An existence result for $p$-Laplace equation with gradient nonlinearity in $\mathbb{R}^N$

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Abstract. We prove the existence of a weak solution to the problem

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla u|^{p-2}\nabla u),$$

$$u(x) > 0 \quad \forall x \in \mathbb{R}^N,$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplace operator, $1 < p < N$ and the non-linearity $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and it depends on gradient of the solution. We use an iterative technique based on the Mountain pass theorem to prove our existence result.

1 Introduction

In this article, we prove the existence of a weak solution to the problem:

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla u|^{p-2}\nabla u),$$

$$u(x) > 0 \quad \forall x \in \mathbb{R}^N,$$

where $1 < p < N$ and the non-linearity $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function.

The Problem (1) is non-variational in nature, as the nonlinearity $f$ depends on gradient of the solution. Such type of problems have been studied widely in literature through non-variational techniques, such as method of sub-solution and super-solution [11], [16], [24], degree theory [20], [22] etc. In 2004, Figueiredo et al. [7] used an iterative technique based on Mountain Pass Theorem to establish the existence of a positive and a negative solution to the problem:

$$-\Delta u = f(x, u, \nabla u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

MSC 2020: 35J20; 35J62; 35J92

Keywords: Gradient dependence; $p$-Laplacian; Iterative methods

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where $\Omega \subseteq \mathbb{R}^n$ is a smooth and bounded domain. Motivated by the techniques used by Figueiredo et al. [7], several authors established existence results for second order elliptic equations with gradient nonlinearities, see for instance [6], [12], [13], [15], [18], [21] and references therein.

This work is motivated the existence results of G.M. Figueiredo [12], where the author obtained existence of a positive solution to (1) with $V(x) = 1$. Recently, an existence result for (1) in case of $p = N$ is discussed by Chen et al. [5]. For some existence results for the problems of the type (1) with potential $V(x)$ and without gradient dependence, we refer to [1], [9], [17] and references therein.

The plan of this article is as follows: In section 2, we state our hypotheses and main result. Section 3 deals with the proof of our main result, i.e., Theorem 2.1.

## 2 Hypotheses and Main Result

In this section, we state hypotheses on the nonlinearity $f$ and the potential $V$. We assume the following conditions on the nonlinearity $f$:

1. $f(t, |\xi|^{p-2}\xi) = 0$ for all $t < 0$, $\xi \in \mathbb{R}^N$.
2. $\lim_{|t| \to 0} \frac{|f(t, |\xi|^{p-2}\xi)|}{|t|^{p-1}} = 0$, $\forall \xi \in \mathbb{R}^N$.
3. There exists $q \in (p, p^*)$ such that $\lim_{|t| \to \infty} \frac{|f(t, |\xi|^{p-2}\xi)|}{|t|^{q-1}} = 0$, $\forall \xi \in \mathbb{R}^N$, where $p^* = \frac{Np}{N - p}$.
4. There exists $\theta > p$ such that
   
   $0 < \theta F(t, |\xi|^{p-2}\xi) \leq tf(t, |\xi|^{p-2}\xi)$,

   for all $t > 0$, $\xi \in \mathbb{R}^N$, where $F(t, |\xi|^{p-2}\xi) = \int_0^t f(s, |\xi|^{p-2}\xi)ds$.
5. There exist positive real numbers $a$ and $b$ such that
   
   $F(t, |\xi|^{p-2}\xi) \geq at^\theta - b$,

   for all $t > 0$, $\xi \in \mathbb{R}^N$.
6. There exist positive constants $L_1$ and $L_2$ such that
   
   $|f(t_1, |\xi|^{p-2}\xi) - f(t_2, |\xi|^{p-2}\xi)| \leq L_1|t_1 - t_2|^{p-1}$

   for all $t_1, t_2 \in [0, \rho_1]$, $|\xi| \leq \rho_2$,

   $|f(t, |\xi_1|^{p-2}\xi_1) - f(t, |\xi_2|^{p-2}\xi_2)| \leq L_2|\xi_1 - \xi_2|^{p-1}$.
for all $t \in [0, \rho_1]$ and $|\xi_1|, |\xi_2| \leq \rho_2$, where $\rho_1, \rho_2$ depend on $q, N$ and $\theta$. Moreover, $L_1$ and $L_2$ satisfy $\left( \frac{L_2}{C_p - L_1} \right)^{1/p - 1} < 1$, where $C_p$ is the constant in the inequality (3).

