COOPERATIVE GAMES ON SIMPLICIAL COMPLEXES

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ABSTRACT. In this work, we define cooperative games on simplicial complexes, generalizing the study of probabilistic values of Weber [Web88] and quasi-probabilistic values of Bilbao, Driessen, Jiménez Losada and Lebrón [BDJLL01].
Applications to Multi-Touch Attribution and the interpretability of the Machine-Learning prediction models motivate these new developments [LL17, RSG16, SK13, SPK17, DSZ16, BBM15].
We deal with the axiomatization provided by the \(\lambda\)-dummy and the monotonicity requirements together with a probabilistic form of the symmetric and the efficiency axioms.
We also characterize combinatorially the set of probabilistic participation influences as the facet polytope of the simplicial complex.

A cooperative game is a pair \((n, v)\) where \(n\) is a positive integer and \(v\) is the worth function \(v : 2^n \to \mathbb{R}\), where \(2^n\) is the power set of \([n] \equiv \{1, \ldots, n\}\). We assume that \(v(\emptyset) = 0\). The elements of \([n]\) are players of the game that may join in coalitions; a coalition \(T\) is a subset of \([n]\) and \(v(T)\) is the number of payoff of \(T\) in the cooperative game.
For every player \(i\) an individual value \(\phi_i(v)\) is a (linear) function measuring the additional worth that \(i\) provides to a coalition during the cooperative game \((n, v)\). The study of such values was extremely relevant for the community and we would like to highlight here a few important works of Shapley [Sha53, Sha72] and Weber [Web88] that have influenced the author.
Recently, quite a lot of effort has been done to study cooperative games on matroids [BDJLL01, BDJLL02, MTMZ19, MZ11, FV11, NZKI97, Zha99]. Inspired by this recent articles, in this manuscript we define cooperative games on simplicial complexes and we study quasi-probabilistic values for such games.

A simplicial complex is a family \(\Delta\) of subsets of \([n]\) such that if \(X \in \Delta\), then every subset \(Y \subseteq X\) will also belong to \(\Delta\). For instance, every graph and the full power set \(2^n\) are simplicial complex. In the latter case, \(\Delta = 2^n\) is called a \((n - 1)\)-dimensional simplex. The reader may find more examples all along the paper, but also highlighted in Figures 2, 1a, and 1b.
A cooperative game on \(\Delta\) is defined by a worth function \(v:\)

\[
v : \Delta \to \mathbb{R}
\]

with the usual constrain that \(v(\emptyset) = 0\). The traditional game \((n, v)\) can be seen as the cooperative game on the \((n - 1)\)-dimensional simplex \((2^n, v)\), where the function \(v\) is the defines as in the classical case.
In other words, in a cooperative game on a a simplicial complex \(\Delta\) a player \(i\) in \([n]\) may join a coalition \(T\) only if \(T \cup i \in \Delta\). In such case, the coalition is feasible and, unfeasible otherwise. Similarly as in the classical case, the individual function \(\phi_i(v)\) measures the additional value that \(i\) provide to a feasible coalition during the cooperative game \((\Delta, v)\).

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Presentation of the results

We are going to present the new research developments and after this we provide several motivation for this work. Further, we explain carefully a concrete application to Multi-Touch Attribution and for this prototype example we show explicitly all the new introduced objects.

A player $i$ is dummy for the cooperative game $(\Delta, v)$ if the player does not provide better results to the coalition then its own worth $v(\{i\})$. In mathematical terms, $i$ is a dummy player in the cooperative game $(\Delta, v)$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $i \notin S$ and $S \cup \{i\} \in \Delta$. For simplicity we are going to neglect the set brackets for the singleton and write $S \cup \{i\}$ instead of $S \cup \{i\}$. Moreover, we say that the worth function $v$ is monotone if $S \subseteq T \in \Delta$ implies $v(S) \leq v(T)$. We denote by $\mathbb{R}(\Delta)$ the set of all cooperative game on the simplicial complex $\Delta$.

Theorem 2 and 3 of [Web88] characterize the individual values that satisfies the dummy and the monotonicity axioms. Theorem 3.1 of [BDJLL01] generalizes such results for cooperative game on a matroids. Every matroid is a simplicial complex but not vice versa, see for instance the example in Figure 2. Next theorem moves further and extends the results to every simplicial complex $\Delta$.

We denote by $0 \leq \lambda_i \leq 1$ the rate of participation of the player $i$ in the cooperative game $(\Delta, v)$ and let us rewrite the two main conditions in Theorem 3.1 of [BDJLL01]:

- **$\lambda_i$-Dummy axiom:** If the player $i$ is dummy for $(\Delta, v)$, then $\phi_i(v) = \lambda_i v(\{i\})$;
- **Monotonicity axiom:** If $v$ is monotone, then $\phi_i(v) \geq 0$.

**Theorem 2.1.** Let $\Delta$ be a simplicial complex on $n$ vertices and let $\phi_i$ be an individual value for a player $i$. The individual value $\phi_i$ is a $\mathbb{R}$-linear function satisfying the $\lambda_i$-Dummy axiom and the Monotonicity axiom if and only if there exists a collection of non-negative real numbers $\{p_T : T \in \text{Link}_\Delta i\}$ with

$$\sum_{T \in \text{Link}_\Delta i} p_T = \lambda_i$$

such that for all $v$ in $\mathbb{R}(\Delta)$,

$$\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T (v(T \cup i) - v(T)).$$

The necessary and sufficient condition of the previous theorem hints a natural generalization of probabilistic values for cooperative games on simplicial complexes. For this reason, Bilbao, Driessen, Jiménez Losada and Lebrón [BDJLL01] called, in the case of matroids, those values quasi-probabilistic, see Section 3.

