CHAOTIC MOTION OF CHARGED PARTICLES IN AN ELECTROMAGNETIC FIELD SURROUNDING A ROTATING BLACK HOLE

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ABSTRACT

The observational data from some black hole candidates suggest the importance of electromagnetic fields in the vicinity of a black hole. Highly magnetized disk accretion may play an importance rule, and large-scale magnetic field may be formed above the disk surface. Then, we expect that the nature of the black hole spacetime would be revealed by magnetic phenomena near the black hole. We will start investigating the motion of a charged test particle which depends on the initial parameter setting in the black hole dipole magnetic field, which is a test field on the Kerr spacetime. Particularly, we study the spin effects of a rotating black hole on the motion of the charged test particle trapped in magnetic field lines. We make detailed analysis for the particle’s trajectories by using the Poincaré map method, and show the chaotic properties that depend on the black hole spin. We find that the dragging effects of the spacetime by a rotating black hole weaken the chaotic properties and generate regular trajectories for some sets of initial parameters, while the chaotic properties dominate on the trajectories for slowly rotating black hole cases. The dragging effects can generate the fourth adiabatic invariant on the particle motion approximately.

Key words: magnetic fields – black hole physics – relativity

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1. INTRODUCTION

In the central region of active galactic nuclei (AGNs), compact X-ray sources and gamma ray bursts (GRBs), many black hole candidates are reported. The observational data from their active regions would include the information about the gravitational and electromagnetic fields. However, there are no direct observational evidences for the existence of a black hole in the center of the compact active regions. So, it is very important how one finds the information on the curved spacetime from observational data, and how its nature is interpreted. The inner part of a gas disk around a supermassive black hole emits nonthermal radiation; this may be explained by synchrotron emission from ultra-relativistic electrons in strong magnetic field regions around a black hole. In the central black hole system, however, the emission process is very little understood, although the consensus is in favor of emission due to plasma process. The emission from such particles would give us the information about the black hole spacetime and/or the electromagnetic field.

In a realistic model of a black-hole–disk system, we should consider interactions between a plasma particle and a magnetic field in the curved spacetime. The plausible electromagnetic field around a black-hole–disk system is still unknown, although some authors have discussed this problem in the framework of magnetohydrodynamics (e.g., Blandford & Znajek 1977; Camenzind 1987; Nitta et al. 1991; Tomimatsu & Takahashi 2001). In general, the task for solving the structure of magnetic fields in a black hole system is very hard. Then, to make a comprehensive analysis, we will consider a vacuum external magnetic field around a rotating black hole, and investigate in detail the motion of a charged test particle confined in the magnetic field. The stationary and axisymmetric vacuum electromagnetic fields around a black hole are discussed by Petterson (1975), Chitre & Vishveshwara (1975), Li (2000), Ghosh (2000), and Tomimatsu & Takahashi (2001). We apply the black hole dipole magnetic field in the Kerr geometry, which is a solution of the vacuum Maxwell’s equations, where the dipole magnetic field is a test field on the Kerr spacetime and hence any back-reaction on the Kerr geometry is treated as negligible. Although the dipole magnetic field could correspond to the intrinsic field of a compact object such as a neutron star, in the case of black holes it results from current rings exterior to the event horizon, but the current rings can be very close to the event horizon (Prasanna & Vishveshwara 1978). As known in the Earth’s magnetosphere, the dipole magnetic field can trap charged particles within its magnetic bottle (i.e., the Van Allen belts). The particle gyrates around the magnetic field line, and drifts in the toroidal direction (see, e.g., Goedbloed & Poedts 2004). Furthermore, in the poloidal plane, the particle oscillates along the magnetic field line. In the black hole case, we can also expect similar situations. In this paper, we treat the case in which a charged test particle, moving on the black hole background with superposed electromagnetic field, is also nonperturbing for the background.

