Singular Yamabe Metrics by Equivariant Reduction

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Abstract
We construct singular solutions to the Yamabe equation using a reduction of the problem in an equivariant setting. This provides a non-trivial geometric example for which the analysis is simpler than in Mazzeo-Pacard program. Our construction provides also a non-trivial example of a weak solution to the Yamabe problem involving an equation with (smooth) coefficients.

Keywords Singular solution · Yamabe problem · Warped product manifold · Equivariant solution

Mathematics Subject Classification Primary: 35J60 · Secondary: 35C20 · 58J60

1 Introduction
We consider the semilinear elliptic equation

\[- \Delta_g u + hu = u^p, \quad u > 0, \quad \text{on } (\mathcal{M}, g)\]  

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where \((\mathcal{M}, g)\) is a \(n\)--dimensional compact Riemannian manifold without boundary, \(p > 1\) and \(h\) is a \(C^1\)--real function on \(\mathcal{M}\) s.t. \(-\Delta_g + h\) is coercive i.e. its first eigenvalue is strictly positive.

We are interested in finding solutions which are singular at \(k\)--dimensional manifolds for some integer \(k \geq 1\).

In the critical case, i.e. \(p = 2_n^* - 1 \equiv \frac{n+2}{n-2}\), when \(h = R_g (R_g\) is the scalar curvature of \(g)\) the equation \((1)\) coincides with the celebrated Yamabe equation (omitting the harmless constant \(\frac{4(n-1)}{n-2}\) in front of the laplacian). The Yamabe problem \([17]\) consists in finding a metric \(\tilde{g}\) conformal to \(g\) whose scalar curvature is constant. If we write \(\tilde{g} = u^{\frac{4}{n-2}} g\) with \(u \in C^\infty (M)\) and \(u > 0\), from an analytic point of view it is equivalent to find a smooth positive solution to \((1)\). The problem has been completely solved through the works of Aubin \([1]\), Schoen \([13]\) and Trudinger \([16]\).

The so-called singular Yamabe problem, which is motivated by a deep theorem by Schoen and Yau on the structure theory of locally conformally flat manifolds \([15]\) asserts to solve equation \((1)\) on \(\mathcal{M} \setminus \Sigma\) where \(\Sigma\) is a closed set in \(\mathcal{M}\) and \(u(x)\) blows up as \(x\) approaches \(\Sigma\). The existence of such singular solutions to \((1)\) has been solved by Mazzeo and Pacard in \([5]\) in the case where \(\Sigma\) is assumed to be a smooth submanifold which is a finite disjoint union of smooth submanifolds \(\Sigma_i\) whose dimensions \(\kappa_i\) satisfy \(n - \frac{2p+2}{p-1} < \kappa_i \leq n - \frac{2p}{p-1}\). We refer also the reader to \([7,10,11]\) for related results. The case of isolated singularities has been considered in the seminal paper of Schoen \([14]\) (see also \([6]\)). In all the previous results, one wants to prescribe the scalar curvature to be \(+1\), which is the difficult case in this kind of problem. The singular solutions correspond to metrics of (positive) constant scalar curvature which are complete in a neighbourhood of the singular set.

The proof in \([5]\) is quite involved and it is somehow desirable to have a simpler setting, providing geometrically meaningful singular solutions. In the present work, we provide a non trivial example of a geometric singular solution, in a much simpler setting than the original construction in \([5]\). Our idea is to rely on an equivariant approach. Additionally we would like to emphasize that the construction of Mazzeo and Pacard relies on the a priori smoothness of the singular set \(\Sigma\). This is due to the fact that they use Fermi coordinates. In our case, one can assume much less regularity on the singular set.

For any integer \(0 \leq k \leq n - 3\) let \(2^*_{n,k} = \frac{2(n-k)}{n-k-2}\) be the \((k+1)\)--st critical exponent. We remark that \(2^*_{n,k} = 2^*_{n-k,0}\) is nothing but the critical exponent for the Sobolev embedding \(H^1_g (\mathcal{M}) \hookrightarrow L^2_g (\mathcal{M})\), when \((\mathcal{M}, g)\) is a \((n-k)\)--dimensional Riemannian manifold. In particular, \(2^*_{n,0} = \frac{2n}{n-2}\) is the usual Sobolev critical exponent.

In order to reduce the problem, we will consider the background manifold \(\mathcal{M}\) to be given by a warped product. Let \((M, g)\) and \((K, \kappa)\) be two Riemannian manifolds of dimensions \(N\) and \(k\), respectively. Let \(\omega \in C^2 (M)\), \(\omega > 0\) be a given function. The warped product \(\mathcal{M} = M \times_\omega K\) is the product (differentiable) \(n\)--dimensional \((n = N + k)\) manifold \(M \times K\) endowed with the Riemannian metric \(g = g + \omega^2 \kappa\). The function \(\omega\) is called the warping function. For example, every surface of revolution (not crossing the axis of revolution) is isometric to a warped product, with \(M\) the generating curve, \(K = S^1\) and \(\omega(x)\) the distance from \(x \in M\) to the axis of revolution.
It is not difficult to check that if \( u \in C^2(M \times \omega K) \) then
\[
\Delta_g u = \Delta_g u + \frac{k}{\omega} g(\nabla_g u, \nabla_g \omega) + \frac{1}{\omega^2} \Delta_k u. \tag{2}
\]
Assume \( h \) is invariant with respect to \( K \), i.e. \( h(x, y) = h(x) \) for any \((x, y) \in M \times K\). If we look for solutions to (1) which are invariant with respect to \( K \), i.e. \( u(x, y) = v(x) \) then by (2) we immediately deduce that \( u \) solves (1) if and only if \( v \) solves
\[
-\Delta_g v - \frac{k}{\omega} g(\nabla_g v, \nabla_g \omega) + h v = v^p \quad \text{in} \quad (M, g), \tag{3}
\]
or equivalently
\[
-\text{div}_g (\omega^k \nabla g v) + \omega^k h v = \omega^k v^p, \quad v > 0 \quad \text{in} \quad (M, g).
\]
It is clear that if \( v \) is a solution to problem (3) which is singular at a point \( \xi_0 \in M \) then \( u(x, y) = v(x) \) is a solution to problem (1) which is singular only on the fiber \( \{\xi_0\} \times K \), which is a \( k \)-dimensional submanifold of \( M \times \omega K \). It is important to notice that the fiber \( \{\xi_0\} \times K \) is totally geodesic in \( M \times \omega K \) (and in particular a minimal submanifold of \( M \times \omega K \)) if \( \xi_0 \) is a critical point of the warping function \( \omega \).

Therefore, we are led to consider the more general anisotropic problem
\[
-\text{div}_g (a \nabla g v) + ah v = av^p, \quad u > 0 \quad \text{in} \quad (M, g) \tag{4}
\]
where \((M, g)\) is a \( N \)-dimensional compact Riemannian manifold, \( p > 1, h \in C^1(M) \) and \( a \in C^2(M) \) with \( \min_M a > 0 \).

Our main result reads as follows.

**Theorem 1.1** Assume that the anisotropic operator 
\(-\text{div}_g (a \nabla g \cdot) + ah\) is coercive in 
\(H^1(M)\). If \(\frac{N}{N-2} < p < \frac{N+2}{N-2}\), then the problem (4) has a solution which is singular at a point \( \xi_0 \in M \).

As a consequence of the previous theorem and the above discussion, we deduce

**Theorem 1.2** Assume that \((M, g)\) is a warped product \( M \times \omega K \) and that the scalar curvature \( R_g \) depends only on \( x \) and satisfies \( R_g > 0 \). If \( 0 < k < \frac{n-2}{2} \) then there exists a solution invariant with respect to \( K \) of
\[
-\Delta_g u + R_g u = u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{in} \quad (M, g)
\]
which is singular on \( \Sigma = \{\xi_0\} \times K \), where \( \xi_0 \) is any point on \( M \). Furthermore, if \( \xi_0 \) is a critical point of \( \omega \) then the submanifold \( \{\xi_0\} \times K \) is minimal in \( M \).

**Proofs of Theorem 1.1 and 1.2.** The proofs of both theorems follow the same lines as in [5], Section 10.2. Notice first that the construction in Theorem 1.1 is itself a consequence of Theorem 1.3. Indeed the linearization around an approximate solution has
the same structure as the one in Theorem 1.3. Under rescalings, possible extra terms in the linearization disappear and one ends up with the model treated in Theorem 1.3. To be sure that the true solutions (for both theorems) remain positive on all of \( M \setminus \Sigma \) where \( \Sigma \) is the singular set, one just needs to check the maximum principle for those equations. This is true under the coercivity assumption in Theorem 1.1, which follows itself from the assumption \( R_g > 0 \) in Theorem 1.2.

\[ \square \]

Remark 1 Note that the regularity of the manifold \( K \), hence the singular set \( \Sigma \) plays almost no role here in the construction, except for the reduction itself. This is a non-trivial example of a singular solution on a non-smooth singular set.

