M-theory backgrounds in the form of unwarped compactifications with or without fluxes are considered. We construct the bilinear forms of supergravity Killing spinors for different choices of spinor inner products on these backgrounds. The equations satisfied by the bilinear forms and their decompositions into product manifolds are obtained for different inner product choices. It is found that the $AdS$ solutions can only appear for some special choices of spinor inner products on product manifolds. The reduction of bilinears of supergravity Killing spinors into the hidden symmetries of product manifolds which are Killing-Yano and closed conformal Killing-Yano forms for $AdS$ solutions is shown. These hidden symmetries are lifted to eleven-dimensional backgrounds to find the hidden symmetries on them. The relation between the choices of spinor inner products, $AdS$ solutions and hidden symmetries on M-theory backgrounds are investigated.

I. INTRODUCTION

Low energy limits of ten-dimensional string theories and eleven-dimensional M-theory correspond to the supergravity theories in those dimensions. The solutions of the bosonic sector of supergravity theories determine the backgrounds that strings and branes can propagate. A special class of backgrounds are obtained by considering compactifications into smaller dimensions in the presence or absence of fluxes defined in relevant supergravity theories [1–10]. One can consider warped or unwarped products for compactifications and determine the field equations on the product manifolds. The supersymmetry parameters in different bosonic supergravity theories satisfy various supergravity Killing spinor equations arising from the variation of the gravitino field. The bilinear forms of these supergravity Killing spinors can be constructed by using inner products on the spinor space and these bilinears are used in the classification of string and M-theory backgrounds [11, 12]. Moreover, these bilinears can have Lie algebra structures in some special cases [13]. However, one can define various spinor inner products depending on the dimension and the signature of the background and the corresponding bilinears will be different for different choices of spinor inner products [14, 15]. In the literature, only some special choices of spinor inner products are considered and there is no exhaustive investigation for all types of inner products and the bilinear forms constructed out of them. On the other hand, supergravity Killing spinors reduce to geometric Killing spinors or parallel spinors on compactified backgrounds depending on the geometric properties of the product manifolds and the existence of these special types of spinors are related to the special holonomy structures of manifolds [16–19]. So, the bilinear forms of supergravity Killing forms can reduce to special types of differential forms on product manifolds and the investigation of these reductions can have implications on the classification problem of string and M-theory backgrounds in all dimensions.

In this paper, we consider eleven-dimensional M-theory backgrounds in the form of unwarped compactifications with or without fluxes. For the eleven-dimensional background $M_{11}$, the unwarped product structures $M_4 \times M_7$, $M_7 \times M_4$, $M_5 \times M_6$, $M_6 \times M_5$ and $M_3 \times M_8$ are considered. We determine the decompositions of field equations and supergravity Killing spinor equation onto product manifolds and summarize the possible solutions. We construct bilinear forms of supergravity Killing spinors for both types of spinor inner products on $M_{11}$ and find the equations satisfied by those bilinears. It is found that the non-zero bilinear forms are dependent on the choice of the inner product. We also find the decompositions of the bilinear form equations onto product manifolds which are also highly dependent on the choice of the spinor inner products on product manifolds. An important result obtained in the paper is the fact that while Minkowski solutions appear for all types of spinor inner products, $AdS$ solutions can only appear for some special choices of spinor inner products on product manifolds. Moreover, while the supergravity Killing form bilinears of Minkowski solutions reduce to parallel forms on product manifolds, the bilinears of $AdS$
solutions reduce to special Killing-Yano (KY) or special closed conformal Killing-Yano (CCKY) forms depending on the choices of the spinor inner products. KY forms are antisymmetric generalizations of Killing vector fields to higher degree differential forms and CCKY forms are a subset of antisymmetric generalizations of conformal Killing vector fields to higher degree forms. These special forms are called the hidden symmetries of manifolds. We also obtain KY and CCKY forms of eleven-dimensional backgrounds by lifting the hidden symmetries on product manifolds. So, we determine the relations between hidden symmetries, AdS solutions and choices of spinor inner products by exhausting all possibilities for spinor inner product choices. This may be considered as a first step of a classification of backgrounds in terms of spinor inner products.

The paper is organized as follows. In Section II, we summarize the equations for the bosonic sector of eleven-dimensional supergravity. Section III deals with $M_4 \times M_7$ type backgrounds. We find the decompositions of field equations and supergravity Killing spinor equation and construct the bilinear form equations for both types of spinor inner products with their decompositions onto product manifolds. In Section IV, the same steps are achieved for $M_7 \times M_4$ type backgrounds. Section V includes the situation for other types of backgrounds. In Section VI, the relation between hidden symmetries and AdS solutions are summarized and the lifts of hidden symmetries to eleven-dimensional backgrounds are considered. Section VII concludes the paper. There are also three appendices containing the topics of inner product classes of spinor spaces, Clifford algebra conventions and Clifford bracket and KY forms.

II. ELEVEN-DIMENSIONAL SUPERGRAVITY

Let us consider an eleven-dimensional Lorentzian spin manifold $M_{11}$, with a metric $g$ and a closed 4-form $F$. $F$ is called the flux 4-form and the bosonic sector of the eleven-dimensional supergravity theory defined on $M_{11}$ is given by the following action

$$S = \frac{1}{12\kappa_{11}} \int \left( R_{AB} \wedge *_{11}e^Ae^B - \frac{1}{2} F \wedge *_{11}F - \frac{1}{6} \mathcal{A} \wedge F \wedge F \right)$$

where $\kappa_{11}$ is the eleven-dimensional gravitational coupling constant, capital letter indices take values $A, B = 0, 1, 2, ..., 9, 10$ and $*_{11}$ is the eleven-dimensional Hodge star operator. $R_{AB}$ are the curvature 2-forms, $e^A$ are co-frame basis and $\mathcal{A}$ is the 3-form potential of the flux 4-form $F = dA$. The first term in (1) corresponds to the gravitational term and second and third terms are Maxwell-like and Chern-Simons terms, respectively. The field equations of the eleven-dimensional bosonic supergravity results from the above action by considering the variations of $e^A$ and $\mathcal{A}$ as follows

$$*_{11}(i_{X_A}P_A) = \frac{1}{2} i_{X_A}F \wedge *_{11}i_{X_A}F - \frac{1}{6} g_{AB}F \wedge *_{11}F \quad \text{(Einstein)}$$

$$d *_{11}F = \frac{1}{2} F \wedge F \quad \text{(Maxwell)}$$

$$dF = 0 \quad \text{(Closure)}$$

where $i_{X_A}$ denotes the interior derivative or contraction operator with respect to the vector field $X_A$, $g_{AB}$ are components of the metric and $P_A$ are the Ricci 1-forms defined from the curvature 2-forms as $P_A = i_{X_A}R_{BA}$. The last equation (4) is the integrability condition for the definition of the flux form $F$. Moreover, the variation of the gravitino field in the fermionic sector will also lead to a condition on the spinor $\epsilon$ which is the supersymmetry parameter and in the bosonic sector it gives the following supergravity Killing spinor equation

$$\nabla_{X_A} \epsilon = -\frac{1}{24} (e^A F - 3F, e^A) \cdot \epsilon$$

where $\nabla_{X_A}$ corresponds the spinor covariant derivative and $\cdot$ denotes the Clifford multiplication. The co-frame basis $e^A$ define a basis of the Clifford algebra bundle $Cl_{10,1}$ on $M_{11}$ with the following equality

$$e^A, e^B + e^B, e^A = 2g^{AB}$$

where $g^{AB}$ are the components of the inverse metric. The supersymmetry parameter $\epsilon$ is an element of the spinor bundle $S$ which corresponds to $\mathbb{R}^{32}$ on $M_{11}$ and hence $\epsilon$ is a Majorana spinor.

In the following chapters, we consider various types of unwarped compactifications of supergravity backgrounds which are the solutions of the field equations (2)-(4) of type $M = M_4 \times M_{11-4}$. By constructing the bilinear forms of supergravity Killing spinors defined in (5) in those backgrounds, we show that the reduction or non-reduction of those bilinear forms into KY and CCKY forms on product manifolds require the existence or non-existence of AdS or Minkowski type solutions with or without internal and external fluxes. Moreover, we determine the correspondences between the choices of spinor inner products on product manifolds and the types of possible supergravity backgrounds. This gives a classification of unwarped compactifications of supergravity backgrounds in terms of spinor inner products.
III. $M_4 \times M_7$ TYPE BACKGROUNDS

We first consider the case that the eleven-dimensional supergravity background $M_{11}$ has the product structure $M_{11} = M_4 \times M_7$ where $M_4$ is a Lorentzian spin 4-manifold and $M_7$ is a Riemannian spin 7-manifold. The frame and co-frame basis indices appeared in the previous equations will split into two parts $A = \{a, \alpha\}$ with $a = 0, 1, 2, 3$ and $\alpha = 4, 5, \ldots, 9, 10$. The Clifford algebra basis $e^A$ will decompose as

$$e^A = \{ e^a \otimes 1, e^a \otimes e^\alpha \}$$  \hspace{1cm} (7)

where $e^a$ are the Clifford algebra basis on $M_4$, $z_4$ is the volume form on $M_4$, $1_7$ is the identity on $M_7$ and $e^\alpha$ are the Clifford algebra basis on $M_7$ which are pure imaginary. By considering the equalities $e^a.e^b + e^b.e^a = 2g^{ab}$, $e^\alpha.e^\beta + e^\beta.e^\alpha = 2g^{\alpha\beta}$ and the properties $z_4^2 = -1$ and $z_4$ anticommutes with all 1-forms on $M_4$, one can obtain the defining relation (6) from (7). Here, the inverse metric is decomposed as $\epsilon^\alpha\beta = \{ g^{ab}, g^{\alpha\beta} \}$ and there is no warped product factor. Similarly, the flux 4-form $F$ will decompose as

$$F = \{ \lambda z_4, \mu \phi \}$$  \hspace{1cm} (8)

where $\lambda$ and $\mu$ are constants and $\phi$ is a 4-form on $M_7$. The flux components on $M_4$ and $M_7$ are called external and internal fluxes respectively and the constants $\lambda$ and $\mu$ determine the existence or non-existence of external and internal flux components. The supersymmetry parameter $\epsilon$ will be constructed from four-dimensional and seven-dimensional spinors $\epsilon_4$ and $\epsilon_7$ as

$$\epsilon = \epsilon_4 \otimes \epsilon_7.$$  \hspace{1cm} (9)

For the product structure $M_4 \times M_7$, the field equations (2)-(4) will decompose into four-dimensional and seven-dimensional equations. For the Maxwell equation (3) and the closure condition (4), we can use the decomposition of the flux 4-form $F$ in (8). For any product structure $M_n = M_p \times M_q$, the Hodge star operator $*_n$ satisfies the following equality

$$*_n (\alpha \wedge \beta) = (-1)^{(p-k)}. \star_p \alpha \wedge \star_q \beta$$  \hspace{1cm} (10)

where $\alpha$ is a $k$-form on $M_p$ and $\beta$ is an $l$-form on $M_q$ \cite{20}. So, in our case, we have

$$*_11 F = i \lambda \star_{11} (z_4 \wedge 1_7) + \mu \star_{11} (1_4 \wedge \phi)$$

$$= i \lambda (\star_{14} z_4 \wedge \star_7 1_7) + \mu (\star_{14} 1_4 \wedge \star_7 \phi)$$

$$= -i \lambda z_7 + \mu z_4 \wedge \star_7 \phi$$  \hspace{1cm} (11)

where we have used $z_4 = \star_{4} 1_4$, $z_4 \star_4 = -1$ and $z_7 = \star_7 1_7$. Its exterior derivative gives

$$d *_{11} F = \mu z_4 \wedge d \star_7 \phi$$  \hspace{1cm} (12)

and the right hand side of (3) is

$$F \wedge F = 2 i \lambda \mu z_4 \wedge \phi$$  \hspace{1cm} (13)

since we have $dz_4 = 0$, $z_4 \wedge z_4 = 0$ and $\phi \wedge \phi = 0$ because of the fact that $\phi \wedge \phi$ is a 8-form on $M_7$. On the other hand, the closure condition $dF = 0$ gives $d\phi = 0$ and hence we obtain the following equalities from equations (3) and (4)

$$d \star_7 \phi = i \lambda \phi$$

$$d \phi = 0.$$  \hspace{1cm} (14)

These equalities define a weak $G_2$ structure on $M_7$ and $\phi$ corresponds to the coassociative 4-form on it. So, $M_7$ will correspond to a proper weak $G_2$ manifold, a Sasaki-Einstein manifold or a 3-Sasaki manifold \cite{21}. Moreover, (14) means that $\phi$ is a CCKY 4-form on $M_7$ and hence it must be generated from a geometric Killing spinor \cite{15}. Since $z_4$ is the volume form on $M_4$, it corresponds to a KY form on $M_4$ and as a result, the flux 4-form $F$ in (8) is generated by KY and CCKY forms on $M_4$ and $M_7$ for $\lambda \neq 0$ and $\mu \neq 0$.