Recall that $\text{Facets} \Delta$ is the set of facets of the simplicial complex $\Delta$, these are sets in $\Delta$ that are maximal by inclusion. Moreover, $\text{Facets}_\Delta i$ collects all the facets containing $i$. In addition, we denote by $\bar{F}$ the simplex on the vertices in $F$, i.e. $\bar{F} = 2^F$.

**Theorem 2.2.** Let $\Delta$ be a simplicial complex and let $\phi_i$ be an individual value for a player $i$ in $[n]$.

The individual value $\phi_i$ is a quasi-probabilistic value if and only if there exists a probability distribution $\{P^i(F_1), \ldots, P^i(F_k)\}$ on Facets $\Delta$ such that

$$\sum_{F \in \text{Facets}_\Delta i} P^i(F) = \lambda_i,$$
and for every $F \in \text{Facets}_\Delta$ there exists an individual (classical) probabilistic value $\phi^F_i$ defined on the simplex $\bar{F}$ such that for all $v$ in $\mathbb{R}(\Delta)$,

$$\phi_i(v) = \sum_{F \in \text{Facets}_\Delta} P^i(F) \phi^F_i(v|_F),$$

where $v|_F$ is the restriction of the cooperative game $(\Delta, v)$ to $(F, v|_F)$.

The previous statement extends Theorem 3.2 in [BDJLL01] to every simplicial complex.

As in the traditional case, we collect all individual values together, in the group value $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$. For sake of presentation of the results, let us assume here that we work in the efficient scenario, that is the individual values are nor optimistic or pessimistic. A group value $\phi$ for the cooperative game $(\Delta, v)$ is reducible if there exists a probability distribution $P$ on the the facets of $\Delta$ such that

$$\phi_i(v) = \sum_{F \in \text{Facets}_\Delta} P(F) \phi^F_i(v|_F),$$

where $\phi^F_i$ is a probabilistic value for a cooperative game on the simplex $\bar{F}$ and $v|_F$ is the restriction of the indicator function $v$ on the simplex $F$, that is $v|_F(S) = v(S)$ for every subset $S$ of $F$.

This notion was introduced in [BDJLL01] as basic value. Our choice for the adjective reducible is made in view of the results in [Mar20a, Mar20b].

We denote by $\text{Prob} \Delta$ the set of probability distribution over the set of facets of $\Delta$. Following Section 4 of [BDJLL01], we also define the probabilistic participation influence $w^P(T)$ for the coalition $T$ as

$$w^P(T) \overset{\text{def}}{=} \sum_{T \subseteq F \in \text{Facets} \Delta} P(F)$$

where $P$ is a probability distribution over the set of facets of $\Delta$ and $Q_\Delta$ is the convex hull in $\mathbb{R}^n$ of vectors $e_F = \sum_{i \in F} e_i$ for every facet $F$ in Facets $\Delta$, where $e_i$ is a standard orthonormal basis of $\mathbb{R}^n$, see Definition 4.2.

While the equality on the left does not hold for simplicial complexes, we are able to prove that $Q_\Delta = \{w^P : P \in \text{Prob} \Delta\}$.

All these results together allow us to generalize the main theorem of [BDJLL01] to pure simplicial complex, i.e. simplicial complex where all the facets have the same cardinality. The next statement characterizes when an individual value is reducible to classical cooperative games defined on the facets of the pure simplicial complex. To do this, we need two additional constrains:

**Substitution for carrier games:** For every feasible coalition $T$ and for every pair of players, one has $\phi_i(v_T) = \phi_j(v_T)$ where $v_T$ is the carrier game defined in Definition 1.2;

**Probabilistic efficiency:** For every cooperative game $(\Delta, v)$, $\sum_i \phi_i(v) = \sum_{F \in \text{Facets} \Delta} P(F)v(F)$. 

3
Theorem 5.1. Let $\Delta$ be a pure simplicial complex on $n$ vertices and and let $\phi_i$ be the individual value for a player $i$. The group value $\phi$ is reducible and it decomposes as the weighted sum of Shapley values,

$$\phi_i(v) = \sum_{F \in Facets_{\Delta} i} P(F) Shapley^F(v|F).$$

if and only if each $\phi_i$ is a linear function that satisfies the $w^P(i)$-dummy axiom and the group value fulfills the Substitution for carrier games and the Probabilistic efficiency axioms.

The substitution axiom is the one enforcing that $\Delta$ is a pure simplicial complex; the probabilistic efficiency provides, instead, the weights of the decomposition of $\phi_i$ as sum of Shapley values.

Motivations

It is worth to mention a few reasons why the generalization provided is relevant.

i) Not every pure simplicial complex is a matroid. Figure 1a and 1b show two pure simplicial complex and only the one in the left is a matroid. Thus, the new results extend quite substantially the results in Theorem 3.1, Theorem 3.2 and Theorem 4.2 of [BDJLL01].

ii) Aside the generalization from matroids to simplicial complexes in it-self, our proofs of the results show that the main matroidal properties are not useful in this context [Sta12, Sta96, Sta84, Oxl11] and underline the importance of the link of the player, see Definition 1.1. This will be crucial in the subsequent works [Mar20a, Mar20b].

iii) A ground-breaking application of the Shapley value methodology is in the interpretability of the Machine-Learning prediction models, see for instance [LL17] and the Phyton package SHAP [Github repository], see [RSG16, SK13, SPK17, DSZ16, BBM+15, LC01].

iv) Last but not the least, another very important application is that Google has started to use it in its own multi-touch attribution system offered in Google 360. (This is the marketing platform developed and offered by Google.)

