Dimensional crossover of Bose-Einstein condensation phenomena in quantum gases confined within slab geometries

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We investigate systems of interacting bosonic particles confined within slab-like boxes of size $L^2 \times Z$ with $Z \ll L$, at their three-dimensional (3D) BEC transition temperature $T_c$, and below $T_c$ where they experience a quasi-2D Berezinskii-Kosterlitz-Thouless transition (at $T_{BKT} < T_c$ depending on the thickness $Z$). The low-temperature phase below $T_{BKT}$ shows quasi-long-range order: the planar correlations decay algebraically as predicted by the 2D spin-wave theory. This dimensional crossover, from a 3D behavior for $T \gtrsim T_c$ to a quasi-2D critical behavior for $T \lesssim T_{BKT}$, can be described by a transverse finite-size scaling limit in slab geometries. We also extend the discussion to the off-equilibrium behavior arising from slow time variations of the temperature across the BEC transition. Numerical evidence of the 3D→2D dimensional crossover is presented for the Bose-Hubbard model defined in anisotropic $L^2 \times Z$ lattices with $Z \ll L$.

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I. INTRODUCTION

The Bose-Einstein condensation (BEC) characterizes the low-temperature behavior of three-dimensional (3D) bosonic gases, below a finite-temperature BEC phase transition separating the high-temperature normal phase and the low-temperature superfluid BEC phase. The phase coherence properties of the BEC phase have been observed by several experiments, see e.g. Refs. [11–10]. Several theoretical and experimental studies have also investigated the critical properties at the BEC transition, when the condensate begins forming, see, e.g., Refs. [11–33]. Both the phase-coherence properties of the BEC phase and the critical properties at the BEC transition turn out to be particularly sensitive to the inhomogeneous conditions arising from spatially-dependent confining potentials, and/or the geometry of the atomic-gas system. Inhomogeneous conditions due to space-dependent trapping potentials give rise to a universal distortion of the homogeneous critical behavior, which can be cast in terms of a universal trap-size scaling [14, 26, 27], controlled by the same universality class of the 3D BEC transition. In the case of homogeneous traps, such as those experimentally realized in Refs. [28, 30, 31, 33], the geometry of the trap may lead to quite different phase-coherence properties, when passing from 3D, to quasi-2D, or quasi-1D systems. For example, atomic gases in elongated homogeneous boxes [34] and harmonic traps [6, 7, 35, 38] show a dimensional crossover from a high-temperature 3D behavior to a low-temperature quasi-1D behavior.

In this paper we consider bosonic particle systems confined within slab geometries, i.e. within boxes of size $L^2 \times Z$ with $Z \ll L$. We investigate their behavior at the BEC transition temperature $T_c$ (this is the critical temperature of the 3D system in the thermodynamic limit, i.e. when all system sizes tend to infinity) and at lower temperatures. Their low-temperature behavior ($T < T_c$) is further characterized by the possibility of undergoing a finite-temperature transition to a quasi-long range order (QLRO) phase, with long-range planar correlations which decay algebraically. This is the well-known Berezinskii-Kosterlitz-Thouless (BKT) transition [39–42], which occurs in 2D statistical systems with a global U(1) symmetry. Experimental evidences of BKT transitions have been also reported for quasi-2D trapped atomic gases [43–49].

The behavior of homogeneous gases in slab geometries can be described in terms of a dimensional crossover, from 3D behaviors for $T \gtrsim T_c$ to a quasi-2D critical behavior for $T \lesssim T_{BKT}$. In the limit of large thickness $Z$, the quasi-2D BKT transition temperature approaches that of the 3D BEC transition, i.e. $T_{BKT} \rightarrow T_c$ for $Z \rightarrow \infty$ (assuming the thermodynamic limit for the planar directions, i.e. $L \gg Z$). The interplay of the BEC and BKT critical modes gives rise to a quite complex behavior. We show that this can be described by a transverse finite-size scaling (TFSS) limit for systems in slab geometries [50–51], i.e., $Z \rightarrow \infty$ and $T \rightarrow T_c$, keeping the product $(T-T_c)Z^{1/\nu}$ fixed, where $\nu$ is the correlation-length exponent at the 3D BEC transition. In this TFSS limit the BKT transition below $T_c$ appears as an essential singularity of the TFSS functions. The dimensional-crossover scenario is expected to apply to any quantum gas of interacting bosonic particles confined in boxes or lattice structures with slab geometries. Analogous arguments apply to $^4$He systems in film geometries [52], and to 3D XY spin models defined in lattices with slab geometries [53–56].

We also extend the discussion to the off-equilibrium behavior arising from slow time variations of the temperature across the BEC transition. The behavior of weakly interacting atomic gas confined in quasi-2D geometries has been experimentally investigated under time-dependent protocols across the BEC regime, see, e.g., Refs. [51–53], to verify the Kibble-Zurek mechanism of defect production [54–55]. In gases confined within slab geometries, the off-equilibrium behavior aris-
ing from the slow variation of the temperature across the BEC transition point is made particularly complex by the presence of the quasi-2D BKT transition at $T_{\text{BKT}} \lesssim T_c$. Thus, disentangling the behaviors corresponding to 3D BEC and quasi-2D BKT transitions may be quite hard in experimental or numerical analyses. To describe this complex behavior, we put forward the emergence of an off-equilibrium transverse finite-size scaling for bosonic gases confined within slab-like homogeneous traps.

We provide evidence of the dimensional-crossover scenario in quantum gases by a numerical study of the Bose-Hubbard (BH) model [59], which models gases of bosonic atoms in optical lattices [10, 60]. We show that the predictions of the 3D→2D dimensional crossover are realized when considering anisotropic slab-like lattices $L^2 \times Z$ with $Z \ll L$. With decreasing $T$ from the high-temperature normal phase, we first meet a quasi-BEC transition where the critical length scale $\xi$ gets large, but it does not diverge being limited by $\xi \sim Z$ (keeping $Z$ fixed). Then we observe a BKT transition to a QLRO phase, where the system develops planar critical correlations essentially described by a Gaussian spin-wave theory. The dimensional crossover explains the apparently complex behavior of the one-particle correlation functions and the corresponding length scale, when decreasing the temperature from $T > T_c$, where $T_c$ is the 3D BEC transition temperature, to $T < T_{\text{BKT}} < T_c$, where $T_{\text{BKT}}$ depends on the thickness $Z$. The results are also consistent with the scaling predictions of the TFSS theory for systems in slab geometries.

The paper is organized as follows. In Sec. II we introduce the BH model that we use as a paradigmatic model of Bose gases showing the phenomenon of dimensional crossover in slab geometries. In Sec. III we present the general theory of the dimensional crossover in slab geometries. In Sec. IV we discuss the new features arising from the presence of a harmonic trap along the shorter transverse direction. Sec. V reports some exact spin-wave results for the phase-coherence correlations within the low-temperature phase of quasi-2D systems with $U(1)$ symmetry. In Sec. VI we discuss the off-equilibrium behavior arising from slow time variations of the temperature across the BEC transition. In Sec. VII we provide numerical evidences of the dimensional crossover in 3D BH models defined on lattices with slab geometries. Finally, we summarize our results in Sec. VIII.

II. THE BOSE-Hubbard MODEL IN SLAB GEOMETRIES

Lattice BH models [59] are interesting examples of interacting Bose gases undergoing BEC transitions. They provide realistic models of gases of bosonic atoms in optical lattices [60]. In the following discussions we use the BH model as a paradigmatic model of Bose gases showing the dimensional crossover in slab geometries.

