SIX-DIMENSIONAL PAINLEVÉ SYSTEMS AND THEIR PARTICULAR SOLUTIONS
IN TERMS OF HYPERGEOMETRIC FUNCTIONS

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Abstract. In this article, we propose a class of six-dimensional Painlevé systems given as the monodromy preserving deformations of the Fuchsian systems. They are expressed as polynomial Hamiltonian systems of sixth order. We also discuss their particular solutions in terms of the hypergeometric functions defined by fourth order rigid systems.

Key Words: Painlevé system, Hypergeometric function, Monodromy.

2000 Mathematics Subject Classification: 34M55, 33C70, 34M35.

1. Introduction

Recently, higher order generalizations of the sixth Painlevé equation ($P_{VI}$) has been studied from a viewpoint of the monodromy preserving deformations of Fuchsian systems. It is shown in [14] [15] [16] that irreducible Fuchsian systems with a fixed number of accessory parameters can be reduced to finite types of systems by using the Katz’s two operations, addition and middle convolution [12]. It is also shown in [8] that the isomonodromy deformation equation is invariant under the Katz’s two operations. These facts allow us to construct a classification theory of Painlevé systems given as the isomonodromy deformation equations.

The Fuchsian systems with two accessory parameters were classified by Kostov [14]. According to it, they are reduced to the systems with the following spectral types:

4 singularities 11, 11, 11, 11
3 singularities 111, 111, 111 22, 1111, 1111 33, 222, 111111

The system with the spectral type {11, 11, 11, 11} gives $P_{VI}$ as the monodromy preserving deformation [11]. The other three systems have no deformation parameters, thus we can not derive isomonodromy deformation equations from them.

In general, Fuchsian systems can be classified with the aid of algorithm proposed by Oshima [15] [16]. The systems with four accessory parameters are reduced as follows:

5 singularities 11, 11, 11, 11, 11
4 singularities 21, 21, 111, 111 31, 22, 22, 1111 22, 22, 22, 211

In addition to them, there exist nine systems which have three singularities; we do not list here. Sakai investigated their monodromy preserving deformations systematically and derived the four-dimensional Painlevé systems in [17].
An aim of this article is to investigate the monodromy preserving deformations of the Fuchsian systems with six accessory parameters. They are reduced as follows:

| 6 singularities | 11, 11, 11, 11, 11, 11 |
|-----------------|-------------------------|
| 5 singularities  | 21, 21, 21, 21, 111     |
| 4 singularities  | 21, 111, 111, 111, 111  |
|                 | 31, 31, 31, 31, 31, 31  |
|                 | 42, 33, 33, 33, 33, 222 |

In addition to them, there exist 24 systems which have three singularities; we do not list here. Among those 12 systems, the following ones have already investigated in [7], [23], [13], [2] respectively:

\{11, 11, 11, 11, 11, 11\}, \{31, 31, 1111, 1111\}, \{33, 33, 33, 31\}, \{51, 33, 33, 111111\}.

In this article, we investigate for the other eight Fuchsian systems and derive six-dimensional Painlevé systems.

It is known that \( P_{VI} \) can be expressed as the Hamiltonian system

\[
\frac{\partial q}{\partial t} = \frac{\partial H_{VI}}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial H_{VI}}{\partial q},
\]

with

\[
H_{VI}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4; q, p; t) = q(q-1)(q-t)p \left( p - \frac{\alpha_1 - 1}{q - t} - \frac{\alpha_3}{q - 1} - \frac{\alpha_4}{q} \right) + \alpha_2(\alpha_0 + \alpha_2)q,
\]

where \( \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1 \). Such a property holds even in higher dimensional cases. The 2\( n \)-dimensional Painlevé systems, which have been already derived, can be expressed as 2\( n \)-th order Hamiltonian systems

\[
\mathcal{H}^m : \quad t(t - 1) \frac{\partial q_j}{\partial t_i} = [H_i^m, q_j], \quad t(t - 1) \frac{\partial p_j}{\partial t_i} = [H_i^m, p_j] \quad (i = 1, \ldots, N; j = 1, \ldots, n),
\]

with the Poisson bracket defined by

\[
\{q_i, p_j\} = -\delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0,
\]

where \( \delta_{i,j} \) stands for the Kronecker’s delta. Here the corresponding Fuchsian systems, whose spectral type are \( m \), have \( N + 3 \) singularities and \( 2n \) accessory parameters. In the case \( N = 1 \), we denote \( t_1 \) and \( H_1 \) by \( t \) and \( H \) respectively. In this article, we obtain explicit formulas of the following Hamiltonians; see Section [2]

\[
H_i^{21,21,21,21}, H_i^{31,31,31,31}, H_i^{22,22,22,22}, H_i^{31,22,22,22}, H_i^{22,22,22,22}, H_i^{42,33,33,33}, H_i^{51,33,22,22}, H_i^{51,33,22,22}.
\]

**Remark 1.1.** The systems \( \mathcal{H}^{21,21,1111,1111} \) and \( \mathcal{H}^{31,31,1111,1111} \) were first considered by Tsuda as similarity reductions of his UC hierarchy in [23, 24]. Independently, they were derived from similarity reductions of the Drinfeld-Sokolov hierarchy in [5, 21]. The relationship between those two origins is clarified with the aid of a Laplace transformation in [6].
Remark 1.2. The systems $\mathcal{H}^{31,22,11,111}$ and $\mathcal{H}^{51,33,33,11111}$ were first discovered by Sasano via a generalization of the affine Weyl group symmetry and the Okamoto’s initial value space for $P_{VI}$ in \cite{18}. Afterward, they were derived from similarity reductions of the Drinfeld-Sokolov hierarchy in \cite{4}.

Remark 1.3. The system $\mathcal{H}^{51,33,33,222}$ has been already derived from a similarity reduction of the Drinfeld-Sokolov hierarchy in \cite{3}.

The other aim of this article is to give particular solutions of the six-dimensional Painlevé systems in terms of the hypergeometric functions defined by fourth order rigid systems. It is known that $P_{VI}$ and the four-dimensional Painlevé systems admit particular solutions as follows:

| Painlevé system | Rigid system | HGF | Ref. |
|-----------------|-------------|-----|-----|
| $\mathcal{H}^{11,11,11,11}$ | $11,11,11$ | $2F_1$ | \cite{11} |
| $\mathcal{H}^{11,11,11,11}$ (H$^{21,21,21,21}$) | $21,21,21,21$ | $P_3(F_1)$ | \cite{7,9} |
| $\mathcal{H}^{21,21,11,111}$ | $21,111,111$ | $3F_2$ | \cite{22,25} |

Note that the system $\mathcal{H}^{21,21,21,21}$ is equivalent to $\mathcal{H}^{11,11,11,11}$ because the corresponding Fuchsian systems are mutually transformed by the Katz’s two operations. And, for the six-dimensional ones, we obtain the following results:

| Painlevé system | Rigid system | HGF | Ref. |
|-----------------|-------------|-----|-----|
| $\mathcal{H}^{11,11,11,11}$ (H$^{21,31,31,31,31}$) | $31,31,31,31$ | $P_4(F_D)$ | \cite{7,9} |
| $\mathcal{H}^{21,21,21,21}$ (H$^{31,31,31,22,21}$) | $31,31,22,211$ | $II_2^*$ | Sec. 5.1 |
| $\mathcal{H}^{31,31,22,22,22}$ | $31,22,22,22$ | $P_{4,4}$ | Sec. 5.2 |
| $\mathcal{H}^{21,111,111,111}$ (H$^{31,211,211,111}$) | $211,211,211$ | $II_2$ | Sec. 5.3 |
| $\mathcal{H}^{21,21,11,1111}$ | $22,211,1111$ | $EO_4$ | Sec. 5.4 |
| $\mathcal{H}^{31,31,11111}$ | $31,1111,1111$ | $4F_3$ | \cite{22,25} |

The symbols $P_3$, $P_4$ and $P_{4,4}$ stand for the Jordan-Pochhammer family and its generalization (cf. \cite{16}). And $EO_4$ stands for the even-four hypergeometric function which is in Simpson’s list \cite{19}. Moreover $II_2$ and $II_2^*$ are in Yokoyama’s list \cite{26}.

**Remark 1.4.** As is seen in above, for $n = 1, 2, 3$, the $2n$-dimensional Painlevé system $\mathcal{H}^{n1,m_2,\ldots,m_{n+3}}$ admits a particular solution in terms of the $(n+1)$-th order rigid system with a spectral type \{m_2, \ldots, m_{n+3}\}. Such a relationship is satisfied for a more general case; see Section 5.

This article is organized as follows. In Section 2 and Appendix A we give explicit formulas of the six-dimensional Painlevé systems. In Section 3 we recall the Schlesinger system and its Poisson structure. In Section 4 we discuss derivations of the Painlevé systems from the Schlesinger systems. In Section 5 we give particular solutions of the six-dimensional Painlevé systems in terms of the hypergeometric functions. In Appendix B we recall the four-dimensional Painlevé systems which have been classified by Sakai.
2. List of Hamiltonians obtained in this article

In this section, we give explicit formulas of the following Hamiltonians:

\[
\begin{align*}
H_i^{21,21,21,111}, & \quad H_i^{31,31,22,22,22}, \quad (i = 1, 2), \\
H^{21,111,111,111}, & \quad H^{31,22,21,111}, \quad H^{22,22,21,111}, \quad H^{22,22,22,111}, \quad H^{31,31,31,31,31,31,31}, \quad H^{31,31,31,31,31,31,31}. 
\end{align*}
\]

The following ones are given in Appendix A

\[
H_i^{11,11,11,11,11,11}, \quad (i = 1, 2, 3), \quad H^{31,111,111,111}, \quad H^{33,33,33,31}, \quad H^{51,33,33,1111111}. 
\]

And the following ones are given in Appendix B

\[
H_i^{11,11,11,11,11,11}, \quad (i = 1, 2), \quad H^{21,21,111,111,111}, \quad H^{22,22,22,111}, \quad H^{31,22,22,111}. 
\]

2.1. Spectral type 21, 21, 21, 21, 111.

\[
H_1^{21,21,21,111,111} = H_1^{11,11,11,11,11,11} + \frac{t_1 - 1}{t_1 - t_2} q_3 (q_3 - t_2) (q_3 - t_1) p_3 \left( p_3 - \frac{\alpha_0 + \alpha_3 + \alpha_5}{q_3 - t_1} - \frac{\alpha_0 + \alpha_4 + \rho_3 - \alpha_1}{q_3 - t_2} \right)
\]

\[
- (\alpha_0 + \alpha_3 + 1) q_3 p_3 - \frac{t_1 - 1}{t_1 - t_2} \alpha_2 \rho_3 q_3 + \frac{1}{t_2} q_2 q_3 (q_1 p_1 + q_2 p_2 + \alpha_0 + \alpha_3 + 1) (q_3 p_3 - \rho_3)
\]

\[
+ q_1 q_3 p_3 (q_1 p_1 + q_2 p_2 + \alpha_0 + \alpha_3 + 1) + \frac{1}{t_2 (t_1 - t_2)} q_2 q_3 [t_1 (t_2 - 1) p_1 - (t_1 - 1) t_2 p_2] (q_3 p_3 - \rho_3)
\]

\[
+ \frac{1}{t_1 - t_2} q_3 p_3 \{-(t_1^2 + t_1 - 2 t_2) q_1 p_1 - t_1 (t_1 - 1) q_2 p_1 + (t_1 - 1) t_2 q_1 p_2 + t_1 (t_2 - 1) q_2 p_2
\]

\[
+ \frac{t_1}{t_1 - t_2} p_3 (t_1 - 1) t_2 q_1 p_1 - (t_1 - 1) t_2 q_1 p_2 + (t_1 - t_2) q_3 p_1\},
\]

and

\[
H_2^{21,21,21,111,111} = H_2^{11,11,11,11,11,11} + \rho_3 q_2 ((q_2 - 1) p_2 + q_1 p_1 + \alpha_0 + \alpha_5 + 1)
\]

\[
+ \frac{t_2 - 1}{t_2 - t_1} q_3 (q_3 - t_1) (q_3 - t_2) p_3 \left( p_3 - \frac{\alpha_0 + \alpha_4 + \rho_3 - 1}{q_3 - t_1} - \frac{\alpha_0 + \alpha_3 + \alpha_5 + 1}{q_3 - t_2} - \frac{\alpha_1}{q_3} \right)
\]

\[
- (\alpha_0 + \alpha_5 + 1) q_3 p_3 - \frac{t_2 - 1}{t_2 - t_1} \alpha_2 \rho_3 q_3 + q_2 q_3 p_3 (q_2 p_2 + q_1 p_1 + \alpha_0 + \alpha_5 + 1)
\]

\[
- \frac{t_2 - 1}{t_2 - t_1} q_2 q_3 (p_2 - p_1) (q_3 p_3 - \rho_3) + t_2 q_1 p_3 (q_2 p_2 + q_1 p_1 + \alpha_0 + \alpha_5 + 1)
\]

\[
+ \frac{1}{t_2 - t_1} q_3 p_3 \{t_2^2 + t_2 - 2 t_1) q_2 p_2 + t_2 (t_2 - 1) q_1 p_2 - (t_2 - 1) t_1 q_2 p_1 - t_2 (t_1 - 1) q_1 p_1
\]

\[
+ \frac{t_2}{t_2 - t_1} p_3 \{t_2 (t_1 - 1) q_1 p_2 + (t_2 - 1) t_1 q_1 p_1 - (t_2 - t_1) q_3 p_2\},
\]

where

\[
\alpha_0 = -\rho_2, \quad \alpha_1 = -\theta_4, \quad \alpha_2 = -\theta_2 - \rho_3, \quad \alpha_3 = -\theta_4, \quad \alpha_4 = -\rho_1 + \rho_2 + 1, \quad \alpha_5 = -\theta_3 - 1,
\]
and $\theta_1 + \theta_2 + \theta_3 + \theta_4 + \rho_1 + \rho_2 + \rho_3 = 0$.

