MULTIPURITY BOUNDS FOR QUADRATIC MONOMIAL IDEALS

MANOJ KUMMINI

Abstract. We prove the multiplicity bounds conjectured by Herzog-Huneke-Srinivasan and Herzog-Srinivasan in the following cases: the strong conjecture for edge ideals of bipartite graphs, and the weaker Taylor bound conjecture for all quadratic monomial ideals. We attach a directed graph to a bipartite graph with perfect matching, and describe operations on the directed graph that would reduce the problem to a Cohen-Macaulay bipartite graph. We determine when equality holds in the conjectured bound for edge ideals of bipartite graphs, and verify that when equality holds, the resolution is pure. We characterize bipartite graphs that have Cohen-Macaulay edge ideals and quasi-pure resolutions.

1. Introduction

Let $V$ be a finite set, and let $R = k[V]$ be a polynomial ring, over a field $k$, treating the elements of $V$ as indeterminates. We make $R$ into a graded ring by setting $\deg x = 1$ for all $x \in V$. Let $f_1, \ldots, f_m \in R$ be homogeneous polynomials, and let $I = (f_1, \ldots, f_m)$. Set $c = \text{ht } I$. Let $e(R/I)$ denote the Hilbert-Samuel multiplicity of $R/I$.

Let $F_s$ be a minimal graded free resolution of $R/I$ over $R$. Let $M_l := M_l(I)$ be the largest twist with which $R$ appears in $F_l, 1 \leq l \leq \text{pd } R/I$. Let $m_l := m_l(I)$ be the smallest twist with which $R$ appears in $F_l$. These do not depend on the choice of the resolution: since the Tor$^R_{l}(k, R/I)$ are graded, we can define the (graded) Betti numbers $\beta_{l,j}(R/I) = \dim k \text{Tor}^R_{l}(k, R/I)_{j}$. Then $m_l = \min \{ j : \beta_{l,j}(R/I) \neq 0 \}$ and $M_l = \max \{ j : \beta_{l,j}(R/I) \neq 0 \}$. However the Tor$^R_{l}(k, R/I)$ are independent of the choice of the resolution of $R/I$. Herzog-Huneke-Srinivasan $[HS98]$ conjectured that:

Conjecture (HHSu). For a homogeneous ideal $I$,

$$e(R/I) \leq \frac{M_1 M_2 \cdots M_c}{c!}.$$

This has subsequently been proved in various cases. A survey appears in $[FS07]$. Some newer results include $[HZ06, KW06, MR06, NS06, Put06]$.

Herzog-Huneke-Srinivasan further conjectured that:

Conjecture (HHSI). Assume that $R/I$ is Cohen-Macaulay. Then

$$e(R/I) \geq \frac{m_1 m_2 \cdots m_c}{c!}.$$

2000 Mathematics Subject Classification. Primary: 13H15, 13F55.
We say that $R/I$ has a pure resolution if for each $l$, there is a unique twist in the free module $F_l$, or, equivalently, $M_l = m_l$. We say that $R/I$ has a quasi-pure resolution if for each $l$, $m_{l+1} \leq M_l$. Huneke-Miller [HMS85] proved that if $R/I$ is Cohen-Macaulay and has a pure resolution, then the above conjectures hold, with equality. Migliore-Nagel-Römer [MNR05] conjectured that:

**Conjecture (HHSI).** If equality holds in Conjecture (HHSn) or in Conjecture (HHSn), then $R/I$ is Cohen-Macaulay with a pure resolution.

Herzog-Srinivasan [HS98] proved that all the three conjectures above are true when $R/I$ is Cohen-Macaulay and has a quasi-pure resolution.

If additionally $f_1, \ldots, f_m$ are monomials, then $R/I$ has another resolution $T_*$, called the Taylor resolution; see, e.g., [Eis95] Ex. 17.11. Let $T_i := T_i(I)$ be the largest twist with which $R$ appears in $T_i$. Then $T_i = \max\{\deg \lcm(f_s, \ldots, f_n) : 1 \leq s_1 < \cdots < s_l \leq m\}$. Herzog-Srinivasan [HS04] conjectured that:

**Conjecture (HHSI).** For a monomial ideal $I$,

$$e(R/I) \leq \frac{T_1 T_2 \cdots T_c}{c!}.$$ 

In general $T_i \geq M_i$ for all $1 \leq l \leq c$; hence Conjecture (HHSn) is weaker than Conjecture (HHSn). In this paper we first prove Conjecture (HHSn) for all ideals generated by quadratic monomials:

**Theorem 1.1.** Let $I \subseteq R$ be generated by monomials of degree 2. Then

$$e(R/I) \leq \frac{T_1 T_2 \cdots T_c}{c!}.$$ 

If $I$ is generated by square-free monomials of degree 2, we define a graph $G$ on $V$ by setting, for all $x, y \in V$, $xy$ to be an edge of $G$ if and only if $xy$ is a minimal generator of $I$, and say that $I$ is the edge ideal of $G$. See [Yao01] Chapter 6] for a discussion on edge ideals. Note that such a graph $G$ is simple, i.e., it has no loops and multiple edges between any pair of vertices.

If $xy$ is an edge of $G$, then we say that $x$ and $y$ are neighbours of each other. An edge is incident on its vertices. We say that an edge $xy$ is isolated if there are no other edges incident on $x$ or on $y$. A vertex $x$ is a leaf vertex if there is a unique $y \in V$ such that $xy$ is an edge that is not isolated; in this case, we call $y$ a stem vertex, and refer to the edge $xy$ as a leaf. The degree of a vertex $x$, denoted $\deg_G x$, is the number of edges incident on $x$. A tree is a connected acyclic graph, and a forest is a graph in which each connected component is a tree. A graph $G$ is bipartite, if there is a partition $V = V_1 \bigcup V_2$ and every edge of $G$ is of the form $xy$ where $x \in V_1$ and $y \in V_2$. (In this paper, $\bigcup$ denotes a disjoint union.) Recall that a graph $G$ is bipartite if and only if it does not contain odd cycles; in particular, forests are bipartite. A path is a tree in which every vertex has degree at most two. A cycle is a connected graph in which every vertex has degree exactly two. We have that Conjecture (HHSn) holds for edge ideals of bipartite graphs:

**Theorem 1.2.** Let $I \subseteq R$ be the edge ideal of a bipartite graph. Then

$$e(R/I) \leq \frac{M_1 M_2 \cdots M_c}{c!}.$$ 

Let $I$ be the edge ideal of a graph $G$. We say that $G$ is Cohen-Macaulay if $R/I$ is Cohen-Macaulay. Herzog-Hibi [HH05] Lemma 3.3 and Theorem 3.4] (see Theorem 4.5 below) characterized Cohen-Macaulay bipartite graphs.

February 1, 2008
Theorem 1.3. Let $I$ be the edge ideal of a bipartite graph. If equality holds in Conjecture (HHSn), then $R/I$ is a complete intersection, or is Cohen-Macaulay with $\reg R/I = 1$. In either of the cases, $R/I$ is Cohen-Macaulay and has a pure resolution.

Kubitzke-Welker [KW06] proved that the Conjecture (HHSn) holds for Stanley-Reisner ideals of barycentric subdivisions of simplicial complexes; such ideals are square-free quadratic monomial ideals, but most often they are not bipartite. Novik-Swartz [NS06] establishes Conjecture (HHSn) when $\dim R/I = n - c$ is small and studies the behaviour of $m_j$ in the context of Conjecture (HHSn).

The paper is organized as follows. In Section 2 we make some reductions. In Section 3 we discuss some properties of Taylor resolutions, and prove Theorem 1.1. Section 4 is devoted to reducing the proof of Conjecture (HHSn) to the Cohen-Macaulay case. In Section 5 we prove Theorem 1.2. Finally, in Section 6 following a proof of Theorem 1.3, we determine when a bipartite graph is Cohen-Macaulay and its edge ideal has a quasi-pure resolution.

2. Preliminary Remarks

Using polarization we first reduce to the case of $I$ generated by square-free monomials; see [MS05, Exercise 3.15] for details on polarization. Herzog-Srinivasan [HS04] showed that we can do this in the context of Conjecture (HHSn). In order to see that it works for Conjecture (HHSn), suppose that $\tilde{R}$ is the polarization of $R$, in a larger ring $\tilde{R}$. Moving from $\tilde{I}$ to $I$ preserves numerical data of a free resolution; in particular, $\beta_{l,j}(\tilde{R}/\tilde{I}) = \beta_{l,j}(R/I)$, for all $l, j$. On the other hand, the graded Betti numbers determine the Hilbert function of $R/I$ and hence $e(R/I)$. Thus for the rest of the paper, we assume that $I$ is square-free, i.e., $f_1, \ldots, f_m$ are square-free monomials. For most part, this section does not assume that the $f_i$ are quadratic.

If $p \subseteq R$ is a prime ideal such that $\height p = c = \height I$ and $I \subseteq p$, then we say that $p$ is an unmixed prime of $R/I$. Denote the set of unmixed primes of $R/I$ by $\Unm R/I$. Unmixed primes are necessarily minimal over $I$, so $\Unm R/I \subseteq \Ass R/I$; we say that $I$ is unmixed if $\Unm R/I = \Ass R/I$. Since $I$ is square-free, $e(R/I)$ is the number of unmixed primes of $R/I$. We add, parenthetically, that for the edge ideal of a graph, unmixed primes correspond to minimal vertex covers of size $c$ [Vil01, Proposition 6.1.16].

We will need the following lemma several times in this paper:

Lemma 2.1 ([HZ06, Proof of Theorem 2.1, step (i)]). Let $I \subseteq R$ be a homogeneous ideal. Then, for $2 \leq l \leq c$, $M_l(I) > M_{l-1}(I)$. \qed

We briefly discuss multigraded resolutions and Hochster’s formula. By $\mathbb{N}^V$, we denote the set of functions from $V$ to the set of natural numbers, $\mathbb{N}$, and by $e_x$, the function that sends $x \mapsto 1$ and $y \mapsto 0$ for all $y \neq x$, for all $x, y \in V$. We treat $R$ as $\mathbb{N}^V$-graded, by setting, for all $x \in V$, $\deg x = e_x$. When an $R$-module is $\mathbb{N}^V$-graded, we will say that it is multigraded. Since $I$ is a monomial ideal, $R/I$ is multigraded, and so are the $\Tor^R_i(k, R/I)$. A multidegree $\sigma$ is an element of $\mathbb{N}^V$. We can then define multigraded Betti numbers $\beta_{l, \sigma}(R/I) := \dim_k \Tor^R_i(k, R/I)_{\sigma}$. If $\sigma \in \mathbb{N}^V$, we also represent the multidegree $\sigma$ as the monomial $\prod_{x \in V} x^{\sigma(x)}$.

