CHERN-MOSER OPERATORS AND WEIGHTED JET DETERMINATION PROBLEMS IN HIGHER CODIMENSION

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Abstract. Counterexamples to the 2−jet determination Chern-Moser Theorem in codimension \( d > 2 \) have recently been constructed [16]. We extend the Chern-Moser approach for hypersurfaces to real submanifolds of higher codimension in complex space to derive results on jet determination for their automorphism group. Using these techniques, we show that the 2−jet determination Chern-Moser Theorem holds in codimension 2.

1. Introduction

The local equivalence problem for real submanifolds in complex spaces is a very natural question, which was started on in complex dimension 2 by H. Poincaré and was then explored in the hypersurface case. Two real submanifolds \( M \) and \( M' \) are said to be locally equivalent at \( p \in M \) and \( p' \in M' \), respectively, if there exist neighborhoods \( V \) of \( p \) and \( V' \) of \( p' \), and a biholomorphic mapping \( F : V \to V' \) such that \( F(V \cap M) = V' \cap M' \).

Local equivalence of submanifolds is obviously a very restrictive condition. Symmetrically, if such a \( F \) exists, it is submitted to strong constraints, as is shown in the hypersurface case by the following classical statement due to Chern and Moser [10]: if \((M, p)\) and \((M', p')\) are smooth Levi nondegenerate real hypersurfaces, then biholomorphic germs of equivalence are uniquely determined by their jet of order 2 at point \( p \).

In higher codimension, finite jet determination problems also attracted much attention. We refer in particular to the contributions [25, 2, 3, 13, 21, 14, 15, 23] in the real analytic case, [11, 12, 17, 20] in the \( C^\infty \) case, [5, 6, 24] in the finitely smooth case.

Here we consider a generic submanifold \( M \subset \mathbb{C}^{n+d} \) of codimension \( d \), of finite type in the sense of Kohn and Bloom-Graham at a given point \( p \) [1, 8, 18]; this includes in particular the case of Levi nondegenerate hypersurfaces. We will view \( M \) as a \( C^r \)-smooth perturbation of the generic homogeneous submanifold \( M_H \) (called the model of \( M \), see Section 2 for details) associated to \( M \) and given by:

\[
\begin{align*}
M_H &= \{ \text{Im } w = P(z, \bar{z}, \text{Re } w) \}, \\
M &= \{ \text{Im } w = P(z, \bar{z}, \text{Re } w) + \ldots \},
\end{align*}
\]
where $P = (P_1, \ldots, P_d)$ with $P_j$ a (non-zero) weighted homogeneous polynomial of degree $m_j$ with no pluriharmonic terms, and the dots corresponding to $P_j$ being $C^\infty$-smooth functions whose derivatives of weighted order less or equal to $m_j$ vanish.

Our first goal was to recover the 2-jet determination theorem stated by Beloshapka in any codimension [4] since the original proof did not work, as we explained in [7]. Our strategy was to extend the theory of the generalized Chern-Moser operator [19, 20] to smooth generic submanifolds of higher codimension of finite type: indeed, the kernel of the generalized Chern-Moser operator reflects the link between the weighted grading of $\text{hol}(M_H,p)$, the Lie algebra of the real-analytic infinitesimal CR automorphisms at $p$ of the model $M_H$ of $M$, and the weighted jet determination problem for the stability group $\text{Aut}(M,p)$.

Using these techniques, we prove that the 2-jet determination Chern-Moser Theorem for hypersurfaces still holds in codimension 2. More precisely, we get

**Theorem 1.1.** Let $M \subset \mathbb{C}^{n+2}$ be a $C^3$-smooth generic submanifold of codimension $d = 2$ that is of finite type $m = (2)$ at $0 \in M$ with a nondegenerate Levi map at $0$. Then any $h = (z + f, w + g) \in \text{Aut}(M,0)$ is uniquely determined by the following partial derivatives

- the first complex tangential derivatives $\frac{\partial f_j}{\partial z_k}(0)$, $j, k = 1, \ldots, n$,
- the first and second order normal derivatives $\frac{\partial f_j}{\partial w_l}(0), \frac{\partial g_u}{\partial w_l}(0), \frac{\partial^2 g_u}{\partial w_j \partial w_l}(0), j = 1, \ldots, n, u, l, s = 1, 2$.

Note that codimension 2 is a specific case, since the same techniques actually led the second author to an example of a quadric of codimension 5 in $\mathbb{C}^9$ for which 4-jet determination (and not less) for biholomorphisms holds [22]. Later on, Jan Gregorovic and the second author [16] gave examples of quadrics with jet determination of arbitrarily high order in codimension $d > 2$.

