Random walk on a randomly oriented honeycomb lattice

Gianluca Bosi and Massimo Campanino
Università degli Studi di Bologna
Piazza di Porta San Donato 5, 40126
Bologna, Italy.
Università degli Studi di Bologna
Piazza di Porta San Donato 5, 40126
Bologna, Italy.
E-mail address: gianluca.bosi4@unibo.it, massimo.campanino@unibo.it

Abstract. We study the recurrence behaviour of random walks on partially oriented honeycomb lattices. The vertical edges are undirected while the orientation of the horizontal edges is random: depending on their distribution, we prove a.s. transience in some cases, and a.s. recurrence in other ones. The results extend those obtained for the partially oriented square grid lattices (Campanino and Petritis (2003), Campanino and Petritis (2014)).

1. Introduction

1.1. Motivation. The behaviour of random walks on oriented two-dimensional graphs has been object in the last years of several works. In particular new methods have to be devised to settle the question of their recurrence or transience. The main result of Campanino and Petritis (2003) concerns the random walk on the two-dimensional square lattice where the vertical edges are undirected while the edges on each horizontal line are all oriented randomly and independently to the right or to the left with equal probability. The result is that this random walk is transient almost surely with respect to the environment; in Campanino and Petritis (2014) the study is generalized to a more general class of environments, in particular showing that the random walk is recurrent provided that the lines are oriented through periodic functions. Subsequently other works have investigated this and related models. A common characteristic of the considered random walks is that one can split them, apart from a time change, into a "horizontal" and "vertical" component, where the latter is independent from the former. Moreover all these studies deal with the square grid lattice, and one can ask if these recurrence properties still hold as the geometry of the underlying two-dimensional graph changes: motivated by this question, in the present work we consider the random walk on the honeycomb randomly directed lattice. Here the "vertical motion" obtained from the splitting is no longer independent from the "horizontal" one: as a result, the subsequent
steps of the "vertical motion" have a Markovian dependency. For this reason, while some of the techniques used in Campanino and Petritis (2003) and Campanino and Petritis (2014) can be properly extended and adopted in our study, in several parts of the proof we need to develop new ideas and techniques. While we consider just a specific example also for reasons of simplicity, we expect in the future to be able to extend our approach to a more general setting.

1.2. Notation and results. Let $L := (\mathbb{Z}^2, E)$ be the square grid lattice, i.e. $E$ is the set of nearest neighbours in $\mathbb{Z}^2$. The honeycomb lattice can be defined as the sub-graph obtained from $L$ by eliminating the following set of edges (see figure 1.1)

$$
\{(2j, 2k), (2j, 2k + 1)\mid j, k \in \mathbb{Z}\} \\
\{(2j + 1, 2k + 1), (2j + 1, 2k + 2)\mid j, k \in \mathbb{Z}\} \\
\{(2j + 1, 2k + 2), (2j + 1, 2k + 1)\mid j, k \in \mathbb{Z}\}. 
$$

Then we can define partially oriented versions of the lattice by imposing a certain orientation on the horizontal edges (this can be done either deterministically or randomly), while keeping the vertical edges unoriented. Our work is devoted to the study of the recurrence (transience) behaviour of a simple random walk $(M_n)_{n \geq 0}$ on such oriented lattices. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The first result we prove is the following:

**Theorem 1.1.** Let $(\epsilon_y)_{y \in \mathbb{Z}}$ be a i.i.d. family of $\{-1, 1\}$-valued Rademacher random variables and denote by $H_\epsilon$ the honeycomb lattice oriented randomly as follows: if $\epsilon_y = 1$ there is only a right-directed edge between $(x, y)$ and $(x + 1, y)$, $\forall x \in \mathbb{Z}$; if $\epsilon_y = -1$, only a left-directed one. Then the random walk $(M_n)_{n \geq 0}$ on $H_\epsilon$ is a.s. transient.

Since the transience behaviour is caused by the presence and the size of fluctuations in the orientations, we impose periodic orientations and prove the following result.

**Theorem 1.2.** Let $Q > 1$ be an even integer and, given a $Q$-periodic function $f : \mathbb{Z} \to \{-1, 1\}$ such that $\sum_{k=0}^{Q-1} f(k) = 0$, consider the oriented honeycomb
lattice $H_f$ whose horizontal vertexes are oriented according to the value of $f$: that is, if $f(k) = 1$ then the edges with ordinate $k$ are right-directed, otherwise they are left-directed. Then the random walk $(M_n)_{n \geq 0}$ on $H_f$ is recurrent.

Finally, we consider the case of horizontal orientations prescribed by a random perturbation of a periodic function.

**Theorem 1.3.** Let $(\epsilon_y)_{y \in \mathbb{Z}}$ and $f$ as above and define for every $y \in \mathbb{Z}$

$$\tau_y := (1 - \lambda_y)f(y) + \lambda_y \epsilon_y$$

where $\lambda = (\lambda_y)_{y \in \mathbb{Z}}$ is a $\{0,1\}$-valued sequence of independent r.v., independent of $\epsilon$ s.t.

$$P(\lambda_y = 1) = \frac{c}{|y|^b}.$$  

Denote by $H_{\tau, \lambda}$ the honeycomb lattice with such random orientations. Then

(i) If $\beta < 1$, the random walk $M_n$ on $H_{\tau, \lambda}$ is $(\epsilon_y, \lambda_y)$-a.s. transient.

(ii) If $\beta > 1$, the random walk $M_n$ on $H_{\tau, \lambda}$ is $(\epsilon_y, \lambda_y)$-a.s. recurrent.

2. Technical preliminaries

2.1. Decomposition of the random walk. Following Campanino and Petritis (2003), we decompose the random walk into two components that, if sampled on a particular sequence of random times, have the same recurrence behaviour of $(M_n)$.

We begin with the following observation. Let $\xi$ be a random variable with geometric distribution of parameter $\frac{1}{2}$, and consider the event of an even or null outcome $A := \bigcup_{m \in \mathbb{N} \cup \{0\}} \{\xi = 2m\}$. We have

$$P(A) = \sum_{m=0}^{\infty} P(\xi = 2m) = \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} = \frac{2}{3},$$

Obviously $P(A^c) = \frac{1}{3}$. Now, if we interpret $\xi$ as the absolute value of the first horizontal displacement of $(M_n)$ (i.e. the number of subsequent horizontal steps before the first vertical one), we immediately see that after an odd outcome the random walk will perform a vertical down-directed step, otherwise a up-directed one. With this observation in mind, we give the following definition.

**Definition 2.1.** The vertical skeleton of $(M_n)_{n \geq 0}$ is the Markov process $(Y_n, \nu_n)_{n \geq 0}$ with values in $\mathbb{Z} \times \{-1, 1\}$ defined by the following transition probabilities:

$$P(y, 1, (y+1, 1)) = P(y, -1, (y-1, 1)) = \frac{1}{3},$$

$$P(y, 1, (y-1, -1)) = P(y, -1, (y+1, 1)) = \frac{2}{3}$$

for any $y \in \mathbb{Z}$, and $P((Y_0, \nu_0) = (0, 1)) = 1$.

The first component of the vertical skeleton represents the projection on the $y$-axis of the position of the random walk, seen at the times of successive vertical steps, while the second component represents the speed -or direction- of the latter step.

**Remark 2.2.** By definition, the skeleton random walk $(Y_k, \nu_k)$ satisfies

$$Y_k = \sum_{i=0}^{k} \nu_k,$$
and the transition matrix of the Markov chain \((\nu_k)_{k \geq 0}\) has the following form:

\[
\pi_\nu = \begin{pmatrix}
q & 1-q \\
1-q & q
\end{pmatrix}, \text{ with } q = \frac{1}{3}.
\]

Now we define the occupation measure of the vertical skeleton and the embedded random walk.

**Definition 2.3.** Let \(n \geq 0\). Let for every \(y \in \mathbb{Z}\)

\[\eta_n(y, \pm 1) := \sum_{k=0}^{n} 1_{\{(Y_k, \nu_k) = (y, \pm 1)\}}.\]

We define the total occupation measure \(\eta_n\) at \(y\) by

\[\eta_n(y) := \eta_n(y, 1) + \eta_n(y, -1).\]

**Definition 2.4.** Let \((\xi_i)_{i \geq 0, y \in \mathbb{Z}}\) be a family of i.i.d. geometric random variables with parameter \(p = \frac{1}{2}\), defined on \((\Omega, \mathcal{A}, P)\). We call embedded random walk the process \((X_n)_{n \geq 0}\) defined by

\[X_n := \sum_{y \in \mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)},\]

with the convention that \(\sum_{i} \) vanishes whenever \(\eta_{n-1}(y) = 0\).

\(X_n\) represents the horizontal displacement of the random walk \((M_n)_{n \geq 0}\) after the \(n\)-th vertical movement has been performed.

