HIGHER-ORDER POLYNOMIAL INVARIANTS OF
3-MANIFOLDS GIVING LOWER BOUNDS FOR THE
THURSTON NORM

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1. Introduction

1.1. Summary of Results. In this paper we define new 3-manifold invariants and show that they give new information about the topology of the 3-manifold. Given a 3-dimensional manifold $X$ and a cohomology class $\psi \in H^1(X; \mathbb{Z})$ we define a sequence of invariants $\delta_n(\psi)$ which arise as degrees of “higher-order Alexander polynomials.” These integers measure the “size” of the successive quotients, $G_r(n+1)/G_r(n+2)$, of the terms of the (rational) derived series of $G = \pi_1(X)$. Loosely speaking, $\delta_n(\psi)$ is the degree of a polynomial that kills the elements of the first homology of the regular $G/G_r(n+1)$-cover of $X$. The precise definitions are given in Section 3. In the case of knot exteriors and zero surgery on knots, these covering spaces were studied by T. Cochran, K. Orr, and P. Teichner in [COT] and [C]. They defined similar generalized Alexander modules and were able to obtain important new results on knot concordance.

Although these invariants are defined algebraically, they have many exciting topological applications. We show that the degree $\delta_n$ of each of our family of polynomials gives a lower bound for the Thurston norm of a class in $H_2(X, \partial X; \mathbb{Z}) \cong H^1(X; \mathbb{Z})$ of a 3-dimensional manifold. We show that these invariants can give much more precise estimates of the Thurston norm than previously known computable invariants. We also show that the $\delta_n$ give obstructions to a 3-manifold fibering over a circle and to a 3-manifold being Seifert fibered. Moreover, we show that the $\delta_n$ give computable algebraic obstructions to a 4-manifold of the form $X \times S^3$ admitting a symplectic structure even when the obstructions given by the Seiberg-Witten invariants fail. Some other applications are to the minimal ropelength and genera of knots and links in $S^3$.

Note that $G_r(n+1)/G_r(n+2)$ is a module over $\mathbb{Z}[G/G_r(n+1)]$. When $n = 0$, $G_r^{(1)}/G_r^{(2)}$ is the classical Alexander module. Since $G_r^{(1)}/G_r^{(2)}$ is the (torsion-free) abelianization of $G$, $G_r^{(1)}/G_r^{(2)}$ is a module over the commutative polynomial ring in several variables $\mathbb{Z}[G/G_r^{(1)}]$. These modules have been studied thoroughly and with much success. For general $n$, however, these “higher-order Alexander modules” are modules over non-commutative rings. Very little was previously known in this case due to the difficulty of classifying such modules.

Let $X$ be a compact, connected, orientable 3-manifold and let $\psi \in H^1(X; \mathbb{Z})$. There is a Poincaré duality isomorphism $H^1(X; \mathbb{Z}) \cong H_2(X, \partial X; \mathbb{Z})$. If an oriented surface $F$ in $X$ represents a class $[F] \in H_2(X, \partial X; \mathbb{Z})$ that corresponds to $\psi$ under

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this isomorphism, we say that $F$ is dual to $\psi$ and vice versa. We measure the complexity of $X$ via the Thurston norm which is defined in [T] as follows. If $F$ is any compact connected surface, let $\chi(X)$ be its Euler characteristic and let $\chi_*(F) = |\chi(F)|$ if $\chi(F) \leq 0$ and equal 0 otherwise. For a surface $F = \coprod F_i$ with multiple components, let $\chi_*(F) = \Sigma \chi_*(F_i)$. Note that $-\chi(F) \leq \chi_*(F)$ in all cases.

The Thurston norm of $\psi \in H^1(X; \mathbb{Z})$ is

$$\|\psi\|_T = \inf \{\chi_*(F) | F \text{ is a properly embedded oriented surface dual to } \psi\}.$$ 

This norm extends continuously to all of $H^1(X; \mathbb{R})$. This norm is difficult to compute except for in the simplest of examples because it is a minimum over an unknown set.

Thurston showed that the unit ball of the norm is a finite sided polyhedron and that the set of classes of $H_2(X, \partial X; \mathbb{R})$ representable by a fiber of a fibration over $S^1$ corresponds to lattice points lying in the cone of the union of some open faces of this polyhedron [T]. This norm has been useful in the resolution of many open questions in 3-manifolds. D. Gabai used the Thurston norm to show the existence of taut, finite-depth codimension one foliations 3-manifolds (see [G1, G2, G3]). In particular, he shows that if $X$ is a compact, connected, irreducible and oriented 3-manifold and $F$ is any norm minimizing surface then there is a taut foliation of finite depth containing $F$ as a compact leaf. Corollaries of Gabai’s existence theorems are that the Property R and Poenaru conjectures are true.

In a recent paper, C. McMullen defined the Alexander norm of a cohomology class of a 3-manifold via the Alexander polynomial and proved that it is a lower bound for the Thurston norm [Mc]. This theorem has also been recently proved by S. Vidussi [V2] using Seiberg-Witten theory and the work of P. Kronheimer, T. Mrowka, G. Meng, and C. Taubes [K1, K2, KM, MT]. We prove in Section 10 that the (unrefined) higher-order degrees $\delta_n$ also give lower bounds for the Thurston norm. When $n = 0$, $\overline{\delta}_0(\psi) = \|\psi\|_A$ hence this gives another proof of McMullen’s theorem.

**Theorem 10.1.** Let $X$ be a compact, orientable 3-manifold (possibly with boundary). For all $\psi \in H^1(X; \mathbb{Z})$ and $n \geq 0$

$$\overline{\delta}_n(\psi) \leq \|\psi\|_T$$

except for the case when $\beta_1(X) = 1$, $n = 0$, $X \not\cong S^1 \times S^2$, and $X \not\cong S^1 \times D^2$. In this case, $\overline{\delta}_0(\psi) \leq \|\psi\|_T + 1 + \beta_3(X)$ whenever $\psi$ is a generator of $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, equality holds in all cases when $\psi : \pi_1(X) \to \mathbb{Z}$ can be represented by a fibration $X \to S^1$.

This theorem generalizes the classical result that for a knot complement, the degree of the Alexander polynomial is less than or equal to twice the genus of the knot. It also generalizes McMullen’s theorem. We remark that $\delta_n = \overline{\delta}_n$ except for some cases where $\overline{\delta}_0 = 0$. In fact, for most of the cases that we are interested in, the $\overline{\delta}_n$ in Theorem 10.1 can be replaced with $\delta_n$.

Not only do the $\delta_n$ give lower bounds for the Thurston norm, but we construct 3-manifolds for which $\delta_n$ give much sharper bounds for the Thurston norm than bounds given by the Alexander norm. In Theorem 11.2 we start with a 3-manifold $X$ and subtly alter it to obtain a new 3-manifold $X'$. The resulting $X'$ cannot be distinguished from $X$ using the $i$th-order Alexander modules for $i < n$ but the
Corollary 12.2. For each $m \geq 1$ and $\mu \geq 2$ there exists a 3-manifold $X$ with $\beta_1 (X) = \mu$ such that
\[
\|\psi\|_A = \delta_0 (\psi) < \delta_1 (\psi) < \cdots < \delta_m (\psi) \leq \|\psi\|_T,
\]
for all $\psi \in H^1 (X; \mathbb{Z})$. Moreover, $X$ can be chosen so that it is closed, irreducible and has the same classical Alexander module as a 3-manifold that fibers over $S^1$.

An interesting application of Theorem 10.1 is to show that the $\delta_n$ give new obstructions to a 3-manifold fibering over $S^1$. The previously known algebraic obstructions to a 3-manifold fibering over $S^1$ are that the Alexander module is finitely generated and (when $\beta_1 (X) = 1$) the Alexander polynomial is monic. For $i, j, n \geq 0$ let $d_{ij} = \delta_i - \delta_j$ and let $r_n (X)$ be the $n^{th}$-order rank of the module $G^{r_{(n+1)}}_r / G^{r_{(n+2)}}_r$ (see Section 3).

Theorem 12.1. Let $X$ be a compact, orientable 3-manifold. If at least one of the following conditions is satisfied then $X$ does not fiber over $S^1$.
(1) $r_n (X) \neq 0$ for some $n \geq 0$,
(2) $\beta_1 (X) \geq 2$ and there exists $i, j \geq 0$ such that $d_{ij} (\psi) \neq 0$ for all $\psi \in H^1 (X; \mathbb{Z})$,
(3) $\beta_1 (X) = 1$ and $d_{ij} (\psi) \neq 0$ for some $i, j \geq 1$ and $\psi \in H^1 (X; \mathbb{Z})$,
(4) $\beta_1 (X) = 1$, $X \not\cong S^1 \times S^2$, $X \not\cong S^1 \times D^2$ and $d_{ij} (\psi) \neq 1 + \beta_3 (X)$ for some $j \geq 1$ where $\psi$ is a generator of $H^1 (X; \mathbb{Z})$.

As a corollary, we see that the examples in Theorem 12.1 cannot fiber over $S^1$ but have the same classical Alexander module and polynomial as a fibered 3-manifold.

Corollary 12.2. For each $\mu \geq 1$, Theorem 12.1 gives an infinite family of closed irreducible 3-manifolds $X$ where $\beta_1 (X) = \mu$, $X$ does not fiber over $S^1$, and $X$ cannot be distinguished from a fibered 3-manifold using the classical Alexander module.

A second application of Theorem 12.1 is to show that the $\delta_n$ give obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure. Recently, S. Vidussi has extended the work of P. Kronheimer to show that if a 4-manifold of the form $X \times S^1$ (X irreducible) admits a symplectic structure then there is a face of the Thurston norm ball of $X$ that is contained in a face of the Alexander norm ball of $X$. We use his work to prove the following.

Theorem 12.5. Let $X$ be a closed irreducible 3-manifold. If at least one of the following conditions is satisfied then $X \times S^1$ does not admit a symplectic structure.
(1) $\beta_1 (X) \geq 2$ and there exists an $n \geq 1$ such that $\delta_n (\psi) > \delta_0 (\psi)$ for all $\psi \in H^1 (X; \mathbb{Z})$,
(2) $\beta_1 (X) = 1$, $\psi$ is a generator of $H^1 (X; \mathbb{Z})$, and $\delta_n (\psi) > \delta_0 (\psi) - 2$ for some $n \geq 1$.

Hence if $X$ is one of the examples in Theorem 12.1 then $X \times S^1$ cannot admit a symplectic structure. We note that $X$ has the same Alexander module as a fibered 3-manifold hence $X \times S^1$ cannot be distinguished from a symplectic 4-manifold using the Seiberg-Witten invariants.

Corollary 12.6. For each $\mu \geq 1$, Theorem 12.1 gives an infinite family of 4-manifolds $X \times S^1$ where $\beta_1 (X) = \mu$, $X \times S^1$ does not admit a symplectic structure, and $X$ cannot be distinguished from fibered 3-manifold using the classical Alexander module.
Another application of Theorem 10.1 is to give computable lower bounds for the ropelength of knots and links. The ropelength of a link is the quotient of its length by its thickness. In [CKS], J. Cantarella, R. Kusner, and J. Sullivan show that the minimal ropelength \( R(L_i) \) of the \( i \)th component of a link \( L = \bigsqcup L_i \) is bounded from below by \( 2\pi \left( 1 + \sqrt{\|\psi_i\|_T} \right) \). Here \( \psi_i \) is the cohomology class that evaluates to 1 on the meridian of \( L_i \) and 0 on the meridian of every other component of \( L \). In Example 8.3, we use Corollary 10.5 to estimate ropelength for a specific link from [CKS].

**Corollary 10.5.** Let \( X = S^3 - L \) and \( \psi_i \) be as defined above. For each \( n \geq 0 \),

\[
R(L_i) \geq 2\pi \left( 1 + \sqrt{\delta_n (\psi_i)} \right).
\]

Moreover, if \( \beta_1 (X) \geq 2 \) or \( n \geq 1 \) (or both) then

\[
R(L_i) \geq 2\pi \left( 1 + \sqrt{\overline{\delta_n} (\psi_i)} \right).
\]

Lastly, we remark that the higher-order degrees give obstructions to a 3-manifold admitting a Seifert fibration.

**Proposition 8.5.** Let \( X \) be a compact, orientable Seifert fibered manifold that does not fiber over \( S^1 \). If \( \beta_1 (X) \geq 2 \) or \( n \geq 1 \) then

\[
\overline{\delta_n} (\psi) = 0
\]

for all \( \psi \in H^1 (X; \mathbb{Z}) \).

### 1.2. Outline of Paper

In Section 2, we review the classical Alexander module, the multivariable Alexander polynomial, and the Alexander norm of a 3-manifold. In Section 3 we define the rational derived series of a group. This series is a slight modification of the derived series so that successive quotients are torsion free. This series will be used to define the higher-order covers of a 3-manifold, the first homology of which will be the chief object of study in this paper.

In Section 4 we define certain skew Laurent polynomial rings \( \mathbb{K}_n [t^{\pm 1}] \) which contain \( \mathbb{Z}\Gamma_n \) and depend on a class in the first cohomology of the 3-manifold. Here, \( \Gamma_n \) is the group of deck translations of the higher-order covers. These will be extremely important in our investigations. Of particular importance is the fact that they are noncommutative (left and right) principal ideal domains. Similar rings were used in the work of [COT], where it was essential that the rings were PIDs.

In Section 5 we define the new higher-order invariants. If \( X \) is any topological space, we define the higher-order Alexander module and rank of \( X \). Finally, if \( \psi \in H^1 (X; \mathbb{Z}) \) we define the higher-order degrees \( \delta_n (\psi) \) and \( \overline{\delta_n} (\psi) \).

Section 6 is devoted to the computation of these invariants using Fox’s Free Calculus. That is, the higher-order invariants can be computed directly from a finite presentation of \( \pi_1 (X) \). The reader familiar with Fox’s Free Calculus should be aware that the classical definitions must be slightly altered since we are using right instead of left modules.

In Section 7, we give a finite presentation of the homology of \( X \) with coefficients in \( \mathbb{K}_n [t^{\pm 1}] \). This will be crucial to prove that \( \overline{\delta_n} \) is bounded above by the Thurston norm. In Section 8, we compute the higher-order invariants of some well known
3-manifolds and give some topological properties of the invariants. The most important computation in this section is the computation of the higher-order degrees and ranks for 3-manifolds that fiber over $S^1$.

Section 9 contains the algebra concerning the rank of a torsion module over a skew (Laurent) polynomial ring. Proposition 9.1 will be used in the proof of Theorem 10.1. In Section 10, we show that the higher-degrees are lower bounds for the Thurston norm. We also prove a theorem relating higher-order degrees of a cohomology class $\psi$ to the first Betti number of a surface dual to $\psi$, and prove a result for links in $S^3$.

In Section 11 we prove the Realization Theorem and construct examples of 3-manifolds whose higher degrees increase. We finish the paper by investigating the applications of Theorem 10.1 to 3-manifolds that fiber over $S^1$ and symplectic 4-manifolds of the form $X \times S^1$ in Section 12.

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2. The Alexander Polynomial

In this section, we define the Alexander polynomial, the Alexander module, and the Alexander norm of a 3-manifold. For more information about the Alexander polynomial we refer the reader to [CS], [Hi], [Ka] and [R].

Let $G$ be a finitely presented group and let $X$ be a finite CW complex with $\pi_1(X,x_0)$ isomorphic to $G$. We can assume that $X$ has one 0-cell, $x_0$. Let $X_0$ be the universal torsion free abelian cover of $X$ and $\tilde{x}_0$ be the inverse image of $x_0$ in $X_0$. That is, $X_0$ is the cover induced by the homomorphism from $G$ onto $ab\left(G/\{G,G\}\right) = \mathbb{Z}^{\mu}$ where $\mu = \beta_1(X)$ is the first betti number of $X$. (The reason for the “0” in $X_0$ will become apparent later in the paper.)

The Alexander Module of $X$ is defined to be

$$A_X = H_1(X_0,\tilde{x}_0;\mathbb{Z})$$

considered as a $\mathbb{Z}[ab(G)]$-module. After choosing a basis $\{x_1,\ldots, x_\mu\}$ for $H_1(X)$ the ring $\mathbb{Z}[ab(G)]$ can be identified with the ring of Laurent polynomials in several variables $x_1,\ldots, x_\mu$ with integral coefficients. The ring $\mathbb{Z}[ab(G)]$ has no zero divisors and is in fact a unique factorization domain. We note that $A_X$ is finitely presented as

$$\mathbb{Z}[ab(G)]^r \xrightarrow{\partial_1^*} \mathbb{Z}[ab(G)]^s \rightarrow A_X$$

where $r$ is the number of relations and $s$ is the number of generators of a presentation of $G$. This presentation is obtained by lifting each cell of $X$ to ab($G$) cells of the torsion free abelian cover, $X_0$.

Let $\Lambda$ be a finitely generated free abelian group. For any finitely presented $\mathbb{Z}[\Lambda]$-module $M$ with presentation

$$\mathbb{Z}[\Lambda]^r \xrightarrow{P} \mathbb{Z}[\Lambda]^s \rightarrow M$$

we define the $i^{th}$ elementary ideal $E_i(M) \subseteq \mathbb{Z}[F]$ to be the ideal generated by the $(s-i) \times (s-i)$ minors of the matrix $P$. This ideal is independent of the presentation of $M$. The Alexander ideal is $I(X) = E_1(A_X)$, the first elementary ideal of $A_X$. 
The Alexander polynomial \( \Delta_X \) of \( X \) is the greatest common divisor of the elements of the Alexander ideal. Equivalently, we could have defined \( \Delta_X \) to be a generator of the smallest principal ideal containing \( I(X) \). Note that \( \Delta_X \in \mathbb{Z}[ab(G)] \) and is well-defined up to units in \( \mathbb{Z}[ab(G)] \). We point out the necessity that \( \mathbb{Z}[ab(G)] \) be a UFD in the definition of \( \Delta_X \).

Now let \( \psi \in H^1(X;\mathbb{Z}) \). Let \( \Delta_X = \sum_{i=1}^{m} a_i g_i \) for \( a_i \in \mathbb{Z} \setminus \{0\} \) and \( g_i \in ab(G) \). The Alexander norm of \( \psi \in H^1(X;\mathbb{R}) \) is defined to be

\[
\|\psi\|_A = \sup_{i,j} \psi(g_i - g_j)
\]

where \( \psi \) is a homomorphism from \( G \) to \( \mathbb{Z} \). In this paper, we view \( \mathbb{Z} \) as the multiplicative group generated by \( t \). Hence the Alexander norm is equal to the degree of the one-variable polynomial \( \sum_{i=1}^{m} \psi(g_i) \) corresponding to \( \psi \).

We note that the Alexander (as well as the Thurston) norm is actually semi-norms since it can be zero on a non-zero vector of \( H^1(X;\mathbb{R}) \).

### 3. Rational Derived Series

This paper investigates the homology of the covering spaces of a 3-manifold corresponding to the rational derived series of a group. We begin by defining the rational derived series of \( G \) and proving some of the properties of the quotient \( G/G^{(r+1)} \). The most important for our purposes will be that \( G/G^{(r+1)} \) is solvable and its successive quotients \( G^{(r)}/G^{(r+1)} \) are \( \mathbb{Z} \)-torsion-free and abelian.

**Definition 3.1.** Let \( G^{(0)}_r = G \). For \( n \geq 1 \) define \( G^{(n)}_r = \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] P_{n-1} \)

where \( P_{n-1} = \left\{ g \in G^{(n-1)}_r \mid g^k \in \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \text{ for some } k \in \mathbb{Z} \setminus \{0\} \right\} \) to be the \( n \)th term of the rational derived series of \( G \).

We denote by \( \Gamma_n \) the quotient \( G/G^{(n+1)}_r \) and by \( \phi_n \) the quotient map \( G \to \Gamma_n \).

By the following lemma, \( \Gamma_n \) is a group. Note that if \( G \) is a finite group then \( G^{(n)}_r = G \) hence \( \Gamma_n = \{1\} \) for all \( n \geq 0 \). Hence, in this paper we will only be interested in groups with \( \beta_1(G) \geq 1 \).

**Lemma 3.2.** \( G^{(n)}_r \) is a normal subgroup of \( G^{(i)}_r \) for \( 0 \leq i \leq n \).

**Proof.** We show that \( \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \) and \( P_{n-1} \) are both normal subgroups of \( G \). Since \( G^{(i+1)}_r \subseteq G^{(i)}_r \) for all \( i \geq 0 \), \( \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \subseteq G^{(n-1)}_r \), and \( P_{n-1} \subseteq G^{(n-1)}_r \) it follows that \( \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \) and \( P_{n-1} \) are normal in \( G^{(i)}_r \) for \( 0 \leq i \leq n \).

\( G^{(n)}_r \) is a normal subgroup of \( G^{(i)}_r \) for \( 0 \leq i \leq n \). Let \( N \) be a normal subgroup of \( G \). Then \( [N,N] \) is normal in \( G \) since \( g \left( \prod [n_1, n_2] g^{-1} = \prod [g n_1 g^{-1}, g n_2 g^{-1}] \right) \).

Therefore \( \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \) is normal in \( G \) by induction on \( n \). Now we show that \( P_{n-1} \) is a closed under multiplication. Let \( p_1, p_2 \in P_{n-1} \) then for some \( k_1, k_2 \neq 0 \), \( p_1^{k_1}, p_2^{k_2} \in \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \). Now for any two elements \( w_1, w_2 \in G^{(n-1)}_r \), we have \( w_1 w_2 = w_2 w_1 c \) where \( c = [w_1^{-1}, w_2^{-1}] \in \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \). Hence \( (p_1 p_2)^{k_1 k_2} = (p_1^{k_1})^{k_2} (p_2^{k_2})^{k_1} \prod c_i \) where \( c_i \in \left[ G^{(n-1)}_r, G^{(n-1)}_r \right] \) so \( p_1 p_2 \in P_{n-1} \), which shows that \( P_{n-1} \) is a subgroup of \( G \). Now if \( g \in G \) then \( (g p_1 g^{-1})^{k_1} = g p_1^{k_1} g^{-1} \in \)}
\[
\left[ G_r^{(n-1)}, G_r^{(n-1)} \right] \text{ since } \left[ G_r^{(n-1)}, G_r^{(n-1)} \right] \text{ is normal in } G. \text{ Therefore } P_{n-1} \text{ is a normal subgroup of } G.
\]

**Definition 3.3.** A group \( \Gamma \) is poly-torsion-free-abelian (PTFA) if it admits a normal series \( \{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = \Gamma \) such that each of the factors \( G_{i+1}/G_i \) is torsion-free abelian. (In the group theory literature only a subnormal series is required.)

**Remark 3.4.** If \( A \lhd G \) is torsion-free-abelian and \( G/A \) is PTFA then \( G \) is PTFA. Any PTFA group is torsion-free and solvable (the converse is not true). Any subgroup of a PTFA group is a PTFA group [\[ Lemma 2.4, p.421].

We show that the successive quotients of the rational derived series are torsion-free abelian. In fact, the following lemma implies that if \( N \) is a normal subgroup of \( G \), then \( G/N \) torsion-free then the rational derived series agrees with the derived series. In general, every PTFA group \( G \) is PTFA.

**Lemma 3.5.** \( G_r^{(i)}/G_r^{(i+1)} \) is isomorphic to \( \left( G_r^{(i)}/G_r^{(i)} \right)/\{\mathbb{Z} - \text{torsion}\} \) for all \( i \geq 0 \).

**Proof.** Since \( \left[ G_r^{(i)}, G_r^{(i)} \right] \subseteq G_r^{(i+1)} \), we can extend the natural projection \( p: G_r^{(i)} \rightarrow G_r^{(i)}/G_r^{(i+1)} \) to a surjective map \( p_1: G_r^{(i)}/\left[ G_r^{(i)}, G_r^{(i)} \right] \rightarrow G_r^{(i)}/G_r^{(i+1)} \). If \( [g] \) is a torsion element in \( G_r^{(i)}/\left[ G_r^{(i)}, G_r^{(i)} \right] \) then \( [g]^k = [g^k] = 1 \) so \( g \in P_i \subseteq G_r^{(i+1)} \). Hence we can extend \( p_1 \) to \( p_2 : \left( G_r^{(i)}/\left[ G_r^{(i)}, G_r^{(i)} \right] \right)/T \rightarrow G_r^{(i)}/G_r^{(i+1)} \) where \( T \) is the torsion subgroup of \( G_r^{(i)}/\left[ G_r^{(i)}, G_r^{(i)} \right] \). We show that \( p_2 \) is injective hence is an isomorphism. Suppose \( p_2(g_2) = 1 \) then \( p(g) = 1 \) for any \( g \) such that \( q_2(q_1(g)) = g_2 \). Hence \( g = fh \) where \( f \in \left[ G_r^{(i)}, G_r^{(i)} \right] \) and \( h \in G_r^{(i)}, G_r^{(i)} \) for some \( k \neq 0 \). Therefore \( (g_2)^k = (q_2(q_1(fh)))^k = (q_2(q_1(f)q_1(h)))^k = (q_2(q_1(h)))^k = q_2(q_1(h)) = q_2(1) = 1 \) which implies that \( g_2 = 1 \).

