( \mathcal{Q} \) reads \( \text{Sync1} \) again: For \( \mathcal{K} \) to reset \( \text{Sync1} \) in line 58 \( \mathcal{K} \) must satisfy the if-condition of line 56 and thus \( \mathcal{K} \) must execute an unsuccessful \( \text{Sync2.CAS}(\perp, \mathcal{K}) \) operation in line 56. Since \( \text{Sync2} = \perp \) at time \( I_2^- \) (Part (a)), and only a releaser can change \( \text{Sync2} \) (Claim \( \text{D.15} \)), and \( \mathcal{Q} \) is busy-waiting in line 14 it follows that \( \text{Sync2} = \perp \) at \( I_2^- \). Thus \( \mathcal{K} \)'s \( \text{Sync2.CAS}(\perp, \mathcal{K}) \) operation in line 56 is successful and we get a contradiction.

**Case b -** some other process becomes a releaser and resets \( \text{Sync1} \) before \( \mathcal{Q} \) reads \( \text{Sync1} \) again: From Part (c) it follows that during \((I_2^-, I_2^+)\) a process can become a releaser of \( \mathcal{L} \) only if it is promoted (by condition (R2)). Since a process is promoted only by a releaser of \( \mathcal{L} \) and \( \mathcal{K} \) is the only other releaser of \( \mathcal{L} \) apart from \( \mathcal{Q} \), it follows that \( \mathcal{K} \) promotes some process before \( \mathcal{Q} \) reads \( \text{Sync1} \) again. As argued in **Case a**, \( \mathcal{K} \) executes a successful \( \text{Sync2.CAS}(\perp, \mathcal{L}) \) operation in line 56. Then from the code structure, \( \mathcal{K} \) does not call \( \text{doPromote}_{\mathcal{K}}() \) in line 62 and thus \( \mathcal{K} \) does not promote any process. Hence, we have a contradiction.

**Proof of (i):** Since \( \mathcal{Q} \) is the process that increases \( \text{Ctr} \) from 1 to 2 at time \( I_2^- \), and since \( \mathcal{Q} \) can increase \( \text{Ctr} \) only by executing a \( \text{Ctr.inc()} \) operation in line 5 (by Claim \( \text{D.21} \)), \( \mathcal{Q} \) set \( \text{Role}[\mathcal{K}] = 1 = \text{QUEEN} \) at \( I_2^- \). Then from the code structure, \( \mathcal{Q} \) does not execute lines 7-9 and does not repeat the role-loopp; instead, it proceeds to line 13 and then proceeds to busy-wait in the spin loop of line 14.

**Case a -** \( \mathcal{Q} \) does not receive a signal to abort while busy-waiting in line 14. From Part (a), \( \mathcal{Q} \) does not busy-wait indefinitely in line 14 and eventually breaks out. Since \( \mathcal{Q} \) breaks out of the spin loop of line 14 it reads non-\( \perp \) from \( \text{Sync1} \) and then from the code structure it follows that \( \mathcal{Q} \) goes on to return that non-\( \perp \) value in line 17. Consequently \( \mathcal{Q} \) calls \( \text{release}_{\mathcal{Q}}(j) \) (follows from conditions 1 and 4). Consider \( \mathcal{Q} \)’s call to \( \text{release}_{\mathcal{Q}}(j) \). Since \( \mathcal{Q} \) last changed \( \text{Role}[\mathcal{Q}] \) only in line 5 \( \text{Role}[\mathcal{Q}] = \text{QUEEN} \) at \( I_2^- \). Since \( \text{Role}[\mathcal{Q}] \) is unchanged during \( \text{release}_{\mathcal{Q}}() \) (Claim \( \text{D.15} \)), it follows that \( \text{Role}[\mathcal{Q}] = \text{QUEEN} \) throughout \( \text{release}_{\mathcal{Q}}() \). Then from the code structure it follows that \( \mathcal{Q} \) executes only lines 31-35, 42-45, and 49-50. Then \( \mathcal{Q} \) calls \( \text{helpRelease}_{\mathcal{Q}}() \) only in line 43 and since \( I_2^- = I_2^+ = \frac{50}{2} < \frac{43}{2} \), our claim holds.

**Case b -** \( \mathcal{Q} \) receives a signal to abort while busy-waiting in line 14. Then \( \mathcal{Q} \) calls \( \text{abort}_{\mathcal{Q}}() \), and from the code structure \( \mathcal{Q} \) executes lines 18-20 and then line 26. If \( \mathcal{Q} \) fails the \( \text{Sync1.CAS}(\perp, \infty) \) operation of line 26, then \( \text{Sync1} \neq \perp \) at \( I_2^- \). From Claim \( \text{D.15} \), only a releasers of \( \mathcal{L} \) can change \( \text{Sync1} \) to a non-\( \perp \) value, and since \( \mathcal{K} \) and \( \mathcal{Q} \) are the only releasers of \( \mathcal{L} \), it follows that \( \mathcal{K} \) changed \( \text{Sync1} \) to a non-\( \perp \) value. Then \( \mathcal{Q} \) satisfies the if-condition of line 26 and returns the non-\( \perp \) value written by \( \mathcal{K} \) to \( \text{Sync1} \) in line 27. Consequently \( \mathcal{Q} \) calls \( \text{release}_{\mathcal{Q}}(j) \) (follows from conditions 1 and 4), and as argued in **Case a**, \( \mathcal{Q} \) executes only lines 31-35, 42-45, and 49-50 and \( \mathcal{Q} \) calls \( \text{helpRelease}_{\mathcal{Q}}() \) only in line 43 Since \( I_2^- = I_2^+ = \frac{50}{2} < \frac{43}{2} \), our claim holds.

If \( \mathcal{Q} \)’s \( \text{Sync1.CAS}(\perp, \infty) \) operation is successful, then \( \mathcal{Q} \) goes on to call \( \text{doCollect}_{\mathcal{Q}}() \) in line 26, calls \( \text{helpRelease}_{\mathcal{Q}}() \) in line 30, then executes lines 32-33 and finally returns \( \perp \) in line 33. Since \( I_2^- = I_2^+ = \frac{50}{2} < \frac{30}{2} \), our claim holds.

Define \( \lambda \) to be the first point in time when \( \text{Sync2} \) is changed to a non-\( \perp \) value, and if \( \text{Sync2} \) is never changed to non-\( \perp \) then \( \lambda = \infty \). Define \( \gamma \) to be the first point in time when a \texttt{PawnSet.promote()} operation is executed, and if a \texttt{PawnSet.promote()} operation is never executed
then $\gamma = \infty$. From Claims D.24(e) and D.24(i), both $K$ and $Q$ execute $\text{helpRelease}_K()$ and $\text{helpRelease}_Q()$, respectively, after time $I^2$. Let $A \in \{K, Q\}$ be the first process among them to execute line 56 and let $B \in \{K, Q\} - \{A\}$ be the other process, i.e., $t^A_A < t^B_B$.

Claim D.25. If $I_2 \neq \emptyset$ and $R(I^-_0) = \emptyset$ and at $I^-_0$, $\text{Sync}1 = \text{Sync}2 = \perp$ and $\text{PawnSet}$ is candidate-empty, then the following claims hold:

(a) $I^-_2 < \lambda = [56]$ and for all $t \in [I^-_2, \lambda)$, $R(t) = \{K, Q\}$ and $\text{Sync}2 = \perp$ throughout $[I^-_2, \lambda)$, and cease-release event $\tau_A$ occurs at $\lambda$.

(b) If $K$ and $Q$ take enough steps, then $A$ executes lines of code of $\text{helpRelease}_A()$ starting with line 56 as depicted in Figure 12.

(c) If $K$ and $Q$ take enough steps, then $B$ executes lines of code of $\text{helpRelease}_B()$ and $\text{doPromote}_B()$ as depicted in Figures 13 and 14, respectively.

(d) $\lambda < \gamma = [56]$.

(e) $\forall t \in [\lambda, \gamma)$, $R(t) = \{B\}$.

(f) At time $\gamma$, $\text{Sync}1 = \text{Sync}2 = \perp$.

(g) No promotion event occurs at lock $L$ during $[I^-_2, \gamma)$.
(h) The PawnSet.promote() operation at time $\gamma$ does not return a value in $\{(a, b) | a \in \{K, Q\}, b \in \mathbb{N}\}$.

(i) If the PawnSet.promote() operation at time $\gamma$ returns a non-$\perp$ value then $B$’s cease-release event $\tau_B$ occurs at time $\gamma$.

(j) If the PawnSet.promote() operation at time $\gamma$ returns value $\langle \perp, \perp \rangle$ then $B$’s cease-release event $\theta_B$ occurs at time $t' = \frac{\theta_B}{\theta_B} \geq \gamma$, and throughout $[\gamma, t')$ no process is promoted, and $\forall t \in [\gamma, t'), R(t) = \{B\}$.

(k) Either $K$ or $Q$ calls doCollect(), specifically during $[I_2^-, \gamma]$.

Proof. Proof of (a): We first show that for all $t \in [I_2^-, \lambda_A^{\perp}]$, $R(t) = \{K, Q\}$ and then show that $\lambda = \lambda_A^{\perp}$. From Claim D.24(a) $K$ calls helpRelease$_K()$ in line 39 after time $I_2^-$. From Claim D.24(a), $K$ is a releaser of $L$ at time $I_2^-$. From an inspection of Figures 8 and 9 throughout $[I_1, R_K^\perp]$, $K$ does not execute a call to helpRelease$_K()$ or doPromote$_K()$. Also from an inspection, $K$ fails to decrease $Ctr$ from 1 to 0 at $R_A$, thus $K$’s cease-release event $\phi_K$ does not occur. Since $K$’s cease-release events $\pi_K, \pi_Q$ and $\theta_K$ only occur during helpRelease$_K()$ or doPromote$_K()$ (Claims D.7(e) and D.7(f)), it follows that $K$ is a releaser of $L$ throughout $[I_1, R_K^\perp]$.

From Claim D.24(a), $Q$ calls helpRelease$_Q()$, respectively either in line 30 or line 43 after time $I_2^-$. From Claim D.24(a), $Q$ is a releaser of $L$ at time $I_2^-$. From an inspection of Figures 10 throughout $[I_2, R_Q^\perp]$, $Q$ does not execute a call to helpRelease$_Q()$ or doPromote$_Q()$. Also from an inspection, $Q$ does not execute a $\text{Ctr.CAS}(1, 0)$ operation in line 38 and thus $Q$’s cease-release event $\phi_Q$ does not occur. Since $Q$’s cease-release events $\tau_Q, \pi_Q$ and $\theta_Q$ only occur during helpRelease$_Q()$ or doPromote$_Q()$ (Claims D.7(e) and D.7(f)), it follows that $Q$ is a releaser of $L$ throughout $[I_2^-, \lambda_Q^{\perp}]$.

Then for all $t \in [I_2^-, \lambda_A^{\perp}]$, $\{K, Q\} \subseteq R(t)$ since $I_1^- < I_2^-$ and $\lambda_A^{\perp} = \min(\lambda_K^{\perp}, \lambda_Q^{\perp})$. From Claim D.24(a), it follows that a process can become a releaser during $I_2^-$ only if it is promoted by a releaser of $L$. Then to show that for all $t \in [I_2^-, \lambda_A^{\perp}]$, $R(t) = \{K, Q\}$, we need to show that no process is promoted by $K$ or $Q$ during $[I_2^-, \lambda_A^{\perp}]$. If a process was promoted by $K$ or $Q$ during $[I_2^-, \lambda_A^{\perp}]$ then by definition cease-release events $\pi_K$ or $\pi_Q$ would have occurred during $[I_2^-, \lambda_A^{\perp}]$, but as shown above this does not happen.

From Claim D.24(a), $\text{Sync2} = \perp$ at time $I_2^-$. From a code inspection, $\text{Sync2}$ is changed to a non-$\perp$ value only in line 56 (during $\text{helpRelease}()$), moreover only by a releaser of $L$ (from Claim D.15). Since for all $t \in [I_2^-, \lambda_A^{\perp}]$, $R(t) = \{K, Q\}$ and $\lambda_A^{\perp} = \min(\lambda_K^{\perp}, \lambda_Q^{\perp})$, it follows then that $\text{Sync2} = \perp$ throughout $[I_2^-, \lambda_A^{\perp}]$ and $A$ executes a successful $\text{Sync2.CAS}(\perp, A)$ operation in line 56. Thus $A$’s cease-release event $\tau_A$ occurs at $\lambda_A^{\perp}$.

Since $\text{Sync2} = \perp$ throughout $[I_0^-, I_1^+]$ (Claims D.23(a) and D.23(g)) and throughout $[I_2^-, \lambda_A^{\perp}]$, it follows that $\text{Sync2}$ was changed to a non-$\perp$ value for the first time at $\lambda_A^{\perp}$, thus $\lambda = \lambda_A^{\perp}$. Then it follows for all $t \in [I_2^-, \lambda)$, $R(t) = \{K, Q\}$, and $\text{Sync2} = \perp$ throughout $[I_2^-, \lambda)$.

Proof of (b): From Part (a), $A$’s cease-release event $\tau_A$ occurs at $\lambda = \lambda_A^{\perp}$, and thus $A$’s $\text{Sync2.CAS}(\perp, A)$ operation in line 56 succeeds. Then from the code structure $A$ does not satisfy the if-condition on line 56 and returns from its call to $\text{helpRelease}_A()$. Thus, Figure 12 follows.

Proof of (c), (d), (e), (f), (g), (h), (i) and (j): From Part (a), $\lambda = \lambda_A^{\perp}$ and for all $t \in [I_2^-, \lambda)$, $R(t) = \{K, Q\}$ and $\text{Sync2} = \perp$ throughout $[I_2^-, \lambda)$ and cease-release event $\tau_A$ occurs at $\lambda$. Then $A$ ceases to be a releaser of $L$ at $\lambda$, and thus $R(\lambda) = \{B\}$ and $\text{Sync2} = A \neq \perp$ at $\lambda$. From
Claim [D.24] it follows that $B$ will continue to be the only releaser of $L$ until the point when $B$ ceases to be a releaser of $L$ or promotes another process. Let $t > \lambda$ be the point in time when $B$ ceases to be a releaser of $L$. Since $B$ ceases to be a releaser of $L$ if it promotes another process (by definition of cease-release event $\pi_B$), it follows that $B$ is the only releaser of $L$ throughout $[\lambda, t)$. Then from Claim [D.15] it follows that $B$ has exclusive write-access to $\text{Sync}1, \text{Sync}2$ and exclusive registration-access to $\text{PawnSet}$ throughout $[\lambda, t)$.