In the following, we state conditions on the potential $V$:

$(V_1)$ $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;

$(V_2)$ $V(x)$ is a continuous 1-periodic function, i.e., $V(x + y) = V(x)$, $\forall y \in \mathbb{Z}^N$ and $\forall x \in \mathbb{R}^N$.

For further details about the periodic potential $V$, we refer to [1] and references therein. Let

$$W = \{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx < \infty \}. $$

$W$ is a reflexive Banach space with the norm

$$||u|| = \left( \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx \right)^{1/p}. $$

Moreover, we have the continuous inclusions $W \hookrightarrow W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [p, p^*]$. For the details, we refer to [9, Lemma 2.1].

Next, in the spirit of Figueiredo et al. [7], we associate with (1), a family of problems with no dependence on the gradient of solution. To be precise, for every, $w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N)$ with $0 < \beta < 1$, we consider the problem

$$-\Delta_p u + V(x)|u|^{p-2}u = f(u, |\nabla w|^{p-2}\nabla w),$$

$$u(x) > 0 \quad \forall x \in \mathbb{R}^N.$$  

(2)

Problem (2) is variational in nature and the critical points of the functional

$$I_w(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx \quad + \quad \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx \quad - \quad \int_{\mathbb{R}^N} F(u, |\nabla w|^{p-2}\nabla w)dx$$

are the weak solutions to (2).

To prove our main result, we will use the following inequality [10] :

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq C_p |x - y|^p,$$  

(3)

for all $x, y \in \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^N$. Now, we state our main result:

**Theorem 2.1.** Suppose that the conditions $(f_1) - (f_6)$ and $(V_1), (V_2)$ are satisfied. Then, there exists a positive solution to (1).
3 Proof of Theorem 2.1

This section deals with the proof of Theorem 2.1. The proof is divided in a series of lemmas.

**Lemma 3.1.** Let \( w \in W \cap C^{1,\beta}_{\text{loc}}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \). Then there exist positive real numbers \( \alpha \) and \( \rho \) independent of \( w \) such that

\[
I_w(u) \geq \alpha > 0, \quad \forall u \in W \text{ such that } \|u\| = \rho.
\]

**Proof.** From \((f_2)\) we have, for any \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) such that

\[
|f(s,|\xi|^{p-2}\xi)| \leq \epsilon |s|^{p-1}, \quad \forall |s| < \delta_1, \xi \in \mathbb{R}^N. \tag{4}
\]

From \((f_3)\) we have, for any \( \epsilon > 0 \), there exists \( \delta_2 > 0 \) such that

\[
|f(s,|\xi|^{p-2}\xi)| \leq \epsilon |s|^{q-1}, \quad \forall |s| > \delta_2, \xi \in \mathbb{R}^N. \tag{5}
\]

By (4) and (5) we have,

\[
|F(u,|\nabla w|^{p-2}\nabla w)| \leq \frac{1}{p} \epsilon |u|^p + \frac{1}{q} \epsilon |u|^q, \quad \forall u \in W. \tag{6}
\]

Thus,

\[
I_w(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} F(u,|\nabla w|^{p-2}\nabla w) dx.
\]

It follows from (6) and embedding result that

\[
I_w(u) \geq \left( \frac{1}{p} - c_1 \epsilon \right) \|u\|^p - c_2 \epsilon |u|^q.
\]

Now choose \( \epsilon \) such that \( \frac{1}{p} - c_1 \epsilon > 0 \) and \( \rho \leq \left( \frac{1}{p-c_1 \epsilon} \right)^{\frac{1}{q-p}} \). This completes the proof. \( \blacksquare \)

**Lemma 3.2.** Let \( w \in W \cap C^{1,\beta}_{\text{loc}}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \). Fix \( v_0 \in C^\infty_0(\mathbb{R}^N) \) with \( \|v_0\| = 1 \). Then \( \exists \ t_0 > 0, \) independent of \( w \), such that

\[
I_w(tv_0) \leq 0, \quad \forall t \geq t_0.
\]

**Proof.** By \((f_5)\) we get,

\[
I_w(tv_0) \leq \frac{t^p}{p} - \int_{\text{Supp}(v_0)} (at^\theta v_0^\theta - b) dx
\]

\[
= \frac{t^p}{p} - at^\theta \int_{\text{Supp}(v_0)} v_0^\theta dx + b|\text{Supp}(v_0)|,
\]

Since \( \theta > p \), the result follows. \( \blacksquare \)
Lemma 3.3. Let conditions $(f_1) - (f_5)$ and $(V_1), (V_2)$ hold. Then, the Problem (2) admits a positive solution $u_w \in W.$