Einstein field equations given in (2) can also be decomposed into $M_4$ and $M_7$ components. From the flux 4-form $F$ in (8), one can find the terms on the right hand side of (2) with similar calculations to the above as follows

$$F \wedge *_{11} F = (\lambda^2 + \mu^2 g_4(\phi, \phi)) z_{11}$$  \hspace{1cm} (15)

$$i_{X_4} F \wedge *_{11} i_{X_4} F = \{- \lambda^2 g_{ab} z_{11}, \mu^2 g_4(\phi, \phi) g_{\alpha\beta} z_{11} \}$$  \hspace{1cm} (16)
where \( g_p \) denotes the metric on \( p \)-forms. Here, we have used the definition of Hodge star in terms of the \( p \)-form metric; for any \( p \)-forms \( \alpha \) and \( \beta \) we have \( \alpha \wedge * \beta = g_p(\alpha, \beta) * 1 \). Hence, we have

\[
\phi \wedge *7\phi = g_4(\phi, \phi) z_7
\]

\[
= \left( (iX_aix_\alpha iX_\beta \phi) iX_\delta iX_\epsilon iX_\theta \phi \right) z_7.
\]

The left hand side of (2) corresponds to \( iX_\alpha P_A = \{ iX_\alpha P_A, iX_\beta P_B \} \) where \( P_A \) and \( P_\alpha \) are Ricci 1-forms on \( M_4 \) and \( M_7 \), respectively. So, the Einstein field equations decompose into \( M_4 \) and \( M_7 \) as follows

\[
iX_\alpha P_A = -\frac{1}{3} \left( 2\lambda^2 + \frac{\mu^2}{2} g_4(\phi, \phi) \right) g_{ab}
\]

\[
iX_\beta P_\alpha = -\frac{1}{6} \left( \lambda^2 + \mu^2 g_4(\phi, \phi) \right) g_{a\beta} + \frac{\mu^2}{2} g_4( iX_\alpha \phi, iX_\beta \phi).
\]

This means that for \( \lambda = \mu = 0 \), both \( M_4 \) and \( M_7 \) are Ricci-flat manifolds and for the special case of \( \lambda \neq 0 \) and \( \mu = 0 \), \( M_4 \) is a negative curvature manifold and \( M_7 \) is a positive curvature Einstein manifolds (since the basis 1-forms are pure imaginary on \( M_7 \), the metric components \( g_{a\beta} = g(e_\alpha, e_\beta) \) will have an extra minus sign).

We will also analyze the decomposition of supergravity Killing spinor equation (5) into product manifolds. From (9), the left hand side of (5) corresponds to

\[
\nabla_X^A \epsilon = \nabla_X e^A \otimes \epsilon_T + \epsilon_A \otimes \nabla_X \epsilon_T
\]

and by using the decompositions in (7) and (8), the right hand side of (5) gives

\[
(e^A.F - 3F.e^A).\epsilon = i\lambda(e^a.z_4 - 3z_4.e^a).\epsilon_4 \otimes \epsilon_T - 2e^a.\epsilon_4 \otimes \mu \phi.\epsilon_T
\]

\[
-2\lambda e_4 \otimes e^a.\epsilon_T + i\lambda e_4 \otimes e^a.\epsilon_T - 3\phi.e^a.\epsilon_T.
\]

So, the supergravity Killing spinor equation can be written as

\[
\nabla_X e^A \otimes \epsilon_T + \epsilon_A \otimes \nabla_X \epsilon_T
\]

\[
= \pm \frac{1}{6} \lambda e^a \epsilon_4 \otimes \epsilon_T + \frac{1}{12} e^a.\epsilon_4 \otimes \mu \phi.\epsilon_T
\]

\[
+ \frac{1}{12} \lambda e_4 \otimes e^a.\epsilon_T \pm \frac{1}{24} e_4 \otimes (e^a.\phi - 3\phi.e^a).\epsilon_T
\]

\[
(20)
\]

where we have used that the volume form \( z_4 \) anticommutes with basis 1-forms on even dimensions that is \( z_4.e^a = -e^a.z_4 \) and on a Lorentzian 4-manifold it satisfies \( (iz_4)^2 = 1 \), so we have \( iz_4.e_4 = \pm e_4 \). The decompositions of supergravity Killing spinor equation on \( M_4 \) and \( M_7 \) have to be considered separately for the cases of existence or nonexistence of internal and external fluxes. For the fluxless case \( \lambda = \mu = 0 \), we have

\[
\nabla_X \epsilon_4 = 0
\]

\[
\nabla_X \epsilon_T = 0
\]

\[
(21)
\]

and this means that \( \epsilon_4 \) and \( \epsilon_T \) are parallel spinors on \( M_4 \) and \( M_7 \), respectively. This is consistent with the Ricci-flatness property in (17) and (18). 7-dimensional Riemannian manifolds admitting parallel spinors correspond to \( G_2 \) holonomy manifolds \([17]\). 4-dimensional Lorentzian manifolds admitting parallel spinors can be Minkowski or plane-wave spacetimes. However, Ricci-flatness property restricts the case to the Minkowski spacetime. Then, this case corresponds to the solution \( \text{Mink}_4 \times G_2 \). For the existence of only the external flux \( \lambda \neq 0 \) and \( \mu = 0 \), we have

\[
\nabla_X \epsilon_4 = \pm \frac{1}{6} \lambda e^a.\epsilon_4
\]

\[
\nabla_X \epsilon_T = -\frac{1}{12} \lambda e^a.\epsilon_T
\]

\[
(22)
\]

and this corresponds to the case that \( \epsilon_4 \) and \( \epsilon_T \) are geometric Killing spinors on \( M_4 \) and \( M_7 \) respectively which is consistent with being Einstein manifolds from (17) and (18). The geometric Killing spinors on \( M_4 \) and \( M_7 \) are real and imaginary Killing spinors, respectively \([22]\). 7-dimensional Riemannian manifolds admitting imaginary Killing spinors correspond to \( G_2 \) manifolds. In the case of admitting one Killing spinor, it is a proper weak \( G_2 \) manifold. For the existence of two and three Killing spinors, it corresponds to Sasaki-Einstein and 3-Sasaki manifolds, respectively. If there are maximal number of Killing spinors, then \( M_7 \) is a round sphere \( S^7 \). 4-dimensional Einstein manifolds with negative curvature admitting real Killing spinors correspond to \( AdS_4 \) spacetimes. Then, the solutions in that case
Indeed, the equations satisfied by 4-form defined on it. In that case, $g$ spinors. If $M$ of a solution for the unwarped case if the internal flux component $\phi$ satisfies a specific condition. We know that the internal flux $\phi$ satisfies the CCKY form equations (14) which means that they are constructed from geometric Killing spinors. If $\phi$ satisfies the condition $\phi.\epsilon_7 = \pm \frac{1}{2}\epsilon_7$, then the supergravity Killing spinor equation decomposes into the following equations

$$\nabla_{X^a}\epsilon_4 = -\frac{1}{6}\left(\pm \lambda + \frac{\mu}{4}\right)e^\alpha.\epsilon_4$$

(23)

$$\nabla_{X^a}\epsilon_7 = \frac{1}{12}\left(\lambda \pm \frac{\mu}{4}\right)e^\alpha.\epsilon_7 \pm \frac{\mu}{8}\phi.e^\alpha.\epsilon_7.$$  

(24)

Moreover, one can write the Clifford product of a 1-form $e^\alpha$ with an arbitrary form $\omega$ in terms of the wedge product and interior derivative as follows

$$e^\alpha.\omega = e^\alpha \wedge \omega + i_{X^a}\omega$$

$$\omega.e^\alpha = e^\alpha \wedge \eta \omega - i_{X^a}\eta \omega$$

(25)

where the automorphism $\eta$ acts on a p-form $\omega$ as $\eta \omega = (-1)^p\omega$. Then, we have

$$\phi.e^\alpha = e^\alpha.\phi - 2i_{X^a}\phi.$$  

(26)

By applying the interior derivative operator $i_{X^a}$ to the equations (14), one can see that $\phi$ satisfies

$$di_{X^a} * \phi = -\frac{3i\lambda}{4}i_{X^a}\phi$$

$$i_{X^a}d * \phi = i\lambda i_{X^a}\phi$$

(27)

and from the definition of the Lie derivative $L_{X^a} = di_{X^a} + i_{X^a}d$ on forms, one obtains

$$L_{X^a} * \phi = \frac{i\lambda}{4}i_{X^a}\phi.$$  

(28)

If the following condition on $\phi$ is satisfied

$$(L_{X^a} * \phi) . \epsilon_7 = i\lambda e^\alpha.\epsilon_7$$

(29)

then the equation (24) is transformed into

$$\nabla_{X^a}\epsilon_7 = \frac{1}{12}\left(\lambda \pm \frac{25}{2}\mu\right)e^\alpha.\epsilon_7.$$  

(30)

Now, if we choose the constant $\mu$ as $\mu = \pm \frac{5}{2}$, then the supergravity Killing spinor equation decomposes into the following equations from (23) and (30)

$$\nabla_{X^a}\epsilon_4 = \pm \frac{7}{40}\lambda e^\alpha.\epsilon_4$$

(31)

$$\nabla_{X^a}\epsilon_7 = -\frac{1}{8}\lambda e^\alpha.\epsilon_7$$

(32)

which correspond to geometric Killing spinors on $M_4$ and $M_7$. This is consistent with the condition $\phi.\epsilon_7 = \pm \frac{1}{2}\epsilon_7$, since if $\phi$ is constructed from $\epsilon_7$ as a bilinear 4-form, then it automatically satisfies this condition from Fierz identities [17]. So, the only restriction on $\phi$ to obtain geometric Killing spinors on product manifolds is the condition (28). Indeed, the equations satisfied by $\phi$ correspond to the case that $M_7$ is a weak $G_2$ manifold and $\phi$ is the coassociative 4-form defined on it. In that case, $g_4(\phi, \phi)$ in (17) and (18) is constant and $g_3(i_{X^a}\phi, i_{X^a}\phi)$ is proportional to $g_{\alpha \beta}$ [23].

So, equations (17) and (18) imply that $M_4$ and $M_7$ are Einstein manifolds. Then, the case $\lambda \neq 0$ and $\mu \neq 0$, for the special choice of $\mu = \pm \frac{5}{2}$, also corresponds to the solutions $AdS_4 \times S^7$ and $AdS_4 \times \text{weak } G_2$. But, for the general case of $\lambda \neq 0$ and $\mu \neq 0$, the supergravity Killing spinor equation (20) cannot be decomposed into $M_4$ and $M_7$ components and one cannot find a general solution. For the final case of $\lambda = 0$ and $\mu \neq 0$ which corresponds to the existence of only the internal flux, the equations will be similar to the previous case. However, if we take $\lambda = 0$ in (17), (18), (31) and (32), then (17) and (18) imply that $M_4$ and $M_7$ are Einstein manifolds, but (31) and (32) imply that they must admit parallel spinors which is inconsistent. So, $\lambda = 0$ and $\mu \neq 0$ case does not correspond to a solution.
A. Bilinear forms

Now, we will construct bilinear forms of supergravity Killing spinors by using the defining equation (5). The spinor bilinear of a spinor $\psi$ is defined in terms of the spinor inner product $(,)$ and co-frame basis as a sum of different degree differential forms as follows

$$\epsilon\psi = (\psi, \psi) + (\psi, e_a\epsilon)e^a + (\psi, e_{ab}\epsilon)e^{a+b} + \ldots + (\psi, e_{a_1a_2\ldots a_{n-1}}\epsilon)e^{a_1a_2\ldots a_n} + \ldots + (-1)^{[n/2]}(\psi, z\epsilon)z$$

where $e^{a_1a_2\ldots a_n}$ is the dual spinor space. Since the connection $\nabla$ is compatible with the spinor inner product $(,)$ and preserves the degree of a form, it is also compatible with the projection operation $(,)_p$ on p-form bilinears and we can write for a supergravity Killing spinor $\psi$ as

$$\nabla_X(\epsilon\psi)_p = ((\nabla_X\epsilon)_p + (\epsilon\nabla_X\psi)_p$$

$$= -\frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$

$$= \frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$

where we have used (5). For any spinor $\psi$, the dual spinor $\overline{\psi}$ can be written in terms of the involution operation $\overline{\psi}$ as $\overline{\psi} = \psi^(-1)$. Since we have $\overline{\psi} = \xi\psi$, for any Clifford form $\omega$ and spinor $\psi$, we have $\overline{\omega\psi} = (\omega, \psi)\xi\psi = \psi^\xi\omega\xi\psi = \psi^\xi\omega$. Then, we can write

$$\overline{(\tilde{X}F - 3F\tilde{X} \xi)} = \epsilon.(\psi(F\tilde{X} - 3\tilde{X}F)\xi)$$

$$= \epsilon.(F\xi, \tilde{X}F - 3\tilde{X}F\xi)$$

$$= \tilde{\epsilon}.(F\tilde{X} - 3F\tilde{X})$$

where we have used $F\xi = F$ and $\tilde{X}\xi = -\tilde{X}$. By using this equality in (34), we obtain

$$\nabla_X(\epsilon\psi)_p = -\frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p + \frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$

$$= \frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$

If we add and subtract the term $\frac{1}{6}(\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$ to the right hand side, we find

$$\nabla_X(\epsilon\psi)_p = -\frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p + \frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p + \frac{1}{6} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$