The prototype example

We provide here a prototype motivating application of the new results: the Multi-Touch Attribution in Marketing. In Marketing attribution the individual values are used to understand what set of advertisements have influenced a person toward a desired behavior, typically becoming
Figure 2. The simplicial complex describing the feasible coalitions for the cooperative game of the prototype example.

a costumer. This is (classically) described as the following cooperative game. \( \Delta = 2^\mathcal{N} \) is the simplex with vertices in the set of marketing channels, labeled for simplicity from 1 to \( n \). To each subset \( T \) of \( [n] \), the worth function \( v \) associates the number of conversions \( v(T) \), people who became costumers, influenced by the marketing channels in \( T \). In other words, \( v(T) \) counts the number of individuals who decided to buy being effected by the advertisements collected in the set \( T \).

Unfortunately, the marketing team cannot always track down \( v(T) \) for every subset of channels, and setting \( v(T) = 0 \) for those untraceable coalitions may results in technical problems such as \( v \) is not anymore monotone, that is \( v(S) \leq v(T) \) for every \( S \subseteq T \subseteq [n] \). The idea we propose is to mark such coalition as unfeasible and to take them out from \( \Delta \). We still assume that if \( T \) is a feasible coalition, then every subset of \( T \) is still feasible. Thus, \( \Delta \) is a simplicial complex, see Section 1.1.

**Example.** The advertising campaign of a certain store is made of six different marketing channels: the distribution of flyers (F) and an advertising stand (S) in the weekly market of the district, together with social network (FB), email (E), TV and search engine (G) advertisements. The shop also offers a discount for on-line purchases that can be retrieved with a promotion code in the flyers or by request in the stand.

Because of the nature of the data, the marketing team cannot analyze if conversions (individuals who make a purchase decision) have been exposed by all the channels and in particular, they can only track the following ones: every subsets of the channels \( \{FB, S, F\} \) and every subsets of \( \{FB, TV, E, G\} \). Figure 2 shows the simplicial complex \( \Delta \) of feasible coalitions:

\[
\Delta = \{\emptyset, \{F\}, \{S\}, \{FB\}, \{E\}, \{TV\}, \{S, F\}, \{FB, F\}, \{FB, S\}, \{FB, S, F\}, \{FB, E\}, \{FB, TV\}, \{TV, E\}, \{FB, TV, E\}, \{FB, G\}, \{TV, G\}, \{FB, TV, G\}, \{E, G\}, \{FB, E, G\}, \{TV, E, G\}\}.
\]

**Breaking the prototype example down**

Before moving to the preliminary definitions and proving all the presented results, we want to provide the reader a concrete example by computing the quasi-probabilistic values for the cooperative games in the previous example.

Assume that the cooperative game satisfies the conditions in Theorem 2.1 and enumerate the set of marketing channels as \( \{F, S, FB, TV, E, G\} = \{1, 2, 3, 4, 5, 6\} \).
If we suppress the brackets notation for set and write, for instance 345 for the set \{3, 4, 5\}. Then we write down the individual values as:

\[
\begin{align*}
\phi_1(v) &= p_0^1(v(1) + p_1^1(v(12) - v(2)) + p_2^1(v(13) - v(3)) + p_{23}^1(v(123) - v(23)); \\
\phi_2(v) &= p_0^2(v(2) + p_1^2(v(12) - v(1)) + p_3^2(v(23) - v(3)) + p_{13}^2(v(123) - v(13)); \\
\phi_3(v) &= p_0^3(v(3) + p_1^3(v(13) - v(1)) + p_2^3(v(23) - v(2)) + p_{12}^3(v(123) - v(12)) + \\
&+ p_3^3(v(43) - v(4)) + p_4^3(v(35) - v(5)) + p_6^3(v(36) - v(6)) + \\
&+ p_{45}^3(v(345) - v(45)) + p_{46}^3(v(346) - v(46)) + p_{56}^3(v(356) - v(56)); \\
\phi_4(v) &= p_0^4(v(4) + p_3^4(v(34) - v(3)) + p_5^4(v(45) - v(5)) + p_6^4(v(46) - v(6)) + \\
&+ p_{34}^4(v(345) - v(34)) + p_{35}^4(v(346) - v(36)) + p_{36}^4(v(356) - v(56)); \\
\phi_5(v) &= p_0^5(v(5) + p_3^5(v(35) - v(3)) + p_4^5(v(45) - v(4)) + p_5^5(v(56) - v(6)) + \\
&+ p_{34}^5(v(345) - v(34)) + p_{35}^5(v(356) - v(36)) + p_{46}^5(v(456) - v(46)); \\
\phi_6(v) &= p_0^6(v(6) + p_3^6(v(36) - v(3)) + p_5^6(v(56) - v(5)) + p_6^6(v(46) - v(6)) + \\
&+ p_{35}^6(v(356) - v(35)) + p_{36}^6(v(346) - v(36)) + p_{45}^6(v(456) - v(45)).
\end{align*}
\]

Since \( \Delta \) is not pure, \( \phi \) cannot be reduced as sum of Shapley values. It is also worth to mention that the face polytope \( Q_\Delta \) that describes the convex all of probabilistic participation influence is a polytope in \( \mathbb{R}^6 \) defined by \( 20 = 6 + 9 + 5 + 2 \) points.

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1. **Preliminaries**

Let \( n \) be a positive integer and we denote by \([n] \) \( [1, \ldots, n] \).