The proof of Theorem (1.1) follows the same strategy developed in [5]. In particular, we will replace the \( N \)−dimensional manifold \( M \) by a bounded smooth domain \( \Omega \) in \( \mathbb{R}^N \) and we will focus on the Dirichlet boundary problem

\[
\begin{aligned}
-\text{div} (a \nabla u) + ah u &= au^p \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
u > 0 &\quad \text{in } \Omega.
\end{aligned}
\]

Here \( h \in C^1(M) \), \( a \in C^2(\bar{\Omega}) \) with \( \min_\Omega a > 0 \) and the anisotropic operator \(-\text{div}(a \nabla u) + ah u\) is coercive in \( H^1_0(\Omega) \). We will show the following result

**Theorem 1.3** If \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \), then the problem (5) has a solution which is singular at a point \( \xi_0 \in \Omega \).

The modification in the arguments to solve the problem on the manifold instead of in the domain are minor and are described in the last section of [5].

The paper is organized as follows. The proof of Theorem 1.3 is carried out in Sect. 3 and relies on the linear theory studied in Sect. 4 together with a contraction mapping argument developed in Sect. 5. All the necessary technical tools are contained in Sect. 2 and in the Appendix 1.

2 Preliminaries

2.1 Function Spaces

For \( \sigma > 0 \) we let \( N_\sigma \) to be the ball \( B_\sigma(\xi_0) \). For \( \alpha \in (0, 1), s \in (0, \sigma), k \in \mathbb{N} \cup \{0\} \) and \( v \in \mathbb{R} \) we define the seminorms

\[
|w|_{k,\alpha,s} := \sum_{j=0}^{k} s^j \sup_{N_\sigma \setminus N_\sigma^{2s}} |\nabla^j w| + s^{k+\alpha} \sup_{x,x' \in N_\sigma \setminus N_\sigma^{2s}} \frac{|\nabla^k w(x) - \nabla^k w(x')|}{|x - x'|^\alpha},
\]

and the weighted Hölder norm (\( \sigma > 0 \) is fixed)

\[
\|w\|_{C^{k,\alpha}_v} := |w|_{C^{k,\alpha}(\bar{\Omega} \setminus N_\sigma^{2s})} + \sup_{0<s<\sigma} s^{-v}|w|_{k,\alpha,s}.
\]
The weighted Hölder space \( C^{k,\alpha}_{\nu}(\Omega \setminus \Sigma) \) is defined by (here \( \Sigma = \{\xi_0\} \))

\[
C^{k,\alpha}_{\nu}(\Omega \setminus \Sigma) := \left\{ w \in C^{k,\alpha}_{loc}(\tilde{\Omega} \setminus \Sigma) : \| w \|_{C^{k,\alpha}_{\nu}} < \infty \right\}.
\]

The subspace of \( C^{k,\alpha}_{\nu}(\Omega \setminus \Sigma) \) with Dirichlet boundary conditions will be denoted by

\[
C^{k,\alpha}_{\nu,D}(\Omega \setminus \Sigma) := \left\{ w \in C^{k,\alpha}_{\nu}(\Omega \setminus \Sigma) : w = 0 \text{ on } \partial \Omega \right\}.
\]

The space \( C^{k,\alpha}_{\nu,\nu'}(\mathbb{R}^N \setminus \{0\}) \) is defined by

\[
\| w \|_{C^{k,\alpha}_{\nu,\nu'}(\mathbb{R}^N \setminus \{0\})} := \| w \|_{C^{k,\alpha}_{\nu}(B_2 \setminus \{0\})} + \sup_{r \geq 1} (r^{-\nu'} \| w(r \cdot) \|_{C^{k,\alpha}_{\nu}(B_2 \setminus B_1)}).
\]

We now list some useful properties of the space \( C^{k,\alpha}_{\nu}(\Omega \setminus \Sigma) \), see e.g. [5] and the book [12].

**Lemma 2.1** The following properties hold.

(i) If \( w \in C^{k+1,\alpha}_{\gamma}(\Omega \setminus \Sigma) \) then \( \nabla w \in C^{k,\alpha}_{\gamma-1}(\Omega \setminus \Sigma) \).

(ii) If \( w \in C^{k+1,0}_{\gamma}(\Omega \setminus \Sigma) \) then \( w \in C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) \) for every \( \alpha \in [0, 1) \).

(iii) For every \( w_i \in C^{k,\alpha}_{\gamma_i}(\Omega \setminus \Sigma) \), \( i=1,2 \), we have

\[
\| w_1 w_2 \|_{k,\gamma_1+\gamma_2,\alpha} \leq C \| w_1 \|_{k,\gamma_1,\alpha} \| w_2 \|_{k,\gamma_2,\alpha},
\]

for some \( C > 0 \) independent of \( w_1, w_2 \).

(iv) There exists \( C > 0 \) such that for every \( w \in C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) \) with \( w > 0 \) in \( \tilde{\Omega} \setminus \Sigma \) we have

\[
\| w^p \|_{k,\gamma,\alpha} \leq C \| w \|_{k,\gamma,\alpha}^p.
\]

**2.2 The Singular Solution**

The building block for our theory is the existence of a singular solution with different behaviour at the origin and at infinity. The following theorem provides such a solution.

**Theorem 2.2** ([5]). Suppose that \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \). Then for every \( \beta > 0 \) there exists a unique radial solution \( u \) to

\[
\begin{align*}
-\Delta u &= u^p \quad \text{in } \mathbb{R}^N \setminus \{0\} \\
u &> 0 \quad \text{in } \mathbb{R}^N \setminus \{0\} \\
limit_{|x| \to 0} u(x) &= \infty,
\end{align*}
\]

such that

\[
\lim_{r \to \infty} r^{N-2} u(r) = \beta, \quad \lim_{r \to 0^+} r^{\frac{2}{p-1}} u(r) = c_p := [k(p, N)]^{\frac{1}{p-1}},
\]

[Springer]
where

\[ k(p, N) = \frac{2}{p - 1} \left( N - \frac{2p}{p - 1} \right). \]

Let \( u \) be a singular radial solution to (7). Then \( u_\epsilon(x) := \epsilon^{-\frac{2}{p-1}} u(\frac{x}{\epsilon}) \) is also a solution to (7). Note that

\[ u_\epsilon(x) \leq C(\delta, u) \epsilon^{N-2 - \frac{2}{p-1}} \text{ for } |x| \geq \delta, \]

which shows that \( u_\epsilon \to 0 \) locally uniformly in \( \mathbb{R}^N \setminus \{0\} \). Due to this scaling and the asymptotic behavior of \( u \) at infinity, for a given \( \alpha > 0 \), we can find a solution \( u_1 \) such that

\[ r^2 u_1^{p-1}(r) \leq \alpha \text{ on } (1, \infty). \]

### 2.3 The Linearized Operator Around the Singular Solution

We consider the linearized operator

\[ L_1 = \Delta + pu_1^{p-1} \]

where in polar coordinates we denote

\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\theta. \]

Following [5], we recall that \( \gamma_j \) is an indicial root of \( L_1 \) at 0 if \( L_1(|x|^\gamma_j \varphi_j) = o(|x|^{\gamma_j - 2}) \), where \( \varphi_j \) is the \( j \)-th eigenfunction of \(-\Delta_\theta\) on \( S^{N-1} \), that is \(-\Delta_\theta \varphi_j = \lambda_j \varphi_j\),

\[ \lambda_0 = 0, \quad \lambda_j = N - 1, \quad \text{for } j = 1, \ldots, N, \]

and so on. Setting

\[ A_p := p \lim_{r \to 0} r^2 u_1^{p-1}(r) = pk(p, N), \]

we have that

\[ \gamma_j^\pm = \frac{1}{2} \left[ 2 - N \pm \sqrt{(N - 2)^2 + 4(\lambda_j - A_p)} \right]. \]

For \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \) we have that (\( \Re \) denotes the real part)

\[ 2 - N < -\frac{2}{p-1} < \Re(\gamma_0^-) \leq \frac{2 - N}{2} \leq \Re(\gamma_0^+) < 0, \]
and
\[ \gamma_j^- < -\frac{2}{p-1} \text{ for } j \geq 1. \]

Since \( \lim_{r \to \infty} r^2 u_1^{p-1}(r) = 0 \), the indicial roots of \( L_1 \) at infinity are the same as for the \( \Delta \) itself. These values are given by
\[ \tilde{\gamma}_j^\pm = \frac{1}{2} \left[ 2 - N \pm \sqrt{(N-2)^2 + 4\lambda_j} \right]. \]

We shall choose \( \mu, \nu \) in the region
\[ \frac{-2}{p-1} < \nu < \min \left\{ \frac{-2}{p-1} + 1, \Re(\gamma_0^-) \right\} \leq \frac{2 - N}{2} \leq \Re(\gamma_0^+) < \mu < 0, \tag{9} \]
so that \( \mu + \nu = 2 - N \).