So, the bilinear form equation of supergravity Killing spinor $\psi$ which is also called the supergravity Killing form equation can be written as

$$\nabla_X(\epsilon\psi)_p = -\frac{1}{24} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p + \frac{1}{6} (\epsilon(F\tilde{X} - 3F\tilde{X})\psi)_p$$
where \( [ , ]_{Cl} \) denotes the Clifford bracket. Since we can write
\[
\tilde{X}.F = \tilde{X} \wedge F + i_X F
\]
\[
F.\tilde{X} = \tilde{X} \wedge F - i_X F
\]
and so
\[
\tilde{X}.F - 3F.\tilde{X} = -2\tilde{X} \wedge F + 4i_X F
\]
\[
F.\tilde{X} - \tilde{X}.F = -2i_X F
\]
the supergravity Killing form equation (38) turns into
\[
\nabla_X(\epsilon_\tau)_p = \frac{1}{12} \left( [\tilde{X} \wedge F, \epsilon_\tau]_{Cl} \right)_p - \frac{1}{6} (i_X F, \epsilon_\tau)_{Cl}_p - \frac{1}{3} (\epsilon_\tau, i_X F)_p.
\]
(41)
The only non-zero bilinear forms of a spinor on an eleven-dimensional Lorentzian manifold are 1-, 2-, 5-, 6-, 9- and 10-forms as can be seen from Table XVII in Appendix A. So, the spinor bilinear of the supergravity Killing spinor \( \epsilon \) is
\[
\epsilon_\tau = (\epsilon_\tau)_1 + (\epsilon_\tau)_2 + (\epsilon_\tau)_5 + (\epsilon_\tau)_6 + (\epsilon_\tau)_9 + (\epsilon_\tau)_{10}.
\]
(42)
We can find the equations satisfied by all of the bilinear forms by considering the definition of the Clifford bracket and projection operation given in (B9). For \( p = 1 \), we have the following equation for the bilinear 1-form \( (\epsilon_\tau)_1 \) from (41)
\[
\nabla_{X_A}(\epsilon_\tau)_1 = \frac{1}{144} F \wedge \frac{1}{4} i_{X_A}(\epsilon_\tau)_6 - \frac{1}{6} i_{X_A} F \wedge \frac{1}{2} (\epsilon_\tau)_2
\]
(43)
where we have used the definition of the contracted wedge product given in (B8). If we use the definitions of the exterior derivative and coderivative in terms of the covariant derivative as \( d = e^A \wedge \nabla_{X_A} \) and \( \delta = -i_{X_A} \nabla_{X_A} \) for zero torsion, we obtain
\[
d(\epsilon_\tau)_1 = \frac{1}{72} F \wedge \frac{1}{4} (\epsilon_\tau)_6 - \frac{1}{3} F \wedge \frac{2}{3} (\epsilon_\tau)_2
\]
(44)
\[
\delta(\epsilon_\tau)_1 = 0.
\]
By comparing the equations (43) and (44), one can easily see that \( (\epsilon_\tau)_1 \) satisfies the equation
\[
\nabla_{X_A}(\epsilon_\tau)_1 = \frac{1}{2} i_{X_A} d(\epsilon_\tau)_1
\]
(45)
and hence \( (\epsilon_\tau)_1 \) is a KY 1-form. Consequently, the vector field which is metric dual to the 1-form \( (\epsilon_\tau)_1 \) is a Killing vector field. The definition and properties of KY forms can be found in Appendix C. For \( p = 2 \), the bilinear form equation (41) gives
\[
\nabla_{X_A}(\epsilon_\tau)_2 = \frac{1}{36} F \wedge \frac{1}{3} i_{X_A}(\epsilon_\tau)_5 + \frac{1}{144} e_A \wedge (F \wedge (\epsilon_\tau)_5) - \frac{1}{3} (\epsilon_\tau)_1 \wedge i_{X_A} F + \frac{1}{18} (\epsilon_\tau)_5 \wedge i_{X_A} F
\]
(46)
and the exterior and co-derivatives are
\[
d(\epsilon_\tau)_2 = (\epsilon_\tau)_1 \wedge F
\]
\[
\delta(\epsilon_\tau)_2 = \frac{11}{72} F \wedge (\epsilon_\tau)_5.
\]
(47)
So, \( (\epsilon_\tau)_2 \) does not satisfy the KY equation. For \( p = 5 \), the bilinear form equation gives
\[
\nabla_{X_A}(\epsilon_\tau)_5 = \frac{1}{6} (e_A \wedge F) \wedge (\epsilon_\tau)_2 - \frac{1}{3} i_{X_A} F \wedge (\epsilon_\tau)_2 + \frac{1}{6} i_{X_A} F \wedge (\epsilon_\tau)_6
\]
\[
- \frac{1}{36} (e_A \wedge F) \wedge (\epsilon_\tau)_6 + \frac{1}{6} (e_A \wedge F) \wedge (\epsilon_\tau)_{10}
\]
(48)
where the terms on the right hand side can also be written in a more explicit way by using the identity
\[(\tilde{X} \wedge F) \wedge \alpha = kF \wedge_{k-1} i_X \alpha + (-1)^k \tilde{X} \wedge (F \wedge \alpha). \tag{49}\]

(48) implies
\[
d(\epsilon^5) = -F \wedge (\epsilon^2) + \frac{1}{24} F \wedge (\epsilon^10)
\]
\[
\delta(\epsilon^5) = \frac{2}{3} F \wedge (\epsilon^2) - \frac{1}{18} (\epsilon^6) \wedge_3 F. \tag{50}\]

Similarly, for \(p = 6\), we have
\[
\nabla_{X_A}(\epsilon^6) = \frac{1}{144} (e_A \wedge F) \wedge_4 (\epsilon^9) - \frac{1}{3} (\epsilon^5) \wedge_1 i_{X_A} F + \frac{1}{18} (\epsilon^6) \wedge_3 i_{X_A} F + \frac{1}{6} (e_A \wedge F) \wedge_2 (\epsilon^5) \tag{51}\]

and
\[
d(\epsilon^6) = \frac{1}{3} F \wedge (\epsilon^5) + \frac{1}{6} F \wedge (\epsilon^1)
\]
\[
\delta(\epsilon^6) = -\frac{1}{144} F \wedge (\epsilon^9) + \frac{7}{6} F \wedge (\epsilon^1) + F \wedge (\epsilon^5). \tag{52}\]

5- and 6-form bilinears also do not satisfy the KY form equation. For the case of \(p = 9\), (41) gives
\[
\nabla_{X_A}(\epsilon^9) = \frac{1}{6} (e_A \wedge F) \wedge_1 (\epsilon^6) - \frac{1}{36} (e_A \wedge F) \wedge_3 (\epsilon^10) - \frac{1}{3} i_{X_A} F \wedge_2 (\epsilon^6) + \frac{1}{6} F \wedge_3 (\epsilon^10) \tag{53}\]

and
\[
d(\epsilon^9) = -\frac{1}{3} \left( F \wedge (\epsilon^6) + F \wedge_2 (\epsilon^10) \right)
\]
\[
\delta(\epsilon^9) = \frac{1}{6} F \wedge_3 (\epsilon^10). \tag{54}\]

For \(p = 10\), we have
\[
\nabla_{X_A}(\epsilon^{10}) = -\frac{1}{12} (e_A \wedge F) \wedge_2 (\epsilon^9) - \frac{1}{3} (\epsilon^5) \wedge_1 i_{X_A} F + \frac{1}{6} (e_A \wedge F) \wedge_5 (\epsilon^5) \tag{55}\]

and
\[
d(\epsilon^{10}) = -\frac{1}{3} F \wedge_1 (\epsilon^9)
\]
\[
\delta(\epsilon^{10}) = -\frac{1}{3} \left( F \wedge (\epsilon^5) + F \wedge_2 (\epsilon^9) \right). \tag{56}\]

So, except the 1-form bilinear, all the higher degree bilinear forms of supergravity Killing spinors do not correspond to KY forms and satisfy different types of equations.

Now, we can consider the decomposition of bilinear forms onto product manifolds \(M_4 \) and \(M_7\). Since the supergravity Killing spinor \(\epsilon\) decomposes as in (9), the spinor bilinears decompose as \(\epsilon = \{\epsilon^{(4)}, \epsilon^{(7)}\}\). By considering the definitions \(\epsilon^{(4)} := \epsilon^4\tilde{\epsilon}^4\) and \(\epsilon^{(7)} := \epsilon^7\tilde{\epsilon}^7\), the \(p\)-form bilinears on product manifolds correspond to
\[
(\epsilon^p) = \{(\epsilon^{(4)})_p, (\epsilon^{(7)})_p\}. \tag{57}\]

Since, the degree of differential forms cannot be greater than the volume form, from (42) we have
\[
\epsilon^{(4)} = (\epsilon^{(4)})_1 + (\epsilon^{(4)})_2
\]
\[
\epsilon^{(7)} = (\epsilon^{(7)})_1 + (\epsilon^{(7)})_2 + (\epsilon^{(7)})_5 + (\epsilon^{(7)})_6. \tag{58}\]
inner product \[ \begin{array}{c|cccc} \hline \text{inner product} & 1 & 2 & 5 & 6 \\ \hline i) & M_4: \mathbb{C}^*-\text{sym} & & & \checkmark \\ & M_7: \mathbb{R}^\text{-skew} & \times & \checkmark & \checkmark \\ \hline ii) & M_4: \mathbb{C}^*-\text{sym} & & & \checkmark \\ & M_7: \mathbb{R}^\text{-sym} & \times & \times & \times \\ \hline iii) & M_4: \mathbb{C}^-\text{skew} & & & \checkmark \\ & M_7: \mathbb{R}^\text{-sym} & \times & \checkmark & \checkmark \\ \hline iv) & M_4: \mathbb{C}^-\text{skew} & & & \checkmark \\ & M_7: \mathbb{R}^\text{-skew} & \times & \checkmark & \checkmark \\ \hline \end{array} \]

Table I: Properties of nonzero bilinears for different spinor inner product choices on Lorentzian \( M_4 \) and Riemannian \( M_7 \) for \( \mathbb{R}^\text{-skew} \) inner product on \( M_{11} \).

Moreover, depending on the spinor inner product choices on \( M_4 \) and \( M_7 \), one can determine the properties of nonzero bilinears constructed out of \( \epsilon_4 \) and \( \epsilon_7 \). \( M_4 \) is a Lorentzian manifold, the spinor space corresponds to \( \mathbb{C}^2 \oplus \mathbb{C}^2 \) and the spinors are Dirac-Weyl spinors. \( M_7 \) is a Riemannian manifold, the spinor space corresponds to \( \mathbb{R}^8 \) and the spinors are Majorana spinors. So, from Table XVII in Appendix A, we have the bilinears for the chosen inner products given in Table I.

So, we have to consider four different cases separately in the decomposition of bilinear forms onto product manifolds.

i) \( M_4: \mathbb{C}^*-\text{sym} \) and \( M_7: \mathbb{R}^\text{-skew} \);

By considering the nonzero bilinears in the above table and the decomposition of the 4-form flux \( F \) in (8), the 1-form bilinear equation (43) decomposes as

\[
\nabla_{\xi_a}(\varepsilon(4))_1 = -\frac{i\lambda}{6} x_a z_4 \wedge (\varepsilon(4))_2 \\
0 = -\frac{\mu}{144} \phi \wedge i x_a (\varepsilon(7))_6 - \frac{\mu}{6} i x_a \phi \wedge (\varepsilon(7))_2
\]

and the second equality implies that \( \mu = 0 \) while the first equality implies that \( \lambda \) is real (since \( (\varepsilon(4))_1 \) is real and \( (\varepsilon(4))_2 \) is pure imaginary from the above table). From the first equality, one can obtain

\[
d(\varepsilon(4))_1 = -\frac{i\lambda}{3} z_4 \wedge (\varepsilon(4))_2 \\
\delta(\varepsilon(4))_1 = 0.
\]

Then, the reduction of the 1-form bilinear onto \( M_4 \) is also a KY 1-form

\[
\nabla_{\xi_a}(\varepsilon(4))_1 = \frac{1}{2} i x_a d(\varepsilon(4))_1.
\]

Similarly, the 2-form bilinear equation (46) decomposes as

\[
\nabla_{\xi_a}(\varepsilon(4))_2 = -\frac{i\lambda}{3} (\varepsilon(4))_1 \wedge i x_a z_4 \\
\nabla_{\xi_a}(\varepsilon(7))_2 = 0.
\]

From the first equality, we have

\[
d(\varepsilon(4))_2 = i\lambda z_4 \wedge (\varepsilon(4))_1 \\
\delta(\varepsilon(4))_2 = 0
\]

and so \( (\varepsilon(4))_2 \) is a KY 2-form

\[
\nabla_{\xi_a}(\varepsilon(4))_2 = \frac{1}{3} i x_a d(\varepsilon(4))_2.
\]

