Game saturation of intersecting families

Balázs Patkós*, Máté Vizer†

1. Introduction

A very much studied notion of extremal combinatorics is that of saturation. Let \( P \) be a property of hypergraphs such that whenever a hypergraph \( H \) possesses the property \( P \), then so do all subhypergraphs of \( P \). We say that the hypergraph \( H \) is saturated with respect to \( P \) if \( H \) has the property \( P \), but for any hyperedge \( e \in E(H) \) the hypergraph \( H \_ e \) does not have the property \( P \) anymore. A typical problem in extremal combinatorics is to determine \( \text{ex}(n,P) \) (\( \text{sat}(n,P) \)) the most (least) number of hyperedges that a hypergraph on \( n \) vertices may contain provided it is saturated with respect to \( P \).

Lots of combinatorial problems have their game theoretical analogs. For a survey on combinatorial games see Fraenkel’s paper [7]. For topics focusing on positional games, we refer the reader to the book of Beck [1] and the forthcoming book of Hefetz, Krivelevich, Stojaković and Szabó [11]. There are two types of combinatorial games that are related to saturation problems. One of them originates from Hajnal’s triangle game [13, 15], but its more general form is as follows [6, 8]; given a family \( F \) of excluded subgraphs and a host graph \( G \), two players pick the edges of \( G \) alternately so that the set of all claimed edges forms an \( F \)-free subgraph \( H \) of \( G \) (i.e., no \( F \in F \) occurs as a subgraph in \( H \)). Whenever \( H \) becomes

* E-mail: patkos.balazs@renyi.mta.hu
† E-mail: vizer.mate@renyi.mta.hu
\( I \)-saturated, the player on turn cannot make a move and loses/wins (depending on the rules of the game). In Hajnal’s triangle game the family \( I \) consists only of the triangle graph.

In this paper we study a game played by two players, Fast and Slow, such that Fast’s aim is to create a maximal hypergraph the size of which is as close to the saturation number as possible while Slow’s aim is to create a maximal hypergraph the size of which is as close to the extremal number as possible. More formally, the saturation game \((X, \mathcal{D})\) is played on the board \( X \) according to the rule \( \mathcal{D} \subseteq 2^X \), where \( \mathcal{D} \) is downward closed (or decreasing) family of subsets of \( X \), that is \( E \subseteq D \subseteq X \) implies \( E \subseteq D \). Two players pick one unclaimed element of the board at each turn alternately so that at any time \( i \) during the game, the set \( C_i \) of all elements claimed thus far belongs to \( \mathcal{D} \). The elements \( x \in X \setminus C_i \) for which \( \{ x \} \cup C_i \subseteq \mathcal{D} \) will be called the legal moves at time \( i \). If these are the elements of the board that can be claimed by a player on turn. The game ends when there are no more legal moves, that is when \( C_i \) is a maximal set in \( \mathcal{D} \) and the score of the game is the size of \( C_i \). The aim of Fast is to finish the game as soon as possible and thus obtain a score as low as possible while the aim of Slow is to keep the game going as long as possible. The game saturation number is the score of the game when both players play according to an optimal strategy. To be precise we have to distinguish two cases depending on which player takes the first move. Let \( g_{sat}(\mathcal{D}) \) and \( g_{sat}(\mathcal{D}) \) denote the score of the saturation game \((X, \mathcal{D})\) when both players play according to an optimal strategy and the game starts with Fast’s or Slow’s move, respectively. In most cases, the board \( X \) is either \( \binom{n}{2} \) for some \( 1 \leq k \leq n \) or \( 2^k \). Clearly, the inequalities \( sat(\mathcal{D}) \leq g_{sat}(\mathcal{D}), g_{sat}(\mathcal{D}) \leq ex(\mathcal{D}) \) hold.

The first result concerning saturation games is due to Füredi, Reimer and Seress [9]. They considered the case when the board \( X \) is the edge set of the complete graph on \( n \) vertices and \( \mathcal{D} = D_{n,k} \) is the family of all triangle-free subgraphs of \( K_n \). They established the lower bound \( (n/2) \log n \leq g_{sat}(D_{n,k}), g_{sat}(D_{n,k}) \) and claimed without proof an upper bound \( n^2/5 \) via personal communication with Paul Erdős. Their paper mentions that the first step of Fast’s strategy is to build a \( C_3 \)-factor. However, as it was recently pointed out by Heefetz, Krivelevich and Stojaković [10], Slow can prevent this to happen. Indeed, in his first \( \lfloor (n-1)/2 \rfloor \) moves Slow can create a vertex \( x \) with degree \( \lfloor (n-1)/2 \rfloor \), whose neighborhood remains an independent set throughout the game due to the triangle-free property. But clearly, a graph that contains a \( C_3 \)-factor cannot have an independent set larger than \( 2n/5 + 4 \). Recently, Cranston, Kinnersley, O and West [3] considered the saturation game when the board \( X = X_G \) is the edge set of a graph \( G \) and \( \mathcal{D} = D_G \) consists of all (partial) matchings of \( G \).

