FRAMED AND ORIENTED LINKS OF CODIMENSION 2

JIANHUA WANG

Abstract. Sanderson [12] gave an isomorphism \( \theta : \pi_m(\bigvee_{i=1}^r S_i^2) \cong \pi_m(\bigvee_{i=1}^{r+1} C_{P_i}^\infty) \).
In this paper we construct for any subset \( \sigma \subseteq \{1, 2, \cdots, r\} \) an isomorphism \( \theta_{\sigma} \) from \( \pi_m(\bigvee_{i=1}^r S_i^2) \) to \( \pi_m(\bigvee_{i=1}^{r+1} C_{P_i}^\infty) \). The inclusion \( S^2 \vee S^2 \hookrightarrow C_{P_i}^\infty \vee C_{P_i}^\infty \) induces a homomorphism \( f : \pi_m(S^2 \vee S^2) \to \pi_m(C_{P_i}^\infty \vee C_{P_i}^\infty) \). We also compute \( f \) by evaluating \( f \) on each factor in the Hilton splitting of \( \pi_m(S^2 \vee S^2) \), the results in [12] concerning the case \( m = 4 \) are generalized.

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1. Introduction

A link \( M_1 \sqcup M_2 \sqcup \cdots \sqcup M_r \subseteq \mathbb{R}^m \) is an ordered disjoint union of closed smooth submanifolds, \( m \geq 3 \). If these submanifolds are oriented then we call it an oriented link. A framing of a \( k \)-codimensional submanifold in \( \mathbb{R}^m \) is a trivialization of its normal vector bundle, or equivalently, an ordered set of \( k \) linearly independent normal vector fields. If every component of the link has a given framing then we call it a framed link. We will assume that the codimensions of the components are 2, when nothing else is stated. The bordism groups of framed and oriented links with \( r \) components of codimension 2 in \( \mathbb{R}^m \) are denoted by \( FL^2_{m,r} \) and \( L^2_{m,r} \) respectively. The facts

\[
FL^2_{m,r} \cong \pi_m(\bigvee_{i=1}^r S_i^2), \quad L^2_{m,r} \cong \pi_m(\bigvee_{i=1}^r C_{P_i}^\infty)
\]

are well known by Pontryagin-Thom construction.

The main purpose of this paper is to discuss the relationship between \( FL^2_{m,r} \) and \( L^2_{m,r+1} \) and to compute the homomorphism

\[
f : \pi_m(S^2 \vee S^2) \to \pi_m(C_{P_i}^\infty \vee C_{P_i}^\infty) \cong \pi_m(S^2).
\]

Geometrically, \( f \) is given by forgetting the framing of \([M_1 \sqcup M_2]_r \in FL^2_{m,2}\) and keeping the orientation determined by the framing, so we may call \( f \) a forgetful homomorphism.

Main results and the organization of this paper: In §2 we give a formula of reframing, suggested by Koschorke, and discuss briefly the role of framing in the Hilton splitting. Let \( \sigma \subseteq \{1, 2, \cdots, r\} \) be any subset. We construct in §3 an isomorphism \( \theta_{\sigma} : FL^2_{m,r} \to L^2_{m,r+1} \). We recover Sanderson’s isomorphism \( \theta \) by taking \( \sigma = \{1, 2, \cdots, r\} \). The forgetful homomorphism \( f \) is computed in §4 by using the inverse of \( \theta_{\sigma} \) and by
choosing $\sigma = \phi$, in particular the following result of [12] is generalized: in case $m = 4$ it holds $f \circ \gamma_* \neq 0$ for $\gamma = [\iota_1, \iota_2], [\iota_1, [\iota_1, \iota_2]]$ and $[\iota_2, [\iota_1, \iota_2]]$, $f \circ \gamma_* = 0$ for $\gamma = \iota_1$ and $\iota_2$.

We work in the category of smooth manifolds.

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2. Framing

Let $M^{m-2} \subset \mathbb{R}^m$ be a closed submanifold with framing $\mathcal{F} = (v_1, v_2)$, and let 

$$s : M \to S^1 \cong SO(2)$$

be a continuous map. For $x \in M$ we can represent $s(x) \in S^1$ by an orthonormal matrix

$$
\begin{pmatrix}
  a_{11}(x) & a_{12}(x) \\
  a_{21}(x) & a_{22}(x)
\end{pmatrix}.
$$

Define $v'_1$ and $v'_2$ by

$$
\begin{align*}
v'_1(x) &= a_{11}(x)v_1(x) + a_{12}(x)v_2(x), \\
v'_2(x) &= a_{21}(x)v_1(x) + a_{22}(x)v_2(x).
\end{align*}
$$

$(v'_1, v'_2)$ is a new framing of $M$ and is denoted by $s\mathcal{F}$. Up to homotopy we may assume $s$ is differential. Let $-1 \in S^1$ be a regular value of $s$ and consider $Z = s^{-1}(-1) \subset M$. Let $Z \times [-1, 1] \subset M$ be a small tubular neighbourhood of $Z$ such that the positive direction of $[-1, 1]$ is in agreement with the usual orientation of $S^1$. Up to homotopy $s\mathcal{F}$ is in fact the $2\pi$-rotation of $\mathcal{F}$ in this neighbourhood, namely outside this neighbourhood it is the same as $(v_1, v_2)$ and inside it

$$
\begin{align*}
v'_1(z, t) &= v_1(z, t) \cos(t + 1)\pi + v_2(z, t) \sin(t + 1)\pi, \\
v'_2(z, t) &= -v_1(z, t) \sin(t + 1)\pi + v_2(z, t) \cos(t + 1)\pi,
\end{align*}
$$

(1) (2)

where $z \in Z$ and $t \in [-1, 1]$.

Let $v_3$ be the normal vector field of $Z \subset M$, determined by the orientation of $S^1$, provide $Z \subset \mathbb{R}^m$ with the framing $(v_1, v_2, v_3)$. We define

$$[Z, \mathcal{F}, s] = [Z, (v_1, v_2, v_3)] \in \pi_m(S^3).$$

Let $\eta : S^3 \to S^2$ be the Hopf map, Koschorke observed that $[Z, \mathcal{F}, s]$ is the only obstruction to homotope $\mathcal{F}$ to $s\mathcal{F}$ and conjectured that $\eta_*[Z, \mathcal{F}, s]$ and the difference

$$[M, s\mathcal{F}] - [M, \mathcal{F}] \in \pi_m(S^2)$$

should be related by some formula.
It’s well known that $M$ bounds a Seifert surface $F$. If the framing of $M$ induced by $F$ is homotopic to the given framing $\mathcal{F}$ of $M$, then we say $M$ is $S$-framed. In this case we have $[M, \mathcal{F}] = 0$, and Turaev [15] proved $[M, s\mathcal{F}] = \eta_*[Z, \mathcal{F}, s]$. Note that if $M$ is oriented then the $S$-framing compatible with the orientation is unique up to homotopy.