If \( G = \pi_1(X) \) this shows that \( G_r^{(n)}/G_r^{(n+1)} \cong H_1(X_{\Gamma_{n-1}})/\{\mathbb{Z} - \text{torsion}\} \) where \( X_{\Gamma_{n-1}} \) is the regular \( \Gamma_{n-1} \)-cover of \( X \). When \( n = 0 \), note that \( G/G_r^{(1)} = G_r^{(0)}/G_r^{(1)} \cong H_1(X)/\{\mathbb{Z} - \text{torsion}\} \cong \mathbb{Z}^{\beta_1(X)} \).

**Corollary 3.6.** \( \Gamma_n \) is a PTFA group.

**Proof.** Consider the subnormal series

\[
1 = G_r^{(n+1)} \subset G_r^{(n)} \subset \cdots \subset G_r^{(i)} \subset G_r^{(n)} \subset \cdots \subset G_r^{(0)} = \Gamma_n.
\]

\( G_r^{(i)} \) is a normal subgroup of \( G_r^{(j)} \) for \( 0 \leq j \leq i \) hence \( G_r^{(i)}/G_r^{(n+1)} \) is a normal subgroup of \( G_r^{(j)}/G_r^{(n+1)} \). From the lemma above, \( \left( G_r^{(i)}/G_r^{(n+1)} \right)/\left( G_r^{(i)}/G_r^{(n+1)} \right) \cong G_r^{(i)}/G_r^{(i+1)} \) is isomorphic to \( \left( G_r^{(i)}/G_r^{(i)} \right)/\{\mathbb{Z} - \text{torsion}\} \) hence is torsion-free abelian.

We next show that if the successive quotients of the derived series of \( G \) are torsion-free then the rational derived series agrees with the derived series. In general we only know that \( G_r^{(i)} \subseteq G_r^{(i)} \) for all \( i \geq 0 \).
Corollary 3.7. If $G^{(i)}/G^{(i+1)}$ is $\mathbb{Z}$-torsion-free for all $i$ then $G_r^{(i)} = G^{(i)}$ for all $i$.

Proof. We prove this by induction on $i$. First, we know that $G_r^{(0)} = G^{(0)} = G$. Now assume that $G_r^{(i)} = G^{(i)}$, then by assumption

$$G_r^{(i)} / \left[ G_r^{(i)}, G_r^{(i)} \right] = G^{(i)}/G^{(i+1)}$$

is $\mathbb{Z}$-torsion-free. Hence Lemma 3.3 gives us $G_r^{(i)}/G^{(i+1)} = G_r^{(i)}/\left[ G_r^{(i)}, G_r^{(i)} \right]$ hence

$$G_r^{(i+1)} = \left[ G_r^{(i)}, G_r^{(i)} \right] = G^{(i+1)}.$$  

R. Strebel showed that if $G$ is the fundamental group of a (classical) knot exterior then the quotients of successive terms of the derived series are torsion-free abelian \footnote{SHELLY L. HARVEY}. Hence for knot exteriors we have $G_r^{(1)} = G^{(1)}$. This is also well known to be true for free groups. Since any non-compact surface has free fundamental group, this also holds for all orientable surface groups.

4. Skew Laurent Polynomial Rings

In this section we define some skew Laurent polynomial rings, $\mathbb{K}_n \left[ t^{\pm 1} \right]$, which are obtained from $\mathbb{Z}\Gamma_n$ by inverting elements of the ring that are “independent” of $\psi \in H^1(G;\mathbb{Z})$. Very similar rings were used in the work of Cochran, Orr and Teichner \cite{COT}. Skew polynomial rings with coefficients in a (skew) field are known to be left and right principal ideal domains as is discussed herein.

Let $\Gamma$ be a PTFA group as defined in the previous section. A crucial property of $\mathbb{Z}\Gamma$ is that it has a (skew) quotient field. Recall that if $R$ is a commutative integral domain then $R$ embeds in its field of quotients. However, if $R$ is non-commutative domain then this is no longer always possible (and is certainly not as trivial if it does exist). We discuss conditions which guarantee the existence of such a (skew) field.

Let $R$ be a ring and $S$ be a subset of $R$. $S$ is a right divisor set of $R$ if the following properties hold.

1. $0 \notin S$, $1 \in S$.
2. $S$ is multiplicatively closed.
3. Given $r \in R$, $s \in S$ there exists $r_1 \in R$, $s_1 \in S$ with $rs_1 = sr_1$.

It is known that if $S \subseteq R$ is a right divisor set then the right quotient ring $RS^{-1}$ exists (\cite{P} p.146 or \cite{Str} p.52)). By $RS^{-1}$ we mean a ring containing $R$ with the property that

1. Every element of $S$ has an inverse in $RS^{-1}$.
2. Every element of $RS^{-1}$ is of the form $rs^{-1}$ with $r \in R$, $s \in S$.

If $R$ has no zero-divisors and $S = R \setminus \{0\}$ is a right divisor set then $R$ is called an Ore domain. If $R$ is an Ore domain, $RS^{-1}$ is a skew field, called the classical right ring of quotients of $R$ (see \cite{Str}). It is observed in \cite{COT}, Proposition 2.5] that the group ring of a PTFA group has a right ring of quotients.

Proposition 4.1 (pp. 591-592,611 of \cite{F}). If $\Gamma$ is PTFA then $\mathbb{Q}\Gamma$ is a right (and left) Ore domain; i.e. $\mathbb{Q}\Gamma$ embeds in its classical right ring of quotients $\mathbb{K}$, which is a skew field.

If $K$ is the (right) ring of quotients of $\mathbb{Z}\Gamma$, it is a $K$-bimodule and a $\mathbb{Z}\Gamma$-bimodule. Note that $K = \mathbb{Z}\Gamma (\mathbb{Z}\Gamma \setminus \{0\})^{-1}$ as above. We list a some properties of $K$. 
Remark 4.2. If $R$ is an Ore domain and $S$ is a right divisor set then then $RS^{-1}$ is flat as a left $R$-module [Ste, Proposition II.3.5]. In particular, $K$ is a flat left $Z\Gamma$-module, i.e. $\cdot \otimes_{Z\Gamma} K$ is exact.

Remark 4.3. Every module over $K$ is a free module [Ste, Proposition I.2.3] and such modules have a well defined rank $rk_K$ which is additive on short exact sequences [Co1, p. 48].

If $M$ is a right $R$-module with $R$ an Ore domain then the rank of $M$ is defined as $\text{rank } M = rk_K (M \otimes_R K)$. Combining Remarks 4.2 and 4.3 we have the following

Remark 4.4. If $C$ is a non-negative finite chain complex of finitely generated free right $Z\Gamma$-modules then the Euler characteristic $\chi(C) = \sum_{i=0}^{\infty} (-1)^i \text{rank } C_i$ is defined and is equal to $\sum_{i=0}^{\infty} (-1)^i \text{rank } H_i(C)$.

The rest of this section will be devoted to the rings $K_n[t^{\pm 1}]$. Consider the group $\Gamma_n = G/G^{(n+1)}$ for $n \geq 0$. Since $\Gamma_n$ is PTFA (Corollary 3.6), $Z\Gamma_n$ embeds in its right ring of quotients, which we denote by $K_n$. Let $\psi \in H^1(G;Z)$ be primitive. Since $H^1(G;Z) \cong \text{Hom}_Z(G,Z)$, $\psi$ can be considered as an epimorphism from $G$ to $Z$. In particular, $\psi$ is trivial on $G^{(n+1)}$ so it induces a well defined homomorphism $\overline{\psi} : \Gamma_n \rightarrow Z$. Let $\Gamma_n'$ be the kernel of $\overline{\psi}$. Since $\Gamma_n'$ is a subgroup of $\Gamma_n$, $\Gamma_n'$ is PTFA by Remark 3.4. Therefore $Z\Gamma_n'$ is an Ore domain and $S_n = Z\Gamma_n' - \{0\}$ is a right divisor set of $Z\Gamma_n$ [F, p.609]. Let $K_n = (Z\Gamma_n')S_n^{-1}$ be the right ring of quotients of $Z\Gamma_n$, $g_\psi : Z\Gamma_n' \rightarrow K_n$ be the embedding of $Z\Gamma_n'$ into $K_n$, and $R_n = (Z\Gamma_n)S_n^{-1}$. We will show that $R_n$ is isomorphic to a certain skew Laurent polynomial ring $K_n[t^{\pm 1}]$ (defined below) which is a non-commutative principal right and left ideal domain by [Co2, 2.1.1. p.49]. That is, $K_n[t^{\pm 1}]$ has no zero divisors and every right and left ideal is principal.

We recall the definition of a skew Laurent polynomial ring. If $K$ is a skew field, $\alpha$ is an automorphism of $K$ and $t$ is an indeterminate, the skew (Laurent) polynomial ring in $t$ over $K$ associated with $\alpha$ is the ring consisting of all expressions of the form

$$t^{-m}a_{-m} + \cdots + t^{-1}a_{-1} + a_0 + ta_1 + \cdots + t^na_n$$

where $a_i \in K$. The operations are coordinate-wise addition and multiplication defined by the usual multiplication for polynomials and the rule $at = t\alpha(a)$ [Co1, p. 54].

Consider the split short exact sequence

$$0 \rightarrow \ker (\overline{\psi}) \rightarrow \Gamma_n \xrightarrow{\psi} Z \rightarrow 0.$$ 

Choose a splitting $\xi : Z \rightarrow \Gamma_n$. Then $\xi$ induces an automorphism of $\Gamma_n' = \ker (\overline{\psi})$ by $g \mapsto \xi (t)^{-1} g \xi (t)$. This induces a ring automorphism of $Z\Gamma_n'$ and hence a field automorphism $\alpha$ of $K_n$ by $\alpha (rs^{-1}) = \alpha (r) \alpha (s)^{-1}$. This defines $K_n[t^{\pm 1}]$ as above.

Proposition 4.5. The embedding $g_\psi : Z\Gamma_n' \rightarrow K_n$ extends to an isomorphism $R_n \rightarrow K_n[t^{\pm 1}]$

Proof. Any element of $\Gamma_n$ can be written uniquely as $\xi(t)^m a_m$ for some $m \in Z$ and $a_m \in Z\Gamma_n'$. It follows that $Z\Gamma_n$ is isomorphic to the skew (Laurent) polynomial ring
$\mathbb{Z}[x^{\pm 1}]$ by sending $\xi(t)^m a_m$ to $x^m a_m$. The automorphism of $\mathbb{Z}[x^{\pm 1}]$ is induced by conjugation, $a \rightarrow x^{-1}ax$ since $a \xi(t) = \xi(t)^{-1} a \xi(t)$.

Hence there is an obvious ring homomorphism of $\mathbb{Z}[x^{\pm 1}]$ extending $g_{\psi}$. Note that the automorphism $g \rightarrow \xi(t)^{-1} g \xi(t)$ defining $K_n[t^{\pm 1}]$ agrees with conjugation in $\Gamma$ so this map is a ring homomorphism. The non-zero elements of $\mathbb{Z}[x^{\pm 1}]$ map to invertible elements in $K_n[t^{\pm 1}]$. It is then easy to show that $R_n \cong K_n[t^{\pm 1}]$.

We note that $(\mathbb{Z})_n S^{-1}$ depends on the (primitive) class $\psi \in H^1(G; \mathbb{Z})$. Moreover, the isomorphism of $(\mathbb{Z})_n S^{-1}$ with $K_n[t^{\pm 1}]$ depends on the splitting $\xi : \mathbb{Z} \rightarrow \Gamma_n$. For any $\psi \in H^1(X; \mathbb{Z})$ we have

$$(\mathbb{Z})_n \cong K_n[t^{\pm 1}] \hookrightarrow K_n.$$  

One should note that the first and last rings only depend on the group $G$ while the middle ring $K_n[t^{\pm 1}]$ depends on the homomorphism $\psi : G \rightarrow \mathbb{Z}$ and splitting $\xi : \mathbb{Z} \rightarrow \Gamma_n$. Often we write $K_n[t^{\pm 1}]$ to emphasize the class $\psi$ on which $K_n[t^{\pm 1}]$ is dependent. From Remark 4.2 we have the following.

**Remark 4.6.** $K_n[t^{\pm 1}]$ and $K_n$ are flat left $\mathbb{Z}[x^{\pm 1}]$-modules.

5. **Definition of Invariants**

Suppose $X$ is a connected CW-complex with $A \subseteq X$ and $x_0 \in A$ a basepoint. Let $\phi : \pi_1(X) \rightarrow \Gamma$ be a homomorphism and $X_{\Gamma} \rightarrow X$ denote the regular $\Gamma$-cover of $X$ associated to $\phi$. That is, $X_{\Gamma}$ is defined to be the pullback of the universal cover of $K(\Gamma, 1)$. We note that there is an induced coefficient system on $A$, $\phi \circ i_* : \pi_1(A) \rightarrow \Gamma$ where $i$ is the inclusion map of $A$ into $X$. Thus, we have a regular covering map of the pair $(X_{\Gamma}, A_{\Gamma})$ to $(X, A)$. If $\phi$ is not surjective then $X_{\Gamma}$ is the disjoint union of $\Gamma/\text{Im}(\phi)$ copies of the regular $\Gamma$-cover corresponding to $\phi : \pi_1(X) \rightarrow \text{Im}(\phi)$.

There is a natural homomorphism $\tau : \Gamma \rightarrow G(X_{\Gamma})$ where $G(X_{\Gamma})$ is the group of deck transformations of $X_{\Gamma}$ (see [Ha] or [H] for more details). We note that $\tau$ is an isomorphism when $\phi$ is surjective. This map is defined by sending $\gamma$ to the deck transformation $\tau_\gamma$ that takes $x_0$ to $\gamma(1)$ where $\gamma$ is the unique lift of $\gamma$ starting at $x_0$. $\tau$ gives us a left $\Gamma$ action on $X_{\Gamma}$ by $x \gamma = \tau_{\gamma}(x)$. We make this into a right action by defining $\bar{x} \gamma = \gamma^{-1} \bar{x}$. Hence $\bar{x} \gamma = \tau_{\gamma^{-1}}(\bar{x})$.

The right action of $\Gamma$ on $X_{\Gamma}$ induces a right action on the group $C_*(X_{\Gamma})$ of singular $n$-chains on $X_{\Gamma}$, by sending a singular $n$-simplex $\sigma : \Delta^n \rightarrow X_{\Gamma}$ to the composition $\Delta^n \rightarrow X_{\Gamma} \rightarrow X_{\Gamma}$. The action of $\Gamma$ on $C_*(X_{\Gamma})$ makes $C_*(X_{\Gamma})$ a right $\mathbb{Z}[x^{\pm 1}]$-module.

Let $\mathcal{M}$ be a $\mathbb{Z}[x^{\pm 1}]$-bimodule. The equivariant homology of $X$ and $(X, A)$ are defined as follows.

**Definition 5.1.** Given $X$, $A$, $\phi$, $\mathcal{M}$ as above, let

$$H_*(X; \mathcal{M}) \equiv H_*(C_*(X_{\Gamma}) \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathcal{M})$$

and

$$H_*(X, A; \mathcal{M}) \equiv H_*(C_*(X_{\Gamma}, A_{\Gamma}) \otimes_{\mathbb{Z}[x^{\pm 1}]} \mathcal{M})$$

as right $\mathbb{Z}[x^{\pm 1}]$-modules.
These are well-known to be isomorphic to the homology of $X$ and $(X,A)$ with coefficient system induced by $\phi$ [3, Theorem VI 3.4].

We now restrict to the case when $\Gamma$ is PTFA. We state the following useful proposition. A proof of this can be found in [3, Section 3]. We remark that the finiteness condition in Proposition 5.2 is necessary.

**Proposition 5.2.** Suppose $\pi_1(X)$ is finitely generated and $\phi: \pi_1(X) \to \Gamma$ is non-trivial. Then

$$\text{rank } H_1(X;\mathbb{Z}\Gamma) \leq \beta_1(X) - 1.$$  

Let $G = \pi_1(X,x_0)$. Define the $n^{th}$-order cover $X_n \xrightarrow{p_n} X$ of $X$ to be the regular $\Gamma_n$-cover corresponding to the coefficient system $\phi_n: G \to \Gamma_n$ where $\Gamma_n = G/G_r^{(n+1)}$ is as defined in Section 3. Recall that $\mathbb{Z}\Gamma_n$ has a (skew) field of quotients $\mathcal{K}_n$. If $R$ is any ring with $\mathbb{Z}\Gamma_n \subseteq R \subseteq \mathcal{K}_n$ then $R$ is a $\mathbb{Z}\Gamma_n$-bimodule. Moreover, $H_n(X;R)$ can be considered as a right $R$-module. We will be interested in the cases when $R$ is $\mathbb{Z}\Gamma_n$, $\mathcal{K}_n$ and $\mathbb{K}_n[t^{\pm 1}]$ where $\mathbb{K}_n[t^{\pm 1}]$ is as described in the Section 5.1. If $M$ is a right (left) $R$-module where $R$ is an Ore domain then we let $T_RM$ be the $R$-torsion submodule of $M$. When there is no confusion we suppress the $R$ and just write $TM$.

We define the higher-order modules. The integral invariants that we can extract from these modules will be our chief interest for the rest of this paper.

**Definition 5.3.** The $n^{th}$-order Alexander module of a CW-complex $X$ is

$$\mathcal{A}_n(X) \equiv T_{\mathbb{Z}\Gamma_n} H_1(X;\mathbb{Z}\Gamma_n)$$

considered as right $\mathbb{Z}\Gamma_n$-module. Similarly, we define

$$\overline{\mathcal{A}}_n(X) \equiv H_1(X;\mathbb{Z}\Gamma_n)$$

considered as right $\mathbb{Z}\Gamma_n$-module.

Let $X$ and $Y$ be 3-manifolds with $G = \pi_1(X)$ and $H = \pi_1(Y)$. Suppose that $G$ is isomorphic to $H$. We would like for their higher-order Alexander modules to be “the same.” However, they are modules over different (albeit isomorphic) rings. We remedy this dilemma with the following definition. It is easy to verify that the following defines an equivalence relation.

**Definition 5.4.** Let $M$ and $N$ be right (left) $R$ and $S$-modules respectively and $f: R \to S$ be an isomorphism. $N$ can be made into a right (left) $R$-module via $f$. We say that $M$ is equivalent to $N$ provided $N$ is isomorphic to $M$ as a right (left) $R$-module.

Let $X$ be a topological space with $G = \pi_1(X)$. The higher-order Alexander modules can be defined group theoretically. Define a right $\mathbb{Z} \left[ \frac{G}{G_r^{(n+1)}} \right]$-module structure on $G_r^{(n+1)}/[G_r^{(n+1)}, G_r^{(n+1)}]$ by $[h][g] = [g^{-1}hg]$ for $h \in G_r^{(n+1)}$ and $g \in G$. We see that

$$\overline{\mathcal{A}}_n(X) \cong \frac{G_r^{(n+1)}}{[G_r^{(n+1)}, G_r^{(n+1)}]}$$

can be considered as a right $\mathbb{Z} \left[ \frac{G}{G_r^{(n+1)}} \right]$-module. We also note that

$$\overline{\mathcal{A}}_n(X)/\{\mathbb{Z}-\text{torsion}\} \cong \frac{G_r^{(n+1)}}{G_r^{(n+2)}}$$
by Proposition 3.5. Suppose that $Y$ is homeomorphic to $X$, then $\pi_1(Y)$ is isomorphic to $G$. It is easy to verify that the isomorphism of groups leads to an equivalence of $A_n(X)$ and $A_n(Y)$. Therefore the equivalence classes of the higher-order Alexander modules are topological invariants. Similarly, one can easily verify that the rest of the definitions in this section are invariants of $X$ or a pair $(X, \psi)$.

**Definition 5.5.** The $n^{th}$-order rank of $X$ is

$$r_n(X) = \text{rk}_{\mathcal{R}_n} H_1(X; \mathcal{K}_n).$$

In the literature, the classical Alexander module of a 3-manifold is often defined as $H_1(X, x_0; \mathbb{Z}[\Gamma])$ (see Section 2) and $\alpha(X) = \text{rk} H_1(X, x_0; \mathbb{Z}[\Gamma])$ is called the nullity of $X$. We will now show that $H_1(X; \mathbb{Z}[\Gamma])$ and $H_1(X, x_0; \mathbb{Z}[\Gamma])$ are related by $r_n(X) = \text{rk}_{\mathcal{R}_n} H_1(X, x_0; \mathcal{K}_n) - 1$ and $T_{2\mathbb{Z}[\Gamma]} H_1(X; \mathbb{Z}[\Gamma]) = T_{2\mathbb{Z}[\Gamma]} H_1(X, x_0; \mathbb{Z}[\Gamma])$. Hence, we could have defined $A_n(X)$ and $r_n(X)$ using homology rel basepoint as well.

**Proposition 5.6.** Let $\Gamma$ be PTFA, and $\phi : \pi_1(X, x_0) \to \Gamma$ be non-trivial. Then

$$\text{rk}_{\mathcal{R}} H_1(X; \mathcal{K}) = \text{rk}_{\mathcal{R}_1} H_1(X, x_0; \mathcal{K}) - 1$$

and

$$T_{\mathbb{R}} H_1(X; R) \cong T_{\mathbb{R}} H_1(X, x_0; R)$$

for any ring $R$ such that $\mathbb{Z}[\Gamma] \subseteq R \subseteq \mathcal{K}$, where $\mathcal{K}$ is the (skew) field of quotients of $\mathbb{Z}[\Gamma]$.

**Proof.** To prove (5.1), consider the long exact sequence the pair $(X, x_0)$,

$$0 \to H_1(X; \mathcal{K}) \overset{\partial}{\to} H_1(X, x_0; \mathcal{K}) \overset{\partial}{\to} H_0(x_0; \mathcal{K}) \to H_0(X; \mathcal{K}).$$

Since $\phi : \pi_1(X) \to \Gamma$ is non-trivial, $H_0(X; \mathcal{K}) = 0$ by the following Lemma. The first result follows since $H_0(x_0; \mathcal{K}) \cong \mathcal{K}$.

**Lemma 5.7.** Suppose $X$ is a connected CW complex. If $\phi : \pi_1(X) \to \Gamma$ is a non-trivial coefficient system and $\Gamma$ is PTFA then $H_0(X; \mathcal{K}) = 0$.

**Proof.** By [18, p. 275] and [19, p.34], $H_0(X; \mathcal{K})$ is isomorphic to the cofixed set $\mathcal{K}/KI$ where $I$ is the augmentation ideal of $\mathbb{Z}[\pi_1(X)]$ acting via $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\Gamma] \to \mathcal{K}$. If $\phi$ is non-trivial then the composition is non-trivial. Thus $I$ contains an element that is a unit hence $KI = \mathcal{K}$. 

We show that the map $\rho : H_1(X; R) \to H_1(X, x_0; R)$ restricts to an isomorphism from $TH_1(X; R)$ onto $TH_1(X, x_0; R)$. Certainly $\rho : TH_1(X; R) \to TH_1(X, x_0; R)$ is a monomorphism. Let $\sigma \in TH_1(X, x_0; R)$ with $\sigma r = 0$ where $r \neq 0$. Since $H_0(x_0; R) \cong R$ is $R$-torsion-free, $\partial(\sigma) = 0$ so there exists $\theta \in H_1(X; R)$ with $\rho(\theta) = \sigma$. We see that $\theta$ is $R$-torsion since $\rho(\theta r) = \rho(\theta) r = \sigma r = 0$ and $\rho$ is a monomorphism. Therefore $\rho : TH_1(X; R) \to TH_1(X, x_0; R)$ is surjective.

For any primitive class $\psi \in H^1(X; \mathbb{Z})$ and splitting $\xi : \mathbb{Z} \to \Gamma_\alpha$ we consider the skew Laurent polynomial ring $\mathbb{K}_n[t^\pm 1]$. We note that $TH_1(X; \mathbb{K}_n[t^\pm 1])$ is a finitely generated right $\mathbb{K}_n$-module. Moreover, any module over $\mathbb{K}_n$ has a well defined rank which is additive on short exact sequences by Remark 1.3.

