Accelerating Branes and Brane Temperature

J. G. Russo

Institució Catalana de Recerca i Estudis Avançats (ICREA),
Departament ECM and Institut de Ciencies del Cosmos,
Facultat de Física, Universitat de Barcelona,
Diagonal 647, E-08028 Barcelona, Spain.

P. K. Townsend

Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge
Wilberforce Road, Cambridge, CB3 0WA, UK.

Abstract: We define the local acceleration and jerk of a relativistic brane in an ambient spacetime, and construct from them a dimensionless parameter $\lambda$ that must be small for an interpretation of brane acceleration as local Unruh temperature. As examples, we discuss (i) open rotating branes, for which $\lambda > 1$ (ii) closed spherical branes expanding in Minkowski spacetime, for which $\lambda = 0$ when the worldvolume is either an Einstein static universe or de Sitter space, in which case the brane temperature equals the Gibbons-Hawking temperature, (iii) closed spherical branes in anti-de Sitter spacetime, for which a maximally symmetric worldvolume is anti-de Sitter, Minkowski or de Sitter according to whether the magnitude of the brane acceleration is less than, equal to or greater than a ‘critical’ value, and (iv) the BTZ black hole, viewed as a membrane.

Keywords: branes, acceleration, jerk, temperature.
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1. Introduction

Following Hawking’s discovery that asymptotically-flat stationary black holes emit thermal radiation \(1\), Unruh showed that there is a similar effect in flat space for a particle detector undergoing constant proper acceleration \(A\): the detector appears to be in a heat bath at the Unruh temperature \(4\) (see \(3\) for a recent review)

\[
T = \frac{A}{2\pi}. \tag{1.1}
\]

As Unruh pointed out, the two phenomena are related. Near the black hole event horizon, the metric is approximately the direct product of a 2-dimensional Rindler spacetime with the round 2-sphere metric, and an application of the Unruh formula to the Rindler spacetime yields the Hawking temperature \(T_H\) of the black hole blue-shifted according to the Tolman law for thermodynamic equilibrium in stationary spacetimes with timelike Killing vector field \(\xi\):

\[
T = T_H/\sqrt{-\xi^2}. \tag{1.2}
\]

A more direct connection between the Hawking and Unruh temperatures was established by Deser and Levin \(4, 5\). They considered the global embedding of the Schwarzschild metric in a six-dimensional Minkowski spacetime \(6\). A static observer, at fixed distance from the horizon, undergoes constant acceleration in the ambient flat spacetime such that the Unruh temperature equals the local temperature of the observer. This is possible because the acceleration has both a component tangent to the spacetime and a component orthogonal to it; near the horizon the acceleration is tangential but at infinity it is orthogonal. These results have since been extended to other black holes \(7, 8, 9, 10, 11\).

The idea of an Unruh-type interpretation of local temperature via a global embedding in a higher-dimensional spacetime first arose in the context of the Gibbons-Hawking (GH) temperature of de Sitter (dS) space \(12\) (see also \(13\)), which can be viewed as a hypersurface in a Minkowski spacetime of one higher dimension. In that case, an application of Unruh’s formula yields \(14\)

\[
2\pi T = A = \sqrt{a^2 + R^{-2}}, \tag{1.3}
\]

where \(a\) is the magnitude of the acceleration in de Sitter space and \(R\) is the de Sitter radius. One recovers the GH temperature by considering the class of observers with \(a = 0\). The general application of this idea is often referred to as the GEMS (Global Embedding in Minkowski Spacetime) program, but it may be necessary to consider higher-dimensional flat spacetimes that are not Minkowski\(^1\). A simple example is \(d\)-dimensional anti-de Sitter space, which is a hypersurface in \(\mathbb{E}^{(2,d-1)}\); using this embedding, one can show that an observer undergoing a constant acceleration \(a\) in the anti-de Sitter space has an acceleration \(A\) in the flat embedding space such that

\[
A^2 = a^2 - R^{-2}. \tag{1.4}
\]

\(^1\)We are not aware of any theorem that guarantees the existence of a global isometric embedding of a generic pseudo-Riemannian metric in a flat spacetime (in contrast to the Riemannian case) although local embeddings of this type always exist \(15\).
This is negative if \( a < 1/R \), which is possible when the embedding spacetime has two time dimensions. In such cases, Unruh’s result applies only to spacelike \( A \), so we deduce that the temperature experienced by an observer undergoing constant acceleration \( a \geq 1/R \) in anti de Sitter space of radius \( R \) is given by

\[
2\pi T = \sqrt{a^2 - R^{-2}}.
\]

The temperature experienced by an observer undergoing constant acceleration of magnitude \( a \leq 1/R \) is zero [17].

The main aim of this paper is to apply these ideas to charged relativistic branes by considering the motion of a ‘probe’ \( p \)-brane in a \( D \)-dimensional Minkowski, de Sitter or anti de Sitter spacetime. In the latter two cases we may view the motion in the context of the flat \((D + 1)\)-dimensional spacetime in which the (anti) de Sitter spacetime is globally embedded, so in all cases we have a flat ‘ambient’ spacetime. By ‘relativistic’ we mean a \( p \)-brane with energy density equal to its tension \( \mu \), and by ‘charged’ we mean that it couples minimally to an external \((p + 1)\)-form potential \( C \) with \((p + 2)\)-form field strength \( F = dC \). Given arbitrary local worldvolume coordinates \( \sigma^i \) \((i = 0, 1, \ldots, p)\), the effective low-energy dynamics is governed by the Dirac-type action

\[
S = -\mu \int d^{p+1}\sigma \left[ \sqrt{-\det g} + C \right],
\]

where \( g \) is the induced worldvolume metric, and \( C \) is the worldvolume Hodge dual of the worldvolume \((p + 1)\)-form induced by \( C \). In cartesian coordinates \( X^\mu \) for the flat ambient spacetime, the equation of motion is

\[
g^{ij} K_{ij}^\mu + F_{(ext)}^\mu = 0, \quad \mu = 0, \ldots, D - 1,
\]

where \( K_{ij}^\mu \) is the extrinsic curvature tensor of the worldvolume, and \( F_{(ext)}^\mu \) the ‘external’ force per unit \( p \)-volume exerted by the external form field:

\[
F_{(ext)}^\mu = \frac{1}{(p + 1)!\sqrt{-\det g}} \varepsilon^{i_1 \ldots i_{p+1}} \partial_{i_1} X^{\nu_1} \ldots \partial_{i_{p+1}} X^{\nu_{p+1}} F_{\mu \nu_1 \ldots \nu_{p+1}}.
\]

Observe that \( \partial_i X \cdot F_{(ext)} = 0 \), as required for consistency since the extrinsic curvature satisfies the ‘Brane Bianchi’ identity

\[
K_{ij} \cdot \partial_\mu X \equiv 0.
\]

1.1 Brane acceleration

An immediate problem for our program is that the brane equation of motion (1.7) does not involve the concept of acceleration in any obvious way. One way around it would be to introduce a congruence of timelike worldvolume worldlines corresponding to the trajectories of ‘test’ particles with worldvolume velocity field \( v^i \) and worldvolume acceleration field \( a^i(v) = v^j \partial_j v \); recall that \( v^2 \equiv -1 \) and \( v_i a^i(v) \equiv 0 \). The \( D \)-velocity in the ambient flat spacetime of these test particles is then \( V = v^i \partial_i X \), and the \( D \)-acceleration is

\[
A(V) \equiv v^i \partial_i V = K_{vv} + a^i(v)\partial_i X, \quad K_{vv} \equiv v^i v^j K_{ij}.
\]
As a consequence of the Brane Bianchi identity, it follows that
\[ A^2(v) = K^2_{vv} + a^2(v). \] (1.11)

Now consider a solution of (1.7) for which
\[ K_{ij} = -\frac{1}{p+1} g_{ij} F_{(ext)}^\mu. \] (1.12)

Recalling that the (intrinsic) Riemann curvature tensor is given in terms of the extrinsic curvature tensor by the formula
\[ R_{ijkl} = K_{ik} \cdot K_{jl} - K_{il} \cdot K_{jk}, \] (1.13)
we see that the intrinsic curvature tensor corresponding to (1.12) is
\[ R_{ijkl} = \frac{F_{(ext)}^2}{{(p+1)}^2} (g_{ik}g_{jl} - g_{il}g_{jk}), \] (1.14)
from which it follows that the worldvolume geometry is conformal to dS if \( F_{(ext)} \) is spacelike, and conformal to adS if \( F_{(ext)} \) is timelike (which is possible if the ambient spacetime has two time dimensions); it is Minkowski if \( F_{(ext)} \) is null (which allows non-zero \( F_{(ext)} \) if the ambient spacetime has two time dimensions). Using eq. (1.12) in (1.11) we deduce that
\[ A^2(v) = a^2(v) + \frac{F_{(ext)}^2}{{(p+1)}^2}. \] (1.15)

For the special case of a constant uniform electric-type field of strength \( E \), the force field \( F_{(ext)} \) is a fixed \( D \)-vector of magnitude \( |E| \), and hence the world-volume geometry is either de Sitter or anti-de Sitter with radius \( R = (p+1)/|E| \), or Minkowski when \( R = \infty \). The acceleration in such cases is
\[ A^2(v) = a^2(v) \pm R^{-2}, \] (1.16)
where the plus sign is for de Sitter and the minus sign is for anti de Sitter. An application of the Unruh formula (1.1) for constant \( a(v) = a \) yields the results quoted earlier for the temperature of an observer undergoing constant acceleration \( a \) in (a)dS. Related observations have been made in [18] and [19].

However, the introduction of test particles on the brane is unsatisfactory because the brane does not come supplied with such particles. Moreover, their introduction obscures the extent to which the results are intrinsic to the brane. We therefore propose a different approach, related to the Hamiltonian formalism that we review in the following section. In this approach, the worldvolume is assumed to be foliated by spacelike hypersurfaces, with local coordinates \( \{\sigma^a; a = 1, \ldots, p\} \), and the leaves of the foliation are parametrized by some arbitrary worldvolume time \( t \). The induced worldvolume metric is then written as
\[ g_{ij} d\sigma^i d\sigma^j = g_{tt} dt^2 + 2g_{ta} dt d\sigma^a + h_{ab} d\sigma^a d\sigma^b, \] (1.17)
so that \( h_{ab} \) are the components of the induced metric on the brane at fixed time; we denote by \( h^{ab} \) the components of its inverse. The inverse to this worldvolume metric is
\[ g^{ij} \partial_i \partial_j = -u^i u^j \partial_i \partial_j + h^{ab} \partial_a \partial_b \] (1.18)
where the worldvolume one-form dual to \( u^i \partial_i \) is
\[
u_i d\xi^i = -\Delta dt, \quad \Delta \equiv \sqrt{-g_{tt} + g_{ta}h^{ab}g_{tb}}, \tag{1.19}\]
which means that \( u^2 = -1 \). We interpret \( u \) as a worldvolume velocity field, and
\[
U^\mu = u^i \partial_i X^\mu \tag{1.20}
\]
as the \( D \)-velocity of the brane, as defined by the chosen foliation of the worldvolume. The brane momentum density is then
\[
P^\mu = \mu \sqrt{\det h} U^\mu, \tag{1.21}
\]
and we define the brane acceleration by
\[
A^\mu = u^i \partial_i U^\mu. \tag{1.22}
\]
For the special case of \( p = 0 \) these expressions reduce to the standard definitions of relativistic velocity, momentum and acceleration of a point particle in Minkowski spacetime.

