An effective method for computing Grothendieck point residue mappings

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Abstract

Grothendieck point residue is considered in the context of computational complex analysis. A new effective method is proposed for computing Grothendieck point residues mappings and residues. Basic ideas of our approach are the use of Grothendieck local duality and a transformation law for local cohomology classes. A new tool is devised for efficiency to solve the extended ideal membership problems in local rings. The resulting algorithms are described with an example to illustrate them. An extension of the proposed method to parametric cases is also discussed as an application.

Key words: Grothendieck local residues mapping, algebraic local cohomology, transformation law

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1 Introduction

The theory of Grothendieck residue and duality is a cornerstone of algebraic geometry and complex analysis. Griffiths and Harris [1978], Grothendieck [1957].

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It has been used and applied in diverse problems of several different fields of mathematics (Baum and Bott, 1972; Bykov et al., 1991; Cardinal and Mourrain, 1996; Dickenstein and Sessa, 1991; Griffiths, 1976; Lehmann, 1991; O’Brien, 1973; Perotti, 1998; Suwa, 2005). In the global situation, methods for computing the total sum of Grothendieck residues have been extensively studied and applied by several authors (Bykov et al., 1991; Cattani et al., 1996; Kytmanov, 1988; Yushakov, 1984). The concept of Grothendieck local residue together with the local duality theory also play quite important roles in complex analysis, especially in singularity theory (Brasselet et al., 2009; Cherveny, 2018; Corrêa et al., 2016; Klehn, 2002; O’Brien, 1975; Suwa, 1988). Computing Grothendieck local residues is therefore of fundamental importance. However, since the problem is local in nature, it is difficult in general to compute Grothendieck local residues (O’Brien, 1977). In fact, a direct use of the classical transformation law described in (Hartshorne, 1966) only gives algorithms which lack efficiency. Compared to the global situation, despite the importance, much less work has been done on algorithmic aspects of computing Grothendieck local residues (Elkadi and Mourrain, 2007; Mourrain, 1997; Ohara and Tajima, 2019ab; Tajima and Nakamura, 2005b). Grothendieck local residues with parameters are useful in the study of singularity theory, for example, deformations of singularity and unfoldings of holomorphic foliations (Kulikov, 1998; Saito, 1983; Varchenko, 1986). However, to the best of our knowledge, existing algorithm of computing Grothendieck local residues are not designed to be able to treat parametric cases.

In this paper, we consider methods for computing Grothendieck point residues from the point of view of complex analysis and singularity theory. We propose a new effective method for computing Grothendieck point residues mappings and residues, which can be extended to treat parametric cases.

Let $X \subseteq \mathbb{C}^n$ be an open neighborhood of the origin $O \in \mathbb{C}^n$ and let $f_1(z), f_2(z), \ldots, f_n(z)$ be $n$ holomorphic functions defined on $X$, where $z = (z_1, z_2, \ldots, z_n) \in X$. Assume that their common locus in $X$ is the origin $O$: $\{ z \in X \mid f_1(z) = f_2(z) = \cdots = f_n(z) = 0 \} = \{ O \}$.

Then, for a given germ $h(z)$ of holomorphic function at $O$, the Grothendieck point residue at the origin $O$, denoted by

$$\text{res}_{\{O\}} \left( \frac{h(z)dz}{f_1(z)f_2(z)\cdots f_n(z)} \right),$$

of the differential form $\frac{h(z)dz}{f_1(z)f_2(z)\cdots f_n(z)}$ can be expressed, or defined, as
the integral
\[ \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int \cdots \int_{\gamma} \frac{h(z)dz}{f_1(z)f_2(z) \cdots f_n(z)}, \]
where \( dz = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \), and where \( \gamma_\epsilon \) is a real \( n \)-dimensional cycle:
\[ \gamma_\epsilon = \{ z \in X \mid |f_1(z)| = |f_2(z)| = \cdots = |f_n(z)| = \epsilon \}, \]
with \( 0 < \epsilon \ll 1 \). (See for instance, [Baum and Bott, 1972; Griffiths and Harris, 1978; Tong, 1973]).

Let
\[ h(z) \longrightarrow \text{res}_{O} \left( \frac{h(z)dz}{f_1(z)f_2(z) \cdots f_n(z)} \right) \]
be the Grothendieck point residue mapping that assigns to a holomorphic function \( h(z) \) the value of the Grothendieck point residue. We show that, based on the concept of local cohomology, the use of Grothendieck local duality and a transformation law for local cohomology classes given by J. Lipman ([Lipman, 1984]) allows us to design an effective method for computing Grothendieck local residue mappings and another one for computing Grothendieck local residues. Note that the classical transformation law on Grothendieck residue is of no avail for computing Grothendieck local residue mappings. Since we compute Grothendieck local residue mappings, our method is applicable when the holomorphic function \( h(z) \) in the numerator is computable, that is the case when the coefficients of the Taylor expansion of \( h(z) \) is computable. This is an advantage of our approach. We also show that the proposed method can be extended to treat parametric cases. This is another advantage of our approach.

In Section 2, we recall the transformation law for local cohomology classes and Grothendieck local duality. In Section 3, we fix our notation and we briefly recall our basic tool, an algorithm for computing Grothendieck local duality. We devise, in the context of exact computation, a new tool which plays a key role in the resulting algorithm. In Section 4, we describe the resulting algorithm for computing Grothendieck point residue mappings and the algorithm for computing Grothendieck point residues. In Section 5, as an application, we generalize the proposed method to treat parametric cases and we show, by using an example, an algorithm for computing Grothendieck point residues associated to a \( \mu \)-constant deformation of quasi homogeneous isolated hypersurface singularities.
2 Local analytic residues

The concept of Grothendieck point residue was introduced by A. Grothendieck in terms of derived categories and local cohomology. In this section, we briefly recall some basics on transformation law for local cohomology classes and Grothendieck local duality.

Let $X \subset \mathbb{C}^n$ be an open neighborhood of the origin $O \in \mathbb{C}^n$. Let $\mathcal{O}_X$ be the sheaf on $X$ of holomorphic functions, and $\Omega^n_X$ the sheaf of holomorphic $n$-forms. Let $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ (resp. $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$) denote the local cohomology supported at $O$ of $\mathcal{O}_X$ (resp. $\Omega^n_X$).