**Definition 5.8.** Let $X$ be a finite CW-complex. For each primitive $\psi \in H^1(X; \mathbb{Z})$ and $n \geq 0$ we define the refined $n^{th}$-order Alexander module corresponding to $\psi$ to be $A_n^\psi(X) = T_{\mathbb{K}_n[t^\pm 1]} H_1(X; \mathbb{K}_n[t^\pm 1])$ viewed as a right $\mathbb{K}_n[t^\pm 1]$-module.
Since $\mathcal{A}_n^\psi (X)$ is a finitely generated module over the principal ideal domain $\mathbb{K}_n \left[ t^{\pm 1} \right]$, 

$$\mathcal{A}_n^\psi (X) \cong \bigoplus_{i=1}^{m} \frac{\mathbb{K}_n \left[ t^{\pm 1} \right]}{p_i(t) \mathbb{K}_n \left[ t^{\pm 1} \right]}$$

for some nonzero $p_i(t) \in \mathbb{K}_n \left[ t^{\pm 1} \right]$ [3]. We define the refined $n^{\text{th}}$-order degree of $\psi$ to be the degree of the polynomial $\prod p_i(t)$. One can verify that this is equal to the rank of $\mathcal{A}_n^\psi (X)$ as a $\mathbb{K}_n$-module.

**Definition 5.9.** Let $X$ be a finite CW-complex. For each primitive $\psi \in H^1 (X; \mathbb{Z})$ and $n \geq 0$ we define the refined $n^{\text{th}}$-order degree of $\psi$ to be 

$$\delta_n (\psi) = \text{rk}_{\mathbb{K}_n} \mathcal{A}_n^\psi (X)$$

We extend by linearity to define $\delta_n (\psi)$ for non-primitive classes $\psi$.

Similarly we define the unrefined higher-order Alexander modules and degrees.

**Definition 5.10.** Let $X$ is a finite CW-complex. For each primitive $\psi \in H^1 (X; \mathbb{Z})$ and $n \geq 0$ we define the unrefined $n^{\text{th}}$-order Alexander module corresponding to $\psi$ to be $\overline{\mathcal{A}}_n^\psi (X) = H_1 (X; \mathbb{K}_n \left[ t^{\pm 1} \right])$ viewed as a right $\mathbb{K}_n \left[ t^{\pm 1} \right]$-module. The unrefined $n^{\text{th}}$-order degree of $\psi$ is 

$$\overline{\delta}_n (\psi) = \text{rk}_{\mathbb{K}_n} \overline{\mathcal{A}}_n^\psi (X)$$

if $\text{rk}_{\mathbb{K}_n} \overline{\mathcal{A}}_n^\psi (X)$ is finite and 0 otherwise. We extend by linearity to define $\overline{\delta}_n (\psi)$ for non-primitive classes $\psi$.

We note that 

$$\overline{\mathcal{A}}_n^\psi (X) \cong \left( \bigoplus_{i=1}^{m} \frac{\mathbb{K}_n \left[ t^{\pm 1} \right]}{p_i(t) \mathbb{K}_n \left[ t^{\pm 1} \right]} \right) \bigoplus \mathbb{K}_n \left[ t^{\pm 1} \right]^{r_n(X)}.$$  

Hence $\text{rk}_{\mathbb{K}_n} \overline{\mathcal{A}}_n^\psi (X)$ is finite if and only if $r_n(X) = 0$.

**Remark 5.11.** If $r_n(X) = 0$ then $\overline{\delta}_n (\psi) = \delta_n (\psi)$ otherwise $0 = \overline{\delta}_n (\psi) \leq \delta_n (\psi)$.

We now show that $\overline{\delta}_0 (\psi)$ is equal to the Alexander norm of $\psi$ hence $\overline{\delta}_0 (\psi)$ is a convex function.

**Proposition 5.12.** $\overline{\delta}_0 (\psi) = \| \psi \|_A$ for all $\psi \in H^1 (X; \mathbb{Z})$.

**Proof.** Recall that $\Gamma_0 = \mathbb{Z}^{\beta_1 (X)}$ hence $\mathbb{Z} \Gamma_0$ is isomorphic to the polynomial ring in several variables. Let $\nu : \mathbb{Z} \Gamma_0 \hookrightarrow \mathbb{K}_0 \left[ t^{\pm 1} \right]$ be the embedding of $\mathbb{Z} \Gamma_0$ into the principal ideal domain $\mathbb{K}_0 \left[ t^{\pm 1} \right]$ and $\Delta_X = \sum n_g g$ be the Alexander polynomial of $X$. We begin by showing that $\| \psi \|_A = \deg \nu (\Delta_X)$. For all $j$ consider the polynomial $\Delta_X^j = \sum_{|\psi| = j} n_g g$. Note that any such $g$ can be written (using the splitting $\xi$) as $h_g \tau^j$ where $\psi(\tau) = t$. We see that $\nu \left( \sum_{|\psi| = j} n_g g \right) = \sum_{|\psi| = j} n_g h_g \tau^j$ where $c_j \equiv \sum n_g h_g \in \mathbb{Z} [\ker \nu]$. Since $\nu$ is a monomorphism we have $c_j \neq 0$ unless $n_g = 0$ for all $g$ with $\psi(g) = t$. It follows that $\deg \nu (\Delta_X) = \deg \nu \left( \sum \Delta_X^j \right) = \deg \sum c_j t^j = \| \psi \|_A$. After choosing a group presentation for $G$, Fox’s Free Calculus (Section 3) gives us a presentation matrix $M$ for $H_1 (X, x_0; \mathbb{Z} \left[ \Gamma_0 \right]) = H_1 \left( \overline{X}, \overline{x_0} \right)$ where $\overline{X}$ is the torsion-free abelian cover of
X. Moreover a presentation of $H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}])$ is also given by $M''$, that is we consider each entry in as an element of $\mathbb{K}_0 [t^{\pm 1}]$. If $s$ is the number of generators in the presentation of $G$ then $\Delta_X = \gcd (E_1 (H_1 (X, x_0; \mathbb{Z} \Gamma_0))) = \gcd \{ d_1, \ldots, d_k \}$ where $\{ d_1, \ldots, d_k \}$ is the set of determinants of the $(s-1) \times (s-1)$ minors of $M$ (Section 3). Note that $\Gamma_0$ is free abelian so $\mathbb{K}_0$ is a commutative field and hence and $\mathbb{Z} \Gamma_0$ and $\mathbb{K}_0 [t^{\pm 1}]$ are unique factorization domains, since any principal ideal domain is a unique factorization domain. We compute $\gcd (E_1 (H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}]))) = \gcd \{ \nu (d_1), \ldots, \nu (d_k) \}$.

Since $\nu$ is an embedding can checks that the degrees of $\nu (\gcd \{ d_1, \ldots, d_k \})$ and $\nu (d_1), \ldots, \nu (d_k)$ are equal. It follows that

$$\| \psi \|_A = \deg (\Delta_X)$$

$$= \deg (\gcd \{ d_1, \ldots, d_k \})$$

$$= \deg (\gcd \{ \nu (d_1), \ldots, \nu (d_k) \})$$

$$= \deg (E_1 (H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}])))$$

so to complete the proof it suffices to show that $\deg (\gcd \{ \nu (d_1), \ldots, \nu (d_k) \}) = \delta_0 (\psi)$. Since $\mathbb{K}_0 [t^{\pm 1}]$ is a principal ideal domain, $H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}])$ is isomorphic to a direct sum of cyclic $\mathbb{K}_0 [t^{\pm 1}]$-modules. That is, $M''$ is equivalent to a matrix of the form

$$\begin{pmatrix} p_1 (t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{s-1} (t) \end{pmatrix}$$

where $p_i (t)$ is zero for some $i$ if and only if $r_0 (X) > 0$. We note that the last row of the matrix can be assumed to be zero since the $\text{rk}_{\mathbb{K}_0} H_1 (X, x_0; \mathbb{K}_0) = \text{rk}_{\mathbb{K}_0} H_1 (X; \mathbb{K}_0) + 1$. Hence if $r_0 (X) = 0$, $H_1 (X; \mathbb{K}_0 [t^{\pm 1}]) = \mathbb{K} [t^{-1} (X, x_0; \mathbb{K}_0 [t^{\pm 1}])$ so $\delta_0 (\psi) = \deg (p_1 (t) \cdots p_{s-1} (t))$. Otherwise $p_i (t) = 0$ for some $i$ so we have $\delta_0 (\psi) = 0 = \deg (p_1 (t) \cdots p_{s-1} (t))$. Using the latter presentation of $H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}])$ we compute $\gcd (E_1 (H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}]))) = p_1 (t) \cdots p_{s-1} (t)$ so

$$\| \psi \|_A = \deg \gcd \{ E_1 (H_1 (X, x_0; \mathbb{K}_0 [t^{\pm 1}])) \}$$

$$= \deg (p_1 (t) \cdots p_{s-1} (t))$$

$$= \delta_0 (\psi).$$

\[\square\]

6. Computing $\delta_1$ and $\mathcal{A}_n^\psi$ via Fox's Free Calculus

We will describe a method of computing the higher-order invariants using Fox's Free Calculus. We remark that this is slightly different than the classically defined free derivatives because we are working with right (instead of the usual left) modules. We refer the reader to [F1], [F2], [CF] and [H2] for more on the Free Calculus (for left modules).

Let $G$ be any finitely presented group with presentation

$$P = \langle x_1, \ldots, x_l | r_1, \ldots, r_m \rangle,$$
$F = \langle x_1, \ldots, x_l \rangle$ be the free group on $l$ generators and $\chi : F \to G$. For each $x_i$ there is a mapping $\frac{\partial}{\partial x_i} : F \to \mathbb{Z}F$ called the $i$th free derivative. This map is determined by the two conditions

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}$$

$$\frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}.$$  

From these, one can prove that

$$\frac{\partial u^{-1}}{\partial x_i} = -u^{-1} \frac{\partial u}{\partial x_i}.$$  

The map $\chi : F \to G$ extends by linearity to a map $\chi : \mathbb{Z}F \to \mathbb{Z}G$. The matrix

$$\left( \frac{\partial r_j}{\partial x_i} \right)^\chi = \begin{pmatrix} \chi \left( \frac{\partial r_1}{\partial x_1} \right) & \cdots & \chi \left( \frac{\partial r_n}{\partial x_1} \right) \\ \vdots & \ddots & \vdots \\ \chi \left( \frac{\partial r_1}{\partial x_m} \right) & \cdots & \chi \left( \frac{\partial r_n}{\partial x_m} \right) \end{pmatrix}$$  

with entries in $\mathbb{Z}G$ is called the Jacobian of the presentation $P$. We note that this matrix is dependent on the presentation.

Suppose $X$ be a finite CW-complex, $G \cong \pi_1(X, x_0)$ and $\phi : G \to \Gamma$. We can assume that $X$ has one 0-cell, $x_0$. Hence the chain complex of $(X, \tilde{x}_0)$ is

$$\cdots \to \mathbb{Z} \Gamma^m \overset{\partial_2}{\to} \mathbb{Z} \Gamma^l \to 0$$

where $l$ and $m$ are the number of one and two cells of $X$ respectively. We define an involution on the group ring $\mathbb{Z}F$ by

$$\sum m_i f_i = \sum m_i f_i^{-1}$$

and extend $\phi : G \to \Gamma$ to $\phi : \mathbb{Z}G \to \mathbb{Z} \Gamma$ by linearity. It is straightforward to verify that $\tilde{\partial}_2 = \left( \frac{\partial x_j}{\partial x_i} \right)^\chi^\phi$. Hence $H_1(X, x_0; \mathbb{Z} \Gamma)$ is finitely presented as $\left( \frac{\partial r_j}{\partial x_i} \right)^\chi^\phi$. We remark that the existence of the involution in the presentation of $H_1$ is necessary since we chose to work with right rather than left modules. In the case that $\Gamma$ is abelian, the involution is not necessary.

Let $\iota : \mathbb{Z} \Gamma \to R$ be a ring homomorphism. Then $R$ is a $\mathbb{Z} \Gamma$-$R$-bimodule and we can consider the right $R$-module $H_1(X, x_0; R)$. The chain complex for $(X, x_0; R)$ is

$$\cdots \to \mathbb{Z} \Gamma^m \otimes_{\mathbb{Z} \Gamma} R \overset{\partial_2 \otimes id}{\to} \mathbb{Z} \Gamma^l \otimes_{\mathbb{Z} \Gamma} R \to 0.$$  

Since $\mathbb{Z} \Gamma^k \otimes_{\mathbb{Z} \Gamma} R \cong R^k$, it follows that $H_1(X, x_0; R)$ is finitely presented as

$$(6.1) \left( \frac{\partial r_j}{\partial x_i} \right)^{\chi^\phi}.$$  

Now let $\phi = \phi_n : G \to \Gamma_n$ be as defined in Section 3 and $\psi : G \to \mathbb{Z}$. Choose a splitting $\xi : \mathbb{Z} \to \Gamma_n$ and let $R = \mathbb{K}_n [t^{\pm 1}]$. We can use (6.1) to show that $H_1(X, x_0; \mathbb{K}_n [t^{\pm 1}])$ is finitely presented as $\left( \frac{\partial r_j}{\partial x_i} \right)^{\phi_n \xi^\psi}$ where $\iota : \mathbb{Z} \Gamma_n \to \mathbb{K}_n [t^{\pm 1}]$ is the embedding of $\mathbb{Z} \Gamma_n$ into $\mathbb{K}_n [t^{\pm 1}].$
Moreover, we compute the $\delta_n(\psi)$ as follows. Since $K_n[t^{\pm 1}]$ is a principal ideal domain, $\frac{\partial r_j}{\partial x_i}$ is equivalent to a diagonal presentation matrix of the form $\{p_1(t), \ldots, p_\lambda(t), 0_{r,s}\}$ [Theorem 16, p. 43] and $0_{r,s}$ is a $r \times s$ size matrix of zeros. Proposition 5.6 implies that $r_n(X) = \text{rk} K_n[H_1(X, x_0; K_n)] - 1$ hence $r_n(X) = r - 1$.

The above presentation implies that $TH_1(X, x_0; K_n[t^{\pm 1}])$ has a diagonal presentation matrix of the form $\{p_1(t), \ldots, p_\lambda(t)\}$. Moreover, Proposition 5.6 gives $TH_1(X; K_n[t^{\pm 1}]) \cong TH_1(X, x_0; K_n[t^{\pm 1}])$. Thus we have used Fox’s Free Calculus to derive a presentation matrix for $A^\psi_n$ and we have shown that $\delta_n(\psi) = \deg \prod_{1 \leq i \leq \lambda} p_i(t)$.

7. A Presentation of $A^\psi_n$ in Terms of a Surface Dual to $\psi$

In the previous section, we used Fox’s Free Calculus to find a presentation matrix of the higher-order Alexander module, $A^\psi_n(X)$. When $X$ is a 3-manifold, we will show that the localized modules $A^\psi_n(X)$ are finitely presented and that the presentation matrix has topological significance. The matrix will depend on the surface dual to a cohomology class. The presentation will be the higher-order analog of the presentation obtained from a Seifert matrix for knot complements. The presentation obtained will be the main tool that we use in Section 10 to prove that the higher-order degrees give lower bounds for the Thurston norm.

Let $X^3$ be a compact, orientable 3-manifold (possibly with boundary), $G = \pi_1(X, x_0)$ and $\psi \in H^1(X; \mathbb{Z})$. Let $\phi : G \to \Gamma$ be a non-trivial coefficient system and $X_\Gamma \xrightarrow{\psi} X$ be the regular $\Gamma$ cover of $X$. For any $\psi$ as above, there exists a properly embedded surface $F$ in $X$ such that the class $[F] \in H_2(X, \partial X; \mathbb{Z})$ is Poincaré dual to $\psi$. We say that $F$ is dual to $\psi$.

Let $F$ be a surface dual to $\psi$, $Y = X - (F \times (0, 1))$, $F_+ = F \times \{1\}$ (see Figure 1 for an example), and $x_0$ be a point of $F = F \times \{0\} \subset Y$. Let $R$ be a ring $\tau : \mathbb{Z}\Gamma \to R$ be a ring homomorphism defining $R$ as a $\mathbb{Z}\Gamma$-bimodule. We will exhibit a presentation of $H_1(X; R)$ in terms of $H_1(F; R)$ and $H_1(Y; R)$. First we remark that it makes sense to speak of the homology of $F$ with coefficients

![Figure 1. The Whitehead manifold cut open along $F$](image)
in $R$. By this, we mean the homology corresponding to the coefficient system

$$\pi_1(F,x_0) \xrightarrow{\gamma} \pi_1(X,x_0) \xrightarrow{\alpha} \Gamma.$$  

Similarly, we have coefficient systems for $\pi_1(Y,x_0)$ and the other terms that are involved in Proposition 7.3 below.

Before we state Proposition 7.1 we need the following notation. Let $c$ be a path in $Y$ with initial point $c(0) = (x_0,1)$ and endpoint $c(1) = (x_0,0)$. Let $c_+ = (x_0,s)$ and $\alpha$ be the closed curve $c_0 \cdot c$ based at $x_0$. Let $i_\pm : F \to Y$ include $F$ into $Y$ by $i_- (f) = (f,0)$ and $i_+ (f) = (f,1)$. Finally let $j : Y \to X$ be the inclusion of $Y$ into $X$.

**Proposition 7.1.** Suppose $\gamma = \phi(\alpha)$ is a non-zero element of $\Gamma$ and either some element of the augmentation ideal of $\mathbb{Z}[\pi_1(F)]$ is invertible (under $\sigma \circ \phi \circ i_\pm$) in $R$ or $\pi_1(F) = 1$. Then the sequence

$$H_1(F; R) \xrightarrow{\eta} H_1(Y; R) \xrightarrow{\alpha} H_1(X; R) \to H_0(F \cup F_+ \cup c; R)$$

is exact where $\eta = (i_+) - (i_-) \gamma$.

**Proof.** For convenience, we will omit the $R$ in $H_1(-;R)$ in this proof. Let $U = (F \times I) \cup c$ where $I = [0,1]$. Then $X = U \cup Y$ and $U \cap Y = F \cup F_+ \cup c$. Consider the homology Mayer-Vietoris sequence for $(U, Y)$ with coefficients in $R \mathbb{Z}$.

$$H_1(F \cup F_+ \cup c) \to H_1((F \times I) \cup c) \oplus H_1(Y) \to H_1(X) \to H_0(F \cup F_+ \cup c) \to .$$

We examine the $H_1$ terms involving $F$ in (7.1). We will compute the homology of these term using the Mayer-Vietoris sequences for $(F, F_+ \cup c)$,

$$0 \to H_1(F) \oplus H_1(F_+ \cup c) \to H_1(F \cup F_+ \cup c) \to H_0(x_0) \to$$

and $(F \times I, \alpha)$,

$$0 \to H_1(F \times I) \oplus H_1(\alpha) \to H_1(F \times I \cup c) \to H_0(x_0) \to$$

The idea behind the rest of the proof in both of the cases (stated in the hypothesis) are similar however the proof when $\pi_1(F) \neq 1$ is more technical. Hence we will first consider the special case when $\pi_1(F) = 1$. Since $\pi_1(F)$ is trivial, both $H_0(x_0) \to H_0(F) \oplus H_0(F_+ \cup c)$ and $H_0(x_0) \to H_0(F \times I) \oplus H_0(\alpha)$ are injective. Hence, $H_1(F \cup F_+ \cup c; M) \cong H_1(F; M) \oplus H_1(F_+ \cup c; M)$ and $H_1((F \times I) \cup c; M) \cong H_1((F \times I) \oplus H_1(\alpha)$.

Since $\gamma$ is non-trivial in $\Gamma$, the curve $\alpha$ does not lift to the $\Gamma$-cover of $X$. Therefore $H_1(\alpha) = 0$ and hence $H_1((F \times I \cup c)) \cong H_1(F \times I)$. Moreover, $H_1((F \times I)) \cong H_1(F)$ where the isomorphism is induced by the map which sends $(f,s)$ to $(f,0)$.

We analyze the first term in the sequence. The isomorphism $\pi_1(F,x_0) \to \pi_1(F_+ \cup c,x_0)$ given by $[\beta] \to [c \cdot i_+ \beta \cdot \bar{\gamma}]$ induces an isomorphism $g : H_1(F) \to H_1(F_+ \cup c)$. By $\gamma$ we mean the curve defined by $\gamma(s) = c(1-s)$. Note that $[c \cdot i_+ \beta \cdot \bar{\gamma}] = \alpha^{-1}[\beta \cdot \bar{\gamma}]$ in $\pi_1(F \times I, x_0)$. Therefore $H_1(F \cup F_+ \cup c) \cong H_1(F) \oplus H_1(F)$. We note that the composition

$$H_1(F) \to H_1(F_+ \cup c) \to H_1(F \times I \cup c) \to H_1(F)$$

sends $\sigma$ to $\sigma \gamma$ and the composition $H_1(F) \to H_1(F \times I \cup c) \to H_1(F)$ is the identity.
Using the isomorphisms above, we rewrite (7.1) as
\[ H_1(F) \oplus H_1(F) \xrightarrow{(f_r,f_Y)} H_1(F) \oplus H_1(Y) \to H_1(X) \to H_0(F \cup F_+ \cup c) \to \]
where \( f_r(\sigma_1,\sigma_2) = \sigma_1 + \sigma_2 \gamma \) and \( f_Y(\sigma_1,\sigma_2) = -(i_-)_*(\sigma_1) + (i_+)_*(\sigma_2) \). It follows from Lemma 7.3 that the sequence
\[ H_1(F) \xrightarrow{\eta} H_1(Y) \xrightarrow{i} H_1(X) \to H_0(F_+ \cup F) \]
is exact with \( \eta(\sigma) = -(i_-)_*(\sigma) + (i_+)_*(-\sigma) = ((i_+)_* - (i_-)_*)\gamma(\sigma) \). The proof of Lemma 7.3 is straightforward hence omitted.

**Lemma 7.2.** Suppose \( A \oplus A \xrightarrow{(f_A,f_B)} A \oplus B \xrightarrow{g} C \xrightarrow{h} D \) is an exact sequence of right \( R \)-modules with \( f_A(a_1,a_2) = a_1 + a_2r \) where \( r \in R \), then \( A \xrightarrow{f'_A} B \xrightarrow{g} C \xrightarrow{h} D \) is an exact sequence of right \( R \)-modules with \( f'(a) \equiv f_B(ar, -ar) \) and \( g'(b) \equiv g(0,b) \).

Now we assume that some element of the augmentation ideal of \( \mathbb{Z}[\pi_1(F)] \) is invertible in \( R \). By \([W] \) p. 275 and \([P] \) p. 34, \( H_0(F) \) is isomorphic to the cofixed set \( R/RJ \) where \( J \) is the augmentation ideal of \( \mathbb{Z}[\pi_1(F)] \). Therefore \( H_0(F) = H_0(F_+ \cup c) = 0 \). We note that \( H_0(x_0) \) is the free \( R \)-module of rank one generated by \([x_0] \). Choose a splitting \( \xi_1 \) for short exact sequence in (7.2) to get
\[ H_1(F \cup F_+ \cup c) \cong M \oplus H_1(F) \oplus H_1(F_+ \cup c) \]
where \( M \) is the free \( R \)-module of rank one generated by \( \xi_1([x_0]) \). Let \( \beta \) be a curve in \( F \cup F_+ \cup c \) representing \( \xi_1([x_0]) \).

Consider the sequence in (7.2). We note that \( H_0(\alpha) = R/\langle \gamma \rangle \). Since \( \gamma \) is non-trivial, \( \text{Im}(H_1(F \times I \cup \alpha) \to H_0(x_0)) \) is the free \( R \)-module of rank one generated by \( \langle \gamma \rangle \) \([x_0] \). Moreover, \( \beta \to u(\gamma - 1)[x_0] \) where \( u \) is a unit of \( R \) under the boundary homomorphism. We choose the splitting \( \xi_2 : u(\gamma - 1)[x_0] \to [\beta] \) to get
\[ H_1(F \times I \cup \alpha) \cong N \oplus H_1(F \times I) \oplus H_1(\alpha) \]
where \( N \) is the free \( R \)-module of rank one generated by \( [\beta] \).

Using the isomorphisms in (7.4) and (7.5), we can rewrite (7.2) as
\[ M \oplus H_1(F) \oplus H_1(F_+ \cup c) \to N \oplus H_1(F \times I) \oplus H_1(\alpha) \oplus H_1(Y) \to H_1(X) \to H_0(F_+ \cup F) \to . \]

We use Lemma 7.3 to get an exact sequence without the \( M \) and \( N \) terms as in the case when \( \pi_1(F) = 1 \). To complete the proof of the proposition, we use the same argument as in the case when \( \pi_1(F) = 1 \). \( \square \)

**Lemma 7.3.** Let \( M \) and \( N \) be free right \( R \)-modules of rank one generated by \( m \) and \( n \) respectively. Suppose \( M \oplus A \xrightarrow{f} N \oplus B \xrightarrow{g} C \xrightarrow{h} D \) is an exact sequence of right \( R \)-modules with \( f(rm,a) = (rn,f_2(rm,a)) \) for some \( f_2 : M \oplus A \to B \). Let \( \eta : 0 \oplus B \to B \) be the isomorphism defined by \( (0,b) \to b \). Then \( A \xrightarrow{f'} B \xrightarrow{g'} C \xrightarrow{h} D \) is an exact sequence of right \( R \)-modules where \( f' \) and \( g' \) are defined by \( f'(a) = \eta(f(0,a)) \) and \( g'(b) = (g(0,b)) \).