(We will use the same notation $\beta_l$ for Betti numbers in both the standard grading and multigrading. Notice that $\beta_{l,j} := \beta_{l,j}(\cdot) = \sum \beta_{l, \sigma}(\cdot)$, where the sum is taken over the set of $\sigma$ such that $\sum_{x \in V} \sigma(x) = j$.)
Let $\Delta$ be the Stanley-Reisner complex of $I$. The correspondence between non-faces of $\Delta$ and monomials in $I$ can also be expressed as follows: for any monomial prime ideal $p \in \text{Spec} R$, $J \subseteq p$ if and only if $p = (F)R$, the ideal generated by $F := V \setminus F$, for some $F \in \Delta$ [MS05, Theorem 1.7]. Thus, minimal prime ideals of $R/I$ correspond to complements of maximal faces of $\Delta$. If $I$ is the edge ideal of a graph $G$, then $\Delta$ is the clique complex of the complement graph $\overline{G}$ [Vil01, Section 6.2]. In this case, we say that $\Delta$ is the coclique complex of $G$.

Proposition 2.2 below will show that most multidegrees of interest in this paper are square-free, i.e., $\sigma(x) \in \{0, 1\}$ for all $x \in V$. When a multidegree $\sigma$ is square-free, we will also use subsets of $V$ to represent multidegrees, by identifying $\sigma \in \mathbb{N}^V$ with $\prod_{x, \sigma(x) = 1} x$ and with $\{x \in V : \sigma(x) = 1\}$. For $\sigma \subseteq V$, $\Delta|_{\sigma}$ denotes the simplicial complex obtained by taking all the faces of $\Delta$ whose vertices belong to $\sigma$. Similarly, we define $G|_{\sigma}$ to be the the subgraph induced on $\sigma$, i.e., the subgraph of $G$ obtained by taking all the edges between pairs of vertices in $\sigma$.

**Proposition 2.2** (Hochster’s Formula [MS05 Corollary 5.12]). Suppose that $I$ is a square-free monomial ideal. Let $\Delta$ be the Stanley-Reisner complex of $I$. The multidegrees $\sigma$ with $\beta_{l, \sigma}(R/I) \neq 0$ are square-free, and

$$\beta_{l, \sigma}(R/I) = \overline{H}_{|\sigma| - l - 1}(\Delta|_{\sigma}; \mathbb{k}).$$

Here $|\cdot|$ denotes the cardinality of a set; later, in Section 3 we will also use it to denote the underlying undirected graph of a directed graph. We now describe how the graded Betti numbers change under restriction to a subset of the variables and under taking colons.

**Lemma 2.3.** Let $I \subseteq R = k[V]$ be a square-free monomial ideal, $x \in V$, $l, j \in \mathbb{N}$ and $\sigma \subseteq V$ with $|\sigma| = j$. Then

(a) Let $W \subseteq V$ and $J = (I \cap k[W])R$. Then,

$$\beta_{l, \sigma}(R/J) = \begin{cases} 0, & \sigma \nsubseteq W; \\ \beta_{l, \sigma}(R/I), & \sigma \subseteq W. \end{cases}$$

In particular, $\beta_{l, j}(R/J) \leq \beta_{l, j}(R/I)$.

(b) If $\beta_{l, \sigma}(R/(I : x)) \neq 0$, then $\beta_{l, \sigma}(R/I) \neq 0$ or $\beta_{l, \sigma \cup \{x\}}(R/I) \neq 0$.

**Proof.** (a): The second assertion follows from the first, which we now prove. Let $\hat{\Delta}$ be the Stanley-Reisner complex of $J$. Since for all $x \in V \setminus W$, $x$ does not belong to any minimal prime ideal of $R/J$, we see that every maximal face of $\hat{\Delta}$ is contained in $V \setminus W$. Hence if $\sigma \nsubseteq W$, then for all $x \in \sigma \setminus W$, $\hat{\Delta}|_{\sigma}$ is a cone with vertex $x$, which, being contractible, does not have any homology. Applying Proposition 2.2 we see that $\beta_{l, \sigma}(R/J) = 0$.

Now let $\sigma \subseteq W$ and $F \subseteq V$. Then $F \in \Delta|_{\sigma}$ if and only if $I \subseteq (F)R$ and $F \subseteq \sigma$ if and only if $J \subseteq (F)R$ and $F \subseteq \sigma$ if and only if $F \in \hat{\Delta}|_{\sigma}$. Apply Proposition 2.2 again to get

$$\beta_{l, \sigma}(R/J) = \overline{H}_{|\sigma| - l - 1}(\hat{\Delta}|_{\sigma}; \mathbb{k}) = \overline{H}_{|\sigma| - l - 1}(\Delta|_{\sigma}; \mathbb{k}) = \beta_{l, \sigma}(R/I).$$

(b): We take the multigraded exact sequence of $R$-modules:

$$0 \longrightarrow R / (I : x) \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I} \longrightarrow 0.$$
The corresponding multigraded long exact sequence of Tor is
\[ \cdots \to \text{Tor}_{l+1}(k, \frac{R}{(I,x)}) \to \text{Tor}_{l}(k, \frac{R}{(I,x)}(-x)) \to \text{Tor}_{l}(k, \frac{R}{J}) \to \cdots. \]

Let \( W = V \setminus \{x\} \) and \( J = (I \cap k[W])R \). Since \( \beta_{l,\sigma}(R/(I \cdot x)) \neq 0 \) and \( x \) does not divide any monomial minimal generator of \((I : x)\), we have, by the same argument as in \([\text{a}]\), \( \sigma \subseteq W \). Let \( \tau = \sigma \cup \{x\} \). First observe that
\[ \text{Tor}_{l}\left( k, \frac{R}{(I : x)} \right)_{\sigma} \cong \text{Tor}_{l}\left( k, \frac{R}{I : x}(-x) \right)_{\tau}. \]

Let us assume that \( \beta_{l,\tau}(R/I) = 0 \), because, if \( \beta_{l,\tau}(R/I) \neq 0 \), there is nothing to prove. Restricting the above long exact sequence to multidegree \( \tau \), we see that 
\[ \text{Tor}_{l+1}(k, \frac{R}{(I,x)}) \neq 0. \]
Now, since \((I, x) = (J, x)\), we see further \( \text{Tor}_{l+1}(k, \frac{R}{(J,x)}) \neq 0 \).

Since \( x \) is a non zerodivisor on \( R/J \), we have a multigraded short exact sequence
\[ 0 \to \frac{R}{J}(-x) \to \frac{R}{J} \to \frac{R}{(J,x)} \to 0, \]
which gives the following long exact sequence of Tor:
\[ \cdots \to \text{Tor}_{l+1}(k, \frac{R}{J}) \to \text{Tor}_{l+1}(k, \frac{R}{(J,x)}) \to \text{Tor}_{l}(k, \frac{R}{J}(-x)) \to \cdots. \]

Since \( x \) does not divide any minimal monomial generator of \( J \), \( \beta_{l+1,\tau}(R/J) = 0 \). Therefore \( \text{Tor}_{l}(k, \frac{R}{J}(-x))_{\tau} = 0 \), or, equivalently, \( \text{Tor}_{l}(k, \frac{R}{J})_{\tau} = 0 \). By \([\text{a}]\) above, \( \beta_{l,\tau}(R/I) \neq 0 \). \(\square\)

Remark 2.4. Lyubeznik showed that, with notation as above, \( \text{depth} \frac{R}{(I : x)} \geq \text{depth} \frac{R}{I} \). \([\text{Lyu88}]\), Lemma 2.3[\text{a}] gives another proof.

Proposition 2.5. Let \( 1 \leq l \leq c \). Then for all \( x \in V \),
\begin{enumerate}
\item[(a)] \( M_l((I, x)) \leq M_l(I) \) and \( M_l((I : x)) \leq M_l(I) \).
\item[(b)] \( T_l((I, x)) \leq T_l(I) \) and \( T_l((I : x)) \leq T_l(I) \).
\end{enumerate}

Proof. Let \( W = V \setminus \{x\} \) and \( J = (I \cap k[W])R \). Then \((I, x) = (J, x)\) and \( x \) is a non zerodivisor on \( R/J \); hence \( c - 1 \leq \text{ht} \ J \leq c \).

[\text{a}]: Let \( \mathbb{G}_{\bullet} \) be a minimal graded free resolution of \( R/J \). Denote the (graded) Koszul complex on \( x \) by \( \mathbb{K}_{\bullet} \). Then \( \mathbb{G}_{\bullet} \otimes \mathbb{K}_{\bullet} \) is a minimal graded free resolution of \( R/(I, x) \); in particular, \( M_l((I, x)) = M_l((I, x)) = \max\{M_l(J), M_{l-1}(J) + 1\} \).

Since \( \text{ht} \ J \geq c - 1 \), we conclude using Lemmas 2.1 and 2.3[\text{a}] that, for \( 1 \leq l \leq c - 1 \), \( M_l(I, x) = M_l(J) \leq M_l(I) \). If \( M_{c}(J) > M_{c-1}(J) \), then \( M_{c}((I, x)) = M_{c}(J) \leq M_{c}(I) \); otherwise, \( M_{c}((I, x)) = M_{c-1}(J) + 1 \leq M_{c-1}(I) + 1 \leq M_{c}(I) \).

Lemma 2.3[\text{a}] implies that \( M_l((I : x)) \leq M_l(I) \).

[\text{a}]: Recall that \( I \) is generated by square-free monomials \( f_1, \ldots, f_m \). Let
\[ (f_j : x) := \begin{cases} J_x, & \text{if } x \text{ divides } f_j, \\ f_j, & \text{otherwise} \end{cases} \]
Since \((I : x) = ((f_1 : x), \ldots, (f_m : x))\) and \((I, x) = (J, x)\), the conclusions follow easily from the definition of \( T_l \). \(\square\)

Consider \( I \cap R_1 \), the vector space generated by the linear forms in \( I \). Suppose that \( \text{dim}_k(I \cap R_1) > 0 \); then, since \( I \) is a monomial ideal, there exists \( x \in V \) such that \( x \in I \). Write \( J = (I \cap k[V \setminus \{x\}])R \). Then \( \text{ht} \ J = c - 1 \) and \( I = (J, x) \).

