Examples of PT Phase Transition : QM to QFT

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Abstract.
Parity Time Reversal (PT) phase transition is a typical characteristic of most of the PT symmetric non-Hermitian (NH) systems. Depending on the theory, a particular system and spacetime dimensionality PT phase transition has various interesting features. In this article we review some of our works on PT phase transitions in quantum mechanics (QM) as well as in Quantum Field theory (QFT). We demonstrate typical characteristics of PT phase transition with the help of several analytically solved examples. In one dimensional QM, we consider examples with exactly as well as quasi exactly solvable (QES) models to capture essential features of PT phase transition. The discrete symmetries have rich structures in higher dimensions which are used to explore the PT phase transition in higher dimensional systems. We consider anisotropic SHOs in two and three dimensions to realize some connection between the symmetry of original hermitian Hamiltonian and the unbroken phase of the NH system. We consider the 2+1 dimensional massless Dirac particle in the external magnetic field with PT symmetric non-Hermitian spin-orbit interaction in the background of the Dirac oscillator potential to show the PT phase transition in a relativistic system. A small mass gap, consistent with the other approaches and experimental observations is generated only in the unbroken phase of the system. Finally we develop the NH formulation in an SU(N) gauge field theoretic model by using the natural but unconventional Hermiticity properties of the ghost fields. Deconfinement to confinement phase transition has been realized as PT phase transition in such a non-hermitian model.

1. Introduction
The use of NH theories to address various problems in physics has a very long and rich history [1]. However no attempts were made to develop a consistent quantum theory with NH hamiltonians until towards the end of last century. It was shown that the NH systems which respect PT symmetry can yield a complete real spectrum [2]. Consequently it has been shown that a fully consistent quantum theory with real spectrum, a complete set of orthonormal eigenfunctions with positive definite norms and unitary time evolution can be developed for certain class of NH systems in a modified Hilbert space [3, 4, 5, 6]. These exciting theoretical works towards the consistency of NH quantum mechanics (NHQM) strongly paved the way forward for NHQM to be the topic of frontier research in different areas in and interdisciplinary branches of physics during the last two decades [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Due to the analogy of the Schrodinger equation
with certain wave equations in optics, the phenomena of NHQM can also be mapped to the analogous phenomena in optics. This leads to the possibility of experimental observation of the theoretical predictions of NHQM. This has been indeed the case and some of the predictions of NHQM have been observed in optics \[37, 38, 39, 40\]. The realizations of NHQM phenomena have ignited huge interest to study the subject both theoretically and experimentally.

PT symmetric NH systems generally exhibit a phase transition\(^1\) that separates two parametric regions (i) region of the unbroken PT symmetry in which the entire spectrum is real and eigenstates of the system respect PT symmetry and (ii) a region of broken PT symmetry in which the whole spectrum (or a part of it) appears as complex conjugate pairs and eigenstates of the Hamiltonian do not respect PT symmetry. Very rich physics is associated with this phase transition point. Even though most of the PT symmetric NH systems pass through PT phase transition, there are systems which do not show any such transition and remain either in broken or in unbroken phase for the entire parameter region. Many of the well known phase transitions have also been realized as PT phase transition in the NH version of those systems. This particular topic has been developed enormously over the past two decades \[41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65\]. It is impossible to attempt a review of all those in this work. In this article we intend to review some typical features of PT phase transition in non-relativistic as well as in relativistic quantum mechanics and in NH quantum field theoretical models. We mainly will focus on our own works \[61, 62, 63, 64, 65\] in these topics, which are solved analytically. In the first model in one dimensional quantum mechanics, we consider a system of a spin half particle in external magnetic field coupled to an oscillator through PT symmetric NH interaction. We found that, except the ground state, all other states occur in doublet due to the interaction of oscillator with two levels system. Such doublets form invariant subspace of the system. The energy eigenvalues and associated eigenfunctions corresponding to each doublet are solved analytically. Under certain conditions on the parameters (frequency of the oscillator, strength of the magnetic field and strength of the NH coupling) we found that we have real energy eigenvalues and the corresponding eigenstates respect the PT symmetry in each invariant subspace. Thus we observe PT phase transition in each invariant subspace of the system in this exactly solvable model \[61\]. However PT phase transition is also observed in quasi exactly solvable (QES) models \[62, 66, 67\]. To demonstrate this we consider a couple of real QES systems and analyse them after making them NH in a PT symmetric way. We explicitly show the occurrence of PT phase transition even in QES systems through analytic calculations \[62\].

Even though numerous NH systems have been studied in 1-d QM, the study of NH system in higher dimensions is restricted to few models \[68, 69, 70, 71, 72, 73, 63\]. Therefore next we consider a couple of oscillator models in higher dimensional QM \[63\]. In the first model, we consider an anisotropic SHO with a simple PT symmetric NH interaction which can be solved analytically. Very interestingly we observe the system to undergo PT phase transition as long as the oscillator is anisotropic in \(x\) and \(y\) direction. The isotropic oscillator with the same NH interaction remains always in the broken phase and it is not possible to find any parameter space for the unbroken phase. Exactly similar conclusion is reached with the three dimensional model. In this case we start with an isotropic SHO in the external magnetic field. External magnetic field induces anisotropy in the system and again we have PT phase transition in the 3-d model. These two examples provide enough indications about some hidden connection between

\(^1\) It is more appropriate to call this as ”PT symmetry breaking transition” since ”phase transition” is typically reserved for non-analyticity of free energy in classical statistical mechanics and applies to thermodynamic-limit systems.
the symmetry of the hermitian part of the hamiltonian and the PT phase transition. 

(2+1) dimensional relativistic QM has become extremely important due to discovery of graphene, a single layer of carbon atoms arranged in a honeycomb lattice. To investigate PT phase transition in a relativistic model, we consider a massless Dirac particle in (2+1) dimension in the external transverse magnetic field in the background of Dirac oscillator (DO). Further we make it NH by introducing a Rashba type spin-orbit coupling with imaginary coupling constant. By using the various structure of parity and time reversal transformation in relativistic QM, we show that such model is PT symmetric and can be solved exactly with analytic calculations. The time reversal of this system represents the other valley of the massless particle. We show the existence of PT phase transition in both the valleys depending on the strength of the magnetic field and value of the spin orbit coupling. More interestingly a small mass gap which is consistent with other approaches is generated only in the unbroken phase of the system. Mass gap vanishes at the phase transition point. Details of this work is provided in Ref.

Finally we discuss NH theories in QFT. In particular we discuss how PT phase transition can occur in a model of free charged scalar with NH mass matrix. However it is more interesting to consider NH models in gauge theories. Using the natural but unconventional Hermiticity properties of ghost fields, SU(N) QCD is cast as NH model of QCD. A new quadratic gauge was recently introduced in Ref to study Abelian projection. In this gauge SU(N) gauge theoretic model has two distinct phases, namely the normal or deconfinement phase and ghost condensed phase or confinement phase. The PT properties of this model have been investigated in both phases to realize that deconfinement to confinement phase transition is actually a PT phase transition. Appropriate C operator for this theory has been constructed.

Now we indicate the plan of this present manuscript. In the next section, we discuss various forms parity and time reversal symmetry in one, two and three dimensional quantum theory, in (2+1) dimensional relativistic quantum mechanics and in quantum field theory. PT phase transition in 1-d exactly solvable as well as in QES systems are reviewed with explicit examples in Sec III. In section IV. we demonstrate the connection between symmetry and broken phases with the help of anisotropic oscillators in two and three dimensional quantum mechanics. PT phase transition in (2+1) dimensional relativistic system is considered Sec V. Section VI. is devoted to discuss phase transition in SU(N) QCD. Summary and conclusions are presented in Sec VII.

2. Parity and Time Reversal symmetries

In this section we will discuss the various form of parity and time reversal symmetry applicable in the different contexts.

