GEOMETRIC CONSTRUCTION OF QUOTIENTS $G/H$
IN SUPERSYMMETRY

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ABSTRACT. It was proved by the first-named author and Zubkov [10] that given an affine algebraic supergroup $G$ and a closed sub-supergroup $H$ over an arbitrary field of characteristic $\neq 2$, the faisceau $G/H$ (in the fppf topology) is a superscheme, and is, therefore, the quotient superscheme $G/H$, which has desirable properties, in fact. We reprove this, by constructing directly the latter superscheme $G/H$. Our proof describes explicitly the structure sheaf of $G/H$, and reveals some geometric features of the quotient.

Key Words: affine algebraic supergroup, Hopf superalgebra, superscheme, faisceau

Mathematics Subject Classification (2000): 14L15, 14M30, 16T05

1. Introduction

Throughout in this paper we work over a fixed, arbitrary field $k$ of characteristic $\neq 2$. Algebras, Hopf algebras, schemes and so on, together with their super-analogues, all are those over $k$. The unadorned $\otimes$ means the tensor product $\otimes_k$ over $k$.

1.1. Quotients $G/H$. Given a group $G$ and a subgroup $H \subset G$, one has the set $G/H$ of cosets. This elementary fact which one learns at the first Algebra Course immediately turns into a difficult question in the context of schemes, in which $G$ is an affine algebraic group scheme and $H$ is a closed subgroup scheme of $G$. But we already know the answer that there exists uniquely a scheme $G/H$ which fits in with the natural co-equalizer diagram $G \times H \rightrightarrows G \to G/H$ of schemes, and which has desirable properties, such as being Noetherian; see [2] Part I, Sections 5.6–5.7, for example.

It is easy to pose the same question in the generalized, super situation. But it came into our interest only in these 12 years, when J. Brundan [1] assumed the existence of such supersymmetric quotients with desirable properties, and then the first named author and A. Zubkov [10] proved that existence. The objective of this paper is to reconstruct the quotient more directly, describing its structure sheaf explicitly.

1.2. Supersymmetry. The word “super” is a synonym of “graded by the order-2-group $\mathbb{Z}/(2) = \{ 0, 1 \}$”; the 0 (resp., the 1) in $\mathbb{Z}/(2)$ is called even (resp., odd). A super-vector spaces is thus a vector space $V$ which is $\mathbb{Z}/(2)$-graded so as $V = V_0 \oplus V_1$; $V$ is said to be purely even (resp., purely odd) if $V = V_0$ (resp., if $V = V_1$). The super-vector spaces $V, W, \ldots$ all together form a symmetric tensor category with respect to the natural tensor product.
V ⊗ W, the unit object k and the supersymmetry

\[ c = c_{V,W} : V \otimes W \longrightarrow W \otimes V, \quad c(v \otimes w) = (-1)^{|v||w|} w \otimes v, \]

where \( v \) and \( w \) are supposed to be homogeneous elements of degree \( |v|, |w| \), respectively. Ordinary objects, such as algebra, commutative algebra, Hopf algebra, which are defined in the symmetric tensor category of vector spaces, equipped with the trivial symmetry \( V \otimes W \cong - \rightarrow W \otimes V \), \( v \otimes w \mapsto w \otimes v \), are generalized by super-objects defined in symmetric tensor category of super-vector spaces; the objects are called with “super” attached, so as superalgebra, super-commutative superalgebra, Hopf superalgebra. Ordinary objects are precisely purely even super-objects.

In what follows, superalgebras \( \mathcal{A} \) (and Hopf superalgebras as well) are assumed to be super-commutative, unless otherwise stated; the assumption means that \( \mathcal{A}_0 \) is a central subalgebra of \( \mathcal{A} \), and we have \( ab = -ba \) for all \( a, b \in \mathcal{A}_1 \). Accordingly, (Hopf) algebras are assumed to be commutative.

