Local-in-Time Existence and Uniqueness of Solutions
to the Prandtl Equations by Energy Methods

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Abstract
We prove local existence and uniqueness for the two-dimensional Prandtl system in weighted Sobolev spaces under Oleinik’s monotonicity assumption. In particular we do not use the Crocco transform or any change of variables. Our proof is based on a new nonlinear energy estimate for the Prandtl system. This new energy estimate is based on a cancellation property that is valid under the monotonicity assumption. To construct the solution, we use a regularization of the system that preserves this nonlinear structure. This new nonlinear structure may give some insight into the convergence properties from the Navier-Stokes system to the Euler system when the viscosity goes to 0. © 2015 Wiley Periodicals, Inc.
1 Introduction

The zero-viscosity limit of the incompressible Navier-Stokes system that is in a bounded domain, with a Dirichlet boundary condition, is one of the most challenging open problems in fluid mechanics. This is due to the formation of a boundary layer that appears because we cannot impose the same Dirichlet boundary condition for the Euler equations. This boundary layer satisfies formally the Prandtl system. Indeed, in 1904, Prandtl [32] suggested that there exists a thin layer called the boundary layer, where the solution $\tilde{u}$ undergoes a sharp transition from a solution to the Euler system to the no-slip boundary condition $\tilde{u} = 0$ on $\partial \Omega$ of the Navier-Stokes system. In other words, Prandtl proved formally that the solution $\tilde{u}$ of the Navier-Stokes system can be written as $\tilde{u} = \tilde{U} + \tilde{u}_{BL}$, where $\tilde{U}$ solves the Euler system with $\tilde{U} \cdot n = 0$ on the boundary and $\tilde{u}_{BL}$ is small except near the boundary. In rescaled variables $\tilde{U} + \tilde{u}_{BL}$ solves the Prandtl system. When studying this problem, there are at least three main questions:

(a) the local well-posedness of the Prandtl system;
(b) proving the convergence of solutions of the Navier-Stokes system towards a solution of the Euler system;
(c) the justification of the boundary layer expansion.

In full generality, these questions are still open except in the analytic case where (a)–(c) can be proved [21, 22, 33, 34]. See also the recent work [23] for the case that the initial vorticity is located away from the boundary.

Concerning (a), the main existence result is due to Oleinik, who proved the local existence for the Prandtl system [29, 30] under a monotonicity assumption and using the Crocco transform (see also [31]). These solutions can be extended as global weak solutions if the pressure gradient is favorable ($\partial_x p \leq 0$) [40, 41]. However, E and Engquist [7] proved a blowup result for the Prandtl system for some special type of initial data. More recently Gérard-Varet and Dormy [8] proved ill-posedness for the linearized Prandtl equations around a nonmonotonic shear flow (see also [10, 12–15]). We also point out that the method developed in this paper was applied to prove the well-posedness for Prandtl equations in other classes of data in [9, 20] very recently.

Concerning (b), the main result is a convergence criterion due to Kato [18] that basically says that convergence is equivalent to the fact that there is no dissipation in a very thin layer (of size $O(\nu)$ where $\nu$ is the viscosity). This criterion was extended in different directions (see [5, 19, 36, 38]). Also, in [24], it is proved that the convergence holds if the horizontal viscosity goes to 0 slower than the vertical one. It is worth noting that the Prandtl system is the same in this case.

Concerning (c), there is a negative result by Grenier [11], who proves that the expansion does not hold in $W^{1, \infty}$. Of course, this does not prevent (b) from holding.

There are many review papers about the inviscid limit of the Navier-Stokes system in a bounded domain and the Prandtl system from different aspects (see [4, 6]...
Let us also mention that when considered in the whole space \([17, 26, 35]\) or with other boundary conditions such as the Navier boundary condition \([2, 16, 27, 39]\) or incoming flow \([37]\), the convergence problem becomes simpler since there is no boundary layer or the boundary layer is stable.

The prime objective of this paper is to prove the local existence and uniqueness for the two-dimensional Prandtl system under Oleinik’s monotonicity assumption in certain weighted energy spaces without using the Crocco transform or any change of variables. A precise statement will be provided in Section 2. In addition to giving a very simple understanding of the monotonicity assumptions, our result may give us a better understanding about questions (b) and (c) since it is given in physical space. Nevertheless, we are still not able to use our new nonlinear energy to study the convergence problem (b) or (c). In spirit this paper is similar to our previous paper about the hydrostatic Euler equations \([28]\), where we gave a proof of existence and uniqueness in physical space under a convexity assumption of the profile. The previous known proof of Brenier \([3]\) uses semi-Lagrangian coordinates and requires more assumptions on the initial data.

Finally, let us mention that the new preprint \([1]\) also considers the existence for the Prandtl system in physical space. The methods of proof are very different.

Let us end this introduction by outlining the structure of this paper. In Section 2 we will state our main result, that is, Theorem 2.2. Explanations of our approach and approximate scheme will be provided in Sections 3 and 4. Assuming the solvability of approximate systems, we will derive our new weighted a priori estimates in Section 5. Using these weighted estimates, we will complete the proof of our main theorem, Theorem 2.2, in Section 6. In Section 7 we will solve the approximate systems. For the sake of self-containedness, we will also provide several elementary proofs and computations in the appendices.

## 2 Main Result

In this section we will first introduce the Prandtl equations and then describe our solution spaces as well as our main result. Main difficulties and a brief explanation of our approach will be given in Section 5.

Throughout this paper, we are concerned with the two-dimensional Prandtl equations in a periodic domain \(\mathbb{T} \times \mathbb{R}^+ := \{(x, y) : x \in \mathbb{R}, 0 \leq y < +\infty\}:

\begin{equation}
\begin{cases}
\partial_t u + u \partial_x u + v \partial_y u = \partial_y^2 u - \partial_x p & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\
\partial_x u + \partial_y v = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\
|u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+, \\
u|_{y=0} = v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T}, \\
\lim_{y \to +\infty} u(t, x, y) = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T},
\end{cases}
\end{equation}

where the velocity field \((u, v) := (u(t, x, y), v(t, x, y))\) is an unknown, and the initial data \(u_0 := u_0(x, y)\) and the outer flow \(U := U(t, x)\) are given and satisfy

\[25\).
the compatibility conditions:
\[(2.2) \quad u_0|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} u_0 = U|_{t=0}.\]

Furthermore, the given scalar pressure \( p := p(t, x) \) and the outer flow \( U \) satisfy the well-known Bernoulli’s law:
\[(2.3) \quad \partial_t U + U \partial_x U = -\partial_x p.\]

In this work, we will consider system (2.1) under Oleinik’s monotonicity assumption:
\[(2.4) \quad \omega := \partial_y u > 0.\]

Under this hypothesis, one must further assume \( U > 0 \).

Let us first introduce the function space in which the Prandtl equations (2.1) will be solved. Denoting the vorticity \( \omega := \partial_y u \), we define the space \( H^{s,\gamma}_{\sigma,\delta} \) for \( \omega \) by
\[H^{s,\gamma}_{\sigma,\delta} := \left\{ \omega : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R} : \|\omega\|_{H^{s,\gamma}} < +\infty, (1 + y)^\gamma \omega \geq \delta, \text{ and } \sum_{|\alpha| \leq 2} |(1 + y)^{\sigma+\alpha_2} D^\sigma \omega|^2 \leq \frac{1}{\delta^2} \right\}
\]
where \( s \geq 4, \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, 1), D^\sigma := \partial_x^{\alpha_1} \partial_y^{\alpha_2} \), and the weighted \( H^{s,\gamma} \) norm \( \|\cdot\|_{H^{s,\gamma}} \) is defined by
\[(2.5) \quad \|\omega\|^2_{H^{s,\gamma}} := \sum_{|\alpha| \leq s} \|(1 + y)^{\gamma+\alpha_2} D^\sigma \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2.
\]

Here, the main idea is adding an extra weight \((1 + y)\) for each \( y \)-derivative. This corresponds to the weight \( \frac{1}{y} \) in the Hardy-type inequality. Furthermore, we also denote
\[H^{s,\gamma}_{\sigma,\delta} := \left\{ \omega : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R} : \|\omega\|_{H^{s,\gamma}} < +\infty \right\}.
\]

**Remark 2.1** (Requirement: \( \sigma > \gamma + \frac{1}{2} \)). If \( \sigma \leq \gamma + \frac{1}{2} \), then one may check that \( H^{s,\gamma}_{\sigma,\delta} \) is an empty set. Thus, we must have the hypothesis \( \sigma > \gamma + \frac{1}{2} \).

Now, we can state our main result:

**Theorem 2.2** (Local \( H^{s,\gamma}_{\sigma,\delta} \) Existence and Uniqueness to the Prandtl Equations (2.1)). Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \) and \( \delta \in (0, \frac{1}{2}) \). For simplicity we suppose that the outer flow \( U \) satisfies
\[(2.6) \quad \sup_t \|U\|_{s+9,\infty} := \sup_t \sum_{l=0}^{[\frac{s+9}{2}]} \|\partial_t^l U\|_{W^{s-2l+9,\infty}(\mathbb{T})} < +\infty.
\]

\(^1\) The regularity hypothesis on the outer flow \( U \) is obviously not optimal in the viewpoint of our a priori weighted energy estimates. One may further loosen the regularity requirement on \( U \) by applying other approximate schemes. We leave this to the interested reader.
Assume that \( u_0 - U \big|_{t=0} \in H^{s,\gamma-1} \) and the initial vorticity \( \omega_0 := \partial_y u_0 \in H^{s,\gamma}_{\sigma,2\delta} \).
In addition, when \( s = 4 \), we further assume that \( \delta > 0 \) is chosen small enough such that
\[
\|\omega_0\|_{H^{s,\gamma}_{\sigma,2\delta}} \leq C \delta^{-1}
\]
where the norm \( \|\cdot\|_{H^{s,\gamma}_{\sigma,2\delta}} \) will be defined in (3.1) and \( C \) is a universal constant. Then there exist a time \( T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}_{\sigma,2\delta}}, U) > 0 \) and a unique classical solution \( (u, v) \) to the Prandtl equations \((2.1)\) such that
\[
u - U \in L^\infty([0, T]; H^{s,\gamma-1}) \cap C([0, T]; H^s - w)
\]
and the vorticity
\[
\omega := \partial_y u \in L^\infty([0, T]; H^{s,\gamma}_{\sigma,2\delta}) \cap C([0, T]; H^s - w),
\]
where \( H^s - w \) is the space \( H^s \) endowed with its weak topology.

Furthermore, all estimates stated in Propositions 5.3 and 6.1 hold for the vorticity \( \omega \) as well.

Remark 2.3 \((U \equiv \text{const})\). When the outer flow \( U \) is a constant, one may show that the life span \( T \) stated in Theorem 2.2 is independent of \( U \). See Remark 6.3 for the details.

The proof of Theorem 2.2 is based on our new weighted energy estimates (see Proposition 5.3 and Section 5.1), which rely on a nonlinear cancellation property (see Section 5.1 and in particular, the weighted \( L^2 \) estimate for \( g_x^s \)) that holds under Oleinik’s monotonicity assumption \((2.4)\). An outline of our proof will be given in Sections 5 and 6 and the detailed analysis will be provided in Sections 7 through 7.

Before we proceed, let us comment on our notation. Throughout this paper, all constants \( C \) may be different in different lines. Subscript(s) of a constant illustrates the dependence of the constant, for example, \( C_s \) is a constant depending on \( s \) only.

### 3 Difficulties and Outline of Our Approach

The aim of this section is to explain main difficulties of proving Theorem 2.2 as well as our strategies for overcoming them. Let us begin by stating the main difficulties as follows.

In order to solve the Prandtl equations \((2.1)\) in certain \( H^s \)-spaces, we have to overcome the following three difficulties:

(i) the vertical velocity \( v := -\partial_x^{-1} \partial_x u \) creates a loss of \( x \)-derivative, so the standard energy estimates do not apply;

(ii) the unboundedness of the underlying physical domain \( T \times \mathbb{R}^+ \) allows the growth of certain quantities at \( y = +\infty \) even if the solution is smooth or bounded in \( H^s \);
(iii) the lack of higher-order boundary conditions at \( y = 0 \) prevents us from applying the integration by parts in the \( y \)-variable, but it is a standard and crucial step to deal with the operator \( \partial_t - \partial_y^2 \).

Indeed, difficulty (i) is the major problem for the Prandtl equations \((2.1)\), and it explains why there are just a few existence results in the literature. The key ingredient of the current work is to develop an \( H^s \) control by considering a new \( H^s \)-norm (see \((3.1)\) below) that can avoid the regularity loss created by \( v \). Difficulty (ii) is somewhat based on the fact that the Poincaré inequality does not hold for the unbounded domain \( T \times \mathbb{R}^+ \). However, one may overcome this technical problem by replacing the Poincaré-type inequalities by Hardy-type inequalities. This is our main reason for adding the weight \( 1.1 \cdot \partial_s^y \) for each \( y \)-derivative to our \( H^s \) energies \((2.5) \) and \((3.1) \). Difficulty (iii) seems to be an obstacle, but it is not. A reconstruction argument for the higher-order boundary conditions can fix this technical difficulty when \( s \) is even; see Lemma \( 5.9 \) for more details.

Now, let us explain our new weighted energy, which is the main novelty in this paper.

Judging from the nonlinear cancellation, the weighted norm \((2.5) \) is not suitable for estimating solutions of the Prandtl equations \((2.1) \). Thus, we introduce another weighted norm for the vorticity \( \omega \), namely,

\[
\| \omega \|_{H^s_{g} (T \times \mathbb{R}^+)}^2 := \|(1+y)^s g_s\|_{L^2(T \times \mathbb{R}^+)}^2 + \sum_{|\alpha| \leq s} \|(1+y)^{s+|\alpha|} D^\alpha \omega\|_{L^2(T \times \mathbb{R}^+)}^2
\]

where

\[
g_s := \partial_s^y \omega - \frac{\partial_y \omega}{\omega} \partial_s^y (u-U) \quad \text{and} \quad u(t,x,y) := \int_0^y \omega(t,x,y) d\tilde{y}
\]

provided that \( \omega := \partial_y u > 0 \). The difference between norms \((2.5) \) and \((3.1) \) is that we replace the weighted \( L^2 \)-norm of \( \partial_s^y \omega \) by that of \( g_s \), which is a better quantity because \( g_s \) can avoid the loss of \( x \)-derivative (i.e., difficulty (i) above) by the nonlinear cancellation. See Section \( 5.1 \) and, in particular, the weighted \( L^2 \) estimate for \( g_s \) for further explanation. The norm \((3.1) \) is important and new.

The first important observation is that as long as \( \omega \in H^s_{g} \), we can show that the new weighted norm \((3.1) \) is almost equivalent to the weighed \( H^s \)-norm \((2.5) \), that is,

\[
\| \omega \|_{H^s_{g}} \lesssim \| \omega \|_{H^s_{g} \circ \delta} + \| u - U \|_{H^{s+1}} \lesssim \| \omega \|_{H^s_{g}} + \| \partial_x^s U \|_{L^2(T)}
\]

provided that \( \omega = \partial_y u, u|_{y=0} = 0 \), and \( \lim_{y \to +\infty} u = U \). The proof of \((3.2) \) is elementary and will be given in Appendix \( A \). In the spirit of \((3.2) \), we will estimate \( \| \omega \|_{H^s_{g}} \) instead of \( \| \omega \|_{H^s_{g} \circ \delta} \).

The second important observation is that due to the nonlinear cancellation, the loss of \( x \)-derivative is avoided by using the norm \( \| \cdot \|_{H^s_{g}} \), so one can simply
derive a priori energy estimates for \( \omega \) by applying the standard energy methods. These weighted energy estimates can indeed be extended to \( \omega^\varepsilon \) := \( \partial_y u^\varepsilon \), which is the regularized vorticity of the regularized Prandtl equations (4.1) below, because the regularization (4.1) preserves the nonlinear structure of the original Prandtl equations (2.1). See Section 5.1 for the detailed analysis.

Once we have obtained the weighted energy estimates, it remains to derive weighted \( L^\infty \) controls on the lower-order derivatives of \( \omega \) so that we can close our estimates in the function space \( H^{s, \gamma} \). The derivation of these \( L^\infty \) estimates is standard: “viewing” the evolution equations of the lower-order derivatives as “linear” parabolic equations with coefficients involving higher-order terms that can be bounded by the weighted energies, we can obtain the desired estimates by the classical maximum principles since we have already controlled the weighted energies. These weighted \( L^\infty \)-estimates can also be extended to the regularized vorticity \( \omega^\varepsilon \); see Section 5.2 for further details.

In order to prove the existence, we will construct an approximate scheme that keeps the a priori estimates described above. Due to the nonlinear cancellation, our a priori estimates are complicated in a certain sense, so the construction of the approximate scheme is tricky. An outline of this construction will be given in Section 4.

For the uniqueness, it is an immediate consequence of an \( L^2 \) comparison principle (see Proposition 6.4), whose proof relies on a nonlinear cancellation that is similar to the one applied in the energy estimates. See Section 6.2 for the details.

### 4 Approximate Scheme

The main purpose of this section is to outline the approximate systems that we apply to prove the existence. Since our weighted \( H^{s} \) a priori estimates are somewhat more nonlinear than usual, the approximate scheme is slightly more complicated.

Our approximate scheme has three different levels and will be explained as follows.

The first approximation of (2.1) is the regularized Prandtl equations: for any \( \varepsilon > 0 \),

\[
\begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon &= \varepsilon^2 \partial_x^2 u^\varepsilon + \partial_y^2 u^\varepsilon - \partial_x p^\varepsilon \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0 \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\
u^\varepsilon |_{t=0} &= u_0 \quad \text{on } \mathbb{T} \times \mathbb{R}^+, \\
v^\varepsilon |_{y=0} &= v^\varepsilon |_{y=0} = 0 \quad \text{on } [0, T] \times \mathbb{T}, \\
\lim_{y \to +\infty} u^\varepsilon (t, x, y) &= U \quad \text{for all } (t, x) \in [0, T] \times \mathbb{T},
\end{align*}
\]

where \( p^\varepsilon \) and \( U \) satisfy a regularized Bernoulli’s law:

\[
\partial_t U + U \partial_x U = \varepsilon^2 \partial_x^2 U - \partial_x p^\varepsilon.
\]
Or equivalently, the regularized vorticity \( \omega^\varepsilon := \partial_y u^\varepsilon \) satisfies the following regularized vorticity system: for any \( \varepsilon > 0 \),

\[
\begin{aligned}
\partial_t \omega^\varepsilon + u^\varepsilon \partial_x \omega^\varepsilon + v^\varepsilon \partial_y \omega^\varepsilon &= \varepsilon^2 \partial_x^2 \omega^\varepsilon + \partial_y^2 \omega^\varepsilon \quad \text{in} \ [0, T] \times \mathbb{T} \times \mathbb{R}^+ , \\
\omega^\varepsilon|_{t=0} &= \omega_0 := \partial_y u_0 \quad \text{on} \ T \times \mathbb{R}^+ , \\
\partial_y \omega^\varepsilon|_{y=0} &= \partial_x p^\varepsilon \quad \text{on} \ [0, T] \times \mathbb{T} ,
\end{aligned}
\]

where the velocity field \((u^\varepsilon, v^\varepsilon)\) is given by

\[
\begin{aligned}
u^\varepsilon(t, x, y) &= -\int_0^y \partial_x u^\varepsilon(t, x, \tilde{y}) d \tilde{y} , \\
u^\varepsilon(t, x, y) &= -\int_{-1}^y \partial_x u^\varepsilon(t, x, \tilde{y}) d \tilde{y} ,
\end{aligned}
\]

The main idea of this approximation is adding the viscous terms \( \varepsilon^2 \partial_x^2 u^\varepsilon \) and \( \varepsilon^2 \partial_y^2 \omega^\varepsilon \) to avoid the loss of \( x \)-derivative. The advantage of this regularization is that our new weighted \( \mathcal{H}^s \) and \( \mathcal{L}^1 \) a priori estimates (see Proposition 5.3) also hold for \( \omega^\varepsilon \), and it is the main reason why we can derive the uniform (in \( \varepsilon \)) estimates in Section 5. The price that we pay is the appearance of extra terms \( \partial_x^2 \omega^\varepsilon \), \( \partial_y^2 \omega^\varepsilon \), \( \partial_x \partial_y \omega^\varepsilon \), and \( \partial_x^2 \omega^\varepsilon \) during the estimation, but these terms can be controlled in the function space \( \mathcal{C}([0, T]; \mathcal{H}^{s,\delta}) \).

Before going to the next level, we should also emphasize that replacing Bernoulli’s law (2.3) by the regularized Bernoulli’s law (4.2) is crucial here; otherwise the conditions \( u^\varepsilon|_{y=0} = 0 \) and \( \lim_{y \to +\infty} u^\varepsilon = U \) cannot be satisfied simultaneously. Although the approximate system (4.3)–(4.4) seems to be nice, its existence in the function space \( \mathcal{H}^{s,\delta} \) is not obvious, so we will further approximate it by the next approximate system.

The second level of approximation is the truncated and regularized vorticity system: for any \( \varepsilon > 0 \) and \( R \geq 1 \),

\[
\begin{aligned}
\partial_t \omega_R + \chi_R \{ u_R \partial_x \omega_R + v_R \partial_y \omega_R \} &= \varepsilon^2 \partial_x^2 \omega_R + \partial_y^2 \omega_R \quad \text{in} \ [0, T] \times \mathbb{T} \times \mathbb{R}^+ , \\
\omega_R|_{t=0} &= \omega_0 := \partial_y u_0 \quad \text{on} \ T \times \mathbb{R}^+ , \\
\partial_y \omega_R|_{y=0} &= \partial_x p^\varepsilon \quad \text{on} \ [0, T] \times \mathbb{T} ,
\end{aligned}
\]

where the velocity field \((u_R, v_R)\) is given by

\[
\begin{aligned}
u_R(t, x, y) &= -\int_0^y \partial_x u_R(t, x, \tilde{y}) d \tilde{y} , \\
u_R(t, x, y) &= -\int_{-1}^y \partial_x u_R(t, x, \tilde{y}) d \tilde{y} ,
\end{aligned}
\]

Here, \( p^\varepsilon \) and \( U \) still satisfy the regularized Bernoulli’s law (4.2). The cutoff function \( \chi_R \) is defined by \( \chi_R(y) := \chi \left( \frac{y}{R} \right) \) where \( \chi \in C^\infty_c([0, +\infty)) \) satisfies the
following properties:

\[(4.7) \quad 0 \leq \chi \leq 1, \quad \chi|_{[0,1]} \equiv 1, \quad \text{supp} \chi \subseteq [0, 2], \quad \text{and} \quad -2 \leq \chi' \leq 0.\]

The main disadvantage of approximate system (4.5)–(4.6) is that the truncation on the convection term \(u_R \partial_x \omega_R + v_R \partial_y \omega_R\) destroys the boundary condition \(u_R|_{y=0} = 0\) as well as our weighted \(H^s\) a priori estimate. However, it still keeps the weighted \(L^\infty\) controls.

To compensate for the lack of our new weighted \(H^s\) estimates, one may apply the standard \(H^s\) energy estimates because the system (4.5)–(4.6) does not have the problem of \(x\)-derivative loss. These estimates depend on \(\epsilon\) but not on \(R\). Thus, we can pass to the limit \(R \to +\infty\) for the solution of (4.5)–(4.6) to that of (4.3)–(4.4). The reason for doing this approximation is to prepare for our next approximate system.

The third level of approximation is the linearized, truncated, and regularized vorticity system: for any \(\epsilon > 0\), \(R \geq 1\), and \(n \in \mathbb{N}\),

\[
(4.8) \quad \begin{cases}
\partial_t \omega^{n+1} + \chi R \{ u^n \partial_x \omega^n + v^n \partial_y \omega^n \} = \epsilon^2 \partial_x^2 \omega^{n+1} + \partial_y^2 \omega^{n+1} & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+,
\omega^{n+1}|_{t=0} = \omega_0 := \partial_y u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+,
\partial_y \omega^{n+1}|_{y=0} = \partial_x p^n & \text{on } [0, T] \times \mathbb{T},
\end{cases}
\]

where the velocity field \((u^n, v^n)\) is given by

\[
(4.9) \quad \begin{cases}
u^n(t, x, y) := U - \int_y^{+\infty} \omega^n(t, x, \tilde{y}) d\tilde{y}, \\
u^n(t, x, y) := -\int_0^y \partial_x u^n(t, x, \tilde{y}) d\tilde{y}.
\end{cases}
\]

In other words, (4.8)–(4.9) is a linearization of (4.5)–(4.6).

