Investigation of quantum transport by means of \( O(N) \) real-space methods

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We investigate quantum transport by a suitable development of the Kubo formula on a basis of orthogonal polynomials and using real space recursion approaches. We show that the method enable to treat systems close to a metal-insulator transition. On quantum Hall systems, results are given in the context of recent new universal relations between transport coefficients. RKKY mesoscopic interaction is also evaluated for two-dimensional quasiperiodic systems and insight of the usefulness of our method is given for large scale computational problems.

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I. INTRODUCTION

Quantum transport in high dimensional non periodic systems, is usually investigated by way of scaling analysis through diagonalization of periodic Hamiltonian. However, if \( N \) is the number of states, exact diagonalization requires a CPU time scaling as \( O(N^3) \), and memory scaling as \( O(N^2) \). For sparse Hamiltonians, the use of Lanczos algorithm reduces memory and CPU time requirements to \( O(N) \). In the present work, quantum transport in presence of magnetic field using a development of Kubo formula on orthogonal polynomials, is investigated. This approach has been developed in the context of strongly disordered, quasiperiodic and inhomogeneous systems where electronic susceptibility and conductivity were successfully estimated. The key point of the algorithm is to rescale the density of states after an evaluation of the upper and lower bounds on energy and then to make a polynomial expansion of the associated Kubo formula. In principle, any orthogonal polynomials may be used, but it turns out that manipulations of Chebyshev polynomials are particularly suitable as they are isomorphic to Fourier series. Possible applications to nanostructures will be mentioned in this context.

In two-dimensional systems, the study of localization in a magnetic field enables to address problems related to the quantum Hall effect and metal insulator transitions. A magnetic field in a pure system introduces further topological complication of the electronic spectrum, which turns out to be a degenerate ensemble of discrete Landau levels. For disordered systems, in the limit of strong magnetic field, perturbational, numerical and field theoretical approaches have depicted a comprehensive view of the corresponding physical phenomena. In real materials, due to disorder, Landau levels are enlarged and overlap to form Landau bands but extended states still persist at the center of each Landau bands, as revealed for instance by the divergence of the localization length near these critical energies. The studies of the associated eigenstates inferred a complicated nature described by multifractality.

The concept of multifractal states, also proposed by Kohmoto for quasiperiodic systems, is an important related issue. Indeed, anomalous quantum diffusion in quantum Hall systems (QHS) and in quasicrystals may have some direct relations with the observed physical properties. The possibility of an experimental observation of multifractal exponents in the QHS has been proposed recently by Brandes et al. To understand properly the effects of these complicated eigenstates, one has to develop suitable methods.

In this context, after discussing briefly the spectrum of the pure systems with magnetic field, we present a new method to evaluate the quantum transport coefficient in various situations. For QHS, the method brings complementary results for studying the transition regions when \( \sigma_{xy} \) goes from one quantized plateau to the next, while \( \sigma_{xx} \) goes through a peaked value and decrease again. It also enables a study of the phase diagram of the integer quantum Hall effect (QHE) through the calculation of the transport coefficient as a function of disorder and magnetic strengths. Besides, we show that physical phenomena indirectly related to electronic propagation, such as RKKY interaction, may be also investigated in high-dimensional aperiodic systems, by using similar algorithms. Finally, applications to large-scale computational methods is suggested.

II. NON INTERACTING ELECTRONS IN 2D SYSTEMS WITH MAGNETIC FIELD

A. Spectral properties for pure systems

The density of states on 2D disordered systems with magnetic field is a well established result which can also be investigated by recursion method. In the zero-
disorder limit, one gets a Landau level type spectrum where the number of gaps are defined by the dimensionless measure (magnetic strength) \( \alpha = eBa^2/2\pi\hbar \) (a: the lattice unit, B magnetic field). To investigate spectral and transport properties by recursion method, let’s consider the tight-binding representation of the Hamiltonian

\[
H = \sum_{n_x,n_y} \varepsilon_{n_x,n_y} |n_x,n_y\rangle \langle n_x,n_y| + \\
\sum_{n_x,n_y} t(|n_x+1,n_y| + |n_x-1,n_y| + e^{2i\pi\alpha n_x}|n_x,n_y+1|e^{-2i\pi\alpha n_x}|n_x,n_y-1|)\langle n_x,n_y|
\]