1.1. **Simplicial Complexes.** A (finite) simplicial complex \( \Delta \) over \( n \) vertices is a family of subsets in \( 2^{[n]} \) such that if a set \( T \) is in \( \Delta \), then any of its subsets \( S \subset T \) will be in \( \Delta \) too. There is a natural rank function, \( \text{rank}_\Delta \), on \( \Delta \) provided by the cardinality of its sets. The elements of \( \Delta \) that are maximal by inclusions are called facets. The maximal value of this rank function is the rank of the simplicial complex rank \( \Delta \).

All along this work \( \Delta \) is a finite simplicial complex over \( n \) vertices of rank \( r \) \( \text{rank}_\Delta \). Let Facets \( \Delta \) be the set of facets of \( \Delta \), Facets \( \Delta = \{F_1, \ldots, F_k\} \) and for every \( T \in \Delta \) let Facets\( _\Delta T \) be the set of facets in \( \Delta \) that contain \( S \).

If \( S \) is an element in \( \Delta \), then \( \bar{S} \) \( \{T : T \subseteq S\} \) is the simplicial complex obtained by all sets contained in \( S \). The next definition is extremely important for this work.

**Definition 1.1.** The link of of an element \( S \) in a simplicial complex \( \Delta \) is made by the subsets \( A \) of \( T \in \Delta \), such that \( T \) is disjoint by \( S \) and can be completed by \( S, S \cup T, \) to an element in \( \Delta \):

\[
\text{Link}_\Delta S = \{A : A \in \bar{T} \text{ with } T \in \Delta \text{ such that } S \cap T = \emptyset, S \cup T \in \Delta \}.
\]

The case when \( S \) is the singleton \( \{i\} \) will be extremely relevant in our work: \( \text{Link}_\Delta i \) is the set of simplex \( T \) in \( \Delta \) with \( i \notin T \) such that \( T \cup i \in \Delta \):

\[
\text{Link}_\Delta i = \{T \in \Delta : i \notin T \text{ and } T \cup i \in \Delta \}.
\]
1.2. Cooperative games on simplicial complexes. A cooperative game on the simplicial complex $\Delta$ is the pair $(\Delta, v)$ where $v$ is a worth function $v : \Delta \to \mathbb{R}$ under the constrain $v(\emptyset) = 0$. (Here we mean that the function is defined on $\text{Set}(\Delta)$, that is the collection of elements of $\Delta$ forgetting the partial order.) The verticies $[n]$ of the simplicial complex are the players of the cooperative game and a coalition $T$ is feasible if $T \in \Delta$. The set $\mathbb{R}(\Delta)$ of indicator functions on $\Delta$ is naturally a real vector space.

Two types of games have a very important role for the theory of probabilistic values, see [Web88]. By abuse of notation we call both families carrier games even if the notion usually refer only to the first family:

$$C = \{v_T : \emptyset \neq T \subset [n]\}, \quad \hat{C} = \{\hat{v}_T : \emptyset \neq T \subset [n]\},$$

where $v_T$ and $\hat{v}_T$ are so defined:

$$v_T(S) = \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise} \end{cases}, \quad \hat{v}_T(S) = \begin{cases} 1 & T \subseteq S \\ 0 & \text{otherwise} \end{cases}.$$

We generalize this definition for any element $T$ of a simplicial complex. Indeed for every partially order set $(P, \leq_P)$ and every element $q$ in $P$ we consider the following function:

$$u^P_q(s) \overset{\text{def}}{=} \begin{cases} 1 & q \leq_P s \\ 0 & \text{otherwise} \end{cases}, \quad \hat{u}^P_q(s) \overset{\text{def}}{=} \begin{cases} 1 & q <_P s \\ 0 & \text{otherwise} \end{cases}.$$

Thus, we define

$$v_T(S) \overset{\text{def}}{=} u^\Delta_T(S), \quad \hat{v}_T(S) \overset{\text{def}}{=} \hat{u}^\Delta_T(S).$$

It is easy to see that in the classical case (when $\Delta$ is a full simplex on $n$ verticies) these functions reproduce the carrier games.

**Definition 1.2.** Let $\Delta$ be a simplicial complex. The sets of carrier games are so defined:

$$C = \{v_T : \emptyset \neq T \in \Delta\}, \quad \hat{C} = \{\hat{v}_T : \emptyset \neq T \in \Delta\},$$

where $v_T(S) \overset{\text{def}}{=} u^\Delta_T(S)$ and $\hat{v}_T(S) \overset{\text{def}}{=} \hat{u}^\Delta_T(S)$.

When $T$ is the empty set, we define $\hat{v}_\emptyset \overset{\text{def}}{=} \hat{u}^\Delta_\emptyset$.

We also denote by $\mathbb{1}_T$ the indicator function of the set $T$ in $\Delta$, that is:

$$\mathbb{1}_T(S) \overset{\text{def}}{=} \begin{cases} 1 & T = S \in \Delta \\ 0 & \text{otherwise} \end{cases}.$$

We observe that $\mathbb{1}_T = v_T - \hat{v}_T$.

**Definition 1.3.** An individual value for a player $i$ in $[n]$ is a function $\phi_i : \mathbb{R}(\Delta) \to \mathbb{R}$. The first axiom in the theory of probabilistic values is that these functions are linear, see Section 3 of [Web88]. We are going to only consider linear individual values.
2. Individual values for simplicial complexes

Recall that the player $i$ is dummy in the cooperative game $(\Delta, v)$ if $v(S \cup i) = v(S) + v(i)$ for every $i \notin S$ and $S \in \Delta$. Moreover, we say that $v$ is monotone if provided $S \subseteq T \in \Delta$, then $v(S) \leq v(T)$.