**Proposition 2.1.** Let $M \subset \mathbb{R}^m$ be a closed submanifold with framing $\mathcal{F} = (v_1, v_2)$ and $s : M \to S^1$ be a map. Let $u\mathcal{F} = (-v_1, v_2)$ and $M^{sh}$ be a small shift of $M$ along $v_1$ provided with the framing $s\mathcal{F} = (v'_1, v'_2)$. Then it holds

$$\left([M \sqcup M^{sh}), (u\mathcal{F} \sqcup s\mathcal{F})\right] = \eta_*[Z, \mathcal{F}, s].$$

**Proof.** Writing $(M \sqcup M^{sh})$ we mean it is considered as submanifold rather than a link of two components. Without loss of generality we may assume that $\mathcal{F} = (v_1, v_2)$ is smooth and orthogonal. Let $\hat{W} = M \times [0, 1]$ be the trace of a shift from $M$ to $M^{sh}$ along $v_1$. Cut out a small $\varepsilon$-neighbourhood $U$ of $Z \times \{\frac{1}{2}\}$ in $\hat{W}$, and define $W = \hat{W} \setminus U$, where $Z = s^{-1}(-1)$. See Fig.1.

![Figure 1](image-url)

Provide $W \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+1}$ with the framing $\mathcal{G} = (e_{m+1}, v_2)$, where $e_{m+1}$ is the last vector in the usual base of $\mathbb{R}^{m+1}$. Let $Z \times [-\varepsilon, \varepsilon] \subset M$ be a $\varepsilon$-neighbourhood of $Z$. Up to homotopy we may assume $s$ maps $M \setminus Z \times [-\varepsilon, \varepsilon]$ to the base point $1 \in S^1$. We get now a well defined map $\hat{s} : W \to S^1$, given by

$$\hat{s}(x, t) = \begin{cases} 
    s(x) & : t \geq \frac{1}{2}, \\
    1 & : t \leq \frac{1}{2}.
\end{cases}$$

So we obtain a new framing $\hat{s}\mathcal{G}$ of $W \subset \mathbb{R}^{m+1}$. In addition, it holds $\partial W = M \sqcup M^{sh} \sqcup \partial U$, where $\partial U \approx Z \times S^1$ is the boundary of $U$. We construct now a diffeotopy of $\mathbb{R}^{m+1}$ which deforms $(W, \hat{s}\mathcal{G})$ to a framed bordism.

Let $\nu(M^{sh})$ be the normal vector bundle of $M^{sh} \subset \mathbb{R}^{m+1}$, framed by $(v_1, e_{m+1}, v_2)$. A homotopy of $\nu(M^{sh})$

$$F_1 : \nu(M^{sh}) \times [0, 1] \to \nu(M^{sh})$$
is given by rotating $e_{m+1}$ to $v_1$, $v_1$ to $-e_{m+1}$ and meanwhile keeping $v_2$ fixed. Define

$$F_1' : M^{sh} \times [0, 1] \rightarrow \mathbb{R}^{m+1}$$

by $F_1'(x, t) = (x, -t)$. Let $U_1$ be a $\delta$-neighbourhood of $M^{sh} \subset \mathbb{R}^{m+1}$ with $\delta \ll \epsilon$. From $F_1, F_1'$ we get an isotopy $H_1 : U_1 \times [0, 1] \rightarrow \mathbb{R}^{m+1}$, given by

$$H_1(x + r_1 v_1(x) + r_2 e_{m+1} + r_3 v_2(x), t) = F_1'(x, t) + r_1 F_1(v_1(x), t) + r_2 F_1(e_{m+1}, t) + r_3 F_1(v_2(x), t),$$

where $x \in M^{sh}$ and $x + r_1 v_1(x) + r_2 e_{m+1} + r_3 v_2(x) \in U_1$. It holds clearly

$$H_1(M^{sh}, 1) = M^{sh} \times \{-1\} \subset \mathbb{R}^m \times \{-1\}.$$  

Since $(e_{m+1}, v_2)$ is deformed to $(v_1, v_2)$ and $\hat{s}|_{M^{sh}} = s$ ($s$ is defined on $M^{sh}$ by identifying $M^{sh}$ with $M$ in the natural way), $\hat{s}G|_{M^{sh}}$ is homotoped to $sF$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}

Let $U_0$ be a $\delta$-neighbourhood of $M \subset \mathbb{R}^{m+1}$. Similarly we have an isotopy

$$H_0 : U_0 \times [0, 1] \rightarrow \mathbb{R}^{m+1}$$

which deforms $M$ to $M \times \{-1\} \subset \mathbb{R}^m \times \{-1\}$ and $(e_{m+1}, v_2)$ to $(-v_1, v_2) = uF$. We have used the homotopy $F_0 : \nu(M) \times [0, 1] \rightarrow \nu(M)$ which rotates $e_{m+1}$ to $-v_1$ and keeps $v_2$ fixed.

Let $\nu(\partial U)$ be the normal vector bundle of $\partial U \subset \mathbb{R}^{m+1}$, framed by $(u_1, e_{m+1}, v_2)$, where $u_1$ is the normal vector field of $\partial U \subset W$ pointing inwards. Let

$$F_2 : \nu(\partial U) \times [0, 1] \rightarrow \nu(\partial U)$$
be the homotopy given by rotating \( e_{m+1} \) to \( u_1 \) and keeping \( v_2 \) fixed; and define

\[
F_2' : \partial U \times [0, 1] \rightarrow \mathbb{R}^{m+1}
\]

by \( F_2'(x, t) = (x, t) \). From \( F_2 \) and \( F_2' \) we obtain an isotopy \( H_2 : U_2 \times [0, 1] \rightarrow \mathbb{R}^{m+1} \), where \( U_2 \) is a \( \delta \)-neighbourhood of \( \partial U \subset \mathbb{R}^{m+1} \). \( H_2 \) isotopes \( \partial U \times \{1\} \subset \mathbb{R}^m \times \{1\} \) and homotopes \((e_{m+1}, v_2)\) to \((u_1, v_2)\). So \( \hat{sG}|_{\partial U} \) is homotoped to \( \hat{s}(u_1, v_2) \). It is not difficult to see that \( \hat{s}(u_1, v_2) \) is homotopic to the \( 2\pi \)-rotation of the \( S \)-framing \((u_1, v_2)\) of \( \partial U \subset \mathbb{R}^m \times \{1\} \), and therefore

\[
(\partial U, \hat{s}(u_1, v_2)) = (Z \times S^1, \hat{s}(u_1, v_2))
\]

is just the fibre-wise embedding of the framed circle \( S^1 \subset \mathbb{R}^3 \) representing \( \eta : S^3 \rightarrow S^2 \). It follows \( [\partial U, \hat{s}(u_1, v_2)] = \eta_* [Z, \mathcal{F}, s] \).

