KASTELEYN COKERNELS AND PERFECT MATCHINGS ON
PLANAR BIPARTITE GRAPHS

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Abstract. The determinant method of Kasteleyn gives a method of computing
the number of perfect matchings of a planar bipartite graph. In addition, results
of Bernardi exhibit a bijection between spanning trees of a planar bipartite graph
and elements of its Jacobian. In this paper, we explore an analogue of Bernardi’s
results, providing a canonical simply transitive group action of the Kasteleyn
cokernel of a planar bipartite graph on its set of perfect matchings, when the
planar bipartite graph in question is of the form $G^+$, as defined by Kenyon,
Propp and Wilson.

1. Introduction

In general, counting matchings of a bipartite graph is a $\#P$-complete problem;
this was proved by Valiant in [6]. In the case that the graph is planar, however,
Kasteleyn’s theorem gives a method of enumerating matchings in polynomial time.
The enumeration involves calculating the determinant of a certain signed adjacency
matrix, called the Kasteleyn matrix of the graph; equivalently, the number of per-
fected matchings is equal to the order of the Kasteleyn cokernel, which is defined as
the cokernel of the Kasteleyn matrix. Kuperberg discusses in [5] the possibility
of a natural bijection between the matchings of the graph and the elements of its
Kasteleyn cokernel. He suggests that it may be better to consider a quantum bi-
jection between these two sets, that is, a unitary isomorphism between their formal
complex linear spans. In the case that the graph is of the form $G^+$, as defined in [3],
we prove that the situation is simpler: there is a canonical simply transitive group
action of the Kasteleyn cokernel on the set of perfect matchings. Our description is
analogous to certain families of combinatorial bijections between the spanning trees
of a planar graph and elements of its Jacobian.

2. Background

All graphs in this paper will be assumed to be finite and connected, possibly with
multi-edges but without self-loops.

Let $G$ be a directed graph on $n$ vertices. Its $n \times n$ signed adjacency matrix $A$ is
defined to to have its $(i, j)$-th entry in $A$ equal to 1 if there is a directed edge from
$v_i$ to $v_j$; $-1$ if there is a directed edge from $v_j$ to $v_i$; and 0 if no edge exists between
the two vertices. If there are multiple edges between $v_i$ and $v_j$, then the matrix
entry is equal to the number of edges oriented $v_i$ to $v_j$ minus the number of edges
oriented $v_j$ to $v_i$.
A divisor on a graph $G = (V, E)$ is a function $D : V \rightarrow \mathbb{Z}$. The set of all divisors of a graph $G$ is denoted $\text{Div}(G)$. Any $d \in \text{Div}(G)$ can be written as $d = \sum_{v \in V} a_v(v)$, for $a_v \in \mathbb{Z}$. The degree of the divisor $d$ is defined as $\deg(d) = \sum_{v \in V} a_v$. The set of all divisors of degree $k$ is denoted $\text{Div}^k(G)$. There is a (non-canonical) map from $\text{Div}^k(G) \rightarrow \text{Div}^0(G)$ defined by $d \mapsto d - d_0$ for some fixed reference divisor $d_0 \in \text{Div}^k(G)$.

The Laplacian operator on a graph $G$ is denoted $\Delta : \mathbb{Z}(G) \rightarrow \text{Div}(G)$, where $\mathbb{Z}(G)$ is the set of integer-valued functions on the set $V(G)$. Whenever $f \in \mathbb{Z}(G)$, the Laplacian operator is defined by $\Delta(f) = \sum_{v \in V(G)} \Delta_v(f)(v)$, where

$$\Delta_v(f) = \sum_{(v,w) \in E(G)} (f(v) - f(w)).$$

The group $\text{Prin}(G)$ of principal divisors is the image of the Laplacian operator. It is obvious that $\text{Prin}(G) \subseteq \text{Div}^0(G)$. Both $\text{Prin}(G)$ and $\text{Div}^0(G)$ are free abelian groups of rank $n-1$, so

$$\text{Jac}(G) = \text{Div}^0(G)/\text{Prin}(G)$$

is a finite group called the Jacobian of $G$.