Next statement extends Theorem 3.1 of [BDJLL01] and Theorem 2 and 3 of [Web88] to every simplicial complex.

We denote by $0 \leq \lambda_i \leq 1$ the rate of participation of the player $i$ in the cooperative game $(\Delta, v)$ and let us rewrite the two main conditions in Theorem 3.1 of [BDJLL01]:

**$\lambda_i$-Dummy axiom:** If the player $i$ is dummy for $(\Delta, v)$, then $\phi_i(v) = \lambda_i v(i)$;

**Monotonicity axiom:** If $v$ is monotone, then $\phi_i(v) \geq 0$.

**Theorem 2.1.** Let $\Delta$ be a simplicial complex on $n$ vertices and let $\phi_i$ be an individual value for a player $i$. The individual value $\phi_i$ is a $\mathbb{R}$-linear function satisfying the $\lambda_i$-Dummy axiom and the Monotonicity axiom if and only if there exists a collection of positive real numbers $\{p_T^i : T \in \text{Link}_\Delta i\}$ with

\[
\sum_{T \in \text{Link}_\Delta i} p_T^i = \lambda_i,
\]

such that for all $v$ in $\mathbb{R}(\Delta)$,

\[
\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T^i (v(T \cup i) - v(T)).
\]

**Proof.** Let us first show the if part. The set $\left\{1_T\right\}_{T \in \Delta}$ is a basis for the vector space $\mathbb{R}(\Delta)$, so every real valued function $v$ can be written uniquely as $v = \sum_{T \in \Delta} 1_T v(T)$. By the linearity of the function $\phi_i$, we have

\[
\phi_i(v) = \phi_i \left( \sum_{T \in \Delta} 1_T v(T) \right) = \sum_{T \in \Delta} \phi_i(1_T) v(T).
\]

We first start by showing that $\phi_i(1_T) = 0$ if $T \in \Delta$ but $T \cup i \notin \Delta$. Note that $1_T = v_T - \hat{v}_T$, that $\phi_i(\hat{v}_T) = \phi_i(1_{T \cup i}) v(T \cup i)$, and that $\phi_i(v_T) = v_T(i) = 0$, because $i$ is dummy for $v_T$. Therefore we get $\phi_i(1_T) = \phi_i(v_T - \hat{v}_T) = 0$.

If $i \in T$, then $\phi_i(1_T) = \phi_i(1_T) = \phi_i(1_{T \cup i})$; thus formula (2) follows from by setting

\[
p_T^i \overset{\text{def}}{=} \begin{cases} 
\phi_i(1_{T \cup i}) & \text{if } i \in T \\
-\phi_i(1_T) & \text{otherwise.}
\end{cases}
\]

Now, let $i$ be a dummy player for $v$, from (2), we get

\[
\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T^i v(i) = \left( \sum_{T \in \text{Link}_\Delta i} p_T^i \right) v(i).
\]

Equation in (1) is obtained by comparing the previous equality with $\phi_i(v) = \lambda_i v(i)$, coming from the $\lambda_i$-dummy property.

It remains to show that every $p_T^i$ is a real number greater or equal than zero. For this, consider the monotone function $\hat{v}_T$. It is easy to see from (2) that

\[
\phi_i(\hat{v}_T) = p_T^i,
\]

8
and the latter is non-negative because \( \hat{v} \) is a monotone game.

Let us show the only if part. From (2), we know that \( \phi_i \) is a linear function. Moreover, if \( i \) is a dummy player, then \( v(S \cup i) = v(S) + v(i) \), so

\[
\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T^i v(i) = v(i) \sum_{T \in \text{Link}_\Delta i} p_T^i = \lambda_i v(i).
\]

Finally, if \( v \) is monotone, then \( v(S \cup i) \geq v(S) \) and, so \( \phi_i(v) \geq 0 \). □

3. Quasi-probabilistic values

The necessary and sufficient condition of the previous theorem hints a natural generalization of probabilistic value for simplicial complex, given for matroids in [BDJLL01].

An individual value \( \phi_i \) is a quasi-probabilistic value if there exists a collection of positive real numbers

\[
\{p_T^i : T \in \text{Star}_\Delta i\}
\]

with

\[
\sum_{T \in \text{Link}_\Delta i} p_T^i = \lambda_i,
\]

such that for all \( v \) in \( \mathbb{R}(\Delta) \),

\[
\phi_i(v) = \sum_{T \in \text{Link}_\Delta i} p_T^i(v(T \cup i) - v(T)).
\]

The next statement extends Theorem 3.2 in [BDJLL01] to every simplicial complex, even not pure ones.

**Theorem 3.1.** Let \( \Delta \) be a simplicial complex and let \( \phi_i \) be an individual value for a player \( i \) in \([n]\).

The individual value \( \phi_i \) is quasi-probabilistic if and only if there exists a probability distribution \( \{P^i(F_1), \ldots, P^i(F_k)\} \) on Facets \( \Delta \) such that

\[
(3) \quad \sum_{F \in \text{Facets}_\Delta i} P^i(F) = \lambda_i
\]

and for every \( F \in \text{Facets}_\Delta i \) there exists a probabilistic value \( \phi_i^F \) defined on the simplex \( \bar{F} \) such that for all \( v \) in \( \mathbb{R}(\Delta) \),

\[
(4) \quad \phi_i(v) = \sum_{F \in \text{Facets}_\Delta i} P^i(F)\phi_i^F(v|_F),
\]

where \( v|_F \) is the restriction of the cooperative game \((\Delta, v)\) to \((F, v|_F)\).