A recursive construction of an orthogonal basis \(|\Psi_n\rangle\) such that

\[
\mathcal{H}|\Psi_n\rangle = a_n|\Psi_n\rangle + b_n|\Psi_{n+1}\rangle + b_{n-1}|\Psi_{n-1}\rangle
\]

\[
a_n = \langle\Psi_n|\mathcal{H}|\Psi_n\rangle
\]

\[
b_n = \langle\Psi_{n+1}|\mathcal{H}|\Psi_n\rangle
\]

tridiagonalizes the Hamiltonian and enables to evaluate the spectral properties quickly and accurately \cite{3}. In the one-band case, the series \(a_n, b_n\) usually converge after about 20 recursion steps, and the limit \(a_\infty, b_\infty\) give the final form of the Jacobi matrix, e.g. the convergence of the continuous fraction development of the Green’s function is achieved.

\[
J = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & a_1 & b_1 & 0 & \ldots \\
0 & b_2 & a_2 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & a_\infty \\
0 & \ldots & \ldots & 0 & a_\infty
\end{pmatrix}
\]

Introducing a magnetic field with rational flux \( \alpha = \frac{\nu}{q} \) leads to Q bands, separated by gaps. The asymptotic regime is given by \( q \) distinct values \( b_q, q = 1, \ldots, Q \), moreover the terminator of the continuous fraction expansion can be evaluated exactly:

\[
\mathcal{T}(z) = \frac{1}{b_0^2} \left( \frac{1}{z - b_1^2} \right) \ldots \left( \frac{1}{z - b_q^2} \right)
\]

\[
\mathcal{T}(z) = \frac{\alpha_q + \beta_q \mathcal{T}(z)}{\gamma_q + \delta_q \mathcal{T}(z)}
\]

For \( n = 2, \ldots, q \) and initial values \( \alpha_1 = 1, \beta_1 = 0, \gamma_1 = z, \delta_1 = -b_1^2 \). The explicit form of the terminator and Green’s function are thus given by:

\[
\langle \psi_0 | G(z) | \psi_0 \rangle = \frac{1}{z - \frac{b_0^2}{b_1^2}} \ldots \frac{1}{z - \frac{b_1^2}{b_2^2}} \ldots \frac{1}{z - \frac{b_q^2}{b_{q+1}}} \mathcal{T}(z)
\]

so that the exact density of states can be determined. \cite{13} On Fig. 1, the recursion coefficients \( b_n \) and the corresponding TDoS for \( \alpha = 0, 1/3, 1/8 \) are presented. The relation between the number of bands and the asymptotic behavior of recursion coefficients becomes obvious.

We now introduce the disorder through the site energies, which are chosen randomly (with uniform probability) within the interval \([-W/2, W/2]\). The presence of disorder will smear out the gaps. We check that for \( W = 6t \) all the gaps have disappeared.

In the following, after discussing current open problems of the integer quantum Hall effect, the method we used to investigate Kubo formula \cite{19}, different from usual Landauer method \cite{20,21}, is presented. It was initially proposed by Mayou \cite{3} for studying frequency dependent conductivity \( \sigma(\omega) \) in disordered systems. An investigation of Kubo conductivity and anomalous quantum diffusion in 3D quasiperiodic systems has been carried out by developing a similar algorithm for static conductivity at zero temperature. \cite{3}

**B. Effect of disorder on quantum transport in magnetic fields**

In the presence of disorder, extended states exist only at energies for which localization length diverges as \( \xi \sim (E - E_c)^{-\nu} \), \( \nu = 2.4 \pm 0.1 \). \cite{9} At the critical value \( E_c \), some controversies about the universality of the absolute value for the diagonal conductivity still remain. Experimentally \( \sigma_{xx}^c \sim 0.2 - 0.5 \) which is in agreement with some numerical scaling analysis of \( \sigma_{xx}^L(E, \eta) \). \cite{22,24}

