Frictional coupling between sliding and spinning motion

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We show that the friction force and torque, acting at a dry contact of two objects moving and rotating relative to each other, are inherently coupled. As a simple test system, a sliding and spinning disk on a horizontal flat surface is considered. We calculate, and also measure, how the disk is slowing down, and find that it always stops its sliding and spinning motion at the same moment. We discuss the impact of this coupling between friction force and torque on the physics of granular materials.

Sliding friction and incomplete normal restitution are normally the main dissipation mechanisms at the contact between two solid grains. They are largely responsible for the fact that the flow properties of granular media differ from those for liquids and solids. Their microscopic origins are currently under intense investigation (see e.g. [1]). On large scales compared to the grain diameter they sometimes transform into unexpected phenomenological friction laws, which have recently been discussed [2, 3, 4, 5, 6, 7]. As the influence of rolling and torsion friction is commonly regarded as negligible, they have been much less investigated so far. However, it turns out that in certain situations they may become crucial, for instance for the stabilization of pores in cohesive powders [8, 9]. Another striking phenomenon will be discussed below.

In fact, rolling and torsion friction are indispensable for a unified view of the dissipation mechanisms at the contact of two viscoelastic spheres, because on the one hand incomplete restitution and rolling friction, and on the other hand sliding and torsion friction are coupled, which will be the main point of this letter. These four dissipation mechanisms correspond to the six degrees of freedom of the relative motion at the contact between two solid spheres. The relative motion of two solid spheres has three translational degrees of freedom, characterized by a velocity vector with one normal component \( \mathbf{v}_n \) (deformation mode) and two tangential components \( \mathbf{v}_t \) (sliding mode), and three rotational ones, characterized by an angular velocity vector, again with two tangential components \( \omega_t \) (rolling mode) and one normal component \( \omega_n \) (spinning mode). While the viscoelastic dissipation mechanism of normal restitution and rolling friction couples \( \mathbf{v}_n \) and \( \omega_n \) [10, 11], the dissipation due to sliding and torsion friction couples \( \mathbf{v}_t \) and \( \omega_t \).

In this Letter we focus on the coupling between \( \mathbf{v}_t \) and \( \omega_t \). For viscoelastic spheres these are the dominant dissipation channels in the quasistatic limit, where \( (1 - \text{restitution coefficient}) \) [11, 12], as well as the coefficient of rolling friction vanish [13]. The reason, why torsion friction, i.e., the torque leading to a decrease of \( \omega_n \), is often neglected, is that it involves the radius of the contact area between the two spheres and hence is small.

Therefore, in order to make our point more clear, instead of the contact between two spheres we consider a flat disk on a horizontal flat surface with nonzero initial translational and angular velocity. The disk is lying on one of its sides, and we assume that this side is in full contact with the table during the motion [see Fig. 1(a)]. The friction force and torque acting on the disk will slow down the sliding and spinning motion until the disk stops moving. We address two questions: (i) how are the friction force and torque related to each other, (ii) what does this imply for the coupling of sliding and spinning motion?

First we calculate the friction force and torque acting on the disk as a function of its instantaneous velocity and angular velocity. We apply the Coulomb friction law, which says that the magnitude of the friction force is proportional to the normal force, while its direction is the opposite of the direction of the surfaces’ relative velocity. Assuming that the mass distribution of the disk is homogeneous, the friction force is

\[
\mathbf{F} = \frac{\mu F_n}{R^2 \pi} \int_{r \in A} \frac{\mathbf{v} + \omega \times \mathbf{r}}{|\mathbf{v} + \omega \times \mathbf{r}|} r^2 \, dr,
\]

where \( R \) is the radius, \( \mathbf{v} \) is the velocity, and \( \omega \) is the angular velocity of the disk, \( \mu \) is the friction coefficient, and the integration extends over the area of the disk with \( \mathbf{r} \) vectors starting at the center. \( F_n \) is the normal component of the force pressing the objects together at the
contact, in our case \( F_n = mg \), where \( m \) is the mass of the disk and \( g \) is the gravitational acceleration. We found it useful to introduce the dimensionless quantity \( \varepsilon = v/R\omega \) with \( v = |v| \) and \( \omega = |\omega| \), because the friction force depends on \( v \) and \( \omega \) only through this combination:

