A sup + C inf inequality on a domain of $\mathbb{R}^2$.

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Abstract

Under some conditions, we give an inequality of type sup + C inf on open set of $\mathbb{R}^2$ for the prescribed scalar curvature equation.

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary. Let’s consider a sequence of functions solutions to:

$$-\Delta u_i = V_i(x)e^{u_i} \quad (E)$$

with,

$$0 \leq V_i(x) \leq b, \quad (*)$$

$$\int_{\Omega} e^{u_i} dx \leq C, \quad (**)$$

According to Brezis-Merle result, see [3], we have:

**Theorem A.** We have the following alternative:

1) $(u_i)_i$ is uniformly locally bounded,

or,

2) $u_i \to -\infty$ on each compact set of $\Omega$,

or,

3) There is a finite set (of blow-up points) $S = \{x_0, x_1, \ldots, x_m\} \subset \Omega$ and sequences of points $(x^i_k)_i$ such that,

$$x^i_k \to x_k, \quad u_i(x^i_k) \to +\infty$$

and, in the sense of distributions we have:

$$V_i e^{u_i} \to \sum_{k=0}^{m} \alpha_k \delta_{x_k}, \quad \alpha_k \geq 4\pi,$$

and,

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$u_i \to -\infty$, on each compact set of $\Omega - S$.

Here, we are interested by to know if, we can have an inequality of type:

$$\sup_{\Omega} u_i + C_1 \inf_K u_i \geq -C_2,$$

where $C_1$ and $C_2$ are two positive constant which depend only on $b, C, K$ and $\Omega$.

First, we give an example of a coercif operator on a manifold with boundary for which the lower bound of the Green function on each compact set of the unit ball tends to 0.

Nullity of the lower bound of the Green function for a manifold with boundary

We have the following result which express that we can not apply the Nash-Moser iterate scheme to prove the existence of a lower bound for $\sup + \inf$ inequality. We need other arguments to obtain such estimate. Note that, in dimension $n \geq 3$, and, in the case of a compact riemannian manifold, we need the existence of a lower bound of the Green function to have a $\sup \times \inf$ inequality.

**Example 1.1** Consider the sequence of blow-up functions :

$$u_i(r) = \log \frac{8i^2}{(1 + i^2r^2)^2},$$

which satisfy

$$u_i(0) \to +\infty, \text{ and, } u_i(x) \to -\infty \forall x \neq 0,$$

and, for all $0 < k \leq 1$,

$$u_i(0) + u_i(k) \to c_\epsilon \in (-\infty, +\infty),$$

Let’s consider the Green function $G_i$ of the following coercif operator in $H_0^1(B_1(0))$:

$$-\Delta + \epsilon_i, \text{ and, } \epsilon_i = 1 + |\nabla u_i|^2,$$

with the following properties:

$$u_i'(r) = -\frac{4i^2r}{(1 + i^2r^2)},$$

$$||\nabla u_i||_{L^\infty(B_1(0) - B_r(0))} \leq C_r < +\infty, \text{ and, } |\nabla u_i(1/i)| = 2i \to +\infty,$$

then for all $0 < k < 1$,

$$\lim_{i} \left( \inf_{x,y \in B_k(0)} G_i(x, y) \right) = 0,$$

We have the following theorem.
Theorem 1.2 Let \((u_i)\) and \((V_i)\) two sequences of functions solutions to the previous problem, then

1) For all compact \(K\) of \(\Omega\) we have:

\[ \sup_K u_i \to -\infty \]

or,

2) For all compact \(K \subset \Omega\), there are two positive constants \(C_1 = C_1(\text{dist}(K, \partial \Omega), b, C)\) and \(C_2 = C_2(K, \Omega, b, C)\) such that:

\[ \sup_{\Omega} u_i + C_1 \inf_K u_i \geq -C_2 \]

2 Proof of the results:

Proof of the Example 1: Nullity of the lower bound of the Green function.

