A SOUND INTERPRETATION OF LEŚNIEWSKI’S EPSILON IN MODAL LOGIC KTB

Abstract

In this paper, we shall show that the following translation $I^M$ from the propositional fragment $L_1$ of Leśniewski’s ontology to modal logic $KTB$ is sound: for any formula $\phi$ and $\psi$ of $L_1$, it is defined as

\begin{align*}
(M1) \quad I^M(\phi \lor \psi) &= I^M(\phi) \lor I^M(\psi), \\
(M2) \quad I^M(\neg \phi) &= \neg I^M(\phi), \\
(M3) \quad I^M(\epsilon ab) &= \diamond p_a \supset p_b \land \Box p_a \supset \Box p_b \land \Box p_b \supset p_a,
\end{align*}

where $p_a$ and $p_b$ are propositional variables corresponding to the name variables $a$ and $b$, respectively. In the last section, we shall give some comments including some open problems and my conjectures.

Keywords: Leśniewski’s ontology, propositional ontology, translation, interpretation, modal logic, KTB, soundness, Grzegorczyk’s modal logic.

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1. Introduction and $I^M$

Inoué [9] initiated a study of interpretations of Leśniewski’s epsilon $\epsilon$ in the modal logic $K$ and its certain extensions. That is, Ishimoto’s propositional fragment $L_1$ (Ishimoto [12]) of Leśniewski’s ontology $L$ (refer to Urbaniak [19]) is partially embedded in $K$ and in the extensions, respectively, by the following translation $I$ from $L_1$ to them: for any formula $\phi$ and $\psi$ of $L_1$, it is defined as
\[(1) \ I(\phi \lor \psi) = I(\phi) \lor I(\psi), \]

\[(2) \ I(\neg \phi) = \neg I(\phi), \]

\[(3) \ I(\epsilon ab) = p_a \land \Box(p_a \equiv p_b), \]

where \(p_a\) and \(p_b\) are propositional variables corresponding to the name variables \(a\) and \(b\), respectively. Here, \(L_1\) is partially embedded in \(K\) by \(I\)” means that for any formula \(\phi\) of a certain decidable nonempty set of formulas of \(L_1\) (i.e. decent formulas (see §3 of Inoué [10])), \(\phi\) is a theorem of \(L_1\) if and only if \(I(\phi)\) is a theorem of \(K\). Note that \(I\) is sound. The paper [10] also proposed similar partial interpretations of Leśniewski’s epsilon in certain von Wright-type deontic logics, that is, ten Smiley-Hanson systems of monadic deontic logic and in provability logic \(GL\), respectively. (See Åqvist [1] and Boolos [3] for those logics.)

The interpretation \(I\) is however not faithful. A counterexample for the faithfulness is, for example, \(eac \land ebc \vdash \epsilon ab \lor ecc\) (for the details, see [10]). Blass [2] gave a modification of the interpretation and showed that his interpretation \(T\) is faithful, using Kripke models. Inoué [11] called the translation \(Blass translation\) (for short, \(B\)-translation) or \(Blass interpretation\) (for short, \(B\)-interpretation). The translation \(B\) from \(L_1\) to \(K\) is defined as follows: for any formula \(\phi\) and \(\psi\) of \(L_1\),

\[(B1) \ B(\phi \lor \psi) = B(\phi) \lor B(\psi), \]

\[(B2) \ B(\neg \phi) = \neg B(\phi), \]

\[(B3) \ B(\epsilon ab) = p_a \land \Box(p_a \supset p_b) \land p_b \supset \Box(p_b \supset p_a), \]

where \(p_a\) and \(p_b\) are propositional variables corresponding to the name variables \(a\) and \(b\), respectively. Inoué [11] extended Blass’s faithfulness result for many normal modal logics, provability logic and von Wright-type deontic logics including \(K4, KD, KB, KD4\), etc, \(GL\) and ten Smiley-Hanson systems of monadic deontic logic, using model constructions based on Hintikka formula (cf. Kobayashi and Ishimoto [13]).

In this paper, we first propose a translation \(I^M\) from \(L_1\) in modal logic \(KTB\), which will be specified in §2.

**Definition 1.1.** A translation \(I^M\) of Leśniewski’s propositional ontology \(L_1\) in modal logic \(KTB\) is defined as follows: for any formula \(\phi\) and \(\psi\) of \(L_1\),

\[(M1) \ I^M(\phi \lor \psi) = I^M(\phi) \lor I^M(\psi), \]
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(M2) $I^M(\neg \phi) = \neg I^M(\phi)$,
(M3) $I^M(eab) = \Diamond p_a \supset p_a \land \Box p_a \supset \Diamond p_b \land \Diamond p_b \supset p_a$,

where $p_a$ and $p_b$ are propositional variables corresponding to the name variables $a$ and $b$, respectively.

We call $I^M$ to be $M$-translation or $M$-interpretation.

In the following §2, we shall collect the basic preliminaries for this paper. In §3, using proof theory, we shall show that $I^M$ is sound, as the main theorem of this paper. In §4, we shall give some comments including some open problems and my conjectures.