The main advantage of the iterative scheme (4.8)–(4.9) is that (4.8) is an inhomogeneous heat equation in \(\mathbb{T} \times \mathbb{R}^+\), so its explicit solution formula can be obtained by the method of reflection. Using the explicit solution formula and the fact that \(\chi R \{ u^n \partial_x \omega^n + v^n \partial_y \omega^n \}\) has compact support, one may prove that there exists a uniform (in \(n\)) life span \(T > 0\) for the approximate sequence \(\{\omega^n\}_{n \in \mathbb{N}}\) provided that \(\omega_0 \in H^{s, \gamma} \sigma_{2,\delta}\). This gives us a starting point so that we can solve the approximate systems and derive estimates.

Solving the above approximate systems in a reverse order and deriving appropriate estimates, we can prove the existence of the Prandtl equations (2.1). Detailed analysis for solving the regularization (4.1) as well as other approximate systems (4.3)–(4.4), (4.5)–(4.6), and (4.8)–(4.9) will be given in Section 7. Assuming that \((u^\epsilon, v^\epsilon, \omega^\epsilon)\) solves (4.1)–(4.4), we will derive uniform (in \(\epsilon\)) weighted estimates in Section 5. Based on these uniform estimates, we will complete the proof of our main theorem, Theorem 2.2, in Section 6.
Uniform Estimates for the Regularized Prandtl Equations

In this section and the next we are going to complete the proof of our main theorem, Theorem 2.2, provided that the regularized Prandtl equations (4.1) are solvable. In this section we will derive uniform estimates for the regularized Prandtl equations (4.1) by using the new weighted energy (3.1) introduced in Section 3. These new a priori estimates, which will be stated in Proposition 5.3 below, are the main novelties of this paper. Then we will finish the proof of Theorem 2.2 in Section 6. After that, an outline for solving the regularized Prandtl equations (4.1) will be provided in Section 7.

Our starting point is that we can solve the regularized Prandtl equations (4.1). More precisely, let us assume Proposition 5.1, which will be shown in Section 7, below for the moment.

**PROPOSITION 5.1 (Local Existence of the Regularized Prandtl Equations).** Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, \frac{1}{2}), \) and \( \epsilon \in (0, 1]. \) If \( \omega_0 \in H^{s+12,\gamma}_{\sigma,\delta}, \) \( U \) and \( p^\epsilon \) are given and satisfy the regularized Bernoulli’s law (4.2) and the regularity assumption (2.6), then there exist a time \( T := T(s, \gamma, \sigma, \delta, \epsilon, \|\omega_0\|_{H^{s+4,\gamma}}, U) > 0 \) and a solution \( \omega^\epsilon \in C([0, T]; H^{s+4,\gamma}_{\sigma,\delta}) \cap C^1([0, T]; H^{s+2,\gamma}_{\sigma,\delta}) \) to the regularized vorticity system (4.3)–(4.4).

Furthermore, the regularized Prandtl equations (4.1) are satisfied by the velocity \( (u^\epsilon, v^\epsilon) \) defined by (4.4).

**Remark 5.2 (Initial Data).** The \( H^{s,\gamma}_{\sigma,\delta} \) functions can be approximated by \( H^{s+12,\gamma}_{\sigma,\delta} \) functions in the norm \( \| \cdot \|_{H^{s,\gamma}} \), so by the standard density argument, the hypothesis \( \omega_0 \in H^{s+12,\gamma}_{\sigma,\delta} \) can be reduced to be \( \omega_0 \in H^{s,\gamma}_{\sigma,\delta} \) in our final result.

According to Proposition 5.1, the life span \( T(s, \gamma, \sigma, \delta, \epsilon, \omega_0, U) \) of \( \omega^\epsilon \) depends on \( \epsilon, \) so the aim of this section is to remove the \( \epsilon \)-dependence by deriving uniform (in \( \epsilon \)) estimates for \( \omega^\epsilon. \) In other words, we will prove the following new a priori estimates.

**PROPOSITION 5.3 (Uniform Estimates for the Regularized Prandtl Equations).** Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, 1), \) and \( \epsilon \in [0, 1]. \) If \( \omega^\epsilon \in C([0, T]; H^{s+4,\gamma}_{\sigma,\delta}) \cap C^1([0, T]; H^{s+2,\gamma}_{\sigma,\delta}) \) and \( (u^\epsilon, v^\epsilon, \omega^\epsilon) \) solves (4.1)–(4.4), then we have the following uniform (in \( \epsilon \)) estimates:

1. **Weighted \( H^s \) Estimate:**

   \[
   \|\omega^\epsilon(t)\|_{H^s_{\epsilon,\gamma}} \leq \left\|\omega_0\|_{H^s_{\epsilon,\gamma}}^2 + \int_0^t F(\tau) d\tau \right\|^\frac{1}{2} \cdot \left\{1 - C(s,\gamma,\sigma,\delta) \left(\|\omega_0\|_{H^s_{\epsilon,\gamma}}^2 + \int_0^t F(\tau) d\tau \right)^\frac{s-2}{s} \right\}^{-\frac{1}{s-2}}
   \]  

(5.1)
as long as the quantity within the second set of braces on the right-hand side of (5.1) is positive, where the weighted norm \( \| \cdot \|_{H^s_{\gamma}} \) is defined in (3.1), the positive constant \( C_{s,\gamma,\sigma,\delta} \) depends on \( s, \gamma, \sigma, \) and \( \delta \) only, and \( F : [0, T] \to \mathbb{R}^+ \) is defined by

\[
F := C_{s,\gamma,\sigma,\delta} \left\{ 1 + \| \vartheta^2 + 1 \|_{L^\infty(T)}^4 \right\} + C_s \sum_{l=0}^{s/2} \| \partial_t^l \partial_x \rho \|_{H^{s-2l}(T)}^2.
\]

(ii) Weighted \( L^\infty \) Estimates: Define \( I(t) := \sum_{|\alpha| \leq 2} (1 + y)^\sigma \partial^\alpha \omega \partial^\alpha (t)^2. \)

For any \( s \geq 4, \)

\[
\| I(t) \|_{L^\infty(T \times \mathbb{R}^+)} \leq \max \left\{ \| I(0) \|_{L^\infty(T \times \mathbb{R}^+)}, 6C^2 \Omega(t)^2 \right\} e^{C_{s,\gamma,\sigma,\delta}(1 + G(t)) t}
\]

where the universal constant \( C \) is the same as the one in inequality (B.3), and \( \Omega \) and \( G : [0, T] \to \mathbb{R}^+ \) are defined by

\[
\Omega(t) := \sup_{[0, t]} \| \omega \|_{H^s_{\gamma}} \quad \text{and} \quad G(t) := \sup_{[0, t]} \| \omega \|_{H^s_{\gamma}} + \sup_{[0, t]} \| \vartheta^2 \|_{L^2(T)}.
\]

In addition, if \( s \geq 6, \) then we also have

\[
\| I(t) \|_{L^\infty(T \times \mathbb{R}^+)} \leq C_{s,\gamma,\sigma,\delta} \{ 1 + \Omega(t) \} \Omega(t)^2 t e^{C_{s,\gamma,\sigma,\delta}(1 + G(t)) t}.
\]

For \( s \geq 4, \) we have the following lower bound estimate:

\[
\min_{T \times \mathbb{R}^+} (1 + y)^\sigma \omega \partial^\alpha \omega (t) \geq (1 - C_{s,\gamma,\sigma,\delta} \{ 1 + G(t) \} t e^{C_{s,\gamma,\sigma,\delta}(1 + G(t)) t})
\]

\[
\cdot (\min_{T \times \mathbb{R}^+} (1 + y)^\sigma \omega_0 - C_{s,\gamma} \Omega(t) t)
\]

provided that \( \min_{T \times \mathbb{R}^+} (1 + y)^\sigma \omega_0 - C_{s,\gamma} \Omega(t) t \geq 0, \) where \( C_{s,\gamma,\sigma,\delta} \) is a positive constant depending on \( s, \gamma, \sigma, \) and \( \delta \) only.

Remark 5.4 (Two \( L^\infty \) Estimates for \( I \)). In Proposition 5.3, we stated two \( L^\infty \) controls on the quantity \( I(t) \), namely, estimates (5.3) and (5.5). Indeed, (5.5) is a better estimate within a short time, but it only holds for \( s \geq 6. \) Thanks to this better estimate, we can derive the uniform weighted \( L^\infty \) bound (6.2) without any additional assumption when \( s \geq 6. \) In contrast, we are required to impose an extra initial hypothesis (2.7) for the case \( s = 4 \) since we only have the weaker estimate (5.3) in this case. See Proposition 6.1 for the details.

Remark 5.5 (A Priori Estimates for the Prandtl Equations). When \( \epsilon = 0, \) the regularized Prandtl equations (4.1) become the Prandtl equations (2.1), and hence Proposition 5.3 also provides a priori estimates for the Prandtl equations (2.1). A similar situation occurs in Proposition 6.1 as well.

The proof of Proposition 5.3 will be given in Sections 5.1 and 5.2 below.
5.1 Weighted Energy Estimates

The objective of this subsection is to derive the uniform (in $\epsilon$) weighted $H^s$ estimate (5.1) for $\omega^\epsilon$. This estimate, which is the main novelty of this paper, includes (i) weighted $L^2$ estimates for $D^\alpha \omega^\epsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$ and (ii) a weighted $L^2$ estimate for $g^\epsilon$. We will combine these two estimates to obtain the uniform weighted energy estimate (5.1). This will complete the proof of part (i) of Proposition 5.3.

Weighted $L^2$ Estimates for $D^\alpha \omega^\epsilon$

Using the standard energy methods, we are going to derive weighted $L^2$ estimates for $D^\alpha \omega^\epsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$. It works because we are allowed to lose at least one $x$-regularity in these cases.

More specifically, we will prove the following:

**Proposition 5.6** ($L^2$ Controls on $\omega^\epsilon$).

Under the hypotheses of Proposition 5.3, we have the following estimate:

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq s, \alpha_1 \leq s - 1} \| (1 + y)^{\gamma + \sigma_2} D^\alpha \omega^\epsilon \|^2_{L^2}$$

$$\leq -\epsilon^2 \sum_{|\alpha| \leq s, \alpha_1 \leq s - 1} \| (1 + y)^{\gamma + \sigma_2} \partial_x D^\alpha \omega^\epsilon \|^2_{L^2}$$

$$- \frac{1}{2} \sum_{|\alpha| \leq s, \alpha_1 \leq s - 1} \| (1 + y)^{\gamma + \sigma_2} \partial_y D^\alpha \omega^\epsilon \|^2_{L^2}$$

$$+ C_{s,\gamma,\sigma,\delta} \| \omega^\epsilon \|_{H^{s,\gamma}} + \| \partial_x^s U \|_{L^{\infty}(T)} \| \omega^\epsilon \|_{H^{s,\gamma}}^2$$

$$+ C_{s,\gamma,\sigma,\delta} \{ 1 + \| \omega^\epsilon \|_{H^{s,\gamma}} \}^{s-2} \| \omega^\epsilon \|_{H^{s,\gamma}}^2$$

$$+ C_s \sum_{l=0}^{s/2} \| \partial_y^l \partial_x p^\epsilon \|_{H^{s-2l}(T)}^2,$$

where the positive constants $C_s$ and $C_{s,\gamma,\sigma,\delta}$ are independent of $\epsilon$.

**Remark 5.7** (Boundary Terms at $y = +\infty$). In the proof of Proposition 5.6 and that of Proposition 5.10 below, we will ignore the boundary terms at $y = +\infty$ while we are integrating by parts in the $y$-variable. Skipping these boundary terms is just for the convenience of presentation, and ignoring these technicalities is harmless. Indeed, one may deal with these boundary terms by either one of the following two methods:

(i) Since $\omega^\epsilon \in H^{s+4,\gamma}_{\sigma,\delta}$, by Proposition C.1, we have nice pointwise decays (C.1) for $\omega^\epsilon$ and its spatial derivatives. Therefore, when $\sigma$ is much larger
than $\gamma$, one may easily check that those terms that we will omit actually vanish.

(ii) As long as $\omega^\epsilon(t) \in H^{s,\gamma}_{r,\delta}$, the norm $\|\omega^\epsilon\|_{H^{s,\gamma}} < +\infty$ provides certain integrability of the underlying quantities. Thus, one may overcome the technical difficulty by first multiplying by a nice cutoff function $\chi_R(y) := \chi(y/R)$ during the estimation, and then passing to the limit $R \to +\infty$. The main advantage of this approach is that it only requires $\sigma > \gamma + 1/2$ and $\omega^\epsilon(t)$ in $H^{s,\gamma}_{r,\delta}$, but not in $H^{s+4,\gamma}_{r,\delta}$. As a demonstration, we will apply this argument in the proof of Proposition 6.4 for the reader’s convenience.

In conclusion, the proofs of Proposition 5.6 and 5.10 are absolutely correct, even if we ignore the boundary terms at $y = C_1$.

**Proof of Proposition 5.6.** Differentiating the vorticity equation (4.3) with respect to $x$ $\alpha_1$ times and $y$ $\alpha_2$ times, we obtain the evolution equation for $D^\alpha \omega^\epsilon$:

$$(5.8) \quad \{\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2\} D^\alpha \omega^\epsilon =$$

$$- \sum_{0 < \beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) \{D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon\}.$$

Multiplying (5.8) by $(1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon$, and then integrating over $\mathbb{T} \times \mathbb{R}^+$, we have

$$\frac{1}{2} \frac{d}{dt} \| (1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon \|_{L^2}^2$$

$$= \varepsilon^2 \int (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \partial_x D^\alpha \omega^\epsilon$$

$$+ \int (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon$$

$$- \int (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \{u^\epsilon \partial_x D^\alpha \omega^\epsilon + v^\epsilon \partial_y D^\alpha \omega^\epsilon\}$$

$$- \sum_{0 < \beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) \int \int (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon$$

$$\cdot \{D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon\}.$$ (5.9)

Now, we can apply integration by parts and the standard Sobolev-type estimates for trilinear forms to control the right-hand side of (5.9) as follows.

**Claim 5.8.** There exist positive constants $C_{s,\gamma}$ and $C_{s,\gamma,\sigma,\delta}$ such that for any $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$,

$$(5.10) \quad \varepsilon^2 \int (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \partial_x^2 D^\alpha \omega^\epsilon = -\varepsilon^2 \| (1 + y)^{\gamma + \alpha_2} \partial_x D^\alpha \omega^\epsilon \|_{L^2}^2.$$
\[ \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \partial_y^2 D^\alpha \omega^\epsilon \leq -\frac{3}{4}(1 + y)^{\gamma + \alpha_2} \partial_x D^\alpha \omega^\epsilon \|_{L^2_T} - \int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} + C_{s,y} \|\omega^\epsilon\|_{H^{s,y}}^2. \]  

(5.11)

\[ \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \{u^\epsilon \partial_x D^\alpha \omega^\epsilon + v^\epsilon \partial_y D^\alpha \omega^\epsilon\} \leq \sum_{0 < \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\epsilon \quad \cdot \{D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon\} \]  

(5.12)

\[ \leq C_{s,y,\sigma,\delta} \|\omega^\epsilon\|_{H^{s,y}} + \| \partial_x U \|_{L^2} \|\omega^\epsilon\|_{H^{s,y}}^2. \]  

(5.13)

Assuming Claim 5.8, which will be shown later, for the moment we can apply inequalities (5.10)–(5.13) to the equality (5.9) and obtain

\[ \frac{1}{2} \frac{d}{dt} \|1 + y\|^{\gamma + \alpha_2} D^\alpha \omega^\epsilon \|^2_{L^2_T} \leq -\epsilon^2 \|1 + y\|^{\gamma + \alpha_2} \partial_x D^\alpha \omega^\epsilon \|^2_{L^2_T} - \frac{3}{4} \|1 + y\|^{\gamma + \alpha_2} \partial_y D^\alpha \omega^\epsilon \|^2_{L^2_T} \]  

(5.14)

\[ - \int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} + C_{s,y} \|\omega^\epsilon\|_{H^{s,y}}^2 + C_{s,y,\sigma,\delta} \|\omega^\epsilon\|_{H^{s,y}} + \| \partial_x U \|_{L^\infty} \|\omega^\epsilon\|_{H^{s,y}}^2. \]  

(5.15)

When \(|\alpha| \leq s - 1\), we can apply the simple trace estimate

\[ \int_T |f| \, dx \bigg|_{y=0} \leq C \int_0^1 \int_T |f| \, dx \, dy + \int_0^1 \int_T |\partial_y f| \, dx \, dy \]  

(5.15)

to control the boundary integral \( \int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} \) as follows:

\[ \int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} \leq \]  

(5.16)

\[ \frac{1}{12} \|1 + y\|^{\gamma + \alpha_2 + 1} \partial_y^2 D^\alpha \omega^\epsilon \|^2_{L^2_T} + C \|\omega^\epsilon\|_{H^{s,y}}^2. \]  

However, when \(|\alpha| = s\), a main difficulty arises: the order of \( \partial_y D^\alpha \omega^\epsilon \big|_{y=0} \) is too high so that we cannot control the boundary integral \( \int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} \)
by the simple trace estimate \( (5.15) \). In order to make use of \( (5.15) \), we must reduce the order of the problematic term \( \partial_y D^\alpha \omega^\epsilon \mid_{y=0} \). When \( s \) is even, this can be done by a boundary reduction argument as follows.

At this moment, let us state without proof the following boundary reduction lemma, which will be proven later.

**Lemma 5.9 (Reduction of Boundary Data).** Under the hypotheses of Proposition 5.3 we have at the boundary \( y = 0 \),
\[
\begin{align*}
\frac{\partial_y \omega^\epsilon}{y=0} &= \partial_x p^\epsilon, \\
\frac{\partial^2_y \omega^\epsilon}{y=0} &= (\partial_t - \epsilon^2 \partial_x^2) \partial_x p^\epsilon + \omega^\epsilon \partial_x \omega^\epsilon \mid_{y=0}.
\end{align*}
\]

For any \( 2 \leq k \leq \frac{s}{2} \), there are some constants \( C_{k, \ell, \rho^1, \rho^2, \ldots, \rho^j} \), which do not depend on \( \epsilon \) or \( (u^\epsilon, v^\epsilon, \omega^\epsilon) \), such that
\[
\frac{\partial^{2k+1}_y \omega^\epsilon}{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x p^\epsilon
\]
\[
+ \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max(2, k-1)} \sum_{\rho \in A^j_{\ell, l}} C_{k, \ell, \rho^1, \rho^2, \ldots, \rho^j} \prod_{i=1}^j D^{\rho^i} \omega^\epsilon \mid_{y=0}
\]
where \( A^j_{k, \ell} := \{ \rho := (\rho^1, \rho^2, \ldots, \rho^j) \in \mathbb{N}^{2j} : 3 \sum_{i=1}^{j} \rho^i_1 + \sum_{i=1}^{j} \rho^i_2 = 2k + 4\ell + 1, \sum_{i=1}^{j} \rho^i_1 \leq k + 2\ell - 1, \sum_{i=1}^{j} \rho^i_2 \leq 2k - 2\ell - 2, \text{ and } |\rho^i| \leq 2k - \ell - 1 \text{ for all } i = 1, 2, \ldots, j \} \).

Now, we are going to apply Lemma 5.9 to control the boundary integral
\[
\int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \ dx \mid_{y=0}
\]
for \( |\alpha| = s \) with \( 0 \leq \alpha_1 \leq s - 1 \) in the following two cases:

**Case 1.** \( \alpha_2 \) is even. When \( \alpha_2 := 2k \) for some \( k \in \mathbb{N} \), we can apply boundary reduction Lemma 5.9 to \( \partial_y D^\alpha \omega^\epsilon \mid_{y=0} \) and obtain
\[
\int_T D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \ dx \mid_{y=0}
\]
\[
= \int_T D^\alpha \omega^\epsilon (\partial_t - \epsilon^2 \partial_x^2)^{k+1} \partial_x^2 p^\epsilon \ dx \mid_{y=0}
\]
\[
+ \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max(2, k-1)} \sum_{\rho \in A^j_{\ell, l}} C_{k, \ell, \rho^1, \rho^2, \ldots, \rho^j} \int_T D^\alpha \omega^\epsilon \partial_x^{\rho^1} \left( \prod_{i=1}^j D^{\rho^i} \omega^\epsilon \right) \ dx \mid_{y=0}.
\]

According to the definition of \( A^j_{k, \ell} \), one may check by using the restrictions on indices that the largest possible order for \( \partial_x^{\rho^1} D^{\rho^j} \omega^\epsilon \) is \( s - 1 \) and at most one of \( \partial_x^{\rho^1} D^{\rho^j} \omega^\epsilon \) can attain the order \( s - 1 \); namely, the orders of other terms are smaller.
than or equal to \( s - 2 \). Therefore, we can apply the simple trace estimate (5.15) and Proposition B.3 to the identity (5.19) to obtain

\[
\left| \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \right|_{y=0} \leq \frac{1}{12} \| (1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha \omega^\epsilon \|^2_{L^2} \\
+ C_s \sum_{l=0}^{s/2} \| \partial^l_t \partial_x P^\epsilon \|^2_{H^{s-2l}(\mathbb{T})} \\
+ C_{s,\gamma,\alpha,\delta} \{ 1 + \| \omega^\epsilon \|_{H^{s,\gamma}_k} \}^{s-2} \| \omega^\epsilon \|^2_{H^{s,\gamma}_k}.
\] (5.20)

**Case 2.** \( \alpha_2 \) is odd. When \( \alpha_2 := 2k + 1 \) for some \( k \in \mathbb{N} \), since \( \alpha_1 + \alpha_2 = s \) is assumed to be even, we know that \( \alpha_1 \geq 1 \). Using integration by parts in the \( x \)-variable, we have

\[
\int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \bigg|_{y=0} = -\int_{\mathbb{T}} \partial_x D^\alpha \omega^\epsilon \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega^\epsilon \\cdot dx \bigg|_{y=0}.
\] (5.21)

The term \( \partial_x D^\alpha \omega^\epsilon \big|_{y=0} = \partial_x^{\alpha_1+1} \partial_y^{2k+1} \omega^\epsilon \big|_{y=0} \) has an odd number of \( y \)-derivatives, and hence, we can apply the boundary reduction Lemma 5.9 to \( \partial_x D^\alpha \omega^\epsilon \big|_{y=0} \) to further reduce the order of the right hand side of (5.21). Similar to Case 1, we can further apply the simple trace estimate (5.15) and Proposition B.3 to eventually obtain the following estimate:

\[
\left| \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \, dx \right|_{y=0} \leq \frac{1}{12} \| (1 + y)^{\gamma + \alpha_2} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega^\epsilon \|^2_{L^2} \\
+ C_s \sum_{l=0}^{s/2} \| \partial^l_t \partial_x P^\epsilon \|^2_{H^{s-2l}(\mathbb{T})} \\
+ C_{s,\gamma,\alpha,\delta} \{ 1 + \| \omega^\epsilon \|_{H^{s,\gamma}_k} \}^{s-2} \| \omega^\epsilon \|^2_{H^{s,\gamma}_k}.
\] (5.22)

Finally, combining estimates (5.14), (5.16), (5.20), and (5.22), and summing over \( \alpha \), we prove inequality (5.7). \( \square \)

In order to complete the proof of Proposition 5.6, it remains to show Claim 5.8 and the boundary reduction Lemma 5.9. Let us first prove Claim 5.8 as follows.

**Proof of Claim 5.8.**

The equality (5.10) follows immediately from an integration by parts in the \( x \)-variable.
Proof of (5.11). Integrating by parts in the $y$-variable (cf. Remark 5.7), we have

$$\iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\varepsilon \partial_y D^\alpha \omega^\varepsilon$$

$$= -\| (1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha \omega^\varepsilon \|^2_{L_2} - \int T D^\alpha \omega^\varepsilon \partial_y D^\alpha \omega^\varepsilon \, dx \Big|_{y=0}$$

$$- 2(\gamma + \alpha_2) \iint (1 + y)^{2\gamma + 2\alpha_2 - 1} D^\alpha \omega^\varepsilon \partial_y D^\alpha \omega^\varepsilon$$

$$\leq -\frac{3}{4} \| (1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha \omega^\varepsilon \|^2_{L_2} - \int T D^\alpha \omega^\varepsilon \partial_y D^\alpha \omega^\varepsilon \, dx \Big|_{y=0}$$

$$+ C_{x,y} \| \omega^\varepsilon \|^2_{H_{x,y}}.$$ 

which is inequality (5.11).