Let \(T_n = n + \sum_{y \in \mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}\) be the time just after the random walk \((M_n)\) has performed its \(n\)-th vertical move. Then it’s straightforward to see that \(M_{T_n} = (X_n, Y_n)\), where \((Y_n)\) is the first component of the vertical skeleton \((Y_n, \nu_n)\). Now denote by \(\sigma_n\) the sequence of consecutive returns to 0 of \((Y_n)\) (by lemma 2.5 below it follows that \(\sigma_n < \infty\) almost surely \(\forall n\).) Obviously, \(M_{\sigma_n} = (X_{\sigma_n}, 0)\).

Let \(F_n := \sigma(\nu_k, k \leq n), G := \sigma(\epsilon_y, y \in \mathbb{Z})\) and \(F \vee G = \sigma(F \cup G)\). We shall need the following lemmas.

**Lemma 2.5.** As \(n \to \infty\)

\[P_0(Y_{2n} = 0) \sim \frac{C}{\sqrt{n}},\]

with \(C > 0\). In particular, \(Y_n\) is recurrent.

**Proof:** Since \(Y_{2n} := \sum_{k=1}^{2n} \nu_k\), the result follows by the local limit theorem for ergodic Markov chain with finite state space (see Kolmogorov (1949)). \(\square\)

**Lemma 2.6.** [Campanino and Petritis (2003), lemma 2.3] \((X_{\sigma_n})_{n \geq 0}\) is transient \(\implies (M_n)_{n \geq 0}\) is transient, i.e.

\[\sum_{n=0}^{\infty} P_0(X_{\sigma_n} = 0 | F \vee G) < \infty \ a.s \implies \sum_{l=0}^{\infty} P_0(M_l = (0, 0) | G) < \infty \ a.s .\]
2.2. Characteristic function of the embedded random walk. Let \( n \in \mathbb{N}, y \in \mathbb{Z} \) and define

\[
m_{n,o}(y) := \sum_{k=0}^{n} 1\{Y_k=y,v_k=v_{k+1}\},
\]

\[
m_{n,e}(y) := \sum_{k=0}^{n} 1\{Y_k=y,v_k\neq v_{k+1}\}.
\]

They satisfy \( m_{n,o}(y)+m_{n,e}(y) = \eta_n(y) \). So we can decompose the embedded random walk \( X_n \) as follows

\[
X_n = \sum_{y \in \mathbb{Z}} \epsilon_y \left( \sum_{i=1}^{m_{n-1,o}(y)} \xi^{(y)}_{i,o} + \sum_{i=1}^{m_{n-1,e}(y)} \xi^{(y)}_{i,c} \right),
\]

where \( \xi^{(y)}_{i,o} \) and \( \xi^{(y)}_{i,c} \) are two independent families of i.i.d. random variables having, respectively, the law of a geometric random variable taking only odd integer values and only null or even integer values; precisely, it is easy to see that

\[
P(\xi^{(y)}_{i,o} = 2k+1) = P(\xi^{(y)}_{i,c} = 2k) = \frac{1}{2} \left( \frac{1}{2} \right)^{2k+2}
\]

for every \( k \in \mathbb{N} \). In the present work, we shall say that a random variable is even geometric if it has the same law of \( \xi^{(0)}_{1,e}, \) and odd geometric if it has the same law of \( \xi^{(0)}_{1,o} \). Their characteristic function are, respectively,

\[
\chi_o(\theta) := \mathbb{E}(\exp(i\theta \xi^{(y)}_{i,o})) = \frac{3e^{i\theta}}{4 - e^{2i\theta}}
\]

and \( \chi_e(\theta) := \mathbb{E}(\exp(i\theta \xi^{(y)}_{i,e})) = e^{-i\theta} \chi_o(\theta) \). Observe that \( r_e(\theta) := |\chi_e(\theta)| = |e^{-i\theta} \chi_o(\theta)| = |\chi_o(\theta)| = r_o(\theta) \). Moreover, note that \( r_o(\theta) \) is an even function.

Lemma 2.7. The characteristic function of \( X_n \) is

\[
\mathbb{E}(\exp(i\theta X_n)) = \mathbb{E} \left( \prod_{y \in \mathbb{Z}} \chi_o(\theta \epsilon_y)^{m_{n-1,o}(y)} \chi_e(\theta \epsilon_y)^{m_{n-1,e}(y)} \right).
\]

Proof: We have

\[
\mathbb{E}(\exp(i\theta X_n)) = \mathbb{E} \left( \mathbb{E} \left( \exp(i\theta \sum_{y \in \mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi^{(y)}_{i} \mid \mathcal{F}_n \vee \mathcal{G} \right) \right)
\]

\[
= \mathbb{E} \left( \mathbb{E} \left( \prod_{y \in \mathbb{Z}} \exp(i\theta \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi^{(y)}_{i}) \mid \mathcal{F}_n \vee \mathcal{G} \right) \right)
\]

\[
= \mathbb{E} \left( \prod_{y \in \mathbb{Z}} \exp(i\theta \epsilon_y \sum_{i=1}^{\eta_{n-1,o}(y)} \xi^{(y)}_{i,o} + \sum_{i=1}^{\eta_{n-1,e}(y)} \xi^{(y)}_{i,c}) \mid \mathcal{F}_n \vee \mathcal{G} \right)
\]

\[
= \prod_{y \in \mathbb{Z}} \chi_o(\theta \epsilon_y)^{m_{n-1,o}(y)} \chi_e(\theta \epsilon_y)^{m_{n-1,e}(y)}
\]

\( \square \)
3. Proofs

3.1. The random walk on the $\mathbb{H}_c$ lattice. This section is devoted to the proof of theorem 1.1. As in Campanino and Petritis (2003) we define, for $n \geq 0$, the following families of events:

\[ A_{n,1} := \{ \max_{0 \leq k \leq 2^n} |Y_k| < n^{\frac{1}{2} + \delta_1} \}, \delta_1 > 0 \]

\[ A_{n,2} := \{ \max_{y \in \mathbb{Z}} \eta_{2n-1}(y) < n^{\frac{1}{2} + \delta_2} \}, \delta_2 > 0 \]

\[ A_n := A_{n,1} \cap A_{n,2} \]

\[ B_n := A_n \cap \{ |\sum_{y \in \mathbb{Z}} \epsilon_y \eta_{2n-1}(y)| > n^{\frac{1}{2} + \delta_3} \}, \delta_3 > 0 \]

where $\delta_1$, $\delta_2$ and $\delta_3$ will be chosen later. Observe that, for every $n$, $A_n \in \mathcal{F}_{2n}$ and $B_n \subset A_n$, $B_n \in \mathcal{F}_{2n} \lor \mathcal{G}$. Thus, we have $p_n = p_{n,1} + p_{n,2} + p_{n,3}$, where

\[ p_n = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0) \]

\[ p_{n,1} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, B_n) \]

\[ p_{n,2} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A_n / B_n) \]

\[ p_{n,3} = \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A_n^c). \]

In order to prove transience, we will provide estimates of respectively $p_{n,1}$, $p_{n,2}$ and $p_{n,3}$, from which we will deduce that $\sum_{n \geq 0} p_n$ is convergent. Then the result will follow at once thanks to the following lemma.

\textbf{Lemma 3.1.} If $\sum_{n \geq 0} p_n < \infty$, then $(M_n)_{n \geq 0}$ is transient.

\textit{Proof:} From the trivial majorization

\[ \sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0) = \sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0, Y_{\sigma_n} = 0) \leq \sum_{n \geq 0} \mathbb{P}(X_{2n} = 0, Y_{2n} = 0), \]

we deduce that $\sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0) < \infty$ and hence also $\sum_{n \geq 0} \mathbb{P}(X_{\sigma_n} = 0 \mid \mathcal{F}_n \lor \mathcal{G}) < \infty$ a.s. By lemma 2.6, this implies the a.s. transience of $(M_n)_{n \geq 0}$. \hfill \Box

3.1.1. Estimate of $p_{n,1}$. Define

\[ N_0^+ := \sum_{k=1}^{2n} 1_{\{\epsilon_{2k-1} = 1\}} 1_{\{\nu_k = \nu_{k+1}\}}, \]

\[ N_e^+ := \sum_{k=1}^{2n} 1_{\{\epsilon_{2k-1} = 1\}} 1_{\{\nu_k \neq \nu_{k+1}\}}, \]

\[ N_0^- := \sum_{k=1}^{2n} 1_{\{\epsilon_{2k-1} = -1\}} 1_{\{\nu_k = \nu_{k+1}\}}, \]

\[ N_e^- := \sum_{k=1}^{2n} 1_{\{\epsilon_{2k-1} = -1\}} 1_{\{\nu_k \neq \nu_{k+1}\}}, \]

and

\[ \Delta_{n,o} := N_0^+ - N_0^-, \]
\[ \Delta_{n,e} := N_e^+ - N_e^-. \]
Lemma 3.2. We have
\[
\mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \exp \left( t(m_o \Delta_{n,o} + m_e \Delta_{n,e}) + \frac{t^2}{2} \left( s_o^2 \Sigma_{n,o} + s_e^2 \Sigma_{n,e} \right) + O(t^3 n) \right).
\]