The Hamiltonian of BH models reads

$$
H_{\text{BH}} = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i,
$$

where $b_i$ is a bosonic operator, $n_i \equiv b_i^\dagger b_i$ is the particle density operator, the sums run over the bonds $\langle ij \rangle$ and the sites $i$ of a cubic $L_1 \times L_2 \times L_3$ lattice, $a = 1$ is the lattice spacing. The phase coherence properties can be inferred from the one-particle correlation function

$$
G(r_1, r_2) = \frac{\text{Tr} \, b_{r_1}^\dagger b_{r_2} e^{-H_{\text{BH}}/T}}{\text{Tr} \, e^{-H_{\text{BH}}/T}}.
$$

We set the hopping parameter $t = 1$, so that all energies are expressed in units of $t$, and the Planck constant $\hbar = 1$.

The phase diagram of 3D BH models and their critical behaviors have been much investigated, see e.g. Refs. [25–27, 29]. The BEC phase extends below $U = 0$ (corresponding to half filling), where the occupation site number is limited to the cases $n = 0, 1$. The condensate wave function provides the complex order parameter of the BEC transition, whose critical behavior belongs to the $U(1)$-symmetric $XY$ universality class. This implies that the length scale $\xi$ of the critical modes diverges at $T_c$ as $\xi \sim (T - T_c)^{-\nu}$, $\nu = 0.6717(1)$. This has been accurately verified by numerical studies, see, e.g., Refs. [25]. The BEC phase extends below the BEC transition line. In particular, in the hard-core limit $U \to \infty$ and for $\mu = 0$ (corresponding to half filling), the BEC transition occurs at $T_c = 2.01599(5)$.

We consider BH lattice gases in anisotropic slab-like geometries, i.e. $L^2 \times Z$ lattices with $Z \ll L$. We consider open boundary conditions (OBC) along the transverse $Z$-direction; we label the corresponding coordinate as $-(Z - 1)/2 \leq z \leq (Z - 1)/2$, so that the innermost plane is the $z = 0$ plane. This choice is motivated by the fact that OBC correspond to gas systems trapped by hard walls, such as the experimental systems of Refs. [28, 30, 31, 53]. Since the thickness $Z$ of the slab is generally considered as much smaller than the size $L$ of the planar directions, and in most cases we consider the 2D thermodynamic $L \to \infty$ limit keeping $Z$ fixed, the boundary conditions along the planar directions are generally irrelevant for our study around $T_c$. However, they become relevant at the BKT transition where the planar correlation length diverges. In the following we consider the most convenient periodic boundary conditions (PBC)
The BEC phase is restricted to a finite region between \( \mu = -6 \) and \( \mu = 6 \). It is bounded by a BEC transition line \( T_c(\mu) \), which satisfies \( T_c(\mu) = T_c(-\mu) \) due to a particle-hole symmetry. Its maximum occurs at the vacuum phase \((\mu < 0)\) and the incompressible \( n = 1 \) Mott phase \((\mu > 6)\).

As we shall argue, BH systems below \( T_c \) are expected to develop quasi-2D critical modes, leading to a BKT transition with a diverging planar correlation length, and a low-temperature QLRO phase.

To study this phenomenon, and in particular how the critical behavior changes if we consider a quasi-2D thermodynamic limit, without the emergence of a nonvanishing order parameter \([71, 72]\). When approaching the BKT transition point \( T_{\text{BKT}} \), from the high-temperature normal phase, these systems develop an exponentially divergent correlation length:

\[
\xi \sim \exp(c/\sqrt{T}), \quad \tau \equiv T/T_{\text{BKT}} - 1,
\]

where \( c \) is a universal constant and \( \omega = 0.785(20) \) is the scaling-correction exponent associated with the leading irrelevant perturbation at the XY fixed point \([62, 64, 66]\). Note that the universal constant \( R_Z \) depends on the boundary conditions along the transverse direction (the boundary conditions along the planar directions are irrelevant since we assume \( L \gg Z \) and \( \xi \sim Z \)).

However, we should also take into account that 2D or quasi-2D systems with a global U(1) symmetry may undergo a finite-temperature transition described by the BKT theory \([39–42, 69]\). The BKT transition separates a high-temperature normal phase and a low-temperature phase characterized by QLRO, where correlations decay algebraically at large distances, without the emergence of a nonvanishing order parameter \([71, 72]\). When approaching the BKT transition point \( T_{\text{BKT}} \), the behavior along the planar directions for \( T < T_{\text{BKT}} \) is a universal constant and \( Z \) is the free energy, \( \phi_a \) are twist angles along one of the planar directions. Note that \( Y_1 = Y_2 \) by symmetry for \( L^2 \times Z \) systems.

As we shall see, the quantities

\[
Y \equiv Y_a, \quad R_L \equiv \xi/L,
\]

are particularly useful to check the effective spin-wave behavior along the planar directions for \( T < T_{\text{BKT}} < T_c \).

### III. DIMENSIONAL CROSSOVER OF BOSE GASES IN SLAB GEOMETRIES

#### A. Phase diagram for a finite thickness \( Z \)

The 3D scenario sketched in Fig. 1 substantially changes if we consider a quasi-2D thermodynamic limit, i.e., \( L \to \infty \) keeping \( Z \) fixed. Indeed the length scale \( \xi \) remains finite at the BEC transition point when \( Z \) is kept fixed. Of course the full 3D critical behavior must be somehow recovered when \( Z \to \infty \), for which one expects \( \xi(Z) \sim Z \) more precisely, defining

\[
R_Z = \lim_{L \to \infty} \xi/Z,
\]

standard FSS arguments \([51, 52]\) predict that at the 3D critical point \( T_c \)

\[
R_Z(T_c) = R_Z^* + O(Z^{-\omega})
\]

where \( R_Z^* \) is a universal constant and \( \omega = 0.785(20) \) is the scaling-correction exponent associated with the leading irrelevant perturbation at the XY fixed point \([62, 64, 66]\). Note that the universal constant \( R_Z^* \) depends on the boundary conditions along the transverse direction (the boundary conditions along the planar directions are irrelevant since we assume \( L \gg Z \) and \( \xi \sim Z \)).

However, we should also take into account that 2D or quasi-2D systems with a global U(1) symmetry may undergo a finite-temperature transition described by the BKT theory \([39–42]\). The BKT transition separates a high-temperature normal phase and a low-temperature phase characterized by QLRO, where correlations decay algebraically at large distances, without the emergence of a nonvanishing order parameter \([71, 72]\). When approaching the BKT transition point \( T_{\text{BKT}} \) from the high-temperature normal phase, these systems develop an exponentially divergent correlation length:

\[
\xi \sim \exp(c/\sqrt{T}), \quad \tau \equiv T/T_{\text{BKT}} - 1,
\]
where \( c \) is a nonuniversal constant. The magnetic susceptibility diverges as \( \chi \sim \xi^{7/4} \), corresponding to the critical exponent \( \eta = 1/4 \).

Consistently with the above picture, 2D BH systems [corresponding to the Hamiltonian (1) with \( Z = 1 \)] undergo a BKT transition. Fig. 2 shows a sketch of the phase diagram of 2D BH systems in the hard-core \( U \to \infty \) limit. The finite-temperature BKT transition of BH models has been numerically investigated by several studies, see e.g. Refs. 23, 73, 70. In particular, \( T_{\text{BKT}} = 0.6877(2) \) in the hard-core \( U \to \infty \) limit and for \( \mu = 0 \) 73. Note that the 2D BH systems do not show a real BEC below the critical temperature \( T_{\text{BKT}} \), but QLRO where the phase-coherence correlations decay algebraically.

