On the sum of squares of the coefficients of Bloch functions

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Abstract

In this article several types of inequalities for weighted sums of the moduli of Taylor coefficients for Bloch functions are proved.

1 Introduction.

It was one of the most famous items in function theory, when A. Bloch proved in the following theorem

**Theorem A.** Let \( F \) be analytic in the unit disc \( D \) and \( F'(0) = 1 \). Let further \( B_F \) be the supremum of all numbers \( r > 0 \) with the following property. There exist a complex number \( a \) and a domain \( \Omega \subset D \) such that \( F \) is injective on \( \Omega \) and \( F(\Omega) = \{ z \mid |z - a| < r \} \). The infimum of these numbers \( B_F \) is positive.

Since then, this infimum has been called Bloch’s constant \( B \). In other words, the range of any function of the above type covers a schlicht disc with radius \( B \). The exact value of Bloch’s constant is not known.

In [12], E. Landau proved that it suffices for estimations of \( B \) to consider only those functions that satisfy in addition to the above conditions the inequality

\[
|F'(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in D.
\]  

This was the beginning of research on the class of Bloch functions that are analytic in \( D \) and satisfy the condition

\[
\sup_{z \in D}(1 - |z|^2)|F'(z)| < \infty.
\]

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This class becomes a normed vector space, the so called Bloch space, if it is endowed with the Bloch norm

$$\| F \| := |F(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|F'(z)|.$$  

Ch. Pommerenke and his co-authors proved a number of important theorems on the Bloch space (compare for example [1] and [13]), among them some theorems on the behaviour of the coefficients $b_k, k \to \infty$, in the expansions

$$F(z) = \sum_{k=0}^{\infty} b_k z^k,$$

where $F$ is in the Bloch space.

Another area of research was related more closely to the condition (1). It concerned the class

$$\mathcal{B} := \{ F \mid |F'(z)| \leq \frac{1}{1 - |z|^2}, z \in \mathbb{D} \}$$

and the behaviour of its elements. In [14] the following problem was posed

**Problem N:** (see [14]) If $f$ is analytic in the unit disc $\mathbb{D}$ and

$$|f(re^{i\theta})| < \frac{1}{1 - r^2},$$

for what positive real numbers $\alpha$, is it true that

$$|f'(re^{i\theta})| < \frac{\alpha}{(1 - r^2)^2}?$$

It is known that $\alpha = 4$ will suffice.

What is the coefficient region for this class?

The first part of this problem was solved in [19] and [2] with different methods. Partial answers to the second question can be found in [19], [4], [6], [16], and [18]. In the following we shall use

**Theorem B.** (see [19] and [4]) Let $F \in \mathcal{B}$ and $x \in [\sqrt{3}/2, \sqrt{3}]$ such that

$$|b_1| = \frac{3\sqrt{3}}{2}x(1 - x^2).$$

Then this equation and the inequality

$$|b_2| \leq \frac{3\sqrt{3}}{4}(1 - 3x^2)(1 - x^2)$$

describe the coefficient region $\{(b_1, b_2) \mid F \in \mathcal{B}\}$.

In [4] and [6] an intimate relation between knowledges on this coefficient region of $\mathcal{B}$ and calculations of estimates for Bloch’s constant is revealed.
Another way to get information on the coefficient region of $B$ consists in the consideration of weighted sums of moduli of Taylor coefficients. One example for an estimate of this type is the use of Parseval’s formula (see f. i. [13])

$$
\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k-2} = \frac{1}{2\pi} \int_0^{2\pi} \left| F' \left( r e^{i\theta} \right) \right|^2 d\theta \leq \frac{1}{(1-r^2)^2}, \quad r \in (0, 1). \quad (2)
$$

Another inequality of this type can easily proved using the maximum principle (compare [17]).