**Proof.** The proof is straightforward hence omitted. \( \square \)
Now we consider the presentation of \( H_1 \left( X; \mathbb{K}_n \left[ t^{\pm 1} \right] \right) \) where \( \mathbb{K}_n \) is the skew field of fractions of \( \mathbb{Z} \Gamma'_n \) as defined before. Let \( F \) and \( Y \) be as defined above. Since \( \pi_1 (F, x_0) \) and \( \pi_1 (Y, x_0) \) are contained in the kernel of \( \psi \), we can consider the homology of \( F \) and \( Y \) with coefficients in \( \mathbb{Z} \Gamma' \) and \( \mathbb{K}_n \). Since \( F \) and \( K \) are finite CW-complexes, \( H_1 (F; \mathbb{K}_n) \) and \( H_1 (Y; \mathbb{K}_n) \) are finitely generated free modules hence are isomorphic to \( \mathbb{K}_n^1 \) and \( \mathbb{K}_n^m \) respectively. Thus the \( \mathbb{K}_n \)-module homomorphisms \( i_\pm : H_1 (F; \mathbb{K}_n) \to H_1 (Y; \mathbb{K}_n) \) can be represented by \( l \times m \) matrices \( V_\pm \) with coefficients in \( \mathbb{K}_n \). We will show that the higher-order module corresponding to \( \psi \) is presented by \( V_+ - V_- t \).

**Proposition 7.4.** \( \text{Im} (j_\gamma) \) is finitely presented as \( \mathbb{K}_n \left[ t^{\pm 1} \right] P \to \mathbb{K}_n \left[ t^{\pm 1} \right] \gamma \to \mathbb{K}_n \left[ t^{\pm 1} \right] \). Moreover, if \( \psi \big|_{\pi_1 (F)} \) is nontrivial for each component \( F^j \) of \( F \) then \( \mathcal{A}_n^\psi (X) \) is finitely presented as

\[
\mathbb{K}_n \left[ t^{\pm 1} \right] P \to \mathbb{K}_n \left[ t^{\pm 1} \right] \gamma \to \mathcal{A}_n^\psi (X).
\]

**Proof.** Let \( \gamma = \phi_n (\alpha) \) be as in Proposition 7.5. We note that \( \psi (\gamma) = t \). Choose the splitting \( \xi : t \to \gamma \). Since \( \pi_1 (F, x_0) \subset \Gamma'_n \), we have

\[
C_\ast (F \Gamma_n) \otimes_{\mathbb{Z} \Gamma_n} \mathbb{K}_n \left[ t^{\pm 1} \right] \cong \left( C_\ast (F \Gamma_n) \otimes_{\mathbb{Z} \Gamma_n} \mathbb{K}_n \right) \otimes_{\mathbb{K}_n} \mathbb{K}_n \left[ t^{\pm 1} \right].
\]

Moreover, \( \mathbb{K}_n \left[ t^{\pm 1} \right] \) is a direct sum of free \( \mathbb{K}_n \)-modules \( \left( \mathbb{K}_n \left[ t^{\pm 1} \right] \cong \oplus_{i = -\infty}^{\infty} \mathbb{K}_n \right). \) Therefore \( \mathbb{K}_n \left[ t^{\pm 1} \right] \) is a flat left \( \mathbb{K}_n \)-module. Thus

\[
H_1 \left( F; \mathbb{K}_n \left[ t^{\pm 1} \right] \right) \cong H_1 \left( F; \mathbb{K}_n \right) \otimes_{\mathbb{K}_n} \mathbb{K}_n \left[ t^{\pm 1} \right]
\]

\[
\cong \mathbb{K}_n \otimes_{\mathbb{K}_n} \mathbb{K}_n \left[ t^{\pm 1} \right]
\]

\[
\cong \mathbb{K}_n \left[ t^{\pm 1} \right].
\]

Similarly, we have \( H_1 \left( F; \mathbb{K}_n \left[ t^{\pm 1} \right] \right) \cong \mathbb{K}_n \left[ t^{\pm 1} \right] \). The first result follows from Proposition 7.5. If \( \psi \big|_{\pi_1 (F)} \) is nontrivial for all \( j \) then \( H_0 \left( F \cup F \cup c; \mathbb{K}_n \left[ t^{\pm 1} \right] \right) = 0 \) so \( \text{Im} (j_\gamma) = \mathcal{A}_n^\psi (X) \).

We use the following lemma to show that it suffices to use the presentation matrix \( V_+ - V_- t \) when computing \( \mathcal{A}_n^\psi (X) \).

**Lemma 7.5.** Suppose \( B \overset{g}{\to} C \overset{h}{\to} D \) is an exact sequence of right \( R \)-modules where \( D \) is \( R \)-torsion free and \( R \) is an Ore domain then \( TRC = TR \text{Im} (g) \).

**Proof.** Since \( \text{Im} (g) \subseteq C \), it is easy to verify \( T \text{Im} (g) \subseteq TC \). Let \( c \in TC \) then there exists a non-zero \( r \in R \) such that \( cr = 0 \). This says that \( h (c) r = h (cr) = 0 \) so that \( h (c) \) is \( R \)-torsion in \( D \) hence \( h (c) = 0 \). By exactness at \( C \) we see that \( c \in \text{Im} (g) \) and \( cr = 0 \) so \( TC \subseteq T \text{Im} (g) \).

**Proposition 7.6.** \( \mathcal{A}_n^\psi (X) \) is isomorphic to the \( \mathbb{K}_n \left[ t^{\pm 1} \right] \)-torsion submodule of \( \text{cok} (V_+ - V_- t) \).

**Proof.** Recall that \( \mathcal{A}_n^\psi (X) \cong T_{\mathbb{K}_n \left[ t^{\pm 1} \right]} \mathcal{A}_n^\psi (X) \). The result follows immediately from Lemma 7.5 and Proposition 7.4.

\[ \square \]
8. Examples

In this section we will compute $r_n$, $\delta_n$, and $\delta_0$ for some well known 3-manifolds and relate their values to those given by the Thurston norm. In each of the examples we denote the fundamental group of $X$ by $G$. Of particular importance will be the 3-manifolds which fiber over $S^1$ and those which are Seifert fibered. We start with the standard examples.

**Example 8.1.** 3-torus

Let $X = S^1 \times S^1 \times S^1$ then $G^{(1)} = \{1\}$ hence $r_n (X) = \delta_n (\psi) = 0$ for all $\psi$ and $n \geq 0$. Note that since $H_2 (X; \mathbb{Z})$ is generated by tori, the Thurston norm is zero for all $\psi \in H^1 (X; \mathbb{Z})$. More generally, if $G$ is any finitely generated abelian group (with $\beta_1 (G) \geq 1$) then $G^{(n)} = T$ where $T$ is the torsion subgroup of $G$. Hence $r_n (X) = \delta_n (\psi) = 0$.

**Example 8.2.** $X = \#_i S^2 \times S^1$

Let $X = \#_i S^2 \times S^1$ for $m \geq 1$ then $G = F_m$, the free group on $m$ generators. Since $H_2 (X; \mathbb{Z})$ is generated by spheres $\|\psi\|_{T} = 0$ for all $\psi$. Moreover, every class in $H_2 (X; \mathbb{Z})$ can be represented by a disjoint union of embedded spheres. Hence there exists a surface $F$ dual to $\psi$ such that $H_1 (F; K_n [ t^{\pm 1} ]) = 0$. By Proposition 7.6 we have $A_\psi (X) = T \cok (0)$. Therefore $\delta_n (\psi) = 0$ for all $\psi$ and $n \geq 0$.

Since $r_n$ only depends on the group $G$ we can assume that $X$ is a wedge of $m$ circles. By Remark 4.4, $1 - m = \chi (X) = \sum_{i=0}^{3} rk_{K_n} H_i (X; K_n)$. Since $\phi_n$ is a non-trivial homomorphism, by Lemma 5.7, $H_0 (X; K_n) = 0$. Therefore $r_n (X) = m - 1$.

There is a large class of 3-manifolds for which $r_0 (X) \geq 1$. Recall that a boundary link is a link $L$ in $S^3$ such that the components admit mutually disjoint Seifert surfaces. It is easy to see that each of these surfaces lifts to the universal abelian cover of $X = S^3 - L$. By Proposition 10.6, $r_0 (X) \geq 1$. Hence the Alexander norm for $X$ is always trivial. It is often true that the refined Alexander norm, $\delta_0$, is non-trivial. We compute an example below where this is the case.

Let $L$ be the link pictured in Figure 2. Let $F$ be a Seifert surface of one of the components of $L$ as in Figure representing the minimal $\chi$- and $\psi$ be dual to $F$. We will show in Example 8.3 that $\delta_0 (\psi) = 4$. Moreover, each component bounds a once punctured genus two surface hence $\|\psi\|_{T} \leq 3$. Hence by Corollary 10.4, we get $\delta_0 \leq \|\psi\|_{T} + 1$.

![Figure 2](image)

Each component of $L$ has minimal ropelength at least $2\pi (1 + \sqrt{3})$. 


We conclude that \( \|\psi\|_T = 3 \). Hence, even for \( n = 0 \), \( \delta_n \) gives a sharper bound for the Thurston norm that the Alexander norm. This also shows that the minimal ropelength of each of the components of \( L \) is at least \( 2\pi (1 + \sqrt{3}) \). For more information on the ropelength of knots and links, see [CKS, Corollary 22].

**Example 8.3.** Let \( L \) be the link in Figure 2 and \( X = S^3 - \tilde{L} \). We use the techniques of Section 8 to compute \( \delta_0 (\psi) \). Using a Wirtinger presentation, we present \( G = \pi (X) \) as

\[
\langle a, b, c, d, e, f, g, h, i, j, k, l \mid b g^{-1} i c^{-1} i^{-1} g, c j^{-1} l a^{-1} l^{-1} j, f e^{-1} h g^{-1} h^{-1} e, \\
i h^{-1} k j^{-1} k^{-1} h, l k^{-1} e d^{-1} e^{-1} k, d a^{-1} e^{-1} a, e b f^{-1} b^{-1}, \\
g b^{-1} h^{-1} b, h c i^{-1} c^{-1}, j c^{-1} k^{-1} c, k a l^{-1} a^{-1} \rangle .
\]

Using Fox’s Free Calculus we obtain a presentation matrix \( M \) for \( H_1 (X_0, \tilde{x}_0) \) (see below). Here, \( x \) is the meridian of \( a \) and \( y \) is the abelianization of \( d \). Since we used a Wirtinger presentation for \( G \), \( x \) and \( y \) represent the meridians of \( L \).

\[
\begin{pmatrix}
0 & -y & 0 & 0 & 0 & 1 - y & 0 & 0 & 0 & y - 1 \\
y & 0 & 0 & 0 & 0 & y - 1 & 1 - y & 0 & 0 & 0 \\
- y & y & 0 & 0 & 0 & 0 & 0 & y - 1 & 1 - y & 0 \\
0 & 0 & 0 & 0 & - y & x & 0 & 0 & 0 & 0 \\
0 & 0 & 1 - y & 0 & y - 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & y & 0 & 0 & 0 & - x & 0 & 0 & 0 \\
1 - x & 0 & - y & 0 & 0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & y - 1 & 1 - y & 0 & 0 & 0 & -1 & 1 & 0 \\
x - 1 & 0 & 0 & y & 0 & 0 & 0 & 0 & - x & 0 \\
0 & 1 - x & 0 & - y & 0 & 0 & 0 & 0 & 0 & x \\
0 & 0 & y - 1 & 1 - y & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & x - 1 & 0 & 0 & y & 0 & 0 & 0 & 0 & - x
\end{pmatrix}
\]

This is equivalent (using the moves in Lemma 9.2) to the matrix

\[
\begin{pmatrix}
1 - x - y & 0 & 0 & 0 & 0 \\
0 & 1 - x - y & 0 & 0 & 0 \\
0 & 0 & x y - x - y & 0 & 0 \\
0 & 0 & 0 & x y - x - y & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence \( r_0 (X) = 1 \) and

\[
\mathcal{A}_0 (X) = \frac{\mathbb{Z} [x, y]}{\langle x + y \rangle} \oplus \frac{\mathbb{Z} [x, y]}{\langle x + y \rangle} \oplus \frac{\mathbb{Z} [x, y]}{\langle x + y \rangle} \oplus \frac{\mathbb{Z} [x, y]}{\langle x + y \rangle} \oplus \frac{\mathbb{Z} [x, y]}{\langle x + y \rangle}.
\]

Let \( \psi \) be dual to a Seifert surface for one of the components of \( L \). Then \( \psi \) maps one of the generators of \( H_1 (X) \) to \( t \) and the other to 1. The link is symmetric so either choice will suffice. Say \( x \mapsto t \) and \( y \mapsto 1 \). Choose the splitting \( t \mapsto x \). Each of the polynomials in the latter matrix has degree 1 in \( K_0 [x, y] \) since \( 1 - x - y \mapsto (1 - y) + t \) and \( x y - x - y \mapsto t (y - 1) - y \). Therefore \( \delta_0 (\psi) = 4 \) as desired. In fact, if \( \psi \) maps
$x \mapsto t^m$ and $y \mapsto t^n$ then

$$\delta_0(\psi) = \deg \left( t^{2n} + t^{3m+n} + t^{2m+2n} + t^{2m+n} + t^{m+n} + t^{2m} + \right.$$  
$$t^{13m} + t^{3n} + t^{4m} + t^{4n} + t^{3m+2n} + t^{3m+n} + t^{2m+3n} + t^{m+3n} + t^{4m+2n} + t^{4m+n} + t^{3m+3n} + t^{4n+2m} + t^{4n+m} \right).$$

whereas $\overline{\delta}_0(\psi) = 0$.

Although the invariants are defined algebraically, they respect much of the topology of the 3-manifold. We begin by considering those 3-manifolds which fiber over $S^1$. In this case the higher-order invariants behave in a very special manner.

**Proposition 8.4.** If $X$ is a compact, orientable 3-manifold that fibers over $S^1$ then

$$r_n(X) = 0.$$ 

Let $\psi$ be dual to a fibered surface. If $n = 0$, $\beta_1(X) = 1$, $X \not\equiv S^1 \times S^2$ and $X \not\equiv S^1 \times D^2$ then $\delta_n(\psi) = \|\psi\|_T + 1 + \beta_3(X)$. Otherwise,

$$\delta_n(\psi) = \|\psi\|_T.$$ 

**Proof.** Consider $X \to S^1$ with fiber surface $F$ and $\psi$ be the element of $H^1(X;\mathbb{Z})$ which is dual to $F$. The $\Gamma_n$-cover of $X$ factors through the infinite cyclic cover corresponding to $\psi$ with total space $F \times \mathbb{R}$. Hence $X_n$ is homeomorphic to $F_n \times \mathbb{R}$ where $F_n$ is a regular cover of $F$. It follows that $H_*(X;\mathbb{K}_n[t^{\pm 1}])$ is isomorphic to $H_*(F;\mathbb{K}_n)$ as a $\mathbb{K}_n$-module. In particular, $\delta_n(\psi) = \text{rk}_{\mathbb{K}_n} H_1(F;\mathbb{K}_n)$. Moreover, since $H_1(F;\mathbb{K}_n)$ is a finitely generated $\mathbb{K}_n$-module, $r_n(X) = 0$ for all $n$. That is, $H_1(X;\mathbb{K}_n[t^{\pm 1}])$ is a torsion module for all $n \geq 0$.

We restrict to the case that $n = 0$ and $\beta_1(X) = 1$. We note that $F_0 = F$ and $\mathbb{K}_0 = \mathbb{Q}$ so that $\text{rk}_{\mathbb{Q}} H_1(F;\mathbb{K}_0) \cong \beta_1(F)$. Thus $\delta_0(\psi) = \beta_1(F) = -\chi(F) + 1 + \beta_3(X)$. As long as $X \not\equiv S^1 \times S^2$ and $X \not\equiv S^1 \times D^2$ the Euler characteristic of $F$ is non-positive hence $\|\psi\|_T = -\chi(F)$. Therefore $\delta_0(\psi) = \|\psi\|_T + 1 + \beta_3(X)$.

Note that if the Euler characteristic of $F$ is ever positive then $\pi_1(F) = 1$. Thus we have $H_1(F;\mathbb{K}_n) = 0$ for all $n \geq 0$. Therefore both $\delta_n(\psi)$ and $\|\psi\|$ are zero and hence equal for all $n \geq 0$.

Otherwise, $F_n$ factors through a non-trivial free abelian cover of $F$. By Lemma 2.7. $H_0(F;\mathbb{K}_n) = 0$. Since $F_n$ is noncompact, $H_2(F;\mathbb{K}_n) = 0$. It follows that $\delta_n(\psi) = \text{rk}_{\mathbb{K}_n} H_1(F;\mathbb{K}_n) = -\chi(F)$. Moreover, $F$ has non-positive Euler characteristic so $-\chi(F) = \|\psi\|_T$. $
$

As with the first two examples in this section, there is a large class of 3-manifolds which have vanishing (unrefined) higher-order degrees. This is the class of Seifert fibered manifolds that do not fiber over $S^1$. We remark that the condition that $X$ not fiber over $S^1$ is necessary by the previous proposition. Some good references on Seifert fibered manifolds are [Ha2, Ch. 2], [H1, Ch. 12], and [Ja, Ch. VI].

**Proposition 8.5.** Let $X$ be a compact, orientable Seifert fibered manifold that does not fiber over $S^1$. If $\beta_1(X) \geq 2$ or $n \geq 1$ then

$$\overline{\delta}_n(\psi) = 0$$

for all $\psi \in H^1(X;\mathbb{Z})$. 

Proof. This is most easily proven using Theorem 10.1 and some well known results about Seifert fibered 3-manifolds. By Theorem VI.34 of [1], we see that any two-sided incompressible surface in \( X \) must be a disc, annulus, or a torus. Therefore the Thurston norm of \( X \) is trivial. Theorem 10.1 implies that \( \delta_n(\psi) \leq \|\psi\|_T = 0 \) whenever \( \beta_1(X) \) or \( n \geq 1 \).

We end this section by showing that under the connected sum of 3-manifolds, the degrees are additive and the ranks plus 1 are additive. The following is not at all obvious because the fundamental groups of the spaces involved are completely different!

**Proposition 8.6.** Suppose \( X = X_1 \# X_2, \beta_1(X_i) \geq 1 \) and \( \psi \in H^1(X; \mathbb{Z}) \). Then
\[
(r_n(X) = r_n(X_1) + r_n(X_2) + 1)
\]
and
\[
\delta_n(\psi) = \delta_n(\psi_1) + \delta_n(\psi_2)
\]
where \( \psi = \psi_1 \oplus \psi_2 \).

**Proof.** We begin by showing that \( r_n(X) = r_n(X_1) + r_n(X_2) + 1 \). Consider the following Mayer-Vietoris sequence of \( R \)-modules for any ring \( R \) with \( \mathbb{Z}_n \subseteq R \subseteq K_n \).

By \( \Gamma_n \) we mean the quotient of \( G = \pi_1(X) \) by the \((n+1)\)st term of the rational derived series of \( G \).

\[
\begin{align*}
0 \to H_1(X_1; R) \oplus H_1(X_2; R) \overset{\phi}{\to} H_1(X; R) \overset{\delta}{\to} H_0(S^2; R) \to 0
\end{align*}
\]

We note that \( H_0(S^2; R) \cong R \) since \( S^2 \) is simply connected. For \( j = 1, 2 \) let \( i_j : G_j \to G \) be the inclusion map, \( \text{pr}_j : G \to G_j \) be the projection onto the \( j \)th factor, and \( \Gamma_n \cong (G_j)/(G_j)^{(n+1)} \) where \( G_j = \pi_1(X_j) \). Since \( G = G_1 * G_2 \), \( \text{pr} \circ i_j = id_{G_j} \). Hence the induced maps \( \Gamma_n^j \to \Gamma_n \) are also the identity making \( \Gamma_n^j \) a monomorphism. Thus the \( \Gamma_n^j \) cover of \( X_j, X_{jn} \), can be constructed as the regular cover corresponding to the map \( \phi_j \circ i_j : G_j \to \text{Im}(\phi_j \circ i_j) \). We extend \( \Gamma_n^j \to \text{Im}(\phi_j \circ i_j) \) to a ring monomorphism \( \overline{\pi}_j : \mathbb{Z}_n \Gamma_n^j \to \mathbb{Z}_n \Gamma_n \).

The map \( G_j \to G/G_{r(n+1)} \) is the zero map if and only if \( G_j/(G_j)^{(n+1)} = 0 \). We assumed \( \beta_1(X_j) > 0 \) hence \( G_j/(G_j)^{(n+1)} \neq 0 \). By Lemma 7.3, \( H_0(X; K_n) = 0 \).

Replacing \( R \) by \( K_n \) in (8.1) we have
\[
r_n(X) = \text{rk}_{K_n} H_1(X_1; K_n) + \text{rk}_{K_n} H_1(X_2; K_n) + 1.
\]

We will show that \( \text{rk}_{K_n} H_1(X_1; K_n) = \text{rk}_{K_n} H_1(X_1; K_n^j) \) hence
\[
r_n(X) = r_n(X_1) + r_n(X_2) + 1.
\]

Let \( \overline{X}_{jn} \) be the cover of \( X_j \) corresponding to \( G_j \to \Gamma_n \). Then \( X'_{jn} \) is a disjoint union of \( \Gamma_n/\Gamma_n^j \) copies of \( X_{jn} \). The extension of \( \overline{\pi}_j \) to a ring homomorphism \( \overline{\pi}_j : \mathbb{Z}_n \Gamma_n \to \mathbb{Z}_n \Gamma_n^j \) gives \( \mathbb{Z}_n \Gamma_n^j \) the structure as a \( \mathbb{Z}_n \)-bimodule. Moreover, since \( \overline{\pi}_j \circ \overline{\pi}_j = id_{\mathbb{Z}_n^j} \otimes_{\mathbb{Z}_n} (\mathbb{Z}_n \otimes_{\mathbb{Z}_n} \mathbb{Z}_n^j) \) acts trivially on any right \( \mathbb{Z}_n^j \)-module. Therefore
\[
(8.2) \quad C_*(\overline{X}_{jn}) \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \Gamma_n^j \cong \left( C_*(X_{jn}) \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \Gamma_n \right) \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \Gamma_n^j
\]
\[
\cong \left( C_*(X_{jn}) \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \right) \otimes_{\mathbb{Z}_n} \mathbb{Z}_n \Gamma_n^j
\]
\[
\cong C_*(X_{jn}).
\]
\[ \bar{t}_j : \mathbb{Z} \Gamma_n^j \rightarrow \mathbb{Z} \Gamma_n \] is a monomorphism, hence we can extend \( \bar{t}_j \) to the right ring of quotients of \( \mathbb{Z} \Gamma_n^j \) and \( \mathbb{Z} \Gamma_n \), \( \bar{t}_j : \mathcal{K}_n^j \rightarrow \mathcal{K}_n \). Therefore \( \mathcal{K}_n \) is a flat left \( \mathcal{K}_n^j \)-module by the following lemma.

**Lemma 8.7.** Suppose that \( R \) is a right and left principal ideal domain, \( S \) has no zero divisors, and \( f : R \rightarrow S \) is a ring monomorphism. Then \( S \) is a flat left \( R \)-module.

**Proof.** Let \( s \in S \) and \( r \in R \) with \( s \neq 0 \). Suppose that \( f(r)s = 0 \). \( S \) has no zero-divisors hence \( f(r) = 0 \). Moreover, \( f \) is a monomorphism so \( r = 0 \). Therefore \( S \) is \( R \)-torsion-free. Since \( R \) is a PID, every finitely generated torsion-free \( R \)-module is free hence flat. Every module is the direct limit of its finitely generated submodules. Hence \( S \) is the direct limit of flat modules. Thus, by [Ste, Prop. 10.3], \( S \) is flat. \( \square \)

We apply \( - \otimes_{\mathbb{Z} \Gamma_n} \mathcal{K}_n \) to (8.2) to get

\[ C_* \left( \overline{X_{jn}} \right) \otimes_{\mathbb{Z} \Gamma_n} \mathcal{K}_n \cong C_* \left( X_{jn} \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n. \]

Since \( C_* \left( X_{jn} \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n \cong C_* \left( X_{jn} \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n^j \otimes_{\mathcal{K}_n} \mathcal{K}_n \) and \( \mathcal{K}_n \) is a flat left \( \mathcal{K}_n^j \)-module,

\[ H_* \left( C_* \left( \overline{X_{jn}} \right) \otimes_{\mathbb{Z} \Gamma_n} \mathcal{K}_n \right) \cong H_* \left( C_* \left( X_{jn} \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n^j \otimes_{\mathcal{K}_n} \mathcal{K}_n \right). \]

Thus \( \text{rk}_{\mathcal{K}_n} H_1 \left( X_1; \mathcal{K}_n^j \right) = \text{rk}_{\mathcal{K}_n} H_1 \left( X_1; \mathcal{K}_n \right) \) as desired.