**Remark 2.1.** The system $H_{31}^{21,21,21,111}$ reduces to $H_{11}^{11,11,11,11}$ via a specialization $p_3 = \rho_3 = 0$.

### 2.2. Spectral type $31, 31, 22, 22, 22$.

$$H_i^{31,31,22,22,22} = H_i^{11,11,11,11,11} - (\alpha_1 - \alpha_2) \frac{t_i}{t_1 - t_2} q_2 \{(t_i - 1)p_i - (t_2 - 1)p_{i+1}\}$$

$$+ \delta_{i,2} (\alpha_1 - \alpha_2) t_2 (q_2 - 1)p_2 - (t_1 + 1)q_3 p_3^2 + \{\alpha_1 + \alpha_3 - 1 + (\alpha_1 + \alpha_4)t_i\}q_3 p_3$$

$$+ q_3 p_3 (2q_1 p_1 + 2q_2 p_2 + q_3 p_3 + 2\alpha_0 + \alpha_5 + 1)$$

$$- q_3 p_{i+1}(2q_1 p_1 + q_{i+1} p_{i+1} + 2q_3 p_3 + 2\alpha_0 + \alpha_2 + \alpha_5 + 1)$$

$$- 2(t_i + 1)q_3 q_i p_3 + t_i q_{i+1} p_3 (q_3 p_3 - \alpha_1 + \alpha_2)$$

$$+ 2t_i q_3 p_i p_3 + \frac{1}{t_i - t_{i+1}} q_3 \{(t_i - 1)p_i^2 - 2t_i(t_{i+1} - 1)p_i p_2 + (t_i - 1)t_{i+1} p_{i+1}^2\},$$

for $i \in \mathbb{Z}/2\mathbb{Z}$, where

$$\alpha_0 = \frac{\theta_1 - \theta_2 - 2\rho_2}{2}, \ \alpha_1 = -\theta_1, \ \alpha_2 = -\theta_1 + \theta_2, \ \alpha_3 = -\theta_4, \ \alpha_4 = -\rho_1 + \rho_2 + 1, \ \alpha_5 = -\theta_3 - 1,$$

and $\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 + 2\rho_1 + 2\rho_2 = 0$.

**Remark 2.2.** The system $H_i^{31,31,22,22,22}$ reduces to $H_i^{11,11,11,11,11}$ with $\alpha_1 = \alpha_2$ via a specialization $q_3 = \theta_1 + \theta_2 = 0$.

### 2.3. Spectral type $21, 111, 111, 111$.

$$H_{21}^{11,11,11,11,11} = H_{21}^{11,11,11,11,11} + q_1 q_3 p_3 (2q_1 p_1 + q_2 p_2 + q_3 p_3 + \alpha_1 + \eta)$$

$$- q_2 q_3 p_3 (q_1 p_1 + q_3 p_3 - \alpha_5 + \eta) + tq_2 q_3 p_3 (q_1 p_1 + q_2 p_2 + q_3 p_3 + \eta)$$

$$+ (q_1 - q_2) (tp_1 - p_3) (q_3 p_3 + \theta_{2,1}) - q_1 q_3 p_3 (tp_1 + p_2)$$

$$- tq_3 p_1 (q_1 p_1 + q_2 p_2 - \alpha_3 + \eta),$$

where

$$\alpha_0 = \rho_2, \ \alpha_1 = -\theta_{3,2} - \rho_2, \ \alpha_2 = -\theta_{3,1} + \theta_{3,2}, \ \alpha_3 = \theta_{2,1} + \theta_{3,1} + \rho_1,$$

$$\alpha_4 = -\rho_1 + \rho_3 + 1, \ \alpha_5 = -\theta_{2,1} - \rho_3, \ \eta = \theta_1 + \theta_{2,1} + \theta_{3,1} + \rho_1,$$

and $\theta_1 + \theta_{2,1} + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + \rho_3 = 0$.

**Remark 2.3.** The system $H_i^{21,11,11,11,11}$ reduces to $H_{21}^{11,11,11,11,11}$ via a specialization $q_3 = \theta_{2,1} = 0$.

### 2.4. Spectral type $31, 22, 211, 1111$.

$$H_{31}^{21,21,21,111} = H_{21}^{21,21,111} - \rho_4 (\alpha_5 - \eta + \rho_4) q_2$$

$$+ H_{VI}(\alpha_5 + \eta - 2\rho_4, \alpha_0, \alpha_2 + \alpha_3 + \alpha_4 + \eta, \alpha_1; q_3, p_3, t)$$

$$- \frac{1}{t} q_1 q_3 (q_3 p_3 - \rho_4) (q_3 p_3 + \alpha_5 - \eta + \rho_4)$$

$$- q_1 q_3 p_3 (2q_1 p_1 + q_2 p_2 - q_3 p_3 + q_5 + \alpha_1 + \eta) + q_2 q_3 p_3 (2q_1 p_1 + q_2 p_2 + q_3 p_3 + \alpha_5)$$

$$+ tq_2 q_3 (q_1 p_1 - q_2 p_2 - q_3 p_3 + \alpha_1) + q_3 p_3 (t + 1) q_1 p_1 - 2t q_2 p_1 + q_1 p_2 - q_2 p_2$$

$$- tp_1 p_3 (q_1 - tq_2),$$
where

\[ \alpha_0 = \theta_{3,2}, \quad \alpha_1 = -\theta_2 - \theta_{3,2} - \rho_2 - \rho_4, \quad \alpha_2 = \theta_2 + \theta_{3,1} + \rho_2 + \rho_4, \]

\[ \alpha_3 = -\theta_2 - \theta_{3,1} - \rho_1 - \rho_4, \quad \alpha_4 = \rho_1 - \rho_3 + 1, \quad \alpha_5 = \theta_2 + \rho_3 + \rho_4, \quad \eta = \rho_3 - \rho_4, \]

and \( \theta_1 + 2\theta_2 + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + \rho_3 + \rho_4 = 0. \)

**Remark 2.4.** The system \( \mathcal{H}^{31,22,21,1111} \) reduces to \( \mathcal{H}^{21,21,111111} \) via a specialization \( p_3 = \rho_4 = 0. \) It also reduces to \( \mathcal{H}^{31,22,211111} \) via a specialization \( (q_1 - t)p_1 - q_3p_3 - \alpha_2 = \alpha_1 + \alpha_2 = 0. \)

### 2.5. Spectral type 22, 22, 211, 211.

\[
H^{22,22,211111} = \text{tr}(Q(Q - 1)P(Q - t)P - (\alpha_1 - 1)Q(Q - 1)P - \alpha_3Q(Q - t)P - \alpha_4(Q - 1)(Q - t)P + \alpha_2(\alpha_0 + \alpha_2)Q + (t - 1)RQP - \alpha_4),
\]

with

\[
P = -\frac{1}{t} \begin{pmatrix} p_1 & p_2 \\ q_2p_2 - \alpha_2 - \alpha_5 & p_3 \end{pmatrix}, \quad Q = -t \begin{pmatrix} q_1 & 1 \\ q_2 & q_3 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -(q_1 - q_3)p_2 + p_1 - p_3 \\ 0 & \alpha_1 + \theta_{3,2} - 1 \end{pmatrix},
\]

where

\[ \alpha_0 = -\theta_2, \quad \alpha_1 = -\theta_{3,1} + 1, \quad \alpha_2 = -\rho_3, \]

\[ \alpha_3 = \theta_1 + \theta_2 + \theta_{3,1} + 2\rho_3, \quad \alpha_4 = -\theta_1, \quad \alpha_5 = -\theta_1 - \theta_2 - \theta_{3,2} - \rho_2, \]

and \( 2\theta_1 + 2\theta_2 + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + 2\rho_3 = 0. \)

**Remark 2.5.** The system \( \mathcal{H}^{22,22,211111} \) reduces to \( \mathcal{H}^{22,22,22111} \) via a specialization \( R = O. \)

### 2.6. Spectral type 22, 22, 22, 1111.

\[
H^{22,22,22,1111} = \text{tr}(Q(Q - 1)P(Q - t)P - (\alpha_1 - 1)Q(Q - 1)P - \alpha_3Q(Q - t)P - \alpha_4(Q - 1)(Q - t)P + \alpha_2(\alpha_0 + \alpha_2)Q - R(QP + PQ)(Q - t) - (\alpha_0 + \alpha_2 - \rho_4)RQ),
\]

with

\[
P = -\frac{1}{t} \begin{pmatrix} p_1 & p_2 \\ q_2p_2 - \alpha_2 - \alpha_5 & p_3 \end{pmatrix}, \quad Q = -t \begin{pmatrix} q_3 & 1 \\ q_2 & q_1 \end{pmatrix}, \quad R = \begin{pmatrix} \alpha_2 + \rho_4 & -(q_1 - q_3)p_2 + p_1 - p_3 \\ 0 & 0 \end{pmatrix},
\]

where

\[ \alpha_0 = -\theta_2, \quad \alpha_1 = -\theta_{3,1} + 1, \quad \alpha_2 = -\rho_3, \]

\[ \alpha_3 = \theta_1 + \theta_2 + \theta_{3,1} + 2\rho_3, \quad \alpha_4 = -\theta_1, \quad \alpha_5 = -\theta_1 - \theta_2 - \theta_{3,2} - \rho_2, \]

and \( 2\theta_1 + 2\theta_2 + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + 2\rho_3 = 0. \)

**Remark 2.6.** The system \( \mathcal{H}^{22,22,22,1111} \) reduces to \( \mathcal{H}^{22,22,22,211} \) via a specialization \( R = O. \)
2.7. Spectral type 42, 33, 33, 222.

\[ H^{42,33,33,222} = H_{V_1}(\theta_1 + \theta_2 + \rho_2, \theta_1 + 2\theta_2 + \theta_3 + \rho_2 + 2\rho_3 + 1, -\theta_1 - \theta_3, -\theta_1 - \theta_2 - 2\rho_2 - 2\rho_3; q_1, p_1; t) \]
\[ + q_1 p_1 (t p_1 + \rho_2) - \frac{\theta_1 - \theta_2 + 2\rho_2}{2} q_2 p_2 \]
\[ + H_{V_1}(t \frac{\theta_1 - \theta_2 + 2\rho_2}{2}, \frac{\theta_1 + \theta_2 + 2\rho_3 + 1, 2\rho_2; q_2, p_2; t}) \]
\[ - 2(t + 1) q_3^2 p_3^2 + \{ -\theta_1 - \theta_2 - 2\theta_3 - 1 + (3\theta_1 + 3\theta_2 + 2\theta_3 + 4\rho_2 + 4\rho_3 + 1)t \} q_3 p_3 \]
\[ - q_1 q_3 (q_1 p_1 - \theta_2)(q_1 p_1 + \theta_1 + \rho_2) \]
\[ - q_1 q_2 p_1 (q_1 p_1 + \frac{\theta_1 - \theta_2 + 2\rho_2}{2}) + q_2 q_3 p_3 (2q_2 p_2 + q_3 p_3 - \theta_1 + 3\rho_2 + 2\rho_3) \]
\[ + t q_3 p_2 (-2q_1 p_1 + 3q_2 p_2 + 4q_3 p_3 - 2\theta_1 - 2\theta_2 - 2\rho_2 - 4\rho_3) \]
\[ + q_2 p_3 (2q_1 p_1 - q_3 p_3 + \frac{\theta_1 - \theta_2 + 2\rho_2}{2}) + 2(t + 1) q_3 p_3 (q_1 p_1 - q_2 p_2) \]
\[ - 2t (t + 1) q_3^2 p_3^2 + p_1 p_2 (q_1 + q_2) + q_3 p_3 (p_1 - 2p_2) \] + \[2t p_1 ((t + 1) p_2 - p_3) \]

where \( 3\theta_1 + 3\theta_2 + 2\theta_3 + 2\rho_1 + 2\rho_2 + 2\rho_3 = 0 \).

2.8. Spectral type 51, 33, 222, 222.

\[ H^{51,33,222,222} = H_{V_1}(\alpha_3, -\alpha_1 - 2\alpha_2 - 2\alpha_3 + 1, \alpha_1, \alpha_3; q_1, p_1; t) \]
\[ + H_{V_1}(\alpha_3, -2\alpha_3 - 2\alpha_4 - \alpha_5 + 1, \alpha_5, \alpha_3; q_2, p_2; t) \]
\[ + H_{V_1}(\alpha_3, -\alpha_0 - 2\alpha_3 - 2\alpha_6 + 1, \alpha_0, \alpha_3, q_3, p_3; t) \]
\[ + \{(q_1 - 1)p_1 + \alpha_2\}(q_2 - 1)p_2 + \alpha_3(q_1 q_2 + t) \]
\[ + \{(q_2 - 1)p_2 + \alpha_4\}(q_3 - 1)p_3 + \alpha_6(q_2 q_3 + t) \]
\[ + \{(q_3 - 1)p_3 + \alpha_6\}(q_1 - 1)p_1 + \alpha_2(q_3 q_1 + t), \]

where
\[ \alpha_0 = \theta_1 + \theta_3 + \theta_2 + \rho_2 + \rho_3 + 1, \quad \alpha_1 = \theta_1 + \rho_2, \quad \alpha_2 = -\theta_1, \quad \alpha_3 = -\rho_3, \]
\[ \alpha_4 = \frac{\theta_1}{2} - \frac{\theta_2}{2} - \theta_3 - \rho_2, \quad \alpha_5 = -\theta_3 + \theta_2, \quad \alpha_6 = \frac{\theta_1}{2} + \frac{\theta_2}{2} + \rho_3, \]

and \( 3\theta_1 + \theta_2 + 2\theta_3 + 2\rho_1 + 2\rho_2 + 2\rho_3 = 0 \).