Now \( U_\delta = U_0 \cup U_1 \cup U_2 \) is a \( \delta \)-neighbourhood of \( \partial W \subset \mathbb{R}^{m+1} \). The isotopies \( H_0, H_1 \) and \( H_2 \) together define an isotopy \( H : U_\delta \times [0, 1] \rightarrow \mathbb{R}^{m+1} \). According to a well known theorem in Differential Topology \( H \) determine a diffeotopy \( \tilde{H} \) of \( \mathbb{R}^{m+1} \) which deforms \((W, \hat{sG})\) to a framed bordism from \([(M \sqcup M^{sh}), (u\mathcal{F} \sqcup s\mathcal{F})]\) to \( \eta_* [Z, \mathcal{F}, s] \), see Fig.2. \( \square \)

**Corollary 2.2.** (i) [Turaev, 1985]: It holds \([M, s\mathcal{F}] = \eta_* [Z, \mathcal{F}, s] \), if \( \mathcal{F} \) is the \( S \)-framing of the submanifold \( M \);

(ii) \( E[M, s\mathcal{F}] - E[M, \mathcal{F}] = E\eta_* [Z, \mathcal{F}, s] \), where \( E \) is the suspension homomorphism.

**Proof.** (i) Let \( F \) be a Seifert surface of \( M \) giving the \( S \)-framing \( \mathcal{F} \). Because \( M^{sh} \) is a small shift of \( M \) along \( \mathcal{F} \), we have \( M^{sh} \pitchfork F = \phi \). Using \( F \) and \( M^{sh} \times [0, 1] \) we obtain easily a framed bordism from \([(M \sqcup M^{sh}), (u\mathcal{F} \sqcup s\mathcal{F})]\) to \([M, s\mathcal{F}]\). The assertion follows now from Proposition 2.1. See also Turaev [15].

(ii) We have clearly

\[
E[(M \sqcup M^{sh}), (u\mathcal{F} \sqcup s\mathcal{F})] = E[M, u\mathcal{F}] + E[M, s\mathcal{F}]
\]

\[
= -E[M, \mathcal{F}] + E[M, s\mathcal{F}].
\]

The statement follows from Proposition 2.1. \( \square \)

In general it is a subtle problem to measure the difference \([M, s\mathcal{F}] - [M, \mathcal{F}]\). Hilton-Hopf invariant up to order 3 are involved. Consider the framed link \((M, u\mathcal{F}) \sqcup (M^{sh}, s\mathcal{F})\) representing an element \( \alpha \) in

\[
\pi_m(S^2_1 \vee S^2_2) \cong \bigoplus_\gamma \pi_m(S^{q(\gamma)+1}),(\gamma)
\]

where \( \gamma \) runs through a system \( \Gamma \) of basic Whitehead products in \( \iota_1 < \iota_2 \), and \( q(\gamma) \) is the height of \( \gamma \in \Gamma \), see Hilton [2]. So we have the splitting \( \alpha = \oplus_\gamma \alpha_\gamma \) with \( \alpha_\gamma \in \pi_m(S^{q(\gamma)+1}) \).

Let \( \alpha_1, \alpha_2, \alpha_3 \) be the Hilton coefficients of \( \alpha \) corresponding to \([\iota_1, \iota_2], [\iota_1, [\iota_1, \iota_2]] \) and
If we map the wedge $S^2_1 \vee S^2_2$ canonically to the sphere $S^2$, then $\iota_1$ and $\iota_2$ are both identified with the identity $\iota$ of $S^2$, and $\alpha$ is mapped to the element
\[
[(M \sqcup M^{sh}), (uF \sqcup sF)] \in \pi_m(S^2).
\]
The basic Whitehead products in $\iota_1 < \iota_2$ are mapped to Whitehead products in $\iota$. Because the Whitehead products in $\iota$ with weight $> 3$ are zero homotopic, we have
\[
[(M \sqcup M^{sh}), (uF \sqcup sF)] = [M, uF] + [M^{sh}, sF] + [\iota, \iota]_* \alpha_1 + [\iota, [\iota, \iota]]_* (\alpha_2 + \alpha_3).
\]
Let $\alpha'_1$ be the first nontrivial Hilton-Hopf invariant of $[M, F]$, namely the one corresponding to $[\iota_1, \iota_2]$, then it holds (see Hilton [2])
\[
[M, uF] = -[M, F] + [\iota, \iota]_* \alpha'_1.
\]
By this and Proposition 2.1 we get

**Corollary 2.3.** It holds
\[
[M, sF] - [M, F] = \eta_* [Z, F, s] - [\iota, \iota]_* (\alpha_1 + \alpha'_1) - [\iota, [\iota, \iota]]_* (\alpha_2 + \alpha_3).
\]
Note that $3[\iota, [\iota, \iota]] = 0$ by Jacobi-identity.

Consider now a framed link $(M_1, F_1) \sqcup (M_2, F_2) \subset \mathbb{R}^m$ of codimensions $k_1, k_2 \geq 2$. The bordism group $FL_{m}^{k_1,k_2}$ of such links is isomorphic to $\pi_m(S^{k_1} \vee S^{k_2})$. We try to understand the role of the framings $F_1, F_2$ in the Hilton splitting of $(M_1, F_1) \sqcup (M_2, F_2)$.

Let $i = 1, 2$. A Seifert surface of $M_i$ is a compact oriented submanifold $F_i \subset \mathbb{R}^m$ with boundary $M_i$. If $M_i$ has a framed Seifert surface $F_i$ such that the framing of $M_i$ as the boundary of $F_i$ is homotopic to the original framing $F_i$, then we say $F_i$ is an $S$-framing of $M_i$ and $M_i$ is $S$-framed. We call $F_i$ a suitably framed Seifert surface. Note that two $S$-framings must not be homotopic.

**Proposition 2.4.** Let $(M_1, F_1) \sqcup (M_2, F_2) \subset \mathbb{R}^m$ be an $S$-framed link representing $\alpha \in \pi_m(S^{k_1} \vee S^{k_2})$, and let $F_1, F_2$ be the corresponding suitably framed Seifert surfaces.

(i) Up to involution it holds $[M_1 \natural F_2] = [F_1 \natural M_2]$. An involution is an isomorphism $u$ of the target group with $u \circ u = \text{id}$.

(ii) Let $\alpha = \oplus_\gamma \alpha_\gamma$ be the Hilton splitting of $\alpha$. Up to involution $\alpha_{[\iota_1, \iota_2]}$ is given by $[M_1 \natural F_2]$, all other Hilton coefficients $\alpha_\gamma$ are zero.

