Standard Hausdorff spectrum of compact $\mathbb{F}_p[[t]]$-analytic groups

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Abstract

We prove that the $\mathbb{F}_p[[t]]$-standard Hausdorff spectrum of a compact $\mathbb{F}_p[[t]]$-analytic group contains a real interval and that it coincides with the full unit interval when the group is soluble. Moreover, we show that the $\mathbb{F}_p[[t]]$-standard Hausdorff spectrum of classical Chevalley groups over $\mathbb{F}_p[[t]]$ is not full, since 1 is an isolated point thereof.

1 Introduction

The concept of Hausdorff dimension arose as a generalisation of the notion of topological dimension. This dimension can be defined in any metric space, and in the specific group theoretical context, the study of the Hausdorff dimension in profinite groups has attracted much attention.

If $G$ is a countably based profinite infinite group, a filtration series of $G$ is a family $\{G_n\}_{n \in \mathbb{N}}$ of descending open subgroups which is a neighbourhood system of the identity, that is, $\bigcap_{n \in \mathbb{N}} G_n = \{1\}$. Such a filtration defines a metric on $G$ by letting

$$d(x, y) = \inf \{|G : G_n|^{-1} \mid xy^{-1} \in G_n\}.$$  

This notion of distance makes $G$ into a metric space and so one can define the Hausdorff dimension of a subset $X \subseteq G$ with respect to that filtration (cf. [1, Section 2] and [5, Chapter 3]); it will be denoted by $\text{hdim}_{\{G_n\}}(X)$ or $\text{hdim}(X)$ if there is no risk of confusion. Further, when a filtration consists of normal subgroups it is called normal filtration. It was proved in [1, Theorem 2.4] that when the filtration is normal and $H$ is a closed subgroup of $G$ then one can compute the Hausdorff dimension by the following formula:

$$\text{hdim}_{\{G_n\}}(H) = \lim_{n \to \infty} \inf \frac{\log |HG_n : G_n|}{\log |G : G_n|}.$$  

(1)

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It has been repeatedly pointed out that the Hausdorff dimension may depend on the chosen filtration. Furthermore, for a fixed filtration \( \{G_n\}_{n\in\mathbb{N}} \) we can consider the collection of all the values \( \text{hdim}_{(G_n)}(H) \) as \( H \) ranges over closed subgroups of \( G \), that is, the set
\[
\text{hspec}_{(G_n)}(G) = \{ \text{hdim}_{(G_n)}(H) \mid H \leq_G G \},
\]
which is called the Hausdorff spectrum of \( G \) with respect to the filtration series \( \{G_n\}_{n\in\mathbb{N}} \). It turns out that these families may have little or no resemblance as one changes the filtration. For example, consider the additive \( p \)-adic analytic group \( \mathbb{Z}_p \oplus \mathbb{Z}_p \). For finitely generated pro-\( p \) groups of this kind there exists a natural filtration series, namely the \( p \)-power filtration given by \( G_n = G^{p^n} \). It is immediate to see that with respect to this filtration one has \( \text{hspec}_{(G_n)}(\mathbb{Z}_p \oplus \mathbb{Z}_p) = \{0, 1/2, 1\} \), and so in particular it is finite.

However, in [11, Theorem 1.3] it is shown that there exists a filtration series \( \{G_n\}_{n\in\mathbb{N}} \) such that \( \text{hspec}_{(G_n)}(\mathbb{Z}_p \oplus \mathbb{Z}_p) \) contains the real interval \( \left[\frac{1}{p+1}, \frac{p-1}{p+1}\right] \). Thus, even the finiteness of the Hausdorff spectrum is not filtration invariant.

According to [11, Corollary 1.2] \( \text{hspec}_{(G^{p^n})}(G) \) is finite for any \( p \)-adic analytic pro-\( p \) group \( G \), which suggests the following classical question (cf. [11, Problem 1]):

**Question 1.** Let \( G \) be a finitely generated pro-\( p \) group such that \( \text{hspec}_{(G^{p^n})}(G) \) is finite. Is \( G \) \( p \)-adic analytic?

Clearly, although the conjecture is stated here for the \( p \)-power filtration, it can also be posed for many other different non-pathological filtrations (some results in this direction can be found in [11]).

We will work in the setting of \( R \)-analytic groups where \( R \) is a pro-\( p \) domain; these comprise an abstract group together with an \( R \)-analytic manifold structure in such a way that both structures are compatible in the sense that the multiplication map and the inversion map are \( R \)-analytic functions. They are thoroughly studied in [4] and [16].

It can be proved that an \( R \)-analytic group is profinite if and only if it is compact, and thus formula (1) (with respect to any normal filtration) holds for compact \( R \)-analytic groups.

In this family of groups the \( p \)-power filtration series can not be used in general. Indeed, \( G^{p^n} \) will normally not be an open subgroup of a compact \( R \)-analytic group \( G \). However, they possess a canonical filtration series, called \( R \)-standard filtration series, which depends only on the \( R \)-analytic manifold structure of \( G \). The Hausdorff dimension relative to this filtration series – which is introduced insightfully in Section 3 – is called \( R \)-standard Hausdorff dimension.

In the present paper, we shall mostly restrict to the case \( R = \mathbb{F}_p[[t]] \), and the main findings of this investigation are:

**Theorem 1.1.** If \( G \) is a soluble compact \( \mathbb{F}_p[[t]] \)-analytic group then the Hausdorff spectrum of \( G \) with respect to the \( \mathbb{F}_p[[t]] \)-standard filtration is \([0, 1]\).
Theorem 1.2. If $G$ is a compact $\mathbb{F}_p[[t]]$-analytic group then the Hausdorff spectrum of $G$ with respect to the $\mathbb{F}_p[[t]]$-standard filtration contains the real interval $[0, \alpha]$ for some $\alpha \geq 1/\dim G$.

In the latter result, $\dim G$ denotes the analytic dimension of $G$ as an $\mathbb{F}_p[[t]]$-analytic manifold. In addition, in Corollary 5.4 the $\alpha$ occurring in the statement of Theorem 1.2 is described more accurately for classical Chevalley groups over $\mathbb{F}_p[[t]]$; in particular we shall show that they always satisfy $\alpha \geq 1/2$. Furthermore, in Corollary 5.9 we shall prove that for most of these groups $1$ is an isolated point in the spectrum, providing some examples of compact $\mathbb{F}_p[[t]]$-analytic groups whose spectrum with respect to the $\mathbb{F}_p[[t]]$-standard filtration series is not full.

Finally, we outline a consequence which can be derived from Theorems 1.1 and 1.2. An $R$-analytic subgroup is a structure which occurs both as a subgroup and a submanifold (for the latter we adopt Serre’s definition in [16, Part II, Section III.11]); for example any open subgroup is an $R$-analytic subgroup of maximal dimension. According to [6, Main Theorem], the $R$-standard Hausdorff dimension of an $R$-analytic subgroup can only take finitely many rational values. However, it follows from Theorem 1.2 that the $\mathbb{F}_p[[t]]$-standard spectrum of compact $\mathbb{F}_p[[t]]$-analytic groups is uncountable; hence showing that there are numerous closed subgroups that are not $\mathbb{F}_p[[t]]$-analytic.

Notation. Most of the notation is standard except $X^{(n)}$, which denotes the $n$-Cartesian power of the set $X$. Apart from that, $\mathbb{N}$ is the set of natural numbers (including 0), $p$ is a prime number, $\mathbb{F}_p$ is the finite field of $p$ elements, $\mathbb{Z}_p$ is the ring of $p$-adic integers and $R[[X]]$ is the power series ring with coefficients in the ring $R$. Moreover, $H \leq_0 G$ (resp. $H \leq_c G$) means that $H$ is an open (resp. closed) subgroup of a topological group $G$.

