PRESSURE IN CHERN-SIMONS FIELD THEORY

TO THREE-LOOP ORDER

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Abstract

We calculate perturbatively the pressure of a dilute gas of anyons through second order in the anyon coupling constant, as described by Chern-Simons field theory. Near Bose statistics, the divergences in the perturbative expansion are exactly cancelled by a two-body δ-function potential which is not required near Fermi statistics. To the order considered, we find no need for a non-hermitian Hamiltonian.
This letter is about a two-dimensional non relativistic system of N particles in the thermodynamic limit, interacting with a Chern-Simons gauge field, and interacting among themselves via a two-body $\delta$-function potential. This system has been shown to obey fractional statistics [1]. From a quantum mechanical point of view, a system of anyons can be studied by introducing a flux tube at the position of each particle, whose magnitude $\theta$ will determine the phase associated with relative particle motion. The value $\theta = 0$ corresponds to the anyons being bosons, whereas $\theta = \pi$ corresponds to fermions. In the dilute gas regime, when the mean interparticle spacing $n^{-1/2}$ is much larger than the thermal wavelength, $\lambda T = (2\pi/mT)^{1/2}$, the first correction to the pressure comes from the reduced two-body hamiltonian. This can be exactly solved in terms of the Aharonov-Böhm phase shifts [2]. Since the full spectrum of the N-anyon system is not known, a computation of the pressure based on N-cluster expansion [3] is not possible.

Nevertheless, full knowledge of the spectrum is not necessary when the anyon model is considered from the point of view of quantum field theory. One needs the thermodynamical potential $\Omega(\mu, T, V) = -PV$ as a power series in the fugacity $z = \exp(\mu/T)$, which can be computed via perturbation theory. This procedure is reliable when the field-theoretical model accounts for the fractional statistics of anyons. Thus, the question one would like to ask is this: which field-theoretical model can be used to reliably compute higher virial coefficients (perturbatively in $\theta$)?

This question has been considered in a series of papers [4-6], where a two-body non-hermitian interaction is introduced in order to handle the divergences due to the statistical gauge field. As a consequence, the perturbative treatment must be carefully done, using either a harmonic regulator, or considering the system in a box. The source of the non-hermitian interaction arises as follows: In the regular gauge where the statistics is standard, the anyonic gas is described by a Hamiltonian where three-body interactions of order $\theta^2$ are present. These lead to divergences in the perturbative expansion. However, the three-body interactions can be removed from the Hamiltonian by a complex gauge transformation [5], leaving a new two-body Hamiltonian with a non-hermitian term of order $|\theta|$. Then the gauge transformation might be understood as performing a self-adjoint extension [9] on the original Hamiltonian, representing an additional interaction at order $\theta$, namely a two-body $\delta$-function.
Another viewpoint is afforded if one consider the anyonic gas as described by a Chern-Simons gauge field coupled to bosonic or fermionic matter. In this case, the singular three-body interactions are kept and they correspond to the last vertex in Table 1. Even so a perturbative expansion in $\theta$ will be reliable if divergences cancel out due a two-body $\delta$-function. It is not a priori clear whether both approaches must lead to the same results.

The purpose of this letter is to calculate the contributions to the pressure to order $\theta^2$ in Chern-Simons field theory. This involves a set of three-loop diagrams, whose computation to second order in the fugacity is our main new result. Contributions of order $e^{3\mu/T}$ are neglected and will be reported in future work. An obvious prerequisite for the validity of such computation is the reproduction of the second virial coefficient to order $\theta^2$. We show that Chern-Simons field theory with the inclusion of a repulsive $\delta$-function potential among particles is a model which exactly reproduces the correct pressure when the fiducial statistics is bosonic. The strength of this interaction must be adjusted to cancel UV divergences. No additional interaction is required when the fiducial statistics is fermionic. Therefore, regardless of whether one works with the non-hermitian Hamiltonian of refs. [4-6] or with Chern-Simons field theory, the result for the second virial coefficient is the same.

We shall consider a spinless non relativistic self-coupled field interacting with a Chern-Simons gauge field, with Lagrangian density given by

$$\mathcal{L}(t, x) = -\frac{1}{2\kappa} \partial_t a \times a + \frac{1}{\kappa} a_0 B - \frac{1}{2\rho} (\nabla \cdot a)^2 + \psi^\dagger iD_0 \psi - \frac{1}{2m} |D \psi|^2 + \mu \psi^\dagger \psi - \frac{\alpha}{4} (\psi^\dagger \psi)^2,$$

$$D_0 = \partial_t + ia_0,$$  
$$D = \nabla - i a \tag{1}$$

Here $B = \nabla \times a$ is the magnetic field, $\mu$ is the chemical potential and $\rho$ is a gauge fixing parameter. The Coulomb gauge to be used refers to the choice $\rho = 0$. Finally, $\alpha = O(\kappa)$ is the strength of the two-body contact interaction of the form

$$V(x_1 - x_2) = \frac{\alpha}{2} \delta(x_1 - x_2). \tag{2}$$

When the starting statistics is fermionic we will set $\alpha = 0$.