2.1. Generalized Temperley Bijection. In \[3\], Kenyon, Propp, and Wilson create a method for obtaining a planar bipartite graph from an arbitrary planar graph and exhibit a bijection, called the Temperley bijection, between the spanning trees of the original graph and the matchings of the new bipartite graph. This section is a summary of this method. Throughout, we denote the starting planar graph as $G$ and the resulting bipartite graph as $G^+$.

Fix an embedding of $G$ in the plane (this process does depend on the chosen embedding). Choose a vertex of $G$ which is adjacent to the infinite face of $G$ with respect to this embedding and call it $q$.

Overlay $G$ with its planar dual $G^\vee$ in the plane and denote as $q^*$ the vertex of $G^\vee$ corresponding to the infinite face of $G$. At each intersection between an edge of $G$ and an edge of $G^\vee$, add a vertex in order to create a bipartite graph. To complete the construction, delete $q$, $q^*$, and all the edges incident to either $q$ or $q^*$. The resulting graph is called $G^+$. The vertices of $G^+$ are partitioned into white and black. The white vertices are those corresponding to edges of $G$ and the black vertices are those corresponding to vertices of either $G$ or $G^\vee$. Each edge in $G^+$ is a half-edge in either $G$ or $G^\vee$. See Figures 1, 2, and 3 for an example of this method.

Kenyon, Propp and Wilson in \[3\] produce a bijection between the set of spanning trees of $G$ and the set of perfect matchings of $G^+$, denoted $T(G)$ and $M(G^+)$ respectively. The starting information for this bijection is a spanning tree $T$ of $G$ and the root vertex $q$. One then constructs a $q$-connected orientation of $G$ associated to $T$. (By a $q$-connected orientation, we mean that for each $v \in V(G)$, there exists a directed path from $q$ to $v$.) This orientation is constructed by first orienting all edges $e \in T$ away from $q$. Then each $e \in T^\ast$ is oriented counterclockwise with respect to its fundamental cycle in $T$. For any spanning tree $T_i \in T(G)$, we denote
the associated $q$-connected orientation of $G$ by $O_i$. It will turn out that every $q$-connected orientation of $G$ arises in this way from some spanning tree (see [2]), so the set of $q$-connected orientations of $G$ is in bijection with the set of spanning trees of $G$.  

Figure 1. $G$

Figure 2. $G \cup G^\vee$

Figure 3. $G^+$
Throughout, edges in $G^+$ which are oriented white $\rightarrow$ black will be referred to as positively oriented and edges oriented black $\rightarrow$ white as negatively oriented.

Now we can construct a matching in $M(G^+)$ from an element of $T(G)$; this construction will produce the desired bijection. Start with some spanning tree of $G$, which we will call $T_0$, and the $q$-connected orientation $O_0$ constructed from $T_0$. Orient the edges of $G^\vee$ counterclockwise from the orientation (in $O_0$) of the edge they intersect in $G$. For each white vertex $v_w$ corresponding to some edge in $T_0$, choose the positively oriented half-edge of $G$ incident to $v_w$ and add it to $M(G^+)$. For each $v_w$ corresponding to some edge in the complement of $T_0$, choose the positively oriented half-edge of $G^\vee$ incident to $v_w$ and add it to $M(G^+)$. See Figures 4 and 5 for an example.

**Remark 2.1.** The orientation of $G^+$ constructed as described above will turn out to be a Kasteleyn orientation; these orientations will be defined in Section 2.3.