Then, $\mathcal{O}_{X,O}$, the stalk at $O$ of the sheaf $\mathcal{O}_X$, and the local cohomology $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ are mutually dual as locally convex topological vector spaces (Bănică and Stănășilă, 1974). The duality is given by the point residue pairing:

$$\text{res}_{\{O\}}(\ast, \ast): \mathcal{O}_{X,O} \times \mathcal{H}^n_{\{O\}}(\Omega^n_X) \longrightarrow \mathbb{C}$$

Let $F = [f_1(z), f_2(z), \ldots, f_n(z)]$ be an $n$-tuple of $n$ holomorphic functions defined on $X$. Assume that their common locus $\{z \in X \mid f_1(z) = f_2(z) = \cdots = f_n(z) = 0\}$ in $X$ is the origin $O$. Let $I_F$ denote the ideal in $\mathcal{O}_{X,O}$ generated by $f_1(z), f_2(z), \ldots, f_n(z)$. Let $\omega_F$ denote a local cohomology class $\omega_F = \left[ \frac{dz}{f_1(z)f_2(z)\cdots f_n(z)} \right]$ in $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$, where $dz = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, and $[\ ]$ stands for Grothendieck symbol (Hartshorne, 1966; Grothendieck, 1967). Residue theory says that, for $h(z)$ in $\mathcal{O}_{X,O}$, one has

$$\text{res}_{\{O\}} \left( \frac{h(z)dz}{f_1(z)f_2(z)\cdots f_n(z)} \right) = \text{res}_{\{O\}}(h(z), \omega_F).$$

2.1 Transformation law

Since $V(I_F) \cap X = \{O\}$, there exists, for each $i = 1, 2, \ldots, n$, a positive integer $m_i$ such that $z_i^{m_i} \in I_F$. There exists an $n$-tuple of holomorphic functions $a_{i,1}(z), a_{i,2}(z), \ldots, a_{i,n}(z)$ such that

$$z_i^{m_i} = a_{i,1}(z)f_1(z) + a_{i,2}(z)f_2(z) + \cdots + a_{i,n}(z)f_n(z), \quad i = 1, 2, \ldots, n.$$ 

Set $A(z) = \det(a_{i,j}(z))_{1 \leq i, j \leq n}$. 

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We have the following key lemma (Lipman, 1984).

**Lemma 1** (Transformation law for local cohomology classes). In $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$, the following formula holds.

$$\omega_F = \begin{bmatrix} A(z)dz \\ z_1^{m_1}z_2^{m_2} \cdots z_n^{m_n} \end{bmatrix}.$$  

For the proof of the result above, we refer the reader to (Kunz, 2009; Lipman, 1984). Note that the formula above implies the classical transformation law

$$\text{res}_{\{O\}}\left( \frac{h(z)dz}{f_1(z)f_2(z) \cdots f_n(z)} \right) = \text{res}_{\{O\}}\left( \frac{h(z)A(z)dz}{z_1^{m_1}z_2^{m_2} \cdots z_n^{m_n}} \right)$$

for point residues described in (Hartshorne, 1966). See also (Baum and Bott, 1972; Boyer and Hickel, 1997; Griffiths and Harris, 1978; Kytmanov, 1988).

### 2.2 Grothendieck local duality

We define $W_F$ to be the set of local cohomology classes in $\mathcal{H}^n_{\{O\}}(\Omega^n_X)$ that are killed by $I_F$:

$$W_F = \{ \omega \in \mathcal{H}^n_{\{O\}}(\Omega^n_X) \mid f_1(z)\omega = f_2(z)\omega = \cdots = f_n(z)\omega = 0 \}.$$  

Then, according to Grothendieck local duality, the pairing

$$\text{res}_{\{O\}}(\ast, \ast) : \mathcal{O}_{X,O}/I_F \times W_F \rightarrow \mathbb{C}$$

induced by the residue mapping is non-degenerate (Altman and Kleiman, 1970; Grothendieck, 1957; Hartshorne, 1966; Lipman, 2002).

Let $\succ^{-1}$ be a local term ordering on the local ring $\mathcal{O}_{X,O}$ and let $\{z^\alpha \mid \alpha \in \Lambda_F\}$ denote the monomial basis of the quotient space $\mathcal{O}_{X,O}/I_F$ with respect to the local term ordering $\succ^{-1}$, where $\Lambda_F \subset \mathbb{N}^n$ is the set of exponents $\alpha$ of basis monomials $z^\alpha$.

Let $\{\omega_\alpha \in W_F \mid \alpha \in \Lambda_F\}$ denote the dual basis of $\{z^\alpha \mid \alpha \in \Lambda_F\}$ with respect to the Grothendieck point residue. Then, we have

(i) $\mathcal{O}_{X,O}/I_F \cong \text{Span}_\mathbb{C}\{z^\alpha \mid \alpha \in \Lambda_F\}$,

(ii) $W_F = \text{Span}_\mathbb{C}\{\omega_\alpha \mid \alpha \in \Lambda_F\}$,
(iii) \( \text{res}_{(O)}(z^\alpha, \omega_\beta) = \begin{cases} 
1, & \alpha = \beta, \\
0, & \alpha \neq \beta, \quad \alpha, \beta \in \Lambda_F. 
\end{cases} \)

2.3 Residue mapping

Since \( \omega_F \) satisfies \( f_1(z)\omega_F = f_2(z)\omega_F = \cdots = f_n(z)\omega_F = 0 \), the local cohomology class \( \omega_F \) is in \( W_F \). Therefore \( \omega_F \) can be expressed as a linear combination of the basis \( \{ \omega_\alpha \mid \alpha \in \Lambda_F \} \).

Assume that, for the moment, we have the following expression:

\[
\omega_F = \sum_{\alpha \in \Lambda_F} b_\alpha \omega_\alpha, \quad b_\alpha \in \mathbb{C}.
\]

Now let

\[
\text{NF}_{\prec -1}(h)(z) = \sum_{\alpha \in \Lambda_F} h_\alpha z^\alpha, \quad h_\alpha \in \mathbb{C}
\]

be the normal form of the given holomorphic function \( h(z) \). Then, we have the following.

