The mapping class group orbits in the framings of compact surfaces

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Abstract

We compute the mapping class group orbits in the homotopy set of framings of a compact connected oriented surface with non-empty boundary. In the case \( g \geq 2 \) the computation is some modification of Johnson’s results \cite{8,9} and certain arguments on the Arf invariant, while we need an extra invariant for the genus 1 case. In addition, we discuss how this invariant behaves in the relative case, which Randal-Williams \cite{14} studied for \( g \geq 2 \).

Introduction

Let \( \Sigma \) be a compact connected oriented smooth \((C^\infty)\) surface with non-empty boundary. Then the tangent bundle \( T\Sigma \) is a trivial bundle. Its orientation-preserving global trivializations \( T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2 \) are called framings of the surface \( \Sigma \), which play important roles in surface topology. The mod2 reduction of a framing can be regarded as a spin structure on the surface \( \Sigma \). A spin structure on a closed surface is called a theta characteristic in a classical context, and the mapping class group orbits in the set of theta characteristics are described by the Arf invariant \cite{3}.

We denote by \( F(\Sigma) \) the set of homotopy classes of framings of \( \Sigma \), and fix a Riemannian metric \( \| \cdot \| \) on the tangent bundle \( \pi : T\Sigma \to \Sigma \). The unit tangent bundle \( U\Sigma := \{ e \in T\Sigma; \| e \| = 1 \} \xrightarrow{\cong} \Sigma \) is a principal \( S^1 \) bundle over \( \Sigma \). A framing defines a continuous map \( U\Sigma \to S^1 \) whose restriction to each fiber is homotopic to the identity \( 1_{S^1} \). Taking the pull-back of the positive generator of \( H^1(S^1; \mathbb{Z}) \), we obtain an element of \( H^1(U\Sigma; \mathbb{Z}) \). This defines a natural embedding \( F(\Sigma) \hookrightarrow H^1(U\Sigma; \mathbb{Z}) \). More precisely, \( F(\Sigma) \) is an affine set modeled by the abelian group \( \mathbb{Z} \times H^1(\Sigma; \mathbb{Z}) \) (See \S 2.1). In particular, the difference \( f_1 - f_0 \) of two framings \( f_0 \) and \( f_1 \in F(\Sigma) \) defines a unique element of \( H^1(\Sigma; \mathbb{Z}) \).

In this paper we consider the mapping class group of \( \Sigma \) fixing the boundary pointwise

\[
\mathcal{M}(\Sigma) := \pi_0 \text{Diff}_+(\Sigma, \text{id on } \partial \Sigma) = \text{Diff}_+(\Sigma, \text{id on } \partial \Sigma)/\text{isotopy},
\]

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which acts on the set $F(\Sigma)$ from the right in a natural way. If we fix an element $f_0 \in F(\Sigma)$, then the map

$$k(f_0) : M(\Sigma) \to H^1(\Sigma; \mathbb{Z}), \quad \varphi \mapsto f_0 \circ \varphi - f_0,$$

is a twisted cocycle of the group $M(\Sigma)$. The cohomology class $k := [k(f_0)] \in H^1(M(\Sigma); H^1(\Sigma; \mathbb{Z}))$ does not depend on the choice of $f_0$, is called the Earle class [6] or the Chillingworth class [4] [15], and generates the cohomology group in the case when the boundary $\partial \Sigma$ is connected and the genus of $\Sigma$ is greater than 1 [11]. For the case where the boundary is not connected, see [10] Theorem 1.A. The construction of $k$ stated here is due to M. Furuta [13] § 4. The Morita trace [12] and its refinement, the Enomoto-Satoh trace [7], are higher analogues of the class $k$. In the author’s joint paper with Alekseev, Kuno and Naef [1], we clarify topological and Lie theoretical meanings of the Enomoto-Satoh trace. The formalism problem of a variant of the Turaev cobracket for an immersed loop on the surface, the Enomoto-Satoh trace and the Kashiwara-Vergne problem in Lie theory are closely related to each other. We need the rotation number of the immersed loop with respect to a framing to define this variant of the Turaev cobracket. This is the reason why we describe the orbit set $F(\Sigma)/M(\Sigma)$ in this paper.

The homotopy set $F(\Sigma)$ we study in this paper is absolute, namely, we allow framings to move on the boundary. In fact, the rotation number of an immersed loop with respect to a framing is invariant under any moves of $f$ on the boundary $\partial \Sigma$. On the other hand, we can consider a relative version of the homotopy set $F(\Sigma, \delta)$ for a fixed framing on the boundary $\delta : T\Sigma|_{\partial \Sigma} \to \partial \Sigma \times \mathbb{R}^2$. Here we make framings on $\partial \Sigma$ equal the given datum $\delta$. We need the latter version to define the rotation number of an arc connecting two boundary components. Randal-Williams [14] computes the mapping class group orbits in the set of ($r$-)spin structures for any genus in the relative version and those in the homotopy set $F(\Sigma, \delta)$ for $g \geq 2$. It is interesting that the (generalized) Arf invariant is defined in any $F(\Sigma, \delta)$ [14], while it is not defined in some absolute cases as in §1 of this paper. In particular, the computations in this paper are different from those by Randal-Williams [14]. In the case $g \geq 2$, the formality of the Turaev cobracket holds good for any choice of a framing. But, if $g = 1$, it depends on the choice of a framing, so that the formality problem is reduced to the computation of the mapping class group orbits in the set $F(\Sigma)$. It is controlled by an extra invariant $\tilde{A}(f)$ introduced in this paper (Corollary 2.10). All these results are proved in [2].

Anyway, following Whitney [16], we consider the rotation number $\text{rot}_f(\ell) \in \mathbb{Z}$ of a smooth immersion $\ell : S^1 \to \Sigma$ with respect to a framing $f \in F(\Sigma)$. We number the boundary components as $\partial \Sigma = \bigsqcup_{j=0}^n \partial_j \Sigma$. The rotation numbers $\text{rot}_f(\partial_j \Sigma)$, $0 \leq j \leq n$, are invariant under the action of the group $M(\Sigma)$. Here we endow each $\partial_j \Sigma$ with the orientation induced by $\Sigma$. By the Poincaré-Hopf theorem (Lemma 2.3), we have

$$\sum_{j=1}^n \text{rot}_f(\partial_j \Sigma) = \chi(\Sigma) = 1 - 2g - n.$$

Our description of the orbit set $F(\Sigma)/M(\Sigma)$ depends on the genus $g(\Sigma)$ of the surface $\Sigma$. First we consider the case $g(\Sigma) = 0$. Clearly we have
Lemma 0.1 (Equation (18)). Suppose $g(\Sigma) = 0$. Then two framings $f_1$ and $f_2 \in F(\Sigma)$ are homotopic to each other, if and only if

$$\text{rot}_{f_1}(\partial_j \Sigma) = \text{rot}_{f_2}(\partial_j \Sigma)$$

for any $0 \leq j \leq n$.

Next we discuss the positive genus case: $g = g(\Sigma) \geq 1$. Choose a system of simple closed curves $\{\alpha_i, \beta_i\}_{i=1}^g$ on $\Sigma$ as in Figure 1. The Arf invariant of the mod2 reduction of $f$ is defined in the case where all the numbers $\text{rot}_f(\partial_j \Sigma)$, $0 \leq j \leq n$, are odd. Then the Arf invariant of the spin structure is defined by

$$\text{Arf}(f) \equiv \sum_{i=1}^g (\text{rot}_f(\alpha_i) + 1)(\text{rot}_f(\beta_i) + 1) \pmod{2}$$

In the case $g(\Sigma) \geq 2$, we have the following.

Theorem 0.2 (Theorem 2.5). Suppose $g(\Sigma) \geq 2$, and $f_1, f_2 \in F(\Sigma)$. Then $f_1$ and $f_2$ belong to the same $\mathcal{M}(\Sigma)$-orbit, if and only if

(i) $\text{rot}_{f_1}(\partial_j \Sigma) = \text{rot}_{f_2}(\partial_j \Sigma)$ for any $0 \leq j \leq n$.

(ii) If all the numbers $\text{rot}_{f_1}(\partial_j \Sigma) = \text{rot}_{f_2}(\partial_j \Sigma)$, $0 \leq j \leq n$, are odd, then $\text{Arf}(f_1) = \text{Arf}(f_2)$.

The proof given in §2.2 is some modification of Johnson’s arguments [8, 9].

