BOUNDARY GROUND RING AND DISC CORRELATION FUNCTIONS IN LIOUVILLE QUANTUM GRAVITY*

IVAN KOSTOV
Service de Physique Théorique, CNRS – URA 2306, C.E.A. - Saclay, F-91191 Gif-Sur-Yvette, France;
Associate member of INRNE - BAS, Sofia, Bulgaria
kostov@sph.tsiclay.cea.fr

We construct the boundary ground ring in $c \leq 1$ open string theories with non-zero boundary cosmological constant (FZZT brane), using the Coulomb gas representation. The ring relations yield an over-determined set of functional recurrence equations for the boundary correlation functions, which involve shifts of the the target space momenta of the boundary fields as well as the boundary parameters on the different segments of the boundary.

1. Introduction

The non-critical string theory, known also as 2D quantum gravity, has two complementary descriptions: the world sheet description in terms of Liouville CFT and the target space description in terms of random matrix models (for reviews see [1, 2, 3, 4, 5]). It has been observed in the early 90’s that both descriptions possess integrable structures, the KP and Toda integrable hierarchies for the matrix models and the ‘ground ring’ for the world sheet CFT. There are some similarities between the integrable structures on both sides, which might be used, if well understood, to build a more direct connection between the two dual descriptions of 2D string theory more direct. Such a connection could be very helpful when studying strong perturbations of the 2D string theory as the 2D Euclidean black hole.

The integrable structure of the world sheet CFT [6, 7, 8] is based on the so called ground ring of operators. It has been shown by Witten [6] that there exists a ring of dimensionless ghost-number zero operators, with respect to which the local observables form a module. The action of ground

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ring leads to to a set of recurrence equations \[2\] for the correlation functions of the primary fields (the closed string ‘tachyons’). It has been also observed \[6, 9\] that the ground ring relations resemble the string equations in the corresponding matrix models\(^a\). The ground ring structure was generalized in \[10\] to open string theory, described by a boundary CFT. The action of the ‘boundary ground ring’ yields a set of recurrence relations for the correlation functions of boundary operators, or open string ‘tachyons’. These (over determined) relations reproduced elegantly the results obtained previously by Coulomb gas integrals \[12, 13, 14, 15\].

Originally the ring relations were applied only for the so called ‘resonant’ amplitudes. These are the correlation functions for vanishing cosmological constant, when the Liouville interaction is not taken into account. In the full theory, the resonant amplitudes give the residues of the ‘on-mass-shell’ poles of the exact amplitudes. It was conjectured \[6\] that the perturbations like the cosmological term are described by certain deformations of the ground ring structure, but the exact field-theoretical meaning of this conjecture remained obscure. Only after the impressive developments in Liouville CFT from the last several years \[16, 17, 18, 19, 20\] it has been realized that the ground ring structure is an exact symmetry of the full CFT \[21, 11\]. The recurrence relations that follow from the ground ring are the string theory version of the relations obtained by the conformal bootstrap in Liouville theory.

In these notes we will focus on boundary 2D quantum gravity, which is characterized by two cosmological constants: the bulk constant \(\mu\) coupled to the area of the world sheet and the boundary constant \(\mu_B\) coupled to the length of the boundary. In the CFT description these are the coupling constants for the bulk and boundary Liouville interaction. In Liouville theory the boundary term can be as well introduced as a non-homogeneous Neumann boundary condition \[18\], which is also referred to as FZZT brane.

A deformation of the boundary ground ring structure by bulk and boundary Liouville interactions was proposed in \[22\]. It was shown that the ring structure yields a set of functional recurrence equations, which completely determine all boundary correlation functions on a disc. Crucial role in the derivation plays the precise definition of Liouville degenerate boundary primaries given in \[18\], which we sketch below.

