Entanglement Entropy in Closed String Theory

Usman Naseer

Department of Physics and Astronomy, Uppsala University,
Box 516, SE-751 20 Uppsala, Sweden

usman.naseer@physics.uu.se

Abstract

In local quantum field theory on a background spacetime, the entanglement entropy of a region is divergent due to the arbitrary short-wavelength correlations near the boundary of the region. Quantum gravitational fluctuations are expected to cut off the entropy of the ultraviolet modes. We study the entanglement entropy in closed string theory using the framework of string field theory. In particular, we compute the one-loop Renyi partition functions by considering the theory on a simple branched cover of the configuration space of closed strings. The short-wavelength modes are cut off at the string scale and the one-loop entanglement entropy is ultraviolet-finite. A non-canonical kinetic term, required to produce the correct one-loop vacuum amplitude, plays a key role.
1. Introduction

Over the last decade concepts from quantum information theory have given remarkable insights into the structure of quantum field theory and quantum gravity. One of the most important quantities in this context is the entanglement entropy (see [1,2] and references therein for a review). In a local quantum field theory with a background spacetime, the entanglement entropy has UV divergences due to the correlations between degrees of freedom near the entangling surface. This leads to the so-called ‘area-law’ of entanglement entropy for typical vacuum states

$$S = c_0 \frac{A}{\epsilon^{D-2}} + \cdots + S_{\text{finite}}. \quad (1.1)$$

The leading divergence in entanglement entropy is proportional to the area $A$ of the entangling surface. Here $D$ is the spacetime dimension, $\epsilon$ is the UV-cutoff and $c_0$ is a constant which depends on the details of the regularization scheme. ‘\cdots’ denote subleading divergences. The finite part $S_{\text{finite}}$ depend on the regularization scheme. In even dimensions there are also logarithmic divergences whose coefficients are regularization scheme independent.

Arguments based on black hole physics, however, suggest that the entanglement entropy is finite in a quantum theory of gravity [3–5]. The basic idea is the following (see [6] for a recent review of related issues). In a black hole background with matter outside the horizon, the total entropy must include contributions from the black hole entropy $S_{\text{BH}} = \frac{A}{4G}$ and the entropy $S_{\text{out}}$ of the fields outside the horizon [7, 8]. $S_{\text{BH}}$ depends on the UV-cutoff via the renormalization of the gravitational coupling constant while $S_{\text{out}}$ depends on the UV-cutoff as in eq. (1.1). If the same regularization scheme is used to compute $S_{\text{out}}$ and the renormalized Newton’s constant then
divergences in $S_{\text{out}}$ match the ones that renormalize Newton’s constant. More concretely

$$\frac{A}{4G_{\text{bare}}} + c_0 \frac{A}{e^{D-2}} = \frac{A}{4G_{\text{ren}}}.$$  \hspace{1cm} (1.2)

So the total entropy $S_{\text{BH}} + S_{\text{out}}$ is renormalization group invariant. Gravity becomes strongly coupled at the Planck scale (or the smallest possible length scale in quantum gravity) and $\frac{1}{\ell_p} \to 0$. So the total entropy is just the entanglement entropy at the Planck scale, which has become finite due to quantum gravitational effects. The partitioning of the total entropy into $S_{\text{BH}}$ and $S_{\text{out}}$ and their behavior under the renormalization group was studied in detail in [9, 10] providing further evidence that the entanglement entropy is a well-defined observable in quantum gravity. Indeed assuming this finiteness was crucial to derive the Einstein’s equations from the maximal vacuum entanglement hypothesis [11]. Motivated by these ideas we study entanglement entropy in the closed string theory.

A generic tool to compute entanglement entropy in field theory is the replica method. This requires computing the partition function of the theory on an $n$-fold branched cover of the spacetime and then analytically continuing in $n$. An important issue in the application of this method to the worldsheet string theory is that there is no known conformal field theory with a target space which is a branched cover. Nevertheless, in earlier works [3, 12, 13], it was proposed that one can use a $\mathbb{Z}_N$ orbifold with the formal identification $n = \frac{1}{N}$ as a background and then analytically continue in $N$. Recently, subtleties regarding this approach for open strings are discussed and clarified in [14]. Such progress had remained elusive for the case of closed string theory: it is not known how to write the orbifold partition function as an analytic function of $N$.

An alternative approach is to study entanglement properties using the formalism of string field theory (SFT). The framework of SFT was developed to obtain non-perturbative insights into the dynamics of strings [19]. In recent years it has become clear that a consistent perturbative formulation of string theory also needs the framework of SFT. Two instructive examples are mass renormalization [20, 21] and the vacuum shift [22] which cannot be addressed using conventional methods based on a worldsheet conformal field theory. Worldsheet conformal invariance imposes the tree level on-shell conditions and vacuum expectation values. Loop-corrections can change the physical mass and generate new terms in the potential so that the new vacuum does not satisfy the classical equations of motion and hence cannot be described by a worldsheet conformal field theory. Nevertheless, mass renormalization and vacuum shift are standard problems in quantum field theory and SFT provides a systematic procedure to address these. The ability of SFT to deal with off-shell backgrounds makes it a promising framework to study the entanglement entropy in string theory. In the language of [3], SFT provides a prescription to compute the off-shell generating functional which is needed for replica method. Since entanglement entropy is a physical quantity one expects it to be independent of the prescription.

In [23] entanglement entropy for open strings was studied using SFT. The result was found to be consistent with the effective field theory: the entanglement entropy is equal to the sum of

\footnote{This should be contrasted with the thermodynamic entropy which can be computed by studying string theory on a background where the time direction is a circle with radius proportional to the inverse temperature [15–18]. Since the sigma model with a compact time direction in the target space is a conformal field theory one does not encounter the subtleties associated with the computation of the entanglement entropy.}
entanglement entropies of all fields in the spectrum of open string theory. In this paper we study
entanglement entropy in closed string theory using the free light-cone SFT. A completely rigorous
treatment of the problem would require using the covariant formalism with small string coupling
but we limit ourselves to a simpler task in the first part of the paper: demonstrating that the
entanglement entropy is UV-finite in closed string theory. We compute the vacuum entanglement
entropy using the replica method and show that it is indeed UV-finite. At a technical level, the
mechanism responsible for the finiteness is the same one which makes one-loop vacuum amplitude
finite in closed string theory. The one-loop vacuum amplitude involves an integration over the
moduli space of the torus. It avoids the problematic regions which give rise to UV-divergences in
a quantum field theory. The UV-finite vacuum amplitude can be obtained in SFT by employing
a non-canonical kinetic term for string-fields. Computing the entanglement entropy with this non-
canonical kinetic term then gives a finite result\(^2\).

This paper is organized as follows: In section 2 we discuss and review aspects of the replica
method to compute entanglement entropy in generic field theories. In section 3 we use the replica
method to compute entanglement entropy in open SFT. In section 4 we introduce basic elements of
closed SFT and compute entanglement entropy. There are various subtleties in defining subregions,
entanglement and algebraic structure in a quantum theory of gravity. We discuss these issues in
light of our computation in section 5 and conclude with future directions.

2. Preliminaries

In this section we review some basic facts about the entanglement entropy and the replica
method in quantum field theories. We start by considering a scalar field theory on ‘spacetime’ \( \mathcal{M} \)
of dimension \( d_{\mathcal{M}} \) described by the action
\[
I = \int d^{d_{\mathcal{M}}} x \phi \, O (\partial_\mu, x^\mu) \phi.
\]
We choose \( x^\mu, \mu = 0, 1, \cdots, d_{\mathcal{M}} - 1 \) as coordinates on \( \mathcal{M} \) and \( x^0 \) is treated as Euclidean time
direction. \( O (\partial_\mu, x^\mu) \) is a positive semi-definite operator whose eigen-functions span the function
space on \( \mathcal{M} \). In general \( O (\partial_\mu, x^\mu) \) can involve derivatives, explicit position dependence and other
constant parameters such as length or mass scales. The zero-modes of \( O (\partial_\mu, x^\mu) \) are on-shell field
configurations for which the action is zero. The partition function of the theory can be computed by
expanding the field in eigen-functions of \( O (\partial_\mu, x^\mu) \) and then performing the Gaussian integration
over non-zero modes. This results into
\[
Z = (\det O (\partial_\mu, x^\mu))^{-\frac{1}{2}},
\]
\(^2\)Since the one-loop vacuum amplitude is independent of the string coupling our result should be understood as
computing the one-loop contribution to vacuum entanglement entropy. At finite coupling one would also expect a
‘classical’ contribution which is proportional to \( \frac{1}{g_s} \) and is expected to produce the black hole entropy. We thank
Raghu Mahajan for emphasizing this point.