**Proof.** The desired framed bordism in the first part is given by $F_1 \natural F_2$.

Let $Z = M_1 \natural F_2$ and let $U \cong Z \times D^{k_1}$ be an open tubular neighbourhood of $Z \subset F_2$. Frame $\partial U$ so that $F_2 \setminus U$ gives rise to a framed bordism between $M_2$ and $\partial U$. Because $F_2 \setminus U$ is disjoint from $M_1$, $M_1 \sqcup M_2$ is framed bordant to $M_1 \sqcup \partial U$, see Fig.3.
If $U$ is small enough, then it holds $F_1 \cap \bar{U} \cong Z \times [0,1]$ with $Z = Z \times \{0\}$. Let $v$ be the normal vector field of $Z \subset F_1 \cap \bar{U}$ pointing inwards. It’s easy to see that $Z' = Z \times \{1\} = F_1 \cap \partial U$ is just a small shift of $Z$ along $v$. Let $U'$ be a small tubular neighbourhood of $Z' \subset F_1$ such that $U' \cap \partial U = Z'$. Frame $\partial U'$ so that $F_1 \setminus U'$ gives rise to a framed bordism between $M_1$ and $\partial U'$. Because $F_1 \setminus U'$ is disjoint from $\partial U$ it follows $[M_1 \sqcup \partial U] = [\partial U' \sqcup \partial U]$, see Fig. 3 again.

In addition, $\partial U' \sqcup \partial U = Z \times S^{k_2 - 1} \sqcup Z \times S^{k_1 - 1}$ is just the fibre-wise embedding of the standard framed Hopf link $S^{k_2 - 1} \sqcup S^{k_1 - 1} \subset \mathbb{R}^{k_1 + k_2 - 1}$ (at least up to involution of the framings) into a small tubular neighbourhood of the framed intersection $Z \subset \mathbb{R}^m$. It follows

$$[M_1 \sqcup M_2] = [\iota_1, \iota_2]_*[Z]$$

at least up to involution. Since the Hilton splitting of $[\iota_1, \iota_2]_*[Z]$ has the form

$$0 + 0 + [Z] + 0 + \cdots,$$

the assertion follows. \hfill \square

**Corollary 2.5.** Let $\gamma$ be a basic Whitehead product in $\iota_1 < \iota_2$ of weight $\geq 3$, and let $M_1 \sqcup M_2 \subset \mathbb{R}^{q(\gamma) + 1}$ be a framed link representing $\gamma$. At least one component of this link is not $S$-framed.

**Proof.** If this is not the case then according to the above result we have $H_\gamma[M_1 \sqcup M_2] = 0$, a contradiction to the fact $H_\gamma[M_1 \sqcup M_2] = \pm 1 \in \pi_{q(\gamma) + 1}(S^{q(\gamma) + 1}) \cong Z$, where $H_\gamma$ is the Hilton homomorphism corresponding to the basic Whitehead product $\gamma$. \hfill \square

### 3. Isomorphisms between $FL^2_{m,r}$ and $L^2_{m,r+1}$

Sanderson [12] gave an isomorphism $\theta : FL^2_{m,r} \rightarrow L^2_{m,r+1}$. We construct here for each subset $\sigma \subset \{1, 2, \ldots, r\}$ such an isomorphism $\theta_\sigma$.

Let $L = (M_1, F_1) \sqcup \cdots \sqcup (M_r, F_r) \subset \mathbb{R}^m$ be a framed link of codimension 2. For $i \in \sigma$ define $\hat{M}_i = -M_i^{sh}$, where $-M_i^{sh}$ is the negative oriented $M_i^{sh}$. Note that a
framed submanifold in \( \mathbb{R}^m \) is canonically oriented. For \( i \not\in \sigma \) consider the framed intersection \( Z_i = M_i \cap \tilde{F}_i \), where \( \tilde{F}_i \) is a Seifert surface of \( M_i^{sh} \). Let \( U_i \) be a small tubular neighbourhood of \( Z_i \subset \tilde{F}_i \). If we orient \( \partial U_i \) as the boundary of \( U_i \) then \( \tilde{F}_i \setminus U_i \) is a Seifert surface of \( (-M_i^{sh}) \cup (-\partial U_i) \). For \( i \not\in \sigma \) we define \( \tilde{M}_i = \partial U_i \) and \( M_{r+1} = \bigcup_{i=1}^{r} \tilde{M}_i \). \( \theta_\sigma \) is given by the assignment

\[
\theta_\sigma[L] = [M_1 \cup \cdots \cup M_r \cup M_{r+1}]_{or}.
\]

**Lemma 3.1.** \( \theta_\sigma : FL^2_{m,r} \rightarrow L^2_{m,r+1} \) is a well defined homomorphism.

**Proof.** To prove this let \( W_1 \cup \cdots \cup W_r \subset \mathbb{R}^m \times [0,1] \) be a framed bordism between

\[
L = \bigsqcup_{i=1}^{r}(M_i, F_i), \quad L' = \bigsqcup_{i=1}^{r}(M'_i, F'_i).
\]

Take \( \tilde{W}_i = W_i^{sh} \) and let \( \tilde{F}_i, F_i' \) be Seifert surfaces of \( M_i^{sh} \) and \( M_i^{sh'} \). We define for \( i \not\in \sigma \)

\[
\tilde{W}_i = \tilde{W}_i \cup \tilde{F}_i \times \{0\} \cup F'_i \times \{1\}
\]

and orient \( \tilde{W}_i \) so that its orientation is in agreement with \( \tilde{W}_i \). Let \( \tilde{F}_{\tilde{W}_i} \) be a Seifert surface of \( \tilde{W}_i \), consider \( \tilde{Z}_i = \tilde{W}_i \cap \tilde{F}_{\tilde{W}_i} \) with boundary \( \partial \tilde{Z}_i = Z_i \times \{0\} \sqcup Z'_i \times \{1\} \). Cut out a small tubular neighbourhood \( U_i \) of \( \tilde{Z}_i \subset \tilde{W}_i \) and orient \( \partial U_i \) so that \( \partial U_i \) is an oriented bordism from \( \partial U_i \) to \( \partial U_i' \). Define

\[
W_{r+1} = (\sqcup_{i \in \sigma} \partial U_i) \sqcup (-\sqcup_{i \in \sigma} W_i^{sh}).
\]

\( W_1 \sqcup \cdots \sqcup W_r \sqcup W_{r+1} \) is clearly an oriented bordism between \( \theta_\sigma(L) \) and \( \theta_\sigma(L') \). It is clear that \( \theta_\sigma \) respects the addition. \( \square \)

To prove that \( \theta_\sigma \) is an isomorphism we construct now a homomorphism \( \zeta_\sigma : L^2_{m,r+1} \rightarrow FL^2_{m,r} \) and show it’s in fact the inverse of \( \theta_\sigma \).