**Proof.** Let \( i \) be a player and let \( P^i \) be a probability distribution on Facets\( \Delta i \). Assume that for every \( F \) facets in Facets\( \Delta i \) and for every cooperative game \((F, w)\), the individual value \( \phi_i^F \) is a probabilistic value and, then, by Theorem 9 in [Web88], \( \phi_i^F \) is defined as

\[
\phi_i^F(w) = \sum_{T \subseteq F \setminus i} p_{F,T}^i (w(T \cup i) - w(T)).
\]
where \( p_{F,T}^i \) are non-negative real numbers such that

\[
\sum_{T \in F \setminus i} p_{F,T}^i = 1.
\]

We note that \( T \subset F \setminus i \) if and only if \( T \in \text{Link}_F i \cap F \).

Then one can show, similarly as in the proof of Theorem 3.2 in [BDJLL01], that for all \( T \) in \( \text{Link}_F i \)

\[
p_T^i = \sum_{F \in \text{Facets}_A T \cup i} p(F) p_{F,T}^i.
\]

and also

\[
\sum_{T \in \text{Link}_F i} p_T^i = \sum_{F \in \text{Facets}_A T \cup i} p(F).
\]

From the last two equations, one can show easily that the provided conditions (3) and (4) are necessary.

Let us now focus on the sufficiency of the conditions. Consider the restriction \( v|_F \) of \( v \) to a generic facet \( F \) of \( \Delta \) and consider an individual probabilistic value \( \phi^F_i \) defined on the simplex \( \bar{F} \).

By definition one has that

\[
\phi^F_i (v|_F = \sum_{T \subseteq F \setminus i} p_{F,T}^i (v|_F (T \cup i) - v|_F (T)).
\]

where \( \sum_{T \in \text{Link}_F i} p_{F,T}^i = 1 \). Consider also a probability distribution \( P^i \) on \( \text{Facets}_A i \), that is \( P^i(F) \geq 0 \) and \( \sum_{F \in \text{Facets}_A i} P^i(F) = 1 \). Finally, from the hypothesis, we know that \( \phi_i(v) = \sum_{T \subseteq F \setminus i} p_{F,T}^i (v(T \cup i) - v(T)), \) where \( \sum_{T \in \text{Link}_i} p_{F,T}^i = \lambda_i. \) From equation (6), one has that if such probability distribution \( \{P^i\} \) exists then \( \sum_{F \in \text{Facets}_A T \cup i} P^i(F) \) has to be equal \( \sum_{T \in \text{Star}_i} p_{F,T}^i \) and, the latter, in the case of a probabilistic value is precisely \( \lambda_i \).

It remains to show that \( \phi_i(v) = \sum_{F \in \text{Facets}_A T \cup i} P^i(F) \phi^F_i (v|_F) \) and for this we need to prove that \( P^i, p_S^i \) and \( p_{F,S}^i \) satisfy equation (5).

Given \( S \in \Delta \), let \( m_S(\Delta) \) be the number of facets \( F \) of \( \Delta \) containing \( S \). For all facets \( F \) in \( \text{Facets}_A i \) denote by

\[
P^i(F) \overset{\text{def}}{=} \sum_{T \in \text{Link}_i \cap F} \frac{p_T^i}{m_{T \cup i}};
\]

we also observe that the sum can we simply taken over \( \text{Link}_F i \) because this set equals \( \text{Link}_A i \cap F \). Moreover, for all facets \( F \) in \( \text{Facets}_A T \cup i \) and for every \( T \in \text{Link}_F i \), we define

\[
p_{F,T}^i \overset{\text{def}}{=} \frac{p_T^i}{m_{T \cup i} P^i(F)}
\]
Now we substitute the previous value in equation (6) and one gets

\[
p_i^T = \sum_{F \in \text{Facets}_S T \cup i} P^i(F) p_i^F_T
\]

\[
= \sum_{F \in \text{Facets}_S T \cup i} P^i(F) \frac{p_i^T}{m_{T \cup i}} P^i(F)
\]

\[
= \sum_{F \in \text{Facets}_S T \cup i} \frac{p_i^T}{m_{T \cup i}}
\]

\[
= \frac{p_i^T}{m_{T \cup i}} \sum_{F \in \text{Facets}_S T \cup i} 1.
\]

The equality holds because by definition

\[
m_{T \cup i} = \sum_{F \in \text{Facets}_S T \cup i} 1 = |\text{Facets}_S T \cup i|.
\]

As we have done in the introduction, it is important to highlight the discrepancy and the improvement between these results and Theorem 3.2 of [BDJLL01]:

(i) Matroids are pure simplicial complexes that satisfies the base exchange properties [Sta12, Sta96, Sta84, Oxl11, Mar18, BM19]. In particular, every facet has the same cardinality, i.e. if the matroid have rank \( r \), then every facet has cardinality \( r \). In this specific case, facets are called bases. The integers \( m_\Delta(S) \) in the proof of Theorem 3.2 in [BDJLL01] counts the number of bases (i.e. facets) of the matroid containing \( S \). While the concept is the same, in our proof of Theorem 3.1, \( m_\Delta(S) \) takes in consideration facet of different cardinality. It is quite remarkable that the probability distribution \( \{P(F_i)\} \) in Theorem 3.1 can be defined in the same way regardless of the difference of rank.

(ii) Not every pure simplicial complex is a matroid. Hence our theorem applies also to the simplicial complex described in Figure 1b. Simplicial complexes that are not matroids are extremely important in Mathematics; few example can be found in [Mar15, GM16, GM18].