3. Schlesinger system

In this section, we recall the Schlesinger system and its Poisson structure following the previous work [10, 11, 17, 20].

Let \( A_1, \ldots, A_{N+2} \in M_L(\mathbb{C}) \). We consider a Fuchsian system on \( \mathbb{P}^1(\mathbb{C}) \)

\[ \frac{\partial}{\partial x} Y(x) = \sum_{i=1}^{N+2} \frac{A_i}{x - t_i} Y(x), \] (3.1)
with regular singularities \( x = t_1, \ldots, t_N, t_{N+1} = 1, t_{N+2} = 0, \infty \). Here we assume that each \( A_i \) 
\((i = 1, \ldots, N + 2)\) can be diagonalized and \( A_\infty := -\sum_{i=1}^{N+2} A_i \) is a diagonal matrix. Then the monodromy preserving deformation of (3.1) gives a Schlesinger system

\[
\frac{\partial A_j}{\partial t_i} = \frac{[A_i, A_j]}{t_i - t_j}, \quad \frac{\partial A_i}{\partial t_i} = -\sum_{j=1, j\neq i}^{N+2} \frac{[A_i, A_j]}{t_i - t_j} \quad (i = 1, \ldots, N; j = 1, \ldots, N + 2; j \neq i).
\]

It is also expressed as a Hamiltonian system

\[
\frac{\partial A_j}{\partial t_i} = \{H_i, A_j\}, \quad H_i = \sum_{j=1, j\neq i}^{N+2} \frac{\text{tr} A_i A_j}{t_i - t_j} \quad (i = 1, \ldots, N; j = 1, \ldots, N + 2),
\]

with a Poisson bracket defined by

\[
\{(A_i)_{k,l}, (A_j)_{r,s}\} = \delta_{ij}[\delta_{r,l}(A_i)_{k,s} - \delta_{k,s}(A_i)_{r,l}].
\]

Thanks to the method established in [10], the Schlesinger system can be rewritten to a canonical Hamiltonian system. Consider a decomposition of matrices \( A_i \) as

\[
A_i = B_i C_i, \quad B_i = (b^{(0)}_{k,l})_{k,l} \in M_{L, \text{rank} A_i}(\mathbb{C}), \quad C_i = (c^{(0)}_{l,k})_{l,k} \in M_{\text{rank} A_i, L}(\mathbb{C}).
\]

Then the variables \( b^{(0)}_{k,l}, c^{(0)}_{l,k} \) can be regarded as canonical ones. In fact, the Poisson bracket

\[
\{b^{(0)}_{k,l}, c^{(0)}_{l,k}\} = -1, \quad \{\text{otherwise}\} = 0,
\]

implies the above one (3.3). In terms of those variables, the system (3.2) is expressed as a Hamiltonian system

\[
\frac{\partial B_j}{\partial t_i} = \{H_i, B_j\}, \quad \frac{\partial C_j}{\partial t_i} = \{H_i, C_j\}, \quad H_i = \sum_{j=1, j\neq i}^{N+2} \frac{\text{tr} B_i C_i B_j C_j}{t_i - t_j} \quad (i = 1, \ldots, N; j = 1, \ldots, N + 2),
\]

with a symplectic form

\[
\omega = \sum_{i=1}^{N+2} \text{tr}(dB_i \wedge dC_i) = \sum_{i=1}^{N+2} \sum_{k=1}^{L, \text{rank} A_i} \sum_{l=1}^{\text{rank} A_i} db^{(0)}_{k,l} \wedge dc^{(0)}_{l,k}.
\]

It remains to find a canonical variables which is suitable for the number of accessory parameters of (3.1). We denote the multiplicity data of eigenvalues of \( A_1, \ldots, A_{N+2}, A_\infty \), called a spectral type, by a \((N + 3)\)-tuples of partitions of natural number \( L \)

\[
\{(m_1, \ldots, m_{j_1}), \ldots, (m_{N+2,1}, \ldots, m_{N+2,j_{N+2}}), (m_{\infty,1}, \ldots, m_{\infty,j_\infty})\}.
\]

Then the number of accessory parameters of (3.1) is given by

\[
(N + 1)L^2 - \sum_{i=1}^{N+2} j_i \sum_{j=1}^{L_i} m^2_{i,j} - \sum_{j=1}^{L_\infty} m^2_{\infty,j} + 2.
\]

And it is generally less than the dimension of a space of matrices \((B_1, C_1, \ldots, B_{N+2}, C_{N+2})\). When we reduce the number of dependent variables of (3.4) to the suitable one, the following proposition plays an important role.
Proposition 3.1 ([17]). Let $P \in GL_L(\mathbb{C})$ and $Q_j \in GL_{\text{rank} A_j}(\mathbb{C})$ $(j = 1, \ldots, N + 2)$.

1. If \( \text{tr}(A_\infty(dP)P^{-1} \wedge (dP)P^{-1}) = 0 \), then \( \omega = \sum_{i=1}^{N+2} \text{tr}(dP^{-1}B_i \wedge dC_iP) \).
2. If \( dQ_jC_iB_jQ_j^{-1} = 0 \) and \( \text{tr}(Q_jC_iB_jQ_j^{-1}(dQ_j)Q_j^{-1} \wedge (dQ_j)Q_j^{-1}) = 0 \), then \( \omega = \text{tr}(dB_jQ_j^{-1} \wedge dQ_jC_j) + \sum_{i=1;i\neq j}^{N+2} \text{tr}(dB_i \wedge dC_i) \).

4. Derivation of the Painlevé system

In this section, we derive six-dimensional Painlevé systems from the Schlesinger system (3.4) associated with the following spectral types:

\[
\{21, 21, 21, 21, 111\}, \quad \{31, 31, 22, 22, 22\}, \quad \{21, 111, 111, 111\}, \quad \{31, 22, 211, 1111\}, \quad \{22, 22, 211, 211\}, \quad \{22, 22, 22, 1111\}, \quad \{42, 33, 33, 222\}, \quad \{51, 33, 222, 222\}.
\]

4.1. Spectral type 21, 21, 21, 21, 111. We consider a Fuchsian system

\[
\frac{\partial}{\partial x} Y(x) = \left( \begin{array}{c} A_1 \\ x - t_1 \\ A_2 \\ x - t_2 \\ A_3 \\ x - 1 \\ A_4 \\ x \end{array} \right) Y(x), \quad (4.1)
\]

with a Riemann scheme

\[
\begin{pmatrix}
 x = t_1 & x = t_2 & x = 1 & x = 0 & x = \infty \\
 \theta_1 & \theta_2 & \theta_3 & \theta_4 & \rho_1 \\
 0 & 0 & 0 & 0 & \rho_2 \\
 0 & 0 & 0 & 0 & \rho_3
\end{pmatrix}.
\]

Note that a Fuchsian relation \( \theta_1 + \theta_2 + \theta_3 + \theta_4 + \rho_1 + \rho_2 + \rho_3 = 0 \) is satisfied. The residue matrices are expressed as

\[
A_i = B_iC_i, \quad B_i = \begin{pmatrix} b_1^{(i)} \\ b_2^{(i)} \\ b_3^{(i)} \end{pmatrix}, \quad C_i = \begin{pmatrix} c_1^{(i)} & c_2^{(i)} & c_3^{(i)} \end{pmatrix} \quad (i = 1, 2, 3, 4),
\]

where \( b_1^{(i)}c_1^{(i)} + b_2^{(i)}c_2^{(i)} + b_3^{(i)}c_3^{(i)} = \theta_i \), and

\[
A_\infty := -\sum_{i=1}^{4} A_i = \begin{pmatrix} \rho_1 & 0 & 0 \\
 0 & \rho_2 & 0 \\
 0 & 0 & \rho_3 \end{pmatrix}.
\]

Under the Schlesinger system (3.4) associated with the Fuchsian one (4.1), we consider a gauge transformation

\[
\tilde{A}_i = \tilde{B}_i\tilde{C}_i, \quad \tilde{B}_i = P^{-1}B_iQ_i^{-1}, \quad \tilde{C}_i = Q_iC_iP \quad (i = 1, 2, 3, 4),
\]

where

\[
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_i = \begin{pmatrix} b_1^{(i)} \\ b_2^{(i)} \\ b_3^{(i)} \\ b_4^{(i)} \end{pmatrix} \quad (i = 1, 2, 3, 4).\]
Then the residue matrices are transformed into
\[
\tilde{A}_1 = \begin{pmatrix} 1 & b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} \theta_1 - b_1 c_1 - b_2 c_2 & c_1 & c_2 \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} 1 \\ b_4 \end{pmatrix} \begin{pmatrix} \theta_2 - b_3 c_3 - b_4 c_4 & c_3 & c_4 \end{pmatrix},
\]
\[
\tilde{A}_3 = \begin{pmatrix} 1 & a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} \theta_3 - a_1 - a_2 & 1 & 1 \end{pmatrix}, \quad \tilde{A}_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \theta_4 & a_3 & a_4 \end{pmatrix},
\]
and
\[
\tilde{A}_n := - \sum_{i=1}^{4} \tilde{A}_i = \begin{pmatrix} \rho_1 & 0 & 0 \\ a_5 & \rho_2 & 0 \\ a_6 & 0 & \rho_3 \end{pmatrix}.
\]

Note that each component is rational in \((b_j^{(i)}, c_j^{(i)})\); we do not give its explicit formula here. Furthermore, the relation (4.2) implies
\[
\begin{align*}
    a_1 &= -\rho_2 - b_1 c_1 - b_3 c_3, \\
    a_2 &= -\rho_3 - b_2 c_2 - b_4 c_4, \\
    a_3 &= -c_1 - c_3 - 1, \\
    a_4 &= -c_2 - c_4 - 1, \\
    a_5 &= -b_1(\theta_1 - b_1 c_1 - b_2 c_2) - b_3(\theta_2 - b_3 c_3 - b_4 c_4) \\
    &\quad + (\rho_2 + b_1 c_1 + b_3 c_3)(\theta_3 + \rho_2 + \rho_3 + b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4), \\
    a_6 &= -b_2(\theta_1 - b_1 c_1 - b_2 c_2) - b_4(\theta_2 - b_3 c_3 - b_4 c_4) \\
    &\quad + (\rho_3 + b_2 c_2 + b_4 c_4)(\theta_3 + \rho_2 + \rho_3 + b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4),
\end{align*}
\]
and
\[
\begin{align*}
    b_1(c_1 - c_2) + b_3(c_3 - c_4) + \rho_2 &= 0, \\
    b_2(c_2 - c_1) + b_4(c_4 - c_3) + \rho_3 &= 0.
\end{align*}
\]
Hence the components of \(\tilde{A}_1, \ldots, \tilde{A}_4\) turn out to be polynomials in \((b_j, c_j)\). Then, thanks to Proposition 3.1 we obtain

**Proposition 4.1.** The dependent variables \(b_j, c_j\) \((j = 1, 2, 3, 4)\) satisfy a Hamiltonian system of eighth order
\[
\begin{align*}
    \frac{\partial b_j}{\partial t_i} &= \{H_i, b_j\}, & \frac{\partial c_j}{\partial t_i} &= \{H_i, c_j\} & (i = 1, 2), \\
    H_1 &= \frac{\text{tr} \tilde{A}_1 \tilde{A}_2}{t_1 - t_2} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_3}{t_1 - 1} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_4}{t_1}, & H_2 &= \frac{\text{tr} \tilde{A}_2 \tilde{A}_1}{t_2 - t_1} + \frac{\text{tr} \tilde{A}_2 \tilde{A}_3}{t_2 - 1} + \frac{\text{tr} \tilde{A}_2 \tilde{A}_4}{t_2},
\end{align*}
\]
with a symplectic form
\[
\omega = \sum_{j=1}^{4} db_j \wedge dc_j,
\]
and the relation (4.3).
We next reduce the Hamiltonian system (4.4) to the one of sixth order. Substituting the second relation of (4.3) to (4.5), we obtain
\[
\omega = d(b_1 + b_2) \wedge \text{dc}_1 + db_2 \wedge d(c_2 - c_1) + d(b_3 + b_4) \wedge \text{dc}_3 + db_4 \wedge d(c_4 - c_3)
\]
\[
= d(b_1 + b_2) \wedge \text{dc}_1 + db_2 \wedge d(c_2 - c_1) + d(b_3 + b_4) \wedge \text{dc}_3 - \frac{b_2(c_2 - c_1)}{c_4 - c_3} \wedge d(c_4 - c_3)
\]
\[
= d(b_1 + b_2) \wedge \text{dc}_1 + db_2(c_4 - c_3) \wedge \frac{c_2 - c_1}{c_4 - c_3} + d(b_3 + b_4) \wedge \text{dc}_3.
\]
Hence we can take a six-dimensional canonical coordinate system by
\[
\frac{q_1}{t_1} = -c_1, \quad t_1 p_1 = b_1 + b_2, \quad \frac{q_2}{t_2} = -c_3, \quad t_2 p_2 = b_3 + b_4,
\]
\[
\frac{q_3}{t_1} = -\frac{c_2 - c_1}{c_4 - c_3}, \quad t_1 p_3 = b_2(c_4 - c_3).
\]
Let
\[
\vec{H}_1 = H_1 + \frac{q_1 p_1}{t_1}, \quad \vec{H}_2 = H_2 + \frac{q_2 p_2}{t_2}.
\]
Then it is easy to verify that the Hamiltonian \(\vec{H}_1\) is just equivalent to the one \(H_i^{21,21,21,21,111}\), which was given in Section 2 for each \(i = 1, 2\). Note that the variables \(b_j, c_j (j = 1, \ldots, 4)\) are described in terms of the canonical coordinates as
\[
b_1 = t_1 p_1 - t_1 p_3 \frac{t_2 p_2 - q_3 p_1}{\rho_2 - p_3}, \quad c_1 = -\frac{q_1}{t_1},
\]
\[
b_2 = t_1 p_3 \frac{t_2 p_2 - q_3 p_1}{\rho_2 - \rho_3}, \quad c_2 = -\frac{q_3}{t_1} - \frac{\rho_2 - \rho_3}{t_1 t_2 p_2 - p_1 q_3} - \frac{q_3}{t_1},
\]
\[
b_3 = t_2 p_2 - (q_3 p_3 - \rho_3) \frac{t_2 p_2 + q_3 p_1}{\rho_2 - \rho_3}, \quad c_3 = -\frac{q_2}{t_2},
\]
\[
b_4 = (q_3 p_3 - \rho_3) \frac{t_2 p_2 - q_3 p_1}{\rho_2 - \rho_3}, \quad c_4 = -\frac{q_2}{t_2} + \frac{\rho_2 - \rho_3}{t_2 p_2 - q_3 p_1}.
\]
Although the components of the matrices \(\vec{A}_1, \ldots, \vec{A}_4\) are rational in \((q_j, p_j)\), the Hamiltonians \(\vec{H}_1, \vec{H}_2\) turn out to be polynomials in \((q_j, p_j)\).

