Viscosity solutions of obstacle problems for Fully nonlinear path-dependent PDEs

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May 30, 2014

Abstract

In this article, we adapt the definition of viscosity solutions to the obstacle problem for fully nonlinear path-dependent PDEs with data uniformly continuous in \((t, \omega)\), and generator Lipschitz continuous in \((y, z, \gamma)\). We prove that our definition of viscosity solutions is consistent with the classical solutions, and satisfy a stability result. We show that the value functional defined via the second order reflected backward stochastic differential equation is the unique viscosity solution of the variational inequalities.

Key words: Path-dependent PDEs, viscosity solutions, reflected backward stochastic differential equations, variational inequalities.

AMS 2000 subject classifications: 35D40, 35K10, 60H10, 60H30.

∗The author would like to thank Nizar Touzi and Jianfeng Zhang for helpful discussions.
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1 Introduction

Since the seminal work of Pardoux and Peng [15], backward stochastic differential equations (BSDEs) have found many areas of application. In [11], El Karoui et al. introduced a new kind of BSDE, called reflected BSDEs, where the solution is forced to stay above a barrier. In the Markovian framework, they have proven that the value function defined through the RBSDE is the unique viscosity solution of an obstacle problem for a semi-linear parabolic partial differential equation, hence extending the well-known Feynman-Kac formula to the associated variational inequalities.

In his recent work [6], Dupire gives the definition of derivatives on the path space and proves a functional Itô’s formula. Using those derivatives, in [3] and [2], the authors derive and study functional differential equations, which extends the Feynman-Kac formula to a non-Markovian case. In [16], Peng proposes the notion of path-dependent partial differential equations (PPDEs) in nonlinear framework. In [7], [9], [10], Ekren et al. proposed a definition for viscosity solution of PPDEs.

Our objective in this paper is to adapt the definition of viscosity solutions of PPDEs given in [9] to an obstacle problem for a fully nonlinear PPDE, for which we give assumption under which wellposedness holds. In order to achieve our objective, especially to tackle with the lack of local compactness of the space of paths, we will use similar ideas as in [10]. In our case, the main difficulty is to produce a sequence of ”smooth” subsolutions and supersolutions of the obstacle problem that converge to the value functional, however in general the solutions of obstacle problem of PDEs do not have $C^{1,2}$ regularity. To overcome this difficulty, we use a penalization approach and a change of variable which allows us to have ”smooth” solution to the obstacle problem.

The paper is organized as follows. In section 2, we introduce the main notations and assumptions that we will use. In section 3 we introduce our functional of interest as the supremum of solutions of RBSDEs and give some regularity results for this functional. In section 4, we introduce the PPDE we want to study and give the definition of viscosity solution for this kind of PPDEs and prove some preliminary results on viscosity solutions. We then prove that our value functional of interest is a viscosity solution of this PPDE. Starting from section 5, we treat the wellposedness of the PPDE. We first prove our partial comparison result at section 5. In section 6, a stability theorem is proven. In section 7 we prove, using a modification of Perron’s approach, the general comparison result (without requiring any smoothness of the sub and supersolutions).
2 Notations

We fix $T > 0$, the time maturity, and an integer $d > 0$. We denote by $\mathbb{S}^d$ the set of symmetric $d$-dimensional square matrices. For $x \in \mathbb{R}^d$, $|x|$ is the norm of $x$, and for $A, B \in \mathbb{S}^d$, $A : B := \text{trace}(AB)$. For any matrix $M$, $M^\ast$ denotes its transpose. We work on the canonical space $\Omega := \{\omega \in C([0,T], \mathbb{R}^d) : \omega_0 = 0\}$ of $d$-dimensional continuous paths. $B$ denotes the canonical process on this space, $\mathbb{F} = \{\mathbb{F}_s\}_{s \in [0,T]}$ is the filtration generated by $B$, and $\mathbb{P}_0$ is the Wiener measure. For $\omega \in \Omega$ and $t \in [0,T]$, the stopped path $\omega_{\wedge t} \in \Omega$ is defined as follows:

$$\omega_{\wedge t}(s) = \omega_s, \quad \text{for } 0 \leq s \leq t,$$

$$\omega_{\wedge t}(s) = \omega_t, \quad \text{for } t \leq s \leq T.$$ 

We denote $\Lambda := \{(t, \omega_{\wedge t}) : 0 \leq t \leq T, \omega \in \Omega\}$. In the sequel, we will denote a generic element of $\Lambda$ as $(t, \omega) \in [0,T] \times \Omega$. By this notation, we mean that we ignore the values of $\omega$ after $t$ and identify $(t, \omega') \in [0,T] \times \Omega$ with $(t, \omega) \in [0,T] \times \Omega$ if $\omega_{\wedge t} = \omega'_{\wedge t}$. We define the following $||.||_T$ and $d_\infty$ metrics on respectively $\Omega$ and $\Lambda$:

$$||\omega||_T := \sup_{s \in [0,T]} |\omega_s|, \quad \text{for } \omega \in \Omega,$$

$$d_\infty((t, \omega), (t', \omega')) = |t - t'| + ||\omega_{\wedge t} - \omega'_{\wedge t'}||_T, \quad \text{for } (t, \omega), (t', \omega') \in \Lambda,$$

then $(\Omega, ||.||_T)$ and $(\Lambda, d_\infty)$ are complete metric spaces. $L^0(\mathbb{F}_T, \mathbb{K})$ and $L^0(\Lambda, \mathbb{K})$ (where $\mathbb{K} = \mathbb{R}, \mathbb{R}^d$ or $\mathbb{S}^d$) denote respectively the space of $\mathbb{F}_T$ measurable $\mathbb{K}$-valued random variables and $\mathbb{F}$—progressively measurable $\mathbb{K}$-valued processes. When $\mathbb{K} = \mathbb{R}$, we omit the symbol $\mathbb{R}$.

2.1 Shifted Spaces

For fixed $s \leq t \in [0,T]$, we define the following shifted objects:

- $\Omega^t := \{\omega \in C([t,T], \mathbb{R}^d) : \omega_t = 0\}$.
- $B^t$ is the canonical process on $\Omega^t$.
- $\mathbb{F}^t = \{\mathbb{F}_s^t\}_{s \in [t,T]}$ is the filtration generated by $B^t$.
- $\mathbb{P}^t_0$ is the Wiener measure on $\Omega^t$, $E^t_0$ is the expectation under $\mathbb{P}^t_0$.
- We define similarly $\Lambda^t$, $||.||^t_T$, $d^t_\infty$, $L^0(\mathbb{F}_T^t)$ etc. In these definitions, the superscripts will generally stand for the shifted space (i.e. the beginning of times) and subscripts for the final time related to the notation.
- For $\omega \in \Omega^s$ and $\omega' \in \Omega^t$, we define $\omega \otimes_t \omega' \in \Omega^s$ the concatenation of $\omega$ and $\omega'$ at $t$ by:

$$(\omega \otimes_t \omega')(r) := \omega_r 1_{[s,t]}(r) + (\omega_t + \omega'_r)1_{[t,T]}(r), \quad \text{for all } r \in [s,T].$$
• For $\xi \in \mathbb{L}^0(\mathbb{F}^T)$, $X \in \mathbb{L}^0(\Lambda)$, and a fixed path $\omega \in \Omega$, we define the shifted random variable $\xi^{t,\omega} \in \mathbb{L}^0(\mathbb{F}^T)$ and process $X^{t,\omega} \in \mathbb{L}^0(\Lambda)$ by:

\[ \xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad X^{t,\omega}(\omega') := X(\omega \otimes_t \omega'), \text{ for all } \omega' \in \Omega. \]

• Finally, we denote by $\mathcal{T}$ the set of $\mathbb{F}$-stopping times, $\mathcal{T}_+$ the set of positive $\mathbb{F}$-stopping times, and $\mathcal{H} \subset \mathcal{T}$ the subset of those hitting times $h$ of the form

\[ h := \inf\{t : B_t \in O^c\} \wedge t_0 = \inf\{t : d(\omega_t, O^c) = 0\} \wedge t_0, \]

for some $0 < t_0 \leq T$, and some open and convex set $O \subset \mathbb{R}^d$ containing $0$ with $O^c := \mathbb{R}^d \setminus O$.

$\mathcal{H} > 0$, $\mathcal{H}$ is lower semi-continuous, and $h_1 \wedge h_2 \in \mathcal{H}$ for any $h_1, h_2 \in \mathcal{H}$.

$\mathcal{T}^t$ and $\mathcal{H}^t$ are defined in the same spirit. For any $\tau \in \mathcal{T}$ (resp. $h \in \mathcal{H}$) and any $(t, \omega) \in \Lambda$ such that $t < \tau(\omega)$ (resp. $t < h(\omega)$), it is clear that $\tau^{t,\omega} \in \mathcal{T}^t$ (resp. $h^{t,\omega} \in \mathcal{H}^t$). For $t \in [0, T]$ and $\delta > 0$ we define the hitting times $h_\delta^t \in \mathcal{H}^t$, that we will use several times, as follows:

\[ h_\delta^t := \inf\{s \geq t : \|B_s\| = \delta\} \wedge (t + \delta) \wedge T. \]

Notice that $O$ is open and contains $0$, and $t_0 > t$. Therefore, for all $h \in \mathcal{H}^t$ there exist $\delta > 0$ such that

\[ t < h_\delta^t \leq h. \]

The class $\mathcal{H}$ and especially, the stopping time $h_\delta^t$ will be our mains tools for studying process locally to the right in time in the space $\Lambda$.

We shall use the following type of regularity, which is stronger than the right continuity of a process in a standard stochastic analysis sense.

**Definition 2.1** We say a process $u \in \mathbb{L}^0(\Lambda)$ is right continuous under $d_\infty$ if for any $(t, \omega) \in \Lambda$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $(s, \tilde{\omega}) \in \Lambda^t$ satisfying $d_\infty((s, \tilde{\omega}), (t, 0)) \leq \delta$, we have $|u^{t,\omega}(s, \tilde{\omega}) - u(t, \omega)| \leq \varepsilon$.

We now define the following sets of functionals which are the equivalents of semi continuous and continuous functions in the viscosity solutions theory of PDEs. Notice that for a mapping $u : \Lambda \to \mathbb{K}$, $\mathbb{F}$-progressive measurability implies that $u(t, \omega) = u(t, \omega, \Lambda)$ for all $(t, \omega) \in [0, T] \times \Omega$. 

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Definition 2.2  (i) \( \mathcal{U} \subset L^0(\Lambda) \) is the set of processes \( u \) that are bounded from above, right continuous under \( d_\infty \), and such that there exist a modulus of continuity \( \rho \) verifying for any \( (t, \omega), (t', \omega') \in \Lambda \):

\[
    u(t, \omega) - u(t', \omega') \leq \rho \left( d_\infty((t, \omega), (t', \omega')) \right) \quad \text{whenever } t \leq t'.
\]  
(2.4)

(ii) \( \overline{\mathcal{U}} \subset L^0(\Lambda) \) is the set of processes \( u \) such that \( -u \in \mathcal{U} \).

(iii) \( C^0(\Lambda, \mathbb{K}) \) (respectively, \( C^0_b(\Lambda, \mathbb{K}) \), \( UC^0_b(\Lambda, \mathbb{K}) \)) is the set of \( \mathbb{F} \)-progressively measurable processes with values in \( \mathbb{K} \) that are continuous (respectively, continuous and bounded, uniformly continuous and bounded) in \( (t, \omega) \) under the \( d_\infty \) metric. When \( \mathbb{K} = \mathbb{R} \), we simply write these sets as \( C^0(\Lambda), C^0_b(\Lambda) \) and \( UC^0_b(\Lambda) \).

Remark 2.3  It is clear that \( \mathcal{U} \cap \overline{\mathcal{U}} = UC^0_b(\Lambda) \). We also recall from [8] Remark 3.2 the condition (2.4) implies that \( u \) has left-limits and \( u_{t-} \leq u_t \) for all \( t \in (0, T] \).  

Remark 2.4  The inequality (2.4) is needed to apply the results in [8]. More powerful results then the one in [8] are available in the literature if one wants only to study the obstacle problem for semi-linear PPDEs (see [13]). In this particular case, it is possible to prove the comparison theorem if we define \( \mathcal{U} \) as the class of cadlag process that are left upper semi-continuous. Notice that this last definition would not require regularity in \( \omega \), hence the comparison result would be more powerful than the one proven in Section 6.

We define \( \overline{\mathcal{U}}^t, \mathcal{U}^t, C^0(\Lambda^t, \mathbb{K}), C^0_b(\Lambda^t, \mathbb{K}) \) and \( UC^0_b(\Lambda^t, \mathbb{K}) \) in the obvious way. It is clear that, for any \( (t, \omega) \in \Lambda \) and any \( u \in C^0(\Lambda) \), we have \( u^{t, \omega} \in C^0(\Lambda^t) \). The other spaces introduced before enjoy the same property. Notice also that for \( u \in C^0(\Lambda, \mathbb{K}) \), the sample paths of \( \{u(t, B)\}_{t \in [0, T]} \) are continuous.

2.2 Nonlinear expectation

We now give a quick description of the the probability sets and associated capacities that we will need to define viscosity solutions of PPDEs. These sets are the ones used in [9].

For every constant \( L > 0 \), we denote by \( \mathcal{P}_L \) the collection of all continuous semimartingale measures \( \mathbb{P} \) on \( \Omega \) whose drift and diffusion characteristics are bounded by \( L \) and \( \sqrt{2L} \), respectively. To be precise, let \( \tilde{\Omega} := \Omega^3 \) be an enlarged canonical space, \( \tilde{B} := (B, A, M) \) be the canonical processes, and \( \tilde{\omega} = (\omega, a, m) \in \tilde{\Omega} \) be the paths. For any \( \mathbb{P} \in \mathcal{P}_L \), there exists an extension \( Q \) on \( \tilde{\Omega} \) such that:

\[
    B = A + M, \quad A \text{ is absolutely continuous}, M \text{ is a martingale},
\]

\[
    |\alpha^\mathbb{P}| \leq L, \quad \frac{1}{2} \text{tr} ( (\beta^\mathbb{P})^2 ) \leq L, \quad \text{where } \alpha^\mathbb{P}_t := \frac{dA_t}{dt}, \beta^\mathbb{P}_t := \frac{dM_t}{dt}, Q\text{-a.s.} \quad (2.5)
\]
Similarly, for any $t \in [0,T)$, we may define $\mathcal{P}_L^t$ on $\Omega^t$.

We denote by $\mathbb{L}^1(\mathbb{F}_T^t, \mathcal{P}_L^t)$ the set of $\xi \in \mathbb{L}^0(\mathbb{F}_T^t)$ satisfying $\sup_{P \in \mathcal{P}_L^t} \mathbb{E}^P[|\xi|] < \infty$. The following nonlinear expectation will play a crucial role:

$$\mathcal{E}^L_t[\xi] := \sup_{P \in \mathcal{P}_L^t} \mathbb{E}^P[\xi] \quad \text{and} \quad \mathcal{E}^L_t[\xi] := \inf_{P \in \mathcal{P}_L^t} \mathbb{E}^P[\xi] = -\mathcal{E}^L_t[-\xi] \quad \text{for all} \quad \xi \in \mathbb{L}^1(\mathbb{F}_T^t, \mathcal{P}_L^t). \quad (2.6)$$

We recall the following lemma whose proof can be found in [9].

**Lemma 2.5** For any $h \in \mathcal{H}$ and any $L > 0$, we have $\mathcal{E}^L_0[h] > 0$.

**Definition 2.6** Let $X \in \mathbb{L}^0(\Lambda)$ such that $X_t \in \mathbb{L}^1(\mathbb{F}_t^t, \mathcal{P}_L^t)$ for all $0 \leq t \leq T$. We say that $X$ is an $\mathcal{E}^L$-supermartingale (resp. submartingale, martingale) if, for any $(t, \omega) \in \Lambda$ and any $\tau \in \mathbb{T}^t$, $\mathcal{E}^L_t[X_{\tau}^\omega] \leq (\text{resp.} \geq, =) X_t(\omega)$.

We now state an important result for our subsequent analysis. Given a bounded process $X \in \mathbb{L}^0(\Lambda)$, consider the nonlinear optimal stopping problem

$$\hat{\mathcal{S}}_t^L[X](\omega) := \sup_{\tau \in \mathbb{T}^t} \mathcal{E}^L_t[X_{\tau}^\omega] \quad \text{and} \quad \hat{\mathcal{S}}_t^L[X](\omega) := \inf_{\tau \in \mathbb{T}^t} \mathcal{E}^L_t[X_{\tau}^\omega], \quad (t, \omega) \in \Lambda. \quad (2.7)$$

By definition, we have $\hat{\mathcal{S}}_t^L[X] \geq X$ and $\hat{\mathcal{S}}_T^L[X] = X_T$. The following nonlinear Snell envelope characterization of the optimal stopping time is proven in [8].

**Theorem 2.7** Let $X \in \mathcal{U}$ be bounded, $h \in \mathcal{H}$, and set $\hat{X}_t := X_t 1_{\{t < h\}} + X_h 1_{\{t \geq h\}}$. Define

$$Y := \hat{\mathcal{S}}_t^L[\hat{X}] \quad \text{and} \quad \tau^* := \inf\{t \geq 0 : Y_t = \hat{X}_t\}.$$

Then $Y_{\tau^*} = \hat{X}_{\tau^*}$, $Y$ is an $\mathcal{E}^L$-supermartingale on $[0,h]$, and an $\mathcal{E}^L$-martingale on $[0,\tau^*]$. Consequently, $\tau^*$ is an optimal stopping time.

We define $\mathcal{P}_\infty := \cup_{L > 0} \mathcal{P}_L$. Our test processes will be smooth in the following sense.

**Definition 2.8** (i) Let $u \in C^0(\Lambda, \mathbb{K})$, its right time-derivative $\partial_t u$, if it exists, is defined as:

$$\partial_t u(t, \omega) := \lim_{\delta \downarrow 0} \frac{u(t + \delta, \omega, A_t) - u(t, \omega)}{\delta}, \quad \text{for} \quad (t, \omega) \in [0,T) \times \Omega,$$

$$\partial_t u(T, \omega) := \lim_{t \uparrow T} \partial_t u(t, \omega) \quad \text{for} \quad \omega \in \Omega.$$

(ii) We say that $u$ is in $C^{1,2}(\Lambda)$ if $u, \partial_t u \in C^0(\Lambda)$ and there exist $\partial_{\omega} u \in C^0(\Lambda, \mathbb{K}^d)$, $\partial_{\omega \omega} u \in C^0(\Lambda, \mathbb{S}^d)$ such that the process $u_t := u(t, B)$, verify:

$$du_t = \partial_t u_t dt + \frac{1}{2} \partial_{\omega \omega} u_t : d\omega + \partial_{\omega} u_t dB_t, \quad \mathbb{P}-a.s.,$$

for every $\mathbb{P} \in \mathcal{P}_\infty$. 


The previous notation \( u_t := u(t, B) \) will be our convention. If we compose a functional \( u \) with the canonical process \( B \), we will use the notation \( u_t \). If we need to compose \( u \) with other processes (for example \( X \)), we will explicitly write \( u(t, X) \).

**Remark 2.9** The requirement of continuity of the derivatives and the fact that the support of \( \mathbb{P}_0 \) is \( \Omega \) show that if \( u \in C^{1,2}(\Lambda) \) then its derivatives are uniquely defined.