We want to prove by contradiction that:

For all \(1 > \epsilon > 0\) and \(\gamma > 0\), there is a constant \(c = c(b, C, \epsilon, \gamma)\) such that:

\[ \gamma u_i(0) + u_i(\epsilon) \to +\infty, \]

For this, we consider the function:

\[ v_i = e^{u_i} \]

\[ -\Delta v_i + |\nabla u_i|^2 v_i = v_i^2. \]

Also, we can write:

\[ -\Delta v_i + (1 + |\nabla u_i|^2) v_i = v_i^2 + v_i, \]

And,

\[ \inf_{B_k(0)} v_i \geq \int_{B_{k+\epsilon}(0)} G_i(y, y)(v_i^2 + v_i)dy \]

Thus, by contradiction, we suppose that:

\[ \lim_{i} \left( \inf_{x,y \in B_{k+\epsilon}(0)} G_i(x, y) \right) > c > 0, \]

Thus,

\[ \sup_{B_k(0)} e^{\gamma u_i} \times v_i(k) \geq c \int_{B_{k+\epsilon}(0)} (v_i^{2+\gamma} + v_i^{1+\gamma})dy, \]

If we argue by contradiction, and suppose that \(\sup + \inf\) is finite:

\[ \int_{B_{k+\epsilon}(0)} v_i^{1+\gamma}dy \leq C_1, \quad (1) \]
and,

\[ \int_{B_{k+\epsilon}(0)} v_i^{2+\gamma} dy \leq C_2, \quad (2) \]

In particular, we have, using the Nash-Moser iterate scheme, the following inequality:

\[ \int_{B_{k+\epsilon/2}(0)} |\nabla(\eta v_i^{1+\gamma})|^2 \leq C_3, \quad (3) \]

where \( \eta \) is a cutoff function. With the Sobolev embedding, we obtain:

\[ \|v_i\|_{L^q(B_{k+\epsilon/4})} \leq C_q \quad (4) \]

Also, we choose (because of Brezis-Merle result) \( r_0 \) such that, \( u_i \to -\infty \) on \( \partial B_{r_0} \) and thus \( v_i \to 0 \) on \( \partial B_{r_0} \).

We use the Green representation formula to have:

\[ v_i(0) = \int_{B_{r_0}(0)} G_0(0, y)(v_i^2 - |\nabla u_i|^2 v_i) dy + \int_{\partial B_{r_0}(0)} \partial_n G_0(0, s) v_i ds \]

Here, \( G_0 \) is the Green function of the Laplacian with Dirichlet condition on \( B_{r_0} \). We can write, \( G_0(x, y) = -\frac{1}{2\pi} \log |x-y| + H_0(x, y) \). For \( x \in B_{1/2+\epsilon/6} \) and \( y \in B_{r_0}, G_0 \in L^q, \forall q \geq 1 \).

We use Holder inequality and (1), (2) or 4 to obtain:

\[ v_i(0) \leq ||G_0(0, .)||_{L^q} ||v_i||_{L^p} + C_4 ||v_i||_{L^\infty(\partial B_{r_0})} \leq C_5 \]

which contradict the fact that :

\[ v_i(0) = e^{u_i(0)} \to +\infty \]

Thus, for all \( \gamma > 0 \):

\[ \gamma u_i(0) + u_i(\epsilon) \to +\infty. \]

This means that, for our particular sequence:

\[ u_i(0) + u_i(1) \to +\infty, \]

it is impossible. Thus:

\[ \lim_{i} \left( \inf_{x,y \in B_{k+\epsilon}(0)} G_i(x, y) \right) = 0. \]

Proof of the theorem 1

Let’s consider the sequence \( (u_i)_i \), on the ball of radius 2.

Now, suppose that, we are in the case 3) of the Brezis-Merle result. We assume that, \( \Omega = B_2(0) \), and, \( 0 \leq |x_0| \leq |x_1| \leq \ldots \leq |x_m| \leq 1/3 \)
We denote by $G$ the Green function of the laplacian on the unit ball. We use the Green representation formula for $u_i$, we obtain:

$$u_i(y_i) = \max_\Omega u_i = \int_{B_1(0)} G(y_i, y) V_i e^{u_i(y)} dy + \int_{\partial B_1(0)} \partial_y G(y_i, s) u_i ds$$