**Theorem 4.2.** The dependent variables \(q_j, p_j (j = 1, 2, 3)\) satisfy the system \(H^{21,21,21,21,111}\).

4.2. Spectral type 31, 31, 22, 22, 22. In this case, we consider a Fuchsian system
\[
\frac{\partial}{\partial x} Y(x) = \left(\frac{A_1}{x - t_1} + \frac{A_2}{x - t_2} + \frac{A_3}{x - 1} + \frac{A_4}{x}\right) Y(x),
\]
with a Riemann scheme
\[
\begin{cases}
  x = t_1 & x = t_2 & x = 1 & x = 0 & x = \infty \\
  \theta_1 & \theta_2 & \theta_3 & \theta_4 & \rho_1 \\
  0 & 0 & \theta_3 & \theta_4 & \rho_2 \\
  0 & 0 & 0 & 0 & \rho_2
\end{cases}.
\]
Note that a Fuchsian relation \(\theta_1 + \theta_2 + 2\theta_3 + 2\theta_4 + 2\rho_1 + 2\rho_2 = 0\) is satisfied.
In a similar manner as Section 4.1, the residue matrices are transformed to

\[
\tilde{A}_1 = \begin{pmatrix}
1 & \theta_1 - b_1 c_1 - b_2 c_2 & a_1 & c_2 \\
0 & b_1 & a_1 & c_1 \\
\end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix}
1 & \theta_2 - a_2 - b_3 c_3 & 1 & c_3 \\
0 & a_2 & \theta_3 & a_4 \\
\end{pmatrix},
\]

\[
\tilde{A}_3 = \begin{pmatrix}
1 & 0 & \theta_3 - a_4 & -a_5 & 1 & 0 \\
0 & 1 & a_4 & a_5 & -a_6 & \theta_3 - a_7 \\
\end{pmatrix}, \quad \tilde{A}_4 = \begin{pmatrix}
1 & 0 & \theta_4 & 0 & a_8 & a_9 \\
0 & 1 & 0 & \theta_4 & a_{10} & a_{11} \\
\end{pmatrix},
\]

and

\[
\tilde{A}_{\omega} := -\sum_{i=1}^{4} \tilde{A}_i = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_1 & 0 & 0 \\
\end{pmatrix}.
\]

By using (4.7), we can show that the variables \(a_1, \ldots, a_{15}\) are polynomials in \((b_j, c_j)\); we do not give their explicit formulas here. Then the dependent variables \(b_j, c_j (j = 1, 2, 3)\) satisfy a Hamiltonian system

\[
\frac{\partial b_j}{\partial t_i} = \{H_i, b_j\}, \quad \frac{\partial c_j}{\partial t_i} = \{H_i, c_j\} \quad (i = 1, 2),
\]

\[
H_1 = \frac{\text{tr}\tilde{A}_1\tilde{A}_2}{t_1 - t_2} + \frac{\text{tr}\tilde{A}_1\tilde{A}_3}{t_1 - 1} + \frac{\text{tr}\tilde{A}_1\tilde{A}_4}{t_1}, \quad H_2 = \frac{\text{tr}\tilde{A}_2\tilde{A}_1}{t_2 - t_1} + \frac{\text{tr}\tilde{A}_2\tilde{A}_3}{t_2 - 1} + \frac{\text{tr}\tilde{A}_2\tilde{A}_4}{t_2},
\]

with a symplectic form \(\omega = \sum_{j=1}^{3} db_j \wedge dc_j\).

Under the system (4.8), we consider a canonical transformation

\[
\frac{q_1}{t_1} = -c_1, \quad t_1 p_1 = b_1 + b_3 c_3, \quad \frac{q_2}{t_2} = -c_3, \quad t_2 p_2 = b_3 + b_2 c_1,
\]

\[
\frac{q_3}{t_1 t_2} = -c_2 + c_1 c_3, \quad t_1 t_2 p_3 = b_2.
\]

Then, by a direct computation, we arrive at

**Theorem 4.3.** The dependent variables \(q_j, p_j (j = 1, 2, 3)\) satisfy the system \(\mathcal{H}^{31,31,22,22,22}\).

**4.3. Spectral type 21, 111, 111, 111.** In this case, we consider a Fuchsian system

\[
\frac{\partial}{\partial x} Y(x) = \left( \frac{A_1}{x - t} + \frac{A_2}{x - 1} + \frac{A_3}{x} \right) Y(x),
\]

with a Riemann scheme

\[
\begin{pmatrix}
x = t & x = 1 & x = 0 & x = \infty \\
\theta_1 & \theta_{2,1} & \theta_{3,1} & \rho_1 \\
0 & \theta_{2,2} & \theta_{3,2} & \rho_2 \\
0 & 0 & 0 & \rho_3 \\
\end{pmatrix}.
\]

Note that a Fuchsian relation \(\theta_1 + \theta_{2,1} + \theta_{2,2} + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + \rho_3 = 0\) is satisfied.
In a similar manner as Section 4.1, the residue matrices are transformed to
\[
\tilde{A}_1 = \begin{pmatrix} 1 \\ b_1 \\ b_2 \end{pmatrix} (\theta_1 - b_1 c_1 - b_2 c_2, c_1, c_2), \quad \tilde{A}_2 = \begin{pmatrix} 1 \\ b_3 \\ b_4 \\ b_5 \\ a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_3 & c_3 & c_4 \\ a_4 & 1 & 1 \end{pmatrix},
\]
\[
\tilde{A}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_3,1 & a_5 & a_6 \\ 0 & \theta_3,2 & a_7 \end{pmatrix},
\]
where
\[
\begin{pmatrix} a_3 & c_3 & c_4 \\ a_4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} \theta_2,1 & 0 \\ 0 & \theta_2,2 \end{pmatrix},
\]
and
\[
\tilde{A}_\infty := -\sum_{i=1}^3 \tilde{A}_i = \begin{pmatrix} \rho_1 & 0 & 0 \\ a_8 & \rho_2 & 0 \\ a_9 & a_{10} & \rho_3 \end{pmatrix}.
\]
By using (4.9) and (4.10), we can show that the variables \(a_1, \ldots, a_{10}\) are rational in \((b_j, c_j)\); we do not give their explicit formulas here. Furthermore, we obtain
\[
\begin{align*}
  b_1 c_1 + b_2 c_2 + b_3 (c_3 + 1) + b_4 (c_4 + 1) + \theta_{2,2} + \theta_{3,2} + \rho_2 + \rho_3 &= 0, \\
  \{b_2 c_2 + b_4 (c_4 + 1) + \rho_3\} (c_3 - c_4) + \theta_{2,2} c_3 + \theta_{2,1} &= 0.
\end{align*}
\]
Then the dependent variables \(b_j, c_j\) \((j = 1, \ldots, 4)\) satisfy a Hamiltonian system
\[
\frac{\partial b_j}{\partial t} = [H, b_j], \quad \frac{\partial c_j}{\partial t} = [H, c_j], \quad H = \frac{\text{tr} \tilde{A}_1 \tilde{A}_2}{t - 1} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_3}{t},
\]
with a symplectic form
\[
\omega = \sum_{j=1}^4 db_j \wedge dc_j,
\]
and the relation (4.11). Note that the Hamiltonian \(H\) turns out to be a polynomial in \((b_j, c_j)\), although the components of the matrices \(A_1, A_2, A_3\) are rational.

We reduce the Hamiltonian system (4.12) to the one of sixth order. Substituting the first relation of (4.11) to (4.13), we obtain
\[
\omega = db_1 \wedge dc_1 + db_2 \wedge dc_2 + db_3 \wedge d(c_3 + 1) + db_4 \wedge d(c_4 + 1)
\]
\[
= db_1 \wedge dc_1 + db_2 \wedge dc_2 + db_3 \wedge d(c_3 + 1) - d \frac{b_1 c_1 + b_2 c_2 + b_3 (c_3 + 1)}{c_4 + 1} \wedge d(c_4 + 1)
\]
\[
= db_1 (c_4 + 1) \wedge d \frac{c_1}{c_4 + 1} + db_2 (c_4 + 1) \wedge d \frac{c_2}{c_4 + 1} + db_3 (c_4 + 1) \wedge d \frac{c_3 + 1}{c_4 + 1}.
\]
Hence we can take a six-dimensional canonical coordinate system by
\[
\begin{align*}
  \frac{q_1}{t} &= -\frac{c_1}{c_4 + 1}, \quad tp_1 = b_1 (c_4 + 1), \\
  \frac{q_2}{t} &= -\frac{c_2}{c_4 + 1}, \quad tp_2 = b_2 (c_4 + 1), \\
  q_3 - 1 &= -\frac{c_3 + 1}{c_4 + 1}, \quad p_3 = b_3 (c_4 + 1).
\end{align*}
\]
Note that the variables $b_j, c_j$ ($j = 1, \ldots, 4$) are rational in $(q_j, p_j)$; we do not give their explicit formulas here. Then, in a similar manner as Section 4.1 we arrive at

**Theorem 4.4.** The dependent variables $q_j, p_j$ ($j = 1, 2, 3$) satisfy the system $\mathcal{H}^{21,111,111,111}$.

4.4. **Spectral type** 31, 22, 211, 1111. In this case, we consider a Fuchsian system

$$\frac{\partial}{\partial x} Y(x) = \left( \frac{A_1}{x-t} + \frac{A_2}{x-1} + \frac{A_3}{x} \right) Y(x),$$

with a Riemann scheme

$$\begin{cases}
  x = t \\ x = 1 \\ x = 0 \\ x = \infty
\end{cases}
\begin{cases}
  \theta_1 & \theta_2 & \theta_{3,1} & \rho_1 \\
  0 & \theta_2 & \theta_{3,2} & \rho_2 \\
  0 & 0 & 0 & \rho_3 \\
  0 & 0 & 0 & \rho_4
\end{cases}.$$

Note that a Fuchsian relation $\theta_1 + 2\theta_2 + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + \rho_3 + \rho_4 = 0$ is satisfied.

In a similar manner as Section 4.1 the residue matrices are transformed to

$$\tilde{A}_1 = \begin{pmatrix}
  1 \\
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix}
\begin{pmatrix}
  \theta_1 - b_1 c_1 - b_2 c_2 - b_3 c_3 & c_1 & c_2 & c_3
\end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix}
  1 & 0 \\
  a_1 & a_2 \\
  a_3 & b_4
\end{pmatrix}
\begin{pmatrix}
  \theta_2 - a_1 - a_3 & -a_2 - b_4 & 1 & 1 \\
  -a_1 - c_4 a_3 & \theta_2 - a_2 - b_4 c_4 & 1 & c_4
\end{pmatrix},$$

$$\tilde{A}_3 = \begin{pmatrix}
  1 & 0 \\
  0 & 1 \\
  0 & 0 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  \theta_{3,1} & a_4 & a_5 & a_6 \\
  0 & \theta_{3,2} & a_7 & a_8
\end{pmatrix},$$

and

$$\tilde{A}_{\infty} := -\sum_{i=1}^{3} \tilde{A}_i = \begin{pmatrix}
  \rho_1 & 0 & 0 & 0 \\
  a_9 & \rho_2 & 0 & 0 \\
  a_{10} & a_{11} & \rho_3 & 0 \\
  a_{12} & a_{13} & 0 & \rho_4
\end{pmatrix}. \tag{4.15}$$

By using (4.15), we can show that the variables $a_1, \ldots, a_{13}$ are polynomials in $(b_j, c_j)$; we do not give their explicit formulas here. Furthermore, we obtain

\begin{align*}
(b_1 c_1 - b_3 c_4 + \theta_2 + \theta_{3,2} + \rho_2)(c_4 - 1) - b_2 c_2 - c_3 - \rho_3 &= 0, \\
\sum_{j=1}^{4} \partial b_j / \partial t &= (H, b_j), \quad \sum_{j=1}^{4} \partial c_j / \partial t = (H, c_j), \\
H &= \frac{\text{tr} \tilde{A}_1 \tilde{A}_2}{t - 1} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_3}{t}, \tag{4.17}
\end{align*}

with a symplectic form

$$\omega = \sum_{j=1}^{4} db_j \wedge dc_j, \tag{4.18}$$

and the relation (4.16).
We reduce the Hamiltonian system (4.17) to the one of sixth order. Substituting the second relation of (4.16) to (4.18), we obtain
\[
\omega = db_1 \wedge dc_1 + d(b_2 + b_3) \wedge dc_2 + db_3 \wedge d(c_3 - c_2) + db_4 \wedge d(c_4 - 1)
\]
\[
= db_1 \wedge dc_1 + d(b_2 + b_3) \wedge dc_2 + db_3 \wedge d(c_3 - c_2) - d\frac{b_3(c_3 - c_2)}{c_4 - 1} \wedge d(c_4 - 1)
\]
\[
= db_1 \wedge dc_1 + d(b_2 + b_3) \wedge dc_2 + db_3(c_4 - 1) \wedge d\frac{c_3 - c_2}{c_4 - 1}.
\]

Hence we can take a six-dimensional canonical coordinate system by

\[
\lambda_1 = b_1, \quad \mu_1 = c_1, \quad \lambda_2 = -c_2, \quad \mu_2 = b_2 + b_3, \quad \lambda_3 = -\frac{c_3 - c_2}{c_4 - 1}, \quad \mu_3 = b_3(c_4 - 1).
\]

Furthermore, we consider a canonical transformation

\[
\frac{q_1}{t} = \frac{1}{\lambda_1}, \quad tp_1 = -\lambda_1(\lambda_1\mu_1 - \lambda_3\mu_3 + \theta_2 + \theta_{3,2} + \rho_2 + \rho_4),
\]
\[
\frac{q_2}{t} = \lambda_2, \quad tp_2 = \mu_2, \quad \frac{q_3}{t} = \lambda_1\lambda_3, \quad tp_3 = \frac{\mu_3}{\lambda_1}.
\]

Note that the variables \(b_j, c_j\) (\(j = 1, \ldots, 4\)) are rational in \((q_j, p_j)\); we do not give their explicit formulas here. Then, in a similar manner as Section 4.1, we arrive at

**Theorem 4.5.** The dependent variables \(q_j, p_j\) (\(j = 1, 2, 3\)) satisfy the system \(\mathcal{H}^{31,22,211,211}\).

