A general class of relative optimization problems

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Abstract
We consider relative or subjective optimization problems where the goal function and feasible set are dependent on the current state of the system under consideration. In general, they are formulated as quasi-equilibrium problems, hence finding their solutions may be rather difficult. We describe a rather general class of relative optimization problems in metric spaces, which in addition depend on the starting state. We also utilize quasi-equilibrium type formulations of these problems and show that they admit rather simple descent solution methods. This approach gives suitable trajectories tending to a relatively optimal state. We describe several examples of applications of these problems. Preliminary results of computational experiments confirmed efficiency of the proposed method.

Keywords Relative optimization · Quasi-equilibrium problems · Metric spaces · Descent methods · Solution trajectories

Mathematics Subject Classification 90B50 · 90C33 · 91A40 · 91B06 · 65K10

1 Introduction

The usual requirement to choose the best variant in various decision making problems naturally leads to their optimization formulations. That is, one then has to find an element attributed to a decision from some given feasible set $D$ that yields the maximal (or minimal) value of some goal (utility) function $\varphi$. For brevity, we write this problem as

$$\max_{y \in D} \varphi(y).$$ (1)
However, due to incomplete and inexact knowledge about the goal function and feasible set this simple formulation usually needs certain corrections; see e.g. (Hlaváček et al. 2004; Ben-Tal et al. 2009). Recently, a new approach to this problem was proposed in Konnov (2019) where it was supposed that the presentation of the goal and constraints defining the system model may vary together with the changes of the system state and that only some limited information about the goal and constraints may be known at each state. It was proposed to consider such mathematical models as relative or subjective optimization problems with respect to system states and to formulate them as (quasi-)equilibrium problems. This means that the goal function is replaced with a bi-function \( \phi(x, y) \) so that \( \phi(x, \cdot) \) is the goal function attributed to a current state \( x \). Similarly, the feasible set \( D \) may also depend on the states and is replaced with a set-valued mapping \( x \mapsto D(x) \). That is, we have only restricted knowledge about the problem at each point. A relatively optimal state \( x^* \) should give the maximal value of the current goal function which is compared with all the other feasible states evaluated at the current state \( x^* \), i.e. one has to find \( x^* \in D(x^*) \) such that

\[
\phi(x^*, x^*) \geq \phi(x^*, y) \quad \forall y \in D(x^*). \tag{2}
\]

It follows that the above concept gives certain restricted optimality. Nevertheless, it can be used in order to decide whether the current state is suitable or should be changed, thus implementing a weaker solution concept. We observe that (2) is nothing but the so-called quasi-equilibrium problem (QEP for short); see (Bensoussan and Lions 1984; Harker 1991; Aubin 1998). Finding a solution of quasi-equilibrium problems may be rather difficult because of the presence of the moving feasible set.

In this paper, we describe a rather general class of relative optimization problems, which in addition depend on the starting state. We also take quasi-equilibrium type formulations of these problems and propose simple descent solution methods for creating suitable trajectories to a relatively optimal state. We establish existence results for these problems under mild conditions and give illustrative examples of applications and computational experiments.

2 Basic problem formulations

We first describe a general model of a system whose possible states are contained in a set \( X \subseteq E \) where \( E \) is a metric space. The starting state \( x^0 \in X \) is known. Given a state \( x \in X \), one can define the set of feasible states \( D(x) \). Therefore, the system can move from \( x \) to any \( y \in D(x) \) and the utility estimate \( \varphi(x, y) \) of any state \( y \in D(x) \) is known at \( x \), i.e. \( D(x) \) stands for a “trust region” at \( x \). We suppose that the estimate \( u(x) = \varphi(x, x) \) is precise, but the value \( u(y) = \varphi(y, y) \) is not supposed to be known at \( x \). It follows that \( x \in D(x) \) for any \( x \in X \). Next, each move \( (x \rightarrow y) \) requires certain expenses \( c(x, y) \). We suppose that \( c(x, y) \) is non-negative and known at \( x \) for any \( y \in D(x) \). Hence, we can define the estimate of pure expenses for the move \( (x \rightarrow y) \) as follows

\[
f(x, y) = \varphi(x, x) + c(x, y) - \varphi(x, y),
\]
as well as the precise pure expenses for this move

\[ e(x, y) = u(x) + c(x, y) - u(y). \]

Choice of the set \( D(x) \) at \( x \in X \) should also guarantee that the estimates have some sufficient precision. We will say that a sequence \( \{x^k\} \subset X \) is a feasible trajectory if \( x^{k+1} \in D(x^k) \) for each number \( k \). Then we can define two relative optimization problems.

**Problem (P1)** Find a point \( x^* \in X \) such that

\[ f(x^*, y) \geq 0 \quad \forall y \in D(x^*). \]  \hspace{1cm} (3)

**Problem (P2)** Find a feasible trajectory \( \{x^k\} \) with the initial state \( x^0 \in X \) and non-positive pure expenses estimates such that it either terminates at a solution of Problem (P1) or its limit points are solutions of Problem (P1).

It is clear that (3) coincides with (2) if \( c(x, x) = 0 \) and we set

\[ \phi(x, y) = \varphi(x, y) - c(x, y). \]

We observe that Problem (P1) is stationary since it does not depend on the initial state whereas Problem (P2) depends on the initial state essentially. In fact, then one also has to take a feasible trajectory \( \{x^k\} \) such that \( f(x^{k-1}, x^k) \leq 0 \) for each \( k \). Then we have

\[ f(x^0, x^1) + f(x^1, x^2) + \ldots + f(x^{k-1}, x^k) \leq 0 \]

for each \( k \), i.e. we intend to move the system from the current state to a relatively optimal state without expenses. Existence of a solution of Problem (P2) means that sequential taking some moves without expenses can yield a relatively optimal state. It should be noted that Problem (P2) differs from the usual global discrete time optimal control problems; see e.g. (Zaslavski 2006).

