Large time asymptotics for the fluctuation SPDE in the Kuramoto synchronization model

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Abstract

We investigate the long-time asymptotics of the fluctuation SPDE in the Kuramoto synchronization model. We establish the linear behavior for large time and weak disorder of the quenched limit fluctuations of the empirical measure of the particles around its McKean-Vlasov limit. This is carried out through a spectral analysis of the underlying unbounded evolution operator, using arguments of perturbation of self-adjoint operators and analytic semigroups. We state in particular a Jordan decomposition of the evolution operator which is the key point in order to show that the fluctuations of the disordered Kuramoto model are not self-averaging.

Keywords: Stochastic partial differential equations, perturbation of analytic operators, Jordan decomposition, Kuramoto model, synchronization, disordered systems, self-averaging

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1. Introduction

1.1. Synchronization of heterogeneous oscillators

Collective behavior of oscillators and synchronization phenomenon are the subject of a vast literature either in biological (neuronal models, collective behavior of insects, cells, etc.) or in physical contexts (see [28, 43] and references therein). While a precise description of each of the different instances in which synchronization emerges demands specific, possibly very complex models, the Kuramoto model [1] has emerged as capturing some of the fundamental aspects of synchronization.

The disordered Kuramoto model concerns a family of heterogeneous oscillators (or rotators) on the circle $S := \mathbb{R}/2\pi \mathbb{Z}$ in a noisy mean-field interaction (that is the dynamics is perturbed by thermal noise). Each rotator obeys to its own natural frequency which may differ from one rotator to another. Those frequencies are chosen at random and independently for each rotator according to a probability distribution $\mu$ on $\mathbb{R}$; hence, this supplementary source of randomness will be considered as a disorder.

One of the main characteristics of the Kuramoto model is that it presents a phase transition, as the coupling strength between rotators increases, from an incoherent state where the rotators are not synchronized to a synchronous one where the phases of the rotators concentrate around a common value (for a review on the subject, see [1]). In this context, the question of how the random frequencies influence synchronization has been raised by many authors, not only in the Kuramoto model ([43]) but also for more general models of weakly interacting diffusions (e.g. neuronal models, see [1] and references therein).
1.1.1. The continuous model

The disordered Kuramoto model [28, 1], in the limit of an infinite population of rotators, is described by the following nonlinear Fokker-Planck equation (or McKean-Vlasov equation):

\[
\partial_t q_t(\theta, \omega) = \frac{1}{2} \partial^2_{\theta^2} q_t(\theta, \omega) - \partial_\theta \left( q_t(\theta, \omega) \left( \langle J \ast q_t \rangle_\mu(\theta) + \omega \right) \right), \quad t > 0, \; \theta \in S, \; \omega \in \text{Supp}(\mu),
\]

with periodic boundary conditions and initial condition given by

\[
\forall \omega \in \text{Supp}(\mu), \quad q_t(\theta, \omega) \, d\theta \xrightarrow{t \to 0} \gamma(d\theta),
\]

for some probability law \(\gamma\) on the circle \(S\). Here,

\[
\langle J \ast q_t \rangle_\mu(\theta) = -K \int_S \int_{\mathbb{R}} \sin(\varphi) q_t(\theta - \varphi, \omega) \, d\varphi \, \mu(d\omega),
\]

stands for the convolution of \(J(\cdot) := -K \sin(\cdot)\) with \(q_t\), averaged with respect to \(\omega\) and \(K\) is the positive coupling strength between rotators. Note that we are looking for solutions \((t, \theta, \omega) \mapsto q_t(\theta, \omega)\) that are probability densities for all fixed \(t\) and \(\omega\): \(q_t(\cdot, \cdot) \geq 0\) for all \(t > 0\) with \(\int_S q_t(\theta, \omega) \, d\theta = 1\) for all \(\omega \in \text{Supp}(\mu)\).

The physical interpretation of (1.1)–(1.2) is the following: \(\theta \in S\) is the phase of the rotators, \(\gamma\) is their initial distribution on \(S\), \(\mu\) is the probability distribution of the frequencies, and \(q_t(\theta, \omega)\) is the density of rotators with phase \(\theta\) and frequency \(\omega\) at time \(t > 0\).

Uniqueness of a solution to (1.1)–(1.2) follows from standard arguments concerning fundamental solutions of parabolic equations ([23, 21]) and has been rigorously established in [23, § A]. Another proof of uniqueness can be found in [15] on the basis of heat kernel estimates under regularity assumptions on the initial condition.

1.1.2. The microscopic model

Existence of a solution to (1.1) can be seen as a consequence of the following probabilistic interpretation: for all \(N \geq 1\), consider the following system of \(N\) stochastic differential equations in a mean-field interaction

\[
d\theta_{j,t} = \omega_j \, dt - \frac{K}{N} \sum_{i=1}^N \sin(\theta_{j,t} - \theta_{i,t}) \, dt + dB_{j,t}, \quad j = 1, \ldots, N, \; t \geq 0,
\]

where at time \(t = 0\), the rotators \(\theta_{j,0}\) are i.i.d. with law \(\gamma\), \((\omega_i)_{i=1,\ldots,N}\) are i.i.d. with law \(\mu\) and \(\{B_{j}\}_{j=1,\ldots,N}\) are \(N\) standard independent Brownian motions. Evolution (1.1) appears naturally as the large \(N\)-limit of the system (1.4) in the following way: if one defines the empirical measure \(\nu_N(t)\) of both rotators and frequencies as

\[
t \mapsto \nu_N(t) := \frac{1}{N} \sum_{j=1}^N \delta(\theta_{j,t}, \omega_j) \in \mathcal{C}([0, +\infty), \mathcal{M}_1(S \times \mathbb{R})),
\]

where \(\delta(\theta, \omega)\) is the Dirac measure in \((\theta, \omega)\) and \(\mathcal{M}_1(S \times \mathbb{R})\) the set of probability measures on \(S \times \mathbb{R}\), it can be shown (see [15, 21]) that, under mild hypotheses, the sequence \(\nu_N(t)\) converges as \(N\) goes to \(+\infty\) in law (as a process), to the deterministic limit \(t \mapsto \nu_t\) such that

\[
\begin{cases}
\nu_0(d\theta, d\omega) = \gamma(d\theta) \otimes \mu(d\omega) \\
\nu_t(d\theta, d\omega) = q_t(\theta, \omega) \, d\theta \mu(d\omega), \quad t > 0,
\end{cases}
\]

where \(q_t\) is solution of the McKean-Vlasov equation (1.1).

Remark 1.1. Due to the mean-field character of (1.4), there is a self-averaging phenomenon (see [31, Th. 2.5]): the above convergence is true for almost every choice of the frequencies \((\omega_j)_{j \geq 1}\) and the limit \(\nu\) does not depend on this initial choice.

This law of large numbers is a disordered generalization of known results about mean-field interacting diffusions (see e.g. [22, 26, 34, 35] for similar situations without disorder). Note also that this convergence is also valid for more general models (see e.g. the recent work on the Winfree model [29] or FitzHugh-Nagumo and Hodgkin-Huxley models of neuronal oscillators [3]).
1.2. The fluctuation SPDE

In this paper, we investigate the asymptotic behavior as \( t \to +\infty \) of the following stochastic partial differential equation (SPDE):

\[
\eta_t = \eta_0 + \int_0^t L_q \eta_s \, ds + W_t,
\]  

(1.7)

where \( L_q \) is the linearized operator around the solution \( t \mapsto q_t \) of nonlinear evolution (1.1):

\[
L_q \varphi(\theta, \omega) := \frac{1}{2} \partial_{\theta^2}^2 \varphi(\theta, \omega) - \partial_\theta \left( \varphi(\theta, \omega) \left( \langle J * q_t \rangle_\mu(\theta) + \omega \right) + q_t(\theta, \omega) \langle J * \varphi \rangle_\mu(\theta) \right),
\]  

(1.8)

where \( \varphi \) is a regular function, \( W \) is a Gaussian process, indexed by functions \( \varphi(\theta, \omega) \) such that \( \partial_\theta \varphi(\cdot, \omega) \in L^2(S) \) for all \( \omega \in \text{Supp}(\mu) \), with covariance

\[
\forall \varphi_1, \varphi_2, \quad \mathbb{E}(W_t(\varphi_1)W_s(\varphi_2)) = \int_0^t \int_S \int_R \partial_\theta \varphi_1(\theta, \omega) \partial_\theta \varphi_2(\theta, \omega) q_u(\theta, \omega) \, d\theta \mu(\,d\omega) \, du,
\]  

(1.9)

and where the initial condition \( \eta_0 \) is independent of \( W \).

1.2.1. The SPDE (1.7) as the limit of the fluctuation process

The SPDE (1.7) is the natural limit object in the Central Limit Theorem associated to the convergence as \( N \to +\infty \) of the empirical measure \( \nu_N \) towards its limit \( \nu \) (1.10). Namely, the object of a previous work [30, Th. 2.10] was to prove that the fluctuation process

\[
t \geq 0 \mapsto \eta_{N,t} := \sqrt{N} (\nu_{N,t} - \nu), \quad N \geq 1,
\]  

(1.10)

converges as \( N \to \infty \), in a weak sense, in an appropriate space of distributions on \( S \times R \), to the solution \( \eta \) of (1.7).

Similar fluctuation results for interacting diffusions had already been considered in the literature (10, 20). The particularity of the above result is that it is a quenched notion of fluctuation, which still keeps track of the influence of the disorder \( (\omega_1, \ldots, \omega_N) \) as \( N \to \infty \). The precise notion of convergence used in [30] is not really relevant for the purpose of this paper; more details can be found in [30, Th. 2.10]. What we only need to retain here is that the limit \( \eta = (\eta^\omega)_{\omega} \) captures the dependence in the disorder through its mean-value: there exists a Gaussian process \( \omega \mapsto C(\omega) \) with covariance

\[
\forall \varphi_1, \varphi_2 : S \times R \to R, \quad \Gamma_C(\varphi_1, \varphi_2) = \text{Cov}_\mu \left( \int_S \varphi_1(\cdot, \omega) \, d\gamma, \int_S \varphi_2(\cdot, \omega) \, d\gamma \right)
\]  

(1.11)

such that for fixed \( \omega \), the initial condition \( \eta^\omega_0 \) in (1.7) may be written as

\[
\eta^\omega_0 = X + C(\omega),
\]  

(1.12)

where \( X \) is an explicit centered Gaussian process. The mean-value \( C(\cdot) \) has an interpretation in terms of the microscopic system (1.4): it models in law the asymmetry in the initial choice of the frequencies \( (\omega_1, \ldots, \omega_N) \) as \( N \to \infty \) (see §2.3.1 for further details).

1.2.2. Finite size effects in the Kuramoto model: non self-averaging phenomenon

The motivation of this work is to study the influence of a typical realization of the frequencies \( (\omega_j)_{j \geq 1} \) (quenched model) on the behavior of (1.4) for large but finite \( N \). Indeed, (as shown numerically in [5]), at the level of the microscopic system (1.4), fluctuations of the frequencies \( (\omega_i)_{i=1,\ldots,N} \) compete with the fluctuations of the thermal noise and make the whole system rotate: even in the simple case of \( \mu = \frac{1}{2}(\delta_{-1} + \delta_{1}) \), fluctuations in a finite sample \( (\omega_1, \ldots, \omega_N) \in \{\pm 1\}^N \) may lead to a majority of +1 with respect to -1, so that the rotators with positive frequency induce a global rotation of the whole system in the direction of the majority. Direction and speed
of rotation depend on this initial random choice of the disorder (Fig. 1 and 2a). This can be noticed by computing the order parameters \((r_{N,t}, \psi_{N,t})\) (recall (1.5)):

\[
    r_{N,t} e^{i \psi_{N,t}} = \frac{1}{N} \sum_{j=1}^{N} e^{i \theta_{j,t}} = \int_{S \times \mathbb{R}} e^{i \theta} \, d\nu_{N,t}(\theta, \omega), \quad N \geq 1, \quad t \geq 0, \tag{1.13}
\]

Here \(r_{N,t} \in [-1, 1]\) gives a notion of synchronization of the system (e.g., \(r_{N,t} = 1\) if the oscillators \(\theta_{j,t}\) are all equal) and \(\psi_{N,t}\) captures the position of the center of synchronization (see Figure 1). One can see on Figure 2a that \(t \mapsto \psi_{N,t}\) has an approximately linear behavior whose slope depends on the choice of the disorder. Note that this disorder-induced phenomenon does not happen at the level of the nonlinear Fokker-Planck equation (1.1), but only at the level of fluctuations (1.10) (the speed of rotation is of order \(N^{-1/2}\)).

![Figure 1: Evolution of the marginal on \(S\) of \(\nu_N\) (\(N = 600, \mu = \frac{1}{2}(\delta_{-1} + \delta_1), K = 6\)). The rotators are initially independent and uniformly distributed on \([0, 2\pi]\) and independent of the disorder. First the dynamics leads to synchronization (\(t = 6\)) to a profile close to a nontrivial stationary solution of (1.1). Secondly, we observe that the center \(\psi_{N,t}\) of this density moves to the right with an approximately constant speed; this speed of fluctuation turns out to be sample-dependent (see Fig. 2a).](image)

1.2.3. Long-time asymptotics of the fluctuation process

What makes evolution (1.7) relevant here is that its solution \(\eta\) still captures this disorder-dependent rotation: at least numerically, one observes trajectories of the process \(\eta\) that are compatible with the ones observed for the finite-size system (1.4) (see Figure 2).

Hence, a way to understand the phenomenon described in §1.2.2 is to analyze the dependence of the fluctuation process \(\eta\) (1.7) in its mean-value \(C\) (which, as we said, captures the initial asymmetry of the disorder). The key point of this paper is to understand how different initial conditions in evolution (1.7) may lead to distinct approximately linear trajectories of the fluctuation process, as in Figure 2b.

Namely, in Theorem 2.10 we prove the following convergence for the solution \(\eta\) of (1.7), in an appropriate space of distribution: for fixed \(\omega\)

\[
    \frac{\eta^\omega_t}{t} \xrightarrow{\text{in law}} V(\omega), \tag{1.14}
\]

where the speed \(V(\omega)\) (which depends on the initial condition \(C(\omega)\)) has an explicit nontrivial law. This result relies on a detailed spectral analysis of the unbounded evolution operator \(L_q\) defined in (1.8), using arguments from perturbation theory of self-adjoint operators (27) and of analytic semigroups (32, 33) and usual techniques about SPDEs in Hilbert spaces (14). The main ingredient for this result consists in proving the existence of a nontrivial Jordan block for the eigenvalue 0 for the operator \(L_q\), relying on \textit{a priori} estimates on the Dirichlet form associated to \(L_q\) and an extension of Lax-Milgram Theorem.
1.3. Conclusion and perspectives

The main conclusion of this work is that the Kuramoto model is not self-averaging at the level of fluctuations: the dynamics of the quenched fluctuations of (1.4) are still disorder-dependent, contrary to the dynamics of the nonlinear Fokker-Planck equation (1.1). However, in order to derive rigorously the exact speed of the rotation of synchronized solutions described in Figure 2, it would be necessary to study (1.4) on larger time scales (e.g., time scales of order $N^{-1/2}$). In Figure 2, trajectories of the fluctuation process $\eta(t)$ are sample-dependent and compatible with the behavior described in Figure 2.

This work addresses the behavior as $t \to \infty$ of the fluctuation SPDE (1.7). It would be hopeless to review the vast literature (since [14, 44]) on long-time behavior of SPDEs (existence of invariant measures or random attractors have been studied for many models e.g., [13, 40, 20]). In our framework, the main difficulty of the long-time analysis of fluctuation for interacting diffusions (see e.g., [12]) lies in the fact that the dynamics of such systems is deeply related to the linear stability of their equilibria, which is, as we said, often hard to characterize and establish.

Concerning possible generalizations of this work, the results presented here should certainly
be applicable to other disordered models of diffusions, provided sufficient information is known about characterization and linear stability of stationary states. In view of the recent work [28], the issue of whether not similar non self-averaging results hold for the Winfree model is an intriguing question and would require further analysis.

1.4. Organization of the paper

The paper is organized as follows: in Section 2, we precise the main set-up for the study of the SPDE (1.7) and state the main results. In particular, Theorem 2.6 and Theorem 2.8 deal with the spectral properties of the evolution operator $L_q$, at least when the disorder is small. Secondly, we state the main result of this paper: Theorem 2.10 establishes the linear asymptotics of the fluctuation process solution of (1.7). Section 3 is devoted to prove Theorem 2.6. In Section 4 we prove Theorem 2.8 whereas the main result of the paper, Theorem 2.10 is proved in Section 5.