Proof of (5.12). Integrating by parts (cf. Remark 5.7) and using $\partial_x u^\varepsilon + \partial_y v^\varepsilon = 0$, we have

$$\iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\varepsilon \{ u^\varepsilon \partial_x D^\alpha \omega^\varepsilon + v^\varepsilon \partial_y D^\alpha \omega^\varepsilon \} =$$

$$- (\gamma + \alpha_2) \iint (1 + y)^{2\gamma + 2\alpha_2 - 1} v^\varepsilon |D^\alpha \omega^\varepsilon|^2.$$ 

This and inequality (B.9) imply inequality (5.12).

Proof of (5.13). Using the facts that $\partial_y v^\varepsilon = -\partial_x u^\varepsilon$ and $\partial_y u^\varepsilon = \omega^\varepsilon$, one may check that all terms on the left-hand side of (5.13) are one of the following three types, we denote $e_1 := (1,0)$ and $e_2 := (0,1)$, for $\eta \in \mathbb{N}$ and $\kappa \in \mathbb{N}^2$:

**TYPE I:**

$$J_1 := \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\varepsilon \partial_\kappa v^\varepsilon D^\kappa \omega^\varepsilon$$

where $1 \leq \eta \leq s - 1$ and $\eta e_1 + \kappa = \alpha + e_2$;

**TYPE II:**

$$J_2 := \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\varepsilon \partial_\kappa u^\varepsilon D^\kappa \omega^\varepsilon$$

where $1 \leq \eta \leq s$ and $\eta e_1 + \kappa = \alpha + e_1$; and

**TYPE III:**

$$J_3 := \iint (1 + y)^{2\gamma + 2\alpha_2} D^\alpha \omega^\varepsilon D^\theta \omega^\varepsilon D^\kappa \omega^\varepsilon$$

where $|\theta| \leq s - 2$ and $\theta + \kappa = \alpha + e_1 - e_2$.

Thus, it suffices to control $J_1$, $J_2$, and $J_3$ by the right-hand side of (5.13) as follows:
ESTIMATES FOR TYPE I: When $1 \leq \eta \leq s - 2$, applying Proposition \[B.3\] we have, since $\kappa_2 = \alpha_2 + 1$,

$$|J_1| \leq \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \frac{\partial_x u^\epsilon}{1 + y} \right\|_{L^\infty} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$\leq C_{s, \gamma, \sigma, \delta} \{\|\omega^\epsilon\|_{H^s_x} + \|\partial_x^2 U\|_{L^2(\mathbb{T})}\}\|\omega^\epsilon\|^2_{H^s_{x, y}}.$$ 

When $\eta = s - 1$, by the triangle inequality and Proposition \[B.3\] we have, since $\kappa_2 = \alpha_2 + 1$,

$$|J_1| \leq \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \frac{\partial_x u^\epsilon}{1 + y} \right\|_{L^2} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$+ \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \partial_x^2 U\right\|_{L^\infty(\mathbb{T})} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$\leq C_{s, \gamma, \sigma, \delta} \{\|\omega^\epsilon\|_{H^s_x} + \|\partial_x^2 U\|_{L^\infty(\mathbb{T})}\}\|\omega^\epsilon\|^2_{H^s_{x, y}}.$$ 

In conclusion, $J_1$ can be controlled by the right-hand side of (5.13).

ESTIMATES FOR TYPE II: When $1 \leq \eta \leq s - 1$, applying Proposition \[B.3\] we have, since $\kappa_2 = \alpha_2$.

$$|J_2| \leq \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \partial_x u^\epsilon \right\|_{L^\infty} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$\leq C_{s, \gamma, \sigma, \delta} \{\|\omega^\epsilon\|_{H^s_x} + \|\partial_x^2 U\|_{L^2(\mathbb{T})}\}\|\omega^\epsilon\|^2_{H^s_{x, y}}.$$ 

When $\eta = s$, by the triangle inequality and Proposition \[B.3\] we have, since $\kappa = (0, \alpha_2)$ and $0 \leq \alpha_2 \leq 1$,

$$|J_2| \leq \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \partial_x (u - U) \right\|_{L^\infty} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$+ \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \partial_x U\right\|_{L^\infty(\mathbb{T})} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$\leq C_{s, \gamma, \sigma, \delta} \{\|\omega^\epsilon\|_{H^s_x} + \|\partial_x U\|_{L^\infty(\mathbb{T})}\}\|\omega^\epsilon\|^2_{H^s_{x, y}}.$$ 

In conclusion, $J_2$ can also be controlled by the right-hand side of (5.13).

ESTIMATES FOR TYPE III: Applying Proposition \[B.3\] we have, since $\theta_2 + \kappa_2 = \alpha_2 - 1$ and $|\theta| \leq s - 2$.

$$|J_3| \leq \|(1 + y)^{\gamma + \alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|(1 + y)^{1+\theta_2} \omega^\epsilon\|_{L^\infty} \|(1 + y)^{\gamma + \kappa_2} D^\kappa \omega^\epsilon\|_{L^2}$$

$$\leq C_{s, \gamma, \sigma, \delta} \{\|\omega^\epsilon\|_{H^s_x} + \|\partial_x^2 U\|_{L^2(\mathbb{T})}\}\|\omega^\epsilon\|^2_{H^s_{x, y}}.$$ 

Combining all estimates for types I through III, we prove inequality (5.13). \[\square\]

Lastly, we will prove the boundary reduction Lemma [5.9] as follows.

PROOF OF LEMMA [5.9] First of all, let us mention that equality (5.17) is exactly the same as the given boundary condition (4.3). Furthermore, differentiating the vorticity (4.3) with respect to $y$ and then evaluating at $y = 0$, we obtain equality (5.17) by using (5.17) and $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$. Thus, it remains to prove formula (5.18).
In order to illustrate the idea, let us first derive formula (5.18) for the case \( k = 2 \) as follows:

Differentiating the vorticity equation (4.3) with respect to \( y \) thrice, and then evaluating at \( y = 0 \), we obtain, by using (5.17) and \( u^y|_{y=0} = v^y|_{y=0} \equiv 0 \),

\[
\partial_y^2 \omega^y|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^2 \partial_x p^y + (\partial_t - \epsilon^2 \partial_x^2) (\omega^x \partial_x \omega^y)|_{y=0} \\
+ 3 \omega^y \partial_x \partial_x^2 \omega^y|_{y=0} + 2 \partial_y \omega^y \partial_x \partial_y \omega^y|_{y=0} - 2 \partial_x \omega^y \partial_y^2 \omega^y|_{y=0}.
\]

(5.23)

Since the last three terms on the right-hand side are in the desired form, we only need to deal with the terms \((\partial_t - \epsilon^2 \partial_x^2) (\omega^x \partial_x \omega^y)|_{y=0}\). Using the evolution equations for \( \omega^x \) and \( \partial_x \omega^y \) as well as \( u^y|_{y=0} = v^y|_{y=0} \equiv 0 \), one may check that

\[
(\partial_t - \epsilon^2 \partial_x^2) (\omega^x \partial_x \omega^y)|_{y=0} = \\
\omega^y \partial_x \partial_x^2 \omega^y|_{y=0} + \partial_x \omega^y \partial_x^2 \omega^y|_{y=0} - 2 \epsilon^2 \partial_x \omega^y \partial_y^2 \omega^y|_{y=0}.
\]

(5.24)

where all terms on the right-hand side of (5.24) are also in the desired form. Substituting (5.24) into (5.23), we justify the formula (5.18) for \( k = 2 \).

Now, using the same algorithm, we are going to prove formula (5.18) by induction on \( k \). For notational convenience, we denote

\[
\mathcal{A}_k := \left\{ \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max(2,k-l)} \sum_{\rho \in A_{k,l}} C_{k,l,\rho} \prod_{i=1}^{j} D^{\rho_i} \omega^y|_{y=0} \right\}.
\]

Under this notation, we will prove \( \partial_y^{2k+1} \omega^y|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x p^y \in \mathcal{A}_k \).

Assuming that formula (5.18) holds for \( k = n \), we will show that it also holds for \( k = n + 1 \) as follows.

In order to reduce the order of \( \partial_y^{2n+3} \omega^y|_{y=0} \), we first differentiate the vorticity equation (4.3) with respect to \( y \) \( 2n + 1 \) times and then evaluate the resulting equation at \( y = 0 \) to obtain

\[
\partial_y^{2n+3} \omega^y|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^{2n+1} \omega^y|_{y=0} \\
+ \sum_{m=1}^{2n+1} \binom{2n+1}{m} \partial_y^{m-1} \omega^y \partial_x^{2n-m+1} \omega^y|_{y=0} \\
- \sum_{m=2}^{2n+1} \binom{2n+1}{m} \partial_x \partial_y^{m-2} \omega^y \partial_y^{2n-m+2} \omega^y|_{y=0}.
\]

(5.25)

By routine checking, one may show that the last two terms of (5.25) belong to \( \mathcal{A}_{n+1} \), so it only remains to deal with the term \((\partial_t - \epsilon^2 \partial_x^2)^{2n+1} \omega^y|_{y=0}\).
Thanks to the induction hypothesis, there exist constants $C_{n,l,\rho^1,\rho^2,\ldots,\rho^j}$ such that
\[ \partial_y^{2n+1} \omega^\varepsilon \big|_{y=0} = (\partial_t - \varepsilon^2 \partial_x^2)^n \partial_x p^\varepsilon 
+ \sum_{l=0}^{n-1} \varepsilon^{2l} \sum_{j=2}^{\max\{2,n-l\}} \sum_{\rho \in A_{n,l}} C_{n,l,\rho^1,\rho^2,\ldots,\rho^j} \prod_{i=1}^j D^{\rho^i} \omega^\varepsilon \big|_{y=0}, \]
so we have, up to a relabeling of the indices $\rho^j$,
\[ (\partial_t - \varepsilon^2 \partial_x^2)^{2n+1} \omega^\varepsilon \big|_{y=0} 
= (\partial_t - \varepsilon^2 \partial_x^2)^{n+1} \partial_x p^\varepsilon 
+ \sum_{l=0}^{n-1} \varepsilon^{2l} \sum_{j=2}^{\max\{2,n-l\}} \sum_{\rho \in A_{n,l}} C_{n,l,\rho^1,\rho^2,\ldots,\rho^j} (\partial_t - \varepsilon^2 \partial_x^2)^j D^{\rho^i} \omega^\varepsilon \prod_{i=2}^j D^{\rho^i} \omega^\varepsilon \big|_{y=0}, \]
(5.26)

where $\tilde{C}_{n,l,\rho^1,\rho^2,\ldots,\rho^j}$ and $\tilde{\omega}^\varepsilon_{n,l,\rho^1,\rho^2,\ldots,\rho^j}$ are some new constants depending on $C_{n,l,\rho^1,\rho^2,\ldots,\rho^j}$. It is worth noting that the last term on the right-hand side of (5.26) belongs to $A_{n+1}$, so it remains to check whether the second term on right-hand side of (5.26) also belongs to $A_{n+1}$.

Differentiating the vorticity equation (4.31) with respect to $x \rho_1^1$ times and $y \rho_2^1$ times, and then evaluating at $y = 0$, we have, by using $u^\varepsilon|_{y=0} = v^\varepsilon|_{y=0} = 0$ and denoting $e_2 := (0, 1)$,
\[ (\partial_t - \varepsilon^2 \partial_x^2)^2 D^{\rho^1} \omega^\varepsilon|_{y=0} \]
\[ = - \sum_{\beta \leq \rho^1 \atop \beta_2 \geq 1} \left( \frac{\rho_1}{\beta} \right) D^{\beta - e_2} \omega^\varepsilon \partial_x D^{\rho^1 - \beta} \omega^\varepsilon|_{y=0} 
+ \sum_{\beta \leq \rho^1 \atop \beta_2 \geq 2} \left( \frac{\rho_1}{\beta} \right) \partial_x D^{\beta - 2e_2} \omega^\varepsilon \partial_y D^{\rho^1 - \beta} \omega^\varepsilon|_{y=0} + \partial_y^2 D^{\rho^1} \omega^\varepsilon|_{y=0}. \]
(5.27)

Using (5.27), one may justify by a routine counting of indices that the second term on the right-hand side of (5.26) belongs to $A_{n+1}$. This completes the proof of Lemma 5.9.

**Weighted $L^2$ Estimate for $g^\varepsilon$**

Now we are going to derive the $L^2$ estimate for $(1 + y)^y g^\varepsilon$ by using the standard energy methods. This can be done since the quantity $g^\varepsilon := \partial_x \omega^\varepsilon - \partial_y \omega^\varepsilon \partial_x (u^\varepsilon - U)$
avoids the loss of $x$-derivative by a *nonlinear cancellation*, which is one of the key observations in this paper and will be explained as follows.

Let us begin by writing down the evolution equations for $\omega^\varepsilon$ and $u^\varepsilon - U$:

\[
\begin{aligned}
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y)\omega^\varepsilon &= \epsilon^2 \partial_x^2 \omega^\varepsilon + \partial_y^2 \omega^\varepsilon \\
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y)(u^\varepsilon - U) &= \epsilon^2 \partial_x^2 (u^\varepsilon - U) + \partial_y^2 (u^\varepsilon - U) - (u^\varepsilon - U) \partial_x U,
\end{aligned}
\]

(5.28)

where we applied the regularized Bernoulli’s law (4.2) in the derivation of (5.28). Since our aim is to control the $H^s$-norm of $\omega^\varepsilon$ (or $u^\varepsilon - U$), let us differentiate (5.28) $s$ times with respect to $x$. Then we have

\[
\begin{aligned}
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y)\partial_x^s \omega^\varepsilon + \partial_y^2 \omega^\varepsilon &= \epsilon^2 \partial_x^{s+2} \omega^\varepsilon + \partial_y \partial_x^2 \omega^\varepsilon + \cdots \\
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y)\partial_x^s (u^\varepsilon - U) + \partial_y^2 \omega^\varepsilon &= \epsilon^2 \partial_x^{s+2} (u^\varepsilon - U) + \partial_y \partial_x^2 (u^\varepsilon - U) + \cdots,
\end{aligned}
\]

(5.29)

where we applied the fact that $\omega^\varepsilon = \partial_y (u^\varepsilon - U)$ and the symbol $\cdots$ represents the lower-order terms that we want the reader to ignore at this moment.

The main obstacle in (5.29) is the term $\partial_x^s v^\varepsilon = -\partial_x^{s+1} \partial_y^{-1} u^\varepsilon$, which has $s + 1$ $x$-derivatives so that the standard energy estimates cannot apply. However, since there are two equations in (5.29), we can eliminate the problematic term $\partial_x^s v^\varepsilon$ by subtracting them in an appropriate way.

Subtracting $\partial_x \omega^\varepsilon \times (5.29)_2$ from (5.29)$_1$, we obtain

\[
\begin{aligned}
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2)g^\varepsilon_s &= 2\epsilon^2 \left\{ \partial_x^{s+1} (u^\varepsilon - U) - \frac{\partial_x \omega^\varepsilon}{\omega^\varepsilon} \partial_x^s (u^\varepsilon - U) \right\} \partial_x a^\varepsilon \\
&\quad+ 2g^\varepsilon_j \partial_x a^\varepsilon - g^\varepsilon_j \partial_x^s U - \sum_{j=1}^{s-1} \binom{s}{j} g_{j+1}^\varepsilon \partial_x^{s-j-1} U \\
&\quad- \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\varepsilon \left\{ \partial_x^j \partial_y \omega^\varepsilon - a^\varepsilon \partial_x^j \omega^\varepsilon \right\} \\
&\quad+ a^\varepsilon \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^j (u^\varepsilon - U) \partial_x^{s-j+1} U,
\end{aligned}
\]

(5.30)

where $g^\varepsilon_k := \partial_x^k \omega^\varepsilon - a^\varepsilon \partial_x^k (u^\varepsilon - U)$ and $a^\varepsilon := \frac{\partial_x \omega^\varepsilon}{\omega^\varepsilon}$. This subtraction is the nonlinear cancellation that we mentioned. Here, the main reason that we can apply this nonlinear cancellation is Oleinik’s monotonicity assumption (i.e., $\omega^\varepsilon > 0$), which is ensured in our solution class $H^{s, \gamma}_{a, \varepsilon}$. For the justification of (5.30), see Appendix [D].

Now, we are going to derive the following weighted energy estimate for $g^\varepsilon_s$. 

Proposition 5.10 ($L^2$ Control on $(1 + y)^r g^\varepsilon_s$). Under the hypotheses of Proposition 5.3, we have the following estimate:

$$\frac{1}{2} \frac{d}{dt} \|(1 + y)^r g^\varepsilon_s\|_{L^2}^2 \leq -\frac{1}{2} \varepsilon^2 \|(1 + y)^r \partial_y g^\varepsilon_s\|_{L^2}^2 - \frac{1}{2} \|(1 + y)^r \partial_y g^\varepsilon_s\|_{L^2}^2$$

(5.31)

$$+ C \|\partial_x^{r+1} p^\varepsilon\|_{L^2(T)}^2 + C \|\partial_x U\|_{L^\infty(T)}^2 \|\omega^\varepsilon\|_{H_x^{r+\gamma}}^2 + C \|\partial_x g^\varepsilon_{s;\gamma}\|_{L^\infty(T)}^2 \|\partial_x ^{r+1} U\|_{L^\infty(T)} \|\omega^\varepsilon\|_{H_x^{r+\gamma}}^2,$$

where the positive constants $C$, $C_{r,\gamma}$, and $C_{r,\gamma,\sigma,\delta}$ are independent of $\varepsilon$.

The proof of Proposition 5.10 is almost a straightforward application of energy methods except that the estimation (5.36) below is slightly tricky.

**Proof of Proposition 5.10.** Multiplying the evolution equation (5.30) by $(1 + y)^r g^\varepsilon_s$ and then integrating over $T \times \mathbb{R}^+$, we have

$$\frac{1}{2} \frac{d}{dt} \|(1 + y)^r g^\varepsilon_s\|_{L^2}^2 = \varepsilon^2 \iint (1 + y)^{2r} g^\varepsilon_s \partial_y g^\varepsilon_s + \iint (1 + y)^r g^\varepsilon_{s;\gamma} \partial_y g^\varepsilon_s$$

$$- \iint (1 + y)^{2r} g^\varepsilon_s u^\varepsilon \partial_x g^\varepsilon_s + v^\varepsilon \partial_y g^\varepsilon_s$$

$$+ 2\varepsilon^2 \iint (1 + y)^{2r} g^\varepsilon_s \left\{ \partial_x^{r+1} (u^\varepsilon - U) - \frac{\partial_x \omega^\varepsilon}{\omega^\varepsilon} \partial_x (u^\varepsilon - U) \right\} \partial_x \omega^\varepsilon$$

$$+ 2 \iint (1 + y)^{2r} g^\varepsilon_{s;\gamma} \partial_y a^\varepsilon - \iint (1 + y)^{2r} g^\varepsilon_{s;1} \partial_x U$$

$$- \sum_{j=1}^{s-1} \left( \begin{array}{c} s \\downarrow \end{array} j \right) \iint (1 + y)^{2r} g^\varepsilon_{j+1} \partial_x \partial_y \partial_x^{s-j} u^\varepsilon$$

$$- \sum_{j=1}^{s-1} \left( \begin{array}{c} s \\downarrow \end{array} j \right) \iint (1 + y)^{2r} \partial_x^{s-j} \frac{2}{\omega^\varepsilon} \partial_y \omega^\varepsilon - a^\varepsilon \partial_x \omega^\varepsilon g^\varepsilon_s$$

$$+ \sum_{j=0}^{s-1} \left( \begin{array}{c} s \\downarrow \end{array} j \right) \iint (1 + y)^{2r} g^\varepsilon_s a^\varepsilon \partial_x \partial_x^{s-j} (u^\varepsilon - U) \partial_x^{s-j+1} U.$$

Indeed, all terms on the right-hand side of (5.32) can be controlled by using integration by parts and the standard Sobolev-type estimates for multilinear forms. More precisely, we have the following:
CLAIM 5.11. There exist positive constants $C$, $C_\delta$, $C_{\gamma,\delta}$, and $C_{s,\gamma,\sigma,\delta}$ such that

\begin{equation}
\epsilon^2 \iint (1 + y)^{2\gamma} g_s^\epsilon \partial_x^2 g_s^\epsilon = -\epsilon^2 \| (1 + y)^{\gamma} \partial_x g_s^\epsilon \|_{L^2}^2.
\end{equation}

\begin{equation}
\iint (1 + y)^{2\gamma} g_s^\epsilon \partial_y g_s^\epsilon
\end{equation}

\begin{equation}
\leq -\frac{1}{2} \|(1 + y)^{\gamma} \partial_y g_s^\epsilon \|_{L^2}^2 + C_{\gamma,\delta} \| 1 + \| \partial_x^\gamma U \|_{L^\infty(\mathbb{T})} \| \omega^\epsilon \|_{H^{\gamma,\delta}}^2
\end{equation}

\begin{equation}
+ C \| \partial_x^{n+1} p^\epsilon \|_{L^2(\mathbb{T})}^2.
\end{equation}

\begin{equation}
\left| \iint (1 + y)^{2\gamma} g_s^\epsilon (u^\epsilon \partial_x g_s^\epsilon + v^\epsilon \partial_y g_s^\epsilon) \right| \leq C_{s,\gamma,\sigma,\delta} \| \omega^\epsilon \|_{H^{\gamma,\delta}} + \| \partial_x^\gamma U \|_{L^2(\mathbb{T})} \| \omega^\epsilon \|_{H^{\gamma,\delta}}.
\end{equation}

\begin{equation}
2 \iint (1 + y)^{2\gamma} g_s^\epsilon |\partial_x^\epsilon g_s^\epsilon | \leq C_{\delta} \| \omega^\epsilon \|_{H^{\gamma,\delta}}^2.
\end{equation}

\begin{equation}
\left| \iint (1 + y)^{2\gamma} g_s^\epsilon \partial_x^\gamma U \right| \leq C_{s,\gamma,\sigma,\delta} \| \partial_x^\gamma U \|_{L^\infty(\mathbb{T})} \left( \| \omega^\epsilon \|_{H^{\gamma,\delta}} + \| \partial_x^\gamma U \|_{L^2(\mathbb{T})} \right) \| \omega^\epsilon \|_{H^{\gamma,\delta}}.
\end{equation}

\begin{equation}
\left| \sum_{j=1}^{s-1} \binom{s}{j} \iint (1 + y)^{2\gamma} g_{j+1}^\epsilon \partial_x^{s-j} u^\epsilon \right| \leq C_{s,\gamma,\sigma,\delta} \left( \| \omega^\epsilon \|_{H^{\gamma,\delta}} + \| \partial_x^\gamma U \|_{L^2(\mathbb{T})} \right)^2 \| \omega^\epsilon \|_{H^{\gamma,\delta}}.
\end{equation}

\begin{equation}
\left| \sum_{j=1}^{s-1} \binom{s}{j} \iint (1 + y)^{2\gamma} \partial_x^{s-j} v^\epsilon \{ \partial_x^\epsilon \partial_y \omega^\epsilon - a^\epsilon \partial_x^\epsilon \omega^\epsilon \} g_s^\epsilon \right| \leq C_{s,\gamma,\sigma,\delta} \left( \| \omega^\epsilon \|_{H^{\gamma,\delta}} + \| \partial_x^\gamma U \|_{L^\infty(\mathbb{T})} \right) \| \omega^\epsilon \|_{H^{\gamma,\delta}}.
\end{equation}
\[ (5.41) \quad \left| \sum_{j=0}^{s-1} \binom{s}{j} \int \int (1 + y)^{2\gamma} g^s \partial_y (u^\varepsilon - U) \partial_x^{s-j+1} U \right| \leq C_{s, y, \sigma, \delta} \left\| \partial_x^{s+1} U \right\|_{L^\infty(T)} \left\| \omega^\varepsilon \right\|_{H^{2, y}_x} + \left\| \partial_x^s U \right\|_{L^2(T)} \left\| \omega^\varepsilon \right\|_{H^{2, y}_x}. \]

Assuming Claim 5.11, which will be proven later, for the moment we can apply (5.33)–(5.41) to (5.32) to obtain inequality (5.31) because \( \varepsilon \in [0, 1] \) and

\[ \left\| \partial_x^s U \right\|_{L^2(T)} \leq \left\| \partial_x^s U \right\|_{L^\infty(T)} \leq \left\| \partial_x^{s+1} U \right\|_{L^2(T)} \leq \left\| \partial_x^s U \right\|_{L^\infty(T)}. \]

To complete the proof of Proposition 5.10, we will show Claim 5.11 as follows.