The phase diagram of quasi-2D systems with finite thickness \( Z > 1 \) is expected to be analogous to that of 2D BH systems, with a BKT transition at \( T_{\text{BKT}} \) depending on the thickness \( Z \). Analogously to 2D systems, they are expected to show a QLRO phase below \( T_{\text{BKT}} \), where correlation functions show power-law decays along the planar directions, as described by the 2D spin-wave theory.

### B. Dimensional crossover limit

The above scenario can be interpreted as a dimensional crossover from a 3D behavior when \( T \gtrsim T_c \), and \( \xi \) is finite (in particular the anisotropy of the system is not locally relevant when \( \xi \ll Z \)), to an effective 2D critical behavior at \( T \lesssim T_{\text{BKT}}(Z) \) where the planar correlation length \( \xi \) diverges.

Such a dimensional crossover can be described by an appropriate TFSS limit, defined as \( \delta \equiv 1 - T/T_c \to 0 \) and \( Z \to \infty \), keeping \( \xi Z^{1/\nu} \) fixed. In this TFSS limit 50, 51

\[
R_Z = \xi Z \approx \mathcal{R}(X), \quad X = Z^{1/\nu} \delta, \quad (13)
\]

where \( \mathcal{R}(X) \) is a universal function (apart from a trivial normalization of the argument \( X \)), but depending on the boundary conditions along the \( Z \) direction. Scaling corrections are suppressed as \( Z^{-\omega} \), analogously to Eq. (11).

In this TFSS framework the BKT transition appears as an essential singularity of the scaling function \( \mathcal{R}(X) \):

\[
\mathcal{R}(X) \sim \exp \left( \frac{b}{\sqrt{X_{\text{BKT}} - X}} \right) \quad \text{for} \quad X \to X_{\text{BKT}}^- , \quad (14)
\]

where \( X_{\text{BKT}} \) is the value of the scaling variable \( X \) corresponding to the BKT transition point

\[
\delta_{\text{BKT}}(Z) \equiv \frac{T_x - T_{\text{BKT}}(Z)}{T_c}, \quad (15)
\]

i.e.,

\[
X_{\text{BKT}} = \lim_{Z \to \infty} Z^{1/\nu} \delta_{\text{BKT}}(Z) > 0. \quad (16)
\]

The constant \( b \) in Eq. (14) is a nonuniversal constant depending on the normalization of the scaling variable \( X \). \( \mathcal{R}(X) \) is not defined for \( X \gtrsim X_{\text{BKT}} \). Note that the above scaling equations predict that

\[
\delta_{\text{BKT}}(Z) \sim Z^{-1/\nu} \quad (17)
\]

in the large-\( Z \) limit.

The TFSS of the planar two-point function (11) is given by

\[
g(x, Z) \approx Z^{-(1+\eta)} g(x/Z, X), \quad (18)
\]

where \( \eta = 0.0381(2) \) is the critical exponent of the 3D XY universality class 61, associated with the power-law decay of the two-point function at \( T_c \). Eq. (18) also implies that the planar susceptibility defined as in Eq. (3) behaves as

\[
\chi \approx Z^{1-\eta} f_\chi(X). \quad (19)
\]

It is important to note that the above features are shared with any quasi-2D statistical system with a global U(1) symmetry, and in particular standard O(2)-symmetric spin models. Numerical analyses of dimensional crossover issues for the XY model are reported in Refs. 52, 50.

### IV. BOSE GASES CONFINED BY A TRANSVERSE HARMONIC TRAP

We now discuss the case of quasi-2D gases trapped by a harmonic potential along the transverse direction, analogously to the experimental setup of Ref. 31.

#### A. The BH model in a transverse harmonic trap

In the case of the BH model the presence of a space-dependent trapping potential can be taken into account
by adding a further Hamiltonian term to Eq. (1), i.e.
\[ H_{bBH} = H_{BH} + \sum_i V(z_i) n_i, \]
where \( z_i \) is the distance of the site \( i \) from the central plane, \( p > 0 \), and \( \ell \) can be considered as the transverse trap size. The harmonic potential corresponds to \( p = 2 \). The transverse trapping potential coupled to the particle density turns out to be equivalent to an effective chemical potential depending on the transverse coordinate \( z \),
\[ \mu_c(\mu, z) \equiv \mu - V(z). \]

Far from the central \( z = 0 \) plane, the potential \( V(z) \) diverges, thus \( \mu_c \to -\infty \) therefore \( \langle n_i \rangle \) vanishes and the particles are trapped along the transverse direction.

We discuss the behavior of the system in the limit of infinite size of the planar dimensions, along which the system appears as homogeneous. For practical realizations, this regime may be realized by considering hard-wall traps along the planar directions with size \( L \gg \ell \) (more precisely \( L \gg \ell^\theta \) where the exponent \( \theta < 1 \) is given below).

The planar correlation functions, for example along the \( z = 0 \) plane, are expected to behave similarly to the case of transverse hard-wall traps. With decreasing \( T \) from the high-temperature normal phase, the length scale \( \xi \) gets large around the BEC transition temperature \( T_c \) (i.e. the critical temperature of the BEC transition of the corresponding homogeneous 3D system). But it does not diverge, since \( \xi \sim \ell^\theta \) where \( \theta \) is an appropriate exponent, see below. Then one may observe a BKT transition to a QLRO phase around the \( z = 0 \) plane, at \( T_{\text{BKT}} < T_c \) depending on \( \ell \). In particular, in the extreme \( \ell \to 0 \) limit, where all particles are confined within the \( z = 0 \) plane, we recover the homogeneous 2D BH model, i.e. the model \( \text{BH} \) with \( Z = 1 \). On the other hand, in the opposite \( \ell \to \infty \) limit, we again expect that \( T_{\text{BKT}}(\ell) \to T_{c,\ell} \), analogously to the homogeneous case. Therefore, similarly to the homogeneous case, the system passes from a high-temperature 3D behavior to a quasi-2D critical temperature at low temperature. This change of regime may be also related to a transverse condensation phenomenon \[31, 32, 33, 108, 109].

B. Transverse trap-size scaling

Like homogeneous systems with transverse hard-wall boundary conditions, the critical behavior of the 3D system must be somehow recovered in the large-\( \ell \) limit, in a spatial region sufficiently close to the central \( z = 0 \) plane. We argue that this limit can be described by a universal transverse-trap-size scaling (TTSS), similar to the TFSS limit discussed in Sec. III B. To derive the TTSS laws for the case at hand, we can exploit the same arguments used to derive the trap-size scaling for isotropic traps \[14, 24, 26].

The trapping potential \( V(z) \) coupled to the particle density significantly affects the critical modes, introducing another length scale \( \ell \). Like general critical phenomena, see, e.g., Ref. [22], the asymptotic scaling behavior of the length scale at \( T_c \) is expected to be characterized by a power law:
\[ \xi_\ell \sim \ell^\theta. \]

The exponent \( \theta \) can be determined by a scaling analysis of the perturbation associated with the external potential coupled to the particle density. Its derivation is identical to that reported in Refs. \[14, 26\] for isotropic traps. The exponent \( \theta \) turns out to be related to the correlation-length exponent \( \nu \) of the universality class of the critical behavior of the homogeneous BEC transition, i.e.,
\[ \theta = \frac{p\nu}{1 + p\nu}, \]
where \( \nu = 0.6717(1) \) is the correlation-length exponent of the 3D XY universality class. For harmonic transverse traps, i.e. \( p = 2, \theta = 0.57327(4) \).

On the basis of these TTSS arguments, we expect that the asymptotic large-\( \ell \) behavior of the two-point function around the central \( z = 0 \) plane, and in particular the correlation function defined as in Eq. (11), behaves as
\[ g(x, \ell) \approx \xi_\ell^{-1+\nu} G_p(x/\xi_\ell, \delta\xi_\ell^{1/\nu}), \]
where \( \xi_\ell \sim \ell^\theta \), \( \delta \equiv 1 - T/T_c \), and we have assumed that the planar sizes are infinite. Actually, one may also take into account the planer size \( L \) by adding a further scaling variable \( L/\ell^\theta \); the \( L \to \infty \) scaling behavior is recovered when \( L/\ell^\theta \gg 1 \).