Now consider $B$’s $\text{helpRelease}_B()$ call. Since $\lambda = t_B^5 < t_B^6$ and $\text{Sync}2 \neq \perp$ at $\lambda$ and $B$ has exclusive write-access to $\text{Sync}2$ throughout $[\lambda, t)$, $B$ fails the $\text{Sync}2.\text{CAS}(\perp, B)$ operation at $t_B^6$, and thus satisfies the if-condition of line [56]. It then executes lines [57]–[62] and calls $\text{doPromote}_B()$ in line [63]. Then Figures [13] and [14] and Part (c) follows immediately.

We now show that $\gamma = t_B^6 \leq t$. Since $\lambda = t_B^6$ and $t_B^6 < t_B^7 < t_B^6$, it would follow that $\lambda < \gamma$, and hence we would have proved Part (i). And since $B$ is the only releaser of $L$ throughout $[\lambda, t)$, we would have proved Part (i) as well, i.e., $B$ is the only releaser throughout $[\lambda, \gamma)$.

During $\text{doPromote}_B()$, $B$ executes a $\text{PawnSet.promote}()$ operation in line [65]. Since $\mathcal{K}$ and $\mathcal{Q}$ are the first two releasers of $L$ during $T$ (Claim [D.21(b)]), and only a releaser executes a $\text{PawnSet.promote}()$ operation (Claim [D.18]), and $\mathcal{A}$ ceased to be a releaser at $t_A^6$, it follows that $B$’s $\text{PawnSet.promote}()$ operation in line [65] is the first $\text{PawnSet.promote}()$ operation, and thus $\gamma = t_B^6$. Since none of $B$’s cease-release events occur during $[t_B^5, t_B^6]$, $t \geq t_B^5$.

During $[t_B^5, t_B^6)$, $B$ resets $\text{Sync}1$ and $\text{Sync}2$ in lines [58] and [60] respectively, and since $B$ has exclusive write-access to $\text{Sync}1$ and $\text{Sync}2$ throughout $[t_B^5, t_B^6)$, at time $\gamma = t_B^6$, $\text{Sync}1 = \text{Sync}2 = \perp$. Thus, Part (i) follows.

By definition $\gamma$ is the point in time when the first $\text{PawnSet.promote}()$ operation occurs. Since a promotion event occurs only when a $\text{PawnSet.promote}()$ operation returns a non-$\perp$ value, it follows that no promotion event occurs during $[t_B^5, \gamma)$. Hence, Part (i) follows.

Since $B$ has exclusive write-access to $\text{Sync}2$ throughout $[\lambda, t_B^6)$, and $\text{Sync}2 = \mathcal{A}$ at $\lambda = t_B^6$, $B$ reads the value $\mathcal{A}$ from $\text{Sync}2$ in line [59] and executes a $\text{PawnSet.remove}(\mathcal{A})$ operation in line [61]. Since $B$ executes $\text{PawnSet.remove}(\mathcal{A})$ and $\text{PawnSet.remove}(B)$ in lines [61] and [64] during $[\lambda, \gamma)$ and $B$ has exclusive registration-access to $\text{PawnSet}$ during $[\lambda, \gamma)$, it follows from the semantics of the $\text{AbortableProArray}_n$ object that $B$’s $\text{PawnSet.promote}()$ operation at time $\gamma$ does not return values in $\{(a, b) | a \in \{\mathcal{A}, B\}, b \in \mathbb{N}\}$. Hence, Part (i) holds.

Case a - $B$’s $\text{PawnSet.promote}()$ operation returns a non-$\perp$ value: Then $B$’s cease-release event $\pi_B$ occurs at $t_B^6 = \gamma$ (Claim [D.18]), and thus Part (i) holds.

Case b - $B$’s $\text{PawnSet.promote}()$ operation in line [65] returns $\perp$. Then $B$ did not find any process to promote, and thus cease-release event $\pi_B$ did not occur. From the code structure $B$ goes on to execute a $\text{Ctr.CAS}(2, 0)$ operation in line [68]. Since $\text{Ctr} = 2$ throughout $I_2$, it follows that $B$’s $\text{Ctr.CAS}(2, 0)$ operation succeeds, and thus $B$’s cease-release event $\theta_B$ occurs at $t_B^8$ and the intervals $I_2$ and $T$ end. Therefore $t' = t_B^8 > t_B^6 = \gamma$. Clearly, $B$ does not promote any process in $[t_B^6, t_B^8] = [\gamma, t')$, and thus Part (i) holds.

Proof of (k): From an inspection of Figure [9] $\mathcal{K}$ executes a $\text{Sync1.CAS}(\perp, \mathcal{K})$ operation in line [37]. Since $I_2 < t_K^7 < t_K^8 < t_K^6 < \gamma$ (from Parts (m) and (p)), it follows that $[t_K^7] \in [I_2, \gamma)$. From an inspection of Figures [10] and [11] $\mathcal{Q}$ may or may not execute a $\text{Sync1.CAS}(\perp, \infty)$ operation in line [26]. If $\mathcal{Q}$ executes a $\text{Sync1.CAS}(\perp, \infty)$ operation in line [26], since $I_2 < t_Q^6 < t_Q^6 < \gamma$, it follows that $[t_Q^6] \in [I_2, \gamma]$.

Since for all $t \in [I_2, \gamma)$, $R(t) \subseteq \{\mathcal{K}, \mathcal{Q}\}$ (from Parts (m) and (p)), and only releasers of $L$ have write-access to $\text{Sync}1$ (Claim [D.15]), and $\text{Sync}1 = \perp$ at $I_2$ (Claim [D.24(a)]), it follows that either $\mathcal{K}$ or $\mathcal{Q}$ executes a successful $\text{CAS}()$ operation on $\text{Sync}1$. Then from the code structure it
follows that either $K$ or $Q$ executed a call to $\text{doCollect()}$ in lines 28 or 29 respectively. Since $\ell_K < k_0 = k_1 < \gamma$ and $\ell_Q < q_0 < \gamma$, $K$ or $Q$ executed a call to $\text{doCollect()}$ during $[I_2, \gamma]$. \hfill $\Box$

**Claim D.26.** If a process $p$ is promoted at time $t' \in T$ and a $\text{PawnSet.reset()}$ has not been executed during $[I_0, t']$, then $p$ did not execute a $\text{PawnSet.abort}(p, s)$ operation during $[I_0, t']$, where $s \in \mathbb{N}$.

**Proof.** Suppose not, i.e., $p$ executed a $\text{PawnSet.abort}(p, s)$ operation at time $t < t'$. Since $p$ has not been promoted before $t' > t$ it follows that a $\text{PawnSet.promote()}$ operation that returns $\langle p, \cdot \rangle$ has not been executed before $t$. Then from Claim D.25 and the semantics of $\text{PawnSet}$, it follows that the $p$-th entry of $\text{PawnSet}$ is not at value $\langle \text{ABORT}, s \rangle = \langle 3, s \rangle$ to the $p$-entry of $\text{PawnSet}$. Then for $p$ to be promoted at $t' > t$, it follows from the semantics of $\text{PawnSet}$ and Claim D.25, that during $[t, t')$ a $\text{PawnSet.reset()}$ operation and then a $\text{PawnSet.collect}(A)$ operation where $A[p] = s$, must be executed, followed by a $\text{PawnSet.promote()}$ of $p$ and $\text{PawnSet.reset()}$ that returns $\langle p, s \rangle$. This is a contradiction to the assumption that a $\text{PawnSet.reset()}$ is not executed during $[I_0, t']$. \hfill $\Box$

Let $\ell$ be the number of times a promotion occurs during $T$. For all $i \in \{1, \ldots, \ell\}$, define $\Omega_i$ to be the $i$-th interval $[\Omega_i^{-}, \Omega_i^{+}]$ that begins when the $i$-th promotion occurs during $T$ and ends when the promoted process ceases to be a releaser of $L$. Let $P_i$ be the process promoted at $\Omega_i^{-}$.

**Claim D.27.** If $I_2 \neq \emptyset$ and $R(I_0) = \emptyset$ and at time $I_0^-$, $\text{Sync} = \text{Sync} = \bot$ and $\text{PawnSet is candidate-empty}$, then the following claims hold for all $i \in \{1, \ldots, \ell\}$:

(a) If $\ell \geq 1$, then $\gamma = \Omega_1^{-}$ and $R(\Omega_1^{-}) = \{P_1\}$, and $\text{Sync} = \text{Sync} = \bot$ at $\Omega_1^{-}$, and no $\text{PawnSet.reset()}$ operation has been executed during $[I_0^-, \Omega_1^{-}]$.

(b) If $R(\Omega_i^{-}) = \{P_i\}$, then for all $t \in [\Omega_i^{-}, \Omega_i^{+}]$, $R(t) = \{P_i\}$. (i.e., $P_i$ is the only releaser throughout $\Omega_i$)

(c) If $i \neq \ell$ and $R(\Omega_i^{-}) = \{P_i\}$, then $\Omega_i^{+} = \Omega_{i+1}^{-}$ and $R(\Omega_{i+1}^{-}) = \{P_{i+1}\}$. (i.e., $P_{i+1}$ is the only releaser at $\Omega_{i+1}^{-}$)

(d) If $i \neq \ell$, then $\Omega_i^{+} = \Omega_{i+1}^{-}$ and $R(\Omega_{i+1}^{-}) = \{P_{i+1}\}$.

(e) For all $t \in [\Omega_i^{-}, \Omega_i^{+}], R(t) = \{P_i\}$. (i.e., $P_i$ is the only releaser throughout $\Omega_i$)

**Proof.** Proof of (a): If the $\text{PawnSet.promote()}$ operation at time $\gamma$ returns value $\langle \bot, \bot \rangle$, then from Claims D.25 and D.25, it follows that no promotion occurs during $T$, which is a contradiction to $\ell \geq 1$. Thus, the $\text{PawnSet.promote()}$ operation at time $\gamma$ returns a non-$\langle \bot, \bot \rangle$ value. By definition $\gamma$ is the point when the first $\text{PawnSet.promote()}$ operation occurs, and $\Omega_1^-$ is the point when the first promotion occurs and $P_1$ is the process promoted at $\Omega_1^-$. Then $\gamma = \Omega_1^-$, and $P_1$ is the first promoted process. From Claim D.25, $B$ is the only releaser of $L$ at the point in time immediately before time $\gamma$. Then from Claim D.25, it follows that $B$ promotes $P_1$ at time $\gamma = \Omega_1^-$, and $B$ ceases to be a releaser of $L$ at $\gamma$, therefore $R(\gamma) = \{P_1\}$. From Claim D.25, it follows that $\text{Sync} = \text{Sync} = \bot$ at $\Omega_1^-$. From an inspection of the code, a $\text{PawnSet.reset()}$ is executed only in line 67 and it can be executed only after a $\text{PawnSet.promote()}$ is executed in line 65. Since $\gamma$ is the first point when a $\text{PawnSet.promote()}$ is executed, it follows that no $\text{PawnSet.reset()}$ operation was executed during $[I_0^-, \gamma]$. 47
Proof of (b): Since $R(\Omega^-_i) = \{P_i\}$, and $\Omega^+_i$ is the point when $P_i$ ceases to be a releaser of $L_i$ for all $t \in [\Omega^-_i, \Omega^+_i)$, $\{P_i\} \subseteq R(t)$. To show that for all $t \in [\Omega^-_i, \Omega^+_i)$, $R(t) = \{P_i\}$, we need to show that no other process becomes a releaser of $L_i$ during $[\Omega^-_i, \Omega^+_i)$. Suppose some process $q \neq P_i$ becomes a releaser of $L_i$ some time during that interval. Since $\Omega^-_i > \Omega^-_i = \gamma > I_2^-$, from Claim D.24(c) it follows that $P_i$ promotes $q$ during $[\Omega^-_i, \Omega^+_i)$. Then from Claim D.18 $P_i$’s cease-release event $\pi_{P_i}$ occurs during $[\Omega^-_i, \Omega^+_i)$, and thus $P_i$ ceases to be a releaser of $L_i$ during $[\Omega^-_i, \Omega^+_i)$. Hence a contradiction.

Proof of (c): Since $i < \ell$, it follows that there exists a process $P_{i+1}$ that becomes a releaser of $L_i$ during $T$. By definition, $P_i$ and $P_{i+1}$ are the $i$-th and $(i+1)$-th promoted processes during $T$, respectively. Since $\Omega^-_{i+1} > \Omega^-_i > \Omega^-_i = \gamma > I_2^-$, from Claim D.24(c) it follows that no other process becomes a releaser after $P_i$ became a releaser and before $P_{i+1}$ becomes a releaser, i.e., during $[\Omega^-_i, \Omega^-_{i+1})$. Moreover, since $R(\Omega^-_i) = \{P_i\}$, it follows that the next process to be promoted, i.e., $P_{i+1}$, is promoted by the only releaser of $L_i$, $P_i$. Then from Claim D.18 it follows that $P_i$ promotes $P_{i+1}$ by executing a PawnSet.promote() in line 65 that returns $(P_{i+1}, s)$, where $s \in N$, and event $\pi_{P_i}$ occurs at $t_{P_i}$. Then $P_i$ ceases to be a releaser of $L_i$ at $t_{P_i}$ and thus $\Omega^+_i = t_{P_i}$. Since $\Omega^+_{i+1}$ is the point when $P_{i+1}$ becomes a releaser of $L_i$, it follows that $\Omega^+_i = \Omega^+_{i+1}$, and thus $R(\Omega^+_{i+1}) = \{P_{i+1}\}$.

Proof of (d): We prove by induction that for all $k < \ell$, $R(\Omega^-_{k+1}) = \{P_{k+1}\}$ and $\Omega^+_k = \Omega^-_{k+1}$. Basis ($k = 1$) From Part (a), $P_1$ is the only releaser of $L_i$ at $\Omega^-_1$, and clearly $\ell > k = 1$. Then from Part (a), $\Omega^-_1 = \Omega^-_\ell$ and $R(\Omega^-_1) = \{P_1\}$.

