Lower bound on the compactness of isotropic ultra-compact objects

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Horizonless spacetimes describing spatially regular ultra-compact objects which, like black-hole spacetimes, possess closed null circular geodesics (light rings) have recently attracted much attention from physicists and mathematicians. In the present paper we raise the following physically intriguing question: How compact is an ultra-compact object? Using analytical techniques, we prove that ultracompact isotropic matter configurations with light rings are characterized by the dimensionless lower bound \( \max_r \{2m(r)/r\} > 7/12 \) on their global compactness parameter.

I. INTRODUCTION

Null circular geodesics on which massless particles (gravitons and photons) can orbit a central astrophysical object are a generic feature of highly compact matter configurations [1–3]. In particular, the compact theorem presented in [4] has revealed the fact that spherically symmetric black-hole spacetimes with a non-positive energy-momentum trace \( T \leq 0 \) [5, 6] must posses at least one closed null circular geodesic.

It is well known, however, that horizonless matter configurations whose characteristic global compactness parameter \( \max_r \{2m(r)/r\} \geq 3 \) is less than that of classical black-hole spacetimes [0], may also possess null circular geodesics (see [10–17] and references therein). As possible spatially regular exotic alternatives to the canonical black-hole spacetimes, the physical and mathematical properties of these horizonless curved spacetimes with light rings have attracted much attention in recent years [10–17].

Spatially regular matter configurations which, like black-hole spacetimes, possess null circular geodesics are usually known in the physics literature by the exotic name ultra-compact objects [10–17]. In the present paper we would like to raise the following physically interesting question: How compact is an ultra-compact object which possesses light rings?

In particular, one naturally wonders whether it is possible, within the framework of general relativity, to give a more precise quantitative meaning to the important physical term ‘ultra-compact object’? In this context, it is worth mentioning that the theorem presented in [11] has revealed the fact that spatially regular matter configurations with a non-negative energy-momentum trace \( T \geq 0 \) are necessarily ultra-compact. That is, the corresponding spherically symmetric horizonless curved spacetimes must possess (at least) one light ring [13]. Moreover, it has been proved in [11] that the characteristic global compactness parameter of these horizonless matter configurations is bounded from below by the simple dimensionless relation

\[
\max_r \left\{ \frac{2m(r)}{r} \right\} > \frac{2}{3} \quad \text{for} \quad T \geq 0 ,
\]

where \( m(r) \) is the gravitational mass of the matter fields contained within a sphere of areal radius \( r \). To the best of our knowledge, no analoguous lower bound on the dimensionless global compactness parameter \( \max_r \{2m(r)/r\} \) of ultra-compact objects with light rings and a negative energy-momentum trace [5] has thus far been presented in the physics literature.

The main goal of the present paper is to study analytically the physical and mathematical properties of spherically symmetric ultra-compact objects whose horizonless spacetimes possess closed light rings (null circular geodesics). In particular, below we shall explicitly prove that the global compactness parameter of spatially regular ultra-compact isotropic matter configurations with a negative energy-momentum trace is bounded from below by the dimensionless relation \( \max_r \{2m(r)/r\} > 7/12 \).

II. DESCRIPTION OF THE SYSTEM

We study the physical properties of horizonless spatially regular isotropic matter configurations which possess closed light rings (null circular geodesics). These ultra-compact objects are described by the spherically symmetric line element [2, 19]

\[
ds^2 = -e^{-2\delta}(1 - C)dt^2 + (1 - C)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) ,
\]
where, for spatially regular asymptotically flat spacetimes, the radially dependent metric functions \( \{C, \delta\} \) are characterized by the small-\( r \) \[ C(r \to 0) = O(r^2) \quad \text{and} \quad \delta(0) < \infty \] and large-\( r \) \[ C(r \to \infty) \to 0 \quad \text{and} \quad \delta(r \to \infty) \to 0 \] functional behaviors.

The spatial behavior of these radially-dependent static metric functions is determined by the Einstein equations \( G_{\mu}^{\nu} = 8\pi T_{\mu}^{\nu} \). In particular, denoting the components of the spherically symmetric energy-momentum tensor by \[ \rho \equiv -T_{t}^{t}, \quad \text{and} \quad \rho \equiv T_{r}^{r} = T_{\theta}^{\theta} = T_{\phi}^{\phi}, \] where \( \rho \) and \( p \) are respectively the energy density and the isotropic pressure of the spatially regular matter configurations, one can express the Einstein-matter field equations in the form \[ C' = 8\pi r \rho - C \] and

\[ \delta' = -\frac{4\pi r (\rho + p)}{1 - C}. \]

The components of the energy-momentum tensor which characterize the static matter configurations are assumed to satisfy the dominant energy condition \[ 0 \leq |p| \leq \rho. \]

The gravitational mass contained within a sphere of radius \( r \) is given by the integral relation \[ m(r) = 4\pi \int_0^r x^2 \rho(x) dx, \] which, taking cognizance of Eq. (3), yields the relation \[ C(r) = \frac{2m(r)}{r} \]

for the characteristic radially dependent dimensionless compactness parameter of the spatially regular self-gravitating matter configurations.