Our main tool is given by the following theorem (see also Theorem 3.10 for a more explicit statement)

**Theorem 1.2.** Let $M_H$ be the generic homogeneous submanifold given by (1). Assume that $M_H$ is holomorphically nondegenerate. Then

1. There exists $k$ such that $\text{hol}(M_H,0)$ admits the weighted grading

   \[ \text{hol}(M_H,0) = \oplus_{\mu \leq k} G_{\mu}, \quad G_k \neq \{0\}. \]

2. For any sufficiently finitely smooth perturbation of $M_H$ given by (2), there exists a constant $k_0 \geq 1$ depending only on $M_H$ such that any $h \in \text{Aut}(M,0)$ is uniquely determined by its $k + k_0$ weighted jets at 0.
We should mention that part (1) in Theorem 1.2 is an immediate consequence of [2]. Also, an inspection of the proof of part (2), based on the Taylor expansion of the holomorphic map $h$, shows that Theorem 1.2 also holds for formal maps sending $M$ to $M$, where $M$ is a smooth generic submanifold (or formal submanifold) with holomorphically nondegenerate model $M_H$.

Theorem 1.1 then comes from the explicit computation of $k$ for which part (1) holds in Theorem 1.2. A crucial point is to use ”integrations” of a vector field (Lemma 4.7 and Definition 4.8). This notion was introduced in [19] for the hypersurface case and was the key point to get the counterexample given in [22].

The paper is organized as follows. In Section 2, we recall the notion of Bloom-Graham finite type and its basic properties. We also define the notions of model generic submanifold $M_H$ associated to $M$, and of weighted coordinates associated to $M_H$. In Section 3, we show how to reduce the study of the weighted jet determination problem for $\text{Aut}(M, p)$, the stability group of $M$, to the study of $\text{hol}(M_H, p)$, the set of real-analytic infinitesimal CR automorphisms of $M_H$ at $p$ (see Theorem 3.10). In Section 4, we introduce the notion of rigid vector fields and prove results regarding the jet determination problem for $\text{hol}(M_H, p)$ (see Proposition 4.4). In Section 5, we discuss the quadric model case $Q$ and use Theorem 3.10 to prove Theorem 1.1 by describing $\text{hol}(M_H, p)$ when $d = 2$.

2. Preliminaries

Usually, the study of a local CR equivalence problem begins with the choice of appropriate coordinates, merged into bunches according to their geometric contributions, each bunch being assigned a (numerical) weight. For instance, in the case of a Levi nondegenerate hypersurface, the complex normal direction is assigned the weight 2 while the complex tangential directions are assigned the weight 1 (see [10]). In the case of finite multitype in the sense of Catlin [9] at a given point, the complex normal direction is assigned the weight 1 while the complex tangential directions are assigned (possibly different) rational weights $\mu_j$, in order to study the generalized Chern-Moser operator, as done recently [10, 19, 20].

Let $M \subseteq \mathbb{C}^{n+d}$ be a smooth generic submanifold of real codimension $d > 1$ and $p \in M$ be a point of finite type $m = (m_1, \ldots, m_k)$, in the sense of Kohn and Bloom-Graham [1, 8, 18], where $m_1 < \cdots < m_k$ are the Hörmander numbers. We consider local holomorphic coordinates $(z, w)$ vanishing at $p$, where $z = (z_1, z_2, \ldots, z_n)$ and $z_j = x_j + iy_j, w \in \mathbb{C}^d$ and $w = u + iv$. Assuming that the tangent space to $M$ at 0 is given by $\{v = 0\}$, $M$ is described near 0 as the graph of a uniquely determined real vector valued function

$$v = \psi(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, u), \quad d\psi(0) = 0.$$
Writing \( w = (w_{m_1}, \ldots, w_{m_k}) \), where \( w_{m_j} \) are vectors of length \( l_j \) (such that \( d = \sum_{j=1}^{k} l_j m_j \)), we may assume that this \( d \)-dimensional equation is actually

\[
M : \begin{cases}
v_{m_1} = \psi_{m_1}(z, \bar{z}, u) = P_{m_1}(z, \bar{z}) + \ldots \\
v_{m_2} = \psi_{m_2}(z, \bar{z}, u) = P_{m_2}(z, \bar{z}, u_{m_1}) + \ldots \\
& \vdots \\
v_{m_k} = \psi_{m_k}(z, \bar{z}, u) = P_{m_k}(z, \bar{z}, u_{m_1}, \ldots, u_{m_{k-1}}) + \ldots 
\end{cases}
\]

where \( P_{m_j}(z, \bar{z}, u_{m_1}, \ldots, u_{m_{j-1}}) \) are real vector valued polynomials of length \( l_j \) satisfying the following conditions of normalization

- Each one of the \( l_j \) components of \( P_{m_j} \) is a homogeneous polynomial of degree \( m_j \), that is,

\[
P_{m_j}(t z, t \bar{z}, t^m_1 u_{m_1}, \ldots, t^{m_{j-1}} u_{m_{j-1}}) \equiv t^{m_j} P_{m_j}(z, \bar{z}, u_{m_1}, \ldots, u_{m_{j-1}});
\]

- \( P_{m_j}(z, 0, u_{m_1}, \ldots, u_{m_{j-1}}) \equiv 0 \);

- There are no terms of the form \( u_{m_k}^{\alpha_k} \ldots u_{m_{j-1}}^{\alpha_{j-1}} P_{m_k} \) in \( P_{m_j} \) for \( k < j \) (see condition (6.2.6) of Theorem (6.2) in [8]), and the dots terms are sums of monomials of order strictly bigger than \( m_j \) in the formal Taylor expansion of \( \psi_{m_j} \).