Induction step ($k > 1$) By definition $P_k$ is the promoted process at $\Omega^-_k$, and since $|R(\Omega^-_{k-1})| = 1$ and $\Omega^-_{k-1} = \Omega^-_k$ (by the induction hypothesis), it follows that $P_k$ is the only releaser of $L_i$ at $\Omega^-_k$. Then from Part (a), $\Omega^+_k = \Omega^-_{k+1}$ and $R(\Omega^-_{k+1}) = \{P_{k+1}\}$.

Proof of (e): From Part (a), $R(\Omega^-_i) = \{P_i\}$, and thus from Part (b), for all $t \in [\Omega^-_i, \Omega^+_i)$, $R(t) = \{P_i\}$. From Part (b), for all $i > 1$, $R(\Omega^-_i) = \{P_i\}$, and thus from Part (b), for all $t \in [\Omega^-_i, \Omega^+_i)$, $R(t) = \{P_i\}$. Hence, our claim follows.

Claim D.28. If $I_2 \neq \emptyset$ and $R(I_2^-) = \emptyset$ and at time $I_2^-$, $\text{Sync}1 = \text{Sync}2 = \perp$ and PawnSet is candidate-empty, then the following claims hold for all $i \in \{1, \ldots, \ell\}$:

(a) A PawnSet.reset() operation is not executed during $[I_2^-, \Omega^-_i]$.

(b) $P_i$ executes lines of code of lock$P_i$() starting with line 3 as depicted in Figure 15.

(c) $P_i$’s call to lock$P_i$() returns $\infty$, and $P_i$ finishes lock$P_i$() during $T$, and Role$[P_i] = \text{PAWN\_P}$ when $P_i$’s call to lock$P_i$() returns.

(d) Exactly one cease-release event among $\pi_{P_i}$ and $\theta_{P_i}$ occurs during $P_i$’s call to doPromote$P_i$().

(e) $P_i$ executes lines of code of release$P_i$() starting with line 32 as depicted in Figure 16.

(f) $P_i$ does not write to $\text{Sync}1$ or $\text{Sync}2$ during $[\Omega^-_i, \Omega^+_i]$.

(g) $t_{P_i} < \Omega^-_i < t_{P_i}^{34} < \Omega^+_i < t_{P_i}^{49}$, and $\Omega^+_i \leq I_2^+.$

(h) If $i \neq \ell$, then a PawnSet.reset() operation is not executed during $[I_2^-, \Omega^+_i]$.

(i) Throughout $[\gamma, \Omega^+_i)$, $\text{Sync}1 = \text{Sync}2 = \perp$.

(j) If $\ell > 1$, $I_2^+ = \Omega^+_\ell$.

(k) For all $t \in [\gamma, I_2^+]$, $|R(t)| = 1.$
Figure 15: $P_i$'s call to \texttt{lock}(P_i)

Figure 16: $P_i$'s call to \texttt{release}(P_i)

(1) $R(I_i^+) = \emptyset$ and at $I_2^+$, Sync1 = Sync2 = ⊥ and PawnSet is candidate-empty.

Proof. Proof of (a)-(h): We prove Parts (a)-(h) by induction on $i$. First, we prove Part (a) for $i = 1$. Second, we show that if Part (a) is true for a fixed $i$, then Parts (b)-(h) are true for $i$. Finally, we show that if Parts (a)-(h) are true for $i$, then Part (a) is true for $i + 1$, thus completing the proof.

From Claim D.27(a), no PawnSet.reset() operation has been executed during $[I_0^-, \Omega_i^-]$. Hence, Part (a) for $i = 1$ holds.

Now we show that if Part (a) is true for a fixed $i$, then Parts (b)-(h) follow for $i$.

Proof of Parts (b) and (c) if Part (a) for $i$ is true: Let $q$ be the process that promotes $P_i$ at $\Omega_i^-$. Then $q$’s PawnSet.promote() operation in line 65 returned value $\langle P_i, s \rangle$, where $s \in \mathbb{N}$, and $\Omega_i^- = t_65$.

Then from the semantics of the PawnSet object it follows that the $P_i$-th entry of PawnSet was $\langle \text{REG}, s \rangle = (1, s)$ immediately before $\Omega_i^-$. Then from Claim D.5(b) it follows that some process (say $r$) executed a PawnSet.collect($A$) operation in line 55 where $A[P_i] = s$. Then from the code structure, $r$ read apply[$P_i$] = $\langle \text{REG}, s \rangle$ in line 52. By Claim D.6(a) apply[$P_i$] is set to value $\langle \text{REG}, s \rangle$ only by process $P_i$ when it executes a successful apply[$P_i$].CAS($\langle \bot, \bot \rangle$, $\langle \text{REG}, s \rangle$), therefore $P_i$ executed the same and broke out of the spin loop of line 2.

Note that $t_2 < t_52 < t_65$.

Since $\text{Ctr} = 0$ throughout $I_0$, $\text{Ctr} = 1$ throughout $I_1$ and $\text{Ctr} = 2$ throughout $I_2$, it follows that $\text{Ctr}$ is increased only at points $I_i^-$ and $I_2^+$ during $T$. Since $K$ and $Q$ are the first two releasers of $L$ and they increased $\text{Ctr}$ to 1 and 2, respectively, at $I_i^-$ and $I_2^+$, respectively, it follows that no other process apart from $K$ and $Q$ increases the value of $\text{Ctr}$ during $T$. Since $\Omega_i^- \geq \Omega_i^+ = \gamma > I_2^+$ (by Claims D.25(a) and D.25(b) and D.27(a)), $P_i$ becomes a releaser of $L$ only after $I_2^+$ (the point at which $Q$ became a releaser of $L$). Thus, $P_i$ is not among the first two releasers of $L$, thus $P_i \notin \{K, Q\}$. Then it follows that $P_i$ does not increase $\text{Ctr}$. Therefore $P_i$’s $\text{Ctr}.\text{inc}()$ operation in
line 5 returns value 2 = PAWN, and thus \( P_i \) sets \( \text{Role}[P_i] \) to 2 = PAWN in line 6. Then from the code structure \( P_i \) satisfies the if-condition of line 6 and proceeds to spin in line 7.

**Case a - \( P_i \) receives a signal to abort while busy-waiting in line 7** Then \( P_i \) stops spinning in line 7 and executes \( \text{abort}_{P_i}() \). Since \( P_i \) last set \( \text{Role}[P_i] \) to PAWN in line 5 it then follows from the code structure that \( P_i \) proceeds to execute lines 18-20 and satisfies the if-condition of line 20 and then executes a \( \text{PawSet.abort}(P_i, s) \) operation in line 21.

Since a \( \text{PawSet.reset}() \) operation has not been executed during \([I_0, \Omega_i^-]\), from Claim D.20 it follows that \( P_i \) did not execute a \( \text{PawSet.abort}(P_i, s) \) operation in line 21 during \([I_0, \Omega_i^-]\), thus \( \Omega_i^- > \Omega_i^- \). Since \( P_i \) has exclusive-registration access to \( \text{PawSet} \) during \([\Omega_i^-, \Omega_i^+]\), and \( p \) has not executed any of its cease-release events or reset \( \text{PawSet} \) during \([t_{P_i}, t_{P_i}]\), and \( t_{P_i} < \Omega_i^- \), it then follows that \( \text{PawSet} \) was not reset during \([\Omega_i^-, \Omega_i^+]\). Then since the \( P_i \)-th entry of \( \text{PawSet} \) was last changed to \( \langle \text{PRO}, s \rangle = \langle 2, s \rangle \) at \( \Omega_i^+ \), it remains \( \langle \text{PRO}, s \rangle \) throughout \([\Omega_i^-, \Omega_i^+]\). Then \( P_i \)'s \( \text{PawSet.abort}(P_i, s) \) operation in line 21 returns \textit{false} by the semantics of the \( \text{PawSet} \) object. Then \( p \) satisfies the if-condition of line 21 proceeds to set \( \text{Role}[P_i] \) to PAWN in line 22 and then returns \textit{false} from its call to \( \text{abort}_{P_i}() \) and \( \text{lock}_{P_i}() \).

**Case b - \( P_i \) does not receive a signal to abort while busy-waiting in line 7**

Recall that process \( q \) promotes \( P_i \) at \( \Omega_i^- \) by executing a \( \text{PawSet.promote}() \) operation in line 65 that returns value \( \langle \text{PRO}, s \rangle \), where \( s \in \mathbb{N} \). Since processes in the system continue to take steps, process \( q \) sets its local variable \( j \) to value \( \text{PRO} \) in line 65 and proceeds to fail the if-condition of line 66 and then executes line 70 where \( \langle j, seq \rangle = \langle \text{PRO}, s \rangle \). Then \( q \) executes a \( \text{apply}[P_i].\text{cas}((\langle \text{REG}, s \rangle, (\text{PRO}, s)) \) operation in line 70.

Recall that process \( r \) read value \( \text{apply}[P_i] = \langle \text{REG}, s \rangle \) in line 52 and \( \Omega_i^- < t_{P_i} < \Omega_i^- = \Omega_i^- \). From an inspection of the code, \( \text{apply}[P_i] \) can change from value \( \langle \text{REG}, s \rangle \) only to value \( \langle \text{PRO}, s \rangle \) and from value \( \langle \text{PRO}, s \rangle \) only to value \( \langle \bot, \bot \rangle \). Also, \( \text{apply}[P_i] \) can be changed from \( \langle \text{PRO}, s \rangle \) to \( \langle \bot, \bot \rangle \), only if \( p \) executes line 32 or 49. Since \( p \) is spinning in line 7 it follows that a \( \text{apply}[P_i].\text{cas}((\langle \text{PRO}, s \rangle, (\bot, \bot)) \) operation is not executed during \([\Omega_i^-, \Omega_i^-] \), and thus \( \text{apply}[P_i] = \langle \text{REG}, s \rangle \) throughout \([\Omega_i^-, \Omega_i^-]\). Therefore, \( q \) executes a successful \( \text{apply}[P_i].\text{cas}((\langle \text{REG}, s \rangle, (\text{PRO}, s)) \) operation in line 70 and thus \( \text{apply}[P_i] = \langle \text{PRO}, s \rangle \) at \( \Omega_i^- \).

Since \( P_i \) is busy-waiting in line 7 for \( \text{apply}[P_i] \) to change to \( \langle \text{PRO}, s \rangle \), it then follows that \( P_i \) busy-wait throughout \([\Omega_i^-, \Omega_i^-]\), and reads \( \text{apply}[P_i] = \langle \text{PRO}, s \rangle \) when it executes line 7 for the first time after \( \Omega_i^- \). Then \( P_i \) breaks out of the spin loop, and then from the code structure, \( P_i \) proceeds to set \( \text{Role}[P_i] \) to PAWN in line 9 breaks out of the role-loop in line 12 executes line 13 and fails the if-condition of line 13 and executes lines 16-17 and returns from \( \text{lock}_{P_i}() \) in line 17 with value \( \infty \). Note that \( \Omega_i^- < \Omega_i^- \).

**Proof of Parts (c), (e) and (f) if Part (a) for \( i \) is true:** Since \( P_i \) is the only releaser of L throughout \([\Omega_i^-, \Omega_i^+]\) (Claim D.27(a)), it follows from Claim D.15(5) that \( P_i \) has exclusive write-access to objects Sync1 and Sync2 and exclusive registration-access to \( \text{PawSet} \) throughout \([\Omega_i^-, \Omega_i^+]\).

Since \( P_i \) returns from its call to \( \text{lock}_{P_i}() \) with value \( \infty \) (by Part (a)), \( P_i \) executes a call to \( \text{release}_{P_i}() \) (follows from conditions (d) and (f)).

Since \( \text{Role}[P_i] = \text{PAWN} \) when \( P_i \)’s call to \( \text{lock}_{P_i}() \) returns (by Part (a)), \( \text{Role}[P_i] = \text{PAWN} \) at \( t_{P_i} \). Since \( \text{Role}[P_i] \) is unchanged during \([t_{P_i}, t_{P_i}]\) (follows from Claim D.4(b)), it follows from the code structure that during \( P_i \)'s call to \( \text{release}_{P_i}(j) \), \( P_i \) only executes lines 34-35, 42 and 45-50. Then Figure 10 follows.

From an inspection of Figures 15 and 16, \( P_i \) does not execute a call to \( \text{helpRelease}_{P_i}() \) or execute a \( \text{cas}((1, 0)) \) operation in line 46 during \( \text{release}_{P_i}() \). Then from Claims D.7(a) and D.7(b) \( P_i \)'s cease-release events \( \phi_{P_i} \) and \( \tau_{P_i} \) do not occur. Since \( P_i \) executes a call to \( \text{doPromote}_{P_i}() \) only
in line 46 it follows from Claim D.19 that exactly one cease-release event among \( \pi_i \) and \( \theta_i \) occurs during \( P_i \)'s call to \( \text{doPromote}_{P_i}() \). Hence, Part (i) follows. Then \( \Omega_i^+ \) is the point when cease-release event \( \pi_i \) or \( \theta_i \) occurs. From an inspection of Figures 15 and 16 and the code, it is clear that \( P_i \) does not change Sync1 or Sync2 during \( \text{lock}_{P_i}() \) and \( \text{release}_{P_i}() \). Therefore, \( P_i \) does not change Sync1 or Sync2 during \( [\Omega_i^-, \Omega_i^+] \).

**Proof of Part (m) if Part (a) for \( i \) is true:** As argued in Part (i) and (ii), \( \theta_i < \Omega_i^- \), and \( \Omega_i^- < \theta_i \). Since \( \theta_i < \Omega_i^- \) and \( \Omega_i^- < \theta_i \), it then follows that \( \theta_i < \Omega_i^- < \Omega_i^- \).

