Group Theoretical Derivation of Consistent Massless Particle Theories

Giuseppe Nisticò¹,²

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Abstract

Current theories of massless free particle assume unitary space inversion and anti-unitary time reversal operators. In so doing robust classes of possible theories are discarded. In the present work theories of massless systems are derived through a strictly deductive development from the principle of relativistic invariance, so that a kind of space inversion or time reversal operator is ruled out only if it causes inconsistencies. As results, new classes of consistent theories for massless isolated systems are explicitly determined. On the other hand, the approach determines definite constraints implied by the invariance principle; they were ignored by some past investigations that, as a consequence, turn out to be not consistent with the invariance principle. Also the problem of the localizability for massless systems is reconsidered within the new theoretical framework, obtaining a generalization and a deeper detailing of previous results.

Keywords Quantum theory of massless particle · Localizability · Implications of relativistic invariance · Non-localizability for non-zero helicity systems

1 Introduction

Relativistic quantum theories of single free particle have been derived by exploiting the implications of relativistic invariance, the principle that establishes the invariance of the physical theory of an isolated system with respect to relativistic changes of inertial reference frames [1–4]. The basic implication states that each symmetry transformation can be assigned a unitary or anti-unitary operator able to realize the corresponding transformation in the quantum theory of the system according to Wigner relation [5]. Moreover, such an assignment gives rise to a unitary representation $U$ of $\hat{\mathcal{P}}_\uparrow^+, t h e$
universal covering group of the proper orthochronus Poincaré group $\mathcal{P}^+_\uparrow$. Also space inversion and time reversal are transformations that should be not excluded \textit{apriori} from the class of symmetry transformations. So, the structure of the quantum theory of an isolated system must contain a triplet $(U, \mathcal{S}, \mathcal{T})$ formed by a unitary representation $U$ of $\mathcal{P}^+_\uparrow$, and by two operators $\mathcal{S}$ and $\mathcal{T}$, each of them unitary or anti-unitary, which can realize all quantum transformations associated to elements of Poincaré group.

Contributions from the literature [1,6–8] in general discard triplets with $\mathcal{S}$ anti-unitary or $\mathcal{T}$ unitary [3], so that the class of the possible quantum theories of isolated system undergoes a drastic reduction. One reason put forward for this exclusion [4,7,9] is that anti-unitarity space inversion operators, or unitary time reversal operators, change the sign of the spectral values of the self-adjoint generator $P_0$ of time translations, that in these studies is identified with energy, and it is usually assumed that this change cannot happen (see [9], p. 135). However, in [10] it is shown that for particles with “non-zero” mass, perfectly consistent theories can be derived from the invariance principle, characterized by anti-unitary space inversion operator $\mathcal{S}$ or unitary time reversal operator $\mathcal{T}$; in particular, theories for Klein-Gordon type particle are identified, without the inconsistencies suffered by the early theories [11–13].

In the present work consistent quantum theories of “massless” isolated systems are derived from the invariance principle, through a strictly deductive development to avoid \textit{apriori} preclusions about the unitary or anti-unitary character of $\mathcal{S}$ or $\mathcal{T}$. Obviously, this methodological commitment is incompatible with a quantum field theory approach. A kind of space inversion or time reversal operator will be ruled out only if it causes inconsistencies; on the other hand, definite constraints for the triplet $(U, \mathcal{S}, \mathcal{T})$ are determined, ignored by past investigations [6,8]. As results, new classes of consistent theories for massless isolated systems are explicitly determined, while some previous theory turns out to be inconsistent with the invariance principle.

The present extension requires to reconsider the \textit{localizability} problem for massless systems, that is to say the problem of ascertaining whether a unique position operator exists or not within each possible theory for massless systems. The investigations about this problem trace back to the paper of Newton and Wigner [14], but the problem was addressed by many researchers following different approaches, e.g. in [3,8,15]. We address the localizability problem in the present extended theoretical framework, obtaining a generalization and a deeper detailing of previous results.

The work makes use of basic mathematical notions and results outlined in Sect. 2. In Sects. 3.1 and 3.2 the existence of a transformer triplet $(U, \mathcal{S}, \mathcal{T})$ is derived within general implications that hold for every isolated system. In Sect. 3.3 the further conditions are identified to be obeyed by the theory of an isolated system in order to be the theory of a massless elementary free particle, such as the irreducibility of the transformer triplet of the theory.

Since unbounded helicities have never been observed in Nature, the irreducibility condition indicates that the identification of the empirically meaningful possible theories of a massless elementary particle entails the identification of the class $I$ of all irreducible transformer triplets with bounded helicity. Section 4 classifies these irreducible triplets into three subclasses $I(u)$, $I(d)$, $I(s)$; each of them turns out to
be characterized by specific combination of the unitary or anti-unitary characters of $\mathcal{S}$ and $\mathcal{T}$.

In Sect. 5 the triplets of the classes $\mathcal{I}(u)$ and $\mathcal{I}(d)$ are identified and investigated. It is found in particular that all triplets are constrained to have zero helicity.

In Sect. 6 the empirically meaningful irreducible triplets of the class $\mathcal{I}(s)$ are identified. Consistent theories are found with anti-unitary $\mathcal{S}$ or unitary $\mathcal{T}$, ignored by the past literature. Among the results, it is found that irreducible triplets with non-zero but opposite values of the helicity exist in this class.

Hence, the class of the possible theories of massless isolated system turns out to be structurally modified: it is extended to admit theories with anti-unitary $\mathcal{S}$ and unitary $\mathcal{T}$. This modification requires to re-consider the problem of the localizability of massless particles. This task is addressed in Sect. 7. The localizability of massless particle with zero helicity is proved to hold also for particles with triplets in $\mathcal{I}(s)$; three inequivalent theories are explicitly identified. With regard to systems with non-zero helicity, past investigations proved non-localizability by making use of triplets in $\mathcal{I}(u)$ or $\mathcal{I}(d)$; but Sect. 5 proved that helicity is zero in these cases, so that the proofs are not effective [8]. In Sect. 7.2 an altripletative general proof of non-localizability is given without these shortcomings.

2 Starting Notions

In this section theoretical and mathematical concepts are reviewed, necessary to the development of the work.

2.1 General Quantum Formalism

The quantum theory of a physical system, formulated in a complex and separable Hilbert space $\mathcal{H}$, is based on the following mathematical structures.

- The set $\Omega(\mathcal{H})$ of all self-adjoint operators of $\mathcal{H}$, which represent quantum observables.
- The complete, ortho-complemented lattice $\Pi(\mathcal{H})$ of all projections operators of $\mathcal{H}$, i.e. quantum observables with possible outcomes in $\{0, 1\}$.
- The set $\Pi_1(\mathcal{H})$ of all rank one orthogonal projections of $\mathcal{H}$.
- The set $\mathcal{S}(\mathcal{H})$ of all density operators of $\mathcal{H}$, which represent quantum states.
- The set $\mathcal{V}(\mathcal{H})$ of all unitary or anti-unitary operators of the Hilbert space $\mathcal{H}$.
- The set $\mathcal{U}(\mathcal{H})$ of all unitary operators of $\mathcal{H}$; trivially, $\mathcal{U}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{H})$ holds.

2.2 Poincaré Groups and Representations

Given any vector $\chi = (x_0, \mathbf{x}) \in \mathbb{R}^4$, we call $x_0$ the time component of $\chi$ and $\mathbf{x} = (x_1, x_2, x_3)$ the spatial component of $\chi$. The proper orthochronous Poincaré group $\mathcal{P}_\uparrow$ is the separable locally compact group of all transformations of $\mathbb{R}^4$ generated by the ten one-parameter sub-groups $T_0, T_j, R_j, B_j, j = 1, 2, 3$, of time translations, spatial translations, proper spatial rotations and Lorentz boosts, respectively, relative
to the axes $x_j$. The Euclidean group $E$ is the sub-group generated by all $T_j$ and $R_j$. The sub-group generated by all $R_j$, $B_j$ is the proper orthochronous Lorentz group $L^+_\uparrow$ [16]. It does not include time reversal $\mathcal{C}$ and space inversion $\mathcal{S}$. Time reversal $\mathcal{C}$ transforms $x = (x_0, \mathbf{x})$ into $(-x_0, \mathbf{x})$; space inversion $\mathcal{S}$ transforms $x = (x_0, \mathbf{x})$ into $(x_0, -\mathbf{x})$. The group generated by $\{ P^\uparrow_+, \mathcal{C}, \mathcal{S} \}$ is the separable and locally compact Poincaré group $P$.

By $L^+_\uparrow$ we denote the subgroup generated by $L^+_\uparrow$ and $\mathcal{C}$, while $L^\uparrow$ denotes the subgroup generated by $L^+_\uparrow$ and $\mathcal{S}$; analogously, $P^\uparrow_+$ denotes the subgroup generated by $P^\uparrow_+ \uparrow_\mathcal{S}$ and $\mathcal{C}$, while $P^\uparrow$ is the subgroup generated by $P^\uparrow_+ \uparrow_\mathcal{S}$.

The sub-group generated by $\mathcal{S}$ and $\mathcal{C}$ is $D_0 = \{ e, \mathcal{S}, \mathcal{C}, \mathcal{C}\mathcal{S} \}$. Since $D_0 \cap P^\uparrow_+ = \{ e \}$, every $g \in P$ is the product $g = g_0 g_1$ of a unique pair $(g_0, g_1) \in D_0 \times P^\uparrow_+$.

In our investigation an important role is played by the semidirect product $\tilde{P}^\uparrow_+ = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ of the additive group $\mathbb{R}^4$ and the group $SL(2, \mathbb{C}) = \{ \Delta \in GL(2, \mathbb{C}) \mid \det \Delta = 1 \}$. $\tilde{P}^\uparrow_+$ is the universal covering group of $P^\uparrow_+$. Accordingly, $\tilde{P}^\uparrow_+$ is simply connected and there is a canonical homomorphism $h : \tilde{P}^\uparrow_+ \to P^\uparrow_+, g \to h(g) \in P$ that becomes an isomorphism when restricted to a suitable neighborhood of the identity $(0, \mathbb{I}^{C_2})$ of $\tilde{P}^\uparrow_+$.

We denote by $\tilde{T}_0, \tilde{T}_j, \tilde{R}_j, \tilde{B}_j, \tilde{L}_j$ the subgroups of $\tilde{P}^\uparrow_+$ that correspond to the subgroups $T_0, T_j, R_j, B_j, L_j$ of $P^\uparrow_+$, respectively, through the homomorphism $h$. All one-parameter abelian subgroups $\tilde{T}_0, \tilde{T}_j, \tilde{R}_j, \tilde{B}_j$ of $\tilde{P}^\uparrow_+$ are additive; in fact, $\tilde{B}_j$ is not additive with respect to the parameter relative velocity $u$, but it is additive with respect to the parameter $\varphi(u) = \frac{1}{2} \ln \frac{1+u}{1-u}$.

The following definition collects general notions concerning group representations.

**Definition 2.1** Let $G$ be a separable, locally compact group with identity element $e$. A correspondence $U : G \to \mathcal{V}(\mathcal{H}), g \to U_g$, with $U_e = \mathbb{I}$, is a generalized projective representation of $G$ if the following conditions are satisfied.

