ON STRICT CATEGORY WEIGHT
AND THE ARNOLD CONJECTURE

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Abstract. In [R2] and [RO] the Arnold conjecture for symplectic manifolds \((M, \omega)\) with \(\pi_2(M) = 0\) was proved. This proof used surgery and cobordism theory. Here we give a purely cohomological proof of this result.

INTRODUCTION

Given a smooth (\(\geq C^\infty\)) manifold \(M\), we set \(\text{Crit } M := \min \{ \text{crit } f \} \) where \(f\) runs over all smooth functions \(M \to \mathbb{R}\).

Let \((M, \omega)\) be a symplectic manifold. Given a smooth function \(f : M \to \mathbb{R}\), let \(\text{sgrad } f\) denote the symplectic gradient of \(f\), i.e., the vector field defined as follows:

\[
\omega(\text{sgrad } f, \xi) = -\text{d}f(\xi)
\]

for every vector field \(\xi\).

In [A] Arnold proposed the following remarkable conjecture. Let \((M, \omega)\) be a closed symplectic manifold, and let \(H : M \times \mathbb{R} \to \mathbb{R}\) be a smooth function such that \(H(m, t) = H(m, t + 1)\) for every \(m \in M, t \in \mathbb{R}\). We define \(H_t : M \to \mathbb{R}\) by setting \(H_t(m) := H(m, t)\). Consider the time-dependent differential equation

\[
\dot{x} = \text{sgrad } H_t(x(t)).
\]

(*)

The Arnold conjecture claims that the number of 1-periodic solutions of (*) is at least \(\text{Crit } M\).

This conjecture admits another interpretation. The equation (*) yields a family \(\varphi_t : M \to M, t \in \mathbb{R}\), where, for every \(x \in M\), \(\varphi_t(x)\) is the integral curve of (*). A Hamiltonian symplectomorphism is a diffeomorphism \(\phi : M \to M\) which has the form \(\phi = \varphi_1\) for some function \(H : M \times \mathbb{R} \to \mathbb{R}\) as above. So, the Arnold conjecture can be reformulated as follows:

\[
\text{Fix } \phi \geq \text{Crit } M
\]

for every Hamiltonian symplectomorphism \(\phi\), where \(\text{Fix } \phi\) denotes the number of fixed points of \(\phi\).

This conjecture was proved for many special cases, see [MS] and [HZ] for a survey. Here we notice the following result of Floer [Fl] and Hofer [H]: the number \(\text{Fix } \phi\) can be estimated from below by the cup-length of \(M\). So, here we have a weak form of the Arnold conjecture.
In [R2] and [RO] the Arnold conjecture was proved for every closed connected symplectic manifold \((M, \omega)\) with \(\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}\). In greater detail, in [R2] the conjecture was proved under the additional condition \(\text{cat} M = \dim M\), and it was proved in [RO] that the condition \(\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}\) implies the condition \(\text{cat} M = \dim M\). Because of the last result, it turns out to be that \(\text{Crit} M = 2n + 1\) provided \(\omega|_{\pi_2(M)} = 0\), and actually we have the inequality \(\text{Fix} \phi \geq 2n + 1\).

The proof of the Arnold conjecture in [R] uses surgery and cobordism theory. Here we give another proof of the Arnold conjecture (under the same restriction \(\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}\)). This proof uses the ordinary cohomology \(H^*\) only; probably, it is more convenient for people which work in the area of dynamical systems and are not very familiar with extraordinary cohomology. The main line of the proof follows Rudyak–Oprea [RO], but here we use the strict category weight instead the category weight.

Remarks. 1. Hofer and Zehnder [HZ, p.250] mentioned that the Theorem 2.2 below is true without the restriction \(c_1|_{\pi_2(M)} = 0\). Because of this, the Arnold conjecture turns out to be valid for all closed connected symplectic manifolds with \(\omega|_{\pi_2(M)} = 0\).

2. Actually, in Theorem 2.2 below the number \(\text{Fix} \phi\) is the number of 1-periodic solutions of the equation \((*)\), while \(\text{Rest} \Phi\) is the number of contractible 1-periodic solutions of this equation. So, here (as well as in [R2] and [RO]) it is proved that the number of contractible 1-periodic solutions of \((*)\) is at least \(2n + 1\).