We have the following propositions whose proofs can be found in [5]

**Proposition 2.3** Let \( w \in C^{2,\alpha}_\mu,0(\mathbb{R}^N \setminus \{0\}) \) be a solution to \( L_1 w = 0 \). Then \( w \equiv 0 \).

(see Proposition 3 in [5]).

**Proposition 2.4** Let \( w \in C^{2,\alpha}_{\gamma,\gamma}(\mathbb{R}^N \setminus \{0\}) \) be a solution to
\[ \Delta w + \frac{A_p}{r^2} w = 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \]
where \( A_p \) is given by (8). If \( \gamma \) is not an indicial root of the operator \( \Delta + \frac{A_p}{r^2} \) then \( w \equiv 0 \).

(See the proof of Lemma 8 in [5]).

### 3 A Scheme of the Proof

Let \( \Sigma = \{\xi_0\} \subset \Omega \). To construct a solution to (5) which is singular precisely at the point \( \xi_0 \), we start by constructing an approximate solution to (5) which is singular exactly on \( \Sigma \). For \( \sigma > 0 \) small (to be chosen later) let us first fix a non-negative cut-off function \( \chi \in C^\infty_\sigma(B_\sigma(\xi_0)) \) such that \( \chi = 1 \) in \( B_\frac{\sigma}{2}(\xi_0) \). An approximate solution \( \tilde{u}_\varepsilon \) is defined by
\[ \tilde{u}_\varepsilon(x) = \chi(x)u_\varepsilon(x - \xi_0) = \varepsilon^{-\frac{2}{p-1}} \chi(x)u_1(\frac{x - \xi_0}{\varepsilon}). \]

We shall look for positive solutions of the form \( u = \tilde{u}_\varepsilon + v \). Then, \( v \) has to satisfy
\[ L_\varepsilon v + f_\varepsilon + Q(v) = 0, \tag{10} \]
where the linear operator $L_\varepsilon$ is
\[ L_\varepsilon v := \text{div} (a \nabla v) + a [p \bar{u}_\varepsilon^{p-1} - h] v, \]
the error term is
\[ f_\varepsilon := \text{div} (a \nabla \bar{u}_\varepsilon) - ah \bar{u}_\varepsilon + a \bar{u}_\varepsilon^p, \]
and the non-linear term $Q$ is
\[ Q(v) = a [\bar{u}_\varepsilon + v]^p - \bar{u}_\varepsilon^p - p \bar{u}_\varepsilon^{p-1} v]. \]

To prove existence of solution to (10) we will use a fixed point argument on the space $C^{2,\alpha}_{2,\alpha}(\Omega \setminus \Sigma)$ for a suitable $\nu$. We note that if $v \in C^{2,\alpha}_{2,\alpha}(\Omega \setminus \Sigma)$ solves (10) then by maximum principle we have that $\bar{u}_\varepsilon + v > 0$ in $\Omega$. This is a simple consequence of the fact that we will choose $\nu > -\frac{2}{p-1}$, and therefore, $\bar{u}_\varepsilon + v > 0$ in a small neighborhood of $\Sigma$, thanks to the asymptotic behavior of $\bar{u}_\varepsilon$, around the origin and the coercivity assumption on $h$.

In the following, given two functions $a_\varepsilon$ and $b_\varepsilon$ we write $a_\varepsilon \approx b_\varepsilon$ if for some constants $c_1, c_2 > 0$ (independent of $\varepsilon$) we have $a_\varepsilon \leq c_1 b_\varepsilon \leq c_2 a_\varepsilon$.

First of all, we estimate the size of the error term.

**Lemma 3.1** The error $f_\varepsilon$ satisfies
\[ \|f_\varepsilon\|_{C^{0,\alpha}_{\gamma-2}} \leq C_\gamma \max \left\{ \varepsilon^{1-\gamma-\frac{2}{p-1}}, \varepsilon^{N-\frac{2p}{p-1}} \right\} \quad \text{for } \varepsilon > 0 \text{ small}, \]
for every $\gamma < 1 - \frac{2}{p-1}$.

**Proof** We only estimate the first term in (6), the estimate for the second term should having the same order as of the first one. It follows that
\[ u_\varepsilon(x - \xi_0) \approx \begin{cases} |x - \xi_0|^{-\frac{2}{p-1}} & \quad \text{for } |x - \xi_0| \leq \varepsilon \\ \varepsilon^{N-\frac{2p}{p-1}} |x - \xi_0|^{N-2} & \quad \text{for } |x - \xi_0| \geq \varepsilon, \end{cases} \]
and
\[ |\nabla u_\varepsilon(x - \xi_0)| \leq C \begin{cases} |x - \xi_0|^{-\frac{2}{p-1}-1} & \quad \text{for } |x - \xi_0| \leq \varepsilon \\ \varepsilon^{N-\frac{2p}{p-1}} |x - \xi_0|^{N-1} & \quad \text{for } |x - \xi_0| \geq \varepsilon. \end{cases} \]

We write
\[ f_\varepsilon = a [\Delta \bar{u}_\varepsilon + \bar{u}_\varepsilon^p] + \nabla \bar{u}_\varepsilon \cdot \nabla a - ah \bar{u}_\varepsilon. \]
Since $\chi \equiv 1$ in a small neighborhood of $\xi_0$, we have

$$a|\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon^p| \leq C\varepsilon^{N-\frac{2p}{p-1}}$$

and $u_\varepsilon(x - \xi_0)|\nabla \chi \cdot \nabla a| \leq \varepsilon^{N-\frac{2p}{p-1}}$.

Moreover,\[\begin{align*}
|x - \xi_0|^2 \chi &|\nabla u_\varepsilon(x - \xi_0)| |\nabla a(x)| \leq C\varepsilon^{1-\gamma - \frac{2}{p-1}}, \\
|x - \xi_0|^2 \gamma \tilde{u}_\varepsilon(x - \xi_0) &\leq C\varepsilon^{2-\gamma - \frac{2}{p-1}}.
\end{align*}\]

The lemma follows. \qed

Next, we use the linear theory of $L_\varepsilon$ developed in the Sect. 4 and, applying the inverse of $L_\varepsilon$, that is $G_\varepsilon$, we rewrite the above equation (10) as

$$v + G_\varepsilon f_\varepsilon + G_\varepsilon Q(v) = 0.$$

The crucial fact we shall use is that the norm of $G_\varepsilon$ is uniformly bounded if $\varepsilon$ is sufficiently small.

By Lemma 3.1, the error $f_\varepsilon$ satisfies the estimate

$$\|f_\varepsilon\|_{0,\alpha,\nu,2} \leq C\varepsilon^q, \quad q := \min\left\{N - \frac{2p}{p-1}, 1 - \nu - \frac{2}{p-1}\right\}.$$

Then, there exists $C_0 > 0$ such that $\|G_\varepsilon f_\varepsilon\|_{2,\alpha,\nu} \leq C_0 \varepsilon^q$. This suggests to work on the ball

$$B_{\varepsilon,M} = \left\{v \in C^{2,\alpha}_v : \|v\|_{2,\alpha,\nu} \leq M\varepsilon^q\right\},$$

for some $M > 2C_0$ large. In Sect. 5 we shall show that the map $v \mapsto G_\varepsilon[f_\varepsilon + Q(v)]$ is a contraction on the ball $B_{\varepsilon,M}$ when $M$ is large and $\varepsilon$ is small enough. That will conclude our proof.

4 The Linear Operator $L_\varepsilon$

4.1 Injectivity of $L_\varepsilon$ on $C^{2,\alpha}_{\mu,D}(\Omega \setminus \Sigma)$

In this section we study injectivity of the linearized operator

$$L_\varepsilon w := \text{div}(a \nabla w) + a[p\tilde{u}_\varepsilon^{p-1} - h]w.$$

We shall use the following notations:

$$\Omega_\varepsilon := \Omega \setminus B_\varepsilon(\xi_0), \quad f^+ := \max\{f, 0\}, \quad f^- := \min\{f, 0\}.$$
Lemma 4.1 After a suitable normalization of $u_1$, the operator $L_\varepsilon$ satisfies the maximum principle in $\Omega_\varepsilon$ for $\varepsilon > 0$ small. More precisely, if $w \in H^1(\Omega_\varepsilon)$ satisfies

$$\begin{cases}
L_\varepsilon w \geq 0 & \text{in } \Omega_\varepsilon \\
w \leq 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}$$

then $w \leq 0$ in $\Omega_\varepsilon$.