The TTSS of the two-point function implies that the planar second-moment correlation length along the \( z = 0 \) plane, defined as in eq. (11), behaves asymptotically as
\[ \xi_\ell \sim \ell^\theta R_p(\mathcal{X}), \quad \mathcal{X} \equiv \delta^{\ell^\theta/\nu}. \]

In particular, we recover \( \xi_\ell \sim \ell^\theta \) at \( T_c \). Note that this scaling behavior is analogous to that of hard-wall traps, cf. Eq. (13), with the transverse size \( Z \) replaced by \( \ell^\theta \). The leading corrections to the above asymptotic TTSS are \( O(\ell^{-\omega}) \).

Note that the trap-exponent \( \theta \) reported in Eq. (24) is identical to that of isotropic traps \[26\], i.e. it does not depend on the number of coordinates entering the space-dependence of the inhomogeneous power-law potential coupled to the particle density. However, the scaling functions \( G_p \) and \( R_p \), entering Eqs. (25) and (26) must definitely differ. Actually, in the \( p \to \infty \) limit we must recover the TFSS behavior, i.e. that of the homogeneous conditions along the transverse direction with OBC, see Sec. III B. Since \( \theta \to 1 \) for \( p \to \infty \), \( \ell \sim Z \) of the transverse hard-wall conditions.
The TTSS functions must present a singularity related to the BKT transition for \( T_{\text{BKT}} < T_c \), unlike those of the isotropic TSS because no such transition occurs for isotropic traps. In particular, TTSS implies that

\[
\delta_{\text{BKT}}(\ell) \equiv 1 - T_{\text{BKT}}(\ell)/T_c \sim \ell^{-\theta/\nu},
\]

and the TTSS function \( f_\xi \) of (20) must show a BKT-like singularity at

\[
\lambda_{\text{BKT}} = \lim_{\ell \to \infty} \delta_{\text{BKT}}(\ell) \ell^{\theta/\nu},
\]

such as that reported in Eq. (14).

### C. Criticality at the boundary of the BEC region

Other interesting features arise at the boundary of the BEC region in atomic gases confined by a transverse harmonic trap. If the trap is sufficiently large and the temperature is sufficiently low, different phases may coexist in different space regions, when moving from the central \( z = 0 \) plane of the trap. Indeed, due to the fact that the effective chemical potential \( \mu_e(z) \), cf. Eq. (22), decreases with increasing \( z \), the BEC region is generally spatially limited. When moving from the \( z = 0 \) plane, the quantum gas passes from the BEC phase around the center of the trap (where space coherence is essentially described by spin waves) to a normal phase far from the center. The atomic gas is expected to develop a peculiar critical behavior at the boundary of the BEC region, with a nontrivial scaling behavior controlled by the universality class of the homogenous BEC transition in the presence of an effective linear external potential coupled to the particle density (22).

This occurs around the planes where the distance \( |z| \) from the \( z = 0 \) plane is such that \( T[\mu_e(\mu, z)] \) is equal to the BEC transition temperature at the local chemical potential \( \mu_e(\mu, z) = \mu - (z/\ell)^2 \), i.e. where

\[
T_e[\mu_e(\mu, z)] \approx T < T_c(\mu).
\]

For example, consider the hard-core BH lattice gas (20) for \( \mu \leq 0 \) and \( T < T_c(\mu) \), see Fig. 1. Since \( T_e(\mu) \) decreases with decreasing \( \mu \), a plane exists at distance \( z = z_b \) such that \( T_e[\mu_e(\mu, z_b)] = T \), thus

\[
z_b = \ell \sqrt{\mu - \bar{\mu}},
\]

where \( T_e(\bar{\mu}) = T \). This plane separates the BEC region from the normal-fluid region. As argued in Ref. [22], in the limit of large \( \ell \), the correlation functions around the surface where \( T_e[\mu_e(\mathbf{r})] = T \) are expected to develop a peculiar critical behavior in the presence of an external effectively linear potential coupled to the particle density.

Around \( z = z_b \)

\[
V(z) = V(z_b) + \Delta z/\ell_b + O[(\Delta z/\ell_b)^2]
\]

with

\[
\ell_b = \frac{\ell}{2\sqrt{\mu - \bar{\mu}}}
\]

such that the critical behavior at the critical planes \( z = z_b \) is essentially determined by the linear term

\[
V_b = \Delta z/\ell_b, \quad \Delta z \equiv z - z_b.
\]

The critical behavior at the critical planes \( z = z_b \) can be derived using the same arguments of Ref. [32], applying them to the particular case of slab geometries where the harmonic potential is only applied along the transverse direction, while the system is translationally invariant along the planar directions. The system develops critical correlations around the planes \( z = z_b \), with a length scale

\[
\xi_b \ell_b, \quad \theta_b = \frac{\nu}{1 + \nu} = 0.40181(3).
\]

For example, the one-particle correlation function along a transverse direction is expected to scale as

\[
G[(\mathbf{x}, z_1), (\mathbf{x}, z_2)] \approx \xi_b^{-1-\eta} \theta_b^{\Delta z_1/\xi_b, \Delta z_2/\xi_b}.
\]

V. LOW-TEMPERATURE BEHAVIOR OF QUASI-2D BOSONIC GASES

This section summarizes some exact results which are expected to characterize the low-temperature QLRO phase of quasi-2D interacting bosonic gases up to the BKT transition.

### A. The QLRO phase below the BKT transition

The general universal features of the QLRO phase of quasi-2D systems with a U(1) symmetry are described by the Gaussian spin-wave theory

\[
H_{\text{sw}} = \frac{\beta}{2} \int d^2x (\nabla \phi)^2.
\]

For \( \beta \geq 2/\pi \), corresponding to \( 0 \leq \eta \leq 1/4 \), this spin-wave theory describes the QLRO phase. The values \( \beta = 2/\pi \) and \( \eta = 1/4 \) correspond to the BKT transition (29).

The spin-wave correlation function

\[
G_{\text{sw}}(\mathbf{x}_1 - \mathbf{x}_2) = \langle e^{-i\phi(\mathbf{x}_1)} e^{i\phi(\mathbf{x}_2)} \rangle
\]

is expected to provide the asymptotic large-\( L \) behavior of the two-point function of 2D interacting bosonic gases within the QLRO phase. For \( |\mathbf{x}_1 - \mathbf{x}_2| \ll L \),

\[
G_{\text{sw}}(\mathbf{x}_1, \mathbf{x}_2) \sim \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^\eta}.
\]
where the exponent \( \eta \) is related to the coupling \( \beta \) by
\[
\eta = \frac{1}{2\pi \beta}.
\]  

(39)

The general size dependence of \( G_{sw} \) on a square \( L^2 \) box with PBC is also known: \[73, 79, 81\]
\[
G_{sw}(x, L) = C(x, L)^\eta \times E(x, L),
\]  

(40)

\[
C(x, L) = e^{\pi y_0^2 \theta_1'(0, e^{-\pi})},
\]
\[
E(x, L) = \sum_{n_1, n_2=-\infty}^{\infty} W(n_1, n_2) \cos[2\pi(n_1 x_1 + n_2 x_2)]
\]
\[
\sum_{n_1, n_2=-\infty}^{\infty} W(n_1, n_2),
\]

where \( x \equiv (x_1, x_2), y_i \equiv x_i/L, \theta_1(u, q) \) and \( \theta_1'(u, q) \) are \( \theta \) functions [82].