This construction gives rise to a bijection between spanning trees of $G$ and perfect matchings of $G^+$, provided that $q$ is incident to $q^*$ (by which we mean that $q$ is adjacent to the infinite face). If $q$ is not incident to $q^*$, then the set $M(G^+)$ may be strictly larger than the set $T(G)$, in which case this map produces an injection $T(G) \rightarrow M(G^+)$. Throughout, we assume that $q$ and $q^*$ are incident.
2.2. Orientations of Spanning Trees. In this section, we will describe the Bernardi bijection between \( T(G) \) and elements of \( \text{Jac}(G) \) and show that it factors through the set of equivalence classes of orientations of \( G \). The bijection begins with some spanning tree \( T_i \) of \( G \). Then a \( q \)-connected orientation is constructed as described in Section 2.1. An equivalence relation can be defined on the orientations of \( G \) by considering two orientations \( O_i \) and \( O_j \) to be equivalent if one can be obtained from the other by a sequence of directed cycle and cut reversals. It is clear that this is an equivalence relation. We will denote the set of equivalence classes of orientations of \( G \) by \( O(G) \). The following proposition is proved in [4]:

**Proposition 2.2.** Each equivalence class in \( O(G) \) contains a unique \( q \)-connected orientation.

Bernardi provides a bijective proof of this fact in [2] by proving that \( T(G) \leftrightarrow O(G) \). Since spanning trees of \( G \) are in bijection with \( q \)-connected orientations, the \( q \)-connected orientations make a natural choice of representatives for the equivalence classes in \( O(G) \).

The bijection between \( O(G) \) and \( \text{Jac}(G) \) maps an orientation in \( O(G) \) into \( \text{Div}^{g-1}(G) \) by placing a coefficient of \( \text{indeg}(v) - 1 \) on each vertex \( v \), where \( g \) denotes the combinatorial genus of \( G \) and \( \text{indeg}(v) \) denotes the indegree of \( v \). The resulting divisor has degree \( \Sigma_{v \in V}(\text{indeg}(v) - 1) = g - 1 \), so this does give a map \( O(G) \to \text{Div}^{g-1}(G) \).

The map from \( \text{Div}^{g-1}(G) \) to \( \text{Jac}(G) \) is defined by subtracting a reference divisor \( d_0 \in \text{Div}^{g-1}(G) \) from each \( d \in \text{Div}^{g-1}(G) \). The reference divisor is taken to be the divisor associated to some spanning tree \( T_0 \) of \( G \), which allows \( T_0 \) to be considered as an “identity element” of \( T(G) \) in this bijection. This produces a bijection between \( \text{Div}^{g-1}(G)/\text{Prin}(G) \) and \( \text{Div}^{0}(G)/\text{Prin}(G) \cong \text{Jac}(G) \). Thus the Bernardi map gives a combinatorially defined bijection between spanning trees of \( G \) and elements of \( \text{Jac}(G) \), which factors as \( T(G) \to O(G) \to \text{Div}^{g-1}(G)/\text{Prin}(G) \to \text{Jac}(G) \). Note that this bijection is not canonical, as it depends on the choice of \( T_0 \) and the choice of \( q \). However, Bernardi also proves that there is an associated group action of the so-called break divisors of \( G \) on the set of spanning trees, and the break divisors are in bijection with \( q \)-connected orientations. Yuen proves in [7] that this group action is independent of the reference tree, and depends only on the choice of \( q \).

By abuse of terminology, we will also refer to the truncated map \( O(G) \to \text{Div}^{g-1}(G)/\text{Prin}(G) \to \text{Jac}(G) \) from equivalence classes of orientations to the Jacobian of \( G \) as the Bernardi bijection.

2.3. Kasteleyn Cokernels and Jacobians. The Kasteleyn cokernel is closely related to the Kasteleyn orientations of a planar bipartite graph. These objects arise as analogues of \( \text{Jac}(G) \) and \( q \)-connected orientations of \( G \), respectively.

**Definition 2.3.** A Kasteleyn orientation on a planar bipartite graph \( G \) is an orientation of \( G \) such that every cycle in the graph has an odd number of clockwise-oriented edges.
(Note that this definition is dependent on the drawing of $G$ in the plane.) The condition for an orientation to be Kasteleyn is equivalent to having an odd number of positively oriented edges in every cycle with length $\ell \equiv 0 \pmod{4}$ and an even number of positively oriented edges in every cycle with length $\ell \equiv 2 \pmod{4}$.