2.1. PT transformations in 1-d QM

Parity and time reversal transformations are defined in 1-d QM as

\[ P: \quad x \to -x, \quad p \to -p; \quad T: \quad x \to x, \quad p \to -p, \quad i \to -i \]  

Here parity is simply space reflection about the origin and time reversal is anti-linear operator. In general if we consider reflection about \( x = a \), then parity transformation will be \( x \to 2a - x, \quad p \to -p \). This general parity transformation will be used in Sec. III.B
2.2. PT transformations in higher dimensional QM

The parity transformation in 2-d, has typical characteristics. Since, parity transformation is an improper Lorentz transformation, in two dimensions it can not be written as $x' = -x, \ y' = -y$. The determinant of the transformation matrix for this is +1. The above transformation represents rotation about z-axis with rotation angle $\theta = \pi$ [85]. The discrete parity transformation in 2-d QM therefore can be defined in few alternative ways, as for example,

$$P_1 : x' = -x, \ y' = y; \quad p_x' = -p_x, \ p_y' = p_y \quad (2)$$

$$P_2 : x' = x, \ y' = -y; \quad p_x' = p_x, \ p_y' = -p_y \quad (3)$$

Both of these forms $P_1$ and $P_2$ are equivalent and one can use either of these while checking PT symmetry of a NH system in two dimensions. In 2-d parity transformation has also been realized as [86]

$$P_3 : x' = y, \ y' = x; \quad p_x' = p_y, \ p_y' = p_x \quad (4)$$

In fact in the study of NH theories, one only requires $P$ to be an involution such that overall PT becomes an anti-linear operator.

The structure of parity transformation in 3-d QM is much more rich. Various different versions of the anti-linear PT transformation due to the rich structure of parity transformation in 3-d have been addressed in details in Ref. [87]. However in three or in any odd dimension commonly used parity transformation is given as space inversion.

$$x' = -x, \ y' = -y, \ z' = -z; \quad p_x' = -p_x, \ p_y' = -p_y, \ p_z' = -p_z \quad (5)$$

In the even dimension, one has to be careful in defining parity transformation as the determinant of transformation matrix should be -1. We will using these transformation properties in Sec IV. Time reversal has same structure in higher dimension as in 1-d.

2.3. PT transformations in relativistic QM

Now we discuss the symmetry properties in relativistic QM. In (3+1) dimension the P and T which leave Dirac equation invariant, for the Dirac wave function are given by

$$P\psi(x,y,z,t) = \gamma_0\psi(-x,-y,-z,t); \quad T\psi(x,y,z,t) = -i\gamma_1\gamma_3\psi^*(x,y,z,-t) \quad (6)$$

where $\gamma_i \ i = 0, 1, 2, 3$ are well known Dirac gamma matrices. Since the P transformation in 2-d is bit different (as discussed in Sec II.B) the Dirac wavefunctions in (2+1) dimension transform as

$$P_1\psi(x,y,t) = \sigma_y\psi(-x,y,t); \quad P_2\psi(x,y,t) = \sigma_x\psi(x,-y,t). \quad (7)$$

Free Dirac equation remains invariant under such $P_1$ and $P_2$ transformations. The T transformation ($t \rightarrow -t, \ i \rightarrow -i, \ p_x \rightarrow -p_x, \ p_y \rightarrow -p_y$) in (2+1) dimension Dirac theory is defined as $T = i\sigma_y\tilde{K}$, where $\tilde{K}$ is complex conjugation operation such that,

$$T\psi(x,y,t) = i\sigma_y\tilde{K}\psi(x,y,-t) = i\sigma_y\psi^*(x,y,-t). \quad (8)$$

We will be using these in the later section to study PT phase transition in (2+1) relativistic QM.
2.4. PT transformations in QFT

Detail discussion in discrete symmetries for various quantum field theoretic systems can be found in standard text books [88]. In this section we list down only those equations which will be used in Section VI for the investigation of PT phase transition in QFT. For charged scalar field \( \Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \), P and T respectively are defined on the doublet as follows

\[
\Phi \xrightarrow{P} P_{2 \times 2} \Phi; \quad \Phi \xrightarrow{T} T_{2 \times 2} \Phi^*
\]  

(9)

From the analogy of the P transformation in 2-d, where \( x \to x \) and \( y \to -y \) we can say that the field \( \phi_1 \) transforms as a scalar and the other, \( \phi_2 \) transforms as a pseudo scalar. Thus the \( P_{2 \times 2} \) is

\[
P_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z
\]

(10)

This implies the choice for the time reversal, \( T_{2 \times 2} \) such that the Lagrangian for free complex scalar field is PT-invariant. We must choose \( T_{2 \times 2} = 1 \) [89, 91]. One can in principle swap the roles of \( P_{2 \times 2} \) and \( T_{2 \times 2} \) as long as symmetry is concern, however in order to interpret this PT-symmetric theory in terms of a coupled system with gain and loss, one should take \( T = 1 \) [89, 91].

Now we discuss these transformation properties of fields in SU(N) field theory as we will be focusing that for our NH study. Parity and time reversal properties of the gluons are well defined and is given by

\[
A^a_i(x,t) \xrightarrow{P} -A^a_i(-x,t); \quad A^a_0(x,t) \xrightarrow{P} A^a_0(-x,t).
\]

(11)

The rule for parity is same for all gluons as it is a linear operator. The SU(N) QCD Lagrangian is invariant under P if we choose ghosts to be scalars/pseudo scalars. However, we consider ghost fields transform as pseudo-scalar under P is more appropriate as ghosts are anti-commuting scalar

\[
c^a(x,t) \xrightarrow{P} -c^a(-x,t); \quad \bar{c}(x,t) \xrightarrow{P} -\bar{c}(-x,t).
\]

(12)

Since some of the generators of \( SU(N) \) are purely imaginary, the time reversal (which is an anti-linear transformation) property is not same for all gluons. Suppose \( a = \{q\} \) corresponds to purely imaginary generators and \( a = \{p\} \) corresponds to real generators. Then for SU(3) \( q = 2, 5, 7 \) and \( p = 1, 3, 4, 6, 8 \). And T for SU(N) gluons is given by

\[
A^p_i(x,t) \xrightarrow{T} -A^p_i(x,-t); \quad A^p_0(x,t) \xrightarrow{T} A^p_0(x,-t),
\]

\[
A^q_i(x,t) \xrightarrow{T} A^q_i(x,-t); \quad A^q_0(x,t) \xrightarrow{T} -A^q_0(x,-t),
\]

(13)

Thus the field strength with any spacetime and group indices can utmost change up to overall negative sign i.e., \( F^a_{\mu\nu} \xrightarrow{T} \pm F^a_{\mu\nu} \). The time reversal property for ghosts is then defined in the following manner,

\[
c^p(x,t) \xrightarrow{T} i\epsilon^p(x,-t); \quad \bar{c}(x,t) \xrightarrow{T} i\bar{c}(x,-t)
\]

\[
c^q(x,t) \xrightarrow{T} c^q(x,-t); \quad \bar{c}(x,t) \xrightarrow{T} \bar{c}(x,-t)
\]

(14)

such that \( SU(N) \) QCD Lagrangian remains invariant under PT.
3. PT Phase transition in 1-d QM

In this section we will demonstrate PT phase transition in 1-d QM with the help of two simple models with exact analytical calculations. We chose one exactly solvable and another one QES models.