Proof. Lemmas 3.1 and 3.2 tell us that the functional $I_w$ satisfies the geometric conditions of the Mountain Pass Theorem. Hence, by the version of Mountain Pass Theorem without (PS) conditions [23], there exist a sequence $\{u_n\} \subset W$ such that

$$I_w(u_n) \to c_w \quad \text{and} \quad I'_w(u_n) \to 0,$$

as $n \to \infty$

where

$$c_w = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_w(\gamma(t)) > 0,$$

with

$$\Gamma = \{ \gamma \in C([0, 1], W) : \gamma(0) = 0, \gamma(1) = t_0 v_0 \}$$

where $t_0$ and $v_0$ are as in Lemma 3.2.

By $(f_4)$, we have $c \|u_n\|^p \leq c_w + \|u_n\|.$ This implies that $\{u_n\}$ is bounded in $W$, hence there exists its subsequence still denoted by $\{u_n\}$, as

$$u_n \rightharpoonup u_w \quad \text{in} \quad W;$$

$$u_n \to u_w \quad \text{in} \quad L^s_{\text{loc}} \quad \text{for} \quad p \leq s < p^*.$$

On following the arguments from [8, Proposition 4.4], we obtain

$$\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u_w}{\partial x_i}(x) \quad \text{a.e. in} \quad \mathbb{R}^N.$$

This implies,

$$\nabla u_n(x) \to \nabla u_w(x) \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Using (10), we get

$$|\nabla u_n|^{p-2} \nabla u_n \to |\nabla u_w|^{p-2} \nabla u_w \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Since $\{|\nabla u_n|^{p-2} \nabla u_n\}$ is bounded in $L^{p/(p-1)}$, we get,

$$|\nabla u_n|^{p-2} \nabla u_n \rightharpoonup |\nabla u_w|^{p-2} \nabla u_w \quad \text{in} \quad L^{p/(p-1)}(\mathbb{R}^N).$$

By the definition of weak convergence, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \ dx \to \int_{\mathbb{R}^N} |\nabla u_w|^{p-2} \nabla u_w \nabla \varphi \ dx \quad \text{for all} \ \varphi \in W.$$

In view of Brezis-Lieb lemma [4], we have

$$\int_{\mathbb{R}^N} V(x)|u_n|^{p-2} u_n \varphi \ dx \to \int_{\mathbb{R}^N} V(x)|u_w|^{p-2} u_w \varphi \ dx \quad \text{for all} \ \varphi \in W.$$
By the help of [4] and Lebesgue Generalized Theorem [3], we get
\[\int_{\mathbb{R}^N} f(u_n, |\nabla w|^{p-2}\nabla w)\varphi \, dx \rightarrow \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)\varphi \, dx \quad \text{for all } \varphi \in W.\]

Therefore, we have
\[I'_w(u_w)\varphi = \int_{\mathbb{R}^N} |\nabla u_w|^{p-2}\nabla u_w \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}u_w\varphi \, dx - \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)\varphi \, dx = 0, \quad \text{for all } \varphi \in W. \tag{11}\]

This implies, \( u \) is the weak solution of (2).

Let \( u_w \neq 0 \). Next, we show that \( u_w > 0 \). By taking \( \varphi = u_w^- \) in (11), we get
\[\int_{\mathbb{R}^N} |\nabla u_w|^{p-2}(u_w^+ - u_w^-)\nabla u_w \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^+ - u_w^-)u_w^- \, dx = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w^- \, dx,\]

which gives
\[- \int_{\mathbb{R}^N} |\nabla u_w|^{p-2}\nabla u_w^- \, dx = \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w^- \, dx + \int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^-)^2 \, dx.\]

Thus,
\[\int_{\mathbb{R}^N} V(x)|u_w|^{p-2}(u_w^-)^2 = 0.\]

This implies, \( |u_w|^{p-2}(u_w^-)^2 = |u_w^+ - u_w^-|^{p-2}(u_w^-)^2 = 0 \) as \( V(x) > 0 \). Therefore, we have
\[0 = |u_w^+(x) - u_w^-(x)|^{p-2}(u_w^-(x))^2 = \begin{cases} 0 ; & u_w^-(x) \geq 0 \\ |u_w^-|^p ; & u_w^-(x) < 0. \end{cases} \]

Hence, \( u_w = u_w^+ - u_w^- = u_w^+ \geq 0 \). Moreover, Harnack inequality implies that \( u_w(x) > 0 \) for all \( x \in \mathbb{R}^N \).