The genus 1 case is different from the others. We need to introduce an invariant $\tilde{A}(f) \in \mathbb{Z}_{\geq 0}$ for $f \in F(\Sigma)$. It is defined to be the generator of the ideal in $\mathbb{Z}$ generated by the set $\{\text{rot}_f(\gamma); \gamma$ is a non-separating simple closed curve on $\Sigma\}$. We have

$$\text{Arf}(f) \equiv \tilde{A}(f) + 1 \pmod{2}$$

On the other hand, if $g \geq 2$, we have $\tilde{A}(f) = 1$ for any $f \in F(\Sigma)$ (Lemma 2.4).
Theorem 0.3 (Theorem 2.8). Suppose $g(\Sigma) = 1$, and $f_1, f_2 \in F(\Sigma)$. Then $f_1$ and $f_2$ belong to the same $\mathcal{M}(\Sigma)$-orbit, if and only if

(i) $\text{rot}_{f_1}(\partial_j \Sigma) = \text{rot}_{f_2}(\partial_j \Sigma)$ for any $0 \leq j \leq n$.

(ii) $\tilde{A}(f_1) = \tilde{A}(f_2) \in \mathbb{Z}_{\geq 0}$.

For the sake of non-experts on topology who are interested only in the Kashiwara-Vergne problem, this paper is self-contained except the results by Johnson [8] and §2.4. In particular, we will give an elementary proof of the Poincaré-Hopf theorem on the surface $\Sigma$ (Lemma 2.3). In §1, following Johnson [8], we study the mapping class orbits in the set of spin structures on any compact surface $\Sigma$ with non-empty boundary. Generalities on framings are discussed in §2.1. Our computation for the case $g(\Sigma) \geq 2$ in §2.2 is some modification of Johnson’s paper [9]. We need some extra invariant $\tilde{A}(f)$ for the case $g(\Sigma) = 1$ in §2.3. It is introduced in the end of §2.1. In §2.4, we prove that the invariant $\tilde{A}(f)$ and the generalized Arf invariant introduced in [14] classify the mapping class orbits in the relative genus 1 case (Theorem 2.11).

In this paper we denote by $H_1(-)$ and $H^1(-)$ the first $\mathbb{Z}$-(co)homology groups, and by $H_1(-)^{(2)}$ and $H^1(-)^{(2)}$ the first $\mathbb{Z}/2$-(co)homology groups. On $H_1(\Sigma)$ and $H_1(\Sigma)^{(2)}$, we have the (algebraic) intersection forms $\cdot : H_1(\Sigma)^\otimes 2 \to \mathbb{Z}$, $a \otimes b \mapsto a \cdot b$, and $\cdot : (H_1(\Sigma)^{(2)})^\otimes 2 \to \mathbb{Z}/2$, $a \otimes b \mapsto a \cdot b$.

By the classification of surfaces, any compact connected oriented smooth surface $\Sigma$ is classified by the genus and the number of the boundary components. We denote by $\Sigma_{g,n+1}$ a compact connected oriented smooth surface of genus $g$ with $n+1$ boundary components for $g,n \geq 0$. It is uniquely determined up to diffeomorphism. Throughout this paper, we fix a system of simple closed curves $\{\alpha_i, \beta_i\}_{i=1}^g$ on the surface $\Sigma_{g,n+1}$ shown in Figure 1. By $\Sigma_{g,0}$ we mean a compact connected oriented surface of genus $g$.

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1 Spin structures

In this section, following Johnson [8], we compute the mapping class group orbits in the set of spin structures on any compact connected oriented surface \( \Sigma \) with non-empty boundary \( \partial \Sigma \).

A spin structure on \( \Sigma \) is, by definition, an unramified double covering of the unit tangent bundle \( U\Sigma \) whose restriction to each fiber is non-trivial. In a natural way, the set (of isomorphism classes) of such double coverings is isomorphic to the complement \( H^1(U\Sigma)^{(2)} \setminus H^1(\Sigma)^{(2)} \) in the exact sequence

\[
0 \to H^1(\Sigma)^{(2)} \xrightarrow{\varpi^*} H^1(U\Sigma)^{(2)} \xrightarrow{\iota^*} \mathbb{Z}/2 \to 0
\]  

associated with the fibration \( S^1 \xrightarrow{i} U\Sigma \xrightarrow{\varpi} \Sigma \). Here we identify \( H^1(\Sigma)^{(2)} \) with its image under \( \varpi^* \). The canonical lifting \( H^1(\Sigma)^{(2)} \to H^1(U\Sigma)^{(2)}, a \mapsto \tilde{a} \), \( a \mapsto \tilde{a} \) is constructed in the same way as the original one for a closed surface by Johnson [8]. In particular, if \( \gamma : \bigsqcup_{i=1}^m S^1 \to \Sigma \) is a smooth embedding, then we have

\[
[\tilde{\gamma}] = \tilde{\gamma} + m_\gamma(1) \in H^1(U\Sigma)^{(2)},
\]

where \( \tilde{\gamma} : \bigsqcup_{i=1}^m S^1 \to U\Sigma \) is the (normalized) velocity vector of \( \gamma \), and \( \iota^* \) is the dual of \( \varpi^* \) in the sequence (1). As was shown in Theorem 1B in [8], we have \( \tilde{a} + b + (a \cdot b) \iota^*(1) \) for any \( a, b \in H^1(\Sigma)^{(2)} \). For any \( \xi \) in the complement \( H^1(U\Sigma)^{(2)} \setminus H^1(\Sigma)^{(2)} \), a quadratic form \( \omega_\xi : H^1(\Sigma)^{(2)} \to \mathbb{Z}/2 \) is defined by \( \omega_\xi(a) := \langle \xi, \tilde{a} \rangle \in \mathbb{Z}/2 \) for any \( a \in H^1(\Sigma)^{(2)} \). By a quadratic form we mean a function \( H^1(\Sigma)^{(2)} \to \mathbb{Z}/2 \) satisfying \( \omega(a + b) = \omega(a) + \omega(b) + a \cdot b \) for any \( a \) and \( b \in H^1(\Sigma)^{(2)} \). We denote by \( \text{Quad}(\Sigma) \) the set of quadratic forms on \( H^1(\Sigma)^{(2)} \). We remark \( \omega_2 - \omega_1 : H^1(\Sigma)^{(2)} \to \mathbb{Z}/2 \) is a homomorphism, so that it can be regarded as an element of \( H^1(\Sigma)^{(2)} \) for any \( \omega_1 \) and \( \omega_2 \in \text{Quad}(\Sigma) \). More precisely, the group \( H^1(\Sigma)^{(2)} \) acts on the set \( \text{Quad}(\Sigma) \) freely and transitively, i.e., the set \( \text{Quad}(\Sigma) \) is an affine set modeled by the abelian group \( H^1(\Sigma)^{(2)} \).

The mapping class group \( \mathcal{M}(\Sigma) \) acts on the sets \( H^1(U\Sigma)^{(2)} \setminus H^1(\Sigma)^{(2)} \) and \( \text{Quad}(\Sigma) \) in a natural way. The map \( \xi \mapsto \omega_\xi \) defines an \( \mathcal{M}(\Sigma) \)-equivariant isomorphism between the sets \( H^1(U\Sigma)^{(2)} \setminus H^1(\Sigma)^{(2)} \) and \( \text{Quad}(\Sigma) \).

For the rest of this section we compute the mapping class group orbits in the set of quadratic forms, \( \text{Quad}(\Sigma) \). We begin by recalling some elementary facts on the (co)homology of the surface \( \Sigma \). The cohomology exact sequence

\[
H^1(\Sigma, \partial\Sigma)^{(2)} \xrightarrow{j^*} H^1(\Sigma)^{(2)} \xrightarrow{\iota^*} H^1(\partial\Sigma)^{(2)}
\]
is compatible with the action of the mapping class group $\mathcal{M}(\Sigma)$. In particular, the subgroup $\text{im} \ j^* = \ker i^* \subset H^1(\Sigma)(2)$ is stable under the action of $\mathcal{M}(\Sigma)$, and equals the image of the map $H_1(\Sigma)(2) \to H^1(\Sigma)(2)$, $x \mapsto \omega x$, from the Poincaré-Lefschetz duality.

**Lemma 1.1.** Any homology class in $H_1(\Sigma)(2)$ is represented by a simple closed curve.

**Proof.** The four elements in $H_1(\Sigma_{1,0})(2)$ are represented by simple closed curves. Similarly all elements in $H_1(\Sigma_{0,n+1})(2)$ are represented by simple closed curves. Any element in $H_1(\Sigma_{g,n+1})(2)$ can be represented by the connected sum of some of these elements. This proves the lemma.