2. Main definitions and results

2.1. Long-time analysis of the McKean-Vlasov equation

Before going into the details of the analysis of the SPDE (1.7), let us recall some results concerning the nonlinear Fokker-Planck equation (1.1). Section 3 is devoted to prove Theorem 2.6. In Section 4 we state the main results. In particular, Theorem 2.6 and Theorem 2.8 deal with evolution (1.1) are captured by the order parameters $u_t$ and $\psi_t$ (the continuous equivalents of $(r_N, \psi_N)$ in (1.13)) defined by:

$$r_t e^{i\psi_t} = \int_{S \times \mathbb{R}} e^{i\theta} q_t(\theta, \omega) \, d\theta \, d\mu(\omega), \quad t \geq 0. \quad (2.1)$$

The quantity $r_t$ captures the degree of synchronization of a solution (the profile $q_t \equiv \frac{1}{2\pi}$ for example corresponds to $r_t = 0$ and represents a total lack of synchronization) and $\psi_t$ identifies the center of synchronization: this is true and rather intuitive for unimodal profiles. In this framework, synchronization reads in the existence of nontrivial stationary solutions $q$ to (1.1): following [41], if $\mu$ is symmetric, any equilibrium in (1.1) may be written as $q(\cdot + \theta_0, \omega)$ for any fixed $\theta_0 \in S$ where

$$q(\theta, \omega) := \frac{S(\theta, \omega; 2Kr)}{Z(\omega, 2Kr)}, \quad (2.2)$$

for

$$S(\theta, \omega, x) := e^{G(\theta, \omega, x)} \left[ (1 - e^{4\pi \omega}) \int_0^\theta e^{-G(u, \omega, x)} \, du + e^{4\pi \omega} \int_0^{2\pi} e^{-G(u, \omega, x)} \, du \right], \quad (2.3)$$

where $G(u, \omega, x) = x \cos(u) + 2\omega u$, $Z(\omega, x) = \int_S S(\theta, \omega, x) \, d\theta$ a normalization constant. The parameter $r \in [0, 1)$ in (2.2) must satisfy the fixed-point relation (2.1):

$$r = \Psi_{\mu}(2Kr), \quad \text{where} \quad \Psi_{\mu}(x) := \int_{\mathbb{R}} \frac{\int_S \cos(\theta) S(\theta, \omega, x) \, d\theta}{Z(\omega, x)} \mu(\, d\omega). \quad (2.4)$$

One can distinguish between two kinds of stationary solutions, depending on admissible solutions $r$ of (2.1):
\[
\bullet \text{ } r = 0 \text{ is always a solution to (2.2), and corresponds to the constant density } q \equiv \frac{1}{2\pi};
\]
\[
\bullet \text{ Any solution } q \text{ with } r > 0 \text{ is called a synchronized solution. An easy calculation of the derivative of } \Psi(\cdot) \text{ at } 0 \text{ shows that such solutions exist at least when the coupling strength } K \text{ is greater than } \hat{K} := \left( \int_{\mathbb{R}} \frac{\mu(d\omega)}{1 + \omega^2} \right)^{-1}. \text{ In that case, due to the rotation invariance (Remark 2.4), each solution } r > 0 \text{ of (2.9) corresponds to a whole circle of synchronized stationary solutions } \{q(\cdot + \theta_0, \omega); \theta_0 \in S\}. \]

2.1.2. The case with no disorder

In the non-disordered case (\(\mu = \delta_0\), (1.1)) reduces to:
\[
\partial_t q_\mu(\theta) = \frac{1}{2} \partial^2 \theta q_\mu(\theta) - \partial_\theta (q_\mu(\theta)(J * q_\mu)(\theta)), \quad (2.5)
\]
and any stationary profile can be written as \(q_0(\theta + \theta_0)\) for
\[
q_0(\theta) := \frac{e^{2K \theta_0 \cos(\theta)}}{\int_{S} e^{2K \cos(\theta)} d\mu} = \frac{e^{2K \theta_0 \cos(\theta)}}{Z_0(2Kr_0)}, \quad (2.6)
\]
where \(r_0\) solves
\[
r_0 = \Psi_0(2Kr_0), \text{ where } \Psi_0(x) := \frac{\int_{S} \cos(\theta) e^{x \cos(\theta)} d\theta}{\int_{S} e^{x \cos(\theta)} d\theta}. \quad (2.7)
\]
Here, since \(\Psi_0\) is strictly concave ([39, Lem. 4]) and \(\partial_\theta \Psi_0(2Kr_0) = K\), the phase transition is obvious: for \(K \leq 1\), \(r_0 = 0\) is the only solution to (2.7) and \(\frac{1}{2\pi}\) is the only stationary solution whereas for \(K > 1\) this solution coexists with a unique (up to rotation) synchronized solution (corresponding to the unique \(r_0 > 0\) solution to (2.7)).

2.2. The evolution operator \(L_q\)

The dynamics of the SPDE (1.7) as \(t \to +\infty\) is deeply linked to the spectral properties of the operator \(L_q\). We will restrict ourselves to the stationary case, that is when \(q_{|t=0} = q_0\) is equal to the synchronized (nontrivial) stationary solution \(q\) of evolution (1.1). In this case, the object of interest is the stationary version of (1.8):
\[
Lh(\theta, \omega) := \frac{1}{2} \partial^2 \theta h(\theta, \omega) - \partial_\theta (h(\theta, \omega) ((J * q)_{\mu}(\theta) + \omega) + q_\mu(\theta)(J * h_{\mu})(\theta)). \quad (2.8)
\]
The domain \(D\) of the operator \(L\) is given by:
\[
D := \left\{ h(\theta, \omega); \forall \omega, \theta \mapsto h(\theta, \omega) \in C^2(S), \int_{S} h(\theta, \omega) d\theta \mu(d\omega) = 0 \right\}. \quad (2.9)
\]

Remark 2.2. The choice of the domain \(D\) of \(L\) is crucial for the study of evolution (1.7). One encounters the same operator \(L\) for the linear stability of the stationary solution \(q\) since the linearized evolution of (1.1) around \(q\) is precisely given by \(\partial_t h_\tau = Lh_\tau\). The natural domain for this latter evolution (see (2.2)) is
\[
\left\{ h(\theta, \omega); \forall \omega, \theta \mapsto h(\theta, \omega) \in C^2(S), \forall \omega, \int_{S} h(\theta, \omega) d\theta = 0 \right\}. \quad (2.10)
\]
Indeed for all \(\omega, q(\cdot, \omega)\) is a probability density on \(S\) so that perturbing by elements of domain (2.10) enables to remain within the set of functions with integral 1 on \(S\). Here, evolution (1.7) does not live in domain (2.10) since \(\eta\) has a nontrivial mean-value \(C(\omega)\) for fixed \(\omega\) (recall (1.12)). We will see that the non self-averaging phenomenon holds in (2.9) and not in (2.10) (see Remark 2.7).
For the rest of this paper, we fix $K > 1$ and we restrict ourselves to the case where

$$\mu = \frac{1}{2} (\delta_{-\omega_0} + \delta_{\omega_0}),$$

(2.11)

where $\omega_0 > 0$ is a fixed parameter. This assumption on $\mu$ appears to be quite restrictive, but generalizing parts of the results we present here to more general distributions $\mu$ does not seem to be straightforward. We refer to §2.8 for a discussion on this topic.

In what follows, the following standard notations will be used: for an operator $F$, we will denote by $\rho(F)$ the set of all complex numbers $\lambda$ for which $\lambda - F$ is invertible, and by $R(\lambda, F) := (\lambda - F)^{-1}$, $\lambda \in \rho(F)$ the resolvent of $F$. The spectrum of $F$ will be denoted as $\sigma(F)$.

The first goal of this paper is to state a spectral decomposition of the operator $L$ defined in §2.8, based on perturbation arguments from the non-disordered case $\mu = \delta_0$ (see §2.1.2 and §2.3).

2.3. Distribution spaces

The spectral analysis of the operator $L$ (§2.8) will be mostly carried out in spaces of distribution that have $H^{-1}$ regularity w.r.t. $\theta$. But the precise study of $L$ requires to introduce weighted version of $H^{-1}$ that we define here. We first focus on weighted Sobolev spaces of functions $\theta \mapsto h(\theta)$ on $S$ (§2.3.1) and then introduce the corresponding spaces for functions with disorder $(\theta, \omega) \mapsto h(\theta, \omega)$ on $S \times \text{Supp}(\mu)$ (§2.3.2):

2.3.1. Weighted Sobolev spaces

For any bounded positive weight function $k(\cdot)$ on $S$ such that $\int_S h(\theta) \, d\theta = 1$, we may consider the space $L^2_k$ closure of $C(S)$ w.r.t. the norm:

$$\| h \|_{2,k} := \left( \int_S h^2(\theta) k(\theta) \, d\theta \right)^{\frac{1}{2}}.$$

(2.12)

The decomposition of $h$ into the sum of $\text{Span}(k)$ and its orthogonal supplementary in $L^2_k$ may be written as:

$$h = \left( \int_S h \right) \cdot k + h_0,$$

(2.13)

where $\int_S h_0 = 0$. Since $h_0$ is with zero mean value, each of its primitives are $2\pi$-periodic. In particular, we can consider $H^{-1}_k$ the closure of $C(S)$ with respect to the following weighted Sobolev norm:

$$\| h \|_{-1,k} := \left( \left( \int_S h \right)^2 + \int_S \frac{\mathcal{H}^2_0}{k} \right)^{\frac{1}{2}},$$

(2.14)

where $\mathcal{H}_0$ is the primitive of $h_0$ on $S$ such that $\int_S \frac{\mathcal{H}_0}{k} = 0$. Note that one can understand the spaces $H^{-1}_k$ as part of a Gelfand-triple construction (see Appendix A for a precise definition). In particular, we will make a constant use of the space $H^{-1}_{q_0}$ (that is for $k(\cdot) = q_0(\cdot)$ where $q_0$ is the stationary solution (2.6) of the non-disordered system) which is the natural space (see Prop. 2.5) for the study on the Kuramoto operator $L_{q_0}$ (2.18) in the non-disordered case.

**Remark 2.3.** In the case of a constant weight $k(\cdot) \equiv \frac{1}{2\pi}$, we will write $(L^2, \| \cdot \|_2)$ and $(H^{-1}, \| \cdot \|_{-1})$ instead of $(L^2_{\frac{1}{2\pi}}, \| \cdot \|_{2, \frac{1}{2\pi}})$ and $(H^{-1}_{\frac{1}{2\pi}}, \| \cdot \|_{-1, \frac{1}{2\pi}})$.

2.3.2. Weighted Sobolev spaces (with disorder)

The natural space in which to study the operator $L$ is the space of functions $h$ in $D$ such that each component $h(\cdot, \omega)$ lives in a certain $H^{-1}_{k(\cdot, \omega)}$ for a weight $k(\cdot, \omega)$ (which may depend on $\omega \in \text{Supp}(\mu)$). More precisely, for any family of positive weight functions $(k(\cdot, \omega))_{\omega \in \text{Supp}(\mu)}$, we denote as $H^{-1}_{\mu,k}$ the closure of $D$ w.r.t. the norm:

$$\| h \|_{\mu,-1,k} := \left( \int_R [h(\cdot, \omega)]^2 \mu(\omega) (d\omega) \right)^{\frac{1}{2}} = \left( \int_R \left( \int_S h(\theta) \, d\theta \right)^2 \mu + \int_R \int_S \frac{\mathcal{H}^2_0}{k} \, d\theta \, d\mu \right)^{\frac{1}{2}}.$$

(2.15)
We will also consider the analogous averaged weighted $L^2$-spaces, that is the space $L_{μ,k}^2$ given by the norm:

$$
\|h\|_{μ,k} := \left( \int_\mathbb{R} \int_\mathbb{S} h(θ,ω)^2 \, dθ \, dμ(ω) \right)^{\frac{1}{2}}. \tag{2.16}
$$

**Remark 2.4.** In the particular case of $k(\cdot,ω) = \frac{1}{ω}$ for all $ω ∈ \text{Supp}(μ)$, we will write

$$
H_{μ,1}^{-1} := H_{μ,1}^{-1}
$$

and the corresponding norm will be denoted as $\| \cdot \|_{H_{μ,1}}$. We will also write $(L_{μ,1}^2, \| \cdot \|_{μ,2})$ instead of $(L_{μ,1}^2, \| \cdot \|_{μ,2})$.

The main theorem concerning the operator $L$ (Theorem 2.8) will be stated in $H_{μ,1}^{-1}$ for the ease of exposition but its proof will require the introduction of weighted Sobolev spaces $H_{μ,k}^{-1}$ for nontrivial weights $k$.

2.4. The non-disordered case

In the context of the Kuramoto model without disorder, the linearized operator $L_{q_0}$ around stationary solution $q_0$ (see § 2.1.2), with domain

$$
\mathcal{D}_0 := \left\{ u ∈ C^2(\mathbb{S}); \int_\mathbb{S} u = 0 \right\} \tag{2.17}
$$

is:

$$
L_{q_0}u := \frac{1}{2} \partial_\theta^2 u - \partial_θ [q_0(J*u) + u(J*q_0)]. \tag{2.18}
$$

In [7], it is mainly proved that $L_{q_0}$ is essentially self-adjoint in $H_{q_0}^{-1}$:

**Proposition 2.5 ([7, Th. 1.8]).** $(L_{q_0}, \mathcal{D}_0)$ is essentially self-adjoint in $H_{q_0}^{-1}$. The spectrum of (the self-adjoint extension of) $L_{q_0}$ is pure point lying in $(-∞,0)$; 0 is in the spectrum, with one-dimensional eigenspace (spanned by $\partial_θ q_0$). Moreover, the distance $λ_K(L_{q_0})$ between the eigenvalue 0 and the rest of the spectrum is strictly positive.

2.5. Non self-averaging phenomenon for the operator $L$ and existence of a Jordan block

Linear trajectories that depend on the initial condition as observed in Figure 2a are reminiscent of an analogous deterministic finite-dimensional example: consider the 2-dimensional evolution $(x(t),y(t)) = L(x(t),y(t))$, for $L = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. It is trivial to see that the solutions of this system are linear in time: $\frac{x(t)}{t} → y_0$ as $t → ∞$. The existence of such a Jordan block is precisely equivalent to the existence of $x$ and $y$ such that $Lx = 0$ and $Ly = x$. The purpose of the first main theorem of this paper is to prove an analogous existence of a Jordan block for the operator $L$ in (2.8):

**Theorem 2.6.** For any fixed $ω_0 > 0$, if $q$ is the stationary solution in (2.2), then

$$
LΩq = 0. \tag{2.19}
$$

Moreover, there exists $p ∈ \mathcal{D}$ such that

$$
∀θ ∈ \mathbb{S}, ∀ω ∈ \text{Supp}(μ), \quad Lp(θ,ω) = Ωq(θ,ω). \tag{2.20}
$$

In particular, the characteristic space of $L$ in 0 is at least of dimension 2.

**Remark 2.7.** Equality (2.19) is a direct consequence of the rotation invariance in (1.4) (Remark 2.1). Note also that $p(\cdot,ω)$ found in (2.20) is with nontrivial mean value for all $ω ∈ \text{Supp}(μ)$. We believe in fact that $\int_\mathbb{S} p(\cdot,ω) = -\frac{1}{ω}$; this fact is derived from non-rigorous computations and verified by numerical simulations. In other terms, such a $p$ (and the corresponding Jordan block $\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ in the matrix representation (2.21) of the operator $L$) do not exist on the domain (2.10).

Theorem 2.6 is proved in Section 3.
2.6. Spectral properties of $L$ and position of the spectrum

The second goal of this paper is to prove that $L$ generates an analytic semi-group of operators with spectrum lying in the complex half-plane with negative real part:

**Theorem 2.8.** In the Hilbert space $H_{\mu}^{-1}$ defined in Remark 2.4, the operator $(L,D)$ is densely defined, closable, its closed extension having compact resolvent. In particular, its spectrum consists of isolated eigenvalues with finite multiplicities.

Moreover, for all $K > 1$, for all $\alpha \in (0, \frac{\pi}{2})$, for all $\rho \in (0, 1)$, there exists $\omega_\star = \omega_\star(K,\alpha,\rho) > 0$ such that, for all $0 < \omega_0 < \omega_\star$, the following is true:

- The spectrum of $L$ lies in a cone $C_{\alpha}$ with vertex 0 and angle $\alpha$

  $$C_{\alpha} := \left\{ \lambda \in \mathbb{C}; \frac{\pi}{2} + \alpha \leq \arg(\lambda) \leq \frac{3\pi}{2} - \alpha \right\} \subseteq \{ z \in \mathbb{C}; \Re(z) \leq 0 \}; \quad (2.21)$$

- There exists $\alpha' \in (0, \frac{\pi}{2})$ such that $L$ is the infinitesimal generator of an analytic semi-group defined on a sector $\Delta_{\alpha'} := \{ \lambda \in \mathbb{C}, |\arg(\lambda)| < \alpha' \}$;

- the dimension of the characteristic space in 0 is exactly 2, spanned by $\partial_0 q$ and $p$, where $p$ is defined in Theorem 2.6;

- the eigenvalue 0 is separated from the rest of the spectrum at a distance $\lambda_K(L) = \lambda(L,K,\rho)$ at least equal to $\rho \cdot \min(\lambda_K(L_{\partial_0 q}), \frac{1}{2})$, where $L_{\partial_0 q}$ and $r_0$ are defined in § 2.1.2.