2 Preliminaries

Throughout this article, relating the Hausdorff dimension of a countably based profinite group to that of its subgroups and quotients will be of vital importance. Therefore, it is sometimes convenient to use the notation $\text{hdim}^G_{\{G_n\}}$ to emphasize that the dimension, with respect to the filtration series $\{G_n\}_{n \in \mathbb{N}}$, is calculated within the group $G$. The following result is known for subgroups (cf. [11, Lemma 5.3]), and it will be stated here for the convenience of the reader.

Lemma 2.1. Let $G$ be a countably based profinite group, $\{G_n\}_{n \in \mathbb{N}}$ a normal filtration series and $H \leq_0 G$ a closed subgroup whose Hausdorff dimension is given by a proper limit. Then

$$\text{hdim}^G_{\{G_n\}}(K) = \text{hdim}^G_{\{G_n\}}(H) \text{hdim}^H_{\{H \cap G_n\}}(K)$$

for all $K \leq_c H$. 

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The Hausdorff dimension of $H$ above being a proper limit means that

$$\text{hdim}_{\{G_n\}}(H) = \lim_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}.$$ 

Moreover, for quotients of countably based profinite groups we have the following result.

**Lemma 2.2.** Let $G$ be a countably based profinite group, $\{G_n\}_{n \in \mathbb{N}}$ a normal filtration series of $G$ and $N \trianglelefteq G$ a closed normal subgroup. Assume that $\text{hdim}_{\{G_n\}}(N)$ is given by a proper limit. Then for every subgroup $H \subseteq G$ containing $N$ one has

$$\text{hdim}_{\{G_n\}}^G(H) = \left(1 - \text{hdim}_{\{G_n\}}^G(N)\right) \text{hdim}_{\{G_n/N\}}^{G/N}(H/N) + \text{hdim}_{\{G_n\}}^G(N).$$

**Proof.** We observe that

$$\frac{\log |HG_n : G_nN|}{\log |G : G_n|} = \frac{\log |G : G_nN|}{\log |G : G_n|} \cdot \frac{\log |HG_n : G_nN|}{\log |G : G_n|} = \frac{\log |G : G_n| - \log |G_n : G_nN|}{\log |G : G_n|} \cdot \frac{\log |HG_n : G_nN|}{\log |G : G_nN|} = \left(1 - \frac{\log |G_nN : G_n|}{\log |G : G_n|}\right) \frac{\log |HG_n : G_nN|}{\log |G : G_nN|}.$$ 

Therefore, since $\text{hdim}_{\{G_n\}}^G(N) = \eta$ is given by a proper limit

$$\text{dim}_{\{G_n\}}^G(H) = \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|} = \liminf_{n \to \infty} \left(1 - \frac{\log |G_nN : G_n|}{\log |G : G_n|}\right) \frac{\log |HG_n : G_nN|}{\log |G : G_nN|} + \eta = (1 - \eta) \liminf_{n \to \infty} \frac{\log |HG_n/N : G_nN/N|}{\log |G/N : G_nN/N|} + \eta = (1 - \eta) \text{hdim}_{\{G_n/N\}}^{G/N}(H/N) + \eta,$$

as required. \qed

**Corollary 2.3.** Let $G$ be a countably based profinite group with normal filtration series $\{G_n\}_{n \in \mathbb{N}}$ and let $N \trianglelefteq G$ be a finite normal subgroup. Then

$$\hspec_{\{G_n\}}(G) = \hspec_{\{G_n/N\}}(G/N).$$

**Proof.** Since $\text{hdim}_{\{G_n\}}^G(N) = 0$ is given by a proper limit, the inclusion

$$\hspec_{\{G_n/N\}}(G/N) \subseteq \hspec_{\{G_n\}}(G)$$
is a direct consequence of the Correspondence Theorem and Lemma 2.2.

For the converse, consider $\eta \in \hspec_{(G_n)}(G)$; then there exists $H \leq G$ such that $\hdim^G_{(G_n)}(H) = \eta$. Thus, since $N$ is finite and the right multiplication is an isometry by Lemma 2.2 one has

$$\hdim^G_{(G_n)}(H) = \hdim^G_{(G_n)}\left( \bigcup_{n \in N} Hn \right) = \left. \hdim^G_{(G_n)}(HN) = \hdim^G_{(G_nN/N)}(HN/N), \right.$$ 

as required.

Finally, the combination of the above results yields the following corollary.

**Corollary 2.4.** Let $G$ be a countably based profinite group, $\{G_n\}_{n \in \mathbb{N}}$ a normal filtration series and let $N \triangleleft K \triangleleft G$ be closed subgroups such that $\hdim^G_{(G_n)}(N) = \eta$ and $\hdim^G_{(G_n)}(K) = \kappa$ are given by proper limits. If $\hspec_{(\frac{K}{G_nN})}(K/N) = [0,1]$ then $[\eta, \kappa] \subseteq \hspec_{(G_n)}(G)$.

**Proof.** Firstly, by Lemma 2.1 it follows that $\hdim^K_{(K \cap G_n)}(N) = \eta/\kappa$, and using the Correspondence Theorem and Lemma 2.2 we obtain

$$[\eta/\kappa, 1] = \left\{ (1 - \eta/\kappa) \alpha + \eta/\kappa \mid \alpha \in \hspec_{(\frac{K}{G_nN})}(K/N) \right\} \subseteq \hspec_{(K \cap G_n)}(K).$$

By another application of Lemma 2.1 one concludes $[\eta, \kappa] \subseteq \hspec_{(G_n)}(G)$. □

### 3 $R$-standard Hausdorff dimension

An $R$-analytic group $S$ is called $R$-standard of level $N$ and dimension $d$ when there exist a homeomorphism $\phi: S \to (m^N)^{(d)}$ such that $\phi(1) = 0$, and a formal group law $F$ over $R$ such that

$$\phi(xy) = F(\phi(x), \phi(y))$$

for every $x, y \in S$.

In that case, we usually write $(S, \phi)$ to denote the standard group, in order to emphasise the rôle of $\phi$. Any $R$-analytic group contains, by [4, Theorem 13.20], an open $R$-standard subgroup. In addition, by [4, Proposition 13.22], $R$-standard groups are pro-$p$ groups and so they are compact.

**Remark 3.1.** Let $X$ and $Y$ be two $d$-tuples of indeterminates. Since the formal group law $F \in R[[X, Y]]^{(d)}$ defines a group structure it is straightforward (cf. [4 Proposition 13.16(i)]) to see that it has the form

$$F(X, Y) = X + Y + G(X, Y),$$

where every monomial involved in $G$ has total degree at least 2 and contains a non-zero power of $X_i$ and $Y_j$ for some $i, j \in \{1, \ldots, d\}$. 

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In the context of compact $R$-analytic groups a natural filtration is available. Indeed, let $G$ be a compact $R$-analytic group and let $(S, \phi)$ be an open $R$-standard subgroup. An $R$-standard filtration of $G$ (the one induced by $S$) is the filtration $\{S_n\}_{n \in \mathbb{N}}$ defined by

$$S_n := \phi^{-1}\left(\left(m^{N+n}\right)^{(d)}\right) \quad \forall n \in \mathbb{N}.$$ 

It is immediate to see that an $R$-standard filtration is indeed a filtration. Furthermore, by [4, Proposition 13.22] one has that $S_n \subseteq S$ for any $n \in \mathbb{N}$ and thus formula (1) holds for $R$-standard groups with the above filtration.