We can use the fact that (Brezis-Merle) $u_i \to -\infty$ on $\Omega - \bigcup_{k=1}^m B(x_k, r_k)$, to have:

$$\int_{B_1(0)} G(y_i, y) V_i e^{u_i(y)} dy = \sum_{k=1}^m \int_{B(x_k, r_k)} G(y_i, y) V_i e^{u_i(y)} dy + \int_{\Omega - \bigcup_{k=1}^m B(x_k, r_k)} G(x_i, y) V_i e^{u_i(y)} dy =$$

$$= \sum_{k=1}^m \int_{B(x_k, r_k)} G(y_i, y) V_i e^{u_i(y)} dy + o(1),$$

Without loss of generality, we can assume that $y_i \to x_0$. Thus, for $k \neq 0$, $G(y_i, y) \to \beta_k > 0$ for $y \in B(x_k, r_k)$. We are concerned by the case $y \in B(x_0, r_0)$. In this case, we can write:

$$\int_{B(x_0, r_0)} G(y_i, y) V_i e^{u_i(y)} dy \leq \int_{B(y_i, r_0 + \epsilon)} G(y_i, y) V_i e^{u_i(y)} dy = \int_{B(y_i, r_0 + \epsilon)} \frac{1}{2\pi} \log \frac{|1 - \bar{y}_i y|}{|y_i - y|} V_i e^{u_i(y)} dy$$

But,

$$|1 - \bar{y}_i y| \to \beta_0 > 0, y \in B(x_0, r_0),$$

Thus,

$$\int_{B(y_i, r_0 + \epsilon)} G(y_i, y) V_i e^{u_i(y)} dy \leq C(\beta, b, C) + \int_{B(y_i, r_0 + \epsilon)} -\frac{1}{2\pi} \log |y_i - y| V_i e^{u_i(y)} dy$$

We do a blow-up around $y_i$, we write

$$y = y_i + x e^{-u_i(y_i)/2}$$

$$\tilde{u}_i(x) = u_i(y_i + x e^{-u_i(y_i)/2}) - u_i(y_i)$$

and,

$$\tilde{V}_i(x) = V_i(y_i + x e^{-u_i(y_i)/2}).$$

We have,

$$\int_{B(y_i, r_0 + \epsilon)} -\frac{1}{2\pi} \log |y_i - y| V_i e^{u_i(y)} dy = \frac{u_i(y_i)}{4\pi} \int_{B(0, (r_0 + \epsilon)e^{u_i(y_i)/2})} \tilde{V}_i e^{\tilde{u}_i} dx +$$

$$+ \int_{B(0, (r_0 + \epsilon)e^{u_i(y_i)/2})} -\frac{1}{2\pi} \log |x| \tilde{V}_i e^{\tilde{u}_i} dx,$$
The fact that $e^{\tilde{u}_i} \leq 1$ and $-\log |x| \leq 0$ for $|x| \geq 1$, we can write:

$$\int_{B(0,(r_0+\varepsilon)e^{u_i(y_i)/2})} \tilde{V}_i e^{\tilde{u}_i} dx \leq C,$$

Also, we can see that, in the case $V_i \to V$ in $C^0(\Omega)$, by the result of YY.Li and I. Shafrir, we have:

$$\int_{B(0,(y_i,r_{0}+\varepsilon))} \tilde{V}_i e^{\tilde{u}_i} dx = \int_{B(y_i,r_0+\varepsilon)} V_i e^{u_i} dy \to 8\pi m \leq bC, \ m \in \mathbb{N}^*$$

Finally, we can write:

$$u_i(y_i) \leq (2m+\varepsilon)u_i(y_i) + \sup_{\partial B_1(0)} u_i + C_1,$$

Thus,

$$C_2(bC) \sup_{\Omega} u_i + \sup_{\partial B_1(0)} u_i \geq -C_3,$$

But,

$$\sup_{\partial B_1(0)} u_i \to -\infty,$$

Thus, by the classical Harnack inequality, we can have:

$$\sup_{\partial B_1(0)} u_i \leq C_4 \inf_{\partial B_1(0)} u_i + C_5,$$

Finally, for all compact set $K$, there are two positive constants $C_1^p = C_1'(dist(K, \partial \Omega), b, C)$ and $C_2^p = C_2'(K, \Omega, b, C)$ such that:

$$\sup_{\Omega} u_i + C_1' \inf_{K} u_i \geq -C_2^p,$$

In general, the condition $V_i \to V$ in $C^0(\Omega)$ can be removed and replaced by the Brezis-Merle consequence of blow-up phenomenon.