Proof The crucial fact we shall use is that the operator

$$v \mapsto -\text{div}(a \nabla v) + ahv,$$

is coercive, that is

$$\int_{\Omega} a|\nabla v|^2 dx + \int_{\Omega} ahv^2 dx \geq \lambda_1 \int_{\Omega} v^2 dx \text{ for every } v \in H^1_0(\Omega), \lambda_1 > 0.$$

Consequently, for some $c_0 > 0$ we have

$$\int_{\Omega} a[|\nabla v|^2 + hv^2] dx \geq c_0 \int_{\Omega} |\nabla v|^2 dx \text{ for every } v \in H^1_0(\Omega).$$

Since $w \leq 0$ on the boundary $\partial \Omega_\varepsilon$, by extending $w^+$ by 0 on $B_\varepsilon(\xi_0)$ we see that $w^+ \in H^1_0(\Omega)$. Multiplying the inequality $L_\varepsilon w \geq 0$ by $w^+$, and then integrating by parts we obtain

$$\int_{\Omega} a[|\nabla w^+|^2 + h(w^+)^2 - p\tilde{u}_\varepsilon^{p-1}(w^+)^2] dx = 0.$$

We also have

$$\int_{\Omega} \frac{(w^+)^2}{|x - \xi_0|^2} dx \leq \frac{4}{(N-2)^2} \int_{\Omega} |\nabla w^+|^2 dx.$$

If we normalize $u_1$ in such a way that

$$pa\tilde{u}_\varepsilon^{p-1} \leq c_0 \frac{(N-2)^2}{8} \frac{1}{|x - \xi_0|^2} \text{ on } \Omega_\varepsilon,$$

then we have

$$\int_{\Omega} |\nabla w^+|^2 dx = 0.$$

We conclude the lemma. \qed

Remark 2 The above proof shows that $L_\varepsilon$ satisfies the maximum principle in $B_\sigma(\xi_0) \setminus B_\varepsilon(\xi_0)$ for $\varepsilon > 0$ small.
Lemma 4.2 Fix $\varepsilon_0 > 0$ such that $L_\varepsilon$ satisfies the maximum principle on $\Omega_\varepsilon$. Let $2 - N < \gamma < 0$ be fixed. Let $w_\varepsilon$ be a solution to $L_\varepsilon w_\varepsilon = f_\varepsilon$ on $\Omega_\varepsilon$ for some $f_\varepsilon \in C^{0,\alpha}_\gamma(\Omega_\varepsilon)$, and $0 < \varepsilon \leq \varepsilon_0$. Assume that $w_\varepsilon = 0$ on $\partial \Omega$. Then there exists $C > 0$ such that

$$\|w_\varepsilon\|_{2,\alpha,\gamma} \leq C \left( \|f_\varepsilon\|_{0,\alpha,\gamma-2} + \varepsilon^{-\gamma} \|w_\varepsilon\|_{C^0(\partial B_\varepsilon(\xi_0))} \right).$$

(14)

Proof For $\phi(x) := |x - \xi_0|^{\gamma}$ we have

$$\Delta \phi(x) = c_{N,\gamma}|x - \xi_0|^{\gamma-2}, \quad c_{N,\gamma} := \gamma(N + \gamma - 2) < 0.$$

Since

$$\nabla a \cdot \nabla \phi - ah \phi = O(|x - \xi_0|^{\gamma-1}),$$

for $\sigma > 0$ small we have that

$$a \Delta \phi + \nabla a \cdot \nabla \phi - ah \phi \leq \frac{c_{N,\gamma}}{2} a|x - \xi_0|^{\gamma-2} \text{ on } B_\sigma(\xi_0).$$

This shows that for a suitable choice of $u_1$, we have for some $\delta > 0$

$$L_\varepsilon \phi(x) \leq -\delta |x - \xi_0|^{\gamma-2} \text{ on } \Omega := B_\sigma(\xi_0) \setminus B_\varepsilon(\xi_0).$$

Therefore, we can choose $c_{1,\varepsilon} \approx \|f_\varepsilon\|_{0,\alpha,\gamma-2}$ so that

$$L_\varepsilon (w_\varepsilon + c_{1,\varepsilon} \phi) \leq 0 \text{ on } \Omega.$$

We can also choose

$$c_{2,\varepsilon} \approx \varepsilon^{-\gamma} \|w_\varepsilon\|_{C^0(\partial B_\varepsilon(\xi_0))} + \|w_\varepsilon\|_{C^0(\partial B_\sigma(\xi_0))} =: c_{3,\varepsilon} + c_{4,\varepsilon},$$

so that

$$w_\varepsilon + (c_{1,\varepsilon} + c_{2,\varepsilon}) \phi \geq 0 \text{ on } \partial \Omega.$$

Then by the maximum principle we have that (to get the other inequality use $-\phi$)

$$|w_\varepsilon| \leq (c_{1,\varepsilon} + c_{2,\varepsilon}) \phi \text{ in } \Omega.$$

Since, $L_\varepsilon w_\varepsilon = f_\varepsilon$ in $\Omega_\varepsilon \setminus \Omega$, and $w_\varepsilon = 0$ on $\partial \Omega$, we get that

$$|w_\varepsilon(x)| \lesssim (c_{1,\varepsilon} + c_{2,\varepsilon}) \text{ for } x \in \Omega_\varepsilon \setminus \Omega.$$

We claim that

$$c_{4,\varepsilon} \lesssim c_{3,\varepsilon} + \|f_\varepsilon\|_{0,\alpha,\gamma-2}.$$
We assume by contradiction that the above claim is false. Then there exists a family of solutions $w_\ell = w_{\varepsilon \ell}$ to $L_{\varepsilon \ell} w_\ell = f_\ell$ with $0 < \varepsilon_\ell < \varepsilon_0$, $f_\ell \in C^{0,\alpha}_0(\Omega_{\varepsilon \ell})$, $w_\ell = 0$ on $\partial \Omega$ such that

$$c_{4,\varepsilon \ell} = 1 \quad \text{and} \quad c_{3,\varepsilon \ell} + \| f_\ell \|_{0,\alpha,\gamma - 2} \to 0.$$  \hfill (15)

Then, up to a subsequence, $\Omega_{\varepsilon \ell} \to \tilde{\Omega}_{\varepsilon}$, where

$$\tilde{\Omega}_{\varepsilon} = \Omega \setminus \{ \xi_0 \} \quad \text{if} \quad \varepsilon > 0, \quad \text{and} \quad \tilde{\Omega}_{\varepsilon} = \Omega \setminus B_\varepsilon(\xi_0) \quad \text{if} \quad \varepsilon > 0.$$  

From the estimates on $w_\ell$ we see that $w_\ell \to w$ in $\tilde{\Omega}_{\varepsilon}$. Moreover, $w$ satisfies

$$\begin{cases} L_\varepsilon w = \text{div} (a \nabla w) - ahw + p\bar{u}_\varepsilon^{p-1}w = 0 & \text{in} \ \tilde{\Omega}_{\varepsilon} \\ w = 0 & \text{on} \ \partial \tilde{\Omega}_{\varepsilon}. \end{cases}$$

Here, for $\varepsilon = 0$ the function $\bar{u}_\varepsilon$ is considered to be identically zero.

If $\varepsilon > 0$ then by Lemma 4.1 we get that $w \equiv 0$. Next we consider the case $\varepsilon = 0$. We have that $w(x) = O(|x - \xi_0|^\gamma)$, and hence the singularity at $\xi_0$ is removable (note that $\gamma > 2 - N$), that is, $L_\varepsilon w = 0$ weakly in $\Omega$. Thus, we can use coercivity hypothesis on $h$ to conclude that $w \equiv 0$. This contradicts the first condition in (15).

In this way we have that there exists $C > 0$ independent of $\varepsilon$, but depending only on the right hand side of (14) such that

$$|w_\varepsilon| \leq C\phi \quad \text{in} \ \Omega_\varepsilon.$$  

The desired estimate follows from Lemma 5.5 and a scaling argument (see e.g. [12, Chapter 2.2.1]). \hfill \square

**Lemma 4.3** There exists $\varepsilon_0 > 0$ sufficiently small such that if $\varepsilon < \varepsilon_0$ then

$$L_\varepsilon : C^{2,\alpha}_\mu(\Omega \setminus \Sigma) \to C^{0,\alpha}_\mu(\Omega \setminus \Sigma)$$

is injective.