(i) A complex function $\sigma : G \times G \to \mathbb{C}$, called multiplier, exists such that $U_{g_1 g_2} = \sigma(g_1, g_2) U_{g_1} U_{g_2}$; the modulus $|\sigma(g_1, g_2)|$ is always 1, of course;

(ii) for all $\phi, \psi \in \mathcal{H}$, the mapping $g \to |U_g \phi \mid \psi$ is a Borel function in $g$.

If $U_g$ is unitary for all $g \in G$, then $U$ is called projective representation, or $\sigma$-representation; if also $\sigma(g_1, g_2) = 1$ holds for all $g_1, g_2$, $U$ reduces to a (ordinary) unitary representation.

A generalized projective representation is said to be continuous if for any fixed $\psi \in \mathcal{H}$ the mapping $g \to U_g \psi$ from $G$ to $\mathcal{H}$ is continuous with respect to $g$.

If $U : G \to \mathcal{V}(\mathcal{H})$ is a generalized projective representation and $\theta(g) \in \mathbb{R}$ for all $g \in G$, with $e^{i\theta(e)} = \mathbb{I}$, then the generalized projective representation $\tilde{U} : G \to \mathcal{V}(\mathcal{H}), g \to \tilde{U}_g = e^{i\theta(g)} U_g$ is said equivalent to $U : G \to \mathcal{V}(\mathcal{H})$.

In [17] we have proved that the following statement holds.

**Proposition 2.1** Let $G$ be a separable locally compact group and let $U : G \to \mathcal{V}(\mathcal{H})$ be a continuous generalized projective representation of $G$. 

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(i) If G is a connected group, then U is a projective representation, i.e. \( U_g \in U(\mathcal{H}) \), for all \( g \in G \).

(ii) If G is connected and simply connected, then the phase exponent \( \theta(g) \) can be chosen so that U is equivalent to a continuous ordinary unitary representation of G.

3 General Implications of Poincaré Invariance

Let \( \mathcal{F} \) be the class of the (inertial) reference frames that move uniformly with respect to each other. The following statement establishes the physical principle that characterizes an isolated system.

\( \text{Sym} \) The theory of an isolated system is invariant with respect to changes of frames within the class \( \mathcal{F} \).

In Sect. 3.1 it is outlined how this invariance principle implies the existence of a transformation \( S_g \) acting on quantum observables, called quantum transformation, in correspondence with each \( g \in \mathcal{P} \). The structure of the quantum theory of an isolated system is then identified in a transformer triplet, whose mathematical properties of interest are stated in Sect. 3.2. In Sect. 3.3 the concept of massless elementary free particle is introduced as a particular isolated localizable system.

3.1 Quantum Transformations

Given a frame \( \Sigma \) in \( \mathcal{F} \) and \( g \in \mathcal{P} \), by \( \Sigma_g \) we denote the frame related to \( \Sigma \) by such \( g \). Let \( \mathcal{M}_A \) denote a procedure to perform the measurement of an observable represented by the self-adjoint operator \( A \). The principle \( \text{Sym} \) implies that another measuring procedure \( \mathcal{M}' \) must exist, which is with respect to \( \Sigma_g \) identical to what is \( \mathcal{M}_A \) with respect to \( \Sigma \), otherwise the invariance established by \( \text{Sym} \) would fail. Then we denote by \( S_g[A] \) the self-adjoint operator that represents the observable measured by \( \mathcal{M}' \). In so doing for every \( g \in \mathcal{P} \) the mapping

\[
S_g : \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}) \ , \ \ A \rightarrow S_g[A]
\]

is defined, called quantum transformation associated to \( g \). In [17] it is proved that the following properties (S.1)–(S.3) of every quantum transformation can be implied from the principle \( \text{Sym} \).

(S.1) Every \( S_g : \Omega(\mathcal{H}) \rightarrow \Omega(\mathcal{H}) \) is bijective;

(S.2) for every \( A \in \Omega(\mathcal{H}) \) and every real function \( f \) such that \( f(A) \in \Omega(\mathcal{H}) \), the equality \( S_g[f(A)] = f(S_g[A]) \) holds.

(S.3) \( S_g h[A] = S_g[S_h[A]] \), for all \( g, h \in \mathcal{P} \) and every \( A \in \Omega(\mathcal{H}) \).

To every element \( \tilde{g} \) of the covering group \( \tilde{\mathcal{P}}_+ \) we can associate the quantum transformation \( S_{h(\tilde{g})} \) through the canonical homomorphism \( h \). Without introducing ambiguity, \( S_{h(\tilde{g})} \) can be denoted by \( S_{\tilde{g}} \).
Following [17], for any \( S_\tilde{g} \) a unitary or anti-unitary operator \( \tilde{U}_g \) must exist such that \( S_\tilde{g}[A] = \tilde{U}_g A \tilde{U}_g^{-1} = e^{i\theta(\tilde{g})}\tilde{U}_g A(e^{i\theta(\tilde{g})}U_\tilde{g})^{-1} \), where \( \theta \) is an arbitrary real function. Since \( h \) is a surjective homomorphism, \( S_{\tilde{g}_1\tilde{g}_2}[A] = S_{\tilde{g}_1} S_{\tilde{g}_2}[A] \) holds, and this implies that \( \tilde{g} \to e^{i\theta(\tilde{g})}\tilde{U}_g \) is a generalized projective representation, provided that \( e^{i\theta(e)}\tilde{U}_e = I \), \( e \) being the neutral element of \( \tilde{P}_+ \).

The idea that small changes of reference frame, i.e. small transformations \( A \to S_\tilde{g}[A] \) motivates the assumption that the mapping \( \tilde{g} \to S_\tilde{g} \) is continuous according to Bargmann’s topology [17]. Then, according to [17], the function \( \theta \) can be chosen so that \( \tilde{g} \to e^{i\theta(\tilde{g})}\tilde{U}_g \) is continuous. Therefore, by Prop. 2.1 a further change of \( \theta \) makes \( \tilde{g} \to e^{i\theta(\til{g})}\tilde{U}_g = U_\tilde{g} \) a continuous unitary ordinary representation of \( \til{P}_+ \).

Since the space inversion \( \mathcal{S} \) and the time reversal \( \mathcal{T} \) are transformations not connected with the identity transformation \( e \in \mathcal{P} \), the operator \( \mathcal{S} \) and \( \mathcal{T} \) that realize \( S_\mathcal{S} \) and \( S_\mathcal{T} \) can be unitary or anti-unitary.

Thus, the principle \( Sym \) ultimately implies that the quantum theory of an isolated system must admit a triplet \(( U, \mathcal{S}, \mathcal{T}) \), called the transformer triplet of the theory, formed by a continuous representation \( U \) of \( \til{P}_+ \) and by two operators \( \mathcal{S}, \mathcal{T} \) such that

\[
S_{\tilde{g}}[A] = U_{\tilde{g}} A U_{\tilde{g}}^{-1}, \quad \mathcal{S} A \mathcal{S}^{-1} = S_{\mathcal{S}}[A], \quad \mathcal{T} A \mathcal{T}^{-1} = S_{\mathcal{T}}[A]
\]

(1)

### 3.2 The Algebra of Generators; Helicity

According to Stone’s theorem [18], for any continuous unitary representation of \( \til{P}_+ \), ten self-adjoint operators \( P_0, P_j, K_j, j = 1, 2, 3 \) exist, which generate the ten one-parameter unitary subgroups

\[
\{ e^{i P_0 t} \}, \quad \{ e^{-i P_j a_j}, a \in \mathbb{R} \}, \quad \{ e^{-i J_j \theta_j}, \theta_j \in \mathbb{R} \}, \quad \{ e^{-i K_j \varphi(u_j)}, u_j \in \mathbb{R} \}
\]

(2)

of \( \mathcal{U}(\mathcal{H}) \), which represent the one-parameter sub-groups \( \til{T}_0, \til{T}_j, \til{K}_j, \til{B}_j \) according to the representation \( U \). The mathematical structural properties of \( \til{P}_+ \) as a Lie group imply that these generators satisfy the following commutation relations [1,19].

\[
(i) \quad [P_j, P_k] = 0, \quad (ii) \quad [J_j, P_k] = i \epsilon_{jkl} P_l, \quad (iii) \quad [J_j, K_k] = i \epsilon_{jkl} J_l,
\]

\[
(iv) \quad [J_j, K_k] = i \epsilon_{jkl} K_l, \quad (v) \quad [K_j, K_k] = -i \epsilon_{jkl} J_l, \quad (vi) \quad [K_j, P_k] = i \delta_{jk} P_0,
\]

\[
(vii) \quad [P_j, P_0] = 0, \quad (viii) \quad [J_j, P_0] = 0, \quad (ix) \quad [J_j, P_0] = 0, \quad (ixi) \quad [K_j, P_0] = 0, \quad (ixii) \quad [P_j, P_0] = 0,
\]

(3)

where \( \epsilon_{jkl} \) is the Levi-Civita symbol \( \epsilon_{jkl} \) restricted by the condition \( j \neq l \neq k \).

Given a reference frame \( \Sigma \) in \( \mathcal{F} \) and \( g \in \mathcal{P} \), we introduce the mapping \( g : \mathbb{R}^4 \to \mathbb{R}^4 \) such that if \( x = (t, x_1, x_2, x_3) \equiv (x_0, x) \) is the vector of the time-space coordinates of an event with respect to \( \Sigma \), then \( g(x) \) is the vector of the time-space coordinates of that event with respect to the frame \( \Sigma_g \) related to \( \Sigma \) just by \( g \). Once introduced the four-operator \( P = (P_0, P_1, P_2, P_3) \equiv (P_0, \mathbf{P}) \), from (3) it can be deduced that the
following statement holds for all $\tilde{g} \in \mathcal{L}_+^\dagger$.

$$U_{\tilde{g}} P U_{\tilde{g}}^{-1} = g(P), \quad \text{where } g = h(\tilde{g}). \quad (4)$$

The mathematical structural properties of the full Poincaré group $\mathcal{P}$ allow to extend (3) to include $\mathcal{S}, \mathcal{T}$ according to the following statements [1,19]:

If $\mathcal{S}$ is unitary, then its phase factor can be chosen so that $\mathcal{S}^2 = I$, and

$$[\mathcal{S}, P_0] = \mathbf{0}, \quad \mathcal{S} P_j = -P_j \mathcal{S}, \quad [\mathcal{S}, J_k] = \mathbf{0}, \quad \mathcal{S} K_j = -K_j \mathcal{S}; \quad (5)$$

If $\mathcal{S}$ is anti-unitary, then $\mathcal{S}^2 = c I$, so that $\mathcal{S}^{-1} = c \mathcal{S}$, where $c = 1$ or $c = -1$, and

$$\mathcal{S} P_0 = -P_0 \mathcal{S}, \quad [\mathcal{S}, P_j] = \mathbf{0}, \quad \mathcal{S} J_k = -J_k \mathcal{S}, \quad \mathcal{S} K_j = K_j \mathcal{S}, \quad (6)$$

If $\mathcal{T}$ is unitary, then its phase factor can be chosen so that $\mathcal{T}^2 = I$, and

$$\mathcal{T} P_0 = -P_0 \mathcal{T}, \quad [\mathcal{T}, P_j] = \mathbf{0}, \quad [\mathcal{T}, J_k] = \mathbf{0}, \quad \mathcal{T} K_j = -K_j \mathcal{T}; \quad (7)$$

If $\mathcal{T}$ is anti-unitary, then $\mathcal{T}^2 = c I$, so that $\mathcal{T}^{-1} = c \mathcal{T}$, either $c = 1$ or $c = -1$, and

$$\mathcal{T} P_0 = P_0 \mathcal{T}, \quad \mathcal{T} P_j = -P_j \mathcal{T}, \quad \mathcal{T} J_k = J_k \mathcal{T}, \quad \mathcal{T} K_j = K_j \mathcal{T}, \quad (8)$$

$$\mathcal{S} \mathcal{T} = \omega \mathcal{T} \mathcal{S}, \quad \text{with } \omega \in \mathbb{C} \text{ and } |\omega| = 1 \quad (9)$$

The helicity operator is defined by $\hat{\lambda} = \frac{\mathbf{J} \cdot \mathbf{P}}{P}$, where $P = \sqrt{P^2_1 + P^2_2 + P^2_3}$. The following relation

$$\mathcal{S} \hat{\lambda} \mathcal{S}^{-1} = -\hat{\lambda} \quad (10)$$

is implied by (5) but also by (6). Therefore it holds independently of the unitary or anti-unitary character of $\mathcal{S}$.