The motions in the black hole magnetosphere have been discussed by some authors. The innermost stable orbits of charged particles in a uniform magnetic field around a rotating black hole were discussed by Aliev & Özdemir (2002). By numerically solving equations of motion of a charged particle moving in some electromagnetic fields, the gyration around the magnetic field line and the drift in the toroidal direction were shown by Prasanna & Vishveshwara (1978) and Prasanna (1978, 1980). For a dipole magnetic field, the multiperiodic off-equatorial motion in the poloidal plane was also presented. Such an orbit is very complicated in both poloidal and toroidal directions. In this paper, although a stationary and axisymmetric black hole magnetosphere is assumed, there are two conserved quantities; that is, the energy and angular momentum of the charged particle. In addition to these, the rest mass of the particle...
is the third invariable quantity. However, the fourth invariable quantity related to the azimuthal motion of the particle does not exist when the electromagnetic field is considered around a black hole. Thus, the motion of a test charge in a black hole magnetosphere is not an integrable system. Then, the chaotic motion is expected (Nakamura & Ishizuka 1993). To examine the trajectory, we plot the Poincaré map in the two-dimensional $r - p_r$ plane, which shows intersections of the particles’ trajectories with the surface of section in phase space (Lichtenberg & Liberman 1992). If the plotted points form a closed curve, the motion is regular (not chaotic). This is because a regular trajectory moves on a torus in the phase space and the curve is a cross section of the torus. On the other hand, if the plotted points are distributed randomly, the motion is irregular (chaotic). From the distribution of the points in the Poincaré map, we can judge whether or not the motion is chaotic. Then, we know chaotic behavior on the trajectory of a charged particle in a black hole magnetosphere. Furthermore, we find the black hole’s spin effects on that chaotic motion.

This study will be related to the source of radiation near a black hole. We will start this work as a basic step before considering the plasma as a magnetized fluid. Although we find chaotic and/or regular motion in a black hole magnetosphere, the problem is how to find the phenomena in the observed spectrum. At this stage, however, we cannot show the relation with the observational data in this work. The main purpose of this study is to understand the black hole spin effects (the dragging effects by a rotating black hole) on a test charge, which will be considered as the source plasma of high-energy radiation in future.

In Section 2, we review the basic equations for the motion of a charged particle in a black hole magnetosphere, using the Hamilton–Jacobi formalism. To understand the nature of orbits of a charged particle in the black hole magnetosphere, we also consider the effective potential including the four-vector potential of the magnetic field in Section 3. We present the equations written in terms of $x^\mu$ and $\pi_\mu$:

$$\frac{dx^\mu}{d\lambda} = g^{\mu\nu}(\pi_\nu - q A_\nu),$$  \hspace{1cm} (2)

where $\pi_\mu$ is the canonical momentum and $A_\mu = A_\mu(r, \theta)$ is the four-vector potential of the electromagnetic field. The four-momentum of a charged particle is

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} = g^{\mu\nu}(\pi_\nu - q A_\nu),$$  \hspace{1cm} (3)

In a stationary and axisymmetric black hole magnetosphere, the magnetic field is specified by a scalar function $\Psi$ of position. The function $\Psi$ is proportional to the poloidal component of the vector potential, $A_\theta$. In this paper, we set up a vacuum magnetosphere, so that no total poloidal current exists, and then the toroidal component of the magnetic field is zero.

From the stationary and axial symmetry of both electromagnetic field and spacetime geometry, $E \equiv \pi_t = p_t + q A_t$ and $L \equiv -\pi_\phi = -(p_\phi + q A_\phi)$ are constants of motion corresponding to the integrable coordinates $t$ and $\phi$ in Equation (2). The third constant of motion is the particle’s rest mass $m = (g^{\mu\nu}p_\mu p_\nu)^{1/2}$. In general, four constants of motion are needed to determine uniquely the orbit of a particle through four-dimensional spacetime. However, a test charge motion in a magnetic field around a black hole possesses only three obvious constants. We can expect chaotic behavior for its motions (Lichtenberg & Liberman 1992; Karas & Vokrouhlický 1992) because such a system is nonintegrable. Note that without the magnetic field the fourth constant of motion exists. This constant is known as Carter’s constant of the motion

$$Q = p_\theta^2 + \cos^2 \theta \left[ a^2(m^2 - E^2) + \frac{L^2}{\sin^2 \theta} \right],$$  \hspace{1cm} (6)

which arises as a separation-of-variables constant in the Hamilton–Jacobi derivation of equation of motion (see Misner et al. 1973). That is, the motion of a test particle in Kerr spacetime without magnetic field is an integrable system.