2. Propositional ontology $L_1$ and modal logic $KTB$

Let us recall a formulation of $L_1$, which was introduced in [12]. The Hilbert-style system of it, denoted again by $L_1$, consists of the following axiom-schemata with a formulation of classical propositional logic $CP$ as its axiomatic basis:

(Ax1) $eab \supset eaa$,
(Ax2) $eab \land ebc \supset eac$,
(Ax3) $eab \land ebc \supset eba$,

where we note that every atomic formula of $L_1$ is of the form $eab$ for some name variables $a$ and $b$ and a possible intuitive interpretation of $eab$ is ‘the $a$ is $b$’. We note that (Ax1), (Ax2) and (Ax3) are theorems of Leśniewski’s ontology (see Slupecki [17]).

The modal logic $K$ is the smallest logic which contains all instances of classical tautology and all formulas of the forms $\Box(\phi \supset \psi) \supset \Box \phi \supset \Box \psi$ being closed under modus ponens and the rule of necessitation (for $K$ and basics for modal logic, see Bull and Segerberg [4], Chagrov and Zakharyaschev [5], Fitting [6], Hughes and Cresswell [8] and so on).

We recall the naming of modal logics as follows (refer to e.g. Poggiolesi [15] and Ono [14], also see Bull and Segerberg [4]):

$\text{KT: } K + \Box \phi \supset \phi$ (T, reflexive relation)
$\text{KB: } K + \phi \supset \Box \Diamond \phi$ (B, symmetric relation)
$\text{KTB: } KT + B$ (reflexive and symmetric relation).
3. The soundness of $I^M$

**Theorem 3.1. (Soundness)** For any formula $\phi$ of $L_1$, we have

$$\vdash_{L_1} \phi \Rightarrow \vdash_{KTB} I^M(\phi).$$

**Proof:** Let $\phi$ be a formula of $L_1$. We shall prove the meta-implication by induction on derivation.

**Basis.**

(Case 1) We shall first treat the case for (Ax1). Let $a$ and $b$ be name variables. Then we have the following inferences in $KTB$:

$(\ast) \ I^M(eab)$ (Assumption)

(1.1) $\lozenge p_a \supset p_a$ from $(\ast)$ and Definition 1.1) $\dagger$

(1.2) $\Box p_a \supset \Box p_a$ (true in $K$) $\dagger$

(1.3) $\lozenge p_a \supset p_a \land \lozenge p_a \supset p_a$ (from (1.1) and (1.2))

(1.4) $I^M(\epsilon aa)$ (from (1.3) and Definition 1.1)

(1.5) $I^M(\epsilon ab \supset \epsilon aa)$ (from $(\ast)$, (1.4) and Definition 1.1).

(Case 2) Next we shall deal with the case of (Ax2). Let $a$, $b$ and $c$ be name variables. Then we have the following inferences in $KTB$:

$(\ast\ast) \ I^M(eab \land \epsilon bc)$ (Assumption)

(2.1) $I^M(eab)$ (from $(\ast\ast)$ and Definition 1.1)

(2.2) $I^M(\epsilon bc)$ (from $(\ast\ast)$ and Definition 1.1)

(2.3) $\lozenge p_a \supset p_a \land \lozenge p_a \supset p_b \land \lozenge p_a \supset p_a$ (from (2.1) and Def 1.1)

(2.4) $\lozenge p_b \supset p_b \land \lozenge p_b \supset \Box p_c \land \lozenge p_b \supset p_b$ (from (2.2) and Def 1.1)

(2.5) $\Box p_a \supset p_a$ (from (2.3)) $\dagger$

(2.6) $\lozenge p_a \supset \Box p_b$ (from (2.3))

(2.7) $\Box p_b \supset \Box p_c$ (from (2.4))

(2.8) $\lozenge p_a \supset \Box p_c$ (from (2.6) and (2.7)) $\dagger$

(2.9) $\Box p_b \supset p_a$ (from (2.3))

(2.10) $\Box (\lozenge p_b \supset p_a)$ (from (2.9) and the rule of necessitation)

(2.11) $\lozenge p_b \supset \Box p_a$ (from (2.10) with a true inference in $K$)

(2.12) $\Box p_a \supset p_a$ (true in $KT$)

(2.13) $\lozenge p_b \supset p_a$ (from (2.11) and (2.12))
(2.14) \( p_b \supset \Box \Diamond p_b \) (true in \( \mathbf{KB} \))
(2.15) \( \Diamond p_c \supset p_b \) (from (2.4))
(2.16) \( \Diamond p_a \supset p_a \) (from (2.13) and (2.14) and (2.15)) †
(2.17) \( \Diamond p_a \supset p_a \land \Box p_a \supset \Box p_a \land \Diamond p_c \supset p_a \) (from (2.5), (2.8) and (2.16))
(2.18) \( I^M(\epsilon ac) \) (from (2.17) and Definition 1.1)
(2.19) \( I^M(\epsilon ab \land \epsilon bc \supset \epsilon ac) \) (from (**), (2.18) and Definition 1.1).