Proof: Consider the generating function \( \phi_o(t) = \mathbb{E}(\exp(t\xi_o)) \), defined in \( t \in ]-\infty, \ln 2[ \), the largest domain in which \( \phi_o(t) < \infty \). We have
\[
\phi_o(t) = \sum_{k \geq 0} \mathbb{P}(\xi_o = k) e^{tk} = \sum_{k \geq 0} \mathbb{P}(\xi_o = k) \left( 1 + kt + \frac{(kt)^2}{2} + O(t^3) \right)
= 1 + \mathbb{E}(t\xi_o) + \frac{1}{2} \mathbb{E} \left( (t\xi_o)^2 \right) + O(t^3)
= \exp \left( tm_o + t^2 \frac{s_o^2}{2} + O(t^3) \right)
\]

Analogously, we define \( \phi_e(t) \) to be the generating function of \( \xi_e \), and observe that its generating function behaves as \( \phi_e(t) = \exp \left( tm_e + t^2 \frac{s_e^2}{2} + O(t^3) \right) \). Note that also \( \phi_e(t) \) is finite if and only if \( t \in ]-\infty, \ln 2[ \). Finally, putting all together, we have
\[
\mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathbb{E} \left( \exp(t \sum_{k=1}^{2n} 1_{\{\xi_k = 1\}} \xi_k - t \sum_{k=1}^{2n} 1_{\{\xi_k = -1\}} \xi_k) \mid \mathcal{F}_{2n} \vee \mathcal{G} \right)
= \phi_o(t)^{N_o^+} \phi_e(t)^{N_e^+} \phi_o(t)^{N_o^-} \phi_e(t)^{N_e^-}
= \exp \left( t(m_o \Delta_{n,o} + m_e \Delta_{n,e}) + \frac{t^2}{2} \left( s_o^2 \Sigma_{n,o} + s_e^2 \Sigma_{n,e} \right) + O(t^3 n) \right).
\]

Proposition 3.3. For large \( n \), on the set \( B_n \), we have
\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O}(\exp(-n^{\delta'}))
\]
for any \( \delta' \in ]0, 2\delta[ \).

Proof: Using Markov inequality and lemma 3.2, we have for \( t < 0 \),
\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \mathbb{P}(X_{2n} \leq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G})
= \mathbb{P}(tX_{2n} \geq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G})
= \mathbb{P}(\exp(tX_{2n}) \geq 1 \mid \mathcal{F}_{2n} \vee \mathcal{G})
\leq \mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G})
\]

\[
\leq \exp \left( t(m_o \Delta_{n,o} + m_e \Delta_{n,e}) + \frac{t^2}{2} \left( s_o^2 \Sigma_{n,o} + s_e^2 \Sigma_{n,e} \right) + O(t^3 n) \right).
\]
For $0 < t < \ln 2$, we obtain analogously the same bound

\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \mathbb{P}(X_{2n} \geq 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \mathbb{E}(\exp(tX_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G})
\]

\[
= \exp \left( t(m_o\Delta_{n,o} + m_e\Delta_{n,e}) + \frac{t^2}{2} \left( \frac{s_o^2}{m_o} \Sigma_{n,1} + \frac{s_e^2}{m_e} \Sigma_{n,-1} \right) + \mathcal{O}(t^3n) \right)
\]

\[
\leq \exp(t \max\{m_o, m_e\}(\Delta_{n,o} + \Delta_{n,e}) + \frac{t^2}{2} \max\{s_o^2, s_e^2\}(\Sigma_{n,o} + \Sigma_{n,e}) + \mathcal{O}(t^3n))
\]

\[
\leq \exp(t \max\{m_o, m_e\}\Delta_n + t^2 \max\{s_o^2, s_e^2\}n + \mathcal{O}(t^3n))
\]

\[
= \exp(tm\Delta_n + t^2s^2n + \mathcal{O}(t^3n)),
\]

where $m := \max\{m_o, m_e\}$ and $s := \max\{s_o, s_e\}$. Then, for the case $\Delta_n > n^{\frac{3}{2}+\delta}$, we choose $t = \frac{m}{2s}n^{\frac{\delta}{2}}$ and get

\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \exp \left( \frac{-m^2}{2s^2}n^{2\delta} + \frac{m^2}{4s^2}n^{2\delta - 1}s^2n + \mathcal{O}((n^{\delta - \frac{1}{2}})^3n) \right)
\]

\[
\leq \exp \left( \frac{-m^2}{4s^2}n^{2\delta} + \mathcal{O}(n^{3\delta - \frac{1}{2}}) \right)
\]

Finally, for the case $\Delta_n < -n^{\frac{3}{2}+\delta}$ we choose $t = \frac{m}{2s}n^{\frac{\delta}{2}}$ and get exactly the same bound. \qed

**Corollary 3.4.**

\[
\sum_{n \in \mathbb{N}} p_{n,1} < \infty.
\]

**Proof:** Observe that

\[
\mathbb{P}(X_{2n} = 0, Y_{2n} = 0, B_n \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq 1_{B_n} \mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}). \quad (3.1)
\]

In proposition 3.3 we proved that, on $B_n$, $\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O}(\exp(-n^{\delta'}))$ for $\delta' \in [0, 2\delta]$. Thus, taking expectations on both sides of (3.1) we obtain

\[
p_{n,1} \leq \mathbb{E}(\mathcal{O}(\exp(-n^{\delta'}))1_{B_n}) = \mathcal{O}(\exp(-n^{\delta'}))\mathbb{E}(1_{B_n}) \leq \mathcal{O}(\exp(-n^{\delta'})).
\]

Thus, $p_{n,1}$ is summable. \qed

### 3.1.2. Estimate of $p_{n,2}$.

**Lemma 3.5.** We have

\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).
\]

**Proof:** In lemma 2.7 we saw that the conditional characteristic function of $X_{2n}$ w.r.t. $\mathcal{F}_{2n} \vee \mathcal{G}$ takes on the following form:

\[
\chi(\theta) := \mathbb{E}(\exp(i\theta X_{2n}) \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \prod_{\gamma \in \mathbb{Z}} \chi_{o}(\theta \epsilon_{y})^{n(\gamma_{2n-1})_{o}} \chi_{e}(\theta \epsilon_{y})^{n(\gamma_{2n-1})_{e}}.
\]

We have, by the inversion formula

\[
\mathbb{P}(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi(\theta)d\theta.
\]
Define $r(\theta) := |\chi_c(\theta)| = |\chi_o(\theta)| = \frac{3}{\sqrt{17-8\cos(2\theta)}}$. Thus

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \chi(\theta) d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{y \in \mathbb{Z}} |\chi_o(\theta e_y)|^{m_{2n-1}(y)} |\chi_c(\theta e_y)|^{m_{2n-1}(y)} d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{y \in \mathbb{Z}} |\chi_o(\theta e_y)|^{n_{2n-1}(y)} d\theta
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\theta) \sum_{y \in \mathbb{Z}} n_{2n-1}(y) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\theta)^{2n} d\theta.
\]

Now we use the parity of $r(\theta)$ and the fact that $r(\theta) < 1$ in $\theta \in [0, \pi] \cup [\pi, 2\pi]$ to bound with $K < 1$ the function $r(\theta)$ in the interval $[\frac{\pi}{4}, \frac{3\pi}{4}] \cup [-\frac{3\pi}{4}, -\frac{\pi}{4}]$. We obtain

\[
P(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G}) \leq \frac{1}{\pi} \int_{0}^{\pi} r(\theta)^{2n} d\theta + \frac{1}{\pi} \int_{\pi}^{2\pi} r(\theta)^{2n} d\theta + O(K^{2n})
\]
\[
= \frac{2}{\pi} \int_{0}^{\pi} r(\theta)^{2n} d\theta + O(K^{2n}).
\]

Now, we have $r(\theta) = 1 - \frac{8}{5} \theta^2 + O(\theta^3)$ and so for large $n$

\[
\int_{0}^{\frac{\pi}{4}} r(\theta)^{2n} d\theta \sim \int_{0}^{\frac{\pi}{4}} \left(e^{-\frac{8}{5} \theta^2}\right)^{2n} d\theta \sim \int_{0}^{\infty} \left(e^{-\frac{8}{5} \pi n \theta^2}\right) d\theta \sim \frac{c}{\sqrt{n}},
\]

with $c = \sqrt{\frac{9\pi}{16}}$.

Finally we need the following lemma, whose proof can be found in the cited paper.