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where the determinant only involves positive eigenvalues of \( O(\partial_\mu, x^\mu) \). This can also be written as
\[
\log Z = \int_0^\infty \frac{dt}{2t} \text{Tr} \mathcal{K}_O(t),
\] (2.3)
where \( \mathcal{K}_O(t, x : x') \) is the heat kernel associated with the operator \( O(\partial_\mu, x^\mu) \), i.e.,
\[
\langle x|e^{-itO}|x'\rangle \equiv \mathcal{K}_O(t, x : x'), \quad \text{Tr} \mathcal{K}_O(t) = \int d^dM \mathcal{K}_O(t, x : x). \] (2.4)

To study the spatial entanglement of a given state in this theory we proceed as follows: We choose a codimension-1 ‘Cauchy’ surface \( \Sigma \) inside \( \mathcal{M} \) and partition it into two regions \( R \) and its complement \( R^c \): \( \Sigma = R \cup R^c \), where \( R \) is the region of interest. Given a state described by a density matrix, we then trace out the degrees of freedom in \( R^c \) to get the reduced density matrix \( \rho \) on \( R \). Entanglement entropy is then just the von Neumann entropy of the reduced density matrix.

A convenient way to carry out this procedure for the vacuum state is the replica method. We place the theory on \( \mathcal{M}_n \) which is an \( n \)-fold branched cover of \( \mathcal{M} \). The branching is along the codimension-2 entangling surface \( \partial R \). The \( n \)th Renyi partition function \( Z(n) \) is the partition function of the theory on \( \mathcal{M}_n \) and it is proportional to \( \text{Tr} \rho_0^n \), where \( \rho_0 \) is the reduced density matrix of the vacuum state. Entanglement entropy is then computed by analytically continuing in \( n \)
\[
S = -\lim_{n \to 1} \left( n \partial_n - 1 \right) \log Z(n) = -\lim_{n \to 1} \left( n \partial_n - 1 \right) \int_0^\infty \frac{dt}{2t} \text{Tr} \mathcal{K}_O^{(n)}(t),
\] (2.5)
where \( \mathcal{K}_O^{(n)}(t, x : y) \) is the appropriate heat-kernel on \( \mathcal{M}_n \).

In this paper, we restrict to Cauchy surfaces given by a constant time slice and the region \( R \) given by the half space, i.e.,
\[
\Sigma = \{ x \in \mathcal{M} | x^0 = 0 \}, \quad R = \{ x \in \Sigma | x^1 \geq 0 \}. \] (2.6)

The relevant branched cover \( \mathcal{M}_n \) has a conical singularity in the \( (x^0, x^1) \)-plane. If we parameterize this plane using polar coordinates \( (r, \phi) \) then the polar angle is \( 2\pi n \) periodic on \( \mathcal{M}_n \). If the theory has \( SO(2) \sim U(1) \) symmetry in the \( (x^0, x^1) \)-plane\(^4\) then the heat kernel \( \mathcal{K}_O(t, x : x') \) depends on the difference of the polar angles \( \phi - \phi' \) and is \( 2\pi \) periodic. The heat kernel on \( \mathcal{M}_n \) is \( 2\pi n \) periodic and is given by [25]
\[
\mathcal{K}_O^{(n)}(t, \phi - \phi') = \mathcal{K}_O(t, \phi - \phi') + \frac{1}{4i\pi n} \int_C dz \cot \left( \frac{z}{2n} \right) \mathcal{K}_O(t, \phi - \phi' + z),
\] (2.7)
where the contour \( C \) consists of two vertical lines: the first one going from \( -\pi - i\infty \) to \( -\pi + i\infty \) and the second going from \( \pi + i\infty \) to \( \pi - i\infty \). In the above expression, we have suppressed the

\(^3\)Strictly speaking, this is not well-defined in a local quantum field theory because of the type-III property of the algebra of observables [24]. We do not worry about such subtleties here as in practice this procedure leads to physically interesting results.

\(^4\)This means that \( \mathcal{M} \) has an isometry which rotates \( (x^0, x^1) \)-plane and the kinetic operator \( O(\partial_\mu, x^\mu) \) is also invariant under the rotation.
dependence of the heat kernel on all coordinates except the polar angle. The entanglement entropy can then be computed by finding the trace of the heat kernel and analytically continuing in $n$. For a Lorentz invariant theory, i.e., $\mathcal{M} = \mathbb{R}^D$ and $\mathcal{O}(\partial_\mu, x^\mu) = \mathcal{O}(\partial_\mu \partial^\mu)$ we give a detailed derivation in appendix A where we show that the partition function on $\mathcal{M}_n$ is given by

$$\log Z(n) = \frac{A}{12 (4\pi)^{D/2 - 1} \Gamma (D/2 - 1)} \frac{1 - n^2}{n} \int_0^\infty \frac{dt}{t} \int_0^\infty dp \, p^{D-3} e^{-t \mathcal{O}(-p^2)} + \ldots. \quad (2.8)$$

The ‘$\ldots$’ represents a term proportional to $n$ which drops out in the computation of entanglement entropy

$$S = \frac{A}{6 (4\pi)^{D/2 - 1} \Gamma (D/2 - 1)} \int_0^\infty \frac{dt}{t} \int_0^\infty dp \, p^{D-3} e^{-t \mathcal{O}(-p^2)}. \quad (2.9)$$

Here $A = \int d^{D-2} x$ is the area of the entangling surface $\partial R$. For generic Lorentz-invariant kinetic terms, the integral over $t$ diverges which leads to the divergent entanglement entropy: $S \sim \frac{A}{\epsilon^{D/2}}$, where $\epsilon$ is a UV-cutoff.

Now consider the theory on a spacetime of the form $\mathcal{M} = \mathbb{R}^D \times \mathcal{X}$, where $\mathcal{X}$ is some $d_X$-dimensional space. The fields now depend on the coordinates $x^\mu$ on $\mathbb{R}^D$ and coordinates along $\mathcal{X}$ which we do not specify. We take the kinetic operator to be $\mathcal{O}(\partial_\mu \partial^\mu) + \mathcal{O}_X$ where $\mathcal{O}_X X$ is some differential operator on $\mathcal{X}$. The heat kernel of the sum of the operator is simple a product of the heat kernels for the two operators. If we are interested in computing the entanglement entropy of the region defined in eq. (2.6), i.e., Cauchy surface and the subregion are defined by imposing appropriate conditions on $\mathbb{R}^D$, then the heat kernel on the relevant branched cover also factorizes.

$$K^{(n)}_{\mathcal{O} + \mathcal{O}_X} = K^{(n)}_{\mathcal{O}} K_{\mathcal{O}_X}. \quad (2.10)$$

Upon analytically continuing in $n$ we see that the entanglement entropy of the half space for the theory on $\mathcal{M} = \mathbb{R}^D \times \mathcal{X}$ is

$$S = \frac{A}{6 (4\pi)^{D/2 - 1} \Gamma (D/2 - 1)} \int_0^\infty \frac{dt}{t} \int_0^\infty dp \, p^{D-3} e^{-t \mathcal{O}(-p^2)} \text{Tr} K_{\mathcal{O}_X}(t). \quad (2.11)$$

In the case where the operator $\mathcal{O}(-p^2) = \frac{p^2}{2}$, i.e., the usual kinetic operator, the above expression simplifies to

$$S = \frac{A}{12 (2\pi)^{D/2 - 1}} \int_0^\infty \frac{dt}{t^{D/2}} \text{Tr} K_{\mathcal{O}_X}(t). \quad (2.12)$$

3. Entanglement entropy in open string theory

In this section we compute entanglement entropy in open SFT using our result in eq. (2.12). This has been previously computed by [23] using algebraic methods. Our approach is based on the replica method and sheds a different light on the problem. It is the replica method approach which we then generalize to the case of closed strings.
In the light-cone gauge, the quadratic part of the open SFT action takes the form [26]

\[ I_{OSFT} = \int [DX]_{open} \Phi \left( \partial_+ \partial_- + \frac{\pi}{2} \int_0^\pi d\sigma \left( -\frac{\delta}{\delta X^I(\sigma)} \frac{\delta}{\delta X^I(\sigma)} + \frac{1}{4\pi^2\alpha'^2} \partial_\sigma X^I(\sigma) \partial_\sigma X^I(\sigma) \right) \right) \Phi. \] (3.1)

Various terms appearing in this action are to be understood as follows. \( x^+ \equiv X^0 \pm X^{D-1} \sqrt{2} \) is the light-cone time coordinate\(^5\) and \( x^- \) is the zero mode associated with the other linearly independent combination

\[ \frac{X^0 - X^{D-1}}{\sqrt{2}} = x^- + \cdots. \] (3.2)

Here ‘\( \cdots \)’ denotes \( \sigma \)-dependent terms which are determined in terms of the mode expansion of the transverse coordinates \( X^I(\sigma) \) for \( I = 1, 2, \cdots, D-2 \). The transverse coordinates have the mode expansion

\[ X^I(\sigma) = x^I + \sqrt{2} \sum_{m=1}^\infty x^I_m \cos(m\sigma). \] (3.3)

The differential operator \( \frac{\delta}{\delta X^I(\sigma)} \) acts on the string-field \( \Phi \) as follows

\[ \frac{\delta}{\delta X^I(\sigma)} \equiv \frac{1}{\pi} \left( \frac{\partial}{\partial x^I} + \sqrt{2} \sum_{m=1}^\infty \cos(m\sigma) \frac{\partial}{\partial x^I_m} \right). \] (3.4)

