GLOBAL RESIDUE FORMULA FOR LOGARITHMIC INDICES OF ONE-DIMENSIONAL FOLIATIONS

MAURÍCIO CORRÉA AND DIÓGO DA SILVA MACHADO

Abstract. We prove a global residual formula in terms of logarithmic indices for one-dimensional holomorphic foliations, with isolated singularities, and logarithmic along normal crossings divisors. We also give a formula for the total sum of the logarithmic indices if the singular set of the foliation is contained in the invariant divisor. As an application, we provide a formula for the number of singularities in the complement of the invariant divisor on complex projective spaces. Finally, we obtain a Poincaré-Hopf type formula for singular normal projective varieties.

1. Introduction

A complex $\partial$-manifold [24] is a complex manifold of the form $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold and $D \subset X$ is a divisor which is called the boundary divisor. S. Iitaka in [20] proposed a version of Gauss-Bonnet theorem for $\partial$-manifold [24]. Y. Norimatsu [24], R. Silvotti [27] and P. Aluffi [2] have proved independently such Gauss-Bonnet type theorem for the case when the boundary divisor $D$ has normal crossings singularities. More precisely, they proved the following formula

$$\int_X c_n(T_X(-\log D)) = \chi(\tilde{X}),$$

where $T_X(-\log D)$ denotes the vector bundle of logarithmic vector fields along $D$ and $\chi(\tilde{X})$ is the Euler characteristic of $\tilde{X}$. In [11] the authors have proved the following Poincaré-Hopf type index theorem

$$\int_X c_n(T_X(-\log D)) = \chi(\tilde{X}) = \sum_{x \in \operatorname{Sing}(v) \cap \tilde{X}} \mu_x(v),$$

where $v$ is a global holomorphic vector field on $X$ tangent to the divisor $D$, $\mu_x(v)$ is the Milnor number of $v$ at $x$ and under the hypothesis that Aleksandrov’s logarithmic indices of $v$ along $D$ vanish. Let us recall Aleksandrov’s definition of logarithmic index [1] in the context of one-dimensional holomorphic foliations:

Let $\mathcal{F}$ be a one-dimensional holomorphic foliation, with isolated singularities, on a complex manifold $X$ and logarithmic along a divisor $D$, we refer to Section 2
for this notion. Let $\text{Sing}(\mathcal{F})$ be the singular set of $\mathcal{F}$. Let $v \in T_X(-\log D)|_U$ be a germ of vector field on $(U, x)$ tangent to $\mathcal{F}$. The interior multiplication $i_v$ induces the complex of logarithmic differential forms

$$0 \longrightarrow \Omega^n_{X,x}(\log D) \xrightarrow{i_v} \Omega^{n-1}_{X,x}(\log D) \xrightarrow{i_v} \cdots \xrightarrow{i_v} \Omega^1_{X,x}(\log D) \xrightarrow{i_v} \mathcal{O}_{n,x} \longrightarrow 0.$$ 

Since all singularities of $v$ are isolated, the $i_v$-homology groups of the complex $\Omega^\bullet_{X,x}(\log D)$ are finite-dimensional vector spaces. Thus, the Euler characteristic

$$\chi(\Omega^\bullet_{X,x}(\log D), i_v) = \sum_{i=0}^n (-1)^i \dim H^i(\Omega^\bullet_{X,x}(\log D), i_v)$$

of the complex of logarithmic differential forms is well defined. Since this number does not depend on local representative $v$ of the foliation $\mathcal{F}$ at $x$, we define the logarithmic index of $\mathcal{F}$ at the point $x$ by

$$\text{Log}(\mathcal{F}, D, x) := \chi(\Omega^\bullet_{X,x}(\log D), i_v).$$

This index is inspired by Gómez-Mont’s homological index [17].

We prove the following Baum-Bott type residual formula [4] in terms of the logarithmic indices for one-dimensional holomorphic foliations, with isolated singularities, and logarithmic along normal crossings divisors:

**Theorem 1.** Let $\mathcal{F}$ be a one-dimensional foliation with isolated singularities and logarithmic along a normal crossings divisor $D$ on a complex compact manifold $X$. Then

$$\int_X c_n(T_X(-\log D) - T_{\mathcal{F}}) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X \setminus D)} \mu_x(\mathcal{F}) + \sum_{x \in \text{Sing}(\mathcal{F}) \cap D} \text{Log}(\mathcal{F}, D, x),$$

where $T_{\mathcal{F}}$ denotes the tangent bundle of $\mathcal{F}$ and $\mu_x(\mathcal{F})$ is the Milnor number of $\mathcal{F}$ at $x$.

We observe that Theorem 1 generalizes our previous result in [11], where we have assumed that the singularities of $\mathcal{F}$ are non-degenerate. In [8, Theorem 1, (ii)] we have proved an analogous result for global holomorphic vector fields on $X$ such that $D$ has isolated singularities.

We obtain the following consequence.

**Corollary 1.1.** Let $\mathcal{F}$ be a one-dimensional foliation on $X$, with isolated singularities and logarithmic along a normal crossings divisor $D$.

(i) If $\text{Log}(\mathcal{F}, D, x) = 0$, for all $x \in \text{Sing}(\mathcal{F}) \cap D$, then

$$\int_X c_n(T_X(-\log D) - T_{\mathcal{F}}) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X \setminus D)} \mu_x(\mathcal{F}).$$
If \( \text{Sing}(F) \subset D \), then
\[
\int_X c_n(T_X(- \log D) - T_F) = \sum_{x \in \text{Sing}(F) \cap D} \text{Log}(F, D, x).
\]

Now, let \( F \) be a one-dimensional foliation on the projective space \( \mathbb{P}^n \). As a consequence of the above formulas we can provide a formula for the number of singularities of the foliation \( F \) in the complement of the invariant divisor \( D \). Moreover, under the assumption \( \text{Sing}(F) \subset D \) we also have a formula in terms of the total sum of the logarithmic index along \( D \).