The question of the disappearance of these extended states is important. In a study by Yang and Bhatt \cite{24},
a critical strength of randomness $W_c$ by which all the extended states vanish, was found to be independent of magnetic field ($W_c \sim 6 \times t$, with $t$ the constant hopping term). Since, in their calculations, $W_c$ still persist in the weak magnetic field limit, the driven mechanism of the metal-insulator transition was consistent with the floating up picture of extended states, as proposed theoretically by Khmel’nitckii and Laughlin. According to their argument, the energies of extended states within a given energy range in the vicinity of the center of a Landau band $E_n = (n + 1/2) \hbar \omega_c + \omega_c t (\omega_c t)^2$ tend to infinity if magnetic field is vanishingly small (as expected by Anderson theory of localization). This scenario turns out to be crucial for the global phase diagram proposed by Kivelson, Lee and Zhang (KLZ). The KLZ approach leads to a physical relation between different quantum Hall liquids, known as the law of corresponding states. Accordingly, transitions from a quantum Hall states to insulators are allowed for certain values of $\nu = 1, 1/3, ..., $ and forbidden for others ($\nu = 2, 3, 4, ..., \nu = 2/5, ...$).

However, this picture has been recently contradicted by recent experiments by Song et al. where transition from $\nu = 2$-state to insulator were observed. Shen and Weng have also reported numerical support of a continuous disappearance of IQHE in a tight-binding model, and related to merging of extended states separating different plateaus. The authors claim that their results in the weak magnetic field limit are not consistent with the floating up picture. In addition, they found that at the metal-insulator transition dissipative and non-dissipative conductivities are equal $\sigma_{xx}(E_c, W_c) = \sigma_{xx}(E_c, W_c)$. The physical explanation of this phenomenon still remains unclear and in the following we will show that their numerical calculations can be criticized.

1. Kubo formula by recursion: diagonal conductivity

The real-space calculation of the diagonal Kubo formula of the electronic conductivity may be considered as an alternative for usual Landauer conductance calculations, or diagonalization methods. The use of Landauer formula for investigating quantum zero temperature transport is usually associated with free escape boundary conditions. One direction of the system is periodic whereas the other, of size $L$, is connected to metallic leads with different chemical potentials. Scaling analysis is performed through $L$. By recursion method, we avoid exact diagonalization of the Hamiltonian, so that we can treat in principle larger and more complex systems. To reduce the possible numerical instability at boundary conditions induced by the velocity operator (periodic boundary conditions will indeed generate a short-circuit across the sample), we transform the Kubo formula in the following way $\langle \dot{X}(t) \rangle = e^{i \eta t/\hbar} \dot{X} e^{-i \eta t/\hbar}$ and $\dot{X}$ is the component along direction $x$ of the position operator, $\Omega$ the volume of the system:

$$\sigma_{xx}(E_F) = \frac{2\hbar e^2 \pi}{\Omega} \lim_{t \to \infty} \frac{\text{Tr}[\delta(E_F - H) (\dot{X}(t) - \dot{X}(0))^2]}{t}$$

and we keep control of the asymptotic behavior of the quantum diffusion of the wave-packets. The conductivity reads:

$$\frac{2\hbar e^2 \pi}{\Omega} \sum_{j_x, j_y} D_j(t) \times \Im m_{\eta \to 0} < \Phi_j(t) | G(E_F + i \eta) | \Phi_j(t) >$$

where $| \Phi_j(t) > = \dot{X} e^{-i \eta t/\hbar} | j >$ and $| \Phi_j(t) >$ is normalized. The summation should be done over the total basis of states $| j_x, j_y >$, but it turns out that a limited number of initial sites is sufficient to achieve convergence of the calculation. The time-dependent evolution of a wave-packet initially localized at $| j_x, j_y >$ is also evaluated by polynomial expansion of the evolution operator $e^{-i \eta t/\hbar} = \sum_{n} (\int dE \mathcal{P}_n e^{-i E t/\hbar}) \mathcal{P}_n (\mathcal{H})$ where we choose Chebyshev polynomials of first kind (see appendix). Finally, one could also define a scaling parameter from the construction of the recursion basis. Indeed, after computing $N$ recursion steps, the energy resolution is roughly $\Gamma \sim \frac{W}{N}$ where $W$ is the total bandwidth of the DoS. Starting from a localized state in $| \Psi_0 > = | j_x, j_y >$ the time corresponding to the “propagation” of the N states $| \Psi_N > = \mathcal{P}_N | \Psi_0 >$ is $\tau \sim \frac{W}{N}$. Accordingly, a scaling analysis as a function of $N$ may be done. Besides, the use of Kubo formula implicitly requires taking the limit of an infinite system, as the spectrum of any finite system is discrete. By retaining a finite $\eta$ imaginary part in Green’s function, one replaces the delta function by a peaked smooth function of width $\eta$, which must be greater than the level spacing in the finite system. The thermodynamic limit is achieved with increasing system size $L \to \infty$, and $\eta \to 0$ in order to retain all the contributions from the spectrum. In our calculation, we find that the thermodynamic limit is achieved for a finite number of initial states $| j_x, j_y >$. Concerning the transition regions mentioned earlier, one notes that the effect of finite temperature or frequency is to smear the QH-metal-QH phase transitions.