Coordinates which provide such a description will be called **standard coordinates**, and \( M \) given by (4) is said to be written in **standard form**.

We assign natural weights to the variables: the tangential variables \( z_1, \ldots, z_n \) are given weight \( \frac{1}{m_1} \) while the component variables of \( w_{m_j} \) are given weight \( \frac{m_j}{m_1} \).

**Definition 2.1.** The weighted degree \( \kappa \) of a monomial

\[
q(z, \bar{z}, \ldots, u_{m_1}, \ldots, u_{m_k}) = c_{\alpha \beta \lambda} z^\alpha \bar{z}^\beta u_{m_1}^\lambda_1 \ldots u_{m_k}^\lambda_k
\]

is defined as

\[
\kappa := \sum_{j=1}^{k} |\lambda_j| \frac{m_j}{m_1} + \frac{1}{m_1} \sum_{i=1}^{n} (\alpha_i + \beta_i).
\]

We obtain then the notion of weighted homogeneous polynomial

**Definition 2.2.** A polynomial \( Q(z, \bar{z}, u) \) is weighted homogeneous of weighted degree \( \kappa \) if it is a sum of monomials of weighted degree \( \kappa \).

**Remark 2.3.** Note that according to this definition, \( P_{m_j} \) is a vector valued weighted homogeneous polynomial of weighted degree \( \frac{m_j}{m_1} \), while the dots terms are made of weighted degree bigger than \( \frac{m_j}{m_1} \).

**Definition 2.4.** In this setting, we say that the generic submanifold of codimension \( d \) given by

\[
M_H = \{(z, w) \in \mathbb{C}^{n+d} \mid v_{m_j} = P_{m_j}(z, \bar{z}, u_{m_1}, \ldots, u_{m_{j-1}}), j = 1, \ldots, k \}
\]

is the model submanifold of \( M \) associated to the standard form (4).
Note that standard coordinates are not unique. For instance, in the case of \( m = (m_1) \), all models are equivalent by a linear action as it is shown in the next section.

### 3. THE BASIC IDENTITIES

In this section, \( M \) is assumed to be given by (4), with \( M_H \) the associated model submanifold. We follow the same approach as in \([19]\) and \([20]\), where the hypersurface case is analyzed.

**Definition 3.1.** We denote by \( \text{Aut}(M,0) \) the set of germs at 0 of biholomorphisms mapping \( M \) into itself and fixing 0.

**Lemma 3.2.** Let \( h \in \text{Aut}(M,0) \). Then \( h \) is of the form
\[
\begin{align*}
    z' &= z + f(z, w) \\
    w_{m_j}' &= w_{m_j} + g_{m_j}(z, w),
\end{align*}
\]
where \( g_{m_j}(z, w) \) (resp. \( f(z, w) \)) is a sum of terms of weighted degree bigger or equal to \( \frac{m_j}{m_1} \) (resp. bigger or equal to \( \frac{1}{m_1} \)).

**Proof.** The statement is obvious for \( f(z, w) \) and \( g_{m_1}(z, w) \). Indeed, using (4), we have
\[
v_{m_1} = \psi_{m_1}(z, \bar{z}, u) = P_{m_1}(z, \bar{z}) + \ldots.
\]
Therefore, we obtain
\[
g_{m_1}(z, w) - \overline{g_{m_1}(z, w)} = 2iP_{m_1}(f, \bar{f}) + \ldots.
\]
Hence, \( g_{m_1}(z, w) \) contains no term of weight less than one.

Suppose now that the statement is true for \( g_{m_k}(z, w) \), \( k < j \), and suppose by contradiction that there is in \( g_{m_j}(z, w) \) a term of minimal order
\[
a_{\alpha}(z)w_{m_i}^{\alpha_i} \ldots w_{m_k}^{\alpha_k}
\]
of weighted degree less than \( \frac{m_j}{m_1} \). If \( a_{\alpha}(0) = 0 \), this leads to a contradiction since one gets a term of the form
\[
a_{\alpha}(z)u_{m_i}^{\alpha_i} \ldots u_{m_k}^{\alpha_k}
\]
of weighted degree less than \( \frac{m_j}{m_1} \), which is not possible using the conditions of normalization and the induction. If \( a_{\alpha}(0) \neq 0 \), then either we obtain a term of the form
\[
a_{\alpha}(0)u_{m_i}^{\alpha_i} \ldots u_{m_k}^{\alpha_k}
\]
or a term of the form
\[
a_{\alpha}(0)ciu_{m_i}^{\alpha_i - 1} \ldots u_{m_k}^{\alpha_k}P_{m_i},
\]
where \( c \) is a real constant. But this cannot cancel with any other term, using the conditions of normalization of \( P_{m_s}, s = 1 \ldots k \).