### 2.3 Second order reflected BSDEs

We refer to [14] for various properties of second order reflected BSDEs (2RBSDEs). The 2RBSDEs that we study have 3 components:

- A final condition: \( \xi : \Omega \to \mathbb{R} \).
- A generator: \( G : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R} \).
- An obstacle: \( h : \Lambda \to \mathbb{R} \).

#### 2.3.1 The Generator

Let \( \mathcal{K} \) be a measurable set with its sigma algebra \( \mathcal{M}_\mathcal{K} \) and 2 mappings:

\[
F : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R} \\
\sigma : \Lambda \times \mathcal{K} \to \mathbb{S}^d.
\]

We consider the following generator

\[
G : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R} \tag{2.8}
\]

\[
G(t, \omega, y, z, \gamma) = \sup_{k \in \mathcal{K}} \left[ \frac{1}{2} \sigma(t, \omega, k)^2 : \gamma + F(t, \omega, y, \sigma(t, \omega, k)z, k) \right] \tag{2.9}
\]

We make the following assumption on the data of the 2RBSDE:

**Assumption 2.10** There exist \( L_0, M_0 \geq 0 \) and \( \rho_0 \) a modulus of continuity with at most polynomial growth verifying the following points.

(i) **Boundedness**: \( \xi, h, \) and \( F(\cdot, 0, \cdot, \cdot) \) are bounded by \( M_0 \).

(ii) **Assumptions on \( F \) and \( G \)**: \( F(\cdot, y, z, k) \) and \( G(\cdot, y, z, \gamma) \) are right continuous under \( d_\infty \) metric in the sense of the definition (2.1). \( F(t, \omega, y, \cdot) \) is Lipschitz continuous in \( \Lambda \) with Lipschitz constant \( L_0 \).

(iii) **Assumption on \( h \)**: \( h \) is uniformly continuous under \( d_\infty \) with modulus of continuity \( \rho_0 \).

(iv) **Assumption on \( \xi \)**: \( \xi \) is uniformly continuous under the \( ||.||_T \) norm with modulus of continuity \( \rho_0 \). For all \( \omega \in \Omega \), \( \xi(\omega) \geq h(T, \omega) \).
(v) **Assumption on $\sigma$** : For all $(t, \omega) \in \Lambda$, $\inf_{k \in K} \sigma(t, \omega, k) > 0$, $|\sigma(t, \omega, k)| \leq \sqrt{2L_0}$, $\sigma(t, .., k)$ is Lipschitz continuous with Lipschitz constant $L_0$, and $\sigma(., k)$ is right continuous under $d_{\infty}$.

We will also need the following additional assumption for our well-posedness results.

**Assumption 2.11** $\sigma$ does not depend on $(t, \omega)$, $F(., y, z, k)$ is uniformly continuous with modulus of continuity $\rho_0$.

**Remark 2.12** The Assumption (2.11) will only be used to prove the Lemma 7.1 and under this additional assumption the operator $G$ is uniformly non-degenerate in $\gamma$.

### 3 Introduction of the value functional

For $t \in [0, T]$, we denote by $K^t$, the set of $\mathbb{R}$-progressively measurable and $K$ valued processes. Under the assumptions (2.10), for fixed $(t, \omega) \in \Lambda$, and $k \in K^t$, by the Lipschitz continuity of $\sigma$ in $\omega$, there exists a unique strong solution $X_{t,\omega,k}$ of the following equation under $P^t_0$:

$$X_{s}^{t,\omega,k} = \int_{t}^{s} \sigma^{t,\omega}(r, X_{r}^{t,\omega,k}, k_r) dB^r_t, \text{ for } s \in [t, T].$$

(3.10)

Additionally, by the classical estimates on SDEs, for $(t, \omega), (t, \omega') \in \Lambda$:

$$\mathbb{E}_0^t \left[ (|X_{t}^{t,\omega,k}|_{T})^p \right] \leq C_p, \text{ for all } p > 0,$$

$$\mathbb{E}_0^t \left[ (|X_{t}^{t,\omega,k} - X_{t}^{t,\omega',k}|_{T})^2 \right] \leq C ||\omega - \omega'||^2.$$ 

(3.11)

At the previous inequality, as it will be the case in the sequel, $C$ is a constant that may change from line to line, however it only depends on $d, M_0, T, L_0$, and $\rho_0$.

We define $\mathbb{P}^{t,\omega,k} := \mathbb{P}_0^t \circ (X^{t,\omega,k})^{-1} \in \mathcal{P}_{L_0}^t$. Notice that, the lemma 2.2 of [19] shows that there exists a mapping $\tilde{k} \in L^0(\Lambda; K)$ such that $\tilde{k}(s, X^{t,\omega,k}) = k(s, B^t)$, $ds \times \mathbb{P}_0^t$-a.s. By rewriting (3.10) under $\mathbb{P}_0^t$:

$$X_{s}^{t,\omega,k} = \int_{t}^{s} \sigma^{t,\omega}(r, X_{r}^{t,\omega,k}, \tilde{k}(r, X_{r}^{t,\omega,k})) dB^r_t, \text{ for } s \in [t, T].$$

(3.12)

Therefore, $\{\sigma^{t,\omega}_{r}(\tilde{k}_{r})^{-1} dB^r_t\}_{r \in [t, T]}$ (recall that $\sigma^{t,\omega}_{r}(\tilde{k}_{r}) = \sigma^{t,\omega}(r, B^t, \tilde{k}(r, B^t))$) is the increment of a Brownian motion under $\mathbb{P}^{t,\omega,k}$. Hence, for fixed $\tau \in \mathcal{T}^t$ and $\mathbb{F}_\tau^t$ measurable
and bounded random variable $\zeta$, one can define $(Y^{t,\omega,k}(\tau,\zeta), Z^{t,\omega,k}(\tau,\zeta), K^{t,\omega,k}(\tau,\zeta))_{s \in [t,\tau]}$ solution to the reflected BSDE on $\Omega^t$, with data $(F_t^{t,\omega}(\cdot,\cdot,\tilde{k}_s), h^{t,\omega}, \zeta)$ under $\mathbb{P}^{t,\omega,k}$:

$$Y_s^{t,\omega,k} = \zeta(B_t) + \int_s^T F_r^{t,\omega}(Y_r^{t,\omega,k}, Z_r^{t,\omega,k}, \tilde{K}_r)dr$$

$$- \int_s^T (Z_r^{t,\omega,k})^* \sigma_r^{t,\omega}(\tilde{k}_r)^{-1} dB_r^t + K_s^{t,\omega,k},$$

$$\forall s \in [t,\tau],$$

$$Y_s^{t,\omega,k} \geq h^{t,\omega}_{s},$$

$$(K^{t,\omega,k})_{s \in [t,\tau]}$$ is increasing in $s$, $K_s^{t,\omega,k} = 0$ and $\int_s^T \left[ Y_{s}^{t,\omega,k} - h^{t,\omega}_{s} \right] dK_s^{t,\omega,k} = 0.$

When $(\tau, \zeta) = (T, \xi)$, we denote

$$(Y_t^{t,\omega,k}, Z_t^{t,\omega,k}, K_t^{t,\omega,k}) = (Y_s^{t,\omega,k}(T,\xi), Z_s^{t,\omega,k}(T,\xi), K_s^{t,\omega,k}(T,\xi)).$$

To make easier our notations, we also define the following reflected BSDE, under $\mathbb{P}_0^t$:

$$\tilde{Y}_s^{t,\omega,k} = \xi(X^{t,\omega,k}) + \int_s^T F^{t,\omega}(s, X^{t,\omega,k}, \tilde{Y}_r^{t,\omega,k}, \tilde{Z}_r^{t,\omega,k}, \tilde{K}_r(r, X^{t,\omega,k}))dr$$

$$- \int_s^T (\tilde{Z}_r^{t,\omega,k})^* \sigma_r^{t,\omega}(\tilde{k}_r)^{-1} dB_r^t + \tilde{K}_s^{t,\omega,k},$$

$$\tilde{Y}_s^{t,\omega,k} \geq h^{t,\omega}_{s}(s, X^{t,\omega,k}),$$

$$(K^{t,\omega,k})_{s \in [t,T]}$$ is increasing in $s$, $\tilde{K}_t^{t,\omega,k} = 0$ and $\int_s^T \left[ \tilde{Y}_{s}^{t,\omega,k} - h^{t,\omega}_{s}(s, X^{t,\omega,k}) \right] d\tilde{K}_s^{t,\omega,k} = 0.$

For all $s \in [t,T]$, $Y_s^{t,\omega,k}$ and $\tilde{Y}_s^{t,\omega,k}$ are $\mathbb{P}_s^{t,\omega,k}$ measurable, so $Y_t^{t,\omega,k}$ and $\tilde{Y}_t^{t,\omega,k}$ are constant. Additionally, the family:

$$(\xi^{t,\omega}(B_t), F_s^{t,\omega}(y, z, \tilde{k}_s), h_s^{t,\omega}, \sigma_s^{t,\omega}(\tilde{k}_s)^{-1} dB_s^t)_{s \in [t,T]}$$

under $\mathbb{P}^{t,\omega,k}$ has the same distribution as the family

$$(\xi^{t,\omega}(X^{t,\omega,k}), F_s^{t,\omega}(s, X^{t,\omega,k}, y, z, \tilde{k}(s, X^{t,\omega,k})), h_s^{t,\omega}(s, X^{t,\omega,k}), dB_s^t)_{s \in [t,T]}$$

under $\mathbb{P}_0^t$. So $Y_t^{t,\omega,k} = \tilde{Y}_t^{t,\omega,k}$. We define the following process which is our value functional of interest:

$$u^0(t, \omega) := \sup_{k \in K^t} Y_t^{t,\omega,k} = \sup_{k \in K^t} \tilde{Y}_t^{t,\omega,k}, \text{ for } (t, \omega) \in \Lambda.$$  \hspace{1cm} (3.16)

### 3.1 Regularity of the value functional

**Proposition 3.1** $u^0$ is bounded and uniformly continuous under the $d_\infty$ metric in $\Lambda$.  

---

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Proof. Under assumptions (2.10), the data of the problem verifies the assumptions of [11] which gives the following a priori estimate:

$$
\mathbb{E}^{t,\omega,k} \left( \sup_{t \leq s \leq T} |Y^t_{s,\omega,k}|^2 + \int_t^T |Z^t_{r,\omega,k}|^2 dr + (K^t_{r,\omega,k})^2 \right) \leq C.
$$

Additionally, for $(t, \omega), (t, \omega') \in \Lambda$, notice that $||\omega \otimes_t X^t_{\omega,k} - \omega' \otimes_t X^t_{\omega',k}||_T \leq ||\omega - \omega'||_t + ||X^t_{\omega,k} - X^t_{\omega',k}||_T$, therefore under our boundedness and regularity assumptions the estimates in [11] gives:

$$
|\tilde{Y}^t_{\omega,k} - \tilde{Y}^t_{\omega',k}|^2 \\
\leq C \mathbb{E}^{t,\omega,k}_0 \left[ \mathbb{E}^{t,\omega,k}_0 \left[ \xi^t_{\omega}(X^t_{\omega,k}) - \xi^t_{\omega'}(X^t_{\omega',k}) \right]^2 \right] \\
+ C \mathbb{E}^{t,\omega,k}_0 \left[ \int_t^T |F^t_{\omega}(s, X^t_{\omega,k}, Y^t_{\omega,k}, Z^t_{\omega,k}, k_s) - F^t_{\omega'}(s, X^t_{\omega',k}, Y^t_{\omega',k}, Z^t_{\omega',k}, k_s)|^2 ds \right] \\
+ C \mathbb{E}^{t,\omega,k}_0 \left[ \sup_{t \leq s \leq T} |h^t_{\omega}(s, X^t_{\omega,k}) - h^t_{\omega'}(s, X^t_{\omega',k})|^2 \right]^{1/2} \\
\leq C \mathbb{E}^{t,\omega,k}_0 \left( \rho_0^2(||\omega - \omega||_t + ||X^t_{\omega,k} - X^t_{\omega',k}||_T) \right). \tag{3.17}
$$

$\rho_0$ has at most polynomial growth, denote $p_0 > 0$ this growth power. For fixed $\delta > 0$, we can estimates the difference $|\tilde{Y}^t_{\omega,k} - \tilde{Y}^t_{\omega',k}|$ as follows:

$$
|\tilde{Y}^t_{\omega,k} - \tilde{Y}^t_{\omega',k}|^2 \\
\leq C \mathbb{E}^{t,\omega,k}_0 \left( \rho_0^2(||\omega - \omega||_t + ||X^t_{\omega,k} - X^t_{\omega',k}||_T) \mathbb{1}_{\{||X^t_{\omega,k} - X^t_{\omega',k}||_T > \delta \}} \right) \\
+ C \mathbb{E}^{t,\omega,k}_0 \left( \rho_0^2(||\omega - \omega||_t + \delta) \mathbb{1}_{\{||X^t_{\omega,k} - X^t_{\omega',k}||_T \leq \delta \}} \right) \\
\leq C \left( \mathbb{E}^{t,\omega,k}_0 \left( \frac{||X^t_{\omega,k} - X^t_{\omega',k}||_T^2}{\delta} \right) \right)^{1/2} \left( 1 + ||\omega - \omega'||_t^{2p_0} \right) + C \rho_0^2(||\omega' - \omega||_t + \delta) \\
\leq C \left( \mathbb{E}^{t,\omega,k}_0 \left( \frac{||X^t_{\omega,k} - X^t_{\omega',k}||_T^2}{\delta} \right) \right)^{1/2} \left( 1 + ||\omega - \omega'||_t^{2p_0} \right) + \rho_0^2(||\omega' - \omega||_t + \delta) \right). \tag{3.18}
$$

If we choose $\delta := \sqrt{||\omega - \omega'||_t}$, then the last line becomes a modulus of continuity $\rho_1$ with at most polynomial growth.

First of all, the previous estimates gives that $Y^t_{\omega,k}$ is bounded by a constant that only depends on $M_0, T, L_0$, and $\rho_0$. With a passage to supremum in $k$, we see that $u^0$ is bounded. Additionally

$$
|u^0(t, \omega) - u^0(t, \omega')| \leq \sup_{k \in K^t} |\tilde{Y}^t_{\omega,k} - \tilde{Y}^t_{\omega',k}| \leq \rho_1(||\omega - \omega'||_t), \tag{3.19}
$$

which show that for fixed $t$, $u^0$ is uniformly continuous in $\omega$ uniformly in $t$.  

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Fix $0 \leq t \leq t_1 \leq T$. Given the uniform continuity of $u^0$ in $\omega$ for fixed times, one can proceed as in Lemma 4.1 of [8] to obtain the following dynamic programming principle at deterministic times:

$$u^0(t, \omega) = \sup_{k \in K^t} \mathcal{Y}^{t, \omega, k}_t (t_1, u^0(t_1, \omega \otimes_t B^t)), \quad (3.20)$$

where $\mathcal{Y}^{t, \omega, k}_t (t_1, u^0(t_1, \omega \otimes_t B^t))$ (denoted only by $\mathcal{Y}^{t, \omega, k}_t$ for simplicity at this section) is defined at (3.13). We estimate the variation in time for $Y$ where

We define the following optimal stopping time for $Y$.

Fix $0 \leq t \leq t_1 \leq T$. Given the uniform continuity of $u^0$ in $\omega$ for fixed times, one can proceed as in Lemma 4.1 of [8] to obtain the following dynamic programming principle at deterministic times:

$$u^0(t, \omega) = \sup_{k \in K^t} \mathcal{Y}^{t, \omega, k}_t (t_1, u^0(t_1, \omega \otimes_t B^t)),$$

where $\mathcal{Y}^{t, \omega, k}_t (t_1, u^0(t_1, \omega \otimes_t B^t))$ (denoted only by $\mathcal{Y}^{t, \omega, k}_t$ for simplicity at this section) is defined at (3.13). We estimate the variation in time for $k \in K^t$ and under $\mathbb{P}^{t, \omega, k}$

$$u^0(t_1, \omega) - u^0(t, \omega) = u^0(t_1, \omega) - u^0(t_1, \omega \otimes_t B^t) - \int_t^{t_1} F^{t, \omega}(r, B^t, \mathcal{Y}^{t, \omega, k}_r, \mathcal{Z}^{t, \omega, k}_r, \tilde{k}_r) dr$$

$$+ \int_t^{t_1} \mathcal{Z}^{t, \omega, k}_r \cdot (\sigma^{t, \omega}_r (\tilde{k}_r))^{-1} dB^t_r - k^{t, \omega, k}_{t_1} + \mathcal{Y}^{t, \omega, k}_t - u^0(t, \omega)$$

We take the expectation under $\mathbb{P}^{t, \omega, k}$ to have:

$$u^0(t_1, \omega) - u^0(t, \omega) \leq \mathbb{E}^{t, \omega, k} [(\rho_1 (||\omega - \omega \otimes_t B^t||_{t_1}) + ||\mathcal{Y}^{t, \omega, k}_t - u^0(t, \omega)||]$$

$$+ C(t_1 - t) + L_0 \int_t^{t_1} \mathbb{E}^{t, \omega, k} [||\mathcal{Z}^{t, \omega, k}_r||] dr.$$

Finally by taking a sequence $k_n$ such that $\mathcal{Y}^{t, \omega, k_n}_t \rightarrow u^0(t, \omega)$ and using estimates on the RBSDEs we have:

$$u^0(t_1, \omega) - u^0(t, \omega) \leq C\sqrt{t_1 - t} + \rho(||\omega \wedge t - \omega||_{t_1}),$$

for some modulus of continuity $\rho$.

We define the following optimal stopping time for $\mathcal{Y}^{t, \omega, k}_t$,

$$D^{t, \omega, k}_t := \inf \{ s \in [t, t_1] : \mathcal{Y}^{t, \omega, k}_s = h^{t, \omega, k}_s \} \wedge t_1.$$ 

Then

$$u^0(t_1, \omega) - \mathcal{Y}^{t, \omega, k}_t = \mathbb{E}^{t, \omega, k} \left[ u^0(t_1, \omega) - (u^0)_{t_1} + \mathbb{1}_{D^{t, \omega, k}_t < t_1} (u^0)_{t_1} - h^{t, \omega, k}_{D^{t, \omega, k}_t} \right]$$

Therefore, we can control the right hand side uniformly in $k$.
Finally combining all the previous results, we obtain that there is a modulus of continuity
\( \tilde{\rho}_0 \), which only depends on, \( M_0, L_0, \rho_0, T \), such that
\[
|u^0(t, \omega) - u^0(t', \omega')| \leq \tilde{\rho}_0(d_\infty((t, \omega), (t', \omega'))).
\] (3.21)

4 Viscosity solutions to path-dependent PDEs

For any \( L \geq 0 \) and \( (t, \omega) \in [0, T) \times \Omega \), and \( u \in \mathcal{U} \), define:
\[
\mathcal{L}^L u(t, \omega) := \left\{ \varphi \in C^{1,2}_b(\Lambda^t) : \text{there exists } h \in \mathcal{H}^t \text{ such that} \right. \\
0 = \varphi(t, 0) - u(t, \omega) = S^L_t [(\varphi - u^0) \cdot h](0) \right\};
\] (4.22)

and for all, \( u \in \mathcal{U} \):
\[
\mathcal{X}^L u(t, \omega) := \left\{ \varphi \in C^{1,2}_b(\Lambda^t) : \text{there exists } h \in \mathcal{H}^t \text{ such that} \right. \\
0 = \varphi(t, 0) - u(t, \omega) = \overline{S}^L_t [(\varphi - u^0) \cdot h](0) \right\}.
\] (4.23)

Those sets are the equivalents of sub/superjets in our theory.

