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Abstract

In this paper we obtain the $L^p$-boundedness of Riesz transforms for the Dunkl transform for all $1 < p < \infty$.

1. Introduction

On the Euclidean space $\mathbb{R}^N$, $N \geq 1$, the ordinary Riesz transform $R_j$, $j = 1, \ldots, N$ is defined as the multiplier operator

$$\hat{R}_j(f)(\xi) = -i \frac{\xi_j}{\|\xi\|} \hat{f}(\xi).$$

(1.1)

It can also be defined by the principal value of the singular integral

$$R_j(f)(x) = d_0 \lim_{\varepsilon \to 0} \int_{\|x-y\| > \varepsilon} \frac{x_j - y_j}{\|x-y\|} f(y) dy$$

where $d_0 = 2^N \frac{\Gamma(N+1)}{\sqrt{\pi}}$. It follows from the general theory of singular integrals that Riesz transforms are bounded on $L^p(\mathbb{R}^N, dx)$ for all $1 < p < \infty$. What is done in this paper is to extend this result to the context of Dunkl theory where a similar operator is already defined.

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl’s seminal work [3] and was further developed by several mathematicians. See for instance the surveys [5, 6, 7, 9] and the

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references cited therein. The study of the $L^p$-boundedness of Riesz transforms for Dunkl transform on $\mathbb{R}^N$ goes back to the work of S. Thangavelu and Y. Xu [10] where they established boundedness result only in a very special case of $N = 1$. It has been noted in [10] that the difficulty arises in the application of the classical $L^p$-theory of Caldéron-Zygmund, since Riesz transforms are singular integral operators. In this paper we describe how this theory can be adapted in Dunkl setting and gives an $L^p$-result for Riesz transforms for all $1 < p < \infty$. More precisely, through the fundamental result of M. Rösler [6] for the Dunkl translation of radial functions, we reformulate a Hörmander type condition for singular integral operators. The Riesz kernel is given by acting Dunkl operator on Dunkl translation of radial function.

This paper is organized as follows. In Section 2 we present some definitions and fundamental results from Dunkl’s analysis. The Section 3 is devoted to proving $L^p$-boundedness of Riesz transforms. As applications, we will prove a generalized Riesz and Sobolev inequalities. Throughout this paper $C$ denotes a constant which can vary from line to line.

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2. Preliminaries

In this section we collect notations and definitions and recall some basic facts. We refer to [5, 3, 6, 7, 9].

Let $G \subset O(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system $R$ and $k : R \to [0, +\infty)$ be a $G$-invariant function (called multiplicity function). Let $R_+$ be a positive root subsystem. We shall assume that $R$ is normalized in the sense that $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^N$.

The Dunkl operators $T_{\xi}$, $\xi \in \mathbb{R}^N$ are the following $k$–deformations of directional derivatives $\partial_\xi$ by difference operators:

$$T_{\xi}f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_{\alpha}x)}{\langle \alpha, x \rangle}, \quad x \in \mathbb{R}^N$$
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where \( \sigma_\alpha \) denotes the reflection with respect to the hyperplane orthogonal to \( \alpha \). For the standard basis vectors of \( \mathbb{R}^N \), we simply write \( T_j = T_{e_j} \).

The operators \( \partial_\xi \) and \( T_\xi \) are intertwined by a Laplace–type operator

\[
V_k f(x) = \int_{\mathbb{R}^N} f(y) \, d\mu_x(y),
\]

associated to a family of compactly supported probability measures

\[
\{ \mu_x \mid x \in \mathbb{R}^N \}.
\]

Specifically, \( \mu_x \) is supported in the the convex hull \( \text{co}(G \cdot x) \).

For every \( \lambda \in \mathbb{C}^N \), the simultaneous eigenfunction problem,

\[
T_\xi f = \langle \lambda, \xi \rangle f, \quad \xi \in \mathbb{R}^N
\]

has a unique solution \( f(x) = E_k(\lambda, x) \) such that \( E_k(\lambda, 0) = 1 \), which is given by

\[
E_k(\lambda, x) = V_k(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} \, d\mu_x(y), \quad x \in \mathbb{R}^N.
\]

Furthermore \( \lambda \mapsto E_k(\lambda, x) \) extends to a holomorphic function on \( \mathbb{C}^N \).