Using Eq. (40), one can easily compute the universal function \( R_L(\eta) \), where \( R_L \equiv \xi/L \) and \( \xi \) is the second-moment correlation length defined as
\[
\xi^2 = \frac{L^2}{4\pi^2} \left( \frac{\chi}{\chi_1} - 1 \right),
\]  

(41)

\[
\chi = \int d^2 x G_{sw}(x), \quad \chi_1 = \int d^2 x \cos \left( \frac{2\pi x_1}{L} \right) G_{sw}(x).
\]

Analogous results are obtained for the helicity modulus [81]
\[
Y(\eta) = \frac{1}{2\pi \eta} - \frac{\sum_{n=\infty}^{\infty} n^2 \exp(-\pi n^2/\eta)}{\sum_{n=\infty}^{\infty} \exp(-\pi n^2/\eta)}.
\]  

(42)

The above asymptotic large-\( L \) behaviors (at fixed \( T \) or \( \eta \) are approached with power-law corrections, indeed
\[
R_L(L, \eta) \equiv \frac{\xi}{L} = R_L(\eta) + aL^{-\varepsilon},
\]  

(43)

\[
Y(L, \eta) = Y(\eta) + aL^{-\zeta},
\]  

(44)

respectively, where \( \varepsilon \) and \( \zeta \) are the exponents associated with the expected leading corrections: [83, 84]
\[
\varepsilon = \text{Min}[2, -\eta, \kappa], \quad \zeta = \text{Min}[2, \kappa],
\]  

(45)

\[
\kappa = 1/\eta - 4 + O[(1/\eta - 4)^2].
\]  

(46)

With increasing \( T \) within the QLRO phase, the critical exponent \( \eta \) of the two-point function, cf. Eq. [38], increases up to \( \eta = 1/4 \) corresponding to the BKT transition. Therefore, close to the BKT transition, i.e. for \( T \ll T_{BKT} \), we may expand the universal curves \( R_L(\eta) \) and \( Y(\eta) \) around \( \eta = 1/4 \), obtaining
\[
R_L(\eta) = 0.7506912222 + 1.699451 \left( \frac{1}{4} - \eta \right) + \ldots,
\]  

(47)

\[
Y(\eta) = 0.6365081782 + 2.551196 \left( \frac{1}{4} - \eta \right) + \ldots
\]  

(48)

We expect that the above universal behaviors are also realized in the low-temperature phase of BH models within slab geometries, for \( T < T_{BKT} \), by the two-point functions \( g(x) \), cf. Eq. [4], and the quantities \( R_L \equiv \xi/L \) and \( Y \) defined in Eq. [9].

B. Finite-size behavior at the BKT transition

The BKT transition is characterized by logarithmic corrections to the asymptotic behavior, due to the presence of marginal renormalization-group (RG) perturbations at the BKT fixed point [81, 83, 85–88].

The asymptotic behaviors at the BKT transition for \( R_L \) and \( Y \) can be obtained by replacing [81, 83]
\[
1/4 - \eta \approx \frac{1}{8w}, \quad w = \ln \frac{L}{\Lambda} + \frac{1}{2} \ln \ln \frac{L}{\Lambda},
\]  

(49)

into Eqs. (47) and (48). The nonuniversal details that characterize the model (such as the thickness \( Z \) of the quasi-2D BH models) are encoded in the model-dependent scale \( \Lambda \). Thus one obtains the asymptotic large-\( L \) behavior
\[
R(L, T_{BKT}) = R^* + C_R w^{-1} + O(w^{-2}).
\]  

(50)

for both \( R = Y, R_L, \) with
\[
Y^* = 0.6365081789, \quad C_Y = 0.31889945, \quad R_L^* = 0.7506912222, \quad C_{R_L} = 0.21243137,
\]  

(51)

for PBC.

In numerical analyses, Eq. (50) may be used to locate the BKT transition point, i.e. by requiring that the finite-size dependence of the data matches it. However we note that this straightforward approach is subject to systematic errors which get suppressed only logarithmically with increasing \( L \). This makes the accuracy of the numerical or experimental determination of the critical parameters quite problematic. This problem can be overcome by the so-called matching method [56, 73, 81, 86, 87, 89, 90], which allows us to control the whole pattern of the logarithmic corrections, leaving only power-law corrections.

The matching method exploits the fact that the finite-size behavior of RG invariant quantities \( R \), such as \( R_L \) and \( Y \), of different models at their BKT transition shares the same logarithmic corrections apart from a nonuniversal normalization of the scale. Indeed, the \( L \)-dependence of two models at their BKT transition is related by the asymptotic relation
\[
R^{(1)}(L_1, T^{(1)}_{BKT}) \approx R^{(2)}(L_2 = \lambda L_1, T^{(2)}_{BKT}),
\]  

(53)

apart from power-law corrections, which are \( O(L^{-2}) \) for the helicity modulus \( Y \) and \( O(L^{-7/4}) \) for the ratio \( R_L \). The matching parameter \( \lambda \) is the only free parameter, but it does not depend on the particular choice of the RG invariant quantity. The matching method consists in finding the optimal value of \( T \) matching the finite-size behavior of \( Y \) and \( R_L \) of the 2D XY model whose value of \( T_{BKT} \) is known with high accuracy [81, 83]. The complete expression of \( R_L \) and \( Y \) of the 2D XY model have been numerically obtained by high-precision numerical studies [81, 83] and by extrapolations using RG results for the asymptotic behavior. For example, the \( L \)-dependence of
the helicity modulus $Y$ at the BKT transition of the 2D $XY$ model is accurately reconstructed by the following expression \[20\]
\[
\tilde{Y}_{XY}(L) \equiv Y_{XY}(T_{BKT}, L) = \frac{L^{z\nu} + \sum_{i=1}^{\infty} \frac{1}{(i L)^{z\nu}}}{L^{(1+\eta)z\nu}}
\]
\[
= 0.6365081782 + 0.318899454w^{-1} + 2.0319176w^{-2} - 40.492461w^{-3} + 325.66533w^{-4} - 874.77113w^{-5} + 8.43794L^{-2} + 79.1227L^{-4} - 210.217L^{-6},
\]
where $w$ is given in Eq. (49) with $L = L_{XY} = 0.31$.

The matching method has been already applied \[7,3\] to the 2D BH models \[1\], obtaining the accurate estimate $T_{BKT} = 0.6877(2)$ in the hard-core $U \to \infty$ limit and at half filling ($\mu = 0$).

VI. OFF-EQUILIBRIUM SLOW DYNAMICS AND DIMENSIONAL CROSSOVER

The dynamical behavior of statistical systems driven across phase transitions is a typical off-equilibrium phenomenon. Indeed, the large-scale modes present at the transition are unable to reach equilibrium as the system changes phase, even when the time scale $t_s$ of the variation of the system parameters is very large. Such phenomena are of great interest in many different physical contexts, at both first-order and continuous transitions, where one may observe hysteresis and coarsening phenomena, the Kibble-Zurek (KZ) defect production, etc, see, e.g., Refs. [30, 33, 57, 58, 91, 102]. The correlation functions obey general off-equilibrium scaling (OS) laws in the limit of large time scale $t_s$ of the variations across the transition, which are controlled by the universal static and dynamic exponents of the equilibrium transition \[94, 96\].