The string-field \( \Phi \) is defined on the configuration space of all open strings \( \mathcal{M}_{open} \) which is infinite dimensional. In the light-cone gauge this space is parameterized by \( x^+, x^- \) and \( X^I(\sigma) \) or equivalently

\[ \mathcal{M}_{open} = \mathbb{R}^D \times \mathcal{X}, \] (3.5)

where \( \mathcal{X} \) is an infinite dimensional space parameterized by the coordinates \( x^I_m \). The measure \([DX]_{open}\) is an infinite dimensional integration measure on \( \mathcal{M}_{open} \) which can be written in terms of the mode expansion as

\[ [DX]_{open} \equiv dx^- dx^+ d^{D-2}x \times \prod_{m=1}^\infty d^{D-2}x_m = d^Dx \times \prod_{m=1}^\infty d^{D-2}x_m. \] (3.6)

Using the mode expansion, the action can be recasted into a more familiar form

\[ I_{OSFT} = \int d^Dx \prod_{m=1}^\infty d^{D-2}x_m \Phi \left( \partial_+ \partial_- - \frac{1}{2} \partial_I \partial^I - \frac{1}{2} \sum_{m=1}^\infty \left( \frac{\partial^2}{\partial x^I_m \partial x^I_m} - \frac{m^2}{2\alpha'} x^I_m x^I_m \right) \right) \Phi. \] (3.7)

Along \( \mathbb{R}^D \) the kinetic operator is the usual Laplacian. Along \( \mathcal{X} \), the kinetic operator is the sum of time-independent Schrodinger operators for simple harmonic oscillators with frequency \( \frac{m^2}{2\alpha'} \) for \( m = 1, 2, \cdots \).

\(^5\)There are subtleties associated with light-cone quantization such as well-posedness of the Cauchy problem, causality, etc. There is a well known method to avoid these subtleties by shifting the coordinates such that the constant \( x^+ \) surfaces are space like [27]. For the computation of partition functions in Euclidean signature such subtleties can be safely ignored.
Let $K_m(t, x_I^I : y_I^I)$ denote the heat kernel for the oscillator part of the kinetic term. Then according to eq. (2.12), the entanglement entropy for open strings is

$$S_{\text{open}} = \frac{A}{12(2\pi)^{D/2-1}} \int_0^\infty \frac{dt}{t^{D/2}} \prod_{I=1}^{D-2} \prod_{m=1}^\infty \text{Tr} K_m(t, x_I^I : x_I^I).$$

(3.8)

The properly normalized heat kernel for the simple harmonic oscillator can be found in [28]. The trace gives the partition function of the simple harmonic oscillator with frequency $\frac{m}{2\alpha'}$.

$$\text{Tr} K_m(t) = \frac{e^{-\frac{mt}{4\alpha'}}}{1 - e^{-\frac{mt}{2\alpha'}}}.$$  

(3.9)

After performing the product over $m$ and $I$ and setting $D = 26$ we get

$$\prod_{I=1}^{D-2} \prod_{m=1}^\infty \text{Tr} K_m(t) = \eta \left(\frac{it}{4\pi\alpha'}\right)^{-24}.$$  

(3.10)

We scale the integration variable $t \to t \times 4\pi\alpha'$ to write the entanglement entropy as

$$S_{\text{open}} = \frac{A}{12(8\pi^2\alpha')^{12}} \int_0^\infty \frac{dt}{t^{13}} \eta(it)^{-24}.$$  

(3.11)

This matches the result of [23] and is also consistent with the effective field theory expectation. An important difference here is that we used the replica method in our approach. It was argued in [23] that their algebraic method does not encounter the ‘contact terms’ of conical entropy which appear in the worldsheet computation using the replica method [13]. Here we see that the replica method does not encounter the aforementioned contact-terms as well. These contact-terms are related to contributions from the edge modes. Any analysis based on the light-cone gauge fixes the gauge freedom completely. We are essentially studying the dynamics of a collection of scalar degrees of freedom and hence do not encounter any contact-terms.

4. Entanglement entropy in closed string theory

In this section we compute the entanglement entropy in closed string theory. We start by a generalization of the open SFT action. The kinetic term that appears in it is a simple generalization of the open SFT kinetic term and includes extra oscillator modes gives a UV-divergent Renyi partition functions and entanglement entropy. This kinetic term also does not lead to the correct one-loop amplitude for closed strings [16]. We then propose a non-local kinetic term which does not modify the on-shell physics but gives the correct one-loop amplitude. The entanglement entropy computed from the closed SFT with this non-canonical kinetic term is UV-finite.

4.1. Closed SFT with the canonical kinetic term

The development of closed SFT has an exciting and rich history. Light-cone SFT was developed in [29, 30]. The covariant formalism for closed bosonic strings was developed in the seminal work
of Zwiebach [31](see [32] for a recent review of covariant formalism for closed superstrings). Since we are interested in the free light-cone theory, we only need some elementary ingredients which can also be ‘derived’ from first quantized theory. In the light-cone gauge the action takes the form

\[ \mathcal{I}_{\text{CSFT}} = \int [DX]_{\text{closed}} \Phi \left( \partial_+ \partial_- + \pi \int_0^{2\pi} d\sigma \left( -\frac{\delta}{\delta X^I(\sigma)} \frac{\delta}{\delta X^I(\sigma')} + \frac{1}{4\pi^2\alpha'^2} \partial_\sigma X^I(\sigma) \partial_\sigma X^J(\sigma) \right) \right) \Phi, \]

(4.1)

with the mode expansions now given by

\[ X^I(\sigma) = x^I + \sqrt{2} \sum_{m=1}^{\infty} x^I_m \cos(m\sigma) + \tilde{x}^I_m \sin(m\sigma), \]

(4.2)

In addition to the equations of motion resulting from the above action, the closed string-field is also subject to a constraint which has its origins in the level matching condition. This can be expressed as

\[ \int_0^{2\pi} d\sigma X''^I(\sigma) \frac{\delta}{\delta X^I(\sigma)} \Phi = 0, \]

(4.3)

As before, one can expand the action and constraints in terms of the coordinates \( x^I_m \) and \( \tilde{x}^I_m \). In particular, the constraint takes the following form

\[ \sum_{m=1}^{\infty} m \left( x^I_m \frac{\partial}{\partial x^I_m} - \tilde{x}^I_m \frac{\partial}{\partial \tilde{x}^I_m} \right) \Phi = 0. \]

(4.4)

It seems difficult to gain much insight from this form of the constraint. Let’s consider an eigenfunction of the kinetic operator along the oscillator directions. These are simply the wavefunctions of an infinite collection of simple harmonics oscillators. The vacuum satisfies the constraint but it is not obvious which excited wavefunctions also satisfy the constraint. To explore the consequences of the constraint further it is useful to change variables.

The key idea is to note that \( X^I(\sigma) \) and \( \frac{\delta}{\delta X^J(\sigma')} \) act on the string-field as operators which are canonically conjugate to each other,

\[ \left[ X^I(\sigma), \frac{\delta}{\delta X^J(\sigma')} \right] = -\delta^{IJ} \delta(\sigma - \sigma'). \]

(4.5)

By a functional Fourier transform, one can switch the role of the ‘coordinate’ variable and the ‘momentum’ variable. The mode expansion given above provides an infinite set of canonical pairs.

\[ \left[ \tilde{x}^I_m, \frac{\partial}{\partial \tilde{x}^J_m} \right] = -\delta_{mn} \delta^{IJ}, \quad \left[ x^I_n, \frac{\partial}{\partial x^J_m} \right] = -\delta_{mn} \delta^{IJ}. \]

(4.6)
The choice is not unique\(^6\). We can equally well assign the ‘coordinate’ interpretation to \(\frac{\partial}{\partial x^I_m} \equiv x^I_m\) with \(-x^I_m \equiv \frac{\partial}{\partial x^I_m}\) being its derivative. We now utilize this freedom make the following change of variables which does not modify the commutation relations given above.

\[\begin{align*}
x^I_m &\to \frac{1}{\sqrt{2}} (x^I_m + \tilde{x}^I_m), \\
\tilde{x}^I_m &\to \frac{i\alpha'}{m\sqrt{2}} \left( \frac{\partial}{\partial x^I_m} - \frac{\partial}{\partial \tilde{x}^I_m} \right), \\
\frac{\partial}{\partial x^I_m} &\to \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^I_m} + \frac{\partial}{\partial \tilde{x}^I_m} \right), \\
\frac{\partial}{\partial \tilde{x}^I_m} &\to \frac{i m}{\sqrt{2} \alpha'} \left( x^I_m - \tilde{x}^I_m \right) .
\end{align*}\] (4.7)

(4.8)

In terms of this new mode expansion, we can write the action as

\[I_{CSFT} = \int [DX]_{\text{closed}} \Phi \mathcal{O} \Phi.\] (4.9)

The kinetic operator \(\mathcal{O}\) is given by

\[\mathcal{O} = \partial_+ \partial_- - \frac{1}{2} \partial_I \partial^I + \mathcal{O}_\mathcal{X} + \mathcal{O}_{\tilde\mathcal{X}},\] (4.10)

where we have defined the operators

\[\begin{align*}
\mathcal{O}_\mathcal{X} &= \frac{1}{2} \sum_{m=1}^{\infty} \left( - \frac{\partial^2}{\partial x^I_m \partial x^I_m} + \left( \frac{m}{\alpha'} \right)^2 x^I_m x^I_m \right), \\
\mathcal{O}_{\tilde{X}} &= \frac{1}{2} \sum_{m=1}^{\infty} \left( - \frac{\partial^2}{\partial \tilde{x}^I_m \partial \tilde{x}^I_m} + \left( \frac{m}{\alpha'} \right)^2 \tilde{x}^I_m \tilde{x}^I_m \right).
\end{align*}\] (4.11)

