Dynamical Invariants for Generalized Coherent States via Complex Quantum Hydrodynamics

Moise Bonilla-Licea * and Dieter Schuch †

Abstract: For time dependent Hamiltonians like the parametric oscillator with time-dependent frequency, the energy is no longer a constant of motion. Nevertheless, in 1880, Ermakov found a dynamical invariant for this system using the corresponding Newtonian equation of motion and an auxiliary equation. In this paper it is shown that the same invariant can be obtained from Bohmian mechanics using complex Hamiltonian equations of motion in position and momentum space and corresponding complex Riccati equations. It is pointed out that this invariant is equivalent to the conservation of angular momentum for the motion in the complex plane. Furthermore, the effect of a linear potential on the Ermakov invariant is analysed.

Keywords: Ermakov invariant; Bohmian mechanics; complex Bohmian quantities; complex Riccati equations; position and momentum representation

1. Introduction

Dynamical invariants, also called constants of motion, are of great interest and importance in physics, partly due to their relations to symmetries in space and time, according to Noether’s theorem. For conservative systems, such invariants can be angular momentum and energy, the latter corresponding to systems with a constant Hamiltonian; this applies to classical, as well as quantum, mechanics. The situation is more involved for time-dependent Hamiltonians. Such Hamiltonians do not necessarily need to be dissipative due to a friction force, their time-dependence might also originate from a time-dependent potential, like in the case of a parametric oscillator, that is an oscillator with time-dependent frequency $\omega(t)$.

Although in this case, the Hamiltonian is no longer a constant of motion, Ermakov [1] was already able to show, in 1880, that with the help of an auxiliary equation (now called Ermakov equation), the frequency $\omega(t)$ could be eliminated and a dynamical invariant, in the literature known as the Ermakov invariant, could be obtained that essentially has the dimension of an action.

It took until 1967 to find a quantum mechanical version of this system [2]. Lewis, who found this version, showed that factorization of the operator, corresponding to the Ermakov invariant, provides generalized creation and annihilation operators that have similar properties as the ones known from the harmonic oscillator. In particular, one can obtain generalized coherent states, i.e., Gaussian wave packets with time-dependent width, as eigenstates of the generalized annihilation operator with a complex eigenvalue. Even if the work of Lewis on the dynamical invariant associated with the Ermakov equation dates back to 1967, the importance of this invariant is still of present interest. Recently its mathematical properties are still studied [3], as well as its relation to quantum mechanical dynamics. Indeed, “it turns out that the evolution of the quantum states can be reduced to the solutions of the classical Ermakov-Milne-Pinney (EMP) equation” [4], especially for time-dependent quadratic potentials [5,6] and coupled oscillators [7]. The interest in this invariant is due to the possibility of studying the parametric oscillator in a suitable way.
The invariant is present in several different technological applications [8]: “The extensions from single harmonic oscillators to coupled time dependent harmonic oscillators may be found in ion-laser interactions [9–11], quantized fields propagating through dielectric media [12], shortcuts to adiabaticity [13], the Casimir effect [14] to name some”. For further details on the historical development of the Ermakov invariant, see [15] and the literature quoted therein.

Considering the important role that dynamical invariants play in classical and quantum mechanics, it is rather surprising that Bohmian mechanics has neglected this topic from its agenda. A reason might be that the current version of this approach is more aimed at simulations and numerical calculations of Bohmian trajectories instead of its analytical structure.

However, a recent formulation of Bohmian mechanics in terms of complex quantities [16] did not only allow for a momentum space representation of this approach, but also an extension to a complex quantum hydrodynamics [17] was possible. The results in this paper are used to demonstrate how to obtain the above-mentioned Ermakov invariant for the generalized coherent states without making use of the classical Newtonian equation of motion and the auxiliary equation for the parametric oscillator. We instead use complex Hamiltonian equations of motion in position and momentum space. This supports and strengthens the role of Bohmian mechanics as a useful analytical tool in quantum mechanics. It should be noticed that in our derivation, no Bohmian postulates (like the guidance equation) and no aspects (like deterministic interpretation) of quantum mechanics are involved.

In Section 2, the complex Bohmian quantities are defined for the generalized coherent state (in one dimension) in position space. Then the Ermakov invariant is obtained, starting from the Hamiltonian equation for complex momentum. Following the same line of reasoning, the same Ermakov invariant can also be obtained, based on the complex Bohmian quantities in momentum space, starting from the Hamiltonian equation for the complex position, as shown in Section 3.

In Section 4, the effects of a linear potential on the Ermakov invariant is studied. It is found that the Ermakov quantity, for generalized coherent states, is not invariant if the linear potential is stationary. Moreover, the strength of the linear potential must be inversely proportional to the position uncertainty so that the Ermakov quantity remains invariant.

Finally, in Section 5, the results are summarized and some conclusions are drawn.

2. Position Representation

2.1. Dynamics of Bohmian Quantities

The formal basis of Bohmian mechanics is the decomposition of the linear complex Schrödinger equation:

$$\frac{\hbar}{i} \frac{\partial}{\partial t} \psi(x, t) = \left\{ -\frac{\hbar^2}{2m} \Delta x + V(x, t) \right\} \psi(x, t),$$

(1)

into two real equations that have similarity with hydrodynamic equations, namely the continuity equation:

$$\frac{\partial}{\partial t} \rho_x + \nabla_x \left[ \rho_x \frac{\nabla_x S_x}{m} \right] = 0,$$

(2)

and a modified Hamilton–Jacobi equation:

$$\frac{\partial}{\partial t} S_x + \frac{1}{2m} (\nabla_x S_x)^2 + V + V_{qu} = 0,$$

(3)

by using a polar ansatz for the wave function $\psi$, i.e., in coordinate representation $\psi(x, t) = \langle \tilde{x} | \psi(t) \rangle = \sqrt{\rho_x} \exp \left( \frac{1}{\hbar} S_x \right)$, where $\rho_x = \psi^* \psi$ and the subscript indicates the representation that is considered. The appearance of the “quantum potential” $V_{qu} = -\frac{\hbar^2}{2m} \Delta x \sqrt{\rho_x}$ distinguishes the modified Hamilton–Jacobi Equation (3) from the classical Hamilton–Jacobi
The result is, in general, complex; however, the mean value of the imaginary part always vanishes \( \langle \Im \psi(t) \rangle = 0 \), i.e., it cannot be observed directly. This is obvious from the definition of the imaginary part of the momentum, \( \langle p \rangle \), in position space, or the imaginary part of the position, \( \langle x \rangle \), in momentum space. In both cases the mean values provide the corresponding probability density at \( \pm \infty \), which always vanishes. Similarly the mean value of the imaginary part of the Hamiltonian is zero in any representation, as it is the time-derivative of the (constant) normalization integral (for further details, see [16,17]).