Because of the dependence of $\text{hdim}$ on the chosen filtration we should not assume a priori that the Hausdorff dimension of a subgroup of a compact $R$-analytic group is the same when computed with respect to two different $R$-standard filtrations. However, the following result (cf. [6, Theorem 3.1]) shows that the $R$-standard Hausdorff dimension is independent of the standard subgroup.

**Theorem 3.2.** Let $G$ be a compact $R$-analytic group and let $(S, \phi)$ and $(T, \psi)$ be two open $R$-standard subgroups of $G$. Then

$$\text{hdim}_{(S, \phi)}(H) = \text{hdim}_{(T, \psi)}(H)$$

for every closed subgroup $H \leq G$.

This Hausdorff dimension, which we will denote by $\text{hdim}_{st}$, is called standard or $R$-standard Hausdorff dimension of $H$ and

$$\text{hspec}_{st}(G) = \{\text{hdim}_{st}(H) \mid H \leq c \leq G\}$$

is the standard or $R$-standard Hausdorff spectrum of $G$.

Note that an $R$-analytic subgroup $H$ of a compact $R$-analytic group $G$ is a compact $R$-analytic group in its own right, since it is a locally closed topological subgroup of a compact group; and thus its Hausdorff dimension can be computed. In particular, an $R$-standard filtration $\{S_n\}_{n \in \mathbb{N}}$ defines a Hausdorff dimension in both $G$ and the open $R$-standard subgroup $S$. In the notation of the preceding section, these dimensions are denoted respectively by $\text{hdim}_{G}(S_n)$ and $\text{hdim}^S_{(S, \phi)}$.

**Lemma 3.3.** Let $G$ be a compact $R$-analytic group with open $R$-standard subgroup $(S, \phi)$. Then

$$\text{hdim}_{G}(S_n)(H) = \text{hdim}^S_{(S, \phi)}(H \cap S)$$

for every closed subgroup $H \leq G$. 

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Proof. Let $d_G$ and $d_S$ be the metrics induced by the filtration $\{S_n\}_{n \in \mathbb{N}}$ in $G$ and in $S$ respectively. Then $d_G(x,y) = |G:S|^{-1}d_S(x,y)$ and so the inclusion map from $(S,d_S)$ to $(G,d_G)$ is bi-Lipschitz. Hence by [3] Proposition 3.3 it follows that

$$hdim_{\{S_n\}}^S(H \cap S) = hdim_{\{S_n\}}^G(H \cap S).$$

Moreover, since $H \cap S$ is an open subgroup of $H$ by [3] Lemma 2.4 we deduce that

$$hdim_{\{S_n\}}^G(H \cap S) = hdim_{\{S_n\}}^G(H),$$

as required.

Thus, we have the following immediate consequence.

**Corollary 3.4.** Let $G$ be a compact $R$-analytic group with an open $R$-standard subgroup $(S, \phi)$. Then $hspec_{st}(G) = hspec_{st}(S)$.

Accordingly, in order to study the standard Hausdorff spectrum of a compact $R$-analytic group we can assume that the original group $G$ is itself an $R$-standard group.

Finally, we shall study the standard Hausdorff dimension of subgroups and quotients. The following lemma relates $hdim^G$ with the Hausdorff dimension on $H$ induced in the natural way by an standard filtration $\{S_n\}_{n \in \mathbb{N}}$ of $G$, i.e., $hdim^H_{(H \cap S_n)}$.

**Lemma 3.5.** Let $G$ be a compact $R$-analytic group and $H$ an $R$-analytic subgroup of $G$. Then $hdim^H_{(H \cap S_n)}(K) = hdim^H_{st}(K)$ for all $K \leq_H H$, where $\{S_n\}_{n \in \mathbb{N}}$ is an $R$-standard filtration of $G$.

Proof. Firstly, let $\{S_n\}_{n \in \mathbb{N}}$ and $\{T_n\}_{n \in \mathbb{N}}$ be two $R$-standard filtrations of $G$. By Lemma 2.1 and Theorem 3.2 it is straightforward that

$$hdim^H_{(H \cap S_n)}(K) = hdim^H_{(H \cap T_n)}(K), \forall K \leq_H H. \tag{3}$$

Secondly, we shall show that there exists an open $R$-standard subgroup $S$ of $G$ such that $\{H \cap S_n\}_{n \in \mathbb{N}}$ is an $R$-standard filtration of $H$. Then for any $R$-standard filtration $\{T_n\}_{n \in \mathbb{N}}$ of $G$, by 3 we have that

$$hdim^H_{(H \cap T_n)}(K) = hdim^H_{(H \cap S_n)}(K) = hdim^H_{st}(K)$$

for all $K \leq_H H$, as desired.

Let $d = \dim G$ and $k = \dim H$, since $H$ is an $R$-analytic subgroup there exists an $R$-chart $(U, \phi)$ of 1 in $G$ such that

$$\phi(H \cap U) = \{(x_1, \ldots, x_d) \in (m^N)^{(d)} \mid x_{k+1} = \cdots = x_d = 0\} = (m^N)^{(k)} \times \{0\},$$

for some $N \geq 1$, and $\phi(1) = 0$. Furthermore, since $U$ is open in $G$, from the proof of [4] Theorem 13.20 there exists an open $R$-standard subgroup $S$ of $G$, of level $L \geq N$, contained in $U$ and with homeomorphism $\phi|_S$. Then

$$\phi(H \cap S) = \phi(S) \cap \phi(H \cap U) = (m^L)^{(d)} \cap (m^N)^{(k)} \times \{0\} = (m^L)^{(k)} \times \{0\},$$

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Therefore, if \( \pi: (m^L)^{(k)} \times \{0\} \to (m^L)^{(k)} \) is the natural homeomorphism, then \((H \cap S, \psi)\), where \( \psi = \pi \circ \phi_{|H \cap S} \), is an open \( R \)-standard subgroup of \( H \). Thus,

\[
\psi(H \cap S_n) = \pi(\phi(H \cap U) \cap \phi(S_n)) = (m^{L+n})^{(k)},
\]

and one concludes that \( \{H \cap S_n\}_{n \in \mathbb{N}} \) is an \( R \)-standard filtration of \( H \).

We will focus on the case \( R = \mathbb{F}_p[[t]] \) for quotients, since it is known (cf. [16, Part II, Section IV.5, Remarks 2]) that if \( G \) is an \( \mathbb{F}_p[[t]] \)-analytic group and \( N \leq G \) is a normal \( \mathbb{F}_p[[t]] \)-analytic subgroup, then \( G/N \) is an \( \mathbb{F}_p[[t]] \)-analytic group. Hence, we shall relate the standard spectrum of the group and the spectrum of its analytic quotients.

**Lemma 3.6.** Let \( G \) a compact \( \mathbb{F}_p[[t]] \)-analytic group, \( \{S_n\}_{n \in \mathbb{N}} \) an \( \mathbb{F}_p[[t]] \)-standard filtration of \( G \) and \( N \leq G \) a normal \( \mathbb{F}_p[[t]] \)-analytic subgroup of \( G \). Then

\[
\text{hdim}_{st}(H) = \text{hdim}_{\{s_n N\}}(H),
\]

for every \( H \leq G/N \).

**Proof.** Let us fix some notation: let \( R \) be the pro-\( p \) domain \( \mathbb{F}_p[[t]] \) with maximal ideal \( m = (t) \), \( d = \dim G \) and \( e = \dim G/N \); let \( \pi \) be the quotient map and let \( \text{pr}: m^{(d)} \to m^{(e)} \) be the projection onto the last \( e \) coordinates.