Note that Theorem 2.8 relies on perturbation arguments of the non-disordered case mentioned in § 2.4. In particular, the spectral gap $\lambda_K(L)$ found in Theorem 2.8 depends on the spectral gap $\lambda_K(L_{\partial_0 q})$ for the non-disordered case.

As a consequence of Theorem 2.8, there exists a decomposition of $H_{\mu}^{-1}$ into the direct sum

$$H_{\mu}^{-1} = G_0 \oplus G_{\alpha}, \quad (2.22)$$

where $G_0$ is of dimension 2 (spanned by $\partial_0 q$ and $p$) such that the restriction of the operator $L$ to $G_0$ has spectrum $\{0\}$ and the restriction of $L$ to $G_{\alpha}$ has spectrum $\sigma(L) \setminus \{0\} \subseteq \{ \lambda \in \mathbb{C}; \Re(\lambda) < 0 \}$. We will denote as $P_0$ the corresponding projection on $G_0$ along to $G_{\alpha}$, and $P_\alpha = 1 - P_0$. In particular, there exist unique continuous linear forms $\ell_{\partial_0 q}$ and $\ell_p$ such that for all $h \in H_{\mu}^{-1}$

$$P_0 h := \ell_{\partial_0 q}(h)\partial_0 q + \ell_p(h)p. \quad (2.23)$$

To fix ideas, one may think of the following infinite matrix representation for the operator $L$:

$$L = \left( \begin{array}{cc} P_0 LP_0 & P_0 LP_{\alpha} \\ P_{\alpha} LP_0 & P_{\alpha} LP_{\alpha} \end{array} \right) = \left( \begin{array}{ccc} 0 & 1 & \ell_{\partial_0 q}(LP_{\alpha}) \\ 0 & 0 & 0 \\ 0 & \vdots & \vdots \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} \partial_0 q \\ p \end{array} \right) \begin{array}{c} G_0 \\ G_{\alpha} \end{array} \quad (2.24)$$

Note that the second line in the matrix representation (2.24) of $L$ is indeed equally zero since for all $h \in H_{\mu}^{-1}$, $Lh$ is of zero mean value on $S$; in particular $\ell_p(Lh) = 0$ for all $h \in H_{\mu}^{-1}$.
Remark 2.9. Any element \( h = (h(\theta, \omega))_{\theta \in \mathcal{S}, \omega \in \text{Supp}(\mu)} \) can be identified in our binary case (2.11) with a couple \((h_+(\theta), h_-(\theta))_{\theta \in \mathcal{S}}\). Moreover, any \( h \in \mathcal{H}_p^1 \) can be decomposed according to (2.22):

\[ h = \ell \partial q(h) \partial q + \ell \partial p(h) p + P_{o} h. \]

Let us integrate the latter decomposition w.r.t. \( \eta \). Since \( \int_{\mathcal{S}} L u = 0 \) for all \( u \in \mathcal{D} \), we have \( \int_{\mathcal{S}} P_{o} h = 0 \) so that one can actually find an explicit formulation for the functional \( \ell_p \):

\[ \ell_p(h) = \frac{\int_{\mathcal{S}} h_+}{\int_{\mathcal{S}} p_+} = \frac{\int_{\mathcal{S}} h_-}{\int_{\mathcal{S}} p_-}. \]

The last equality in (2.25) is due to the fact that \( \int_{\mathcal{S}} (h_+ + h_-) = \int_{\mathcal{S}} (p_+ + p_-) = 0 \).

2.7. Long time evolution of the fluctuation SPDE

We now turn to the main result of the paper, which concerns the asymptotic behavior of the fluctuation process \( \eta \) defined in (1.7):

Theorem 2.10. Under the hypothesis of Theorem 2.8, there exists a unique weak solution \( \eta \) to (1.7) in \( \mathcal{H}_p^1 \). Moreover, \( \eta \) satisfies the following asymptotic linear behavior: for fixed initial condition \( \eta_0^\omega = X + C(\omega) \), there exists \( v(\omega) \in \mathbb{R} \) such that

\[ \frac{\eta_t}{t} \underset{t \to \infty}{\xrightarrow{\text{in law}}} v(\omega) \partial q, \quad \text{as } t \to +\infty. \]

Moreover, \( \omega \mapsto v(\omega) \) is a Gaussian random variable with variance

\[ \sigma_v^2 := 2\left( \int_{\mathcal{S}} p_+ (\theta) \, d\theta \right)^{-2}, \]

where \( p_+ (\theta) := p(\theta, \omega_0) \) is defined by (2.20).

2.8. Comments on Theorem 2.10

2.8.1. Initial asymmetry of the disorder

As we will see in the proof of Theorem 2.10, the speed \( v(\omega) \) in (2.26) depends explicitly on the mean-value of the initial condition \( C(\omega) \) (recall (1.12)): \( \eta_0^\omega = X + C(\omega) \). Let us be more explicit on this dependence. At time \( t = 0 \), for \( N \geq 1 \) and \( \varphi : \mathcal{S} \times \mathbb{R} \to \mathbb{R} \), \( \eta_{N,0}^\varphi \) defined by (1.10) may be written as

\[ \eta_{N,0}^\varphi = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \varphi(\theta_j, \omega_j) + \int_{\mathcal{S} \times \mathbb{R}} \varphi(\theta, \omega) \gamma(\, d\theta) \mu(\, d\omega) \right). \]

Considering the binary case, the solution \( \eta_{N,0}^\varphi \) for the initial condition \( \eta_0^\omega = X + C(\omega) \) can be written as

\[ \eta_{N,0}^\varphi = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \varphi(\theta_j, \omega_j) + \int_{\mathcal{S} \times \mathbb{R}} \varphi(\theta, \omega) \gamma(\, d\theta) \right) + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left( \int_{\mathcal{S}} \varphi(\theta, \omega) \gamma(\, d\theta) - \int_{\mathcal{S} \times \mathbb{R}} \varphi(\theta, \omega) \gamma(\, d\theta) \mu(\, d\omega) \right). \]

The process \( X_N \) captures the initial fluctuations of the rotators whereas \( C_N \) captures the fluctuations of the disorder. It is easily seen that \( C_N \) converges in law (w.r.t. the disorder) to the process \( C \) with covariance given by (1.11). As we will see in the proof of Theorem 2.10, \( v(\cdot) \) actually depends on the process \( C_+ \) (indexed by functions \( \psi : \mathcal{S} \to \mathbb{R} \)) that is the restriction of the process \( C \) to the component on \( \tau_{\omega_0} \) (recall (2.11)):

\[ \forall \psi : \mathcal{S} \to \mathbb{R}, \quad C_{\psi,\psi} := C_{\psi_{1,\omega_0}}. \]
Thanks to (2.28), $C_+$ is the limit in law of the microscopic process $C_{N,+}$ defined by

$$\forall \psi, \ C_{N,+}(\psi) := \left( \int_{\mathcal{S}} \psi(\cdot) \, d\gamma \right) \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( I_{(\omega_i = \omega_0)} - \frac{1}{2} \right) = \left( \int_{\mathcal{S}} \psi(\cdot) \, d\gamma \right) \frac{\alpha_N}{\sqrt{N}} \quad (2.30)$$

Here, $\alpha_N$ is exactly the (centered) number of frequencies among $(\omega_1, \ldots, \omega_N)$ that are positive, so that $C_{N,+}$ captures the lack of symmetry of the initial chosen disorder: $\alpha_N > 0$ (resp. $\alpha_N < 0$) represents the case of an asymmetry in favor of positive (resp. negative) frequencies.

In [5, §10.2, p. 47], it is observed numerically that if we get rid artificially\(^1\) of the asymmetry between frequencies, there is no rotation in (1.4), no matter how the frequencies are sampled. We actually retrieve this phenomenon in Theorem 2.10 in the case where $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$, since in that case the quantity $\alpha_{2N}$ in (2.30) is equally zero for all $N \geq 1$ and so is the consequent limit speed $v$.

2.8.2. Perspectives

One could hope to generalize the results of the paper in at least two directions. Firstly, we have restricted ourselves to the binary case $\mu \in C_{\delta_{-\omega_0} + \delta_{\omega_0}}$. Note that the proof of Theorem 2.6 concerning the existence of a Jordan block (although written in this particular case for the reader’s convenience) is not specific to this case: one could easily rewrite the same proof for more general distributions $\mu$ (even with unbounded support), satisfying appropriate integrability conditions in $0$ and in $\infty$.

The main restriction on $\mu$ concerns Theorem 2.8: the hypothesis $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$ is critical for its proof. Indeed, the key argument of the proof is based on the fact that perturbing a finite dimensional kernel of an operator $A$ by a sufficiently small perturbation $B$ leads to a kernel for the operator $A + B$ with the same finite dimension. But for distributions more general than (2.11), the kernel of $L$ is likely to become of infinite dimension, so that similar perturbation arguments cannot be applied.

Secondly, Theorem 2.8 is only proved for small disorder $\omega_0$ whereas one would expect it to be true even for large disorder. It is indeed natural to believe that the non self-averaging phenomenon seen in Figure 2 not only holds for large disorder but would even be more noticeable in that case. However, since Theorem 2.8 relies on perturbation arguments, proving similar results for large $\omega_0$ seems to require alternative methods.

3. On the existence of a Jordan block for $L$ (Proof of Theorem 2.6)

The purpose of this section is to prove Theorem 2.6 i.e. the fact that the operator $L$ defined in (2.8) has a Jordan block of size at least 2. The symmetry of the system (Remark 2.1) leads to consider the set of distributions which are odd w.r.t. $(\theta, \omega) \in \mathcal{S} \times \text{Supp}(\mu)$:

$$\mathcal{O} := \{ h; \forall (\theta, \omega) \in \mathcal{S} \times \text{Supp}(\mu), \ h(-\theta, -\omega) = -h(\theta, \omega) \}. \quad (3.1)$$

We also denote by $\mathcal{N}$ the set of functions with zero mean-value for all $\omega \in \text{Supp}(\mu)$ (recall the definition of $\mathcal{D}$ in (2.9)):

$$\mathcal{N} := \left\{ h \in \mathcal{D}; \forall \omega \in \text{Supp}(\mu), \int_{\mathcal{S}} h(\theta, \omega) \, d\theta = 0 \right\}. \quad (3.2)$$

In the following straightforward lemma, whose proof is left to the reader, we sum-up the basic properties of the stationary solution $q$ (2.7) and the operator $L$ (2.8):

**Lemma 3.1.** The following statements are true:

\(^1\)for $N \geq 1$, choose $2N$ frequencies by sampling the first $N$ according to $\mu$ and choose the $N$ remaining frequencies as the exact opposite of the first ones.
1. \( \partial \omega q \in \mathcal{O} \cap \mathcal{N} \).
2. If \( h \in \mathcal{O} \) then \( L h \in \mathcal{O} \).
3. For all \( h \in \mathcal{D} \), \( L h \in \mathcal{N} \).
4. There exist \( 0 < c < C \) such that for all \( \theta \in \mathcal{S} \), \( \omega \in \text{Supp}(\mu) \), \( 0 < c \leq q(\theta, \omega) \leq C \).
5. For all \( \theta \in \mathcal{S} \), \( \omega \in \text{Supp}(\mu) \),

\[
\frac{1}{2} \partial \omega q(\theta, \omega) = q(\theta, \omega) (\langle J \ast q \rangle_\mu + \omega) + \kappa(\omega),
\]

where \( \kappa(\omega) = \frac{1 - e^{4\omega}}{2Z(\omega)} \), and \( Z(\omega) = Z(\omega, 2Kr) \) is the normalization constant defined in (2.2).

The fact that \( \partial \omega q \in \mathcal{O} \) can be seen as a consequence of Remark 2.1. A direct calculation shows that \( \partial \omega q \) is in the kernel of \( L \) (it corresponds to the rotation invariance of the problem). The rest of this section is devoted to prove the existence of an element \( p \in \mathcal{D} \) such that \( L p = \partial \omega q \).

We recall here the definition of the weighted Sobolev spaces introduced in § 2.3.2: we use here the spaces \( (H_{\mu, q}^{-1}, \langle \cdot, \cdot \rangle_{\mu, -1, q}) \) defined in (2.1.3) in the case of \( k = q \) and \( (L_{\mu}^2, \langle \cdot, \cdot \rangle_{\mu, 2}) \) defined in Remark 2.3. The main result is the following:

**Proposition 3.2.** For every \( \omega_0 > 0 \), in the binary case (2.1.1), for every \( v \in H_{\mu, q}^{-1} \cap \mathcal{O} \) (and in particular for \( v = \partial \omega q \)), there exists some \( p \in \mathcal{L}_\mu^2 \cap \mathcal{O} \) such that

\[
\forall l \in H_{\mu, q}^{-1} \cap \mathcal{O}, \ (L p, l)_{\mu, -1, q} = \langle v, l \rangle_{\mu, -1, q}.
\]

Moreover, in the case \( v = \partial \omega q \), any \( p \) that satisfies (3.3) is in fact a regular function (\( p(\cdot, \omega) \in C^\infty(\mathcal{S}) \) for all \( \omega \in \text{Supp}(\mu) \)) and is a classical solution to (2.2.2).

**Remark 3.3.** The scope of Proposition 3.2 is more general than the restrictive case of a binary law \( \mu = \frac{1}{4} (\delta_{\omega_0} + \delta_{\omega_0}) \); the following proof works for more general distributions \( \mu \), the only additional requirement being integrability conditions\(^2\) in \( 0 \) and \( +\infty \), see Remark 3.3.

Proof of Proposition 3.2 relies on several lemmas:

**Lemma 3.4.** For \( h \in \mathcal{O} \cap \mathcal{D} \), \( l \in \mathcal{O} \cap \mathcal{D} \), let us introduce the Dirichlet form

\[
\mathcal{E}_L(h, l) := \langle L h, l \rangle_{\mu, -1, q}.
\]

\( \mathcal{E}_L(\cdot, \cdot) \) is well defined on \( \mathcal{D}(\mathcal{E}_L) := (L_{\mu}^2 \cap \mathcal{O}) \times (H_{\mu, q}^{-1} \cap \mathcal{O}) \) and one can decompose \( \mathcal{E}_L(\cdot, \cdot) \) into:

\[
\forall (h, l) \in \mathcal{D}(\mathcal{E}_L), \quad \mathcal{E}_L(h, l) = \Gamma(h, l) + K \ell(h) \cdot \ell(l),
\]

where \( \Gamma(\cdot, \cdot) \), bilinear form on \( \mathcal{D}(\mathcal{E}_L) \) and \( \ell(\cdot) \) linear form on \( H_{\mu, q}^{-1} \cap \mathcal{O} \), are defined as follows:

\[
\forall (h, l) \in \mathcal{D}(\mathcal{E}_L), \quad \Gamma(h, l) := -\frac{1}{2} \int_{\mathbb{S} \times \mathbb{R}} \frac{h l}{q} d\lambda d\mu + \int_{\mathbb{S} \times \mathbb{R}} \kappa(\cdot) \frac{h L}{q^2} d\lambda d\mu,
\]

\[
\forall l \in H_{\mu, q}^{-1} \cap \mathcal{O}, \quad \ell(l) := \int_{\mathbb{S} \times \mathbb{R}} l \sin(\cdot) d\lambda d\mu,
\]

where \( \kappa \) and \( L \) in (3.7) are respectively defined in (3.3) and as the primitive of \( l \in H_{\mu, q}^{-1} \) such that \( \int_{\mathbb{S}} \frac{L(\omega)}{q(\omega)} = 0 \) for all \( \omega \in \text{Supp}(\mu) \) (recall § 2.3.2).  

**Lemma 3.5.** For all continuous linear form \( f \) on \( H_{\mu, q}^{-1} \cap \mathcal{O} \), there exists some \( p_1 \in \mathcal{L}_\mu^2 \cap \mathcal{O} \) such that for \( l \in H_{\mu, q}^{-1} \cap \mathcal{O} \)

\[
\Gamma(p_1, l) = f(l).
\]

\(^2\)Those conditions are obviously satisfied in the binary case (2.1.1).
Lemma 3.6. The linear form $\ell(\cdot)$ defined in (3.8) can be expressed as a scalar product on $H^{-1}_{\mu, q} \cap \mathcal{O}$: there exists $p_2 \in D \cap \mathcal{O}$, for all $l \in H^{-1}_{\mu, q} \cap \mathcal{O}$

$$\ell(l) = (Lp_2, l)_{\mu, -1, q}.$$  

(3.10)

Let us admit for a moment Lemmas 3.4, 3.5 and 3.6 and let us prove Proposition 3.2.

