FUNCTIONAL INEQUALITIES
FOR SOME GENERALISED MEHLER SEMIGROUPS

LUCIANA ANGIULI, SIMONE FERRARI, DIEGO PALLARA

Abstract. We consider generalised Mehler semigroups and, assuming the existence of an associated invariant measure \( \sigma \), we prove functional integral inequalities with respect to \( \sigma \), such as logarithmic Sobolev and Poincaré type. Consequently, some integrability properties of exponential functions with respect to \( \sigma \) are deduced.

1. Introduction

Generalised Mehler semigroups are defined for real-valued, bounded and Borel measurable functions \( f : X \to \mathbb{R} \), i.e. \( f \in B_b(X) \), by the formula

\[
(P_t f)(x) = \int_X f(T_t x + y) \mu_t(dy),
\]

where \( X \) is a (finite or infinite dimensional) Banach space, \((T_t)_{t \geq 0}\) is a strongly continuous semigroup of bounded operators on \( X \) and \((\mu_t)_{t \geq 0}\) is a family of Borel probability measures on \( X \) verifying \( \mu_0 = \delta_0 \) and \( \mu_{t+s} = (\mu_t \circ T_s^{-1}) \ast \mu_s \) for any \( s, t \geq 0 \). The semigroup (1.1) is related to the stochastic differential equation

\[
\begin{cases}
   dZ(t) = AZ(t)dt + dY(t), & t > 0; \\
   Z(0) = x \in X;
\end{cases}
\]

where \( A : D(A) \subseteq X \to X \) is the infinitesimal generator of \( T_t \) and \( Y(t) \) is a Lévy process in \( X \), i.e., a stochastic process with càdlàg trajectories starting at 0 and having stationary and independent increments. For \( \xi \in X^* \), \( t > 0 \) we have \( \mathbb{E}[\xi Y(t)] = \exp(-\langle \lambda(\xi) \rangle) \) and \( \mu_t \) is defined through its characteristic function \( \hat{\mu}_t(\xi) = \exp(-\int_0^t \lambda(T_s^* \xi) ds) \), see [12]. By the Lévy-Khinchine theorem, the function \( \lambda \) is determined by its characteristics \([b, Q, M]\) with \( b \in X \), \( Q \) is a nonnegative definite symmetric trace-class operator on \( X \) and \( M \) is a Lévy measure, see (2.5) below. The semigroup \( P_t \) is related to (1.2) by

\[
P_t f(x) = \mathbb{E}[f(Z(t, x))], \quad t \geq 0, \ x \in X, \ f \in B_b(X);
\]

where \( Z(t, x) \) is the (mild or weak) solution of (1.2). If \( Y(t) = \sqrt{Q} W(t) \), where \( W(t) \) is a Brownian motion (i.e. \( M \equiv 0 \)), then, setting \( Q_x = \int_0^t T_s Q T_s^* ds \), \( M_t = N(0, Q_t) \) and \( P_t \) is the Ornstein–Uhlenbeck semigroup given by the classical Mehler formula. In this case the trajectories are continuous, whereas in the general case \( Y(t) \) may have jumps giving rise to nonlocal effects. Indeed, the (weak) generator of the semigroup \( P_t \) is in general a nonlocal, or pseudodifferential operator (see [28] and the example in Subsection 6.1) and is given by

\[
\mathcal{L} f(x) = \frac{1}{2} \text{Tr}[Q D^2 f](x) + \langle x, A^* Df(x) \rangle + \int_X [f(x + y) - f(x) - (Df(x), y) \chi_{B_1}(y)] M(dy),
\]

on regular functions. We refer to Section 2 for a more detailed explanation, to [38] for a general introduction to these topics, to [5, 9, 12, 15, 20, 28, 29, 39, 40, 41, 45] as more specific basic references to generalised Mehler semigroups and to the very recent [32] and the reference therein for an updated account on the regularity theory, which we do not discuss here.

Date: January 9, 2022.
2020 Mathematics Subject Classification. 35R15, 47D07, 60J60.
Key words and phrases. Generalised Mehler semigroups, Functional inequalities, Lévy processes.
In this paper we always assume that $X$ is a separable Hilbert space and that there exists a unique invariant measure $\sigma$ associated to $P_t$, keeping the conditions given in [20], and look for functional inequalities with respect to $\sigma$. The most classical ones are the logarithmic Sobolev inequalities coming back to [22, 23] and [19], a theory widely developed in the Wiener case $M \equiv 0$. We refer to [11, 27, 42, 45] and the reference therein, as well as to [1, 41, 45, 46] for more recent results. For the general case little is known and we refer to [11] for a general discussion of functional inequalities related to entropy. In the general case of processes with jumps such estimates are not available, as pointed out e.g. in [11, 11, 15, 16, 21]. Therefore, as done in the quoted papers, we study modifications of such estimates. In particular, we estimate the entropy of positive measurable functions $f$ by the integral of some relative increments of $f$ with respect to the Lévy measure $M$, which is charged to take into account the nonlocal effects. Accordingly, our estimates hold true for positive functions whose infimum is far from 0, see Theorem [5.3]. From these modified logarithmic Sobolev type inequalities we derive Poincaré inequalities on a suitable class of functions and we study the exponential integrability of Lipschitz continuous functions, which in our framework appears to be the natural counterpart of the classical Fernique theorem in a Gaussian context. As further consequences of the basic estimates, comparisons of moments of the measures $M$ and $\sigma$ are provided.

In order to simplify the presentation, we have performed all the computations assuming that $Q = 0$ in the above recalled representation of the function $\lambda$, which amounts to saying that there is no diffusion term in the generator, see [1.3] and [2.4]. We stress that this is not restrictive, because the general case can be recovered by standard arguments. Indeed, at the end of Sections 4, 5, 6 we discuss the adaptation of the proofs and the results presented in each section needed to extend them to the general case. In particular, in Remark [4.9] we describe the new invariant measure and how the entropy estimates must be modified, in Remark [5.4] we sketch how the statement and the proof of Theorem [5.3] must be modified to get exponential integrability with respect to the new invariant measure and in Remark [5.2] we point out that also the examples can easily be generalised to the general case.

The paper is organised as follows. In Section 2 we recall the notation we use and collect the main results on generalised Mehler semigroups concerning the weak generator, the measures $\mu_t$ and the exponential function $\lambda$ that appears in connection to the Lévy process $Y$. In Section 3 we recall a condition ensuring the existence of an invariant measure $\sigma$ for $P_t$, extend the semigroup to the $L^p(X, \sigma)$ spaces and describe its asymptotic behaviour. In Section 4 we prove the main logarithmic Sobolev type integral inequalities, in Section 5 we study the exponential integrability of Lipschitz continuous functions. In particular, we deduce from the estimates in Section 4 an estimate on the size of the tail of the distribution of Lipschitz continuous functions. Finally, in Section 6 some examples of semigroups to which our results apply are presented.

Acknowledgements. S.F. has been partially supported by the OK-INSAID project ARS01-00917. The authors are members of G.N.A.M.P.A. of the Italian Istituto Nazionale di Alta Matematica (INdAM) and have been partially supported by the PRIN 2015 MIUR project 2015233N54. The authors are grateful to Alessandra Lunardi and Enrico Priola for many helpful conversations.

2. Notation and Preliminaries

For any $a, b \in \mathbb{R}$ we set $a \lor b := \max\{a, b\}$ and $a \land b := \min\{a, b\}$. Let $X$ be a real separable Hilbert space, that can be either finite or infinite dimensional, with inner product $\langle \cdot, \cdot \rangle_X$ and associated norm $||X||$. and let $X^*$ be its topological dual. When there is no risk of confusion we drop the $X$ from the symbols. $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra of $X$ and $B_b(X)$ the space of real-valued bounded Borel functions on $X$. We denote $B_1$ the open unit ball centred at the origin in $X$. If $f : X \to \mathbb{R}$ is a Fréchet differentiable function, we denote by $Df$ its Fréchet derivative.

The symbol $\mathcal{L}(X)$ denotes the space of bounded linear operators from $X$ to itself and $I$ denotes the identity operator. The domain of a linear operator $A$ on $X$ is denoted $D(A)$ and its range
Ran(A). An operator \( T \in \mathcal{L}(X) \) is Hilbert–Schmidt if
\[
\sum_{n=1}^{\infty} |T_{en}|^2 < \infty,
\]
for some (hence all) orthonormal basis \( \{e_n \mid n \in \mathbb{N}\} \) of \( X \). An operator \( T \in \mathcal{L}(X) \) is trace-class if it is compact and the series \( \sum_k |\lambda_k| \) of its eigenvalues \( (\lambda_k)_{k \in \mathbb{N}} \), counted with their multiplicity, is convergent.

The Sazonov topology on \( X \) is the topology generated by the family of seminorms \( x \mapsto |Tx| \), where \( T \) ranges over all Hilbert–Schmidt operators on \( X \) and it plays an important role in the definition of Lévy processes and generalised Mehler semigroups. We refer to [10, 35] for an in-depth study of all this notions.

If \( \mu \) and \( \gamma \) are two finite Borel measures on \( X \), we denote by \( \hat{\mu} \) the characteristic function of \( \mu \) and by \( \mu \ast \gamma \) the convolution measure defined by
\[
\hat{\mu}(\xi) := \int_X e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in X^*;
\]
\[
[\mu \ast \gamma](E) := \int_X \mu(E - x) \gamma(dx), \quad E \in \mathcal{B}(X).
\]

If \( T \in \mathcal{L}(X) \) we denote by \( \mu \circ T^{-1} \) the image measure defined as \( (\mu \circ T^{-1})(B) := \mu(T^{-1}(B)) \) for any \( B \in \mathcal{B}(X) \).

A generalised Mehler semigroup on \( X \) is defined by the formula
\[
(P_t f)(x) = \int_X f(T_t x + y) \mu_t(dy), \quad f \in B_b(X), \tag{2.1}
\]
where \( (T_t)_{t \geq 0} \) is a strongly continuous semigroup of linear operators on \( X \) and \( (\mu_t)_{t \geq 0} \) is a family of Borel probability measures on \( X \). The semigroup law for \( (P_t)_{t \geq 0} \) is equivalent to the following property of the family \( (\mu_t)_{t \geq 0} \):
\[
\mu_0 = \delta_0, \quad \mu_{t+s} = (\mu_t \circ T_s^{-1}) \ast \mu_s \quad \text{for all } s, t \geq 0, \tag{2.2}
\]
see [12, Proposition 2.2]. We recall, see [12, Lemma 2.6], that if for any \( \xi \in X^* \) the function \( t \mapsto \hat{\mu}_t(\xi) \) is absolutely continuous on \( [0, \infty) \) and differentiable at \( t = 0 \) then, setting
\[
\lambda(\xi) := -\frac{d}{dt} \hat{\mu}_t(\xi)_{t=0},
\]
the function \( t \mapsto \lambda(T_t^*\xi) \) belongs to \( L^1_{\text{loc}}([0, \infty)) \), hence [22] is equivalent to
\[
\hat{\mu}_t(\xi) = \exp \left( -\int_0^t \lambda (T_s^*\xi) ds \right), \quad t \geq 0, \quad \xi \in X^*. \tag{2.3}
\]
In this case \( \lambda \) is negative definite, i.e., the matrices whose entries are \( (\lambda(\xi_i - \xi_j))_{i,j=1,...,n} \) are negative definite for every \( n \in \mathbb{N} \) and for every \( n \)-tuple \( (\xi_1, \ldots, \xi_n) \in X^* \). Throughout the paper we assume that \( \lambda \) is also Sazonov continuous on \( X^* \). This implies that, for every \( t \geq 0 \), the functions \( e^{-t\lambda} \) are positive definite (see [19]) and Sazonov continuous. Therefore, by [44, Theorem VI.1.1], they are characteristic functions of probability measures on \( X \). This implies that \( e^{-t\lambda} \) is the characteristic function of an infinitely divisible probability measure on \( X \). Using the Lévy–Khinchine theorem, (see [35, Theorem VI.4.10]), there are \( b \in X \), a nonnegative self-adjoint trace-class operator \( Q \in \mathcal{L}(X) \) and a Lévy measure \( M \), that is a Borel measure satisfying
\[
M(\{0\}) = 0, \quad \int_X (1 \wedge |x|^2)M(dx) < \infty, \tag{2.4}
\]
such that \( \lambda \) can be written in the form
\[
\lambda(\xi) = -i \langle \xi, b \rangle + \frac{1}{2} \langle Q \xi, \xi \rangle - \int_X \left( e^{i \langle x, \xi \rangle} - 1 - i \langle x, \xi \rangle \chi_{B_1}(x) \right) M(dx). \tag{2.5}
\]
In the sequel we use the symbol $\leftrightarrow$ to associate the triple $[b, Q, M]$ with $\lambda, \mu_t$ and $\tilde{\mu}_t$, according to (2.1), (2.3). It is immediate to check that $P_t$ maps $C_0(X)$ into itself and $\|P_tf\|_\infty \leq \|f\|_\infty$, $t > 0$, $f \in C_0(X)$, but, in general, $P_t$ is not strongly continuous in $C_0(X)$. The continuity of the map $(t, x) \rightarrow P_tf(x)$, $f \in C_0(X)$ allows us to define the weak generator $\mathcal{L}$ through its resolvent

$$[R(\gamma, \mathcal{L})f](x) = \int_0^\infty e^{-\gamma t}P_tf(x)dt$$

for any $\gamma > 0$, $f \in C_0(X)$ and $x \in X$. Let $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of the semigroup $(P_t)_{t \geq 0}$. We recall that by [21] p. 40 if $Q = 0$ we have

$$\mathcal{L}f(x) = \langle Ax, Df(x) \rangle + \int_X [f(x + y) - f(x) - \langle Df(x), y \rangle \chi_{B_1}(y)] \, M(dy),$$

(2.6)

for any $f \in \mathcal{F}C^2_\infty(X)$. Note that for such functions, the integral in (2.6) is well defined by the Taylor formula.