Note that \( c(R/J) = c(R/I) \). From Lemma 2.3[\text{a}] we know that \( M_l(J) \leq M_l(I) \) for February 1, 2008
1 ≤ l ≤ c − 1. From the definition of $T_i$, we see that $T_i(J) ≤ T_i(I)$ for 1 ≤ l ≤ c − 1. Therefore it is enough to prove Conjectures (HHSu) and (HHSu) hold for (HHSu) by induction on height. We can also look at (HHSu) as an ideal in n − 1 variables. On the other hand, if $ht(I, x) > c$, then $e(R/I) = e(R/(I: x))$; we then replace I by (I : x) which is an ideal in n − 1 variables. In this case, we can prove the conjectures using induction on the number of variables. Therefore, we reduce to the case of 

**Hypothesis 2.7.** For all $x ∈ V$, $ht(I : x) = c = ht(I, x)$; consequently, $e(R/I) = e(R/(I: x)) + e(R/(I : x))$.

The remark about $e(R/I)$ follows from (HHSu). This hypothesis is equivalent to the assumption that for all $x ∈ V$, there exist $p, q ∈ \text{Unm } R/I$ such that $x ∈ p \setminus q$. Moreover, while proving the conjectures, we will assume, inductively, that Conjectures (HHSu) and (HHSu) hold for (I : x) and (I, x).

We now restrict our attention to (square-free) quadratic monomial ideals, i.e., $f_1, \cdots, f_m$ are square-free monomials of degree 2. Since I is square-free, I is the edge ideal of a graph $G$. For any $x ∈ V$, (I, x) is essentially of height ≤ c − 1, and (I : x) is essentially of height ≤ c − δ, where δ is the degree of the vertex $x$ in $G$. If $G$ is bipartite, then Hypothesis 2.7 means that $G$ is perfectly matched; see Section 4.

**Discussion 2.8.** For $ρ, γ ∈ \mathbb{N}$, let

$$μ(ρ, γ) := \begin{cases} \frac{(2ρ + 1)(γ + ρ)}{(ρ + 1)(γ + 1)}, & ρ < γ \\ 1, & \text{otherwise.} \end{cases}$$

Note that $μ(γ − 1, γ) = (2γ − 1)/γ < 2 = 2μ(γ, γ)$. For any $ρ < γ − 1$,

$$\frac{μ(ρ + 1, γ)}{μ(ρ, γ)} = \frac{(ρ + 1)(γ + ρ + 1)}{(2ρ + 1)(2ρ + 2)} > \frac{1}{2}.$$

Combining these, we conclude that

$$2μ(ρ + 1, γ) > μ(ρ, γ), \text{ for all } ρ < γ ∈ \mathbb{N}.$$  

We conclude this section with a crucial lemma that captures the main numerical argument in the proofs of Theorems 1.1 and 1.2

**Lemma 2.9.** Let $ρ, γ, γ_1 ∈ \mathbb{N}$ such that $2 ≤ ρ < γ ≤ ργ_1$ and $ρ − 1 ≤ γ − γ_1$. Then

$$2^p μ(ρ, γ − 1) + 2^{p−1} μ(ρ − 1, γ − γ_1) < 2^p μ(ρ, γ).$$

**Proof.** Since $μ(ρ, γ) − μ(ρ, γ − 1) = ρμ(ρ, γ − 1)/γ$, we must show that

$$μ(ρ − 1, γ − γ_1) < \frac{2pμ(ρ, γ − 1)}{γ}.$$

We first reduce the problem to the case of $γ = ργ_1$ as follows. If $γ < ργ_1$, and if we replace $γ$ by $γ + 1$, the left-hand-side and right-hand-side of (3) change by...
Lemma 3.1. Assume Hypothesis 2.7. Then, for all \( \rho, \gamma \) which is equivalent to the following sequence of equivalent statements:

\[
\mu(\rho - 1, \rho\gamma - \gamma_1) < \frac{2\rho\mu(\rho, \rho\gamma_1 - 1)}{\rho\gamma_1}.
\]

The hypothesis gives that \( \gamma_1 > 1 \), so we need to show that

\[
\frac{(2\rho - 1)(2\rho) \cdots (\rho\gamma_1 - \gamma_1 + \rho - 1)}{\rho(\rho + 1) \cdots (\rho\gamma_1 - \gamma_1)} < \frac{2\rho}{\rho\gamma_1} \frac{(2\rho + 1)(2\rho + 2) \cdots (\rho\gamma_1 + \rho - 1)}{(\rho + 1)(\rho + 2) \cdots (\rho\gamma_1 - 1)}.
\]

We can verify this by hand for \( (\rho, \gamma_1) = (2, 2), (2, 3) \) and \( (3, 2) \). For all other values of \( \rho, \gamma_1, \rho + 1 \leq \rho\gamma_1 - \gamma_1 - 1 \) and we rewrite the above equation as

\[
\frac{(2\rho - 1)2\rho (2\rho + 1) \cdots (\rho\gamma_1 - \gamma_1 - 1 + \rho)}{(\rho\gamma_1 - 1 + \rho)} \frac{1}{\rho\gamma_1} < \frac{2\rho}{\rho\gamma_1} \frac{(2\rho + 1) \cdots (\rho\gamma_1 - 1 + \rho)}{(\rho\gamma_1 - 1)}
\]

which is equivalent to the following sequence of equivalent statements:

\[
\frac{(2\rho - 1)}{\rho} \frac{1}{(\rho\gamma_1 - \gamma_1)} \frac{(\rho\gamma_1 - \gamma_1 + \rho)}{(\rho\gamma_1 - 1 + \rho)} \cdots \frac{1}{(\rho\gamma_1 - 1)\rho}\gamma_1 < \frac{\rho}{\rho\gamma_1} \frac{(\rho\gamma_1 - 1 + \rho)}{(\rho\gamma_1 - 1 + \rho)} \cdots \frac{1}{(\rho\gamma_1 - 1)}
\]

This is indeed true, since there are \( \gamma_1 \) terms on the right-hand-side and each of them is at least as large as \( \left(1 + \frac{\rho - 1}{\rho\gamma_1}\right) \). Recall that \( \gamma_1 > 1 \).

\[\square\]

3. Proof of Theorem 1.1

We first make some observations on how \( T_l \) changes with \( l \). Let \( \rho(I) \) be the length of the longest \( R \)-regular sequence in \( \{f_1, \ldots, f_m\} \).

Lemma 3.1. Assume Hypothesis 2.7. Then, for all \( 1 \leq l \leq m \), if \( T_l < n \), then \( T_l > T_{l-1} \). For all \( 2 \leq l \leq m - 1 \), we have \( T_l - T_{l-1} \geq T_{l+1} - T_l \). Consequently,

\[
T_l = \begin{cases} 
2l, & 1 \leq l \leq \rho(I) \\
\min\{\rho(I) + l, n\} & \rho(I) \leq l \leq c
\end{cases}
\]

Moreover, for all \( x \in V \), \( \rho((I : x)) < \rho(I) \).

Proof. A consequence of Hypothesis 2.7 is that for every \( x \in V \) there is a monomial minimal generator \( f_j \) such that \( x \) divides \( f_j \), from which the first assertion follows.

To prove the second assertion, assume, by way of contradiction, and by induction on \( m \), that \( m \) is the smallest integer \( m' \) such that there exists an ideal generated by \( m' \) quadratic monomials such that the conclusion does not hold. Write \( \delta_l = T_l - T_{l-1} \); it is clear that \( 0 \leq \delta_l \leq 2 \). Pick \( l \) smallest such that \( \delta_l < \delta_{l+1} \). If \( \delta_l = 0 \), then \( T_{l+1} = T_l = n \). Hence \( \delta_l = 1 \) and \( \delta_{l+1} = 2 \).

We now claim that \( l = m - 1 \). For, assume, without loss of generality, that \( T_{l+1} = \deg \text{lcm}(f_1, \ldots, f_{l+1}) \). Let \( J = (f_1, \ldots, f_{l+1}) \). Then \( T_l(J) \leq T_l(I) = T_{l+1}(I) - 2 \). If \( m > l + 1 \), then, by minimality of \( m \), \( \delta_2(J) = \cdots = \delta_{l+1}(J) = 2 \).
Hence $f_1, \cdots, f_{t+1}$ is a regular sequence, and, therefore, $T_j(I) = 2j, \forall j \leq l + 1$ and $\delta_2(I) = \cdots = \delta_{t+1}(I) = 2$ contradicting the choice of $l$. Therefore $l = m - 1$.

Assume that $T_{m-1}(I) = \deg \lcm(f_1, \cdots, f_{m-1})$. Let $J = (f_1, \cdots, f_{m-1})$. If $T_{m-2}(J) < T_{m-2}(I)$, then $\delta_{m-1}(J) = 2$, and hence $J$ is generated by a regular sequence of $m - 1$ quadratic monomials. Therefore $T_j(J) = 2j, 1 \leq j \leq m - 1$.

Since $T_j(J) \leq T_j(I) \leq 2j, T_j(I) = 2j, 1 \leq j \leq m - 1$, contradicting the assumption that $\delta_{m-1}(I) = 1$; hence $T_{m-2}(J) = T_{m-2}(I)$. We may assume that $T_{m-2}(J) = \deg \lcm(f_2, \cdots, f_{m-1})$. Then $T_{m-1}((f_2, \cdots, f_m)) = 2 + T_{m-2}(J) > T_{m-1}(J) = T_{m-1}(I)$, leading to a contradiction.

From the above discussion, and since $T_1 = 2$, clearly there exists $\rho$ such that $T_l = \begin{cases} 2t, & 1 \leq l \leq \rho \\ \min\{\rho + l, n\} & \rho \leq l < c \end{cases}$

What we need to show is that $\rho$ is the length of the longest $R$-regular sequence in \{f_1, \cdots, f_m\}. If $f_j, \cdots, f_{j+1}$ form a regular sequence, then $T_l = 2t$, so $\rho \geq t$. Conversely, since $T_\rho = 2\rho$, there exists a regular sequence of length $\rho$ in \{f_1, \cdots, f_m\}.