**Lemma 3.6.** [Campanino and Petritis (2003), Prop. 4.3] For large $n$, we have

\[
P(A_n \setminus B_n \mid \mathcal{F}_{2n}) = O(n^{-\frac{3}{4} + \frac{2k}{2k+4}}).
\]

**Corollary 3.7.**

\[
\sum_{n \in \mathbb{N}} p_{n,2} < \infty.
\]

**Proof:**

\[
p_{n,2} = P(X_{2n} = 0, Y_{2n} = 0, A_n \setminus B_n)
\]
\[
= P(Y_{2n} = 0)P(X_{2n} = 0, A_n \setminus B_n)
\]
\[
= E(Y_{2n})E(X_{2n} = 0 \mid A_n \setminus B_n)
\]
\[
= E(Y_{2n})E(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G})
\]
\[
= E(Y_{2n})E(X_{2n} = 0 \mid \mathcal{F}_{2n} \vee \mathcal{G})
\]
\[
= O(n^{-\frac{3}{4} n^{-\frac{3}{4} + \frac{2k}{2k+4}}} = O(n^{-\frac{3}{4} + \frac{2k}{2k+4}}).
\]

Where we used the estimates of lemma 2.5, lemma 3.5 and lemma 3.6. Now it’s enough to choose $2\delta_3 + \delta_1 < \frac{1}{2}$.

$$\square$$
3.1.3. Estimate of \( p_{n,3} \). Notice that \( A^c_n = A^c_{n,1} \cup A^c_{n,2} \). We are going to provide exponential estimates of both \( \mathbb{P}(A^c_{n,1} \mid Y_{2n} = 0) \) and \( \mathbb{P}(A^c_{n,2} \mid Y_{2n} = 0) \).

**Lemma 3.8.** We have, for large \( n \) and for every \( t > 0 \)
\[
\mathbb{E}(e^{tY_{2n}}) \sim c(\lambda_1(t))^{2n},
\]
where \( c > 0 \) and \( \lambda_1(t) > 0 \).

**Proof:** We have, by the Markov property,
\[
\mathbb{E}_{\nu_0}(e^{tY_{2n}}) = e^{t\nu_0} \int \pi_{\nu}(\nu_0, dy_1)e^{ty_1} \int \pi_{\nu}(y_1, dy_2)e^{ty_2} \cdots \int \pi_{\nu}(y_{2n-1}, dy_{2n})e^{ty_{2n}}. 
\]
(3.2)
It is now easy to see that can compute the quantity (3.2) by means of the 2n-th power of the matrix
\[
\pi_{\nu,t} := \left( \begin{array}{ccc} qe^t & (1-q)e^{-t} \\ (1-q)e^t & qe^{-t} \end{array} \right)
\]
which has the following eigenvalues
\[
\lambda_{1,2}(t) = \frac{q(e^t - e^{-t}) \pm \sqrt{q^2(e^t + e^{-t})^2 - 4(2q - 1)}}{2}
\]
By the spectral decomposition, we know that \((\pi_{\nu,t})^{2n} \sim \lambda_1^{2n}(t)h_1h_1^T \) for large \( n \), where \( \lambda_1 \) is the largest eigenvalue. Hence for large \( n \)
\[
\mathbb{E}(e^{tY_{2n}}) = \sum_{y \in \{1,-1\}} (\pi_{\nu,t})^{2n}(\nu_0, y) \sim c\lambda_1^{2n}(t), \ c > 0.
\]

**Proposition 3.9.** For large \( n \), there exist \( \delta > 0 \) such that
\[
\mathbb{P}(A^c_{n,1} \mid Y_{2n} = 0) = \mathcal{O}(\exp(-n^\delta)).
\]

**Proof:** Let \( a_n = [n^{\frac{1}{2}+\delta}] \); we have
\[
\mathbb{P}(\max_{0 \leq k \leq 2n} Y_k \geq a_n \mid Y_{2n} = 0) = \sum_{y \in \{a_n, a_{n+1}, \ldots, n\}} \frac{\mathbb{P}(\max_{0 \leq k \leq 2n} Y_k = y, Y_{2n} = 0)}{\mathbb{P}(Y_{2n} = 0)}
\]
The estimate for \( \mathbb{P}(\min_{0 \leq k \leq 2n} Y_k \leq -a_n \mid Y_{2n} = 0) \) can be obtained by the same argument, so we shall omit it. By the reflection principle (note that the probability of any reflected path is equal to a multiplicative constant times the probability of the original path, this constant being 1/2 or 2)
\[
\sum_{y \in \{a_n, a_{n+1}, \ldots, n\}} \mathbb{P}(\max_{0 \leq k \leq 2n} Y_k = y, Y_{2n} = 0) \leq 2 \sum_{y \in \{a_n, a_{n+1}, \ldots, n\}} \mathbb{P}(Y_{2n} = 2y)
\]
\[
= 2\mathbb{P}(Y_{2n} \geq 2a_n)
\]
\[
\leq 2\inf_{t > 0} \mathbb{P}(\exp(tY_{2n}) \geq \exp(2ta_n))
\]
\[
\leq 2\inf_{t > 0} \frac{\mathbb{E}(e^{tY_{2n}})}{e^{2ta_n}}
\]
By the previous lemma, we have that for large \( n \)
\[
\mathbb{E}(e^{tY_{2n}}) \sim c \left( \frac{q(e^t - e^{-t}) + \sqrt{q^2(e^t + e^{-t})^2 - 4(2q - 1)}}{2} \right)^{2n}
\]
Now by the Taylor expansion at \( t = 0 \), and substituting \( q = \frac{1}{3} \), we have
\[
\frac{q(e^t - e^{-t}) + \sqrt{q^2(e^t + e^{-t})^2 - 4(2q - 1)}}{2} = \frac{2 + \sqrt{7}}{6} + \left(1 + \frac{1}{3\sqrt{7}}\right) t^2 + o(t^2)
\]
\[
< 1 + \left(1 + \frac{1}{3\sqrt{7}}\right) t^2,
\]
where the inequality holds for \( t \leq t^* \) for sufficiently small \( t^* \). Hence \( \mathbb{E}(e^{Y_{2n}}) < c(1 + 2nst^2) < c \exp(2nst^2) \), with \( s = \frac{1}{6} + \frac{1}{3\sqrt{7}} \). So for large \( n \)
\[
\inf_{t > 0} \frac{\mathbb{E}(e^{Y_{2n}})}{e^{2ta_n}} < c \inf_{t > 0, t \leq t^*} \exp(-2ta_n) \exp(2nst^2) = c \exp \left( -\frac{a_n^2}{2sn} \right) = c \exp \left( -\frac{n^{2\delta_1}}{2s} \right).
\]
and, in the first equality, we used the fact that the minimum is attained at \( t = \frac{a_n}{2sn} \), which goes to 0 as \( n \) tends to infinite. Then, putting all together and using lemma 2.5, we obtain
\[
\mathbb{P}(A_{n,1}^c | Y_{2n} = 0) = O \left( nn^{-\delta} \exp \left( -\frac{n^{2\delta_1}}{2s} \right) \right).
\]

\[\square\]

**Lemma 3.10.** Let \( \sigma_{a,a} \) the time of first return to state \( a \in \mathbb{Z} \times \{-1, 1\} \) of the Markov chain \( (Y_n, \nu_n) \) starting at \( a \). We have
\[
\mathbb{E}(e^{-\sigma_{a,a}}) \sim \exp(-c\sqrt{t}),
\]
with \( c > 0 \) (i.e. \( \lim_{t \to 0} \frac{\mathbb{E}(e^{-\sigma_{a,a}})}{\exp(-c\sqrt{t})} = 1 \)).

**Proof:** By lemma 2.5 we have \( p_{a,n} \sim \frac{C}{\sqrt{n}} \) as \( n \to \infty \) with \( C > 0 \). And this implies, by the Tauberian theorem (Feller \(1966\), p.447, th.5), that \( G_{a,a}(s) \sim \frac{C_1}{\sqrt{1-s}} \) as \( s \to 1 \), \( C_1 > 0 \). Then, using a standard result from the theory of Markov chain (e.g. cfr. Woess \(2009\)), we see that as \( s \to 1 \)
\[
\mathbb{E}(s^{\sigma_{a,a}}) = 1 - \frac{1}{G_{a,a}(s)} \sim 1 - c\sqrt{1-s},
\]
where \( c = C_1^{-1} \). Then, if we write \( s = e^{-t} \), we have for \( t \to 0 \)
\[
\mathbb{E}(e^{-\sigma_{a,a}}) \sim 1 - c\sqrt{1-e^{-t}} \sim 1 - c\sqrt{t} \sim e^{-c\sqrt{t}}.
\]
\[\square\]

**Proposition 3.11.** For large \( n \) there exist \( \delta' > 0 \) such that
\[
\mathbb{P}(A_{n,2}^c | Y_{2n} = 0) = O \left( \exp(-n^{\delta'}) \right).
\]

**Proof:** We have
\[
\mathbb{P}(A_{n,2}^c | Y_{2n} = 0) = \mathbb{P} \left( \max_{y \in \mathbb{Z}} \eta_{2n-1}(y) \geq n^{1/2 + \delta_2} | Y_{2n} = 0 \right) \leq \sum_{y \in \mathbb{Z}} \frac{\mathbb{P} \left( \eta_{2n-1}(y) \geq n^{1/2 + \delta_2} \right)}{\mathbb{P}(Y_{2n} = 0)}.
\]

