A FAMILY OF RELATIVISTIC BARGMANN-TYPE TRANSFORMS ATTACHED TO MAASS LAPLACIANS ON THE POINCARÉ DISK

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ABSTRACT. We construct a set of coherent states through special superpositions of eigenstates of a relativistic pseudoharmonic oscillator. In each superposition the coefficients are chosen to be $L^2$-eigenfunctions of a $\sigma$-weight Maass Laplacian on the Poincaré disk, which are associated with the eigenvalue $4m(\sigma - 1 + m)$, $m = 0, 1, \ldots, \lfloor (\sigma - 1)/2 \rfloor$. For each integer $m$ the obtained coherent states transform constitutes a relativistic Bargmann-type transform whose integral kernel is expressed in terms of a special Appel-Kampé de Fériet’s hypergeometric function.

1. INTRODUCTION

In [1] V. Bargmann has introduced a second transform labeled by a parameter $\delta > 0$ as

$$ B_\delta : L^2 \left( \mathbb{R}_+, \frac{x^\delta}{\Gamma(1 + \delta)} \, dx \right) \to A^{\delta + 1} (\mathbb{D}) \tag{1.1} $$

defined by

$$ B_\delta [\phi] (z) := \frac{\left( \frac{\delta}{\pi} \right)^{\frac{1}{2}}}{\Gamma (\delta + 1) (1 - z)^{\delta + 1}} \int_0^{+\infty} \exp \left( -\frac{x}{2} \left( \frac{1 + z}{1 - z} \right) \right) \phi (x) x^\delta \, dx \tag{1.2} $$

where

$$ A^{(\delta + 1)} (\mathbb{D}) := \left\{ \psi \text{ analytic on } \mathbb{D}, \int_\mathbb{D} |\psi(z)|^2 \left( 1 - |z|^2 \right)^{\delta - 1} \, d\mu (z) < +\infty \right\} \tag{1.3} $$

denotes the weighted Bergman space on the unit disk $\mathbb{D} = \{ z \in \mathbb{C}; |z| < 1 \}$ and $d\mu (z)$ being the Lebesgue measure on it. Thus, setting $\delta = \sigma - 1 > 0$ and using the isometry form $L^2 (\mathbb{R}_+, dx)$ onto $L^2 (\mathbb{R}_+, \Gamma^{-1} (\sigma) x^{\sigma - 1} \, dx)$, one extends $B_\delta$ to the transform

$$ B^{\Pi}_\sigma : L^2 (\mathbb{R}_+, dx) \to A^\sigma (\mathbb{D}) \tag{1.4} $$

defined by

$$ B^{\Pi}_\sigma [\phi] (z) := \frac{\left( \frac{\sigma - 1}{\pi \Gamma (\sigma)} \right)^{\frac{1}{2}}}{(1 - z)^{\sigma}} \int_0^{+\infty} \exp \left( -\frac{x}{2} \left( \frac{1 + z}{1 - z} \right) \right) \phi (x) x^{\frac{1}{2}(\sigma - 1)} \, dx. \tag{1.5} $$

The involved kernel function in (1.5) corresponds to the generating function of Laguerre polynomials [2]. The latter ones turn out to be fundamental pieces in expressing wave functions of the eigenstates of the pseudoharmonic oscillator Hamiltonian (see [3] and references therein). Now, after being observed that the role of the Laguerre polynomials can be replaced by the continuous dual Hahn polynomials [4] which are involved in the wave functions of the eigenstates of a relativistic pseudoharmonic oscillator Hamiltonian [5], here we propose a “relativistic” version of the second Bargmann transform (1.4), which also...
will include a generalization of the arrival space \( \mathcal{A}_m^\sigma \) in (1.4). The latter one will be replaced by the eigenspace (6,7):

\[
\mathcal{A}_m^\sigma (\mathbb{D}) := \left\{ \psi : \mathbb{D} \to \mathbb{C}, \Delta_\sigma \psi = e_m^\sigma \psi, \int_{\mathbb{D}} |\psi(z)|^2 (1-\sigma z)^{-2} d\mu(z) < +\infty \right\}
\]  

(1.6)
of the second order differential operator

\[
\Delta_\sigma := -4 (1-\sigma z) \left( (1-\sigma z) \frac{\partial^2}{\partial z \partial \overline{z}} - \sigma \frac{\partial}{\partial \overline{z}} \right)
\]  

(1.7)

with the eigenvalue (hyperbolic Landau level):

\[
e_m^\sigma := 4m (\sigma - 1 - m), m = 0, 1, 2, \cdots, \left[ \frac{\sigma - 1}{2} \right],
\]  

(1.8)

where \( [x] \) denotes the greatest integer less than \( x \). The operator in (1.6) can be unitarily intertwined to represent the Schrödinger operator of a charged particle evolving in the Poincaré disk under influence of a uniform magnetic field with a strength proportional to \( \sigma \). For \( m = 0 \), the space \( \mathcal{A}_0^\sigma (\mathbb{D}) \) in (1.5) coincides with the Bergman space \( \mathcal{A}_0^\sigma (\mathbb{D}) \) in (1.3) where \( \delta = \sigma - 1 \). In this paper, we precisely construct a family of integral transforms of the form \( \mathcal{B}_{c,m}^{II} : L^2 (\mathbb{R}_+) \to \mathcal{A}^{2(\gamma + m)} (\mathbb{D}) \) defined by

\[
\mathcal{B}_{c,m}^{II} [f] (z) := \frac{(2\gamma - 1)\frac{1}{2}}{\sqrt{\pi m!}} \frac{\sqrt{2\Gamma (m + 2\gamma) (i)^\gamma}}{\Gamma (2\gamma) \Gamma \left( \gamma + \frac{1}{2} \right)} \left( \frac{z - 1}{(1-z)(1-z\overline{z})} \right)^m \int_0^{+\infty} \frac{\Gamma^2 (\gamma + i\xi) \Gamma (c-4) i^{\xi}}{\Gamma (i\xi)} F_5 \left( \begin{array}{c} \gamma + i\xi, \gamma - i\xi, 2\gamma + m; \gamma + \frac{1}{2} \end{array} \right| \frac{1-\xi z}{(1-z)(z-1)}, \frac{1}{1-\overline{z}} \right) f(\xi) d\xi,
\]  

(1.9)

where \( \sigma = 2 (\gamma + m), c > 0, \gamma = \gamma (c) = (1 + \sqrt{1 + 2c^2})/2 \) and \( F_5 \) is a special Appel-Kampé de Fériet hypergeometric function [8]. Our method in constructing the transform (1.8) is based on a coherent states analysis by adopting a general probabilistic scheme “à la Gazeau” [9]. In the analytic case which corresponds to the particular value \( m = 0 \), we prove that the transform (1.9) reduces to the following one

\[
\mathcal{B}_{c,0}^{II} : L^2 (\mathbb{R}_+) \to \mathcal{A}^{2\gamma} (\mathbb{D})
\]  

(1.10)
declared by

\[
\mathcal{B}_{c,0}^{II} [f] (z) := \frac{\sqrt{2} (-i)^\gamma \left( \frac{2\gamma - 1}{\pi (2\gamma)} \right)^{\frac{1}{2}}}{\Gamma \left( \gamma + \frac{1}{2} \right)} \int_0^{+\infty} \frac{\Gamma^2 (\gamma - i\xi) \Gamma (c-4) i^{\xi}}{(c-4) i^{\xi} (1-z\overline{z})^{i\xi}} \left( \gamma - i\xi, \gamma + \frac{1}{2} \right| z \right) f(\xi) d\xi,
\]  

(1.11)

where \( 2F_1 (\cdot) \) denotes Gauss hypergeometric function [10].