On Figure 2, the diagonal conductivities obtained for a disorder and magnetic strength $W = 2, 5$ $\alpha = 1/3$ and $\eta = 0.15$, are similar to those from reference. The increasing of disordered strength leads to a reduction of the conductivity, as well as a shift of extended states towards the center of the band (as indicated by the arrows.
on Figure 2). Ando first described numerically this effect by studying the density of states and Thouless numbers. A physical interpretation of this levitation mechanism associated to Landau-level mixing has been proposed for high magnetic field.

The question of universality of the diagonal conductance at critical energies is also addressed in Figure 3. As the finite imaginary part of Green’s function tends to zero, one clearly sees that the $e^2/2h$ limit is approached with our method. One notes that the figures (η = 0.15, 0.09, 0.05), in the inset, are obtained for only one initial state $|j_x,j_y\rangle$, so that the fluctuations of the exact shape are artificial. The central figure is an average result on ten different sites.

We also discuss the numerical results obtained in 23 for $\alpha = 1/16$ and different values of the disorder strength. To study the universal relation between the transport coefficient, the authors have computed $\sigma_{xx}(W, E_F)$ and $\sigma_{xy}(W, E_F)$ for $E_F = -2.75$ in proper units. On Fig. 4, $\sigma_{xx}(W, E_F = -2.75)$ for different disorder strengths is shown. The Fermi energy $E_F = -2.75$ considered in 23 turns out to lie within a gap of the spectrum. The results from 23 are then inconsistent with the physics of the problem. Indeed, if the Fermi energy is located within a gap of the pure system, by increasing disorder, a consequent enhancement of the conductivity may be induced for a finite system, as it is illustrated on Figure 5. However, if the Fermi level lies within the center of a Landau band, such a behavior is not seen in our results for the Fermi energy in the center of a Landau band. The authors of 23 may have not consider the correct location of the Fermi energy (remind that the position of the extended state will be affected by the disorder) and accordingly, their statement is at least numerically questionable.

On figure 6, we show the $\sigma_{xx}(E_F)$ for two different values of disorder. The black curves are the results obtained for one site $D_j(t) \times 3m_q \rightarrow \Phi_j(t) | G(E_F + i\eta) | \Phi_j(t)\rangle$, whereas the red one is the averaged result. One can see that the fluctuations of the components, entering in the sum of the Kubo expression in real space, is not critical, so that finite number of sites is sufficient to achieve convergence.

2. Hall Kubo conductivity

The most spectacular result for 2D electrons in magnetic field is given by the quantization of the Hall conductance in units of $e^2/\pi$, when the Fermi level lies within a gap. This result has been well established and related to topological invariant known as the Chern numbers, which count the exact number of extended states up to the Fermi energy. Contribution to electronic conductivity of a given eigenstate $|k\rangle$ can be calculated from

$$\sigma_{xy}^k = \frac{i e^2 \hbar}{A} \sum_{q \neq k} < q | \tilde{V}_0 | k > - < k | \tilde{V}_x | q > - < q | \tilde{V}_y | k > \left( \epsilon_q - \epsilon_k \right)^2$$

$$< \sigma_{xy}^k > = \frac{1}{4\pi^2} \int d\varphi_1 d\varphi_2 \sigma_{xy}^k(\varphi_1, \varphi_2) = \frac{e^2}{h} N_k$$

On finite size systems, Hall conductance (boundary condition average) enables us to identify if $< \sigma_{xy}^k >$ is a so-called non-zero Chern number $N_k$ or not, thus if the corresponding $|k\rangle$ is an extended state or localized state (formulation is due to Thouless, Kohmoto, Nightingale and den Nijs’s 33,34). Recently, non-commutative geometry has provided an interesting framework to investigate Kubo formula and quantum Hall effect.

