TRANSPARENT CONNECTIONS OVER NEGATIVELY CURVED SURFACES

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Abstract. Let \((M, g)\) be a closed oriented negatively curved surface. A unitary connection on a Hermitian vector bundle over \(M\) is said to be transparent if its parallel transport along the closed geodesics of \(g\) is the identity. We study the space of such connections modulo gauge and we prove a classification result in terms of the solutions of certain PDE that arises naturally in the problem. We also show a local uniqueness result for the trivial connection and that there is a transparent \(SU(2)\)-connection associated to each meromorphic function on \(M\).

1. Introduction

Let \((M, g)\) be a closed Riemannian manifold and let \(E \to M\) be a Hermitian vector bundle of rank \(n\) over \(M\). A unitary connection \(\nabla\) on \(E\) is said to be transparent if its parallel transport along every closed geodesic of \(g\) is the identity. These connections are “ghosts” or “invisible” from the point of view of the closed geodesics of \(g\). Clearly, if \(\nabla\) is transparent any other connection gauge equivalent to it will also be transparent. The goal of the present paper is the study of transparent connections modulo gauge transformations when \((M, g)\) is a closed oriented negatively curved surface.

The motivation for studying this problem comes from several a priori unrelated quarters. Transparent connections on \(S^2\) (and \(\mathbb{RP}^2\)) arise in a natural way in the theory of integrable systems and solitons when studying the Bogomolny equations \(D\Phi = \star F\) in \((2 + 1)\)-dimensional Minkowski space \([17, 18]\). Here \(\Phi\) is the Higgs field, \(F\) is the curvature of the connection, \(\star\) is the Hodge star operator of the metric and \(D\) is the induced connection on the endomorphism bundle. The condition of having trivial holonomy along the closed geodesics of a compactified space-like plane picks up finite dimensional families of solutions and enables the use of methods from twistor theory over a compact twistor space. \([1]\). In fact, using a more refined twistor correspondence, L. Mason has recently classified all transparent connections on \(S^2\) and \(\mathbb{RP}^2\) with the standard round metric \([12]\). For the case of \(S^2\), his results say that the space of transparent connections modulo gauge is in 1-1 correspondence with holomorphic vector bundles \(W \to (\mathbb{CP}^2)^*\) and positive definite Hermitian metrics on \(W\) restricted to the real slice \((\mathbb{RP}^2)^*\). Similar results are obtained for anti-self-dual Yang-Mills connections over \(S^2 \times S^2\) with split signature, see \([11]\).

The problem of determining a connection from its parallel transport along geodesics is a natural integral-geometry problem that can be considered also in the case of manifolds with boundary or \(\mathbb{R}^d\) with appropriate decay conditions at infinity. It arises for example when one considers the wave equation associated to the Schrödinger...
equation with external Yang-Mills potential $A$ and the inverse problem of determining the potential $A$ from the Dirichlet-to-Neumann map $\Lambda_A$. There are various results known for the integral-geometry problem. Local uniqueness theorems under various assumptions on the connection or its curvature are proved by V.A. Sharafutdinov [15], R. Novikov [14] and D. Finch and G. Uhlmann [6]. A global uniqueness result for connections with compact support is proved by G. Eskin in [5]. In the case of $\mathbb{R}^2$, Novikov shows (building on the work of Ward previously mentioned) that global uniqueness may fail and in fact, his construction gives non-trivial transparent connections over $\mathbb{R}P^2$. He also shows global uniqueness (with reconstruction) for $d \geq 3$ without assuming compact support.

As we mentioned before, in the present paper we will discuss transparent connections when the metric is negatively curved, or more generally, when its geodesic flow is Anosov. While our main focus here will be in the non-abelian case, we should mention that the abelian case $n = 1$ is also of interest, but it can be reduced to known results (see Theorem 3.2) to obtain that transparent connections, when they exist, are unique up to gauge equivalence. The abelian case arises also when discussing positivity of entropy production in dissipative geodesic flows or thermostats [3], thus showing that the problem of understanding transparent connections also pops up naturally in dynamical systems and non-equilibrium statistical mechanics.

Our first result (Theorem 3.1) asserts that not all bundles over a surface of genus $g$ carry transparent connections. In fact we show that a complex vector bundle $E$ over $M$ admits a transparent connection if and only if $2 - 2g$ divides its first Chern class $c_1(E)$. This result, and subsequent ones, are based on the classical Livsic theorem for non-abelian cocycles which is recalled in Section 2.

One of the obvious differences with the abelian case is the appearance of the following ghosts. Let $K$ be the canonical line bundle and $K^s$ with $s \in \mathbb{Z}$ be its tensor powers (if $s = 0$ we get the trivial bundle). The powers $K^s$ for $s \neq 0$ carry the Levi-Civita connection which we denote by $\nabla^{s}_\ell$. If $s = 0$ we understand that this is the trivial connection. Note that the Levi-Civita connection on $TM (= K^{-1})$ is transparent, since the parallel transport along a closed geodesic $\gamma$ must fix $\dot{\gamma}(0)$ and consequently any vector orthogonal to it since the parallel transport is an isometry and the surface is orientable. Thus any $\nabla^{s}_\ell$ is transparent. Given an $n$-tuple of integers $S := (s_1, \ldots, s_n)$, the connection

$$\nabla^{S}_\ell := \nabla^{s_1}_\ell \oplus \cdots \oplus \nabla^{s_n}_\ell$$

defines a transparent unitary connection on the bundle $E_S := K^{s_1} \oplus \cdots \oplus K^{s_n}$. Clearly $c_1(E_S) = (2g - 2)(s_1 + \cdots + s_n)$ and any complex vector bundle $E$ supporting a transparent connection is isomorphic to $E_S$ for $S$ such that $c_1(E) = c_1(E_S)$.

Now let $E$ be a Hermitian vector bundle and consider a unitary isomorphism $\tau : E \to E_S$, where $S$ is such that $c_1(E) = c_1(E_S)$. The unitary connection $\tau^* \nabla^{S}_\ell$ is a transparent connection on $E$ and its gauge equivalence class, denoted by $[S]$, is independent of $\tau$. Note that $[S_1] = [S_2]$ if and only if $S_1$ and $S_2$ coincide up to a permutation. However, as we shall see below, these will not be the only ghosts.
Given two transparent connections $\nabla^1$ and $\nabla^2$ write $\nabla^2 = \nabla^1 + A$ where $A \in \Omega^1(M, \text{ad} E)$. Let $\pi : SM \to M$ be the unit circle bundle and $X$ the vector field of the geodesic flow of the metric. The Livsic theorem will provide solutions $u \in \Omega^0(SM, \text{Aut} \pi^* E)$ of $D_X u + A u = 0$, where $D$ is the connection on the bundle of endomorphisms of $\pi^* E$ induced by $\nabla^1$. Inspired by the methods in [5], we will show that these solutions must have a finite Fourier expansion (cf. Theorem 5.1), in other words, $u$ must be a polynomial in the velocities. The degree of this polynomial will allow us to define a distance function on the set of transparent connections modulo gauge. As a consequence of these results we will derive the following local uniqueness statement for the trivial connection on the trivial bundle:

**Theorem A.** Consider the trivial bundle and let $\nabla$ be a transparent connection with curvature $F_T$. Let $K < 0$ be the Gaussian curvature of the surface and suppose that the Hermitian matrix $\pm i \ast F_T(x) - K(x) \text{Id}$ is positive definite for all $x \in M$. Then $\nabla$ is gauge equivalent to the trivial connection.

Thus, if a transparent connection has small enough curvature, it must be gauge equivalent to the trivial connection. Note that Theorem A is sharp, since a ghost genus $g$ discussed at the end of the paper, Subsection 6.5.

Our second main result is a classification of the set $\mathcal{T}$ of transparent connections on $E$ modulo gauge in terms of the solutions of certain non-linear PDE that arises naturally in the problem. In order to describe this PDE, recall that the unit sphere bundle $SM$ of $M$ has a canonical frame $\{X, H, V\}$ where $X$ is the geodesic vector field, $V$ is the vertical vector field and $H = [V, X]$ is the horizontal vector field. Let $f : SM \to u(n)$ be a smooth function, where $u(n)$ denotes the set of all $n \times n$ skew-Hermitian matrices. Consider the PDE:

$$H(f) + VX(f) = [X(f), f].$$

(1) Observe that the set $\mathcal{H}$ of solutions to (1) is invariant under the action of $U(n)$ given by $f \mapsto q^{-1} f g$, where $q \in U(n)$.

We shall say that two functions $f, h : SM \to u(n)$ are $V$-cohomologous if there exists a smooth function $u : SM \to U(n)$ such that $f = u^{-1} V(u) + u^{-1} h u$.

Given a constant matrix $c \in u(n)$ with $e^{2 \pi c} = \text{Id}$ we consider the $U(n)$-invariant subset $\mathcal{H}_c \subset \mathcal{H}$ given by those solutions $f$ which are $V$-cohomologous to $c$. The set $\mathcal{H}_c$ only depends on $\text{tr}(c)$ (see Lemma 6.2).

**Theorem B.** Let $E$ be a Hermitian vector bundle over a closed oriented surface of genus $g$ whose geodesic flow is Anosov. Suppose that $2 - 2g$ divides $c_1(E)$ and let $c \in u(n)$ be a constant matrix with $e^{2 \pi c} = \text{Id}$ and $c_1(E) = (2g - 2) \text{tr}(ic)$. Then, there is a 1-1 correspondence between $\mathcal{T}$ and $\mathcal{H}_c/U(n)$.