Let \( \overline{\sigma} = \sigma \cup \{r+1\} \), and let \( L = M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1} \subset \mathbb{R}^m \) be an oriented link of codimension 2. Take a Seifert surface \( F_{\sigma} \) of \( M_\sigma = \sqcup_{i \in \sigma} M_i \). Denote by \( u_1 \) the normal vector field of \( M_\sigma \subset F_\sigma \) pointing outwards and by \( u_2 \) the normal vector field of \( F_\sigma \) determined by the orientation. For \( i \in \sigma \) take \( F_i = (u_1, u_2)|_{M_i} \) as a framing of \( M_i \). If \( i \not\in \sigma \) consider the intersection \( Z_i = M_i \cap F_\sigma \) and let \( Z_i \times [-1,1] \subset M_i \) be a small tubular neighbourhood of \( Z_i \subset M_i \) such that the positive direction of \( [-1,1] \) is in agreement with \( u_2|_{Z_i} \). We assume here that \( M_i \) intersects \( F_\sigma \) perpendicularly. Let \( (v_1^S, v_2^S) \) be the \( S \)-framing of \( M_i \) determined by the orientation. Define \( F_i = (v_1, v_2) \) to be the \( 2\pi \)-rotation of \( (v_1^S, v_2^S) \) in \( Z_i \times [-1,1] \), see (1) and (2) in §2. By doing this we have provided \( M_i \) with the framing \( F_i \) for \( 1 \leq i \leq r \). Now \( \zeta_\sigma : L^2_{m,r+1} \rightarrow FL^2_{m,r} \) is defined by the following assignment

\[
\zeta_\sigma[M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1}]_{or} = [(M_1, F_1) \sqcup \cdots \sqcup (M_r, F_r)].
\]

**Lemma 3.2.** \( \zeta_\sigma \) is a well defined homomorphism.
Theorem 3.3. \(\theta : FL^2_{m,r} \rightarrow L^2_{m,r+1}\) is an isomorphism, in fact its inverse is \(\zeta\).

Proof. (1) \(\zeta \circ \theta = id\). Consider

\[
\zeta \circ \theta [(M_1, F_1) \sqcup \cdots \sqcup (M_r, F_r)] = \zeta [M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1}]_{or} = [(M_1, F_1') \sqcup \cdots \sqcup (M_r, F_r')].
\]

For \(M_{r+1}\) and the framings \(F_i'\) see the definitions of \(\theta\) and \(\zeta\). Denote by \(M_i \times [0, \varepsilon]\) the trace of a small \(\varepsilon\)-shift from \(M_i\) to \(M_i^{sh}\). Orient

\[
F_\sigma = (\sqcup_{i \notin \sigma} U_i) \sqcup (\sqcup_{i \in \sigma} M_i \times [0, \varepsilon])
\]

so that \(F_\sigma\) is a Seifert surface of \(M_\sigma\). According to the definition of \(\zeta\) we see easily that up to homotopy it holds \(F_i' = F_i\) for \(i \in \sigma\). In addition, for \(i \notin \sigma\) the intersection \(M_i \cap F_\sigma = M_i \cap U_i\) is exactly the submanifold \(Z_i = M_i \cap \tilde{F}_i\), because \(U_i\) is a small tubular neighbourhood of \(Z_i \subset \tilde{F}_i\), see the definition of \(\theta\). This implies that \(F_i\) and \(F_i'\) are homotopic, since both \(F_i\) and \(F_i'\) are essentially the \(2\pi\)-rotation of the \(S\)-framing of \(M_i\) in a small tubular neighbourhood of \(Z_i \subset M_i\). It follows \(\zeta \circ \theta = id\).
(2) \( \theta_\sigma \circ \zeta_\sigma = id \). Consider

\[
\theta_\sigma \circ \zeta_\sigma [M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1}] = \theta_\sigma [(M_1, F_1) \sqcup \cdots \sqcup (M_r, F_r)] = [M_1 \sqcup \cdots \sqcup M_r \sqcup M'_{r+1}]_{or}.
\]

For \( M'_{r+1} \) and the framings see the definitions of \( \theta_\sigma \) and \( \zeta_\sigma \). Let \( F_\sigma \) be an oriented Seifert surface of \( M_\sigma \). Cut out a small tubular neighbourhood \( V_\sigma \) of \( M_\sigma \subset F_\sigma \) to get \( F'_\sigma \) with

\[
\partial F'_\sigma = \partial (F_\sigma \setminus V_\sigma) = M_{r+1} \sqcup (\sqcup_{i \in \sigma} M^{sh}_i).
\]

For \( i \not\in \sigma \) let \( \tilde{F}_i \) be an oriented Seifert surface of \( M_i^{sh} \). According to the definitions of \( \theta_\sigma \) and \( \zeta_\sigma \) we may assume

\[
Z_i = M_i \cap \tilde{F}_i = M_i \cap F'_\sigma
\]

for \( i \not\in \sigma \), because \( \partial \tilde{F}_i = M_i^{sh} \) is a shift of \( M_i \) along the framing \( F_i \) ( \( F_i \) is given by the \( 2\pi \)-rotation of the \( S \)-framing in a small tubular neighbourhood of \( M_i \cap F'_\sigma \subset M_i \)), and because \( M_i \cap \tilde{F}_i \) is just where the \( 2\pi \)-rotation of the \( S \)-framing takes place. In addition, we may assume \( M_i \) intersects \( \tilde{F}_i \) perpendicularly. This implies \( U_i \subset F'_\sigma \) after a small isotopy of \( F'_\sigma \) fixing boundary, where \( U_i \subset \tilde{F}_i \) is a small tubular neighbourhood of \( Z_i = M_i \cap \tilde{F}_i \). Define

\[
\tilde{F}_\sigma = F'_\sigma \setminus (\sqcup_{i \not\in \sigma} U_i)
\]

which is a Seifert surface of

\[
M_{r+1} \sqcup (\sqcup_{i \in \sigma} M_i^{sh}) \sqcup (\sqcup_{i \not\in \sigma} \partial U_i) = M_{r+1} \sqcup (-M'_{r+1}).
\]

Because \( \tilde{F}_\sigma \) is disjoint from all \( M_i \), \( 1 \leq i \leq r \), there is an embedding \( W_{r+1} \subset \mathbb{R}^m \times [0, 1] \) of \( \tilde{F}_\sigma \) such that \( W_{r+1} \) is an oriented bordism from \( M_{r+1} \) to \( M'_{r+1} \) and such that \( W_{r+1} \) is disjoint from \( M_i \times [0, 1] \subset \mathbb{R}^m \times [0, 1] \) for all \( 1 \leq i \leq r \). So we obtain an oriented bordism

\[
M_1 \times [0, 1] \sqcup \cdots \sqcup M_r \times [0, 1] \sqcup W_{r+1} \subset \mathbb{R}^m \times [0, 1]
\]

from \( M_1 \sqcup \cdots \sqcup M_r \sqcup M_{r+1} \) to \( M_1 \sqcup \cdots \sqcup M_r \sqcup M'_{r+1} \). It follows \( \theta_\sigma \circ \zeta_\sigma = id \). \( \square \)

**Example 3.4.** If \( \sigma, \sigma' \subset \{1, \cdots, r\} \) are different then \( \theta_\sigma \neq \theta_{\sigma'} \) in general. Without loss of generality we assume \( 1 \in \sigma \) and \( 1 \notin \sigma' \). Consider the framed Hopf link

\[
L = S^1_1 \sqcup S^1_2 \sqcup \phi \sqcup \cdots \sqcup \phi \subset \mathbb{R}^3.
\]