### 2.2. Parametric Oscillator

In order to obtain exact analytical expressions, the following discussion is restricted to generalized coherent states as quantum states \( |\psi(t)\rangle \), i.e., to Gaussian wave packets with time-dependent widths in position space. These functions are solutions to the time-dependent Schrödinger equation with potentials that are, at most, quadratic in the position variable. Without loss of generality, we consider the one-dimensional case, i.e.,

\[
\text{i} \hbar \frac{\partial}{\partial t} \psi(x,t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right\} \psi(x,t),
\]

with \( \psi(x,t) = \langle x | \psi(t) \rangle \), i.e. the quantum state \( |\psi(t)\rangle \) in position representation; the momentum representation is discussed in the next Section. As we restrict our discussion to quadratic potentials and assume a constant mass \( m \), the time-dependence of the potential \( V(x,t) \) originates from the time-dependence of the oscillator frequency \( \omega(t) \) and has the form \( V(x,t) = \frac{1}{2} m \omega^2(t) x^2 \), describing the so-called parametric oscillator. The solutions of the Schrödinger equation strongly depend on the specific time-dependence of \( \omega(t) \). Few frequencies have analytical solutions that exist, e.g., for \( \omega(t) = \frac{1}{2 \pi |t+b|} \) (see, e.g., [21,22]). Nevertheless, the dynamical invariant derived in the following exists for any \( \omega(t) \)!

Another example for a quadratic potential with time-dependent frequency is the Paul trap, that can be used to confine a Bose-Einstein condensate. This situation can be described effectively by a nonlinear modification of the Schrödinger equation, the so-called Gross–Pitaevskii equation. Although this equation is for non-integrable time-dependent potential, the dynamics of the system can be determined via the “momentum method” (see [23–25]), leading to an Ermakov equation (see [26]), like Equation (10), below.

In the following, we use a generalized Gaussian wave packet ansatz (6) for the solution of Equation (5) that provides the classical Newtonian equation of motion and the parametric Newton-type equation (Ermakov equation) that was used by Ermakov to eliminate the frequency \( \omega(t) \) and to obtain the dynamical invariant. We will not follow this procedure, but instead use the wave packet (6) to define complex Bohmian quantities. We show that these quantities now fulfill complex Hamiltonian equations of motion. The frequency \( \omega(t) \) is then eliminated using a complex nonlinear Riccati equation. Rearranging the resulting equations provides the desired invariant.
The Gaussian solution can be written in the form:

$$\langle x | \psi(t) \rangle = N_e(t) \exp \left[ \frac{i}{\hbar} \left( \frac{m}{2} \bar{x}^2 + \langle p \rangle \bar{x} + f(t) \right) \right], \quad (6)$$

where $\bar{x} = x - \langle x \rangle = x - \eta(t)$, i.e., the maximum of the wave function is at $\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x,t) \psi(x,t) \, dx$. This is in agreement with Ehrenfest’s theorem that the mean value $\langle x \rangle$ follows the classical trajectory, here denoted as $\eta(t)$. The coefficient of the quadratic term, $C = C(t)$, is a complex function of time and $\langle p \rangle = m\eta$, where the overdots denote time-derivatives. Inserting (6) into (5) provides two equations of motion for $\eta(t)$ and $C(t)$,

$$\dot{\eta} + \omega^2(t) \eta = 0, \quad (7)$$

and:

$$\dot{C} + C^2 + \omega^2(t) = 0. \quad (8)$$

Equation (7) is Newton’s equation of motion for the trajectory, Equation (8) is a complex nonlinear Riccati equation that is connected with the time-dependence of the wave packet width. This connection becomes clearer by introducing a new variable $\alpha(t)$ and expressing $C(t)$ in terms of $\alpha$ and its time-derivative as:

$$C \doteq \frac{\dot{\alpha}}{\alpha} + i \frac{1}{\alpha^2}. \quad (9)$$

The real part $C_R = \frac{\dot{\alpha}}{\alpha}$ follows from inserting $C_I = \frac{1}{\alpha^2}$ into the imaginary part of the complex Riccati equation (8). With these expressions for $C_R$ and $C_I$, the real part of this equation turns into:

$$\ddot{\alpha} + \omega^2(t) \alpha = \frac{1}{\alpha^2}, \quad (10)$$

a real nonlinear differential equation known as the Ermakov equation.

The imaginary part of $C$ shows that $\alpha(t)$ is directly proportional to the wave packet width, as $C_I = \frac{1}{\alpha^2} = \frac{\hbar}{2m} \frac{1}{\sigma_x^2}$ with $\sigma_x^2 = \langle \bar{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ being the mean square deviation in position, so $\alpha(t)$ represents, up to a constant factor, the position uncertainty.

At this point, a short deviation might be allowed. There are several approaches to link Bohmian mechanics with cosmology, e.g., [27–29], and the literature cited therein. To our opinion, the closest connection was achieved by Lidsey [30] who could show that the Friedmann–Lemaître equations could be brought into the form of the Ermakov Equation (10), only replacing the position uncertainty $\alpha(t)$ by the scale factor $a(t)$. Corresponding “coherent states of the universe” could be constructed using adequate creation and annihilation operators (for details, see Section 7.7 of [15]).

According to (4), the Bohmian quantities for position, momentum, potential energy and kinetic energy for the coherent state (6) are obtained as:

$$X = x \quad \text{real}, \quad (11)$$

$$P = mC \bar{x} + \langle p \rangle \quad \text{complex}, \quad (12)$$

$$V = \frac{m}{2} \omega^2(t) \bar{x}^2, \quad (13)$$

$$T = \frac{1}{2m} \left[ (P)^2 + \frac{\hbar}{i} \frac{\partial}{\partial x} \langle p \rangle \right] = \frac{1}{2m} \left[ (P)^2 - i\hbar m C \right]. \quad (14)$$

The occurrence of the term $\frac{\hbar}{i} \frac{\partial}{\partial x} \langle p \rangle$ in (14), a term that in general does not vanish, reflects the non-locality of the theory. In the case of our Gaussian wave packets, this term is proportional to $C(t)$, the quantity that fulfills the complex Riccati equation (8) that is equivalent to the Ermakov equation (10), describing the dynamics of the wave packet width and thus of the position uncertainty, which is actually the non-classical non-local effect.
It should be mentioned that this non-locality (in space) is different from a non-locality that appears in the context of non-local-in-time complexified Lagrangians \cite{31,32}.

From (14), it also follows that the real part, whose mean value does not vanish, can be written as:

$$T_R = \frac{1}{2m} \left( \frac{\partial}{\partial x} S_x \right)^2 + V_{qu},$$

(15)

with:

$$V_{qu} = -\frac{1}{2m} \left[ p_I^2 - \hbar \frac{\partial}{\partial x} p_I \right],$$

(16)

i.e., $V_{qu}$ is completely determined by the imaginary part of our complex momentum, and originates from the kinetic energy, therefore the name “quantum potential” is rather misleading.