The paper is organized as follows. In Section 2, we recall briefly some needed tools from the spectral theory of the \( \sigma \)-weight Maass Laplacians on the Poincaré disk. Section 3 deals with the construction of a set of coherent states in the framework of a probabilistic Hilbertian schema without specifying the corresponding Hamiltonian system. In section 4 we summarize some required information on a relativistic model for the pseudoharmonic oscillator. In section 5 we particularize the constructed coherent states for the relativistic
pseudoharmonic oscillator and we obtain expressions for their wave functions in an explicit way. Section 6 is devoted to establish the corresponding coherent states transforms.

2. MAASS LAPLACIANS ON THE POINCARÉ DISK

Let $\mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \}$ be unit disk endowed with its usual Khâler metric $ds^2 = -\partial \overline{\partial} \log (1 - z \overline{z}) \, dz \otimes d\overline{z}$. The Bergman distance on $\mathbb{D}$ is given by

$$\cosh^2 d(z, w) = \frac{(1 - z \overline{w})(1 - z w)}{(1 - z \overline{z})(1 - w \overline{w})}$$

and the volume element reads

$$d\mu(z) = \frac{1}{(1 - z \overline{z})^2} dv(z)$$

with the Lebesgue measure $dv(z)$. Let us consider the differential 1–form on $\mathbb{D}$ defined by $\theta = -i \left( \partial - \overline{\partial} \right) \log (1 - z \overline{z})$ to which the Schrödinger operator

$$H_\sigma := \left( d + i \sigma \text{ ext} (\theta) \right)^* \left( d + i \sigma \text{ ext} (\theta) \right)$$

can be associated. Here $\sigma \geq 0$ is a fixed number, $d$ denotes the usual exterior derivative on differential forms on $\mathbb{D}$ and $\text{ ext} (\theta)$ is the exterior multiplication by $\theta$ while the symbol $\ast$ stands for the adjoint operator with respect to the Hermitian scalar product induced by the Bergman metric $ds^2$ on differential forms. Actually, the operator $H_\sigma$ is acting on the Hilbert space $L^2(\mathbb{D}, d\mu(z))$ and can be unitarily intertwined as

$$(1 - z \overline{z})^{1/2} \Delta_\sigma (1 - z \overline{z})^{-1/2} = H_\sigma$$

in terms of the second order differential operator $\Delta_\sigma$ introduced in (1.7). The latter one is acting on the Hilbert space $L^{2,\sigma} (\mathbb{D}) = L^2(\mathbb{D}, (1 - z \overline{z})^{\sigma-2} dv(z))$. Note that this operator is an elliptic densely defined operator on $L^{2,\sigma} (\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by $\Delta_\sigma$. The part of its spectrum is not empty if and only if $\sigma > 1$. This discrete part consists of eigenvalues occurring with infinite multiplicities and having the expression $\epsilon_m^\sigma = 4m(\sigma - m - 1)$ in (1.8) for varying $m = 0, 1, \cdots, \lfloor (\sigma - 1)/2 \rfloor$. Moreover, it is well known ([6,7,11]) that the functions given in terms of Jacobi polynomials [2] by

$$\Phi_k^{\sigma,m}(z) := \sqrt{\frac{(\sigma - 2m - 1) \Gamma(\sigma - m) k!}{\pi m! \Gamma(\sigma - 2m + k)}} \frac{(-1)^k z^{m-k}}{(1 - z \overline{z})^m} P_k^{(m-k,\sigma-2m-1)}(1 - 2z \overline{z})$$

constitute an orthonormal basis of the eigenspace

$$A_m^\sigma (\mathbb{D}) := \left\{ \phi \in L^{2,\sigma} (\mathbb{D}), \Delta_\sigma \phi = \epsilon_m^\sigma \phi \right\}.$$  (2.6)

of $\Delta_\sigma$ associated with the eigenvalue $\epsilon_m^\sigma$ in (2.6). Finally, the $L^2$–eigenspace $A_0^\sigma (\mathbb{D}) = \left\{ \phi \in L^{2,\sigma} (\mathbb{D}), \Delta_\sigma \phi = 0 \right\}$ corresponding to $m = 0$ and associated to $\epsilon_0^\sigma = 0$ in (2.7) reduces further to the weighted Bergman space consisting of holomorphic functions $\phi$: $\mathbb{D} \rightarrow \mathbb{C}$ with the growth condition

$$\int_{\mathbb{D}} |\phi(z)|^2 (1 - z \overline{z})^{\sigma-2} dv(z) < +\infty.$$  (2.7)

This is why the eigenspaces in (2.6) are also called generalized Bergman spaces on the complex unit disk.
Remark 2.1. The spectral analysis of $\Delta_\sigma$ have been studied by many authors, see [6] and references therein and it can also be obtained from the $\sigma-$weight Maass Laplacian $y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - i \sigma y \partial_x$ on the Poincaré upper half-plane [12]. The condition $\sigma > 1$ ensuring the existence of the eigenvalues $e^{\nu}_m$ in (1.8) should imply that the magnetic field $B = \sigma \Omega(z)$, where $\Omega$ stands for the Kähler 2-form on $\mathbb{D}$, has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the classical orbit of the particle will intercept the disk boundary whose points stand for $\{ \infty \}$ which means escaping to infinity (see [13, p.189]).

3. Coherent states in a Hilbertian probabilistic scheme

The negative binomial states [14] are labeled by points $z \in \mathbb{D}$ and are of the form

$$|z, \sigma, 0 \rangle := (1 - z \bar{z})^{1/2} \sum_{k=0}^{+\infty} \sqrt{\Gamma(\sigma + k)} \Gamma(\sigma) k! z^k |\psi_k \rangle$$

(3.1)

where $\sigma > 1$ is a fixed parameter and the kets $|\psi_k \rangle$ are for instance elements of an abstract Hilbert space $\mathcal{H}$. Their photon-counting probability distribution is given by

$$\Pr(X = k) := |\langle \psi_k | z, \sigma, 0 \rangle|^2 = (1 - z \bar{z})^{\sigma} (z \bar{z})^k \Gamma(\sigma + k) \Gamma(\sigma) k!$$

which obeys the negative binomial probability distribution [15]. Observe that the coefficients in the superposition (3.1):

$$\Phi^{\sigma,0}_k(z) := \sqrt{\frac{\Gamma(\sigma + k)}{\pi \Gamma(\sigma) k!}} z^k, k = 0, 1, 2, \cdots ,$$

(3.2)

constitute an orthonormal basis of the eigenspace $\mathcal{A}_0^\sigma(\mathbb{D})$ associated with the first eigenvalue $e^{\nu}_0 = 0$ and consisting of analytic functions on $\mathbb{D}$ with the growth condition (2.7). For instance, let $\sigma > 1$ and $m = 0, 1, \cdots , [(\sigma - 1) / 2]$ be fixed parameters and let $\{|\psi_k \rangle\}_{k=0}^\infty$ be a set of Fock states in a Hilbert space $\mathcal{H}$. Then, adopting the Hilbertian probabilistic scheme of coherent states in ([9, p.74, Eq. (58)]), we state the following.