Let \( m_k \) be the measure on \( \mathbb{R}^N \), given by

\[
dm_k(x) = \prod_{\alpha \in \mathbb{R}_+} |\langle \alpha, x \rangle|^{2k(\alpha)} \, dx.
\]

For \( f \in L^1(m_k) \) (the Lebesgue space with respect to the measure \( m_k \)) the Dunkl transform is defined by

\[
\mathcal{F}_k(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x) \, E_k(-i \xi, x) \, dm_k(x), \quad c_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} \, dm_k(x).
\]

This new transform shares many analogous properties of the Fourier transform.

(i) The Dunkl transform is a topological automorphism of \( \mathcal{S}(\mathbb{R}^N) \) (Schwartz space).

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of \( L^2(m_k) \).

(iii) (Inversion formula) For every \( f \in L^1(m_k) \) such that \( \mathcal{F}_k f \in L^1(m_k) \), we have

\[
f(x) = \mathcal{F}_k^2 f(-x), \quad x \in \mathbb{R}^N.
\]
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(iv) For all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$
\[ F_k(T_\xi(f))(x) = \langle i\xi, x \rangle F_k(f)(x), \quad x \in \mathbb{R}^N. \]  
(2.1)

Let $x \in \mathbb{R}^N$, the Dunkl translation operator $\tau_x$ is defined on $L^2(m_k)$ by,
\[ F_k(\tau_x(f))(y) = E_k(ix,y)F_kf(y), \quad y \in \mathbb{R}^N. \]  
(2.2)

If $f$ is a continuous radial function in $L^2(m_k)$ with $f(y) = \tilde{f}(\|y\|)$, then
\[ \tau_x(f)(y) = \int_{\mathbb{R}^N} \tilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2 < y, \eta >})d\mu_x(\eta). \]  
(2.3)

This formula is first proved by M. Rösler [6] for $f \in \mathcal{S}(\mathbb{R}^N)$ and recently is extended to continuous functions by F. and H. Dai Wang [2].

We collect below some useful facts :

(i) For all $x, y \in \mathbb{R}^N$,
\[ \tau_x(f)(y) = \tau_y(f)(x). \]  
(2.4)

(ii) For all $x, \xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$,
\[ T_\xi \tau_x(f) = \tau_x T_\xi(f). \]  
(2.5)

(iii) For all $x \in \mathbb{R}^N$ and $f, g \in L^2(m_k)$,
\[ \int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)dm_k(y) = \int_{\mathbb{R}^N} f(y)\tau_x g(-y)dm_k(y). \]  
(2.6)

(iv) For all $x \in \mathbb{R}^N$ and $1 \leq p \leq 2$, the operator $\tau_x$ can be extended to all radial functions $f$ in $L^p(m_k)$ and the following holds
\[ \|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}. \]  
(2.7)

$\|\cdot\|_{p,k}$ is the usual norm of $L^p(m_k)$.

3. Riesz transforms for the Dunkl transform.

In Dunkl setting the Riesz transforms (see [10]) are the operators $\mathcal{R}_j$, $j = 1...N$ defined on $L^2(m_k)$ by
\[ \mathcal{R}_j(f)(x) = d_k \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}}dm_k(y), \quad x \in \mathbb{R}^N. \]  
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where

\[ d_k = 2^{p_k - \frac{1}{2}} \frac{\Gamma\left(\frac{p_k}{2}\right)}{\sqrt{\pi}}; \quad p_k = 2\gamma_k + N + 1 \quad \text{and} \quad \gamma_k = \sum_{\alpha \in \mathbb{R}_+} k(\alpha). \]

It has been proved by S. Thangavelu and Y. Xu [10], that \( R_j \) is a multiplier operator given by

\[ \mathcal{F}_k(R_j(f))(\xi) = -i \frac{\xi_j}{\|\xi\|} \mathcal{F}_k(f)(\xi), \quad f \in \mathcal{S}(\mathbb{R}^N), \quad \xi \in \mathbb{R}^N, \quad (3.1) \]

