DARK MATTER OR A NEW FORCE?

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Abstract

We show that a new force, which appeared in a five-dimensional generalization of general relativity, implies precisely the flat circular velocity curves of luminous matter observed in the outer parts of spiral galaxies. This is attributed nowadays to a halo of dark matter exerting a Newtonian force on the luminous matter. The same result could also be obtained if general relativity were arbitrarily supplemented by a similar 4-tensor new force. Several other properties of this new force are discussed. First, it acts also on light and enhances the usual gravitational lensing by the luminous matter according to general relativity. Thus it acts like the hypothetical dark matter in this respect also. Second, an intriguing feature of the new force is that its energy tensor enters the field equations for the metric with a sign opposite to that of ordinary matter. Hence a cosmological solution for the new force (not the galactic solution treated in this paper) might explain the cosmic negative, or repulsive, gravitation responsible for the accelerated expansion of the universe, attributed nowadays to “dark energy” and sometimes modelled by Einstein’s cosmological constant.

Subject headings: galaxies — dark matter — flat rotation curves — new force.
1. INTRODUCTION

That the circular velocity $v_c$ of luminous matter departs from its Keplerian form in the outer parts of many spiral galaxies and approximately approaches a limit with distance ("flat rotation curve") has been known for about thirty years. This limiting value (Tully-Fisher relation) is

$$v_c = 220(L/L_\star)^{0.22} \text{ km s}^{-1} ,$$

where $L$ is the galaxy luminosity and $L_\star$, a certain standard luminosity. Nowadays this is generally ascribed to dark matter distributed in a halo around the galaxy and exerting a Newtonian force on the luminous matter. What this postulated dark matter is is not clear even today. Some experts, however, are not completely convinced by the dark matter hypothesis. Peebles for example (Peebles 1993) writes “But since this subject is still being explored, it is well to bear in mind the alternative that we are not using the right physics.”

A new force, which turned up first in a 5-D (five-dimensional) version of general relativity (Ingraham 1979), enforces precisely this flat circular velocity curve. This calculation is given below. This new force $f_{\alpha\beta} = -f_{\beta\alpha}$ is formally just like the electromagnetic force tensor, but is an extra form of gravitation beyond that given by the metric. Some details of this theory are given in §2.1.

This new force also acts on light, and produces a deflection of light passing through or around a galaxy in addition to that due to the mass of the galaxy predicted by GR (general relativity hereafter). So it would also act like the dark matter halo in this respect. However, a new feature is the following: this extra deflection does not have the full rotational symmetry of the GR deflection. This can be seen intuitively because the new force acts formally like the electromagnetic force in the motion equations of matter and light. We plan to examine the data on gravitational lensing to see if such an asymmetry shows up.

An intriguing feature of this new force is that it enters the field equations for the metric with the “wrong” sign, that is, like a sort of negative energy as opposed to the positive energy of ordinary matter or radiation. This sign is unambiguous because the new force’s stress-energy-momentum tensor $T_{\alpha\beta}$ enters the metric field equations without any coupling constant, to be contrasted with the stress-energy-momentum of ordinary matter, which
is coupled to the metric by Newton’s gravitational constant. This was noted in the original article as anomalous (Ingraham 1979, bottom p. 243). However, today such a new force should be quite welcome as a candidate for the so-called “dark energy” (Cowan 2001). Whether some cosmological solution for $f_{\alpha \beta}$ could explain the observed accelerating expansion of the universe is difficult as yet to say because of the difficulty of solving the cosmological field equations. These are now partial differential equations in the two independent variables $x^0 \equiv \text{time}$ and $x^5 \equiv \lambda$, the fifth coordinate. But the theory satisfies one stringent necessary condition precisely because it is five-dimensional. Namely, due to the isotropy demand for cosmological solutions, no purely space-time components $f_{\mu \nu}, \mu, \nu = 0, 1, 2, 3$, could survive because they obviously would define preferred directions in the spatial universe. But in five dimensions we can look for a solution with only $f_{05} \neq 0$.

