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On the Classification of Generalized Cartan Matrices of Rank 3

Kyriakos Papadopoulos
kxp878@bham.ac.uk
Foreword

In this talk we describe very briefly how Gritsenko and Nikulin classified in [3] the generalised Cartan matrices of rank 3, of elliptic type, which are twisted to symmetric generalised Cartan matrices. We will then introduce a conjecture on this classification and propose an algorithm for attacking it.

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0.1 Reflection Groups of Integral Hyperbolic Lattices and Generalised Cartan Matrices

In this section we will give all the basic definitions on generalised Cartan matrices, and show how they are related to reflection groups of integral hyperbolic lattices.

Definition 1 For a countable set of indices $I$, a finite-rank matrix $A = (a_{ij})$ is called a generalised Cartan matrix, if and only if:

(i) $a_{ii} = 2$;

(ii) $a_{ij} \in \mathbb{Z}^-$, $i \neq j$ and

(iii) $a_{ij} = 0 \Rightarrow a_{ji} = 0$.

We consider such a matrix to be indecomposable, i.e. there does not exist a decomposition $I = I_1 \cup I_2$, such that $I_1 \neq \emptyset$, $I_2 \neq \emptyset$ and $a_{ij} = 0$, for $i \in I_1$, $j \in I_2$.

Definition 2 A generalised Cartan matrix $A$ is symmetrizable, if there exists an invertible diagonal matrix $D = \text{diag}(\ldots \epsilon_i \ldots)$ and symmetric matrix $B = (b_{ij})$, such that:

$$A = DB \text{ or } (a_{ij}) = (\epsilon_i b_{ij}).$$

\footnote{The presentation of this section is based on [1], [2] and [3].}
where $\epsilon_i \in \mathbb{Q}$, $\epsilon_i > 0$, $b_{ij} \in \mathbb{Z}^-$, $b_{ii} \in 2\mathbb{Z}$ and $b_{ii} > 0$.

The matrices $D$ and $B$ are defined uniquely, up to a multiplicative constant, and $B$ is called \textit{symmetrised generalised Cartan matrix}.

\textbf{Remark 1} A symmetrizable generalised Cartan matrix $A = (a_{ij})$ and its symmetrised generalised Cartan matrix $B = (b_{ij})$ are related as follows:

\[ (a_{ij}) = \left( \frac{2b_{ij}}{b_{ii}} \right), \]

where $b_{ii}|2b_{ij}$.

\textbf{Definition 3} A symmetrizable generalised Cartan matrix $A$ is called \textit{hyperbolic}, if its symmetrised generalised Cartan matrix $B$ has exactly one negative square (or $A$ has exactly one negative eigenvalue).

Let us now recall the definition for \textit{hyperbolic integral quadratic form} $S$:

\textbf{Definition 4} A \textit{hyperbolic integral symmetric bilinear form} (or \textit{hyperbolic integral quadratic form}) on a finite rank free $\mathbb{Z}$–module $M$, of dimension $n$, over the ring of integers, is a map:

\[ S : M \times M \to \mathbb{Z}, \]

satisfying the following conditions:

(i) $S(\alpha m_1 + \beta m_2, m_3) = \alpha S(m_1, m_3) + \beta S(m_2, m_3)$;

(ii) $S(m_3, \alpha m_1 + \beta m_2) = \alpha S(m_3, m_1) + \beta S(m_3, m_2)$;

(iii) $S(m_1, m_2) = S(m_2, m_1)$ (symmetry) and

(iv) signature=$\langle n, 1 \rangle$; i.e. in a suitable basis, the corresponding matrix of $S$ is a diagonal matrix with $n$ positive squares and one negative square in the diagonal,

where $\alpha, \beta \in \mathbb{Z}$ and $m_1, m_2, m_3 \in M$.

We now consider an \textit{integral hyperbolic lattice} $(M, S)$, i.e. a pair of a free $\mathbb{Z}$–module $M$ and a hyperbolic integral symmetric bilinear form $S$.

By considering the corresponding cone:

\[ V(M) = \{ x \in M \otimes \mathbb{R} : (x, x) < 0 \}, \]
and by choosing its half-cone \( V^+(M) \), we can define the corresponding hyperbolic (or Lobachevskii) space:

\[
\Lambda^+(M) = V^+(M)/\mathbb{R}^{++},
\]

as a section (slice) of the cone (a cone is by definition the set of rays with origin at zero), by a hyperplane.