Now we show that

\[ (8.3) \quad T_R \left( H_1 \left( X_1; R \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n^j \right) \cong T_R \left( H_1 \left( X_1; \mathcal{K}_n^j \right) \right). \]

First we note that \( T \left( H_1 \left( X_1; R \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n^j \right) \cong T \left( H_1 \left( X_1; \mathcal{K}_n^j \right) \right) \). Consider the restriction of \( \nu \) in (8.1) to the torsion submodule of \( H_1 \left( X_1; R \right) \otimes_{\mathbb{Z} \Gamma_n^j} \mathcal{K}_n^j \)

\[ \nu_T : T \left( H_1 \left( X_1; R \right) \right) \rightarrow T \left( H_1 \left( X_1; \mathcal{K}_n^j \right) \right). \]

We show that \( \nu_T \) is an isomorphism. It is immediate that \( \nu_T \) is a monomorphism since \( \nu \) is a monomorphism. To show that \( \nu_T \) is surjective, let \( x \in T \left( H_1 \left( X_1; R \right) \right) \) and \( 0 \neq r \in R \) with \( x r = 0 \). Since \( H_0 \left( S^2; R \right) \) is \( R \)-torsion free, \( d_1 \left( x r \right) = 0 \) hence there exists \( y \in H_1 \left( X_1; R \right) \) such that \( \nu_T \left( y \right) = x \). Moreover, \( \nu_T \left( y r \right) = r x r = 0 \). Since \( \nu_T \) is a monomorphism \( y r = 0 \). Hence \( y \) is in \( T \left( H_1 \left( X_1; R \right) \right) \).

Since \( H^1 \left( X_1; \mathbb{Z} \right) \cong H^1 \left( X_1; \mathbb{Z} \right) \), \( \psi \) can be uniquely extended to \( \psi_1 \oplus \psi_2 \) where \( \psi_j \in H^1 \left( X_1; \mathbb{Z} \right) \). Note that \( \psi_j \) need not be a primitive class in \( H^1 \left( X_1; \mathbb{Z} \right) \). For each \( j \), let \( d_j \) be the largest divisor of \( \psi_j \). Hence, there exist \( \psi_j \) primitive with \( d_j \psi_j = \psi \). Recall that \( \ker \psi_j = \ker \psi_j \) and \( \delta_n \left( \psi_j \right) = d_j \delta_n \left( \psi_j \right) \).

Substitute \( R = \mathbb{Z} \Gamma_n \left( \mathbb{Z} \ker \psi \right)^{-1} \) into (8.3). Then

\[ \delta_n \left( \psi \right) = \text{rk}_{\mathbb{K}_n} T_{H_1} \left( X_1; \mathbb{Z} \Gamma_n \left( \mathbb{Z} \ker \psi \right)^{-1} \right) \oplus \text{rk}_{\mathbb{K}_n} T_{H_1} \left( X_2; \mathbb{Z} \Gamma_n \left( \mathbb{Z} \ker \psi \right)^{-1} \right) \]

where \( \mathbb{K}_n = \mathbb{Z} \ker \psi \left( \mathbb{Z} \ker \psi - \{0\} \right)^{-1} \) is the right ring of quotients of \( \mathbb{Z} \ker \psi \). Recall that if \( S \) is a right divisor set then \( \text{RS}^{-1} \) exists and is the ring obtained by inverting all of the elements in \( S \). Let \( R_n = \mathbb{Z} \Gamma_n \left( \mathbb{Z} \ker \psi - \{0\} \right)^{-1} \), \( R_n^j = \mathbb{Z} \Gamma_n \left( \mathbb{Z} \ker \psi_j - \{0\} \right)^{-1} \), and \( \mathbb{K}_n^j = \mathbb{Z} \ker \psi_j \left( \mathbb{Z} \ker \psi_j - \{0\} \right)^{-1} \). To complete the proof we must show that

\[ \text{rk}_{\mathbb{K}_n^j} T_{H_1} \left( X_1; R_n \right) \cong \text{rk}_{\mathbb{K}_n} T_{R_n^j} \left( X_1; R_n^j \right). \]
Since $\psi = \psi_j \circ \tilde{t}_j$, $\tilde{t}_j (\ker \psi_j) \subset \ker \psi$, we can extend $\tilde{t}_j$ to $\tilde{t}_j : R^j_n \rightarrow R_n$. By Lemma 8.7, $R_n$ is a flat left $R^j_n$-module. Therefore

\begin{equation}
H_* \left( C_* \left( \tilde{X}_{jn} \right) \otimes _{Z^j R_n} R_n \right) \cong H_* \left( C_* \left( X_{jn} \right) \otimes _{Z^j R_n^j} R^j_n \right) \otimes _{R^j_n} R_n.
\end{equation}

Let $M = H_1 \left( X_j ; R^j_n \right)$ then $M \cong \left( R^j_n \right)^m \oplus T_{R^j_n} M$ since $M$ is finitely generated and $R^j_n$ is a principal ideal domain. It is straightforward to show that

$$T_{R^j_n} \left( M \otimes _{R^j_n} R_n \right) \cong T_{R^j_n} M \otimes _{R^j_n} R_n.$$ 

Moreover, $T_{R^j_n} M \cong \frac{R^j_n}{\langle r_1 \rangle} \oplus \cdots \oplus \frac{R^j_n}{\langle r_k \rangle}$ so it suffices to show that $\text{rk}_{K_n} \frac{R_n}{\langle \tilde{t}_j (r_i) \rangle} = d_j \text{rk}_{K_n} \frac{R^j_n}{\langle r_i \rangle}$ for any nonzero $r \in R^j_n$. Note that this would imply

$$\text{rk}_{K_n} \left( T_{R^j_n} M \otimes _{R^j_n} R_n \right) = \text{rk}_{K_n} \left( \frac{R_n}{\langle \tilde{t}_j (r_1) \rangle} \oplus \cdots \oplus \frac{R_n}{\langle \tilde{t}_j (r_k) \rangle} \right) = \text{rk}_{K_n} \frac{R_n}{\langle \tilde{t}_j (r_1) \rangle} + \cdots + \text{rk}_{K_n} \frac{R_n}{\langle \tilde{t}_j (r_k) \rangle} = d_j \text{rk}_{K_n} \frac{R^j_n}{\langle r_1 \rangle} + \cdots + d_j \text{rk}_{K_n} \frac{R^j_n}{\langle r_k \rangle}.$$

Let $T \in \Gamma_n^j$ such that $\psi_j (T) = t^{d_j}$. We can write any element in $\Gamma_n^j$ as $T^m \kappa$ where $\kappa \in \ker \psi_j$. Hence $r$ can be written as a nonconstant (Laurent) polynomial in $T$ with coefficients in $K_j$. We can assume that $r = a_0 + Ta_1 + \cdots + T^q a_q$ with $a_0 \neq 0$.

Since $\psi$ is surjective, there is an $S \in \Gamma_n$ such that $\psi (S) = t$. We can write any element in $\Gamma_n$ as $S^f$ where $f \in \ker \psi$. In particular, any element of $\Gamma_n$ that maps $t^{d_j}$ under $\psi$ can be written as $S^d_j f$. Since $\psi (\tilde{t}_j (T)) = \psi (T) = t^{d_j}$, $\tilde{t}_j (T) = S^d_j f$ for some $f \in \ker \Gamma_n$. Hence

$$\tilde{t}_j (r) = \tilde{t}_j (a_0) + \tilde{t}_j (T) \tilde{t}_j (a_1) + \cdots + \tilde{t}_j (T)^q \tilde{t}_j (a_q) = \tilde{t}_j (a_0) + S^d_j f \tilde{t}_j (a_1) + \cdots + (S^d_j f)^q \tilde{t}_j (a_q) = \tilde{t}_j (a_0) + S^d_j g_1 \tilde{t}_j (a_1) + \cdots + S^d_j g_k \tilde{t}_j (a_q)$$

for some $g_i \in \ker \Gamma_n$. We note that $\tilde{t}_j (a_i) \in \ker \Gamma_n$ which gives us our desired result. This completes the proof that $\delta_n (\psi) = \delta_n (\psi_1) + \delta_n (\psi_2)$. \qed

An immediate consequence is that $r_n (X) \geq 1$ whenever the hypotheses in Proposition 8.6 are satisfied. In particular, we have $\overline{\delta} (\psi) = 0$ for all $\psi \in H^1 (X; \mathbb{Z})$. We note that if $\beta_1 (X_j) = 0$ for some $j$ then $\delta_n (\psi) = \delta_n (\psi)$ and $r_n (X) = r_n (X_2)$ (similarly if $\beta_1 (X_2) = 0$).

**Corollary 8.8.** Let $X$ be a compact, orientable 3-manifold with $r_k (X) = 0$ for some $k \geq 0$. Suppose that $G = \pi_1 (X)$ does not satisfy both $\frac{G}{\pi_1 (X)} \cong \mathbb{Z}$ and $\frac{G}{\pi_1 (X)} = 0$. Then there exists an irreducible 3-manifold $Y$ with $H = \pi_1 (Y)$ such that

$$\frac{G}{G_{r(n+1)}} = \frac{H}{H_{r(n+1)}}$$

for all $n \geq 0$. 

Proposition 9.1. If \( R \) is a division ring then \( K \) over \( \pi_1(Y) \). It is easy to verify that \( \frac{\beta_1(X)}{\beta_1(X)} = \frac{H}{H_r} \). Moreover, the hypothesis on \( G \) guarantees that \( Y \neq S^2 \times S^1 \). Therefore \( Y \) is irreducible.

9. Rank of Torsion Modules over Skew Polynomial Rings

In this section we will show that the rank of a (torsion) module presented by an \( m \times m \) matrix of the form \( A + tB \) (where \( A, B \) have coefficients in \( \mathbb{K} \)) has rank at most \( m \) as a \( \mathbb{K} \) vector space. This is well known when \( \mathbb{K} \) is a commutative field. In this case the rank of \( M \) over \( \mathbb{K} \) is the determinant of \( A + tB \) which is a polynomial with degree less than or equal to \( l \). We will use this result in the proof that the higher-order degrees give lower bounds for the Thurston norm in the next section. For a first read, the reader may wish to only read the statements in Proposition 9.1 and Lemma 9.2 before proceeding to the next section.

Let \( M \) be a right \( \mathbb{K}[t^{\pm 1}] \)-module with presentation matrix of the form \( A + tB \) where \( A \) and \( B \) are \( l \times m \) matrices with coefficients in \( \mathbb{K} \), \( l \) is the number of generators of \( M \) and \( \mathbb{K} \) is a (skew) field. We denote by \( TM \) the \( \mathbb{K}[t^{\pm 1}] \)-torsion submodule of \( M \). Using the embedding \( \mathbb{K} \to \mathbb{K}[t^{\pm 1}] \) by \( k \mapsto k \cdot 1 \) we consider \( TM \) as a module over \( \mathbb{K} \).

**Proposition 9.1.** If \( M \) is a right \( \mathbb{K}[t^{\pm 1}] \)-module with presentation matrix of the form \( A + tB \) where \( A \) and \( B \) are \( l \times m \) matrices with coefficients in \( \mathbb{K} \) and \( \mathbb{K} \) is a division ring then

\[
\text{rank}_\mathbb{K} TM \leq \min \{l, m\}.
\]

We begin by stating when two presentation matrices of a finitely presented right \( R \)-module \( H \) are equivalent.

**Lemma 9.2 (pp. 117-120 of Za).** Two presentation matrices of \( H \) are related by a finite sequence of the following operations.

1. Interchange two rows or two columns.
2. Multiply a row (on left) or column (on right) by a unit of \( R \).
3. Add to any row a \( R \)-linear combination of other rows (multiplying a row by a unit of \( R \) on left) or to any column a \( R \)-linear combination of other columns (multiplying a column by a unit of \( R \) on right).
4. \( P \to \begin{pmatrix} P & \ast \end{pmatrix} \), where \( \ast \) is a \( R \)-linear combination of columns of \( P \).
5. \( P \to \begin{pmatrix} P & \ast \end{pmatrix} \), where \( \ast \) is an arbitrary column.

We will find the following lemmas useful in the proof of Proposition 9.1.

**Lemma 9.3.** A presentation matrix of the form \( \begin{pmatrix} A_1 + tI_s & A_2 \\ A_3 & A_4 \end{pmatrix} \) where \( A_i \) has entries in \( \mathbb{K} \) (a non-commutative division ring) is related (in the sense of Lemma 9.2) to a matrix of the form \( \begin{pmatrix} A_1' + tI_s & A_2' \\ A_3' & 0 \end{pmatrix} \) for some \( r \geq 0 \).

**Proof.** Since \( A_4 \) is a matrix over \( \mathbb{K} \) [1], Corollary to Theorem 16, p.43] there are \( C \) and \( D \) such that \( CA_4D = \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \). Here, \( C \) and \( D \) are units in the rings of
$l \times l$ and $m \times m$ matrices with entries in $K$ respectively. Hence we can get the new presentation matrix

$$
\begin{pmatrix}
I & 0 \\
0 & C
\end{pmatrix}
\begin{pmatrix}
A_1 + tI_s & A_2 \\
A_3 & A_4
\end{pmatrix}
\begin{pmatrix}
0 & I \\
D & 0
\end{pmatrix}
= \begin{pmatrix}
A_1 + tI_s & A_2 D \\
CA_3 & CA_4 D
\end{pmatrix}
= \begin{pmatrix}
A_1 + tI_s & A_2 D \\
CA_3 & 0 \\
0 & 0
\end{pmatrix}.
$$

Now we can make the last $r$ rows of the matrix of the form $(0 \ I_r)$ by adding (column $(m - (l - i)) \cdot (-a_{i,j})$ to column $j$ for each non-zero entry $a_{i,j}$ in the last $r$ rows of $CA_3$. In general this will change $A_1 + tI_s$ to $A'_1 + tI_s$ for some $A'_1$ whose entries lie in $K$. Using operation 5, we delete the last $r$ rows and columns to obtain our desired result.

\[\] Proof. Let $A = \begin{pmatrix}
A_1 & A_2 \\
A_3 & 0
\end{pmatrix}$ and $a_{k,i}$ be the $(k, i)$ entry of $A$. By permuting rows in $A_3$ we can assume that the last row has a non-zero element. Suppose that the first non-zero element in this row occurs in the $i^\text{th}$ column. We can assume that this element is 1. Now if $a_{i,j}$ is any other non-zero entry in the last row ($i < j \leq s$) we add (column $i$) · $(-a_{i,j})$ to (column $j$) to get a presentation with a zero in column $j$ of the last row. However, this changes the $(i,j)$ the entry of our matrix to $(a_{i,j} - a_{i,i}a_{i,j}) - ta_{i,j}$ which does not lie in $K$. To remedy this, we add $(ta_{i,j}t^{-1}) \cdot (row j)$ to (row $i$). Performing these two steps for all non-zero $a_{i,j}$ gives us a matrix whose last row is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$. By cyclically permuting columns $i$ through $m$ (so that the $i^\text{th}$ column becomes the $m^\text{th}$ column) and using the operation of type 5 in Lemma 9.2, we see that this matrix is related to the matrix obtained by deleting column $i$ and row $l$. We note that all the entries in row $i$ lie in $K$. For the final step we cyclically permute rows $i$ through $s$ (so that the $i^\text{th}$ row becomes the $s^\text{th}$ row) and use Lemma 9.3 to get our desired result.

Proof of Proposition 9.1. Let $P = A + tB$ be a presentation matrix of $M$. As in the proof of lemma 9.3 there are $C_{l \times l}$ and $D_{m \times m}$ (units in the rings of matrices over $K$) such that $CBD = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$, where $s \leq \min \{l, m\}$. Now if $C^t \equiv tCt^{-1}$ we have

$$
C^t PD = C^t (A + tB) D = C^t AD + tCBD.
$$

Hence $M$ has a presentation matrix of the form

$$
(9.2) \quad A + t \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} + t & \cdots & a_{1,s} \\ \vdots & \ddots & \vdots \\ a_{s,1} & \cdots & a_{s,s} + t \\ * & \cdots & * \end{pmatrix}_{l \times m}
$$

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where by $A$ is now $C^t AD$, a matrix with entries in $\mathbb{K}$. We can now use Lemma 9.3 and Lemma 1.4 to get a new presentation matrix $\begin{pmatrix} A'_1 + tI_{s'} & A'_2 \\ 0 & 0 \end{pmatrix}$ where $s' \leq s$. It follows that $TM$ has a presentation matrix

\begin{equation}
\begin{pmatrix} A'_1 + tI_{s'} & A'_2 \end{pmatrix}
\end{equation}

where $A'_i$ are matrices in $\mathbb{K}$. Let $\sigma_i (1 \leq i \leq s')$ be the generators of $TM$ corresponding to $\{s_i\}$. We show that these generate $TM$ as a $\mathbb{K}$-module. Let $a_{i,j}$ be the $(i,j)$ the entry of $A'_1$ then we have the relations $\sigma_1 a_{1,j} + \cdots + \sigma_j (a_{j,j} + t) + \cdots + \sigma_{s'} a_{s',j} = 0$ for $j \leq s'$. Hence $\sigma_i t = - \sum \sigma_k a_{k,i}$ is in the span of $\{\sigma_i\}$. We prove by induction on $n$ that $\sigma_i t^n$ is in the span of $\{\sigma_i\}$. Suppose $\sigma_i t^n = \sum \sigma_k b_{k,i}$ where $b_{k,i} \in \mathbb{K}$ then

\[
\sigma_i t^{n+1} = \left( \sum_k \sigma_k b_{k,i} \right) t^n = \sum_k \sigma_k t \left( t^{-1} b_{k,i} t \right) = \sum_k \left( \sum_l \sigma_l b_{l,k} \right) \left( t^{-1} b_{k,i} t \right) = \sum_l \sigma_l \left( \sum_k b_{l,k} \left( t^{-1} b_{k,i} t \right) \right)
\]

for all $i \leq s'$. Therefore any element $\sum \sigma_i p_i(t)$ with $p_i(t) \in \mathbb{K}[t^{\pm 1}]$ can be written as a linear combination of $\sigma_i$ with coefficients in $\mathbb{K}$. It follows that $\text{rk}_\mathbb{K} TM \leq s' \leq s \leq \min \{l, m\}$.

10. Relationships of $\delta_n$ and $\delta_n$ to the Thurston Norm

In this section, we will prove one of the main theorems of this paper. We show that the higher-order degrees of a 3-manifold give lower bounds for the Thurston norm. The result when $X$ is a knot complement appears in [8] although it uses some of our work.

**Theorem 10.1.** Let $X$ be a compact, orientable 3-manifold (possibly with boundary). For all $\psi \in H^1(X; \mathbb{Z})$ and $n \geq 0$

\[
\overline{\delta}_n(\psi) \leq \|\psi\|_T
\]

except for the case when $\beta_1(X) = 1$, $n = 0$, $X \not\cong S^1 \times S^2$, and $X \not\cong S^1 \times D^2$. In this case, $\overline{\delta}_0(\psi) \leq \|\psi\|_T + 1 + \beta_3(X)$ whenever $\psi$ is a generator of $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, equality holds in all cases when $\psi : \pi_1(X) \to \mathbb{Z}$ can be represented by a fibration $X \to S^1$.

The proof of this Theorem will follow almost directly from Proposition 9.4 and Proposition 7.1. However, because of some technical details we postpone the proof until after Corollary 10.7. We will begin the section by proving a more generalized (but less applicable) version of Theorem 10.1. We first introduce some notation.

Let $X$ be a 3-manifold, $\psi \in H^1(X; \mathbb{Z})$, $G = \pi_1(X)$, $\Gamma_n = G/G_s^{(n+1)}$. Recall that if $F$ is an embedded surface dual to $\psi$, we can consider the homology of $F$ with
coefficients in $\mathbb{K}_n$, where $\mathbb{K}_n$ is the field of fractions of $\mathbb{Z}\Gamma'_n$. Define the higher-order Betti numbers of $F$ to be
\[
b_i^n (F) = \text{rk}_{\mathbb{K}_n} H_i (F; \mathbb{K}_n) .
\]
By Remark 4.4 we see that the Euler characteristic of $F$ can be computed using $b_i^n$,
\[
\chi (F) = \sum (-1)^i b_i^n (F)
\]
for any $n \geq 0$.

Now we consider the collection of Thurston norm minimizing surfaces dual to $\psi$, $\mathcal{F}_\psi$. It is very possible that a surface in $\mathcal{F}_\psi$ is highly disconnected. One could ask, “What is the minimal number of components of a surface in $\mathcal{F}_\psi$?” For our purposes, it will turn out to be important to compute the number of components of surface in $\mathcal{F}_\psi$ that lift to the $n^{th}$ order cover of $X$. To be precise we make the following definitions.

Let $F = \coprod F^i$ be a (possibly disconnected) surface. We define $N_n (F)$ to be the number of components of $F$ with $i_* (\pi_1 (F^i)) \subseteq G_{r(n+1)}$ and $N_n^c (F)$ to be the number of closed components of $F$ with $i_* (\pi_1 (F^i)) \subseteq G_r^{(n+1)}$. Finally, we define
\[
N_n (\psi) = \min_{F \in \mathcal{F}_\psi} \{ N_n (F) + N_n^c (F) \} .
\]

**Theorem 10.2.** Let $X$ be a compact, orientable 3-manifold (possibly with boundary). For all $\psi \in H^1 (X; \mathbb{Z})$ and $n \geq 0$
\[
\delta_n (\psi) \leq \| \psi \|_T + N_n (\psi) .
\]

**Proof.** Let $F$ be a Thurston norm minimizing surface dual to $\psi$ that minimizes $N_n (F) + N_n^c (F)$. We remark that a connected surface has $b_0^n (F) = 0$ if and only if the coefficient system $\phi \circ i_* : \pi_1 (F) \to G/G_r^{(n+1)}$ is trivial by Lemma 5.7. Therefore $N_n (F) = b_0^n (F)$. Similarly, we have $N_n^c (F) = b_2^n (F)$. By (10.1),
\[
b_0^n (F) = -\chi (F) + N_n (F) + N_n^c (F) \\
\leq \| \psi \|_T + N_n (\psi) .
\]

To complete the proof, we show that $\delta_n (\psi) \leq b_0^n (F)$. By Proposition 7.6, $\mathcal{A}_n^w (X)$ has a presentation matrix of the form $A + tB$ of size $(b_2^n (F) \times m)$ where $m = \text{rk}_{\mathbb{K}_n} H_1 (Y; \mathbb{K}_n)$. Thus, by Proposition 9.1 we have
\[
\delta_n (\psi) = \text{rk}_{\mathbb{K}_n} \mathcal{A}_n^w (X) \leq \min \{ b_0^n (F), m \} \leq b_0^n (F) .
\]

We note that the term $N_n (\psi)$ is an invariant of the pair $(X, \psi)$. However, in a general, $N_n (\psi)$ may be difficult to compute. Fortunately, in some cases, we may be able bound this term by a constant.

Suppose that we are interested in the genera of knots or links. More generally, suppose we are only interested in the connected surfaces embedded in a 3-manifold. Then it is reasonable to measure the complexity of the surface by its first Betti number. Using the proof of Theorem 10.2 we can find a lower bound for the first Betti number of $F$ that has no “extra term.”

**Corollary 10.3.** If $F$ is any surface dual to $\psi$ then $\delta_n (\psi) \leq \beta_1 (F)$. 
Proof. Since $N_n(F) \leq \beta_0(F)$ and $N_n^c(F) \leq \beta_2(F)$,
\[ b^s_n(F) = -\chi(F) + N_n(F) + N_n^c(F) = \beta_1(X) + (N_n(F) - \beta_0(F)) + (N_n^c(F) - \beta_2(F)) \leq \beta_1(X) . \]
Therefore, $\delta_n(\psi) \leq b^s_n(F) \leq \beta_1(X)$. \qed

We consider the case when $X$ is the complement of a link $L$ in $S^3$. If $L$ has $m$ components then $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^m$ generated by the $m$ meridians $\mu_i$. Let $\psi_i$ be defined by $\psi_i(\mu_i) = t^{i+j}$. That is, $\psi_i$ is dual to any surface that algebraically intersects the $i^{th}$ meridian once and the $j^{th}$ meridian zero times for $j \neq i$. We will show that a Thurston norm minimizing surface dual to $\psi_i$ can be chosen to be connected and hence we can bound the term $N_n(\psi_i)$ by 1.