The constraint eq. (4.3) now takes the form

\[(\mathcal{O}_\mathcal{X} - \mathcal{O}_{\tilde{X}}) \Phi = 0.\] (4.12)

The space of configurations of closed string in the light-cone gauge is given by

\[\mathcal{M}_{\text{closed}} = \mathbb{R}^D \times \left( \mathcal{X} \times \tilde{\mathcal{X}} \right),\] (4.13)

where \(\mathcal{X}\) and \(\tilde{\mathcal{X}}\) are two infinite dimensional spaces encoding the dependence on the oscillator directions \(x^I_n\) and \(\tilde{x}^I_n\). The integration measure on this space is

\[\left[DX\right]_{\text{closed}} \equiv d^D x \times \prod_{m=1}^{\infty} d^{D-2} x_m d^{D-2} \tilde{x}_m.\] (4.14)

So far we have insisted on imposing the constraint of eq. (4.12) ‘by hand’. We can implement

\(^6\)This non-uniqueness is related to the ambiguity in identifying canonically conjugate variables for the Poincaré conserved charges in string theory. We discuss this in more detail in section 5.1.
the constraint (4.12) as a consequence of equations of motion\(^7\) by introducing the operator

\[
\mathcal{P} = \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \, e^{2\pi i \alpha' s} = \frac{\sin (\alpha' \pi \mathcal{O}_-)}{\alpha' \pi \mathcal{O}_-},
\]

(4.15)

which projects onto the space of functions which are annihilated by \(\mathcal{O}_- = \mathcal{O}_X - \mathcal{O}_{\tilde{X}}\). To see this, note that the operator \(\mathcal{O}_-\) has eigenvalues which are integer multiples of \(\frac{1}{\alpha'}\). The operator \(\mathcal{P}\) acts as zero on the functions with non-zero eigenvalues of \(\mathcal{O}_-\) while it acts as identity on the functions with zero eigenvalue of \(\mathcal{O}_-\). We can now decompose an arbitrary field \(\Phi\) as

\[
\Phi = \Psi + \varphi,
\]

(4.16)
such that \(\Psi\) is annihilated by \(\mathcal{O}_-\), i.e., \(\mathcal{P} \Psi = \Psi\) and \(\varphi\) is not, i.e., \(\mathcal{P} \varphi = 0\). The closed SFT action can now be written as

\[
I_{\text{CSFT}} = \int [D\mathcal{X}]_{\text{closed}} \Psi \mathcal{O} \Psi + \varphi \varphi,
\]

(4.17)

Fields \(\varphi\) which are not annihilated by \(\mathcal{O}_-\) are set equal to zero by the equation of motion and only the fields \(\Psi\) which are annihilated by \(\mathcal{O}_-\) furnish consistent on-shell configurations. Using the projection operator \(\mathcal{P}\) and the orthogonality of \(\Psi\) and \(\varphi\) we can write the above action in terms of the unrestricted fields \(\Phi\) as follows

\[
I_{\text{CSFT}} = \int [D\mathcal{X}]_{\text{closed}} \Phi \mathcal{O} \Phi + \Phi \left(1 - \mathcal{P}\right) \Phi.
\]

(4.18)

Upon expanding the above quadratic operator there will be cross terms between \(\Psi\) and \(\varphi\) but they integrate to zero because of the orthogonality. The operator that is appearing in the above action is actually equal to the operator \(\mathcal{O} \mathcal{P}\). To see this note that we can rewrite \(\mathcal{O} \mathcal{P} = \exp (\mathcal{P} \log \mathcal{O})\). Since \(\mathcal{P}\) and \(\mathcal{O}\) commute, this rewriting is unambiguous. Next we expand the exponential in powers of \(\mathcal{P} \log \mathcal{O}\)

\[
\exp (\mathcal{P} \log \mathcal{O}) = \sum_{n=0}^{\infty} \frac{\mathcal{P}^n (\log \mathcal{O})^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\mathcal{P} (\log \mathcal{O})^n}{n!} = 1 + \mathcal{P} (\exp (\log \mathcal{O}) - 1) = \mathcal{O} \mathcal{P} + (1 - \mathcal{P}).
\]

(4.19)

So we finally arrive at the action

\[
I_{\text{CSFT}} = \int [D\mathcal{X}]_{\text{closed}} \Phi \mathcal{O} \mathcal{P} \Phi.
\]

(4.20)

Given this action, the computation of entanglement entropy is a straightforward generalization of the open string case. On the \(n\)-fold branched cover of \(\mathcal{M}_{\text{closed}}\) the heat kernel factorizes. After computing the trace over the ‘zero-mode’ directions the entanglement entropy takes the form

\[
S_{\text{closed}} = \frac{A}{12 (2\pi)^{D/2-1}} \int \frac{dt}{t^{D/2}} \text{Tr} \mathcal{P} \mathcal{K}_{\mathcal{O}_X + \mathcal{O}_{\tilde{X}}}.
\]

(4.21)

\(^7\)This is not possible in the covariant closed SFT as suitable kinetic term, gauge invariance and the inclusion of interaction demand that the string field is subject to a set of subsidiary conditions which include \(\mathcal{O}_- \Phi = 0\) [31].
Expressing the heat kernel in terms of the operators and after scaling \( t \) appropriately the above expression becomes

\[
S_{\text{closed}} = \frac{A}{12 (4\pi^2 \alpha')^{12}} \int dt \frac{1}{t^{11/2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \, \text{Tr} \, q^{\alpha' \mathcal{O} X} \bar{q}^{\alpha' \mathcal{O} \bar{X}},
\]

(4.22)

where \( q = e^{2\pi i (s + it)} \equiv e^{2\pi i \tau} \). The trace over the oscillators gives familiar factors of Dedekind eta function

\[
\text{Tr} \, q^{\alpha' \mathcal{O} X} \bar{q}^{\alpha' \mathcal{O} \bar{X}} = \left| \eta(\tau) \right|^{-48}.
\]

(4.23)

Finally we write the integral in terms of the complex variable \( \tau \) to get

\[
S_{\text{closed}} = \frac{A}{24 (4\pi^2 \alpha')^{12}} \int_S \frac{d\tau d\bar{\tau}}{(\text{Im} \, \tau)^{13}} \left| \eta(\tau) \right|^{-48},
\]

(4.24)

where the integral is over the strip \( S = \{ \tau \mid \text{Im} \, \tau \geq 0, -\frac{1}{2} \leq \text{Re} \, \tau < \frac{1}{2} \} \). This resembles the torus amplitude in closed strings. However, if we interpret \( \tau \) as the modulus of the torus, the integration region is not restricted to the fundamental domain. The entanglement entropy computed above is divergent. In the \( \text{Im} \, \tau \to \infty \) region of the integration, it has divergences due to the closed string tachyon but this is not interesting. We expect such a divergence to go away in a consistent string theory. It also has the usual UV-divergence of quantum field theory in the \( \text{Im} \, \tau \to 0 \) limit of integration region. In fact, the result can be written as a sum over entanglement entropies of different fields in the closed string spectrum. In that sense, one may argue that the this is indeed the expected result and hence also has usual UV-divergences.

But one must stress that the physics of closed string theory, even in the free limit, is different than that of a collection of free fields in the closed string spectrum. The most lucid example of this fact is the UV-finiteness of one-loop vacuum amplitude in closed string theory. The vacuum amplitude is not just a sum over individual vacuum amplitudes of all fields in the closed string spectrum. The canonical kinetic term that appeared in the closed SFT action above, does not produce the correct vacuum amplitude for closed string theory. We discuss this in more detail in the next section.