By making use of (3)–(8) it can be proved that the following relations hold.

$$[V, P_0^2 - \mathbf{P}^2] = \mathbf{0}, \quad [V, W_0^2 - \mathbf{W}^2] = \mathbf{0}, \quad \text{for all } V \in U(\mathcal{P}_+^\dagger) \cup \{\mathcal{T}, \mathcal{S}\} \quad (11)$$

where $W_0 = \mathbf{J} \cdot \mathbf{P}$ and $\mathbf{W} = P_0 \mathbf{J} + \mathbf{P} \wedge \mathbf{K}$ form the Pauli-Lubański four-operator $(W_0, \mathbf{W})$.

### 3.3 Massless Elementary Free Particles

An isolated system is said to be localizable if its quantum theory is endowed with a unique position observable, that is to say with a unique triplet $(Q_1, Q_2, Q_3) \equiv \mathbf{Q}$ of self-adjoint operators, whose components $Q_j$ are called coordinate operators, characterized by the following conditions.\footnote{The commutativity condition $[Q_j, Q_k] = 0$ establishes the possibility of performing a measurement that yields all three values of the position coordinates. The nonexistence of commutative position operators...
(Q.1) \( [Q_j, Q_k] = 0 \), for all \( j, k = 1, 2, 3 \). This condition establishes that a measurement of position yields all three values of the coordinates of the same specimen of the system.

(Q.2) For every \( g \in \mathcal{P} \), the triplet \( (Q_1, Q_2, Q_3) \equiv Q \) and the transformed position operators \( S_g(Q) = (S_g(Q_1), S_g(Q_2), S_g(Q_3)) \) satisfy the specific relations implied by the transformation properties of position with respect to \( g \).

According to (Q.2), the following specific relations hold.

\[
\begin{align*}
S_{\epsilon}[Q] &= Q \text{ and } S_{\epsilon}[Q] = -Q, \text{ i.e. } ^TQ = Q^T \text{ and } \epsilon Q = -Q \epsilon, \quad (12.1) \\
S_g(Q) &= U_g Q U_g^{-1} = g(Q) \text{ for every } g \in \mathcal{E}, \text{ which imply} \\
[Q_k, P_j] &= i \delta_{jk} \text{ and } [J_j, Q_k] = i \delta_{jkl} Q_l. \quad (12.2)
\end{align*}
\]

A localizable isolated system will be also called free particle. Following a customary procedure, a free particle will be qualified as elementary if the system of operators \( \{U(\hat{P}_+^\dagger), \mathcal{S}, ^T; Q\} \) is irreducible.

For an elementary free particle, the transformer triplet \( (U, \mathcal{S}, ^T) \) of its quantum theory must be irreducible. Let us explain why. If \( (U, \mathcal{S}, ^T) \) were reducible, then a unitary operator \( V \) would exist such that \([\mathcal{S}, V] = [^T, V] = [V, U^g] = 0 \) for all \( g \in \mathcal{P}_+ \), but \([V, Q_j] \neq 0 \) for some \( j \) if \([V, Q_k] = 0 \) held for all \( k \), then \([U(\hat{P}_+^\dagger), \mathcal{S}, ^T; Q] \) would be reducible, and this is not possible for elementary particles. Hence, if we define \( \tilde{\mathcal{Q}}_k = V Q_k V^{-1} \), then \( \tilde{Q} \neq Q \), while \( VU^g V^{-1} = U^g \) for all \( g \in \mathcal{P}_+ \). The mathematical relations between the operators \( \tilde{\mathcal{Q}} = VQV^{-1} \) and each operator \( VBV^{-1} \), where \( B \) belongs to the triplet \( (U, \mathcal{S}, ^T) \) must be the same as the mathematical relations between \( Q \) and that \( B \), because \([U(\hat{P}_+^\dagger), \mathcal{S}, ^T; Q] \) and \( V(U(\hat{P}_+^\dagger), \mathcal{S}, ^T; Q)V^{-1} \) are unitarily isomorphic. Then the triplet \( \tilde{\mathcal{Q}} \) satisfies (Q.2), because \( Q \) does, and therefore \( \tilde{\mathcal{Q}} \) would be a position operator in all respects. Thus, for the same elementary particle two different position operators would exist, in contradiction with the required uniqueness.

Since the representation \( U \) of \( \hat{P}_+^\dagger \) can be reconstructed from its generators according to (2), a transformer triplet will be irreducible if and only if the operators \( P_0, P_j, J_j, K_j, \mathcal{S}, ^T, j = 1, 2, 3 \) form an irreducible system; therefore, by a straightforward application of Schur lemma, from (11) we imply that the quantum theory of an elementary free particle is characterized by two real numbers \( \eta, \sigma \) such that

\[
P_0^2 - P^2 = \eta \Pi, \quad W^2 \equiv W_0^2 - (W_1^2 + W_2^2 + W_3^2) = \sigma \Pi. \quad (13)
\]

In the present work we investigate the theories of those elementary isolated systems for which \( \eta = 0 \), i.e. \( P = |P_0| \), called massless systems.

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4 Identification of Theories of Massless Systems

Because of the necessary irreducibility of the transformer triplet of every quantum theory of a massless elementary free particle, the first step for identifying these theories is to identify the class of all irreducible transformer triplets with \( \eta = 0 \). Since a reducible triplet is decomposable in terms of irreducible triplets, in so doing we identify also the class of all triplets any theory of massless isolated system is built up.

Now, the irreducible components of the representation \( U \) in such a transformer triplet are irreducible representations of \( \tilde{P}_\uparrow \) with \( \eta = 0 \), which are well known. Each of them is characterized by a non-negative value of a parameter \( r_0 \) related to the Pauli-Lubanski constant \( \varpi \) in (13). If \( r_0 > 0 \), then the helicity operator turns out to have unbounded spectrum \([1,3]\), therefore the set of the possible values of the helicity \( \hat{\lambda} \) is unbounded. This feature has never been observed in Nature; the present investigation is restricted to the empirically meaningful theories, that is to say to theories with bounded helicity. Accordingly, we shall identify the class \( I \) of the irreducible triplets whose representation has all irreducible components with \( \eta = 0 \) and \( r_0 = 0 \). To this aim, we shall make use of the following classification of these triplets in terms of spectral properties of the representation \( U \).

4.1 Three Kinds of Theories

The irreducibility condition for the transformer triplet \( (U, \mathcal{S}, \mathcal{T}) \) imposes strong constraints to the spectrum \( \sigma(P) \) of the four-operator \( P \). Namely there are three mutually exclusive possibilities for \( \sigma(P) \), and each possibility turns out to be related to the unitary or anti-unitary character of the space inversion and time reversal operators \( \mathcal{S} \) and \( \mathcal{T} \). A clear formulation of this fact requires some notion of spectral theory.

Let \( E^{(\alpha)} : \mathcal{B}(\mathbb{R}) \rightarrow \Pi(\mathcal{H}) \) be the spectral measure \([25]\) of each generator \( P_\alpha, \alpha = 0, 1, 2, 3 \). Since the operators \( P_\alpha \) commute with each other, a common spectral measure \( E : \mathcal{B}(\mathbb{R}^4) \rightarrow \Pi(\mathcal{H}) \) is generated by defining it for hyperrectangles \( \Delta = \Delta_0 \times \Delta_1 \times \Delta_2 \times \Delta_3 \), where \( \Delta_\alpha = (a_\alpha, b_\alpha) \), as \( E(\Delta) = E(0)E(1)E(2)E(3) \); then

\[
P = \int p \, dE_p ,
\]

with \( dE_p = E((p_0, p_0 + dp_0) \times (p_1, p_1 + dp_1) \times (p_2, p_2 + dp_2) \times (p_3, p_3 + dp_3)) \).

Once introduced the four-operator \( \mathcal{P} = (P_0, P_1, P_2, P_3) \equiv (P_0, P) \), the spectrum of \( \mathcal{P} \) can be defined as the following subset of \( \mathbb{R}^4 \).

\[
\sigma(\mathcal{P}) = \{ p = (p_0, p) \in \mathbb{R}^4 \mid E(\Delta_p) \neq \emptyset \text{ for every neighborhood } \Delta_p \text{ of } p \}. \quad (15)
\]

**Proposition 4.1** Let \( U : \tilde{P}_\uparrow \rightarrow \mathcal{U}(\mathcal{H}) \) be a continuous unitary representation, and let \( \mathcal{S}, \mathcal{T} \in \mathcal{V}(\mathcal{H}) \) satisfy (5)–(9). If \( (U, \mathcal{S}, \mathcal{T}) \) is irreducible with \( \eta = 0 \), then there are only the following mutually exclusive possibilities for the spectra \( \sigma(\mathcal{P}) \) and \( \sigma(P_0) \).
\( (u) \) \( \sigma(P) = S_0^+ \) and \( \sigma(P_0) = (0, \infty) \), with \( \mathcal{S} \) unitary and \( T^- \) anti-unitary,

\( (d) \) \( \sigma(P) = S_0^+ \) and \( \sigma(P_0) = (-\infty, 0) \), with \( \mathcal{S} \) unitary and \( T^- \) anti-unitary,

\( (s) \) \( \sigma(P) = S_0^+ \cup S_0^- \) and \( \sigma(P_0) = \mathbb{R} \setminus \{0\} \), with \( \mathcal{S} \) anti-unitary or \( T^- \) unitary, where \( S_0^+ = \{ p \mid p_0^2 - p^2 = 0, p_0 > 0 \} \) and \( S_0^- = \{ p \mid p_0^2 - p^2 = 0^2, p_0 < 0 \} \) are the positive and the negative hypercones, or zero-mass orbits, in \( \mathbb{R}^3 \).

A proof can be found in [26] and references therein.