In the abelian case $n = 1$, it is not hard to see that the only solutions to (1) are the constants provided $K < 0$. This can be shown using the energy estimates method (the Pestov identity) which also gives some information about (1) for $n \geq 2$. This is discussed at the end of the paper, Subsection 6.5.
For $n \geq 2$, the constant solutions in $\mathcal{H}_c$ correspond precisely to the Levi-Civita ghosts $[S]$, but as we mentioned before, there are other ghosts besides $[S]$ and these have to come from non-constants solutions to (1). To see that this is the case we consider functions $f$ which only depend on the base point $x$. Under such assumption, it is easy to see that (1) turns into $2 \star df = [df, f]$, which only depends on the conformal class of the metric $g$. We discuss this equation in Subsection 6.4 for the case of $SU(2)$ and we show that its non-zero solutions correspond precisely with the set of holomorphic maps $f : M \to \mathbb{C}P^1$. We also show that all these maps are $V$-cohomologous to the zero matrix. In this way, via Theorem B, we virtually obtain as many $SU(2)$-transparent connections on the trivial bundle (modulo gauge) as meromorphic functions on $M$; these are all the transparent connections at distance one from the trivial connection (cf. Corollary 6.7 for the precise statement).

There are several questions which are worthy of further consideration. In particular, it would be interesting to exhibit elements in $\mathcal{H}_c$ which have dependence on the velocities. It seems that one can deal with this issue using an appropriate Bäcklund transformation, but we leave it as the subject of a future paper. The inclusion of a Higgs field $\Phi$ and the problem of understanding transparent pairs $(\nabla, \Phi)$ is also of interest, but it will also be discussed elsewhere.

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2. The Livsic theorem for non-abelian cocycles

Let $X$ be a closed manifold and $\phi_t : X \to X$ a smooth transitive Anosov flow. Recall that $\phi_t$ is Anosov if there is a continuous splitting $TX = E^0 \oplus E^u \oplus E^s$, where $E^0$ is the flow direction, and there are constants $C > 0$ and $0 < \rho < 1 < \eta$ such that for all $t > 0$ we have

$$\|d\phi_{-t}|_{E^u}\| \leq C \eta^{-t} \quad \text{and} \quad \|d\phi_{t}|_{E^s}\| \leq C \rho^t.$$ 

It is very well known that the geodesic flow of a closed negatively curved Riemannian manifold is a transitive Anosov flow.

Let $G$ be a compact Lie group; for the purposes of this paper it is enough to think of $G$ as $U(n)$.

**Definition 2.1.** A $G$-valued cocycle over the flow $\phi_t$ is a map $C : X \times \mathbb{R} \to G$ that satisfies

$$C(x, t + s) = C(\phi_t x, s) C(x, t)$$

for all $x \in X$ and $s, t \in \mathbb{R}$.

In this paper the cocycles will always be smooth. In this case $C$ is determined by its infinitesimal generator $A : X \to \mathfrak{g}$ given by

$$A(x) := \left. \frac{d}{dt} \right|_{t=0} C(x, t).$$
The cocycle can be recovered from $A$ as the unique solution to
\[ \frac{d}{dt} C(x, t) = dR_{C(x,t)}(A(x,t)), \quad C(x, 0) = \text{Id}, \]
where $R_g$ is right translation by $g \in G$.

**Theorem 2.2** (The smooth Livsic periodic data theorem). Let $C$ be a smooth cocycle such that $C(x,T) = \text{Id}$ whenever $\phi_T x = x$. Then, there exists a smooth function $u : X \rightarrow G$ such that
\[ C(x, t) = u(\phi_t x) u(x)^{-1}. \]

The existence of a Hölder continuous function $u$ (assuming $A$ is Hölder) was proved by Livšic [9, 10]. Smoothness of $u$ was proved by Niţică and Török [13].

In our applications we will need to consider non trivial vector bundles. Suppose $E$ is a rank $n$ Hermitian vector bundle over $X$ and $\phi_t : X \rightarrow X$ is as above, a smooth transitive Anosov flow.

**Definition 2.3.** A cocycle over $\phi_t$ is an action of $\mathbb{R}$ by bundle automorphisms which covers $\phi_t$. In other words, for each $(x, t) \in X \times \mathbb{R}$, we have a unitary map $C(x, t) : E_x \rightarrow E_{\phi_t x}$ such that $C(x, t+s) = C(\phi_t x, s) C(x, t)$.

If $E$ admits a unitary trivialization $f : E \rightarrow X \times \mathbb{C}^n$, then
\[ f C(x, t) f^{-1}(x, a) = (\phi_t x, D(x, t)a), \]
where $D : X \times \mathbb{R} \rightarrow U(n)$ is a cocycle as in Definition [2.1].

Let $E^*$ denote the dual vector bundle to $E$. If $E$ carries a Hermitian metric $h$, we have a conjugate isomorphism $\ell_h : E \rightarrow E^*$, which induces a Hermitian metric $h^*$ on $E^*$. Given a cocycle $C$ on $E$, $C^* := \ell_h C \ell_h^{-1}$ is a cocycle on $(E^*, h^*)$.

**Proposition 2.4.** Let $E$ be a Hermitian vector bundle over $X$ such that $E \oplus E^*$ is a trivial vector bundle. Let $C$ be a smooth cocycle on $E$ such that $C(x,T) = \text{Id}$ whenever $\phi_T x = x$. Then $E$ is a trivial vector bundle.

**Proof.** As explained above, the cocycle $C$ on $E$ induces a cocycle $C^*$ on $E^*$. On the trivial vector bundle $E \oplus E^*$ we consider the cocycle $C \oplus C^*$. Clearly $C \oplus C^*(x, T) = \text{Id}$ everytime that $\phi_T x = x$. Choose a unitary trivialization $f : E \oplus E^* \rightarrow X \times \mathbb{C}^{2n}$ and write
\[ f C \oplus C^*(x, t) f^{-1}(x, a) = (\phi_t x, D(x, t)a). \]
By Theorem 2.2 there exists a smooth function $u : X \rightarrow U(2n)$ such that $D(x, t) = u(\phi_t x) u^{-1}(x)$. Since $\phi_t$ is a transitive flow, we may choose $x_0 \in X$ with a dense orbit and without loss of generality we may suppose that $u(x_0) = \text{Id}$. Let
\[ \{e_1(x_0), \ldots, e_n(x_0)\} \]
be a unitary frame at $E_{x_0}$. Write $f(x_0, e_i(x_0)) = (x_0, a_i)$, where $a_i \in \mathbb{C}^{2n}$ and let
\[ e_i(x) := f^{-1}(x, u(x)a_i). \]
Clearly at every $x \in X$, $\{e_1(x), \ldots, e_n(x)\}$ is a smooth unitary $n$-frame of $E_x \oplus E_x^*$. We claim that in fact $e_i(x) \in E_x$ for all $x \in X$. This, of course, implies the triviality of $E$. Note that

$$e_i(\phi_t x_0) = f^{-1}(\phi_t x_0, u(\phi_t x_0) a_i) = f^{-1}(\phi_t x_0, D(x_0, t) a_i) = C \oplus C^*(x_0, t) e_i(x_0).$$

But $e_i(x_0) \in E_{x_0}$, thus $e_i(\phi_t x_0) \in E_{\phi_t x_0}$. It follows that $e_i(x) \in E_x$ for a dense set of points in $X$. By continuity of $e_1$, $e_i(x) \in E_x$ for all $x \in X$.

\[\square\]

**Remark 2.5.** The hypothesis of $E \oplus E^*$ being trivial is not needed in Proposition 2.4. Ralf Spatzier has informed the author that it is possible to adapt the proof of the usual Livsic periodic data theorem to show directly that $E$ is trivial. However, this weaker version is all that we will need in this paper.

A proof of the measurable Livsic theorem for bundles (which we do not use here) may be found in [7].

### 3. Transparent connections and the Livsic theorem

Let $M^d$ be a closed orientable Riemannian manifold whose geodesic flow $\phi_t$ is Anosov. The geodesic flow acts on the unit sphere bundle $SM$ and we let $\pi: SM \to \overline{M}$ be the footpoint projection.

Let $E \to M$ be a Hermitian vector bundle of rank $n$ and let $\nabla$ be a unitary connection on $E$. Given a geodesic $\gamma: \mathbb{R} \to M$ with initial conditions $(x, v) \in SM$, we can consider the parallel transport of $\nabla$ along $\gamma$. The parallel transport $P_{x, \gamma(t)}: E_x \to E_{\gamma(t)}$ is an isometry and defines a smooth cocycle $C$ over the geodesic flow on the Hermitian vector bundle over $SM$ given by the pull-back bundle $\pi^*E$. The connection $\nabla$ is transparent if and only if $C(x, v, T) = \text{Id}$ every time that $\phi_T(x, v) = (x, v)$.

#### 3.1. Arbitrary bundles over an Anosov surface.

Suppose $d = 2$. In this case, complex vector bundles $E$ over $M$ are classified topologically by the first Chern class $c_1(E) \in H^2(M, \mathbb{Z}) = \mathbb{Z}$. Since $c_1(E^*) = -c_1(E)$ and $c_1$ is additive with respect to direct sums, we see that $E \oplus E^*$ is the trivial bundle and therefore we will be able to apply Proposition 2.4. In fact we will show:

**Theorem 3.1.** Let $M$ be a closed orientable surface of genus $g$ whose geodesic flow is Anosov. A complex vector bundle $E$ over $M$ admits a transparent connection if and only if $2 - 2g$ divides $c_1(E)$.