However, the second equality in (61) implies that \( (\varepsilon(7))_2 \) is a parallel form and hence must be constructed from the parallel spinor \( \epsilon_7 \). This implies from (22) that \( \lambda \) must also vanish \( \lambda = 0 \). Then, the bilinears on \( M_4 \) also correspond
to parallel forms and $\epsilon_4$ is also a parallel spinor. So, the choice of inner product forces the flux $F$ to vanish and the decompositions of 5-form and 6-form bilinear equations also imply this. Then, as a result, the first inner product choice allows only Mink$4 \times G_2$ solutions.

ii) $M_4 : \mathbb{C}^\ast$-sym $\xi$ and $M_7 : \mathbb{R}$-sym $\xi\eta$:

In this case, all bilinear forms on $M_7$ which appear in the bilinear form equations are automatically zero as can be seen from the above table. So, the seven-dimensional parts of the decompositions are trivial and this does not give a restriction on $\mu$. The inner product choice for $M_4$ is the same as for the first case and hence the four-dimensional parts of the bilinears correspond to KY forms

$$\nabla_{X_a}(\mathbf{e}^{(4)})_1 = \frac{1}{2} i_{X_a} d(\mathbf{e}^{(4)})_1$$

$$\nabla_{X_a}(\mathbf{e}^{(4)})_2 = \frac{1}{3} i_{X_a} d(\mathbf{e}^{(4)})_2$$  (65)

and moreover they correspond to special KY forms. By direct computation, one can see that

$$\nabla_{X_a} d(\mathbf{e}^{(4)})_1 = \frac{2}{9} \lambda^2 e_a \wedge (\mathbf{e}^{(4)})_1$$

$$\nabla_{X_a} d(\mathbf{e}^{(4)})_2 = \frac{1}{3} \lambda^2 e_a \wedge (\mathbf{e}^{(4)})_2$$  (66)

and this implies that $\epsilon_4$ must correspond to a geometric Killing spinor. This also does not put a restriction on $\lambda$ and hence all types of solutions for this inner product choice is possible; $AdS_4 \times$ weak $G_2, AdS_4 \times S^7$ and Mink$4 \times G_2$.

iii) $M_4 : \mathbb{C}$-skew $\xi\eta$ and $M_7 : \mathbb{R}$-skew $\xi$:

The choice of spinor inner product on $M_7$ is same as in the first case. So, this choice also implies that $\lambda = 0$ and $\mu = 0$ and hence all the bilinears on $M_4$ and $M_7$ correspond to parallel forms. The only solution is Mink$4 \times G_2$.

iv) $M_4 : \mathbb{C}$-skew $\xi\eta$ and $M_7 : \mathbb{R}$-sym $\xi\eta$:

The choice of spinor inner product on $M_7$ is same as in the second case. So, there is no restriction on $\lambda$ and $\mu$ and bilinear forms on $M_4$ correspond to special KY forms. However, the choice of inner product on $M_4$ does not determine the real or pure imaginary character of the bilinear 1- and 2-forms and hence only if $\lambda$ can be chosen as real, we can have the solutions $AdS_4 \times$ weak $G_2$ and $AdS_4 \times S^7$. Mink$4 \times G_2$ solution already exists since $\lambda = 0 = \mu$ for it.

In summary, the relation between spinor inner product choices and $M_4 \times M_7$ solutions is as given in Table II.

| $M_{11} : \mathbb{R}$-skew $\xi\eta$ | solutions |
|------------------------------------------|-----------|
| $M_4 : \mathbb{C}^\ast$-sym $\xi$       | Mink$4 \times G_2$ |
| $M_7 : \mathbb{R}$-skew $\xi$           |           |
| $M_4 : \mathbb{C}^\ast$-sym $\xi\eta$   | Mink$4 \times G_2$ |
| $M_7 : \mathbb{R}$-sym $\xi\eta$        | $AdS_4 \times S^7, AdS_4 \times$ weak $G_2$ |
| $M_4 : \mathbb{C}$-skew $\xi\eta$       | Mink$4 \times G_2$ |
| $M_7 : \mathbb{R}$-skew $\xi$           |           |
| $M_4 : \mathbb{C}$-skew $\xi\eta$       | $AdS_4 \times S^7, AdS_4 \times$ weak $G_2$ (if $\lambda$ is real) |
| $M_7 : \mathbb{R}$-sym $\xi\eta$        |           |

Table II: The relation between the choice of spinor inner products and $M_4 \times M_7$ solutions for $\mathbb{R}$-skew $\xi\eta$ inner product on $M_{11}$.

Note that $AdS$ solutions can exist only for the inner product choices for which the supergravity Killing forms decompose into special KY forms on product manifolds.

2. $\mathbb{R}$-sym $\xi$ inner product

In the second case, we choose the spinor inner product on the eleven-dimensional Lorentzian manifold $M_{11}$ as $\mathbb{R}$-sym with $\xi$ involution and consider the decomposition of bilinear forms in that case. $p$-form bilinear equation is the same as in (34)

$$\nabla_X (\mathbf{e}_p)_{\mathbf{e}} = -\frac{1}{24} (\overline{X.F - 3F.X} \cdot \mathbf{e}_{\mathbf{e}})_{\mathbf{e}} - \frac{1}{24} (\epsilon (\overline{X.F - 3F.X}) \cdot \mathbf{e}_{\mathbf{e}})_{\mathbf{e}},$$  (67)
we can write
\[(X, F - 3F, X). \epsilon = \tau(F, X - 3X, F)\] (68)
where we have used \(F^\xi = F\) and \(\bar{X}^\xi = \bar{X}\). By adding and subtracting the term \(\frac{1}{6} (\epsilon, F, X - 3X). F\) to (67), we find
\[\nabla_X (\epsilon, \tau) = -\frac{1}{24} \left((X, F - 3F, \bar{X}) \cdot \epsilon, \tau + C\right)_p - \frac{1}{6} \left(\epsilon, F, X\right)_{C\tau} p\] (69)
where \([,]_{C\tau}\) denotes the Clifford anticommutator which is defined in (B11) and (B12). In terms of wedge product and interior derivative, the supergravity Killing form equation can also be written from (40) as
\[\nabla_X (\epsilon, \tau) = \frac{1}{12} \left((X, F, \epsilon, \tau + C\right)_p - \frac{1}{6} \left(\epsilon, F, X\right)_{C\tau} p + \frac{1}{3} \left(\epsilon, F, X\right)_{p}.\] (70)
The nonzero bilinear forms for R-sym \(\xi\) inner product are 0-, 1-, 4-, 5-, 8- and 9-forms and the spinor bilinear of the supergravity Killing spinor \(\epsilon\) corresponds to
\[\epsilon = (\epsilon)_0 + (\epsilon)_1 + (\epsilon)_{4} + (\epsilon)_{5} + (\epsilon)_{8} + (\epsilon)_{9}.\] (71)
From (70), we can find the bilinear form equations for different degrees. For \(p = 0\), we have
\[\nabla_{X_A} (\epsilon, \tau)_0 = \frac{1}{144} F \wedge (\epsilon, \tau)_{4}\] (72)
and
\[d(\epsilon, \tau)_0 = \frac{1}{144} F \wedge (\epsilon, \tau)_4\]
\[\delta(\epsilon, \tau)_0 = 0.\] (73)
Then \((\epsilon, \tau)_0\) satisfies \(\nabla_{X_A} (\epsilon, \tau)_0 = i_{X_A} d(\epsilon, \tau)_0\) and hence is a KY 0-form. For \(p = 1\), the bilinear form equation corresponds to
\[\nabla_{X_A} (\epsilon, \tau)_1 = \frac{1}{144} (e_{A, F})_4 (\epsilon, \tau)_4 + \frac{1}{18} (\epsilon, \tau)_4 \wedge i_{X_A} F\] (74)
and
\[d(\epsilon, \tau)_1 = -\frac{1}{36} F \wedge (\epsilon, \tau)_4\]
\[\delta(\epsilon, \tau)_1 = \frac{1}{144} F \wedge (\epsilon, \tau)_4.\] (75)
For \(p = 4\), (70) gives
\[\nabla_{X_A} (\epsilon, \tau)_4 = \frac{1}{6} (e_{A, F})_1 (\epsilon, \tau)_1 - \frac{1}{36} (e_{A, F})_3 (\epsilon, \tau)_5 + \frac{1}{144} F \wedge i_{X_A} (\epsilon, \tau)_9\]
\[+\frac{1}{3} (\epsilon, \tau)_1 \wedge i_{X_A} F + \frac{1}{6} i_{X_A} F \wedge (\epsilon, \tau)_5\] (76)
and
\[d(\epsilon, \tau)_4 = \frac{7}{6} F \wedge (\epsilon, \tau)_1 + \frac{1}{12} F \wedge (\epsilon, \tau)_5 + \frac{5}{144} F \wedge (\epsilon, \tau)_9\]
\[\delta(\epsilon, \tau)_4 = \frac{5}{6} F \wedge (\epsilon, \tau)_1 - \frac{1}{18} F \wedge (\epsilon, \tau)_5.\] (77)
For \(p = 5\), we have
\[\nabla_{X_A} (\epsilon, \tau)_5 = \frac{1}{6} (e_{A, F})_1 (\epsilon, \tau)_0 - \frac{1}{12} (e_{A, F})_2 (\epsilon, \tau)_4 + \frac{1}{144} (e_{A, F})_4 (\epsilon, \tau)_8\]
\[-\frac{1}{3} (\epsilon, \tau)_4 \wedge i_{X_A} F + \frac{1}{18} (\epsilon, \tau)_8 \wedge i_{X_A} F\] (78)
and
\[ d(\mathcal{e}_5) = \frac{1}{2} F \wedge (\mathcal{e}_4) + \frac{1}{12} F \wedge (\mathcal{e}_3) \]
\[ \delta(\mathcal{e}_5) = -\frac{7}{6} F \wedge (\mathcal{e}_0) + \frac{13}{12} F \wedge (\mathcal{e}_4) - \frac{19}{144} F \wedge (\mathcal{e}_3). \]

Similarly, for \( p = 8 \), (70) gives
\[ \nabla_{X_A} (\mathcal{e}_8) = \frac{1}{6} (e_A \wedge F) \wedge (\mathcal{e}_5) - \frac{1}{36} (e_A \wedge F) \wedge (\mathcal{e}_9) - \frac{1}{3} i_{X_A} F \wedge (\mathcal{e}_5) + \frac{1}{6} i_{X_A} F \wedge (\mathcal{e}_9) \]
and
\[ d(\mathcal{e}_8) = -\frac{1}{2} F \wedge (\mathcal{e}_5) - \frac{1}{4} F \wedge (\mathcal{e}_9) \]
\[ \delta(\mathcal{e}_8) = \frac{1}{6} F \wedge (\mathcal{e}_5) + \frac{5}{36} F \wedge (\mathcal{e}_9). \]

For \( p = 9 \), we have
\[ \nabla_{X_A} (\mathcal{e}_9) = \frac{1}{6} (e_A \wedge F) \wedge (\mathcal{e}_5) - \frac{1}{12} (e_A \wedge F) \wedge (\mathcal{e}_9) - \frac{1}{3} (\mathcal{e}_5) \wedge i_{X_A} F \]
and
\[ d(\mathcal{e}_9) = -\frac{1}{6} F \wedge (\mathcal{e}_5) \]
\[ \delta(\mathcal{e}_9) = \frac{1}{2} F \wedge (\mathcal{e}_5) + \frac{3}{4} F \wedge (\mathcal{e}_9). \]

So, only the 0-form bilinear correspond to a KY form and other higher degree bilinears satisfy different types of equations.

Now, we can decompose the bilinear form equations onto product manifolds \( M_4 \) and \( M_7 \). We have the following bilinear forms on product manifolds
\[ \mathcal{e}_4(4) = (\mathcal{e}_4(4))_0 + (\mathcal{e}_4(4))_1 + (\mathcal{e}_4(4))_4 \]
\[ \mathcal{e}_7(7) = (\mathcal{e}_7(7))_0 + (\mathcal{e}_7(7))_1 + (\mathcal{e}_7(7))_4 + (\mathcal{e}_7(7))_5 \]
and from Table XVII in Appendix A, the properties of bilinear forms depending on the choice of the spinor inner product are given in Table III.

| inner product      | 0 1 4 5 |
|--------------------|--------|
| i) \( M_4 : C^\ast\)-sym \( \xi \) | \( R R R \) |
| \( M_7 : R\)-skew \( \xi \) | \( \times \times \times \) |
| ii) \( M_4 : C^\ast\)-sym \( \xi \) | \( R R R \) |
| \( M_7 : R\)-sym \( \eta \) | \( \checkmark \times \checkmark \) |
| iii) \( M_4 : C\)-skew \( \xi \eta \) | \( \times \checkmark \times \) |
| \( M_7 : R\)-skew \( \xi \) | \( \times \times \times \) |
| iv) \( M_4 : C\)-skew \( \xi \eta \) | \( \checkmark \checkmark \times \) |
| \( M_7 : R\)-sym \( \xi \eta \) | \( \checkmark \checkmark \times \) |

Table III: Properties of nonzero bilinears for different spinor inner product choices on Lorentzian \( M_4 \) and Riemannian \( M_7 \) for \( R\)-sym \( \xi \) inner product on \( M_{11} \).