**Corollary 1.2.** Let \( F \) be a one-dimensional foliation on \( \mathbb{P}^n \), of degree \( d > 1 \), with isolated singularities and logarithmic along a normal crossings divisor \( D = \bigcup_{i=1}^k D_i \). Denote \( d_i := \deg(D_i) \).

(i) If \( \text{Log}(F, D, x) = 0 \), for all \( x \in \text{Sing}(F) \cap D \), then
\[
\sum_{x \in \text{Sing}(F) \cap (X \setminus D)} \mu_x(F) = \sum_{i=0}^n \binom{n+1}{i} \sigma_{n-i}(d_1, \ldots, d_k, d-1),
\]
where \( \sigma_{n-i} \) is the complete symmetric function of degree \( (n-i) \) in the variables \((d_1, \ldots, d_k, d-1)\).

If, in addition, \( d = 2 \), \( n \geq k \) and \( d_i = 1 \) for all \( i \), then \( \text{Sing}(F) \subset D \). Otherwise, we have that \( \text{Sing}(F) \cap (X \setminus D) \neq \emptyset \).

(ii) If \( \text{Sing}(F) \subset D \), then
\[
\sum_{x \in \text{Sing}(F) \cap D} \text{Log}(F, D, x) = \sum_{i=0}^n \binom{n+1}{i} \sigma_{n-i}(d_1, \ldots, d_k, d-1).
\]

If, in addition, \( d \neq 2 \) then \( F \) has at least one degenerate singular point.

Cukierman, Soares, and Vainsencher in [15] have provided a formula prescribing the number of isolated singularities of a codimension one logarithmic foliation on \( \mathbb{P}^n \). We notice that the Corollary 1.2 can be seen as a version for one-dimensional foliations of this result which provides the same number of singularities for \( n = 2 \) and if the logarithmic indices along the divisor vanish. Also, the part (i) of Corollary 1.2 generalizes and improves [11, Theorem 3].

Another interesting application of Theorem 1 is a Poincaré-Hopf type formula for singular varieties. As usual we denote by \( T_Y := \mathcal{H}om(\Omega^1_Y, \mathcal{O}_Y) \) the tangent sheaf of \( Y \), where \( \Omega^1_Y \) is the sheaf of Kähler differentials of \( Y \). We obtain the following result.

**Corollary 1.3.** Let \( Y \) be a normal variety and \( v \in H^0(Y, T_Y) \) a holomorphic vector field with isolated singularities such that \( \text{Sing}(v) \cap \text{Sing}(Y) = \emptyset \). If \( \pi : (X, D) \to (Y, \emptyset) \) is a functorial log resolution of \( Y \), then
\[
\int_X c_n(T_X(\log D)) = \sum_{x \in \text{Sing}(v)} \mu_x(v),
\]
where $\mu_x(v)$ is the Milnor number of $v$ at $x$.

Acknowledgments. The first named author is partially supported by the Università degli Studi di Bari and by the PRIN 2022MWPMAB- "Interactions between Geometric Structures and Function Theories" and he is a member of INdAM-GNSAGA; he was partially supported by CNPq grant numbers 202374/2018-1, 400821/2016-8 and Fapemig grant numbers APQ-02674-21, APQ-00798-18, APQ-00056-20; he is grateful to the University of Oxford for its hospitality during his visit, when the work on this project had begun. The authors also thank anonymous referees for giving many suggestions that helped to improve the presentation of the article.

2. Preliminaries

2.1. Logarithmic forms and vector fields. Let $X$ be a complex manifold of dimension $n$ and let $D$ be a reduced hypersurface on $X$. Given a meromorphic $q$-form $\omega$ on $X$, we say that $\omega$ is a logarithmic $q$-form along $D$ at $x \in X$ if the following conditions occurs:

(i) $\omega$ is holomorphic on $X - D$;

(ii) If $f = 0$ is a reduced equation of $D$, locally at $x$, then $f \omega$ and $f d\omega$ are holomorphic.

Denoting by $\Omega^q_X(x, \log D)$ the set of germs of logarithmic $q$-form along $D$ at $x$, we define the following coherent sheaf of $O_X$-modules

$$\Omega_X^q(\log D) := \bigcup_{x \in X} \Omega^q_{X,x}(\log D),$$

which is called by sheaf of logarithmic $q$-forms along $D$. See [16], [21] and [25] for details.

Now, given $x \in X$, let $v \in T_{X,x}$ be germ at $x$ of a holomorphic vector field on $X$. We say that $v$ is a logarithmic vector field along of $D$ at $x$ if the following condition: if $f = 0$ is a equation of $D$, locally at $x$, then the derivation $v(f)$ belongs to the ideal $(f_x)O_{X,x}$. Denoting by $T_{X,x}(- \log D)$ the set of germs of logarithmic vector field along of $D$ at $x$, we define the following coherent sheaf of $O_X$-modules

$$T_X(- \log D) := \bigcup_{x \in X} T_{X,x}(- \log D),$$

which is called sheaf of logarithmic vector fields along $D$. It follows from [25] that if $D$ is an analytic hypersurface with normal crossings singularities, then the sheaves $\Omega_X^1(\log D)$ and $T_X(- \log D)$ are locally free, furthermore, the Poincaré residue map

$$\text{Res} : \Omega_X^1(\log D) \to O_D \cong \bigoplus_{i=1}^k O_{D,i}$$
give the following exact sequence of sheaves on $X$

$$0 \rightarrow \Omega^1_X \rightarrow \Omega^1_X(\log D) \xrightarrow{\text{Res}} \bigoplus_{i=1}^k \mathcal{O}_{D_i} \rightarrow 0,$$

where $\Omega_X^1$ is the sheaf of holomorphic 1-forms on $X$ and $D_1, \ldots, D_k$ are the irreducible components of $D$.