For any $a \in H_1(\Sigma)(2)$ we introduce a map $T_a : H_1(\Sigma)(2) \to H_1(\Sigma)(2)$ defined by $x \mapsto x - (a \cdot x)a$. If $\gamma$ represents the element $a$, the map $T_a$ is induced by the right-handed Dehn twist along $\gamma$ denoted by $t_\gamma \in \mathcal{M}(\Sigma)$. In particular, $T_a$ respects the intersection form. We denote by $G(\Sigma) \subset \text{Aut}(H_1(\Sigma)(2))$ the subgroup generated by $\{T_a; a \in H_1(\Sigma)(2)\}$. From the Dehn-Lickorish theorem and Lemma 1.1 it equals the image of the mapping class group $\mathcal{M}(\Sigma)$ in the group $\text{Aut}(H_1(\Sigma)(2))$. In particular, the $\mathcal{M}(\Sigma)$-orbits in the set $\text{Quad}(\Sigma)$ are the same as the $G(\Sigma)$-orbits.

For a quadratic form $\omega : H_1(\Sigma)(2) \to \mathbb{Z}/2$, we define a map $m_\omega : G(\Sigma) \to H^1(\Sigma)(2)$ by $S \mapsto m_\omega(S) := \omega S - \omega$. Then we have

$$m_\omega(S_1S_2) = m_\omega(S_1)S_2 + m_\omega(S_2) \quad (5)$$

for any $S_1$ and $S_2 \in G(\Sigma)$. One can compute $(m_\omega(T_a), x) = \omega(T_a x) - \omega(x) = \omega(x - (x \cdot a)a) - \omega(x) = (x \cdot a)\omega(a) + (x \cdot a)^2 = (x \cdot a)(\omega(a) + 1)$ for $a, x \in H_1(\Sigma)(2)$. This means

$$m_\omega(T_a) = (\omega(a) + 1)a. \in \text{im} j^* \subset H^1(\Sigma)(2) \quad (6)$$

Hence we obtain a 1-cocycle $m_\omega : G(\Sigma) \to \text{im} j^* (\subset H^1(\Sigma)(2))$.

**Theorem 1.2.** Let $\omega_1$ and $\omega_2 : H_1(\Sigma)(2) \to \mathbb{Z}/2$ be quadratic forms. Then $\omega_1$ and $\omega_2$ belong to the same $\mathcal{M}(\Sigma)$-orbit if and only if

$$\exists x \in H_1(\Sigma)(2) \text{ s.t. } \omega_1(x) = 0, \quad \omega_2 - \omega_1 = x \cdot \exists \im j^* \quad (\exists)$$

**Proof.** We denote by $\omega_1 \sim \omega_2$ the assertion that $\omega_1$ and $\omega_2$ satisfy the condition $(\exists)$, and begin the proof by checking that the relation $\sim$ is an equivalence relation on the set $\text{Quad}(\Sigma)$. The reflexivity $\omega \sim \omega$ follows from $\omega(0) = 0$. If $x \in H_1(\Sigma)(2)$ satisfies $\omega_1(x) = 0$, then we have $(\omega + x \cdot \cdot (x) = \omega_1(x) + x \cdot x = 0$, which proves the symmetry: $(\omega_1 \sim \omega_2)$ implies $(\omega_2 \sim \omega_1)$. Assume $\omega_1 \sim \omega_2$ and $\omega_2 \sim \omega_3$. This means there exist $x_1$ and $x_2 \in H_1(\Sigma)(2)$ such that $\omega_1(x_1) = \omega_2(x_2) = 0$, $\omega_2 - \omega_1 = x_1$, and $\omega_3 - \omega_2 = x_2$. Then we have $\omega_3 - \omega_1 = (x_1 + x_2)$, and $\omega_1(x_1 + x_2) = \omega_1(x_1) + x_1 \cdot x_2 + \omega_1(x_2) = \omega_1(x_1) + \omega_2(x_2) = 0$. Hence we obtain $\omega_1 \sim \omega_3$. This proves the transitivity.

Next we assume $\omega_2 = \omega_1 T_a$ for some $a \in H_1(\Sigma)(2)$. Then, by the formula $(\exists)$, we have $\omega_2 - \omega_1 = m_\omega(T_a) = (\omega_1(a) + 1)a$, while $\omega_1((\omega_1(a) + 1)a) = (\omega_1(a) + 1)\omega_1(a) = 0$. This implies $\omega_1 \sim \omega_1 T_a$. The relation $\sim$ is an equivalence relation, and $G(\Sigma)$ is generated by $T_a$'s. Hence, if $\omega_1$ and $\omega_2$ belong to the same $G(\Sigma)$-orbit, then we have $\omega_1 \sim \omega_2$. 

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In the last case, the two orbits are distinguished by the Arf invariant $Z$.

For any $\omega$ and $\omega$ for any $G(\Sigma)$ in $\text{Aut}(\Sigma)$, we have the intersection form vanishes on $\omega$ and $\omega$. Hence we have the restriction map

$$i^* : \text{Quad}(\Sigma) \to H^1(\partial\Sigma)^{(2)}$$

(7)

The kernel ker $i_*$ is spanned by the $\mathbb{Z}/2$-fundamental class $[\partial\Sigma]^2 \in H_1(\partial\Sigma)^{(2)}$. Hence, if $h \in H^1(\partial\Sigma)^{(2)}$ satisfies $\iota[h][\partial\Sigma]^2 = 0$, then it induces a homomorphism on $i_*H_1(\partial\Sigma)^{(2)}$, and extended to the element of $H^1(\Sigma)^{(2)}$ satisfying $h([\alpha_i]) = h([\beta_i]) = 0$ for any $1 \leq i \leq g$. Here $\alpha_i$ and $\beta_i$ are the simple closed curves shown in Figure 1. Moreover we define a map $\omega^{0,h} : H_1(\Sigma)^{(2)} \to \mathbb{Z}/2$ by

$$\omega^{0,h}(x) := \sum_{i=1}^{g} (x \cdot [\alpha_i])(x \cdot [\beta_i]) + h(x)$$

for $x \in H_1(\Sigma)^{(2)}$. It is easy to check $\omega^{0,h}$ is a quadratic form, and $i^*\omega^{0,h} = h$. If a quadratic form $\omega \in \text{Quad}(\Sigma)$ satisfies $i^*\omega = 0 \in H^1(\partial\Sigma)^{(2)}$, then the Arf invariant $\text{Arf}(\omega)$ is defined by

$$\text{Arf}(\omega) := \sum_{i=1}^{g} \omega([\alpha_i])\omega([\beta_i]) \in \mathbb{Z}/2$$

(9)

For any $x \in H_1(\Sigma)^{(2)}$, we have

$$\text{Arf}(\omega^{0,0} + x) = \sum_{i=1}^{g} (x \cdot [\alpha_i])(x \cdot [\beta_i]) = \omega^{0,0}(x).$$

(10)

In particular, the Arf invariant $\text{Arf}$ is $G(\Sigma)$-invariant, namely, we have $\text{Arf}(\omega S) = \text{Arf}(\omega)$ for any $\omega \in (i^*)^{-1}(0)$ and $S \in G(\Sigma)$. In fact, there are $x_0$ and $x_1 \in H_1(\Sigma)^{(2)}$ such that $\omega = \omega^{0,0} + x_0$, $\omega S - \omega = x_1$, and $\omega(x_1) = 0$. Then we have $\text{Arf}(\omega S) = \omega^{0,0}(x_0 + x_1) = \omega^{0,0}(x_0) + x_0 \cdot x_1 + \omega^{0,0}(x_1) = \text{Arf}(\omega) + \omega(x_1) = \text{Arf}(\omega)$.

Now recall $m_\omega(G(\Sigma)) \subset \ker(i^* : H_1(\Sigma)^{(2)} \to H^1(\partial\Sigma)^{(2)})$ and $G(\Sigma)$ is the image of $\mathcal{M}(\Sigma)$ in $\text{Aut}(H_1(\Sigma)^{(2)})$. Hence the restriction map $i^*$ induces the map

$$\rho_2 : \text{Quad}(\Sigma)/\mathcal{M}(\Sigma) \to H^1(\partial\Sigma)^{(2)}, \quad \omega \mod G(\Sigma) \mapsto i^*\omega.$$  

(11)

**Theorem 1.3.** For any $h \in H^1(\partial\Sigma)^{(2)}$, the cardinality of the set $\rho_2^{-1}(h)$ is given by

$$\#\rho_2^{-1}(h) = \begin{cases} 0, & \text{if } h[\partial\Sigma]^2 \neq 0, \\
1, & \text{if } h[\partial\Sigma]^2 = 0 \text{ and } (h \neq 0 \text{ or } g = 0), \\
2, & \text{if } h = 0 \text{ and } g \geq 1. \end{cases}$$

In the last case, the two orbits are distinguished by the Arf invariant $\text{Arf} : (i^*)^{-1}(0) \to \mathbb{Z}/2$. 