A $q$-connected orientation on $G$ gives rise to a Kasteleyn orientation on $G^+$ using the same method as in the Temperley bijection. First recall that all edges of $G^+$ are either half-edges of $G$ or half-edges of $G^\vee$. Orient each half-edge of $G$ the same way as in the original $q$-connected orientation of $G$ and orient each half-edge of $G^\vee$ counterclockwise from its corresponding edge in $G$. (This orientation was used in the Temperley bijection to produce a perfect matching on $G^+$ from a spanning tree on $G$.) Figure 5 shows the matching and orientation coming from $T_0$, and one can verify that the orientation induced on $G^+$ is a Kasteleyn orientation.

The signed bipartite adjacency matrix of a graph is constructed with white vertices indexing the columns and black vertices indexing the rows. A signed bipartite adjacency matrix arising from a Kasteleyn signing is called a Kasteleyn matrix. The Kasteleyn cokernel is defined as follows.

**Definition 2.4.** The Kasteleyn cokernel of a planar bipartite graph $H$ is the finite abelian group $K(H) = \text{Div}(H)/\text{Prin}(H)$, where $\text{Div}(H)$ is the free abelian group on the white vertices and $\text{Prin}(H)$ is the column span of a Kasteleyn matrix of $H$.

In general, Kasteleyn signings are far from being unique. However, the Kasteleyn cokernel is independent of the Kasteleyn signing chosen; see [5].

Jacobson proves the following theorem in [1], which was originally conjectured by Kuperberg:

**Theorem 2.5.** The Kasteleyn cokernel of $G^+$ is isomorphic to the Jacobian of $G$, i.e. $K(G^+) \cong \text{Jac}(G)$.

In order to explicitly describe the isomorphism, first note that elements $a, b \in \text{Jac}(G)$ are equivalent if and only if one can be obtained from the other by a sequence of chip-firing moves. A chip-firing move from a fixed vertex $v_0$ has the form

$$\Delta v_0 = -|N(v_0)|; \Delta v = 1 \text{ for } v \in N(v_0)$$

where $N(v_0)$ denotes the neighborhood of $v_0$. It is easy to see that chip-firing equivalence is an equivalence relation on the divisors of $G$ and that the degree of the divisor will not change in a chip-firing move.

The isomorphism $\phi : K(G^+) \to \text{Jac}(G)$ is defined as follows. Let $\text{Div}(G^+)$ denote the set of linear equivalence classes of white vertices of $G^+$, where two divisors are said to be linearly equivalent if they differ by something in the column span of a Kasteleyn matrix of $G^+$.

Let $k \in \text{Div}(G^+)$, and choose a representative for its equivalence class. Denote the integer on a given white vertex $v_w$ as $d$. Let $e$ denote the edge of $G$ associated with $v_w$, and place $d$ chips at the head of $e$ and $-d$ chips at the tail of $e$, where the heads and tails of each edge are determined by a fixed $q$-connected orientation on $G$. Extend by linearity, and take the linear equivalence class of the resulting
divisor on $G$ to be the image of $\phi(k)$. The linear equivalence class of the resulting divisor is independent of the representative chosen for $k$, so $\phi$ is well-defined. Then $\phi : \text{Div}(G^+) \to \text{Jac}(G)$ is a surjection whose kernel is exactly $\text{Prin}(G^+)$, which gives the desired isomorphism.

3. Group action of the Kasteleyn cokernel on matchings

In this section, we establish the following theorem.

**Theorem 3.1.** There is a canonical simply transitive group action of the Kasteleyn cokernel of $G^+$ on the set of perfect matchings of $G^+$.