**Remark 2.1** We note that the usual decision making approaches require the choice of the best variant with respect to some given optimality criterion even in the presence of uncertainty factors. That is, such a solution must be globally optimal with respect to all the variants. However, we think that the “globally marginal” behaviour is not so suitable in the case of inexact and incomplete data. The above relaxed optimality concepts give an alternative approach, which enables one to only evaluate the necessity to change the current state of the system. This means that the optimization formulation is then restricted within a variable feasible set containing only the states whose estimates at the current state are sufficiently precise. Besides, evaluation of a real system with inexact and incomplete information may require to investigate Problems (P1) and (P2) before implementation of a decision procedure. Then \( D(x) \) is treated as the largest possible estimate of the feasible set at \( x \), \( \varphi(x, y) \) is the maximal possible estimate of the value \( u(y) \) at \( x \), and \( c(x, y) \) is the minimal possible estimate of the expenses for the move \( (x \rightarrow y) \).
3 The basic method and its convergence

We will use the following set of basic assumptions.

(A1) The set $X \subseteq E$ is nonempty and closed, the bi-function $\varphi : X \times X \to \mathbb{R}$ is continuous, the bi-function $c : X \times X \to \mathbb{R}$ is non-negative and continuous, and $c(x, x) = 0$ for each $x \in X$.

(A2) For any number $\alpha$ the set $X_\alpha = \{ x \in X \mid u(x) \geq \alpha \}$ is compact, and for any bounded set $\tilde{X} \subset X$ there exists a number $\beta$ such that $\tilde{X} \subseteq X_\beta$.

(A3) The mapping $D : X \to \Pi(X)$ is lower semi-continuous on $X$ and $x \in D(x)$ for each $x \in X$.

Here $\Pi(A)$ denotes the family of all subsets of a set $A$. We recall that a set-valued mapping $T : X \to \Pi(X)$ is said to be lower semi-continuous at a point $z \in X$ on a set $X$ if, for any sequence $\{x^k\} \to z$, $x^k \in X$, and any $t \in T(z)$ there exists a sequence $\{t^k\} \to t$, $t^k \in T(x^k)$. The mapping $T$ is said to be lower semi-continuous on the set $X$ if it is lower semi-continuous at any point of $X$.

Clearly, (A2) is a general coercivity condition, which together with continuity of $u$ implies that the usual optimization problem

$$\max_{x \in X} u(x) \quad (4)$$

has a solution and that

$$u^* = \max_{x \in X} u(x) < +\infty.$$

We now describe a general threshold descent method (TDM) for Problem (P2) and hence for (P1) as well.

Method (TDM).

Initialization: Take the given point $x^0$, choose a sequence $\{\delta_l\} \downarrow 0$. Set $l = 1$, $k = 0$, $z^0 = x^0$.

Step 1: Given a point $z^k \in X$, find a point $z^{k+1} \in D(z^k)$ such that

$$f(z^k, z^{k+1}) < -\delta_l, \quad (5)$$

set $k = k + 1$ and go to the beginning of Step 1. Otherwise, i.e., if this point does not exist, go to Step 2.

Step 2: Set $x^l = z^k$, $l = l + 1$ and go to Step 1.

Therefore, $\delta_l$ stands for the current descent threshold, which determines the sufficient profit for the movement. The index $l$ is a counter for the number of restarts (threshold changes). The choice of $z^{k+1}$ in accordance with (5) depends on peculiarities of the problem. In particular, it can be based on a solution of the auxiliary
problem

\[
\min_{z \in D(z^k)} \rightarrow \{c(z^k, z) - \varphi(z^k, z)\}. \tag{6}
\]

In order to guarantee convergence of (TDM) we need additional conditions for the accuracy of utility estimates related to feasible system moves expenses. For brevity, set \([\alpha]_+ = \max\{\alpha, 0\}\) for a number \(\alpha\) and

\[
b(x, y) = (\varphi(x, y) - u(y))_+.
\]

That is, \(b(x, y)\) is the utility over-estimate of the state \(y\) at \(x\).

(A4)

(i) For any feasible trajectory \(\{z^k\}\) it holds that

\[
\lim_{k \to \infty} [b(z^k, z^{k+1}) - c(z^k, z^{k+1})]_+ = 0; \tag{7}
\]

(ii) For any unbounded feasible trajectory \(\{z^k\}\) it holds that

\[
\sum_{k=0}^{\infty} [b(z^k, z^{k+1}) - c(z^k, z^{k+1})]_+ < \infty.
\]

Theorem 3.1 Let assumptions (A1)–(A4) be fulfilled. Then the sequence \(\{x^l\}\) generated by Method (TDM) has limit points, all these limit points are solutions of Problem (P1), and the sequence \(\{z^k\}\) solves Problem (P2).

Proof The assertion will be proved in several steps.

Step 1: For each \(l\) the number of changes of the index \(k\) is finite. Using (5), we have

\[
e(z^k, z^{k+1}) = f(z^k, z^{k+1}) + (\varphi(z^k, z^{k+1}) - u(z^{k+1})) \leq f(z^k, z^{k+1}) + b(z^k, z^{k+1})
\]

\[
< -\delta_l + b(z^k, z^{k+1}),
\]

hence

\[
u(z^{k+1}) - u(z^k) > \delta_l - [b(z^k, z^{k+1}) - c(z^k, z^{k+1})]_+ \tag{7}
\]

for each fixed index \(l\). If the number of changes of the index \(k\) is infinite for some \(l\), (A4) (i) and (7) imply \(u(z^k) \to +\infty\) as \(k \to \infty\), which is a contradiction with (A2).

Step 2: The sequence \(\{z^k\}\) is bounded.

Suppose \(\{z^k\}\) is unbounded. Then we have by (7) that

\[
u(z^{k+1}) - u(z^0) > - \sum_{s=0}^{k} [b(z^s, z^{s+1}) - c(z^s, z^{s+1})]_+, \quad \forall k = 0, 1, \ldots
\]
The latter along with (A4) (ii) implies
\[ u(z^{k+1}) \geq u(z^0) - \sum_{s=0}^{k} [b(z^s, z^{s+1}) - c(z^s, z^{s+1})]_+ > -\infty, \quad \forall k = 0, 1, \ldots \] (8)

Hence, there exists a number \( \alpha \) such that \( z^k \in X_\alpha \) for any \( k \). It follows that the sequence \( \{z^k\} \) is contained in the compact set \( X_\alpha \), which is a contradiction.