**Corollary 10.4.** Let $X = S^3 - L$ and $\psi_i$ be as defined above. Then
\[ \delta_n(\psi_i) \leq \|\psi_i\|_T + 1 \]
for all $n \geq 0$.

**Proof.** We show that for all $i \in \{1, \ldots, m\}$ there exists a Thurston minimizing surface $F_i$ which is connected and has non-trivial boundary. Hence $N_n(\psi_i) \leq 1$ and $N_n^c(\psi_i) = 0$. The result follows from Theorem 10.2.

Let $F = \bigsqcup F^j$ be a Thurston norm minimizing surface dual to $\psi_i$. Let $\{\alpha_k\}$ be the set of boundary components of $F$. Suppose that $\alpha_k$ and $\alpha_l$ are parallel and have opposite orientation. Then we can glue an annulus along $\alpha_k$ and $\alpha_l$ to get a new surface whose relative homology class and $\chi$ are unchanged. Altering our surface in this way, we can assume that there is exactly one $k_0$ such that $\alpha_{k_0} \cdot \mu_i = 1$ and $\alpha_{k_0} \cdot \mu_j = 0$ for all $k$ whenever $j \neq i$. Secondly, we can assume that all the components of $F$ have boundary since every closed surface is zero in $H_2(X,\partial X;\mathbb{Z})$. Now, let $F_i$ be the connected component of $F$ having $\alpha_{k_0}$ as one of its boundary components. Then $F_i$ represents the same relative homology class as $F$ and $\chi(F_i) \leq \chi(F)$. Thus $F_i$ is a Thurston norm minimizing surface dual to $\psi$ which is connected and has non-trivial boundary. \qed

The ropelength of a link is the quotient of its length by its thickness. In [CKS], J. Cantarella, R. Kusner, and J. Sullivan show that the minimal ropelength $R(L_i)$ of the $i^{th}$ component of a link $L = \bigsqcup L_i$ is bounded from below by $2\pi \left(1 + \sqrt{\|\psi_i\|_T}\right)$. Hence the higher-order degrees give computable lower bounds for the ropelength of knots and links.

**Corollary 10.5.** Let $X = S^3 - L$ and $\psi_i$ be as defined above. For each $n \geq 0$,
\[ R(L_i) \geq 2\pi \left(1 + \sqrt{\delta_n(\psi_i)} - 1\right) . \]
Moreover, if $\beta_1(X) \geq 2$ or $n \geq 1$ (or both) then
\[ R(L_i) \geq 2\pi \left(1 + \sqrt{\delta_n(\psi_i)}\right) . \]

**Proof.** The first (respectively second) statement follows from the bound given in [CKS] and Corollary 10.4 (respectively Theorem 10.1). \qed
Although it seems that the second statement in the Corollary is “stronger”, in practice the first statement is often more useful. That is, $\overline{S}_n = 0$ whenever the rank is positive hence gives no new information. We exemplify this phenomena in Example 8.3.

We would like to determine conditions that will guarantee that a surface will not lift to the $n^{th}$-order cover of $X$. We show that if $\beta_1 (X) \geq 2$ then $r_n (X) = 0$ guarantees that no homologically essential surface can lift to the $n^{th}$-order cover of $X$. In particular, if $r_0 (X) = 0$ then $i_* \pi_1 (F) \not\subseteq G_r^{(1)}$ so that $i_* \pi_1 (F) \not\subseteq G_r^{(n+1)}$ for all $n \geq 0$. If $\beta_1 (X) = 1$ a surface representing the generator of $H_2 (X, \partial X; \mathbb{Z})$ can only lift if the rational derived series stabilizes at the first step, i.e. $G_r^{(1)} = G_r^{(2)} = \ldots = G_r^{(n+1)}$.

**Proposition 10.6.** If there exists a compact, connected, orientable, two-sided properly embedded surface $F \subseteq X$ with $\beta_1 (X) \geq 2$ such that $0 \neq [F] \in H_2 (X, \partial X; \mathbb{Z})$ and $i_* \pi_1 (F) \subseteq G_r^{(n+1)}$ then $r_n (X) \geq 1$.

**Proof.** Let $Y = X \setminus (F \times I)$, since $[F] \neq 0$ $F$ does not separate $X$. Hence $Y$ is connected. Let $\gamma$ be a oriented simple closed curve that intersects $F$ exactly once, then $G = \pi_1 (X) = \langle \pi_1 (Y), \gamma \rangle$ relations from $\pi_1 (F)$). If $\pi_1 (Y) \subseteq G_r^{(1)}$ then $G/G_r^{(1)} = \langle \gamma \rangle$ which contradicts $\beta_1 (X) \geq 2$. This implies that $\pi_1 (Y) \not\subseteq G_r^{(1)}$ hence $\pi_1 (Y) \not\subseteq G_r^{(n+1)}$ for all $n \geq 0$. Now we consider the Mayer-Vietoris sequence

$$0 \rightarrow \text{Im} (i_* + j_*) \rightarrow H_1 (X; \mathcal{K}_n) \rightarrow H_0 (F_+; \mathcal{K}_n) \rightarrow H_0 (F \times I; \mathcal{K}_n) \oplus H_0 (Y; \mathcal{K}_n) \rightarrow H_0 (X; \mathcal{K}_n) \rightarrow 0.$$ 

Since $\pi_1 (Y) \not\subseteq G_r^{(n+1)}$, $\pi_1 (Y) \rightarrow G \rightarrow \Gamma_n$ is a non-trivial coefficient system. Therefore we have $H_0 (Y; \mathcal{K}_n) = 0$ and $H_0 (X; \mathcal{K}_n) = 0$ by Lemma 5.7. We note that $\text{rk}_{\mathcal{K}_n} H_0 (F; \mathcal{K}_n) = 1$ since $\pi_1 (F) \subseteq G_r^{(n+1)}$. It follows that

$$r_n (X) = \text{rk}_{\mathcal{K}_n} H_1 (X; \mathcal{K}_n)$$

$$= \text{rk}_{\mathcal{K}_n} H_0 (F; \mathcal{K}_n) + \text{rk}_{\mathcal{K}_n} \text{Im} (i_* + j_*) \geq 1.$$ 

In particular, if there is a non-trivial surface that lifts to the $n^{th}$ cover then $r_n (X) \geq 1$.

**Corollary 10.7.** If there exists a compact, connected, orientable, two-sided properly embedded surface $F \subseteq X$ with $\beta_1 (X) \geq 2$ such that $0 \neq [F] \in H_2 (X, \partial X; \mathbb{Z})$ and $i_* \pi_1 (F) \subseteq G_r^{(n+1)}$ then $\overline{s}_n (\psi) = 0$ for all $\psi \in H^1 (X; \mathbb{Z})$.

**Proof.** $r_n (X) \geq 1$ implies that $\overline{s}_n (\psi) = 0$ for all $\psi \in H^1 (X; \mathbb{Z})$.

We now prove the main theorem of this section.

**Proof of Theorem 10.4.** We break the proof up into two cases.

Case 1: Let $X$ be a 3-manifold with $\beta_1 (X) \geq 2$. Let $F = \cup F_i$ be a surface dual to $\psi$ that is minimal with respect to $\| \cdot \|_T$. We can assume that $[F_i] \neq 0$ for all $i$. If any component of $F$, say $F_j$ lifts to the $n^{th}$ rational derived cover of $X$, i.e. $\pi_1 (F_j) \subseteq G_r^{(n+1)}$ then $\overline{s}_n (\psi) = 0$ by Corollary 10.7. Otherwise $\mathcal{N}_n (\psi) = 0$ so by Theorem 10.1 we have $\overline{s}_n (\psi) \leq \delta_n (\psi) \leq \| \psi \|_T$. 


Case 2: Let $X$ be a 3-manifold with $\beta_1 (X) = 1$ and $\psi$ be a generator of $H^1 (X; \mathbb{Z})$. Let $F = \cup F_i$ be a surface dual to $\psi$ that is minimal with respect to $\|\cdot\|_r$. Since $\beta_1 (\ker \psi) < \infty$, we can represent $F$ as a connected surface with $\beta_2 (F) = \beta_3 (X)$ [Mc. Proposition 6.1]. Therefore $N_0 (\psi) \leq 1 + \beta_3 (X)$ so by Theorem 10.2 we have $\delta_0 (\psi) = \delta_0 (\psi) \leq \|\psi\|_r + 1 + \beta_3 (X)$. Now suppose $n \geq 1$. If $\pi_1 (F) \not\subseteq G_r^{(2)}$ (hence $\pi_1 (F) \not\subseteq G_r^{(n+1)}$) then $N_n (\psi) = 0$ so the result follows from Theorem 10.2. Otherwise, by Proposition 10.9, $\delta_n (\psi) = \delta_n (\psi) = 0$. We remark that if the higher-order degrees of $S^1 \times S^2$ and $S^1 \times D^2$ are zero.

The last sentence in the theorem follows from the calculations in Proposition 8.4. Note that $r_n (X) = 0$ for fibered 3-manifolds so that $\mathfrak{F}_n (\psi) = \delta_n (\psi)$ for all $\mathfrak{n}$.

To complete the proof of Theorem 10.1, we need prove Proposition 10.9 which states that if a homologically essential surface dual to $\psi$ lifts to the $n^{th}$ order cover then $\mathfrak{A}^n_{\mathfrak{r}} (X) = 0$ for $i < n$. This will be our main objective for the rest of this section.

We begin by showing that $\mathfrak{A}^n_{\mathfrak{r}} (X)$ is generated by $H_1 (F; \mathbb{K}_n [t^{\pm 1}])$. The idea behind the proof is simple. If $\alpha \neq 0$ is $\mathbb{K}_n [t^{\pm 1}]$-torsion then there exists a $p (t) \in \mathbb{K}_n [t^{\pm 1}]$ such that $\alpha p (t) = 0$. Moreover, since $\mathbb{K}$ is a (skew) field, we can assume that $p (t) = 1 + t a_1 + \cdots + t^m a_m$ where $a_m \neq 0$. Thus $\alpha$ and $\alpha t a_1 + \cdots + \alpha t^m a_m$ cobound a surface, $S$ in $X_n$ (see Figure 3). Since the power of $t$ on each term of the latter sum is positive, $S$ must intersect a lift of the surface $F$. Hence $\alpha$ is homologous to the intersection of $S$ with the lift of $F$. Note that in Figure 3, $\alpha$ is homologous to $\beta_1 + \beta_2$.

\begin{center}
\textbf{Figure 3.} $\mathfrak{A}^n_{\mathfrak{r}} (X)$ is generated by $H_1 (F; \mathbb{K}_n [t^{\pm 1}])$
\end{center}

Lemma 10.8. $\mathfrak{A}^n_{\mathfrak{r}} (X) \subseteq \text{Im} (i_*)$ where $i_* : H_1 (F; \mathbb{K}_n [t^{\pm 1}]) \to H_1 (X; \mathbb{K}_n [t^{\pm 1}])$.

Proof. By Proposition 7.6, $\mathfrak{A}^n_{\mathfrak{r}} (X) \subseteq T \text{Im} (j_*)$ where $j_* : H_1 (Y; \mathbb{K}_n [t^{\pm 1}]) \to H_1 (X; \mathbb{K}_n [t^{\pm 1}])$. We will show that $T \text{Im} (j_*) \subseteq \text{Im} (i_*)$ which completes the proof.

Let $\sigma_X \in T \text{Im} (j_*)$ with $j_* (\sigma_Y) = \sigma_X$. Since $\sigma_X$ is $\mathbb{K}_n [t^{\pm 1}]$-torsion, there exists $p (t) \in \mathbb{K}_n [t^{\pm 1}]$ such that $\sigma_X p (t) = 0$. We have $j_* (\sigma_Y p (t)) = j_* (\sigma_Y) p (t) =
\[ \sigma_X p(t) = 0 \] so there exists \( \sigma_F \in H_1(F; \mathbb{K}_n[t^{\pm 1}]) \) such that \( \eta(\sigma_F) = \sigma_Y p(t) \).

We can assume that \( p(t) = 1 + t c_1 + \cdots + t^n c_n \) since \( \sigma_X p(t) = 0 \) if and only if \( \sigma_X p(t) u = 0 \) for any unit \( u \in \mathbb{K}_n[t^{\pm 1}] \). Now \( H_1(F; \mathbb{K}_n[t^{\pm 1}]) \cong H_1(F; \mathbb{K}_n) \otimes \mathbb{K}_n \mathbb{K}_n[t^{\pm 1}] \) and \( H_1(Y; \mathbb{K}_n[t^{\pm 1}]) \cong H_1(Y; \mathbb{K}_n) \otimes \mathbb{K}_n \mathbb{K}_n[t^{\pm 1}] \) so every element in \( H_1(F; \mathbb{K}_n[t^{\pm 1}]) \) (resp. \( H_1(Y; \mathbb{K}_n[t^{\pm 1}]) \)) has the form \( \sum_{i=-\infty}^{\infty} \alpha_i \otimes t^i \) (resp. \( \sum_{i=-\infty}^{\infty} \beta_i \otimes t^i \)) such that \( \alpha_i \in H_1(F; \mathbb{K}_n) \) (resp. \( \beta_i \in H_1(Y; \mathbb{K}_n) \)) and there are only finitely many non-zero \( \alpha_i \) (resp. \( \beta_i \)). We write \( \sigma_F = \sum_{i=-\infty}^{k} \alpha_i \otimes t^i \) and (as with \( p(t) \)) we may write \( \sigma_Y = \sum_{i=0}^{k} \beta_i \otimes t^i \). Using this notation we now have

\[
\sigma_Y p(t) = \sum_{i=0}^{k} \left( \sum_{j=0}^{m} \beta_i \otimes t^j \right) \sum_{j=0}^{m} t^j c_j
\]

\[
= \sum_{i=0}^{k+m} \left( \sum_{j=0}^{m} \beta_i (c_j) t^{i+j} \right)
\]

\[
= \sum_{i=0}^{k+m} \left( \sum_{j=0}^{m} \beta_i (c_j) t^j \right) \otimes t^i
\]

and

\[
\eta(\sigma_F) = \sum_{i=-\infty}^{\infty} \eta(\alpha_i \otimes t^i)
\]

\[
= \sum_{i=-\infty}^{\infty} \left( (i-) (\alpha_i \otimes t^i) - (i+) (\alpha_i \otimes t^i) \right) t
\]

\[
= \sum_{i=-\infty}^{\infty} \left( (i-) (\alpha_i) \otimes t^i - (i+) (\alpha_i) \otimes t^{i+1} \right)
\]

\[
= \sum_{i=-\infty}^{\infty} \left( (i-) (\alpha_i) \otimes t^i - (i+) (\alpha_{i-1}) \otimes t^{i} \right)
\]

\[
= \sum_{i=-\infty}^{\infty} \left( (i-) (\alpha_i) - (i+) (\alpha_{i-1}) \right) \otimes t^i.
\]

Recall that \( \eta(\sigma_F) = \sigma_Y p(t) \) which implies that \( \sum_{i+j=l} \beta_i (a_j) t^l = (i-) (\alpha_i) - (i+) (\alpha_{i-1}) \) for all \( 0 \leq l \leq k + m \). In particular \( c_0 = 1 \) so when \( l \leq k \) we can write \( \beta_i \) as a combination of \( \beta_i \) and \( (i-) (\alpha_i) + (i+) (\alpha_{i-1}) \) with \( i < l \). That is,

\[
\beta_i \otimes t^l = (i-) (\alpha_i) - (i+) (\alpha_{i-1}) \otimes t^l - \sum_{i+j=l, i < l} \beta_i (c_j) t^l \otimes t^i.
\]

We will prove by induction that \( j_* (\beta_i \otimes t^l) \in \text{Im}(i_*) \) for each \( l \) implying that \( \sigma_X = j_* (\sigma_Y) = j_* \left( \sum_{0 \leq l \leq k} \beta_i \otimes t^l \right) \in \text{Im}(i_*) \) which completes the proof. We first
Proposition 10.9. The fundamental groups all include into the (\(G\) gets mapped into
\(X\)) similar for \(\rho\)).

Proof. Consider the following diagram of abelian groups:

\[
\begin{array}{ccc}
H_1(F_{\Gamma_1}) & \xrightarrow{i_*} & H_1(X_{\Gamma_1}) \\
\rho_F & & \rho_X \\
H_1(F;\mathbb{K}_i [t^{\pm 1}]) & \xrightarrow{i_*} & H_1(X;\mathbb{K}_i [t^{\pm 1}])
\end{array}
\]

where \(i_*\) is induced by the inclusion map \(i : F \to X\) and \(\rho_F ([\sigma]) = [\sigma \otimes 1]\) (similarly for \(\rho_X\)). First we observe that if \(i \leq n - 1\), every class in \(\pi_1 (F_{\Gamma_1})\) gets mapped into \(G^{(n+1)}_i \subseteq G^{(i+2)}_i\) by \(i_* \circ p_*\) hence is zero in \(G^{(i+1)}_i / G^{(i+2)}_i = H_1(X_{\Gamma_1}) / \{\mathbb{Z} - \text{torsion}\}\). Therefore \(i_* : H_1(F_{\Gamma_1}) \to H_1(X_{\Gamma_1})\) maps \(H_1(F_{\Gamma_1})\) to the \(\mathbb{Z}\)-torsion subgroup of \(H_1(X_{\Gamma_1})\). Since \(H_1(X;\mathbb{K}_i [t^{\pm 1}])\) is \(\mathbb{Z}\)-torsion free, \(\rho_X \circ i_* = 0\).

Since \((10.2)\) commutes, the image of \(\rho_F\) goes to zero under \(i_*\). Therefore if \([\sigma \otimes p(t)]\) is an element of \(H_1(F;\mathbb{K}_i [t^{\pm 1}])\), \(i_* ([\sigma \otimes p(t)]) = i_* ([\sigma \otimes 1]) p(t) = 0 p(t) = 0 (i_*

We can use this to show that if \(\psi\) is dual to a union of surfaces in \(X\) whose fundamental groups all include into the \((n + 1)^{st}\) rational derived subgroup of \(G = \pi_1 (X)\) then \(\delta_1 (\psi) = 0\) for \(i \leq n - 1\).

Proposition 10.9. If there exists a union of properly embedded surfaces \(F = \bigcup F_j\) in \(X\) with \([F] \in H_2 (X, \partial X; \mathbb{Z})\) dual to \(\psi \in H^1 (X; \mathbb{Z})\) such that for all \(j\), \(\pi_1 (F_j) \subseteq G^{(n+1)}_i\) then \(A^i (X) = 0\) whenever \(0 \leq i \leq n - 1\).

Proof. Consider the following diagram of abelian groups:
is a $\mathbb{K}_i \lbrack t^{\pm 1} \rbrack$-module homomorphism). By Lemma 10.8, $A^1_v (X)$ is generated by $\text{Im} (i_*) = 0$ hence $A^1_v (X) = 0$.

**Corollary 10.10.** Let $X$ be a 3-manifold with $\beta_1 (X) = 1$ and $F$ a surface dual to a generator of $H^1 (X; \mathbb{Z})$. If $\pi_1 (F) \subseteq G_r^{(2)}$ then $G_r^{(i)} = G_r^{(i+1)}$ for all $i \geq 1$.

**Proof.** Since $\beta_1 (X) = 1$, $r_n (X) = 0$ by Proposition 5.2. That is, $H_1 (X; \mathbb{K}_i \lbrack t^{\pm 1} \rbrack)$ is a torsion module. When $i = 0$, $\mathbb{K}_0 = \mathbb{Q}$ so that

$$TH_1 (X; \mathbb{K}_0 \lbrack t^{\pm 1} \rbrack) = H_1 (X; \mathbb{Q} \lbrack t^{\pm 1} \rbrack) = H_1 (X_{\Gamma_0}) \otimes \mathbb{Q} = G_r^{(1)} / G_r^{(2)} \otimes \mathbb{Q}.$$  

If $\pi_1 (F) \subseteq G_r^{(2)}$, Proposition 10.9 implies $TH_1 (X; \mathbb{K}_0 \lbrack t^{\pm 1} \rbrack) = 0$. Since $G_r^{(1)} / G_r^{(2)}$ is $\mathbb{Z}$-torsion free, $G_r^{(1)} / G_r^{(2)} \to G_r^{(1)} / G_r^{(2)} \otimes \mathbb{Q}$ sending $g \mapsto g \otimes 1$ is a monomorphism. Therefore $G_r^{(i)} = G_r^{(i+1)}$ for all $i \geq 1$.

**11. Realization Theorem**

We are ready to prove that the invariants $\delta_n$ give much more information than the classical invariants. In fact, we subtly alter 3-manifolds to obtain new 3-manifolds with striking behavior. Cochran proves this result when $\beta_1 (X) = 1$.

**Theorem 11.1.** For each $m \geq 1$ and $\mu \geq 2$ there exists a 3-manifold $X$ with $\beta_1 (X) = \mu$ such that

$$\| \psi \|_A = \delta_0 (\psi) < \delta_1 (\psi) < \cdots < \delta_m (\psi) \leq \| \psi \|_T,$$

for all $\psi \in H^1 (X; \mathbb{Z})$. Moreover, $X$ can be chosen so that it is closed, irreducible and has the same classical Alexander module as a 3-manifold that fibers over $S^1$.

The proof of this will be an application of the following more technical theorems. We will postpone the proof until later in the section. Theorem 11.2 is a tool that will allow us to subtly alter 3-manifolds in order to construct new 3-manifolds whose degrees are unchanged up to the $n$th stage but increase at the $n$th stage.

**Theorem 11.2 (Realization Theorem).** Let $X$ be a compact, orientable 3-manifold with $G = \pi_1 (X)$ and $G_r^{(n)} / G_r^{(n+1)} \neq 0$ for some $n \geq 0$. Let $[x]$ be a primitive class in $H_1 (X; \mathbb{Z})$. Then for any positive integer $k$, there exists a 3-manifold $X (n, k)$ homology cobordant to $X$ such that

\[
\frac{G}{G_r^{(i+1)}} \cong \frac{H}{H_r^{(i+1)}} \quad \text{for} \quad 0 \leq i \leq n - 1
\]

where $H = \pi_1 (X (n, k))$ and

\[
\delta_n^{X(n,k)} (\psi) \geq \delta_n^X (\psi) + k |p|.
\]

for any $\psi \in H^1 (X (n, k); \mathbb{Z})$ with $\psi (x) = t^p$.

**Proof.** Let $X$ be a compact 3-manifold with $G_r^{(n)} / G_r^{(n+1)} \neq 0$. $G = G_r^{(1)}$ implies that $G_r^{(n)} = G_r^{(n+1)}$ hence our hypothesis guarantees that $\beta_1 (X) \geq 1$. Since $[x]$ is a primitive class in $H_1 (X; \mathbb{Z})$, we can present $G$ as

$$G \cong \langle x_1, \ldots, x_p, y_1, \ldots, y_l | R_1, \ldots, R_m \rangle$$

where $y_i \in G_r^{(1)}$, $x_1 = x$, and $\{[x_1], \ldots, [x_p] \}$ is a basis for $G / G_r^{(1)}$. 


Begin by adding a 1-handle to \( X \times I \) to obtain a 4-manifold \( V \) with boundary \((X \sqcup X') \cup \partial(X \times I)\) where \( X' \) is obtained from \( X \) by taking the connect sum with \( S^1 \times S^2 \). Then \( \pi_1(V) \cong \pi_1(X') \cong G \ast \langle z \rangle \) where \( z \) is the generator of \( \pi_1(S^1 \times S^2) \).

Choose a non-trivial element \( B \in G_r^{(n)} - G_r^{(n+1)} \), and let \( w = z x^{-1} \) and \( \alpha = [A_k, B] \) where \( A_k \) is defined inductively as

\[
A_1 = w \\
A_k = [A_{k-1}, x] \quad \text{for } k \geq 2.
\]

Now add a 2-handle to \( V \) along a curve \( c \) (any framing) embedded in \( X' \) representing \( w[x, \alpha] \) to obtain a 4-manifold \( W \) with boundary \((X \sqcup X(n, k)) \cup (\partial X \times I)\). Let \( E = \pi_1(W) \), \( H = \pi_1(X(n, k)) \) and denote by \( i \) and \( j \) the inclusion maps of \( X \) and \( X(n, k) \) into \( W \) respectively.