**Theorem 2.**

\[
\text{res}_{(O)} \left( \frac{h(z)dz}{f_1(z)f_2(z)\cdots f_n(z)} \right) = \sum_{\alpha \in \Lambda_F} h_\alpha b_\alpha
\]

**Proof.** Since \( h - \text{NF}_{\prec -1}(h) \in I_F \), we have

\[
\text{res}_{(O)}(h(z), \omega_F) = \text{res}_{(O)}(\text{NF}_{\prec -1}(h)(z), \omega_F).
\]

Therefore,

\[
\text{res}_{(O)} \left( \frac{h(z)dz}{f_1(z)f_2(z)\cdots f_n(z)} \right) = \text{res}_{(O)} \left( \sum_{\alpha \in \Lambda_F} h_\alpha z^\alpha, \sum_{\beta \in \Lambda_F} b_\beta \omega_\beta \right),
\]

which is equal to

\[
\sum_{\alpha, \beta \in \Lambda_F} h_\alpha b_\beta \text{res}_{(O)}(z^\alpha, \omega_\beta) = \sum_{\alpha \in \Lambda_F} h_\alpha b_\alpha.
\]

This completes the proof.
Let us consider a method for computing Grothendieck point residues in the context of symbolic computation. We start by recalling some basics on an algorithm for computing Grothendieck local duality given in (Tajima and Nakamura, 2009; Tajima et al., 2009).

Let $K = \mathbb{Q}$ be the field of rational numbers and let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. Let $H^n_{(O)}(K[z])$ denote the algebraic local cohomology defined to be

$$H^n_{(O)}(K[z]) = \lim_{k \to \infty} \text{Ext}^n_{K[z]}(K[z]/m^k, \Omega^n_X),$$

where $m$ is the maximal ideal $m = (z_1, z_2, \ldots, z_n)$ in $K[z] = K[z_1, z_2, \ldots, z_n]$.

We adopt the notation used in (Nabeshima and Tajima, 2015a,b, 2016a,b) to handle local cohomology classes. For instance, a polynomial $\sum \lambda c_\lambda \xi^\lambda$ in $K[\xi] = K[\xi_1, \xi_2, \ldots, \xi_n]$ represents the local cohomology class of the form

$$\sum_{\lambda=(\ell_1, \ell_2, \ldots, \ell_n)} c_\lambda \left[ \frac{1}{z_1^{\ell_1+1} z_2^{\ell_2+1} \cdots z_n^{\ell_n+1}} \right].$$

Note that a multiplication on $\xi^\beta$ by $z^\alpha$ is

$$z^\alpha \cdot \xi^\beta = \begin{cases} \xi^{\beta+\alpha}, & \beta \geq \alpha. \\ 0, & \text{otherwise}. \end{cases}$$

Let $\succ$ be a term ordering on $K[\xi]$. For a local cohomology class $\psi = c_\alpha \xi^\alpha + \sum_{\xi^\gamma > \xi^\alpha} c_\gamma \xi^\gamma$, we call $\xi^\alpha$ the head monomial of $\psi$, and $\alpha \in \mathbb{N}^n$ the head exponent of $\psi$.

Let $F = [f_1(z), f_2(z), \ldots, f_n(z)]$ be a list of $n$ polynomials $f_1, \ldots, f_n$ in $K[z]$. We also assume as in the previous section that there exists an open neighborhood $X$ of the origin $O$ such that their common locus is the origin: $\{z \in X \mid f_1(z) = f_2(z) = \cdots = f_n(z) = 0\} = \{O\}$.

We set

$$H_F = \{ \psi \in H^n_{(O)}(K[z]) \mid f_1(z) \cdot \psi = f_2(z) \cdot \psi = \cdots = f_n(z) \cdot \psi = 0 \}.$$

3.1 Algorithm for computing Grothendieck local duality

In (Nabeshima and Tajima, 2017; Tajima et al., 2009), an algorithm for computing bases of $H_F$ is introduced. Let $\Psi_F$ denote an output of the algorithm. Then,

$$W_F = \text{Span}_\mathbb{C}\{\psi dz \mid \psi \in \Psi_F\}$$
holds. Furthermore, the algorithm computes Grothendieck local duality with respect to the Grothendieck local residue pairing. Here we recall some basic properties of the algorithm.

An output of the algorithm, say $Ψ_F$, a basis of the vector space $H_F$, has the following form:

$$Ψ_F = \left\{ \psi_α \mid ψ_α = ξ^α + \sum_{ξ^α > ξ^γ} c_γ ξ^γ, \quad α ∈ Λ_F \right\},$$

where $Λ_F ⊂ \mathbb{N}^n$ is the set of the head exponents of local cohomology classes in $Ψ_F$.

Let $L_F$ denote the set of lower exponents of local cohomology classes in $Ψ_F$:

$$L_F = \left\{ γ ∈ \mathbb{N}^n \mid ∃ ψ_α = ξ^α + \sum_{ξ^α > ξ^γ} c_γ ξ^γ ∈ Ψ_F \text{ such that } c_γ ≠ 0 \right\}.$$

Set $E_F = Λ_F ∪ L_F$ and $T_F = \{ ξ_λ \mid λ ∈ E_F \}$. Now let $ℓ_{F,i} = \max\{ ℓ \mid ξ^ℓ_ℓ ∈ T_F \}$. Then Grothendieck local duality implies the following.

**Lemma 3.** Set $m_i = ℓ_{F,i} + 1$. Then $z_i^{m_i} ∈ I_F$ holds, where $I_F$ is the ideal in the local ring $K\{z\}$ generated by $f_1(z), f_2(z), \ldots, f_n(z)$.

**Proof.** Since $z_i^{m_i} * ψ_α = 0$ and $ψ_α ∈ Ψ_F$ hold, we have $z_i^{m_i} * ψ = 0$ for $ψ ∈ H_F$. It follows from the Grothendieck local duality that $z_i^{m_i}$ is in $I_F$. $\square$

Now let us consider the set of monomials $M_F$ in $K\{z\}$ defined to be $M_F = \{ z^α \mid α ∈ Λ_F \}$. Let $>^{-1}$ denote the local term ordering on $K\{z\}$ defined as the inverse ordering of $>$. Then, $M_F$ constitutes a monomial basis of the quotient $K\{z\}/I_F$ with respect to the local term ordering $>^{-1}$. Furthermore, we have the following result (Tajima and Nakamura, 2005a, 2009).

**Theorem 4.** Let $Ψ_F, M_F$ be as above. Then, $Ψ_F$ is the dual basis of the basis $M_F$ with respect to Grothendieck local residue pairing. That is, for $z^α ∈ M_F$ and for $ψ_β ∈ Ψ_F$,

$$\text{res}_{(O)}(z^α, ψ_β dz) = \begin{cases} 1, & α = β, \\ 0, & α ≠ β, \end{cases}$$

holds.