Definition 3.1. A class of coherent states can be defined as

$$|z, \sigma, m \rangle := (\mathcal{N}_{\sigma,m}(z))^{-1/2} \sum_{k=0}^{+\infty} \Phi^{\sigma,m}_k(z) |\psi_k \rangle$$

(3.3)

where $\mathcal{N}_{\sigma,m}(z)$ is a normalization factor and $\{\Phi^{\sigma,m}_k(z)\}_{k=0}^\infty$ is the orthonormal basis (2.5) of the generalized Bergman space $\mathcal{A}_m^\sigma(\mathbb{D})$.

Now, one of the important tasks to do is to determine the overlap relation between two coherent states.

Proposition 3.2. Let $\sigma > 1$ and $m = 0, 1, \cdots , [(\sigma - 1) / 2]$. Then, for every $z, w \in \mathbb{D}$, the overlap relation between two coherent states is given through the scalar product

$$<w, \sigma, m | z, \sigma, m >_\mathcal{H} = \frac{(\sigma - 2m - 1)\Gamma(\sigma - m) \mathcal{N}(z) \mathcal{N}(w)^{-1/2}}{\pi m! (-1)^m \Gamma(\sigma - 2m) (1 - z \bar{w})^\sigma}
\times \left( \frac{(1 - z \bar{w})(1 - \bar{w}z)}{(1 - z \bar{z})(1 - w \bar{z})} \right)^m _2F_1 \left( -m, \sigma - m, \sigma - 2m; \frac{(1 - z \bar{z})(1 - w \bar{w})}{(1 - z \bar{w})(1 - w \bar{z})} \right)$$

(3.4)

where $_2F_1$ is a terminating Gauss hypergeometric sum.
\textbf{Proof.} In view of Eq. (3.3), the scalar product of two coherent states \(|z, \sigma, m\rangle\) and \(|w, \sigma, m\rangle\) in \(\mathcal{H}\) reads

\[<w, \sigma, m | z, \sigma, m \rangle_{\mathcal{H}} = (\mathcal{N}_{\sigma, m}(z)\mathcal{N}_{\sigma, m}(w))^\frac{-1}{2} \sum_{k=0}^{\infty} \Phi_k^{\sigma, m}(z) \Phi_k^{\sigma, m}(w) = \frac{(\sigma - 2m - 1)(\mathcal{N}(z)\mathcal{N}(w))^\frac{-1}{2}}{\pi ((1-z\overline{z}) (1-w\overline{w}))^m} S_{z,w}^{\sigma, m}, \tag{3.5}\]

where

\[S_{z,w}^{\sigma, m} = \frac{\Gamma (\sigma - m) (z\overline{w})^m}{m! \Gamma (\sigma - 2m)} \sum_{k=0}^{\infty} \frac{k!}{(\sigma - 2m)_k} \left( \frac{1}{z\overline{w}} \right)^k \times P_k^{(m-k, \sigma-2m-1)} (1-2z\overline{z}) P_k^{(m-k, \sigma-2m-1)} (1-2w\overline{w}). \tag{3.6}\]

Making use of the following identity due to A. Sirvastava and A. B. Rao ([16, p.1329]):

\[\sum_{n=0}^{\infty} \frac{n! t^n}{(1+\alpha)_n} P_n^{(\gamma-n, \alpha)} (x) P_n^{(\gamma-n, \alpha)} (y) = \left( 1 - \frac{1}{4} (x-1)(y-1) t \right)^{1+\gamma+\alpha} \times (1-t)^{-2F1} \left( 1 + \gamma + \alpha, -\gamma, 1+\alpha; -\frac{(x+1)(y+1)t}{(1-t)(4-(x-1)(y-1))} \right) \tag{3.7}\]

for \(n = k, t = 1/zw, \gamma = m, \alpha = \sigma - 2m - 1, x = 1 - 2z\overline{z}\) and \(y = 1 - 2w\overline{w}\), we obtain, after calculations, the expression

\[S_{z,w}^{\sigma, m} = \frac{\Gamma (\sigma - m) (-1)^m ((1-z\overline{w})(1-w\overline{z}))^m}{m! \Gamma (\sigma - 2m) (1-z\overline{w})^\sigma} \times 2F1 \left( -m, \sigma - m, \sigma - 2m; \frac{(1-z\overline{w})(1-w\overline{w})}{(1-z\overline{w})(1-w\overline{z})} \right). \tag{3.8}\]

Returning back to Eq. (3.5) and inserting the expression (3.8) we arrive at the announced formula. \(\square\)

\textbf{Corollary 3.3.} The normalization factor in (3.3) is given by

\[\mathcal{N}_{\sigma, m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1-z\overline{z})^{\sigma}}, \tag{3.9}\]

for every \(z \in \mathbb{D}\).

\textbf{Proof.} We first make appeal to the relation ([17, p.212]):

\[2F1 \left( -n, n + \kappa + q + 1, 1 + \kappa; \frac{1 - \tau}{2} \right) = \frac{n! \Gamma (1+\kappa)}{\Gamma (1+\kappa+n)} P_n^{(\kappa, q)} (\tau) \tag{3.10}\]

connecting the \(2F1\)–sum with the Jacobi polynomial for the parameters \(n = m, \kappa = \sigma - 2m - 1, q = 0\) and the variable

\[\tau = 1 - 2\frac{(1-z\overline{z})(1-w\overline{w})}{(1-z\overline{w})(1-w\overline{z})} \tag{3.11}\]
to rewrite Eq. (3.4) as
\[
\sqrt{\mathcal{N}_{\sigma,m}(z)\mathcal{N}_{\sigma,m}(w)} = \frac{(-1)^m (\sigma - 2m - 1) (1 - z\bar{w})^{-\sigma}}{\pi < w, \sigma, m | z, \sigma, m > \mathcal{H}} (3.12)
\]
\[
\times P_m^{(\sigma-2m-1,0)} \left(1 - 2 \frac{(1 - z\bar{w}) (1 - w\bar{w})}{(1 - z\bar{w}) (1 - w\bar{z})} \right).
\]