After defining the hyperbolic space, we can work in hyperbolic geometry, by defining the distance \( \rho \) between two points \( \mathbb{R}^{++}x \) and \( \mathbb{R}^{++}y \) in \( \Lambda^+(M) \), as follows:

\[
\cosh \rho (\mathbb{R}^{++}x, \mathbb{R}^{++}y) = \frac{-S(x, y)}{\sqrt{S(x, x)S(y, y)}}
\]

Obviously these two points in hyperbolic space are rays in the half-cone \( V^+(M) \).

**Remark 2** By definition, when we use signature \((n, 1)\), the square of a vector which is outside the cone \( V(M) \) is strictly greater than zero and the square of a vector which is inside the cone is strictly negative. Furthermore, if a vector lies on the surface of the cone, then its square is zero. Obviously, we put a minus sign in the numerator of the definition of distance in hyperbolic space, because the hyperbolic cosine should be always positive.

Now, each element \( \alpha_i \in M \otimes \mathbb{R} \), with \((\alpha_i, \alpha_i) > 0\), defines the half spaces:

\[
H^+_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) \leq 0\},
\]

\[
H^-_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) > 0\},
\]

which are bounded by the hyperplane:

\[
H_{\alpha_i} = \{\mathbb{R}^{++}x \in \Lambda^+(M) : (x, \alpha_i) = 0\},
\]

where \( \alpha_i \in M \otimes \mathbb{R} \) is defined up to multiplication on elements of \( \mathbb{R}^{++} \). The hyperplane \( H_{\alpha_i} \) is also called mirror of symmetry.

Let us denote by \( O(M) \) the group of automorphisms, which preserves the cone \( V(M) \). Its subgroup \( O^+(M) \subseteq O(M) \) is of index 2, and fixes the half-cone \( V^+(M) \). Furthermore, \( O^+(M) \) is discrete in \( \Lambda^+(M) \), and has fundamental domain of finite volume.

**Proposition 1** The index \([O(M) : O^+(M)]\) is always equal to 2.

**Proof.** The group \( O(M) \) acts on two elements which are the half cones \( V^+ \) and its opposite \( V^- \). So, the kernel of this action, \( O^+(M) \), cannot have index greater than 2. Also, the element \(-1\), from \( O(M) \), changes \( V^+ \) with \( V^- \). Thus the index is exactly equal to 2. \(\square\)
Definition 5 For \((\alpha_i, \alpha_i) > 0\), by \(s_{\alpha_i} \in O^+(M)\) we define reflection in a hyperplane \(H_{\alpha_i}\), of \(\Lambda^+(M)\), as follows:

\[
s_{\alpha_i}(x) = x - \frac{2(x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i,
\]

where \(x \in M\) and \(\alpha_i \in M\).

Remark 3 Why this equation, that we gave for reflection in our hyperbolic lattice, works? Answer: because of the following two facts:

(i) for \(x = \alpha_i\), we get that \(s_{\alpha_i}(\alpha_i) = -\alpha_i\) and

(ii) for \(x\) perpendicular to \(\alpha_i\), we have that \(s_{\alpha_i}(x) = x\),

both of which show that our formula cannot work if we omit number 2 from the numerator.

An obvious remark is that a reflection \(s_{\alpha_i}\) changes place between the half-spaces \(H_{\alpha_i}^+\) and \(H_{\alpha_i}^-\).

Going a bit further, if an orthogonal vector \(\alpha_i \in M\), \((\alpha_i, \alpha_i) > 0\), in a hyperplane \(H_{\alpha_i}\) of \(\Lambda^+(M)\), is a primitive root, i.e. its coordinates are coprime numbers, then:

\[
\frac{2(M, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i \subseteq M \iff (\alpha_i, \alpha_i) | 2(M, \alpha_i)
\]

Definition 6 Any subgroup of \(O(M)\) (the corresponding discrete group of motions of \(\Lambda(M)\)), generated by reflections, is called reflection group.

We denote by \(W(M)\) the subgroup of \(O^+(M)\) generated by all reflections of \(M\), of elements with positive squares (always for signature \((n, 1)\)).

We will also denote by \(W\) the subgroup of \(W(M)\) generated by reflections in a set of elements of \(M\). Obviously, \(W \subseteq W(M) \subseteq O^+(M)\) is a subgroup of finite index.