From Part (ii), exactly one cease-release event among \( \pi_i \) and \( \theta_i \) occurs during \( P_i \)'s call to \( \text{doPromote}_{P_i}() \). If cease-release event \( \theta_i \) occurs then \( \Omega_i^+ \) is the point when \( P_i \)'s cease-release event \( \theta_i \) occurs, i.e., \( \Omega_i^+ = \theta_i \). Then \( P_i \) changes \( \text{Ctr} \) to 0 and the \( \text{Ctr} \)-cycle interval \( T \) ends at \( \Omega_i^+ = \theta_i \).

If cease-release event \( \pi_i \) occurs then \( \Omega_i^+ \) is the point when \( P_i \)'s cease-release event \( \pi_i \) occurs, i.e, \( \Omega_i^+ = \pi_i \). Since \( P_i \) calls \( \text{doPromote}_{P_i}() \) only in line 46 (by inspection of Figure 16), it then follows that \( \Omega_i^+ \in \{ \theta_i, \pi_i \} < \theta_i \). Thus, Part (m) holds.

**Proof of Part (i) if Part (m) for \( i \) is true:** As argued in Part (i), exactly one cease-release event among \( \pi_i \) and \( \theta_i \) occurs during \( P_i \)'s call to \( \text{doPromote}_{P_i}() \). If cease-release event \( \theta_i \) occurs then \( \Omega_i^- \) is the point when \( P_i \)'s cease-release event \( \theta_i \) occurs, i.e, \( \Omega_i^- = \theta_i \). Then \( P_i \) changes \( \text{Ctr} \) to 0 and the \( \text{Ctr} \)-cycle interval \( T \) ends at \( \Omega_i^- = \theta_i \), and thus \( \ell = i \). This is a contradiction to the assumption \( i \neq \ell \). Then \( P_i \)'s cease-release event \( \pi_i \) occurs during \( P_i \)'s call to \( \text{doPromote}_{P_i}() \). Then \( \Omega_i^+ \) is the point when \( P_i \)'s cease-release event \( \pi_i \) occurs, i.e, \( \Omega_i^+ = \pi_i \). From an inspection of Figures 15 and 16 and the code, it follows that \( P_i \) does not execute a \( \text{PawnSet} \cdot \text{reset}() \) operation during \( [\pi_i, \theta_i] \), and \( P_i \) calls \( \text{doPromote}_{P_i}() \) only in line 46. Since \( \Omega_i^+ = \pi_i \), from an inspection of the code of \( \text{doPromote}_{P_i}() \), \( P_i \) does not execute a \( \text{PawnSet} \cdot \text{reset}() \) operation during \( [\pi_i, \pi_i] \). Then \( P_i \) does not execute a \( \text{PawnSet} \cdot \text{reset}() \) operation during \( [\Omega_i^-, \Omega_i^+] \).

Since \( P_i \) is the only releaser of \( L \) throughout \( [\Omega_i^-, \Omega_i^+] \) (Claim D.27(m)), it follows from Claim D.15 that \( P_i \) has exclusive registration-access to \( \text{PawnSet} \) throughout \( [\Omega_i^-, \Omega_i^+] \). Then since no \( \text{PawnSet} \cdot \text{reset}() \) operation was executed during \( [I_0, \Omega_i^-] \), and \( P_i \) does not execute a \( \text{PawnSet} \cdot \text{reset}() \) operation during \( [\Omega_i^-, \Omega_i^+] \), it follows that no \( \text{PawnSet} \cdot \text{reset}() \) operation is executed during \( [I_0, \Omega_i^+] \). Hence, Part (i) holds.

Finally, we show that if Parts (a)- (i) are true for \( i \), then Part (ii) is true for \( i+1 \), thus completing the proof. From Part (i) for \( i \), no \( \text{PawnSet} \cdot \text{reset}() \) operation has been executed during \( [\Omega_i^-, \Omega_i^+] \). From Claim D.27(iii), \( \Omega_i^+ = \Omega_{i+1}^- \). Then Part (ii) for \( i+1 \) holds.

**Proof of (i):** From Claim D.27(iv), \( \text{Sync1} = \text{Sync2} = \bot \) at \( \Omega_i^- = \gamma \). From Claims D.27(iv) and D.27(iii), it follows that \( \gamma = \Omega_i^- < \Omega_i^+ = \Omega_1^- < \Omega_2^- < \Omega_3^- \ldots < \Omega_i^- = \Omega_i^- < \Omega_i^+ \).

From Claim D.27(iii), for all \( t \in [\Omega_i^- , \Omega_i^+] \), \( \text{R}(t) \in \{ P_i \} \). Then \( P_i \) has exclusive write-access to \( \text{Sync1} \) and \( \text{Sync2} \) throughout \( [\Omega_i^-, \Omega_i^+] \). Since \( P_i \) does not change \( \text{Sync1} \) or \( \text{Sync2} \) during \( [\Omega_i^-, \Omega_i^+] \) (Part (i)), it then follows that \( \text{Sync1} = \text{Sync2} = \bot \) throughout \( [\Omega_i^-, \Omega_i^+] = [\gamma, \Omega_i^+] \).

**Proof of (j):** As argued in Part (i), exactly one cease-release event among \( \pi_i \) and \( \theta_i \) occurs during \( P_i \)'s call to \( \text{doPromote}_{P_i}() \). If cease-release event \( \pi_i \) occurs then \( P_i \) promotes some process, and thus the number of processes that get promoted during \( T \) is larger than \( \ell \), which contradicts the definition of \( \ell \). Hence, cease-release event \( \theta_i \) occurs during \( \text{doPromote}_{P_i}() \) and \( \Omega_i^+ \) is the point when cease-release event \( \theta_i \) occurs, i.e, \( \Omega_i^+ = \pi_i \). Since \( \text{Ctr} \) is changed from 2 to 0 when \( \theta_i \) occurs, the \( \text{Ctr} \)-cycle interval \( T \) ends at \( \Omega_i^+ = \pi_i \), and thus \( I_i^+ = \pi_i \).
Case a - $\ell = 0$ : Consider the first $\text{PawnSet.promote()}$ operation at $\gamma$. Since $\ell = 0$, the $\text{PawnSet.promote()}$ operation at $\gamma$ returns value $(\bot, \bot)$. Then from Claim D.25[1], it follows that $B$’s cease-release event $\theta_B$ occurs at $t' = \frac{68}{2} \geq \gamma$, and throughout $[\gamma, t')$ no process is promoted, and for all $t \in [\gamma, t')$, $R(t) = \{B\}$. Since $\text{Ctr}$ is changed from 2 to 0 when $\theta_B$ occurs, the $\text{Ctr}$-cycle interval $T$ ends at $t' = \frac{68}{2}$, and thus $I_2^+ = \frac{68}{2} = t'$. Then for all $t \in [\gamma, t') = [\gamma, I_2^+]$, $|R(t)| = 1$.

From an inspection of Figure 14 and the code, it follows that $B$ executed a $\text{PawnSet.reset()}$ operation in line 67 during $[\gamma, t')$, and thus $\text{PawnSet}$ is candidate-empty immediately after. Since for all $t \in [\gamma, t')$, $R(t) = \{B\}$, $B$ has exclusive registration-access to $\text{PawnSet}$ throughout $[\gamma, t')$ (follows from Claim D.15). Then it follows that $\text{PawnSet}$ is candidate-empty at $t' = I_2^+$.

Since for all $t \in [\gamma, t')$, $R(t) = \{B\}$, $B$ has exclusive write-access to $\text{Sync1}$ and $\text{Sync2}$ throughout $[\gamma, t')$ (follows from Claim D.15). Since $\text{Sync1} = \text{Sync2} = \bot$ at $\gamma$ (Claim D.25[1]), and $B$ does not write to $\text{Sync1}$ and $\text{Sync2}$ during $[\gamma, t')$, it follows that $\text{Sync1} = \text{Sync2} = \bot$ throughout $[\gamma, t') = [\gamma, I_2^+]$.

Case b - $\ell \geq 1$ : From Part i, $I_{2}^{+} = \Omega_{\ell}^{+} = \frac{68}{\ell}$. Then from Part ii, it follows that $\text{Sync1} = \text{Sync2} = \bot$ throughout $[\gamma, I_{2}^{+}]$, and from Claim D.27[c], it follows that for all $t \in [\Omega_{1}^{+}, \Omega_{\ell}^{+}] = [\gamma, I_{2}^{+}]$, $|R(t)| = 1$. Since $\mathcal{P}_c$ ceases to be a releaser of $L$ at $\Omega_{\ell}^{+}$, $R(I_{2}^{+}) = \emptyset$.

Since $\Omega_{\ell}^{+} = \frac{68}{\ell}$, $\mathcal{P}_c$ executed line 68 and before that line 67. Hence, $\mathcal{P}_c$ executed a $\text{PawnSet.reset()}$ operation at $t_{67} < \Omega_{\ell}^{+}$. Since $t_{67} = \frac{68}{\ell} > t_{34}^{1}$ and $t_{67} = \frac{68}{\ell} > \Omega_{\ell}^{+}$ (by Part ii), it follows that $t_{67} > \Omega_{\ell}^{+}$. Hence, $\mathcal{P}_c$ executed a $\text{PawnSet.reset()}$ operation at $t_{67} \in [\Omega_{\ell}^{+}, \Omega_{\ell}^{+}]$. Since $\mathcal{P}_c$ is the only releaser of $L$ throughout $[\Omega_{\ell}^{+}, \Omega_{\ell}^{+}]$ (Claim D.27[g]), it follows from Claim D.15 that $\mathcal{P}_c$ has exclusive registration-access to $\text{PawnSet}$ throughout $[\Omega_{\ell}^{+}, \Omega_{\ell}^{+}]$. Then it follows that $\text{PawnSet}$ is candidate-empty at $\Omega_{\ell}^{+} = I_{2}^{+}$.

Claim D.29. $R(I_0^-) = \emptyset$ and at $I_0^-$, $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty for any $\text{Ctr}$-cycle interval $T$ during history $H$.

Proof. Let $T^k$ denote the $k$-th $\text{Ctr}$-cycle interval $T$ during history $H$. We give a proof by induction over the integer $k$. Basis - At $I_0^-$ for $T^1$, the claim holds trivially since all variables are at their initial values ($\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty).

Induction Step - By the induction hypothesis, at $I_0^-$ for $T^{k-1}$, $R(I_0^-) = \emptyset$, and $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty. Since $T^k$ begins immediately after $T^{k-1}$ ends, to prove our claim we need to show that, when $T^{k-1}$ ends, there are no releasers of $L$ and $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty. The time interval $T^{k-1}$ ends either at time $I_1^+$ or time $I_2^+$.

Case a - $T^{k-1}$ ends at time $I_1^+$ : Then $I_2 = \emptyset$. From Claim D.23[1] it follows that $\mathcal{K}$ is the only releaser of $L$ during $I_1$. Since $I_2 = \emptyset$, it then follows from Claim D.23[1], that $\mathcal{K}$’s $\text{Ctr.CAS}(1,0)$ operation in line 36 is successful, and the interval $I_1$ as well as $T^{k-1}$ ends at time $\frac{36}{6}$. Then $\mathcal{K}$’s cease-release event $\phi_K$ occurs at $\frac{36}{6} = I_1^+$, and thus there are no releasers of $L$ immediately after $T^{k-1}$ ends. And from Claim D.23[1], it follows that $\text{Sync1} = \text{Sync2} = \bot$ and $\text{PawnSet}$ is candidate-empty when $T^{k-1}$ ends.

Case b - $T^{k-1}$ ends at time $I_2^+$ : Then $I_2 \neq \emptyset$. Then our proof obligation follows immediately from Claim D.28[1].

Note that in the following claims, notations $I_0, I_1, I_2, \lambda, \gamma, \Omega_i, K, Q$ and $P_i$ are defined relative to a $\text{Ctr}$-cycle interval, as was defined previously in pages 38, 43 and 17. The exact $\text{Ctr}$-cycle interval is clear from the context of the discussion.

Lemma D.30. The mutual exclusion property holds during history $H$. 

52
Proof. For the purpose of a contradiction assume that at time \( t \), two processes (say \( p \) and \( q \)) are poised to execute a call to \( L \).\text{release}() . From Claim [D.13] , it follows that both \( p \) and \( q \) are releasers of \( L \) at \( t \). Consider the Ctr-cycle interval \( T \) such that \( t \in T \).

From Claim [D.29] it follows that at \( I^0_T\) , \( \text{Sync1} = \text{Sync2} = \bot \) and \( \text{PawnSet} \) is candidate-empty, and \( R(I^0_T) = \emptyset \). Then from Claims [D.23](a), [D.23](b), [D.25](a) and [D.25](b) , it follows that during \( T \), lock \( L \) has two releasers only during \([I^-_2, \lambda)\) . Then \( t \in [I^-_2, \lambda)\) . Also from Claim [D.25](a) , for all \( t \in [I^-_2, \lambda)\) , \( R(t) = \{\mathcal{K}, \mathcal{Q}\} \) . Then \( \langle p, q \rangle = \{\mathcal{K}, \mathcal{Q}\} \) . Let \( p = \mathcal{K} \) and \( q = \mathcal{Q} \) without loss of generality.

Recall that \( I^0_T\) is the point in time when \( \mathcal{Q} \) increases \( \text{Ctr} \) from 1 to 2 and sets \( \text{Role}[\mathcal{Q}] \) to Queen in line 3. Since \( q \)'s call to \text{lock}() returned a non-\( \bot \) value, it follows from an inspection of Figure 10 that \( \mathcal{Q} \) returned either in line 17 or line 27. Then \( \mathcal{Q} \) either read a non-\( \bot \) value from \( \text{Sync1} \) in line 14 or \( \mathcal{Q} \) failed the \text{Sync1}.\text{cas}(\bot, \infty) \) operation in line 26. Since \( \text{Sync1} = \bot \) at \( I^-_2 \) (by Claim [D.24](a)) , and \( I^-_2 = \mathbb{F}_Q \) , it then follows that \( \text{Sync1} \) is changed to a non-\( \bot \) value during \([I^-_2, t]\) . Clearly, \( \mathcal{Q} \) does not change \( \text{Sync1} \) during \([I^-_2, t]\) .