3.1. Non-Hermitian interaction of Spin-half particle with SHO

A spin $\frac{1}{2}$ particle in the external magnetic field, $\vec{B}$ coupled to an oscillator through some NH interaction is described by the Hamiltonian

$$H = \mu \vec{A} \cdot \vec{B} + \hbar \omega a^{\dagger}a + \rho(\sigma_{+}a - \sigma_{-}a^{\dagger}).$$  \hspace{1cm} (15)

Where $\sigma$'s are Pauli matrices, $\rho$ is some arbitrary real parameter and $\sigma_{\pm} \equiv \frac{1}{2}[\sigma_{x} \pm i\sigma_{y}]$ are spin projection operators. Usual creation and annihilation operators for the oscillator states $a, a^{\dagger}$ are defined as $a = \frac{\sigma^{y} + im_{\omega x}}{\sqrt{2m_{\omega}\hbar}}, \quad a^{\dagger} = \frac{\sigma^{y} - im_{\omega x}}{\sqrt{2m_{\omega}\hbar}}$ with $a|n>= \sqrt{n}|n-1>$, and $a^{\dagger}|n>= \sqrt{n+1}|n+1>$ where the notation $|n>$ for number eigenvectors for the oscillator has been adopted. We choose the external magnetic field in $z$-direction, $\vec{B} = B_{0}\hat{z}$ and the Hamiltonian for the system as given in Eq.(15) is reduced to,

$$H = \frac{\epsilon}{2}\sigma_{z} + \hbar\omega a^{\dagger}a + \rho(\sigma_{+}a - \sigma_{-}a^{\dagger}),$$  \hspace{1cm} (16)

where $\epsilon = 2\mu B_{0}$. Note that this Hamiltonian is not hermitian as,

$$H^{\dagger} = \frac{\epsilon}{2}\sigma_{z} + \hbar\omega a^{\dagger}a - \rho(\sigma_{+}a - \sigma_{-}a^{\dagger}), \neq H$$  \hspace{1cm} (17)

as $\sigma_{\pm} = \sigma_{\mp}$. However this hamiltonian is invariant under PT transformation. To see the PT phase transition in this system we need to find the energy eigenvalues and corresponding eigenvectors of the system. We adopt the notation for the state as, $|n,\frac{1}{2}m_{s}>$ where $n$ is eigenvalue for the number operator $a^{\dagger}a$ i.e. $a^{\dagger}a|n>= n|n>$ and $m_{s} = \pm 1$ are the eigenvalues of the operator $\sigma_{z}$. $|\frac{1}{2}m_{s}>= \frac{1}{\sqrt{2}}(|\frac{1}{2}m_{s}>)$. It is readily seen that $|0, -\frac{1}{2}>$ is a ground state of the Hamiltonian with eigenvalue $-\frac{\epsilon}{2}$ and it is non-degenerate.

$$H|0, -\frac{1}{2}> = -\frac{\epsilon}{2}|0, -\frac{1}{2}>.$$  \hspace{1cm} (18)

We observe that the next possible states $|0, \frac{1}{2}>$ is not a eigenstate of the Hamiltonian,

$$H|0, \frac{1}{2}> = -\frac{\epsilon}{2}|0, \frac{1}{2}> - \rho|0, -\frac{1}{2}>,$$  \hspace{1cm} (19)

The spin projection operators $\sigma_{\pm}$ have the usual properties $\sigma_{\pm}|n, \pm \frac{1}{2}> = 0; \quad \sigma_{\pm}|n, \mp \frac{1}{2}> = |n, \pm \frac{1}{2}>$. However this state along with the state $|1, -\frac{1}{2}> close under the action of the Hamiltonian and form a invariant subspace in the space of states as,

$$H|1, \frac{1}{2}> = (\hbar\omega - \frac{\epsilon}{2})|1, \frac{1}{2}> + \rho|0, \frac{1}{2}>,$$  \hspace{1cm} (20)

First two excited states belong to this sector spanned by these two states, $|0, \frac{1}{2}> and |1, -\frac{1}{2}> wherein the Hamiltonian matrix is given by

$$H_{1} = \begin{bmatrix}
\frac{\epsilon}{2} & -\rho \\
-\rho & -\frac{\epsilon}{2} + \hbar\omega
\end{bmatrix}.$$
The eigenvalues of this Hamiltonian matrix are given by \( \lambda^{I,II}_{1} = \frac{1}{2} \left[ \hbar \omega \pm \sqrt{(\hbar \omega - \epsilon)^2 - 4\rho^2} \right] \).

Note these eigenvalues are real provided \( |\hbar \omega - \epsilon| \geq 2\rho \). Putting \( 2\rho = (\hbar \omega - \epsilon) \sin \theta_{1} \) we find the eigenvectors corresponding to this doublet are

\[
|\Psi_{1}^{I}\rangle = \cos \frac{\theta_{1}}{2} |0, \frac{1}{2}\rangle + \sin \frac{\theta_{1}}{2} |1, -\frac{1}{2}\rangle, \quad \text{for } \lambda_{1}^{I} = \frac{\hbar \omega}{2} (1 + \cos \theta_{1}) - \frac{\epsilon}{2} \cos \theta_{1},
\]

\[
|\Psi_{1}^{II}\rangle = \sin \frac{\theta_{1}}{2} |0, \frac{1}{2}\rangle + \cos \frac{\theta_{1}}{2} |1, -\frac{1}{2}\rangle, \quad \text{for } \lambda_{1}^{II} = \frac{\hbar \omega}{2} (1 - \cos \theta_{1}) + \frac{\epsilon}{2} \cos \theta_{1}.
\]

It may be observed that these two states are not orthogonal to each other nor do they have to be as \( H \neq H^\dagger \). The system can be solved by considering the subsequent subspaces and the full spectrum is obtained. The general states of the sector spanned by \(|n, \frac{1}{2}\rangle\) and \(|n+1, -\frac{1}{2}\rangle\) wherein the Hamiltonian matrix is given by,

\[
H_{n+1} = \begin{bmatrix}
\frac{\epsilon}{2} + n\hbar \omega & \rho \sqrt{n+1} \\
-\rho \sqrt{n+1} & -\frac{\epsilon}{2} + (n+1)\hbar \omega
\end{bmatrix}.
\]

Now we have the eigenvalues of this Hamiltonian matrix are given by

\[
\lambda^{I,II}_{n+1} = \frac{1}{2} \left[ (2n+1)\hbar \omega \pm \sqrt{(\hbar \omega - \epsilon)^2 - 4\rho^2(n+1)} \right]. \tag{21}
\]

These eigenvalues are real provided \( |\hbar \omega - \epsilon| \geq 2\rho \sqrt{n+1} \). Now putting \( 2\rho \sqrt{n+1} = (\hbar \omega - \epsilon) \sin \theta_{n+1} \), we find the eigenvectors corresponding to this doublet are

\[
|\Psi_{n+1}^{I}\rangle = \cos \frac{\theta_{n+1}}{2} |n, \frac{1}{2}\rangle + \sin \frac{\theta_{n+1}}{2} |n+1, -\frac{1}{2}\rangle,
\]

for \( \lambda_{n+1}^{I} = \frac{\hbar \omega}{2} (2n+1 + \cos \theta_{n+1}) - \frac{\epsilon}{2} \cos \theta_{n+1} \),

\[
|\Psi_{n+1}^{II}\rangle = \sin \frac{\theta_{n+1}}{2} |n, \frac{1}{2}\rangle + \cos \frac{\theta_{n+1}}{2} |n+1, -\frac{1}{2}\rangle,
\]

for \( \lambda_{n+1}^{II} = \frac{\hbar \omega}{2} (2n+1 - \cos \theta_{n+1}) + \frac{\epsilon}{2} \cos \theta_{n+1} \). \tag{22}

We observe that these eigenstates respect PT symmetry \( PT|\Psi^{I,II}_{n+1}\rangle = (-1)^n|\Psi^{I,II}_{n+1}\rangle \) as \( P|n, \pm \frac{1}{2}\rangle = (-1)^n|n, \pm \frac{1}{2}\rangle \) and \( \sigma_z |n, \pm \frac{1}{2}\rangle = \pm |n, \pm \frac{1}{2}\rangle \). Thus we have real eigenvalues when the symmetry is not broken. In the regime \( |\hbar \omega - \epsilon| < 2\rho \sqrt{n+1} \) the eigenvalues for a particular doublet become complex conjugate. For \( n+1 \) th doublet, the complex eigenvalues are \( \frac{1}{2} \left[ \hbar \omega (2n+1) + \frac{\epsilon}{2} \sqrt{4\rho^2(n+1) - (\hbar \omega - \epsilon)^2} \right] \).