The observables in Liouville CFT with boundary are meromorphic functions of the boundary cosmological constant \(\mu_B\) with a cut along the semi-

\(^a\)This analogy explained and developed further in the recent paper \[11\].
infinite interval \([-\infty, -\mu_B^0]\), where \(\mu_B^0 \sim \sqrt{\pi}\). The branch point singularity can be resolved by the uniformization map

\[
\mu_B = \mu_B^0 \cosh(\pi b s),
\]

where \(b\) is the Liouville coupling constant and \(s\) is a dimensionless parameter. It happens that it is more appropriate to label the Liouville boundary conditions by the values of \(s\) instead of \(\mu_B\). Thus a Liouville boundary operator is specified by three numbers: the Liouville charge \(\beta\) and the two boundary parameters \(s_{\text{left}}\) and \(s_{\text{right}}\). In [18], Fateev, Zamolodchikov and Zamolodchikov (FZZ) observed that the correctly defined lowest degenerate boundary operators introduce shifts of the boundary parameter \(s \to s \pm ib\) or \(s \to s \pm ib^{-1}\) at the points of the boundary where they are inserted\(^b\).

The fusion rules with such an operator yield a functional equation for the boundary two-point function (see the concluding section of [18]), which can be written as a difference equation with respect to one of the boundary parameters. It was subsequently found [23] that all boundary Liouville structure constants satisfy similar difference equations.

In these notes we review and slightly generalize the result of [22] (there the matter field obeys \(U(1)\) fusion rules) by adding the two screening charges to the effective action. The ring relations lead to a set of functional recurrence identities for all boundary correlators, from which we derive a pair of difference equations with respect to one of the Liouville boundary parameters. We will see that the screening charges add an extra term to the functional equations of [22], but the difference equations remain the same.

2. Preliminaries: world-sheet CFT for the disc amplitudes

2.1. The effective action

We consider the gaussian field realization [24] of the matter CFT, which is very convenient for actual calculations. Then the Euclidean 2D string theory is defined in terms of a Liouville field \(\phi\) with background charge \(Q\) and a gaussian matter field \(\chi\) with background charge \(e_0\). The simplest form of the effective action is achieved by mapping the disc to the upper half plane \(\{\text{Im} x \geq 0\}\). Then the bulk and boundary curvatures are concentrated at

\(^b\)The general validity of this suggestion, which was proven in [18] only for the two-point function, follows from the formula for the three point function derived subsequently in [19].
infinity and the effective action takes the form \[18, 30\]

\[
\mathcal{A}[\phi, \chi] = \int_{\text{Im } x \geq 0} d^2 x \left(\frac{1}{4\pi} (\nabla \phi)^2 + (\nabla \chi)^2 \right) + \int_{-\infty}^{\infty} dx \, \mu_B \, e^{b\phi} + \text{ghosts} + \int_{-\infty}^{\infty} \mu e^{b\phi} + \text{ghosts}
\]

(2.1)

The couplings \(\mu\) and \(\mu_B\) are referred to as bulk and the boundary cosmological constants. The matter and Liouville background charges

\[e_0 = 1/b - b, \quad Q = 1/b + b\]

(2.2)

are introduced through the asymptotics of the fields at infinity

\[
\phi(x, \bar{x}) \sim -Q \log |x|^2, \quad \chi(x, \bar{x}) \sim -e_0 \log |x|^2.
\]

(2.3)

The two background charges satisfy \(Q^2 - e_0^2 = 4\), which is equivalent to the balance of the central charge \(c_{\text{tot}} \equiv c_{\phi} + c_{\chi} + c_{\text{ghosts}} = (1 + 6Q^2) + (1 - 6e_0^2) - 26 = 0\).

The boundary term encodes a inhomogeneous Neumann boundary condition for the Liouville field

\[i(\partial - \bar{\partial})\phi(x, \bar{x}) = 4\pi \mu_B \, e^{b\phi(x, \bar{x})} \quad (\bar{x} = x),\]

(2.4)

the so called FZZT brane. The matter field is assumed to satisfy pure Neumann boundary condition:

\[i(\partial - \bar{\partial})\chi(x, \bar{x}) = 0 \quad (\bar{x} = x),\]

(2.5)

which is of course Dirichlet boundary condition for the T-dual field \(\tilde{\chi}(x, \bar{x})\).