In the sequel it will be useful to consider a core for the generator of $P_t$ in $C_0(X)$, i.e., the finest locally convex topology that agrees on norm bounded sets with the topology of uniform convergence on compacts (see [21] for a more in-depth discussion about this topology). To do that we state the following hypothesis, see [41].

**Hypothesis 2.1.** There exists an orthonormal basis $\{b_n | n \in \mathbb{N}\}$ of $X$ consisting of eigenvectors of $A^*$ and

$$\int_{B_1^*} |x|M(dx) < \infty.$$  

(2.7)

Following [5] (see also [39, Remark 5.11]), we say that $f \in C^2_\infty(X)$ if $f \in C_0(X)$ belongs to $C^2(X)$, its first and second order derivatives are uniformly bounded and uniformly continuous on bounded subsets of $X$, $\text{Ran}(Df) \subseteq D(A^*)$ and $x \rightarrow \langle x, A^*Df(x) \rangle \in C_0(X)$. We say $F \in \mathcal{F}C^2_\infty(X)$ if there exist $n \in \mathbb{N}$ and $f \in C^2_\infty(\mathbb{R}^n)$ such that

$$F(x) = f((x,h_1),\ldots,(x,h_n)), \quad x \in X.$$

In [5, Theorem 5.2], see also [39, Remark 5.11], it is shown that $\mathcal{F}C^2_\infty(X)$, under Hypothesis 2.1, is a core for the generator of $P_t$ in $C_0(X)$ equipped with the mixed topology. Recall that a core of an operator $A : D(A) \subseteq X \rightarrow X$ is a subspace $C \subseteq D(A)$ which is dense in $D(A)$ with respect to the graph norm $\|\cdot\|_A := \|\cdot\| + \|A\cdot\|$. In the next section we use this result to prove that $\mathcal{F}C^2_\infty(X)$ is also a core for $\mathcal{L}$ in $L^2(X, \sigma)$ as well, when an invariant measure $\sigma$ exists for the semigroup.

### 3. Invariant Measure

In this section we recall some conditions implying the existence of a unique invariant measure. A Borel probability measure $\sigma$ on $X$ is an invariant measure for $(P_t)_{t \geq 0}$ if

$$\int_X P_t f \, d\sigma = \int_X f \, d\sigma, \quad t \geq 0, \ f \in B_\sigma(X);$$

(3.1)

or, equivalently, $\sigma = (\sigma \circ T_t^{-1})*\mu_t$ for any $t > 0$, where $\mu_t$ are the measures in (2.3). Throughout this section we consider $\lambda \leftrightarrow [b, 0, M]$, with $\tilde{\mu}_t \leftrightarrow [b_1, 0, M_t]$ according to (2.2) and (2.5), where

$$b_t := \int_0^t T_s b \, ds + \int_0^t \int_X T_s x \left(\chi_{B_1}(T_s x) - \chi_{B_1}(x)\right) M(dx) \, ds$$  

(3.2)

and $M_t$ are Borel measures defined setting $M_t(\{0\}) = 0$ and

$$M_t(B) := \int_0^t M(T_s^{-1}(B)) \, ds, \quad B \in \mathcal{B}(X), 0 \notin B.$$  

(3.3)
Note that $M_t$ are Lévy measures. Indeed, as $T_t$ is strongly continuous, there exist $K \geq 1$ and $\omega \in \mathbb{R}$ such that $|T_t x| \leq Ke^{\omega |x|}$ for any $t > 0$ and $x \in X$. Hence

$$
\int_X (1 \wedge |x|^2)M_t(dx) = \int_0^\infty \int_X (1 \wedge |T_s x|^2)M(dx)ds \\
\leq \frac{K^2}{2\omega}(e^{2\omega t} - 1) \int_X (1 \wedge |x|^2)M(dx) < \infty. 
$$

(3.4)

Following [20] Theorem 3.1] we assume the following hypotheses that guarantee the existence and the uniqueness of an invariant measure for $(P_t)_{t \geq 0}$.

**Hypotheses 3.1.** Let $\lambda \leftrightarrow [b,0,M]$, let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup and let $b_t, M_t$ be given in (3.2) and (3.3). Assume

(i) there exists $b_\infty := \lim_{t \to \infty} b_t$ in $X$;

(ii) setting $M_\infty := \sup_{t \geq 0} M_t$ (i.e., $M_\infty(\{0\}) = 0$ and $M_\infty(B) = \int_0^\infty M(T_s^{-1}(B))ds$, $B \in \mathcal{B}(X)$, $0 \notin B$), it holds that

$$
\int_0^\infty \int_X (1 \wedge |T_s x|^2)M(dx)ds < \infty;
$$

(iii) $\lim_{t \to \infty} T_t x = 0$ in $X$ for every $x \in X$.

The following result can be found in [20, Section 3] and it is fundamental in most of the results of this paper.

**Theorem 3.2.** Under Hypotheses 3.1 and 3.3, $M_\infty$ is a Lévy measure and the measure $\sigma \leftrightarrow [b_\infty,0,M_\infty]$ is invariant for $P_t$. In addition, if Hypothesis 3.1(iii) holds true, then $\sigma$ is unique and

$$
\int_X \mathcal{L}f d\sigma = 0 
$$

(3.5)

for any $f \in \mathcal{F}C^2_b(X)$. Moreover $\mu_t$ converges weakly-star to $\sigma$ as $t \to \infty$.

Let us show that $\mathcal{F}C^2_b(X)$ is a core for $\mathcal{L}$ in $L^2(X,\sigma)$.

**Lemma 3.3.** If Hypotheses 3.1 and 3.3 hold true, then $\mathcal{F}C^2_b(X)$ is invariant for $P_t$ and it is a core for $\mathcal{L}$ in $L^2(X,\sigma)$.

**Proof.** We point out that $\mathcal{F}C^2_b(X)$ is invariant with respect to $P_t$ (see [5] Theorem 5.2 and [33] Remark 5.11), so to conclude we just need to show that $\mathcal{F}C^2_b(X)$ is contained in the domain of the generator $\mathcal{L}$ in $L^2(X,\sigma)$ and that it is dense in $L^2(X,\sigma)$.

By [3] Theorem 5.1(2)] the space $\mathcal{F}C^2_b(X)$ is contained in $D(\mathcal{L}_m)$, the domain of the generator of $P_t$ in $C_b(X)$ equipped with the mixed topology. This means that for any $F \in \mathcal{F}C^2_b(X)$ there exists $G \in C_b(X)$ such that

$$
\tau_m \lim_{t \to 0} \frac{P_tF - F}{t} = G.
$$

Let $\{t_n\}_{n \in \mathbb{N}} \in (0,\infty)$ be a sequence converging to zero. By [21] Proposition 2.3] the sequence $((1/t_n)(P_{t_n}F - F) - G)_{n \in \mathbb{N}}$ is uniformly convergent to zero on compact subsets of $X$ and

$$
\sup_{n \in \mathbb{N}} \left\| \frac{P_{t_n}F - F}{t_n} - G \right\|_\infty < \infty.
$$

By the dominated convergence theorem we get that the sequence $((1/t_n)(P_{t_n}F - F) - G)_{n \in \mathbb{N}}$ converges to zero in $L^2(X,\sigma)$. Since the argument is independent on the choice of the sequence $(t_n)_{n \in \mathbb{N}}$ we obtain

$$
\lim_{t \to 0} \left\| \frac{P_tF - F}{t} - G \right\|_{L^2(X,\sigma)} = 0,
$$

hence $F$ belongs to the domain of $\mathcal{L}$ in $L^2(X,\sigma)$ and $\mathcal{L}F = \mathcal{L}_mF$.

The fact that $\mathcal{F}C^2_b(X)$ is dense in $L^2(X,\sigma)$ can be proved by using that $\mathcal{F}C^2_b(X)$ is $\tau_{m}$-sequentially dense in $C_b(X)$ (see [21] Lemma 2.6) and the same arguments as above. □
The following equality will be useful later on.

**Lemma 3.4.** Assume that Hypotheses [2.1] and [3.1] hold true. For every \( f \in \mathcal{C}_A^2(X) \) and every \( \Phi \in C^2(\mathbb{R}) \) we have

\[
\int_X \Phi'(f) \cdot \mathcal{L} f \, d\sigma = \int_X \int_X \left[ \Phi(f(x)) - \Phi(f(x + y)) + \Phi'(f(x)) \left( f(x + y) - f(x) \right) \right] M(dy) \sigma(dx). \tag{3.6}
\]

**Proof.** By the invariance relation (3.5) it suffices to prove that

\[
\mathcal{L}(\Phi \circ f) = (\Phi' \circ f)(\mathcal{L} f) + \int_X \left[ (\Phi \circ f)(\cdot + y) - (\Phi \circ f) - (\Phi' \circ f) \left( f(\cdot + y) - f \right) \right] M(dy) \tag{3.7}
\]

and to observe that \( \Phi \circ f \) belongs to \( \mathcal{C}_A^2(X) \). Formula (3.7) easily follows from (2.6). Indeed, we have

\[
[\mathcal{L}(\Phi \circ f)](x) = [(\Phi' \circ f)(x)](Ax, Df(x)) + \int_X \left[ (\Phi \circ f)(x + y) - (\Phi \circ f)(x) - [(\Phi' \circ f)(x)](Df(x), y) \chi_{B_1}(y) \right] M(dy).
\]

Now adding and subtracting \( \int_X [(\Phi' \circ f)(x)](f(x + y) - f(x)) M(dy) \) we get (3.7). \( \square \)

In the following proposition we collect the main properties of the semigroup \( P_t \) in the space \( L^p(X, \sigma) \), \( p \in [1, \infty) \).

**Proposition 3.5.** Assume that Hypotheses [2.1] and [3.1] hold true. The semigroup \( P_t \) can be extended to a contractive strongly continuous semigroup (still denoted by \( P_t \)) on \( L^p(X, \sigma) \) for any \( 1 \leq p < \infty \).

**Proof.** The Jensen inequality, formula (2.1) and the invariance property (3.1) yield that, for any \( f \in C_b(X) \)

\[
\int_X |P_t f|^p \, d\sigma \leq \int_X |f|^p \, d\sigma
\]

whence \( \|P_t f\|_{L^p(X, \sigma)} \leq \|f\|_{L^p(X, \sigma)} \) for any \( f \in C_b(X) \). Moreover, since the measure \( \sigma \) is a probability Borel measure, the space \( C_b(X) \) is dense in \( L^p(X, \sigma) \) for any \( p \in [1, \infty) \) (see Lemma A.1). Thus we can extend \( P_t \) to a bounded linear operator in \( L^p(X, \sigma) \) with \( \|P_t\|_{\mathcal{L}(L^p(X, \sigma))} \leq 1 \). Now, let us prove that \( P_t \) is strongly continuous in \( L^p(X, \sigma) \). To this aim, notice that for \( f \in C_b(X) \) the function \( (t, x) \mapsto P_t f(x) \) is continuous in \([0, \infty) \times X \) (see [12, Lemma 2.1]). This fact, estimate \( \|P_t f\|_\infty \leq \|f\|_\infty \) together with the dominated convergence theorem imply that \( \|P_t f - f\|_{L^p(X, \sigma)} \) vanishes as \( t \to 0^+ \) for any \( f \in C_b(X) \) and \( p \in [1, \infty) \). To conclude we argue by approximation. Let \( f \in L^p(X, \sigma) \) and \( (f_n)_n \subseteq C_b(X) \) converging to \( f \) in \( L^p(X, \sigma) \) as \( n \to \infty \).