(Case 3) Lastly we shall proceed to the case of (Ax3). Let \( a, b \) and \( c \) be name variables. Then we also have the following inferences in \( \mathbf{KTB} \):

(***) \( I^M(\epsilon ab \land \epsilon bc) \) (Assumption)
(3.1) \( I^M(\epsilon ab) \) (from (***) and Definition 1.1)
(3.2) \( I^M(\epsilon bc) \) (from (***) and Definition 1.1)
(3.3) \( \Diamond p_a \supset p_a \land \Box p_a \supset \Box p_a \land \Diamond p_b \supset p_a \) (from (3.1) and Def 1.1)
(3.4) \( \Diamond p_b \supset p_b \land \Box p_b \supset \Box p_b \land \Diamond p_c \supset p_b \) (from (3.2) and Def 1.1)
(3.5) \( \Diamond p_b \supset p_b \) (from (3.4)) †
(3.6) \( \Diamond p_b \supset p_a \) (from (3.3))
(3.7) \( \Box(\Diamond p_b \supset p_a) \) (from (3.6) and the rule of necessitation)
(3.8) \( \Box(\Diamond p_b \supset p_a) \) (from (3.7) with a true inference in \( \mathbf{K} \))
(3.9) \( p_b \supset \Box \Diamond p_b \) (true in \( \mathbf{KB} \))
(3.10) \( p_b \supset p_b \) (true in \( \mathbf{KT} \))
(3.11) \( \Box p_b \supset \Box p_b \) (from (3.8) and (3.9) and (3.10)) †
(3.12) \( \Diamond p_a \supset p_a \) (from (3.3))
(3.13) \( p_a \supset \Box \Diamond p_a \) (true in \( \mathbf{KB} \))
(3.14) \( \Diamond p_a \supset \Box \Diamond p_a \) (from (3.12) and (3.13))
(3.15) \( \Box(\Diamond p_a \supset p_a) \) (from (3.12) and the rule of necessitation)
(3.16) \( \Box(\Diamond p_a \supset \Box p_a) \) (from (3.15) with a true inference in \( \mathbf{K} \))
(3.17) \( \Diamond p_a \supset \Box p_a \) (from (3.14) and (3.16))
(3.18) \( \Box p_b \supset \Box p_b \) (from (3.3))
(3.19) \( \Diamond p_a \supset \Box p_b \) (from (3.17) and (3.18))
(3.20) \( \Box p_b \supset p_b \) (true in \( \mathbf{KT} \))
(3.21) \( \Diamond p_a \supset p_b \) (from (3.19) and (3.20)) †
\[(3.22) \Diamond p_b \supset p_b \land \Box p_b \supset \Box p_a \land \Diamond p_a \supset p_b \]

(from (3.5), (3.11) and (3.21))

\[(3.23) I^M(\epsilon ba) \text{ (from (3.22) and Definition 1.1)}
\]

\[(3.24) I^M(\epsilon ab \land \epsilon bc \supset \epsilon ba) \text{ (from (***)}, (3.23) \text{ and Definition 1.1).} \]

**Induction Steps.** The induction step is easily dealt with. Suppose that \( \phi \) and \( \phi \supset \psi \) are theorems of \( L_1 \). By induction hypothesis, \( I^M(\phi) \) and \( I^M(\phi \supset \psi) \leftrightarrow I^M(\phi) \supset I^M(\psi) \) are theorems of \( KTB \). By modus ponens, we obtain \( \vdash_{KTB} I^M(\psi) \). Thus this completes the proof the theorem. \( \square \)

### 4. Comments

One motive from which I wrote [9] and [10] is that I wished to understand Leśniewski’s epsilon \( \epsilon \) on the basis of my recognition that Leśniewski’s epsilon would be a variant of truth-functional equivalence \( \equiv \). Namely, my original approach to the interpretation of \( \epsilon \) was to express the deflection of \( \epsilon \) from \( \equiv \) in terms of Kripke models. Another (hidden) motive of mine for \( I^M \) is to interpret \( L_1 \) in intuitionistic logic and bi-modal logic. It is well-known that Leśniewski’s epsilon can be interpreted by the Russellian-type definite description in classical first-order predicate logic with equality (see [12]). Takano [18] proposed a natural set-theoretic interpretation for the epsilon. To repeat, I do not deny the interpretation using the Russellian-type definite description and a set-theoretic one. I wish to obtain another interpretation of Leśniewski’s epsilon having a more propositional character. We have the following direct open problems.

**Open problem 1:** Is \( I^M \) faithful?

**Open problem 2:** Find the set of other translations and modal logics in which \( L_1 \) is embedded. I think that there seems to be many possibilities.

**Open problem 3:** Can \( L_1 \) be embedded in \( S4.2 \)? (See e.g. Hamkins and Löwe [7].)

**Open problem 4:** Can \( L_1 \) be embedded in Grzegorczyk’s modal Logic? (See e.g. Savateev and Shamkanov [16])

My conjectures are the following.

**Conjecture 4.1.** \( I^M \) is faithful.

**Conjecture 4.2.** It seems that \( L_1 \) cannot be embedded in intuitionistic propositional logic.
Conjecture 4.3. It seems that $L_1$ can well be embedded in intuitionistic modal propositional logic.

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