In this paper, we will be interested in intersecting families. That is, the board \( X = X_{n,k} \) will be the edge set of the complete graph on \( n \) vertices and \( \mathcal{D} = D_{n,k} \) is the set of all intersecting families, \( D_{n,k} = \{ F \subseteq X_{n,k} : F \cap G \neq \emptyset, F, G \in \mathcal{F} \} \). Note that by the celebrated theorem of Erdős, Ko and Rado [5] we have \( ex(I_{n,k}) = \left( \begin{array}{c} n-k+1 \\ k-1 \end{array} \right) \) provided \( 2k \leq n \). The saturation number \( sat(I_{n,k}) \) is not known. Meyer [14] conjectured this to be \( k^2 - k + 1 \) whenever a projective plane of order \( k - 1 \) exists. This was disproved by Füredi [8] by constructing a maximal intersecting family of size \( 3k^2/4 \) provided a projective plane of order \( k/2 \) exists, and this upper bound was later improved by Börös, Füredi, and Kahn to \( k^2/2 + O(k) \) [2] provided a projective plane of order \( k - 1 \) exists. The best known lower bound on \( sat(I_{n,k}) \) is \( 3k \), due to Dow, Drake, Füredi, and Larson [4]; it holds for all values of \( k \).

We mentioned earlier that the game saturation number might depend on which player starts the game. This is indeed the case for intersecting families. If \( k = 2 \), then the first two moves of the already claimed edges are two sides of a triangle. Thus, if Fast is the next to move, he can claim the last edge of this triangle and the game is finished, so \( g_{sat}(I_{n,2}) = 3 \). On the other hand, if Slow can claim the third edge, then he can pick an edge containing the intersection point of the first two edges and then all such edges will be claimed by one and we obtain \( g_{sat}(I_{n,3}) = n - 1 \).

The main result of the paper is the following theorem that bounds both \( g_{sat}(I_{n,k}) \) and \( g_{sat}(I_{n,k}) \) away from \( ex(I_{n,k}) \) and \( sat(I_{n,k}) \) if \( n \) is large enough compared to \( k \).

**Theorem 1.1.**

For all \( k \geq 2 \),

\[
\Omega_k \left( n^{k/3} \right) \leq g_{sat}(I_{n,k}), g_{sat}(I_{n,k}) \leq O_k \left( n^{k-v/2} \right).
\]

In Theorem 1.1 and later on in the paper, by \( g(n, k) = \Omega_k (f(n, k)) \) and \( g(n, k) = O(f(n, k)) \) we mean that for any positive integer \( k \) there exists a positive constant \( c_k \) such that \( c_k f(n, k) \leq g(n, k) \) and \( c_k f(n, k) \geq g(n, k) \), respectively. Furthermore, \( g(n, k) = \Theta_k (f(n, k)) \) denotes the fact that both of the previous inequalities hold.
2. Proof of Theorem 1.1

We start this section by defining an auxiliary game that will enable us to prove Theorem 1.1. We say that a set $S$ covers a family $\mathcal{F}$ of sets if $S \cap F \neq \emptyset$ for every set $F \in \mathcal{F}$. The covering number $\tau(\mathcal{F})$ is the minimum size of a set $S$ that covers $\mathcal{F}$. Note that if $\mathcal{F}$ is an intersecting family of $k$-sets, then $\tau(\mathcal{F}) \leq k$ as by the intersecting property any set $F \in \mathcal{F}$ covers $\mathcal{F}$. The following proposition is folklore, but for the sake of self-containedness we present its proof.