The factor $\mathcal{N}_{\sigma,m}(z)$ should be such that $< z, \sigma, m | z, \sigma, m > \mathcal{H} = 1$. So that we put $z = w$ in (3.12) and we use the well known symmetry identity satisfied by the Jacobi polynomials $P_m^{(\sigma,\gamma)}(\xi) = (-1)^m P_m^{(\sigma,\eta)}(-\xi)$ to obtain the expression
\[
\mathcal{N}_{\sigma,m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\bar{z})^\sigma} P_m^{(0,\sigma-2m-1)}(1) (3.13)
\]

Finally, we apply the fact that (17, p.209):
\[
P_m^{(\alpha,\nu)}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} (3.14)
\]
in the case of $\alpha = 0, n = m$ and $\nu = \sigma - 2m - 1$. This ends the proof. \hfill \Box

**Proposition 3.4.** Let $\sigma > 1$ and $m = 0, 1, \cdots, [(\sigma - 1)/2]$. Then, the states in (3.3) satisfy the following resolution of the identity
\[
1_{\mathcal{H}} = \int_D | z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z) (3.15)
\]
where $1_{\mathcal{H}}$ is the identity operator, $d\mu_{\sigma,m}(z)$ is a measure given by
\[
d\mu_{\sigma,m}(z) := \pi^{-1} (\sigma - 2m - 1) (1 - z\bar{z})^{-2} dv(z), (3.16)
\]
and $dv(z)$ being the Lebesgue measure on $D$.

**Proof.** Let us assume that the measure takes the form $d\mu_{\sigma,m}(z) = \mathcal{N}_{\sigma,m}(z) \Omega(z) dv(z)$ where $\Omega(z)$ is an auxiliary density to be determined. Let $\varphi \in \mathcal{H}$ and let us start by writing the following action
\[
\mathcal{O} [\varphi] := \left( \int_D | z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z) \right) [\varphi] (3.17)
\]
\[
= \int_D < \varphi | z, \sigma, m > < z, \sigma, m | d\mu_{\sigma,m}(z). (3.18)
\]

Making use Eq. (3.3), we obtain successively
\[
\mathcal{O} [\varphi] = \int_D < \varphi | (\mathcal{N}_{\sigma,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_k^{\sigma,m}(z) | \psi_k >> < z, \sigma, m | d\mu_{\sigma,m}(z) (3.19)
\]
\[
= \left( \sum_{j,k=0}^{+\infty} \int_D \Phi_j^{\sigma,m}(z) \Phi_k^{\sigma,m}(z) | \psi_k >> < \psi_j | (\mathcal{N}_{\sigma,m}(z))^{-1} d\mu_{\sigma,m}(z) \right) [\varphi]. (3.20)
\]

We replace the measure $d\mu_{\sigma,m}(z)$ by the expression $\mathcal{N}_{\sigma,m}(z)\Omega(z)dv(z)$, then Eq. (3.20) can be written without $\varphi$ as follows
\[
\mathcal{O} = \sum_{j,k=0}^{+\infty} \left[ \int_D \Phi_j^{\sigma,m}(z) \Phi_k^{\sigma,m}(z) \Omega(z) dv(z) \right] | \psi_j >> < \psi_k |. (3.21)
\]
Therefore, we need to have
\[ \int_{\mathcal{D}} \Phi_{j}^{\sigma, m}(z) \Phi_{k}^{\sigma, m}(z) \Omega(z) dv(z) = \delta_{jk}. \tag{3.22} \]
For this we recall the orthogonality relation of the \( \Phi_{k}^{\sigma, m}(z) \) in the Hilbert space \( L^{2,\sigma}(\mathcal{D}) \), which reads
\[ \int_{\mathcal{D}} \Phi_{j}^{\sigma, m}(z) \Phi_{k}^{\sigma, m}(z) (1 - z\bar{z})^{\sigma - 2} dv(z) = \delta_{jk}. \tag{3.23} \]
This suggests us to set \( \Omega(z) := (1 - z\bar{z})^{\sigma - 2} \). Therefore, we get that
\[ d\mu_{\sigma, m}(z) = (\sigma - 2m - 1) \pi (1 - z\bar{z}) d\nu(z). \tag{3.24} \]
we arrive at the announced expression of the measure. Therefore, Eq. (3.21) reduces to
\[ \mathcal{O} = \sum_{j, k=0}^{+\infty} \delta_{jk} | \psi_{j} \rangle \langle \psi_{k} | = 1_{\mathcal{H}}. \tag{3.25} \]
The proof is finished.

\[ \square \]

**Proposition 3.5.** Let \( \sigma > 1 \) and \( m = 0, 1, \ldots, \lfloor (\sigma - 1) / 2 \rfloor \). Then, the states \( | z, \sigma, m > \) satisfy the continuity property with respect to the label \( z \in \mathcal{D} \). That is, the norm of the difference of two states
\[ d_{\sigma, m}(z, w) := \| (| z, \sigma, m > - | w, \sigma, m > ) \|_{\mathcal{H}} \tag{3.26} \]
goes to zero whenever \( z \to w \).

**Proof.** By using the fact that any state \( | z, \sigma, m > \) is normalized by the factor given in (3.9), direct calculations enable us to write the square of the quantity in (3.26) as
\[ d_{\sigma, m}^{2}(z, w) = 2 (1 - \Re(<z, \sigma, m | w, \sigma, m>). \tag{3.27} \]
Next, we use of the expression of the scalar product in (3.9) form which it is clear that the overlap takes the value 1 as \( z \to w \) and consequently \( d_{\sigma, m}(z, w) \to 0 \). \[ \square \]

We have verified that the basic minimum properties for the constructed states to be considered as coherent states are satisfied. Namely, the conditions which have been formulated by Klauder [18]: (a) the continuity of labeling, (b) the fact that these states are normalizable but not orthogonal and (c) these states fulfilled the resolution of the identity with a positive weight function. As we can see, these coherent states are independent of the basis \( | \psi_{k} > \) we use and the only condition which is implicitly fulfilled is the orthonormality of the basis vectors of \( \mathcal{H} \). But if we want to attach these coherent states to a concrete quantum system then a Hamiltonian operator should be specified together with a corresponding explicit eigenstates basis. This will be the goal of the next section.