1.3. Geometrical vs. functorial viewpoints. The article [10] showed that the circumstance around schemes is directly generalized to the super situation, as follows. The notion of superschemes is defined in two ways, from geometrical viewpoint and from functorial viewpoint; the notion from the latter will be called a functorial superscheme in this paper. Roughly speaking, a superscheme is a topological space, equipped with a structure sheaf of superalgebras, which is covered by some affine open sub-superschemes; an affine superscheme, \( \text{Spec} \mathcal{A} \), is uniquely given by a superalgebra, say \( \mathcal{A} \), so that the underlying topological space is the the spectrum \( \text{Spec}(\mathcal{A}_0) \) of the algebra \( \mathcal{A}_0 \), and the superalgebra \( \mathcal{O}_{\text{Spec} \mathcal{A}}(\text{Spec}(\mathcal{A}_0)) \) of global sections is \( \mathcal{A} \). A functorial superscheme is a set-valued functor defined on the category of superalgebras, which is the union of some affine open sub-functors; a functorial affine superscheme, \( \text{Sp} \mathcal{A} \), is a representable functor, which thus is uniquely represented by a superalgebra, say \( \mathcal{A} \). The Comparison Theorem [10, Theorem 5.14] states that \( \text{Spec} \mathcal{A} \mapsto \text{Sp} \mathcal{A} \) naturally extends to an equivalence from the category of superschemes to the category of functorial superschemes. An advantage of the functorial viewpoint is in that the latter category is included in the tractable category of faisceaux; a faisceau is a functor which behaves like a sheaf with respect to the so-called fppf-coverings of superalgebras.

Group-objects in the category of (functorial) superschemes are called supergroup schemes. But we treat only affine supergroup schemes in this paper. In addition, when we discuss affine (super)group schemes (not affine (super)schemes), we omit the word “scheme”, and say affine (super)groups, following the widely known custom of Jantzen [4].

1.4. Main result and consequences. Let \( G = \text{Spec} \mathcal{C} \) be an affine algebraic group, and \( H = \text{Spec} \mathcal{D} \) a closed sub-supergroup. Thus, \( \mathcal{C} \) is a finitely generated Hopf superalgebra, and \( \mathcal{D} \) is a quotient Hopf superalgebra of \( \mathcal{C} \). It is easy to construct the quotient \( G/H \) is the category of faisceaux. One principle is that if the faisceau \( G/H \) happens to be a functorial superscheme, we have the quotient \( G/H \) in the category of superschemes by the Comparison Theorem. In fact, the article [10] referred to in Section 1.1 has proved that
the assumption is satisfied, to obtain the conclusion. But we only depend on the principle in the restricted situation that the quotient is affine. Being more on the geometrical side, we construct the superscheme $G/H$ directly, as follows.

One sees that $G$ (resp., $H$) includes an affine algebraic group $G = \text{Spec } C$ (resp., $H = \text{Spec } D$) as the largest purely even closed sub-supergroup. We remark that $G$ and $G$ (resp., $H$ and $H$) has the same underlying topological space, so that $|G| = |G| \supset |H| = |H|$, whence $G \supset H$. Let $\pi : G \rightarrow G/H$ be the quotient morphism; to this, known results can apply. Choose arbitrarily an affine open subset $\emptyset \neq U \subset |G/H|$. Then $\pi^{-1}(U)$ is an $H$-stable affine open subscheme of $G$ such that $\pi^{-1}(U)/H = U$. Note that $\pi^{-1}(U)$ is an open subset of $G$, as well. The key of ours is to construct an $H$-equivariant embedding of some right $D$-super-comodule superalgebra onto $\pi^{-1}(U)$ in $G$. Such an embedding is in the form $\text{Spec}(\omega)$, where $\omega : C \rightarrow A$ is a map of right $D$-super-comodule superalgebras; the question is, therefore, to find an appropriate right $D$-super-comodule superalgebra $A$ together with $\omega$ such as above. Indeed, Hopf-algebraic techniques enable us to find out very useful ones; see Proposition 4.8 and Corollary 4.10. The result is that the $\pi^{-1}(U)$ in $G$ is an $H$-stable affine open subscheme of $G$, such that $\pi^{-1}(U)/H$ exists, and is an affine superscheme. Our main theorem, Theorem 4.12, shows that the thus obtained affine superschemes, when $U$ ranges over all affine open subsets of $|G/H|$, are uniquely glued into a superscheme with the underlying topological space $|G/H|$, and the resulting superscheme is indeed the quotient $G/H$: the underlying topological space $|G/H|$ is thus the same as $|G/H|$. The proof will give a new description of the structure sheaf $O_{G/H}$ (Remark 4.13): $O_{G/H}$ is locally isomorphic to the sheaf

$$\wedge_{O_{G/H}}(\pi_*O_G \Box_D Z),$$

where $\pi_*O_G \Box_D Z$ is a locally free $O_{G/H}$-module sheaf.