**Proposition 2.1.**
If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a maximal intersecting family with covering number $\tau$, then

$$\binom{n-\tau}{k-\tau} \leq |\mathcal{F}| \leq k^\tau \binom{n-\tau}{k-\tau}.$$

**Proof.** The first inequality follows from the following observation: if $S$ covers $\mathcal{F}$, then all $k$-subsets of $[n]$ that contain $S$ belong to $\mathcal{F}$ due to maximality. To obtain the second inequality, note that denoting the maximal degree of $\mathcal{F}$ by $\Delta(\mathcal{F})$, the inequality $|\mathcal{F}| \leq k\Delta(\mathcal{F})$ holds. Indeed, by the intersecting property we have $|\mathcal{F}| \leq \sum_{x \in \mathcal{F}} d(x)$ for any set $F \in \mathcal{F}$ and the right hand side is clearly not greater than $k\Delta(\mathcal{F})$. Let $d_i(\mathcal{F})$ denote the maximum number of sets in $\mathcal{F}$ that contain the same $j$-subset, thus $d_i(\mathcal{F}) = \Delta(\mathcal{F})$ by definition. For any $j < \tau$ and $j$-subset $J$ that is contained in some $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ with $J \cap F' = \emptyset$. Thus, $d_j(\mathcal{F}) \leq kd_{\tau-1}(\mathcal{F})$ is true. Since $d_i(\mathcal{F}) \leq \binom{n-i}{k-i}$, we obtain $d_i(\mathcal{F}) \leq k^{\tau-1} \binom{n-i}{k-i}$ and the second inequality follows.

The main message of Proposition 2.1 is that if $k$ is fixed and $n$ tends to infinity, then the order of magnitude of the size of a maximal intersecting family $\mathcal{F}$ is determined by its covering number. As $|\mathcal{F}| = \Theta_k(n^{k-\tau(\mathcal{F})})$, a strategy in the saturation game that maximizes $\tau(\mathcal{F})$ is optimal for Fast up to a constant factor, and a strategy that minimizes $\tau(\mathcal{F})$ is optimal for Slow up to a constant factor.

Therefore, from now on we will consider the $\tau$-game in which two players: minimizer and Maximizer take unclaimed elements of $X = X_{n,k} = \binom{[n]}{k}$ alternately so that at any time during the game the set of all claimed elements forms an intersecting family. The game stops when the claimed elements form a maximal intersecting family $\mathcal{F}$. The score of the game is the covering number $\tau(\mathcal{F})$ and the aim of minimizer is to keep the score as low as possible while Maximizer’s aim is to reach a score as high as possible. Let $\tau_m(n,k)$ (resp. $\tau_M(n,k)$) denote the score of the game when both players play according to their optimal strategy and the first move is taken by minimizer (resp. Maximizer). The following simple observation will be used to define strategies.

**Proposition 2.2.**
Let $\mathcal{G} = \{G_1, \ldots, G_{k+1}\}$ be an intersecting family of $k$-sets. Assume there exists a set $C$ such that $G_1 \setminus C, \ldots, G_{k+1} \setminus C$ are non-empty and pairwise disjoint. Then $\tau(\mathcal{F}) \leq |C|$ for any intersecting family $\mathcal{F} \supseteq \mathcal{G}$ of $k$-sets.

To see the above statement, observe that if a $k$-set $F$ is disjoint from $C$, then it cannot meet all $k+1$ of the sets $G_1 \setminus C, G_2 \setminus C, \ldots, G_k \setminus C$. Theorem 1.1 will stem from the following two lemmas and Proposition 2.1.

**Lemma 2.3.**
For any positive integer $k$, there exists $n_0 = n_0(k)$ such that if $n \geq n_0$, then

$$\tau_m(n,k), \tau_M(n,k) \leq \left\lceil \frac{2k}{3} \right\rceil + 4.$$

**Proof.** We have to provide a strategy for minimizer that ensures the covering number of the resulting family to be small. Let us first assume that minimizer starts the game and let $m_0, M_1, m_1, M_2, m_2, \ldots$ denote the $k$-sets claimed during the game. Minimizer’s strategy will involve sets $A_i, C_i$ for $2 \leq i \leq k$ with the following properties:
(a) $A_i \subseteq m_0$, $C_{i-1} \subseteq C_i$, $|C_i| \leq |C_{i-1}| + 1$;
(b) the sets $m_0 \setminus (A_i \cup C_i)$ and $m_1 \setminus \ldots \setminus m_i \setminus C_i$ are non-empty and pairwise disjoint;
(c) $A_i \cup C_i$ meets all sets $m_j$, $M_j$ for $j \leq i$;
(d) $C_i$ meets all $m_j$ and all but at most one of the $M_j$ for $j \leq i$.