**Proof.** Suppose $E$ admits a transparent connection. As explained above we may apply Proposition 2.4 to deduce that $\pi^*E$ is a trivial bundle and since $c_1(\pi^*E) = \pi^*c_1(E)$ we conclude that $\pi^*c_1(E) = 0$. Consider now the Gysin sequence of the unit circle bundle $\pi: SM \to \overline{M}$,

$$0 \to H^1(M, \mathbb{Z}) \xrightarrow{\pi^*} H^1(SM, \mathbb{Z}) \xrightarrow{0} H^0(M, \mathbb{Z}) \xrightarrow{\times (2 - 2g)} H^2(M, \mathbb{Z}) \xrightarrow{\pi^*} H^2(SM, \mathbb{Z}) \to \cdots .$$

We see that $\pi^*c_1(E) = 0$ if and only if $c_1(E)$ is in the image of the map $H^0(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ given by cup product with the Euler class of the unit circle bundle. Equivalently, $2 - 2g$ must divide $c_1(E)$. 

Let $K$ be the canonical line bundle of $M$. We can think of $K$ as the cotangent bundle to $M$; it has $c_1(K) = 2g - 2$. The tensor powers $K^s$ of $K$ (positive and negative) generate all possible line bundles with first Chern class divisible by $2 - 2g$ and they all carry the unitary connection induced by the Levi-Civita connection of the Riemannian metric on $M$. All these connections are clearly transparent. Topologically, all complex vector bundles over $M$ whose first Chern class is divisible by $2 - 2g$ are of the form $K^s \oplus \epsilon$, where $\epsilon$ is the trivial vector bundle. Since the trivial connection on the trivial bundle is obviously transparent, it follows that every complex vector bundle whose first Chern class is divisible by $2 - 2g$ admits a transparent connection.

3.2. Arbitrary bundles over an Anosov 3-manifold. Suppose $M$ is a closed 3-manifold whose geodesic flow is Anosov. Complex vector bundles $E$ over $M$ are also classified topologically by $c_1(E) \in H^2(M, \mathbb{Z})$, hence as in the two dimensional case, $E \oplus E^*$ is the trivial bundle. Thus if $E$ admits a transparent connection, Proposition 2.4 implies that $\pi^* E$ is trivial. However now the argument with the Gysin sequence that we explained in the proof of Theorem 3.1 shows that $\pi^*: H^2(M, \mathbb{Z}) \to H^2(SM, \mathbb{Z})$ is injective and thus $c_1(E) = 0$. Therefore if $E$ admits a transparent connection, it must be trivial. This shows that the problem for 3-manifolds is in some sense simpler than the problem for surfaces, at least, there are no obvious transparent connections besides the trivial one.

3.3. The abelian case. The goal of this subsection is to show the following result.

**Theorem 3.2.** Let $M$ be a closed orientable Riemannian manifold whose geodesic flow is Anosov and let $E$ be a Hermitian line bundle over $M$. Then, any two transparent connections are gauge equivalent.

**Proof.** Let $\nabla^1$ and $\nabla^2$ be transparent connections. We may write $\nabla^2 = \nabla^1 + A$, where $A \in \Omega^1(M, \text{ad } E)$. Since $E$ is a line bundle, $A = i\theta$, where $\theta$ is a real-valued 1-form in $M$. Since $\nabla^1$ and $\nabla^2$ are transparent,

\begin{equation}
\int_\gamma \theta \in 2\pi \mathbb{Z}
\end{equation}

for every closed geodesic $\gamma$. Consider the cocycle over $\phi_t$, $C : SM \times \mathbb{R} \to S^1$ defined as follows. Given $(x, v) \in SM$, let $\gamma : \mathbb{R} \to M$ be the unique geodesic with initial conditions $(x, v)$. Set

$$C(x, v, t) := \exp \left( i \int_0^t \theta_{\gamma(s)}(\dot{\gamma}(s)) \, ds \right).$$

By (2), the cocycle $C$ has the property that $C(x, v, T) = 1$, every time that $\phi_T(x, v) = (x, v)$, thus by Theorem 2.2 there is a smooth function $u : SM \to S^1$ such that $C(x, v, t) = u(\phi_t(x, v))u^{-1}(x, v)$. If we differentiate this equality with respect to $t$ and set $t = 0$ we obtain

\begin{equation}
ui\theta = du(X),
\end{equation}
where \( X \) is the geodesic vector field. The function \( u \) gives rise to a real-valued closed 1-form in \( SM \) given by \( \varphi := \frac{du}{iu} \). Since \( \pi^* : H^1(M, \mathbb{R}) \to H^1(SM, \mathbb{R}) \) is an isomorphism (this follows easily from the Gysin sequence, since \( M \) cannot be the 2-torus), there exists a closed 1-form \( \omega \) in \( M \) and a smooth function \( f : SM \to \mathbb{R} \) such that

\[
\varphi = \pi^* \omega + df.
\]

When this equality is applied to \( X \) and combined with (3) one obtains

\[
\theta_x(v) - \omega_x(v) = df(X(x,v))
\]

for all \((x, v) \in SM\). This cohomological equation is actually equivalent -via the classic Livsic theorem for \( \mathbb{R} \)-values cocycles- to saying that

\[
\int_{\gamma} \theta - \omega = 0
\]

for every closed geodesic \( \gamma \). It is known that this implies that \( \theta - \omega \) is exact. This was proved by V. Guillemin and D. Kazhdan \([8]\) for surfaces of negative curvature, by C. Croke and Sharafutdinov \([2]\) for arbitrary manifolds of negative curvature and by N.S. Dairbekov and Sharafutdinov \([1]\) for manifolds whose geodesic flows is Anosov.

If \( \theta - \omega \) is exact, \( \theta \) must be closed and by (2), \( [\theta]/2\pi \in H^1(M, \mathbb{Z}) \). Thus there exists a smooth function \( g : M \to S^1 \) such that \( \theta = dg/ig \). This is precisely the statement that \( \nabla_1 \) and \( \nabla_2 \) are gauge equivalent.

4. Setting up the Fourier analysis

Let \( E \) be a complex Hermitian vector bundle over \( M \) and let \( \nabla \) be a unitary connection on \( E \). If \( \pi : SM \to M \) denotes the canonical projection, then \( \nabla \) induces a unitary connection on the pull-back bundle \( \pi^*E \) that we denote by \( \pi^*\nabla \). This pull-back connection induces in turn a unitary connection on the bundle \( \text{End}\pi^*E \) of complex endomorphisms of \( \pi^*E \), which we denote by \( D \). Note that \( \text{End}\pi^*E \) naturally inherits a Hermitian metric determined by the trace \( \text{tr}(uw^*) \) where \( u, w \in \Omega^0(SM, \text{End}\pi^*E) \). This Hermitian metric together with the Liouville measure \( \mu \) of \( SM \) combine to give an \( L^2 \)-inner product of sections

\[
\langle u, w^* \rangle = \int_{SM} \text{tr}(uw^*) d\mu.
\]

Let \( F_\nabla \in \Omega^2(M, \text{ad} E) \) be the curvature of \( \nabla \). Then \( F_{\pi^*\nabla} \in \Omega^2(SM, \text{ad} \pi^*E) \) is given by \( F_{\pi^*\nabla} = \pi^*F_\nabla \) and \( F_D \in \Omega^2(SM, \text{ad End} \pi^*E) \) is given by \( F_D = [F_{\pi^*\nabla}, \cdot] \). Note that if \( \star \) denotes the Hodge star operator of the metric, then \( \star F_\nabla \in \Omega^0(M, \text{ad} E) \).

The vertical vector field \( V \) and the connection \( D \) induce a first order differential operator

\[
D_V : \Omega^0(SM, \text{End} \pi^*E) \to \Omega^0(SM, \text{End} \pi^*E)
\]

which in fact is independent of \( \nabla \), for if \( \nabla' \) is another connection and we write \( \nabla' = \nabla + A \), then \( D' = D + [\pi^*A, \cdot] \) and \( D_V = D_V \) since \( \pi^*A(V) = 0 \).

Note that \( -iD_V \) is self-adjoint, since \( V \) preserves the Liouville measure \( \mu \). Indeed, observe that the compatibility of \( D \) with the Hermitian metric implies \( V\langle u, w \rangle = \)
\[ \langle D_V u, w \rangle + \langle u, D_V w \rangle. \] Since the integral of \( V \langle u, w \rangle \) with respect to \( \mu \) vanishes, \( (D_V)^* = -D_V \). We also have an orthogonal decomposition

\[ L^2(SM, \text{End} \pi^*E) = \bigoplus_{n \in \mathbb{Z}} H_n \]

such that \(-iD_V\) acts as \( n \text{Id} \) on \( H_n \). To see this, triangulate \( M \) in such a way that both \( SM \to M \) and \( E \to M \) are trivial over each face \( M_r \) of the triangulation. Since \( L^2(SM, \text{End} \pi^*E) \) is isomorphic to \( \bigoplus_r L^2(SM_r, \text{End} \pi^*E) \) we are reduced to the case of both bundles being trivial in which case the claim is clear because \( D_V u = V(u) \), where \( u \) is a matrix valued function on \( M_r \times S^1 \).