4.1.1 Implications of the Irreducibility of \( U \)

If just the representation \( U \) of a transformer triplet \( (U; \mathcal{S}, T^-) \) is irreducible, then the whole triplet is irreducible, of course. In such a case, however, only the cases \( (u) \) or \( (d) \) of Prop. 4.1 can occur, because of the following proposition.

**Proposition 4.2** Once defined the projection operators \( E^+ = \int_{S_0^+} dE_p \equiv \chi_{S_0^+}(P) \), where \( \chi_{S_0^+} \) is the characteristic functional of \( S_0^+ \), the relations (3.i), (3.vii) imply that \( E^+ \) commutes with \( P_0 \) and with all \( P_j \), and therefore \( \mathcal{H}^+ \) is left invariant by \( U_{\tilde{g}} \), if \( \tilde{g} \in T_0 \cup T_1 \cup T_2 \cup T_3 \). Now we show that \( \mathcal{H}^+ \) is left invariant by \( U_{\tilde{g}} \) for every \( \tilde{g} \in \tilde{L}^+ \), too, and hence for all \( \tilde{g} \in \tilde{P}^+_+ \). Relation (4) implies that \( U_{\tilde{g}} E(\Delta) U_{\tilde{g}}^{-1} = E (g^{-1}(\Delta)) \) holds, where \( g = h(\tilde{g}) \); therefore, if \( \psi \in \mathcal{H}^+ \), then for every \( \tilde{g} \in \tilde{L}^+ \) we have \( U_{\tilde{g}} \psi = U_{\tilde{g}} \int_{S_0^+} dE_p \psi = \int_{S_0^+} dE_p U_{\tilde{g}} E_p U_{\tilde{g}}^{-1}(U_{\tilde{g}} \psi) = \int_{S_0^+} dE_p g^{-1}(p)(U_{\tilde{g}} \psi) \). The last integral is a vector of \( \mathcal{H}^+ = E^+ \mathcal{H} \), because \( p' = g^{-1}(p) \in S_0^+ \) if \( p \in S_0^+ \) for \( \tilde{g} \in \tilde{L}^+ \). The proof of \( [E^-, U_{\tilde{g}}] = 0 \) is quite analogous.

\( \square \)

Hence, according to Prop. 4.2, if \( U \) is irreducible then either the case \( E^+ = \mathcal{I} \), \( E^- = \mathcal{O} \) or the case \( E^+ = \mathcal{O} \) and \( E^- = \mathcal{I} \) must occur. If the first case occurs, then the relation \( P = E^+ P = \chi_{S_0^+}(P) P \) holds, which implies \( \sigma(P) = \{ p \in \mathbb{R}^4 \mid \chi_{S_0^+}(p) = p \} \), i.e. \( \sigma(P) = S_0^+ \). Analogously, in the atripletative case where \( E^- = \mathcal{I} \) occurs, \( \sigma(P) = S_0^- \) is obtained.

According to the classification operated by Prop. 4.1, the class \( \mathcal{I} \) is the union \( \mathcal{I} = \bigcup \mathcal{I}(u) \cup \bigcup \mathcal{I}(d) \cup \bigcup \mathcal{I}(s) \) of three non-overlapping subclasses; the subclass \( \mathcal{I}(u) \) (resp., \( \mathcal{I}(d), \mathcal{I}(s) \)) is formed by those irreducible triplets in \( \mathcal{I} \) for which \( \sigma(P) = S_0^+ \) (resp., \( S_0^-, S_0^0 \cup S_0^+ \)). So, there can be three kinds of theories, according to the sub-class \( \mathcal{I}(u) \), \( \mathcal{I}(d) \) or \( \mathcal{I}(s) \) the transformer triplet of the theory belongs to. The representation \( U \) in a triplet of \( \mathcal{I} \) is the direct sum or integral of irreducible representations \( U^\alpha \). In Sect. 5 we determine the features of the transformer triplets of \( \mathcal{I}(u) \) and \( \mathcal{I}(d) \). The triplets of \( \mathcal{I}(s) \) will be investigated in Sect. 6.
5 Theories with Triplets in $\mathcal{I}(u)$ and $\mathcal{I}(d)$

The representation $U$ of a transformer triplet $(U; \mathcal{S}, \mathcal{T})$ in the class $\mathcal{I}(u)$ or $\mathcal{I}(d)$ can be reducible or not, though the triplet is irreducible. In Sect. 5.1 we determine conditions to be satisfied by an irreducible representation in order to give rise to a theory based on $\mathcal{I}(u)$. In Sect. 5.2 these conditions are proved to apply to the theories with $U$ reducible too. In the same section the this treatment is extended to $\mathcal{I}(d)$.

5.1 Theories Based on Triplets with Irreducible $U$

According to Prop. 4.2, a transformer triplet with $U$ irreducible cannot belong to $\mathcal{I}(s)$. If $r_0 = 0$, then within unitary isomorphisms each irreducible representation of $\mathcal{H}^d_+$ with $\sigma(P) = S^+_0$ or $\sigma(P) = S^-_0$ turns out to be completely identified by a number $m \in \mathbb{Z}$. Representations with different values of this parameter are unitarily inequivalent. Modulo unitary isomorphisms, the Hilbert space of the representation is $\mathcal{H} = L_2(\mathbb{R}^3, d\nu)$, i.e. the space of the complex functions on $\mathbb{R}^3$ square integrable with respect to the invariant measure $d\nu = \frac{dp_1 dp_2 dp_3}{p_0}$, with $p_0 = \sqrt{p^2}$.

The generators of the representation with $\sigma(P) = S^+_0$ and a given $m$ are the following operators.

$$
(P_j \psi)(p) = p_j \psi(p), \quad P_0 \psi(p) = p_0 \psi(p) \quad (16)
$$

$$
J_j = J^{(0)}_j + j_j, \quad K_j = K^{(0)}_j + k_j, \quad (17)
$$

where $J^{(0)}_j = -i \left( p_k \frac{\partial}{\partial p_l} - p_l \frac{\partial}{\partial p_k} \right), (j, k, l)$ being cyclic, $K^{(0)}_j = ip_0 \frac{\partial}{\partial p_j}$.

$$
J_1 = \frac{m}{2} \frac{p_1 p_0}{p_1^2 + p_2^2}, \quad J_2 = \frac{m}{2} \frac{p_2 p_0}{p_1^2 + p_2^2}, \quad J_3 = 0,
$$

$$
k_1 = -\frac{m}{2} \frac{p_2 p_3}{p_1^2 + p_2^2}, \quad k_2 = \frac{m}{2} \frac{p_3 p_1}{p_1^2 + p_2^2}, \quad k_3 = 0. \quad (18)
$$

If $\sigma(P) = S^-_0$ the generators are

$$
(P_j \psi)(p) = p_j \psi(p), \quad P_0 \psi(p) = -p_0 \psi(p) \quad (19)
$$

$$
J_j = J^{(0)}_j + j_j, \quad K_j = -(K^{(0)}_j + k_j). \quad (20)
$$

A simple computation shows that $(16)$–$(20)$ imply $\hat{\lambda} = \frac{m}{2} \mathbb{I}$, so that any representation in the class we are studying turns out to be identified by the value of the helicity.

Possible theories are obtained by identifying the space inversion operator $\mathcal{S}$ and the time reversal operator $\mathcal{T}$ consistent with $(16)$, $(17)$ to form a triplet of $\mathcal{I}(u)$, or consistent with $(19)$, $(20)$ to form a triplet of $\mathcal{I}(d)$. According to Prop. 4.1 the operator $\mathcal{S}$ must be unitary and $\mathcal{T}$ must be anti-unitary. Relations $(5)$ implies $\mathcal{S} \hat{\lambda} \mathcal{S}^{-1} = -\hat{\lambda}$, i.e., $\frac{m}{2} = -\frac{m}{2}$. Therefore, all triplets in $\mathcal{I}(u)$ or $\mathcal{I}(d)$ with $U$ irreducible have $m = 0$ and hence zero helicity.
To explicitly identify the unitary operator $\mathcal{S}$, let us define $\hat{S} = \Upsilon \mathcal{S}$. Then $[\mathcal{S}, P_0] = \mathbf{0}$ and $\mathcal{S} P_j = -P_j \mathcal{S}$ in (5) imply $[\hat{S}, P_j] = \mathbf{0}$; hence $\hat{S} \psi(p) = s(p) \psi(p)$ where $s$ is a complex function of $p$ because $(P_1, P_2, P_3)$ in (16) form a complete system of operators in $L_2(\mathbb{R}^3, dv)$. Taking into account (16)–(20), the last equation in (5) becomes $K^{(0)} \tau s = -\Upsilon s K^{(0)}$, which implies that $s$ is a constant function, hence we can set $s = 1$; so that $\mathcal{S} = \Upsilon$.

To explicitly find $^\ast \Upsilon$ we introduce the unitary operator $\hat{T} = K \Upsilon^\ast$, where $K$ is the anti-unitary complex conjugation operator defined by $\hat{K} \psi(p) = \overline{\psi(-p)}$ and $\Upsilon$ is the unitary parity operator defined by $\Upsilon \psi(p) = \psi(-p)$. By applying (8) to the operators of (16)–(20) we find $[\hat{T}, P_0] = [\hat{T}, P_j] = [\hat{T}, J_j] = [\hat{T}, K_j] = \mathbf{0}$ for all $j$. Since $U$ is irreducible, the set of its generators is irreducible too, so that $\hat{T}$ is a constant operator $\hat{T} = e^{i \theta_0} I$; thus we can choose the phase factor so that $^\ast \Upsilon = K \Upsilon$.

Thus, modulo unitary isomorphisms, each sub-class $\mathcal{I}(u)$ or $\mathcal{I}(d)$ contains only one irreducible transformer triplet with $r_0 = 0$ and $U$ irreducible, identified

by (16) and $J_j = J_j^{(0)}$, $K_j = K_j^{(0)}$, $\mathcal{S} = \Upsilon$, $^\ast \Upsilon = K \Upsilon$, in $\mathcal{I}(u)$, and

by (19) and $J_j = J_j^{(0)}$, $K_j = -K_j^{(0)}$, $\mathcal{S} = \Upsilon$, $^\ast \Upsilon = K \Upsilon$, in $\mathcal{I}(d)$.

### 5.2 Constraints for Triplets of $\mathcal{I}(u)$ or $\mathcal{I}(d)$ with Reducible $U$

Since the irreducible representations of $\mathcal{P}_+$ with $r_0 = 0$ and $\eta = 0$ form a discrete set, the representation $U$ in a triplet of $\mathcal{I}(u)$ (resp., $\mathcal{I}(d)$) must be the direct sum $U = \bigoplus \alpha U^\alpha$ of representations identified by (16)–(18) (resp., (19)–(20)), so that the Hilbert space of the theory is the direct sum $\mathcal{H} = \bigoplus \alpha \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ is the Hilbert space of the irreducible component $U^\alpha$. In this section we prove that also in this case the helicity must be zero.

#### Proposition 5.1

The helicity of a theory with irreducible triplet in $\mathcal{I}(u) \cup \mathcal{I}(d)$ is zero, independently of the reducibility of $U$.

**Proof** Every irreducible component $U^\alpha$ of $U$ has $r_0 = 0$, and hence it is characterized by a specific value $m_\alpha$ of the parameter $m$ in (16)–(20). We prove that $m_\alpha = 0$, for every $\alpha$.