Firstly, if \( \{S_n\}_{n \in \mathbb{N}} \) and \( \{T_n\}_{n \in \mathbb{N}} \) are two \( R \)-standard filtrations of \( G \), as in the proof of [15, Theorem 3.1] it can be seen that

\[
\text{hdim}_{\{s_n N\}}(H) = \text{hdim}_{\{T_n N\}}(H), \quad \forall H \leq G/N.
\]

Hence by [14] it suffices to find an open \( R \)-standard subgroup \( S \) of \( G \) such that \( \{S_n N/N\}_{n \in \mathbb{N}} \) is an \( R \)-standard filtration of \( G/N \). According to [16, Part II, Section III.12] there exists an \( R \)-chart \((U, \phi)\) of 1 in \( G \) adapted to \( N \), that is, \( \phi(1) = 0 \) and \( \text{pr} \circ \phi(x) = \text{pr} \circ \phi(y) \) if and only if \( xy^{-1} \in N \). Since \( U \) is open in \( G \), from the proof of [14, Theorem 13.20] there exists an open \( R \)-standard subgroup \( S \), of level \( L \), contained in \( U \) with homeomorphism \( \phi_{|S} \). Let \( \sigma: \pi(S) \to S \) be a continuous section such that \( \sigma(1N) = 1 \) (which exists by [15, Proposition 2.2.2]), then \( \pi(S) \) is an \( R \)-standard subgroup of \( G/N \), with level \( L \), dimension \( e \) and homeomorphism \( \psi = \text{pr} \circ \phi \circ \sigma \). Note that since \((U, \phi)\) is an adapted \( R \)-chart, the definition of \( \psi \) is independent of the selected section and \( \psi(S_n N/N) = \text{pr} \circ \phi(S_n) = (m^{L+n})^{(e)} \), so \( \{S_n N/N\}_{n \in \mathbb{N}} \) is an \( R \)-standard filtration of \( G/N \).

\[
\text{Proof.}
\]

4 Soluble compact \( R \)-analytic groups

This section is devoted to proving Theorem [14].
4.1 Abelian compact $R$-analytic groups

Before dealing with soluble groups, we will prove the analogous result in the abelian case, where $R$ is a general pro-$p$ domain of characteristic $p$. We will use the following technical lemma (cf. [6, Lemma 2.3]).

**Lemma 4.1.** Let $(S, φ)$ be an $R$-standard group of dimension $d$. Then there exists a non-constant polynomial $f$ such that $|S : S_n| = p^{q(n)}$ for large enough $n$.

**Proposition 4.2.** Let $R$ be a pro-$p$ domain of characteristic $p$ and let $(S, φ)$ be an abelian $R$-standard group. Then $\hspec_{st}(S) = [0, 1]$.

**Proof.** By [11, Theorem 5.4] it suffices to prove that every finitely generated subgroup $H \leq_c S$ satisfies $\hdim_{st}(H) = 0$. Let $d$ be the dimension of $S$ and let $H \leq S$ be a topologically $r$-generated closed subgroup. Since the group operation in $S$ is given by a formal group law, by (2) whenever $x \in S_n$ we have

$$φ(x^n) = p φ(x) \equiv 0 \mod (m^{2n})^{(d)},$$

and thus $x^n \equiv 1 \mod S_{2n}$. Therefore $S_n/S_{2n}$ is an elementary abelian $p$-group.

Since $S$ abelian then $H/(H \cap S_n)$ is an abelian $p$-group of exponent $p^e$ where $e \leq [\log_2(n)]$. Moreover, since $H$ is topologically $r$-generated, it follows that $H/(H \cap S_n)$ is $r$-generated and so

$$|H : H \cap S_n| \leq p^{er} \leq p^{[\log_2(n)]r}.$$

Accordingly, by Lemma 4.1

$$\hdim_{st}(H) = \liminf_{n \to \infty} \frac{\log_p |H : H \cap S_n|}{\log_p |S : S_n|} \leq \liminf_{n \to \infty} \frac{r[\log_2(n)]}{df(n)} = 0,$$

as desired. □

Clearly, in view of Corollary 3.4, this result can be generalised to compact abelian $R$-analytic groups.

**Corollary 4.3.** Let $R$ be a pro-$p$ domain of characteristic $p$. If $G$ is an abelian compact $R$-analytic group, then $\hspec_{st}(G) = [0, 1]$.

Furthermore, it is known that any $R$-standard group of dimension one is abelian (cf. [7, Theorem 1.6.7]), and we thus have the following:

**Corollary 4.4.** Let $R$ be a pro-$p$ domain of characteristic $p$ and let $G$ be a compact $R$-analytic group of dimension one. Then $\hspec_{st}(G) = [0, 1]$.  

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4.2 $\mathbb{F}_p[[t]]$-analytic subgroups

Now we will mainly turn to the case when $R = \mathbb{F}_p[[t]]$. The main strategy to prove Theorem 1.1 lies in adding successive intervals to the spectrum, using the consecutive abelian quotients of a subnormal series. In fact, we have the following result.

**Lemma 4.5.** Let $G$ be a compact $\mathbb{F}_p[[t]]$-analytic group and let $N \triangleleft K \triangleleft G$ be $\mathbb{F}_p[[t]]$-analytic subgroups such that $\text{hspec}_{st}(K/N) = [0,1]$. Then

$$\left[ \frac{\dim N}{\dim G}, \frac{\dim K}{\dim G} \right] = [\text{hdim}_{st}(N), \text{hdim}_{st}(K)] \subseteq \text{hspec}_{st}(G).$$

**Proof.** By [6, Main Theorem] one has $\text{hdim}_{st}(H) = \dim H/\dim G$ for any $\mathbb{F}_p[[t]]$-analytic subgroup $H$ and a closer scrutiny of its proof reveals that such dimension is given by a proper limit; so the result is straightforward from Corollary 2.4, Lemma 3.5 and Lemma 3.6. \qed

Thus, we shall establish a useful criterion for finding $\mathbb{F}_p[[t]]$-analytic subgroups of a compact $\mathbb{F}_p[[t]]$-analytic group. The main obstacle compared with classical Lie theory arises here: it is well-known that any closed subgroup of a real ($p$-adic) Lie group is a real ($p$-adic) Lie subgroup; nevertheless for $R$-analytic groups, closedness is a necessary condition, but not sufficient. For example, the additive group $\mathbb{F}_p[[t]]$ is an $\mathbb{F}_p[[t]]$-standard group and $\mathbb{F}_p[[t^2]]$ is a closed subgroup with its own $\mathbb{F}_p[[t]]$-standard group structure. However, those manifold structures are not compatible and $\mathbb{F}_p[[t^2]]$ is not an $\mathbb{F}_p[[t]]$-analytic subgroup of $\mathbb{F}_p[[t]]$.

**Remark 4.6.** Denote by $\mathbb{F}_p((t))$ the local field of characteristic $p$ and valuation ring $\mathbb{F}_p[[t]]$, and let $M$ be an $\mathbb{F}_p[[t]]$-analytic manifold. Since $\mathbb{F}_p[[t]]$ is a discrete valuation ring, $M$ has also an $\mathbb{F}_p((t))$-analytic manifold structure (cf. [4, Section 13.1]).

