Modeling, identification, and optimization of violin bridges

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We present the derivation and implementation of a mathematical model for the linear elastic structure of a violin bridge. In favour of getting highly accurate simulation results, special effort has been taken to reconstruct the computational geometry from a \(\mu\)-CT scan. After laying out our approach for this in detail, we also present first vibroacoustical analysis like solving the eigenvalue problem of elasticity for the orthotropic material law of the wooden bridge and simulation of its dynamical behaviour. We then conclude with explaining our ongoing work of identifying material parameters by an inverse problem. With this project we are aiming to optimize a violin bridge regarding its material parameters and its shape.

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1 Problem Setting/Mathematical Model and Tools (IGA)

We investigate the vibroacoustical properties of a violin bridge which is a small and slender piece of wood, i.e. of orthotropic material structure. Therefore we consider the eigenvalue problem of linear elasticity

\[
-\text{div} (\sigma(u)) = \rho \lambda u
\]

with \(u\) denoting the displacement vector, \(\sigma(u)\) the Cauchy stress tensor and \(\rho > 0\) the mass density. After discretization, the dynamical behaviour of the bridge subjected to a force \(f\) is described by

\[
M\ddot{u}(t) + C\dot{u}(t) + Au(t) = f(t)
\]

where \(M\) indicates the mass, \(A\) the stiffness and \(C\) the damping matrix of our system.

We discretize the computational geometry with isogeometric elements \([1]\) and decompose it into 16 patches which we couple weakly by means of mortar methods \([3]\).

2 Derivation of the computational geometry from a \(\mu\)-CT scan

Because of the small size of a violin bridge and the high sensitivity of its eigenvalues regarding geometrical changes, it is crucial for correct simulations to base them on a highly precise geometry. For this reason a \(\mu\)-CT scan of the bridge was made which provides us with a point cloud contained in an STL file. We reconstruct the geometry by a total least squares fit. Given a set of data points \(\{ (x_i, v_i) \}_{i=1}^{N}\) we are looking for parametric points \(\{ \xi_i \}_{i=1}^{N}\) and control points \(\{ C_j \}_{j=1}^{J} = (C_j^x, C_j^y)_T\), that minimize the difference between a 2D spline curve and the data:

\[
\sum_{i=1}^{N} \left\| (u_i, v_i)^T - \left( \sum_{j=1}^{J} N_j^p(\xi_i) \cdot C_j \right) \right\|_2^2 \rightarrow \text{min},
\]

Here \(N_j^p\) are the spline basis functions of degree \(p\). This will give us the fit for the boundary splines of each patch. By introducing the matrix \(N_j\) defined as

\[
N_j(\xi) := \begin{pmatrix}
N_1(\xi_1) & \ldots & N_J(\xi_1) \\
\vdots & \ddots & \vdots \\
N_1(\xi_N) & \ldots & N_J(\xi_N)
\end{pmatrix}
\]

we can treat the minimization problem as a matrix equation and formally eliminate the controlpoints such that we end up minimizing

\[
\left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} N_j(\xi)N_j^T(\xi) \\ \cdot \cdot \cdot \\ N_j(\xi)N_j^T(\xi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|_2^2 \rightarrow \text{min},
\]

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where we are left with only finding the optimal $\{\xi_i\}_{i=1}^{N}$ which we tackle with a Gauss-Newton approach. The algorithm can be further improved by introducing penalties on the spacing of the control points to reach a desired distribution of them, as well as penalties to enforce the exact interpolation of end points and interpolatory break points.

With the boundary splines on hand we can construct the spline surfaces by applying a Coons patch. Afterwards we go over to the final step, where we lift the surfaces up to spline volumes by a linear least squares fit between the $z$-component $C_{j}^z$ of the control points and the $z$-component $w_i$ of the data points:

$$
\min_{C_{j}^z} \sum_{i=1}^{N} \left( \sum_{j=1}^{J} \sum_{k=1}^{K} N_j(\xi_i)N_k(\eta_i) \cdot C_{j}^z - z_i \right)^2.
$$

(5)

The parameter values $({\xi}_i, {\eta}_i)$ belonging to a data point $(u_i, v_i, w_i)$ need to be determined by inverting the fitted spline surface at $(u_i, v_i)$.

3 Vibroacoustical Simulations

3.1 Eigenvalue problem of linear elasticity

Based on the derived geometry we solve the eigenvalue problem of elasticity (1). Two of the computed eigenmodes are shown in figure 1. The first one is an out-of-plane mode and the second one an in-plane mode, the latter one being most likely to cause resonance peaks [4].

![Fig. 1: Two typical eigenmodes of a violin bridge](image1)

![Fig. 2: Damped displacement of the violin bridge](image2)

3.2 Dynamical behaviour

Furthermore, we model the damped structure of the violin bridge which is described by equation (2). Figure 2 shows the computed displacement at one selected node of the computational grid. It clearly illustrates the damped harmonic oscillation of the bridge.

4 Outlook

Currently we are working on an inverse problem, were we would like to derive the material parameters contained in the Cauchy stress tensor $\sigma(u)$ and the damping matrix $C$ from measurements of the frequency response (FR). For this aim we model the FR numerically and minimize its distance to the measured FR. Furthermore, we are planning to do uncertainty quantification for this measurements, e.g. to determine appropriate measurement points for the experimental setup. In the future also the consideration and coupling of further parts of the violin like the top plate are planned.

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