The authors state that if \( N = 1 \) and \( 2\gamma_k \in \mathbb{N} \) the operator \( R_j \) is bounded on \( L^p(m_k) \), \( 1 < p < \infty \). In [1] this result is improved by removing \( 2\gamma_k \in \mathbb{N} \), where Riesz transform is called Hilbert transform. If \( \gamma_k = 0 \) \((k = 0)\), this operator coincides with the usual Riesz transform \( R_j \) given by (1.1). Our interest is to prove the boundedness of this operator for \( N \geq 2 \) and \( k \geq 0 \). To do this, we invoke the theory of singular integrals. Our basic is the following,

**Theorem 3.1.** Let \( \mathcal{K} \) be a measurable function on \( \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x, g.x); \ x \in \mathbb{R}^N, \ g \in G\} \) and \( S \) be a bounded operator from \( L^2(m_k) \) into itself, associated with a kernel \( \mathcal{K} \) in the sense that

\[ S(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x, y)f(y)dm_k(y), \quad (3.2) \]

for all compactly supported function \( f \) in \( L^2(m_k) \) and for a.e \( x \in \mathbb{R}^N \) satisfying \( g.x \notin \text{supp}(f) \), for all \( g \in G \). If \( \mathcal{K} \) satisfies

\[ \int_{\min_{g \in G} \|g.x-y\|>2\|y-y_0\|} |\mathcal{K}(x, y) - \mathcal{K}(x, y_0)|dm_k(x) \leq C, \quad y, y_0 \in \mathbb{R}^N, \quad (3.3) \]

then \( S \) extends to a bounded operator from \( L^p(m_k) \) into itself for all \( 1 < p \leq 2 \).

**Proof.** We first note that \((\mathbb{R}^N, m_k)\) is a space of homogenous type, that is, there is a fixed constant \( C > 0 \) such that

\[ m_k(B(x, 2r)) \leq C m_k(B(x, r)), \quad \forall \ x \in \mathbb{R}^N, \ r > 0 \quad (3.4) \]

where \( B(x, r) \) is the closed ball of radius \( r \) centered at \( x \) (see [8], Ch 1). Then we can adapt to our context the classical technic which consist to show that \( S \) is weak type \((1,1)\) and conclude by Marcinkiewicz interpolation theorem.
In fact, the Calderón-Zygmund decomposition says that for all \( f \in L^1(m_k) \cap L^2(m_k) \) and \( \lambda > 0 \), there exist a decomposition of \( f \), \( f = h + b \) with \( b = \sum_j b_j \) and a sequence of balls \((B(y_j, r_j))_j = (B_j)_j\) such that for some constant \( C \), depending only on the multiplicity function \( k \)

\[
\begin{align*}
(\text{i}) & \quad \|h\|_{\infty} \leq C\lambda; \\
(\text{ii}) & \quad \text{supp}(b_j) \subset B_j; \\
(\text{iii}) & \quad \int_{B_j} b_j(x)dm_k(x) = 0; \\
(\text{iv}) & \quad \|b_j\|_{1,k} \leq C \lambda m_k(B_j); \\
(\text{v}) & \quad \sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda}.
\end{align*}
\]

The proof consists in showing the following inequality hold for \( w = h \) and \( w = b \):

\[
\rho_{\lambda}(S(w)) = m_k \left( \{ x \in \mathbb{R}^N; |S(w)(x)| > \frac{\lambda}{2} \} \right) \leq C \frac{\|f\|_{1,k}}{\lambda}. \tag{3.5}
\]

By using the \( L^2 \)-boundedness of \( S \) we get

\[
\rho_{\lambda}(S(h)) \leq \frac{4}{\lambda^2} \int_{\mathbb{R}^N} |S(h)(x)|^2dm_k(x) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^N} |h(x)|^2dm_k(x). \tag{3.6}
\]

From (i) and (v),

\[
\int_{\cup B_j} |h(x)|^2dm_k(x) \leq C \lambda^2 \mu_k(\cup B_j) \leq C \lambda \|f\|_{1,k}. \tag{3.7}
\]

Since on \((\cup B_j)^c\), \( f(x) = h(x) \), then

\[
\int_{(\cup B_j)^c} |h(x)|^2dm_k(x) \leq C \lambda \|f\|_{1,k}. \tag{3.8}
\]

From (3.6), (3.7) and (3.8), the inequality (3.5) is satisfied for \( h \).