**Proof** We assume by contradiction that $L_\varepsilon$ is not injective for some $\varepsilon_\ell \to 0$. Let $w_\ell \in C^{2,\alpha}_\mu(\Omega \setminus \Sigma)$ be a non-trivial solution to $L_\varepsilon w_\ell = 0$. We normalize $w_\ell$ so that

$$\max_{\partial \Omega_\varepsilon} \rho(x)^{-\mu} |w_\ell(x)| = (\varepsilon_\ell)^{-\mu} \max_{\partial \Omega_\varepsilon} |w_\ell(x)| = 1,$$  \hfill (16)

where $\rho(x) = |x - \xi_0|$ in a small neighborhood of $\xi_0$, and outside it is a smooth positive function. Then by Lemma 4.2 we get that

$$\sup_{\Omega_\varepsilon} \left( \rho(x)^{-\mu} |w_\ell(x)| + \rho(x)^{-\mu + 1} |\nabla w_\ell(x)| \right) \leq C.$$  \hfill (17)
We set
\[ \tilde{w}_\ell(x) = (\varepsilon_\ell)^{-\mu} w_\ell(\varepsilon_\ell^2 x + \xi_0), \quad |x| < R_\ell := \frac{\sigma}{2\varepsilon_\ell}. \]

Then
\[ \Delta \tilde{w}_\ell(x) + pu_1^{p-1} \tilde{w}_\ell(x) = f_\ell(x), \]

where
\[ f_\ell(x) := (\varepsilon_\ell)^2 h(y) \tilde{w}_\ell(x) - \varepsilon_\ell a(y)^{-1} \nabla a(y) \cdot \nabla \tilde{w}_\ell(x). \]

It follows from (17) that \( \tilde{w}_\ell \to \tilde{w}_\infty \) and \( f_\ell \to 0 \) in \( C^{2}_{loc}(\mathbb{R}^N \setminus B_1) \) and \( C^{1}_{loc}(\mathbb{R}^N \setminus B_1) \) respectively.

Next we show that \( \tilde{w}_\ell \) is bounded in \( C^{2}_{loc}(B_2 \setminus \{0\}) \). To this end it suffices to prove that
\[ S_\ell = \sup_{B_2} \left( |x|^{-\mu} |\tilde{w}_\ell(x)| + |x|^{-\mu+1} |\nabla \tilde{w}_\ell(x)| \right) \leq C. \]

We assume by contradiction that the above supremum is not uniformly bounded. Let \( 0 \neq x_\ell \in B_2 \) be such that
\[ S_\ell \approx |x_\ell|^{-\mu} |\tilde{w}_\ell(x_\ell)| + |x_\ell|^{1-\mu} |\nabla \tilde{w}_\ell(x_\ell)|. \]

We claim that \( |x_\ell| \to 0. \) On the contrary, if \( x_\ell \to x_\infty \neq 0 \), then setting \( \tilde{w}_\ell = \frac{w_\ell}{S_\ell} \) we see that \( \tilde{w}_\ell \to \tilde{w}_\infty \), where
\[ L_1 \tilde{w}_\infty = 0 \quad \text{in} \quad B_2 \setminus \{0\}, \quad \tilde{w}_\infty \equiv 0 \quad \text{in} \quad B_2 \setminus B_1. \]

Therefore, \( \tilde{w}_\infty \equiv 0 \) in \( B_2 \), which contradicts to
\[ |x_\infty|^{-\mu} |\tilde{w}_\infty(x_\infty)| + |x_\infty|^{1-\mu} |\nabla \tilde{w}_\infty(x_\infty)| \approx 1. \]

Thus we get that \( x_\ell \to 0. \)

Now we set
\[ v_\ell(x) = \frac{r_\ell^{-\mu} \tilde{w}_\ell(r_\ell x)}{S_\ell}, \quad r_\ell := |x_\ell|. \]

Then, for every \( \delta > 0 \) and \( \ell \) large we have
\[ L_1 v_\ell = o_\ell(1), \quad |x|^\mu |v_\ell| + |x|^{1+\mu} |\nabla v_\ell| \leq C \quad \text{for} \quad \delta \leq |x| \leq \frac{1}{\delta}. \]
Therefore, up to a subsequence, \( v_\ell \to v_\infty \) where \( v_\infty \) satisfies

\[
\Delta v_\infty + \frac{p \kappa(p, N)}{|x|^2} v_\infty = 0, \quad |v_\infty(x)| \leq C |x|^{\mu} \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

Hence, by Proposition 2.4 we have \( v_\infty \equiv 0 \), a contradiction to \( \max_{\partial B_1} (|v_\infty| + |\nabla v_\infty|) \approx 1 \). This proves that \( S_\ell \leq C \), and consequently we obtain that \( \tilde{w}_\ell \to \tilde{w}_\infty \) in \( C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \). Then the limit function \( \tilde{w}_\infty \) would satisfy

\[
\Delta \tilde{w}_\infty + p u_{\frac{p-1}{2}} \tilde{w}_\infty = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad \tilde{w}_\infty \in C^{2,\alpha}_{\mu,\mu}(\mathbb{R}^N \setminus \{0\}).
\]

Then by Proposition 2.3 we have \( \tilde{w}_\infty \equiv 0 \), a contradiction to (16). \( \square \)

### 4.2 Uniform Surjectivity of \( L_\varepsilon \) on \( C^{2,\alpha}_{\mu,D}(\Omega \setminus \Sigma) \)

Instead of using the general theory of edge operators as developed in [4], we shall use the notes of Pacard [8,9] and Pacard-Rivière [12] for edge operators with point singularity. Denoting \( \rho(x) := |x - \xi_0| \), the weighted space \( L^2_\delta(\Omega \setminus \Sigma) \) is defined by (we may also simply write \( L^2_\delta \) or \( L^2_\delta(\Omega \setminus \Sigma) \))

\[
L^2_\delta(\Omega \setminus \Sigma) := \left\{ w \in L^2_{\text{loc}}(\Omega \setminus \Sigma) : \int_\Omega \rho^{-2-2\delta} |w|^2 dx < \infty \right\}.
\]

Let \( L^2_{-\delta}(\Omega \setminus \Sigma) \) be the dual of \( L^2_\delta(\Omega \setminus \Sigma) \) with respect to the pairing

\[
L^2_\delta(\Omega \setminus \Sigma) \times L^2_{-\delta}(\Omega \setminus \Sigma) \ni (w_1, w_2) \mapsto \int_\Omega w_1 w_2 \rho^{-2} dx.
\]

We note that the following embedding is continuous

\[
C^{k,\alpha}_\gamma(\Omega \setminus \Sigma) \hookrightarrow L^2_\delta(\Omega \setminus \Sigma) \quad \text{for } \delta < \gamma + \frac{N-2}{2}.
\]

**Lemma 4.4** Let \( w \in L^2_\delta \) be a solution to

\[
L_\varepsilon w = 0 \quad \text{in } \Omega \setminus \{\xi_0\}.
\]

Then \( w \in C^{2,\alpha}_{\delta-N-2}(\tilde{\Omega}) \) for every \( \tilde{\Omega} \subset \subset \Omega \).

**Proof** For \( x_0 \in \Omega \) with \( d(x_0, \partial \Omega) \geq |x_0 - \xi_0| \) we set

\[
v(x) = w(x_0 + Rx), \quad R = \frac{1}{2} |x - \xi_0|, \quad |x| \leq 1.
\]

\( \square \) Springer
Then using the elliptic regularity for \( v \), namely,
\[
\| v \|_{C^0(B_{1/2}^\infty)} \leq C \| v \|_{L^2(B_1)},
\]
one obtains that \( |w(x)| \leq C|x|^{\delta - \frac{N-2}{2}} \) for \( x \) in a small neighborhood of \( \xi_0 \). In fact, by elliptic regularity, this estimate also holds on compact sets in \( \Omega \). The lemma follows by a scaling argument and Schauder regularity.

The natural domain \( D(L_r) \) of the operator \( L_r \) is the set of functions \( w \in L^2_\delta \) such that the distributional derivative \( L_r w \) is in \( L^2_{\delta-2} \). More precisely, \( w \in D(L_r) \) if there exists \( f \in L^2_{\delta-2} \) such that \( w \) satisfies \( L_r w = f \) in the sense of distributions in \( \Omega \setminus \Sigma \). However, in order to identify the adjoint of \( L_r \) in a natural way, one has to consider a smaller space including the boundary condition \( w = 0 \) on \( \partial \Omega \).

Together with Lemma 5.4 and a rescaling argument (see e.g., [8, Proposition 1.2.1]) one can show that the following elliptic estimate holds: for \( r_0 > 0 \) with \( B_{2r_0}(\xi_0) \subset \Omega \)
\[
\sum_{\ell = 0}^{2} \| \nabla^\ell w \|_{L^2_{\delta-\ell}(B_{r_0}(\xi_0))} \leq C(\| f \|_{L^2_{\delta-2}(B_{2r_0}(\xi_0))} + \| w \|_{L^2_{\delta}(B_{2r_0}(\xi_0))}). \tag{18}
\]

In our next lemma we bound the weighted norm \( \| w \|_{L^2_\delta} \) by \( L^2 \) norm of \( w \) and the weighted norm \( \| f \|_{L^2_{\delta-2}} \) for some values of \( \delta \).