It does happen that the sheaves $O_{G/H}$ and $\wedge_{O_{G/H}}(\pi_*O_G \Box_D Z)$ are not globally isomorphic; see Remark 4.20. On the other hand, Proposition 4.19 gives some sufficient conditions for the two sheaves to be globally isomorphic. The new description above shows that $G/H$ has desirable properties; see Proposition 4.16. They include the remarkable one: an open subset of $|G/H| (= |G/H|)$ is affine in $G/H$ if and only if it is affine in $G/H$.

The results looked over above are contained in Section 4. The preceding two sections are devoted to preliminaries. Section 2 summarizes basic facts on super-(co)algebras ans superschemes; they includes the Comparison Theorem, Theorem 2.3 referred to above. Section 3 mostly reviews known results on affine supergroups and Hopf superalgebras.

2. Superalgebras and superschemes

This preliminary section summarizes basic facts on super-(co)algebras and on superschemes in Sections 2.1–2.4 and in Sections 2.5–2.7, respectively.

2.1. Super vs. non-super situations. Super-(co)algebras are regarded as ordinary (co)algebras, with the $\mathbb{Z}/(2)$-grading forgotten. A right, say, supermodule $M$ over a superalgebra $B$ is (faithfully) flat as an ordinary right $B$-module if and only if the functor $M \otimes_B$ defined on the category
of left $\mathcal{B}$-supermodules is (faithfully) exact [6, Lemma 5.1 (1)]. Similarly, a (left or right) super-comodule over a super-coalgebra $\mathcal{C}$ is injective (or equivalently, coflat) as an ordinary $\mathcal{C}$-comodule if and only if it is so in the category of $\mathcal{C}$-super-comodules. If the equivalent conditions are satisfied we say simply that the object in question is (faithfully) flat or injective.

Recall that given a left $\mathcal{C}$-super-comodule $L = (L, \lambda_L : L \to \mathcal{C} \otimes L)$ and a right $\mathcal{C}$-super-comodule $M = (M, \rho_M : M \to M \otimes \mathcal{C})$, the co-tensor product $M \square_\mathcal{C} L$ is the super-vector space defined as the equalizer of $\text{id}_M \otimes \lambda_L$ and $\rho_M \otimes \text{id}_L$. The functor $M \square_\mathcal{C}$ (resp., $\square_\mathcal{C} L$) defined on the category of left (resp., right) $\mathcal{C}$-(super-)comodules is left exact. If it is exact, then $M$ (resp., $L$) is said to be coflat. The condition is equivalent to the $\mathcal{C}$-(super-)comodule being injective, as noted above; see [17, Proposition A.2.1].

2.2. **Superalgebras.** Recall that all superalgebras are assumed to be super-commutative. Given a superalgebra $\mathcal{B}$, left $\mathcal{B}$-supermodules and right $\mathcal{B}$-supermodules are identified by a canonical category-isomorphism [6, Lemma 5.2 (2)]. It follows that a $\mathcal{B}$-superalgebra $\mathcal{A}$ is faithfully flat as a left $\mathcal{B}$-(super)module if and only if it is so as a right $\mathcal{B}$-(super)module [6, Lemma 5.3 (2)]. In this case we say that $\mathcal{A}$ is faithfully flat over $\mathcal{B}$, or $\mathcal{B} \to \mathcal{A}$ is faithfully flat. We say that $\mathcal{A}$ is fppf (fidélement plat de présentation finite) over $\mathcal{B}$ if it is faithfully flat and finitely presented. Recall that $\mathcal{A}$ is said to be finitely presented over $\mathcal{B}$ if it is in the form $\mathcal{B}[X,Y]/I$, where $\mathcal{B}[X,Y] = \mathcal{B}[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$ is a polynomial superalgebra in finitely many even variables $\underline{X} = (X_i)_i$ and odd variables $\underline{Y} = (Y_i)_i$, and $I$ is a finitely generated super-ideal.