We will show that the group action depends only on the choice of root vertex $q$, which we consider to be part of the structure of $G^+$. To describe the action, start by fixing a reference matching $M_0$ on a planar bipartite graph $G^+$, and fix the Kasteleyn orientation coming from $M_0$ via the corresponding $T_0 \in T(G)$. (Later, we will show that the group action is independent of the choice of $M_0$.) Define the alternating cycles of $G^+$ (with respect to $M_0$) to be cycles whose edges alternate between edges $e \in M_0$ and $e \not\in M_0$. For any matching $M_j$ in $M(G^+)$, the symmetric difference $M_0 \triangle M_j$ is some disjoint union of alternating cycles. More generally, for any 2 matchings $M_i$ and $M_j$, the symmetric difference $M_i \triangle M_j$ is a disjoint union of cycles of $G^+$. We define $L_{ij} := M_i \triangle M_j$.

We define a map $\Psi_0 : M(G^+) \to K(G^+)$ as follows. For each white vertex $v$ in the support of $L_{0i}$, if the 2 edges in $L_{0i}$ incident to $v$ have the same orientation (that is, both are positively oriented or both are negatively oriented), then a 0 is placed on $v$. If the 2 edges have opposite orientations (that is, one positive and the other negative), then a 1 is placed on $v$. For any $v$ not in the support of $L_{0i}$, a 0 is placed on $v$. This gives a divisor on $G^+$; take $\Psi_0(M_i)$ to be the linear equivalence class of this divisor, which gives an element of $K(G^+)$. Note that $\Psi_0(M_0) = 0$. See Figures 4, 5, 6, 7 and 8 for an example of two spanning trees and the disjoint union of alternating cycles arising from the corresponding matchings on $G^+$.

$\Psi_0$ naturally extends to a map $\psi_0 : M(G^+) \times M(G^+) \to K(G^+)$ by letting $\psi_0(M_i, M_j) = \Psi_0(M_i) - \Psi_0(M_j)$. Both $\Psi_0(M_i)$ and $\Psi_0(M_j)$ are elements of the Kasteleyn group, and the subtraction is performed in $K(G^+)$. It is clear from the definition that $\psi_0$ defines a group action of $K(G^+)$ on $M(G^+)$. We will prove that the action is simply transitive by proving that $\Psi_0$ is a bijection. We will later prove that this action is independent of the reference data, i.e. the choice of $M_0$ and the corresponding Kasteleyn orientation, so in fact this group action is canonical.

**Theorem 3.2.** The map $\Psi_0 : M(G^+) \to K(G^+)$ is a bijection between $K(G^+)$ and $M(G^+)$. 

Given a matching $M_i \in M(G^+)$, denote as $T_i$ the corresponding spanning tree of $G$ under the Temperley bijection, and denote as $O_i$ the $q$-connected orientation...
of $G$ associated to $T_1$. We will show that the map $\Psi_0$ makes the following diagram commute:

$$
\begin{array}{c}
T(G) \xrightarrow{Temperley} \xrightarrow{\Psi_0} K(G^+) \\
\downarrow \phi \\
M(G^+) \xrightarrow{\Psi_0} K(G^+)
\end{array}
$$

Recall that since $T(G)$ is in natural bijection with the set $O(G)$ of equivalence classes of orientations of $G$ and the Bernardi bijection factors through this set, it is equivalent to state that the map $\Psi_0$ makes the following diagram commute:

$$
\begin{array}{c}
O(G) \xrightarrow{Bernardi} \xrightarrow{\Psi_0} K(G^+) \\
\downarrow \psi \\
M(G^+) \xrightarrow{\Psi_0} K(G^+)
\end{array}
$$

Lemma 3.3. The map $\Psi_0$ is the same as the map induced by placing a 1 on each edge of $G$ which changes orientation between $O_0$ and $O_1$.

Proof. The statement holds by the isomorphism $\phi : \text{Jac}(G) \to K(G^+)$, since flipping the orientation of an edge $e \in G$ decreases the indegree of one endpoint of $e$ while increasing the indegree of the other endpoint of $e$. Therefore $\Psi$ coincides with the map $O(G) \to \text{Jac}(G) \to K(G^+)$ given by composing the Bernardi bijection $O(G) \to \text{Jac}(G)$ with Jacobson’s bijection $\phi : \text{Jac}(G) \to K(G^+)$. $\square$
For an example, see Figures 4, 6 and 8. Note that the edges which have different orientations in \( O_0 \) and \( O_1 \) are exactly those which have 1’s placed on them when \( \Psi_0 \) is applied to \( L_1 \).

