Saturation of turbulent helical dynamos

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The presence of large scale magnetic fields in nature is often attributed to the inverse cascade of magnetic helicity driven by turbulent helical dynamos. In this work we show that in turbulent helical dynamos the inverse flux of magnetic helicity towards the large scales \( \Pi_H \) is bounded by \( |\Pi_H| \leq c \epsilon_k \), where \( \epsilon \) is the energy injection rate and \( k \) is the Kolmogorov magnetic dissipation wavenumber. Assuming the classical isotropic turbulence scaling for which \( \epsilon_k \propto \ell_f^{-1} Rm^{3/4} \max[1, Pr]^{-1/4} \), with \( \ell_f \) the forcing lengthscale, implies that the inverse flux of magnetic helicity decreases with \( Rm \) at least as \( \Pi_H \leq \epsilon c \ell_f Rm^{-3/4} \max[Pr, 1]^{1/4} \), where \( c \) is an order one constant. We demonstrate this scaling with \( Rm \) using direct numerical simulations of turbulent dynamos forced at intermediate scales. The results further indicate that nonlinear saturation is achieved by a balance between the inverse cascade and dissipation at domain size scales \( L \) and the saturation values of magnetic energy are bounded by \( \mathcal{E}_m \leq c L (\epsilon \ell_f)^{3/4} Rm^{1/4} \max[1, Pr]^{1/4} \). Numerical simulations also demonstrate this bound.

I. INTRODUCTION

Magnetic fields are observed in a plethora of astrophysical objects from planetary to galactic scales [1–3]. Their generation and sustainment is often attributed to dynamo action: their self-amplification by a continuous stretching and refolding of magnetic field lines due to the underlying (turbulent in most cases) flow [4]. In many cases the magnetic structures formed span the entire astrophysical object, reaching scales much larger than the small scale turbulence that generates them. The pioneering work of [5] showed that large scale magnetic fields can be generated from small scale flows if the advecting flow is helical. This result is based on an expansion for large scale separation and is referred as alpha-dynamo. However, such expansion can be formally done only below a critical value of \( Rm \) (the ratio of Ohmic to dynamic time scales). Above this critical value small scale dynamo action begins and the expansion ceases to be valid [6–8].

The validity of the alpha model is further questioned in the nonlinear regime for which the magnetic field feeds back to the velocity field through the Lorentz force. In [9, 10] it was argued that the growth of alpha dynamos saturates when the large scale magnetic field \( \mathbf{B} \) becomes larger than \( |\mathbf{B}| < U_{rms} Rm^{-1/2} \), where \( U_{rms} \) is the root mean square value of the velocity fluctuations. This gives a very weak magnetic field for most astrophysical applications for which \( Rm \gg 1 \).

An alternative way of explaining the formation of large magnetic fields is through the inverse cascade of helicity [11, 12]. This intrinsically non-linear mechanism (that is however compatible with alpha dynamos) predicts that magnetic helicity will be transferred by nonlinear interactions to larger scales. Indeed, several works that followed [13–16] demonstrated with numerical simulations that when magnetic helicity is injected in a flow (by a dynamo or other mechanism), it cascades inversely to larger scales. These results however were performed on moderate values of \( Rm \) and the inverse cascade of magnetic helicity at large \( Rm \) is not tested. To our knowledge a quantitative understanding of how turbulent helical dynamos saturate with clear predictions of the saturating amplitude does not exist.

In this work we demonstrate by analytical arguments and numerical simulations that in dynamo flows, the inverse cascade of magnetic helicity is bounded from above by a decreasing power of \( Rm \). Thus it cannot survive the infinite \( Rm \) limit. This leads to a prediction for the saturation amplitude of the magnetic field that we test with numerical simulations.

II. THEORETICAL ARGUMENTS

We begin by considering the MHD equations for the incompressible velocity \( \mathbf{u} \) and magnetic field \( \mathbf{b} \) given by

