FINITISTIC TEST IDEALS ON NUMERICALLY $\mathbb{Q}$-GORENSTEIN VARIETIES

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Dedicated to Professor Craig Huneke on the occasion of his sixty-fifth birthday.

Abstract. We prove that the finitistic test ideal $\tau_{fg}(R, \Delta, a^t)$ coincides with the big test ideal $\tau_b(R, \Delta, a^t)$ if the pair $(R, \Delta)$ is numerically log $\mathbb{Q}$-Gorenstein.

1. Introduction

Let $R$ be an excellent Noetherian domain of characteristic $p > 0$. The notion of test ideals has its origin in the theory of tight closure, which was introduced about 30 years ago by Hochster and Huneke [16]. Tight closure is a closure operation performed on ideals in $R$, or more generally, on inclusions of $R$-modules. The test ideal defined in [16] is generated by test elements, or those elements which can be used to test the tight closure membership for all inclusions of finitely generated $R$-modules. This is the etymology of the name “test ideal”. This test ideal is nowadays called the finitistic test ideal of $R$ and denoted by $\tau_{fg}(R)$.

Later, another test ideal, called the big test ideal of $R$ and denoted by $\tau_b(R)$, was introduced. It is generated by big test elements, or those elements used to test the tight closure membership for all inclusions of (not necessarily finitely generated) $R$-modules. The big test ideal fits better into the theory of Frobenius splittings and can be characterized without using tight closure. Indeed, Schwede [22] proved that (if $R$ is $F$-finite) $\tau_b(R)$ is the unique smallest nonzero ideal $J \subseteq R$ such that $\varphi(J^{1/p^e}) \subseteq J$ for all $e \in \mathbb{N}$ and all $\varphi \in \text{Hom}_R(R^{1/p^e}, R)$. The big test ideal quickly began finding applications in its own right. Now this notion is becoming a fundamental tool in birational geometry and singularity theory in positive characteristic.

It is clear from definition that $\tau_b(R)$ is contained in $\tau_{fg}(R)$, and they are conjectured to be equal.

Conjecture 1.1 ([24, Conjecture 5.14]). $\tau_b(R) = \tau_{fg}(R)$.

Conjecture 1.1 is considered to be one of the most important conjectures in tight closure theory. When $R$ is normal and $\mathbb{Q}$-Gorenstein, Conjecture 1.1 is known to hold (see [2], [3], [11], [14], [25], [26]). What if $R$ is normal but not $\mathbb{Q}$-Gorenstein? Various results on $\tau_b(R)$ in the $\mathbb{Q}$-Gorenstein setting were generalized in [7] to the case where the anti-canonical ring $\bigoplus_{i \geq 0} \mathcal{O}_X(-iK_X)$ of $X = \text{Spec} R$ (that is, the symbolic Rees...
algebra of the anti-canonical ideal of $R$) is finitely generated. In particular, they gave an affirmative answer to Conjecture 1.1 in this case.

In this paper, as another generalization of $\mathbb{Q}$-Gorensteinness, we consider the condition of being numerically $\mathbb{Q}$-Gorenstein. The notion of numerically $\mathbb{Q}$-Gorenstein varieties was introduced in [5], [6] when the base field is of characteristic zero, and is generalized to arbitrary characteristic in [8] by making use of regular alterations instead of resolutions of singularities. Being numerically $\mathbb{Q}$-Gorenstein is a much weaker condition than being $\mathbb{Q}$-Gorenstein. For example, every normal surface is numerically $\mathbb{Q}$-Gorenstein. We also note that this condition is somewhat orthogonal to the finite generalization of the anti-canonical ring, because if $R$ satisfies both conditions, then it has to be $\mathbb{Q}$-Gorenstein (see Lemma 3.5).

We give an affirmative answer to Conjecture 1.1 when $R$ is numerically $\mathbb{Q}$-Gorenstein. More generally, we can prove the equality for a generalization of test ideals. Inspired by the theory of multiplier ideals, Hara-Yoshida [14] and Takagi [26] generalized the big and finitistic test ideals $\tau_b(R), \tau_{fg}(R)$ to the ideals $\tau_b(R, \Delta, a^t), \tau_{fg}(R, \Delta, a^t)$ associated to a triple $(R, \Delta, a^t)$, where $\Delta$ is an effective $\mathbb{Q}$-Weil divisor on $\text{Spec } R$, $a \subset R$ is an ideal, and $t > 0$ is a real number. Our main result is stated for the ideals $\tau_b(R, \Delta, a^t)$ and $\tau_{fg}(R, \Delta, a^t)$.

**Main Theorem** (Theorem 4.4, Corollary 4.6). Let $(R, \Delta, a^t)$ be a triple where $R$ is an $F$-finite normal domain and set $X = \text{Spec } R$. Suppose that one of the following conditions is satisfied:

(a) $R$ is essentially of finite type over a field and $K_X + \Delta$ is numerically $\mathbb{Q}$-Cartier, or

(b) $R$ is of dimension two.

Then $\tau_b(R, \Delta, a^t) = \tau_{fg}(R, \Delta, a^t)$.

For the proof, we employ the strategy of MacCrimmon [21] and Williams [27]. The assertion follows from valuative characterizations of tight closure and numerically $\mathbb{Q}$-Cartier divisors (Proposition 4.1 and Lemma 3.7).

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**Notation.** Throughout this paper all rings are commutative rings with unity. Given a ring $R$, we denote by $R^\circ$ the set of elements of $R$ which are not in any minimal prime ideal. Suppose that $R$ is a normal domain. Given a $\mathbb{Q}$-Weil divisor $\Delta = \sum_i d_i D_i$ on $X = \text{Spec } R$, the round up (resp. the round down) of $\Delta$ is $\lceil \Delta \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. $\lfloor \Delta \rfloor = \sum_i \lfloor d_i \rfloor D_i$) where $\lceil d_i \rceil$ denotes the smallest integer greater than or equal to $d_i$ (resp. $\lfloor d_i \rfloor$ denotes the largest integer less than or equal to $d_i$). For an integral Weil divisor $D$ on $X$, we will use the notation $R(D)$ to denote $H^0(X, \mathcal{O}_X(D))$.