\( \square \)
Definition 3.3. We denote by $\text{hol}(M, 0)$ the set of germs of real-analytic infinitesimal CR automorphisms of $M$ at 0.

Remark 3.4. (II) Recall that $X \in \text{hol}(M_H, 0)$ if and only if there exists a germ $Z$ at 0 of a holomorphic vector field in $\mathbb{C}^{n+d}$ such that $\text{Re}Z$ is tangent to $M_H$ and $X = \text{Re}Z|_{M_H}$. By abuse of notation, we also say that $Z \in \text{hol}(M_H, 0)$.

We decompose the formal Taylor expansion of $\psi_{m_j}$, denoted by $\Psi_{m_j}$, into weighted homogeneous polynomials $\Psi_{m_j, \nu}$ of weighted degree $\nu$,

$$\Psi_{m_j} = \sum \Psi_{m_j, \nu}.$$ 

Let $h = (z_j', w') \in \text{Aut}(M, 0)$ given by (6).

Putting $f = (f_1, \ldots, f_n)$, and $g = (g_{m_1}, \ldots, g_{m_k})$, we consider the mapping given by

$$(7) \quad T = (f, g),$$

and, again, decompose each power series $f_j$ and $g_{m_j}$ into weighted homogeneous polynomials $f_{j, \mu}$ and $g_{m_j, \mu}$ of weighted degree $\mu$,

$$f_j = \sum f_{j, \mu}, \quad g_j = \sum g_{m_j, \mu}.$$ 

Since $h \in \text{Aut}(M, 0)$, substituting (6) into $v' = \psi(z', \bar{z}', u')$ we obtain the transformation formula

$$(8) \quad \psi(z + f(z, u + i\psi(z, \bar{z}, u)), z + f(z, u + i\psi(z, \bar{z}, u)), u + \text{Re} g(z, u + i\psi(z, \bar{z}, u)) = \psi(z, \bar{z}, u) + \text{Im} g(z, u + i\psi(z, \bar{z}, u)).$$

Expanding (8) we consider terms of weight $\mu > 1$. We get

$$(9) \quad 2\text{Re} \sum_{j=1}^{n} P_{m_1, z_j}(z, \bar{z}) f_{j, \mu-1 + \frac{1}{m_1}}(z, u + iP(z, \bar{z})) +$$

$$(9) \quad 2\text{Re} \sum_{j=1}^{k-1} P_{m_1, w_j}(z, \bar{z}) g_{m_j, \mu-1 + \frac{m_j}{m_1}}(z, u + iP(z, \bar{z})) =$$

$$= \text{Im} g_{m_1, \mu-1 + \frac{m_1}{m_1}}(z, u + iP(z, \bar{z})) + \ldots$$

where dots denote terms depending on $f_j, \nu - 1 + \frac{1}{m_1}$, $g_{m_j, \nu - 1 + \frac{m_j}{m_1}}, \psi_{\nu}$, for $\nu < \mu$.

Proposition 3.5. Let $h = (z + f, w + g) \in \text{Aut}(M, 0)$ be given by (6). Let

$$(f, g) = \sum_{\mu} (f, g)_{\mu}$$

where

$$(f, g)_{\mu} = (f_{\mu-1 + \frac{1}{m_1}}, g_{m_1, \mu-1 + \frac{m_1}{m_1}}, \ldots, g_{m_k, \mu-1 + \frac{m_k}{m_1}}),$$

and $\mu_0$ be minimal such that $(f, g)_{\mu_0} \neq 0$. 
If $\mu_0 > 1$, the (non trivial) vector field
\[(10)\quad Y = \sum_{j=1}^{n} f_j, \mu_0-1+\frac{1}{m_1} \frac{\partial}{\partial z_j} + \sum_{j=1}^{k} g_m, \mu_0-1+\frac{m_j}{m_1} \cdot \frac{\partial}{\partial w_m}\]
lies in $\text{hol}(M_H,0)$, where $M_H$ is given by (5).

Here the notation $\frac{\partial}{\partial w_m}$ stands for the $l_j$-dimensional vector of the corresponding partial derivatives, and $g_m \cdot \frac{\partial}{\partial w_m}$ for the usual dot product.