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Proof. (0) If $h[\partial \Sigma]_2 \neq 0$, we have $(i^*)^{-1}(h) = \emptyset$ since $i_*[\partial \Sigma]_2 = 0$.

(1) Suppose $h[\partial \Sigma]_2 = 0$ and $g = 0$. Then $(i^*)^{-1}(h) = \{\omega^{0,h}\}$ is a one-point set.

Next suppose $h[\partial \Sigma]_2 = 0$, $h \neq 0$ and $g \geq 1$. Then $\omega^{0,h} \in (i^*)^{-1}(h) \neq \emptyset$. For any $\omega \in (i^*)^{-1}(h)$ we have $\omega - \omega^{0,h} \in \ker i^* \cong \text{im} j^*$, so that $\omega - \omega^{0,h} = x_0 \cdot \in H^1(\Sigma)^{(2)}$ for some $x_0 \in H_1(\Sigma)^{(2)}$. Since $h \neq 0$, we have $\omega(x_0) = h(x_1)$ for some $x_1 \in H_1(\partial \Sigma)^{(2)}$. Then $(x_0+x_1) \cdot = x_0 \cdot = \omega - \omega^{0,h}$ and $\omega(x_0)+x_0 \cdot x_1 = \omega(x_0)+x_0 \cdot x_1 = h(x_1)+0+h(x_1) = 0$.

By Theorem 1.2 we have $\omega = \omega^{0,h}S$ for some $S \in G(\Sigma)$. This proves $\sharp \rho_2^{-1}(h) = 1$.

(2) Suppose $h = 0$ and $g \geq 1$. Then $\omega^{0,0} \in (i^*)^{-1}(0) \neq \emptyset$, and we have $\omega^{0,0}(x_0) = 1$ for some $x_0 \in H_1(\Sigma)^{(2)}$. For any $\omega \in (i^*)^{-1}(0)$ there exists some $x \in H_1(\Sigma)^{(2)}$ such that $\omega - \omega^{0,0} = x \cdot \in H^1(\Sigma)^{(2)}$. If $\omega^{0,0}(x) = \text{Arf}(\omega) = 0$, then, by Theorem 2.2 we have $\omega = \omega^{0,0}S$ for some $S \in G(\Sigma)$. On the other hand, if $\omega^{0,0}(x) = \text{Arf}(\omega) = 1$, then we have $\omega - (\omega^{0,0} + x_0 \cdot ) = (x-x_0)$ and $(\omega^{0,0} + x_0 \cdot )(x-x_0) = \omega^{0,0}(x-x_0) + x_0 \cdot x = \omega^{0,0}(x) - \omega^{0,0}(x_0) = 0$. This implies $\omega = (\omega^{0,0} + x_0 \cdot )S$ for some $S \in G(\Sigma)$. This proves $\sharp \rho_2^{-1}(0) = 2$.

This completes the proof of the theorem. □

As was proved by Randal-Williams in [13] Theorem 2.9, the cardinality of the mapping class group orbit sets in the set of spin structures for the relative version is always 2, and does not depend on the boundary value. In particular, the (generalized) Arf invariant can be defined in any cases. The situation is similar for framings in the case $g \geq 2$ (Theorem 2.3).

2 Framings

2.1 Generalities

Let $\Sigma$ be a compact connected oriented smooth surface with non-empty boundary as before. In this paper, we denote by $F(\Sigma)$ the set of homotopy classes of framings, i.e., orientation-preserving global trivializations $T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$ of the tangent bundle $T\Sigma$.

In this paper, the composite of such an trivialization and the second projection, $T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2 \xrightarrow{pr_2} \mathbb{R}^2$, is also called a framing. The group $[\Sigma, S^1] = H^1(\Sigma) = H^1(\Sigma; \mathbb{Z})$ acts on the set $F(\Sigma)$ freely and transitively. In fact, the difference of any two framings gives a continuous map $\Sigma \to GL^+(2; \mathbb{R}) \cong S^1$. The mapping class group $M(\Sigma)$ acts on the set $F(\Sigma)$ from the right in a natural way.

Consider the inclusion map $i : S^1 \hookrightarrow U\Sigma$ and the projection $\pi : U\Sigma \to \Sigma$ as in the preceding section. Then we have $M(\Sigma)$-equivariant exact sequences

\begin{align}
0 & \to \mathbb{Z} \xrightarrow{i_*} H_1(U\Sigma) \xrightarrow{\pi_*} H^1(\Sigma) \to 0, \quad \text{and} \quad (12) \\
0 & \to H^1(\Sigma) \xrightarrow{\pi^*} H^1(U\Sigma) \xrightarrow{i^*} \mathbb{Z} \to 0 \quad \text{(13)}
\end{align}

in the integral (co)homology. The group $H^1(\Sigma)$ obviously acts on the inverse image $(i^*)^{-1}(1)$ of $1 \in \mathbb{Z}$ freely and transitively. For a framing $f \in F(\Sigma)$ we denote by $\xi(f) \in H^1(U\Sigma)$ the pull-back of the positive generator of $H^1(S^1)$ by the map $f : U\Sigma \to S^1$. It is clear that $i^*\xi(f) = 1 \in \mathbb{Z}$. Then the map $F(\Sigma) \to (i^*)^{-1}(1), f \mapsto \xi(f)$, is
equivariant under the actions of the groups $\mathcal{M}(\Sigma)$ and $H^1(\Sigma)$. In particular, it is an $\mathcal{M}(\Sigma)$-equivariant isomorphism $F(\Sigma) \cong (\iota^*)^{-1}(1)$, by which we identify these two sets with each other.

An immersion $\ell : S^1 \to \Sigma$ lifts to its (normalized) velocity vector $\vec{\ell} : S^1 \to U\Sigma$, $t \mapsto \ell(t)/\|\ell(t)\|$. The rotation number of $\ell$ with respect to a framing $f$ is defined by

$$\text{rot}_f \ell := \langle \xi(f), [\vec{\ell}] \rangle = \deg(f \circ \vec{\ell} : S^1 \to S^1) \in \mathbb{Z}$$

(14)

Lemma 2.1. For any $\varphi \in \mathcal{M}(\Sigma)$ we have

$$\text{rot}_{f \varphi} \ell = \text{rot}_f (\varphi \circ \ell).$$

(15)

**Lemma 2.2.** For any smooth embedding $\ell : S^1 \to \Sigma$, we have

$$\omega_f ([\ell]) = \text{rot}_f (\ell) + 1 \in \mathbb{Z}/2.$$

**Proof.** Recall the canonical lifting in [8] is given by $\tilde{\ell} = \ell + \iota_*(1) \in H_1(U\Sigma)^{(2)}$. Hence we have

$$\omega_f ([\ell]) = \langle \xi_2(f), [\tilde{\ell}] \rangle = \langle \xi_2(f), [\tilde{\ell}] \rangle + 1 = \text{rot}_f (\ell) + 1 \in \mathbb{Z}/2.$$

This proves the lemma.

The following is a straight-forward consequence of the Poincaré-Hopf theorem. But we will give its elementary proof for the convenience of non-experts on topology.

**Lemma 2.3.** Let $S \subset \Sigma$ be a compact smooth subsurface. We number the boundary components of $S$: $\partial S = \bigsqcup_{k=1}^N \partial_k S$. Then we have

$$\sum_{k=1}^N \text{rot}_f (\partial_k S) = \chi(S)$$

for any $f \in F(\Sigma)$. Here we endow each $\partial_k S$ with the orientation induced by $S$, and $\chi(S)$ is the Euler characteristic of the surface $S$. 