The Coulomb gas mapping requires also to add to the action the bulk and boundary screening charges

\[Q_+ = \mu_+ \int_{\text{Im } x \geq 0} d^2 x \, e^{-2ib\chi}, \quad Q_- = \mu_- \int_{\text{Im } x \geq 0} d^2 x \, e^{2i\chi/b},\]

(2.6)

\[Q_B^+ = \mu_B^+ \int_{-\infty}^{\infty} dx e^{-ib\chi}, \quad Q_B^- = \mu_B^- \int_{-\infty}^{\infty} dx \, e^{i\chi/b}\]

(2.7)

The values of the ‘couplings’ \(\mu_{\pm}\) and \(\mu_{\pm}^B\) are related to the normalization of the correlation functions. Below we will choose particular values for which the fusion rules acquire simplest form.
2.2. Bulk and boundary vertex operators

The bulk primary fields (closed string tachyons) $V^{(\pm)}_P(x, \bar{x})$ are defined as the bulk vertex operators

\[
V^{(+)}_P = \frac{1}{\pi} \gamma(b_P) e^{i(e_0 - P)x + (Q-P)\phi}
\]

\[
V^{(-)}_P = \frac{1}{\pi} \gamma(-\frac{1}{b} P) e^{i(e_0 - P)x + (Q+P)\phi}
\]

(2.8)

where $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$. This normalization removes the external ‘leg pole’ factors in the correlation functions and is the one to be used when comparing with the microscopic realizations of 2D QG. The boundary primary fields $B^{(\pm)}_P(x)$, or left/right moving on-mass-shell open string tachyons, are defined as the boundary vertex operators

\[
B^{(+)}_P = \frac{1}{\pi} \Gamma(2b_P) e^{i(\frac{1}{2}e_0 - P)x + (\frac{1}{2}Q-P)\phi}
\]

\[
B^{(-)}_P = \frac{1}{\pi} \Gamma(-\frac{2}{b} P) e^{i(\frac{1}{2}e_0 - P)x + (\frac{1}{2}Q+P)\phi}
\]

(2.9)

As any CFT boundary operator, the open string tachyon is completely defined only after the left and right boundary conditions are specified [25]. The matter component is assumed to satisfy homogeneous Neumann boundary condition, and the Liouville left and right boundary conditions are determined by the values of the uniformization parameter $s$ defined as

\[
\mu_B = \mu_B^0 \cosh(\pi bs).
\]

(2.10)

Following [18], we will denote such an operator as $s_1[B^{(\pm)}_P]s_2$.

The tachyons of opposite chiralities are related by the Liouville bulk and boundary reflection amplitudes

\[
V^{(+)}_P = S^{(+)}_P V^{(-)}_P, \quad s_1[B^{(\pm)}_P]s_2 = D^{(+-)}_P(s_1, s_2) s_1[B^{(-)}_P]s_2.
\]

(2.11)

For each value of the momentum there is only one vertex operator that corresponds to a physical state. The physical operators are

\[
V_P = \begin{cases} 
V^{(+)}_P & \text{if } P < 0, \\
V^{(-)}_P & \text{if } P > 0.
\end{cases}
\]

\[
B_P = \begin{cases} 
B^{(+)}_P & \text{if } P < 0, \\
B^{(-)}_P & \text{if } P > 0.
\end{cases}
\]

(2.12)

The “wrongly dressed” operators are related to the physical ones by (2.11).

