Analyticity Properties of Scattering Amplitude in Theories with Compactified Space Dimensions: The Proof of Dispersion Relations

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Abstract

The analyticity properties of the scattering amplitude for a massive scalar field is reviewed in this article where the spacetime geometry is $R^{3,1} \otimes S^1$ i.e. one spatial dimension is compact. Khuri investigated the analyticity of scattering amplitude in a nonrelativistic potential model in three dimensions with an additional compact dimension. He showed that, under certain circumstances, the forward amplitude is nonanalytic. He argued that in high energy scattering if such a behaviour persists it would be in conflicts with the established results of quantum field theory and LHC might observe such behaviors. We envisage a real scalar massive field in flat Minkowski spacetime in five dimensions. The Kaluza-Klein (KK) compactification is implemented on a circle. The resulting four dimensional manifold is $R^{3,1} \otimes S^1$. The LSZ formalism is adopted to study the analyticity of the scattering amplitude. The nonforward dispersion relation is proved. In addition the Jin-Martin bound and an analog of the Froissart-Martín bound are proved. A novel proposal is presented to look for evidence of the large-radius-compactification scenario. A seemingly violation of Froissart-Martín bound at LHC energy might hint that an extra dimension might be decompactified. However, we find no evidence for violation of the bound in our analysis.

1Dedicated in fond memories of André Martin.
1 Introduction

The purpose of this review is to present the study of the analyticity properties of scattering amplitude for a massive hermitian scalar field theory in four dimensional spacetime with one additional compactified spatial dimension. This extra dimension is a circle i.e. we consider $S^1$ compactification. The axiomatic field theory formulation of Lehmann, Symanzik and Zimmermann (LSZ) is adopted to prove the dispersion relations for the four point amplitude. Our motivation to undertake this investigation stems from several discussion with André Martin in 2018. Khuri, in 1995, had studied the analyticity property of forward scattering amplitude in a nonrelativistic potential model in three dimensions with an extra compact spatial dimension (he considered $S^1$ compactification). He concluded that, under certain circumstances, as will be elaborated later, the amplitude does not satisfy the same analyticity property as enjoyed by the amplitudes of conventional potential models in three dimensional space. Martin raised an important question: What are the analyticity properties of an amplitude in a relativistic quantum field theory where one spatial dimension is compact? Should the established analyticity properties of an amplitude, derived rigorously in field theories in $D = 4$, be violated in the compactified-spatial-dimension theory it would lead to grave consequences. The fallout would be that some of the fundamental axioms of local quantum field theories would be questioned. We had undertaken an investigation to address these issues. The details will be elaborated in the sequel. This article is presented in a pedagogical style. Thus the reader, with a background in relativistic quantum field theory, can work out the steps if interested. The approach presented here for the problem at hand, to the best of the knowledge of the author, has not been reported previously. Therefore, we have expended on our earlier investigations [1, 2] in order to make this article accessible to a wider audience. We begin with a few remarks to motivate the reader.

There are two approaches to study scattering processes. In the perturbative formulation, we decompose the Lagrangian into a free part and an interaction part. The free field equations are exactly solvable. The procedures of perturbation theory allow us to compute the S-matrix elements order by order. We encounter divergences in computations. The renormalization prescription consistently removes the infinities at each order. Therefore, in this approach, the renormalizable theories are able to give us finite results which are subjected to verifications against experimental data. The crossing symmetry is maintained at each order as we include all Feynman diagrams. In other words, when we consider all the Feynman diagrams in a given order, we include all direct channel diagrams and all possible crossed channel diagrams. The diagrammatic technique already has built in crossing symmetry. Furthermore, the unitarity property of the amplitude is to be ensured order by order in the perturbation expansions. There are well laid down prescriptions to test the analyticity properties. The success of quantum electrodynamic (QED) in computing the anomalous magnetic moments of charged leptons and in computing Lamb shift with unprecedented
accuracy tell the success of perturbative formulation of renormalizable quantum field theories. Moreover, the predictions of the standard model of particle physics have been tested to great degree of accuracy against experimental data.

The S-matrix proposal of Heisenberg [3] is radically different from the perturbation theoretic formulations. He argued that in a scattering experiment the initial states are in the remote past. The projectile hits the target and the experimentalists observe the outgoing particles in remote future. Therefore, the initial state consists of free particles characterized by their physical mass, angular momenta and spins etc. Similarly, the attributes of final states are observed in the detector. We may imagine the initial state to be a vector which is prepared and final state to be another vector. However, each complete set of vectors span the Hilbert space. Therefore, there must be a unitary operator connecting the two sets in order to ensure the conservation of probability. He designated it as the scattering matrix or S-matrix. The initial and final states contain particles whose masses are observed quantities. Therefore, there are no divergence difficulties as encountered in perturbative approach. This is a very naive and qualitative way of introducing the concept of his S-matrix. Thus there is a different philosophical approach. The idea of Heisenberg was built on rigorous foundation in subsequent years. The axiomatic formulation of Lehmann, Symanzik and Zimmermann (LSZ) [4] is a landmark in relativistic quantum field theories (QFT). Wightman [5] proposed that field theories be studied in terms of vacuum expectation values of product of field operators and he introduced a set of axioms. There are very important theorems on the analyticity properties of scattering amplitude which have been proved from the frameworks of general axiomatic field theories [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

We shall adopt the axiomatic approach of LSZ in this article and this formulation will be elaborated in the next section. They introduced the notion of asymptotic fields and interacting fields. Moreover, a Lagrangian is not introduced explicitly. Therefore, there is no proposal of separating the theory into a free and an interacting theory. Some of the axioms include existence of a Hilbert space, Lorentz invariance, microcausality to mention a few (see next section for details). It is quite startling that the formulation enables computation of a scattering amplitude. Furthermore, the axioms lead to a set of linear relationships among various amplitudes. It is the hallmark of the axiomatic field theory. It must be emphasized that, within this linear framework, several important attributes of the scattering amplitude are derived. Of special importance is the proof of dispersion relations. As we shall discuss in subsequent sections, the dispersion relations are derived from general requirements such as Lorentz invariance and microcausality. Indeed, there is a deep and intimate relationship between analyticity of scattering amplitude and causality when we consider local quantum field theories. The unitarity of the S-matrix, which is a nonlinear relation, is not invoked in order to arrive at the linear relations. We mention en pasant that the unitarity of S-matrix is proved in the LSZ framework [17, 19]. Indeed, the analyticity properties of scattering amplitudes are derived rigorously in the LSZ framework.
formalism. One of most celebrated result is the Froissart-Martin bound \([20, 21]\) on total cross sections

$$\sigma_t(s) \leq \frac{4\pi}{t_0} \log^2\left(\frac{s}{s_0}\right)$$

(1)
The essential interpretation of the bound is as follows. We may ask what is the energy dependence of a total cross section, \(\sigma_t(s)\), at asymptotic energies? The above bound implies that its energy dependence is not arbitrary. Moreover, the constant prefactor appearing in the r.h.s. of (1) was determined from first principles by Martin \([21]\). We need to introduce a constant, \(s_0\), in order that the argument of the \(\log\) is dimensionless. However, it cannot be determined from first principles. The necessary ingredients for derivation of the upper bound are: (i) Analyticity of the scattering amplitude, (ii) polynomial boundedness of the scattering amplitude as the energy tends to asymptotic values and (iii) the unitarity bounds on partial wave amplitudes. All the three properties have been proved from axiomatic field theories. The total cross sections, \(\sigma_t\), measured in high energy experiment respect the Froissart-Martin bound. Should there be conclusive experimental evidence of the violation of this bound the fundamental axioms of local relativistic quantum field theories would be questioned. There is a host of results, usually presented as upper and lower bounds on experimentally measurable parameters, which have been subjected to experimental tests. There are no evidences for violation of any these rigorous bounds \([7, 12, 13]\).

We live in four spacetime dimensions. All the experiments are carried out in laboratories and the theories are constructed in four dimensions. The fundamental theories have been subjected to experimental tests. It is now an accepted idea that there might exist deeper fundamental theories which live in higher spacetime dimensions, \(D > 4\). There are well defined theories such as supersymmetric theories, supergravity theories and string theories which are defined in higher dimensions. The string theories hold the prospect of unifying the four fundamental interactions. Considerable attentions have been focused on string theories over last few decades. Therefore, it is pertinent to ask what relevance these theories have for physics in four spacetime dimensions. The proposal of Kaluza and Klein (KK) \([22, 23]\) are invoked in the context of such higher dimensional theories. Kaluza and Klein \([22, 23]\) envisaged a five dimensional theory of pure gravity which is a generalized version of Einstein’s theory. They argued that the 5th dimension is compactified on a circle. Therefore, the length scale probed in that era cannot resolve the size of the extra compact dimension. They carried out what is now known as the KK compactification scheme. It was shown that the effective four dimensional theory, in its massless sector, corresponds to a Maxwell-Einstein theory in four dimensions. Moreover, there is a tower of massive states (the KK states) and the mass in each level is proportional to \(\frac{1}{R}\), where \(R\) is the radius of the circle. If the radius of the circle is very small then these states become very massive and they cannot be observed by the experimental techniques prevailing those days. The proposal of Kaluza and Klein were employed to the compactification of supergravity theories in 1970’s \([24]\). There was a lot of interest in the KK
compactification after the second superstring revolution. In a rapid development, more elegant and sophisticated compactification schemes were developed [25, 26]. In the early phase of the string compactification era, it was generally believed that the radius of compactification of the compact dimensions would be in the vicinity of the Planck length. Therefore, the string excitations of the compactified theory would be so heavy that their observation will be out of reach of any accelerator. Antoniadis [27] proposed a scenario where the compactification scale is in the TeV range and therefore, the KK excitations associated with string theories might be detected in future accelerators. Antoniadis, Munoz and Quiros [28] pursued this idea further. Arkani-Hamed, Dimopoulos and Dvali [29] proposed a large-radius-compactification scheme of theirs and worked out the phenomenological implications. Subsequently, Antoniadis, Arkani-Hamed, Dimopoulos and Dvali [30] advanced the idea of large radius compactification proposal further. There were a lot of activities, in subsequent years, to investigate details of phenomenology of these proposals. The LHC was going to be commissioned in near future. There was optimism that KK states of string would be observed at LHC and it would be an experimental confirmation of the ideas of string theories. A review of the theoretical progress in this direction will be found in [31, 32]. So far the LHC experiments have established only limits on the scale of compactification in the light of the large radius compactification paradigm [33, 34]. In an interesting paper Khuri [35], investigated the analyticity of scattering amplitude where a spatial dimension is compactified on a circle. He envisaged a nonrelativistic potential model in three spatial dimensions and with an extra compact spatial dimension. The perturbative Greens function technique was employed to compute the quantum mechanical scattering amplitude. The additional feature was the existence of KK states. Thus the standard integral equations which are employed in potential models of scattering were modified appropriately. In the conventional study of scattering, we consider incoming plain waves before the scattering. The Green’s function formalism enables us to extract the scattering amplitude. For the problem at hand, the wave function is characterized by its momentum, \( \mathbf{k} \) and an integer, \( n \) due to the presence of a compact coordinate, \( \Phi \); \( n \) is interpreted as the KK quantum number. Therefore, the initial state is designated as \((\mathbf{k}, n = 0)\) evolving to a final state \((\mathbf{k}', n' = 0)\). Moreover, there are conservation laws which are to be respected. Khuri found that when one considers the scattering process where a state \((\mathbf{k}, n = 0)\) scatters into \((\mathbf{k}', n' = 0)\) then the scattering amplitude exhibits the analyticity properties which are known for a long time [37, 36, 38]. The situation is different when one considers the process \((\mathbf{k}, n) \rightarrow (\mathbf{k}', n')\). The Green’s function technique developed by Khuri was employed. He showed that for the scattering process \((\mathbf{k}, n = 1) \rightarrow (\mathbf{k}', n = 1)\), the forward scattering amplitude exhibits a nonanalytic behavior when it is computed to second order. In other words, when one considers scattering states having momentum \( |\mathbf{k}| \) and KK quantum number \( n = 1 \), the forward amplitude develops nonanalytic behavior whereas the amplitude for scattering in the \( n = 0 \) sector exhibits no such attribute. Moreover, Khuri [35] remarked that this
phenomena will have very serious consequences if such KK states are produced in the LHC experiments. Indeed, he cited the works on Antoniadis \cite{27} and argued that the KK states might be produced in the large-compactification-radius scenario. It must be emphasized that the rigorous results of Khuri was derived in the frameworks of nonrelativistic quantum mechanics where the perturbative Green’s function technique was employed. Should such nonanalytic behavior of the scattering amplitude continues to be exhibited in a relativistic field theory it would be a matter of concern. We have mentioned that analyticity and causality are closely related while deriving results from axiomatic field theories. The analyticity and crossing properties of scattering amplitude were investigated, for $D > 4$, in the LSZ formulation only recently. We summarize the essential conclusions of \cite{39} which will be utilized in the study of analyticity of scattering amplitudes in compactified theories. It was shown, in the LSZ formalism, that the scattering amplitude has desire attributes in the following sense: (i) We proved the generalization of the Jost-Lehmann-Dyson theorem for the retarded function \cite{40, 41} for the $D > 4$ case \cite{42}. (ii) Subsequently, we showed the existence of the Lehmann-Martin ellipse for such a theory. (iii) Thus a dispersion relation can be written down in $s$ for fixed $t$ when the momentum transfer squared lies inside Lehmann-Martin ellipse \cite{45, 46}. (iv) The analog of Martin’s theorem can be derived in the sense that the scattering amplitude is analytic the product domain $D_s \otimes D_t$ where $D_s$ is the cut $s$-plane and $D_t$ is a domain in the $t$-plane such that the scattering amplitude is analytic inside a disk, $|t| < \tilde{R}$, $\tilde{R}$ is radius of the disk and it is independent of $s$. Thus the partial wave expansion converges inside this bigger domain. (v) We also derived the analog of Jin-Martin \cite{47} upper bound on the scattering amplitude which states that the fixed $t$ dispersion relation in $s$ does not require more than two subtractions. (vi) Consequently, a generalized Froissart-Martin bound was be proved.

In order to accomplish our goal for a $D = 4$ theory which arises from $S^1$ compactification of a $D = 5$ theory i.e. to prove nonforward dispersion relations, we have to establish the results (i) to (iv) for this theory. It is important to point out, at this juncture, that (to be elaborated in the sequel) the spectrum of the theory consists of a massive particle of the original five dimensional theory and a tower of Kaluza-Klein states. Thus the requisite results (i)-(iv) are to obtained in this context in contrast to the results of the D-dimensional theory with a single massive neutral scalar field. The developments in this case are similar to the ones derived for $D = 4$ theories. However, certain subtle issues had to be surmounted in order to prove analyticity and crossing properties for theories defined in higher dimensions, $D > 4$. The author was drawn into the topic through discussions with André Martin (Martin private discussions). He expressed his concern that if the analyticity would be violated in a compactified field theory then several rigorous results derived from axiomatic field theories will be questioned. In particular, what would be the fate of Froissart-Martin bound for such a theory? The author undertook the study of analyticity of scattering amplitude in a field theory with a compact spatial dimension. It is necessary to
start from fundamental axioms of LSZ for an uncompactified field theory in higher spacetime dimension and compactify a spatial dimension and examine the analyticity properties of the scattering amplitude. This is the topic to be discussed in this article.