The expression (1.22) can be rewritten as
\[
A^\mu = K^\mu_{uu} + a^i \partial_i X^\mu, \quad K^\mu_{uu} \equiv u^i u^j K^\mu_{ij}, \tag{1.23}
\]
where the worldvolume acceleration is
\[
a^i = u^j D_j u^i \equiv u^j \partial_j u^i + u^j u^k \Gamma^i_{jk}. \tag{1.24}
\]
Observe that \( a \) is orthogonal to \( u \), as expected, and also that
\[
A^2 = K^2_{uu} + a^2, \tag{1.25}
\]
as a consequence of the Brane Bianchi identity. These results are formally the same as we found above by introducing a congruence of timelike worldlines with \( D \)-velocity field \( v \), but with the difference that \( u \) is determined, given a foliation of the worldvolume, by the solution of the brane equations. Assuming that \( A^2 \geq 0 \), application of the Unruh formula gives
\[
T = \frac{|A|}{2\pi} = \frac{1}{2\pi} \sqrt{K^2_{uu} + a^2}, \tag{1.26}
\]
for the brane temperature.

1.2 Limitations
There are several reasons why the formula (1.20) cannot be universally valid. To begin with, what we mean by ‘the brane’ depends on how the worldvolume is foliated by spacelike hypersurfaces. This foliation provides a preferred class of observers, those on orbits of the vector field \( u \), but \( u \) depends on the choice of foliation because it does not transform as a vector field under general worldvolume coordinate transformations. Specifically, an infinitesimal change of worldvolume coordinates that induces \( X \rightarrow X + \zeta^i \partial_i X \) leads to \( u \rightarrow u + \delta u \), where
\[
\delta u^i \partial_i = [\zeta, u]^i \partial_i - h^{ab} \partial_b \zeta^i \Delta \partial_a, \tag{1.27}
\]
where the first term, involving the commutator of the vector fields $\zeta$ and $u$, is the standard infinitesimal transformation of a worldvolume vector field. Thus, $u$ fails to transform as a vector field under space-dependent time reparametrizations, and the brane temperature could depend on precisely what is meant by ‘the brane’. However, temperature is an equilibrium concept so we should not attempt to interpret brane acceleration as temperature unless the worldvolume metric is stationary, or nearly so. This means that there should be some timelike worldvolume vector field $\xi$ that is, at least approximately, Killing. In such cases there is a ‘preferred’ foliation of the worldvolume: that for which $u = \xi/\sqrt{-\xi^2}$. In some of the examples that we shall meet, one may choose the foliation such that

$$u = \Delta^{-1} \xi,$$  

(1.28)

where the timelike vector field $\xi$ is either Killing or asymptotically Killing, and the Tolman law (1.2) becomes

$$T = T_0/\Delta.$$  

(1.29)

In some of the examples we consider in this paper, the local Unruh temperature, given by (1.26), takes the form (1.29), consistent with thermal equilibrium. In other examples the local Unruh temperature does not take this form and is therefore not consistent with thermal equilibrium, but in all such cases the Unruh formula turns out to be inapplicable. Recall that the Unruh formula was derived assuming constant (time-independent) proper acceleration. It is physically reasonable to allow the ($D$-vector) acceleration to change slowly with time but then we need a measure of how slow this change is. In non-relativistic particle mechanics the time rate of change of acceleration is called ‘jerk’. In the context of a congruence of timelike worldlines that are orbits of a vector field $v$, as discussed above, the natural relativistic generalization of jerk would be $j = v^i D_j a^i(v)$, but this is non-zero even when $a^2$ is constant because $v \cdot j = -a^2$. This suggests that

$$\varsigma^k = j^k - a^2 v^k, \quad j^k = v^i D_i a^k$$  

(1.30)

is a better definition of relativistic jerk, since $a \cdot \varsigma \equiv a \cdot j$ but $v \cdot \varsigma \equiv 0$. There is a similar problem with $u^i \partial_i A$ as the definition of ‘brane jerk’, which is resolved by considering instead the $D$-vector valued worldvolume field

$$\Sigma^\mu = u^i \partial_i A^\mu - A^2 U^\mu.$$  

(1.31)

We will argue in the following section that

$$\lambda = \sqrt{\Sigma^2}/A^2$$  

(1.32)

is the relevant dimensionless parameter, in the sense that (1.26) should be valid whenever $\lambda \ll 1$. For all the examples we consider, the local Unruh temperature is consistent with (approximate) thermal equilibrium whenever this condition is satisfied.

### 1.3 Organization

In the following section we present further aspects, and technical details, of the above ideas. We then consider a series of illustrative examples. We start with an open $p$-brane in a $(2p + 1)$-dimensional Minkowski spacetime, rotating freely, and rigidly, in $p$ orthogonal planes. This is a
case in which the (constant but non-uniform) brane acceleration is everywhere tangential to the (static) worldvolume. The local Unruh temperature is not of the Tolman form but we also find that the parameter $\lambda$ of (1.32) is everywhere greater than unity, whereas the Unruh formula is valid only if $\lambda \ll 1$.

Most of our other examples involve a spherical $p$-brane, possibly coupled to a constant uniform electric-type field, expanding or contracting in a maximally symmetric spacetime. We consider first the case in which the maximally-symmetric spacetime is $(p+2)$-dimensional Minkowski spacetime. In such cases the worldvolume has co-dimension one in the ambient flat spacetime, and the brane may be viewed as a spherical domain wall. For a non-zero electric-type field, there is an equilibrium solution of the brane equation of motion (1.7) for which the external force is cancelled by the ‘internal’ force due to tension; this solution has $A \equiv 0$ and its brane temperature is therefore zero. The equilibrium is unstable because a small uniform perturbation leads either to collapse or to a runaway solution in which the brane expands indefinitely. This runaway solution approaches an asymptotic solution with non-zero uniform $A$, and $\lambda = 0$; its worldvolume geometry is de Sitter and the brane temperature is the Gibbons-Hawking temperature.

We then turn to spherical $p$-branes undergoing constant uniform acceleration in either $dS_{p+2}$ or $adS_{p+2}$. As $(a)dS_{p+2}$ may be globally embedded in a $(p+3)$-dimensional flat spacetime, we are effectively considering motion in a flat spacetime but now with a worldvolume that has co-dimension two rather than co-dimension one. Consider the special case of a zero-brane undergoing constant acceleration $a$ in $dS_2$. The formula (1.3) applies, and this can also be found by an application of the Unruh formula to the zero-brane’s acceleration $A$ in Minkowski, since the two non-zero components of $A$ are $a$ and $1/R$. From the above discussion of brane acceleration for $p > 0$, it should be clear that the same argument applies to a $p$-brane undergoing constant uniform acceleration in $dS_{p+2} \hookrightarrow \text{Minkowski}_{p+3}$, and hence that the formula (1.3) still applies. This makes sense because the constant uniform acceleration implies that the worldvolume geometry is de Sitter. Note, however, that $a$ is now to be interpreted as the brane’s acceleration in the ambient $(a)dS$ spacetime, rather than the worldvolume acceleration of some test particle on the brane. Similarly, the formula (1.4), and hence (1.3) when $a \geq 1/R$, applies to a zero-brane in $adS_2$ but generalizes to a $p$-brane in $adS_{p+2}$. This again makes sense because, as we show, constant uniform acceleration $a$ implies that the worldvolume geometry is $adS_{p+1}$ if $a < 1/R$, Minkowski$_{p+1}$ if $a = 1/R$, and $dS_{p+1}$ if $a > 1/R$. This is a purely kinematical statement but all these cases occur as solutions to the equations of motion of a $p$-brane coupled to a constant uniform electric-type field; one finds that $a < 1/R$ for a ‘sub-critical’ electric field, $a = 1/R$ for critical electric field, and $a > 1/R$ for super-critical electric field.

Our final example is the BTZ black hole [20, 21]. The global embedding of the BTZ metric in a flat $(2 + 2)$-dimensional spacetime was given in [3], where it was used to discuss thermal properties, although the local temperature used there is the non-equilibrium temperature of ZAMOs (zero angular momentum observers), which fails to agree with the Unruh temperature except near the horizon. Our results, obtained by viewing the BTZ black hole as a brane, yield similar results because our definition of brane acceleration effectively picks out the ZAMOs as ‘preferred’ observers. We show that $\lambda$ is small for such observers only near the horizon. However, we further show that observers who are rotating with the same angular velocity as the black hole horizon have $\lambda = 0$ everywhere, and their Unruh temperature precisely matches the equilibrium
black hole temperature found by applying the Tolman law to the timelike Killing vector field that becomes null on the horizon. The ‘jerk-free’ observers are those on orbits of this Killing vector field.

We leave to a final section some further discussion of the implications our results.

2. Brane Kinematics

Here we give further details related to some of the ideas discussed above. We begin with a derivation of (1.25) from the Hamiltonian formulation, and consider in more detail the issue of coordinate-dependence of the brane acceleration. We then recall some formulae of extrinsic geometry, and investigate the circumstances under which the brane acceleration can be expressed in terms of intrinsic worldvolume geometry. Finally we elaborate on the concept of brane jerk and explain its relevance to the validity of the Unruh formula.

2.1 Hamiltonian formalism

For simplicity, we suppose that the $p$-brane is a closed $p$-surface, and that $C = 0$; in which case the Hamiltonian form of the action is

$$S = \int dt \int d^p\sigma \left\{ \dot{X} \cdot P - \frac{1}{2} \ell \left( P^2 + \mu^2 \det h \right) - s^a P \cdot \partial_a X \right\}, \quad (2.1)$$

where $(\ell, s^a)$ are the ‘lapse’ and ‘shift’ Lagrange multipliers for the Hamiltonian and worldspace diffeomorphism constraints; note that the latter implies that the $D$-momentum $P$ is everywhere orthogonal to the brane. To verify the equivalence to the Dirac form of the action, first eliminate $P$ by its equation of motion

$$P = \ell^{-1} \left( \dot{X} - s^a \partial_a X \right) = \mu \sqrt{\det h} u^i \partial_i X, \quad (2.2)$$

where the second equality follows from the definition

$$u^i \partial_i \equiv \frac{\ell^{-1}}{\mu \sqrt{\det h}} (\partial_t - s^a \partial_a). \quad (2.3)$$

If we substitute for $P$ in the action and vary the resulting action with respect to $u$, we find that

$$u = \Delta^{-1} (\partial_t - s^a \partial_a), \quad s^a = h^{ab} g_{tb}, \quad (2.4)$$

where we recall from (1.19) that

$$\Delta \equiv \sqrt{-g_{tt} + g_{ta} h^{ab} g_{tb}} = \frac{1}{\sqrt{-g^{tt}}}. \quad (2.5)$$

One may now verify that the dual worldvolume one-form is given by (1.19). When this result is used to eliminate $u$ from the action, the action (1.6) (with $C = 0$) is recovered upon use of the identity

$$\sqrt{-\det g} = \Delta \sqrt{\det h}. \quad (2.6)$$
As noted in the introduction, \( u^2 = -1 \), from which follows the interpretation of \( u \) as a world-volume velocity field. Also, if (2.4) is used in (2.2) then we recover the formula (1.21) for brane momentum density, with the brane velocity \( U \) given by (1.20).