### 4.5. Spectral type 22, 22, 211, 211

In this case, we consider a Fuchsian system

\[
\frac{\partial}{\partial x} Y(x) = \left( \frac{A_1}{x - t} + \frac{A_2}{x - 1} + \frac{A_3}{x} \right) Y(x),
\]

with a Riemann scheme

\[
\begin{bmatrix}
  x = t & x = 1 & x = 0 & x = \infty \\
  \theta_1 & \theta_2 & \theta_{3,1} & \rho_1 \\
  \theta_1 & \theta_2 & \theta_{3,2} & \rho_2 \\
  0 & 0 & 0 & \rho_3 \\
  0 & 0 & 0 & \rho_3
\end{bmatrix}.
\]

Note that a Fuchsian relation \(2\theta_1 + 2\theta_2 + \theta_{3,1} + \theta_{3,2} + \rho_1 + \rho_2 + 2\rho_3 = 0\) is satisfied.

In a similar manner as Section 4.1, the residue matrices are transformed to

\[
\tilde{A}_1 = \begin{pmatrix}
  I_2 & \theta_1 I_2 - \tilde{C}_1 \tilde{B}_1 & \tilde{C}_1, \\
  \tilde{B}_1 & b_1 & b_2 \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix}
  b_1 & b_2 \\
  a_1 & b_3
\end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix}
  c_1 & 1 \\
  c_2 & c_3
\end{pmatrix},
\]
\[
\tilde{A}_2 = \begin{pmatrix}
  I_2 & \theta_2 I_2 - \tilde{B}_2 & I_2, \\
  \tilde{B}_2 & a_2 & a_3 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix}
  a_2 & a_3 \\
  a_4 & a_5
\end{pmatrix},
\]
\[
\tilde{A}_3 = \begin{pmatrix}
  I_2 & \tilde{C}_{3,1} & \tilde{C}_{3,2} \\
  \tilde{C}_{3,1} & \tilde{C}_{3,2}
\end{pmatrix}, \quad \tilde{C}_{3,1} = \begin{pmatrix}
  \theta_{3,1} & a_6 \\
  0 & \theta_{3,2}
\end{pmatrix}, \quad \tilde{C}_{3,2} = \begin{pmatrix}
  a_7 & a_8 \\
  a_9 & a_{10}
\end{pmatrix}.
\]
and

$$\overline{A}_\infty := - \sum_{i=1}^{3} \overline{A}_i = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ a_{11} & \rho_2 & 0 & 0 \\ a_{12} & a_{13} & \rho_3 & 0 \\ a_{14} & a_{15} & 0 & \rho_3 \end{pmatrix}. \quad (4.19)$$

By using (4.19), we can show that the variables $a_1, \ldots, a_{15}$ are polynomials in $(b_j, c_j)$; we do not give their explicit formulas here. Then the dependent variables $b_j, c_j$ ($j = 1, 2, 3$) satisfy a Hamiltonian system

$$\frac{\partial b_j}{\partial t} = \{H, b_j\}, \quad \frac{\partial c_j}{\partial t} = \{H, c_j\}, \quad H = \frac{\text{tr} \overline{A}_1 \overline{A}_2}{t-1} + \frac{\text{tr} \overline{A}_1 \overline{A}_3}{t}, \quad (4.20)$$

with a symplectic form $\omega = \sum_{j=1}^{3} db_j \wedge dc_j$.

The system (4.20) is transformed into the one $\mathcal{H}^{22,22,111,111}$ as follows. The Hamiltonian $H$ is described as

$$H = \frac{1}{t-1} \text{tr} (\theta_2 I_2 - \overline{B}_2(I_2 - \overline{C}_1) I_2 - \overline{C}_1) + \frac{1}{t} \text{tr} (\overline{C}_3, I_2 - \overline{C}_1) \overline{B}_2) + \frac{1}{t} \text{tr} (\beta_2 C_3, I_2 - \overline{C}_1) \overline{B}_2(I_2 + \overline{C}_1)$$

$$= \frac{1}{t-1} \text{tr} [(\theta_2 + \rho_3) I_2 - \overline{B}_1(I_2 - \overline{C}_1) I_2 - \overline{C}_1] \text{tr} (\theta_2 + \rho_3) - \frac{1}{t-1} \text{tr} (\theta_2 + 2 \rho_3) \overline{B}_1 C_1(I_2 + \overline{C}_1)$$

$$+ \frac{1}{t(t-1)} \text{tr} (\theta_2) B_1(I_2 + \overline{C}_1) - \frac{1}{t(t-1)} \text{tr} (\theta_2) B_1 C_1(I_2 + \overline{C}_1) - \frac{1}{t} \text{tr} (\overline{C}_1) \overline{B}_1(I_2 - \overline{C}_1).$$

Here we set

$$q_1 = c_1, \quad p_1 = -b_1, \quad q_2 = c_2, \quad p_2 = -b_2, \quad q_3 = c_3, \quad p_3 = -b_3,$$

and

$$P = \frac{1}{t} \overline{B}_1, \quad Q = -t \overline{C}_1, \quad R = -\overline{\theta}_{3,1} I_2 + \overline{C}_3, \overline{I}_3.$$

Then it is easy to verify that the Hamiltonian $H$ is just equivalent to the one $H^{22,22,111,111}$, which was given in Section [2]. Note that

$$a_1 = q_2 p_2 - \theta_1 - \theta_2 - \theta_{3,2} - \rho_2 - \rho_3, \quad a_6 = -(q_1 - q_3) p_2 + p_1 - p_3.$$

**Theorem 4.6.** The dependent variables $q_j, p_j$ ($j = 1, 2, 3$) satisfy the system $\mathcal{H}^{22,22,111,111}$.

4.6. **Spectral type** 22, 22, 22, 11111. In this case, we consider a Fuchsian system

$$\frac{\partial}{\partial x} Y(x) = \left( \frac{A_1}{x-t} \right) Y(x),$$

$$= \left( \frac{A_2}{x-1} \right) Y(x),$$

$$= \left( \frac{A_3}{x} \right) Y(x),$$
with a Riemann scheme

\[
\begin{cases}
x = t & x = 1 \\
\theta_1 & \theta_2 \\
0 & 0 \\
0 & 0 \\
\theta_1 & \theta_2 \\
0 & 0 \\
0 & 0 \\
\theta_3 & \theta_3 \\
\rho_1 & \rho_2 \\
\rho_3 & \rho_4 \\
\rho_4 & \rho_4
\end{cases}
\]

Note that a Fuchsian relation \(2\theta_1 + 2\theta_2 + 2\theta_3 + \rho_1 + \rho_2 + \rho_3 + \rho_4 = 0\) is satisfied.

In a similar manner as Section 4.1, the residue matrices are transformed to

\[
\begin{align*}
\tilde{A}_1 &= \left( \frac{L_1}{B_1} \right) \left( \begin{array}{c} \theta_1 I_2 - \bar{C}_1 \bar{B}_1 - \bar{C}_1 \end{array} \right), \\
\tilde{B}_1 &= \left( \begin{array}{cc} b_1 & b_2 \\
b_3 & b_4 \end{array} \right), \\
\bar{C}_1 &= \left( \begin{array}{cc} c_1 & c_3 \\
c_2 & c_4 \end{array} \right), \\
\tilde{A}_2 &= \left( \frac{L_2}{B_2} \right) \left( \begin{array}{c} \theta_2 I_2 - \bar{C}_2 \bar{B}_2 - \bar{C}_2 \end{array} \right), \\
\tilde{B}_2 &= \left( \begin{array}{cc} a_1 & b_5 \\
a_2 & a_3 \end{array} \right), \\
\bar{C}_2 &= \left( \begin{array}{cc} 1 & 0 \\
c_5 & 1 \end{array} \right), \\
\tilde{A}_3 &= \left( \frac{L_3}{O} \right) \left( \begin{array}{c} \theta_3 I_2 - \bar{C}_3 \bar{B}_3 - \bar{C}_3 \end{array} \right), \\
\bar{C}_3 &= \left( \begin{array}{cc} a_4 & a_5 \\
a_6 & a_7 \end{array} \right),
\end{align*}
\]

and

\[
\tilde{A}_\infty := -\sum_{i=1}^{3} \tilde{A}_i = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ a_8 & a_9 & \rho_3 & 0 \\ a_{10} & a_{11} & 0 & \rho_4 \end{pmatrix}.
\] (4.21)

By using (4.21), we can show that the variables \(a_1, \ldots, a_{11}\) are polynomials in \((b_j, c_j)\); we do not give their explicit formulas here. Furthermore, we obtain

\[
\begin{align*}
b_2(c_2 - c_1) + b_4(c_4 - c_3) + b_5(c_5 - 1) - \theta_1 - \theta_2 - \theta_3 - \rho_2 &= 0, \\
b_1(c_1 c_5 - c_2) + b_2 c_2(c_5 - 1) + b_3(c_3 - c_4) + b_5 c_5(c_5 - 1) - \rho_3 c_5 + \theta_1 + \theta_2 + \theta_3 + \rho_2 - \rho_4 &= 0, \\
b_1(c_1 - c_3) + b_2(c_2 - c_4) + b_5(c_5 - 1) - \rho_3 &= 0, \\
b_2 c_2(c_5 - 1) + b_3(c_3 - c_1) + b_4(c_4 c_5 - c_2) + b_5 c_5(c_5 - 1) - (\theta_1 + \theta_2 + \theta_3 + \rho_2)(c_5 - 1) - \rho_4 &= 0.
\end{align*}
\] (4.22)

Then the dependent variables \(b_j, c_j\) \((j = 1, \ldots, 5)\) satisfy a Hamiltonian system

\[
\frac{\partial b_j}{\partial t} = \{H, b_j\}, \quad \frac{\partial c_j}{\partial t} = \{H, c_j\}, \quad H = \frac{\text{tr} \tilde{A}_1 \tilde{A}_2}{t - 1} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_3}{t},
\] (4.23)

with a symplectic form

\[
\omega = \sum_{j=1}^{5} db_j \wedge dc_j,
\] (4.24)

and the relation (4.22).

We derive a six-dimensional canonical coordinate system in advance. The first and third relation of (4.22) are rewritten to

\[
\begin{align*}
(b_1 + b_2)(c_1 - c_3) - (b_4 + b_2)(c_4 - c_3) + \theta_1 + \theta_2 + \theta_3 + \rho_2 - \rho_3 &= 0, \\
(b_1 + b_2)(c_1 - c_3) + b_2(c_2 - c_4 - c_1 + c_3) + b_5(c_5 - 1) - \rho_3 &= 0.
\end{align*}
\]
Substituting them to (4.24), we obtain
\[
\omega = d(b_1 + b_2) \land d(c_1 - c_3) + db_2 \land d(c_2 - c_4 - c_1 + c_3) + d(b_3 + b_4 + b_1 + b_2) \land dc_3 + d(b_4 + b_2) \land d(c_4 - c_3) + db_5 \land d(c_5 - 1) = d(b_1 + b_2) \land d(c_1 - c_3) + db_2 \land d(c_2 - c_4 - c_1 + c_3) + d(b_3 + b_4 + b_1 + b_2) \land dc_3 + d(b_4 + b_2)(c_1 - c_3) \land d(c_4 - c_3) - d((b_1 + b_2)(c_1 - c_3) + b_2(c_2 - c_4 - c_1 + c_3)) \land d(c_5 - 1) = d(b_1 + b_2)(c_5 - 1) \land d(c_1 - c_3)(c_4 - c_3) \land d(c_5 - 1) + db_2(c_5 - 1) \land d(c_2 - c_4 - c_1 + c_3) \land c_5 - 1 + d(b_3 + b_4 + b_1 + b_2) \land dc_3.
\]

Hence we can take
\[
q_1 = c_3, \quad p_1 = -b_3 - b_4 - b_1 - b_2, \quad q_2 = \frac{(c_1 - c_3)(c_4 - c_3)}{c_5 - 1}, \quad p_2 + \frac{\theta_1 + \theta_2 + \theta_3 + \rho_2 + \rho_1}{q_2} = -\frac{(b_1 + b_2)(c_5 - 1)}{c_4 - c_3}, \quad q_3 = \frac{c_2 - c_4 - c_1 + c_3}{c_5 - 1}, \quad p_3 = -b_2(c_5 - 1).
\]