The stationary and axisymmetric electromagnetic fields around a rotating black hole in source-free regions were derived by Petterson (1975; see also Hanni & Ruffini 1973; Prasanna & Varma 1977, for a Schwarzschild black hole case). Here, we consider the solution that denotes dipole magnetic fields at...
distant regions, and call this solution “black hole dipole electromagnetic field.” This dipole electromagnetic field is a special case of the Petterson multipolar solution following the approach of Teukolsky to black hole perturbations and in the context of Newman–Penrose complex formalism. In this black hole magnetosphere, we expect that the dipole magnetic field configuration can trap a charged particle within their magnetic bottle, even if it extends close to the black hole. The four-vector potential of the black hole dipole has only two nonzero components, $A_t$ and $A_\phi$; the dipole magnetic field and an induced quadrupole electric field in the Kerr geometry are given by (see, e.g., Prasanna & Vishveshwara 1978; Dhurandhar & Dadhich 1984)

$$A_t = \frac{-3\alpha \mu}{2\gamma^2 \Sigma} \left[ r(r - M) + (a^2 - Mr) \cos^2 \theta \right] - \frac{1}{2\gamma} \ln \left( \frac{r - r_+}{r - r_-} \right) - (r - M \cos^2 \theta),$$

$$A_\phi = \frac{-3\alpha \sin^2 \theta}{4\gamma^2 \Sigma} \left[ r(r - M)a^2 \cos^2 \theta + r(r^2 + Mr + 2a^2) - [r(r^2 - 2Ma^2 + a^2) + \Delta a^2 \cos^2 \theta] \right] - \frac{1}{2\gamma} \ln \left( \frac{r - r_+}{r - r_-} \right),$$

where $\gamma \equiv (M^2 - a^2)^{1/2}$, $r_{\pm} = M \pm \gamma$ and the dipole moment $\mu$ is taken to be antiparallel to the rotation axis. In the $a \to M$ limit, we obtain

$$A_t = \frac{-M \mu [r \sin^2 \theta - 2(r - M) \cos^2 \theta]}{2(r^2 + M^2 \cos^2 \theta)(r - M)^2},$$

$$A_\phi = \frac{-\mu \sin^2 \theta [-2r^3 + M(r-M)(r + M \cos^2 \theta)]}{2(2r^2 + 2M^2 \cos^2 \theta)(r - M)^2}.$$

In the limit of $a \to M$, $|A_t|$ and $|A_\phi|$ diverge as $(r - M)^{-2}$, while in the case of $a < M$, $|A_t|$ and $|A_\phi|$ diverge logarithmically at the event horizon. Although the source of the magnetic field should be outside the event horizon, the current loops could locate just outside the event horizon (arbitrarily close to it) as the source of such dipole magnetic fields as considered in this work (Prasanna 1978). In spite of this singularity, we will use this configuration for the study of the chaotic motion of a test charge around a black hole. The appearance of the singularity does not imply that the dipole field is invalid. It is valid in the regions considered outside the event horizon. In a realistic situation for the black hole magnetosphere as an astrophysical model, to avoid the singularity at the event horizon, the infinite sum of multipole fields is necessary (Li 2000; Tomimatsu & Takahashi 2001). The plausible magnetic field configuration around a black hole should be investigated as a future work.

In the next section, we will introduce the effective potential to see the trapping regions of a charged test particle in a vacuum black hole magnetosphere. Furthermore, the possibility of the trapping test charges in the magnetic field will be discussed in the Appendix.

### 3. ORBITS OF A CHARGED PARTICLE IN A MAGNETOSPHERE

Now, we will discuss charged particle motions off the equatorial plane of the black hole dipole field. The concrete expressions of Equations (4) and (5) including electromagnetic field terms are very complicated, and are not particularly informative. However, by considering an effective potential in the poloidal plane, we can get a general picture of the orbits, although we need to integrate the equations of motion numerically to obtain the practical orbits.

Here, we consider off-equatorial motion for a charged test particle in the dipole magnetic field. By using the condition $g^{\mu \nu} p_\mu p_\nu = m^2$ with the constants of motion $E$ and $L$, the equation of motion in the poloidal plane can be obtained. Then, without the kinetic terms of the poloidal motion ($p' = p^\phi = 0$), we can define the effective potential (Esteban & Medina 1990) as

$$V_{\text{eff}}(r, \theta) = \frac{E_{\text{min}}}{m} = \frac{q A_t}{g^\phi} \left[ \left( \frac{L}{m} + \frac{q}{m} A_\phi \right)^2 + g^\phi \right]^{1/2},$$

where $\rho_{\text{inj}}^2 \equiv \Delta \sin^2 \theta$ and $E_{\text{min}}$ is the allowed minimum energy for a particle at the injection point. Although the distribution of the effective potential depends on the value of $L/m$ and $q/m$ of the charged particle and the electromagnetic field configuration $A_t$ and $A_\phi$, we will study the chaotic behavior we are interested in the case that a charged test particle is trapped in a bound orbit. When the energy of the particle is not so large ($V_{\text{eff}} < E/m < 1$), the particle is bound in the orbit in the region of the potential well. That is, the particle has turning points, which correspond to the inner and outer envelopes of the Larmor motion in the poloidal plane.