Adding the 2-handle to \( X' \) kills the element \( w[x, \alpha] \) in \( G \ast \langle z \rangle \cong G \ast \langle w \rangle \) so \( E \cong \langle G, w[x, \alpha] \rangle = \langle x_1, \ldots, x_i, y_1, \ldots, y_i, w[R_1, \ldots, R_m, w[x, \alpha]] \rangle \) where we see that \( X(n, k) \) is the 3-manifold obtained by performing Dehn surgery (with integer surgery coefficient corresponding to the framing of the 2-handle) along the curve \( c \). Let \( \gamma \) be the meridian curve to \( c \) in \( X(n, k) \). The dual handle decomposition of \( \text{rel } X(n, k) \) is obtained by adding to \( X(n, k) \) a 0-framed 2-handle along \( \gamma \) and 3-handle. This gives us \( E \cong \langle H|\gamma \rangle \).

We show that

\[
(11.1) \quad \frac{G}{G_r^{(i)}} \xrightarrow{\tau} \frac{E}{E_r^{(i)}} \xleftarrow{\sim} \frac{H}{H_r^{(i)}}
\]

for \( 0 \leq i \leq n \). Using Lemma 3.5, this will imply that

\[
(11.2) \quad \frac{G}{G_r^{(i+1)}} \xrightarrow{\tau} \frac{E}{E_r^{(i+1)}} \xleftarrow{\sim} \frac{H}{H_r^{(i+1)}}.
\]

There is a surjective map \( \text{pr} : \langle G, w[x, \alpha] \rangle \twoheadrightarrow G \) defined by killing \( w \) so that \( \text{pr} \circ \iota_x = \text{id}_G \). Consider the induced maps

\[
\frac{G}{G_r^{(i)}} \xrightarrow{\tau} \frac{E}{E_r^{(i)}} \xleftarrow{\sim} \frac{G}{G_r^{(i)}}.
\]

We will show by induction that \( w \in \left[ E_r^{(i)}, E_r^{(i)} \right] \) for \( 0 \leq i \leq n \). Since \( w = \left[ [A_k, B], x \right] \), it is clear that \( w \in \left[ E_r^{(i)}, E_r^{(i)} \right] \). Now suppose that \( w \in \left[ E_r^{(i-1)}, E_r^{(i-1)} \right] \) for some \( i \leq n \). Since \( A_k = [A_{k-1}, x] \) and \( A_1 = w \), we have \( A_k \in \left[ E_r^{(i-1)}, E_r^{(i-1)} \right] \subseteq E_r^{(i)} \). Moreover, since \( B \in G_r^{(n)} \), we have \( B \in E_r^{(i)} \) for \( i \leq n \). Therefore \( [A_k, B] \in \left[ E_r^{(i)}, E_r^{(i)} \right] \) for \( i \leq n \) and hence \( w \in \left[ E_r^{(i)}, E_r^{(i)} \right] \subseteq E_r^{(i+1)} \). It follows that \( \overline{\text{pr}} \) is an isomorphism. Since \( \overline{\text{pr}} \circ \iota_x \) is an isomorphism, \( \iota_x \) is an isomorphism for \( 0 \leq i \leq n \).

Now consider the maps

\[
\frac{H}{H_r^{(i)}} \xrightarrow{\tau} \frac{E_r^{(i)}, E_r^{(i)}}{H} \xleftarrow{\sim} \frac{G_r^{(i)}, G_r^{(i)}}{G} \quad \text{where now we are considering}
\]

\( E \) as the group \( \langle H|\gamma \rangle \). We will show that \( \gamma \in \left[ H_r^{(i)}, H_r^{(i)} \right] \) for \( 0 \leq i \leq n \) hence the above map will be an isomorphism. Recall that \( X(n, k) \) can be obtained from \( X \) by first doing Dehn surgery on a 0-framed unlinked trivial knot in \( X \) to get the manifold \( X' = X \# (S^1 \times S^2) \) and then performing Dehn surgery along a curve
c representing \( w[x, \alpha] \) in \( X' \). Let \( Y = X' - N(c) \) be the 3-manifold obtained by removing a regular neighborhood of \( c \) in \( X' \). We use the notations \( P = \pi_1(Y) \), \( K = \pi_1(X') \cong G \ast \langle z \rangle \), and \( l : Y \to X' \) be the inclusion map. Let \( \gamma \) be the meridian of \( c \) based at \( x_0 \) as in Figure 4. We show that \( \gamma \in \left[ P^{(i)}_r, P^{(i)}_r \right] \) for \( 0 \leq i \leq n \) which implies that \( \gamma \in \left[ H^{(i)}_r, H^{(i)}_r \right] \) since \( P \to H \).

![Figure 4. \( \gamma \) and the \( \gamma_i^{+1} \)'s cobound a punctured sphere](image)

To begin, we will show that

\[
\gamma = [\gamma, u_1^{w_1}] \cdots [\gamma, u_{2k}]^{w_{2k}} \lambda_1^{w_1} \cdots \lambda_m^{w_m/2}
\]  

(11.3)

where \( l_* (u_j) \in K^{(n)}_r \) and \( \lambda_j \in Ncl \langle \gamma \rangle = \ker (l_* : P \to K) \). Using this, we will show that the induced map \( l_* : P^{(i)}_r \to K^{(i)}_r \) is surjective for \( 0 \leq i \leq n + 1 \). Assuming these two statements to be true for now, we shall prove by induction on \( i \) that \( \gamma \in \left[ P^{(i)}_r, P^{(i)}_r \right] \) for \( 0 \leq i \leq n \) as desired. It is clear from (11.3) that \( \gamma \in \left[ P^{(0)}_r, P^{(0)}_r \right] \). Now suppose that \( \gamma \in \left[ P^{(i-1)}_r, P^{(i-1)}_r \right] \subseteq P^{(i)}_r \) for some \( i \leq n \). Then \( \lambda_j \in P^{(i)}_r \) for all \( j \). Since \( l_* : P^{(i)}_r \to K^{(i)}_r \) is surjective and \( l_* (u_j) \in K^{(n)}_r \), it follows that \( u_j = p_j \lambda_j \) where \( p_j \in P^{(i)}_r \) and \( \lambda_j \in Ncl \langle \gamma \rangle \subseteq P^{(i)}_r \). Thus by (11.3), \( \gamma \in \left[ P^{(i)}_r, P^{(i)}_r \right] \) for \( 0 \leq i \leq n \).

Let \( c \) be a curve representing the element

\[
w[x, \alpha] = z \alpha x^{-1} \alpha^{-1} = z A_k B A_k^{-1} B^{-1} x^{-1} B A_k B^{-1} A_k^{-1}.
\]

For simplicity, we prove the case when \( c \) intersects the cosphere (belt sphere) of the 1-handle attached to \( X \times I \) exactly \( 1 + 2k \) times. The proof can be modified
for the case where $c$ intersects the cosphere more than $1 + 2^{k+1}$ times. The $2^{k+1}$ intersections are a result of the $2^{k-1}$ occurrences of $z$ and $z^{-1}$ in $A_k$ and the first intersection is a result the first $z$ that occurs in $\alpha x^{-1} \alpha^{-1}$. Note that the only occurrences of $z$ in $\alpha x^{-1} \alpha^{-1}$ show up in $A_k$ since $B$ is an element of $G$. Let $\gamma_j$ be the meridian of $c$ based at $x_0$ corresponding to the $j^{th}$ occurrence of $z$ or $z^{-1}$ in $\alpha x^{-1} \alpha^{-1}$. Note that the only occurrences of $z$ in $\alpha x^{-1} \alpha^{-1}$ show up in $A_k$ since $B$ is an element of $G$. Let $\gamma_j$ be the meridian of $c$ based at $x_0$ corresponding to the $j^{th}$ occurrence of $z$ or $z^{-1}$ in $\alpha x^{-1} \alpha^{-1}$ as shown in Figure 4.

Before proceeding, we sketch the idea of the next part of the proof. First we note that $\gamma$ is a product of $\gamma_j^\pm 1$. We can pair each $\gamma_j$ with $\gamma_{2^{k+1} - j}$ since they bound an annulus as in Figure 5. Thus they are related by $\gamma_{2^{k+1} - j} = u^{-1} \gamma_j u$ and hence $\gamma_j \gamma_{2^{k+1} - j} = [\gamma_j, u^{-1}]$. We show that $l_*(u)$ is in the $n^{th}$ term of the derived series of $K$. Since $\gamma_j$ is a conjugate of $\gamma$, $\gamma$ is a product of commutators which can be written as $[\gamma, u]^v$ with $l_*(u) \in K^{(n)}$.

![Figure 5. $\gamma_j$ and $\gamma_{2^{k+1} - j}$ differ by a conjugation](image)

Let $a^b = b^{-1} a b$. Since the longitude of the unknot is trivial in $P$, we see that $\gamma$ is equal to a product of the $\gamma_j^\pm 1$ as in Figure 5. Moreover, we can order the $\gamma_j$ as we choose since switching $\gamma_i$ and $\gamma_j$ only changes the element by a commutator of elements of $Ncl \langle \gamma \rangle$. For example,

$$
\lambda_1 \gamma_i \gamma_j \lambda_2 = \lambda_1 \gamma_j \gamma_i \left[ \gamma_i^{-1}, \gamma_j^{-1} \right] \lambda_2
= \lambda_1 \gamma_j \gamma_i \lambda_2 \left[ \gamma_i^{-1}, \gamma_j^{-1} \right] \lambda_2.
$$

Hence we see that

$$
\gamma = \left( \prod_{j=1}^{2^{k-1}} \gamma_j^{\pm 1} \gamma_{2^{k+1} - j}^{\pm 1} \right) \left( \prod_{j=1}^{2^{k-1}} \gamma_j^{\pm 1} \gamma_{2^{k+1} + j}^{\pm 1} \gamma_{2^{k+1} + 1 - j}^{\pm 1} \right) \left[ \lambda_1, \lambda_2 \right]^{w_1} \cdots \left[ \lambda_{m-1}, \lambda_m \right]^{w_m/2}
$$

where $\lambda_j \in Ncl \langle \gamma \rangle$. The chosen ordering will become clear in the next paragraph.
If the $j^{th}$ occurrence of a $z^{±1}$ is a $z$ then $γ_j = γ^p_j$ where $l_*(p_j) = zω_j$ and $ω_j$ is the word that occurs in $αx^{-1}α^{-1}$ up to but not including the $j^{th}$ $z$. Whereas, if the $j^{th}$ occurrence of a $z^{±1}$ is a $z^{-1}$ then $γ_j = γ^{-p_j}$ where $l_*(p_j) = zω_jz^{-1}$ where $ω_j$ is the word that occurs in $αx^{-1}α^{-1}$ up to but not including the $j^{th}$ $z^{-1}$. Now we consider the case $1 ≤ j ≤ 2^k-1$. The $j^{th}$ occurrence of $z^{±1}$ in $αx^{-1}α^{-1} = A_kBA_k^{-1}B^{-1}x^{-1}BA_kB^{-1}A_k^{-1}$ occurs in the first $A_k$ and the $(2^k-j+1)^{th}$ occurrence of $z^{±1}$ occurs in the first $A_k^{-1}$ as the opposite power as the $j^{th}$ occurrence. Hence if $p_j = zω_j$ then $p_{2^k+1-j} = zA_kBA_k^{-1}$ and if $p_j = zω_jz^{-1}$ then $p_{2^k+1-j} = zA_kB^{-1}A_k^{-1}ω_jz^{-1}$. Moreover, the term $γ_j^{±1}γ_{2^k+1-j}^{±1}$ in the formula above will always be of the form $(γ_jγ_{2^k+1-j})^{±1}$. Similarly, the $(2^k+j)^{th}$ occurrence of $z^{±1}$ in $αx^{-1}α^{-1}$ occurs in the second $A_k$ and the $(2^k+1-j)^{th}$ occurrence of $z^{±1}$ occurs in the second $A_k^{-1}$ as the opposite power as the $(2^k+j)^{th}$ occurrence. Thus, if $p_{2^k+j} = zαx^{-1}Bω_j$ then $p_{2^k+1+j-1} = zαx^{-1}α^{-1}ω_j$, if $p_{2^k+j} = zαx^{-1}Bω_jz^{-1}$ then $p_{2^k+1+j-1} = zαx^{-1}α^{-1}ω_jz^{-1}$, and $γ_{2^k+j}γ_{2^k+1+j-1} = (γ_{2^k+j}γ_{2^k+1+j-1})^{±1}$. In either case we see that for $1 ≤ j ≤ 2^{k-1}$

$$γ_jγ_{2^k+1-j} = (γ_j)^{p_{2^k+1-j}} = [γ, p_j p_{2^k+1-j}]^{p_j}$$

and

$$γ_{2^k+j}γ_{2^k+1+j-1} = (γ_{2^k+j})^{p_{2^k+1+j-1}} = [γ, p_{2^k+j} p_{2^k+1+j-1}]^{p_{2^k+j}}$$

where $l_*(p_{2^k+1-j}) = l_*(BA_k^{-1}) ∊ K_r^{(n)}$ and $l_*(p_{2^k+j} p_{2^k+1+j-1}) = l_*(BA_k) ∊ K_r^{(n)}$. Therefore

$$γ = [γ, u_1]^{w_1} ... [γ, u_{2^k+m}]^{w_{2^k+m}} [λ_1, λ_2]^{w_1} ... [λ_{m-1}, λ_m]^{w_{m/2}}$$

where $l_*(u_j) ∊ K_r^{(n)}$ and $λ_j ∊ Ncl(γ)$ as desired.

Before proceeding, we note that $γ$ can be simplified to the form

$$(11.4) \quad γ = [γ, u_1]^{w_1} ... [γ, u_{2^k+m}]^{w_{2^k+m}}$$

where $l_*(u_j) ∊ K_r^{(n)}$, since if $λ_1, λ_2 ∊ Ncl(γ)$ then $[λ_1, λ_2]$ is a product of elements of the form $[γ, u_l]$ where $l_*(u_l) = 1$. This is easily verified using the relation

$$[ab, c] = [b, c]^a [a, c].$$

We prove by induction that $l_* : P_r(i) → K_r(i)$ is surjective for $0 ≤ i ≤ n + 1$. It is clear that $l_* : P_r(0) → K_r(0)$ is surjective. Now assume that $l_* : P_r(i) → K_r(i)$ for $i ≤ n$ and let $g ∊ K_r(i+1)$. We note that if $G → H$ is surjective then $[G, G] → [H, H]$ is surjective. Therefore it suffices to consider $g$ such that $g^k ∊ [K_r(i), K_r(i)]$ for some $k ≠ 0$. Since $P_r(i) → K_r(i)$ is surjective, $l_* : [P_r(i), P_r(i)] → [K_r(i), K_r(i)]$ is surjective and hence there exists an $f ∊ [P_r(0), P_r(0)]$ such that $l_*(f) = g^k$. Moreover, since $P → K$ there exists a $p ∊ P$ such that $l_*(p) = g$. It follows that $l_*(f) = l_*(g^k)$
and hence \( p^k = f \lambda \) where \( \lambda \in Ncl(\langle \gamma \rangle) \). Since \( i \leq n \), \( l_* (u_j) \in K_r^{(i)} \). Hence, by the induction hypothesis, there exist \( q_j \in P_r^{(i)} \) with \( u_j = q_j A_j \) for \( A_j \in Ncl(\langle \gamma \rangle) \). Using (11.4) we have

\[
\gamma = [\gamma, q_1 A_1] v_1 \cdots [\gamma, q_{2k+m} A_{2k+m}] v_{2k+m}
\]

hence \( \gamma \in \left[ P_r^{(i)}, P_r^{(i)} \right] \). Since \( \lambda \in Ncl(\langle \gamma \rangle) \), \( \lambda \in \left[ P_r^{(i)}, P_r^{(i)} \right] \) so that \( p^k = f \lambda \in \left[ P_r^{(i)}, P_r^{(i)} \right] \). Thus \( p \in P_r^{(i+1)} \) and \( l_* (p) = g \) which implies that \( l_* : P_r^{(i+1)} \to K_r^{(i+1)} \) is surjective for \( i \leq n \). This concludes the proof of (11.2).

The isomorphisms in (11.1) and (11.2) imply the following three statements. First, we can obtain the \( G/G_r^{(n+1)} \) and \( H/H_r^{(n+1)} \)-regular covers of \( X \) and \( X (n,k) \) respectively by restricting to the boundary of the \( E/E_r^{(n+1)} \)-regular cover of \( W \). Secondly, when \( i = 0 \) the inclusion maps \( i \) and \( j \) induce isomorphisms on \( H_1 (-, Z) \) hence on \( H^1 (-, Z) \). In fact, \( i \) and \( j \) induce isomorphisms on all integral homology groups. Thus, we can consider the homomorphism \( \psi \) as a homomorphism on \( E \) and \( H \) as well as \( G \). Lastly,

\[
\frac{G_r^{(i)}}{G_r^{(i)} \cdot G_r^{(i)}} \cong \frac{E_r^{(i)}}{E_r^{(i)} \cdot E_r^{(i)}} \cong \frac{H_r^{(i)}}{H_r^{(i)} \cdot H_r^{(i)}}
\]

for \( 0 \leq i \leq n - 1 \). In particular this implies that for any \( \psi \in H^1 (X (n,k), Z) \) and \( 0 \leq i \leq n - 1 \), \( \delta_r^{X(n,k)}(\psi) = \delta_r^W(\psi) = \delta_r^X(\psi) \).

We use the presentation given by the Fox Free Calculus (Section \([3]\)) to compute \( \delta_r^W(\psi) \). Let \( F = F \langle x_1, \ldots, x_\mu, y_1, \ldots, y_l, w \rangle \), \( \chi : F \to G \). Recall that

\[
\frac{\partial [C, D]}{\partial w} = (1 - [C, D]) \frac{\partial C}{\partial w} + (C - [C, D]) \frac{\partial D}{\partial w}
\]

for any \( C, D \in F \). We compute \( \frac{\partial A_k}{\partial w} = (1 - A_k x) \frac{\partial A_{k-1}}{\partial w} \) and \( \frac{\partial A_k}{\partial w} = 1 \) so that

\[
\frac{\partial A_k}{\partial w} = (1 - A_k x) \cdots (1 - A_2 x).
\]

It follows that

\[
\frac{\partial w [x, \alpha]}{\partial w} = 1 + w (x - [x, \alpha]) \frac{\partial \alpha}{\partial w}
\]

\[
= 1 + w (x - [x, \alpha]) \left( (1 - \alpha B) \frac{\partial A_k}{\partial w} + (A_k - \alpha) \frac{\partial B}{\partial w} \right)
\]

\[
= 1 + w (x - [x, \alpha]) \left( (1 - \alpha B)(1 - A_k x) \cdots (1 - A_2 x) + (A_k - \alpha) \frac{\partial B}{\partial w} \right).
\]

Similarly we compute \( \frac{\partial w [x, \alpha]}{\partial x} \) and \( \frac{\partial w [x, \alpha]}{\partial v} \) when \( v \in \{ x_2, \ldots, x_\mu, y_1, \ldots, y_l \} \):

\[
\frac{\partial w [x, \alpha]}{\partial x} = w \left[ (1 - [x, \alpha]) + (x - [x, \alpha]) \left( (1 - \alpha B) \frac{\partial A_k}{\partial x} + (A_k - \alpha) \frac{\partial B}{\partial x} \right) \right],
\]

\[
\frac{\partial w [x, \alpha]}{\partial v} = w (x - [x, \alpha]) \left( (1 - \alpha B) \frac{\partial A_k}{\partial v} + (A_k - \alpha) \frac{\partial B}{\partial v} \right).
\]
We note that $\frac{\partial A_k}{\partial x_v} = 0$ since $A_k$ does not involve $v$. Moreover $\frac{\partial A_k}{\partial x_v} \chi_v \phi^E_\alpha = 0$ since $\partial A_k = 0$ and $\frac{\partial A_k}{\partial x_v} = (1 - [A_{k-1}, x]) \frac{\partial A_{k-1}}{\partial x} + (A_{k-1} - [A_{k-1}, x])$.

Using the involution and projecting to $Z \left[ E/E_r^{(n+1)} \right]$ we get

$$\frac{\partial \phi^E_\alpha}{\partial x_v} \chi_v \phi^E_\alpha = 1 + (x - 1)(1 - B)(1 - x)^{k-1}$$

$$\frac{\partial \phi^E_\alpha}{\partial x_i} = 0$$

Thus

$$(11.5) \left( \frac{\partial \phi^E_\alpha}{\partial x_v} \chi_v \phi^E_\alpha \right) \otimes Z \left[ E/E_r^{(n+1)} \right] i d \gamma_{[t^\pm]}$$

is a presentation of $H_1(W, s; \mathbb{K}_n^{[t^\pm]})$. 

Let $\psi$ be a primitive class in $H^1(W, \mathbb{Z})$ with $\psi(x) = t^p$ and let $\xi$ be a splitting of $\psi_n : E/E_r \to \mathbb{Z}$. We rewrite $a = 1 + (x - 1)(1 - B)(1 - x)^{k-1}$ as a polynomial in $t$. The lowest and highest degree terms of $a$ are $B$ and $t^{kp} (\xi(t)^{kp} k^k - k (1 - B) x^{k-1})$ respectively. Our assumption that $B \notin G_r^{(n+1)}$ (hence $B \notin E_r^{(n+1)}$ guarantees that $B - 1$ is a unit in $\mathbb{K}_n^{[t^\pm]}$). Therefore the degree of $a$ is $k$. Moreover, $\deg \sum v a_i = \deg \sum t^p a_i$ hence

$$\deg 1 + (x - 1)(1 - B)(1 - x)^{k-1} = kp.$$ 

Lastly, $H_1(X, s; \mathbb{K}_n^{[t^\pm]})$ is presented as $\left( \frac{\partial \phi^G_\alpha}{\partial x_v} \chi_v \phi^G_\alpha \right) \otimes Z [G/G_r^{(n+1)}] i d \gamma_{[t^\pm]}$ therefore

$$\delta_n^W(\psi) = \delta_n^X(\psi) + kp.$$ 

To finish the proof we will show that $\delta_n^{X(n,k)}(\psi) \geq \delta_n^W(\psi)$ for any $\psi$. Since $(W, X(n,k))$ has only 2 and 3-handles $H_1(W, X(n,k); R) = 0$. By Lemma 11.3, $H_2(W, X(n,k); R)$ is $R$-torsion. We have the following long exact sequence of pairs:

$$\to TH_2(W, X(n,k); R) \xrightarrow{\partial} H_1(X(n,k); R) \xrightarrow{j_*} H_1(W; R) \to 0$$

Since $j_* (TH_1(X(n,k); R)) \subseteq TH_1(W; R)$ we can consider the homomorphism

$$TH_1(X(n,k); R) \xrightarrow{j_*} TH_1(W; R).$$

We show that this map is surjective. Let $\sigma \in TH_1(W; R)$ and $\theta \in H_1(X(n,k); R)$ such that $j_* (\sigma) = \sigma$. There exists $r \in R$ such that $\sigma r = 0$ so $j_* (\sigma r) = j_* (\sigma) r = \sigma r = 0$. By exactness, this implies that $\sigma r \in \text{Im} \partial$. Hence there exists $\nu \in TH_2(W, X(n,k); R)$ with $\partial(\nu) = \sigma r$ and $\nu s = 0$ for some non-zero $s \in \mathbb{R}$. Therefore $\theta(\nu s) = (\theta r) s = \theta(\nu) s = \partial(\nu s) = \partial(0) = 0$ which implies $\theta \in TH_1(X(n,k); R)$. Lastly, let $R = \mathbb{K}_n^{[t^\pm]}$ then $TH_1(X(n,k); \mathbb{K}_n^{[t^\pm]})$ and
$TH_1 (W; \mathbb{K}_n [t^\pm 1])$ can be considered as free $\mathbb{K}_n$-modules with finite rank. We have $j_*$ surjective so

$$\delta_n^{X(n,k)} (\psi) = \text{rk}_{\mathbb{K}_n} TH_1 (X(n,k); \mathbb{K}_n [t^\pm 1]) \geq \text{rk}_{\mathbb{K}_n} TH_1 (W; \mathbb{K}_n [t^\pm 1]) = \delta_n^W (\psi).$$

Before proceeding, we will construct a specific example. We will begin with zero surgery on the trivial link with 2 components and subtly alter the manifold to increase $\delta_1$.

**Example 11.3.** A specific example of $X(1,1)$ when $X = S^1 \times S^2 \# S^1 \times S^2$. Let $X$ be 0-surgery on the 2-component trivial link. Then $X = S^1 \times S^2 \# S^1 \times S^2$ with $\pi_1$ generated by $x$ and $y$ (Figure 6). Let $B = [x,y]$ and construct $X(1,1)$ as in the theorem by doing $k$-framed surgery on $c$ (see Figure 7). For all $\psi \in H^1 (X(1,1))$,

$$\delta_0^{X(1,1)} (\psi) = \delta_0^X (\psi) = 0.$$

Moreover, we have

$$\delta_1^{X(1,1)} (\psi_x) \geq \delta_1^X (\psi_x) + 1 = 1.$$

We can assume that the manifolds that we have constructed to be irreducible.