Step 3: The sequence \( \{x^l\} \) has limit points, all these limit points are solutions of Problem (P1).

From Steps 1–2 it follows that the sequence \( \{x^l\} \) is infinite and bounded, hence it is contained in a compact set \( X_\beta \) due to (A2). It follows that \( \{x^l\} \) has limit points. For each \( l \) from the definition we have
\[ f(x^l, y) \geq -\delta_l \quad \forall y \in D(x^l). \] (9)

Let \( \bar{x} \) be an arbitrary limit point of \( \{x^l\} \), i.e. \( \{x^l_s\} \to \bar{x} \). Then \( \bar{x} \in X \) since \( X \) is closed. Take any \( \bar{y} \in D(\bar{x}) \), then there exists a sequence of points \( \{y^l_s\}, \{y^l_s\} \to \bar{y} \) such that \( y^l_s \in D(x^l_s) \) since the mapping \( D \) is lower semi-continuous on \( X \). Setting \( l = l_s \) and \( y = y^l_s \) in (9) and taking the limit \( s \to \infty \) give
\[ f(\bar{x}, \bar{y}) \geq 0, \]

i.e. \( \bar{x} \) is a solution of Problem (P1). Since \( f(z^k, z^{k+1}) < 0, \{z^k\} \) is a solution of Problem (P2).

Clearly, Theorem 3.1 implies existence of solutions of Problems (P1) and (P2) under assumptions (A1)–(A4). We observe that a solution of the optimization problem (4) is not in general a solution of Problem (P1) under assumptions (A1)–(A4) as the following simple examples illustrate.

Example 3.1 Let \( X = [0, 1], u(x) = 1 - x/4, c(x, y) \equiv 0, D(x) = [x, x + 0.1(1 - x)], \varphi(x, y) = (1 - y/4) + 0.6(0.5 - x)(y - x). \) Hence \( \varphi(x, y) = u(y) \) if \( x = 0.5 \). Then the point \( x^0 = 0 \) is a unique solution of (4) since \( u(x^0) = 1. \) But it is not a solution of (P1) since \( x^1 = 0.1 \in D(x^0) \) and
\[ f(x^0, x^1) = x^1(0.25 - 0.3) < 0. \]

The point \( \bar{x} = 1/12 \) is the solution of (P1) closest to \( x^0 \) since
\[ f(\bar{x}, y) = 0 \quad \forall y \in D(\bar{x}), \]

and \( f(x, x') < 0 \) for any \( x \in (0, 1/12) \) and \( x' \in (x, 1/12] \). At the same time, we conclude that all the assumptions in (A1)–(A4) are fulfilled. In fact, any feasible trajectory \( \{z^k\} \) is bounded and \( z^k \leq z^{k+1} \). Hence it converges to a point in \( X \), which implies
\[ \lim_{k \to \infty} b(z^k, z^{k+1}) = 0. \]
Example 3.2 Let $X = [0, 1]$, $u(x) = 1 - x/4$, $c(x, y) \equiv 0$,

$$\varphi(x, y) = (1 - y/4) + [0.5 - x]_+(y - x),$$

$$D(x) = [x - 0.1[x - 0.5]_+, x + 0.1(2 - x)] \cap X.$$ 

Here $D(x)$ is not a singleton at any point $x \in X$ and $\varphi(x, y) = u(y)$ if $x \geq 0.5$. Again the point $x^0 = 0$ is a unique solution of (4) since $u(x^0) = 1$. But it is not a solution of $(P1)$ since $x^1 = 0.2 \in D(x^0)$ and

$$f(x^0, x^1) = x^1(0.25 - 0.5) < 0.$$ 

The point $\bar{x} = 0.25$ is the solution of $(P1)$ closest to $x^0$ since

$$f(\bar{x}, y) = 0 \ \forall y \in D(\bar{x}),$$ 

and $f(x, x') < 0$ for any $x \in (0, 0.25)$ and $x' \in (x, 0.25)$. Also, all the assumptions in (A1)–(A4) are fulfilled. It suffices to check (A4) (i). Let us take any feasible trajectory $\{z^k\}$. If $z^k \leq 0.5$, then $z^k \leq z^{k+1}$ and

$$b(z^k, z^{k+1}) = (0.5 - z^k)(z^{k+1} - z^k).$$

Hence, if the trajectory $\{z^k\}$ is contained completely in $[0, 0.5]$, we have

$$\lim_{k \to \infty} b(z^k, z^{k+1}) = 0. \quad (10)$$

However, if $z^k \geq 0.5$, then $z^{k+1} \geq 0.5$ and $b(z^k, z^{k+1}) = 0$. It follows that only one transition ($z^k < 0.5$) $\to$ ($z^{k+1} \geq 0.5$) is possible for any trajectory $\{z^k\}$. So, if the infinite number of elements of $\{z^k\}$ is contained in $[0.5, 1]$, (10) clearly holds. Therefore, (A4) (i) is fulfilled.

4 Discussion of conditions and modifications

We observe that conditions (A1)–(A3) seem rather natural and simple. They even do not involve convexity/monotonicity properties and do not impose restrictions on the values of the mapping $x \mapsto D(x)$. Therefore, the set of assumptions is somewhat different from the custom ones; cf. e.g. (Bensoussan and Lions 1984; Yuan and Tan 1997; Aubin 1998). We now discuss the assumptions in (A4) which in fact indicate the precision bounds for utility estimates of any state $y \in D(x)$ at $x$. In the general case the cost value $c(z^k, z^{k+1})$ is known at $z^k$ by assumption. Hence, the proper choice of the set $D(z^k)$ needs certain concordance of the utility over-estimate and move expenses for providing the relation

$$[b(z^k, z^{k+1}) - c(z^k, z^{k+1})]_+ \approx 0$$
and attaining the convergence. In other words, the difference between the utility over-
estimate and move expenses should tend to zero along any infinite feasible trajectory
and this convergence should be rather rapid if the trajectory is unbounded.