2.3. **Graded superalgebras.** Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra. The super-ideal $I_\mathcal{A} = (\mathcal{A}_1)$ generated by the odd component $\mathcal{A}_1$ is the smallest super-deal such that the quotient

\[(2.1) \quad A := \mathcal{A}/I_\mathcal{A} (= \mathcal{A}_0/I_\mathcal{A}^2)\]

is an ordinary (commutative) algebra. This last algebra is said to be associated with the original superalgebra, denoted by the corresponding normal capital letter. The descending chain $\mathcal{A} \supset I_\mathcal{A} \supset I_\mathcal{A}^2 \supset \cdots$ of super-ideals constructs the graded superalgebra

\[(2.2) \quad \text{gr } \mathcal{A} := \bigoplus_{n \geq 0} I_\mathcal{A}^n/I_\mathcal{A}^{n+1}\]

associated with $\mathcal{A}$. By a graded superalgebra we mean an algebra graded by $\mathbb{N} = \{0, 1, 2, \ldots\}$ which, regarded as $\mathbb{Z}/(2)$-graded by mod-2 reduction, is a super-commutative superalgebra. Note that the $A$-module $(\text{gr } \mathcal{A})(1) = I_\mathcal{A}/I_\mathcal{A}^2 (= \mathcal{A}_1/I_\mathcal{A}^3)$ is purely odd, and the embedding $I_\mathcal{A}/I_\mathcal{A}^2 \hookrightarrow \text{gr } \mathcal{A}$ induces a surjection of graded superalgebras

\[\wedge_A(I_\mathcal{A}/I_\mathcal{A}^2) \to \text{gr } \mathcal{A}\]

from the exterior $A$-algebra on the $A$-module $I_\mathcal{A}/I_\mathcal{A}^2$. 

2.4. Noetherian superalgebras. Retain the notation as above. We say that \( A \) is Noetherian if its super-ideals satisfy the ACC. The condition is easily seen to be equivalent to each of the following:

(i) The commutative algebra \( A_0 \) is Noetherian, and the \( A_0 \)-algebra \( A \) is generated by finitely many odd elements;
(ii) \( A_0 \) is Noetherian, and the \( A_0 \)-module \( A_1 \) is finitely generated;
(iii) \( A \) is Noetherian, and the \( A \)-module \( I_A/I_A^2 \) is finitely generated;
(iv) The superalgebra \( \wedge_A(I_A/I_A^2) \) is Noetherian;
(v) The superalgebra \( \text{gr} A \) is Noetherian.

See \cite[Section A.1]{[11]}. Given a Noetherian superalgebra \( B \), a finitely generated \( B \)-superalgebra is finitely presented over \( B \), and is Noetherian.

Let \( A \) and \( B \) be Noetherian superalgebras. Then \( \text{gr} A \) and \( \text{gr} B \) are finitely graded in the sense that \( (\text{gr} A)(n) = 0 = (\text{gr} B)(n) \) for \( n \gg 0 \). It follows that a superalgebra map \( f : A \to B \) is surjective/injective if and only if the associated graded superalgebra map \( \text{gr} f : \text{gr} A \to \text{gr} B \) is so.

2.5. Superschemes. A super- ringed space (over \( k \)) is a topological space equipped with a sheaf of superalgebras (over \( k \)) on it. It is said to be local if the stalk at every point is local; see below.

Let \( A = A_0 \oplus A_1 \) be a superalgebra with associated algebra \( A = A_0/(A_1) \). The affine superscheme \( \text{Spec} A \) associated with \( A \) is a local-super-ringed space. Its underlying topological space is the spectrum \( \text{Spec}(A_0) \) of the algebra \( A_0 \); it is naturally identified with the spectrum \( \text{Spec} A \) of \( A \), since \( A = A_0/A_1^2 \) and \( A_1^2 \subset \sqrt{0} \). Note that every super-ideal \( P \) of \( A \) such that \( A/P \) is an integral domain is uniquely in the form \( P = P \oplus A_1 \) with \( P \in \text{Spec}(A_0) \). Similarly, every proper super-ideal of \( A \) that is maximal with respect to inclusion is uniquely in the form \( P \oplus A_1 \), where \( P \subset A_0 \) is a maximal ideal. We say that \( A \) is local if it has a unique maximal super-ideal, or equivalently, if \( A_0 \) is local. The localization \( S^{-1}A \) by a multiplicative set \( S \subset A_0 \) is the base extension \( A \otimes_{A_0} S^{-1}A_0 \) of the \( A_0 \)-algebra \( A \) along the localization \( A_0 \to S^{-1}A_0 \). If \( S = A_0 \setminus P \) with \( P \in \text{Spec}(A_0) \) (resp., if \( S = \{1, x, x^2, \ldots \} \) with \( x \in A_0 \)), then \( S^{-1}A \) is denoted \( A_P \) (resp., \( A_x \)), as usual. Note that \( A_P \) is local. The structure sheaf \( \mathcal{O}_{\text{Spec} A} \) of \( A \) is the unique sheaf of superalgebras that assigns \( A_x \) to every principal open set \( D(x) = \{ P \mid x \notin P \} \). The stalk \( \mathcal{O}_{\text{Spec} A, P} \) at \( P \) is \( A_P \). In \cite{[11]}, \( \text{Spec} A \) is alternatively denoted \( S\text{Spec} A \).