2.3. Normalization of the bulk and boundary cosmological constants and the screening operators

It is convenient to redefine the cosmological constants $\mu \rightarrow \Lambda$ and $\mu_B \rightarrow z$ according to the normalizations [25] of the vertex operators:

\[
\mu e^{2b\phi} = \Lambda V^{(\pm)}_{e_0}, \quad \mu_B e^{b\phi} = z B^{(\pm)}_{e_0/2}.
\]

(2.13)
The new cosmological constants are related to the old ones by
\[ \Lambda = \pi \gamma (b^2) \mu, \quad z = \pi \Gamma(1 - b^2)^{-1} \mu_B. \] (2.14)

Now the uniformization map 2.10 reads
\[ z = M \cosh(\pi bs), \quad M = \sqrt{\Lambda}. \] (2.15)

Also, it will be convenient to normalize the screening charges as
\[ Q_{\pm} = \int_{\text{Im } x > 0} d^2x \, V_{\pm Q}, \quad Q^B_{\pm} = \int_{-\infty}^{\infty} dx \, V_{\pm Q/2}. \] (2.16)

In the normalization (2.13) the self-duality property [18] of Liouville quantum gravity reads
\[ b \to \tilde{b} = 1/b, \quad \Lambda \to \tilde{\Lambda}, \quad M \to \tilde{M}, \] (2.17)

The boundary parameter \( s \) is self-dual. The duality transformation for the correlation functions follows from the duality transformation of left and right chiral tachyons
\[ \tilde{V}_{(\pm)} = V_{(\mp)}, \quad s_1 [\tilde{B}_{(\pm)}] s_2 = s_1 [B_{(\mp)}] s_2, \] (2.18)

where the bar means complex conjugation.

### 2.4. Degenerate fields

By degenerate fields in Liouville quantum gravity we will understand the gravitationally dressed degenerate matter fields. In our case these are the on-shell vertex operators (2.12) with degenerate matter components
\[ V_{rs} = V_{r/b - sb}, \quad B_{rs} = B_{(r/b - sb)/2} \quad (r, s \in \mathbb{N}). \] (2.19)

The flat scaling dimensions of these fields are
\[ \Delta_{rs} = \frac{(r/b - sb)^2 - e_0^2}{4}. \]

The fusion rules for the degenerate in Liouville quantum gravity are the same as the fusion rules for the degenerate fields in the matter CFT [20]. Note that the Liouville components of these fields are not degenerate Liouville fields.
3. Bulk ground ring

3.1. The ring relations

Before considering the boundary ground ring, we will give a short review of the ground ring structure for a theory without boundary. The operators that span the ground ring are obtained by applying a Virasoro raising operator of level $rs-1$ to the product of two degenerate matter and Liouville fields with Kac labels $r, s$. The resulting operators have conformal weights $\Delta = \bar{\Delta} = 0$. The ring is generated by the first two lowest operators [6]

$$a_+ = -|bc - b\partial_z (\phi - i\chi)|^2 e^{-b^{-1}(\phi + i\chi)}$$

$$a_- = -|bc - b^{-1}\partial_z (\phi + i\chi)|^2 e^{-b(\phi - i\chi)}$$

(3.20)

where $b, c$ are the reparametrization ghost and anti-ghost fields. The ground ring is spanned on the polynomials $(a_+)^m(a_-)^n$ with $m, n \in \mathbb{Z}_+$. In the case of non-rational $b^2$ the ground ring contains no other relations and has an infinite number of elements labeled by the integers $r, s \geq 1$.

A crucial property of the operators $a_\pm$ is that their derivatives $\partial_x a_\pm$ and $\partial\bar{x} a_\pm$ are BRST exact: $\partial_x a_- = \{Q_{\text{BRST}}, b a_-\}$. Therefore, any amplitude that involves $a_\pm$ and other BRST-invariant operators does not depend on the position of $a_\pm$. This property allows to write recurrence equations for the correlation functions from the OPE of $a_\pm$ and the other BRST-invariant fields.