Proof of Proposition 3.2. Let $v$ be a fixed element of $H^{-1}_{\mu, q} \cap \mathcal{O}$. Applying Lemma 3.5 to the continuous linear form $f(l) = \langle v, l \rangle_{\mu, -1, q}$, there exists some $p_1 \in L^2_{\mu} \cap \mathcal{O}$ such that $\Gamma(p_1, l) = \langle v, l \rangle_{\mu, -1, q}$, which gives, using Lemma 3.4 and Lemma 3.6

$$\langle v, l \rangle_{\mu, -1, q} = \Gamma(p_1, l),$$

$$= (Lp_1, l)_{\mu, -1, q} - K \ell(p_1) \ell(l),$$

$$= (Lp_1, l)_{\mu, -1, q} - K \ell(p_1) (Lp_2, l)_{\mu, -1, q}.$$ We can conclude that the variational formula (3.4) is verified for the following choice of $p$:

$$p := p_1 - K \ell(p_1) p_2 \in L^2_{\mu} \cap \mathcal{O}.$$  

(3.11)

Let us prove now that such $p$ is in fact a regular function in $\theta$: since $p_2 \in D$ is regular in $\theta$, it suffices to prove that for all $\omega \in \text{Supp}(\mu)$, $\theta \mapsto p_1(\theta, \omega)$ is $\mathcal{C}^2$ (in fact $\mathcal{C}^\infty$) in $\theta$. We start from the definition of $p_1$:

$$\forall l \in H^{-1}_{\mu, q} \cap \mathcal{O}, \quad \Gamma(p_1, l) = (\partial_\theta q, l)_{\mu, -1, q}.$$  

Since this true for all $l \in H^{-1}_{\mu, q} \cap \mathcal{O}$, thanks to the expression of $\Gamma$ in (3.7), we obtain that for any fixed $\omega \in \text{Supp}(\mu)$, for Lebesgue-almost every $\theta \in S$:

$$\frac{1}{4} \int_S \frac{p_1(\theta, \omega)}{q(\theta, \omega)} = -\kappa(\omega) \left( \int_0^\theta p_1(u, \omega) \frac{u}{q(u, \omega)^2} du \right) + \int_0^\theta Q(u, \omega) q(u, \omega) du,$$  

(3.12)

where $Q(\cdot, \omega)$ is the primitive of $\partial_\theta q(\cdot, \omega)$ such that $\int_S \frac{Q(\cdot, \omega)}{q(\cdot, \omega)} = 0$. Using that $q$ is bounded and $\mathcal{C}^\infty$ in $\theta$ and that $p_1(\cdot, \omega) \in L^2$, we see that the primitive $\int_0^\theta \frac{Q(u, \omega)}{q(u, \omega)} du$ has a $\mathcal{C}^1$ version. Thanks to (3.12), $p_1$ has a $\mathcal{C}^1$ version. So, the right-hand side of (3.12) is at least $\mathcal{C}^2$, and so does $p_1$. The same repeated argument shows that $p_1$ is $\mathcal{C}^\infty$ in $\theta$. That concludes the proof of Proposition 3.2.

It now remains to prove the three lemmas:

Proof of Lemma 3.4. Let us prove equality (3.10): since $L$ is a primitive of $l$, one has

$$\frac{1}{2} \int_S \frac{(\partial_\theta h)L}{q} = -\frac{1}{2} \int_S \frac{h l}{q} + \frac{1}{2} \int_S \frac{h L}{q^2} \partial_\theta q.$$  

Using (3.10), for $\omega \in \text{Supp}(\mu)$

$$\frac{1}{2} \int_S \frac{(\partial_\theta h)L}{q} = -\frac{1}{2} \int_S \frac{h l}{q} + \int_S \frac{h L}{q^2} (\langle J * q \rangle_{\mu}(\cdot) + \kappa(\cdot)) \int_S \frac{h L}{q^2}.$$  

Thanks to the expression of $Lh$ in (2.8), we obtain

$$E_L(h, l) = -\frac{1}{2} \int_{S \times R} \frac{h l}{q} + \int_{S \times R} \kappa(\cdot) \frac{h}{q^2} L - \int_{S \times R} \langle J * h \rangle_{\mu} L,$$  

(3.13)

$$= \Gamma(h, l) - \int_{S \times R} \langle J * h \rangle_{\mu} L.$$
Lastly, integrating by parts the last term in (3.13) and expanding the cosine function (recall $J(\cdot) = -K\sin(\cdot)$), we obtain:

$$
- \int_{S^*R} (J + h) \mu, L = K \left( \int_{S^*R} \cos(\cdot) t d\lambda d\mu \right) \left( \int_{S^*R} \cos(\cdot) h d\lambda d\mu \right) + K \left( \int_{S^*R} \sin(\cdot) t d\lambda d\mu \right) \left( \int_{S^*R} \sin(\cdot) h d\lambda d\mu \right).
$$

But, since $l \in \mathcal{O}$, the first term in the latter expression is zero. The result (3.10) follows.

\textbf{Proof of Lemma 3.3.} In this proof, we use the following extension to Lax-Milgram Theorem:

\textbf{Proposition 3.7 (12 chap. III).} Let $\mathcal{H}, |.|$ be a Hilbert space and $\mathcal{G}, |.|$ a normed linear space. Suppose $\Gamma : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$ is bilinear and that $\Gamma(\cdot, \varphi)$ is continuous for each $\varphi \in \mathcal{G}$. If there exists some constant $C > 0$ such that

$$
\inf_{|\varphi| = 1} \sup_{|l| \leq 1} |\Gamma(h, \varphi)| \geq C, \quad \text{(weak coercivity), (3.14)}
$$

Then for each $f \in \mathcal{G}$ there exists some $p \in \mathcal{H}$ such that $\Gamma(p, \varphi) = f(\varphi)$ for all $\varphi \in \mathcal{G}$.

The principle of the proof of Lemma 3.3 is to show that the bilinear function $\Gamma$ defined in (3.7) satisfies Proposition 3.7 for

$$
\mathcal{H} := L^2_{\mu} \cap \mathcal{O}, \text{ endowed with } \|\cdot\|_{\mu, 2}, \quad (3.15)
$$

$$
\mathcal{G} := H^{-1}_{\mu, q} \cap \mathcal{O} \cap L^\infty(S \times R), \text{ endowed with } \|\cdot\|_{\mu, -1, q}, \quad (3.16)
$$

Namely, we have the following:

1. For each $l \in \mathcal{G}$, $\Gamma(\cdot, l)$ is continuous on $L^2_{\mu} \cap \mathcal{O}$: indeed, for the first term of $\Gamma(h, l)$, we have

$$
\left| \int_{S^*R} \frac{hl}{q} d\lambda d\mu \right| \leq C \| l \|_{\infty} \int_{S^*R} |h| d\lambda d\mu \leq C \| h \|_{\mu, 2}.
$$

And for the second term, using the boundedness of $q$:

$$
\int_{S^*R} \left| \kappa(\cdot) \frac{h}{q} L \right| d\lambda d\mu \leq C \int_{S^*R} |h L| d\lambda d\mu \leq C \| h \|_{\mu, 2} \| l \|_{\mu, -1, q}.
$$

2. $\Gamma$ is weakly coercive: let us fix $l \in \mathcal{G}$ such that $\| l \|_{\mu, -1, q} = 1$.

Let us choose $h = gL \in L^2_{\mu} \cap \mathcal{O}$, where for all $\omega \in \text{Supp}(\mu)$, $g(\cdot, \omega)$ is a $2\pi$-periodic function to be defined later. Then, by integration by parts in the equality $\Gamma(h, l)$

$$
\Gamma(h, l) = \frac{1}{2} \int_{S^*R} \frac{g}{q} L + \int_{S^*R} \kappa(\cdot) \frac{g}{q} L^2 = \int_{S^*R} \left\{ \frac{1}{4} \partial_{\theta} \left( \frac{g}{q} \right) + \kappa(\cdot) \frac{g}{q^2} \right\} L^2.
$$

Consider now for fixed $\omega \in \text{Supp}(\mu)$ the following first order ODE, with periodic boundary condition:

$$
\frac{1}{4} \partial_{\theta} f(\cdot, \omega) + \kappa(\omega) \frac{f(\cdot, \omega)}{q(\cdot, \omega)} = \frac{1}{q(\cdot, \omega)}, \quad \text{with } f(0, \omega) = f(2\pi, \omega).
$$

Then for any $\omega \in \text{Supp}(\mu) \setminus \{0\}$, an explicit calculation (left to the reader) shows that there exists a unique solution to (3.18), $\theta \mapsto f(\theta, \omega)$.

\textbf{Remark 3.8.} In the case $\omega = 0$, (3.18) reduces to $\frac{1}{4} \partial_{\theta} f = \frac{1}{q_0}$ which is incompatible with the condition $f(0) = f(2\pi)$, since $\int_{\theta_0}^{\theta_0} \frac{1}{q_0} d\theta > 0$: there is no such $2\pi$-periodic solution in the case $\omega = 0$.

Moreover, it is straightforward to see that $\| \int_{\mathbb{R}} |f(\cdot, \omega)| d\mu \|_{\infty, S} \leq C$, for some constant $C > 0$. 

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Remark 3.9. It is easy to see that $f(\cdot,\omega)$ is not bounded as $\omega \to 0$ and $\omega \to +\infty$; thus, for general distributions $\mu$, the same control on $f$ requires additional integrability assumptions in $0$ and $+\infty$ (namely $\int_{\mathbb{R}} \max\left(\frac{1}{|\xi|^d}, e^{\lambda_\omega}\right) \mu(\,d\omega) < \infty$ for some constant $c > 0$).

If we choose $\mathbf{h}$ such that $h = g \cdot \mathcal{L}$ with $g(\cdot,\omega) = q(\cdot,\omega) f(\cdot,\omega)$, we have the following:

- By construction of $f$, using (3.18) in (3.17), $\Gamma(h,l) = \|l\|_{\mu,-1,q}^2 = 1$,
- $\|h\|_{\mu,2}^2 \leq C \int_{\mathbb{R}} f^2 \frac{e^{2|\omega|}}{7} \, d\lambda \leq C$. So, $\sup_{\|l\|_{\mu,1,2,1} \leq 1} \|\Gamma(h,l)\|_{\mu} \geq \frac{1}{2}$, where $C$ is independent of $l \in \mathcal{G}$ such that $\|l\|_{\mu,-1,q} = 1$.

Applying Proposition 3.7 we obtain the existence of some $p_1 \in L^2_{\mu} \cap \mathcal{O}$ such that $\Gamma(l_1,1) = f(l)$, for all $l \in \mathcal{G}$. But by density, this is also true for $l \in H^{\mu,-1}_{\mu,q} \cap \mathcal{O}$.

Proof of Lemma 3.6. Since there exists some constants $C, c > 0$ such that for all $\omega \in \text{Supp}(\mu)$, $\theta \in S$, $0 < c \leq q(\theta,\omega) \leq C$, $\ell$ is continuous on $H^{-1}_{\mu,q} \cap \mathcal{O}$ (as well as on $L^2_{\mu} \cap \mathcal{O}$). More precisely, by Riesz theorem, there exists a unique $\mu \in H^{-1}_{\mu,q} \cap \mathcal{O}$ such that for all $l \in H^{-1}_{\mu,q} \cap \mathcal{O}$, $\ell(l) = \{e, l\}_{\mu,-1,q}$.

One can be more explicit: a simple calculation shows that this $(\theta,\omega) \mapsto \ell(\theta,\omega)$ corresponds to the primitive $E(\theta,\omega) = -q(\theta,\omega) \cos(\theta)$, that is:

\[ \forall \theta \in S, \omega \in \text{Supp}(\mu), \quad e(\theta,\omega) = -\partial_\theta q(\theta,\omega) \cos(\theta) + q(\theta,\omega) \sin(\theta). \quad (3.19) \]

Let us introduce the following function $p_2 \in L^2_{\mu} \cap \mathcal{O}$:

\[ p_2(\theta,\omega) = \frac{e^{-B(\theta,\omega)}}{1 - e^{4\pi\omega}} \int_{S} e^{B(u,\omega)+4\pi u} \, du + \int_{0}^{\theta} e^{B(u,\omega)-B(\theta,\omega)} \, du, \quad (3.20) \]

for

\[ B(\theta,\omega) = -2(K_{r}(\cos(\theta) - 1) + \omega \theta). \]

Then one readily verifies that $Lp_2$ is proportional to $e$.

4. Global spectral properties of operator $L$ (Proof of Theorem 2.8)

The purpose of this section is to prove Theorem 2.8. The main idea of the proof is to decompose the operator $L$ defined by (2.8) on the domain $\mathcal{D}$ given by (2.0) into the sum of a self-adjoint operator $A$ (in a weighted Sobolev space for appropriate weights) and a perturbation $B$ which will be considered to be small w.r.t. $A$. Namely, one can decompose $L(2.8)$ into $L = A + B$ where, for all $h \in \mathcal{D}$, for all $\omega \in \text{Supp}(\mu)$,

\[ Ah(\theta,\omega) := \frac{1}{2} \partial_\theta^2 h(\theta,\omega) - \partial_\theta \left( h(\theta,\omega)(J * q_0)(\theta) + q_0(\theta)(J * h)\mu \right), \quad (4.1) \]

and,

\[ Bh(\theta,\omega) := -\partial_\theta \left( h(\theta,\omega)(J * (q - q_0))\mu(\theta) + \omega \right) + (q(\theta,\omega) - q_0(\theta))(J * h)\mu. \quad (4.2) \]

We divide the proof of Theorem 2.8 into three parts: in § 4.1 we prove that $A$ is essentially self-adjoint (and thus generates an analytic semigroup) in some weighted Sobolev space (recall § 2.3) for an appropriate choice of weights. Note that this section strongly relies on the fact that $\mu$ is a binary distribution.

The purpose of § 4.2 is to establish precise control of the size of the perturbation $B$ w.r.t. $A$. The last step of the proof (§ 4.3) consists in deriving similar spectral properties for $L = A + B$, especially the fact that the spectrum of $L$ lies in the complex half-plane with negative real part.
4.1. Spectral properties of the operator $A$

In this paragraph, we prove mainly that $A$ defined in (4.1) is essentially self-adjoint for a Sobolev norm that is equivalent to the norm $\| \cdot \|_{L^2}$ defined in §2.3.2.

Since we are working in the domain $\mathcal{D}$ (recall (2.9)), the test functions $h$ are such that $\int h(\cdot, \omega) \, d\mu = \frac{1}{2} (h(\cdot, +\omega_0) + h(\cdot, -\omega_0))$ has zero mean value on $\mathcal{S}$. The idea of this paragraph is to reformulate the operator $\tilde{A}$ in terms of the sum $\frac{1}{2} (h(\cdot, +\omega_0) + h(\cdot, -\omega_0))$ and the difference $\frac{1}{2} (h(\cdot, +\omega_0) - h(\cdot, -\omega_0))$; namely, we define the following $2 \times 2$ invertible matrix:

$$M := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and for $h \in \mathcal{D}$, let $(w) := M \cdot h$, namely

$$\begin{align*}
  u(\cdot) & := \frac{1}{2} (h(\cdot, +\omega_0) + h(\cdot, -\omega_0)), \\
  v(\cdot) & := \frac{1}{2} (h(\cdot, +\omega_0) - h(\cdot, -\omega_0)).
\end{align*}$$

We are now able to define the following operator: $\hat{A} := M \circ A \circ M^{-1}$, defined on the domain $\hat{\mathcal{D}}$

$$\hat{\mathcal{D}} := \left\{(u, v) \in C^2(\mathcal{S}) \times C^2(\mathcal{S}); \int_{\mathcal{S}} u(\theta) \, d\theta = 0 \right\},$$

given by

$$\forall (u, v) \in \hat{\mathcal{D}}, \quad \hat{A}(u, v) := \begin{pmatrix} \hat{A}_1 u \\ \hat{A}_2 v \end{pmatrix} := \begin{pmatrix} \frac{1}{2} \partial_{\theta^2} u - \partial_{\theta^2} [u(J * q_0) + q_0(J * u)] \\ \frac{1}{2} \partial_{\theta^2} v - \partial_{\theta^2} [v(J * q_0)] \end{pmatrix}. $$

The remarkable observation is that operator $\hat{A}$ is now uncoupled w.r.t. variables $u$ and $v$; consequently, in order to diagonalize $\hat{A}$, it suffices to diagonalize both components of $\hat{A}$, namely $\hat{A}_1$ and $\hat{A}_2$. This is the purpose of Propositions 4.1 and 4.2 below.

We use here the weighted Sobolev norms $\| \cdot \|_{-1, k}$ defined in (2.14) for different choices of $k(\cdot)$. Concerning the first component, $\hat{A}_1 = L_{q_0}$ (with domain $\{u \in C^2(\mathcal{S}); \int_{\mathcal{S}} u = 0\}$) is equal to the McKean-Vlasov operator with no disorder defined in (2.15). Following §2.3 the natural space for the study of $\hat{A}_1$ is $H_{q_0}^{-1}$ defined in (2.14), for the weight $k(\cdot) = q_0(\cdot)$ (recall (2.6)). In this space, we have

**Proposition 4.1.** In $H_{q_0}^{-1}$, $\hat{A}_1$ is essentially self-adjoint with compact resolvent and spectrum in the negative part of the real axis. $0$ is a one-dimensional eigenvalue, spanned by $\partial_{\theta} q_0$. The spectral gap $\lambda_K(\hat{A}_1) = \lambda_K(L_{q_0})$ between $0$ and the rest of the spectrum is strictly positive.