**Proof of Claim 5.11.**

**Proof of (5.33).** The equality (5.33) follows directly from an integration by parts in the \( x \)-variable.

**Proof of (5.34).** Integrating by parts in the \( y \)-variable (cf. Remark 5.7), we have

\[ \int \int (1 + y)^{2\gamma} g^s \partial_y g^s \]

\[ = -\left\| (1 + y)^{\gamma} \partial_y g^s \right\|_{L^2}^2 - \int_T \partial_y g^s \partial_y g^s \, dx \bigg|_{y=0} \]

\[ - 2\gamma \int \int (1 + y)^{2\gamma-1} g^s \partial_y g^s \]

\[ \leq -\frac{3}{4} \left\| (1 + y)^{\gamma} \partial_y g^s \right\|_{L^2}^2 - \int_T \partial_y g^s \partial_y g^s \, dx \bigg|_{y=0} + C_{\gamma} \left\| \omega^\varepsilon \right\|_{H^{2, y}_x}^2. \]

In order to deal with the boundary integral \( \int_T g^s \partial_y g^s \, dx \bigg|_{y=0} \), one may apply the boundary conditions (4.1) \( 4 \) and (4.3) \( 3 \) to justify that

\[ \partial_y g^s \bigg|_{y=0} = -\partial_x^{s+1} p^\varepsilon + \frac{\partial_x^s \omega^\varepsilon}{\omega^\varepsilon} \partial_x^s U \bigg|_{y=0} - a^\varepsilon g^s \bigg|_{y=0}. \]

This boundary condition allows us to reduce the order of \( \partial_y g^s \big|_{y=0} \), and hence, using the simple trace estimate (5.15) and the facts that \( \omega^\varepsilon \big|_{y=0} \geq \delta \) and \( \left\| a^\varepsilon \right\|_{L^\infty} \leq \delta^{-2} \), one may prove that

\[ \int_T g^s \partial_y g^s \, dx \bigg|_{y=0} \]

\[ \leq \frac{1}{4} \left\| (1 + y)^{\gamma} \partial_y g^s \right\|_{L^2}^2 + C_{\delta} \left\{ 1 + \left\| \partial_x^s U \right\|_{L^\infty(T)}^2 \right\} \left\| \omega^\varepsilon \right\|_{H^{2, y}_x}^2 \]

\[ + C \left\| \partial_x^{s+1} p^\varepsilon \right\|_{L^2(T)}^2. \]

Substituting (5.43) into (5.42), we obtain (5.34).
Proof of (5.35). Integrating by parts (cf. Remark 5.7) and using $\to u_x + \to y v = 0$, we have
\[
\iint (1 + y)^{2\gamma} g_s (u^e \to x g_s^e + v^e \to y g_s^e) = -\gamma \iint (1 + y)^{2\gamma-1} v^e |g_s^e|^2.
\]
This and inequality (B.9) imply inequality (5.35).

Proof of (5.36). Since $\omega^e \in C([0, T]; H^{s+4,2})$, it follows from the definition of $H^{s+4,2}$ that $(1 + y)^{\alpha} \omega^e \geq \delta$ and $|(1 + y)^{\alpha + \alpha_2} D^\alpha \omega^e| \leq \delta^{-1}$ for all $|\alpha| \leq 2$. Thus, we have $\|(1 + y)^{\gamma} \to_x u^e\|_{L^2} \leq \delta^{-2} + \delta^{-4}$ and $\|	o_x \omega^e / \omega^e\|_{L^\infty} \leq \delta^{-2}$, and hence
\[
\left\| (1 + y)^{\gamma-1} \to^{s+1}_x (u^e - U) \right\|_{L^2} \leq 2\epsilon^2 C_\delta \left\{ \|(1 + y)^{2\gamma} g_s^e\|_{L^2 (\Omega)} \right\} \left\{ \|(1 + y)^{2\gamma-1} \to^{s+1}_x (u^e - U)\|_{L^2} \right\} + \|(1 + y)^{2\gamma-1} \to^{s}_x (u^e - U)\|_{L^2} \}
\]
(5.44)

Now, we require the following inequality:
\[
\|(1 + y)^{\gamma-1} \to^{s+1}_x (u^e - U)\|_{L^2} \leq C_{\gamma, \alpha, \delta} \left\{ \|\to^{s+1}_x U\|_{L^2 (\Omega)} + \|(1 + y)^{2\gamma} \to_x g_s^e\|_{L^2} \right\} + \|(1 + y)^{2\gamma-1} \to^{s}_x (u^e - U)\|_{L^2} \}
\]
(5.45)
Assuming (5.45) for the moment, we can apply it and Proposition B.3 to (5.44), and obtain
\[
\begin{align*}
2\epsilon^2 \iint (1 + y)^{2\gamma} g_s^e \left\{ \to^{s+1}_x (u^e - U) - \frac{\to_x \omega^e}{\omega^e} \to^{s}_x (u^e - U) \right\} \partial_x a^e \\
\leq 2\epsilon^2 C_{\gamma, \alpha, \delta} \left\{ \|\omega^e\|_{H^{s+\gamma}_x} + \|\partial_x g_s^e\|_{L^2 (\Omega)} \right\} \left\{ \|\partial^{s+1}_x U\|_{L^2 (\Omega)} + \|\partial^{s}_x U\|_{L^2 (\Omega)} \right\} \\
+ \|(1 + y)^{2\gamma} \to_x g_s^e\|_{L^2 (\Omega)} \|\omega^e\|_{H^{s+\gamma}_x} \}
\end{align*}
\]
which implies (5.36) by Cauchy’s inequality and the inequality $\|\partial^{s+1}_x U\|_{L^2 (\Omega)} \leq \pi \|\partial^{s+1}_x U\|_{L^2 (\Omega)}$.

To complete the justification of (5.36), we have to verify (5.45) as follows.
Since $\delta \leq (1 + y)^{\alpha} \omega^e \leq \delta^{-1}$ and $u^e |_{y=0} = 0$, applying part (ii) of Lemma B.1 we have
\[
\|(1 + y)^{\gamma-1} \to^{s+1}_x (u^e - U)\|_{L^2} \leq \delta^{-1} \left\| (1 + y)^{\gamma-\sigma-1} \to^{s+1}_x (u^e - U) \right\|_{L^2} \}
\]
(5.46)
\[
C_{\gamma, \alpha, \delta} \left\{ \|\partial^{s+1}_x U\|_{L^2 (\Omega)} + \left\| (1 + y)^{2\gamma} \omega^e \partial_y \left( \frac{\partial^{s+1}_x (u^e - U)}{\omega^e} \right) \right\|_{L^2} \}
\]
It is worth noting that

\[
\omega^\varepsilon \partial_y \left( \frac{\partial_x^{s+1} (u^\varepsilon - U)}{\omega^\varepsilon} \right) = g^\varepsilon_{s+1} = \partial_x g^\varepsilon_s + \partial_x a^\varepsilon \partial_x^2 (u^\varepsilon - U),
\]

so by (5.46), we have

\[
\|(1 + y)^{y^{-1}} \partial_x^{s+1} (u^\varepsilon - U)\|_{L^2} \leq C_{y, \sigma, \delta} \{ \|\partial_x^{s+1} U\|_{L^2(T)} + \|(1 + y)^y \partial_x g^\varepsilon_s \|_{L^2} + \|(1 + y) \partial_x a^\varepsilon \|_{L^\infty} \|(1 + y)^{y^{-1}} \partial_x^2 (u^\varepsilon - U)\|_{L^2}\},
\]

which implies inequality (5.45) because \(\|(1 + y) \partial_x a^\varepsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}\).

**Proof of (5.37).** According to the definition of \(H_{\sigma, \delta, \gamma}^{s+4, y}\), \(\omega^\varepsilon \in C(\mathbb{R}; H_{\sigma, \delta, \gamma}^{s+4, y})\) satisfies \((1 + y)^y \omega^\varepsilon \geq \delta\) and \(|(1 + y)^y D^{\sigma+\alpha} \omega^\varepsilon| \leq \delta^{-1}\) for all \(|\alpha| \leq 2\), so we also have \(\|\partial_y \omega^\varepsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}\).

The inequality (5.37) follows from the fact that \(\|\partial_y \omega^\varepsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}\) and the definition of \(\|\omega^\varepsilon\|_{H_{\sigma, \gamma}^{s+4, y}}\).

**Proof of (5.38) and (5.39).** Both inequalities (5.38) and (5.39) follow from Hölder's inequality and Proposition B.3.

**Proof of (5.40).** For \(j = 2, 3, \ldots, s - 1\), using \(\|(1 + y) a^\varepsilon\|_{L^\infty} \leq \delta^{-2}\) and Proposition B.3, we have

\[
\left| \left\langle 1 + y \right\rangle^{2\gamma} \partial_x^{s-j} v^\varepsilon \{ \partial_x \partial_y \omega^\varepsilon - a^\varepsilon \partial_x \omega^\varepsilon \} g^\varepsilon_s \right| \cdot \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \leq \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \|(1 + y)^{\gamma+1} \partial_x \partial_y \omega^\varepsilon \|_{L^2} + \|(1 + y)^{\gamma+1} \partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \|(1 + y)^{y} \partial_x \omega^\varepsilon \|_{L^\infty} \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \leq \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \|\partial_x \partial_y \omega^\varepsilon \|_{L^\infty} \leq C_{s, y, \sigma, \delta} \{ \|\omega^\varepsilon\|_{H_{\sigma, \gamma}^{s+4, y}} + \|\partial_x^2 U\|_{L^2(T)} \} \|\omega^\varepsilon\|_{H_{\sigma, \gamma}^{s+4, y}}^2.
\]
When \( j = 1 \), using Proposition \( \text{B.3} \) and \( \|(1 + y)\omega^\epsilon\|_{L^\infty} \leq \delta^{-2} \) again, we have

\[
\left| \iint (1 + y)^{-\gamma} \partial_x^{-\gamma} u^\epsilon \{ \partial_x \partial_y \omega^\epsilon - a^\epsilon \partial_x \omega^\epsilon \} g_x^\epsilon \right|
\leq \left\| \frac{\partial_x^{-\gamma} u^\epsilon + y \partial_x^\epsilon U}{1 + y} \right\|_{L^2} \\
\cdot \left( \|(1 + y)^{-\gamma} \partial_x \partial_y \omega^\epsilon\|_{L^\infty} + \|(1 + y) a^\epsilon\|_{L^\infty} \|(1 + y)^{\gamma} \partial_x \omega^\epsilon\|_{L^\infty} \right)
+ \left\| \partial_x^\epsilon U \|_{L^\infty(T)} \right\| \|(1 + y)^{\gamma} \partial_x \omega^\epsilon\|_{L^2}
+ \left\| \partial^+ u^\epsilon \|_{L^\infty(T)} \right\| \|(1 + y)^{\gamma} \partial_x \omega^\epsilon\|_{L^2}
\]

(5.48)

\[
\leq C_{s,y,\sigma,\delta} \|\omega^\epsilon\|_{H^{s,\gamma}_x} + \|\partial_x^\epsilon U \|_{L^\infty(T)} \|\omega^\epsilon\|^2_{H^{s,\gamma}_x}.
\]

Combining estimates (5.47) and (5.48), and summing over \( j \), we prove inequality (5.40).

**Proof of (5.41).** For any \( j = 0, 1, \ldots, s - 1 \), by Proposition \( \text{B.3} \) \( \|(1 + y) a^\epsilon\|_{L^\infty} \leq \delta^{-2} \), and the fact that \( \|\partial_x^\epsilon U \|_{L^\infty(T)} \leq \|\partial_x^\epsilon U \|_{L^\infty(T)} \leq \cdots \leq \|\partial_x^+ u^\epsilon \|_{L^\infty(T)} \), we have

\[
\left| \iint (1 + y)^{-\gamma} g_x^\epsilon \partial_x^j (u^\epsilon - U) \partial_x^{-j+1} U \right|
\leq \|(1 + y)^{\gamma} \partial_x^\epsilon g_x^\epsilon \|_{L^2} \|\partial^\epsilon u^\epsilon\|_{L^\infty} \left\| (1 + y)^{\gamma - 1} \partial_x^j (u^\epsilon - U) \right\|_{L^2} \|\partial_x^{-j+1} U \|_{L^\infty(T)}
\leq C_{s,y,\sigma,\delta} \|\partial_x^j U \|_{L^\infty(T)} \|\omega^\epsilon\|_{H^{s,\gamma}_x} + \|\partial_x^\epsilon U \|_{L^\infty(T)} \|\omega^\epsilon\|_{H^{s,\gamma}_x},
\]

which implies inequality (5.41) by summing over \( j \).

**Weighted \( H^s \) Estimate for \( \omega^\epsilon \).**

The aim here is to combine the estimates in Proposition \( \text{5.6} \) and Proposition \( \text{5.10} \) to derive the growth rate control (5.1) on the weighted \( H^s \) energy of \( \omega^\epsilon \).

According to Propositions \( \text{5.6} \) and \( \text{5.10} \), we know from the definition of \( \|\cdot\|_{H^{s,\gamma}_x} \) that

\[
\frac{d}{dt}\|\omega^\epsilon\|^2_{H^{s,\gamma}_x}
\leq C_{s,y,\sigma,\delta} \left\{ 1 + \|\omega^\epsilon\|_{H^{s,\gamma}_x} + \|\partial_x^\epsilon U \|_{L^\infty(T)} \right\}
\cdot \|\omega^\epsilon\|_{H^{s,\gamma}_x} + \|\partial_x^{s+1} U \|_{L^\infty(T)} \|\omega^\epsilon\|_{H^{s,\gamma}_x}
+ C_{y,\delta} \|\partial_x^\epsilon U \|_{L^\infty(T)} \|\omega^\epsilon\|^2_{H^{s,\gamma}_x} + C_{s,y,\sigma,\delta} \left\{ 1 + \|\omega^\epsilon\|_{H^{s,\gamma}_x} \right\}^{s-2} \|\omega^\epsilon\|^2_{H^{s,\gamma}_x}
+ C_s \sum_{l=0}^{s/2} \|\partial_x^l \partial_x p^\epsilon\|^2_{H^{s-2l}(T)} \leq
\]

...
\[ \leq C_{s,y,\sigma,\delta} \| \omega^\varepsilon \|_{H^s,\gamma}^s + C_{s,y,\sigma,\delta} \{ 1 + \| \delta^{s+1} U \|_{L^\infty(\mathbb{T})} \} \]
\[ + C_s \sum_{l=0}^{s/2} \| \delta^l \partial_x p^\varepsilon \|_{H^{s-2l}(\mathbb{T})}, \]

and hence, it follows from the comparison principle of ordinary differential equations that

\[ \| \omega^\varepsilon \|_{H^s,\gamma}^2 \leq \left\{ \| \omega_0 \|_{H^s,\gamma}^2 + \int_0^t F(\tau) \, d\tau \right\} \]
\[ \cdot \left\{ 1 - \left( \frac{s}{2} - 1 \right) C_{s,y,\sigma,\delta} \left( \| \omega_0 \|_{H^s,\gamma}^2 + \int_0^t F(\tau) \, d\tau \right)^{\frac{s-2}{2}} \right\}^{\frac{2}{s-2}} \]

as long as the quantity within the second set of braces on the right-hand side of the above inequality is positive, where \( F : [0, T] \to \mathbb{R}^+ \) is defined by (5.2). This proves inequality (5.1).

5.2 Weighted \( L^\infty \) Estimates for Lower Order Terms

In this section we will derive uniform (in \( \varepsilon \)) weighted \( L^\infty \)-estimates for \( D^\alpha \omega^\varepsilon \) for \( |\alpha| \leq 2 \) by using the classical maximum principles. The key idea is to “view” the evolution equation of \( D^\alpha \omega^\varepsilon \) as a “linear” parabolic equation with coefficients involving higher-order terms of \( u^\varepsilon, v^\varepsilon, \) and \( \omega^\varepsilon \), which can be controlled by Proposition B.3 provided that \( \| \omega^\varepsilon \|_{H^s,\gamma} < +\infty \).

More precisely, we will prove part (ii) of Proposition 5.3 as follows.

Proof of Part (ii) of Proposition 5.3. This proof, based on simple applications of the classical maximum principles for parabolic equations, will be divided into two steps. In the first step, we will derive weighted \( L^\infty \) controls on \( D^\alpha \omega^\varepsilon \) by using the maximum principles stated in Appendix E. These controls will rely on the boundary values of \( D^\alpha \omega^\varepsilon \) at \( y = 0 \), so we will also derive estimates for the boundary values of \( D^\alpha \omega^\varepsilon \) by using Sobolev embedding or a growth rate control argument in the second step.

Step 1. Maximum Principle Argument. First of all, let us derive an \( L^\infty \)-estimate for

\[ I := \sum_{|\alpha| \leq 2} |(1 + y)^{\sigma + \alpha_2} D^\alpha \omega^\varepsilon |^2 \]

as follows.

For notational convenience, let us denote, for \( |\alpha| \leq 2 \),

\[ B_\alpha := (1 + y)^{\sigma + \alpha_2} D^\alpha \omega^\varepsilon, \]

By a direct computation, \( B_\alpha \) satisfies

\[ \{ \partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2 \} B_\alpha = Q_\alpha \partial_y B_\alpha + R_\alpha B_\alpha + S_\alpha, \]

where the quantities \( Q_\alpha, R_\alpha, \) and \( S_\alpha \) are given explicitly by

\[ Q_\alpha := \frac{-2(\sigma + \alpha_2)}{1 + y}, \]
\[ R_\alpha := \frac{\sigma + \alpha_2}{1 + y} v^\varepsilon + \frac{(\sigma + \alpha_2)(\sigma + \alpha_2 + 1)}{(1 + y)^2}, \]
Here, $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Using Proposition B.3, we have the following pointwise controls on $Q_a$, $R_a$, and $S_a$: for $|\alpha| \leq 2$,

\[
(5.50) \begin{cases}
|Q_a| \leq C_\alpha, \\
|R_a| \leq C_{\alpha,y,\sigma,\delta} \left( 1 + \|o\|_{H^{\gamma}\|y\| T}^2 + \|\phi_y^* U\|_{L^2(T)} \right), \\
|S_a| \leq C_{\alpha,y,\sigma,\delta} \left( 1 + \|\phi_y^* U\|_{L^2(T)} \right) \sum_{\alpha < \beta \leq \alpha} \|B_{\alpha-\beta+e_1}\| + |B_{\alpha-\beta+e_2}|.
\end{cases}
\]

where $C_\sigma$ and $C_{\alpha,y,\sigma,\delta}$ are some universal constants that are independent of the solution $\omega^\varepsilon$.

Let us recall from the definition that $I := \sum_{|\alpha| \leq 2} |B_{\alpha}|^2$, so using (5.49) and (5.50), we have

\[
\{\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2\} I
\]

\[
= -2 \sum_{|\alpha| \leq 2} \{\varepsilon^2 |\partial_x B_{\alpha}|^2 + |\partial_y B_{\alpha}|^2\}
+ 2 \sum_{|\alpha| \leq 2} \{Q_a B_{\alpha} \partial_y B_{\alpha} + R_a |B_{\alpha}|^2 + S_{\alpha} B_{\alpha}\}
\leq C_{\alpha,y,\sigma,\delta} \left( 1 + \|\omega^\varepsilon\|_{H^{\gamma}\|y\| T}^2 + \|\phi_x^* U\|_{L^2(T)} \right) I.
\]

Applying the classical maximum principle for parabolic equations (see Lemma E.1 for instance) to the quantity $I$, we have, after using definition (5.4) of $G$,

\[
\|I(t)\|_{L^\infty(T \times \mathbb{R}^+)}
\leq \max \left\{ e^{C_{\alpha,y,\sigma,\delta} \left( 1 + G(t) \right) t} \|I(0)\|_{L^\infty(T \times \mathbb{R}^+)} \right\},
\]

\[
\max_{\tau \in [0,t]} \left\{ e^{C_{\alpha,y,\sigma,\delta} \left( 1 + G(t) \right) (t-\tau)} \|I(\tau)|_{y=0}\|_{L^\infty(T)} \right\}'.
\]

Next, we are going to derive a lower bound estimate for $B_{(0,0)} := (1 + y)^\sigma \omega^\varepsilon$.

To do this, let us recall that $B_{(0,0)}$ satisfies

\[
\{\partial_t + u^\varepsilon \partial_x + (v^\varepsilon - Q_{(0,0)}) \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2\} B_{(0,0)} = R_{(0,0)} B_{(0,0)}.
\]

Using the classical maximum principle (see Lemma E.2 for instance) and (5.50), we obtain

\[
\min_{T \times \mathbb{R}^+} (1 + y)^\sigma \omega^\varepsilon(t) \geq \left( 1 - C_{\alpha,y,\sigma,\delta} \left( 1 + G(t) \right) t e^{C_{\alpha,y,\sigma,\delta} \left( 1 + G(t) \right) t} \right) \cdot \min_{T \times \mathbb{R}^+} (1 + y)^\sigma \omega^\varepsilon |_{y=0}.
\]

\[
(5.52)
\]

Step 2. Controls on Boundary Values. Given inequalities (5.51) and (5.52), we have already controlled the underlying quantities $I$ and $B_{(0,0)} := (1 + y)^\sigma \omega^\varepsilon$ by...
their initial and boundary values. However, their boundary values are not given in the problem, so we will estimate them in this step.

In order to control $I_{y=0} := \sum_{|\alpha|\leq 2} |D^\alpha \omega^\epsilon|_{y=0}$, we will apply a Sobolev embedding argument and growth rate control argument in the cases $s \geq 4$ and $s \geq 6$, respectively. Combining the boundary estimates for $I$ with (5.51), we will finally obtain inequalities (5.3) and (5.5).

To derive inequality (5.3), we first apply Lemma B.2 to obtain

$$
\| I_{y=0} \|_{L^\infty(T)} \leq C s \| \omega^\epsilon \|_{H^s_{x,y}}.
$$

This and inequality (5.51) imply inequality (5.3).

To derive inequality (5.5), let us begin by writing down the evolution equation for $D^\alpha \omega^\epsilon \big|_{y=0}$: for any $|\alpha| \leq 2$,

$$
(5.53) \quad \partial_t D^\alpha \omega^\epsilon \big|_{y=0} = (\epsilon^2 \partial_x^2 + \partial_y^2) D^\alpha \omega^\epsilon \big|_{y=0} + E_{\alpha} \big|_{y=0}
$$

where the term $E_{\alpha}$ is given explicitly by

$$
E_{\alpha} := \begin{cases}
0 & \text{if } \alpha_2 = 0, \\
-\alpha_2 \omega^\epsilon \partial_x \partial_y^{\alpha_2-1} \omega^\epsilon & \text{if } \alpha_1 = 0 \text{ and } 1 \leq \alpha_2 \leq 2, \\
-|\partial_x \omega^\epsilon|^2 - \omega^\epsilon \partial_x^2 \omega^\epsilon & \text{if } \alpha = (1, 1).
\end{cases}
$$

Here, the identity (5.53) follows from a direct differentiation on the vorticity equation (4.3) and the boundary condition $u^\epsilon \big|_{y=0} = v^\epsilon \big|_{y=0} \equiv 0$. Furthermore, by Proposition B.3 if $s \geq 4$, then

$$
(5.54) \quad \| E_{\alpha} \|_{y=0} \|_{L^\infty(T)} \leq C_{s,y} \| \omega^\epsilon \|_{H^s_{x,y}}.
$$

In addition, if $s \geq |\alpha| + 4$, then by Proposition B.3 again, for $\epsilon \in [0, 1]$,

$$
(5.55) \quad \| (\epsilon^2 \partial_x^2 + \partial_y^2) D^\alpha \omega^\epsilon \big|_{y=0} \|_{L^\infty(T)} \leq C_{s,y} \| \omega^\epsilon \|_{H^s_{x,y}}.
$$

Therefore, using (5.53)-(5.55) and inequality (B.10), we have, for $s \geq 6$,

$$
\| \partial_t I \|_{y=0} \|_{L^\infty(T)} \leq C_{s,y} \{ 1 + \| \omega^\epsilon \|_{H^s_{x,y}} \} \| \omega^\epsilon \|_{H^s_{x,y}},
$$

and hence, by direction integration and definition (5.4) of $\Omega$, we obtain

$$
(5.56) \quad \| I(t) \|_{y=0} \|_{L^\infty(T)} \leq \| I(0) \|_{y=0} \|_{L^\infty(T)} + C_{s,y} \{ 1 + \Omega(t) \} \Omega(t)^2 t.
$$

Combining estimates (5.51) and (5.56), we prove the desired estimate (5.5).

