Completely Uncoupled User Association Algorithms for State Dependent Networks

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**Abstract**—We study a distributed user association algorithm for a heterogeneous wireless network with the objective of maximizing the sum of the utilities (on the received throughput) of wireless users. We consider a state-dependent wireless network, where the rate achieved by the users is a function of their user associations as well as the state of the system. We consider four different scenarios depending on the state evolution and the users’ knowledge of the system state. In this context, we present completely uncoupled user association algorithms for utility maximization, where the users’ association is entirely a function of its past associations and its received throughput. In particular, the user is oblivious to the association of the other users in the network. Using the theory of perturbed Markov chains, we show the optimality of our algorithms under appropriate scenarios.

**Index Terms**—Completely uncoupled, Distributed Resource allocation, Heterogenous Network, State Dependent Networks, User Association.

**I. INTRODUCTION**

In the present wireless scenario, a mobile user has the luxury to choose between several Access Points (APs), that are possibly enabled with different access technologies (e.g., WiFi, WiMAX, LTE, etc). The APs could be further heterogeneous in terms of their size (e.g., cellular, femto-cell, WiFi AP), and could be owned by different service providers. Thus, it may not be possible to expect a centralized coordination among different APs.

In the above context, we are interested in designing distributed user association algorithms with the objective of optimizing system utilization. A key challenge in such design is the unavailability of information with each user regarding other users’ behaviour (in terms of their association and the utilities they receive). Further, due to privacy concerns (since different service providers are involved), the APs may be reluctant to share some system level parameters (e.g., their transmit powers, pricing strategy, etc.) with the users. Thus, user association algorithms are expected to be **completely uncoupled** [2], whereby a user’s association-decision is entirely based on its past decisions and the utilities it received in the past. In this paper, we design such a completely uncoupled user association algorithm for a state based system, comprising of finite number of states, BSs, and users.

In a state based network, the throughput or pay-off received by a user depends on the system state, in addition to the users’ association choices. The state could represent any background process, which the users do not have any control over. The following examples illustrate the need for state based model considered in this paper.

1) Delayed pay-off: The pay-off received by a user is delayed by a fixed unknown time. In this case, the state represents a moving window of previous associations.

2) Network Response: In the user association problem, the base station could employ channel selection or power control algorithms. These algorithms could be a function of the association choices. Here, the state is modelled as an independent random process depending on the action. In a more interesting case, these algorithms could depend on the previous state in addition to the current association choices. In this case, the state is modelled as a controlled Markov process, where the control action correspond to the association choices.

3) Wireless Channel: The state could represent fading in wireless channel. In typical wireless network, fading is generally modelled as an ergodic random process [3].

In this paper, we consider four cases depending on user’s knowledge of the state and state transition model.

We start with the case where the state is unknown to the users. When the state is unknown, we consider two state transition models. First, we assume that state transition is a deterministic function of the prior state and the current association vector. Second, we consider the case where the state is an iid random variable depending on the association vector chosen. We then consider the case where the state is known to users. For the case where state is known, we first consider a more general controlled Markov state transition model. Finally, we consider the case with ergodic state transitions. A formal description of the considered system model will be presented in Section [II]. Before proceeding further, we end this section with a brief survey of related literature.

**Related Work:** Utility maximization is known to achieve notions of fairness [4]. For example, log utility is known to achieve proportional fairness [4]. In [5], Kushner and Whiting showed the convergence of gradient algorithms in a time varying environment. The above algorithms are centralized in nature i.e., they require information about all the nodes in the network.
In [6], Jiang and Walrand proposed CSMA based distributed scheduling algorithms for a conflict graph model, for which proof of optimality was shown in [7]. In [8] Kauffmann et al., proposed distributed channel selection and user association algorithms for IEEE 802.11 networks using Gibbs sampler for some tailored utilities. In [9], Borst et al., showed that maximizing utilities using Gibbs sampling require two hop information, if the utility depends on one hop neighbours.

Uncoupled learning algorithms were popularized by Young in the context of coordination games, in his seminal work [10]. Over the years, several variants of the algorithm in [10] have been studied. For instance, Pradelski and Young [11] proposed an algorithm for achieving efficient Nash equilibrium in general n person games satisfying interdependence property, while the problem of obtaining pareto optimal solution has been considered by Marden et al., in [2]. Also, in [12], Borowski and Marden proposed a completely uncoupled algorithm for achieving efficient correlated equilibrium under interdependence assumption. Algorithms for state space based potential games have been studied in [13]. In contrast, we study utility maximization in state based networks satisfying interdependence property.

In the context of wireless networks, algorithms for user association are available in the literature (see e.g., [14]–[16]). However, these are either centralized [14], or require message passing within the network [15], [16]. Singh and Chaporkar [17] were the first to design uncoupled user association algorithm for wireless networks. Similar to the objective in [2], the authors in [17] consider the problem of maximizing the sum of user payoffs. The algorithm in [17] is essentially based on the algorithm proposed in [2]. Similarly, in our prior work [18] we have adapted the algorithm in [2] to obtain a distributed algorithm for maximizing the sum of user utilities. However, in [18] we assume that the utilities are a function of the long-term throughput achieved by the users, rather than the instantaneous throughput as considered in [17].

In this work, we generalize the setting in [18] by incorporating a state evolution into the model. To the best of our knowledge, the particular setting we consider, and the corresponding optimality result we obtain is not available in the literature.

**Paper Outline:** In Section II, we formally discuss our system model. In Section III, we propose distributed algorithm and show optimality under deterministic state evolution. Then, we consider iid state evolution depending on the association vector in Section IV, where the users are oblivious to the system state. Under complete state knowledge, we propose an optimal distributed user association algorithm, when the state evolution is a controlled Markov process in Section V and any ergodic process in Section VI

### II. System Model

We consider a wireless system comprising $M$ Access Points (APs) and $N$ users. Let $\mathcal{M}$ and $\mathcal{N}$ denote the set of APs and users, respectively. The APs could be heterogeneous in terms of their wireless technology (e.g., WiFi, WiMAX, LTE) and size (e.g., cellular, femto-cell, WiFi AP). We assume that each user can associate with a subset of these APs. Such a limitation could arise, possibly, because of the proximity of a user to only some APs, or due to the limited wireless technologies available on their user-equipments. Specifically, let $\mathcal{A}_i \subseteq \mathcal{M}$ be the subset of APs with which user $i \in \mathcal{N}$ can associate.

We assume a time slotted system. In time slot $t \in \mathbb{N}$, user $i \in \mathcal{N}$ is associated with a single AP $a_i(t)$ where $a_i(t) \in \mathcal{A}_i$. Let $\mathbf{a}(t) := (a_1(t), \ldots, a_N(t))$ denote the vector of associations of all users. The set of all possible association vectors is denoted as $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$.

It is usually assumed that the rate achieved by a user in a given time-slot is a function of the vector of associations in that slot (see e.g., [17], [18]). In our work, we generalize the above setting by introducing a finite set, $\mathcal{S}$, of system states, and assume that the users’ rate is a function of the system state and the association vector in the current time slot. Thus, if $s(t) \in \mathcal{S}$ is the system state at time $t$, then the rate, $r_i(t)$, achieved by user-$i$ in slot $t$ is given by,

$$r_i(t) = f_i(s(t), \mathbf{a}(t)), \quad (1)$$

where, $f_i : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}_+$, for all $i \in \mathcal{N}$. Without loss of generality, we assume that $r_i(t)$ lies between 0 and 1.

