A NOTE ON THE COEFFICIENTS OF RAWNSLEY’S EPSILON FUNCTION OF CARTAN–HARTOGS DOMAINS

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ABSTRACT. We extend the result of Z. Feng and Z. Tu in [5] by showing that if one of the coefficients \( a_j \), \( 2 \leq j \leq n \), of Rawnlsey’s epsilon function associated to a \( n \)-dimensional Cartan–Hartogs domain is constant, then the domain is biholomorphically equivalent to the complex hyperbolic space.

1. Introduction and statement of main result

Consider an \( n \)-dimensional complex manifold \((M, g)\) endowed with a Kähler metric \( g \) and assume that there exists a globally defined Kähler potential \( \varphi : M \to \mathbb{R} \) for \( g \), i.e. if \( \omega \) is the Kähler form associated to \( g \), we have \( \omega = \frac{i}{2} \partial \bar{\partial} \varphi \). Let \( \mathcal{H}_\alpha \) be the weighted Bergman space of square integrable holomorphic functions on \( M \) with respect to the measure \( e^{-\alpha \varphi} \omega^n/n! \), i.e.:

\[
\mathcal{H}_\alpha = \left\{ f \in \text{Hol}(M) \mid \int_M e^{-\alpha \varphi} |f|^2 \omega^n/n! < \infty \right\}.
\]

If \( \mathcal{H}_\alpha \neq \{0\} \), choose an orthonormal basis \( \{f_j\} \) with respect to the product:

\[
(f, h)_\alpha = \int_M e^{-\alpha \varphi} f \bar{h} \omega^n/n!,
\]

and denote by \( K_\alpha(x, y) \) the reproducing kernel of \( \mathcal{H}_\alpha \), namely:

\[
K_\alpha(x, y) = \sum_j f_j(x) \bar{f}_j(y), \quad x, y \in M.
\]

Define the \( \epsilon \)-function associated to \( g \) to be the function:

\[
\epsilon_{\alpha g}(x) = e^{-\alpha \varphi(x)} K_\alpha(x, x), \quad x \in M.
\]
In the literature the function $\epsilon_{\alpha g}$ was first introduced by J. Rawnsley under the name of $\eta$-function in [7] and later as $\theta$-function in [2].

We say that $\epsilon_{\alpha g}$ admits the Engliš expansion:

$$
\epsilon_{\alpha g}(x) \sim \sum_{j=0}^{\infty} a_j(x)\alpha^{n-j}, \quad x \in M,
$$

for $\alpha \to +\infty$, if for every integers $l$, $r$ and every compact $H \subseteq M$,

$$
\|\epsilon_{\alpha}(x) - \sum_{j=0}^{l} a_j(x)\alpha^{n-j}\|_{C^r} \leq \frac{C(l, r, H)}{\alpha^{l+1}},
$$

for some constant $C(l, r, H) > 0$. Such expansion is the counterpart for non-compact manifolds of the celebrated TYZ (Tian-Yau-Zelditch) expansion of Kempf’s distortion function for polarized compact Kähler manifolds (see [9] and also [1]). In [4] M. Engliš proved that each of the coefficients $a_j(x)$ in (1) is a polynomial of the curvature of the metric $g$ and its covariant derivatives at $x$, which can be found by finitely many steps of algebraic operations, and gives an explicit expression of the coefficients $a_j$ for $j \leq 3$.

In this paper we consider the case of Cartan-Hartogs domains, which are defined as follows. Consider a Cartan domain $\Omega \subset C^d$, i.e. an irreducible bounded symmetric domain, of rank $r$ and numerical invariants $a$, $b$. Recall that the triple $\{r, a, b\}$ uniquely determines $\Omega$ and in particular it defines the dimension $d = \frac{r(r-1)}{2} a + rb + r$ and the genus $\gamma = (r-1) a + b + 2$ of $\Omega$. Let $K(z, z)$ be the Bergman kernel of $\Omega$ and $N_\Omega(z, z)$ its generic norm, i.e.

$$
N_\Omega(z, z) = (V(\Omega)K(z, z))^{-\frac{1}{d}},
$$

where $V(\Omega)$ is the total volume of $\Omega$ with respect to the Euclidean measure of the ambient complex Euclidean space.