Lastly, it remains to show inequality (5.6). Let us begin by deriving an estimate for $\omega^\epsilon \big|_{y=0}$.
Using identity (5.53), inequality (5.55), and definition (5.4) of $\Omega$, we have, for any $s \geq 4$,
$$\|\partial_t \omega^\varepsilon|_{y=0}\|_{L^\infty(T)} \leq C_{s,y} \Omega(t),$$
so a direct integration yields
\begin{equation}
\min_{T} \omega^\varepsilon(t)|_{y=0} \geq \min_{T} \omega_0|_{y=0} - C_{s,y} \Omega(t) t. \tag{5.57}
\end{equation}
Combining estimates (5.52) and (5.57), we prove (5.6). \qed

6 Proof of the Main Theorem

The purpose of this section is to complete the proof of our main theorem, Theorem 2.2. In other words, we will prove existence and uniqueness of the Prandtl equations (2.1) in Sections 6.1 and 6.2, respectively.

6.1 Existence for the Prandtl Equations

In this subsection we will construct the solution to the Prandtl equations (2.1) by passing to the limit $\varepsilon \to 0^+$ in the regularized Prandtl equations (4.1). Our proof will be based on the uniform (in $s$) weighted estimates derived in Proposition 5.3. Using these estimates, we will first derive uniform bounds and life span for $\omega^\varepsilon$, and then prove convergence of $\omega^\varepsilon$ and consistency of the limit $\omega$.

Uniform Bounds and Life Span for $\omega^\varepsilon$

According to Proposition 5.1, a solution $\omega^\varepsilon$ to the regularized Prandtl equations (4.1) exists up to a time interval $[0, T_{s,y,\sigma,\delta,\varepsilon,\omega_0,U}]$, which may depend on $\varepsilon$ as well. However, in Proposition 5.3 we have already derived uniform (in $s$) estimates for $\omega^\varepsilon$, so one may apply the standard continuous induction argument to further solve the regularized Prandtl equations (4.1) up to a time interval that is independent of $\varepsilon$. As a result, we have the following:

**Proposition 6.1 (Uniform Life Span and Estimates for $\omega^\varepsilon$).** In addition to the hypotheses of Proposition 5.1 when $s = 4$, we further assume that $\delta > 0$ is chosen small enough such that the initial hypothesis (2.7) holds. Then there exists a uniform life span $T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,y}_x}, U) > 0$, which is independent of $\varepsilon$, such that the regularized vorticity system (4.3)–(4.4) has a solution $\omega^\varepsilon \in C([0, T]; R^{s,2,y}) \cap C^1([0, T]; H^{s-2,y}_x)$ with the following uniform (in $\varepsilon$) estimates:

(i) (Uniform Weighted $H^s$ Estimate) For any $\varepsilon \in [0, 1]$ and any $t \in [0, T]$,
\begin{equation}
\|\omega^\varepsilon(t)\|_{H^{s,y}_x} \leq 4 \|\omega_0\|_{H^{s,y}_x}, \tag{6.1}
\end{equation}
where the weighted norm $\|\cdot\|_{H^{s,y}_x}$ is defined in (3.1).

(ii) (Uniform Weighted $L^\infty$ Bound) For any $\varepsilon \in [0, 1]$ and $t \in [0, T]$,
\begin{equation}
\left\| \sum_{|\sigma| \leq 2} |(1 + y)\sigma_0 + \sigma_2 D^\sigma \omega^\varepsilon(t)| \right\|_{L^\infty(T \times \mathbb{R}^+)} \leq \frac{1}{\delta^2}. \tag{6.2}
\end{equation}
(iii) \((\text{Uniform Weighted } L^\infty \text{ Lower Bound})\) For any \(\epsilon \in [0, 1]\) and \(t \in [0, T]\),
\[
\min_{T \times \mathbb{R}^+} (1 + y)\omega^\epsilon(t) \geq \delta.
\]

\text{PROOF.} The uniform life span \(T := T(s, y, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}_y}, U)\) can be guaranteed by using the standard continuous induction argument and the uniform estimates \((6.1)-(6.3)\), so it suffices to verify these estimates. Indeed, the life span \(T\) can be chosen as \(\min\{T_1, T_2, T_3\}\) where \(T_1, T_2, \) and \(T_3\) will be defined below.

(i) According to the definition \((5.2)\) of \(F\) and the regularized Bernoulli’s law \((4.2)\),
\[
\|F\|_{L^\infty} \leq C_{s, y, \sigma, \delta} \left(1 + \sum_{l=0}^{\frac{s}{2}+1} \|\partial_t^l U\|^2_{H^{s-2l+2}_y(T)}\right)^{\frac{1}{2}} \leq C_{s, y, \sigma, \delta} M_U
\]
where \(M_U := \sup_t \{1 + \sum_{l=0}^{(s/2)+1} \|\partial_t^l U\|^2_{H^{s-2l+2}_y(T)}\} < +\infty\), so if we choose
\[
T_1 := \min \left\{ \frac{3\|\omega_0\|^2_{H^{s,\gamma}_y}}{C_{s, y, \sigma, \delta} M_U}, \frac{1 - 2^{-s+2}}{2^{s-2} C_{s, y, \sigma, \delta} \|\omega_0\|_{H^{s-2,\gamma}_y}} \right\},
\]
then by inequality \((5.1)\), estimate \((6.1)\) holds for all \(t \in [0, T_1]\).

(ii) When \(s \geq 6\), using part (i) of Proposition \((6.1)\) we know from definition \((5.4)\) of \(\Omega\) and \(G\) that for any \(t \in [0, T_1]\) where \(T_1\) is defined in part (i),
\[
\Omega(t) \leq 4\|\omega_0\|_{H^{s,\gamma}_y} \quad \text{and} \quad G(t) \leq 4\|\omega_0\|_{H^{s,\gamma}_y} + M_U =: K.
\]
Thus, if we choose
\[
T_2 := \min \left\{ T_1, \frac{1}{64\delta^2 C_{s, y, \sigma} (1 + 4\|\omega_0\|_{H^{s,\gamma}_y})^\alpha \omega_0^2_{H^{s,\gamma}_y}}, \frac{\ln 2}{C_{s, y, \sigma, \delta}(1 + K)} \right\},
\]
then using inequality \((5.2)\) and the initial assumption
\[
\sum_{|\alpha| \leq 2} |(1 + y)^\sigma + \sigma_2 D^\sigma \omega_0|^2 \leq \frac{1}{4\delta^2},
\]
we have the upper bound \((6.2)\) for all \(t \in [0, T_2]\).

When \(s = 4\), using inequality \((5.3)\), estimate \((6.4)\) and the initial hypothesis \((2.7)\), we also have the upper bound \((6.2)\) for all \(t \in [0, T_2]\).

(iii) Let us choose
\[
T_3 := \min \left\{ T_1, \frac{\delta}{8C_{s, y, \sigma} \omega_0^2_{H^{s,\gamma}_y}}, \frac{1}{6C_{s, y, \sigma, \delta}(1 + K)} \right\}.
\]
Then using inequalities \((5.6)\) and \((6.4)\), we know that the lower bound \((6.3)\) holds for all \(t \in [0, T_3]\).
Convergence and Consistency

Using almost equivalence relation (A.1) and uniform weighted $H^s$ estimate (6.1), we have

\begin{equation}
(6.5) \sup_{0 \leq t \leq T} \left( \| \omega^\varepsilon \|_{H^{s,\gamma}} + \| u^\varepsilon - U \|_{H^{s,\gamma-1}} \right) \leq C_{s,\gamma, \sigma, \delta} \left( \| \omega_0 \|_{H^{s,\gamma}_k} + \sup_{0 \leq t \leq T} \| \partial_x^\sigma U \|_{L^2(\mathbb{R})} \right) < +\infty.
\end{equation}

Furthermore, using evolution equations (5.28), uniform $H^s$ bound (6.5), and Proposition B.3, one also finds that $\partial_t \omega$ and $\partial_t (u^\varepsilon - U)$ are uniformly (in $\varepsilon$) bounded in $L^\infty([0, T]; H^{s-2,\gamma})$ and $L^\infty([0, T]; H^{s-2,\gamma-1})$, respectively. By the Lions-Aubin lemma and the compact embedding of $H^{s,\gamma}$ in $H^{s,\gamma}_0$ stated in Lemma 6.2, we have, after taking a subsequence, as $\varepsilon_k \to 0^+$,

\begin{equation}
(6.6)
\begin{cases}
\omega^{\varepsilon_k} \to \omega & \text{in } L^\infty([0, T]; H^{s,\gamma}), \\
\omega^{\varepsilon_k} \to 0 & \text{in } C([0, T]; H^{s,\gamma}_0), \\
u^{\varepsilon_k} - U \to u - U & \text{in } L^\infty([0, T]; H^{s,\gamma-1}), \\
u^{\varepsilon_k} \to u & \text{in } C([0, T]; H^{s,\gamma}_0),
\end{cases}
\end{equation}

for all $s' < s$, where $\omega = \partial_t u \in L^\infty([0, T]; H^{s,\gamma}) \cap \bigcap_{s' < s} C([0, T]; H^{s,\gamma}_0)$ and $u - U \in L^\infty([0, T]; H^{s,\gamma-1}) \cap \bigcap_{s' < s} C([0, T]; H^{s,\gamma}_0)$. Using the local uniform convergence of $\partial_x u^{\varepsilon_k}$, we also have the pointwise convergence of $v^{\varepsilon_k}$: as $\varepsilon_k \to 0^+$,

\begin{equation}
(6.7) v^{\varepsilon_k} = -\int_0^y \partial_x u^{\varepsilon_k} \, dy \to -\int_0^y \partial_x u \, dy =: v.
\end{equation}

Combining (6.6)–(6.7), one may justify the pointwise convergence of all terms in the regularized Prandtl equations (4.1)–(4.4). Thus, passing to the limit $\varepsilon_k \to 0^+$ in (4.1)–(4.4)4 and the regularized Bernoulli’s law (4.2), we know that the limit $(u, v)$ solves the Prandtl equations (2.1)–(2.4) with the Bernoulli’s law (2.3) in the classical sense.

Lastly, in order to complete the proof of consistency, it remains to justify that $\omega \in L^\infty([0, T]; H^{s,\gamma}_\sigma)$ and the matching condition (2.1)5. Since $D^\sigma \omega^{\varepsilon_k}$ converges pointwise to $D^\sigma \omega$ for all $|\alpha| \leq 2$ and $\omega^{\varepsilon_k}$ satisfies

\begin{equation}
(6.8) \sum_{|\alpha| \leq 2} (1 + y)^{s+\alpha_2} D^\sigma \omega^{\varepsilon_k} \leq \frac{1}{\delta^2} \quad \text{and} \quad (1 + y)^{s} \omega^{\varepsilon_k} \geq \delta,
\end{equation}

we deduce that (6.8) still holds for $\omega$, and hence $\omega \in L^\infty([0, T]; H^{s,\gamma}_\sigma)$. Also, by Lebesgue’s dominated convergence theorem,

\[ \int_0^{+\infty} \omega \, dy = \lim_{\varepsilon_k \to 0^+} \int_0^{+\infty} \omega^{\varepsilon_k} \, dy = U, \]

which is equivalent to the matching condition (2.1)5.
To complete the proof of existence, let us state and prove the following:

**Lemma 6.2.** Let $s$ be a positive integer, $\gamma' \geq 0$, and $M < +\infty$. Assume

\begin{equation}
\|f^\epsilon\|_{H^{s,\gamma'}} \leq M
\end{equation}

for all $\epsilon \in (0, 1]$. Then there exist a function $f \in H^{s,\gamma'}$ and a sequence $\{\epsilon_k\}_{k \in \mathbb{N}} \subseteq (0, 1]$ with $\lim_{k \to +\infty} \epsilon_k = 0^+$ such that as $\epsilon_k \to 0^+$,

\begin{equation}
\epsilon_k \to f \quad \text{and} \quad f^\epsilon_k \to f^{s'} \quad \text{for all } s' < s.
\end{equation}

**Proof of Lemma 6.2.** First of all, let us mention that $H^{s,\gamma'}$ has an inner product structure:

$$
\langle \phi, \psi \rangle_{H^{s,\gamma'}} := \sum_{|\alpha| \leq s} \int_0^{+\infty} (1 + \gamma')^{2\gamma' + 2\alpha_2} D^\alpha \phi D^\alpha \psi,
$$

so the uniform bound (6.9) implies the weak convergence of $f^\epsilon_k$ in (6.10) via the Banach-Alaoglu theorem.

Next, by the definition of $\|f\|_{H^{s,\gamma'}}$, $\|f^\epsilon\|_{H^s} \leq \|f^\epsilon\|_{H^{s,\gamma'}} \leq M$. This implies the local $H^{s'}$-norm convergence in (6.10) for all $s' < s$ because of the standard compactness of $H^s(\mathbb{T} \times \mathbb{R}^+)$.

Finally, let us end this subsection by giving the following:

**Remark 6.3 (U ≜ const).** In the special case that $U > 0$ is a constant, one may prove that the life span $T$ stated in Theorem 2.2 is independent of the constant value of $U$, that is, $T := T(s, \gamma, \sigma, \delta, \|w_0\|_{H^{s,\gamma'}})$. Let us explain as follows.

When $U ≜ const$, it follows from the regularized Bernoulli’s law (4.2) that $\partial_x p^\epsilon = 0$, so by definitions (5.2) and (5.4), we have $F = C_{s,\gamma,\sigma,\delta}$ and $G(t) = \Omega(t) = \|w_0\|_{H^{s,\gamma'}}$ where all of $F$, $G$, and $\Omega$ are independent of $U$. As a result, all of our weighted estimates (5.1), (5.3), (5.5), and (5.6) are independent of $U$. Therefore, one may slightly modify the proof of Proposition 6.1 to show that uniform weighted estimates (6.1)–(6.3) hold in a time interval $[0, T_{s,\gamma,\sigma,\delta,\|w_0\|_{H^{s,\gamma'}}}]$, which is independent of $U$ as well. According to our proof of convergence and consistency in Section 6.1, we can solve the Prandtl equations (2.1) in the same time interval, and hence the life span $T$ stated in Theorem 2.2 is also independent of $U$.

### 6.2 Uniqueness for the Prandtl Equations

The aim of this subsection is to prove the uniqueness of $H^{s,\gamma'}_{\sigma,\delta}$ solutions constructed in Section 6.1. To show the uniqueness, we will generalize the nonlinear
cancellation applied in Section 5.1 to the $L^2$ comparison of two $H^{s,\gamma}_{\sigma,\delta}$ solutions. This motivates us to consider the quantity $\tilde{g}$ below.

Specifically, the uniqueness of $H^{s,\gamma}_{\sigma,\delta}$ solutions to the Prandtl equations (2.1) is a direct consequence of the following $L^2$ comparison principle.

**PROPOSITION 6.4 ($L^2$ Comparison Principle).** For any $s \geq 4$, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, and $\delta \in (0,1)$, let $(u_i, v_i)$ solve the Prandtl equations (2.1) with the vorticity $\omega_i := \partial_y u_i \in C([0,T]; H^{s,\gamma}_{\sigma,\delta}) \cap C^1([0,T]; H^{s-2,\gamma})$ for $i = 1, 2$. Define $\tilde{g} := \omega_1 - \omega_2 - \frac{\partial_x \omega_2}{\omega_2} (u_1 - u_2)$. Then we have

\[
(6.11) \quad \|\tilde{g}(t)\|_{L^2}^2 + \int_0^t \|\partial_y \tilde{g}\|_{L^2}^2 \leq \|\tilde{g}(0)\|_{L^2}^2 + C_{\gamma,\sigma,\delta,\omega, U} \int_0^t \|\tilde{g}\|_{L^2}^2
\]

where the positive constant $C_{\gamma,\sigma,\delta,\omega, U}$ depends on $\gamma$, $\sigma$, $\delta$, $\|\omega_1\|_{H^{s+\gamma}}$, $\|\omega_2\|_{H^{s+\gamma}}$, and $\|\partial_x \omega U\|_{L^2(T)}$ only.

Assuming Proposition 6.4, which will be shown later in this subsection, for the moment we will prove the uniqueness as follows.

Applying Gronwall’s lemma to (6.11), we obtain

\[
\|\tilde{g}(t)\|_{L^2}^2 \leq \|\tilde{g}(0)\|_{L^2}^2 e^{C_{\gamma,\sigma,\delta,\omega, U} t},
\]

which implies $\tilde{g} \equiv 0$ provided that $u_1|_{t=0} \equiv u_2|_{t=0}$. Since $\omega_2 \partial_y \left( \frac{u_1 - u_2}{\omega_2} \right) = \tilde{g} \equiv 0$, we have

\[
(6.12) \quad u_1 - u_2 = q \omega_2
\]

for some function $q := q(t,x)$. Using the Oleinik’s monotonicity assumption $\omega_2 \geq 0$ and the Dirichlet boundary condition $u_i|_{y=0} \equiv 0$ for $i = 1, 2$, we know via (6.12) that $q \equiv 0$ and hence $u_1 \equiv u_2$. Since $v_i$ can be uniquely determined by $u_i$, we also have $v_1 \equiv v_2$. This proves the uniqueness of $H^{s,\gamma}_{\sigma,\delta}$ solutions.

In the rest of this subsection, we will prove Proposition 6.4 as follows.

**PROOF OF PROPOSITION 6.4.** Let us denote $(\tilde{u}, \tilde{v}) := (u_1 - u_2, v_2 - v_1)$, $\tilde{\omega} := \omega_1 - \omega_2$ and $\tilde{a}_2 := \frac{\partial_x \omega_2}{\omega_2}$. Then one may check that $\tilde{g} = \tilde{\omega} - \tilde{a}_2 \tilde{u} = \omega_2 \partial_y \left( \frac{\tilde{u}}{\omega_2} \right)$ and satisfies

\[
(6.13) \quad (\partial_t + u_1 \partial_x + v_1 \partial_y - \partial_y^2)\tilde{g} = -2 \tilde{\omega} \partial_y \tilde{a}_2 - \tilde{u} \{ \tilde{u} \partial_x \tilde{a}_2 + \tilde{v} \partial_y \tilde{a}_2 + 2 \tilde{a}_2 \partial_y \tilde{a}_2 \}.
\]

To derive the $L^2$-estimate for $\tilde{g}$, let us first recall that we define the cutoff function $\chi_R(y) := \chi \left( \frac{y}{R} \right)$ for any $R \geq 1$, where $\chi \in C_c^\infty((0, +\infty))$ satisfies the properties (4.7). Then $\chi_R$ has the following pointwise properties: as $R \to +\infty$,

\[
\chi_R \to 1_{\mathbb{R}^+}, \quad \left| \chi'_R \right| \leq \frac{2}{R} \to 0^+, \quad \text{and} \quad \left| \chi''_R \right| \leq O \left( \frac{1}{R^2} \right) \to 0^+.
\]
For any \( t \in (0, T] \), multiplying equation (6.13) by \( 2\chi_R \bar{g} \) and then integrating over \([0, t] \times \mathbb{T} \times \mathbb{R}^+\), we obtain, via integration by parts,

\[
\begin{align*}
\iint \chi_R \bar{g}^2(t)dy\,dx &- \iint \chi_R \bar{g}^2|_{t=0}dy\,dx \\
= -2\int_0^t \iint \chi_R |\partial_y \bar{g}|^2 - 2\int_0^t \int_T \bar{g} \partial_y\bar{g}|_{y=0} \,dx \\
&- 4\int_0^t \iint \chi_R \bar{g} \partial_y a_2 \\
&- 2\int_0^t \iint \chi_R \bar{g} \{ \bar{u} \partial_x a_2 + \bar{v} \partial_y a_2 + 2a_2 \partial_y a_2 \} \\
&+ \mathcal{R}_1 + \mathcal{R}_2 \\
\leq -2\int_0^t \iint \chi_R |\partial_y \bar{g}|^2 - 2\int_0^t \int_T \bar{g} \partial_y\bar{g}|_{y=0} \,dx \\
&+ 4\|\partial_y a_2\|_{L^\infty} \int_0^t \|\bar{g}\|_{L^2} \|\bar{u}\|_{L^2} \\
&+ 2\|(1+y)\{ \bar{u} \partial_x a_2 + \bar{v} \partial_y a_2 + 2a_2 \partial_y a_2 \}\|_{L^\infty} \\
&\cdot \int_0^t \|\bar{g}\|_{L^2} \left\| \frac{\bar{u}}{1+y} \right\|_{L^2} \\
&+ \mathcal{R}_1 + \mathcal{R}_2, 
\end{align*}
\tag{6.14}
\]

where the remainder terms \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are defined by

\[
\mathcal{R}_1 := \int_0^t \iint \chi_R' v_1 \bar{g}^2 \quad \text{and} \quad \mathcal{R}_2 := \int_0^t \iint \chi_R'' \bar{g}^2.
\]

Now, the first technical problem is to deal with the boundary integral

\[
\int_0^t \int_T \bar{g} \partial_y \bar{g}|_{y=0} \,dx.
\]
Since $\partial_y \vec{g}|_{y=0} = -a_2 \vec{g}|_{y=0}$, we have, after applying the simple trace estimate (5.15),
\begin{align*}
\left| \int_0^t \int_T \tilde{g} \partial_y \vec{g} |_{y=0} \, dx \right| & \leq \frac{1}{2} \int_0^t \int \chi_R |\partial_y \vec{g}|^2 \\
& + C \{\|a_2\|_{L^\infty} + \|a_2\|_{L^\infty}^2 + \|\partial_y a_2\|_{L^\infty}\} \int_0^t \int \tilde{g}^2.
\end{align*}
(6.15)

Furthermore, since $\omega_2 \in H^{s,\nu}_{\sigma,\delta}$, it follows from the weighted $L^\infty$ bounds for $\omega_2$ that
\begin{align*}
\| (1 + y) a_2 \|_{L^\infty} & \leq \delta^{-2}, \\
\| (1 + y) \partial_x a_2 \|_{L^\infty}, \| (1 + y)^2 \partial_y a_2 \|_{L^\infty} & \leq \delta^{-2} + \delta^{-4}.
\end{align*}
(6.16)

so by Proposition B.3
\begin{align*}
\| (1 + y) \{ \tilde{u} \partial_x a_2 + \tilde{v} \partial_y a_2 + 2 a_2 \partial_y a_2 \} \|_{L^\infty} & \leq \\
C_{\gamma,\sigma,\delta} \{ 1 + \| \omega_1 \|_{H^{4,\nu}_{E^2}} + \| \omega_2 \|_{H^{4,\nu}_{E^2}} + \| \partial_x^4 U \|_{L^2(T)} \}. \\
\end{align*}
(6.17)

Combining estimates (6.14)–(6.17), we obtain
\begin{align*}
\int \int \chi_R \tilde{g}^2(t) dy \, dx - \int \int \chi_R \tilde{g}^2|_{t=0} dy \, dx & \leq - \int_0^t \int \chi_R |\partial_y \vec{g}|^2 + C_\delta \int_0^t \| \tilde{g} \|_{L^2} \\
& + C_{\gamma,\sigma,\delta,\omega_1} \| \omega_1 \|_{H^{4,\nu}_{E^2}}, \| \omega_2 \|_{H^{4,\nu}_{E^2}}, \| \partial_x^4 U \|_{L^2} \\
& \cdot \left\{ \| \tilde{u} \|_{L^2} + \left\| \frac{\tilde{u}}{1 + y} \right\|_{L^2} \right\} + R_1 + R_2.
\end{align*}
(6.18)

Next, we emphasize that both $\tilde{\omega}$ and $\frac{\tilde{u}}{1+y}$ can be controlled by $\tilde{g}$.