All schemes are assumed to be Noetherian and separated. A variety over a field $k$ means an integral scheme of finite type over $k$. 
2. A quick review on test ideals

In this section, we briefly review the definitions and basic properties of tight closure and test ideals.

Let \( R \) be a normal domain of characteristic \( p > 0 \). Given an ideal \( I \) of \( R \) and an integer \( e \in \mathbb{N} \), the \( e \)-th Frobenius power \( I^{[p^e]} \) of \( I \) is the ideal of \( R \) generated by all \( p^e \)-th powers of elements of \( I \).

Fix an algebraic closure \( L \) of the quotient field \( \text{Frac}(R) \) of \( R \). Given a fractional ideal \( J \) of \( R \) and an integer \( e \in \mathbb{N} \), set \( J^{1/p^e} = \{ y \in L \mid y^{p^e} \in J \} \) and denote an element \( y \in J^{1/p^e} \) by \( x^{1/p^e} \) where \( x = y^{p^e} \in J \). We say that \( R \) is \( F \)-finite if \( R^{1/p} \) is a finitely generated \( R \)-module. By definition, a field \( K \) is \( F \)-finite if and only if \( [K : K^p] < \infty \). More generally, if \( R \) is a complete local ring with \( F \)-finite residue field or is essentially of finite type over an \( F \)-finite field, then it is \( F \)-finite.

**Definition 2.1.** In this paper, we say that \((R, \Delta, \mathfrak{a}^t)\) is a **triple** in characteristic \( p > 0 \) if \( R \) is an excellent normal domain of characteristic \( p > 0 \), \( \Delta \) is an effective \( \mathbb{Q} \)-Weil divisor on \( \text{Spec} R \), \( \mathfrak{a} \) is a nonzero ideal of \( R \) and \( t \geq 0 \) is a real number. We say that the triple \((R, \Delta, \mathfrak{a}^t)\) is \( F \)-finite if so is \( R \).

**Definition 2.2** ([3, Definition 3.1]). Let \((R, \Delta, \mathfrak{a}^t)\) be a triple in characteristic \( p > 0 \).

(i) Let \( I \) be an ideal in \( R \). An element \( x \in R \) is said to be in the \((\Delta, \mathfrak{a}^t)\)-tight closure \( I^{* (\Delta, \mathfrak{a}^t)} \) of \( I \) if there exists \( c \in R^\circ \) such that

\[
ca^{[t(q-1)]}x^q \subset I^{[q]}R((q-1)\Delta)
\]

for all large \( q = p^e \). When \( a = R \) (resp. \( \Delta = 0 \) and \( a = R \)), we simply denote the ideal \( I^{* (\Delta, \mathfrak{a}^t)} \) by \( I^* \Delta \) (resp. \( I^* \)).

(ii) Let \( M \) be an \( R \)-module. An element \( z \in M \) is said to be in the \((\Delta, \mathfrak{a}^t)\)-tight closure \( 0^{* (\Delta, \mathfrak{a}^t)}_M \) of the zero submodule in \( M \) if there exists \( c \in R^\circ \) such that

\[
(c\mathfrak{a}^{[t(q-1)]})^{1/q} \otimes z = 0 \quad \text{in} \quad R([q-1]\Delta)^{1/q} \otimes_R M
\]

for all large \( q = p^e \). We say that \( z \) is in the **finitistic tight closure** \( 0^{* (\Delta, \mathfrak{a}^t)}_M \) of the zero submodule in \( M \) if \( M \) exists a finitely generated \( R \)-submodule \( N \subset M \) such that \( z \in 0^{* (\Delta, \mathfrak{a}^t)}_N \). In other words,

\[
0^{* (\Delta, \mathfrak{a}^t)}_M = \bigcup_{N \subset M} 0^{* (\Delta, \mathfrak{a}^t)}_N
\]

where \( N \) runs through all finitely generated \( R \)-submodules of \( M \).

(iii) We say that an element \( c \in R^\circ \) is a **big sharp test element** for \((R, \Delta, \mathfrak{a}^t)\) if for all \( R \)-modules \( M \) and all \( z \in 0^{* (\Delta, \mathfrak{a}^t)}_M \), we have

\[
(c\mathfrak{a}^{[t(q-1)]})^{1/q} \otimes z = 0 \quad \text{in} \quad R([q-1]\Delta)^{1/q} \otimes_R M
\]

for every \( q = p^e \). When \( \Delta = 0 \) and \( a = R \), we refer to such elements as big test elements for \( R \).

**Remark 2.3.** Definition 2.2 (i), (ii) do not change even if we replace \( \mathfrak{a}^{[t(q-1)]} \) (resp. \( R([q-1]\Delta]) \) by \( \mathfrak{a}^{[tq]} \) (resp. \( R([q\Delta]) \)). However, Definition 2.2 (iii) does change if we make such a replacement.
Big sharp test elements always exist if the ring is \( F \)-finite.

**Lemma 2.4** ([22, Lemma 2.17]). Let \((R, \Delta, a^t)\) be an \( F \)-finite triple. If we choose an element \( c \in R^2 \) such that the localization \( R_c \) is regular, \( \div(c) \geq \Delta \) and \( aR_c = R_c \), then some power \( c^\alpha \) of \( c \) is a big sharp test element for \((R, \Delta, a^t)\).

Now we introduce test ideals. There are two kinds of test ideals, finitistic test ideals and big test ideals.