Let us suppose that $m_\alpha \equiv m \neq 0$. Given any vector $\psi \in \mathcal{H}_\alpha$ with $||\psi|| = 1$, the vector $\hat{\psi} = \mathcal{S} \psi$ must be not null because $\mathcal{S}$ is unitary. Being $\hat{\psi}$ in $\mathcal{H} = \bigoplus \alpha \mathcal{H}_\alpha$, it can be decomposed as $\hat{\psi} = \sum c_\alpha \psi_\alpha$, with $\psi_\alpha \in \mathcal{H}_\alpha$ and $c_\alpha \in \mathbb{C}$. According to (10), $\mathcal{S} \hat{\psi} = \mathcal{S} \hat{\psi} = \hat{\psi}$ holds, which implies

$$
\hat{\lambda} \hat{\psi} = c_0 \lambda \psi_0 + \sum_{\alpha \neq \alpha_0} c_\alpha \lambda \psi_\alpha = \frac{m}{2} c_0 \psi_0 + \sum_{\alpha \neq \alpha_0} (\frac{m_\alpha}{2}) c_\alpha \psi_\alpha
$$

$$
= \mathcal{S} \mathcal{S}^{-1} \hat{\lambda} \mathcal{S} \psi = \mathcal{S} (-\hat{\lambda}) \psi = -\frac{m}{2} \mathcal{S} \psi = -\frac{m}{2} \hat{\psi} = -\frac{m}{2} c_0 \psi_0 - \sum_{\alpha \neq \alpha_0} \frac{m}{2} c_\alpha \psi_\alpha.
$$
Hence
\[
\frac{m}{2}c_0\varphi_0 + \sum_{\alpha \neq \alpha_0} \frac{m_\alpha}{2}c_\alpha \varphi_\alpha = -\frac{m}{2}c_0\varphi_0 - \sum_{\alpha \neq \alpha_0} \frac{m}{2}c_\alpha \varphi_\alpha;
\]

since \( \langle \varphi_\alpha | \varphi_\beta \rangle = \delta_{\alpha\beta} \), we have to conclude that \( c_0 = 0 \) and if \( c_\alpha \neq 0 \) then \( m_\alpha = -m \). Therefore, if there are component representations \( U^{\alpha_0}_\alpha \) with \( m^{\alpha_0}_\alpha = m \neq 0 \), then there must be also component representations with \( m_\alpha = -m \). Let \( \{U^{\alpha_j}_\alpha\} \) and \( \{U^{\alpha_k}_\alpha\} \) be the set of all irreducible components with \( m^{\alpha_j}_\alpha = m \) and \( m^{\alpha_k}_\alpha = -m \), and let us define the representations \( U^{(+)} = \bigoplus_j U^{\alpha_j}_\alpha \) and \( U^{(-)} = \bigoplus_k U^{\alpha_k}_\alpha \), which act of the Hilbert spaces and \( \mathcal{H}^{(+)} = \bigoplus_j \mathcal{H}^{\alpha_j} \) and \( \mathcal{H}^{(-)} = \bigoplus_k \mathcal{H}^{\alpha_k} \) respectively.

Every vector \( \psi \in \mathcal{H} \) can be decomposed as \( \psi = \psi^\perp + \psi^- + \psi_0 \), where \( \psi^\perp \in \mathcal{H}^{(+)} \), \( \psi^- \in \mathcal{H}^{(-)} \) and \( \psi_0 \in \mathcal{H}^{(0)} = \bigoplus_{\alpha \neq \alpha_0} \mathcal{H}^{\alpha} \). Of course, \( \mathcal{H}^{(0)} \perp (\mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)}) \). Such a decomposition induces a matrix representation where the vector \( \psi \) is represented by a column vector \( \psi \equiv \begin{bmatrix} \psi^\perp \\ \psi^- \\ \psi_0 \end{bmatrix} \) and any linear or anti-linear operator \( A \) is represented by a matrix \( A \equiv \begin{bmatrix} A^+ \\ A^- \\ A^0 \end{bmatrix} \). Accordingly, in the case that the triplet belongs to \( \mathcal{I}(u) \), the generators are represented by

\[
\begin{align*}
P_0 &= \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{bmatrix}, \\
P_j &= \begin{bmatrix} p_j & 0 & 0 \\ 0 & p_j & 0 \\ 0 & 0 & p_j \end{bmatrix}, \\
J_j &= \begin{bmatrix} (J_j^{(0)} + j_j) & 0 & 0 \\ 0 & (J_j^{(0)} - j_j) & 0 \\ 0 & 0 & J_{j00} \end{bmatrix}, \\
K_j &= \begin{bmatrix} (K_j^{(0)} + k_j) & 0 & 0 \\ 0 & (K_j^{(0)} - k_j) & 0 \\ 0 & 0 & K_{j00} \end{bmatrix}.
\end{align*}
\]

Now we show that no space inversion operator exists if \( m \neq 0 \). Let us introduce the unitary operator \( \hat{S} = Y \cdot \mathcal{S} \). By imposing (5) for the matrices \( P_j \), by the completeness of \( p_j \) in \( L^2(\mathbb{R}^3, d\nu) \) we imply that \( \hat{S} = \begin{bmatrix} 0 & S_1(p) & 0 \\ S_2(p) & 0 & 0 \\ 0 & 0 & S_{00}(p) \end{bmatrix} \), where \( S_1 \) and \( S_2 \) are functions of \( p \).
For the matrices $K_j$, $j = 1, 2$ condition (5) implies $2i S_1(p) p_0 \frac{\partial}{\partial p_j} = 2k_j - i p_0 \frac{\partial S_1(p)}{\partial p_j}$, $2i S_2(p) p_0 \frac{\partial}{\partial p_j} = -2k_j + i p_0 \frac{\partial S_2(p)}{\partial p_j}$, which cannot be satisfied unless $S_1 = S_2 = 0$. The proof for triplets in $I(d)$ is quite analogous.

According to this result, in a theory based on a transformer triplet in $I(u) \cup I(d)$ with $U$ reducible, all irreducible components $U^\mu$ must be identical to each other, up unitary isomorphism, and with $m_a = 0$.

**Example 5.1** Let us consider the Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$, with $\mathcal{H}^{(1)} = \mathcal{H}^{(2)} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, dv)$; let every vector $\psi = \psi_1 + \psi_2$ in $\mathcal{H}$, with $\psi_1, \psi_2 \in L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, dv)$ be represented as the column vector $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$. The operators represented by the matrices

\[
P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad J_k = \begin{bmatrix} J_k^{(0)} & 0 \\ 0 & J_k^{(0)} \end{bmatrix}, \quad K_j = \begin{bmatrix} K_j^{(0)} & 0 \\ 0 & K_j^{(0)} \end{bmatrix},
\]

satisfy (3). Hence, a reducible representation $U : \tilde{P}_+^\dagger \rightarrow \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$, where $\mathcal{H}^{(1)} = \mathcal{H}^{(2)} L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, dv)$, is determined. The operators

\[
\mathcal{S} = \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{T} = \gamma K \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

(23) satisfy conditions (5), (8), (9). Therefore, an irreducible transformer triplet of $I(u)$ is identified. Moreover, if $A$ is any self-adjoint operator of $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, dv) \oplus L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, dv)$, then the conditions $[A, P_0] = [A, P_j] = [A, J_k] = [A, K_j] = [A, \mathcal{S}] = [A, \mathcal{T}] = 0$ imply $A = a I$, and therefore the triplet is irreducible.

**Example 5.2** An irreducible triplet of $I(d)$ with $U$ reducible can be obtained from Example 5.1, by changing only $P_0$ and $K_j$ into

\[
P_0 = \begin{bmatrix} -p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad K_j = \begin{bmatrix} -K_j^{(0)} & 0 \\ 0 & -K_j^{(0)} \end{bmatrix},
\]

(24)

**Remark 5.1** The necessity for theories based on transformer triplets with reducible $U$ arose from the empirical evidence that specific massless particles, namely photons, occur with positive and negative helicities [27]. Then theories were taken into account based on irreducible triplets whose representation $U$ is the direct sum of two irreducible representaions of $\tilde{P}_+^\dagger$ [6,8,15], characterized by opposite non-zero values of the helicity, but with $\mathcal{S}$ unitary and $\mathcal{T}$ anti-unitary. Prop. 4.1 entails that these attempts cannot lead to theories consistent with the invariance principle. Indeed, according to Prop. 4.1 if $\mathcal{S}$ is unitary and $\mathcal{T}$ anti-unitary, then the triplet must belong either to $I(u)$ or to $I(d)$; then the constraint stated by Prop. 5.1 forces the helicity to be zero.
6 Theories Based on $\mathcal{I}(s)$

According to Remark 5.1, to search for possible theories of massless elementary particle with non-zero helicity we have to turn on triplets of $\mathcal{I}(s)$. In this section the features of these triplets are investigated, and as a result we find also that this search is successful. In Sect. 6.1 we prove that the component $U^-$ of a triplet in $\mathcal{I}(s)$ is constrained to be the “mirrored version” of $U^+$; in particular, $U^-$ is irreducible if and only if $U^+$ does. The possible theories with $U^+$ irreducible and zero helicity are identified in Sect. 6.2.1. In Sect. 6.2.2 it is proved that for every non zero value $m_+ \in \mathbb{ZZ}$ there are two possible theories, each of them with opposite values $m_+$ and $-m_+$ of the helicity.

6.1 Constraints Between $U^+$ and $U^-$

In a triplet of $\mathcal{I}(s)$, since $\sigma(P) = S_0^+ \cup S_0^-$, the projection operators $E^+$ and $E^-$ of Prop. 4.2 are both different from $I$, so that the representation $U$ of $\mathcal{P}_s^+$ is always reducible: $U = U^+ \oplus U^-$, where $U^+ = E^+UE^+$ and $U^- = E^-UE^-$ are the components of $U$ reduced by $\mathcal{H}^+$ and $\mathcal{H}^-$ respectively. These reduced components, $U^+$ or $U^-$, can be reducible or not. The following proposition implies that in a theory with transformer triplet in $\mathcal{I}(s)$ the reducibility of $U^+$ is equivalent to the reducibility of $U^-$. 

**Proposition 6.1** Let $(U, \mathcal{S}, \mathcal{T})$ be a transformer triplet in $\mathcal{I}(s)$, and let $F_+$ be a projection operator that reduces $U^+$; then the following statements hold.

(i) In the case that $\mathcal{T}$ is unitary, the projection operator $F_\pm = \mathcal{T}F_+\mathcal{T}$ reduces $U^-$, and $F_\pm = F_+ + F_-^\pm$ reduces $U$;

(ii) in the case that $\mathcal{S}$ is anti-unitary, the projection operator $F_\pm = \mathcal{S}F_+\mathcal{S}$ reduces $U^-$, and $F_\pm = F_+ + F_-^\pm$ reduces $U$.

(iii) $U^+$ is reducible if and only if $U^-$ is reducible.