**Proposition 1.** Let $F \in B$. Then

$$
\sum_{k=n+1}^{\infty} k^2 |b_k|^2 r^{2k-2} \leq \frac{(n+2)^{n+2}}{4n^n} r^{2n}, \quad 0 \leq r \leq \sqrt{\frac{n}{n+2}} = r_n.
$$

In this estimate, equality is attained for

$$
F_n'(z) = \frac{n+2}{2} \left( \frac{n+2}{n} \right)^{\frac{n}{2}} z^n.
$$

**Proof.** Let us set

$$
\Phi_n(z) = \sum_{k=n+1}^{\infty} k^2 |b_k|^2 z^{2k-2}.
$$

From (2) it follows that

$$
|\Phi_n(z)| \leq \frac{1}{(1-r^2)^2} = \frac{(n+2)^{n+2}}{4n^n} r^{2n}, \quad |z| = r = r_n.
$$

Consequently, by the maximum principle

$$
\left| \frac{\Phi_n(z)}{z^{2n}} \right| \leq \frac{(n+2)^{n+2}}{4n^n}, \quad |z| \leq r_n.
$$

Setting $z = r \leq r_n$ we prove Proposition 1.

**Remark 1.** If Proposition 1 is compared with (2), it is obvious that in (2) equality is attained for $r = r_n$ and $F = F_n$. It is an open question whether there exist other values of $r$ such that in (2) occurs equality.

Inequalities of different type for weighted sums can be found in [11].

In the paper [8], another application of such sums was revealed. Namely, it was shown that the function

$$
\Phi(r) := \sup_{F \in B} (1-r^2) \sum_{k=1}^{\infty} k |b_k|^2 r^{2(k-1)}
$$
is a decreasing function in \( r \) and \( \Phi(0.4) < 0.9 \). Moreover, in the cited paper it was demonstrated that

\[
\sigma_F^2 \leq \Phi(r), \quad r \in [0, 1),
\]

where

\[
\sigma_F^2 = \limsup_{r \to 1} \frac{1}{2\pi|\log(1-r)|} \int_{|z|=r} |F(z)|^2 |dz|
\]

is the asymptotic variance of the Bloch function \( F \). The asymptotic variance plays an important role in the probabilistic behaviour of the Bloch function \( F \). Namely, Makarov’s law of the iterated logarithm (see [10], and also [8], [9]) asserts that

\[
\limsup_{r \to 1} \frac{|F'(re^{i\theta})|}{\sqrt{\log \log \log \frac{1}{r}}} \leq \sup_{F \in B} \sqrt{\sigma_F^2} \quad \text{for almost all } \theta \in [0, 2\pi).
\]

The present paper is dedicated to further estimates that generalize and sharpen some of the above ones.

2 Statement and proofs of the results.

**Theorem 1** Let \( F(z) = \sum_{k=0}^{\infty} b_k z^k \in B, b_1 = a \in (0, 1), \)

\[
a = \frac{3\sqrt{3}}{2} x(1-x^2), \quad x \in (0, 1/\sqrt{3}),
\]

\[
0 < r \leq \sqrt{\frac{1/3 - x}{1 - x\sqrt{1/3}}}. \quad (3)
\]

Then the sharp inequalities

\[
\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k} \leq \frac{27r^2(1-x^2)^2((r^2+x^2)(1-r^2x^2)^2 - 6r^2x^2(1-x^2)(1-r^2))}{4(1-r^2x^2)^5}, \quad (4)
\]

and

\[
\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq \frac{27r^2(1-x^2)^2(3x^2(1-r^2)^2 + (1-x^2r^2)(r^2-x^2))}{8(1-r^2x^2)^4} \quad (5)
\]

are valid.
Proof. Using the rotation $e^{-i\theta}F(e^{i\theta}z)$ it is easy to check that
\[
\{ F'(z) \mid |z| = r, F \text{ as above} \} = \{ F'(r) \mid F \text{ as above} \}.
\]
Following [4], Satz 2.2.1, this set is given by the bounded region whose boundary is the Jordan curve
\[
\{ G'(re^{i\phi}) \mid \phi \in [0, 2\pi] \}
\]
where
\[
G'(z) = -\frac{a}{x} \frac{z - x}{(1 - zx)^3} = \sum_{k=1}^{\infty} kA_k z^{k-1}
\]
\[
= \frac{a}{2} \sum_{k=1}^{\infty} kx^{k-3} (2x^2 + (k - 1)(x^2 - 1)) z^{k-1}.
\]

From [4], Lemma 2.3.3 and Lemma 2.3.5 it follows that $G'(rz)$ maps the closed unit disc $\overline{D}$ univalently. Since $F'(0) = G'(0)$ and $F'(r\overline{D}) \subset G'(r\overline{D})$, this means that $F'(rz)$ is subordinate to $G'(rz)$. According to a famous Lemma of Rogosinski (see [15]), this implies
\[
\sum_{k=1}^{n} |b_k|^2 \rho^{2(k-1)} \leq \sum_{k=1}^{n} |A_k|^2 \rho^{2(k-1)}, \rho \in [0, r].
\]
The limiting process $n \to \infty$ and a straightforward calculation delivers (4). If we let $u = \rho^2$ and integrate this inequality with respect to $u$ from 0 to $r^2$, we get the assertion (5).