**Claim 6.5.**
\begin{align*}
\| \tilde{\omega} \|_{L^2}, \left\| \frac{\tilde{u}}{1+y} \right\|_{L^2} & \leq C_{\sigma,\delta} \| \tilde{g} \|_{L^2}.
\end{align*}
(6.19)

The proof of Claim 6.5 is very similar to that of Lemma A.2, so we will only outline it at the end of this subsection. Assuming Claim 6.5 for the moment, we
can apply (6.19) to (6.18) to obtain
\[
\begin{aligned}
\int_0^T \int_R \mathcal{R} \tilde{g}^2(t) dy \ dx - \int_0^T \int_R \mathcal{R} \tilde{g}^2|_{t=0} dy \ dx \\
\leq - \int_0^T \int_R \mathcal{R} |\partial_y \tilde{g}|^2 + C_{y, \sigma, \delta} \|\omega_1\|_{H^2_g, \gamma} \|\omega_2\|_{H^2_g, \gamma} \|\partial_y U\|_{L^2} \int_0^T \|\tilde{g}\|_{L^2}^2 \\
+ \mathcal{R}_1 + \mathcal{R}_2.
\end{aligned}
\] (6.20)

Finally, both integrands of \(\mathcal{R}_1\) and \(\mathcal{R}_2\) can be controlled by a multiple of \(\tilde{g}^2\), which belongs to \(L^1([0, T]; \mathbb{T} \times \mathbb{R}^+)\), so applying Lebesgue’s dominated convergence theorem, we have
\[
\lim_{R \to +\infty} \mathcal{R}_i = 0 \quad \text{for } i = 1, 2.
\] (6.21)

Using the monotone convergence theorem and (6.21), we can pass to the limit \(R \to +\infty\) in (6.20) to obtain (6.11). \(\square\)

Lastly, we will justify Claim 6.5 as follows.

**Proof of Claim 6.5** Using the triangle inequality and (6.16), we have
\[
\|\tilde{u}\|_{L^2} \leq \|\tilde{g}\|_{L^2} + \delta^{-2} \|\tilde{u}/(1 + y)\|_{L^2},
\]
so it suffices to control \(\tilde{u}\).

Since \(\delta \leq (1 + y)^\sigma \omega_2 \leq \delta^{-1}\) and \(\tilde{u}|_{y=0} \equiv 0\), applying part (ii) of Lemma B.1, we obtain
\[
\left\| \frac{\tilde{u}}{1 + y} \right\|_{L^2} \leq (1 + y)^{-\sigma} \left\| \frac{\tilde{u}}{\omega_2} \right\|_{L^2} \leq C_{\sigma, \delta} \left\| (1 + y)^{-\sigma} \partial_y \left( \frac{\tilde{u}}{\omega_2} \right) \right\|_{L^2} \leq C_{\sigma, \delta} \|\tilde{g}\|_{L^2}
\]
because \(\tilde{g} = \omega_2 \partial_y \left( \frac{\tilde{u}}{\omega_2} \right)\). \(\square\)

### 7 Existence for the Regularized Prandtl Equations

The aim of this section is to solve the regularized Prandtl equations (4.1), or, equivalently, its vorticity system (4.3)–(4.4). In other words, we will prove Proposition 5.1 according to the plan described in Section 4. However, we will only sketch our proof because the methods for solving intermediate approximate systems (4.3)–(4.4), (4.5)–(4.6), and (4.8)–(4.9) are standard.

Before we proceed, let us remark that we will solve the approximate systems (4.8)–(4.9), (4.5)–(4.6), and (4.3)–(4.4) with a decreasing order of regularities. The main reason for this technical arrangement is to derive our estimates in a rigorous way so that we can differentiate the intermediate equations rigorously and have enough pointwise decay at \(y = +\infty\) according to Proposition C.1.
7.1 Solvability of Inhomogeneous Heat Equation

In this first subsection we will solve an inhomogeneous heat equation in the weighted space $H^{s,\gamma}_{\sigma,\delta}$. This existence result will be applied to solve the linearized, truncated, and regularized vorticity system \( \text{(4.8)} - \text{(4.9)} \) in the next subsection.

Let us consider the following inhomogeneous heat equation: for any $\epsilon > 0$,

\[
\begin{align*}
\partial_t W + F_R &= \epsilon^2 \partial_x^2 W + \partial_y^2 W \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\
W|_{t=0} &= W_0 \quad \text{ on } \mathbb{T} \times \mathbb{R}^+, \\
\partial_y W|_{y=0} &= \partial_x p^\epsilon \quad \text{ on } [0, T] \times \mathbb{T},
\end{align*}
\]

(7.1)

where $W$ is an unknown, $W_0$ and $\partial_x p^\epsilon$ are given initial and boundary data, and $F_R$ is a given inhomogeneous term with compact support in $[0, T] \times \mathbb{T} \times [0, 2\mathbb{R}]$. Since (7.1) is just a standard inhomogeneous heat equation, we can solve it by classical methods and obtain the following:

**Proposition 7.1 (Existence of Inhomogeneous Heat Equation).** Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, \frac{1}{2})$, and $\epsilon \in (0, 1]$. If $W_0 \in H^{s+12,\gamma}_{\sigma,\delta}$ and $F_R \subseteq [0, T] \times \mathbb{T} \times [0, 2 \mathbb{R}]$, then there exist a time $T := T(s, \gamma, \sigma, \delta, R, \|W_0\|_{H^{s+8,\gamma}}, \|F_R\|_{C^2}) > 0$ and a solution

\[
W \in C([0, T]; H^{s+8,\gamma}_{\sigma,\delta}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)
\]

(7.1)

to the inhomogeneous heat equation (7.1).

Furthermore, we have the following pointwise decay at $y = +\infty$: for any $l = 0, 1, \ldots, \frac{s}{2} + 4$, for any $|\alpha| \leq s - 2l + 9$,

\[
\partial_x^l D^\alpha W = \begin{cases} 
O((1 + y)^{-\sigma - \alpha^2}) & \text{if } |\alpha| + 2l \leq 2, \\
O((1 + y)^{-\frac{\sigma + 2|\alpha| + 2|\alpha| - 1}{2|\alpha| + 2|\alpha| - 2} - \alpha^2}) & \text{if } 2 \leq |\alpha| + 2l \leq s + 9,
\end{cases}
\]

and energy estimate

\[
\frac{d}{dt} \|W\|^2_{s+8,\gamma} + \epsilon^2 \|\partial_x W\|^2_{s+8,\gamma} + \|\partial_y W\|^2_{s+8,\gamma} \leq C_{s,\gamma} \|W\|^2_{s+8,\gamma} + C_s \|W\|^2_{s+8,\gamma} \|F_R\|_{s+8,\gamma} + C_\delta \|F_R\|^2_{s+7,\gamma} + C_{\epsilon} \|\partial_x p^\epsilon\|^2_{s+8},
\]

(7.3)

where the norms $\|\cdot\|_{s,\gamma}$ and $\|\cdot\|_{s',\gamma}$ are defined in Definition \( \text{(B.4)} \).

**Outline of the Proof.** Using the method of reflection and Duhamel’s principle, one may express the unique global-in-time $C([0, T] \times \mathbb{T} \times \mathbb{R}^+) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ solution to (7.1) by an explicit solution formula

\[
W = K_W + K_{\partial_x p^\epsilon} + K_{F_R}
\]

(7.4)

where the terms $K_W$, $K_{\partial_x p^\epsilon}$, and $K_{F_R}$ can be written explicitly by using the Gaussian (i.e., the heat kernel) and depend on $W_0$, $\partial_x p^\epsilon$, and $F_R$, respectively. Since the solution formula (7.4) is explicit, based on the properties of the Gaussian
and the fact that $F_R$ has compact support, one may prove the following two facts:
as $y \to +\infty$,

(i) both $K_{s,p}$ and $K_{F,R}$ as well as their derivatives decay exponentially fast;

(ii) the term $D^\alpha K W_0 \lesssim (1 + y)^{-\beta \alpha}$ provided that $D^\alpha W_0$ is.

Using quantitative versions of facts (i) and (ii), one can justify by (7.4) that $W$ fulfills all weighted $L^\infty$ controls for $H^{s+8,\gamma}_{\sigma,\delta}$ within a short time interval $[0, T_{s,\gamma,\sigma,\delta, R, ||W_0||_{H^{s+8,\gamma}}, ||F_R||_{C^2}}]$.

Furthermore, applying Proposition [C.1] with $s' = s + 8$, we know that $W_0$ satisfies the pointwise decay (7.2) for $l = 0$ and $|\alpha| \leq s + 9$, and hence by (7.4) and facts (i) and (ii) again, $W$ satisfies the same decay estimate as well. Using the heat equation (7.1) repeatedly, we also obtain the pointwise decay (7.2) in the desired ranges of $l$ and $\alpha$.

Finally, it remains to show the energy estimate (7.3), but its proof just follows from the standard energy methods, so we will omit the proof here. However, during the estimation, one must apply the integration by parts in the $y$-variable to deal with the operator $\partial_y^2$, so we would like to give the following two remarks regarding the boundary values of $W$:

(I) (Boundary Values at $y = 0$) The boundary values of $W$ as well as its derivatives at $y = 0$ can be reconstructed by using the boundary reduction formula

$$\partial_y^{2k+1} W|_{y=0} = \left( \partial_t - \epsilon^2 \partial_x^2 \right)^{k} \partial_x p + \sum_{j=0}^{k-1} \left( \partial_t - \epsilon^2 \partial_x^2 \right)^{k-j-1} \partial_y^{2j+1} F_R \right|_{y=0},$$

which reduces the order of the boundary terms so that we can control the boundary integral at $y = 0$ via the simple trace estimate (5.15).

(II) (Boundary Values at $y = +\infty$) All boundary terms of $W$ as well as its derivatives required for deriving energy estimate (7.3) actually vanish fast enough at $y = +\infty$ because of the pointwise decay estimate (7.2). Thus, all required boundary integrals at $y = +\infty$ are equal to zero. \hfill \Box

### 7.2 Solvability of Linearized, Truncated, and Regularized Vorticity System

Using the $H^{s,\gamma}_{\sigma,\delta}$ solutions to the inhomogeneous heat equation (7.1) derived in Section [7.1], we will construct a sequence of solutions to the linearized, truncated, and regularized vorticity system (4.8)–(4.9) with uniform bounds in this subsection. This sequence of solutions as well as their uniform bounds will be the foundation for solving the truncated and regularized vorticity system (4.5)–(4.6) in the next subsection.

Let us begin by defining an iterative sequence $\{(u^n, v^n, \omega^n)\}_{n \in \mathbb{N}}$ as follows:

(i) $\omega^0(t, x, y) := \omega_0(x, y)$;

(ii) $(u^n, v^n)$ is defined by formulae (4.9) for all $n \in \mathbb{N}$;
(iii) $\omega^{n+1}$ is defined to be the $C([0, T]; H^{s+8, \gamma}_{\sigma, \delta}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ solution to the linearized, truncated, and regularized vorticity system (4.8)–(4.9) for all $n \in \mathbb{N}$.

The natural question is whether the iterative sequence $\{(u^n, v^n, \omega^n)\}_{n \in \mathbb{N}}$ is well-defined, and the answer is affirmative because of the following:

**Proposition 7.2 (Existence of Linearized, Truncated, and Regularized Vorticity System).** Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, \frac{1}{2})$, $\epsilon \in (0, 1]$, and $R \geq 1$. If $\omega_0 \in H^{s+12, \gamma}_{\sigma, 2\delta}$ and $\sup_n \|U\|_{s+9, \infty} < +\infty$ where the norm $\|\cdot\|_{s', \infty}$ is defined as in Definition B.4 then there exist a uniform (in $n$) life span $T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \|\omega_0\|_{H^{s+8, \gamma}}, \sup_n \|U\|_{s+9, \infty}) > 0$ and a sequence of solutions $\{\omega^n\}_{n \in \mathbb{N}} \subseteq C([0, T]; H^{s+8, \gamma}_{\sigma, \delta}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ to the linearized, truncated, and regularized vorticity system (4.8)–(4.9).

Furthermore, the pointwise decay estimate (7.2) holds for $W := \omega^n$ for all $n \in \mathbb{N}$, and we have the following uniform (in $n$) energy estimate: for all $n \in \mathbb{N}$ and for all $t \in [0, T]$,

$$
(7.5) \quad \|\omega^n\|_{s+8, \gamma}^2 + 2 \int_0^t \|\partial_x \omega^n\|_{s+8, \gamma}^2 + \int_0^t \|\partial_y \omega^n\|_{s+8, \gamma}^2 \leq \mathcal{Q}_{s+10}(\|\omega_0\|_{H^{s+8, \gamma}})
$$

where the norm $\|\cdot\|_{s', \gamma}$ is defined in Definition B.4 and $\mathcal{Q}_l$ is a degree $l$ polynomial with nonnegative coefficients that depend on $s$, $\gamma$, $\chi$, and $\|U\|_{s+8, \infty}$ only.

**Outline of the Proof.** For a given $\omega_0^n \in C([0, T]; H^{s+8, \gamma}_{\sigma, \delta}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$, local-in-time solvability of $\omega^{n+1}$ in the same function space and decay estimate (7.2) for $\omega^{n+1}$ follow directly by applying Proposition 7.1 with $W := \omega^n$ and $F_R := \chi_R \{u^n \partial_x \omega^n + v^n \partial_y \omega^n\}$, although the life span $T$ may depend on $n$. However, the uniform (in $n$) energy estimate (7.5) guarantee the uniform (in $n$) life span $T$ by the standard continuous induction argument, so it suffices to prove (7.5).

In order to derive the energy estimate (7.5), we have to control $\|F_R\|_{s+7, \gamma}$ and $\|F_R\|_{s+8, \gamma}$ for $F_R := \chi_R \{u^n \partial_x \omega^n + v^n \partial_y \omega^n\}$. Using the triangle inequality, Proposition B.5 and Proposition B.6 one may check that

$$
(7.6) \quad \left\{ \begin{array}{l}
\|F_R\|_{s+7, \gamma} \leq C_{s, \gamma, \chi, \|\omega^n\|_{s+8, \gamma}}^2 + \|U\|_{s+8, \infty}^2,
\|F_R\|_{s+8, \gamma} \leq C_{s, \gamma, \chi, R} \{\|\omega^n\|_{s+8, \gamma} + \|U\|_{s+9, \infty}\} \cdot \{\|\partial_x \omega^n\|_{s+8, \gamma} + \|\partial_y \omega^n\|_{s+8, \gamma}\},
\end{array} \right.
$$

where the norm $\|\cdot\|_{s', \infty}$ is defined in Definition B.4.

Using (7.3) and (7.6), one can easily show that as long as $\|\omega^n\|_{s+8, \gamma}|_{t=0} \leq L$, there exists a uniform (in $n$) time interval $[0, T; s, \gamma, \sigma, \delta, \epsilon, \chi, R, \sup_t \|U\|_{s+9, \infty}, L]$ such
that

\[
\|\omega^n\|_{s+8,\gamma}^2 + c^2 \int_0^t \|\partial_x \omega^n\|_{s+8,\gamma}^2 + \int_0^t \|\partial_y \omega^n\|_{s+8,\gamma}^2 \leq 4L^2
\]

for all \(n \in \mathbb{N}\) because all constants in (7.3) and (7.6) are independent of \(n\). Thus, it remains to derive a uniform (in \(n\)) control on the initial data \(\|\omega^n\|_{s+8,\gamma}|_{t=0}\).

To estimate \(\|\omega^n\|_{s+8,\gamma}|_{t=0}\), let us first state without proof the following fact:

\[
(7.8) \quad \|\partial_l \omega^n\|_{H^{s-2l+8,\gamma}}|_{t=0} \leq P_{l+1}(\|\omega_0\|_{H^{s+8,\gamma}})
\]

where \(P_{l+1}\) is a degree \(l + 1\) polynomial defined by

\[
P_1(Z) := Z \quad \text{and} \quad P_{l+1} := P_1 + C_{s,\gamma,x} \sum_{j=0}^{l-1} (P_{j+1} + \|U\|_{s+8,\infty})P_{l-j}
\]

for all \(l \geq 1\).

The fact (7.8) can be proved by induction on \((n, l)\) together with the following estimate:

\[
(7.9) \quad Y_{n+1,l+1} \leq Y_{n+1,l} + C_{s,\gamma,x} \sum_{j=0}^{l} (Y_{n,j} + \|U\|_{s+8,\infty})Y_{n,l-j}
\]

where \(Y_{n,l} := \|\partial_l \omega^n\|_{H^{s-2l+8,\gamma}}|_{t=0}\). The derivation of (7.9), which is based on (4.8) \(1\), Proposition B.5 and Proposition B.6 will be left to the reader.

Combining estimates (7.7) and (7.8), we show the uniform energy estimate (7.5) for \(Q_{s+10} := 4 \sum_{l=0}^{(s/2)+4} P_{l+1}^2\).

### 7.3 Solvability of Truncated and Regularized Vorticity System

The aim of this subsection is to construct a solution to the truncated and regularized vorticity system (4.5)–(4.6) by passing to the limit in its linearized version (4.8)–(4.9), which has been solved with uniform bounds in Section 7.2.

In other words, we will prove the following:

**Proposition 7.3 (Existence of Truncated and Regularized Vorticity System).** Let \(s \geq 4\) be an even integer, \(\gamma \geq 1\), \(\sigma > \gamma + \frac{1}{2}\), \(\delta \in (0, \frac{1}{2})\), \(\epsilon \in (0, 1]\), and \(R \geq 1\). If \(\omega_0 \in H^{s+12,\gamma}_{\sigma,28}\) and \(\sup_{t} \|U\|_{s+9,\infty} < +\infty\) where \(\|\cdot\|_{s',\infty}\) is defined in Definition 3.4, then there exist a time

\[
T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \|\omega_0\|_{H^{s+8,\gamma}}, \sup_{t} \|U\|_{s+9,\infty}) > 0
\]

and a solution

\[
\omega_R \in C([0, T]; H^{s+8,\gamma}_{\sigma,\delta}) \cap \bigcap_{l=1}^{\frac{s}{2}+4} C^l([0, T]; H^{s-2l+8,\gamma})
\]

to the truncated and regularized vorticity system (4.5)–(4.6).
Furthermore, we have the following uniform (in $R$) weighted energy estimate:

$$
\frac{d}{dt} \left\| \omega_R \right\|^2_{H^{s+4}, \gamma} + \epsilon^2 \left\| \partial_x \omega_R \right\|^2_{H^{s+4}, \gamma} + \left\| \partial_y \omega_R \right\|^2_{H^{s+4}, \gamma} 
\leq C_{s, \gamma, \epsilon, \chi} \left\{ 1 + \left\| \omega_R \right\|_{H^{s+4}, \gamma} + \left\| U \right\|_{H^{s+6}, \gamma} \right\} \left\| \omega_R \right\|^2_{H^{s+4}, \gamma} 
+ C_s \left\{ 1 + \left\| U \right\|_{H^{s+6}, \gamma} \right\} \left\| U \right\|^2_{H^{s+6}}
$$

(7.10)

and the following weighted $L^\infty$ estimates:

$$
\left\| I_R(t) \right\|_{L^\infty(T \times R^+)} \leq \left( \left\| I_R(0) \right\|_{L^\infty(T \times R^+)} + C\Lambda(t) \sup_{[0,t]} \left\| \omega_R \right\|_{H^{s+4}, \gamma} \right) e^{C_\sigma \Lambda(t) \gamma},
$$

(7.11)

$$
\min_{T \times R^+} \left( 1 + y \right)^\sigma \omega_R(t) \geq (1 - C_\sigma \Lambda(t)) e^{C_\sigma \Lambda(t) \gamma} \left( \min_{T \times R^+} \left( 1 + y \right)^\sigma \omega_0 - C\Lambda(t) \sup_{[0,t]} \left\| \omega_R \right\|_{H^{s+4}, \gamma} t \right).
$$

(7.12)

where the norms $\left\| \cdot \right\|_{s', \gamma}$ and $\left\| \cdot \right\|_{s'}$ are defined in Definition 3.4 and the quantities $I_R$ and $\Lambda$ are defined by

$I_R(t) := \sum_{|\alpha| \leq 2} \left| (1 + y)^{\sigma + \alpha_2} D^\alpha \omega_R(t) \right|^2$,

$\Lambda(t) := 1 + \sup_{[0,t]} \left\| \omega_R \right\|_{H^{s+4}, \gamma} + \sup_{[0,t]} \left\| U \right\|_{C^1(T)}$.

**Outline of the Proof.** According to Proposition 7.2, the sequence of solutions $\{\omega^n\}_{n \in \mathbb{N}}$ to the linearized, truncated, and regularized vorticity system (4.8–4.9) has a uniform (in $n$) life span $[0, T, \gamma, \sigma, \delta, \epsilon, \chi, R, \omega_0, U]$, in which $\left\| \omega^n \right\|_{s+8, \gamma}$ is uniformly bounded by estimate (7.5). Based on this uniform bound, one may apply the standard energy methods to $\omega^{n+1} - \omega^n$ to prove that the approximate sequence $\{\omega^n\}_{n \in \mathbb{N}}$ is indeed Cauchy in the norm $\sup_{t \in [0,T]} \left\| \cdot \right\|_{s+6, \gamma}$, where the time $T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \omega_0)_{s+8, \gamma}$, $\sup_{t} \left\| U \right\|_{s+9, \infty}$ > 0 is independent of $n$. As a result, we can pass to the limit $n \to +\infty$ in (4.8–4.9) to obtain a solution $\omega_R := \lim_{n \to +\infty} \omega^n$ to the truncated and regularized vorticity system (4.5–4.6). Moreover, $\omega_R$ belongs to $C([0, T]; H^{s+8, \gamma}_{\sigma, \delta}) \cap \bigcap_{i=1}^{\frac{s+4}{2}} C^l([0, T]; H^{s-2i+8, \gamma})$ because $\omega^n$ does.

The uniform energy estimate (7.10) follows from the standard energy methods, so its proof will be omitted here. It is worth mentioning that unlike the estimates in Section 7.2, all constants in (7.10) are independent of $R$. This improvement is based on applying integration by parts appropriately to the integral involving the convection term $\int_{R} \{ \mu_R \partial_x \omega_R + v_R \partial_y \omega_R \}$, but it does not exist in (4.8–4.9) because the linearization destroys this structure.
The weighted $L^\infty$ controls (7.11) and (7.12) can be derived by the classical maximum principles (see Lemmas E.1 and E.2 for instance) as in Section 5.2. We leave this to the interested reader.

7.4 Solvability of Regularized Vorticity System and Regularized Prandtl Equations

In this subsection we will construct a solution $\omega^\varepsilon$ to the regularized vorticity system (4.3)–(4.4) by passing to the limit in its truncated version (4.5)–(4.6), whose local-in-time solvability and uniform bounds have been shown in Section 7.3. Furthermore, we will also justify that the velocity $(u^\varepsilon, v^\varepsilon)$ defined by (4.4) solves the regularized Prandtl equations (4.1).

More precisely, we will complete the proof of Proposition 5.1 as follows.

Outlines of the Proof of Proposition 5.1. To solve the regularized vorticity system (4.3)–(4.4), we first choose any function $\chi$ with the properties (4.7) in the truncated and regularized vorticity system (4.5)–(4.6). Then by Proposition 7.3, we have a local-in-time solution $\omega_R$ to (4.5)–(4.6) and uniform bounds (7.10)–(7.12) for $\omega_R$. Since the estimates (7.10)–(7.12) are independent of $R$, one can show that there exists a uniform (in $R$) time $T := T(s, \gamma, \sigma, \delta, \varepsilon, \|\omega_0\|_{H^{s+4, \gamma}}, U) > 0$

such that

$$\{\omega_R\}_{R \geq 1} \subseteq C([0, T]; H^{s+4, \gamma}_{\sigma, \delta}) \cap C^1([0, T]; H^{s+2, \gamma})$$

and

$$\|\omega_R\|_{H^{s+4, \gamma}} \leq C_{s, \gamma, \sigma, \delta, \varepsilon, \|\omega_0\|_{H^{s+4, \gamma}}, U}$$

for all $R \geq 1$. Therefore, by the standard compactness argument, there exist a function $\omega^\varepsilon \in C([0, T]; H^{s+4, \gamma}_{\sigma, \delta}) \cap C^1([0, T]; H^{s+2, \gamma})$ and a subsequence $\{R_k\}_{k \in \mathbb{N}}$ with $\lim_{k \to +\infty} R_k = +\infty$ such that $\omega_{R_k}$ converges to $\omega^\varepsilon$ in $C([0, T]; H^{s+2}_{\text{loc}})$ as $R_k \to +\infty$. As a result, we can pass to the limit $R_k \to +\infty$ in (4.5)–(4.6) and prove that $\omega^\varepsilon$ solves the regularized vorticity system (4.3)–(4.4) in a classical sense.