Next we turn to the inequality (3.5) for the function \( b \). Consider

\[
B_j^* = B(y_j, 2r_j); \quad \text{and} \quad Q_j^* = \bigcup_{g \in G} g.B_j^*.
\]

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Then
\[ \rho_\lambda(S(b)) \leq m_k\left( \bigcup_j Q^*_j \right) + m_k \{ x \in \left( \bigcup_j Q^*_j \right)^c ; |S(b)(x)| > \frac{\lambda}{2} \}. \]

Now by (3.4) and (v)
\[ m_k\left( \bigcup_j Q^*_j \right) \leq |G| \sum_j m_k(B^*_j) \leq C \sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda}. \]

Furthermore if \( x \notin Q^*_j \), we have
\[ \min_{g \in G} \| g.x - y_j \| > 2\|y - y_j\|, \quad y \in B_j. \]

Thus, from (3.2), (iii), (ii), (3.3), (iv) and (v)
\[ \int_{(\bigcup_j Q^*_j)^c} |S(b)(x)| dm_k(x) \]
\[ \leq \sum_j \int_{Q^*_j} |S(b_j)(x)| dm_k(x) \]
\[ = \sum_j \int_{Q^*_j} \int_{\mathbb{R}^N} |K(x, y)b_j(y)dm_k(y)| dm_k(x) \]
\[ = \sum_j \int_{Q^*_j} \int_{\mathbb{R}^N} b_j(y) |K(x, y) - K(x, y_j)| dm_k(y) dm_k(x) \]
\[ \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{Q^*_j} |K(x, y) - K(x, y_j)| dm_k(x) dm_k(y) \]
\[ \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{\min_{g \in G} \|g.x - y_j\| > 2\|y - y_j\|} |K(x, y) - K(x, y_j)| dm_k(x) dm_k(y) \]
\[ \leq C \sum_j \|b_j\|_{1,k} \]
\[ \leq C \|f\|_{1,k}. \]

Therefore,
\[ m_k \{ x \in \left( \bigcup_j Q^*_j \right)^c ; |S(b)(x)| > \frac{\lambda}{2} \} \]
\[ \leq \frac{2}{\lambda} \int_{(\bigcup_j Q^*_j)^c} |S(b)(x)| dm_k(x) \leq C \frac{\|f\|_{1,k}}{\lambda}. \]
This achieves the proof of (3.5) for $b$. □

Now, we will give an integral representation for the Riesz transform $\mathcal{R}_j$. For this end, we put for $x, y \in \mathbb{R}^N$ and $\eta \in \text{co}(G.x)$

$$A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 < y, \eta >} = \sqrt{\|y - \eta\|^2 + \|x\|^2 - \|\eta\|^2}.$$  

It is easy to check that

$$\min_{g \in G} \|g.x - y\| \leq A(x, y, \eta) \leq \max_{g \in G} \|g.x - y\|. \quad (3.9)$$

The following inequalities are clear

$$\left| \frac{\partial \ell}{\partial y_r}(x, y, \eta) \right| \leq C A^s(x, y, \eta),$$

$$\left| \frac{\partial^2 \ell}{\partial y_r \partial y_s}(x, y, \eta) \right| \leq C A^{s-2}(x, y, \eta) \quad (3.10)$$

and for $\alpha \in \mathbb{R}_+$,

$$\left| \frac{\partial \ell}{\partial y_r}(x, \sigma \alpha.y, \eta) \right| \leq C A^{s-1}(x, \sigma \alpha.y, \eta),$$

$$\left| \frac{\partial^2 \ell}{\partial y_r \partial y_s}(x, \sigma \alpha.y, \eta) \right| \leq C A^{s-2}(x, \sigma \alpha.y, \eta), \quad (3.11)$$

where $r, s = 1, \ldots, N$ and $\ell \in \mathbb{R}$.

Let us set

$$\mathcal{K}_j^{(1)}(x, y) = \int_{\mathbb{R}^N} \frac{\eta_j - y_j}{A^{pk}(x, y, \eta)} d\mu_x(\eta)$$

$$\mathcal{K}_j^{(\alpha)}(x, y) = \frac{1}{y, \alpha} \int_{\mathbb{R}^N} \left[ \frac{1}{A^{pk-2}(x, y, \eta)} - \frac{1}{A^{pk-2}(x, \sigma \alpha.y, \eta)} \right] d\mu_x(\eta),$$

$$\mathcal{K}_j(x, y) = d_k \left\{ \mathcal{K}_j^{(1)}(x, y) + \sum_{\alpha \in \mathbb{R}_+} \frac{k(\alpha)\alpha_j}{pk - 2} \mathcal{K}_j^{(\alpha)}(x, y) \right\},$$

where $\alpha \in \mathbb{R}_+$.