Remark 2. If we assume $F'(0) = ea$, $|e| = 1$, we may apply the above reasoning to $\tau F(z)$ and $\tau F'(z)$ and we see that (4) and (5) are valid likewise.

Remark 3. Another subordination theorem for Bloch functions may be found in [5].

Theorem 2 Let $F \in B$. Then the sharp inequality
\[
\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k} \leq \frac{27}{4} r^4, \quad \sqrt{4/15} \leq r \leq \sqrt{1/3},
\]
is valid.

Proof. If we let $r = \frac{1}{\sqrt{3}}$ in (2), we get
\[
\sum_{k=3}^{\infty} k^2 |b_k|^2 \frac{1}{3^{k-1}} \leq \frac{9}{4} - |b_1|^2 - (4/3)|b_2|^2.
\]
According to Remark 1, this inequality is sharp. From here we see that
\[
\sum_{k=3}^{\infty} k^2 |b_k|^2 r^{2(k-1)} \leq 9r^4 \left( \frac{9}{4} - |b_1|^2 - (4/3)|b_2|^2 \right).
\]
and consequently
\[
\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2(k-1)} \leq |b_1|^2 + 4|b_2|^2 r^2 + 9r^4 \left( \frac{9}{4} - |b_1|^2 - (4/3)|b_2|^2 \right) = (1 - 9r^4)|b_1|^2 + (4r^2 - 12r^4)|b_2|^2 + \frac{81r^4}{4} \leq \frac{27r^2}{4}, \quad r \geq \sqrt{4/15}.
\]
From Theorem B we know that it is sufficient to verify this inequality in the case when
\[
b_1 = \frac{3\sqrt{3}}{2} x(1 - x^2), \quad x \in (0, 1/\sqrt{3}),
\]
and
\[
b_2 = \frac{3\sqrt{3}}{4} (1 - x^2)(1 - 3x^2).
\]
We have
\[
(1 - 9r^4)|b_1|^2 + (4r^2 - 12r^4)|b_2|^2 + \frac{81r^4}{4} - \frac{27r^2}{4} = (27/4)(1 - 3x^2)x^2(1 - 2x^2 + x^4 + r^2(-5 + 16x^2 - 21x^4 + 9x^6)).
\]
It remains to show that
\[
1 - 2x^2 + x^4 + r^2(-5 + 16x^2 - 21x^4 + 9x^6) \leq 0, \quad 1/\sqrt{3} \geq r \geq \sqrt{4/15}.
\]
Obviously, the first term 1 - 2x^2 + x^4 is positive, whereas the second one is negative for x \in (0, 1/\sqrt{3}). Hence, it is evidently enough to verify this inequality at r = \sqrt{4/15}. In this case we have
\[
1 - 2x^2 + x^4 + r^2(-5 + 16x^2 - 21x^4 + 9x^6) = (1/15)(-1 + 3x^2)(-5 + 4x^2) \leq 0.
\]
Theorem 2 is proved.

**Problem 1.** Which is the biggest interval \([c_1, \frac{1}{\sqrt{3}}]\), c_1 > 0, such that the inequality of Theorem 2 remains valid in this interval?

If one adds to the inequality (1) the slightly stronger inequality
\[
|b_1| + \sum_{k=2}^{\infty} k b_k z^{k-1} \leq \frac{1}{1 - |z|^2}, \quad |z| = 1/\sqrt{3}, \quad (6)
\]
it is possible to improve the length of the interval from Theorem 2.
Theorem 3 Suppose that a function $F \in B$ satisfies (7). Then the sharp inequality

$$\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k} \leq \frac{27}{4} r^4, \quad \sqrt{(9 - \sqrt{65})/6} \leq r \leq \sqrt{1/3},$$

is valid.