Let \( L_\sigma, L_{\sigma'} \) be the oriented links representing \( \theta_\sigma[L] \) and \( \theta_{\sigma'}[L] \) respectively. It is not difficult to verify that the linking number between the second and the last components of \( L_\sigma \) is \( \pm 1 \), and the linking number between the second and the last components of \( L_{\sigma'} \) is 0. This shows \( \theta_\sigma \neq \theta_{\sigma'} \). In the case \( \sigma = \{1, \cdots, r\} \) it holds \( \theta_{\sigma} = \theta \), where \( \theta \) is the isomorphism given by Sanderson [12]. Some computations may be simplified by suitably choosing \( \theta_\sigma \) or \( \zeta_\sigma \). In next section we will choose \( \zeta_\sigma = \zeta_\phi \), namely \( \sigma = \phi \).
4. COMPUTATION OF THE FORGETFUL HOMOMORPHISM

Let $f : FL^2_{m,2} \rightarrow L^2_{m,2}$ be the forgetful homomorphism given by forgetting the framings. To compute $f$ we compute the composition

$$f' = \zeta_\sigma \circ f : FL^2_{m,2} \rightarrow I^2_{m,2} \xrightarrow{\phi} FL^2_{m,1}$$

where and throughout this section $\sigma = \phi$.

Let $M^{m-2} \subset \mathbb{R}^m$ be a framed or an oriented submanifold. Denote by $M^S$ the same submanifold but provided with the $S$-framing. Let $M_1 \sqcup M_2 \subset \mathbb{R}^m$ be a framed submanifold but provided with the $S$-framing, where

Let $\eta : S^3 \rightarrow S^2$ is the Hopf map and $F_2$ is a Seifert surface of $M_2$. So we only need to compute $M^S_1 \sqcup F_2$.

It is easily seen that $f' \circ \gamma_* = 0$ for $\gamma = \iota_1$ and $\gamma = \iota_2$. So let $\gamma$ be a basic Whitehead product in $\iota_1 < \iota_2$ of weight $\geq 2$, and $\eta_\gamma : S^{q(\gamma)+1} \rightarrow S^3$ be the map determined by $M^S_1(\gamma) \sqcup F_2(\gamma)$, where $M_1(\gamma) \sqcup M_2(\gamma)$ is a framed link representing $\gamma$ and $F_2(\gamma)$ is a framed Seifert surface of $M_2(\gamma)$. Given $[Z] \in \pi_m(S^{q(\gamma)+1})$, by fibre-wise embedding it follows easily that $f' \circ \gamma_*[Z] = \eta_\gamma[Z]$ which is represented by $Z \times (M^S_1(\gamma) \sqcup F_2(\gamma))$. Therefore we only need to compute $M^S_1(\gamma) \sqcup F_2(\gamma)$.

For $\gamma = [\iota_1, \iota_2]$ we see easily that $M^S_1(\gamma) \sqcup F_2(\gamma)$ is a framed point, this means $\eta_\gamma$ is the identity up to sign in this case. Let $\gamma_w = [\iota_2, [\iota_2, \iota_1, \iota_2], \cdots]$ be a basic Whitehead product of weight $w \geq 3$, clearly $q(\gamma_w) = w$. The following framed link in $\mathbb{R}^{w+1}$ represents $\gamma_w$

$$M_1(w) = S^1_w \times S^1_{w-1} \times \cdots \times S^1_3 \times S^1_2,$$

$$M_2(w) = S^1_w \times S^1_{w-1} \times \cdots \times S^1_3 \times S^1 \sqcup S^1_w \times S^1_{w-1} \times \cdots \times S^2 \sqcup \cdots \sqcup S^1_w \times S^{w-2} \sqcup S^{w-1} = N_{2,1} \sqcup \cdots \sqcup N_{2,w-1},$$

where $S^1_2 \sqcup S^1_1$, $S^1_1 \sqcup S^{i-1}$ are $S$-framed Hopf links, $3 \leq i \leq w$, and all products are given by fibre-wise embeddings. $M_1(w)$ is clearly $S$-framed, but $M_2(w)$ not, according to Corollary 2.5. In addition, we can assume $M_1(w) \subset \mathbb{R}^w \times \{0\} \subset \mathbb{R}^{w+1}$.

Consider now the oriented submanifolds of $M_1(w)$

$$Z_i = S^1_w \times \cdots \times S^1_{i+1} \times \{pt\} \times S^1_{i-1} \cdots \times S^1_3 \times S^1_2,$$
\[ 2 \leq i \leq w \text{ and } \{pt\} \text{ is a set of a single point.} \]

\[ \bigcup_{i=2}^{w} Z_i \subset M_1(w) \text{ is a transversally immersed submanifold.} \]

Using the classical trick in Fig.4 we can successively dissolve the multi-points to get an embedded submanifold \( Z(w) \subset M_1(w) \) of codimension 1. Let \( v_1, v_2 \) be the normal vector fields of \( Z(w) \subset M_1(w) \) and of \( M_1(w) \subset \mathbb{R}^w \times \{0\} \) respectively. Define \( F_w = (v_1, v_2|_{Z(w)}) \).

\[ \text{Figure 4.} \]

For the proof of Proposition 4.1 we need to consider the following situation. Let

\[ N^{m-2} = N_1 \sqcup N_2 \subset \mathbb{R}^m \]

be an oriented closed submanifold and let \( F_1, F_2 \) be Seifert surfaces of \( N_1 \) and \( N_2 \) respectively. We construct now a Seifert surface \( F \) of \( N \) from \( F_1 \) and \( F_2 \). Cut out
tubular neighborhoods of \( N_1 \subset F_1 \) and \( N_2 \subset F_2 \) to get \( F_1^* \) and \( F_2^* \), and cut out a tubular neighbourhood \( U(C) \) of
\[
C = F_1^* \cap F_2^* \subset F_1^* \cup F_2^*.
\]
Then we can use the trick in Fig.5 to sew \( F_1^* \setminus U(C) \) and \( F_2^* \setminus U(C) \) together to get \( F^* \) with \( \partial F^* = N_1^{sh} \cup N_2^{sh} \), where \( N_1^{sh} \) is essentially a small shift of \( N_1 \) along the framing given by the \( 2\pi \)-rotation of the \( S \)-framing of \( N_1 \) in a tubular neighbourhood of \( N_1 \cap F_2 \subset N_1 \), similarly \( N_2^{sh} \). Let \( T_1, T_2 \) be the traces of these shifts. \( F^* \cup T_1 \cup T_2 \) is oriented, we can make it smooth and get a Seifert surface \( F \) of \( N \). Note that, shown in Fig.5 is the locus near \( F_1 \cap N_2 \), for the locus \( N_1 \cap F_2 \) it is completely similar; away from \( F_1 \cap N_2 \) and \( N_1 \cap F_2 \) the method shown in Fig.4 applies. For details see [16], p.14–17.