In the imaginary-time formalism of thermal field theory [7], the functional integral expression for the grand partition function involves an integration over imaginary time from $0$ to $\beta = T^{-1}$,

$$Z(\mu, T, V) = \int_{\text{antiperiodic}} D a_0 D a D \psi^\dagger D \psi \exp \left( \int_0^\beta d\tau \int d^2x \mathcal{L}(t = -i\tau, x) \right). \tag{3}$$
Here (anti)periodic means that the integration over fields is constrained so that \( \psi(x, \beta) = \pm \psi(x, 0) \) where the (lower) upper sign refers to (fermions) bosons.

Now we can proceed to expand in a power series in \( \kappa \), by using a set of diagrammatic rules which follows from (1). These are listed in Table 1. The diagrams describing the perturbative series for \( \ln Z \) have the form of connected closed loops. It should be noted that there will be a factor of \( \beta V \) left over for each graph, corresponding to the extensivity of \( \ln Z \). This factor cancel out in expressions for the pressure.

The non-zero graphs contributing to the pressure up to order \( \kappa^2 \) are shown in fig.1. Note that all graphs of order \( \kappa \) vanish due to the index summations associated with vertices. The same argument applies to graphs of order \( \alpha \kappa \), although at higher orders this is not necessarily so. In order to keep the correct order of the operators according to their \( \tau \)-values, one must insert a factor \( e^{i\omega_n \eta} \) whenever a particle line either closes on itself or is joined by the same instantaneous interaction line. We take \( \eta \to 0^+ \) at the end of the calculation. As a consequence, terms involving three bubbles in \( P^{(2)}_\alpha \) and in \( P^{(2)}_\kappa \) are of order \( e^{3\mu \beta} \). This follows simply from power counting with the formulae

\[
\frac{1}{\beta} \sum_n \frac{e^{i\omega_n \eta}}{\omega_n + \mu - q^2/(2m)} = -\zeta \frac{1}{\exp \left[ \beta \left( \frac{q^2}{2m} - \mu \right) \right] - \zeta} = -\zeta n_\zeta(q),
\]

(4)

\[
\frac{1}{\beta} \sum_n \frac{1}{i\omega_n + \mu - q^2/(2m)} = -\frac{1}{2} - \zeta n_\zeta(q),
\]

(5)

where \( \zeta = 1 \) for bosons and \( \zeta = -1 \) for fermions. Therefore, graphs (c) and (g) can be neglected, since we are computing to order \( e^{2\mu \beta} \).

We are now in position to derive the perturbative corrections to the pressure

\[
P = \frac{T}{V} \ln Z.
\]

(6)

Graph (a) gives

\[
P^{(1)}_\alpha = -\frac{\alpha}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} n_1(p) n_1(q) = -\frac{\alpha e^{2\beta \mu}}{2\lambda^4_T} + O(e^{3\beta \mu}),
\]

(7)

and there are not first order correction with fermionic statistics.

The contribution from graph (b) may be written

\[
P^{(2)}_\alpha = \frac{\alpha^2}{8\beta} \sum_n \int \frac{d^2 q}{(2\pi)^2} \left[ \Pi^0_{00}(\nu_n, q) \right]^2,
\]

(8)
\[ \Pi^0_{00}(\nu_n, q) = -\frac{1}{\beta} \sum_{\omega_1} \int \frac{d^2p}{(2\pi)^2} \mathcal{G}^0(\omega_1, p) \mathcal{G}^0(\omega_1 + \nu_n, q + p) \]

\[
= -\int \frac{d^2p}{(2\pi)^2} \frac{n_\zeta(p + q) - n_\zeta(p)}{i\nu_n - \varepsilon_{p+q} + \varepsilon_p}.
\]

represents the lowest-order density-density correlator (see fig. 2). At high temperature and low density, distribution functions reduce to

\[ n_\zeta(p) = e^{\beta \mu} e^{-\beta p^2/2m}. \]