The system (4.23) is transformed into the one \(\mathcal{H}^{22,22,22,1111}\) as follows. We set
\[
P = \frac{1}{t} \Gamma^{-1} \bar{B}_1 \bar{C}_2 \Gamma, \quad Q = -t \Gamma^{-1} \bar{C}_2^{-1} \bar{B}_1 \Gamma, \quad R = \Gamma^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -\rho_3 + \rho_4 \end{pmatrix} \Gamma,
\]
where
\[
\Gamma = \begin{pmatrix} 0 & -1 \\ c_1 - c_3 & 1 \end{pmatrix}.
\]

Then, by using (4.22) and (4.25), we can show that the components of the matrices \(P, Q, R\) are polynomials in \((q_j, p_j)\); we do not give their explicit formulas here. And, in a similar manner as Section 4.5, we arrive at

**Theorem 4.7.** The dependent variables \(q_j, p_j (j = 1, 2, 3)\) satisfy the system \(\mathcal{H}^{22,22,22,1111}\).

### 4.7. Spectral type 42, 33, 33, 222

In this case, we consider a Fuchsian system
\[
\frac{d}{dx} Y(x) = \left( \frac{A_1}{x - 1} + \frac{A_2}{x} + \frac{A_3}{x} \right) Y(x),
\]
with a Riemann scheme
\[
\begin{cases}
  x = t & x = 1 \\
  \theta_1 & \theta_2 \\
  \theta_1 & \theta_2 \\
  \theta_1 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 
\end{cases} \quad \begin{cases}
  x = 0 & x = \infty \\
  \theta_3 & \rho_1 \\
  \theta_3 & \rho_1 \\
  0 & \rho_2 \\
  0 & \rho_2 \\
  0 & \rho_3 \\
  0 & \rho_3 
\end{cases}.
\]

Note that a Fuchsian relation \(3\theta_1 + 3\theta_2 + 2\theta_3 + 2\rho_1 + 2\rho_2 + 2\rho_3 = 0\) is satisfied.
In a similar manner as Section 4.1, the residue matrices are transformed to

\[ \tilde{A}_1 = \left( I_3 \frac{I_1}{B_1} \right) \left( \theta_1 I_3 - \tilde{C}_1 \tilde{B}_1 \tilde{C}_1 \right), \quad \tilde{B}_1 = \begin{pmatrix} 0 & b_1 & 0 \\ b_2 & a_1 & b_3 \\ b_4 & b_2 & b_5 \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} a_3 & c_2 & c_4 \\ c_1 & 0 & 1 \\ a_4 & c_3 & c_5 \end{pmatrix}, \]

\[ \tilde{A}_2 = \left( I_3 \frac{I_2}{B_2} \right) \left( \theta_2 I_3 - \tilde{C}_2 \tilde{B}_2 \tilde{C}_2 \right), \quad \tilde{B}_2 = \begin{pmatrix} a_5 & a_6 & 1 \\ a_7 & a_8 & a_6 \\ a_9 & a_{10} & a_7 \end{pmatrix}, \quad \tilde{C}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ a_{11} & c_6 & c_7 \end{pmatrix}, \]

\[ \tilde{A}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_3 & 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & \theta_3 & a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix}. \]

and

\[
\tilde{A}_{\infty} := -\sum_{i=1}^{3} \tilde{A}_i = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_1 & 0 & 0 & 0 & 0 & 0 \\
a_{20} & a_{21} & \rho_2 & 0 & 0 & 0 & 0 \\
a_{22} & a_{23} & 0 & \rho_2 & 0 & 0 & 0 \\
a_{24} & a_{25} & 0 & 0 & \rho_3 & 0 & 0 \\
a_{26} & a_{27} & 0 & 0 & 0 & \rho_3 & 0 
\end{pmatrix}.
\]  

(4.26)

Now we can find 35 relations in (4.26). Among them, the relations derived from the following matrix components are used to determine the variables \(a_{12}, \ldots, a_{27}\) as polynomials in the other variables:

\[
(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (4,1), (4,2), (5,1), (5,2), (6,1), (6,2). \]

And the following ones are used to determine the variables \(a_1, \ldots, a_{11}\) as rational expressions in \((b_j, c_j)\):

\[
(1,2), (2,1), (2,2), (3,3), (3,4), (4,4), (4,5), (5,5), (5,6), (6,5), (6,6). \]

We do not give their explicit formulas here. Then the dependent variables \(b_j, c_j (j = 1, \ldots, 7)\) satisfy a rational Hamiltonian system

\[
\frac{\partial b_j}{\partial t} = \{H, b_j\}, \quad \frac{\partial c_j}{\partial t} = \{H, c_j\}, \quad H = \frac{\text{tr}\tilde{A}_1\tilde{A}_2}{t-1} + \frac{\text{tr}\tilde{A}_1\tilde{A}_3}{t}, \]

(4.27)

with a symplectic form

\[
\omega = \sum_{j=1}^{7} db_j \wedge dc_j. \]

(4.28)

Furthermore, we have 8 relations which is derived from the matrix components

\[
(3,5), (3,6), (4,3), (4,6), (5,3), (5,4), (6,3), (6,4). \]
In order to derive the Hamiltonian system of sixth order, we use the first four relations, whose explicit formulas are given as

\[
\begin{align*}
    c_3 + c_6 &= 0, \\
    c_5 + c_7 &= 0, \\
    b_1c_1 - b_1c_5 + \theta_2 + \rho_2 &= 0, \\
    b_1(c_1 - 1) - b_3c_3 - b_5c_5 - b6c_6 - b_7(c_7 + 1) + \theta_1 + \theta_2 + 2\rho_2 &= 0.
\end{align*}
\]  

(4.29)

We reduce the Hamiltonian system (4.27) to the one of sixth order. Substituting the first and second relation of (4.29) to (4.28), we obtain

\[
\omega = db_1 \land d(c_1 - 1) + db_2 \land dc_2 + d(b_3 - b_6) \land dc_3 + db_4 \land dc_4 + d(b_5 - b_7 - 1) \land dc_5.
\]

Hence we can take

\[
\lambda_1 = -c_1 + 1, \quad \mu_1 = b_1, \quad \lambda_2 = -c_2, \quad \mu_2 = b_2, \quad \lambda_3 = -c_3, \quad \mu_3 = b_3 - b_6,
\]

\[
\lambda_4 = -c_4, \quad \mu_4 = b_4, \quad \lambda_5 = -c_5, \quad \mu_5 = b_5 - b_7 - 1.
\]

Those variables satisfy a rational Hamiltonian system of fifth order with a symplectic form \(\omega = \sum_{j=1}^{5} d\lambda_j \land d\mu_j\); we do not give its explicit formula here. Then the third and fourth relation of (4.29) are described as

\[
\lambda_1\mu_1 + \mu_1\mu_5 - \theta_2 - \rho_2 = 0, \quad \lambda_1\mu_1 - \lambda_3\mu_3 - \lambda_5\mu_5 - \theta_1 - \theta_2 - 2\rho_2 = 0.
\]

Substituting them to the symplectic form \(\omega\) again, we obtain

\[
\omega = d\lambda_1 \land d\mu_1 + d\lambda_2 \land d\mu_2 + d\lambda_3 \land d\left(\frac{\lambda_1\mu_1 + \lambda_1\lambda_5 - (\theta_2 + \rho_2)\lambda_5}{\mu_1} - \theta_1 - \theta_2 - 2\rho_2\right)
\]

\[
\quad + d\lambda_4 \land d\mu_4 + d\lambda_5 \land d\left(-\lambda_1 + \frac{\theta_2 + \rho_2}{\mu_1}\right)
\]

\[
= d\lambda_3 \left(\lambda_1 - \frac{\theta_2 + \rho_2}{\mu_1}\right) \land d\left(\frac{\mu_1 + \lambda_5}{\lambda_3}\right) + d\lambda_2 \land d\mu_2 + d\lambda_4 \land d\mu_4.
\]

Hence we can take

\[
q_1 = \frac{\mu_1 + \lambda_5}{\lambda_3}, \quad p_1 = -\lambda_3 \left(\lambda_1 - \frac{\theta_2 + \rho_2}{\mu_1}\right), \quad q_2 = \lambda_4, \quad p_2 = \mu_4, \quad \frac{q_3}{t} = -\lambda_2, \quad tp_3 = -\mu_2.
\]

Then, in a similar manner as Section 4.1, we arrive at

**Theorem 4.8.** The dependent variables \(p_j, q_j\) \((j = 1, 2, 3)\) satisfy the system \(\mathcal{H}^{42,33,33,222}\).

In the above, we used 31 relations of (4.26) to derive the system \(\mathcal{H}^{42,33,33,222}\). And the rest 4 relations have not been used yet. They are used to determine the variables \(b_6, b_7, c_3, c_5\) as rational expressions in \((q_j, p_j)\). Hence we can show that the components of the matrices \(A_1, A_2, A_3\) are rational in \((q_j, p_j)\); we do not give their explicit formulas here.
4.8. **Spectral type** 51, 33, 222, 222. In this case, we consider a Fuchsian system

\[
\frac{\partial}{\partial x} Y(x) = \left( \frac{A_1}{x-t} + \frac{A_2}{x-1} + \frac{A_3}{x} \right) Y(x),
\]

with a Riemann scheme

\[
\begin{pmatrix}
  x = t & x = 1 & x = 0 & x = \infty \\
  \theta_1 & \theta_2 & \theta_{3,1} & \rho_1 \\
  \theta_1 & 0 & \theta_{3,1} & \rho_1 \\
  \theta_1 & 0 & \theta_{3,2} & \rho_2 \\
  0 & 0 & \theta_{3,2} & \rho_2 \\
  0 & 0 & 0 & \rho_3 \\
  0 & 0 & 0 & \rho_3
\end{pmatrix}.
\]

Note that a Fuchsian relation 3\(\theta_1 + \theta_2 + 2\theta_{3,1} + 2\theta_{3,2} + 2\rho_1 + 2\rho_2 + 2\rho_3 = 0\) is satisfied.

In a similar manner as Section 4.4.1 the residue matrices are transformed to

\[
\tilde{A}_1 = \begin{pmatrix} I_3 \\ \tilde{B}_1 \end{pmatrix} \left( \theta_1 I_3 - \tilde{C}_1 \tilde{B}_1 \tilde{C}_1 \right), \quad \tilde{B}_1 = \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & b_1 & b_2 \\ a_4 & a_5 & b_3 \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & c_1 & 0 \\ a_6 & c_2 & c_3 \end{pmatrix},
\]

\[
\tilde{A}_2 = \begin{pmatrix} 1 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{pmatrix} \left( \theta_2 - a_7 - a_8 - a_{10} \right), \quad \tilde{A}_3 = \begin{pmatrix} \theta_{3,1} & 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & \theta_{3,1} & a_{16} & a_{17} & a_{18} & a_{19} \\ 0 & 0 & \theta_{3,2} & 0 & a_{20} & a_{21} \\ 0 & 0 & 0 & \theta_{3,2} & a_{22} & a_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

and

\[
\tilde{A}_\infty := -\sum_{i=1}^{3} \tilde{A}_i = \begin{pmatrix} \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 & 0 \\ a_{24} & a_{25} & \rho_2 & 0 & 0 \\ a_{26} & a_{27} & 0 & \rho_2 & 0 \\ a_{28} & a_{29} & a_{30} & a_{31} & \rho_3 \\ a_{32} & a_{33} & a_{34} & a_{35} & 0 & \rho_3 \end{pmatrix}.
\]

By using (4.30), we can show that the variables \(a_1, \ldots, a_{35}\) are rational in \((b_j, c_j)\); we do not give their explicit formulas here. Then the dependent variables \(b_j, c_j\) \((j = 1, 2, 3)\) satisfy a Hamiltonian system

\[
\frac{\partial b_j}{\partial t} = [H, b_j], \quad \frac{\partial c_j}{\partial t} = [H, c_j], \quad H = \frac{\text{tr} \tilde{A}_1 \tilde{A}_2}{t - 1} + \frac{\text{tr} \tilde{A}_1 \tilde{A}_3}{t},
\]

with a symplectic form \(\omega = \sum_{j=1}^{3} db_j \wedge dc_j\). Note that the Hamiltonian \(H\) turns out to be a polynomial in \((b_j, c_j)\), although the components of the matrices \(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\) are rational.

Under the system (4.31), we consider a canonical transformation

\[
q_1 = -b_2, \quad p_1 = -c_2, \quad q_2 = \frac{1}{b_3}, \quad p_2 = -b_3 \left( b_3 c_3 - \frac{\theta_1}{2} - \frac{\theta_2}{2} - \theta_{3,2} - \rho_2 \right),
\]

\[
q_3 = -\frac{1}{b_1}, \quad tp_3 = b_1 \left( b_1 c_1 + \frac{\theta_1}{2} + \frac{\theta_2}{2} + \rho_3 \right).
\]
Then, by a direct computation, we arrive at

**Theorem 4.9.** The dependent variables \( p_j, q_j \) \((j = 1, 2, 3)\) satisfy the system \( \mathcal{H}^{51,33,222,222} \).

5. **Particular Solutions**

In this section, we give particular solutions of the six-dimensional Painlevé systems in terms of the hypergeometric functions.