**Definition 2.5** ([3, Definition-Propositions 3.2 and 3.3], cf. [16]). Let \((R, \Delta, a^t)\) be a triple in characteristic \( p > 0 \) and \( E = \bigoplus_m E_R(R/m) \) be the direct sum, taken over all maximal ideals \( m \) of \( R \), of the injective hulls \( E_R(R/m) \) of the residue fields \( R/m \).

(i) The **finitistic test ideal** \( \tau_{fg}(R, \Delta, a^t) \) associated to \((R, \Delta, a^t)\) is defined by

\[
\tau_{fg}(R, \Delta, a^t) = \Ann_R(0^e_{E}(\Delta, a^t)) = \bigcap_M \Ann_R(0^e_{M}(\Delta, a^t)),
\]

where \( M \) runs through all finitely generated \( R \)-submodules of \( E \). When \( a = R \) (resp. \( \Delta = 0 \) and \( a = R \)), we denote it by \( \tau_{fg}(R, \Delta) \) (resp. \( \tau_{fg}(R) \)).

(ii) The **big test ideal** \( \tau_b(R, \Delta, a^t) \) associated to \((R, \Delta, a^t)\) is defined by

\[
\tau_b(R, \Delta, a^t) = \Ann_R(0^e_{E}(\Delta, a^t)).
\]

When \( a = R \) (resp. \( \Delta = 0 \) and \( a = R \)), we denote it by \( \tau_b(R, \Delta) \) (resp. \( \tau_b(R) \)).

(iii) Suppose in addition that \( R \) is \( F \)-finite. The ring \( R \) is said to be **strongly \( F \)-regular** if \( \tau_b(R) = R \).

**Remark 2.6.** The finitistic test ideal was introduced by Hochster-Huneke [16] about 30 years ago, but the notations used for test ideals vary in the literature. The notation \( \tau(R) \) was used for the finitistic test ideal \( \tau_{fg}(R) \) originally, but nowadays is more often used for the big test ideal \( \tau_b(R) \), which is sometimes denoted by \( \tilde{\tau}(R) \) in the literature.

**Lemma 2.7** (cf. [16, Proposition 8.23 (c)]). Let \((R, \Delta, a^t)\) be a triple where \((R, m)\) is an \( F \)-finite local ring of characteristic \( p > 0 \). Let \( \widehat{R} \) denote the \( m \)-adic completion of \( R \) and \( \widehat{\Delta} \) denote the pullback of \( \Delta \) to \( \text{Spec} \, \widehat{R} \). Then

\[
\tau_{fg}(R, \Delta, a^t) = \tau_{fg}(\widehat{R}, \widehat{\Delta}, (a\widehat{R})^t) \cap R.
\]

**Proof.** Since excellent reduced local rings are approximately Gorenstein by [15], there exists a sequence \( \{I_t\} \) of \( m \)-primary irreducible ideals in \( R \) cofinal with the powers of \( m \). Then by the same argument as the proof of [16, Proposition 8.23 (f)], one has

\[
\tau_{fg}(R, \Delta, a^t) = \bigcap_t (I_t \colon_R I_t^{*}(\Delta, a^t)),
\]

\[
\tau_{fg}(\widehat{R}, \widehat{\Delta}, (a\widehat{R})^t) = \bigcap_t (I_t \widehat{R} \colon_{\widehat{R}} (I_t \widehat{R})^{*}(\widehat{\Delta}, (a\widehat{R})^t)).
\]

Thus, it suffices to prove that \( I_t^{*}(\Delta, a^t) \widehat{R} = (I_t \widehat{R})^{*}(\widehat{\Delta}, (a\widehat{R})^t) \). Since \( (I_t \widehat{R})^{*}(\widehat{\Delta}, (a\widehat{R})^t) \) is an \( m\widehat{R} \)-primary ideal or the unit ideal of \( \widehat{R} \), there exists an ideal \( J \) in \( R \) such that \( J \widehat{R} = (I_t \widehat{R})^{*}(\widehat{\Delta}, (a\widehat{R})^t) \). It is enough to show that \( I_t^{*}(\Delta, a^t) = J \), that is, an element
\( x \in R \) lies in \( I_t^{(\Delta, a_t)} \) if and only if \( x \in (I_t R)^{t(\Delta, (a_t))} \). Let \( c \in R^* \) be a big sharp test element both for \((R, \Delta, a_t)\) and for \((\hat{R}, \hat{\Delta}, (a\hat{R})^t)\) (Lemma 2.4 produces such an element). By definition, \( x \in I_t^{(\Delta, a_t)} \) if and only if \( cx^q a^{t(\Delta - 1)} \subset I_t^{[q(R(\{(q - 1)\Delta\})]} \) for all \( q = p^e \). However, since \( R \hookrightarrow \hat{R} \) is faithfully flat, this is equivalent to saying that \( cx^q (\hat{a\hat{R}})^{t(\Delta - 1)} \subset I_t^{[q(\hat{R}(\{(q - 1)\hat{\Delta}\})]} \) for all \( q = p^e \), which implies that \( x \in (I_t \hat{R})^{t(\hat{\Delta}, (a\hat{R})^t)} \).

**Remark 2.8.** (1) When \( \Delta = 0 \) and \( a = R \), it follows from [16, Proposition 8.23 (e)] that Lemma 2.7 holds for arbitrary excellent reduced local rings of characteristic \( p > 0 \), not necessarily \( F \)-finite.

(2) ([3, Remark 3.6], [12, Proposition 3.2]) Suppose that we are in the setting of Lemma 2.7. In the case of big test ideals, a similar but stronger statement holds:

\[
\tau_b(R, \Delta, a_t) R = \tau_b(\hat{R}, \hat{\Delta}, (a\hat{R})^t).
\]

**3. Numerically \( \mathbb{Q} \)-Cartier divisors**

In this section, we recall the notion of numerically \( \mathbb{Q} \)-Cartier divisors, introduced in [5] and [6], which is a natural generalization of \( \mathbb{Q} \)-Cartier divisors.

First we note that the negativity lemma holds for normal varieties over any field.