As we are interested, in particular, in computing the non-dissipative conductivity for energies in the transition regions, (where $\sigma_{xy}$ is not quantized), one has to adopt a different strategy. To that end, we propose a real-space approach of Hall conductivity. Starting from the general off-diagonal of the Kubo conductivity, one shows that the proper algorithm allowing the expansion of the Hall conductance in a real-space basis is given by:

$$\sigma_{xy} = -\frac{i e^2 \hbar}{2\Omega} \int_{E_1 > E_F} \int_{E_1 < E_F} dE_1 dE_2 \frac{f(E_1) - f(E_2)}{E_1 - E_2} \text{Tr}[\delta(E_1 - H) \hat{Y} \delta(E_2 - H)]$$

enables us to expand the spectral measure on the basis of Chebyshev polynomials. After some simple algebra, one finds two parts to be evaluated separately:

$$\sigma_{xy} = -\frac{i e^2 \hbar}{2\Omega} \sum_{m,n,i} I_{mn} \times \langle i_x, i_y | P_n(H) \hat{Y} P_m(H) \hat{Y} | i_x, i_y \rangle$$

where $I_{mn}$ is analytical and depends on the choice of the polynomial basis. In our case, it corresponds to $(A_F = \text{Arcos}(\frac{E_2 - E_1}{2\Omega})$, and $a.b$ associated to the weight function of Chebyshev polynomials):

$$I_{mn} = \frac{1}{\pi b} \left\{ \frac{\sin(m + n + 3)A_F}{(m + n + 3)} - \frac{\sin(m + n + 1)A_F}{(m + n + 1)} \right\}$$

The other part implies the calculation of the coefficients $\langle j_x, j_y | P_n(H) \hat{Y} P_m(H) \hat{Y} | j_x, j_y \rangle$. A reasonable number of initial sites $|j_x, j_y\rangle$ should be considered. The sum over $m$ and $n$-indice is, given the form of the $I_{mn}$ factors, limited by some appropriate cut-off. One notes that the computational time is however much larger compared to...
the $\sigma_{xx}$ calculation. Further work is in progress for testing the accuracy as well as the gain over conventional diagonalization procedures.

III. RKKY MAGNETIC INTERACTION IN NON-PERIODIC SYSTEMS

Amongst interesting phenomena related to electronic propagation, the Rudermann Kittel Kasuya Yosida interaction (RKKY) [36] between magnetic sites in disordered systems has been subject to great attention. In particular it has been shown to be very important for understanding the spin glass transition or more recently giant magnetoresistance effects in magnetic multilayers.

The RKKY interaction is generically given by

$$I_{RKKY}(r_i, r_j, E) = J^2 \chi(r_i, r_j, E) S_r S_r,$$

where $J$ is the interaction between the localized moment $S_r$ and the spin of the itinerant electrons, and $\chi(r_i, r_j, E)$ contains the sum of all the electron-hole propagation paths from $|r_i\rangle$ to $|r_j\rangle$. The susceptibility can be written down as

$$\chi(r_i, r_j) = 2 \Re \int_{E > E_F} \int_{E' < E_F} dE dE' \langle r_i | \delta(E - \mathcal{H}) | r_j \rangle \langle r_j | \delta(E' - \mathcal{H}) | r_i \rangle,$$

and by development of spectral measure on Chebyshev polynomials one gets [33]

$$\chi_{i,j} = 2 \Re \sum_{m,n} I_{mn} \langle r_i | P_n(\mathcal{H}) | r_j \rangle \langle r_j | P_m(\mathcal{H}) | r_i \rangle,$$

$$I_{mn} = \int_{E > E_F} \int_{E' < E_F} N(E)N(E') \frac{P_m(E)P_n(E')}{E - E'} dE dE'$$

where the coefficients $I_{mn}$ have been previously defined for the Hall conductance.