5.1. **Spectral type** \(21,21,21,21,111\). Under the system \( \mathcal{H}^{21,21,21,21,111} \), we consider a specialization

\[
p_1 - \frac{\alpha_1}{q_1} = p_2 = q_3 = \alpha_0 + \alpha_1 + \alpha_5 + 1 = 0.
\]

Also we set

\[
y_1 \bigg/ y_0 = q_1, \quad y_2 \bigg/ y_0 = q_2, \quad y_3 \bigg/ y_0 = -t_2 q_1 p_3,
\]

where the variable \( y_0 \) satisfies a Pfaff system

\[
t_1(t_1 - 1) \frac{\partial}{\partial t_1} \log y_0 = (\alpha_5 + 1) q_1 + \alpha_1 t_1, \quad t_2(t_2 - 1) \frac{\partial}{\partial t_2} \log y_0 = t_2 q_1 p_3 + (\alpha_5 - \rho_3 + 1) q_2 - \alpha_2 t_2.
\]

Then we have

**Theorem 5.1.** A vector of variables \( y = (y_0, y_1, y_2, y_3) \) satisfies a rigid system

\[
\frac{\partial y}{\partial t_i} = \left( \frac{M_i^{(i)}}{t_i - t_{i+1}} + \frac{M_1^{(i)}}{t_i - 1} + \frac{M_0^{(i)}}{t_i} \right) y \quad (i \in \mathbb{Z}/2\mathbb{Z}), \tag{5.1}
\]

with matrices

\[
M_i^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\alpha_2 & -\alpha_1 & -1 \\
0 & \alpha_2 & \alpha_1 & 1 \\
0 & -\alpha_2 \rho_3 & -\alpha_1 \rho_3 & -\rho_3
\end{pmatrix}, \quad
M_1^{(1)} = \begin{pmatrix}
\alpha_1 & \alpha_5 + 1 & 0 & 0 \\
\alpha_1 & \alpha_5 + 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha_1 \rho_3 & \alpha_1 + \alpha_5 + 1
\end{pmatrix},
\]

\[
M_0^{(1)} = \begin{pmatrix}
0 & -\alpha_5 - 1 & 0 & 0 \\
0 & \alpha_1 - \alpha_5 + 1 & 0 & 0 \\
0 & -\alpha_2 & 0 & 0 \\
0 & \alpha_2 \rho_3 & 0 & 0
\end{pmatrix},
\]

and

\[
M_i^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\alpha_2 & -\alpha_1 & -1 \\
0 & \alpha_2 & \alpha_1 & 1 \\
0 & -\alpha_2 \rho_3 & -\alpha_1 \rho_3 & -\rho_3
\end{pmatrix}, \quad
M_1^{(2)} = \begin{pmatrix}
-\alpha_2 & 0 & \alpha_5 - \rho_3 + 1 & -1 \\
0 & 0 & 0 & 0 \\
-\alpha_2 & 0 & \alpha_5 - \rho_3 + 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
M_0^{(2)} = \begin{pmatrix}
0 & 0 & -\alpha_5 + \rho_3 - 1 & 1 \\
0 & 0 & \alpha_1 & 1 \\
0 & -\alpha_2 - \alpha_5 + \rho_3 + 1 & 0 \\
0 & 0 & -\alpha_2 - \alpha_5 + \rho_3 + 1 & 0
\end{pmatrix}.
\]
The Riemann scheme of the system (5.1) is given by
\[
\begin{cases}
t_1 = t_2 & t_1 = 1 & t_1 = 0 & t_1 = \infty \\
\alpha_1 - \alpha_2 - \rho_3 & \alpha_1 + \alpha_5 + 1 & \alpha_1 - \alpha_3 + 1 & -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_5 - 2 \\
0 & \alpha_1 + \alpha_5 + 1 & 0 & -\alpha_1 - \alpha_5 + \rho_3 - 1 \\
0 & 0 & 0 & -\alpha_1 \\
0 & 0 & 0 & -\alpha_1
\end{cases},
\]
and
\[
\begin{cases}
t_2 = t_1 & t_2 = 1 & t_2 = 0 & t_2 = \infty \\
\alpha_1 - \alpha_2 - \rho_3 & -\alpha_2 + \alpha_5 - \rho_3 + 1 & -\alpha_2 - \alpha_3 + \rho_3 + 1 & -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_5 - 2 \\
0 & 0 & -\alpha_2 - \alpha_3 + \rho_3 + 1 & \alpha_2 + \alpha_3 - 1 \\
0 & 0 & 0 & \alpha_2 \\
0 & 0 & 0 & \alpha_2
\end{cases}.
\]
Namely, for each $i \in \mathbb{Z}/2\mathbb{Z}$, the system (5.1) is a Fuchsian one with a spectral type $31, 31, 22, 211$.

5.2. **Spectral type** 31, 31, 22, 22, 22. Under the system $\mathcal{H}^{31,31,22,22,22}$, we consider a specialization $q_1 = q_2 = q_3 = \alpha_1 = 0$.

Note that such a specialization implies $\tilde{A}_1 = O$. We also set
\[
\frac{y_1}{y_0} = -t_1 p_1, \quad \frac{y_2}{y_0} = -t_2 p_2, \quad \frac{y_3}{y_0} = t_1 t_2 (p_1 p_2 - \alpha_2 p_3),
\]
where the variable $y_0$ satisfies a Pfaff system
\[
t_i(t_i - 1) \frac{\partial}{\partial t_i} \log y_0 = t_i p_i - (\alpha_0 + \alpha_5 + 1)t_i - \alpha_3(t_i - 1) \quad (i \in \mathbb{Z}/2\mathbb{Z}).
\]
Then we have

**Theorem 5.2.** A vector of variables $y = ^t[y_0, y_1, y_2, y_3]$ satisfies a rigid system
\[
\frac{\partial y}{\partial t_i} = \left(\frac{M_i^{(i)}}{t_i - t_{i+1}} + \frac{M_1^{(i)}}{t_i - 1} + \frac{M_0^{(i)}}{t_i}\right) y \quad (i \in \mathbb{Z}/2\mathbb{Z}),
\]
with matrices
\[
M_i^{(i)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_2 & 0 \\
0 & -\alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
M_1^{(i)} = \begin{pmatrix}
-(\alpha_0 + \alpha_5 + 1) & -1 & 0 & 0 \\
\alpha_0(\alpha_0 + \alpha_5 + 1) & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & (\alpha_0 + \alpha_2)(\alpha_0 + \alpha_2 + \alpha_5 + 1) & \alpha_0 + \alpha_2
\end{pmatrix},
\]
\[
M_0^{(i)} = \begin{pmatrix}
-\alpha_3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha_2 & -\alpha_3 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
and

\[ M_i^{(2)} = E_{23} M_i^{(1)} E_{23}, \quad M_1^{(2)} = E_{23} M_1^{(1)} E_{23}, \quad M_0^{(2)} = E_{23} M_0^{(1)} E_{23}, \quad E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The Riemann scheme of the system (5.2) is given by

\[
\begin{align*}
\begin{cases}
t_1 = t_2 & t_1 = 1 & t_1 = 0 & t_1 = \infty \\
2\alpha_2 & -\alpha_5 - 1 & -\alpha_3 & \alpha_0 + \alpha_3 + \alpha_5 + 1 \\
0 & -\alpha_5 - 1 & -\alpha_3 & \alpha_0 + \alpha_3 + \alpha_5 + 1 \\
0 & 0 & 0 & -\alpha_0 - \alpha_2 \\
0 & 0 & 0 & -\alpha_0 - \alpha_2
\end{cases},
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
t_2 = t_1 & t_2 = 1 & t_2 = 0 & t_2 = \infty \\
2\alpha_2 & -\alpha_5 - 1 & -\alpha_3 & \alpha_0 + \alpha_3 + \alpha_5 + 1 \\
0 & -\alpha_5 - 1 & -\alpha_3 & \alpha_0 + \alpha_3 + \alpha_5 + 1 \\
0 & 0 & 0 & -\alpha_0 - \alpha_2 \\
0 & 0 & 0 & -\alpha_0 - \alpha_2
\end{cases}.
\end{align*}
\]

Namely, for each \( i \in \mathbb{Z}/2\mathbb{Z} \), the system (5.2) is a Fuchsian one with a spectral type \( 31, 22, 22, 22 \).

5.3. **Spectral type** 21, 111, 111, 111. Under the system \( \mathcal{H}^{21,111,111,111} \), we consider a specialization

\[ p_1 = p_2 = p_3 = \eta = 0. \]

Also we set

\[ \frac{y_1}{y_0} = \frac{q_1}{t}, \quad \frac{y_2}{y_0} = \frac{q_2}{t}, \quad \frac{y_3}{y_0} = q_3, \]

where the variable \( y_0 \) satisfies a Pfaff system

\[ t(t - 1) \frac{d}{dt} \log y_0 = -\alpha_1 q_1 - \alpha_3 q_2 - \alpha_3 t. \]

Then we have

**Theorem 5.3.** A vector of variables \( y = t[y_0, y_1, y_2, y_3] \) satisfies a rigid system

\[
\frac{dy}{dt} = \left( \frac{M_i}{t - 1} + \frac{M_0}{t} \right) y,
\]

with matrices

\[
M_i = \begin{pmatrix}
-\alpha_3 & -\alpha_1 & -\alpha_5 & 0 \\
-\alpha_3 & -\alpha_1 + \theta_{2,1} & -\alpha_5 - \theta_{2,1} & \alpha_3 \\
-\alpha_3 & -\alpha_1 & -\alpha_5 & 0 \\
0 & -\theta_{2,1} & \theta_{2,1} & -\alpha_3
\end{pmatrix}, \quad M_0 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\alpha_3 & -\alpha_2 - \alpha_3 & 0 & -\alpha_3 \\
\alpha_3 & \alpha_1 & \alpha_4 + \alpha_5 - 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
The Riemann scheme of the system (5.3) is given by

\[
\begin{pmatrix}
 t = 1 \\
 t = 0 \\
 t = \infty
\end{pmatrix}
\begin{pmatrix}
 -a_1 - a_3 - a_5 & a_4 + a_5 - 1 & -a_4 + 1 \\
 -a_3 + \theta_{2,1} & -a_2 - a_3 & a_1 + a_2 + a_3 - \theta_{2,1} \\
 0 & 0 & a_3
\end{pmatrix}.
\]

Namely, the system (5.3) is a Fuchsian one with a spectral type 211, 211, 211.

5.4. Spectral type 31, 22, 211, 1111. Under the system \( H^{31,22,211,1111} \), we consider a specialization

\[ q_1 p_1 - a_1 = q_2 = q_3 = a_1 + a_3 - \eta = 0. \]

Note that such a specialization implies \( \widetilde{A}_1 = O \). We also set

\[
\frac{y_1}{y_0} = t p_1, \quad \frac{y_2}{y_0} = t p_2 - \frac{t^2 p_1 p_3}{\eta}, \quad \frac{y_3}{y_0} = \frac{t^2 p_1 p_3}{\eta}.
\]

where the variable \( y_0 \) satisfies a Pfaff system

\[
t(t - 1) \frac{d}{dt} \log y_0 = t p_1 + t p_2 + (a_1 - a_5 - \rho_4) t - (a_0 + a_1 + a_2)(t - 1)
\]

Then we have

**Theorem 5.4.** A vector of variables \( y = [y_0, y_1, y_2, y_3] \) satisfies a rigid system

\[
\frac{dy}{dt} = \left( \frac{M_1}{t - 1} + \frac{M_0}{t} \right) y,
\]

with matrices

\[
M_1 = \begin{pmatrix}
 a_1 - a_5 - \rho_4 & 1 & 1 & 1 \\
 a_1 (a_1 - \eta) & -a_1 - a_5 + \eta - \rho_4 & -a_1 + \eta & -\eta - a_1 + \eta \\
 \eta (a_5 - a_5 + \rho_4) & 0 & 0 & 0
\end{pmatrix},
\]

\[
M_0 = \begin{pmatrix}
 -a_0 - a_1 - a_2 & 1 & 1 & 1 \\
 0 & -a_0 & a_1 & a_1 - \eta \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The Riemann scheme of the system (5.4) is given by

\[
\begin{pmatrix}
 t = 1 & t = 0 & t = \infty
\end{pmatrix}
\begin{pmatrix}
 -a_5 + \eta - 2\rho_4 & -a_0 - a_1 - a_2 & a_0 + a_2 + a_5 + \rho_4 \\
 -a_5 + \eta - 2\rho_4 & -a_0 & a_0 + a_1 + a_5 - \eta + \rho_4 \\
 0 & 0 & -\eta + \rho_4
\end{pmatrix}.
\]

Namely, the system (5.4) is a Fuchsian one with a spectral type 22, 211, 1111.
5.5. Remark: Theoretical Background. Let $m = \{m_2, \ldots, m_{N+3}\}$, where $m_i = (m_{i,1}, \ldots, m_{i,l_i})$, be a $(N + 2)$-tuples of partitions of natural number $L$ such that

$$NL^2 - \sum_{i=2}^{N+3} \sum_{j=1}^{l_i} m_{i,j}^2 + 2 = 0.$$  

Also let $\tilde{m} = \{m_1, m_2, \ldots, m_{N+3}\}$, where $m_1 = (m_{1,1}, m_{1,2})$, be a $(N + 3)$-tuples of partitions of $L$. Note that a Fuchsian system with a spectral type $m$ (or resp. $\tilde{m}$) contains 0 (or resp. $2m_{1,1}m_{1,2}$) accessory parameters. We consider a Schlesinger system (5.4) associated with a spectral type $\tilde{m}$, which is rewritten to

$$\frac{\partial B_j}{\partial t_i} = \frac{A_j}{t_i - t_j} B_j, \quad \frac{\partial C_j}{\partial t_i} = -C_j \frac{A_j}{t_i - t_j}, \quad \frac{\partial B_i}{\partial t_i} = -\sum_{j=1, j\neq i}^{N+2} \frac{A_j}{t_i - t_j} B_i, \quad \frac{\partial C_i}{\partial t_i} = C_i \sum_{j=1, j\neq i}^{N+2} \frac{A_j}{t_i - t_j} (5.5)$$

($i = 1, \ldots, N; j = 1, \ldots, N + 2; j \neq i$).