Returning to the action (2.1), we observe that the \( X \) equation of motion is

\[
\dot{P} - \partial_a (s^a P) = \mu^2 \partial_a \left( \ell \det h h^{ab} \partial_b X \right). \tag{2.7}
\]

Using (2.4) and (2.6), we may rewrite this as

\[
\dot{P} - \partial_a (s^a P) = \sqrt{-\det g} F_{\text{int}}, \tag{2.8}
\]

where the ‘internal’ force density, due to the brane tension, is

\[
\sqrt{-\det g} F_{\text{int}} = \mu \partial_a \left( \sqrt{-\det g} h^{ab} \partial_b X \right). \tag{2.9}
\]

Now we use (2.2), and then (2.4), to deduce that

\[
\dot{P} - \partial_a (s^a P) = \mu \partial_i \left( \sqrt{-\det g} u^i U \right). \tag{2.10}
\]

Equivalently,

\[
\dot{P} - \partial_a (s^a P) = \mu \sqrt{-\det g} A + \mu \partial_i \left( \sqrt{-\det g} u^i \right) U, \tag{2.11}
\]

where \( A = u^i \partial_i U \) is the brane acceleration as defined in the introduction. We now see that (2.8) is equivalent to

\[
F_{\text{int}} = \mu A + \mu \left( D_i u^i \right) U. \tag{2.12}
\]

In the presence of an external force, this equation still holds but with

\[
F_{\text{int}} \rightarrow F_{\text{int}} + F_{\text{ext}} \equiv F. \tag{2.13}
\]

Equation (2.12) is essentially Newton’s second law. For \( p = 0 \) it reduces to the standard form of this law for a relativistic particle of rest-mass \( \mu \). For \( p > 0 \) there is an extra term proportional to the covariant divergence of \( u \), as a consequence of which we deduce for a closed brane that

\[
\oint d^p \sigma \Delta P \cdot F_{\text{int}} = -\mu \dot{M}, \quad M \equiv \mu \oint d^p \sigma \sqrt{\det h}. \tag{2.14}
\]

It is instructive to compare this with the example of a relativistic particle with variable rest-mass \( m(t) \) and velocity \( v(t) \), and hence momentum \( p = mv \). Given that \( F = dp/d\tau \), where \( \tau \) is proper time, one has \( p \cdot F = -m \, dm/d\tau \), in close analogy with (2.14). The reason should be clear: a non-vanishing divergence of the velocity field implies a non-conservation of local rest-mass. Globally, the brane behaves like a particle at its centre of mass, and the rest-mass of this ‘particle’ is conserved because it includes the kinetic energy of the brane.
2.2 Extrinsic vs Intrinsic

We first recall some basic formulae of extrinsic geometry. We suppose that a \((p+1)\)-dimensional spacetime, which we call the ‘worldvolume’, is globally embedded in a flat \(D\)-dimensional spacetime, which may have more than one time but is such that the induced worldvolume metric has Lorentzian signature. We may write the flat \(D\)-metric as

\[
ds^2_D = \eta_{\mu\nu} dX^\mu dX^\nu = -(dX^0)^2 + (dX^1)^2 + \cdots \tag{2.15}
\]

where the dots indicate additional dimensions, one of which may be an additional time dimension. Let \(\{\xi^i; i = 0, 1, \ldots, p\}\) be local worldvolume coordinates. The induced worldvolume metric in these coordinates is

\[
g_{ij} = \eta_{\mu\nu} \partial_i X^\mu \partial_j X^\nu. \tag{2.16}
\]

The extrinsic curvature of the worldvolume is the \(D\)-vector valued symmetric worldvolume tensor

\[
K^\mu_{ij} = \partial_i \partial_j X^\mu - \Gamma^k_{ij} \partial_k X^\mu, \tag{2.17}
\]

where \(\Gamma^k_{ij}\) is the Levi-Civita connexion for the induced worldvolume metric. This extrinsic curvature tensor satisfies the ‘Brane Bianchi’ identity \((1.9)\), which implies that the only non-zero components of \(K\) are those orthogonal to the worldvolume (but this could include a tangential component if the flat \(D\)-dimensional spacetime has more than one time dimension).

It is now straightforward to derive the formula \((1.23)\) for \(A\), starting from the formula

\[
A^\mu = u^i \partial_i U^\mu = u^i \partial_i \left(u^j \partial_j X^\mu\right). \tag{2.18}
\]

In the special case that \(a = 0\), we find that

\[
A^\mu = K^\mu_{uu} \equiv u^i u^j K^\mu_{ij}. \tag{2.19}
\]

Thus, in such cases, \(A\) is defined in terms of the extrinsic worldvolume geometry. We now aim to investigate the conditions under which it can be re-expressed entirely in terms of the intrinsic worldvolume geometry. We begin by observing that \((1.13)\) implies the following formula for the Ricci tensor:

\[
R_{ij} = K_{ij} \cdot \left(g^{kl} K_{kl}\right) - g^{kl} K_{ik} \cdot K_{lj}. \tag{2.20}
\]

Using the brane equation of motion \((1.7)\), and eq. \((1.18)\), we can rewrite this as

\[
R_{ij} = -K_{ij} \cdot F - h^{ab} K_{ai} \cdot K_{bj} + u^k u^l K_{ki} \cdot K_{lj}. \tag{2.21}
\]

By contracting with \(u^i u^j\), we find that

\[
R_{uu} \equiv u^i u^j R_{ij} = K_{uu}^2 - K_{uu} \cdot F - h^{ab} \left[u^i K_{ia}\right] \cdot \left[u^j K_{jb}\right]. \tag{2.22}
\]

It follows that a necessary condition for \(K_{uu}^\mu\) to be expressible in terms the intrinsic geometry is that

\[
h^{ab} \left[u^i K_{ia}\right] \cdot \left[u^j K_{jb}\right] = 0 \tag{2.23}
\]
This is satisfied, in particular, for
\[ u^i K^\mu_{ia} = 0. \quad (2.24) \]
When the world-volume is co-dimension one, the condition (2.24) is equivalent to (2.23). In this case
\[ (2K_{uu} - F)^2 = 4R_{uu} + F^2. \quad (2.25) \]
In principle the $D$-vector field $(2K_{uu} - F)$ could depend on the extrinsic geometry in such a way that the magnitude is independent of the extrinsic geometry. In the case of a worldvolume of co-dimension one this cannot happen because then $(2K_{uu} - F)$ has only one component.

### 2.3 Brane Jerk

Jerk is the time rate of change of acceleration. The $D$-vector valued worldvolume field $u^i \partial_i A$ is the natural definition of ‘brane jerk’ but this does not vanish when $A^2$ is constant. We therefore define
\[ \Sigma = u^i \partial_i A - A^2 U. \quad (2.26) \]
As a consequence of the identity $U \cdot A \equiv 0$, we have the further identity
\[ U \cdot \Sigma \equiv 0. \quad (2.27) \]
We can decompose $\Sigma$ into a component orthogonal to the worldvolume and a component tangent to it:
\[ \Sigma^\mu = \Sigma^\mu_\perp + \zeta^i \partial_i X^\mu, \quad (2.28) \]
where $\zeta^i$ is the world-volume jerk as defined in (1.30), and
\[ \Sigma_\perp^\mu = u^i u^j u^k \left[ D_i K^\mu_{jk} - \partial_k X^\mu (A \cdot K_{ij}) \right] + 3a^i u^j K^\mu_{ij}. \quad (2.29) \]

An interesting example is the case of maximally symmetric spaces of radius $R$, for which
\[ K^\mu_{ij} = \frac{1}{R} \delta_{ij} n^\mu, \quad n^2 = \pm 1, \quad (2.30) \]
where the plus (minus) sign corresponds to (anti) de Sitter. In this case
\[ \partial_i n^\mu = -\frac{n^2}{R} \partial_i X^\mu. \quad (2.31) \]
Using these expressions, one finds that the orthogonal component of $\Sigma$ is identically zero and hence that $\Sigma^\mu = \zeta^i \partial_i X^\mu$. Therefore $\zeta = 0$ implies $\Sigma = 0$ for maximally symmetric world-volumes.

For the purposes of the Unruh formula, the time rate of change of the acceleration and hence of the temperature, will be “small” if $|\Sigma| = \sqrt{\Sigma^2}$ is small. Given $A^2 > 0$, we expect $|A| = \sqrt{A^2}$ to be the typical frequency associated with the Unruh temperature (in fundamental units), and this should be much greater than $\dot{T}/T$. This amounts to requiring
\[ \lambda \equiv |\Sigma|/A^2 \ll 1. \quad (2.32) \]
\[ ^2 \text{Remarkably, this is also the condition for } K^\mu_{uu} \text{ to transform, infinitesimally, as a scalar under reparametrizations of the worldvolume.} \]
3. Open rotating branes

Consider a \((2p + 1)\)-dimensional Minkowski space
\begin{equation}
    ds^2 = -dT^2 + \sum_{\alpha=1}^{p} [dX_{\alpha}^2 + dY_{\alpha}^2] \tag{3.1}
\end{equation}

Now set
\begin{align*}
    T &= t, \\
    X_{\alpha} &= \sigma_{\alpha} \cos (\phi_{\alpha} - \omega_{\alpha} t), \\
    Y_{\alpha} &= \sigma_{\alpha} \sin (\phi_{\alpha} - \omega_{\alpha} t). \tag{3.2}
\end{align*}

The Minkowski metric in the new coordinates \((t; \sigma_1, \phi_1; \ldots; \sigma_p, \phi_p)\) is
\begin{equation}
    ds^2 = - \left( 1 - \sum_{\alpha=1}^{p} \omega_{\alpha}^2 \sigma_{\alpha}^2 \right) dt^2 + \sum_{\alpha=1}^{p} \left[ \sigma_{\alpha}^2 d\phi_{\alpha}^2 - 2\omega_{\alpha} \sigma_{\alpha}^2 d\phi_{\alpha} dt + d\sigma_{\alpha}^2 \right]. \tag{3.3}
\end{equation}

We now have the Minkowski metric in coordinates adapted to a frame rotating in \(p\) orthogonal planes with angular velocity vector \((\omega_1, \ldots, \omega_p)\). The vector field
\begin{equation}
    k = \partial_t \tag{3.4}
\end{equation}
is a Killing vector field, but it is not the same Killing vector field as \(\partial_T\):
\begin{equation}
    \partial_T = \partial_t + \sum_{\alpha=1}^{p} \omega_{\alpha} \frac{\partial}{\partial \phi_{\alpha}}. \tag{3.5}
\end{equation}

In contrast to \(\partial_T\), the vector field \(\partial_t\) has a horizon, on the ellipsoid
\begin{equation}
    \sum_{\alpha=1}^{p} \omega_{\alpha}^2 \sigma_{\alpha}^2 = 1. \tag{3.6}
\end{equation}

The metric is singular on this ellipsoid, but this is, of course, just a coordinate singularity.

Now consider an open \(p\)-brane in this spacetime, and identify \((t; \sigma_1, \ldots, \sigma_p)\) with the world-volume coordinates, and suppose further that it is rigidly rotating, so that all \(p\) angles \(\phi_{\alpha}\) are constant. It is straightforward to verify that this is a solution of the equation of motion, making use of the fact that the induced worldvolume metric is
\begin{equation}
    ds^2_{\text{ind}} = -\Delta^2 dt^2 + \sum_{\alpha=1}^{p} d\sigma_{\alpha}^2, \quad \Delta = \sqrt{1 - \sum_{\alpha=1}^{p} \omega_{\alpha}^2 \sigma_{\alpha}^2}. \tag{3.7}
\end{equation}

This metric is static, with Killing vector field \(\partial_t\), but has a curvature singularity on the \((p - 1)\)-ellipsoid (3.6), which we identify as the boundary of the \(p\)-brane. All points on this boundary move at the speed of light. From the induced metric we see that
\begin{equation}
    u = \frac{1}{\sqrt{1 - \sum_{\alpha=1}^{p} \omega_{\alpha}^2 \sigma_{\alpha}^2}} \partial_t \tag{3.8}
\end{equation}

Further direct computation shows that
\begin{equation}
    a = -\Delta^{-2} \sum_{\alpha=1}^{p} \omega_{\alpha}^2 \sigma_{\alpha} \frac{\partial}{\partial \sigma_{\alpha}} \tag{3.9}
\end{equation}
and also that
\[ \varsigma = \Delta^{-5} \left( \sum_{\alpha=1}^{p} \omega_{\alpha}^{3} \sigma_{\alpha}^{2} \right) \partial_{t} - a^{2} u = 0. \]  
(3.10)

The last equality, which is expected from the time-independence of \( a^{2} \), shows that the brane jerk \( \Sigma \), which we compute below, is orthogonal to the worldvolume.