**Sketch of the proof.** Since the algorithm outputs a reduced basis of $H_F$, we have $Λ_F ∩ L_F = \emptyset$, which implies the result. $\square$
3.2 A key tool

Let \( m_i \) be an integer such that \( z_i^{m_i} \) is in the ideal \( I_F = (f_1, f_2, \ldots, f_n) \) in the local ring. Then there exist germs \( a_{i,1}(z), a_{i,2}(z), \ldots, a_{i,n}(z) \) of holomorphic functions such that

\[
z_i^{m_i} = a_{i,1}(z)f_1(z) + a_{i,2}(z)f_2(z) + \cdots + a_{i,n}(z)f_n(z), \quad i = 1, 2, \ldots, n.
\]

Theory of symbolic computation asserts that such \( n \)-tuple of holomorphic functions can be obtained by computing syzygies in the local ring \( K\{z\} \). Whereas, since the cost of computation of syzygies in local rings is high, a direct use of the classical algorithm of computing syzygy is not appropriate in actual computations. In fact, it is difficult to obtain these holomorphic functions. In previous papers [Nabeshima and Tajima, 2016b], the authors of the present paper have proposed a new effective method to overcome this type of difficulty.

We adopt the proposed method mentioned above and devise a new, much more efficient algorithm by improving the previous algorithm presented in [Nabeshima and Tajima, 2015b, 2016b]. We start by recalling the main idea given in [Nabeshima and Tajima, 2016b]. Let \( J_F = (f_1(z), f_2(z), \ldots, f_n(z)) \) denote the ideal in the polynomial ring \( K[z] \) generated by \( f_1(z), f_2(z), \ldots, f_n(z) \). Let \( J_{F,O} \) be the primary component of \( J_F \) whose associated prime is the maximal ideal \( \mathfrak{m} = \langle z_1, z_2, \ldots, z_n \rangle \), and \( G_Q \) a Gröbner basis of the ideal quotient \( Q = J_F : J_{F,O} \subset K[z] \). Then there is in \( G_Q \) a polynomial, say \( q(z) \), such that \( q(O) \neq 0 \).

Now let \( r(z) \in J_{F,O} \). Then, since \( q(z)r(z) \in J_F \), there exists an \( n \)-tuple of polynomials \( p_1(z), p_2(z), \ldots, p_n(z) \) in \( K[z] \), such that

\[
q(z)r(z) = p_1(z)f_1(z) + p_2(z)f_2(z) + \cdots + p_n(z)f_n(z).
\]

Since, \( q(O) \neq 0 \), we have a following expression in the local ring \( K\{z\} : \)

\[
r(z) = \frac{p_1(z)}{q(z)}f_1(z) + \frac{p_2(z)}{q(z)}f_2(z) + \cdots + \frac{p_n(z)}{q(z)}f_n(z).
\]

Since \( I_F = K\{z\} \otimes J_{F,O} \) and \( z_i^{m_i} \in I_F \), \( z_i^{m_i} \in J_{F,O} \) holds. Therefore, the argument above can be applied to compute germs \( a_{i,1}(z), a_{i,2}(z), \ldots, a_{i,n}(z) \) of holomorphic functions. Note also that, since \( J_{F,O} = \{p(z) \in K[z] \mid p(z) * \psi_{\alpha} = 0, \ \psi_{\alpha} \in \Psi_F\} \), the primary ideal \( J_{F,O} \) can be computed by using \( \Psi_F \).

Let \( G_F = \{g_1, g_2, \ldots, g_p\} \) be a Gröbner basis of \( J_F \). Let \( R_F \) be a list of relations between \( g_j \) and \( F = [f_1, f_2, \ldots, f_n] : \)

\[
g_j = r_{1,j}f_1 + r_{2,j}f_2 + \cdots + r_{n,j}f_n,
\]

where \( r_{1,j}, r_{2,j}, \ldots, r_{n,j} \) are elements of the field. The Gröbner basis \( G_F \) can be computed by using the proposed method.
where $r_{i,j} \in K[z]$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, \nu$. Let $S_F$ be a Gröbner basis of the module of syzygies among $F$:

$$s_1f_1 + s_2f_2 + \cdots + s_nf_n = 0,$$

where $s_i \in K[z]$, $i = 1, 2, \ldots, n$. Let $q$ be a polynomial in $G_Q$ such that $q(O) \neq 0$.

Now we are ready to present a new tool.

**Algorithm 1. localexpression**

**Input:** $G_F, R_F, S_F, q, r$.

**Output:** $[p_1, p_2, \ldots, p_n]$ such that $q(z)r(z) = p_1(z)f_1(z) + p_2(z)f_2(z) + \cdots + p_n(z)f_n(z)$.

**BEGIN**

step 1: divide $qr$ by the Gröbner basis $G_F = \{g_1, g_2, \ldots, g_\nu\}$:

$$qr = e_1g_1 + e_2g_2 + \cdots + e_\nu g_\nu;$$

step 2: rewrite the relation above by using $R_F$:

$$qr = \left(\sum_j r_{j,1}e_j\right)f_1 + \left(\sum_j r_{j,2}e_j\right)f_2 + \cdots + \left(\sum_j r_{j,\nu}e_j\right)f_\nu;$$

step 3: simplify the expression above by using $S_F$:

$$q(z)r(z) = p_1(z)f_1(z) + p_2(z)f_2(z) + \cdots + p_n(z)f_n(z);$$

return $[p_1, p_2, \ldots, p_n]$;

**END**

**Example 5** ($E_{12}$ singularity). Let $f(x, y) = x^3 + y^7 + xy^5$ and let $F = [\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)]$. Note that $f(x, y)$ is a semi quasi-homogeneous function with respect to the weight vector $(7, 3)$. Let $\succ$ be the weighted degree lexicographical ordering on $K[\xi, \eta]$ with respect to the weight vector $(7, 3)$, where $\xi, \eta$ correspond to $x, y$.