Such a system admits a particular solution given by a rigid system.

**Lemma 5.5.** The system (5.5) admits a specialization $C_1 = 0$. Then a matrix of variables $B_1$ satisfies

$$\frac{\partial B_1}{\partial t_1} = -\sum_{j=2}^{N+2} \frac{A_j}{t_1 - t_j} B_1, \quad \frac{\partial B_i}{\partial t_i} = \frac{A_j}{t_i - t_j} B_1 (i = 2, \ldots, N).$$

In the previous subsections, we have given specializations and Pfaff systems in order to derive rigid systems. Their origins can be clarified with the aid of this lemma. Furthermore, this lemma suggests that we can always give a particular solutions of a Painlevé system $H^{\tilde{m}}$ by a rigid system with a spectral type $m$.

**Appendix A. Hamiltonians of the six-dimensional Painlevé system**

In this section, we recall the Hamiltonian of the six-dimensional Painlevé systems which have been already derived in [7, 23, 13, 2].

**A.1. Spectral type 11, 11, 11, 11, 11, 11.**

$$H_i^{11,11,11,11,11,11} = H_{\Sigma i}(\pi_6 + 1, \pi_{i+1} + \pi_{i-1} + \pi_4, \pi_5, \pi_i, p_i; t_i) + q_i q_{i+1} p_{i+1} (2q_i p_i + q_{i+1} p_{i+1} + 2\pi_0 + \pi_6 + 1)$$

$$- \frac{1}{t_i - t_{i+1}} q_i q_{i+1} (t_i - 1)p_i^2 - 2t_i(t_i+1 - 1)p_i p_{i+1} + (t_i - 1)p_{i+1}^2)$$

$$+ \alpha_i \frac{t_i}{t_i - t_{i+1}} (t_i - 1) p_i - (t_i - 1) p_{i+1} - \alpha_i \frac{(t_i - 1)t_{i+1}}{t_i - t_{i+1}} q_i (p_i - p_{i+1})$$

$$+ q_i q_{i-1} p_{i-1} (2q_i p_i + q_{i-1} p_{i-1} + 2\pi_0 + \pi_6 + 1)$$

$$- \frac{1}{t_i - t_{i-1}} q_i q_{i-1} (t_i - 1)p_i^2 - 2t_i(t_{i-1} - 1)p_i p_{i-1} + (t_i - 1)p_{i-1}^2)$$

$$+ \alpha_i \frac{t_i}{t_i - t_{i-1}} (t_i - 1) p_i - (t_i - 1) p_{i-1} - \alpha_i \frac{(t_i - 1)t_{i-1}}{t_i - t_{i-1}} q_i (p_i - p_{i-1}),$$

26
for $i \in \mathbb{Z}/3\mathbb{Z}$, where $2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0$.

A.2. **Spectral type** 31, 31, 1111, 1111.

$$H^{31,31,1111,1111} = H_{\mathcal{V}1}(-\alpha_1 + \eta, \alpha_2 + \alpha_4 + \alpha_6, \alpha_0, \alpha_3 + \alpha_5 + \alpha_7 - \eta; q_i, p_i; t)$$

$$+ H_{\mathcal{V}1}(-\alpha_3 + \eta, \alpha_4 + \alpha_6, \alpha_0 + \alpha_2, \alpha_1 + \alpha_5 + \alpha_7 - \eta; q_i, p_i; t)$$

$$+ H_{\mathcal{V}1}(-\alpha_5 + \eta, \alpha_6, \alpha_0 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_7 - \eta; q_i, p_i; t)$$

$$+ (q_1 - 1)(q_2 - t)((q_1 p_1 + \alpha_1)p_2 + p_1(p_2 q_2 + \alpha_3))$$

$$+ (q_1 - 1)(q_3 - t)((q_1 p_1 + \alpha_1)p_3 + p_1(p_3 q_3 + \alpha_5))$$

$$+ (q_2 - 1)(q_3 - t)((q_2 p_2 + \alpha_3)p_3 + p_2(p_3 q_3 + \alpha_5)),$$

where $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = 1$.

A.3. **Spectral type** 33, 33, 33, 321.

$$H^{33,33,33,321} = \tr(Q(Q - 1)P(Q - t)P - (\alpha_1 - 1)Q(Q - 1)P$$

$$- \alpha_5 Q(Q - t)P - \alpha_4 (Q - 1)(Q - t)P + \alpha_2 (\alpha_0 + \alpha_2)Q)$$

where $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ and

$$P = -\frac{1}{t} \begin{pmatrix}
\frac{1}{3} p_1 + q_2^3 p_3 & \frac{2}{3} q_2 p_2 + (q_3 - q_1^3) p_3 + 2\alpha_2 + 2\alpha_5 & \frac{1}{3} (q_3 - q_1^3) p_2 + 2q_2^2 p_3 \\
\frac{1}{3} p_1 + (q_2 + q_1^2) p_3 & \frac{1}{3} q_2 p_2 + (q_3 - q_1^3) p_3 + \alpha_2 + \alpha_5 \\
\frac{1}{3} p_1 - (q_2 - q_1^3) p_3
\end{pmatrix},$$

$$Q = -t \begin{pmatrix}
q_1 & 2q_2 & q_3 - q_1^3 \\
1 & q_1 & q_2 \\
0 & 1 & q_1
\end{pmatrix}.$$

A.4. **Spectral type** 51, 33, 33, 111111.

$$H^{51,33,33,111111} = H_{\mathcal{V}1}(\alpha_0, \alpha_1, \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7, \alpha_3 + \alpha_5 + \alpha_8; q_1, p_1; t)$$

$$+ H_{\mathcal{V}1}(\alpha_0 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7, \alpha_5 + \alpha_8; q_2, p_2; t)$$

$$+ H_{\mathcal{V}1}(\alpha_0 + 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_5, 2\alpha_6 + \alpha_7, \alpha_8; q_3, p_3; t)$$

$$+ 2(q_1 - t)p_1 q_2((q_2 - 1)p_2 + \alpha_4) + 2(q_1 - t)p_1 q_3((q_3 - 1)p_3 + \alpha_6)$$

$$+ 2(q_2 - t)p_2 q_3((q_3 - 1)p_3 + \alpha_6),$$

where $\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 = 1$.

**Appendix B. Hamiltonians of the four-dimensional Painlevé system**

In this section, we recall the Hamiltonian of the four-dimensional Painlevé systems which have been classified by Sakai [17].
B.1. Spectral type 11, 11, 11, 11, 11.

\[ H_{11,11,11,11,11}^{11} = H_{V1}(\alpha_5 + 1, \alpha_{i+1} + \alpha_3, \alpha_4, \alpha_i; q_i, p_i; t_i) + q_1 q_2 p_{i+1} (2q_i p_i + q_{i+1} p_{i+1} + 2\alpha_0 + \alpha_5 + 1) \]
\[ - \frac{1}{t_i - t_{i+1}} q_1 q_2 [t_i(t_i - 1) p_i^2 - 2t_i(t_{i+1} - 1) p_i p_{i+1} + (t_i - 1) t_{i+1} p_{i+1}^2] \]
\[ + \alpha_i \frac{t_i}{t_i - t_{i+1}} q_{i+1} [(t_i - 1) p_i - (t_{i+1} - 1) p_{i+1}] - \alpha_{i+1} \frac{(t_i - 1) t_{i+1}}{t_i - t_{i+1}} q_i (p_i - p_{i+1}), \]

for \( i \in \mathbb{Z}/2\mathbb{Z} \), where \( 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0 \).

B.2. Spectral type 21, 21, 111, 111.

\[ H_{21,111,111}^{21} = H_{V1}(-\alpha_1 + \eta, \alpha_2, \alpha_0 + \alpha_4, \alpha_3 + \alpha_5 - \eta; q_1, p_1; t) \]
\[ + H_{V1}(-\alpha_5 + \eta, \alpha_0 + \alpha_2, \alpha_4, \alpha_1 + \alpha_3 - \eta; q_2, p_2; t) \]
\[ + (q_1 - t)(q_2 - 1)((q_1 p_1 + \alpha_1)p_2 + p_1(q_2 p_2 + \alpha_5)), \]

where \( \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1 \).

B.3. Spectral type 22, 22, 22, 211.

\[ H_{22,22,22,211}^{22} = \text{tr}(Q(Q - 1)P(Q - t)P - (\alpha_1 - 1)Q(Q - 1)P \]
\[ - \alpha_2 Q(Q - t)P - \alpha_4 (Q - 1)(Q - t)P + \alpha_2 (\alpha_0 + \alpha_2)Q), \]

where \( \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1 \) and

\[ P = \frac{1}{t} \begin{pmatrix} \frac{1}{2} p_1 & -p_2 \\ q_2 p_2 + \alpha_2 + \alpha_5 & \frac{1}{2} p_1 \end{pmatrix}, \quad Q = t \begin{pmatrix} -q_1 & -1 \\ q_2 & -q_1 \end{pmatrix}. \]

B.4. Spectral type 31, 22, 22, 1111.

\[ H_{31,22,22,1111}^{31} = H_{V1}(\alpha_0, \alpha_1, \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_3 + \alpha_6; q_1, p_1; t) \]
\[ + H_{V1}(\alpha_0 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_5, \alpha_6; q_2, p_2; t) \]
\[ + 2(q_1 - t) p_1 q_2 [(q_2 - 1)p_2 + \alpha_4], \]

where \( \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 = 1 \).

Acknowledgement

The author would like to express his gratitude to the collaborator in the previous work [2], Dr. Kenta Fuji, Mr. Keisuke Inoue and Mr. Keisuke Shinomiya. The author is also grateful to Professors Kazuki Hiroe, Hiroshi Kawakami, Hajime Nagoya, Masatoshi Noumi, Toshio Oshima, Hidetaka Sakai, Teruhisa Tsuda and Yasuhiko Yamada for valuable discussions and advices.
REFERENCES

[1] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, Math. Ann. 63 (1907) 301-321.

[2] K. Fuji, K. Inoue, K. Shinomiya and T. Suzuki, Higher order Painlevé system of type $D_{2n+2}^{(1)}$ and monodromy preserving deformation, J. Nonlinear Math. Phys. 20 (2013) 57-69.

[3] K. Fuji and T. Suzuki, Coupled Painlevé VI system with $E_{6}^{(1)}$-symmetry, J. Phys. A: Math. Theor. 42 (2009) 145205.

[4] K. Fuji and T. Suzuki, Higher order Painlevé system of type $D_{2n+2}^{(1)}$ arising from integrable hierarchy, Int. Math. Res. Not. 1 (2008) rnm129.

[5] K. Fuji and T. Suzuki, Drinfeld-Sokolov hierarchies of type $A$ and fourth order Painlevé systems, Funkcial. Ekvac. 53 (2010) 143-167.

[6] T. Suzuki and K. Fuji, Higher order Painlevé systems of type $A$, Drinfeld-Sokolov hierarchies and Fuchsian systems, RIMS Kokyuroku Bessatsu B30 (2012) 181-208.

[7] R. Garnier, Sur des équations différentielles du troisième ordre dont l’intégrale est uniform et sur une classe d’équations nouvelles d’ordre supérieur dont l’intégrale générale a ses point critiques fixés, Ann. Sci. École Norm. Sup. 29 (1912) 1-126.

[8] Y. Haraoka and G. M. Filipuk, Middle convolution and deformation for Fuchsian systems, J. Lond. Math. Soc. 76 (2007) 438-450.

[9] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé: A Modern Theory of Special Functions, Aspects of Mathematics E16 (Vieweg, 1991).

[10] M. Jimbo, T. Miwa, Y. Mori and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica D (1980) 80-158.

[11] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I, Physica 2D (1981) 306-352.

[12] N. M. Katz, Rigid Local Systems, Annals of Mathematics Studies 139 (Princeton University Press, 1995).

[13] H. Kawakami, Private communication.

[14] V. P. Kostov, The Deligne-Simpson problem for zero index of rigidity, Perspective in Complex Analysis, Differential Geometry and Mathematical Physics (World Scientific, 2001) 1-35.

[15] T. Oshima, Classification of Fuchsian systems and their connection problem, RIMS Kokyuroku Bessatsu B37 (2013) 163-192.

[16] T. Oshima, Fractional calculus of Weyl algebra and Fuchsian differential equations, MSJ Memoirs 28 (2012).

[17] H. Sakai, Isomonodromic deformation and 4-dimensional Painlevé type equations, UTMS 2010-17 (Univ. of Tokyo, 2010) 1-21.

[18] Y. Sasano, Higher order Painlevé equations of type $D_{1}^{(1)}$, RIMS Koukyuroku 1473 (2006) 143-163.

[19] C. T. Simpson, Products of Matrices, Canadian Math. Soc. Conference Proceedings 12 (AMS, 1991) 157-185.

[20] L. Schlesinger, Über eine classe von differentialsystemen beliebiger ordnung mit festen kritischen punkten, J. Reine Angew. Math. 141 (1912) 96-145.

[21] T. Suzuki, A class of higher order Painlevé systems arising from integrable hierarchies of type $A$, AMS Contemp. Math. 593 (2013) 125-141.

[22] T. Suzuki, A particular solution of a Painlevé system in terms of the hypergeometric function $n+1F_{n}$, SIGMA 6 (2010) 078.

[23] T. Tsuda, From KP/UC hierarchies to Painlevé equations, Int. J. Math. 23 (2012) 1250010.

[24] T. Tsuda, UC hierarchy and monodromy preserving deformation, J. Reine Angew. Math., in press.

[25] T. Tsuda, Hypergeometric solution of a certain polynomial Hamiltonian system of isomonodromy type, Quart. J. Math. 63 (2012) 489-505.

[26] T. Yokoyama, On an irreducibility condition for hypergeometric systems, Funkcial. Ekvac. 38 (1995) 11-19.

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