**Lemma 4.5** Assume that \( \delta = \frac{N-2}{2} \notin \{ j/\alpha_j : j = 0, 1, \ldots \} \). Then there exists a compact set \( K \subset \tilde{\Omega} \setminus \{ \xi_0 \} \) and \( r > 0 \) such that
\[
\| w \|_{L^2_\delta(B_r(\xi_0))} \leq C(\| f \|_{L^2_{\delta-2}(\Omega)} + \| w \|_{L^2(\Omega)}). \tag{19}
\]

**Proof** Let \( R > 0 \) be such that \( B_{4R}(\xi_0) \subset \Omega \). Applying Lemma 5.3 on the ball \( B_R(\xi_0) \) we get that
\[
\| u \|_{L^2_\delta(B_R(\xi_0))} \leq C_1 \left[ \| h \|_{L^2_{\delta-2}(B_R(\xi_0))} + \| \nabla a \cdot \nabla w \|_{L^2_{\delta-2}(B_R(\xi_0))} + \| u \|_{L^2(\Omega)} \right],
\]
for some \( C_1 > 0 \) and for some compact set \( K \subset \tilde{B}_R(\xi_0) \setminus \{ \xi_0 \} \). Since \( |\nabla a| \in L^\infty(\Omega) \), it follows that
\[
\lim_{r \to 0} \frac{\| \nabla a \cdot \nabla w \|_{L^2_{\delta-2}(B_r(\xi_0))}}{\| \nabla w \|_{L^2_{\delta-1}(B_r(\xi_0))}} = 0.
\]
Therefore, for \( r > 0 \) small enough, the weighted norm \( \| \nabla a \cdot \nabla w \|_{L^2_{\delta-2}(B_r(\xi_0))} \) can be absorbed one the left hand side, thanks to (18). On the region \( B_R(\xi_0) \setminus \tilde{B}_r(\xi_0) \), the weighted norm \( \| \nabla a \cdot \nabla w \|_{L^2_{\delta-2}} \) is equivalent to \( \| \nabla a \cdot \nabla w \|_{L^2} \), and this can be controlled by
\[
\| w \|_{L^2(B_{2R}(\xi_0) \setminus B_R(\xi_0))} + \| f \|_{L^2(B_{2R}(\xi_0) \setminus B_R(\xi_0))}.
\]
We conclude the lemma. □

As a consequence of (18)–(19) one can prove the following lemma (see e.g., Chapter 9, [9]).

**Lemma 4.6** The operator \( L_\varepsilon : L^2_\delta \rightarrow L^2_{\delta-2} \) is Fredholm, provided \( \delta - \frac{N+2}{2} \notin \{ \mathbb{N} \gamma_j^\pm : j = 0, 1, \ldots \} \).

We shall fix \( \delta > 0 \) slightly bigger than \( \mu + \frac{N-2}{2} \), where \( \mu \) is fixed according to (9). Thanks to the previous comment on the domain of \( L_\varepsilon \), the adjoint of the operator

\[
L_\varepsilon : L^2_{-\delta} \rightarrow L^2_{-\delta-2} \tag{20}
\]

is given by

\[
L^2_{\delta+2} \rightarrow L^2_\delta, \quad w \mapsto \rho^2 L_\varepsilon (w \rho^{-2}). \tag{21}
\]

Then the adjoint operator (21) is injective, and \( L_\varepsilon \) in (20) is surjective. Using the isomorphism

\[
\rho^{2\delta} : L^2_\delta \rightarrow L^2_{2\delta+\delta}, \quad w \mapsto \rho^{2\delta} w,
\]

we identify the adjoint operator as

\[
L^\ast_\varepsilon : L^2_{-\delta+2} \rightarrow L^2_{-\delta}, \quad w \mapsto \rho^{2-2\delta} L_\varepsilon (w \rho^{2\delta-2}).
\]

Now we consider the composition

\[
\mathcal{L} = L_\varepsilon \circ L^\ast_\varepsilon : L^2_{-\delta+2} \rightarrow L^2_{-\delta-2}, \quad w \mapsto L_\varepsilon [\rho^{2-2\delta} L_\varepsilon (w \rho^{2\delta-2})].
\]

Then \( \mathcal{L} \) is an isomorphism, and hence there exists a two sided inverse

\[
G_\varepsilon : L^2_{-\delta-2} \rightarrow L^2_{-\delta+2}.
\]

Consequently, the right inverse of \( L_\varepsilon \) is given by \( G_\varepsilon := L^\ast_\varepsilon G_\varepsilon \). It follows that

\[
G_\varepsilon : C^0_{v-2}(\Omega \setminus \Sigma) \rightarrow C^2_{v,D}(\Omega \setminus \Sigma)
\]

is bounded.

**Lemma 4.7** Let \( \varepsilon_0 > 0 \) be as in Lemma 4.3. Then for \( 0 < \varepsilon < \varepsilon_0 \) the system \( L_\varepsilon w_1 = 0, \ w_1 = L^\ast_\varepsilon w_2 \) with \( w_1 \in C^2_{v,D}(\Omega \setminus \Sigma) \) and \( w_2 \in C^4_{v+2,D}(\Omega \setminus \Sigma) \) has only the trivial solution.

**Proof** We set \( w = \rho^{2\delta-2} w_2 \). Then \( L_\varepsilon [\rho^{2-2\delta} L_\varepsilon w] = 0 \). Multiplying the equation by \( w \) and then integrating by parts we get

\[
0 = \int_{\Omega} \rho^{2-2\delta} |L_\varepsilon w|^2 dx.
\]
Since \( v + 2\delta > \mu \), we have \( w \in C^{2,\alpha}_{v+2\delta} (\Omega \setminus \Sigma) \subset C^{2,\alpha}_\mu (\Omega \setminus \Sigma) \). Then by Lemma 4.3 we get that \( w = 0 \), equivalently \( w_1 = w_2 = 0 \).

**Lemma 4.8** There exists \( \varepsilon_0 > 0 \) small such that if \( 0 < \varepsilon < \varepsilon_0 \), then the sequence of solutions \( (w_{1,\varepsilon}) \subset C^{2,\alpha}_{v,\varepsilon} (\Omega \setminus \Sigma) \cap L^*_\varepsilon [C^{4,\alpha}_{v+2,\varepsilon} (\Omega \setminus \Sigma)] \) to \( L_{\varepsilon} w_{1,\varepsilon} = f_{\varepsilon} \) is uniformly bounded in \( C^{2,\alpha}_{v} (\Omega \setminus \Sigma) \), provided \( (f_{\varepsilon}) \) is uniformly bounded in \( C^{0,\alpha}_{v} (\Omega \setminus \Sigma) \).

**Proof** Assume by contradiction that the lemma is false. Then there exists a sequence \( \varepsilon_\ell \to 0 \) and \( w_{1,\varepsilon_\ell} \in C^{2,\alpha}_{v,\varepsilon_\ell} (\Omega \setminus \Sigma) \cap L^*_\varepsilon_\ell [C^{4,\alpha}_{v+2,\varepsilon} (\Omega \setminus \Sigma)] \) with \( L_{\varepsilon_\ell} w_{1,\varepsilon_\ell} = f_{\varepsilon_\ell} \) such that \( \| f_{\varepsilon_\ell} \|_{C^{0,\alpha}_{v-2,\varepsilon} (\Omega \setminus \Sigma)} \leq C \), and \((w_{1,\varepsilon_\ell})\) is not bounded in \( C^{2,\alpha}_{v} (\Omega \setminus \Sigma) \). By Lemma 4.2

\[
\|w_{1,\varepsilon_\ell}\|_{C^{2,\alpha}_{v} (\Omega \setminus \varepsilon_\ell)} \leq C + C \max_{\partial B_\varepsilon (\xi_0)} (\varepsilon_\ell)^{-\nu} \left( |w_{1,\varepsilon_\ell}| + \varepsilon_\ell |\nabla w_{1,\varepsilon_\ell}| \right) =: C + CS_{\varepsilon_\ell}.
\]

We distinguish the following two cases.