Definition 7 A lattice \(M\) is called reflective, if index \([O(M) : W(M)]\) is finite. In other words, \(W(M)\) has fundamental polyhedron of finite volume, in \(\Lambda(M)\).

Talking a bit more about lattices, we consider again our integral hyperbolic lattice \(S : M \times M \to \mathbb{Z}\). For \(m \in \mathbb{Q}\) we denote by \(S(m)\) the lattice which one gets, if multiplying \(S\) by \(m\). If \(S\) is reflective, then \(S(m)\) is reflective, too.

Furthermore, \(S\) is called even lattice, if \(S(x, x)\) is even, \(x \in M\). Otherwise, \(S\) is called odd.

Last, but not least, \(S\) is called primitive lattice (or even primitive), if \(S(\frac{1}{m})\) is not lattice (or even lattice), \(m \in \mathbb{N}, m \geq 2\).
Definition 8 A convex polyhedron $\mathcal{M}$, in $\Lambda(M)$, is an intersection:

$$\mathcal{M} = \bigcap_{\alpha_i} H^+_\alpha$$

of several half-spaces orthogonal to elements $\alpha_i \in M$, $(\alpha_i, \alpha_i) > 0$.

This convex polyhedron is the fundamental chamber for a reflection group $W(M)$, with reflections generated by (primitive) roots, in $M$, each of them orthogonal to exactly one side of this polyhedron. Moreover, we get this chamber if we remove all mirrors of reflection, and take the connected components of the complement (with the boundary). The fundamental chamber acts simply transitively, because if we consider an element $w \in W(M)$, then $w(\mathcal{M}_1) = \mathcal{M}_2$, in other words it fills in our (hyperbolic) space with congruent polyhedra.

The polyhedron $\mathcal{M}$ belongs to the cone:

$$\mathbb{R}^+ \mathcal{M} = \{ x \in V^+(M) : (x, \alpha_i) \leq 0 \},$$

where $\alpha_i \in P(M) = \{ \alpha_i : i \in I \}$; set of orthogonal vectors to $\mathcal{M}$, where exactly one element $\alpha_i$ is orthogonal to each face of $\mathcal{M}$.

We say that $P(M)$ is acceptable, if each of its elements is a (primitive) root, which is perpendicular to exactly one side of a convex polyhedron, in $\Lambda^+(M)$.

Also, $\mathcal{M}$ is non-degenerate, if it contains a non-empty open subset of $\Lambda^+(M)$ and elliptic, if it is a convex envelope of a finite set of points in $\Lambda^+(M)$ or at infinity of $\Lambda^+(M)$.

Let us now return back to the theory for generalised Cartan matrices, and relate it with the material that we introduced for reflection groups of integral hyperbolic lattices.

Definition 9 An indecomposable hyperbolic generalised Cartan matrix $A$ is equivalent to a triplet:

$$A \sim (M, W, P(M)), $$

where $S : M \times M \to \mathbb{Z}$ is a hyperbolic integral symmetric bilinear form, $W \subseteq W(M) \subseteq O^+(M)$, $W$ is a subgroup of reflections in a set of elements of $M$, with positive squares, $W(M)$ is the subgroup of reflections in all elements of $\mathcal{M}$ (with positive squares), $O^+(M)$ is the group of automorphisms, which fixes the half-cone $V^+(M)$, and $P(M) = \{ \alpha_i : i \in I \}$, $(\alpha_i, \alpha_i) > 0$, $A = \left\{ \frac{2(\alpha', \alpha)}{(\alpha, \alpha)} \right\}$, where $\alpha, \alpha' \in P(M)$, $\mathcal{M}$ is a locally finite polyhedron in $\Lambda^+(M)$, $\mathcal{M} = \bigcap_{\alpha} H^+_{\alpha}$ and $H^+_\alpha = \{ \mathbb{R}^+ x \in \Lambda^+(M) : (x, \alpha_i) \leq 0 \}$.

The triplet $(M, W, P(M))$ is called geometric realisation of $A$.6
Let us now consider \( \lambda(\alpha) \in \mathbb{N}, \alpha \in P(M) \) and \( \gcd(\{\lambda(\alpha) : \alpha \in P(M)\}) = 1 \).

\[
\tilde{\alpha} = \lambda(\alpha) \alpha, \quad (\tilde{\alpha}, \tilde{\alpha}) | 2(\tilde{\alpha}', \tilde{\alpha}) \iff \lambda(\alpha,\alpha) | 2\lambda(\alpha')(\alpha',\alpha)
\]

Then, \( \tilde{A} = \left(2(\tilde{\alpha}', \tilde{\alpha})/(\tilde{\alpha}, \tilde{\alpha})\right) = \left(2\lambda(\alpha')(\alpha',\alpha)/\lambda(\alpha)(\alpha,\alpha)\right), \alpha, \alpha' \in P(M), \)

where \( \tilde{A} \) is a twisted, to \( A \), hyperbolic generalised Cartan matrix and \( \lambda(\alpha) \) are called twisted coefficients of \( \alpha \).