Conditions (A1)–(A4) admit various modifications. Let us first take the following
pair of conditions instead of (A2) and (A4).

(A2') For any number \( \alpha \) the set
\[
X_\alpha = \{ x \in X \mid u(x) \geq \alpha \}
\]
is compact.

(A4') For any feasible trajectory \( \{ z^k \} \) it holds that
\[
\sum_{k=0}^{\infty} [b(z^k, z^{k+1}) - c(z^k, z^{k+1})]_+ < \infty.
\]

**Corollary 4.1** Let assumptions (A1), (A2'), (A3), and (A4') be fulfilled. Then the
sequence \( \{ x^l \} \) generated by Method (TDM) has limit points, all these limit points
are solutions of Problem (P1), and the sequence \( \{ z^k \} \) solves Problem (P2).

**Proof** Along the lines of Steps 1–2 of Theorem 3.1 we obtain (7), whereas (8) now
holds for any sequence \( \{ z^k \} \). Hence, for each \( l \) the number of changes of the index \( k \)
is finite, besides, the sequence \( \{ z^k \} \) is contained in the compact set \( X_\alpha \) for some number
\( \alpha \). Therefore, the sequence \( \{ x^l \} \) has limit points. The rest part of the proof is the same
as in Step 3 of Theorem 3.1. \( \square \)

Here (A2') is weaker than (A2), but (A4') is stronger than (A4). Nevertheless, this
is the case if the utility over-estimate of a state \( y \in D(x) \) at \( x \) appears to be less than the
move expenses \( c(x, y) \) due to our subjective choice of the set \( D(x) \). Then we can
in turn replace (A2') and (A4') with the following.

(A2'') For some number \( \alpha \leq u(x^0) \) the set \( X_\alpha \) is compact.

(A4'') For any \( x \in X \) it holds that
\[
b(x, y) \leq c(x, y) \quad \forall y \in D(x).
\]

**Corollary 4.2** Let assumptions (A1), (A2''), (A3), and (A4'') be fulfilled. Then the
sequence \( \{ x^l \} \) generated by Method (TDM) has limit points, all these limit points
are solutions of Problem (P1), and the sequence \( \{ z^k \} \) solves Problem (P2).

In fact, now the whole sequence \( \{ z^k \} \) is contained in the compact set \( X_\alpha \), where \( \alpha \) is
taken from (A2''), since (7) gives
\[
u(z^{k+1}) - u(z^k) > \delta_l.
\]

Let us now suppose that the cost bi-function \( c \) satisfies (A1) without any additional
assumptions. Then (A4) should be modified as follows.

(A5)
(i) For any feasible trajectory \( \{z^k\} \) it holds that
\[
\lim_{k \to \infty} b(z^k, z^{k+1}) = 0;
\]

(ii) For any unbounded feasible trajectory \( \{z^k\} \) it holds that
\[
\sum_{k=0}^{\infty} b(z^k, z^{k+1}) < \infty.
\]

This means that only the utility over-estimates tend to zero along any infinite feasible trajectory and that this convergence is rather rapid if the trajectory is unbounded. Since (A5) implies (A4), we obtain the same assertions.

**Corollary 4.3** Let assumptions (A1)–(A3) and (A5) be fulfilled. Then the sequence \( \{x^k\} \) generated by Method (TDM) has limit points, all these limit points are solutions of Problem (P1), and the sequence \( \{z^k\} \) solves Problem (P2).

As above, we can use proper modifications of (A5) by analogy with (A4′) and (A4″). For instance, the assumptions in (A5) clearly hold true if there is no any over-estimate, i.e. when \( \varphi(x, y) \leq u(y) \) for any \( y \in D(x) \). In this case (A2) can be replaced with (A2″). Then the assertions of Theorem 3.1 remain also true. It should be noted that all the above properties are based on the proper choice of the feasible mapping \( x \mapsto D(x) \).

Let us take the simple descent method (SDM) for Problem (P2):
\[
x^{k+1} \in D(x^k), \quad f(x^k, x^{k+1}) < 0 \quad \text{for } k = 0, 1, \ldots
\]
(11)

Unlike (TDM), it does not converge to a solution under more strong assumptions as the following simple example illustrates.

**Example 4.1** Let \( X = [0, 1] \), \( u(x) = x \), \( c(x, y) \equiv 0 \), \( \varphi(x, y) = u(y) \), \( D(x) \equiv X \). Then the process
\[
x^{k+1} = x^k + 2^{-(k+2)}, \quad k = 0, 1, \ldots, \quad x^0 = 0,
\]
which corresponds to (11), clearly converges to \( \tilde{x} = 0.5 \) instead of the unique solution \( x^* = 1 \).

However, (SDM) can be useful in the particular case where the total number of states is countable and there exists a lower positive threshold for move expenses. Then we can remove all the continuity assumptions and modify the conditions in (A1)–(A4) as follows.

(B1) The set \( X \subseteq E \) is nonempty and countable, \( x \in D(x) \) for each \( x \in X \), \( c(x, x) = 0 \) for each \( x \in X \), and there exists a number \( \delta > 0 \) such that \( c(x, y) \geq \delta \) for all \( x, y \in X \), \( x \neq y \).

(B2)
(i) It holds that

\[ u^* = \sup_{x \in X} u(x) < +\infty; \]

(ii) For any feasible trajectory \( \{z^k\} \) it holds that

\[ \lim_{k \to \infty} b(z^k, z^{k+1}) = 0. \]

**Proposition 4.1** Let assumptions (B1)–(B2) be fulfilled. Then the sequence \( \{z^k\} \) generated by Method (SDM) solves Problem (P2). It is finite and stops at a solution of Problem (P1).

**Proof** It suffices to prove the finiteness of Method (SDM). Now we have

\[ e(z^k, z^{k+1}) = f(z^k, z^{k+1}) + (\varphi(z^k, z^{k+1}) - u(z^{k+1})) \leq f(z^k, z^{k+1}) + b(z^k, z^{k+1}) < b(z^k, z^{k+1}), \]

hence

\[ u(z^{k+1}) - u(z^k) > \delta - b(z^k, z^{k+1}) \]

for each \( k \). If the sequence \( \{z^k\} \) is infinite, (B2) (ii) now implies \( u(x^k) \to +\infty \) as \( k \to \infty \), which is a contradiction with (B2) (i).