A similar computation of the full brane velocity and acceleration yields the result that \( U = u \) and \( A = a \), as vector fields, so this is a case in which the brane acceleration is purely tangential; equivalently, \( K_{uu} = 0 \), as a computation confirms. An application of the Unruh formula would give
\[ 2\pi T = \Delta^{-2} \sqrt{\sum_{\alpha} \omega_{\alpha}^{4} \sigma_{\alpha}^{2}}, \]  
(3.11)

but this is not of the form (1.2) required for thermal equilibrium. This feature could have been anticipated from the fact that the null hypersurface swept out by the brane’s boundary is a curvature singularity of the worldvolume metric. Physically, we should expect the brane to radiate as a consequence of its acceleration, and there is no guarantee that thermal equilibrium can be achieved by immersion in a heat bath. In view of this, we should ask whether application of the Unruh formula is justified in this case.

A direct computation of the brane jerk, shows that
\[ \Sigma = \Delta^{-3} \sum_{\alpha=1}^{p} \omega_{\alpha}^{3} \partial_{\phi_{\alpha}} - \Delta^{-5} \left( \sum_{\alpha=1}^{p} \omega_{\alpha}^{4} \sigma_{\alpha}^{2} \right) \partial_{t}, \]  
(3.12)

which yields
\[ \lambda^{2} = \frac{\sum_{\alpha} \omega_{\alpha}^{6} \sigma_{\alpha}^{2}}{\left( \sum_{\alpha} \omega_{\alpha}^{4} \sigma_{\alpha}^{2} \right)^{2}} + \left[ 1 - \frac{\left( \sum_{\beta} \omega_{\beta}^{2} \sigma_{\beta}^{2} \right) \left( \sum_{\alpha} \omega_{\alpha}^{6} \sigma_{\alpha}^{2} \right)}{\left( \sum_{\gamma} \omega_{\gamma}^{4} \sigma_{\gamma}^{2} \right)^{2}} \right] \]  
(3.13)

where \( \lambda \) is the parameter of (2.32). This can be shown to be equivalent to
\[ \lambda^{2} \left( 1 - \Delta^{2} \right) = 1 + \frac{\Delta^{2}}{\left( \sum_{\gamma} \omega_{\gamma}^{4} \sigma_{\gamma}^{2} \right)^{2}} \sum_{\alpha \neq \beta} \left[ \left( \omega_{\alpha}^{2} - \omega_{\beta}^{2} \right) \omega_{\alpha} \omega_{\beta} \sigma_{\alpha} \sigma_{\beta} \right]^{2}. \]  
(3.14)

It follows that
\[ \lambda^{2} \geq \left( 1 - \Delta^{2} \right)^{-1} \geq 1, \]  
(3.15)

where the second inequality follows from the fact that \( \Delta^{2} < 1 \). This violates (2.32), so there is no region of the brane in which we can use the Unruh formula!

4. Spherical branes in flat space

A simple class of solutions of the \( p \)-brane equations of motion (1.7) is obtained by considering the motion of a spherical \( p \)-brane in a \((p + 2)\)-dimensional Minkowski spacetime. In this case we can think of the brane as an effective description of a spherical domain wall, with a worldvolume that is a hypersurface in the Minkowski spacetime. In the absence of an external force, all
solutions of (1.7) describe minimal Lorentzian surfaces in which a spherical brane oscillates between expansion and contraction due to the ‘internal’ force provided by its tension. The inclusion of an external, constant and uniform, force allows for a static, but unstable, spherical brane with a definite, equilibrium radius of curvature. A radial perturbation causes such a brane to either collapse or to expand indefinitely, in which case it approaches an asymptotic solution for which the worldvolume geometry is de Sitter.

4.1 Kinematics

As these examples are all to do with the motion of a spherical $p$-brane, it is convenient to choose spherical polar coordinates for the flat ambient space, such that the Minkowski spacetime metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega^2_p,$$

where $d\Omega^2_p$ is the unit $SO(p+1)$-invariant metric on the $p$-sphere. As the notation suggests, we choose $t$ to coincide with worldvolume time. We may also choose the brane coordinates to coincide with the angular coordinates of the $p$-sphere. If the brane has radius $r(t)$ at time time $t$ then the induced worldvolume metric is

$$ds^2_{ind} = -(1 - \dot{r}^2) dt^2 + r^2 d\Omega^2_p,$$

where the overdot indicates differentiation with respect to $t$. It follows that the worldvolume velocity field $u$ is

$$u = \Delta^{-1} \partial_t, \quad \Delta = \sqrt{1 - \dot{r}^2}$$

and a straightforward calculation shows that the worldvolume acceleration is zero. The brane acceleration is therefore orthogonal to the worldvolume in these examples. Note that $\partial_t$ is not Killing unless $\dot{r} = 0$, so only in this case does $u$ takes the form (1.28).

Using (4.3) one may deduce that $U$ and $A$, as vector fields on Minkowski spacetime, are given by

$$U = \Delta^{-1} (\partial_t + \dot{r} \partial_r), \quad A = \dot{r} \Delta^{-4} (\partial_t + \partial_r).$$

It follows that

$$A^2 = \frac{\dot{r}^2}{(1 - \dot{r}^2)^3}. \quad (4.5)$$

One can similarly compute the brane jerk:

$$\Sigma = \Delta^{-3} (\ddot{r} - 3\Delta^{-2} \dot{r}^2 \dot{r}) (\dot{r} \partial_t + \partial_r).$$

This leads to

$$\lambda = \frac{1 - \dot{r}^2}{\dot{r}^2} \left[ (1 - \dot{r}^2) + 3\dot{r}^2 \dot{r}^2 \right] = \frac{(1 - \dot{r}^2)^2}{2\dot{r}} A^{-2} dA^2/dt. \quad (4.7)$$

As expected, $\lambda = 0$ when $\dot{r} = 0$, and in this case the brane temperature is well-defined, but zero. In addition $\lambda = 0$ when $A^2$ is constant, and in such cases the local Unruh temperature is

$$T = \frac{\left| \dot{r} \right|}{2\pi (1 - \dot{r}^2)}. \quad (4.8)$$
This would appear to be inconsistent with (1.29), but here we should recall that the vector field \( \xi \) defined by \( u = \Delta^{-1} \xi \) is not Killing unless \( \dot{r} = 0 \), so that (1.29) cannot be applied in the coordinates that we are using. It could be that there are other coordinates such that \( u = \Delta^{-1} \xi \) for Killing vector field \( \xi \), so that (1.29) is applicable. As we shall see, this is the case, and (11.8) is consistent with (1.29) in these new coordinates.

4.2 Dynamics

For our discussion of the dynamics, we will wish to allow for a uniform electric-type field in the Minkowski background; i.e.

\[
F_{p+2} = Er^p dt \wedge dr \wedge dV_p
\]  \hspace{1cm} (4.9)

where \( E \) is a constant and \( dV_p \) is the volume element on the \( p \)-sphere. We may choose a gauge in which the \((p+1)\)-form potential is

\[
C_{p+1} = -\frac{E}{p+1} r^{p+1} dt \wedge dV_p,
\]  \hspace{1cm} (4.10)

in which case the worldvolume Hodge dual of the \((p+1)\)-form induced by \( C_{p+1} \) is

\[
\mathcal{C} = -\frac{E}{p+1} r^{p+1}.
\]  \hspace{1cm} (4.11)

Putting things together, we see that the action (1.6) becomes

\[
S[r] = \mu V_p \int dt \left\{ -r^p \sqrt{1 - \dot{r}^2} + \frac{E}{(p+1)} r^{p+1} \right\},
\]  \hspace{1cm} (4.12)

where \( V_p \) is the volume of the unit \( p \)-sphere. The Lagrangian has no explicit dependence on \( t \) so there is a corresponding conserved quantity \( \mathcal{E} \), and the first integral of the equations of motion is

\[
\dot{r}^2 = 1 - \frac{(p+1)^2 \dot{r}^{2p}}{(\mathcal{E} + Er^{p+1})^2}.
\]  \hspace{1cm} (4.13)

Using this equation to eliminate \( \dot{r} \) from (4.13), we find that

\[
\mathcal{A}^2 = \frac{1}{(p+1)^2} \left( E - \frac{p \mathcal{E}}{r^{p+1}} \right)^2.
\]  \hspace{1cm} (4.14)

We may also use (4.13) to deduce from (4.7) that

\[
\lambda = \frac{p (p+1)^3 \dot{r}^{2p} \mathcal{E} \dot{r}}{(\mathcal{E} + Er^{p+1}) (Er^{p+1} - p \mathcal{E})^2}.
\]  \hspace{1cm} (4.15)

Notice that this vanishes in precisely two cases: (i) \( \dot{r} = 0 \) and (ii) \( \mathcal{E} = 0 \).
4.3 Zero external force

In the absence of an electric field, i.e. $E = 0$, we can rewrite (4.13) as

$$
\dot{r}^2 = 1 - \left(\frac{r}{r_{\text{max}}}\right)^{2p},
$$

where the integration constant $r_{\text{max}}$ is the maximum value of $r$. For $p = 1$ the solution is $r = r_{\text{max}} \sin t$, and quite generally we may choose $t$ such that $r = 0$ at $t = 0$. Even without having to integrate, we can substitute for $\dot{r}$ in (4.2) to see that the worldvolume metric is

$$
ds_{p+1}^2 = -\left(\frac{r(t)}{r_{\text{max}}}\right)^{2p} dt^2 + r^2(t) d\Omega_p^2.
$$

In this case we may compute the brane acceleration directly from the ‘intrinsic’ formula (2.25) for $E = 0$, which yields

$$
A^2 = R_{uu} = -R_{tt}/g_{tt} = \frac{|\dot{r}|^2}{(1 - \dot{r}^2)^3} = \left(\frac{p r_{\text{max}}^{p+1}}{r^{p+1}}\right)^2,
$$

in agreement with (4.14), for $E = 0$. From (4.15) we find that

$$
\lambda = \frac{p + 1}{p} \left(\frac{r}{r_{\text{max}}}\right)^{2p} \sqrt{1 - \left(\frac{r}{r_{\text{max}}}\right)^{2p}}.
$$

We have $\lambda \ll 1$ as $r \to r_{\text{max}}$, so the Unruh formula should apply, and we deduce that the brane has a temperature $T \approx p/(2\pi r_{\text{max}})$. We also have $\lambda \ll 1$ as $r \to 0$, but in this case the temperature is rapidly increasing without bound and we expect other physics to intervene.

4.4 Non-zero external force

For positive $E$, there is an unstable static solution of (4.13) with

$$
r = p/E \equiv r_0, \quad \mathcal{E} = r_0^p.
$$

The induced metric is that of an Einstein Static Universe:

$$
ds_{\text{ind}}^2 = -\frac{R}{R^2 + t^2} dt^2 + r_0^2 d\Omega_p^2.
$$

The brane acceleration according to (4.14) is $A = 0$, in agreement with (2.25) since $R_{uu} = 0$ in this case. Since $\dot{r} = 0$, we get $\lambda = 0$ from (4.15). The Unruh formula applies in this case but the brane has zero temperature.

4.4.1 The de Sitter solution

All solutions of (4.13) for which $r \to \infty$ are asymptotic to the solution

$$
r = \sqrt{R^2 + t^2}, \quad \mathcal{E} = 0 \quad (R = (p+1)/E),
$$

for which the induced metric is

$$
ds_{\text{ind}}^2 = -\frac{R^2}{R^2 + t^2} dt^2 + (R^2 + t^2) d\Omega_p^2.
$$
For $p = 1$, this solution was found in [22]; apart from the extension to $p > 1$, the main new observation here is that the worldvolume geometry of this asymptotic solution is de Sitter. To see this, we substitute the solution into (4.2) and then introduce a new time coordinate $\tau$ such that

$$t = R \sinh \tau .$$

(4.24)

The resulting worldvolume metric is

$$ds^2_{\text{ind}} = R^2 \left[ -d\tau^2 + (\cosh \tau)^2 d\Omega^2_p \right],$$

(4.25)

which is a standard parametrization of the metric of a de Sitter spacetime with radius of curvature $R$. Note that

$$u^i \partial_i = R^{-1} \partial_\tau$$

(4.26)

so the brane velocity and acceleration are effectively defined as first and second derivatives with respect to the time parameter $\tau$. From (4.14) we see that, for this solution

$$A^2 = 1/R^2 ,$$

(4.27)

so the brane acceleration is constant, with magnitude $1/R$. Since $\lambda = 0$ we may apply the Unruh formula to deduce that the brane temperature is $T = 1/(2\pi R)$, which is the Gibbons-Hawking temperature of de Sitter space.