$$\int_{B(0,(r_0+\varepsilon)e^{u_i(y_i)/2})} \tilde{V}_i e^{\tilde{u}_i} dx = \int_{B(y_i,r_0+\varepsilon)} V_i e^{u_i} dy \to 4\pi \tilde{\alpha}_0 \leq bC, \ \tilde{\alpha}_0 \geq 1.$$

**Remark:** an example of the situation 1 of the theorem 2:

Let’s consider the following sequence of functions, on $\Omega = B_1(0)$;

$$u_\varepsilon(r) = \log \left( \frac{1}{(\mu_\varepsilon^2 + r^2)^2} \right) \text{ with } \mu_\varepsilon \to +\infty$$

Then, for $K = B_k(0)$ with $0 < k < 1$, we have:

$$-\Delta u_\varepsilon = e^{u_\varepsilon} \text{ and,}$$
\[
\sup_{K} u_{\epsilon} \to -\infty,
\]
\[
\sup_{\Omega} u_{\epsilon} + \inf_{K} u_{\epsilon} \to -\infty
\]

**Questions:** 1) a) Can we have the following sharp inequality:

\[
\sup_{\Omega} u_i \geq C_1 = C_1(b, C, K, \Omega)
\]

We look to the following example of I. Shafrir, see [7]:

\[
u(r) = \begin{cases} 
2 \log \left( \frac{2 \beta r^{\beta - 1}}{1 + r^2} \right) & \text{if } r > 1 \\
2 \log \beta + 2 \log \left( \frac{2}{1 + r^2} \right) & \text{if } r \leq 1.
\end{cases}
\]

We take \(u_i(r) = u(ir) + 2 \log i\). Then, if we take \(\beta > 1\), we have:

\[-\Delta u_i = \begin{cases} 
2e^{u_i} & \text{if } r > 1/i \\
\frac{2}{\beta^2}e^{u_i} & \text{if } r \leq 1/i.
\end{cases}\]

\[u_i(0) \to +\infty,\]

\[\forall 0 < k \leq 1, \ u_i(k) \to -\infty,\]

b) Perhaps with more regularity on \(V_i\)?

With the previous example, we can use the example of Brezis-Li-Sahfrir, see [2], to construct two sequences of functions \((u_i)_i\), and \((V_i)_i (1 < \beta = \beta_i \searrow 1)\), such that:

\[-\Delta u_i = V_i e^{u_i} \text{ in } B_1(0),\]

\[C^1(B_1(0)) \ni V_i \to V \equiv 2 \text{ in } C^0(B_1(0)),\]

\[u_i(0) \to +\infty,\]

\[\forall 0 < k \leq 1, \ u_i(k) \to -\infty,\]

and we have:

\[\forall 0 < k \leq 1, \ u_i(0) + u_i(k) \to -\infty.\]

c) Perhaps, if we consider the solutions to the following equation:

\[-\Delta u_i = e^{u_i} \text{ in } \Omega\]
Consider the situation of X.Chen, see [5]. First we can have a blowing-up sequence \( \tilde{u}_i \), with two blow-up points, for example \( z_0 = 0 \) and \( z_1 = 1 \) and associated radii \( r_i \to +\infty \), such that:

\[
e^{\tilde{u}_i} \to \sum_{k=0}^{1} 8\pi \delta_{z_k},
\]

and,

\[
\tilde{u}_i \to -\infty, \text{ on each compact set of } B_{r_i} - \{0, 1\}.
\]

\[
\int_{B_{r_i}} e^{\tilde{u}_i} dx \to 16\pi.
\]

It is clear that we can choose the radius \( r_i \) such that:

\[
r_i << \tilde{u}_i(0).
\]

Now, in the situation of the paper of X.Chen, we take the following sequence:

\[
u_i(y) = \tilde{u}_i(r_iy) + 2 \log r_i.
\]