As mentioned earlier, the analyticity property of the amplitude in nonrelativistic potential scattering has been investigated long ago. The result of Khuri was that scattering amplitude for a potential with a compact coordinate violates analyticity was a surprise. However, we should carefully analyze the implications of Khuri’s result. We recall that in QFT the analyticity of an amplitude and causality are intimately related. The relativistic invariance of the theory implies that no signal can travel faster than the velocity of light. Therefore, two local (bosonic) operators commute when they are separated by spacelike distance. As we shall discuss later, the key ingredient to prove analyticity of the amplitude is the axiom of microcausality. In the context of nonrelativistic potential scatterings, the theory is invariant under Galilean transformations. Consequently, the concept of microcausality is not envisaged in potential scattering. Therefore, the lack of analyticity of an amplitude, in certain cases, is not so serious an issue as would be the case if analyticity is not respected in a relativistic QFT. We shall proceed while keeping in mind the preceding remarks.

The article is organized as follows. In the next section, Section 2, we present a very brief account of Khuri’s results to familiarize the reader with his formulation of the problem for a potential which has a compact spatial dimension. The third section, Section 3, is devoted to a short review of LSZ formalism. We present the LSZ reduction technique for a massive neutral scalar field theory in higher dimensions, i.e in five spacetime dimensions, $D = 5$. All the requisite ingredients to prove dispersion relations are summarized here. We briefly discuss crossing symmetry and touch upon derivation of the Lehmann ellipses. We need these two results to write down dispersion relations. Next we discuss the $S^1$ compactification of the flat space five dimensional theory. The $R^{4,1}$ manifold is compactified to $R^{3,1} \otimes S^1$. The starting point is to consider a single massive scalar field theory defined in a flat five dimensional manifold, $R^{4,1}$. Thus there is one massive scalar of mass $m_0$ living in $D = 5$. When we compactify one spatial coordinate on $S^1$, the resulting theory defined on the manifold $R^{3,1} \otimes S^1$ is endowed with the following spectrum. There is a massive scalar of mass $m_0$. In addition, there is a tower of KK states whose mass spectrum is $m_n^2 = \left( \frac{n}{R} \right)^2$ where $R$ is the compactification radius and $n \in \mathbb{Z}$. In fact each KK state is endowed with an integer KK charge, $q_n \in \mathbb{Z}$ which is conserved. Therefore, the compactified QFT has various features which differ from a nonrelativistic potential model. The next section, Section 4, is devoted to investigate analyticity properties of the scattering amplitude for the theory alluded to above. We systematically derive the spectral representation for the absorptive amplitudes. Then discuss the crossing properties. We touch upon the Jost-Lehmann-Dyson theorem for this case which has been derived for a field theory defined in higher dimensions, $D > 4$. However, it is essential to consider the existence of Lehmann ellipses. The proof of dispersion relation requires the existence of Lehmann ellipses, especially, the Large Lehmann Ellipse
Subsequently, we write down the unsubtracted, fixed-t dispersion relations. In fact, the elastic scattering amplitude for $n = 1$ states is considered. It is shown that there is no violation of analyticity in this case. Indeed, our proof goes beyond the results of Khuri since we have proven the nonforward dispersion relations. We derive a few corollaries based on our main results.

Section 5 is devoted to prove the generalized unitarity relation in the LSZ formulation for the theory under considerations. It has two purposes. First, we note that the unitarity constraint already provides a preview of crossing as will be discussed. We have not proved crossing explicitly since it is not our main goal. The second important result is that the unitarity of S-matrix implies that only a finite number KK excited states contribute to the spectral representations as physical intermediate states. We draw attention of the reader to a very important observation that only the physical states appear as intermediate states in the spectral representation. It is unitarity, the nonlinear relation, which cuts off the sum to a finite number of terms when we sum over the KK towers. Notice that when we derive the spectral representation for the matrix element of the causal commutator of the source currents the sum over intermediate states is the entire KK tower. There is no way to conclude, in the linear program, that the sum could be over finite number of KK states.

We had proposed another novel way to look for the evidence large-radius-compactification proposal. In Section 6 we proceed to examine that idea. We argue that precise measurement of $\sigma_t$ at LHC energy and beyond might provide a clue to look for evidence for the large-radius-compactification hypothesis. If a theory is defined in higher dimensional flat space, $D > 4$, then the Froissart bound on $\sigma_t$ is modified. The proof is derived from LSZ axioms. Suppose, one extra dimension is decompactified at LHC energies and the total cross section has an energy dependence which violates the $D = 4$ Froissart-Martin bound. In the light of above remark, one should conclude immediately that some of the axioms of local field theories might be violated. Instead, we should interpret the observed energy dependence a signal of decompactification of extra dimensions. We have analyzed the data recently. However, we conclude that there is no conclusive evidence for violation of the Froissart-Martin bound. We feel that more precise measurements of $\sigma_t$ might provide some hints on this issue.

2. Non-relativistic Potential Scattering for $R^3 \otimes S^1$ Geometry

In this section, we shall discuss the essential results of Khri where he considered a nonrelativstic potential scattering. He introduced a spherically symmetric potential in three dimensions with an additional compact coordinate. Let us consider the setup for potential scattering in the framework of nonrelativistic quantum mechanics. The potential, $V(r)$, is spherically symmetric, where $r = |r|$. It is chosen to be a short range potential with good behaviors for large $r$, see for details. The starting point

I thank Luis Alvarez Gaume for raising this question.

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is the Schrödinger equation

\[ \left[ \nabla^2 + k^2 - V(r) \right] \Psi(r) = 0 \] (2)

This equation is expressed in the dimensionless form so that the mass and Planck’s constant do not appear. The solution to the above equation is chosen such that for large \( r \) there is a plane wave part and an outgoing spherical wave component. Therefore,

\[ \Psi(r) = e^{ikr} + \int d^3r' G_0(r - r')V(r')\Psi(r') \] (3)

with the free plane wave solution \( \frac{1}{2\pi^2}e^{ikx} \) and the Green’s function \( G_0(r - r') \) satisfies the free Schrödinger equation

\[ \left[ \nabla^2 + k^2 \right] G_0(r - r') = \delta^3(r - r') \] (4)

The solution satisfies the desired boundary conditions. The asymptotic form of the above solution (3) is

\[ \Psi(r) \approx e^{ikr} + \frac{1}{|r|}e^{i|kr|}f(k, \cos\theta) \] (5)

Here \( r = |r| \) and \( \cos\theta \) is the center of mass scattering angle. The coefficient of the spherical wave component, \( f(k, \cos\theta) \) is defined to be the scattering amplitude. This is the Born amplitude and we iterate this procedure to get the higher order correction. Moreover, Khuri [37] went through a rigorous procedure to study the analyticity of the scattering amplitude. He proved that the scattering amplitude is analytic in the upper half \( k \)-plane for fixed \( \cos\theta \). Furthermore, it is bounded in the upper half plane and also on the real axis for a general class of potentials which have good convergent behavior as \( r \to \infty \). It was an important result at that juncture. It was also a surprising and unexpected outcome since nonrelativistic theories are not endowed with principle of microcausality as is the case in the relativistic theories. The velocity of light is the limiting velocity for latter theories. Therefore, analyticity and causality are intimately connected only in relativistic theories.

Now we turn attention to Khuri’s study of analyticity of scattering amplitude in a nonrelativistic theory in three spatial dimension which also has one compactified spatial coordinate. Khuri [35], in 1995, envisaged scattering of a particle in a space with \( R^3 \otimes S^1 \) topology. We provide a brief account of his work and incorporate his important conclusions. We refer to the original paper to the interested reader. The notations of [35] will be followed. The compactified coordinate is a circle of radius \( R \) and it is assumed that the radius is small i.e. \( \frac{1}{R} \gg 1 \) where dimensionless units were used. We mention here that the five dimensional theory is defined in a flat Minkowski space. The only mass scale available to us is the mass of the particle; therefore, \( \frac{1}{R} \)
is much larger than this scale. The potential, $V(r, \Phi)$, is such that it is periodic in the angular coordinate, $\Phi$, of $S^1$; $r \in \mathbb{R}^3$ and $r = |r|$. The potential, $V(r, \Phi)$, belongs to a broad class such that for large $r$ this class of potentials fall off like $e^{-\mu r}/r$ as $r \to \infty$. Moreover, $V(r, \Phi) = V(r, \Phi + 2\pi)$. The scattering amplitude depends on three variables - the momentum of the particle, $k$, the scattering angle $\theta$, and an integer $n$ which appears due to the periodicity of the $\Phi$-coordinate. Thus forward scattering amplitude is denoted by $T_{nn}(K)$, where $K^2 = k^2 + n^2/R^2$. The starting point is the Schrödinger equation

\[
\left[\nabla^2 + \frac{1}{R^2} \frac{\partial^2}{\partial \Phi^2} + K^2 - V(r, \Phi)\right] \Psi(r, \Phi) = 0
\]  

(6)

The free plane wave solutions are

\[
\Psi_0(x, \Phi) = \frac{1}{(2\pi)^2} e^{ikx} e^{in\Phi}
\]  

(7)

and $n \in \mathbb{Z}$. The total energy is defined to be

\[
K^2 = k^2 + \frac{n^2}{R^2}
\]  

(8)

The free Green’s function (in the presence of a compact coordinate) assumes the following form

\[
G_0(K; x : x' ; \Phi - \Phi') = \frac{1}{(2\pi)^4} \sum_{n=-\infty}^{n=+\infty} \int d^3p \frac{e^{i\mathbf{p} \cdot (x-x')} e^{in(\Phi-\Phi')}}{[p^2 + \frac{n^2}{R^2} - K^2 - i\epsilon]}
\]  

(9)

The free Green’s function satisfies the free Schrödinger equation

\[
\left[\nabla^2 + \frac{1}{R^2} \frac{\partial^2}{\partial \Phi^2} + K^2\right] G_0(K; x : x' ; \Phi - \Phi') = \delta^3(x - x') \delta(\Phi - \Phi')
\]  

(10)

The $d^3p$ integration can be performed in the expression (9) leading to

\[
G_0(K; x - x' ; \Phi - \Phi') = \frac{1}{(8\pi^2)} \sum_{n=-\infty}^{n=+\infty} \frac{e^{i\sqrt{K^2 - (n^2/R^2)|x-x'|}} e^{in(\Phi-\Phi')}}{|x-x'|}
\]  

(11)

Khuri introduced the following prescription for $\sqrt{K^2 - n^2/R^2}$. It is defined in such a way that when $n^2/R^2 > K^2$

\[
i\sqrt{K^2 - n^2/R^2} \to -\sqrt{n^2/R^2 - K^2}, \quad n^2 > K^2 R^2
\]  

(12)

Note that the series expansion for $G_0(K; x - x' ; \Phi - \Phi')$ as expressed in (11) is strongly damped for large enough $|n|$. A careful analysis, as was carried out in ref. [35], shows
that the Green’s function is well defined and bounded, except for $|x - x'| \to 0$; similar
to the properties of Green’s functions in potential scattering for a fixed $K^2$. Khuri
[37] expressed the scattering integral equation for the potential $V(r, \Phi)$ as

$$\Psi_{k,n}(x, \Phi) = e^{ik \cdot x} e^{i n \Phi} + \int_0^{2\pi} d\phi' \int d^3x' G_0(K; |x - x'|; |\phi - \phi'|) V(x', \Phi') \Psi_{k,n}(x', \Phi')$$  \hspace{1cm} (13)$$

The expression for the scattering amplitude is extracted from the large $|x|$ limit when
one looks at the asymptotic behavior of the wave function,

$$\Psi_{k,n} \to e^{ik \cdot x} e^{i n \Phi} + \sum_{m = -[KR]}^{+[KR]} T(k', m : k, n) \frac{e^{ik'_m |x|}}{|x|} e^{im \Phi}$$  \hspace{1cm} (14)$$

where $[KR]$ is the largest integer less than $KR$ and

$$k'_m = \sqrt{k^2 + \frac{n^2}{R^2} - \frac{m^2}{R^2}}$$  \hspace{1cm} (15)$$

He also identifies a conservation rule: $K^2 = k^2 + \left(\frac{n^2}{R^2}\right) = k'^2 + \left(\frac{n^2}{R^2}\right)$. Thus it
is argued that that the scattered wave has only $(2[KR] + 1)$ components and those
states with $(m^2/R^2) > k^2 + \left(\frac{n^2}{R^2}\right)$ are exponentially damped for large $|x|$ and
consequently, these do not appear in the scattered wave (see eq. (12)). Now the scattering amplitude is extracted from equations (13) and (14) to be

$$T(k', n'; k, n) = -\frac{1}{8\pi^2} \int d^3x' \int_0^{2\pi} d\phi' e^{-ik' \cdot x'} e^{-in' \phi'} V(x', \Phi') \Psi_{k,n}(x', \Phi)$$  \hspace{1cm} (16)$$

The condition, $k'^2 + \frac{n'^2}{R^2} = k^2 + \frac{n^2}{R^2}$ is to be satisfied. Thus the scattering amplitude describes the process where incoming wave $|k, n>$ is scattered to final state $|k', n'>$.

Remark: Reader should pay attention to the expression for the discussion of scattering processes in relativistic QFT in the sequel and note the similarities and differences as discussed in subsequent sections.

Formally, the amplitude assumes the following form for the full Green’s function

$$T(k', n'; k, n) - T_B = -\frac{1}{8\pi^2} \int d^3x \int d^3x' \int d\phi \int d\phi' e^{-i(k' \cdot x' + n' \phi')} V(x', \Phi')$$
$$G(K; x', x; \Phi', \Phi) V(x, \Phi) e^{i(k \cdot x + n \phi)}$$  \hspace{1cm} (17)$$

Here $T_B$ is the Born term.

$$T_B = -\frac{1}{8\pi^2} \int d^3x \int_0^{2\pi} d\phi e^{i(k-\bar{k}) \cdot x} V(x, \Phi) e^{i(n-n') \phi}$$  \hspace{1cm} (18)$$
Full Green’s function satisfied an equation with the full Hamiltonian

\[
\left( \nabla^2 + \frac{1}{R^2} \frac{\partial^2}{\partial \Phi^2} + K^2 - V(x, \Phi) \right) G(K; x, x', \Phi, \Phi') = \delta^3(x - x') \delta(\Phi - \Phi') \tag{19}
\]

This is the starting point of computing scattering amplitude perturbatively in potential scattering [36]. Khuri [35] proceeds to study the analyticity properties of the amplitude and it is a parallel development similar to investigations done in the past. In the context of a theory with compact space dimension he analysed the amplitude \( T_{nn}(K) \) to the second order in the Born approximation for \( n = 1 \).