We now consider the off-equilibrium behavior arising from slow time variations of the temperature $T$ across the BEC transition. We assume a standard linear protocol, varying $T$ so that
\[
\delta(t) \equiv 1 - T(t)/T_c = t/t_s,
\]
starting at a time $t_i < 0$ in the high-$T$ phase and ending at $t_f > 0$ in the low-$T$ phase. $t_s$ is the time scale of the temperature variation. The BEC transition point corresponds to $t = 0$ (however this is not strictly required, it is only convenient for our discussion). Several experiments implementing off-equilibrium time-dependent protocols in cold-atom systems have been reported, see, e.g., Refs. [30, 31, 33, 105, 107].

Beside the static critical exponent \[64\] $\nu = 0.6717(1)$ of the 3D $XY$ universality class, we also need information on the dynamic exponent $z$ of the critical modes at the BEC transition. This is characterized by the dynamic exponent $z = d/2$, thus $z = 3/2$ in 3D, associated with the model-$F$ dynamics \[103, 104\] which is conjectured to describe the dynamic universality class of the 3D BEC transition.

In the standard thermodynamic limit of cubic-like boxes, with $L_1 \sim L_2 \sim L_3 \sim L$ and $L \to \infty$, one defines the OS limit as the large-time-scale limit, $t_s \to \infty$, keeping the OS scaling variables
\[
T \equiv t/t_s^\kappa, \quad x_s \equiv x/t_s^{\zeta},
\]
fixed. Scaling arguments allow us to determine the appropriate exponents $\kappa$ and $\zeta$, obtaining \[57, 58, 96\]
\[
\kappa = \frac{z\nu}{1 + z\nu}, \quad \zeta = \frac{\nu}{1 + z\nu},
\]
where $\nu$ and $z$ are the static correlation-length and dynamic exponents. In particular, by inserting the values of $\nu$ and $z$, we obtain $\kappa = 0.50188(4)$ and $\zeta = 0.33459(3)$. We may apply these OS arguments to the equal-time two-point correlation function, measured after a time $t$ and averaged over the initial Gibbs distribution at a given initial temperature $T > T_c$. Standard scaling arguments lead to the OS asymptotic behaviors \[90\]
\[
G(x, t, t_s) \approx t_s^{-(1+\eta)}G_0(x_s, T).
\]
Moreover, we expect
\[
\xi(t, t_s) \approx t_s^{\zeta}R_0(T),
\]
for any length scale associated with the critical modes. Experimental studies of this dynamic behavior, and the related KZ defect production, led to the estimate \[30\] $\zeta = 0.35(4)$, which is in good agreement with the theoretical result \[57\].

We now discuss how this off-equilibrium behavior may change in quantum gases confined within slab geometries with $Z \ll L$, and in particular with a finite thickness $Z$ and infinite $L \to \infty$ planar sizes. Analogous experiments with quasi-2D cold-atom systems constrained in slab geometries have been reported in Refs. [31, 33] (homogeneous hard-wall traps along the planar directions and harmonic along the transverse direction). They observe the emergence of coherence when cooling the atomic gas through the BEC temperature.

The off-equilibrium behavior arising from the slow variation of the temperature across the BEC transition point is made particularly complex by the presence of a close quasi-2D BKT transition. Thus, disentangling the behaviors corresponding to BEC and BKT is quite hard in experimental or numerical analyses. The authors of \[31, 33\] interpreted the observed behavior as a transverse condensation phenomenon \[31, 33, 108, 109\]. In the following we put forward an alternative framework to describe the dimensional crossover in slab geometries, based on an off-equilibrium FSS (OFSS).

As already said, for a finite thickness $Z$, even though $L \to \infty$, the system does not develop a diverging correlation length at the 3D BEC transition temperature $T_c$, but $\xi$ remains of the order of the transverse size $Z$. Thus the systems can evolve adiabatically, i.e. its evolution can be perfomed by passing through quasi-equilibrium
where $S(T)$ replacing the same TTSS arguments of Sec. IV. Apart from the case of a transverse harmonic trap, extending to the case of a transverse harmonic trap, unscaled functions must somehow show the off-equilibrium behavior. Eq. (13). In particular, at $t = 0$ corresponding to $T(t) = T_c$, we expect to recover the equilibrium result $\xi(z) \approx Z$ when $t_s \gg Z^{1/\delta}$. Note however that the equilibrium limit is not well defined for any $X_o$, because it diverges when $X_o \geq X_{BKT}$, cf. Eq. (10), corresponding to the BKT transition. Around $X_{BKT}$ the behavior of the scaling functions must somehow show the off-equilibrium singularities associated with a slow passage thorough a BKT transition.

The above scaling behaviors can be straightforwardly extended to the case of a transverse harmonic trap, using the same TTSS arguments of Sec. IV. Apart from replacing $Z$ with $\delta^{\theta}$, the main features of the OS behavior remain the same.

We mention that experiments under analogous time-dependent protocols crossing the BEC transition have been performed with atomic gases confined in slab-like traps with a transverse harmonic trapping potential. They were able to check the initial 3D behavior, without a clear identification of the subsequent quasi-2D behavior. The computation of the defect production arising from the Kibble-Zurek mechanism is further complicated by later-time coarsening phenomena.

However things become quite involved when the thickness $Z$ becomes large because the BKT transition gets very close to the BEC temperature $T_c$, cf. Eq. (15). Therefore, the analysis of numerical and experimental data may become hard, and straightforward power-law fits may turn out to be misleading.

In order to check the dimensional crossover scenario discussed in the previous section, we present a numerical study of the equations of the BH model in the hard-core $U \to \infty$ limit and at zero chemical potential, corresponding to half filling, i.e., $\langle n_o \rangle = 1/2$ for any $T$. In the hard-core limit and for $\mu = 0$, the 3D BEC transition occurs at $T_c = 2.01599(5)$ and the 2D BKT transition at $T_{\text{BKT}} = 0.6877(2)$.

**VII. NUMERICAL RESULTS FOR THE BH MODEL**

![QMC data of the planar correlation length $\xi$ for the hard-core $U \to \infty$ BH model at zero chemical potential, for various values of $L$, $Z$ and $T$. The dashed vertical line indicates our estimates of the BKT transition temperature for $Z = 5, 9, 13$ (from the left to right). The statistical errors of the data are so small to be hardly visible.](image)

Numerical results are obtained by quantum Monte Carlo (QMC) simulations using the directed operator-loop algorithm. We consider slab geometries, i.e., $L^2 \times Z$ lattices with $Z \ll L$, with OBC along the transverse directions, and PBC along the planar directions. We present numerical results for some values of the thickness $Z$, in particular $Z = 5, 9, 13$, various planar sizes up to $L \approx 100$, and several values of the temperature $T \leq T_c$. The maximum size $Z$ of our numerical study is limited by the fact that the computational effort of QMC rapidly increases, because they also require larger values of the planar sizes.

We compute the observables defined in Sec. III. In QMC simulations the helicity modulus is obtained from the lin-
ear winding number $W_a$ along the $a^{th}$ direction, i.e.

$$Y = Y_a = \langle W_a^2 \rangle, \quad W_a = \frac{N_a^+ - N_a^-}{L} \quad (63)$$

where $N_a^+$ and $N_a^-$ are the numbers of non-diagonal operators which move the particles respectively in the positive and negative $a^{th}$ direction.

Figure 3 shows data for the planar second-moment correlation length $\xi$ defined in Eq. (7), for $Z = 5, 9, 13$ and $T \lesssim T_c$. We observe that $\xi$ is small for $T > T_c$, and apparently $L$- and $Z$-independent (for sufficiently large $L$ and $Z$), indicating that it remains finite in the large-$L$ and large-$Z$ limit. Around $T_c$ the data of $\xi$ appear to converge to a finite value when increasing $L$ at fixed $Z$; however, they show that $\xi$ increases with increasing $Z$, approximately as $\xi \sim Z$. Then, for sufficiently small values of $T$, the data begin showing a significant dependence on $L$. At low temperature we observe $\xi \sim L$ at fixed $T$, suggesting that $\xi$ diverges with increasing $L$ even when keeping $Z$ fixed. In the following we show that this apparently complicated behavior can be explained by the dimensional crossover scenario put forward in the previous sections.