Also, \( \tilde{A} = (\tilde{M}, \tilde{W}, \tilde{P}(\tilde{M})) = (M \supseteq \{\lambda(\alpha) : \alpha \in P(M)\}, W, \{\lambda(\alpha) : \alpha \in P(M)\}) \). In other words, \( W \) and \( M \) are the same for \( \tilde{A} \) and \( A \).

Obviously, \( A \) is untwisted, if it cannot be twisted to any generalised Cartan matrix, different from itself.

Our purpose is to work on hyperbolic generalised Cartan matrices of elliptic type, so we need to introduce some more material:

**Definition 10** Let \( A \) be a hyperbolic generalised Cartan matrix, \( A \sim (M, W, P(M)) \).

We define the group of symmetries of \( A \) (or \( P(M) \)) as follows:

\[ \text{Sym}(A) = \text{Sym}(P(M)) = \{ g \in O^+(M) : g(P(M)) = P(M) \} \]

**Definition 11** A hyperbolic generalised Cartan matrix \( A \) has restricted arithmetic type, if it is not empty and the semi-direct product of \( W \) with \( \text{Sym}(A) \), which is equivalent to the semi-direct product of \( W \) with \( \text{Sym}(P(M)) \), has finite index in \( O^+(M) \).

**Remark 4** For \( W = (w_1, s_1), S = (w_2, s_2) \mathbb{Z} \)-modules, the semi-direct product of \( W \) with \( S \) is equal to \(((s_2w_1)w_2, s_1s_2)\).

**Definition 12** A hyperbolic generalised Cartan matrix \( A \) has lattice Weyl vector, if there exists \( \rho \in M \otimes \mathbb{Q} \), such that:

\[ (\rho, \alpha) = -(\alpha, \alpha)/2, \alpha \in P(M) \]

Additionally, \( A \) has generalised lattice Weyl vector, if there exists \( 0 \neq \rho \in M \otimes \mathbb{Q} \), such that for constant \( N > 0 \):

\[ 0 \leq -(\rho, \alpha) \leq N \]

We can think of \( \rho \) in \( \Lambda^+(M) \) as being the centre of the inscribed circle to \( M \), where \( M \) is the fundamental chamber of a reflection group \( W \).

And we can now give the last, and very important, definition of this introductory section:

**Definition 13** A hyperbolic generalised Cartan matrix \( A \) has elliptic type, if it has restricted arithmetic type and generalised lattice Weyl vector \( \rho \), such that \( (\rho, \rho) < 0 \). In other words, \( |O(M) : W| < \infty \) or \( \text{vol}(M) < \infty \) or \( P(M) < \infty \).
0.2 The Classification of Generalised Cartan Matrices of Rank 3, of Elliptic Type, with the Lattice Weyl Vector, which are twisted to Symmetric Generalised Cartan Matrices

In this section we are going to describe very briefly how Gritsenko and Nikulin classified in [3] the generalised Cartan matrices of rank 3, of elliptic type (so they have the generalised lattice Weyl vector), which are twisted to symmetric generalised Cartan matrices. We will then introduce a conjecture on this classification and propose an algorithm for attacking it.

Let \( A \) be a generalised Cartan matrix of elliptic type, twisted to a symmetric generalised Cartan matrix \( \tilde{A} \). This automatically implies that \( \tilde{A} \) is of elliptic type, too. Let also \( G(A) = (M, W, P(M)) \) be the geometric realisation of \( A \), where rank of \( A \) is equal to 3.

Furthermore, for \( \alpha \in P(M) \), we set:

\[
\alpha = \lambda(\alpha) \delta(\alpha),
\]

where \( \lambda(\alpha) \in \mathbb{N} \) are the twisted coefficients of \( \alpha \), and \( (\delta(\alpha), \delta(\alpha)) = 2 \). So, \( \tilde{P}(M) = \{ \delta(\alpha) = \alpha/\lambda(\alpha) : \alpha \in P(M) \} \).