The paper is organized as follows. In §1 we discuss strict category weight, in §2 we use Floer’s results in order to reduce the Arnold conjecture to a certain topological problem, in §3 we prove main results, in Appendix we discuss an analog of the Arnold conjecture for locally Hamiltonian symplectomorphisms.

The cohomology group \(H^n(X; G)\) is always defined to be the Alexander–Spanier cohomology group with coefficient group \(G\), see [M] or [S] for the details.

We reserve the term “map” for continuous functions.

“Connected” always means path connected.

§1. STRICT CATEGORY WEIGHT

1.1. Definition ([LS], [Fox], [F], [BG]). Given a map \(\varphi: A \to X\), we define the Lusternik–Schnirelmann category \(\text{cat} \varphi\) of \(\varphi\) to be the minimal number \(k\) with the following property: \(A\) can be covered by open sets \(A_1, \ldots, A_{k+1}\) such that \(\varphi|A_i\) is null-homotopic for every \(i\). Furthermore, we define the Lusternik–Schnirelmann category \(\text{cat} X\) of a space \(X\) by setting \(\text{cat} X := \text{cat} 1_X\).

1.2. Proposition ([BG]). (i) For every diagram \(A \xrightarrow{\varphi} Y \xrightarrow{f} X\) we have \(\text{cat} f \varphi \leq \min\{\text{cat} \varphi, \text{cat} f\}\). In particular, \(\text{cat} f \leq \min\{\text{cat} X, \text{cat} Y\}\).

(ii) If \(\varphi \simeq \psi: A \to X\) then \(\text{cat} \varphi = \text{cat} \psi\).

(iii) If \(h: Y \to X\) is a homotopy equivalence then \(\text{cat} \varphi = \text{cat} h \varphi\) for every \(\varphi: A \to X\). □

Given a connected pointed space \(X\), let \(e: S\Omega X \to X\) be the map adjoint to \(1\), here \(\Omega X\) is the loop space of \(X\) and \(S\) denotes the suspension, see e.g. [Sw].
1.3. **Theorem** ([Sv, Theorems 3, 19' and 21]). Let \( \varphi : A \to X \) be a map of connected Hausdorff paracompact spaces. Then \( \text{cat} \varphi < 2 \) iff there is a map \( \psi : A \to S\Omega X \) such that \( \varepsilon \psi = \varphi \). □

1.4. **Definition** ([R1]). Let \( X \) be a Hausdorff paracompact space, and let \( u \in H^q(X;G) \) be an arbitrary element. We define the **strict category weight** of \( u \) (denoted by \( \text{swgt} u \)) by setting

\[
\text{swgt} u = \sup\{k \mid \varphi^* u = 0 \text{ for every map } \varphi : A \to X \text{ with } \text{cat} \varphi < k\}
\]

where \( A \) runs over all Hausdorff paracompact spaces.

We use the term “strict category weight”, since the term “category weight” is already used (introduced) by Fadell–Hussein [FH]. Concerning the relation between category weight and strict category weight, see [R1].

We remark that \( \text{swgt} u = \infty \) if \( u = 0 \).

1.5. **Theorem.** Let \( X \) and \( Y \) be two Hausdorff paracompact spaces. Then for every \( u \in H^*(X) \) the following hold:

(i) for every map \( f : Y \to X \) we have \( \text{cat} f \geq \text{swgt} u \) provided \( f^* u \neq 0 \). Furthermore, if \( X \) is connected then \( \text{swgt} u \geq 1 \) whenever \( u \in H^*(X) \);

(ii) for every map \( f : Y \to X \) we have \( \text{swgt} f^* u \geq \text{swgt} u \);

(iii) for every \( u, v \in H^*(X) \) we have \( \text{swgt}(uv) \geq \text{swgt} u + \text{swgt} v \).

**Proof.** (i) This follows from the definition of \( \text{swgt} \).

(ii) This follows from 1.2(i).