Then,

\[
\|P_t f - f\|_{L^p(X, \sigma)} \leq \|P_t (f - f_n)\|_{L^p(X, \sigma)} + \|P_t f_n - f_n\|_{L^p(X, \sigma)} + \|f_n - f\|_{L^p(X, \sigma)} \leq \|P_t f_n - f_n\|_{L^p(X, \sigma)} + 2\|f_n - f\|_{L^p(X, \sigma)}, \tag{3.8}
\]

where in the last line we used the contractivity of \( P_t \) in \( L^p(X, \sigma) \). Fix \( \varepsilon > 0 \) and let \( n_0 \in \mathbb{N} \) be such that \( \|f_{n_0} - f\|_{L^p(X, \sigma)} \leq \varepsilon/4 \). The first part of the proof yields the existence of \( t_0 > 0 \) such that \( \|P_t f_{n_0} - f_{n_0}\|_{L^p(X, \sigma)} \leq \varepsilon/2 \) for any \( t \in (0, t_0) \). Thus, writing estimate (3.8) with \( n \) replaced by \( n_0 \), we conclude that \( \|P_t f - f\|_{L^p(X, \sigma)} \leq \varepsilon \) for any \( t \in (0, t_0) \) and this completes the proof. \( \square \)

The next result concerns the asymptotic behaviour of \( P_t \) as \( t \to \infty \). For any \( f \in L^1(X, \sigma) \) we denote by \( m_\sigma(f) \) the mean of \( f \) with respect to \( \sigma \), i.e.,

\[
m_\sigma(f) := \int_X f \, d\sigma.
\]
Proof. Let \( f > 0 \) and if \( \mu_t \) have \( \text{Hypotheses 4.1}. \) Analogously, as \( T_t x \) vanishes as \( t \to \infty \), there is \( t_1 > 0 \) such that for every \( t \geq t_0 \)

\[
\| f \| \leq \epsilon \]

(3.11)

where \( L \) is the Lipschitz constant of \( f \). Thus, for every \( t \geq \max \{ t_0, t_1 \} \), by (3.10) and (3.11), we have

\[
\begin{align*}
|P_t f(x) - \int_X f \, d\sigma| & = \left| \int_X f(T_t x + z) \mu_t(dz) - \int_X f(z) \sigma(dz) \right| \\
& \leq \left| \int_X (f(T_t x + z) - f(z)) \mu_t(dz) \right| + \left| \int_X f(z) \mu_t(dz) - \int_X f(z) \sigma(dz) \right| \\
& \leq \int_X \left| f(T_t x + z) - f(z) \right| \mu_t(dz) + \epsilon \\
& \leq L \int_X |T_t x| \mu_t(dz) + \frac{\epsilon}{2} < \epsilon.
\end{align*}
\]

If \( f > 0 \) the function \( x \mapsto (P_t f)(x) \log((P_t f)(x)) \) is well defined and bounded by \( \| f \|_\infty \log \| f \|_\infty \), as \( P_t \) is contractive and preserves positivity. Therefore, by the dominated convergence theorem we get (3.9). \( \square \)

Corollary 3.7. Assume that Hypotheses \( \ref{hypothesis} \) and \( \ref{hypothesis2} \) hold true. For any \( f \in L^p(X, \sigma) \), it holds that

\[
\lim_{t \to \infty} \| P_t f - m_\sigma(f) \|_{L^p(X, \sigma)} = 0.
\]

(3.12)

Proof. Formula (3.12) easily follows from the dominated convergence theorem for bounded and Lipschitz continuous functions. The general case follows by approximation from Proposition A.2. \( \square \)

4. A LOGARITHMIC SOBOLEV TYPE INEQUALITY AND ITS CONSEQUENCES

In this section we prove a logarithmic Sobolev type inequality satisfied by \( \sigma \), the unique invariant measure for \( P_t \) provided by Theorem \( \ref{theo1} \) For any positive function on \( X \) we denote by

\[
\text{Ent}_\sigma(f) := \left( \int_X f \log f \, d\sigma \right) - m_\sigma(f) \log m_\sigma(f)
\]

the entropy of \( f \) with respect to \( \sigma \). In order to prove the desired logarithmic Sobolev type inequality we need further assumptions on the Lévy measure \( M \). Similar assumptions are considered in [11] Hypothesis (H4) and Lemma 2.1(2)].

Hypotheses 4.1. We assume that \( M \) is a Lévy measure on \( X \) and

(i) there exists a function \( h : (0, \infty) \to (0, \infty) \) such that, for every \( t > 0 \),

\[
h(t)M - M \circ T_t^{-1}
\]

is a positive measure;
(ii) the function $h$ belongs to $L^1((0, \infty))$.

In the following lemma we prove an estimate that plays the role of pointwise gradient estimates in the local case.

**Lemma 4.2.** Assume that Hypotheses (2.1) and (4.1) hold true. Then,

$$
\int_X \left| (P_t f)(x + y) - (P_t f)(x) \right|^p M(dy) \leq h(t) \int_X \left| P_t \left( f(\cdot + y) - f(\cdot) \right)(x) \right|^p M(dy). \tag{4.1}
$$

for every $p \in [1, \infty)$, $t > 0$, $x \in X$ and $f \in L^1(X)$. 

**Proof.** Using (2.1), the Jensen inequality and Hypothesis (4.1) we get

$$
\begin{align*}
\int_X \left| (P_t f)(x + y) - (P_t f)(x) \right|^p M(dy) \\
= \int_X \int_X \left( f(T_1 x + T_1 y + z) - f(T_1 x + z) \right) \mu_t(z) \left| \right|^p M(dy) \\
= \int_X \int_X \left( f(T_1 x + w + z) - f(T_1 x + z) \right) \mu_t(dz) \left| \right|^p d(M \circ T_1^{-1})(w) \\
\leq h(t) \int_X \int_X \left( f(T_1 x + w + z) - f(T_1 x + z) \right) \mu_t(dz) \left| \right|^p M(dw) \\
= h(t) \int_X \left| P_t \left( f(\cdot + w) - f(\cdot) \right)(x) \right|^p M(dw).
\end{align*}
$$

The main result of this section is the following estimate of the entropy of $f$. As pointed out in the Introduction, on the right hand side the gradient of $f$, typical of the logarithmic Sobolev inequalities available in the context of semigroups generated by local operators, has to be replaced by the integral of the relative increment because of the nonlocal effects.

**Theorem 4.3.** Assume that Hypotheses (2.1) and (4.1) hold true. Then, for every $p \in [1, \infty)$ and $f \in \mathcal{F}C^2(X)$ with positive infimum, the following estimate

$$
\text{Ent}_\sigma(f^p) \leq C \int_X \int_X \frac{\left| f^p(x + y) - f^p(x) \right|^2}{f^p(x)} M(dy) \sigma(dx), \tag{4.2}
$$

holds true with $C = \|h\|_{L^1((0, \infty))}$. 

**Proof.** Let $f$ be as in the statement. It is not restrictive to assume also that $\sup f \leq 1$. Indeed, if this is not the case we consider $f/\|f\|_\infty$ in place of $f$. Thus, consider the function $F : (0, \infty) \rightarrow \mathbb{R}$ defined as

$$
F(t) := \int_X (P_t f^p) \log(P_t f^p) d\sigma.
$$

The function $(t, x) \mapsto \psi(t, x) := (P_t f^p(x)) \log(P_t f^p(x))$ is bounded and continuously differentiable in $[0, \infty) \times X$ since $P_t f^p$ belongs to $\mathcal{F}C^2(X)$ and takes values in $[(\inf f)^p, 1]$ for any $t, x$ as above (see (2.1)). Moreover, since

$$
\frac{\partial}{\partial t} \psi(t, \cdot) = (\log(P_t f^p) + 1) \mathcal{L}(P_t f^p)
$$

belongs to $\mathcal{F}C_b(X)$, the function $F$ is differentiable and its derivative is given by

$$
F'(t) = \int_X (\mathcal{L}(P_t f^p)) \log(P_t f^p) d\sigma + \int_X \mathcal{L}(P_t f^p) d\sigma = \int_X (\mathcal{L}(P_t f^p)) \log(P_t f^p) d\sigma
$$

$$
= - \int_X \int_X \left[ (P_t f^p)(x + y) \log(P_t f^p)(x + y) - (P_t f^p)(x + y) - (P_t f^p)(x) \log(P_t f^p)(x) \\
+ (P_t f^p)(x) - \left( (P_t f^p)(x + y) - (P_t f^p)(x) \right) \log(P_t f^p)(x) \right] M(dy) \sigma(dx).
$$
where we used (3.5) and (4.6) with $\Phi(\xi) = \xi \log \xi - \xi$. Notice that for every $r, s > 0$ the following inequality holds

$$r \log r - r - s \log s + s - (r - s) \log s \leq \frac{(r-s)^2}{s}.$$ 

Indeed, multiplying by $s^{-1}$ and setting $t := rs^{-1}$ it is reduced to the elementary estimate $\log t \leq 1 - 1, t > 0$. By this last inequality and (4.1) we get

$$F'(t) \geq -h(t) \int_X (P_t f^p)(x) \leq \left( t \right) \frac{|P_t f^p(x + y) - f^p(x)|}{f_p(x)} M(dy) \sigma(dx).$$

(4.3)

The Hölder inequality yields

$$|P_t f^p(x + y) - f^p(x)| \leq \left( t \right) \frac{|f^p(x + y) - f^p(x)|}{f_p(x)} (P_t f^p(x))^{1/2},$$

(4.4)

for every $x, y \in X$, hence combining (4.3) with (4.4) we get

$$F'(t) \geq -h(t) \int_X \int_X P_t \left( \frac{|f^p(x + y) - f^p(x)|}{f_p(x)} \right) M(dx) \sigma(dy).$$

By the Fubini theorem and the invariance of $\sigma$ with respect to $P_t$ we get

$$F'(t) \geq -h(t) \int_X \int_X \frac{|f^p(x + y) - f^p(x)|}{f_p(x)} M(dx) \sigma(dy).$$

Now integrating the previous inequality from 0 to $t$ we get

$$F(t) - F(0) \geq -\|h\|_{L^1((0, \infty))} \int_X \int_X \frac{|f^p(x + y) - f^p(x)|}{f_p(x)} M(dy) \sigma(dx)$$

or equivalently

$$\int_X P_t f^p \log(P_t f^p) d\sigma - \int_X f^p \log f^p d\sigma \geq -\|h\|_{L^1((0, \infty))} \int_X \int_X \frac{|f^p(x + y) - f^p(x)|^2}{f_p(x)} M(dy) \sigma(dx).$$

Letting $t$ to infinity, and recalling (3.9) we get

$$\int_X f^p \log f^p d\sigma - \left( \int_X f^p d\sigma \right) \log \left( \int_X f^p d\sigma \right) \leq C \int_X \int_X \frac{|f^p(x + y) - f^p(x)|^2}{f_p(x)} M(dy) \sigma(dx),$$

where $C := \|h\|_{L^1((0, \infty))}$, whence the claim. $\square$

Now, let us denote by $\mathcal{F}^p$ the Banach space completion of $\mathcal{F}^2_A(X)$ with respect to the norm

$$\|f\|_{\mathcal{F}^p} := \|f\|_{L^p(X, \sigma)} + \left( \int_X \int_X |f(x + y) - f(x)|^2 M(dy) \sigma(dx) \right)^{1/2}.$$

Observe that, since $M$ is a Lévy measure, for every $f$ belonging to $\mathcal{F}^2_A(X)$

$$\|f\|_{\mathcal{F}^p} \leq (1 + 2\sqrt{M(B_1^2)})\|f\|_{\infty} + \|Df\|_{\infty} \left( \int_{B_1} |y|^2 M(dy) \right)^{1/2} < \infty.$$ 

An immediate consequence of (4.2) is the Poincaré inequality (4.3). Similar estimates have already been proved in [11 Corollary 1.4]. But, we derive them from the logarithmic Sobolev type inequality (4.2), while in [11], as these were not available when $M \neq 0$, they are derived by using an idea due to Bakry and Ledoux which consists in differentiating the map $s \mapsto P_{ts}(P_s f)^2$ (see [9]) in order to get (4.4).

**Proposition 4.4.** Under the hypotheses of Theorem 4.3, the estimate

$$\|f - m_\sigma(f)\|_{L^2(X, \sigma)} \leq 2C \left( \int_X \int_X |f(x + y) - f(x)|^2 M(dy) \sigma(dx) \right)^{1/2}$$