4.2. A non-canonical kinetic term for closed SFT

On-shell information contained in the Lagrangian of a field theory is the spectrum of the theory and various S-matrix elements. Different Lagrangians, related by field redefinitions, may have different sets of Green’s functions but will yield the same values for on-shell observables. For a free theory, the on-shell content is merely the spectrum of the theory. On the other hand, contribution to one-loop amplitude comes only from the off-shell states. It is therefore possible to have different vacuum amplitudes but the same on-shell content. In this section we pursue this possibility for free closed SFT.
The vacuum amplitude resulting from the closed SFT action (4.20) is given by

\[ \mathcal{V} = -\log Z = \frac{1}{2} \text{Tr} \log \mathcal{O}^P, \]
\[ = -\frac{1}{4 \left(4\pi^2\alpha'\right)^{13}} \int_S \left(\text{Im} \tau\right)^{14} \left|\eta(\tau)\right|^{-48}, \]

which is not the same as the one obtained from the worldsheet computation [16]. The difference is precisely that of the integration region in the complex \( \tau \)-plane. The world sheet computation has an integration over the fundamental domain \( \mathcal{F} \) of the torus and hence gives a finite (up to tachyonic divergences) result. If we interpret \( \tau \) as the modulus of the torus then the strip \( \mathcal{S} \) covers the moduli space of torus an infinite number of times and this over counting leads to an infinite answer. In an ordinary quantum field theory, vacuum amplitude such as the one computed above only gives an overall phase and hence drops out in all expectation values. However, when one considers the coupling to gravity, the one-loop vacuum amplitude sources the geometry and hence it is an observable. With this understanding, we proceed by asking: what modification of the free closed SFT action gives a vacuum amplitude consistent with the worldsheet computation?\(^8\)

Since we are limiting ourselves to the free closed SFT, we look for field redefinitions which do not introduce any cubic or higher-order terms in fields. The eventual consequence of such a field redefinition is to change the kinetic term so that

\[ I_{\text{CSFT}} = \int [DX] \Phi \mathcal{O}^P \times f (\mathcal{O}, \mathcal{O}_-) \Phi. \]

Here \( f (\mathcal{O}, \mathcal{O}_-) \) is a differential operator which must be chosen so that

1. The on-shell content of the theory does not change. This can be guaranteed by showing that \( \mathcal{O}^P \) and \( \mathcal{O}^P f (\mathcal{O}, \mathcal{O}_-) \) have the same kernel.
2. The inverse of the kinetic operator only has simple poles at \( \mathcal{O}^P = 0 \), i.e., at on-shell states. This can be ensured by arguing that \( f \) does not vanish on-shell.
3. The vacuum amplitude from the modified action coincides with the worldsheet computation.

Note that these three requirements do not determine the kinetic operator uniquely. One can modify it further without spoiling the above conditions. We emphasize, however, that this ambiguity does not lead to an ambiguity in the computation of entanglement entropy. To see this more explicitly, consider two operators \( \mathcal{O}^P f (\mathcal{O}, \mathcal{O}_-) \) and \( \mathcal{O}^P g (\mathcal{O}, \mathcal{O}_-) \) which satisfies the above three requirements. Since the vacuum amplitude resulting from the both kinetic operators must be the same this implies

\[ \text{Tr} \log g (\mathcal{O}, \mathcal{O}_-) = 0. \]

It then follows that such modification would not change the result of the path integral over the replicated manifold and would not affect the result of the entanglement entropy.

\(^8\)In the covariant closed SFT one can obtain a finite contribution to the vacuum amplitude from a modified kinetic term. The finite contribution involves integration of the worldsheet conformal field theory partition function over a strip in the moduli space of the torus. It seems necessary to add a constant term in the action to get the correct vacuum amplitude which involves integration over the fundamental domain of the torus. We thank Barton Zwiebach for discussion on this point.
We now find the function \( f(\mathcal{O}, \mathcal{O}_-) \) so that the third requirement is satisfied by construction. We will then demonstrate that our choice also satisfies the first two requirements. The vacuum amplitude as computed from worldsheet methods takes the following form

\[
V_{w.s} = \frac{1}{4} \frac{V}{(4\pi^2\alpha')^{1/3}} \int \frac{d\tau d\bar{\tau}}{(2\pi\text{Im}\tau)^{14}} \left| \eta(\tau) \right|^{-1/2} \quad (4.28)
\]

Using \( \tau = s + it \) the above expression can be written in terms of a trace involving operators \( \mathcal{O} \) and \( \mathcal{O}_- \). Note that

\[
\text{Tr} e^{2\pi i\alpha's\mathcal{O}} e^{-2\pi\alpha't\mathcal{O}} = \text{Tr} q^{\alpha'O} \bar{q}^{\alpha'O} \bar{x}^{\alpha'} e^{+\pi\alpha't\partial_x \partial^x}.
\]

The trace over the oscillators, i.e., the first two factors is given in eq. (4.23) and the trace of the last factor is computed in eq. (A.2). We get

\[
\text{Tr} e^{2\pi i\alpha's\mathcal{O}} e^{-2\pi\alpha't\mathcal{O}} = \left| \eta(\tau) \right|^{-1/2} \frac{V}{(4\pi^2\alpha')^{1/3}}, \quad (4.30)
\]

We can now write the worldsheet vacuum amplitude as

\[
V_{w.s} = -\frac{1}{2} \text{Tr} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds e^{2\pi i\alpha's\mathcal{O}} \int_{\sqrt{1-s^2}}^\infty dt \frac{V}{(4\pi^2\alpha')^{1/3}} e^{-2\pi\alpha'tO}, \quad (4.31)
\]

where \( E_1(x) \) is the exponential integral defined (for values of \( x \) off-the negative real axis) as

\[
E_1(x) = \int_1^\infty \frac{e^{-tx}}{t} dt = -\gamma - \log x - \sum_{n=1}^{\infty} \frac{(-x)^n}{nn!}. \quad (4.32)
\]

Comparing the last line of eq. (4.31) with the vacuum amplitude \( \frac{1}{2} \text{Tr} \log \mathcal{O}^P f(\mathcal{O}, \mathcal{O}_-) \) obtained from the closed SFT action in eq. (4.9) we find that the modified kinetic operator is

\[
\mathcal{O}^P f = \exp \left( -\int_{-\frac{1}{2}}^{\frac{1}{2}} ds e^{2\pi i\alpha's\mathcal{O}} E_1 \left( 2\pi\alpha'\sqrt{1-s^2O} \right) \right). \quad (4.33)
\]

Using the series representation of the exponential integral eq. (4.32) we can also find a power series expansion for \( \log f \)

\[
\log f = \mathcal{P} \log \left( 2\pi\alpha'e^\gamma \right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} ds e^{2\pi i s\mathcal{O}} \left[ \log \sqrt{1-s^2} + \sum_{n=1}^{\infty} \frac{\left( -2\pi\alpha'\sqrt{1-s^2O} \right)^n}{nn!} \right] \quad (4.34)
\]

\(^9\)It is instructive to compare this kinetic operator with the one obtain by Zwiebach and Sen in sec. (6.3) of [34]. Their kinetic operator can be written as \( \exp \left( -E_1(2\alpha'\mathcal{O}) \right) \). Upon computing the trace this gives the integral of the worldsheet partition function over a strip in the moduli space of the torus as mentioned in footnote 8. Geometrically this corresponds to including all tori built by joining opposite ends of a cylinder of length greater than or equal to \( 2a \).
For any finite value of $O$ the function $\log f$ is finite. The modified kinetic operator is highly non-local as it involves infinite number of derivatives. But we emphasize that the operator satisfies the above three requirements and does not change any of the on-shell physics.

First we show that the kernel is the same as that of $O P$. If $P = 1$ and $O = 0$ then $f$ is a finite quantity and hence the modified operator $O f = 0$. To prove the converse let’s assume $O f = 0$ but $O \neq 0$. This can happen in two ways: (1). $P = 0$. It is then clear that $\log f$ is finite for any finite value of $O$ and hence $O f \neq 0$ leading to a contradiction. (2). $P = 1$ but $O \neq 0$. In this case $O f = O f$. Now, again for any finite value of $O$, $f$ is a finite quantity so $O f$ must vanish, satisfying the first requirement. The second requirement is also satisfied because $f$ is a finite quantity as $O \to 0$ so $1/O f$ only has simple poles at $O = 0$. The third requirement is satisfied by construction.

4.3. Entanglement entropy with modified kinetic operator

We next use the modified kinetic term to compute entanglement entropy of the half space. We need the representation of $\log O f$ on the $n$-fold branch cover of $M_{\text{closed}}$. As before, the trace factorizes naturally into contributions from oscillators and zero-modes

\[
\log Z = -\frac{1}{2} \text{Tr} \log O f = \frac{1}{2} \text{Tr} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds e^{2\pi \alpha' s O} \int_{\sqrt{1-s^2}}^{\infty} dt \frac{e^{-2\pi \alpha' t O}}{t},
\]

\[
= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{\sqrt{1-s^2}}^{\infty} dt \frac{e^{-2\pi \alpha' t O}}{t} \text{Tr} X (q^{\alpha' O} X) \text{Tr} \tilde{X} (q^{\alpha' O} X) \text{Tr} R_{26} e^{\pi \alpha' \partial_\mu \partial^\mu}.
\]

To compute the trace over the branched cover one only needs to replace the zero-mode factor, i.e., $R_{26}$ by the appropriate branched cover. The trace of $e^{-t O (\partial_\mu \partial^\mu)}$ on that branched cover is computed in the appendix (see eq. (A.12)). The trace over the oscillators gives well known factors of Dedekind eta function. The logarithm of the Renyi partition function is

\[
\log Z (n) = \frac{A}{48 (4\pi^2 \alpha')^{12}} \frac{1 - n^2}{n} \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im } \tau)^{13}} \left| \eta (\tau) \right|^{-48} + n \log Z.
\]

The entanglement entropy computed from this is

\[
S = \frac{A}{24 (4\pi^2 \alpha')^{12}} \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im } \tau)^{13}} \left| \eta (\tau) \right|^{-48}.
\]

The above expression is finite up to tachyonic divergences which will be absent in the superstrings as the tachyon is projected out of the spectrum. Moreover, if one interprets the parameter $\tau$ as modulus of a torus then the result for entanglement entropy is not modular invariant. Here we use the term modular invariance in the same sense in which the vacuum amplitude of closed strings is modular invariant: the answer can be written as an integral of a modular invariant function over the fundamental domain of the torus with a modular invariant integration measure. This is related to the fact that we are considering an off-shell background to compute the entanglement entropy. We believe that the modular invariance can be restored by a careful analysis of the contribution of the edge-modes to the entanglement entropy [35].