**Proof** We recall that if $T$ is a unitary or anti-unitary operator such that $T A T^{-1} = f(A)$, where $A$ is a self-adjoint operator with spectral measure $E^A$ and $f$ is a continuous bijection of $\mathbb{R}$, then $T E^A(\Delta) T^{-1} = E^A(f^{-1}(\Delta))$, for every Borel set $\Delta \subseteq \mathbb{R}$.

Now, if $\mathcal{T}$ is unitary, then $\mathcal{T}^{-1} = \mathcal{T}$ and $\mathcal{T}P_0 \mathcal{T} = -P_0$ follow from (11); this implies $\mathcal{T} E^+ T = \mathcal{T} \chi_{[0,\infty)}(P_0) \mathcal{T} = \chi_{(-\infty,0]}(P_0) = E^-$. If $F_+$ is a projection operator that reduces $U^+$, and hence $\mathcal{T} < F_+ < E^+$, then $F_- E^- = (\mathcal{T} F_+ \mathcal{T}) E^- = (\mathcal{T} F_+ \mathcal{T}) \mathcal{T} E^+ \mathcal{T} = \mathcal{T} F_+ E^+ \mathcal{T} = \mathcal{T} F_+ \mathcal{T}$ since $F_+ < E^+$. Therefore, $\mathcal{T} < F_- E^-$. Now we show that $[F_\pm, P_0^-] = [F_\pm, P_f^-] = [F_\pm, K_f^-] = [F_\pm, J_f^-] = \mathcal{O}$, i.e. that $F_-^\pm$ reduces $U^-$. Since $P_0^- = E^- P_0 E^-$ and $[F_+, P_0] = [F_+, P_0^+] = \mathcal{O}$, we have

$$
P_0^- F_-^\pm = P_0^- F_-^\pm E^- = E^- P_0^- E^+ F_+ \mathcal{T} E^- = E^- P_0^- E^+ \mathcal{T} F_+ \mathcal{T} E^- = -E^- \mathcal{T} P_0 E^+ F_+ \mathcal{T} E^- = -E^- \mathcal{T} E^+ P_0 F_+ \mathcal{T} E^- = -E^- \mathcal{T} F_+ \mathcal{T} P_0 E^- = -E^- F_-^\pm P_0 E^- = F_-^\pm P_0^-.
$$
A similar derivation shows that \([F_\pm, P_j] = [F_\pm, K_j^-] = [F_\pm, J_k^-] = 0\); therefore \(F_\pm\) reduces \(U^-\). Now we see that \(F_\pm = F_+ + F_-\) reduces \(U\mid_{\mathcal{P}_+}\). The equalities \(F_\pm P_0 = (F_+ + F_\mp)P_0 = P_0(F_+ + F_\mp) = P_0 F_\pm\) immediately follow from \(P_0 = E_+ P_0 E_+ + E_- P_0 E_-\) and \(F_\pm E_- = F_\pm, F_+ E_+ = F_\mp, F_+ E_- = F_- E_+ = 0\). Similarly, \([F_\pm, P_j] = [F_\pm, J_k] = [F_\pm, K_j] = 0\) hold. Hence, \(F_\pm\) reduces \(U\mid_{\mathcal{P}_+}\). Moreover, \(F_\pm T_\mp = F_+ T_\mp F_- T_\mp = T_\mp F_+ T_\mp F_- T_\mp = T_\mp F_- T_\mp F_+ T_\mp = T_\mp F_\pm\). Therefore, \(F_\pm\) reduces also \(U\mid_{\mathcal{P}_-}\). A quite similar argument proves statement (ii).

Statement (iii) is a direct consequence of statements (ii) and (ii). \(\square\)

So, in a triplet of \(\mathcal{I}(s)\) the component \(U^-\) is the “mirrored” version of \(U^+\). According to Prop. 6.1, \(\mathcal{I}(s)\) can be decomposed as \(\mathcal{I}(s) = \mathcal{I}_{irred}(s) \cup \mathcal{I}_{red}(s)\), with obvious meaning of the notation.

### 6.2 \(U^+\) and \(U^-\) Irreducible

In this section we derive theories based on triplets in \(\mathcal{I}_{irred}(s)\). In these triplets the irreducible component \(U^+\) is identified, according to (16)–(18) in Sect. 5.1, by a value \(m^+\) of the parameter \(m\); analogously, \(U^-\) is identified by the value \(m^-\) of \(m\), according to (19), (20). Therefore, the Hilbert space of the representation \(U = U^+ \oplus U^-\) is \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,\) where \(\mathcal{H}^+ = L_2(\mathbb{R}^3, dv) = \mathcal{H}^-\). If every vector \(\psi = \psi^+ + \psi^- \in \mathcal{H}\), with \(\psi^\pm \in \mathcal{H}^\pm\) is represented by the column vector \(\psi \equiv \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix}\), then the generators are represented by the following matrices.

\[
P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad J_j = \begin{bmatrix} J_j^{(0)} + j_j^+ & 0 \\ 0 & J_j^{(0)} + j_j^- \end{bmatrix}, \quad K_j = \begin{bmatrix} K_j^{(0)} + k_j^+ & 0 \\ 0 & -(K_j^{(0)} + k_j^-) \end{bmatrix}, \quad (25.\text{i})
\]

where \(j_j^\pm\) and \(k_j^\pm\) are given by (18) with the value \(m^\pm\) of \(m\). Whenever \(\psi^\pm \in \mathcal{H}^\pm\), the relation \(\lambda = \frac{m^\pm}{2}\) holds, of course.

**Proposition 6.2** If \(U\) is a representation of a triplet in \(\mathcal{I}(s)\) with generators given by (25), then \(m^- = -m^+\); furthermore, whenever \(m^+ \neq 0, \psi \in \mathcal{H}^\pm\) implies \(\mathcal{S}\psi \in \mathcal{H}^\mp\).

**Proof** Given any non vanishing vector \(\psi \in \mathcal{H}^+,\) let us define \(\tilde{\psi} = \mathcal{S}\psi\). Of course, \(\tilde{\psi} = \varphi^+ + \varphi^-\), with \(\varphi^\pm \in \mathcal{H}^\pm\). By making use of (10) we find

\[
\hat{\lambda} \tilde{\psi} = \hat{\lambda} \mathcal{S} \psi = \mathcal{S} \mathcal{S}^{-1} \hat{\lambda} \mathcal{S} \psi = \mathcal{S} (-\hat{\lambda}) \psi = -\frac{m^+}{2} \tilde{\psi} = -\frac{m^+}{2} \varphi^+ - \frac{m^+}{2} \varphi^-. \quad (26)
\]

On the other hand \(\hat{\lambda} \tilde{\psi} = \hat{\lambda} (\varphi^+ + \varphi^-) = \frac{m^+}{2} \varphi^+ + \frac{m^-}{2} \varphi^-\), hence by (26) we obtain

\[
-m^+ \varphi^+ - \frac{m^+}{2} \varphi^- = \frac{m^+}{2} \varphi^+ + \frac{m^-}{2} \varphi^- . \quad (27)
\]
If \( m^+ \neq 0 \), then (27) implies \( \varphi^+ = 0 \), i.e. \( \mathcal{S}\psi \in \mathcal{H}^- \), and \( m^- = -m^+ \).

If \( m^- \neq 0 \), then the previous derivation, carried out starting from a vector \( \psi \in \mathcal{H}^- \), leads to conclude that \( \varphi^- = 0 \), i.e. \( \mathcal{S}\psi \in \mathcal{H}^+ \) and \( m^+ = -m^- \) hold. As a consequence, \( m^+ = 0 \) if and only if \( m^- = 0 \). \( \square \)

Thus, in every triplet of \( \mathcal{I}(s) \) with \( U^+ \) irreducible the operator \( \mathcal{S} \) is identified by a matrix of the from

\[
\mathcal{S} = \begin{bmatrix} 0 & \mathcal{S}_1 \\ \mathcal{S}_2 & 0 \end{bmatrix},
\]

and the representation \( U \) is characterized by a value of the parameter \( m \) such that

\[
J_j = \begin{bmatrix} J^{(0)}_j + j_j & 0 \\ 0 & J^{(0)}_j - j_j \end{bmatrix}, \quad K_j = \begin{bmatrix} K^{(0)}_j + k_j & 0 \\ 0 & -K^{(0)}_j + k_j \end{bmatrix},
\]

(25.iii)

where \( j_j \) and \( k_j \) are given by (18). Now we address the problem of determining the complete triplets, that is to say the operators \( \mathcal{S} \) and \( \hat{\mathcal{S}} \). Since \( \sigma(P) = S^+_0 \cup S^-_0 \), according to Prop. 4.1 the possible combinations are

- (U.U) \( \mathcal{S} \) unitary and \( \hat{\mathcal{S}} \) unitary,
- (A.U) \( \mathcal{S} \) anti-unitary and \( \hat{\mathcal{S}} \) unitary,
- (A.A) \( \mathcal{S} \) anti-unitary and \( \hat{\mathcal{S}} \) anti-unitary.

### 6.2.1 Theories with Zero Helicity

First we determine \( \mathcal{S} \) and \( \hat{\mathcal{S}} \) for the triplets with \( m = 0 \), whose generators are given by

\[
P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad J_j = \begin{bmatrix} J^{(0)}_j & 0 \\ 0 & J^{(0)}_j \end{bmatrix}, \quad K_j = \begin{bmatrix} K^{(0)}_j & 0 \\ 0 & -K^{(0)}_j \end{bmatrix}.
\]

(29)

**Combination (U.U).** Let \( (U, \mathcal{S}, \hat{\mathcal{S}}) \) be a triplet with generators given by (29) and \( \mathcal{S} \) unitary. Let us define the unitary operator \( \hat{\mathcal{S}} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \Upsilon \mathcal{S} \), so that \( \mathcal{S} = \Upsilon \hat{\mathcal{S}} \).

By making use of \( \mathcal{S}P_j = -P_j\mathcal{S} \) in (5) and of the completeness of \( P \) in \( L_2(\mathbb{R}^3, d\nu) \), we find that the entries \( S_{mn} \) of \( \hat{\mathcal{S}} \) must be complex functions of \( \mathbf{p} \): \( S_{mn} = S_{mn}(\mathbf{p}) \). By making use of \( [\mathcal{S}, P_0] = 0 \) in (5) we find \( S_{12} = S_{21} = 0 \). Finally, \( \mathcal{S}K_j = -K_j\mathcal{S} \) in (5) implies \( S_{11}(\mathbf{p}) = \text{constant}, S_{22}(\mathbf{p}) = \text{constant} \). The further condition \( \mathcal{S}^2 = \mathbb{I} \) in (5) implies that there are two possibilities for \( \mathcal{S} \): \( \mathcal{S} = \Upsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) or \( \mathcal{S} = \Upsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

In order to determine \( \hat{\mathcal{S}} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \), we make use of the relations in (7), analogously to what done for determining \( \mathcal{S} \) making use of the relations in (5). As a result we find \( \hat{\mathcal{S}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Therefore, there are two inequivalent triplets with \( \mathcal{S} \) unitary and \( \hat{\mathcal{S}} \) unitary.
Combination (A.U). To determine the anti-unitary \( \mathcal{S} \) we define the unitary operator \( \hat{S} = \mathcal{K} \mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \). By making use of the relations (6) we find \( \mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) when \( \mathcal{S}^2 = 1 \) and \( \mathcal{S} = \mathcal{K} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) when \( \mathcal{S}^2 = -1 \).