Moreover, the self-adjoint extension of $\hat{A}_1$ is the infinitesimal generator of a strongly continuous semi-group of contractions $\hat{T}_1(t)$ on $H_{q_0}^{-1}$. For every $0 < \alpha < \frac{\pi}{2}$, this semigroup can be extended to an analytic semi-group $\hat{T}_1(z)$ defined on $\Delta_\alpha = \{ \lambda \in \mathbb{C}; \arg(\lambda) < \frac{\pi}{2} + \alpha \} \cup \{0\}$:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right), \ \forall \lambda \in \Sigma_\alpha, \ \| R(\lambda, \hat{A}_1) \|_{-1, q_0} \leq \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|^2}. $$

The second component $\hat{A}_2$ is a second order ordinary differential operator, with domain $C^2(\mathcal{S})$. The natural space in which to study $\hat{A}_2$ (see §1.1.2) is $H_w^{-1}$, for the choice of the weight function $\theta \mapsto w(\theta) = \frac{\sin(\theta)}{\sin(\theta)}$, with

$$\Phi(\theta) := -2K r_0 \cos(\theta),$$

where $r_0$ is given by (2.7). Namely, we have
Proposition 4.2. The operator $(\tilde{A}_2, C^2(\mathbf{S}))$ is essentially self-adjoint in $\mathbf{H}_w^{-1}$ and has compact resolvent. Hence, its spectrum consists of isolated eigenvalues with finite multiplicities. The kernel of $\tilde{A}_2$ is of dimension 1, spanned by $w(\theta) = e^{i\theta}$. Moreover, we have the following spectral gap estimation:

$$\forall v \in C^2(\mathbf{S}), \quad \langle \tilde{A}_2 v, v \rangle_{-1, w} \geq \frac{e^{-4K_0}}{2} \left\| v - \left( \int_{\mathbf{S}} v \right) w \right\|_{-1, w},$$

(4.8)

so that the spectrum of $\tilde{A}_2$ lies in the negative part of the real axis and the distance between 0 and the rest of the spectrum $\lambda_K(\tilde{A}_2)$ is at least equal to $\frac{e^{-4K_0}}{2}$. One also has explicit estimate on the resolvent of $\tilde{A}_2$:

$$\forall \alpha \in (0, \frac{\pi}{2}), \forall \lambda \in \Sigma_\alpha, \quad \left\| R(\lambda, \tilde{A}_2) \right\|_{-1, w} \leq \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}.$$  

(4.9)

Putting things together, the natural norm for the operator $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$ is the Hilbert-norm:

$$\left( \left\| u \right\|_{-1, q_0}^2 + \left\| v \right\|_{-1, w}^2 \right)^{\frac{1}{2}}, (u, v) \in \tilde{D}.$$  

But since $\tilde{A}$ is the conjugate of $A$ through the invertible matrix $M$, to say that $\tilde{A}$ is essentially self-adjoint for the previous norm is equivalent to say that $A$ is essentially self-adjoint for the corresponding conjugate norm:

$$\forall h \in \mathcal{D}, \quad \| h \|_{H_w} := \left( \left\| \frac{1}{2} (h(\cdot, +\omega_0) + h(\cdot, -\omega_0)) \right\|_{-1, q_0}^2 + \left\| \frac{1}{2} (h(\cdot, +\omega_0) - h(\cdot, -\omega_0)) \right\|_{-1, w}^2 \right)^{\frac{1}{2}}.$$  

(4.10)

The results of § 4.1 can be summed-up in the following proposition, which is an easy consequence of Propositions 4.1.1 and 4.2.

Proposition 4.3. For the norm $\| \cdot \|_{H_w}$ defined in (4.10), the operator $(A, D)$ is essentially self-adjoint, with compact resolvent. The spectrum of (the self-adjoint extension of) $A$ is pure-point, and consists of eigenvalues with finite multiplicities. Moreover it lies in the negative part of the real-axis and $A$ is the infinitesimal generator of an analytic semigroup of operators $T_t(A(z)$ defined on a domain $\Delta_\alpha = \{ z \in \mathbb{C}; |\arg(z)| < \alpha \}$, for any $0 < \alpha < \frac{\pi}{2}$. One also has the following estimate about the resolvent of $A$:

$$\forall \alpha \in (0, \frac{\pi}{2}), \forall \lambda \in \Sigma_\alpha, \quad \left\| R(\lambda, A) \right\|_{H_w} \leq \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}.$$  

(4.11)

The kernel of $A$ is of dimension 2, spanned by $\{ \partial_\theta q_0 + \frac{e^{i\theta}}{\tau} \partial_\phi q_0, \partial_\theta q_0 - \frac{e^{i\theta}}{\tau} \partial_\phi q_0 \}$ and the eigenvalue 0 is separated from the rest of the spectrum with a distance $\lambda_K(A) := \min \{ \lambda_K(\tilde{A}_1), \lambda_K(\tilde{A}_2) \}$, where $\lambda_K(\tilde{A}_1)$ and $\lambda_K(\tilde{A}_2)$ are defined in Propositions 4.1.1 and 4.2 respectively.

Remark 4.4. The norm $\| \cdot \|_{H_w}$ is equivalent to the norm $\| \cdot \|_{H_w}$ defined in § 2.3.2 since the weights $q_0$ and $w$ are bounded above and below. In $\mathbf{H}_w^{-1}$, the operator $A$ (although no longer self-adjoint) still generates an analytic semi-group with the same spectrum and the same spectral gap.

The aim of paragraphs § 4.1.1 (resp. § 4.1.2) is to prove Proposition 4.1 (resp. Proposition 4.2).

4.1.1. Spectral properties of $\tilde{A}_1$: proof of Proposition 4.1

As $\tilde{A}_1 = L_{q_0}$ corresponds to the linear evolution operator of the non-disordered Kuramoto model studied in [7], we know from Proposition 2.5 that $\tilde{A}_1$ is essentially self-adjoint and dissipative in $\mathbf{H}_{q_0}^{-1}$. It remains to prove that $\tilde{A}_1$ generates an analytic semigroup $\tilde{T}_1(t)$ in an appropriate domain. We refer to classical references [27, 32, 38] for detailed definitions of analytic semigroups of operators defined on a sector of the complex plane. We recall the following result about analytic extensions of strongly continuous semigroups.
Proposition 4.5 ([38 Th 5.2, p.61]). Let $T(t)$ a uniformly bounded strongly continuous semigroup, whose infinitesimal generator $F$ is such that $0 \in \rho(F)$ and let $\alpha \in (0, \frac{\pi}{2})$. The following statements are equivalent:

1. $T(t)$ can be extended to an analytic semigroup in the sector $\Delta_\alpha = \{ \lambda \in \mathbb{C}; |\arg(\lambda)| < \alpha \}$ and $\| T(z) \|$ is uniformly bounded in every closed sub-sector $\Delta_{\alpha'}$, $\alpha' < \alpha$, of $\Delta_\alpha$.

2. There exists $M > 0$ such that

$$\rho(F) \supset \Sigma_\alpha = \left\{ \lambda \in \mathbb{C}; |\arg(\lambda)| < \frac{\pi}{2} + \alpha \right\} \cup \{0\},$$

and

$$\| R(\lambda, F) \| \leq \frac{M}{|\lambda|}, \quad \lambda \in \Sigma, \lambda \neq 0.$$ (4.13)

We are now in position to prove the rest of Proposition [4.1] we know from [7, Prop. 2.3, Prop. 2.6] that for any $\lambda > 0$, $\lambda - \hat{A}_1$ is positive with range $H_{\alpha_0}^{-1}$. Consequently, we can apply Lumer-Phillips Theorem (see [38 Th 4.3 p.14]): $\hat{A}_1$ is the infinitesimal generator of a $C_0$ semi-group of contractions denoted by $T_{1}(t)$.

The rest of the proof is devoted to show the existence of an analytic extension of this semigroup in a proper sector. We follow here the lines of the proof of [38 Th 5.2, p. 61-62], but with explicit estimates on the resolvent: let us first replace the operator $\hat{A}_1$ by a small perturbation: for all $\varepsilon > 0$, let $\hat{A}_{1,\varepsilon} := \hat{A}_1 - \varepsilon$, so that 0 belongs to $\rho(\hat{A}_{1,\varepsilon})$. As $\hat{A}_1$, the operator $\hat{A}_{1,\varepsilon}$ is self-adjoint and generates a strongly continuous semigroup of operators (which is $\hat{T}_{1,\varepsilon}(t) = \hat{T}_1(t)e^{-\varepsilon t}$). Moreover, since $\hat{A}_{1,\varepsilon}$ is self-adjoint, we have

$$\forall \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \| R(\lambda, \hat{A}_{1,\varepsilon}) \|_{-1,\varepsilon} \leq \frac{1}{\|\lambda\|},$$

and since the spectrum of $\hat{A}_{1,\varepsilon}$ is negative, for every $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > 0$, we have

$$\| R(\lambda, \hat{A}_{1,\varepsilon}) \|_{-1,\varepsilon} \leq \frac{1}{|\lambda|}.$$ (4.15)

Let us prove that for $\lambda \in \Sigma_\alpha$,

$$\| R(\lambda, \hat{A}_{1,\varepsilon}) \|_{-1,\varepsilon} \leq \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}.$$ (4.16)

Note that (4.14) is clear from (4.13) and (4.15) when $\Re(\lambda) \geq 0$. Let us prove it for $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$. Consider $\sigma > 0, \tau \in \mathbb{R}$ to be chosen appropriately later and write the following Taylor expansion for $R(\lambda, \hat{A}_{\varepsilon})$ around $\sigma + i\tau$ (at least well defined in a neighborhood of $\sigma + i\tau$ since $\sigma > 0$):

$$R(\lambda, \hat{A}_{1,\varepsilon}) = \sum_{n=0}^{\infty} R(\sigma + i\tau, \hat{A}_{1,\varepsilon})^{n+1}((\sigma + i\tau) - \lambda)^n.$$ (4.17)

This series $R(\cdot, \hat{A}_{1,\varepsilon})$ is well defined in $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$ if one can choose $\sigma, \tau$ and $k$ in $(0, 1)$ such that $\| R(\sigma + i\tau, \hat{A}_{1,\varepsilon}) \|_{-1,\varepsilon} |\lambda - (\sigma + i\tau)| \leq k < 1$. In particular, using (4.13), it suffices to have $|\lambda - (\sigma + i\tau)| \leq k|\tau|$ and since $\sigma > 0$ is arbitrary, it suffices to find $k \in (0, 1)$ and $\tau$ with $|\lambda - i\tau| \leq k|\tau|$ to obtain the convergence of (4.17). For this $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$, let us define $\lambda'$ and $\tau$ as in Figure [3] Then, $|\lambda - i\tau| \leq |\lambda' - i\tau| = |\sin(\alpha)||\tau|$ with $\sin(\alpha) \in (0, 1)$. So the series converges for $\lambda \in \Sigma_\alpha$ and one has, using again (4.14),

$$\| R(\lambda, \hat{A}_{1,\varepsilon}) \|_{-1,\varepsilon} \leq \frac{1}{(1 - \sin(\alpha))|\tau|} \leq \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}.$$ (4.18)

The fact that $\hat{T}_{1,\varepsilon}(t)$ can be extended to an analytic semigroup $\hat{T}_{1,\varepsilon}(z)$ on the domain $\Delta_\alpha$ is a simple application of (4.16) and Proposition 4.5 with $M := \frac{1}{1 - \sin(\alpha)}$. Let us then define $\hat{T}_{1}(z) := e^{z\hat{T}_{1,\varepsilon}(z)}$, for $z \in \Delta_\alpha$, so that $\hat{T}_{1}(z)$ is an analytic extension of $\hat{T}_{1}(t)$ (an argument of analyticity shows that $\hat{T}_{1}(z)$ does not depend on $\varepsilon$).

Note that estimation (4.16) can be obtained by letting $\varepsilon \to 0$ in (4.16).
4.1.2. Spectral properties of $\tilde{A}_2$: proof of Proposition 4.2

$\tilde{A}_2$ may be written as

$$\tilde{A}_2 v = \frac{1}{2} \partial \theta v + \partial \theta (vKv_0 \sin(\cdot)),$$

(4.19)

where $r_0 = \Psi_0(2Kr_0)$ (recall (2.7)). One recognizes in $\tilde{A}_2$ a Fokker-Planck operator on $C^2(S)$ with a sine potential. This operator can easily be seen, by integrations by parts in an appropriate weighted $L^2$-space, as a Sturm-Liouville operator ([17, 11]) acting on $C^2$, $2\pi$-periodic functions.

The problem is that a $L^2$-norm is not appropriate for the future study of the SPDE (1.7); a look at the covariance structure of the noise $W$ (see (1.9)) shows that $W$ naturally lives in a $H^{-1}$-space instead of a $L^2$-space.

An easy calculation shows that $\tilde{A}_2$ can be rewritten in terms of the weight function $\Phi$ defined in (4.7):

$$\tilde{A}_2 v = \frac{1}{2} \partial \theta (e^{-\Phi} \partial \theta (e^{\Phi} v)).$$

(4.20)

Let $w$ be:

$$w(\theta) := \frac{e^{-\Phi(\theta)}}{\int_S e^{-\Phi}}.$$ 

(4.21)

One directly sees from (4.20) that $w$ lies in the kernel of $\tilde{A}_2$: $\tilde{A}_2 w = 0$. We are now in position to prove Proposition 4.2; we place ourselves in the framework of the weighted Sobolev spaces $(L^2_w, \langle \cdot, \cdot \rangle_{2,w})$ and $(H^{-1}_w, \langle \cdot, \cdot \rangle_{-1,w})$, in the particular case of $k(\cdot) = w(\cdot)$.

**Proof of Proposition 4.2** In $H^{-1}_w$, the operator $(\tilde{A}_2, C^2(S))$ is formally symmetric: for $u, v \in C^2(S)$, for $u_0$ ad $v_0$ defined by (2.13), we have successively,

$$\langle \tilde{A}_2 u, v \rangle_{-1,w} = \left( \int_S e^{-\Phi} \right) \int_S e^{\Phi} \frac{1}{2} e^{-\Phi} \partial \theta (e^{\Phi} u) v_0 = -\frac{1}{2} \left( \int_S e^{-\Phi} \right) \int_S e^{\Phi} uv_0,$$

$$= -\frac{1}{2} \int_S uv_0 \frac{1}{w} - \frac{1}{2} \int_S u \int_S v_0 = \frac{1}{2} \int_S \frac{u_0 v_0}{w}.$$ 

(4.22)

Let us prove that $(\tilde{A}_2, C^2(S))$ is essentially self-adjoint: let $\mathcal{E}_2$ be the following Dirichlet form

$$\mathcal{E}_2(u, v) := \langle u, (1 - \tilde{A}_2) v \rangle_{-1,w} = \int_S u \int_S v + \int_S V_0 \frac{u_0 v_0}{w} + \frac{1}{2} \int_S \frac{u_0 v_0}{w}.$$ 

(4.23)
Then it is easy to see that $\mathcal{E}_2$ is a continuous bilinear form on $L^2_w$ (thanks to Poincaré inequality). Moreover $\mathcal{E}_2$ is coercive: for all $u \in L^2_w$

$$\mathcal{E}_2(u, u) = \left( \int_S u \right)^2 + \int_S \frac{u^2}{w} + \frac{1}{2} \int_S \frac{u^2}{w} \geq \left( \int_S u \right)^2 + \frac{1}{2} \left\| u - \left( \int_S u \right) \right\|^2_{2,w} \geq \frac{1}{2} \left\| u \right\|^2_{2,w}. \quad (4.24)$$

Since for all $f \in H^{-1}_w$, the linear form $v \mapsto \langle v, f \rangle_{-1,w}$ is continuous on $L^2_w$, an application of Lax-Milgram Theorem shows that for such an $f \in H^{-1}_w$ there exists an unique $u \in L^2_w$ such that for all $v \in L^2_w$

$$\mathcal{E}_2(v, u) = \langle v, f \rangle_{-1,w}. \quad (4.25)$$

It is then easy to see that $\int_S f = \int_S u$ and that for almost every $\theta \in S$,

$$\frac{1}{2} \frac{u_0(\theta)}{w(\theta)} = -\int_0^\theta \frac{\mathcal{F}_0}{w} + \int_0^\theta \frac{\mathcal{U}_0}{w}. \quad (4.26)$$

Since $u \in L^2_w$, $\mathcal{U}_0$ admits a $C^1$-version and if we assume that $f$ is square-integrable, the same argument holds for the first term of the right-hand side of (4.26). So, if $f$ is square integrable, $u_0$ admits a $C^2$-version. To sum-up, if we suppose that $f$ is continuous, there exists $u \in C^2(S)$ such that, applying $\partial_v (e^{-\Phi} \partial_0 (\cdot))$ to (4.26):

$$f = f_0 + \left( \int_S f \right) w = -\frac{1}{2} \partial_v (e^{-\Phi} \partial_0 (e^{\Phi} u_0)) + u_0 + p_w(u) w,$$

$$= -\tilde{A}_2 u_0 + u = (1 - \tilde{A}_2) u. \quad (4.27)$$

But since those functions $f$ are dense in $H^{-1}_w$, we see that the range of $1 - \tilde{A}_2$ is dense so that $\tilde{A}_2$ is essentially self-adjoint.