The detailed motion in the equatorial plane ($p^\phi = 0$) in the Kerr background with the dipole magnetic field has been discussed by Prasanna (1980). We will also see the projection onto the equatorial plane of the off-equatorial particle’s motions. From the $\phi$-component of the equation of motion, we obtain

$$p^\phi = g^\phi (E - q A_t) - g^\phi (L + q A_\phi).$$

For a nonrotating black hole case, the first term is negligible. When the signatures of parameters $L$ and $q \mu$ are the same, the particle does not gyrate because of $p^\phi \neq 0$. To observe the gyration motion of a charged particle, where $p^\theta = 0$ is achieved periodically, we will choose $L < 0$ and $Q_A \equiv (q/m) \mu > 0$ in the following numerical calculations. To see the off-equatorial motion of a charged particle, the initial value of $p^\phi$ or $p^\theta$ should be specified. In this paper, we inject the particle into the magnetosphere with the injection angle $\psi_{\text{inj}}$, which is the angle from the origin of the coordinate axes in the poloidal plane and is related to the ratio of the initial $p^\phi$ and $p^\theta$-values, where we choose $0 < \psi_{\text{inj}} < 0.5 \pi$ for the calculations. The label “ini” indicate the quantity at the injection point. The equations are solved by using the fourth-order Runge–Kutta–Gill method.

Figure 1 shows the hole’s spin dependence on the effective potential $V_{\text{eff}}(r, \theta)$ for the poloidal motion of a charged test particle in the black hole dipole magnetic field. We see the valley-like structure in the lower-level potential region (named “potential valley”), which is across the northern and southern hemispheres nearly along a dipole magnetic field line, wherein the particle with lower energy can be trapped inside it. We also see a divide of the potential valley just on the equatorial plane, Figure 2 shows the $r$ and $\theta$ dependences of the local minimum of the effective potential in the $r$-direction, which is obtained from $\partial V_{\text{eff}}(r, \theta)/\partial \theta = 0$ and corresponds to the bottom along the valley, for various values of the spin parameter $a$. In this plot, we see the local minimum in the middle-latitude region in the $\theta$-direction.
In the nonrotating black hole case (see Figure 1 (a)), the value of the local minimum of the effective potential decreases from the equator to the event horizon along almost the dipole magnetic field line in both hemispheres, and has the minimum value at the event horizon. Thus, the potential valley is opened narrower toward the event horizon. So, although a particle can be trapped within the potential valley for some time, the particle will fall into the black hole sooner or later. Figure 3 is an example of the orbit for a nonrotating black hole, where the projection on the equatorial plane (left panel) and the poloidal motion (right panel) are shown; see also Prasanna & Varma (1977) for a case of Schwarzschild black hole. The particle gyrates around the magnetic field line, and drifts in the toroidal direction. Furthermore, the particle oscillates in the poloidal plane along the magnetic field lines. In this dipole magnetic field case, the particle entering the regions of higher magnetic field strengths reflects back into the regions of a smaller magnetic field strength. This is the so-called “mirror effects.” Although the dipole field traps a particle between two mirrors, the particle falls into the black hole in the end, for almost nonrotating black hole cases. In Figure 3, we show the case of $\psi_{inj} = -0.22 \pi$, but it is easy to fall into the black hole sooner for the larger value of $\psi_{inj}$.

Next, for a rotating black hole case (see Figure 1 (b)), the centrifugal barrier by the hole’s spin effects arises and the potential valley is disconnected from the event horizon. This is because the centrifugal barrier due to the hole’s spin becomes higher with increasing the black hole spin. Thus, the dragging effects of spacetime enhance the mirror effects. Then, the particle can be trapped within the potential valley without falling onto the black hole. Furthermore, we see the “double-well potential” along the valley. Basically, the particle oscillates between the northern and southern hemispheres, but sometimes a particle with lower energy may be trapped in one hemisphere. Such a double-well potential may be related to the chaotic motion as discussed later. For the extreme rotating case shown in Figure 1(c), the spin effects are more effective. We see an almost single-well potential along the valley. Then, we can expect that the particle would oscillate between the northern and southern hemispheres periodically. The trajectory for the particle trapped in the potential valley will be discussed in Section 5 again.