We illustrate the above assumptions by a network example.

**Example 4.2** Let us given a graph with an infinite (countable) set of nodes \( \mathcal{N} = \{1, 2, \ldots, \} \) and a set of oriented arcs \( \mathcal{A} \), which join some of the nodes, so that an arc \( a = (i, j) \) has the origin \( i \) and destination \( j \). We treat each node as a state of a system, hence arc \( a = (i, j) \) is associated with the system move \((i \to j)\). The starting state \( b \in \mathcal{N} \) is supposed to be fixed. Let \( u(i) \) denote the utility of the system state \( i \) and let \( c_{ij} \) denote the cost of the move \((i \to j)\) so that \( c_{ii} = 0 \). Given a state \( i \), we define the current set of feasible states

\[ D(j) = \{ j \in \mathcal{N} \mid (i, j) \in \mathcal{A} \}. \]

We suppose that the precise value \( u(j) \) is known at \( i \) if \( j \in D(i) \). If

\[ u(i) \leq C < +\infty \quad \forall i \in \mathcal{N} \quad \text{and} \quad c_{ij} \geq \delta > 0 \quad \forall (i, j) \in \mathcal{A}, \]

then assumptions (B1)–(B2) are fulfilled. Problem (P1) gives a locally optimal state, whereas (P2) requires to find a trajectory to such a state without reduction of the utility. It follows from Proposition 4.1 that procedure (SDM) creates a finite sequence that appears a solution of Problem (P2), i.e., it finds a solution of Problem (P1).
The basic assumptions of Theorem 3.1 can be modified in a complete metric space setting. Then we can remove the compactness assumption.

**C1** The set $X \subseteq E$ is nonempty and closed, $E$ is a complete metric space with the metric bi-function $d : X \times X \rightarrow \mathbb{R}$.

**C2** The bi-functions $\varphi : X \times X \rightarrow \mathbb{R}$ and $c : X \times X \rightarrow \mathbb{R}$ are continuous,

$$u^* = \sup_{x \in X} u(x) < +\infty.$$  

**C3** The bi-function $c : X \times X \rightarrow \mathbb{R}$ satisfies the triangle inequality, i.e.,

$$c(x, z) + c(z, y) \geq c(x, y) \ \forall x, y, z \in X;$$

there exists an increasing continuous function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\theta(0) = 0$ and that for all $x, y \in X$ we have $\theta(d(x, y)) \leq c(x, y)$.

**C4** For any feasible trajectory $\{z^k\}$ it holds that

$$\sum_{k=0}^{\infty} b(z^k, z^{k+1}) < \infty.$$  

We need also an auxiliary property of numerical sequences from Lemma 1 in (Gol’shtein and Tret’yakov 1989, Chapter III).

**Lemma 4.1** Let a numerical sequence $\{\mu_k\}$ be bounded below and

$$\mu_{k+1} \leq \mu_k + \varepsilon_k, \varepsilon_k \geq 0, \ k = 0, 1, \ldots$$

If

$$\sum_{k=0}^{\infty} \varepsilon_k < \infty,$$  

then $\{\mu_k\}$ converges to a number $\mu'$.

**Theorem 4.1** Let assumptions (C1)–(C4) and (A3) be fulfilled. Then the sequence $\{x^l\}$ generated by Method (TDM) converges to a solution of Problem (P1), and the sequence $\{z^k\}$ solves Problem (P2).

**Proof** The assertion will be proved in several steps.

**Step 1:** For each $l$ the number of changes of the index $k$ is finite.

Using (5), we now have

$$u(z^{k+1}) - u(z^k) = -f(z^k, z^{k+1}) + c(z^k, z^{k+1}) + u(z^{k+1}) - \varphi(z^k, z^{k+1})$$

$$\geq \delta_l + c(z^k, z^{k+1}) - b(z^k, z^{k+1})$$

$$\geq \delta_l - b(z^k, z^{k+1}).$$  

(12)
If the number of changes of the index $k$ is infinite for some fixed $l$, (C4) and (12) imply $u(z^k) \to +\infty$ as $k \to \infty$, which contradicts (C2).

Step 2: The sequence $\{z^k\}$ converges to a point $\bar{x} \in X$.

Setting $\mu_k = -u(z^k)$ and using (C2) and (C4), we have that the numerical sequence $\{\mu_k\}$ satisfies the conditions of Lemma 4.1, hence

$$\lim_{k \to \infty} u(z^k) = \tilde{u} < +\infty.$$

(13)

It also follows from (12) that

$$c(z^k, z^{k+1}) \leq u(z^{k+1}) - u(z^k) + b(z^k, z^{k+1}).$$

Take any indices $k$ and $m = k + p$, then we have

$$\theta[d(z^k, z^{k+p})] \leq c(z^k, z^{k+p}) \leq c(z^k, z^{k+1}) + \ldots + c(z^{k+p-1}, z^{k+p})$$

$$\leq u(z^{k+p}) - u(z^k) + \sum_{i=k}^{k+p-1} b(z^i, z^{i+1}).$$

On account of (C3), (C4) and (13) we now obtain that for any number $\alpha > 0$ there exists an index $k'$ such that $d(z^k, z^m) < \alpha$ if $\min\{k, m\} = k > k'$. Hence, $\{z^k\}$ is a Cauchy sequence and it converges to a point $\bar{x} \in X$ since $X$ is closed.

Step 3: The sequence $\{x^l\}$ converges to a point $\bar{x} \in X$, which is a solution of Problem (P1).

Since the sequence $\{x^l\}$ is contained in $\{z^k\}$ and is infinite due to Step 1, Step 2 implies that $\{x^l\}$ converges to the point $\bar{x} \in X$. The rest part of the proof is the same as in Step 3 of Theorem 3.1.

\[\square\]

Remark 4.1 The basic technique for obtaining the assertion of Step 2 of Theorem 4.1 resembles that of the Caristi fixed point theorem; see e.g. (Aubin 1998, Section 1.8). However, $c$ need not be a metric bi-function, besides, we do not determine a choice mapping, since the mapping $D$ only imposes restrictions on the choice at a current point, which should also conform to the descent rule. For this reason, the set of assumptions is somewhat different.