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5. Discussion, caveats and outlook

In this paper we have taken initial steps towards understanding entanglement entropy in closed string theory using the formalism of SFT. We only needed some elementary ingredients from closed light-cone SFT and we demonstrated that the resulting entanglement entropy is UV-finite. An important role is played by a non-canonical kinetic term which gives a finite one-loop amplitude. Thus the mechanism responsible for the finiteness of the entanglement entropy is the same that makes the one-loop vacuum amplitude UV-finite. While this is encouraging progress, we have so far neglected subtleties associated with defining subregions and entanglement entropy in string theory as well as the algebraic aspects of finiteness of entanglement entropy. In the rest of this paper we discuss these issues in light of our computation.

5.1. Ambiguities in defining spacetime and subregions

There are conceptual issues related to entanglement and the definition of subregions in a theory of gravity. In a theory with a gauge symmetry degrees of freedom in two regions cannot be factorized unambiguously. The extended phase space approach of [35, 36] addresses this issue by introducing extra degrees of freedom at the entangling surface, the so called edge-modes, which contribute non-trivially to the entanglement entropy. The analysis of [23] for the case of open strings suggest that a similar picture should hold in closed string theory. The covariant SFT (at least in the $g_s \to 0$ limit) can be used to shed some light on the extended phase space of closed strings. While the issues related to gauge invariance and edge-modes present challenging technical problems here we point out inherently stringy ambiguities in defining the spacetime and subregions.

Let us discuss two important choices that we had to make in order to define and compute entanglement entropy. Even though the detailed expression for entanglement entropy certainly depends on these choices, we believe that the general structure, i.e., the finiteness of entanglement entropy is independent of them. First of all we had to make a choice of physical time to define a Cauchy surface in the space of strings and a component of space to define a subregion. In the light-cone gauge, these choices may seem natural but in string theory there is no unambiguous way to make these choices. String theory has a well-defined notion of momenta $p^\mu$ as the conserved charges associated with translations in $X^\mu(\sigma)$. There is, however, no unique way to choose the coordinate $x^\mu$ to label the spacetime $\mathbb{R}^D$. One way is to impose the canonical commutation relation

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}. \quad (5.1)$$

Let us define $x^\mu$ by integrating $X^\mu(\sigma)$ over the length of the string against a projection $\Pi^\mu_\nu$ such that

$$x^\mu = \int_0^{2\pi} d\sigma \Pi^\mu_\nu X^\nu(\sigma), \quad \int_0^{2\pi} d\sigma \Pi^\mu_\nu = \delta^\mu_\nu. \quad (5.2)$$

All such $x^\mu$ satisfy the canonical commutation relation of eq. (5.1). The definition (5.2) picks out the center of the mass of the string if we choose $\Pi^\mu_\nu = \frac{1}{2\pi} \delta^\mu_\nu$. But as is obvious by now, it is not the only choice. One can easily arrange to have $x^\mu$ equal to a linear combination of oscillator modes of various frequencies. It is also possible to have different choices along different directions. For example, $x^0$ given by the center of mass of $X^0(\sigma)$ and $x^1$ given by oscillator mode of $X^1(\sigma)$. If we make such a choice, then we do not have the U(1) symmetry in the $(x^0, x^1)$-plane which was
necessary in our computation of heat kernels on the branched cover.

Second key choice that we made in our analysis is the extension of the Cauchy slice and a surface in the c.o.m space to the full configuration space of strings. We used a trivial extension as in [23] but it is clear that different choices will give different expressions.

The ambiguities in defining spacetime and subregions are the stringy manifestation of the subtleties familiar in diffeomorphism theories.

5.2. Non-locality and the algebraic structure

Another important conceptual question is what does the finiteness of entanglement entropy teach us about the algebraic structure of quantum gravity? Entanglement entropy in our computation is rendered finite by non-local nature of closed strings. Since entanglement entropy is a property of the algebra of observables of the quantum system this suggests that the underlying algebraic structure in quantum gravity is different than that of a local quantum field theory. For perturbative quantum gravity such non-locality and its consequences for the underlying algebraic structure are discussed in [37–39]. It was argued in these papers that diffeomorphism invariant observables have non-local commutators which fail to vanish at spacelike separations. The non-locality appears at first order in the Newton’s constant. Analogous results can be obtained in string theory using the framework of SFT [40]. The commutator of string fields fail to vanish at space-like separations at first order in the string-coupling. This might seem counterintuitive given the fact that the scale of non-locality in string theory is controlled by the string length $\sqrt{\alpha'}$ and not the string coupling. Moreover, our result for one-loop entanglement entropy is independent of the string-coupling but it depends on $\alpha'$. But a careful analysis show that the failure of spacelike commutators to vanish is due to the non-local interactions between string-fields and hence depends on the string-length in non-perturbative way. The observables that were considered by [40] and [37–39] only see the non-locality when interactions are involved. While a better understanding of the algebraic structure in string theory is desirable we focus here on a simple example to explain essential features.

We consider a theory which has non-local interactions but a local kinetic term. We show that this leads to non-vanishing commutators at spacelike separations. We then do a field redefinition such as the kinetic term becomes non-local but the interaction stay local. In the new variables the non-locality manifests itself through off-shell quantities such as the propagator and the free energy.

Consider a scalar field theory with non-local interactions described by the Lagrangian\textsuperscript{10}

$$I = - \int d^4x \, \partial_\mu \phi \partial^\mu \phi + \frac{g}{3!} \left( e^{-\alpha' \partial^2} \phi \right)^3. \quad (5.3)$$

Our aim here is to show how non-local interactions can lead to the non-vanishing commutators between operators at space like separations. Let $\phi_I(x)$ denote the field operator in the interaction picture which satisfies the free field equation of motion and the commutator

$$[\phi_I(x), \phi_I(y)] = -ig \, G(x-y), \quad (5.4)$$

where $G(x-y)$ is the Green’s function for operator $\partial^2$ and it vanishes for spacelike separation $x-y$.\textsuperscript{10}

\textsuperscript{10}This is inspired by the truncation of the complete SFT action to the tachyonic sector with a cubic interaction.
This can be represented as

\[
G(x - y) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{-1}{\left(k^0 - |\vec{k}|\right)\left(k^0 + |\vec{k}|\right)} e^{i k \cdot (x - y)},
\]

(5.5)

where \( C \) is the contour of integration in the complex \( k^0 \)-plane which surrounds both poles. The field operator in the Heisenberg picture \( \phi(x) \) differs from \( \phi_I(x) \) because \( \phi(x) \) is evolved with an interacting Hamiltonian. Equivalently, one can obtain \( \phi(x) \) by solving the equations of motion resulting from the Lagrangian (5.3)

\[
\partial^2 \phi = \frac{g}{2} e^{-\alpha' \partial^2} \left(e^{-\alpha' \partial^2} \phi\right)^2.
\]

(5.6)

We want to compute the commutator of the Heisenberg field operators at spacelike separations to first order in coupling \( g \). Without loss of generality we can choose the time \( t = 0 \) when the field operators in two pictures agree\(^{11}\)

\[
\phi(0, \vec{x}) = \phi_I(0, \vec{x}).
\]

(5.7)

We expand the field as \( \phi(x) = \phi_0(x) + g\phi_1(x) + O(g^2) \) and compute the commutator

\[
[\phi(x), \phi(0)] = [\phi_I(x), \phi_I(0)] + g [\phi_1(x), \phi_I(0)].
\]

(5.8)

For spacelike \( x \) the first term on the r.h.s vanishes and we want to investigate the behavior of the second term. We can find \( \phi_1(x) \) by solving eq. (5.6) perturbively. At first order in \( g \) we get

\[
\phi_1(x) = -\frac{1}{2} \int_{x^0 > 0} d^4 x' e^{\alpha' \partial^2} G_R(x - x') \phi_I^2(x'),
\]

(5.9)

where \( G_R(x - x') \) is the retarded Green’s function. Restricting the integral to be over \( x^0 > 0 \) ensures that \( \phi_1(0, \vec{x}) = 0 \) as required by eq. (5.7). The derivatives inside the integrand are with respect to \( x' \). For spacelike separations the commutator of interest is

\[
[\phi(x), \phi(0)] = ig \int_{x^0 > 0} d^4 x' \left[e^{-\alpha' \partial^2} G_R(x - x')\right] G(x') \phi_I(x').
\]

(5.10)

Since \( x^0 \geq 0 \), the Green’s function \( G(x') \) inside the integrand is non-vanishing for \( x' \geq |\vec{x}'| \). In the absence of non-local interactions, i.e., \( \alpha' = 0 \), the first term in the square brackets on r.h.s is non-zero when \( x - x' \) is timelike, i.e.,

\[
\left[e^{-\alpha' \partial^2} G_R(x - x')\right] \neq 0 \implies x^0 - x'^0 \geq |\vec{x} - \vec{x}'|.
\]

(5.11)

Combine the two inequalities we see that the commutator is non-vanishing only when

\[
x^0 \geq |\vec{x} - \vec{x}'| + |\vec{x}'| \geq |\vec{x} - \vec{y}|.
\]

(5.12)

\(^{11}\)Conventionally one chooses the field operators in the two pictures to agree at past infinity.
In the presence of non-local interactions, however, the statement (5.11) is not true. The most straightforward way to see this is to use the representation of the retarded Green’s function in the momentum space given by

\[ e^{\alpha' \partial^2} G_R (x - x') = \int \frac{d^3 \vec{k}}{(2\pi)^3} \int C_R \frac{dk^0}{2\pi} \frac{e^{-\alpha'(k^0)^2} e^{-i\vec{k} \cdot (x^0 - x'^0)}}{\left(k^0 - |\vec{k}|\right) \left(k^0 + |\vec{k}|\right)} e^{-\alpha |\vec{k}|^2} e^{i\vec{k} \cdot (x - x')}, \]  

(5.13)

where \( C_R \) is the contour for integration over \( k^0 \) in the complex \( k^0 \)-plane as shown in fig. 1. In the presence of the non-local interactions, one cannot use the usual contour deformations arguments which give a non vanishing result only for timelike \( x - x' \).