2.2. Singular one-dimensional holomorphic foliations. A one-dimensional holomorphic foliation $\mathcal{F}$ on $X$ is given by the following data:

(i) an open covering $\mathcal{U} = \{U_\alpha\}$ of $X$;
(ii) for each $U_\alpha$, a holomorphic vector field $v_\alpha \in T_X|_{U_\alpha}$;
(iii) for every non-empty intersection, $U_\alpha \cap U_\beta \neq \emptyset$, a holomorphic function $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$;

such that $v_\alpha = g_{\alpha\beta} v_\beta$ in $U_\alpha \cap U_\beta$ and $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$.

We denote by $K_\mathcal{F}$ the line bundle defined by the cocycle $\{g_{\alpha\beta}\} \in H^1(X, \mathcal{O}^*)$. Thus, a one-dimensional holomorphic foliation $\mathcal{F}$ on $X$ induces a global holomorphic section $\vartheta_\mathcal{F} \in H^0(X, \mathcal{T}_X \otimes K_\mathcal{F})$. The line bundle $T_\mathcal{F} := (K_\mathcal{F})^*$ is called the tangent bundle of $\mathcal{F}$. The singular set of $\mathcal{F}$ is $\text{Sing}(\mathcal{F}) = \{\vartheta_\mathcal{F} = 0\}$.

Throughout this paper we will assume that $\text{cod}(\text{Sing}(\mathcal{F})) \geq 2$.

Given $\mathcal{F}$ a one-dimensional holomorphic foliation on $X$, we say that $\mathcal{F}$ is logarithmic along $D$ if it satisfies the following condition: for all $x \in D - \text{Sing}(D)$, the vector $v_\alpha(x)$ belongs to $T_xD$, with $x \in U_\alpha$.

2.2.1. Projective holomorphic foliations. A foliation on a complex projective space $\mathbb{P}^n$ is called a projective foliation. Let $\mathcal{F}$ be a projective foliation of dimension one with tangent bundle $T_\mathcal{F} = \mathcal{O}_\mathbb{P}^n(r)$. The integer $d := r + 1$ is called the degree of $\mathcal{F}$. A projective foliation of dimension one of degree $d$ can be induced by a polynomial vector field on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d$, see for instance [12, 13, 7].

2.3. Logarithmic and homological indices. Let $D \subset X$ be a reduced hypersurface with a local equation $f \in \mathcal{O}_{X,x}$ in a neighborhood of a point $x \in D$. Consider the $\mathcal{O}_{D,x}$-module of germs of regular differentials of order $i$ on $D$:

$$\Omega^i_{D,x} = \frac{\Omega^i_{X,x}}{df \wedge \Omega^{i-1}_{X,x}}.$$

Now, let $\mathcal{F}$ be a one-dimensional holomorphic foliation on $X$, with isolated singularities, logarithmic along $D$. Let $x \in \text{Sing}(\mathcal{F})$ be and consider a germ of vector field $v \in T_X(-\log D)|_U$ on $(U, x)$ tangent to $\mathcal{F}$, where $U$ is a neighborhood of $x$.

Since $v$ is also tangent to $(D, x)$ the interior multiplication $i_v$ induces the complex

$$0 \rightarrow \Omega^{n-1}_{D,x} \xrightarrow{i_v} \Omega^{n-2}_{D,x} \xrightarrow{i_v} \cdots \xrightarrow{i_v} \Omega^1_{D,x} \xrightarrow{i_v} \mathcal{O}_{D,x} \rightarrow 0.$$
The homological index is defined as the Euler characteristic of the complex \((\Omega^\bullet_{D,x}, i_v)\)

\[
\text{Ind}_{\text{hom}}(\mathcal{F}, D, x) = \sum_{i=0}^{n-1} (-1)^i \dim H_i(\Omega^\bullet_{D,x}, i_v).
\]

Since the vector field \(v\) has an isolated singularity at \(x\), then the \(i_v\)-homology groups of the complex \(\Omega^\bullet_{D,x}\) are finite-dimensional vector spaces and the Euler characteristic is well defined. Similarly, the Euler characteristic of the Koszul complex \((\Omega^\bullet_{X,x}, i_v)\) associated to \(v\) is well defined and the Milnor number of \(F\) at \(x\) is defined by

\[
\mu(F,x) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega^\bullet_{X,x}, i_v).
\]

We recall that Milnor number of \(F\) at \(x\) coincides with the Poincaré Hopf index of \(v\) at \(x\). For more details on residues of holomorphic foliations see [4, 6, 9, 14, 28].

The homological index was introduced by Gómez-Mont in [17] and it coincides with the GSV-index introduced by Gómez-Mont, Seade and Verjovsky in [19]. The concept of GSV-index has been extended to more general contexts, we refer to the works [26, 23, 5, 6, 28, 10].

Aleksandrov has proved in [11, Proposition 1] that the following formula holds

\[
\text{Log}(\mathcal{F}, D, x) = \mu_x(\mathcal{F}) - \text{Ind}_{\text{hom}}(\mathcal{F}, D, x),
\]

where \(\mu_x(\mathcal{F})\) is the Milnor number of \(\mathcal{F}\) at \(x\). Denote by \(D_{\text{reg}} := D \setminus \text{Sing}(D)\) the regular part of \(D\). If \(x \in \text{Sing}(\mathcal{F}) \cap D_{\text{reg}}\) then

\[
\text{Log}(\mathcal{F}, D, x) = \mu_x(\mathcal{F}) - \mu_x(\mathcal{F}|_{D_{\text{reg}}}),
\]

since in this case the GSV-index of \(\mathcal{F}\) at \(x\) coincides with the Milnor number of the restriction \(\mathcal{F}|_{D_{\text{reg}}}\) at \(x\). Furthermore,

\[
\text{Log}(\mathcal{F}, D, x) = 0,
\]

whenever \(x\) is a non-degenerate singularity of \(\mathcal{F}\).