In metallic systems, the interaction is calculated exactly $I_{RKKY}(r, E) \sim A(r) \cos(2k_F r + \delta(r))/r^3$, contrary to quasiperiodic or disorder systems where there is no simple analytical form. However, in weakly disordered systems, one can evaluate the quantum fluctuations that arise in the higher moments of the interaction. [37] It is found that only even moments lead to significant contributions:

$$\langle \chi^{2p}(|r_i - r_j|) \rangle \approx \Omega_p \left( \frac{\rho^2(E_F)}{|r_i - r_j|^2} \right)^p \sim \left( \langle \chi^2(|r_i - r_j|) \rangle \right)^p$$

$$\langle \chi^{2p+1}(|r_i - r_j|) \rangle \approx \exp(- \frac{|r_i - r_j|}{l_m})$$

with $\rho(E_F)$ the DoS at Fermi level, $l_m$ the mean free path, whereas $\Omega_p$ is constant independent of the parameter of the hamiltonian.

Quasiperiodic systems cannot be described by such averaging process, so that the use of recursion method gives here some interesting quantitative informations. [38] On Fig. 7 and 8, the TDoS for a 2D quasiperiodic Fibonacci quasilattice, as well as typical signature of aperiodic long range order are depicted. The strength of the quasiperiodic potential is $V_{qp} = 0.4t$ (with $t$ the constant hopping integral between first neighbors) and the susceptibility is given in a-units, with a the lattice spacing. In the Fig. 8, typical features of the interaction are shown, but due to the complicated nature of quasiperiodic potential, no Fermi wavelength can be properly defined and oscillations exhibit resurgences that are absent from the periodic potential. [39]

IV. POSSIBLE APPLICATIONS FOR LARGE SCALE COMPUTATIONAL METHODS

The expansion of any operator on orthogonal polynomials can also be applied in the context of large scale computational methods, which aim at reducing large memory and CPU time costs, for investigating more realistic models. For instance, spectral properties and optical spectra for realistic model of Silicon quantum dots have been evaluated and quantum confinement investigated. [40] One can also consider thermodynamical properties of quantum systems thanks to the development of the partition functions $Z(\beta) = \text{Tr}[e^{-\beta \mathcal{H}}] = \sum_{p} \sum_{n} \langle dE \text{e}^{-\beta E} \mathcal{P}_n(E) \rangle \times \mathcal{P}_n(\mathcal{H})$, with $\{\mathcal{P}_n\}$ a suitable basis of orthogonal polynomials. Indeed, one needs to consider average quantities as $\langle \hat{A} \rangle = \text{Tr}[\hat{A}\mathcal{P}]/\text{Tr}[\mathcal{P}]$ ($\hat{A}$ the density operator), but if the expansion of the partition function can be easily done in the case of a scalar argument, when applied to an operator (for large scale systems), one may not be able to switch into the eigen-representation of the Hamiltonian since full diagonalization is practically impossible. [41] Consequently, polynomial expansions of operator are the only available alternative technique.

Finally, in order to perform realistic atomic calculations of binding energies and interatomic forces, real-space methods turn out to be efficient and accurate $O(N)$-methods. [42]

V. CONCLUSION

Results on quantum transport by means of a real-space methods have been presented. The generality of the use of orthogonal polynomials for high-dimensional
non periodic systems has been discussed. In two-dimensional disordered systems with magnetic field, global phase diagram and new universalities of the quantum Hall effect may be studied by this method. One notes that, nowadays, lots of efforts are devoted to improve the efficiency of available numerical algorithms. The well-known Car-Parinello method has opened new perspectives for ab-initio electronic structure calculations, but the development of the so-called “order-N scheme” is of a major importance to investigate complex molecular systems. Quantum transport has also opened new challenges since the development of nanosstructures (quantum dots, nanotubes,...) has become crucial in the field of electronic devices. In this context, further “order-N” schemes to investigate electronic propagation also need to be steadily improved.

