Correlated-PCA: Principal Components’ Analysis when Data and Noise are Correlated

Namrata Vaswani and Han Guo
Iowa State University, Ames, IA, USA
Email: {namrata,hanguo}@iastate.edu

Abstract

Given a matrix of observed data, Principal Components Analysis (PCA) computes a small number of orthogonal directions that contain most of its variability. Probably accurate solutions for PCA have been in use for a long time. However, to the best of our knowledge, all existing theoretical guarantees for it assume that the data and the corrupting noise are mutually independent, or at least uncorrelated. This is valid in practice often, but not always. In this paper, we study the PCA problem in the setting where the data and noise can be correlated. Such noise is often also referred to as “data-dependent noise”. We obtain a correctness result for the standard eigenvalue decomposition (EVD) based solution to PCA under simple assumptions on the data-noise correlation. We also develop and analyze a generalization of EVD, cluster-EVD, that improves upon EVD in certain regimes.

1 Introduction

Principal Components Analysis (PCA) is among the most frequently used tools for dimension reduction. Given a matrix of data, it computes a small number of orthogonal directions that contain all (or most) of the variability of the data. The subspace spanned by these directions is the “principal subspace”. To use PCA for dimension reduction, one projects the observed data onto this subspace. The standard solution to PCA is to compute the reduced singular value decomposition (SVD) of the data matrix, or, equivalently, to compute the reduced eigenvalue decomposition (EVD) of the empirical covariance matrix of the data. If all eigenvalues are nonzero, a threshold is used and all eigenvectors with eigenvalues above the threshold are retained. This solution, which we henceforth refer to as simple EVD, or just EVD, has been used for many decades and is well-studied in literature, e.g., see [1] and references therein. However, to the best of our knowledge, all existing results for it assume that the true data and the corrupting noise in the observed data are independent, or, at least, uncorrelated. This is valid in practice often, but not always. Here, we study the PCA problem in the setting where the data and noise vectors may be correlated (correlated-PCA). Such noise is sometimes called “data-dependent” noise.

Contributions. (1) Under a boundedness assumption on the true data vectors, and some other assumptions, for a fixed desired subspace error level, we show that the sample complexity of simple-EVD for correlated-PCA scales as \( f^2 r^2 \log n \) where \( n \) is the data vector length, \( f \) is the condition number of the true data covariance matrix and \( r \) is its rank. Here “sample complexity” refers to the number of samples needed to get a small enough subspace recovery error with high probability (whp). The dependence on \( f^2 \) is problematic for datasets with large condition numbers, and, especially in the high dimensional setting where \( n \) is large. (2) To address this, we also develop and analyze a generalization of simple-EVD, called cluster-EVD. Under an eigenvalues’ “clustering” assumption, cluster-EVD weakens the dependence on \( f \).

To our best knowledge, the correlated-PCA problem has not been explicitly studied. We first encountered it while solving the dynamic robust PCA problem in the Recursive Projected Compressive Sensing (ReProCS) framework [2-5]. The version of correlated-PCA studied here is motivated...
by these works. Some other somewhat related recent works include [6, 7] that study stochastic optimization based techniques for PCA; and [8, 9, 10, 11] that study online PCA.

**Notation.** We use the interval notation $[a, b]$ to mean all of the integers between $a$ and $b$, inclusive, and similarly for $[a, b)$ etc. We use $\| \cdot \|$ to denote the $l_2$ norm of a vector or the induced $l_2$ norm of a matrix. For other $l_p$ norms, we use $\| \cdot \|_p$. For a set $T$, $I_T$ refers to an $n \times |T|$ matrix of columns of the identity matrix indexed by entries in $T$. For a matrix $A$, $A_T := AI_T$. A tall matrix with orthonormal columns is referred to as a *basis matrix*. For basis matrices $\hat{P}$ and $P$, we quantify the subspace error (SE) between their range spaces using

$$SE(\hat{P}, P) := \| (I - \hat{P}P) \|.$$

1.1 Correlated-PCA: Problem Definition

We are given a time sequence of data vectors, $y_t$, that satisfy

$$y_t = \ell_t + w_t, \text{ with } w_t = M_t \ell_t \text{ and } \ell_t = P a_t$$

(2)

where $P$ is an $n \times r$ basis matrix with $r \ll n$. Here $\ell_t$ is the true data vector that lies in a low dimensional subspace of $\mathbb{R}^n$, $\text{range}(P)$; $a_t$ is its projection into this $r$-dimensional subspace; and $w_t$ is the data-dependent noise. We refer to $M_t$ as the correlation / data-dependency matrix. The goal is to estimate $\text{range}(P)$. We make the following assumptions on $\ell_t$ and $M_t$.

**Assumption 1.1.** The subspace projection coefficients, $a_t$, are zero mean, mutually independent and bounded random vectors (r.v.’s), with a diagonal covariance matrix $\Lambda$. Define $\lambda^\gamma := \lambda_{\text{min}}(\Lambda)$, $\lambda^+ := \lambda_{\text{max}}(\Lambda)$ and $f := \frac{\lambda^+}{\lambda^\gamma}$. Since the $a_t$’s are bounded, we can also define a finite constant $\eta := \max_{j=1,2,...,r} \max_t (\alpha_j)^2 \lambda_j$. Thus, $(\alpha_t)^2 \leq \eta \lambda_j$.

For most bounded distributions, $\eta$ will be a small constant more than one, e.g., if the distribution of all entries of $a_t$ is iid zero mean uniform, then $\eta = 3$. From Assumption 1.1 clearly, the $\ell_t$’s are also zero mean, bounded, and mutually independent r.v.’s with a rank $r$ covariance matrix $\Sigma = \text{EVD} \, P \Lambda P^\top$. In the model, for simplicity, we assume $\Lambda$ to be fixed. However, even if we replace $\Lambda$ by $\Lambda_t$ and define $\lambda^\gamma = \min_t \lambda_{\text{min}}(\Lambda_t)$ and $\lambda^+ = \lambda_{\text{max}}(\Lambda_t)$, all our results will still hold.

**Assumption 1.2.** Decompose $M_t$ as $M_t = M_{2,t} M_{1,t}$. Assume that

$$\| M_{1,t} P \| \leq q < 1, \| M_{2,t} \| \leq 1,$$

(3)

and, for any sequence of positive semi-definite Hermitian matrices, $A_t$, the following holds

$$\text{for a } \beta < \alpha, \quad \left\| \frac{1}{\alpha} \sum_{t=1}^{\alpha} M_{2,t} A_t M_{2,t} \right\| \leq \frac{\beta}{\alpha} \max_{\alpha \in [1,\alpha]} \| A_t \|.$$

(4)

We will need the above to hold for all $\alpha \geq \alpha_0$ and for all $\beta \leq c_0 \alpha$ with a $c_0 < 1$. We set $\alpha_0$ and $c_0$ in Theorems 2.1 and 3.3; both will depend on $q$. Observe that, using (3), $\| w_t \| \leq q$, and so $q$ is an upper bound on the signal-to-noise ratio (SNR).

To understand the assumption on $M_{2,t}$, notice that, if we allow $\beta = \alpha$, then (4) always holds and is not an assumption. Let $B$ denote the matrix on the LHS of (4). One example situation when (4) holds with a $\beta \ll \alpha$ is if $B$ is block-diagonal with blocks $A_t$. In this case, it holds with $\beta = 1$. In fact, it also holds with $\beta = 1$ if $B$ is permutation-similar to a block diagonal matrix. The matrix $B$ will be of this form if $M_{2,t} = I_{T_t}$ with all the sets $T_t$ being mutually disjoint. More generally, if $B$ is permutation-similar to a block-diagonal matrix with blocks given by the summation of $A_t$’s over at most $\beta_0 < \alpha$ time instants, then (4) holds with $\beta = \beta_0$. This will happen if $M_{2,t} = I_{T_t}$ with $T_t = T^k$ for at most $\beta_0$ time instants and if sets $T^k$ are mutually disjoint for different $k$. Finally, the $T^k$’s need not even be mutually disjoint. As long as they are such that $B$ is a matrix with nonzero blocks on only the main diagonal and on a few diagonals near it, e.g., if it is block tri-diagonal, it can be shown that the above assumption holds. This example is generalized in Assumption 1.3 given below.

1.2 Examples of correlated-PCA problems

One key example of correlated-PCA is the PCA with missing data (PCA-missing) problem. Let $T_t$ denote the set of missing entries at time $t$. Suppose, we set the missing entries of $y_t$ to zero. Then,

$$y_t = \ell_t - I_{T_t} I_{T_t} \ell_t.$$

(5)
In this case $M_{2,t} = I_{T_t}$ and $M_{1,t} = -I_{T_t}$. Thus, $q$ is an upper bound on $\|I_{T_t}P\|$. Clearly, it will be small if the columns of $P$ are dense vectors. For the reader familiar with low-rank matrix completion (MC), e.g., [12, 13], PCA-missing can also be solved by first solving the low-rank matrix completion problem to recover $L$, followed by PCA on the completed matrix. This would, of course, be much more expensive than directly solving PCA-missing and would need more assumptions.

Another example where correlated-PCA occurs is that of robust PCA (low-rank + sparse formulation) [14, 15, 16] when the sparse component’s magnitude is correlated with $\ell_t$. Let $T_t$ denote the support set of $\omega_t$ and let $x_t$ be the $|T_t|$-length vector of its nonzero entries. If we assume linear dependency of $x_t$ on $\ell_t$, we can write out $y_t$ as

$$y_t = \ell_t + I_{T_t}x_t = \ell_t + I_{T_t}M_{s,t}\ell_t.$$  \hspace{1cm} (6)

Thus $M_{2,t} = I_{T_t}$ and $M_{1,t} = M_{s,t}$ and so $q$ is an upper bound on $\|M_{s,t}P\|$. In the rest of the paper, we refer to this problem as “PCA with sparse data-dependent corruptions (PCA-SDDC)”.

A key application where it occurs is in foreground-background separation for videos consisting of a slow changing background sequence (modeled as lying close to a low-dimensional subspace) and a sparse foreground image sequence consisting typically of one or more moving objects [14]. The PCA-SDDC problem is to estimate the background sequence’s subspace. In this case, $\ell_t$ is the background image at time $t$, $T_t$ is the support set of the foreground image at $t$, and $x_t$ is the difference between foreground and background intensities on $T_t$. An alternative solution approach for PCA-SDDC is to use an RPCA solution such as principal components’ pursuit (PCP) [14, 15] or Alternating-Minimization (Alt-Min-RPCA) [17] to first recover the matrix $L$ followed by PCA on $L$. However, as shown in Sec. 3 Table I this approach will be much slower; and it will work only if its required incoherence assumptions hold. For example, if the columns of $P$ are sparse, it fails.

For both problems above, a solution for PCA will work only when the corrupting noise $\omega_t$ is small compared to $\ell_t$. A sufficient condition for this is that $q$ is small.

A third example where correlated-PCA and its generalization, correlated-PCA with partial subspace knowledge, occurs is in the subspace update step of Recursive Projected Compressive Sensing (Re-ProsC) for dynamic robust PCA [3, 5]. We refer the reader to [18] to understand this application.

In all three of the above applications, the assumptions on the data-noise correlation matrix given in Assumption 1.2 hold if there are enough changes of a certain type in the set of missing or corrupted entries, $T_t$. One example where this is true is in case of a 1D object of length $s$ or less that remains static for at most $\beta$ frames at a time. When it moves, it moves by at least a certain fraction of $s$ pixels. The following assumption is inspired by the object’s support.

**Assumption 1.3.** Let $l$ denote the number of times the set $T_t$ changes in the interval $[1, \alpha]$ (or in any given interval of length $\alpha$ in case of dynamic robust PCA). So $0 \leq l \leq \alpha - 1$. Let $t^0 = 1; t^l, t^{l+1}$, with $t^k < t^{k+1}$, denote the time instants in this interval at which $T_t$ changes; and let $T^{[k]}$ denote the distinct sets. In other words, $T_t = T^{[k]}$ for $t \in [t^k, t^{k+1})$, for each $k = 1, 2, \ldots, l$. Assume that the following hold with a $\beta < \alpha$:

1. $(t^{k+1} - t^k) \leq \beta$ and $|T^{[k]}| \leq s$;
2. $\rho^2 \beta \leq \beta$ where $\rho$ is the smallest positive integer so that, for any $0 \leq k \leq l$, $T^{[k]}$ and $T^{[k+\rho]}$ are disjoint;
3. for any $k_1, k_2$ satisfying $0 \leq k_1 < k_2 \leq l$, the sets $(T^{[k_1]} \setminus T^{[k_1+1]})$ and $(T^{[k_2]} \setminus T^{[k_2+1]})$ are disjoint.

An implicit assumption for condition 3 holds is that $\sum_{k=0}^{l} |T^{[k]} \setminus T^{[k+1]}| \leq n$. Observe that conditions 2 and 5 enforce an upper bound on the maximum support size $s$.

To connect Assumption 1.3 with the moving object example given above, condition 1 holds if the object’s size is at most $s$ and if it moves at least once every $\beta$ frames. Condition 2 holds, if, every time it moves, it moves in the same direction and by at least $\frac{\rho}{\beta}$ pixels. Condition 3 holds if, every time it moves, it moves in the same direction and by at most $d_0 \geq \frac{\rho}{\beta}$ pixels, with $d_0 \alpha \leq n$ (or, more generally, the motion is such that, if the object were to move at each frame, and if it started at the top of the frame, it does not reach the bottom of the frame in a time interval of length $\alpha$).