Khuri [35] explicitly computed the second born term \( T^{(2)} \) for the forward amplitude, for the choice \( n = 1 \). He has discovered that the analyticity of the forward amplitude breaks down with a counter example; where \( T_{nn}(k) \) does not satisfy dispersion relations for a class of Yukawa-type potentials of the form

\[
V(r, \Phi) = u_0(r) + 2 \sum_{m=1}^{N} u_m(r) \cos(m\Phi) \tag{20}
\]

where \( u_m(r) = \lambda_m e^{-\mu r} \). Khuri noted an important feature of his studies that in the case when scattering theory was applied perturbatively in \( R^3 \) space the resulting amplitude satisfied analyticity properties for similar Yukawa-type potentials. Thus there has been concerns[3] when non-analyticity of the aforementioned scattering amplitude was discovered in the non-relativistic quantum mechanics by Khuri in the space with the topology \( R^3 \otimes S^1 \).

We shall describe the framework of our investigation in the next section. We remark in passing that the analyticity of scattering amplitude in nonrelativitivit scattering is not such a profound property as in the relativistic QFT although the analyticity in non-relativistic potential scattering has been investigated quite thoroughly in the past [36]. However, it is to be noted that in absence a limiting velocity (in relativistic case velocity of light, \( c \), profoundly influences the study of the analyticity of amplitudes) the microcausality is not enforced in nonrelativistic processes. As we shall show (and has been emphasized in many classic books) there is indeed a deep connection between microcausality and analyticity. When a spatial dimension is compactified on \( S^1 \), the coordinate on the circle is periodic; we can understand microcausality as follows. The compact coordinate \( y \) is periodic. Therefore, we can define spacelike separation between two points. We keep this aspect in mind and we shall undertake a systematic study of the analyticity of scattering amplitude in the sequel.

3. Quantum Field Theory with Compact Spatial Dimensions

First, we present the LSZ formalism for a \( D = 5 \) massive theory in the flat Minkowski spacetime. Subsequently, we discuss the \( S^1 \) compactification in detail.

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3Andre Martin brought the work of Khuri [35] to my attention and persuaded me to undertake this investigation.
3.1 Quantum Field Theory in $D = 5$ Spacetime.

We have shown in [1] (henceforth referred as I) that the forward scattering amplitude of a theory, defined on the manifold $R^{3.1} \otimes S^1$, satisfied dispersion relations. This result was obtained in the frame works of the LSZ formalism. Thus the proof of the forward dispersion relations will not be presented in this review. The interested reader can get the details in I. We summarize, in this subsection, the starting points of I as stated below.

We considered a neutral, scalar field theory with mass $m_0$ in flat five dimensional Minkowski space $R^{4.1}$. It is assumed that the particle is stable and there are no bound states. The notation is that the spacetime coordinates are, $\hat{x}$, and all operators are denoted with a hat when they are defined in the five dimensional space where the spatial coordinates are noncompact. The LSZ axioms are [1]:

A1. The states of the system are represented in a Hilbert space, $\hat{H}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{H}$.

A2. The theory is invariant under inhomogeneous Lorentz transformations.

A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz and translation invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a}, \hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are Hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_\mu$, with $\hat{\mu} = 0, 1, 2, 3, 4$ which have following properties:

$$\left[ \hat{P}_\mu, \hat{P}_\nu \right] = 0$$

(21)

If $\hat{F}(\hat{x})$ is any Heisenberg operator then its commutator with $\hat{P}_\mu$ is

$$\left[ \hat{P}_\mu, \hat{F}(\hat{x}) \right] = i\partial_\mu \hat{F}(\hat{x})$$

(22)

It is assumed that the operator does not explicitly depend on spacetime coordinates. If we choose a representation where the translation operators, $\hat{P}_\mu$, are diagonal and the basis vectors $|\hat{p}, \hat{\alpha}>$ span the Hilbert space, $\mathcal{H}$,

$$\hat{P}_\mu |\hat{p}, \hat{\alpha}> = \hat{p}_\mu |\hat{p}, \hat{\alpha}>$$

(23)

then we are in a position to make more precise statements:

• Existence of the vacuum: there is a unique invariant vacuum state $|0>$ which has the property

$$\hat{U}(\hat{a}, \hat{\Lambda})|0> = |0>$$

(24)

The vacuum is unique and is Poincaré invariant.

• The eigenvalue of $\hat{P}_\mu$, $\hat{p}_\mu$, is light-like, with $\hat{p}_0 > 0$. We are concerned only with massive stated in this discussion. If we implement infinitesimal Poincaré transformation
on the vacuum state then

\[ \hat{P}_\mu |0 > = 0, \quad \text{and} \quad \hat{M}_{\mu \nu} |0 > = 0 \quad (25) \]

from above postulates and note that \( \hat{M}_{\mu \nu} \) are the generators of Lorentz transformations.

**A4.** The locality of theory implies that a (bosonic) local operator at spacetime point \( \hat{x}^0 \) commutes with another (bosonic) local operator at \( \hat{x}'^0 \) when their separation is spacelike i.e. if \((\hat{x} - \hat{x}')^2 < 0\). Our Minkowski metric convention is as follows: the inner product of two 5-vectors is given by \( \hat{x} \cdot \hat{y} = \hat{x}^0 \hat{y}^0 - \hat{x}^1 \hat{y}^1 - ... - \hat{x}^4 \hat{y}^4 \). Since we are dealing with a neutral scalar field, for the field operator \( \hat{\phi}(\hat{x}) : \hat{\phi}(\hat{x})^\dagger = \hat{\phi}(\hat{x}) \) i.e. \( \hat{\phi}(\hat{x}) \) is Hermitian. By definition it transforms as a scalar under inhomogeneous Lorentz transformations

\[ \hat{U}(\hat{a}, \hat{A}) \hat{\phi}(\hat{x}) \hat{U}(\hat{a}, \hat{A})^{-1} = \hat{\phi}(\hat{A} \hat{x} + \hat{a}) \quad (26) \]

The micro causality, for two local field operators, is stated to be

\[ \left[ \hat{\phi}(\hat{x}), \hat{\phi}(\hat{x}') \right] = 0, \quad \text{for} \quad (\hat{x} - \hat{x}')^2 < 0 \quad (27) \]

It is well known that, in the LSZ formalism, we are concerned with vacuum expectation values of time ordered products of operators as well as with the the retarded product of fields. The requirements of the above listed axioms lead to certain relationship, for example, between vacuum expectation values of R-products of operators. Such a set of relations are termed as the linear relations and the importance of the above listed axioms is manifested through these relations. In contrast, unitarity imposes nonlinear constraints on amplitude. For example, if we expand an amplitude in partial waves, unitarity demands certain positivity conditions to be satisfied by the partial wave amplitudes.

We summarize below some of the important aspects of LSZ formalism as we utilize them throughout the present investigation. Moreover, the conventions and definitions of I will be followed for the conveniences of the reader.

(i) The asymptotic condition: According to LSZ the field theory accounts for the asymptotic observables. These correspond to particles of definite mass, charge and spin etc. \( \hat{\phi}^{\text{in}}(\hat{x}) \) represents the free field in the remote past. A Fock space is generated by the field operator. The physical observable can be expressed in terms of these fields.

(ii) \( \hat{\phi}(\hat{x}) \) is the interacting field. LSZ technique incorporates a prescription to relate the interacting field, \( \hat{\phi}(\hat{x}) \), with \( \hat{\phi}^{\text{in}}(\hat{x}) \); consequently, the asymptotic fields are defined with a suitable limiting procedure. Thus we introduce the notion of the adiabatic switching off of the interaction. A cutoff adiabatic function is postulated such that this function controls the interactions. It is \( 1 \) at finite interval of time and it has a smooth limit of passing to zero as \(|t| \to \infty \). It is argued that when adiabatic
Another important property of the R-product is that the R-product is hermitian for hermitian fields \( \hat{\phi} \) it is located in its position. We list below some of the important properties of the R-products for future use:

(i) The fields \( \hat{\phi}^{in}(\hat{x}) \) and \( \hat{\phi}(\hat{x}) \) are related as follows:

\[
\hat{x}_0 \to -\infty \quad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2} \hat{\phi}^{in}(\hat{x})
\]

By the first postulate, \( \hat{\phi}^{in}(\hat{x}) \) creates free particle states. However, in general \( \hat{\phi}(\hat{x}) \) will create multi particle states besides the single particle one since it is the interacting field. Moreover, \( <1|\hat{\phi}^{in}(\hat{x})|0> \) and \( <1|\hat{\phi}(\hat{x})|0> \) carry same functional dependence in \( \hat{x} \). If the factor of \( \hat{Z} \) were not the scaling relation between the two fields, then canonical commutation relation for each of the two fields (i.e. \( \hat{\phi}^{in}(\hat{x}) \) and \( \hat{\phi}(\hat{x}) \)) will be the same. Thus in the absence of \( \hat{Z} \) the two theories will be identical. Moreover, the postulate of asymptotic condition states that in the remote future

\[
\hat{x}_0 \to \infty \quad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2} \hat{\phi}^{out}(\hat{x}).
\]

We may as well construct a Fock space utilizing \( \hat{\phi}^{out}(\hat{x}) \) as we could with \( \hat{\phi}(\hat{x})^{in} \). Furthermore, the vacuum is unique for \( \hat{\phi}^{in}, \hat{\phi}^{out} \) and \( \hat{\phi}(\hat{x}) \). The normalizable single particle states are the same i.e. \( \hat{\phi}^{in}|0> = \hat{\phi}^{out}|0> \). We do not display \( \hat{Z} \) from now on. If at all any need arises, \( \hat{Z} \) can be introduced in the relevant expressions.

We define creation and annihilation operators for \( \hat{\phi}^{in}, \hat{\phi}^{out} \). We recall that \( \hat{\phi}(\hat{x}) \) is not a free field. Whereas the fields \( \hat{\phi}^{in,out}(\hat{x}) \) satisfy the free field equations \( [\square_5 + m_0^2] \hat{\phi}^{in,out}(\hat{x}) = 0 \), the interacting field satisfies an equation of motion which is endowed with a source current: \( [\square_5 + m_0^2] \hat{\phi}(\hat{x}) = \hat{j}(\hat{x}) \). We may use the plane wave basis for simplicity in certain computations; however, in a more formal approach, it is desirable to use wave packets.

The relevant vacuum expectation values of the products of operators in LSZ formalism are either the time ordered products (the T-products) or the retarded products (the R-products). We shall mostly use the R-products and we use them extensively throughout this investigation. It is defined as

\[
R \hat{\phi}(\hat{x}) \hat{\phi}_1(\hat{x}_1) ... \hat{\phi}_n(\hat{x}_n) = (-1)^n \sum_P \theta(\hat{x}_0 - \hat{x}_{10}) \theta(\hat{x}_{10} - \hat{x}_{20}) ... \theta(\hat{x}_{n-10} - \hat{x}_{n0})
\]

\[
[[[...[\hat{\phi}(\hat{x}), \hat{\phi}_{i_1}(\hat{x}_{i_1})], \hat{\phi}_{i_2}(\hat{x}_{i_2})]...], \hat{\phi}_{i_n}(\hat{x}_{i_n})]
\]

note that \( R\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x}) \) and \( P \) stands for all the permutations \( i_1,...,i_n \) of \( 1,2,...n \). The R-product is hermitian for hermitian fields \( \hat{\phi}_i(\hat{x}_i) \) and the product is symmetric under exchange of any fields \( \hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) \). Notice that the field \( \hat{\phi}(\hat{x}) \) is kept where it is located in its position. We list below some of the important properties of the R-product for future use:

(i) \( R \hat{\phi}(\hat{x}) \hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) \neq 0 \) only if \( \hat{x}_0 > \max \{\hat{x}_{10},...\hat{x}_{n0}\} \).

(ii) Another important property of the R-product is that

\[
R \hat{\phi}(\hat{x}) \hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = 0
\]
whenever the time component $\hat{x}_0$, appearing in the argument of $\hat{\phi}(\hat{x})$ whose position is held fix, is less than time component of any of the four vectors $(\hat{x}_1, \ldots, \hat{x}_n)$ appearing in the arguments of $\hat{\phi}(\hat{x}_1) \ldots \hat{\phi}(\hat{x}_n)$.

(iii) We recall that

$$\hat{\phi}(\hat{x}_i) \rightarrow \hat{\phi}(\hat{\Lambda}\hat{x}_i) = \hat{U}(\hat{\Lambda}, 0)\hat{\phi}(\hat{x}_i)\hat{U}(\hat{\Lambda}, 0)^{-1} \quad (32)$$

Under Lorentz transformation $\hat{U}(\hat{\Lambda}, 0)$. Therefore,

$$R \hat{\phi}(\hat{\Lambda}\hat{x})\hat{\phi}(\hat{\Lambda}\hat{x}_i) \ldots \hat{\phi}_n(\hat{\Lambda}\hat{x}_n) = \hat{U}(\hat{\Lambda}, 0)R \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1) \ldots \hat{\phi}_n(\hat{x}_n)\hat{U}(\hat{\Lambda}, 0)^{-1} \quad (33)$$

And

$$\hat{\phi}_1(\hat{x}_i) \rightarrow \hat{\phi}_1(\hat{x}_i + \hat{\alpha}) = e^{i\hat{\alpha}.\hat{P}}\hat{\phi}_1(\hat{x}_i)e^{-i\hat{\alpha}.\hat{P}} \quad (34)$$

under spacetime translations. Consequently,

$$R \hat{\phi}(\hat{x} + \hat{\alpha})\hat{\phi}(\hat{x}_i + \hat{\alpha}) \ldots \hat{\phi}_n(\hat{x}_n + \hat{\alpha}) = e^{i\hat{\alpha}.\hat{P}}R \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1) \ldots \hat{\phi}_n(\hat{x}_n)e^{-i\hat{\alpha}.\hat{P}} \quad (35)$$

Therefore, the vacuum expectation value of the R-product dependents only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: $\xi_1 = \hat{x}_1 - \hat{x}$, $\xi_2 = \hat{x}_2 - \hat{x}_1 \ldots \xi_n = \hat{x}_{n-1} - \hat{x}_n$ as a consequence of translational invariance.