### 3.2. PT Phase transition in 1-d QES system

There are only limited number of systems in QM which can be solved exactly. However there are Hamiltonians which preserve the finite dimensional Hilbert space and the spectrums for those systems are calculated exactly for a finite part in a closed form. Such systems are known as QES\[95, 96, 97\]. We consider the QES system described by the NH but PT-invariant Hamiltonian

\[
H = p^2 - (\zeta \cosh 2x - iM)^2, \tag{23}
\]
where the parameter $\zeta$ is real, parameter $M$ has only integer values and $\hbar = 2m = 1$. This NH system is PT symmetric. Here we consider parity as $x \to i\pi/2 - x$, $p \to -p$. Like corresponding Hermitian model ( removing $i$ from the Hamiltonian), this NH system is also shown to be QES system. The QES levels for these systems are calculated using Bender and Dunne [92] method for different integer values of $M$. For positive integer values of $M (= J)$, first $J$ levels for this systems can be calculated exactly in the closed form. We observed very different scenarios for the even and odd values of $J$. Hence the cases for even and odd $M$ are presented separately.

**Case-I** $M = 2$ : The QES eigenvalues and corresponding eigenvalues are given as

$$E_{\pm} = 3 - \zeta^2 \pm 2i\zeta.$$  
QES eigenvalues are in complex conjugate pair and can never be real for non-zero $\zeta$ and PT symmetry is broken spontaneously as the QES eigenfunctions are not eigenstates of PT. Because under the PT transformation as defined above, $\cosh x \to -i\sinh x$, $\sinh x \to i\cosh x$ while $i\cosh 2x$ remains invariant so that the two wave functions are not invariant under PT.

**Case-II**: $M = 4$: The four QES eigenvalues are

$$E_{\pm}^1 = 11 - \zeta^2 - 2i\zeta \pm 4\sqrt{1 - i\zeta - \zeta^2} ; \quad E_{\pm}^2 = 11 - \zeta^2 + 2i\zeta \pm 4\sqrt{1 + i\zeta - \zeta^2} .$$  
and the corresponding eigenfunctions are

$$\psi_{+}^1(x) = e^{i\frac{x}{2}\cosh 2x}\phi(x) (A\sinh 3x + B\sinh x), \quad \frac{B}{A} = \frac{E - 7 + \zeta^2}{2i\zeta};$$  
$$\psi_{-}^1(x) = e^{i\frac{x}{2}\cosh 2x}\phi(x) (A\cosh 3x + B\cosh x), \quad \frac{B}{A} = \frac{E - 7 + \zeta^2}{2i\zeta} ,$$  

(24)  

(25)  

(26)  

It is easy to check that PT symmetry is broken spontaneously. This is always true for any value of even $M$. However on the other hand, if $M$ is an odd integer, then the eigenvalues are real provided $\zeta^2 < \zeta_c^2$ and PT symmetry is respected by the eigenstates.

**Case-III** $M = 1$: The QES level is at $E = 1 - \zeta^2$ with the eigenfunction $\psi(x) = e^{i\frac{\zeta}{2}\cosh 2x}$. Here $PT\psi = \psi$ and hence PT symmetry remains unbroken for any value of $\zeta$. This is expected as there is no possible to have a complex conjugate pair of eigenvalues for $M = 1$.

**Case IV**: $M = 3$: The three QES eigenvalues and eigenfunctions are

$$E = 5 - \zeta^2 \quad E_{\pm} = 7 - \zeta^2 \pm 2\sqrt{1 - 4\zeta^2}$$

$$\psi = e^{i\frac{\zeta}{2}\cosh 2x}\sinh 2x \quad \psi_{\pm} = e^{i\frac{\zeta}{2}\cosh 2x}(A\cosh 2x + iB)$$  

(27)  

where $B/A = \frac{4\zeta}{E - 9 + \zeta^2}$. These QES levels are real if $\zeta^2 < \zeta_c^2 = 1/4$ and at $\zeta = \zeta_c$, the two highest QES levels are degenerate. It is easily checked that the corresponding wavefunction $\psi$ is indeed an eigenfunctions of PT with eigenvalue 1 while the other two wave functions ($\psi_{\pm}$) are eigenfunctions with eigenvalue -1. Thus the PT symmetry remains unbroken. It has been checked for higher odd values of $M$ numerically and same result is true. Thus we have PT phase transition in this QES system for odd integers of $M$. For even $M$ the QES system always in broken phase.

Further, using the anti-isospectral transformation QES eigenvalues and eigenfunctions of the PT-invariant potential, $V(\theta) = (\zeta \cos 2\theta - iM)^2$ are related to those of the $(\zeta \cosh 2x - iM)$
by $\tilde{E}_k = -E_{M-1-k}$, $\tilde{\psi}_k(\theta) = \psi_{M-1-k}(ix)$. In particular, the QES eigenvalues of this periodic potential are real if $M$ is an odd integer and $\zeta \leq \zeta_c$ and further in this case QES eigenstates are also eigenstate of PT. Therefore like the earlier case we have PT phase transition in this NH QES case. On the other hand, when $M$ is an even integer, then the eigenvalues are complex conjugate pairs and PT symmetry is spontaneously broken. The system remains in broken phase always.

There are many examples of PT phase transition in 1-d quantum mechanics but now we will focus on some models in two, three dimension to get some new features of PT phase transition.

4. PT Phase transition in higher dimension

In this section we consider two examples, one in two dimensions and other one in three dimensions to demonstrate the PT phase transition in higher dimension. Few other works in PT phase transition in higher dimension can also be found in[68, 69, 70, 71, 72, 73]

4.1. Role of anisotropy in PT Phase transition in 2-d

We consider an PT symmetric NH anisotropic oscillator in 2-d described by the Hamiltonian,

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_x^2x^2 + \frac{1}{2}m\omega_y^2y^2 + i\lambda xy, = H_0 + H_{nh}$$

(28)

$\lambda$ is real and $\omega_x \neq \omega_y$. The NH interaction term is invariant under $P_xT$, $P_yT$ and $P_3T$. However we will not use $P_3T$ as $H_0$ is not invariant under it. $P_1T(i\lambda xy) = P_2T(i\lambda xy) = i\lambda xy$

The energy eigenvalues and eigenfunctions for this system are written as

$$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar C_1 + (n_2 + \frac{1}{2})\hbar C_2, \quad C_1^2 = \frac{1}{2}\left[\omega_+^2 - \frac{\omega_+^2}{k}\right]; \quad C_2^2 = \frac{1}{2}\left[\omega_+^2 + \frac{\omega_+^2}{k}\right].$$

$$\psi_{n_1,n_2}(x,y) = N \exp\left[-\frac{m}{2\hbar}[C_1(x^2 + y^2) + (C_2 - C_1)2i\lambda kx]ight]$$

$$H_{n_1}[\alpha_1(\sqrt{\frac{k}{2} - 1} - i\sqrt{\frac{k}{2} - 1})]H_{n_2}[\alpha_2(\sqrt{\frac{k}{2} + 1} + i\sqrt{\frac{k}{2} + 1})]$$

(29)

$$\alpha_1^2 = \frac{mc_1}{\hbar} \text{ and } \alpha_2^2 = \frac{mc_2}{\hbar}. \text{ And where, } \omega_x^2 = \omega_x^2 \pm \omega_y^2; \quad \omega_\perp = \sqrt{\frac{\omega_x^2 + \omega_y^2}{k}}$$

Now we consider the case when $k$ is real i.e. $|\lambda| \leq \frac{|\omega_x^2 - \omega_y^2|}{2}$. In this case,

$$C_1^2 = \frac{1}{2}[\omega_\perp^2(1 + \frac{1}{k}) + \omega_\perp^2(1 - \frac{1}{k})] > 0, \quad C_2^2 = \frac{1}{2}[\omega_\perp^2(1 - \frac{1}{k}) + \omega_\perp^2(1 + \frac{1}{k})] > 0,$$  