2. DERIVATION OF THE FLAT CIRCULAR VELOCITY CURVE

We model the spiral galaxy as a spherically symmetric core of mass $M$ with negligible mass outside the core. The field equations for the 5-D metric $\gamma_{\alpha \beta}$ outside the core (Ingraham 1979, eqs. (2.19a,b) or (2.21a,b)) will not be needed here. The field equation for the new force $f_{\alpha \beta}$ outside the core is $\nabla_{\alpha} f^{\alpha \beta} = 0$, where $\nabla_{\alpha}$ is the covariant derivative, or equivalently

$$\partial_{\alpha} (\sqrt{\gamma} f^{\alpha \beta}) = 0, \quad \gamma \equiv \det \gamma_{\alpha \beta}.$$  

For this galactic problem we assume that $f_{\alpha \beta}$ is small of first order. Then it perturbs the metric only to second order, since $T_{\alpha \beta}$ is quadratic in $f_{\alpha \beta}$. Hence we can use a spherically symmetric O-order line element of Schwarzschild form:

$$d\Theta^2 = -\lambda^{-2} g_{\alpha \beta} \, dx^{\alpha} \, dx^{\beta}, \quad \alpha, \beta = 0, 1, 2, 3, 5,$$

$$g_{\alpha \beta} \, dx^{\alpha} \, dx^{\beta} = e^{2\mu} \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) - e^{2\nu} \, dt^2 + e^{2\xi} \, d\lambda^2,$$

$$x^1 \equiv r, \, x^2 \equiv \theta, \, x^3 = \phi, \, x^0 = t, \, x^5 \equiv \lambda \quad \text{(units: } c = 1),$$

where the metric coefficients $\mu, \nu, \text{ and } \xi$ are functions of $r$ and $\lambda$ only.

Here $d\Theta$ has the geometrical meaning of the infinitesimal angle of intersection of the space-time spheres $x^{\alpha}$ and $x^{\alpha} + dx^{\alpha}$ where $x^{\mu}, \mu = 0, 1, 2, 3$, is the center and $x^5 \equiv \lambda$ is the radius of this sphere (Ingraham 1998). The
15-parameter conformal group is the symmetry group of this angle metric in the gravitation-free case. It is convenient to factor a $-\lambda^{-2}$ out of the angle metric and deal with the $g_{\alpha\beta}$ in most problems because the $g_{\alpha\beta}$ reduce to constants in Cartesian coordinates in the gravitation-free world.

The ordinary Schwarzschild metric in GR,

$$e^{2\nu} = e^{-2\mu} = 1 - 2m/r \ , \ e^{2\xi} = 1 ,$$

is an exact solution of the metric field equations with $f_{\alpha\beta} \equiv 0$. Here $m \equiv GM/c^2$, $M \equiv$ mass of the galactic core, is the geometric mass. It makes sense to use this $\lambda$-independent solution for the $g_{\alpha\beta}$ here because a $\lambda$-dependence of the $g_{\alpha\beta}$ is forced only by $T_{\alpha\beta}$, and we are neglecting that here.

The motion equations are in general

$$\frac{d^2x^\alpha}{d\Theta^2} + \left\{ \begin{array}{l} \alpha \\ \beta \gamma \end{array} \right\} \frac{dx^\beta}{d\Theta} \frac{dx^\gamma}{d\Theta} - f_{\alpha \beta}^\gamma \frac{dx^\gamma}{d\Theta} = 0 .$$

(5)

Note the universal form of eq. (5): there is no dimensional strength constant multiplying $f_{\alpha \beta}^\gamma \frac{dx^\gamma}{d\Theta}$, so that in the angle geometry the new force accelerates all bodies the same, just as does the force from the metric. This form is mathematically forced. Since $T_{\alpha\beta}$ occurs in the metric field equations without a coupling constant, $f_{\alpha\beta}$ must have dimensions (length)$^{-2}$ in cartesian coordinates (we write this $f_{\alpha\beta} \sim$ (length)$^{-2}$ hereafter). Thus $f_{\alpha \beta}^\gamma \equiv \gamma^\alpha^\beta f_{\delta\gamma}$ is dimensionless. This fact plus the fact that $d\Theta$ is dimensionless forbids any dimensional strength constant, as asserted.