**Proposition 4.1.** (a) Let \( \gamma_w, M_1(w), M_2(w), Z(w) \) and \( \mathcal{F}_w \) be as above, and assume \( w \geq 3 \). There exists a Seifert surface \( F_2(w) \) of \( M_2(w) \) such that the following holds

(i) \( M_1(w) \cap F_2(w) = Z(w) \),

(ii) \([M_1^S(w) \cap F_2(w)] = \pm E[Z(w), \mathcal{F}_w]\).

(b) Up to sign the framed submanifold \( M_1^S(w) \cap F_2(w) \subset \mathbb{R}^{w+1} \) represents the map
\[
\eta_\gamma = (E\eta) \circ (E^2\eta) \circ \cdots \circ (E^{w-3}\eta) \circ (E^{w-2}\eta) : S^{w+1} \rightarrow S^3.
\]

**Proof.** (a) The case \( w = 3 \) is easy. Assume inductively that the assertion is true for \( \gamma_{w-1} \), and let \( F_2(w-1) \) be a Seifert surface of \( M_2(w-1) \) with the desired property.

By fibre-wise embedding we get a Seifert surface \( S_1^w \times F_2(w-1) \) of \( N_{2,1} \sqcup \cdots \sqcup N_{2,w-2} \). \( N_{2,w-1} = S^{w-1} \) bounds a ball \( D^w \). From \( D^w \) and \( S_1^w \times F_2(w-1) \) we obtain a Seifert surface \( F_2(w) \) of \( M_2(w) \). To compute \( M_1(w) \cap F_2(w) \) we look at
\[
Z' = M_1(w) \cap S_1^w \times F_2(w-1)
\]
\[
= S_1^w \times (M_1(w-1) \cap F_2(w-1))
\]
\[
= S_1^w \times Z(w-1),
\]
\[
Z'' = M_1(w) \cap D^w
\]
\[
= \{pt\} \times S_{w-1}^1 \times \cdots \times S_3^1 \times S_2^1
\]
\[
= Z_w.
\]

Considered in \( M_1(w) \) we have the transversal intersection
\[
Q = Z' \cap Z'' = \{pt\} \times Z(w-1)
\]
of codimension 2. Because \( Q \) is disjoint from the boundaries of \( S_1^w \times F_2(w-1) \) and \( D^w \) we see that in the construction of \( F_2(w) \) we have just dissolved \( Q \) as Fig.4. This means \( M_1(w) \cap F_2(w) = Z(w) \). Part (i) follows. Because \( M_1(w) \) is \( S \)-framed, at least up to sign \( \mathcal{F}_w \) is the framing of \( M_1^S(w) \cap F_2(w) = Z(w) \), part (ii) follows.
(b) Up to involution we have

\[ [M_1^S(w) \cap F_2(w)] = [F_1(w) \cap M_2^S(w)], \]

see Proposition 2.4. Consider

\[ Z'(w) = F_1(w) \cap M_2(w) \]
\[ = F_1(w) \cap N_{2,1} \]
\[ = S_1^w \times S_{w-1}^1 \times \cdots \times S_3^1 \times \{pt\} \times \{pt\}, \]

where \( D_2 \) is a disk with boundary \( S_2^1 \). Let \( (v_1, v_2) \) be the \( S \)-framing of \( M_2(w) \) and \( v_3 \) be the normal vector field of \( F_1(w) \subset \mathbb{R}^{w+1} \). Define \( \mathcal{F}_w' = (v_1, v_2, v_3)|_{Z'(w)} \). So we have

\[ [Z'(w), \mathcal{F}_w'] = [F_1(w) \cap M_2^S(w)], \]

In addition, \( (v_1, v_2)|_{N_{2,1}} \) is given by the \( 2\pi \)-rotations of the \( S \)-framing \( (u_1, u_2) \) of \( N_{2,1} \) in the tubular neighborhoods of all

\[ S_1^w \times \cdots \times S_{i+1}^1 \times \{pt\} \times S_{i-1}^1 \times \cdots \times S_3^1 \times S_1^1 \subset N_{2,1}, \]

\( 3 \leq i \leq w \) and \( \{pt\} \) denotes a set of a single point. We will get \( (Z'(w), \mathcal{F}_w') \) when we take a regular value in \( S^3 \) of the map

\[ (E\eta) \circ (E^2\eta) \circ \cdots \circ (E^{w-3}\eta) \circ (E^{w-2}\eta) : S^w \to S^3 \]

and perform the Pontryagin-Thom construction. The statement follows.

\[ \square \]

**Proposition 4.2.** Let \( \gamma = [\tau_2, [\tau_2, \cdots [\tau_2, [\tau_1, \cdots [\tau_1, \tau_2] \cdots ]] \cdots ]] \) be a basic Whitehead product in \( \tau_1 < \tau_2 \) of weight \( w \geq 3 \), and let \( M_1 \sqcup M_2 \subset \mathbb{R}^{w+1} \) be a framed link representing \( \gamma \). The following map

\[ \eta_{\gamma} = (E\eta) \circ (E^2\eta) \circ \cdots \circ (E^{w-3}\eta) \circ (E^{w-2}\eta) : S^{w+1} \to S^3 \]

is represented by \( M_1^S \cap F_2 \) up to sign, where \( F_2 \) is a Seifert surface of \( M_2 \).
Proof. Assume \( \nu_i \) appears \( w_i \)-times in \( \gamma \), \( i = 1, 2 \). The following framed link \( M_1 \sqcup M_2 \) in \( \mathbb{R}^{w+1} \) representing \( \gamma \)

\[
M_1 = S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+2} \times S^1_{w_1+1} \cdots \times S^1_3 \times S^1_2 \sqcup \\
S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+2} \times S^1_{w_1+1} \cdots \times S^2 \sqcup \\
\cdots \cdots \sqcup \\
S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+2} \times S^{w_1} \\
= N_{1,1} \sqcup \cdots \sqcup N_{1,w_1}
\]

\[
M_2 = S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+3} \times S^1_{w_1+2} \times \cdots \times S^3_1 \times S^1_1 \sqcup \\
S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+3} \times S^{w_1+1} \sqcup \\
\cdots \cdots \sqcup \\
S^1_w \times S^{w-2} \sqcup \\
S^{w-1}
\]

\[
= N_{2,1} \sqcup \cdots \sqcup N_{2,w_2}.
\]

where \( S^1_2 \sqcup S^1_1, S^1_3 \sqcup S^2, \cdots, S^1_u \sqcup S^{w-1} \) are usually framed Hopf links and all products are given by fibre-wise embeddings.