Below we shall analyze the spatial behavior of the dimensionless isotropic pressure function \[ P(r) \equiv r^2 p(r). \]

Taking cognizance of Eqs. (3), (9), and (11), one finds that, for spatially regular asymptotically flat spacetimes, this radially dependent pressure function is characterized by the simple small-\( r \) \[ P(r \to 0) \to 0 \] and large-\( r \) \[ r P(r \to \infty) \to 0 \] functional behaviors. Substituting the Einstein differential equations (6) and (7) into the conservation equation \[ T_{\mu}^{\mu} = 0, \]

one obtains the gradient relation \[ \frac{2}{r} P'(r) = \frac{\mathcal{R}(\rho + p)}{1 - C} + 2(-\rho + p) \]

for the dimensionless pressure function (11), where \[ \mathcal{R}(r) = 2 - 3C - 8\pi P. \]
III. NULL CIRCULAR GEODESICS OF SPHERICALLY SYMMETRIC CURVED SPACETIMES

The characteristic radial equation for the null circular geodesics of spherically symmetric curved spacetimes has been derived in [2, 20, 24]. For completeness of the presentation, we shall give in the present section a brief sketch of the derivation. We first note that, as explicitly proved in [2, 20, 24], the null circular geodesics of the spherically symmetric spacetime (2) are characterized by the relations [25]

\[ V_r = E^2 \quad \text{and} \quad V'_r = 0 , \tag{17} \]

where the effective radial potential that governs the null trajectories is given by [2, 20, 24]

\[ V_r = (1 - e^{2\delta})E^2 + (1 - C)L^2 \frac{r}{r^2} . \tag{18} \]

Here the conserved physical quantities \( \{ E, L \} \) are respectively the energy and the angular momentum along the null trajectories [2, 20, 24].

Taking cognizance of the Einstein equations (6) and (7), one obtains from Eqs. (17) and (18) the radial equation [2, 20, 24]

\[ \mathcal{R}(r = r_\gamma) = 0 \tag{19} \]

which, for compact enough matter configurations [see Eq. (28) below], determines the discrete radii of the null circular geodesics (light rings) that characterize the spherically symmetric curved spacetime (2). Interestingly, it has recently been proved [11, 15, 16] that spatially regular horizonless spacetimes generally possess an even number of light rings [26].

IV. LOWER BOUND ON THE LOCAL COMPACTNESS PARAMETER AT THE OUTER LIGHT RING

In the present section we shall explicitly prove that, using analytical techniques, one can derive a lower bound on the dimensionless compactness parameter \( C(r_\text{out}^\gamma) \) [see Eq. (10)] which characterizes the outermost null circular geodesic (outermost light ring) of the spatially regular isotropic ultra-compact objects.

We first point out that, taking cognizance of Eqs. (3), (4), (12), and (13), one finds that the dimensionless radial function \( \mathcal{R}(r) \) [see Eq. (16)] is characterized by the relations

\[ \mathcal{R}(r = 0) = 2 \quad \text{and} \quad \mathcal{R}(r \to \infty) \to 2 . \tag{20} \]

From Eqs. (19) and (20) one deduces that the outermost null circular geodesic of the spherically symmetric horizonless ultra-compact objects is characterized by the gradient relation

\[ \mathcal{R}'(r = r_\gamma^\text{out}) \geq 0 . \tag{21} \]

In addition, taking cognizance of Eqs. (5), (15), (16), and (19), one finds the functional relation

\[ \mathcal{R}'(r = r_\gamma) = \frac{2}{r_\gamma} \left[ 1 - 8\pi r_\gamma^2 (\rho + p) \right] , \tag{22} \]

which yields the characteristic inequality [see Eq. (21)]

\[ 8\pi (r_\gamma^\text{out})^2 (\rho + p) \leq 1 \tag{23} \]

at the outermost light ring of the ultra-compact matter configurations.