There is another choice of coordinates for which the de Sitter spacetime is static:

$$ds^2_{p+1} = \left( 1 - \frac{\tilde{r}^2}{R^2} \right) dt^2 + \frac{R^2 d\tilde{r}^2}{R^2 - \tilde{r}^2} + \tilde{r}^2 d\Omega^2_{p-1} ,$$

(4.28)

The coordinate singularity at $\tilde{r} = R$ is the event horizon of an observer at $\tilde{r} = 0$. The Killing vector field $k = \partial_t$ is such that $k^2 = -1$ at this observer, who therefore plays a role analogous to that of an observer at infinity in a black hole spacetime. The Euclidean metric with $\tilde{t} = iR\theta$ is non-singular at $\tilde{r} = R$ if $\theta$ is identified with $\theta + 2\pi$, which means that the metric is periodic in imaginary time with period $2\pi R$, which is interpreted as the inverse of the (Gibbons-Hawking) temperature of the observer at the origin [12]:

$$T_{\text{GH}} = \frac{1}{2\pi R} .$$

(4.29)

In these static coordinates for the de Sitter metric, the worldvolume velocity field is again $u = \Delta^{-1} \partial_t$, but now $\Delta = \sqrt{1 - \tilde{r}^2/R^2}$ and $\partial_t$ is a Killing vector field.

In static coordinates, de Sitter spacetime does not have an obvious interpretation as a brane, but it is still embedded in Minkowski spacetime of one higher dimension. The embedding is

$$X_0 \pm X_{p+1} = \pm \sqrt{R^2 - \tilde{r}^2} e^{\pm i/R}, \quad (X_1, \ldots, X_p) = (x_1, \ldots x_p)$$

(4.30)

where $x_1^2 + \cdots + x_p^2 = \tilde{r}^2$. Defining $U$, $A$ and $\Sigma$ as before, one finds that

$$A^2 = \frac{1}{R^2 - \tilde{r}^2}$$

(4.31)
One also finds that $\Sigma \equiv 0$, so we may apply the Unruh formula to find that

$$T = \frac{1}{2\pi} \left[ \frac{1}{R^2} + a^2(\tilde{r}) \right], \quad a(\tilde{r}) = \frac{\ddot{\tilde{r}}}{R\sqrt{R^2 - \tilde{r}^2}}$$

(4.32)

where $a(\tilde{r})$ is the acceleration within the de Sitter space of an observer at fixed $\tilde{r}$.

This example illustrates that a change of coordinates, leading to a change in the world-volume velocity, can change the brane acceleration $A$ (cf. eqs. (4.27), (4.31)). In general, one should expect $\Sigma$ to change too. However, for de Sitter world-volume, we have shown previously that $\Sigma$ is determined by $\varsigma$ which is zero whenever $a$ is constant, as it is in the above examples.

### 4.4.2 Higher-dimensional Rindler spacetime

The same de Sitter solution may be found by another method. As we shall use this method extensively in the next section, we illustrate it here. We write the Minkowski metric, or rather part of it, as a foliation by dS hypersurfaces. This leads to a natural generalization of the $(1 + 1)$-dimensional Rindler metric. Specifically, we define new coordinates $(\tau, \rho)$ by

$$t = \rho \sinh \tau, \quad r = \rho \cosh \tau,$$

(4.33)

to get the Minkowski spacetime metric in the form

$$ds_{p+2}^2 = d\rho^2 + \rho^2 \left[ -d\tau^2 + \cosh^2 \tau \, d\Omega_p^2 \right].$$

(4.34)

In these coordinates,

$$F = \rho^{p+1} (\cosh \tau)^p \, d\rho \wedge d\tau \wedge d\Omega_p,$$

(4.35)

and the effective Lagrangian is

$$L_{\text{eff}} = -\rho^p (\cosh \tau)^p \sqrt{\rho^2 - (\partial_\tau \rho)^2} + R^{-1} \frac{(p+1)}{p+2} \rho^{p+2} (\cosh \tau)^p.$$

(4.36)

This is time-dependent, which complicates the study of general solutions, but if we focus on solutions with $\partial_\tau \rho = 0$ then we need only consider the effective potential

$$V_{\text{eff}} \equiv -L_{\text{eff}}|_{\dot{\rho} = 0} = (\cosh \tau)^p \left[ \rho^{p+1} - \frac{p+1}{R(p+2)} \rho^{p+2} \right],$$

(4.37)

which is minimized when $\rho = R$.

### 4.4.3 General solution

Let us now consider the general solution, with non-zero $\mathcal{E}$, that approaches the dS solution at late times. For such solutions, $r$ increases indefinitely, so that

$$\dot{r} \sim 1 - \frac{R^2}{2\rho^2} + \ldots$$

(4.38)

where $R = (p+1)/\mathcal{E}$ is the radius of the eventual dS spacetime. We then have

$$|A| \sim R^{-1} \left[ 1 - \frac{p\mathcal{E}}{2E_{p+1}} + \ldots \right], \quad \lambda \sim \frac{pR^3\mathcal{E}}{r^{p+3}} + \ldots$$

(4.39)
Since $\lambda$ is small and the worldvolume geometry is approximately de Sitter, there is a local temperature given approximately by

$$T = T_{GH} \left[ 1 - \frac{pE}{2Er^{p+1}} + \ldots \right], \quad T_{GH} = \frac{1}{2\pi R} \tag{4.40}$$

If we consider a spherical $p$-brane that starts close to the static solution with $r = p/E$ and then expands to approach, asymptotically, the dS solution, we start with zero temperature and end with the non-zero Gibbons-Hawking temperature $T_{GH}$. In the intermediate phase it may not make sense to attribute a temperature to the brane but eventually we may consider the brane to have a temperature slightly less than $T_{GH}$ which is approached asymptotically.

5. Spherical branes in (a)dS

We now consider a spherical $p$-brane in a maximally symmetric $(p + 2)$-dimensional spacetime that is not Minkowski; in other words in either $dS_{p+2}$ or $adS_{p+2}$. As these may be globally embedded in a flat $(p + 3)$-dimensional spacetime, we could analyse the problem from this perspective, in which case we would be dealing with a spherical brane in a flat spacetime but such that the worldvolume has co-dimension two rather than co-dimension one (as was the case in the previous section).

We will focus on those cases for which the worldvolume is itself a maximally symmetric spacetime. Such solutions are possible in the presence of a constant uniform force produced by a non-zero constant uniform electric-type field, which we henceforth assume. For $p = 0$ this means that we are considering a particle undergoing constant acceleration $a$ in $(a)dS_2$, and we know from the earlier work summarized in the introduction that $A^2 = a^2 \pm R^{-2}$, where $R$ is the $(a)dS$ radius. Here we use our concept of brane acceleration to show how this formula also applies to $p$-branes undergoing constant uniform acceleration in a $(p + 2)$-dimension $(a)dS$ space, except that now we must specify that we consider the solution for which the worldvolume geometry is maximally symmetric because other solutions are possible.

Because of this focus on maximally symmetric worldvolumes, the simplest way to find solutions is to consider the possible foliations of $(a)dS$ by a family of $(a)dS$ hypersurfaces, parametrized by some radial coordinate, and to seek solutions for which this radial coordinate is constant on the brane’s worldvolume. This reduces the problem to looking for stationary points of an effective potential, as illustrated in the previous section for the ‘dS brane’ expanding in Minkowski spacetime.

5.1 Accelerating brane in de Sitter space

We begin by considering a $p$-brane undergoing uniform acceleration in a $(p + 2)$-dimensional dS spacetime of radius $R$. As observed above, we could embed the dS spacetime in an ‘auxiliary’ $(p + 3)$-dimensional Minkowski spacetime, and thereby convert the problem into one of a brane moving in a flat spacetime, but it is as simple to consider the curved ‘physical’ dS spacetime directly. A global foliation of dS by leaves that are maximally symmetric is possible only if the leaves of the foliation are also de Sitter spaces, so we take the spacetime de Sitter metric in the form

$$ds^2_{p+2} = R^2 \left[ dy^2 + \cos^2(y)g^{dS}_{ij} d\sigma^i d\sigma^j \right], \tag{5.1}$$
where \( \bar{g}^{dS}_{ij} \) is the metric on a unit radius de Sitter space of dimension \((p + 1)\), with coordinates \( \sigma^i \) \((i = 0, 1, \ldots , p)\). As the notation suggests, we will identify these coordinates with the worldvolume coordinates of the \( p \)-brane. The induced worldvolume metric is then

\[
ds_{\text{ind}}^2 = R^2 \cos^2 y \left[ \bar{g}^{dS}_{ij} + \partial_i y \partial_j y \right] d\sigma^i d\sigma^j.
\]

We assume a constant uniform electric-type field of strength \( E \), so that

\[
F_{p+2} = ER^{p+2} \cos^{p+1} y dy \wedge dV_{p+1}(dS)
\]

where \( dV_{p+1}(dS) \) is the volume \((p + 1)\)-form of the unit radius dS space. We may choose a gauge for which the electric \((p + 1)\)-form field is

\[
C_{p+1} = ER^{p+2} f(y) dV_{p+1}(dS), \quad f'(y) = \cos^{p+1} y.\]

Using these results in (5.6) we find the effective action

\[
S[y] = -\mu R^{p+1} \int dV_{p+1}(dS) \left\{ \cos^p y \sqrt{\cos^2 y + \bar{g}^{dS}_{ij} \partial_i y \partial_j y - ER f(y)} \right\}.
\]

Solutions of the equations of motion with constant \( y \) are easily found by extremizing the effective potential

\[
V_{\text{eff}}(y) = \mu R^{p+1} \left[ \cos^{p+1} y - ER f(y) \right].
\]

There is a maximum of this potential when

\[
\tan y = -\frac{ER}{p + 1}.
\]

and in this case the induced worldvolume metric is dS with radius \( R_{\text{ind}} \) such that

\[
\frac{1}{R_{\text{ind}}^2} = \frac{1}{R^2} + \left( \frac{E}{p + 1} \right)^2.
\]

As the worldvolume is a dS space, its acceleration \( A \) in the higher-dimensional flat space is such that \( A^2 = R_{\text{ind}}^{-2} \). Applying the Unruh formula, we see that the brane temperature is given

\[
2\pi T = \sqrt{a^2 + R^{-2}}, \quad a = E/(p + 1)
\]

which is (1.3) but with \( a \) interpreted as the acceleration of the \( p \)-brane in the \((p + 2)\)-dimensional dS spacetime. The latter has an acceleration \( 1/R \) in the \((p + 2)\)-dimensional Minkowski spacetime, which provides another way of understanding our result for \( A^2 \).

Observe that the brane temperature equals that of the dS spacetime in which it is embedded only if \( E = 0 \). In this case the brane is in mechanical equilibrium and thermal equilibrium with its surroundings. A non-zero electric field causes the brane to accelerate in the surrounding dS spacetime, so in this sense it is no longer in mechanical equilibrium. As we have seen, this also increases the brane temperature so the brane is also out of thermal equilibrium with the surrounding dS spacetime.
5.2 Accelerating branes in anti de Sitter space

We now consider a $p$-brane undergoing uniform acceleration in a $(p+2)$-dimensional adS spacetime of radius $R$. As in the dS case, there exist foliations of adS in which the leaves are maximally symmetric, but this worldvolume geometry could now be dS, Minkowski, or adS.