Proof. We have

$$\sum_{k=2}^{\infty} k^2 |b_k|^2 \frac{1}{3^{k-1}} \leq \left(\frac{3}{2} - |b_1|\right)^2$$

and hence

$$\sum_{k=3}^{\infty} k^2 |b_k|^2 \frac{1}{3^{k-1}} \leq \left(\frac{3}{2} - |b_1|\right)^2 - (4/3)|b_2|^2.$$ 

From here we see that

$$\sum_{k=3}^{\infty} k^2 |b_k|^2 r^{2(k-1)} \leq 9r^4 \left[ \left(\frac{3}{2} - |b_1|\right)^2 - (4/3)|b_2|^2 \right].$$

and consequently

$$\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2(k-1)} \leq |b_1|^2 + 4|b_2|^2 r^2 + 9r^4 \left[ \left(\frac{3}{2} - |b_1|\right)^2 - (4/3)|b_2|^2 \right] \leq \frac{27r^2}{4}.$$ 

In view of Theorem B it is enough to verify this inequality in the case when

$$b_1 = \frac{3\sqrt{3}}{2} x(1-x^2), \quad x \in (0, 1/\sqrt{3}),$$

and

$$b_2 = \frac{3\sqrt{3}}{4} (1-x^2)(1-3x^2).$$

Also we may suppose that $r = \sqrt{(9 - \sqrt{65})/6}$. We have

$$|b_1|^2 + 4|b_2|^2 r^2 + 9r^4 \left[ \left(\frac{3}{2} - |b_1|\right)^2 - (4/3)|b_2|^2 \right] - \frac{27r^2}{4} =$$

$$-\frac{9x}{8} (2\sqrt{3}(73 - 9\sqrt{65}) + (-737 + 91\sqrt{65})x + 2\sqrt{3}(-73 + 9\sqrt{65})x^2 +$$

$$(1858 - 230\sqrt{65})x^3 + 3(-587 + 73\sqrt{65})x^5 - 72(-8 + \sqrt{65})x^7) =$$

$$(1-\sqrt{3}x)^2 \left(\frac{9}{4} \sqrt{3}(-73 + 9\sqrt{65})x + \frac{9}{8}(-139 + 17\sqrt{65})x^2 - \frac{9}{8} \sqrt{3}(-153 + 19\sqrt{65})x^3 -$$
\[-\frac{9}{8}(-395 + 49\sqrt{65})x^4 + 18\sqrt{3}(-8 + \sqrt{65})x^5 + 27(-8 + \sqrt{65})x^6 \] 
\[= (1 - \sqrt{3}x)(-1.71348 \ldots x - 2.18432 \ldots x^2 - 0.712771 \ldots x^3 - 
-0.0569854 \ldots x^4 + 1.941 \ldots x^5 + 1.68096 \ldots x^6). \]

From here we see that negative coefficients are dominating. Consequently, this polynomial is negative in the interval \((0, 1/\sqrt{3})\). Theorem 3 is proved.

**Problem 2.** Which is the biggest interval \([c_2, \frac{1}{\sqrt{3}}]\), \(c_2 > 0\), such that the inequality of Theorem 3 remains valid in this interval?

Now we are going to study the behavior of the area functional

\[\sum_{k=1}^{\infty} k|b_k|^2 r^{2k}.\]

A simple integration of the inequality (2) gives us the inequality

\[\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq \frac{r^2}{1 - r^2}\]

which is, unfortunately, not sharp for all \(r \in (0, 1)\).

**Theorem 4** Let \(F \in \mathcal{B}\) be as in Theorem 1 and let \(r \leq \frac{1}{\sqrt{3}}\).

Then for any \(n \in \mathbb{N} \setminus \{1\}\) the inequality

\[\sum_{k=2}^{n} k|b_k|^2 r^{2k} \leq \frac{3(9 - 4a^2)^2}{64a^4} \sum_{k=2}^{n} \frac{1}{k} \left(\frac{4a^2 r^2}{3}\right)^k \]

is valid.