Then, we can consider four different cases in the decomposition.

i) \( M_4 : C^\ast\)-sym \( \xi \) and \( M_7 : R\)-skew \( \xi \):

For this inner product choice, the bilinear form equations on \( M_4 \) correspond to
\[ \nabla_{X_a} (\mathcal{e}_4(4))_0 = 0 \]
\[ \nabla_{X_a} (\mathcal{e}_4(4))_1 = \frac{i \lambda}{36} z_4 \wedge i_{X_a} (\mathcal{e}_4(4))_4 + \frac{i \lambda}{144} e_a \wedge (z_4 \wedge (\mathcal{e}_4(4))_4) + \frac{i \lambda}{18} (\mathcal{e}_4(4))_4 \wedge i_{X_a} z_4 \]
\[ \nabla_{X_a} (\mathcal{e}_4(4))_4 = \frac{i \lambda}{3} (\mathcal{e}_4(4))_1 \wedge i_{X_a} z_4. \]
So, \((\epsilon \bar{\epsilon}^{(4)})_n\) is constant and we can write the exterior and coderivatives of bilinear forms as
\[
d((\epsilon \bar{\epsilon}^{(4)})_1) = 0 \\
d((\epsilon \bar{\epsilon}^{(4)})_4) = 0 \\
\delta((\epsilon \bar{\epsilon}^{(4)})_1) = \frac{i\lambda}{18} z_4 \wedge (\epsilon \bar{\epsilon}^{(4)})_4 \\
\delta((\epsilon \bar{\epsilon}^{(4)})_4) = -\frac{i\lambda}{3} (\epsilon \bar{\epsilon}^{(4)})_1 \wedge z_4.
\]
(86)
and
\[
d((\epsilon \bar{\epsilon}^{(4)})_1) = 0 \\
d((\epsilon \bar{\epsilon}^{(4)})_4) = 0 \\
\delta((\epsilon \bar{\epsilon}^{(4)})_1) = \frac{4\lambda^2}{9} iX_\alpha (\epsilon \bar{\epsilon}^{(4)})_1 \\
\delta((\epsilon \bar{\epsilon}^{(4)})_4) = -\frac{\lambda^2}{9} iX_\alpha (\epsilon \bar{\epsilon}^{(4)})_4.
\]
(87)

Then, by comparing (85) with (86) and (87), one can see that they satisfy the CCKY equation
\[
\nabla_{X_\alpha} ((\epsilon \bar{\epsilon}^{(4)})_1) = -\frac{1}{4} e_\alpha \wedge (\epsilon \bar{\epsilon}^{(4)})_1 \\
\nabla_{X_\alpha} ((\epsilon \bar{\epsilon}^{(4)})_4) = -e_\alpha \wedge (\epsilon \bar{\epsilon}^{(4)})_4.
\]
(88)

Moreover, they correspond to special CCKY forms
\[
\nabla_{X_\alpha} \delta((\epsilon \bar{\epsilon}^{(4)})_1) = -\frac{4\lambda^2}{9} iX_\alpha (\epsilon \bar{\epsilon}^{(4)})_1 \\
\nabla_{X_\alpha} \delta((\epsilon \bar{\epsilon}^{(4)})_4) = -\frac{\lambda^2}{9} iX_\alpha (\epsilon \bar{\epsilon}^{(4)})_4
\]
(89)

So, \(\epsilon_4\) is a geometric Killing spinor generating the supergravity Killing forms which correspond to special CCKY forms. All of the bilinear form equations on \(M_7\) are trivial and hence we have all types of solutions for this inner product choice. Namely, \(AdS_4 \times \text{weak } G_2, AdS_4 \times S^7\) and \(\text{Mink}_4 \times G_2\).

ii) \(M_4 : \mathbb{C}^n\text{-sym } \xi\) and \(M_7 : \mathbb{R}\text{-sym } \xi\eta\):

In this case, the situation for \(M_4\) is the same as in the previous case and hence \((\epsilon \bar{\epsilon}^{(4)})_1\) and \((\epsilon \bar{\epsilon}^{(4)})_4\) are special CCKY forms. For \(M_7\), we have the following equalities
\[
\nabla_{X_\alpha} ((\epsilon \bar{\epsilon}^{(7)})_0) = 0 \\
0 = \frac{\mu}{36} \phi \wedge iX_\alpha (\epsilon \bar{\epsilon}^{(7)})_4 + \frac{\mu}{18} (\epsilon \bar{\epsilon}^{(7)})_4 \wedge iX_\alpha \phi + \frac{\mu}{144} e_\alpha \wedge (\phi \wedge (\epsilon \bar{\epsilon}^{(7)})_4 \\
\nabla_{X_\alpha} ((\epsilon \bar{\epsilon}^{(7)})_4) = 0 \\
0 = \frac{\mu}{6} e_\alpha \wedge \phi \wedge (\epsilon \bar{\epsilon}^{(7)})_0 - \frac{\mu}{12} (e_\alpha \wedge \phi) \wedge (\epsilon \bar{\epsilon}^{(7)})_4 - \frac{\mu}{3} (\epsilon \bar{\epsilon}^{(7)})_4 \wedge iX_\alpha \phi.
\]
(90)

So, we have \(\mu = 0\) and 0- and 4-forms are parallel. Then, we have the solutions \(AdS_4 \times \text{weak } G_2, AdS_4 \times S^7\) for \(\lambda \neq 0\) and \(\mu = 0\). For \(\lambda = \mu = 0\), we have \(\text{Mink}_4 \times G_2\).

iii) \(M_4 : \mathbb{C}\text{-skew } \xi\eta\) and \(M_7 : \mathbb{R}\text{-skew } \xi\);

This case gives
\[
\nabla_{X_\alpha} ((\epsilon \bar{\epsilon}^{(4)})_1) = 0 \\
0 = \frac{i\lambda}{3} (\epsilon \bar{\epsilon}^{(4)})_1 \wedge iX_\alpha z_4
\]
(91)
on \(M_4\) and we have \(\lambda = 0\). The seven-dimensional equations on \(M_7\) are all trivial and the only solution is \(\text{Mink}_4 \times G_2\).

iv) \(M_4 : \mathbb{C}\text{-skew } \xi\eta\) and \(M_7 : \mathbb{R}\text{-sym } \xi\eta\);

The case for \(M_4\) is the same as the previous case and for \(M_7\) it is the same with case ii. So, both \(\lambda\) and \(\mu\) vanishes and we have \(\text{Mink}_4 \times G_2\) solution.

In summary, for the inner product choice of \(\mathbb{R}\text{-sym } \xi\) on \(M_{11}\), the solutions that appear for different types of inner product choices on \(M_4\) and \(M_7\) can be given as in Table IV.

Note that, when the \(AdS\) solutions exist for the relevant choices of spinor inner products, the supergravity Killing forms decompose into special CCKY forms on product manifolds.
Table IV: The relation between the choice of spinor inner products and $M_4 \times M_7$ solutions for $\mathbb{R}$-sym $\xi$ inner product on $M_{11}$.

| $M_{11}$ | $\mathbb{R}$-sym $\xi$ | solutions |
|---------|----------------|-----------|
| $M_4$  | $\mathbb{C}^*$-sym $\xi$ | $\text{Mink}_{4} \times G_2$ |
| $M_7$  | $\mathbb{R}$-skew $\xi$ | $AdS_4 \times S^7$, $AdS_4 \times \text{weak} \; G_2$ |
| $M_4$  | $\mathbb{C}^*$-sym $\xi$ | $\text{Mink}_{4} \times G_2$ |
| $M_7$  | $\mathbb{R}$-sym $\xi \eta$ | $AdS_4 \times S^7$, $AdS_4 \times \text{weak} \; G_2$ |
| $M_4$  | $\mathbb{C}$-skew $\xi \eta$ | $\text{Mink}_{4} \times G_2$ |
| $M_7$  | $\mathbb{R}$-sym $\xi \eta$ | $\text{Mink}_{4} \times G_2$ |

\[ \Phi_a = e^A \Phi_a^\lambda = 1_7 \otimes e^\lambda \otimes z_4 \]

where $1_7$ is the identity on $M_7$, $e^a$ are the Clifford algebra basis on $M_7$, $z_4$ is the volume form on $M_4$ and $e^\alpha$ are the Clifford algebra basis on $M_4$. The flux 4-form is decomposed as

\[ F = \{ \lambda \phi, \mu z_4 \} \]

where $\lambda$ and $\mu$ are constants and $\phi$ is a 4-form on $M_7$. Similarly, the supersymmetry parameter can be written as

\[ \epsilon = \epsilon_7 \otimes \epsilon_4. \]

By decomposing the field equations similar to the case in Section III, the Maxwell-like field equations give

\[ d *_7 \phi = \mu \phi \]
\[ d \phi = 0. \]

This means that $\phi$ is a CCKY 4-form on $M_4$ and must be generated from a geometric Killing spinor. Since the volume form $z_4$ is also a KY 4-form, the flux form $F$ is generated by KY and CCKY forms and hence by geometric Killing spinors for $\lambda \neq 0$ and $\mu \neq 0$.

The decomposition of Einstein field equations will give the following equalities on $M_4$ and $M_7$ respectively

\[ i_{X_a} P_a = \frac{\lambda^2}{2} \left( g_4(i_{X_a} \phi, i_{X_a} \phi) - \frac{1}{3} g_4(\phi, \phi) g_{ab} \right) - \frac{\mu^2}{6} g_{ab} \]

\[ i_{X_a} P_\alpha = \frac{1}{3} \left( \mu^2 - \frac{\lambda^2}{2} g_4(\phi, \phi) \right) g_{\alpha \beta}. \]

This means that for $\lambda = \mu = 0$, both $M_7$ and $M_4$ are Ricci-flat manifolds and for the special case of $\lambda = 0$ and $\mu \neq 0$, $M_7$ is a negative curvature and $M_4$ is a positive curvature Einstein manifolds.

The decomposition of the supergravity Killing spinor equation into product manifolds can be found as follows

\[ \nabla_{X^a} \epsilon_7 \otimes \epsilon_4 + \epsilon_7 \otimes \nabla_{X^a} \epsilon_4 = \frac{1}{12} \lambda \Phi_a \epsilon_7 \otimes e^\alpha \epsilon_4 + \frac{1}{12} \mu e^\alpha \epsilon_7 \otimes \epsilon_4 \]
\[ + \frac{1}{6} \epsilon_7 \otimes \epsilon_4 \]
\[ + \frac{1}{24} \lambda (e^\alpha \Phi_7 - 3 \Phi_7 e^\alpha) \epsilon_7 \otimes \epsilon_4 \]

where we have used that $z_4 e^\alpha = -e^\alpha z_4$ and $z_4^2 = 1$ for the Riemannian manifold $M_4$, so we have $z_4 \epsilon_4 = \pm \epsilon_4$. Then, for the fluxless case $\lambda = \mu = 0$, we have two equations on product manifolds

\[ \nabla_{X^a} \epsilon_7 = 0 \]
\[ \nabla_{X^a} \epsilon_4 = 0 \]
Table V: Properties of nonzero bilinears for different spinor inner product choices on Lorentzian $M_7$ and Riemannian $M_4$ for $\mathbb{R}$-skew $\xi$ inner product on $M_{11}$.

| inner product | 1 | 2 | 5 | 6 |
|---------------|---|---|---|---|
| i) $M_7: \mathbb{H}^-$-sym $\xi$ | $R$ | $V$ | $R$ | $V$ |
| $M_4$: $\mathbb{H}$-swap $\xi$ | $\checkmark$ | $\times$ |       |       |
| ii) $M_7: \mathbb{H}^-$-sym $\xi$ | $R$ | $V$ | $R$ | $V$ |
| $M_4$: $\mathbb{H}^-$-sym $\oplus \mathbb{H}^-$-sym $\xi_\eta$ | $V$ | $V$ |       |       |
| iii) $M_7: \mathbb{H}^-$-sym $\xi_\eta$ | $(-)$ | $(-)$ | $(-)$ | $(-)$ |
| $M_4$: $\mathbb{H}$-swap $\xi$ | $\checkmark$ | $\times$ |       |       |
| iv) $M_7: \mathbb{H}^-$-sym $\xi_\eta$ | $(-)$ | $(-)$ | $(-)$ | $(-)$ |
| $M_4$: $\mathbb{H}^-$-sym $\oplus \mathbb{H}^-$-sym $\xi_\eta$ | $V$ | $V$ |       |       |

and this means that $\epsilon_7$ and $\epsilon_4$ are parallel spinors on $M_7$ and $M_4$, respectively and both $M_7$ and $M_4$ are Ricci-flat manifolds. 4-dimensional Riemannian manifolds admitting parallel spinors correspond to Calabi-Yau manifolds with $SU(2)$ holonomy (and also hyperk"{a}hler manifolds with $Sp(1)$ holonomy but they are equivalent to Calabi-Yau manifolds with $SU(2)$ holonomy). 7-dimensional Ricci-flat Lorentzian manifolds admitting parallel spinors can be Minkowski spacetimes. So, fluxless case corresponds to $M_{11} \times CY_2$. For the existence of only internal flux $\lambda = 0$ and $\mu \neq 0$, we have the following equations

\[
\nabla_{X^a}\epsilon_7 = \frac{\mu}{12} e^a \epsilon_7
\]

\[
\nabla_{X^a}\epsilon_4 = \mp \frac{\mu}{6} e^a \epsilon_4
\]