5.2.1 adS worldvolume

Let us first consider the adS foliations because this case is analogous to the foliation of dS by dS spaces. We write the spacetime adS metric in the form

$$ds^2_{p+2} = R^2 \left[ dy^2 + \cosh^2(y) \tilde{g}_{ij}^{adS} d\sigma^i d\sigma^j \right],$$

(5.10)

where $\tilde{g}_{ij}^{adS}$ is the metric on a unit radius anti-de Sitter space of dimension $(p+1)$, with coordinates $\sigma^i \ (i = 0, 1, \ldots, p)$. As the notation suggests, we will identify these coordinates with the worldvolume coordinates of the $p$-brane. The induced worldvolume metric is then

$$ds^2_{ind} = R^2 \left[ \cosh^2(y) \tilde{g}_{ij}^{adS} + \partial_i y \partial_j y \right] d\sigma^i d\sigma^j.$$

(5.11)

We assume a constant uniform electric-type field of strength $E$, so that

$$F_{p+2} = ER^{p+2} \cosh^{p+1} y dy \wedge dV_{p+1}(adS)$$

(5.12)

where $dV_{p+1}(adS)$ is the volume $(p+1)$-form of the unit radius adS space. We may choose a gauge for which the electric $(p+1)$-form field is

$$C_{p+1} = ER^{p+2} f(y) dV_{p+1}(dS), \quad f'(y) = \cosh^{p+1} y.$$

(5.13)

Using these results in (1.6) we find the effective action

$$S[y] = -\mu R^{p+1} \int dV_{p+1}(adS) \left\{ \cosh^p y \sqrt{\cosh^2 y + \tilde{g}_{ij}^{adS} \partial_i y \partial_j y} - ER f(y) \right\}.$$

(5.14)

Solutions of the equations of motion with constant $y$ are easily found by extremizing the effective potential

$$V_{eff}(y) = \mu R^{p+1} \left[ \cosh^{p+1} y - ER f(y) \right].$$

(5.15)

There is a minimum of this potential when

$$\tanh y = \frac{ER}{p+1},$$

(5.16)

and in this case the induced worldvolume metric is adS with radius $R_{ind}$ given by

$$\frac{1}{R^2_{ind}} = \frac{1}{R^2} - \left( \frac{E}{p+1} \right)^2.$$

(5.17)

This makes sense only if

$$|E| < E_{crit} = \frac{p+1}{R}.$$

(5.18)
Otherwise, the assumption that the worldvolume has adS geometry must be false. One can guess that if $|E| > E_{\text{crit}}$ the worldvolume geometry will be dS rather than adS, and we verify this shortly. For the moment, we must restrict $E$ to be ‘subcritical’. As the worldvolume is an adS space, its acceleration $A$ in the higher-dimensional flat space is such that

$$A^2 = -R_{\text{ind}}^{-2} = -R^{-2} + a^2, \quad a = |E|/(p + 1),$$

(5.19)

where

$$a < a_{\text{crit}} \equiv E_{\text{crit}}/(p + 1) = R^{-1}.$$  

(5.20)

Because $A^2 < 0$, the brane temperature is zero.

### 5.2.2 dS worldvolume

We now consider a dS foliation of the adS spacetime by writing the adS metric in the form

$$ds_{p+2}^2 = R^2 \left[ dy^2 + \sinh^2 y \bar{g}^{ij}_{dS} d\sigma^i d\sigma^j \right].$$

(5.21)

As before we identify the unit-radius dS coordinates $\sigma^i (i = 0, 1, \ldots, p)$ with the worldvolume coordinates of the $p$-brane, in which case the induced worldvolume metric is

$$ds_{\text{ind}}^2 = R^2 \left[ \sinh^2 y \bar{g}^{ij}_{dS} + \partial_i y \partial_j y \right] d\sigma^i d\sigma^j.$$  

(5.22)

As before, we assume a constant uniform electric-type field of strength $E$, so that

$$F_{p+2} = ER^{p+2} \sinh^{p+1} y dy \wedge dV_{p+1}(dS)$$

(5.23)

where $dV_{p+1}(dS)$ is the volume $(p + 1)$-form of the unit radius dS space. We may choose a gauge for which the electric $(p + 1)$-form field is

$$C_{p+1} = ER^{p+2} f(y) dV_{p+1}(dS), \quad f'(y) = \sinh^{p+1} y.$$  

(5.24)

Using these results in (1.6) we find the effective action

$$S[y] = -\mu R^{p+1} \int dV_{p+1}(adS) \left\{ \sinh^p y \sqrt{\sinh^2 y + \bar{g}^{ij}_{adS} \partial_i y \partial_j y - ER f(y)} \right\}.$$  

(5.25)

Solutions of the equations of motion with constant $y$ are easily found by extremizing the effective potential

$$V_{\text{eff}}(y) = \mu R^{p+1} \left[ \sinh^{p+1} y - ER f(y) \right].$$  

(5.26)

There is a maximum of this potential when

$$\coth y = \frac{ER}{p + 1},$$  

(5.27)

and in this case the induced worldvolume metric is dS with radius $R_{\text{ind}}$ such that

$$\frac{1}{R_{\text{ind}}^2} = \left( \frac{E}{p + 1} \right)^2 - R^{-2}.$$  

(5.28)

This makes sense only for $|E| > E_{\text{crit}}$, which we now assume because otherwise the initial assumption of dS worldvolume geometry is false. Since the worldvolume geometry is dS, the brane acceleration $A$ is such that $A^2 = R_{\text{ind}}^{-2}$, and hence the brane temperature is given by

$$2\pi T = \sqrt{a^2 - R^{-2}}, \quad a = |E|/(p + 1) > R^{-1} \equiv a_{\text{crit}}.$$  

(5.29)
5.2.3 Minkowski worldvolume

One may guess from the above results that when $|E| = E_{\text{crit}}$ the worldvolume geometry will be flat, and hence locally Minkowski. The simplest way to confirm this is to write the adS spacetime metric as

$$ds^2_{p+2} = R^2 [dy^2 + e^{2y} \eta_{ij} d\sigma^i d\sigma^j]$$

where $\eta$ is the flat Minkowski metric, in local coordinates $\sigma^i$ ($i = 0, 1, \ldots, p$). Again, we identify these coordinates with the brane worldvolume coordinates, in which case the induced metric is

$$ds^2_{\text{ind}} = R^2 [e^{2y} \eta_{ij} + \partial_i y \partial_j y] d\sigma^i d\sigma^j.$$  (5.31)

We assume a constant uniform electric-type field of strength $E$, so that

$$F_{p+2} = ER^{p+2} e^{(p+1)y} dy \wedge d\xi^0 \wedge d\xi^1 \wedge \cdots \wedge d\xi^p,$$  (5.32)

and we may then choose a gauge such that

$$C_{p+1} = \left( \frac{ER}{p+1} \right) R^{p+1} e^{(p+1)y} d\sigma^0 \wedge d\sigma^1 \wedge \cdots \wedge d\sigma^p.$$  (5.33)

Using these results in (1.6) we find the effective action

$$S[y] = -\mu R^{p+1} \int d^{p+1} \xi e^{(p+1)y} \left\{ \sqrt{1 + e^{2y} \eta_{ij} \partial_i y \partial_j y - \frac{ER}{p+1} } \right\}.$$  (5.34)

The effective potential for this case is

$$V_{\text{eff}} = \frac{\mu R^{p+1} e^{(p+1)y}}{p+1} \left[ E_{\text{crit}} - E \right],$$  (5.35)

where $E_{\text{crit}} = (p+1)/R$, as in (5.18). This potential has no extrema, except when $E = E_{\text{crit}}$, in which case $V_{\text{eff}} \equiv 0$, so that there is a solution for any constant $y$. For all these solutions the induced metric is locally Minkowski, but the brane acceleration is nevertheless non-zero. One can show that $a = R^{-1}$, the critical acceleration, as expected. If the same problem is analysed in terms of the brane moving in the ambient flat space of dimension $(p+3)$, with two time dimensions, in which the adS spacetime is embedded then one finds that the acceleration vector $A^\mu$ is null, and hence $A^2 = 0$ consistent with the zero brane temperature.

6. BTZ black hole

As a final application of our formalism, we consider the stationary ‘BTZ’ metric representing a $(1 + 2)$-dimensional black hole spacetime [20, 21]. This was found as a solution of the $(1 + 2)$-dimensional Einstein’s equations with negative cosmological constant but it can be globally embedded in a flat $(2 + 2)$-dimensional spacetime [5] and could therefore be interpreted as the worldvolume of a membrane, albeit with dynamics that differs from that assumed so far. Of course, any metric that can be globally embedded in a higher-dimensional flat spacetime can be viewed as a brane worldvolume for kinematical purposes. The BTZ black hole provides a
simple illustration of this point, and it allows us to explore some issues arising for metrics that are stationary but not static.

Also, the BTZ metric is locally diffeomorphic to \( \text{adS}_3 \). We have previously considered strings in \( \text{adS}_3 \) as a special case of \( p \)-branes in \( \text{adS}_{p+2} \), but the global identifications of \( \text{adS}_3 \) needed to get the BTZ metric allows a further possibility for closed strings that we also examine.

6.1 Non-rotating black hole

We begin with a brief discussion of the non-rotating BTZ black hole of mass \( M \), for which the metric is

\[
ds^2 = -\left( \frac{r^2}{R^2} - M \right) dt^2 + \left( \frac{r^2}{R^2} - M \right)^{-1} dr^2 + r^2 d\phi^2 ,
\]

(6.1)

where \( \phi \) is an angular coordinate with the standard \( 2\pi \) identification. This metric is static, with Killing vector field \( k = \partial_t \). The singularity of the metric at \( r = R\sqrt{M} \) is a coordinate singularity at a horizon of \( k \), and the Euclidean continuation of the metric is non-singular if the imaginary time is identified with period \( 2\pi R/\sqrt{M} \), so the Hawking temperature is \( T_H = \sqrt{M} / (2\pi R) \). Actually, there is no absolute meaning to this temperature (in contrast to the Hawking temperature for asymptotically flat black holes) because \( k^2 \to 0 \) as \( r \to \infty \) and there is therefore no natural normalization for \( k \). However, a rescaling of \( k \) leads to a rescaling of \( T_H \) such that the local temperature, satisfying the Tolman law (1.2) is unchanged. This local temperature is

\[
T = \frac{T_H}{\sqrt{-k^2}} = \frac{1}{2\pi} \sqrt{\frac{M}{r^2 - MR^2}} .
\]

(6.2)

This result can be reproduced [5] by applying the Unruh formula to the global embedding of the BTZ metric in a flat spacetime, which we give for the general stationary BTZ black hole in the following subsection. A shortcut is to observe that since the BTZ metric is locally diffeomorphic to \( \text{adS}_3 \), the temperature is given by the formula (1.5), with \( a \) the acceleration of an observer in the BTZ metric at fixed \((r, \phi)\), which is

\[
a = \frac{r}{R\sqrt{r^2 - MR^2}} .
\]

(6.3)

Using this in (1.5) we recover (6.2).