On the other hand we have
\[
\mathbb{P} \left( \eta_{2n-1}(y) \geq a_n \right) \leq \mathbb{P} \left( \eta_{2n-1}(y, 1) \geq \frac{a_n}{2} \right) + \mathbb{P} \left( \eta_{2n-1}(y, -1) \geq \frac{a_n}{2} \right).
\]
\[\text{(3.3)}\]
Now let $\sigma^{(k)}_{a,a}$ be the time of $k$-th return to point $a$ for the process $(Y_n, \nu_n)_{n \geq 0}$ starting at $a$. Observe that $\mathbb{P}(\eta_{2n-1}(a) \geq a_n) \leq \mathbb{P}_a(\sigma^{(a_n)}_{a,a} \leq 2n)$ and consider the first term at the right-hand-side of (3.3). Notice that by lemma 3.10, $\mathbb{E}(e^{-t\sigma_{a,a}})^m \sim \exp(-cm\sqrt{t})$ for every $m \in \mathbb{N}$; then, for $C > 1$ there exists $t^*$ s.t. for every $t < t^*$, $\mathbb{E}(e^{-t\sigma_{a,a}})^m \leq C \exp(-cm\sqrt{t})$. Hence for sufficiently large $n$

$$\mathbb{P}\left(\eta_{2n-1}(y,1) \geq \frac{a_n}{2}\right) \leq \inf_{t > 0} \mathbb{P}_y\left(\exp\left(-t\sigma^{(1)}_{(y,1), (y,1)}\right) \geq \exp(-2nt)\right) \leq \inf_{t > 0} \exp(2nt) \left(\mathbb{E}\left(\exp\left(-t\sigma^{(1)}_{(y,1), (y,1)}\right)\right)\right)^{\frac{2nt}{\sigma(2)}} \leq C \inf_{t > 0, t < t^*} \exp\left(2nt - \frac{a_n}{2}c\sqrt{t}\right) = C \exp\left(-\frac{C^2a^2_n}{32n}\right) = C \exp\left(-c'n^{2\delta_2}\right)$$

with $c' = \frac{C^2}{32}$, where we used the fact that the minimum is attained at $t = \left(\frac{a_n}{2}\right)^2$.

Since we can provide, with the same procedure, an exponential estimate also for $\eta_{2n-1}(y,-1)$, we finally obtain by lemma 2.5

$$\mathbb{P}(A^\circ_n \mid Y_{2n} = 0) \leq \sum_{y \in \mathbb{Z}} \frac{\mathbb{P}(\eta_{2n-1}(y) \geq n^{\frac{1}{4}+\delta_2})}{\mathbb{P}(Y_{2n} = 0)} = O\left(nn^{\frac{1}{2}} \exp(-cn^{\delta_2})\right).$$

$\square$

**Corollary 3.12.**

$$\sum_{n \in \mathbb{N}} p_{n,3} < \infty.$$  

**Proof:** Combining proposition 3.9 and 3.11, we know that for large $n$

$$\mathbb{P}(A^\circ_n \mid Y_{2n} = 0) = O(\exp(-n^{\min(\delta, \delta')}).$$

Then the result follows by the trivial majorization

$$p_{n,3} := \mathbb{P}(X_{2n} = 0, Y_{2n} = 0, A^\circ_n) \leq \mathbb{P}(Y_{2n} = 0, A^\circ_n) \leq \mathbb{P}(A^\circ_n \mid Y_{2n} = 0).$$

$\square$

This completes the proof of theorem 1.1.

### 3.2. The random walk on the $H_f$ lattice.

This section is devoted to the proof of theorem 1.2. We begin with some preliminary observations regarding the particular properties of the vertical skeleton in this case.

Let $\mathbb{Z}_Q = \mathbb{Z}/Q$ and, for every $y \in \mathbb{Z}$, we write $\overline{y} = y \mod Q$. Consider the process

$$(\overline{Y_n}, \nu_n; \overline{Y_{n+1}}, \nu_{n+1})_{n \geq 0},$$

where $\overline{Y_n} = Y_n \mod Q$. Note that it is clearly a Markov chain, it has $4Q$ different states that enclose the information of the last three movements of $Y_n$: this will be useful later, since we will need an estimate of the number of times $(Y_n)_{n \geq 0}$ “changes direction” before it returns to the origin. We will assume with no restrictions that the vertical skeleton starts from

$$(Y_{-1}, \nu_{-1}, Y_0, \nu_0) = (-1, -1; 0, 1),$$
and then the MC starts from the correspondent class, namely \( s_1 := (-1, -1; 0, 1) \).

**Lemma 3.13.** The process \((\overline{Y}_n, \nu_n; \overline{Y}_{n+1}, \nu_{n+1})_{n \geq 0}\) is a one-class recurrent Markov chain on a finite state space with \(4Q\) different states, with period 2. Its stationary distribution \( \pi \) is defined as follows

\[
\begin{align*}
\pi(\overline{y}, \nu; \overline{y}', \nu') &= \frac{1}{3} \quad \text{if } \nu \neq \nu', \overline{y} = \overline{y}' + \nu \\
\pi(\overline{y}, \nu; \overline{y}', \nu') &= \frac{1}{3} \quad \text{if } \nu = \nu', \overline{y} = \overline{y}' + \nu
\end{align*}
\]

**Proof:** It is easy to verify that \((\overline{Y}_n, \nu_n)_{n \geq 0}\) is a Markov chain with \(2Q\) states and period 2, and that its stationary distribution is \( \tilde{\pi}(\overline{y}, \nu) = \frac{1}{2Q}, \forall (\overline{y}, \nu) \in \mathbb{Z}_Q \times \{-1, 1\} \). Then \((\overline{Y}_n, \nu_n; \overline{Y}_{n+1}, \nu_{n+1})_{n \geq 0}\) is again a MC, whose stationary distribution \( \pi \) is directly derived from \( \tilde{\pi} \) by defining

\[
\pi(\overline{y}, \nu; \overline{y}', \nu') := \tilde{\pi}(\overline{y}, \nu)p(\overline{y}, \nu); (\overline{y}', \nu').
\]

The others statements are straightforward to verify. \( \square \)

To simplify our notation, from now on we shall denote by \( s_1, s_2, ..., s_{4Q} \) the states of the MC \((\overline{Y}_{n-1}, \nu_{n-1}, \overline{Y}_n, \nu_n) \). Accordingly we define

\[
\pi = (\pi(1), ..., \pi(4Q))
\]

to be the vector where the \( i \)-th component is the value that the stationary distribution takes at state \( s_i \), and the occupation measure of the MC

\[
\overline{\eta}_n = (\overline{\eta}_n(1), ..., \overline{\eta}_n(4Q)),
\]

where

\[
\overline{\eta}_n(i) := \sum_{i=1}^{n} \mathbf{1}_{\{(\overline{Y}_n, \nu_n; \overline{Y}_{n+1}, \nu_{n+1}) = s_i\}}, \quad 1 \leq i \leq 4Q.
\]

Define the event

\[
\overline{S} := \{(\overline{Y}_{2n-1}, \nu_{2n-1}, \overline{Y}_{2n}, \nu_{2n}) = s_1\},
\]

that the MC returns to its initial state after \(2n\) steps, and the event

\[
S := \{(Y_{2n-1}, \nu_{2n-1}, Y_{2n}, \nu_{2n}) = (-1, -1; 0, 1)\} = \left\{ \overline{S}, \sum_{i=0}^{2n} \nu_i = 0 \right\}.
\]

Finally, for every \( n \in \mathbb{N} \) define

\[
\overline{\pi} := (\overline{\eta}_{2n}, \sum_{i=0}^{2n} \nu_i).
\]

**Proposition 3.14.** We have, for large even \( n \) and positive \( C \)

\[
\mathbb{P}(\|\overline{\eta}_{2n} - 2n\pi\| \leq Cn^{\frac{1}{2}} \mid S) \geq \delta_C > 0.
\]

**Proof:** Let \( 0 < \tau_1 < ... < \tau_n < ... \) be the sequence of times at which the MC returns to the initial state \( s_1 \). Define

\[
\overline{\tau}_n := \left( \overline{\eta}_{\tau_n}, \sum_{i=0}^{\tau_n} \nu_i \right),
\]

...
and
\[ \delta(n) := r_n - r_{n-1}. \]

It is easy to see that \((\delta(n))_{n \geq 1}\) are i.i.d. random variables with finite moments of the first and second order (for example, see [Kolmogorov (1949) lemma 4, lemma 5]. Actually they have finite moments of all orders) and precisely
\[ \mu := E(\delta(1)) = (\pi/\pi_1, 0) = \left( \frac{\pi_1}{\pi_1}, \frac{\pi_2}{\pi_1}, ..., \frac{\pi_{4Q}}{\pi_1}, 0 \right). \]

Let moreover \(Z \subset \mathbb{Z}^{4Q+1}\) be the so-called fundamental lattice, that is the minimal group generated by the set of vectors \(x\) such that \(P(\delta(1) = x) > 0\) and let \(L\) be the smallest vector space containing \(Z\); finally, denote by \(L_1\) the space of all vectors in \(L\) such that their first component is equal to 0, and \(r := \dim(L_1)\). In [Kolmogorov (1949)] it is shown that \(L_1\) coincides to the space generated by the set of vectors \(x\) such that \(x = x' - x''\) with \(P(\delta(1) = x') > 0\) and \(P(\delta(1) = x'') > 0\).