holds true for any $f \in \mathcal{F}^2$. Here $C$ is the constant appearing in (4.2).
Proof. Consider first \( f \in \mathcal{F}_A^2(X) \) with \( m_\sigma(f) = 0 \). For \( 0 < \varepsilon < (2\|f\|_\infty)^{-1} \), the function \( f_\varepsilon := 1 + \varepsilon f \) is greater or equal to 1/2. Thus, estimate \[ (1.2) \] with \( p = 2 \) yields
\[
\int_X f_\varepsilon^2 \log(f_\varepsilon^2) \, d\sigma - m_\sigma(f_\varepsilon^2) \log(m_\sigma(f_\varepsilon^2)) \leq C \int_X \int_X \frac{|f_\varepsilon^2(x+y) - f_\varepsilon^2(x)|^2}{f_\varepsilon^2(x)} M(dy)\sigma(dx).
\]
Observing that
\[
\int_X f_\varepsilon^2 \log(f_\varepsilon^2) \, d\sigma - m_\sigma(f_\varepsilon^2) \log(m_\sigma(f_\varepsilon^2)) = 2\varepsilon^2 \|f\|^2_{L^2(X,\sigma)} + o(\varepsilon^2), \quad \varepsilon \to 0^+,
\]
and that
\[
\int_X \int_X \frac{|f_\varepsilon^2(x+y) - f_\varepsilon^2(x)|^2}{f_\varepsilon^2(x)} M(dy)\sigma(dx)
= \int_X \int_X \left( \varepsilon(f(x+y) - f(x)) + 2\varepsilon(f(x+y) - f(x)) \right)^2 \frac{(1+\varepsilon f(x))^2}{M(dy)\sigma(dx)}
= 4\varepsilon^2 \int_X \int_X \frac{|f(x+y) - f(x)|^2}{(1+\varepsilon f(x))^2} M(dy)\sigma(dx) + \int_X \int_X \frac{g_\varepsilon(x,y)}{(1+\varepsilon f(x))^2} M(dy)\sigma(dx),
\]
where
\[
g_\varepsilon(x,y) = \varepsilon^4(f^2(x+y) - f^2(x))^2 + 4\varepsilon^3(f^2(x+y) - f^2(x))(f(x+y) - f(x)),
\]
we get
\[
2\varepsilon^2 \|f\|^2_{L^2(X,\sigma)} + o(\varepsilon^2) \leq 4\varepsilon^2 C \int_X \int_X \frac{|f(x+y) - f(x)|^2}{(1+\varepsilon f(x))^2} M(dy)\sigma(dx)
+ C \int_X \int_X \frac{g_\varepsilon(x,y)}{(1+\varepsilon f(x))^2} M(dy)\sigma(dx).
\]
Using the assumptions on \( f \) and \( \varepsilon \) we can estimate
\[
\frac{|f(x+y) - f(x)|^2}{f_\varepsilon^2(x)} \leq \frac{1}{4} \left( 2\|f\|_\infty^2 \chi_{B_1^i}(y) + \|Df\|_\infty^2 \|\chi_{B_1^i}(y)\right), \quad x, y \in X.
\]
and, analogously
\[
g_\varepsilon(x,y) \leq \frac{1}{4} \left( C_1 \chi_{B_1^i}(y) + C_2 \|\chi_{B_1^i}(y)|^2 \right), \quad x, y \in X,
\]
for some positive constants \( C_1 \) depending on \( \|f\|_\infty \) and \( C_2 \) depending on \( \|f\|_\infty \) and \( \|Df\|_\infty \). As \( M \) is a Lévy measure, letting \( \varepsilon \to 0 \) in \((4.6)\) by the dominated convergence theorem we get
\[
\|f\|^2_{L^2(X,\sigma)} \leq 2C \int_X \int_X |f(x+y) - f(x)|^2 M(dy)\sigma(dx).
\]
For a general \( f \in \mathcal{F}_A^2(X) \), applying \((4.7)\) to \( f - m_\sigma(f) \), we deduce
\[
\|f - m_\sigma(f)\|^2_{L^2(X,\sigma)} \leq 2C \int_X \int_X |f(x+y) - f(x)|^2 M(dy)\sigma(dx).
\]
To conclude, let us consider \( f \in \mathcal{F}^2 \) and let \( (f_n) \subseteq \mathcal{F}^2_A(X) \) converging to \( f \) in \( \|\cdot\|_{\mathbb{L}^2} \). Then writing \((4.5)\) with \( f_n \) in place of \( f \) and letting \( n \to \infty \) we get the claim. Indeed
\[
\|f_n - m_\sigma(f_n)\|^2_{L^2(X,\sigma)} = \|f_n\|^2_{L^2(X,\sigma)} - (m_\sigma(f_n))^2
\]
converges to \( \|f\|^2_{L^2(X,\sigma)} - (m_\sigma(f))^2 \) by the dominated convergence theorem. Further, since
\[
|f_n(x+y) - f_n(x)| = |f_n(x+y) - f(x)| \leq |f_n(x+y) - f(x) - f_n(x) + f(x)|,
\]
for \( M\)-a.e. \( y \in X \), \( \sigma \)-a.e. \( x \in X \) and the right hand side of the previous inequality vanishes as \( n \to \infty \) for \( M\)-a.e. \( y \in X \) and \( \sigma \)-a.e. \( x \in X \), coming back to \((4.8)\) with \( f_n \) in place of \( f \) and letting \( n \to \infty \) we conclude the proof.
For $p \in [1, \infty)$, we denote by
\[
W^p = \left\{ f : X \to \mathbb{R} \left| \int_X \int_X |f|^p(x + y) - |f|^p(x)|M(dy)\sigma(dx) < \infty \right. \right\}.
\]
In the following proposition we use a bootstrap procedure similar the one in [3] in order to obtain estimates looking like (4.5) for $p > 2$.

**Proposition 4.5.** Assume Hypotheses 2.1 and 3.1 hold true. For any $f \in W^2$ it holds that
\[
\|f - m_\sigma(f)\|_{L^2(X,\sigma)} \leq c \left( \int_X \int_X |f|^2(x + y) - |f|^2(x)|M(dy)\sigma(dx) \right)^{1/2}
\]
for some positive constant $c$. Then, for every $p \in (2, \infty)$, there exists a positive constant $c_p$ such that
\[
\|f\|_{L^p(X,\sigma)}^p \leq c_p \int_X \int_X |f|^p(x + y) - |f|^p(x)|M(dy)\sigma(dx)
\]
for any $f \in W^p$ with $m_\sigma(f) = 0$.

**Proof.** Let $f \in W^p$ with $m_\sigma(f) = 0$. Since $p > 2$, the function $f^{p/2}$ belongs to $W^2$. Then, applying estimate (4.9) to $f^{p/2}$ we deduce
\[
\|f\|_{L^p(X,\sigma)}^p \leq \|f\|_{L^2(X,\sigma)}^2 = \|f^{p/2} - m_\sigma(f^{p/2})\|_{L^2(X,\sigma)}^2
\]
for any $f \in W^p$ with $m_\sigma(f) = 0$.

Now, if $p \in (2, 4]$, using that $\|f\|_{L^p(X,\sigma)} \leq \|f\|_{L^2(X,\sigma)}$, from (4.9) and (4.11) we obtain
\[
\int_X \int_X |f|^2(x + y) - |f|^2(x)|M(dy)\sigma(dx) \leq c \left( \int_X \int_X |f|^2(x + y) - |f|^2(x)|M(dy)\sigma(dx) \right)^{p/2}
\]
where in the last line we used the Jensen inequality taking into account that $\sigma$ is a Borel probability measure. Furthermore, multiplying and dividing by $|y|^{2(p-2)/p}$ in $B_1$, using the Hölder inequality and using that $M$ is a Lévy measure, we can estimate
\[
\left( \int_X |f|^2(x + y) - |f|^2(x)|M(dy) \right)^{p/2} \leq \left( \int_{B_1} |f|^2(x + y) - |f|^2(x)|M(dy) \right)^{p/2}
\]
for any $f \in W^p$. Finally, using the fact that $\|f\|_{L^2(X,\sigma)} \leq \|f\|_{L^p(X,\sigma)}$ for any $p \in (2, \infty)$, we can conclude that
\[
\|f\|_{L^p(X,\sigma)} \leq c_p \int_X \int_X |f|^p(x + y) - |f|^p(x)|M(dy)\sigma(dx)
\]
\[ \begin{align*}
2 \int_{B_1} |y|^2 M(dy) + \frac{2}{|y|^{p-2}} \int_{B_1} |f|^p(x + y) - |f|^p(x) M(dy)
+ (2M(B_1^2))^{\frac{p}{p-2}} \int_{B_1} |f|^p(x + y) - |f|^p(x) M(dy)
\end{align*} \]

where in the last inequality we used estimate \(|a| - |b| \leq |a| - |b|^{p-1}\) which holds true for \(a, b \in \mathbb{R}, p > 1\). Indeed, assuming \(|a| > |b| > 0\) and setting \(t = |a| - |b|^{p-1}\), it suffices to prove that \((1, \infty) \ni t \mapsto g(t) := (t - 1)^p - t^{p-1} + 1\) is nonpositive. But, \(g'(t) = p(t - 1)^{p-1} - t^{p-1} \geq 0\) and then \(g(t) \leq g(1) = 0\) for any \(t \in (1, \infty)\). Thus, summing up in (4.12) we get (4.10) for \(p \in (2, 4]\). Now, let \(p \in (4, 8]\), then \(p/2 \in (2, 4]\). Thus, starting from (4.11) and using estimate (4.10) with \(p/2\) in place of \(p\) we deduce

\[ \begin{align*}
\|f\|_{L^p(X, \sigma)}^p \leq & \|f\|_{L^{p/2}(X, \sigma)}^p + C \int_X \int_X \|f|^p(x + y) - |f|^p(x) M(dy) \sigma(dx)
\end{align*} \]

\[ \begin{align*}
&\leq \frac{2}{p/2} \left( \int_X \int_{B_1} \|f|^{p/2}(x + y) - |f|^{p/2}(x)|M(dy) \sigma(dx)
+ \int_X \int_{B_1} \|f|^{p/2}(x + y) - |f|^{p/2}(x)| M(dy) \sigma(dx) \right)^2
\end{align*} \]

\[ \begin{align*}
&\leq 2c^2_{p/2} \left( \int_X \int_{B_1} \|f|^{p/2}(x + y) - |f|^{p/2}(x)| M(dy) \sigma(dx) \right)^2
+ 2c^2_{p/2} \left( \int_X \int_{B_1} \|f|^{p/2}(x + y) - |f|^{p/2}(x)| M(dy) \sigma(dx) \right)^2
\end{align*} \]

\[ \begin{align*}
&\leq 2c^2_{p/2} \int_X \int_{B_1} \|f|^{p}(x + y) - |f|^{p}(x)| M(dy) \sigma(dx)
+ 2c^2_{p/2} \int_X \int_{B_1} \|f|^{p}(x + y) - |f|^{p}(x)| M(dy) \sigma(dx)
\end{align*} \]

\[ \begin{align*}
&\leq 2c^2_{p/2} \int_X \int_{B_1} \|f|^{p}(x + y) - |f|^{p}(x)| M(dy) \sigma(dx)
+ 2c^2_{p/2} \int_X \int_{B_1} \|f|^{p}(x + y) - |f|^{p}(x)| M(dy) \sigma(dx)
\end{align*} \]

getting again (4.10) for \(p \in (4, 8]\). Iterating this procedure we complete the proof. \(\square\)

**Remark 4.6.** Note that (4.9) is implied by (4.3). Therefore, under the hypotheses of Theorem 4.4 inequality (4.10) holds true.

The arguments used in the proof of Proposition 4.5 can be used to deduce the integrability of functions with polynomial growth with respect to \(\sigma\) from their integrability with respect to \(M\). We discuss this in term of the moments of \(M\) and \(\sigma\). If \(\mu\) is a Borel measure on \(X\), we denote by

\[ \mu_A(p) := \int_A |x|^p \mu(dx), \quad A \in \mathcal{B}(X) \]

the moment of order \(p\) of \(\mu\) on \(A\). In the case \(A = X\) we simply write \(\mu(p)\).

**Proposition 4.7.** Assume that the hypotheses of Theorem 4.3 are satisfied and that \(\sigma(1), M(1) < \infty\). Then (i) if \(M(2) < \infty\), then \(\sigma(2) < \infty\) and (ii) if \(\sigma(2) < \infty\) and \(M(p) < \infty\) for some \(p > 2\), then \(\sigma(p) < \infty\).
Under the hypotheses of Theorem 4.3, for any \( f \) satisfying \(|f| \in X \) with positive infimum. In this case estimate (4.15) holds true with \( c = \frac{\epsilon}{p} > 0 \). Let \( H \) be the closure of \([c, 2]\) for a general \( f \) in \( X \) and let us set \( \sigma \) be the closure of \( \sigma \) for every \( p \). We deduce, for \( p \geq 2 \),

\[
\int_X \int_X |x + y|^p - |x|^p M(dy) \sigma(dx) \leq \frac{c}{p} \int_X \int_X |y|(|x + y|^{p-1} + |x|^{p-1}) M(dy) \sigma(dx) \leq \frac{c}{p} (1 + 2^{p-2}) M(1) \sigma(p - 1) + 2^{p-2} p M(p). \tag{4.13}
\]

Estimates (4.11) and (4.13) prove assertion (i).

Now, take \( p > 2 \). Applying (4.12) to \( f(x) = |x| \) and using (4.13) we have

\[
\sigma(p) \leq \sigma(2)^{p/2} + c(1 + 2^{p-2}) M(1) \sigma(p - 1) + 2^{p-2} c p M(p). \tag{4.14}
\]

The Young inequality yields

\[
\sigma(p - 1) \leq \epsilon^{p-1} \sigma(p) + \frac{1}{p} \epsilon \sigma(p)
\]

for any \( \epsilon > 0 \). Hence, from (4.14) we get

\[
\sigma(p) \leq \sigma(2)^{p/2} + c(1 + 2^{p-2}) \epsilon(p - 1) M(1) \sigma(p) + \frac{c(1 + 2^{p-2})}{\epsilon^{p-1}} M(1) + 2^{p-2} c p M(p),
\]

and choosing \( \epsilon > 0 \) small enough we conclude that

\[
\sigma(p) \leq C_1 \sigma(2)^{p/2} + C_2 M(p) + C_3 M(1),
\]

for some positive constants \( C_1, C_2 \) and \( C_3 \).

Now we introduce an appropriate class of functions that satisfies (4.12) in a “nicer” way. For every \( c > 0 \) consider the space

\[
[\mathcal{F}_A^2(X)]_c := \{ f \in \mathcal{F}_A^2(X) \mid |f| \geq c \}.
\]

Let \( \mathcal{H}_c^2 \) be the closure of \([\mathcal{F}_A^2(X)]_c\) in \( \mathcal{H}_c^2 \) and let us set

\[
\mathcal{H}_c^2 := \bigcup_{c > 0} \mathcal{H}_c^2.
\]

**Proposition 4.8.** Under the hypotheses of Theorem 4.3, for any \( f \in \mathcal{H}_c^2 \) there exists a positive constant \( c_f \), depending on \( f \), such that

\[
\text{Ent}_\sigma(|f|) \leq C c_f \int_X \int_X |f(x + y) - f(x)|^2 M(dy) \sigma(dx).
\tag{4.15}
\]

Proof. First note that from (4.12) we immediately deduce estimate (4.13) for every \( f \) in \( \mathcal{F}_A^2(X) \) with positive infimum. In this case estimate (4.15) holds true with \( c_f^{-1} = \inf f \). To deduce (4.15) for a general \( f \) in \([\mathcal{F}_A^2(X)]_c\), consider the sequence \((f_n)\) defined by \( f_n := (f^2 + 1/n)^{1/2} \) satisfying \( f_n \geq c \) for every \( n \in \mathbb{N} \). Writing (4.15) with \( f_n \) in place of \( f \) we get

\[
\int_X f_n \log f_n \sigma(dx) - m_\sigma(f_n) \log m_\sigma(f_n) \leq C c_f^{-1} \int_X \int_X |f_n(x + y) - f_n(x)|^2 M(dy) \sigma(dx).
\]

Observing that \( f_n \) converges pointwise to \( |f| \) as \( n \to \infty \), by the dominated convergence theorem we deduce estimate (4.15) for functions in \([\mathcal{F}_A^2(X)]_c\).