(iv) The retarded property of R-function and the asymptotic conditions lead to the following relations.

$$[R \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1) \ldots \hat{\phi}_n(\hat{x}_n), \hat{\phi}^{\text{in}}_l(\hat{y}_l)] = i \int d^\prime \hat{y}_l \Delta(\hat{y}_l - \hat{y}_l') (\Box_{55'} + \hat{\bar{m}}_l^2) \times R \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1) \ldots \hat{\phi}_n(\hat{x}_n)\hat{\phi}_l(\hat{y}_l) \quad (36)$$

Note: here $\hat{m}_l$ stands for the mass of a field in five dimensions. We may define 'in' and 'out' states in terms of the creation operators associated with 'in' and 'out' fields as follows

$$|\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n \text{ in} > = \hat{a}^{\dagger}_\text{in}(\hat{k}_1)\hat{a}^{\dagger}_\text{in}(\hat{k}_2) \ldots \hat{a}^{\dagger}_\text{in}(\hat{k}_n)|0> \quad (37)$$

$$|\hat{k}_1, \hat{k}_2, \ldots, \hat{k}_n \text{ out} > = \hat{a}^{\dagger}_\text{out}(\hat{k}_1)\hat{a}^{\dagger}_\text{out}(\hat{k}_2) \ldots \hat{a}^{\dagger}_\text{out}(\hat{k}_n)|0> \quad (38)$$

We can construct a complete set of states either starting from 'in' field operators or the 'out' field operators and each complete set will span the Hilbert space, $\hat{H}$. Therefore, a unitary operator will relate the two sets of states in this Hilbert space. This is a heuristic way of introducing the concept of the S-matrix. We shall define S-matrix elements through LSZ reduction technique in subsequent section.

We shall not distinguish between notations like $\hat{\phi}^{\text{out, in}}$ or $\hat{\phi}^{\text{out, in}}$ and therefore, there
might be use of the sloppy notation in this regard.
We record the following important remark *en passant*. The generic matrix element

\[ \langle \alpha | \hat{\phi}(\hat{x}_1) \hat{\phi}(\hat{x}_2) \ldots | \beta \rangle \]

is not an ordinary function but a distribution. Thus it is to be always understood as smeared with a Schwartz type test function \( f \in S \). The test function is infinitely differentiable and it goes to zero along with all its derivatives faster than any power of its argument. We shall formally derive expressions for scattering amplitudes and the absorptive parts by employing the LSZ technique. It is to be understood that these are generalized functions and such matrix elements are properly defined with smeared out test functions.

We obtain below the expression for the Källen-Lehmann representation for the five dimensional theory. It will help us to transparently expose, as we shall recall in the next section, the consequences of \( S^1 \)-compactification. Let us consider the vacuum expectation value (VEV) of the commutator of two fields in the \( D = 5 \) theory:

\[ < 0 | [\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})] | 0 > \]

We introduce a complete set of states between product of the fields after opening up the commutator. Thus we arrive at the following expression by adopting the standard arguments,

\[ < 0 | [\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})] | 0 > = \sum_{\alpha} \left( < 0 | \hat{\phi}(0) \hat{\alpha} > e^{-i \hat{p}_\alpha \cdot (\hat{x} - \hat{y})} < \hat{\alpha} | \hat{\phi}(0) | 0 > - (\hat{x} \leftrightarrow \hat{y}) \right) \quad (39) \]

Let us define

\[ \hat{\rho}(\hat{q}) = (2\pi)^4 \sum_{\alpha} \delta^5(\hat{q} - \hat{p}_\alpha) | < 0 | \hat{\phi}(0) | \hat{\alpha} > |^2 \quad (40) \]

Note that \( \hat{\rho}(\hat{q}) \) is positive, and \( \hat{\rho} = 0 \) when \( \hat{q} \) is not in the light cone. It is also Lorentz invariant. Thus we write

\[ \hat{\rho}(\hat{q}) = \hat{\sigma}(\hat{q}^2) \theta(\hat{q}_0), \quad \hat{\sigma}(\hat{q}^2) = 0, \quad \text{if} \quad \hat{q}^2 < 0 \quad (41) \]

This is a positive measure. We may separate the expression for the VEV of the commutator (39) into two parts: the single particle state contribution and the rest. Moreover, we use the asymptotic state condition to arrive at

\[ < 0 | [\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})] | 0 > = i \hat{Z} \hat{\Delta}(\hat{x}, \hat{y}; m_0) + i \int_{\tilde{m}_1^2}^{\infty} d\tilde{m}'^2 \hat{\Delta}(\hat{x}, \hat{y}; \tilde{m}') \quad (42) \]

where \( \hat{\Delta}(\hat{x}, \hat{y}; m_0) \) is the VEV of the free field commutator, \( m_0 \) is the mass of the scalar. \( \tilde{m}_1^2 > M^2 \), the multiple particle threshold.

We are in a position to study several attributes of scattering amplitudes in the five dimensional theory such as proving existence of the Lehmann-Martin ellipse, give a proof of fixed t dispersion relation to mention a few. However, these properties have been derived in a general setting recently [?] for D-dimensional theories. The purpose of incorporating the expression for the VEV of the commutator of two fields in the
5-dimensional theory is to provide a prelude to the modification of similar expressions when we compactify the theory on $S^1$ as we shall see in the next section.

3.2. The Compactification of Scalar Field Theory: $\mathbb{R}^{4.1} \rightarrow \mathbb{R}^{3.1} \otimes S^1$

In this subsection, $S^1$ compactification of a spatial coordinate of the five dimensional theory is considered. To start with, decompose the five dimensional spacetime coordinates, $\hat{x}^\mu$, as follows:

$$\hat{x}^\mu = (x^\mu, y)$$

where $x^\mu$ are the four dimensional Minkowski space coordinates; $y$ is the compact coordinate on $S^1$ with periodicity $y + 2\pi R = y$, $R$ being the radius of $S^1$. We summarize below the attributes of this $S^1$ compactification. The neutral scalar field of mass $m_0$ defined in $D = 5$ manifold is now described in the geometry $\mathbb{R}^{3.1} \otimes S^1$. We focus on the free field version such as the ‘in’ and ‘out’ field, $\hat{\phi}^{\text{in, out}}(\hat{x})$. The equation of motion is $[\Box_5 + m_0^2] \hat{\phi}^{\text{in, out}}(\hat{x}) = 0$. We expand the field

$$\hat{\phi}^{\text{in, out}}(\hat{x}) = \hat{\phi}^{\text{in, out}}(x, y) = \phi_0^{\text{in, out}}(x) + \sum_{n=-\infty, n \neq 0}^{+\infty} \phi_n^{\text{in, out}}(x) e^{iny/R}$$

Note that $\phi_0^{\text{in, out}}(x)$, the so called zero mode, has no $y$-dependence. The terms in rest of the series satisfy periodicity in $y$. The five dimensional Laplacian, $\Box_5$, is decomposed as sum two operators: $\Box_4$ and $\frac{\partial}{\partial y^2}$. The equation of motion is

$$[\Box_4 - \frac{\partial}{\partial y^2} + m_n^2] \phi_n^{\text{in, out}}(x, y) = 0$$

where $\phi_n^{\text{in, out}}(x, y) = \phi_n^{\text{in, out}} e^{iny/R}$ and $n = 0$ term has no $y$-dependence being $\phi_0(x)$; from now on $\Box_4 = \Box$. Here $m_n^2 = m_0^2 + \frac{n^2}{R^2}$. Thus we have tower of massive states. The momentum associated in the $y$-direction is $q_n = n/R$ and is quantized in the units of $1/R$ and it is an additive conserved quantum number. We term it as Kaluza-Klein (KK) charge although there is no gravitational interaction in the five dimensional theory; we still call it KK reduction. For the interacting field $\hat{\phi}(\hat{x})$, we can adopt a similar mode expansion.

$$\hat{\phi}(\hat{x}) = \hat{\phi}(x, y) = \phi_0(x) + \sum_{n=-\infty, n \neq 0}^{+\infty} \phi_n(x) e^{iny/R}$$

The equation of motion for the interacting fields is endowed with a source term. Thus source current would be expanded as is the expansion (46). Each field $\phi_n(x)$ will have a current, $J_n(x)$ associated with it and source current will be expanded as

$$\hat{j}(x, y) = j_0(x) + \sum_{n=-\infty, n \neq 0}^{+\infty} J_n(x) e^{iny/R}$$
Note that the set of currents, \( \{ J_n(x) \} \), are the source currents associated with the tower of interacting fields \( \{ \phi_n(x) \} \). These fields carry the discrete KK charge, \( n \). Therefore, \( J_n(x) \) also carries the same KK charge. We should keep this aspect in mind when we consider matrix element of such currents between stated. In future, we might not explicitly display the charge of the current; however, it becomes quite obvious in the context.

The zero mode, \( \phi_{0,0}^{in,out} \), create their Fock spaces. Similarly, each of the fields \( \phi_n^{in,out}(x) \) create their Fock spaces as well. For example a state with spatial momentum, \( p \), energy, \( p_0 \) and discrete momentum \( q_n \) (in \( y \)-direction) is created by

\[
A^\dagger(p, q_n)|0> = |p, q_n>, \ p_0 > 0
\]  

(48)

**Remark:** The five dimensional theory has a neutral, massive scalar field. After the \( S^1 \) compactification to the \( R^{3,1} \otimes S^1 \), the spectrum of the resulting theory consists of a massive field of mass \( m_0 \), associated with the zero mode and tower of Kaluza-Klein (KK) states characterized by a mass and a 'charge', \( (m_n, q_n) \), respectively. We now discuss the structure of the Hilbert space of the compactified theory.

**The Decomposition of the Hilbert space \( \hat{H} \):** The Hilbert space associated with the five dimensional theory is \( \hat{H} \). It is now decomposed as a direct sum of Hilbert spaces

\[
\hat{H} = \sum \oplus \mathcal{H}_n
\]  

(49)

Thus \( \mathcal{H}_0 \) is the Hilbert space constructed from \( \phi_0^{in,out} \) with charge \( q_n=0 \). This space is built by the actions of the creation operators \( \{ a^\dagger(k) \} \) acting on the vacuum and these states span \( \mathcal{H}_0 \). A single particle state is \( a^\dagger(k)|0> = |k> \) and multiparticle states are created using the procedure outlined in (37) and (38). We can create Fock spaces by the actions of fields \( \phi_n(x, y) \) with charge \( q_n \) on the vacuum. This space is constructed through the action of creation operators \( \{ A^\dagger(p, q_n) \} \). Now two state vectors with different 'charges' are orthogonal to one another

\[
< p, q_n| p', q_{n'} > = \delta^2(p - p') \delta_{n,n'}
\]  

(50)

**Remark:** We assume that there are no bound states in the theory and all particles are stable as mentioned. There exists a possibility that a particle with charge \( 2n \) and mass \( m_{2n}^2 = m_0^2 + 4n^2 \frac{R^2}{\alpha} \) could be a bound state of two particles of charge \( n \) and masses \( m_n \) each under certain circumstances. We have excluded such possibilities from the present investigation.

The LSZ formalism can be adopted for the compactified theory. If we keep in mind the steps introduced above, it is possible to envisage field operators \( \phi_n^{in}(x) \) and \( \phi_n^{out}(x) \) for each of the fields for a given \( n \). Therefore, each Hilbert space, \( \mathcal{H}_n \) will be spanned by the state vectors created by operators \( a^\dagger(k) \), for \( n = 0 \) and \( A^\dagger(p, q_n) \), for \( n \neq 0 \). Moreover, we are in a position to define corresponding set of interacting field \( \{ \phi_n(x) \} \)
which will interpolate into 'in' and 'out' fields in the asymptotic limits.

Remark: Note that in (44) sum over \{n\} runs over positive and negative integers. If there is a parity symmetry \( y \rightarrow -y \) under which the field is invariant we can reduce the sum to positive \( n \) only. However, since \( q_n \) is an additive discrete quantum number, a state with \( q_n > 0 \) could be designated as a particle and the corresponding state \( q_n < 0 \) can be interpreted as its antiparticle. Thus a two particle state \( |p, q_n > |p, -q_n > \), \( q_n > 0 \) and \( p_0 > 0 \) is a particle antiparticle state, \( q_n = 0 \); in other words the sum of the total charges of the two states is zero. Thus it has the quantum number of the vacuum. For example, it could be two particle state of \( \phi_0 \) satisfying energy momentum conservation, especially if they appear as intermediate states.

Now return to the Källen-Lehmann representation (39) in the present context and utilize the expansion (46) in the expression for the VEV of the commutator of two fields defined in \( D = 5 \):

\[
< 0 | [\hat{\phi}(x), \hat{\phi}(x')] | 0 > = < 0 | [\phi_0(x) + \sum_{-\infty}^{+\infty} \phi_n(x, y), \phi_0(x') + \sum_{-\infty}^{+\infty} \phi_l(x', y')] | 0 > \quad (51)
\]

The VEV of a commutator of two fields given by the spectral representation (39) will be decomposed into sum of several commutators whose VEV will appear:

\[
< 0 | [\phi_0(x), \phi_0(x')] | 0 >, \quad < 0 | [\phi_n(x), \phi_{-n}(x')] | 0 >, \quad ... \quad (52)
\]

Since the vacuum carries zero KK charge, \( q_{vac} = 0 = q_0 \), the commutator of two fields (with \( n \neq 0 \)) should give rise to zero-charge and only \( \phi_n \) and \( \phi_{-n} \) commutators will appear. Moreover, commutator of fields with different \( q_n \) vanish since the operators act on states of different Hilbert spaces. Thus we already note the consequences of compactification. When we wish to evaluate the VEV and insert complete set of intermediate states in the product of two operators after opening up the commutators, we note that all states of the entire KK tower can appear as intermediate states as long as they respect all conservation laws. This will be an important feature in all our computations in what follows.

3.3. Definitions and Kinematical Variables

The purpose of this investigation is to derive analyticity property of the fixed-\( t \) dispersion relations for scattering of the KK states carrying nonzero charge i.e. scattering in the \( q_n \neq 0 \) sector. However, we mention in passing the other possible processes. These are (i) scattering of states with \( q_n = 0 \) states, i.e. scattering of zero modes. (ii) The scattering of a state carrying charge \( q_n = 0 \) with a state with non-zero KK charge. We have studied reactions (i) and (ii) in I and therefore, we do not wish to dwell upon them.

We shall define the kinematical variables below. The states carrying \( q_n \neq 0 \) are denoted by \( \chi_n \) (from now on a state carrying charge is defined with a subscript \( n \)
and momenta carried by external particles are denoted as \( p_a, p_b, \ldots \). Moreover, we shall consider elastic scattering of states carrying equal charge; the elastic scattering of unequal charge particles is just elastic scattering of unequal mass states due to mass-charge relationship for the KK states.