(30)

as $k \geq 1$. This further leads to entire real spectrum. Further the eigenfunctions respect both $P_1T$ and $P_2T$ i.e. $P_1T\psi_{n_1,n_2}(x,y) = (-1)^i\psi_{n_1,n_2}(x,y); \quad i = 1, 2; \quad s_1 = n_1 + n_2, \quad s_2 = 1$. Therefore PT symmetry is unbroken as long as $|\lambda| \leq \frac{|\omega_x^2 - \omega_y^2|}{2}$. On the other hand for $|\lambda| > \frac{|\omega_x^2 - \omega_y^2|}{2}$, $k$ is imaginary and hence $P_1T\psi_{n_1,n_2}(x,y) \neq \pm\psi_{n_1,n_2}(x,y)$ for $i = 1, 2$ and the spectrum is no longer real. Some of the eigenvalues occur in complex conjugate pairs, the spectrum in this situation is written as

$$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar(A - iB) + (n_2 + \frac{1}{2})\hbar(A + iB)$$

(31)
Where A and B are real. It is clear from the above equation that $E_{n_1, n_2}$ and $E_{n_2, n_1}$ are complex conjugate to each other for $n_1 \neq n_2$ and energy eigenvalue are real when $n_1 = n_2$. The critical value of the coupling $\lambda_c = \frac{m\omega^2}{2}$ depends on the anisotropic of the system. If the system is more anisotropic the span of the PT unbroken phase is longer. When the system becomes isotropic, i.e. $\omega_x = \omega_y$, i.e. $\lambda_c = 0$, the system will always lie in the broken PT phase and it will not be possible to have entire spectrum real for any condition on the parameters. This result leads to a very important realization for this particular systems. Rotational symmetry which leads to isotropy of original Hermitian system $H_0$ is complementary to unbroken PT symmetry of the PT symmetric NH system $H$. As long as the original system $H_0$ has rotational invariance, $H$ can not have unbroken PT symmetry. The moment rotational symmetry of $H_0$ breaks the NH system becomes capable of going through a PT phase transition. The role of degeneracies in the Hermitian part of the PT Hamiltonian in determining a nonzero threshold has been extensively studied in Refs [70, 71].

4.2. Three-dimensional isotropic harmonic oscillator in an external imaginary magnetic field

We consider an isotropic simple harmonic oscillator in external imaginary magnetic field $(iB)$ with $i\mu_\parallel B$ coupling in 3-d as

$$H = \frac{1}{2m} (p_x^2 - \frac{iqA}{c})^2 + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) + i\mu_\parallel B$$

(32)

where $q$ is the charge of the oscillator. The vector potential in symmetric gauge for a uniform magnetic field in $z$ direction is written as $A = \{-B_y, B_z, 0\}$. Then the above Hamiltonian is reduced to

$$H = \frac{1}{2m}(p_x^2 + \frac{iqyB}{2c})^2 + \frac{1}{2m}(p_y^2 - \frac{iqxB}{2c})^2 + \frac{1}{2m}(p_z^2) + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) + i\mu_\parallel B$$

(33)

This PT invariant Hamiltonian further simplified to

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2} m\omega_1^2 (x^2 + y^2) + \frac{1}{2} m\omega_2^2 z^2$$

(34)

with $\omega_1^2 = \omega^2 - \frac{q^2 B^2}{4mc^2}$, $\omega_2^2 = \omega^2 - \frac{\omega_1^2}{4}$, $\omega_1 = \frac{qB}{mc}$ is usual cyclotron frequency. Anisotropy is induced in the system due to external magnetic field in some specific direction. The energy eigenvalue and eigenfunction for this systems are calculated as,

$$E_{n_xn_y,n_z} = (n_x + n_y + 1)\hbar \omega_1 + (n_z + \frac{1}{2})\hbar \omega_2$$

(35)

$$\psi_{n_xn_y,n_z}(x, y, z) = \exp \left[-\frac{\alpha_1^2}{2}(x^2 + y^2) + \frac{\alpha_2^2 z^2}{2}\right] H_{n_x}(\alpha x) H_{n_y}(\alpha y) H_{n_z}(\alpha z)$$

(36)

where, $\alpha_1^2 = \frac{m\omega_1}{\hbar}$, $\alpha_2^2 = \frac{m\omega_2}{\hbar}$. Now we will show that this system undergoes a PT phase transition, if $B \leq \frac{2m\omega_2}{q}$ for fixed $\omega$ or for a fixed magnetic field $\omega \geq \frac{\omega_1}{2}$, then $\omega_1$ is real and hence the entire spectrum is real and $\text{PT}\psi_{n_xn_y,n_z}(x, y, z) = (-1)^{n_x+n_y+n_z}\psi_{n_xn_y,n_z}(x, y, z)$, indicating the system is in unbroken PT phase. Alternatively if $B > B_c = \frac{2m\omega_2}{q}$ or $\omega \leq \frac{\omega_1}{2}$, then $\omega_1 = \pm \sqrt{\omega^2 - \frac{q^2 B^2}{4mc^2}}$, becomes complex, we have pairs of complex conjugate eigenvalues given as $E_{n_xn_y,n_z} = \pm i((n_x + n_y + 1)\hbar \omega + (n_z + \frac{1}{2})\hbar \omega)$ and system is in broken phase of PT. This means if we gradually change the strength of the external magnetic field for fixed oscillator frequency or if we change the oscillator frequency $\omega$ for fixed magnetic field, the system undergoes a PT phase transition. The details on this can be found in Ref. [63].
5. **PT Phase transition in Relativistic QM**

In the previous sections we have considered several examples to show the PT phase transitions in non-relativistic quantum mechanics in one and higher dimensions. Now we are going to consider PT symmetric NH systems in relativistic quantum mechanics to highlight some features of PT phase transition.

5.1. **The Model**

Let us consider a massless Dirac particle in \((2 + 1)d\) with imaginary spin-orbit interaction in the background of DO potential as described by the Hamiltonian

\[
H = v_f |\vec{\sigma}.(\vec{p} + e\vec{A}/c) - iK_1 \vec{r}\beta| + i\lambda(\vec{\sigma} \times \vec{\Pi}).\hat{z},
\]

where \(K_1, \lambda\) are real constants and \(v_f\) is Fermi velocity. The term linear in \(\vec{r}\) is DO potential [79] as in the non-relativistic limit, it reduces to SHO potential with strong spin-orbit coupling. Discussions on DO have been found in [80, 81, 82, 83]. We assume the magnetic field along \(z\)-direction, \(B = B_0 \hat{k}\) for simplicity and choose the vector potential in the symmetric gauge as \(\vec{A} = (-\frac{B_0 y}{2}, \frac{B_0 x}{2}, 0)\). The Hamiltonian in component form is written as

\[
H = v_f(\sigma_x \Pi_x + \sigma_y \Pi_y) - iK_1 v_f(\sigma_x x + \sigma_y y)\beta + i\lambda(\sigma_x \Pi_y - \sigma_y \Pi_x)
\]  

\[\text{(38)}\]

where, \(\Pi_x = p_x + \frac{eB_0 y}{2c}\) and \(\Pi_y = p_y + \frac{eB_0 x}{2c}\). It is straight forward to check that \(H^\dagger = v_f(\sigma_x \Pi_x + \sigma_y \Pi_y) - iK_1 v_f(\sigma_x x + \sigma_y y)\beta - i\lambda(\sigma_x \Pi_y - \sigma_y \Pi_x) \neq H\) 

This system, however is invariant under both the \(PT\) as \(P_iT(P_iT)^{-1} = H, \quad i = 1, 2\). The graphene has two distinct valleys in its electronic structure in which the electrons have the same energy. Similar to that, the time reversal of this Hamiltonian for massless particle defines the system to other valley and is given by,

\[
\tilde{H} = THT^{-1} = v_f(\sigma_x \tilde{\Pi}_x + \sigma_y \tilde{\Pi}_y) - iK_1 v_f(\sigma_x x + \sigma_y y)\beta - i\lambda(\sigma_x \tilde{\Pi}_y - \sigma_y \tilde{\Pi}_x)
\]

\[\text{(39)}\]

with \(\tilde{\Pi}_x = p_x + \frac{eB_0 y}{2c}\) and \(\tilde{\Pi}_y = p_y - \frac{eB_0 x}{2c}\). Further note \(\tilde{H} \neq \tilde{H}^\dagger\) and \([\tilde{H}, P_iT] = 0 = [\tilde{H}, P_2T]\). Now we proceed to show the PT phase transition and mass gap generation as a consequence of PT phase transition in both valleys of the system.