**Proof.** Since \(|F'(z)| \leq \frac{\frac{1}{1-|z|^2}}{1-|z|^2} \leq \frac{a}{2}\) for \(r \leq \frac{1}{\sqrt{3}}\), the image of the unit disc under \(F'(z/\sqrt{3})\) lies in the disc with radius \(3/2\) around the origin. Hence, this function is subordinate to the function

\[H(z) = \frac{3z + \frac{2a}{3}}{2 + \frac{2a}{3}z}.\]

As

\[F'\left(\frac{z}{\sqrt{3}}\right) = a + \sum_{k=2}^{\infty} k b_k \left(\frac{z}{\sqrt{3}}\right)^{k-1}\]

and

\[H(z) = a + \sum_{k=2}^{\infty} \frac{3}{2} \frac{9 - 4a^2}{9} \left(\frac{-\frac{2a}{3}}{3}\right)^{k-2} z^{k-1},\]
we get from Rogosinski’s theorem (see again [15])

\[
\sum_{k=2}^{n} k^2 |b_k|^2 \frac{1}{3^{k-1}} \leq \frac{9(9-4a^2)^2}{64a^4} \sum_{k=2}^{n} \left( \frac{4a^2}{9} \right)^k .
\]

Now we summands of both sums by \(\lambda_k = \frac{(3r^2)^k}{k} \). Since this is a decreasing sequence of nonegative numbers, using properties of the Abel transformation (see also [7], Theorem 6.3), we get

\[
3 \sum_{k=2}^{n} k |b_k|^2 r^{2k} \leq \frac{9(9-4a^2)^2}{64a^4} \sum_{k=2}^{n} \left( \frac{4a^2}{9} \right)^k r^{2k} .
\]

This results in the inequality [7].

**Corollary 1.** The limiting process \(n \to \infty\) in this theorem results in the inequality

\[
\sum_{k=2}^{\infty} k |b_k|^2 r^{2k} \leq \frac{3(9-4a^2)^2}{64a^4} \left( -\log \left( 1 - \frac{4a^2 r^2}{3} \right) - \frac{4a^2 r^2}{3} \right) = B_a(r),
\]

where \(r \leq \frac{1}{\sqrt{3}}\).

An immediate consequence of Corollary 1 is

**Corollary 2.** For \(F \in \mathcal{B}, r \leq \frac{1}{\sqrt{3}},\) the inequality

\[
\sum_{k=2}^{\infty} k |b_k|^2 r^{2k} \leq \frac{27r^4}{8}
\]

is valid.

**Proof.** We have to prove that

\[
B_a(r) - \frac{27r^4}{8} \leq 0, \quad r \leq \frac{1}{\sqrt{3}}.
\]

To abbreviate the calculations, we let \(w = \frac{4a^2 r^2}{3}\). With this abbreviation the inequality to prove is the following

\[
H_a(w) = \left( 1 - \frac{4a^2}{9} \right)^2 \left( -\log(1-w) - w \right) - \frac{w^2}{2} \leq 0, \quad w \in \left[ \frac{4a^2}{9}, 0 \right] .
\]

Since \(H_a(0) = 0, H'_a(0) = 0, H'_a(w) < 0\) for sufficiently small positive values of \(w\), and \(H'\) has only one positive zero, it is sufficient to prove the above
inequality for \( w = \frac{4a^2}{9} \). If we let \( v = \frac{4a^2}{9} \), this task reduces to the proof of the inequality

\[
- \log(1 - v) - v - \frac{v^2}{2(1 - v)^2} \leq 0, \quad v \in \left[0, \frac{4}{9}\right].
\]

This proof can be done by elementary calculations.

**Remark 4:** Corollary 2 follows immediately from the case \( n = 1 \) of Proposition 1 using \( k \leq k^2/2 \) for \( k \geq 2 \).

**Theorem 5** Let \( F \in \mathcal{B} \). For

\[
R = \frac{1}{4\sqrt{3}} \sqrt{59 - \sqrt{2713}} \leq r \leq \frac{1}{\sqrt{3}}
\]

the following sharp inequality holds

\[
(1 - |b_1|^2) \sum_{k=1}^{\infty} k |b_k|^2 r^{2k} \leq \frac{27}{8} r^4.
\]

Proof. At first let us remark it is enough to prove the theorem for the cases \( r = R \) and \( r = \sqrt{1/3} \). Indeed, for the analytic function

\[
\Psi(z) = (1 - |b_1|^2) \sum_{k=1}^{\infty} k |b_k|^2 z^{2k}
\]

we have

\[
|\Psi(z)/z^4| \leq \frac{27}{8}
\]

for \( r = R \) and \( r = \sqrt{1/3} \) and therefore the inequality holds inside the ring \( R \leq r \leq \sqrt{1/3} \).