The task of finding $\mathbb{F}_p((t))$-analytic subgroups will be carried out by using a generalisation from [9] which shows that homogeneous subsets have manifold structure over the local field $\mathbb{F}_p((t))$. According to the definition therein (cf. [9, Section 4]) a set $X \subseteq M$ is an analytic subset if for each $x \in X$ there exist an open neighbourhood $U$ of $x$ and some $\mathbb{F}_p((t))$-analytic functions $f_1, \ldots, f_r$ defined on $U$ (for some $r = r_x$) such that

$$X \cap U = \{ y \in U \mid f_i(y) = 0 \forall i = 1, \ldots, r \}.$$

In other words, an analytic subset is locally the nullset of some analytic functions. We then have (cf. [9, Corollary 4.2]):

**Theorem 4.7.** Let $G$ be an $\mathbb{F}_p[[t]]$-analytic group and let $H$ be both a subgroup of $G$ and an analytic subset of $G$. Then $H$ is an $\mathbb{F}_p[[t]]$-analytic subgroup of $G$.
Corollary 4.8. Let $G$ be an $\mathbb{F}_p[[t]]$-standard group and $a \in G$. Then $Z(G)$ and $C_G(a)$ are $\mathbb{F}_p[[t]]$-analytic subgroups.

Proof. By the previous theorem it is enough to show that $Z(G)$ and $C_G(a)$ are analytic subsets. The former is proved in [9, Corollary 4.3], while the latter follows the same spirit. Indeed, since $G$ is $\mathbb{F}_p[[t]]$-standard of level $N$ and dimension $d$, then it can be identified with $(t^N)^{(d)}$. Hence, for every $x \in G$, one has that $\mathbb{F}_p[[t]][[X_1, \ldots, X_d]]$ is a subring of the local ring of functions at $x$. For $(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ define $|\alpha| = \sum_{i=1}^d \alpha_i$. Since the group operation is given by a formal group law, by [2] there exist some $g_{i,\alpha} \in \mathbb{F}_p[[t]][[X_1, \ldots, X_d]]$ such that
\[
\pi_i \left( y^{-1}ay \right) = a_i + \sum_{|\alpha| > 1} g_{i,\alpha}(a)y_{1}^{\alpha_1} \cdots y_{d}^{\alpha_d} = a_i + h_i(y),
\]
where the map $\pi_i : (t^N)^{(d)} \to (t^N)$ is the projection to the $i$th coordinate. Then the maps $h_i(y) = \sum_{|\alpha| > 1} g_{i,\alpha}(a)y_{1}^{\alpha_1} \cdots y_{d}^{\alpha_d}$ are clearly $\mathbb{F}_p[[t]]$-analytic. Therefore
\[
C_G(a) = \{ y \in G \mid \pi_i \left( y^{-1}ay \right) = a_i \ \forall i = 1, \ldots, d \} = \{ y \in G \mid h_i(y) = 0 \ \forall i = 1, \ldots, d \},
\]
and $C_G(a)$ is an analytic subset. \qed

Corollary 4.9. Let $G \subseteq \text{GL}_n(\mathbb{F}_p[[t]])$ be a linear $\mathbb{F}_p[[t]]$-analytic group and let $\mathcal{H}$ be a Zariski closed subgroup of $\text{GL}_n(\mathbb{F}_p[[t]])$. Then $\mathcal{H} \cap G$ is an $\mathbb{F}_p[[t]]$-analytic subgroup of $G$.

Proof. Since $\mathcal{H}$ is closed in the Zariski topology it is an affine set, that is, there exists a subset $I$ of $\mathbb{F}_p[[t]][[\mathbf{X}]]$, where $\mathbf{X}$ is a tuple of $n^2$ variables, such that
\[
\mathcal{H} = \{ y \in \text{GL}_n(\mathbb{F}_p[[t]]) \mid f_i(y) = 0 \ \forall f_i \in I \}.
\]
But since $\mathbb{F}_p[[t]][[\mathbf{X}]]$ is Noetherian we can assume $I$ to be finite, and thus
\[
\mathcal{H} \cap G = \{ y \in G \mid f_i(y) = 0 \ \forall f_i \in I \}
\]
is an analytic subset, so it is an $\mathbb{F}_p[[t]]$-analytic subgroup by Theorem 4.7. \qed

We are now in position to prove the main theorem by using the previous results:

Proof of Theorem 4.11. By Corollary 3.3 we can assume without loss of generality that $G$ is $\mathbb{F}_p[[t]]$-standard. We first prove the theorem for the case when $G$ is linear over $\mathbb{F}_p[[t]]$, that is, $G \subseteq \text{GL}_n(\mathbb{F}_p[[t]])$. Let $\mathcal{G}$ be the Zariski closure of $G$ in $\text{GL}_n(\mathbb{F}_p[[t]])$. According to [13, Theorem 5.11] $\mathcal{G}$ is a soluble algebraic group, so there exists a soluble series
\[
\mathcal{G} = \mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \cdots \supseteq \mathcal{H}_{k-1} \supseteq \mathcal{H}_k = \{1\}.
\]
of Zariski closed subgroups. Then

\[ G = \mathcal{H}_1 \cap G \supset \mathcal{H}_2 \cap G \supset \cdots \supset \mathcal{H}_{k-1} \cap G \supset \mathcal{H}_k \cap G = \{1\} \]

is a soluble series of \( G \) given by \( \mathbb{F}_p[[t]] \)-analytic subgroups, by Corollary 4.9.

Denote \( H_i = \mathcal{H}_i \cap G \). Since each \( H_i \) is an \( \mathbb{F}_p[[t]] \)-analytic subgroup of \( G \) then \( H_{i-1}/H_i \) is a compact abelian \( \mathbb{F}_p[[t]] \)-analytic group for all \( i \in \{2, \ldots, k\} \), so by Corollary 4.3 it follows that \( \text{hspec}_{st}(H_i/H_{i-1}) = [0,1] \). Hence by Lemma 4.9 one has \( [\text{hdim}_{st}(H_i), \text{hdim}_{st}(H_{i-1})] \subseteq \text{hspec}_{st}(G) \) for all \( i \in \{2, \ldots, k\} \) and thus \( \text{hspec}_{st}(G) = [0,1] \).

Let us finally turn to the general case. By Corollary 4.8 \( Z(G) \) is an abelian \( \mathbb{F}_p[[t]] \)-analytic subgroup of \( G \) and thus by Corollary 4.3 and Lemma 4.5

\[ [0, \text{hdim}_{st} Z(G)] \subseteq \text{hspec}_{st}(G). \]

Moreover, by [10, Proposition 5.1] one has that \( G/Z(G) \) is a compact soluble linear \( \mathbb{F}_p[[t]] \)-analytic group. Hence

\[ \text{hspec}_{st}(S_n Z(G)/Z(G)) \subseteq \text{hspec}_{st}(G/Z(G)) = [0,1], \]

and so by Corollary 2.4

\[ [\text{hdim}_{st} Z(G), 1] \subseteq \text{hspec}_{st}(G), \]

thus obtaining the whole interval in the spectrum.

\[ \Box \]

5 Compact \( \mathbb{F}_p[[t]] \)-analytic groups

In this section, we first prove Theorem 1.2 and subsequently we study the Hausdorff spectrum of some classical Chevalley groups.

5.1 Proof of Theorem 1.2

The previous section suggests that a suitable way to find an interval in the \( \mathbb{F}_p[[t]] \)-standard Hausdorff spectrum of a compact \( \mathbb{F}_p[[t]] \)-analytic group \( G \) is looking for a soluble \( \mathbb{F}_p[[t]] \)-analytic subgroup. This search will rely heavily on the topological analogous of the Tits alternative. But we first observe the following:

Lemma 5.1. Let \( G \) be an \( \mathbb{F}_p[[t]] \)-standard group. Suppose that either

(i) \( Z(G) \) is infinite or

(ii) \( G \) contains an element \( x \) of infinite order.