The vertex operators [28] form a module under the ground ring:

$$a_+ V^{(+)}_P = -V^{(+)}_{P+\frac{1}{b}}, \quad a_- V^{(-)}_P = -V^{(-)}_{P-\frac{1}{b}}.$$  

(3.21)

and also

$$a_+ V^{(-)}_P = a_- V^{(+)}_P = 0.$$  

(3.22)

Both relations [3.21] and [3.22] follow from the free field OPE and are true up to commutators with the BRST charge. While the first one survives in the correlation functions, the second one receives non-linear corrections. Due to the contact terms, the last relation should be modified in presence of an integrated vertex operator to

$$a_+ V^{(-)}_P \int d^2 z \ V^{(-)}_{P+b} = V^{(-)}_{P+b+b}.$$  

(3.23)

$$a_- V^{(+)}_P \int d^2 z \ V^{(+)}_{P-b} = V^{(+)}_{P-b-\frac{1}{b}}.$$  

(3.24)
These relations imply a set of recurrence equations \([9, 10, 26]\), which determine completely the resonant tachyon amplitudes.

After taking into account the Liouville interaction and the screening charges second relation \((3.22)\) gets deformed. Remarkably, the deformation can be calculated perturbatively by applying \((3.23)\) and \((3.24)\): \(F\) does not depend on \(a\), but this has been justified by various self-consistency checks in Liouville theory \([10, 17]\). The ground ring structure in presence of screening charges has been considered in \([27, 28]\).

### 3.2. Functional equations for the bulk correlators

Using the ring relations \((3.21)\) and \((3.25)\) one can obtain a set of recurrence equations for the correlation functions of the bulk tachyons

\[
G(P_1, \ldots, P_n | P_{n+1}, \ldots, P_{n+m}) = \left\langle \left[ \prod_{k=1}^{n-1} \int \mathcal{V}_{P_k}^{(-)} \right] \mathcal{C} \mathcal{V}_{P_n}^{(-)}(0) \times \right. \\
\times \mathcal{C} \mathcal{V}_{P_{n+1}}^{(+)}(1) \left[ \prod_{j=n+2}^{n+m-1} \int \mathcal{V}_{P_j}^{(+)} \right] \mathcal{C} \mathcal{V}_{P_{n+m}}^{(+)}(\infty) \right\rangle. \tag{3.26}
\]

Consider the auxiliary function

\[
F(x, \bar{x} | P_1, \ldots, P_n | P_{n+1}, \ldots, P_{n+m}) = \left\langle \left[ \prod_{k=1}^{n-1} \int \mathcal{V}_{P_k}^{(-)} \right] \mathcal{C} \mathcal{V}_{P_{n+b}}^{(-)}(0) \times \right. \\
\times a_-(x, \bar{x}) \left[ \prod_{j=n+2}^{n+m-1} \int \mathcal{C} \mathcal{V}_{P_j}^{(+)} \right] \mathcal{C} \mathcal{V}_{P_{n+m}}^{(+)}(\infty) \right\rangle. \tag{3.27}
\]

The function \(F\) does not depend on \(x\) and \(\bar{x}\). This can be proved by using \(\partial_x a_-(=\{Q_{\text{BRST}}, b_1 a_-\})\) and deforming the contour, commuting \(Q_{\text{BRST}}\) with the other operators in \((3.21)\) \([10]\). Therefore one can evaluate this function at \(x = 0\) and \(x = 1\) by using the fusion rules \((3.21)\) and \((3.25)\). As a result one obtains the recurrence relation

\[
G(P_1, \ldots, P_n | P_{n+1}, \ldots, P_{n+m}) = \Lambda G(P_1, \ldots, P_n + b | P_{n+1} - b, \ldots, P_{n+m}) - \\
+ G(P_1, \ldots, P_n - b | P_{n+1} - b, \ldots, P_{n+m}) - \\
- \sum_{j=1}^{m-1} G(P_1, \ldots, P_n + b | P_{n+2}, \ldots, P_{n+j} + P_{n+m} - \frac{b}{2}, \ldots, P_{n+m}). \tag{3.28}
\]

Similarly, by inserting \(a_+\) we get the dual recurrence relation

\[
G(P_1, \ldots, P_n | P_{n+1}, \ldots, P_{n+m}) = \tilde{\Lambda} G(P_1, \ldots, P_n + \frac{b}{2} | P_{n+1} - \frac{b}{2}, \ldots, P_{n+m}) - \\
+ G(P_1, \ldots, P_n - \frac{b}{2} | P_{n+1} - \frac{b}{2}, \ldots, P_{n+m}) - \\
- \sum_{k=1}^{m-1} G(P_1, \ldots, P_k + P_n + b, \ldots, P_{n-1} | P_{n+1}, \ldots, P_{n+m}). \tag{3.29}
\]
Note that the three-point function $m + n = 3$ coincides with the corresponding Liouville three-point function. The latter has been evaluated by using a very similar argument in [30].