Notation: from now on, \( \delta(\alpha_i) = \delta_i \) and \( \lambda(\alpha_i) = \lambda_i \).

Now, \( A \) and its geometric realisation are equivalent to a \((1 + \lceil n/2 \rceil) \times n\) matrix \( G(A) : \)

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
-\delta_1 & -\delta_2 & -\delta_3 & -\delta_4 & -\delta_5 \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\
\end{array}
\]

Let us illustrate this by giving a specific example:

**Example 1** Let \( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5 \) be elements with positive squares, each of them orthogonal to exactly one side of a convex polytope, of five sides, in hyperbolic space. So, \( 1 \leq i \leq \lceil 5/2 \rceil \) and:

\[
G(A) = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\
\end{pmatrix}
\]

*The presentation of this section is based on [3].*
Our problem is to find all matrices $G(A)$, having the lattice Weyl vector $\rho$, $\rho \in M \otimes \mathbb{Q}$, such that $(\rho, \alpha) = -\frac{(\alpha, \alpha)}{2} \Leftrightarrow (\rho, \delta_i) = -\lambda_i$, $i = 1, \ldots, n$. The answer has been presented by Gritsenko and Nikulin in the following theorem (1.2.1. from [3]):

**Theorem 1** All geometric realisations $G(A)$, of hyperbolic generalised Cartan matrices $A$ of rank 3, of elliptic type, with the lattice Weyl vector, which are twisted to symmetric generalised Cartan matrices, all twisting coefficients $\lambda_i$ satisfy:

$$\lambda_i \leq 12$$

are given in Table 1 (from [3]).

**Remark 5** Table 1 (from [3]) gives 60 matrices. 7 of them are of the compact case (they represent a convex polytope of finite volume in hyperbolic space), the 4 of which are untwisted. The rest 53 matrices are of the non-compact case.

All these matrices can be found in pages 166-168, from [3].

The main aim of this section is to discuss about the conjecture that follows the above theorem, which we give below:

**Conjecture 1** Table 1 (from [3]) gives the complete list of hyperbolic generalised Cartan matrices $A$ of rank 3, of elliptic type, with the lattice Weyl vector, which are twisted to symmetric generalised Cartan matrices.

In other words, one can drop the inequality $\lambda_i \leq 12$ from theorem 1.2.1., from [3].

Gritsenko and Nikulin give, in the same paper, the following arguments for supporting the conjecture:

(i) The number of all hyperbolic generalised Cartan matrices of elliptic type, with the lattice Weyl vector, is finite, for rank greater than or equal to 3.

So, there exists an absolute constant $m$, such that $\lambda_i \leq m$.

(ii) Calculations were done for all $\lambda_i \leq 12$, but result has only matrices with all $\lambda_i \leq 6$.

So, there do not exist new solutions between 6 and 12.

We will now propose a way of attacking the mentioned conjecture, in four steps, giving the complete solution for the first proposed step.
FIRST STEP: We consider a triangle in hyperbolic space, with sides $a$, $b$ and $c$. The angle between $a$ and $b$ is $\pi/2$, between $a$ and $c$ is $\pi/3$ and between $c$ and $b$ is 0 radians, that is, the vertex which is created from the intersection of $c$ and $b$ is at infinity of our space. This triangle will be the fundamental chamber for reflection in $\Lambda^+(M)$, and the reflections will cover all space, tending to infinity.

In the proof of theorem 1.2.1, the authors gave the following relations, which we will use:

$$0 \leq (\delta_1, \delta_2) \leq 2, \quad 0 \leq (\delta_1, \delta_3) < 14, \quad 0 \leq (\delta_2, \delta_3) \leq 2 \quad (\ast)$$

for $\delta_1, \delta_2, \delta_3 \in \mathcal{P}(M)$ being orthogonal vectors to three consecutive sides, of a polygon $A_1 \ldots A_n$ in $\Lambda^+(M)$, namely $A_1A_2$, $A_2A_3$, $A_3A_4$.

Our suggestion is to find all these $\delta_1, \delta_2, \delta_3$ satisfying $(\ast)$, for the group generated by reflections in $\Lambda^+(M)$, with fundamental chamber the triangle with sides $a$, $b$, $c$.