Before proving how minimizer is able to pick his $k$-sets $m_0, m_1, \ldots, m_k$ with the above properties, let us explain why it is good for him. Maximizer would like to utilize Proposition 2.2 to claim that no matter how the players continue to play, after choosing $m_{k+1}$, he can be sure that the resulting maximal intersecting family will have low covering number. As minimizer cannot control Maximizer’s moves, he will apply Proposition 2.2 with $k + 1$ of his own sets playing the role of $\mathcal{S}$. The set $C_i$ will be a temporary approximation of a future covering set $C$: it meets all previously claimed sets but at most one and its union with the auxiliary set $A_i$ does indeed meet all sets of the game until round $i$. Whenever minimizer decides that an element $x$ is included in $C_i$, then $x$ stays there forever. This is condition (a) saying $C_{i-1} \subseteq C_i$.

Such strategy would be easy to follow without the sets $A_i$ and still fulfilling the second part of condition (a), namely, that the covering set can have at most one new element in each round. Indeed, minimizer in the $(i+1)$-st round could claim a set $m_{i+1}$ containing $C_i$ and an element $x$ from Maximizer’s last move $M_i$ and let $C_{i+1} = C_i \cup \{x\}$. This would be a legal move as $m_{i+1}$ meets all sets $m_0, M_1, m_1, \ldots, M_{i-1}$, $m_i$ as $m_{i+1}$ contains $C_i$ and meets $M_i$ as they both contain $x$.

At the end of the proof of Lemma 2.3 we show how the auxiliary sets $A_i$ can help so that in many rounds the covering set does not need to grow at all. But first we make sure that minimizer is able to claim $k$-sets $m_0, m_1, \ldots, m_k$ so that sets $A_i, C_i$ with the above properties exist. Minimizer can claim an arbitrary $m_0$, and after Maximizer’s first move $M_1$, he can pick $a_1 \in m_0 \cap M_1$ and claim $m_1 = \{a_1\} \cup N_1$ where $N_1$ is a $(k-1)$-set disjoint from $m_0 \cup M_1$. Minimizer’s strategy distinguishes two cases for claiming $m_2$ depending on Maximizer’s second move $M_2$. If $a_1 \notin M_2$, then minimizer claims $m_2 = \{a_1\} \cup N_2$ with $N_2$ being a $(k-1)$-set disjoint from $m_0 \cup m_1 \cup M_1 \cup M_2$ and we define $A_2 = \emptyset$, $C_2 = \{a_1\}$. If $a_1 \in M_2$, then by the intersecting property there exists $c_2 \in M_1 \cap M_2$. Let minimizer claim $m_2 = \{a_1, c_2\} \cup N_2$ with $N_2$ being a $(k-2)$-set disjoint from $m_0 \cup m_1 \cup M_1 \cup M_2$ and put $A_2 = \emptyset$, $C_2 = \{a_1, c_2\}$. In both cases, properties (a)–(d) hold.

Let us assume that minimizer is able to claim $k$-sets $m_0, m_1, \ldots, m_k$ and define sets $A_i, C_i$. The strategy of minimizer will distinguish several cases depending on Maximizer’s move $M_i$. In all cases minimizer’s set $m_i$ will consist of elements of $C_{i-1}$, a possible element $a_i$ and elements of a set $N_i$ that is disjoint from all previously claimed sets. As we are interested in not more than $2(k+2)$ sets, minimizer will always be able to choose $N_i$ if $n \geq 2k(k+2)$.

**Case 1**: $C_{i-1}$ meets all previously claimed $k$-sets.

- If $M_i \cap C_{i-1} \neq \emptyset$, then let
  - $m_i = C_{i-1} \cup N_i$ with $|N_i| = \max\{k-|C_{i-1}|, 0\}$ and $N_i \cap \left( \bigcup_{j=0}^{i-1} m_j \cup \bigcup_{j=1}^{i} M_j \right) = \emptyset$;
  - $C_i = C_{i-1}$, $A_i = A_{i-1}$.

  The set $m_i$ is a legal move for minimizer in this subcase as it contains $C_{i-1}$. Now observe that (a) is satisfied as it was satisfied at step $i - 1$; (b) is satisfied as it was satisfied at step $i - 1$ and by the choice of $N_i$; (c) is satisfied as it was satisfied at step $i - 1$ and by the fact that we are in the $M_i \cap C_{i-1} \neq \emptyset$ subcase and we choose $m_i$ to contain $C_{i-1}$; (d) is satisfied by the assumptions that $C_i = C_{i-1}$ meets all previously claimed sets and that $C_{i-1} \cap M_{i-1} \neq \emptyset$. Note that if step $i$ is in this subcase, then at step $i + 1$ we are still in Case 1.