4. The boundary ground ring

4.1. The ring relations

Similarly to the bulk case the two generators of the boundary ground ring are defined as

\[
A_+ = -[bc - \frac{1}{2}b \partial_x (\phi - i \chi)] e^{-\frac{1}{2}b^{-1}i(\phi + i \chi)} \\
A_- = -[bc - \frac{1}{2}b^{-1} \partial_x (\phi + i \chi)] e^{-\frac{1}{2}b(i\phi - i\chi)},
\]

(4.1)

where $b, c$ are the boundary reparametrization ghosts. The two operators are related by a duality transformation combined with complex conjugation:

\[
\hat{A}_+ = \overline{A}_- , \quad \hat{A}_- = \overline{A}_+.
\]

(4.2)

These operators are BRST closed: $\partial_x A_{\pm} = \{Q_{\text{BRST}}, b_{-1} A_{\pm}\}$ and have $\Delta = 0$. The open string tachyons [2.9] form a module with respect to the ring generated by these two operators.

Let us first consider the simplest case of pure Neumann b.c. for the Liouville field ($\mu_B = 0$), following [10]. Then the action of the ring on the tachyon modules is generated by the relations

\[
A_+ B_P^{(+)} = B_P^{(+)} \frac{1}{2}, \quad A_- B_P^{(-)} = B_P^{(-)} \frac{1}{2}
\]

(4.3)

\[
A_+ B_P^{(-)} = A_- B_P^{(+)} = 0.
\]

(4.4)

Again, the first relation (4.3) is exact and the second relation (4.4) gets deformed in presence of integrated boundary tachyon fields [10]:

\[
A_{-} B_{P_{1}}^{(+)} \int dx B_{P_{2}}^{(+)} = \frac{1}{\sin 2\pi b P_{1}} B_{P_{1}+P_{2}-\frac{1}{2}}^{(+)}.
\]

(4.5)

\[
\int dx B_{P_{1}}^{(+)} A_{-} B_{P_{2}}^{(+)} = \frac{\sin 2\pi b (P_{1} + P_{2})}{\sin 2\pi b P_{1} \sin 2\pi b P_{2}} B_{P_{1}+P_{2}-\frac{1}{2}}^{(+)}.
\]

(4.6)

\[
A_{+} B_{P_{1}}^{(-)} \int dx B_{P_{2}}^{(-)} = \frac{1}{\sin \frac{\pi}{b} P_{1}} B_{P_{1}+P_{2}+\frac{1}{2}}^{(-)}
\]

(4.7)
The coefficients on the r.h.s. are obtained as standard Coulomb integrals. As in the bulk case, these relations yield an over-determined set of recurrence equations for the $n$-point open string amplitudes [10].

4.2. Deformation of the ring relations by the boundary Liouville term and the screening charges

With the screening charges (2.16) added to the effective action, the action of the ring on the vertex operators gets deformed according to the general relations (4.3) and (4.5). Namely the relations (4.4) become

$$A_+ B_+^{(-)} = n_P \cdot B_+^{(-)} \quad A_- B_-^{(-)} = \tilde{n}_P \cdot B_-^{(-)}$$

where the coefficients $m_P$ and $\tilde{m}_P$ are given by

$$n_P = \tan(\pi b P) - \tan(\pi b e_0/2)$$
$$\tilde{n}_P = \tan(\pi P/b) + \tan(\pi e_0/2b).$$