Then \( [0, \frac{1}{\text{dim} G}] \subseteq \text{hspec}(G) \).
Proof. Under the first hypothesis, by Corollary 4.8 $Z(G)$ is an abelian infinite $F_p[[t]]$-analytic subgroup. Similarly, under the second hypothesis $Z(C_G(x))$ is an abelian $F_p[[t]]$-analytic subgroup which is infinite, because $\langle x \rangle \leq Z(C_G(x))$. In both cases, since $G$ is compact, there exists an abelian $F_p[[t]]$-analytic subgroup of positive dimension whose $F_p[[t]]$-standard spectrum is the whole interval $[0, 1]$, according to Proposition 4.2, thus the result follows by Lemma 4.5.

Proof of Theorem 5.2. First, observe that when $Z(G)$ is infinite the result follows by Lemma 5.1(i), so we shall deal with the case when $Z(G)$ is finite. But then $G/Z(G)$ is an $F_p[[t]]$-analytic group of dimension $\dim G$ and according to Corollary 2.3 it follows that

$$\text{hspec}_{\text{at}}(G) = \text{hspec}_{\text{at}}(G/Z(G)).$$

Furthermore, using [9] Proposition 5.1 we have that $G/Z(G)$ is an $F_p((t))$-analytic group which is linear over $F_p((t))$. Hence by the topological Tits alternative (cf. [3] Theorem 1.3) it follows that $G/Z(G)$ contains either an open solvable subgroup, say $H$, or contains a dense free subgroup. In the former case, $H$ is a solvable $F_p[[t]]$-analytic group of dimension $\dim G/Z(G) = 0$, and thus

$$\text{hspec}_{\text{at}}(G/Z(G)) = [0, 1].$$

In the latter case $G/Z(G)$ contains an element of infinite order and the statement follows from Lemma 5.1(ii).

5.2 Classical Chevalley groups

Even though the previous result ensures the existence of a real interval of type $[0, \alpha]$ in the standard Hausdorff spectrum, there is no general method to find the maximum value of $\alpha$.

However, when $G \subseteq \text{GL}_n(F_p[[t]])$ is linear over $F_p[[t]]$ we can use the theory of algebraic groups. Indeed, the Borel subgroup of $\text{GL}_n(F_p[[t]])$ - i.e. a maximal connected soluble algebraic subgroup of $\text{GL}_n(F_p[[t]])$, which is unique up to conjugation in $\text{GL}_n(F_p[[t]])$ (cf. [8] Theorem 21.3) - is the set of invertible $n \times n$ upper triangular matrices, say $B = T_n(F_p[[t]])$. Since $B$ is an algebraic group, by Corollary 4.9 it follows that $B(G) := B \cap G$ is a soluble $F_p[[t]]$-analytic subgroup of $G$. In particular, we can use this fact in order to describe the Hausdorff spectrum of the classical Chevalley groups with coefficients in the ring $F_p[[t]]$.

The Chevalley group over $F_p[[t]]$ associated to a root system of type $A_n$ ($n \geq 1$) is $\text{SL}_{n+1}(F_p[[t]])$, referred to as the special linear group. It is well-known that $\text{SL}_n(F_p[[t]])$ is an $F_p[[t]]$-analytic group containing as an open subgroup of finite index the $F_p[[t]]$-standard group

$$\text{SL}_n^1(F_p[[t]]) := \ker\{\text{SL}_n(F_p[[t]]) \rightarrow \text{SL}_n(F_p[[t]]/tF_p[[t]])\}$$
(cf. [4] Exercise 13.9). Accordingly, \( \text{SL}_n(\mathbb{F}_p[[t]]) \) is a compact \( \mathbb{F}_p[[t]] \)-analytic group of dimension \( n^2 - 1 \). For this first classical group we recover the following description of its \( \mathbb{F}_p[[t]] \)-standard spectrum, already proved in [1] for the congruence subgroup filtration of \( \text{SL}_n(\mathbb{F}_p[[t]]) \) — which is indeed an \( \mathbb{F}_p[[t]] \)-standard filtration of \( \text{SL}_n(\mathbb{F}_p[[t]]) \).

**Corollary 5.2.** (cf. [1, Proposition 4.4]) The \( \mathbb{F}_p[[t]] \)-standard Hausdorff spectrum of \( \text{SL}_n(\mathbb{F}_p[[t]]) \) contains the real interval \( [0, \frac{n(n+1)-2}{2n^2-2}] \).

**Proof.** Note that \( \mathcal{B}(\text{SL}_n(\mathbb{F}_p[[t]])) = \mathcal{B} \cap \text{SL}_n(\mathbb{F}_p[[t]]) \) is the soluble subgroup of matrices with determinant 1 and entries in \( \mathbb{F}_p[[t]] \), which is an \( \mathbb{F}_p[[t]] \)-analytic subgroup of dimension \( \frac{n(n+1)}{2} - 1 \). The result follows by Theorem 1.1 and Lemma 4.5.

This method can be also used with the remaining classical Chevalley groups over a general pro-\( p \) domain \( R \).

- A root system of type \( B_n \) (\( n \geq 2 \)) defines the odd special orthogonal group

  \[
  \text{SO}_{2n+1}(R) := \{ A \in M_{2n+1}(R) \mid A^t K_{2n+1} A = K_{2n+1} \},
  \]

  where \( K_n = \begin{pmatrix}
  0 & \cdots & 0 & 1 \\
  0 & \cdots & 1 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  1 & \cdots & 0 & 0
  \end{pmatrix} \in M_n(R) \), which is an \( R \)-analytic group of dimension \( n(2n+1) \).

- A root system of type \( C_n \) (\( n \geq 3 \)) defines the symplectic group

  \[
  \text{Sp}_{2n}(R) := \{ A \in M_{2n}(R) \mid A^t J_{2n} A = J_{2n} \},
  \]

  where \( J_{2n} = \begin{pmatrix}
  0 & K_n \\
  -K_n & 0
  \end{pmatrix} \), which is an \( R \)-analytic group of dimension \( n(2n+1) \).

- A root system of type \( D_n \) (\( n \geq 4 \)) defines the even special orthogonal group

  \[
  \text{SO}_{2n}(R) := \{ A \in M_{2n}(R) \mid A^t K_{2n} A = K_{2n} \},
  \]

  which is \( R \)-analytic of dimension \( n(2n-1) \).

Again, all these groups contain an \( R \)-standard subgroup of the same dimension (cf. [4] Exercise 13.11), say \( S \). Then \( S \) is an open \( R \)-standard subgroup, and since according to the following lemma they are compact \( R \)-analytic groups it follows that they are in fact profinite groups.

**Lemma 5.3.** Let \( R \) be a pro-\( p \) domain. Then \( \text{SO}_n(R) \) and \( \text{Sp}_{2n}(R) \) are compact topological spaces.
Proof. Since \( R/\mathfrak{m} \) is finite then \( \widehat{R} \), the completion of \( R \) in the \( \mathfrak{m} \)-adic topology, is compact. But \( R \) is complete so \( R = \widehat{R} \) is compact with the \( \mathfrak{m} \)-adic topology, and so \( M_n(R) = R^{(n^2)} \) is a compact topological space. Hence, the closed subgroups \( \text{SO}_n(R) \) and \( \text{Sp}_{2n}(R) \) are compact.

We are now in a position for describing the Hausdorff spectrum of those profinite groups.

**Corollary 5.4.** For any \( n \geq 1 \)

(i) \( \text{hspec}_{\text{st}}(\text{Sp}_{2n}(\mathbb{F}_p[[t]])) \) contains the real interval \( \left[ 0, \frac{n+1}{2n+1} \right] \),

(ii) \( \text{hspec}_{\text{st}}(\text{SO}_{2n}(\mathbb{F}_p[[t]])) \) contains the real interval \( \left[ 0, \frac{n}{2n-1} \right] \) and

(iii) \( \text{hspec}_{\text{st}}(\text{SO}_{2n+1}(\mathbb{F}_p[[t]])) \) contains the real interval \( \left[ 0, \frac{n+1}{2n+1} \right] \).