Figures 4–6 show the orbits of a charged particle around a rotating black hole. Charged particles are trapped within the potential valley (i.e., a magnetic bottle) without falling onto a black hole. So, we can carry out long-time calculations for chaos studies. The particle oscillates quasiperiodically between...
Figure 3. Motion of a charged particle in the equatorial plane (left) and in the poloidal plane (right). The particle finally falls into the event horizon from the polar region of the black hole. The parameters of the motion are given as \( a = 0, \frac{E}{m} = 0.920, L = -7.0 M, Q_d = 70 M^2, \psi_{ini} = -0.22\pi, x_{ini} = 10.5 M, \) and \( \theta_{ini} = \pi/2. \) In the poloidal plane, the dipole magnetic field lines (thin gray curves) and the effective potential (thin curves) are also plotted. (A color version of this figure is available in the online journal.)

Figure 4. Example of the random trajectory of a test charge. Left panel shows the projections on the equatorial plane, and the right panel shows the projections on the poloidal plane. The parameters of the motion are set as \( a = 0.3 M, \frac{E}{m} = 0.900, \psi_{ini} = -0.20\pi, L/m = -7.0 M, Q_d = 70.0 M^2, r_{ini} = 10.5 M \) and \( \theta_{ini} = \pi/2. \) (A color version of this figure is available in the online journal.)

4. REGULAR AND CHAOTIC ORBITS

The particle motions are common in the sense that they are the combinations of three types of motions (gyration, bouncing, and drifting). Here we analyze the property of such complicated motions using the Poincaré map, which shows intersections of a trajectory with the surface of section in phase space. The motion of a point in phase space could be followed over hundreds of thousands of oscillation periods. The Poincaré map is an useful tool to classify visually whether the motions are regular (nonchaotic) or irregular (chaotic). To make the Poincaré map, we adopt the equatorial plane \( (\theta = \pi/2) \) as a Poincaré surface and plot the point \( (r, p_r) \) when the particle crosses the Poincaré section with \( (p_\theta > 0) \). We obtain that, for a slowly rotating black hole case, a trajectory plots a lot of points on the map randomly, while for a rapidly rotating case there are some regular trajectories shown by tori. The different tori indicate the different initial injection angle \( \psi_{ini} \) on the Poincaré map. The appearance of regular trajectories suggests that the dragging effects of the black hole generate a nearly integrable system in spite of the existence of a magnetic field in Kerr spacetime (the details will be discussed in Section 5).

Figures 7–10 show several typical examples of the Poincaré map for various hole’s spin and particle’s energies. We find the spin effects on the trajectories in the Poincaré map. For slowly rotating black hole cases, the region of random trajectories fill a
finite portion of the energy surface in phase space (see Figure 7). The intersections of a single random trajectory with the surface of section fill a finite area. For a rotating black hole cases, both types of trajectories are mixed (see Figures 8 and 9). That is, the distribution on the Poincaré map makes wide rings by random trajectories, and closed curves by regular trajectories. These differences on their trajectories depend on the initial ejection angle $\psi_{ini}$. Specially, for rapidly rotating black hole cases, the ratio of the regular orbits increases rather than mildly rotating black hole case (see Figure 9). The hole’s spin effects weaken the chaotic motion in the black hole magnetosphere. In fact, for the maximally rotating black hole case, for a wide range of $\psi_{ini}$-values the regular orbits are observed, although random trajectories also appear for larger energy particle motions (see Figure 10).

Figure 5. Examples of the trajectories of a test charge for $a = 0.6 M$. The left column shows the projections on the equatorial plane, and right column shows the projections on the poloidal plane. The parameters of the motion are set as (a, b) $E/m = 0.895, \psi_{ini} = -0.10\pi, (c, d) E/m = 0.890, \psi_{ini} = -0.30\pi$, with $L/m = -7.0 M, Q_d = 70.0 M^2, r_{ini} = 10.5 M$ and $\theta_{ini} = \pi/2$. (A color version of this figure is available in the online journal.)

Next, let us see the energy dependences on the Poincaré map. For a slowly rotating black hole case (see Figure 7), the random nature of intersections on the Poincaré map is almost independent of particle’s energy. The outer boundary of cross sections plotted by the trajectory of $p_{th}^m = 0$ only shrinks with decreasing particle’s energy; it depends on the effective potential well specified by the values of $Q_d, L,$ and $a$ (Karas & Vokrouhlický 1992). We also see a few void regions on the map in some cases. On the other hand, we find the energy dependence of the Poincaré map for mildly and rapidly rotating black hole cases. In Figures 8(a) and (c) (where $a = 0.6 M$), most trajectories are chaotic, but in Figures 8(b) and (d), the regular trajectories appear on the map (less chaotic). Figures 9 and 10 show the cases of a rapidly rotating black hole case. The regular trajectory is obtained when the elevation of the particle ejected is small (somewhat parallel to the equator) or large (somewhat perpendicular to the equator), while the random trajectory is observed when the elevation has an intermediate value of them. For example, in Figure 9 (a), when the initial value of $p_r$ is in the range of A–B or C–D, the regular trajectories are obtained, while the range of B–C shows random trajectories.