Let us consider a generic 4-body reaction (all states carry non-zero \( n \))

\[
a + b \rightarrow c + d
\]  

(53)

The particles \((a, b, c, d)\) (the corresponding fields being \( \chi_a, \chi_b, \chi_c, \chi_d \)) respectively carrying momenta \( \tilde{p}_a, \tilde{p}_b, \tilde{p}_c, \tilde{p}_d \); these particles may correspond to the KK zero modes (with KK momentum \( q = 0 \)) or particles might carry nonzero KK charge. We shall consider only elastic scatterings. The Lorentz invariant Mandelstam variables are

\[
s = (p_a + p_b)^2 = (p_c + p_d)^2, \quad t = (p_a - p_d)^2 = (p_b - p_c)^2, \quad u = (p_a - p_c)^2 = (p_b - p_d)^2
\]  

(54)

and \( \sum p_a^2 + p_b^2 + p_c^2 + p_d^2 = m_a^2 + m_b^2 + m_c^2 + m_d^2 \). The independent identities of the four particles will facilitate the computation of the amplitude so that to keep track of the fields reduced using LSZ procedure. We list below some relevant (kinematic) variables which will be required in future

\[
M_a^2, \quad M_b^2, \quad M_c^2, \quad M_d^2
\]  

(55)

These correspond to lowest mass two or more particle states which carry the same quantum number as that of particle \( a, b, c \) and \( d \) respectively. We define below six more variables

\[
(M_{ab}, M_{cd}), \quad (M_{ac}, M_{bd}), \quad (M_{ad}, M_{bc})
\]  

(56)

The variable \( M_{ab} \) carries the same quantum number as \((a and b)\) and it corresponds to two or more particle states. Similar definition holds for the other five variables introduced above. We define two types of thresholds: (i) the physical threshold, \( s_{\text{phys}} \), and \( s_{\text{thr}} \). In absence of anomalous thresholds (and equal mass scattering) \( s_{\text{thr}} = s_{\text{phys}} \). Similarly, we may define \( u_{\text{phys}} \) and \( u_{\text{thr}} \) which will be useful when we discuss dispersion relations. We assume from now on that \( s_{\text{thr}} = s_{\text{phys}} \) and \( u_{\text{thr}} = u_{\text{phys}} \). Now we outline the derivation of the expression a four point function in the LSZ formalism.

We start with \(|p_d, p_c \text{ out } >\) and \(|p_b, p_a \text{ in } >\) and considers the matrix element \(< p_d, p_c \text{ out } | p_b, p_a \text{ in } >\). Next we subtract out the matrix element \(< p_d, p_c \text{ in } | p_b, p_a \text{ in } >\) to define the S-matrix element.

\[
< p_d, p_d \text{ out } | p_b, p_a \text{ in } > = \delta^3(p_d - p_b)\delta^3(p_c - p_a) - \frac{i}{(2\pi)^3} \int d^4x \int d^4x' e^{-i(p_a-x-p_c x')} K_x K_{x'} < p_d \text{ out } | R(x', x) | p_b \text{ in } >
\]  

(57)

where \( K_x \) and \( K_{x'} \) are the four dimensional Klein-Gordon operators and

\[
R(x, x') = -i\theta(x_0 - x'_0) [\chi_a(x), \chi_c(x')]
\]  

(58)
We have reduced fields associated with $a$ and $c$ in \([57]\). In the next step we may reduce all the four fields and in such a reduction we shall get VEV of the R-product of four fields which will be operated upon by four K-G operators. However, the latter form of LSZ reduction (when all fields are reduced) is not very useful when we want to investigate the analyticity property of the amplitude in the present context. In particular our intent is to write the dispersion relation. Thus we abandon the idea of reducing all the four fields.

Remark: Note that on the right hand side of the equation \([57]\) the operators act on $R\chi_a(x)\chi_c(x')$ and there is a $\theta$-function in the definition of the R-product. Consequently, the action of $K_xK_{x'}$ on $R\chi_a(x)\chi_c(x')$ will produce a term $RJ_a(x)J_c(x')$. In addition the operation of the two K-G operators will give rise to $\delta$-functions and derivatives of $\delta$-functions and some equal time commutators i.e. there will terms whose coefficients are $\delta(x_0 - x'_0)$. When we consider Fourier transforms of the derivatives of these $\delta$-function derivative terms they will be transformed to momentum variables. However, the amplitude is a function of Lorentz invariant quantities. Thus one will get only finite polynomials of such variables, as has been argued by Symanzik \([53]\).

His arguments is that in a local quantum field theory only finite number of derivatives of $\delta$-functions can appear. Moreover, in addition, there are some equal time commutators and many of them vanish when we invoke locality arguments. Therefore, we shall use the relation

$$K_xK_{x'}R\chi(x)\chi_c(x') = RJ_a(x)J_c(x')$$ \(59\)

keeping in mind that there are derivatives of $\delta$-functions and some equal time commutation relations which might be present. Moreover, since the derivative terms give rise to polynomials in Lorentz invariant variables, the analyticity properties of the amplitude are not affected due to the presence of such terms. This will be understood whenever we write an equation like \((59)\).

4. Nonforward Elastic Scattering of $n \neq 0$ Kaluza-Klein States

We envisage elastic scattering of two equal mass, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$, hence equal charge KK particles and we take $n$ positive. Our first step is to define the scattering amplitude for this reaction (see \([57]\))

$$<p_d, p_c \text{ out}|p_b, p_a \text{ in} > = 4p_0^0p_0^0\delta^3(P_d - P_b)\delta^3(P_a - P_c) - \frac{i}{(2\pi)^3} \int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times \tilde{K}_x\tilde{K}_{x'} <p_d \text{ out}|\hat{R}(x'; x)|p_b \text{ in} >$$ \(60\)

where

$$\hat{R}(x'; x) = -i\theta(x_0 - x'_0)[\chi_a(x), \chi_c(x')]$$ \(61\)
and $\tilde{K}_x = (\Box + m_n^2)$. We let the two KG operators act on $\tilde{R}(x; x')$ in the VEV and resulting equation is

$$< p_d, p_c \text{ out} | p_b, p_a \text{ in } > = < p_d, p_c \text{ in} | p_b, p_a \text{ in } > - \frac{1}{(2\pi)^3} \int d^4x \int d^4x' e^{-i(p_a \cdot x - p_c \cdot x')} \times$$

$$< p_d | \theta(x_0 - x_0) [J_a(x), J_a(x)] | p_b > \quad (62)$$

Here $J_a(x)$ and $J_a(x')$ are the source currents associated with the fields $\chi_a(x)$ and $\chi_a(x')$ respectively. We arrive at (62) from (60) with the understanding that the r.h.s. of (62) contains additional terms; however, these terms do not affect the study of the analyticity properties of the amplitude as alluded to earlier. We shall define three distributions which are matrix elements of the product of current. The importance of these functions will be evident in sequel

$$F_R(q) = \int_{-\infty}^{+\infty} d^4z e^{iq \cdot z} \theta(z_0) < Q_f [ [J_a(z/2), J_c(-z/2)] ] Q_i > \quad (63)$$

$$F_A(q) = -\int_{-\infty}^{+\infty} d^4z e^{iq \cdot z} \theta(-z_0) < Q_f [ [J_a(z/2), J_c(-z/2)] ] Q_i > \quad (64)$$

and

$$F_C(q) = \int_{-\infty}^{+\infty} d^4z e^{iq \cdot z} < Q_f [ [J_a(z/2), J_c(-z/2)] ] Q_i > \quad (65)$$

Moreover,

$$F_C(q) = F_R(q) - F_A(q) \quad (66)$$

$|Q_i >$ and $|Q_f >$ are states which carry four momenta and these momenta are held fixed. At this stage we treat them as parameter; it is elaborated in ensuing discussions. Let us focus attention on the matrix element of the causal commutator defined in (65). We open up the commutator of the currents and introduce a complete set of physical states. Let us assign KK charge $n$ to each of the states. Thus the conservation of KK charge only permits those intermediate states which respect the charge conservation laws. The physical complete sets are: $\sum_n | P_n \tilde{\alpha}_n > < P_n \tilde{\alpha}_n | = 1$ and $\sum_{n'} | P_{n'} \tilde{\beta}_{n'} > < P_{n'} \tilde{\beta}_{n'} | = 1$. Here $\{\tilde{\alpha}_n, \tilde{\beta}_{n'}\}$ stand for quantum numbers that are permitted for the intermediate states. The matrix element defining $F_C(q)$, (66), assumes the following form

$$\int d^4z e^{iq \cdot z} \left[ \sum_n \left( \int d^4 P_n < Q_f [J_a(z/2)] | P_n \tilde{\alpha}_n > | P_n \tilde{\alpha}_n | J_a(-z/2) > | Q_i > \right) - \sum_{n'} \left( \int d^4 P_{n'} < Q_f [J_a(z/2)] | P_{n'} \tilde{\beta}_{n'} > | P_{n'} \tilde{\beta}_{n'} | J_a(z/2) > | Q_i > \right) \right] \quad (67)$$

We proceed as follows at this point. Let us use translation operations judiciously so that the currents do not carry any dependence in the $z$-variables in their arguments. Subsequently, we integrate over $d^4z$ which leads to $\delta$-functions.
The expression for $F_C(q)$ now takes the form

$$F_C(q) = \sum_n \left( <Q_f | j_a(0) | P_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n > \times 
<\tilde{\alpha}_n, P_n = \frac{(Q_i + Q_f)}{2} - q|j_c(0)|Q_i > \right)$$

$$- \sum_{n'} \left( <Q_f | j_c(0) | P_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} > \times 
<\tilde{\beta}_{n'}, P_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_a(0)|Q_i > \right)$$

(68)

A few explanatory comments are in order: The momentum of the intermediate state $P_n$ appearing in first term in (68) is constrained to $P_n = \frac{(Q_i + Q_f)}{2} - q$ after the $d^4z$ integration. Similarly, $P_{n'} = \frac{(Q_i + Q_f)}{2} + q$ in the second term of (68). The second point is that, in the derivation of the spectral representation line (68) for a theory with single scalar field, the physical intermediate states correspond to the multiparticle states consistent with energy momentum conservation (physical states). For the case at hand, the intermediate states consist of the entire KK tower as long as these states satisfy energy momentum conservation constraints and the KK charge conservation rules. We shall discuss the consequences of this aspect in the sequel.

Let us define

$$2A_s(q) = \sum_{n'} \left( <Q_f | j_a(0) | P_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} > \times 
<\tilde{\beta}_{n'}, P_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_c(0)|Q_i > \right)$$

and

$$2A_u = \sum_n \left( <Q_f | j_c(0) | P_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n > \times 
<\tilde{\alpha}_n, P_n = \frac{(Q_i + Q_f)}{2} - q|j_a(0)|Q_i > \right)$$

(69)

(70)

Consequences of microcausality: The Fourier transform of $F_C(q), \bar{F}_C(z)$, vanishes outside the light cone. We recall that,

$$F_C(q) = \frac{1}{2} (A_u(q) - A_s(q))$$

Moreover, $F_C(q)$ will also vanish as function of $q$ wherever, both $A_s(q)$ and $A_u(q)$ vanish simultaneously. We recall that the the intermediate states are physical states and their four momenta lie in the forward light cone, $V^+$, as a consequence

$$\left(\frac{Q_i + Q_f}{2} + q\right)^2 \geq 0, \quad \left(\frac{Q_i + Q_f}{2}\right)_0 + q_0 \geq 0$$

(71)
and
\[ \left( \frac{Q_i + Q_f}{2} - q \right)^2 \geq 0, \quad \left( \frac{Q_i + Q_f}{2} \right)_0 - q_0 \geq 0 \] (73)

The above two conditions, for nonvanishing of \( A_u(q) \) and \( A_s(q) \) implies existence of minimum mass parameters
(i) \( (Q_i, Q_f) \geq M^2_+ \) and (ii) \( (Q_i, -Q_f) \geq M_-. \)

The matrix elements for \( A_s(q) \) and \( A_u(q) \) will not vanish and if the two conditions stated above, pertinent to each of them, are fulfilled.

We would like to draw the attentions of the reader to the following facts in the context a theory with compactified spatial dimension. In the case where there is only one scalar field, the sum over intermediate physical states as given in (69) and (70) is the multiparticles states permitted by energy momentum conservations. However, in the present situation, the contributions to the intermediate states are those which come from the KK towers as allowed by the charge conservation rules (depending on what charges we assign to \( Q_i > 0 \) and \( Q_f > 0 \) for the elastic scattering) and energy momentum conservation. For example, if the initial states have change \( n = 1 \), then the tower of multiple particle intermediate states should have one unit of KK charge. Thus the question is whether the infinity tower of KK states would contribute? It looks like that at the present stage, when we are in the 'linear programme’ framework of the general field theoretic formalism, this issue cannot be resolved. As we shall discuss subsequently, when unitarity constraint is invoked there are only contributions from finite number of terms as long as \( s \) is finite but can be taken to be very large.

In order to derive a fixed\(-t\) dispersion relation we have to identify a domain which is free from singularities in the \( t \)-plane. The first step is to obtain the Jost-Lehmann-Dyson representation for the causal commutator, \( \tilde{F}_C(q) \). We are considering elastic scattering of equal mass particles i.e. all particles carry same KK charge. Therefore, the technique of Jost and Lehmann [40] is quite adequate; we do not have to resort to more elegant and general approach of Dyson [41] (see [?] for detail discussions). We shall adhere to notations and discussions of reference I and present those results in a concise manner. As noted in (72) and (73), \( \tilde{F}_C(q) \) is nonvanishing in those domains. We designate this region as \( \mathbb{R} \),

\[ \mathbb{R} : \left\{ (Q + q)^2 \geq \mathcal{M}_+^2, Q + q \in V^+ \quad \text{and} \quad (Q - q)^2 \geq \mathcal{M}_-^2, Q - q \in V^+ \right\} \] (74)

where \( Q = \frac{Q_i + Q_f}{2} \) and \( V^+ \) being the future light cone. We need not repeat derivation of the Jost-Lehmann representation here. The present case differs from the case where only one field is present in the following way. Here we are looking for the nearest singularity to determine the singularity free region. For the case at hand, the presence of the towers of KK states is to be envisaged in the following perspective. Since we consider equal mass scattering the location of nearest singularity will be decided by the lowest values of \( \mathcal{M}_+ \) and \( \mathcal{M}_- \). Let us elaborate this point. We recall that there
is the tower of KK states appearing as intermediate states (see (69) and (70)). Thus each new threshold could create region of singularity of $F_C(q)$. We are concerned about the identification of the singularity free domain. Thus the lowest threshold of two particle intermediate state, consistent with desired constraints, control the determination of this domain of analyticity. Therefore, for the equal mass case, the Jost-Lehmann representation for $F_C(q)$ is such that it is nonzero in the region $\mathbb{R}$,

$$F_C(q) = \int_S d^4u \int_0^\infty d\chi^2 \epsilon(q_0 - u_0)\delta[(q - u)^2 - \chi^2] \Phi(u, Q, \chi^2)$$  \hspace{1cm} (75)

Note that $u$ is also a 4-dimensional vector (not the Mandelstam variable $u$). The domain of integration of $u$ is the region $S$ specified below

$$S : \left\{ q + u \in V^+, \ Q - u \in V^+, \ Max \ [0, \mathcal{M}_+ - \sqrt{(Q + u)^2}, \mathcal{M}_- - \sqrt{(Q - u)^2}] \leq \chi \right\}$$ \hspace{1cm} (76)

and $\Phi(u, Q, \chi^2)$ arbitrary. Here $\chi^2$ is to be interpreted like a mass parameter. Moreover, recall that the assumptions about the features of the causal function stated above are the properties we have listed earlier and $Q$ is already defined above. Since the retarded commutator involves a $\theta$-function, if we use integral representation for it (see [40]) we derive an expression for the regarded function,

$$F_R(q) = \frac{i}{2\pi} \int d^4q' \delta^3(q' - q) \frac{1}{q_0' - q_0} F_C(q'), \ \text{Im} \ q_0 > 0$$ \hspace{1cm} (77)