**Proof.** Using (9) and the definition of $\mu_0$, we obtain
\[(11)\quad 2\text{Re} \sum_{j=1}^{n} P_{m_j, z_j}(z, \bar{z}) f_j, \mu_0-1+\frac{1}{m_1} (z, u+P(z, \bar{z}))+
\]
\[(12)\quad 2\text{Re} \sum_{j=1}^{k-1} P_{m_j, w_j}(z, \bar{z}) g_m, \mu_0-1+\frac{m_j}{m_1} (z, u+P(z, \bar{z})) =
\]
\[= \text{Im} g_m, \mu_0-1+\frac{m_j}{m_1} (z, u+iP(z, \bar{z})).\]

Applying $Y$ to $v - P$ and using (11), we obtain
\[(13)\quad \text{Re} Y(v_m - P_m)|_{M_H} =
\]
\[-\text{Re} \sum_{j=1}^{n} P_{m_j, z_j}(z, \bar{z}) f_j, \mu_0-1+\frac{1}{m_1} (z, u+iP(z, \bar{z}))+
\]
\[-\text{Re} \sum_{j=1}^{l-1} P_{m_j, w_j}(z, \bar{z}) g_m, \mu_0-1+\frac{m_j}{m_1} (z, u+iP(z, \bar{z}))-\]
\[+\frac{1}{2} \text{Im} g_m, \mu_0-1+\frac{m_j}{m_1} (z, u+iP(z, \bar{z})) = 0.\]

**Definition 3.6.** We say that the vector field
\[Y = \sum_{j=1}^{n} F_j(z, w) \frac{\partial}{\partial z_j} + \sum_{j=1}^{k} G_m(z, w) \cdot \frac{\partial}{\partial w_m}\]
has homogeneous weight $\mu \geq -\frac{m_k}{m_1}$ if $F_j$ is a weighted homogeneous polynomial of weighted degree $\mu + \frac{1}{m_1}$, and $G_j$ is a homogeneous polynomial of weighted degree $\mu + \frac{m_j}{m_1}$.

**Remark 3.7.** We write
\[(14)\quad \text{hol}(M_H,0) = \oplus_{\mu \geq -\frac{m_k}{m_1}} G_\mu,\]
where $G_\mu$ consists of weighted homogeneous vector fields of weight $\mu$. Note that each weighted homogeneous component $X_\mu$ of $X$ is in $\text{hol}(M_H, 0)$ if $X \in \text{hol}(M_H, 0)$.

**Example 3.8.** The vector fields $W_{m_k,j}, j = 1, \ldots, l_k$, given by

\[(15) \quad W_{m_k,j} = \frac{\partial}{\partial w_{m_k,j}}\]

lie in $G_{- \frac{m_k}{m_1}}$.

**Example 3.9.** The vector field defined by

\[(16) \quad E = \frac{1}{m_1} \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{k} \frac{m_j}{m_1} w_{m_j} \cdot \frac{\partial}{\partial w_{m_j}}\]

lies in $G_0$.

**Theorem 3.10.** Let $M \subset \mathbb{C}^{n+d}$ be a smooth generic submanifold of codimension $d$ that is of finite type at $0$ given by (4). Let $M_H$ be the model hypersurface given by (5). Let $\mu_0(> \frac{m_k}{m_1})$ such that

\[(17) \quad \text{hol}(M_H, 0) = \bigoplus_{- \frac{m_k}{m_1} \leq \mu < \mu_0 - \frac{m_k}{m_1}} G_\mu\]

Then any $h = (z + f, w + g) \in \text{Aut}(M, 0)$ given by (6) such that $(f, g)_\mu = 0$ for $\mu < \mu_0$ is the identity map.

**Proof.** Using Examples 3.8 and 3.9, we see that $- \frac{m_k}{m_1} \leq \mu < \mu_0 - \frac{m_k}{m_1}$, with $\mu_0 > \frac{m_k}{m_1}$. Then we apply Proposition 3.5. \qed

**Proof of Theorem (1.2)** An inspection of the proof of Proposition 3.5 shows that the conclusion of Theorem 3.10 holds if $M$ is assumed to be of class $C^{m_k+1}$.

**Remark 3.11.** If $m = (2)$, the conclusion of Theorem 3.10 holds if $M$ is assumed to be of class $C^3$.

4. The components $G_\mu$.

**Definition 4.1.** We denote by $G_\mu^R$ the set of vector fields in $G_\mu$ that are rigid, that is, whose coefficients depend only on $z$.

**Remark 4.2.** Note that $W_{m_k,j}$ are rigid, while $E$ is not.

We recall the following definition

**Definition 4.3.** A real-analytic submanifold $M \subset \mathbb{C}^N$ is holomorphically nondegenerate at $p \in M$ if there is no germ at $p$ of a holomorphic vector field $X$ tangent to $M$. 

Proposition 4.4. Let $M \subset \mathbb{C}^{n+d}$ be a smooth generic submanifold of codimension $d$ that is of finite type at 0, written in standard form. If the associated model $M_H$ is holomorphically non degenerate, then $G_{\mu}^R = \{0\}$ for $\mu \geq \frac{m_k - 1}{m_1}$.