**Proposition 3.2.** If $f \in L^2(m_k)$ with compact support, then for all $x \in \mathbb{R}^N$ satisfying $g.x \notin \text{supp}(f)$ for all $g \in G$, we have

$$\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y).$$
Proof. Let \( f \in L^2(m_k) \) be a compact supported function and \( x \in \mathbb{R}^N \), such that \( g.x \notin \text{supp}(f) \) for all \( g \in G \). For \( 0 < \varepsilon < \min_{g \in G} \min_{y \in \text{supp}(f)} |g.x - y| \) and \( n \in \mathbb{N} \), we consider \( \tilde{\varphi}_{n,\varepsilon} \) a \( C^\infty \)-function on \( \mathbb{R} \), such that:

- \( \tilde{\varphi}_{n,\varepsilon} \) is odd .
- \( \tilde{\varphi}_{n,\varepsilon} \) is supported in \( \{ t \in \mathbb{R}; \varepsilon \leq |t| \leq n+1 \} \).
- \( \tilde{\varphi}_{n,\varepsilon} = 1 \) in \( \{ t \in \mathbb{R}; \varepsilon + \frac{1}{n} \leq t \leq n \} \).
- \( |\tilde{\varphi}_{n,\varepsilon}| \leq 1 \).

Let \( \tilde{\varphi}_{n,\varepsilon}(t) = \int_{-\infty}^{t} \tilde{\varphi}_{n,\varepsilon}(u) \frac{du}{|u|^{p-1}} \) and \( \phi_{n,\varepsilon}(y) = \tilde{\varphi}_{n,\varepsilon}(|y|) \), for \( t \in \mathbb{R} \) and \( y \in \mathbb{R}^N \). Clearly, \( \phi_{n,\varepsilon} \) is a \( C^\infty \) radial function supported in the ball \( B(0, n+1) \) and

\[ \lim_{n \to +\infty} \tilde{\varphi}_{n,\varepsilon}(|y|) = 1, \quad \forall \, y \in \mathbb{R}^N, \, |y| > \varepsilon. \]

The dominated convergence theorem, (2.5) and (2.6) yield

\[
\int_{|y| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{|y|^{p_k}} d\mu_k(y) = \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) \frac{y_j}{|y|^{p_k}} \tilde{\varphi}_{n,\varepsilon}(|y|) d\mu_k(y) = \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y)T_j(\phi_{n,\varepsilon})(y) d\mu_k(y) = \lim_{n \to \infty} \int_{\mathbb{R}^N} f(y)T_j\tau_x(\phi_{n,\varepsilon})(-y) d\mu_k(y).
\]

Now we have

\[
T_j\tau_x(\phi_{n,\varepsilon})(-y) = \int_{\mathbb{R}^N} (\eta_j - y_j)\tilde{\varphi}_{n,\varepsilon}(A(x, y, \eta)) \frac{A^p(x, y, \eta)}{A^p_k(x, y, \eta)} d\mu_x(\eta) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_j \int_{\mathbb{R}^N} \tilde{\varphi}_{n,\varepsilon}(A(x, \sigma\alpha y, \eta)) - \tilde{\varphi}_{n,\varepsilon}(A(x, y, \eta)) < y, \alpha > d\mu_x(\eta),
\]

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where from (3.9)

\[ \varepsilon < A(x, y, \eta) ; \quad \varepsilon < A(x, \sigma, y, \eta), \quad y \in \text{supp}(f), \quad \eta \in \text{co}(G.x). \]

Then with the aid of dominated convergence theorem

\[ \lim_{n \to \infty} T_j \tau_x(\phi_n, \varepsilon)(-y) = \frac{1}{d_k} K_j(x, y), \]

and

\[ d_k \int_{\|y\| \geq \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^p} dm_k(y) = \int_{\mathbb{R}^N} K_j(x, y) f(y) dm_k(y). \]

Letting \( \varepsilon \to 0 \), it follows that

\[ \mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} K_j(x, y) f(y) dm_k(y), \]

which proves the result. \( \square \)

Now, we are able to state our main result.