Then, keeping terms to second order in the fugacity, eq.(8) gives

\[ P^{(2)}_\alpha = \frac{\alpha^2 m e^{2\beta \mu}}{8\pi \lambda^4_\delta} \int_0^x dx \Phi(x), \]

where \( x^* = \sqrt{\frac{\beta}{4m}} q^* \) is a cutoff and \( \Phi(x) \) is the plasma dispersion function [8],

\[ \Phi(x) = 2 \int_0^x dt \frac{e^{-t^2}}{\sqrt{x^2 - t^2}} = 2e^{-x^2} \int_0^x dt e^{t^2}. \]

\( \Phi(x) \) has the following limiting behavior

\[ \Phi(x) = 2x + O(x^3) \quad x \ll 1, \]

\[ \Phi(x) = x^{-1} + O(x^{-3}) \quad x \gg 1. \]

We have found a UV logarithmic divergence. This corresponds to the divergence in the second Born approximation for the scattering amplitude with a \( \delta \)-function potential [9].

Now we consider the ring contribution from graph (d) of fig. 1,

\[ P^d_n = \frac{k^2}{2\beta} \sum_{\nu_n} \int \frac{d^2q}{(2\pi)^2} \frac{\Pi^0(\nu_n, q) \Xi^0(\nu_n, q)}{q^2}, \]

where \( \Xi^0(\nu_n, q) \) is the two-dimensional transverse component of the lowest-order current-current correlation given by (see fig.2)

\[ \Pi^0_{ij}(\nu_n, q) = -\frac{\zeta}{m^2} \frac{1}{\beta} \sum_{\omega_1} \int \frac{d^2p}{(2\pi)^2} \left[ (p + q)^i (p + q)^j \mathcal{G}^0(\omega_1, p) \mathcal{G}^0(\omega_1 + \nu_n, q + p) \right. \]

\[ + m \delta_{ij} e^{i\omega_1 \eta} \mathcal{G}^0(\omega_1, p) \left] \right. \]

\[ = -\Pi^0_{00}(\nu_n, q) \frac{\nu_n q_i q_j}{q^2} + \Xi^0(\nu_n, q) (\delta_{ij} - \frac{q_i q_j}{q^2}). \]
In the classical limit, when \( n_\zeta(p) = e^{\beta\mu} e^{-\beta p^2/2m} \),
\[
\Xi^0(\nu_n, q) = \frac{1}{m\beta} \left( \Pi^0_{00}(\nu_n, q) - \frac{\beta e^{\beta\mu}}{\lambda_T^4} \right) + O(e^{2\beta\mu}),
\]
and eq. (13) gives
\[
P^d_\kappa = \frac{\kappa^2 e^{2\beta\mu}}{8\pi m \lambda_T^4} \int_0^x dx \left[ \frac{\Phi(x)}{x^2} - \frac{2}{x} \right].
\]
Again a UV logarithmic divergence appears coming from the ring loop with a seagull vertex.

We still have to compute the exchange graphs (e) and (f) of fig. 1. These give
\[
P^e_\kappa = -\zeta \frac{\kappa^2}{4m^2 \beta^3} \sum_{\nu \omega_1 \omega_2} \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \mathcal{G}^0(\omega_1, p) \mathcal{G}^0(\omega_1 + \nu, q + p) \times \mathcal{G}^0(\omega_2 - \nu, k - q) \mathcal{G}^0(\omega_2, k) \frac{[\mathbf{p} - \mathbf{k} + \mathbf{q}] \mathbf{q}}{[\mathbf{p} - \mathbf{k} + \mathbf{q}]^2 \mathbf{q}^2}, \tag{17}
\]
\[
P^f_\kappa = -\zeta \frac{\kappa^2}{m \beta^3} \sum_{\nu \omega_1 \omega_2} \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} e^{i\omega_1 \eta} \mathcal{G}^0(\omega_1, p) e^{i\omega_2 \eta} \mathcal{G}^0(\omega_2, k) \times \mathcal{G}^0(\omega_1 + \nu, p + q) \frac{[\mathbf{p} - \mathbf{k} + \mathbf{q}] \cdot \mathbf{q}}{[\mathbf{p} - \mathbf{k} + \mathbf{q}]^2 \mathbf{q}^2} \tag{18}
\]
A detailed evaluation of these terms to order \( e^{2\beta\mu} \) yields the divergent contribution
\[
P^e_\kappa + P^f_\kappa = -\zeta \frac{\kappa^2 e^{2\beta\mu}}{4\pi m \lambda_T^4} \int_0^x dx \Phi(x), \tag{19}
\]
where the divergence comes again from the graph (f) with a seagull vertex.