Proof. Let $X \in G_{\mu}^R$, $\mu \geq \frac{m_k - 1}{m_1}$, be given by

$$X = \sum_{j=1}^{n} f_j(z) \frac{\partial}{\partial z_j} + \sum_{l=1}^{k} g_{m_l}(z) \frac{\partial}{\partial w_{m_l}}. \tag{18}$$

We prove that $X$ itself is complex tangent to $M_H$. First, using (4) and in particular the fact that no $P_{m_r}$ contains pluriharmonic terms, we obtain that $g_{m_r} = 0$.

Then we get by assumption that for any $r = 1, \ldots, k$,

$$\left(\text{Re} \sum_{j=1}^{n} f_j(z) \frac{\partial}{\partial z_j}\right) \left(v_{m_r} - P_{m_r}(z, \bar{z}, u)\right) = 0. \tag{19}$$

By the reality of $P_{m_r}$, we may rewrite (19) as

$$\text{Re} \left(\sum_{j=1}^{n} f_j(z) \frac{\partial P_{m_r}}{\partial z_j}(z, \bar{z}, u)\right) = 0. \tag{20}$$

Write

$$\sum_{j=1}^{n} f_j(z) \frac{\partial P_{m_r}}{\partial z_j}(z, \bar{z}, u) = \sum_{\alpha, \hat{\alpha}} B_{\alpha \hat{\alpha}, r} z^\alpha \bar{z}^{\hat{\alpha}} u^s, \tag{21}$$

Using (20), we obtain

$$B_{\alpha \hat{\alpha}, r} = -\overline{B_{\alpha \hat{\alpha}, r}}. \tag{22}$$

On the other hand, since $P_{m_r}$ is of weighted degree $\frac{m_r}{m_1}$, we have

$$\text{weight}(\frac{\partial P_{m_r}}{\partial z_j}) = \frac{m_r}{m_1} - \frac{1}{m_1}. \tag{23}$$

First we claim that $B_{\alpha \hat{\alpha}, r}$ are zero for all $\alpha, \hat{\alpha}, s, r$. By contradiction, assume there is $\alpha, \hat{\alpha}, s, r$, with $B_{\alpha \hat{\alpha}, r} \neq 0$. By assumption, $|\alpha| \geq \frac{m_r}{m_1}$ whereas $|\hat{\alpha}| < \frac{m_r}{m_1}$, using (23). On the other hand, by (22), we obtain that there exists a nonzero term with weight in $z$ less than $\frac{m_r}{m_1}$, and in $\bar{z}$ greater than or equal $\frac{m_r}{m_1}$. That gives a contradiction, hence all $B_{\alpha \hat{\alpha}, r}$ are zero. Therefore we obtain that $X$ is complex tangent to $M_H$, and since $M_H$ is holomorphically nondegenerate, $X = 0$. $\square$
Corollary 4.5. Proposition 4.4 yields

Let \( g \) be holomorphically non degenerate, then there exist \( N_1, N_2, N_3 \) such that \( g \) is uniquely determined by the following set of derivatives

\[
\{ \frac{\partial^{\alpha} f}{\partial z^{\alpha}}, \frac{\partial^{\beta+\gamma} g}{\partial z^{\beta}\partial w^{\gamma}}, |\alpha| \leq N_1, |\beta| \leq N_2, 0 < |\gamma| \leq N_3 \}.
\]

Proposition 4.3 yields

**Corollary 4.5.** Let \( M \subset \mathbb{C}^{n+d} \) be a smooth generic submanifold of codimension \( d \) that is of finite type at 0 written in standard form, such that the associated model \( M_H \) is holomorphically non degenerate. Let \( h = (z + f, w + g) \in Aut(M, 0), N_1, N_2, N_3 \) as above. Then \( N_1 \leq m_k - 1 \).

**Remark 4.6.** Note that if \( m = (m_1) \),

\[ \{W_{m_1,j}, j = 1, \ldots, d\} = G_{-1}^R = G_{-1} \]

We have the following lemma whose easy proof is left to the reader.

**Lemma 4.7.** Let \( M \subset \mathbb{C}^{n+d} \) be a smooth generic submanifold of codimension \( d \) that is of finite type at 0 with \( m = (m_1) \), written in standard form. Let \( Y \in G_{\mu} \setminus G_{\mu}^R \) and let \( \{W_{m_1,j}, j = 1, \ldots, d\} \) be given by (15). For every \( 1 \leq j \leq d \), there exist an integer \( k_j \geq 0 \) and a vector field denoted by \( D^{(k_j)}(Y) \in hol(M_H, 0) \), \( (D^{0}(Y) = Y) \), whose coefficients do not depend on \( w_{m_1,j} \) such that \([\ldots[[Y;W_{m_1,j}];W_{m_1,j}];\ldots];W_{m_1,j}] = D^{(k_j)}(Y) \), where the string of brackets is of length \( k_j \).

This leads to the following definition in the case \( m = (m_1) \).