Although the charged particle is trapped in the effective potential well, when the energy has an almost minimum value at the injection point; that is, $E/m \sim V_{eff}(r_{ini}, \theta_{ini})$, the intersections of the whole trajectories are bounded by outer and inner closed curves, which make a narrow layer; see Figures 9(f) and (d). The inner boundary of the cross section is plotted for the orbit of $p_{th}^m = 0$. Note that, in the case of $a \gtrsim 0.6 M$, some orbits
with their almost minimum energy show the random trajectories on the Poincaré map, but the region of the random trajectories is restricted within a narrow torus-like belt. The width of this belt becomes narrow when the particle’s energy has its minimum energy that is specified as the value of the effective potential at the particle’s injected point; for a slowly rotating black hole case, such a beltlike distribution does not appear on the map. In summary, we see the tendency that random trajectories are independent of the initial injection angles and their energy in a slowly rotating black hole spacetime. For a rapidly rotating black hole case, however, we also see regular trajectories, which depend on the initial injected angle and also their energy.

5. DISCUSSION

In Section 2, we have mentioned that in a stationary and axisymmetric black hole magnetosphere there are only three constants of motion, \( \pi_t = E \), \( \pi_\phi = -L \), and \( m \), and as a consequence of the nonexistence of the forth integral of motion, \( \pi_\theta \), the chaos would appear in the system. In spite of being a nonintegrable system, however, we have found the regular trajectories on the Poincaré map, although we have also seen the chaotic trajectories (for the different injection angles). In this section, we discuss the reason.

As shown in Figures 4–6, a particle in the black hole dipole magnetic field is trapped when a black hole rotates. It bounces back and forth between the northern and southern mirrors (also called magnetic bottle). Since the regular or random trajectories depend discontinuously on a choice of initial angle, their presence does not imply the existence of a global invariant of the system. However, if regular trajectories exist, they should represent some kind of invariants of the motion. These trajectories are conditionally periodic with angle variables. With this motion an adiabatic invariant would be associated, and that is related to the longitudinal motion. Then, we define the action integral as (see, e.g., Goedbloed & Poedts 2004)

\[
J_\theta \equiv \frac{1}{\ell} \oint \sqrt{-p_\theta p^\theta} \, d\lambda,
\]

where the integral is taken over one cycle of the oscillation in time and \( \ell \) is the length of one cycle of the path. In general, the value of \( J_\theta \) is not a constant for each cycle of the oscillation of a test charge in the black hole dipole magnetic field around a central object, so that we will see random trajectories (chaos).
In fact, for a slowly rotating black hole, we see that most orbits with the various injection angles are chaotic, and their $J_0$ values are not constants; they vary irregularly. However, we can also observe the regular trajectories, when the action integral of $J_0$ is nearly constant for a long-time orbital motion. For a rotating black hole case, we can find the regular trajectory where the value of $J_0$ becomes approximately constant, while it oscillates slightly in a few cycles. In this case, the motion shows a regular trajectory on the Poincaré map. Thus, we find that the motion of a test charge in the magnetosphere around a rotating black hole can be nearly integrable.

The region of random trajectories fill a finite portion of the surface of a section in phase space. Figure 11(a) shows that the typical intersections of a single random trajectory with the surface of a section fill a finite area, where the value of $J_0$ varies irregularly; see Figure 11(b). The time series of $\pi/2 - \theta$ also shows an irregular feature in Figure 11(c). In Figure 11(d), we see an annular layer randomly filled with intersections of a single trajectory lying between two invariant curves. We also see periodic trajectories that are composed by small regular trajectories inside these randomly filled regions (see, e.g., Figures 8(d) and 9(d)). Such a structure may be related to resonances of the orbits that play a crucial role in the appearance of random motions in near-integrable systems (Lichtenberg & Liberman 1992). In this paper, however, we do not discuss about the details. In this annular layer case, we also see that the value of $J_0$ varies irregularly (see Figure 11(e)). On the other hand, in Figure 11(g), we see a generic trajectory that covers the surface of the torus. The motion along the dipole magnetic field line is almost periodic in the $\theta$-direction (see Figure 11(i)). The intersections of the trajectory with the surface of a section at a value of $\psi_m$ lie on a closed invariant curve and densely cover the curve over long periods of time. The value of $J_0$ is nearly constant as shown in Figure 11(h). Figure 12 shows the other characteristic trajectories in a rotating black hole spacetime; see also Figure 6. In Figure 12 (top), the crescent regular trajectory is presented, where the value of $J_0$ is approximately constant. On the other hand, in Figure 12 (bottom), many islands of a trajectory distribute on a curve. This shows a regular trajectory. However, the value of $J_0$ is not a constant, and it oscillates periodically. When we consider the average of $J_0$ during dozens of cycles of oscillations, the value is almost constant over long periods of cycles. It seems that this feature is enough to guarantee the generation of the regular trajectory.