The operator \( \mathcal{T} \), being unitary, is \( \mathcal{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) as in combination (U.U). Therefore, also in this case there are two inequivalent triplets.

Combination (A.A). The operator \( \mathcal{S} \), being anti-unitary, is \( \mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), as in combination (A.U). To determine the anti-unitary \( \mathcal{T} \) we define the unitary operator \( \hat{T} = \mathcal{K} \mathcal{T} \mathcal{S} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \). By making use of the relations (8) we find \( \mathcal{T} = \mathcal{K} \mathcal{T} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \). By transforming every operator \( A \) into \( WAW^{-1} \), where \( W = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \), we obtain an equivalent theory. In so doing all generators are left invariant, while \( \mathcal{T} \) transforms into \( \mathcal{T} = \mathcal{K} \mathcal{T} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). So there are two inequivalent triplets.

Thus, in \( \mathcal{I}(s) \) there are six inequivalent triplets with zero helicity and \( U^+ \) irreducible.

6.2.2 Terms with \( m \neq 0 \)

For the triplets in \( \mathcal{I}(s) \) with representations characterized by \( m \neq 0 \), the generators are given by (25.i,iii). Equation (28) implies that \( \mathcal{S} \) cannot be unitary; in this case, indeed, once introduced the unitary operator \( \hat{S} = \mathcal{T} \mathcal{S} \), the relations \( \mathcal{S} P_j = -P_j \mathcal{S} \) and \( [\mathcal{S}, P_0] = 0 \) in (5) would imply \( \hat{S} = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} \) while (28) implies \( \hat{S} = \begin{bmatrix} 0 & S_{11} \\ S_{22} & 0 \end{bmatrix} \), so that \( \mathcal{S} \) should be \( 0 \). Therefore combinations (U.U) and (U.A) cannot occur.

Now we see that if \( m \neq 0 \) then \( \mathcal{T} \) cannot be unitary. Let \( \mathcal{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \). Conditions \([P_j, \mathcal{T}] = 0\) and \( P_0 \mathcal{T} = -\mathcal{T} P_0 \) in (7) imply \( \mathcal{T} = \begin{bmatrix} 0 & T_{11}(p) \\ T_{22}(p) & 0 \end{bmatrix} \). Now, the condition \([J_j, \mathcal{T}] = 0\) in (7) implies \([J_j, T_1] = -2j T_1 \) and \([J_j, T_2] = 2j T_2 \). In particular, for \( j = 3 \) we have

\[
i \left( p_2 \frac{\partial T_n}{\partial p_1} - p_1 \frac{\partial T_n}{\partial p_2} \right) = 0, \quad n = 1, 2. \quad (30)\]

On the other hand, the condition \( K_j \mathcal{T} = -\mathcal{T} K_j \) in (7) leads to the equations \( i p_0 \frac{\partial T_1}{\partial p_j} = -2k_j T_1 \) and \( i p_0 \frac{\partial T_2}{\partial p_j} = 2k_j T_2 \); in particular

\[
\frac{\partial T_n}{\partial p_1} = -2i \frac{p_2 p_3}{p_1^2 + p_2^2} \frac{1}{p_0} T_n, \quad \frac{\partial T_n}{\partial p_2} = 2i \frac{p_1 p_3}{p_1^2 + p_2^2} \frac{1}{p_0} T_n. \quad (31)\]

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By making use of (31) in (30) we obtain $2p_0^3 T_n = 0$, which can hold only if $T_n = 0$. This means that $\mathcal{T}$ cannot be unitary.

Now we show that the combination (A.A), where both $\mathcal{S}$ and $\mathcal{T}$ are anti-unitary, can occur. If $\mathcal{S}$ is anti-unitary, once defined $\hat{S} = \mathcal{K} \mathcal{S} = \begin{bmatrix} 0 & S_1 \\ S_2 & 0 \end{bmatrix}$, the relation $[\mathcal{S}, P_j] = \mathcal{O}$ in (6) implies $S_1 = S_1(\mathbf{p})$ and $S_2 = S_2(\mathbf{p})$. The condition $[\mathcal{S}, K_j] = \mathcal{O}$ of (6), where (16) is used, implies the equalities $S_1(\mathbf{p}) = \text{constant}$ and $S_2(\mathbf{p}) = \text{constant}$, which are consistent with the condition $[\mathcal{S}, P_0] = \mathcal{O}$ in (6). Hence we can set $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The condition $\mathcal{S}^2 = \pm 1$ implies that $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if $\mathcal{S}^2 = 1$ and $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ if $\mathcal{S}^2 = -1$.

Let $\mathcal{T}$ be anti-unitary. If we define $\hat{T} = \mathcal{K} \gamma \mathcal{T}$, then conditions $\mathcal{T} P_j = -P_j \mathcal{T}$ and $[P_0, \mathcal{T}] = \mathcal{O}$ in (8) imply $\mathcal{T} = \mathcal{K} \gamma \begin{bmatrix} T_1(\mathbf{p}) & 0 \\ 0 & T_2(\mathbf{p}) \end{bmatrix}$.

By making use of this result and of (25.3) in condition $[\mathcal{T}, K_j] = \mathcal{O}$ in (8), we obtain $T_1(\mathbf{p}) = \text{constant}$, $T_2(\mathbf{p}) = \text{constant}$, i.e. $\mathcal{T} = \mathcal{K} \gamma \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$. By transforming every operator $A$ into $W A W^{-1}$, where $W = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$, an equivalent theory is obtained. In so doing, all generators are left invariant whereas $\mathcal{T}$ is transformed into $\mathcal{K} \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Thus, + that for every $m \neq 0$ there are two inequivalent triplets in $\mathcal{I}(s)$, whose generators are given by (25.1) and (25.3): a triplet with $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathcal{T} = \mathcal{K} \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and another triplet with $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\mathcal{T} = \mathcal{K} \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

### 6.3 $U^+$ Reducible

The triplets so far identified do not exhaust all possible irreducible triplets. Indeed, the class of irreducible triplets of $\mathcal{I}(s)$ with $U^+$, and hence $U^-$ reducible is not empty. Now we present an instance of these triplets. The Hilbert space of the triplet is $\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \oplus \mathcal{H}^{(4)}$, where $\mathcal{H}^{(n)} = L_2(\mathbb{R}^3, dv)$, $n = 1, 2, 3, 4$. Every vector $\psi \in \mathcal{H}$ is represented as a column vector $\psi = \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \\ \psi^{(3)} \\ \psi^{(4)} \end{bmatrix}$, with $\psi^{(m)} \in L_2(\mathbb{R}^3, dv)$.  

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The projections $E^+$, $E^-$ and the self-adjoint generators satisfying (3) are

$$E^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & -p_0 \end{bmatrix},$$

$$P_j = \begin{bmatrix} p_j & 0 & 0 & 0 \\ 0 & p_j & 0 & 0 \\ 0 & 0 & p_j & 0 \\ 0 & 0 & 0 & p_j \end{bmatrix}, \quad J_k = \begin{bmatrix} j_k & 0 & 0 & 0 \\ 0 & j_k & 0 & 0 \\ 0 & 0 & j_k & 0 \\ 0 & 0 & 0 & j_k \end{bmatrix}, \quad K_j = \begin{bmatrix} k_j & 0 & 0 & 0 \\ 0 & k_j & 0 & 0 \\ 0 & 0 & k_j & 0 \\ 0 & 0 & 0 & -k_j \end{bmatrix}.$$

So, we have a reducible representation of $P_+$. Now we extend it to an irreducible transformer triplet, by introducing a unitary $\sim_T$ and an anti-unitary $\sim_S$ satisfying (5)–(9), as

$$\sim_T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \sim_S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, let $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$ be any self-adjoint operator of $H$; the conditions $[A, P_0] = [A, P_j] = [A, J_k] = [A, K_j] = [A, \sim_T] = [A, \sim_S] = \mathcal{O}$ are satisfied if and only if $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \equiv a \mathbb{I}$ with $a \in \mathbb{R}$. Thus the triplet $(U, \sim_S, \sim_T)$ is irreducible.

### 6.4 Summary

Let us summarize the class of possible theories of massless particle based on irreducible triplets with $r_0 = 0$.

In $\mathcal{I}(u)$ (resp. $\mathcal{I}(d)$),

- if $U$ is irreducible then there is just one triplet, identified by (21) (resp. (22)), and $\hat{\lambda} = 0$;
- if $U$ is reducible, then each irreducible component is identical to the representation with zero helicity.

In $\mathcal{I}(s)$, if $U^+$ is irreducible, then

- there are six possible theories with zero helicity, determined by (29), two for each combination $(U,U)$, $(A,U)$, $(A,A)$, according to Sect. 6.2.1.

For every $m \in \mathbb{Z} \setminus \{0\}$ there are two triplets with non-zero helicity, determined by (25.i,iii); they have both $\sim_S$ and $\sim_T$ anti-unitary and opposite value of the helicity,
\( \pm m^2 \), according to Sect. 6.2.2. In view of these results, whenever a non-zero value of the helicity is measured on an isolated system, then its theory must be developed with one of these two new triplets, while triplets with \( S \) unitary and \( T \) anti-unitary are inconsistent with the relativistic invariance principle. As a consequence, also the opposite value must occur. For instance, the theory of the photon, which experimentally exhibited \( \pm 1 \) helicity [27], must be based on one of the two triplets with \( m = \pm 2 \).

7 Identification of Particle Theories

The identification of massless elementary particle theories can be addressed by checking which of the transformer triplets identified in Sects. 5 and 6 admit a position operator \( Q = (Q_1, Q_2, Q_3) \) according to Sect. 3.3.

7.1 Localizability of Zero Helicity Systems

Massless isolated systems with zero helicity whose theory is based on a triplet of \( I(u) \) or \( I(d) \) are always localizable [8,14]. Following [8], indeed, if the representation \( U \) of the triplet in \( I(u) \) is irreducible, then the Newton and Wigner three-operator \( F = (F_1, F_2, F_3) \), defined by \( F_j = i \frac{\partial}{\partial p_j} - \frac{i}{2p_0} p_j \), is proved to be the unique three-operator such that (12) hold. This conclusion holds also for the triplet in \( I(d) \). Now we extend this result to a theory with transformer triplet in \( I(s) \).

7.1.1 Particle Theories Theories with Triplets in \( I(s) \) and Zero Helicity

According to Sect. 6.2, there are six inequivalent triplets in \( I(s) \) with zero helicity and \( U^+ \) irreducible. The Hilbert space of these triplets is \( \mathcal{H} = L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv) \) and the generators are given by (29). We check whether a position operator \( \hat{Q} \) with components \( \hat{Q}_j = \begin{bmatrix} Q_{j11} & Q_{j12} \\ Q_{j21} & Q_{j22} \end{bmatrix} \) exists such that the conditions (12) hold. Hence, we introduce the operator \( \hat{F}_j = \begin{bmatrix} F_j & 0 \\ 0 & F_j \end{bmatrix} \), where the \( F_j \)’s form the Newton and Wigner three-operator, and the operators \( \hat{D}_j = \hat{Q}_j - \hat{F}_j \).