**Lemma 3.1** (Negativity lemma). Let \( h : Z \rightarrow Y \) be a birational morphism between normal varieties over a field \( k \). Suppose that \( -B \) is an \( h \)-nef \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-Weil divisor on \( Z \). Then \( B \) is effective if and only if \( h_* B \) is.

**Proof.** After extending the base field and taking the normalization, we can reduce to the case where the base field is algebraically closed. The assertion now follows from [4, 23].

Using regular alterations, we define numerically \( \mathbb{Q} \)-Cartier divisors. Here an alteration of a variety \( X \) over a field is a generically finite proper surjective morphism \( \pi : Y \rightarrow X \) and a regular alteration of \( X \) is an alteration \( \rho : Z \rightarrow X \) from a regular variety \( Z \), which always exists by [10].

**Definition 3.2** ([8, Definition 15], cf. [6, Definition 5.2]). Let \( X \) be a normal variety over a field.

(i) Suppose that \( D \) is a \( \mathbb{Q} \)-Weil divisor on \( X \) and \( x \in X \) is a (not necessarily closed) point of \( X \). We say that \( D \) is numerically \( \mathbb{Q} \)-Cartier at \( x \) if there exist an open neighborhood \( U \) of \( x \), a regular alteration \( \pi : V \rightarrow U \), and a \( \pi \)-numerically trivial \( \mathbb{Q} \)-divisor \( D_V \) on \( V \) such that \( D|_U = \pi_* D_V \). We also say that \( D \) is numerically \( \mathbb{Q} \)-Cartier if \( D \) is numerically \( \mathbb{Q} \)-Cartier at all points of \( X \).

(ii) \( X \) is said to be numerically \( \mathbb{Q} \)-Gorenstein if \( K_X \) is numerically \( \mathbb{Q} \)-Cartier. \( X \) is said to be numerically \( \mathbb{Q} \)-factorial if every \( \mathbb{Q} \)-Weil divisor on it is numerically \( \mathbb{Q} \)-Cartier.

**Remark 3.3.** Definition 3.2 (i) is equivalent to saying that there exists a open neighborhood \( U \) of \( x \), a regular alteration \( \pi : V \rightarrow U \), and a \( \mathbb{Q} \)-divisor \( D'_V \) on \( V \) such that \( D'_V \) is \( f \)-numerically trivial and \( f_* D'_V = g^* D|_U \), where \( \pi : V \xrightarrow{\pi} W \xrightarrow{g} U \) is the stein
factorization of $\pi$. The uniqueness of $D'_V$ can be verified by applying Lemma 3.1 to $f$. Therefore, we denote $D'_V$ by $\pi^*_{\text{num}}D|_U$ and call it the numerical pullback of $D|_U$ to $V$. It follows from essentially the same argument as in [6, Proposition 5.3] that the numerical pullback of $D|_U$ can be defined for an arbitrary regular alteration of $U$.

**Example 3.4.** (1) $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisors are clearly numerically $\mathbb{Q}$-Cartier.

(2) Every two-dimensional excellent normal integral scheme is numerically $\mathbb{Q}$-factorial, because Mumford’s pullback for $\mathbb{Q}$-Weil divisors exists by the Hodge index theorem [19, Theorem 10.1].

The following lemma tells us that the condition of $D$ being numerically $\mathbb{Q}$-Cartier is somewhat orthogonal to the finite generation of the section ring $\bigoplus_{i \geq 0} \mathcal{O}_X([iD])$.

**Lemma 3.5.** Let $x \in X$ be a point of a normal variety $X$ over a field and $D$ be a $\mathbb{Q}$-Weil divisor on $X$. Suppose that $\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{O}_X([iD])$ is Noetherian. Then $D$ is numerically $\mathbb{Q}$-Cartier at $x$ if and only if it is $\mathbb{Q}$-Cartier at $x$.

**Proof.** Suppose that $D$ is numerically $\mathbb{Q}$-Cartier at $x$. After shrinking $X$ if necessary, we can define the numerical pullback of $D$ to an arbitrary regular alteration of $X$. Let $\rho : Z = \text{Proj} \mathcal{R} \to X$ be the structure morphism. Then $\rho$ is a small morphism and $\rho_*^{-1}D$ is a $\rho$-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on $Z$. We will show that $\rho$ is an isomorphism.

Take a regular alteration $\mu : Y \to Z$, and then the composite morphism $\pi : Y \xrightarrow{\mu} Z \xrightarrow{\rho} X$ is a regular alteration of $X$. Since $\mu_*\pi^*_{\text{num}}D = (\deg \mu)\rho_*^{-1}D$, it follows from the uniqueness of the numerical pullback of $\rho_*^{-1}D$ (see Remark 3.3) that $\pi^*_{\text{num}}D = \mu^*\rho_*^{-1}D$.

Suppose to the contrary that $\rho$ is not an isomorphism. Then

$$(\pi^*_{\text{num}}D, \mu_*^{-1}C) = (\mu^*\rho_*^{-1}D, \mu_*^{-1}C) = (\rho_*^{-1}D, C) > 0,$$

because $\rho_*^{-1}D$ is $\rho$-ample. This, however, contradicts the fact that $\pi^*_{\text{num}}D$ is numerically $\pi$-trivial. \hfill \Box

**Example 3.6** (cf. [6, Example 5.8]). Let $x \in X$ be a closed point of a three-dimensional strongly $F$-regular (affine) variety over an algebraically closed field of characteristic $p > 5$ and let $D$ be a $\mathbb{Q}$-Weil divisor on $X$. It follows from [23, Theorem 4.3] and [13, Theorem 3.3] that there exists an effective $\mathbb{Q}$-Weil divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $(X, \Delta)$ is klt. Then $\bigoplus_{i \geq 0} \mathcal{O}_X([iD])$ is Noetherian by [7, Lemma 2.27]. Applying Lemma 3.5, we see that $D$ is numerically $\mathbb{Q}$-Cartier at $x$ if and only if it is $\mathbb{Q}$-Cartier at $x$.