Let $y \in V$ be such that $xy \in I$. If $f_1, \cdots, f_s$ are all the quadratic minimal generators of $(I : x)$, then none of them involves $x$ and $y$; therefore, to any regular sequence in \{f_1, \cdots, f_s\}, one can add $xy$, to get a longer regular sequence. The last statement follows immediately. \[ \square \]

**Lemma 3.2.** With notation as above, $\rho(I) \geq \frac{e}{2}$.

**Proof.** Since $\rho(I) \geq 1$, this holds when $c = 1$. By induction on $c$, we may assume that for all square-free monomial ideals $J$ with $\operatorname{ht} J < c$, $\rho(J) > \frac{e}{2}$. Take a minimal generator $xy$ of $I$. Let $J = (I \cap (V \setminus \{x, y\}))R$. Since $xy$ is a non-zerodivisor on $R/J$, $\rho(J) = \rho(I) - 1$, and, further, since, $(J, xy) \subseteq I$, $\operatorname{ht} J < \operatorname{ht}(J, xy) \leq \operatorname{ht} I$ and since $(I, x, y) = (J, x, y)$, $\operatorname{ht} J \geq c - 2$. By induction, $\rho(J) \geq \frac{e}{2}$, and, therefore, $\rho(I) \geq \frac{e}{2}$. \[ \square \]

We now prove that Conjecture \[\text{(HHSn)}\] holds for quadratic monomial ideals.

**Theorem 1.3** Let $I \subseteq R$ be generated by monomials of degree $2$. Then $e(R/I) \leq \frac{T_1T_2 \cdots T_c}{c!}$.

**Proof.** We proceed by induction on $c$. If $c = 2$, the Taylor bound holds for $I$ \[\text{(HSS04)}\] Corollary 4.3, so let $c \geq 3$. As discussed in the previous section, we take $I$ to be the edge ideal of a graph $G$ and assume that Hypothesis \[\square\] holds.

For all $x \in V$, notice that $e(R/(I, x))$ is the number of unmixed primes $p$ of $R/I$ containing $x$. Since each such prime has height $c$, in the sum $\sum_{x \in V} e(R/(I, x))$, it is counted $c$ times. Therefore $e(R/I) = \frac{1}{c} \sum_{x \in V} e(R/(I, x))$.

Now suppose $T_c = n$. As noted earlier, $(I, x)$ is essentially of height $\leq c - 1$. Therefore, by induction and by Proposition 2.3.10, $e(R/I) \leq \frac{n}{c} \frac{T_1T_2 \cdots T_{c-1}}{(c - 1)!} = \frac{T_1T_2 \cdots T_c}{c!}$. Therefore we may further assume that $T_c = c + \rho(I) < n$. February 1, 2008
We now reduce to the case that $\rho(I) < c$. If $\rho(I) = c$ then, without loss of generality, take $f_1, \ldots, f_c$ to be a regular sequence. Write $J = (f_1, \ldots, f_c)$. Since $J \subseteq I$ and $\text{ht } J = c = \text{ht } I$, we see that $e(R/I) \leq e(R/J) = 2^c$. From Lemma 3.1, $T_i = 2l$ for all $1 \leq l \leq c$. Hence

$$e(R/I) \leq \frac{T_1 T_2 \cdots T_c}{c!}.$$  

In particular $G$ is not a collection of $c$ isolated edges, which would have given $\rho(I) = c$ and $|V| = 2c$. We pick $x \in V$ such that $\deg_G x \geq 2$. Then $(I : x)$ is essentially of height $\leq c - 2$. Moreover $\rho((I : x)) < \rho(I)$, by Lemma 3.1. We noted earlier that $(I, x)$ is essentially of height $\leq c - 1$. Let $\rho' := \rho((I, x))$. Hence, by induction on $c$ and by Hypothesis (2), we have

$$e(R/(I, x)) \leq \frac{2 \cdot 4 \cdots 2\rho' \cdot (2\rho' + 1) \cdots (c + \rho' - 1)}{(c - 1)!} = 2^{\rho'} \mu(\rho', c - 1),$$

which gives, after successive application of (2), (which is permitted since $\rho(I) < c$), $e(R/(I, x)) \leq 2^{\rho(I)} \mu(\rho(I), c - 1)$. Since $\deg_G x \geq 2$ and $\rho((I : x)) \leq \rho(I) - 1$, we can conclude, by a similar argument, that $e(R/(I, x)) \leq 2^{\rho(I) - 1} \mu(\rho(I) - 1, c - 2)$. (Notice that since $\rho(I) - 1 \leq c - 2$, we can apply (2).)

We must show that

$$e(R/I) \leq \frac{2 \cdot 4 \cdots 2\rho(I) \cdot (2\rho(I) + 1) \cdots (c + \rho(I))}{c!} = 2^{\rho(I)} \mu(\rho(I), c).$$

Since $e(R/I) = e(R/(I, x)) + e(R/(I : x))$, it suffices to show that

$$2^{\rho(I)} \mu(\rho(I), c - 1) + 2^{\rho(I) - 1} \mu(\rho(I) - 1, c - 2) < 2^{\rho(I)} \mu(\rho(I), c).$$

Set $\rho = \rho(I)$, $\gamma = c$, $\gamma_1 = 2$. Since $2 \leq \rho < c$, and $c \geq 3$, we see that $2 \leq \rho < \gamma \leq \rho \gamma_1$ and $\rho - 1 \leq \gamma - \gamma_1$. Applying Lemma 2.4 now finishes the proof. \qed

4. Reduction to the Cohen-Macaulay Case of Theorem 1.2

Let $I$ be the edge ideal of a bipartite graph $G$ on $V = V_1 \cup V_2$. In order to prove Theorem 1.2 for $I$, we will first reduce to the case of $G$ having perfect matching, and, by associating a certain directed graph to $G$, show that only Cohen-Macaulay bipartite graphs matter. The next section is devoted to proving the theorem for Cohen-Macaulay bipartite graphs.

A matching in $G$ is a maximal (under inclusion) set $m$ of edges such that for all $x \in V$, at most one edge in $m$ is incident on $x$. Edges in a matching form a regular sequence on $R$. We say that $G$ has perfect matching, or, is perfectly matched, if there is a matching $m$ such that for all $x \in V$, there is exactly one edge in $m$ is incident on $x$. König’s theorem [Vil01, Section 6.4] states that the maximum size of any matching equals the minimum size of any vertex cover. In the language of algebra, we can restate it as that the maximum length of a regular sequence in the set of monomial minimal generators of the edge ideal equals the height of the ideal.

Lemma 4.1. With notation as above, $G$ is perfectly matched if and only if $|V_1| = |V_2| = \text{ht } I = c$.

Proof. If $G$ is perfectly matched, then, first, $|V_1| = |V_2|$. Secondly, the matching gives a regular sequence of length $|V_1|$ in $I$, so $\text{ht } I \geq |V_1|$. Since $V_1$ is a vertex cover for $G$, $\text{ht } I \leq |V_1|$. Hence $|V_1| = |V_2| = \text{ht } I = c$. Conversely, assume that
towards (respectively, away from) it. We say that a set $A$ vertex $i$ for all $i, j \subseteq A$ of antichains in transitively closed $c$ cocliques in $\succ$ the order $i$ is an edge for all $i, j \subseteq V_2$, so $G$ is perfectly matched.

**Proposition 4.2.** Let $I$ be the edge ideal of a bipartite graph $G$ on $V = V_1 \bigcup V_2$. Then Hypothesis 2.7 holds for $I$ if and only if $G$ is perfectly matched.

**Proof.** If $G$ is perfectly matched, then let $p := (x : x \in V_1)$ and $q := (x : x \in V_2)$. By Lemma 4.1 $ht(p) = q = c$. For all $x \in V_1$, $(I, x) \subseteq q$ and $(I : x) \subseteq q$; the case of $x \in V_2$ is similar. Hence we see that Hypothesis 2.7 holds for $I$.

Conversely, assume that $G$ is not perfectly matched. Since $V_1$ and $V_2$ are minimal vertex covers for $G$, we see that $|V_1| \geq c$ and that $|V_2| \geq c$. In light of Lemma 4.1 we may assume, without loss of generality, that $|V_1| > c$. In the paragraph preceding Lemma 4.1 we noted that there is a matching with $c$ edges. Let $\{x_1, \ldots, x_c\} \subseteq I_1, \{y_1, \ldots, y_c\} \subseteq V_2$ be such that $x_1y_1, \ldots, x_cy_c$ is a matching of $G$. Pick $x \in V_1 \setminus \{x_1, \ldots, x_c\}$. Then $x_1y_1, \ldots, x_cy_c, x$ is a regular sequence in $(I, x)$, giving $ht(I, x) > c$. Hence Hypothesis 2.7 does not hold.

**Remark 4.3.** The proof above shows that, if $I$ is the edge ideal of a bipartite graph such that $ht(I, x) = c$ for all $x \in V$, then $ht(I : x) = c$, for all $x \in V$. This is not true for arbitrary square-free monomial ideals.

**Discussion 4.4.** For the rest of this section, we restrict our attention to bipartite graphs $G$ with perfect matching. Let $V_1 = \{x_1, \ldots, x_c\}$ and $V_2 = \{y_1, \ldots, y_c\}$. We abbreviate $\{1, \ldots, c\}$ as $[c]$. After relabelling the vertices, we will assume that $x_1y_1$ is an edge for all $i \in [c]$. We associate $G$ with a directed graph $G[d]$ on $[c]$ defined as follows: for $i, j \in [c], ij$ is an edge of $G[d]$ if and only if $x_1y_i$ is an edge of $G$. (Here, by $ij$, we mean the the directed edge from $i$ to $j$.) We will write $j \succ i$ if there is a directed path from $i$ to $j$ in $d$. By $j \succ i$ we mean that $j \succ i$ or $j = i$. Let $d$ be any directed graph on $[c]$, and denote the underlying undirected graph of $d$ by $|d|$. A vertex $i$ of $d$ is called a source (respectively, sink) vertex if it has no edge directed towards (respectively, away from) it. We say that a set $A \subseteq [c]$ is an antichain if for all $i, j \in A$, there is no directed path from $i$ to $j$ in $d$, and, by $A_d$, denote the set of antichains in $d$. We consider $\emptyset$ as an antichain. A coclique of $|d|$ is a set $A \subseteq [c]$ such that for all $i \neq j \in A$, $i$ and $j$ are not neighbours in $|d|$. Antichains in $d$ are cocliques in $|d|$, but the converse is not, in general, true. We say that $d$ is acyclic if there are no directed cycles, and transitively closed if, for all $i, j, k \in [c]$, whenever $ij$ and $jk$ are (directed) edges in $d$, $ik$ is an edge. Observe that $d$ is a poset under the order $\succ$ (if and only if) it is acyclic and transitively closed. In this case, for all $A \subseteq [c]$, $A$ is an antichain in $d$ if and only if $A$ is a coclique in $|d|$. Let $\kappa(d)$ denote the largest size of any coclique in $|d|_G$.