4. A RELATIVISTIC PSEUDOHARMONIC OSCILLATOR

In this section, we recall some needed results which have been developed in [5], where the authors considered a model for the relativistic pseudo-harmonic oscillator with the following interaction potential
\[ U_{m, \omega, \lambda, g}(x) := \left( \frac{1}{2} m_{*} \omega^{2} x (x + i\lambda) + \frac{g}{x (x + i\lambda)} \right) e^{ip\partial_{x}} \tag{4.1} \]
where \( \omega \) is a frequency, \( g \geq 0 \) is a real quantity and \( \lambda = \hbar / m_{*}c \) denotes the Compton wavelength defined by the ratio of Planck’s constant \( \hbar \) by the mass \( m_{*} \) times the speed.
of light $c$. The corresponding stationary Schrödinger equation is described by the finite-difference equation

$$
\left( mc^2 \cosh i\beta \partial_\xi + U_{m,\omega,\beta,\xi} (x) \right) \varphi(\xi) = E \varphi(\xi) \tag{4.2}
$$

with the boundary conditions for the wave function $\varphi(0) = 0$ and $\varphi(\infty) = 0$. As in [5], we will restrict ourself to the interval $0 \leq x < \infty$ and in terms of dimensionless variable $\bar{\xi} = x/\lambda$ and parameters $\omega_0 = \hbar \omega / m_c^2, g_0 = m_s g / \hbar^2$ the equation (4.2) takes the form

$$
\left( \cosh i\beta \bar{\xi} + \frac{1}{2} \omega_0^2 \bar{\xi} (\bar{\xi}^2 + i) e^{i\beta \bar{\xi}} + \frac{g_0}{\bar{\xi} (\bar{\xi}^2 + i)} \right) \varphi(\bar{\xi}) = \frac{E}{m_s c^2} \varphi(\bar{\xi}) \tag{4.3}
$$

The authors in [5] have obtained the energy spectrum of the Schrödinger operator in (4.2) as

$$
E_k := \hbar \omega (2k + \alpha_+ + \alpha_-), \quad k = 0, 1, 2, \cdots, \tag{4.4}
$$

where

$$
2\alpha_\pm - 1 = \sqrt{1 + \frac{2}{\omega_0^2}} \left( 1 \pm \sqrt{1 - 8g_0\omega_0^2} \right), \tag{4.5}
$$

For our purpose to ensure that $E_k$ are real we choose $1 - 8g_0\omega_0^2 = 0$ which means, in system of units $\hbar = m_s = \omega = 1$, the choice $g = c^4/8$. In this case $\alpha_+ = \alpha_- = \gamma_c = (1 + \sqrt{1 + 2c^4})/2 \equiv \gamma$. So that we will be concerned with eigenstates of the form (5):

$$
\varphi_k^\gamma(\xi) := \frac{\sqrt{2i\gamma} (c^{-4})}{} \frac{\Gamma^2 \left( \gamma + i\xi \right)}{\Gamma \left( k + \gamma + \frac{1}{2} \right) \sqrt{k!} \Gamma(k + 2\gamma)} S_k \left( \xi^2; a, b, c \right), \tag{4.6}
$$

where $S_k(\xi^2, a, b, c)$ denotes the continuous dual Hahn polynomial [4, p.331]), which can be defined in terms of the $3F_2$-sum as

$$
S_n \left( \xi^2; a, b, c \right) := (a + b)_n (a + c)_n \cdot 3F_2 \left( \begin{array}{c} -n, a + i\xi, a - i\xi \\ a + b, a + c \end{array} \right). \tag{4.7}
$$

Finally, we note that the wave functions in (4.6) satisfy the relations:

$$
\int_0^{+\infty} \varphi_k^\gamma(\xi) \varphi_j^\gamma(\xi) d\xi = \delta_{k,j} \tag{4.8}
$$

which means that they constitute an orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+, d\xi)$.

**5. A FAMILY OF COHERENT STATES FOR THE RELATIVISTIC PSEUDOHARMONIC OSCILLATOR**

We now adopt the Hilbertian probabilistic schem discussed above in Section 3 to define a class of coherent states as follows.

**Definition 5.1.** For $\sigma > 1$ and $m = 0, 1, \cdots$, $\left[ \frac{\sigma-1}{2} \right]$ a class of coherent states for the relativistic pseudoharmonic oscillator (4.2) are defined by

$$
\varphi_{\sigma,\gamma,m}(\cdot) := (\mathcal{N}_{\sigma,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_k^{\sigma,m}(z) | \varphi_k^\gamma(\cdot) > \tag{5.1}
$$

where $\mathcal{N}_{\sigma,m}(z)$ is the factor in (3.9), $\Phi_k^{\sigma,m}(z)$ are given by (2.5) and $\varphi_k^\gamma(\cdot)$ are the eigenstates in (4.6).
Proposition 5.2. Let $\sigma = 2(m + \gamma)$ and $z \in \mathbb{D}$ be a fixed labeling point. Then, the wave functions of the states $\Phi_k^\gamma(z)$ are of the form

$$\varphi_{z,\gamma,m}(\xi) = \frac{\sqrt{2\Gamma(m+2\gamma)}i^\gamma(c^{-4})i^\gamma}{\Gamma(2\gamma)\Gamma(\gamma + \frac{1}{2})\sqrt{m!}} \frac{\Gamma(\gamma + i^\xi)}{\Gamma(i^\xi)}$$

$$\times \frac{(1-z\xi)^\gamma}{(1-z)^{2\gamma}} \left( \frac{z-1}{1-z} \right)^m F_5 \left( \gamma + i^\xi, \gamma - i^\xi : 2\gamma + m, 2\gamma \right| \frac{1-z\xi}{(1-z)(z-1)}, \frac{1}{1-z} \right)$$

for any $\xi \in \mathbb{R}_+$, where $F_5$ is a special Appel-Kampé de Fériet hypergeometric function.

Proof. We start from Eq. (5.1) by replacing the coefficients $\Phi_k^\gamma(z)$ by their expressions taking into account Eq. (2.5). This leads to the expression

$$\varphi_{z,\gamma,m}(\xi) = \left( \frac{(\sigma - 2m - 1)}{\pi(1-z\xi)^{\sigma - m}} \right) \sum_{k=0}^{\infty} \frac{(\sigma - 2m - 1)k!\Gamma(\sigma - m)}{\pi m!\Gamma(\sigma - 2m + k)} \times (1-z\xi)^{-m} (-1)^k z^{m-k} P_k^{(m-k,\sigma-2m-1)}(1-2z\xi) \varphi_k^\gamma(\xi).$$

Next, introducing the variable $u := 2z\xi - 1$ and inserting the expression of $\varphi_k^\gamma(\xi)$ in Eq. (5.3), we obtain that

$$\varphi_{z,\gamma,m}(\xi) = z^m (1-z\xi)^{\frac{\sigma - m}{2}} \sqrt{\frac{2\Gamma(\sigma - m)}{m!}} c^{-4} i^\gamma \frac{\Gamma(\gamma + i^\xi)}{\Gamma(i^\xi)}$$

$$\times \sum_{k=0}^{\infty} \sqrt{\frac{k!}{\Gamma(\sigma - 2m + k)}} P_k^{(\sigma - 2m - 1, m - k)}(u) S_k(\xi^2, \gamma, 1/2) \sqrt{k!} \Gamma(k + 2\gamma).$$

We use the notation $\sigma - 2m = 2\gamma$ and we focus on the sum in (5.4):

$$\mathcal{G} := \sum_{k=0}^{\infty} \sqrt{\frac{k!}{\Gamma(2\gamma + k)}} P_k^{(2\gamma - 1, m - k)}(u) S_k(\xi^2, \gamma, 1/2) \sqrt{k!} \Gamma(k + 2\gamma).$$