We will use the following fact in the proof of the main theorem.

**Lemma 3.7** ([6, Proposition 5.9]). Let $z \in Z$ be a point of a normal variety $Z$ over a field and let $B$ be a Weil divisor on $Z$ which is numerically $\mathbb{Q}$-Cartier at $z$. Then for every divisorial valuation $\nu$ on $Z$ centered at $z$, we have

$$\lim_{m \to \infty} \frac{\nu(\mathcal{O}_Z(mB)\mathcal{O}_Z(-mB))}{m} = 0,$$

where $\nu(\mathcal{O}_Z(mB)\mathcal{O}_Z(-mB)) = \min\{\nu(f) \mid f \in \mathcal{O}_Z(mB)\mathcal{O}_Z(-mB), f \subset \mathcal{O}_{Z,z}\}$. 

Proof. After shrinking $Z$ if necessary, we may assume that $B$ is a numerically $\mathbb{Q}$-Cartier Weil divisor on $Z$. The assertion then follows from essentially the same argument as in [6, Proposition 5.9]. \hfill \Box

4. Main Theorem

In this section, we will prove the main theorem. The following valuative characterization of tight closure plays an important role in the proof.

**Proposition 4.1** (cf. [17, Theorem 3.1]). Let $(R, \Delta, a')$ be a triple where $(R, \mathfrak{m})$ is a $F$-finite complete local ring of characteristic $p > 0$. Suppose that $S$ is a normal domain which is a module-finite extension of $R$ and $p^s\Delta$ is an integral divisor on $Z = \text{Spec } S$, where $\rho : Z \to X$ is the morphism induced by the inclusion $R \hookrightarrow S$. Let $u$ be an element of $R$ and $I$ be an ideal of $R$, and fix a $\mathbb{Q}$-valued valuation $\nu$ on $S$ which is nonnegative on $R$ and positive on $\mathfrak{m}$. Then $u \in I^{(\Delta, a')}$ if and only if there exists a sequence of nonzero elements $c_e \in S$ satisfying the following three conditions:

(i) $\{\nu(c_e)/p^s\}$ is a monotonically decreasing sequence,
(ii) $\lim_{e \to \infty} \nu(c_e)/p^s = 0$,
(iii) $c_e u^{p^s} a^{[\nu_e]} S \subset I^{[\nu_e]} S(p^s \rho^s \Delta)$ for all $e$.

**Proof.** Suppose that there exists a sequence of nonzero elements $c_e \in S$ satisfying the conditions (i)-(iii). First note that $\nu$ can be extended to a $\mathbb{Q}$-valued valuation on the absolute integral closure $R^+$ of $R$ (we use the same letter $\nu$ to denote this valuation on $R^+$). It follows from [17, Theorem 3.3] that there exist a real number $\lambda > 0$ and an integer $r > 0$ such that for every element $s \in R^+$ with $\nu(s) < \lambda$, there is an $R$-linear map $\varphi : R^+ \to R$ such that $\varphi(s) \notin \mathfrak{m}^r$. Fix any power $p = p^e$ of $p$, and choose $q' = p^{e'}$ so large that $\nu(c_e^{1/q'}) = \nu(c_e)/q' \leq \nu(c_e)q/q' < \lambda$, where the second inequality follows from (i) and the third one does from (ii). We have $c_{ee'}u^{q'} a^{[\nu_{ee'}]} S \subset I^{[\nu_{ee'}]} S(qq' \rho^s \Delta)$ by (iii) and taking the $q'$-th root on both sides yields that

$$c_{ee'}^{1/q'} u^{q'} a^{[\nu_{ee'}]} S^{1/q'} \subset I^{[\nu_{ee'}]} S(qq' \rho^s \Delta)^{1/q'}.$$  

(\clubsuit)

By the choice of $\lambda$ and $r$, there exists an $R$-linear map $\bar{\varphi} : S^{1/q'} \to R$ such that $\bar{\varphi}(c_{ee'}^{1/q'}) \notin \mathfrak{m}^r$, which can be extended to an $R$-linear map $S(qq' \rho^s \Delta)^{1/q'} \to R([q\Delta])$ (we use the same letter $\bar{\varphi}$ to denote this map). Setting $\bar{c}_q = \bar{\varphi}(c_{ee'}^{1/q'}) \in R \setminus \mathfrak{m}^r$ and applying $\bar{\varphi}$ to (\clubsuit), we have $\bar{c}_q u^{q'} a^{[\nu_{ee'}]} \subset I^{[\nu_{ee'}]} R([q\Delta])$ for all $q = p^e$.

Set $J_q = ((I^{[\nu_{ee'}]} :_R u^{q'} a^{[\nu_{ee'}]}))$ for each $q = p^e$. Note that $\bar{c}_q \in J_q$, because $I^{[\nu_{ee'}]} R([q\Delta]) \cap R$ is contained in $(I^{[\nu_{ee'}]})^{[\nu_{ee'}]}$. Then $\{J_q\}$ is a decreasing sequence of ideals of $R$. Indeed, if $xu^{q'} a^{[\nu_{ee'}]} \subset (I^{[\nu_{ee'}]})^{[\nu_{ee'}]}$ with $x \in R$, then there exists a nonzero element $b \in R$ such that

$$b(xu^{q})^{pQ}(a^{[\nu_{ee'}]})^{[pQ]} \subset b(xu^{q})^{pQ}(a^{[\nu_{ee'}]})^{[pQ]} \subset I^{[pQ]} R(Q[pq\Delta]) \subset I^{[pQ]} R(pQ[pq\Delta]).$$