### 3. Proof of Theorem [11]

If \(x \in \text{Sing}(\mathcal{F}) \cap D_{\text{reg}}\), then it follows from [11] that

\[
\text{Log}(\mathcal{F}, D, x) = \mu_x(\mathcal{F}) - \mu_x(\mathcal{F}|_{D_{\text{reg}}}).
\]

In order to simplify the proof we also will adopt the notation

\[
\text{Log}(\mathcal{F}, D, x) = \mu_x(\mathcal{F}),
\]

if \(x \in X \setminus D\).

Let \(D = D_1 \cup \ldots \cup D_k\) be the decomposition of \(D\) into irreducible components. Fixing an irreducible component, let us say \(D_k\), we define \(\hat{D}_k := D_1 \cup \ldots \cup D_{k-1}\). The intersection \(\hat{D}_k \cap D_k = (D_1 \cap D_k) \cup \ldots \cup (D_{k-1} \cap D_k)\) is an analytic hypersurface with normal crossings singularities on the \((n-1)\)-dimensional smooth submanifold
Lemma 3.1. Let $X, D = D_1 \cup \ldots \cup D_k$ and $\mathcal{F}$ be as described above. Then, for all $x \in \hat{D}_k \cap D_k$, we have

\begin{equation}
\text{Log}(\mathcal{F}, D, x) = \text{Log}(\mathcal{F}, \hat{D}_k, x) - \text{Log}(\mathcal{F}|_{D_k}, \hat{D}_k \cap D_k, x). \tag{5}
\end{equation}

Proof. Given $x \in \hat{D}_k \cap D_k$, let $(z_1, \ldots, z_n)$ be a local coordinate system in a neighborhood of $x$, such that each $D_j$ is locally defined by equation $z_j = 0$, so $D$ is defined by the equation $z_1 \cdots z_k = 0$. Moreover, we have that the set

$$\left\{ \frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n \right\}$$

constitutes a system of $\mathcal{O}_{X,x}$-free basis for $\Omega^1_{X,x}(\log D)$. In general, for all $q = 1, 2, \ldots, n$, we have that

$$\Omega^q_{X,x}(\log D) = \sum_{i_1, \ldots, i_q} \mathcal{O}_{X,x} \omega_{i_1} \wedge \ldots \wedge \omega_{i_q},$$

where $\omega_1 = \frac{dz_1}{z_1}, \ldots, \omega_k = \frac{dz_k}{z_k}, \omega_{k+1} = dz_{k+1}, \ldots, \omega_n = dz_n$, see [25, pg. 270]. Thus, we have the map of complexes

$$\Omega^i_{X,x}(\log D) \xrightarrow{i^*} \Omega^i_{X,x}(\log \hat{D}_k)$$

which defines the following exact sequence

$$0 \longrightarrow (\Omega^i_{X,x}(\log D)) \xrightarrow{i^*} (\Omega^i_{X,x}(\log \hat{D}_k)) \longrightarrow (\Omega^i_{D_1,x}(\log (\hat{D}_k \cap D_k))) \longrightarrow 0,$$

where $i^*$ denotes the pullback of inclusion map $i : D_k \hookrightarrow X$. Therefore, the result follows from the additivity of the holomorphic Euler characteristic.

\[\square\]

Lemma 3.2. Let $X, D = D_1 \cup \ldots \cup D_k$ and $\mathcal{F}$ be as described above. Then, for all $x \in \hat{D}_k - (\hat{D}_k \cap D_k)$, we have that

\begin{equation}
\text{Ind}_{\text{hom}}(\mathcal{F}, \hat{D}_k, x) = \text{Ind}_{\text{hom}}(\mathcal{F}, D, x). \tag{6}
\end{equation}

Proof. Given $x \in D$, since $D$ is a hypersurface with normal crossings singularities, we can choose a local coordinate system in a neighborhood of $x$ such that each $D_j$ is locally defined by the equation $z_j = 0$. If $x \in \hat{D}_k - (\hat{D}_k \cap D_k)$, then $\hat{D}_k$ and $D$ are both defined by the same equation $z_1 \cdots z_{k-1} = 0$. Thus, for all $q = 0, 1, \ldots, n,$
we have that \( \Omega^2_{D_{k,x}} \cong \Omega^2_{D,x} \), and, consequently, we obtain

\[
Ind_{\text{hom}}(\mathcal{F}, D, x) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega^\bullet_{D_{k,x}}, i_{D_{k,x}}) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega^\bullet_{D,x}, i_{D}) = Ind_{\text{hom}}(\mathcal{F}, D, x).
\]

\[\Box\]

We will use the following multiple index notation: for each multi-index \( J = (j_1, \ldots, j_k) \) and \( J' = (j'_1, \ldots, j'_{k-1}) \), with \( 1 \leq j_l, j'_l \leq n \), we denote

\[
c_1(D)^J = c_1([D_1])^{j_1} \cdots c_1([D_k])^{j_k},
\]

\[
c_1(\hat{D}_k)^{J'} = c_1([D_1])^{j'_1} \cdots c_1([D_{k-1}])^{j'_{k-1}}.
\]

**Lemma 3.3.** In the above conditions, for each \( i = 1, \ldots, n \), we have

\[
c_i(\Omega^1_X(\log D)) = \sum_{l=0}^{i} \sum_{|J|=l} c_{i-l}(\Omega^1_X)c_1(D)^J.
\]