\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + (\nabla \times \mathbf{b}) \times \mathbf{b} - \nu \nabla^2 \mathbf{u} + \mathbf{f},
\]

\[
\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \eta \nabla^2 \mathbf{b},
\]

in a cubic periodic domain of size \( 2\pi L \), with \( \nu \) being the viscosity \( \eta \) the magnetic diffusivity and \( \mathbf{f} \) an external mechanical force. The non-linear terms of this equation conserve the total energy \( \mathcal{E} = \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{b}|^2) \) and magnetic helicity \( \mathcal{H} = \frac{1}{2} \mathbf{a} \cdot (\nabla \times \mathbf{b}) \), where the angular brackets stand for spatial average and \( \mathbf{a} = -\nabla^{-2} \nabla \times \mathbf{b} \) is the vector potential. Their balance reads

\[
\partial_t \mathcal{E} = \mathcal{I}_\mathcal{E} - \epsilon, \quad \partial_t \mathcal{H} = -\epsilon_\mathcal{H},
\]

where \( \mathcal{I}_\mathcal{E} = \langle \mathbf{f} \cdot \mathbf{u} \rangle \) is the energy injection rate, \( \epsilon = \epsilon_u + \epsilon_b \) is the energy dissipation rate with \( \epsilon_u = \nu \langle |\nabla \mathbf{u}|^2 \rangle \) the viscous dissipation rate and \( \epsilon_b = \eta \langle |\nabla \mathbf{b}|^2 \rangle \) the Ohmic dissipation rate. Finally \( \epsilon_\mathcal{H} = \eta \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle \) is the helicity generation/dissipation rate. The forcing \( \mathbf{f} \) is assumed to

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act on a scale $\ell_f \ll L$ while dissipation acts at the smaller viscous scale $\ell_v$ and Ohmic scale $\ell_\eta$. For large Reynolds number $Re = \epsilon^{1/3} \ell_f^{4/3}/\nu$ and large magnetic Reynolds number $Rm = \epsilon^{1/3} \ell_f^{4/3}/\eta$, the dissipation length scales $\ell_v$, $\ell_\eta$ scale like [17]

$$\ell_v \propto \ell_f R^{-3/4} \quad \text{and} \quad \ell_\eta \propto \ell_f R^{-3/4} \frac{Pm}{\nu} \quad (4)$$

for $Pm = \nu/\eta \leq 1$ while for $Pr \geq 1$

$$\ell_v \propto \ell_f R^{-3/4} \quad \text{and} \quad \ell_\eta \propto \ell_f R^{-3/4} Pm^{1/4}. \quad (5)$$

At intermediate scales $\ell$ (so called inertial scales $\ell_f \ll \ell \ll \ell_v, \ell_\eta$), there is a constant flux of energy across scales given by

$$\Pi_E(k) = \langle \mathbf{u}_\mathbf{k}^C \cdot (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{b} \cdot \nabla \mathbf{b}) - \mathbf{b}_\mathbf{k}^C \cdot (\nabla \times \mathbf{u} \times \mathbf{b}) \rangle, \quad (6)$$

where $\mathbf{u}_\mathbf{k}^C, \mathbf{b}_\mathbf{k}^C$ stand for the filtered velocity and magnetic field respectively so that only Fourier modes with wavenumbers of norm smaller than $k \ll 1/\ell$ are kept [18]. Conservation of energy by the non-linear terms implies that the energy flux at the inertial scales is constant in $k$ and equals the energy dissipation rate $\Pi_E(k) = \epsilon$.

Similarly, there is a flux of magnetic helicity

$$\Pi_H(k) = -\langle \mathbf{b}_\mathbf{k}^C \cdot (\mathbf{u} \times \mathbf{b}) \rangle \quad (7)$$

that also has to be constant at scales that dissipation plays no role. However, unlike energy, the forcing does not inject magnetic helicity which can be only generated or destroyed by the Ohmic dissipation at rate $\epsilon_\eta$. Nonetheless if the forcing is helical the flow can transport magnetic helicity from the small Ohmic scales to ever larger scale $L' > \ell_f$ up until the domain size reached $L' \approx L$ where a helical condensate will form. Conservation of magnetic helicity by the non-linear terms implies again that the flux of helicity at scales $L' \gg \ell \gg \ell_\eta$ has to be constant in $k \propto 1/\ell$ with $\Pi_H(k) = \epsilon_\eta$.

The two cascades, energy and helicity, are not independent and the first limits the later [19]. To show that, we write the magnetic field in Fourier space $\mathbf{b}_\mathbf{k}$ using the helical basis $\mathbf{b}_\mathbf{k} = b_\mathbf{k}^+ \mathbf{h}^+ + b_\mathbf{k}^- \mathbf{h}^-$ where

$$\mathbf{h}^\pm = \frac{\mathbf{e} \times \mathbf{k} \times \mathbf{k}}{\sqrt{2|\mathbf{e} \times \mathbf{k} \times \mathbf{k}|}} \pm i \frac{\mathbf{e} \times \mathbf{k}}{\sqrt{2|\mathbf{e} \times \mathbf{k}|}} \quad (8)$$

are the eigenvectors of the curl operator with $\mathbf{e}$ an arbitrary vector [20, 21]. By doing that we can write the magnetic energy spectrum as $E_b^+(k) = E_b^+(k) + E_b^-(k)$ and the magnetic helicity spectrum as $H(k) = (E_b^+(k) - E_b^-(k))/k$ where $E_b^+(k)dk = \sum_{k \leq |k| < k + dk} |b_k^+|^2$ is the sum of the energy of the $b^\pm$ Fourier modes on a spherical shell of width $dk$ and radius $k$.