Moreover, for the retarded function, $F_R(q)$, the corresponding Jost-Lehmann representation reads [40]

$$F_R(q) = \frac{i}{2\pi} \int_S d^4u \int_0^\infty d\chi^2 \frac{\Phi(u, Q, \chi^2)}{(q - u)^2 - \chi^2}$$ \hspace{1cm} (78)

We mention in passing that these integral representations are written under the assumption that the functions appearing inside the integral are such that the integral converges. However, if there are polynomial growths asymptotically then subtraction procedure can be invoked to tame the divergences. It is to be borne in mind that these expressions can have only polynomial behaviors for asymptotic values of the argument as we have argued earlier. The polynomial behaviors will not affect the study of analyticity properties. One important observation is that the singularities lie in the complex $q$-plane. We provide below a short and transparent discussion for the sake of completeness. The locations of the singularities are found by examining where the denominator (78) vanishes,

$$(q_0 - u_0)^2 - (q_1 - u_1)^2 - (q_2 - u_2)^2 - (q_3 - u_3)^2 = \chi^2$$  \hspace{1cm} (79)

\[\text{see Itzykson and Zubber [8] and Sommer [11] for elaborate discussions}\]
We conclude that the singularities lie on the hyperboloid given by (79) and those points are in domain $S$ as defined in (76). There are points in the hyperboloid which belong to the domain $S$. These are called admissible. Moreover, according to our earlier definition, the domain $\mathbf{R}$ is where $F_C(q)$ is nonvanishing (see (74)). Then there is a domain which contains a set of real points where $F_C(q)$ vanishes, call it $\bar{R}$ and this is complimentary to real elements of $\mathbf{R}$. From the above arguments, we arrive at the conclusion that $F_C(q) = 0$ for every real point belonging to $\bar{R}$ (the complimentary of $\mathbf{R}$). Thus these are the real points in the $q$-plane where $F_R(q) = F_A(q)$ since $F_C(q) = 0$ there. Recall the definition of $\mathbf{R}$, (74). A border is defined by the upper branch of the parabola given by the equation $(Q + q)^2 = M_+^2$ and the other one is given by the equation for another parabola $(Q - q)^2 = M_-^2$. Now we identify the coincidence region to be the domain bordered by the two parabolae. It is obvious from the above discussions that the set $S$ is defined by the range of values $u$ and $\chi^2$ assume in the admissible parabola. Now we see that those set of values belong to a subset of $(u, \chi^2)$ of all parabolas (recall equation (79)) [11] and [40, 41]. In order to transparently discuss the location of a singularity, let us go through a few short steps as the prescription to illustrate essential points. We discussed about the identification of admissible parabola. The amplitude is function of Lorentz invariant kinematical variables; therefore, it is desirable to express the constraints and equations in terms of those variables eventually. Let us focus on $Q \in V^+$ and choose a Lorentz frame such that four vector $Q = (Q_0, 0)$ where 0 stands for the three spatial components of $Q$. Next step is to choose four vector $q$ appropriately to exhibit the location of singularity in a simple way. This is achieved as follows: choose one spatial component of $q$ in order to identify the position of the singularity in this variable and treat $q_0$ and the rest of the components of $q$ as parameters and hold them fixed [11]. We remind the reader that all the variables appearing in the Jost-Lehmann representation for $F_C(q)$ and $F_R(q)$ are Lorentz invariant objects. Thus going to a specific frame will not alter the general attributes of the generalized functions. If we solve for $q_1^2$ in (79) after obtaining an expression for $q_1^2$

$$q_1 = u_1 \pm i \sqrt{\chi^2_{\text{min}}(u) - (q_0 - u_0)^2 + (q_2 - u_2)^2 + (q_3 - u_3)^2 + \rho, \rho > 0}$$  

(80)

We remind that the set of points $\{u_0, u_1, u_2, u_3; \chi^2_{\text{min}} = \text{min} \chi^2\}$ lie in $S$. The above exercise has enabled us to identify the domain where the singularities might lie with the choice for the variables $Q$ and $u$ we have made. We are dealing with the equal mass case and note that the location of the singularities are symmetric with respect to the real axis. We now examine a further simplified scenario where the coincidence region is bounded by two branches of hyperboloids so that $M^2_+ = M^2_- = M^2$. Now the singular points are

$$q_1 = u_1 \pm i \sqrt{\text{min} \left[\chi^2_{\text{min}} - u_0^2 + u_2^2 + u_3^2\right] + \rho, \rho > 0}$$  

(81)
For the case under considerations: \((Q + q)^2 = (Q - q)^2 = M^2\), and
\[
q_1 = u_1 \pm i \sqrt{(M - \sqrt{Q^2 - u_1^2})^2 + \rho, \rho > 0}
\]  
(82)

Now we can utilize this analysis to present a derivation of the Lehmann ellipse. The essential difference between the present investigation in this context with the known results is that now we have to deal with several thresholds for identification of the coincidence regions. These thresholds are the multiparticle states in various channels as discussed earlier as introduced in Section 3 through the two equations (55) and (56). Their relevance is already reflected in the spectral representations, (69) and (70), when we introduced complete set of intermediate states. We remark in passing that the presence of the excited KK states do not shrink the singularity free regions. Therefore, the domain we have obtained is the smallest domain of analyticity; nevertheless, we feel that in order to arrive at this conclusion the entire issue had to be examined with care.

**The Lehmann Ellipses**

Our goal is to derive fixed-\(t\) dispersion relations. We have noted that as \(s \rightarrow s_{\text{thr}}\), \(\cos \theta\) goes out of the physical region \(-1 \leq \cos \theta \leq +1\), \(\theta\) being the c.m. angle when we wish to hold \(t\) fixed. We choose the following kinematical configuration in order to derive the Lehmann ellipse. For the case at hand i.e. elastic scattering of equal (nonzero) charge KK states, hence particles of equal mass. Here \((a, b)\) and \((c, d)\) are respectively the incoming and outgoing particles. They are assigned the following energies and momenta in the c.m. frame:
\[
p_a = (E_a, \mathbf{k}), \quad p_b = (E_b, -\mathbf{k}), \quad p_c = (E_c, \mathbf{k}'), \quad p_d = (E_d, -\mathbf{k}')
\]  
(83)

\(\mathbf{k}\) is the c.m. momentum, \(|\mathbf{k}| = |\mathbf{k}'|\), \(E_a = \sqrt{(m_a^2 + k^2)}\), \(E_b = \sqrt{(m_b^2 + k^2)}\), \(E_c = \sqrt{(m_c^2 + k'^2)}\) and \(E_d = \sqrt{(m_d^2 + k'^2)}\). Although all the particles, \((a, b, c, d)\), are identical, we keep labeling them as individual one for the purpose which will be clear shortly. Thus \(E_a = E_b\) and \(E_c = E_d\) and \(\mathbf{k} \cdot \mathbf{k}' = \cos \theta\). It is convenient to choose the following coordinate frame for the ensuing discussions.
\[
p_a = (\sqrt{s}, +\mathbf{k}, 0), \quad p_b = (\sqrt{s}, -\mathbf{k}, 0)
\]  
(84)

0 is the two spatial components of vector \(\mathbf{k}\) and
\[
p_c = (\sqrt{s}, +k\cos \theta, +k\sin \theta, 0), \quad p_d = (\sqrt{s}, -k\cos \theta, -k\sin \theta, 0)
\]  
(85)

with \(k = |\mathbf{k}| = |\mathbf{k}'|\). Thus, \(s = (p_a + p_b)^2 = (p_c + p_d)^2\)
\[
q = \frac{1}{2}(p_d - p_c) = (0, -k\cos \theta, -k\sin \theta, 0), \quad P = \frac{1}{2}(p_a + p_b) = (\sqrt{s}, 0, 0, 0)
\]  
(86)
With these definitions of $q$ and $P$, when we examine the conditions for nonvanishing of the spectral representations of $A_s$ and $A_u$ we arrive at

$$(P + q)^2 > M_+^2, \text{ for } A_s \neq 0, \quad (P - q)^2 > M_-^2, \text{ for } A_u \neq 0$$

Thus the coincidence region is given by the condition

$$(P + q)^2 < M_+^2, \quad (P - q)^2 < M_-^2$$

We are dealing with the equal mass case; therefore, $M_+^2 = M_-^2 = M^2$. We conclude from the energy momentum conservation constraints (use the expressions for $P$ and $q$) that $p_c^2 = (P - q)^2 < M_c^2$ and $p_d^2 = (P + q)^2 < M_d^2$ in this region. Moreover, $(p_a - p_c)^2 = (P - q - p_a)^2 < M_{ac}^2$ and $(p_a + p_d)^2 = (P - q - p_d)^2 < M_{ad}^2$. We also note that $(P - q) \in V^+$ and $(P + q) \in V^+$. The admissible hyperboloid is $(q - u)^2 = \chi_{min}^2 + \rho, \rho > 0$ with $(p_a + p_b) \pm u \in V^+$. $\chi_{min}^2$ assumes the following form,

$$\chi_{min}^2 = Max \left\{ 0, M - \sqrt{\frac{(p_a + p_b)}{2} + u}, M - \sqrt{\frac{(p_a + p_b)}{2} - u} \right\}$$

Notice that $M$ appearing in the second term of the curly in (89) is the mass of two or more multiparticle states carrying the quantum numbers of particle $c$; whereas $M$ appearing in the third term inside the curly bracket is the mass of two or more multiparticle states carrying the quantum numbers of particle $d$. In the present case $M$ has the same quantum number as that of the incoming state carrying KK charge $n$. Thus, in this sector, we can proceed to show the existence of the small Lehmann Ellipse (SLE). It is not necessary to present the entire derivation here. The extremum of the ellipse is given by

$$\cos \theta_0 = \left( 1 + \frac{(M_c^2 - m_n^2)(M_d^2 - m_0^2)}{k^2(s - M_c^2 - M_d^2)} \right)^{1/2}$$

We note that $M_c = \sqrt{m_n^2 + m_0^2}$ is the mass of the lowest multiparticle state (one particle with KK charge one and another with KK charge zero; moreover, $M_c = M_d$. Thus the denominator is $k^2 s$.

$$\cos \theta_0 = \left( 1 + \frac{9m_0^4}{k^2 s} \right)^{1/2}$$

It will be a straightforward work to derive the properties of the large Lehmann Ellipse (LLE) by reducing all the four fields in the expression for the four point function as is the standard prescription. also note that the value of $\cos \theta(s)$ depends on $s$. A natural question to ask is: what is the role of the KK towers?

**Important Remark:** The first point to note is that in the presence of the other states of KK tower, we have to carry out the same analysis as above for each sector. Notice,
however, each multiparticle state composed of KK towers has to have the quantum numbers of $c$ (same as $d$ since we consider elastic channels of equal mass scattering). Thus if $c$ carries charge $n$, then a possible KK state could be $q + l + m = n$ since KK charges can be positive and negative. The second point is when we derive the value of $\cos\theta_0$, for each such case, it is rather easy to work out that value will be away from original expression (90). Thus the nearest singularity in $\cos\theta$ plane is given by the expression (91) although there will be Lehmann ellipses associated with higher KK towers.

Consequently, when we expand the scattering amplitude in partial waves (in the Legendre polynomial basis) the domain of convergence is to be identified. This domain of analyticity is enlarged (earlier it was only physically permitted values of $-1 \leq \cos\theta \leq +1$) to a region which is an ellipse whose semimajor axis is given by (91). Moreover, the absorptive part of the scattering amplitude has a domain of convergence beyond $\cos\theta = \pm 1$; it converges inside the large Lehmann ellipse (LLE). Therefore, we are able to write fixed-$t$ dispersion relations as long as $t$ lies in the following domain

$$|t| + |t + 4k^2| < 4k^2 \cos\theta_0$$

(92)

The absorptive parts $A_s$ and $A_u$ defined on the right hand and left hand cuts respectively, for $s' > s_{thr}$ and $u' > u_{thr}$ are holomorphic in the LLE. Thus, assuming no subtractions

$$F(s, t) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds'}{s' - s} A_s(s', t) + \frac{1}{\pi} \int_{u_{thr}}^{\infty} \frac{du'}{u' + s - 4m^2 + t} A_u(u', t)$$

(93)

We shall discuss the issue of subtractions in sequel. We remark in passing that crossing has not been proved explicitly in this investigation. However, it is quite obvious from the preceding developments, it will not be hard to prove crossing either from the prescriptions of Bremmermann, Oehme and Taylor [50] or from the procedures of Bross, Epstein and Glaser [51].

5. Unitarity and Asymptotic Behavior of the Amplitude

In this section we shall explore the consequences of unitarity as mentioned earlier. The investigation so far has followed what is known as the linear program in axiomatic field theory. All our conclusions about the analyticity properties of the scattering amplitude are derived from micro causality, Lorentz invariance, translational invariance and axioms of LSZ. Note that unitarity of the $S$-matrix is a nonlinear relationship and it is quite powerful. For example, the positivity properties of the partial wave amplitude follows as a consequence. First we utilize unitarity in a new context in view of the fact that there are infinite towers of KK states in the spectral representation of $F_C(q)$ and the representation for $F_R(q)$. 

29
Let us define the T-matrix as follows:

\[ S = 1 - iT \]  

(94)

The unitarity of the S-matrix, \( SS^\dagger = S^\dagger S = 1 \) yields

\[ (T^\dagger - T) = iT^\dagger T \]  

(95)

In the present context, we consider the matrix element for the reaction \( a + b \rightarrow c + d \).