**Proof.** By using the exact sequence \([\square]\), we get

\[
c_i(\Omega^1_X(\log D)) = \sum_{l=0}^{i} c_{i-l}(\Omega^1_X)c_1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{D_i}\right)
= \sum_{l=0}^{i} c_{i-l}(\Omega^1_X) \left( \sum_{j_1+\cdots+j_k=l} c_{j_1}(\mathcal{O}_{D_1}) \cdots c_{j_k}(\mathcal{O}_{D_k}) \right)
= \sum_{l=0}^{i} c_{i-l}(\Omega^1_X) \left( \sum_{j_1+\cdots+j_k=l} c_1([D_1])^{j_1} \cdots c_1([D_k])^{j_k} \right),
\]

where in the last equality we use the relations

\[
c_j(\mathcal{O}_{D_i}) = c_j(\mathcal{O}_X - \mathcal{O}(-D_i)) = c_1([D_i])^{j'}, \quad j = 1, \ldots, n.
\]

\[\Box\]

**Lemma 3.4.** In the above conditions, for each irreducible component \( D_j \), we have that

\[
c_i(\Omega^1_X|_{D_j}) = c_i(\Omega^1_{D_j}) - c_{i-1}(\Omega^1_{D_j})c_1([D_j])|_{D_j}, \quad \forall i = 1, \ldots, n - 1.
\]

**Proof.** This follows by taking the total Chern class in the exact sequence

\[
0 \to T_{D_j} \to T_X|_{D_j} \to [D_j]|_{D_j} \to 0.
\]

\[\Box\]
Remark 3.5. We recall that the total Chern class of the virtual vector bundle $T_X(-\log D) - L$ is defined by

$$c(T_X(-\log D) - L) = \frac{c(T_X(-\log D))}{c(L)}$$

and the $n$-th Chern class of $T_X(-\log D) - L$ is by definition the component of $\frac{c(T_X(-\log D))}{c(L)}$ in dimension $2n$, see [4, section 4]. Moreover, we have that

$$c_n(T_X(-\log D) - L) = c_n(T_X(-\log D) \otimes L^*).$$

Lemma 3.6. In the above conditions, if $L$ is a holomorphic line bundle on $X$, then the following holds:

(7) \[ \int_X c_n(T_X(-\log D) - L) = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|j|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^j c_1(L^*)^l. \]

In particular, we have that

(8) \[ \int_X c_n(T_X(-\log \hat{D}_k) - L) = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|j|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) \hat{c}_1(\hat{D}_k)^j c_1(L^*)^l. \]

and

(9) \[ \int_{\hat{D}_k} c_{n-1}(T_{\hat{D}_k}(-\log (\hat{D}_k \cap D_k)) - L|_{\hat{D}_k}) = \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-j} \sum_{|j|=l} \int_{\hat{D}_k} (-1)^{n-1-j} c_{n-1-j-l}(\Omega^1_{\hat{D}_k}) \hat{c}_1(\hat{D}_k)^j c_1(L^*)^l. \]

Proof. On the one hand, since by remark 3.5 we have that

$$c_n(T_X(-\log D) - L) = c_n(T_X(-\log D) \otimes L^*) = \sum_{j=0}^{n} c_{n-j}(T_X(-\log D)) c_1(L^*)^j$$

and $c_{n-j}(T_X(-\log D)) = (-1)^{n-j} c_{n-j}(\Omega^1_X(\log D))$ we get

$$\int_X c_n(T_X(-\log D) - L) = \int_X \sum_{j=0}^{n} (-1)^{n-j} c_{n-j}(\Omega^1_X(\log D)) c_1(L^*)^j.$$

On the other hand, by Lemma 3.3 we obtain

$$c_{n-j}(\Omega^1_X(\log D)) = \sum_{l=0}^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^l.$$

Substituting this, we obtain (7). We get the relation (8) by taking $D = \hat{D}_k$ in the relation (7). Analogously, applying the relation (8), we can obtain (9) by taking
$X = D_k$ as a complex manifold of dimension $n - 1$ and $D = \hat{D}_k \cap D_k$ as an analytic subvariety of $D_k$ with normal crossings singularities. □

**Proposition 3.7.** In the above conditions, if $L$ is a holomorphic line bundle on $X$, then

$$\int_X c_n(T_X(-\log D) - L) = \int_X c_n(T_X(-\log \hat{D}_k) - L) - \int_{D_k} c_{n-1}(T_{D_k}(-\log \hat{D}_k \cap D_k) - L|_{D_k}).$$

In particular, if $D$ is irreducible we get

$$\int_X c_n(T_X(-\log D) - L) = \int_X c_n(T_X - L) - \int_D c_{n-1}(T_D - L|_D).$$

**Proof.** By Lemma 3.6 it is enough to show that the following equality occurs

$$\sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^j c_1(L^*)_j =$$

$$\sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(\hat{D}_k)^j c_1(L^*)_j -$$

$$-\sum_{j=0}^{n-1} \sum_{l=0}^{n-1-j} \sum_{|J'|=l} \int_{D_k} (-1)^{n-1-j} c_{n-1-j-l}(\Omega^1_{D_k}) c_1(\hat{D}_k)^j c_1(L^*)_j.$$

Indeed, we can decompose the sum on the left hand side into the terms with $l = 0$ and those with $l \geq 1$ as follows:

$$\sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^j c_1(L^*)_j =$$

$$= \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(L^*)_j + \sum_{j=0}^{n-1} \sum_{l=0}^{n-j} \sum_{|J'|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^j c_1(L^*)_j.$$

The second sum on the right hand side can be computed as follows:

$$\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(D)^j c_1(L^*)_j =$$

$$\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|J'|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1(\hat{D}_k)^j c_1(L^*)_j +$$

$$+ \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|J'=l|} \int_X (-1)^{n-j} c_{n-j-l}(\Omega^1_X) c_1([D_1])^{j_1} \cdots c_1([D_k])^{j_k} c_1(L^*)_j.$$
By using that \( c_1([D_k]) \) is the Poincaré dual to the fundamental class of \( D_k \), we obtain:

\[
\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(D)^j c_1(L^*)^j = \\
\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(\tilde{D}_k)^j c_1(L^*)^j + \\
+ \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1([D_1])^{j_1} \ldots c_1([D_k])^{j_k-1} c_1(L^*)^j.
\]