By making use of (29) it can be verified that \( [F_j, P_k] = i \delta_{jk} \) and \( [J^{(0)}_j, F_k] = i \hat{\epsilon}_{jkl} F_l \) hold. On the other hand the conditions (12.ii) hold, so that \( [\hat{D}_j, P_k] = i \delta_{jk} \) and \( [J^{(0)}_j, \hat{D}_k] = i \hat{\epsilon}_{jkl} \hat{D}_l \). These relations imply that \( \hat{D}_j = \hat{d}(p_0) p_j \), where \( \hat{d}(p_0) = \begin{bmatrix} d_{11}(p_0) & d_{12}(p_0) \\ d_{21}(p_0) & d_{22}(p_0) \end{bmatrix} \).

Since \( S \hat{F} = -\hat{F} S \) and \( T \hat{F} = \hat{F} T \) hold for all six triplets, (12.i) implies that the following relations must always hold.

\[ S \hat{D} = -\hat{D} S, \quad [\hat{T}, \hat{D}] = 0 \] (32)
By making use of (32) in the six inequivalent triplets we find that

\[
\text{if } S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ then } \hat{D}_j = \begin{bmatrix} d_1(p_0) & d_2(p_0) \\ d_2(p_0) & d_1(p_0) \end{bmatrix} p_j; \tag{33.i}
\]

\[
\text{if } S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ then } \hat{D}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} p_j; \tag{33.ii}
\]

\[
\text{if } S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ then } \hat{D}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} p_j; \tag{33.iii}
\]

\[
\text{if } S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ then } \hat{D}_j = \begin{bmatrix} d(p_0) & 0 \\ 0 & -id(p_0) \end{bmatrix} p_j; \tag{33.iv}
\]

\[
\text{if } S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ then } \hat{D}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} p_j. \tag{33.vi}
\]

Thus, there are three inequivalent theories, identified by (33.ii), (33.iii) and (33.vi), where there is a unique position operator \( \hat{Q} = \hat{F} \) because \( \hat{D} = \Theta \). In particular, the theory corresponding to (33.iii) has \( S \) anti-unitary and \( \Upsilon \) unitary.

### 7.2 Non-localizability of Non-zero Helicity Particles

The several investigations about the localizability of massless particle, carried out through different approaches, agree in concluding that massless particles with non-zero helicity are not localizable. However, the theoretical structures for which the non-existence of a position operator is proved are triplets where \( S \) is unitary and \( \Upsilon \) is anti-unitary [8]. In view of the analysis performed in the present work, this is a serious shortcoming, because according to Prop. 4.1 these structures must be triplets in \( I(\mathbf{u}) \) or \( I(\mathbf{d}) \). But in Sect. 6 it is proved that irreducible triplets with non-zero helicity can exists only in \( I(\mathbf{s}) \). Therefore, these non-localizability proofs do not apply. Now we present an altripletative proof independent of \( S \) and \( \Upsilon \), so that it avoids the shortcomings of the previous ones.

**Lemma 7.1** If \( m \neq 0 \) then the following system of equations has no solution for the functions \( d_j(p) \), \( j = 1, 2, 3 \).

\[
p_3 \frac{\partial d_3}{\partial p_1} - p_1 \frac{\partial d_3}{\partial p_3} = -d_1 + \frac{m}{2} \frac{p_2 p_3}{p_0(p_1^2 + p_2^2)}; \tag{d.1}
\]

\[
p_1 \frac{\partial d_3}{\partial p_2} = p_2 \frac{\partial d_3}{\partial p_1}; \tag{d.2}
\]

\[
p_2 \frac{\partial d_2}{\partial p_3} - p_3 \frac{\partial d_2}{\partial p_2} = -d_3 - \frac{m}{2} \left( \frac{p_0 p_1 p_2}{(p_1^2 + p_2^2)^2} - \frac{p_1 p_2}{p_0(p_1^2 + p_2^2)} \right); \tag{d.3}
\]

\[
p_3 \frac{\partial d_1}{\partial p_1} - p_1 \frac{\partial d_1}{\partial p_3} = d_3 - \frac{m}{2} \left( \frac{p_0 p_1 p_2}{(p_1^2 + p_2^2)^2} - \frac{p_1 p_2}{p_0(p_1^2 + p_2^2)} \right); \tag{d.4}
\]
\[
\begin{align*}
p_2 \frac{\partial d_1}{\partial p_3} - p_3 \frac{\partial d_1}{\partial p_2} &= -\frac{m}{2} \left( \frac{p_0 p_1^2}{(p_1^2 + p_2^2)^2} - \frac{p_0}{p_1^2 + p_2^2} - \frac{p_1^2}{p_0(p_1^2 + p_2^2)} \right) \quad \text{(d.5)} \\
p_1 \frac{\partial d_2}{\partial p_2} - p_2 \frac{\partial d_2}{\partial p_1} &= d_1 \quad \text{(d.6)} \\
p_3 \frac{\partial d_2}{\partial p_1} - p_1 \frac{\partial d_2}{\partial p_3} &= -\frac{m}{2} \left( \frac{p_0 p_2^2}{(p_1^2 + p_2^2)^2} - \frac{p_0}{p_1^2 + p_2^2} - \frac{p_2^2}{p_0(p_1^2 + p_2^2)} \right) \quad \text{(d.7)} \\
p_1 \frac{\partial d_1}{\partial p_2} - p_2 \frac{\partial d_1}{\partial p_1} &= d_2 \quad \text{(d.8)} \\
p_2 \frac{\partial d_3}{\partial p_3} - p_3 \frac{\partial d_3}{\partial p_2} &= d_2 + \frac{m}{2} \frac{p_3 p_1}{p_0(p_1^2 + p_2^2)}. \quad \text{(d.9)}
\end{align*}
\]

**Proof** Once defined \( \zeta(\mathbf{p}) = \mathbf{p} \cdot \mathbf{d}(\mathbf{p}) = \sum_j p_j d_j(\mathbf{p}) \), making use of (d.1)-(d.9) we obtain, after a certain amount of computation,

\[
\begin{align*}
p_2 \frac{\partial \zeta}{\partial p_3} - p_3 \frac{\partial \zeta}{\partial p_2} &= \frac{m}{2} \frac{p_1 p_0}{p_1^2 + p_2^2} \quad \text{(34)} \\
p_3 \frac{\partial \zeta}{\partial p_1} - p_1 \frac{\partial \zeta}{\partial p_3} &= \frac{m}{2} \frac{p_2 p_0}{p_1^2 + p_2^2} \quad \text{(35)} \\
p_1 \frac{\partial \zeta}{\partial p_2} - p_2 \frac{\partial \zeta}{\partial p_1} &= 0 \quad \text{(36)}
\end{align*}
\]

Now, Eqs. (36) and (35) imply \( \frac{\partial \zeta}{\partial p_3} = -\frac{m}{2} \frac{p_0}{p_1^2 + p_2^2} \frac{p_3}{p_1} \), which becomes \( \frac{m}{2} \frac{p_0}{p_1} = 0 \) by (34).

If \( m \) were not zero, then the last equation would imply \( p_0 = 0 \). Therefore, there is no solution for \( \zeta \), and hence for \( d_j \). \( \square \)

**Proposition 7.1** If the theory of a massless isolated system is based on a triplet \((U, \mathcal{S}, \varphi^0)\) of non-zero helicity with \( r_0 = 0 \), then there is no position operator.

**Proof** If \( U \) is irreducible then its triplet must belong to \( \mathcal{I}(\mathbf{u}) \) or \( \mathcal{I}(\mathbf{d}) \); hence, according to Prop. 5.1 the helicity must be zero. Therefore, it is sufficient to prove the proposition for the case that \( U \) is reducible. If the helicity is not zero, then an irreducible component \( U^{(1)} \) must exists with non zero helicity; suppose that such \( U^{(1)} \) belongs to \( \mathcal{I}(\mathbf{u}) \), so that its Hilbert space is \( \mathcal{H}^{(1)} = L_2(\mathbb{R}^3, dv) \) and the generators are given by (16)–(18). Hence \( U = U^{(1)} \oplus U^{(2)} \), where \( U^{(2)} \) can be reducible. The Hilbert space of \( U \) decomposes as \( \mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \), where \( \mathcal{H}^{(2)} \) is the Hilbert space of \( U^{(2)} \). According to such a decomposition, the generators are

\[
\begin{align*}
P_0 &= \begin{bmatrix} p_0 & 0 \\ 0 & p_0^{(2)} \end{bmatrix}, \quad P_j &= \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad \text{(37.i)} \\
J_j &= \begin{bmatrix} J_j^{(0)} + i j & 0 \\ 0 & J_j^{(2)} \end{bmatrix}, \quad K_j &= \begin{bmatrix} K_j^{(0)} + k_j & 0 \\ 0 & K_j^{(2)} \end{bmatrix}. \quad \text{(37.ii)}
\end{align*}
\]
If a position operator \( \hat{Q} = (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3) \) exists, then each coordinate operator \( \hat{Q}_j \) must have the form \( \hat{Q}_j = \left[ \begin{array}{c} \hat{Q}_{j11} \\ \hat{Q}_{j12} \\ \hat{Q}_{j21} \\ \hat{Q}_{j22} \end{array} \right] \), where \( \hat{Q}_{j11} \) is a self-adjoint operator of \( L_2(\mathbb{R}^3, dv) \). Let us define the self-adjoint operators

\[
\hat{D}_j = \left[ \begin{array}{cc} \hat{Q}_{j11} - F_j & \hat{Q}_{j12} \\ \hat{Q}_{j21} & \hat{Q}_{j22} - F_j \end{array} \right] = \left[ \begin{array}{cc} d_j & D_{j12}(p_0) \\ D_{j21} & D_{j22} \end{array} \right],
\]

where \( F_j \) is the \( j \)-th component of the Newton and Wigner operator. We prove that (12) implies conditions for \( d_j \) that cannot be satisfied, and thus no position operator exists.

Since \([F_j, p_k] = i \delta_{jk}\) holds, (12.iii) implies \([\hat{D}_j, P_k] = 0\), in particular \([d_j, p_k] = 0\); therefore each \( d_j \) is a function of \( p \): \( d_j = d_j(p) \). Since \([J_j^{(0)}, F_k] = i \hat{\epsilon}_{jkl} F_l\), (12.iii) implies \([J_j^{(0)} + j, d_k(p)] = i \hat{\epsilon}_{jkl} d_l(p) - [j, F_k]\), i.e.,

\[
[J_j^{(0)}, d_k(p)] = i \hat{\epsilon}_{jkl} d_l(p) - [j, F_k]. \tag{38}
\]

Making use of \( J_j^{(0)} = -i \left( p_k \frac{\partial}{\partial p_l} - p_l \frac{\partial}{\partial p_k} \right) \) and (18), the computation of \([j, F_k]\) for all \((j, k)\) yields equations (d.1)-(d.9). Lemma 7.1. proves that no solution \( d(p) \) exists for these equations if \( m \neq 0 \). Thus, no position operator can exist. \( \square \)

It must be stressed how this proof of non-localizability is carried out without making use of the operators \( \mathcal{S} \) and \( \mathcal{T} \), so that it holds also if space inversion or time reversal are not assumed to be symmetries of the system.

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