Recall that \( I^1_T\) is the point in time when \( \mathcal{K} \) increases \( \text{Ctr} \) from 0 to 1 and sets \( \text{Role}[\mathcal{K}] \) to King in line 3. It follows from an inspection of Figure 8 that \( \mathcal{K} \) does not change \( \text{Sync1} \) during \text{lock}(\_\_\_\_\_) , and thus during \([I^1_2, t]\) . Since \( \text{Sync1} \) is changed to a non-\( \bot \) only by a releaser of \( L \) (by Claim [D.15]) and \( \text{Sync1} = \bot \) at \( I^2_2 \) , and the only releasers of \( L \) during \([I^-_2, t]\) do not change \( \text{Sync1} \) , it then follows that \( \text{Sync1} = \bot \) throughout \([I^2_2, t]\) . Hence, a contradiction.

\[\text{Proof. Given (a): From Claim [D.29] it follows that at } I^0_T, \text{ Sync1} = \text{Sync2} = \bot \text{ and PawnSet is candidate-empty, and } R(I^0_T) = \emptyset. \text{ Then from Claim [D.25](a), it follows that exactly one call to } \text{doCollect}() \text{ is executed during } T \text{ by a process } q \in \{\mathcal{K}, \mathcal{Q}\}. \text{ Since processes are collected only during a call to } \text{doCollect}(), q \in \{\mathcal{K}, \mathcal{Q}\} \text{ collects } p \text{ during } \text{doCollect}() \text{ during } T. \text{ And } q \text{ does so by executing a PawnSet.cas}(A) \text{ operation in line 55, where } A[p] = s \in \mathbb{N}, \text{ and sets the } p\text{-th entry of PawnSet to } (\text{REG}, s). \text{ Since a PawnSet.promote}() \text{ that returns } \langle \bot, \bot \rangle \text{ is executed at } A[p], \text{ during } T, \text{ it then follows from the semantics of the PawnSet object that } p \text{ was promoted during } T. \text{ Then } p = \mathcal{P}_i, \text{ for some } i \leq \ell. \text{ Note that } T \text{ does not end during } [\Omega^-_i, \Omega^+_i].

\text{We now show that } p \text{ is also notified of its promotion during } T. \text{ The process (say } r) \text{ that promoted } p \text{ by executing a PawnSet.promote}() \text{ operation in line 65 also goes on to notify } p \text{ of its promotion by executing a } \text{apply}[p].\text{cas}((\text{REG}, s),(\text{PRO}, s)) \text{ operation in line 70. Since } p \text{ does not abort, it follows from an inspection of Figure 17 and the code, that } p \text{ spins on } \text{apply}[p] \text{ in line 17 until its notification. Then } p \text{ executes line 17 at } t \in \mathbb{N} \text{ and spins on variable } \text{apply}[p] \text{ and } \text{Ctr} \text{ in line 17. Since } \text{Ctr} \text{ is only changed to 0 at the end of } T, \text{ it follows that } \text{Ctr} = 2 \text{ throughout } [0, I^+_2]. \text{ Then } p \text{ busy-waits in the spin loop of line 17 until the end of } T, \text{ or if it reads value } (\text{PRO}, s), \text{ for some } s \in \mathbb{N}, \text{ from } \text{apply}[p] \text{ in line 17 during } T. \text{ Now, } \text{apply}[p] \text{ is changed to value } (\text{PRO}, s) \text{ by some process other than } p, \text{ only if that process notifies } p, \text{ i.e., executes a successful } \text{apply}[p].\text{cas}((\text{REG}, s),(\text{PRO}, s)) \text{ operation in line 70. We now show that } p \text{ is notified during } T.\]
From Claim D.29 it follows that at $I_0$, Sync1 = Sync2 = \perp and PawnSet is candidate-empty, and $R(I_0) = \emptyset$. Then from Claim D.25, it follows that exactly one call to doCollect() is executed during $T$ by a process $q \in \{K, Q\}$. Consider the point when $q$ reads apply[p] in line 52. If $q$ reads a value different from (REG, s), then some process must have notified $p$ during $[I_0, q]$ and since $I_0 < p$ and $I_q \in T$, our claim holds. If $q$ reads the value (REG, s) from apply[p], then $q$ collects $p$ during $T$ by executing a PawnSet.collect(A) operation, where $A[p] = s$, in line 55 during $T$. Thus, our claim follows from Part (a).

Claim D.32. If $p$ registered itself in line 14 and incurred $O(1)$ RMRs in the process, and $p$ does not abort, and all processes in the system continue to take steps, then

(a) $p$ finishes its call to lock$_p()$ and returns a non-$\perp$ value.

(b) $p$ incurs $O(1)$ RMRs in expectation during its call to lock$_p()$.

Proof. Proof of (a) and (b). From an inspection of the code of lock$_p()$, $p$ incurs a constant number of RMRs while executing all other lines of lock$_p()$ except while busy-waiting in lines 2, 7, and 14.

Consider $p$'s call to lock$_p()$. By assumption of the claim, $p$ registered itself in line 2 by executing a successful apply[p].CAS((\perp, \perp),(REG, s)) operation in line 2 and incurred $O(1)$ RMRs in the process. Then $p$ proceeds to execute a Ctr.inc() operation in line 5 and stores the returned value in Role[p]. A Ctr.inc() operation returns values in \{KING, QUEEN, PAWN, $\perp$\}. If it returns $\perp$, $p$ repeats the role-loop, and executes another Ctr.inc() operation in line 5. From Claim A.2 it follows that $p$ repeats the role-loop only a constant number of times before its Ctr.inc() operation returns a non-$\perp$ value.

Case a - $p$ executes a Ctr.inc() operation in line 5 that returns KING. Then $p$ sets Role[p] = KING in line 5. Then from the code structure, $p$ does not busy-wait on any variables, and proceeds to return $\infty$ in line 17, and thus incurs only $O(1)$ RMRs. Hence, (a) and (b) hold.

Case b - $p$ executes a Ctr.inc() operation in line 5 that returns QUEEN. Then $p$ increments Ctr from 1 to 2 in line 5 and sets Role[p] = QUEEN in line 5. Then from the code structure, $p$ proceeds to busy-wait on Sync1 in line 14. Since $p$ increased Ctr from 1 to 2, $I_{14} = I_2^5$ for some Ctr-cycle interval $T$. From Claim D.29 it follows that at $I_0$, Sync1 = Sync2 = $\perp$ and PawnSet is candidate-empty, and $R(I_0^5) = \emptyset$. Then from Claim D.24 it follows that $p$ does not starve in line 14. Since $p$ does not abort, it follows from an inspection of Figure 10 and the code, that $p$ returns a non-$\perp$ value in line 14, and $p$ does not change Sync1. Hence, we have shown that Part (a) holds. Apart from $p$, the only releasers of L during $T$ are $\{K, P_1, \ldots, P_1\}$, where $\ell$ is the number of promotions during $T$. From an inspection of Figures 8, 9, 15, 16 and the code, it follows that only $K$ possibly writes a non-$\perp$ value to Sync1 during $T$ in line 37. Since Sync1 is written to only be a releaser of L, and $I_4^4 \in T$, it then follows that Sync1 is changed to a non-$\perp$ value at most once during $T$. Then $p$ incurs at most one RMR while busy-waiting on Sync1. Hence, we have shown that Part (b) holds.

Case c - $p$ executes a Ctr.inc() operation in line 5 that returns PAWN. Then $p$ found Ctr to be 2 in line 5 and set Role[p] = PAWN in line 5. Then from the code structure, $p$ proceeds to busy-wait on apply[p] and Ctr in line 7.

We now show that $p$ does not starve while busy-waiting in line 7. Since Ctr = 2 at $I_p$, it follows that $I_p \in T$ for some Ctr-cycle interval $T$.

Subcase (i) - apply[p] = (REG, s) at $I_0$ during $T$, for some $s \in N$. Then from Claim D.31, $p$ is notified during $T$. Since $p$ is notified during $T$ and $p$ does not abort, it follows that $p$ does
not change \texttt{apply}[p], and thus \texttt{apply}[p] is changed from \langle \textsc{reg}, s \rangle to \langle \textsc{pro}, s \rangle when \( p \) is notified. Since \texttt{apply}[p] is changed from \langle \textsc{pro}, s \rangle to some other value only by \( p \), it then follows that \texttt{apply}[p] remains \langle \textsc{pro}, s \rangle when \( p \) reads \texttt{apply}[p] for the first time after \( p \) was notified. Then \( p \) incurs one RMR when it reads \texttt{apply}[p] in line 7 after its notification, breaks out of the spin loop of line 4, proceeds to satisfy the \texttt{if}-condition of line 8, and sets \texttt{role}[p] = \textsc{pawn}_p \) in line 9, and proceeds to return \( \infty \) in line 17. Then we have shown Parts (a) and (b) hold.

**Subcase (ii) - \texttt{apply}[p] \neq \langle \textsc{reg}, s \rangle at \( I_{p}^{-} \) during \( T \), for some \( s \in \mathbb{N} \).** Consider the only call to \texttt{doCollect()} during \( T \) by \( q \in \{ \textsc{k}, \textsc{q} \} \). If \( p \) registered itself (i.e., executed its \texttt{apply}[p].\texttt{cas}((\langle \bot, \bot \rangle, \langle \textsc{reg}, s \rangle)) \) operation in line 2 before \( q \) reads \texttt{apply}[p] in line 52 during \texttt{doCollect}_q()\), then \( q \) collects \( p \) during \( T \). Then from Claim \ref{31(b)}, \( p \) is collected and promoted during \( T \), and eventually notified. Then Parts (a) and (b) hold as argued in **Subcase (i)**.

If \( p \) registers itself after \( q \) attempts to acknowledge \( p \) during \( T \), then no process changes \texttt{apply}[p] during \( T \). Then \( p \) continues to busy-wait in line 7 until the \( \text{ctr-} \)cycle interval \( T \) ends and \( \text{ctr} \) is reset to 0.

If \( \text{ctr} \) is increased to 2 before \( p \) reads \( \text{ctr} \) again in line 7, then let \( T' \) be the \( \text{ctr-} \)cycle interval that starts when \( \text{ctr} \) was reset to 0 at the end of \( T \). Since \texttt{apply}[p] was changed to a non-\langle \textsc{reg}, s \rangle \rangle value before the start of \( T' \), it follows that \texttt{apply}[p] = \langle \textsc{reg}, s \rangle at the start of \( T' \). Then from Claim \ref{31(b)}, \( p \) is acknowledged, collected, promoted during \( T'' \), and eventually notified. Then Parts (a) and (b) hold as argued in **Subcase (i)**.

If \( \text{ctr} \neq 2 \) when \( p \) reads \( \text{ctr} \) again in line 7, then \( p \) incurs one RMR in line 7 breaks out of the spin loop, and proceeds to execute line 8. If \( p \) satisfies the \texttt{if}-condition of line 8 then \( p \) has been acknowledged during some \( \text{ctr-} \)cycle interval \( T'' \). Then from Claim \ref{31(b)}, \( p \) is collected, promoted during \( T'' \), and eventually notified. Then Parts (a) and (b) hold as argued in **Subcase (i)**. If \( p \) fails the \texttt{if}-condition of line 8 then \( p \) proceeds to repeat the role-loop. Consider \( p \)'s second iteration of the role-loop. If \( p \) sets \texttt{role}[p] = \{ \textsc{k}, \textsc{q} \} \) in line 5 then Parts (a) and (b) hold as argued in **Case a** and **Case b**. If \( p \) sets \texttt{role}[p] = \textsc{paw} \ in line 5 then it follows that \( \|_p \in T'' \), for some \( \text{ctr-} \)cycle interval \( T'' \), such that \texttt{apply}[p] = \langle \textsc{reg}, s \rangle at \( I_{p}^{-} \) for \( T'' \). Parts (a) and (b) hold as argued in **Case c(i)**.

**Lemma D.33.** If all processes in the system continue to take steps and \( p \) does not abort, then

(a) \( p \) finishes its call to \texttt{lock}_p() and returns a non-\( \bot \) value.

(b) \( p \) incurs \( O(1) \) RMRs in expectation during its call to \texttt{lock}_p().

**Proof.** From an inspection of \texttt{lock}_p(), \( p \) incurs a constant number of RMRs while executing all other lines of \texttt{lock}_p() except while busy-waiting in lines 2, 7, and 11.

Consider \( p \)'s call to \texttt{lock}_p(). Process \( p \) first attempts to register itself in line 2 by attempting to execute an \texttt{apply}[p].\texttt{cas}((\langle \bot, \bot \rangle, \langle \textsc{reg}, s \rangle)) operation. Now, \texttt{apply}[p] is changed from \langle \langle \bot, \bot \rangle \rangle to a non-\langle \langle \bot, \bot \rangle \rangle value only by \( p \) \) (Claim D.6(a)). If \( \texttt{apply}[p] = \langle \bot, \bot \rangle \rangle \) at \( t \), then \( p \) executes a successful \texttt{apply}[p].\texttt{cas}((\langle \bot, \bot \rangle, \langle \textsc{reg}, s \rangle)) operation in line 2 and incurs only one RMR. Then our claims follow immediately from Claims D.32(a) and D.32(b).

If \( \texttt{apply}[p] \neq \langle \bot, \bot \rangle \rangle t \), it follows that some process \( p' \) executed a successful \texttt{apply}[p].\texttt{cas}((\langle \bot, \bot \rangle, \langle \textsc{reg}, s' \rangle)) \) in line 2 during \texttt{lock}_p(), and \texttt{apply}[p] \neq \langle \bot, \bot \rangle \rangle \) throughout \([2, 2 \langle \bot, \bot \rangle\rangle \). Since calls to \texttt{lock}_p() are not executed concurrently, it follows that \( p' \) has completed its call to \texttt{lock}_p() during \([2, 2 \langle \bot, \bot \rangle\rangle \). \(**Case 1** - \( p' \)'s call to \texttt{lock}_p() returned \( \bot \). Then it follows from the code structure that \( p' \) executed a call to \texttt{abort}_p() and returned from line 18 or 33. Since \( p \) executed a successful
apply\([p].CAS(\langle \bot, \bot \rangle, (\text{REG}, s'))\) in line 2, \(p'\) could not have aborted while busy-waiting on line 2 and thus \(p'\) aborted while busy-waiting in line 7 or 14. Then \(p'\) executed line 3 and set its local variable \(p'.\flag\) to \text{true}, and thus \(p\) could not have returned \(\bot\) from line 13 during \(\text{abort}_p()\). Then \(p'\) returned \(\bot\) in line 19 and thus \(p'\) executed operations \(\text{apply}[p].CAS((\text{REG}, s'), (\text{PRO}, s'))\) (in line 19), and \(\text{apply}[p].CAS((\text{PRO}, s'), (\bot, \bot))\) (in line 32). Since, \(\text{apply}[p]\) can be changed from \(\langle \text{REG}, s' \rangle\) only to \(\langle \text{PRO}, s' \rangle\), and from \(\langle \text{PRO}, s' \rangle\) only to \(\langle \bot, \bot \rangle\), it then follows that \(p'\) executes a successful \(\text{apply}[p].CAS((\text{PRO}, s'), (\bot, \bot))\) (in line 32). Then \(p'\) eventually resets \(\text{apply}[p]\) during its \(\text{lock}_p()\) call. Since \(\text{apply}[p] \neq (\bot, \bot)\) throughout \([2, p']\) and \(p'\) completed its call to \(\text{lock}_p()\) during \([2, p']\), we have a contradiction.