**Theorem 3.3.** The Riesz transform \( \mathcal{R}_j \), \( j = 1 \ldots N \), is a bounded operator from \( L^p(m_k) \) into itself, for all \( 1 < p < \infty \).

**Proof.** Clearly, from (3.1) and Plancherel’s theorem \( \mathcal{R}_j \) is bounded from \( L^2(m_k) \) into itself, with adjoint operator \( \mathcal{R}_j^* = -\mathcal{R}_j \). Thus, via duality it’s enough to consider the range \( 1 < p \leq 2 \) and apply Theorem 3.1. In view of Proposition 3.2 it only remains to show that \( K_j \) satisfies condition (3.3).

Let \( y, y_0 \in \mathbb{R}^N, y \neq y_0 \) and \( x \in \mathbb{R}^N \), such that

\[ \min_{g \in G} \|g.x - y\| > 2\|y - y_0\|. \quad (3.12) \]

By mean value theorem,
\[
|K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)| = \left| \sum_{i=0}^{N} (y_i - (y_0)_i) \int_{0}^{1} \frac{\partial K_j^{(1)}}{\partial y_i}(x, y_t) \, dt \right|
= \left| \sum_{i=0}^{N} (y_i - (y_0)_i) \int_{0}^{1} \int_{\mathbb{R}^N} \left( \frac{\delta_{i,j}}{Ap_k(x, y_t, \eta)} + \frac{p_k((y_t)_i - \eta_i)(\eta_j - (y_t)_j)}{Ap_k+2(x, y_t, \eta)} \right) \, d\mu_x(\eta) \right|
\leq C \|y - y_0\| \int_{0}^{1} \int_{\mathbb{R}^N} \frac{1}{Ap_k(x, y_t, \eta)} \, d\mu_x(\eta) dt.
\]

where \( y_t = y_0 + t(y - y_0) \) and \( \delta_{i,j} \) is the Kronecker symbol.

In view of (3.9) and (3.12), we obtain
\[
\|y - y_0\| < A(x, y_t, \eta), \quad \eta \in co(G.x).
\]

Therefore,
\[
|K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)|
\leq C \|y - y_0\| \int_{0}^{1} \int_{\mathbb{R}^N} \frac{1}{\left( \|y - y_0\|^2 + A^2(x, y_t, \eta) \right)^{\frac{p_k}{2}}} \, d\mu_x(\eta) dt.
\]

where \( \psi \) is the function defined by
\[
\psi(z) = \frac{1}{\left( \|y - y_0\|^2 + \|z\|^2 \right)^{\frac{p_k}{2}}}, \quad z \in \mathbb{R}^N.
\]

Using Fubini’s theorem, (2.4) and (2.7), we get
\[
\int_{\min_{\eta \in G} \|g.x - y\| > 2\|y - y_0\|} |K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)| dm_k(x)
\leq C \|y - y_0\| \int_{0}^{1} \int_{\mathbb{R}^N} \tau_{-y_t}(\psi)(x) dm_k(x) \, dt
\leq C \|y - y_0\| \int_{\mathbb{R}^N} \psi(z) dm_k(z) = C \int_{\mathbb{R}^N} \frac{du}{(1 + u^2)^{\frac{p_k}{2}}} = C'.
\]

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This established the condition (3.3) for $K^{(1)}_j$.

To deal with $K^{(a)}_j, \alpha \in R_+, \text{we put for } x, y \in \mathbb{R}^N, \eta \in co(G.x)$ and $t \in [0, 1]$

\[
U(x, y, \eta) = A^{2p_k-4}(x, y, \eta), \quad V_\alpha(x, y, \eta) = A^{p_k-2}A_\alpha^{p_k-2}(A^{p_k-2} + A_\alpha^{p_k-2}), \\
h_{\alpha, t}(y) = y + t(\sigma_\alpha y - y) = y - t < y, \alpha>
\]