Proof. Denote \( R = \mathbb{F}_p[[t]] \).

(i) Note that \( B \text{pSp}_{2n}(\mathbb{F}_p[[t]]) = \text{Sp}_{2n}(\mathbb{F}_p[[t]]) \cap T_{2n}(R) \), which after a simple computation (cf. Example 6.7(4)) can be seen to coincide with the product of the set

\[
\left\{ \begin{pmatrix} A & 0 \\ 0 & K_n A^{-1} K_n \end{pmatrix} \bigg| A \in T_{n}(R) \right\}
\]

with \( \left\{ \begin{pmatrix} I_n & K_n S \\ 0 & I_n \end{pmatrix} \bigg| S \in M_n(R) \text{ is symmetric} \right\} \).

Hence, \( B(\text{Sp}_{2n}(R)) \) is a soluble \( \mathbb{F}_p[[t]] \)-analytic subgroup of dimension \( n^2 + n \), and thus by Theorem 1.1 and Lemma 4.5 we have

\[
\left[ 0, \frac{n+1}{2n+1} \right] = \left[ 0, \frac{\dim B(\text{Sp}_{2n}(R))}{\dim \text{Sp}_{2n}(R)} \right] \subseteq \text{hspec}_{\text{st}}(\text{Sp}_{2n}(R)).
\]

(ii) In much the same way one has \( B(\text{SO}_{2n}(R)) = \text{SO}_{2n}(R) \cap T_{2n}(R) \), which is a soluble \( \mathbb{F}_p[[t]] \)-analytic subgroup of dimension \( n^2 \), so

\[
\left[ 0, \frac{n}{2n-1} \right] = \left[ 0, \frac{\dim B(\text{SO}_{2n}(R))}{\dim \text{SO}_{2n}(R)} \right] \subseteq \text{hspec}_{\text{st}}(\text{SO}_{2n}(R)).
\]

(iii) Similarly one has \( B(\text{SO}_{2n+1}(R)) = \text{SO}_{2n+1}(R) \cap T_{2n+1}(R) \), which is a soluble \( \mathbb{F}_p[[t]] \)-analytic subgroup of dimension \( n^2 + n \), so

\[
\left[ 0, \frac{n+1}{2n+1} \right] = \left[ 0, \frac{\dim B(\text{SO}_{2n+1}(R))}{\dim \text{SO}_{2n+1}(R)} \right] \subseteq \text{hspec}_{\text{st}}(\text{SO}_{2n+1}(R)). \quad \square
\]

Note in passing that for any classical Chevalley group one has \( \alpha \geq 1/2 \).
Finally, we shall provide examples of compact \( \mathbb{F}_p[[t]] \)-analytic groups whose spectrum is not the whole interval. More precisely, we will show that in classical Chevalley groups \( 1 \) is an isolated point in the spectrum, thus proving that \( \alpha < 1 \).

In passing, we note that in \([1, \text{Theorem 1.4}]\) it is proved that if \( p > 2 \) then

\[
\text{hspec}_{st} \left( \text{SL}_n^1(\mathbb{F}_p[[t]]) \right) \cap \left( 1 - \frac{1}{n + 1}, 1 \right) = \emptyset.
\]

We will prove an analogous result for the other classical Chevalley groups following the same technique and working in the corresponding graded Lie algebra. Given an \( R \)-analytic group and a \( p \)-central series \( \{G_n\}_{n \in \mathbb{N}} \) (note that by \([4, \text{Proposition 13.22}]\) any \( R \)-standard filtration is a \( p \)-central series), we can define the restricted graded Lie \( \mathbb{F}_p \)-algebra \( L \)

\[
L = \bigoplus_{n \geq 0} \frac{(H \cap G_n)G_{n+1}}{G_{n+1}}.
\]

Although every closed subgroup defines a graded subalgebra, there might be graded subalgebras that do not arise in this way.

**Notation.** Since \( \text{dim} \) usually denotes the analytic dimension of a manifold, henceforth \( \text{dim}_F \) will be used to denote the \( F \)-vector space dimension.

Given a graded \( \mathbb{F}_p \)-algebra \( L = \bigoplus_{n \geq 0} L_n \) and a graded \( \mathbb{F}_p \)-subalgebra \( K = \bigoplus_{n \geq 0} K_n \), the **Hausdorff density** is defined as follows

\[
h \text{D}(K) := \liminf_{n \to \infty} \frac{\sum_{m \leq n} \text{dim}_F K_m}{\sum_{m \leq n} \text{dim}_F L_m}.
\]

Clearly, in view of the preceding definitions for any closed subgroup \( H \) we have \( h \text{D}(L(H)) = \text{hdim}_{(G_n)}(H) \) (cf. \([1, \text{Lemma 5.1}]\)).

Let now \( F \) be a field and \( \mathcal{G} \) a finite dimensional perfect (i.e. \( [\mathcal{G}, \mathcal{G}] = \mathcal{G} \)) \( F \)-algebra; then we can consider the infinite dimensional \( F \)-algebra \( \mathcal{G} \otimes_F t \mathbb{F}_p[t] \) with Lie bracket defined by \( [A \otimes t^n, B \otimes t^m] := [A, B]_{\mathcal{G}} \otimes t^{n+m} \) on elementary tensors. We now note the following:

**Lemma 5.5.** Let \( L = \mathcal{G} \otimes_F t \mathbb{F}_p[t] \) be as above. Then, any graded \( F \)-subalgebra of infinite codimension is contained in a graded \( F \)-subalgebra of infinite codimension, maximal with respect to that property.

**Proof.** Firstly, \( L \) is a finitely generated \( F \)-algebra. Indeed, let \( \{x_1, \ldots, x_m\} \) be a generating set of \( \mathcal{G} \), then \( S = \{x_1, \ldots, x_m, x_1 \otimes t, \ldots, x_m \otimes t\} \) generates \( L \). Indeed, \( \langle S \rangle_F \) contains \( \mathcal{G} \) and \( \mathcal{G} \otimes t \); and assume by induction that \( \langle S \rangle_F \) contains \( \mathcal{G} \otimes t^{n-1} \). Then, since \( \mathcal{G} \) is perfect

\[
\mathcal{G} \otimes t^n = [\mathcal{G}, \mathcal{G}] \otimes t^n = [\mathcal{G} \otimes t^{n-1}, \mathcal{G} \otimes t] \subseteq \langle S \rangle_F.
\]
Now, the result follows by Zorn’s Lemma. Indeed, consider the set of graded $F$-subalgebras of infinite codimension, which is partially ordered under inclusion. Let $\{H_i\}_{i \in I}$ be a totally ordered subset of graded $F$-algebras of infinite codimension and consider $H = \cup_{i \in I} H_i$, which is a graded $F$-subalgebra of $L$. Suppose by contradiction that $H$ has finite codimension in $L$, and so that it is a finitely generated $F$-algebra. Assume that $H = \langle h_1, \ldots, h_r \rangle_F$, then there exists an $i_0 \in I$ such that $h_k \in H_{i_0}$ for all $k \in \{1, \ldots, r\}$ and so $H = H_{i_0}$ has infinite codimension in $L$, which is a contradiction. Hence $\{H_i\}_{i \in I}$ has a maximal member with respect to inclusion, which concludes the proof. \[ \square \]

If one requires central simplicity, we have the following result (cf. \[4\] Corollary 5.3) bounding the Hausdorff density of graded subalgebras that are maximal with respect to having infinite codimension.