Finally, we will justify that $(u^\varepsilon, v^\varepsilon)$ defined by (4.4) satisfies the regularized Prandtl equations (4.1) as follows.

First of all, the matching condition (4.1)$_5$, the Dirichlet boundary condition $v^\varepsilon|_{y=0} = 0$, and the initial condition (4.1)$_3$ follows immediately from the formulae (4.4), $\omega_0 := \partial_y u_0$, and the compatibility condition (2.2). Then by direct differentiations on (4.4), we also have the incompressibility condition (4.1)$_2$ and $\omega^\varepsilon = \partial_y u^\varepsilon$.

To justify equation (4.1)$_1$, we substitute $\omega^\varepsilon = \partial_y u^\varepsilon$ into (4.3)$_1$ and obtain, via using (4.1)$_2$,

$$(7.13) \quad \partial_y \{\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \omega^\varepsilon\} = \partial_y \{\varepsilon^2 \partial_x^2 u^\varepsilon + \partial_y \omega^\varepsilon\}.$$
Then one may derive (4.1) by integrating (7.13) with respect to \(y\) over \([y, +\infty)\) and using the following pointwise convergence: as \(y \to +\infty\),

\[
(7.14) \quad \begin{cases} 
\nu \omega^\varepsilon, \partial_y \omega^\varepsilon \to 0, \\
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon \to \partial_t U + U \partial_x U - \varepsilon^2 \partial_x^2 U = -\partial_x p^\varepsilon.
\end{cases}
\]

The pointwise convergence (7.14) can be shown easily provided that

\[
(7.14) \quad \frac{u^\varepsilon}{y} \to u, \quad \text{as} \quad y \to C_1,
\]

so we leave this to the interested reader.

Lastly, it remains to show the Dirichlet boundary condition \(u^\varepsilon|_{y=0} = 0\). To prove this, we evaluate the evolution equation (4.1) at \(y = 0\) and apply the boundary conditions \(u^\varepsilon|_{y=0} = 0\) and (4.3) to obtain that \(u^\varepsilon|_{y=0}\) satisfies the viscous Burgers’ equation:

\[
(7.15) \quad \partial_t (u^\varepsilon|_{y=0}) + (u^\varepsilon|_{y=0}) \partial_x (u^\varepsilon|_{y=0}) = \varepsilon^2 \partial_x^2 (u^\varepsilon|_{y=0}).
\]

It follows from the classical uniqueness result for the viscous Burgers’ equation (7.15) that \(u^\varepsilon|_{y=0} \equiv 0\) since it does initially according to the compatibility condition (2.2).

\textbf{Appendix A Almost Equivalence of Weighted Norms}

The purpose of this appendix is to justify the almost equivalence relation (3.2). In other words, we will prove the following:

\textbf{Proposition A.1 (Almost Equivalence of Weighted }H^s\textbf{ Norms). Let }s \geq 4\textbf{ be an integer, }\gamma \geq 1\textbf{, }\sigma > \gamma + \frac{1}{2}\textbf{ and }\delta \in (0, 1). \textbf{For any }\omega \in H^{s,\gamma}_{\sigma,\delta}(\mathbb{T} \times \mathbb{R}^+), \textbf{we have the following inequality: there exist positive constants }c_\delta \textbf{ and }C_{s,\gamma,\sigma,\delta} \textbf{ such that}

\[
(\text{A.1}) \quad c_\delta \|\omega\|_{H^{s,\gamma}_{\sigma,\delta}} \leq \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H^{s,\gamma}_{\sigma,\delta}} + \|\partial_x^\delta U\|_{L^2(\mathbb{T})}\}
\]

\textbf{provided that }\omega = \partial_y u, \ u|_{y=0} = 0, \textbf{ and }\lim_{y \to +\infty} u = U, \textbf{where the weighted }H^s\textbf{-norms }\|\cdot\|_{H^s,\gamma}\textbf{ and }\|\cdot\|_{H^{s,\gamma}_{\sigma,\delta}}\textbf{ are defined by (2.5) and (3.1), respectively.}

\textbf{Proof.} Without loss of generality, we only need to prove inequality (A.1) for any smooth function \(\omega\) because the general case can be obtained by the standard density argument. First of all, it follows from the definitions of \(\|u - U\|_{H^{s,\gamma-1}}\) and \(\|\omega\|_{H^{s,\gamma}}\) that

\[
(\text{A.2}) \quad \|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1 + y)^{\gamma - 1} \partial_x^k (u - U)\|_{L^2} \leq \|\omega\|_{H^{s,\gamma-1}} + \|u - U\|_{H^{s,\gamma-1}} \leq 2 \{\|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1 + y)^{\gamma - 1} \partial_x^k (u - U)\|_{L^2}\}.
\]
Furthermore, applying Wirtinger’s inequality in the $x$-variable repeatedly and part (i) of Lemma B.1, we have

\[
\frac{1}{2} \sum_{k=1}^{s} \left\| (1 + y)^{\gamma - 1} \partial_{x}^{k} (u - U) \right\|_{L^2} \leq \frac{1 + \gamma - 2s}{1 - \gamma - 2s} \left\| (1 + y)^{\gamma - 1} \partial_{x}^{s} (u - U) \right\|_{L^2},
\]

\[
\left\| (1 + y)^{\gamma - 1} (u - U) \right\|_{L^2} \leq \frac{2}{2 \gamma - 1} \left\| (1 + y)^{\gamma} \omega \right\|_{L^2} \leq \frac{2}{2 \gamma - 1} \left\| \omega \right\|_{H^{\gamma, \gamma}}.
\]

and hence there exists a constant $C_{s, \gamma} > 0$ such that

\[
\left\| \partial_{x}^{s} (u - U) \right\|_{L^2} \leq C_{s, \gamma} \left\| \omega \right\|_{H^{\gamma, \gamma}} + \left\| (1 + y)^{\gamma - 1} \partial_{x}^{s} (u - U) \right\|_{L^2}.
\]

Therefore, according to inequalities (A.2) and (A.3), it suffices to prove

\[
c_{s, \gamma} \left\| \partial_{x}^{s} (u - U) \right\|_{L^2} \leq \left\| \partial_{x}^{s} (u - U) \right\|_{L^2}
\]

(A.4)

for some constants $c_{s, \gamma}$ and $C_{s, \gamma, \delta} > 0$.

The key idea of proving (A.4) is the following:

**Lemma A.2 (L^2 Comparison of %MathType!M!\text{\textstyle\partial}_x^k (u - U), \partial_x^k \omega, and g_k).** Let $s \geq 4$ be an integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}$, and $\delta \in (0, 1)$. If $\omega \in H^{\epsilon, \gamma}_{\sigma, \delta}(\mathbb{T} \times \mathbb{R}^+)$, then for any $k = 1, 2, \ldots, s$,

\[
\left\| (1 + y)^{\gamma} g_k \right\|_{L^2} \leq \left\| (1 + y)^{\gamma} \partial_x^k \omega \right\|_{L^2} + \delta^{-2} \left\| (1 + y)^{\gamma - 1} \partial_x^k (u - U) \right\|_{L^2}
\]

(A.5)

where $g_k := \partial_{x}^{k} \omega - \partial_{x}^{k} \omega_{\partial_{x}^{k} (u - U)}$. In addition, if $u|_{y=0} = 0$, then for any $k = 1, 2, \ldots, s$,

\[
\left\| (1 + y)^{\gamma} \partial_x^k \omega \right\|_{L^2} + \left\| (1 + y)^{\gamma - 1} \partial_x^k (u - U) \right\|_{L^2} \leq C_{\gamma, \sigma, \delta} \left\{ (1 + y)^{\gamma} g_k \right\}_{L^2} + \left\| \partial_x^k U \right\|_{L^2(T)}
\]

(A.6)

where $C_{\gamma, \sigma, \delta}$ is a constant depending on $\gamma, \sigma, \sigma$ and $\delta$ only.

Assuming Lemma $A.2$, which will be shown at the end of this appendix, for the moment we can show inequality (A.4) as follows.

Applying Lemma $A.2$ for $k = s$, we obtain, from (A.5) and (A.6),

\[
\frac{1}{2} \partial_{x}^{s} \left\| (1 + y)^{\gamma} g_s \right\|_{L^2} \leq \left\| (1 + y)^{\gamma} \partial_x^s \omega \right\|_{L^2} + \left\| (1 + y)^{\gamma - 1} \partial_x^s (u - U) \right\|_{L^2} \leq C_{\gamma, \sigma, \delta} \left\{ (1 + y)^{\gamma} g_s \right\}_{L^2} + \left\| \partial_x^s U \right\|_{L^2(T)}.
\]

(A.7)
Adding
\[ \sum_{|\alpha| \leq s} \|(1 + y)^{\nu + \sigma_2} D^{\sigma_1} \omega \|_{L^2}^2 \]
to (A.7), we have
\[ \frac{1}{2} \delta^4 \| \omega \|_{H^{\nu,\nu}}^2 \leq \| \omega \|_{H^{\nu,\nu}}^2 + \|(1 + y)^{\nu-1} \partial_x^s (u - U) \|_{L^2}^2 \]
\[ \leq C_{\gamma,\sigma,\delta} \left\{ \| \omega \|_{H^{\nu,\nu}}^2 + \| \partial_x^s U \|_{L^2(T)}^2 \right\}, \]
which implies inequality (A.4).

Finally, in order to complete the justification of the almost equivalence relation (A.1), we will prove Lemma A.2.

PROOF OF LEMMA A.2. To prove (A.5), let us first recall from the definition of
\[ H^{\nu,\nu}_{\sigma,\delta} \]
that
\[ (A.8) \]
\[ \begin{cases} \delta(1 + y)^{-\sigma} \leq \omega \leq \delta^{-1}(1 + y)^{-\sigma}, \\ |\partial_y \omega| \leq \delta^{-1}(1 + y)^{-\sigma-1}, \end{cases} \]
so \( |\partial_y \omega| \leq \delta^{-2}(1 + y)^{-1} \), and hence, for any \( k = 1, 2, \ldots, s \),
\[ \|(1 + y)^{\nu} g_k \|_{L^2} \]
\[ \leq \|(1 + y)^{\nu} \partial_x^k \omega \|_{L^2} + \left\| (1 + y)^{\nu-1} \partial_x^k (u - U) \right\|_{L^2} \]
\[ \leq \|(1 + y)^{\nu} \partial_x^k \omega \|_{L^2} + \delta^{-2} \|(1 + y)^{\nu-1} \partial_x^k (u - U) \|_{L^2}, \]
which is inequality (A.5).

Next, we are going to show (A.6). The main observation is that we can rewrite
\[ g_k = \omega \partial_y \left( \partial_x^k (u - U) / \omega \right) \]
Thus, applying (A.8) and part (ii) of Lemma B.1, we have
\[ \|(1 + y)^{\nu-1} \partial_x^k (u - U) \|_{L^2} \]
\[ \leq \delta^{-1} \left\| (1 + y)^{\nu-1-\sigma} \partial_x^k (u - U) / \omega \right\|_{L^2} \]
\[ \leq C_{\gamma,\sigma,\delta} \left\{ \left\| \partial_x^k U \right\|_{L^2(T)} + \left\| (1 + y)^{\nu} \partial_x^k g_k \right\|_{L^2} \right\}, \]
(A.9)
Now, using the triangle inequality, (A.8), and (A.9), we also have
\[ \|(1 + y)^{\nu} \partial_x^k \omega \|_{L^2} \]
\[ \leq \|(1 + y)^{\nu} g_k \|_{L^2} + \delta^{-2} \|(1 + y)^{\nu-1} \partial_x^k (u - U) \|_{L^2} \]
(A.10)
Summing up (A.9) and (A.10), we prove (A.6).

**Appendix B  Calculus Inequalities**

In this appendix we will introduce calculus inequalities for the incompressible velocity field \((u, v)\), the vorticity \(\omega\), and the quantity \(g_k\). These inequalities are not related to any equations; they hold just because of elementary computations.

**B.1 Basic Inequalities**

In this subsection we will state without proof two elementary inequalities (i.e., Lemma B.1 and Lemma B.2 below).

**Lemma B.1 (Hardy Type Inequalities).** Let \(f: \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}\). Then

(i) if \(\lambda > -\frac{1}{2}\) and \(\lim_{y \to +\infty} f(x, y) = 0\), then

\[
\|(1 + y)^{-\lambda} f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq \frac{2}{2\lambda + 1} \|(1 + y)^{\lambda + 1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)};
\]

(ii) if \(\lambda < -\frac{1}{2}\), then

\[
\|(1 + y)^{\lambda} f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq \sqrt{\frac{2}{2\lambda + 1} \|f|_{y=0}\|_{L^2(\mathbb{T})} - \frac{2}{2\lambda + 1} \|(1 + y)^{\lambda + 1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}.
\]

The proof of Lemma B.1 is elementary, so we leave it to the reader.

Next, we will state the following Sobolev-type inequality.

**Lemma B.2 (Sobolev-Type Inequality).** Let \(f: \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}\). Then there exists a universal constant \(C > 0\) such that

\[
\|f\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq C \{\|f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_x f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_y^2 f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}\}.
\]

To prove Lemma B.2, one may extend the domain of \(f\) to \(\mathbb{R}^2\) via the standard extension argument. Then inequality (B.3) follows easily by the Fourier inversion formula. We leave this to the reader as well.

**B.2 Estimates for \(H_{\sigma, \delta}^{k, \nu}\) Functions**

In this subsection we will use the weighted norm \(\cdot\|_{H_{\sigma, \delta}^{k, \nu}}\) to control certain \(L^2\)- and \(L^\infty\)-norms of \(u, v, \omega, g_k\), and their derivatives. To derive these estimates, we shall apply Lemma B.1 and Lemma B.2 which were introduced previously in Section B.1.
Our aim is to prove the following:

**Proposition B.3** ($L^2$ and $L^\infty$ Controls on $u, v, \omega$, and $g_k$). Let the vector field $(u, v)$ defined on $\mathbb{T} \times \mathbb{R}^+$ satisfy the incompressibility condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $u|_{y=0} = v|_{y=0} = 0$, and $\lim_{y \to +\infty} u = U$. If the vorticity $\omega := \partial_y u \in H^{s, 0}_{\sigma, \delta}$ for some integer $s \geq 4$ and for some constants $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}$, and $\delta \in (0, 1)$, then we have the following estimates: there exists a constant $C_{s, \gamma, \sigma, \delta} > 0$ such that the following hold:

**Weighted $L^2$ Estimates**

(i) For all $k = 0, 1, \ldots, s$,

$$(B.4) \quad \| (1 + y)^{\gamma-1} \partial_x^k (u - U) \|_{L^2} \leq C_{s, \gamma, \sigma, \delta} \left\{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \right\}.$$

(ii) For all $k = 0, 1, \ldots, s - 1$,

$$(B.5) \quad \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1 + y} \right\|_{L^2} \leq C_{s, \gamma, \sigma, \delta} \left\{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \right\}.$$

(iii) For all $|\alpha| \leq s$,

$$(B.6) \quad \| (1 + y)^{\gamma+\alpha} D^\alpha \omega \|_{L^2} \leq \begin{cases} C_{s, \gamma, \sigma, \delta} \{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \} & \text{if } \alpha = (s, 0), \\ \| \omega \|_{H^{\gamma, s}_{\sigma}} & \text{if } \alpha \neq (s, 0). \end{cases}$$

(iv) For all $k = 1, 2, \ldots, s$,

$$(B.7) \quad \| (1 + y)^{\gamma} g_k \|_{L^2(\mathbb{T})} \leq \begin{cases} C_{s, \gamma, \sigma, \delta} \{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \} & \text{if } k = 1, 2, \ldots, s - 1, \\ \| \omega \|_{H^{\gamma, s}_{\sigma}} & \text{if } k = s, \end{cases}$$

where the quantity $g_k := \partial_x^k \omega - \frac{\partial_x \omega \cdot \partial_x^k u}{\omega} (u - U)$.

**Weighted $L^\infty$ Estimates**

(v) For all $k = 0, 1, \ldots, s - 1$,

$$(B.8) \quad \| \partial_x^k u \|_{L^\infty} \leq C_{s, \gamma, \sigma, \delta} \{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \}.$$

(vi) For all $k = 0, 1, \ldots, s - 2$,

$$(B.9) \quad \left\| \frac{\partial_x^k v}{1 + y} \right\|_{L^\infty} \leq C_{s, \gamma, \sigma, \delta} \left\{ \| \omega \|_{H^{\gamma, s}_{\sigma}} + \| \partial_x^k U \|_{L^2(\mathbb{T})} \right\}.$$

(vii) For all $|\alpha| \leq s - 2$,

$$(B.10) \quad \| (1 + y)^{\gamma+\alpha^2} D^\alpha \omega \|_{L^\infty} \leq C_{s, \gamma} \| \omega \|_{H^{\gamma, s}_{\sigma}}.$$
PROOF.

(i) It follows from the definition of $\| \cdot \|_{H^{s,y-1}}$ that $\| (1 + y)^{y-1} \partial_x^k (u - U) \|_{L^2} \leq \| u - U \|_{H^{s,y-1}}$, so inequality (B.4) is a direct consequence of the almost equivalence inequality (A.1).

(ii) Applying part (ii) of Lemma B.1 and inequality (B.4), we have
\[
\frac{\| \partial_x^k v + y \partial_x^{k+1} U \|_{L^2}}{1 + y} \leq 2 \| \partial_x^{k+1} (u - U) \|_{L^2} \leq C_{s,y,\sigma,\delta} \{ \| \omega \|_{H^{s,y}} + \| \partial_x^k U \|_{L^2(T)} \}
\]
which is inequality (B.5).

(iii) Inequality (B.6) follows directly from the definition of $\| \cdot \|_{H^{s,y}}$ and inequality (A.1).

(iv) Using inequalities (A.5), (B.4), and (B.6), we have
\[
\| (1 + y)^{y} g_k \|_{L^2} \leq \| (1 + y)^{y} \partial_x^{k} \omega \|_{L^2} + \| (1 + y)^{y-1} \partial_x (u - U) \|_{L^2} \leq C_{s,y,\sigma,\delta} \{ \| \omega \|_{H^{s,y}} + \| \partial_x^k U \|_{L^2(T)} \},
\]
which is inequality (B.7) for $k = 1, 2, \ldots, s - 1$. When $k = s$, the better upper bound in (B.7) follows directly from the definition of $\| \cdot \|_{H^{s,y}}$.

(v) For any $k = 1, 2, \ldots, s - 1$, applying Lemma B.2, inequalities (B.4), and (B.6), we have
\[
\| \partial_x^k (u - U) \|_{L^\infty} \leq C \{ \| \partial_x^k (u - U) \|_{L^2} + \| \partial_x^{k+1} (u - U) \|_{L^2} + \| \partial_x^k \partial_y \omega \|_{L^2} \} \leq C_{s,y,\sigma,\delta} \{ \| \omega \|_{H^{s,y}} + \| \partial_x^k U \|_{L^2(T)} \},
\]
and hence, by the triangle inequality and $\| \partial_x^k U \|_{L^\infty(T)} \leq C_s \| \partial_x^k U \|_{L^2(T)}$, we justify (B.8).

For the case $k = 0$, let us first recall from the hypothesis that $\omega := \partial_y u > 0$, so
\[
(B.11) \quad 0 \leq u \leq U = \int_0^{+\infty} \omega \, dy.
\]

Thus, using the Cauchy-Schwarz inequality and estimate (B.6), we have
\[
\| U \|_{L^2(T)}^2 = \int_T \left[ \int_0^{+\infty} \omega \, dy \right]^2 \, dx \leq \frac{1}{2y - 1} \| (1 + y)^y \omega \|_{L^2} \leq \frac{1}{2y - 1} \| \omega \|_{H^{s,y}}^2,
\]
and hence, by (B.11), the Sobolev inequality, and $\| \partial_x U \|_{L^2(T)} \leq C_s \| \partial_x^s U \|_{L^2(T)}$, we obtain
\[
\| u \|_{L^\infty} \leq \| U \|_{L^\infty(T)} \leq C \{ \| U \|_{L^2(T)} + \| \partial_x U \|_{L^2(T)} \} \leq C_{s,y,\sigma,\delta} \{ \| \omega \|_{H^{s,y}} + \| \partial_x^s U \|_{L^2(T)} \},
\]
which is inequality (B.8).
(vi) Applying the triangle inequality, Sobolev inequality, Lemma B.2, \( \partial_x u + \partial_y v = 0 \), and \( \omega = \partial_y u \), we have

\[
\| \frac{\partial^s_v}{1+y} \|_{L^\infty} \leq \left\| \frac{y \partial^{s+1}_x U}{1+y} \right\|_{L^\infty} + \left\| \frac{\partial^s_x v + y \partial^{s+1}_x U}{1+y} \right\|_{L^\infty} \leq C \left\{ \left\| \partial^{s+1}_x U \right\|_{L^2(T)} + \left\| \partial^{s+2}_x U \right\|_{L^2(T)} + \left\| \frac{\partial^s_x v + y \partial^{s+1}_x U}{1+y} \right\|_{L^2} + \left\| \frac{\partial^{s+1}_x u - U}{1+y} \right\|_{L^2} + \left\| \frac{\partial^{s+1}_x \omega}{1+y} \right\|_{L^2} \right\} \]

which implies inequality (B.9) because of Wirtinger’s inequality and (B.4)–(B.6).

(vii) Inequality (B.10) follows directly from Lemma B.2 and inequality (B.6).

---

B.3 Estimates for Functions Vanishing at Infinity

In this subsection we will first define certain weighted norms involving time and spatial derivatives. Then we will state without proof two basic inequalities about these norms; see Proposition B.5 below. Finally, in Proposition B.6 we will control the weighted norms of \( u \) and \( v \) by that of \( \omega \) provided that \( u - U \) and its derivatives vanish at \( y = +\infty \). The vanishing hypotheses are usually guaranteed by Proposition C.1 in applications.

Let us begin by defining the weighted norms.

**Definition B.4 (Weighted Norms).** For any \( s' \in \mathbb{N} \) and \( y \in \mathbb{R} \), we define

\[
\| \cdot \|_{s',y}^2 := \sum_{l=0}^{[\frac{s'}{2}]} \| \partial^l_t \cdot \|_{H^{s'-2l},y}^2, \quad \| \cdot \|_{s'}^2 := \sum_{l=0}^{[\frac{s}{2}]} \| \partial^l_t \cdot \|_{H^{s'-2l}(T)}^2, \quad \| \cdot \|_{s',\infty}^2 := \sum_{l=0}^{[\frac{s'}{2}]} \sum_{|\alpha| \leq s-2l} \| (1+y)^{y+\alpha} \partial^l_t D^\alpha \cdot \|_{L^\infty(T \times \mathbb{R}^+)}^2, \quad \| \cdot \|_{s',\infty}^2 := \sum_{l=0}^{[\frac{s}{2}]} \| \partial^l_t \cdot \|_{W^{s'-2l},\infty(T)}^2,
\]

where \( [\frac{s'}{2}] \) denotes the largest integer that is less than or equal to \( \frac{s'}{2} \).