By mean value theorem we have

\[
K^{(a)}_j(x, y) = \int_{\mathbb{R}^N} \frac{1}{< y, \alpha>} \frac{U(x, \sigma_\alpha y, \eta) - U(x, y, \eta)}{V_\alpha(x, y, \eta)} d\mu_x(\eta)
\]

\[
= - \int_{\mathbb{R}^N} \int_0^1 \frac{\partial \alpha U(x, h_{\alpha, t}(\cdot), \eta)}{V_\alpha(x, y, \eta)} dt d\mu_x(\eta)
\]

and

\[
K^{(a)}_j(x, y) - K^{(a)}_j(x, y_0)
\]

\[
= \int_{\mathbb{R}^N} \int_0^1 \int_0^1 \partial_y - y_0 \left( \frac{\partial \alpha U(x, h_{\alpha, t}(\cdot), \eta)}{V_\alpha(x, \cdot, \eta)} \right) (y_\theta) d\theta dt d\mu_x(\eta). \tag{3.13}
\]

Here the derivations are taken with respect to the variable $y$.

To simplify, let us denote by

\[
A = A(x, y_\theta, \eta); \quad A_\alpha = A(x, \sigma_\alpha y_\theta, \eta)
\]

Then using (3.10) and the fact

\[
\|\eta - h_{\alpha, t}(y_\theta)\| \leq \max(\|\eta - y_\theta\|, \|\eta - \sigma_\alpha(y_\theta)\|)
\]

we obtain

\[
\left| \frac{\partial U}{\partial y_r}(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C\left( A^{2p_k-5} + A_\alpha^{2p_k-5} \right)
\]

\[
\left| \frac{\partial^2 U}{\partial y_r \partial y_s}(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C\left( A^{2p_k-6} + A_\alpha^{2p_k-6} \right), \quad r, s = 1, \ldots, N.
\]

This gives us the following estimates

\[
\left| \partial \alpha U(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C\left( A^{2p_k-5} + A_\alpha^{2p_k-5} \right), \tag{3.14}
\]
\[ \left| \partial_{y - y_0} \left( \partial_{\alpha} U(x, h_{\alpha, \cdot}(\cdot), \eta) \right)(y_0) \right| \leq C \| y - y_0 \| \left( A^{2p_k - 6} + A^{2p_k - 6}_\alpha \right). \] (3.15)

By (3.10) and (3.11), we also have
\[ \left| \frac{\partial V_\alpha}{\partial y_r}((x, y_\theta, \eta)) \right| \leq CA^{p_k - 3}A^{p_k - 3}_\alpha(A^{p_k - 2} + A^{p_k - 2}_\alpha)(A + A_\alpha). \]

The elementary inequality \( \frac{u + v}{u^\ell + v^\ell} \leq \frac{3}{u^{\ell - 1} + v^{\ell - 1}}, \ u, v > 0, \ \ell \geq 1, \) leads to
\[ \left| \frac{\partial_{y - y_0} V_\alpha(x, y_\theta, \eta)}{V_\alpha^2(x, y_\theta, \eta)} \right| \leq C \| y - y_0 \| \frac{A_\alpha + A}{A^{p_k - 1}A^{p_k - 1}_\alpha(A^{p_k - 2} + A^{p_k - 2}_\alpha)} \]
\[ \leq C \| y - y_0 \| \frac{1}{A^{p_k - 1}A^{p_k - 1}_\alpha(A^{p_k - 3} + A^{p_k - 3}_\alpha)}. \] (3.16)

Now (3.14), (3.15) and (3.16) yield
\[ \left| \partial_{y - y_0} \left( \frac{\partial_{\alpha} U(x, h_{\alpha, \cdot}(\cdot), \eta)}{V_\alpha(x, \cdot, \eta)} \right)(y_0) \right| \]
\[ \leq C \| y - y_0 \| \left( \frac{A^{2p_k - 6} + A^{2p_k - 6}_\alpha}{A^{p_k - 2}A^{p_k - 2}_\alpha(A^{p_k - 2} + A^{p_k - 2}_\alpha)} \right) \]
\[ + C \| y - y_0 \| \left( \frac{A^{2p_k - 5} + A^{2p_k - 5}_\alpha}{A^{p_k - 1}A^{p_k - 1}_\alpha(A^{p_k - 3} + A^{p_k - 3}_\alpha)} \right) \]
\[ \leq C \| y - y_0 \| \left( \frac{1}{A^2A^{p_k - 2}_\alpha} + \frac{1}{A^{p_k - 2}A^2_\alpha} \right) \]
\[ + C \| y - y_0 \| \left( \frac{1}{A^2A^{p_k - 1}_\alpha} + \frac{1}{A^{p_k - 1}A_\alpha} \right) \]
\[ \leq C \| y - y_0 \| \left( \frac{1}{A^{p_k}} + \frac{1}{A^{p_k}_\alpha} \right) \]