Following Guillemin and Kazhdan in [8] we introduce the following first order differential operators

\[ \eta_+, \eta_- : \Omega^0(SM, \text{End} \pi^*E) \to \Omega^0(SM, \text{End} \pi^*E) \]

given by

\[ \eta_+ := \frac{D_X - iD_H}{2}, \]
\[ \eta_- := \frac{D_X + iD_H}{2}, \]

where \( H = [V, X] \). We recall the other two structure equations of the Riemannian metric: \( X = -[V, H] \) and \( [X, H] = KV \), where \( K \) is the Gaussian curvature of the surface.

The next lemma describes the commutation relations between these operators.

**Lemma 4.1.** We have

\[ [-iD_V, \eta_+] = \eta_+, \]
\[ [-iD_V, \eta_-] = -\eta_-, \]
\[ [\eta_+, \eta_-] = \frac{i}{2} (K D_V + [\ast F_\nabla, \cdot]). \]

**Proof.** In order to prove the lemma we need the following general preliminary observation: if \( U \) and \( W \) are vector fields in \( SM \) then

\[ F_D(U, W) = [D_U, D_W] - D_{[U,W]}. \]

As noted before

\[ F_D = [\pi^*F_\nabla, \cdot]. \]

Thus for any vector field \( U, F_D(V, U) = 0 \) (\( V \) is vertical) and hence \([D_V, D_U] = D_{[V,U]}\). If we now take \( U = X, H \) and we use the commutation relations \([V, X] = H \) and \([V, H] = -X \) we obtain \([D_V, D_X] = D_H \) and \([D_V, D_H] = -D_X \). The first two commutation relations in the lemma follow easily from this. To prove the third relation note that \( 2[\eta_+, \eta_-] = i[D_X, D_H] \). Using (4) and (5) together with \([X, H] = KV \) we see that \([D_X, D_H] = F_D(X, H) + K D_V = [\ast F_\nabla, \cdot] + K D_V \) and the third commutation relation follows.
Let us set $\Omega_n := H_n \cap \Omega^0(SM, \text{End } \pi^*E)$. The first two commutation relations in the lemma imply that $\eta_+: \Omega_n \to \Omega_{n+1}$ and $\eta_-: \Omega_n \to \Omega_{n-1}$. It also follows easily from the fact that $X$ and $H$ preserve $\mu$ and the definitions that $\eta^*_+ = -\eta_-$ and $\eta^*_- = -\eta^*_+$. Indeed, like $D_V$, $D_X$ and $D_H$ are skew-Hermitian since both $X$ and $H$ preserve the Liouville measure $\mu$.

4.1. Modified operators. We will now modify the operators $\eta_+$ and $\eta_-$ to suit our purposes. Consider a second unitary connection $\nabla^0$ and write $\nabla^0 = \nabla + A$, where $A \in \Omega^1(M, \text{ad } E)$. We may regard $A$ and $\star A$ as elements of $\Omega^0(SM, \text{ad } \pi^*E)$ and if we do so, then $D_V A = \star A$ since $(D_V A)(x, v) = A(x, iv) = (\star A)(x, v)$. This certainly implies that $D^2_V A = -A$. Having this in mind, we decompose $A$ as $A = A_{-1} + A_1$ where

$$A_1 := \frac{A - iD_V A}{2} \in \Omega_1, \quad A_{-1} := \frac{A + iD_V A}{2} \in \Omega_{-1}.$$ 

Observe that this decomposition corresponds precisely with the usual decomposition of 1-forms on a surface:

$$\Omega^1(M, \text{ad } E) \otimes \mathbb{C} = \Omega_{1,0}^1(M, \text{ad } E) \oplus \Omega^{0,1}(M, \text{ad } E),$$

given by the eigenvalues $\pm i$ of the Hodge star operator.

We now set $\mu_+ := \eta_+ + A_1$ and $\mu_- := \eta_- + A_{-1}$. It is straightforward to check that $\mu_+ : \Omega_n \to \Omega_{n+1}$ and $\mu_- : \Omega_n \to \Omega_{n-1}$. It is also easy to check that $\mu^*_+ = -\mu_-$ and $\mu^*_- = -\mu^*_+$. We will need the following auxiliary lemma.

**Lemma 4.2.** The following relation holds

$$\frac{i}{2} \star (\nabla A + A \wedge A) = \eta_+(A_{-1}) - \eta_-(A_1) + A_1 A_{-1} - A_{-1} A_1.$$ 

**Proof.** Using the definitions we derive

$$A_1 A_{-1} - A_{-1} A_1 = \frac{i}{2} (AD_V(A) - D_V(A) A),$$

$$\eta_+(A_{-1}) - \eta_-(A_1) = \frac{i}{2} (D_X D_V A - D_H A).$$

But it is easy to check that

$$\star (A \wedge A) = AD_V(A) - D_V(A) A,$$

and since

$$\star (\nabla A) = D_X (\pi^* A)(X, H)$$

$$= D_X (\pi^* A(X)) - D_H(\pi^* A(X)) - \pi^* A([X, H])$$

$$= D_X D_V A - D_H A,$$

the lemma is proved. \qed
The next lemma describes the commutator $[\mu_+, \mu_-]$.

**Lemma 4.3.** Given $u \in \Omega^0(SM, \text{End} \pi^*E)$ we have

$$[\mu_+, \mu_-]u = \frac{i}{2} (K D_V u + (\ast F_{\nabla_0}) u - u(\ast F_{\nabla})) .$$

**Proof.** Using the definitions we compute

$$[\mu_+, \mu_-]u = [\eta_+, \eta_-]u + (\eta_+(A_{-1}) - \eta_-(A_1))u + (A_1 A_{-1} - A_{-1} A_1)u .$$

Using Lemmas 4.1 and 4.2 we obtain

$$[\mu_+, \mu_-]u = \frac{i}{2} (K D_V u + [\ast F_{\nabla}, u]) + \frac{i}{2} (\nabla A + A \wedge A)u .$$

The lemma is now a consequence of the fact that $F_{\nabla_0} = F_{\nabla} + \nabla A + A \wedge A$.

Let $|\cdot|$ stand for the $L^2$-norm in $\Omega^0(SM, \text{End} \pi^*E)$.

**Corollary 4.4.** Given $u \in \Omega^0(SM, \text{End} \pi^*E)$ we have

$$|\mu_+ u|^2 = |\mu_- u|^2 + \frac{i}{2} (\langle K D_V u, u \rangle + \langle (\ast F_{\nabla_0}) u, u \rangle - \langle u(\ast F_{\nabla}), u \rangle) .$$

**Proof.** Using that $\mu_+^* = -\mu_-$ and $\mu_-^* = -\mu_+$ we derive

$$|\mu_+ u|^2 = \langle \mu_+ u, \mu_+ u \rangle = \langle (\mu_+)^* \mu_+ u, u \rangle = \langle -\mu_- \mu_+ u, u \rangle = \langle -\mu_+ \mu_- u, u \rangle + \langle [\mu_+, \mu_-] u, u \rangle = \langle (\mu_-)^* \mu_- u, u \rangle + \langle [\mu_+, \mu_-] u, u \rangle = |\mu_- u|^2 + \langle [\mu_+, \mu_-] u, u \rangle$$

and the corollary follows from the previous lemma.

**5. A distance between transparent connections**

Let $\nabla^1$ and $\nabla^2$ be two unitary connections. We may write $\nabla^2 = \nabla^1 + A$, where $A \in \Omega^1(M, \text{ad} E)$.

If $\pi : SM \to M$ denotes the canonical projection, we obtain unitary connections on the pull-back bundle $\pi^*E$ which are related by

$$\pi^* \nabla^2 = \pi^* \nabla^1 + \pi^* A ,$$

where $\pi^* A \in \Omega^1(SM, \text{ad} \pi^* E)$. These connections on $\pi^*E$ induce in turn connections $D^1$ and $D^2$ on the bundle End $\pi^*E$ of complex endomorphisms of $\pi^*E$ and are related by

$$D^2 = D^1 + [\pi^* A, \cdot] .$$
Suppose now that both $\nabla^1$ and $\nabla^2$ are transparent. As explained in Section 3, they induce smooth cocycles $C_1$ and $C_2$ on $\pi^*E$. By Proposition 2.4, $\pi^*E$ is trivial and via a unitary trivialization we may use the Livsic theorem 2.2 to conclude that there exists a smooth $u \in \Omega^0(SM, \text{Aut} \pi^*E)$ such that

$$C_2(x, v, t) = u(\phi_t(x, v))C_1(x, v, t)u^{-1}(x, v).$$

Take $\xi \in E_x$. Since $C_1(x, v, t)\xi$ (resp. $C_2(x, v, t)\xi$) is $\nabla^1$-parallel (resp. $\nabla^2$-parallel) along the geodesic determined by $(x, v)$, if we apply $\nabla^1$ to the previous equality and set $t = 0$ we obtain at $(x, v)$:

$$-A\xi = (D_Xu)u^{-1}\xi$$

where $D := D^1$, and since this holds for all $\xi$ we derive

$$D_Xu + Au = 0. \quad (6)$$

The main result that we will prove about the solutions $u$ of (6) is that they have a finite Fourier expansion.

Given an element $u \in \Omega^0(SM, \text{End} \pi^*E)$, we write $u = \sum_{m \in \mathbb{Z}} u_m$, where $u_m \in \Omega^m$. We will say that $u$ has degree $N$, if $N$ is the smallest non-negative integer such that $u_m = 0$ for all $m$ with $|m| \geq N + 1$.