2.3. Dynamical Invariant

In quantum mechanics, a dynamical invariant $\hat{\mathcal{I}}$, for a system with Hamiltonian operator $\hat{\mathcal{H}}$ has to satisfy the relation:

$$\frac{d}{dt} \langle \hat{\mathcal{I}} \rangle = -\frac{i}{\hbar} \langle [\hat{\mathcal{I}}, \hat{\mathcal{H}}] \rangle + \langle \frac{\partial}{\partial t} \hat{\mathcal{I}} \rangle = 0,$$

(17)

with $[\ ,\ ]_-$ being the commutator.

Finding such an invariant is particularly difficult when the Hamiltonian is explicitly time-dependent, as in Equation (5) where the time-dependence is introduced via the time-dependent frequency $\omega(t)$ in the potential. Already in 1880, Vladimir Ermakov solved this problem for the classical case \cite{1} by eliminating $\omega(t)$ between Equations (7) and (10), leading to an invariant named after him in the literature.

In the following, it is shown how to obtain the same invariant without the requirement of the real Newton-type equations of motion (7) and (10). Instead, our derivation is based on Hamiltonian equations for complex Bohmian quantities and complex nonlinear Riccati equations.

As shown in \cite{17}, in position space, the Hamiltonian equation for the complex momentum $P$ is given by:

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial x} \mathcal{H},$$

(18)

with the expressions for $P$, as given for the generalized coherent states in Section 2.2, one obtains:

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial x} \left( \frac{1}{2m} p^2 - \frac{i\hbar}{2} \mathcal{C} + \frac{m}{2} \omega^2(t)x^2 \right).$$

(19)

Taking into account that $\mathcal{C}(t)$ does not depend on position, and replacing $\omega^2(t)$ using the Riccati Equation (8), leads to:

$$\frac{\partial}{\partial t} P = -PC + m\mathcal{C}x + m\mathcal{C}^2x.$$  

(20)

These terms can be rearranged to yield:

$$\frac{\partial}{\partial t} \ln \left( P - m\mathcal{C}x \right) = -\mathcal{C} = -\frac{\partial}{\partial t} \ln \alpha - \frac{1}{\alpha^2}.$$  

(21)

Recalling that the logarithm of a complex number $w = |w| \exp(i\phi)$ is given by $\ln w = \ln |w| + i\phi$, Equation (21) can be rewritten as,

$$\frac{\partial}{\partial t} \ln \left| P - m\mathcal{C}x \right| + i \frac{\partial}{\partial t} \tan^{-1} \left( \frac{P_I - m\mathcal{C}x}{P_R - m\mathcal{C}x} \right) = -\frac{\partial}{\partial t} \ln \alpha - \frac{1}{\alpha^2}.$$  

(22)
Equating real and imaginary parts leads to two relations that have to be fulfilled:

\[
\frac{\partial}{\partial t} \ln |P - mCx| = \frac{\partial}{\partial t} \ln \frac{1}{\alpha},
\]

and:

\[
\frac{\partial}{\partial t} \tan^{-1} \left( \frac{P_I - mC_Ix}{P_R - mC_Rx} \right) = -\frac{1}{\alpha^2}. \tag{24}
\]

After integration, the real part leads to:

\[
|P - mCx| = \frac{\alpha_0}{\alpha}, \tag{25}
\]

or:

\[
\alpha \left| P - mCx \right| = \alpha_0 \left| P_0 - mC_0x \right|, \tag{26}
\]

and the imaginary part to:

\[
\tan^{-1} \left( \frac{P_I - mC_Ix}{P_R - mC_Rx} \right) = \tan^{-1} \left( \frac{P_I(0) - mC_I(0)x}{P_R(0) - mC_R(0)x} \right) - \int_0^t \frac{1}{\alpha^2} dt', \tag{27}
\]

where the lhs corresponds to the phase angle \( \phi \) of the complex quantity \( w \). The physical meaning of this angle will be discussed in the next Subsection.

Squaring the expressions in Equation (26) leads to the quadratic invariant:

\[
\alpha^2 \left| P - mCx \right|^2 = \alpha_0^2 \left| P_0 - mC_0x \right|^2. \tag{28}
\]

Bearing in mind that according to (12), \( P - mCx = -mC \eta + m \dot{\eta} \), and Equation (28) can be written as:

\[
\alpha^2 \left[ (-mC_R \eta + m \dot{\eta})^2 + (-mC_I \eta)^2 \right] = \alpha_0^2 \left[ (-mC_R(0) \eta_0 + m \dot{\eta}_0)^2 + (-mC_I(0) \eta_0)^2 \right], \tag{29}
\]

and can be reduced to:

\[
(\alpha \eta - \dot{\eta})^2 + \left( \frac{\eta}{\alpha} \right)^2 = (\alpha_0 \eta_0 - \dot{\eta}_0)^2 + \left( \frac{\eta_0}{\alpha_0} \right)^2. \tag{30}
\]

In most cases, the classical initial position \( \eta_0 \), and the initial spreading \( \dot{\eta}_0 \), can be taken equal to zero \( \eta_0 = 0, \dot{\eta}_0 = 0 \), and \( \eta_0 \) can, in our case, be expressed via the initial momentum as \( \eta_0 = \frac{P_0}{m} \). Then the rhs of (30) can be simplified to yield:

\[
(\alpha \eta - \dot{\eta})^2 + \left( \frac{\eta}{\alpha} \right)^2 = \left( \frac{\alpha_0 P_0}{m} \right)^2. \tag{31}
\]

The lhs is, apart from a factor of \( \frac{1}{2} \), identical to the invariant found by Ermakov, but here, neither work without making use of the Newtonian equation for \( \eta \) or the nonlinear equation for \( \alpha \).

Instead, the ingredients that were needed for our derivation of this invariant are the Hamiltonian equation of motion (18) for the complex Bohmian quantities and the complex Riccati Equation (8).

2.4. Phase Angles of the Complex Quantities

In the last Subsection, the quantity \( P - mCx \) was identified with a complex number \( w = |w| \exp (i \phi) \) that can also be written in Cartesian coordinates as \( w = w_R + i w_I = |w| \cos \phi + i |w| \sin \phi \) with \( \frac{w_I}{w_R} = \tan \phi \). Therefore, the lhs of Equation (27) is just the phase
angle $\phi$ of our complex quantity. In order to get an idea what the meaning of this angle is, the corresponding angle $-\int_0^1 \frac{1}{\alpha} \, dt'$ on the rhs of Equation (27) is now considered. How can this be related to the quantities we already know?