\[
\mathbf{F} = -\mu F_n \frac{\mathbf{e}_v + \mathbf{e}_\omega \times \hat{r}}{|\mathbf{e}_v + \mathbf{e}_\omega \times \hat{r}|} d^2 r,
\]

where \( \mathbf{e}_v = v/v, \mathbf{e}_\omega = \omega/\omega, \hat{r} = r/R, \) and \( A_1 \) is the area of the unit disk. Figure 3(b)-(d) show local relative velocities on the surface of the disk for various values of \( \varepsilon \). Note that the local friction force does not depend on the absolute value of the relative velocity, only on its direction. After evaluating the integral in Eq. (1) one gets \( \mathbf{F} = -\mu F_n \mathcal{F}(\varepsilon) \mathbf{e}_v \), where

\[
\mathcal{F}(\varepsilon) = \begin{cases} 
\frac{4}{3} (\varepsilon^2 + 1) E(\varepsilon) + (\varepsilon^2 - 1) K(\varepsilon), & \varepsilon \leq 1 \\
\frac{4}{\varepsilon \pi} (\varepsilon^2 + 1) E\left(\frac{1}{\varepsilon}\right) - (\varepsilon^2 - 1) K\left(\frac{1}{\varepsilon}\right), & \varepsilon \geq 1.
\end{cases}
\]

Here \( K(\varepsilon) \) and \( E(\varepsilon) \) are the complete elliptic integral functions of the first and the second kind, respectively [13]. This calculation and the others below were performed using the mathematics software Maple [14]. The two parts of \( \mathcal{F}(\varepsilon) \) are smoothly connected at \( \varepsilon = 1 \), since \( \lim_{\varepsilon \to 1} \mathcal{F}(\varepsilon) = 8/3\pi \) and \( \lim_{\varepsilon \to 1} \mathcal{F}'(\varepsilon) = 4/3\pi \) from both the left and the right hand side. Here prime denotes differentiation w.r.t. \( \varepsilon \). The limiting values are \( \mathcal{F}(0) = 0 \) and \( \lim_{\varepsilon \to \infty} \mathcal{F}(\varepsilon) = 1 \).

The friction torque is

\[
\mathbf{T} = -\mu F_n R T(\varepsilon) \mathbf{e}_\omega,
\]

and after calculating the integral we get \( \mathbf{T} = -\mu F_n R T(\varepsilon) \mathbf{e}_\omega \), where

\[
T(\varepsilon) = \begin{cases} 
\frac{4}{9} (4 - 2\varepsilon^2) E(\varepsilon) + (\varepsilon^2 - 1) K(\varepsilon), & \varepsilon \leq 1 \\
\frac{4}{\varepsilon \pi} (4 - 2\varepsilon^2) E\left(\frac{1}{\varepsilon}\right) + (2\varepsilon^2 - 5 + \frac{8}{\varepsilon^2}) K\left(\frac{1}{\varepsilon}\right), & \varepsilon \geq 1.
\end{cases}
\]

The two parts of this function are also smoothly connected, as \( \lim_{\varepsilon \to 1} T(\varepsilon) = 8/9\pi \) and \( \lim_{\varepsilon \to 1} T'(\varepsilon) = -4/3\pi \) from both the left and the right hand side. The limiting values are \( T(0) = 2/3 \) and \( \lim_{\varepsilon \to \infty} T(\varepsilon) = 0 \). Figure 3 shows \( \mathcal{F}(\varepsilon) \) and \( T(\varepsilon) \), and also the \( T(\mathcal{F}) \) function. This latter exists and is invertable because both \( \mathcal{F}(\varepsilon) \) and \( T(\varepsilon) \) are strictly monotonic functions.