- If $M_i \cap C_{i-1} = \emptyset$ and there exists $a_i \in (M_i \cap m_0) \setminus A_{i-1}$, then let
  - $m_i = C_{i-1} \cup \{a_i\} \cup N_i$ with $|N_i| = \max\{k-|C_{i-1}| - 1, 0\}$ and $N_i \cap \left( \bigcup_{j=0}^{i-1} m_j \cup \bigcup_{j=1}^{i-1} M_j \right) = \emptyset$;
  - $C_i = C_{i-1}$, $A_i = A_{i-1} \cup \{a_i\}$.

  The set $m_i$ is a legal move for minimizer in this subcase as $m_i$ contains $C_{i-1}$ and $a_i$. Now observe that (a) is satisfied as it was satisfied at step $i - 1$ and by the choice $a_i \in m_0$; (b) is satisfied as it was satisfied at step $i - 1$ and by the choice of $N_i$ and $a_i$; (c) is satisfied as it was satisfied at step $i - 1$ and by the fact that $a_i \in M_i \cap m_1$; (d) is satisfied as $C_{i-1} = C_{i-1}$ meets all previously claimed sets. Note that if step $i$ is in this subcase, then at step $i + 1$ we are not in Case 1 as $M_i$ is not met by $C_{i-1} = C_i$. 

• If none of the above subcases happen, then we must have $M_i \cap C_{i-1} = \emptyset$ and $\emptyset \neq M_i \cap m_0 \subset A_{i-1}$, as Maximizer must pick $M_i$ so that it intersects all previously claimed $k$-sets, in particular, it intersects $m_0$. Let $a \in M_i \cap m_0$ and thus $a \in A_{i-1}$ and let

- $m_i = C_{i-1} \cup \{a\} \cup N_i$ with $|N_i| = k - |C_{i-1}| - 1$ and $N_i \cap (\bigcup_{j=0}^{i-1} m_j \cup \bigcup_{j=1}^{i-1} M_j) = \emptyset$;
- $A_i = A_{i-1} \setminus \{a\} \cup \{c\}$.

The set $m_i$ is a legal move for minimizer in this subcase as $m_i$ contains $C_{i-1}$ and $a$. Now observe that (a) is satisfied as it was satisfied at step $i - 1$; (b) is satisfied as it was satisfied at step $i - 1$, $A_{i-1} \cup C_{i-1} = A_i \cup C_i$ and by the choice of $N_i$ and $a$; (c) is satisfied as $A_{i-1} \cup C_{i-1}$ meets all sets $m_0, M_1, \ldots, M_{i-1}$, and in this subcase we have $A_{i-1} \cup C_{i-1} = A_i \cup C_i$ and $a \in M_i$; (d) is satisfied as $C_{i-1} \subset C_i$ meets all previously claimed sets and $a \in m_i \cap C_i$. Note that if step $i$ is in this subcase, then at step $i + 1$ we are still in Case I.

**Case II:** There exists $M_j$, $j \leq i - 1$, with $M_j \cap C_{i-1} = \emptyset$.

As Maximizer picks $M_i$ so that it meets all previously claimed $k$-sets, there exists an element $c \in M_i \cap M_j$. By (c), we have that $C_{i-1} \cup \{c\}$ meets all previously claimed $k$-sets, then let

- $m_i = C_{i-1} \cup \{c\} \cup N_i$ with $|N_i| = k - |C_{i-1}| - 1$ and $N_i \cap (\bigcup_{j=0}^{i-1} m_j \cup \bigcup_{j=1}^{i-1} M_j) = \emptyset$;
- $A_i = A_{i-1} \setminus \{c\} \cup \{c\}$.

The set $m_i$ is a legal move for minimizer in this subcase as $m_i$ contains $C_{i-1}$ and $c$. Now observe that (a) is satisfied as it was satisfied at step $i - 1$ thus $A_i \subset A_{i-1} \subset m_i$; (b) is satisfied as it was satisfied at step $i - 1$, $A_{i-1} \cup C_{i-1} = A_i \cup C_i$ and by the choice of $N_i$ and $c$; (c) is satisfied as it was satisfied at step $i - 1$, $A_{i-1} \cup C_{i-1} = A_i \cup C_i$ and $c \in M_i \cap M_j$; (d) is satisfied due to the fact that $C_{i-1} \subset C_i$ meets all previously claimed sets but $M_j, c \in M_i \cap M_j$ and $C_i = C_{i-1} \cup \{c\}$. Note that if step $i$ is in Case II, then step $i + 1$ is in Case I.