In the fermionic case, the total contribution from the gauge field is finite by itself, giving
\[
P^{(2)}_\kappa = P^d_\kappa + P^e_\kappa + P^f_\kappa
= \frac{\kappa^2 e^{2\beta\mu}}{8\pi m \lambda_T^4} \int_0^\infty dx \left[ \frac{\Phi(x)}{x^2} - \frac{2}{x} + 2 \Phi(x) \right]
= \frac{\kappa^2 e^{2\beta\mu}}{8\pi m \lambda_T^4} \int_0^\infty dx \frac{d}{dx} \left[ \frac{\Phi(x)}{x} \right]
= \frac{\kappa^2 e^{2\beta\mu}}{4\pi m \lambda_T^4}. \tag{20}
\]
In the bosonic case, the divergences in the total contribution to the pressure to order $\kappa^2$ cancel out only if $\alpha = \pm 2\kappa/m$, and we find

\[
P_{\kappa}^{(2)} = P_{\kappa}^d + P_{\kappa}^e + P_{\kappa}^f + P_{\alpha}^{(2)} = \frac{\kappa^2 e^{2\beta \mu}}{8\pi m \lambda_T^4} \int_0^\infty dx \left[ \frac{\Phi(x)}{x^2} - \frac{2}{x} - 2\Phi(x) + 4\Phi(x) \right] = \frac{\kappa^2 e^{2\beta \mu}}{4\pi m \lambda_T^4},
\]

which is the same result as in fermionic case.

Finally, using the pressure for two-dimensional ideal quantum gases

\[
P^{(0)}(\mu, T) = \zeta T \lambda_T^{-2} \text{Li}_2(\zeta e^{\mu/T}),
\]

where $\text{Li}_2$ denotes the dilogarithm function, we obtain the cluster expansion

\[
\frac{P(\mu, T)}{T} = \frac{1}{\lambda_T^2} \sum_{l=1}^\infty b_l z^l = T^{-1} \left( P^{(0)}(\mu, T) + P_{\alpha}^{(1)} + P_{\kappa}^{(2)} + O(\kappa^3) \right)
\]

\[
= \frac{z}{\lambda_T^2} + \frac{z^2}{\lambda_T^2} \left[ \frac{\zeta}{4} + \frac{(1+\zeta)\kappa}{4\pi} + \frac{\kappa^2}{8\pi^2} \right] + O(z^3).
\]

When the fiducial statistics is bosonic this formula reproduces the correct second virial coefficient by taking the upper sign corresponding to a repulsive contact interaction of strength $\alpha = 2\kappa/m$ and putting $\kappa = 2\theta$. The other possible choice, $\alpha = -2\kappa/m$, corresponds to the nonrelativistic model admitting classical self-dual solutions [10]. When the fiducial statistics is fermionic, the suitable choice is $\kappa = 2(\theta - \pi)$, where $\theta$ is the statistical parameter in both cases.

To conclude, we have shown that standard perturbative Chern-Simons field theory coupled to nonrelativistic matter exactly reproduces the second virial coefficient. Through second order in $\theta$, all divergences cancel, and there is no need for renormalization or resummation. A non-hermitian Hamiltonian is not required to account for the statistical interaction between anyons.

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Table 1. *Bare propagators and vertices*

\[ \mathcal{G}^0(\omega_n, p) = \frac{1}{i\omega_n + \mu - \frac{p^2}{2m}} \]

\[ \omega_n = \frac{2n\pi}{\beta} \quad \text{in boson propagator} \]

\[ \omega_n = \frac{(2n + 1)\pi}{\beta} \quad \text{in fermion propagator} \]

\[ \mathcal{D}_{0j}(q) = \frac{i\kappa \epsilon_{jm} q_m}{q^2} \]

\[ \mathcal{D}_{00}(q) = \mathcal{D}_{ij}(q) = 0 \quad \text{in the Coulomb gauge, } \rho = 0 \]

\[ \Gamma^\alpha = -\alpha \]

\[ \Gamma_0 = -1 \]

\[ \Gamma_j(p, k) = \frac{p_j + k_j}{2m} \]

\[ \Delta_{ij} = \frac{1}{m} \delta_{ij} \]
Figure Captions

Fig. 1. Non zero diagrams contributing to the pressure to the second order in $\kappa$. The sign $\pm$ refers to Bose or Fermi propagators. Graphs (c) and (g) contribute to the third virial coefficient. Combinatoric factors are shown in the diagram.

Fig. 2. The self-energy of the statistical gauge field at the one-loop level in terms of the functional derivative of the (zero) pressure to lowest order. 1PI means that only the one-particle irreducible diagrams contribute to $\Pi^0$. 