Using Hölder’s inequality and the Sobolev inequality, one can easily show the following:
PROPOSITION B.5 (Basic Inequalities for Weighted Norms).
(i) For any $s' \in \mathbb{N}$ and $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma = \gamma_1 + \gamma_2$, 
\[
|||F_1 F_2|||_{s', \gamma} \leq C_{s'} |||F_1|||_{s', \gamma_1} |||F_2|||_{s', \gamma_2}.
\]
(ii) For any $s' \geq 5$ and $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma = \gamma_1 + \gamma_2$, 
\[
|||F_1 F_2|||_{s', \gamma} \leq C_{s', \gamma_1, \gamma_2} (|||F_1|||_{s', \gamma_1} |||F_2|||_{s', \gamma_2} + |||F_1|||_{s', \gamma_1} |||F_2|||_{s', \gamma_2}).
\]

Next, we will state the weighted controls on $u$ and $v$ as follows:

PROPOSITION B.6 (Weighted Controls on $u$ and $v$). For any $s' \geq 4$ and $\gamma \geq 1$, let the vector field $(u, v)$ defined on $\mathbb{T} \times \mathbb{R}^+$ satisfying the incompressibility condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $v|_{y=0} = 0$, and 
\[
\lim_{y \to +\infty} \partial_t \partial_x^{k} u = \partial_t \partial_x^{k} U \text{ for all } l = 0, 1, \ldots, \lfloor \frac{s'}{2} \rfloor \text{ and } k = 0, 1, \ldots, s' - 2l + 1.
\]
Denote the vorticity $\omega := \partial_y u$. Then there exists a universal constant $C > 0$ such that 
\[
\|u - U|||_{s', 0} \leq C \|\omega|||_{s', \gamma} \quad \text{and} \quad \|v + y \partial_x U|||_{s', -1} \leq C \|\partial_x \omega|||_{s', \gamma}.
\]

OUTLINE OF THE PROOF. The hypotheses of Proposition B.6 allow us to apply Lemma B.1 to $\partial_t \partial_x^{k} (u - U)$ and $\partial_t \partial_x^{k} (v + y \partial_x U)$ provided that $2l + k \leq s'$, so we obtain 
\[
\|u - U|||_{s', 0} \leq C \|\omega|||_{s', 1} \quad \text{and} \quad \|v + y \partial_x U|||_{s', -1} \leq C \|\partial_x \omega|||_{s', 1},
\]
which imply (B.12) since $\gamma \geq 1$. □

Appendix C Decay Rates for $H^{s', \gamma}_{\sigma, \delta}$ Functions

The aim of this appendix is to prove that the actual pointwise decay rates of $H^{s', \gamma}_{\sigma, \delta}$ functions at $y = +\infty$ are better than the decay rates obtained by the Sobolev embeddings. The proof relies on a pointwise interpolation argument (see Lemma C.3 below), which is a direct consequence of a Taylor series expansion.

More specifically, we will prove the decay property of $D^\alpha \omega$ as $y$ goes to $+\infty$ as follows.

PROPOSITION C.1 (Decay Rates for $H^{s', \gamma}_{\sigma, \delta}$ Functions). Let $s' \geq 4$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, and $\delta \in (0, 1)$. If $\omega \in H^{s' + 4, \gamma}_{\sigma, \delta}$, then $\omega$ is $s' + 2$ times continuously differentiable and there exists a constant $C_{s', \gamma, \delta, \|\omega\|_{H^{s', \gamma}}}$ such that for all $|\alpha| \leq s' + 2$,
\[
|D^\alpha \omega| \leq C_{s', \gamma, \delta, \|\omega\|_{H^{s', \gamma} + 4, \gamma}} (1 + y)^{-b_\alpha} \quad \text{in} \quad \mathbb{T} \times \mathbb{R}^+
\]
where the exponent
\[
b_\alpha := \begin{cases} 
\frac{\sigma + \alpha_2}{2|\alpha| - 2} \text{ if } |\alpha| \leq 2, \\
\frac{\sigma + (2|\alpha| - 2)\gamma}{2(2|\alpha| - 2)} + \alpha_2 \text{ if } 2 \leq |\alpha| \leq s' + 1, \\
\gamma + \alpha_2 \text{ if } |\alpha| = s' + 2.
\end{cases}
\]
**Remark C.2 (Decay Rates from Sobolev Embeddings).** Using the standard Sobolev embedding argument and the definition of $H_{s',\delta}^{\sigma,\gamma}$, one may prove that if $\omega \in H_{s,\delta}^{s',\gamma}$, then $\omega$ is $s' + 2$ times continuously differentiable and there exists a constant $C_{s,\gamma} > 0$ such that

$$
(D^x \omega) \leq \begin{cases} 
\delta^{-1} (1 + y)^{-\sigma - \alpha_2} & \text{if } |\alpha| \leq 2, \\
C_{s,\gamma} \|\omega\|_{H_{s',\gamma}^{\sigma,\gamma}} (1 + y)^{-\gamma - \alpha_2} & \text{if } 2 \leq |\alpha| \leq s' + 2.
\end{cases}
$$

Thus, the interesting part of Proposition C.1 is that the decay rate of (C.1) is better than that of (C.3). This slightly better pointwise decay will help us to deal with the boundary terms at $y = +\infty$ while we are integrating by parts in the $y$-variable (cf. Remark 5.7).

**Proof of Proposition C.1.** According to Remark C.2, we are only required to justify the inequality (C.1) with the decay rate defined in (C.2).

First of all, let us state without proof the following calculus lemma.

**Lemma C.3 (Pointwise Interpolation).** Let $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function. Then we have the following:

(i) If there exist constants $C_0$, $C_2$, $b_0$, and $b_2$ such that $|\partial_x^i f| \leq C_i (1 + y)^{-b_i}$ for all $i = 0, 2$, then

$$
|\partial_x f| \leq 2\sqrt{C_0 C_2 (1 + y)^{-\frac{b_0 + b_2}{2}}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+.
$$

(ii) If there exist nonnegative constants $C_0$, $C_2$, $b_0$, and $b_2$ such that $|\partial_y^i f| \leq C_i (1 + y)^{-b_i}$ for all $i = 0, 2$, then

$$
|\partial_y f| \leq 2\sqrt{C_0 C_2 (1 + y)^{-\frac{b_0 + b_2}{2}}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+.
$$

The proof of Lemma C.3 is based on a standard Taylor series expansion technique and will be omitted here.

Now, applying Lemma C.3 to $D^x \omega$ inductively on $|\alpha| = 3, 4, \ldots, s' + 1$ with the inequality (C.3), we prove (C.1) with the exponent $b_\alpha$ defined in (C.2).

**Remark C.4 (Further Improvement on the Decay Rate).** The decay rate $b_\alpha$ defined in (C.2) is obviously not optimal because one can apply the pointwise interpolation Lemma C.3 again to further improve it. However, we do not intend to optimize it here. Indeed, repeatedly applying the pointwise interpolation Lemma C.3, one may improve the decay rate $b_\alpha$ as

$$
b_\alpha := \begin{cases} 
\sigma + \alpha_2 & \text{if } |\alpha| \leq 2, \\
(\sigma' + 2 - |\alpha|)\sigma + (|\alpha| - 2)\gamma & \text{if } 3 \leq |\alpha| \leq s' + 1, \\
\gamma + \alpha_2 & \text{if } |\alpha| = s' + 2.
\end{cases}
$$

We leave this proof to the interested reader.
Appendix D Equations for $a^\epsilon$ and $g_s^\epsilon$

In this appendix we will derive the evolution equations for $a^\epsilon := \frac{\partial_y \omega^\epsilon}{\partial x}$ and $g_s^\epsilon := \frac{\partial_s^2 \omega^\epsilon}{\partial x} - a^\epsilon \frac{\partial_s^2 \omega^\epsilon}{\partial x} (u^\epsilon - U)$ provided that $\omega^\epsilon > 0$, and $(u^\epsilon, v^\epsilon, \omega^\epsilon)$ and $(p^\epsilon, U)$ satisfy (4.1)–(4.4). These derivations just follow from direct computations.

**Equation for $a^\epsilon$:** Differentiating the vorticity equation (4.3) with respect to $y$ once, we obtain

\[(D.1) \ (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_y \omega^\epsilon = \epsilon^2 \frac{\partial_x^2 \partial_y \omega^\epsilon}{\omega^\epsilon} + \partial_x^3 \omega^\epsilon - \omega^\epsilon \partial_x \omega^\epsilon + \partial_x u^\epsilon \partial_y \omega^\epsilon.\]

Using (D.1) and the vorticity equation (4.3), we can compute

\[(\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) a^\epsilon = \frac{(\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_y \omega^\epsilon}{\omega^\epsilon} = \frac{\partial_x \partial_y \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_x^2 \omega^\epsilon}{\omega^\epsilon} - \partial_x \omega^\epsilon + a^\epsilon \partial_x a^\epsilon.\]

On the other hand, by direct differentiations only, one may check that

\[(D.3) \ \begin{cases} \partial_x^2 a^\epsilon = \frac{\partial_x^2 \partial_y \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_x^2 \omega^\epsilon}{\omega^\epsilon} - 2 \partial_x \omega^\epsilon \partial_x a^\epsilon, \\ \partial_y^2 a^\epsilon = \frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon} - 2 a^\epsilon \partial_y a^\epsilon. \end{cases}\]

Substituting (D.3) into (D.2), we obtain an equation for $a^\epsilon$:

\[(D.4) \ (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) a^\epsilon = \frac{2 \epsilon^2 \partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x a^\epsilon + 2 a^\epsilon \partial_y a^\epsilon - g_s^\epsilon + a^\epsilon \partial_x U,\]

where $g_s^\epsilon := \partial_x \omega^\epsilon - a^\epsilon \partial_x (u^\epsilon - U)$.

**Equation for $g_s^\epsilon$:** (derivation of equation (5.30)): Differentiating $s$ times the evolution equations for $\omega^\epsilon$ and $u^\epsilon - U$ (i.e., equations (5.28)) with respect to $x$, we have

\[(D.5) \ \begin{cases} (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) a^\epsilon + \partial_x^s v^\epsilon \partial_y \omega^\epsilon = - \sum_{j=0}^{s-1} \frac{\partial_x^s \partial_x^j u^\epsilon}{\partial x} \partial_x^j \omega^\epsilon - \sum_{j=1}^{s-1} (\frac{s}{j}) \partial_x^s - j \partial_x^j \partial_x^j v^\epsilon \partial_y \omega^\epsilon \\ (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) \omega^\epsilon = - \sum_{j=0}^{s-1} (\frac{s}{j}) \partial_x^s - j \partial_x^j u^\epsilon \partial_x^j + \partial_x^s \partial_x^j (u^\epsilon - U) + \partial_x^s v^\epsilon \omega^\epsilon = - \sum_{j=0}^{s-1} (\frac{s}{j}) \partial_x^s - j \partial_x^j (u^\epsilon - U) - \sum_{j=1}^{s-1} (\frac{s}{j}) \partial_x^s - j \partial_x^j v^\epsilon \partial_x^j \omega^\epsilon \\ - \sum_{j=0}^{s-1} (\frac{s}{j}) \partial_x^s - j \partial_x^j (u^\epsilon - U) \partial_x^s - j + 1 U. \end{cases}\]
To eliminate the problematic term $\partial^2_x v^\varepsilon$, we subtract $a^\varepsilon \times \frac{\partial}{\partial t} \frac{\partial^2_x v^\varepsilon}{g_s}$ from $\frac{D.5}{2}$ and obtain

\[
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial^2_x - \partial^2_y)g^\varepsilon_x
+ \{(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial^2_x - \partial^2_y)\partial^s_x (u^\varepsilon - U)
= 2\varepsilon^2 \partial_x^{s+1} (u^\varepsilon - U) \partial_x a^\varepsilon + 2\partial_x^2 \omega^\varepsilon \partial_y a^\varepsilon - \sum_{j=0}^{s-1} \binom{s}{j} b_j \partial_x^{s-j} u^\varepsilon
\]

\[
- \sum_{j=1}^{s-1} \left( \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\varepsilon \{ \partial_x^j \partial_y \omega^\varepsilon - a^\varepsilon \partial_x^j \omega^\varepsilon \}ight)
+ a^\varepsilon \sum_{j=0}^{s} \binom{s}{j} \partial_x^j (u^\varepsilon - U) \partial_x^{s-j+1} U.
\]

Substituting (D.4) into (D.6), we obtain equation (5.30).

### Appendix E Classical Maximum Principles

The main purpose of this appendix is to state two classical maximum principles that are useful in Section 5.2, for parabolic equations.

The first lemma is the maximum principle for bounded solutions to parabolic equations.

**Lemma E.1 (Maximum Principle for Parabolic Equations).** Let $\varepsilon \geq 0$. If $H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}^+) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}^+))$ is a bounded function that satisfies the differential inequality

\[
\{ \partial_t + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial^2_x - \partial^2_y \} H \leq f H \text{ in } [0, T] \times \mathbb{T} \times \mathbb{R}^+,
\]

where the coefficients $b_1$, $b_2$, and $f$ are continuous and satisfy

\[
\| \frac{b_2}{1 + y} \|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}^+)} < +\infty \quad \text{and} \quad \| f \|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}^+)} \leq \lambda,
\]

then for any $\varepsilon \in [0, T]$, $\varepsilon$

\[
\sup_{\mathbb{T} \times \mathbb{R}^+} H(t) \leq \max\{ e^{\lambda t} \| H(0) \|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} , \max_{\tau \in [0, t]} \{ e^{\lambda (t-\tau)} \| H(\tau) \|_{y=0} \|_{L^\infty(\mathbb{T})} \} \}.
\]

The proof of Lemma E.1 is a direct application of the classical maximum principle. For the reader’s convenience, we will outline its proof as follows.

**Outline of Proof of Lemma E.1.** For any $\mu > 0$, let us define

\[
\mathcal{H} := e^{-\lambda t} H - \mu \left\| \frac{b_2}{1 + y} \right\|_{L^\infty} t - \mu \ln(1 + y).
\]
Then one may check that for any \( \tilde{t} \in (0, T] \),
\[
\{ \partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 + (\lambda - f) \} h < 0 \quad \text{in} \quad [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+,
\]
so by the classical maximum principle for parabolic equations, we have
\[
\max_{[0,\tilde{t}] \times \mathbb{T} \times [0, R]} h \leq \max_{[0,\tilde{t}] \times [0, R]} \left\{ \| H(0) \|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \right\}, \max_{\tau \in [0,\tilde{t}]} \left\{ e^{-\frac{\lambda}{\mu} \tau} \| H(\tau) \|_{L^\infty(\mathbb{T})} \right\}
\]
provided that \( R \geq \exp(\frac{1}{\mu} \| H \|_{L^\infty}) - 1 \). Therefore, for any \((x, y) \in \mathbb{T} \times \mathbb{R}^+\), we have
\[
H(\tilde{t}, x, y) = - \mu \left( \frac{b_2}{1 + y} \right) e^{\lambda \tilde{t}} - \mu e^{\lambda \tilde{t}} \ln(1 + y) \leq \max_{\tau \in [0,\tilde{t}]} \left\{ e^{\lambda \tilde{t}} \| H(0) \|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \right\}, \max_{\tau \in [0,\tilde{t}]} \left\{ e^{\lambda (\tilde{t} - \tau)} \| H(\tau) \|_{L^\infty(\mathbb{T})} \right\}
\]
which implies (E.2) if we pass to the limit \( \mu \to 0^+ \) and replace the arbitrary time \( \tilde{t} \) by \( t \).

\( \Box \)

The second lemma is a lower bound estimate for bounded solutions to parabolic equations.

**Lemma E.2 (Minimum Principle for Parabolic Equations).** Let \( \epsilon \geq 0 \). If \( H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}^+)) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}^+)) \) is a bounded function with
\[
\kappa(t) := \min \left\{ \min_{T \times \mathbb{R}^+} H(0), \min_{[0, t] \times \mathbb{T}} H \big|_{y=0} \right\} \geq 0
\]
and satisfies
\[
\{ \partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 \} H = f H
\]
where the coefficients \( b_1, b_2, \) and \( f \) are continuous and satisfy (E.1), then for any \( t \in [0, T] \),
\[\text{(E.3)} \quad \min_{T \times \mathbb{R}^+} H(t) \geq (1 - \lambda t e^{\lambda t}) \kappa(t). \]

The proof of Lemma E.2 is also standard and very similar to that of Lemma E.1. We will outline it here for the reader’s convenience as well.

**Outline of Proof of Lemma E.2.** For any fixed \( \tilde{t} \in [0, T] \) and \( \mu > 0 \), let us define
\[
h := e^{-\lambda t} \{ H - \kappa(\tilde{t}) \} + \left\{ \lambda \kappa(\tilde{t}) + \mu \left( \frac{b_2}{1 + y} \right) L^\infty \right\} t + \mu \ln(1 + y).
\]
Then one may check that
\[
\{ \partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 + (\lambda - f) \} h > 0 \quad \text{in} \quad [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+,
\]
so by the classical maximum principle for parabolic equations, we have
\[
\min_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} h \geq 0
\]
provided that $R \geq \exp\left(\frac{1}{\mu}\{\|H\|_{L^\infty} + \kappa(\tilde{t})\}\right) - 1$. Passing to the limit $R \to +\infty$, and then $\mu \to 0^+$, we obtain

$$H(\tilde{t}) \geq (1 - \lambda \tilde{t} e^{\lambda \tilde{t}})\kappa(\tilde{t}),$$

which implies inequality (E.3) if we replace the arbitrary time $\tilde{t}$ by $t$. \hfill \Box

**Acknowledgment.** Nader Masmoudi was partially supported by NSF-DMS Grant 0703145. Tak Kwong Wong was partially supported by the Croucher Foundation.

**Bibliography**

[1] Alexandre, R.; Wang, Y.-G.; Xu, C.-J.; Yang, T. Well-posedness of the Prandtl equation in Sobolev spaces. Preprint, 2012. [arXiv:1203.5991]

[2] Beirão da Veiga, H.; Crispo, F. Concerning the $W^{k,p}$-inviscid limit for 3-D flows under a slip boundary condition. *J. Math. Fluid Mech.* 13 (2011), no. 1, 117–135. doi:10.1007/s00021-009-0012-3

[3] Brenier, Y. Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity* 12 (1999), no. 3, 495–512. doi:10.1088/0951-7715/12/3/004

[4] Caflisch, R. E.; Sammartino, M. Existence and singularities for the Prandtl boundary layer equations. *ZAMM Z. Angew. Math. Mech.* 80 (2000), no. 11-12, 733–744. doi:10.1002/1521-4001(200011)80:11<733::AID-ZAMM733>3.0.CO;2-L

[5] Constantin, P.; Kukavica, I.; Vicol, V. On the inviscid limit of the Navier-Stokes equations. Preprint, 2014. [arXiv:1403.5748]

[6] E, W. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)* 16 (2000), no. 2, 207–218. doi:10.1007/s101140000034

[7] E, W.; Engquist, B. Blowup of solutions of the unsteady Prandtl’s equation. *Comm. Pure Appl. Math.* 50 (1997), no. 12, 1287–1293. doi:10.1002/(SICI)1097-0312(199712)50:12<1287::AID-CPA4>3.0.CO;2-4

[8] Gérard-Varet, D.; Dormy, E. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.* 23 (2010), no. 2, 591–609. doi:10.1090/S0894-0347-09-00652-3

[9] Gérard-Varet, D.; Masmoudi, N. Well-posedness for the Prandtl system without analyticity or monotonicity. Preprint, 2013. [arXiv:1305.0221]

[10] Gérard-Varet, D.; Nguyen, T. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* 77 (2012), no. 1-2, 71–88.

[11] Grenier, E. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* 53 (2000), no. 9, 1067–1091. doi:10.1002/(SICI)1097-0312(200009)53:9<1067::AID-CPA1>3.0.CO;2-H

[12] Grenier, E.; Guo, Y.; Nguyen, T. Spectral instability of characteristic boundary layer flows. Preprint, 2014. [arXiv:1406.3862]

[13] Grenier, E.; Guo, Y.; Nguyen, T. Spectral instability of symmetric shear flows in a two-dimensional channel. Preprint, 2014. [arXiv:1402.1395]

[14] Grenier, E.; Guo, Y.; Nguyen, T. Spectral stability of Prandtl boundary layers: an overview. Preprint, 2014. [arXiv:1406.4452]

[15] Guo, Y.; Nguyen, T. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.* 64 (2011), no. 10, 1416–1438. doi:10.1002/cpa.20377

[16] Iftimie, D.; Sueur, F. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Anal.* 199 (2011), no. 1, 145–175. doi:10.1007/s00205-010-0320-2
[17] Kato, T. Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Functional Analysis* 9 (1972), 296–305.

[18] Kato, T. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. *Seminar on nonlinear partial differential equations* (Berkeley, Calif., 1983), 85–98. Mathematical Sciences Research Institute Publications, 2. Springer, New York, 1984. doi:10.1007/978-1-4612-1110-5_6

[19] Kelliher, J. P. On Kato’s conditions for vanishing viscosity. *Indiana Univ. Math. J.* 56 (2007), no. 4, 1711–1721. doi:10.1512/iumj.2007.56.3080

[20] Kukavica, I.; Masmoudi, N.; Vicol, V.; Wong, T. K. On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* 46 (2014), no. 6, 3865–3890. doi:10.1137/140956440

[21] Kukavica, I.; Vicol, V. On the local existence of analytic solutions to the Prandtl boundary layer equations. *Comm. Math. Sci.* 11 (2013), no. 1, 269–292. doi:10.4310/CMS.2013.v11.n1.a8

[22] Lombardo, M. C.; Cannone, M.; Sammartino, M. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.* 35 (2003), no. 4, 987–1004 (electronic). doi:10.1137/S0036141002412057

[23] Maekawa, Y. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* 67 (2014), no. 7, 1045–1128. doi:10.1002/cpa.21516

[24] Masmoudi, N. The Euler limit of the Navier-Stokes equations, and rotating fluids with boundary. *Arch. Rational Mech. Anal.* 142 (1998), no. 4, 375–394. doi:10.1007/s002050050097

[25] Masmoudi, N. Examples of singular limits in hydrodynamics. *Handbook of differential equations: evolutionary equations. Vol. III*, 195–275. Handbook of Differential Equations. Elsevier/North-Holland, Amsterdam, 2007.

[26] Masmoudi, N. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.* 270 (2007), no. 3, 777–788. doi:10.1007/s00220-006-0171-5

[27] Masmoudi, N.; Rousset, F. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.* 203 (2012), no. 2, 529–575. doi:10.1007/s00205-011-0456-5

[28] Masmoudi, N.; Wong, T. K. On the $H^s$ theory of hydrostatic Euler equations. *Arch. Ration. Mech. Anal.* 204 (2012), no. 1, 231–271. doi:10.1007/s00205-011-0485-0

[29] Oleinik, O. A. On the system of Prandtl equations in boundary-layer theory. *Dokl. Akad. Nauk SSSR* 150 (1963), 28–31.

[30] Oleinik, O. A. On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *Prikl. Mat. Meh.* 30, 801–821 (Russian); translated as *J. Appl. Math. Mech.* 30 (1966), 951–974. doi:10.1016/0021-8928(66)90001-3

[31] Oleinik, O. A.; Samokhin, V. N. *Mathematical models in boundary layer theory*. Applied Mathematics and Mathematical Computation, 15. Chapman & Hall/CRC, Boca Raton, Fla., 1999.

[32] Prandtl, L. Über Flüssigkeits-bewegung bei sehr kleiner Reibung. *Actes du 3ème Congrès international des Mathématiciens*, 484–491. Heidelberg, Teubner, Leipzig, 1904.

[33] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* 192 (1998), no. 2, 433–461. doi:10.1007/s002200050304

[34] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solutions. *Comm. Math. Phys.* 192 (1998), no. 2, 463–491. doi:10.1007/s002200050305

[35] Swann, H. S. G. The convergence with vanishing viscosity of nonstationary Navier-Stokes flows to ideal flow in $\mathbb{R}^3$. *Trans. Amer. Math. Soc.* 157 (1971), 373–397. doi:10.2307/1995853

[36] Temam, R.; Wang, X. On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 25 (1997), no. 3-4, 807–828 (1998).
[37] Temam, R.; Wang, X. Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case. *J. Differential Equations* **179** (2002), no. 2, 647–686. [doi:10.1006/jdeq.2001.4038](https://doi.org/10.1006/jdeq.2001.4038)

[38] Wang, X. A Kato type theorem on zero viscosity limit of Navier-Stokes flows. *Indiana Univ. Math. J.* **50** (2001), Special Issue, 223–241. [doi:10.1512/iumj.2001.50.2098](https://doi.org/10.1512/iumj.2001.50.2098)

[39] Xiao, Y.; Xin, Z. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. *Comm. Pure Appl. Math.* **60** (2007), no. 7, 1027–1055. [doi:10.1002/cpa.20187](https://doi.org/10.1002/cpa.20187)

[40] Xin, Z.; Zhang, L. On the global existence of solutions to the Prandtl’s system. *Adv. Math.* **181** (2004), no. 1, 88–133. [doi:10.1016/S0001-8708(03)00046-X](https://doi.org/10.1016/S0001-8708(03)00046-X)

[41] Xin, Z.; Zhang, L.; Zhao, J. Global well-posedness and regularity of weak solutions to the Prandtl’s system. Preprint.

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Received February 2013.