(100)

and hence $\epsilon_7$ and $\epsilon_4$ correspond to geometric Killing spinors on $M_7$ and $M_4$, respectively. So, $M_7$ and $M_4$ are Einstein manifolds. The only four dimensional Riemannian manifold admitting geometric Killing spinors is the four-sphere $S^4$ and the solution in this case corresponds to $AdS_7 \times S^4$. If both internal and external fluxes are non-zero $\lambda \neq 0$ and $\mu \neq 0$ and $\phi$ satisfies $\phi \epsilon_7 = \pm \frac{1}{4} \epsilon_7$, then the supergravity Killing spinor equation decomposes into

\[
\nabla_{X^a}\epsilon_7 = \frac{1}{12} \left( \mu - \frac{\lambda}{4} \right) e^a \epsilon_7 \pm \frac{\lambda}{8} \phi e^a \epsilon_7
\]

(101)

\[
\nabla_{X^a}\epsilon_4 = \pm \frac{1}{6} \left( \frac{\lambda}{4} - \mu \right) e^a \epsilon_4.
\]

(102)

By doing similar calculations as in Section III, one can find that if $\phi$ satisfies the condition

\[
(\mathcal{L}_{X^a} \ast_7 \phi) \epsilon_7 = \mu e^a \epsilon_7
\]

(103)

then (100) transforms into a geometric Killing spinor equation and both $\epsilon_7$ and $\epsilon_4$ are geometric Killing spinors. However, this case does not give a new solution and also corresponds to $AdS_7 \times S^4$ solution. For the case of $\lambda \neq 0$ and $\mu = 0$, the field equations and Killing spinor equations give an inconsistency and hence this case does not correspond to a solution.

The decomposition of bilinear forms of supergravity Killing spinors have to be investigated separately for different choices of spinor inner products. For the choice of spinor inner product $\mathbb{R}$-skew $\xi$ on $M_{11}$, the supergravity Killing forms $(\mathcal{E})_p = \{(\mathcal{E}^{(7)}), (\mathcal{E}^{(4)})_p\}$ satisfy (41) and non-zero bilinear forms correspond to

\[
\mathcal{E}^{(7)} = (\mathcal{E}^{(7)})_1 + (\mathcal{E}^{(7)})_2 + (\mathcal{E}^{(7)})_5 + (\mathcal{E}^{(7)})_6
\]

\[
\mathcal{E}^{(4)} = (\mathcal{E}^{(4)})_1 + (\mathcal{E}^{(4)})_2
\]

on $M_7$ and $M_4$, respectively. $M_7$ is a Lorentzian 7-manifold, so the spinor space is $\mathbb{H}^4$ and the spinors are symplectic Majorana spinors. $M_4$ is a Riemannian 4-manifold, so the spinor space is $\mathbb{H} \oplus \mathbb{H}$ and the spinors are symplectic Majorana-Weyl spinors. From Table XVII in Appendix A, we have the bilinears for the chosen inner products given in Table V.

We consider four different inner product choices in the decomposition of supergravity Killing forms.

i) $M_7: \mathbb{H}^-$-sym $\xi$ and $M_4: \mathbb{H}$-swap $\xi$;
In that case, the bilinear form equations for \((\epsilon\sigma)_1\), \((\epsilon\sigma)_2\), \((\epsilon\sigma)_5\), \((\epsilon\sigma)_6\), \((\epsilon\sigma)_9\) and \((\epsilon\sigma)_{10}\) on \(M_7\) corresponds to

\[
\nabla_{X_a}(\epsilon\sigma^{(7)})_1 = \frac{\lambda}{144} \phi \frac{i}{4} X_a(\epsilon\sigma^{(7)})_6 - \frac{\lambda}{6} X_a \phi \frac{1}{2} (\epsilon\sigma^{(7)})_2
\]

\[
\nabla_{X_a}(\epsilon\sigma^{(7)})_2 = \frac{\lambda}{36} \phi \frac{i}{3} X_a(\epsilon\sigma^{(7)})_5 + \frac{\lambda}{144} e_a \phi \frac{1}{4} (\epsilon\sigma^{(7)})_5
\]

\[
\nabla_{X_a}(\epsilon\sigma^{(7)})_5 = \frac{\lambda}{6} (e_a \phi) \frac{1}{1} (\epsilon\sigma^{(7)})_2 - \frac{\lambda}{3} X_a \phi \frac{1}{2} (\epsilon\sigma^{(7)})_2
\]

\[
\nabla_{X_a}(\epsilon\sigma^{(7)})_6 = -\frac{\lambda}{3} (\epsilon\sigma^{(7)})_5 \frac{i}{1} X_a \phi + \frac{\lambda}{6} (e_a \phi) \frac{1}{2} (\epsilon\sigma^{(7)})_1 - \frac{\lambda}{12} (e_a \phi) \frac{1}{2} (\epsilon\sigma^{(7)})_5
\]

\[
0 = \frac{\lambda}{6} (e_a \phi) \frac{1}{1} (\epsilon\sigma^{(7)})_6 - \frac{\lambda}{3} X_a \phi \frac{1}{2} (\epsilon\sigma^{(7)})_6
\]

\[
0 = \frac{\lambda}{6} (e_a \phi) (\epsilon\sigma^{(7)})_5
\]

and on \(M_4\), we have

\[
\nabla_{X_a}(\epsilon\sigma^{(4)})_1 = 0
\]

\[
0 = -\frac{\mu}{3} (\epsilon\sigma^{(4)})_1 \frac{i}{1} X_a z_4.
\]

As can be seen from the last two equalities of (104) and the second equality in (105), both \(\lambda\) and \(\mu\) have to vanish, \(\lambda = \mu = 0\). Then bilinear forms are parallel and they are constructed from parallel spinors \(\epsilon_7\) and \(\epsilon_4\). Hence, this case corresponds to the fluxless case and we have \(\text{Mink} \times CY_2\) solution.

ii) \(M_7 : \mathbb{H}^\text{-sym} \xi\) and \(M_4 : \mathbb{H}^\text{-sym} \otimes \mathbb{H}^\text{-sym} \xi_8\):

For this choice, the bilinear form equations on \(M_7\) are same as in case (i) since the inner product is same. The equations on \(M_4\) are as follows

\[
\nabla_{X_a}(\epsilon\sigma^{(4)})_1 = -\frac{\mu}{6} i X_a z_4 \frac{1}{2} (\epsilon\sigma^{(4)})_2
\]

\[
\nabla_{X_a}(\epsilon\sigma^{(4)})_2 = -\frac{\mu}{3} (\epsilon\sigma^{(4)})_1 \frac{i}{1} X_a z_4.
\]

Hence, we have \(\lambda = 0\) and \(\mu \neq 0\). Since \((\epsilon\sigma^{(4)})_1\) and \((\epsilon\sigma^{(4)})_2\) are pure vector quantities from the inner product table, \(\mu\) corresponds to a real number. Moreover, from (106) and (107), we can write

\[
d((\epsilon\sigma^{(4)})_1) = -\frac{\mu}{3} z_4 \frac{1}{2} (\epsilon\sigma^{(4)})_2
\]

\[
\delta((\epsilon\sigma^{(4)})_1) = 0
\]

and

\[
d((\epsilon\sigma^{(4)})_2) = \mu((\epsilon\sigma^{(4)})_1 \frac{1}{1} z_4
\]

\[
\delta((\epsilon\sigma^{(4)})_2) = 0.
\]

So, \((\epsilon\sigma^{(4)})_1\) and \((\epsilon\sigma^{(4)})_2\) correspond to KY forms

\[
\nabla_{X_a}((\epsilon\sigma^{(4)})_1) = \frac{1}{2} i X_a d((\epsilon\sigma^{(4)})_1)
\]

\[
\nabla_{X_a}((\epsilon\sigma^{(4)})_2) = \frac{1}{3} i X_a d((\epsilon\sigma^{(4)})_2)
\]

and in fact they are special KY forms

\[
\nabla_{X_a} d((\epsilon\sigma^{(4)})_1) = \frac{2 \mu^2}{9} e_a \wedge ((\epsilon\sigma^{(4)})_1)
\]

\[
\nabla_{X_a} d((\epsilon\sigma^{(4)})_2) = \frac{\mu^2}{3} e_a \wedge ((\epsilon\sigma^{(4)})_2).
\]
Hence, this inner product choice corresponds to $AdS_7 \times S^4$ solution and the geometric Killing spinor $\epsilon_7$ generates the flux component $\phi$ which is a special CCKY form and the geometric Killing spinor $\epsilon_4$ generates the bilinear forms $(\epsilon^{(4)})_1$ and $(\epsilon^{(4)})_2$ which are special KY forms. If we choose $\mu = 0$, then the solution reduces to Mink$\times$CY$_2$ case and the bilinear forms correspond to parallel forms.

iii) $M_7 : \mathbb{H}^- \text{-sym } \xi \eta$ and $M_4 : \mathbb{H}^- \text{-swap } \xi$;

Since all the bilinear forms on $M_7$ are also nonzero in this inner product choice, the equations satisfied by the bilinear forms are the same as in the previous inner product choices. Similarly, the equations on $M_4$ are the same with case (i) because of the same inner product choice and we have $\lambda = \mu = 0$ in that case. So, $\epsilon_7$ and $\epsilon_4$ are parallel spinors and this choice corresponds to Mink$\times$CY$_2$.

iv) $M_7 : \mathbb{H}^- \text{-sym } \xi \eta$ and $M_4 : \mathbb{H}^- \text{-sym } \oplus \mathbb{H}^- \text{-sym } \xi \eta$;

The bilinear form equations in this choice are exactly the same as in case (ii) and we have $\lambda = 0$ and $\mu \neq 0$. So, the geometric Killing spinor $\epsilon_7$ generates the flux component $\phi$ and the geometric Killing spinor $\epsilon_4$ generates the bilinear forms $(\epsilon^{(4)})_1$ and $(\epsilon^{(4)})_2$ which are special KY forms. So, this case corresponds to $AdS_7 \times S^4$ and Mink$\times$CY$_2$ solutions.

As a result, for the inner product choice of $\mathbb{R}$-skew $\xi \eta$ on $M_{11}$, the relation between the solutions $AdS_7 \times S^4$ and Mink$\times$CY$_2$ and the inner product choices on $M_7$ and $M_4$ can be described as in Table VI.

Note that in the presence of $AdS$ solutions, supergravity Killing forms decompose into special KY forms. For the choice of spinor inner product $\mathbb{R}$-sym $\xi$ on $M$, the supergravity Killing forms $(\epsilon^{(7)})_p = \{(\epsilon^{(7)})_p, (\epsilon^{(4)})_p\}$ satisfy (70) and non-zero bilinear forms are

\[
\epsilon^{(7)} = (\epsilon^{(7)})_0 + (\epsilon^{(7)})_1 + (\epsilon^{(7)})_4 + (\epsilon^{(7)})_5
\]

\[
\epsilon^{(4)} = (\epsilon^{(4)})_0 + (\epsilon^{(4)})_1 + (\epsilon^{(4)})_4
\]

on $M_7$ and $M_4$, respectively. For this inner product choice on $M_{11}$, we have the properties of bilinear forms for the chosen inner products on $M_7$ and $M_4$ as given in Table VII.