VI. APPENDIX ON ORTHOGONAL POLYNOMIALS

The principle of the real-space algorithms, used in this article, is based on expansion of spectral measures as well as the key points of this approach. For any spectral measures, the moments as well as the key points of this approach. For any spectral measure belonging to the spectral subset of \( \mathcal{P}_n(E) \), there exists a family of orthogonal polynomials \( \{ \rho(E) \}_{n=0}^{\infty} \) of degree \( n \) such that

\[
\int_{-\infty}^{+\infty} N(E) \rho_n(E) \rho_m(E) dE = \delta_{nm},
\]

\[
\rho(E) \sum_n \rho_n(E) \rho_n(E') = \delta(E - E')
\]

for \( E, E' \) belonging to the spectral subset of \( \rho(E) \) and \( \rho(E) \neq \rho(E') \). These polynomials generally satisfy a three-term recursive relation \( E \rho_n(E) = a_n \rho_{n-1}(E) + b_n \rho_{n+1}(E) \) with \( b_{-1} = 0, b_n > 0 \). However, the expansion of spectral measure can be also done in an arbitrary basis of polynomials, for instance on Chebyshev polynomials \( \{ Q_n \} \):

\[
\delta(E - H) = \hat{\rho}(E) \sum_n Q_n(E) Q_n(H)
\]

where \( \hat{\rho}(E) = 1/\pi \sqrt{4b^2 - (E - a)^2} \) and \( \rho(E) \) the total density of states. Here, \( a \) and \( b \) define the band edges and bandwidth of the spectral function associated to Chebyshev polynomials. Green’s functions can be obtained by Hilbert transformation of \( \rho(E) \) (for absolute continuous spectrum):

\[
G(z) = \int dE \frac{\rho(E)}{(z - E)}
\]

\[
\rho(E) = \lim_{\eta \to 0^+} -\frac{1}{\pi} \text{Im} G(E + i\eta)
\]

and general analytic properties of the Green’s function can be investigated through the properties of the considered polynomials.

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VIII. FIGURES CAPTIONS

Fig. 1. Recursion coefficients $b_n$ and corresponding total density of states for three different strengths of the magnetic field.

Fig. 2. Diagonal conductivity for quantum Hall systems with different disorder strengths ($\alpha = 1/3, \eta = 0.15$) in $e^2/h$ unit.

Fig. 3. Diagonal conductivity versus Fermi energy and finite imaginary part of the Green’s function $\eta$ (inset).

Fig. 4. Diagonal conductivity versus Fermi energy and disorder strength for $\alpha = 1/16$.

Fig. 5. Dissipative conductivity as a function of disorder strength. Stars are associated with a Fermi level in the gap of the pure system (no disorder) whereas Fermi level lies within the middle of a Landau band for diamonds.

Fig. 6. Components of the dissipative conductivity as a function of disorder strength and Fermi energy (black curves), and averaged result (red curve).

Fig. 7. Total density of states for a two-dimensional quasiperiodic lattice ($V_{qp}$ is the strength of the Fibonacci quasiperiodic potential). In the onset is given the energy interval for which corresponding susceptibility are calculated.

Fig. 8. Electronic susceptibility (in a-units, a lattice parameter) as a function of Fermi energy with 1.a, 2.a, 2.b, 2.c, 3.a, 3.b, 3.c corresponding respectively to Fermi energy $-1.9 + 0.1 \times \lambda$, $\lambda = 0, 1, 2, 3, 4, 5, 6$. 

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\[ \alpha = 0 \]

\[ \alpha = \frac{1}{3} \]

\[ \alpha = \frac{1}{8} \]

\[ \text{TDoS} \]
\[ \sigma_{xx}(E) \]

- Light gray line: \( W=2 \)
- Black line: \( W=5 \)

Energy range from -6 to 6
$\sigma_{xx}(E)$ vs Energy for different values of $W$:
- Solid line: $W=1.5$
- Dashed line: $W=3.5$
- Dash-dotted line: $W=6.5$

$\alpha = 1/16$
The graph shows the behavior of $\sigma_{xx}(W)$ with respect to $W$, the disorder strength.
$\sigma_{xx}(E,W)$

$W=1.5$

$W=3.5$
$\text{TDoS}(E)$

$V_{qp}=0.4$
\[
\chi_{(dij)}
\]

\[V_{qp}=0.4\]