**Definition 4.8.** Let \( X \in G_{\mu}^R \), and \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d \). We say that \( Y \in G_{\mu + \sum_{j=1}^{d} k_j} \) is a \( \kappa \)-integration of \( X \) if

\[
D^{(\kappa_1)}(\ldots D^{(\kappa_d)}(Y) \ldots) = X.
\]

We denote \( Y \) by \( D^{-(\kappa)}(X) \).

By abuse of notation, we will also refer to \( D^{-k}(X) \) where \( k = |\kappa| = \sum_{i=1}^{d} \kappa_i \), since the resulting integrated vector fields will be treated similarly.

5. The Components \( G_{\mu} \) for the Quadric.

We wish to discuss Theorem 3.10 in the case of a smooth generic submanifold of codimension \( d \) that is of finite type \( m = (2) \). Once written in standard form, we get that the model submanifold is a quadric \( Q \), that is,

\[
Q = \left\{ \begin{array}{l}
v_1 = t \bar{z} A_1 z, \\
v_2 = t \bar{z} A_2 z, \\
\ldots, \\
v_j = t \bar{z} A_j z, \\
\ldots, \\
v_d = t \bar{z} A_d z
\end{array} \right.
\]
with \( A_j, j = 1, \ldots, d \) being linearly independent Hermitian matrices.

**Remark 5.1.** In coordinates, \( A = (A_1, \ldots, A_d) \) corresponds to the Levi map of \( M \) at 0. The linear independence of the \( A_i \) is actually equivalent to being of finite type \( m = 2 \) at 0 for \( M \subset \mathbb{C}^{n+d} \) a smooth generic submanifold of codimension \( d \).

The properties of the standard form also give the following lemma.

**Lemma 5.2.** Let \( M \subset \mathbb{C}^{n+d} \) be a smooth generic submanifold of codimension \( d \) that is of finite type at \( 0 \), written in standard form with its model quadric given by (26). Then
\[
\cap \ker (A_j) = \{0\}
\]
if and only if there is no holomorphic tangent vector field to \( Q \).

A direct application of Proposition 4.4 yields

**Corollary 5.3.** If the conditions in Lemma 5.2 are satisfied, then \( G_\mu \mathcal{R} = \{0\} \) for \( \mu > 0 \).

**Remark 5.4.** Note that \( G_0 \mathcal{R} \neq \{0\} \), since the vector field
\[
\sum_{j=1}^n iz_j \frac{\partial}{\partial z_j} \in G_0 \mathcal{R}.
\]

**Remark 5.5.** We also get that \( \mathcal{D}^{-1}(G_0 \mathcal{R}) \neq \{0\} \) can only happen if the codimension \( d \) is bigger than 2, since for \( d = 1 \) Chern Moser’s Theorem [10] shows that the mixed derivatives \( \frac{\partial^2 f_j}{\partial w \partial z_k}, j, k = 1, \ldots, n \), are not needed. And it actually happens, as in the following example.

**Example 5.6.** Let \( M \) be given by
\[
M = \{(z_1, z_2, z_3, z_4, w_1, w_2, w_3) \in \mathbb{C}^7 | v_1 = z_3 \bar{z}_3, v_2 = z_4 \bar{z}_4, v_3 = z_1 \bar{z}_3 + z_3 \bar{z}_1 + z_2 \bar{z}_4 + z_4 \bar{z}_2 \}\.
\]
The associated matrices are
\[
A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

Let \( X \) and \( Y \) be the vector fields
\[
X = iz_3 \frac{\partial}{\partial z_1}, \quad Y = -iz_4 \frac{\partial}{\partial z_2}
\]
It is easy to check that
\[ w_1Y + w_2X \in \mathcal{D}^{-1}(G_0^R) \]

**Theorem 5.7.** Let \( M \subset \mathbb{C}^{n+2} \) be a smooth generic submanifold of codimension \( d = 2 \) that is of finite type \( m = (2) \) at 0, written in standard form. Assume the associated model quadric \( Q \) is holomorphically non degenerate, then

- (i) \( \mathcal{D}^{-2}(G_0^R) \backslash (\mathcal{D}^{-1}(G_0^R) \cap \mathcal{D}^{-2}(G_{-1}^R)) = \{0\} \)
- (ii) \( \mathcal{D}^{-2}(G_{-1}^R) = \{0\} \)
- (iii) \( \mathcal{D}^{-1}(G_{-1}^R) \neq 0, \mathcal{D}^{-3}(G_{-1}^R) = \{0\} \)

Before proving Theorem 5.7, let us explain how it leads to theorem 1.1, providing a generalization of the 2-jet determination Chern-Moser Theorem in the case of codimension 2.