We have just seen that minimizer is able to claim $k$-sets $m_1, m_2, \ldots, m_k$ so that there exist sets $A_i, C_i$, $2 \leq i \leq k$, satisfying properties (a)–(d). The following claim states that in at least one third of the rounds minimizer does not need to increase $C_i$ and thus obtains a small covering set.

**Claim 2.4.**

For any $2 \leq i \leq k$, $|C_i| \leq 3 + [2(i - 2)/3]$.

**Proof.** Let $\alpha_i = |\{j : 2 < j \leq i, |C_{j-1}| = |C_j|\}|$, i.e., the number of steps when we are in the first two subcases of Case I. Let $\beta_i = |\{j : 2 < j \leq i, |C_{j-1}| + 1, j\text{'th turn is in Case I}\}|$ and $\gamma_i = |\{j : 2 < j \leq i, j\text{'th turn is in Case II}\}|$. Clearly, we have $\alpha_i + \beta_i + \gamma_i = i - 2$. If the $j$th turn is in the last subcase of Case I or in Case II, then $C_i$ meets all previously claimed $k$-subsets and $m_j$, as well. Thus we obtain $\gamma_i + \beta_i \leq \alpha_i + \beta_i + 1$ and, therefore, $\gamma_i \leq \alpha_i + 1$. Also, as in the last subcase of Case I the size of $A_i$ decreases, and this size only increases if we are in the first two subcases of Case I, we obtain $\beta_i \leq \alpha_i$. From these three inequalities it follows that $1 + (i - 2)/3 \leq \alpha_i$ and we get the claim. \[\Box\]

Let $M_{k+1}$ be the next move of Maximizer. By property (d), there can be at most one set $M_i$ that is disjoint from $C_i$. If minimizer picks an element $m$ of $M_{k+1} \cap M_j$ and claims the $k$-set $m_{k+1} = C_k \cup \{m\}$ with $|N_{k+1}| = k - |C_k| - 1$ and $N_i \cap (\bigcup_{j=0}^{i} m_j \cup \bigcup_{j=1}^{i} M_j) = \emptyset$, then the set $C = C_k \cup \{m\}$ and $m_1, \ldots, m_{k+1}$ satisfy the conditions of Proposition 2.2. This proves $\tau_{\min}(n, k) \leq |C| \leq |2k/3 + 3| + 3$.

If Maximizer starts the game, then minimizer can imitate his previous strategy to obtain a sequence of moves $M_1, m_1, M_2, m_2, \ldots, M_k, m_k$ with the following slight modification: the sets $A_i, C_i$ and the moves $m_i$ still satisfy properties (b)–(d), but property (a) is replaced with (a’) $A_i \subset m_0, C_{i-1} \subset C_i, |C_i| \leq |C_{i-1}| + 1$.

The proof is identical to the one when minimizer starts the game. In this way, minimizer obtains $C_k$ with the same size as before and in his $(k + 1)$st and $(k + 2)$nd moves, he can add two more elements to obtain a set $C'$ that is just one larger and satisfies the conditions of Proposition 2.2 together with $m_2, \ldots, m_{k+1}, m_{k+2}$. Therefore, we obtain $\tau_{\max}(n, k) \leq |C'| \leq |2k/3 + 4|$. \[\Box\]
Now we turn our attention to the upper bound on $\tau_a(n, k)$ and $\tau_b(n, k)$. In the following proof we will use the following notations: the degree of a vertex $x$ in a family $F$ of sets is $\deg_F(x) = |\{ F : x \in F \in F \}|$. Also, we will write $M_i = \{ M_j : j \leq i \}$ for the family of $k$-sets that Maximizer picks until step $i$.

**Lemma 2.5.**
For any positive integer $k$, if $k^{1/2} \leq n$, then

$$\frac{1}{2} \sqrt{k} \leq \tau_a(n, k), \tau_b(n, k).$$

**Proof.** Note that

$$\max_{x \in X} |F| \leq \tau(F)$$

for any set $X$ and a family $F$ of subsets of $X$. Therefore, a possible strategy for Maximizer is to keep $\max_{x \in [n]} \deg_{M_i}(s)$ as small as possible. If Maximizer is able to do so long enough, then already the sets claimed by him will ensure that the covering number of the resulting family is large. We claim that Maximizer can choose legal steps $M_1, \ldots, M_{\lceil k^{1/2} \rceil}$ so that