**Average Rate and Utilities:** Let $s(0) = s_0$ be the initial state of the system. Then, given a sequence of association vectors $\{\mathbf{a}(t) : t \in \mathbb{N}\}$, the long-term average rate received by user $i$ can be written as,

$$\tau_i(s_0, \{\mathbf{a}(t)\}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} f_i(s(t), \mathbf{a}(t)) \quad (2)$$

Different sequences of association vectors can yield possibly different long-term rate vectors. Let $\mathcal{R}(s_0)$ denote the set of all such feasible long-term average rate vectors $\{\{\tau_1, \cdots, \tau_N\}\}$.

The utility achieved by user-$i$ is measured using a utility function $U_i$, which is a function of the average rate, $\tau_i$. We assume that the utility functions are continuous and satisfy, for all $i \in \mathcal{N}, 0 \leq U_i(\tau_i) \leq u_{\text{max}} < 1$ for all possible $\tau_i \in [0, 1]$ (i.e., the utility functions are continuous and bounded, but are general otherwise).

**Optimization Framework:** We are interested in maximizing the sum of utilities of all users. Formally, we consider:

$$\begin{align*}
\text{Maximize:} & \quad \sum_{i \in \mathcal{N}} U_i(\tau_i) \\
\text{Subject to:} & \quad s(0) = s_0, \quad \forall i \in \mathcal{N}, t \geq 0 \\
& \quad s(t) \in \mathcal{S} \\
& \quad r_i(t) = f_i(s(t), \mathbf{a}(t)) \\
& \quad \tau_i = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} r_i(t).
\end{align*} \quad (3)$$

The problem in (3) can be solved in a centralized manner, assuming that the state is known before the association decision is made [3]. Our goal is to solve (3) in a completely distributed manner. In the following sections, we shall consider four scenarios depending on the state evolution and user’s knowledge of the states.

First we shall assume that the users do not know the system state. Under this assumption, we shall consider two cases: (i)
State evolves deterministically and (ii) the State depends only on the action and evolves iid over time. For deterministic state evolution, we will propose a distributed algorithm with stationary performance close to the optimal utility. Also, this optimal sum utility is no worse compared to the centralized solution with complete state knowledge. For the iid case, we propose an algorithm that maximizes the sum utility, where the utility is a function of the expected average rate (w.r.t state). This is optimal under the assumption that the users do not know the system state.

Next, we shall assume that the users have complete knowledge of the state. Under this assumption, we shall consider two cases: (i) State evolves as a controlled Markov process and (ii) State evolves as an Ergodic random process. For the controlled Markov evolution, we propose a distributed algorithm that maximizes the sum utility, where the utility is a function of the expected average rate (w.r.t the stationary distribution of the state). For the ergodic evolution, we propose a distributed utility maximization algorithm that is optimal.

### III. Unknown State: Deterministic Evolution

In this section, we assume that the system state evolves deterministically and is unknown to the users. Formally, the state transition is as follows,

\[ s(t + 1) = g(s(t), a(t)), \]

where, \( g : S \times A \rightarrow S \). The rate functions \((f_j(\cdot))\) and the state transitions \((g(\cdot))\) are deterministic functions of the current state and the current association-vector. We make the following irreducibility assumption about the wireless system.

**Assumption 1:** (Irreducibility) Given any pair of states \( s, s' \in S \), there exists a finite sequence of association vectors \((a^{(1)}, \ldots, a^{(n)})\) such that \( s^{(1)} = s, s^{(j+1)} = g(s^{(j)}, a^{(j)}) \) for all \( j = 1, \ldots, n \), and \( s^{(n+1)} = s' \). The above assumption insitu that all the states can be visited by choosing an appropriate sequence of association vectors. Thus, it follows that the set of feasible long-term average rates in (2) is independent of the initial state of the system, i.e., \( \mathcal{R}_{\pi}(s_0) = \mathcal{R} \). The set \( \mathcal{R} \) is usually referred to as the rate region of the wireless system. The formulation in (5) could be re-written as,

\[
\text{Maximize: } \sum_{i \in \mathcal{N}} U_i(\pi_i). \tag{5}
\]

The formulation in (5) requires us to seek an optimal sequence of association-vectors from the set of all possible infinite length sequences. We simplify this formulation in (5) (in Section III-A) before proceeding to the design and analysis of an optimal user association algorithm (Sections III-B and III-C).

#### A. Configuration Cycles

In this section, we identify cycles of state and association-vector pairs such that convex combinations of these cycles can achieve any rate vector in the rate region \( \mathcal{R} \). This representation will enable us to simplify the formulation in (5). We begin with the following definitions.

A pair \((s, a)\) of state \( s \in S \), and association-vector \( a \in A \), is referred to as a **configuration**. A sequence of configurations, \( c = (c^{(1)}, \ldots, c^{(n)}) \), where \( c^{(j)} = (s^{(j)}, a^{(j)}) \) for \( j = 1, 2, \ldots, n \), is said to be a **configuration cycle** (or simply cycle) if it satisfies:

\[ s^{(j+1)} = g(c^{(j)}) \text{ for all } j = 1, \ldots, n - 1 \text{, and } g(c^{(n)}) = s^{(1)} \].

If the sequence of configurations are distinct, i.e., if \( c^{(1)}, \ldots, c^{(n)} \) are distinct elements, then the configuration cycle is called a basic configuration cycle. Clearly, the length of any basic configuration cycle is restricted to be not more than \(|S \times A|\).

Let \( C \) denote the set of all basic configuration cycles. Clearly, the set \( C \) is non-empty and finite. Given a basic configuration cycle \( c \in C \) of length \(|c|\), the average rate achieved in the cycle \( c \) by user-\( i \), denoted \( r_i(c) \), is defined as

\[
 r_i(c) := \frac{1}{|c|} \sum_{j=1}^{|c|} f_i(c^{(j)}) \tag{6}
\]

Let \((r_1(c), \ldots, r_N(c))\) denote the vector of user rates achievable in the cycle \( c \). The following lemma relates the set of rates achievable using the basic configuration cycles and the rate region \( \mathcal{R} \) of the wireless system.

**Lemma 1:** Let \( \mathcal{R}_C \) denote the convex hull of the rate vectors achievable using the basic configuration cycles, i.e.,

\[ \mathcal{R}_C = \left\{ \left(\tau_1, \ldots, \tau_N\right) \mid \mathcal{T}_i = \sum_{\epsilon \in c} p_{\epsilon} r_i(c), \quad p_{\epsilon} \geq 0 \quad \forall \epsilon \in C, \sum_{\epsilon \in C} p_{\epsilon} \leq 1 \right\} \]

Then, \( \mathcal{R}_C = \mathcal{R} \).

The above lemma permits us to propose an equivalent formulation of the optimization problem (5) in terms of the basic configuration cycles.

\[
\begin{aligned}
\text{Maximize: } & \sum_{i \in \mathcal{N}} U_i(\tau_i) \\
\text{Subject to: } & \mathcal{T}_i = \sum_{\epsilon \in c} p_{\epsilon} r_i(c), \quad p_{\epsilon} \geq 0, \quad \sum_{\epsilon \in C} p_{\epsilon} \leq 1
\end{aligned} \tag{7}
\]

In the subsequent section, we will discuss a user association algorithm that can achieve any time average of the basic configuration cycles that optimises the above formulation.