Before we proceed, we need the characterization of Cohen-Macaulay bipartite graphs, due to Herzog-Hibi.

**Theorem 4.5.** ([HH05]) Let $G$ be a bipartite graph on $V_1 \bigcup V_2$, with edge ideal $I$. Then $G$ is Cohen-Macaulay if and only if $|V_1| = |V_2| = c = ht(I)$ and we can write $V_1 = \{x_1, \ldots, x_c\}$ and $V_2 = \{y_1, \ldots, y_c\}$ such that

(a) For all $1 \leq i \leq n$, $x_iy_i$ is an edge of $G$.
(b) For all $1 \leq i, j \leq n$, if $x_iy_i$ is an edge of $G$, then $j \geq i$.
(c) For all $1 \leq i, j, k \leq n$, if $x_iy_j$ and $x_jy_k$ are edges of $G$, then $x_iy_k$ is an edge of $G$.

February 1, 2008
Remark 4.6. When we say that $G$ is a Cohen-Macaulay bipartite graph on the vertex set $\{x_1, \ldots, x_c\} \cup \{y_1, \ldots, y_c\}$, we will assume that the variables have already been relabelled so that the conditions of Theorem 4.5 hold. It is clear that $G$ is Cohen-Macaulay if and only if $\mathcal{D}_G$ is a poset.

Lemma 4.7. Let $G$, $I$, and $\mathcal{D}_G$ be as in Discussion 4.4. Let $j \succ i$. Then for all $p \in \text{Unm} R/I$, if $y_i \in p$, then $y_j \in p$.

Proof. Applying induction on the length of a directed path from $i$ to $j$, we may assume, without loss of generality, that $ij$ is a directed edge of $\mathcal{D}_G$. Let $p \in \text{Unm} R/I$ and $k \in [c]$. Since $x_ky_k \in I$, $x_k \in p$ or $y_k \in p$. Since $ht \mathfrak{p} = c$, in fact, $x_k \in p$ if and only if $y_k \notin p$. Now since $y_i \in p$, $x_i \notin p$, so $(I : x_i) \subseteq p$. Note that since $x_iy_j$ is an edge of $G$, $y_j \in (I : x_i)$.

Discussion 4.8 (Collapsing directed graphs). Suppose that $\mathcal{D}_G$ has a directed cycle, i.e., a sequence of directed edges $i_0i_1, i_1i_2, \ldots, i_{p-1}i_p, i_p0$, for some $p \geq 1$. Set $B = \{i_1, \ldots, i_p\}$. We collapse $\mathcal{D}_G$ to obtain a new directed graph $\mathcal{D}$ on the vertex set $[c] \setminus B$ as follows: if $ij$ an edge of $\mathcal{D}_G$ for some $i,j \notin B$, then $ij$ is an edge of $\mathcal{D}$. For all $1 \leq s \leq p$, and for all $j \notin B$, if $i_sj$ (respectively, $ji_s$) is an edge of $\mathcal{D}_G$, then set $i_sj$ (respectively, $ji_s$) to be an edge of $\mathcal{D}$. Any cycle of $\mathcal{D}$ comes from a cycle of $\mathcal{D}_G$; hence the total number of cycles decreases. Let $\hat{G}$ be the bipartite graph associated to $\mathcal{D}$. If $A \subseteq [c] \setminus B$ is a coclique in $[\mathcal{D}]$, then it is clearly a coclique in $[\mathcal{D}_G]$; hence $\kappa(\hat{G}) \leq \kappa(G)$. Write $\hat{I} \subseteq R$ for the edge ideal of $\hat{G}$. Let $p \in \text{Unm} R/I$. It is an immediate corollary to Lemma 4.7 that $x_0 \in p$ if and only if $(x_0, x_1, \ldots, x_p) \subseteq p$, which holds if and only if $y_j \notin p$ for all $0 \leq s \leq p$. Let $\hat{p} := (p \cap \mathbb{k}\{x_i, y_j : i \notin B\})R$. We first claim that $\hat{I} \subseteq \hat{p}$. To prove this, we only need to consider the new edges introduced in $G$, which are of the form $x_iy_j$ or $x_jy_i$ for some $j \in [c] \setminus B$. For the edge $x_iy_j$, if $x_i \notin \hat{p}$, then $x_i \notin p$; by the above observation, we see that $x_i \notin p$ for all $0 \leq s \leq p$. Hence $y_j \notin p$, giving $y_j \notin p$. The case of $x_jy_i$ is similar. Now since $ht \hat{p} = ht \hat{I} = c - |B|$, we conclude that $\hat{p} \in \text{Unm} R/\hat{I}$. The map $\text{Unm} R/I \to \text{Unm} R/\hat{I}$ sending $p \mapsto \hat{p}$ is injective. Conversely, let $\hat{q} \in \text{Unm} R/\hat{I}$. Set $q := \begin{cases} \hat{q} + (x_i, \ldots, x_p) & \text{if } x_0 \in \hat{q} \\ \hat{q} + (y_j, \ldots, y_p) & \text{if } y_0 \in \hat{q} \end{cases}$.

Then we get an injective map $\text{Unm} R/\hat{I} \to \text{Unm} R/I$. Therefore, $e(R/I) = e(\hat{R}/\hat{I})$.

Discussion 4.9 (Closing directed graphs under transitivity). Suppose that $ij$ and $jk$ are edges of $\mathcal{D}_G$; then we add an edge $ik$. Call the new graph $\mathcal{D}$ and let $\hat{G}$ be the bipartite graph associated to $\mathcal{D}$. Let $\hat{I}$ be the edge ideal of $\hat{G}$. Since $I \subseteq \hat{I}$ and $ht I = ht \hat{I}$, we have that $e(R/I) \geq e(\hat{R}/\hat{I})$. In order to show that $e(R/I) = e(\hat{R}/\hat{I})$, it suffices to show that $x_iy_k \in p$, for all $p \in \text{Unm} R/I$. Let $p \in \text{Unm} R/I$ be such that $x_i \notin p$. Then, since $k \succ i$, by Lemma 4.7, $y_k \in p$, and therefore, $x_iy_k \in p$. Moreover, any coclique in $|\mathcal{D}|$ is a coclique in $|\mathcal{D}_G|$, so $\kappa(\hat{G}) \leq \kappa(G)$.

The significance of $\kappa(G)$ is that it gives a lower bound on the (Castelnuovo-Mumford) regularity, $\text{reg} R/I$. Following Zheng [Zhe04], we say that two edges $vw$ and $v'w'$ of a graph $G$ are disconnected if they are no more edges between the four vertices $v, v', w, w'$. The edges in any pairwise disconnected set form a regular sequence in $R$; in fact, a set $A$ of edges is pairwise disconnected if and only if $(I \cap \mathbb{k}[V_A])R$ is generated by the regular sequence of edges in $A$, where by $V_A$, we mean the February 1, 2008
set of vertices on which the edges in $a$ are incident. The latter condition holds if and only if the subgraph of $G$ induced on $V_a$, denoted as $G|_{V_a}$, is a collection of $|a|$ isolated edges. Set $r(I) := \max\{|a| : a$ is a set of pairwise disconnected edges in $G\}$. If $G$ is a forest with edge ideal $I$, then $\text{reg } R/I = r(I)$ [Zhe04, Theorem 2.18].

**Lemma 4.10.** With notation as above, $r(I) \geq \kappa(G) \geq \max\{|A| : A \in A_0\}$.  

**Proof.** If $A \subseteq [c]$ is a coclique of $|G|$, we easily see that the edges $\{x_iy_i : i \in A\}$ are pairwise disconnected in $G$. The assertion now follows from the observation, that we made in Discussion 4.1, that any antichain in $|G|$ is a coclique of $|G|$. \qed

**Lemma 4.11.** With notation as above, for $1 \leq l \leq r(I)$, $M_l(I) = 2l$ and for $r(I) \leq l \leq c$, $M_l(I) \geq l + r(I)$. Hence, for all $1 \leq l \leq c$, $M_l(I) > l$.

**Proof.** Let $a$ be a set of pairwise disconnected edges with $|a| = r(I)$. Then, with the notation as above, $(I \cap \mathbb{K}[V_a])R$ is generated by a regular sequence of length $r(I)$. From Lemma 2.3(i), we have that, for $1 \leq l \leq r(I)$, $M_l(I) \geq 2l$. From the Taylor resolution of $R/I$, it follows that $M_l \leq 2l$. Hence $M_l = 2l$ for all $1 \leq l \leq r(I)$. For $l > r(I)$, we see from Lemma 2.1 that $M_l \geq l + r(I)$. \qed

**Discussion 4.12** (Reduction to the Cohen-Macaulay case). Now let $G$ be any perfectly matched bipartite graph. We first collapse $\mathcal{G}_a$, repeatedly if necessary, to get a directed acyclic graph, which we denote $\vec{G}$. We now close $\vec{G}$ under transitivity, and call it $\vec{G}$. Denote the corresponding bipartite graph by $\vec{G}$, and its edge ideal by $\vec{I}$. Notice that $\vec{G}$ is Cohen-Macaulay, from Remark 4.6. From the discussion, we see that $ht(\vec{I}) \leq c$, $e(\vec{R}/\vec{I}) = e(R/I)$ and that $\kappa(\vec{G}) \leq \kappa(G)$.

Here is the outline of the rest of the proof: since $\vec{G}$ is Cohen-Macaulay, $r(\vec{I}) = \kappa(\vec{G})$ and that equality must also hold for $\vec{I}$ in Lemma 4.11. Hence $M_l(\vec{I}) \leq M_l(I)$ for $1 \leq l \leq ht(\vec{I})$. Now, since $ht(\vec{I}) = c = ht(I, x_1)$, the conjectured bound for $I$ would be established, if it can be established for $\vec{I}$.

5. **COHEN-MACLAURAY BIPARTITE GRAPHS**

Lemma 4.11 gives that $\text{reg } R/I \geq r(I)$. For arbitrary bipartite graphs, this might be a strict inequality, (consider, e.g., the edge ideal of the cycle on 8 vertices), but we have:

**Proposition 5.1.** Let $I$ be the edge ideal of be a Cohen-Macaulay bipartite graph $G$ on the vertex set $\{x_1, \ldots, x_c\} \sqcup \{y_1, \ldots, y_c\}$. Then $\text{reg } R/I = r(I)$.

