Symmetric units in integral group rings

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Abstract. In this paper, we study the question of when the symmetric units in an integral group ring $K\mathbb{G}$ form a multiplicative group. When $\mathbb{G}$ is periodic, necessary and sufficient conditions are given for this to occur.

1. Introduction

Let $U(K\mathbb{G})$ be the group of units of the group ring $K\mathbb{G}$ of the group $\mathbb{G}$ over a commutative ring $K$. The anti-automorphism $g \rightarrow g^{-1}$ of $\mathbb{G}$ extends linearly to an anti-automorphism $a \rightarrow a^*$ of $K\mathbb{G}$. Let $S_*(K\mathbb{G}) = \{x \in U(K\mathbb{G}) \mid x^* = x\}$ be the set of all symmetric units of $U(K\mathbb{G})$.

The subgroup $U_*(K\mathbb{G}) = \{x \in U(K\mathbb{G}) \mid xx^* = 1\}$ is called the unitary subgroup of $U(K\mathbb{G})$. It is easy to see ([4], Proposition 1.3) that if $K = \mathbb{Z}$ then $U_*(\mathbb{Z}\mathbb{G})$ is trivial, i.e. $U_*(\mathbb{Z}\mathbb{G}) = \pm 1$. If $U(\mathbb{Z}\mathbb{G}) \neq \pm 1$, then in $U(\mathbb{Z}\mathbb{G})$ there always exist nontrivial symmetric units, for example $xx^*$ where $x$ is a nontrivial unit in $U(\mathbb{Z}\mathbb{G})$.

In this paper we answer the question: for which groups $\mathbb{G}$ do the symmetric units of the integral group ring $\mathbb{Z}\mathbb{G}$ form a multiplicative group? If $K$ is a commutative ring of characteristic $p$ and $\mathbb{G}$ is a locally finite $p$-group this question for $K\mathbb{G}$ was described in [2].

Lemma (see [2]). Let $K$ be a commutative ring and $\mathbb{G}$ be an arbitrary group. If $S_*(K\mathbb{G})$ is a subgroup in $U(K\mathbb{G})$ then $S_*(K\mathbb{G})$ is abelian and normal in $U(K\mathbb{G})$.

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Theorem. If \( S_* \) is a subgroup in \( U(ZG) \), then the set \( t(G) \) of elements of \( G \) of finite order is a subgroup in \( G \). Every subgroup of \( t(G) \) is normal in \( G \) and \( t(G) \) is either abelian or a hamiltonian 2-group. Conversely, suppose that the group \( G \) satisfies the above conditions and \( G/t(G) \) is a right ordered group. Then \( S_* \) is a subgroup in \( U(ZG) \).

2. Proof of the theorem

If the subgroup \( t(G) \) of the group \( G \) has the given properties and the quotient group \( G/t(G) \) is right ordered, then by Theorem 5.2 [1]

\[
V(ZG) = G \cdot V(Zt(G)).
\]

Hence, every element \( u \in S_* \) can be written as \( bw \), where \( b \) is an element of \( G \) and \( w \in U(Zt(G)) \). Suppose that \( b \) is of infinite order and \( w = \alpha_1 g_1 + \ldots + \alpha_s g_s \). Then \( bw = w^i b^{-1} \) and \( \text{Supp}(bw) = \{bg_1 b, \ldots, bg_s b\} = \{g_1^{-1}, \ldots, g_s^{-1}\} \). Thus \( bg_1 b = g_i^{-1} \) and \( (bg_1)^2 = g_i^{-1} g_1 \) is an element of finite order, which is a contradiction.

We conclude that \( S_* \subseteq U(Zt(G)) \). If \( t(G) \) is abelian then \( S_* \) is a subgroup. On the other hand, if \( t(G) \) is a hamiltonian 2-group then by Corollary 2.3 in [4], \( V(Zt(G)) = t(G) \) and so \( S_* \) coincides with the centre of \( t(G) \) and is again a subgroup.

So now we assume that \( S_* \) is a subgroup in \( U(ZG) \). We first show that any subgroup of \( t(G) \) is normal in \( G \) (this also proves that \( t(G) \) is a subgroup of \( G \)). If not, then there exist \( x \in t(G), y \in G \) with \( y^{-1} x y \notin \langle x \rangle \). But then \( u = 1 + (1 - x)y \) is a nontrivial bicyclic unit in \( ZG \) (where \( \hat{x} = 1 + x + \ldots + x^{n-1}, n = o(x) \)), and MARCINIAK and SEHGAL proved in [3] that \( \langle u, u^* \rangle \) is a nonabelian free subgroup of \( U(ZG) \). In particular, this means that \( uu^* \neq u^* u \) and that \( uu^* \) and \( u^* u \) do not commute with each other. Since \( uu^* \) and \( u^* u \) are in \( S_* \), this contradicts the lemma.