Since magnetic helicity is primarily generated at Ohmic wavenumbers $k_\eta = 1/\ell_\eta$ for any wavenumber $k$ in the range $k \leq k_\eta$ we can write

$$|\Pi_H(k)| = |\epsilon_\eta| \approx \eta \left| \int_{k}^{\infty} q |E_b^+(q) - E_b^-(q)| dq \right| \leq \eta k^{-1} \int_{k}^{\infty} q^2 |E_b^+(q) + E_b^-(q)| dq \leq k^{-1} \epsilon_b \quad (9)$$

Choosing as $k$ the largest allowed $k \approx k_\eta$ and adding the viscous damping $\epsilon_a/k_\eta$ we obtain our final result

$$|\Pi_H(k)| \leq \epsilon/k_\eta. \quad (10)$$

In other words the flux of magnetic helicity is bounded by the energy injection rate divided by the Ohmic dissipation wavenumber $k_\eta$. Note that this bound is saturated if the magnetic field at small scales is fully helical. If not, $|\Pi_H|$ can be much smaller than (10). Using the estimates for $k_\eta$ for isotropic MHD turbulence we obtain

$$|\Pi_H(k)| \leq c k_f^{-1} Rm^{-3/4} \max[1, Pr^{1/4}], \quad (11)$$

where $c$ is an order one constant. Given the very large values of $Rm$ in nature this gives very little hope of observing such fluxes.

However, despite having a diminishing flux of magnetic helicity for large $Rm$, this does not mean that large scale dynamos cannot be observed. In a finite domain of size $L$ a magnetic helicity condensate will form. Its magnetic field amplitude $B$ will be determined by a balance of magnetic helicity flux with the magnetic helicity dissipation at that scale so that $\Pi_H \propto \eta B^2/L$. Using the previous estimate for the flux leads to the prediction for the large scale energy $E_m = B^2$ given by

$$E_m \lesssim c^{2/3} L k_f^{1/3} Rm^{1/4} \max[1, Pr]^{1/4}. \quad (12)$$

We note that a very long time $T \sim (B^2 L) / \Pi_H \propto Rm$ would be required for such a field to be formed. Furthermore if there is an other $\eta$-independent mechanism for magnetic helicity saturation, like magnetic helicity expulsion [22, 23], then the amplitude of the large scale magnetic field will diminish to zero as $Rm \rightarrow \infty$.

Finally we note that this result is based on strong turbulence scaling of $k_\eta$. One can argue that as the large scale magnetic field builds up the relation between $k_\eta$ and $Rm$ can change from that of strong turbulence to that of weak turbulence [24] or turbulence driven by the large scale magnetic shear [25]. Both of these options lead to a faster increase of $E_m$ with $Rm$ as $E_m \lesssim c^{2/3} L^{5/6} k_f^{8/15} Rm^{2/5}$ that is not found however in the numerical studies that follow.

### III. NUMERICAL SIMULATIONS

To demonstrate the above arguments we perform a series of numerical simulations that solve the MHD equations (1–2) using the pseudo-spectral code GHOST [26] in
a cubic domain of side $2\pi L$ with a fully helical random delta correlated forcing at wavenumber $k_f = 1/\ell_f$ that fixes the energy injection rate $\epsilon$. The Prandtl number $Prm = \nu/\eta$ was set to unity for all runs. The resolution used varied from $128^3$ grid points to $1024^3$ grid points for the largest $Rm$. The resolution was chosen so that the largest inertial range is obtained while remaining well resolved with a clear dissipation wavenumber range.

As a first step to accommodate for the the large scale pile up of magnetic helicity we introduce a magnetic hypo-dissipation term $\eta_h \nabla^{-2} b$ in (2) that arrests large scale magnetic helicity. With the inclusion of this term, the simulations reach quickly a steady state where the helicity generated at the small scales by $\epsilon H$ is transported and dissipated at the largest scales by hypo-dissipation.