Under these conditions, in [Kolmogorov (1949), lemma 16] Kolmogorov establishes a LLT type result for the sums \(\delta(1) + \delta(2) + ...\). Precisely, he proves that if \(m \in Z\) then, as \(m_1 \to \infty\)
\[
(m_1)^{\ast}P(\overline{m} = m, \overline{S}) - c_1 \exp \left\{ Q \left( \frac{m^T - m_1\mu^T}{\sqrt{m_1}} \right) \right\}
\]
tends to 0, where \(c_1 > 0\) is the density of the points of \(Z\) in \(L_1\), \(Q\) is the quadratic form associated to the inverse of the covariance matrix in the space \(L_1\), and the convergence is uniform for \(m\) such that
\[
|m_i - m_1\pi_i| \leq C\sqrt{m_1}, \forall 1 \leq i \leq 4Q, \quad \text{and} \quad |m_{4Q+1}| \leq C\sqrt{m_1},
\]
with any fixed \(C\). Note that under condition (3.5) we have
\[
2n = \frac{m_1}{\pi_1} + O(\sqrt{m_1}). \tag{3.6}
\]

Now let \(m \in Z\) s.t. \(\sum_{i=1}^{4Q} m_i = 2n\) and \(m_{4Q+1} = 0\), and write \(m = (x, 0)\), where \(x := (m_1, ..., m_{4Q})\). We have
\[
P(\overline{m}_2n = x \mid S) = \frac{P(\overline{m}_2n = x, S)}{P(S)} = \frac{P(\overline{m} = m, \overline{S})}{P(S)}
\]

Now, by lemma 2.5 we can find \(\tilde{C} > 0\) such that \(P(S) \sim \frac{C}{\sqrt{n}}\) for large \(n\). Then, as \(n \to \infty\), thanks to (3.4) and (3.6), we have
\[
P(\overline{m}_2n = x \mid S) \sim \frac{\sqrt{n}}{C}P(\overline{m} = m, \overline{S}_0) \sim \frac{c_1}{C(n)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2}Q \left( \frac{(x, 0)^T - m_1(\pi/\pi_1, 0)^T}{\sqrt{m_1}} \right) \right\}
\]
and thus we obtain
\[
P(|\overline{m}_2n - \frac{m_1}{\pi_1}\pi| \leq C\sqrt{m_1} \mid S) = \sum_{m \in Z, m = (x, 0), \|x - \frac{m_1}{\pi_1}\pi\| \leq C\sqrt{m_1}} P(\overline{m}_2n = x \mid S) \geq \delta_0 > 0
\]
for appropriate \(c' > 0\). On the other hand, if \(|\overline{m}_2n - \frac{m_1}{\pi_1}\pi| \leq C\sqrt{m_1}\), then \(|\overline{m}_2n - 2n\pi| \leq C'\sqrt{n}\) for an appropriate \(C' > 0\); in fact, by (3.6) we have
\[
|\overline{m}_2n - 2n\pi| \leq |\overline{m}_2n - \frac{m_1}{\pi_1}\pi| + |\frac{m_1}{\pi_1}\pi - 2n\pi| \leq C_1\sqrt{m_1} \sim C_2\sqrt{n\pi},
\]
This completes the proof. □

3.2.1. Proof of recurrence. To prove recurrence it is enough to show that \( \sum_{k=1}^{\infty} \mathbb{P}(X_{2k} = 0, Y_{2k} = 0) = \infty \), since
\[
\sum_{k=1}^{\infty} \mathbb{P}(M_{2k} = (0, 0)) \geq \sum_{k=1}^{\infty} \mathbb{P}(X_{2k} = 0, Y_{2k} = 0).
\]

**Definition 3.15.** We define the random sums
\[
S_{n,e} := \sum_{i=1}^{2n} f_e(\overline{y}_{i-1}, \nu_{i-1}, Y_i, \nu_i) := \sum_{i=1}^{2n} f(\overline{y}_{i-1})1_{\{\nu_{i-1} \neq \nu_i\}},
\]
\[
S_{n,o} := \sum_{i=1}^{2n} f_o(\overline{y}_{i-1}, \nu_{i-1}, Y_i, \nu_i) := \sum_{i=1}^{2n} f(\overline{y}_{i-1})1_{\{\nu_{i-1} = \nu_i\}}.
\]

Note that \( \mathbb{E}_\pi(S_{n,e}) = \sum_{i=1}^{2n} 2 \frac{1}{2^Q} \sum_{\overline{y} \in \mathbb{Z}^Q} f(\overline{y}) = 0 \), and analogously \( \mathbb{E}_\pi(S_{n,o}) = 0 \).

**Lemma 3.16.** We have, for large \( n \) and positive \( C \)
\[
\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq Cn^{\frac{1}{2}} \mid \mathcal{S}) \geq \delta_C > 0.
\]

**Proof:** Define \( u \in \{-1, 0, 1\}^{4Q} \) to be the vector such that \( u_i \) equals to the value that \( f_e \) takes on the \( i \)-th state. Then \( S_{n,e} = \sum_{i=1}^{4Q} \overline{y}_{2n}(i) u_i = u \overline{y}_{2n} \). Analogously, we construct a vector \( v \) such that \( S_{n,o} = u \overline{y}_{2n}^T \). Clearly \( u(2n\pi)^T = v(2n\pi)^T = 0 \).

Now, if \( ||\overline{y}_{2n} - 2n\pi|| \leq C'n^{\frac{1}{2}} \) then we have
\[
|S_{n,e}| + |S_{n,o}| = |u \overline{y}_{2n}^T| + |u \overline{y}_{2n}^T| = |u(\overline{y}_{2n} - 2n\pi^T + 2n\pi^T)| + |v(\overline{y}_{2n} - 2n\pi^T + 2n\pi^T)|
\]
\[
= |u(\overline{y}_{2n} - 2n\pi^T) + u(2n\pi^T)| + |v(\overline{y}_{2n} - 2n\pi^T) + v(2n\pi^T)|
\]
\[
= |u(\overline{y}_{2n} - 2n\pi^T)| + |v(\overline{y}_{2n} - 2n\pi^T)|
\]
\[
\leq \sum_{i=1}^{4Q} |\overline{y}_{2n}(i) - 2n\pi_i| \leq 4QC'\sqrt{n}.
\]

and hence the result follows from proposition 3.14. □

Define a set of constrained paths
\[
\text{Constr}(n, f) := \{ (\gamma, q) : \{-1, 0, 1, ..., 2n\} \rightarrow \mathbb{Z} \times \{-1, 1\} \text{ s.t. } \forall i, \gamma(i) = \gamma(i-1) \pm 1,
\]
\[
(\gamma(-1), q(-1), q(0), q(0)) = (\gamma(2n - 1), q(2n - 1), q(2n), q(2n)) = s_1,
\]
\[
\left| \sum_{i=1}^{2n} f(\overline{y}_{i-1})1_{\{\nu_{i-1} \neq \nu_i\}} \right| + \left| \sum_{i=1}^{2n} f(\overline{y}_{i-1})1_{\{\nu_{i-1} = \nu_i\}} \right| \leq C\sqrt{n} \right\}.
\]

Now, let \( (\gamma, q) \in \text{Constr}(n, f) \). If we prove that
\[
\mathbb{P}(X_{2n} = 0 \mid (Y, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n) \geq \frac{c}{\sqrt{n}}
\]
\[(3.7)\]
then the recurrence of the random walk will follow. In fact, if this is the case, thanks to (3.7) and to lemma 3.16 we have for large \( n \)

\[
P(X_{2n} = 0, Y_{2n} = 0) \geq \mathbb{P}\left( X_{2n} = 0, |S_{n,e}| + |S_{n,o}| \leq Cn^{\frac{1}{2}}, S \right)
\]

\[
= \sum_{(\gamma, q) \in \text{Constr}(n, f)} \mathbb{P}(X_{2n} = 0 \mid (Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n)
\]

\[
\times \mathbb{P}((Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n)
\]

\[
\geq \frac{c}{\sqrt{n}} \sum_{(\gamma, q) \in \text{Constr}(n, f)} \mathbb{P}((Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n)
\]

\[
= \frac{c}{\sqrt{n}} \mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq Cn^{\frac{1}{2}}, S)
\]

\[
\geq \frac{c'}{\sqrt{n}} \mathbb{P}(S) \sim \frac{C'}{n},
\]

where \( C, C' > 0 \) are appropriate constants.