We consider four different choices for the decomposition of supergravity Killing forms.

i) $M_7 : \mathbb{H}^- \text{-sym } \xi$ and $M_4 : \mathbb{H}^- \text{-swap } \xi$;
The bilinear form equations for \((\epsilon \tau)_0, (\epsilon \tau)_1, (\epsilon \tau)_4, (\epsilon \tau)_5, (\epsilon \tau)_8\) and \((\epsilon \tau)_9\) on \(M_7\) can be found as follows

\[
\nabla_{X_a} (\epsilon \tau^{(7)})_0 = \frac{\lambda}{144} \phi \wedge i_{X_a} (\epsilon \tau^{(7)})_5 \\
\nabla_{X_a} (\epsilon \tau^{(7)})_1 = \frac{\lambda}{144} (e_{a} \wedge \phi) \wedge (\epsilon \tau^{(7)})_4 + \frac{\lambda}{18} (\epsilon \tau^{(7)})_4 \wedge i_{X_a} \phi \\
\nabla_{X_a} (\epsilon \tau^{(7)})_4 = \frac{\lambda}{6} (e_{a} \wedge \phi) \wedge (\epsilon \tau^{(7)})_1 - \frac{\lambda}{36} (e_{a} \wedge \phi) \wedge 3 (\epsilon \tau^{(7)})_5 \\
\quad + \frac{\lambda}{3} (\epsilon \tau^{(7)})_1 \wedge i_{X_a} \phi + \frac{\lambda}{6} i_{X_a} \phi \wedge (\epsilon \tau^{(7)})_5 \\
\nabla_{X_a} (\epsilon \tau^{(7)})_5 = \frac{\lambda}{6} e_{a} \wedge \phi \wedge (\epsilon \tau^{(7)})_0 - \frac{\lambda}{12} (e_{a} \wedge \phi) \wedge 2 (\epsilon \tau^{(7)})_4 - \frac{\lambda}{3} (\epsilon \tau^{(7)})_4 \wedge i_{X_a} \phi \\
\quad 0 = \frac{\lambda}{6} (e_{a} \wedge \phi) \wedge 1 (\epsilon \tau^{(7)})_5 - \frac{\lambda}{3} i_{X_a} \phi \wedge (\epsilon \tau^{(7)})_5 \\
\quad 0 = \frac{\lambda}{6} e_{a} \wedge \phi \wedge (\epsilon \tau^{(7)})_4
\]

and on \(M_4\), we have

\[
\nabla_{X_a} (\epsilon \tau^{(4)})_0 = 0 \\
\nabla_{X_a} (\epsilon \tau^{(4)})_1 = \frac{\mu}{18} (\epsilon \tau^{(4)})_4 \wedge i_{X_a} z_4 \\
\nabla_{X_a} (\epsilon \tau^{(4)})_4 = \frac{\mu}{3} (\epsilon \tau^{(4)})_1 \wedge i_{X_a} z_4.
\]

The last two equalities of (113) gives \(\lambda = 0\). From (114), \((\epsilon \tau^{(4)})_0\) is a constant and from (115) and (116), one finds

\[
\begin{align*}
    d(\epsilon \tau^{(4)})_1 &= 0 \\
    \delta(\epsilon \tau^{(4)})_1 &= \frac{\mu}{18} (\epsilon \tau^{(4)})_4 \wedge z_4
\end{align*}
\]

and

\[
\begin{align*}
    d(\epsilon \tau^{(4)})_4 &= 0 \\
    \delta(\epsilon \tau^{(4)})_4 &= -\frac{\mu}{3} (\epsilon \tau^{(4)})_1 \wedge z_4.
\end{align*}
\]

So, by using the equality (49), one can see that \((\epsilon \tau^{(4)})_1\) and \((\epsilon \tau^{(4)})_4\) correspond to CCKY forms

\[
\begin{align*}
    \nabla_{X_a} (\epsilon \tau^{(4)})_1 &= -\frac{1}{3} e_{a} \wedge \delta(\epsilon \tau^{(4)})_1 \\
    \nabla_{X_a} (\epsilon \tau^{(4)})_4 &= -e_{a} \wedge \delta(\epsilon \tau^{(4)})_4.
\end{align*}
\]

Moreover, they correspond to special CCKY forms;

\[
\begin{align*}
    \nabla_{X_a} \delta(\epsilon \tau^{(4)})_1 &= -\frac{4\mu^2}{9} i_{X_a} (\epsilon \tau^{(4)})_1 \\
    \nabla_{X_a} \delta(\epsilon \tau^{(4)})_4 &= -\frac{\mu^2}{9} i_{X_a} (\epsilon \tau^{(4)})_4.
\end{align*}
\]

Hence, this inner product choice corresponds to \(AdS_7 \times S^4\) solution and the geometric Killing spinor \(\epsilon_7\) generates the flux component \(\phi\) which is a special CCKY form and the geometric Killing spinor \(\epsilon_4\) generates the bilinear forms \((\epsilon \tau^{(4)})_0\), \((\epsilon \tau^{(4)})_1\) and \((\epsilon \tau^{(4)})_4\) which are special CCKY forms. If we choose \(\mu = 0\), then the solution reduces to \(\text{Mink}_7 \times CY_2\) case and the bilinear forms correspond to parallel forms.

ii) \(M_7 : \mathbb{H}^+\)-sym \(\xi\) and \(M_4 : \mathbb{H}^+\)-sym \(\mathbb{H}^+\)-sym \(\xi\):

For that choice of inner products, the bilinear form equations on \(M_7\) are the same as in (113) and we have \(\lambda = 0\). Since, we have the same nonzero bilinears as in case (i), the bilinear form equations on \(M_4\) are also the same as in (114)-(116). However, for this choice the bilinear forms \((\epsilon \tau^{(4)})_0\) and \((\epsilon \tau^{(4)})_4\) are real quantities while \((\epsilon \tau^{(4)})_1\) is a vector quaternion. So, the equalities (115) and (116) imply that \(\mu\) must be a vector quaternion. Thus, if we choose
the flux 4-form \( F \) as a real quantity, then the consistency of (115) and (116) can only be achieved by taking \( \mu = 0 \). Hence, the only solution corresponding to this choice is Mink_7 \times CY_2.

iii) \( M_7 : \mathbb{H}^-\text{-sym} \xi \eta \) and \( M_4 : \mathbb{H}^-\text{-swap} \xi \);

Since all the bilinear forms on \( M_7 \) are also nonzero in this inner product choice, the equations satisfied by the bilinear forms are the same as in (113). Similarly, the equations on \( M_4 \) are the same with case (i) and we have \( \lambda = 0 \) and \( \mu \neq 0 \) with special CCKY forms \((\mathfrak{e}^{(4)}_0), (\mathfrak{e}^{(4)}_1)\) and \((\mathfrak{e}^{(4)}_2)\) on \( M_4 \) in that case. So, this choice corresponds to \( AdS_7 \times S^4 \) solution and for the special case of \( \mu = 0 \), we have Mink_7 \times CY_2 solution.

iv) \( M_7 : \mathbb{H}^-\text{-sym} \xi \eta \) and \( M_4 : \mathbb{H}^-\text{-sym} \oplus \mathbb{H}^-\text{-sym} \xi \);

The bilinear form equations on \( M_7 \) corresponds to (113) and the same situation as in case (ii) appears on \( M_4 \) in this choice and we have \( \lambda = \mu = 0 \). So, the parallel spinors \( \epsilon_7 \) ad \( \epsilon_4 \) generate parallel forms and we have Mink_7 \times CY_2 solution.

As a result, for the inner product choice of \( \mathbb{R}\text{-sym} \xi \) on \( M_{11} \), the relation between the solutions \( AdS_7 \times S^4 \) and Mink_7 \times CY_2 and the inner product choices on \( M_7 \) and \( M_4 \) can be described as in Table VIII.

Note that in the presence of \( AdS \) solutions, supergravity Killing forms decompose into special CCKY forms.

V. \( M_5 \times M_6, M_6 \times M_5 \) AND \( M_3 \times M_8 \) TYPE BACKGROUNDS

We can also consider different types of decompositions into product manifolds for \( M_{11} \) other than \( M_4 \times M_7 \) and \( M_7 \times M_4 \) decompositions. For example, we can investigate \( M_5 \times M_6, M_6 \times M_5 \) and \( M_3 \times M_8 \) type backgrounds. Remember that we only consider the unwarped product manifolds and in that case these types of backgrounds will not give interesting examples for the reduction of supergravity Killing form bilinears into KY and CCKY forms by choosing different spinor inner products. The reason for that is the fact that the \( AdS \) solutions can only appear for these types of backgrounds in the presence of a warp factor and in the unwarped case we do not have \( AdS \) solutions.

In \( M_5 \times M_6 \) case, we have the following decompositions

\[
\begin{align*}
e^A &= \{ e^\alpha \otimes iz_6, 1_5 \otimes e^\alpha \} \\
F &= \{ \lambda \psi, \mu \phi \} \\
\epsilon &= \epsilon_5 \otimes \epsilon_6
\end{align*}
\]

(125)

where \( \psi \) is a 4-form on \( M_5 \), \( \phi \) is a 4-form on \( M_6 \) and \( \lambda \) and \( \mu \) are constants. However, for these choices, the consistent decompositions of Maxwell-like, Einstein and supergravity Killing spinor equations can only be possible for the fluxless case \( \lambda = 0 = \mu \). Hence, in that case the solution for all types of spinor inner products is Mink_5 \times CY_3.

For \( M_6 \times M_5 \) case, the situation is similar and the only consistent decomposition corresponds to the fluxless case. However, in that case we do not have any solution since there are no five-dimensional compact Riemannian manifolds admitting parallel spinors.

In \( M_3 \times M_8 \), we have the following decompositions

\[
\begin{align*}
e^A &= \{ e^\alpha \otimes z_8, 1_3 \otimes e^\alpha \} \\
F &= \{ 0, \mu \phi \} \\
\epsilon &= \epsilon_3 \otimes \epsilon_8
\end{align*}
\]

(126)

where \( \phi \) is a 4-form on \( M_8 \). Similarly, the only consistent decomposition is in the fluxless case \( \mu = 0 \) and for all types of spinor inner products the only solution is Mink_3 \times Spin(7).
VI. REDUCTION AND LIFT OF KY AND CCKY FORMS

The existence of AdS solutions for the unwarped $M_4 \times M_7$ and $M_7 \times M_4$ type backgrounds is highly dependent on the choice of spinor inner products on product manifolds. As we have seen in sections III and IV, only some special choices of spinor inner products allow the AdS solutions. Moreover, we have shown that, for the AdS solutions, there is a relation between supergravity Killing forms on $M_{11}$ and the hidden symmetries on product manifolds. The type of hidden symmetries on product manifolds is also dependent on the choice of the spinor inner product on $M_{11}$. For the choice of $\mathbb{R}$-skew $\xi \eta$ inner product on $M_{11}$, supergravity Killing forms reduce onto special KY 1- and 2-forms on $M_4$. If one chooses $\mathbb{R}$-sym $\xi$ inner product on $M_{11}$, then the supergravity Killing forms reduce onto special CCKY 1- and 4-forms on $M_4$. These are correct for both $M_4 \times M_7$ and $M_7 \times M_4$ type backgrounds. The situation can be summarized as in Table IX.

KY and CCKY forms on product manifolds which are reduced from the supergravity Killing forms on $M_{11}$ can also be lifted to hidden symmetries on $M_{11}$. For any manifold $M$ with a product structure $M = M_m \times M_n$ and metric

$$g_{AB} = \{\bar{g}_{\alpha\beta}, \bar{g}_{\alpha\beta}\},$$

one can construct KY and CCKY forms on $M$ by using the KY and CCKY forms on $\overline{M}$. For a KY $p$-form $\overline{\omega}$ on $\overline{M}$ and a CCKY $q$-form $\overline{\nu}$ on $\overline{M}$, the following forms

$$\omega = \overline{\omega}$$
$$\nu = z_{\overline{M}} \wedge \overline{\nu}$$

are KY $p$-forms and CCKY $(m + q)$-forms on $M$, respectively. Here $z_{\overline{M}}$ is the volume form on $\overline{M}$. So, for the solutions $AdS_4 \times S^7$ and $AdS_4 \times \text{weak } G_2$, the internal component of the flux which is the 4-form $\phi$ is a CCKY 4-form and the following form

$$\nu = z_4 \wedge \phi$$

is a CCKY 8-form on $M_{11}$. For the spinor inner product $\mathbb{R}$-skew $\xi \eta$ on $M_{11}$ and the solution $AdS_7 \times S^4$, we have special KY forms $(\omega^{(4)})_1$ and $(\omega^{(4)})_2$ on $S^4$. So, we have the following KY 1- and 2-forms on $M_{11}$

$$\omega_1 = (\omega^{(4)})_1$$
$$\omega_2 = (\omega^{(4)})_2.$$  \hspace{1cm} (130)

However, these do not need to be special KY forms. For the spinor inner product $\mathbb{R}$-sym $\xi$ on $M_{11}$ and the solution $AdS_7 \times S^4$, we have the special CCKY forms $(\nu^{(4)})_1$ and $(\nu^{(4)})_4$ on $S^4$. So, we have the following CCKY 8- and 11-forms on $M_{11}$

$$\nu_1 = z_7 \wedge (\nu^{(4)})_1$$
$$\nu_2 = z_7 \wedge (\nu^{(4)})_4.$$  \hspace{1cm} (131)

Again, these do not need to be special CCKY forms. As a result, supergravity Killing forms constructed out of supergravity Killing spinors induce KY and CCKY forms on $AdS$ backgrounds of eleven-dimensional supergravity.

VII. CONCLUSION

We show that the choices of spinor inner products play a central role for the M-theory backgrounds corresponding to unwarped compactifications. Especially, the existence of $AdS$ solutions depends on the choice of some special types
of spinor inner products on product manifolds. For AdS solutions, supergravity Killing forms which are bilinear forms of supergravity Killing spinors reduce onto the hidden symmetries on product manifolds. These hidden symmetries correspond to special KY and special CCKY forms. Moreover, this reduction gives rise to the lift of hidden symmetries onto eleven-dimensional backgrounds and we find KY and CCKY forms on M-theory backgrounds. The methods leading to the relations between AdS solutions, choices of spinor inner products and reduction to hidden symmetries can be seen as a first step of a classification procedure for general string and M-theory backgrounds in terms of spinor inner products.

One can also investigate the situation for warped product compactifications of M-theory backgrounds. Obviously, the field and bilinear form equations will be different from the unwarped case since they will include the warp factor in that case. On the other hand, these investigations can also be extended into ten-dimensional string backgrounds and their dependence on the choices of spinor inner products can be determined. So, by finding the relations between solutions, spinor inner products and reduction of bilinear forms, possible classification schemes can be obtained in that way. The conformal field theory equivalent of the choice of spinor inner products can also be investigated in the framework of AdS/CFT correspondence. These may be considered as motivations for future investigations about the topic of the paper.