To begin with, we investigate the nature of the low-temperature behavior where the planar correlation length $\xi$ appears to diverge with increasing $L$. According to the arguments of the previous sections, at low temperature BH systems for any thickness $Z$ should show a quasi-2D QLRO phase, whose behavior is essentially described by the 2D spin-wave theory, see in particular Sec. V. As discussed in Sec. V A this implies universal relations among the ratio $R_L \equiv \xi / L$, the quasi-2D helicity modulus $Y$ and the exponent $\eta$ characterizing the planar two-point

**FIG. 4:** $R_L \equiv \xi / L$ versus $Y$ for $Z = 5$ (bottom) and $Z = 9$ (top), and for several values of $L$ and $T$. The full line shows the spin-wave curve $R_L(Y)$ which is expected to be asymptotically approached for $L \to \infty$ within the QLRO phase; its end point corresponds to the BKT transition. In particular, for $Z = 5$ the values of $T$ of the data shown in the bottom figure are (from right to left) $T = 1.1858, 1.3518, 1.5179, 1.64, 1.65, 1.67, 1.6839, 1.85$. The behavior of the data close to the BKT point suggests $T_{\text{BKT}}(Z = 5) \approx 1.65$. Analogously for $Z = 9$ the data are for $T = 1.8, 1.81, 1.82, 1.83, 1.84, 1.85, 1.9, 2$; they suggest that $T_{\text{BKT}}(Z = 9) \approx 1.83$. The statistical errors of the data are so small to be hardly visible.

**FIG. 5:** Data of $Y$ for $Z = 9$ (bottom) and $Z = 13$ (top) around the corresponding BKT temperatures. Analogous results have been obtained for $Z = 5$. The dashed horizontal line indicates the BKT value $Y^* = 0.6365...$. The dotted vertical lines indicate the interval corresponding to our best estimates of $T_{\text{BKT}}$, i.e. $T_{\text{BKT}} = 1.829(1)$ for $Z = 9$ and $T_{\text{BKT}} = 1.899(1)$ for $Z = 13$, obtained by the matching procedure.
correlation function. In Fig. 6 we plot data of $R_L$ versus those of $Y$, comparing them with the universal curve $R_L(Y)$ which can be easily obtained from the spin-wave results reported in Sec. VII A. This curve ends at the BKT point $(Y^*, R_L^*) = (0.6365... , 0.7506...)$. For sufficiently small $T$, depending on the value of $Z$, the data approach the universal spin-wave curve $R_L(Y)$ with increasing $L$. Extrapolations using the expected power-law corrections, cf. Eqs. (13) and (14), turn out to be consistent with the exact spin-wave results. Therefore, the numerical results nicely support the existence of a QLRO phase for any $Z$, with the expected universal spin-wave behaviors.

We also note that above a given temperature, depending on the thickness $Z$, the data do not approach the spin-wave curve $R_L(Y)$ anymore, as it is expected to occur for $T > T_{BKT}$ where both $R_L$ and $Y$ vanish in the large-$L$ limit. Therefore, the data of Fig. 6 allow us to approximately locate $T_{BKT}$ between the temperature values of the data closest to the BKT point $(Y^*, R_L^*)$ which respectively approach the spin-wave curve and deviate from it. We already note that $T_{BKT}$ increases with increasing $Z$. This can be also inferred by the data of the helicity modulus $Y$ versus the temperature, see Fig. 6. They are generally decreasing, and for sufficiently large $T$ they appear to cross the value $Y = Y^* \approx 0.6365$ corresponding to the BKT transition, indicating that those values of $T$ are larger than $T_{BKT}$.

More accurate estimates of $T_{BKT}$ can be obtained by looking for the optimal values of $T$ achieving the matching of the available data of $Y$ and $R_L$ with the finite-size dependence of the 2d XY model at its BKT transition, see Sec. VII B. In particular, $T_{BKT}(Z)$ is given by the value of $T$ providing the optimal matching of the data of $Y(Z, L, T)$ with the finite-size dependence of the helicity modulus of the 2D XY model, i.e.

$$Y(Z, L, T) = \tilde{Y}_{XY}[\lambda(Z)L] + O(L^{-2}),$$

with $\tilde{Y}_{XY}$ given by Eq. (71). Some matching procedures are described in Ref. [73]. This numerical analysis largely suppresses the systematic error, because it is not affected by logarithmic corrections, but only $O(L^{-2})$ power-law corrections. For $Z = 1$ the optimal matching led to the estimate $T_{BKT}(Z = 1) = 0.6877(2)$ and $\lambda(Z = 1) \approx 1.5$.

We determine the optimal values of $T$ and $\lambda(Z)$ satisfying the scaling relation (34). We skip most details of the numerical matching procedures, which can be found in Ref. [72]. We only mention that we use QMC data from $L = 20$ to $L = 100$, for sufficiently close values of $T$ to obtain reliable estimates for any $T$ by interpolation, see Fig. 6. Our estimates for the optimal matching parameters are $T_{BKT}(Z = 5) = 1.645(2)$, $T_{BKT}(Z = 9) = 1.829(1)$, and $T_{BKT}(13) = 1.899(1)$; correspondingly we obtain $\lambda(Z = 5) = 0.4(2)$, $\lambda(Z = 9) = 0.20(5)$, $\lambda(Z = 13) = 0.14(2)$. The statistical error of the analysis is estimated using bootstrap methods. The error reported above takes also into account the variations of the results when changing the procedure to obtain the optimal matching, for example when considering or not the $O(L^{-2})$ scaling corrections, and varying the minimum size $L$ of the data used in the analysis.

The quality of the matching can be inferred from Fig. 6 which shows the data at the optimal matching values of $T_{BKT}$ versus the ratio $L/\lambda(Z)$ with $\lambda(Z) = \Lambda_{XY}/\lambda(Z)$, so that all data of $Y$, for any $Z$, are expected to follow the same curve $\tilde{Y}_{XY}$ versus $L/\lambda_{XY}$ with $\Lambda_{XY} = 0.31$. This is indeed what we observe, apart from some scaling corrections at the smallest values of $L$, which are expected to get suppressed as $O(L^{-2})$. We consider the results of the matching analysis of the $Y$ data as our best estimates of $T_{BKT}$. Note also that the values of $\lambda(Z)$ are decreasing, as expected because the value $\lambda(Z)L$ is somehow related to the equivalent planar size of the lattice, and for slab geometries one may expect that this is approximately given by the aspect ratio $L/Z$, thus $\lambda(Z) \sim 1/Z$ roughly.

An analogous numerical analysis can be done using the data of $R_L$. However it turns out to be less accurate due to larger scaling corrections. As also observed in Ref. [73], $R_L$ is subject to significantly large power-law scaling corrections, which decrease as $L^{-7/4}$. The $XY$ curve of $R_L$ is reported in Ref. [73]. Note that once determined $T_{BKT}$ and $\lambda(Z)$, there are no other free parameters to optimize the matching. The inset of Fig. 6 shows the data and their comparison with the $XY$ curve using the values of $T_{BKT}$ and $\lambda(Z)$ obtained from the analysis of the data of $Y$. The data appear to approach the asymptotic curve with increasing $L$, therefore they are consistent with the theoretical predictions. However, as already mentioned, we note that the approach to the expected asymptotic behavior is characterized by larger scaling corrections, thus requiring larger lattice sizes to obtain independent
estimates of $T_{BKT}$ as accurate as those obtained using the data of $Y$.