**Theorem 5.1.** Let $u \in \Omega^0(SM, \text{End} \pi^*E)$ be a smooth solution to (6). Then $u$ has degree $N < \infty$. Moreover $N \leq l - 1$ where $l$ is the smallest positive integer such that the Hermitian operators

$$\text{End } E_x \ni \alpha \mapsto -lK(x)\alpha + \alpha (i \star F_{\nabla^2}(x)), \quad \text{End } E_x \ni \alpha \mapsto -lK(x)\alpha - \alpha (i \star F_{\nabla^2}(x))$$

are positive definite for all $x \in M$.

**Proof.** Since $D_X = \eta_+ + \eta_-$, equation (6) may be rewritten as

$$\mu_+(u) + \mu_-(u) = 0.$$  

Projecting onto $\Omega_m$-components we obtain

$$\mu_+(u_{m-1}) + \mu_-(u_{m+1}) = 0 \quad (7)$$

for all $m \in \mathbb{Z}$. Since $K < 0$, there exists a positive integer $l$ such that the Hermitian operators

$$u \mapsto -lKu + (i \star F_{\nabla^2})u - u(i \star F_{\nabla^1}), \quad u \mapsto -lKu - (i \star F_{\nabla^2})u + u(i \star F_{\nabla^1})$$

are positive definite for all $x \in M$. Using Corollary 4.4, we can find a constant $c > 0$ such that

$$|\mu_+(u_m)|^2 \geq |\mu_-(u_m)|^2 + c|u_m|^2 \quad (8)$$

for all $m \geq l$. There is also a constant $d > 0$ such that

$$|\mu_-(u_m)|^2 \geq |\mu_+(u_m)|^2 + d|u_m|^2 \quad (9)$$
for all $m \leq -l$. Combining (7) and (8) we obtain
\[
|\mu_+(u_{m+1})| \geq |\mu_+(u_{m-1})|
\]
for all $m \geq l - 1$. Similarly, it follows from (7) and (9) that
\[
|\mu_-(u_{m-1})| \geq |\mu_-(u_{m+1})|
\]
for all $m \geq -l + 1$. Since the function $u$ is smooth, $\mu_+(u_m)$ must tend to zero in the $L^2$-topology as $m \to \infty$. It follow from (10) that $\mu_+(u_m) = 0$ for $m \geq l - 2$. However, (8) implies that $\mu_+$ is injective for $m \geq l$ and thus $u_m = 0$ for $m \geq l$. Similarly, using (9) and (11) we deduce that $u_m = 0$ for $m \leq -l$. This shows that $u$ has finite degree $N \leq l - 1$.

Let $\mathcal{T}$ denote the space of transparent connections modulo gauge transformations. Using the previous theorem we can introduce a distance function on $\mathcal{T}$ as follows.

Given $[\nabla^1], [\nabla^2] \in \mathcal{T}$ we set $d([\nabla^1], [\nabla^2]) := N$, where $N$ is the smallest degree of $u \in \Omega^0(SM, Aut \pi^*E)$ which solves $D_Xu + Au = 0$. It is easy to check that this definition does not depend on the chosen representatives as we now explain. Let $\varphi, \psi \in \Omega^0(M, Aut E)$ and write $\varphi^*\nabla^2 = \psi^*\nabla^1 + A$. One checks that

$$\tilde{A} = \varphi^{-1}A\varphi + \varphi^{-1}\nabla^1\varphi - \psi^{-1}\nabla^1\psi$$

and using this one also checks that $\varphi^{-1}u\psi$ solves $(\tilde{D}_X + \tilde{A})(\cdot) = 0$, where $\tilde{D}$ is the connection induced by $\psi^*\nabla^1$. Since $\varphi^{-1}u\psi$ has the same degree as $u$, $d$ is well defined.

**Proposition 5.2.** $d([\nabla^1], [\nabla^2])$ defines a distance function on $\mathcal{T}$.

**Proof.** Suppose $d([\nabla^1], [\nabla^2]) = 0$. This means that there exists $u \in \Omega^0(SM, Aut \pi^*E)$ which solves $D_Xu + Au = 0$ and $D_Xu = 0$. But this last equality means that $u(x, v)$ is independent of $v$. Indeed, consider a unitary trivialization of $E$ over a neighbourhood $V$ of $x$ and write $\nabla^1 = d + C$, where $C$ is a $u(n)$-valued 1-form on $V$. Then

$$0 = D_Xu = V(u) + [\pi^*C(V), u] = V(u).$$

This implies that we may write $u = w \circ \pi$, where $w \in \Omega^0(M, Aut E)$. But

$$(D_X^1u)(x, v) = (D_X^1w \circ \pi)(x, v) = (\nabla_{d\pi(X)}w)(x) = \nabla^1w.$$

Thus $\nabla^1w + Aw = 0$, which combined with $\nabla^2w = \nabla^1w + [A, w]$, implies that $w^*\nabla^2 = \nabla^1$. Hence $[\nabla^1] = [\nabla^2]$ as desired.

To show that $d$ is symmetric, it suffices to note that if $u$ solves $D_X^1u + Au = 0$, then $u^*$ solves $D_X^2u^* - Au^* = 0$ and that $u$ and $u^*$ have the same degree.

In order to prove the triangle inequality, consider $\nabla^2 = \nabla^1 + A$ with $D_X^1u + Au = 0$, and $\nabla^3 = \nabla^1 + B$ with $D_X^1w + Bw = 0$. Obviously $\nabla^3 = \nabla^2 + (B - A)$. Using that $D^2 = D^1 + [\pi^*A, \cdot]$ and that $D_X^1u^* = u^*A$, we compute

$$D_X^2(wu^*) = (D_X^2w)u^* + w(D_X^2u^*)$$

$$= (D_X^1w + Aw - wA)u^* + w(D_X^1u^* + Au^* - u^*A)$$

$$= (A - B)wu^*$$
and since $\deg(wu^*) \leq \deg(u) + \deg(w)$ it follows that $d([\nabla^3], [\nabla^2]) \leq d([\nabla^3], [\nabla^1]) + d([\nabla^2], [\nabla^1])$.

\begin{proof}[Proof of Theorem A] Let us apply Theorem \[5.1\] when $\nabla = \nabla^2$ and $\nabla^1$ is the trivial connection $d$. The hypothesis of $\pm i \ast F_\nabla(x) = K(x)\Id$ being positive definite for all $x \in M$ implies that $d([\nabla], [d]) = 0$. Thus $\nabla$ is gauge equivalent to the trivial connection.
\end{proof}

6. Proof of Theorem B

6.1. Levi-Civita ghosts. As in the introduction, let $K$ be the canonical line bundle and $K^s$ with $s \in \mathbb{Z}$ be its tensor powers (if $s = 0$ we get the trivial bundle). The powers $K^s$ for $s \neq 0$ carry the Levi-Civita connection which we denote by $\nabla^s_K$. If $s = 0$ we understand that this is the trivial connection. Given an $n$-tuple of integers $S := (s_1, \ldots, s_n)$, the connection

$$\nabla^S = \nabla^{s_1} \oplus \cdots \oplus \nabla^{s_n}$$

defines a transparent unitary connection on the bundle $E_S := K^{s_1} \oplus \cdots \oplus K^{s_n}$. Clearly $c_1(E_S) = (2g-2)(s_1 + \cdots + s_n)$ and any complex vector bundle $E$ supporting a transparent connection is isomorphic to $E_S$ for $S$ such that $c_1(E) = c_1(E_S)$.

Now let $E$ be a Hermitian vector bundle and consider a unitary isomorphism $\tau : E \to E_S$, where $S$ is such that $c_1(E) = c_1(E_S)$. The unitary connection $\tau^* \nabla^S_\ell$ is a transparent connection on $E$ and its gauge equivalence class, denoted by $[S]$, is independent of $\tau$. Note that $[S_1] = [S_2]$ if and only if $S_1$ and $S_2$ coincide up to a permutation.

The next lemma will allow us to see these ghosts in a different form, more appropriate for our purposes.

Let $L$ be a $\mathfrak{u}(n)$-valued 1-form on $SM$. It defines a unitary connection $d_L := d + L$ on the trivial bundle $SM \times \mathbb{C}^n$.

**Lemma 6.1.** Suppose $L(X) = L(H) = 0$ and $L(V) = c$, where $c \in \mathfrak{u}(n)$ is a constant matrix such that $e^{2\pi c} = \Id$. Let $is_1, \ldots, is_n$ be the eigenvalues of $-c$, where $s_k \in \mathbb{Z}$. Set $S = (s_1, \ldots, s_n)$. Then, there exists a unitary trivialization $\psi : \pi^*E_S \to SM \times \mathbb{C}^n$ such that $\psi^*(d_L) = \pi^* \nabla^S_\ell$.

**Proof.** Since $c \in \mathfrak{u}(n)$, we can find a matrix $a \in U(n)$ such that $-a^{-1}ca$ is a diagonal matrix with entries $is_1, \ldots, is_n$. Hence we might as well assume that $-c$ has already this diagonal form. It is now clear that it suffices to prove the lemma for the case $n = 1$ and we let $s := s_1 \in \mathbb{Z}$. The case $s = 0$ is obvious and we will prove the lemma for $s < 0$ (the case $s > 0$ is similar). If $s < 0$, then $K^s$ is just $(TM)^{\otimes m}$, where $m = -s > 0$. Given $v \in T_xM$, we let $v^m \in (TM)^{\otimes m}$ be the tensor product of $v$ with itself $m$ times.