Note that on L.H.S of (64) it is taken between \( T^\dagger - T \). We introduce a complete set of physical states between \( T^\dagger T \). For the elastic case with all particles of KK charge, \( n \), the unitarity relation is

\[ <p_d, p_c \text{ in} |T^\dagger - T|p_b, p_a \text{ in}> = i \sum_N \langle N|T^\dagger|N\rangle <N|T|p_b, p_c \text{ in}> \]  

(96)

The complete of states stands for \( |N\rangle = |\mathcal{P}_n \tilde{\alpha}_n\rangle \). The unitarity relation reads,

\[ T^*(p_a, p_b; p_c, p_d) - T(p_d, p_c; p_b, p_a) = 2\pi i \sum_N \delta(p_d + p_c - p_n)T^*(n; p_c, p_d)T(n; p_b, p_a) \]  

(97)

We arrive at an expression like the second term of the R.H.S of (60) after reducing two fields. If we reduce a single field as the first step (as is worked out in text books) there will be a single KG operator acting on the field and eventually we obtain matrix element of only a single current. The R.H.S. of (66) has matrix element like (for example) \( p_a + p_b \rightarrow p_n \). Thus we can express it as

\[ \delta(p_n - p_a - p_b)T(n; p_b, p_a) = (2\pi)^{3/2} \delta(p_n - p_a - p_b) \]  

(98)

After carrying out the computations we arrive at

\[ T(p_d, p_c; p_b, p_a) - T^*(p_d, p_c; p_b, p_a) = \sum_N \left[ \delta(p_d + p_c - p_n) \times \right. \\
\left. T(p_d, p_c; n)T^*(n; p_b, p_a) - \delta(p_a - p_c - p_n) \times \right. \\
\left. T(p_d, -p_c; n)T^*(p_d, -p_c; n) \right] \]  

(99)

Let consider the scattering amplitude for the reaction under considerations.

\[ F(s, t) = i \int d^4x e^{i(p_a + p_c)\cdot x} \theta(x_0) <p_d|[J_a(x/2), J_c(-x'/2)]|p_b> \]  

(100)

\footnote{We adopt the arguments and procedures of Gasiorowicz in these derivations}
We evaluate the imaginary part of this amplitude, $F(s,t)$

$$\text{Im } F(s,t) = \frac{1}{2i} (F - F^*)$$

$$= \frac{1}{2} \int d^4 x e^{i(p_a + p_c) \cdot x} < p_d | [J_a(x/2), J_c(-x/2)] | p_b >$$

(101)

Note that $F^*$ is invariant under interchange $p_b \rightarrow p_d$ and also $p_d \rightarrow p_b$; moreover, $\theta(x_0) + \theta(-x_0) = 1$. We open up the commutator of the two currents in (70). Then introduce a complete set of physical states $\sum_N |N><N| = 1$. Next we implement translation operations in each of the (expanded) matrix elements to express arguments of each current as $J_a(0)$ and $J_c(0)$ and finally integrate over $d^4 x$ to get the $\delta$-functions.

As a consequence (70) assumes the form

$$F(p_d, p_c; p_b, p_a) - F^*(p_b, p_a; p_c, p_d) = 2\pi i \sum_N \left[ \delta(p_d + p_c - p_n) F(p_d, p_c; n) F^*(p_a, p_b; n) 
- \delta(p_a - p_c - p_n) \times 
F(p_d, -p_a; n) F^*(p_b, -p_c; n) \right]$$

(102)

This is the generalized unitarity relation where all external particles are on the mass shell. Notice that the first term on the $R.H.S$ of the above equation is identical in form to the $R.H.S.$ of (68); the unitarity relation for $T$-matrix. The first term in (71) has the following interpretation: the presence of the $\delta$-function and total energy momentum conservation implies $p_d + p_c = p_n = p_a + p_b$. We identify it as the $s$-channel process $p_a + p_b \rightarrow p_c + p_d$.

Let us examine the second term of (71). Recall that the unitarity holds for the $S$-matrix when all external particles are on shell (as is true for the $T$-matrix). The presence of the $\delta$-function in the expression ensures that the intermediate physical states will contribute for

$$p_b + (-p_c) = p_n = p_d + (-p_a)$$

(103)

The masses of the intermediate states must satisfy

$$M^2_n = p^2_n = (p_b - p_c)^2$$

(104)

It becomes physically transparent if we choose the Lorentz frame where particle 'b' is at rest i.e. $p_b = (m_b, 0)$; thus

$$M^2_n = 2m_b (m_b - p^0_c), \quad p^0_c > 0$$

(105)

since $m_b = m_c$ and $p^0_c = \sqrt{m_c^2 + p^2_c} = \sqrt{m_b^2 + p^2_c}; \quad M^2_n < 0$ in this case. We recall that all particles carry KK charge $n$ and hence the mass is $m_b^2 = m_n^2 = m_0^2 + n^2$. The intermediate state must carry that quantum number. In conclusion, the second
term of (71) does not contribute to the s-channel reaction. There is an important implication of the generalized unitarity equation: Let us look at the crossed channel reaction

\[ p_b + (-p_c) \to p_d + (-p_a); \quad -p_a^0 > 0, \text{ and } -p_c^0 > 0 \]  

Here \( p_b \) and \( p_c \) are incoming (hence the negative sign for \( p_c \)) and \( p_d \) and \( p_a \) are outgoing. The second matrix element in (71) contributes to the above process in the configurations of the four momenta of these particle; whereas the first term in that equation does not if we follow the arguments for the s-channel process.

Remark: We notice the glimpses of crossing symmetry here. Indeed, the starting point will be to define \( F_C(q) \) and look for the coincidence region. Notice that \( q \) is related to physical momenta of external particles when \( |Q_i> \) and \( |Q_f> \) are identified with the momenta of the ‘unreduced’ fields. Indeed, we could proceed to prove crossing symmetry for the scattering process; however, it is not our present goal.

An important observation is in order:
We could ask whether entire Kaluza-Klein tower of states would appear as intermediate states in the unitarity equation. It is obvious from the unitarity equation (71) that for the s-channel process, due to the presence of the energy momentum conserving \( \delta \)-function, \( p_n^2 = \mathcal{M}_n^2 = (p_a + p_b)^2 \); consequently, not all states of the infinite KK tower will contribute to the reaction in this, (s), channel. Therefore the sum would terminate after finite number of terms, even for very large \( s \) as long as it is finite. Same argument also holds for the crossed channel process. Thus unitarity constraint settles the issue of the contributions of KK towers as we alluded to in the previous section in the context of the spectral representation of \( F_R(q) \), \( F_A(q) \) and \( F_C(q) \).

Let us turn the attention to the partial wave expansion of the amplitude and the power of the positivity property of absorptive part of the amplitude. We recall that the scattering amplitude admits a partial wave expansion

\[ F(s, t) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l + 1)f_l(s)P_l(\cos \theta) \]  

(107)

where \( k = |\mathbf{k}| \) and \( \theta \) is the c.m. scattering angle. The expansion converges inside the Lehmann ellipse with with focii at \( \pm 1 \) and semimajor axis \( 1 + \frac{\text{const}}{2k^2} \). Unitarity leads to the positivity constraints on the partial wave amplitudes

\[ 0 \leq |f_l(s)|^2 \leq \text{Im } f_l(s) \leq 1 \]  

(108)

As is well known, the semimajor axis of the Lehmann ellipse shrinks and \( s \) grows. Recall that derivation of the Lehmann ellipse is based on the \textit{linear program}. Martin [46] has proved an important theorem. It is known as the procedure for the enlargement of the domain of analyticity. He demonstrated that the scattering amplitude is analytic in the topological product of the domains \( D_s \otimes D_t \). This domain is defined by
$|t| < \tilde{R}$, $\tilde{R}$ being independent of $s$ and $s$ is outside the cut $s_{\text{thr}} + \lambda = 4m_n^2 + \lambda$, $\lambda > 0$. In order to recognize the importance of this result, we briefly recall the theorem of BEG \[52\]. It is essentially the study of the analyticity property of the scattering amplitude $F(s,t)$. It was shown that in the neighborhood of any point $s_0$, $t_0 - T < t_0 \leq 0$, $s_0$ outside the cuts, there is analyticity in $s$, and $t$ in a region

$$|s - s_0| < \eta_0(s_0,t_0), \quad |t - t_0| < \eta_0(s_0,t_0)$$

(109)

The amplitude is analytic. Note the following features of BEG theorem: it identifies the domain of analyticity; however, the size of this domain may vary as $s_0$ and $t_0$ vary. Furthermore, the size of this domain might shrink to zero; in other words, as $s \to 0$, $\eta(s)$ may tend to zero. The importance of Martin’s theorem lies in his proof that $\eta(s)$ is bounded from below i.e. $\eta(s) \geq \tilde{R}$ and $\tilde{R}$ is $s$-independent. It is unnecessary to repeat the proof of Martin’s theorem here. Instead, we shall summarize the conditions to be satisfied by the amplitude as stated by Martin \[46\].

**Statement of Martin’s Theorem:** If following requirements are satisfied by the elastic amplitude

I. $F(s,t)$ satisfies fixed-$t$ dispersion relation in $s$ with finite number of subtractions ($-T_0 \leq t \leq 0$).

II. $F(s,t)$ is an analytic function of the two Mandelstam variables, $s$ and $t$, in a neighborhood of $\bar{s}$ in an interval below the threshold, $4m_n^2 - \rho < \bar{s} < 4m_n^2$ and also in some neighborhood of $t = 0$, $|t| < R(\bar{s})$. This statement hold due to the work of Bros, Epstein and Glaser \[51, 52\].

III. Holomorphicity of $A_s(s',t)$ and $A_u(u',t)$: The absorptive parts of $F(s,t)$ on the right hand and left hand cuts with $s' > 4m_n^2$ and $u' > 4m_n^2$ are holomorphic in the LLE.

IV. The absorptive parts $A_s(s',t)$ and $A_u(u',t)$, for $s' > 4m_n^2$ and $u' > 4m_n^2$ satisfy the following positivity properties

$$\left| \left( \frac{\partial}{\partial t} A_s(s',t) \right)^n \right| \leq \left. \left( \frac{\partial}{\partial t} A_s(s',t) \right)^n \right|_{t=0}, \quad -4k^2 \leq t \leq 0$$

(110)

and

$$\left| \left( \frac{\partial}{\partial t} A_u(u',t) \right)^n \right| \leq \left. \left( \frac{\partial}{\partial t} A_u(u',t) \right)^n \right|_{t=0}, \quad -4k^2 \leq t \leq 0$$

(111)

where $k$ is the c.m. momentum. Then $F(s,t)$ is analytic in the quasi topological product of the domains $D_s \otimes D_t$. (i) $s \in \text{cut plane: } s \neq 4m_n^2 + \rho, \rho > 0$ and (ii) $|t| < \tilde{R}$, there exists some $\tilde{R}$ such that dispersion relations are valid for $|t| < \tilde{R}$, independent of $s$. We may follow the standard method to determine $\tilde{R}$. The polynomial boundedness, in $s$, can be asserted by invoking the simple arguments presented earlier. Consequently, a dispersion relation can be written down for $F(s,t)$ in the domain $D_s \otimes D_t$. The importance of Martin’s theorem is appreciated from the fact that it implies that the $\eta$ of BEG is bounded from below by an $s$-independent
Moreover, value of $\tilde{R}$ can be determined by the procedure of Martin (see [11] for the derivations).

We shall list a few more results as corollary without providing detailed computations:

(i) It can be proved that the partial wave expansion can be expressed as sum of two terms

$$F(s, t) = S_1 + S_2$$

where

$$S_1 = \frac{\sqrt{s}}{k} \sum_{l=0}^{L} (2l + 1)f_l(s)P_l(1 + \frac{t}{2k^2})$$

and

$$S_2 = \frac{\sqrt{s}}{k} \sum_{L+1}^{\infty} (2l + 1)f_l(s)P_l(1 + \frac{t}{2k^2})$$

where $L = \text{const.} \sqrt{\log s}$ is the cut off which is derived from the convergence of the partial wave expansion inside the Lehmann-Martin ellipse and the polynomial boundedness of the amplitude. The partial sum $S_2$ has subleading contributions to the amplitude compared to $S_1$; in fact $\frac{S_1}{S_2} \to (\log s)^{-1/4}$ for asymptotic $s$ apart from some innocent $t$-dependent prefactor; as is well known.

(ii) The Bound on $\sigma_t$: The analog of Froissart-Martin bound can be [2] obtained in that $\sigma_t(s) \leq \text{const.} (\log s)^2$. The constants appearing determining $L$ and in derivation of the Froissart-Martin bound can be determined in terms of $\tilde{R}$ and we have refrained from giving those details.

(iii) Number of subtractions: Once we have derived (i) and (ii) it is easy to prove the Jin-Martin [47] bound which states that the amplitude requires at most two subtractions. This is achieved by appealing to the existence of fixed-$t$ dispersion relation relations and to Phragman-Lindelof theorem.

We would like to draw the attention of the reader to the fact that a field theory defined on the manifold $R^3_1 \otimes S^1$ whose spectrum consists of a massive scalar field and a tower of Kaluza-Klein states satisfies nonforward dispersion relations. This statements begs certain clarifications. The theory satisfies LSZ axioms. The analyticity properties can be derived in the linear program of axiomatic field theory which leads to the proof of the existence of the Lehmann ellipses. The role of the KK tower is to be assessed in this program. Once we invoke unitarity constraint stronger results follow and the enlargement of the domain of analyticity in $s$ and $t$ variables can be established.

6. Proposal to Explore Decompactification of Extra Dimensions

In this section we would like to examine a possibility of exploring the signature of large extra dimension in high energy collisions of hadrons at LHC. There has been intense phenomenological activities to look for evidence of extra spatial dimensions.
If the radius of extra dimension is large then excited states would be produced in proton-proton scatterings at LHC. There has been proliferation of model suggesting detection of these exotic particles. There is no conclusive experimental evidence so far to confirm that there are compact spatial dimensions with large radius of compactification. We refer to the papers cited in the introduction section [31, 32]. It was suggested sometime ago [48] that precision measurements of total cross sections at high energy might be another way to explore whether extra compact dimensions are decompactified at LHC energy and beyond. The proposal is based on the following idea. It is well known that the total cross section, $\sigma_t$, should respect the Froissart-Martin bound i.e. it cannot grow faster than $\log^2 s$ at asymptotic energies. If this bound is violated then axioms of axiomatic local quantum field theory would face serious problems. There is rigorous derivation of a bound on total cross section in field theories which live in higher dimensions [39], $D > 4$. A hermitian massive field theory was considered in $D$-dimensions. The axiomatic LSZ technique was adopted to investigate the analyticity properties of the four point amplitude [39] as has been alluded to in the introduction section. A bound was derived

$$\sigma_t \leq \tilde{C} \log^{D-2} \frac{s}{s_0}$$

(115)

where $D$ is the number of spacetime dimensions and $\tilde{C}$ is a constant, determined from the first principles. Note that, for $D = 4$ we recover the Froissart-Martin bound. Now consider the following situation. Suppose an extra compact dimension decompactifies at the LHC energy regime. Then the energy dependence of total cross section is not necessarily bounded by $\log^2 (s)$ and one would conclude that Froissart-Martin bound is violated. However, for a five dimensional flat Minkowski spacetime, the bound on total cross section is $\sigma_t^{D=5} \leq \tilde{C} \log^3 (s)$. This bound is derived from LSZ axioms. In such a situation, should energy dependence of $\sigma_t$ exhibit a behavior violating the Froissart-Martin we should refrain from challenging axioms of local field theory. The reason is that the Froissart-bound-violating-behavior of $\sigma_t$ might have a different origin.