![Figure 1: Integration contour for the retarded Green's function in the complex \( k^0 \)-plane. In the absence of non-local interaction, for \( x^0 < y^0 \) one can close the contour in the upper half plane and the integral vanishes. For \( x^0 > y^0 \) the contour can be closed below picking up contribution from both poles. In the presence of the non-local interaction the contour cannot be closed and the integral must be split into a contribution from the two poles and a principle valued component.](image)

Let us now change to field variable in which the interaction become local and the non-locality appears in the kinetic term. In terms of the field \( \tilde{\phi} = e^{-\alpha' \partial^2} \phi \) the Lagrangian (5.3) becomes

\[ I = \int d^4 x \left[ \frac{1}{2} \tilde{\phi} e^{2\alpha' \partial^2} \partial^2 \tilde{\phi} - \frac{g}{3!} \tilde{\phi}^3 \right]. \]  

(5.14)

The kernel of the new kinetic operator is the same as that of \( \partial^2 \) and the inverse of the kinetic operator only has simple poles as \( \partial^2 \to 0 \). Moreover, one can verify that the S-matrix elements agree. In the sense of section 4.2 the above field redefinition does not change the on-shell physics. The non-locality in these variables manifest itself in off-shell quantities. For example Green’s function or the propagator now takes the following form in the momentum space

\[ \frac{e^{2\alpha' k^2}}{k^2}. \]  

(5.15)
Due to the non-locality this propagator does not have the interpretation of the correlation function of time ordered product of two $\tilde{\phi}$ operators. Nevertheless, it captures the important non-local behavior which, for example, is responsible for the finiteness of the loop integrals in the theory. Similarly, the path integral over the fields $\tilde{\phi}$ and $\phi$ will lead to different free energies.

5.3. Future directions

The objective of this paper was to show that the entanglement entropy in closed string theory is UV-finite and it can be computed using the framework of string field theory. We briefly commented on subtleties in defining spacetime and subregions in string theory. There are many directions in which our analysis can be extended to gain a better understanding. We close this paper by mentioning a few avenues for further study.

Entanglement entropy in superstring theory

A natural extension of this work is to compute the entanglement entropy in superstring theory. Field theory of superstrings have been formulated relatively recently (see [32] for a review) and has already lead to interesting applications. As we observed for the case of bosonic strings, the entanglement entropy involved the oscillator partition function as a term in the integrand. If the same pattern persists for the case of superstrings entanglement entropy would be zero. This has been argued in [41] but it seems unlikely to us. The vanishing of the vacuum amplitude in superstrings is a result of the target space supersymmetry and the computation of entanglement entropy via the replica method breaks all spacetime supersymmetries. Secondly the entanglement entropy is a manifestly positive quantity. Adding more degrees of freedom in a system should only increase the entropy. Moreover, divergences in entanglement entropy renormalize the gravitational coupling constant and loops of both fermions and bosons contribute to this renormalization with the same sign [4]. Therefore, it will be very interesting to study the entanglement entropy for superstrings and see which of the points of views gets validated.

The algebraic method

It would also be interesting to compute the entanglement entropy in closed string field theory using the algebraic method and canonical quantization. This requires identifying a set of canonical pairs on some Cauchy slice in the configuration space of closed strings and then imposing canonical commutation relations. The non-canonical kinetic term presents a major obstacle in this regard and it is not obvious how to define canonical momenta in a higher-derivative theory. The non-canonical kinetic term, however, will modify the form of the two-point correlation functions of the string-field which will eventually lead to a finite entanglement entropy.

Interactions

Another interesting direction is to include interactions. Although the full action of the closed SFT has an infinite number of terms, important insights can be obtained by just including the fundamental three string vertex. Within the replica method frame of work, the correction due to interactions boils down to computing amplitudes on the branched cover [42]. Since the branched cover is not an on-shell string background, the definition of off-shell amplitudes in string theory [43]
is expected to play a role here. Another approach will be to use the Susskind-Uglum prescription [3] for computing the off-shell generating functional at higher genus.

*Covariant SFT*

It will be very instructive to compute the entanglement entropy using covariant SFT. This would help illuminate the role played by ghosts and also circumvent issues associated with the light-cone quantization. This would be the natural framework to address issues related to the stringy edge modes. Moreover, the UV-finiteness of the vacuum amplitude has a rather intricate resolution in the covariant SFT. This can give qualitatively different insight into the UV-finiteness of entanglement entropy.

*Worldsheet perspective*

Another important conceptual point is to make contact with the worldsheet description of entanglement entropy. It would be interesting to see if our result can be understood in terms of Susskind and Uglum’s prescription to compute string theory partition function on off-shell backgrounds [3]. Another intriguing proposal in this regard is [44] which introduces the idea of target space entanglement entropy which seems a natural notion to study entanglement entropy in string theory from a worldsheet point of view.

We hope that this work will lead to further study in some of the above issues and a better understanding of entanglement entropy in closed string theory.

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*A. Trace of the heat kernel on the n-fold branched cover*

In this appendix we provide details regarding the computation of the trace of Lorentz-invariant heat kernel on the n-fold branched cove. In this case the theory has $U(1)$ symmetry in the $(x^1, x^2)$-plane and the heat kernel on $\mathcal{M}_n$ can be written in terms of the heat kernel on $\mathcal{M}$ as in eq. (2.7).

For a Lorentz invariant theory $\mathcal{O} = \mathcal{O}(\partial_{\mu} \partial^{\mu})$. By inserting $1 = \int \frac{dp}{(2\pi)^D} |p\rangle \langle p|$ in the definition of the heat kernel we get the momentum space expression

$$K_{\mathcal{O}}(t, x : x') = \int \frac{dp}{(2\pi)^D} e^{ip.(x-x')} e^{-t\mathcal{O}(-p^2)}$$  \hspace{1cm} (A.1)

Trace of the above quantity is

$$\text{Tr} K_{\mathcal{O}}(t) = \int d^D x K_{\mathcal{O}}(t, x : x) = V \Omega_{D-1} \int_0^\infty \frac{dp}{(2\pi)^D} p^{D-1} e^{-t\mathcal{O}(-p^2)} = \frac{V}{(2\pi)^2}.$$  \hspace{1cm} (A.2)
where \( V \) is the volume of the spacetime and \( \Omega_n = \frac{2\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} \) is the surface area of \( n \)-sphere with unit radius. The first term that appears in the heat kernel on \( \mathcal{M}_n \) as given in eq. (2.7) is \( \mathcal{K}_O(t, x : x') \). The trace of this term on \( \mathcal{M}_n \) is just \( n \) times the expression derived above in eq. (A.2). The second term in \( \mathcal{K}_O^{(n)} \) is

\[
T_2 = \frac{1}{4\pi n} \int_\mathcal{C} dz \cot\left(\frac{z}{2n}\right) \mathcal{K}_O(t, \phi - \phi' + z), \tag{A.3}
\]

where the dependence on rest of the coordinates is suppressed and only the polar angle is shown. To find the trace of the second term we start by writing eq. (A.1) using spherical coordinates in momentum space. We set up the coordinates so that the angle between \( p^\mu \) and \( (x - x')^\mu \) is \( \theta \). Then the dependence on the rest of \( D - 2 \) angular coordinates drops out and integration over those simply give a factor of \( \Omega_{D-2} \). We have

\[
\mathcal{K}_O(t, x : x') = \frac{\Omega_{D-2}}{(2\pi)^{D-1}} \int_0^\infty dp \int_0^\pi d\theta p^{D-1} \sin^{D-2} \theta e^{i p |x-x'| \cos \theta} e^{-t \mathcal{O}(-p^2)} \tag{A.4}
\]