The evaluation of the deformation of the ring relations by the boundary Liouville interaction is more subtle. It is based on the observation made in [18] and subsequently confirmed in [19, 23] that a level-$n$ degenerate boundary Liouville field $e^{-n b \phi}$ has vanishing null-vector and therefore a truncated OPE with the other primary fields if either $s_{\text{left}} - s_{\text{right}} = ibk$ or $s_{\text{left}} + s_{\text{right}} = ibk$, with $k = -n/2, -n/2 + 1, ..., n/2$. By the duality property of Liouville theory, the degenerate boundary fields $e^{-n b \phi}$ exhibit a similar property with $b$ replaced by $1/b$. No direct proof is supplied for this statement, but it was shown to be consistent with the exact results obtained in boundary Liouville theory. In particular, the above property leads to a pair of functional equations for the Liouville boundary reflection amplitude, whose unique solution coincides with the result of the standard conformal bootstrap [18].

According to [18], the operators (4.1) should be defined by

$$A_+ \rightarrow s[A_+]^{s \pm i/b}$$
$$A_- \rightarrow s[A_-]^{s \pm i/b}$$

and the relations (4.8), (4.9), (4.11) and (4.12) understood as

$$s_1 [B_+^{(-)}]^{s \pm i/b} = - s_1 [B_-^{(-)}]^{s \pm i/b}$$

(4.12)
etc. Taking into account the three possible insertions of the boundary Liouville interaction boundary interaction one finds \[22\] that the relations deform further to
\[ s_1[A_+]^{s_1} s_2 = m_+^{s_1} b + n P \cdot s_2 \]  
Eq. (4.13)

\[ s_1[A_-]^{s_1} s_2 = \tilde{m}_+^{s_1} b + \tilde{n} P \cdot s_2 \]  
Eq. (4.14)

with the coefficients \(m_+^{s_1} = m_+^{s_1}(s_1, s_2)\) and \(\tilde{m}_+^{s_1} = \tilde{m}_+^{s_1}(s_1, s_2)\) given by
\[ m_+^{s_1} = -\frac{M}{b} \frac{\cosh[\pi b (s_1 \pm 2i P)] + \cosh(\pi b s_2)}{\sin 2\pi b P}. \]  
Eq. (4.15)

\[ \tilde{m}_+^{s_1} = -\frac{M^{1/b}}{b} \frac{\cosh[\pi (s_1 \pm 2i P)/b] + \cosh(\pi s_2/b)}{\sin 2\frac{\pi}{b} P}. \]  
Eq. (4.16)

The derivation of (4.13) and (4.14) essentially repeats the one presented in \[18\] for pure Liouville vertex operators and leads to the same fusion coefficients.

### 4.3. Functional and difference equations for the correlation functions of boundary primary fields

Consider a boundary correlation function of the form
\[ W_{P_1, P_2, \ldots}(s_1, s, s_2, s_3, \ldots) = \langle \ s_1 [B_{P_1}^{(s_1)}] [B_{P_2}^{(s_2)}] [B_{P_3}^{(s_3)}] \ldots \rangle \]  
Eq. (4.17)

The realization of the physical boundary fields depends on the sign of the momenta and is given by \[24\]. Our notations do not distinguish between integrated and non-integrated fields; it is assumed that three of the integrations should be replaced by ghost insertions. The amplitude (4.17) is analytic in the boundary parameters \(s, s_1, \ldots\) but not in the target space momenta, since the fields with positive and negative momenta are realized by different vertex operators. By the momentum conservation the correlation function is zero unless \(\sum_k (\frac{1}{2} e_0 - P_k) = e_0 + m/b - nb\) for some positive integers \(m\) and \(n\). Equation (2.18) implies the identity
\[ W_{P_1, P_2, \ldots} = W_{-P_1, -P_2, \ldots} = W_{-P_1, -P_2, \ldots}. \]  
Eq. (4.18)

Let us assume that \(P_1 < 0, P_2 > 0\). Then the amplitude (4.17) is realized as
\[ W_{P_1, P_2, P_3, \ldots}(s_1, s, s_2, s_3, \ldots) = \langle \ s_1 [B_{P_1}^{(-s_1)}] [B_{P_2}^{(-s_2)}] [B_{P_3}^{(+s_3)}] \ldots \rangle \]  
Eq. (4.19)
By the symmetry (4.18) the amplitude can be written also as
\[ W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = \langle \ldots [B_{-P_3}^{-}]^{s_2} [B_{-P_2}^{-}]^{s_1} [B_{-P_1}^{+}]^{s_1} \rangle. \] (4.20)

We will use these two representations to derive two independent functional identities for \( W \).