Figure 1 shows the magnetic helicity flux (top) and energy flux (bottom) for a series of runs varying $Rm$ as shown in the legend for $k_f L = 8$. The energy flux shows its classical behavior increasing as $Rm$ is increased, approaching its maximal value $\Pi_e \sim \epsilon$ in the inertial range. On the other hand, $\Pi_H(k)$ first starts to increase with $Rm$ once the dynamo onset is crossed, reaches a maximum and then starts to decrease again. The magnetic helicity generation spectrum $\eta k^2 H(k)$ for the largest $Rm$ is shown in figure 2. Solid line corresponds to positive values of $\eta k^2 H(k)$ while dashed line corresponds to negative values of $\eta k^2 H(k)$, thus negative magnetic helicity is generated at the smallest scales. In the same plot we show the normalized magnetic energy spectrum $\eta k E_b(k)$ that bounds $\eta k^2 H(k) \leq \eta k E_b(k)$ with the equality corresponding to a fully helical magnetic field.

The dependence of the magnetic helicity cascade with $Rm$ is best seen in figure 3 where the magnetic helicity generation rate $\epsilon_H$ (circles) and magnetic flux $\Pi_H(k_f/2)$ (crosses) as a function of the magnetic Reynolds number $Rm$ for three different values of $k_f L = 4$ (blue), $k_f L = 8$ (green) and $k_f L = 16$ (red). FIG. 1. Magnetic helicity flux $\Pi_H$ (top) and total energy flux $\Pi_E$ (bottom) for several values of the magnetic Reynolds number $Rm$ for $k_f/k_0 = 8$.

FIG. 2. Magnetic helicity dissipation/generation spectrum $\eta k^2 H(k)$ and normalized magnetic energy spectrum $\eta k E_b(k)$ for the case with hypo-dissipation with $k_f/k_0 = 8$ and the largest attained magnetic Reynolds number $Rm$.

FIG. 3. Magnetic helicity generation rate $\epsilon_H$ (circles) and magnetic flux $\Pi_H(k_f/2)$ (crosses) as a function of the magnetic Reynolds number $Rm$ for three different values of $k_f L = 4$ (blue), $k_f L = 8$ (green) and $k_f L = 16$ (red).
Note that while the large scales are fully helical small scales are less. The amplitude of the large scale condensate (at the smallest $k = k_f/8$) is so large that despite the small value of $\eta$ the negative magnetic helicity generated at small scales by Ohmic diffusion is balanced by the positive helicity generated at the largest scale again by Ohmic diffusion. This leads to the flux of (negative) helicity from small to large scales shown in the lower panel of the same figure.

The balance between the magnetic helicity generated at small scales and dissipated at the large leads to the prediction (12) of a weak power-law increase of magnetic energy $E_m$ with $Rm$. Figure 5 shows $E_m$ as a function of $Rm$ in a log-log plot. The last three points appear to agree with the predicted power-law. The range of values of $Rm$ compatible with this law is however rather limited and smaller power-laws or logarithmic increase cannot be excluded. In the inset of the same figure we show the same data in a lin-log plot demonstrating that the data could also be fitted to a logarithmic increase. A logarithmic increase, if true, is still compatible with the bound (12) but does not saturate it, and would imply that small scales become less and less helical as $Rm$ is increased.

**IV. CONCLUSION**

The present work gives for the first time estimates for the flux of magnetic helicity and the saturation of the magnetic field for the nonlinear state of a large $Rm$ turbulent helical dynamo. Remarkably, even for asymptotically large values of $Rm$ it has been shown that the magnetic helicity flux still depends on $Rm$ and in fact it has to decrease at least as fast as $\Pi_H \sim \epsilon/k_\eta \sim \epsilon/k_f Rm^{-3/4}$. This analytical result has also been clearly demonstrated by numerical simulations that are shown to follow this upper bound scaling. This excludes any inviscid nonlinear theory for large scale dynamos from being realized!

Furthermore it was shown that saturation is achieved by a balance of the inverse magnetic helicity flux with the helicity dissipation at the condensate scale. This has lead to the prediction that the magnetic field amplitude at steady state is smaller than $Rm^{1/4}$. Numerical simulations are compatible with this result although the covered range of $Rm$ cannot exclude other small power-laws or logarithmic increase with $Rm$.

It is important to note that these results are based on magnetic helicity conservation and are independent of the actual dynamo mechanism involved (alpha or other). They are thus rather general. Finally, we stress that if an $\eta$–independent mechanism exists at large scales to saturate the large scale magnetic helicity (like magnetic flux expulsion), our result (10) implies that no dominant large scale magnetic field will be present in the $Rm \to \infty$ limit.

The present results have critical implications for large magnetic fields in astrophysical systems and their origin.
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