**Case 2 -** \(p'\)'s call to \(\text{lock}_p()\) returned a non-\(\bot\) value. Then from the code structure \(p'\)'s executed operations \(\text{apply}[p].CAS((\text{REG}, s'), (\text{PRO}, s'))\) (in line 16 or line 19) before returning from its call to \(\text{lock}_p()\). Since \(\text{apply}[p]\) can be changed from \(\langle \text{REG}, s' \rangle\) only to \(\langle \text{PRO}, s' \rangle\), and from \(\langle \text{PRO}, s' \rangle\) only to \(\langle \bot, \bot \rangle\) and only by a process with pseudo-ID \(p\), it then follows that \(\text{apply}[p] = (\text{PRO}, s')\) when \(p'\)'s \(\text{lock}_p()\) returns. Then it also follows that \(\text{apply}[p] = (\text{PRO}, s')\) until a process with pseudo-ID \(p\) executes an \(\text{apply}[p].CAS((\text{PRO}, s'), (\bot, \bot))\) operation.

Since \(p'\) won the lock \(L\), it follows that some process, say \(r\), eventually executes a call to \(\text{release}_p(j)\), for some integer \(j\). Since a call to \(\text{release}_p(j)\) is wait-free and all processes continue to take steps, it follows that eventually \(r\) executes lines 18 and 19 where it reads value \(\langle \text{PRO}, s' \rangle\) from \(\text{apply}[p]\) in line 18 and resets \(\text{apply}[p]\) with a \(\text{apply}[p].CAS((\text{PRO}, s'), (\bot, \bot))\) operation in line 19. Since \(p\) does not abort, and no other process calls \(\text{lock}_p()\) concurrently, it then follows that eventually \(p\) executes a successful \(\text{apply}[p].CAS((\bot, \bot), (\text{REG}, s))\) operation in line 2. Since \(\text{apply}[p]\) changed only once from \(\langle \text{PRO}, s' \rangle\) to \(\langle \bot, \bot \rangle\) while \(p\) busy-waited in line 2 it follows that \(p\) incurs \(O(1)\) RMRs during the entire process. Then our claims follow immediately from Claims \([D.32][\text{a}], [D.32][\text{b}]\).

**Lemma D.34.** The abort-way is wait-free.

**Proof.** The abort-way is defined to be all steps taken by a process (say \(p\)) after it receives a signal to abort and breaks out of one of the busy-wait cycles of lines 2, 7 or 14. After \(p\) breaks out of one of the busy-wait cycles of lines 2, 7 or 14 \(p\) executes a call to \(\text{abort}_p()\). If \(p\)'s call to \(\text{abort}_p()\) returns \(\bot\), then \(p\)'s passage ends, or else \(p\)'s \(\text{lock}_p()\) returns non-\(\bot\) value and \(p\) calls \(\text{release}_p()\) and \(p\)'s passage ends when the \(\text{release}_p()\) method returns. Since \(\text{abort}_p()\) and \(\text{release}_p()\) are both wait-free (by Lemma \([D.2]\)), our claim follows.

**Lemma D.35.** The starvation freedom property holds during history \(H\).

**Proof.** Consider a process \(p\) that begins to execute its passage. From Lemma \([D.33][\text{a}]\), it follows that if \(p\) does not abort during \(\text{lock}_p()\) and all processes continue to take steps then \(p\) eventually returns from \(\text{lock}_p()\) with a non-\(\bot\) value. Then \(p\) eventually calls \(\text{release}_p()\), and since \(\text{release}_p()\) is wait-free, \(p\) eventually completes its passage. If \(p\) receives a signal to abort during \(\text{lock}_p()\), then \(p\) executes its abort-way. Since the abort-way is wait-free (by Lemma \([D.34]\)), \(p\) eventually completes its passage.

**Lemma D.36.** If a call to \(\text{release}_p(j)\) returns \text{true}, then there exists a concurrent call to \(\text{lock}()\) that eventually returns \(j\).

**Proof.** The only operations that write a value to \(\text{Sync}1\) are \(\text{Sync}1.CAS(\langle \bot, \infty \rangle, (\text{REG}, s'))\) in line 20 and \(\text{Sync}1.CAS(\langle \bot, j \rangle)\) in line 37. From Claim \([D.15]\) \(\text{Sync}1\) is written to only by a releaser of \(L\). From Claim \([D.29]\) it follows that at \(I_0\), \(\text{Sync}1 = \text{Sync}2 = \bot\) and \(\text{PawnSet}\) is candidate-empty, and \(R(I_0) = \emptyset\). Then from Claims \([D.28][\text{a}], [D.28][\text{b}], [D.25][\text{a}], [D.25][\text{c}], [D.28][\text{k}], [D.28][l]\), the only
releasers of L during a Ctr-cycle interval T, are \{κ, Q, P₁, ..., Pₖ\}. Then from an inspection of Figures 8, 9, 10, 11, 15 and 16 it follows that only κ and Q can write to Sync1 during Ctr-cycle interval T.

Since p returns true, it then follows from an inspection of the code that p executed a successful Sync1.CAS(⊥, j) operation in line 58, and thus failed the Ctr.CAS(1, 0) operation in line 59 and Role[p] = KING at tₐ. Then p = κ for some Ctr-cycle interval T. Since κ failed the Ctr.CAS(1, 0) operation in line 59, it then follows that Ctr was increased to 1 by process Q during T, and I₂ = \(\frac{tₐ}{Q} < \frac{tₐ}{κ}\). Since \(I₁ = \frac{tₐ}{κ}\) and \(I₁ < I₂\), it then follows that Q’s lock_Q() call is concurrent to κ’s release_K(j) call.

From Claim D.29 it follows that at \(I₀\), Sync1 = Sync2 = ⊥ and PawnSet is candidate-empty, and R(\(I₀\)) = \∅. Then from Claim D.25(k), Sync1 = ⊥ at \(I₂\), and κ and Q are the only two releasers of L during \([I₂, \lambda]\), where \(\lambda\) is the first point in time when T is changed to a non-⊥ value, and \(\lambda = \min(\frac{tₐ}{κ}, \frac{tₐ}{Q})\).

Now, Sync1 is reset only in line 58 and since \(tₐ < tₐ\) ≥ \(\lambda\) and \(tₐ > tₐ\) ≥ \(\lambda\), it then follows that κ and Q do not reset Sync1 during \([I₂, \lambda]\). Since κ and Q are the only processes with write-access to Sync1, Sync1 is not reset during \([I₂, \lambda]\).

Consider Q’s lock() call (see Figure 10). Since κ executed a successful Sync1.CAS(⊥, j) operation and Sync1 is not reset during \([I₂, \lambda]\), it then follows that if Q executes the Sync1.CAS(⊥, ∞) operation in line 20 then the operation fails. From an inspection of Figure 10 Q either returns from its lock() call in line 17 or line 27. In both these lines, Q returns the non-⊥ value stored in Sync1. Since κ is the only process apart from Q that can write to Sync1, Q returns the value j that κ wrote during its release_K(j) call.

Now consider an implementation of object ALockArray_n, where instance PawnSet is implemented using object SFMSUnivConst(AbortableProArray_n), and the operations in lines 55, 61, 65, 64, and 67 are executed using the doFast() method, while the operation in line 24 is executed using the doSlow().

**Claim D.37.** Lines 64, 65, 67 of doPromote(), all lines of doCollect(), and lines 57, 62 are not executed concurrently.

**Proof.** From Claim D.13(b), it follows that only a releaser of L can execute any of these lines. From Claim D.29 it follows that at \(I₀\), Sync1 = Sync2 = ⊥ and PawnSet is candidate-empty, and R(\(I₀\)) = \∅. Then from Claims D.23(a), D.23(f), D.25(a), D.25(c), D.28(k), and D.28(l) it follows that L has more than one releaser only during \([I₂, \lambda]\) for some Ctr-cycle interval T. More specifically, there are two releasers of L only during \([I₂, \lambda]\), and the releasers are κ and Q. From Claim D.25(k) it follows that a doCollect() is executed only by κ or Q but not both. Then it follows immediately that lines of doCollect() are not executed concurrently. Since \(\lambda = \min(\frac{tₐ}{κ}, \frac{tₐ}{Q})\), it follows from an inspection of Figures 8, 9, 10, 11, 15 and 16 that the code, that processes κ and Q have not executed a call to doPromote() or lines 57, 62 of helpRelease(), before \(tₐ\) and \(tₐ\) respectively. Then none of the lines chosen in the claim are executed concurrently, and thus our claim holds.

**Lemma D.38.** (a) Both helpRelease_p() and doPromote_p() have O(1) RMR complexity.

(b) doCollect_p() has O(n) RMR complexity.

(c) abort_p() has O(n) RMR complexity.

(d) If a call to release_p(j) returns true, then p incurs O(n) RMRs during release_p(j).
(e) If a call to \texttt{release}_p(j) returns \texttt{false}, then \( p \) incurs \( O(1) \) RMRs during \texttt{release}_p(j).

\textbf{Proof.} Proof of (\ref{part:lemma-e1}) and (\ref{part:lemma-e2}): As per the properties of object SFMSUnivConst(\texttt{AbortableProArray}_n) (Lemma 2.2), an operation performed using the \texttt{doFast()} method has \( O(1) \) RMR complexity, as long as it is not executed concurrently with another \texttt{doFast()} method call. Since \texttt{PawnSet} is an instance of object SFMSUnivConst(\texttt{AbortableProArray}_n), where operations in lines 25, 61, 63 and 67 are executed using the \texttt{doFast()} method, and each of these operations are not executed concurrently (by Claim \( D.37 \)), it then follows that all of these operations have \( O(1) \) RMR complexity. Then \texttt{Part (a)} follows immediately from an inspection of methods \texttt{helpRelease()} and \texttt{doPromote()}. Since method \texttt{doCollect()} has a loop of size \( n \) that incurs a constant number of RMRs in each iteration, Part \( \text{[b]} \) follows.

Proof of (\ref{part:lemma-e3}), (\ref{part:lemma-e4}) and (\ref{part:lemma-e5}): As per the properties of object SFMSUnivConst(\texttt{AbortableProArray}_n) (Lemma 2.2), an operation performed using the \texttt{doSlow()} method has \( O(n) \) RMR complexity, where \( n \) is the maximum number of processes that can access the object concurrently. Since the operation in line 21 is executed using the \texttt{doSlow()} method, the operation has \( O(n) \) RMR complexity. Since \texttt{helpRelease()} and \texttt{doPromote()} have an RMR complexity of \( O(1) \) (by Part \( \text{[a]} \)), and \texttt{doCollect()} has an RMR complexity of \( O(n) \) (by Part \( \text{[b]} \)), it then follows from an inspection of \texttt{abort()}, that a call to \texttt{abort()} has an RMR complexity of \( O(n) \). Thus Part \( \text{[b]} \) follows.

If a call to \texttt{release}_p(j) returns \texttt{true}, then \( p \) does execute a call to \texttt{doCollect}_p() in line 38, else it does not. Then from an inspection of \texttt{release}_p(j), Parts \( \text{[d]} \) and \( \text{[e]} \) follow immediately. \( \Box \)

Lemma 5.1 follows from Lemmas D.2, D.30, D.33, D.34, D.35, D.36 and D.38.

\section{The Tree Based Randomized Abortable Lock}

\subsection{Implementation / Low Level Description}

We assume that the tree structure \( T \) provides a function \texttt{getNode()}, such that, for a leaf node \texttt{leaf} and integer \( \ell \), the function \texttt{getNode}(\texttt{leaf}, \ell) returns a pair \( \langle u, i \rangle \), where \( u \) is the \( \ell \)-th node on the path from \texttt{leaf} to the root node, and \( i \) is the index of the child node of \texttt{u} that lies on the path.

We now describe the implementation of the abortable lock (see Figure 17).

\textbf{Description of the lock}_p() method. Suppose process \( p \) executes a call to \texttt{lock}_p(). With every iteration of the while-loop, process \( p \) captures at least one node on its path from \texttt{leaf}_p to \( T.\text{root} \). Suppose \( p \) executes an iteration of while-loop (lines 1-10) and \( \ell_p = k \) at line 1 for some arbitrary integer \( k \). In line 2, process \( p \) determines the \( k \)-th node (say \( u \)) on \( \text{path}_p \) and the index (say \( r \)) of \( u \)'s child node that lies on \( \text{path}_p \), and stores them in local variables \( v_p \) and \( i_p \). The variables \( v_p \) and \( i_p \) are unchanged during the rest of the iteration. In line 3, process \( p \) attempts to capture \( u.L \), and thus node \( u \) by executing a call to \texttt{u.L.lock()} with pseudo-ID \( r \). If \( p \)'s \texttt{u.L.lock()} returns an integer value (say \( j \)) then \( p \) has been transferred all nodes on its path up to height \( j \) (we ensure \( j \geq h_u \)). If \( p \)'s \texttt{u.L.lock()} returns \( \infty \) then \( p \) has captured lock \( u.L \). In lines 4 and 5, \( p \) stores the height of the highest captured node in its local variable \( \ell_p \). In line 6, \( p \) checks whether it has received a signal to abort. In this case \( p \) releases all its captured nodes by executing a call to \texttt{release}_p() in line 7 and then returns from its call to \texttt{lock}_p() in line 8 with value \( \bot \). Otherwise \( p \) continues its while-loop. On completing its while-loop, \( p \) owns the root node, and thus returns with value \( \infty \) in line 11 to indicate a successful \texttt{lock()} call.