To prove (3.7) we proceed as follows. From now on we fix \((\gamma, q) \in \text{Constr}(n, f)\) and every probability will be taken conditionally to \( \{(Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n\} \), although, in order to simplify the notation, we will omit to write it every time. Let \( N^+_n \) and \( N^-_n \) be, respectively, the number of right (left) directed even steps of the embedded random walk up to time \( 2n \), and \( N^+_o, N^-_o(n) \) the analogous quantities for the odd steps. Observe that \( S_{n,e} = N^+_n - N^-_n \) and \( S_{n,o} = N^+_o - N^-_o \). In particular, since \((\gamma, q) \in \text{Constr}(n, f)\), we have

\[
|N^+_n - N^-_n| + |N^+_o - N^-_o| \leq C \sqrt{n}.
\]

Lemma 3.17. Conditionally to \( \{(Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n\} \), we have \( \mathbb{E}(X_{2n}) = \mathcal{O}(\sqrt{n}) \) and \( \sigma^2(X_{2n}) \sim Cn \), \( C > 0 \).

Proof: We have

\[
\mathbb{E}(X_{2n}) = \mathbb{E}(\xi_{i,e})(N^+_e - N^-_e) + \mathbb{E}(\xi_{i,o})(N^+_o - N^-_o) \leq \max\{\mathbb{E}(\xi_{i,o}), \mathbb{E}(\xi_{i,e})\}C \sqrt{n}
\]

On the other hand, the variance of \( X_{2n} \) is, by independence, the sum of the variances of the even and odd geometric random variables, and so since both of them has finite variance we obtain \( \sigma^2(X_{2n}) \sim Cn \) for some \( C > 0 \). \( \square \)

Lemma 3.18. There exists \( c > 0 \) such that, for every large \( n \) and conditionally to \( \{(Y_i, \nu_i) = (\gamma(i), q(i)) \forall i \leq 2n\} \) we have

\[
P(X_{2n} = 0) \geq \frac{c}{\sqrt{n}}.
\]

(3.8)

Proof: For every \( k \in \mathbb{N} \) let \( \xi_k \) be the random variable that represents the \( k \)-th step of the horizontal random walk \( X_n \). We write

\[
X_{2n} = \sum_{k=1}^{2n} \xi_k = \sum_{i=1}^{N^+_n} \xi_{l,e} + \sum_{i=1}^{N^-_n} \xi_{l,o} - \sum_{i=N^+_n+1}^{N^+_n+N^-_n} \xi_{l,e} - \sum_{i=N^-_n+1}^{N^+_n+N^-_n} \xi_{l,o}
\]

and for every \( k \) let \( a_k := \mathbb{E}(\xi_k), b_k^2 := \sigma^2(\xi_k) \) and

\[
A_n := \sum_{i=1}^{2n} \mathbb{E}(\xi_k), \quad B_n^2 := \sum_{i=1}^{2n} \sigma^2(\xi_k).
\]
First, we are going to show that
\[ B_n \mathbb{P}(X_{2n} = 0) - \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\Delta_n}{B_n} \right)^2} \to 0, \quad (3.9) \]
as \( n \to \infty \). Then, thanks to the previous lemma, we obtain the estimate (3.8). Notice that, if \( \{Y_{2n}, \nu_{2n}\} = (0,1) \), which happens in our case since we are conditioning to \( \{(Y_i, \nu_i) = (\gamma(i), \varphi(i)) \forall i \leq 2n\} \), then the value taken by \( X_{2n} \) is either a null or a even integer (see figure 1.1): then \( \mathbb{P}(X_{2n} = 0) > 0 \).

To prove (3.9), we follow the strategy due to Gnedenko (cfr Gnedenko (1962)). Let \( \phi_\xi(t) = \mathbb{E}(e^{it\xi}) \), and \( \phi_{X_{2n}}(t) = \mathbb{E}(e^{itX_{2n}}) = \phi_{\sum_{k=1}^{2n} \xi_k}(t) = \prod_{k=1}^{2n} \phi_{\xi_k}(t) \), and precisely
\[ \phi_{X_{2n}}(t) = \chi_e(t)^N \chi_o(t)^N \chi_e(-t)^N \chi_o(-t)^N, \]
where we recall that \( \chi_o(t) = \frac{e^{-\frac{1}{2} t^2}}{\sqrt{2\pi}} \) and \( \chi_e(t) = e^{-itX_0}(t) \). In particular note that
\[ |\chi_o(t)| = |\chi_e(t)| = |\chi_o(-t)| = |\chi_e(-t)| = 1 \text{ for } t = 0 \text{ and } t = \pi, \text{ and } \leq 2 \text{ otherwise}. \]
Now, since \( \sum_{k=-\infty}^{\infty} \mathbb{P}(X_{2n} = 2k)e^{ikt} = \phi_{X_{2n}}(t) \), if we integrate both sides of this equation from \( -\pi/2 \) to \( \pi/2 \) we obtain
\[ \pi \mathbb{P}(X_{2n} = 0) = \int_{-\pi/2}^{\pi/2} \phi_{X_{2n}}(x)dx. \]
The following equality is easily proved for every \( z \in \mathbb{R} \).
\[ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} = \frac{1}{2\pi} \int e^{-itz - \frac{t^2}{2}} dt. \]
In particular, in our case, we take \( z := -\frac{\Delta_n}{B_n} \). We write
\[ R_n := 2\pi \left[ \frac{B_n}{2} \mathbb{P}(X_{2n} = 0) - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\Delta_n}{B_n} \right)^2} \right] = J_1 + J_2 + J_3 + J_4, \quad (3.10) \]
where
\[ J_1 = \int_{-A}^{A} e^{it \frac{\Delta_n}{B_n}} \left[ \phi_{X_{2n}-\Delta_n}(t) - e^{-\frac{t^2}{2}} \right] dt \]
\[ J_2 = -\int_{|t|>A} e^{it \frac{\Delta_n}{B_n}} \frac{t^2}{2} dt \]
\[ J_3 = \int_{eB_n < |t| < \pi B_n/2} e^{it \frac{\Delta_n}{B_n}} \phi_{X_{2n}-\Delta_n}(t) dt \]
\[ J_4 = \int_{eB_n < |t| < \pi B_n} e^{it \frac{\Delta_n}{B_n}} \phi_{X_{2n}-\Delta_n}(t) dt \]
So to complete the proof we must show that these quantities tend to 0 as \( n \to \infty \) and for sufficiently large \( A \) and small \( \epsilon \).

First, we show that the sequence \( (\xi_k)_{k \geq 1} \) satisfies the Lyapunov condition with \( \delta = 1 \), that is
\[ \lim_{n \to \infty} \frac{1}{B_n^{2+\delta}} \sum_{k=1}^{2n} \mathbb{E}|\xi_k-a_k|^{2+\delta} = 0. \]
In fact, by the previous lemma, $B_n^2 \sim Cn$ with $C > 0$ and the $\xi_k$s clearly have finite moment of the third order, so for appropriate $C' > 0$

$$\frac{1}{B_n^2} \sum_{k=1}^{2n} E|\xi_k - a_k|^3 \sim \frac{1}{Cn^{3/2}} \sum_{k=1}^{2n} E|\xi_k - a_k|^3 \leq \frac{C'n}{n^{3/2}} \sim \frac{C'}{n^{1/2}}.$$  

Then by the CLT we have that, as $n \to \infty$,

$$\frac{\phi_{x_n - A_n}}{B_n} (t) \to e^{-\frac{t^2}{2}}$$

which implies $|J_1| \to 0$.

We have

$$|J_2| \leq \int_{|t| > A} |e^{-it\frac{\phi_{x_n - A_n}}{B_n}} ||e^{-\frac{t^2}{2}}| dt = \int_{|t| > A} |e^{-\frac{t^2}{2}}| \leq \frac{2}{A} e^{-\frac{t^2}{2}}$$

and so by choosing a sufficiently large $A$ we can make $J_2$ arbitrarily small.

For every $k$, $\phi_{t} (t)$ is either $\chi_e (t)$, $\chi_\omega (t)$ or $\chi_\omega (-t)$. Since for $\epsilon < |t| < \pi/2$ we have $|\phi_{t} (t)| < 1$, we can find $c > 0$ such that $|\phi_{t} (t)| \leq e^{-c} < 1$ for every $k$.

Then, if $eB_n < |t| < \pi B_n/2$, we have

$$|\phi_{x_n - A_n} (t)| = \prod_{k=1}^{2n} |\phi_{t_k} - a_k (t/B_n)| = \prod_{k=1}^{2n} |e^{-iak/B_n}||\phi_{t_k} (t/B_n)|$$

$$= \prod_{k=1}^{2n} |\phi_{t_k} (t/B_n)| \leq \prod_{k=1}^{2n} e^{-c} = e^{-cn},$$

which tends to 0 as $n \to \infty$. This implies $|J_3| \to 0$ as $n \to \infty$.