In this case, we have one exterior blow-up point \( 0 \) and two interior blow-up points \( 0 \) and \( z_i = 1/(r_i) \), in the unit ball, and,

\[
e^{u_i} \to 16\pi \delta_0,
\]

Let’s consider the Green function of the unit ball, \( G \):

\[
G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|}.
\]

We write:

\[
u_i(0) = \int_{B_1(0)} G(0, y)e^{u_i} dy + \int_{\partial B_1(0)} \partial_\nu G(0, \sigma)u_i(\sigma) d\sigma,
\]

hence,

\[
u_i(0) \geq \int_{B_{r_i^0}(0)} -\frac{1}{2\pi} \log |y|e^{u_i} dy + \int_{B_{r_i^1}(z_i)} -\frac{1}{2\pi} \log |y|e^{u_i} dy + \inf_{\partial B_1(0)} u_i,
\]

where,

\[
r_i^0 = e^{-u_i(0)/2},
\]

\[
r_i^1 = e^{-u_i(z_i)/2},
\]

and,

\[
l_{r_i}^0, l_{r_i}^1 \leq \frac{1}{2} |z_i|,
\]
we can choose $t_0^i$ as in the paper of CC. Chen and C.S Lin, see [4] (see also the formulation of Li-Shafrir, [6]), to obtain the following estimates:

$$
\int_{B_{\sqrt{2}r_i^i}(0)} \frac{1}{2\pi} \log |y| e^{u_i} dy = \frac{u_i(0)}{4\pi} \int_{B_{r_i^i}} e^{v_i} dt + \int_{B_{r_i^i}} \frac{1}{2\pi} \log |t| e^{v_i} dt,
$$

with,

$$
\int_{B_{r_i^i}} e^{v_i} dt \geq 8\pi - \frac{C}{L_i^i} - \frac{C}{u_i(0)},
$$

and,

$$
\int_{B_{r_i^i}} \frac{1}{2\pi} \log |t| e^{v_i} dt \leq C,
$$

where, $v_i$ is the blow-up function around 0:

$$
v_i(t) = u_i(e^{-u_i(0)/2}t) - u_i(0).
$$

With,

$$(L_i)^* = ((1/2)|z_i|e^{u_i(0)/2})^* = ((1/2)r_i e^{\tilde{u}_i(0)/2})^* = (1/2)e^{\tilde{u}_i(0)/2} \gg \tilde{u}_i(0) + 2\log r_i = u_i(0).$$

For the term:

$$
\int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |y| e^{u_i} dy.
$$

We take,

$$
w_i(t) = u_i(e^{-u_i(z_i)/2}t + z_i) - u_i(z_i),
$$

we have:

$$
\int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |y| e^{u_i} dy = \int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |t r_i^i + z_i| e^{w_i} dt,
$$

which we can write as:

$$
\int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |t r_i^i + z_i| e^{w_i} dt = -\frac{1}{2\pi} \log |z_i| \int_{B_{r_i^i}(0)} e^{w_i} dt + \int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |z_i| e^{w_i} dt + \int_{B_{r_i^i}(0)} \frac{1}{2\pi} \log |t r_i^i + 1| e^{w_i} dt,
$$

but,

$$
l_i^i r_i^i \leq \frac{1}{2}|z_i|,
$$

9
Thus the term \( \log \left| \frac{t^{1/2}}{z_i} + 1 \right| \) is bounded. And we have:
\[
\int_{B_{\frac{1}{2}^{1/2}}(z_i)} -\frac{1}{2\pi} \log |y| e^{u_i} \, dy \geq -\frac{1}{2\pi} \log |z_i| \int_{B_{1}^{1/2}(0)} e^{u_i} \, dt - C,
\]
because:
\[
\int_{B_{1}^{1/2}(0)} e^{u_i} \, dt \to 8\pi.
\]
Finally, we obtain the following inequality:
\[
u_i(0) \geq 2u_i(0) + \inf_{\partial B_i(0)} u_i - (4 - \epsilon) \log |z_i| - C,
\]
which we can write as:
\[
u_i(0) + \inf_{\partial B_i(0)} u_i \leq (4 - \epsilon) \log |z_i| + C \to -\infty,
\]
Thus,
\[
\sup_{\Omega} u_i + \inf_{B_i(0)} u_i = u_i(0) + \inf_{\partial B_i(0)} u_i \to -\infty.
\]
Here, we have chosen the Green function of the unit ball. The same argument may be applied to a ball of radius \( 1 \geq r > 0 \), to have:
\[
\sup_{\Omega} u_i + \inf_{B_r(0)} u_i = u_i(0) + \inf_{\partial B_r(0)} u_i \to -\infty.
\]
Thus, we have an example of blowing-up sequence with finite volume (conformal volume) and the \( \sup + \inf \) is not bounded below by a constant.