#### B. User Association Algorithm

In this section, we present a **completely uncoupled** user association algorithm for a state dependent wireless network, where the state transitions are deterministic. In a completely uncoupled scenario, a user can observe only its past actions and received utilities; the actions and utilities of other users are not known. In fact, a user can be completely oblivious about the existence of other users in the system. In this case, we will further assume that the users cannot observe the system state as well. We note that the algorithm presented here generalizes the techniques studied in work such as [2] and [18].

Our user association algorithm is presented in Algorithm 1. In the following, we describe the working principles of our
Algorithm 1: User Association Algorithm

Initialize:

Fix $z > N$, $K_{\text{max}} \in \mathbb{Z^+}$ and $\epsilon > 0$.
For all $i \in \mathcal{N}$, set $K_i(0)$ uniformly from $\{1, \ldots, K_{\text{max}}\}$.
For all $i \in \mathcal{N}$, set $q_i(0) = 0$.

Update for user association at time $t$:

if $(q_i(t-1) = 1)$ then
  $a_i(t) = \begin{cases} a_i(t-K_i(t-1)) & \text{w.p. } 1-\epsilon^z \\ a_i \in A_i & \text{w.p. } \frac{\epsilon}{|A_i|} \end{cases}$
else
  $a_i(t) = a_i \in A_i \text{ w.p. } \frac{1}{|A_i|}$
end if

Update for $q_i(\cdot)$ and $K_i(\cdot)$ at time $t$:

if $(q_i(t-1) = 1)$ and $(a_i(t) = a_i(t-K_i(t-1)))$ then
  $K_i(t) = K_i(t-1)$
  $q_i(t) = 1$
else
  Pick $K_i(t)$ uniformly from $\{1, \ldots, K_{\text{max}}\}$
  $q_i(t) = \begin{cases} 1 & \text{w.p. } 1-U_i \left( \frac{1}{N} \sum_{j=1-K_i(t)+1}^{t} r_i(j) \right) \\ 0 & \text{w.p. } 1 - \epsilon \left( \frac{1}{N} \sum_{j=1-K_i(t)+1}^{t} r_i(j) \right) \end{cases}$
end if

algorithm. Suppose, at every time $t - 1$, each user $i \in \mathcal{N}$ maintains its past associations $(a_i(1), \ldots, a_i(t-1))$ and throughputs received $(r_i(1), \ldots, r_i(t-1))$. Further, let the users maintain an internal “satisfaction” variable $q_i(t-1)$, and an averaging window size $K_i(t-1)$. We let $q_i(\cdot)$ take values from the binary set $\{0, 1\}$, where $q_i(\cdot) = 1$ represents a state of “content” with the choice of user association and the average throughput received (in the previous $K_i(\cdot)$ slots), while $q_i(\cdot) = 0$ represents a state of “discontent” for the user. The averaging window size, $K_i(\cdot)$, is used to average the received throughput (and also identifies the sequence length of actions that are repeated) with $K_i(\cdot)$ taking values from the set $\{1, \ldots, K_{\text{max}}\}$ where $K_{\text{max}} \in \mathbb{N}$ is fixed.

The choice of user association, $a_i(t)$, made at the beginning of the slot $t$ is entirely a function of the internal satisfaction variable $q_i(t-1)$. When a user $i$ is content at the beginning of slot $t$, i.e., when $q_i(t-1) = 1$, the user repeats an earlier action, here $a_i(t-K_i(t-1))$, with high probability $1-\epsilon^z$ (where $z$ is a parameter satisfying $z > N$, the number of users). When a user $i$ is discontent at the beginning of slot $t$, i.e., when $q_i(t-1) = 0$, then the user selects an association uniformly from $A_i$.

The internal satisfaction variable $q_i(t)$ and the averaging window size $K_i(t)$ are updated at the end of slot $t$. If the user $i$ was content in slot $t-1$ (i.e., when $q_i(t-1) = 1$), then, the user continues to remain content in slot $t$ if it had repeated an earlier action (i.e., if $a_i(t-1) = a_i(t-K_i(t-1))$, which happens with high probability) and if it had received the same throughput as in the slot $t-K_i(t-1)$, i.e., $r_i(t) = r_i(t-K_i(t-1))$ (which could happen when the vector of user associations and the system state remains unchanged). Otherwise, a user becomes content with a very low probability depending on the utility ($U_i$) of the average throughput received by the user in the previous $K_i(t-1)$ time slots.

We note that when all the users are content, i.e., when $q_i(\cdot) = 1$ for all $i$, then, the users repeat their last $K_i(\cdot)$ actions (in synchrony) and continue to receive a constant average throughput (if the corresponding action sequence length of $K_i$ is a configuration cycle) based on the sequence of actions and the system states. A sequence of actions is preferred depending on the average user throughput corresponding to the $K_i(\cdot)$ association sequence and the user utilities, and is a function of $\epsilon$ as well. In the following section, we will show that Algorithm 1 chooses an action sequence that optimises the formulation in (7) as $\epsilon \to 0$ and as $K_{\text{max}} \to \infty$.

C. Optimality Results

In this section, we show that Algorithm 1 selects a sequence of associations for users that tends to optimize the formulation in (7). Define $X_\epsilon(t)$ as

$$X_\epsilon(t) = (c(t-K_{\text{max}} + 1), \ldots, c(t), K(t), q(t))$$

where $c(t) = (s(t), a(t), K(t) = (K_1(t), \ldots, K_N(t)))$ and $q(t) = (q_1(t), \ldots, q_N(t))$. Let $\mathcal{K}$ and $\mathcal{Q}$ denote the state space of $K(t)$ and $q(t)$. $X_\epsilon(t)$ corresponds to the relevant configuration states of the system, the vector of averaging window sizes and the satisfaction variables of the $N$ users in the current slot $t$. The following lemma shows that the random process $\{X_\epsilon(t) : t \in \mathbb{N}\}$ is a regular perturbed Markov chain (perturbed by the algorithm parameter $\epsilon$) with a positive stationary distribution.

Definition 1: $\{X_\epsilon(t)\}$ is a regular perturbed Markov process (perturbed by $\epsilon$) if the following conditions are satisfied (see [10]).

1) $\forall \epsilon > 0$, $\{X_\epsilon(t)\}$ is an ergodic Markov Process
2) $\forall \omega, \omega' \in \Omega$, $\lim_{\epsilon \to 0} \mathbb{P}_\epsilon(\omega, \omega') = \mathbb{P}_0(\omega, \omega')$
3) $\forall \omega, \omega' \in \Omega$, if $\mathbb{P}_\epsilon(\omega, \omega') > 0$ for some $\epsilon > 0$, then,
$$0 < \lim_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(\omega, \omega')}{\mathbb{P}_\epsilon(\omega', \omega)} < \infty$$

for some $r(\omega, \omega') > 0$ and $r(\omega, \omega')$ is called the resistance of the one-step transition $\omega, \omega'$.

Lemma 2: $\{X_\epsilon(t)\}$ induced by Algorithm 1 is a regular perturbed Markov chain (perturbed by $\epsilon$) over the state space $\Omega = (\mathcal{S} \times \mathcal{A})^{K_{\text{max}}} \times \mathcal{K} \times \mathcal{Q}$ with a positive stationary distribution $\pi_\epsilon$.

Proof: See Appendix IX-A.

The stationary distribution of the Markov chain $\{X_\epsilon(t)\}$ characterizes the user associations (the configuration states) and the long term average throughput received by the users with the Algorithm 1. In our work, we seek to characterize the stationary distribution of the Markov chain $\{X_\epsilon(t)\}$ especially for small $\epsilon > 0$. The following definition helps identify the stationary distribution for small $\epsilon$ (and the user associations and the average throughput that occur for a significant fraction of time).