For all positive real numbers $\mu$, a Cartan–Hartogs domains is given by $(M^d_\Omega(\mu), g(\mu))$ where:

$$
M^d_\Omega(\mu) = \left\{ (z, w) \in \Omega \times C^d, \quad ||w||^2 < N_\Omega(z, z)^{\mu} \right\},
$$

and $g(\mu)$ is the Kähler metric whose associated Kähler form $\omega(\mu)$ can be described by the (globally defined) Kähler potential centered at the origin:

$$
\Phi(z, w) = -\log(N_\Omega(z, z)^{\mu} - ||w||^2).
$$

The domain $\Omega$ is called the base of the Cartan–Hartogs domain $M^d_\Omega(\mu)$ (one also says that $M^d_\Omega(\mu)$ is based on $\Omega$). These domains have been considered
by several authors (see e.g. [8] and references therein). In [8] it is shown that for \( \mu_0 = \gamma/(d + 1) \), \((M^1_\Omega(\mu_0), g(\mu_0))\) is a complete Kähler-Einstein manifold which is homogeneous if and only if \( \Omega \) is the complex hyperbolic space. In [6] the authors of the present paper proved that for \( \Omega \neq \mathbb{CH}^d \), the metric \( \alpha g(\mu) \) on \( M^1_\Omega(\mu) \) is projectively induced for all positive real number \( \alpha \geq \frac{(r-1)a}{2\mu} \), exhibiting the first example of complete, noncompact, nonhomogeneous and projectively induced Kähler-Einstein metric. In [10] the author of the present paper proved that for \( d_0 = 1 \), \( g(\mu) \) is extremal (in the sense of Calabi [3]) if and only if it is Kähler–Einstein and that the coefficient \( a_2 \) of Englisch expansion of the \( \epsilon \)-function associated to a Cartan–Hartogs domain \((M^1_\Omega(\mu), g(\mu))\) is constant, then \((M^1_\Omega(\mu), g(\mu))\) is Kähler–Einstein, conjecturing also that the converse was true. In [5], Z. Feng and Z. Tu generalize that theorem to generic \( d_0 \) and proved that conjecture. More precisely, they prove the following:

**Theorem 1** (Z. Feng, Z. Tu [5, Th. 1.3]). The coefficient \( a_2 \) of the Rawnsley’s \( \epsilon \)-function expansion is a constant on \( M^1_\Omega(\mu) \) if and only if \((M^1_\Omega(\mu), g(\mu))\) is biholomorphically isometric to the complex hyperbolic space \((\mathbb{CH}^{d+d_0}, g_{\text{hyp}})\).

Notice that \( g_{\text{hyp}} \) denotes the hyperbolic metric on \( \mathbb{CH}^{d+d_0} \) and

\[
(\mathbb{CH}^{d+d_0}, g_{\text{hyp}}) = (M^1_{\mathbb{CH}^d}(1), g(1)).
\]

The prove of the previous theorem is based on the explicit formula for the \( \epsilon \)-function \( \epsilon_{\alpha g(\mu)} \) of Cartan–Hartogs domains:

\[
\epsilon_{\alpha g(\mu)}(z, w) = \frac{1}{\mu^d} \sum_{k=0}^{d} \frac{D^k \tilde{\chi}(d)}{k!} \left( 1 - \frac{|w|^2}{N_\Omega(z, z)^{\mu}} \right)^{d-k} \frac{\Gamma(\alpha - d + k)}{\Gamma(\alpha - d - d_0)} \Gamma(\alpha - d + k)
\]

for

\[
D^k \tilde{\chi}(d) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \tilde{\chi}(d - j)
\]

and

\[
\tilde{\chi}(d - j) = \prod_{j=1}^{r} \frac{\Gamma(\mu(d - j) - \gamma - (j + 1)\frac{d}{2} + 2 + b + ra)}{\Gamma(\mu(d - j) - \gamma + 1 + (j - 1)\frac{d}{2})}
\]

where \( \Gamma \) is the usual Gamma-function. Observe that formula (1) shows that English expansion of the \( \epsilon \)-function of Cartan–Hartogs domains is finite. In [11] the author of this paper uses this formula to prove the existence of a Berezin-Englisch quantization for Cartan–Hartogs domains.

The aim of this paper is to generalize Theorem 1 above (see next section) by proving the following:
Theorem 2. For all \( j = 2, \ldots, d + d_0 \), any coefficient \( a_j \) of the Rawnsley’s \( \epsilon \)-function expansion is a constant on \( M^{d_0}_\Omega(\mu) \) if and only if \( (M^{d_0}_\Omega(\mu), g(\mu)) \) is biholomorphically isometric to the complex hyperbolic space \( (CH^{d + d_0}, g_{\text{hyp}}) \).