If \( u_w \equiv 0 \), then there exist a sequence \( \{z_n\} \subset \mathbb{R}^N \) and \( \delta, R > 0 \) such that
\[\int_{B_R(z_n)} |u_n|^p \, dx \geq \delta. \tag{12}\]

For, if on the contrary
\[\limsup_{n \to \infty} \int_{B_R(x)} |u_n|^p \, dx = 0,\]

then by using [14, Lemma 1.1], \( u_n \to 0 \) in \( L^s(\mathbb{R}^N) \) with \( p < s < p^* \), which implies that \( I_w(u_n) \to 0 \) as \( n \to \infty \). It contradicts the fact that \( c_w > 0 \).
Let us define \( v_n(x) = u_n(x + z_n) \). Since \( V(x) \) is a 1-periodic function, we can use the invariance of \( \mathbb{R}^N \) under translations to conclude that \( I_w(v_n) \to c_w \) and \( I_w'(v_n) \to 0 \). Moreover, up to a subsequence, \( v_n \to v_w \) in \( W \) and \( v_n \to v_w \) in \( L^p(B_R(0)) \), where \( v_w \) is a critical point of \( I_w \). By (12), we conclude that \( v_w \) is non zero. Arguing as above, we get that \( v_w \) is a positive solution to (2). This completes the proof.

\[ \blacksquare \]

**Lemma 3.4.** Let \( w \in W \cap C^{1,\beta}_{loc}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \). Then there exists positive real number \( \eta \) independent of \( w \), such that \[ ||u_w|| \leq \eta, \] where \( u_w \) is the solution of (2) obtained in Lemma 3.3.

**Proof.** Using the characterization of \( c_w \), we have
\[ c_w \leq \max_{t \geq 0} (tu). \]

Fix \( v \in W \) such that \( ||v|| = 1 \). By (f5), we have
\[ c_w \leq \max_{t \geq 0} I_w(tv) \leq \max_{t \geq 0} \left( \frac{t^p}{p} - c_6 t^\theta - c_7 \right) = \eta_0. \]

By (f4), we have
\[ I_w(u_w) \geq \frac{1}{p} ||u_w||^p - \frac{1}{\theta} \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w dx. \]

Also, we have
\[ I_w'(u_w)(u_w) = ||u_w||^p - \int_{\mathbb{R}^N} f(u_w, |\nabla w|^{p-2}\nabla w)u_w dx. \]

By (13) and (14), we obtain
\[ \left( \frac{1}{p} - \frac{1}{\theta} \right) ||u_w||^p \leq I_w(u_w) - \frac{1}{\theta} I_w'(u_w)(u_w). \]

Next, by using the fact that \( I_w'(u_w)(u_w) = 0 \) and \( I_w(u_w) = c_w \) we get
\[ \left( \frac{1}{p} - \frac{1}{\theta} \right) ||u_w||^p \leq c_w \leq \eta_0, \]
and the proof is complete.

\[ \blacksquare \]

**Lemma 3.5.** If \( u_w \) is a positive solution of the equation (2) obtained in Lemma 3.3, then \( u_w \in C^{1,\beta}_{loc} \cap L_\infty^{p}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \). Moreover, there exist positive numbers \( \rho_1 \) and \( \rho_2 \), independent of \( w \), such that \( ||u_w||_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_1 \) and \( ||\nabla u_w||_{C^{0,\beta}_{loc}(\mathbb{R}^N)} \leq \rho_2 \).
Proof. By using the fact that \( V(x) \geq V_0 \) and \( u_w > 0 \), we have
\[
f(u_w, \nabla w|^{p-2}\nabla w) - V(x)|u_w|^{p-2}u_w \leq f(u_w, \nabla w|^{p-2}\nabla w) - V_0|u_w|^{p-1}.
\]
By the help of \((f_2)\) and \((f_3)\), one gets
\[
|f(u_w, \nabla w|^{p-2}\nabla w)| \leq \epsilon|u_w|^{p-1} + \epsilon|u_w|^{q-1} + V_0|u_w|^{p-1} \leq (\epsilon + V_0)(|u_w|^{p-1} + |u_w|^{q-1}).
\]
By \([19, \text{Theorem 2.2}]\), for any compact set \( K \subseteq \mathbb{R}^N \), we have \( \|u_w\|_{L^\infty(K)} \leq C \), where the constant \( C \) depends on \( p, q, N \) and \( \|u_w\|_{L^p(K)} \). By Sobolev embedding theorem and Lemma 3.4, there exist \( C_0 \) independent of \( w \) such that \( \|u_w\|_{L^\infty(K)} \leq C_0 \). By \([2, \text{Theorem 1}]\), \( \|\nabla u_w\|_{L^\infty(K)} \leq C_1 \), for some constant \( C_1 \) dependent on \( p, q, N \) and \( \|u_w\|_{L^\infty(K)} \). Hence there exists a constant \( C_2 \) independent of \( w \) such that \( \|\nabla u_w\|_{L^\infty(K)} \leq C_2 \).