5.2. **PT phase transition and generation of mass gap**

In this section we analytically solve the Dirac equations corresponding to \(H\) and \(\tilde{H}\) for the massless particle explicitly to obtain energy levels and eigenfunctions. This will help us to demonstrate PT phase transition in \((2+1)\) dimensional in this relativistic system. We analytically showed that mass gap which is consistent with experimental finding is generated due to imaginary Rashba interaction in the background of DO potential as long as the system is in the unbroken phase. We further show how the different solutions of Dirac equation for both the valleys in this particular model are interrelated. For this purpose we write the Hamiltonians in Eqs.(38) and (39) in a compact form on a complex plane \(z = (x + iy)\) as,

\[
H = \begin{pmatrix}
0 & A\Pi_z + iC_1\bar{z} \\
B\Pi_z + iC_2\bar{z} & 0
\end{pmatrix}, \quad \tilde{H} = \begin{pmatrix}
0 & B\Pi_z + iC_2\bar{z} \\
A\Pi_z + iC_1\bar{z} & 0
\end{pmatrix}
\]

\[\text{(40)}\]
with, \( A = 2(v_f - \lambda) \), \( B = 2(v_f + \lambda) \), \( C_1 = K_1v_f - (v_f - \lambda)\frac{\partial}{\partial z} \) and \( C_2 = -K_1v_f + (v_f + \lambda)\frac{\partial}{\partial z} \) are constants and canonical conjugate momenta in complex plane \( \Pi_z \) and \( \Pi_{\bar{z}} \) are defined as \( \Pi_z = -i\hbar \frac{d}{dz} = \frac{1}{2}(p_x - ip_y) \), \( \Pi_{\bar{z}} = -i\hbar \frac{d}{d\bar{z}} = \frac{1}{2}(p_x + ip_y) \). It can be verified in a straightforward manner as

\[
[\bar{z}, \Pi_{\bar{z}}] = i\hbar; \quad [z, \Pi_z] = 0; \quad [\bar{z}, \Pi_z] = 0; \quad [\Pi_{\bar{z}}, \Pi_z] = 0; \quad [z, \Pi] = i\hbar
\]  

(41)

5.3. Solution for the system \( H \) and \( \hat{H} \)

We present the solutions for DE corresponding to the Hamiltonian \( H \) only and solution for \( \hat{H} \) can be written using symmetry properties. For this purpose we assume a two component trial solution as

\[
\psi = \begin{pmatrix} \xi(z, \bar{z})e^{d_1 z \bar{z}} \\ i\eta(z, \bar{z})e^{d_1 z \bar{z}} \end{pmatrix}
\]  

(42)

The solutions are given as,

\[
E^2_n = (n + 1)(AC_2 - BC_1)
\]

(43)

\[
\psi_n(z, \bar{z}, t) = \alpha_n^I \left( \frac{z^n}{i z^{n+1}} \right) e^{\frac{C_1}{\hbar}z \bar{z}} e^{-\frac{i}{\hbar}E_n t} \text{ for } d_1 = \frac{C_1}{A\hbar}
\]

(44)

and \( E^2_n = (n + 1)(BC_1 - AC_2) = -E^2_n \)

(45)

\[
\psi_n(z, \bar{z}, t) = \alpha_n^I \left( \frac{z^{n+1}}{i z^n} \right) e^{\frac{C_2}{\hbar}z \bar{z}} e^{-\frac{i}{\hbar}E_n t} \text{ for } d_1 = \frac{C_2}{B\hbar}
\]

(46)

We observe the following parameter symmetry to obtain \( \hat{H} \) from \( H \)

\[
H \rightarrow -\frac{A+B}{C_1+C_2} \rightarrow \hat{H}
\]

Interchanging \( A \leftrightarrow B \), \( C_1 \leftrightarrow C_2 \) is equivalent to \( \lambda \rightarrow -\lambda, K_1 \rightarrow -K_1 \) and \( B_0 \rightarrow -B_0 \). Using these changes of parameters in the solutions for Hamiltonian \( H \), the solutions for the time reversal system, \( \hat{H} \) are obtained

\[
E^2_n = (n + 1)(AC_2 - BC_1)
\]

(47)

\[
\tilde{\psi}_n^I(z, \bar{z}, t) = \tilde{\alpha}_n^I \left( \frac{\bar{z}^n}{i \bar{z}^{n+1}} \right) e^{\frac{C_1}{\hbar}z \bar{z}} e^{-\frac{i}{\hbar}E_n t} \text{ for } \tilde{d}_1 = \frac{C_1}{A\hbar}
\]

(48)

and \( E^2_n = (n + 1)(BC_1 - AC_2) = -E^2_n \)

(49)

\[
\tilde{\psi}_n^I(z, \bar{z}, t) = \tilde{\alpha}_n^I \left( \frac{\bar{z}^{n+1}}{i \bar{z}^n} \right) e^{\frac{C_2}{\hbar}z \bar{z}} e^{-\frac{i}{\hbar}E_n t} \text{ for } \tilde{d}_1 = \frac{C_2}{B\hbar}
\]

(50)

Now will proceed to discuss PT phase transition in the relativistic system with these solutions.
5.4. PT phase transition

Putting the values of the constant we see that the energy eigenvalues for the solutions corresponding to \( d_1 = \tilde{d}_1 = \frac{C_1}{\mathcal{M}} \) in both the valleys are,

\[
E_n = \pm \sqrt{(n + 1) \left[ 2(v_f^2 - \lambda^2) \frac{B_0 e \hbar}{c} - 4K_1 v_f^2 \hbar \right]}
\]  

(51)

The mass gap between positive and negative energy solutions in the system is given by

\[
\Delta_0 = \sqrt{2(v_f^2 - \lambda^2) \frac{B_0 e \hbar}{c} - 4K_1 v_f^2 \hbar} 
\]  

(52)

The condition for real spectrum

(i) \( \lambda^2 \leq v_f^2 (1 - \frac{2K_1 c}{B_0 e}) \equiv \lambda_c^2 \) OR

(ii) \( B_0 > 2K_1 \frac{v_f^2 c}{(v_f^2 - \lambda^2) e} \equiv B_0^c \)

(53)

And under this condition, \( \psi_n^I \) and \( \tilde{\psi}_n^I \) respect at both the valleys as \( P_1 T \) and \( P_2 T \) symmetry as \( P_1 T \psi_n^I(z, \bar{z}, t) = -i(-1)^n \psi_n^I(z, \bar{z}, t); \) and \( P_2 T \tilde{\psi}_n^I(z, \bar{z}, t) = -i(-1)^n \tilde{\psi}_n^I(z, \bar{z}, t); \) \( j = 1, 2. \)

Now we have few interesting comments regarding the phase transition of this system.