Next, we set $t := 1/z$ and we rewrite (5.5) in a simple form as

$$\mathcal{G} = \sum_{k=0}^{\infty} t^k P_k^{(2\gamma - 1, m - k)}(u) S_k(\xi^2, \gamma, 1/2) \frac{\Gamma(2\gamma + k)}{\Gamma(k + 2\gamma + \frac{1}{2})}.$$

Now, we need the definition of the continuous dual Hahn polynomials by the hypergeometric terminating $3\!F_2$-sum (4 p.331):

$$S_k(\xi^2, \gamma, 1/2) = \frac{\Gamma(2\gamma + k) \Gamma\left(\gamma + \frac{1}{2} + k\right)}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} 3\!F_2\left( -k, \gamma + i^\xi, \gamma - i^\xi \right| 2\gamma, \gamma + \frac{1}{2} | 1 \right).$$

So that the sum (5.6) becomes

$$\mathcal{G} = \frac{1}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} \sum_{k=0}^{\infty} t^k P_k^{(2\gamma - 1, m - k)}(u) 3\!F_2\left( -k, \gamma + i^\xi, \gamma - i^\xi \right| 2\gamma, \gamma + \frac{1}{2} | 1 \right).$$

Now, we make use of the integral representation (119, p.84):

$$3\!F_2\left( \alpha, \beta, \rho \right| \frac{\alpha}{\tau}, \rho + \omega | 1 \right) = \frac{\Gamma(\rho + \omega)}{\Gamma(\rho) \Gamma(\omega)} \int_0^1 x^{\rho - 1} (1-x)^{\omega - 1} \cdot F_1\left( \alpha, \beta \right| \tau | x \right) dx$$
provided that $\Re(\rho) > 0$, $\Re(\omega) > 0$ and $\Re(\tau - n - \alpha - \beta) > 0$ for the parameters $\alpha = -k, \beta = \gamma + i\zeta, \rho = \gamma - i\zeta, \tau = 2\gamma, \omega = \frac{1}{2} + i\zeta$ to rewrite the $3F_2$-sum in (5.8) as

$$3F_2\left(\frac{-k, \gamma + i\zeta, \gamma - i\zeta}{2\gamma, \gamma + \frac{1}{2}} \mid 1\right) = \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma - i\zeta\right) \Gamma\left(\frac{1}{2} + i\zeta\right)} \times \int_0^1 x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta} \cdot _2F_1\left(\frac{-k, \gamma + i\zeta}{2\gamma} \mid x\right) dx. \tag{5.10}$$

Therefore, Eq. (5.8) transforms successively as

$$\mathcal{G} = \frac{1}{\Gamma(2\gamma)} \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma\left(\gamma - i\zeta\right) \Gamma\left(\frac{1}{2} + i\zeta\right)} \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1,m-k)}(u) \times \int_0^1 x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta} \cdot _2F_1\left(\frac{-k, \gamma + i\zeta}{2\gamma} \mid x\right) dx, \tag{5.11}$$

where

$$\Phi(x, \zeta) := \frac{x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta}}{\Gamma(\gamma - i\zeta) \Gamma\left(\frac{1}{2} + i\zeta\right)}. \tag{5.13}$$

Now, we look closely at the sum in (5.12):

$$\Xi(x) := \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1,m-k)}(u) \cdot _2F_1\left(\frac{-k, \gamma + i\zeta}{2\gamma} \mid x\right). \tag{5.14}$$

We exploit the connection formula ([20, p.63]):

$$P_n^{(\alpha, \beta)}(u) = \left(1 - \frac{u}{2}\right)^\alpha P_n^{(-2\alpha - \beta - 1, \beta)}\left(\frac{u + 3}{u - 1}\right) \tag{5.15}$$

for the parameters $\alpha = 2\gamma - 1, \beta = m - k$ and $n = k$ to rewrite the Jacobi polynomial in (5.14) as

$$P_k^{(2\gamma-1,m-k)}(u) = (1 - z\zeta)^k P_k^{(-2\gamma-m-k,m-k)}\left(\frac{z\zeta + 1}{z\zeta - 1}\right). \tag{5.16}$$

Therefore Eq. (5.14) can be rewritten as

$$\Xi = \sum_{k=0}^{\infty} \left(\frac{1 - z\zeta}{z}\right)^k P_k^{(-2\gamma-m-k,m-k)}\left(\frac{z\zeta + 1}{z\zeta - 1}\right) \cdot _2F_1\left(\frac{-k, \gamma + i\zeta}{2\gamma} \mid x\right). \tag{5.17}$$

We introduce the variables

$$\theta := \frac{1 - z\zeta}{z}, V := \frac{z\zeta + 1}{z\zeta - 1} \tag{5.18}$$

in terms of which Eq. (5.17) also reads

$$\Xi = \sum_{k=0}^{\infty} \left(\frac{2\gamma}{2\gamma}\right)^k \theta^k P_k^{(-2\gamma-m-k,m-k)}(V) \cdot _2F_1\left(\frac{-k, \gamma + i\zeta}{2\gamma} \mid x\right). \tag{5.19}$$
We are now in position to apply the bilinear generating formula due to S. Saran ([21, p.14]):

\[\sum_{k=0}^{\infty} \frac{(b)_k}{(d)_k} \theta^k p^{(a-k,\beta-k)} (V)_2 F_1 \left( \begin{array}{c} -k, c \\ b \end{array} \mid \gamma y \right) \] (5.20)

\[= (1 - y)^{-c} F_8 \left( b, b, b, c, -\alpha, -\beta; b, d, d; \frac{y}{y-1}, -\frac{1}{2} (V + 1) \theta, -\frac{1}{2} (V - 1) \theta \right).\]

In our context \( b = d = 2\gamma \), this implies that the Lauricella triple hypergeometric series \( F_8 \), which is denoted \( F_G \) by S. Saran, reduces to the expression

\[\left[ 1 + \frac{(V + 1) \theta}{2} \right]^a \left[ 1 + \frac{(V - 1) \theta}{2} \right]^b (1 - y)^c \times F_1 \left( c, -\alpha, -\beta; b; \frac{y (V + 1) \theta}{2 + (V + 1) \theta}, \frac{y (V - 1) \theta}{2 + (V - 1) \theta} \right),\] (5.21)

in terms of the first \( F_1 \) Appell’s hypergeometric function ([19], p.265). Therefore, in terms of our parameters, the sum (5.19) also has the following expression

\[\Xi = \left[ 1 + \frac{(V + 1) \theta}{2} \right]^{-2\gamma - m} \left[ 1 + \frac{(V - 1) \theta}{2} \right]^m \times F_1 \left( \gamma + i\xi, 2\gamma + m, -m; 2\gamma; \frac{x (V + 1) \theta}{2 + (V + 1) \theta}, \frac{x (V - 1) \theta}{2 + (V - 1) \theta} \right).\] (5.22)