4.1 The PPDE

For fixed \( (t, \tilde{\omega}) \in \Lambda \), we define the differential operator \( \mathcal{L}^t \tilde{\omega} \) on \( C^{1,2}(\Lambda^t) \):
\[
\mathcal{L}^t \tilde{\omega} \phi(s, \omega) := -\partial_t \phi(s, \omega) - \mathcal{G}^t \tilde{\omega}(s, \omega, \phi(s, \omega), \partial_\omega \phi(s, \omega), \partial_{\omega \omega} \phi(s, \omega)).
\] (4.24)

When \( t = 0 \) the operator is simply written \( \mathcal{L} \). The functional \( u^0 \) defined by (3.16) is related, as it is the case in the Markovian case, to the following PPDE:
\[
\min\{\mathcal{L}u(t, \omega); (u - h)(t, \omega)\} = 0, \text{ for all } (t, \omega) \in [0, T) \times \Omega,
\] (4.25)
\[
u(T, \omega) = \xi(\omega) \text{ for all } \omega \in \Omega.
\] (4.26)

4.2 Viscosity solution of PPDEs

We give the following definition of viscosity solution.
Definition 4.1 (i) For any $L \geq 0$, we say $u \in U$ is a viscosity $L$-supersolution of PPDE (4.25) if, for any $(t, \omega) \in [0, T) \times \Omega$ and any $\varphi \in \mathcal{A}_L^u(t, \omega)$, it holds that

$$u(t, \omega) - h(t, \omega) \geq 0, \text{ and } (L_t^\omega \varphi)(t, 0) \geq 0,$$

or equivalently $\min\{L_t^\omega \varphi(t, 0); u(t, \omega) - h(t, \omega)\} \geq 0$.

(ii) We say $u \in U$ is a viscosity $L$-subsolution of PPDE (4.25) if, for any $(t, \omega) \in [0, T) \times \Omega$ such that $u(t, \omega) - h(t, \omega) > 0$ and any $\varphi \in \mathcal{A}_L^u(t, \omega)$, it holds that

$$(L_t^\omega \varphi)(t, 0) \leq 0.$$

(iii) We say $u$ is a viscosity subsolution (resp. supersolution) of PPDE (4.25) if $u$ is viscosity $L$-subsolution (resp. $L$-supersolution) of PPDE (4.25) for some $L \geq 0$.

(iv) We say $u$ is a viscosity solution of PPDE (4.25) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 4.2 For $0 \leq L_1 \leq L_2$ and $(t, \omega) \in [0, T) \times \Omega$, we have $\mathcal{E}_L^{L_2}; \leq \mathcal{E}_L^{L_1};$ and $\mathcal{A}_L^{L_2}u(t, \omega) \subset \mathcal{A}_L^{L_1}u(t, \omega)$. If $u$ is a viscosity $L_1$-subsolution then $u$ is a viscosity $L_2$-subsolution. Same statement also holds supersolutions.

Remark 4.3 The definition of viscosity solution property is local in the following sense. For any $(t, \omega) \in [0, T) \times \Omega$, to check the viscosity property of $u$ at $(t, \omega)$, it suffices to know the value of $u_t^\omega$ on $[t, h^\delta_t]$ for an arbitrarily small $\delta > 0$. The hitting times $h^\delta_t$ are our tools of localization.

Remark 4.4 We have some flexibility to choose the set of test functionals. All the results in this paper still hold true if we replace the $\mathcal{A}_L^u$ with the $\mathcal{A}_L^{L_1}u$

$$\mathcal{A}_L^{L_1}u(t, \omega) := \left\{ \varphi \in C^{1,2}(\Lambda^t) : \exists \, h \in \mathcal{H}^t \text{ such that, for all } \tau' \in T_+^t, \right.$$

$$(\varphi - u_t^\omega)(0) = 0 < \mathcal{E}_L^t[(\varphi - u_t^\omega)_{\tau' \Lambda^t}] \right\}. \quad (4.27)$$

4.3 Consistency

Proposition 4.5 Assume $u \in C^{1,2}(\Lambda, \mathbb{R})$ then $u$ is a viscosity subsolution (respectively, supersolution) of the PPDE (4.25) if and only if $u$ is a classical subsolution (respectively, supersolution) of the same equation.
Therefore:

Then \( P(d) \) and the data is right continuous under the viscosity subsolution, a fortiori it is not a viscosity \( L_0 \)-subsolution. Therefore, there exist \((t, \omega) \in [0, T) \times \Omega \) and \( \phi \in A^{L_0} u(t, \omega) \) with the associated \( h \in \mathcal{H}^t \) such that \((u - h)(t, \omega) > 0 \) and \( c := L_t^\omega \phi(t, \omega) > 0 \). The processes \( G(\phi_t, \partial_\omega \phi_t, \partial_{\omega \omega} \phi_t), \phi, u_t^\omega \) are right continuous under the \( d_\infty \) metric, so there exist \( \delta > 0 \) such that for \((s, \tilde{\omega}) \in [t, h^t] \times \Omega^t \), the following inequalities holds:

\[
|G(t, \omega, \phi_t, \partial_\omega \phi_t, \partial_{\omega \omega} \phi_t) - G(s, \omega \otimes \tilde{\omega}, \phi_t, \partial_\omega \phi_t, \partial_{\omega \omega} \phi_t)| \leq c/4, \tag{4.28}
\]

\[
|\partial_t \phi_t - \partial_t \phi_s| + L_0|\phi_t - \phi_s| + M_0 L_0|\partial_\omega \phi_t - \partial_\omega \phi_s| + L_0|\phi_s - u_t^\omega| + L_0|\partial_{\omega \omega} \phi_t - \partial_{\omega \omega} \phi_s| \leq c/4
\]

Then \( L_t^\omega \phi_s - L_0|\phi_s - u_t^h| \geq c/2 \) for \( s \in [t, h^t] \). \( u \) is a subsolution of the PPDE (4.25) and the data is right continuous under \( d_\infty \), so we can choose a constant process \( k \in K^t \) and \( \delta > 0 \) small enough such that for all \( s \in [t, h^t] \):

\[
\partial_t u_t^s + \frac{1}{2} \partial_{\omega \omega} u_t^s : (\sigma_s(k_s))^2 + F_s(k_s) \partial_\omega (\phi - u_t^\omega)_s \geq -c/4. \tag{4.29}
\]

Notice that for \( k \) constant the equation (4.11) has strong solutions. Applying Itô’s formula under \( \mathbb{P}^{t, \omega, k} \):

\[
0 = \left( \phi - u_t^\omega \right)_t = (\phi - u_t^\omega)_{h^t}
\]

\[
- \int_t^{h^t} \partial_t (\phi - u_t^\omega)_s + \frac{1}{2} \partial_{\omega \omega} (\phi - u_t^\omega)_s : \sigma_s(k_s)^2 ds - \int_t^{h^t} \partial_\omega (\phi - u_t^\omega)_s dB_t^s
\]

\[
\geq (\phi - u_t^\omega)_{h^t} + \int_t^{h^t} L_t^\omega \phi_s ds + \int_t^{h^t} -c/4 ds
\]

For some \( r_s \) and \( \alpha_s \) progressively measurable \( r \in \mathbb{R}, \alpha \in \mathbb{R}^d \), with \(|r_s| \leq L_0\) and \(|\alpha_s| \leq L_0\).

Therefore:

\[
0 \geq (\phi - u_t^\omega)_{h^t} + \int_t^{h^t} L_t^\omega \phi_s + r_s (\phi - u_t^\omega)_s ds
\]

\[
- \frac{(h^t - t) c}{4} - \int_t^{h^t} \partial_\omega (\phi - u_t^\omega)^*_s (dB_t^s - \sigma_s(k_s) \alpha_s ds)
\]

\[
\geq (\phi - u)_{h^t} + \frac{(h^t - t) c}{4} - \int_t^{h^t} \partial_\omega (\phi - u_t^\omega)^*_s (dB_t^s - \sigma_s(k_s) \alpha_s ds)
\]
Notice that by Girsanov’s theorem, there exists $P \in \mathcal{P}^t_{L^0}$ equivalent to $\mathbb{P}^{t,\omega,k}$ such that the last integral is a martingale under $P$. Therefore, we have the following inequalities that contradicts the assumption that $\phi \in A^t_{L^0} u(t, \omega)$:

\[
0 > -\frac{c}{4} \mathbb{E}^P[H^t_{s} - t] \geq \mathbb{E}^P[(\phi - u)_{H^t_{s}}] \geq \mathbb{E}^P_{L^0}[(\phi - u)_{H^t_{s}}].
\]

4.4 A change of variable formula

We will need the following change of variable formula in our subsequent analysis.

**Proposition 4.6** Let $C, \lambda, \mu \in \mathbb{R}$ be constants and $u \in \mathcal{U}$ then $u$ is a viscosity $L$-subsolution of the PPDE (4.25) with data $(G, \xi, h)$ if and only if $u' := e^{\lambda t} u + C e^{\mu t} t$ is a viscosity $L$-subsolution of the PPDE (4.25) with data $(G', \xi', h')$ where:

\[
G'(t, \omega, y, z, \gamma) := -C e^{\mu t}(1 + (\mu - \lambda) t) - \lambda y + e^{\lambda t} G(t, \omega, e^{-\lambda t} y - C e^{(\mu - \lambda) t} t, e^{-\lambda t} z, e^{-\lambda t} \gamma),
\]

\[
\xi' := e^{\lambda T} \xi + C e^{\mu T} T,
\]

\[
h'_t := e^{\lambda t} h_t + C e^{\mu t} t.
\]

The same statement holds also for $L$-supersolutions.

**Proof** We will only prove the subsolution case. Assume that $u$ is a viscosity $L$-subsolution with data $(G, \xi, h)$. We want to show that $u'$ is a viscosity $L$-subsolution with data $(G', \xi', h')$. Take $(t, \omega) \in [0, T) \times \Omega$ such that $u'(t, \omega) - h'(t, \omega) > 0$ (notice that this is equivalent to $u(t, \omega) - h(t, \omega) > 0$) and $\phi' \in \mathcal{A}^t_{L^0} u'(t, \omega)$, with the corresponding hitting time $H \in \mathcal{H}$. For fixed $\varepsilon > 0$, we define

\[
\phi^\varepsilon_s := e^{-\lambda s} \phi'_s - C e^{(\mu - \lambda) s} s + \varepsilon(s - t).
\]

Notice that $\phi'_t = u'_t = e^{\lambda t} u_t + C e^{\mu t} t$ and for $s \geq t$:

\[
\begin{align*}
\phi^\varepsilon_s - u'_s t^\omega &- e^{-\lambda t} (\phi'_s - (u'_s t^\omega)) \\
&= (e^{-\lambda s} - e^{-\lambda t})((\phi'_s - C e^{\mu s} s) - (\phi'_t - C e^{\mu t} t)) + (e^{\lambda (s-t)} - 1)(u'_s t^\omega - u'_t t^\omega) \\
&+ (e^{\lambda (t-s)} + e^{\lambda (s-t)} - 2)u'_t t^\omega + \varepsilon(s - t)
\end{align*}
\]
There exists a constant $K > 1$ which may depend on $\lambda, t, T$ but not in $s \in (t, T]$, and $\varepsilon > 0$ such that:

$$
0 \leq |(e^{-\lambda s} - e^{-\lambda t})| \leq K(s - t)
$$

$$
0 \leq e^{\lambda(s-t)} - 1 \leq K(s - t)
$$

$$
0 \leq |e^{\lambda(t-s)} + e^{\lambda(s-t)} - 2| \leq K(s - t)^2
$$

Additionally $u \in \mathcal{U}$ and $\phi$ is continuous in under the $d_\infty$ metric, so there exist $\delta$(depending on $\varepsilon$) such that on $[t, H^0_t]$:

$$
R_0 := 1 \lor \sup_s u_s < \infty,
$$

$$
u^{t,\omega}_s - u^{t,\omega}_t \geq -\frac{\varepsilon}{3KR_0}
$$

$$
|\phi_s - Ce^{\mu s} - (\phi_t - Ce^{\mu t})| \leq \frac{\varepsilon}{3KR_0}
$$

$$
0 \leq s - t \leq \frac{\varepsilon}{3KR_0}.
$$

Combining the previous inequalities:

$$
\phi_s - u^{t,\omega}_s - e^{-\lambda t}(\phi_t - (u^{t,\omega}_s)^t) \geq -\varepsilon(s - t) + \varepsilon(s - t) \geq 0.
$$

Then for all $\tau \in \mathcal{T}_t$, such that $\tau \leq H^0_t$ it holds that:

$$
\phi^\varepsilon_{\tau} - u^{t,\omega}_{\tau} \geq e^{-\lambda t}(\phi^\varepsilon_{\tau} - (u^{t,\omega}_{\tau})^t),
$$

therefore:

$$
\mathcal{E}^L_t[\phi^\varepsilon_{\tau} - u^{t,\omega}_{\tau}] \geq e^{-\lambda t}\mathcal{E}^L_t[\phi^\varepsilon_{\tau} - (u^{t,\omega}_{\tau})^t] \geq 0 = \phi^\varepsilon_{\tau} - u^{t,\omega}_{\tau}.
$$

Which shows that $\phi^\varepsilon \in A^\varepsilon_{u(t, \omega)}$, and $u(t, \omega) - h(t, \omega) > 0$, by the definition of viscosity subsolutions, $0 \geq L^{t,\omega}\phi^\varepsilon$. Taking the limit as $\varepsilon$ goes to 0, we obtain:

$$
0 \geq L^{t,\omega}\phi = -\partial_t \phi_t + G(t, \omega, \phi_t, \partial_\omega \phi_t, \partial_{\omega\omega} \phi_t).
$$

**Remark 4.7**: Notice that after the change of variable the function $G'$ can be written as:

$$
G'(t, \omega, y, z, \gamma) = \sup_{k \in \mathcal{K}} \left( \frac{\sigma(t, \omega, k)^2}{2} + F'(t, \omega, \sigma(t, \omega, k)z, k) \right)
$$

where $F'(t, \omega, y, z, k) := e^{\lambda t}F(t, \omega, e^{-\lambda t}y - Ce^{(\mu-\lambda)t}z, e^{-\lambda t}z, k) - Ce^{\mu t}(1 + (\mu - \lambda)t) - \lambda y$.
We will make the following choices for the constants: \( \lambda = L_0 + 1 \), \( \mu = 0 \) and \( C = -2e^{(L_0+1)T}\). With this change of variable, the data of the problem verify the following properties:

$$
G'(t, \omega, y, z, \gamma) \geq G'(t, \omega, y + \delta, z, \gamma) + \delta, \text{ for all } \delta > 0, \text{ and any } (t, \omega, y, z, \gamma), \quad (4.30)
$$

$$
F'(t, \omega, y, z, k) \geq F'(t, \omega, y + \delta, z, k) + \delta, \text{ for all } \delta > 0, \text{ and any } (t, \omega, y, z, k),
$$

$$
F'(t, \omega, h'(t, \omega), 0, k) = -C - \lambda e^{Lt} + e^{Lt}F(t, \omega, h(t, \omega), 0, k) \geq 0 \text{ for all } (t, \omega) \in \Lambda.
$$

When needed, we will assume, that \( F, G \) and \( h \) verify (4.30). This change of variable formula will be useful at subsection (A.2.1).

### 4.5 Viscosity solution property of the value functional

Before starting to study the viscosity solution property of \( u^0 \), we give the following dynamic programming principal on random times. Its proof is similar to the proof of Theorem 4.3 of [8]. With the notation introduced at (3.13), for all \( \tau \in \mathcal{T}^t \), the following dynamic programming at stopping times holds

$$
u^0(t, \omega) = \sup_{k \in \mathcal{K}} Y^\delta_{\tau, \omega, k}(\tau, \nu^0(t, \omega, t) \otimes B^t) \quad (4.31)
$$

**Theorem 4.8** Under the assumptions (2.10) on the data of the problem, the value functional \( u^0 \) defined at (3.16) is viscosity solution of the PPDE (4.25).

### 4.5.1 Subsolution property of the value functional

We assume without loss of generality that

\( G \) and \( F \) are increasing in \( y \).

We reason by contradiction by assuming that \( u^0 \) is not a viscosity \( L_0 \)-subsolution, so there exist \( (t, \omega) \in [0, T) \times \Omega \) such that \( u^0(t, \omega) > h(t, \omega) \), and \( \phi \in \mathcal{A}^{L_0} u^0(t, \omega) \), with the associated \( h \in \mathcal{H}^t \) and verifying:

$$
c = \min\{\mathcal{L}^{t, \omega} \phi(t, 0), u^0(t, \omega) - h(t, \omega)\} > 0.
$$

Without loss of generality we will assume that \( (t, \omega) = (0, 0) \). Recall that \( u^0 \) and \( h \) are uniformly continuous. Therefore there exist \( \delta > 0 \) such that for all \( s \in [0, h_\delta] \),

$$
|u^0_s - u^0_0| \leq c/4, \quad |h_0 - h_s| \leq c/4.
$$
Denote, for $s \in [0, T]$ and $k \in \mathcal{K}^0$:

$$\delta Y^k_s := \phi_s - Y^{0,0,k}_s, \quad \delta Z^k_s := \partial_\omega \phi_s - (\sigma_s(k_s))^{-1} Z^{0,0,k}_s$$

$$G_s(u^0, \partial_\omega \phi) := G_s(u^0, \partial_\omega \phi(s), \partial_\omega \phi(s, B)),$$

$$F^k_s(y, z) := F_s(y, z, k_s),$$

By the continuity of $\phi$ and the right continuity of $G$, we can take $\delta > 0$ small enough, to have for $s \in [0, H_s]$:

$$-\partial_t \phi_s - G_s(u^0, \partial_\omega \phi_s, \partial_\omega \phi_s) \geq c/2.$$  

$G$ is defined as the supremum in (2.8), then for all $k \in \mathcal{K}^0$, the following inequality holds:

$$-\partial_t \phi_s - \frac{1}{2} \partial_\omega \partial_\omega \phi_s : \sigma_s(k_s)^2 - F_s(u^0, \sigma_s(k_s) \partial_\omega \phi_s, k_s) \geq c/2.$$  

For $k \in \mathcal{K}^0$, we apply functional Itô's formula to $\phi$ and use the definition of $Y^{0,0,k}$ in (3.14) to obtain under $\mathbb{P}^{0,0,k}$:

$$d(\delta Y^k)_s = [\partial_t \phi_s + \frac{1}{2} \partial_\omega \partial_\omega \phi_s : \sigma_s(k_s)^2 + F^k_s(u^0, \partial_\omega \phi_s)] ds$$

$$+ [F^k_s(Y^{0,0,k}_s, Z^{0,0,k}_s) - F^k_s(u^0, \phi)] ds + (\delta Z^k_s)^* dB_s + dK^{0,0,k}_s.$$  

Therefore for all $k, \mathbb{P}^{0,0,k}$-a.s.:

$$\delta Y^k_{H_s} - \delta Y^k_0 = \int_t^{H_s} [\partial_t \phi_s + \frac{1}{2} \partial_\omega \partial_\omega \phi(s, B^t) : \sigma_s(k_s)^2 + F^k_s(u^0, \phi)] ds$$

$$+ \int_0^{H_s} (F^k_s(Y^{0,0,k}_s, Z^{0,0,k}_s) - F^k_s(u^0, \phi)) ds + \int_0^{H_s} (\delta Z^k_s)^* dB_s + K^{0,0,k}_{H_s}$$

$$\leq \frac{-cH_s}{2} + \int_0^{H_s} (F^k_s(Y^{0,0,k}_s, Z^{0,0,k}_s) - F^k_s(u^0, \phi)) ds + \int_0^{H_s} (\delta Z^k_s)^* dB_s + K^{0,0,k}_{H_s}$$

We have assumed that $F$ is increasing in $y$ and $u^0_s \geq Y^{0,0,k}_s$ therefore for all $k, \mathbb{P}^{0,0,k}$-a.s.:

$$(\phi - u^0)_{H_s} - (u^0 - Y^{0,0,k})_0 = (\phi - u^0)_{H_s} - (\phi - Y^{0,0,k})_0$$

$$\leq (\phi - Y^{0,0,k})_{H_s} - (\phi - Y^{0,0,k})_0 = \delta Y^k_{H_s} - \delta Y^k_0$$

$$\leq \frac{-cH_s}{2} + \int_0^{H_s} (F^k_s(u^0, Z^{0,0,k}_s) - F^k_s(u^0, \phi)) ds + \int_0^{H_s} (\delta Z^k_s)^* dB_s + K^{0,0,k}_{H_s}$$

$$= \frac{-cH_s}{2} + \int_0^{H_s} (\delta Z^k_s)^* (dB_s + \sigma_s(k_s) \alpha_s ds) + K^{t,\omega,k}_{H_s}$$

where $|\alpha_s| \leq L_0$.