To answer this question, we use the fact that a nonlinear Riccati equation can always be linearized, replacing the Riccati-variable by a logarithmic derivative. In our case, an ansatz

$$ C = \frac{\dot{\lambda}}{\lambda}, $$ (32)

leads to the linear second order Newtonian-type differential equation:

$$ \ddot{\lambda} + \omega^2(t) \lambda = 0, $$ (33)

for the complex variable $\lambda = \alpha \exp (i \phi) = u + iz = \alpha \cos \phi + i \alpha \sin \phi$. The choice of $\alpha$ for the amplitude of $\lambda$ is not by chance, as can be seen by inserting the polar form of $\lambda$ into (32), leading to:

$$ C = \frac{\dot{\alpha}}{\alpha} + i \dot{\phi}, $$ (34)

which would be in agreement with (9), if the relation:

$$ \dot{\phi} = \frac{1}{\alpha^2}, $$ (35)

is fulfilled. This can be proven by inserting (34) into the complex Riccati equation (8), and checking the imaginary part of the equation. Relation (35), connecting absolute value and phase angle of $\lambda$, written in the form $\alpha^2 \dot{\phi} = 1$, looks again, like a conservation law and can actually be compared with a well-known physical conservation law. To demonstrate this, the two-dimensional motion of a particle in a real plane (in contrast to the dynamics of $\lambda$ in the complex plane) shall be considered (see also [33]).

The corresponding Lagrangian function in polar coordinates for the motion under the influence of a central force field,

$$ L(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r), $$ (36)

does not depend on the variable $\theta$. Therefore, from the Euler-Lagrange equation it follows that:

$$ \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = \text{const.} = \ell, $$ (37)

i.e., the conservation of angular momentum $\ell$. For constant mass $m$, condition (37) can be fulfilled by $\dot{\theta} = \frac{1}{r}$. Thus, Equation (35) represents the conservation of angular momentum for the motion of $\lambda(t)$ in a complex plane, a property with no classical analogue.

The connection with the Ermakov invariant (31) is even closer than expected at first sight. For this purpose, Equation (35) can be written in terms of the Cartesian coordinates $u$ and $z$ of $\lambda$ and their time-derivatives $\dot{u}$ and $\dot{z}$, as shown later in Equation (69). It is well-known that canonical transformations in phase space can be described (for a one-dimensional problem) by the so-called two-dimensional real symplectic group $Sp(2, \mathbb{R})$, represented by $2 \times 2$ matrices with determinant equal to 1.

It has been shown in [15,34] that the representation of these transformations in time-dependent quantum mechanics can be obtained by replacing the elements of the matrices by $u, z, \dot{u},$ and $\dot{z}$ in a way that the determinant fulfills Equation (69). With the help of the Wigner function, it was also possible to show that the Ermakov invariant and the conservation law (35) or (69) are up to a constant identical factor (for details see [34] and Section 2.9 of [15]).

The relation between the angles on the lhs and rhs of Equation (27) can further be clarified. For this purpose we use the fact that the wave packet (6) can also be obtained by applying a time-dependent Green function $G(x, x', t, t')$ on an initial Gaussian wave packet,
ψ(x, t) = ∫ dx' G(x, x', t, t') ψ(x', t'). The time-dependent parameters occurring in the Green function can entirely be expressed in terms of u(t) and z(t), the real and imaginary parts of λ(t). Comparing the wave packet obtained in this way with the form in (4) shows that:

\[ z = \frac{m}{a_0 p_0}(x) = \frac{m}{a_0 p_0} \eta(t), \]  

(38)
is valid (for further details, see [15]).

Multiplying Equation (31) by \( \left( \frac{m}{a_0 p_0} \right)^2 \) and using \( z = \alpha \sin \varphi \) yields:

\[ \left( \frac{m}{a_0 p_0} \right)^2 \left( \dot{\eta} \alpha - \eta \dot{\alpha} \right)^2 + \sin^2 \varphi = 1. \]  

(39)

Therefore, the first term on the lhs must be \( \cos^2 \varphi = \left( \frac{u}{\bar{u}} \right)^2 \), leading to (up to a ± sign):

\[ u = \frac{m}{a_0 p_0} (\dot{\eta} \alpha^2 - \eta \dot{\alpha} \alpha), \]  

(40)

and hence to:

\[ \tan \varphi = \frac{z}{u} = \frac{\eta}{\dot{\eta} \alpha^2 - \eta \dot{\alpha} \alpha}. \]  

(41)

Returning now to Equation (27), the use of expression (12) for P and the form (9) for C in combination with the condition employed for Equation (31), i.e., \( \eta_0 = 0 \), yields the following expression,

\[ \tan^{-1} \left( \frac{P_I - m C_I x}{P_R - m C_R x} \right) = \tan^{-1} \left( \frac{-m C_I \eta}{-m C_R \eta + m \eta} \right) \]

\[ = - \tan^{-1} \left( \frac{\eta}{\eta \alpha^2 - \eta \dot{\alpha} \alpha} \right) = - \varphi, \]  

(42)
in agreement with \( - \int \frac{1}{\alpha^2} dt' = - \int \dot{\varphi} dt' \) on the rhs of (27).

To summarize, the phase angle \( \varphi \) of the complex quantity \( P - m C x = \omega = |\omega| \exp(i \varphi) \) is identical with the phase angle \( \varphi \) of the complex quantity \( \lambda = \alpha \exp(i \varphi) \) whose logarithmic time-derivative is the variable of the Riccati equation (8), whose amplitude is proportional to the wave packet width, and whose motion in the complex plane has an "angular momentum"-type invariant according to \( \alpha^2 \dot{\varphi} = 1. \)

3. Momentum Representation

3.1. Parametric Oscillator

In this Section it is shown how to recover the results in momentum space that were obtained in position space in the previous Section. This is important, as the validity of a constant of motion should be independent of the chosen representation.

The quantum state corresponding to (6) in position space can be obtained by a Fourier transformation and can be written in the form:

\[ \langle p | \psi(t) \rangle = N_p(t) \exp \left[ - \frac{i}{\hbar} \left( \frac{1}{2m} \mathcal{U} \tilde{p}^2 + \eta \tilde{p} + g(t) \right) \right], \]  

(43)

where \( \tilde{p} = p - \langle p \rangle = p - m \dot{\eta} \) and the complex coefficient of \( \tilde{p}^2 \) is the inverse of the quantity \( \mathcal{C}(t) \), fulfilling the Riccati Equation (8), i.e., \( \mathcal{U} = \mathcal{C}^{-1}(t) \), the new symbol is used just for convenience. Furthermore, the dynamics of \( \mathcal{U} \), are ruled by a complex Riccati equation,

\[ - \dot{\mathcal{U}} + \omega^2(t) \mathcal{U} \mathcal{U}^2 + 1 = 0, \]  

(44)
that turns into Equation (8) if \( \mathcal{U} \) is replaced by \( C^{-1} \). Due to this connection between \( \mathcal{U} \) and \( C \), the sign of the derivative term in the Riccati equation changes, going from one representation to the other. This change of sign will also appear when comparing the final results of the two representations.