Then, $\dim_K(H_F) = 12$, the Milnor number at the origin $(0, 0)$ of the curve $\{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$. The algorithm for computing Grothendieck local duality, mentioned in this section, outputs a basis $\Psi_F$ that consists of the following 12 local cohomology classes:

$$1, \eta, \xi, \eta^2, \xi\eta, \eta^3, \xi\eta^2, \eta^4, \eta^5 - \frac{1}{4}\xi^2, \xi\eta^3, \xi^2 - \frac{5}{7}\eta^6 + \frac{5}{21}\xi^2\eta, \xi\eta^5 - \frac{5}{7}\eta^7 - \frac{1}{3}\xi^3 + \frac{5}{21}\xi^2\eta^2.$$

Note for instance that the local cohomology class $\begin{bmatrix} 1 \\ xy^6 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ x^3y \end{bmatrix}$ repre-
sented by $\psi_{(0,5)} = \eta^5 - \frac{1}{3} \xi^2$ above acts on a holomorphic function $h(x, y) = \sum_{(i, j)} c_{(i, j)} x^i y^j$ by

$$\text{res}_O(h(x, y), \psi_{(0,5)} dx \wedge dy) = c_{(0,5)} - \frac{1}{3} c_{(2,0)}.$$ 

The output implies that

$$\Lambda_F = \{(0, 0), (0, 1), (0, 2), (1, 0), (0, 3), (1, 1), (0, 4), (1, 2), (0, 5), (1, 3), (1, 4), (1, 5)\}$$

and $M_F = \{x^iy^j \mid (i, j) \in \Lambda_F\}$ is the monomial basis of the quotient space $K\{x, y\}/I_F$ with respect to the local term ordering $\succ^{-1}$ on $K\{x, y\}$, where $I_F$ denote the ideal in $K\{x, y\}$ generated by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Furthermore $W_F = \{\psi dx \wedge dy \mid \psi \in \Psi_F\}$ is the dual basis of the monomial basis $M_F$ with respect to the Grothendieck local residue pairing. Since $\lambda_F = (3, 7)$, we have $x^4, y^8 \in I_F$.

Let $J_F$ be the ideal in $K[x, y]$ generated by the two polynomials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Let $J_{F,O}$ be the primary component of $J_F$ whose associated prime is the maximal ideal $\langle x, y \rangle$. A Gröbner basis of the ideal quotient $J_{F,O} : J_F$ is

$$3125x + 151263, 25y + 147.$$ 

Set $q(x, y) = 25y + 147$. Then, the algorithm `localexpression` outputs the following:

$$q(x, y)x^4 = (49x^2 + 25/3x^2y - 49/3y^5)\frac{\partial f}{\partial x} + (-5/3xy^2 + 7/3y^4)\frac{\partial f}{\partial y},$$

$$q(x, y)y^8 = 25y^4\frac{\partial f}{\partial x} + (-15x + 21y^2)\frac{\partial f}{\partial y}.$$ 

4 Algorithms

Let $\tau_F$ denote the local cohomology class in $H_F$ defined to be

$$\tau_F = \begin{bmatrix} 1 \\ f_1(z)f_2(z) \cdots f_n(z) \end{bmatrix}.$$ 

Since $\omega_F = \tau_F dz$, the local cohomology class $\tau_F$ is the kernel function of the point residue mapping.

Let

$$q(z)z_i^{m_i} = p_{i,1}(z)f_1(z) + p_{i,2}(z)f_2(z) + \cdots + p_{i,n}(z)f_n(z), \quad i = 1, 2, \ldots, n,$$

and set $\text{Det}(z) = \det(p_{i,j}(z))_{1 \leq i, j \leq n}$. 


Let $I_M$ be the ideal in $K[z]$ generated by $z_1^{m_1}, z_2^{m_2}, \ldots, z_n^{m_n}$. Let $u(z) \in K[z]$ be a polynomial such that $u(z)q(z) - 1 \in I_M$.

Since $A(z) = \det(p_{i,j}(z)/q(z))_{1 \leq i,j \leq n}$ is equal to $\frac{1}{q(z)^n}\det(z)$, the transformation law implies the following

$$\tau_F = \begin{bmatrix} u(z)^n\det(z) \\ z_1^{m_1}z_2^{m_2} \cdots z_n^{m_n} \end{bmatrix}.$$

Let $\lambda_F = (\ell_{F,1}, \ell_{F,2}, \ldots, \ell_{F,n})$. Since $m_i = \ell_{F,i} + 1$, the formula above can be rewritten as $\tau_F = u(z)^n\det(z) * \xi^{\lambda_F}$.

Note that, according to an algorithm in [Sato and Suzuki, 2009] discovered by Y. Sato and A. Suzuki, the inverse $u(z)$ of $q(z)$ in $K[z]/I_M$ can be obtained by using Gröbner basis computation.

The following algorithm computes a representation of the local cohomology class $\tau_F$, the kernel function of the point residue mapping.

---

**Algorithm 2. tau**

**Input:** $V = [z_1, z_2, \ldots, z_n], \succ, F = [f_1(z), f_2(z), \ldots, f_n(z)]$.

/* $V$: a list of variables, $\succ$: a term order */

**Output:** $\tau_F = \sum_{\alpha \in \Lambda_F} b_{\alpha} \psi_{\alpha}$.

**BEGIN**

step 1: compute a basis $\Psi_F = \{\psi_\alpha \mid \alpha \in \Lambda_F\}$ of the space $H_F$;

/* $\Lambda_F$: the set of head terms of $\Psi_F$ */

step 2: compute $\ell_{F,i} = \max\{\ell | \xi^\ell \in T_F\}$ and set $m_i = \ell_{F,i} + 1$, $i = 1, 2, \ldots, n$;

/* $T_F = \{\xi^\lambda | \lambda \in E_F\}$ */

step 3: compute a Gröbner basis of the ideal $J_{F,O} = \{p(z) \in K[z] \mid p(z) * \psi_\alpha = 0, \alpha \in \Lambda_F\}$.

step 4: compute $G_F, R_F, S_F$;