By the definition of $u^0$ there exists a sequence $k^n \in \mathcal{K}^0$ such that $Y^{0,0,k^n}_0 \uparrow u^0(0, \mathbf{0})$ as $n$ goes to infinity, and $Y^{0,0,k^n}_0 \geq u^0_s - c/4$ for all $n$. Define the optimal stopping time $D^k$ for $Y^{0,0,k}$ by $D^k = \inf\{s \geq 0 : Y^{0,0,k}_s = H_s \wedge T\}$. We can write

$$Y^{0,0,k}_0 = \mathbb{E}^{0,0,k} \left[ \int_0^{D^k \wedge H_s} F^k_r(Y^{0,0,k}_r, Z^{0,0,k}_r) dr + h_{D^k} 1_{\{D^k < H_s\}} + u^0_{H_s} 1_{\{D^k \geq H_s\}} \right].$$
Using the uniform bounds on \((Y^{0,0,k},Z^{0,0,k})\), we have that
\[
\mathbb{E}^{0,0,k}\left[\int_0^{D^k \land H_\delta} |F_r^{k}(Y_0^{0,0,k},Z_0^{0,0,k})|dr\right] \leq C \sqrt{\delta},
\]
uniformly in \(k\). We choose \(\delta\) small such that the previous term is dominated by \(\frac{\varepsilon}{4}\). Recall also that \(h_{D^k n} \leq u_0^0 - \frac{\varepsilon}{4}\) on \(\{D^k n < H_\delta\}\) and \(u_0^0 \succeq u_0^0 - \frac{\varepsilon}{4}\).

Then, for all \(n\), \(\mathbb{P}^{0,0,k_n}(D^k n < H_\delta) = 0\). Therefore \(K^0_{n,k^n} = 0\), \(\mathbb{P}^{0,0,k_n}\text{-a.s. for all } n\).

Injecting this into the previous inequalities, the following holds under \(\mathbb{P}^{0,0,k_n}\):
\[
(\phi - u^0)_{H_\delta} + \frac{cH_\delta}{2} \leq (u^0 - Y^{0,0,k_n})_0 + \int_0^{H_\delta} (\delta Z^{k_n}_s) \alpha_s + \sigma_s(k^n_s) \alpha_s ds.
\]

There exists a probability \(\mathbb{P}\) such that \(\mathbb{P}^{0,0,k_n}\) such that the previous integral is a \(\mathbb{P}\) martingale. Taking the expectation \(\mathbb{E}^{\mathbb{P}}\):
\[
\mathbb{E}^{\mathbb{P}}[(\phi - u^0)_{H_\delta}] + \mathbb{E}^{\mathbb{P}}\left[\frac{cH_\delta}{2}\right] \leq (u^0 - Y^{0,0,k_n})_0
\]
\(\phi \in A^{L_0}(0,0)\) implies that \(0 \leq \mathbb{E}^{L_0}[(\phi - u^0)_{H_\delta}] \leq \mathbb{E}^{\mathbb{P}}[(\phi - u^0)_{H_\delta}]\). Therefore :
\[
0 < \frac{c}{2} \mathbb{E}^{L_0}(H_\delta) \leq \mathbb{E}^{\mathbb{P}}\left[\frac{cH_\delta}{2}\right] \leq (u^0 - Y^{0,0,k_n})_0.
\]
Taking the limit as \(n\) goes to infinity we arrive to the contradiction \(0 < \frac{\varepsilon}{2} \mathbb{E}^{L_0}(H_\delta) \leq 0\).

### 4.5.2 Supersolution property of the value functional

Without loss of generality, we can assume that
\[ F \text{ and } G \text{ are decreasing in } y. \]

We will again reason by contradiction. Assume that \(u^0\) is not a viscosity supersolution. A fortiori, it is not a viscosity \(L_0\)-supersolutions. So there exist \((t,\omega) \in [0,T] \times \Omega\), and \(\phi \in \tilde{A}^{L_0} u^0(t,\omega)\) with the associated \(h \in H^t\) such that \(-c := \min(\mathcal{L}^{t,\omega}\phi(t,0),\phi(t,0) - h(t,\omega)) < 0\). Notice that \(0 = \phi(t,0) - u^0(t,\omega)\) so \(\phi(t,0) \geq h(t,\omega)\). Therefore \(-c = \mathcal{L}^{t,\omega}\phi(t,0)\).

Without loss of generality we assume \((t,\omega) = (0,0)\). Similarly to the previous case there exist \(\delta > 0\), such that for \(s \in [0,H_\delta]\) it holds that :
\[
\partial_t \phi_s + G_s(u^0,\partial_\omega \phi) \geq c/2.
\]
By the definition of \(G\) (in \((2.3)\)) and the right continuity of the processes involved, there exists a constant process \(k^0 \in K^0\) such that by taking \(\delta > 0\) small enough, for all \(s \in [0,H_\delta]\) the following inequality holds :
\[
\partial_t \phi_s + \frac{1}{2} \partial_{\omega \omega} \phi_s : \sigma_s(k^0_s)^2 + F^k_s(u^0_s,\sigma_s(k^0_s)\partial_\omega \phi) \geq c/3.
\]
We use (4.31) with \( \tau = \epsilon_{\delta} \) and denote

\[
(\mathcal{Y}, Z, K) := (\mathcal{Y}^{0,0,k_0}(\epsilon_{\delta}, u^0_{H\delta}), Z^{0,0,k_0}(\epsilon_{\delta}, u^0_{H\delta}), K^{0,0,k_0}(\epsilon_{\delta}, u^0_{H\delta}))
\]

and with the obvious modifications of the notations of the subsolution case, under \( \mathbb{P}^{0,0,k_0} \) we have:

\[
d(\phi - \mathcal{Y}) = \left[ (\partial_t \phi_s + \frac{1}{2} \partial_{\omega \omega} \phi_s : \sigma_s(k^0_s) + F^k_s(u^0_s, \sigma_s(k^0_s) \partial_{\omega} \phi_s) \right] ds
\]

\[
+ \left[ F^k_s(\mathcal{Y}_s, Z_s) - F^k_s(u^0_s, \sigma_s(k^0_s) \partial_{\omega} \phi_s) \right] ds + (\delta Z^k_s)^* dB_s + dK^{0,0,k_0}_s
\]

\[
\geq \frac{c}{6} ds + \left[ F^k_s(u^0_s, Z_s) - F^k_s(u^0_s, \sigma_s(k^0_s) \partial_{\omega} \phi_s) \right] ds + (\delta Z^k_s)^* dB_s
\]

\[
\geq \frac{c}{6} ds + (\delta Z^k_s)^* (dB_s + \sigma_s(k^0_s) \alpha_s ds)
\]

for some \(|\alpha_s| \leq L_0\).

Therefore under \( \mathbb{P}^{0,0,k_0} \):

\[
(\phi - u^0)_{H\delta} - (u^0 - \mathcal{Y})_0
\]

\[
= (\phi - u^0)_{H\delta} + (u^0 - \mathcal{Y})_{H\delta} - (u^0 - \mathcal{Y})_0
\]

\[
= (\phi - \mathcal{Y})_{H\delta} - (u^0 - \mathcal{Y})_0
\]

\[
= (\phi - \mathcal{Y})_{H\delta} - (\phi - \mathcal{Y})_0 \geq \frac{cH^\delta}{6} + \int_0^{H^\delta} (Z^k_s)^* (dB_s - \sigma_s(k^0_s) \alpha_s ds)
\]

Recall that the DPP (4.31) gives \((u^0 - \mathcal{Y})_0 \geq 0\) therefore:

\[
(\phi - u^0)_{H\delta} \geq \frac{cH^\delta}{6} + \int_0^{H^\delta} (Z^k_s)^* (dB_s - \sigma_s(k^0_s) \alpha_s ds)
\]

Similarly to the subsolution case, there exist \( \mathbb{P} \in \mathcal{P}_{L_0} \) equivalent to \( \mathbb{P}^{0,0,k_0} \) such that the last integral is a \( \mathbb{P} \) martingale and by assumption \( \phi \in \mathcal{X}^{L_0} u^0(0,0) \). Therefore

\[
0 \geq \mathcal{E}^{L_0}[(\phi - u^0)] \geq \mathbb{E}^p[(\phi - u^0)] \geq \frac{c}{6} \mathbb{E}^p[H] \geq \frac{c}{6} \mathcal{E}^{L_0}[H] > 0
\]

which is impossible.

5 Partial comparison

Following the same method as [10], we will first prove a weaker version of the comparison principle, when, one of the functionals is "smoother". Then we will extend it to general sub/supersolution. In our case the 2RBSDE provides us with a representation formula,
therefore our set of "smoother" functionals is simpler than the one in [10]. Another difference comes from our definition of subsolutions that is only required when the functional does not touch the barrier. Except these points the proofs are the same as the ones in [10]. We define the following classes of "smoother" processes.

**Definition 5.1** Let $u \in \mathbb{L}^0(\Lambda)$, we say that $u$ is in $C^{1,2,-}(\Lambda)$ (respectively, $C^{1,2,+}(\Lambda)$) if:

(i) $u \in \mathcal{U}$ (respectively $u \in \overline{\mathcal{U}}$),

(ii) There exists a sequence of hitting times $\{H_i\}_{i \in \mathbb{N}}$, such that $0 = H_0 \leq H_1 \leq \ldots$ and for all $\omega$ the set $\{i \in \mathbb{N} : H_i(\omega) < T\}$ is finite.

(ii) For all $(t, \omega) \in \Lambda$, $t < T$, and $i$ such that $H_i(\omega) \leq t < H_{i+1}(\omega)$, $u^t,\omega \in C^{1,2}(\Lambda(\omega))$, where $\Lambda(\omega) := \{(s, \tilde{\omega}) : \tilde{\omega} \in \Lambda : H_{i+1}(\tilde{\omega}) > s\}.

**Theorem 5.2** Let $u_1 \in \mathcal{U}$ and $u_2 \in \overline{\mathcal{U}}$ be respectively a viscosity subsolution and a supersolution of (4.25) such that for all $\omega \in \Omega$, $u^1(T, \omega) \leq u^2(T, \omega)$. Assume further that $u_1 \in C^{1,2,-}$ or $u_2 \in C^{1,2,+}$, then for all $(t, \omega) \in \Lambda$

$$u^1(t, \omega) \leq u^2(t, \omega).$$

**Proof** To avoid repeating same arguments as in [10], we will use the same notations as in the proof of partial comparison in [10]. We will only point out the differences.

Remark 4.30 allows us, without loss of generality, to assume that $F$ is non-increasing in $y$. We will prove the statement at $(t, \omega) = (0, 0)$, it is also valid for all intermediate $(t, \omega) \in \Lambda$.

Define $\hat{u} := u^1 - u^2$ and denote by $\{H_i\}_{i \in \mathbb{N}}$ the stopping times given by (5.1). We will first prove that

$$\hat{u}^+(\omega) \leq \mathcal{E}_{H_i(\omega)}^L \left[ (\hat{u}^+_{H_{i+1}^{-}})^{H_{i}^{+}}(\omega) \right].$$

We only prove the inequality for $i = 0$, the proof is valid for all $i$. Assume on the contrary that

$$2Tc := \hat{u}^+_0 - \mathcal{E}_0^L \left[ \hat{u}^+_0 \right] > 0.$$

and define $X \in \mathcal{U}$ by:

$$X : \Lambda \to \mathbb{R},$$

$$X(t, \omega) := (\hat{u})^+(t, \omega) + ct.$$

and $\hat{X} := X1_{[0,H_1]} + X_{H_1-1}[H_1,T]$, $Y := \mathcal{S}^t[\hat{X}]$, $\tau^* := \inf\{s \geq 0 : Y_t = \hat{X}_t\}.$

Similarly as in [10], there exists $\omega^* \in \Omega$ such that $t^* = \tau^*(\omega^*) < H_1(\omega^*)$ and

$$0 < (u^1 - u^2)^{+}(\omega^*) = (u^1 - u^2)^{+}(\omega^*).$$

(5.32)
There are 2 cases to treat.

- Assume that $u^2 \in C^{1,2,+}$. Then, by definition of $C^{1,2,+}$, there exist $i$ such that $H_i(\omega^*) \leq t^* < H_{i+1}(\omega^*)$ and $\phi_i := (u^2)_t(t^*, \omega^*) - ct \in C^{1,2}(A_{H_i}(H_{i+1}))$.

By taking $\delta > 0$ smaller to have $H_\delta^* \leq H_{i+1}^*$ then for all $\tau \in T^{t^*}$ we have:

$$
(u^1)_t(t^*, \omega^*) - \phi \geq E_t^{\omega^*}[X_t^{(t^*, \omega^*)}],
$$

which shows that $\phi \in A_t^{u^1}(t^*, \omega^*)$. Additionally $u^2(t^*, \omega^*) = h(t^*, \omega^*) \geq 0$, so the inequality (5.32) gives that $(u^1)_t(t^*, \omega^*) > h(t^*, \omega^*)$ (this point is the only difference between our proof and the proof in [10]).

which is impossible.

- Assume that $u^1 \in C^{1,2,-}$. This case is the same as the one in [10].

In conclusion,

$$
\hat{u}^+_i(\omega) \leq E^{[H_i]} \left[ \left( \hat{u}^+_{i+1} \right)^{H_i} \right].
$$

Then by the Lemma 5.2 of [10] (which only depends on regularity of $u^1$, $u^2$ and not on their viscosity solution properties), we have that for all $P \in P_L$, it holds that

$$
E^P \left[ \hat{u}^-_i \right] \leq E^{[H_i]} \left[ \hat{u}^+_i \right],
$$

By taking the supremum in $P$ and taking into account the positive sign of the possible jumps of $\hat{u}$ we have that

$$
\hat{u}^+_0 \leq E^{[H]_0} \left[ \hat{u}^+_T \right] = 0
$$

which completes the proof.

6 Stability

In this section, we will prove an extension of Theorem 5.1 of [9] to the PPDE (4.25).
Theorem 6.1 Fix $L > 0$ and for $\varepsilon > 0$, let $(G^\varepsilon, h^\varepsilon, \xi^\varepsilon)$ be a family of data verifying assumptions (2.10) with the same constants $M_0, L_0$ and $\rho_0$ and $u^\varepsilon$ an $L$-subsolution of (4.25). Assume that as $\varepsilon$ goes to 0 the following locally uniform convergences hold:

for all $(t, \omega, y, z, \gamma) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, there exists $\delta$ such that:

$$
(G^\varepsilon)^{t,\omega} \rightarrow G^{t,\omega}, \quad (h^\varepsilon)^{t,\omega} \rightarrow h^{t,\omega}, \quad (\xi^\varepsilon)^{t,\omega} \rightarrow \xi^{t,\omega}, \quad (u^\varepsilon)^{t,\omega} \rightarrow u^{t,\omega},
$$

uniformly on $O^\delta(t, \omega, y, z, \gamma) := \{(s, \omega', y', z', \gamma') \in \Lambda^t \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : d^\infty((s, \omega'), (t, 0)) + |y - y'| + |z - z'| + |\gamma - \gamma'| \leq \delta\}$.

Then $u$ is a viscosity $L$-subsolution of the PPDE (4.25) with data $(G, h, \xi)$.

Remark 6.2 We are not able to prove the stability result when $L$ depends on $\varepsilon$.

Remark 6.3 Except the condition $u(t, \omega) \geq h(t, \omega)$, our definition of viscosity supersolution is the same as the one given in [10]. Therefore their stability result for viscosity supersolutions can directly be applied for the PPDE (4.25).

Proof We will use the same notations as in [9] and only point out the differences. We will prove the viscosity subsolution property at $(0^0, 0)$. We assume that $u(0, 0) > h(0, 0)$, $\phi \in \overline{A}^L u(0, 0)$, and $h \in \mathcal{H}$. The main difference with the proof of Theorem 5.1 in [9] is that we need to take $\varepsilon_\delta > 0$ small enough to have $u^\varepsilon_s > h^\varepsilon_s$ for $s \in [0, H_\delta]$ for all $0 < \varepsilon < \varepsilon_\delta$. Then we have $\phi^\varepsilon_\delta \in \overline{A}^\varepsilon u^\varepsilon(t^*, \omega^*)$ and with our choice of $\varepsilon_\delta$ the process $u^\varepsilon$ does not touch the barrier $h^\varepsilon$. Therefore we can use the viscosity subsolution property of $u^\varepsilon$ for the PPDE with data $(G^\varepsilon, h^\varepsilon, \xi^\varepsilon)$ to obtain the equation (5.3) in [9] and conclude.

7 Comparison

Our objective in this section is to extend the partial comparison result. We will carry out the proof in a similar way as in [10], and for $0 \leq t_1 < t_2 \leq T$, $\zeta \in \Lambda^0(\mathcal{F}_{t_2})$ and $\omega \in \Omega$, define the following sets:

$$
\overline{D}(t, \omega) := \{\phi \in C^{1,2,+}(\Lambda^t) ; \min\{\mathcal{L}^{t,\omega} \phi_s, \phi_s - h_{s}^{t,\omega}\} \geq 0, \phi_T \geq \xi_T^{t,\omega}\},
$$

$$
\underline{D}(t, \omega) := \{\psi \in C^{1,2,-}(\Lambda^t) ; \min\{\mathcal{L}^{t,\omega} \psi_s, \psi_s - h_{s}^{t,\omega}\} \leq 0, \psi_T \leq \xi_T^{t,\omega}\}.
$$

and processes:

$$
\underline{\pi}(t, \omega) := \inf\{\phi(t, 0) : \phi \in \overline{D}(t, \omega)\},
$$

$$
\underline{\gamma}(t, \omega) := \sup\{\psi(t, 0) : \psi \in \underline{D}(t, \omega)\}.
$$
Lemma 7.1 Under Assumptions (2.10), (2.11), the equality $\overline{u} = \underline{u}$ holds.

Proof The proof of this lemma is very technical and requires the introduction of various notations. The construction of the smooth approximating subsolutions and supersolutions are the subject of Appendix A. In the Appendix B, we prove the required regularity of those approximating sequences.