**Theorem 3.4.** The map \( \Psi_0 \) makes the diagrams above commute, and therefore produces a bijection between \( M(G^+) \) and \( K(G^+) \).

**Proof.** Consider the Kasteleyn orientation of \( G^+ \) created from \( O_0(G) \), as described in Section 2.1. Under the Temperley bijection, each edge taken in a matching \( M_i \) is positively oriented in the Kasteleyn orientation of \( G^+ \) arising from \( O_i \). Suppose a 1 is placed on a vertex \( v \) in \( L_i \). Then the orientation of that edge is different in \( O_0 \) and \( O_1 \), since the Temperley bijection always picks up positively oriented edges. Now suppose that there is a 0 on some vertex \( v \) in \( L_i \). Then the orientation of that edge is the same in \( O_0 \) and \( O_1 \). Since it is in the symmetric difference \( M_0 \Delta M_i \), the edge was either in \( T_0 \) but not in \( T_i \), or vice versa, but its orientation remained the same in the two corresponding \( q \)-connected orientations since the same half-edges are still positively oriented.

Therefore, this bijection between \( K(G^+) \) and \( M(G^+) \) makes the diagram commute. Since the other 3 arrows in the diagram are all bijections, \( \Psi_0 \) is as well.

\( \square \)

See Figures 4, 6 and 8 for an illustration of this statement. The central edge and bottom right edge are the ones whose orientations are reversed between
the orientations of $T_0$ and $T_1$, and the white vertices in $G^+$ corresponding to those edges are exactly the vertices which will have 1’s placed on them under the map $\Psi_0$. So in this example, the map $\Psi_0$ does in fact complete the commutative diagram.

Last, we show that the induced group action $\psi_0$ is independent of the reference matching $M_0$. Suppose that some $M_i$ is used as the reference matching instead of $M_0$, so the group action $\psi_i$ is induced by the map $\Psi_i$ sending $M_j \rightarrow L_{ij} \rightarrow K(G^+)$, and the map $L_{ij} \rightarrow K(G^+)$ is defined with respect to the Kasteleyn orientation arising from $M_i$ via the Temperley bijection.

The action described of $K(G^+)$ on $M(G^+)$ is equivalent to the action of break divisors on spanning trees of $G$, and this action is canonical (see [7]), i.e., independent of the reference spanning tree. Therefore the action $\psi_0$ is independent of $M_0$, and in fact depends only on the choice of $q$, which we consider to be part of the data of $G^+$ as a planar graph.

Therefore $\psi_0 = \psi_i$, so this defines a canonical group action, which we denote $\psi$, of $K(G^+)$ on $M(G^+)$. We note that this algorithm does not extend to graphs not of the form $G^+$. It would be interesting to know whether a similar algorithm exists for more general graphs.

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References

[1] Jacobson, B. Critical groups of graphs. Available at http://www-users.math.umn.edu/~reiner/HonorsTheses/Jacobson thesis.pdf (2016).

[2] Bernardi, O. A Characterization of the Tutte polynomial via combinatorial embeddings. Annals of Combinatorics. 12 2008 139-153.

[3] Kenyon, R., Propp, J., and Wilson, D. Trees and Matchings. Journal of Combinatorics. 7 (2001)

[4] Backman, S. Riemann-Roch theory for graph orientations. Advances in Mathematics. 2017 655-691.

[5] Kuperberg, G. Kasteleyn Cokernels Electron. J. Combinatorics. 2002

[6] L.G. Valiant The complexity of computing the permanent Theoretical Computer Science 1979 189-201.

[7] C.H. Yuen Geometric bijections between spanning trees and break divisors. Journal of Combinatorial Theory A 2015