where in the last equality we have used the fact that \( \frac{1}{u^\ell + v^\ell} \leq \frac{1}{u^{\ell - 1} + v^{\ell - 1}}, \ u, v > 0 \) and \( \ell \geq 1. \)
Thus, in view of (3.13),
\[ |K_j^{(\alpha)}(x, y) - K_j^{(\alpha)}(x, y_0)| \]
\[ \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \left[ \frac{1}{A^p_k(x, y_\theta, \eta)} + \frac{1}{A^p_k(x, \sigma_\alpha y_\theta, \eta)} \right] d\mu_x(\eta) d\theta. \]

Then by the same argument as for $K_j^{(1)}$ we obtain
\[ \int_{\min_{g \in G} \|g.x - y\| > 2\|y - y_0\|} |K_j^{(2)}(x, y) - K_j^{(2)}(x, y_0)| dm_k(x) \leq C, \]
which established the condition (3.3) for the kernel $K_j^{(\alpha)}$ and furnishes the proof.

As applications, we will prove a generalized Riesz and Sobolev inequalities

**Corollary 3.4** (Generalized Riesz inequalities). For all $1 < p < \infty$ there exists a constant $C_p$ such that
\[ \|T_rT_s(f)\|_{k,p} \leq C_p \|\Delta_k f\|_{k,p}, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N), \quad (3.17) \]
where $\Delta_k$ is the Dunkl laplacian: $\Delta_k f = \sum_{r=1}^N T_r^2(f)$

**Proof.** From (2.1) and (3.1) one can see that
\[ T_rT_s(f) = \mathcal{R}_r\mathcal{R}_s(-\Delta_k)(f), \quad r, s = 1...N, \quad f \in \mathcal{S}(\mathbb{R}^N). \]
Then (3.17) is concluded by Theorem 3.3.

**Corollary 3.5** (Generalized Sobolev inequality). For all $1 < p \leq q < 2\gamma(k) + N$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2\gamma(k) + N}$, we have
\[ \|f\|_{q,k} \leq C_{p,q} \|\nabla_k f\|_{p,k} \quad (3.18) \]
for all $f \in \mathcal{S}(\mathbb{R}^N)$. Here $\nabla_k f = (T_1 f, ..., T_N f)$ and $|\nabla_k f| = \left( \sum_{r=1}^N |T_r f|^2 \right)^{1/2}$. 

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Proof. For all \( f \in \mathcal{S}(\mathbb{R}^N) \), we write
\[
\mathcal{F}_k(f)(\xi) = \frac{1}{\|\xi\|} \sum_{r=1}^{N} \frac{-i\xi_r}{\|\xi\|} \left( i\xi_r \mathcal{F}_k(f)(\xi) \right)
\]
\[
= \frac{1}{\|\xi\|} \sum_{r=1}^{d} \frac{-i\xi_r}{\|\xi\|} \left( \mathcal{F}_k(T_rf)(\xi) \right).
\]
This yields to the following identity
\[
f = I_k^1 \left( \sum_{j=1}^{N} \mathcal{R}_j(T_jf) \right),
\]
where
\[
I_k^\beta(f)(x) = (d_k^\beta)^{-1} \int_{\mathbb{R}^N} \frac{\tau_y f(x)}{\|y\|^{2\gamma(k)+N-\beta}} dm_k(y),
\]
here
\[
d_k^\beta = 2^{-\gamma(k)-N/2+\beta} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\gamma(k) + \frac{N-\beta}{2})}.
\]
Theorem 1.1 of [4] asserts that \( I_k^\beta \) a bounded operator from \( L^p(m_k) \) to \( L^q(m_k) \). Then (3.18) follows from Theorem 3.3. \( \square \)

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