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Proof. Let \( \{(e_\lambda, \varphi_\lambda : D^{n_\lambda} \to S)\}_{\lambda \in \Lambda} \) be a finite cell decomposition of the surface \( S \) such that each characteristic map \( \varphi_\lambda : D^{n_\lambda} \to \overline{e_\lambda} \subset S \) is a smooth embedding and each 0-cell is located on the boundary \( \partial S \). We denote \( C_i := \# \{ \lambda \in \Lambda : n_\lambda = i \} \), \( 0 \leq i \leq 2 \), so that \( \chi(S) = C_2 - C_1 + C_0 \). Then we compute the sum \( \sum_{n_\lambda=2} \text{rot}_f(\varphi_\lambda(\partial D^{n_\lambda})) \). Since the loop \( \varphi_\lambda(\partial D^2) \) is regular homotopic to a small loop around the center of \( e_\lambda \), the sum equals \( C_2 \). The contribution of both sides of each interior 1-cell cancel each other, while the contribution of the boundary 1-cells equals the sum \( \sum_{N} \text{rot}_f(\partial k S) \). The contribution of a vertex, i.e., a 0-cell \( e_\lambda \) equals \( \frac{1}{2}(d_\lambda - 2) \), where \( d_\lambda \) is the valency at the vertex \( e_\lambda \). See Figure 2. On the other hand, we have \( C_1 = \frac{1}{2} \sum_{n_\lambda=0} d_\lambda \). Hence we obtain

\[
C_2 = \left( \sum_{k=1}^{N} \text{rot}_f(\partial_k S) \right) + \frac{1}{2} \sum_{n_\lambda=0} (d_\lambda - 2) = \left( \sum_{k=1}^{N} \text{rot}_f(\partial_k S) \right) + C_1 - C_0,
\]

which proves the lemma.

Figure 2: the case \( d_\lambda = 5 \)

Now suppose \( \Sigma = \Sigma_{g,n+1} \) for \( g,n \geq 0 \). We number the boundary components: \( \partial \Sigma = \coprod_{i=0}^{n} \partial_i \Sigma \). Since any element of the group \( \mathcal{M}(\Sigma) \) fixes the boundary pointwise, we can define a map

\[
\rho : F(\Sigma) / \mathcal{M}(\Sigma) \to \mathbb{Z}_{n+1}, \quad f \mod \mathcal{M}(\Sigma) \mapsto (\text{rot}_f(\partial_i \Sigma) + 1)^{n}_{i=0}.
\]

Here, taking Lemma \( 2.2 \) into account, we consider \( \text{rot}_f(\partial_i \Sigma) + 1 \) instead of the rotation number itself. By Lemmas \( 2.1 \) and \( 2.3 \) we have

\[
\text{im} \rho = \{(\nu_i)_{i=0}^{n} \in \mathbb{Z}^{n+1} ; \sum_{i=0}^{n} \nu_i = 2 - 2g \}.
\]

In the genus 0 case, i.e., \( \Sigma = \Sigma_{0,n+1} \), these lemmas imply

\[
F(\Sigma) / \mathcal{M}(\Sigma) = F(\Sigma) \cong \{(\nu_i)_{i=0}^{n} \in \mathbb{Z}^{n+1} ; \sum_{i=0}^{n} \nu_i = 2 \}.
\]

We conclude this subsection by introducing an extra invariant for a framing, which will be used for the genus 1 case. For \( f \in F(\Sigma) \) we consider the ideal \( a(f) \in \mathbb{Z} \) generated by the set \( \{ \text{rot}_f(\gamma) ; \gamma \text{ is a non-separating simple closed curve in } \Sigma \} \), and define \( \tilde{A}(f) \in \mathbb{Z}_{\geq 0} \) to be the non-negative generator of the ideal \( a(f) \). It is clear that these are invariants under the action of the mapping class group \( \mathcal{M}(\Sigma) \). But, if \( g \geq 2 \), they are trivial invariants.
Lemma 2.4. If \( g \geq 2 \), we have \( \tilde{A}(f) = 1 \) for any \( f \in F(\Sigma_{g,n+1}) \).

Proof. From the assumption, there is a smooth compact subsurface \( P \subset \Sigma \) diffeomorphic to a pair of pants \( \Sigma_{0,3} \) such that each of the three boundary components \( \partial_i P, 0 \leq i \leq 2 \), is a non-separating curve in \( \Sigma \). Then, from Lemma 2.3, we have \( \text{rot}_f(\partial_0 P) + \text{rot}_f(\partial_1 P) + \text{rot}_f(\partial_2 P) = \chi(P) = -1 \), so that \(-1 \in \mathfrak{a}(f)\). This proves the theorem.

2.2 The case \( g \geq 2 \)

In this subsection we consider \( \Sigma = \Sigma_{g,n+1} \) for the case \( g \geq 2 \). In this case our computation modifies that in [9]. Consider the map \( \rho: F(\Sigma)/\mathcal{M}(\Sigma) \to \mathbb{Z}^{n+1} \) in (16).

Theorem 2.5. Suppose \( g \geq 2 \). Then, for any \( \nu \in \text{im} \rho = \{ (\nu_i)_{i=0}^n \in \mathbb{Z}^{n+1}; \sum_{i=0}^n \nu_i = 2 - 2g \} \), we have \( \sharp \rho^{-1}(\nu) = \begin{cases} 1, & \text{if } \nu \in \text{im} \rho \setminus (2\mathbb{Z})^{n+1}, \\ 2, & \text{if } \nu \in \text{im} \rho \cap (2\mathbb{Z})^{n+1}. \end{cases} \)

In the latter case, the two orbits are distinguished by the Arf invariant of the spin structure \( \xi_2(f) \).

Proof. Let \( f_1 \) and \( f_2 \in F(\Sigma) \) satisfy \( \rho(f_1) = \rho(f_2) \) and \( \text{Arf}(\xi_2(f_1)) = \text{Arf}(\xi_2(f_2)) \) if \( \rho(f_1) = \rho(f_2) \in (2\mathbb{Z})^{n+1} \). Then, by Theorem 1.3, we have

\[
\xi(f_2) - \xi(f_0) = 2\left( \sum_{i=1}^g \lambda_i [\alpha_i] + \sum_{i=1}^g \mu_i [\beta_i] \right) \cdot \in H^1(\Sigma) \tag{19}
\]

for some \( \varphi_0 \in \mathcal{M}(\Sigma) \) and \( \lambda_i, \mu_i \in \mathbb{Z} \). Here \( \alpha_i \) and \( \beta_i \) are the simple closed curves shown in Figure 1. Hence it suffices to construct \( \varphi'_i \) and \( \varphi''_i \in \mathcal{M}(\Sigma) \) for each \( 1 \leq i \leq g \) such that

\[
\xi(f \varphi'_i) - \xi(f) = 2[\alpha_i] \cdot \quad \text{and} \quad \xi(f \varphi''_i) - \xi(f) = 2[\beta_i]; \tag{20}
\]

for any \( f \in F(\Sigma) \). We denote by \( t_\gamma \in \mathcal{M}(\Sigma) \) the right-handed Dehn twist along a simple closed curve \( \gamma \) in \( \Sigma \).

Now from the assumption \( g \geq 2 \) there exist simple closed curves \( \hat{\alpha}_i \) and \( \hat{\beta}_i \) satisfying the conditions

1. \( \alpha_i \) and \( \hat{\alpha}_i \) bound a smooth compact subsurface diffeomorphic to \( \Sigma_{1,2} \).
2. \( \beta_i \) and \( \hat{\beta}_i \) bound a smooth compact subsurface diffeomorphic to \( \Sigma_{1,2} \).
3. \( \hat{\alpha}_i \) and \( \hat{\beta}_i \) are disjoint from \( \{ \alpha_k, \beta_k \}_{k \neq i} \).
4. \( \hat{\alpha}_i \) intersects with \( \beta_i \) transversely at a unique point.
5. \( \hat{\beta}_i \) intersects with \( \alpha_i \) transversely at a unique point.
Choose a point on each component of $\partial \Sigma_{1,2}$. Then, by the disk theorem, two simple arcs connecting these two chosen points are mapped to each other by the action of the group $\mathcal{M}(\Sigma_{1,2})$. Similar transitivity holds also for the surface $\Sigma_{g-2,n+3}$. Hence, by the classification theorem of surfaces, the quadruples $(\Sigma, \alpha_i, \beta_i)$ and $(\Sigma, \beta_i, \hat{\beta}_i, \alpha_i)$ are diffeomorphic to $(\Sigma, \gamma_1, \gamma_2, \gamma_0)$ in Figure 3 (a). Then the simple closed curve $t_{\gamma_2}^{-1}t_{\gamma_1}(\gamma_0)$ is computed as in Figure 3 (b), so that $\gamma_0$ and $t_{\gamma_2}^{-1}t_{\gamma_1}(\gamma_0)$ bound a smooth compact subsurface diffeomorphic to $\Sigma_{1,2}$ By Lemma 2.6, we have

$$\left| \text{rot}_{t_{\gamma_2}^{-1}t_{\gamma_1}}(\gamma_0) - \text{rot}_f(\gamma_0) \right| = |\chi(\Sigma_{1,2})| = 2$$

for any $f \in F(\Sigma)$. It is clear that $\text{rot}_{t_{\gamma_2}^{-1}t_{\gamma_1}}(\gamma_1) = \text{rot}_f(\gamma_1)$. The mapping class $t_{\gamma_2}^{-1}t_{\gamma_1}$ is just a BP-map in $\mathcal{M}$.