Now consider a circular string in the BTZ background, at constant \( r \). This implies that the string is subject to a force that prevents it from shrinking towards the BTZ horizon but just such a force results from a minimal coupling of the string to an appropriate constant uniform electric-type field; we omit the details as they are given in more generality below. The induced worldsheet metric for this circular-string solution is flat but the extrinsic curvature is non-zero. Computing the extrinsic curvature for the circular string embedded in the four-dimensional flat ambient space one gets (see also below)

\[
A = K_{uu} = \sqrt{\frac{M}{r^2 - MR^2}} .
\]

(6.4)

from which we deduce that the string temperature equals the BTZ black hole temperature at \( r = r_0 \), and hence that the string is in thermal equilibrium with its background.
6.2 Rotating black hole

We now turn to the case of the rotating BTZ black hole, which introduces a number of novel features. The metric takes the form

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 \left( d\phi + N^\phi dt \right)^2, \]  

(6.5)

where

\[ N^2 = \frac{1}{r^2 R^2} (r^2 - r_+^2)(r^2 - r_-^2) = \frac{r^2}{R^2} - M + \frac{J^2}{4r^2}, \]
\[ N^\phi = -\frac{r_+ r_-}{r^2 R} = -\frac{J}{2r^2}. \]  

(6.6)

for constants \( r_\pm \), in terms of which the mass and angular momentum are, respectively,

\[ M = \frac{r_+^2 + r_-^2}{R^2}, \quad J = \frac{2r_+ r_-}{R}. \]  

(6.7)

This metric has coordinate singularities at \( r = r_+ \) and \( r = r_- \). Observers at fixed \( r \) that follow orbits of the vector field

\[ \xi = \partial_t - N^\phi(r) \partial_\phi. \]  

(6.8)

are known as ZAMOs, or zero angular momentum observers. The vector field \( \xi \) is not Killing, but the vector fields

\[ \xi_{\pm} = \xi |_{r=r_{\pm}} \]  

(6.9)

are Killing, and the coordinate singularity at \( r = r_{\pm} \) is a Killing horizon of \( \xi_{\pm} \). In particular

\[ \xi_{\pm}^2 = -\frac{(r_+^2 - r_-^2)(r^2 - r_{\pm}^2)}{R^2 r_{\pm}^2}, \]  

(6.10)

which shows that \( \xi_+ \) is timelike for \( r > r_+ \) and null at \( r = r_+ \). Thermal equilibrium for \( r > r_+ \) requires that \( T = T_H/\sqrt{-\xi_+^2} \), where \( T_H \) is the Hawking temperature, as determined by requiring non-singularity of the Euclidean metric (with imaginary \( J \)):

\[ T_H = \frac{r_+^2 - r_-^2}{2\pi r_+ R^2}. \]  

(6.11)

This gives the local temperature

\[ T = \frac{1}{2\pi R} \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_+^2}}. \]  

(6.12)

The BTZ metric can be globally embedded in a flat (2+2)-dimensional spacetime with metric

\[ ds^2 = -(dX_0)^2 + (dX_1)^2 + (dX_2)^2 - (dX_3)^2 \]  

(6.13)

The embedding is [5]

\[ X_0 \pm X_1 = \pm R \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \exp \left[ \pm \frac{r \pm t}{R^2 / R} \right] \]
\[ X_2 \pm X_3 = \pm R \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \exp \left[ \pm \frac{r \phi - r_- t}{R^2 / R} \right]. \]  

(6.14)
In other words, the BTZ metric is the induced metric on the hypersurface specified by this embedding. If we view this hypersurface as the worldvolume of a membrane then we may apply the formalism developed in the previous examples. In particular, we read off from the induced metric that the worldvolume velocity is

\[ u = N^{-1} \xi \]  

(6.15)

where \( \xi \) is the vector field of (6.8). Thus, our definition of brane acceleration picks out ZAMOs as ‘preferred’ observers.

It is now a straightforward exercise to compute the brane velocity \( U \), brane acceleration \( A \) and brane jerk \( \Sigma \). One finds that

\[
U_0 \pm U_1 = \frac{r_+}{r} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \exp \left[ \pm \left( \frac{r_+ t}{R^2} - \frac{r_+ \phi}{R} \right) \right]
\]

\[
U_2 \pm U_3 = -\frac{r_+}{r} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \exp \left[ \pm \left( \frac{r_+ \phi}{R} - \frac{r_+ t}{R^2} \right) \right].
\]

(6.16)

and

\[
A_0 \pm A_1 = \pm \frac{r_+}{R r} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} (U_0 \pm U_1)
\]

\[
A_2 \pm A_3 = \mp \frac{r_-}{R r} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} (U_2 \pm U_3).
\]

(6.17)

One may verify that \( U^2 = -1 \) and \( U \cdot A = 0 \), and a simple computation shows that

\[
A^2 = \frac{r^6 (r_+^2 + r_-^2) - 3r_+^4 r_+^2 r_-^2 + r_+^4 r_-^4}{r^4 R^2 (r_+^2 - r_-^2) (r_+^2 - r_-^2)}.
\]

(6.18)

Similarly, one can show that

\[
\Sigma_0 \pm \Sigma_1 = -\frac{r_+^2 (r^4 - r_+^2 r_-^2)}{R^2 r^4 (r_+^2 - r_-^2)} (U_0 \pm U_1)
\]

\[
\Sigma_2 \pm \Sigma_3 = -\frac{r_-^2 (r^4 - r_+^2 r_-^2)}{R^2 r^4 (r_+^2 - r_-^2)} (U_2 \pm U_3),
\]

(6.19)

and hence that

\[
\lambda \equiv \sqrt{\frac{\Sigma^2}{A^2}} = \frac{r_- r_+ (r^4 - r_+^2 r_-^2) \sqrt{(r^2 - r_+^2) (r^2 - r_-^2)}}{r^6 (r_+^2 + r_+^2) - 3r_+^4 r_+^2 r_-^2 + r_+^4 r_-^4}
\]

(6.20)

We recover the non-rotating case by setting \( r_- = 0 \). In this case \( \lambda = 0 \) and the Unruh formula applies. In addition, \( \lambda \to 0 \) as \( r \to r_+ \), so the Unruh formula applies near the horizon. Comparison with (6.12) shows that

\[
|A| = 2\pi T \left[ 1 + O \left( \sqrt{r - r_+} \right) \right],
\]

(6.21)

so the Unruh formula also gives the temperature predicted by thermal equilibrium, near the horizon. This is entirely as expected because the ZAMOs follow orbits of \( \xi_+ \) near the horizon only, and away from the horizon, the Unruh formula is not applicable.
6.2.1 Jerk-free observers

We have seen that the brane acceleration, as defined by the choice of coordinates in which the BTZ metric is given by (6.3), does not reproduce the local temperature (6.12), except near the horizon, for understandable reasons: the Unruh formula is not valid away from the horizon because the parameter $\lambda$ is small only near the horizon. However, let us consider local observers that follow orbits of the Killing vector field

$$u_+ = \xi_+ / \sqrt{-\xi^2_+}.$$  

(6.22)

These observers are circling the black hole with the angular velocity of the horizon. As noticed in [5], they experience a temperature equal to the local equilibrium temperature (6.12). Following our formalism and taking $u = u_+$, we define

$$U_+ = u_+^i \partial_i X, \quad A_+ = u_+^i \partial_i U, \quad \Sigma_+ = u_+^i \partial_i A - A^2 U_+.$$  

(6.23)

A calculation shows that

$$A^2_+ = \frac{r^2 - r^2_+}{R^2 (r^2 - r^2_+)}, \quad \Sigma_+ \equiv 0.$$  

(6.24)

It follows that the parameter $\lambda$ is zero, so the Unruh formula is applicable; it yields precisely the local temperature (6.12). This result is in agreement with a general principle discussed in [23] that a heat bath in equilibrium with a rotating black hole must itself rotate with the angular velocity of the horizon.

6.3 Circular string

Now we consider a circular string at constant $r$. We may identify $(t, \phi)$ with the worldsheet coordinates. The induced metric is then

$$ds^2_{ind} = -\frac{1}{R^2 (r^2 - r^2_+)} dt^2 - 2\frac{r^+ r}{R} d\phi dt + r^2 d\phi^2$$  

(6.25)

Using this in the Nambu-Goto action, and allowing for a uniform electric-type 2-form field, as in previous examples, one finds a string action from which we obtain the following effective potential:

$$V_{eff} = \mu r \left[ \sqrt{\frac{r^2}{R^2} - M + \frac{J^2}{4r^2} - \frac{e r}{R}} \right], \quad e \equiv \frac{1}{2} |E| R.$$  

(6.26)

There is an $e$, with $e^2 > 1$, such that this potential has a maximum at any $r = r_0 > r_+$, so we may adjust $e$ to get a static string solution at any distance from the black hole horizon. We now apply our previous formalism to find a brane temperature. Firstly, the worldsheet velocity is $u = N^{-1} \xi$, exactly as it was for the BTZ black hole itself, except that $\xi$ is now a worldsheet Killing vector field because $r = r_0$. This means that the calculation of $U, A, \Sigma$ for the string is identical to the calculation just done for the BTZ black hole. In particular, $A^2_{string}$ is given by (6.18) with $r = r_0$. Alternatively, we can compute $A^2$ of the string from the extrinsic curvature of the worldsheet in the Minkowski spacetime, because a simple computation shows that the
worldsheet acceleration is zero. The non-zero components of the extrinsic curvature tensor are

\[
K^0_{uu} \pm K^1_{uu} = \pm \frac{r^2 \mp (r_0^2 - r^2)}{r_0^2 R \sqrt{(r_0^2 - r^2)} (r_0^2 - r^2)} \exp \left[ \pm \left( \frac{r \pm t R}{R^2 - r^2} \right) \right],
\]

\[
K^3_{uu} \pm K^4_{uu} = \pm \frac{r^2 \mp (r_0^2 - r^2)}{r_0^2 R \sqrt{(r_0^2 - r^2)} (r_0^2 - r^2)} \exp \left[ \pm \left( \frac{r \pm \phi R}{R^2 - r^2} \right) \right].
\]  

We may now use \( A^2 = K^2_{uu} \) to verify that \( A^2_{\text{string}} \) is given by (6.18) with \( r = r_0 \).

Exactly the same calculation that led to (6.20) now gives \( \lambda_{\text{string}} \) on setting \( r = r_0 \). Therefore the Unruh formula is applicable only near the horizon \((r_0 \to r_+ \rangle\) where it gives a temperature approximately equal to the local temperature (6.12) of the black hole.

7. Discussion

It is a remarkable fact, established in recent years, that the local temperature of a stationary black hole in thermal equilibrium is the local Unruh temperature associated to acceleration in a flat ambient spacetime in which the black hole spacetime is globally and isometrically embedded. As this interpretation of black hole temperature relies only on the kinematics of global embeddings, it is equally applicable to relativistic branes, for which the dynamics arises not from Einstein’s equations ‘on the brane’ but rather from the forces due to brane tension and minimal coupling to a background \((p + 1)\)-form potential. In this paper we have shown that the Hamiltonian formulation of brane dynamics leads naturally to a ‘brane velocity’ in the ambient spacetime, and a ‘brane acceleration’ to which the Unruh formula may be applied to deduce a ‘brane temperature’.

Although the Unruh effect was originally derived for a particle undergoing constant proper acceleration, it is obvious on physical grounds that the result must continue to apply to a good approximation when the acceleration is allowed to vary sufficiently slowly in time. In the context of brane acceleration, which reduces for \( p = 1 \) to the standard relativistic acceleration, we constructed a dimensionless quantity \( \lambda \) from the acceleration and a relativistic ‘brane jerk’ that provides a measure of the time rate of change of acceleration. When \( \lambda \ll 1 \) we expect the Unruh formula to be applicable, but otherwise it will not apply. Naturally, our definition of brane jerk applies, as a special case, to the standard relativistic mechanics of a particle in \((1 + 3)\)-dimensional Minkowski spacetime with 4-velocity \( u \) and 4-acceleration \( a \), and in that case the 4-vector \( j = da/d\tau \) (which might be considered the natural relativistic generalization of 3-jerk) has the curious feature that it does not vanish even when the 3-acceleration is constant; the 4-vector that measures the rate of change of the 3-acceleration is \( \zeta = j - a^2 u \). We are not aware of any standard name for this quantity, but it would make sense to call this the relativistic jerk. In any case, a similarly defined quantity \( \Sigma \) for brane jerk was needed in our definition of the parameter \( \lambda \).