Taking cognizance of Eqs. (8), (11), (16), and (19), one obtains the dimensionless relations [27]

\[ C(r_\gamma^\text{out}) = \frac{1}{3} \left[ 2 - 8\pi (r_\gamma^\text{out})^2 p \right] \geq \frac{1}{3} \left[ 2 - 4\pi (r_\gamma^\text{out})^2 (\rho + p) \right] . \tag{24} \]

Substituting into (21) the characteristic inequality (23), one finds the lower bound

\[ C(r_\gamma^\text{out}) \geq \frac{1}{2} \tag{25} \]

on the dimensionless compactness parameter at the outermost light ring of the spatially regular horizonless ultra-compact objects.
V. LOWER BOUND ON THE GLOBAL COMPACTNESS PARAMETER OF THE ULTRA-COMPACT OBJECTS

In the present section we shall derive a lower bound on the global compactness parameter $\max_r \{C(r)\}$ which characterizes the spatially regular isotropic ultra-compact objects. As noted above [see Eq. (11)], it has previously been proved that regular self-gravitating matter configurations with a non-negative energy-momentum trace are characterized by the global lower bound [11, 28]

$$\max_r \{C(r)\} > \frac{2}{3} \quad \text{for} \quad T \geq 0 .$$

(26)

We shall now prove that a slightly weaker bound can be derived on the dimensionless compactness parameter $C(r^{\text{out}}_\gamma)$ of ultra-compact objects which are characterized by the opposite (negative) isotropic trace relation [5]

$$T = -\rho + 3p < 0 .$$

(27)

Taking cognizance of Eqs. (11), (16), (19), and (27), one obtains the series of dimensionless inequalities [29]

$$C(r^{\text{out}}_\gamma) = \frac{1}{3} \left[ 2 - 8\pi (r^{\text{out}}_\gamma)^2 p \right] > \frac{1}{3} \left[ 2 - 2\pi (r^{\text{out}}_\gamma)^2 (\rho + p) \right] \geq \frac{1}{3} \left( 2 - \frac{1}{4} \right) = \frac{7}{12}$$

(28)

for $T < 0$ which characterize the outermost null circular geodesic of the self-gravitating isotropic ultra-compact objects.

VI. SUMMARY

Spatially regular ultra-compact objects are described by horizonless curved spacetimes which, like black-hole spacetimes, possess closed light rings (null circular geodesics). These non-singular compact objects may provide exotic alternatives to canonical black-hole spacetimes and their physical properties have therefore been explored extensively in recent years (see [10–17] and references therein). It should be realized, however, that the characteristic dimensionless compactness parameter of these exotic matter configurations is lower than the corresponding compactness parameter $2m(r_H)/r_H = 1$ of spherically symmetric classical black-hole spacetimes.

In the present paper we have studied the physical and mathematical properties of the spatially regular ultra-compact matter configurations. In particular, we have examined the possibility of providing a quantitative meaning to the important physical term ‘ultra-compact object’. We have therefore raised the physically interesting question: How compact is an ultra-compact object?

Using analytical techniques, we have demonstrated that one can obtain a non-trivial lower bound on the characteristic compactness parameter of spherically symmetric ultra-compact objects. In particular, it has been explicitly proved that horizonless isotropic ultra-compact matter configurations which possess light rings are characterized by the compact dimensionless lower bound [see Eqs. (10), (26), and (28)]

$$\max_r \left\{ \frac{2m(r)}{r} \right\} > \frac{7}{12}$$

(29)

on their global compactness parameter.

Null circular geodesics (closed light rings) are usually associated with black-hole spacetimes. Interestingly, though, horizonless compact objects whose global compactness parameter is characterized by the sub-black hole relation $\max_r \{2m(r)/r\} < 1$ [9] may also possess null circular geodesics [10, 17]. In the present analysis we have explicitly proved that, under the assumptions of spherical symmetry and isotropy, closed light rings cannot be associated with arbitrarily dilute matter configurations. In particular, taking cognizance of the analytically derived lower bound (29) and the recently estimated compactness parameter of neutron stars [30], one deduces that astrophysically realistic isotropic neutron stars cannot possess light rings.

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The trace of a spherically symmetric energy-momentum tensor can be expressed in the form \( T = -\rho + p_r + 2p_T \), where \( \{\rho, p_r, p_T\} \) are respectively the energy density, the radial pressure, and the tangential pressure of the matter fields [4, 5].

We shall use natural units in which \( G = c = 1 \).

Here \( m(r) \) is the gravitational mass of the horizonless field configuration which is contained within a sphere of radius \( r \) [see Eq. (9) below].

Spherically symmetric black-hole spacetimes are characterized by the well-known compactness relation \( 2m(r_H)/r_H = 1 \), where \( r_H \) is the horizon radius of the central black hole.

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It is worth noting that these spatially regular horizonless spacetimes generally possess an even number of light rings [11, 15]. However, as explicitly proved in [16], there are special ultra-compact horizonless matter configurations with degenerate null circular geodesics [these special light rings are characterized by the relations \( R(r = r_\gamma) = R'(r = r_\gamma) = 0 \), see Eq. (16) below] which may possess an odd number of null circular geodesics.

Here \((t, r, \theta, \phi)\) are the familiar Schwarzschild spacetime coordinates.

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Here a prime \( ' \) denotes a spatial derivative with respect to the radial coordinate \( r \).