We now have that \( t(G) \) is either abelian or hamiltonian. To finish the proof, we need only to show that if \( Q = \langle a, b | a^4 = 1, a^2 = b^2, ba = a^3 b \rangle \) is the usual quaternion group and \( g \) is of odd prime order \( p \), then \( Q \times \langle g \rangle \) contains a pair of noncommuting symmetric units.

Recall ([4], p. 34) that if \( x \) is of order \( n \) in \( G \) and \( (i, n) = (j, n) = 1 \), and \( ik \equiv 1 \) (mod \( n \)), then

\[
u = (1 + x^j + \ldots + x^{j(i-1)})(1 + x^i + \ldots + x^{i(k-1)}) + \frac{1 - ik}{n} \hat{x}
\]
is a (Hochchmann) unit in \( ZG \).
First assume $p \neq 3$. Then $ag$ and $bg$ are of order $4p$, and setting $i = j = 3$ (and $3k \equiv 1 \pmod{4p}$) we obtain units

$$u = (1 + (ag)^3 + (ag)^6)(1 + (ag)^3 + \ldots + (ag)^{3(k-1)}) + \frac{1 - 3k}{4p} \tilde{a}g$$

$$v = (1 + (bg)^3 + (bg)^6)(1 + (bg)^3 + \ldots + (bg)^{3(k-1)}) + \frac{1 - 3k}{4p} \tilde{b}g.$$ 

Now $u_1 = (ag)^{-2}u$ and $v_1 = (bg)^{-2}v$ are symmetric units. We claim that $u_1$ and $v_1$ do not commute. Since $(ag)^{-2}$ and $(bg)^{-2}$ are central, this is equivalent to showing that $u$ and $v$ do not commute.

Since $\frac{1 - 3k}{4p} \tilde{a}g$ and $\frac{1 - 3k}{4p} \tilde{b}g$ are central, this is equivalent to showing that $u_2$ and $v_2$ do not commute where

$$u_2 = (1 + (ag)^3 + (ag)^6)(1 + (ag)^3 + \ldots + (ag)^{3(k-1)})$$

$$= 1 + 2(ag)^3 + 3(ag)^6 + \ldots + 3(ag)^{3(k-1)} + 2(ag)^{3k} + (ag)^{3(k+1)}$$

$$v_2 = 1 + 2(bg)^3 + 3(bg)^6 + \ldots + 3(bg)^{3(k-1)} + 2(bg)^{3k} + (bg)^{3(k+1)}.$$ 

Since all terms with even exponents are central, this is equivalent to showing that $u_3$ and $v_3$ do not commute where

$$u_3 = 2(ag)^3 + 3(ag)^9 + \ldots + 3(ag)^{3(k-2)} + 2(ag)^{3k}$$

$$v_3 = 2(bg)^3 + 3(bg)^9 + \ldots + 3(bg)^{3(k-2)} + 2(bg)^{3k}.$$ 

But in $u_3v_3$ only 4 products are not divisible by 3. Since $3k \equiv 1 \pmod{4p}$, these reduce to $4abg^6 + 8a^3bg^4 + 4abg^2$. In $v_3u_3$, the same products reduce to $4a^3bg^6 + 8abg^4 + 4a^3bg^2$. Because all other products are divisible by 3, we see $u_3v_3 \neq v_3u_3$.

If $p = 3$, the same argument works with $i = j = k = 5$. In this case, direct calculation shows that if $u$ and $v$ are defined as before, the symmetric units $(ag)^4u$ and $(bg)^4v$ do not commute.

Note that when $G$ is periodic, the theorem shows that $S_\ast(\mathbb{Z}G)$ is a subgroup only in the obvious cases – namely when $G$ is either abelian or a hamiltonian 2-group.

We remark that it is possible to avoid using the result from [3] and to prove that every subgroup of $t(G)$ is normal in $G$ by a direct argument instead. We have decided to use [3] in order to indicate how useful the Marciniak–Sehgal result can be.
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