/* notations are from subsection 3.2 */

step 5: compute a Gröbner basis $G_Q$ of the quotient ideal $Q = J_F : J_{F,O}$ and choose a polynomial $q(z)$ from $G_Q$ such that $q(O) \neq 0$;

step 6: compute

$q(z)z_i^{m_i} = p_{i,1}(z)f_1(z) + p_{i,2}(z)f_2(z) + \cdots + p_{i,n}(z)f_n(z), (i = 1, 2, \ldots, n),$

by using the algorithm **localexpression**;

step 7: compute $\det(z) = \det(p_{i,j}(z))_{1 \leq i,j \leq n}$ and set $ND = NF_{I_M}(\det(z))$, the normal form of $\det(z)$ with respect to $I_M$;
step 8: compute a Gröbner basis of the ideal in $K[z, u]$ generated by

$$1 - q(z)u, z_1^{m_1}, z_2^{m_2}, \ldots, z_n^{m_n}$$

with respect to an elimination ordering to eliminate $u$;

step 9: choose a polynomial of degree one with respect to $u$, of the form $cu + poly(z)$, from the Gröbner basis of step 8 and set

$$\text{Den} = (-c)^n, \text{NU} = NF_{I_M}(poly(z)^n), \text{Num} = NF_{I_M}(ND \times \text{NU});$$

step 10: compute $\psi = \text{Num} \times \xi^{\lambda_F}$ and set $\text{Coeff} = \{c_\alpha \mid \alpha \in \Lambda_F\};$

/* $c_\alpha$ is the coefficient of a term $\xi^\alpha$ of $\psi$, $\alpha \in \Lambda_F$. */

return $[\Lambda_F, \Psi_F, \text{Coeff}, \text{Den}]$;

END

The return of the algorithm above means

$$\tau_F = \frac{1}{\text{Den}} \sum_{\alpha \in \Lambda_F} c_\alpha \psi_\alpha.$$ 

Note that, since,

$$\text{res}_O(h(z)\tau_F dz) = \frac{1}{\text{Den}} \sum_{\alpha \in \Lambda_F} b_\alpha |\text{res}_O(h(z)\psi_\alpha dz)$$

holds, the output of the algorithm above completely describes the Grothendieck point residue mapping

$$h(z) \rightarrow \text{res}_O \left( \frac{h(z)dz}{f_1(z)f_2(z) \cdots f_n(z)} \right).$$

Let $\text{Res}_F = \tauau(V, \succ, F)$ be the output of the algorithm $\tauau$. The following algorithm residues evaluates the value of Grothendieck point residue.

**Algorithm 3. residues**

**Input:** $h \in K[z]$, Res$_F$.

**Output:** $\text{res}_O(h(z)\tau_F dz)$.

**BEGIN**

step 1: compute the normal form of $h$ by using $\Psi_F$, i.e., $NF_{\succ}(h)(z) = \sum_{\alpha \in \Lambda_F} h_\alpha z^\alpha$;

step 2: compute $\text{sum} = \sum_{\alpha \in \Lambda_F} h_\alpha c_\alpha$;

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return \(\sum\frac{\text{Den}}{\text{Den}}\); 
END

Note that \(\text{NF}_\infty(h)\) is computed by the algorithms given in [Tajima and Nakamura, 2009; Tajima et al., 2009]. The algorithm is free from standard bases computation. All the algorithms given in the present paper are implemented in a computer algebra system Risa/Asir (Noro and Takeshima, 1992).

**Example 6** (\(E_{12}\) singularity). Let us continue the computation. Since step 1 to step 6 are done, we start from step 7. From

\[
\begin{pmatrix}
p_{1,1} & p_{1,2} \\
p_{2,1} & p_{2,2}
\end{pmatrix} = \begin{pmatrix}
\frac{25}{3}x^2y + 49x^2 & -\frac{5}{3}xy^2 + 7/3y^4 \\
25y^4 & -15x + 21y^2
\end{pmatrix},
\]

we have the determinant

\[
\text{Det} = \left(-125y - 735\right)x^3 + \left(175y^3 + 1029y^2\right)x^2 + \left(125/3y^6 + 245y^5\right)x - 175/3y^8 - 343y^7.
\]

A Gröbner basis of the ideal in \(K[x, y, u]\) generated by \(1 - uq(x, y), x^4, y^8\) with respect to a elimination ordering \(u \succ x, y\) is

\[
\{x^4, y^8, -6103515625y^7 + 35888671875y^6 - 211025390625y^5 + 124082929875y^4 - 7296076265625y^3 + 42900928441875y^2 - 25225745923225y - 218041257467152161u + 1483273860320763\}.
\]

We have

\[
\text{Num} = (6654091109227055694580078125y^7 - 39126055722255087484130859375y^6 + 2300612076486859914406689453125y^5 - 1352759900963536296711333984375y^4 + 795422821766559342466243828125y^3 - 46770861919873689337016345y^2 + 275012668088857293301656112771125y - 1617074488362480884613737943094215)x^3 + (-322085690705603880169365234375y^7 + 189386386134895015395867578125y^6 - 11135919504731830794527701359375y^5 + 65479206667823165071822883993125y^4 - 385017735324400210622318557879575y^3 + 2263904283707473238459233120331901y^2)x^2 + (15590287306624563112338781903125y^7 - 916708936295243110552037590375y^6 + 53902489454160294871245981031405y^5)x - 7546347612358244281974437343967y^7
\]

and \(\text{Den} = (218041257467152161)^2\).

Since \(b_\alpha = \frac{c_\alpha}{\text{Den}}\), we have \(\tau_F = \frac{1}{\text{Den}}(\text{Num} * (\xi^3\eta^7))\).

Therefore,
\[ \tau_F = \frac{30517578125}{218041257467152161} - \frac{1220703125}{1483273860320763} + 4 \]
\[ \frac{8828125}{10090298369529} \eta^2 = -1953125/68641485507 \eta^2 + 78125/466948881 \eta^4 - 3 \]
\[ 125/3176523 \eta^5 + 125/21609 \eta^6 - 5/147 \eta^7 - 9765625/1441471195647 \xi + 390625/9 \]
\[ 80592651 \xi \eta - 15625/66706983 \xi \eta^2 + 125/453789 \xi \eta^3 - 25/3087 \xi \eta^4 + 1/21 \xi \eta^5 + \]
\[ 3125/9529569 \xi^2 = 125/64827 \xi^2 \eta + 5/441 \xi^2 \eta^2 - 1/63 \xi^3. \]