The following lemma 4 shows that, with Assumption 1.3 on $T_t$, $M_{2,t} = I_{T_t}$ satisfies the assumption on $M_{2,t}$ given in Assumption 1.2. Its proof generalizes the discussion below Assumption 1.2.
Lemma 1.4. [4] Let Assumption 5.2 and 5.3 hold. For any sequence of $|T_1| \times |T_2|$ symmetric positive-semi-definite matrices $A_t$,

$$\sum_{t=1}^{\alpha} I_{T_1} A_t I_{T_2} \leq (\rho^2 \beta) \max_{t \in [1, \alpha]} \| A_t \| \leq \beta \max_{t \in [1, \alpha]} \| A_t \|$$

Thus, if $\| I_{T_1} P \| \leq q < 1$, then the PCA-missing problem satisfies Assumption 1.2. If $\| M_{s,t} P \| \leq q < 1$, then the PCA-SDDC problem satisfies Assumption 1.2.

Assumption 1.3 is one model on $T_1$ that ensures that, if $M_{2,t} = I_{T_2}$, the assumption on $M_{2,t}$ given in Assumption 1.2 holds. For its many generalizations, see Supplementary Material, Sec. 7, or [4].

2 Simple EVD

Simple EVD computes the top eigenvectors of the empirical covariance matrix, $\frac{1}{n} \sum_{t=1}^{\alpha} y_t y_t^T$, of the observed data. The following can be shown.

Theorem 2.1 (simple-EVD result). Let $\tilde{P}$ denote the matrix containing all the eigenvectors of $\frac{1}{n} \sum_{t=1}^{\alpha} y_t y_t^T$ with eigenvalues above a threshold, $\lambda_{\text{thresh}}$, as its columns. Pick $\gamma$ so that $r(\gamma) \leq 0.01$. Suppose that $y_t$’s satisfy (2) and the following hold.

1. Assumption 1.1 on $T_0$ holds. Define $\alpha_0 := C \gamma^2 r^2 \log n \left(0.01 \gamma \right)^2 \max(f, qf, q^2 f)^2$, $C := \frac{32}{0.01^2}$.

2. Assumption 1.2 on $M_t$ holds for any $\alpha = \alpha_0$ and for any $\beta$ satisfying $\beta \leq \frac{1}{\alpha} \left(1 - r(\gamma) \right)^2 \min \left(\frac{(r(\gamma))^2}{1.41(qf)^2}, \frac{r(\gamma)}{q^2 f} \right)$.

3. Set algorithm parameters $\lambda_{\text{thresh}} = 0.95 \lambda$ and $\alpha \geq \alpha_0$.

Then, with probability at least $1 - 6n^{-10}$, $\text{SE}(\tilde{P}, P) \leq r(\gamma)$.

Proof: The proof involves a careful application of the sin $\theta$ theorem [19] to bound the subspace error, followed by using matrix Hoeffding [20] to obtain high probability bounds on each of the terms in the sin $\theta$ bound. It is given in the Supplementary Material, Section 8.

Consider the lower bound on $\alpha$. We refer to this as the “sample complexity”. Since $q < 1$, and $\eta$ is a small constant (e.g., for the uniform distribution, $\eta = 3$), for a fixed error level, $r(\gamma)$, $\alpha_0$ simplifies to $e f^2 r^2 \log n$. Notice that the dependence on $n$ is logarithmic. It is possible to show that the sample complexity scales as $\log n$ because we assume that the $\ell_t$’s are bounded r.v.s. As a result we can apply the matrix Hoeffding inequality [20] to bound the perturbation between the observed data’s empirical covariance matrix and that of the true data. The bounded r.v. assumption is actually a more practical one than the usual Gaussian assumption since most sources of data have finite power.

By replacing matrix Hoeffding by Theorem 5.39 of [21] in places where one can apply a concentration of measure result to $\sum a_{i} a_i^T / \alpha$ (which is at $r \times r$ matrix), and by matrix Bernstein [20] elsewhere, it should be possible to further reduce the sample complexity to $c \max((qf)^2 r \log n, f^2 (r + \log n))$. It should also be possible remove the boundedness assumption and replace it by a Gaussian (or a sub-Gaussian) assumption, but, that would increase the sample complexity to $c(qf)^2 n$.

Consider the upper bound on $\beta / \alpha$. Clearly, the smaller term is the first one. This depends on $1/(qf)^2$. Thus, when $f$ is large and $q$ is not small enough, the bound required may be impractically small. As will be evident from the proof (see Remark 5.2 in Supplementary Material), we get this bound because $w_t$ is correlated with $\ell_t$ and this results in $E[\ell_t w_t^T] \neq 0$.

If $w_t$ and $\ell_t$ were uncorrelated, $qf$ would get replaced by $\lambda_{\max}(\text{Cov}(w_t))$ in the upper bound on $\beta / \alpha$ as well as in the sample complexity.

Application to PCA-missing and PCA-SDDC. By Lemma 1.4 the following is immediate.

Corollary 2.2. Consider the PCA-missing model, (5), and assume that $\max_t \| I_{T_1} P \| \leq q < 1$; or consider the PCA-SDDC model, (6), and assume that $\max_t \| M_{s,t} P \| \leq q < 1$. Assume that everything in Theorem 2.1 holds except that we replace Assumption 1.2 by Assumption 1.3. Then, with probability at least $1 - 6n^{-10}$, $\text{SE}(P, P) \leq r(\gamma)$. 

4
3 Cluster-EVD
To try to relax the strong dependence on \( f^2 \) of the result above, we develop a generalization of simple-EVD that we call cluster-EVD. This requires the clustering assumption.

3.1 Clustering assumption
To state the assumption, define the following partition of the index set \( \{1, 2, \ldots, r\} \) based on the eigenvalues of \( \Sigma \). Let \( \lambda_i \) denote its \( i \)-th largest eigenvalue.

**Definition 3.1** (\( g \)-condition-number partition of \( \{1, 2, \ldots, r\} \)). Define \( G_1 = \{1, 2, \ldots, r_1\} \) where \( r_1 \) is the index for which \( \frac{\lambda_1}{\lambda_{r_1}} \leq g \) and \( \frac{\lambda_1}{\lambda_{r_1+1}} > g \). In words, to define \( G_1 \), start with the index of the first (largest) eigenvalue and keep adding indices of the smaller eigenvalues to the set until the ratio of the maximum to the minimum eigenvalue first exceeds \( g \).

For each \( k > 1 \), define \( G_k = \{r_s+1, r_s+2, \ldots, r_s+r_k\} \) where \( r_s = (\sum_{i=1}^{k-1} r_i) \), \( r_k \) is the index for which \( \frac{\lambda_r}{\lambda_{r+1}} \leq g \) and \( \frac{\lambda_{r+1}}{\lambda_{r+r_k+1}} > g \). In words, to define \( G_k \), start with the index of the \( (r_s + 1) \)-th eigenvalue, and repeat the above procedure.

Stop when \( \lambda_{r_s+r_k+1} = 0 \), i.e., when there are no more nonzero eigenvalues. Define \( \vartheta = k \) as the number of sets in the partition. Thus \( \{G_1, G_2, \ldots, G_{\vartheta}\} \) is the desired partition.

Define \( G_0 = [\] \), \( G_k := (P)G_k, \lambda_k^+ := \max_{i \in G_k} \lambda_i (A), \lambda_k^- := \min_{i \in G_k} \lambda_i (A) \) and
\[
\chi := \max_{k=1,2,\ldots,\vartheta} \frac{\lambda_{k+1}}{\lambda_k}.
\]

\( \chi \) quantifies the “distance” between consecutive sets of the above partition. Moreover, by definition, \( \frac{\lambda_{k+1}}{\lambda_k} \leq g \). Clearly, \( g \geq 1 \) and \( \chi \leq 1 \) always. We assume the following.

**Assumption 3.2.** For a \( 1 \leq g^+ < f \) and a \( 0 \leq \chi^+ < 1 \), assume that there exists a \( g \) satisfying \( 1 \leq g \leq g^+ \) and a \( \chi \) satisfying \( 0 \leq \chi \leq \chi^+ \), for which we can define a \( g \)-condition-number partition of \( \{1, 2, \ldots, r\} \) that satisfies \( \chi \leq \chi^+ \). The number of sets in the partition is \( \vartheta \). When \( g^+ \) and \( \chi^+ \) are small, we say that the eigenvalues are “well-clustered” with “clusters”, \( G_k \).

This assumption can be understood as a generalization of the eigen-gap condition needed by the block power method, which is a fast algorithm for obtaining the \( k \) top eigenvectors of a matrix [22]. We expect it to hold for data that has variability across different scales. The large scale variations would result in the first (largest eigenvalues’) cluster and the smaller scale variations would form the later clusters. This would be true, for example, for video “textures” such as moving waters or waving trees in a forest. We tested this assumption on some such videos. We describe our conclusions here for three videos - “lake” (video of moving lake waters), “waving-tree” (video consisting of waving trees), and “curtain” (video of window curtains moving due to the wind). For each video, we first made it low-rank by keeping the eigenvectors corresponding to the smallest number of eigenvalues that contain at least 90% of the total energy and projecting the video onto this subspace. For the low-rankified lake video, \( f = 74 \) and Assumption 3.2 holds with \( \vartheta = 6 \) clusters, \( g^+ = 2.6 \) and \( \chi^+ = 0.7 \). For the waving-tree video, \( f = 180 \) and Assumption 3.2 holds with \( \vartheta = 6 \), \( g^+ = 9.4 \) and \( \chi^+ = 0.72 \). For the curtain video, \( f = 107 \) and the assumption holds \( \vartheta = 3 \), \( g^+ = 16.1 \) and \( \chi^+ = 0.5 \). We show the clusters of eigenvalues in Fig. 1.

3.2 Cluster-EVD algorithm
The cluster-EVD approach is summarized in Algorithm 1. Its main idea is as follows. We start by computing the empirical covariance matrix of the first set of \( \alpha \) observed data points, \( D_1 := \frac{1}{\alpha} \sum_{i=1}^{\alpha} y_i y_i' \). Let \( \hat{\lambda}_i \) denote its \( i \)-th largest eigenvalue. To estimate the first cluster, \( G_1 \), we start with the index of the first (largest) eigenvalue and keep adding indices of the smaller eigenvalues
Theorem 3.3

enough to ensure that the

Algorithm 1 Cluster-EVD

Parameters: \( \alpha, \hat{g}, \lambda_{\text{thresh}} \).

Set \( \hat{G}_0 \leftarrow [\cdot] \). Set the flag \( \text{Stop} \leftarrow 0 \). Set \( k \leftarrow 1 \). Repeat

1. Let \( \hat{G}_{\text{det}, k} := [\hat{G}_0, \hat{G}_1, \ldots, \hat{G}_{k-1}] \) and let \( \Psi_k := (I - \hat{G}_{\text{det}, k} \hat{G}_{\text{det}, k}') \). Notice that \( \Psi_1 = I \). Compute

\[
\hat{D}_k = \frac{1}{\alpha} \sum_{t=(k-1)\alpha + 1}^{k\alpha} y_t y_t' \Psi_k
\]

2. Find the \( k \)-th cluster, \( \hat{g}_k \): let \( \lambda_i = \lambda_i(\hat{D}_k) \);
   (a) find the index \( \hat{r}_k \) for which \( \hat{\lambda}_{\hat{r}_k} \leq \hat{g} \) and either \( \hat{\lambda}_{\hat{r}_k+1} > \hat{g} \) or \( \hat{\lambda}_{\hat{r}_k+1} < \lambda_{\text{thresh}} \);
   (b) set \( \hat{g}_k = \{\hat{r}_s + 1, \hat{r}_s + 2, \ldots, \hat{r}_s + \hat{r}_k\} \) where \( \hat{r}_s := \sum_{j=1}^{k-1} \hat{r}_j \);
   (c) if \( \hat{\lambda}_{\hat{r}_k+1} < \lambda_{\text{thresh}} \), update the flag \( \text{Stop} \leftarrow 1 \)
3. Compute \( \hat{G}_k \leftarrow \text{eigenvectors}(\hat{D}_k, \hat{r}_k) \); increment \( k \)

Until \( \text{Stop} = 1 \).

Set \( \hat{g} \leftarrow k \), Output \( \hat{P} \leftarrow [\hat{G}_1 \cdots \hat{G}_g] \).

\text{eigenvectors}(\mathcal{M}, r) \) returns a basis matrix for the span of the top \( r \) eigenvectors of \( \mathcal{M} \).

Algorithm [1] is related to, but significantly different from, the ones introduced in [3, 5] for the subspace deletion step of ReProCS. The one introduced in [3] assumed that the clusters were known to the algorithm (which is unrealistic). The one studied in [5] has an automatic cluster estimation approach, but one that needs a larger lower bound on \( \alpha \) compared to what Algorithm [1] needs.