Figure 7 shows $\delta_{BKT}(Z) \equiv 1 - T_{BKT}(Z)/T_c$ versus $Z^{-1/\nu}$, as obtained from the above estimates of $T_{BKT}$. The data turn out to be consistent with the expected asymptotic behavior $\delta_{BKT}(Z) \sim Z^{-1/\nu}$. We also estimate

$$X_{BKT} = \lim_{Z \to \infty} Z^{1/\nu} \delta_{BKT} = 3.2(1),$$

by extrapolating the available data for the product $Z^{1/\nu} \delta_{BKT}$ using the ansatz

$$Z^{1/\nu} \delta_{BKT} = X_{BKT} + c Z^{-\omega},$$

see the inset of Fig. 7 where $\omega = 0.785(20)$ is the leading scaling-correction exponent of the 3D XY universality class.

Finally, we check the TFSS $R_Z \approx f_\xi(X)$ with $X = Z^{1/\nu} \delta$ around $T_c$, in the planar thermodynamic limit, i.e. when $\xi, Z \ll L$. As argued in Sec. III D the scaling function $f_\xi(X)$ is expected to have an essential singularity at $X_{BKT} \approx 3.2$, cf. Eq. (11). In Fig. 8 we show data of $R_Z$ around $T_c$ versus $X \equiv \delta Z^{1/\nu}$. They support the TFSS behavior of $R_Z$. Scaling corrections are expected to decrease as $Z^{-\omega}$. They appear significantly larger for $X > 0$, when approaching the singularity at $X_{BKT}$. By extrapolating the available data at $T_c$ using $R_Z(Z, T_c) = R^*_Z + c Z^{-\omega}$ (see the inset of Fig. 8), we estimate $R^*_Z = 0.372(3)$ for the universal large-$Z$ ratio $R_Z \equiv \xi/\delta$ characterizing the TFSS of the critical planar correlation length. An analogous scaling behavior is expected for the planar susceptibility defined as in Eq. (5). The data shown in Fig. 9 nicely support the corresponding TFSS (19).

**FIG. 7:** Estimates of $\delta_{BKT}(Z) \equiv 1 - T_{BKT}(Z)/T_c$ vs $Z^{-1/\nu}$ with $\nu = 0.6717$. The data are compatible with the expected behavior $\delta_{BKT}(Z) \sim Z^{-1/\nu}$. The inset shows the product $Z^{1/\nu} \delta_{BKT}(Z)$ versus $Z^{-\omega}$ with $\omega = 0.785$ (the dashed line is obtained by a linear fit), which is the expected behavior of the leading scaling corrections.

**FIG. 8:** Scaling of $R_Z$ around $T_c$ (most data are taken for $L = 120$), which is sufficiently large to provide a good approximation of the $L \to \infty$ limit in the range of $Z$ and $X$ values considered, within about 1%. We plot the data versus $X \equiv Z^{1/\nu} \delta$ with $\nu = 0.6717$ and $\delta \equiv 1 - T/T_c$. The inset shows the data of $R_Z$ at $T_c$ versus $Z^{-\omega}$ which is the expected behavior of the leading scaling corrections (the dashed line is).

**FIG. 9:** Scaling of the planar susceptibility $\chi$ around $T_c$. We plot $Z^{-1/\nu} \chi$ versus $X \equiv Z^{1/\nu} \delta$. The data support the asymptotic TFSS $\chi \approx Z^{-\eta} f_\chi(X)$.

**VIII. SUMMARY**

We have studied the phase-coherence properties of Bose gases confined within slab-like boxes of size $L^2 \times Z$ with $Z \ll L$, at the 3D BEC transition temperature $T_c$ and at lower temperatures. Unlike systems confined within cubic-like geometries, i.e. boxes with $L \sim Z$, the low-temperature behavior of gases confined within slab geometries is also characterized by the possibility of undergoing a finite-temperature quasi-2D BKT transition at $T_{BKT} < T_c$ with $T_{BKT}$ depending on the thickness $Z$. Below $T_{BKT}$ the planar one-particle correlations decay algebraically, as predicted by the QLRO of the 2D spin-wave theory.

Therefore, Bose gases in slab geometries experience a dimensional crossover with decreasing $T$, from 3D
haviors for $T \gtrsim T_c$ to a quasi-2D critical behavior for $T \lesssim T_{BKT}$. However, in the limit of large thickness $Z$ the quasi-2D BKT transition temperature approaches that of the 3D BEC transition, i.e. $T_{BKT} \to T_c$, for $Z \to \infty$. The interplay of 3D and quasi-2D critical modes can be described by the TFSS limit for systems on slab geometries: $Z \to \infty$ and $T \to T_c$ keeping the product $(T - T_c)Z^{1/\nu}$ fixed (the planar sizes are assumed to be infinite), where $\nu$ is the correlation-length exponent at the 3D BEC transition. The corresponding TFSS functions present an essential singularity due to the quasi-2D BKT transition below $T_c$. A similar TTSS behavior is also put forward in the case the particles are trapped by a transverse harmonic potential in the limit of large transverse trap size $\ell$. In the TTSS framework the length scale $\xi = \ell^\theta$, where $\theta = 2\nu/(1 + 2\nu) = 0.57327(4)$, plays the same role of the transverse size $Z$ of the TFSS.

We also extend the discussion to the off-equilibrium behavior arising from slow time variations of the temperature $T$ across the BEC transition. In particular we consider the linear protocol $\delta(t) = 1 - T(t)/T_c = t/t_s$ where $t_s$ is a time scale. The corresponding off-equilibrium behavior is made particularly complex by the presence of the close quasi-2D BKT transition at $T_{BKT} < T_c$, which is also crossed during the time-dependent protocol. Thus, disentangling the behaviors corresponding to BEC and BKT is quite hard in experimental or numerical analyses. We argue that the off-equilibrium behavior in the limit of large $t_s$ can be described by an off-equilibrium FSS theory for bosonic gases confined within slab geometries, extending the TFSS of the equilibrium properties.

To provide evidence of the dimensional-crossover scenario in interacting bosonic gases, we present a numerical study of the BH model (11) in anisotropic slab-like lattices $L^2 \times Z$ with $Z \ll L$. With decreasing $T$ from the high-temperature normal phase, we first observe a quasi-BEC transition where the critical length scale $\xi$ gets large, but it does not diverge, being limited by $\xi \sim Z$ (keeping $Z$ fixed). Then a BKT transition occurs to a QLRO phase, where the system develops planar critical correlations essentially described by the 2D Gaussian spin-wave theory. We show that the 3D$\to$2D dimensional-crossover scenario explains the apparently complex dependence on $T$, $Z$, and $L$ of the one-particle correlation functions and the corresponding length scale, when decreasing the temperature from $T > T_c$ to $T < T_{BKT} < T_c$. The results turn out to be consistent with the predictions of the TFSS at the BEC transition.

The dimensional-crossover scenario is expected to apply to any quantum gas of interacting bosonic particles constrained in boxes or lattice structures with slab geometries. Analogous arguments apply to $^4$He systems in film geometries [42], and to 3D XY spin models defined in lattices with slab geometries [53, 54]. We conclude stressing that the above issues related to the dimensional-crossover scenario are of experimental relevance since cold-atom systems confined within slab geometries can be effectively realized, see e.g. Refs. [28, 30, 31, 33]. These experimental setups offer the possibility of investigating the dependence of the phase-coherence properties on the geometry of the cold-atom system. Our study provides a framework to interpret the experimental or numerical data related to the 3D$\to$2D dimensional crossover in Bose gases confined within slab geometries, and in particular their complicated dependence on the thickness $Z$.

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