$$\max_{x \in [n]} \deg_{M_{\lceil k^{1/2} \rceil}}(s) \leq 2.$$

In order to establish the aim above, at the $i$th step Maximizer should choose his set $M_i$ of the form $M_i = M_i^1 \cup M_i^2$, where $M_i^1 \cap M_i^2 = \emptyset$, $|M_i^1| = \lceil k^{1/2} \rceil - 1 = \lfloor l \rfloor$ and $|M_i^2| = k - l + 1$. The sets $M_i^1$ would be chosen independently of how minimizer picks his sets and to ensure $M_i \cap M_j = \emptyset$ for any pair $1 \leq j < i \leq l + 1$. The sets $M_i^2$ would be chosen to ensure that $M_i$ meets all $i$ or $i - 1$ sets that minimizer has claimed by that point of the game (depending on who started the game). We will define these sets inductively.

Let $M_i^1 = [l]$ and assume the elements of $M_i^1, \ldots, M_{l-1}^1$ are enumerated increasing as $v_1^1, \ldots, v_i^1, v_i^2, \ldots, v_{l-1}^1, \ldots, v_{l-1}^i$, then let

$$M_i^2 = \{ v_{l-1}^1, v_{l-1}^2, \ldots, v_{l-1}^i \} \cup \{ u_1, u_2, \ldots, u_i + l - i \}$$

where $u_i = 1 + \sum_{h=0}^{i-1} l - h$. By definition, $M_i^2$ meets all previous $M_j^1$ at exactly one point and all intersection points are different, thus we obtain that the maximum degree is $2$. Also, since $M_i^2$ introduces $l - i + 1$ new points, we have $U = \bigcup_{i=1}^{l+1} M_i = [l(l + 1)/2]$ and $l(l + 1)/2 \leq k/2$.

We will prove by induction that Maximizer can pick $M_i^2$ so that the following three properties hold for $i \leq \lceil k^{1/2} \rceil$ with the notation $M_i^2 = \{ M_{k^2}^j : j \leq i \}$:

(i) $|\{ x : \deg_{M_i^2}(x) = 2 \}| \leq i^2/2$,

(ii) $\{ x : \deg_{M_i^2}(x) \geq 3 \} = \emptyset$, and

(iii) $U \cap M_i^2 = \emptyset$.

The set $M_i^2$ can be chosen arbitrarily with the restriction that it is disjoint from $U$ and if minimizer starts the game, then it meets $m_i$. Note that the latter is possible as $|U| < k/2$ and thus $|m_i \setminus U| > k/2$. Assume Maximizer picked $M_i^1, \ldots, M_{i-1}^1$ for some $1 < i \leq \lceil k^{1/2} \rceil$ satisfying (i)–(iii). By the inductive hypothesis for all $1 \leq h < i$ we have

$$|m_h \setminus \{ x : \deg_{M_{i-1}^2}(x) = 2 \} \cup U | > k - \frac{k}{2} - (i - 1)^2 \geq 2k^{1/2},$$

and the sets $M_i^2 \setminus \{ x : \deg_{M_{i-1}^2}(x) = 2 \}$ with $1 \leq h < i$ are pairwise disjoint. Thus if Maximizer picks $M_i^2$ so that it is disjoint from $\{ x : \deg_{M_{i-1}^2}(x) \geq 2 \} \cup U$, then (ii) and (iii) are clearly satisfied. Let us fix $x_h \in m_h \setminus \{ x : \deg_{M_{i-1}^2}(x) \geq 2 \} \cup U$ for all $1 \leq h < i$ and let $M_i^2 = \{ x : 1 \leq h < i \} \cup M_i^2$, where $|M_i^2| = k - l + 1 - |\{ x_h : 1 \leq h < i \}|$ and $M_i^2 \cap m_h \cap \bigcup_{h=1}^{i-1} (m_h \cup U) = \emptyset$.

It is possible to satisfy the latter condition as $n \geq k^{1/2}$. We see that the new degree-2 elements are $x_h$ and thus there is at most $i - 1$ of them. As $(i - 1)^2 + i - 1 \leq i^2$ we know that $M_i^2$ satisfies (i).