(iii) Let \( \text{swgt} u = k \), \( \text{swgt} v = l \) with \( k, l < \infty \). Given \( f : A \to X \) with \( \text{cat} f < k+l \), we prove that \( f^*(uv) = 0 \). Indeed, \( \text{cat} f < k+l \), and so \( A = A_1 \cup \cdots \cup A_{k+l} \) where each \( A_i \) is open in \( A \) and \( f|A_i \) is null-homotopic. Set \( B = A_1 \cup \cdots \cup A_{k} \) and \( C = A_{k+1} \cup \cdots \cup A_{k+l} \). Then \( \text{cat} f|B < k \) and \( \text{cat} f|C < l \). Hence \( f^* u|B = 0 = f^* v|C \), and so \( f^*(uv)|(B \cup C) = 0 \), i.e., \( f^*(uv) = 0 \).

The case of infinite category weight is leaved to the reader. □

§2. **Floer’s reduction and related things.**

2.1. **Recollection.** A flow on a topological space \( X \) is a family \( \Phi = \{\varphi_t\}, t \in \mathbb{R} \) where each \( \varphi_t : X \to X \) is a self-homeomorphism and \( \varphi_s \varphi_t = \varphi_{s+t} \) for every \( s, t \in \mathbb{R} \) (notice that this implies \( \varphi_0 = 1_X \)).

A flow is called continuous if the function \( X \times \mathbb{R} \to X, (x,t) \mapsto \varphi_t(x) \) is continuous.

A point \( x \in X \) is called a rest point of \( \Phi \) if \( \varphi_t(x) = x \) for every \( t \in \mathbb{R} \). We denote by \( \text{Rest} \Phi \) the number of rest points of \( \Phi \).

A continuous flow \( \Phi = \{\varphi_t\} \) is called gradient-like if there exists a continuous (Lyapunov) function \( F : X \to \mathbb{R} \) with the following property: for every \( x \in X \) we have \( F(\varphi_t(x)) < F(\varphi_s(x)) \) whenever \( t > s \) and \( x \) is not a rest point of \( \Phi \).

The following Theorem can be found in [Fl, Th. 7], cf., also [HZ].
2.2. Theorem. Let \((M, \omega)\) be a closed connected symplectic manifold such that 
\(\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}\), and let \(\phi : M \to M\) be a Hamiltonian symplectomorphism. Then there exists a map \(f : X \to M\) with the following properties:

(i) \(X\) is a compact metric space;
(ii) \(X\) possesses a continuous gradient-like flow \(\Phi\) such that \(\text{Rest } \Phi \leq \text{Fix } \phi\);
(iii) The homomorphism \(f^* : H^n(M; G) \to H^n(X; G)\) is a monomorphism for every coefficient group \(G\). □

The following theorem (of Lusternik–Schnirelmann type) is proved in [R2].

2.3. Theorem. Let \(\Phi\) be a continuous gradient-like flow on a compact metric space \(X\), let \(Y\) be a Hausdorff space which can be covered by open and contractible in \(Y\) subspaces, and let \(f : X \to Y\) be a map. Then

\[\text{Rest } \Phi \geq 1 + \text{cat } f.\] □

We need also the following well-known fact which follows from [LS] and [T].

2.4. Theorem. For every closed smooth manifold \(M\) we have

\[1 + \text{cat } M \leq \text{Crit } M \leq 1 + \text{dim } M.\] □

§3. PROOF OF THE ARNOLD CONJECTURE

3.1. Theorem (cf. [FH], [RO], [St]). Let \(\pi\) be a discrete group. Then for every \(u \in H^k(K(\pi, 1); G)\) with \(k > 1\) we have \(\text{swgt } u \geq 2\).

(Actually, Strom [St] proved that \(\text{swgt } u \geq k\). Moreover, it is easy to see that \(\text{swgt } u \leq k\) provided \(u \neq 0\), and so \(\text{swgt } u = k\) if \(u \neq 0\).)