Now if we denote the parameters and arguments occurring the last Appell \( F_1 \)-sum respectively by \( a = \gamma + i\xi, b = 2\gamma + m, c = -m, d = 2\gamma \),

\[X = \frac{x (V + 1) \theta}{2 + (V + 1) \theta} = \mu_x, Y = \frac{x (V - 1) \theta}{2 + (V - 1) \theta} = \nu_x\] (5.23)

with

\[\mu_x := \frac{(V + 1) \theta}{2 + (V + 1) \theta} = \frac{\Xi}{1-z}, \nu_x := \frac{(V - 1) \theta}{2 + (V - 1) \theta} = \frac{1}{1-z}.\] (5.24)

then this \( F_1 \)-sum can be presented as \( F_1 \) \((a,b,c,d;X,Y)\) with a particularity here consisting on the fact that the parameters \( b, c \) and \( d \) satisfy \( d = b + c \) so that it can be reduced to a Gauss hypergeometric function according to the transformation ([22]):

\[F_1 \left( a, b, c, b + c; X, Y \right) = (1 - Y)^{-a} \cdot 2 F_1 \left( a, b | X - Y \right).\] (5.25)

Therefore, using (5.25), we obtain from (5.22) the following fact

\[F_1 \left( \gamma + i\xi, 2\gamma + m, -m; 2\gamma; \mu_x, \nu_x \right)\]

\[= (1 - \nu_x)\cdot 2 F_1 \left( \gamma + i\xi, 2\gamma + m | \frac{(\mu_x - \nu_x)x}{1 - \nu_x} \right).\] (5.26)

We set

\[\tau_x := \mu_x - \nu_x = \frac{\Xi}{1-z} - \frac{1}{1-z} = \frac{1 - \Xi z}{(1-z)(\Xi - 1)}.\] (5.27)

So that the sum in (5.22) becomes

\[\Xi (x) = \left[ 1 + \frac{(V + 1) \theta}{2} \right]^{-2\gamma - m} \left[ 1 + \frac{(V - 1) \theta}{2} \right]^m \times (1 - \nu_x)^{-\gamma - i\xi} \cdot 2 F_1 \left( \gamma + i\xi, 2\gamma + m | \frac{(\mu_x - \nu_x)x}{1 - \nu_x} \right).\] (5.28)
Next, to compute the prefactor in the last Eq. (5.28), in terms of the fixed labeling point \( z \in \mathbb{D} \), we are of need of the equalities
\[
1 - \bar{z} = 1 + \frac{(V + 1)\theta}{2} \quad \text{and} \quad \frac{z - 1}{z} = 1 + \frac{(V - 1)\theta}{2}.
\] (5.29)
in order to rewrite (5.28) as
\[
\Xi(x) = (1 - \bar{z})^{-2\gamma} z^{-m} \left( \frac{z - 1}{1 - \bar{z}} \right)^m
\] (5.30)
\[
\times (1 - v_zx)^{-\gamma - i\xi} \cdot _2F_1 \left( \frac{\gamma + i\xi, 2\gamma + m}{2\gamma} | \frac{\tau_zx}{1 - v_zx} \right).
\]

Returning back to the sum in (5.12) and inserting \( \Xi(t) \), we get that
\[
\mathcal{G} = \frac{1}{\Gamma(2\gamma) \Gamma(\gamma - i\xi)} \int_0^1 x^{\gamma - i\xi - 1} (1 - x)^{-1 + i\xi} (\Xi(x)) \, dx.
\] (5.31)

Explicitly, this last quantity reads
\[
\mathcal{G} = \frac{1}{\Gamma(2\gamma) \Gamma(\gamma - i\xi)} \left( 1 - \bar{z} \right)^{-2\gamma} z^{-m} \left( \frac{z - 1}{1 - \bar{z}} \right)^m
\] (5.32)
\[
\times \int_0^1 x^{\gamma - i\xi - 1} (1 - x)^{-1 + i\xi} (1 - v_zx)^{-\gamma - i\xi} \cdot _2F_1 \left( \frac{2\gamma + m, \gamma + i\xi}{2\gamma} | \frac{\tau_zx}{1 - v_zx} \right) \, dx.
\]

At this stade we can make use of the integral representation due to S.K. Kulshreshtha ([8, p.137, Eq. (2.2)]), of a very special case of the Appel-Kampé de Fériet’s hypergeometric function of two variables of higher order [23] as follows
\[
_F^5 \left[ \begin{array}{cc} 2 & c & d \\ 1 & a & b \\ 1 & e & . \\ 1 & a' & b \\ \end{array} | \frac{\chi t}{\zeta} \right] \equiv F_5 \left( \begin{array}{ccc} c, & d : & a \\ e : & a' & | \chi t \zeta \end{array} \right)
\] (5.33)
\[
= \frac{\Gamma(e)}{\Gamma(d) \Gamma(e - d)} \int_0^1 t^{d-1} (1 - t)^{e-d-1} (1 - \zeta t)^{-c} \cdot _2F_1 \left( \frac{a, c}{a'} | \frac{\chi t}{1 - \zeta t} \right) \, dt
\] (5.34)
for the parameters \( d = \gamma - i\xi, e = \frac{1}{2} + \gamma, c = \gamma + i\xi, a = 2\gamma + m, a' = 2\gamma, \zeta = v_z, \chi = \tau_z \) and \( t = x \). So that the integral occurring in (5.32) reads
\[
\frac{\Gamma(\gamma - i\xi) \Gamma \left( \frac{1}{2} + i\xi \right)}{\Gamma(\gamma + \frac{1}{2})} F_5 \left( \begin{array}{ccc} \gamma + i\xi, & \gamma - i\xi : & 2\gamma + m \\ \gamma + \frac{1}{2} : & 2\gamma & | \tau_z, v_z \end{array} \right),
\] (5.35)
and therefore the sum in (5.32) takes the form
\[
\mathcal{G} = \frac{(1 - \bar{z})^{-2\gamma}}{\Gamma(\gamma + \frac{1}{2}) \Gamma(2\gamma) z^m} \left( \frac{z - 1}{1 - \bar{z}} \right)^m F_5 \left( \begin{array}{ccc} \gamma + i\xi, & \gamma - i\xi : & 2\gamma + m \\ \gamma + \frac{1}{2} : & 2\gamma & | \tau_z, v_z \end{array} \right).
\] (5.36)
Summarizing the above calculations we arrive at the expression of the wave functions:

\[
\varphi_{z,\gamma,m}(\xi) = (1 - z\overline{z})^\gamma \frac{\sqrt{2\Gamma(m + 2\gamma)}}{m!} i^\tau (c^{-4})^{i\xi} \Gamma^2(\gamma + i\xi) \Gamma(i\xi)^{-1}
\]

\[
\times \frac{(1 - z)^{-2\gamma}}{\Gamma(2\gamma) \Gamma(\gamma + \frac{1}{2})} \left( \frac{z - 1}{1 - z} \right)^m F_5 \left( \gamma + i\xi, \gamma - i\xi : 2\gamma + m \right. \left. \frac{1 - z\overline{z}}{1 - \overline{z}} \right)
\]

as announced in the proposition. Replacing the arguments \(v_z\) and \(\tau_z\) by their expressions in (5.24) and (5.27), we end the proof.