\textbf{Description of the release}_p() method. Suppose process \( p \) executes a call to \texttt{release}_p(). Let \( s \) be the highest node \( p \) owns at the beginning of \texttt{release}_p(). We later prove that \( h_s = \ell_p \).
Algorithm: Implementation of the abortable lock

```plaintext
define Node: struct { L: ALockArray_Δ }
shared:    \( T \): complete \( \Delta \)-ary tree of height \( \Delta \) and node type Node
local:     \( v \): Node init ⊥; \( i, \ell, k \): int init 0; abort_signal: boolean init false;

define function \( T\.getNode(Node \ leaf, int \ \ell) \): returns a pair \( ⟨u, i⟩\), where \( u \) is the \( \ell \)-th node on the path from \( \leaf \) to the root node of \( T \), and \( i \) is the index of the child node of \( u \) that lies on the path.

Method lock\(_p\)()
1 while \( \ell < T\.height \) do
2     \( (v, i) \leftarrow T\.getNode(\leaf_p, \ell + 1) \)
3     \( val \leftarrow v\.L.lock_i() \)
4     if \( val = \infty \) then \( \ell \leftarrow \ell + 1 \)
5     if \( val \notin \{⊥, \infty\} \) then \( \ell \leftarrow val \)
6     if abort_signal = true then
7         release\(_p\)()
8         return ⊥
9     end
10 end
11 return \( \infty \)

Method release\(_p\)()
12 while \( k \leq \ell \) do
13     \( (v, i) \leftarrow T\.getNode(\leaf_p, k) \)
14     if \( v\.L.release_i(\ell) \) then
15         break
16     \( k \leftarrow k + 1 \)
17 end
```

Figure 17: Implementation of the abortable lock
During an iteration of the while-loop (lines 12-16), process \( p \) either releases a node on its path from leaf\(_p\) to \( s \), or \( p \) hands over all remaining nodes that it owns to some process.

Consider the execution of an iteration of the while-loop where \( k_p = t \) at line 12 for some integer \( t \leq h_s \). In line 13 process \( p \) determines the \( t \)-th node (say \( u \)) on path\(_p\) and the index (say \( r \)) of \( u \)'s child node that lies on path\(_p\), and stores them in local variables \( v_p \) and \( i_p \). In line 14 process \( p \) releases \( u.L \), and thus node \( u \), by executing a call to \( u.L\text{release}(h_s) \) with pseudo-ID \( r \). If \( p \)'s \( u.L\text{release}_p(h_s) \) returns \text{false} then \( p \) has successfully released lock \( u.L \), and thus node \( u \). If \( p \)'s \( u.L\text{release}_p(h_s) \) returns \text{true} then \( p \) has successfully handed over all nodes from \( u \) to \( s \) on path\(_p\) to some process that is executing a concurrent call to \( u.L\text{lock}() \). If \( p \) has handed over all its nodes, then \( p \) breaks out of the while-loop in line 14 and returns from its call to \( \text{release}_p() \). If \( p \) has not handed over all its nodes then \( p \) increases \( k_p \) in line 15 and continues its while-loop.

Notice that our strategy to release node locks is to climb up the tree until all node locks are released or a hand over of remaining locks is made. Climbing up the tree is necessary (as opposed to climbing down) in order to hand over node locks to a process, say \( q \), such that the handed over nodes lie on path\(_q\). There is however a side effect of this strategy which is as follows: Suppose \( p \) owns nodes \( v \) and \( u \) on path\(_p\) such that \( ⟨u, i⟩ = \text{getNode}(\text{leaf}_p, h_u) \) and \( v \) is the \( i \)-th child on node \( u \). Now suppose \( p \) releases lock \( v.L \) at node \( v \). Since the lock at node \( v \) is now released, some process \( r \neq p \) may now capture lock \( v.L \) and then proceed to call \( u.L\text{lock}() \). If process \( p \) has not yet released \( u.L \) by completing its call to \( u.L\text{release}_i() \), then we have a situation where a call to \( u.L\text{lock}() \) is made before a call to \( u.L\text{release}_i() \) is completed. Since there can be at most one owner of lock \( v.L \) there can be at most one such call to \( u.L\text{lock}_i() \) concurrent to \( u.L\text{release}_i() \). This is precisely the reason why we designed object ALockArray\(_n\) to be accessed by at most \( n + 1 \) processes concurrently.

### E.2 Analysis and Proofs of Correctness

In this section, we formally prove all properties of our abortable lock for the CC model. We first, establish the safety conditions on the usage of the object.

**Condition E.1.** (a) If process \( p \) executes a successful \( \text{lock}_p() \) call, then process \( p \) eventually executes a \( \text{release}_p() \) call.

(b) A process calls method \( \text{release}() \) if and only if its last access of the lock object was a successful \( \text{lock}() \) call.

(c) Methods \( \text{lock}_p() \) and \( \text{release}_p() \) are called only by process \( p \), where \( p \in \{0, \ldots, N - 1\} \).

(d) For every \( \text{release}_p() \) call, there must exist a unique successful \( \text{lock}_p() \) call that has been executed.

**Notations and Definitions.** Let \( H \) be an arbitrary history of an algorithm that accesses an instance \( L \) of our abortable lock where Condition E.1 is satisfied. Consider an arbitrary node \( u \) on the tree \( T \). Let \( h_u \) denote the height of node \( u \).

A node \( u \) is said to be handed over from process \( p \) to process \( q \), when \( p \) executes a \( u.L\text{release}(j) \) call that returns \text{true}, where \( j \geq h_u > h_v \), and \( q \) executes a concurrent \( v.L\text{lock}() \) call that returns \text{false}. Process \( p \) is said to start to own node \( u \) when \( p \) captures \( u.L \) or when it is handed over node \( u \) from the previous owner of node \( u \). Process \( p \) ceases to own node \( u \) when \( p \) releases \( u.L \), or when \( p \) hands over node \( u \) to some other process.

**Claim E.2.** Consider an arbitrary process \( p \) and some node \( u \) on path\(_p\).
(a) If \( p \) executes a \( u.L.lock() \) operation that returns value \( j \notin \{\bot, \infty\} \), then \( j \geq h_u \).

(b) The value of \( \ell_p \) is increased every time \( p \) writes to it.

(c) If \( \ell_p = k \), then process \( p \) owns all nodes on \( path_p \) up to height \( k \).

**Proof of (a):** Then from the properties of object \( ALockArray_\Delta \) (Lemma 3.11), it follows that some process (say \( q \)) executed a concurrent \( u.L.release(j) \) operation. Then from the code structure, \( q \) executed a \( T.getNode(leaf_q, k) \) operation in line 13 that returned \( \langle u, i \rangle \), for some \( i \), such that \( h_u = k_q \) (from the semantics of the \( getNode() \) method). Since \( j = \ell_q \geq k_q = h_u \), our claim follows.

**Proof of (b):** Process \( p \) writes to its local variable \( \ell_p \) only in lines 4 and 5. Clearly, \( p \) increases \( \ell_p \) every time it executes line 4. Now, suppose \( p \) executes line 5 where it writes the value of \( val_p \) to \( \ell_p \), where \( v_p = u \), for some node \( u \). Since \( p \) satisfies the if-condition of line 5 and the \( ALockArray_\Delta \) method \( lock() \) only returns a value in \( \{\bot, \infty\} \cup \mathbb{N} \), it follows that \( p \)'s call to \( u.L.lock() \) returned a non-\( \{\bot, \infty\} \) value. Then from Part (a), \( val_p \geq h_u \). Since \( p \) also executed a \( T.getNode(leaf_p, b) \) operation in line 2 where \( b = \ell_p + 1 \) that returned \( \langle u, i \rangle \), for some \( i \), such that \( h_u = b \) (from the semantics of the \( getNode() \) method), it follows that \( val_p \geq h_u = \ell_p + 1 \). Then, \( p \) increases \( \ell_p \) when \( p \) writes \( val_p \) to \( \ell_p \) in line 5.

**Proof of (c):** Let \( t^i \) be the point in time such that \( p \) writes to its local variable \( \ell_p \) for the \( i \)-th time. We prove our claim by induction over \( i \).

**Basis** \((i = 0)\): Since the initial value of \( \ell_p \) is 0 and \( \ell_p \) is written to for the first time only at \( t^1 > t^0 \), the claim holds.

**Induction step** \((i > 0)\): Let the value of \( \ell_p \) be \( j \) after the \((i - 1)\)-th write to it. Then from the induction hypothesis, \( p \) owns all nodes on \( path_p \) up to height \( j \). Consider the iteration of the while-loop during which \( p \) writes to \( \ell_p \) for the \( i \)-th time, and specifically the \( T.getNode(leaf_p, \ell + 1) \) operation in line 2. Since \( \ell_p = j \), at the beginning of this while-loop iteration, it follows from the semantics of the \( getNode() \) operation, that the operation returned the pair \( \langle u, i \rangle \), for some \( i \), where \( h_u = j + 1 \). Now, process \( p \) writes to its local variable \( \ell_p \) only in lines 4 and 5.

**Case a** - \( p \) writes to \( \ell_p \) in line 4. Then \( p \) increased \( \ell_p \) from \( j \) to \( j + 1 \) in line 4. Then, to prove our claim we need to show that \( p \) owns the node with height \( j + 1 \) on \( path_p \). Since \( p \) satisfies the if-condition of line 4 it follows from the code structure that \( p \)'s \( u.L.lock() \) method in line 8 returned the special value \( \infty \), where \( v_p = u \). Since \( h_u = j + 1 \), and \( p \) successfully captured lock \( u.L \), it follows that \( p \) owns the \( j + 1 \)-th node on \( path_p \).

**Case b** - \( p \) writes to \( \ell_p \) in line 5. Let \( val_p = x \) when \( p \) writes to \( \ell_p \) in line 5. From Part (b), it follows that \( \ell_p \) is increased every time it is written to, and therefore \( val_p = x > \ell_p \) when \( p \) writes to \( \ell_p \) in line 5. Thus, to prove our claim we need to show that \( p \) owns all nodes on \( path_p \) with heights in the range \( \{j, \ldots, x\} \). Since \( p \) satisfies the if-condition of line 5 and the \( ALockArray_\Delta \) method \( lock() \) only returns a value in \( \{\bot, \infty\} \cup \mathbb{N} \), it follows that \( p \)'s call to \( u.L.lock() \) returned a non-\( \{\bot, \infty\} \) value. Thus, \( p \) has captured \( u.L \) and now owns node \( u \). It also follows that \( p \) has been handed over all nodes on \( path_p \) with heights in the range \( \{h_u + 1, \ldots, x\} \). Since \( h_u = j \), our claim follows. \( \square \)

A process is said to attempt to capture node \( u \) if it executes a \( u.L.lock() \) method in line 8.

**Claim E.3.** (a) If two distinct processes \( p \) and \( q \) attempt to capture node \( v \), then their local variables \( i \) have different values.

(b) A node has at most one owner at any point in time.
Proof. We prove our claims for all nodes of height at most $h$, by induction over integer $h$.

**Basis** ($h = 1$) Consider an arbitrary node $u$ of height 1, such that two distinct processes $p$ and $q$ attempt to capture node $u$. Then processes $p$ and $q$ executed a `getNode(⟨leaf$^p$, 1⟩)` and `getNode(⟨leaf$^q$, 1⟩)` in line 2 and received pairs $⟨u, i⟩$ and $⟨u, j⟩$, and set their local variables $i_p$ and $i_q$ to $i$ and $j$ respectively. Since $p$ and $q$ are distinct, $leaf^p$ and $leaf^q$ are distinct leaf nodes of tree $T$, and thus from the semantics of the `getNode()` method it follows that $i ≠ j$, and thus Part (a) follows.

Consider an arbitrary node $u$ of height 1. From Part (a), it follows that no two processes execute a concurrent call to `u.L.locki()` for the same $i$, and thus it follows from the mutual exclusion property of object $ALockArray$, that at most one process captures $u.L$. By definition, a process can become an owner of node $u$ only if it captures $u.L$ or if it is handed over node $u$ from some other process $q$. If a node $u$ is handed over from some other process $q$, then $q$ also ceases to be the owner of node $u$ at that point, and thus the number of owners of $u$ does not increase upon a hand over. Thus it follows that node $u$ has at most one owner at any point in time, and thus Part (b) follows.

**Induction Step** ($h > 1$) Consider an arbitrary node $u$ of height $h$, such that two distinct processes $p$ and $q$ attempt to capture node $u$. Then processes $p$ and $q$ executed a `getNode(⟨leaf$^p$, $h$⟩)` and `getNode(⟨leaf$^q$, $h$⟩)` in line 2 and received pairs $⟨u, i⟩$ and $⟨u, j⟩$, and set their local variables $i_p$ and $i_q$ to $i$ and $j$ respectively. For the purpose of a contradiction, assume $i = j$. From the induction hypothesis of Part (b) for $h - 1$, $w$ has at most one owner at any point in time. Since $\ell_p = \ell_q = h - 1$ when $p$ and $q$ attempt to capture node $u$, it follows from Claim [E.2(c)], that $p$ and $q$ own all nodes up to height $h - 1$ on their individual paths $path_p$ and $path_q$. Then $p$ and $q$ are both the owners of $w$ – a contradiction. Thus, Part (a) follows.