Using polar coordinates in the \((x_1, x_2)\)-plane we can write

\[
|x - x'|^2 = r^2 + r'^2 - 2rr' \cos (\phi - \phi') + |x_\perp - x'_\perp|^2, \tag{A.5}
\]

where \( x_\perp \) denotes the \( D - 2 \) coordinates on the transverse space. In computing the trace of the term (A.3) we set all coordinates equal and then integrate over \( \mathcal{M}_n \). The integral over the transverse coordinates simply give the area \( A \) of the entangling surface. Integral over the polar angle \( \phi \) gives a factor of \( 2\pi n \) and we get

\[
\text{Tr} T_2 = \frac{n A \Omega_{D-2}}{(2\pi)^{D-1}} \int_\mathcal{C} \frac{dz}{4\pi n} \cot\left(\frac{z}{2n}\right) \int_0^\infty dr \int_0^\pi dp \int_0^\pi d\theta p^{D-1} e^{-t \mathcal{O}(-p^2)} e^{2irp \sin(z) \cos \theta} \sin^{D-2} \theta \tag{A.6}
\]

We perform the integral over \( \theta \) using

\[
\int_0^\pi d\theta \sin^n \theta e^{ia \cos \theta} = \sqrt{\pi} \left(\frac{2}{a}\right)^\frac{n}{2} \Gamma\left(\frac{n+1}{2}\right) J_{\frac{n}{2}}(a) \tag{A.7}
\]

and for the \( r \)-integration, we change the variable from \( r \) to \( a \) by setting \( r = \frac{a}{2p \sin(z/2)} \) so that

\[
\text{Tr} T_2 = \frac{n A \Omega_{D-2} \sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right) 2^{D-3}}{(2\pi)^{D-1}} \int_\mathcal{C} \frac{dz}{4\pi n \sin^2\left(\frac{z}{2n}\right)} \int_0^\infty dp \int_0^\pi dp^{D-3} e^{-t \mathcal{O}(-p^2)} \int_0^\infty da \ a^{D-3} J_{\frac{D-1}{2}}(a) \tag{A.8}
\]

Now perform the \( a \)-integral and the contour integral using

\[
\int_0^\infty da \ a^{1-n} J_n(a) = \frac{2^{1-n}}{\Gamma(n)}, \quad \int_\mathcal{C} \frac{dz}{4\pi n \sin^2\left(\frac{z}{2n}\right)} = \frac{1}{3n^2} (1 - n^2) \tag{A.9}
\]
and use the value of $\Omega_{D-2}$ to get
\[
\text{Tr} T_2 = \frac{A\pi^D}{3 (2\pi)^{D-1} \Gamma(\frac{D}{2} - 1)} \left( \frac{1 - n^2}{n^2} \right) \int_0^\infty dp p^{D-3} e^{-t\mathcal{O}(-p^2)}.
\]  
(A.10)

Combining this with the first term, we finally obtain the expression for the trace of heat kernel on $M_n$
\[
\text{Tr} K^{(n)}_\mathcal{O}(t) = \frac{2nV}{(4\pi)^D \Gamma(\frac{D}{2})} \int dp p^{D-1} e^{-t\mathcal{O}(-p^2)} + \frac{A}{6 (4\pi)^{D-1} \Gamma(\frac{D}{2} - 1)} \frac{1 - n^2}{n} \int dp p^{D-3} e^{-t\mathcal{O}(-p^2)}.
\]  
(A.11)

For $\mathcal{O}(-p^2) = \frac{p^2}{2}$ this becomes
\[
\text{Tr} K^{(n)}_{\frac{p^2}{2}}(t) = \frac{nV}{(2\pi t)^\frac{D}{2}} + \frac{A}{12 (2\pi t)^{\frac{D}{2}-1}} \frac{1 - n^2}{n}.
\]  
(A.12)

**B. Second quantization from the first quantization**

In this appendix we briefly motivate the form of the SFT action in the light-cone gauge. The discussion here is inspired by [45](see sec. 11.4).

First we briefly recap some elementary aspects of string dynamics. Motion of relativistic strings in $\mathbb{R}^{1,D-1}$ is described by the maps $X^\mu(\tau,\sigma)$ from a two dimensional world sheet to the target space $\mathbb{R}^{1,D-1}$ subject to the action functional
\[
I = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{(\partial_\tau X^\mu \partial_\sigma X_\mu)^2 - (\partial_\tau X)^2 (\partial_\sigma X)^2}.
\]  
(B.1)

The worldsheet is parametrized by $(\tau,\sigma)$. The above action is invariant under reparameterization of the worldsheet. For open string we take $\sigma \in [0,\pi]$ and for closed strings $\sigma \in [0,2\pi]$ with identification $(\tau,\sigma) \sim (\tau,\sigma + 2\pi)$. The conserved current associated with a constant shift in $X^\mu$ is $\mathcal{P}_\mu^a = \frac{\partial \mathcal{S}}{\partial (\partial_\sigma X_\mu)}$, $a = \tau,\sigma$ denotes worldsheet coordinates. The integral of the $\tau$ component over $\sigma$ give conserved charges $p^\mu$. The equation of motion is
\[
\partial_\sigma \mathcal{P}_\mu^a = 0.
\]  
(B.2)

In light-cone gauge, the $\tau$ parameterization is fixed by choosing
\[
X^+ = \beta \alpha' p^+ \tau \quad \beta = 2(1) \text{ for open(closed) strings}.
\]  
(B.3)

The $\sigma$ parameterization is fixed by demanding the constancy of $\mathcal{P}^{\sigma,+}$ along the string, i.e.,
\[
p^+ = \frac{2\pi}{\beta} \mathcal{P}^{\sigma,+}.
\]  
(B.4)

Conservation of $p^+$ and equations of motion then imply that $\mathcal{P}^{\sigma,+]$ is also constant along the string
and it can be set equal to zero. This leads to the constraint
\[ \partial_\tau X^\mu \partial_\sigma X_\mu = 0 \] (B.5)
which along with the \( \sigma \)-parameterization gives the constraint
\[ \partial_\tau X^2 + (\partial_\sigma X)^2 \] (B.6)
These two constraints can be combined as
\[ (\partial_\tau X^\mu \pm \partial_\sigma X^\mu) (\partial_\tau X_\mu \pm \partial_\sigma X_\mu) = 0, \] (B.7)
which can be used to determine \( X^- (\tau, \sigma) \) in terms of \( X^I (\tau, \sigma) \) up to a constant. The equations of motion for the transverse coordinates are
\[ \partial_\tau^2 X^I - \partial_\sigma^2 X^I = 0. \] (B.8)
For open strings with free end-points, these are solved by
\[ X^I (\tau, \sigma) = x^I + 2\alpha' p^I \tau + \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n} \alpha_n e^{-i n \tau} \cos n \sigma. \] (B.9)
We rearrange various coefficients in the above expansion and define
\[ X^I (\sigma) = X^I (0, \sigma) \equiv x^I + \sqrt{2} \sum_{n=1}^{\infty} x^I_n \cos n \sigma, \]
\[ P^I (\sigma) = \frac{1}{2\pi \alpha'} X^I (0, \sigma) \equiv \frac{1}{\pi} \left( p^I + \sqrt{2} \sum_{n=1}^{\infty} p^I_n \cos n \sigma \right). \] (B.10)
So classically the motion of open strings after gauge fixing is described by \( (X^I (\tau, \sigma), x^-, p^+) \) or equivalently \( (X^I (\sigma), P^I (\sigma), x^-, p^+) \). We can quantize the system by imposing commutation relation between canonical pairs.
\[ [X^I (\sigma), P^J (\sigma')] = i \delta^{IJ} \delta (\sigma - \sigma'), \quad [x^-, p^+] = -i. \] (B.11)
A complete set of commuting observables is \( (p^+, P^I (\sigma)) \) or \( (x^-, X^I (\sigma)) \). However, none of these sets commute with the Hamiltonian
\[ H = \pi \alpha' \int_0^\pi d\sigma \left( p^I p^I + \frac{1}{(2\pi \alpha')^2} X^I X^I' \right). \] (B.12)
We choose to label the states of the quantum string as
\[ |x^-, X^I (\sigma)\rangle. \] (B.13)
Then a generic superposition

$$|\Phi\rangle = \int dx^- \left[ DX^I (\sigma) \right] \Phi \left( x^+, x^-, X^I (\sigma) \right) |x^-, X^I (\sigma)\rangle$$

(B.14)
satisfies the Schrodinger equation

$$i \frac{\partial}{\partial \tau} |\Phi\rangle = H |\Phi\rangle,$$

(B.15)
if the the functional $$\Phi \left( x^+, x^-, X^I (\sigma) \right)$$ satisfies the equation

$$\partial_+ \partial_- + \frac{\pi}{2} \int_0^\pi d\sigma \left( -\frac{\delta}{\delta X^I (\sigma)} \frac{\delta}{\delta X^I (\sigma)} + \frac{1}{4\pi^2\alpha'} X'^I X'^I \right) \Phi \left( x^+, x^-, X^I (\sigma) \right) = 0.$$

(B.16)

We can now proceed to ‘second quantization’ in which we quantize the field $$\Phi \left( x^+, x^-, X^I (\sigma) \right)$$ so that it satisfies the above equation on-shell. This would lead to a quantum theory of string-field operators and states with multiple strings. It is now obvious how to write an action which gives the above equations of motion. A similar analysis can be carried out for closed strings with appropriate modifications.

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