Theorem 7.2 Assume (2.10), and (2.11), and let $u^1 \in \mathcal{U}$ (respectively, $u^2 \in \overline{\mathcal{U}}$) a viscosity subsolution (respectively, supersolution) of (4.25), such that $u^1(T,\omega) \leq \xi(\omega) \leq u^2(T,\omega)$ for all $\omega \in \Omega$, then $u^1(t,\omega) \leq u^2(t,\omega)$ for all $(t,\omega) \in \Lambda$.

Proof For all $(t,\omega) \in \Lambda$, and $\psi, \phi$ belonging respectively to $\mathcal{D}(t,\omega)$ and $\overline{\mathcal{D}}(t,\omega)$, by partial comparison result, $u^1(t,\omega) \leq \phi(t,0)$ and $\psi(t,0) \leq u^2(t,\omega)$. We take the supremum in $\psi$ and the infimum in $\phi$ to have $u^1(t,\omega) \leq \overline{\pi}(t,\omega)$ and $\underline{u}(t,\omega) \leq u^2(t,\omega)$. The lemma 7.1 gives the the equality $\overline{u}(t,\omega) = \underline{u}(t,\omega)$, therefore:

$$u^1(t,\omega) \leq u^2(t,\omega)$$

A Appendix A

In the following 2 subsections we will construct 2 families of processes $\{\Psi^m, \alpha\}_{\alpha>0, m \in \mathbb{N}} \in \mathcal{D}(0,0)$ and $\{\Phi^\alpha\}_{\alpha>0} \in \overline{\mathcal{D}}(0,0)$ that will allow us to show the Lemma 7.1. We will adopt the following strategy to prove the equality $\underline{u} = \overline{u} = u^0$. We will freeze the data of the problem $(F,h,\xi)$, in regions of $\Lambda$ related to the stopping times $H^*_t$. Then, we will show that the functionals defined as the solutions of the problem with frozen data are stepwise Markovian. This will bring us to a PDE problem. The proposition 8.2 of [10] allows us to construct smooth approximation to the solutions of the frozen PDE.

We recall that, for the comparison result, $\sigma$ does not depend on $(t,\omega)$, and the assumptions on the data allows us to claim that

$$c_0 := \inf_{k \in \mathcal{K}} \inf_{|\xi| = 1} \xi^* \sigma(k) \xi > 0$$

(1.36)

and $F$ is uniformly continuous in $(t,\omega)$ with modulus $\rho_0$. Additionally, recall that Remark (4.6) allows us to assume without loss of generality that $F,G$ and $h$ verify (4.30).
We will need the following definitions to carry out this construction. For \( \alpha > 0 \) (that will go to 0) and \( t \in [0, T) \), we define:

\[
\mathcal{O}_\alpha := \{ x \in \mathbb{R}^d : |x| < \alpha \}, \quad \overline{\mathcal{O}}_\alpha := \{ x \in \mathbb{R}^d : |x| \leq \alpha \}, \quad \partial \mathcal{O}_\alpha := \{ x \in \mathbb{R}^d : |x| = \alpha \}
\]

\[
\mathcal{O}_\alpha^t := [t, (t + \alpha) \wedge T) \times \mathcal{O}_\alpha, \quad \overline{\mathcal{O}}_\alpha^t := [t, (t + \alpha) \wedge T] \times \overline{\mathcal{O}}_\alpha, \quad \partial \mathcal{O}_\alpha^t := ((t, (t + \alpha) \wedge T] \times \partial \mathcal{O}_\alpha) \cup \{(t + \alpha) \wedge T) \times \mathcal{O}_\alpha\}.
\]

For \( \{t_i\}_{i\geq 0} \) a nondecreasing sequence in \([0, T]\) with \( t_0 = 0 \), \( \sup_i \{t_{i+1} - t_i\} \leq \alpha \), and \( \{x_i\}_{i\geq 0} \) a sequence in \( \overline{\mathcal{O}}_\alpha \) with \( x_0 = 0 \) and \( n \geq 0 \), we denote \( \pi_n := \{(t_i, x_i)\}_{0\leq i\leq n} \). In the sequel \( \pi_n \) will always verify the previous properties. The sequence \( \{t_i\} \) will represent the successive hitting times of a given level by the canonical process, and \( \{x_i\} \) the direction of variation of between the hitting times. For such \( \pi_n \), and \( (t, x) \in \mathcal{O}_\alpha^t \), we define:

\[
\begin{align*}
\hat{h}_{t-1, \alpha}^l &:= t_n, \\
\hat{h}_0^{x, \alpha} &:= \inf\{s \geq t : |x + B^l_s| = \alpha\} \wedge (t_n + \alpha) \wedge T, \quad \text{and for } i \geq 0, \\
\hat{h}_i^{x, \alpha} &:= \inf\{s \geq \hat{h}_i^{x, \alpha} : |B^l_s - B_{\hat{h}_i^{x, \alpha}}^l| = \alpha\} \wedge (\hat{h}_i^{x, \alpha} + \alpha) \wedge T.
\end{align*}
\]

Notice that we can associate to \( \pi_n \) a path \( \hat{\omega}_n \in \Omega \), which is the linear interpolation of \( (t_i, \sum_{j=0}^i x_j)_{0 \leq i \leq n} \) and \( (T, \sum_{j=0}^n x_j) \) and we can associated to \( \pi_n \), \( (t, x) \in \overline{\mathcal{O}}_\alpha^t \) and a path \( \omega \in \Omega^t \) a path \( \hat{\omega}_n^{\pi_n, t, x, \alpha} \in \Omega \), the linear interpolation of \( (t_i, \sum_{j=0}^i x_j)_{0 \leq i \leq n} \) and of \( (\hat{h}_i^{x, \alpha}(\omega), \sum_{j=0}^n x_j + x + \omega_{\hat{h}_i^{x, \alpha}(\omega)})_{i \geq 0} \). We remark that \( \hat{\omega}_n^{\pi_n, t, x, \alpha} \) is not adapted to the filtration \( (\mathbb{F}^i_s)_{0 \leq s \leq T} \). Indeed to know the value of \( \hat{\omega}_n^{\pi_n, t, x, \alpha} \) after the date \( \hat{h}_i^{x, \alpha}(\omega) \) we need to know the value of \( \omega \) at the date \( \hat{h}_i^{x, \alpha}(\omega) \). However the discretization in time allows us to define the approximated data \( \{1.40\} \) as constant between these hitting times. Thus, the data in \( \{1.40\} \) are adapted. For \( (t, x) \in \overline{\mathcal{O}}_\alpha^t \), the notation \( \pi_n^{t, x} \) means that we add \( (t, x) \) to the sequence \( \pi_n \) as \((n + 1)\)th element, namely \( \pi_n^{t, x} = \{\pi_n, (t, x)\} \).

We will construct \( \{\Psi^{m, \alpha}\} \) and \( \{\Phi^{\alpha}\} \) by approximating the data of the problem by data with constant coefficients between some hitting times of \( B \).

For \( \pi_n \), and \( (t, x) \in \overline{\mathcal{O}}_\alpha^t \), we define the following generator, final condition and barrier for the approximated equations:

\[
\begin{align*}
\hat{F}^{\pi_n, t, x, \alpha} &:= \Lambda^t \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}, \\
\hat{h}^{\pi_n, t, x, \alpha} &:= \Lambda^t \times \mathbb{R} \to \mathbb{R}, \\
\hat{\xi}^{\pi_n, t, x, \alpha} &:= \Omega^t \to \mathbb{R}.
\end{align*}
\]
If \((s, \omega) \in \Lambda_t\), with \(h_i^{t,x,\alpha}(\omega) \leq s < h_{i+1}^{t,x,\alpha}(\omega)\):

\[
\hat{F}_{i,n}^{t,x,\alpha}(s, \omega, y, z, k) = F(h_i^{t,x,\alpha}(\omega), \hat{\omega}_{i,n}^{t,x,\alpha}(s, \omega, y, z, k)),
\]
\[
\hat{h}_{i,n}^{t,x,\alpha}(s, \omega) = h(h_i^{t,x,\alpha}(\omega), \hat{\omega}_{i,n}^{t,x,\alpha}(s, \omega)),
\]
\[
\hat{\xi}_{i,n}^{t,x,\alpha}(\omega) = \xi(\hat{\omega}_{i,n}^{t,x,\alpha}(s, \omega)).
\]

There are 3 important features of this approximation:

- The approximated generator and barrier are still adapted to \(F\).
- They verify the assumptions of [12], therefore we can use the results on RBSDEs.
- Their difference from the original data is less than \(\rho_0(2\alpha)\).

The idea which consists in approximating the data and studying the RBSDE with the approximated data can not allow us to construct a sequence of subsolutions in \(D(t, \omega)\). Indeed the barrier for the approximated problem might have negative jumps therefore the construction produce subsolutions which might not be in \(U\) and we would not be able to use this sequence in partial comparison. However this idea allows to produce supersolutions. Indeed the solutions of the RBSDEs can only have negative jumps which does not generate any problem for supersolutions in partial comparison.

### A.1 Construction of subsolutions by penalization

In this subsection, we will construct \(\{\Psi_{m,\alpha}\}_{m>0, \alpha>0} \in D(0, 0)\). The construction will be done by penalization. It is worth noticing that this approach might even work for more general control problem such as stochastic target problems. However in this more general case, in a non-Markovian framework, the value functional might not be continuous, therefore we don’t expect a characterization of the value functional as a unique solution of a PPDE but only as the minimal supersolution (i.e. \(\underline{u} = u^0\)).

Fix \(\alpha > 0\), \(n, m \in \mathbb{N} - \{0\}\), \(\pi_n\) as previously, \((t, x) \in \mathcal{O}_{t_0}^t, k \in \mathcal{K}_t\), and define \(\mathbb{P}^{t,k}\) as follows:

\[
dX_{s,k}^t = \sigma(k_s)dB_s^t, \quad \text{under } \mathbb{P}_0^t
\]
\[
X_t^{t,k} = 0,
\]
\[
\text{and } \mathbb{P}^{t,k} := \mathbb{P}_0^t \circ (X^{t,k})^{-1}.
\]

Consider \((Y_{s}^{\pi,n,t,x,\alpha,k,m}, Z_{s}^{\pi,n,t,x,\alpha,k,m})_{s \in [t,T]}\) (denoted \((Y_s, Z_s)\) for simplicity), the solution
of the following BSDE under $\mathbb{P}^{t,k}$:
\[
\mathcal{Y}_s = \xi_{\pi_n,t,x,\alpha}(B^t) - \int_s^T \mathcal{Z}_r \sigma^{-1}(\mathbf{k}_r) dB_r^t + \int_s^T \mathcal{F}_r \mathcal{Y}_r, \mathcal{Z}_r, \mathbf{k}_r \right) + m(\mathcal{Y}_r - \mathbf{h}_r)dr,
\]
and define
\[
\theta_{n,m}(\pi_n; t, x) := \sup_{k \in K} \mathcal{Y}_{t,k}^{\pi_n,t,x,\alpha,k,m}.
\]

Notice that, a priori, we don’t know anything on the regularity of this function $\theta_{n,m,\alpha}$. We will prove that it is uniformly continuous with modulus of continuity depending only on $\alpha, m, d$ and bounds on the data of the initial problem and it is a viscosity solution of
\[
-\partial_t \theta_{n,m}(\pi_n; \cdot) - G(t, \pi_n, \theta_{n,m}(\pi_n; \cdot), \partial_x \theta_{n,m}(\pi_n; \cdot), \partial_{xx} \theta_{n,m}(\pi_n; \cdot)) - m(\theta_{n,m}(\pi_n; \cdot) - h(t, \pi_n)) = 0, \quad \text{on } \mathcal{O}_t^{\alpha},
\]
\[
\theta_{n,m}(\pi_n; t, x) = \theta_{n+1,m}(\pi_n(t,x); t, 0), \quad \text{for all } (t, x) \in \partial \mathcal{O}_t^{\alpha}.
\]

We first notice that if $(t, x) \in \partial \mathcal{O}_t^{\alpha}$ then $H_t^{t,x,\alpha} = t$. Therefore $\hat{\omega}_{\pi_n,t,x,\alpha} = \hat{\omega}_{\pi_n(t,x); t,0,\alpha}$ for all $\omega \in \Omega^t$, which implies the equality of the data defining $(\mathcal{Y}_{\pi_n,t,x,\alpha,k,m}, \mathcal{Z}_{\pi_n,t,x,\alpha,k,m})$ and $(\mathcal{Y}_{\pi_n(t,x),t,0,\alpha,k,m}, \mathcal{Z}_{\pi_n(t,x),t,0,\alpha,k,m})$. Therefore:
\[
\theta_{n,m}(\pi_n; t, x) = \theta_{n+1,m}(\pi_n(t,x); t, 0), \quad \text{for all } (t, x) \in \partial \mathcal{O}_t^{\alpha}.
\]

Our main difficulty in the rest of the paper is the fact that the stopping times $\{\mathbf{h}_i^{t,x,\alpha}\}$ does not depend continuously on $(t, x)$. However the regularization effect of the PDE allows us to prove the following result.

**Proposition A.1** For all $\alpha > 0$, $n, m \in \mathbb{N} - \{0\}$, $\pi_n$ as previously, the mapping
\[
(t, x) \in \partial \mathcal{O}_t^{\alpha} \rightarrow \theta_{n+1,m}(\pi_n(t,x); t, 0) = \theta_{n,m}(\pi_n; t, x)
\]
is uniformly continuous with modulus of continuity depending only on $d, m, \alpha, L_0, c_0, M_0, \rho_0, T$.

**Proof** The proof this proposition and the proposition [A.3] is in the Appendix B. ■

Given this result, and the proposition 5.14 of [20] we can represent $\theta_{n,m,\alpha}(\pi_n; \cdot)$ as the supremum of solutions of BSDEs with a final condition $\theta_{n,m,\alpha}(\pi_n; H_0^{t,x,\alpha}, B_0^{t,x,\alpha})$. Define $(\mathcal{Y}_{\pi_n,t,x,\alpha,k,m}, \mathcal{Z}_{\pi_n,t,x,\alpha,k,m})$ the solution of the following BSDE under $\mathbb{P}^{t,k}$:
\[
\mathcal{Y}_{\pi_n,t,x,\alpha,k,m} = \mathcal{Y}_s = \theta_{n,m,\alpha}(\pi_n; H_0^{t,x,\alpha}, B_0^{t,x,\alpha}) - \int_s^T \mathcal{Z}_r \sigma^{-1}(\mathbf{k}_r) dB_r^t + \int_s^T F(t, \pi_n, \mathbf{Y}_r, \mathbf{Z}_r, \mathbf{k}_r) + m(\mathbf{Y}_r - h(t, \pi_n))dr,
\]
and by Proposition 5.14 of [20], $\theta_{n}^{m,\alpha}(\pi_{n}; t, x) = \sup_{\pi \in \mathcal{K}_{t}} \psi_{n}^{\pi_{n},t,x,\alpha,k,m}$, which comes from a classical Markovian 2BSDE. Therefore by the uniqueness of viscosity solutions to PDEs in the class of locally bounded functions $\theta_{n}^{m}(\pi_{n}; \cdot)$ verifies (see chapter 5 of [20] on Markovian 2BSDEs):

$$-\partial_{t}\theta_{n}^{\alpha,m}(\pi_{n}; \cdot) - G(t_{n}, \hat{\pi}_{n}, \theta_{n}^{\alpha,m}(\pi_{n}; \cdot), \partial_{x}\theta_{n}^{\alpha,m}(\pi_{n}; \cdot), \partial_{xx}\theta_{n}^{\alpha,m}(\pi_{n}; \cdot)), (1.45)$$

$$-m(\theta_{n}^{\alpha,m}(\pi_{n}; \cdot) - h(t_{n}, \hat{\pi}_{n}))^{-} = 0, \text{ for all } (t, x) \in \mathcal{O}_{t_{n}}, (1.46)$$

$$\theta_{n}^{m}(\pi_{n}; t, x) = \theta_{n+1}^{m}(\pi_{n}(t,x); t, 0), \text{ for all } (t, x) \in \partial\mathcal{O}_{t_{n}}. (1.47)$$

We want to construct a process in $\mathcal{C}^{1,2,-}$. The functions $\theta_{n}^{\alpha,m}$ might not be $C^{1,2}$. However, the PDE (1.45) verifies the assumptions of the Proposition 8.2 of [10]. Thus, we can approximate the viscosity solution of this PDE with smooth subsolutions. In this approximation we will lose the continuity at dates $\{H_i\}$. However we can manage to have positive jumps which is consistent with the definition of $C^{1,2,-}$. We define 2 other sequences of functions, $\tilde{\theta}_{n}^{\alpha,m}$ and $\tilde{\theta}_{n}^{\alpha,m}$.

Define $\tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot)$ as the unique viscosity solution of

$$-\partial_{t}\tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot) - G(t_{n}, \hat{\pi}_{n}, \tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot), \partial_{x}\tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot), \partial_{xx}\tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot))$$

$$-m(\tilde{\theta}_{n}^{\alpha,m}(\pi_{n}; \cdot) - h(t_{n}, \hat{\pi}_{n}))^{-} = 0, \text{ for all } (t, x) \in \mathcal{O}_{t_{n}},$$

$$\tilde{\theta}_{n}^{m}(\pi_{n}; t, x) = \theta_{n+1}^{m}(\pi_{n}(t,x); t, 0) - \frac{\alpha}{2n}, \text{ for all } (t, x) \in \partial\mathcal{O}_{t_{n}}. (1.47)$$

Notice that the operator $G(t_{n}, \hat{\pi}_{n}, u, \partial_{x}u, \partial_{xx}u) + m(u - h(t_{n}, \hat{\pi}_{n}))^{-}$ is decreasing in $u$. Thus $\theta_{n}^{m}(\pi_{n}; t, x) - \frac{\alpha}{2n} \leq \tilde{\theta}_{n}^{m}(\pi_{n}; t, x) \leq \theta_{n}^{m}(\pi_{n}; t, x)$ for all $(t, x) \in \mathcal{O}_{t_{n}}^{*}$.