Definition 2: (Stochastically stable state [10]) A state $\omega \in \Omega$ of a regular perturbed Markov chain $\{X_i(t)\}$ is said to be stochastically stable, if $\lim_{n \to 0} \pi_n(\omega) > 0$.

We prove optimality by showing that the stochastically stable states of $\{X_i(t)\}$ corresponds to the configuration sequences that maximize the network utility. To proceed in that direction, we require an important assumption on the network called interdependence.

Assumption 2: (Interdependence) For every state $s \in S$ and for any subset of the users $N' \subset N$ and user association vector $a = (a_{N'}, a_{N''})$, there exists a user $j \notin N'$ and a user association vector $a' = (a_{N'}, a_{N''})$ such that $f_j(s, a) \neq f_j(s, a')$.

Remark 1: A key assumption needed for Algorithm 1 to work is the interdependence defined above. We study a completely uncoupled setup where the only feedback to a wireless user on the network configuration is the user’s throughput in the slot. The interdependence assumption ensures that changes in user association by any user(s) can be perceived by other users in the network as a change in their user throughput.

Algorithm 1 exploits this feature where a discontent user changes associations randomly to effect change in throughput of the other users thereby causing discontent to the other users in the network.

Further, from [10], we know that the stochastically stable states of the Markov chain $\{X_i(t)\}$ must necessarily belong to the recurrent classes of Markov chain $\{X_0(t)\}$ (the Markov chain obtained by substituting $\epsilon = 0$ in the transition probabilities). The following lemma characterizes the recurrent classes (and states) of the Markov chain $\{X_0(t)\}$.

Lemma 3: The recurrent classes (and states) of the Markov chain $X_0(t)$ are the following:

1) A state $\omega = (c^{(1)}, \ldots, c^{(K_{\text{max}})}, K, q) \in \Omega$ is part of a recurrent class if $q_i = 1$ for all $i \in N'$, $s^{(j)} = g(c^{(j-1)})$ and if the association values and the throughput received repeat with interval $K_i$, for every user $i$. For example, consider a configuration cycle $(c^{(1)}, \ldots, c^{(K')})$ of length $K' \leq K_{\text{max}}$. Then, $\omega = (c^{(1)}, \ldots, c^{(K')}, c^{(1)}, \ldots, K', 1)$ is a recurrent state. The recurrent class to which the state belongs includes states such as

$$((c^{(2)}, \ldots, c^{(K')}, c^{(1)}, c^{(2)}, \ldots, K', 1),$$

$$((c^{(3)}, \ldots, c^{(K')}, c^{(1)}, c^{(2)}, c^{(3)}, \ldots, K', 1) \ldots$$

Let $B = \{B_1, \ldots, B_L\}$ denote the set of all such recurrent classes. Further, the set $B$ is non-empty if $K_{\text{max}} \geq |A \times S|$. This is because the maximum length of a basic configuration cycle is $|A \times S|$.

2) All states $\omega \in \Omega$ such that $q_i = 0$ for all $i \in N$ form a single recurrent class. Let us denote this class as $O$.

Proof: See Appendix IX-B

We need the following additional definitions from the theory of regular perturbed Markov processes from [10] to complete the discussion on the stochastically stable states of $\{X_i(t)\}$.

1) Consider a sequence of state transitions $\omega_1 \rightarrow \cdots \rightarrow \omega_k$.

The resistance of the path (sequence of transitions) is defined as the sum of the resistances of the one-step transitions in the path, i.e., $r(\omega_1, \omega_2) + \cdots + r(\omega_{k-1}, \omega_k)$.

2) The resistance from state $\omega_i$ to $\omega_j$ is defined as the minimum resistance over all paths from $\omega_i$ to $\omega_j$.

3) The resistance from a recurrent class $B$ to another recurrent class $B'$, $\rho(B, B')$, is defined as the minimum resistance from any state $\omega \in B$ to any state $\omega' \in B'$.

4) Consider a complete directed graph $G$ with the recurrent classes of $\{X_0(t)\}$ as the vertices. We assign weights to the edges as follows, e.g., $\rho(B \rightarrow B')$ is the weight of the directed edge from recurrent class $B$ to $B'$. Now, consider a tree rooted at a recurrent class, say $B_i$, with a directed path from every other vertex to $B_i$. Then, the resistance of the tree is defined as the sum of the weights of the edges of the tree.

5) The stochastic potential $\gamma(B_i)$ of a recurrent class $B_i$ is defined as the minimum resistance over all trees rooted at that recurrent class.

The following lemmas compute the resistance between the recurrent classes of $X_0(t)$ and the stochastic potential of the recurrent classes.

Lemma 4: Consider the recurrent classes $B_1, \ldots, B_L$ and $O$ of $\{X_0(t)\}$. Then,

1) $\rho(B_i \rightarrow O) = z$

2) Let $(c^{(1)}, \ldots, c^{(K_{\text{max}})}, K, 1)$ be a state in $B_i$. Then,

$$\rho(O \rightarrow B_i) = \sum_{i=1}^{N} \left(1 - U_i \left(\frac{f_i(c^{(1)}) + \cdots + f_i(c^{(K_i)})}{K_i}\right)\right)$$

3) For $j \neq i$, $z \leq \rho(B_j \rightarrow B_i) < 2z$.

Proof: See Appendix IX-C.

Lemma 5: The stochastic potential of a recurrent class $B_i$ with state $(c^{(1)}, \ldots, c^{(K_{\text{max}})}, K, 1)$ is given by

$$\gamma(B_i) = z(L - 1) + \sum_{i=1}^{N} \left(1 - U_i \left(\frac{f_i(c^{(1)}) + \cdots + f_i(c^{(K_i)})}{K_i}\right)\right)$$

Proof: Follows from Lemma 4 and Lemma 4.3 in [19].

Lemma 6: The stochastic potential of the recurrent class $O$ is $Lz$ and there exists $i$ such that $\rho(O) > \rho(B_i)$.

The following theorem from [10] identifies the stochastically stable states of the process $\{X_i(t)\}$ from among the recurrent classes of $\{X_0(t)\}$.

Theorem 1: [10]. The stochastically stable states of a regular perturbed Markov chain $\{X_i(t)\}$ are states of the recurrent class having minimum stochastic potential.

The above theorem insists that the stochastically stable classes of the Markov chain $\{X_i(t)\}$ are the states where all users are content and that minimizes $\gamma(B_i)$, i.e.,

$$\sum_{i=1}^{N} \left(1 - U_i \left(\frac{f_i(c^{(1)}) + \cdots + f_i(c^{(K_i)})}{K_i}\right)\right)$$

This implies that the stochastically stable states are those that maximize

$$\sum_{i=1}^{N} U_i \left(\frac{f_i(c^{(1)}) + \cdots + f_i(c^{(K_i)})}{K_i}\right).$$

We note again that any configuration cycle of length $K' \leq K_{\text{max}}$ belongs to the set of recurrent classes of $\{X_0(t)\}$. 
Hence, the stochastically stable states of the Markov chain must achieve a sum utility at least as high as these classes (states). Further, configuration cycles of large lengths permit almost every convex combination of the basic configuration cycles (follows from the irreducibility assumption and the fact that basic configuration cycles are of length at most $|S \times A|$). Thus, as $K_{\text{max}} \to \infty$ and as $\epsilon \to 0$, the stochastically stable states of Algorithm 1 optimize the formulation in (3).