Hence, if we take $\varphi_i'$ to be $t_{\alpha_i}^{-1}t_{\alpha_i}$ or its inverse, then $\text{rot}_{t_{\alpha_i}^{-1}t_{\alpha_i}}(\beta_i) - \text{rot}_f(\beta_i) = 2$ and $\text{rot}_{t_{\alpha_i}^{-1}t_{\alpha_i}}(\alpha_i) - \text{rot}_f(\alpha_i) = 0$. From the condition (ii) above, $\text{rot}_{t_{\alpha_i}^{-1}t_{\alpha_i}}(\alpha_k) - \text{rot}_f(\alpha_k) = 0$ and $\text{rot}_{t_{\alpha_i}^{-1}t_{\alpha_i}}(\beta_k) - \text{rot}_f(\beta_k) = 0$ for $k \neq i$. Hence $\xi(f, \varphi_i') - \xi(f) = 2[\alpha_i]$, as desired in (20). Similarly, if we take $\varphi_i''$ to be $t_{\beta_i}^{-1}t_{\beta_i}$ or its inverse, then $\varphi_i''$ satisfies (20). This proves the theorem.

\section*{2.3 The genus 1 case}

Finally we study the genus 1 case: $\Sigma = \Sigma_{1,n+1}$. We write simply $\alpha = \alpha_1$ and $\beta = \beta_1$ shown in Figure 1, $\nu_j = \nu_j(f) := \text{rot}_f(\partial_j \Sigma) + 1$, $0 \leq j \leq n$, and take a closed regular neighbourhood $\Sigma'$ of the subset $\alpha(S^1) \cup \beta(S^1)$. It is diffeomorphic to $\Sigma_{1,1}$. We begin by computing the invariant $\tilde{A}(f)$ for $f \in F(\Sigma)$.

\begin{lemma}
The ideal in $\mathbb{Z}$ generated by the set $\{\text{rot}_f(\alpha), \text{rot}_f(\beta), \nu_j(f); 0 \leq j \leq n\}$ equals the ideal $\tilde{a}(f)$. In other words, $\tilde{A}(f)$ is the non-negative greatest common divisor of the set.
\end{lemma}

\begin{proof}
We denote the ideal given above by $b(f)$. For each $0 \leq j \leq n$, we choose a band connecting $\alpha$ and $\partial_j \Sigma$ to obtain a non-separating simple closed curve $\alpha^{(j)}$ such that $\alpha$, $\partial_j \Sigma$ and $\alpha^{(j)}$ bound a pair of pants. Then we have $\text{rot}_f(\alpha^{(j)}) = \text{rot}_f(\alpha) + \nu_j$, so that we obtain $b(f) \subset \tilde{a}(f)$.

Let $\gamma$ be any non-separating simple closed curve in $\Sigma$. When the curve $\gamma$ crosses the boundary component $\partial_j \Sigma$, the rotation number changes by $\pm(\text{rot}_f(\partial_j \Sigma) + 1) = \pm \nu_j$.
Hence there exists a non-separating simple closed curve \( \gamma' \) in \( \Sigma' \) such that \( \text{rot}_f(\gamma) - \text{rot}_f(\gamma') \in b(f) \). The curve \( \gamma' \) is mapped to \( \alpha \) by an element of the subgroup generated by the Dehn twists \( t_\alpha \) and \( t_\beta \). For any simple closed curve \( \gamma'' \) in \( \Sigma \), we have

\[
\text{rot}_f(t_\beta(\gamma'')) - \text{rot}_f(\gamma'') = ([\gamma''] \cdot [\beta])\text{rot}_f(\beta) \in b(f) \tag{21}
\]

and \( \text{rot}_f(t_\alpha(\gamma'')) - \text{rot}_f(\gamma'') \in b(f) \). Hence we have \( \text{rot}_f(\gamma') \in \text{rot}_f(\alpha) + b(f) = b(f) \). This proves \( a(f) \subset b(f) \), and completes the proof of the lemma. \( \square \)

Corollary 2.7. If \( \text{rot}_f(\partial_j \Sigma) \) is odd for any \( 0 \leq j \leq n \), we have

\[ \text{Arf}(\xi_2(f)) \equiv \tilde{A}(f) + 1 \pmod{2}. \]

Proof. By Lemma 2.2 we have \( \text{Arf}(\xi_2(f)) \equiv (\text{rot}_f(\alpha) + 1)(\text{rot}_f(\beta) + 1) \pmod{2} \). \( \square \)

Theorem 2.8. Suppose \( g = 1 \), and \( f_1, f_2 \in F(\Sigma_{1,n+1}) \). Then \( f_1 \) and \( f_2 \) belong to the same \( \mathcal{M}(\Sigma_{1,n+1}) \)-orbit, if and only if \( f_1 \) and \( f_2 \) satisfy both of the following conditions

(i) \( \text{rot}_{f_1}(\partial_j \Sigma) = \text{rot}_{f_2}(\partial_j \Sigma) \) for any \( 0 \leq j \leq n \).

(ii) \( \tilde{A}(f_1) = \tilde{A}(f_2) \in \mathbb{Z}_{\geq 0} \).

Proof. If \( f_1 \) and \( f_2 \) belong to the same \( \mathcal{M}(\Sigma) \)-orbit, then it is clear that they satisfy both of the conditions. Hence it suffices to prove the following: For any \( f \in F(\Sigma) \) we have \( (\text{rot}_{f,\varphi}(\alpha), \text{rot}_{f,\varphi}(\beta)) = (\tilde{A}(f), 0) \in \mathbb{Z}^2 \) for some \( \varphi \in \mathcal{M}(\Sigma) \).

From the formula (21) and the similar one for \( t_\alpha \), the actions of \( t_\alpha \) and \( t_\beta \) on the row vectors \( (\text{rot}_f(\alpha), \text{rot}_f(\beta)) \in \mathbb{Z}^2 \) generate the standard right action of \( SL_2(\mathbb{Z}) \) on \( \mathbb{Z}^2 \). By the Euclidean algorithm, the vectors \( (a_1, b_1) \) and \( (a_2, b_2) \) in \( \mathbb{Z}^2 \) belong to the same \( SL_2(\mathbb{Z}) \)-orbit if and only if \( \gcd(a_1, b_1) = \gcd(a_2, b_2) \in \mathbb{Z} \).

We denote \( d := \gcd(\text{rot}_f(\alpha), \text{rot}_f(\beta)) \) and \( c := \gcd(\nu_j(f); 0 \leq j \leq n) \). Then \( \tilde{A}(f) = \gcd(c, d) \). By the Euclidean algorithm, we have \( (\text{rot}_{f,\varphi}(\alpha), \text{rot}_{f,\varphi}(\beta)) = (d, 0) \) for some \( \varphi_1 \in \mathcal{M}(\Sigma) \). Recall the non-separating simple closed curve \( \alpha^{(j)} \) introduced in the proof of Lemma 2.6. For any \( f' \in F(\Sigma) \) we have

\[
\text{rot}_f'(t_\alpha^{-1}t_{\alpha^{(j)}}(\alpha)) = \text{rot}_f'(\alpha), \quad \text{and}
\]

\[
\text{rot}_f'(t_\alpha^{-1}t_{\alpha^{(j)}}(\beta)) = \text{rot}_f'(t_{\alpha^{(j)}}(\beta)) + ([\beta] \cdot [\alpha])\text{rot}_f'(\alpha)
\]

\[
= \text{rot}_f'(\beta) - ([\beta] \cdot [\alpha])\text{rot}_f'(\alpha^{(j)}) + ([\beta] \cdot [\alpha])\text{rot}_f'(\alpha)
\]

\[
= \text{rot}_f'(\beta) - \nu_j(f^j). \]

Hence there exists an element \( \varphi_2 \) in the subgroup generated by the elements \( t_\alpha^{-1}t_{\alpha^{(j)}} \), \( 0 \leq j \leq n \), such that \( (\text{rot}_{f,\varphi_1\varphi_2}(\alpha), \text{rot}_{f,\varphi_1\varphi_2}(\beta)) = (d, c) \). Recall \( \tilde{A}(f) = \gcd(c, d) \). By the Euclidean algorithm, we have \( (\text{rot}_{f,\varphi_1\varphi_2\varphi_3}(\alpha), \text{rot}_{f,\varphi_1\varphi_2\varphi_3}(\beta)) = (\tilde{A}(f), 0) \) for some \( \varphi_3 \in \mathcal{M}(\Sigma) \). This proves the theorem. \( \square \)

Corollary 2.9. For \( \nu = (\nu_j)_{j=0}^n \in \mathbb{Z}^{n+1} \setminus \{0\} \) with \( \sum_{j=0}^n \nu_j = 0 \), the inverse image \( \rho^{-1}(\nu) \) is parametrized by the positive divisors of \( \gcd(\nu_j; 0 \leq j \leq n) \), while \( \rho^{-1}(0) \) by the non-negative integers \( \mathbb{Z}_{\geq 0} \).
Proof. If $\nu \neq 0$, then $\tilde{A}(f)$ is a positive divisor of the $\text{gcd}$. The corollary follows from Lemma 2.4. See also the equation (17).