According to Proposition 4.1 we can assume

\[
Z = F_1 \sqcup M_2 = F_1 \sqcup N_{2,1}
\]

\[
= S^1_w \times S^1_{w-1} \times \cdots \times S^1_{w_1+2} \times Z ( w_1 + 1 ),
\]

where \( Z ( w_1 + 1 ) \) is given by dissolving the multi-points of the following immersion iterated

\[
\bigcup_{i=1, i \neq 2}^{w_1+1} S^1_{w_1+1} \times \cdots \times S^1_{i+1} \times \{ pt \} \times S^1_{i-1} \times \cdots \times S^1_3 \times S^1_1 \\
\subset S^1_{w_1+1} \times \cdots \times S^1_3 \times S^1_1.
\]

Let \( Z' ( w_1 + 1 ) = S^1_{w_1+1} \times \cdots \times S^1_3 \times \{ pt \} \), framed by \( ( v_1, v_2, v_3 ) \) as in the proof of Proposition 4.1, part (b). So we have a framed bordism \( ( W', \mathcal{G}' ) \) from \( Z ( w_1 + 1 ) \) to \( Z' ( w_1 + 1 ) \) (at least up to involution) with \( \mathcal{G}' = ( \bar{v}_1, \bar{v}_2, \bar{v}_3 ) \). Consider the fibre-wise embedding

\[
W = S^1_w \times \cdots \times S^1_{w_1+2} \times W' \subset \mathbb{R}^{w+1} \times [0,1].
\]

We obtain a framing \( \mathcal{G} \) of \( W \) by performing the \( 2\pi \)-rotations of \( ( \bar{v}_2, \bar{v}_3 ) \) in tubular neighborhoods of all

\[
S^1_w \times \cdots \times S^1_{i+1} \times \{ * \} \times S^1_{i-1} \times \cdots \times S^1_{w_1+2} \times W' \subset W,
\]
w_1 + 2 \leq i \leq w. \ (W, \mathcal{G}) \text{ is a framed bordism from } F_1 \sqcup M_2 \rightarrow Z'(w), \text{ according to the proof of Proposition 4.1, part (b), } Z'(w) \text{ represents }
\ \ \ \ \ \ \ \ \ \ \ \ (E^{g_1}) \circ (E^{g_2}) \circ \cdots \circ (E^{w_3}) \circ (E^{w_4}) : S^{w+1} \rightarrow S^3.

The assertion follows by Lemma 2.4.

Note that in case \( m = 4 \) it holds \( f' \circ \gamma_s \neq 0 \) for \( \gamma = [t_1, t_2], \ [t_1, [t_1, t_2]] \text{ and } [t_2, [t_1, t_2]], \)
and \( f' \circ \gamma_s = 0 \) for \( \gamma = t_1 \text{ and } t_2. \text{ Because } \zeta_\phi \text{ is an isomorphism, the same holds if we replace } f' \text{ by } f. \) So we recover the corresponding results of Sanderson [12].

Let \( \Gamma \) be a system of basic Whitehead products in \( t_1 < t_2, \) and \( \gamma = [\alpha, \beta] \in \Gamma \) be such that the weights of \( \alpha \) and \( \beta \) are greater than 1. Take \( \gamma' \) to be one of \( \alpha, \beta \) and \( \gamma, \) and let \( M_1(\gamma') \sqcup M_2(\gamma') \subseteq \mathbb{R}^{q(\gamma') + 1} \) be a framed link representing \( \gamma'. \) Denote by \( \eta_{\gamma'} : S^{q(\gamma') + 1} \rightarrow S^3 \) the map given by \([M_1^S(\gamma') \sqcup F_2(\gamma')]_{fr}, \) where \( F_2(\gamma') \) is a Seifert surface of \( M_2(\gamma'). \)

**Proposition 4.3.** Under the above notations and up to sign we have

\[ \eta_{\gamma} = [\eta_\alpha, \eta_\beta] : S^{q(\alpha) + q(\beta) + 1} = S^{q(\gamma) + 1} \rightarrow S^3. \]

**Proof.** Surely, we may take \( M_1(\gamma) \sqcup M_2(\gamma) \subseteq \mathbb{R}^{q(\gamma) + 1} \) to be the following

\[
\begin{align*}
M_1(\gamma) &= S^{q(\alpha)} \times M_1(\beta) \sqcup S^{q(\beta)} \times M_1(\alpha), \\
M_2(\gamma) &= S^{q(\alpha)} \times M_2(\beta) \sqcup S^{q(\beta)} \times M_2(\alpha),
\end{align*}
\]

where \( S^{q(\alpha)} \sqcup S^{q(\beta)} \subseteq \mathbb{R}^{q(\gamma) + 1} \) is the usually framed Hopf link and all products are given by fibre-wise embeddings. Let \( F_i(\alpha), \ F_i(\beta) \) be Seifert surfaces of \( M_i(\alpha) \) and \( M_i(\beta) \) respectively, \( i = 1, 2, \) then we get a Seifert surface

\[
F_i(\gamma) = S^{q(\alpha)} \times F_i(\beta) \sqcup S^{q(\beta)} \times F_i(\alpha)
\]

of \( M_i(\gamma) \) by fibre-wise embeddings. This implies

\[
\begin{align*}
M_1^S(\gamma) &= S^{q(\alpha)} \times M_1^S(\beta) \sqcup S^{q(\beta)} \times M_1^S(\alpha), \\
M_1^S(\gamma) \sqcup F_2(\gamma) &= S^{q(\alpha)} \times (M_1^S(\beta) \sqcup F_2(\beta)) \sqcup S^{q(\beta)} \times (M_1^S(\alpha) \sqcup F_2(\alpha)).
\end{align*}
\]

Because by induction \( M_1^S(\alpha) \sqcup F_2(\alpha), \ M_1^S(\beta) \sqcup F_2(\beta) \) represent the maps \( \eta_\alpha \text{ and } \eta_\beta \) respectively, the assertion \( \eta_{\gamma} = [\eta_\alpha, \eta_\beta] \) follows.

Combining Propositions 4.1, 4.2 and 4.3 we get a complete computation of \( f' \) and therefore the forgetful homomorphism \( f. \) Given \( \alpha = \oplus \gamma_\alpha, \in \pi_m(S^2 \vee S^2), \ f'(\alpha) \) is the sum of \( f' \circ \gamma_s(\alpha_\gamma), \) and all the homomorphisms \( f' \circ \gamma_s \) are computed above. To get the decomposition \( \alpha = \oplus \gamma_\alpha \) we may use the method in the author’s paper [17].

We end this paper with a question which U. Kaiser mentioned to me. Given a framed link \((M_1, F_1) \sqcup (M_2, F_2) \subseteq \mathbb{R}^m \) of codimension 2, denote by \( F_i^S \) the \( S \)-framing of \( M_i, \)
There is a map $s_i : M_i \rightarrow S^1$ such that $F_i \simeq s_iF^S_i$. Let $q_i : S^1 \rightarrow S^1$ be a map of degree $q_i$ and define $s'_i = q_i \circ s_i$. The assignment
\[
[(M_1, F_1) \sqcup (M_2, F_2)] \mapsto [(M_1, s'_1F^S_1) \sqcup (M_2, s'_2F^S_2)]
\]
determines a well defined homomorphism $(q_1, q_2)_* : FL^2_{m,2} \rightarrow FL^2_{m,2}$. The question is How can one describe the homomorphism $(q_1, q_2)_*$ by using the Hilton splitting of $FL^2_{m,2} \cong \pi_m(S^2 \vee S^2)$?

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Address