**Proposition 11.4.** If $X$ is irreducible and $\partial$-irreducible then $X(n,k)$ is irreducible and $\partial$-irreducible.
Proof. Recall that \( X(n, k) \) can be constructed from \( X \) by first taking a connected sum with \( S^1 \times S^2 \) and then doing integer surgery on a curve \( c \). Recall that the first homology class of \( c \) was equal to \( x^{-1} z \) where \( x \) was a generator of \( H_1(X) \) and \( z \) was the generator of \( S^1 \times S^2 \). Let \( M = (X \# S^1 \times S^2) \). We will show that \( M - c \) is irreducible and \( \partial \)-irreducible. A theorem of Scharlemann \([3]\) p. 481 implies that \( X(n, k) \) is irreducible. It is clear that \( X(n, k) \) is \( \partial \)-irreducible.

First we show that \( M - c \) is \( \partial \)-irreducible. Since \( M \) is \( \partial \)-irreducible, it suffices to show that \( \ker i_* : \pi_1(\partial c) \to \pi_1(M) \) is trivial. Any curve on the boundary of \( c \) that is parallel to \( X \) is irreducible and \( S \) was the generator of \( \pi_1(M) \). We note that \( c \) is irreducible and let \( \Sigma \) be an embedded 2-sphere in \( M \) and \( \partial \Sigma \) bounds a ball in \( M \). Suppose that \( \Sigma \neq \emptyset \). Then \( \Sigma \) bounds a 3-ball \( B \) in \( M \). Since \( \Sigma \) is irreducible, \( \Sigma \) bounds a 3-ball \( B \) in \( M \). Hence \( M = B \cup \Sigma V \). \( f(W) \) is connected and \( f(W) \cap \Sigma = \emptyset \) so either \( f(W) \subset B \) or \( f(W) \subset V \). However, the homology class of \( c \) is equal to \( x^{-1} z \) hence \( f(W) \notin B \). Therefore \( f(W) \subset V \) hence \( \Sigma \) must bound a ball in \( M' \), a contradiction. Thus \( M' \) is irreducible.

Suppose that \( P_1 \) is compressible in \( M' \). Let \( \alpha \) be an curve on \( P_1 \) that bounds an embedded disc \( D \) in \( M' \). \( \alpha \) bounds the discs \( E_1 \) and \( E_2 \) on \( S \). Since \( M \) is irreducible, \( D \cup E_1 \) bound a 3-ball \( B \) in \( M \). If \( B \cap D = \emptyset \) then either \( D \cup E_1 \) or \( D \cup E_2 \) bounds a 3-ball \( B' \) in \( N \). We can use \( B' \) to isotope \( c \) and reduce the number of intersections of \( c \) with \( E_1 \) or \( E_2 \) (see Figure 5). This contradicts the minimality of \( \#(\Sigma \cap P) \). Thus \( M - c \) is irreducible.

Now suppose that \( B \cap B_2 \neq \emptyset \). Using \( B \) we can assume that either \( D \cup E_1 \) or \( D \cup E_2 \) bounds a 3-ball \( B' \) in \( M - B_1 \). Let \( S' = \partial B' \). We note that \( \#(S' \cap c) < \#(S \cap c) \) since \( c \) intersects \( E_1 \) and \( E_2 \). Moreover, \( S' \) and \( \partial B_2 \) cobound an embedded \( S^2 \times I \) in \( N \). Therefore \( S' \) and \( S \) cobound an embedded \( S^2 \times I \) in \( M \) (see Figure 5) which can be used to isotope the curve \( c \) to reduce the number of intersections with \( S \). This contradicts our minimality condition hence \( P_1 \) is incompressible in \( M' \). This completes the proof that \( M - c \) is irreducible.
Lemma 11.5. The manifold $X(n,k)$ is $K_i$-homology cobordant to $X$ for $i \leq n$. That is, $i : X \to W$ and $j : X(n,k) \to W$ induce isomorphisms on homology with $K_i$ coefficients where $W$ is the cobordism between $X$ and $X(n,k)$.

Proof. Consider the relative chain complex $C_\ast(W, X(n,k))$. Since $W$ is obtained from $X(n,k)$ by adding only 2 and 3-handles, $W$ is homotopy equivalent to a cell complex obtained by adding a single 2 and 3-cell. Hence we can assume $C_2(W, X(n,k)) = C_3(W, X(n,k)) \cong \mathbb{Z}$ and $C_j(W, X(n,k)) = 0$ for all other $j$. For all $i \leq n$ we lift the cells of $(W, X(n,k))$ to form the chain complex of
\[
(W, \tilde{X}(n, k)) \quad 0 \to C_3 (W, \tilde{X}(n, k)) \otimes K_i \xrightarrow{\partial_3 \otimes id} C_2 (\tilde{W}, \tilde{X}(n, k)) \otimes K_i \to 0
\]

where \((\tilde{W}, \tilde{X}(n, k))\) are the regular \(E/E^{(i+1)}\)-cover of \((W, X(n, k))\). Since \(\partial_3 \otimes id : K_i \to K_i\), \(\partial_3 \otimes id\) is an isomorphism if and only if \(\partial_3 \otimes id \neq 0\) if and only if \(\partial_3(\sigma) \neq 0\) for some \(\sigma\). But since \(H_*(W, X(n, k)) = 0\), \(\partial_3 : C_3(W, X(n, k)) \to C_2(W, X(n, k))\) is not the zero map, hence \(\partial_3\) is not the zero map. Therefore \(H_*(W, X(n, k); K_i) = 0\) which gives us \(j_* : H_*(X(n, k); K_i) \cong H_*(W; K_i)\). The proof that \(i_* : H_*(X; K_i) \cong H_*(W; K_i)\) follows almost verbatim except that \((W, X)\) has only cells in dimension 1 and 2.

**Lemma 11.6.** For each \(n \geq 1\), if \(\delta_{n-1}(\psi) \neq 0\) for some \(\psi \in H^1(X; \mathbb{Z})\) then \(G^{(n)}_r / G^{(r+1)}_r \neq 0\).

**Proof.** If \(\delta_{n-1}(\psi) \neq 0\) for some \(\psi\) then the rank of \(H_1(X_{n-1})\) as an abelian group is at least 1, hence \(H_1(X_{n-1}) / \{\mathbb{Z}-\text{torsion}\} \neq 0\). But, Lemma 3.3 gives \(G^{(n)}_r / G^{(r+1)}_r = H_1(X_{n-1}) / \{\mathbb{Z}-\text{torsion}\}\) so \(G^{(n)}_r / G^{(r+1)}_r \neq 0\).

We now proceed with the proof of Theorem 11.1.

**Proof of Theorem 11.1.** Let \(X_0\) be a 3-manifold with \(r_0(X_0) = 0\), \(G = \pi_1(X_0)\), \(\mu = \beta_1(X_0)\) and whose universal abelian cover has non-trivial \(\beta_1\). For example, we could let \(X_0\) be any 3-manifold with \(\beta_1(X_0) = \mu\) which fibers over \(S^1\) with fiber a surface of genus \(g \geq 2\) (see Proposition 8.4). Let \(\{x_1, \ldots, x_\mu\}\) be a basis of \(H_1(X_0; \mathbb{Z}) / \{\mathbb{Z}-\text{torsion}\}\) and \(\{\psi_{x_1}, \ldots, \psi_{x_\mu}\}\) the (Hom) dual basis of \(H^1(X_0; \mathbb{Z})\). Since \(\beta_1((X_0)_{\mathbb{C}}) > 0\), \(G^{(1)}_r / G^{(2)}_r \neq 0\) we can use Theorem 11.2 with \(k = 1\) to construct a new manifold \(X_1\) with \(\delta_0^{X_0}(\psi) = \delta_0^{X_1}(\psi)\) and \(\delta_1^{X_0}(\psi) < \delta_1^{X_1}(\psi)\) for all \(\psi \in H^1(X_1; \mathbb{Z})\) and \(\delta_0^{X_1}(\psi_{x_1}) < \delta_1^{X_0}(\psi_{x_1}) + 1 \leq \delta_1^{X_1}(\psi_{x_1})\)

We note that Theorem 11.2 guarantees \(\delta_1^{X_1}(\psi) \geq \delta_1^{X_0}(\psi)\) for all other \(\psi\). Now we continue this for all other basis elements \(\psi_{x_2}, \ldots, \psi_{x_\mu}\) to get a 3-manifold \(X_1 = X_1^\mu\) with \(\delta_0^{X_0}(\psi) = \delta_0^{X_1}(\psi)\) and \(\delta_0^{X_0}(\psi) < \delta_0^{X_1}(\psi)\) for all \(\psi \in H^1(X_1; \mathbb{Z})\).

In particular, \(\delta_1^{X_1}(\psi) > 0\) so by Lemma 11.4 \(G^{(2)}_r / G^{(3)}_r \neq 0\). Hence we can construct \(X_2\) with \(\delta_2^{X_2}(\psi) = \delta_2^{X_1}(\psi)\) when \(i \leq 1\) and \(\delta_2^{X_1}(\psi) < \delta_2^{X_2}(\psi)\) for all \(\psi \in H^1(X_2; \mathbb{Z})\). We continue this process until we obtain a 3-manifold \(X = X_m\) with \(\delta_0^{X}(\psi) < \delta_1^{X}(\psi) < \cdots < \delta_m^{X}(\psi)\) for all \(\psi \in H^1(X; \mathbb{Z})\). Since \(r_0(X_0) = 0\), Lemma 11.3 guarantees that \(r_0(X) = 0\) hence \(\|\psi\|_A = \delta_0(\psi)\) and \(\delta_m(\psi) \leq \|\psi\|_T\).

If we choose \(X_0\) to be closed, then \(X\) will be closed. Finally, to guarantee that \(X\) is irreducible, it suffices to choose \(X_0\) irreducible by Proposition 11.4.

We note that since \(r_i(X) = 0\), \(\delta_i(\psi) = \delta_i(\psi)\) hence we could have stated this theorem in terms of \(\delta_i\) as well as \(\delta_i\).
12. Applications

We show that the higher-order degrees give new computable algebraic obstructions to 3-manifolds fibering over $S^1$ even when the classical Alexander module fails. Moreover, using the work of P. Kronheimer, T. Mrowka, and S. Vidussi we are able to show that the higher-order degrees give new computable algebraic obstructions a 4-manifold of the form $X \times S^4$ admitting a symplectic structure, even when the Seiberg-Witten invariants fail.

12.1. Fibered 3-manifolds. Recall that if $X$ is a compact, orientable 3-manifold that fibers over $S^1$ then by Proposition 8.4, the higher-order ranks must be zero. Moreover, if $\beta_1 (X) \geq 2$ and $\psi$ is dual to a fibered surface then $\delta_n (\psi)$ must be equal to the Thurston norm for all $n$ and hence are constant as a function of $n$. We define the following function of $\psi$. Let $d_{ij} : H^1 (X; \mathbb{Z}) \to \mathbb{Z}$ be defined by $d_{ij} = \delta_i - \delta_j$ for $i, j \geq 0$. Note that $d_{ij} = 0$ if and only if $\delta_i = \delta_j$ for all $\psi \in H^1 (X; \mathbb{Z})$.

Theorem 12.1. Let $X$ be a compact, orientable 3-manifold. If at least one of the following conditions is satisfied then $X$ does not fiber over $S^1$.

1. $r_n (X) \neq 0$ for some $n \geq 0$.
2. $\beta_1 (X) \geq 2$ and there exists $i, j \geq 0$ such that $d_{ij} (\psi) \neq 0$ for all $\psi \in H^1 (X; \mathbb{Z})$.
3. $\beta_1 (X) = 1$ and $d_{ij} (\psi) \neq 0$ for some $i, j \geq 1$ and $\psi \in H^1 (X; \mathbb{Z})$.
4. $\beta_1 (X) = 1, X \not\cong S^1 \times S^2, X \not\cong S^1 \times D^2$ and $d_{ij} (\psi) \neq 1 + \beta_3 (X)$ for some $j \geq 1$ where $\psi$ is a generator of $H^1 (X; \mathbb{Z})$.

Proof. We consider each of the cases separately.

(1) This follow immediately from Proposition 8.4.

To prove that each of last three conditions implies that $X$ does not fiber over $S^1$ we can assume that $r_n (X) = 0$ for all $n \geq 0$. Otherwise the conclusion would be (vacuously) true since $X$ would satisfy condition (1). Hence $\delta_n = \delta_n$ for all $n \geq 0$ by Remark 4.11.

(2) If $X$ fibers over $S^1$ and $\beta_1 (X) \geq 2$ then for all $n \geq 0$, $\delta_n (\psi) = \|\psi\|_T$ for some $\psi \in H^1 (X; \mathbb{Z})$ by Proposition 8.4. Hence $\delta_n (\psi)$ is a constant function of $n$. In particular, $d_{ij} (\psi) = 0$ for all $i, j \geq 0$ which contradicts our hypothesis.

(3) If $X$ fibers over $S^1$ and $\beta_1 (X) = 1$ then for all $n \geq 1$ and $\psi \in H^1 (X; \mathbb{Z})$, $\delta_n (\psi) = \|\psi\|_T$ by Proposition 8.4. Hence $\delta_n (\psi)$ is a constant function of $n$ for $n \geq 1$. In particular, $d_{ij} (\psi) = 0$ for all $i, j \geq 1$ and $\psi \in H^1 (X; \mathbb{Z})$ which contradicts our hypothesis.

(4) If $X$ fibers over $S^1$, $\beta_1 (X) = 1, X \not\cong S^1 \times S^2, X \not\cong S^1 \times D^2$ and $\psi$ is a generator of $H^1 (X; \mathbb{Z})$ then by Proposition 8.4, $\delta_0 (\psi) = \|\psi\|_T + 1 + \beta_3 (X)$. The rest of the proof is similar to the previous two cases.

The previously known algebraic obstructions to a 3-manifold fibering over $S^1$ are that the Alexander module $H_1 (X; \mathbb{Z} \Gamma_0)$ is finitely generated and (when $\beta_1 (X) = 1$) the Alexander polynomial is monic. If $\beta_1 (X) \geq 2$, the Alexander module being finitely generated implies that $r_0 (X) = 0$.

Consider the 3-manifolds in Theorem 11.1. We note that $\delta_1 > \delta_0$ hence they cannot fiber over $S^1$. Moreover, they can be chosen to have the same Alexander module as those of a 3-manifold that fibers over $S^1$ as remarked in the first paragraph in the proof of Theorem 11.1.
Corollary 12.2. For each $\mu \geq 1$, Theorem 11.1 gives an infinite family of closed irreducible 3-manifolds $X$ where $\beta_1(X) = \mu$, $X$ does not fiber over $S^1$, and $X$ cannot be distinguished from a fibered 3-manifold using the classical Alexander module.

12.2. Symplectic 4-manifolds of the form $X \times S^1$. We now turn our attention to symplectic 4-manifolds of the form $X \times S^1$. It is well known that if $X$ is a closed 3-manifold that fibers over $S^1$ then $X \times S^1$ admits a symplectic structure. Taubes conjectures the converse to be true.

Conjecture 12.3 (Taubes). Let $X$ be a 3-manifold such that $X \times S^1$ admits a symplectic structure. Then $X$ admits a fibration over $S^1$.

Using the work of Meng-Taubes and Kronheimer-Mrowka, Vidussi [V2] has recently given a proof of McMullen’s inequality using Seiberg-Witten theory. This generalizes the work of Kronheimer [K2] who dealt with the case that $X$ is the 0-surgery on a knot. Moreover, Vidussi shows that if $X \times S^1$ admits a symplectic structure (and $\beta_1(X) \geq 2$) then the Alexander and Thurston norms of $X$ coincide on a cone over a face of the Thurston norm ball of $X$, supporting the conjecture of Taubes.

Theorem 12.4 ([K2 [V1, V2]). Let $X$ be an closed, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure. If $\beta_1(X) \geq 2$ there exists a $\psi \in H^1(X; \mathbb{Z})$ such that $\|\psi\|_A = \|\psi\|_T$. If $\beta_1(X) = 1$ then for any generator $\psi$ of $H^1(X; \mathbb{Z})$, $\|\psi\|_A = \|\psi\|_T + 2$.

Consequently, we show that the higher-order degrees of a 3-manifold $X$ give new computable algebraic obstructions to a 4-manifold of the form $X \times S^1$ admitting a symplectic structure.

Theorem 12.5. Let $X$ be a closed irreducible 3-manifold. If at least one of the following conditions is satisfied then $X \times S^1$ does not admit a symplectic structure.

1. $\beta_1(X) \geq 2$ and there exists an $n \geq 1$ such that $\delta_n(\psi) > \delta_0(\psi)$ for all $\psi \in H^1(X; \mathbb{Z})$.
2. $\beta_1(X) = 1$, $\psi$ is a generator of $H^1(X; \mathbb{Z})$, and $\delta_n(\psi) > \delta_0(\psi) - 2$ for some $n \geq 1$.

Proof. If $\beta_2(X) \geq 2$, $n \geq 1$, and $X \times S^1$ admits a symplectic structure then by Theorem 10.1, Theorem 12.4, and Proposition 5.12, $\delta_n(\psi) \leq \|\psi\|_T = \delta_0(\psi)$ for some $\psi \in H^1(X; \mathbb{Z})$. If $\beta_2(X) = 1$, $n \geq 1$, $\psi$ is a generator of $H^1(X; \mathbb{Z})$ and $X \times S^1$ admits a symplectic structure then by Theorems 10.1 and 12.4, $\delta_n(\psi) \leq \|\psi\|_T = \delta_0(\psi) - 2$.

Thus, Theorem 11.3 gives examples of 4-manifolds of the form $X \times S^1$ which do not admit a symplectic structure but cannot be distinguished from a symplectic 4-manifold using the invariants of Seiberg-Witten theory.

Corollary 12.6. For each $\mu \geq 1$, Theorem 11.3 gives an infinite family of 4-manifolds $X \times S^1$ where $\beta_1(X) = \mu$, $X \times S^1$ does not admit a symplectic structure, and $X$ cannot be distinguished from fibered 3-manifold using the classical Alexander module.

We note that the conditions in Theorem 12.5 are (strictly) stronger that the conditions in Theorem 12.3. The cause of this discrepancy is our lack of knowledge of the behavior of higher-order degrees when $X \times S^1$ admits a symplectic structure. We make the following conjecture, supporting the conjecture of Taubes.
**Conjecture 12.7.** If $X$ is a closed, orientable, irreducible 3-manifold such that $X \times S^1$ admits a symplectic structure then there exists a $\psi \in H^1(X;\mathbb{Z})$ such that $\delta_n(\psi) = ||\psi||^2$ for all $n \geq 1$.

More interesting would be the possibility of finding an a symplectic 4-manifold of the form $X \times S^1$ such that $\delta_1(\psi) < \delta_0(\psi)$ for all $\psi \in H^1(X;\mathbb{Z})$; giving a counterexample to the conjecture of Taubes [2.3].

We conclude with the remark that Conjecture [12.7] is true when $X$ is a knot complement in $S^3$, [C, Theorem 9.5]. The proof of this relies on the fact that the higher-order degrees are non-decreasing in $n$ [C, Theorem 5.4] and are bounded by the Thurston norm. More precisely, Cochran proves that $\delta_0(\psi) - 1 \leq \delta_1(\psi) \leq \ldots \leq \delta_n(\psi) \ldots$ whenever $X = S^3 - K$ and $\psi$ is a generator $H^1(X;\mathbb{Z})$. Moreover, the proof of Theorem 5.4 in [C] can be modified to prove that higher-order degrees are non-decreasing (in $n$) when $X$ is any finite CW-complex homotopy equivalent to a 2-complex with Euler characteristic zero. Hence Conjecture [12.7] is also true for any 3-manifold with non-empty toroidal boundary.

**References**

[B] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.

[CKS] J. Cantarella, R. B. Kusner, and J. M. Sullivan, *On the minimum ropelength of knots and links*. To appear in Inventiones. math.GT/0103224

[C] T. D. Cochran, *Non-commutative knot theory*. Preprint, 2001. math.GT/0206258

[COT] T. D. Cochran, K. E. Orr, and P. Teichner, *Knot concordance, Whitney towers and $L^2$-signatures*. Preprint, 1999. math.GT/9908117

[Co1] P. M. Cohn, *Free rings and their relations*. Second Edition, London Math. Soc. Monographs, no. 19, Academic Press, London and New York, 1985.

[Co2] P. M. Cohn, *Skew fields*. Cambridge Univ. Press, Cambridge, 1995.

[CF] R. H. Crowell and R.H. Fox, *Introduction to knot theory*. Graduate Texts in Mathematics, 57, Springer-Verlag, 1963.

[CS] R. H. Crowell and D. Strauss, *On the elementary ideals of link modules*. Trans. Amer. Math. Soc., no. 142, 93–109, 1969.

[F1] R. H. Fox, *Free differential calculus I, derivation in the free group ring*. Ann. of Math., 57, 1953, 547–560.

[F2] R. H. Fox, *Free differential calculus II, the isomorphism problem*. Ann. of Math., 59, 1954, 196–210.

[G1] D. Gabai, *Foliations and the topology of 3-manifolds*. J. Differential Geom. 18 (1983), no. 3, 445–503.

[G2] D. Gabai, *Foliations and surgery on knots*. Bull. Amer. Math. Soc. (N.S.) 15 (1986), no. 1, 83–87.

[G3] D. Gabai, *Foliations and the topology of 3-manifolds*. II. J. Differential Geom. 26 (1987), no. 3, 461–478.

[Ha1] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Ha2] A. Hatcher, *Notes on basic 3-manifold topology*. http://www.math.cornell.edu/~hatcher

[H1] J. Hempel, *3-Manifolds*. Ann. of Math. Studies, No. 86. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976.

[H2] J. Hempel, *Intersection calculus on surfaces with applications to 3-manifolds*. Memoirs Amer. Math. Soc., 43, no. 282, 1983.

[Hi] J. A. Hillman, *Alexander ideals of links*. Lecture Notes in Mathematics, Springer-Verlag, 1981.

[Ja] W. Jaco, *Lectures on three-manifold topology*. CBMS Regional Conference Series in Mathematics, 43. American Mathematical Society, Providence, R.I., 1980.

[J] N. Jacobson, *The theory of rings*. AMS Mathematical Surveys and Monographs, v. 2, 1943.

[Ka] A. Kawauchi, *A survey of knot theory*. Birkhauser, 1996.
[K1] P. Kronheimer, *Minimal genus in $S^1 \times M^3$*. Inventiones Mathematicae 134, 1998, 363–400.

[K2] P. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*. Surveys in Differential Geometry, V III (Cambridge, MA, 1996), International Press, 1998, 243–298.

[KM] P. Kronheimer and T. Mrowka, *Scalar curvature and the Thurston norm*. Mathematical Research Letters 4, 1997, 931–937.

[M] W. S. Massey, *A basic course in algebraic topology*. Graduate Texts in Mathematics, 127. Springer-Verlag, New York, 1991.

[Mc] C. T. McMullen, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*. Preprint, 1998. http://www.math.harvard.edu/~ctm

[MT] G. Meng, C. Taubes, *SW = Milnor torsion*. Math. Res. Lett. 3 (1996), no. 5, 661–674.

[P] D. Passman, *The algebraic structure of group rings*. John Wiley and Sons, New York, 1977.

[R] D. Rolfsen, *Knots and links*. Mathematical Lecture Series, no. 7, Publish or Perish Inc., Berkeley, CA, 1976.

[S] M. Scharlemann, *Producing reducible 3-manifolds by surgery on a knot*. Topology 29 (1990), no. 4, 481–500.

[Ste] B. Stenström, *Rings of quotients*. Springer-Verlag, New York, 1975.

[Str] R. Strebel, *Homological methods applied to the derived series of groups*. Comment. Math. 49, 1974, 302–332.

[T] W. P. Thurston, *A norm for the homology of 3-manifolds*. Memoirs of the AMS, no. 339, 1986, 99–130.

[V1] S. Vidussi, *The Alexander norm is smaller than the Thurston norm; a Seiberg-Witten proof*. Prepublication École Polytechnique 6, 1999.

[V2] S. Vidussi, *Norms on the cohomology of a 3-manifold and SW theory*. Preprint, 2001. math.GT/0204211

[W] G. W. Whitehead, *Elements of homotopy theory*. Graduate Texts in Mathematics #61, Springer-Verlag, New York, 1978.

[Za] H. J. Zassenhaus, *The theory of groups*. Chelsea Publ. Co., 1958.

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