- Under the condition in (53) \( d_1 = \tilde{d}_1 = \frac{C_1}{\mathcal{M}} \) the system corresponding to both the valleys are in unbroken PT phase and solutions for \( d_1 = \tilde{d}_1 = \frac{C_2}{\mathcal{M}} \) correspond to the broken PT phase as \( P_1 T \psi_n^{II} \neq a_i \psi_n^{II}, \ P_2 T \psi_n^{II} \neq b_i \psi_n^{II} \) for \( i = 1, 2 \)
- Reverse things happen, when the condition in Eq. 53 is not satisfied i.e. \( \lambda > \lambda_c \) for fixed \( B_0 \) or \( B_0 < B_0^c \) for fixed \( \lambda \). That is the solutions for \( d_1 = \tilde{d}_1 = \frac{C_1}{\mathcal{M}} \) correspond to the unbroken PT phase and the \( d_1 = \tilde{d}_1 = \frac{C_2}{\mathcal{M}} \) will be in broken PT phase.
- Thus the system in both the valleys undergo PT phase transition for either of the values \( d_1 = \tilde{d}_1 = \frac{C_1}{\mathcal{M}} \) or \( d_1 = \tilde{d}_1 = \frac{C_2}{\mathcal{M}} \)
- For \( d_1 = \tilde{d}_1 = \frac{C_1}{\mathcal{M}} \), when \( \lambda < \lambda_c \) or \( B_0 > B_0^c \) we see from Eq.(52) mass gap is always positive and becomes imaginary when \( \lambda > \lambda_c \) or \( B_0 < B_0^c \). Thus a mass gap is generated as long as system is unbroken PT phase and given by the expression in Eq. (52). Further in unbroken phase \( \Delta_0 \) varies as \( \sqrt{B_0} \) for large enough magnetic field. This result is consistent with other approaches.
- For \( d_1 = \tilde{d}_1 = \frac{C_2}{\mathcal{M}} \) when \( \lambda > \lambda_c \) or \( B_0 < B_0^c \) the system is in unbroken phase and again real mass gap is generated which is consistent with other approaches. Thus mass gap is always generated in this theory which is consist with other approaches.

Special cases:

(i) I \( (K_1 = 0) \), i.e. No DO potential, the system passes to broken phase when strength of Rashba interaction exceeds Fermi velocity \( \lambda^2 > v_f^2 \). The mass gap in the unbroken PT phase is given by \( \Delta_0 = \sqrt{2(v_f^2 - \lambda^2) \frac{B_0 e \hbar}{c}} \) which is real as \( \lambda^2 < v_f^2 \) for the unbroken phase. Interestingly the mass gap vanishes at the transition point \( \lambda^2 = v_f^2 \).
(ii) In the absence of spin-orbit coupling ($\lambda = 0$), the system is Hermitian, we can have real energy eigenvalues as long as the magnetic field $B_0 > \frac{2K_1c}{e}$, the mass gap in this case is $\Delta_0 = \sqrt{2v^2f^2_0 - 2K_1}$ which is always real for $B_0 > \frac{2K_1c}{e}$. In both the cases for sufficiently large magnetic field $\Delta_0 \propto \sqrt{B_0}$, which is consistent with other approaches. In this case also mass gap vanishes at the transition point, $B_0 = \frac{2K_1c}{e}$. This critical value of magnetic field is same as obtained in [82]. The factor of 2 is due to the choice of vector potential in the symmetric gauge.

6. PT Phase transition in gauge theories

Before we go into the discussion of NH SU(N) gauge field theoretic model, we want to point out some technique, which is equally applicable for the non-Hermitian scalar field theoretic models.

6.1. Non-Hermitian complex scalars

We consider the following simple complex scalar NH model discussed in Refs. [89, 91].

$$L = \partial_\mu \phi_1^* \partial_\mu \phi_1 + \partial_\mu \phi_2^* \partial_\mu \phi_2 + [\phi_1^* \phi_2^* m_1^2 - \mu^2] [\phi_1 \phi_2^* m_2^2] + \frac{1}{4} \left[ m_1^2 - \mu^2 \right] \left[ m_2^2 - \mu^2 \right]$$

(54)

Here $m_1^2, m_2^2, \mu^2 \geq 0$ and the mass term in the above Lagrangian density is not Hermitian. However considering the PT transformations for field theoretic models as discussed in Sec. II it is straightforward to check that the Lagrangian in Eq. 54 is PT symmetric. The eigenvalues of the mass matrix are given as,

$$\Lambda_\pm^2 = \frac{1}{2}(m_1^2 + m_2^2) \pm \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4}$$

(55)

For $|m_1^2 - m_2^2| \geq 2\mu^2$, we are in the phase of unbroken PT symmetry and when $|m_1^2 - m_2^2| < 2\mu^2$ happens, we step into the region of broken PT-symmetry as eigenvalues turn complex and PT $\psi_\pm = \pm \psi_\pm$ is no longer valid, where $\psi_\pm$ are eigenfunctions of the mass matrix corresponding to eigenvalues $\Lambda_\pm^2$. We shall encounter similar NH mass matrix for gluons in our model with SU(N) gauge theories. The hidden C-symmetry can be associated with the fact, that eigenvalues in Eq. (55) do not change under $\mu^2 \rightarrow -\mu^2$. The charge conjugation is defined as follows $\Phi \rightarrow C\Phi^*$ and in this model C=P [89, 91]. The theory in the region $|m_1^2 - m_2^2| < 2\mu^2$ violates CP also but preserves CT symmetry. Such charge conjugation symmetry exist in the non Abelian model also.

6.2. SU(N) QCD in the quadratic gauge

The resulting effective Lagrangian density with a new quadratic gauge fixing $F^n_a[A^\mu(x)] = A^n_a(x)A^{\mu}a(x)$, [84] for each $a$ is given as,

$$L_Q = -\frac{1}{4} F^n_\mu F^{\mu\nu a} - \frac{1}{2\zeta} (A^n_a A^{\mu}a)^2 - 2\zeta A^{\mu}a(D_\mu c)^a$$

(56)

. $\zeta$ is an arbitrary gauge fixing parameter, the field strength $F^n_\mu = \partial_\mu A^n_a(x) - \partial_a A^n_\mu(x) - gf^{abc} A^n_b(x) A^n_c(x)$ and $(D_\mu c)^a = \partial_\mu c^a - gf^{abc} A^n_b(x) c^c$. The summation over an index $a$ is understood
when it appears repeatedly, including when occurred thrice in the ghost term. The resulting Lagrangian is BRST invariant which is essential for the ghost independence of the green functions and unitarity of the $S$-matrix. This quadratic gauge is very useful to discuss the abelian projection of SU(N) gauge theory.

6.3. Two phases of the theory

This theory has two different phases (i) the normal or deconfined phase described by the Lagrangian in Eq. (56) and (ii) the ghost condensed phase showing the confinement. In this ghost fields are auxiliary fields as they do not have kinetic terms. However play an important role in the IR regime. To demonstrate the significance of ghosts we expand the ghost Lagrangian,

$$\bar{c}^a A^a \mu (D^\mu c^a) = -\bar{c}^a A^a \mu \partial^\mu c^a + g f^{abc} c^b A^c A^a \mu$$

where the summation over indices $a$, $b$ and $c$ each runs independently over 1 to $N^2 - 1$. Now if the ghosts freeze they amount to a non-zero mass matrix for the gluons as follows

$$\left( M^2 \right)_{ab} ^{\text{dyn}} = 2g \sum_{c=1}^{N^2-1} f^{abc} \langle \bar{c}^c \rangle.$$

(57)

We would get masses of gluons by diagonalizing the matrix and finding its eigenvalues. The ghost condensation as a concept was introduced in Ref. [93]. In an $SU(N)$ symmetric state, where all ghost-anti-ghost condensates are identical

$$\langle \bar{c}^1 c^1 \rangle = ... = \langle \bar{c}^{N^2-1} c^{N^2-1} \rangle = ... = \langle \bar{c}^{N^2-1} c^{N^2-1} \rangle = K.$$

(58)

The mass matrix becomes

$$\left( M^2 \right)_{ab} ^{\text{dyn}} = 2gK \sum_{c=1}^{N^2-1} f^{abc}$$

(59)

which is an antisymmetric matrix i.e., non-Hermitian, $(M^2)^\dagger \neq M^2$ due to the antisymmetry of the structure constants.