**Theorem 5.6.** Let $G$ be a central simple algebra over a field $F$ and let $L = G \otimes_F tF[t]$. Then the density of a graded subalgebra that is maximal with respect to having infinite codimension is either $1/q$, where $q$ is a prime, or $\dim_F \mathcal{H}/\dim_F G$, where $\mathcal{H}$ is a maximal graded subalgebra of $G$.

**Remark 5.7.** Recall that a finite dimensional algebra over a field $F$ is called central simple when it is simple and its centroid coincides with $F$. Nevertheless, if $F$ is finite, the assumption of the previous theorem can be weakened to only requiring that $G$ is simple. Indeed, the previous theorem is a corollary of \[2\] Theorem 4.1 and it is pointed out in \[2\] Remark after Theorem 1.1] that when $F$ is finite simplicity of $G$ is enough.

Now, we apply this result to see that $1$ is an isolated point in the standard spectrum of most of the classical Chevalley groups.

**Corollary 5.8.** Let $X_n$ be a root system of type $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 3$) or $D_n$ ($n \geq 4$), let $G = G(X_n)$ be the classical Chevalley group associated to $X_n$ on $F_p[[t]]$, and $L(R)$ the algebra associated to that root system on an arbitrary ring $R$. If $L(F_p)$ is simple, then

$$\text{hspec}_{st}(G) \cap \left(1 - \frac{1}{\dim G}, 1\right) = \emptyset.$$ 

**Proof.** On the one hand, by \[4\] Exercise 13.11] it follows that $G$ contains an open $F_p[[t]]$-standard subgroup, say $S$, such that $L(S) \cong L(F_p[[t]])$. Furthermore, by \[4\] Proposition 13.27] there is an isomorphism $L(S) \cong L_0 \otimes_{F_p} \text{gr} \mathfrak{m}$ as $F_p$-vector spaces where

$$L_0 \cong L(F_p[[t]])/tL(F_p[[t]]) = L(F_p), \quad \text{and} \quad \text{gr} \mathfrak{m} = \bigoplus_{n \geq 1} (t^n)/(t^{n+1}).$$

Hence $L(S) \cong L(F_p) \otimes_{F_p} tF_p[t]$.

On the other hand, let $H \leq_c S$ be a closed subgroup with $\text{hdim}_{st}(H) < 1$. Then $|S : H|$ is infinite and so $L(H)$ has infinite codimension in $L(S)$. Since
$L(F_p)$ is simple and any simple algebra is perfect, according to Lemma 5.5 we have that $L(H)$ is contained in a graded subalgebra of $L(S)$, say $M$, maximal with respect to having infinite codimension. Hence by Theorem 5.6 and Remark 5.7 we have

\[
hdim_{\text{st}}(H) = hD(L(H)) \leq hD(M)
\]

\[
\leq \max \left\{ \frac{1}{2} \frac{\dim_{F_p} \mathcal{H}}{\dim_{F_p} L(F_p)} \right\} \quad \text{maximal subalgebra of } L(F_p)
\]

\[
\leq 1 - \frac{1}{\dim_{F_p} L(F_p)} = 1 - \frac{1}{\dim S},
\]

because $\dim S = \dim G(X_n) = \dim_{F_p} L(F_p)$. Therefore, the result follows since $h\text{spec}_{\text{st}}(G) = h\text{spec}_{\text{st}}(S)$.

Finally, the classical Chevalley algebras $\mathfrak{so}_n(F)$ and $\mathfrak{sp}_{2n}(F)$ over a field of positive characteristic $p$ have been thoroughly studied. When $p = 2$ none of them is simple, but when $p \geq 3$ it is well-known that $\mathfrak{so}_n(F) \quad (n \geq 5)$ and $\mathfrak{sp}_{2n}(F) \quad (n \geq 2)$ are simple algebras (cf. [17]). Hence we deduce that:

**Corollary 5.9.** Let $p \geq 3$ be a prime and assume $G$ is either $\mathfrak{so}_n(F_p[[t]])$ $(n \geq 5)$ or $\mathfrak{sp}_{2n}(F_p[[t]])$ $(n \geq 2)$. Then

\[
h\text{spec}_{\text{st}}(G) \cap \left( 1 - \frac{1}{\dim G}, 1 \right) = \emptyset.
\]

Classical Chevalley groups over the local field $F_p((t))$ are linear simple algebraic groups. More generally, if $G$ is an algebraic group over the local field $F_p((t))$, then the group of $F_p[[t]]$-rational points, $G(F_p[[t]])$, admits naturally an $F_p[[t]]$-analytic manifold structure (cf. [14, Proposition I.2.5.2]). Hence, the above result suggests the following conjecture:

**Conjecture 5.10.** Let $G$ be a linear $F_p((t))$-algebraic semisimple group. Then

\[
h\text{spec}_{\text{st}}(G(F_p[[t]])) \cap \left( 1 - \frac{1}{\dim G(F_p[[t]]), 1} \right) = \emptyset.
\]

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References

[1] Y. Barnea and A. Shalev, Hausdorff dimension, pro-$p$ groups and Kac-Moody algebras, *Trans. Amer. Math. Soc.* **349** (1997), 5073-5091.

[2] Y. Barnea, A. Shalev and E.I. Zelmanov, Graded subalgebras of affine Kac-Moody algebras, *Israel J. Math.* **104** (1998), 321-334.
[3] E. Breuillard, T. Gelander, A topological Tits alternative, *Ann. of Math.* **166** (2007), 427-474.

[4] J.D. Dixon, M.P.F. Du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, 2nd ed, Cambridge University Press, Cambridge, 1999.

[5] K. Falconer, *Fractal geometry: mathematical foundations and applications*, 3rd ed, John Wiley & Sons Ltd, Chichester, 2014.

[6] G.A. Fernández-Alcober, E. Giannelli and J. González-Sánchez, Hausdorff dimension in $R$-analytic profinite groups, *J. of Group Theory* **20** (2017), 579-587.

[7] M. Hazewinkel, *Formal groups and applications*, Academic Press, New York-San Francisco-London, 1978.

[8] J.E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975.

[9] A. Jaikin-Zapirain and B. Klopsch, Analytic groups over general pro-$p$ domains, *J. Lond. Math. Soc.* **76** (2007), 365-383.

[10] A. Jaikin-Zapirain, On linear just infinite pro-$p$ groups, *J. Algebra* **255** (2002), 392-404.

[11] B. Klopsch, A. Thillaisundaram and A. Zugadi-Reizabal, Hausdorff dimension in $p$-adic analytic groups, *Israel J. Math.* **231** (2019), 1-23.

[12] A. Lubotzky and A. Shalev, On some $A$-analytic pro-$p$ groups, *Israel J. Math.* **85** (1994), 307-337.

[13] G. Malle and D. Testerman, *Linear algebraic groups and finite groups of Lie type*, Cambridge University Press, Cambridge, 2011.

[14] G.A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, Berlin-Heidelberg, 1991.

[15] L. Ribes and P. Zalesskii, *Profinite groups*, 2nd edition, Springer-Verlag, Berlin-Heidelberg, 2010.

[16] J.P. Serre, *Lie Algebras and Lie Groups*, Springer-Verlag, Berlin-Heidelberg, 1992.

[17] H. Strade, Simple Lie algebras over fields of positive characteristic, Volume I Structure Theory, De Gruyter, Berlin-Boston, 2004.

[18] B.A.F. Wehrfritz, *Infinite Linear Groups*, Springer-Verlag, Berlin-Heidelberg, 1973.
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