We define a unitary trivialization $\psi : \pi^*(TM)^{\otimes m} \to SM \times \mathbb{C}$ as follows. Given $(x, v) \in SM$ and $w \in (T_xM)^{\otimes m}$ we set $\psi(x, v, w) = (x, v, \lambda)$, where $\lambda \in \mathbb{C}$ is the unique number such that $w = \lambda v^m$. Note that the Riemannian metric on $M$ determines the unitary structure on $(T_xM)^{\otimes m}$. The real 2-dimensional tangent space $T_xM$
carries the complex structure \( iv \) that rotates a vector \( v \in T_xM \) by \( \pi/2 \) according to the orientation of the surface. We will show that \( (\psi^{-1})^*(\pi^*\nabla^*_f) = d_L \).

Let \( \xi \in \Omega^0(SM, \pi^*(T_xM)^{\otimes m}) \) be the section given by \( \xi(x, v) = v^m \). Consider a smooth function \( f : SM \to \mathbb{C} \) and note that \( (\psi^{-1})^*f = f\xi \). By the definition of the Levi-Civita connection

\[
(\pi^*\nabla^*_f)_X(\xi) = (\pi^*\nabla^*_f)_H(\xi) = 0
\]

and thus

\[
(\pi^*\nabla^*_f)_X(f\xi) = X(f)\xi = (\psi^{-1})^*(d_L X f),
\]

\[
(\pi^*\nabla^*_f)_H(f\xi) = H(f)\xi = (\psi^{-1})^*(d_L H f).
\]

We finally check what happens on \( V \). Note that for any affine connection \( \nabla \) on \( TM \) we have \( \pi^*\nabla_V(v) = iv \). Using the definition of the induced connection on a tensor product we deduce

\[
(\pi^*\nabla^*_f)_V(\xi) = m i \xi,
\]

hence

\[
(\pi^*\nabla^*_f)_V(f\xi) = V(f)\xi + f m i \xi.
\]

On the other hand

\[
d_{L,V}(f) = V(f) + imf
\]

and the lemma follows.

The next lemma, like the previous one, does not need any curvature assumption; only that we are working on a surface which is not a torus. The relation of being \( V \)-cohomologous is an equivalence relation and given \( f : SM \to u(n) \), let \([f]_V\) denote the class of \( f \).

**Lemma 6.2.** Let \( c_1, c_2 \in u(n) \) be two matrices such that \( e^{2\pi c_k} = \text{Id} \) for \( k = 1, 2 \). Then \([c_1]_V = [c_2]_V\) if and only if \( \text{tr}(c_1) = \text{tr}(c_2) \).

**Proof.** Suppose first that \( \text{tr}(c_1) = \text{tr}(c_2) \). The matrix \( c_k \) determines a bundle \( E_{S_k} \) and let \( \psi_k : \pi^*E_{S_k} \to SM \times \mathbb{C}^n \) be the unitary trivialization given by the previous lemma. By hypothesis, we may take a unitary isomorphism \( \phi : E_{S_1} \to E_{S_2} \) and let \( \rho : \pi^*E_{S_1} \to \pi^*E_{S_2} \) be the induced isomorphism, \( \rho(x, v, \xi) = (x, v, \phi_x(\xi)) \). Let us write \( \varphi := \psi_2 \circ \rho \circ \psi_1^{-1}(x, v, a) = (x, v, w(x, v) a) \) where \( w : SM \to U(n) \) and \( a \in \mathbb{C}^n \). Let \( G \) be the unique \( u(n) \)-valued 1-form on \( SM \) such that \( \psi_1^*(d_G) = \rho^*\nabla^*_f = \pi^*\nabla^*_f \).

Write \( \phi^*\nabla^*_f = \nabla^*_f + A \). Since \( \pi^*\nabla^*_f = \psi_1^*(d_L) \) we must have \( G = L_1 + \psi_1 \pi^*A \psi_1^{-1} \) which gives \( G(V) = L_1(V) = c_1 \). But \( \varphi^*(d_{L_2}) = d_G \); that is, \( G = w^{-1}dw + w^{-1}L_2w \).

Applying the last equality to \( V \) we derive \( c_1 = w^{-1}V(w) + w^{-1}c_2w \), i.e., \([c_1]_V = [c_2]_V\).

Suppose now that there is \( w : SM \to U(n) \) such that \( c_1 = w^{-1}V(w) + w^{-1}c_2w \).

Taking traces

\[
\text{tr}(c_1) - \text{tr}(c_2) = h^{-1}V(h),
\]

where \( h := \det w : SM \to S^1 \). Arguing as in the proof of Theorem 3.2 the function \( h \) gives rise to a real-valued closed 1-form in \( SM \) given by \( -ih^{-1}dh \). Since \( \pi^* : H^1(M, \mathbb{R}) \to H^1(SM, \mathbb{R}) \) is an isomorphism (this follows easily from the Gysin
sequence, since $M$ is not the 2-torus), there exists a closed 1-form $\omega$ in $M$ and a smooth function $f : SM \to \mathbb{R}$ such that

$$-i\hbar^{-1}dh = \pi^*\omega + df.$$

Applying this equality to $V$ we derive

$$iV(f) = \text{tr}(c_1) - \text{tr}(c_2)$$

which clearly implies $\text{tr}(c_1) = \text{tr}(c_2)$.

The next lemma is not needed in what follows, but it illustrates the distance in $\mathcal{T}$.

**Lemma 6.3.** Suppose $\sum_{k=1}^n s_k = 0$, so that $\nabla^S_\pi$ induces a connection on the trivial bundle. Then $d([S],[d]) = \max |s_k|$, where $d$ is the trivial connection.

**Proof.** Let $\phi : E_S \to M \times \mathbb{C}^n$ be a unitary trivialization and let $\rho : \pi^*E_S \to SM \times \mathbb{C}^n$ be the induced unitary trivialization, $\rho(x,v,\xi) = (x,v,\phi_x(\xi))$. Let $A$ be the unique $u(n)$-valued 1-form on $M$ given by $\phi^*(dA) = \nabla^S_\pi$. By Lemma 6.1 there is a unitary trivialization $\psi : \pi^*E_S \to SM \times \mathbb{C}^n$ such that $\psi^*(dL) = \pi^*\nabla^S_\pi$. Hence $(\rho\psi^{-1})^*(d_{\pi^*}A) = d_L$. In other words, if we write $\rho\psi^{-1}(x,v,\xi) = (x,v,u(x,v)\xi)$, where $u : SM \to U(n)$, then $L = u^{-1}du + u^{-1}\pi^*Au$. Since $L(X) = 0$, $u$ solves $X(u) + Au = 0$. An inspection of the construction of $\psi$ in Lemma 6.1 reveals that $\psi$ has polynomial dependence on the velocities with degree given by $\max |s_k|$. It follows that $\deg(u) = \max |s_k|$ and thus $d([S],[d]) \leq \max |s_k|$. Finally note that equality must hold since if $w$ is another solution of $X(u) + Au = 0$, then $u^*w$ must be constant. Indeed, a simple calculation shows that $X(u^*w) = 0$ and the claim follows from the transitivity of the geodesic flow of $X$.

\[\square\]

6.2. **Proof of Theorem B.** (Forward direction.) The matrix $c$ determines a bundle $E_S$ and by considering a unitary isomorphism $\tau : E \to E_S$ we may suppose $E = E_S$. Let $\nabla$ be a transparent connection on $E_S$ and let $C$ be its associated cocycle in $\pi^*E_S$. Let $\psi : \pi^*E_S \to SM \times \mathbb{C}^n$ be the unitary trivialization given by Lemma 6.1.

Write

$$\psi C(x,v,t) \psi^{-1}(x,v,a) = (\phi_t(x,v),D(x,v,t)a),$$

where $D : SM \times \mathbb{R} \to U(n)$ is a cocycle as in Definition 2.1. By the Livsic theorem [2.2] there exists a smooth function $u : SM \to U(n)$ such that $D(x,v,t) = u(\phi_t(x,v))u^{-1}(x,v)$. Let $\Gamma : \mathbb{R} \to SM$ be $\Gamma(t) = \phi_t(x,v)$. By the definition of $C$, $\Gamma^*\pi^*\nabla(t) \to C(x,v,t)\xi) = 0$ for any $\xi \in E_S(x)$. Now let $G$ be the unique $u(n)$-valued 1-form on $SM$ such that $\psi^*(d_G) = \pi^*\nabla$, where $d_G = d + G$. Then $\Gamma^*d_G(t) \to D(x,v,t)a) = 0$ for all $a \in \mathbb{C}^n$. Equivalently

$$\frac{dD}{dt} + G(X)D = 0$$

and setting $t = 0$, we obtain: $X(u) + G(X)u = 0$. 

\[\square\]
As in the proof of Lemma 6.2 write $\nabla = \nabla_1^S + A$. Since $\pi^*\nabla_1^S = \psi^*(d_L)$ we must have $G = L + \psi^*A\psi^{-1}$ which gives $G(V) = L(V) = c$.

Now let us set $B := u^{-1}du + u^{-1}Gu$. Then $d_G$ and $d_B$ are gauge equivalent, but $B(X) = 0$.