(iii) The proof of the statements does not involve any of the matroidal properties [Sta12, Sta96, Sta84, Oxl11]. As highlighted in the introduction, none of the matroidal property plays a role in the proof of Theorem 2.1 and 3.1. This will be different in Section 5 where we will need our complex to be pure.

4. Group value and the core

The goal of the individual values is to assign the payoff of the grand coalition \( v([n]) \) proportionally to the worth of the player. (This is, somehow, the efficient scenario: we assume that the values are nor optimistic or pessimistic. We are going to focus on this more in details in the subsequent section.)
Therefore, as in the traditional case, we collect all individual values together, in the group value \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \). In [BDJLL01], they introduce the concept of basic value. In view of the results in [Mar20a, Mar20b], we call this property reducible.
Definition 4.1. A group value \( \phi \) on \( \mathbb{R}(\Delta) \) is reducible if there exists a probability distribution \( P \) on the the facets of \( \Delta \), Facets \( \Delta \), such that
\[
\phi_i(v) = \sum_{F \in \text{Facets}_i} P(F)\phi_i^F(v|F),
\]
where \( \phi_i^F \) is a probabilistic value for a cooperative game on the simplex \( \bar{F} \) and \( v|F \) is the restriction of the indicator function \( v \) on the simplex \( F \), that is \( v|F(S) = v(S) \) for every subset \( S \) of \( F \).

In other words, a group value is reducible if the group value can be computed reducing the cooperative game to \( k = |\text{Facets}\Delta| \) cooperative games \( (\bar{F}_1, v|_{\bar{F}_1}), \ldots, (\bar{F}_k, v|_{\bar{F}_k}) \) on the full simplicies \( \bar{F}_1, \ldots, \bar{F}_k \).

While every component of a reducible group value \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \), is a quasi-probabilistic value, Example 3.1 in [BDJLL01] shows that the converse is not true. This is because the distribution \( P \) on the facets of \( \Delta \) needs to be unique for every vertex \( i \).

Given a cooperative game \( (2^n, v) \), the group value \( \phi(v) \) detect a vector in \( \mathbb{R}^n \); we could also assume that \( \phi_i(v) \geq v(i) \), i.e. each player does not accept any redistribution of the payoff \( v([n]) \) if it is less than \( v(i) \), the amount the player could obtain on its own. We should assume the same for every coalition \( T \), that is \( \sum_{i \in T} y_i \geq v(T) \). Such vectors in \( \mathbb{R}^n \) are called imputations.

Let \( x \) and \( y \) be two imputations in \( \mathbb{R}^n \). We say that \( x \) is dominated by \( y \) for the cooperative game \( ([n], v) \), if

(a) there exists a coalition \( T \) such that \( x_i \leq y_i \) for all \( i \in T \);
(b) there exists a player \( i \) in \( T \) such that \( x_i < y_i \);
(c) The coalition \( T \) can adopt \( y \) as imputation, that is \( \sum_{i \in T} y_i \geq v(T) \).

In other words, the imputation \( y \) would be a better deal than \( x \) for the players in \( T \). The core is the set of imputations that are not dominated. This notion is used since latest 19-th century, but also more recently appears in several research works; here just a few examples [Kan92, Gil59, AK17, XDSS15, Ich81, BOn10, AFY17]. Mathematically, this is formulated as it follows. Assume that for any \( (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \), \( x(S) = \sum_{i \in S} x_i \). The core of a cooperative game \( ([n], v) \) is the following set:
\[
\text{core}([n], v) \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x([n]) = v([n]), x(S) \geq v(S) \forall S \subset [n]\}.
\]

Similarly, we define the anticore as
\[
\text{anticore}([n], v) \overset{\text{def}}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x([n]) = v([n]), x(S) \leq v(S) \forall S \subset [n]\}.
\]

Definition 4.2. Given a simplicial complex \( \Delta \) over \( n \) vertices with facets Facets \( \Delta \), the facet polytope \( Q_{\Delta} \) is the convex hull in \( \mathbb{R}^n \) of vectors \( e_F = \sum_{i \in F} e_i \) for every facet \( F \) in Facets \( \Delta \), where \( e_i \) is a standard orthonormal basis of \( \mathbb{R}^n \):
\[
Q_{\Delta} = \text{convex}[e_F : F \in \text{Facets} \Delta].
\]

The facet polytope and the core do not just share the ambient space. Indeed, when \( \Delta = M \) is a matroid, then \( Q_M \) is the base polytope, also called matroid polytope, (see for instance Chapter 9 of [BLVS93]) and Edmonds [Edm70] has shown that \( Q_M \) is the anti-core of of the game \( ([n], r) \), where \( r \) is the rank function of the matroid \( M \),
\[
\text{anticore}([n], r_M) = Q_M.
\]

This was also reproved in Theorem 2.3 of [BDJLL01].
Moreover, when $\Delta = 2^{[n]}$ is the full simplex, Weber [Web88] shows the core of the game is contained in the so called Weber set. Let us recall the definition of Weber set and this result. Let $S_n$ be the set of bijective function from $[n]$ to $[n]$. For every such bijective function $\pi$, we list the image of each element in a unique ordered set $\pi = (\pi(1), \ldots, \pi(n))$ and we denote for any $i \in [n]$ by $\pi^i \defeq \{ j \in [n] : \pi(j) < \pi(i) \}$. Let $(2^n, v)$ be a cooperative game and we define the marginal worth vector $a^\pi(v)$ as the imputation satisfying $a^\pi_j(v) = v(\pi^j \cup i) - v(\pi^i)$ for all $i \in [n]$. Let the Weber set be the set of all imputations which are associated with $v$ by some random-order value (that is equivalent, under the efficiency axiom to an efficient probabilistic group value), that is:

$$\text{Weber}(2^n, v) \defeq \text{convex}\{a^\pi(v) : \pi \in S_n\}$$

**Theorem 4.1** (Thm 14 in [Web88], Shapley [Sha72] and Ichiishi [Ich81]). For any cooperative game $([n], v)$, core$([n], v)$ $\subseteq$ Weber$([n], v)$. The equality holds if the game is convex.