5 Examples of models

We now describe some applied models, which can be formulated within the proposed framework. These models are modifications and extensions of those from (Konnov 2019; Kelly et al. 1998).

Example 5.1 (Treatment of industrial wastes) Let us consider an industrial firm which may utilize $n$ production technologies and have a plant for treatment of its wastes containing $m$ polluted substances. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ be the vector of technology activity levels (activity profile) of the firm. Then $q(x) =$
(q_1(x), \ldots, q_m(x))^\top \in \mathbb{R}^m$ is the corresponding vector of its wastes and $\mu(x)$ is the benefit of this firm. That is, $\mu(x) = \mu_1(x) - \mu_2(x)$, where $\mu_1(x)$ is the income from selling its products and $\mu_2(x)$ is the total resource expenses at the technology activity profile $x$. We denote by $X \subseteq \mathbb{R}_n^+$ the whole feasible activity profile set of the firm, which stands for the set of feasible states. Suppose that the vector $p$ of unit treatment charges depends on the pollution volumes, that is $p = p[q(x)]$. Then the firm has the utility (profit) function

$$u(x) = \mu(x) - \sum_{i=1}^m q_i(x) p_i[q(x)],$$

which is maximized over the set $X$. It is natural to suppose that the exact values of these parameters are not known, hence one can calculate this function values only at the current state $x$ or in some its neighborhood. Also, if $x$ is the current vector of activity levels, then the move $(x \rightarrow y)$ is possible only if $y$ belongs to some neighborhood of $x$ due to production technology restrictions. Besides, changing the activity profile may invoke the necessity to change the treatment technology, which can be also restricted. Collecting all these conditions, we can determine the current feasible set $D(x) \subseteq X$ and the utility (profit) value estimate bi-function $\varphi(x, y)$, such that $\varphi(x, x) = u(x)$. Next, the activity transition $(x \rightarrow y)$ may require new facilities, which were not used before, hence it is related with some additional charge value $c(x, y)$. If we have the utility estimate $\varphi(x, y)$, we can define the estimate of the pure expenses for the move $(x \rightarrow y)$ as follows

$$f(x, y) = \varphi(x, x) + c(x, y) - \varphi(x, y),$$

which coincides with that in Sect. 2. Given the initial activity profile $x^0 \in X$, Problem (P2) will consist in finding a feasible trajectory approximating a solution of Problem (P1). In such a way, one finds a relatively optimal technology activity profile. However, this needs certain further specialization of properties of all the functions and mappings for verification of conditions (A1)–(A4) or their analogues. One possible approach for rather general class of problems including this example is described in Sect. 6.

**Example 5.2 (Resource allocation in telecommunication networks)** We first describe an optimal flow distribution problem in telecommunication data transmission networks. The network contains $n$ transmission links (arcs) and accomplishes some submitted data transmission requirements from $n$ selected pairs of origin-destination vertices within a fixed time period. Denote by $z_i$ and $d_i$ the current and maximal value of data transmission for pair demand $i$, respectively, and by $x_j$ the capacity of link $j$. Each pair demand is associated with a unique data transmission path, hence each link $j$ is associated uniquely with the set $N(j)$ of pairs of origin-destination vertices, whose transmission paths contain this link. For each pair demand $i$ we denote by $\mu_i(z_i)$ the network profit value at the data transmission volume $z_i$. Then we can write
the network profit maximization problem as follows:

\[
\max \rightarrow \mu(z) = \sum_{i=1}^{m} \mu_i(z_i)
\]

subject to

\[
\sum_{i \in N(j)} z_i \leq x_j, \ j = 1, \ldots, n;
0 \leq z_i \leq d_i, \ i = 1, \ldots, m.
\]

Denote by \(u(x)\) the optimal value of this problem depending on the right-hand sides \(x\) of the constraints as parameters. Let \(X\) denote the set of all the feasible capacity profiles, for instance, we can take

\[
X = \left\{ x \in \mathbb{R}^n \mid 0 \leq x_j \leq \alpha_j, \ j = 1, \ldots, n, \sum_{j=1}^{n} \beta_j x_j \leq C \right\}.
\]

That is, \(X\) stands for the set of feasible states. Here \(\alpha_j\) denotes the maximal capacity of link \(j\), \(\beta_j\) denotes the cost of the unit capacity for link \(j\), whereas \(C\) is the total upper bound for the capacity cost of the network. Each capacity profile \(x\) reflects the fixed allocation of network resources, hence, the transition \((x \rightarrow y)\) requires certain expenses \(c(x, y)\). Suppose that one can calculate the values \(c(x, y)\) and \(u(y)\) only if \(y\) belongs to some neighborhood \(D_1(x)\) of \(x\) and that the direct transition \((x \rightarrow y)\) is possible within the fixed time period only if \(y\) belongs to some neighborhood \(D_2(x)\) of \(x\). In fact, some deviations from the current capacity profile may require new facilities, which were not used before, and essential changes in network organization. Then we can set \(D(x) = D_1(x) \cap D_2(x)\). Given a current state \(x^0 \in X\), Problem (P2) will determine a feasible trajectory of allocations tending to a relatively optimal solution.

The above examples show that problems of form (P1) and (P2) appear to be rather natural in the case where one has only restricted and/or local knowledge about the initial problem under solution. Application of Method (TDM) will clearly require more information about the initial problem for obtaining existence and convergence properties. One general approach to implementation of (TDM) is described in the next section.

6 Computational experiments

In order to illustrate the implementation of (TDM) and check its performance we carried out some computational experiments. We first describe one possible approach to the implementation of (TDM) under rather general assumptions.