Secondly, the spectral gap estimation (4.18) holds: for every $u \in C^2(S)$, we have using (4.22) and Poincaré inequality:

$$-\langle \tilde{A}_2 v, v \rangle_{-1,w} = \frac{1}{2} \left( \int_S e^{-\Phi} \right) \left( \int_S e^{\Phi} v_0^2 \right) \geq \frac{1}{2} e^{-2K_0} \left( \int_S e^{-\Phi} \right) \left( \int_S e^{\Phi} \right) \left( \int_S e^{\Phi} v_0^2 \right) = \frac{1}{2} e^{-4K_0} \left\| v - \left( \int_S v \right) w \right\|^2_{-1,w}. \quad (4.28)$$

Moreover, $\tilde{A}_2$ has compact resolvent: it suffices to prove that $\lambda - \tilde{A}_2$ has compact resolvent for at least one value of $\lambda$. We prove it for $\lambda = 1$ which is indeed in the resolvent set, thanks to the beginning of this proof. For $u \in H^{-1}_w$, let us consider $f := (1 - \tilde{A}_2)^{-1} u$ so that $\{ f, (1 - \tilde{A}_2) f \}_{-1,w} = \langle f, u \rangle_{-1,w}$. Using the coercivity of $\mathcal{E}_2$, one has, $c \left\| f \right\|_{2,w} \leq \left\| f \right\|_{-1,w} \leq \left\| f \right\|_{-1,w} \left\| u \right\|_{-1,w}$, for some constant $c$. Using the continuous injection of $L^2_w$ into $H^{-1}_w$ (say $\left\| \cdot \right\|_{-1,w} \leq C \left\| \cdot \right\|_{2,w}$, for some positive constant $C$), one has

$$\left\| f \right\|_{2,w} \leq C \left\| u \right\|_{-1,w}. \quad (4.29)$$

So $(1 - \tilde{A}_2)^{-1}$ maps sequences that are bounded in $H^{-1}_w$ into sequences that are bounded in $L^2_w$. It remains then to prove that the injection of $L^2_w$ into $H^{-1}_w$ is compact. This is indeed true since for every $v \in H^{-1}_w$, one has, by Cauchy-Schwartz inequality

$$\left\| v_0(\theta) - v_0(\theta') \right\| \leq C \left\| v_0 \right\|_{2,w} \sqrt{\left| \theta - \theta' \right|} \leq C \left\| v \right\|_{2,w} \sqrt{\left| \theta - \theta' \right|}.$$ 

That means that, by Ascoli-Arzelà Theorem that the sets $\{ v \in H^{-1}_w; \left\| v \right\|_{2,w} \leq cst \}$ are relatively compact in $C(S)$ and also in $L^2_w$. That completes the proof.
Indeed, for given proof consists in two steps: we first prove that there exists some constant $K$.

4.2. Control on the perturbation $B$

In order to derive spectral properties for the operator $L = A + B$, we need to have a precise estimation about the smallness of the perturbation $B$ w.r.t. operator $A$ studied in the previous paragraph § 4.1

Remark 4.6. For simplicity, we work now with the norm $\| \cdot \|_{H_\mu}$ (recall Remark 2.4): as already mentioned this norm is equivalent to the norm $\| \cdot \|_{H_\mu}$ used in § 4.1. Recall also the definition of the space $(L^2, \| \cdot \|_2)$ defined in Remark 2.3 and of $(L^2, \| \cdot \|_{\mu,2})$ defined in Remark 2.4.

Secondly, since the whole operator $L$ is no longer symmetric in $H_{\mu}^1$, its spectrum need not be real. Thus, we will assume for the rest of this document that we work with the complexified versions of the scalar products defined previously in this paper. The results concerning the operator $A$ are obviously still valid.

The smallness of the perturbation $B$ with respect to $A$ can be quantified in terms of the difference $\| q(\cdot, \omega) - q_0(\cdot) \|_{\infty}, \omega \in \text{Supp}(\mu)$. For the ease of exposition, we do not attempt to derive precise estimations of this difference $\| q(\cdot, \omega) - q_0(\cdot) \|_{\infty}$ (Lemma 4.7) and of coefficients $a(\omega_0)$ and $b(\omega_0)$ (Lemma 4.8), in terms of the coupling strength $K$. $c$ will be a positive constant (depending on $K$) which may change from a line to another.

Lemma 4.7. For $\omega > 0$ and $K > 1$, let us define

$$\| q - q_0 \|_{\infty} := \sup_{\theta \in \mathbb{S}, |\theta| \leq \omega} |q(\theta, u) - q_0(\theta)|.$$ (4.29)

Then $\| q - q_0 \|_{\infty} = O(\omega)$, as $\omega \to 0$.

Proof. This is clear since one can bound $\partial_\omega q(\theta, \omega)$ uniformly in $(\theta, \omega)$, as $\omega \to 0$ (by a constant depending on $K$). □

Proposition 4.8. The operator $B$ is $A$-bounded in the sense that there exist positive constants $a(\omega_0) = a(\omega_0, K)$ and $b(\omega_0) = b(\omega_0, K)$ such that

$$\forall h \in \mathcal{D}, \quad \| Bh \|_{H_\mu} \leq a(\omega_0) \| h \|_{H_\mu} + b(\omega_0) \| Ah \|_{H_\mu},$$ (4.30)

and moreover, for fixed $K > 1$,

$$a(\omega_0) = O(\omega_0), \quad \text{and} \quad b(\omega_0) = O(\omega_0), \quad \text{as} \quad \omega_0 \to 0,$$ (4.31)

Proof of Proposition 4.8. Recall that $\langle h \rangle_\mu(\cdot) = \int h(\cdot, \omega) \mu(d\omega)$ is the averaging of $h(\cdot, \omega)$. The proof consists in two steps: we first prove that there exists some constant $\alpha_{\mu, \omega_0}$ such that for all $h \in \mathcal{D}$,

$$\| Bh \|_{H_\mu, 2} \leq \alpha_{\mu, \omega_0} \| h \|_{\mu, 2},$$ (4.32)

Indeed, for given $h \in \mathcal{D}$, for all $\omega \in \text{Supp}(\mu)$, we have $\| Bh(\cdot, \omega) \|_{-1} = \| Bh(\cdot, \omega) \|_{2}$, where $Bh(\cdot, \omega)$ is the appropriate primitive of $Bh(\cdot, \omega)$ in $H^{-1}$, (recall Remark 2.3):

$$Bh := -h(\langle J(\cdot, \omega) - q_0(\cdot) \rangle_{\mu} + \omega) - (q - q_0) \cdot (J \ast h)_\mu + \int_S h : (\langle J(\cdot, \omega) - q_0(\cdot) \rangle_{\mu} + \omega) + \int_S (q - q_0) : (J \ast h)_\mu.$$

Using the boundedness of $q_0$ and the bounds $| (J \ast \varepsilon) | \leq 4K \varepsilon \| \varepsilon \|_{\infty}$ and $| (J \ast h)_{\mu} | \leq \frac{K}{\mu^2} \| \mu \|_{2}$, it is easy to deduce that, for some constant $c > 0$:

$$| Bh | \leq c(\| q - q_0 \|_{\infty} + \| q_0 \|_{\infty} + \| Bh \|_{\mu} + \| \langle h \rangle_\mu \|_{2}).$$ (4.33)
Consequently,
\[
\| Bh \|_{H_{\mu}} = \left( \| Bh \|_{L^2} \right)^{\frac{1}{2}} \leq c \left( \| q - q_0 \|_{\infty} + \omega_0 \right) \| h \|_{\mu, 2},
\]  
(4.34)
so that (4.32) is satisfied for some coefficient \( \alpha_{K, \omega_0} \) such that (Lemma 4.4) \( \alpha_{K, \omega_0} = O(\omega_0 - a(\omega_0)) \).

The second step of the proof is to control the \( L^2 \)-norm \( \| h \|_{\mu, 2} \) of \( h \) with the \( H^{-1}_{\mu} \)-norms of \( Ah \) and \( h \): namely we prove that there exist constants \( \gamma_K \) and \( \delta_K \) such that
\[
\| h \|_{\mu, 2} \leq \gamma_K \| Ah \|_{H_{\mu}} + \delta_K \| h \|_{H_{\mu}}.
\]  
(4.35)

The proof is based on a usual interpolation argument: for all integer \( n > 1 \), for any \( f \in C^2(S) \), one has
\[
\| \partial_0 f \|_{2} \leq \sqrt{n} \| f \|_{2} \| \partial_2^2 f \|_{\sqrt{n}} \leq \frac{n}{2} \| f \|_{2} + \frac{\| \partial_2^2 f \|_{2}}{2n}.
\]  
(4.36)

Let us use this interpolation relation (4.36) to derive (4.35): for all \( h \in D, \omega \in \text{Supp}(\mu) \), one has
\[
\| h(\cdot, \omega) \|_{2} = \left( \int_{S} h(\cdot, \omega) \right)^{2} + \| h_0(\cdot, \omega) \|_{2}^{2}
\]  
(4.37)

Applying relation (4.36) with \( f(\cdot) = \mathcal{H}_0(\cdot, \omega) \) we obtain
\[
\| h(\cdot, \omega) \|_{2}^{2} \leq \left( \int_{S} h(\cdot, \omega) \right)^{2} + \frac{n}{2} \| \mathcal{H}_0(\cdot, \omega) \|_{2}^{2} + \frac{\| \partial_0 h(\cdot, \omega) \|_{2}}{2n},
\]  
(4.38)

where we used the fact that \( \partial_0 h_0(\cdot, \omega) = \partial_0 h(\cdot, \omega) \). Integrating w.r.t. \( \mu \),
\[
\| h \|_{\mu, 2}^{2} \leq \int_{R} \int_{S} \left( h(\cdot, \omega) \right)^{2} d\mu + \frac{n}{2} \| h_0 \|_{\mu, 2}^{2} + \frac{\| \partial_0 h \|_{\mu, 2}^{2}}{2n}.
\]  
(4.39)

As previously for the operator \( B \), a simple calculation shows that for all \( \omega \in \text{Supp}(\mu) \), we have \( \| Ah(\cdot, \omega) \|_{-1} = \| Ah(\cdot, \omega) \|_{2} \), where \( Ah \) is the appropriate primitive of \( Ah \) in \( H^{-1} \) (recall (4.11)):
\[
Ah = \frac{1}{2} \partial_0 h - h(J * q_0) - q_0(J * h)_\mu + \int_{S} (h(J * q_0) + q_0(J * h)_\mu),
\]  
(4.40)

so that, for some constant \( c > 0 \)
\[
\| \partial_0 h \|_{\mu, 2}^{2} \leq 12 \| Ah \|_{\mu, 2}^{2} + c \| h \|_{\mu, 2}^{2}.
\]  
(4.41)

Injecting this inequality in (4.39), one obtains
\[
\| h \|_{\mu, 2}^{2} \leq \int_{R} \int_{S} \left( h(\cdot, \omega) \right)^{2} d\mu + \frac{n}{2} \| h_0 \|_{\mu, 2}^{2} + \frac{1}{2n} \left( 12 \| Ah \|_{\mu, 2}^{2} + c \| h \|_{\mu, 2}^{2} \right).
\]  
(4.42)

Choosing \( n > 1 \) sufficiently large so that the coefficient in front of \( \| h \|_{\mu, 2}^{2} \) in the right-hand side of (4.42) is lower than \( \frac{1}{2} \) leads to (for some constant \( c > 0 \)):
\[
\| h \|_{\mu, 2}^{2} \leq 2 \int_{R} \int_{S} \left( h(\cdot, \omega) \right)^{2} d\mu + c \| h_0 \|_{\mu, 2}^{2} + c \| Ah \|_{\mu, 2}^{2} \leq c \| h \|_{H_{\mu}}^{2} + c \| Ah \|_{H_{\mu}}^{2},
\]  
which shows (4.35). Putting together estimates (4.32) and (4.35), we find the \( A \)-boundedness of \( B \) (4.30) with coefficients \( a(\omega_0) \) and \( b(\omega_0) \) which satisfy (4.31), thanks to Lemma 4.7.

**Proposition 4.9.** The operator \( B \) is \( A \)-compact, in the sense that for any sequence \( (h_p)_{p \geq 0} \in D^N \) such that \( \| h_p \|_{H_{\mu}} \) and \( \| Ah_p \|_{H_{\mu}} \) are bounded, there exists a convergent subsequence for \( (B h_p)_{p \geq 1} \).
Proof of Proposition 4.9. Let \((h_p)_p \geq 0\) a sequence in \(\mathcal{D}\) such that \(\|h_p\|_{H_p}\) and \(\|Ah_p\|_{H_p}\) are bounded by a constant \(c\). A closer look at the operator \(B\) defined in (4.2) and the definition of the norm \(\|\cdot\|_{H_p}\) in (2.15) shows that it suffices to prove that there exists a subsequence \((h_{p_k})_k \geq 0\) that is convergent in \(L^2_\mu\). In particular, for all \(p \geq 0\), \(\|Ah_p\|_{H_p} \leq c\). Using this boundedness and (4.41), we have \(\|\partial_\theta h_p\|_{\mu,2} \leq c + c \|h_p\|_{\mu,2}\), so that

\[
\|\partial_\theta h_p\|_{\mu,2} \leq c + c \left( \int_{\mathbb{R}} \left| \int_{S} h \right|^2 \, d\mu + \|h_{0,p}\|_{\mu,2}^2 \right),
\]

which we used again (4.36) for \(f = H_{0,p}(\cdot,\omega)\). Choosing a sufficiently large \(n > 1\) leads to \(\|\partial_\theta h_{0,p}\|_{\mu,2} = \|\partial_\theta h_p\|_{\mu,2} \leq c + c \|h_p\|_{H_p} \leq c\) for a constant \(c\) independent of \(p \geq 0\). An easy application of Cauchy-Schwarz inequality leads to \(\|h_{0,p}(\theta,\omega) - h_{0,p}(\theta',\omega)\| \leq \|\partial_\theta h_{0,p}\|_{\mu,2} \sqrt{\theta - \theta'},\) for all \(\omega \in \{\omega_0\}\). Since the functions \((h_{0,p})_p \geq 0\) are such that \(\int_S h_{0,p} = 0\) for all \(p \geq 0\), Ascoli-Arzela Theorem and the previous bound show the existence of a convergent subsequence \((h_{0,p_k})_k\) (for each \(\omega \in \text{Supp}(\mu)\)) in the space of continuous functions on \(S\). In particular, this subsequence is convergent in \(L^2_\mu\) and is renamed \((h_{0,p})_p \geq 0\), with a slight abuse of notations.

The fact that \(\int_R \int_S h \, d\mu \leq c\) shows that one can extract a further subsequence of \((h_p)_p \geq 0\) which is also convergent in \(L^2_\mu\). This concludes the proof.

4.3. Spectral properties of \(L = A + B\)

We are now in position to derive by perturbation results on \(A\) similar spectral properties on \(L = A + B\) using theory of perturbation of operators (27) and analytic semi-groups (38).

4.3.1. The spectrum of \(L\) is pure point

**Proposition 4.10.** For all \(K\), for all \(\omega_0 > 0\),

1. the operator \((L,\mathcal{D})\) is closable. In that case, its closure has the same domain as the closure of \(A\),

2. the closure of \(L\) has compact resolvent. In particular, its spectrum is pure point.

**Remark 4.11.** Note that Proposition 4.10 is valid without any assumption on the smallness of \(\omega_0\), since it relies on the relative compactness of \(B\) with respect to \(A\) (Prop. 4.9).

**Proof of Proposition 4.10.** It is a simple consequence of the relative compactness of \(B\) w.r.t. the self-adjoint operator \(A\). The first assertion of Proposition 4.10 is a consequence of [27] Th. 1.11 p.194 and the second assertion can be found in [33] Lemma 3.6, p.17 for example.

4.3.2. \(L\) generates an analytic operator

We prove that the perturbed operator \(L\) generates an analytic semigroup of operators on an appropriate sector. An immediate corollary is the position of the spectrum in a cone whose vertex is zero. We know (Proposition 4.3) that for all \(0 < \alpha < \frac{\pi}{2}\), \(A\) generates a semigroup of operators on \(\Delta_\alpha = \{z \in \mathbb{C}; |\arg(z)| < \alpha\}\).