3.3 Main result

We give the performance guarantee for Algorithm [1] here. Its parameters are set as follows. We set \( \hat{g} \) to a value that is a little larger than \( g \). This is needed to allow for the fact that \( \lambda_i \) is not equal to the \( i \)-th eigenvalue of \( A \) but is within a small margin of it. For the same reason, we need to also use a nonzero “zeroing” threshold, \( \lambda_{\text{thresh}} \), that is larger than zero but smaller than \( \lambda^- \). We set \( \alpha \) large enough to ensure that \( \text{SE}(\hat{P}, P) \leq r\zeta \) holds with a high enough probability.

**Theorem 3.3** (cluster-EVD result). Consider Algorithm [1]. Pick a \( \zeta \) so that \( r^2 \zeta \leq 0.0001 \), and \( r^2 \zeta f \leq 0.01 \). Suppose that \( y_{i'}, j \)'s satisfy (2) and the following hold.

1. Assumption [1] and Assumption [3, 2] on \( \ell_t \) hold with \( \chi^+ \) satisfying \( \chi^+ \leq \min(1 - r\zeta - \frac{0.08}{0.2}, \frac{g^2}{1.01 g^2 + 0.0001} - 0.0001) \). Define

\[
\alpha_0 := Cn^2 g^2 (11 \log n + \log \theta) \max(g^+, q g^+),
\]

\[
q^2 f, q(r\zeta f), (r\zeta)^2 f, q \sqrt{f g^+}, (r\zeta) \sqrt{f g^+}^2, C := \frac{32 \cdot 16}{0.01^2}.
\]

2. Assumption [3, 2] on \( M_t \) holds with \( \alpha \geq \alpha_0 \) and with \( \beta \) satisfying

\[
\frac{\beta}{\alpha} \leq \left( \frac{1 - r\zeta - \chi^+}{2} \right) \min \left( \frac{(r_k \zeta)^2}{4.1 (q g^+)^2}, \frac{(r_k \zeta)}{q^2 f} \right).
\]

3. Set algorithm parameters \( \hat{g} = 1.01 g^+ + 0.0001 \), \( \lambda_{\text{thresh}} = 0.95 \lambda^- \) and \( \alpha \geq \alpha_0 \).

Then, with probability at least \( 1 - 12n^{-10} \), \( \text{SE}(\hat{P}, P) \leq r\zeta \).
Proof: The proof is given in Section 9 in Supplementary Material.

We can also get corollaries for PCA-missing and PCA-SDDC for cluster-EVD. We have given one specific value for $\hat{g}$ and $\lambda_{\text{thresh}}$ in Theorem 3.3 for simplicity. One can, in fact, set $\hat{g}$ to be anything that satisfies (12) given in Supplementary Material and one can set $\lambda_{\text{thresh}}$ to be anything satisfying $5r\zeta \lambda^2 \leq \lambda_{\text{thresh}} \leq 0.95\lambda^2$. Also, it should be possible to reduce the sample complexity of cluster-EVD to $c\max(q^2(g^+)^2\log n, (g^+)^2(r + \log n))$ using the approach explained in Sec. 2.

4 Discussion

Comparing simple-EVD and cluster-EVD. Consider the lower bounds on $\alpha$. In the cluster-EVD (c-EVD) result, Theorem 3.3 if $q$ is small enough (e.g., if $q \leq 1/\sqrt{f}$), and if $(r\zeta)f \leq 0.01$, it is clear that the maximum in the $\max(\ldots, \ldots)$ expression is achieved by $(g^+)^2$. Thus, in this regime, c-EVD needs $\alpha \geq C\sqrt{11\log n + \log g}g^2$ and its sample complexity is $\vartheta(\alpha)$. In the EVD result (Theorem 2.1), $g^+$ gets replaced by $f$ and $\vartheta$ by 1, and so, its sample complexity, $\alpha \geq C\sqrt{11\log n} f^2$. In situations where the condition number $f$ is very large but $g^+$ is much smaller and $\vartheta$ is small (the clustering assumption holds), the sample complexity of c-EVD will be much smaller than that of simple-EVD. However, notice that, the lower bound on $\alpha$ for simple-EVD holds for any $q < 1$ and for any $\zeta$ with $r\zeta < 0.01$ while the c-EVD lower bound given above holds only when $q$ is small enough, e.g., $q = O(1/\sqrt{f})$, and $\zeta$ is small enough, e.g., $r\zeta = O(1/f)$. This tight bound on $\alpha$ is needed because the error of the $k$-th step of c-EVD depends on the errors of the previous steps times $f$. Therefore, the c-EVD result also needs $\chi^+$ and $\vartheta$ to be small (clustering assumption holds well), whereas, for simple-EVD, by definition, $\chi^+ = 0$ and $\vartheta = 1$. Another thing to note is that the constants in both lower bounds are very large with the c-EVD one being even larger.

To compare the upper bounds on $\beta$, assume that the same $\alpha$ is used by both, i.e., $\alpha = \max(\alpha_0(\text{EVD}), \alpha_0(\text{c-EVD}))$. As long as $r_k$ is large enough, $\chi^+$ is small enough, and $g$ is small enough, the upper bound on $\beta$ needed by the c-EVD result is significantly larger. For example, if $\chi^+ = 0.2$, $\vartheta = 2$, $r_k = r/2$, then c-EVD needs $\beta \leq (0.5 \cdot 0.79 \cdot 0.5)^2 (r\zeta)^2 1.11 \log f \alpha$. As long as $g = 3$ but $f = 100$, clearly the c-EVD bound is looser.

Comparison with other results for PCA-SDDC and PCA-missing. To our knowledge, there is no other result for correlated-PCA. Hence, we provide comparisons of the corollaries given above for the PCA-missing and PCA-SDDC special cases with works that also study these or related problems. An alternative solution for either PCA-missing or PCA-SDDC is to first recover the entire matrix $L$ and then compute its subspace via SVD on the estimated $L$. For the PCA-missing problem, this can be done by using any of the low-rank matrix completion techniques, e.g., nuclear norm minimization (NNM) (13) or alternating minimization (Alt-Min-MC) (23). Similarly, for PCA-SDDC, this can be done by solving any of the recent provably correct RPCA techniques such as principal components’ pursuit (PCP) (14, 15, 16) or alternating minimization (Alt-Min-RPCA) (17).

However, as explained earlier doing the above has two main disadvantages. The first is that it is much slower (see Sec. 5). The difference in speed is most dramatic when solving the matrix-sized convex programs such as NNM or PCP, but even the Alt-Min methods are slower. If we use the time complexity from (17), then finding the span of the top $k$ singular vectors of an $n \times m$ matrix takes $O(nmk)$ time, Thus, if $f$ is a constant, both simple-EVD and c-EVD need $O(nar^2)$ time, whereas, Alt-Min-RPCA needs $O(nar^2)$ time per iteration (17). The second disadvantage is that the above methods for MC or RPCA need more assumptions to provably correctly recover $L$. All the above methods need an incoherence assumption on both the left singular vectors, $P$, and the right singular vectors, $V$, of $L$. Of course, it is possible that, if one studies these methods with the goal of only recovering the column space of $L$ correctly, the incoherence assumption on the right singular vectors is not needed. From simulation experiments (see Sec. 5), the incoherence of the left singular vectors is definitely needed. On the other hand, for the PCA-SDDC problem, simple-EVD or c-EVD do not even need the incoherence assumption on $P$.

The disadvantage of both EVD and c-EVD, or in fact of any solution for the PCA problem, is that they work only when $q$ is small enough (the corrupting noise is small compared to $\ell_1$).

5 Numerical Experiments

We use the PCA-SDDC problem as our case study example. We compare EVD and cluster-EVD (c-EVD) with PCP (15), solved using (24), and with Alt-Min-RPCA (17) (implemented using code.
Thus c-EVD outperforms PCP and AltMinRPCA when columns of constant in the required lower bounds on EVD but the advantage in mean error is not as much as our theorems predict. One reason is that the still only controlled to ensure that

| Mean Subspace Error (SE) | Average Time |
|--------------------------|--------------|
|                          | c-EVD | EVD | PCP | A-M-RPCA | c-EVD | EVD | PCP | A-M-RPCA |
| Expt 1                   | 0.0908 | 0.0911 | 1.0000 | 1.0000 | 0.0549 | 0.0255 | 0.2361 | 0.0810 |
| Expt 2                   | 0.3626 | 0.3821 | 0.4970 | 0.4846 | 0.0613 | 0.0223 | 1.6784 | 5.5144 |

Table 1: Comparison of SE($P, \hat{P}$) and execution time (in seconds). A-M-RPCA: Alt-Min-RPCA. Expt 1: simulated data, Expt 2: lake video with simulated foreground.

from the authors’ webpage). For both PCP and Alt-Min-RPCA, $\hat{P}$ is recovered as the top $r$ eigenvectors of of the estimated $L$. To show the advantage of EVD or c-EVD, we let $\ell_i = Pa_i$ with columns of $P$ being sparse. These were chosen as the first $r = 5$ columns of the identity matrix. We generate $a_i$’s iid uniformly with zero mean and covariance matrix $\Lambda = \text{diag}(100, 100, 100, 0.1, 0.1)$. Thus the condition number $f = 1000$. The clustering assumption holds with $\alpha = 2, g^+ = 1$ and $\chi^+ = 0.001$. The noise $w_i$ is generated as $w_i = I_{\ell_i}M_{\ell_i}\ell_i$ with $\ell_i$ generated to satisfy Assumption 1.3 with $s = 5, \rho = 2$, and $\beta = 1$; and the entries of $M_{\ell_i}$ being iid $N(0, q^2)$ with $q = 0.01$. We used $n = 500$. EVD and c-EVD (Algorithm 1) were implemented with $\alpha = 300, \lambda_{\text{thresh}} = 0.095, g^+ = 3$. 10000-time Monte Carlo averaged values of SE($\hat{P}, P$) and execution time are shown in the first row of Table 1. Since the columns of $P$ are sparse, both PCP and Alt-Min-RPCA fail. Both have average SE close to one whereas the average SE of c-EVD and EVD is 0.0908 and 0.0911 respectively. Also, both EVD and c-EVD are much faster than the other two. We also did an experiment with the settings of this experiment, but with $P$ dense. In this case, EVD and c-EVD errors were similar, but PCP and Alt-Min-RPCA errors were less than $10^{-5}$.

For our second experiment, we used images of a low-rankified real video sequence as $\ell_i$’s. We chose the escalator sequence from [http://perception.12r.a-star.edu.sg/bk_model/bk_index.html](http://perception.12r.a-star.edu.sg/bk_model/bk_index.html) since the video changes are only in the region where the escalator moves (and hence can be modeled as being sparse). We made it exactly low-rank by retaining its top 5 eigenvectors and projecting onto their subspace. This resulted in a data matrix $L$ of size $n \times r$ with $n = 20800$ and $r = 5$. We overlaid a simulated moving foreground block on it. The intensity of the moving block was generated to satisfy Assumption 1.3 with $s = 5, \rho = 2$, and $\beta = 1$; and the entries of $M_{\ell_i}$ being iid $N(0, q^2)$ with $q = 0.01$. We used $n = 500$. EVD and c-EVD (Algorithm 1) were implemented with $\alpha = 300, \lambda_{\text{thresh}} = 0.095, g^+ = 3$. 10000-time Monte Carlo averaged values of SE($\hat{P}, P$) and execution time are shown in the first row of Table 1. Since the columns of $P$ are sparse, both PCP and Alt-Min-RPCA fail. Both have average SE close to one whereas the average SE of c-EVD and EVD is 0.0908 and 0.0911 respectively. Also, both EVD and c-EVD are much faster than the other two. We also did an experiment with the settings of this experiment, but with $P$ dense. In this case, EVD and c-EVD errors were similar, but PCP and Alt-Min-RPCA errors were less than $10^{-5}$.

Thus c-EVD outperforms PCP and AltMinRPCA when columns of $P$ are sparse. It also outperforms EVD but the advantage in mean error is not as much as our theorems predict. One reason is that the constant in the required lower bounds on $\alpha$ is very large. It is hard to pick an $\alpha$ that is this large and still only $O(\log n)$ unless $n$ is very large. Secondly, both guarantees are only sufficient conditions.

6 Conclusions and Future Work

We studied the problem of PCA in noise that is correlated with the data (data-dependent noise). We obtained sample complexity bounds for the most commonly used PCA solution, simple EVD. We also developed and analyzed a generalization of EVD, called cluster-EVD, that has lower sample complexity under extra assumptions. We provided a detailed comparison of our results with those for other approaches to solving its example applications - PCA with missing data and PCA with sparse data-dependent corruptions.

We used the matrix Hoeffding inequality [20] to obtain our results. As explained in Sec. 2 it should be possible to improve the sample complexity bounds if this is replaced by [21] Theorem 5.39 or matrix Bernstein. Moreover, as done in [8] (for ReProCS), the mutual independence of $\ell_i$’s can be easily replaced by a more practical assumption of $\ell_i$’s following autoregressive model with almost no change to our assumptions. Thirdly, by generalizing the proof techniques developed here, we can also study the problem of correlated-PCA with partial subspace knowledge. This is done in [25].