It remains to prove the Theorem 5 in the case \( r = R \).

We set \( r = R \) and consider three cases (we use these there cases due to technical reasons only, probably there exists a shorter proof).

**Case 1:** \( a \leq 3/5 \). Remember that \( a = \frac{3\sqrt{7}}{2} x (1 - x^2) \). From here we see that \( x \leq 0.287 \) so that \( (1/\sqrt{3} - x)/(1 - x/\sqrt{3}) \geq 0.38 \geq R \) and we can apply Theorem 1. In view of Theorem 1 we shall show that

\[
(1 - a^2) \frac{27R^2(1 - x^2)^2(2x^2 + R^2(1 + 2(-3 + R^2)x^2 + x^4))}{8(1 - R^2x^2)^4} - (27/8)R^4 \leq 0
\]

for \( x \leq 1/4 \). We have

\[
(1 - a^2) \frac{27R^2(1 - x^2)^2(2x^2 + R^2(1 + 2(-3 + R^2)x^2 + x^4))}{8(1 - R^2x^2)^4} - (27/8)R^4 =
\]
\[-\frac{27R^2x^4}{32(1 - R^2x^2)^4}(24.5695 \ldots - 103.49 \ldots x^2 + 159.036 \ldots x^4 -
- 99.926 \ldots x^6 + 16.2316 \ldots x^8 + 3.88886 \ldots x^{10})\].

From here we see that the positive coefficients are dominating for \(x \leq 1/4\) and consequently this expression is negative.

**Case 2:** \(3/5 < a \leq 3/4\). In this case from Corollary 1 it follows that

\[
\sum_{k=2}^{\infty} k|b_k|^2r^{2k} \leq \frac{3(9 - 4a^2)^2}{64a^4} \frac{16a^4}{9} \left( \log \frac{1}{1-r^2} - r^2 \right) = \\
\frac{(9 - 4a^2)^2}{12} \left( \log \frac{1}{1-r^2} - r^2 \right).
\]

Consequently

\[
(1 - a^2) \sum_{k=1}^{\infty} k|b_k|^2r^{2k} \leq (1 - a^2) \left( a^2r^2 + \frac{(9 - 4a^2)^2}{12} \left( \log \frac{1}{1-r^2} - r^2 \right) \right).
\]

A little calculus shows that maximum of the last expression is attained at the point \(a = 3/4\) and it is less than \(27r^4/8\).

**Case 3:** \(a \geq 3/4\).

According to Corollary 2 the following inequality holds

\[
\sum_{k=2}^{\infty} k|b_k|^2r^{2k} \leq \frac{27}{8}r^4.
\] (8)

From (8) it follows that

\[
(1 - |b_1|^2) \sum_{k=1}^{\infty} k|b_k|^2r^{2k} \leq \max_{3/4 \leq a \leq 1} (1 - a^2) \left( a^2r^2 + \frac{27}{8}r^4 \right) \leq \frac{27}{8}r^4.
\] (9)

**Remark 5.** Routine and straightforward calculations show that the number \(R\) in Theorem 3 cannot be improved.

**Problem 4.** How can one get inequalities analogous to the above ones in the intervals \([r_n, r_{n+1}]\), \(n \geq 1\)?

**Remark 6.** Computer experiments suggest that the inequality (3) can be replaced by \(r \leq \sqrt{1/3}\). If it is so then simple but routine calculations show that the lower bound \(r = \sqrt{4/15} = 0.5163\ldots\) from Theorem 2 can be replaced by the sharp number \(r = \sqrt{\rho} = 0.39466\ldots\) where \(\rho\) is the positive root of the equation

\[-4 - y + 81y^2 + 642y^3 - 564y^4 + 1188y^5 - 82y^6 - 5809y^7 + 4581y^8 = 0.\]
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