According to Lemma 4.7, we need to study the \( \kappa \)-integrations of any rigid vector field. Theorem 5.7 provides the precise \( \kappa \)-integrations needed, and shows that at most 2-integrations are needed, depending on the rigid vector field. Using Theorem 3.10 and Remark 3.11 we then conclude that any \( h = (z + f, w + g) \in \text{Aut}(M, 0) \) is uniquely determined by the following partial derivatives

- the first complex tangential derivatives \( \frac{\partial f_j}{\partial z_k}(0), j, k = 1, \ldots, n \), corresponding to (i) in Theorem 5.7
- the first and second order normal derivatives \( \frac{\partial f_j}{\partial w_l}(0), \frac{\partial g_u}{\partial w_l}(0), \frac{\partial^2 g_u}{\partial w_s \partial w_l}(0), j = 1, \ldots, n, u, l, s = 1, 2 \)

**Proof.** In this special case of codimension \( d = 2 \) with \( m = (2) \), we refer to (26) by setting \( Q = (Q_1, Q_2) \) where \( Q_i = t \bar{z} A_i z (i = 1, 2) \), instead of \( P_{m_1}, P_{m_2} \).

- **Proof of (i)** - Suppose by contradiction that there exist \( X, Y \in G_0^R \) such that
  \[ w_1X + w_2Y \in \mathcal{D}^{-1}(G_0^R) \]

Without loss of generality, we may assume that \( [X, Y] = 0 \), since otherwise, we have \( w_2[X, Y] \in \mathcal{D}^{-1}(G_0^R) \), which is not possible unless \( [X, Y] = 0 \). Then, by assumption, we obtain the following equation

\[ Q_1X(Q) + Q_2Y(Q) = 0. \]

Using the fact that \( M \) is of finite type \( m = (2) \), (30) leads to

\[ X(Q) = \alpha Q_2, \ Y(Q) = \beta Q_1, \]

\( \alpha \) and \( \beta \) complex valued vectors.

Using the fact that \( [X, Y] = 0 \), and the fact that the model quadric \( Q \) is holomorphically nondegenerate, we obtain that \( X = Y = 0 \).
• Proof of (ii) - Let \( Z_1 \in G_{-\frac{1}{2}}^R \) be of the form

\[
Z_1 = a_1 \frac{\partial}{\partial z_1} + a_2 \frac{\partial}{\partial z_2} + b_1(z) \frac{\partial}{\partial w_1} + b_2(z) \frac{\partial}{\partial w_2},
\]

with \( a_i \in \mathbb{C} \), and \( b_i(z) \) linear. If \( Z_1 \) integrates, we obtain, after a possible permutation of the variables \( w_1 \) and \( w_2 \), an equation of the form

\[
Q_1 \text{Im} \, Z_1(Q) + Q_2 \text{Im} \, Z_2(Q) + \text{Re} X_1(Q) = 0,
\]

where \( Z_2 \in G_{-\frac{1}{2}}^R \) and \( X_1 \) is a vector field of weight \( \frac{1}{2} \). If \( D^{-2}(Z_1) \) exists, then we obtain the following system

\[
\begin{align*}
Q_1 \text{Im} \, X_1(Q) + Q_2 \text{Im} \, X_2(Q) &= 0 \\
Q_1 \text{Im} \, X_2(Q) + Q_2 \text{Im} \, X_3(Q) &= 0,
\end{align*}
\]

where \( X_j \) are vector fields of weight \( \frac{1}{2} \). It is not hard to see that since \( Q_1 \) and \( Q_2 \) are linearly independent Hermitian forms, the only solution to the system \( (34) \) is the trivial solution. Hence, using Corollary 5.3 and \( (33) \), the following system of equations holds

\[
\begin{align*}
Q_1 \, Z_1(Q) + Q_2 \, Z_2(Q) &= 0 \\
Q_1 \, Z_2(Q) + Q_2 \, Z_3(Q) &= 0,
\end{align*}
\]

where \( Z_3 \in G_{-\frac{1}{2}}^R \). Using \( (35) \), we conclude that \( Z_1(Q) = 0 \), and hence \( Z_1 = 0 \), which gives the contradiction.

• Proof of (iii) - By integrating \( W_1 = \frac{\partial}{\partial w_1} \), we obtain an equation of the form

\[
A Q + \text{Re} X(Q) = 0,
\]

where \( A \) is a nonzero \( 2 \times 2 \) real matrix, and \( X \) is a vector field of weight 0. Using the Euler field, we conclude that \( (36) \) holds (with \( A = -I \)). Hence \( D^{-1}(G_{-1}^R) \neq 0 \). If \( D^{-3}(G_{-1}^R) \neq 0 \), we obtain a nontrivial equation of the form

\[
\sum c_{\alpha_1 \alpha_2} Q_1^{\alpha_1} Q_2^{\alpha_2} = 0.
\]

But \( (37) \) is not possible by assumption of finiteness. Indeed, \( (37) \) would imply \( Q_1 = \alpha Q_2 \), which contradicts that \( M \) is of finite type \( m = (2) \).

\[\Box\]
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