The solution to the latter problem helps to greatly simplify the proof of correctness of ReProCS for online dynamic RPCA [13]. Fourthly, the boundedness assumption on $\ell_i$’s can be replaced by a Gaussian or a well-behaved sub-Gaussian assumption but this will increase the sample complexity to $O(n)$. Finally, an open-ended question is how we relax Assumption 1.2 on $M_i$ and still get results similar to Theorem 2.1 or Theorem 3.3.
References

[1] B. Nadler, “Finite sample approximation results for principal component analysis: A matrix perturbation approach,” The Annals of Statistics, vol. 36, no. 6, 2008.
[2] C. Qiu and N. Vaswani, “Real-time robust principal components’ pursuit,” in Allerton Conf. on Communication, Control, and Computing, 2010.
[3] C. Qiu, N. Vaswani, B. Lois, and L. Hogben, “Recursive robust pca or recursive sparse recovery in large but structured noise,” IEEE Trans. Info. Th., pp. 5007–5039, August 2014.
[4] B. Lois and N. Vaswani, “Online matrix completion and online robust pca,” in IEEE Intl. Symp. Info. Th. (ISIT), 2015.
[5] J. Zhan, B. Lois, H. Guo, and N. Vaswani, “Online (and Offline) Robust PCA: Novel Algorithms and Performance Guarantees,” in Intl. Conf. Artif. Intell. and Stat. (AISTATS), 2016.
[6] R. Arora, A. Cotter, and N. Srebro, “Stochastic optimization of pca with capped msg,” in Adv. Neural Info. Proc. Sys. (NIPS), 2013, pp. 1815–1823.
[7] O. Shamir, “A stochastic pca and svd algorithm with an exponential convergence rate,” arXiv:1409.2848, 2014.
[8] C. Boutsidis, D. Garber, Z. Karnin, and E. Liberty, “Online principal components analysis,” in Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA), 2015, pp. 887–901.
[9] A. Balsubramani, S. Dasgupta, and Y. Freund, “The fast convergence of incremental pca,” in Adv. Neural Info. Proc. Sys. (NIPS), 2013, pp. 3174–3182.
[10] Z. Karnin and E. Liberty, “Online pca with spectral bounds,” in Proce. Conference on Computational Learning Theory (COLT), 2015, pp. 505–509.
[11] I. Mitliagkas, C. Caramanis, and P. Jain, “Memory limited, streaming pca,” in Adv. Neural Info. Proc. Sys. (NIPS), 2013, pp. 2886–2894.
[12] M. Fazel, “Matrix rank minimization with applications,” PhD thesis, Stanford Univ, 2002.
[13] E. J. Candes and B. Recht, “Exact matrix completion via convex optimization,” Found. of Comput. Math., no. 9, pp. 717–772, 2008.
[14] E. J. Candès, X. Li, Y. Ma, and J. Wright, “Robust principal component analysis?,” Journal of ACM, vol. 58, no. 3, 2011.
[15] V. Chandrasekaran, S. Sanghavi, P. A. Parrilo, and A. S. Willsky, “Rank-sparsity incoherence for matrix decomposition,” SIAM Journal on Optimization, vol. 21, 2011.
[16] D. Hsu, S.M. Kakade, and T. Zhang, “Robust matrix decomposition with sparse corruptions,” IEEE Trans. Info. Th., Nov. 2011.
[17] P. Netrapalli, U N Niranjan, S. Sanghavi, A. Anandkumar, and P. Jain, “Non-convex robust pca,” in Neural Info. Proc. Sys. (NIPS), 2014.
[18] N. Vaswani, B. Lois, and P. Narayanamurthy, “Provably correct recursive projected compressive sensing (reprocs) for dynamic robust pca: A correlated-pca reformulation,” in http://www.ece.iastate.edu/long_RobSubTrack_3.pdf submitted to ICASSP 2017.
[19] C. Davis and W. M. Kahan, “The rotation of eigenvectors by a perturbation. iii,” SIAM J. Numer. Anal., vol. 7, pp. 1–46, Mar. 1970.
[20] J. A. Tropp, “User-friendly tail bounds for sums of random matrices,” Found. Comput. Math., vol. 12, no. 4, 2012.
[21] R. Vershynin, “Introduction to the non-asymptotic analysis of random matrices,” Compressed sensing, pp. 210–268, 2012.
[22] G. H. Golub and H. A. Van der Vorst, “Eigenvalue computation in the 20th century,” Journal of Computational and Applied Mathematics, vol. 123, no. 1, pp. 35–65, 2000.
[23] P. Netrapalli, P. Jain, and S. Sanghavi, “Low-rank matrix completion using alternating minimization,” in Symposium on Theory of Computing (STOC), 2013.
[24] Z. Lin, M. Chen, and Y. Ma, “Alternating direction algorithms for l1 problems in compressive sensing,” Tech. Rep., University of Illinois at Urbana-Champaign, November 2009.
[25] N. Vaswani and H. Guo, “Correlated-pca: Principal components’ analysis when data and noise are correlated,” submitted to IEEE Trans. Sig. Proc., 2016.
Supplementary Material

7 More examples of Assumption 1.2

Assumption 1.3 is one simple example of a support change model that ensures that, if \( M_{2,t} = I_{T_t} \), the assumption on \( M_{2,t} \) given in Assumption 1.2 holds. If instead of one object, there are \( k \) objects, and each of their supports satisfies Assumption 1.3 then, again, with some modifications, it is possible to show that both the PCA-missing and PCA-SDDC problems satisfy Assumption 1.2. Moreover, notice that Assumption 1.3 does not require the entries in \( T_t \) to be contiguous at all (they need not correspond to the support of one or a few objects). Similarly, we can replace the condition that \( T_t \) be constant for at most \( \beta \) time instants in Assumption 1.3 by \( \| t : T_t = T^t \{ k \} \| \leq \beta \).

Thirdly, the requirement of the object(s) always moving in one direction may seem too stringent. As explained in [2] Lemma 9.4], a Bernoulli-Gaussian “constant velocity with random acceleration” motion model will also work whp. It allows the object to move at each frame with probability \( p \) and not move with probability \( 1 - p \) independent of past or future frames; when the object moves, it moves with an iid Gaussian velocity that has mean \( 1.1s/\rho \) and variance \( \sigma^2; \sigma^2 \) needs to be upper bounded and \( p \) needs to be lower bounded.

Lastly, if \( s < c_1 \alpha \) for \( c_1 \ll 1 \), another model that works is that of an object of length \( s \) or less moving by at least one pixel and at most \( b \) pixels at each time [4, Lemma 9.5].

8 Proof of Theorem 2.1

This result also follows as a corollary of Theorem 5.3. We prove it separately first since its proof is short and and less notation-ally intensive. It will help understand the proof of Theorem 5.3 much more easily. Both results rely on the sin \( \theta \) theorem reviewed next.

8.1 sin \( \theta \) theorem

Davis and Kahan’s sin \( \theta \) theorem [19] studies the rotation of eigenvectors by perturbation.

**Theorem 8.1** (sin \( \theta \) theorem [19]). Consider two Hermitian matrices \( D \) and \( \hat{D} \). Suppose that \( D \) can be decomposed as

\[
D = [E \; E_\perp] \begin{bmatrix} A & 0 \\ 0 & A_\perp \end{bmatrix} \begin{bmatrix} E' \\ E_\perp' \end{bmatrix}
\]

where \([E \; E_\perp]\) is an orthonormal matrix. Suppose that \( \hat{D} \) can be decomposed as

\[
\hat{D} = [F \; F_\perp] \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda_\perp \end{bmatrix} \begin{bmatrix} F' \\ F_\perp' \end{bmatrix}
\]

where \([F \; F_\perp]\) is another orthonormal matrix and is such that \( \text{rank}(F) = \text{rank}(E) \). Let \( H := \hat{D} - D \) denote the perturbation. If \( \lambda_{\min}(A) > \lambda_{\max}(A_\perp) \), then

\[
\| (I - FF')E \| \leq \frac{\| H \|}{\lambda_{\min}(A) - \lambda_{\max}(A_\perp)}.
\]

Let \( r = \text{rank}(E) \). Suppose that \( F \) is the matrix of top \( r \) eigenvectors of \( \hat{D} \). Then \( A \) and \( A_\perp \) are diagonal and \( \lambda_{\max}(A_\perp) = \lambda_{r+1}(D) \leq \lambda_{r+1}(D) + \| H \| \). The inequality follows using Weyl’s inequality. Suppose also that \( \lambda_{\min}(A) > \lambda_{\max}(A_\perp) \). Then, (i) \( \lambda_r(D) = \lambda_{\min}(A) \) and \( \lambda_{r+1}(D) = \lambda_{\max}(A_\perp) \) and (ii) range \( (E) \) is equal to the span of the top \( r \) eigenvectors of \( D \). Thus, \( \lambda_{\max}(A_\perp) \leq \lambda_{\max}(A_\perp) + \| H \| \). With this we have the following corollary.

**Corollary 8.2.** Consider a Hermitian matrix \( D \) and its perturbed version \( \hat{D} \). Suppose that \( D \) can be decomposed as

\[
D = [E \; E_\perp] \begin{bmatrix} A & 0 \\ 0 & A_\perp \end{bmatrix} \begin{bmatrix} E' \\ E_\perp' \end{bmatrix}
\]

where \( E \) is a basis matrix. Let \( F \) denote the matrix containing the top \( \text{rank}(E) \) eigenvectors of \( \hat{D} \). Let \( H := \hat{D} - D \) denote the perturbation. If \( \lambda_{\min}(A) - \lambda_{\max}(A_\perp) - \| H \| > 0 \), then

\[
\| (I - FF')E \| \leq \frac{\| H \|}{\lambda_{\min}(A) - \lambda_{\max}(A_\perp) - \| H \|}.
\]

and range \( (E) \) is equal to the span of the top \( \text{rank}(E) \) eigenvectors of \( D \).
8.2 Proof of Theorem 2.1

We use the sinθ theorem [19] from Corollary 8.2. Apply it with \( \hat{D} = \frac{1}{\alpha} \sum_{i} y_{i} y_{i}' \) and \( D = \frac{1}{\alpha} \sum_{i} t_{i} e_{i}' \). Thus, \( F = \hat{P} \). Recall that \( a_{i} = P_{i}' \ell_{i} \). Then, \( D \) can be decomposed as \( P_{i}(\frac{1}{\alpha} \sum_{i} a_{i} a_{i}')P + P_{\perp} 0P_{\perp} \), and so we have \( E = P, A = \frac{1}{\alpha} \sum_{i} a_{i} a_{i}' \) and \( A_{\perp} = 0 \). Moreover, it is easy to see that the perturbation \( H := \frac{1}{\alpha} \sum_{i} y_{i} y_{i}' - \frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}' \) satisfies

\[
H = \frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}' + \frac{1}{\alpha} \sum_{i} w_{i} e_{i}' + \frac{1}{\alpha} \sum_{i} w_{i} w_{i}'.
\]

(7)

Thus,

\[
\text{SE}(\hat{P}, P) \\
\leq \frac{2 \| \frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}' \| + \| \frac{1}{\alpha} \sum_{i} w_{i} e_{i}' \|}{\lambda_{r}(\frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}') - 2 \| \frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}' \| + \frac{1}{\alpha} \sum_{i} w_{i} w_{i}')}
\]

if the denominator is positive.

Remark 8.3. Because \( w_{i} \) is correlated with \( \ell_{i} \), the \( \ell_{i} e_{i}' \) terms are the dominant ones in the perturbation expression given in (7). If they were uncorrelated, these two terms would be close to zero whp due to law of large numbers and the \( w_{i} w_{i}' \) term would be the dominant one.

In the next lemma, we bound the terms in the bound on \( \text{SE}(\hat{P}, P) \) using the matrix Hoeffding inequality [20].

Lemma 8.4. Let \( \epsilon = 0.01 r_{\zeta} \lambda^{-} \).

1. With probability at least \( 1 - 2 n \exp \left( -\alpha \frac{\epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) \),

\[
\| \frac{1}{\alpha} \sum_{i} \ell_{i} e_{i}' \| \leq q \lambda^{+} \sqrt{\frac{2}{\alpha}} + \epsilon = [q \sqrt{\frac{\beta}{\alpha}} + 0.01 r \zeta] \lambda^{-}
\]

2. With probability at least \( 1 - 2 n \exp \left( -\frac{\alpha \epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) \),

\[
\| \frac{1}{\alpha} \sum_{i} w_{i} e_{i}' \| \leq \frac{\beta \epsilon^{2}}{\alpha} \lambda^{+} + \epsilon = [q^{2} f \sqrt{\frac{\beta}{\alpha}} + 0.01 r \zeta] \lambda^{-}
\]

3. With probability at least \( 1 - 2 n \exp \left( -\frac{\alpha \epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) \),

\[
\lambda_{r}(\frac{1}{\alpha} \sum_{i} e_{i}') \geq (1 - (r_{\zeta})^{2}) \lambda^{-} - \epsilon
\]

Proof. This follows by using Lemma 9.6 given later with \( G_{\text{cur}} \equiv P, G_{\text{det}} \equiv [], G_{\text{undet}} \equiv [] \).