Nayak and Maharana [49] have examined this issue recently. The first point to note is that the fit to high energy cross sections has been presented in the particle data group (PDG) data book [55]. They fit the data which respects Froissart-Martin bound and most of the analysis also fit the data with a term of the form $\log^2 s$ with a constant coefficient. We shall discuss this aspect later in this section. There is an analysis [56] which fitted the data from laboratory energy of 5 GeV to LHC energy and included the cosmic ray data as well. They claim to have fitted the data with a Froissart-bound-violating energy dependent term. It is worth mentioning that the number of data points in the 'low energy range' (i.e. 5 GeV to below ISR energy) are vast compared to the data points from ISR range to LHC. Moreover, those 'low energy region' are measured with better precisions. Therefore, when one adopts a fitting formula and goes for the $\chi^2$ minimization program these data points primarily control the minimization procedure. Furthermore, the fitting procedure and other
techniques adopted by them [56] have been subject to criticism by Block and Halzen [57]. We have no remarks to offer on this issue.

We have adopted a different strategy to test whether Froissart-Martin bound is violated in high energy scattering. We argue that it is best to test the validity of the aforementioned bound starting from an energy domain where $\sigma_t(s)$ start rising with energy. Then one can go all the way up to the cosmic energy domain. Our proposal is to consider a set of data from an energy range where the total cross section starts growing with energy up to LHC energy and beyond. If we focus only on $\sigma_t^{pp}$ then the number of data points are quite limited. We include $\sigma_t^{pp}$ data from ISR, CERN SPS collider, Tevatron and fit the total set of data points which is quite substantial. We justify inclusion of the $\sigma_t^{p\bar{p}}$ along with $\sigma_t^{pp}$ data on the following ground. We invoke the Pomranchuk theorem [58]. The theorem, in its original form, stated that particle-particle and particle-antiparticle total cross sections tend to equal values at asymptotic energies. We recall, Pomeranchuk assumed that the two cross sections attain constant values at high energy. The total cross sections for $pp$ and $p\bar{p}$ started rising from ISR energies. Therefore, the Pomeranchuk theorem had to be reexamined. The bound on total cross section, $\sigma_t \leq \log^2 s$ has been proved from analyticity and unitarity of S-matrix in the axiomatic field theories. It requires additional reasonable assumptions [59] to derive behavior of $\Delta \sigma = \sigma_t^{pp} - \sigma_t^{p\bar{p}}$ for asymptotic s and show under which circumstances $\Delta \sigma \rightarrow 0$ as $s \rightarrow \infty$. The test of Pomeranchuk theorem comes from ISR experiments since it measures $pp$ and $p\bar{p}$ total cross sections in the same energy domain. Note that the SPS ($p\bar{p}$)collider at CERN and the Tevatron at FERMILAB measure $\sigma_t^{p\bar{p}}$. The LHC measures $\sigma_t^{pp}$. It is noteworthy that $\Delta \sigma$ shows the tendency to decrease with energy in the energy regime covered by ISR. Therefore, we feel that it is quite justified to combine the high energy total cross sections of $pp$ and $p\bar{p}$ and fit the total cross sections.

Now we discuss our fits to total cross section [49]. We choose the following parameterization to fit the combined data.

$$\sigma_t = H \log^\alpha \left( \frac{s}{s_0} \right) + P$$

(116)

$H$ and $P$ are the Heisenberg and Pomeranchuk constants, respectively. $P$ is the contribution of the Pomeranchuk trajectory in the Regge pole parlance. The constants $H$, $P$ and $\alpha$, are free parameters and are determined from the fits. We fix $s_0 = 16.00 \text{ GeV}^2$, taking a hint from the PDG fit. PDG adopted the following strategy to fit $\sigma_t$ data. For the fit to $\sigma_t^{pp}$ the chosen energy range was from 5 GeV to cosmic ray regime. The Froissart-bound-saturating energy dependence is assumed in their fitting procedure. Note that in the pre-ISR energy regime the measured cross sections are flat and measured with very good precision. Moreover, Regge pole contributions, with subleading power behaviors in energy, should be included in the pre-ISR energy domain. However, in the energy range starting from ISR, the Regge contributions are negligible. It is worth while to discuss and justify our reasonings for not including
the contributions of subleading Regge poles to \( \sigma_t \) in the energy range starting from ISR point and beyond (where our interests lie). Moreover, the subleading Regge pole contributions are important in the relatively moderate energy range while fitting \( \sigma_t \) data which remains flat. We refer to [12] and to the review article of Leader [60] for detailed discussions. Let us consider the case of pp scattering to get a concrete idea.

The Pomeranchuk trajectory contributes a constant term to \( \sigma_t \) and its intercept is \( \alpha_P(0) = 1 \). Then there are subleading trajectories corresponding to \( \omega, \rho, A_2, \phi \), etc whose energy dependence to the total cross section is like \( (\frac{s}{s_\star})^{\alpha_{R}(0)-1} \); \( \alpha_{R}(0) \) is the intercept of subleading trajectories such as \( \omega, \rho, A_2, \phi \), and it is of the order of \( \frac{1}{2} \).

When a fit to \( \sigma_{pp}^{t} \) was considered by Rarita et al [61], they concluded, from numerical fits, that the \( \omega \) trajectory dominates [60, 61] and the contributions of other Regge trajectories is quite small [62]. They found that the Regge residue (interpreted as the Regge trajectory coupling) is \( R_{\omega pp} \approx 15.5 \text{ mb} \) and \( \alpha_{\omega}(0) \approx 0.45 \) and the Regge scale, to define a dimensionless ratio (say \( \frac{\alpha}{s_{\star}} \)) is \( s_{\star} = 1 \text{ GeV}^2 \). Let us estimate what is the contribution of the \( \omega \)-trajectory to \( \sigma_{pp}^{t} \) at the ISR energy. The contribution of the \( \omega \)-trajectory to \( \sigma_t \) is quite small in the energy range from ISR to LHC. For example, at ISR energy of \( \sqrt{s} = 23.5 \text{ GeV} \), the \( \omega \)-Regge pole contribution to \( \sigma_t \) is approximately \( 0.5 \text{ mb} \) whereas at LHC, for \( \sqrt{s} = 8 \text{ TeV} \), it is \( 0.001 \text{ mb} \); the corresponding \( \sigma_t \) are \( \approx 39 \text{ mb} \) and \( \approx 103 \text{ mb} \) at 23.5 GeV and 8 TeV respectively.

The parameterizations of [61] is used for the above estimates. Consequently, for our purpose, the parameterization (116) is well justified. We considered the combined data of \( \sigma_{pp}^{t} \) and \( \sigma_{\bar{p}p}^{t} \) for the energy range as mentioned earlier. The measured values of cross sections against \( \sqrt{s} \), along with the fitted curve, are shown in Fig. 1. The fitted values for the parameters are \( P = 36.4 \pm 0.3 \text{ mb} \), \( H = 0.22 \pm 0.02 \text{ mb} \), and \( \alpha = 2.07 \pm 0.04 \). The quality of the fit, as reflected by the \( \chi^2/n.d.f. \) is found to be moderate due to inclusion of both \( \sigma_{pp}^{t} \) and \( \sigma_{\bar{p}p}^{t} \) measurements from ISR. A fit excluding \( \sigma_{\bar{p}p}^{t} \) from ISR, as shown in Fig. 1 (lower), improves the fit quality without significantly changing the value of the fit parameters. We have not provided all the references of the experimental papers from where the data were taken for the plot. The interested reader may refer to our paper [49] where detail references are cited. We find no conclusive evidence for the violation of the Froissart bound.

We arrive at the conclusion that the data are consistent with Froissart-Martin bound. Therefore, there is no indirect evidence for decompactification of extra dimensions as far as our proposal goes. If the total cross sections are measured with more precision showing that there is violation of Froissart-Martin bound then one might interpret it as a sign of decompactification. Moreover, our analysis is in qualitative agreement with the experimental lower bound on the radius of compactification in the sense that ATLAS and CMS have not been able to determine the radius of compactification [33, 34]. We might close this section with an optimistic note that there is possibility of gathering experimental evidence in favor of large-radius-compactification scenario if the precision of the measurements of \( \sigma_t \) is improved significantly. Furthermore, the future high energy accelerators might provide evidence for the existence of extra
Figure 1: $\sigma_t^{pp}$ and $\sigma_t^{p\bar{p}}$ against $\sqrt{s}$, measured by various experiments. The data points are fitted to a function defined in (116). The upper plot includes both $pp$ and $p\bar{p}$ data points, while the lower one excludes $p\bar{p}$ data points from ISR experiments.
spatial dimensions besides discovering new phenomena hinting at the existence of fundamental physics.

7. Summary and Discussions

We summarize our results in this section and discuss their implications. The objective of the present work is to investigate the analyticity property of the scattering amplitude in a field theory with a compactified spatial dimension on a circle i.e. the $S^1$ compactification. We were motivated to undertake this work from work of Khuri [35] who considered potential scattering with a compact spatial coordinate. He showed the lack of analyticity of the forward scattering amplitude under certain circumstances. Naturally, it is important to examine what is the situation in relativistic field theories. As has been emphasized by us before, lack of analyticity of scattering amplitude in a QFT will be a matter of concern since analyticity is derived under very general axioms in QFT. Thus a compactified spatial coordinate in a theory with flat Minkowski spacetime coordinates should not lead to unexpected drastic violations of fundamental principles of QFT. In this paper, initially, a five dimensional neutral massive scalar theory of mass, $m_0$, was considered in a flat Minkowski spacetime. Subsequently, we compactified a spatial coordinate on $S^1$ leading to a spacetime manifold $R^{3,1} \otimes S^1$. The particles of the resulting theory are a scalar of mass $m_0$ and the Kaluza-Klein towers. In this work, we have focused on elastic scattering of states carrying nonzero equal KK charges, $n \neq 0$ to prove fixed-$t$ dispersion relations. We have left out the elastic scattering of $n = 0$ states as well as elastic scattering of an $n = 0$ state with an $n \neq 0$ state for nonforward directions. These two cases can be dealt with without much problem from our present work. Moreover, our principal task is to prove analyticity for scattering of $n \neq 0$ states and thus complete the project we started with in order to settle the issue related to analyticity as was raised by Khuri [35] in the context of potential scattering. We showed in I that forward amplitude satisfies dispersion relations. However, it is not enough to prove only the dispersion relations for the forward amplitude but a fixed-$t$ dispersion relation is desirable. We have adopted the LSZ axiomatic formulation, as was the case in I, for this purpose. Our results, consequently, do not rely on perturbation theory whereas, Khuri [35] arrived at his conclusions in the perturbative Greens function techniques as suitable for a nonrelativistic potential model. Thus the work presented here, in some sense, has explored more than what Khuri had investigated in the potential scattering.

We have gone through several steps, as mentioned in the discussion section of I, in order to accomplish our goal. The principal results of this are as follows. First we obtain a spectral representation for the Fourier transform of the causal commutator, $F_c(q)$. We discussed the coincidence region which is important for what followed. In order to identify the singularity free domain, we derived analog of the Jost-Lehmann-Dyson theorem. A departure from the known theorem is that there are several massive states, appearing in the spectral representation, and their presence has to be taken into considerations. Thus, we identified the the singularity free region i.e. the bound-
ary of the domain of analyticity. Next, we derived the existence of the Lehmann ellipse. We were able to write down fixed-$t$ dispersion relations for $|t|$ within the Lehmann ellipse.

We have proceeded further. It is not enough to obtain the Lehmann ellipse since the semimajor axis of the ellipse shrinks as $s$ increases. Thus it is desirable to derive the analog of Martin’s theorem [46]. We appealed to unitarity constraints following Martin and utilized his arguments on the attributes of the absorptive amplitude and showed that indeed Martin’s theorem can be proved for the case at hand. As a consequence, the analog of Froissart-Martin upper bound on total cross sections, for the present case, is obtained. The convergence of partial wave expansions within the Lehmann-Martin ellipse and polynomial boundedness for the amplitude, $F(s, t)$ for $|t|$ within Lehmann-Martin ellipse, lead to the Jin-Martin upper bound [47] for the problem we have addressed here. In other words, the amplitude, $F(s, t)$, does not need more than two subtractions to write fixed $t$ dispersion relations for in the domain $D_s \otimes D_t$.

We have made two assumptions: (i) existence of stable particles in the entire spectrum of the theory defined on $R^{3.1} \otimes S^1$ geometry. Our arguments is based on the conservation of KK discrete charge $q_n = \frac{n}{R}$; it is the momentum along the compatified direction. (ii) The absence of bound states. We have presented some detailed arguments in support of (ii). To put it very concisely, we conveyed that this flat space $D = 4$ theory with an extra compact $S^1$ geometry results from toroidal compactification of five dimensional defined in flat Minkowski space. In absence of gravity in $D = 5$, the lower dimensional theory would not have massless gauge field and consequently, BPS type states are absent. It is unlikely that the massive scalars (even with KK charge) would provide bound states. This is our judicious conjecture.

We have proposed a novel idea to look for indirect evidence of decompactification in the LHC energy regime. As has been elaborated in Section 6, we argued that precision measurement of very high energy total cross section might provide a clue. Suppose, at LHC energy, the energy dependence of $\sigma_t(s)$ shows a departure from the Froissart-Martin bound that total cross section is bounded by $\log^2 s$. On the face of it, one might tend to conclude that some of the axioms of local field theories might not hold. However, on the other hand, if an extra spatial dimension decompactifies then the generalized Froissart-Martin bound in $\log^{D-2}s$ where $D$ is the number of spacetime dimensions [39]. Therefore, in the event of such an observation, we need not question the fundamental axioms. We have fitted the data [49] from ISR energies to the LHC energy and included the cosmic ray data points for $\sigma_t(s)$. We kept the power of $\log s$ as a floating parameter. Our analysis does not indicate conclusive violation of the Froissart-Martin bound.

Another interesting aspect needs further careful consideration. Let us start with a five dimensional Einstein theory minimally coupled to a massive neutral scalar field of mass $m_0$. We are unable to fulfill requirements of LSZ axioms in the case of the five dimensional theory in curved spacetime. Furthermore, let us compactify this
theory to a geometry $R^{3,1} \otimes S^1$. Thus the resulting scalar field lives in flat Minkowski space with a compact dimension. We have an Abelian gauge field in $D = 4$, which arises from $S^1$ compactification of the 5-dimensional Einstein metric. The spectrum of the theory can be identified: (i) There is a massive scalar of mass $m_0$ descending of $D = 5$ theory accompanied by KK tower of states. (ii) A massless gauge boson and its massive KK partners. (iii) If we expand the five dimensional metric around four dimensional Minkowski metric when we compactify on $S^1$, we are likely to have massive spin 2 states (analog of KK towers). We may construct a Hilbert space in $D = 4$ i.e with geometry $R^{3,1} \otimes S^1$. It will be interesting to investigate the analyticity properties of the scattering amplitudes and examine the high energy behaviors. Since only a massless spin 1 particle with Abelian gauge symmetry appears in the spectrum, it looks as if the analyticity of amplitudes will not be affected. However, there might be surprises since a massive spin 2 particle is present in the spectrum.

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