Now let \( f \in \mathcal{H}_c^2 \). Then, there exists \( c > 0 \) such that \( f \in \mathcal{H}_c^2 \) and a sequence \((f_n)_{n \in \mathbb{N}} \subseteq [\mathcal{F}_A^2(X)]_c \), converging to \( f \) in \( \mathcal{H}_c^2 \), as \( n \to \infty \). By the previous step

\[
\int_X |f_n| \log |f_n| \sigma(dx) - m_\sigma(|f_n|) \log m_\sigma(|f_n|) \leq C c_f^{-1} \int_X \int_X |f_n(x + y) - f_n(x)|^2 M(dy) \sigma(dx).
\]
Up to a subsequence we may assume that \((f_n)_{n \in \mathbb{N}}\) converges pointwise \(\sigma\)-a.e. to \(f\). So by the Fatou lemma we have
\[
\int_X |f| \log |f| d\sigma \leq \liminf_{n \to \infty} \int_X |f_n| \log |f_n| d\sigma
\]
and using the convergence in \(H^2\) we obtain
\[
\int_X |f| \log |f| d\sigma \leq \liminf_{n \to \infty} \int_X |f_n| \log |f_n| d\sigma
\]
\[
\leq \liminf_{n \to \infty} \left[ \left( \int_X |f_n| d\sigma \right) \log \left( \int_X |f_n| d\sigma \right) \right]
\]
\[
+ Cc^{-1} \liminf_{n \to \infty} \left[ \int_X \int_X |f_n(x+y) - f_n(x)|^2 M(dy)\sigma(dx) \right]
\]
\[
= m_\sigma(|f|) \log(m_\sigma(|f|)) + Cc^{-1} \int_X \int_X |f(x+y) - f(x)|^2 M(dy)\sigma(dx)
\]
and we conclude. \(\square\)

**Remark 4.9.** Until now, we have assumed \(Q = 0\) in the representation \(\lambda \leftrightarrow [b, Q, M]\) of the function defined in (24). If \(Q \neq 0\) is a nonnegative self-adjoint trace-class operator, then we define the operators \(Q_t = \int_0^t Q_s T_s^2 ds\), that are nonnegative, self-adjoint and trace-class as well. The measures \(\mu_t\) are associated with the triple \([b_t, Q_t, M_t]\) with \(b_t, M_t\) given by (25). Assuming that \(\sup \text{Tr} Q_t < \infty\), the operator \(Q_\infty\) is well defined and under the assumptions of Theorem 3.2 there is a unique invariant measure associated with \(P_t\) given by the convolution between the Gaussian measure \(\gamma : N(0, Q_\infty) \leftrightarrow [0, Q_\infty, 0]\) and the probability measure \(\sigma \leftrightarrow [b_\infty, 0, M_\infty]\). In such case, assume further the estimate
\[
|Q_\infty^{1/2} DP_t f| \leq \psi(t) P_t|Q_\infty^{1/2} Df|,
\]
for any \(t \geq 0\), \(f \in \mathcal{F} C_A^2(X)\) and some nonnegative \(\psi \in L^1((0, \infty))\) (see, for instance, the proof of [18] Proposition 11.2.17) for the Ornstein-Uhlenbeck semigroup and [31] Lemma 2.1 for more general semigroups in the infinite dimensional case and [31] Chapter 6 for the same estimates in finite dimension). Then, by the classical logarithmic Sobolev and Poincaré inequalities for \(\gamma\) and the product property of the entropy and the variance with respect to convolution of measures (see [26] Proposition 2.2), estimates (4) and (7) can be reformulated as
\[
\text{Ent}_{\gamma \sigma^\gamma}(f^p) \leq c \int_X f^{p-2} |Q_\infty^{1/2} Df|^2 d\gamma + C \int_X \int_X \frac{|f^p(x+y) - f^p(x)|^2}{f^p(x)} M(dy)\sigma(dx)
\]
(4.17)
for any \(p \in [1, \infty)\), \(f \in \mathcal{F} C_A^2(X)\) with positive infimum and some positive \(c\) depending on \(\|\psi\|_{L^1((0, \infty))}\), and
\[
\|f - m_\sigma(f)\|_{L^2(X, \gamma \sigma^\gamma)} \leq c \|Q_\infty^{1/2} Df\|_{L^2(X, \gamma)} + \sqrt{2C} \left( \int_X \int_X |f(x+y) - f(x)|^2 M(dy)\sigma(dx) \right)^{1/2}
\]
for any \(f \in \mathcal{F} C^2\) and some positive \(c\). In particular, estimate (4.15) becomes
\[
\text{Ent}_{\gamma \sigma^\gamma}(|f|) \leq c \int_X |f|^{-1} |Q_\infty^{1/2} Df| |f|^2 d\gamma + Cc_f \int_X \int_X |f(x+y) - f(x)|^2 M(dy)\sigma(dx)
\]
for any \(f \in \mathcal{F} C^2_0\) and some positive constant \(c_f\). Here \(C\) and \(c_f\) are the constants appearing in Theorem 4.3.

5. **Exponential integrability of Lipschitz functions**

In this section we provide sufficient conditions for exponentially growing functions to be integrable with respect to the invariant measure \(\sigma\). This type of results are known for various type of measures: for example the classical Fernique theorem (see e.g. [20]) says that functions with exponential growth are integrable with respect to Gaussian measures in infinite dimension, and similar results hold for discrete Bernoulli, or Poisson measures (see [11, 40]).
To get our results, we need to require further properties on the Lévy measure \( M \).

**Hypothesis 5.1.** For any \( s \in (0, \infty) \) it holds that \( M_s := \int_X |y|^2 e^{s|y|} M(dy) \) is finite, and there exist \( C_0 > 0, \gamma \geq 1 \) and \( s_0 > 0 \) such that for any \( s \geq s_0 \)

\[
\psi(s) := \int_{B_1^s} |y| e^{s|y|} M(dy) \geq C_0 e^{-\gamma s}.
\]  

(5.1)

Note that the function \( \psi : (0, \infty) \to (0, \infty) \) defined in (5.1) is a continuous nondecreasing function, hence its inverse is well-defined and it is continuous and nondecreasing, too. Moreover, we stress that, by (2.4), the finiteness of \( M_s \) in Hypothesis 5.1 is equivalent to the finiteness of the same integral on \( B_1^s \).

**Lemma 5.2.** Under the hypotheses of Theorem 4.3, for any \( f \in \text{Lip}_b(X) \), with Lipschitz constant less than or equal to \( \tau \), it holds that

\[
\text{Ent}_\sigma(e^f) \leq C \tau^2 M_{2\tau} m_\sigma(e^f),
\]  

(5.2)

where \( C \) is the constant appearing in (4.2) and \( M_{2\tau} \) is defined in Hypothesis 5.1.

**Proof.** By Proposition A.3 it suffices to prove the claim for \( f \in FC^2_A(X) \) and use the dominated convergence theorem to complete the proof.

For \( f \in FC^2_A(X) \), the function \( e^f \) belongs to \( FC^2_A(X) \) and has positive infimum. Moreover, the mean value theorem together with the fact that \( \|DF\|_\infty \leq \tau \) yield

\[
|e^{f(x+y)} - e^{f(x)}| = e^0|f(x+y) - f(x)| \leq \tau e^0|y|
\]

for any \( x, y \in X \) and some \( \theta \in (f(x+y) \land f(x), f(x+y) \lor f(x)) \). We can then apply (4.2) with \( p = 1 \) to get

\[
\text{Ent}_\sigma(e^f) \leq C \int_X \int_X \frac{|e^{f(x+y)} - e^{f(x)}|^2}{e^{f(x)}} M(dy)\sigma(dx)
\]

\[
\leq C \tau^2 \int_X \int_X |y|^2 e^{2\tau f(x)} M(dy)\sigma(dx)
\]

\[
\leq C \tau^2 \int_X \int_X |y|^2 e^{2(\theta - f(x))} e^{f(x)} M(dy)\sigma(dx)
\]

\[
\leq C \tau^2 \int_X \int_X |y|^2 e^{2|f(x+y) - f(x)|} e^{f(x)} M(dy)\sigma(dx)
\]

\[
\leq C \tau^2 \int_X \int_X |y|^2 e^{2\tau|y|} e^{f(x)} M(dy)\sigma(dx)
\]

\[
= C \tau^2 M_{2\tau} \int_X e^{f(x)} \sigma(dx).
\]

The next result is a Fernique type theorem for the measure \( \sigma \). The key tool is an estimate of the tail of the distribution of a Lipschitz continuous function with respect to \( \sigma \) in terms of the function \( \psi \) introduced in (5.1) (cf. [9] for the Poisson case).

**Theorem 5.3.** Assume that the hypotheses of Theorem 4.3 and Hypothesis 5.1 hold true. Any Lipschitz continuous function \( g : X \to \mathbb{R} \), with Lipschitz constant less than or equal to \( 1 \), belongs to \( L^1(X, \sigma) \) and there exist positive constants \( c_0, c_1, c_2 \) and \( t_0 \) such that

\[
\sigma\left( \{ g \geq m_\sigma(g) + t \} \right) \leq \begin{cases} 
\exp(-c_0 t^2), & t \in (0, t_0); \\
\exp(-c_1 t \psi^{-1}(c_2 t)), & t \in [t_0, \infty). 
\end{cases}
\]

(5.3)

Moreover, for sufficiently small \( c > 0 \),

\[
\int_X e^{c \psi^{-1}(|y|)} d\sigma < \infty.
\]

(5.4)
Proof. We divide the proof in three steps.

Step 1. We start considering \( g \in \text{Lip}_0(X) \), with Lipschitz constant less than or equal to 1. The function \( \tau g \), with \( \tau > 0 \) satisfies the assumptions of Lemma 5.2 and consequently,

\[
\text{Ent}_\sigma(e^{\tau g}) \leq C\tau^2 M_{2r}m_\sigma(e^{\tau g}).
\]  

(5.5)

Moreover, the function \( G(\tau) := m_\sigma(e^{\tau g}) \) is differentiable and, from (5.3), its derivative \( G'(\tau) = m_\sigma(ge^{\tau g}) \) satisfies

\[
\tau G'(\tau) - G(\tau) \log G(\tau) \leq C\tau^2 M_{2r}G(\tau), \quad \tau > 0.
\]  

(5.6)

Thus, set

\[
H(\tau) := \begin{cases} m_\sigma(g), & \tau = 0; \\ \tau^{-1}\log G(\tau), & \tau > 0. \end{cases}
\]

By (5.6) we deduce that \( H'(\tau) \leq CM_{2r} \) whence, integrating from 0 to \( t \), we have

\[
H(t) - H(0) \leq C \int_0^t \int_X |y|^2 e^{2\tau|y|} M(dy)d\tau
\]

\[
= C \int_X |y|^2 \int_0^t e^{2\tau|y|} d\tau M(dy)
\]

\[
= \frac{C}{2} \int_X |y|(e^{2|y|} - 1)M(dy) =: \theta(t)
\]

(5.7)

or, equivalently, \( m_\sigma(e^{\tau g}) \leq \exp \left( t(\theta(t) + m_\sigma(g)) \right) \). Applying the Chebyshev inequality we get

\[
\sigma \left( \{ g \geq m_\sigma(g) + s \} \right) \leq \exp \left( -ts + \frac{Ct}{2} \int_X |y|(e^{2|y|} - 1)M(dy) \right).
\]

Now, using the inequality \( e^{ax} - 1 \leq (e^a - 1)x \) for any \( a > 0 \) and \( x \in (0,1) \), we can estimate

\[
\sigma \left( \{ g \geq m_\sigma(g) + s \} \right) \leq \exp \left( -ts + \frac{Ct}{2}(e^{2t} - 1) \int_B |y|^2 M(dy) + \frac{Ct}{2} \int_{B^c} |y|(e^{2|y|} - 1)M(dy) \right)
\]

\[
= \exp \left( -ts + C_1 t(e^{2t} - 1) + C_2 t \int_{B^c} |y|(e^{2|y|} - 1)M(dy) \right) =: \exp(\varphi(t,s))
\]

(5.8)

for any \( t, s > 0 \). Let us fix \( 0 < \alpha < (C_1 C_0^{-1} e^{s_0(1-\gamma)} + C_2)^{-1} \) and distinguish two cases.

As the first one we take \( s \geq C_0 e^{s_0} \), where \( s_0 > 0 \) (see Hypothesis 5.1) is such that \( \psi(\tau) \geq C_0 e^{\gamma \tau}, \quad \tau \geq s_0 \).