\( r_{\text{det}} = 0, r_{\text{cur}} = 0, r_{\text{cur}} = r, g \equiv f, \chi = 0, \vartheta = 1. \)

Using this lemma to bound the subspace error terms, followed by using the bounds on \( \beta/\alpha \) and \( \zeta \), we conclude the following: w.p. at least \( 1 - 2 n \exp \left( -\alpha \frac{\epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) - 2 n \exp \left( -\frac{\alpha \epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) - 2 n \exp \left( -\frac{\alpha \epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) \),

\[
\text{SE}(\hat{P}, P) \\
\leq \frac{2 q f \sqrt{\frac{\beta}{\alpha}} + q^{2} f \frac{\beta}{\alpha} + 0.03 r \zeta}{1 - (r_{\zeta})^{2} - 0.01 r \zeta - (2 q f \sqrt{\frac{\beta}{\alpha}} + q^{2} f \frac{\beta}{\alpha} + 0.03 r \zeta)} \\
\leq \frac{0.75(1 - r \zeta) r \zeta + 0.03 r \zeta}{1 - r \zeta} < r \zeta
\]

Using the bound \( \alpha \geq \alpha_{0} \) from the theorem, the probability of the above event is at least \( 1 - 6 n^{-10} \). We get this by bounding each of the three negative terms in the probability expression by \( -2 n^{-10} \).

We work this out for the first term: \( \alpha \frac{\epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \geq \frac{32 \cdot 11.1 q r \zeta (\log n)}{(0.01)^{2} (r_{\zeta})^{2}} \geq (0.01 r \zeta) \lambda^{-} = 0.01 \log n \).

Thus, \( 2 n \exp \left( -\alpha \frac{\epsilon^{2}}{32(\eta r q \lambda^{\alpha} + \lambda)} \right) \leq 2 n \exp (-11 \log n) \leq 2 n^{-10} \).
9 Proof of Theorem 3.3
We explain the overall idea of the proof next. In Sec. 9.2 we give a sequence of lemmas in generalized form (so that they can apply to various other problems). The proof of Theorem 3.3 is given in Sec. 9.3 and follows easily by applying these. One of the lemmas of Sec. 9.2 is proved in Sec. [10] while the others are proved there itself.

9.1 Overall idea
We need to bound SE(\( \mathbf{P}, \mathbf{P} \)). From Algorithm 1 \( \mathbf{P} = [\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_\rho] \) where \( \hat{G}_k \) is the matrix of top \( \hat{r}_k \) eigenvectors of \( \hat{D}_k \) defined in Algorithm 1. Also, \( \mathbf{P} = [G_1, G_2, \ldots, G_\rho] \) where \( G_k \) is a basis matrix with \( r_k \) columns.

Definition 9.1. Define \( \zeta_k := \text{SE}(\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_k), \zeta_0 = 0 \). Define \( \zeta_k^+ := r_k \zeta \). Let \( r_0 = 0 \).

It is easy to see that

\[
\text{SE}(\mathbf{P}, \mathbf{P}) \leq \sum_{k=1}^\rho \text{SE}(\mathbf{P}, G_k)
\]

\[
\leq \sum_{k=1}^\rho \text{SE}(\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_k, G_k) = \sum_{k=1}^\rho \zeta_k
\]

(8)

The first inequality is triangle inequality, the second follows because \([\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_k]\) is orthogonal to \([\hat{G}_{k+1}, \ldots, G_\rho] \). Since \( r = \sum_k r_k \), if we can show that \( \zeta_k \leq \zeta_k^+ = r_k \zeta \) for all \( k \) we will be done.

We bound \( \zeta_k \) using induction. The base case is easy and follows just from the definition, \( \zeta_0 = \text{SE}([1, 1]) = 0 = r_0 \zeta \). For bounding \( \zeta_k \), assume that for all \( i = 1, 2, \ldots, k - 1 \), \( \zeta_i \leq r_i \zeta \). This implies that

\[
\text{SE}([\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{k-1}], [G_1, G_2, \ldots, G_{k-1}])
\]

\[
\leq \sum_{i=1}^{k-1} \text{SE}([\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{k-1}], G_i)
\]

\[
\leq \sum_{i=1}^{k-1} \zeta_i \leq \sum_{i=1}^{k-1} r_i \zeta \leq r \zeta
\]

(9)

Using this, we will first show that \( \hat{r}_k = r_k \), and then we will use this and the \( \sin \theta \) result to bound \( \zeta_k \).

Before proceeding further, we simplify notation.

Definition 9.2.
1. Let

\[
G_{\text{det}} := [G_1, G_2, \ldots, G_{k-1}], G_{\text{cur}} := G_k,
\]

\[
G_{\text{undet}} := [\hat{G}_{k+1}, \ldots, G_\rho]
\]

2. Similarly, let \( \hat{G}_{\text{det}} := [\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{k-1}], \hat{G}_{\text{cur}} := \hat{G}_k \).

3. Let \( \mathcal{G}_{\text{det}} := G_1 \cup G_2 \cdots \cup G_{k-1} \) and \( \mathcal{G}_{\text{cur}} = G_k \).

4. Let \( r_{\text{cur}} := r_k = \text{rank}(G_k) \) and \( \hat{r}_{\text{cur}} := \hat{r}_k \).

5. Let \( \lambda_{\text{cur}}^+ := \lambda_k^+, \lambda_{\text{cur}}^- := \lambda_k^- \), \( \lambda_{\text{undet}}^+ := \lambda_{k+1}^+ \).

6. Let \( t_\alpha = k \alpha \).

9.2 Main lemmas - generalized form
In this section, we give a sequence of lemmas that apply to a generic problem where \( y = \ell + w \), \( \ell = \ell + M_\ell \ell \) with \( \ell \) satisfying Assumption 1.1 \( M_\ell \) satisfying Assumption 1.2 and with \( P \) split into three parts as \( P = [G_{\text{det}}, G_{\text{cur}}, G_{\text{undet}}] \). We can correspondingly split \( A \) as \( A = \text{diag}(A_{\text{det}}, A_{\text{cur}}, A_{\text{undet}}) \).
We are given \( \hat{G}_{\text{det}} \) that was computed using (some or all) \( y_t \)'s for \( t \leq t^* \) and that satisfies \( \zeta_{\text{det}} \leq r \zeta \). The goal is to estimate \( \text{range}(G_{\text{cur}}) \) and bound the estimation error. This is done by first estimating \( \hat{r}_{\text{cur}} \) and then computing \( \hat{G}_{\text{cur}} \) as the top \( \hat{r}_{\text{cur}} \) eigenvectors of

\[
\hat{D} := \frac{1}{\alpha} \sum_{t=t^*+1}^{t^*+\alpha} \Psi y_t y_t' \Psi.
\] (10)

To bound the estimation error, we first show that, whp, \( \hat{r}_{\text{cur}} = r_{\text{cur}} \) and so \( \hat{G}_{\text{cur}} = G_{\text{cur}} \); and then we use this to show that \( \zeta_{\text{cur}} \leq r_{\text{cur}} \zeta \).

**Definition 9.3.**

1. Define \( \Psi := I - \hat{G}_{\text{det}} \hat{G}_{\text{det}}' \).
2. Define \( \zeta_{\text{det}} := \text{SE}(\hat{G}_{\text{det}}, G_{\text{det}}) = \| \Psi G_{\text{det}} \| \) and \( \zeta_{\text{det}}^+ = r \zeta \).
3. Define \( \zeta_{\text{cur}} := \text{SE}((\hat{G}_{\text{det}}, \hat{G}_{\text{cur}}), G_{\text{cur}}) \).
4. Let \( (\Psi G_{\text{cur}}) \) QR \( E_{\text{cur}} R_{\text{cur}} \) denote its reduced QR decomposition. Thus \( E_{\text{cur}} \) is a basis matrix whose span equals that of \( (\Psi G_{\text{cur}}) \) and \( R_{\text{cur}} \) is a square upper triangular matrix with \( \| R_{\text{cur}} \| = \| \Psi G_{\text{cur}} \| \leq 1 \).
5. Let \( \lambda_{\text{cur}}^+ = \lambda_{\text{max}}(A_{\text{cur}}) \), \( \lambda_{\text{cur}}^- = \lambda_{\text{min}}(A_{\text{cur}}) \), \( \lambda_{\text{undet}}^+ = \lambda_{\text{max}}(A_{\text{undet}}) \).
6. Let \( r_{\text{cur}} = \text{rank}(G_{\text{cur}}) \). Clearly, \( r_{\text{cur}} \leq r \).

**Remark 9.4.** In special cases, \( G_{\text{det}} \) (and hence \( \hat{G}_{\text{det}} \)) could be empty; and/or \( G_{\text{undet}} \) could be empty.

- Since \( A \) contains eigenvalues in decreasing order, when \( G_{\text{undet}} \) is not empty, \( \lambda^- \leq \lambda_{\text{undet}}^+ \leq \lambda_{\text{cur}}^- \leq \lambda_{\text{cur}}^+ \leq \lambda^+ \).
- When \( G_{\text{undet}} \) is empty, \( \lambda_{\text{undet}}^+ = 0 \) and \( \lambda^- \leq \lambda_{\text{cur}}^- \leq \lambda_{\text{cur}}^+ \leq \lambda^+ \).

Using \( \| R_{\text{cur}} \| = \| \Psi G_{\text{cur}} \| \leq 1 \),

\[
\zeta_{\text{cur}} = \|(I - \hat{G}_{\text{cur}} \hat{G}_{\text{cur}}') \Psi G_{\text{cur}} \|
= \|(I - \hat{G}_{\text{cur}} \hat{G}_{\text{cur}}') E_{\text{cur}} R_{\text{cur}} \|
\leq \|(I - \hat{G}_{\text{cur}} \hat{G}_{\text{cur}}') E_{\text{cur}} \| = \text{SE}(\hat{G}_{\text{cur}}, E_{\text{cur}}).
\]

Thus, to bound \( \zeta_{\text{cur}} \) we need to bound \( \text{SE}(\hat{G}_{\text{cur}}, E_{\text{cur}}) \). \( \hat{G}_{\text{cur}} \) is the matrix of top \( \hat{r}_{\text{cur}} \) eigenvectors of \( \hat{D} \). From its definition, \( E_{\text{cur}} \) is a basis matrix with \( r_{\text{cur}} \) columns. Suppose for a moment that \( \hat{r}_{\text{cur}} = r_{\text{cur}} \). Then, in order to bound \( \text{SE}(\hat{G}_{\text{cur}}, E_{\text{cur}}) \), we can use the sin \( \theta \) result, Corollary 9.2 To do this, we need to define a matrix \( D \) so that, under appropriate assumptions, the span of its top \( r_{\text{cur}} \) eigenvectors equals \( \text{range}(E_{\text{cur}}) \). For the simple EVD proof, we used \( \frac{1}{\alpha} \sum_{t=t^*+1}^{t^*+\alpha} \Psi \ell_t \ell_t' \Psi \) as the matrix \( D \). However, this will not work now since \( E_{\text{cur}} \) is not orthonormal to \( \Psi G_{\text{det}} \) or \( \Psi G_{\text{undet}} \). But, instead we can use

\[
D = E_{\text{cur}} AE_{\text{cur}}' + E_{\text{cur},\perp} A_{\perp} E_{\text{cur},\perp}', \text{ where}
\]

\[
A := E_{\text{cur}}' (\frac{1}{\alpha} \sum_{t=t^*+1}^{t^*+\alpha} \Psi \ell_t \ell_t' \Psi) E_{\text{cur}} \text{ and}
\]

\[
A_{\perp} := E_{\text{cur},\perp}' (\frac{1}{\alpha} \sum_{t=t^*+1}^{t^*+\alpha} \Psi \ell_t \ell_t' \Psi) E_{\text{cur},\perp}.
\]

(11)

Now, by construction, \( D \) is in the desired form.

With the above choice of \( D \), \( H := \hat{D} - D \) satisfies \( \| E_{\text{cur}} E_{\text{cur}}' (\frac{1}{\alpha} \sum_{t} \Psi \ell_t \ell_t' \Psi) E_{\text{cur},\perp} \| E_{\text{cur},\perp} \| = \text{term}1 + \text{term}1' + \text{term}2 + \text{term}3 + \text{term}3' \) where \( \text{term}1 := \frac{1}{\alpha} \sum_t \Psi \ell_t w_t' \Psi \), \( \text{term}2 := \frac{1}{\alpha} \sum_t \Psi w_t w_t' \Psi \) and \( \text{term}3 = E_{\text{cur}} E_{\text{cur}}' (\frac{1}{\alpha} \sum_{t} \Psi \ell_t \ell_t' \Psi) E_{\text{cur},\perp} E_{\text{cur},\perp}' \).

This follows easily by writing \( H = (\hat{D} - \frac{1}{\alpha} \sum_t \Psi \ell_t \ell_t' \Psi) + (\frac{1}{\alpha} \sum_t \Psi \ell_t \ell_t' \Psi - D) \) and using the fact that \( M = (EE' + E_{\perp} E_{\perp}') M (EE' + E_{\perp} E_{\perp}') \) for \( \frac{1}{\alpha} \sum_{t} \Psi \ell_t \ell_t' \Psi \).
Thus, using the above along with Corollary 8.2 we can conclude the following.