As our first example, we considered an open \( p \)-brane undergoing rigid rotation in a \((2p + 1)\)-dimensional Minkowski spacetime, in the absence of any external forces. We remark here that the same solution applies to a ‘folded’ closed \( p \)-brane in which the \((p - 1)\)-dimensional ‘fold’ moves at the speed of light, just like the boundary of an open \( p \)-brane. For \( p = 5 \), this yields
what appears to be a new solution of the M5-brane equations in the 11-dimensional Minkowski background of M-theory. It is convenient to describe this solution in a co-rotating frame in which the obvious timelike Killing vector field has a $p$-dimensional horizon. This Killing horizon of the Minkowski spacetime becomes a curvature singularity of the induced worldvolume metric, albeit one with a well-understood physical interpretation: it is the singularity on the null hypersurface swept out by the $(p - 1)$-dimensional boundary of the open $p$-brane. It is worth noting that this provides an example of a curvature singularity that needs no ‘resolution’ in the context of some more general theory. No doubt this case is exceptional, but it shows that the usual assumption that curvature singularities require new physics is not self-evident.

In this example, the parameter $\lambda$ is always greater than unity, so we cannot expect to associate the centripetal acceleration with a temperature. Indeed, the attempt to do so leads to a local temperature that is not consistent with thermal equilibrium. This is probably related to the fact that the singularity of the worldvolume metric at the $p$-brane boundary is a curvature singularity rather than a coordinate singularity. It is instructive to compare the worldvolume metric of the rigidly-rotating $p$-brane with that of $(p + 1)$-dimensional de Sitter spacetime in static coordinates. Consider $p = 1$ for simplicity. Then, by rescaling coordinates, both metrics can be put in the form

$$\text{d}s^2 = -(1 - \rho^2) \text{d}t^2 + f(\rho) \text{d}\rho^2$$

for a function $f$, with $f = 1$ for the rotating string and $f = 1/(1 - \rho^2)$ for $\text{adS}_2$. Both are singular at $\rho = 1$, and in both cases the local acceleration in the embedding Minkowski spacetime becomes infinite as this singularity is approached. However, in the de Sitter case the singularity is a coordinate singularity and the local acceleration is never zero; it takes a minimum at $\rho = 0$, and the Unruh temperature there is the Gibbons-Hawking temperature. In contrast, the singularity in the rotating brane case is a curvature singularity, and the local acceleration is zero at $\rho = 0$.

For the $p = 1$ case, our rigidly-rotating brane example is closely related to the motion of particles in accelerators. A ‘circular Unruh’ effect was considered long ago in this context [25] (see [26] for an up-to-date account). As already noted in [25], the ‘circular Unruh’ effect is not thermal, and so cannot be simply characterized by a temperature. This is presumably related to the fact that one cannot ignore the jerk of a particle undergoing circular motion.

Most of our other examples involved spherical $p$-branes coupled to a constant and uniform electric-type background field. The simplest of these involve a $p$-brane in a $(p + 2)$-dimensional spacetime. There is a static solution, at an unstable equilibrium point of the ‘effective’ potential of the $p$-brane action. As expected, this has zero brane acceleration and hence zero brane temperature. A slight increase of the $p$-brane radius leads to a runaway expanding brane solution; such solutions were considered previously in the context of a brane generalization of the Schwinger pair-creation process for charged particles in $(1 + 1)$-dimensional electrodynamics [22]. A new observation of this paper is that the asymptotic expanding-brane solution has de Sitter worldvolume geometry. We also show that the brane acceleration of this asymptotic solution is constant and uniform, and that the associated Unruh temperature is the Gibbons-Hawking temperature of de Sitter space.

There is a close similarity of these results to cosmology with the force due to tension playing
the role of gravitational attraction and the electric-type field playing the role of a positive cosmological constant. The static brane solution is both analogous and isometric to the Einstein static universe solution of Einstein’s equations, and the de Sitter brane solution is both analogous and isometric to the de Sitter solution of Einstein’s equations. Recall that the Einstein static universe is unstable, and that there exists a perturbation of that leads to an expanding universe that approaches the de Sitter universe at late times; this interpolating solution of Einstein’s equations is analogous to the solution of the brane equations that interpolates between the static and de Sitter branes. This analogy suggests that the observed small cosmological constant might be explained by a bulk electric field in some braneworld scenario; if so, the question of why the cosmological constant is small could be traded for a similar question for the bulk electric field, but the latter question might have some simple answer (e.g. because electric field is produced by vacuum fluctuations). It would be interesting to find a concrete scenario in which this possibility could be explored.

We have also studied spherical \(p\)-branes in \((p + 2)\)-dimensional de Sitter and anti-de Sitter spacetimes. Since these may be embedded in a flat \((p + 3)\)-dimensional spacetime, albeit with two time dimensions in the adS case, this problem is closely related to one in which a \(p\)-brane is embedded in a flat \((p + 3)\)-dimensional spacetime, so that the worldvolume has co-dimension two in the flat ambient spacetime. Viewed from this perspective, the brane acceleration \(A\), which is orthogonal to the worldvolume, now has two components: one is due to the acceleration \(a\) of the brane in \((a)dS\) and the other to the acceleration of \((a)dS\) in the flat spacetime. This yields \(A^2 = a^2 \pm R^{-2}\) for an \((a)dS\) spacetime of radius \(R\), where the plus sign is for de Sitter and the minus sign for anti-de Sitter. The adS case is more interesting because a non-zero temperature requires \(a > a_{\text{crit}}\) for ‘critical’ acceleration \(a_{\text{crit}} = 1/R\). We have shown that the brane worldvolume is a \((p + 1)\)-dimensional adS spacetime for \(a < a_{\text{crit}}\) whereas it is a \((p + 1)\)-dimensional dS spacetime for \(a > a_{\text{crit}}\), with a local temperature that again equals the Gibbons-Hawking temperature.

It is interesting to ask what significance the de Sitter embeddings in adS might have for the adS/CFT correspondence. Recall that the \(adS_{p+2}\) spacetime has a boundary that is topologically \(S^1 \times S^p\). An expanding bulk \(p\)-brane with de Sitter worldvolume could be the end result of an instanton-induced creation process that leads to an expanding bubble in the bulk that, asymptotically, wraps the \(p\)-sphere at infinity. It is natural to suppose that the asymptotic approach of the brane temperature to the Gibbons-Hawking temperature of a dS spacetime corresponds to some approach to thermal equilibrium on the boundary, and this implies that the boundary temperature is non-zero.

In our final example we reconsidered the BTZ black hole, studied from the GEMS perspective in [5, 10]. As we focus on the kinematics, we may suppose that it is a membrane in \((1 + 3)\)-dimensional Minkowski spacetime, and then ask what its brane acceleration is according to our formalism. We found results that are close to those found in [5] because our definitions effectively select the class of observers known as ZAMOs, which are also those considered in [23] and in earlier discussions of BTZ black holes [24].

As a final comment, we recall that the Minkowski metric in the form (4.34), foliated by de Sitter hypersurfaces, is the natural higher-dimensional generalization of the \((1 + 1)\)-dimensional Rindler spacetime; the analog of the Unruh temperature in Rindler space is the Gibbons-
Hawking temperature on a de Sitter slice. This suggests a new kind of holography because the isometry group of the Rindler-type wedge is $SO(p + 1, 1)$, which is the same as the isometry group of $dS_{p+1}$. Thus, gravitational physics on the Rindler-type wedge with a cut-off at $r = R$ could have a holographic description in terms of a quantum field theory on the de Sitter slice at $r = R$. One may also wonder whether higher-dimensional Rindler spacetimes can arise as near-horizon geometries in the same way that the original, $(1 + 1)$-dimensional, Rindler spacetime arises in the context of the Schwarzschild black hole.

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References

[1] S. W. Hawking, “Particle Creation By Black Holes,” Commun. Math. Phys. 43 (1975) 199 [Erratum-ibid. 46 (1976) 206].
[2] W. G. Unruh, “Notes on black hole evaporation,” Phys. Rev. D 14, 870 (1976).
[3] L. C. B. Crispino, A. Higuchi and G. E. A. Matsas, “The Unruh effect and its applications,” arXiv:0710.5373 [gr-qc].
[4] S. Deser and O. Levin, “Equivalence of Hawking and Unruh temperatures through flat space embeddings,” Class. Quant. Grav. 15, L85 (1998) [arXiv:hep-th/9806223].
[5] S. Deser and O. Levin, “Mapping Hawking into Unruh thermal properties,” Phys. Rev. D 59, 064004 (1999) [arXiv:hep-th/9809159].
[6] C. Fronsdal, “Completion and Embedding of the Schwarzschild Solution,” Phys. Rev. 116 (1959) 778.
[7] H. Z. Chen, Y. Tian, Y. H. Gao and X. C. Song, “The GEMS Approach to Stationary Motions in the Spherically Symmetric Spacetimes,” JHEP 0410, 011 (2004) [arXiv:gr-qc/0409107].
[8] H. Z. Chen and Y. Tian, “A note on the GEMS of charged black holes,” arXiv:gr-qc/0410077.
[9] N. L. Santos, O. J. C. Dias and J. P. S. Lemos, “Global embedding of D-dimensional black holes with a cosmological constant in Minkowskian spacetimes: Matching between Hawking temperature and Unruh temperature,” Phys. Rev. D 70 (2004) 124033 [arXiv:hep-th/0412076].
[10] S. T. Hong, Y. W. Kim and Y. J. Park, “Higher dimensional flat embeddings of (2+1) dimensional black holes,” Phys. Rev. D 62 (2000) 024024 [arXiv:gr-qc/0003097].
[11] E. J. Brynjolfsson and L. Thorlacius, “Taking the Temperature of a Black Hole,” arXiv:0805.1876 [hep-th].
[12] G. W. Gibbons and S. W. Hawking, “Cosmological Event Horizons, Thermodynamics, And Particle Creation,” Phys. Rev. D 15 (1977) 2738.
[13] R. Figari, R. Hoegh-Krohn and C. R. Nappi, “Interacting Relativistic Boson Fields In The De Sitter Universe With Two Space-Time Dimensions,” Commun. Math. Phys. 44 (1975) 265.

[14] H. Narnhofer, I. Peter and W. E. Thirring, “How hot is the de Sitter space?,” Int. J. Mod. Phys. B 10, 1507 (1996).

[15] A. Friedman, “Local isometric embedding of Riemannian manifolds with indefinite metric”, J. Math. Mech. 10, 625 (1961).

[16] S. Deser and O. Levin, “Accelerated detectors and temperature in (anti) de Sitter spaces,” Class. Quant. Grav. 14, L163 (1997) [arXiv:gr-qc/9706018].

[17] J. Bros, H. Epstein and U. Moschella, “Towards a general theory of quantized fields on the anti-de Sitter space-time,” Commun. Math. Phys. 231, 481 (2002) [arXiv:hep-th/0111255].

[18] Y. Tian, “De Sitter Thermodynamics from Diamonds’s Temperature,” JHEP 0506 (2005) 045 [arXiv:gr-qc/0504040].

[19] D. Jennings, “The brane universe as an Unruh observer,” arXiv:hep-th/0508215.

[20] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. 69 (1992) 1849 [arXiv:hep-th/9204099].

[21] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, “Geometry of the (2+1) black hole,” Phys. Rev. D 48 (1993) 1506 [arXiv:gr-qc/9302012].

[22] F. Dowker, J. P. Gauntlett, G. W. Gibbons and G. T. Horowitz, “Nucleation of P-Branes and Fundamental Strings,” Phys. Rev. D 53, 7115 (1996) [arXiv:hep-th/9512154].

[23] J. D. Brown, E. A. Martinez and J. W. . York, “Complex Kerr-Newman geometry and black hole thermodynamics,” Phys. Rev. Lett. 66 (1991) 2281.

[24] S. Carlip, “The (2+1)-Dimensional black hole,” Class. Quant. Grav. 12, 2853 (1995) [arXiv:gr-qc/9506079].

[25] J. S. Bell and J. M. Leinaas, “The Unruh effect and quantum fluctuations of electrons in storage rings”, Nucl. Phys. B 284 (1987) 488.

[26] E. T. Akhmedov and D. Singleton, “On the physical meaning of the Unruh effect,” arXiv:0705.2525 [hep-th].