To compare with present day motion equations it is convenient (in fact necessary) to introduce a length variable $\tau$ via $d\tau/d\Theta = \lambda^2/\ell$, where $\ell$ is a length parameter (Ingraham 1998). This $\tau$ turns out to be the proper time in simple cases, including the present case (see later). The motion equations (5) with the Schwarzschild metric (3) can be rewritten with $\tau$ the independent variable. We note that the driving term in the new force then appears as $\ell f_{\alpha \beta}^\gamma \frac{dx^\gamma}{d\tau}$ in the left members, where $f_{\alpha \beta}^\gamma \equiv g^{\alpha\gamma} f_{\gamma\beta}$. Of these we need only the two angle motion equations here,

$$\ddot{\phi} + 2 \frac{\dot{r} \phi}{r} + 2 \cot \theta \ \dot{\phi} \dot{\theta} + \ell \ f_{\alpha \beta}^\gamma \ \ddot{x}^\beta = 0 , \ (6a)$$

$$\ddot{\theta} + 2 \frac{\dot{r} \dot{\theta}}{r} - \sin \theta \cos \theta \ \dot{\phi}^2 + \ell \ f_{\alpha \beta}^\gamma \ \ddot{x}^\beta = 0 , \ (6b)$$
where the dot means $d/d\tau$. Choose the polar coordinates such that the galactic core’s center is at the origin and the disc of the galaxy lies in the plane $\theta = \pi/2$. Then we hypothesize that the new force has only the component $f_{13}(r, \theta, \lambda) \neq 0$. The field equations (2) for the Schwarzschild metric (3) with $\sqrt{\gamma} = \lambda^{-5} e^{\mu + \nu + \xi} r^2 \sin \theta$ and $f^{\alpha \beta} = \lambda^4 f^{\alpha \beta}_{\text{Schw}}$ reduce to that for $\beta = 3$, which is

$$\partial_1 (\lambda^{-1} r^2 \sin \theta f_{13}^{13}) = 0.$$  

The solution is

$$f_{13}^{13} = C(\lambda, \theta) / r^2, \quad f_{13} = C(\lambda, \theta) e^{2 \mu} \sin^2 \theta,$$

where $C(\lambda, \theta)$ is arbitrary. We choose $C(\lambda, \theta) = C / \sin^2 \theta$, $C = \text{const}$. Then the $\phi$-motion (6a) has the driving term $-\ell C e^{2 \mu} \dot{r} / r^2 \sin^2 \theta$. Multiply eq. (6a) by $r^2 \sin^2 \theta$ and integrate once. The result can be written in the form

$$\sin^2 \theta \cdot r \phi = \ell CG(r) + h / r,$$

$$G(r) \equiv 1 - 2m/r + (2m/r) \log(r/2m - 1), \quad h = \text{const}.$$  

For the $\theta$-motion (6b) $f_{2\beta}^{\beta} \equiv 0$, but there is no integrating factor. However, $\theta = \pi/2$ is a solution. The $t$-motion integrates to $t = a e^{-2 \nu}$, $a = \text{const}$. The $\lambda$-motion can be integrated fully, with the result $\lambda = (\tau^2 + \ell^2)^{1/2}$, and implies

$$\lambda^2 + \ell^2 / \lambda^2 = 1.$$  

If eq. (3) is rewritten to give $d\tau^2$ and the result (10) is used, it is seen that $\tau$ is precisely the proper time. As to the $r$-motion, see below.

So for the motion of material in the galactic disc $\theta = \pi/2$ we have from eq. (9) that the circular velocity $v_c$ is

$$v_c \equiv r d\phi / dt = r \dot{\phi} / \dot{t} = \ell C / a + h / ar \quad \text{(units: } c = 1)$$  

approximately, where we have put $G(r) \approx 1$ and $e^{2 \nu} \approx 1$ since $C$ is already assumed first order small and $2m/r << 1$. Thus $v_c \approx \ell C / a = \text{const}$. if $\ell C$ dominates the usual angular momentum term $h / r$ at the distances of interest, and certainly goes to a constant as $r \to \infty$. Taking $v_c = 220$ km $s^{-1}$ for example, one gets $\ell C \approx v_c / c \approx 7.3 \times 10^{-4}$ for the dimensionless combination $\ell C$ for nonrelativistic material motion ($a \approx 1$).

About the $r$-motion we mention here only that the orbit will differ from the GR result of an ellipse with precession of perihelion (Adler et al.1975) if the new force term $\ell C$ in eq. (11) dominates $h / r$, as observations suggest.
3. CONCLUDING REMARKS

1. This explanation of the flat circular velocity curve of luminous matter in the galactic disc of spiral galaxies could be phrased in terms of GR equipped with an extra new force of gravitation \( F_{\mu\nu} \) without recourse to the 5-D theory as follows.