**Fact 9.5.**

1. If \( \hat{r}_{cur} = r_{cur} \), and \( \lambda_{\min}(A) - \lambda_{\max}(A_{\perp}) - \|H\| > 0 \),
   \[
   \zeta_{\text{cur}} \leq \text{SE}(\hat{G}_{\text{cur}}, E_{\text{cur}}) \leq \frac{\|H\|}{\lambda_{\min}(A) - \lambda_{\max}(A_{\perp}) - \|H\|}.
   \]

2. Let \( Q := E_{\text{cur}} E_{\text{cur}}' \left( \frac{1}{\alpha} \sum_t \Psi_\ell_t \ell_t' \Psi_\ell_t \right) E_{\text{cur}, \perp} E_{\text{cur}, \perp}' \). We have
   \[
   \|H\| \leq 2 \| \frac{1}{\alpha} \sum_t \Psi_\ell_t w_t' \| + \| \frac{1}{\alpha} \sum_t w_t w_t' \| + 2 \|Q\|.
   \]

The next lemma bounds the RHS terms in the above lemma and a few other quantities needed for showing \( \hat{r}_{cur} = r_{cur} \).

**Lemma 9.6.** (1) Assume that \( y_t = \ell_t + w_t = \ell_t + M_t \ell_t \) with \( \ell_t \) satisfying Assumption 1.2 and \( M_t \) satisfying Assumption 1.2.

(2) Assume that we are given \( \hat{G}_{\text{det}} \) that was computed using (some or all) \( y_t \)'s for \( t \leq t_\ast \) and that satisfies \( \zeta_{\text{det}} \leq r_{\zeta} \).

Define \( g := \lambda_{\text{cur}}^+ / \lambda_{\text{cur}}^- \), \( \chi := \lambda_{\text{undet}}^+ / \lambda_{\text{cur}}^- \). Set \( \epsilon := 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^- \).

Then, the following hold:

1. Let \( p_1 := 2n \exp(-\frac{\alpha^2}{2r_{\text{prob}}^2}) \) where \( b_{\text{prob}} := \eta \rho q((r_\zeta) \lambda^+ + \lambda_{\text{cur}}^+ + (r_\zeta) \sqrt{\lambda^+ \lambda_{\text{cur}}^+} + \sqrt{\lambda^+ \lambda_{\text{cur}}^-}) \). Conditioned on \( \{ \zeta_{\text{det}} \leq r_{\zeta} \} \), with probability at least \( 1 - p_1 \),
   \[
   \| \frac{1}{\alpha} \sum_t \Psi_\ell_t w_t' \| \leq q((r_\zeta) \lambda^+ + \lambda_{\text{cur}}^+) \sqrt{\frac{\beta}{\alpha}} + \epsilon \\
   \leq [q(r_\zeta)f \sqrt{\frac{\beta}{\alpha}} + qg \sqrt{\frac{\beta}{\alpha}} + 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^-].
   \]

2. Let \( p_2 := 2n \exp(-\frac{\alpha^2}{2q^2 r_{\text{prob}}^2}) \). Conditioned on \( \{ \zeta_{\text{det}} \leq r_{\zeta} \} \), with probability (w.p.) at least \( 1 - p_2 \),
   \[
   \| \frac{1}{\alpha} \sum_t w_t w_t' \| \leq \frac{\beta}{\alpha} q^2 \lambda^+ + \epsilon \leq [\frac{\beta}{\alpha} q^2 f + 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^-].
   \]

3. Let \( p_3 := 2n \exp(-\frac{\alpha^2}{2q^2 r_{\text{prob}}^2}) \) with \( b_{\text{prob}} := \eta \rho q((r_\zeta)^2 \lambda^+ + \lambda_{\text{cur}}^+ + 2(r_\zeta) \sqrt{\lambda^+ \lambda_{\text{cur}}^+}) \). Conditioned on \( \{ \zeta_{\text{det}} \leq r_{\zeta} \} \), with probability at least \( 1 - p_3 \),
   \[
   \| E_{\text{cur}} E_{\text{cur}}' \left( \frac{1}{\alpha} \Psi_\ell_t \ell_t' \Psi_\ell_t \right) E_{\text{cur}, \perp} E_{\text{cur}, \perp}' \| \\
   \leq (r_\zeta)^2 \lambda^+ + \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \lambda_{\text{undet}} + \epsilon \\
   \leq [(r_\zeta)^2 f + \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \chi + 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^-].
   \]

4. Conditioned on \( \{ \zeta_{\text{det}} \leq r_{\zeta} \} \), w.p. at least \( 1 - p_3 \),
   \[
   \lambda_{\min}(A) \geq (1 - (r_\zeta)^2) \lambda_{\text{cur}}^- - \epsilon \\
   = [1 - (r_\zeta)^2 - 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^-].
   \]

5. Conditioned on \( \{ \zeta_{\text{det}} \leq r_{\zeta} \} \), w.p. at least \( 1 - p_3 \),
   \[
   \lambda_{\max}(A_{\perp}) \leq ((r_\zeta)^2 \lambda^+ + \lambda_{\text{undet}}^+) + \epsilon \\
   \leq [(r_\zeta)^2 f + \chi + 0.01 r_{\text{cur}} \zeta \lambda_{\text{cur}}^-].
   \]
Lemma 9.8. Consider the setting of Corollary 9.7. In addition, also assume that

\[ \lambda_{\max}(A) \leq \lambda_{\max}(A_{\perp}) \geq (1 - (r\zeta)^2) - \frac{(r\zeta)^2}{\sqrt{1 - (r\zeta)^2}} \lambda_{\text{undet}}^+ - \epsilon. \]

7. Conditioned on \( \{\zeta_{\text{det}} \leq r\zeta\} \), w.p. at least \( 1 - p_3 \),

\[ \lambda_{\max}(A) \geq (1 - (r\zeta)^2) \lambda_{\text{cur}}^+ - \epsilon 
\leq [(1 - (r\zeta)^2) g - 0.01 r_{\text{cur}}^2] \lambda_{\text{cur}}^-. \]

8. Conditioned on \( \{\zeta_{\text{det}} \leq r\zeta\} \), w.p. at least \( 1 - p_3 \),

\[ \lambda_{\max}(A) \leq \lambda_{\text{cur}}^+ + (r\zeta)^2 \lambda_{\text{cur}}^+ + \frac{1}{1 - (r\zeta)^2} (r\zeta)^2 \lambda_{\text{undet}}^+ + \epsilon \]
\[ \leq [g + (r\zeta)^2 f + \frac{(r\zeta)^2}{1 - (r\zeta)^2} (r_{\text{cur}}^2) 0.01 r_{\text{cur}}^2] \lambda_{\text{cur}}^- \]

Proof. The proof is in Section 10. \( \Box \)

Corollary 9.7. Consider the setting of Lemma 9.6. Assume

1. \( r(r\zeta) \leq 0.0001 \), and \( r(r\zeta) f \leq 0.01 \). Since \( r_{\text{cur}} \leq r \), this implies that \( r_{\text{cur}}^2 \zeta \leq 0.0001 \), and

2. \( \beta \leq \left( \frac{(1 - r_{\text{cur}}^2 \zeta - \chi)}{2} \right)^2 \min \left( \frac{(r_{\text{cur}}^2)^2}{4 q^2 q}, \frac{(r_{\text{cur}}^2)^2}{q^2} \right) \alpha. \)

Using these and using \( g \geq 1, g \leq f, \chi \leq 1 \) (these hold by definition), with probability at least \( 1 - p_1 - p_2 - 4 p_3 \),

\[ \|H\| \leq \left[ 2.02 q g \sqrt{\frac{\beta}{\alpha} + \frac{\beta}{\alpha} q^2 f} + 0.08 r_{\text{cur}}^2 \zeta \lambda_{\text{cur}}^- \right] \]
\[ \leq \left[ 0.75 (1 - r\zeta - \chi) r_{\text{cur}}^2 \zeta + 0.08 r_{\text{cur}}^2 \zeta \lambda_{\text{cur}}^- \right] \]
\[ \leq 0.83 r_{\text{cur}}^2 \zeta \lambda_{\text{cur}}^- \]
\[ \lambda_{\max}(A_{\perp}) \leq r_{\text{cur}}^2 \zeta \lambda_{\text{cur}}^- \]
\[ \lambda_{\max}(A) \leq \lambda_{\text{cur}}^+ - 0.02 r_{\text{cur}}^2 \zeta \lambda_{\text{cur}}^- \]
\[ \lambda_{\min}(A) \geq [1 - 0.0101 r_{\text{cur}}^2 \zeta] \lambda_{\text{cur}}^- \]
\[ \lambda_{\max}(A) \leq [g + 0.0202 r_{\text{cur}}^2 \zeta] \lambda_{\text{cur}}^- \]
\[ \lambda_{\max}(A) \leq [g - 0.02 r_{\text{cur}}^2 \zeta] \lambda_{\text{cur}}^- \]

Lemma 9.8. Consider the setting of Corollary 9.7. In addition, also assume that

1. \( \hat{g} = 1.01 g + 0.0001 \) and

2. \( \chi \leq \min \left( \frac{g - 0.0001}{1.01 g + 0.0001} - 0.0001, 1 - r_{\text{cur}}^2 \zeta - 0.08 \right) \).

Let \( \hat{\lambda}_i := \lambda_i(\hat{D}) \). Then, with probability at least \( 1 - p_1 - p_2 - 4 p_3 \), the following hold.

1. When \( G_{\text{undet}} \) is not empty: \( \frac{\hat{\lambda}}{\lambda_{\text{cur}}^+ + 1} \geq \hat{g} \), and \( \hat{\lambda}_{\text{cur}}^+ + 1 \geq \lambda_{\text{thresh}} \).

2. When \( G_{\text{undet}} \) is empty: \( \frac{\hat{\lambda}}{\lambda_{\text{cur}}^+ + 1} \geq \hat{g} \) and \( \hat{\lambda}_{\text{cur}}^+ + 1 < \lambda_{\text{thresh}} < \hat{\lambda}_{\text{cur}}^+ \).

3. If \( \hat{r}_{\text{cur}} = r_{\text{cur}} \), then \( \zeta_{\text{cur}} \leq \frac{\|H\|}{\lambda_{\min}(A) - \lambda_{\max}(A_{\perp}) - \|H\|} \leq 0.75 r_{\text{cur}}^2 \zeta + \frac{0.08 r_{\text{cur}}^2 \zeta}{(1 - r_{\text{cur}}^2 \zeta)} \leq r_{\text{cur}} \).

Proof.
**Fact 9.9.** From the bound on \( \chi, \chi \leq 1 - 0.0001 \leq 1 - r_{\text{cur}} \zeta \). Thus, using Corollary 9.7, \( \lambda_{\min}(A) > \lambda_{\max}(A_{\perp}) \) and so \( \lambda_{\text{cur}}(D) = \lambda_{\min}(A), \lambda_{\text{cur}+1}(D) = \lambda_{\max}(A_{\perp}), \) and \( \lambda_1(D) = \lambda_{\max}(A) \). Recall: \( \lambda_1(.) \) is the same as \( \lambda_{\max}(.) \).

**Proof of item 1.** Recall that \( \hat{D} \) and \( D \) are defined in (10) and (11). Using Weyl’s inequality, Fact 9.9 and Corollary 9.7, with the probability given there,

\[
\frac{\hat{\lambda}_1}{\lambda_{\text{cur}}} \leq \frac{\lambda_{\min}(A) + \|H\|}{\lambda_{\min}(A) - \|H\|} \leq \frac{g + 0.85r_{\text{cur}} \zeta}{1 - 0.85r_{\text{cur}} \zeta}
\]

and

\[
\frac{\hat{\lambda}_1}{\lambda_{\text{cur}+1}} > \frac{\lambda_{\min}(A) - \|H\|}{\lambda_{\max}(A_{\perp}) + \|H\|} > \frac{g - 0.85r_{\text{cur}} \zeta}{\chi + 0.85r_{\text{cur}} \zeta}
\]

Thus, if

\[
\frac{g + 0.85r_{\text{cur}} \zeta}{1 - 0.85r_{\text{cur}} \zeta} \leq \hat{g} \leq \frac{g - 0.85r_{\text{cur}} \zeta}{\chi + 0.85r_{\text{cur}} \zeta}
\]

holds, we will be done. The above requires \( \chi \) to be small enough so that the lower bound is not larger than the upper bound and it requires \( \hat{g} \) to be appropriately set. Both are ensured by the assumptions in the lemma.