By \([2, \text{Theorem 2}]\), we obtain \( \|u_w\|_{C^{1,\beta}_{\text{loc}}(\mathbb{R}^N)} \leq C_3 \), where \( C_3 \) is dependent on \( p, q, N \) and \( \|\nabla u_w\|_{L^\infty(K)} \). Thus there exists a positive number \( \rho \) independent of \( w \) such that, \( \|u_w\|_{C^{1,\beta}_{\text{loc}}(\mathbb{R}^N)} \leq \rho \). Subsequently, there exist positive real numbers \( \rho_1 \) and \( \rho_2 \), independent of \( w \), such that \( \|u_w\|_{C^{0,\beta}_{\text{loc}}(\mathbb{R}^N)} \leq \rho_1 \) and \( \|\nabla u_w\|_{C^{0,\beta}_{\text{loc}}(\mathbb{R}^N)} \leq \rho_2 \). This completes the proof.

\[\Box\]

**Lemma 3.6.** Let \( w \in W \cap C^{1,\beta}_{\text{loc}}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \). Then there exists positive real number \( \lambda \) independent of \( w \), such that
\[
\|u_w\| \geq \lambda,
\]
where \( u_w \) is the solution of \((2)\) obtained in Lemma 3.3.

**Proof.** Since \( u_w \) is the weak solution of \((2)\) obtained in Lemma 3.3, for all \( v \in W \), we have \( l_w(u_w)(v) = 0 \). In particular, by putting \( v = u_w \) we get
\[
\int_{\mathbb{R}^N} |\nabla u_w|^p dx + \int_{\mathbb{R}^N} V(x)|u_w|^p dx = \int_{\mathbb{R}^N} f(u_w, \nabla w|^{p-2}\nabla w)u_w dx
\]
\[
\|u_w\|^p = \int_{\mathbb{R}^N} f(u_w, \nabla w|^{p-2}\nabla w)u_w dx.
\]
By using \((4)\) and \((5)\) we have,
\[
\|u_w\|^p \leq c_4 \epsilon \|u_w\|^p + c_5 \epsilon \|u_w\|^q.
\]
Since \( q > p \), we get \( \|u_w\| \geq \left( \frac{1 - c_4 \epsilon}{c_5 \epsilon} \right)^{q-p} \). This completes the proof. \[\Box\]

Now, we are in position to prove Theorem 2.1.

**Proof of Theorem 2.1:** Starting with an arbitrary \( u_0 \in W \cap C^{1,\beta}_{\text{loc}}(\mathbb{R}^N) \) with \( 0 < \beta < 1 \), we construct a sequence \( \{u_n\} \subseteq W \) as solution of
\[
-\Delta u_n + V(x)|u_n|^{p-2}u = f(u_n, \nabla u_{n-1}|^{p-2}\nabla u_{n-1}), \quad \text{in } \mathbb{R}^N \quad (P_n)
\]
obtained in Lemma 3.3. By Lemma 3.5, \( \{u_n\} \subseteq W \cap C^{1,\beta}_{{\text{loc}}}({\mathbb R}^N) \) with \( 0 < \beta < 1 \), \( \|u_n\|_{C^{0,\beta}_{{\text{loc}}}({\mathbb R}^N)} \leq \rho_1 \) and \( \|\nabla u_n\|_{C^{0,\beta}_{{\text{loc}}}({\mathbb R}^N)} \leq \rho_2 \). Since \( u_{n+1} \) is the weak solution of \( (P_{n+1}) \), we have

\[
\int_{\mathbb R^N} |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\nabla \varphi \, dx + \int_{\mathbb R^N} V(x)|u_{n+1}|^{p-2}u_{n+1}\varphi \, dx = \int_{\mathbb R^N} f(u_{n+1},|\nabla u_{n+1}|^{p-2}\nabla u_{n+1})\varphi \, dx, \quad \forall \varphi \in W. \tag{15}
\]