Additionally under the condition $c_{0} > 0$ the mapping

$$(y, z, \gamma) \rightarrow G(t_{n}, \pi_{n}, y, z, \gamma)$$

is convex and uniformly non-degenerate in $\gamma$, therefore, the Proposition (8.2) of [10] gives the existence of

$$\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot) \in C^{1,2}(\mathcal{O}_{t_{n}}^{\alpha}) \cap C(\overline{\mathcal{O}_{t_{n}}^{\alpha}}),$$

$$-\partial_{t}\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot) - G(t_{n}, \hat{\pi}_{n}, \tilde{\theta}_{n}^{m}(\pi_{n}; \cdot), \partial_{x}\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot), \partial_{xx}\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot))$$

$$-m(\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot) - h(t_{n}, \hat{\pi}_{n}))^{-} \leq 0, \text{ for all } (t, x) \in \mathcal{O}_{t_{n}},$$

$$\tilde{\theta}_{n}^{m}(\pi_{n}; t, x) \leq \tilde{\theta}_{n+1}^{m}(\pi_{n}(t,x); t, 0), \text{ for all } (t, x) \in \partial\mathcal{O}_{t_{n}},$$

$$\tilde{\theta}_{n}^{m}(\pi_{n}; \cdot) - \frac{\alpha}{2n} \leq \tilde{\theta}_{n}^{m}(\pi_{n}; \cdot) \leq \tilde{\theta}_{n}^{m}(\pi_{n}; \cdot), \text{ for all } (t, x) \in \mathcal{O}_{t_{n}}^{\alpha}$$

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For all \((t, x) \in \partial \Omega_{t_m}^\alpha\),

\[
\tilde{\theta}^\alpha_m(\pi_n; t, x) \leq \tilde{\theta}^\alpha_n(\pi_n; t, x) = \tilde{\theta}^\alpha_{n+1}(\pi_n(t, x); t, 0) - \frac{\alpha}{2^n} \leq \tilde{\theta}^\alpha_{n+1}(\pi_n(t, x); t, 0) + \frac{\alpha}{2^{n+1}} - \frac{\alpha}{2^n} \leq \tilde{\theta}^\alpha_{n+1}(\pi_n(t, x); t, 0)
\]

Therefore \(\tilde{\theta}^\alpha_m(\pi_n; .)\) is a subsolution of

\[
\min\{-\partial_t \tilde{\theta}^\alpha_m(\pi_n; .) - G(t_n, \hat{\pi}_n, \tilde{\theta}^\alpha_m(\pi_n; .), \partial_x \tilde{\theta}^\alpha_m(\pi_n; .), \partial_{xx} \tilde{\theta}^\alpha_m(\pi_n; .)), \tilde{\theta}^\alpha_n(\pi_n; .) - h(t_n, \hat{\pi}_n)\} \leq 0 \quad \text{for all} \quad (t, x) \in \partial \Omega_{t_m}^\alpha,
\]

\[
\tilde{\theta}^\alpha_n(\pi_n; t, x) \leq \tilde{\theta}^\alpha_{n+1}(\pi_n(t, x); t, 0) \quad \text{for all} \quad (t, x) \in \partial \Omega_{t_m}^\alpha.
\]

We can now define the process \(\{\Psi^{m,\xi}\}\). \(\Pi^{0,0,\alpha}_i\) will be denoted by \(H_i\). For \((s, \omega) \in \Lambda\) with \(H_n(\omega) \leq s < H_{n+1}(\omega)\), \(\pi_n(\omega)\) will stand for \(\{(H_i(\omega), \omega_{H_i(\omega)} - \omega_{H_{i-1}(\omega)})\}_{0 \leq i \leq n}\), and we define:

\[
\Psi^{m,\alpha}(s, \omega) = \tilde{\theta}^\alpha_n(\pi_n(\omega); s, \omega_{H_n(\omega)} - \rho_0(\alpha)).
\]

We now show that ,with the associated sequence of stopping times \(\{H_i\}\), \(\Psi^{m,\xi} \in C^{1,2-}\).

The definition of \(\{\Psi^{m,\alpha}\}\) gives that if \((t, \omega) \in \Lambda\) with \(\omega_{H_n(\omega)} \leq s < \omega_{H_{n+1}(\omega)}\), then denoting

\[
P := \partial_t \tilde{\theta}^\alpha_n(\pi_n(\omega); t, \omega_t - \omega_{H_n(\omega)}),
\]

\[
Q := \tilde{\theta}^\alpha_n(\pi_n(\omega); t, \omega_t - \omega_{H_n(\omega)}),
\]

\[
R := \partial_x \tilde{\theta}^\alpha_n(\pi_n(\omega); t, \omega_t - \omega_{H_n(\omega)}),
\]

\[
S := \partial_{xx} \tilde{\theta}^\alpha_n(\pi_n(\omega); t, \omega_t - \omega_{H_n(\omega)})
\]

\[
\min\{-\partial_t \Psi^{m,\alpha}(t, \omega) - G(t, \omega, \Psi^{m,\alpha}(t, \omega), \partial_\omega \Psi^{m,\alpha}(t, \omega), \partial_{\omega\omega} \Psi^{m,\alpha}(t, \omega)), h(t, \omega)\}
\]

\[
\Psi^{m,\alpha}(t, \omega) - h(t, \omega)
\]

\[
\min\{-P - G(t, \omega, Q - \rho_0(\alpha), R, S), Q - \rho_0(\alpha) - h(t, \omega)\}
\]

\[
\leq \min\{-P - G(t, \omega, Q, R, S) - \rho_0(\alpha), Q - h(\omega_{H_n(\omega)}, \hat{\pi}_n(\omega))\}
\]

\[
\leq \min\{-P - G(H_n(\omega), \pi_n(\omega), Q, R, S), P - h(H_n(\omega), \hat{\pi}_n(\omega))\} \leq 0.
\]

which shows that \(\Psi^{m,\alpha} \in \mathcal{D}(0, 0)\).

**A.2 Construction of supersolutions by approximation**

Fix \(\alpha > 0\), \(\pi_n\), for \((t, x) \in \overline{\Omega}_{t_m}^\alpha\) and \(k \in K^i\), our approximated data defined at (1.37) verifies the assumption of [12], therefore we have the existence of \((\hat{Y}_{\pi_n,t,x,\alpha,k}^\tau, \tilde{Z}_{\pi_n,t,x,\alpha,k}^\tau, \tilde{K}_{\pi_n,t,x,\alpha,k}^\tau)_{s \in [t, T]}\)
(we drop the superscript \( \pi_n, t, x, \alpha, k \) for simplicity of notation) solution of the following RB-SDE under \( \mathbb{P}^{t,k} \):

\[
\hat{Y}_s = \xi_{\pi_n,t,x,\alpha}(B_t) + \int_s^T \hat{F}^\pi_{\nu_{\pi_n,t,x,\alpha}}(\hat{Y}_r, \hat{Z}_r, b_r)dr - \int_s^T \hat{Z}^\pi_{\nu_{\pi_n,t,x,\alpha}}(b_r)dB^t_r + \hat{K}_T - \hat{K}_s,
\]

\[
\hat{Y}_s \geq \hat{h}_{\pi_{\nu_{\pi_n,t,x,\alpha}}}, \quad |\hat{Y}_s - \hat{h}_{\pi_{\nu_{\pi_n,t,x,\alpha}}}|d\hat{K}^c = 0,
\]

\[
\Delta_s\hat{Y} := \hat{Y}_s - \hat{Y}_{s-} = -\left(\hat{h}_{\pi_{\nu_{\pi_n,t,x,\alpha}}} - \hat{Y}_s\right)^+, \quad \hat{K} \text{ non decreasing}, \quad \hat{K}_t = 0.
\]

And similarly we define the following mapping:

\[
\Gamma^c_n(\pi_n; t, x) := \sup_{k \in \mathcal{K}} \hat{Y}_{t,\pi_n,t,x,\alpha,k}.
\]

Notice that if \( \Delta_s\hat{Y} > 0 \) then \( \Delta_s\hat{h}_{\pi_{\nu_{\pi_n,t,x,\alpha}}} > 0 \). Therefore the jumps of \( \hat{Y} \), which are the jumps of the discontinuous part \( \hat{K}^d \) of \( \hat{K} \), can only happen when there is a jump of \( \hat{h}_{\pi_{\nu_{\pi_n,t,x,\alpha}}} \), those possible jump dates are \( \{\mathcal{H}^{t,x,\alpha}_i\} \).

In the literature, there are some estimates of \( d\hat{K}^c \), the continuous part of \( \hat{K} \) when the barrier is a continuous semimartingale. In that case it can be shown that \( 0 \leq d\hat{K}^c_s \leq (\hat{F}^\pi_{\nu_{\pi_n,t,x,\alpha}}(\hat{Y}_s, \hat{Z}_s, \hat{b}_s)ds + dA_s)^- \) where \( A \) is the drift part of the barrier, (notice that in our case \( dA_s = 0 \), excepts at \( \{\mathcal{H}^{t,x,\alpha}_i\} \)). We will extend this result to our case.

### A.2.1 Study of \( K^c \)

At this subsection, we will study the RBSDE defined at (1.50) under \( \mathbb{P}^{t,k} \). We again drop the superscript \( \pi_n, t, x, \alpha, k \) for notational simplicity. We denote by \( (\hat{F}, \hat{\xi}, \hat{h}) \) the data and by \( (\hat{Y}, \hat{Z}, \hat{K}) \) the solution. Recall that by the Remark 1.7 we can assume that \( \hat{F} \) and \( \hat{h} \) verifies (1.30). We then have the following proposition:

**Proposition A.2** Under assumptions (2.10),

\[
\hat{K}^c \equiv 0, \quad \mathbb{P}^{t,k}\text{-a.s.}
\]

**Proof** We differentiate \( (\hat{Y} - \hat{h}) \) in 2 different ways under \( \mathbb{P}^{t,k} \):

\[
d(\hat{Y}_s - \hat{h}_s) = -\hat{F}_s ds - d\hat{K}^c_s - d\hat{K}^d_s - d\hat{h}_s + \hat{Z}^*_s\sigma^{-1}(b_r)dB^t_r,
\]

where \( \hat{F}_s := \hat{F}^{\pi_{\nu_{\pi_n,t,x,\alpha}}}_{\nu_{\pi_n,t,x,\alpha}}(\hat{Y}_s, \hat{Z}_s, \hat{b}_s) \), and \( \hat{h}_s = \hat{h}_{\nu_{\pi_n,t,x,\alpha}}(s, B^t) \). The processes \( \hat{K}^c \) and \( \hat{K}^d \) have integrable variation. Between two successive \( \mathcal{H}^{t,x,\alpha}_i \) (denoted \( \mathcal{H}_i \) for simplicity), \( \hat{h} \) is constant, therefore the variation of \( \hat{h}(\omega) \) is bounded by \( 2(N(\omega) + 1)\rho_0(\varepsilon) < \infty \) for all \( \omega \in \omega' \), where \( N(\omega) = \inf\{i \in \mathbb{N} : \mathcal{H}_i(\omega) = T\} < \infty \). Additionally \( \hat{h} \) is cadlag and constant between
the terms of \( \{H_i\} \), so \( \hat{Y} - \hat{h} \) is a semimartingale, denoting \( L^0 \) its local time at 0, by the Itô-Meyer formula:

\[
\begin{align*}
  d(\hat{Y}_s - \hat{h}_s) &= d(\hat{Y}_s - \hat{h}_s)^+ = \\
  1_{\hat{Y}_s > \hat{h}_s} (-F_s ds - d\hat{K}_s^d - d\hat{h}_s + \hat{Z}_s^* \sigma^{-1}(\hat{k}_s)dB_s^1) & \quad \text{(1.52)} \\
  + \Delta_s(\hat{Y} - \hat{h})^+ - 1_{\hat{Y}_s > \hat{h}_s} \Delta_s(\hat{Y}_s - \hat{h}_s) + \frac{1}{2} L_s^0 & \quad \text{(1.53)} \\
  = 1_{\hat{Y}_s > \hat{h}_s} (-\hat{F}_s ds - d\hat{K}_s^d - d\hat{h}_s + \hat{Z}_s^* \sigma^{-1}(\hat{k}_s)dB_s^1) & \quad \text{(1.54)} \\
  + 1_{\hat{Y}_s = \hat{h}_s} \Delta_s(\hat{Y}_s - \hat{h}_s) + \frac{1}{2} L_s^0. & \quad \text{(1.55)}
\end{align*}
\]

In the previous equality, we used 0 = (\( \hat{Y}_s - \hat{h}_s \))d\( \hat{K}_s^c \) = 1_{\{\hat{Y}_s > \hat{h}_s\}} d\( \hat{K}_s^c \) to eliminate the term 1_{\hat{Y}_s > \hat{h}_s} d\( \hat{K}_s^c \). Regrouping terms:

\[
\begin{align*}
  1_{\hat{Y}_s = \hat{h}_s} (-\hat{F}_s ds - d\hat{K}_s^d - d\hat{h}_s + \hat{Z}_s^* \sigma^{-1}(\hat{k}_s)dB_s^1) & = \\
  1_{\hat{Y}_s = \hat{h}_s} \Delta_s(\hat{Y}_s - \hat{h}_s) + \frac{1}{2} L_s^0.
\end{align*}
\]

We define the set \( \mathcal{J} := \{(s, \omega) : s \neq H_i(\omega) \text{ for all } i\} \) and notice that on \( \mathcal{J} \), \( d\hat{h}_s = d\hat{K}_s^d = \Delta_s(\hat{Y} - \hat{h}) = 0 \), so \( \hat{Y}_s = \hat{Y}_s^- \) and \( \hat{h}_s = \hat{h}_s^- \). By rewriting the previous equality on \( \mathcal{J} \):

\[
1_{\hat{Y}_s = \hat{h}_s} \hat{Z}_s^* \sigma^{-1}(\hat{k}_s)dB_s^1 = d\hat{K}_s^c + 1_{\hat{Y}_s = \hat{h}_s} (\hat{F}_s ds) + \frac{1}{2} L_s^0. \quad \text{(1.57)}
\]

The right term is predictable finite variation and the left term defines a martingale. Therefore on the set \( \{\hat{Y}_s = \hat{h}_s\} \cap \mathcal{J} \), we have:

\[
\hat{Z}_s = 0, \text{ and} \]

\[
0 \leq d\hat{K}_s^c \leq \hat{F}_s^- (\hat{h}_s, \hat{Z}_s, \hat{k}_s) ds = \hat{F}_s^- (\hat{h}_s, 0, \hat{k}_s) ds.
\]

Notice that for \( \omega \in \Omega^f \) using the Remark 4.17:

\[
\hat{F}_s(\hat{h}_s, 0, \hat{k}_s) = \hat{F}_s^{\pi_n, t, x, \alpha}(\hat{h}_s, \pi_n, t, x, \alpha(s, \omega), 0, \hat{k}_s) \\
= F(s, \hat{\omega}^{\pi_n, t, x, \alpha}, h(s, \hat{\omega}^{\pi_n, t, x, \alpha}), 0, \hat{k}_s) \geq 0.
\]

So \( d\hat{K}_s^c = 0 \) on \( \{\hat{Y}_s = \hat{h}_s\} \cap \mathcal{J} \).

On \( \{\hat{Y}_s \neq \hat{h}_s\} \cap \mathcal{J} \), the equality \( \text{(1.57)} \) directly gives \( d\hat{K}_s^c = 0 \).

Therefore on \( \mathcal{J} \), \( d\hat{K}_s^c = 0 \), \( dt \times \mathbb{P}^{t,k} \)-a.s. which shows that \( \hat{K}^c \) is constant between the \( H_i \) so it is always 0.

\( \blacksquare \)

Then \( \hat{K} = \hat{K}^d \) can only jump at the stopping times \( \{H_i^{t,x,\alpha}\} \).
If we rewrite (1.50) up to $H_{0}^{1,x,\alpha}$ for $s < H_{0}^{1,x,\alpha}$ it becomes (without the superscript $(\pi_{n},t,x,\alpha,k)$) under $\mathbb{P}^{k}$:

$$
\dot{Y}_{s} = \dot{Y}_{h_{0}^{1,x,\alpha}} + \Delta_{h_{0}^{1,x,\alpha}}Y := - (\dot{Y}_{h_{0}^{1,x,\alpha}} - \dot{Y}_{h_{0}^{1,x,\alpha}}) = (\dot{h}_{h_{0}^{1,x,\alpha}} - \dot{Y}_{h_{0}^{1,x,\alpha}})\cdot
$$

Therefore for $s < H_{0}^{1,x,\alpha}$:

$$
\dot{Y}_{s} = \max\{\dot{Y}_{h_{0}^{1,x,\alpha}}; h(t_{n}, \hat{\pi}_{n})\}
+ \int_{s}^{H_{0}^{1,x,\alpha}} F_{s}(\dot{Y}_{s}, \dot{Z}_{s}, \dot{k}_{s}) \, dt
- \int_{s}^{H_{0}^{1,x,\alpha}} Z_{s}^{*} \sigma^{-1}(\dot{k}_{s}) \, dB_{s},
$$

This equation is actually a BSDE up to $H_{0}^{1,x,\alpha}$. We need the following results to continue our analysis.

**Proposition A.3** The mapping $M_{n}^{\alpha}$

$$(t,x) \in \partial O_{t_{n}}^{\alpha} \rightarrow \max\{\Gamma_{n+1}(\pi_{n}^{(t,x)}; t,0) ; h(t_{n}, \hat{\pi}_{n})\} = M_{n}^{\alpha}(\pi_{n};t,x),
$$

is uniformly continuous with modulus of continuity depending only at $d, \alpha, T, c_{0}, M_{0}, L_{0}, \rho_{0}$.

**Proof** The proof of this result is the subject of the next Appendix. 

Given the previous regularity result it is easy to prove that for $(t,x) \in O_{t_{n}}^{\alpha}, \Gamma_{n}^{\alpha}(\pi_{n};t,x)$ can also be represented as the following supremum:

$$
\Gamma_{n}^{\alpha}(\pi_{n}; t, x) = \sup_{\pi_{t,x,\alpha,k}^{\pi_{n}}} \dot{Y}_{\pi_{t,x,\alpha,k}^{\pi_{n}}}
$$

where $(\dot{Y}_{\pi_{t,x,\alpha,k}^{\pi_{n}}}, \dot{Z}_{\pi_{t,x,\alpha,k}^{\pi_{n}}})$ solves for $s < H_{0}^{1,x,\alpha}$:

$$
\dot{Y}_{s} = M_{n}^{\alpha}(\pi_{n}; H_{0}^{1,x,\alpha}, B_{h_{0}^{1,x,\alpha}}) + \int_{s}^{H_{0}^{1,x,\alpha}} F_{s}(\dot{Y}_{s}, \dot{Z}_{s}, \dot{k}_{s}) \, dt
- \int_{s}^{H_{0}^{1,x,\alpha}} Z_{s}^{*} \sigma^{-1}(\dot{k}_{s}) \, dB_{s}.
$$

Similarly to the subsolution case, given the regularity of the boundary condition, we have the following proposition:

**Proposition A.4** Under the assumptions (2.10), for fixed $n$ and $\alpha > 0$, the function $\Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot})$ is uniformly continuous and viscosity solution of the following PDE:

$$
- \partial t \Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot}) - G(t_{n}, \hat{\pi}_{n}, \Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot}), \partial_x \Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot}), \partial_{xx} \Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot})) = 0,
$$

for all $(t,x) \in O_{t_{n}}^{\alpha}$, $\Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot}) = h(t_{n}, \hat{\pi}_{n})$ for all $(t,x) \in O_{t_{n}}^{\alpha}$, and

$$
\Gamma_{n}^{\alpha}(\pi_{n}^{\cdot,\cdot}) = \Gamma_{n+1}(\pi_{n}^{(t,x)}; t,0),
$$

for all $(t,x) \in \partial O_{t_{n}}^{\alpha}$. 

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Similarly to the subsolution case we can find $\tilde{\Gamma}_n^\alpha(\pi_n;.) \in C^{1,2}(\mathcal{O}_{2n}^\alpha)$, supersolution of (1.62), and verifying

$$\Gamma_n^\alpha(\pi_n;.) + \alpha \geq \tilde{\Gamma}_n^\alpha(\pi_n;.) \geq \Gamma_n^\alpha(\pi_n;.)$$

We define $\Phi^\alpha \in C^{1,2,+}$ as in the subsolution case. Similarly, $H_t$ stands for $H_{t,n}^{0,0,\alpha}$, for $(s, \omega) \in \Lambda$ with $H_{n}(\omega) \leq s < H_{n+1}(\omega)$, $\pi_n(\omega)$ stands for $\{(H_t(\omega), \omega_{h_t(\omega)} - \omega_{h_{t-1}(\omega)})\}_{0 \leq i \leq n}$, and we define :

$$\Phi^\alpha(s, \omega) := \tilde{\Gamma}_n^\alpha(\pi_n(\omega); s, \omega_s - \omega_{h_n(\omega)}) + \rho_0(\alpha).$$

As it is proven for $\Psi^{m,\alpha}$, using the Remark 4.6, $\Phi^\alpha \in \mathcal{D}(0, 0)$. Finally, we can now prove the lemma 7.1.