**IV. Unknown state: Independent state evolution**

In this section, we assume that the state $s(t)$ is a sequence of independent random variables drawn with probability mass function (pmf) $\mu(\cdot, a)$, where $a$ is the action profile chosen at time $t$. Thus for a fixed action profile $a$, the state is independent and identically distributed with pmf $\mu(\cdot, a)$. We also assume that the users do not know the state $s(t)$. In this setup, we would like to maximize the following formulation:

$$
\max \sum_i U_i(\bar{r}_i)
\quad \text{s.t. } \bar{r}_i \leq \sum_a p(a) E(r_i(a, s)),
\qquad (8)
\sum_a p(a) = 1, \ p(a) \geq 0,
$$

where, the expectation is with respect to the distribution $\mu(\cdot, a)$. Additionally, in this section, we assume that $U_i$‘s are Lipschitz continuous.

**A. Utility maximization algorithm**

In this subsection, we shall propose a distributed algorithm to maximize the formulation in (8). We consider frames of length $L$ slots. Each user $i$ chooses an action $a_i(l)$ at the beginning of every frame $l$ and repeats the same action during the frame. Let $\bar{r}_i(l)$ be the average throughput received during frame $l$. Users maintain satisfaction variable $\bar{r}_i(l)$, which is updated at the end of each frame. We intend to use the algorithm in [18] over these frames. By choosing frames of suitably large length, the time average throughput received over a frame will be close to expected throughput for the chosen action profile (expectation over the state $s(t)$). We formalize the idea above in the following discussion.

If user $i$ is content at the beginning of frame $l$, it repeats the access point it chose $K$ frames earlier with a large probability $1 - \epsilon^2$. If user $i$ is discontent at the beginning of frame $l$, it chooses an access points uniformly from the set $A_i$.

If user $i$ was content in the previous frame and repeats the same associations chosen $K$ frames earlier, the player remains content if the difference between the average throughput of the current frame and the frame $K$ slots earlier is within $\delta$ in magnitude. In other cases, player $i$ becomes content with a small probability $\epsilon^{1 - U_i} \left( \frac{1}{\epsilon} \sum_{j=K+1}^\infty \bar{r}_i(j) \right)$, where $\bar{r}_i(j)$ is the average throughput received during the frame $j$.

**Algorithm 2 : User Association Algorithm**

**Initialize:**

Fix $z > N$, $K, L \in \mathbb{Z}^+$, and $\epsilon > 0$.
For all $i \in \mathcal{N}$, set $q_i(0) = 0$.

**Update for user association at frame $l$:**

if $(q_i(l - 1) = 1)$ then
$$
a_i(l) = \begin{cases} 
a_i(l - K) & \text{w.p. } 1 - \epsilon^2 \\
 a_i \in A_i & \text{w.p. } \frac{1}{|A_i|} 
\end{cases}
$$

else
$$
a_i(l) = a_i \in A_i \text{ w.p. } \frac{1}{|A_i|}
$$
end if

**Update for $q_i(\cdot)$ at time $t$:**

if $(q_i(l - 1) = 1)$ and $(a_i(l) = a_i(l - K))$ and $(|\bar{r}_i(l) - \bar{r}_i(l - K)| < \delta)$ then
$$
q_i(l) = 1
$$
else
$$
q_i(l) = \begin{cases} 
1 \text{ w.p. } \epsilon \\
 0 \text{ w.p. } 1 - \epsilon
\end{cases}
$$
end if

**B. Optimality Results**

In this section, we provide sufficient conditions on $L$ and $\delta$ such that the algorithm maximizes the formulation in (8) as $\epsilon \to 0$. Before proceeding to the analysis, we modify the Interdependence assumption as follows:

**Assumption 3:** (Interdependence) For any subset of the users $\mathcal{N}' \subset \mathcal{N}$ and user association vector $a = (a_{\mathcal{N}'}, a_{\mathcal{N}'}, \ldots)$, there exists a user $j \notin \mathcal{N}'$ and a user association vector $a' = (a_{\mathcal{N}'}, a_{\mathcal{N}'}, \ldots)$ such that $E(r_j(s, a)) \neq E(r_j(s, a'))$.

Choice of $L$ and $\delta$: We choose $L$ and $\delta$ to satisfy the following conditions:

1) $\delta \to 0$ as $\epsilon \to 0$.
2) $L \delta^2 \geq z \log(1/\epsilon)$.
3) $L \to \infty$ as $\epsilon \to 0$.
4) $L^k \to \infty$, for some $k$.

One possible choice of $\delta$ and $L$ satisfying the above is $L = 1/\epsilon$ and $\delta^2 \geq z \log(1/\epsilon)$.

Let $Z_{\epsilon}(l) = (a_i(l - K + 1), \ldots, a_i(l), q_i(l), i = 1, \ldots, N)$. Note that $Z_{\epsilon}(l)$ is a process that changes over frames of length $L$. Now, we have the following lemma.

**Lemma 7:** $Z_{\epsilon}(l)$ is regular perturbed Markov chain on the state space $(A^K \times Q)$. □

As a first step in analysing the performance of $Z_{\epsilon}$, we first identify the recurrent classes of $Z_{\epsilon}(l)$.

**Lemma 8:** The recurrent classes of $Z_{\epsilon}(l)$ are as follows:

1) States where all the users are content. For example, a sequence associations and satisfaction variable pair $(a^{(1)}, \ldots, a^{(K)}, 1)$ belongs to a recurrent class. All cyclic shifts of $(a^{(1)}, \ldots, a^{(K)})$ with all users content also belongs to this class. Let $B_1, \ldots, B_I$ denote the recurrent classes of this type.

2) States where all the users are discontent forms a single recurrent class. We denote this class by $O$. □
In the lemma below, we provide bounds on the resistances between the recurrent classes of $Z_0$.

**Lemma 9:** Let $(a^{(1)}, \ldots, a^{(K)}) \in B_i$. We have the following results,

1) $\rho(B_i, O) = z$. Let $\bar{\tau}_j(l)$ denote the average throughput received by user $j$ in frame $l$, when action profile $a$ is played. Let $\bar{\tau}_j(a)$ denote the expected average throughput received by user $j$ in a frame i.e., $\bar{\tau}_j(a) = E(\bar{\tau}_j(l))$. A transition from $B_i$ happens when a user changes its association sequence with probability $\epsilon^\gamma$ or when the average throughput of a user changes by more than $\delta$ in a frame. The former happens with resistance at least $z$. To calculate the resistance for the latter case, consider,

$$P\{|ar{\tau}_j(l) - \bar{\tau}_j(l - 1)| > \delta\} = P\{|ar{\tau}_j(l) - \bar{\tau}_j(a) + \bar{\tau}_j(a) - \bar{\tau}_j(l - 1)| > \delta\} \leq P\{|ar{\tau}_j(l) - \bar{\tau}_j(a)| > \delta/2\} \leq e^{-\frac{\delta^2}{2}}$$

Where (a) follows from Hoeffding’s lemma and (b) follows from our choice of $L\delta^2 \geq 2z \log(\frac{1}{\epsilon})$. Thus, $\rho(B_i, O) \geq z$. By our choice of $\delta$ and interdependence, once a user becomes discontent every other player becomes discontent with zero resistance. Therefore, $\rho(B_i, O) = z$.