We fix $\delta_2 = a$. Then, we have the following possibilities for $\delta_1$:

$$\delta_1 = c$$

$$s_c(b) = b - 2((b, c)/c^2)c = b + 2c$$

$$s_{b+2c}(a) = a - 2((a, b + 2c)/(b + 2c)^2)(b + 2c) = a + 2b + 4c$$

etc.

In other words:

$$\delta_1 = na + (n + 1)b + 2(n + 1)c$$

Now, the possibilities for $\delta_3$ are:

$$s_b(c) = c - 2((c, b)/b^2)b \Rightarrow \delta_3 = 2b + c$$

$$s_{2b+c}(-b) = -b - 2((b, 2b + c)/(2b + c)^2)(2b + c) \Rightarrow \delta_3 = 3b + 2c$$

Also:

$$s_{3b+2c}(a) = a - 2((a, 3b + 2c)/(3b + 2c)^2)(3b + 2c) = a + 6b + 4c$$

$$s_{a+6b+4c}(-3b-2c) = (-3b-2c) - 2((-3b-2c, a+6b+4c)/(a+6b+4c)^2)(a+6b+4c) = 2a + 9b + 6c$$

etc.

So,

$$\delta_3 = (n + 1)a + (3n + 6)b + (2n + 4)c$$
Remark 6 We fixed $\delta_2 = a$, and we looked for possible $\delta_1$ and $\delta_3$, such that the angle between $\delta_2$ and $\delta_1$ is acute; the same for the angle between $\delta_2$ and $\delta_3$.

Below, we give the list of all possible $\delta_1$ and $\delta_3$, for $\delta_2 = a$, such that $(\delta_1, \delta_3) < 14$ (in the right hand side we give $\delta_3$ and in the left $\delta_1$):

- $c, b$
- $b + 2c, b$
- $a + 2b + 4c, b$
- $2a + 3b + 6c, b$
- $3a + 4b + 8c, b$
- $4a + 5b + 10c, b$
- $5a + 6b + 12c, b$
- $c, 2b + c$
- $c, 3b + 2c$
- $c, a + 6b + 4c$
- $c, 2a + 9b + 6c$
- $c, 3a + 12b + 8c$
- $b + 2c, 2b + c$
- $a + 2b + 4c, 2b + c$
- $2a + 3b + 6c, 2b + c$
- $3a + 4b + 8c, 2b + c$
- $b + 2c, 3b + 2c$
- $a + 2b + 4c, 3b + 2c$
- $2a + 3b + 6c, 3b + 2c$
\[ b + 2c, \ a + 6b + 4c \]

\[ b + 2c, \ 2a + 9b + 6c \]

We now fix \( \delta_2 = b \), and give the list of all possible \( \delta_1 \) and \( \delta_3 \), for \( \delta_2 = b \), such that \((\delta_1, \delta_3) < 14\) (in the right hand side we give \( \delta_3 \) and in the left \( \delta_1 \)):

\[
\begin{align*}
&c, \ a \\
&b + 2c, \ a \\
&2b + 3c, \ a \\
&3b + 4c, \ a \\
&4b + 5c, \ a \\
&5b + 6c, \ a \\
&6b + 7c, \ a \\
&7b + 8c, \ a \\
&8b + 9c, \ a \\
&9b + 10c, \ a \\
&10b + 11c, \ a \\
&11b + 12c, \ a \\
&12b + 13c, \ a \\
&c, \ a + c \\
&c, \ 2a + b + 2c \\
&c, \ 3a + 2b + 3c \\
&c, \ 4a + 3b + 4c \\
&c, \ 5a + 4b + 5c \\
&c, \ 6a + 5b + 6c \\
&c, \ 7a + 6b + 7c
\end{align*}
\]
\[c, \ 8a + 7b + 8c\]
\[c, \ 9a + 8b + 9c\]
\[c, \ 10a + 9b + 10c\]
\[c, \ 11a + 10b + 11c\]
\[c, \ 12a + 11b + 12c\]
\[c, \ 13a + 12b + 13c\]
\[c, \ 14a + 13b + 14c\]
\[c, \ 15a + 14b + 15c\]
\[b + 2c, \ a + c\]
\[b + 2c, \ 2a + b + 2c\]
\[b + 2c, \ 3a + 2b + 3c\]
\[b + 2c, \ 4a + 3b + 4c\]
\[b + 2c, \ 5a + 4b + 5c\]
\[b + 2c, \ 6a + 5b + 6c\]
\[b + 2c, \ 7a + 6b + 7c\]
\[2b + 3c, \ a + c\]
\[2b + 3c, \ 2a + b + 2c\]
\[2b + 3c, \ 3a + 2b + 3c\]
\[2b + 3c, \ 4a + 3b + 4c\]
\[2b + 3c, \ 5a + 4b + 5c\]
\[3b + 4c, \ a + c\]
\[3b + 4c, \ 2a + b + 2c\]
\[3b + 4c, \ 3a + 2b + 3c\]
\[4b + 5c, \ a + c\]
\[4b + 5c, \ 2a + b + 2c\]
$4b + 5c, 3a + 2b + 3c$