2) What’s happen if we remove the condition (***) of the problem:
\[
\int_{\Omega} e^{u_i} \, dx \leq C?
\]
Here, we can use the example:
\[
u(r) = \begin{cases} 
2 \log \left( \frac{2\beta r^{\beta-1}}{1 + r^{2\beta}} \right) & \text{if } r > 1 \\
2 \log \beta + 2 \log \left( \frac{2}{1 + r^2} \right) & \text{if } r \leq 1.
\end{cases}
\]
We take \( u_i(r) = u(ir) + 2 \log i \). Then, if we replace \( \beta \) by \( i \), we have:
\[
-\Delta u_i = \begin{cases} 
2 e^{u_i} & \text{if } r > 1/i \\
2 e^{u_i} & \text{if } r \leq 1/i.
\end{cases}
\]
\[
u_i(0) \to +\infty,
\]
\[
\forall 0 < k \leq 1, u_i(k) \to -\infty.
\]
∀ C > 0, ∀ 0 < k ≤ 1, \( u_i(0) + Cu_i(k) \to -\infty \)

and,

\[
\int_{B_i(0)} e^{u_i} dx \geq C' i \to +\infty,
\]

3) Can we replace it, as in the paper of Brezis-Merle, by the following condition:

\[
0 < a \leq V_i(x) \leq b? \quad (***)
\]

Here, we can use the example:

\[
u_i(r) = 2 \log \left( \frac{2i^{i-1}}{1 + r^{2i}} \right) \text{ if } 1 \leq r \leq 2.
\]

We have:

\[-\Delta u_i = 2e^{u_i} \text{ if } 1 \leq r \leq 2.\]

\[u_i(1) \to +\infty,\]

\[\forall 1 < k \leq 2 \ u_i(k) \to -\infty,\]

\[\forall C > 0, \forall 1 < k \leq 2 \ u_i(1) + Cu_i(k) \to -\infty\]

and,

\[
\int_{\{x, 1 \leq |x| \leq 2\}} e^{u_i} dx \geq C' i \to +\infty,
\]

Case when the sup + inf inequality holds:

Consider on the unit ball, the radial solutions to the previous equation with the following condition:

\[u_i(0) \to +\infty,\]

\[\forall 0 < k \leq 1, \ u_i(k) \to -\infty,\]

\[\int_{B_1(0)} e^{u_i} dx \leq C.\]

Then we have:

\[\inf_{\partial B_i(0)} u_i = \sup_{\partial B_i(0)} u_i = u_i(1),\]

This means:

\[\sup_{\partial B_i(0)} u_i - \inf_{\partial B_i(0)} u_i = 0,\]
All the conditions of the paper of YY.Li, see [7], (on riemannian surfaces) are satisfied and thus, we have:

$$\forall 0 < k \leq 1, \ |u_i(0) + u_i(k)| \leq c = c(C, k),$$

For the general case when we assume that the prescribed scalar curvature $V_i$ uniformly lipschitzian, it is sufficient to prove the YY.Li condition:

$$\sup_{\partial B_{1}(0)} u_i - \inf_{\partial B_{1}(0)} u_i \leq B.$$ 

Note that, in the case of compact riemannian surfaces without boundary, the last condition hold and then, the sup + inf inequality hold.

Now, assume we are on a compact riemannian surface without boundary. Assume that the sequence blow-up and we have the following inequality:

$$\int_M V_i e^{u_i} dx \leq 8\pi.$$ 

Then, we have a sup + inf inequality. (For the 2-sphere, we have another proof of this fact).