Since Part (a) holds for $h$, Part (b) holds for $h$, as argued in the Basis case. □

**Lemma E.4.** The mutual exclusion property is satisfied during history $H$.

**Proof.** Assume two processes $p$ and $q$ are in their Critical Section at the same time, i.e., both processes returned a non-$L$ value from their last `lock()` call. Then both processes executed line 11 and thus $\ell_p = \ell_q = T.height$ holds. Then from Claim [E.2(c)] it follows that both $p$ and $q$ own node $T.root$. But from Claim [E.3(b)], at most one process may own $T.root$ at any point in time – a contradiction. □

**Claim E.5.** Process $p$ repeats the while-loop in `lock()` at most $\Delta$ times.

**Proof.** Consider an arbitrary process $p$ that calls `lock()`. From the code structure of `lock()`, it follows that if $p$ repeats an iteration of the while-loop then $p$ either executed line 1 or line 5 in its previous iteration. Then it follows from Claim [E.2(b)] that $p$ increases $\ell_p$ every time it repeats an iteration of the while-loop. Since the height of the $T$ is $\Delta$, our claim follows. □

**Lemma E.6.** No process starves in history $H$.

**Proof.** Since no two processes execute a concurrent call to `u.L.locki()` for the same $i$ (from Claim [E.3(a)]), it follows from the starvation-freedom property of object $ALockArray$, that a process does not starve during a call to `u.L.lock()` for some node $u$ on its path.

Consider an arbitrary process $p$ that calls `lock()`. Since $p$ repeats the while-loop in `lock()` at most $\Delta$ times before returning from line 11 (follows from Claim [E.5]), it follows that $p$ starves only if $p$ starves during a call to `u.L.lock()` in line 3 for some node $u$. As already argued, this cannot happen, and thus our claim follows. □
Lemma E.7. Process $p$ incurs $O(\Delta)$ RMRs during $\text{release}_p()$.

Proof. Consider $p$’s call to $\text{release()}$. Since $\ell_p \leq T$.height $= \Delta$, it follows from an inspection of the code that during $\text{release()}$, $p$ executes at most $\Delta$ calls to $\text{L.release()}$ (in line 14), and at most one of the $\text{L.release()}$ calls returns true. As per the properties of object $\text{ALockArray}_\Delta$ (Lemma 3.1), a process incurs $O(\Delta)$ RMRs during a call to $\text{L.release()}$, if the call returns true, otherwise $O(1)$ RMRs. Then our claim follows immediately.

Lemma E.8. Process $p$ incurs $O(\Delta)$ RMRs in expectation during $\text{lock}_p()$.

Proof. A process may or may not receive a signal to abort during $\text{lock}_p()$. 

**Case a** - $p$ does not receive a signal to abort during $\text{lock}_p()$. As per the properties of object $\text{ALockArray}_\Delta$ (Lemma 3.1), if a process does not receive a signal to abort during a call to $\text{L.lock()}$, then the process incurs $O(1)$ RMRs in expectation during the call. Since $p$ repeats the while-loop in $\text{lock()}$ at most $\Delta$ times (by Claim E.5), and $p$ does not receive a signal to abort during $\text{lock}_p()$, it follows that $p$ incurs $O(\Delta)$ RMRs in expectation during $\text{lock}_p()$.

**Case b** - $p$ receives a signal to abort during $\text{lock}_p()$. As per the properties of object $\text{ALockArray}_\Delta$ (Theorem 3.1), if a process aborts during a call to $\text{L.lock()}$, then the process incurs $O(\Delta)$ RMRs in expectation during the call. Since $p$ repeats the while-loop in $\text{lock()}$ at most $\Delta$ times (by Claim E.5), and $p$ executes at most one call to $\text{u.L.lock()}$ after having received an abort signal, it follows that $p$ incurs $O(\Delta)$ RMRs in expectation during $\text{lock}_p()$.

Lemma E.9. Method $\text{release()}$ is wait-free.

Proof. As per the bounded exit property of object $\text{ALockArray}_\Delta$, method $\text{release()}$ of the object is wait-free. Then our claim follows immediately from an inspection of the code of $\text{release()}$.

Lemma E.10. The abort-way is wait-free and has $O(\Delta)$ RMR complexity.

Proof. The abort-way of a process $p$ consists of the steps executed by the process after receiving a signal to abort and before completing its passage. From Lemma E.9 and E.7 method $\text{release}_p()$ is wait-free, and has $O(\Delta)$ RMR complexity. From Claim E.5 a process repeats the while-loop in $\text{lock}_p()$ at most $\Delta$ times. Then from an inspection of the code it follows that a process executes all steps during its passage in a wait-free manner, except the call to $\text{u.L.lock()}$ in line 8 and that a process incurs at most $O(\Delta)$ RMRs during all these steps.

To complete our proof we now show that if a process has received a signal to abort and it executes a call to $\text{u.L.lock()}$ in line 8 for some node $u$, then the process executes $\text{u.L.lock()}$ in a wait-free manner and incurs $O(\Delta)$ RMR during the call, and does not call $\text{v.L.lock()}$ for any other node $v$.

Suppose that $p$ has received a signal to abort, and $p$ executes a call to $\text{u.L.lock()}$ call in line 8. Since $p$ has received a signal to abort, it follows that $p$ executes the abort-way of the node lock $u$. As per the properties of object $\text{ALockArray}_\Delta$ (Lemma 3.1), its abort-way is wait-free and has $O(\Delta)$ RMR complexity. Then $p$ executes the $\text{u.L.lock()}$ call in line 8 in a wait-free manner and incurs $O(\Delta)$ RMR complexity. It then goes on to satisfy the if-condition of line 5 and executes a call to $\text{release()}$ in line 7 and returns $\bot$ in line 8 thereby completing its abort-way. Thus, our claim holds.

Theorem 1.1 follows from Lemmas E.4, E.6, E.7, E.8, E.9 and E.10.


F Remaining Proofs of Properties of ALockArray

Claim F.1. Suppose a process $p$ executes a call to $\mathtt{lock}_p()$ during a passage. The value of $\mathtt{Role}[p]$ at various times is as follows.

| Points in time | Value of $\mathtt{Role}[p]$ |
|----------------|-----------------------------|
| $[t_0, t_1)$   | $\{\infty, \mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}\}$ |
| $[t_1, t_2)$   | PAWN                        |
| $[t_2, t_3)$   | PAWN$_p$                    |
| $[t_3, t_4)$   | $\{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}_p\}$ |
| $[t_4, t_5)$   | QUEEN                       |
| $[t_5, t_6)$   | $\{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}_p\}$ |

Proof. Since the values returned by a $\mathtt{Ctr.inc}()$ operation are in $\{\infty, 0, 1, 2\} = \{\infty, \mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}\}$, $\mathtt{Role}[p]$ is set to one of these values in line $5$. Hence, $\mathtt{Role}[p] \in \{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}\}$ at $t_5$. If $p$ satisfies the if-condition of line $6$ then $\mathtt{Role}[p] = \mathtt{PAWN}$, and $p$ changes $\mathtt{Role}[p]$ next only in line $9$. Hence, $\mathtt{Role}[p] = \mathtt{PAWN}$ during $[t_7, t_9]$. In line $9$ $p$ changes $\mathtt{Role}[p]$ to $\mathtt{PAWN}_p$ and does not change $\mathtt{Role}[p]$ thereafter. Hence, $\mathtt{Role}[p] = \mathtt{PAWN}_p$ at $t_9$.

Process $p$ does not change $\mathtt{Role}[p]$ after line $9$. To break out of the getLock loop, $\mathtt{Role}[p] \in \{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}_p\}$ must be satisfied when $p$ executes line $12$. Hence, $\mathtt{Role}[p] = \{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}_p\}$ during $[t_{10}, t_{12}]$. Since $p$ executes line $13$ only after breaking out of the getLock loop, $\mathtt{Role}[p] \in \{\mathtt{KING}, \mathtt{QUEEN}, \mathtt{PAWN}_p\}$ at $t_{13}$. If $p$ satisfies the if-condition of line $13$ then $\mathtt{Role}[p] = \mathtt{QUEEN}$, and since $p$ does not change $\mathtt{Role}[p]$ thereafter, $\mathtt{Role}[p] = \mathtt{QUEEN}$ at $t_{14}$. □

Claim F.2. Suppose a process $p$ executes a call to $\mathtt{abort}_p()$. The value of $\mathtt{Role}[p]$ at various points in time is as follows.

| Points in time | Value of $\mathtt{Role}[p]$ |
|----------------|-----------------------------|
| $[t_{16}, t_{20}]$ | $\{\mathtt{QUEEN}, \mathtt{PAWN}\}$ |
| $[t_{21}, t_{22}]$ | PAWN                        |
| $[t_{22}, t_{23}]$ | PAWN$_p$                    |
| $[t_{23}, t_{24}]$ | QUEEN                       |

Proof. Process $p$ calls $\mathtt{abort}_p()$ only if $p$ has received a signal to abort and $p$ is busy waiting in one of lines $2$, $7$, or $14$. Then, the last line executed by $p$ before calling $\mathtt{abort}_p()$ is line $2$, $7$, or line $14$. From Claim F.1 it follows that $\mathtt{Role}[p] = \mathtt{PAWN}$ at $t_7$ and $\mathtt{Role}[p] = \mathtt{QUEEN}$ at $t_{14}$.

Now, $p$’s local variable $\mathtt{flag}$ is set to value $\mathtt{true}$ for the first time in line $9$. If $p$ fails the if-condition of line $13$, then $p$ must have executed line $8$ and thus $p$ broke out of the busy-wait loop of line $2$. Then, $p$ last executed line $7$ or line $14$ before calling $\mathtt{abort}_p()$. Hence, $\mathtt{Role}[p] \in \{\mathtt{PAWN}, \mathtt{QUEEN}\}$ in $[t_{19}, t_{24}]$, since $p$ changes $\mathtt{Role}[p]$ next only in line $22$.

If $p$ satisfies the if-condition of line $20$ then $\mathtt{Role}[p] = \mathtt{PAWN}$, and $p$ changes $\mathtt{Role}[p]$ next only in line $22$. Hence, $\mathtt{Role}[p] = \mathtt{PAWN}$ at $t_{21}$. In line $22$ $p$ changes $\mathtt{Role}[p]$ to $\mathtt{PAWN}_p$ and $p$ does not change $\mathtt{Role}[p]$ after that. Hence, $\mathtt{Role}[p] = \mathtt{PAWN}_p$ during $[t_{22}, t_{23}]$. If $p$ does not satisfy the if-condition of line $20$ then $\mathtt{Role}[p] = \mathtt{QUEEN}$ at $[t_{26}, t_{30}]$. □

Claim F.3. Suppose a process $p$ executes a call to $\mathtt{release}_p(j)$ during a passage. The value of $\mathtt{Role}[p]$ at various points in time is as follows.

64
Proof. Suppose the point in time \( p^{34} \). Then, \( p \) is executing a call to \( \text{release}_p(j) \), and \( p \) last executed a call to \( \text{lock}_p() \) that returned a non-\( \bot \) value. Then, \( p \)'s call to \( \text{lock}_p() \) either returned from line 17 in \( \text{lock}_p() \) or from line 23 or line 27 in \( \text{abort}_p() \). From Claim F.1, \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN}, \text{PAWN}_P \} \) at time \( p^{17} \), and from Claim F.2 \( \text{Role}[p] = \text{PAWN}_P \) at \( p^{23} \) and \( \text{Role}[p] = \text{QUEEN} \) at \( p^{27} \). Therefore, \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN}, \text{PAWN}_P \} \) at time \( p^{34} \).

From Claim F.3, \( \text{Role}[p] \) is unchanged during \( \text{release}_p() \). Therefore, \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN}, \text{PAWN}_P \} \) during \( [p^{34}, p^{35}] \) and \( [p^{38}, p^{39}] \). Then, from the if-conditions of lines 35, 42 and 43, it follows immediately that \( \text{Role}[p] = \text{KING} \) during \( [p^{36}, p^{39}] \), and \( \text{Role}[p] = \text{QUEEN} \) at \( p^{37} \), and \( \text{Role}[p] = \text{PAWN}_P \) at \( p^{40} \).

\[ \square \]

**Claim F.4.** Suppose a process \( p \) executes a call to \( \text{doCollect}_p() \), \( \text{helpRelease}_p() \) or \( \text{doPromote}_p() \) during a passage. The value of \( \text{Role}[p] \) at various points in time is as follows.

| Points in time | Value of \( \text{Role}[p] \) |
|----------------|-------------------------------|
| \( p^{34} \) - \( p^{35} \) | \{ \text{KING}, \text{QUEEN} \} |
| \( p^{38} \) - \( p^{39} \) | \{ \text{KING}, \text{QUEEN} \} |
| \( p^{40} \) - \( p^{43} \) | \{ \text{KING}, \text{QUEEN}, \text{PAWN}_P \} |

Proof. From the code structure, \( p \) does not change \( \text{Role}[p] \) during \( \text{doPromote}_p() \), \( \text{doCollect}_p() \) and \( \text{helpRelease}_p() \).

From a code inspection, \( \text{doCollect}_p() \) is called by \( p \) only in lines 29 and 38. From Claim F.2 \( \text{Role}[p] = \text{QUEEN} \) at \( p^{29} \) and from Claim F.3 \( \text{Role}[p] = \text{KING} \) at \( p^{38} \). Since \( \text{Role}[p] \) is unchanged during \( \text{doCollect}_p() \), it follows that \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN} \} \) during \( [p^{29}, p^{38}] \).

Now, suppose \( p \) executes a call \( \text{helpRelease}_p() \). From a code inspection, \( \text{helpRelease}_p() \) is called by \( p \) only in lines 30, 39 and 43. From Claim F.2 \( \text{Role}[p] = \text{QUEEN} \) at \( p^{30} \) and from Claim F.3 \( \text{Role}[p] = \text{KING} \) at \( p^{39} \) and \( \text{Role}[p] = \text{QUEEN} \) at \( p^{43} \). Since \( \text{Role}[p] \) is unchanged during \( \text{helpRelease}_p() \), it follows that \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN} \} \) during \( [p^{30}, p^{43}] \).

Now, suppose \( p \) executes a call \( \text{doPromote}_p() \). From a code inspection, \( \text{doPromote}_p() \) is called by \( p \) only in lines 46 and 62. From Claim F.3 \( \text{Role}[p] = \text{PAWN}_P \) at \( p^{46} \) and from earlier in this claim, \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN} \} \) at \( p^{62} \). Since \( \text{Role}[p] \) is unchanged during \( \text{doPromote}_p() \), it follows that \( \text{Role}[p] \in \{ \text{KING}, \text{QUEEN}, \text{PAWN}_P \} \) during \( [p^{46}, p^{71}] \).