Now, by using the relation of Lemma 3.4, we get

\[
\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1([D_1])^{j_1} \ldots c_1([D_k])^{j_k-1} c_1(L^*)^j = \\
= \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(D)^j c_1(L^*)^j - \\
- \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(D)^j c_1(L^*)^j = \\
= \sum_{j=0}^{n-1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(L^*)^j + \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(\tilde{D}_k)^j c_1(L^*)^j = \\
= - \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(\tilde{D}_k)^j c_1(L^*)^j.
\]

Hence,

\[
\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(D)^j c_1(L^*)^j = \\
\sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(\tilde{D}_k)^j c_1(L^*)^j - \\
- \sum_{j=0}^{n-1} \sum_{l=1}^{n-j} \sum_{|j|=1}^{n-j} \sum_{j_k \geq 1} \int_{D_k} (-1)^{n-j} c_{n-j-l}(\Omega_X) c_1(\tilde{D}_k)^j c_1(L^*)^j.
\]
and we complete the calculation of the second sum. Replacing it in the initial equality (11), we obtain

\[
\sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X^1) c_1(D)^j c_1(L^*)^j = \\
= \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j}(\Omega_X^1) c_1(L^*)^j + \sum_{j=0}^{n-1} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X^1) c_1(\widehat{D}_k)^j c_1(L^*)^j - \\
- \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-j} \sum_{|J|=l} \int_{D_k} (-1)^{n-1-j} c_{n-1-j-l}(\Omega_{D_k}^1) c_1(\widehat{D}_k)^j c_1(L^*)^j = \\
= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \sum_{|J|=l} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X^1) c_1(\widehat{D}_k)^j c_1(L^*)^j - \\
- \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-j} \sum_{|J|=l} \int_{D_k} (-1)^{n-1-j} c_{n-1-j-l}(\Omega_{D_k}^1) c_1(\widehat{D}_k)^j c_1(L^*)^j.
\]

Now, if \( D \) is irreducible, we can repeat the same argument that we have used above in order to obtain the following formula:

\[
\sum_{j=0}^{n} \sum_{l=0}^{n-j} \int_X (-1)^{n-j} c_{n-j-l}(\Omega_X^1) c_1([D])^j c_1(L^*)^j = \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j}(\Omega_X^1) c_1(L^*)^j - \\
- \sum_{j=0}^{n-1} \int_D (-1)^{n-1-j} c_{n-1-j}(\Omega_D^1) c_1(L^*)^j.
\]

Thus, we get

\[
\int_X c_n(T_X(-\log D) - L) = \sum_{j=0}^{n} \int_X (-1)^{n-j} c_{n-j}(\Omega_X^1) c_1(L^*)^j - \sum_{j=0}^{n-1} \int_D (-1)^{n-1-j} c_{n-1-j}(\Omega_D^1) c_1(L^*)^j = \\
= \sum_{j=0}^{n} \int_X c_{n-j}(T_X)c_1(L^*)^j - \sum_{j=0}^{n-1} \int_D c_{n-1-j}(T_D)c_1(L^*)^j = \\
= \int_X c_n(T_X - L) - \int_D c_{n-1}(T_D - L|_D).
\]

\(\Box\)

In order to prove the Theorem, we will use the induction principle on the number of irreducible components of \( D \). Indeed, if the number of irreducible component of \( D \) is 1, then we can invoke the formula (10) of the Proposition 3.7 to obtain the
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\[ \int_X c_n(T_X (\log D) - T_{\mathcal{F}}) = \int_X c_n(T_X - T_{\mathcal{F}}) - \int_D c_{n-1}(T_D - T_{\mathcal{F}}) \]

\[ = \sum_{x \in \text{Sing}(\mathcal{F}) \cap X} \mu_x(\mathcal{F}) - \sum_{x \in \text{Sing}(\mathcal{F}) \cap D} \mu_x(\mathcal{F}|_D), \]

where in the last step we apply the Baum-Bott classical formula, see [4]. By using the disjoint decomposition \( X = (X - D) \cup D \), we get

\[ \int_X c_n(T_X (\log D) - T_{\mathcal{F}}) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X - D)} \mu_x(\mathcal{F}) + \]

\[ + \left( \sum_{x \in \text{Sing}(\mathcal{F}) \cap (D)} \mu_x(\mathcal{F}) - \sum_{x \in \text{Sing}(\mathcal{F}) \cap D} \mu_x(\mathcal{F}|_D) \right) \]

But, by relation [4], we have

\[ \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X - D)} \log(\mathcal{F}, D, x) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X - D)} \mu_x(\mathcal{F}). \]

On the other hand, since \( D \) smooth, it follows from equality [3] that

\[ \sum_{x \in \text{Sing}(\mathcal{F}) \cap (D)} \log(\mathcal{F}, D, x) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (D)} \mu_x(\mathcal{F}) - \sum_{x \in \text{Sing}(\mathcal{F}) \cap D} \mu_x(\mathcal{F}|_D). \]

Therefore, we get

\[ \int_X c_n(T_X (\log D) - T_{\mathcal{F}}) = \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, D, x). \]

Let us suppose that for every analytic hypersurface on \( X \), satisfying the hypothesis of Theorem 1 and having \( k - 1 \) irreducible components, the formula of Theorem 1 holds. Let \( D \) be an analytic hypersurface on \( X \) with \( k \) irreducible components, satisfying the hypotheses of the Theorem 1. We will prove that the formula of Theorem 1 holds for \( D \).