The Bohmian quantities [16] for position, momentum and potential energy for these states, obtained according to (4), are given by:

\[
P = p \quad \text{real,} \tag{45}
\]

\[
X = \frac{1}{m} \mathcal{U} \bar{p} + \eta \quad \text{complex,} \tag{46}
\]

\[
V = \frac{1}{2} m \omega^2(t) \left[ (X)^2 - \frac{\hbar}{i} \frac{\partial}{\partial p} X \right] = \frac{1}{2} m \omega^2(t) \left[ (X)^2 + \frac{i\hbar}{m} \mathcal{U} \right]. \tag{47}
\]

As in our approach, \( X \) is complex, the potential \( V \) in (47) is complex, but this is not the potential that enters the Schrödinger equation. Therefore, there should be no confusion with the approaches that discuss non-Hermitian Hamiltonians with imaginary contributions in quantum mechanics as considered, e.g., in [35–37] and the references cited therein.

Furthermore, a few remarks to avoid confusion with complexified trajectories as they appear in various forms of complex Bohmian mechanics [38–48] (for a more detailed review and further references, see also [49]). Most of these approaches write the wave function in terms of a complex action function, which actually goes back to Schrödinger’s original definition. From the complex action, the complex momentum field \( P \) arises. In the above-mentioned approaches, in view of the fact that \( P \) is complex, the (incorrect) conclusion is drawn that also the (independent) position variable must be complex. Various attempts have been made seeking to accomplish this. Either by integrating \( \frac{P}{m} \) (in position space), or replacing the independent position variable \( x \) by a complex one, \( x = x_R + i x_I \), in the Schrödinger equation (leading to a different physical situation than described by the original Schrödinger equation which is already complex due to the complexity of \( \psi \)), or similar techniques. Moreover, an approach using coherent states, like in our approach, but with complex Bohmian trajectories [50] belongs to this category.

The complexity of our Bohmian position variable \( X \), Equation (46), has a totally different origin. It is derived from the complex wave function \( \psi(p, t) \) in momentum space and is essentially different from integrating the complex Bohmian velocity field \( \frac{P}{m} \) see Equation (12) that is derived from the wave function \( \psi(x, t) \) in position space.

3.2. Dynamical Invariants

We now proceed as in Section 2.3, only in momentum space. As shown in [17], in this space the Hamiltonian equation for the complex position is:

\[
\frac{\partial}{\partial t} X = \frac{\partial}{\partial p} \mathcal{H}, \tag{48}
\]

i.e., like the classical Hamiltonian equation of motion, but with complex position \( X \) and Hamiltonian \( \mathcal{H}_p = \frac{p^2}{2m} + V(X, \frac{\partial}{\partial p} X) \). For the coherent state (43), this leads to:

\[
\frac{\partial}{\partial t} X = \frac{p}{m} + m \omega^2(t) X \frac{\partial}{\partial p} X = \frac{p}{m} + \omega^2(t) \mathcal{U} X. \tag{49}
\]

Using \( \mathcal{U} C = 1 \) and eliminating \( \omega^2(t) \) with the help of the Riccati Equation (44), one obtains:

\[
\frac{\partial}{\partial t} X = \frac{p}{m} + (\mathcal{U} - 1) CX. \tag{50}
\]
With expression (46) for the Bohmian position, again using $UC = 1$ and $\dot{U} = -\frac{\dot{C}}{C}$, this can be rearranged to yield:

$$\frac{\partial}{\partial t} \ln \left( X - \frac{p}{m} \right) = -\frac{\partial}{\partial t} \ln C - C, \quad (51)$$

and finally, as $m$ is not time-dependent, this can be expressed in the form:

$$\frac{\partial}{\partial t} \ln \left( mCX - p \right) = -C. \quad (52)$$

Comparing this result with the corresponding one, Equation (21), in position space shows:

$$\frac{\partial}{\partial t} \ln \left( P - mcX \right) = -C \quad \text{for } x\text{-representation}, \quad (53)$$

$$\frac{\partial}{\partial t} \ln \left( mcX - p \right) = -C \quad \text{for } p\text{-representation}, \quad (54)$$

or, using the expressions for the Bohmian quantities $X$ and $P$,

$$\frac{\partial}{\partial t} \ln \left( -mc\eta + m\dot{\eta} \right) = -C \quad \text{for the } x\text{-representation}, \quad (55)$$

$$\frac{\partial}{\partial t} \ln \left( -m\dot{\eta} + mc\eta \right) = -C \quad \text{for the } p\text{-representation}. \quad (56)$$

This means that the argument of the logarithm on the lhs is, up to a ± sign identical. Therefore, the absolute values of these two complex expressions, and thus, the dynamical invariants, as well as the ratio of imaginary and real parts, and thus the phase angles are identical too. This confirms that the invariants we obtained are independent of the representation.

The relation to the invariant $\alpha^2 \dot{\phi} = 1$, connected with the phase angle of the complex quantities discussed in Section 2.4, becomes more obvious if the rhs of Equation (55) is written with help of (32) in the form:

$$-C = -\frac{\dot{\lambda}}{\lambda} = -\frac{\partial}{\partial t} \ln \lambda. \quad (57)$$

Comparing this with the lhs of Equation (55) and using $\frac{\alpha_0 p_0}{m} z = \eta$, Equation (55) can be rewritten as:

$$\frac{\partial}{\partial t} \ln \left( -m + \frac{\alpha_0 p_0}{m} z + m \frac{\alpha_0 p_0}{m} \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda, \quad (58)$$

or:

$$\frac{\partial}{\partial t} \ln \alpha_0 p_0 \left( -\frac{\lambda}{\lambda} z + \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda, \quad (59)$$

and with the properties of the logarithm,

$$\frac{\partial}{\partial t} \ln \alpha_0 p_0 + \frac{\partial}{\partial t} \ln \left( -\frac{\lambda}{\lambda} z + \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda, \quad (60)$$

one obtains,

$$\frac{\partial}{\partial t} \ln \left( -\frac{\lambda}{\lambda} z + \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda. \quad (61)$$

Factorizing $\frac{1}{\lambda}$ and using the same argument as previously, leads to:

$$\frac{\partial}{\partial t} \ln \frac{1}{\lambda} + \frac{\partial}{\partial t} \ln \left( -\lambda z + \lambda \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda, \quad (62)$$

or:

$$-\frac{\partial}{\partial t} \ln \lambda + \frac{\partial}{\partial t} \ln \left( -\lambda z + \lambda \dot{z} \right) = -\frac{\partial}{\partial t} \ln \lambda, \quad (63)$$
hence,
\[
\frac{\partial}{\partial t} \ln \left( -\dot{\lambda}z + \lambda \dot{z} \right) = 0.
\] (64)
This is fulfilled as long as the argument is a constant; therefore,
\[ -\dot{\lambda}z + \lambda \dot{z} = -\dot{\lambda}_0 z_0 + \lambda \dot{z}_0. \] (65)