This yields

\[ \tau_F = \sum_{0 \leq i,j \leq 5} b_{i,j} \psi_{i,j}, \]

where

\begin{align*}
 b_{0,0} &= 30517578125/218041257467152161, \\
 b_{0,1} &= -1220703125/1483273860320763, \\
 b_{0,2} &= 48828125/10090298369529, b_{0,3} = -1953125/68641485507, \\
 b_{0,4} &= 78125/466948881, b_{0,5} = -3125/3176523, \\
 b_{1,0} &= -9765625/1441471195647, b_{1,1} = 390625/9805926501, \\
 b_{1,2} &= -15625/66706983, b_{1,3} = 625/453789, b_{1,4} = -25/3087, b_{1,5} = 1/21.
\end{align*}

and

\begin{align*}
 \psi_{0,0} &= 1, \ \psi_{0,1} = \eta, \ \psi_{0,2} = \eta^2, \ \psi_{0,3} = \eta^3, \ \psi_{0,4} = \eta^4, \ \psi_{0,5} = \eta^5 - \frac{1}{3} \xi^2, \\
 \psi_{1,0} &= \xi, \ \psi_{1,1} = \xi \eta, \ \psi_{1,2} = \xi \eta^2, \ \psi_{1,3} = \xi \eta^3, \\
 \psi_{1,4} &= \xi \eta^4 - \frac{5}{7} \eta^6 + \frac{5}{21} \xi^2 \eta, \ \psi_{1,5} = \xi \eta^5 - \frac{5}{7} \eta^7 - \frac{1}{3} \xi^3 + \frac{5}{21} \xi^2 \eta^2.
\end{align*}

Let \( NF_r(h)(x, y) = \sum_{(i, j) \in \Lambda_F} h_{i,j} x^i y^j. \) Then,

\[ \text{res}_{(O)}(h(x, y), \tau_F dx \wedge dy) = \sum_{(i, j) \in \Lambda_F} h_{i,j} b_{i,j}. \]

We have for instance,

\[ \text{res}_{(O)} \left( \frac{dx \wedge dy}{\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}} \right) = \frac{30517578125}{218041257467152161}. \]

Recall that, as local cohomology class \( \omega_F = \tau_F dx \wedge dy \) is in \( H^2_{\{u,v\}}(\Omega_X^2) \), the cohomology class \( \tau_F \) defines the residue mapping

\[ \text{res}_{(O)}(\ast, \tau_F) : \mathcal{O}_{X, O} \rightarrow \mathbb{C}. \]

Therefore, the formula above is valid for germs of holomorphic functions \( h(x, y) \). More precisely, for a germ of holomorphic function \( h(x, y) = \sum_{(i, j)} c_{i,j} x^i y^j \), we have

\[ \text{res}_{(O)}(h(x, y), \tau_F dx \wedge dy) = c_{0,0} b_{0,0} + c_{0,1} b_{0,1} + c_{0,2} b_{0,2} + c_{1,0} b_{1,0} + c_{0,3} b_{0,3} + c_{1,1} b_{1,1} + c_{0,4} b_{0,4} + c_{1,2} b_{1,2} + (c_{0,5} - \frac{5}{7} c_{2,0}) b_{0,5} + c_{1,3} b_{1,3} + (c_{1,4} - \frac{5}{7} c_{0,6} + \frac{5}{21} c_{2,1}) b_{1,4} + (c_{1,5} - \frac{5}{7} c_{3,0} - \frac{5}{7} c_{0,7} + \frac{5}{21} c_{2,2}) b_{1,5}. \]
In this section, we consider a $\mu$-constant deformation of a quasi homogeneous singularity, a family of semi-quasi homogeneous isolated hypersurface singularities (Greuel, 1986; Lê and Ramanujam, 1976). We give, as an application of the algorithms presented in the previous section, an algorithm for computing Grothendieck point residues associated to a $\mu$-constant deformation of a quasi homogeneous isolated hypersurface singularity. The keys of the resulting algorithm are the use of parametric local cohomology systems and parametric Gröbner systems (comprehensive Gröbner systems).

Let $w = (w_1, w_2, \ldots, w_n) \in \mathbb{N}^n$ be a weight vector for $z = (z_1, z_2, \ldots, z_n)$. Let $d_w(z^\lambda)$ denote the weighted degree of a monomial $z^\lambda = z_{1}^{\ell_{1}} z_{2}^{\ell_{2}} \cdots z_{n}^{\ell_{n}}$ defined to be

$$d_w(z^\lambda) = \ell_{1} w_{1} + \ell_{2} w_{2} + \cdots + \ell_{n} w_{n}.$$

**Definition 7.** (1) A non-zero polynomial $f_0$ is called a weighted homogeneous (or quasi homogeneous) polynomial of type $(d, w)$, if all monomials of $f_0$ have the same weighted degree $d$ with respect to the weight vector $w$, that is $f_0 = \sum_{d_w(z^\lambda) = d} c_\lambda z^\lambda$ where $c_\lambda \in K$.

(2) A polynomial $f(z) = f_0(z) + g(z)$ is called a semi weighted homogeneous (or semi quasi homogeneous) polynomial of type $(d, w)$, if

(i) $f_0$ is weighted homogeneous of type $(d, w)$, and $f_0(z) = 0$ has an isolated singularity at the origin $O$, and

(ii) $g(z) = \sum_{d_w(z^{\beta_j}) > d} b_j z^{\beta_j}$, where $b_j$ are coefficients.

Let $t = (t_1, t_2, \ldots, t_m)$ denote a set of new indeterminates, and let $T = \{t \mid t \in \mathbb{C}^m\}$. Let

$$f_t(z) = f_0(z) + g(z, t), \quad \text{with} \quad g(z, t) = \sum_{d_w(z^{\beta_j}) > d} t_j z^{\beta_j}$$

be a family of semi weighted homogeneous polynomials in $K(t)[z]$, where $t \in T$ is regarded as a deformation parameter. Then $f_t$ is a $\mu$-constant deformation of $f_0$.

Set $F = \left[\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_n}\right]$. Let $I_F$ denote a family of ideals in $K(t)[z]$ generated by $F$ with the parameter $t \in T$ and let

$$H_F = \left\{\psi \in H_{\{O\}}^n(K(t)[z]) \mid \frac{\partial f}{\partial z_1} \ast \psi = \frac{\partial f}{\partial z_2} \ast \psi = \cdots = \frac{\partial f}{\partial z_n} \ast \psi = 0\right\}.$$
set of leading exponents $\Lambda_F$ is independent of $t$ and thus so is the corresponding basis monomial set $M_F$. In our previous papers (Nabeshima and Tajima, \textit{2015d,b}), an algorithm for computing a basis $\Psi_F$ of $H_F$ is given. The algorithm also computes Grothendieck local duality as in the non parametric cases. The other steps, from step 3 to step 10 in the algorithm $\text{tau}$ are also executable by using parametric Gröbner systems. The step 1 and step 2 of the algorithm $\text{residues}$ are also executable.