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2. Proof of Theorem 2

Due to Theorem 1, we need only to prove that if \( a_j \) is constant for some \( j = 3, 4, \ldots, d + d_0 \), then \( a_2 \) is.

Consider first the polynomial \( P(\alpha) \) in the variable \( \alpha \):

\[
P(\alpha) = \frac{\Gamma(\alpha - d + k)}{\Gamma(\alpha - d - d_0)},
\]

and observe that for \( k = d \), \( (d \geq 1) \):

\[
P(\alpha) = \prod_{j=1}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0,
\]

for \( k = d - 1 \), \( (d \geq 1) \):

\[
P(\alpha) = \prod_{j=2}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0 - 1,
\]

for \( k = d - 2 \), \( (d \geq 2) \):

\[
P(\alpha) = \prod_{j=3}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0 - 2,
\]

and so on. Thus, from (1) we get that the factor with \( k = d \) contributes to all the coefficients \( a_0, a_1, \ldots, a_{d+d_0} \) \( (d \geq 1) \), the factor with \( k = d - 1 \) to all from \( a_1 \) to \( a_{d+d_0} \) \( (d \geq 1) \), the factor with \( k = d - 2 \) to all from \( a_2 \) \( (d \geq 2) \), and so on. Obviously the \( j \)-th coefficient is constant iff each one of its factors (except the \( k = d \) one) vanishes, in fact the term \( \left(1 - \frac{||w||^2}{N_\Omega(z, \bar{z})^\mu} \right) \) in each factor has a different power.

In particular, the coefficient \( a_i \), \( i = 1, \ldots, d + d_0 \), contains the factor:

\[
\frac{1}{\mu^d (d-1)!} D^{d-1} \tilde{\chi}(d) \left(1 - \frac{||w||^2}{N_\Omega(z, \bar{z})^\mu} \right) A_i^2, \quad (d \geq 1),
\]

and the coefficient \( a_i \), \( i = 2, \ldots, d + d_0 \), contains the factor:

\[
\frac{1}{\mu^d (d-2)!} D^{d-2} \tilde{\chi}(d) \left(1 - \frac{||w||^2}{N_\Omega(z, \bar{z})^\mu} \right)^2 A_i^{d-1}, \quad (d \geq 2),
\]
where we denote by $A_p^q$ the $p$-th coefficient of the polynomial in $\alpha$:

$$
\prod_{j=q}^{d+d_0} (\alpha - j).
$$

Observe that $A_2^2$ and $A_3^2$ do not vanish. In fact we have:

$$
\prod_{j=2}^{d+d_0} = \alpha^{d+d_0} + e_1(2, \ldots, d + d_0)\alpha^{d+d_0-1} + \cdots + e_{d+d_0}(2, \ldots, d + d_0),
$$

$$
\prod_{j=3}^{d+d_0} = \alpha^{d+d_0-1} + e_1(3, \ldots, d + d_0)\alpha^{d+d_0-2} + \cdots + e_{d+d_0-1}(3, \ldots, d + d_0)
$$

where $e_j(x_1, \ldots, x_n)$ is the elementary symmetric polynomial in the variables $(x_1, \ldots, x_n)$, i.e.:

$$
e_j(x_1, \ldots, x_n) = \sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq n} x_{k_1} \cdots x_{k_j}.
$$

Since in our case $x_j$ are positive integers, $A_2^2 = e_1(2, \ldots, d + d_0)$ and $A_3^2 = e_i(3, \ldots, d + d_0)$ do not vanish.

Thus we have that for $d \geq 2$ and for each $i = 3, \ldots, d$, if $a_i$ is constant then $D^{d-2}\tilde{\chi}(d) = D^{d-1}\tilde{\chi}(d) = 0$, and conclusion follows by [5], where in the proof of Theorem 1.3 it is pointed out that when $d > 1$ we have

$$
D^{d-2}\tilde{\chi}(d) = D^{d-1}\tilde{\chi}(d) = 0,
$$

if and only if $a_2$ is constant.

If $d = 1$, then $r = 1$ and $\Omega = CH^1$, thus we need only to prove that if $a_j$ is constant for some $j = 3, 4, \ldots, d_0 + 1$, then $\mu = 1$. By the discussion above, if $a_j$ is constant for some $j = 3, 4, \ldots, d_0 + 1$ then $D^0\tilde{\chi}(1) = 0$, which by [5] Lemma 3.5] directly implies $\mu = 1$, concluding the proof.

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