We denote by Prob $\Delta$ the set of probability distribution over the set of facets of $\Delta$, that is:

$$\text{Prob } \Delta \defeq \{ P \in \mathbb{R}[\text{Facets } \Delta ] : P(F) \geq 0 \text{ and } \sum_{F \in \text{Facets } \Delta} P(F) = 1 \}. $$

Associated to a probability distribution $P$ in Prob $\Delta$, following Section 4 of [BDJLL01], we also define the probabilistic participation influence $w^P(T)$ of $T$ as

$$w^P(T) = \sum_{T \subseteq F \in \text{Facets } \Delta} P(F).$$

If $T$ is not in $\Delta$ then $w^P(T) = 0$; this can also be seen using the previous definition as $T$ is not a subset of any facet of $\Delta$.

In Proposition 4.1 of [BDJLL01], using Edmonds [Edm70] results in (7), they are able to prove that

$$\text{anticore}([n], \text{rk}_M) \supseteq Q_M = \{ w^P : P \in \text{Prob } M \}.$$  

While the same equality between the anticore and the facet polytope in (7) does not hold also for a generic simplicial complex, we are able to prove that $Q_\Delta = \{ w^P : P \in \text{Prob } \Delta \}$. Aside of the generalization per se of the result in [BDJLL01], one perk of next proposition is that we do not use Edmond result or any connection with the anticore of the cooperative game.

**Theorem 4.2.** Let $\Delta$ be a simplicial complex and let $r$ be its rank function. Then

$$Q_\Delta = \{ w^P : P \in \text{Prob } \Delta \}.$$  

**Proof.** Let $q$ be an element in $Q_\Delta$; $q$ can be written as a convex linear combination of the incidence vectors of the facets $\{ e_F \}$, that is $q = \sum_F \alpha_F e_F$ with $\alpha_F \geq 0$ and $\sum_F \alpha_F = 1$. If we set $P(F) \defeq \alpha_F$, then $q = w^P$. The opposite inclusion follows similarly. \hfill $\Box$

### 5. Reducible quasi-probabilistic values

In the traditional theory, a cooperative game is defined on the full simplex $\Delta = 2^{[n]}$ and the Shapley values are probabilistic values arising from the following common point of view among the players: The player $i$ joins a coalitions of different sizes with the same probability; All coalition of the same size are equally likely.

Thus, the every player has $n$ possibilities to choose the size of a coalition (the joint coalitions may have cardinality $k = 0 \leq k \leq n - 1$) and, further, there are $\binom{n-1}{k}$ choices among all sets
(coalitions) of cardinality $k$ among the other players $[n] \setminus i$. Therefore, one defines the Shapley values for the player $i$ as

$$\text{Shapley}_i(v) = \sum_{T \subseteq [n] \setminus i} \frac{1}{n} \frac{|T|!(n-|T|-1)!}{(n-1)!} (v(T \cup i) - v(T)).$$

The classical Shapley Theorem characterizes Shapley values as follow:

**Theorem 5.1** (Shapley’s Theorem, see for instance [Web88]). Let $(n, v)$ belong to a cone $\mathcal{I}$ of cooperative games containing the carrier games $\mathcal{C}$ and $\hat{\mathcal{C}}$. Assume that if $v \in \mathcal{I}$, then the permuted game $\pi \cdot v$ is also in $\mathcal{I}$ for every permutation $\pi$ of $[n]$. Let $\phi$ be a group value.

If each $\phi_j$ is a linear function that satisfies the dummy axiom and the monotonicity axiom and if the symmetric and the efficiency axioms hold for the groups value $\phi$, then for every cooperative game $(n, v)$ in the domain of $\phi$ and every $i$ in $[n]$,

$$\phi_i(v) = \text{Shapley}_i(v).$$

In [BDJLL01], they characterize when in individual value can be written as weighted sum of Shapley values defined on the bases of a matroids.

All these results together allow us to generalize the main theorem of [BDJLL01] to pure simplicial complex.

**Theorem 5.2.** Let $\Delta$ be a pure simplicial complex on $n$ vertices and and let $\phi_i$ be the individual value for a player $i$. The group value $\phi$ is reducible and it decomposes as the weighted sum of Shapley values,

$$\phi_i(v) = \sum_{F \in \text{Facets}_\Delta i} P(F) \text{Shapley}_F(v|_F)$$

if and only if each $\phi_i$ is a linear function that satisfies the $w^P(i)$-dummy axiom and the group value fulfills the following two axioms:

- **Substitution for carrier games**: For every coalition $T$ and for every pair of players, one has $\phi_i(v_T) = \phi_j(v_T)$;
- **Probabilistic efficiency**: For every cooperative game $(\Delta, v)$, $\sum_i \phi_i(v) = \sum_{F \in \text{Facets}_\Delta} P(F) v(F)$.

**Proof.** Part of the characterization, follows form the ones in Theorem 2.1 and Theorem 3.1.

The Substitution for carrier games axioms imposes that locally each individual value has to be of a Shapley one with maximal facet of the same cardinality. Finally, the Probabilistic efficiency provides the requested weighted sum. $\square$

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