2) Let, $\bar{\tau}_i = \frac{\sum_{j=1}^{K} \bar{\tau}_j(l)}{K}$ and $\bar{\tau}_i = \frac{\sum_{j=1}^{K} \bar{\tau}_j(a^{(j)})}{K}$. Then, we have,

$$\rho(O \rightarrow B_i) = \sum_{i=1}^{N} (1 - U_i(\bar{\tau}_i)).$$

Transition from $O$ to $B_i$ would require all the players to become content which happens with probability $1 - \sum_{i=1}^{N} U_i\left(\left(\frac{\epsilon \sum_{j=1}^{K} \bar{\tau}_j(l)}{K}\right)\right)$.

To prove the above, we need to show that, for all $i$,

$$\lim_{\epsilon \rightarrow 0} U_i(\bar{\tau}_i) - U_i(\bar{\tau}_i) = 1$$

By Lipschitz continuity of $U_i$, we have $\forall \delta_1 > 0$, with probability $1 - e^{-2L\delta_1^2}$, we have,

$$-P\delta_1 \leq U_i(\bar{\tau}_i) - U_i(\bar{\tau}_i) \leq P\delta_1,$$

where $P$ is assumed to be the Lipschitz constant. This implies, for all $\delta_1$, we have,

$$e^{P\delta_1} (1 - e^{-2L\delta_1^2}) \leq e^{U_i(\bar{\tau}_i) - U_i(\bar{\tau}_i)}, \text{ and } e^{U_i(\bar{\tau}_i) - U_i(\bar{\tau}_i)} \leq e^{P\delta_1} (1 - e^{-2L\delta_1^2}) + e^{-2L\delta_1^2}$$

we have the result by taking the limit along $\delta_1 = 1/L^{1/4}$.

3) $\epsilon \leq \rho(B_i, B_j) < 2\epsilon$

The proof for the above statement follow from the arguments in [18].

**Theorem 2:** Under Assumption 3 (Interdependence), the stochastically stable states of the Markov chain induced by the above algorithm are the states which maximize the following formulation,

$$\max \sum_i U_i(\bar{\tau}_i)$$

s.t. $\bar{\tau}_i \leq \sum_a p(a) E(\tau_i(a, s)),$

$$\sum_a p(a) = 1, \ p(a) \in \{0, \frac{1}{K}, \frac{2}{K}, \ldots, 1\}.$$

**Proof:** Follows from Lemma 9 and Lemmas 4, 5 and Theorem 2 in [18].

So far we have assumed that the state is unknown to the users. In the following sections, we shall assume that the state is known to the users. This assumption allows us to work with a more general state evolution model (Section V) and a significant increase in the rate region (Section VI).

**V. KNOWN STATE: CONTROLLED MARKOV EVOLUTION**

In this section, we shall assume that the state is known to the users and evolves as a controlled Markov process, i.e.,

$$P(s(t + 1)|s(j), a(j), 0 \leq j \leq t) = P(s(t + 1)|s(t), a(t))$$

We say that the control $a(t)$ is stationary, if it satisfies

$$a(t) = h(s(t)) = (h_1(s(t)), \ldots, h_N(s(t))),$$

where $h$ is a deterministic function from $S$ to $A$. We assume that the for any stationary control $h$, the controlled Markov process $S(t, h)$ is ergodic with stationary distribution $\mu(.)$, $h$. Further, for a given control $h$, the expected stationary pay-off is given by,

$$r_i(h) = \sum_{s \in S} \mu(s, h(s)) r_i(s, h(s))$$

(9)

Denote by $\mathcal{H}$ the set of stationary controls. Since the set of action profiles and states are finite, the set $\mathcal{H}$ is finite as well. Our objective here is to time share between functions $h$ such that the sum utility is maximized. Formally,

$$\max \sum_i U_i(\bar{\tau}_i)$$

s.t. $\bar{\tau}_i = \sum_{h \in \mathcal{H}} p(h) r_i(h)$

$$\sum_{h} p(h) = 1, \ p(h) \in \{0, \frac{1}{K}, \ldots, 1\}$$

(10)

Note the similarity between the above formulation and (8). In the formulation above, the expected pay-off in (8) is with respect to the stationary distribution $\mu(s, h)$ of the controlled Markov chain, whereas in (9) we assumed that the expectation is with respect to an iid random variable with distribution $\mu$. Hence, to solve the above formulation, we run Algorithm ??, where node $i$ chooses stationary control $h_i$. To establish an estimate on the frame size $L$ and $\delta$, we need the following assumptions on the controlled Markov chain.
for every state \( s \) \{ \}
j\{i, h\}

Now we have the results of Theorem 2 holds for the formulation with the following choices of \( L \) and \( \delta \) with appropriate interdependence assumption (i.e., with \( \alpha \) replaced by \( h \) and \( E(r_i(a, s)) \) replaced by \( r_i(h) \) in Assumption 3).

1. \( \delta \to 0 \) as \( \epsilon \to 0 \).
2. \( L\delta^2 \geq \frac{\epsilon^2}{L \lambda_{\min}} \log(1/\epsilon) \).
3. \( L \to \infty \) as \( \epsilon \to 0 \).
4. \( L\epsilon^k \to \infty \), for some \( k \).

The proof follows by replacing the Hoeffding inequality for \( \epsilon \)-iid random variables with the inequality for Markov chains (See Theorem 2.3 in [20]).

VI. KNOWN STATE: ERGODIC STATE EVOLUTION

In this section, we shall assume that the state \( s(t) \) evolves as an ergodic random process taking values in a finite set \( S \) with time average probabilities \( \mu(\cdot) \). We assume that the users know the state \( s(t) \) prior to choosing their associations at time \( t \). In this setup, we aim to maximise the following formulation:

\[
\max \sum_i U_i(\tilde{r}_i) \\
\text{s.t. } \tilde{r}_i \leq \sum_{a \in S} \mu(s) \sum_{a} p(a, s) r_i(a, s), \\
\sum_{a} p(a, s) = 1, \quad p(a, s) \geq 0, \quad \forall s \in S.
\]

In the following subsections, we describe the proposed algorithm and discuss optimality results.

A. Utility maximization algorithm

We now propose a completely uncoupled utility maximization algorithm, assuming that users know the state prior to choosing access points. Each user has a binary satisfaction variable \( q_i(t) \). The purpose of \( q_i(t) \) is similar to the algorithm with deterministic state transition in Section III-B. A user chooses an access point based on the current and prior state, history of the access points chosen by the user and its satisfaction variable \( q_i \). Let the history of system state, the access points chosen and throughput received by user \( i \) until time \( t \) be \( \{s(l), a_i(l), r_i(l), l = 1, \ldots, t - 1\} \). For each state \( s \), we require the users to keep track of associations and throughput received during the last \( K \) occurrences of state \( s \). We denote by \( (\tilde{a}_i(K, s), \tilde{r}_i(K, s)) \), the access point chosen and throughput received by user \( i \) the previous time when state \( s \) occurred. Let \( \{\tilde{a}_i(j, s), \tilde{r}_i(j, s), j = 1, \ldots, K\} \) denote access points chosen and throughput received by user \( i \) during the \( K \) recent time slots when state \( s \) occurred. We require each user to keep track of the history \( \{\tilde{a}_i(j, s), \tilde{r}_i(j, s), j = 1, \ldots, K\} \) for every state \( s \in S \). If state \( s \) occurred for less than \( K \) times, we set by default, \( \tilde{a}_i(j, s) = a_0 \in A_i \), and \( \tilde{r}_i(j, s) = 0 \), for all \( j \), where state \( s \) has occurred for less than \( K - j + 1 \) times.

We also require each user to keep track of the number of times state \( s \) has occurred and denote it by \( t_s \). Then, \( t_s/t \) denotes the fraction of time state \( s \) has occurred.