**Case 1** \( S_{\varepsilon_\ell} \leq C \).

In this case we proceed as in the proof of Lemma 4.3. Let \( x_\ell \in B_\varepsilon (\xi_0) \) be such that

\[
\sup_{B_\varepsilon (\xi_0)} (\rho^{-\nu} |w_{1,\varepsilon_\ell}| + \rho^{-\nu+1} |\nabla w_{1,\varepsilon_\ell}|) \approx |x_\ell - \xi_0|^{-\nu} \left( |w_{1,\varepsilon_\ell} (x_\ell)| + |x_\ell - \xi_0| |\nabla w_{1,\varepsilon_\ell} (x_\ell)| \right)
\]

\[
=: S_\ell \to \infty.
\]

Then necessarily \( r_\ell := |x_\ell - \xi_0| = o (\varepsilon_\ell) \). Setting

\[
\tilde{w}_{1,\varepsilon_\ell} (x) := \frac{r_\varepsilon^{-\nu} w_{1,\varepsilon_\ell} (r_\ell x + \xi_0)}{S_\ell}
\]

one would get that \( \tilde{w}_{1,\varepsilon_\ell} \to \tilde{w}_1 \neq 0 \) where

\[
\tilde{L}_1 \tilde{w}_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad r^{-\nu} |\tilde{w}_1| \leq C, \quad \tilde{L}_1 := \Delta + \frac{A_p}{r^2},
\]

where \( A_p \) is as in (8). Since \( v \) does not coincide with indicial roots of \( \tilde{L}_1 \), from Proposition 2.4 we get that \( \tilde{w}_1 = 0 \), a contradiction.

**Case 2** \( S_{\varepsilon_\ell} \to \infty \).

In this case we set

\[
\tilde{w}_{1,\varepsilon_\ell} (x) = (\varepsilon_\ell)^{-\nu} w_{1,\varepsilon_\ell} (\varepsilon_\ell x + \xi_0) \frac{S_{\varepsilon_\ell}}{S_{\varepsilon_\ell}}.
\]

Then \( \max_{\partial B_1} (|\tilde{w}_{1,\varepsilon_\ell}| + |\nabla \tilde{w}_{1,\varepsilon_\ell}|) \approx 1 \). Moreover, proceeding as before (see Lemma 4.3) we would get that \( w_{1,\varepsilon_\ell} \to \tilde{w}_1 \neq 0 \) where

\[
L_{1} \tilde{w}_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad r^{-\nu} |\tilde{w}_1| \leq C.
\]
Since \( \tilde{w}_1 \) decays at infinity, its decay rate is determined by the indicial roots of \( L_1 \) (which are exactly the same as \( \Delta \)) at infinity. In fact, \( \tilde{w}_1 \) would be bounded by \( r^{2-N} \) at infinity, see e.g., [5].

Since \( w_{1, \varepsilon} \in L_{e\varepsilon}^* [C^{4, \alpha}_{v+2} \mathcal{D}(\Omega \setminus \Sigma)] \), we have \( w_{1, \varepsilon} = \rho^{2-2\delta} L_{e\varepsilon} w_{2, \varepsilon} \) for some \( w_{2, \varepsilon} \in C^{4, \alpha}_{v+2\delta} \mathcal{D}(\Omega \setminus \Sigma) \). Now we set

\[
\tilde{w}_{2, \varepsilon}(x) := \frac{(e\varepsilon)^{-v-2\delta} w_{2, \varepsilon}(e\varepsilon x + \xi_0)}{S_{e\varepsilon}}.
\]

Using that \( 2\delta + \nu > \mu \), and following the proof of Lemma 4.3, one can show that the family \( \tilde{w}_{2, \varepsilon} \) converges to a limit function \( \tilde{w}_2 \), where

\[
L_1 \tilde{w}_2 = |x|^{2\delta-2} \tilde{w}_1 \ \text{in} \ \mathbb{R}^n \setminus \{0\}, \ |x|^{-v-2\delta} |\tilde{w}_2| \leq C.
\]

Thus, \( L_1[\rho^{2-2\delta} L_1 \tilde{w}_2] = 0 \). We multiply this equation by \( \tilde{w}_2 \) and integrate it on \( \mathbb{R}^N \). Then an integration by parts leads \( L_1 \tilde{w}_2 = 0 \) (this is justified because of the decay of \( \tilde{w}_1 \) at infinity, provided we choose \( \delta > 0 \) sufficiently close to \( \mu + \frac{N-2}{2} \)). Again, as \( 2\delta + \nu > \mu \), by Proposition 2.3 we have \( \tilde{w}_2 = \tilde{w}_1 = 0 \), a contradiction. \( \square \)

## 5 The Non-linear Term \( Q \)

**Lemma 5.1** Let \( M_1 > 1 \) be fixed. Then for \( \varepsilon_0 << 1 \) we have

\[
\|Q(v_1) - Q(v_2)\|_{0, \alpha, \nu-2} \leq \frac{1}{M_1} \|v_1 - v_2\|_{2, \alpha, \nu}
\]

for every \( v_1, v_2 \in B_{\varepsilon, M} := \{v \in C^{2, \alpha}_v : \|v\|_{2, \alpha, \nu} \leq M\varepsilon^q\} \).

**Proof** In Lemma 3.1, the error term \( f_\varepsilon \) is bounded by the maximum of two terms. If the maximum is the second term \( \varepsilon^{-N-2p}/(p-1) \), we argue as in [5]. Let us consider the case when the maximum is the first term. Let

\[
q_1 := \left( N - \frac{2p}{p-1} \right) - \left( 1 - \nu - \frac{2}{p-1} \right) = N + \nu - 3 > 0.
\]

We start by showing that there exists \( \tau > 0 \) small (independent of \( \varepsilon << 1 \)) such that

\[
|v(x)| \leq \frac{1}{10} \bar{u}_\varepsilon(x) \ \text{for every} \ x \in B_{\varepsilon}(\xi_0), \ v \in B_{\varepsilon, M}, \ \text{ (22)}
\]

where

\[
\tau_\varepsilon := \tau \varepsilon^{-q_1/2N} \to 0.
\]
To prove this we recall that there exists $c_1, c_2 > 1$ such that
\[
\frac{1}{c_1} \leq |x|^{\frac{2}{p-1}} u_\varepsilon(x) \leq c_1 \quad \text{for } |x| \leq \varepsilon,
\]
\[
\frac{1}{c_2} \leq \varepsilon^{-N+\frac{2p}{p-1}} |x|^{N-2} u_\varepsilon(x) \leq c_2 \quad \text{for } \varepsilon \leq |x| \leq \tau.
\]

On the other hand,
\[
\varepsilon^{-N+\frac{2p}{p-1}} \rho(x)^{-v} |v(x)| \leq M.
\]

As $\nu > -\frac{2}{p-1}$, we have (22) for some $\tau > 0$ small.

We have
\[
Q(v_1) - Q(v_2) = a \int_0^1 \frac{d}{dt} |\tilde{u}_\varepsilon + v_1 + t(v_1 - v_2)|^p dt - p\tilde{u}_\varepsilon^{p-1} (v_1 - v_2)
\]
\[
= ap(v_1 - v_2) \int_0^1 \left( |\tilde{u}_\varepsilon + v_1 + t(v_1 - v_2)|^{p-1} - \tilde{u}_\varepsilon^{p-1} \right) dt
\]
\[
=: ap(v_1 - v_2) \int_0^1 Q(v_1, v_2) dt.
\]

Next, using that
\[
(1 + r)^{p-1} = 1 + O(|r|) \quad \text{for } |r| \leq \frac{1}{2},
\]
we estimate for $x \in B_\varepsilon(\xi_0)$
\[
|Q(v_1, v_2)|(x) \leq C\tilde{u}_\varepsilon(x)^{p-1} \frac{|v_1|(x) + |v_2|(x)}{\tilde{u}_\varepsilon(x)}
\]
\[
\leq CM \varepsilon^{\frac{2}{p-1} + v + q} \rho^{-2}(x)
\]
\[
= CM \varepsilon \rho(x)^{-2},
\]
and for $x \in B_{\tau_\varepsilon}(\xi_0) \setminus B_\varepsilon(\xi_0)$
\[
|Q(v_1, v_2)|(x) \leq CM \rho(x)^{-2} \max \{\varepsilon, \varepsilon^{(N - \frac{2p}{p-1})(p-2) + q \tau_\varepsilon(2-N)(p-2) + v + 2}\}
\]
\[
= CM \rho(x)^{-2} o_\varepsilon(1).
\]