$5b + 6c, a + c$

$5b + 6c, 2a + b + 2c$

$6b + 7c, a + c$

$6b + 7c, 2a + b + 2c$

$7b + 8c, a + c$

$7b + 8c, 2a + b + 2c$

$8b + 9c, a + c$

$9b + 10c, a + c$

$10b + 11c, a + c$

$11b + 12c, a + c$

$12b + 13c, a + c$

$13b + 14c, a + c$

$14b + 15c, a + c$

Last, we fix $\delta_2 = c$, and give the list of all possible $\delta_1$ and $\delta_3$, for $\delta_2 = c$, such that $(\delta_1, \delta_3) < 14$ (in the right hand side we give $\delta_3$ and in the left $\delta_1$):

$b, a$

$2b + c, a$

$3b + 2c, a$

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4b + 3c, a
5b + 4c, a
6b + 5c, a
7b + 6c, a
8b + 7c, a
9b + 8c, a
10b + 9c, a
11b + 10c, a
12b + 11c, a
13b + 12c, a
14b + 13c, a

b, 2a + b + 2c

b, 8a + 6b + 9c
b, 12a + 9b + 14c
b, 16a + 12b + 19c

2b + c, 2a + b + 2c
2b + c, 3a + 2b + 3c
2b + c, 4a + 3b + 4c
2b + c, 5a + 4b + 5c
2b + c, 6a + 5b + 6c
2b + c, 7a + 6b + 7c
2b + c, 8a + 7b + 8c
2b + c, 9a + 8b + 9c
2b + c, 10a + 9b + 10c
2b + c, 11a + 10b + 11c

3b + 2c, 2a + b + 2c
SECOND STEP: So, we have found 115 triples of elements $\delta_1$, $\delta_2$, $\delta_3$, as we described above, and we now need to find, for each case separately, the corresponding twisting coefficients, $\lambda_1$, $\lambda_2$, $\lambda_3$. So, we introduce three new elements, $\tilde{\delta}_1 = \lambda_1 \delta_1$, $\tilde{\delta}_2 = \lambda_2 \delta_2$, $\tilde{\delta}_3 = \lambda_3 \delta_3$.

As we mentioned in the first section, any $\tilde{\delta}_i$ and $\tilde{\delta}_j$ should satisfy the relations:

\[ \tilde{\delta}_i^2 \mid 2(\tilde{\delta}_i, \tilde{\delta}_j) \Rightarrow 2\lambda_i \mid 2\lambda_j (\delta_i, \delta_j) \] (**) 

\[ \tilde{\delta}_j^2 \mid 2(\tilde{\delta}_i, \tilde{\delta}_j) \Rightarrow 2\lambda_j \mid 2\lambda_i (\delta_i, \delta_j) \]

Now, let us introduce some more notation; so, by $\nu_p(\lambda)$, we denote the power of the prime number $p$, in the prime factorisation of the natural number $\lambda$. For example, $\nu_3(21) = 1$ and $\nu_2(21) = 0$.

Applying this notation to our case, we get that for our $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\delta}_3$, the following conditions should be satisfied:

\[ |\nu_p(\lambda_1) - \nu_p(\lambda_2)| \leq \nu_p(g_{12}) \]
\[ |\nu_p(\lambda_1) - \nu_p(\lambda_3)| \leq \nu_p(g_{13}) \]
\[ |\nu_p(\lambda_2) - \nu_p(\lambda_3)| \leq \nu_p(g_{23}) \]

where $g_{12}$, $g_{13}$, $g_{23}$ are elements of the Gram matrix of $\delta_1$, $\delta_2$, $\delta_3$, with diagonal equal to 2 and (by definition) $g_{ij} = g_{ji}$.