Recall that, we have assumed that every user knows the state before choosing the access point to associate with. If user \( i \) was content in slot \( t - 1 \) and the current state is \( s(t) \), then user \( i \) chooses the access point \( \tilde{a}_i(1, s(t)) \) with a large probability \( 1 - \epsilon^2 \). Here, \( \tilde{a}_i(1, s(t)) \) is the access point chosen by user \( i \) the \( K \)th last time state \( s(t) \) occurred. With a small probability \( \epsilon^2 \), user \( i \) chooses any other access point uniformly at random.

If user \( i \) was discontent in slot \( t - 1 \), then it chooses an access point uniformly at random from \( A_i \), independent of the state \( s(t) \).

User \( i \) updates its satisfaction variable \( q_i(t) \) based on the fraction of time each state has occurred \( \{t_s/t \mid s \in S\} \), the current state \( s(t) \), and its prior satisfaction variable \( q_i(t - 1) \), history \( (\tilde{a}_i(K, s), \tilde{r}_i(K, s)) \), current association \( a_i(t) \) and throughput \( r_i(t) \). If player \( i \) was in content in slot \( t - 1 \), and chose the action \( \tilde{a}_i(1) \) and received the payoff \( \tilde{r}_i(1) \) in slot \( t \), then it remains content \( (q_i(t) = 1) \) with probability \( 1 \). In other cases, player \( i \) becomes content \( (q_i(t) = 1) \) with a small probability \( e^2 \). Here, \( \tilde{r}_i(1) \) is given as follows. Let \( \tilde{r}_i(s) \) denote the average payoff received by player \( i \) over the previous \( K \) slots when state \( s \) occurred i.e., \( \tilde{r}_i(s) = 1/K \sum_{j=2}^{K} \tilde{r}_i(j, s) \). Now \( \tilde{r}_i(s) \) is the weighted average of \( \tilde{r}_i(s) \) weighted by the fraction of time state \( s \) has occurred i.e.,

\[
\tilde{r}_i = \sum_{s \neq s(t)} \frac{t_s}{t} \tilde{r}_i(s) + \frac{t_s}{t} \frac{1}{K} \sum_{j=2}^{K} \tilde{r}_i(j, s(t)) + r_i(t) \cdot
\]

Finally, \( (\tilde{a}_i(j, s(t)), \tilde{r}_i(j, s(t))) \) is updated with the recent action and payoff.

B. Optimality Results

In this subsection, we will study the stationary performance of Algorithm 2 as \( \epsilon \to 0 \). Let \( Y_i(t) = \{(\tilde{a}_i(j, s), q_i(t)), j = 1 \ldots, K, s \in S, i \in N\} \). First, we will show in the lemma below that the algorithm induces a Markov chain.

Lemma 10: \( Y_i(t) \) induces a time non-homogeneous Markov chain on the state space \( A^{K \times |S|} \times Q \). Proof: See Appendix X-D.

Let \( P_\epsilon(t) \) denote the transition probability matrix of \( Y_i(t) \). Also, let \( P_\epsilon \) denote the transition probability matrix of algorithm 2 with \( t_s/t \) replaced by its ensemble average \( \mu(s) \).

In the next lemma we show that the Markov chain is strongly ergodic.

Definition 3: A non-homogeneous Markov chain with transition probability matrix \( P(t) \) is strongly ergodic if there exists a probability distribution \( \pi \), such that, for all \( m \geq 0 \), we have,

\[
\lim_{t \to \infty} \sup_{\mu} \int d\mu P(m, k, \pi) = 0,
\]

where, \( P(m, k) = \prod_{j=m}^{k-1} P(j) \) and \( d\mu(\cdot, \cdot) \) is the total variation distance.

Lemma 11: The Markov chain \( Y_i(t) \) is strongly ergodic.

Proof: By ergodicity of \( s(t) \), we have, \( \lim_{t \to \infty} t_s/t = \mu(s) \).
Algorithm 3 : User Association Algorithm

Initialize:
Fix $z > N$, $K \in \mathbb{Z}^+$ and $\epsilon > 0$.
For all $i \in \mathcal{N}$, $j = 1, \ldots, K$, and $s \in S$, set $\hat{a}_i(j, s) = a_0 \in \mathcal{A}_i$, $\hat{r}_i(j, s) = 0$, $q_i(0) = 0$.

Update for State at time $t$:
$t_{s(t)} = t_{s(t) + 1}$

Update for user association at time $t$:
if $(q_i(t - 1) = 1)$ then
  $a_i(t) = \hat{a}_i(1, t)$ w.p. $1 - \epsilon^z$
else
  $a_i(t) = a_i \in \mathcal{A}_i$ w.p. $\frac{1}{|\mathcal{A}_i|}$
end if

Update for $q_i(t)$ at time $t$:
if $(q_i(t - 1) = 1)$ and $(a_i(t) = \hat{a}_i(1, s(t)))$
and $(r_i(t) = \hat{r}_i(1, s(t)))$ then
  $q_i(t) = 1$
else
  $q_i(t) = \begin{cases} 1 \text{ w.p. } \epsilon^{1 - U_i(\hat{r}_i(t))} \\ 0 \text{ w.p. } 1 - \epsilon^{1 - U_i(\hat{r}_i(t))} \end{cases}$
where,
  \[ \hat{r}_i = \sum_{s \neq s(t)} t_i(s) + \frac{t_{s(t)}}{K} \left( \sum_{j=2}^K \hat{r}_i(j, s(t)) + r_i(t) \right), \]
  \[ \hat{r}_i(s) = \frac{1}{K} \sum_{j=1}^K \hat{r}_i(j, s). \]
end if

Update for $\hat{a}_i(\cdot, \cdot)$ and $\hat{r}_i(\cdot, \cdot)$ at time $t$:
For $j = 1, \ldots, K - 1$, set $\hat{a}_i(j, s(t)) = \hat{a}_i(j + 1, s(t))$, and $\hat{r}_i(j, s(t)) = \hat{r}_i(j + 1, s(t))$.
Set $\hat{a}_i(K, s(t)) = a_i(t)$ and $\hat{r}_i(K, s(t)) = r_i(t)$

Also with continuity of $U_i's$, we have,
\[ \lim_{t \to \infty} |P_{s(t)} - \hat{P}_{s(t)}| = 0 \]

Note that $\hat{P}_{s(t)}$ is an ergodic transition probability matrix. Thus, by Theorem V.4.5 in [21], the Markov chain $Y_{s(t)}$ is strongly ergodic.

The theorem below characterizes the stationary performance of the Markov chain $Y_{s(t)}$ as $\epsilon \to 0$.

**Theorem 3:** Under Assumption $\Box$ (Interdependence), the stochastically stable states of the Markov chain $Y_{s(t)}$ maximizes the following formulation:

\[
\max \sum_i U_i(\hat{r}_i) \\
s.t. \quad \hat{r}_i \leq \mu(s) \sum_a p(a, s) r_i(a, s), \\
\sum_a p(a, s) = 1, \quad p(a, s) \in \left\{ 0, \frac{1}{K}, \frac{2}{K}, \ldots, 1 \right\}, \quad \forall s \in S.
\]

**Proof:** The stochastically stable states of $Y_{s(t)}$ is the stochastically stable states of $\hat{P}_{s(t)}$. The proof follows similar to Theorem 2 in [18] for $\hat{P}_{s(t)}$.