Since $F_{\pi^*\nabla}(\cdot, V) = 0$, we must also have $F_B(\cdot, V) = 0$. Using that $F_B = dB + B \wedge B$ and $B(X) = 0$ we compute

$$F_B(X, V) = dB(X, V) + [B(X), B(V)] = dB(X, V).$$

But

$$dB(X, V) = XB(V) - VB(X) - B([X, V]) = XB(V) + B(H),$$

hence

(12) \hspace{1cm} B(H) = -XB(V).$$

We also compute

$$F_B(H, V) = dB(H, V) + [B(H), B(V)],$$

and

$$dB(H, V) = HB(V) - VB(H) - B([H, V]) = HB(V) - VB(H),$$

hence

(13) \hspace{1cm} HB(V) - VB(H) + [B(H), B(V)] = 0.

Combining (12) and (13) we derive the following non-linear PDE for $f := B(V)$

(14) \hspace{1cm} H(f) + VX(f) - [X(f), f] = 0.

This is precisely equation (11) in the Introduction and since $f = B(V) = u^{-1}V(u) + u^{-1}cu$ it follows that $f \in \mathcal{H}_c$. Note that $f$ is defined exclusively in terms of $u$ and $c$ and $u$ must solve $X(u) + G(X)u = 0$. However, up to right multiplication by an element $q \in U(n)$, there is only one such solution. Indeed, if $w$ is another solution, then $X(u^*w) = 0$ and by transitivity of the geodesic flow there is $q \in U(n)$ such that $w = uq$. This implies that $f$ is uniquely defined in $\mathcal{H}_c/U(n)$. To complete the correspondence in the forward direction, we must check that if we consider a connection gauge equivalent to $\nabla$ we obtain the same $f$. A connection $\nabla^1$ gauge equivalent to $\nabla$ determines a connection $d_{G_1}$ in $SM$ gauge equivalent to $d_G$. In other words, there is a smooth function $r: SM \rightarrow U(n)$ such that $G_1 = r^*dr + r^*Gr$. But if $u$ solves $X(u) + G(X)u = 0$, then $r^*u$ solves $X(w) + G_1(X)w = 0$ (unique up to multiplication on the right by an element in $U(n)$). Next observe that

$$G_1(V) = c = r^*V(r) + r^*cr$$

and

$$f_1 = u^*rV(r^*u) + u^*r^*cr^*u = u^*V(u) + u^*(rV(r^*) + rcr^*)u = f$$

thus obtaining a well defined map $T \mapsto \mathcal{H}_c/U(n)$. 
(Backward direction.) Suppose now that we have a solution $f$ of (14) such that there is $u : SM \to U(n)$ with $f = u^*V(u) + u^*cu$. Define a $u(n)$-valued 1-form $G$ on $SM$ by setting:

$$G(X) = -X(u)u^*,$$

$$G(H) = -uX(f)u^* - H(u)u^*,$$

$$G(V) = c,$$

and define an element $\mathcal{A} \in \Omega^1(SM, \text{ad} \pi^*E_S)$ by

$$\mathcal{A} := \psi^{-1}(G - L)\psi.$$

Clearly $\mathcal{A}(V) = 0$ and we wish to show that there exists $A \in \Omega^1(M, \text{ad} E_S)$ such that $\mathcal{A} = \pi^*A$. For this, it suffices to show that $D_V \mathcal{A}(X) = \mathcal{A}(H)$ and $D_V \mathcal{A}(H) = -\mathcal{A}(X)$, where $D$ here stands for the connection induced by $\nabla^S_f$. Equivalently, using the unitary isomorphism $\psi$, we are required to show that $D^G_V G(X) = G(H)$ and $D^G_V G(H) = -G(X)$, where $D^G$ is induced by $d_L$. Explicitly this means $V(G(X)) + [c, G(X)] = G(H)$ and $V(G(H)) + [c, G(H)] = -G(X)$. Using the definition of $G(X)$, the structure equations of the metric and $uf = V(u) + cu$ we compute:

$$V(G(X)) = -VX(u)u^* - X(u)V(u^*)$$

$$= -XV(u)u^* - H(u)u^* - X(u)V(u^*)$$

$$= -X uf - cu^* - H(u)u^* - X(u)(u^*c - fu^*)$$

$$= -uX(f)u^* - H(u)u^* + [c, X(u)u^*]$$

$$= G(H) - [c, G(X)].$$

Similarly we compute (we omit some of the details)

$$V(G(H)) = -V(u)X(f)u^* - uVX(f)u^* - uX(f)V(u^*) - VH(u)u^* - H(u)V(u^*)$$

$$= X(u)u^* + u([X(f), f] - VX(f) - H(f))u^* + [c, uX(f)u^*] + [c, H(u)u^*]$$

$$= -G(X) - [c, G(H)] + u([X(f), f] - VX(f) - H(f))u^*.$$
Addendum to Theorem B. We claim that \( \text{tr}(f) = \text{tr}(c) \) and thus \( \text{tr}(f) \) is constant and determined by the topology of \( E \). Consider the transparent connections induced by \( \nabla \) and \( \nabla^S \) on the line bundle \( \det E_S \). By Theorem 3.2 they must be gauge equivalent; in other words, there is a smooth function \( g : M \to S^1 \) such that \( \text{tr}(\nabla) = dg/g \). Recall that \( X(u) + G(X)u = 0 \) and \( G(X) = \psi A_x(v) \psi^{-1} \).

Hence \( \text{tr}(G(X)) = \text{tr}(A) = X(g)/g \). Since \( X(\det u) = \det u \text{tr}(u^*X(u)) \) we derive \( X(\det u) = \det u(-X(g)/g) \) and thus \( X(g \det u) = 0 \). By transitivity of the geodesic flow \( g \det u \) is a constant and hence \( V(\det u) = 0 \). But this is equivalent to \( \text{tr}(u^*V(u)) = 0 \). Since \( f = u^*V(u) + u^*cu \), \( \text{tr}(f) = \text{tr}(c) \) as desired.

Remark 6.4. One can also compute the curvature \( F_B(X, H) \) of the connection \( d_B \) from the theorem. Using that \( B(X) = 0 \) we derive:

\[
F_B(X, H) = dB(X, H) = XB(H) - B([X, H]) = -X^2(f) - Kf.
\]

Note that \( F_B(X, H) \) is conjugate to \( *F_\nabla \circ \pi \) via a unitary trivialization.

6.3. Transparent connections at distance one from the trivial connection.
Let \( \nabla \) be a transparent connection on the trivial bundle with \( d([\nabla], [d]) = 1 \). If we follow the proof of Theorem B, we see that in this case \( \psi \) is the identity and if we write \( \nabla = d + A \), then \( G = \pi^*A \). Since \( d([\nabla], [d]) = 1 \), there exists a smooth function \( u : SM \to U(n) \) such that \( u = u_1 + u_0 + u_1 \) and \( X(u) + Au = 0 \). Also, \( B = u^{-1}du + u^{-1}\pi^*Au \) and \( f = B(V) = u^*V(u) = -V(u^*)u \).

Lemma 6.5. \( f \in \Omega_0 \).

Proof. By separating \( X(u) + Au = 0 \) into even and odd parts we deduce

\[
X(u_0) + Au_0 = 0,
\]

\[
X(u_{-1} + u_1) + A(u_{-1} + u_1) = 0.
\]

These two equations yield

\[
X(u_0^*(u_{-1} + u_1)) = u_0^*A(u_{-1} + u_1) + u_0^*(-A)(u_{-1} + u_1) = 0,
\]

and since the geodesic flow is transitive \( u_0^*(u_{-1} + u_1) \) must be constant and thus

\[
u_0^*u_1 = u_0^*u_{-1} = 0.
\]

Using the special form of \( u \) we derive

\[
f = (u_0^*u_{-1} + u_0^*u_1)(-iu_{-1} + iu_1)
\]

\[
= i(u_1^*u_1 - u_{-1}^*u_{-1}) + u_0^*u_1 - u_0^*u_{-1} + u_1^*u_1 - u_1^*u_{-1}).
\]

Using that \( u^*u = \text{Id} \) we see that the terms of degree \( \pm 2 \), \( u_{-1}^*u_1 \in \Omega_2 \) and \( u_1^*u_{-1} \in \Omega_{-2} \) must vanish. Using (16) we obtain

\[
f = i(u_1^*u_1 - u_{-1}^*u_{-1}) \in \Omega_0.
\]

\( \square \)

Corollary 6.6. Suppose the Hermitian matrix \( \pm i *F_\nabla(x) - 2K(x)\text{Id} \) is positive definite for all \( x \in M \). Then \( f \in \Omega_0 \).
Proof. This follows right away from the last lemma and Theorem 5.1 which implies that \( d([d], [\nabla]) \leq 1 \).

Since \( f \in \Omega_0 \) we can think of \( f \) as a function which depends only on the base point \( x \). Thus \( X(f)(x, v) = df_x(v) \) and \( H(f)(x, v) = df_x(iv) \). Since \( VX(f) = XV(f) + H(f) = H(f) \), equation (14) gives \( 2H(f) = [X(f), f] \) and we can rewrite this in terms of matrix valued 1-forms as

\[
2 \star df = [df, f]. \tag{17}
\]

We discuss this equation in the next subsection. Note that if we wish \( f \) to be \( V \)-cohomologous to a matrix \( c \) as in Theorem B we must have \( e^{2\pi f(x)} = \text{Id} \).