The mass matrix has $N(N - 1)$ non-zero eigenvalues only and thus has nullity $N - 1$ which implies that $N(N - 1)$ off-diagonal gluons obtain masses and the $N - 1$ diagonal gluons remain massless. Because of the antisymmetry, eigenvalues of $M^2$ are purely imaginary and in conjugate pairs. The massive off-diagonal gluons are inferred as evidence of Abelian dominance, which is one of signatures of quark confinement. Further, mass squared of the off-diagonal gluon is purely imaginary, hence the pole of the off-diagonal gluon propagator is on imaginary $p^2$ axis which is another important signature of color confinement. The mass for gluons generated through a given dynamical mechanism breaks the gauge symmetry as usual. We note that there exist other mechanisms where the mass can consistently be given to gluons in a gauge invariant manner, a thorough overview of such mechanisms is found in Refs. [94]. Thus, we see that the ghost condensation acts as the QCD vacuum. Therefore, in the ghost condensed phase the Lagrangian can effectively be given as follows

$$\mathcal{L}_{GC} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} - \frac{1}{2\kappa} (A^{a}_{\mu} A^{a}_{\mu})^{2} + M_{a}^{2} A^{a}_{\mu} A^{a}_{\mu}$$

(60)

Here for the diagonal gluons $M_{a}^{2} = 0$, and off diagonal elements occur as $\pm \text{im}^2$. For $SU(3)$, $M_{3}^{2} = M_{8}^{2} = 0$ and the off-diagonal gluons, $M_{1}^{2} = +\text{im}^{2}_{1}, M_{2}^{2} = -\text{im}^{2}_{2}, M_{4}^{2} = +\text{im}^{2}_{3}, M_{5}^{2} = ...$
\[ -im_1^2, \quad M_2^2 = +im_2^2, \quad M_3^2 = -im_3^2 \quad (m_1^2, m_2^2, m_3^2 \text{ are positive real}) \]

So the gluons 1 and 2 can be considered as conjugate of each other. The same is true for other pairs. Hence for SU(3), the last term of the effective Lagrangian in Eq. 60 would be

\[ M_a^2 A^a_\mu A^{\mu a} = +im_1^2 (A^1_\mu A^{\mu 1} - A^2_\mu A^{\mu 2}) + im_2^2 (A^3_\mu A^{\mu 3} - A^6_\mu A^{\mu 6} + im_3^2 (A^5_\mu A^{\mu 5} - A^7_\mu A^{\mu 7}) \quad (61) \]

Now to show the PT phase transition we need to discuss hermiticity properties of the system as PT symmetry becomes meaningful only in non-Hermitian systems. The hermiticity property of fields \( A^a_\mu \) is well defined since they describe real degrees of freedom. Fields must be Hermitian in order to define the real degrees of freedom i.e., \( A^a_\mu = A^\dagger_\mu^a \). However, this is not the case for ghosts as the hermiticity properties remain unclear. As the operation of conjugation in principle transforms particle to its anti-particle, the following is the natural choice of hermiticity property for ghosts [90]

\[ c^{a1} = \overline{c^a}, \quad \overline{c^{\dagger}} = c^a \quad (62) \]

With these hermiticity properties the Lagrangian in the normal phase is not-hermitian since ghost term is not Hermitian as \( (\overline{c^a} \overline{\partial \mu c^a}) \dagger = (\partial \mu \overline{c^a}) c^a \neq (\overline{c^a} \overline{\partial \mu c^a}) \). The effective Lagrangian in the ghost condensed (confinement) phase (60) is also not Hermitian as the mass term for gluons is purely imaginary as explained. Important point here is to note that non hermiticity of the Lagrangian in this ghost condensed phase is free of the hermiticity convention for ghosts as they do not appear in this phase and thus the non hermiticity of the ghost condensed phase is profound. Anti-linearity makes two sets of ghosts transform in a completely different manner. Thus, the theory in normal phase is individually both parity and time reversal invariant. This PT symmetry breaks down spontaneously in the confined phase as we explain now.

It is easy to check that parity is still a symmetry of the theory in the confined phase as given in Eq. (60). However, the time reversal is broken due to pure complex nature of the mass term, \( M_a^2 A^a_\mu A^{\mu a} = \frac{\psi|}{\psi} \rightarrow -M_a^2 A^a_\mu A^{\mu a} \text{ and also PT } \psi \neq \pm \psi \), where \( \psi \)s are eigenfunctions of the mass matrix (59). The first two terms of \( \mathcal{L}_{GC} \) remain unaffected by the time-reversal. Thus, PT symmetry is broken in this phase. We can see that the anti symmetric nature of structure constant appearing in the mass matrix has led to this breaking. The PT symmetry breaking in the confined phase is profound as it is independent of the convention for ghosts. Therefore, the transition from the normal phase to the confinement phase with \( SU(N) \) symmetric ghost condensates is identified as PT phase transition from unbroken to broken PT phase. The association between colour confinement and spontaneous PT breaking is model and mechanism independent even though in this model the link is through ghost condensation since one prime signature of quark confinement, the pole of the propagator on purely imaginary \( p^2 \) axis, inevitably breaks PT symmetry. Usefulness of a consistent model such as one in this paper lies in that it gives valuable insight into a process through which the link can take place.

There is a crucial difference between the non Abelian model and the toy model of complex scalars. Complex scalar theory has the parameter \( \eta \equiv \frac{2\mu^2}{|m_1^2 - m_2^2|} \) whose value separates two phases of the PT symmetry in the theory. There is no such single order parameter in the non Abelian theory which governs the phase transition. Different ghost bilinears \( \overline{c^a} c^c \) (a and c runs over 1 to \( N^2 - 1 \) independently) gradually condensing to the stated \( SU(N) \) symmetric vacuum give rise to the PT phase transition in this non Abelian model. Thus, we have provided a gauge theory in which PT phase transition is explicitly shown for the first time[65].

Finally we would like to comment on the hidden C symmetry, which satisfy \( [H, C] = 0, \quad [PT, C] = 0, \quad C^2 = 1 \). So far, no explicit representation of the C-symmetry is known in the
framework of gauge theories. We realize that the inner automorphism exchanges group indices i.e. for SU(3), 1 ↔ 2, 4 ↔ 5, 6 ↔ 7, 3 ↔ 8 plays the role of C symmetry [84] It is clear that the theory in both the phases is invariant under CPT. In the broken PT phase, the theory also violates CP symmetry but preserves the CT, in complete analogy with the scalar model described in sec. II.

7. Conclusion

We review some of our works on PT phase transition through analytically solvable examples in various space time dimension in quantum mechanics and in quantum field theory. Our examples capture typical features of this phase transitions. In one dimension we realized PT phase transition very early through a QES system. QES levels have real energy eigenvalues and corresponding eigenvectors respect PT symmetry in the unbroken phase. However interestingly for even values of the QES parameter, this particular system always remain in broken phase. PT symmetric NH interaction between two level system and SHO was considered. In this model, except ground state all levels are doubly degenerate at the transition point. PT phase transition is shown to occur in all invariant subspaces of the system. Interesting connection between rotational symmetry of the original Hamiltonian and unbroken phase is observed in 2 and 3 dimensional quantum mechanical systems. Graphene acquires small mass gap due to spin orbit coupling. By considering PT symmetric NH spin orbit coupling for a massless Dirac particle in the background of DO potential, we show that mass gap is only generated in the unbroken PT phase of the system. SU(N) gauge field theoretic models is cast as NH model by adopting the natural but unconventional hermiticity properties of the ghost fields. By using a recently introduced quadratic gauge the phase transition in SU(N) QCD has been studied. We realize deconfinement to confinement phase transition as a PT phase transition in such a model.

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