By the Taylor expansion at $t = 0$

$$|\phi_{x_n - A_n} (t)| = \prod_{k=1}^{2n} |\phi_{t_k} - a_k (t/B_n)| = \prod_{k=1}^{2n} \left| 1 - \frac{\sigma_k^2 t^2}{2B_n^2} + o(t^2) \right|.$$  

Now, if $|t| \leq \epsilon B_n$ for sufficiently small $\epsilon$, we have

$$|\phi_{x_n - A_n} (t)| < \prod_{k=1}^{2n} \left| 1 - \frac{\sigma_k^2 t^2}{2B_n^2} \right| < \prod_{k=1}^{2n} e^{-\frac{\sigma_k^2 t^2}{4B_n^2}} = e^{-t^2/4}.$$  

Then

$$|J_4| \leq 2 \int_A^{\epsilon B_n} e^{-t^2/4} dt < 2 \int_A^{\infty} e^{-t^2/4} dt$$

where the right hand side tends to 0 as $A \to \infty$. So we can make $|J_4|$ arbitrarily small.

The proof of recurrence is now complete.

3.3. The random walk on the $H_{p,\lambda}$ lattice. This section is devoted to the proof of theorem 1.3.

3.3.1. Proof of theorem 1.3 (i). To prove a.s. transience, we can follow the same technique we used for the case of a random environment defining, for $n \geq 0$, the events $A_n$ and $B_n$ just as before (the only difference is that, this time, we write $\sigma_y$ in place of $c_y$). Now, it is clear that many of the estimates we obtained in the case of a random environment still hold: in fact, according to Campanino and Petritis (2014), we only need to provide an estimate on $A_n \setminus B_n$, conditionally to $\mathcal{F} := \sigma (Y_i, v_i) ; n = 1, \ldots, n)$. This estimate is given by the following result, whose proof can be found in the cited paper.
Proposition 3.19 (Proposition 3.2, Campanino and Petritis (2014)). For all $\beta < 1$, there exists a $\delta_\beta > 0$ such that \( \mathbb{P}(A_n \setminus B_n \mid \mathcal{F}) = \mathcal{O}(n^{-\delta_\beta}) \).

Then, exactly as in the case of random environment, we use the estimates to show that $\mathbb{P}(X_{2n} = 0, Y_{2n} = 0)$ is summable. This proves the a.s. transience.

3.3.2. Proof of theorem 1.3 (ii). To prove a.s. recurrence we need to show that $\mathbb{P}(X_{2n} = 0, Y_{2n} = 0 \mid \mathcal{G}) = \infty$, where $\mathcal{G} := \sigma(\tau_y, y \in \mathbb{Z})$. We know from Borel-Cantelli lemma that for almost every realization of the environment, we have only a finite number of randomly perturbed directions around the origin. So, in what follows fix a realization $\epsilon$ such that the number of perturbations is $L < \infty$; we will compute all the probabilities conditionally to $\epsilon$, although we will not always specify that.

Let

$$S_{n,e}^{\leq L} := \sum_{i=1}^{2n} 1_{\{\nu_i \neq \nu_i, |Y_i| \leq L\}} f(Y_i),$$

$$S_{n,e}^{\geq L} := \sum_{i=1}^{2n} 1_{\{\nu_i \neq \nu_i, |Y_i| \geq L\}} f(Y_i),$$

$$S_{n,o}^{\leq L} := \sum_{i=1}^{2n} 1_{\{\nu_i \neq \nu_i, |Y_i| \leq L\}} f(Y_i),$$

$$S_{n,o}^{\geq L} := \sum_{i=1}^{2n} 1_{\{\nu_i \neq \nu_i, |Y_i| \geq L\}} f(Y_i).$$

Note that $S_{n,e}^{\geq L} = S_{n,e}^{\geq L}$. Moreover let

$$S_{n,e} = S_{n,e}^{\leq L} + S_{n,e}^{\geq L},$$

$$S_{n,o} = S_{n,o}^{\leq L} + S_{n,o}^{\geq L}.$$ 

In a completely analogous way we define the quantities correspondent to the odd steps: $S_{n,o}^{\leq L}, S_{n,o}^{\geq L}(n), S_{n,o}^{\leq L}(n), S_{n,o}, S_{n,o}.$

Lemma 3.20. We have

$$|S_{n,e}| \leq 2 \sum_{i=1}^{2n} 1_{\{|Y_i| \leq L\}} + |S_{n,e}|,$$

$$|S_{n,o}| \leq 2 \sum_{i=1}^{2n} 1_{\{|Y_i| \leq L\}} + |S_{n,o}|.$$
Proof: We have

\[
\mathbb{E} \left( \sum_{i=1}^{2n} 1_{(|Y_i| \leq L)} \right) = 2n \sum_{i=1}^{2n} \mathbb{P}(|Y_i| \leq L | S) = \sum_{k=-L}^{L} \sum_{i=1}^{2n} \mathbb{P}(Y_i = k | S).
\]

Now, for all sufficiently large \(i\), the probability \(\mathbb{P}(Y_i = k)\) is maximal for \(k = 0\), in which case is majorised by \(c \sqrt{i}\) for an appropriate \(c > 0\). Hence

\[
\sum_{i=1}^{2n} \mathbb{P}(Y_i = k | S) = \sum_{i=1}^{2n} \mathbb{P}_0(Y_i = k) \mathbb{P}_k((Y_{2n-i-2}, Y_{2n-i-2}, Y_{2n-i}) = (0, -1, 0))
\]

\[
\leq \sum_{i=1}^{2n} \mathbb{P}_0(Y_i = k) \mathbb{P}_k(Y_{2n-i} = 0)
\]

\[
\leq C \sqrt{n} \int_{t=0}^{2n} \frac{1}{\sqrt{t(2n-t)}} dt = C \sqrt{n} \left[ \arcsin \left( \frac{t - 2n}{2n} \right) \right]_0^{2n} \leq c' \sqrt{n}.
\]

\[\square\]

Corollary 3.22. We have

\[
\mathbb{P}(|S_{n,e}| + |S_{n,o}| \leq C \sqrt{n} | S) \geq K_{C,L} > 0
\]

with \(C > 0\) and sufficiently large \(n\).

Proof: By lemma 3.20 we have for large \(n\)

\[
\mathbb{P} \left( \frac{|S_{n,e}| + |S_{n,o}|}{\sqrt{n}} \leq C | S \right) \geq \mathbb{P} \left( \frac{4 \sum_{i=1}^{2n} 1_{(|Y_i| \leq L)} + |S_{n,e}| + |S_{n,o}|}{\sqrt{n}} \leq C | S \right)
\]

\[
\geq \mathbb{P} \left( \frac{|S_{n,e}| + |S_{n,o}|}{\sqrt{n}} \leq C/2, \sum_{i=1}^{2n} 1_{(|Y_i| \leq L)} \leq C/2 | S \right)
\]
Now, by lemma 3.16
\[ \mathbb{P}\left( \frac{|S_n, e| + |S_n, o|}{\sqrt{n}} \leq C/2 \mid \mathcal{S} \right) \geq \delta_C > 0 \]
and by the Markov inequality together with lemma 3.21
\[ \mathbb{P}\left( \frac{\sum_{i=1}^{2n} 1_{\{|Y_i| \leq L\}}}{\sqrt{n}} \leq C/2 \mid \mathcal{S} \right) \geq \delta'_{C, L} > 0, \]
where both \( \delta_C \) and \( \delta'_{C, L} \) tend to 1 as \( C \) grows to infinity. So if we take a sufficiently large \( C \) s.t. \( \delta'_{C, L} > 1 - \delta_C \), the intersection between these two events will have positive probability. \( \square \)

Now, following the same argument used in the proof of theorem 1.2 (we shall not repeat it), one shows recurrence for the random walk conditionally to the environment \( \tau \). But since the choice of \( \tau \) is arbitrary, with the only requirement that there are only a finite number of perturbations around the origin, and since this requirement is satisfied by a.e. realization, the proof of a.s. recurrence is complete.

4. Conclusion

This paper shows that the random walk has the same recurrence behaviour as in the square grid lattice case. It would be desirable to extend our results to a more general class of planar graphs with some undirected and some directed bonds. We are confident that the techniques developed here may be useful for obtaining results on recurrence in a more general setting.

References

M. Campanino and D. Petritis. Random walks on randomly oriented lattices. Mark. Proc. Rel. Fields 9, 391–412 (2003).
M. Campanino and D. Petritis. Type transition of simple random walks on randomly directed regular lattices. J. Appl. Prob. 51, 1065–1080 (2014).
W. Feller. An Introduction to Probability Theory and Its Applications. John Wiley & Sons, New York. (1966).
B.V. Gnedenko. The Theory of Probability. Chelsea Publ. Comp., New York. (1962).
A.N. Kolmogorov. A local limit theorem for classical markov chains. Izv. Akad. Nauk SSSR Sero Mat. 13, 281–300 (1949).
W. Woess. Denumerable Markov Chains. Eur. Math. Soc., Zurich (2009).