In such case, we choose \( t = 2^{-1}\psi^{-1}(\alpha s) \) and get

\[
\varphi(2^{-1}\psi^{-1}(\alpha s), s) = -2^{-1}s\psi^{-1}(\alpha s) + C_1 2^{-1}\psi^{-1}(\alpha s)(e^{\psi^{-1}(\alpha s)} - 1)
\]

\[
+ C_2 2^{-1}\psi^{-1}(\alpha s) \int_{B^c} |y|(e^{\psi^{-1}(\alpha s)|y|} - 1)M(dy)
\]

\[
\leq -2^{-1}s\psi^{-1}(\alpha s) + C_1 2^{-1}\psi^{-1}(\alpha s)e^{\psi^{-1}(\alpha s)} + C_2 2^{-1}\alpha s\psi^{-1}(\alpha s)
\]

\[
= -2^{-1}s\psi^{-1}(\alpha s)(1 - C_1 s^{-1}e^{\psi^{-1}(\alpha s)} - C_2 \alpha).
\]

(5.9)

Using Hypothesis 5.1 we deduce that \( e^{\psi^{-1}(\alpha s)} \leq (s C_0^{-1})^{1/\gamma} \) for any \( s \geq C_0 e^{s_0} \). Applying the last estimate with \( z = \alpha s \) in (5.3), we conclude that \( \varphi(2^{-1}\psi^{-1}(\alpha s), s) \leq -c_1 s\psi^{-1}(\alpha s) \) for some positive \( c_1 \).

As second case, if \( s \leq C_0 e^{s_0} =: s_1 \), choosing \( t = \beta s \) with a suitable \( \beta \in (0,1) \) we can show that \( \varphi(\beta s, s) \leq -\beta s^2 \) for some \( \beta > 0 \). Indeed, using the estimate

\[
e^{\beta s} - 1 \leq s e^{\beta s} - 1 \leq s \eta > 0,
\]
we have
\[ \varphi(\beta s, t) = -\beta s^2 + C_1\beta s(e^{2\beta s} - 1) + C_2\beta s \int_{B_1} |y|(e^{2\beta s}|y| - 1)M(dy) \]
\[ \leq -\beta s^2 + C_1\beta s^2 s_1^{-1}(e^{2\beta s_1} - 1) + C_2\beta s^2 s_1^{-1} \int_{B_1} |y|(e^{2\beta s}|y| - 1)M(dy) \]
\[ = -\beta s^2 \left(1 - C_1 s_1^{-1}(e^{2\beta s_1} - 1) - C_2 s_1^{-1} \int_{B_1} |y|(e^{2\beta s}|y| - 1)M(dy)\right). \]
(5.9)

Now, if \( \beta \in (0, 1) \) is such that
\[ \beta \leq s_1 \left[ C_1 e^{2s_1} + C_2 \int_{B_1} |y|e^{2s_1}|y|M(dy) \right]^{-1} \]
then the term in round brackets in the right hand side of (5.9) is positive and \( \varphi(\beta s, t) \leq -\beta s^2 \) for some \( \beta > 0 \). Summing up, estimate (5.3) follows.

**Step 2.** Here we consider the general case and we approximate any Lipschitz continuous function \( g \), with Lipschitz constant less than or equal to 1, by the sequence of functions \( (g_n)_n \) defined by \( g_n := (-n) \wedge (g \vee n), n \in \mathbb{N} \) which converges pointwise to \( g \). Then, applying the previous step to \(-|g_n|\) we infer that
\[ \sigma(\{|g_n| \leq m_\sigma(|g_n|) - t\}) \leq \exp(-c_1 t \psi^{-1}(c_2 t)) \]
for \( t \) large enough. Choosing \( t_0 \) such that \( \exp(-c_1 t_0 \psi^{-1}(c_2 t_0)) \leq 1/2 \) and \( m \) such that \( \sigma(|g| \geq m) < 1/2 \) and using that \( |g_n| \leq |g| \) we deduce that \( \|g_n\|_{L^1(X, \sigma)} = m_\sigma(|g_n|) \leq m + t_0 \) for any \( n \in \mathbb{N} \). Indeed, by contradiction, if \( m_\sigma(|g_n|) > m + t_0 \), we have
\[ \sigma(|g| < m) \leq \sigma(|g_n| < m) = \sigma(|g_n| + t_0 < m + t_0) \]
\[ \leq \sigma(|g_n| + t_0 < m_\sigma(|g_n|)) = \sigma(|g_n| < m_\sigma(|g_n|) - t_0) \leq 1/2 \]
which yields a contradiction with \( \sigma(|g| \geq m) < 1/2 \), as \( \sigma \) is a probability measure. Hence, from the previous estimate and the monotone convergence theorem we get that \( g \in L^p(X, \sigma) \) for any \( p \geq 1 \) and that \( \|g_n\|_{L^p(X, \sigma)} \) converges to \( \|g\|_{L^p(X, \sigma)} \) as \( n \to \infty \). Moreover, using (5.3) and that \( m_\sigma(|g_n|) \leq m + t_0 \) we obtain
\[ \sigma(|g_n| \geq m + t_0) \leq \exp(-c_1 t \psi^{-1}(c_2 t_0)) \leq \exp(-c_1 t_0 \psi^{-1}(c_2 t_0)) \]
for \( t \geq t_0 \) whence \( \sup_n \|g_n\|_{L^2(X, \sigma)} < \infty \). By a standard compactness argument we get that \( g_n \) converges to \( g \) in \( L^2(X, \sigma) \) as \( n \to \infty \) and hence in measure, i.e., for any \( \varepsilon > 0 \)
\[ \lim_{n \to \infty} \sigma(\{|g_n - g| \geq \varepsilon\}) = 0. \] (5.10)

From (5.10) and (5.3) for \( g_n \) we infer that (5.3) holds true for \( g \) too.

**Step 3.** Here we prove the last statement. Let \( c > 0 \) and \( \varphi_c(x) := c \psi^{-1}(|x|) \). Observing that \( \varphi_c \) is a nondecreasing and invertible function, we have
\[ \int_X e^{c \psi^{-1}(|y|)}d\sigma = \int_X e^{\varphi_c(g)}d\sigma \]
\[ \leq 1 + \int_1^\infty \sigma(\{\varphi_c(g) > s\})ds \]
\[ = 1 + \int_1^\infty \sigma(\{\varphi_c(g) > \log s\})ds \]
\[ = 1 + \int_1^\infty \sigma(\{g > \varphi_c^{-1}(\log s)\})ds \]
\[ \leq K_1 + c \int_T^\infty \sigma(\{g > t\})e^{\varphi_c(t)}(K_2 + \log t)dt, \]
for some $K_1, K_2 > 0$ and $T := \varphi_{c}^{-1}(0) \vee (C_0 e^{\gamma_0})$. Now, performing the change of variable $\tau = t - m_\sigma(g)$ and using the estimate (9.3) we have
\[
\int_T^\infty \sigma\{g > t\} e^{\psi^{-1}(\tau)}(K_2 + \log t) dt
\]
\[
= \int_{T-m_\sigma(g)}^\infty \sigma\{g > \tau + m_\sigma(g)\} e^{\psi^{-1}(\tau + m_\sigma(g))}(K_2 + \log(\tau + m_\sigma(g))) d\tau
\]
\[
\leq C + \int_0^\infty \sigma\{g > \tau + m_\sigma(g)\} e^{\psi^{-1}(\tau + m_\sigma(g))}(K_2 + \log(\tau + m_\sigma(g))) d\tau
\]
\[
\leq C + \int_0^\infty e^{-c_1 \tau}(\tau^2 + \tau + m_\sigma(g))^2 e^{\psi^{-1}(\tau + m_\sigma(g))}(K_2 + \log(\tau + m_\sigma(g))) d\tau,
\]
for some positive $C$. Then, for $c$ small enough the function $e^{\psi^{-1}(\vert g \vert)}$ is summable and the proof is complete. \hfill \Box

**Remark 5.4.** Under the hypotheses of Theorem 5.3 and assuming $\psi(s) = C_0 e^{\gamma s}$ for some $C_0 > 0$, $\gamma \geq 1$ and $s$ big enough, we have
\[
\sigma\{g > m_\sigma(g) + t\} \leq \exp(-c_1 t \log(c_2 t))
\]
for some positive $c_1, c_2$ and $t$ large enough. Viceversa, if Hypothesis 5.1 holds true with a strict inequality in (5.1) then there exist $K < 1$ such that $\psi^{-1}(t) \leq K \log t$ as $t \to \infty$. If in Hypothesis 5.1 the estimate (5.1) holds true for any $\gamma \geq 1$, then we conclude that $\psi^{-1}(t) = o(\log t)$ as $t \to \infty$.

**Corollary 5.5.** Under the hypotheses of Theorem 5.3 it follows that $\sigma(p) < \infty$ for any $p \geq 1$.

**Proof.** It suffices to take $g(x) = |x|$ in (5.4) and observe that there exist $r_0 \in (0, \infty)$ such that $\psi^{-1}(r_0) > 0$ and $c_0(x)\psi^{-1}(g(x)) \geq c_0\psi^{-1}(r_0)|x|$ for any $|x| \geq r_0$.

**Remark 5.6.** Let us show how the results in this section can be reformulated if the Gaussian term $Q$ in the representation of $\lambda \leftrightarrow [h, Q, M]$ does not vanish and satisfies the assumptions in Remark 4.3. We recall that, in this case, under the hypotheses of Theorem 5.3 there is a unique invariant measure associated with $P_t$, given by $\gamma * \sigma$, where $\gamma$ is the Gaussian measure defined in Remark 4.4. Assuming further that estimate (4.10) holds true, let us reformulate the results in Theorem 5.3. Indeed, we can prove that any Lipschitz continuous function $g : X \to \mathbb{R}$, with Lipschitz constant less than or equal to 1, belongs to $L^1(X, \gamma * \sigma)$ and there exist positive constants $\tilde{c}_0, c_1, c_2$ and $\tilde{t}_0$ such that
\[
(\gamma * \sigma)\{g \geq m_\sigma(g) + t\} \leq \begin{cases} \exp(-\tilde{c}_0 t^2), & t \in (0, \tilde{t}_0); \\ \exp(-c_1 t^2), & t \in [\tilde{t}_0, \infty). \end{cases}
\] (5.11)

Moreover, for sufficiently small $\tilde{c} > 0$,
\[
\int_X e^{\gamma \psi^{-1}(\vert g \vert)} d(\gamma * \sigma) < \infty.
\] (5.12)

To prove this fact, it suffices to perform some changes in Step 1 in the proof of Theorem 5.3 as described below. First of all, notice that $m_\gamma(f)$ and $m_\sigma(f)$ can be estimated by $m_{\gamma * \sigma}(f)$ for any nonnegative function $f$. Thanks to estimate (1.17), formula (5.2) becomes
\[
\text{Ent}_{\gamma * \sigma}(e^f) \leq K^2 (M_{2\tau} + 1) m_{\gamma * \sigma}(e^f)
\] (5.13)
for any $f \in \mathcal{C}_0^1(\chi)$ with $\|Df\|_\infty \leq \tau$ and some positive constant $K$ depending also by $\|Q\|_\infty$. Then, considering $\tilde{G}(\tau) := m_{\gamma * \sigma}(e^{\tau g})$ and
\[
\tilde{H}(\tau) := \begin{cases} m_{\gamma * \sigma}(g), & \tau = 0, \\ \tau^{-1} \log \tilde{G}(\tau), & \tau > 0 \end{cases}
\]
in place of $G$ and $H$ (see Step 1 in the proof of Theorem 5.3) and using (5.13) we deduce that $\tilde{H}(\tau) \leq K(M_{2\tau} + 1)$, whence integrating from 0 to $t$ we have
\[
\tilde{H}(t) - \tilde{H}(0) \leq K(\theta(t) + t)
\]
where, up to a constant, θ is defined by $\theta(t) := \int_X |y|(e^{2t|y|} - 1)M(dy)$. At this point, the proof could be repeated slavishly if we prove that there exists a positive constant $c$ such that $t \leq c\theta(t)$ for any $t > 0$. Indeed, in such case we would have $\tilde{H}(t) - \tilde{H}(0) \leq K'\theta(t)$ for some positive $K'$ as in [5.7]. The estimate $t \leq c\theta(t)$ can be proved using that $e^t - 1 \geq x$ for any $x \geq 0$ and observing that

$$\theta(t) = \int_X |y|(e^{2t|y|} - 1)M(dy) \geq 2t \int_X |y|^2M(dy) \geq 2tM,$$

for any $s \in (0, \infty)$, see Hypothesis 5.1. In this way all the results stated for $\sigma$ in Theorem 5.3 are true for $\gamma \ast \sigma$ too. In particular, (5.11) and (5.12) follow.

6. EXAMPLES

Here we provide examples of generalised Mehler semigroups satisfying our assumptions and to which our results can be applied.