6. RELATIVISTIC BARGMANN-TYPE TRANSFORMS

Now, since we have obtained the expression of the wave functions (5.2), we can apply the coherent states transform formalism \([9]\) to obtain an integral transform \(\mathcal{B}^\Pi_{c,m} : L^2(\mathbb{R}_+) \to \mathcal{A}^{2(\gamma + m)}(\mathbb{D})\) defined by

\[
\mathcal{B}^\Pi_{c,m}[f](z) := \left( \mathcal{N}_{2(\gamma + m),m}(z) \right)^{\gamma} \langle f, \varphi_{z,\gamma,m} \rangle_{L^2(\mathbb{R}_+)}.
\]

We precisely state the following precise result.

**Theorem 6.1.** The coherent state transform associated with the wave functions (5.2) is the isometry \(\mathcal{B}^\Pi_{c,m} : L^2(\mathbb{R}_+) \to \mathcal{A}^{2(\gamma + m)}(\mathbb{D})\) defined by

\[
\mathcal{B}^\Pi_{c,m}[f](z) := \frac{(2\gamma - 1)^{\frac{1}{2}} \sqrt{2\Gamma(m + 2\gamma)}(-i)^\gamma}{\sqrt{\pi m!} \Gamma(2\gamma) \Gamma(\gamma + \frac{1}{2})} \left( \frac{z - 1}{1 - z} \right)^m
\]

\[
\times \int_0^{+\infty} \frac{\Gamma^2(\gamma - i\xi) (c^{-4})^{-i\xi}}{\Gamma(-i\xi)} F_5 \left( \gamma - i\xi, \gamma + i\xi : 2\gamma + m \right. \left. \frac{1 - z\overline{z}}{1 - \overline{z}} \right) f(\xi) d\xi.
\]

**Definition 6.2.** Let \(\gamma = (1 + \sqrt{1 + 2c^4})/2\), then the coherent state transform (6.2) will be called a relativistic Bargmann-type transform attached to the hyperbolic Landau level \(c_m^\gamma := 4m(m + 2\gamma - 1)\) on the Poincaré disk.

**Corollary 6.3.** For \(m = 0\), the coherent state transform associated with the wave functions (5.2) is the isometry \(\mathcal{B}^\Pi_{c,0} : L^2(\mathbb{R}_+) \to \mathcal{A}^{2\gamma}(\mathbb{D})\) of holomorphic functions \(\varphi : \mathbb{D} \to \mathbb{C}\) with \(\int_\mathbb{D} |\varphi(z)|^2 (1 - z\overline{z})^{2\gamma - 2} dv(z) < +\infty\) by

\[
\mathcal{B}^\Pi_{c,0}[f](z) = \left( \frac{2\gamma - 1}{\pi} \right)^{\gamma} \frac{\sqrt{2}}{\Gamma(\gamma + \frac{1}{2})} \Gamma(2\gamma)
\]

\[
\times \int_0^{+\infty} \frac{(c^{-4})^{-i\xi} \Gamma^2(\gamma - i\xi)}{(1 - z)^{\gamma} \Gamma(-i\xi)} F_1 \left( \gamma - i\xi, \gamma + \frac{1}{2} - i\xi : z \right) f(\xi) d\xi.
\]

**Proof.** We start by putting \(m = 0\) in the expression (5.2). This gives

\[
\varphi_{z,\gamma,0}(\xi) = \frac{\sqrt{2i^\gamma (c^{-4})^{-i\xi} \Gamma^2(\gamma + i\xi)}}{\sqrt{\Gamma(2\gamma)} \Gamma(\gamma + \frac{1}{2})} \Gamma(i\xi)
\]

\[
\times \left( \frac{1 - z\overline{z}}{1 - \overline{z}} \right)^\gamma F_5 \left( \gamma + i\xi, \gamma - i\xi : 2\gamma \right. \left. \frac{1 - z\overline{z}}{1 - \overline{z}} \right).
\]
We now observe that the two parameters $a$ and $a'$ in the $F_5$-sum as denoted in (5.33) are equal in the case of Eq. (6.4). In this situation the reduction ([8, p.136, Eq. (1.7)]):

$$F_5\left(\begin{array}{c}
c, d : a \\
e : a
c, d : a \\
e : a
\end{array} \mid \chi, \zeta \right) = 2F_1\left(\begin{array}{c}
c, d \\
e : a
c, d \\
e : a
\end{array} \mid \chi + \zeta \right)$$  (6.5)

can be applied and enables us to write

$$F_5\left(\begin{array}{c}
\gamma + i\zeta, \gamma - i\zeta : 2\gamma \\
\gamma + \frac{1}{2} : 2\gamma \mid \tau_z, v_z \end{array} \right) = 2F_1\left(\begin{array}{c}
\gamma + i\zeta, \gamma - i\zeta : 2\gamma \\
\gamma + \frac{1}{2} : 2\gamma \mid \frac{z}{z-1} \end{array} \right),$$  (6.6)

where we have replaced of the quantities $v_z$ and $\tau_z$ by their above expression respectively. Next, we make use appeal to the Pfaff transformation ([4, p.68]):

$$2F_1\left(\begin{array}{c}
a, b \\
c : x
\end{array} \mid \gamma \right) = (1 - x)^{-a} 2F_1\left(\begin{array}{c}
a, c - b \\
c : x \end{array} \mid \gamma \right)$$  (6.7)

to present the Gauss hypergeometric function in right hand side of Eq. (6.6) as

$$(1 - z)^{\gamma + i\zeta} 2F_1\left(\begin{array}{c}
\gamma + i\zeta, \frac{1}{2} + i\zeta \\
\gamma + \frac{1}{2} \mid z \end{array} \right).$$  (6.8)

Returning back to (6.4), and inserting (6.8), we arrive at the expression

$$\phi_{z,\gamma,0}(\xi) = (1 - z\xi)^{\gamma} (1 - z)^{-\gamma + i\zeta} i^{-\gamma} (c^{-4})^{i\zeta} \Gamma^2(\gamma + i\zeta) \Gamma(i\zeta) \times \frac{\sqrt{2}}{\Gamma(\gamma + \frac{1}{2}) \sqrt{\Gamma(2\gamma)}} 2F_1\left(\begin{array}{c}
\gamma + i\zeta, \frac{1}{2} + i\zeta \\
\gamma + \frac{1}{2} \mid z \end{array} \right).$$  (6.9)

Finally, we write the quantity $(\mathcal{N}_{2\gamma,0}(z))^{\frac{1}{2}} \langle f, \phi_{z,\gamma,0} \rangle_{L^2(\mathbb{R}^+)}$ for an arbitrary function $f$ in $L^2(\mathbb{R}^+)$. □

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