**Proof.** It suffices to show that $\text{reg } R/I \leq r(I)$, by induction on the number of vertices. Since the claim is true for a Cohen-Macaulay bipartite graph on 2 vertices, we assume inductively that for all Cohen-Macaulay bipartite graphs on fewer than $2c$ vertices, the claim holds.

Note that $y_1$ is a leaf vertex of $G$. Since $\text{depth } \frac{R}{(I : x_1)} \geq \text{depth } R/I = \dim R/I = \dim \frac{R}{(I : x_1)}$ (Remark 2.4 and Hypothesis 2.7), $\frac{R}{(I : x_1)}$ is Cohen-Macaulay. Further, $(I, x_1)$ is the edge ideal of the deletion of the vertices $x_1$ and $y_1$ in $G$; this graph satisfies the conditions in Theorem 4.4, so $\frac{R}{(I, x_1)}$ is Cohen-Macaulay. Moreover $ht(I : x_1) = c = ht(I, x_1)$. Let $a$ be a set of pairwise disconnected edges in the graph of $(I : x_1)$. Then $a$ is pairwise disconnected in $G$, because the graph $(I : x_1)$ is obtained by deleting the neighbours of $x_1$ from $G$. Hence we may join $x_1y_1$. February 1, 2008
to get a set of pairwise disconnected edges in $G$, so $r((I : x_1)) \leq r(I) - 1$. Let $J = (I \cap \mathbb{k}[x_2, \ldots, x_c, y_2, \ldots, y_c])R$. Then $J$ is the edge ideal of the deletion $G \setminus x_1$ and $(I, x_1) = (J, x_1)$. It is evident that $r((I : x_1)) = r(J) \leq r(I)$.

We need to show that $M_l(I) \leq l + r(I)$ for all $1 \leq l \leq c = \text{ht} I = \text{pd} R/I$. From the exact sequence

$$\cdots \longrightarrow \text{Tor}_i \left( \mathbb{k}, \frac{R}{(I : x_1)} \right) (-1) \longrightarrow \text{Tor}_i (\mathbb{k}, R/I) \longrightarrow \text{Tor}_i \left( \mathbb{k}, \frac{R}{(I, x_1)} \right) \longrightarrow \cdots,$$

we can see that showing

$$\text{Tor}_i \left( \mathbb{k}, \frac{R}{(I : x_1)} \right) (-1) \big|_j = 0 = \text{Tor}_i \left( \mathbb{k}, \frac{R}{(I, x_1)} \right) \big|_j$$

for all $j > l + r(I)$, will suffice. This is equivalent, by the induction hypothesis, to showing that $r((I : x_1)) \leq r(I) - 1$ and that $r((I, x_1)) \leq r(I)$, which we have done.

**Corollary 5.2.** With notation as above, for $1 \leq I \leq r(I)$, $M_l(I) = 2l$ and for $r(I) \leq l \leq c$, $M_l(I) = l + r(I)$.

**Proof.** Follows from Lemma 4.11 and the definition of regularity. 

For the rest of this section, we will take $I$ to be the edge ideal of an arbitrary Cohen-Macaulay bipartite graph $G$ on the vertex set $\{x_1, \ldots, x_c\} \bigsqcup \{y_1, \ldots, y_c\}$. Recall that the labelling of vertices was chosen so that the conditions of Theorem 4.5 hold; see Remark 4.6. Hence for all $i, j \in [c]$, if $j \succ i$ then $j > i$.

**Proposition 5.3.** With notation as above, $r(I) = \max \{|A| : A \in \mathcal{A}_G\}$. Consequently, $r(I) = \kappa(G)$.

**Proof.** Let $A \in \mathcal{A}_G$. Then $\{x_i y_i : i \in A\}$ is a set of pairwise disconnected edges in $G$. Conversely, let $b$ be a set of pairwise disconnected edges such that there exists $j \neq i$ such that $x_i y_j \in b$. Let $a := (b \setminus \{x_i y_j\}) \cup \{x_i y_i\}$. We claim that the edges in $a$ are pairwise disconnected; for, if not, then some edge in $b \setminus \{x_i y_j\}$ is incident on a neighbour of $x_i$ or $y_i$. We claim that this must be on a neighbour of $y_i$, for, if it were on a neighbour of $x_i$, then the set $b$ would not have been pairwise disconnected. Therefore $k < i$ such that $x_k y_i$ is an edge of $G$, and some edge incident on $x_k$ belongs to $a$. However, since $G$ is Cohen-Macaulay, $x_k y_j$ is an edge of $G$ too, contradicting the hypothesis that the edges of $b$ are pairwise disconnected. Repeating this if necessary, we can construct a set $a$ of pairwise disconnected edges in $G$ such that $a = \{x_i y_i : i \in A\}$ for some $A \subseteq [c]$ and $|A| = |b|$. Such a set $A$ is an antichain in $\mathcal{A}_G$.

**Proposition 5.4.** With notation as above, $e(R/I) = |\mathcal{A}_G|$.

**Proof.** Let $p \in \text{Unm} R/I$. Let $A := \{i \in [c] : y_i \in p\}$ and for all $j \in [c]$ with $i \succ j, y_j \notin p$. Note that $A$ is an antichain. This gives a map from $\text{Unm} R/I$ to $\mathcal{A}_G$, which is injective by Lemma 4.7. Conversely, for any antichain $A$ of $\mathcal{A}_G$, the prime ideal $(x_j : j \neq i$ for any $i \in A) + (y_j : j \succ i$ for some $i \in A)$ belongs to $\text{Unm} R/I$. This gives a bijection $\mathcal{A}_G$ and $\text{Unm} R/I$, with the empty set corresponding to $(x_1, \ldots, x_c)$.

**Lemma 5.5.** Let $\mathfrak{d}$ be any poset on $c$ vertices, with order $\succ$, $A$ the set of antichains in $\mathfrak{d}$ and $r = \max \{|A| : A \in \mathcal{A}_G\}$. Then $|A| \leq 2^r \mu(r, c)$. Equality holds above, if and only if $r = 1$ or $r = c$.

February 1, 2008
Proof. We prove this by induction on \( c \). If \( r = 1 \), (in particular, if \( c = 1 \)), \( \mathcal{D} \) is a chain, i.e., for all \( i \neq j \in [c] \), \( i > j \) or \( j > i \). In this case, \( |A| = c + 1 = 2^{\mu(1, c)} \). If \( c = r \geq 2 \), then \( \mathcal{D} \) is a collection of \( c \) isolated vertices, in which every subset of \([c]\) is an antichain, i.e., \( |A| = 2^r = 2^{\mu(c, c)} \). Note that equality holds in both the cases above.

We now have \( c > r \geq 2 \). Pick a vertex \( i \) such that there is an antichain \( A \) with \( i \in A \) and \( |A| = r \). Set \( \tilde{\mathcal{D}} := \{j \in \mathcal{D} : j \neq i \text{ or } i \neq j\} \). Let \( \mathcal{D}' \) be the poset obtained by deleting \( i \) from \( \mathcal{D} \), keeping all the other elements and relations among them. Denote the respective sets of antichains by \( \mathcal{A} \) and \( \mathcal{A}' \). Now for any \( A \subseteq [c], A \in \mathcal{A} \setminus \mathcal{A}' \) if and only if \( i \in A \) and \( A \setminus \{i\} \in \mathcal{A}' \). Therefore \( \mathcal{A} = \mathcal{A}' \cup \{A \cup \{i\} : A \in \mathcal{A}\} \) and \( |A| = |\mathcal{A}'| + |\tilde{\mathcal{A}}| \).

Observe that \( \max\{|A| : A \in \tilde{\mathcal{A}}\} = r - 1 \). Let \( r' := \max\{|A| : A \in \mathcal{A}'\} \). Then \( r' \leq r \). Let \( c_1 := \{|j \in \mathcal{D} : j \nleq i \text{ or } i \nleq j\}\}. Then \( \tilde{\mathcal{D}} \) has \( c - c_1 \) vertices. We note that \( r - 1 \leq c - c_1 \). We assume, by induction on the number of vertices, that the lemma holds for \( \tilde{\mathcal{D}} \) and \( \mathcal{D}' \), yielding

\[
|A| \leq 2^{r'} \mu(r', c - 1) + 2^{r_1 - 1} \mu(r - 1, c - c_1),
\]

and, by repeated application of (2) from Discussion 2.8 (which is permitted since \( r' \leq r < c - 1 \))

\[
(4) \quad |A| \leq 2^r \mu(r, c - 1) + 2^{r_1 - 1} \mu(r - 1, c - c_1). \]

Since \( c > r \geq 2 \), we must show that \( |A_0| < 2^r \mu(r, c) \); to this end, it suffices to show that

\[
2^r \mu(r, c - 1) + 2^{r - 1} \mu(r - 1, c - c_1) \leq 2^r \mu(r, c),
\]

which follows from Lemma 5.3 with \( \rho = 1, \gamma = c, \gamma_1 = c_1 \). Note that by the choice of \( i, c \leq r c_1 \).

\[\square\]

**Theorem 1.2** Let \( I \subseteq R \) be the edge ideal of a bipartite graph. Then

\[
e(R/I) \leq \frac{M_1 M_2 \cdots M_c}{c!}.
\]

Proof. Denote the bipartite graph by \( G \) and its vertex set by \( V = V_1 \sqcup V_2 \). We may assume that Hypothesis 2.4 holds; hence \( G \) is perfectly matched. We reduce the proof to the Cohen-Macaulay case to obtain \( \tilde{G} \) and \( \tilde{I} \) as in Discussion 4.12. Let \( \tilde{c} := \text{ht} \tilde{I} \). The height of the edge ideal does not increase during collapsing the directed graph, while it remains unchanged after closing the directed graph under transitivity, so \( \tilde{c} \leq c \). Again, from Discussion 4.12 \( e(R/I) = e(\tilde{R}/\tilde{I}) \). We already observed that \( r(\tilde{I}) \geq \kappa(G) \geq \kappa(\tilde{G}) = r(\tilde{I}) \); see Lemma 5.3. Now from Lemma 4.11 and Corollary 5.2 we see that, for \( 1 \leq l \leq \tilde{c}, M_l(\tilde{I}) \geq M_l(I) \) and that \( M_l(I) > l \) for all \( c < l \leq c \). Hence it suffices to show that

\[
e(\tilde{R}/\tilde{I}) \leq \frac{M_1(\tilde{I}) \cdots M_c(\tilde{I})}{\tilde{c}!}.
\]

From Proposition 5.4, \( e(R/\tilde{I}) = |A_{\tilde{D}}| \). Corollary 5.2 gives

\[
\frac{M_1(\tilde{I}) \cdots M_c(\tilde{I})}{\tilde{c}!} = 2^{r(\tilde{I})} \mu(r(\tilde{I}), \tilde{c}).
\]

Since, by Proposition 5.3, \( r(\tilde{I}) = \max\{|A| : A \in A_{\tilde{D}}\} \), we apply Lemma 5.3 with \( A = A_{\tilde{D}} \), to finish the proof.

\[\square\]
6. Pure and Quasi-pure Resolutions

When can equality hold for \( I \) in the conjectured bound? The proof Theorem 1.2 above and Lemma 5.5 show that if \( G \) is a Cohen-Macaulay bipartite graph with edge ideal \( I \), and equality holds for \( I \), then \( \reg R/I = c \) or \( \reg R/I = 1 \). We are now ready to prove Theorem 1.3.