A superscheme (over \( k \)) is a local-super-ringed space (over \( k \)) which is locally isomorphic to some affine superscheme. The superschemes form a full subcategory of the category of local-super-ringed spaces. A morphism \( f : X \to Y \) of the latter category is required to be such that the induced superalgebra map \( f_P^* : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P} \) between the stalks is \( \text{local} \), that is, \( f_P^* \) sends the maximal super-ideal of \( \mathcal{O}_{Y,f(P)} \) into that of \( \mathcal{O}_{X,P} \). Basic notions for schemes and their morphisms, such as algebraic/Noetherian scheme, open/closed embedding, affine/faithfully flat/finitely-presented morphism, and relevant basic results are generalized to our super context in the obvious manner.

A superscheme \( X \) is said to be smooth at point \( P \) of the underlying topological space \( |X| \), if the stalk \( \mathcal{O}_{X,P} \) at \( P \) is smooth as a superalgebra; this
means that a superalgebra surjection onto $O_{X,P}$ splits whenever its kernel is a nilpotent super-ideal. A superscheme is said to be smooth if it is smooth at every point. Theorem A.2 of [11] gives some characterizations for a Noetherian affine superscheme to be smooth.

2.6. The associated graded superscheme. Let $X$ be a superscheme with structure sheaf $O_X$. Given a non-empty affine open subset $U \subset |X|$, we have the affine superscheme

$$Y_U := \text{Spec} (\text{gr} O_X(U))$$

given by the graded superalgebra $\text{gr} O_X(U)$ associated with the superalgebra $O_X(U)$. The underlying topological space $|Y_U|$ is naturally identified with that space of $\text{Spec}(O_X(U)/(O_X(U)_1))$, and hence with $U$.

**Lemma-Definition 2.1.** The affine superschemes $Y_U$, where $U$ ranges over non-empty affine open subsets of $|X|$, are uniquely glued into a superscheme with the underlying topological space $|X|$.

We denote the resulting superscheme by $\text{gr} X$, and call it the graded superscheme associated with $X$.

**Proof.** Let $U \supset U'$ be affine open in $X$. Suppose $U = \text{Spec} \, A$, $U' = \text{Spec} \, A'$, and that $U \supset U'$ arise from a superalgebra map $i : A \rightarrow A'$. Let $A = A/(A_1)$ and $A' = A'/A'_1$ be the associated algebras. Choose $P \in \text{Spec}(A'_0)$ arbitrarily, and set $Q = i^{-1}(P) (\in \text{Spec}(A_0))$. Then the map of stalks $i_P : A_Q \rightarrow A'_P$ at $P$ is an isomorphism. Note that the operation $\text{gr}$ commutes with localization, so that $\text{gr}(i_P) = (\text{gr} i)_P$, and

$$(\text{gr} i)_P : \text{gr}(A)_Q \rightarrow \text{gr}(A')_P$$

is an isomorphism. Here we may suppose that $P \in \text{Spec} \, A'$, $Q \in \text{Spec} \, A$, and that the relevant localizations $\text{gr}(A')_P$ and $\text{gr}(A)_Q$ are by those, since the $A'_0$-algebra $\text{gr}(A')_P$ and $A_0$-algebra $\text{gr}(A)_Q$ are, in fact, an $A'$-algebra and an $A$-algebra, respectively. The result remains unchanged if we replace $P$ and $Q$ with the corresponding primes in $\text{Spec}(\text{gr}(A'_0))$ and in $\text{Spec}(\text{gr}(A_0))$, respectively. Indeed, the localizations are then unchanged since one may ignore the deference by nilpotent elements for localizing elements.