At this point, the same initial conditions used for deriving Equation (31) are used for Equation (65). Therefore, \(\eta_0 = 0\) implies that \(z_0 = 0\), see Equation (38); this also means that, according to Equation (40), \(u_0 = a_0\). By recalling the definition of the complex quantity \(\lambda = u + iz\), then \(\lambda_0 = a_0\). After setting these initial conditions, Equation (65) takes the following form,
\[ -\dot{\lambda}z + \lambda \dot{z} = a_0 \frac{m}{\dot{\alpha}_0 \dot{\eta}_0} \eta_0. \] (66)

With \(p_0 = m\dot{\eta}_0\), it is straightforward to see that,
\[ \lambda \dot{z} - \dot{\lambda}z = 1 \] (67)
or:
\[ \lambda z - \dot{\lambda}z = 1 = u\dot{z} - \dot{u}z - i(\dot{z}z - \dot{z}z). \] (68)
Keeping in mind that \(z = a \sin \varphi\) and \(u = a \cos \varphi\) leads to:
\[ u\dot{z} - \dot{u}z = a^2 \varphi = 1, \] (69)
i.e., the conservation law (35) written in the “cartesian coordinates” \(z\) and \(u\) in the complex plane.

As relation (56) for the momentum representation differs from Equation (55) only by a constant factor \(-1\) in the logarithm, the same applies to this expression.

4. Linear Potential

In this Section, the dynamical invariant associated with a linear potential \(\hat{V} = -E\hat{X}\) along the \(x\)-axis is obtained. For that purpose, recall the quantities (11)–(14) computed for a generalized coherent state (6),

\[
X = x \quad \text{real,} \quad (70)
\]
\[
P = mc\bar{x} + \langle p \rangle \quad \text{complex,} \quad (71)
\]
\[
V = -Ex, \quad (72)
\]
\[
T = \frac{1}{2m} \left( (P)^2 - i\hbar mC^\prime \right). \quad (73)
\]

In analogy to Section 2.3 one can proceed by handling Equation (18) taking into account these Bohmian quantities. Therefore,
\[
\frac{\partial}{\partial t} P = -\frac{\partial}{\partial x} \left( \frac{1}{2m} P^2 - \frac{i\hbar}{2} C' - Ex \right), \quad (74)
\]
or
\[
\frac{\partial}{\partial t} P = -PC' + E. \quad (75)
\]
For the case of a linear potential, it is clear that as \(\frac{\partial^2 V}{\partial x^2} = 0\), the potential does not contribute to the Riccati equation; therefore, the Riccati equation imposes \(\dot{C} + C^2 = 0\). This helps to simplify the previous equation by adding a zero:

\[
\frac{\partial}{\partial t} P = -PC + m(C + C^2)x + E, \tag{76}
\]

\[
\frac{\partial}{\partial t} (P - mCx) = -(P - mCx)C + E. \tag{77}
\]

Due to the presence of \(E\) from the linear contribution of the potential, the differential equation is not separable anymore unlike in the case of the parametric oscillator. Indeed, expressing \(C\) as a logarithmic derivative of a complex function \(\lambda\), \(C = \frac{d}{dt} \ln \lambda\) [15] allows us to write:

\[
\frac{\partial}{\partial t} \left( \lambda (P - mCx) \right) = E\lambda. \tag{78}
\]

By direct integration and taking the absolute square:

\[
\left| \lambda (P - mCx) \right|^2 = \left| E \int_0^t \lambda dt' + \lambda_0 \left( P_0 - mC_0x \right) \right|^2, \tag{79}
\]

one can see that the definite integral on the rhs cannot be evaluated until the magnitude and phase of \(\lambda = \alpha \exp \left[ i\phi \right]\) are specified. This means, that the Ermakov quantity is not a constant of motion for the generalized coherent states under the influence of a stationary linear potential.

\[
I_{\text{Ermakov}} = \frac{1}{2} \left| E \int_0^t \lambda dt' + \lambda_0 \left( P_0 - mC_0x \right) \right|^2 \neq \text{const.} \tag{80}
\]

Nevertheless, one may drop the assumption of stationarity of \(E, E = \text{const.}\) and assume, based on Equation \(\text{(78)}\), the following form for the potential parameter \(E\),

\[
E(t) = \kappa \frac{\dot{\phi}}{\dot{\alpha}}, \tag{81}
\]

where \(\phi = \frac{1}{\alpha^2}\) [15] and \(\kappa\) represents the (constant) strength of the potential in the units \(g \cdot cm \cdot s^{-2}\). The form \(\text{(81)}\) of \(E(t)\) is in agreement with a generalization of the Ermakov method to include Hamiltonians of the form:

\[
H(t) = \frac{1}{2m} p^2 + \frac{m}{2} \omega^2 x^2 + \frac{1}{\alpha^2} f \left( \frac{x}{\alpha} \right) \tag{82}
\]

with unchanged equation for \(x(t)\), but from \(\text{(82)}\) follows for \(x\), or in our wave packet context for \(\langle x \rangle = \eta(t)\),

\[
\dot{\eta} + \omega^2(t) \eta = \frac{1}{\alpha^2} f' \left( \frac{\eta}{\alpha} \right), \tag{83}
\]

with \(f' = \frac{d}{d \left( \frac{x}{\alpha} \right)} f\). Consequently, the Ermakov invariant changes into:

\[
I = \left( \alpha \eta - \dot{\eta} \alpha \right)^2 + \left( \frac{\eta}{\alpha} \right)^2 + 2f \left( \frac{\eta}{\alpha} \right), \tag{84}
\]

which is confirmed by Equation \(\text{(92)}\). For further details see also [51–53] and Section 2.12 of reference [15].
With expression (81), Equation (78) turns into,

$$\frac{\partial}{\partial t} \left( \lambda (P - mC_0x) \right) = \kappa \dot{\phi} \lambda,$$

where $$\lambda = \alpha \exp \left[ i \int \frac{1}{\alpha^2} dt' \right] = \alpha \exp \left[ i\phi \right]$$.