**Proof of lemma 7.1** We show the two inequalities separately.

Notice that by partial comparison, for any $\phi \in \mathcal{D}(0, 0)$ and $\psi \in \mathcal{D}(0, 0)$, $\phi(0, 0) \geq \psi(0, 0)$, which by taking the infimum in $\phi$ and supremum in $\psi$ shows that $\underline{u}(0, 0) \geq \overline{u}(0, 0)$. $\Psi^{m,\alpha} \in \mathcal{D}(0, 0)$ and $\Phi^\alpha(0, 0) \in \mathcal{D}(0, 0)$, so

$$\Psi^{m,\alpha}(0, 0) \leq \underline{u}(0, 0) \leq \overline{u}(0, 0) \leq \Phi^\alpha(0, 0).$$

Fix $\delta > 0$ and $\alpha > 0$, then $\Gamma_0^\alpha(\pi_0; 0, 0)$ is the value at 0 of the 2RBSDE with data $(\hat{G}^{0,0,\alpha}, \hat{h}^{0,0,\alpha}, \xi^{0,0,\alpha})$ and $\theta_0^m(\pi_0; 0, 0)$ is the value at 0 of the 2BSDE with generator

$$\dot{G}^{0,0,\alpha}(s, \omega, x, y) - m(y - \hat{h}^{0,0,\alpha})^{-}$$

and final condition $\xi^{0,0,\alpha}$, the convergence of the solutions of the penalized BSDE to the solution of the RBSDE gives that there exists $m_\alpha \in \mathbb{N}$ such that $\theta_0^{m_\alpha}(\pi_0; 0, 0) \geq \Gamma_0^\alpha(\pi_0; 0, 0) - \delta$.

We rewrite these inequalities in terms of $\Psi^{m,\alpha}$ and $\Phi^\alpha$ to have $\Psi^{m,\alpha}(0, 0) + \rho(\alpha) + \alpha \geq \Phi^\alpha(0, 0) - \rho(\alpha) - \alpha - \delta$. By the definition of $\overline{u}$, and $\underline{u}$, this gives

$$\underline{u}(0, 0) + \rho_0(\alpha) + \alpha \geq \overline{u}(0, 0) - \alpha - \rho_0(\alpha) - \delta.$$

We take the limit as $\alpha, \delta$ goes to 0 to have

$$\underline{u}(0, 0) \geq \overline{u}(0, 0).$$

**B Appendix B**

We first prove a lemma on the dependence of the solutions of the approximated problem on $\pi_n$. 

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Lemma B.1 There exist $C > 0$ depending only on $d, L_0, M_0$, and $T$ such that for all $\pi_n = \{(t_i, x_i)\}_{0 \leq i \leq n}$ and $\pi'_n = \{(t'_i, x'_i)\}_{0 \leq i \leq n}$ with $t_n = t'_n,$

$$|\Gamma_n^\alpha(\pi_n'; t_n, 0) - \Gamma_n^\alpha(\pi_n; t_n, 0)| \leq C \rho_0(||\hat{\pi}_n - \hat{\pi}_n'||t_n)$$

holds. For the function $\theta_{m, \alpha}^{n, \alpha}$ the constant $C$ also depend on $m$.

Remark B.2 We have defined $\theta_{m, \alpha}^{n, \alpha}$ with the penalized 2BSDE (see (1.43)). The generator of this 2BSDE is $$\{\hat{F}_{\pi_n, t_n, 0, \alpha}(s, \omega, y, z, k) + m(y - \hat{h}_{\pi_n, t_n, 0, \alpha}(s, \omega))\}_{k \in K^t}$$ whose Lipschitz constant in $y$ might depend on $m$. Therefore in the next lemmas, as it is the case in the previous lemma, the constants $C$ might have an extra dependence in $m$ for the functions $\theta_{m, \alpha}^{n, \alpha}$.

Proof

$\Gamma_n^\alpha(\pi_n'; t_n, 0)$ and $\Gamma_n^\alpha(\pi_n; t_n, 0)$ are defined with the solutions at $t_n$ of RBSDEs with respectively data

$$\{\hat{F}_{\pi_n, t_n, 0, \alpha}(s, \omega, y, z, k), \hat{h}_{\pi_n, t_n, 0, \alpha}(s, \omega), \hat{\xi}_{\pi_n, t_n, 0, \alpha}(\omega)\}$$

and

$$\{\hat{F}_{\pi'_n, t_n, 0, \alpha}(s, \omega, y, z, k), \hat{h}_{\pi'_n, t_n, 0, \alpha}(s, \omega), \hat{\xi}_{\pi'_n, t_n, 0, \alpha}(\omega)\}$$

and the stopping times $\{\tau_i^{t_n, 0, \alpha}\}$ in (1.37) are the same for both of the data. Therefore for all $(s, \omega) \in \Lambda^t$, denoting $\delta = ||\hat{\pi}_n - \hat{\pi}_n'||t_n$, we have:

$$|\hat{F}_{\pi_n, t_n, 0, \alpha}(s, \omega, y, z, k) - \hat{F}_{\pi'_n, t_n, 0, \alpha}(s, \omega, y, z, k)| \leq \rho_0(\delta),\quad (2.63)$$

$$|\hat{h}_{\pi_n, t_n, 0, \alpha}(s, \omega) - \hat{h}_{\pi'_n, t_n, 0, \alpha}(s, \omega)| \leq \rho_0(\delta),\quad (2.64)$$

$$|\hat{\xi}_{\pi_n, t_n, 0, \alpha}(\omega) - \hat{\xi}_{\pi'_n, t_n, 0, \alpha}(\omega)| \leq \rho_0(\delta),\quad (2.65)$$

Given the a priori estimates of RBSDEs and taking the sup in $k \in K^t$, we have that

$$|\Gamma_n^\alpha(\pi_n; t_n, 0) - \Gamma_n^\alpha(\pi_n'; t_n, 0)| \leq C \rho_0(\delta).\quad (2.66)$$

B.0.2 Regularity of the hitting times

To show the regularity of $\Gamma_n^\alpha$ and $\theta_{m, \alpha}^{n, \alpha}$, we cannot rely on the same method as we have used to show the regularity of the value functional $u^0$. Indeed, in order to bring the problem
to the framework of the Markovian RBSDEs, we had to freeze the data (1.40). Because of the lack of continuity of the stopping times \( \{ H_i \} \), the approximated data is not uniformly continuous any more. However the stopping times are continuous on a set of full capacity, and we will be able to claim the uniform continuity of the boundary condition (1.59).

The regularity of \( \theta_n^{\alpha,0} \) is a sub-case of the regularity of \( \Gamma^\alpha \), we only prove the last one. Our first objective is to prove the uniform continuity of \( \Gamma^\alpha (\pi_n;) \) in \( x \) uniformly in \( t \).

For \( t \in [0, T) \), we also define following set of probability and capacity (for which the superscript \( t \) will be omitted),

\[
P_G^t := \{ P^{t,k}; k \in K \}, \quad C_G^t := \sup_{P \in P_G^t} P, \quad \mathcal{E}_{G}^t := \sup_{P \in P_G^t} \mathbb{E}^P [ ] .
\]

Notice that under the additional assumption (2.11), \( \sigma \) does not depend on \( (t, \omega) \), hence the nondegeneracy assumption in (2.10) implies that \( c_0 > 0 \), where \( c_0 \) is defined in (1.36).

We introduce the following hitting times that will simplify our task in giving uniform bounds in \( H_i^{t,x,\alpha} \).

\[
\Pi^{t,x,\alpha,t_0} = \inf \{ s \geq t : |x + B^t_s| = \alpha \} \wedge t_0 \wedge T, \tag{2.67}
\]
\[
H_i^{t,x,\alpha} := H_i^{t,x,\alpha} \vee H_i^{t,0,\alpha}. \tag{2.68}
\]

In giving the estimates on the family \( \{ H_i^{t,x,\alpha} \} \), the hitting times \( \Pi^{t,x,\alpha,t_0} \) will allow us to write \( (H_i^{t,x,\alpha}, H_i^{t,0,\alpha}) \) conditionally on \( (H_i^{t,x,\alpha}, H_i^{t,0,\alpha}) \). However we need to make sure that \( H_i^{t,x,\alpha} \wedge H_i^{t,0,\alpha} \geq H_i^{t,x,\alpha} \vee H_i^{t,0,\alpha} \). We will use Holder continuity to claim this last point. We give the following estimates on the family \( \Pi \) whose proof is based on the proof of Lemma 4.7 in [10].

**Proposition B.3** There exists a constant \( C \) that only depends on \( c_0, T \) such that for all \( \delta \in (0, |t_1 - t_0|) \) it holds that

\[
\mathcal{C}_G^t (| \Pi^{t,0,\alpha,t_0} - \Pi^{t,x,\alpha,t_1} | > \delta) \leq C \frac{|x|}{\sqrt{\delta}} and \tag{2.69}
\]
\[
\mathcal{E}_G^t (| \Pi^{t,0,\alpha,t_0} - \Pi^{t,x,\alpha,t_1} |) \leq |t_1 - t_0| + C |x| \tag{2.70}
\]

**Proof** As in Lemma 4.7 of [10], we fix \( P \in P_G^t \) and define \( A_s := \int_t^s \frac{x}{|x|} \sigma^2 (k_r) \frac{x}{|x|} dr \), \( \tau_s := \inf \{ r \geq 0 : A_r \geq s \} \) and \( M_s := \int_0^s \frac{x}{|x|} dB_r \), which is a \( P \) Brownian motion and \( A \) verifies \( A_s \geq 2c_0 (s-t) \).

Notice that under the constraint \( |t_1 - t_0| > \delta \), we have

\[
\{ \Pi^{t,x,\alpha,t_1} > \Pi^{t,0,\alpha,t_0} + \delta \} \subset \{ \sup_{|x| \leq \Pi^{t,0,\alpha,t_0}} \frac{x}{|x|} \cdot B_s^{t,0,\alpha,t_0} \leq |x| \}
\]
\[
\subset \{ \sup_{|x| \leq \Pi^{t,0,\alpha,t_0} + 2c_0 \delta} (M_s - M_{H_i^{t,0,\alpha,t_0}}) \leq |x| \}
\]
Taking the thee probabilities of the events,
\[ \mathbb{P}(h^{t,x,\alpha}_1 > h^{0,\alpha}_0 + \delta) \leq \mathbb{P}(\sup_{t \leq s \leq 2\alpha t_0} (M_s - M^{t,0,\alpha}_t) \leq |x|) \]
\[ \leq \mathbb{P}(|B_{2\alpha t_0}| \leq |x|) \leq C \frac{|x|}{\sqrt{\delta}} \]

Thus, \( \mathbb{P}(|h^{t,0,\alpha}_t - h^{t,x,\alpha}_t| > \delta) \leq 2C \frac{|x|}{\sqrt{\delta}} \), which gives the 2 bounds. 

We recall that \( h_{i-1}^{t,x,\alpha} = t \).

**Proposition B.4** For \( n > 0 \) define
\[ \Gamma_n^{t,x,\alpha} := \sup_{0 \leq i \leq n} |h^{t,x,\alpha}_i - h^{t,0,\alpha}_0|_{\Gamma_i} \quad \text{and} \]
\[ C_{K,1/3} := \{ \omega \in \Omega^t \, \sup_{t \leq s \leq r \leq T} \left| \frac{\omega_r - \omega_s}{|r - s|^{1/3}} \right| < K \} \]

then for all \( \varepsilon > 0 \) there exists \( K_\varepsilon < \infty \), whose choice is independent in \( t \), such that for all \( \delta > 0 \) small enough and \( n \in \mathbb{N} \) there exists \( q > 0 \), whose choice is independent of \( t \), verifying \( C_G((\Gamma_n^{t,x,\alpha} > \delta) \cup (C_{K,1/3})) \leq \varepsilon \) if \( |x| \leq q \).

**Proof** Notice that the definitions of the hitting times for \( i = 0 \) and \( i > 0 \) are different. We will first give estimates for \( i = 0 \).

By classical results on stochastic analysis, there exist a constant \( p > 0 \) depending only on \( d \) such that \( \mathbb{E}_G \left[ \sup_{t \leq s < r \leq T} \left| B^t_r - B^t_s \right|^{1/3} \right] < \infty \). Then \( C_G((C_{K,1/3})^{\infty}) \leq \mathbb{E}_G \left[ \sup_{t \leq s < r \leq T} \left| B^t_r - B^t_s \right|^{1/3} \right] = 0 \) as \( K \) goes to infinity. Remark that the upper bound depends only on \( d, L_0 \) and \( T \).

Fix \( \varepsilon > 0 \), and \( t \in [0, \alpha/2] \) then, by the previous inequality, there exists \( K_\varepsilon > 0 \), whose choice is independent of \( t \), such that \( C_G((C_{K,1/3})^{\infty}) \leq \varepsilon/2 \).

We fix \( \delta \in (0, \frac{\varepsilon}{8K^3}) \) and \( \eta \in \mathbb{N} \), by definition \( H^{t,x,\alpha}_i = \Pi^{t,x,\alpha}_i \), and using the Proposition (13.3), there exists \( q_{\varepsilon,\delta} \in (0, \alpha/2) \) such that \( C_G((|H^{t,x,\alpha}_0 - H^{t,0,\alpha}_0| > \delta) \leq \frac{\varepsilon}{4} \) if \( |x| \leq q_{\varepsilon,\delta}^0. \) We define \( \Omega_{\varepsilon,\delta} := \{ |H^{t,x,\alpha}_0 - H^{t,0,\alpha}_0| \leq \delta \} \cap C_{K,1/3} \) verifying \( C_G((\Omega_{\varepsilon,\delta})^{\infty}) \leq 3\varepsilon/4 \), for all \( t \in [0, \alpha/2] \), and \( |x| \leq q_{\varepsilon,\delta}^0. \)

Notice the Holder continuity, the upper bound on \( \delta \) and the bound \( |H^{t,x,\alpha}_0 - H^{t,0,\alpha}_0| \leq \delta \) implies that on \( \Omega_{\varepsilon,\delta} \), it holds that \( |B^t_{n_0^{t,x,\alpha}} - B^t_{n_0^{t,0,\alpha}}| \leq \frac{\delta}{4} \) for all \( |x| \leq q_{\varepsilon,\delta}^0. \)

On the event \( \{ H^{t,x,\alpha}_0 \leq H^{t,0,\alpha}_0 + \alpha < T \} \cap \Omega_{\varepsilon,\delta}^{t,x,\alpha} \), we have the following inequality and equalities for all \( |x| \leq q_{\varepsilon,\delta}^0: \)
\[
H^{t,x,\alpha}_1 \land H^{t,0,\alpha}_1 \geq H^{t,x,\alpha}_0 \lor H^{t,0,\alpha}_0 \\
H^{t,0,\alpha}_1 = \Pi^{t,0,\alpha}_1 \land H^{t,0,\alpha}_0 + \alpha \\
H^{t,x,\alpha}_1 = \Pi^{t,0,\alpha}_1 \land (B^t_{n_0^{t,x,\alpha}} - B^t_{n_0^{t,0,\alpha}}) \lor H^{t,0,\alpha}_0 + \alpha
\]
Hence, switching the roles of $x$ and $0$ and using the estimates \((2.69)\), we obtain for all $t \in [0, \alpha/2]$, $\mathbb{P} \in \mathcal{P}_G^t$, and $|x| \leq q_{\varepsilon, \delta}^0$,

$$
\mathbb{E} \left[ |h_1^{t,x,\alpha} - h_1^{t,0,\alpha}| \mathbb{1}_{[\tilde{h}_0^{0,t,\alpha} + \alpha < T] \cap \tilde{h}_0^{0,t,\alpha}} \right] \\
\leq |h_0^{t,x,\alpha} - h_0^{t,0,\alpha}| + C|B_{\tilde{h}_0^{0,t,\alpha}}^{t} - B_{\tilde{h}_0^{0,t,\alpha}}^{t}| \\
\leq (1 + CK_{\varepsilon})|h_0^{t,x,\alpha} - h_0^{t,0,\alpha}|.
$$

Thus, by induction, for all $i = 1, \ldots, n$, $t \in [0, \alpha/2]$, $\mathbb{P} \in \mathcal{P}_G^t$, and $|x| \leq q_{\varepsilon, \delta}^0$, we have:

$$
\mathbb{E} \left[ |h_1^{i,t,x,\alpha} - h_1^{i,t,0,\alpha}| \mathbb{1}_{[\tilde{h}_0^{0,t,\alpha} + \alpha < T] \cap \tilde{h}_0^{0,t,\alpha}} \right] \\
\leq |h_0^{i,t,x,\alpha} - h_0^{i,t,0,\alpha}| + C|B_{\tilde{h}_0^{0,t,\alpha}}^{i} - B_{\tilde{h}_0^{0,t,\alpha}}^{i}| \\
\leq (1 + CK_{\varepsilon})^i|h_0^{i,t,x,\alpha} - h_0^{i,t,0,\alpha}|.
$$

Therefore for all $t \in [0, \alpha/2]$, $\mathbb{P} \in \mathcal{P}_G^t$, and $|x| \leq q_{\varepsilon, \delta}^0$,

$$
\mathbb{P} \left( \{\Gamma_n^{t,x,\alpha} > \delta\} \cap \tilde{\Omega}_{\varepsilon, \delta}^{0,t,x,\alpha} \right) \leq (1 + CK_{\varepsilon})^n \mathbb{E} \left[ |h_0^{t,x,\alpha} - h_0^{t,0,\alpha}| \right].
$$

We now choose $q \in (0, q_{\varepsilon, \delta}^0)$ to have the right hand side smaller than $\varepsilon/4$(notice that this only depends on $d, \alpha, L_0, c_0, T$ and $\varepsilon, \delta, n$) then, for all $t \in [0, \alpha/2]$, $\mathbb{P} \in \mathcal{P}_G^t$, and $|x| \leq q$,

$$
\mathbb{P} \left( \{\Gamma_n^{t,x,\alpha} > \delta\} \cup (\mathcal{C}_{K_{\varepsilon},1/3})^c \right) \\
\mathbb{P} \left( \{\Gamma_n^{t,x,\alpha} > \delta\} \cup (\tilde{\Omega}_{\varepsilon, \delta}^{0,t,x,\alpha})^c \right) \\
\mathbb{P} \left( (\tilde{\Omega}_{\varepsilon, \delta}^{0,t,x,\alpha})^c \right) + \mathbb{P} \left( \{\Gamma_n^{t,x,\alpha} > \delta\} \cap \tilde{\Omega}_{\varepsilon, \delta}^{0,t,x,\alpha} \right) \leq \varepsilon.
$$

\textbf{B.1 Proof of Proposition \([A.1]\) and \([A.3]\)}

\textbf{Proof}  Notice that the proposition \([A.1]\) is a sub-case of \([A.3]\). We only prove the second one. Thanks to Lemma \([B.1]\) without loss of generality we assume that $n = 0$, and $t_n = 0$.