---

VII. Numerical Examples

In this section, we shall present numerical simulation of our proposed algorithms in the context of user association in IEEE 802.11ac WiFi network. The simulations are performed using a ns3/c++ simulator. We assume that access points independently choose their channel and their channel choice is modeled as the state of the network. We consider an IEEE 802.11ac WiFi network with three access points and five users. The access points are placed at the vertices of an equilateral triangle of length 25 meters. We assume that, two orthogonal 20 MHz channels are available and in each time slot, the access points can operate in one of them. We consider three states, where each state corresponds to allocating an orthogonal channel to an access point and the other two access points share a common channel. For example, state 1 corresponds to allocating an orthogonal channel to access point 1, whereas access points 2 and 3 share a common channel. In each time slot, the objective of our algorithm is to choose user association decisions that maximizes the sum utility of the users. In this example, we shall consider the utility $\log(\delta + r_i)$ for a small $\delta > 0$. The log utility is shown to achieve proportional fairness in [4] and we use $\log(\delta + \hat{r}_i)$ to keep the utility function bounded.

For the deterministic state transition case, we assume that orthogonal channel is allocated to the access point with the maximum number of users. We also assume that ties are resolved in a deterministic manner. We run Algorithm II for different values of $\epsilon$ with $K_{\max} = 2$. In Figure 1 we plot the sum utility of users for $\epsilon = 0.05, 0.1, 0.2$, and 0.3. We also plot the performance of the centralized subgradient algorithm for reference.
the orthogonal channel. In every time slot, choosing a state uniformly at random correspond to equal time sharing of the orthogonal channel between access points. Thus, we assume that, the state evolution is i.i.d and uniformly distributed. In the second example, we assume that channel allocation is unknown to the users prior to association. We run Algorithm 2 with $K = 2$, $L = 4000$ and $\delta = 0.05$. We plot the sum utility for $\epsilon = 0.05, 0.1, 0.2$, and $0.3$ in Figure 2. We also plot the performance of a centralized subgradient algorithm for reference.

In the third case, we assume that the channel allocation is known to the users prior to association. We run Algorithm 3 for different values of $\epsilon$ with $K = 2$. We plot the sum utility for $\epsilon = 0.05, 0.1, 0.2$, and $0.3$ in Figure 3. We also plot the sum utility obtained by a centralized subgradient Algorithm for reference.

Fig. 2. Sum Utility of the users obtained by Algorithm 2 for an IEEE 802.11ac WiFi network with 5 users and 3 Access points. The state corresponds to channel allocated to the access points, the state transition is i.i.d and is known to the users. The performance of a centralized subgradient algorithm is shown for reference.

Fig. 3. Sum Utility of the users obtained by Algorithm 3 for an IEEE 802.11ac WiFi network with 5 users and 3 Access points. The state corresponds to channel allocated to the access points, the state transition is i.i.d and is known to the users. The performance of a centralized subgradient algorithm is shown for reference.

In the third case, we assume that the channel allocation is known to the users prior to association. We run Algorithm 3 for different values of $\epsilon$ with $K = 2$. We plot the sum utility for $\epsilon = 0.05, 0.1, 0.2$, and $0.3$ in Figure 3. We also plot the sum utility obtained by a centralized subgradient Algorithm for reference.

VIII. CONCLUSION

In this work, we present completely uncoupled utility maximisation algorithms for a state based network model. We have considered four cases based on the knowledge of the state and its evolution. We further presented the performance of these algorithms for user association, where the state corresponds to channels in which the access points operate.

In our earlier work [22], we have presented a completely uncoupled subgradient algorithm for maximizing concave utilities. We conclude by noting that, with modifications as considered in this paper, we could extend the subgradient algorithm in [22] to a state based model as well.

IX. APPENDIX

A. Proof of Lemma

We know that $\{X_i(t)\}$ is a discrete time, finite state space random process. At time $t+1$ and for any user $i$, the transition probabilities for $a_i(t+1)$ are a function only of $q_i(t), K_i(t)$ and $a_i(t+1 - K_i(t))$ (i.e., the current state $X_i(t)$). And, the transition probabilities for $s(t+1)$ are a function only of $s(t)$ and $a(t+1)$ (and hence a function of the current state $X_i(t)$). Also, the transition probabilities for $q_i(t+1)$ (and $K_i(t+1)$) is a function only of $q_i(t), K_i(t)$ and the configuration states $c(t+2 - K_{\text{max}}), \cdots c(t+1)$ (the throughputs are a deterministic function of the user association vectors and the system states). Hence, we conclude that the transition probabilities of $\{X_i(t)\}$ are independent of the past, given $X_i(t)$. Thus, $\{X_i(t)\}$ is a Markov chain. Also, for any $\epsilon > 0$, $\{X_i(t)\}$ is an irreducible and aperiodic random process (follows from the irreducibility assumption of the system state and the transition probabilities in Algorithm 1). Thus, for any $\epsilon > 0$, $\{X_i(t)\}$ is an ergodic Markov process. Let $\pi_\epsilon$ denote the unique (and positive) stationary distribution of $\{X_i(t)\}$.

From the state transition probabilities listed in Algorithm 1 we clearly see that conditions 2) and 3) are satisfied as well. Hence, $\{X_i(t)\}$ is a regular perturbed Markov chain (perturbed by $\epsilon$).

B. Proof of Lemma

When $\epsilon = 0$, a content user repeats the action it chose $K_i$ slots before. Also, if a content user receives the payoff that it received $K_i$ slots before, then it remains content. Thus, any state $(c^{(1)}, \ldots, c^{(K_{\text{max}})}, K, 1)$ where, all the users are content and the association values and throughput received repeat with interval $K_i$ (for every user $i$) is a recurrent state in $X_0$.

When all the users are discontent, users choose actions uniformly at random. Due to assumption 1 (Irreducibility), there is a positive probability of reaching all possible configurations. Hence, the set $\mathcal{O}$ is a recurrent class.

Consider any state with at least one discontent user. For a content user to remain content, the payoff it receives should
repeat every $K_i$ slots. However, by assumption 2 (Interdependence), the discontent users could choose actions such that a content user(s) experiences a change in payoff forcing the content user(s) to become discontent. Extending this argument, all the users will become discontent with a positive probability. Thus, a state with some content and rest discontent users is not a recurrent class of $X_e$. □

C. Proof of Lemma 4

1) A transition from $B_i$ to $O$ involves at least one user to change its action and hence become discontent. This happens with resistance $z$. Once a user is discontent, every other user could become discontent with zero resistance (due to interdependence). Thus $\rho(B_i, O) = z$.

2) A transition from $O$ to $B_i$ involves all the users becoming content. User $i$ becomes content with resistance $(1 - U_i \left( \frac{f(c_i)}{a_i} \right))$.

3) A transition from $B_i$ at least one user becoming discontent with resistance $z$. The upper bound follows from: $\rho(B_j, B_i) \leq \rho(B_i, O) + \rho(O, B_j)$.

□

D. Proof of Lemma 10

The action chosen at time $t$, $a(t)$, depends on $q(t-1)$ and $\hat{a}$ at time $t-1$. The update of $\hat{a}$ at time $t$ depends only on $\hat{a}$ at time $t-1$ and the action $a(t)$ chosen at time $t$. Also, the satisfaction variable $q(t)$ depends on $q(t-1)$, $a(t)$, fraction of time each state occurred $t_j/t$ and $\hat{a}$. (Note that $r(t) = f(a(t))$ and $\hat{f}(t) = f(\hat{a}(t))$). Thus $Y_e(t)$ is a Markov chain. The Markov chain is time non homogenous due to the explicit time dependence in $t_j/t$.

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