The ratio of the regular trajectories on the Poincaré map increases with the value of the hole’s spin. Specially, for the maximally rotating case of $a = M$, almost trajectories become regular (see Figure 10). It seems that the fourth constant of motion is generated for the system with the black hole dipole magnetic field in the $a \rightarrow M$ limit. One may expect that, by using the dipole magnetic fields given by Equations (9) and (10), the equation of motion can be separable with respect to the $r$ and $\theta$ coordinates, and the “modified Carter constant” including the magnetic field terms may be redefined. However, without the separable treatment on the basic equation, we find the constancy of $J_0$ in this system numerically. Exactly speaking, for the regular trajectories appeared in this system the value of $J_0$ is not necessary to be a constant; in fact it oscillates periodically. However, for a regular trajectory, we can regard the value of $J_0$
Figure 8. Poincaré map of a charged particle orbiting in the dipole magnetic field around a black hole of $a = 0.6 \, M$. The parameters for the motion are (a) $E/m = 0.905$, (b) $E/m = 0.900$, (c) $E/m = 0.895$, (d) $E/m = 0.890$, (e) $E/m = 0.885$, and (f) $E/m = 0.882$ with $L/m = -7.0 \, M$, $Q_d = 70.0 \, M^2$, $r_{\text{ini}} = 10.5 \, M$, and $\theta_{\text{ini}} = \pi/2$.

(A color version of this figure is available in the online journal.)

(or the average on a few cycles) as a constant approximately, comparing with that of chaotic trajectories. Even in the $a \to M$ limit, this constancy can be broken depending on the injection angle. In fact, for larger $E/m$, we can also see an annular layer of randomly filled by trajectories.

Our result suggests that the saddle in the double-well potential plays an important role for chaotic motions of a charged particle. Similar behavior has been investigated in Hamiltonian dynamical systems (Reichl & Zheng 1984). In the limit of $a \to M$, however, we see that the feature of the double-well potential vanishes. Then, the chaotic behavior weakens, where many regular trajectories are observed. The property of chaos and/or regular trajectories in the dipole magnetic field around a rapidly rotating black hole is related to the hole’s spin dependence on the effective potential, which is generated by the combination of the electromagnetic force, the gravitational force, and the centrifugal force. Although in this paper we consider the black hole dipole magnetic field, we
could understand the basic properties of a charged particle by considering the distribution of the effective potential when we treat a motion of a test charge in any arbitrary electromagnetic field around a black hole. It is a future work to confirm the above suggestion by more detailed analysis.

6. CONCLUDING REMARKS

We have discussed the off-equatorial motion of a test charge in the black hole dipole magnetosphere. The charged particle gyrates around a magnetic field line, drifts in the toroidal direction, and oscillates between the northern and southern hemispheres, and then the orbits become very complicated. So, the numerical study by the Poincaré map is effective. Then, we find the spin dependence of the trajectory on the Poincaré map. That is, for a slowly rotating black hole case, the particle’s trajectory shows chaos. More interestingly, we also find that, for a rapidly rotating black hole case, the chaotic behavior of the trajectories weakens and the fourth invariance of the motion, which is an adiabatic invariant related to the longitudinal motion, can be approximately generated.

In this paper, the typical Larmor gyration radii considered in our numerical calculation are roughly a gravitational radius. The strength of such a magnetic field is the order $10^{-3}$ gauss for an
electron (or positron) orbiting a $M \sim 10 \ M_\odot$ black hole, whose strength is much weaker than the maximum field strength on the black hole in astrophysical situations (see, e.g., Thorne et al. 1986). Thus, the backreaction on the Kerr geometry is treated as negligible. Furthermore, we have not considered the interactions between charged particles, and the emission process from the chaotic/regular motion of a charged particle. However, we can expect that the spectrum emitted from such charged particles in periodic motions in the inhomogeneous magnetic field carries the information on the black hole spin and the strength and/or distribution of the electromagnetic field. These studies should be considered in future works.