Therefore, it is straightforward to obtain the following expression,

$$\frac{\partial}{\partial t} \left( \lambda (P - mC_0x) \right) = \kappa \dot{\phi} \alpha \lambda,$$

and after direct integration in time,

$$\lambda \left( P - mC_0x \right) - \lambda_0 \left( P_0 - mC_0x \right) = \frac{\kappa}{i\alpha} \exp \left[ i\phi \right] - \frac{\kappa}{i\alpha_0} \lambda_0,$$

leading to:

$$\lambda \left( P - mC_0x + i\frac{\kappa}{\alpha} \right) = \lambda_0 \left( P_0 - mC_0x + i\frac{\kappa}{\alpha_0} \right).$$

Using, as before, $$P - mC_0x = m\dot{\eta} - mC\eta$$ and $$C$$ in the form (9), after taking the square of the magnitude, one obtains:

$$\alpha^2 \left[ \left( m\dot{\eta} - \frac{\kappa}{\alpha} \eta \right)^2 + \left( \frac{\kappa}{\alpha} - m\frac{1}{\alpha^2} \eta \right)^2 \right] = \text{const},$$

or:

$$\left( m\dot{\eta} - m\ddot{\eta} \right)^2 + \left( m\frac{\eta}{\alpha} \right)^2 + \kappa^2 - 2\kappa m\frac{\eta}{\alpha} = \text{const.}$$

Dividing by $$m^2$$ and taking into account that $$\kappa$$ and $$m$$ are constants, the equation might be rewritten directly as:

$$\left( \ddot{\eta} - \frac{\kappa}{m} \eta \right)^2 + \left( \frac{\eta}{\alpha} \right)^2 - 2\kappa \frac{\eta}{m} = \text{const.}$$

Therefore, if the time-dependent linear potential has the form $$V = -\kappa \dot{\phi} \hat{x}$$, its associated Ermakov invariant $$I_{\text{Ermakov, LP}}$$ differs from the Ermakov invariant of the parametric oscillator (see Equation (31)) $$I_{\text{Ermakov, PO}}$$ in the following way:

$$I_{\text{Ermakov, LP}} = I_{\text{Ermakov, PO}} - \frac{\kappa}{m} \frac{\eta}{\alpha} = \text{const.}$$

This result depends clearly on the choice of $$E(t)$$ to ensure the integrability of Equation (78). For further details see [15] page 61 and [51].

In a nutshell, for the generalized coherent states the Ermakov invariants can be obtained for the parametric oscillator in the form $$I_{\text{Ermakov, PO}} = \frac{1}{2} \left( \left( \ddot{\eta} - \alpha \eta \right)^2 + \left( \frac{\eta}{\alpha} \right)^2 \right)$$. For the case of a linear potential, the Ermakov invariant takes the form $$I_{\text{Ermakov, LP}} = \frac{1}{2} \left( \left( \ddot{\eta} - \alpha \eta \right)^2 + \left( \frac{\eta}{\alpha} \right)^2 \right) - \frac{\kappa}{m} \frac{\eta}{\alpha}$$, as long as the linear potential has the structure $$V = -\kappa \dot{\phi} \hat{x}$$.

5. Conclusions

Dynamical invariants, also called constants of motion, play an important role in analyzing classical, as well as quantum, mechanical systems. For conservative systems, these invariants can usually be related to some symmetries in space and time and some physical quantities that are conserved. For time-dependent Hamiltonians, the situation is more involved. Nevertheless, in cases like the parametric oscillator with time-dependent frequency $$\omega(t)$$, it is also possible to find a corresponding invariant with the dimensions of
action. So far, this invariant was usually obtained by eliminating $\omega(t)$ using the classical Newtonian equation of motion and a similar auxiliary equation, Equation (10), to arrive at the so-called Ermakov-Lewis invariant (describing both the classical and the quantum mechanical case).

In this work, we have shown that the same invariant can also be obtained in the framework of Bohmian mechanics if complex Bohmian quantities are used and the corresponding Hamiltonian equations (in position and momentum space, in both cases) lead to the same invariant. This invariant was obtained from the absolute value of a complex quantity. However, the phase of the complex quantity was also connected with an invariant, namely $a^2 \varphi = 1$, that could be interpreted as the “conservation of angular momentum” for motion in a complex plane. As is shown in [34], this invariant is essentially (up to constant factors) identical to the Ermakov invariant; therefore, this interpretation can also apply to the Ermakov invariant. This makes sense as it has, up to a factor of mass $m$ (what is explained in [15] §2.3), the dimension of action like angular momentum. Hence, the absolute value and phase of the complex quantities in both representations lead to the same result. Therefore, working with complex Bohmian quantities in position and momentum space leads, via the corresponding Hamiltonian equations of motion, to a conserved quantity that could be interpreted as the “conserved angular momentum” for motion in the complex plane.

Moreover, it was shown that the Ermakov quantity, for generalized coherent states, is not invariant if the linear potential is stationary. The strength of the linear potential must be inversely proportional to the position uncertainty so that the Ermakov quantity remains invariant. The Ermakov invariant, in this case (92), is a shifted version of the Ermoakov invariant for the parametric oscillator (see Equation (93)).

This confirms the statement of C. N. Yang in his lecture on the occasion of Schrödinger’s 100th anniversary [54], that the major difference between classical and quantum physics, is the occurrence of complex quantities in quantum mechanics, not just as a tool for computational convenience, but as a “conceptual element of the very foundations of physics”. In this sense, Bohmian mechanics, expressed in complex quantities, appears to be a complementary formulation of quantum mechanics, that can supply useful information without going into any debates concerning its ontological aspects.