Similarly, \( u_n \) is the weak solution of \( (P_n) \), we have

\[
\int_{\mathbb R^N} |\nabla u_n|^{p-2}\nabla u_n\nabla \varphi \, dx + \int_{\mathbb R^N} V(x)|u_n|^{p-2}u_n\varphi \, dx = \int_{\mathbb R^N} f(u_n,|\nabla u_n|^{p-2}\nabla u_n-1)\varphi \, dx, \quad \forall \varphi \in W. \tag{16}
\]

Set \( \varphi = u_{n+1} - u_n \). On subtracting (17) from (15) and by using the inequality (3), we get

\[
\|u_{n+1} - u_n\|^p \leq \frac{1}{C_p} \int_{\mathbb R^N} [f(u_{n+1},|\nabla u_n|^{p-2}\nabla u_n) - f(u_n,|\nabla u_n|^{p-2}\nabla u_n)](u_{n+1} - u_n) \, dx
\]

\[
+ \frac{1}{C_p} \int_{\mathbb R^N} [f(u_n,|\nabla u_n|^{p-2}\nabla u_n) - f(u_n,|\nabla u_n|^{p-2}\nabla u_n-1)](u_{n+1} - u_n) \, dx.
\]

By using \( (f_6) \), we obtain

\[
\|u_{n+1} - u_n\|^p \leq \frac{L_1}{C_p} \int_{\mathbb R^N} |u_{n+1} - u_n|^{p-1}(u_{n+1} - u_n) \, dx + \frac{L_2}{C_p} \int_{\mathbb R^N} |u_n - u_{n-1}|^{p-1}(u_{n+1} - u_n) \, dx.
\]

On simplification, we have

\[
\frac{C_p - L_1}{C_p} \|u_{n+1} - u_n\|^p \leq \frac{L_2}{C_p} \int_{\mathbb R^N} |u_n - u_{n-1}|^{p-1}(u_{n+1} - u_n) \, dx.
\]

Thanks to Hölder inequality, we get

\[
\|u_{n+1} - u_n\| \leq \left( \frac{L_2}{C_p - L_1} \right)^{1/p-1} \|u_n - u_{n-1}\| =: d\|u_n - u_{n-1}\|,
\]

where \( d = \left( \frac{L_2}{C_p - L_1} \right)^{1/p-1} \). Since \( d < 1 \), \( \{u_n\} \) is a Cauchy sequence in \( W \), there exists \( u \in W \) such that \( \{u_n\} \) converges to \( u \) in \( W \). Next, we will prove that \( u \) is a solution of the Problem (1). Since \( \|\nabla u_n\|_{C^{0,\beta}_{{\text{loc}}}({\mathbb R}^N)} \leq \rho_2 \), we have \( \|\nabla u_n|^{p-2}\nabla u_n\nabla \varphi \| \leq \rho_2^{p-1}\|\nabla \varphi \|. \) Then, by the help of Lebesgue’s Dominated Convergence Theorem, we get

\[
\int_{\mathbb R^N} |\nabla u_n|^{p-2}\nabla u_n\nabla \varphi \, dx \to \int_{\mathbb R^N} |\nabla u|^{p-2}\nabla u\nabla \varphi \, dx \quad \text{for all } \varphi \in W.
\]
In view of Brezis-Lieb lemma [4], we have
\[
\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\varphi \, dx \to \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi \, dx \quad \text{for all } \varphi \in W.
\]
By the help of Lebesgue Generalized Theorem [3], we get
\[
\int_{\mathbb{R}^N} f(u_n, |\nabla u_n|^{p-2}\nabla u_n)\varphi \, dx \to \int_{\mathbb{R}^N} f(u, |\nabla u|^{p-2}\nabla u)\varphi \, dx \quad \text{for all } \varphi \in W.
\]
Therefore, as \(n \to \infty\), (17) implies
\[
\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi \, dx - \int_{\mathbb{R}^N} f(u, |\nabla u|^{p-2}\nabla u)\varphi \, dx = 0,
\]
for all \(\varphi \in W\). This implies that \(u\) is the weak solution of Problem (1). By Lemma 3.6, \(u > 0\) in \(\mathbb{R}^N\).

\[\blacksquare\]

Acknowledgement

Authors would like to thank referee for his/her valuable comments and suggestions. The second author is supported by Science and Engineering Research Board, India under the grant no. MTR/2018/000233.

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Received: July 17, 2020  
Accepted for publication: October 27, 2020  
Communicated by: Diana Barseghyan