Take the usual Schwarzschild solution for the O-order 4-D metric \( g_{\mu\nu} \) outside the core. The field equations for \( F_{\mu\nu} \) would be

\[
\partial_\mu (\sqrt{g} F^{\mu\nu}) = 0 \ , \quad \sqrt{g} = r^2 \sin \theta
\]

in analogy to eq. (2). Assuming only \( F_{13}(r, \theta) \neq 0 \), we get the solution

\[
F^{13} = A(\theta)/r^2 \ , \quad F_{13} = A(\theta) e^{2\mu} \sin^2 \theta
\]

where \( A(\theta) \) is arbitrary. Choose \( A(\theta) = A/\sin^2 \theta \), \( A = \text{const} \). The \( \phi \)-motion equation is identical to eq. (6a) except that the driving term is \( \ell F^{3}_\nu \dot{x}^\nu \), and the dot means \( d/d\tau \), \( \tau \equiv \text{proper time} \). Inserting the solution (13), one integrates once to get

\[
\sin^2 \theta \ r \ \dot{\phi} = \ell A G(r) + h/r
\]

in strict analogy with eq.(9). The \( \theta \)-motion again yields the particular solution \( \theta = \pi/2 \), valid for motion in the galactic disc. The \( t \)-motion is as before. Hence we get the result

\[
v_c \approx \ell A/a + h/ar \quad (\text{units}: c = 1)
\]

or a flat circular velocity curve when \( \ell A \) dominates \( h/r \).

While this theory may be more attractive to some theorists because it stays within the bounds of four dimensions and uses the familiar, accepted theory of general relativity, it is open to the charge of being \textit{ad hoc}: there is no \textit{mathematical} necessity of grafting a new field \( F_{\mu\nu} \) onto the beautifully economical structure of GR. The interested reader may well wonder whether the same objection applies to the 5-D theory presented here. The answer is “No”. If we had started with a 5-D \textit{Riemannian} space of metric \( \gamma_{\alpha\beta} \), then equipping it with the extra field \( f_{\alpha\beta} \) would have been equally \textit{ad hoc}. But the field equations of this theory (Ingraham 1979) were derived from varying a simple action of Einsteinian form in a 5-D \textit{projective} space furnished with
a $6 \times 6$ quadric $S_{\rho \tau}$, a function of six “homogeneous” coordinates $X^\rho (\rho, \tau = 1, 2, \cdots 6)$. The resulting field equations for $S_{\rho \tau}$ were then transformed into equations with respect to the five “inhomogeneous” coordinates $x^\alpha$, and the split into the 5-D Riemannian tensors $\gamma_{\alpha \beta}$ and $f_{\alpha \beta}$ occurred as a result of this transformation. This derivation and transformation are given in detail in the cited work (Ingraham 1979, pp. 236-241).

2. The motion equation (5) is clearly universal: all bodies are accelerated the same, there is no constant referring to the particular body in question (its mass, charge, etc.) multiplying $f^\alpha {}_\beta \, dx^\beta /d\Theta$. As we showed, dimensions forbid it. However, when we go to the motion in terms of the proper time $\tau$ via the substitution $d\tau/d\Theta = \lambda^2/\ell$, a dimensional strength constant $\ell$, a length, appears in the motion, cf. eqs. (6). This $\ell$ may refer to the properties of the particular body in question or it may be a universal length; which alternative holds is not clear at the moment. However, the result $v_c/c \approx \ell C$ ($a \approx 1$) shows that this $\ell$ is the same at least for all the forms of luminous matter which obey the Tully-Fisher relation (1).

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FOOTNOTES

1 This 5-D theory is enforced if conformal group symmetry, which generalizes the present day Poincaré group symmetry, is demanded as the fundamental kinematical symmetry of physics (Ingraham 1998). It has nothing to do with Kaluza or Kaluza-Klein theories, has very different physical consequences, etc.
This fifth coordinate has a well-defined role in the dynamics of particles (Ingraham 1998) and should be observable, unlike the hidden extra dimensions in some current particle theories.

If we take $M = 3 - 5 \times 10^{10} \, M_\odot$ for the mass of the core and $r > 5$ kpc for a typical flat circular velocity curve (Peebles 1993, cf. Fig. 3.12), one finds $2m/r < 2 - 3 \times 10^{-7}$.

The length strength constant $\ell$ must appear here by dimensions if we take $F_{\mu\nu} \sim (\text{length})^{-2}$ like $f_{\alpha\beta}$.