Now let us calculate how a sliding and spinning disk is slowing down. Assuming that only gravity and friction forces are acting, the scalar equations of motion are

\[
m\frac{dv}{dt} = -\mu mg \mathcal{F}(\varepsilon), \quad \tag{2}
\]

\[
\frac{1}{2} m R^2 \frac{d\omega}{dt} = -\mu mg R T(\varepsilon). \quad \tag{3}
\]

By introducing dimensionless velocities and time as \( v^* = v/\sqrt{Rg} \mu, \omega^* = \omega/\sqrt{R}/\mu, \) and \( t^* = t/\sqrt{g/R} \), Eqs. (2) and (3) reduce to

\[
\frac{dv^*}{dt^*} = -\mathcal{F}(\varepsilon), \quad \tag{4}
\]

\[
\frac{d\omega^*}{dt^*} = -2 T(\varepsilon). \quad \tag{5}
\]

with \( \varepsilon = v^*/\omega^* \). As \( \mathcal{F}(\varepsilon) \) and \( T(\varepsilon) \) are positive for \( \varepsilon > 0 \), the translational and angular velocities are strictly monotonically decreasing in time, as expected. Now the question arises: Is it possible that any of them reaches zero before the other, i.e., may it happen that an initially sliding and spinning disk after some time is only sliding or spinning? We solved Eqs. (4) and (5) numerically with many different initial conditions. The results indicate, as can be seen in Fig. 3(a), that \( v^* \) and \( \omega^* \) always reach zero together, meaning that the disk always stops its sliding and spinning motion at the same moment. Before we show its proof, let us try to explain qualitatively what is happening. If the velocity is much higher than the angular velocity (\( v \gg R\omega \), i.e., \( \varepsilon \gg 1 \)), then the friction torque is negligible compared to the force, see Fig. 3(a). Therefore, the velocity decreases with a higher rate than the angular velocity, and \( \varepsilon \) decreases. On the other hand, if the angular velocity is much higher than the velocity (\( \varepsilon \ll 1 \)), then the friction torque is higher than the force, and \( \varepsilon \) increases. Thus a negative feedback effectively equilibrates the sliding and spinning motion. Indeed, the
numerical results show this behavior, as \( \varepsilon \) always tends to the same value, \( \varepsilon_0 \approx 0.653 \), when the motion stops [Fig. 3(b)]. This means that \( \omega^* \) and \( \nu^* \) not only reach zero simultaneously, but also that their ratio approaches a universal value, irrespective of the initial conditions.

In order to prove that \( \varepsilon \) always has this value at the end of the motion, we derive an autonomous differential equation for \( \varepsilon \) from equations (3) and (4), using the variable transformation \( x = -\ln \omega^* \):

\[
\frac{d\varepsilon}{dx} = \varepsilon - \frac{\mathcal{F}(\varepsilon)}{2\mathcal{T}(\varepsilon)} = f(\varepsilon).
\]

Note that \( \omega^* \to 0 \), the condition of stopping, now corresponds to \( x \to \infty \) (with the exception of pure sliding motion). For small \( \varepsilon \) the right hand side of Eq. (6) vanishes like \( f(\varepsilon) \approx \varepsilon/4 \), while it behaves asymptotically for \( \varepsilon \to \infty \) like \( f(\varepsilon) \approx -\varepsilon \). In between, at \( \varepsilon_0 \), it changes sign (Fig. 4). Therefore Eq. (6) has three fixed points: Two of them, \( \varepsilon = 0 \) and \( \varepsilon = \infty \), are trivial and correspond to pure spinning or pure sliding motion, respectively. For all other initial conditions (\( 0 < \varepsilon < \infty \)), corresponding to initial sliding and spinning, \( \varepsilon_0 \) is the attractive fixed point, meaning that \( \varepsilon \) has this value just before the disk stops its motion, which is what we wanted to prove.