6.1. Ornstein–Uhlenbeck operators with fractional diffusion. We consider the Ornstein–Uhlenbeck operator defined by

$$(\mathcal{L}u)(x) = \frac{1}{2}[\text{Tr}^s(QD^2u)](x) + \langle Bx, Du(x) \rangle, \quad s \in (0, 1), \ x \in \mathbb{R}^d.$$

We assume that $Q$ is a symmetric nonnegative definite matrix, $B$ is a symmetric nonpositive definite matrix and $\text{Tr}^s(QD^2u)$ is the pseudo-differential operator with symbol $(Q\xi, \xi)^s$. The realisation of $\mathcal{L}$ in $L^2(\mathbb{R}^d)$ has been studied in [4] and in the space of H"older continuous functions in [22]. The associated generalised Mehler semigroup is given by

$$\left( P_t f \right)(x) = \int_{\mathbb{R}^d} f(e^{tB}x + y)\mu_t(dy),$$

where

$$\tilde{\mu}_t(x) = \exp \left(-\int_0^t \lambda(e^{2s}B)\xi d\tau\right) = \exp \left(-\frac{1}{2}\int_0^t |Q^{1/2}e^{sB}\xi|^{2s}d\tau\right), \quad \xi \in \mathbb{R}^d.$$

If $Q$ is invertible, then the Lévy measure $M$ defined in (2.3) has the form

$$M(A) = \frac{1}{\text{det}Q^{1/2}} \int_{\mathbb{R}^d} \frac{\chi_A(y)}{|Q^{-1/2}y|^{2s+d}}dy, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Moreover, from [20] Section 7, Hypotheses 3.1 are satisfied and consequently there exists a unique invariant measure $\sigma$ for $P_t$. Such measure is absolutely continuous with respect to the Lebesgue measure with density $\rho$ satisfying

$$\frac{1}{C(1 + |x|^{d+2s})} \leq \rho(x) \leq \frac{C}{1 + |x|^{d+2s}}$$

for some $C > 1$ and any $x \in \mathbb{R}^d$ (see [8]). It is also clear that the measure $M$ satisfies (2.7) whenever $s \in \left(\frac{1}{2}, 1\right)$. From now on we restrict our attention to this case and prove that, under further conditions, Hypotheses 5.1 are satisfied, too, as the next proposition shows.

**Proposition 6.1.** Assume that $Q$ and $B$ are as above. If $Q$ and $B$ are invertible and commute, then Hypotheses 4.7 hold true whenever

$$\min \{||\lambda_i|| \mid i = 1, \ldots, d\} > \frac{[\text{Tr}B]}{2s + d},$$

where the $\lambda_i$ are the eigenvalues of $B$.

**Proof.** Let $\Lambda := \max \{||\lambda_i|| \mid i = 1, \ldots, d\} < 0$. We have

$$[M \ast e^{-tB}](A) = \frac{1}{\text{det}Q^{1/2}} \int_{\mathbb{R}^d} \frac{\chi_A(e^{tB}y)}{|Q^{-1/2}y|^{2s+d}}dy = \frac{e^{-t\text{Tr}B}}{\text{det}Q^{1/2}} \int_{\mathbb{R}^d} \frac{\chi_A(z)}{|Q^{-1/2}e^{-tB}z|^{2s+d}}dz$$
there exists a finite measure \( \mu \), details. Let us show that if \( 6.2. \) holds as well for any \( f \), holds true for any \( f \) are satisfied. Then the logarithmic Sobolev inequality so the function \( h \) in Hypotheses [4.1] is
\[
h(t) := e^{-t(TB-(2s+d)\lambda)},
\]
that clearly belongs to \( L^1((0, \infty)) \) whenever \( |\lambda| > \frac{|TB|}{2s+d} \). \( \square \)

As examples of matrices satisfying the hypotheses of Proposition 6.1, one can consider as \( \beta \) positive definite invertible matrix and as \( B \) one of the following:
(a) \( B = -\beta I \) for any \( \beta > 0 \);
(b) \( B = -Q^\alpha \) for some \( \alpha > 0 \). In this case, condition [6.1] becomes
\[
r_1 > \left( \frac{1}{2s + d - 1} \sum_{i=2}^{d} r_i \right)^{1/\alpha},
\]
where \( r_1 \) is the minimum eigenvalue of \( Q \).

Under the previous conditions all the assumptions in Theorem 4.3 and Propositions 4.4 and 4.8 are satisfied. Then the logarithmic Sobolev inequality
\[
\text{Ent}_\sigma(|f|) \leq K_{1,f} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dy \sigma(dx)
\]
holds true for any \( f \in \mathcal{E}_0^2 \) and some positive \( K_{1,f} \). The Poincaré inequality
\[
\|f - m_\sigma(f)\|_{L^2(X,\sigma)} \leq K_{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{d+2s}} dy \sigma(dx)
\]
holds as well for any \( f \in \mathcal{E}_0^2 \) and some positive \( K_{2} \).

6.2. An example in infinite dimension. Let \( X \) be a separable Hilbert space and let \( \lambda \leftrightarrow [b, 0, M] \) where \( b \in X \) and \( M \) is an infinitely divisible \( \alpha \)-stable Lévy measure. Consider the semigroup \( (T_t)_{t \geq 0} \) defined by \( T_t x := e^{-\beta t} x \) for some \( \beta > 0 \) and every \( x \in X \) and \( t \geq 0 \).

We recall that an infinitely divisible Lévy measure is \( \alpha \)-stable with \( \alpha \in (0, 2) \) if and only if there exists a finite measure \( \mu \) concentrated on the unit sphere of \( X \) such that for any Borel set \( B \subseteq X \)
\[
M(B) = \int_{0}^{\infty} r^{-1-\alpha} \left( \int_{S_1} \chi_B(rx) \mu(dx) \right) dr,
\]
where \( S_1 = \{ x \in X \mid |x| = 1 \} \) denotes the unit sphere of \( X \), see [30] Theorem 6.2.8 for more details. Let us show that if \( \alpha \in (1, 2) \) and \( \mu \) is a symmetric measure then Hypothesis 2.1 6.1 and 4.1 are all satisfied.

Indeed, observing that
\[
\int_{B_1} |x|M(dx) = \int_{1}^{\infty} r^{-1-\alpha} \left( \int_{S_1} |rx| \mu(dx) \right) dr = \mu(S_1) \int_{1}^{\infty} r^{-\alpha} dr < \infty,
\]
we deduce that Hypothesis 2.1 holds true. Now observe that
\[
\lim_{t \to \infty} \frac{b_t}{\beta} = \lim_{t \to \infty} \left( \int_{0}^{t} T_r bdr + \int_{0}^{t} \int_{X} T_r x(\chi_{B_1}(T_r x) - \chi_{B_1}(x)) M(dx) dr \right)
\]
\[
= \lim_{t \to \infty} \left( \int_{0}^{t} e^{-r \beta} bdr + \int_{0}^{t} \int_{X} s^{-1-\alpha} \int_{S_1} e^{-r \beta} sy(\chi_{B_1}(e^{-r \beta} sy) - \chi_{B_1}(sy)) \mu(dy) ds dr \right)
\]
\[
= \lim_{t \to \infty} \frac{b_t (1 - e^{-t \beta})}{\beta} = \left( \int_{0}^{t} e^{-r \beta} \left( \int_{0}^{s^{-\alpha}} ds + \int_{0}^{1} s^{-\alpha} ds \right) dr \int_{S_1} \mu(dy) \right) = \frac{b_t}{\beta}.
\]
where in the last equality we used the symmetry of $\mu$. Furthermore, arguing as in (3.4) we deduce
\[
\int_0^\infty \int_X (1 \wedge |T_t x|^2) M(dx)dr = \lim_{t \to \infty} \int_0^t \int_X (1 \wedge |T_t x|^2) M(dx)dr \\
\leq \frac{1}{2\beta} \int_X (1 \wedge |x|^2) M(dx) < \infty,
\]
hence Hypothesis 4.1(ii) holds true. Clearly Hypothesis 4.1(iii) holds true too and Theorem 4.3 states that for every $A \sigma$ can be applied to prove the existence of a unique invariant measure hence Hypothesis 3.1(ii) holds true. Clearly Hypothesis 3.1(iii) holds true too and Theorem 3.2

recalling formula (3.2) we have
\[
c > \]
for some $c > 0$ and $\sigma \in (0, 1)$. First of all we show that $P_t$ admits an invariant measure. Indeed, recalling formula (3.2) we have
\[
\lim_{t \to \infty} b_t = \lim_{t \to \infty} \left( \int_0^t T_t b dr + \int_0^t \int_X T_t x (\chi_{B_t}(T_t x) - \chi_{B_t}(x)) M(dx)dr \right) \\
= \lim_{t \to \infty} \left( \frac{b}{\beta} (1 - e^{-\beta t}) + c \int_0^t e^{-\beta x} \int_x^{e^t x} \frac{e^{-x^2}}{|x|^{1+2s}} dx dr - c \int_0^t e^{-\beta x} \int_1^e \frac{e^{-x^2}}{|x|^{1+2s}} dx dr \right) \\
= \lim_{t \to \infty} \frac{b}{\beta} (1 - e^{-\beta t}) = \frac{b}{\beta}. \]
Moreover
\[
\int_0^\infty \int_X (1 \wedge |T_t x|^2) M(dx)dr = c \int_0^\infty \int_X (1 \wedge |e^{-\beta x}|^2) \frac{e^{-x^2}}{|x|^{1+2s}} dx dr \\
= 2c \int_0^\infty \int_{e^{-\beta x}}^{e^{-\beta t}} \frac{e^{-x^2}}{|x|^{1+2s}} dx dr + 2c \int_0^\infty e^{-2\beta x} \int_{-1}^e \frac{x^2 e^{-x^2}}{|x|^{1+2s}} dx dr
\]

6.3. A modified version of a model introduced by Koponen. A tempered stable process is obtained from a one dimensional stable process by “tempering” the large jumps, i.e., by damping exponentially the tails of the Lévy measure. This class of Lévy processes has been introduced by Koponen in [25] for options pricing (see also [36, Section 13.4.3]).

Here we slightly modify the process introduced by Koponen providing a Lévy measure $M$ and consequently a generalised Mehler semigroup $P_t$ which satisfy all our assumptions. For simplicity we consider the one dimensional and centred case but it is not difficult to extend the results for any dimension and in the non-centred case.

Let $X = \mathbb{R}$ and consider the semigroup $T_t x := e^{-\beta t} x$ for some $\beta > 0$. The Lévy process we are interested in is identified by the triple $[b, 0, M]$ where $b \in \mathbb{R}$ and
\[
M(dx) := \frac{e^{-x^2}}{|x|^{1+2s}} dx,
\]
for some $c > 0$ and $s \in (0, 1)$. First of all we show that $P_t$ admits an invariant measure. Indeed,
respectively. Let us start from Hypotheses 4.1. If \( A \subseteq K \subseteq L \), we get
\[
\int_{B_1^c} |x|M(dx) = 2c \int_1^\infty \frac{e^{-x^2}}{x^{1+2s}} dx + 2c \int_0^\infty e^{-2t\beta} \int_0^t e^{-\gamma t} \frac{1}{x^{2s-1}} dx dt
\]
whence Hypothesis 2.1 holds true and \( C_A^2(\mathbb{R}) \) is a core for the generator of the semigroup \( P_t \) in \( L^2(\mathbb{R}, \sigma) \).

In order to apply the results in Sections 4 and 5 we have to verify Hypotheses 4.1 and 5.1, respectively. Let us start from Hypotheses 4.1. If \( A \) be a Borel subset of \( \mathbb{R} \), then
\[
[M \circ T^{-1}_t](A) = \int \chi_{T^{-1}_t(A)}(x)e^{-x^2/2|x|^{1+2s}} dx = ce^{t\beta} \int \chi_{T^{-1}_t(A)}(e^{\beta t} y)e^{-e^{2t\beta}y^2} dy
\]
\[
= ce^{-2st\beta} \int \chi_A(y)\frac{e^{-e^{2t\beta}y^2}}{|y|^{1+2s}} dy = ce^{-2st\beta} \int \chi_A(y)e^{(1-e^{2s\beta})y^2} \frac{e^{-y^2}}{|y|^{1+2s}} dy
\]
\[
\leq ce^{-2st\beta} \int \chi_A(y)\frac{e^{-y^2}}{|y|^{1+2s}} dy = e^{-2st\beta} M(A).
\]
Now, the function \( h(t):= e^{-2st\beta} \) is continuous in \((0, \infty)\) and belongs to \( L^1((0, \infty)) \), hence all the results in Section 4 can be applied and, in particular, Theorem 4.3 states that for every \( f \in C_b(\mathbb{R}) \) with positive infimum and \( p \in [1, \infty) \)
\[
\text{Ent}_\sigma(f^p) \leq C \int \frac{|f^p(x + y) - f^p(x)|^2}{f^p(x)} \frac{e^{-y^2}}{|y|^{1+2s}} dy \sigma(dx).
\]
To conclude, let us consider Hypothesis 5.1. Indeed recalling that for any \( \alpha > 0 \) there exists \( K(\alpha) > 0 \) such that \( y^\alpha e^{\alpha y - y^2} \leq K(\alpha) \) for any \( y > 0 \) we have
\[
\int_{B_1^c} |y|^2 e^{\alpha |y|} M(dy) = 2c \int_1^\infty y^2 e^{\alpha y - y^2} \frac{1}{y^{1+2s}} dy
\]
\[
\leq 2c \int_1^\infty y^2 e^{\alpha y - y^2} dy
\]
\[
\leq 2cK(\alpha) \int_1^\infty \frac{1}{y^2} dy = \frac{2c}{3} K(\alpha),
\]
and, again, for \( \alpha \in (0, \infty) \)
\[
\psi(\alpha) = \int_{B_1^c} |y|^\alpha e^{\alpha |y|} M(dy) = 2c \int_1^\infty ye^{\alpha y - y^2} \frac{e^{-y^2}}{y^{1+2s}} dy \geq 2ce^{\alpha} \int_1^\infty \frac{e^{-y^2}}{y^{2s}} dy = Ce^\alpha.
\]
Thus, all the results in Section 5 can be applied and, in particular, Theorem 5.3 guarantees the exponential integrability of Lipschitz continuous functions with respect to \( \sigma \).