**Proposition 4.12.** For all \(K > 1\), \(\varepsilon > 0\) and \(0 < \alpha < \frac{\pi}{2}\), there exists \(\omega_1 > 0\) (depending on \(\alpha\), \(K\) and \(\varepsilon\)) such that for all \(0 < \omega_0 < \omega_1\), the spectrum of \(L\) lies within the sector \(\Theta_{\varepsilon,\alpha} := \{\lambda \in \mathbb{C}; \frac{\pi}{2} + \alpha \leq \arg(\lambda) \leq \frac{\pi}{2} - \alpha\} \cup \{\lambda \in \mathbb{C}; |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C}; |\lambda| \leq \varepsilon\} \cup \{\lambda \in \mathbb{C}; |\lambda| \leq \varepsilon\} \). For such \(\omega_0\), \(L\) generates an analytic semigroup of operators on \(\Delta_{\alpha'}\), for some \(\alpha' \in (0, \frac{\pi}{2})\).
Proof of Proposition 4.12. Let $0 < \alpha < \frac{\pi}{2}$ be fixed. Thanks to (4.14), there exists a constant $c > 0$ (which comes from the equivalence of the norms $\| \cdot \|_{H^\mu}$ and $\| \cdot \|_{H^\mu}$) such that for every $\lambda \in \Sigma_\alpha := \{ \lambda \in \mathbb{C} ; |\arg(\lambda)| < \frac{\pi}{2} + \alpha \}$:

$$\| R(\lambda, A) \|_{H^\mu} \leq \frac{c}{(1 - \sin(\alpha))|\lambda|} \quad \text{and}, \quad \| AR(\lambda, A) \|_{H^\mu} \leq 1 + \frac{c}{(1 - \sin(\alpha))}. $$

Then for $\lambda \in \Sigma_\alpha$, $h \in \mathcal{D}$:

$$\| BR(\lambda, A) h \|_{H^\mu} \leq a(\omega_0) \| R(\lambda, A) h \|_{H^\mu} + b(\omega_0) \| AR(\lambda, A) h \|_{H^\mu},$$

$$\leq \left( a(\omega_0) \frac{c}{(1 - \sin(\alpha))|\lambda|} + b(\omega_0) \left( 1 + \frac{c}{1 - \sin(\alpha)} \right) \right) \| h \|_{H^\mu}. $$

Let us fix $\varepsilon > 0$ and choose $\omega_1$ such that:

$$\max \left( \frac{4a(\omega_1)c}{\varepsilon(1 - \sin(\alpha))}, \frac{4b(\omega_1)}{\varepsilon(1 - \sin(\alpha))} \right) \leq 1. \quad (4.45)$$

For this choice of $\omega_1$, for all $0 < \omega_0 < \omega_1$, for any $\lambda \in \Sigma_\alpha$ such that $|\lambda| \geq \varepsilon \geq \frac{4a(\omega_1)c}{1 - \sin(\alpha)}$, we have $\| BR(\lambda, A) h \|_{H^\mu} \leq \frac{1}{\varepsilon} \| h \|_{H^\mu}$, and thus the operator $1 - BR(\lambda, A)$ is invertible with $\| 1 - BR(\lambda, A) \|_{H^\mu} \leq 2$. Since it can easily be shown that

$$R(\lambda, A + B) = R(\lambda, A) (1 - BR(\lambda, A))^{-1},$$

one deduces the following estimates about the resolvent of the perturbed operator $L = A + B$:

$$\forall \lambda \in \Sigma_\alpha, |\lambda| \geq \varepsilon, \| R(\lambda, L) \|_{H^\mu} \leq \frac{2}{(1 - \sin(\alpha))|\lambda|}. \quad (4.46)$$

The fact that the spectrum of $L$ lies within $\Theta_{\varepsilon, \alpha}$ is a straightforward consequence of (4.46). Secondly, (4.46) entails that $L$ generates an analytic semigroup of operators on an appropriate sector. Indeed, if one denotes by $L_{2\varepsilon} := L - \varepsilon$, one deduces from (4.46) that $0 \in \rho(L_{2\varepsilon})$ and that for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ (in particular, $|\lambda| < |\lambda + 2\varepsilon|$)

$$\| R(\lambda, L_{2\varepsilon}) \|_{H^\mu} = \| R(\lambda + 2\varepsilon, L) \|_{H^\mu} \leq \frac{2}{(1 - \sin(\alpha))|\lambda + 2\varepsilon|},$$

$$\leq \frac{2}{(1 - \sin(\alpha))|\lambda|}. \quad (4.47)$$

Hence, using the same arguments of Taylor expansion as in the proof of Proposition 4.1 and applying Proposition 4.3, one easily sees that $L_{2\varepsilon}$ generates an analytic semigroup in a (a priori) smaller sector $\Delta_{\alpha'}$, where $\alpha' \in (0, \frac{\pi}{2})$ can be chosen as $\alpha' := \frac{1}{2} \arctan \left( \frac{1 - \sin(\alpha)}{2} \right)$. But if $L_{2\varepsilon}$ generates an analytic semigroup, so does $L$. \hfill\Box

### 4.3.3. 0 is an eigenvalue with multiplicity 2

Let us fix $K > 1$, $\alpha \in (0, \frac{\pi}{2})$, $\rho \in (0, 1)$ and define $\varepsilon = \rho \lambda_K(A)$. Applying Proposition 4.12, we know that for small $\omega_0$ (depending on $K$, $\alpha$, $\rho$), $L$ generates an analytic semigroup on $\Theta_{\varepsilon, \alpha}$. Let $\Theta_0^{\alpha} := \{ \lambda \in \Theta_{\varepsilon, \alpha} ; \Re(\lambda) \geq 0 \}$ be the subset of $\Theta_{\varepsilon, \alpha}$ which lies in the positive part of the complex plane (see Figure 3).

The purpose of this paragraph is to show that one can choose a perturbation $B$ small enough so that no non-zero eigenvalue of $A + B$ remains in the small set $\Theta_0^{\alpha}$. To do so, we proceed by an argument of local perturbation: we know that the distance $\lambda_K(A) = \min(\lambda_K(\tilde{A}_1), \lambda_K(\tilde{A}_2))$ between the eigenvalue 0 and the rest of the spectrum of $A$ is strictly positive. In particular, one can separate 0 from the rest of the spectrum of $A$ by a circle $\mathcal{C}$
centered in 0 with radius \((\frac{\rho+1}{2})\lambda_k(A)\). Note that the choice of \(\varepsilon\) made at the beginning of this paragraph ensures that the interior of \(\mathcal{E}\) contains \(\Theta_{\varepsilon,\alpha}^\pm\) (Figure 4).

The argument is the following: by construction of \(\mathcal{E}\), 0 is the only eigenvalue (with multiplicity 2) of the operator \(A\) lying in the interior of \(\mathcal{E}\). A principle of continuity of eigenvalues shows that, while adding a small enough perturbation \(B\) to \(A\), the interior of \(\mathcal{E}\) still contains either one eigenvalue with algebraic multiplicity 2 or two eigenvalues with multiplicity 1; those perturbed eigenvalues remain close but are \textit{a priori} not equal to the initial eigenvalue 0 (see Figure 4).

Figure 4: The domains \(\Theta_{\varepsilon,\alpha}\) and \(\Theta_{\varepsilon,\alpha}^+\) (in light grey). Note that the two dimensional eigenvalue 0 for the operator \(A\) \((4a)\) may split in two single eigenvalues for the perturbed operator \(L\) \((4b)\). These eigenvalues are the only ones within the circle \(\mathcal{E}\). But since we already know that the eigenspace in 0 of \(L\) is of dimension 2, 0 is still a double eigenvalue for \(L\), by uniqueness.

But we already know that for the perturbed operator \(L = A + B\), 0 is always an eigenvalue (since \(L\partial_\theta q = 0\) and \(Lp = \partial_\theta q\), recall Th. 2.6). Therefore, the algebraic multiplicity of 0 for the operator \(L\) is at least 2. By uniqueness, one can conclude that 0 is the \textit{only} element of the spectrum of \(L\) within \(\mathcal{E}\), and is an eigenvalue with algebraic multiplicity exactly 2. In particular, there is no element of the spectrum in the positive part of the complex plane.

In order to make this argument precise, we need to quantify the appropriate size of the perturbation \(B\), by explicit estimates on the resolvent \(R(\lambda, A)\) on the circle \(\mathcal{E}\):

\textbf{Lemma 4.13.} There exists some explicit constant \(c_\mathcal{E}(K)\) only dependent on \(K\), such that for all \(\lambda \in \mathcal{E}\),

\[\| R(\lambda, A) \|_{H_p} \leq c_\mathcal{E}(K), \quad (4.48)\]

\[\| AR(\lambda, A) \|_{H_p} \leq 1 + \left(\frac{\rho + 1}{2}\right) c_\mathcal{E}(K). \quad (4.49)\]

One can choose \(c_\mathcal{E}(K) = \frac{1}{\lambda_k(A)} \max \left(\frac{2}{\rho+1}, \frac{2}{1-\rho}\right)\).

\textbf{Proof of Lemma 4.13.} Applying the spectral theorem (see \([17\), Th. 3, p.1192]) to the essentially self-adjoint operator \(A\), there exists a spectral measure \(E\) vanishing on the complementary of the spectrum of \(A\) such that \(A = \int_\mathcal{R} \lambda dE(\lambda)\). In that extent, one has for any \(\zeta \in \mathcal{E}\)

\[R(\zeta, A) = \int_\mathcal{R} \frac{dE(\lambda)}{\lambda - \zeta}, \quad (4.50)\]
In particular, for \( \zeta \in \mathcal{C} \)

\[
\left\| R(\zeta, A) \right\|_{\mathbf{H}_\mu} \leq \sup_{\lambda \in \sigma(A)} \frac{1}{\lambda - \zeta} \leq \frac{2}{\rho + 1} \frac{2}{\lambda_K(A)}.
\]

(4.51)

The estimation (4.49) is straightforward.

We are now in position to apply our argument of local continuity of eigenvalues: following [27, Th III-6.17, p.178], there exists a decomposition of the operator \( A \) according to \( \mathbf{H}_\mu^{-1} = F_0 \oplus F_\omega \) (in the sense that \( AF_0 \subset F_0 \), \( AF_\omega \subset F_\omega \) and \( F_0 \subset D \), where \( F_0 \) is the projection on \( F_0 \) along \( F_\omega \)) in such a way that \( A \) restricted to \( F_0 \) has spectrum \( \{0\} \) and \( A \) restricted to \( F_\omega \) has spectrum \( \sigma(A) \setminus \{0\} \subset \{ \lambda \in \mathbb{C} ; \Re(\lambda) < 0 \} \).

Let us note that the dimension of \( F_0 \) is exactly 2, since the characteristic space of \( A \) for the eigenvalue 0 is reduced to its kernel which is of dimension 2 (see Prop. 4.3).

Then, applying [27, Th. IV-3.18, p.214], and using Proposition 4.8, we find that if one chooses \( \omega_2 > 0 \), such that

\[
\sup_{\lambda \in \sigma(A)} \left( a(\omega_2) \left\| R(\lambda, A) \right\|_{\mathbf{H}_\mu} + b(\omega_2) \left\| \lambda R(\lambda, A) \right\|_{\mathbf{H}_\mu} \right) < 1,
\]

(4.52)

then for all \( 0 < \omega_0 < \omega_2 \), the perturbed operator \( L \) is likewise decomposed according to \( \mathbf{H}_\mu^{-1} = G_0 \oplus G_\omega \), in such a way that \( \dim(F_0) = \dim(G_0) = 2 \), and that the spectrum of \( L \) is again separated in two parts by \( \mathcal{C} \).

But thanks to Theorem 2.6, we already know that the characteristic space of the perturbed operator \( L \) according to the eigenvalue 0 is at least of dimension 2 (since \( L(0) = 0 \) and \( Lp = \partial_t q \)). We can conclude, that for such an \( 0 < \omega_0 < \omega_2 \), 0 is the only eigenvalue in \( \mathcal{C} \) and that dim\( (G_0) \) is exactly 2.

Applying Lemma 4.13, we see that condition (4.52) is satisfied if we choose \( \omega_2 > 0 \) so that:

\[
a(\omega_2) c_\mathcal{C}(K) + b(\omega_2) \left( 1 + \frac{\rho + 1}{2} c_\mathcal{C}(K) \right) < 1.
\]

(4.53)

In that case, the spectrum of \( L \) is contained in \( \{ z \in \mathbb{C} ; \Re(z) \leq 0 \} \). Theorem 2.8 is proved.

5. Non self-averaging phenomenon for the fluctuation process (Proof of Theorem 2.10)

The purpose of this section is to prove Theorem 2.10 that is the linear asymptotics (2.20) for the SPDE (1.7).

In our framework, (recall that \( \mu = \frac{1}{2}(\delta_0 + \delta_\omega) \)), the solution \( \eta \) of evolution (1.7) in \( S'(S \times \mathbb{R}) \) acts on test functions \( \varphi \) of the form \( \varphi = (\varphi(\cdot, +\omega_0), \varphi(\cdot, -\omega_0)) \). In particular, one can understand \( \eta \) as an element of \( \mathbf{H}_\mu^{-1} \) by identifying \( \eta \) with \( (\eta_{\omega_0}, \eta_{-\omega_0}) \), where, for any smooth function \( \psi : S \to \mathbb{R} \), \( \eta_{\omega_0}(\psi) := \eta(\psi, 0) \) and \( \eta_{-\omega_0}(\psi) := \eta(0, \psi) \). Defining analogously \( W_{\pm \omega_0} \) for the Wiener process \( W \) in (1.9), the object of interest is then

\[
\forall t > 0, \ \eta_t = \eta_0 + \int_0^t L\eta d\sigma + W_t.
\]

(5.1)

5.1. The noise \( W \) as a cylindrical Brownian Motion

We first focus on the regularity properties of the noise \( W \): in the stationary case \( (q_t = q)_{t>0} = q \) for all \( t > 0 \) the covariance defined in (1.9) becomes, for any regular functions on \( S \times \mathbb{R} \), \( \varphi_1 \) and \( \varphi_2 \), \( s, t > 0 \):

\[
\mathbf{E}(W_s(\varphi_1)W_t(\varphi_2)) = (s \wedge t) \int_S \int_{\mathbb{R}} \partial_\varphi \varphi_1 \partial_\varphi \varphi_2 q d\varphi d\mu.
\]

(5.2)

Consequently, it is easily seen that \( (W_{\pm \omega_0,t}, \omega) \) is a couple of two independent Gaussian processes with covariance (where \( \psi_1, \psi_2 : S \to \mathbb{R} \)):

\[
\forall \omega \in \{ \omega_0, -\omega_0 \}, \ \mathbf{E}(W_s(\psi_1)W_t(\psi_2)) = \frac{1}{2}(s \wedge t) \int_{\mathbb{R}} \partial_\psi \psi_1 \partial_\psi \psi_2 q(\cdot, \omega).
\]

(5.3)
In what follows, we will denote by \( H_0 \) the closed subspace of \( H_0^\mu \) consisting of elements of \( H_0^\mu \) with zero mean-value; in particular the norm \( \| \cdot \|_{H_0^\mu} \) defined in (2.15) coincides on \( H_0 \) with:

\[
\forall h \in H_0, \ |h|_{H_0^\mu} = \left( 2\pi \int_{\mathbb{R}} \int_{S} \mathcal{H}^2 \right)^{\frac{1}{2}}, \tag{5.4}
\]

where we recall that \( \mathcal{H} \) is the primitive of \( h \) such that \( \int_{S} \mathcal{H} = 0 \). Following [14, p. 96], \( W \) has the same law as a \( Q \)-Wiener process in the Hilbert space \( H_0 \); for an appropriate bounded symmetric operator \( Q \) on \( H_0 \): indeed, if one denotes by \( X \) a \( Q \)-Wiener process on \( H_0 \), with the following definition of \( Q \)

\[
\forall h \in H_0, \ Qh := \partial_t(q\mathcal{H}), \tag{5.5}
\]

then one readily verifies that the Gaussian process \( (W(\varphi))_{\varphi} \) has the same law as \( (X(\varphi))_{\varphi} := \left( \left( X_I, \left( \frac{\partial^2 \varphi}{\partial^2 \varphi} \right) \right)_{H_0^\mu}, \left( X_I, \left( \frac{\partial^2 \varphi}{\partial^2 \varphi} \right) \right)_{H_0^\mu} \right)_{\varphi}. \)

The fact that this supplementary weight \( q \) in (5.5) entails some technical complications, but one really has to consider the operator \( Q \) defined in (5.5) only as a perturbation of the case \( Q = I \).

5.2. Existence and uniqueness of a weak solution to the fluctuation equation

We now turn to the existence and uniqueness of a weak solution of (5.1). We recall that any \( H_0^\mu \)-valued predictable process \( \eta_t, t \in [0, T] \) is a weak solution of (5.1) if the trajectories of \( \eta \) are almost-surely Bochner-integrable and if for all \( \varphi \in \mathcal{D}(L^\star) \) and for all \( t \in [0, T] \)

\[
\eta_t(\varphi) = \eta_0(\varphi) + \int_0^t \eta_s(L^\star \varphi) \, ds + W_t(\varphi). \tag{5.6}
\]

**Proposition 5.1.** Equation has a unique weak solution in \( H_0^\mu \), given by the mild formulation

\[
\eta_t = T_L(t) \eta_0 + \int_0^t T_L(t-s) dW_s, \quad t \in [0, T]. \tag{5.7}
\]

To prove Proposition 5.1 one needs to define properly the stochastic convolution \( W_L(t) := \int_0^t T_L(t-s) dW_s \). In this purpose, let use prove firstly that the inverse of \( A \) is of class trace.