We know that \( \mathcal{D}_k \) is an analytic hypersurface on \( X \) and \( \mathcal{D}_k \cap D_k \) is an analytic hypersurface on \( D_k \), both with normal crossings singularities and having exactly \( k - 1 \) irreducible components. Moreover, \( \mathcal{F} \) and its restriction \( \mathcal{F}|_{D_k} \) on \( D_k \) are logarithmic along \( D_k \) and \( \mathcal{D}_k \cap D_k \), respectively. Thus, we can use the induction hypothesis and we obtain

\[ \int_X c_n(T_X (\log \mathcal{D}_k) - T_{\mathcal{F}}) = \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, \mathcal{D}_k, x) \]

and
\[
\int_{D_k} c_{n-1}(T_{D_k}(- \log \hat{D}_k \cap D_k)) - T_{\mathcal{F}|D_k}) = \sum_{x \in \text{Sing}(\mathcal{F}|D_k)} \log(\mathcal{F}|D_k, \hat{D}_k \cap D_k, x).
\]

By the Proposition 3.7, we get
\[
\int \left( T_X(- \log D) - T_{\mathcal{F}} \right) = \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, \hat{D}_k, x) - \sum_{x \in \text{Sing}(\mathcal{F}|D_k)} \log(\mathcal{F}|D_k, \hat{D}_k \cap D_k, x).
\]

Thus, it is enough to show that the following equality holds
\[
\sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, D, x) + \sum_{x \in \text{Sing}(\mathcal{F}|D_k)} \log(\mathcal{F}|D_k, \hat{D}_k \cap D_k, x).
\]

Indeed, by using the following disjoint decomposition
\[
X = (X - D) \cup \{ \hat{D}_k \cap D - (\hat{D}_k \cap D_k) \} \cup [(D - \hat{D}_k) \cap D_k] \cup (\hat{D}_k \cap D_k),
\]
we get
\[
\sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), x \in (X - D)} \log(\mathcal{F}, \hat{D}_k, x) + \sum_{x \in \text{Sing}(\mathcal{F}), x \in [D - (\hat{D}_k \cap D_k)]} \log(\mathcal{F}, \hat{D}_k, x) + \sum_{x \in \text{Sing}(\mathcal{F}), x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, \hat{D}_k, x) + \sum_{x \in \text{Sing}(\mathcal{F}), x \in \hat{D}_k \cap D_k} \log(\mathcal{F}, \hat{D}_k, x).
\]

We can rewrite each of the above sums in a more appropriate way, as follows: in the first one by using the relation \[4\] we obtain
\[
\sum_{x \in \text{Sing}(\mathcal{F}), x \in (X - D)} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), x \in (X - D)} \mu_x(\mathcal{F}) = \sum_{x \in \text{Sing}(\mathcal{F}), x \in (X - D)} \log(\mathcal{F}, D, x).
\]

In the second sum by using the relation \[2\] and the Lemma 3.2 we get
\[
\sum_{x \in \text{Sing}(\mathcal{F}), x \in \hat{D}_k \cap [D - (\hat{D}_k \cap D_k)]} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), x \in \hat{D}_k \cap [D - (\hat{D}_k \cap D_k)]} [\mu_x(\mathcal{F}) - \text{Ind}_{\text{hom}}(\mathcal{F}, \hat{D}_k, x)]
\]
\[
= \sum_{x \in \text{Sing}(\mathcal{F}), x \in \hat{D}_k \cap [D - (\hat{D}_k \cap D_k)]} [\mu_x(\mathcal{F}) - \text{Ind}_{\text{hom}}(\mathcal{F}, D, x)]
\]
\[
= \sum_{x \in \text{Sing}(\mathcal{F}), x \in \hat{D}_k \cap [D - (\hat{D}_k \cap D_k)]} \log(\mathcal{F}, D, x).
\]
In the next sum by using the relation \(4\) again we have
\[
\sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F})
\]
\[
= \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}) - \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}|D) + \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}|D),
\]
and, since \((D - \hat{D}_k) \cap D_k \subset D_{\text{reg}}\), the identity \(5\) gives us
\[
\sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, D, x) + \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}|D).
\]
Since \(\mathcal{F}|D\) and \(\mathcal{F}|D_{\hat{k}}\) coincide locally around \(x \in (D - \hat{D}_k) \cap D_k\), we obtain
\[
\sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, D, x) + \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}|D_{\hat{k}}).
\]
Finally, in the last sum, by Lemma \(3.1\) we get that
\[
\sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}, D, x) + \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}|D), \hat{D}_k \cap D_k, x).
\]
Now, by taking each sum we have
\[
\sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, \hat{D}_k, x) = \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, D, x) +
\]
\[
+ \left[ \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \mu_x(\mathcal{F}|D_{\hat{k}}) + \sum_{x \in \text{Sing}(\mathcal{F}), \; x \in (D - \hat{D}_k) \cap D_k} \log(\mathcal{F}|D_{\hat{k}}, \hat{D}_k \cap D_k, x) \right] =
\]
\[
= \sum_{x \in \text{Sing}(\mathcal{F})} \log(\mathcal{F}, D, x) + \sum_{x \in \text{Sing}(\mathcal{F}|D_{\hat{k}})} \log(\mathcal{F}|D_{\hat{k}}, \hat{D}_k \cap D_k, x),
\]
where in the last step we use the relation \(4\) and the following equality of sets
\((D - \hat{D}_k) \cap D_k = D_k - (\hat{D}_k \cap D_k)\). \(\square\)

4. Proof Corollary \(1.2\)

By Theorem \(11\) we have that
\[
\int \! \! \! \! _{\mathbb{P}^n} c_n(T_{p^n}(-\log D) - \mathcal{O}_{p^n}(1-d)) = \sum_{x \in \text{Sing}(\mathcal{F}) \cap (X \setminus D)} \mu_x(\mathcal{F}) + \sum_{x \in \text{Sing}(\mathcal{F}) \cap D} \log(\mathcal{F}, D, x),
\]
since \( T_\mathcal{F} = \mathcal{O}_{p^n}(1 - d) \), where \( d \) is the degree of \( \mathcal{F} \). We have to prove the positivity of
\[
\int_{p^n} c_n(T_{p^n}(- \log D) - \mathcal{O}_{p^n}(1 - d)).
\]