The following is related to the formality problem of the Turaev cobracket on genus 1 surfaces [1].

**Corollary 2.10.** For $f \in F(\Sigma_{1,n+1})$, there exists a mapping class $\varphi \in \mathcal{M}(\Sigma_{1,n+1})$ satisfying $\text{rot}_{f,\varphi}(\alpha) = \text{rot}_{f,\varphi}(\beta) = 0$, if and only if $\tilde{A}(f) = \text{gcd}(\nu_j; 0 \leq j \leq n)$.

**Proof.** By Lemma 2.11 there exists a framing $f_\bullet \in F(\Sigma_{1,n+1})$ such that $\text{rot}_{f_\bullet}(\alpha) = \text{rot}_{f_\bullet}(\beta) = 0$ and $\nu_j(f_\bullet) = \nu_j(f)$ for any $0 \leq j \leq n$. Then $\tilde{A}(f_\bullet) = \text{gcd}(\nu_j; 0 \leq j \leq n)$ from Lemma 2.6. Hence the corollary follows from Theorem 2.8.

2.4 The relative genus 1 case

We conclude this paper by some discussion about the relative version [14], which we will need to describe to the self-intersection of an immersed path. Here we fix a framing of the tangent bundle restricted to the boundary $\delta : T\Sigma|_{\partial \Sigma} \xrightarrow{\cong} \partial \Sigma \times \mathbb{R}^2$, and consider the set $F(\Sigma, \delta)$ of homotopy classes of framings $f : T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$ which extend the framing $\delta$, where all the homotopies we consider fix $\delta$ pointwise. By Lemma 2.1 and some obstruction theory, the set $F(\Sigma, \delta)$ is not empty if and only if $\sum_{j=0}^n \text{rot}(\partial_j \Sigma) = \chi(\Sigma)$. For the rest, we assume $F(\Sigma, \delta) \neq \emptyset$. In this setting, for any $f \in F(\Sigma, \delta)$, we can consider the rotation number $\text{rot}_f(\ell) \in \mathbb{R}$ of an immersed path $\ell$ connecting two different points on the boundary $\partial \Sigma$. We denote by $1 \in S^1$ the unit element of $S^1 = SO(2)$. The group $\{([\delta], \partial_1 \Sigma) \in (S^1, 1)\} = H^1(\Sigma, \partial \Sigma; \mathbb{Z}) = H^1(\Sigma, \partial \Sigma)$ acts on the set $F(\Sigma, \delta)$ freely and transitively. For $1 \leq j \leq n$, we choose a point $*j \in \partial_j \Sigma$ and a simple arc $\eta_j$ from a point on $\partial_0 \Sigma$ to $*j$ such that each $\eta_j$ is disjoint from $\{[\alpha_i, \beta_i] \}_{i=1}^g \cup \{[\eta_k] \}_{k \neq j}$, and transverse to $\partial_0 \Sigma$ and $\partial_j \Sigma$. Then the homology classes $\{[\alpha_i, \beta_i] \}_{i=1}^g \cup \{[\eta_j] \}_{j=1}^n$ constitute a free basis of $H_1(\Sigma, \partial \Sigma)$. The evaluation map

$$\text{Ev} : F(\Sigma, \delta) \to \mathbb{Z}^{2g+n}, \quad f \mapsto (\text{rot}_f(\alpha_i), \text{rot}_f(\beta_i))_{i=1}^g, ([\text{rot}_f(\eta_j)])_{j=1}^n$$

(22)

is bijective, and compatible with the action of $H^1(\Sigma, \partial \Sigma)$. Here $\text{rot}_f(\eta_j) \in \mathbb{Z}$ is the ceiling of the rotation number $\text{rot}_f(\eta_j) \in \mathbb{R}$. Randal-Williams [14] introduced the generalized Arf invariant $\tilde{\text{Arf}}(f) \in \mathbb{Z}/2$ by

$$\tilde{\text{Arf}}(f) := \sum_{i=1}^g (\text{rot}_f(\alpha_i) + 1)(\text{rot}_f(\beta_i) + 1) + \sum_{j=1}^n \nu_j [\text{rot}_f(\eta_j)] \mod 2 \in \mathbb{Z}/2,$$

(23)

which is denoted by $A(f)$ in the original paper [14]. The mapping class group $\mathcal{M}(\Sigma)$ acts on the set $F(\Sigma, \delta)$ in a natural way. As was proved in [14], the generalized Arf invariant is invariant under the mapping class group action for any $g \geq 0$, and, if $g \geq 2$, the orbit set $F(\Sigma, \delta)/\mathcal{M}(\Sigma)$ is of cardinality 2 or 0 for any $\delta$, and described by the generalized Arf invariant.

Now we consider the case $g = 1$. We use the notation in §2.3. The invariant $\tilde{A}(f)$ is related to the generalized Arf invariant $\tilde{\text{Arf}}(f)$ as follows.

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(1) Suppose \( \tilde{A}(f) \) is even. Then \( \text{rot}_f(\alpha), \text{rot}_f(\beta) \) and all of \( \nu_j \)'s are even. Hence 
\[
\tilde{A}r_f(f) \equiv (\text{rot}_f(\alpha) + 1)(\text{rot}_f(\beta) + 1) \equiv 1 \text{ mod } 2.
\]
If \( f_1 \in F(\Sigma, \delta) \) is given by \( \text{Ev}(f_1) = ((\tilde{A}(f), 0), (0, \ldots, 0)) \), then we have \( \tilde{A}(f_1) = \tilde{A}(f) \).

(2) Next we consider the case \( \tilde{A}(f) \) is odd and \( \tilde{A}r_f(f) \equiv 0 \text{ mod } 2 \). If \( f_2 \in F(\Sigma, \delta) \) is given by \( \text{Ev}(f_2) = ((\tilde{A}(f), 0), (0, \ldots, 0)) \), then we have \( \tilde{A}(f_2) = \tilde{A}(f) \) and \( \tilde{A}r_f(f_2) \equiv 0 \text{ mod } 2 \).

(3) Finally assume \( \tilde{A}(f) \) is even and \( \tilde{A}r_f(f) \equiv 1 \text{ mod } 2 \). Then we have \( \nu_j \equiv 1 \text{ (mod 2)} \) for some \( 1 \leq j \leq n \). If not, \( \text{rot}_f(\alpha) \) or \( \text{rot}_f(\beta) \) are odd, so that \( \tilde{A}r_f(f) \equiv 0 \text{ mod } 2 \). This contradicts the assumption. Let \( j_0 \) be the maximum \( j \) satisfying \( \nu_j \equiv 1 \) (mod 2). If \( f_3 \in F(\Sigma, \delta) \) is given by \( \text{Ev}(f_3) = ((\tilde{A}(f), 0), (0, \ldots, 0, 1, 0, \ldots, 0)) \), then we have \( \tilde{A}(f_3) = \tilde{A}(f) \) and \( \tilde{A}r_f(f_3) \equiv 1 \text{ mod } 2 \).

From Lemma 2.6 the invariant \( \tilde{A}(f) \) can be realized to be any non-negative divisor of \( \gcd(\nu_j; 0 \leq j \leq n) \). Here we agree that any integer is a divisor of 0.

**Theorem 2.11.** Suppose \( g = 1 \) and \( F(\Sigma, \delta) \neq \emptyset \). Then the orbit set \( F(\Sigma, \delta)/M(\Sigma) \) is parametrized by the invariant \( \tilde{A}(f) \) and the generalized Arf invariant \( \tilde{A}r_f(f) \). More precisely, for any \( f \in F(\Sigma, \delta) \), we have \( f = f_k \circ \phi \) for some \( \phi \in M(\Sigma) \) and \( k = 1, 2, 3 \).

Here we choose \( f_k \) according to the invariants \( \tilde{A}(f) \) and \( \tilde{A}r_f(f) \) as stated above.