6.4. Solutions to \( 2 \star df = [df, f] \). If we let \( A := \frac{1}{2} \star df \), then \( 2 \star df = [df, f] \) may be rewritten as \( d_A f = 0 \), so \( f \) is covariant constant relative to the connection \( d_A \). This implies that \( f \) only hits one adjoint orbit of the adjoint action of \( U(n) \) on \( \mathfrak{u}(n) \). To see that this is the case observe first that \( dtr(f^m) = \text{tr}(d_A f^m) = 0 \) for any \( m \) and thus the eigenvalues of \( f(x) \) must be constant (and belong to \( i\mathbb{Z} \) if \( e^{2\pi f(x)} = \text{Id} \)). Also, the multiplicities of the eigenvalues do not change with \( x \). Indeed, let \( \xi \in \mathbb{C}^n \) be an eigenvector of \( f(x) \) with eigenvalue \( \lambda \) and let \( \gamma : [0, 1] \to M \) be a curve connecting \( x \) to \( y \). Let \( \xi(t) \) be the parallel transport of \( \xi \) along \( \gamma \). Since \( f(\gamma(t))\xi(t) \) is also parallel (\( d_A f = 0 \)), it must equal \( \lambda \xi(t) \) and thus \( f(y)\xi(1) = \lambda \xi(1) \) which shows that parallel transport preserves the eigenspaces of \( \lambda \).

Suppose now \( f : M \to \mathfrak{su}(2) \). The discussion above implies that \( f^2 = -\lambda^2 \text{Id} \) for some constant \( \lambda \). This implies that \( df f = -f df \), so we rewrite \( 2 \star df = [df, f] \) as \( \star df = df f \). Applying \( \star \) we derive \( df = \lambda^2 df \). Hence if \( \lambda^2 \neq 1 \), \( f \) must be constant. Let us suppose then that \( f^2 = -\text{Id} \), so \( f \) hits the adjoint orbit of

\[
\begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}
\]

which we denote by \( S \) and we identify with the 2-sphere. For \( g \in S \) and \( X \in T_g S \), let \( J_g(X) := Xg \). Clearly \( J_g^2 = -\text{Id} \), so \( J_g \) is a complex structure in \( S \) and the equation \( \star df = df f \) simply says \( df_x(iv) = J_f(x)(df_x(v)) \), i.e. \( f : M \to S \) is a holomorphic map.

We now wish to show that given such a map \( f : M \to S \), then \( f \) is \( V \)-cohomologous to the zero matrix, that is, there exists \( u : SM \to SU(2) \) such that \( f = u^*V(u) \). This would show that \( \mathcal{H}_0 \cap \Omega_0 \) can be identified with the set of holomorphic maps \( f : M \to \mathbb{C} \mathbb{P}^1 \) as claimed in the introduction.

Consider a map \( f : M \to S \) and let \( L(x) \) (resp. \( U(x) \)) be the eigenspace corresponding to the eigenvalue \( i \) (resp. \( -i \)) of \( f(x) \). We have an orthogonal decomposition \( \mathbb{C}^2 = L(x) \oplus U(x) \) for every \( x \in M \). Consider sections \( \alpha \in \Omega^{1,0}(M, \mathcal{C}) \) and \( \beta \in \Omega^{0,1}(M, \text{Hom}(L, U)) = \Omega^{1,0}(M, L^*U) \) such that \( |\alpha|^2 + |\beta|^2 = 1 \). Such pair of sections always exists; for example, we can choose a section \( \tilde{\beta} \) with a finite number of isolated zeros and then choose \( \tilde{\alpha} \) such that it does not vanish on the zeros of \( \tilde{\beta} \). Then we set \( \alpha := \tilde{\alpha}/(|\tilde{\alpha}|^2 + |\tilde{\beta}|^2)^{1/2} \) and \( \beta := \tilde{\beta}/(|\tilde{\alpha}|^2 + |\tilde{\beta}|^2)^{1/2} \).
Note that $\alpha \in \Omega^{0,1}(M, \mathbb{C})$ and $\beta^* \in \Omega^{0,1}(M, \text{Hom}(U, L)) = \Omega^{0,1}(M, U^* L)$. Using the orthogonal decomposition we define $u : SM \rightarrow SU(2)$ by

$$
 u(x, v) = \left( \begin{array}{cc} \alpha(x, v) & \beta^*(x, v) \\ -\beta(x, v) & \bar{\alpha}(x, v) \end{array} \right).
$$

Clearly $u = u_{-1} + u_1$, where

$$
 u_1 = \left( \begin{array}{c} \alpha \\ -\beta \end{array} \right)
$$

and

$$
 u_{-1} = \left( \begin{array}{c} 0 \\ \beta^* \end{array} \right).
$$

It is straightforward to check that $uf = V(u)$.

Combining the discussion above with Theorem B (and its addendum) we derive:

**Corollary 6.7.** The set of transparent $U(2)$-connections modulo gauge transformations at distance one from the trivial connection is in 1-1 correspondence with holomorphic maps $f : M \rightarrow \mathbb{CP}^1$ up to composition with an orientation preserving isometry of $\mathbb{CP}^1$.

**Remark 6.8.** We can actually compute the distance $d([A], [B])$ where $A$ and $B$ define transparent connections at distance one from the trivial connection. Let $u = u_{-1} + u_1$ solve $X(u) + Au = 0$ and let $w = w_{-1} + w_1$ solve $X(w) + Bw = 0$. Then $r = wu^*$ solves $X(r) + Br - rA = 0$ or equivalently $D_X^r + (B - A)r = 0$. In fact it is easy to check using arguments already used before that $wqu^*$, where $q \in SU(2)$ is a constant matrix, are all the solutions of $X(r) + Br - rA = 0$. Now observe that $wqu^* = w_{-1}qu^*_{-1} + w_1qu^*_1 + w_{-1}qu^*_1 + w_1qu^*_{-1}$. Thus $wqu^*$ has terms only of degree zero or $\pm 2$. It follows that $d([A], [B]) = 2$ unless $[A] = [B]$. Hence the distance induced via Corollary 6.7 on the space of holomorphic maps $f : M \rightarrow \mathbb{CP}^1$ (modulo $SU(2)$) is just the discrete distance.

6.5. **The energy estimates method.** In order to deal with equation (14) one may try to use the energy estimates method (the Pestov identity) in the case of matrix valued functions as done by L.B. Vertgeim [19], Sharafutdinov [15] and Finch and Uhlmann [6]. However in order to control the non-linear term given by the bracket in (14) one ends up requiring some assumption of smallness on the connection or its curvature.

In our case the relevant integral identity takes virtually the same form as in the case of complex valued functions; we omit its proof here which is a straightforward generalization of the case $n = 1$, which may be found in the form below in [16, Lemma 2.1]. Let $f : SM \rightarrow \mathbb{M}_n(\mathbb{C})$ be a smooth function, where $\mathbb{M}_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices. Then

$$
 2 \int_{SM} \langle H(f), VX(f) \rangle \, d\mu = \int_{SM} |H(f)|^2 \, d\mu + \int_{SM} |X(f)|^2 \, d\mu - \int_{SM} K|V(f)|^2 \, d\mu,
$$

where $K$ is a constant depending on the geometry of $M$. This identity allows us to bound certain energy integrals.
where $\langle A, B \rangle = \Re \text{tr}(AB^*)$ for $A, B \in M_n$. If $f$ satisfies equation (14), then (18)

$$2 \int_{SM} \langle H(f), [X(f), f] \rangle d\mu = 3 \int_{SM} |H(f)|^2 d\mu + \int_{SM} |X(f)|^2 d\mu - \int_{SM} K|V(f)|^2 d\mu.$$ 

The last equality gives right away that $f$ is constant if $n = 1$. Indeed, if $K < 0$ the right hand side of the equality is $\geq 0$ and the left hand side vanishes since the bracket must vanish. This implies $H(f) = X(f) = V(f) = 0$ and thus $f$ is constant. For $n \geq 2$ it is not clear how to deal with the term in the left hand side for arbitrary $f$. Here is an attempt in the spirit of [6].

Using the Cauchy-Schwartz inequality we can estimate the left hand side of (18) by

$$2 \int_{SM} \langle H(f), [X(f), f] \rangle d\mu \leq 2 \max ||f|| \int_{SM} (|X(f)|^2 + |H(f)|^2) d\mu,$$

where $||f||$ is the operator norm of $f$. Hence if $2 \max ||f|| \leq 1$, (18) gives when $K < 0$ that $H(f) = V(f) = 0$ and again $f$ must be constant ($X = -[V, H]$). One can now try to estimate $\max ||f||$ using the curvature of $\nabla$ and Remark 6.4. If for example $K = -1$ we can solve the ODE (15) explicitly as

$$2f(x, v) = \int_0^\infty e^{-s}F_B(X, H)(\phi_s(x, v)) ds + \int_{-\infty}^0 e^sF_B(X, H)(\phi_s(x, v)) ds.$$

Hence if the operator norm of $\star F_\nabla$ is everywhere $\leq 1/2$, so is $F_B(X, H)$, and then $\max ||f|| \leq 1/2$. By the argument above $f$ must be constant. However, this seems to give a weaker result than Theorem A.

Finally we note that (18) shows that if $f$ is a solution of (13) which is also odd (i.e. $f(x, -v) = -f(x, v)$), then it must be identically zero. Indeed, in this case $H(f)$, $X(f)$ and $VX(f)$ are even functions, but $[X(f), f]$ is odd. It follows that $[X(f), f] = 0$ and by (18), $f$ must be a constant, and thus identically zero.

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