**Lemma 5.2.** The operator \( A^{-1} \) is of class trace in \( H_0^\mu \). Equivalently, if \( (\lambda_n^{(A)})_{n \geq 1} \) is the sequence of eigenvalues of the self-adjoint operator \( A \), one has

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{(A)}} < \infty. \tag{5.8}
\]

**Proof of Lemma 5.2.** Since \( A \) and \( \hat{A} = M \circ A \circ M^{-1} \) (recall (5.9)) are conjugate, it suffices to prove (5.8) when \( A \) is replaced by \( \hat{A} \). The idea of the proof is that identity (5.8) is true when \( \hat{A} = (\hat{A}_1, \hat{A}_2) \) is replaced by \( (-\Delta, -\Delta) \) and that \( \hat{A} \) is only a relatively-bounded perturbation of this case. More precisely, the proof relies on the following MinMax Principle:

**Proposition 5.3 ([18, p. 1543]).** Let \((F, \mathcal{D}(F))\) a self-adjoint linear operator on a separable Hilbert space \( \mathcal{H} \) such that \( F \) is positive, with compact resolvent. We denote by \( S^n \) the family of \( n \)-dimensional subspace of \( \mathcal{H} \), and for \( n \geq 1 \) we let \( \lambda_n \) the number defined as follows

\[
\lambda_n := \sup_{G \in S^n} \inf_{u \in (\mathcal{G} \cap \mathcal{D}(F)) \setminus \{0\}} \frac{\langle u, Fu \rangle_{\mathcal{H}}}{\langle u, u \rangle_{\mathcal{H}}}. \tag{5.9}
\]

Then there exists a complete orthonormal system \((\psi_n)_{n \geq 1}\) such that

\[
F\psi_n = \lambda_n \psi_n, \quad n \geq 1.
\]

In other words, the sequence \((\lambda_n)_{n \geq 1}\) is the non-decreasing enumeration of the eigenvalues of \( F \), each repeated a number of times equal to its multiplicity. Moreover, the sup in (5.9) is attained for \( G \) equal to the span of \( \{\psi_1, \ldots, \psi_n\} \).
Let us apply Proposition 5.3 to $F = -\Delta$ with domain 
\[ \mathcal{D}(-\Delta) := \left\{ u; u \in C^2(S), \int_S u(\theta) \, d\theta = 0 \right\}, \tag{5.10} \]
in $H^{-1}$ (recall the definition of $(H^{-1}, \langle \cdot, \cdot \rangle_1)$ in Remark 2.3) and let us denote by $\mathcal{E}_0(u,v) := \langle u, -\Delta v \rangle_1 = 2\pi \int_S uv \, d\theta$ the Dirichlet form associated to $-\Delta$. Note that $\mathcal{E}_0$ is well defined on $L^2 \supseteq \mathcal{D}(-\Delta)$. Then, denoting by $(\lambda_n(-\Delta))_{n \geq 1}$ the sequence of eigenvalues associated to $-\Delta$ in $H^{-1}$:
\[ \lambda_n(-\Delta) = \sup_{G \subset S^n} \inf_{u \in (G \cap \mathcal{D}(-\Delta)) \setminus \{0\}} \mathcal{E}_0(u,u). \]
Since the supremum is attained for $G = \{ \psi_1, \ldots, \psi_n \} \subseteq L^2$, one has in fact:
\[ \lambda_n(-\Delta) = \sup_{G \subset S^n} \inf_{u \in (G \cap L^2) \setminus \{0\}} \mathcal{E}_1(u,u). \]
Secondly, note that one does not change the result by considering $1 - A_1$ instead of $-A_1$. Hence, if one denotes by $\mathcal{E}_1(u,v) := \langle u, (1 - A_1) v \rangle_{-1,0}$, the Dirichlet form associated to $1 - A_1$, one deduces from [14, Eq.(247)] that $\mathcal{E}_1$ is well defined on $L^2$ and that it is equivalent to $\mathcal{E}_0$: there exists a constant $C > 0$ such that
\[ \forall u \in L^2, \quad \frac{1}{C} \mathcal{E}_0(u,u) \leq \mathcal{E}_1(u,u) \leq C \mathcal{E}_0(u,u). \]
Then, using again Proposition 5.3
\[ \lambda_n^{(1-A_1)} = \sup_{G \subset S^n} \inf_{u \in (G \cap L^2) \setminus \{0\}} \mathcal{E}_1(u,u). \]
Since the norms $\| \cdot \|_{-1}$ and $\| \cdot \|_{-1,0}$ are equivalent, one directly sees that there exist constants $c,C > 0$ such that, for all $n \geq 1$
\[ c \lambda_n(-\Delta) \leq \lambda_n^{(1-A_1)} \leq C \lambda_n(-\Delta). \tag{5.11} \]
One can prove similar bounds for $A_2$ in the Hilbert space $H^{-1}_w$ in the same way: first notice that any eigenvector which corresponds to a non-zero eigenvalue of $A_2$ is necessarily with zero mean-value, so that it suffices to work on the domain $\{ v \in L^2, \int_S v = 0 \}$. It is then easy to deduce from [4,24] that both Dirichlet forms $\mathcal{E}_0$ and $\mathcal{E}_2$ (recall definition [4,23]) are equivalent on the subspace of $L^2$ with zero mean-value. Using Proposition 5.3 one easily obtains similar bounds as (5.11) for $A_2$ and (5.8) follows.

Following the lines of [14], we deduce that the linear operator $\int_0^t T_L(s)QT_L(s)^* \, ds$ is of class trace: indeed, it is easy to see from [14, 1.24] that $B$ satisfies the assumption (5.59) in [14, p.145], namely, $B$ is a continuous linear operator from $L^2_{\mu}$ into $H_{-1}^1$, and there exists a constant $c > 0$ such that for all $h \in \mathcal{D}$, $\langle Bh, h \rangle_{H_{-1}^1} \leq c \| h \|_{H_{-1}^1}^2$. Since Lemma 5.2 is true, the assumptions of [14] Prop. 5.25, p.145] are satisfied, so that the operator $\int_0^t T_L(s)QT_L(s)^* \, ds$ is of class trace. Then an application of [14, Th. 5.2] shows that the stochastic convolution $W_L(\cdot)$ is well defined as a predictable process in $H_{-1}^1$. The assumptions of [14, Th. 5.4] concerning the existence and uniqueness a weak solution of (5.1) are satisfied and Proposition 5.1 is proved.

5.3. Linear asymptotic behavior of the fluctuation process

We are in position to prove the main statement of Theorem 2.10, that is the asymptotic behavior of the mild solution [2.20] of the mild solution [5.7]. We place ourselves under the hypothesis of Theorems 2.0 and 2.8.
First note that the continuous linear form $\ell_{\dot{\partial} q}(\cdot)$ on $H^{-1}_\mu$ can be represented, by Riesz representation theorem, as a scalar product w.r.t. some vector $\zeta \in H^{-1}_\mu$:

$$\ell_{\dot{\partial} q}(\cdot) = \langle \zeta, \cdot \rangle_{H^{-1}_\mu}.$$  \hfill (5.12)

The convergence (2.20) is a consequence of Remark (2.9) and the following two propositions:

**Proposition 5.4.** The stochastic convolution $W(t)$ satisfies the following linear behavior, as $t \to +\infty$: $W(t) \to 0$, where the convergence is in law.

**Proposition 5.5.** For every initial condition $\eta_0$, $\frac{T(t)\eta_0}{t}$ converges, as $t \to +\infty$, to $\ell_{\dot{\partial} q}(\eta_0)\partial q$.

Before proving these results, let us show how the speed $v(\omega)$ in Theorem 2.10 is computed. The above results give that for fixed $\omega$, the process $\frac{\eta}{t}$ converges in law, as $t \to \infty$ to $\ell_{\dot{\partial} q}(\eta_0)\partial q = \frac{\int_q C_t(\omega)}{\int_q \eta_0}$. Using (1.11), an easy computation shows that $v(\cdot)$ is Gaussian with variance (2.27).

Let us now prove these two propositions:

**Proof of Proposition 5.4.** Recall that $W$ is a $Q$-Wiener process in $H$, which can be decomposed into $H = \text{Span}(\dot{\partial} q) \oplus G_\omega$. Note also that the restriction on $H$ of the projection $P_0$ defined on $H^{-1}_\mu$ by (2.22) coincides with $\ell_{\dot{\partial} q}(\cdot)\partial q$. With a small abuse of notations, we will use the same notation $P_0$ for this restriction on $H$. Let us decompose the stochastic convolution into $W(t) = \int_0^t T(t-s)P_0\,dW_s + \int_0^t T(t-s)P_\omega\,dW_s$, and treat the two terms separately:

For the first term $\frac{1}{t}\int_0^t T(t-s)P_0\,dW_s$, one has, using that $T(t)\partial q = \partial q$ for all $u > 0$

$$\frac{1}{t}\int_0^t T(t-s)P_0\,dW_s = \frac{\partial q}{t} \int_0^t T(t-s)\ell_{\dot{\partial} q}\,dW_s = \frac{\partial q}{t} \int_0^t \ell_{\dot{\partial} q}W_s = \frac{\partial q}{t} \langle \zeta, W(t) \rangle_{H^{-1}_\mu},$$  \hfill (5.13)

Thanks to the $Q$-Wiener structure of $W$ (see § 5.1), one has

$$\mathbb{E}\left(\left\| \frac{1}{t}\int_0^t T(t-s)P_0\,dW_s \right\|^2_{H^{-1}_\mu} \right) = \frac{\|\partial q\|_{H^{-1}_\mu}^2}{t^2} \mathbb{E}\left(\langle \zeta, W(t) \rangle_{H^{-1}_\mu}^2 \right) = \frac{\|\partial q\|_{H^{-1}_\mu}^2}{t} \langle Q\zeta, \zeta \rangle_{H^{-1}_\mu},$$  \hfill (5.14)

which converges to 0 as $t \to +\infty$.

For the second term $\int_0^t T(t-s)P_\omega\,dW_s$, it is easy to see that it is the unique weak solution in $H$ of

$$w_t = \int_0^t Lw_s\,ds + P_\omega W_t.$$  \hfill (5.15)

Let us decompose evolution (5.16) along this decomposition $H = \text{Span}(\dot{\partial} q) \oplus G_\omega$: writing $w_t = P_0 w_t + P_\omega w_t := y_t + z_t$, one has:

$$\begin{cases}
  z_t = \int_0^t P_0 Ly_s\,ds, \\
y_t = \int_0^t P_\omega LP_\omega y_s\,ds + P_\omega W_t.
\end{cases}$$  \hfill (5.17)

Since the operator $P_\omega LP_\omega$ has its spectrum in the negative part of the complex plane with a strictly positive spectral gap $\lambda_K(L)$ and generates an semigroup of operators, it is immediate to see from the covariance estimates of stochastic convolutions (see [14], Th. 5.2, p.119) that there exist some $t_0 > 0$ and a constant $c > 0$ such that for all $t \geq t_0$

$$\mathbb{E}\left(\left\| y_t \right\|^2_{H^{-1}_\mu} \right) \leq \mathbb{E}\left(\left\| y_{t_0} \right\|^2_{H^{-1}_\mu} \right) \leq ce^{-\frac{\lambda_{K}(L)t}{2}}.$$  \hfill (5.18)
Consequently, one has
\[
E\left(\|z_t\|_{H^s}\right) \leq \int_0^t E\left(\|P_0 L y_s\|_{H^s}\right) \, ds = \|\partial \eta_0\|_{H^s} \int_0^t E\left(\|\ell \partial \eta_0 (L y_s)\|\right) \, ds,
\]
\[
= \int_0^t E\left(\|\zeta, L y_s\|_{H^s}, \|\partial \eta_0\|_{H^s}\right) \, ds, \quad \text{(recall (5.12))}
\]
\[
\leq \|\partial \eta_0\|_{H^s} \|L^* \zeta\|_{H^s} \int_0^t E\left(\|y_s\|_{H^s}\right) \, ds,
\]
\[
= \|\partial \eta_0\|_{H^s} \|L^* \zeta\|_{H^s} \left(\int_0^t E\left(\|y_s\|_{H^s}\right) \, ds + \int_0^t E\left(\|y_s\|_{H^s}\right) \, ds\right). \quad (5.19)
\]

It is immediate from (5.18) and (5.19) that the following convergence holds:
\[
E\left(\|z_t\|_{H^s}\right) \rightarrow_{t \to +\infty} 0. \quad (5.20)
\]

Putting together (5.18) and (5.20), Proposition 5.4 is proved.

**Proof of Proposition 5.5**

Let us fix an initial condition \(\eta_0 \in H^{-1}_\mu\). Then \(X(t) := T_L(t)\eta_0\) is the unique solution in \(H^{-1}_\mu\) of
\[
X(t) = \eta_0 + \int_0^t L X_s \, ds. \quad (5.21)
\]

Decompose \(X(t)\) along the direct sum \(G_0 \oplus G_\alpha\), that is \(X(t) = \alpha(t) \partial \eta_0 + \beta(t) p + Y(t)\), with \(Y(t) \in G_\alpha\). Then, projecting on \(\partial \eta_0, p\) and \(G_\alpha\) respectively (see (2.24)), one obtains that (5.21) is equivalent to
\[
\forall t > 0, \quad \begin{cases} \alpha(t) = \ell \partial \eta_0 (\eta_0) + \int_0^t (\beta(s) + \ell \partial \eta_0 (LP_\alpha Y(s))) \, ds, \\ \beta(t) = \ell p (\eta_0), \\ Y(t) = P_\alpha \eta_0 + \int_0^t P_\alpha LP_\alpha Y(s) \, ds. \end{cases} \quad (5.22)
\]

Then, since \(T_{P_\alpha LP_\alpha}(t)\) is a semigroup of contraction whose infinitesimal generator has a strictly positive spectral gap \(\lambda_K(L)\), there exists a constant \(c > 0\) such that \(Y(t) = T_{RP_\alpha LP_\alpha}(t)P_\alpha \eta_0\) and \(\|Y(t)\|_{H^s} \leq c e^{-\lambda_K(t)}\) (in particular, \(\frac{1}{t} \|Y(t)\|_{H^s} \rightarrow_{t \to +\infty} 0\)). Then, using again (5.12),
\[
\frac{\alpha(t)}{t} = \frac{\ell \partial \eta_0 (\eta_0)}{t} + \ell p (\eta_0) + \frac{1}{t} \int_0^t \ell \partial \eta_0 (LP_\alpha Y(s)) \, ds, \quad (5.23)
\]
\[
= \frac{\ell \partial \eta_0 (\eta_0)}{t} + \ell p (\eta_0) + \frac{1}{t} \int_0^t \langle P_\alpha^* L^* \zeta, Y(s) \rangle_{H^s} \, ds. \quad (5.24)
\]

Using the previous exponential bound for \(Y(s)\), it is easy to see that \(\frac{\alpha(t)}{t}\) converges to \(\ell p (\eta_0)\) as \(t \rightarrow +\infty\). The result of Proposition 5.5 follows.

**Appendix A. Gelfand-triple construction**

The construction of the weighted Sobolev spaces defined in §2.3 and used throughout the paper is based on the usual notion of Gelfand-triple (or rigged-Hilbert spaces) that we make precise here. We refer to [3, §2.2] or [4, p.81] for precise definitions.

Namely, one can understand the closure of \(\{h \in C^1(S); \int_S h = 0\}\) w.r.t. the norm \(\|h\|_{-1,k}\) defined in (2.14) as the dual space \(V'\) of the space \(V\) closure of \(\{h \in C^1(S); \int_S h = 0\}\) with respect to the norm
\[
\|h\|_{V'} := \left(\int_S h'(\theta)^2 k(\theta) \, d\theta\right)^{1/2}.
\]
The pivot space is the usual $L^2(\lambda)$, endowed with the Hilbert norm

$$\|h\|_{L^2} := \left( \int_S h(\theta)^2 \, d\theta \right)^\frac{1}{2}.$$ 

One easily sees that the inclusion $V \subseteq L^2(\lambda)$ is dense. Consequently, one can define $T : L^2(\lambda) \to V'$ by setting $Th(v) = \int_S h(\theta) v(\theta) \, d\theta$. One can prove that $T$ continuously injects $L^2(\lambda)$ into $V'$ and that $T(L^2(\lambda))$ is dense in $V'$ so that one can identify $h \in L^2(\lambda)$ with $Th \in V'$. Then for $h \in L^2(\lambda)$,

$$\|h\|_{V'} = \|Th\|_{V'} = \sup_{v \in V} \frac{\int_S \mathcal{H}h' \, v \, d\theta}{\|v\|_{V'}} = \sqrt{\int_S \mathcal{H}^2 k}, \tag{A.1}$$

where we used in (A.1) Cauchy-Schwarz inequality for the lower bound and chose $v' := \frac{\mathcal{H}}{k}$ for the upper bound. This enables to identify $H_k^{-1}$ with $V'$.

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