\textit{Step 1: Regularity in space :} Fix $\varepsilon > 0$, and $\delta > 0$ small enough, then there exist $K_{\varepsilon} < \infty$ such that $\mathcal{C}_G((\mathcal{C}_{K_{\varepsilon},1/3})^c) \leq \varepsilon/2$. There exists $n_{\varepsilon} < \infty$ such that on $\mathcal{C}_{K_{\varepsilon},1/3}$, for all $|x| \leq \alpha/2$, we have $h_{n_{\varepsilon}}^{t,x,\alpha} = T$. Then using again the Proposition \([B.4]\), for all $d > 0$ small enough, there exists $q \in (0, \alpha)$ such that $\mathcal{C}_G(\{\Gamma_{n_{\varepsilon}}^{t,x,\alpha} > \delta\} \cup (\mathcal{C}_{K_{\varepsilon},1/3})^c) \leq \varepsilon$ for all $|x| \leq q$. Notice that the choice of $q$ does not depend on $t$. We define

$$
\Omega_1 := \{\Gamma_{n_{\varepsilon}}^{t,x,\alpha} \leq \delta\} \cap \mathcal{C}_{K_{\varepsilon},1/3}.
$$
On the event $\Omega_1 \cap \{\hat{h}_{i-1}^{t,x,\alpha} + \alpha < T\}$, the following inequalities hold for $|x| \leq q$:

\[
\begin{align*}
|H_i^{t,x,\alpha} - H_i^{t,0,\alpha}| &\leq \delta \\
|H_i^{t,x,\alpha} - H_{i-1}^{t,x,\alpha}| &\geq \frac{\alpha^3}{8K_3^2}.
\end{align*}
\]

The second inequality is the consequence of the Holder continuity of the paths, and the estimates is $\frac{\alpha^3}{8K_3^2}$ for $i = 0$ but $\frac{\alpha^3}{K_3^2}$ for $i > 0$. We take the smallest one.

Given the previous inequalities we can estimate the slope of $\hat{\omega}^{\pi,n,t,x,\alpha}$ on $[H_i^{t,x,\alpha}(\omega), H_i^{t,x,\alpha}(\omega)]$, for all $\omega \in \Omega_1 \cap \{\hat{h}_{i-1}^{t,x,\alpha} + \alpha < T\}$. Chosing $\delta \leq \frac{\alpha^3}{32K_3^2}$, we have that the slope of $\hat{\omega}^{\pi,n,t,x,\alpha}$ on $[H_i^{t,x,\alpha}(\omega), H_i^{t,x,\alpha}(\omega)]$ is less than $\frac{\alpha}{16K_3^2} \leq \frac{\alpha}{\alpha^2} \leq \frac{16K_3^2}{\alpha^3}$.

Therefore, for all $\omega \in \Omega_1$,

\[
||\hat{\omega}^{\pi,n,t,x,\alpha} - \hat{\omega}^{\pi,n,t,0,\alpha}|| \leq \frac{16\delta K_3^3}{\alpha^2} + |x| \leq \frac{16\delta K_3^3}{\alpha^2} + q := l_{\delta,\varepsilon,q},
\]

whenever $|x| \leq q$.

Fix also $k \in \mathcal{K}'$, we introduce the following notations:

\[
\begin{align*}
\hat{Y}_s := \hat{Y}_s^{\pi_n,t,0,\alpha,k}, & \quad \hat{Y}'_s := \hat{Y}_s^{\pi_n,t,x,\alpha,k} \\
\hat{Z}_s := \hat{Z}_s^{\pi_n,t,0,\alpha,k}, & \quad \hat{Z}'_s := \hat{Z}_s^{\pi_n,t,x,\alpha,k} \\
\hat{K}_s := \hat{K}_s^{\pi_n,t,0,\alpha,k}, & \quad \hat{K}'_s := \hat{K}_s^{\pi_n,t,x,\alpha,k} \\
\hat{F}_s := \hat{F}_s^{\pi_n,t,0,\alpha}(s,\omega,\hat{Y}_s,\hat{Z}_s,\hat{k}_r), & \quad \hat{F}'_s := \hat{F}_s^{\pi_n,t,x,\alpha}(s,\omega,\hat{Y}_s,\hat{Z}_s,\hat{k}_r) \\
\hat{h}_s := \hat{h}_s^{\pi_n,t,0,\alpha}(s,\omega), & \quad \hat{h}'_s := \hat{h}_s^{\pi_n,t,x,\alpha}(s,\omega)
\end{align*}
\]

With this notation on $\Omega_1$,

\[
|\hat{\xi} - \hat{\xi}'| \leq \rho_0(l_{\delta,\varepsilon,q}).
\]

Additionally for $s \in (H_i' \lor H_i \lor H_{i+1} \land H_{i+1})$

\[
|\hat{F}_{0,r} - \hat{F}'_{0,r}| \leq \rho_0(\delta + l_{\delta,\varepsilon,q}), \quad (2.77)
\]

\[
|\hat{h}_s - \hat{h}'_s| \leq \rho_0(\delta + l_{\delta,\varepsilon,q}). \quad (2.78)
\]

The classical a priori estimates on RBSDEs control the difference of the solutions with $E[\sup_{s \in [t,T]} |\hat{h}_s - \hat{h}'_s|]$, for some expectation $E$. In our case this estimate is not sharp enough. Indeed, the jumps dates $H_i$ and $H_i'$ will be different in the generic case, and the value of $\sup_{s \in [t,T]} |\hat{h}_s - \hat{h}'_s|$ will be at the order of $\rho_0(\alpha)$, which can not be controlled with $|x|$, thus
rendering the classical estimates useless.
However we can improve those estimates. We only need to improve the upper bound of the term \( \int_t^T (Y_{s_-} - Y'_{s_-}) d(K_s - K'_s) \) under \( \mathbb{P}^{t,k} \), which is easier in our case because of the fact that the \( K^c = 0 \):

\[
\begin{align*}
\int_t^T (Y_{s_-} - Y'_{s_-}) d(K_s - K'_s) &\leq \int_t^T (\dot{h}_{s_-} - \dot{h}'_{s_-}) d(K_s - K'_s) \\
&= \sum_{i=0}^{n_e} 1_{\tau_i' < \tau_i} \left\{ [h(\zeta_{i-1}) - h'(\zeta_{i-1}')] \Delta K_{\tau_i} - [h(\zeta_{i-1}) - h'(\zeta_{i-1}')] \Delta K'_{\tau_i} \right\} \\
&\quad + \sum_{i=0}^{n_e} 1_{\tau_i < \tau_i'} \left\{ [h(\zeta_{i-1}) - h'(\zeta_{i-1}')] \Delta K_{\tau_i} - [h(\zeta_{i}) - h'(\zeta_{i}')] \Delta K'_{\tau_i} \right\} \\
&= \sum_{i=0}^{n_e} 1_{\tau_i' < \tau_i} \left\{ [h(\zeta_{i}) \Delta K'_{\tau_i} - h'(\zeta_{i}')] \Delta K_{\tau_i} + [h'(\zeta_{i-1}) \Delta K_{\tau_i} - h(\zeta_{i-1}) \Delta K'_{\tau_i}'] \right\} \\
&\leq C \sum_{i=0}^{n_e} 1_{\tau_i' < \tau_i} [h(\zeta_{i}) - h'(\zeta_{i}')] + |\Delta K'_{\tau_i} - \Delta K_{\tau_i}|.
\end{align*}
\]

Our RBSDEs verifies the general assumptions in [12], therefore the size of the jumps verify:

\[
|\Delta K'_{\tau_i} - \Delta K_{\tau_i}| \leq |(h'(\zeta_{i-1}) - Y'_{\tau_i})^+ - (h(\zeta_{i-1}) - Y_{\tau_i})^+| \\
\leq |h'(\zeta_{i-1}) - h(\zeta_{i-1})| + |Y'_{\tau_i} - Y_{\tau_i}| \\
\leq \rho_0 (\delta + l_{\delta,\varepsilon,q}) + |Y'_{\tau_i} - Y_{\tau_i}|.
\]

Combining the previous inequalities, and restrict our analysis to \( \Omega_1 \), we obtain on \( \Omega_1 \):

\[
\begin{align*}
\int_t^T (Y_{s_-} - Y'_{s_-}) d(K_s - K'_s) &\leq C (2n_e \rho_0 (\delta + l_{\delta,\varepsilon,q}) + \sum_{i=0}^{n_e} 1_{\tau_i' < \tau_i} |Y'_{\tau_i} - Y_{\tau_i}|).
\end{align*}
\]

(2.79)

Recall that under \( \mathbb{P}^{t,k} \):

\[
\begin{align*}
Y_{\tau_i} &= \max \{ h(\zeta_{i}), Y_{\tau_{i+1}} \} + \int_{\tau_i}^{\tau_{i+1}} \hat{F}_r \, dr - \int_{\tau_i}^{\tau_{i+1}} \sigma^{-1}(\hat{k}_r) dB^t_r, \\
Y'_{\tau_i} &= \max \{ h'(\zeta_{i}), Y'_{\tau_{i+1}} \} + \int_{\tau'_i}^{\tau'_{i+1}} \hat{F}'_r \, dr - \int_{\tau'_i}^{\tau'_{i+1}} \sigma^{-1}(\hat{k}_r) dB^t_r.
\end{align*}
\]

Using the estimates (2.77) between \( (\zeta_i' \lor \zeta_{i+1}, \zeta_{i+1} \land \zeta_{i+1}) \) and the boundedness of the \( \hat{Y} \) and
\( Y^\prime \), the classical estimates on BSDEs give:

\[
\mathbb{E}^{t,k}_{\Omega_1} \left[ |Y^\prime_{\Pi_1} - Y_{\Pi_1}|^2 \right] \\
\leq C \mathbb{E}^{t,k}_{\Omega_1} \left[ \left| h^\prime(h) - h(h_1) \right|^2 + \left| Y^\prime_{\Pi_1} - Y_{\Pi_1} \right|^2 + \left( \int_{\Pi_1 \wedge h_1} \left| \hat{F}_0 - \hat{F}_0 \right|^2 dr \right) \right]
\]

\[
\leq C \mathbb{E}^{t,k}_{\Omega_1} \left\{ \delta + (1 + T) \rho_0^2(\delta + l_{\delta,\epsilon,q}) + 1_{\Omega_1} \left| Y^\prime_{\Pi_1} - Y_{\Pi_1} \right|^2 \right\}
\]

\[
\leq C \left( n_\epsilon + 1 \right) (1 + T) (\delta + \rho_0^2(\delta + l_{\delta,\epsilon,q})).
\]

In the previous inequalities the term \(+\delta\) comes from the fact that the difference between the stopping times \( h^\prime \) and \( h \) is less than \( \delta \) and the integrands can be bounded.

Injecting this to (2.79), we obtain the following inequality on \( \Omega_1 \):

\[
\mathbb{E}^{t,k}_{\Omega_1} \left[ 1_{\Omega_1} \int_T (Y_s - Y^\prime_s) d(K_s - K^\prime_s) \right] \leq C n_\epsilon^2 \rho_0(\delta + l_{\delta,\epsilon,q}),
\]

(2.80)

We can improve the classical estimates as follows:

\[
|Y^\prime_t - Y_t|^2 \leq \mathbb{E}^{t,k}_{\Pi_1} \left[ |\xi' - \xi|^2 + \int_T |(\hat{F}' - \hat{F})_r(Y_r, Z_r)|^2 dr + \int_T (Y_s - Y^\prime_s) d(K_s - K^\prime_s) \right]
\]

\[
\leq C \mathbb{E}^{t,k}_{\Pi_1} \left\{ 1_{\Omega_1} \left[ |\xi' - \xi|^2 + \int_T |(\hat{F}' - \hat{F})_r(Y_r, Z_r)|^2 dr + \int_T (Y_s - Y^\prime_s) d(K_s - K^\prime_s) \right] \right\}
\]

\[
+ C \mathbb{E}^{t,k}_{\Pi_1} \left\{ 1_{\Omega_1} \left[ |\xi' - \xi|^2 + \int_T |(\hat{F}' - \hat{F})_r(Y_r, Z_r)|^2 dr + \int_T (Y_s - Y^\prime_s) d(K_s - K^\prime_s) \right] \right\}
\]

\[
\leq C \left\{ n_\epsilon^2 \rho_0(\delta + l_{\delta,\epsilon,q}) + \mathbb{E}^{t,k}_{\Pi_1} \left[ \int_T |(\hat{F}' - \hat{F})_r(Y_r, Z_r)|^2 dr \right] \right\} + C \epsilon / 2
\]

\[
\leq C \left\{ n_\epsilon^2 \rho_0(\delta + l_{\delta,\epsilon,q}) + \delta n_\epsilon \right\} + C \epsilon / 2.
\]

Taking \( \delta > 0 \) small enough, there exists \( q > 0 \) whose choice depend only on \( d, \alpha, \rho_0, L_0, \)

\( M_0, c_0, T, \epsilon, K_\epsilon, \) and some universal constant (and additionally on \( k \) for \( \theta_n^{k,\alpha} \)), but not on,

\( t \) or \( \pi_n, \) such that if \( |x| \leq q, \) the inequality

\[
|\hat{Y}_{\pi_n,t,x,\alpha,k} - \hat{Y}_{\pi_n,t,0,\alpha,k}|^2 \leq C \epsilon
\]

holds for all \( k \in \mathcal{K}^t \). In conclusion, by taking the sup in \( k \), there exist \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), with

\( \rho(0+) = 0 \) and

\[
|\Gamma^\alpha_n(\pi_n; t, x) - \Gamma^\alpha_n(\pi_n; t, 0)| \leq \rho(|x|),
\]

for \( t \in [t_n; t_n + \alpha/2] \) and \( |x| \leq \alpha/2.\)
Step 2: Regularity in time: We fix \( t_n \leq t < t' \leq t + \alpha/2 \), and define the hitting time \( H_{\alpha/2} := \inf \{ s \geq t : |B_s| = \alpha/2 \} \wedge (t + \alpha/2) \). Notice that on \([t_n, H_{\alpha/2} \wedge t']\), for all \( k \in K^t \), the RBSDE under \( \mathbb{P}^{t,k} \) is actually a BSDE thanks to the fact that \( \hat{K}^c = 0 \). Additionally, the previous regularity in \( x \) allows us to use proposition 5.14 of [20] to have the representation

\[
\Gamma_n^\alpha(\pi_n; t, 0) = \sup_{k \in K^t} \tilde{Y}_{\pi_n, t, 0, \alpha, k},
\]

where \((\tilde{Y}_{\pi_n, t, 0, \alpha, k}, \tilde{Z}_{\pi_n, t, 0, \alpha, k})\) solves (without the superscripts) under \( \mathbb{P}^{t,k} \),

\[
\tilde{Y}_s = \max\{\Gamma_n^\alpha(\pi_n; H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^t), \hat{t}(t_n, \tilde{\pi}_n)\}
+ \int_s^{H_{\alpha/2} \wedge t'} \tilde{F}_r(\tilde{Y}_r, \tilde{Z}_r, \tilde{k}_r) dr - \int_s^{H_{\alpha/2} \wedge t'} \tilde{Z}_r \sigma^{-1}(k_r) dB_r^t.
\]

Notice that \((H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^{t'}) \in \mathcal{O}_t^n \) therefore

\[
\max\{\Gamma_n^\alpha(\pi_n; H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^{t'}), \hat{t}(t_n, \tilde{\pi}_n)\} = \Gamma_n^\alpha(\pi_n; H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^{t'}),
\]

so we can rewrite the previous BSDE under \( \mathbb{P}^{t,k} \):

\[
\tilde{Y}_s = \Gamma_n^\alpha(\pi_n; H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^{t'})
+ \int_s^{H_{\alpha/2} \wedge t'} \tilde{F}_r(\tilde{Y}_r, \tilde{Z}_r, \tilde{k}_r) dr - \int_s^{H_{\alpha/2} \wedge t'} \tilde{Z}_r \sigma^{-1}(k_r) dB_r^t.
\]

Fix \( \varepsilon > 0 \), then there exist \( k \in K^t \) such that \( \Gamma_n^\alpha(\pi_n, t, 0) \leq \tilde{Y}_{\pi_n, t, 0, \alpha, k} + \varepsilon \). Then

\[
\Gamma_n^\alpha(\pi_n, t, 0) - \Gamma_n^\alpha(\pi_n, t', 0) \leq \Gamma_n^\alpha(\pi_n; H_{\alpha/2} \wedge t', B_{H_{\alpha/2} \wedge t'}^{t'}) - \Gamma_n^\alpha(\pi_n, t', 0)
+ \int_s^{H_{\alpha/2} \wedge t'} \tilde{F}_r(\tilde{Y}_r, \tilde{Z}_r, \tilde{k}_r) dr - \int_s^{H_{\alpha/2} \wedge t'} \tilde{Z}_r \sigma^{-1}(k_r) dB_r^t
\]

\[
\leq 1_{\{H_{\alpha/2} \geq t'\}}(\Gamma_n^\alpha(\pi_n, t', B_{H_{\alpha/2} \wedge t'}^{t'}) - \Gamma_n^\alpha(\pi_n, t', 0)) + C 1_{\{H_{\alpha/2} < t'\}}
+ \int_s^{H_{\alpha/2} \wedge t'} \tilde{F}_r(\tilde{Y}_r, \tilde{Z}_r, \tilde{k}_r) dr - \int_s^{H_{\alpha/2} \wedge t'} \tilde{Z}_r \sigma^{-1}(k_r) dB_r^t.
\]

Taking the expectation under \( \mathbb{P}^{t,k} \) and using the previous \( x \) regularity result,

\[
\Gamma_n^\alpha(\pi_n, t, 0) - \Gamma_n^\alpha(\pi_n, t', 0)
\leq \mathbb{E}^{\mathbb{P}^{t,k}}[|\Gamma_n^\alpha(\pi_n, t', B_{H_{\alpha/2} \wedge t'}^{t'}) - \Gamma_n^\alpha(\pi_n, t', 0)|] + C \mathbb{P}^{t,k}[H_{\alpha/2} < t']
+ \mathbb{E}^{\mathbb{P}^{t,k}}\left[ \int_t^{t'} |F_r(0, \tilde{k}(s, B_s^t)) + C + L_0| \tilde{Z}_r|dr \right]
\leq \mathbb{E}^{\mathbb{P}^{t,k}}[\rho(B_{H_{\alpha/2} \wedge t'}^{t'})] + C \mathbb{P}^{t,k}[H_{\alpha/2} < t']
+ C|t - t'| + CE_G\left[ \int_t^{t'} |\tilde{Z}_r|dr \right]
\leq E_G[\rho(B_{H_{\alpha/2} \wedge t'}^{t'})] + C \mathcal{P}_G[H_{\alpha/2} < t'] + C|t - t'| + C\sqrt{|t - t'|}.\]
This inequality controls the variation at one direction and the last term is a modulus of continuity for the variation in $t$. For the other direction we fix a $k \in \mathcal{K}^t$:

$$\Gamma_n^\alpha(\pi_n, t', 0) - \Gamma_n^\alpha(\pi_n, t, 0) \leq \Gamma_n^\alpha(\pi_n, t', 0) - \tilde{Y}_{\pi_n, t, 0, \alpha, k}$$

By similar bounds, $|\Gamma_n^\alpha(\pi_n, t, 0) - \Gamma_n^\alpha(\pi_n, t', 0)| \leq \rho(|t - t'|)$.

In conclusion, the mapping

$$t \in [t_n, t_n + \alpha/2] \to \Gamma_n^\alpha(\pi_n; t, 0)$$

is uniformly continuous, with modulus of continuity depending only on $d, \rho_0, c_0, L_0, M_0, T$ and $\alpha$. Which proves (A.3). Combining this result with the Lemma [B.1], we obtain the uniform continuity of the mapping $(t, x) \in \partial \Omega_{\varepsilon_{n}} \to \Gamma_{\varepsilon_{n+1}}(\pi_{n+1}^{(t,x)}; t, 0)$, with the modulus of continuity depending only on the previously cited parameters. For the function $\theta_{m,n}^{m,\alpha}$ the modulus of continuity may also depend on $m$.  

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