To prove the assertion it suffices to prove that the structure sheaf $O_{Y_U}$ of $Y_U$, pull-backed to $U'$, coincides with $O_{X,U'}$. But this follows from the result just proven. $\square$

The structure sheaf of $\text{gr} X$ is a sheaf of graded superalgebras. One sees that a superscheme $X$ is Noetherian if and only if $\text{gr} X$ is. The argument of the last proof, concentrating in degree zero, shows the following.

**Lemma-Definition 2.2.** The affine schemes

$$\text{Spec}(O_X(U)/(O_X(U)_1))$$

where $U$ ranges over non-empty affine open subsets of $|X|$, are uniquely glued into a scheme with the underlying topological space $|X|$.

We call the resulting scheme the scheme associated with $X$. 

2.7. **Functorial viewpoint.** Let $\text{SAlg}$ and $\text{Set}$ denote the categories of superalgebras (over $k$) and of sets, respectively. A $k$-functor is a functor $\mathcal{F} : \text{SAlg} \to \text{Set}$. Let

$$
\text{Func} = \text{Set}^{\text{SAlg}}
$$

denote the category of $k$-functors and natural transformations. A $k$-functor $\mathcal{F}$ is called a *faisceau* (resp., *faisceau dur*), if it preserves finite direct products and if it turns every equalizer diagram of superalgebras $B \to A \Rightarrow A \otimes B A$ that naturally arises from an fpff (resp., faithfully flat) map $B \to A$ (the paired arrows indicate $a \mapsto a \otimes 1$, $1 \otimes a$) into an equalizer diagram of sets $\mathcal{F}(B) \to \mathcal{F}(A) \Rightarrow \mathcal{F}(A \otimes B A)$.

Given $A \in \text{SAlg}$, we let

$$
\text{Sp} A = \text{SAlg}(A, -) : \text{SAlg} \to \text{Set}, \; T \mapsto \text{SAlg}(A, T)
$$

denote the $k$-functor represented by $A$; this is alternatively denoted $SSp A$ in [10]. Such a representable $k$-functor is called a *functorial affine superscheme*; see the last paragraph of this section. A sub-functor of $\text{Sp} A$ in the form $D(a)$, where $a \subset A$ is a super-ideal and

$$
D(a)(T) = \{ f \in (\text{Sp} A)(T) \mid f(a)T = T \}, \; T \in \text{SAlg},
$$

is said to be open. An open sub-functor $\mathcal{G}$ of a $k$-functor $\mathcal{F}$ in general is a sub-functor such that for every $A \in \text{SAlg}$ and for every morphism $\phi : \text{Sp} A \to \mathcal{F}$ in $\text{Func}$, the sub-functor $\phi^{-1}(\mathcal{G})$ of $\text{Sp} A$ given by $\phi^{-1}(\mathcal{G})(T) := \phi^{-1}_T(\mathcal{G}(T))$, $T \in \text{SAlg}$, is open. A *functorial superscheme* is a $k$-functor $F$ which is the union $\mathcal{F} = \bigcup_i \mathcal{G}_i$ of some affine open sub-functors $\mathcal{G}_i$ in the sense that $\mathcal{F}(K) = \bigcup_i \mathcal{G}_i(K)$ for every field $K$ including $k$.

The category $\text{Func}$ includes full subcategories in the relation:

$$
\left( \begin{array}{c}
\text{functorial} \\
\text{affine superschemes}
\end{array} \right) \subset \left( \begin{array}{c}
\text{functorial} \\
\text{superschemes}
\end{array} \right) \subset \left( \begin{array}{c}
\text{faisceaux dur} \\
\text{faisceaux}
\end{array} \right).
$$

See [10] Proposition 5.15. The cited article [10] puts emphasis on the functorial viewpoint, while we do more on the geometrical viewpoint; the article calls (dur) $k$-sheaves, (affine) superschemes and geometric superschemes what we call faisceaux (dur), functorial (affine) superschemes and superschemes, respectively.