We intend to solve Problems (P1) and (P2) under the following conditions. Let \(X\) be a nonempty, convex, and compact set in the real \(n\)-dimensional space \(\mathbb{R}^n\). At each
point (state) \( x \in X \) we can define the feasible set \( D(x) \), which is a nonempty, convex, and closed subset of \( X \). Besides, for each pair of points \((x, y)\) such that \( x \in X \) and \( y \in D(x) \), we can calculate the value of the non-negative cost bi-function \( c(x, y) \).

Next, in order to maintain more generality, we take a utility function \( u : X \to \mathbb{R} \), which is the sum of two functions, i.e.

\[
    u(x) = u_1(x) + u_2(x),
\]

where the value \( u_1(y) \) of the first function \( u_1 : X \to \mathbb{R} \) is supposed to be calculated rather easily at \( x \) if \( y \in D(x) \). The second function \( u_2 : X \to \mathbb{R} \) is supposed to be smooth, but we can calculate only its value and its gradient at each point \( x \in X \).

Then, we define the utility estimate bi-function as follows:

\[
    \varphi(x, y) = u_1(y) + u_2(x) + \langle u'_2(x), y - x \rangle - \eta(x, y),
\]

for some non-negative bi-function \( \eta(x, y) \) such that \( \eta(x, x) = 0 \), hence, \( \varphi(x, x) = u(x) \) and

\[
    f(x, y) = u_1(x) - u_1(y) + \langle u'_2(x), x - y \rangle + c(x, y) + \eta(x, y).
\]

Problem (P1) is now re-written as follows: Find a point \( x^* \in X \) such that

\[
    u_1(y) - u_1(x^*) + \langle u'_2(x^*), y - x^* \rangle \leq c(x^*, y) + \eta(x^*, y) \quad \forall y \in D(x^*). \tag{14}
\]

Let us verify the basic assumptions that provide implementation and convergence of Method (TDM). It is natural to suppose in addition that the function \( u_1 : X \to \mathbb{R} \) and the bi-function \( \eta : X \times X \to \mathbb{R} \) are continuous. Then assumptions (A1) and (A2') are clearly fulfilled. In order to provide (A3) we can suppose in addition that \( \text{int}X \neq \emptyset \), take a convex and compact set \( V \) such that \( 0 \in \text{int}V \), and define

\[
    D(x) = X \cap (x + V). \tag{15}
\]

Then, the mapping \( x \mapsto D(x) \) is lower semi-continuous on the set \( X \) (see e.g. (Borisovich et al. 1984, Corollary 1.3.10)), and (A3) holds true. Next, let us suppose in addition that the gradient map of the function \( u_2 \) is locally Lipschitz continuous in the sense that for each point \( x \) there exists the Lipschitz constant \( L_x \) such that

\[
    \|u'_2(x') - u'_2(x'')\| \leq L_x \|x' - x''\| \quad \forall x', x'' \in D(x).
\]

Then, we have the inequality

\[
    -u_2(y) \leq -u_2(x) + \langle u'_2(x), x - y \rangle + 0.5L_x \|y - x\|^2 \quad \forall y \in D(x);
\]
see Lemma 1.2 in (Dem’yanov and Rubinov 1968, Chapter III). It follows that
\[ \varphi(x, y) - u(y) = u_2(x) + \langle u'_2(x), y - x \rangle - u_2(y) - \eta(x, y) \leq 0.5L_x \|y - x\|^2 - \eta(x, y) \quad \forall y \in D(x). \]

Hence, \((A4')\) holds if
\[ 0.5L_x \|y - x\|^2 \leq c(x, y) + \eta(x, y) \quad \forall y \in D(x). \quad (16) \]

It follows from Corollary 4.2 that Method (TDM) generates a solution trajectory for Problems \((P1)\) and \((P2)\).

If we apply (TDM) to \((P1)\) and \((P2)\), we can take (6) as the basic auxiliary problem, which is equivalently re-written as follows:
\[ \min_{z \in D(z^k)} \to \{ c(z^k, z) + \eta(z^k, z) - \langle u'_2(z^k), z \rangle - u_1(z) \}. \quad (17) \]

If \(u_1\) is concave, \(c(x, \cdot) + \eta(x, \cdot)\) is strictly convex, then (17) is a convex optimization problem, which has a unique solution, and it can be taken as \(z^{k+1}\). So, the implementation of (TDM) becomes rather simple. It should be also noticed that the bi-function \(\eta(x, y)\), unlike the cost bi-function \(c(x, y)\), can be chosen rather arbitrarily within the above conditions. That is, it can be included in the estimate bi-function \(\varphi(x, y)\) if the single bi-function \(c(x, y)\) is not sufficient to provide \((A4')\). Besides, it can be utilized to guarantee the (strict) convexity of \(c(x, \cdot) + \eta(x, \cdot)\), if necessary. Otherwise, it can be dropped. For instance, if the function \(u_2\) is convex and \(c(x, \cdot)\) is (strictly) convex, setting \(\eta(x, y) \equiv 0\) gives
\[ \varphi(x, y) - u(y) = u_2(x) + \langle u'_2(x), y - x \rangle - u_2(y) \leq 0. \]

**Example 6.1** We now apply the above approach to the industrial firm problem with waste treatment described in Example 5.1. Suppose that the industrial profit function \(\mu(x)\) is rather simple and known within the local feasible set \(D(x)\) given in (15). Then we can set \(u_1(y) = \mu(y)\) for \(y \in D(x)\). Suppose also that the mappings \(p(q)\) and \(q(x)\) are smooth, but at least one of them is known only at the current state \(x\). Then we can set
\[ u_2(x) = - \sum_{i=1}^{m} q_i(x) c_i[q(x)], \]
and take
\[ \varphi(x, y) = u_1(y) + u_2(x) + \langle u'_2(x), y - x \rangle - \eta(x, y), \]
where the bi-function \(\eta(x, y)\) is chosen in accordance with (16). It follows that (TDM) can be applied to this problem within the above conditions.
Now we describe its specialization for computational experiments. We took the sets

\[ X = \{ x \in \mathbb{R}^n \mid 0 \leq x_j \leq 10, \ j = 1, \ldots, n \} \]

and

\[ V = \{ x \in \mathbb{R}^n \mid -\beta \leq x_j \leq \beta, \ j = 1, \ldots, n \}, \]

where \( \beta > 0 \), and the feasible set \( D(x) \) was defined as in (15). We took the linear function \( \mu(x) = \langle b, x \rangle \), and the linear waste mapping \( q(x) = Ax \) where \( A \) is a non-negative \( m \times n \) matrix. The treatment charge mapping was defined to be diagonal, i.e. \( p_i(q) = p_i(q_i) \) for \( i = 1, \ldots, m \). Next, we took \( \eta(x, y) = 0.5\theta'\|x - y\|^2 \) and \( c(x, y) = 0.5\|x - y\|^2 \). We chose the elements of \( A \) so that

\[ \sum_{j=1}^{n} a_{ij} \leq 1, \ i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} a_{ij} \leq 1, \ i = 1, \ldots, n, \]

then \( \|A^\top A\| \leq 1 \).