for all large powers $Q$ of $p$, which implies that $xu^{q'} a^{[\nu_{ee'}]} \subset (I^{[\nu_{ee'}]})^{[\nu_{ee'}]}$. Since the sequence $\{J_q\}$ is decreasing, if it had intersection (0), then Chevalley’s theorem would force $J_q \subset \mathfrak{m}^r$ for sufficiently large $q$. This contradicts the fact that $\bar{c}_q \in J_q \setminus \mathfrak{m}^r$, so we can choose a nonzero element $d \in \bigcap_q J_q$, that is, $du^{q'} a^{[\nu_{ee'}]} \subset (I^{[\nu_{ee'}]})^{[\nu_{ee'}]}$ for all $q = p^e$. Let
$c \in R^\circ$ be a $\Delta$-test element in the sense of [26] (see [26, Definition 2.4] for its definition and [26, Lemma 2.5] for the existence of such an element). It follows from the definition of $\Delta$-test elements that $c(I^{[q]}_{v}\cap \Delta) \subset I^{[q]}_vR([q\Delta])$, so that $cdu^{a^{[q]}_t}\subset I^{[q]}_vR([q\Delta])$ for all $q = p^e$. Therefore, $u \in I^e(\Delta,a^t)$ as required.

In order to prove the main theorem, we employ the strategy of MacCrimmon [21] and Williams [27]. The following notation is due to [27].

**Notation 4.2.** Let $R$ be a Noetherian local ring and $x = x_1, \ldots, x_n$ be a system of parameters for $R$. For an integer $t \geq 1$, we write $x^t = x_1^t, \ldots, x_n^t$ and denote the ideal $(x_1^t, \ldots, x_n^t)$ by $(x^t)$. For every $R$-module $M$, we also denote by $\mathcal{K}(x, t, M)$ the kernel of the map $M/(x^t)M \xrightarrow{x(x_1, \ldots, x_n)^{t-1}} M/(x^t)M$ induced by the multiplication by the element $(x_1 \cdots x_n)^{t-1}$, and we write $\mathcal{K}(x, \infty, M) := \bigcup_{t \geq 1} \mathcal{K}(x, t, M)$.

**Lemma 4.3** (cf. [27]). Let $(R, \Delta, a^t)$ be a triple where $(R, m)$ is an $n$-dimensional $F$-finite complete local ring of characteristic $p > 0$. Suppose that $(S, n)$ is a normal local ring which is a module-finite extension of $R$ and $\rho^* \Delta$ is an integral divisor on $Z = \text{Spec} S$, where $\rho : Z \to X$ is the morphism induced by the inclusion $R \hookrightarrow S$. Let $x = x_1, \ldots, x_n$ be a system of parameters for $R$ and $J$ be a divisorial ideal of $R$, and fix a $\mathbb{Q}$-valued valuation $\nu$ on $S$ which is nonnegative on $R$ and positive on $m$. Then $0^{\nu*}(\Delta, a^t)_m = 0^{\nu*}(\Delta, a^t)_S$ if there exists a sequence of nonzero elements $c_e \in S$ satisfying the following three conditions:

(i) $\{\nu(c_e)/p^e\}$ is a monotonically decreasing sequence,
(ii) $\lim_{e \to \infty} \nu(c_e)/p^e = 0$,
(iii) there exists an integer $t_0 \geq 2$ such that for all $s \geq 1$ and all $q = p^e$,

$$c_e \mathcal{K}(x^{qs}, \infty, J^{[q]}_v S(q^*\Delta)) \subset \mathcal{K}(x^{qs}, t_0, J^{[q]}_v S(q^*\Delta)).$$

**Proof.** Let $\xi \in 0^{\nu*}(\Delta, a^t)_m(J)$, that is, there exists a nonzero element $d \in R$ such that $(d_{t_0}^{[q]} \otimes R \xi = 0$ in $R([q\Delta])^{1/q} \otimes R H^a_m(J)$ for all $q = p^e$. We identify $H^a_m(J)$ with $\text{lim}_{q\to\infty} J/(x^t)J$ and represent $\xi$ by the image of $z \mod (x^s)J \in J/(x^s)J$ for some $z \in J$ and $s \geq 1$. Since the natural map

$$R([q\Delta])^{1/q} \otimes R H^a_m(J) \xrightarrow{\text{lim}_{q\to\infty}} (J^{[q]}_v R([q\Delta]))/((x^{qs})J^{[q]}_v R([q\Delta]))$$

sends $1^{1/q} \otimes \xi$ to the image of $z^q \mod (x^{qs})J^{[q]}_v R([q\Delta])$,

the image of $dz^{q}\mathbf{a}^{[tq]} \mod (x^{qs})J^{[q]}_v R([q\Delta]),$

in $\text{lim}_{q\to\infty} (J^{[q]}_v R([q\Delta]))/((x^{qs})J^{[q]}_v R([q\Delta]))$ for all $q = p^e$. Therefore,

$$ddz^{q}\mathbf{a}^{[tq]} \subset \mathcal{K}(x^{qs}, \infty, J^{[q]}_v S(q^*\Delta))$$

for a nonzero element $d' \in R(-[\Delta])$, which implies by the condition (iii) that

$$c_e ddz^{q}\mathbf{a}^{[tq]} \subset \mathcal{K}(x^{qs}, t_0, J^{[q]}_v S(q^*\Delta)).$$
that is, $c_\circ \eta d d' z' z^1(x_1 \cdots x_n)^{q(t-1)} a [\eta] S \subset ((x^{s t_0})J)[\eta] S(q \rho^t \Delta)$ for all $q = p^\xi$. Applying Proposition 4.1, one has $z(x_1 \cdots x_n)^{q(t-1)} \in ((x^{s t_0})J)^*(\Delta, a')$. This means that $\xi$, which is represented by the image of $z(x_1 \cdots x_n)^{q(t-1)}$ mod $(x^{s t_0})J \in J/(x^{s t_0})J, \xi$ belongs to $0_N^{*(\Delta, a')}$, where $N$ is the submodule of $H^m_m(J)$ generated by the image of $J/(x^{s t_0})J$. Thus, $\xi \in 0_N^{*(\Delta, a') k}$. □

Suppose that $R$ is a normal domain essentially of finite type over a field $k$ and $D$ is a $Q$-Weil divisor on $X = \text{Spec } R$. We say that $D$ is numerically $Q$-Cartier if there exist a normal affine variety $X' = \text{Spec } R'$ over $k$ and a $Q$-Weil divisor $\Delta'$ on $X'$ such that $R$ is a localization of $R'$, $D$ is the pullback of $D'$ to $X$, and $D'$ is numerically $Q$-Cartier at all $x \in X \subset X'$.