Consider the auxiliary correlation function \( F \) with an operator \( A_- \) inserted in the r.h.s of (4.17):
\[ F = \langle \ldots [B_{-P_3}^{-}]^{s_2} [A_-]^{s_1 \pm i b} [B_{P_2}^{+}]^{s_1} [B_{P_3}^{+}]^{s_2} \rangle. \]

Let us assume further that \( P_1 < -\frac{b}{2} \) and \( P_2 > \frac{b}{2} \). Then, evaluating \( F \) by (4.12) and by (4.13) and equating the results we get the following functional identities for the correlation functions of three or more fields:
\[ W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = + m \cdot W_{P_1, P_2, P_3, \ldots} (s_1, s_1, s_2, s_3, \ldots) + n \cdot W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) + \sin^{-1}(2\pi b P_2) \cdot W_{P_1, P_2, P_3, \ldots} (s_1, s_2, s_3, \ldots). \] (4.21)

The last, ‘contact’, term represents a correlation function with one operator less. A dual equation can be obtained in the same way, assuming that \( P_1 < -\frac{1}{2} \) and \( P_2 > \frac{1}{2} \), by evaluating the complex conjugated function \( \overline{F} \) using the representation (4.20) and the relations (4.12).

The symmetric part of (4.21) generalize the recurrence equations of [10] derived in the free-field (\( \mu B = 0 \)) limit. The antisymmetric part equation (4.21) and its dual represents a pair of homogeneous difference equations which do not have analog in the \( \mu B = 0 \) limit. We write them in operator form:
\[ \left[ \sin(b \partial_s) - M \sinh(\pi bs) e^{\frac{\pi i}{b} (\partial_{P_1} - \partial_{P_2})} \right] W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = 0 \] (4.22)
for \( P_1 < -\frac{b}{2} \) and \( P_2 > \frac{b}{2} \), and
\[ \left[ \sin\left(\frac{1}{b} \partial_s\right) - M \frac{\pi i}{b} \sinh\left(\frac{\pi}{b} s\right) e^{\frac{\pi}{b} (\partial_{P_1} - \partial_{P_2})} \right] W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = 0 \] (4.23)
for \( P_1 < -\frac{1}{2} \) and \( P_2 > \frac{1}{2} \).

Note that these equations have the same form for matter CFT with or without screening charges. Difference equations for all values of the momenta can be obtained by the following rule [31][23]. If the sign of one or both momenta changes after the shift, one should apply the reflection
property with the amplitude $D_p^{(+)}(s_1, s_2)$. The details are explained in [23, 22].

5. Discussion

We have constructed the boundary ground ring in the Coulomb gas realization of the matter CFT. Since any matter CFT can be constructed from the Coulomb gas, the validity of the difference equations (4.22) and (4.23) is universal. Unlike underlying functional equations, the form of the difference equations is not altered by adding the screening charges. Remarkably, these equations have their equivalent in the microscopic realization of 2D QG as loop gas on random planar graphs [32]. In this realization the degenerate boundary operators are represented geometrically as sources of open lines with endpoints at the boundary. As a matter of fact, the difference equations (4.22) and (4.23) have been first derived in the microscopic approach [31, 23] by cutting open the sum over triangulated surfaces along one of the lines and then using certain factorization property of the sum over surfaces. Of course, these equations are equivalent to certain word identities in the corresponding matrix model. This confirms the deep relation between the integrable structures in the world sheet and microscopic formulations of 2D QG.

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