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Appendix A: Inner Product Classes of Spinor Spaces

In this appendix, we will give the possible inner product choices for spinor spaces in different dimensions and signatures. Let us consider the real Clifford algebra $Cl_{p,q}$ with $p$ positive and $q$ negative generators in $n = p + q$ dimensions. It is isomorphic to real, complex or quaternionic matrices as given in Table X. In the table, $D(k)$ denotes the $k \times k$ matrices with $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The even subalgebra $Cl^0_{p,q}$ of a Clifford algebra is isomorphic to a Clifford algebra in one lower dimension as follows

$$Cl^0_{p,q} \cong Cl_{q,p-1}. \quad (A1)$$

So, we can write the even subalgebras in different dimensions as in Table XI. If we define the spinor spaces as the representation spaces of even subalgebras, then we obtain the classes of spinors in different dimensions as in Table XII.

One can define different types of inner products $(\phi , \psi)$ on representation spaces of Clifford algebras. If $\psi$ and $\phi$ are elements of representation spaces of Clifford algebras, then we have

$$(\phi , \psi) = \pm (\phi , \psi)^3 \quad (A2)$$
which are called \( D^j \)-symmetric or \( D^j \)-skew inner products respectively where \( j \) denotes the identity for \( D = \mathbb{R} \), identity or complex conjugation (\( ^* \)) for \( D = \mathbb{C} \), quaternionic conjugation (\( ^{\hat{}} \)) or quaternionic reversion (\( ^\dagger \)) for \( D = \mathbb{H} \).

Moreover, for any Clifford form \( \omega \), we have the following property

\[
(\psi, \omega.\phi) = (\omega^j.\psi, \phi) \tag{A3}
\]

where \( J \) corresponds to \( \xi \) or \( \xi \eta \) involutions on the Clifford algebra and . denotes the Clifford product which is defined as in (B1) and (B2). Here \( \xi \) denotes the anti-involution acting on any \( p \)-form \( \omega \) as \( \omega^\xi = (-1)^{p/2} \omega \) and \( \eta \) is the inner automorphism acting as \( \omega^\eta = (-1)^{p/2} \omega \). \( \lfloor \cdot \rfloor \) denotes the floor function which takes the integer part of the argument. So, we have three choices for an inner product; symmetry or anti-symmetry, the involution \( J \) and the induced involution \( j \).

From the detailed analysis of Clifford algebras, one can see that there are ten different types of inner products on real Clifford algebras as in Table XIII [24]. In the table, table means that when the arguments in the inner product are reversed, their semi-spinor space is changed. The inner products induced on Clifford algebra representations in different dimensions can be listed as in Table XIV [24]. In the table, for each dimension, the first row corresponds to the inner product with \( \xi \) involution and the second row corresponds to the inner product with \( \xi \eta \) involution and the numbers in the table corresponds to the inner product classes in the table XIII. The inner product classes \( k \oplus k \) denotes \( k \)th inner product class on each semi-spinor space. The table repeats itself after dimension 7 with respect to mod 8. As the representation spaces of even subalgebras, the inner products on spinor spaces can also be obtained from Table XIV via the isomorphism \( C_{p,q}^0 \cong C_{q,p-1} \) as in Table XV.

On a spin manifold \( M \), the possible inner product choices on the spinor bundle can be determined from the Table XV. So, the manifolds that we consider throughout the text can have the spinor inner products given in Table XVI.

For any spinor field \( \epsilon \), the choice of the inner product determines the properties of the bilinear forms constructed from \( \epsilon \). For a \( p \)-form \( \omega \), we have

\[
(\epsilon, \omega.\epsilon) = \pm (\epsilon, \omega^j.\epsilon)^j \tag{A4}
\]

and if we take \( \omega \) as the \( p \)-form basis, then symmetry or antisymmetry of the inner product and the choice of involution \( J \) determine the properties of the bilinear \( p \)-form. For example, on an eleven-dimensional Lorentzian manifold \( M_{11} \) with the spinor inner product \( \mathbb{R} \)-skew \( \xi \eta \), the 3-form bilinear corresponds to

\[
(\epsilon, (e^c \wedge e^b \wedge e^a).\epsilon) = (\epsilon, (e^c \wedge e^b \wedge e^a)^\xi.\epsilon)
\]

which means that it vanishes automatically. However, for a 2-form bilinear, we have

\[
(\epsilon, (e^b \wedge e^a).\epsilon) = (\epsilon, (e^b \wedge e^a)^\xi.\epsilon)
\]

| \( p - q \) (mod 8) | \( S \) | type of spinors |
|---------------------|-------|----------------|
| 0                   | \( \mathbb{R}^{2(n-2)/2} \oplus \mathbb{R}^{2(n-2)/2} \) | Majorana-Weyl |
| 1, 7                | \( \mathbb{R}^{2(n-1)/2} \) | Majorana |
| 2, 6                | \( \mathbb{C}^{2(n-2)/2} \oplus \mathbb{C}^{2(n-2)/2} \) | Dirac-Weyl |
| 3, 5                | \( \mathbb{H}^{2(n-3)/2} \) | Symplectic Majorana |
| 4                   | \( \mathbb{H}^{2(n-4)/2} \oplus \mathbb{H}^{2(n-4)/2} \) | Symplectic Majorana-Weyl |

Table XII: Spinor spaces \( S \) and the classes of spinors for different \( p \) and \( q \) values.

Table XIII: Types of inner products for real Clifford algebras.

| \( p \) | \( q \) | \( S \) |
|--------|--------|-------|
| 1      | 6      | \( \mathbb{H}^- \)-sym |
| 2      | 7      | \( \mathbb{H} \)-sym |
| 3      | 8      | \( \mathbb{R} \)-swap |
| 4      | 9      | \( \mathbb{H} \)-swap |
| 5      | 10     | \( \mathbb{C} \)-swap |
Table XIV: The inner products induced on Clifford algebra representations for different dimensions where the rows denote the positive generators \( p \) and the columns denote the negative generators \( q \).

and hence it is nonzero. For the inner product choices of the manifolds that are considered in the text, the properties of bilinear \( p \)-forms for different form degrees can be summarized as in the Table XVII. When the induced involution \( j \) is the identity, some of the bilinear forms vanish and these are denoted by \( \times \) in the table while the non-vanishing ones are denoted by \( \checkmark \). When \( j \) is the complex conjugation, the bilinear forms are real or pure imaginary and these are denoted in the table as \( R \) and \( I \), respectively. When \( j \) is the quaternionic conjugation, the bilinear forms are real or vector quaternions and these are denoted in the table as \( R \) and \( V \), respectively. When \( j \) corresponds to the quaternionic reversion, the bilinear forms are symmetric or antisymmetric under reversion operation and these are denoted in the table as \( (+) \) and \( (-) \), respectively. The properties of the bilinear forms resulting from the decompositions of eleven-dimensional supergravity backgrounds in the main text can be deduced from this table.

**Appendix B: Clifford algebra conventions and Clifford bracket**

On the exterior bundle \( \Lambda M \) on an \( n \)-dimensional manifold \( M \), besides the wedge product \( \wedge \), one can also define the Clifford product \( \cdot \). This turns \( \Lambda M \) into a Clifford bundle \( Cl(M) \) on \( M \). Sections of \( Cl(M) \) are called Clifford forms. On the Clifford bundle, the coframe basis \( \{ e^a \} \) satisfy the following Clifford algebra identity

\[
e^a \cdot e^b + e^b \cdot e^a = 2 g^{ab}.
\]  

where \( g^{ab} \) is the inverse metric. The Clifford product can be written in terms of the wedge product and interior derivative. For any \( p \)-form \( \omega \), we have the following identities

\[
e^a \cdot \omega = e^a \wedge \omega + i_X \omega
\]

\[
\omega \cdot e^a = e^a \wedge \eta \omega - i_X \eta \omega
\]  

(B2)
and similarly the wedge product and interior derivative can be written in terms of the Clifford product as

\[
e^a \wedge \omega = \frac{1}{2} (e^a \omega + \eta \omega \cdot e^a)
\]

\[
i_{X} \omega = \frac{1}{2} (e^a \omega - \eta \omega \cdot e^a).
\]

From these equalities, one can also deduce that

\[
e^a \omega \cdot e_a = (n - 2p)\eta \omega.
\]
Table XVII: The properties of bilinear $p$-forms for the inner product choices of the manifolds that are considered in the text where the columns correspond to the value of $p$.

For any two Clifford forms $\alpha$ and $\beta$ which correspond to inhomogeneous differential forms, one can write the Clifford product from (B2) as in the following form

$$\alpha \beta = \sum_{k=0}^{n} \frac{(-1)^{[k/2]}}{k!} (\eta^k i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \alpha) \wedge (i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \beta). \quad (B5)$$

Moreover, we can also define the Clifford commutator $[.,.]_C$ as

$$[\alpha, \beta]_C = \alpha \beta - \beta \alpha \quad (B6)$$

and from (B5), it can be written as

$$[\alpha, \beta]_C = \sum_{k=0}^{n} \frac{(-1)^{[k/2]}}{k!} \left[ (\eta^k i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \alpha) \wedge (i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \beta) 
- (\eta^k i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \beta) \wedge (i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \alpha) \right]. \quad (B7)$$

To write it in a more compact form, we define the contracted wedge product

$$\alpha \wedge_k \beta = i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \alpha \wedge i_{X_{a_1}} i_{X_{a_2}} \ldots i_{X_{a_k}} \beta \quad (B8)$$

and (B7) turns into

$$[\alpha, \beta]_C = \sum_{k=0}^{n} \frac{(-1)^{[k/2]}}{k!} \left[ \eta^k \alpha \wedge_k \beta - \eta^k \beta \wedge_k \alpha \right]. \quad (B9)$$

For example, if we consider the special case where $\alpha$ is a 2-form and $\beta$ arbitrary, then the right hand side of the Clifford commutator only has one nonzero term and we have

$$[\alpha, \beta]_C = -2\alpha \wedge_1 \beta. \quad (B10)$$
If we apply the projection operator \( (\cdot)_p \) to the Clifford commutator, that is \( ([\alpha, \beta]_{\text{Cl}})_p \), then it gives only the \( p \)-form part of the right hand side of (B9). Similarly, we can define the Clifford anticommutator as

\[
[\alpha, \beta]_+ = \alpha \beta + \beta \alpha
\]

and from (B5), it can be written as

\[
[\alpha, \beta]_+ = \sum_{k=0}^{n} \frac{(-1)^{\lfloor k/2 \rfloor}}{k!} \left[ \eta^k \alpha \wedge \beta + \eta^k \beta \wedge \alpha \right].
\]

**Appendix C: Killing-Yano forms**

Killing vector fields correspond to the symmetries of a manifold and the antisymmetric generalizations of them to the higher-degree differential forms are Killing-Yano (KY) forms which are called the hidden symmetries of the manifold. A \( p \)-form \( \omega \) is a KY \( p \)-form if it satisfies the following equation

\[
\nabla_X \omega = \frac{1}{p+1} i_X d\omega
\]

for any vector field \( X \). Similarly, conformal Killing vector fields can also be generalized to higher-degree differential forms and those are called conformal Killing-Yano (CKY) forms. A \( p \)-form \( \omega \) is a CKY \( p \)-form on an \( n \)-dimensional manifold \( M \), if it satisfies the following equation

\[
\nabla_X \omega = \frac{1}{p+1} i_X d\omega - \frac{1}{n-p+1} \bar{X} \wedge \delta \omega
\]

for any vector field \( X \). So, KY forms correspond to coclosed CKY forms satisfying \( \delta \omega = 0 \). Another subset of CKY forms satisfying \( d\omega = 0 \) are called CCKY forms and hence they are solutions of the following equation

\[
\nabla_X \omega = -\frac{1}{n-p+1} \bar{X} \wedge \delta \omega.
\]

We can also define special subsets of the spaces of KY and CCKY forms. A KY \( p \)-form \( \omega \) is called a special KY \( p \)-form if it satisfies the following condition

\[
\nabla_X d\omega = -c(p+1) \bar{X} \wedge \omega
\]

for a constant \( c \) [20]. Similarly, a CCKY \( p \)-form \( \omega \) is called a special CCKY \( p \)-form if it satisfies the following condition

\[
\nabla_X \delta \omega = c(n-p+1) i_X \omega.
\]

The importance of the special KY and CCKY forms is the fact that they have to be generated from geometric Killing spinors as bilinear forms [15]. Non-special KY and CCKY forms cannot be generated by geometric Killing spinors.

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[27] In some papers, the sign convention for the Clifford algebra is chosen as $e^a.e^b + e^b.e^a = -2g^{ab}$ and the real and imaginary Killing spinors appear on $M_7$ and $M_4$, respectively which is reverse to our sign convention $e^a.e^b + e^b.e^a = 2g^{ab}$ which gives real and imaginary Killing spinors on $M_4$ and $M_7$, respectively.