Here we give an example of computation.

\textbf{Example 8 ($E_{12}$ singularity)}. Let us consider $f = x^3 + y^7 + txy^5$ ($t \neq 0$).

\textbf{step 1}: A basis $\Psi_F$ of the vector space $H_F$ with respect to a term ordering $\succ$ compatible with the weight $w = (7,3)$ is

$$\{1, \eta, \eta^2, \xi, \eta^3, \xi \eta, \eta^4, \xi \eta^2, \eta^5 - \frac{t}{3} \xi^2, \xi \eta^3, \xi \eta^4 - \frac{5t}{7} \eta^6 + \frac{5t^2}{21} \xi^2 \eta, \\
\xi \eta^5 - \frac{t}{3} \xi^3 - \frac{5t}{7} \eta^7 + \frac{5t^2}{21} \xi^2 \eta^2\}.$$  

The set $\Lambda_F$ is

$$\Lambda_F = \{(0,0), (0,1), (0,2), (1,0), (0,3), (1,1), (0,4), (1,2), (0,5), (1,3), (1,4), (1,5)\}.$$  

\textbf{step 2}: $x^4, y^8 \in I_F$.

\textbf{step 5}: $q(x, y) = 147 + 25t^3 y \in J_F : J_{F,O}$.

\textbf{step 6}:

$$\begin{pmatrix}
q(x, y)x^4 \\
q(x, y)y^8
\end{pmatrix} = \begin{pmatrix}
(25/3t^3 y + 49)x^2 - 49/3t y^5, -5/3t^3 y^2 x + 7/3t^2 y^4 \\
25t^2 y^4 \\
-15tx + 21y^2
\end{pmatrix} \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right).$$

\textbf{step 7}: $\det(x, y)$ is

$$(-125t^4 y - 735t)x^3 + (175t^3 y^3 + 1029y^2)x^2 + (125/3t^5 y^6 + 245t^2 y^5)x - 175/3t^4 y^8 - 343ty^7.$$  

\textbf{step 8}: A Gröbner basis of $\langle x^4, y^8, 1 - q(x, y)u \rangle$ is

$$\{y^8, x^4 - 6103515625t^{21}y^7 + 35888671875t^{18}y^6 - 211025390625t^{15}y^5 + 1240829296875t^{12}y^4 - 7290076265625t^9y^3 + 42900928401875t^6y^2 - 252257459238225t^3y - 218041257467152161u + 1483273860320763\}.$$
step 9: We have

$$\text{Den} = (218041257467152161)^2,$$

$$poly(x,y) = -6103515625t^{21}y^7 + 35888671875t^{18}y^6 - 211025390625t^{15}y^5 + 1$$
$$240829296875t^{12}y^4 - 7296076265625t^9y^3 + 42900928441875t^6y^2 - 252257549238$$
$$225t^3y + 1483273860320763,$$

$$\text{NU} = -72425481460974755859375000t^{21}y^7 + 37262910211671511889648375$$
$$t^{18}y^6 - 1878050674668244199238281250t^{15}y^5 + 920244830587439657626757812$$
$$5t^{12}y^4 - 432883168308331614947628675000t^9y^3 + 19090147722397424219190345$$
$$1875t^6y^2 - 74833379071797902933261531350t^3y + 22001013447108583461324$$
$$8902169,$$

and

$$\text{Num} = (6654091109227055694580078125t^{22}y^7 - 3912605572255087484130859$$
$$375t^{19}y^6 + 230061207646859914406689453125t^{16}y^5 - 13527599009635362967113$$
$$33984375t^{13}y^4 + 7954228217665593424662643828125t^{10}y^3 - 46770861919873689$$
$$337016345709375t^7y^2 + 27501266808885729330165611277112ttty^3 - 16170744883$$
$$62480884613737943094215(t^3 + (-322085690705603880169365234375t^{15}y^7 + 1$$
$$893863861348950815395867578125t^{12}y^6 - 11135919504731830794527701359375$$
$$t^9y^5 + 6547920668782316507182883393125t^6y^4 - 38501773532440021062231855$$
$$7879575t^3y^2 + 2263904283707473238459233120331901y^2)x^2 + (155902873066245$$
$$63112338781903125t^8y^7 - 9167088936295243100052037590375t^6y^5 + 539024829$$
$$45416029487124598103405t^3y^2 - 7546347676235824412819744373443967ty^7.$$ 

As an output we thus have

$$\tau_F = \sum_{0 \leq i,j \leq 5} b_{i,j}\psi_{i,j},$$

where

$$b_{0,0} = 30517578125t^{22}/218041257467152161,$$
$$b_{0,1} = -1220703125t^{19}/1483273860320763,$$
$$b_{0,2} = 48828125t^{16}/10090298369529, b_{0,3} = -1953125t^{13}/68641485507,$$
$$b_{0,4} = 78125t^{10}/466948891, b_{0,5} = -3125t^7/3176523,$$
$$b_{1,0} = -9765625t^{15}/1441471195647,$$
$$b_{1,1} = 390625t^{12}/9805926501, b_{1,2} = -15625t^9/66706983,$$
$$b_{1,3} = 625t^6/453789, b_{1,4} = -25t^3/3087, b_{1,5} = 1/21.$$

and

$$\psi_{0,0} = 1, \ \psi_{0,1} = \eta, \ \psi_{0,2} = \eta^2, \ \psi_{0,3} = \eta^3, \ \psi_{0,4} = \eta^4, \ \psi_{0,5} = \eta^5 - \frac{t}{3}\xi^2,$$

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\[ \psi_{1,0} = \xi, \quad \psi_{1,1} = \xi \eta, \quad \psi_{1,2} = \xi \eta^2, \quad \psi_{1,3} = \xi \eta^3, \quad \psi_{1,4} = \xi \eta^4 - \frac{5t}{7} \eta^6 + \frac{5t^2}{21} \xi^2 \eta; \]

\[ \psi_{1,5} = \xi \eta^5 - \frac{5t}{7} \eta^7 - \frac{t}{3} \xi^3 + \frac{5t^2}{21} \xi^2 \eta^2. \]

We have, for instance,

\[
\text{res}_{\{O\}} \left( \begin{vmatrix} dx \wedge dy \\ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \end{vmatrix} \right) = \frac{30517578125}{218041257467152161 t^{22}}.
\]

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