We are now ready to prove our main theorem, which is a generalization of [3, Proposition 3.7] (cf. [14, Definition-Theorem 6.5], [26, Theorem 2.8 (2)])

**Theorem 4.4.** Let $(R, \Delta, a')$ be a triple where $R$ is a normal domain essentially of finite type over an $F$-finite field $k$ of characteristic $p > 0$. If $K_X + \Delta$ is numerically $Q$-Cartier where $X = \text{Spec } R$, then $\tau_b(R, \Delta, a') = \tau_{fg}(R, \Delta, a')$.

**Proof.** We may assume that $(R, \mathfrak{m})$ is a local ring by essentially the same argument as the proof of [3, Proposition 3.7]. Choose a point $x$ of a normal affine variety $X'$ over $k$ and a $Q$-Weil divisor $\Delta'$ on $X'$ such that $R \cong \mathcal{O}_{X', x}$, $\Delta \cong \Delta'_x$ and $K_{X'} + \Delta'$ is numerically $Q$-Cartier at $x$. Take an effective Weil divisor $D'$ on $X'$ such that $D' \sim -K_{X'}$. Let $\pi' : Y' \to X'$ be a regular alteration and $Y' \xrightarrow{\mu'} Z' \xrightarrow{\rho'} X'$ denote its Stein factorization. After possibly passing to a further alteration, we may assume that $\rho^*(\Delta' - D')$ is an integral divisor on $Z'$. Fix a point $z \in Z'$ lying over $x \in X'$ and a divisorial valuation $\nu$ on $Z'$ centered at $z$. By Lemma 3.7,

$$\lim_{m \to \infty} \frac{\nu(\mathcal{O}_{Z'}(m \rho' (\Delta' - D')))}{m} = 0.$$ 

Let $\pi : Y \xrightarrow{\mu} Z \xrightarrow{\rho} X = \text{Spec } R$ be the flat base change of $\pi' = \mu' \circ \rho'$ via $X \to X'$ and $D$ be the pullback of $D'$ to $X$. Then $D$ is an effective Weil divisor on $X$ such that $D \sim -K_X$ and $\rho^*(\Delta - D)$ is an integral divisor on $Z$. Set $S = \rho_* \mathcal{O}_Z$, which is a normal domain and a module-finite extension of $R$.

It follows from Lemma 2.7 and Remark 2.8 (2) that

$$\tau_{fg}(R, \Delta, a') = \tau_{fg} (\hat{R}, \hat{\Delta}, (a\hat{R})^l) \cap R,$$

$$\tau_b(R, \Delta, a') = \tau_b (\hat{R}, \hat{\Delta}, (a\hat{R})^l) \cap R,$$

where $\hat{R}$ is the $\mathfrak{m}$-adic completion of $R$ and $\hat{\Delta}$ is the pullback of $\Delta$ to $\text{Spec } \hat{R}$. On the other hand, the $\mathfrak{m}$-adic completion $S \otimes_R \hat{R}$ of $S$ is a finite product $S_1 \times \cdots \times S_r$ of complete normal local rings $(S_i, n_i)$ which are module-finite extensions of $\hat{R}$. Note that after reindexing if necessary, we can assume that $(S_1, n_1)$ is isomorphic to the maximal-ideal-adic completion of $\mathcal{O}_{Z', x}$. Therefore, by [18, Proposition 9.3.5], the divisorial valuation $\nu$ can be extended to a divisorial valuation $\hat{\nu}$ on $\text{Spec } S_1$ centered at $n_1$ such
that
\[
\lim_{m \to \infty} \frac{\nu\left(S(m\rho_1^*(\Delta - \hat{D}))-S(-m\rho_1^*(\Delta - \hat{D}))\right)}{m} = 0,
\]
where \( \rho_1 : \text{Spec } S_1 \to \text{Spec } \hat{R} \) is the finite morphism induced by \( \rho \) and \( \hat{D} \) is the pullback of \( D \) to \( \text{Spec } \hat{R} \). Thus, by passing to completion, we may assume that \((R, \mathfrak{m})\) and \((S, \mathfrak{n})\) are complete local rings of characteristic \( p > 0 \) and of dimension \( n \geq 2 \), \( \nu \) is a divisorial valuation on \( \text{Spec } S \) centered at \( \mathfrak{n} \), and
\[
\lim_{m \to \infty} \frac{\nu\left(S(m\rho^*(\Delta - D))-S(-m\rho^*(\Delta - D))\right)}{m} = 0. \tag{\spadesuit}
\]
Set \( J = R(-D) \subset R \), and we will show that \( 0^{s(\Delta, t)}_{H_{\text{fin}}(J)} = 0^{s(\Delta, t)}_{H_{\text{fin}}(J)} \).