Since \( G_{\text{undet}} \) is not empty, \( \lambda^{+}_{\text{undet}} = \chi \lambda_{\text{cur}} > \lambda^{−} \) Thus, using Weyl’s inequality followed by Corollary 9.7 with the probability given there,

\[
\hat{\lambda}_{\text{cur}+1} \geq \lambda_{\text{cur}+1}(D) - \|H\| = \lambda_{\max}(A_{\perp}) - \|H\|
\]

\[
\geq [\chi - 0.02r_{\text{cur}} \zeta] \lambda_{\text{cur}} - 0.83r_{\text{cur}} \zeta \lambda_{\text{cur}}
\]

\[
\geq (1 - 0.85r_{\text{cur}} \zeta) \lambda^{−} > \lambda_{\text{thresh}}
\]

**Proof of item 2.** Since \( G_{\text{undet}} \) is empty, \( \lambda^{+}_{\text{undet}} = 0 \) and so \( \chi = 0 \). Thus, using Corollary 9.7 with probability given there,

\[
\hat{\lambda}_{\text{cur}+1} \leq \lambda_{\text{cur}+1}(D) + \|H\| = \lambda_{\max}(A_{\perp}) + \|H\|
\]

\[
\leq 0 + 0.02r_{\text{cur}} \zeta \lambda^{−} + \|H\| \leq 0.85r_{\text{cur}} \zeta \lambda^{−} < \lambda_{\text{thresh}}.
\]

\[
\hat{\lambda}_{\text{cur}} \geq \lambda_{\text{cur}}(D) - \|H\| = \lambda_{\min}(A) - \|H\|
\]

\[
\geq \lambda^{−} - 0.05r_{\text{cur}} \zeta \lambda_{\text{cur}} \geq (1 - 0.85r_{\text{cur}} \zeta) \lambda^{−} > \lambda_{\text{thresh}}.
\]

and

\[
\frac{\hat{\lambda}_1}{\lambda_{\text{cur}}} \leq \frac{\lambda_{\max}(A) + \|H\|}{\lambda_{\min}(A) - \|H\|} \leq \frac{g + 0.85r_{\text{cur}} \zeta}{1 - 0.85r_{\text{cur}} \zeta} \leq \hat{g}
\]

**Proof of item 3.** Using Fact 9.3 and Corollary 9.7 since \( \hat{r}_{\text{cur}} = r_{\text{cur}} \) is assumed, we get

\[
\zeta_{\text{cur}} \leq \frac{[0.75(1 - r_{\text{cur}} \zeta - \chi)r_{\text{cur}} \zeta + 0.08r_{\text{cur}} \zeta \lambda^{−}_{\text{cur}}]}{\lambda_{\text{cur}}[1 - 0.0101r_{\text{cur}} \zeta - \chi - 0.02r_{\text{cur}} \zeta - 0.83r_{\text{cur}} \zeta]} \leq \frac{0.75(1 - r_{\text{cur}} \zeta - \chi)r_{\text{cur}} \zeta + 0.08r_{\text{cur}} \zeta}{(1 - r_{\text{cur}} \zeta - \chi)} \leq r_{\text{cur}} \zeta
\]

The last inequality used the bound on \( \chi \).

**9.3 Proof of Theorem 3.3**

The theorem is a direct consequence of using 9.3 and applying Lemma 9.8 for each of the \( k \) steps with the substitutions given in Definition 9.2 along with picking \( \alpha \) appropriately. A detailed proof is in Sec. 11.
10 Proof of Hoeffding lemma, Lemma 9.6

The following lemma, which is a modification of [3, Lemma 8.15], will be used in our proof. It is proved in Sec. 11. The proof uses [3, Lemma 2.10].

Lemma 10.1. Given \( \zeta_{\text{det}} \leq r_\zeta \),

1. \( \| \Psi^{\text{det}} \| \leq r_\zeta \) and \( \| \Psi^{\text{cur}} \| \leq 1 \).
2. \( \sqrt{1 - (r_\zeta)^2} \leq \sigma_i(R_{\text{cur}}) = \sigma_i(\Psi^{\text{cur}}) \leq 1 \) and \( \sqrt{1 - (r_\zeta)^2} \leq \sigma_i(\Psi^{\text{undet}}) \leq 1 \)
3. \( \| E_{\text{cur}}' \Psi^{\text{undet}} \| \leq \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \)
4. \( \Psi \Sigma \Psi = \begin{bmatrix} \Lambda^{\text{det}} & 0 & 0 \\ 0 & \Lambda^{\text{cur}} & 0 \\ 0 & 0 & \Lambda^{\text{undet}} \end{bmatrix} \begin{bmatrix} \Psi^{\text{det}} \\ \Psi^{\text{cur}} \\ \Psi^{\text{undet}} \end{bmatrix} \)

with \( \lambda_{\max}(\Lambda^{\text{det}}) \leq \lambda^+ \), \( \lambda^{\text{cur}}_\zeta \leq \lambda_{\min}(\Lambda^{\text{cur}}) \leq \lambda_{\max}(\Lambda^{\text{cur}}) \leq \lambda^{\text{cur}}_+ \), \( \lambda_{\max}(\Lambda^{\text{undet}}) \leq \lambda^{\text{undet}}_+ \).
5. Using the first four claims, it is easy to see that
   (a) \( \| E_{\text{cur}, \lambda} \Psi \Sigma \Psi E_{\text{cur}, \lambda} \| \leq (r_\zeta)^2 \lambda^+ + \lambda^{\text{undet}}_+ \)
   (b) \( \| E_{\text{cur}, \lambda} \Psi \Sigma \Psi E_{\text{cur}, \lambda} \| \leq (r_\zeta)^2 \lambda^+ + \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \lambda^{\text{undet}}_+ \)
   (c) \( \| \Psi \Sigma \| \leq (r_\zeta) \lambda^+ + \lambda^{\text{cur}}_+ \) and \( \| \Psi \Sigma M_{1, t} \| \leq q((r_\zeta) \lambda^+ + \lambda^{\text{cur}}_+) \)
   (d) \( \| M_{1, t} \Sigma \| \leq \lambda^+ \) and \( \| M_{1, t} \Sigma M_{1, t} \| \leq q^2 \lambda^+ \)

If \( \tilde{G}_{\text{det}} = G_{\text{det}} = \cdot \), then all the terms containing \( (r_\zeta) \) disappear.
6. \( \lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B) \)
7. Let \( a_t := P' \ell_t, a_{t, \text{det}} := G_{\text{det}}' \ell_t, a_{t, \text{cur}} := G_{\text{cur}}' \ell_t \) and \( a_{t, \text{undet}} := G_{\text{undet}}' \ell_t \). Also let \( a_{t, \text{rest}} := [a_{t, \text{cur}}, a_{t, \text{undet}}]' \). Then \( \| a_{t, \text{rest}} \|^2 \leq r\eta \lambda^{\text{cur}}_+ \) and \( \| a_{t, \text{det}} \|^2 \leq \| a_t \|^2 \leq r\eta \lambda^+ \).
8. \( \sigma_{\min}(E_{\text{cur}, \lambda} \Psi G_{\text{undet}}')^2 \geq 1 - (r_\zeta)^2 - \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \)

The following corollaries of the matrix Hoeffding inequality [20], proved in [3], will be used in the proof.

Corollary 10.2. Given an \( \alpha \)-length sequence \( \{ Z_t \} \) of random Hermitian matrices of size \( n \times n \), a rv. \( X \), and a set \( \mathcal{C} \) of values that \( X \) can take. For all \( X \in \mathcal{C} \), (i) \( Z_t \)'s are conditionally independent given \( X \); (ii) \( \mathbb{P}(b_1 I \leq Z_t \leq b_2 I \mid X) = 1 \) and (iii) \( b_3 I \leq \mathbb{E}[\frac{1}{\alpha} \sum_t Z_t \mid X] \leq b_4 I \). For any \( \epsilon > 0 \), for all \( X \in \mathcal{C} \),

\[ \mathbb{P} \left( \lambda_{\max} \left( \frac{1}{\alpha} \sum_t Z_t \right) \leq b_4 + \epsilon \mid X \right) \geq 1 - n \exp \left( \frac{-\alpha \epsilon^2}{8(b_2 - b_1)^2} \right), \]

\[ \mathbb{P} \left( \lambda_{\min} \left( \frac{1}{\alpha} \sum_t Z_t \right) \geq b_3 - \epsilon \mid X \right) \geq 1 - n \exp \left( \frac{-\alpha \epsilon^2}{8(b_2 - b_1)^2} \right). \]

Corollary 10.3. Given an \( \alpha \)-length sequence \( \{ Z_t \} \) of random matrices of size \( n_1 \times n_2 \). For all \( X \in \mathcal{C} \), (i) \( Z_t \)'s are conditionally independent given \( X \); (ii) \( \mathbb{P}(\| Z_t \| \leq b_1 \mid X) = 1 \) and (iii) \( \mathbb{E}[\frac{1}{\alpha} \sum_t Z_t \mid X] \leq b_2 \). For any \( \epsilon > 0 \), for all \( X \in \mathcal{C} \),

\[ \mathbb{P} \left( \left\| \frac{1}{\alpha} \sum_t Z_t \right\| \leq b_2 + \epsilon \mid X \right) \geq 1 - (n_1 + n_2) \exp \left( \frac{-\alpha \epsilon^2}{32b_2^2} \right). \]
Proof of Lemma 9.6. Recall that we are given \( \hat{G}_{\text{det}} \) that was computed using (some or all) \( y_t \)’s for \( t \leq t_* \) and that satisfies \( \zeta_{\text{det}} \leq r\zeta \). From (2), \( y_t \) is a linear function of \( \ell_t \). Thus, we can let \( X := \{\ell_1, \ell_2, \ldots, \ell_t, \ell_{t+1}^{\infty}\} \) denote all the random variables on which the event \( \{\zeta_{\text{det}} \leq r\zeta\} \) depends. In each item of this proof, we need to lower bound the probability of the desired event conditioned on \( \zeta_{\text{det}} \leq r\zeta \). To do this, we first lower bound the probability of the event conditioned on \( X \) that is such that \( X \in \{\zeta_{\text{det}} \leq r\zeta\} \). We get a lower bound that does not depend on \( X \) as long as \( X \in \{\zeta_{\text{det}} \leq r\zeta\} \). Thus, the same probability lower bound holds conditioned on \( \{\zeta_{\text{det}} \leq r\zeta\} \).

Fact 10.4. For an event \( \mathcal{E} \) and random variable \( X \), \( \mathbb{P}(\mathcal{E}|X) \geq p \) for all \( X \in C \) implies that \( \mathbb{P}(\mathcal{E}|X \in C) \geq p \).

Proof of Lemma 9.6, item 1. Let

\[
\text{term} := \frac{1}{\alpha} \sum_{t}^{\alpha} \Psi \ell_t w_t' = \frac{1}{\alpha} \sum_{t}^{\alpha} \Psi \ell_t \ell_t' M_{1,t}' M_{2,t}'
\]

Since \( \Psi \) is a function of \( X \), since \( \ell_t \)’s used in the summation above are independent of \( X \) and \( \mathbb{E}[\ell_t, \ell_t'] = \Sigma \),

\[
\mathbb{E}[\text{term}|X] = \frac{1}{\alpha} \sum_{t}^{\alpha} \Psi \Sigma M_{1,t}' M_{2,t}'
\]

Next, we use Cauchy-Schwartz for matrices:

\[
\left\| \sum_{t=1}^{\alpha} X_t Y_t' \right\|^2 \leq \lambda_{\text{max}} \left( \sum_{t=1}^{\alpha} X_t X_t' \right) \lambda_{\text{max}} \left( \sum_{t=1}^{\alpha} Y_t Y_t' \right) \tag{14}
\]

Using (14), with \( X_t = \Psi \Sigma M_{1,t}' \) and \( Y_t = M_{2,t} \), followed by using \( \sqrt{\|\frac{1}{\alpha} \sum_{t} X_t X_t'\|} \leq \max_{\alpha} \|X_t\| \), Assumption 1.2 with \( A_t = I \), and Lemma 10.1

\[
\|\mathbb{E}[\text{term}|X]\| \leq \max_{t} \|\Psi \Sigma M_{1,t}'\| \sqrt{\frac{\beta}{\alpha}} \leq q((r\zeta)\lambda^+ + \lambda_{\text{cur}}^+) \sqrt{\frac{\beta}{\alpha}}
\]

for all \( X \in \{\zeta_{\text{det}} \leq r\zeta\} \). To bound \( \|\Psi \ell_t w_t'\| \), rewrite it as \( \Psi \ell_t w_t' = [\Psi G_{\text{det}} a_t, \Psi G_{\text{rest}} a_{t,r}] M_{1,t} M_{2,t} \). Thus, using \( \|M_{2,t}\| \leq 1, \|M_{1,t} P\| \leq q < 1 \), and Lemma 10.1

\[
\|\Psi \ell_t w_t'\| \leq q r \eta (r\zeta)\lambda^+ + \lambda_{\text{cur}}^+ + (r\zeta) \sqrt{\lambda^+ \lambda_{\text{cur}}^+} + \sqrt{\lambda^+ \lambda_{\text{cur}}^+}
\]

holds w.p. one when \( \{\zeta_{\text{det}} \leq r\zeta\} \).

Finally, conditioned on \( X \), the individual summands in term are conditionally independent. Using matrix Hoeffding, Corollary 10.3 followed by Fact 10.4 the result follows.