Proof. Because of 1.3, it suffices to prove that \(\varepsilon^* u = 0\) where \(\varepsilon\) is a map from 1.3 and \(\varepsilon^*: H^*(K(\pi, 1); G) \to H^*(S\Omega K(\pi, 1); G)\) is the induced homomorphism. But \(\Omega K(\pi, 1)\) is homotopy equivalent to the discrete space \(\pi\), and so \(S\Omega K(\pi, 1)\) is homotopy equivalent to a wedge of circles. Hence, \(H^i(K(\pi, 1); G) = 0\) for \(k > 1\), and thus \(\varepsilon^* u = 0\). □

3.2. Theorem (cf. [RO]). Let \(Y\) be a connected finite CW-space, and let \(y \in H^2(Y; G)\) be such that \(y|_{\pi_2(Y)} = 0\). Then \(\text{swgt } y \geq 2\).

Proof. Let \(\pi = \pi_1(Y)\), and let \(g : Y \to K(\pi, 1)\) be a map which induces an isomorphism of fundamental groups. First, we prove that

\[y \in \text{Im}(g^*: H^2(K(\pi, 1); R) \to H^2(Y; R)).\]

Indeed, since \(Y\) is a finite CW-space, its singular cohomology coincides with \(H^*\), and so we have the universal coefficient sequence

\[0 \to \text{Ext}(H_1(Y), R) \to H^2(Y; R) \xrightarrow{i} \text{Hom}(H_2(Y), R) \to 0.\]

On the other hand, there is a Hopf exact sequence

\[\pi_1(Y) \to H_1(Y) \to H_1(K(\pi, 1)) \to 0.\]
and so we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{cccccc}
\text{Ext}(H_1(K), R) & \longrightarrow & H^2(K; R) & \longrightarrow & \text{Hom}(H_2(K), R) & \longrightarrow & 0 \\
g' \downarrow \cong & & \downarrow g^* & & \downarrow g'' & & \\
\text{Ext}(H_1(Y), R) & \longrightarrow & H^2(Y; R) & \longrightarrow & \text{Hom}(H_2(Y), R) & \longrightarrow & 0 \\
& & \downarrow & & \text{Hom}(\pi_2(Y), R)
\end{array}
$$

where $K$ denotes $K(\pi, 1)$. Now, since $y|_{\pi_2(Y)} = 0$, we conclude that $l(y) \in \text{Im } g''$. Since $g'$ is an isomorphism, an easy diagram hunting shows that $y \in \text{Im } g^*$.

Thus, by 3.1 and 1.5(ii), swgt $y \geq 2$.

3.3. Corollary. Let $Y$ be a connected finite CW-space, let $R$ be a commutative ring, let $y \in H^2(Y; R)$ be such that $y|_{\pi_2(Y)} = 0$, and let $X$ be a Hausdorff paracompact space. If $f : X \to Y$ is a map with $f^*(y^n) \neq 0$, then $\text{cat } f \geq 2n$.

Proof. By 1.5, $\text{cat } f \geq \text{swgt } y^n \geq n \text{swgt } y \geq 2n$.

3.4. Corollary ([RO]). Let $(M^{2n}, \omega)$ be a closed connected symplectic manifold with $\omega|_{\pi_2(M)} = 0$. Then $\text{cat } M = 2n$ and $\text{Crit } M = 2n + 1$.

Proof. Since $\omega^n \neq 0$, we conclude that, by 3.3, $\text{cat } M = 1_M \geq 2n$. So, $\text{cat } M = 2n$ since $\text{cat } M \leq \dim M$. The second equality follows from 2.4.

3.5. Corollary. Let $(M^{2n}, \omega)$ be a closed symplectic manifold with $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism. Then $\text{Fix } \phi \geq 2n + 1$. In particular, the Arnold conjecture holds for $(M, \omega)$.

Proof. Let $f : X \to M$ and $\Phi$ be a map as in 2.2. Since every closed connected smooth manifold is a finite polyhedron, and since $\omega^n$ yields a non-trivial cohomology class in $H^*(X; \mathbb{R})$, we conclude that, by 3.3, $\text{cat } f \geq 2n$. Now, by 2.2 and 2.3,

$$
\text{Fix } \phi \geq \text{Rest } \Phi \geq 1 + \text{cat } f \geq 2n + 1.
$$

Thus, because of 3.4, the Arnold conjecture holds for $(M, \omega)$.

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