Given a superscheme $X$, the $k$-functor

$$
X^\circ : \text{SAlg} \to \text{Set}, \; X^\circ(T) = \text{Mor}(\text{Spec} T, X),
$$

where $\text{Mor}$ denotes the set of the morphisms of superschemes, is proved to be a functorial superscheme (see [10] Lemma 5.2), and is called the functorial superscheme *represented by* $X$. We say that $X$ *represents* $X^\circ$. For example, the affine superscheme $\text{Spec} A$ represents $\text{Sp} A$; see [10] Lemma 4.1.

Here we reproduce from [10] Theorem 5.14 the Comparison Theorem:

**Theorem 2.3.** $X \mapsto X^\circ$ gives rise to a category-equivalence from the category of superschemes to the category of functorial superschemes.
For an explicit quasi-inverse see [10, Proposition 5.12, Lemma 5.13].
An affine superscheme $X = \text{Spec} \mathbb{A}$ and the assigned, functorial affine superscheme $X^f = \text{Sp} \mathbb{A}$ are both controlled by the superalgebra $\mathbb{A}$, and may not be distinguished in many situations. We will call the latter as well, an affine superscheme, omitting the word “functorial”, as usual. Even when one has to distinguish them, which is meant will be clear from the context or the notation.

3. AFFINE SUPERGROUPS AND HOPF SUPERALGEBRAS

This section is devoted again to preliminaries, which include reproducing two fundamental theorems on affine supergroups and Hopf superalgebras.

3.1. Affine supergroups. One sees just as in the non-super situation that the two categories treated in the last theorem have finite direct products (and more generally, fiber products). Therefore, both of them have group objects, which we call supergroup schemes and functorial supergroup schemes, respectively. The proved category-equivalence induces a category-equivalence between those group objects. But in what follows, we will discuss only affine supergroup schemes; they are precisely affine superschemes

$\text{(3.1)} \quad \text{Spec} \mathbb{D}, \quad \text{Sp} \mathbb{D}$

equipped with group structure, which uniquely arises from a Hopf-superalgebra structure on $\mathbb{D}$. We call the two of (3.1) both an affine supergroup, omitting the word “scheme”, following the custom of Jantzen [4]. It is called an affine algebraic supergroup if the Hopf superalgebra $\mathbb{D}$ is finitely generated as a superalgebra.

Let $G = \text{Spec} \mathbb{D}$ be an affine supergroup. A right $\mathbb{D}$-super-comodule is the same as a left $G$-super-module. Given such a super-comodule $M = (M, \rho_M)$, the super-vector space $M^{\text{co}D}$ of all $\mathbb{D}$-coinvariants in $M$ is defined by

$\text{(3.2)} \quad M^{\text{co}D} = \{ m \in M \mid \rho_M(m) = m \otimes 1 \}$

This is identified with the co-tensor product $M \boxtimes \mathbb{k}$, where $\mathbb{k}$ is the trivial, purely even left $\mathbb{D}$-super-comodule, and also with the super-vector space of all $G$-invariants in $M$.

3.2. Affinity criteria. Let $X = \text{Spec} \mathbb{A}$ be an affine superscheme, and let $G = \text{Spec} \mathbb{D}$ be an affine supergroup. Suppose that $G$ acts on $X$ from the right. This means that there is given a morphism of (functorial) superschemes $X \times G \to X$, called an action by $G$ on $X$, which satisfies the familiar associativity and unit-property. Such an action arises uniquely from a right $\mathbb{D}$-super-comodule superalgebra structure

$\rho_h : \mathbb{A} \to \mathbb{A} \otimes \mathbb{D}$

on $\mathbb{A}$; it is by definition a superalgebra map with which $\mathbb{A}$ is a right $\mathbb{D}$-supercomodule. Let

$\text{(3.3)} \quad \mathbb{B} = \mathbb{A}^{\text{coD}} (= \{ b \in \mathbb{A} \mid \rho_h(b) = b \otimes 1 \})$.

This is a sub-superalgebra of $\mathbb{A}$. Let $\text{SMod}_{\mathbb{B}}$ denote the category of right $\mathbb{B}$-supermodules. A super-vector space $M$ equipped with a right $\mathbb{A}$-supermodule
structure and a right $\mathbb{D}$-super-comodule structure
\[ M \otimes A \to M, \; m \otimes a \mapsto ma; \quad \rho_M : M \to M \otimes \mathbb{D} \]
is called a \((\mathbb{D}, A)\)-Hopf supermodule (see [18, p.454]) if it satisfies
\[ \rho_M(ma) = \rho_M(m) \rho_A(a), \quad m \in M, \; a \in A. \]

Let $\mathbf{SMod}^\mathbb{D}_A$ denote the category of \((\mathbb{D}, A)\)-Hopf supermodules; the morphisms are $A$-supermodule and $\mathbb{D}$-super-comodule maps. Obviously, $A \in \mathbf{SMod}^\mathbb{D}_A$.