Proof of Lemma 9.6, item 2.

\[
\mathbb{E}[\frac{1}{\alpha} \sum_{t} w_t w_t'|X] = \frac{1}{\alpha} \sum_{t} M_{2,t} M_{1,t} \Sigma M_{1,t}' M_{2,t}'
\]

By Lemma 10.1 \( \|M_{1,t} \Sigma M_{1,t}'\| \leq q^2 \lambda^+ \). Thus, using Assumption 1.2 with \( A_t = M_{1,t} \Sigma M_{1,t}' \),

\[
\|\mathbb{E}[\frac{1}{\alpha} \sum_{t} w_t w_t'|X]\| \leq q^2 \lambda^+ \sqrt{\frac{\beta}{\alpha}}
\]

Using Assumption 1.2 and Lemma 10.1

\[
\|w_t w_t'| \leq \|M_{2,t} M_{1,t} P a_t\|^2 \leq q^2 \eta \lambda^+
\]

Conditional independence of the summands holds as before. Thus, using Corollary 10.3 and Fact 10.4 the result follows.
Proof of Lemma 9.6 item 3.

\[
\mathbb{E}\left[ \frac{1}{\alpha} \sum_{t} E_{\text{cur}} E_{\text{cur}}' \Psi \ell_t \ell_t' \Psi E_{\text{cur}, \perp} E_{\text{cur}, \perp}' \right] = E_{\text{cur}} E_{\text{cur}}' \Psi \Sigma \Psi E_{\text{cur}, \perp} E_{\text{cur}, \perp}'
\]

Using Lemma 10.1, \( \|E_{\text{cur}} E_{\text{cur}}' \Psi \Sigma \Psi E_{\text{cur}, \perp} E_{\text{cur}, \perp}' \| \leq (r_\zeta)^2 \lambda^+ + \frac{(r_\zeta)^2}{\sqrt{1 - (r_\zeta)^2}} \lambda_{\text{undet}} \) when \( \zeta_{\text{det}} \leq r_\zeta \). Also, \( \|E_{\text{cur}} E_{\text{cur}}' \Psi \ell_t \ell_t' \Psi E_{\text{cur}, \perp} \| \leq \|\Psi \ell_t \ell_t' \Psi \| \leq \eta r (r_\zeta)^2 \lambda^+ + r_\zeta^2 \lambda_{\text{cur}} \) := \( b_{\text{prob}} \) holds w.p. one when \( \zeta_{\text{det}} \leq r_\zeta \). In the above bound, the first inequality is used to get a loose bound, but one that will also apply for the proofs of the later items given below. The rest is the same as in the proofs of the earlier parts.

Proof of Lemma 9.6 item 4. Using Ostrowski’s theorem,

\[
\lambda_{\text{min}}(\mathbb{E}[A_{\perp} | X]) = \lambda_{\text{min}}(E_{\text{cur}}' \Psi \Sigma \Psi E_{\text{cur}})
\]

\[
= \lambda_{\text{min}}(E_{\text{cur}}' \Psi G_{\text{cur}} A_{\text{cur}} G_{\text{cur}}' \Psi E_{\text{cur}})
\]

\[
\geq \lambda_{\text{min}}(R_{\text{cur}} A_{\text{cur}} R_{\text{cur}}')
\]

\[
\geq \lambda_{\text{min}}(R_{\text{cur}} R_{\text{cur}}') \lambda_{\text{min}}(A_{\text{cur}}) \geq (1 - (r_\zeta)^2) \lambda_{\text{cur}}
\]

for all \( X \in \{ \zeta_{\text{det}} \leq r_\zeta \} \). Ostrowski’s theorem is used to get the second-last inequality, while Lemma 10.1 helps get the last one.

As in the proof of item 3, \( \|E_{\text{cur}} E_{\text{cur}}' \Psi \ell_t \ell_t' \Psi E_{\text{cur}, \perp} \| \leq \|\Psi \ell_t \ell_t' \Psi \| \leq b_{\text{prob}} \) holds w.p. one when \( \zeta_{\text{det}} \leq r_\zeta \). Conditional independence of the summands holds as before. Thus, by matrix Hoeffding, Corollary 10.2 the result follows.

Proof of Lemma 9.6 item 5. By Lemma 10.1,

\[
\lambda_{\text{max}}(\mathbb{E}[A_{\perp} | X]) = \lambda_{\text{max}}(E_{\text{cur}, \perp} E_{\text{cur}, \perp}' \Psi \Sigma \Psi E_{\text{cur}, \perp}')
\]

\[
\leq ((r_\zeta)^2 \lambda^+ + \lambda_{\text{undet}}^+)
\]

when \( \{ \zeta_{\text{det}} \leq r_\zeta \} \). The rest of the proof is the same as that of the previous part.

Proof of Lemma 9.6 item 6. Using Ostrowski’s theorem,

\[
\lambda_{\text{max}}(\mathbb{E}[A_{\perp} | X]) \geq \lambda_{\text{max}}(E_{\text{cur}, \perp}' \Psi G_{\text{undet}} A_{\text{undet}} G_{\text{undet}}' \Psi E_{\text{cur}, \perp}) \geq \lambda_{\text{min}}(E_{\text{cur}, \perp}' \Psi G_{\text{undet}} A_{\text{undet}} G_{\text{undet}}' \Psi E_{\text{cur}, \perp}' \Psi E_{\text{cur}, \perp}) \lambda_{\text{max}}(A_{\text{undet}}).
\]

By definition, \( \lambda_{\text{max}}(A_{\text{undet}}) = \lambda_{\text{undet}}^+ \).

By Lemma 10.1, \( \lambda_{\text{min}}(E_{\text{cur}, \perp}' \Psi G_{\text{undet}} A_{\text{undet}} G_{\text{undet}}' \Psi E_{\text{cur}, \perp}' \Psi E_{\text{cur}, \perp}) = \sigma_{\text{min}}(E_{\text{cur}, \perp}' \Psi G_{\text{undet}}' \Psi E_{\text{cur}, \perp}') \geq (1 - (r_\zeta)^2) \lambda_{\text{undet}}^+ \) when \( \zeta_{\text{det}} \leq r_\zeta \). The rest of the proof is the same as above.

Proof of Lemma 9.6 item 7. Using Ostrowski’s theorem and Lemma 10.1,

\[
\lambda_{\text{max}}(\mathbb{E}[A_{\perp} | X]) \geq \lambda_{\text{max}}(E_{\text{cur}}' \Psi G_{\text{cur}} A_{\text{cur}} G_{\text{cur}}' \Psi E_{\text{cur}}) \geq \lambda_{\text{min}}(R_{\text{cur}} R_{\text{cur}}') \lambda_{\text{max}}(A_{\text{cur}}) \geq (1 - (r_\zeta)^2) \lambda_{\text{cur}}^+
\]

when \( \zeta_{\text{det}} \leq r_\zeta \). The rest of the proof is the same as above.

11 Detailed Proof of Theorem 3.3 and Proof of Lemma 10.1

Proof of Theorem 3.3. Recall that we need to show that \( \zeta_k \leq r_k \zeta \). Assume the substitutions given in Definition 9.2. We will use induction.

Consider a \( k < \vartheta \). For the \( k \)-th step, assume that \( \zeta_i \leq r_i \zeta \) for \( i = 1, 2, \ldots, k - 1 \). Thus, using (9), \( \zeta_{\text{det}} \leq r_\zeta \) and so Lemma 9.8 is applicable. We first show that \( \hat{r}_k = r_k \) and that Algorithm 1 does not stop (proceeds to \((k + 1)\)-th step). From Algorithm 1, \( \hat{r}_k = r_k \) if \( \frac{1}{\lambda_{\lambda_k}} \leq \hat{g} \) and \( \frac{1}{\lambda_{\lambda_k + 1}} > \hat{g} \). Also it will not stop if \( \hat{\lambda}_{\lambda_k + 1} \geq \lambda_{\text{thresh}} \). Since \( k < \vartheta \), \( G_{\text{undet}} \) is not empty. Thus, item 1 of Lemma 9.8 shows that all these hold. Hence \( \hat{r}_k = r_k \) and algorithm does not stop w.p. at least \( 1 - p_1 - p_2 - p_3 \). Thus, by item 3 of the same lemma, with the same probability, \( \zeta_k \leq r_k \zeta \).

Now consider \( k = \vartheta \). We first show \( \hat{r}_k = r_k \) and that Algorithm 1 does stop, i.e., \( \vartheta = \vartheta \). This will be true if \( \frac{1}{\lambda_{\lambda_k}} \leq \hat{g} \) and \( \hat{\lambda}_{\lambda_k + 1} \leq \lambda_{\text{thresh}} \). For \( k = \vartheta \), \( G_{\text{undet}} \) is empty. Thus, item 2 of Lemma 9.8 shows that this holds w.p. at least \( 1 - p_1 - p_2 - p_3 \). Thus, by item 3 of the same lemma, with the same probability, \( \zeta_k \leq r_k \zeta \).
Thus, using the union bound, w.p. at least $1 - \vartheta(p_1 + p_2 + 4p_3)$, $\hat{r}_k = r_k$ and $\zeta_k \leq r_k\zeta$ for all $k$. Using (8), this implies that $\text{SE} \leq r\zeta$ with the same probability.

Finally, the choice $\alpha \geq \alpha_0$, implies that $p_1 \leq \frac{1}{2}2n^{-10}$, $p_2 \leq \frac{1}{2}2n^{-10}$, $p_3 \leq \frac{1}{2}2n^{-10}$. Hence $\text{SE} \leq r\zeta$ w.p. at least $1 - 12n^{-10}$. We work this out for $p_1$ below. The others follow similarly.

Recall that $p_1 = 2n \exp(-\alpha \frac{\epsilon^2}{32\epsilon_{\text{prob}}})$, $\epsilon = 0.01(r\zeta)\lambda^{-}$ and $b_{\text{prob}} = \eta rq((r\zeta)\lambda^{+} + \lambda_{\text{cur}}^{+} + (r\zeta)\sqrt{\lambda^{+}\lambda_{\text{cur}}^{+}} + \sqrt{\lambda^{+}\lambda_{\text{cur}}^{+}})$. Thus, $\frac{b_{\text{prob}}}{\lambda^{-}} \leq (4\eta r \max(q(r\zeta)f, gg, q\sqrt{sf}, q(r\zeta)\sqrt{sf}))^2 \leq 16n\eta^2 r^2 \max(q(r\zeta)f, gg, q\sqrt{sf})^2$.

Thus, $\alpha \frac{\epsilon^2}{32\epsilon_{\text{prob}}} \geq \frac{32}{0.01} \frac{\eta^2 n^2 (11 \log n + \log \vartheta)}{(r\zeta)^2} \max(q(r\zeta)f, gg, q\sqrt{sf}) \geq \frac{11 \log n + \log \vartheta}{0.01(r\zeta)^2}$. Thus, $p_1 \leq \frac{1}{2}2n^{-10}$.

**Proof of Lemma 10.1.** The first claim is obvious. The next two claims follow using the following lemma:

**Lemma 11.1 (3).** Suppose that $P$, $\hat{P}$ and $Q$ are three basis matrices. Also, $P$ and $\hat{P}$ are of the same size, $Q'P = 0$ and $\|Q'\hat{P}P\| = \zeta_*$. Then,

1. $\| (I - \hat{P}P')P\hat{P}' \| = \| (I - P\hat{P}')P\| = \| (I - \hat{P}\hat{P}')P\| = \zeta_*$
2. $\| PP' - \hat{P}\hat{P}' \| \leq 2\| (I - \hat{P}\hat{P}')P\| = 2\zeta_*$
3. $\| \hat{P}'Q \| \leq \zeta_*$
4. $\sqrt{1 - \zeta_*^2} \leq \sigma_1\left( (I - \hat{P}\hat{P}')Q \right) \leq 1$

Use item 4 of Lemma 11.1 and the fact that $G_{\text{det}}'G_{\text{cur}} = 0$ and $G_{\text{det}}'G_{\text{under}} = 0$ to get the second claim.

For the third claim, notice that $E_{\text{cur}}'\Psi'G_{\text{under}} = R_{\text{cur}}^{-1}G_{\text{cur}}'\Psi'G_{\text{under}} = R_{\text{cur}}^{-1}G_{\text{cur}}'\hat{G}_{\text{det}}'G_{\text{under}}$ since $\Psi^2 = \Psi$ and $G_{\text{cur}}'G_{\text{under}} = 0$. Using the second claim, $\| R_{\text{cur}}^{-1} \| \leq \frac{1}{\sigma_{\text{min}}(R_{\text{cur}})} \leq \frac{1}{1-(r\zeta)^2}$. Use item 3 of Lemma 11.1 and the facts that $G_{\text{cur}}'G_{\text{det}} = 0$ and $G_{\text{under}} = 0$ to bound $\| G_{\text{cur}}'\hat{G}_{\text{det}}', \| = \| \hat{G}_{\text{det}}'G_{\text{under}} \|$ respectively.

The fourth claim just uses the definitions. The fifth claim uses the previous claims and the assumptions on $M_1$ from Assumption 1.2. The sixth claim follows using Weyl’s inequality.

The second last claim: We show how to bound $\sigma_{\text{rest}}$: $\| \sigma_{\text{rest}} \| = \| a_{\text{rest}} \|^2 \leq \sum_{\eta \in \text{cur}} \eta_\lambda \lambda_\lambda + \sum_{\eta \in \text{under}} \eta_\lambda \lambda_\lambda \leq r\eta\lambda_{\text{cur}}^{+}$ (since $\lambda_j \leq \lambda_{\text{cur}}^{+}$ for all the $j$’s being summed over). The other bounds follow similarly.

**Last claim:**

$$\sigma_{\text{min}}(E_{\text{cur}}', \Psi'G_{\text{under}}) \frac{2}{\zeta^2}$$

$$= \lambda_{\text{min}}(G_{\text{under}}') E_{\text{cur}}', \Psi'G_{\text{under}}')$$

$$= \lambda_{\text{min}}(G_{\text{under}}' \Psi' \Psi G_{\text{under}}')$$

$$\geq \lambda_{\text{min}}(G_{\text{under}}' \Psi' G_{\text{under}}')$$

$$\lambda_{\text{max}}(G_{\text{under}}' \Psi' E_{\text{cur}}' G_{\text{under}}')$$

$$\geq 1 - (r\zeta)^2 - \frac{1}{\sqrt{1 - (r\zeta)^2}}.$$

The last inequality follows using the second and the third claim.