Now, assume that, on a compact surface without boundary, we have:

$$\int_M V_i e^{u_i} dx \leq C,$$

then,

$$C \sup_M u_i + \inf_M u_i \geq -C_2 = -C_2(b, C, M).$$

If we use YY.Li condition, we can extend this result to the unit ball $B_{1}(0) \subset \mathbb{R}^2$. According to the proof of theorem 2, if we assume for a blowing-up sequence, that:

$$0 < a \leq V_i(x) \leq b,$$

$$\int_{B_{1}(0)} V_i e^{u_i} dx \leq 8\pi,$$

and,

$$\sup_{\partial B_{1}(0)} u_i - \inf_{\partial B_{1}(0)} u_i \leq B.$$ 

Then, the sup + inf inequality hold. We have exactly:

$$\sup_{B_{1}(0)} u_i + \inf_{B_{1}(0)} u_i \geq -C = -C(a, b, B).$$

The last case correspond to the case of one blow-up point.

4) Is it possible to have $C_1$ independant of the compact $K$ ? This is the case if we suppose the YY.Li condition satisfied.
For example, on a compact Riemannian surface $M$ without boundary, if we consider the following equation:

$$-\Delta u_i + R = V_i e^{u_i},$$

with $V_i$ such that:

$$0 < a \leq V_i(x) \leq b, \text{ on } M,$$

we have after integration of the equation that:

$$\forall \Omega \subset M, \ 0 < a \int_{\Omega} e^{u_i} dx \leq a \int_M e^{u_i} dx \leq \int_M R(x) dx \leq |M| \sup_M R,$$

Also, we have:

$$0 \leq \int_M R(x) dx = \int_M V_i e^{u_i} dx \leq b \sup_M u_i,$$

thus,

$$\sup_M u_i \geq \log \left( \frac{\int_M R(x) dx}{b} \right).$$

Thus, we are in the case of theorem 2 and the point 1) of this theorem is not possible. We have the following:

**Corollary 2.1** Let $(u_i)$ and $(V_i)$ two sequences of functions solutions to the previous problem on a Riemannian surface $M$, such that, we have only the following condition:

$$0 < a \leq V_i(x) \leq b, \quad (***)$$

then, there are two positive constants $C_1 = C_1(a, b, M)$ and $C_2 = C_2(a, b, M)$ such that:

$$\sup_M u_i + C_1 \inf_M u_i \geq -C_2,$$

**Question:** Note that in the previous paper, by using the maximum principle, we have proved that if we consider on compact Riemannian surface without boundary the following equation:

$$-\Delta u_1 + R = V_1 e^{u_1},$$

with $V_1$ such that:

$$a = 0 \leq V_1(x) \leq b, \text{ on } M,$$

Then we have an inequality of type:

$$\sup_M u_1 + C_1 \inf_M u_i \geq -C_2,$$

with $C_1 = C_1(b, M)$ and $C_2 = C_2(b, M).$
Can we try to find this result with the previous argument of the theorem 2?

We have,

\[ \int_M V_i e^{u_i} \, dx = \int_M R(x) \, dx \leq |M| \sup_M R = C, \]

We can say that, after using the proof of theorem 2 and the YY.Li inequality around the maximum of \( u_i \):

\[ \forall \Omega, \sup_{\partial \Omega} u_i - \inf_{\partial \Omega} u_i \leq C', \]

that,

**Corollary 2.2** Let \((u_i)\) and \((V_i)\) two sequences of functions solutions to the previous problem on a riemannian surface \( M \) without boundary, such that, we have only the following condition:

\[ 0 \leq V_i(x) \leq b, \]  

\[ (***) \]

then, there is a positive constant \( C_1 = C_1(b, M) \) such that:

\[ \left( \frac{\int_M R}{4\pi} - 1 \right) \sup M u_i + \inf_M u_i \geq -C_1, \]

This is the wellknown inequality in a previous paper. To see this, we have assumed in a previous paper that:

\[ R \equiv k \in \mathbb{R}^*_+, \text{ and } \text{Volume}(M) = 1, \]

thus, the inequality is:

\[ \left( \frac{k - 4\pi}{4\pi} \right) \sup_M u_i + \inf_M u_i \geq -C_1. \]

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