The chaotic motion in a black hole magnetosphere may be related to the origin of cosmic rays. The magnetic mirror concepts play a prominent role in space and plasma astrophysics. Examples are the Van Allen belt in the magnetosphere of the Earth, where high-energy particles are trapped in a magnetic bottle and some kind of acceleration mechanisms are expected. In the case of a black hole magnetosphere, we may expect the formation of the magnetic bottle near the black hole. When such a situation is realized, the particles trapped in the “black hole Van Allen belt” would emit high-energy radiation, or some particles may accelerate to ultra-relativistic velocity by some electromagnetic interactions (e.g., the Fermi acceleration by the perturbed magnetic bottle). Furthermore, some kinds of waves (i.e., Alfvén waves and/or fast waves) in the magnetosphere will give influence to the motion of a particle moving in a magnetized plasma.

Our simple model with the black hole dipole magnetic field presented here could be a preliminary step toward the understanding of the motion of charged particles around a black hole.

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APPENDIX

PLASMA HORIZON

In this paper, we have discussed about a trapping test charge with angular momentum $L$ in a black hole dipole magnetic field. To see the behavior for such a particle motion, the effective potential discussed in Section 3 is very useful. Here, we discuss the minimal local conditions for magnetic dominance and the “plasma horizon” discussed in Damour et al. (1978) and Ruffini (1978).

The minimal local condition for magnetic dominance is given by

$$F \equiv \frac{1}{2} F_{\mu \nu} F^{\mu \nu} > 0 \quad \text{or} \quad B^2 - E^2 > 0,$$

where $F_{\mu \nu}$ is an electromagnetic field tensor and $B$ and $E$ are the magnetic and electric fields in a local inertial frame (see...
Figure 11. Various types of the Poincaré map (left column). Many points on the Poincaré maps (a, d) are plotted by stochasticity (chaos), while that on the Poincaré map (g) shows regular trajectory. The action integral $J_\theta$ (center column) and the time series of $\pi/2 - \theta$ (right column), where the first several cycles are shown, are also presented. Small circles in the time series correspond to the points plotted on the Poincaré map. The values of $J_\theta$ are almost constant for (h) regular trajectories.

Damour et al. 1978). This local condition is a necessary condition only when the Lorentz force dominates the motion of a charged particle expect very near the event horizon. In the region of $F < 0$, the gravity by the black hole dominates to the electromagnetic force, and the particle cannot be trapped in the magnetic field. In the black hole dipole magnetic field, we have confirmed $F(r, \theta) > 0$ around the black hole (including the particle trapping region discussed in this paper) numerically, although in this paper an uncharged black hole is assumed.

Damour et al. (1978) also argued a stronger requirement for trapping a charged particle that the magnetic field force can balance the electric force in direction as well as modulus. The magnetic field can dominate the electric field in direction in the region of

$$V^2 \equiv V^\mu V^\mu > 0 \quad \text{or} \quad |E \times B| > E^2,$$

where $V^\mu \equiv F^{\mu\nu} F_{\nu\sigma} \eta^\sigma$ and $\eta^\mu$ is the four-velocity of an observer at rest in the local inertial frame (i.e., ZAMO). The boundary of this region is called the “plasma horizon.” In a nonrotating black hole case, there is no plasma horizon, because the electric field (or $A_\nu$), which can be generated by the dragging effect, is zero everywhere. On the other hand, in a rotating black
Figure 12. Examples of the Poincaré map (left column) for the \( a = 0.9 \, M \) case. The regular trajectories are shown. The action integral \( J_\theta \) (center column) and the time series of \( \pi/2 - \theta \) (right column), where the first several cycles are shown, are also presented. The parameters of these trajectories are set as \((a, b, c) = (0.9, 0.885, 0.0)\) and \((d, e, f) = (0.2, 70.0, 10.5)\). The other parameters are \( E/m = 0.00, L/m = 7.0 \, M, Q_d = 70.0 \, M^2, r_{\text{ini}} = 10.5 \, M, \) and \( \theta_{\text{ini}} = \pi/2. \)

Figure 13. Plasma horizon in a dipole magnetic field (thick curve) for (a) \( a = 0.1 \, M \) and (b) \( a = 0.9 \, M \) cases. Thin curves show the dipole magnetic field line, while thin arrows show the quadrupole electric field. (A color version of this figure is available in the online journal.)
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