**Theorem 1.3** Let \( I \) be the edge ideal of a bipartite graph. If equality holds in Conjecture (HHSu), then \( R/I \) is a complete intersection, or is Cohen-Macaulay with \( \reg R/I = 1 \). In either of the cases, \( R/I \) is Cohen-Macaulay and has a pure resolution.

**Proof.** Denote the bipartite graph by \( G \). We first reduce to the case that Hypothesis 2.4 holds. We will show that \( \ht(I, x) = c \) for \( x \in V \); this suffices, by Remark 4.3.

Assume, by way of contradiction, that \( x \in V \) is such that \( \ht(I, x) > c \). Then \( \ht(I : x) = c \) and \( e(R/(I : x)) = e(R/I) \). We may assume that \( x \) is not an isolated vertex of \( G \); for otherwise, \( x \) would not have divided any minimal generator of \( I \). Hence \( x \) has at least one neighbour, so \( (I : x) \) is essentially of height at most \( c - 1 \); see the paragraph following Proposition 2.5. Let \( J \subseteq R \) be the ideal generated by the quadratic minimal generators of \( (I : x) \). Observe that \( (I : x) \) is generated by the neighbours of \( x \), modulo \( J \). Hence \( e(R/(I : x)) = e(R/J) \). It follows from Lemma 2.3(a) and Proposition 2.5(a) that \( M_l(J) \leq M_l((I : x)) \leq M_l(I) \) for all \( 1 \leq l \leq c \). Now, \( M_l(J) > t \), for all \( l \). Therefore equality holds for \( J \) in Conjecture (HHSu). Since \( M_l((I : x)) \geq M_l(J) \) and \( \ht J < c = \ht(I : x) \), we see that equality cannot hold for \( (I : x) \), and hence, again by Proposition 2.5(a), for \( I \). Therefore we may assume that Hypothesis 2.7 holds.

By Proposition 4.2, \( G \) has perfect matching. Let \( \delta_G \) be the directed graph associated to \( G \), as in Discussion 4.4. We can now reduce the problem to the Cohen-Macaulay case. Let \( G \) and \( \hat{I} \) be as in the proof of Theorem 1.2. Since equality holds for \( I \), we see that \( \hat{c} = \ht I = c \), \( \hat{M}_l(\hat{I}) = M_l(I) \) for all \( 1 \leq l \leq c \). In particular, since \( \reg R/I = \reg R/\hat{I} \), it follows from Proposition 5.3 and Discussion 4.12 that \( \kappa(\hat{G}) = \kappa(G) \). Moreover, equality must hold for \( \hat{I} \).

Since \( G \) is Cohen-Macaulay, \( A_{\delta_G} \) is a poset, and, from Lemma 5.5, we see that \( r(\hat{I}) = 1 \) or \( r(\hat{I}) = \hat{c} \). If \( r(\hat{I}) = c \), then \( A_{\delta_{\hat{G}}} = A_{\delta_G} \) is a collection of \( c \) isolated vertices, or, equivalently, \( G \) is a collection of \( c \) isolated edges. In this case, \( R/I \) is a complete intersection. Since all the minimal generators of \( I \) have the same degree, \( R/I \) has a pure resolution.

If \( r(\hat{I}) = 1 \), then \( A_{\delta_G} \) and, hence, \( A_{\delta_{\hat{G}}} \) have precisely one source vertex and one sink vertex. With that, \( 1 = r(\hat{I}) = \kappa(\hat{G}) = \kappa(G) \) if and only if \( A_{\delta_{\hat{G}}} \) is a chain, i.e., \( A_{\delta_{\hat{G}}} = A_{\delta_G} \). In other words, \( R/I \) is Cohen-Macaulay with \( \reg R/I = 1 \), which, evidently, has a pure resolution.

A Cohen-Macaulay bipartite graph \( G \) on the vertex set \( \{x_1, \ldots, x_c\} \cup \{y_1, \ldots, y_c\} \) has \( r(I) = 1 \), if and only if, in the context of Theorem 5.3, \( x_jy_j \) is an edge for all \( j \geq i \). Such a graph \( G \) is acyclic (i.e., a forest), if and only if it is a path on three edges.

Suppose that \( G \) is a graph on \( V \) with edge ideal \( I \) with the property that \( r(I) = \reg R/I \). For instance, \( G \) is a forest ([Zhe04 Theorem 2.18]) or a Cohen-Macaulay bipartite graph (Proposition 5.1). If \( r(I) \leq 2 \), then \( R/I \) has a quasi-pure resolution. Now suppose that \( r(I) \geq 3 \) and that \( R/I \) has a quasi-pure resolution. Since \( r(I) \geq 3 \), February 1, 2008
Hence we need to show that $\beta = V_{xy}$ is an edge of $G$.

**Proposition 6.1.** Let $I$ be the edge ideal of a graph $G$. For any multidegree $\sigma$, $\beta_{|\sigma|-1,\sigma}(R/I) \neq 0$ if and only if there exists a partition $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_d$ such that for all $1 \leq i \leq d$, for all $x \in \sigma_i$, and for all $y \in \sigma \setminus \sigma_i$, $xy$ is an edge of $G_{|\sigma}$.

**Proof.** We immediately reduce the problem to the case that $\sigma = V$, noting that, by Lemma 2.3 and the flatness $R$ over $k[\sigma]$, $\beta_{|\sigma|-1,\sigma}(R/I) = \beta_{|\sigma|-1,\sigma}\left( R/(I \cap k[\sigma]) R \right)$.

Hence we need to show that $\beta_{|V|-1,V}(R/I) \neq 0$ if and only if there is a partition $V = V_1 \sqcup \cdots \sqcup V_d$ such that for all $1 \leq i \leq d$, for all $x \in V_i$ and for all $y \in V \setminus V_i$, $xy$ is an edge of $G$. Let $\Delta$ be the coclique complex of $G$. Hochster’s formula gives that $\beta_{|V|-1,V}(R/I) = \dim_k \tilde{H}_0(\Delta; k)$. Hence we must show that $\Delta$ is disconnected if and only if a partition, such as above, exists.

Suppose such a partition exists. Then any coclique of $G$ is contained in $V_i$ for some $1 \leq i \leq d$; hence $\Delta$ is disconnected. Conversely, assume that $\Delta$ is disconnected. Denoting the number of distinct components of $\Delta$ by $d$, we set $V_i, 1 \leq i \leq d,$ be the vertex sets of these components. We see immediately that for $x, y \in V$, whenever $x$ and $y$ are in different components of $\Delta$, there is an edge $xy$ in $G$.

We wish to mention here that this agrees with the result of Novik-Swartz [NS06, Theorem 1.3] that the first skip in the sequence of $m_l$’s is at $n - q_1 + 1$, where $q_1$ is the Cohen-Macaulay connectivity of the 1-dimensional skeleton of the Stanley-Reisner complex of $I$. For the edge ideal of a graph $G$, the 1-dimensional skeleton of its Stanley-Reisner complex is the complement graph $\tilde{G}$. In passing, let us note that if $G$ is a forest, then Proposition 6.1 implies that $\max\{l : m_l(I) = l + 1\} = \max\{\deg_G x : x \in V\}$. More generally, if $G$ is a bipartite graph, then $\max\{l : m_l(I) = l + 1\}$ is the largest cardinality of a complete bipartite subgraph of $G$.

**Discussion 6.2.** We already observed that if $\reg R/I \leq 2$, then $R/I$ has a quasi-pure resolution. Let $G$ be a connected Cohen-Macaulay bipartite graph such that $\reg R/I \geq 3$ and $R/I$ has a quasi-pure resolution. It is easy to see that $G$ is connected if and only if $\mathfrak{d} G$ is a connected poset. Since $\reg R/I = r(I) \geq 3$, $M_3(I) = 6$ by Corollary 5.2, so for $R/I$ to have a quasi-pure resolution, we must have $m_4(I) \geq 6$. This means, by the observation in the last paragraph, that for all $i$, there are at most two elements $j$ such that $j \triangleright i$ (or $i \triangleright j$) in $\mathfrak{d} G$. For $i, j \in [c]$, say that $j$ covers $i$ if $j \triangleright i$ and there does not exist $j'$ such that $j \triangleright j' \triangleright i$. Since $\mathfrak{d} G$ is connected, in every maximal chain, there exists $i, j, j'$ such that $j$ and $j'$ cover $i$ or $i$ covers $j$ and $j'$. From the observation above, it follows that, in the first case, $i$ is a source vertex and that $j$ and $j'$ are sink vertices. Similarly, in the second case, $i$ is a sink vertex and $j$ and $j'$ are source vertices. Hence every maximal chain of $\mathfrak{d} G$ has length at most one; in fact, since $\mathfrak{d} G$ is connected, every maximal chain has length one. Therefore every vertex in $\mathfrak{d} G$ is a source vertex or a sink vertex, but not both. Every source (respectively, sink) vertex in $\mathfrak{d} G$ is covered by (respectively, covers) at most two sink (respectively, source) vertices. For $x_i y_i$ to be a leaf in $G$, it is necessary and sufficient that $i$ is a source vertex or a sink vertex in $\mathfrak{d} G$. Therefore, in our case, $x_i y_i$ is a leaf for all $i$; in
other words, \( G \) is the suspension\(^1\) of its subgraph \( G' \) induced on the set of vertices

\[ V' := \{ x_i : i \text{ is a source vertex of } \partial G \} \cup \{ y_i : i \text{ is a sink vertex of } \partial G \} \subseteq V. \]

The underlying undirected graph \( \partial G \) is, first, bipartite, and, secondly, a path on \( c \) vertices (necessarily, if \( c \) is odd) or a cycle on \( c \) vertices. The subgraph \( G' \) of \( G \) described above is a path (respectively, a cycle) if and only if \( |\partial G| \) is a path (respectively, a cycle).