This inductive construction showed that the maximum degree of $M_{i+1}$ is 2 and thus its covering number is at least $(l + 1)/2 = \lfloor \sqrt{k}/2 \rfloor$. □
Proof of Theorem 1.1. To obtain the upper bound, Fast can use Maximizer’s strategy in the \( \tau \)-game, that is whenever it is his turn, he just copies whatever Maximizer would do in the same situation. According to Lemma 2.5, Fast can make sure that the resulting maximal intersecting family \( \mathcal{F} \) will have covering number at least \( \sqrt{k}/2 \). Thus by Proposition 2.1, we have
\[
|\mathcal{F}| \leq k^{\sqrt{\tau}/2} \left( n - k^{\sqrt{\tau}/2} \right) = O \left( n^{k^{\sqrt{\tau}/2}} \right).
\]
To obtain the lower bound, Slow can imitate minimizer’s strategy to ensure that, by Lemma 2.3, the resulting maximal intersecting family \( \mathcal{F} \) will have covering number at most \( \lceil 2k/3 \rceil + 2 \). Thus, by Proposition 2.1, we have
\[
|\mathcal{F}| \geq k^{\lceil 2k/3 \rceil - 4} = \Omega \left( n^{k^{\lceil 2k/3 \rceil - 4}} \right).
\]

3. Concluding remarks and open problems

The main result of the present paper, Theorem 1.1 states that the exponent of \( gsat(1_k) \) grows linearly in \( k \). We proved that the constant of the linear term is at least \( 1/3 \) and at most \( 1 \). We deduced this result by obtaining lower and upper bounds on the covering number of the family of sets claimed during the game and using a well-known relation between the covering number and the size of an intersecting family. However, we were not able to determine the order of magnitude of the covering number. We conjecture it is linear in \( k \). If it is true, then

Conjecture 3.1.

There exists a constant \( c > 0 \) such that for any \( k \geq 2 \) and \( n \geq n_0(k) \) the inequality \( gsat_r(I_n,k), gsat_s(I_n,k) \leq O \left( n^{(1-c)k} \right) \) holds.

In the proof of Lemma 2.5, there are two reasons for which Maximizer cannot continue his strategy for more than \( \sqrt{k} \) steps. First of all, the union of \( \mathcal{M}^1 \) becomes too large, and second of all, the intersection points of \( \mathcal{M}^2 \) and the sets of minimizer should be disjoint. The first problem can probably be overcome by a result of Kahn [12] who showed the existence of an \( r \)-uniform intersecting family \( \mathcal{F} \), with \( |\mathcal{F}| = O(r) \), \( |\bigcup_{F \in \mathcal{F}} F| = O(r) \) and \( \tau(\mathcal{F}) = r \).

As we mentioned in the introduction, the answers to game saturation problems considered so far did not depend on which player makes the first move, while this is the case for intersecting families if \( k = 2 \). However, if we consider the \( \tau \)-game, both our lower and upper bounds differ by at most one depending on whether it is Maximizer or minimizer to make the first move. Thus we formulate the following conjecture.

Conjecture 3.2.

There exists a constant \( c \) such that \( |\tau_M(n,k) - \tau_s(n,k)| \leq c \) holds independently of \( n \) and \( k \).

Intersecting families are most probably one of the two most studied classes of families in extremal set system theory. The other class is that of Sperner families: families \( \mathcal{F} \) that do not contain two different sets \( F, F' \) with \( F \subseteq F' \). The downset \( \mathcal{S}_n = \{ F \subseteq 2^n : F \text{ is Sperner} \} \) is another example for which the two game saturation numbers differ a lot. Clearly, if Fast starts the game, then he can claim either the empty set or \( |n| \) to finish the game immediately as both \( \emptyset \) and \( \{n\} \) are maximal Sperner families in \( 2^n \), thus we have \( gsat_r(\mathcal{S}_n) = 1 \). It is not very hard to see that if Slow starts with claiming a set \( F \subseteq [n] \) of size \( \lfloor n/2 \rfloor \), then the game will last at least a linear number of turns. Indeed, consider the family \( \mathcal{N}_F = \{ F \setminus \{ x \} : x \in F, y \in [n] \setminus F \} \). \( \mathcal{N}_F \) has size about \( n^2/4 \), while
\[
\max_{G \subseteq 2^n \setminus F} \left| \{ F' \in \mathcal{N}_F : F' \subseteq G \text{ or } G \subseteq F' \} \right| = \left\lfloor \frac{n}{2} \right\rfloor.
\]

This shows that \( gsat_s(\mathcal{S}_n) \geq n/2 \). One can improve this bound, but we were not able to obtain a superpolynomial lower bound nor an upper bound \( o \left( \binom{n}{\lfloor n/2 \rfloor} \right) \) for \( \text{ex}(\mathcal{S}_n) \).
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