In the first series, we chose \( p_i(q_i) = \sigma_i \arctan(q_i), \sigma_i \in (0, 1] \) for \( i = 1, \ldots, m \). Then \( L_x \leq 2 \) and we took \( \theta' = 1 \) to satisfy (16). It follows that assumptions (A1), (A2′′), (A3), and (A4′′) hold true, and that (17) reduces to \( n \) independent one-dimensional problems, each of which is very simple for solution.

In all the series of experiments, we implemented (TDM) in Delphi with double precision arithmetic. We took the starting point \( x_0^j = 1 + \sin j, \ j = 1, \ldots, n \), the starting value \( \delta_0 = 20 \), and the rule \( \delta_{l+1} = v\delta_l \) with \( v = 0.5 \). We set \( \beta = 0.5 \) and made experiments for different values of \( m \) and \( n \). The main goal was to compare the number of iterations that gave the inequality \( \delta_{l+1} \leq \varepsilon = 0.01 \) and the attained accuracy with respect to the gap function

\[ \Delta(x) = \max_{z \in D(x)} \{ u_1(z) - u_1(x) + \langle u_2'(x), z - x \rangle - c(x, z) - \eta(x, z) \}; \]

cf. (14) and (17).

In the second series, we chose \( p_i(q_i) = \sigma_i'q_i + \sigma_i''q_i, \sigma_i' \in (0, 1], \sigma_i'' > 0 \) for \( i = 1, \ldots, m \). Then again \( L_x \leq 2 \), the other parameters were taken as above. We observe that condition (16) is fulfilled in both the cases. The results are given in Table 1, where we indicate the number of iterations (it) and the attained gap value (\( \Delta \)).

**Example 6.2** We also made computational experiments for some other test problems within the above general approach. We took the sets

\[ X = \{ x \in \mathbb{R}^n \mid -5 \leq x_j \leq 5, \ j = 1, \ldots, n \}, \]
Table 1  Computation for two test problems

| m  | n  | it | Δ   | m  | n  | it | Δ   |
|----|----|----|-----|----|----|----|-----|
| 10 | 20 | 28 | 0   | 10 | 23 | 0  | 44  |
| 20 | 30 | 23 | 0   | 20 | 24 | 0  | 89  |
| 50 | 30 | 29 | 0   | 50 | 29 | 0  | 90  |

and

\[ V = \{x \in \mathbb{R}^n \mid -\beta \leq x_j \leq \beta, \ j = 1, \ldots, n\}, \]

where \( \beta > 0 \), and the feasible set \( D(x) \) was defined as in (15). We took the basic function

\[ \mu(x) = 0.5\langle Ax, x \rangle - \langle b, x \rangle + \sum_{i=1}^n \sin x_i, \]

where

\[ A = \begin{pmatrix}
  0 & 1 & \ldots & 0 & 0 \\
  1 & 0 & \ldots & 0 & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & \ldots & 0 & 1 \\
  0 & 0 & \ldots & 1 & 0
\end{pmatrix}, \]

and \( b = (2, \ldots, 2)^\top \in \mathbb{R}^n \).

Note that even the quadratic part of the function \( \mu \) is neither convex nor concave, however, \( \|A\| \leq 1 \), hence the gradient \( \mu' \) has the Lipschitz constant \( L' \leq 2 \). Also, we took \( c(x, y) = \|x - y\|^2 \).

In the first series, we chose \( u_1(x) \equiv 0 \) and \( u(x) = u_2(x) = \mu(x) \). It follows that we can set \( \eta(x, y) \equiv 0 \). We took the starting point \( x_j^0 = \sin j, \ j = 1, \ldots, n \), the starting value \( \delta_0 = 20 \), and the rule \( \delta_{i+1} = v\delta_i \) with \( v = 0.5 \). We set \( n = 100 \) and made experiments for different values \( \beta \). The main goal was to compare the number of iterations that gave the inequality \( \delta_{i+1} \leq \varepsilon = 0.01 \) and the attained accuracy with respect to the gap function

\[ \Delta(x) = \max_{z \in D(x)} \{\langle u'(x), z - x \rangle - c(x, z)\}; \]

cf. (14) and (17). In the second series, we chose \( u_1(x) \equiv 0 \) and \( u(x) = u_2(x) = -\mu(x) \), besides, set \( \eta(x, y) \equiv 0 \). The other parameters were taken as above. We observe that condition (16) is fulfilled in both the cases. The results are given in Table 2, where we indicate the number of iterations (it) and the attained gap value (\( \Delta \)).
The method showed rather rapid convergence. In almost all the cases, it gave points close to relatively optimal solutions.

7 Conclusions

We presented a rather general class of relative optimization problems in metric spaces. The stationary problem is formulated as a quasi-equilibrium problem since the goal function and feasible set are dependent on states. The dynamic problem consists in finding a trajectory attributed to an initial state such that its points tend to a solution of the stationary problem. We proposed simple descent solution methods for creating suitable trajectories to a relatively optimal state under different conditions, which also gave existence results for these problems. The approach was illustrated by applied models and computational experiments.

Investigations of these topics can be continued in several directions. For instance, it seems worthwhile to reveal some other classes of relative optimization problems, which admit rather simple conditions for existence results and efficient solution methods.

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