**Proposition 6.3.** Let \( I \) be the edge ideal of a connected Cohen-Macaulay bipartite graph \( G \). Then the following are equivalent:

(a) \( \text{reg} R/I \geq 3 \) and \( R/I \) has a quasi-pure resolution.
(b) \( G \) is the suspension of the path on five or six vertices or of the 6-cycle.

**Proof.** \( (a) \Rightarrow (b) \): First, if \( c \geq 7 \), then we claim that \( R/I \) cannot have a quasi-pure resolution. Since \( \text{reg} R/I \geq 3 \), \( \partial G \) is such that every vertex is a source vertex or a sink vertex, but not both, and that every source (respectively, sink) vertex in \( \partial G \) is covered by (respectively, covers) at most two sink (respectively, source) vertices. If \( c > 7 \), then restrict \( \partial G \) to one of its connected subgraphs with seven vertices. This corresponds to restricting \( G \) to a Cohen-Macaulay subtree on 14 vertices. If we show that the edge ideal of this subtree does not have a quasi-pure resolution, then, by Lemma 2.3(a), we have that \( R/I \) does not have a quasi-pure resolution. Therefore, by replacing \( G \) by this subgraph, we may assume that \( G \) is a Cohen-Macaulay tree on 14 vertices, such that the length of every maximal path in \( \partial G \) is one, and prove the \( R/I \) does not have a quasi-pure resolution. We may verify this with a computer algebra system, such as [M2], but we give a direct proof below.

We will prove this when \( \partial G \) has four source vertices and three sink vertices. The other case is of \( \partial G \) with three source vertices and four sink vertices; this corresponds to relabelling the partition of the vertex set. Since \( c = 7 \) is odd, \( G \) is the suspension of a path on 7 vertices. We label the source vertices 1, 2, 3, 4 and the sink vertices 5, 6, 7. Then the edges of \( \partial G \) are 15, 25, 26, 36, 37 and 47. Hence \( I = (x_1y_1, \cdots, x_7y_7, x_1y_5, x_2y_5, x_2y_6, x_3y_6, x_3y_7, x_4y_7) \). We saw that \( m_4(I) = 6 \). Since the set of four source vertices in \( \partial G \) form an antichain, \( \text{reg} R/I = 4 \), and hence \( M_4(I) = 8 \); to prove that \( R/I \) does not have a quasi-pure resolution, we need to show that \( m_5(I) \leq 7 \). Let \( \sigma = \{ x_1, y_5, x_2, y_6, x_3, y_7, x_5 \} \), and \( J = (I \cap \mathbb{k}[\sigma])R = (x_1y_5, x_2y_5, x_3y_5, x_2y_6, x_3y_6, x_3y_7) \). We will show that \( \beta_5,7(R/J) \neq 0 \), which will suffice, by Lemma 2.3[31], to show that \( m_5(I) \leq 7 \). We have a short exact sequence of graded \( R \)-modules

\[
\begin{array}{cccccc}
0 & \xrightarrow{R} & R/J & \xrightarrow{R/(J, y_5)} & R/(J, y_6) & 0.
\end{array}
\]

\(^1\)A graph \( G \) is said to be the suspension of a subgraph \( G' \), if \( G \) is obtained by attaching exactly one leaf vertex to every vertex of \( G' \). If \( G \) is the suspension of a subgraph \( G' \), then \( G \) is Cohen-Macaulay. We see this as follows. Let \( V' \subseteq V \) be the set of vertices of \( G' \). Denote the edge ideal of \( G' \) in \( \mathbb{k}[V'] \) by \( I' \). Then \( J \) is the polarization of \( I' + (x^2 : x \in V') \) in the ring \( R = \mathbb{k}[V] \). Since \( \mathbb{k}[V']/(I' + (x^2 : x \in V')) \) is Artinian, we see that \( R/I \) is Cohen-Macaulay. Villarreal showed that a tree \( G \) is Cohen-Macaulay if and only if \( G \) is the suspension of a subgraph \( G' \); see, e.g., [VH03] Theorem 6.5.1].
Since $R/(J, y_5)$ is Cohen-Macaulay and $\text{ht}(J, y_5) = 3$, we see from the associated long exact sequence of $\text{Tor}(k, -)$ that

$$\text{Tor}_5(k, R/(J : y_5)(-1)) \simeq \text{Tor}_5(k, R/J).$$

To complete the argument, we will show that $\beta_{5,6}(R/(J : y_5)) \neq 0$. Since $(J : y_5) = (x_1, y_2, x_3, x_4x_5, x_5x_6)$, this is equivalent to $\beta_{2,3}(R/(x_4x_5, x_5x_6)) \neq 0$, which is true.

We showed so far that $c \leq 6$. Now, if $c < 5$, $\text{reg} R/I < 3$. Hence $c = 5$ or $c = 6$. As we noted in Discussion 6.2 that $G$, therefore, is the suspension of the path or the cycle in five or six vertices or of the 6-cycle.

If $G$ is the suspension of the path or the cycle on $c$ vertices, then $\hat{G}$ is such that every vertex is a source vertex or a sink vertex, but not both, and that every source (respectively, sink) vertex in $\hat{G}$ is covered by (respectively, covers) at most two sink (respectively, source) vertices. Hence $\text{reg} R/I = \left\lceil \frac{c}{2} \right\rceil$. Since $c = 5$ or $c = 6$ in our case, $\text{reg} R/I = 3$. With this, $R/I$ has a quasi-pure resolution if and only if $m_4(I) = 6$, which we now show. If on the other hand, $m_4(I) = 5$, then there exists $\sigma \subseteq V$ and a partition $\sigma = \sigma_1 \sqcup \sigma_2$ (into two sets, since $G$ is bipartite) such that $|\sigma| = 5$ and $G|_\sigma$ is a complete bipartite graph (Proposition 6.1). Recall that $V = V_1 \sqcup V_2$ is the partition of the vertex set $V$ of $G$. We may assume that $\sigma_i \subseteq V_i$, $i = 1, 2$. If $|\sigma_i| = 1$ for any $i$, then $|\sigma| \leq 4$, because $\text{deg}_G x \leq 3$ for all $x \in V$. On the other hand, if, say, $|\sigma_1| \geq 2$, then $|\sigma_2| = 1$, because otherwise, we would get a 4-cycle in $G$, contradicting the fact that $G$ has only a 6-cycle, if any. Now, again, $|\sigma_1| \leq 3$, so $|\sigma| < 5$. Hence $R/I$ has a quasi-pure resolution.

We add, in passing, that the edge ideals $I$ of the suspension of paths and cycles on four or fewer vertices have quasi-pure resolutions, but this follows easily from the fact that $\text{reg} R/I \leq 2$.

Acknowledgments

The author thanks C. Huneke and J. Martin for helpful discussions.

References

[A05] Jürgen Herzog and Takayuki Hibi, Distributive lattices, bipartite graphs and Alexander duality, J. Algebraic Combin. 22 (2005), no. 3, 289–302. MR MR2181367 (2006h:06004)

[B05] David Eisenbud, Commutative algebra, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 97a:13001

[BS07] Chris Francisco and Hema Srinivasan, Multiplicity conjectures, Syzygies and Hilbert Functions (Irena Peeva, ed.), Lecture Notes in Pure and Appl. Math., vol. 254, CRC Press, New York, 2007, pp. 145–178.

[H02] Daniel R. Grayson and Michael E. Stillman, Macaulay 2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/]

[HM85] Craig Huneke and Matthew Miller, A note on the multiplicity of Cohen-Macaulay algebras with pure resolutions, Canad. J. Math. 37 (1985), no. 6, 1149–1162. MR MR828839 (87d:13024)

[HS98] Jürgen Herzog and Hema Srinivasan, Bounds for multiplicities, Trans. Amer. Math. Soc. 350 (1998), no. 7, 2879–2902. MR MR1458304 (99g:13033)

[HS04] Jürgen Herzog and Hema Srinivasan, Multiplicities of monomial ideals, J. Algebra 274 (2004), no. 1, 230–244. MR MR2040872 (2004m:13064)

[HZ06] Jürgen Herzog and Xinxian Zheng, Notes on the multiplicity conjecture, Collect. Math. 57 (2006), no. 2, 211–226. MR MR2223853
[KW06] Martina Kubitzke and Volkmar Welker, *The multiplicity conjecture for barycentric subdivisions*, 2006, [arXiv:math.AC/0606274](http://arxiv.org/abs/math.AC/0606274).

[Lyu88] Gennady Lyubeznik, *The minimal non-Cohen-Macaulay monomial ideals*, J. Pure Appl. Algebra 51 (1988), no. 3, 261–266. MR MR946577 (89h:13031).

[MNR05] Juan Migliore, Uwe Nagel, and Tim Römer, *Extensions of the multiplicity conjecture*, 2005, [arXiv:math.AC/0505229](http://arxiv.org/abs/math.AC/0505229).

[MR06] Rosa M. Miró-Roig, *A note on the multiplicity of determinantal ideals*, J. Algebra 299 (2006), no. 2, 714–724. MR MR2228336 (2007a:13030).

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR MR2110098 (2006d:13001).

[NS06] Isabella Novik and Ed Swartz, *Face ring multiplicity via cm-connectivity sequences*, 2006, [arXiv:math.AC/0606246](http://arxiv.org/abs/math.AC/0606246).

[Put06] Tony J. Puthenpurakal, *On the upper bound of the multiplicity conjecture*, 2006, [http://arxiv.org/abs/math/0701793v1](http://arxiv.org/abs/math/0701793v1).

[Vil01] Rafael H. Villarreal, *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 238, Marcel Dekker Inc., New York, 2001. MR MR1800904 (2002c:13001).

[Zhe04] Xinxian Zheng, *Resolutions of facet ideals*, Comm. Algebra 32 (2004), no. 6, 2301–2324. MR MR2100472 (2006c:13034).

University of Kansas, Lawrence, KS 66045, USA.

E-mail address: kummini@math.ku.edu

February 1, 2008