So, the information that we have is quite enough, for calculating $\lambda_1$, $\lambda_2$, $\lambda_3$, for each of the 115 triples that we calculated in (i), simply by working on the Gram matrix for each case.
Another way to do these calculations is via programming. Here we introduce a programme in MAPLE (with some explanation following the programme):

```maple
with(numtheory);

[GIgcd, bigomega, cfrac, cfracpol, cyclotomic, divisors, factorEQ, factorset, 
fermat, imagunit, index, integralbasis, invcfrac, invphi, issqrfree, 
jacobi, kronecker, lambda, legendre, mcombine, mersenne, migcdex, minkowski, 
mipolys, mlog, mobius, mroot, msqrt, nearestp, nthconver, nthdenom, 
nthnum, nthpow, order, pdexpand, phi, pi, pprimroot, primroot, quadres, 
rootsunit, safeprime, sigma, sq2factor, sum2sqr, tau, thue]

x := 2; y := 4; z := 2; x := 2; y := 2; z := 2;

count := 0;
for x in divisors(lcm(x*y*z)) do
  for y in divisors(lcm(y*x*z)) do
    for z in divisors(lcm(z*x*z)) do
      if modp(x*y*y, x) = 0 and
        modp(x*z*z, x) = 0 and
        modp(y*x*x, y) = 0 and
        modp(y*z*z, y) = 0 and
        modp(z*x*x, z) = 0 and
        igcd(x, y, z) = 1
      then
        print(x, y, z);
        count := count + 1;
      fi;
    od;
  od;
od;
printf ("Number of solutions
1, 1, 1
1, 1, 2
1, 2, 1
1, 2, 2
1, 2, 4
2, 1, 1
2, 1, 2
2, 2, 1
4, 2, 1
Number of solutions 9")
```

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This programme works as follows; for each $\delta_1, \delta_2, \delta_3$ we apply (***) and get six relations of the type:

\[
\begin{align*}
&x|x \cdot y \ast y \\
&x|x \cdot z \ast z \\
&y|y \cdot x \ast x \\
&y|y \cdot z \ast z \\
&z|z \cdot x \ast x \\
&z|z \cdot y \ast y
\end{align*}
\]

where $\gcd(x, y, z) = 1$. It is enough to examine the values of $x$, which belong to the set $S_x$, of the divisors of the $\text{lcm}(x \cdot y \ast x \cdot z)$. In other words, $\text{lcm}(x, y, z) = x$. The same for $y$ and $z$, we have the sets $S_y$ and $S_z$ respectively, and check for which triples $(x, y, z)$ (with $x, y, z$ be longing to $S_x, S_y, S_z$ respectively) the conditions hold.

**THIRD STEP:** The third step is to find for each triple $\delta_1, \delta_2, \delta_3$ the lattice Weyl vector. And we already mentioned, in the first section, that the following relation should be satisfied:

\[(\rho, \delta_i) = -\lambda_i, \ 1 \leq i \leq 3\]

Furthermore, it should be always true that $(\rho, \rho) < 0$.

We are optimistic that we will have a computer programme which can do this calculation, soon.

**FOURTH STEP:** In this step we will try to find more elements $\delta_i$, which will be orthogonal to sides of polytopes in $\Lambda^+(M)$; polytopes to which $\delta_1, \delta_2, \delta_3$ were orthogonal (each one perpendicular to a side of the polytope, respectively).

So, for each case (from steps (i)-(iii)) we should examine if one more side exists, where a new element $\delta_4$ is orthogonal to this side. It will be vital to prove the existence of the lattice Weyl vector.

What we already know is that the determinant of the Gram matrix of $\delta_1, \delta_2, \delta_3, \delta_4$ should be equal to zero, because $\delta_1, \delta_2, \delta_3$ should come from a 3–D lattice, and so the four elements $\delta_1, \delta_2, \delta_3, \delta_4$ should be linearly independent.

Also, (***) should take the form:

\[(\lambda_4 \delta_4)^2 \mid 2(\lambda_4 \delta_4, \lambda_i \delta_i),\]

where $\delta_4 = x_1\delta_1 + x_2\delta_2 + x_3\delta_3, \ x_i \in \mathbb{Q}$. 

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For the cases where $\delta_4$ does not exist, we stop there. If it exist, then we repeat the same thinking for a new element, $\delta_5$ and so on.

We will have proved the conjecture to theorem 1.1.2, from [1], when we find all convex polytopes in the hyperbolic space, which correspond to the matrices from Table 1 (from [3]).
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