Firstly, we will prove the following formula
\[
\int_{p^n} c_n(T_{p^n}(- \log D) - \mathcal{O}_{p^n}(1 - d)) = \sum_{i=0}^{n} \binom{n + 1}{i} \sigma_{n-i}(d_1, \ldots, d_k, d - 1).
\]

It follows from [3, Theorem 4.3] that
\[
0 \rightarrow T_{p^n}(- \log D) \rightarrow \mathcal{O}_{p^n}(1)^{n+1} \oplus \mathcal{O}_{p^n}^{k-1} \rightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{p^n}(d_i) \rightarrow 0.
\]

From this exact sequence we get
\[
c(T_{p^n}(- \log D)) = \frac{(1 + h)^{n+1}}{\prod_{i=1}^{k}(1 + d_i h)^i},
\]
where \( h = c_1(\mathcal{O}_{p^n}(1)) \). Since \( c(T_{p^n}) = 1 + (1 - d)h \) we have
\[
c(T_{p^n}(- \log D) - T_{\mathcal{F}}) = \frac{c(T_{p^n}(- \log D))}{c(T_{\mathcal{F}})} = \frac{(1 + h)^{n+1}}{(1 + (1 - d)h) \prod_{i=1}^{k}(1 + d_i h)^i} = \frac{(1 + h)^{n+1}}{\prod_{i=0}^{k}(1 + m_i h)^i},
\]
where \( m_i := d_{i+1} \), for all \( i = 0, \ldots, k - 1 \) and \( m_k = 1 - d \). Therefore
\[
c_n(T_{p^n}(- \log D) - T_{\mathcal{F}}) = \left[ \frac{(1 + h)^{n+1}}{\prod_{i=0}^{k}(1 + m_i h)^i} \right]_n \sum_{i=0}^{n} (-1)^{n-i} \binom{n + 1}{i} \sigma_{n-i}(m_0, \ldots, m_k) h^n.
\]

Thus, we have that
\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n + 1}{i} \sigma_{n-i}(m_0, \ldots, 1 - d) = \sum_{i=0}^{n} \binom{n + 1}{i} \sigma_{n-i}(m_0, \ldots, d - 1).
\]

Now, suppose that \( d = 2 \) and \( d_i = 1 \), for all \( k \), then
\[
c(T_{p^n}(- \log D) - T_{\mathcal{F}}) = \frac{(1 + h)^{n}}{(1 + h)^{k}}.
\]

Therefore, the conclusion is the same as [15, Example 1.1]. If \( d > 1 \), then it is clear that
\[
0 < \sum_{i=0}^{n} \binom{n + 1}{i} \sigma_{n-i}(d_1, \ldots, d_k, d - 1) = \int_{p^n} c_n(T_{p^n}(- \log D) - T_{\mathcal{F}}).
\]
In order to conclude the item (ii) we recall that \( \operatorname{Log}(\mathcal{F}, D, x) = 0 \) for all non-degenerate singularity \( x \in \operatorname{Sing}(\mathcal{F}) \cap D \). Then

\[
0 < \sum_{i=0}^{n} \binom{n+1}{i} \sigma_{n-i}(d_1, \ldots, d_k, d-1) = \sum_{x \in \operatorname{Sing}(\mathcal{F}) \cap D} \operatorname{Log}(\mathcal{F}, D, x)
\]

says us that there exist at least one degenerate singular point.

5. Proof Corollary 1.3

Consider a functorial log resolution \( \pi : (X, D) \to (Y, \emptyset) \) with exceptional divisor \( D \), see [22, Theorems 3.35, 3.34]. Since the singular locus of \( X \) is invariant with respect to any automorphism it follows from [13, Corollary 4.6]) that the vector field \( v \in H^0(Y, T_Y) \) has a lift \( \tilde{v} \in H^0(X, T_X(-\log D)) \). Denoting by \( \mathcal{F} \) the foliation associated to the vector field \( \tilde{v} \), we have that \( T_{\mathcal{F}} = O_X \), since \( \tilde{v} \in H^0(X, T_X(-\log D)) \subset H^0(X, T_X) \) is a global vector field.

Now, since \( \operatorname{Sing}(v) \cap \operatorname{Sing}(X) = \emptyset \), we have that \( \tilde{v} \) has no zeros along the normal crossings divisor \( D \). By Theorem 1 we have

\[
\int_X c_n(T_X(-\log D) - O_X) = \int_X c_n(T_X(-\log D)) = \sum_{x \in \operatorname{Sing}(v) \cap Y_{\text{reg}}} \mu_{x(v)}(v) = \sum_{x \in \operatorname{Sing}(v)} \mu_x(v)
\]

since \( \mu_{\pi^{-1}(x)}(\tilde{v}) = \mu_x(v) \), for all \( x \in \operatorname{Sing}(v) \).

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GLOBAL RESIDUE FORMULA FOR LOGARITHMIC INDICES OF ONE-DIMENSIONAL FOLIATIONS

Maurício Corrêa,
Università degli Studi di Bari, Dipartimento di Matematica, Via E. Orabona 4, 1-70125 Bari, Italy.
Departamento de Matemática, Universidade Federal de Minas Gerais, Av. Antônio Carlos 6627, 30123-970, Belo Horizonte-MG, Brazil.
Email address: mauriciojr@ufmg.br, mauricio.barros@uniba.it

Diogo da Silva Machado,
Departamento de Matemática, Universidade Federal de Viçosa, Avenida Peter Henry Rolfs, s/n - Campus Universitário, 36570-900 Viçosa - MG, Brazil.
Email address: diogo.machado@ufv.br