We also performed a simple experiment to measure the friction force and torque acting on a sliding and spinning disk. We set a standard writable CD disk (\( R = 6 \text{ cm} \)), with its data carrying side down, into motion manually on a horizontal polyamide fabric surface several times, and recorded its motion with a Sony DCR-VX2000E PAL digital video camera (25 images/second). Then we processed the images to obtain the position and the orientation of the disk as a function of time. We fitted the position and angular data with second degree polynomials to get the instantaneous translational and angular velocity and acceleration. Then, assuming that only gravitational and frictional forces were acting on the disk, using Eqs. (2) and (3), and having only the friction coefficient as a fit parameter, we were able to plot functions \( \mathcal{F}(\varepsilon) \) and \( \mathcal{T}(\varepsilon) \), see Fig. 5. We had, however, a minor complication: We observed that the friction coefficient slightly increased linearly with the number of throws, \( n \). We found that \( \mu = 0.202 + 0.00053n \) was a good fit, and the data presented in Fig. 5 was obtained using this “time-dependent” friction coefficient in the data processing. Investigating the disk we concluded that the reason for the increase of \( \mu \) was probably that the film on the CD was gradually removed. The disk also had a 1.5 cm diameter hole at its center, but a numerical calculation of \( \mathcal{F}(\varepsilon) \) and \( \mathcal{T}(\varepsilon) \) for this geometry showed that deviation from the full disk case is not significant, it is much smaller than the accuracy of the measured data. Our experimental data also showed that the disk always stopped its sliding and spinning motion at the same moment (within error).

We used the sliding and spinning disk as a simple, illustrative example to show how friction force and torque are coupled. In this case we were able to derive all results analytically, because the local pressure is everywhere the same in the contact area. However, in general the pressure distribution over the contact area will be
non-uniform. As an example, if we replace the flat disk by a cylinder standing on one of its flat faces, then the friction force leads to a torque with respect to the center of mass. Provided that the cylinder does not topple, this torque must be compensated by a pressure increase at the front and a pressure decrease at the rear part of the contact area. Therefore the spinning motion induces a friction component perpendicular to the translational motion, in the direction of $v \times \omega$. Hence, in contrast to the straight sliding of a flat spinning disk, the path of the cylinder will be curved in this direction. This resembles the Magnus effect, although the physical origin is completely different.

The pressure distribution can also depend on the shape and elastic properties of the sliding body. For instance, if it is a sphere, linear elasticity theory predicts a $\sqrt{1 - r^2/R^2}$ shaped radial pressure function [15]. We calculated the $F(\varepsilon)$ and $T(\varepsilon)$ curves numerically for this case and found that their qualitative behavior remains the same [16]. Therefore coupling between the friction force and torque is still present: For large $\varepsilon$ torsion friction is suppressed by sliding, for small $\varepsilon$ sliding friction gets reduced by spinning. This may explain, why the translational motion of a fast spinning top is hardly decelerated.

Now let us consider what impact the coupling between sliding and spinning motion may have on the physics of granular media. In one limiting case, when the particles are spherical and very hard, torsion friction typically can be neglected, as the contact area is very small, hence $\varepsilon \gg 1$. However, on the other hand, real particles are usually non-spherical, or they may be very soft, so that the size of the contact area can be comparable to that of the particle. On the other hand, even in the case of hard spherical particles with contact radius $r$ much smaller than the particle radius $R$, the coupling between sliding and spinning motion can have subtle consequences. As the torsion friction is very small in this case, one can expect that the spinning degree of freedom can be easily excited. Typical sliding velocities will be comparable to $\omega_n R$, so that $\varepsilon \approx R/r \gg 1$. In this case, the sliding friction is basically $\mu F_n$. However, when a granular packing relaxes into a static configuration, the coupling between sliding and spinning becomes important. For example, if the initial sliding velocity is zero, but the spinning degree of freedom is excited (i.e., $\varepsilon = 0$), an arbitrarily small force can induce sliding.

An important extension of this work will be to investigate the coupling between the static friction force and torque. As static friction is different from sliding friction, and its theory is somewhat more difficult, we cannot expect that the maximum static friction forces and torques will lie on the friction-torque curve in Fig. 2(b). However, we have preliminary evidence that such coupling is present also in the static case. We expect that the threshold torques and forces needed to turn a sticking contact into a sliding and/or spinning one form a curve which lies above the one in Fig. 2(b). In particular, this implies that the application of a torque at a sticking contact makes it easier to excite the sliding degree of freedom. For hard spherical spheres a very small torque will already have this effect.

Finally, the fact that $F(\varepsilon)$ and $T(\varepsilon)$ depend on the pressure distribution in the contact area raises the question, whether the “inverse problem” has a unique solution, i.e., are the experimentally accessible functions $F(\varepsilon)$ and $T(\varepsilon)$ a fingerprint of the pressure distribution in the contact area? This would also be an interesting question to study in the future.

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