Let \( x_1 \in J \) be an arbitrary nonzero element. We can choose an element \( x_2 \in R(D)R(-D) \) such that \( x_2 \) is \( R/(x_1) \)-regular, because \( (R(D)R(-D) \subset R \) is the unit ideal or has height \( \geq 2 \). We extend \( x_1, x_2 \) to a system of parameters \( x_1, x_2, \ldots, x_n \) for \( R \). For each \( e \geq 1 \), set \( M_e = S(p^e \rho^*(\Delta - D)) \). Then
\[
J^{[q]}(q^e \Delta) \subseteq M_e \subseteq S(q^e \Delta),
\]
\[
x_e^2 M_e \subseteq J^{[q]} R(D)M_e \subseteq J^{[q]} S(q^e \Delta)
\]
for all \( q = p^e \). Since \( x_e^q \in J^{[q]} \), this implies that \((x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)^{q^{e}} M_e \) is contained in \( J^{[q]} S(q^e \Delta) \) for all integers \( s \geq 1 \) and \( 1 \leq i \leq n \). Therefore, the map
\[
H^{n-1}(x^{q^{e} s}; M_e / J^{[q]} S(q^e \Delta)) \to H^{n-1}(x^{q^{e-1} s}; M_e / J^{[q]} S(q^e \Delta))
\]
between Koszul cohomology modules is zero for all \( r, s \geq 1 \) and all \( q = p^e \). On the other hand, fix any power \( q = p^e \) of \( p \) and suppose that \( z \mod (x^q S) M_e \in K(x^q S, \infty, M_e) \) with \( z \in M_e \), that is, there exists an integer \( r \geq 1 \) such that \((x_1 \cdots x_n)^{q^{r-1} s} z \in (x^{q^r s}) M_e \). Since
\[
(x_1 \cdots x_n)^{q^{r-1} s} z S(q^e \Delta) \subseteq (x^{q^r s}) S(q^e \Delta) M_e \subseteq (x^{q^r s}) S,
\]
it follows from the colon-capturing property of tight closure [16, Theorem 4.7] that \( S(q^e \Delta) z \subseteq ((x^{q^r s}) S)^* \). Let \( c \in S \) be a big test element for \( S \) and set \( I_e = S(q^e \Delta) M_e \subset S \) for each \( q = p^e \). Then \( c I_e z \subseteq (x^{q^r s}) M_e \) by the definition of big test elements, which implies that \( c I_e K(x^{q^r s}, \infty, M_e) = 0 \) for all \( s \geq 1 \) and all \( q = p^e \).

Now applying [11, Lemma A.3] (cf. [21]) to the exact sequence
\[
0 \to J^{[q]} S(q^e \Delta) \to M_e \to M_e / J^{[q]} S(q^e \Delta) \to 0,
\]
we see that
\[
c I_e K(x^{q^r s}, \infty, J^{[q]} S(q^e \Delta)) \subset K(x^{q^r s}, 2, J^{[q]} S(q^e \Delta))
\]
for all \( s \geq 1 \) and all \( q = p^e \). Choose an element \( d_e \in I_e \) such that
\[
\nu(d_e) = \min\{\nu(f) \mid f \in I_e\} = \nu(I_e)
\]
for every \( e \geq 0 \), and we will check that the conditions (i)-(iii) in Lemma 4.3 are satisfied for the sequence \( \{cd_e\} \). We have already seen that (iii) is satisfied for \( \{cd_e\} \). Since
\[
\lim_{e \to \infty} \nu(cd_e)/p^e = \lim_{e \to \infty} (\nu(c) + \nu(I_e))/p^e = 0 \text{ by } (\spadesuit), \text{ the condition } (ii) \text{ is also}
\]
satisfied for \( \{cd_e\} \). Finally we use the containment \( I^p_e \subset I_{e+1} \) for every \( e \geq 0 \) to verify the condition (i) for \( \{cd_e\} \). Thus, the assertion follows from Lemma 4.3.

**Remark 4.5.** (1) When \( \Delta = 0 \) and \( a = R \), by applying [16, Proposition 8.23 (e)] (resp. [1, Theorem 3.6 (2)], [17, Theorem 3.1]) instead of Lemma 2.7 (resp. Remark 2.8 (2), Proposition 4.1), one can show that Theorem 4.4 holds without assuming that \( R \) is \( F \)-finite.

(2) As a generalization of \( \mathbb{Q} \)-Gorensteinness, the finite generation of the anti-canonical ring is discussed in [7]. In the setting of Theorem 4.4, assume that the anti-canonical ring \( \bigoplus_{i \geq 0} O_X(-i(K_X + \Delta)) \) is finitely generated instead of assuming that \( K_X + \Delta \) is numerically \( \mathbb{Q} \)-Cartier. It then follows from [7, Corollary 5.7] that \( \tau_t(R, \Delta) = \tau_{fg}(R, \Delta) \).

Since every normal surface is numerically \( \mathbb{Q} \)-factorial, the following holds.

**Corollary 4.6.** Let \( (R, \Delta, a^t) \) be a triple where \( R \) is a two-dimensional \( F \)-finite normal domain of characteristic \( p > 0 \). Then \( \tau_t(R, \Delta, a^t) = \tau_{fg}(R, \Delta, a^t) \).

**Proof.** First, \( \text{Spec } R \) is numerically \( \mathbb{Q} \)-factorial by Example 3.4 (2). Second, using the Hodge index theorem [19, Theorem 10.1], we can verify that Lemma 3.7 holds for all two-dimensional excellent normal integral schemes. Thus, the assertion follows from an argument very similar to the proof of Theorem 4.4.

**Remark 4.7.** (1) When \( \Delta = 0 \) and \( a = R \), Corollary 4.6 follows from [20, Theorem 8.8].

(2) Combining Corollary 4.6 and [8, Theorem 1], we see that if \( X = \text{Spec } R \) is a normal affine surface over an uncountable algebraically closed field of characteristic zero and \( \Delta \) is an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-Weil divisor on \( X \), then the multiplier ideal \( \mathcal{J}(X, \Delta) \) in the sense of [9] coincides, after reduction to characteristic \( p \gg 0 \), with the finitistic test ideal \( \tau_{fg}(R_p, \Delta_p) \) associated to modulo \( p \) reduction \( (R_p, \Delta_p) \) of \( (R, \Delta) \).

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