**Proof.** We may assume each \( \eta_j \) is disjoint from the subsurface \( \Sigma' = (\cong \Sigma_{1,1}) \), a regular neighborhood of \( \alpha(S^1) \cup \beta(S^1) \). There is an element \( \tau \in M(\Sigma) \) whose support is in \( \Sigma' \) such that \( (\text{rot}_f(\alpha), \text{rot}_f(\beta)) = (-\text{rot}_f(\alpha), -\text{rot}_f(\beta)) \) for any \( f \in F(\Sigma, \delta) \). In fact, \( \tau \) can be obtained as some product of Dehn twists \( t_\alpha \) and \( t_\beta \). In particular, we have \( \text{rot}_f(\eta_j) = \text{rot}_f(\eta_j) \) for any \( 0 \leq j \leq n \).

Next we consider a framing \( f \in F(\Sigma, \delta) \), which satisfies \( \text{Ev}(f) = ((A, 0), (\rho_1, \ldots, \rho_n)) \) for some \( \rho_j \in \mathbb{Z} \). Here we assume \( A = \tilde{A}(f) \). We remark that \( A \) divides any \( \nu_j, 0 \leq j \leq n \). Recall the non-separating simple closed curve \( \alpha^{(j)} \) introduced in the proof of Lemma 2.6. Here we choose the band connecting \( \alpha \) and \( \partial_j \Sigma \) to be disjoint from any \( \eta_k, 1 \leq k \leq n \). Then, the curve \( \alpha^{(j)} \) is disjoint from \( \eta_k \) for \( k \neq j \), and we may assume \( \alpha^{(j)} \) and \( \eta_k \) intersect transversely to each other at the unique point. We define \( \psi_j := t_{\alpha^{(j)}} t_{\alpha^{-1}(-\nu_j/A)\partial_j \Sigma}^{-1} \in M(\Sigma) \). Since \( \text{rot}_f(\alpha^{(j)}) = A + \nu_j \) and \( \text{rot}_f(\partial_j \Sigma) = \nu_j - 1 \), we have \( \text{rot}_{f \circ \psi_j}(\eta_j) - \text{rot}_{f}(\eta_j) = -A - \nu_j + \nu_j - 1 = -A - 1 \) and \( \text{rot}_{f \circ \psi_j}(\beta) - \text{rot}_f(\beta) = A + \nu_j - (1 + (\nu_j/A))A = 0 \). Clearly we have \( \text{rot}_{f \circ \psi_j}(\alpha) = \text{rot}_f(\alpha) = A \) and \( \text{rot}_{f \circ \psi_j}(\eta_k) = \text{rot}_f(\eta_k) = 0 \) for \( k \neq j \). Moreover we define \( \psi'_j := \tau t_{\alpha^{(j)}} t_{\alpha^{-1}(-\nu_j/A)\partial_j \Sigma}^{-1} \tau^{-1} \in M(\Sigma) \). Similarly we have \( \text{rot}_{f \circ \psi'_j}(\eta_j) - \text{rot}_f(\eta_j) = -A - 1, \text{rot}_{f \circ \psi'_j}(\alpha) = \text{rot}_f(\alpha) = A, \text{rot}_{f \circ \psi'_j}(\beta) = \text{rot}_f(\beta) \) and \( \text{rot}_{f \circ \psi'_j}(\eta_k) - \text{rot}_f(\eta_k) = 0 \) for \( k \neq j \). As a consequence of the construction of \( \psi_j \) and \( \psi'_j \), there is some \( \varphi_j \in M(\Sigma) \) and \( \epsilon_j \in \{0, 1\} \) such that \( \text{Ev}(f \circ \varphi_j) = ((A, 0), (\rho_1, \ldots, \rho_{j-1}, \epsilon_j, \rho_{j+1}, \ldots, \rho_n)) \) and \( \epsilon_j \equiv \rho_j \text{ (mod 2)} \). In fact, \( \gcd\{-A - 1, A - 1\} \) divides 2.

Now we consider an arbitrary element \( f_0 \in F(\Sigma, \delta) \). We denote \( A = \tilde{A}(f_0) \). From the proof of Theorem 2.3 we have \( \text{Ev}(f_0 \circ \varphi_0) = ((A, 0), (\rho_1^0, \ldots, \rho_n^0)) \) for some \( \varphi_0 \in M(\Sigma) \) and \( \rho_j^0 \in \mathbb{Z} \).
(1) Suppose $A = \tilde{A}(f)$ is even. Then we may assume each $\rho^0_j$ is even. In fact, $
abla \cdot \tilde{A}(f) = \nabla \cdot A$. Hence we have some suitable product $\tilde{\varphi} \in \mathcal{M}(\Sigma)$ of $\varphi_j \in \mathcal{M}(\Sigma)$’s stated above such that $Ev(f_0 \circ \varphi_0 \circ \tilde{\varphi}) = ((A,0),(0,\ldots,0))$. This means $f_0 \circ \varphi_0 \circ \tilde{\varphi} = f_1 \in F(\Sigma,\delta)$, as was desired.

(2) Assume $A = \tilde{A}(f)$ is odd and $\tilde{Arf}(f) = 0 \pmod{2}$. Then we have $0 = \tilde{Arf}(f_0) = \tilde{Arf}(f_0 \circ \varphi_0) \equiv A + 1 + \sum_{j=1}^n \nu_j \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\eta_j) \right] \equiv \sum_{j=1}^n \nu_j \left[ \nabla \cdot A(\eta_j) \right] \equiv 1 \pmod{2}$ and $\nu_j \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\eta_j) \right] \equiv 0 \pmod{2}$ if $j \notin \{j_1,j_2,\ldots,j_{2m}\}$. We choose a band connecting $\partial j_1(\Sigma)$ and $\partial j_2(\Sigma)$ disjoint from $\alpha, \beta$ and $\gamma_k$ for $k \neq j_1,j_2$ to obtain a separating simple closed curve $\lambda$ such that $\partial j_1(\Sigma) \cup \partial j_2(\Sigma)$ and $\lambda$ bound a pair of pants. Then $\nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\lambda) = \nu_{j_1} + \nu_{j_2} - 1$ is odd. Hence we have $\left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0 \circ \varphi')(\eta_{j_1}) \right] \equiv \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0 \circ \varphi')(\eta_{j_2}) \right] \equiv 0 \pmod{2}$. By similar consideration we obtain some $\varphi' \in \mathcal{M}(\Sigma)$ such that $\left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0 \circ \varphi')(\eta_j) \right] \equiv 0 \pmod{2}$ for any $1 \leq j \leq n$. Hence we have some suitable product $\tilde{\varphi} \in \mathcal{M}(\Sigma)$ of $\varphi_j \in \mathcal{M}(\Sigma)$’s such that $Ev(f_0 \circ \varphi_0 \circ \varphi' \circ \tilde{\varphi}) = ((A,0),(0,\ldots,0))$. This means $f_0 \circ \varphi_0 \circ \varphi' \circ \tilde{\varphi} = f_2 \in F(\Sigma,\delta)$, as was desired.

(3) Assume $A = \tilde{A}(f)$ is odd and $\tilde{Arf}(f) = 1 \pmod{2}$. Then $\sum_{j=1}^n \nu_j \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\eta_j) \right] \equiv 1 \pmod{2}$. Hence there are some $1 \leq j_1 < j_2 < \cdots < j_{2m-1} \leq n$ such that $\nu_{j_1} \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\eta_{j_1}) \right] \equiv 1 \pmod{2}$ and $\nu_j \left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0)(\eta_{j_1}) \right] \equiv 0 \pmod{2}$ if $j \notin \{j_1,j_2,\ldots,j_{2m-1}\}$. In a similar way to (2), we obtain some $\varphi' \in \mathcal{M}(\Sigma)$ such that $\left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0 \circ \varphi')(\eta_{j_0}) \right] \equiv 1 \pmod{2}$ and $\left[ \nabla \cdot \tilde{A}(f_0 \circ \varphi_0 \circ \varphi')(\eta_j) \right] \equiv 0 \pmod{2}$ for any $j \neq j_0$. Hence we have some suitable product $\tilde{\varphi} \in \mathcal{M}(\Sigma)$ of $\varphi_j \in \mathcal{M}(\Sigma)$’s such that $Ev(f_0 \circ \varphi_0 \circ \varphi' \circ \tilde{\varphi}) = ((A,0),(0,\ldots,0,\tilde{1},0,\ldots,0))$. This means $f_0 \circ \varphi_0 \circ \varphi' \circ \tilde{\varphi} = f_3 \in F(\Sigma,\delta)$, as was desired.

This completes the proof of the theorem. \qed

The situation for the relative genus 0 case is elementary, but seems too complicated to describe by some simple invariants.

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