**Remark 6.2.** All the examples considered above can be modified adding a Gaussian term satisfying the assumptions in Remark 4.9. In such case, all the estimates and the results stated for \( \sigma \) can be reformulated for \( \gamma * \sigma \), see Remarks 5.3 and 5.4 for a detailed descriptions of the results.

**Appendix A. Approximation of and by Lipschitz functions**

We collect here, for the reader’s convenience, some results we used in the paper, even though their proofs are quite standard.
Lemma A.1. Let $X$ be a metric space, $\theta$ be a finite Radon measure on $X$ and $f : X \to \mathbb{R}$ be a Borel function. For every $\varepsilon > 0$ there exists a bounded uniformly continuous function $g_\varepsilon : X \to \mathbb{R}$ such that

$$\theta(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) < \varepsilon.$$ 

Furthermore $\|g_\varepsilon\|_\infty \leq 2\|f\|_\infty$.

Proof. Consider a compact set $K_0 \subseteq X$ such that $\theta(X \setminus K_0) < \varepsilon/2$. The function $f|_{K_0} : K_0 \to \mathbb{R}$ is a Borel function and by Luzin theorem (see [43, Theorem 2.24]) there exists a continuous function $\tilde{g}_\varepsilon : K_0 \to \mathbb{R}$ such that

$$\theta(\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\}) < \frac{\varepsilon}{2}.$$ 

and

$$\sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| \leq \sup_{x \in K_0} |f|_{K_0}(x)| = \sup_{x \in K_0} |f(x)|.$$ 

The Heine–Cantor theorem says that $\tilde{g}_\varepsilon$ is a bounded and uniformly continuous function on $K_0$. Consider the bounded and uniformly continuous extension (see [33])

$$g_\varepsilon(x) = \begin{cases} \tilde{g}_\varepsilon(x) & x \in K_0; \\ \inf_{y \in K_0} \tilde{g}_\varepsilon(y) & \inf_{y \in K_0} \tilde{g}_\varepsilon(y) = \infty. \end{cases}$$

An easy computation gives that for every $x \notin K_0$

$$|g_\varepsilon(x)| \leq \sup_{z \in X} |f(z)|.$$ 

Eventually we get

$$\sup_{x \in X} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |g_\varepsilon(x)| + \sup_{x \in X \setminus K_0} |g_\varepsilon(x)| \leq \sup_{x \in K_0} |\tilde{g}_\varepsilon(x)| + \sup_{x \in X \setminus K_0} |g_\varepsilon(x)| \leq 2 \sup_{x \in X} |f(x)|.$$ 

Furthermore

$$\theta(\{x \in X \mid f(x) \neq g_\varepsilon(x)\}) \leq \theta(X \setminus K_0) + \theta(\{x \in K_0 \mid f(x) \neq g_\varepsilon(x)\})$$

$$= \theta(X \setminus K_0) + \theta(\{x \in K_0 \mid f|_{K_0}(x) \neq \tilde{g}_\varepsilon(x)\}) < \varepsilon.$$ 

□

Proposition A.2. Let $X$ be a metric space, let $\theta$ be a finite Radon measure on $X$ and $p \geq 1$. The space $\text{Lip}_p(X)$ of bounded and Lipschitz continuous functions on $X$ is dense in $L^p(X, \theta)$.

Proof. Fix a version of a $f \in L^p(X, \theta)$. For every $k \in \mathbb{N}$ set

$$f_k(x) = \begin{cases} k & f(x) > k; \\ f(x) & -k \leq f(x) \leq k; \\ -k & f(x) < -k. \end{cases}$$

Applying Lemma [A.1] for every $k \in \mathbb{N}$ there is a bounded and uniformly continuous function $\tilde{f}_k$ such that

$$\theta(\{x \in X \mid \tilde{f}_k(x) \neq f_k(x)\}) \leq \frac{1}{2^k},$$

and $\sup_{x \in X} |\tilde{f}_k(x)| \leq 2 \sup_{x \in X} |f_k(x)| \leq 2k$. Theorem 1 in [34] gives a function $g_k \in \text{Lip}_p(X)$ such that

$$\|g_k - \tilde{f}_k\|_\infty \leq \frac{1}{2^k}.$$ 

We have

$$\|g_k - f\|_{L^p(X, \theta)} \leq \|g_k - \tilde{f}_k\|_{L^p(X, \theta)} + \|	ilde{f}_k - f_k\|_{L^p(X, \theta)} + \|f_k - f\|_{L^p(X, \theta)}.$$ 

Observe that $f_k$ converges pointwise $\theta$-a.e. to $f$ and $|f_k| \leq |f|$, then by the dominated convergence theorem we get $\lim_{k \to \infty} \|f_k - f\|_{L^p(X, \theta)} = 0$. Furthermore

$$\lim_{k \to \infty} \|g_k - \tilde{f}_k\|_{L^p(X, \theta)} \leq \theta(X) \frac{1}{2^k} \lim_{k \to \infty} \|g_k - \tilde{f}_k\|_\infty = 0,$$
and

$$\lim_{k \to \infty} \| \tilde{f}_k - f_k \|_{L^p(X, \theta)} = \lim_{k \to \infty} \left( \int_{\{ \tilde{f}_k \neq f_k \}} \left| \tilde{f}_k(x) - f_k(x) \right|^p \, d\theta(x) \right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{p-1}{p}} (2^p + 1)^{\frac{1}{p}} \lim_{k \to \infty} k \left( \theta \left( \left\{ x \in X \mid \tilde{f}_k(x) \neq f_k(x) \right\} \right) \right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{p-1}{p}} (2^p + 1)^{\frac{1}{p}} \lim_{k \to \infty} \frac{k}{2^{k/p}} = 0. \quad \square$$

In the following proposition we state an approximation result for Lipschitz continuous and bounded functions by means of cylindrical regular functions. We give just a sketch of the proof of the result emphasizing the construction of the approximant sequence, see [17, Section 2.1] and the proof of [37] Lemma 2.5 for further details.

**Proposition A.3.** Assume that Hypothesis 2.1 holds true and let $g \in \text{Lip}_b(X)$, with $\text{Lip} g \leq L$. Then, there exists a sequence $\{g_{m,n} \mid m, n \in \mathbb{N} \} \subseteq \mathcal{F}_A^2(X)$ such that

$$\lim_{n \to +\infty} \lim_{m \to +\infty} g_{m,n}(x) = g(x), \quad x \in X;$$

and

$$\sup_{m,n \in \mathbb{N}} \| g_{m,n} \|_\infty \leq \| g \|_\infty, \quad \sup_{m,n \in \mathbb{N}} \| Dg_{m,n} \|_\infty \leq L.$$

**Proof.** Let $\{ h_k \mid k \in \mathbb{N} \}$ be the orthonormal basis fixed in Hypothesis 2.1. For every $n \in \mathbb{N}$ consider the function $\psi_n : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\psi_n(\xi) : = g \left( \sum_{k=1}^{n} \xi_k h_k \right), \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Let $\rho \in C_b^0(\mathbb{R}^n)$ with support contained in the unit ball and such that $\int_{\mathbb{R}^n} \rho(\eta) d\eta = 1$. For every $m \in \mathbb{N}$ consider

$$\psi_{m,n}(\xi) : = \int_{\mathbb{R}^n} \psi_n(\xi - m^{-1} \eta) \rho(\eta) d\eta, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$ 

Letting $g_{m,n}(x) : = \psi_{m,n}(x, h_1, \ldots, x, h_n)$, the thesis follows by standard arguments as in [17, Section 2.1] and the proof of [37] Lemma 2.5. \( \square \)

**References**

[1] C. Ané et Al. *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses, vol 10, Soc. Math. de France, 2000.

[2] L. Angiuli, S. Ferrari, D. Pallara, Gradient estimates for perturbed Ornstein–Uhlenbeck semigroups on infinite-dimensional convex domains, *J. Evol. Equ.* 19 (2019), 677–715.

[3] L. Angiuli, L. Lorenzi, A. Lunardi, Hypercontractivity and asymptotic behavior in nonautonomous Kolmogorov equations, *Comm. Part. Differ. Equat.* 38 (2013), 2049–2080.

[4] P. Alphonse, J. Bernier, Smoothing properties of fractional Ornstein-Uhlenbeck semigroups and null-controllability, *Bull. Sci. Math.* 165 (2020).

[5] D. Applebaum, On the Infinitesimal Generators of Ornstein–Uhlenbeck Processes with Jumps in Hilbert Space, *Potential Anal.* 26 (2007), 79–100.

[6] D. Applebaum, Infinite dimensional Ornstein–Uhlenbeck processes driven by Lévy processes, *Probab. Surv.* 12 (2015), 35–54.

[7] D. A. Bigonini, S. Ferrari, On generators of transition semigroups associated to semilinear stochastic partial differential equations, Submitted. *Arxiv e-prints*, 2021. *arXiv*: 2010.03908.

[8] R. M. Blumenthal, R. K. Getoor, Some theorems on stable processes, *Trans. Amer. Math. Soc.* 95, (1960), 263–273.

[9] D. Bakry, M. Ledoux, Lévy-Gromov’s isoperimetric inequality for an infinite dimensional diffusion generator, *Invent. Math.* 123 (1996), 259–281.

[10] C. van den Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer, 1975.

[11] S. G. Bobkov, M. Ledoux, On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures, *J. Funct. Anal.* 156, (1998), 347–365.
M. Ledoux, The concentration of measure phenomenon, Arnesano, I-73100 LECCE, Italy

M. Fuhrman, M. Röckner, Generalized Mehler semigroups: the non-Gaussian case,

I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian

P. Lescot, M. Röckner, Perturbations of generalized Mehler semigroups and applications to stochastic heat equations with Levy noise and singular drift, Potential Anal. 12 (2000), 1–47.

B. Goldys, M. Kocan, Diffusion Semigroups in Spaces of Continuous Functions with Mixed Topology, J. Diff. Eq. 175 (2001), 17–39.

L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975), 1061–1083.

L. Gross, Logarithmic Sobolev inequalities and contractivity properties of semigroups in Dirichlet forms (Varenna, 1992), Lecture Notes in Math. 1563 (1993), Springer, 54–88.

H. Kawabi, A simple proof of log-Sobolev inequalities on a path space with Gibbs measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 321–329.

I. Koponen, Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, Physical Review 52 (1995), 1197–1199.

M. Ledoux, Concentration of measures and logarithmic Sobolev inequalities, Séminaire de Probab. XXXIII, Lecture Notes in Math. 1709, (1999), Springer 120–216.

M. Ledoux, The concentration of measure phenomenon, Amer. Math. Soc. 2001.

P. Lescot, M. Röckner, Generators of Mehler type semigroups as pseudo-differential operators, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002), 297–315.

P. Lescot, M. Röckner, Perturbations of generalized Mehler semigroups and applications to stochastic heat equations with Levy noise and singular drift, Potential Anal. 20 (2004), 317–344.

W. Lindner, Probability in Banach Spaces–stable and Infinitely Divisible Distributions, Wiley, 1986.

L. Lorenzi, Analytical Methods for Markov Semigroups, CRC Press, 2nd ed., 2017.

A. Lunardi, M. Röckner, Schauder theorems for a class of (pseudo-)differential operators on finite and infinite dimensional state spaces, J. London Math. Soc. (2021), to appear.

M. Mandellern, On the uniform continuity of Tietze extensions, Arch. Math. (Basel) 55 (4) (1990), 387–388.

R. Miculescu, Approximations by Lipschitz functions generated by extensions, Real Anal. Exchange 28 (1) (2002/03), 33–40.

K. R. Parthasarathy, Probability measures on Metric Spaces Academic Press, 1967.

A. Pascucci, PDE and Martingale Methods in Option Pricing, Springer-Verlag Italia, 2011.

S. Peszat, J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert spaces, Ann. Probab. 23 (1) (1995), 157–172.

S. Peszat, J. Zabczyk, Stochastic Partial Differential Equations with Lévy noise, Cambridge U.P. 2007.

E. Priola, S. Tracà, On the Cauchy problem for non-local Ornstein–Uhlenbeck operators, Nonlinear Anal. 131 (2016), 182–205.

E. Priola, J. Zabczyk, Harmonic functions for generalized Mehler semigroups, in: Stochastic partial differential equations and applications VII, Lect. Notes Pure Appl. Math. 245, Chapman & Hall/CRC, Boca Raton, FL, 2006, 243-256.

M. Röckner, F.-Yu Wang, Harnack and functional inequalities for generalized Mehler semigroups, J. Funct. Anal. 10 (1) (2003), 237–361.

G. Royer, Une initiation aux inégalités de Sobolev logarithmiques, Cours spécialisés. Soc. Math. de France, 1999.

W. Rudin, Real and complex analysis, McGraw-Hill, 3rd ed., 1987.

N. N. Vakhania, V. I. Tarieladze, S. Chobanyan, Probability Distributions on Banach Spaces, D. Reidel, 1987.

F.-Yu Wang, Functional inequalities, Markov semigroups and spectral theory, Science Press, Beijing 2005.

L. Wu, A new modified logarithmic Sobolev inequality for Poisson point processes and several applications, Probab. Theory Relat. Fields 118 (2000), 427–438.