Here we have used that the second term in the maximum is of the order $\varepsilon^r$ for some $r > 0$. Indeed, from the definition of $\tau_\varepsilon$, $q$ and $q_1$, the exponent of $\varepsilon$ is
\[
\left( N - \frac{2p}{p-1} \right) (p-2) + q + [(2 - N)(p-2) + v + 2] \left( 1 - \frac{1}{N + v - 2} \right)
\]
\[
\begin{align*}
&= 1 - \frac{(2 - N)(p - 2) + v + 2}{N + v - 2} \\
&= \frac{(N - 2)(p - 1) - 2}{N + v - 2} > 0,
\end{align*}
\]

where the last inequality follows from \( p > \frac{N}{N - 2} \) and \( N + v - 2 > 0 \). Finally, as \( \nu > -\frac{2}{p - 1} \), we easily obtain for \( x \in \Omega \setminus B_{\tau_\epsilon}(\xi_0) \)

\[
|Q(v_1, v_2)(x)| \leq C(u_\epsilon^{p-1} + |v_1|^{p-1} + |v_2|^{p-1})(x) \\
\leq C \rho(x)^{-2} \left( u_\epsilon^{p-1}(x) \rho(x)^2 + M \epsilon^{q(p-1)} \right) \\
= o_\epsilon(1) \rho(x)^{-2}.
\]

Combining these estimates we get for \( \epsilon << 1 \)

\[
\|Q(v_1) - Q(v_2)\|_{0,0,\nu-2} = o_\epsilon(1)\|v_1 - v_2\|_{0,0,\nu} = o_\epsilon(1)\|v_1 - v_2\|_{2,\alpha,\nu}.
\]

Next we estimate the weighted Hölder norm of \( Q(v_1) - Q(v_2) \) with Hölder exponent \( \alpha \leq p - 1 \). For \( 0 < s < \sigma \) we write

\[
\begin{align*}
&\sup_{x,x' \in N_s \setminus N_{\frac{s}{2}}} \frac{|[Q(v_1) - Q(v_2)](x) - [Q(v_1) - Q(v_2)](x')|}{|x - x'|^\alpha} \\
&\leq 4\|Q(v_1) - Q(v_2)\|_{0,0,\nu-2} \\
&+ s^{2-v+\alpha} \sup_{x,x' \in N_s \setminus N_{\frac{s}{2}}, |x - x'| \leq \frac{s}{4}} \frac{|[Q(v_1) - Q(v_2)](x) - [Q(v_1) - Q(v_2)](x')|}{|x - x'|^\alpha}.
\end{align*}
\]

Notice that for \( x, x' \in N_s \setminus N_{\frac{s}{2}} \) with \( |x - x'| \leq \frac{s}{4} \), the line segment \([x, y]\) joining \( x \) and \( y \) lies in \( N_{2s} \setminus N_{\frac{3s}{4}} \). The desired estimate follows on the ball \( B_{\tau_\epsilon}(\xi_0) \) by estimating \( Q(v_1, v_2)(x) - Q(v_1, v_2)(x') \) using the following gradient bound (we are using that \( |u_\epsilon + v|^{p-1} \) is \( C^1 \) in this region)

\[
\begin{align*}
\nabla Q(v_1, v_2) &= (p - 1) \left[ (u_\epsilon + v_1 + t(v_1 - v_2))^{p-2} - (u_\epsilon^{p-2})^p \right] \nabla u_\epsilon \\
&+ (p - 1)(u_\epsilon + v_1 + t(v_1 - v_2))^{p-2} \nabla [v_1 + t(v_1 - v_2)] \\
&= O(1)u_\epsilon^{p-2}(|v_1| + |v_2|)\nabla u_\epsilon + O(1)(v_1 + v_2) \nabla [v_1 + t(v_1 - v_2)].
\end{align*}
\]

In fact, gradient bounds can also be used for the region \( B_{\tau_\epsilon}(\xi_0) \) if \( p \geq 2 \). For \( 1 < p \leq 2 \), one can use the following inequality

\[
|\phi|^{p-1}(x) - |\phi|^{p-1}(x')| \leq |\phi(x) - \phi(x')|^{p-1} \leq \|\nabla \phi\|_{C^0([x,x'])}^{p-1}|x - x'|^{p-1},
\]

with \( \phi = u_\epsilon \) and \( \phi = u_\epsilon + v_1 + t(v_1 - v_2) \).
We conclude the lemma. \hfill \Box

Appendix

The following lemma can be proven in the spirit of \cite[Proposition 1.5.1]{8}

**Lemma 5.2** For $d \in \mathbb{R}$ set

$$
\delta_j := \Re\left(\left(\frac{N-2}{2}\right)^2 + \lambda_j - d\right)^{\frac{1}{2}}, \quad j \in \mathbb{N}.
$$

Then for $\delta \in \mathbb{R} \setminus \{\pm \delta_j : j = 0, 1, \ldots\}$ there exists $C = C(N, \delta)$ such that if $u$ is a solution to

$$
\Delta u + \frac{d}{|x|^2} u = f \quad \text{in } B_1 \setminus \{0\},
$$

then

$$
\|u\|_{L^2_\delta(B_1)} \leq C(\|f\|_{L^2_{\delta-2}(B_1)} + \|u\|_{L^2(B_1 \setminus B_\frac{1}{2})}).
$$

**Lemma 5.3** Let $\zeta$ be a continuous function in $\bar{B}_1$. Let $\delta_j$ be given by (23) with $d = \zeta(0)$. Then for $\delta \in \mathbb{R} \setminus \{\pm \delta_j : j = 0, 1, \ldots\}$ there exists a compact set $K \subset \bar{B}_1 \setminus \{0\}$ and a constant $C > 0$ such that for every $u \in L^2_\delta(B_1)$ solving

$$
\Delta u + \frac{\zeta}{|x|^2} u = f \quad \text{in } B_1 \setminus \{0\}, \quad f \in L^2_\delta(B_1),
$$

we have

$$
\|u\|_{L^2_\delta(B_1)} \leq C(\|f\|_{L^2_{\delta-2}(B_1)} + \|u\|_{L^2(K)}).
$$

**Proof** We rewrite the equation as

$$
\Delta u + \frac{\zeta(0)}{|x|^2} u = f + \tilde{f}, \quad \tilde{f} := \frac{\zeta(0) - \zeta}{|x|^2} u.
$$

Then by Lemma 5.2 we get

$$
\|u\|_{L^2_\delta(B_1)} \leq C_1(\|f\|_{L^2_{\delta-2}(B_1)} + \|\tilde{f}\|_{L^2_{\delta-2}(B_1)} + \|u\|_{L^2(B_1 \setminus B_\frac{1}{2})}).
$$

Let $r > 0$ be sufficiently small so that $|\zeta - \zeta(0)| \leq \frac{1}{2C_1}$ on $B_r$. Then

$$
\|\tilde{f}\|_{L^2_{\delta-2}(B_1)} \leq \frac{1}{2C_1} \|u\|_{L^2_\delta(B_r)} + C(r, \|\zeta\|_{L^\infty(B_1)}) \|u\|_{L^2(B_1 \setminus B_r)}.
$$
The proof follows by absorbing the term $\|u\|_{L^2_2(B_r)}$ on the left hand side, and taking $K = \bar{B}_1 \setminus B_r$. \hfill \qed

**Lemma 5.4** ($L^2$ estimate). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Let $b_i \in L^\infty(\Omega)$ with
\[
\|b_i\|_{L^\infty(\Omega)} \leq \Lambda, \quad i = 0, 1, \ldots, n.
\]
Let $u \in W^{1,2}(\Omega)$ be a weak solution solution to
\[
\Delta u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + b_0 u = f \quad \text{in } \Omega,
\]
for some $f \in L^2(\Omega)$. Then for every $\tilde{\Omega} \Subset \Omega$ there exists $C = C(\tilde{\Omega}, \Lambda)$ such that
\[
\|u\|_{W^{2,2}(\tilde{\Omega})} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).
\]
(see for example Theorem 8.8 in [3]).

**Lemma 5.5** (Schauder estimate). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Let $b_i \in C^{0,\alpha}(\Omega)$ with
\[
\|b_i\|_{C^{0,\alpha}(\Omega)} \leq \Lambda, \quad i = 0, 1, \ldots, n.
\]
Let $u$ be a classical solution to
\[
\Delta u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + b_0 u = f \quad \text{in } \Omega,
\]
for some $f \in C^{0,\alpha}(\Omega)$. Then for every $\tilde{\Omega} \Subset \Omega$ there exists $C = C(\tilde{\Omega}, \Lambda)$ such that
\[
\|u\|_{C^{2,\alpha}(\tilde{\Omega})} \leq C(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^{0}(\Omega)}).
\]
Additionally, if $\Omega$ is regular, $\partial \Omega$ has two components $\Gamma_1$ and $\Gamma_2$, and if $u = 0$ on $\Gamma_1$ then for $\tilde{\Omega} \Subset (\Omega \cup \Gamma)$ there exists $C = C(\tilde{\Omega}, \Lambda)$ such that
\[
\|u\|_{C^{2,\alpha}(\tilde{\Omega})} \leq C(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^{0}(\Omega)}).
\]
(see for example Theorem 6.2 in [3]).

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