Author Contributions: Conceptualization, M.B.-L. and D.S.; methodology, M.B.-L. and D.S.; validation, M.B.-L. and D.S.; investigation, M.B.-L. and D.S.; writing—original draft preparation, M.B.-L. and D.S.; writing—review and editing, M.B.-L. and D.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by CONACyT grant number 633437 and by DAAD grant number 91679828.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Ermakov, V.P. Second-order differential equations: conditions of complete integrability. Univ. Izv. Kiev 1880, 20, 123–145. [CrossRef]
2. Lewis, H.R., Jr. Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians. Phys. Rev. Lett. 1967, 18, 510. [CrossRef]
3. Cerveró, J.M.; Estévez, P.G. A Review in Ermakov Systems and Their Symmetries. Symmetry 2021, 13, 493. [CrossRef]
4. Andrzejewski, K. Dynamics of entropy and information of time-dependent quantum systems: exact results. arXiv 2021, arXiv:2108.00975.
5. Soto-Eguiabar, F.; Asenjo, F.A.; Hojman, S.A.; Moya-Cessa, H.M. Bohm potential for the time dependent harmonic oscillator. arXiv 2021, arXiv:2101.05907.
6. Sen, A.; Zurab, S. Ermakov-Lewis invariant in Koopman-von Neumann mechanics. Int. J. Theor. Phys. 2020, 59, 2187. [CrossRef]
7. Mandal, S.; Kora, M.; Bayen, D.K.; Saha, K.C. Exactly Solvable Model of Classical and Quantum Oscillators of Time Dependent Complex Frequencies: Squeezing Properties of Coherent Field. Braz. J. Phys. 2021, 51, 954. [CrossRef]
8. Ramos-Prieto, I.; Urbáñez, A.R.; Fernández-Guasti, M.; Moya-Cessa, H.M. Ernakov-Lewis invariant for two coupled oscillators. J. Phys. Conf. Ser. 2020, 1540, 012009. [CrossRef]
9. Moya-Cessa, H.; Tombesi, P. Filtering number states of the vibrational motion of an ion. Phys. Rev. A 2000, 61, 025401. [CrossRef]
10. Moya-Cessa, H.; Jonathan, D.; Knight, P.L. A family of exact eigenstates for a single trapped ion interacting with a laser field. J. Mod. Opt. 2003, 50, 265. [CrossRef]
11. Casanova, J.; Puebla, R.; Moya-Cessa, H.; Plenio, M.B. Connecting nth order generalised quantum Rabi models: Emergence of nonlinear spin-boson coupling via spin rotations. NPJ Quant. Inf. 2018, 5, 47. [CrossRef]
12. Dodonov, V.V.; Klimov, A.B. Generation and detection of photons in a cavity with a resonantly oscillating boundary. Phys. Rev. A 1996, 53, 2664. [CrossRef] [PubMed]
13. Chen, X.; Ruschhaupt, A.; Schmidt, S.; del Campo, A.; Guéry-Odelin, D.; Muga, J.G. Fast Optimal Frictionless Atom Cooling in Harmonic Traps: Shortcut to Adiabaticity. Phys. Rev. Lett. 2010, 104, 063002. [CrossRef] [PubMed]
14. Román-Ancheyta, R.; Ramos-Prieto, I.; Perez-Leija, A.; Busch, K.; León-Montiel, R.D.J. Dynamical Casimir effect in stochastic systems: Photon harvesting through noise. Phys. Rev. A 2017, 96, 032501. [CrossRef]
15. Schuch, D. Quantum Theory from a Nonlinear Perspective; Springer: Berlin/Heidelberg, Germany, 2018.
16. Bonilla-Licea, M.; Schuch, D. Bohmian mechanics in momentum representation and beyond. Phys. Lett. A 2020, 384, 126671. [CrossRef]
17. Bonilla-Licea, M.; Schuch, D. Quantum hydrodynamics with complex quantities. Phys. Lett. A 2021, 392, 127171. [CrossRef]
18. Madelung, E. Quantentheorie in Hydrodynamischer Form. Z. Phys. 1927, 40, 322. [CrossRef]
19. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables I. Phys. Rev. 1952, 85, 166. [CrossRef]
20. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables II. Phys. Rev. 1952, 85, 180. [CrossRef]
21. Schuch, D. Analytical solutions for the quantum parametric oscillator from corresponding classical dynamics via a complex Riccati equation. J. Phys. Conf. Ser. 2018, 1071, 012020. [CrossRef]
22. Schuch, D. Some remarks on analytical solutions for a damped quantum parametric oscillator. J. Phys. Conf. Ser. 2019, 1075, 012033. [CrossRef]
23. Vlasov, S.N.; Petrishchev, V.A.; Talanov, V.I. Averaged description of wave beams in linear and nonlinear media (the method of moments). Radiophys. Quantum Electron. 1971, 14, 1062. [CrossRef]
24. Pérez-García, V.M.; Porras, M.A.; Vázquez, L. The nonlinear Schrödinger equation with dissipation and the moment method. Phys. Lett. A 1995, 202, 176. [CrossRef]
25. Pérez-García, V.M.; Michinel, H.; Cirac, J.I.; Lewenstein, M.; Zoller, P. Dynamics of Bose-Einstein condensates: Variational solutions of the Gross–Pitaevskii equations. Phys. Rev. A 1997, 56, 1424. [CrossRef]
26. García-Ripoll, J.J.; Pérez-García, V.M.; Torres, P. Extended parametric resonances in nonlinear Schrödinger systems. Phys. Rev. Lett. 1999, 83, 1715. [CrossRef]
27. Vink, J.C.; Uwe, J.W. Gauge fixing on the lattice without ambiguity. Phys. Lett. B 1992, 369, 707. [CrossRef]
28. Marto, J.; Moniz, V. de Broglie–Bohm FRW universes in quantum string cosmology. Phys. Rev. D 2001, 65, 023516. [CrossRef]
29. Ali, A.F.; Saurya, D. Cosmology from quantum potential. Phys. Lett. B 2015, 741, 276. [CrossRef]
30. Lidsey, J.E. Cosmic dynamics of Bose–Einstein condensates. Class. Quantum Gravity 2004, 21, 777. [CrossRef]
31. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in classical mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
32. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in quantum mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
33. El-Nabulsi, R.A. Complex backward–forward derivative operator in non-local-in-time Lagrangians mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
34. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in classical mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
35. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in quantum mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
36. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in classical mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
37. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in quantum mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
38. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in classical mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
39. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in quantum mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
40. El-Nabulsi, R.A. Non-standard non-local-in-time Lagrangians in classical mechanics. Found. Phys. Lett. 2013, 26, 223. [CrossRef]
43. Chou, C.-C.; Wyatt, R.E. Complex-extended Bohmian mechanics. J. Chem. Phys. 2010, 132, 134102. [CrossRef] [PubMed]
44. Poirier, B. Flux continuity and probability conservation in complexified Bohmian mechanics. Phys. Rev. A 2008, 77, 022114. [CrossRef]
45. Goldfarb, Y.; Degani, J.; Tannor, D.J. Bohmian mechanics with complex action: A new trajectory-based formulation of quantum mechanics. J. Chem. Phys. 2006, 125, 231103. [CrossRef] [PubMed]
46. Sanz, Á.S.; Miret-Artés, S. Comment on “Bohmian mechanics with complex action: A new trajectory-based formulation of quantum mechanics. J. Chem. Phys. 2007, 127, 197101. [CrossRef] [PubMed]
47. Goldfarb, Y.; Tannor, D.J. Interference in Bohmian mechanics with complex action. J. Chem. Phys. 2007, 127, 161101. [CrossRef]
48. Sanz, Á.S.; Borondo, F. Miret-Artés Particle diffraction studied using quantum trajectories. J. Phys. Condens. Matter 2002, 14, 6109. [CrossRef]
49. Benseny, A.; Albareda, G.; Sanz, Á.S.; Mompart, J.; Oriols, X. Applied Bohmian Mechanics. Eur. Phys. J. D 2014, 68, 1–42. [CrossRef]
50. John, M.V.; Mathew, K. Coherent states and modified de Broglie-Bohm complex quantum trajectories. Found. Phys. 2013, 43, 859. [CrossRef]
51. Hartley, J.G.; Ray, J.R. Ermakov systems and quantum-mechanical superposition laws. Phys. Rev. A 1981, 24, 2873. [CrossRef]
52. Hartley, J.G.; Ray, J.R. Solutions to the time-dependent Schrödinger equation. Phys. Rev. A 1982, 25, 2388. [CrossRef]
53. Ray, J.R. Minimum-uncertainty coherent states for certain time-dependent systems. Phys. Rev. D 1982, 25, 3417. [CrossRef]
54. Yang, C.N. Schrödinger—Centenary Celebration of a Polymath; Cambridge University Press: Cambridge, UK, 1987; p. 53.