The categories $\mathbf{SMod}_B$ and $\mathbf{SMod}^\mathbb{D}_A$ are both $k$-linear abelian. Given an object $N \in \mathbf{SMod}_B$, the right $A$-supermodule $N \otimes_B A$, equipped with the right $\mathbb{D}$-super-comodule structure $\text{id} \otimes_B \rho_A : N \otimes_B A \to N \otimes_B (A \otimes \mathbb{D}) = (N \otimes_B A) \otimes \mathbb{D}$, turns into a \((\mathbb{D}, A)\)-Hopf supermodule. This construction gives rise to a $k$-linear functor
\[
\mathbf{SMod}_B \to \mathbf{SMod}^\mathbb{D}_A, \; N \mapsto N \otimes_B A,
\]
which is left adjoint to
\[
\mathbf{SMod}^\mathbb{D}_A \to \mathbf{SMod}_B, \; M \mapsto M^{\text{co}\mathbb{D}}.
\]

The following theorem, which is reproduced from [10], is a super-analogue of U. Oberst’s Satz A of [14]; see Remark 3.3 (1) below. Some notion and notation used here will be explained soon below.

**Theorem 3.1** ([10, Theorem 7.1]). Retain the situation as above.

1. The following are equivalent:
   
   (i) The action by $G$ on $X$ is free, and the faisceau dur $\tilde{X}/G$ is an affine superscheme;
   
   (ii) (a) $A$ is injective as a right $\mathbb{D}$-super-comodule, and
   (b) the map
   \[
   \alpha = \alpha_A : A \otimes A \to A \otimes \mathbb{D}, \; \alpha(a \otimes a') = a \rho_A(a')
   \]
   is a surjection;
   
   (iii) (a) $A$ is faithfully flat over $B$, and
   (b) the map
   \[
   \beta : A \otimes_B A \to A \otimes \mathbb{D}, \; \beta(a \otimes_B a') = a \rho_A(a')
   \]
   induced from the map $\alpha$ in (ii) is a bijection.
   
   (iv) The functors \((3.4)\) and \((3.5)\) are (necessarily, mutually quasi-inverse) equivalences.

   If these equivalent conditions are satisfied, then $\tilde{X}/G = \text{Sp} B$.

2. Suppose that $G$ is algebraic, or in other words, $\mathbb{D}$ is finitely generated. Suppose that $A$ is Noetherian. If the equivalent conditions above are satisfied, then $X/G$ is Noetherian (or equivalently, $B$ is Noetherian), and it coincides with the faisceau $\tilde{X}/G$.

In the situation above we say that the action $X \times G \to X$ is free (see Condition (i) above), if for every $T \in \mathbf{SAlg}$, the action $X(T) \times G(T) \to X(T)$, $(x, g) \mapsto x^g$ is free; this last means that $x^g = x$ implies $g = 1$, or equivalently, that $X(T) \times G(T) \to X(T) \times X(T)$, $(x, g) \mapsto (x, x^g)$ is injective. Obviously, the action $X \times G \to X$ is free if Condition (ii)(b) above is satisfied.
Given a \(k\)-functor \(F \in \text{Func}\), there exists uniquely a faisceau \(\tilde{\mathcal{F}}\) equipped with a morphism \(F \to \tilde{\mathcal{F}}\) in \(\text{Func}\) such that for any faisceau \(\mathcal{G}\), the map \(\text{Func}(\tilde{\mathcal{F}}, \mathcal{G}) \to \text{Func}(\mathcal{F}, \mathcal{G})\) induced by the morphism is a bijection; see [10, Proposition 3.6]. We have the faisceau dur \(\tilde{\mathcal{F}}\) with the analogous universality for faisceaux dur. If \(\mathcal{F}\) preserves finite direct products and has the property that if \(S \to T\) is fppf (resp., faithfully flat), \(\tilde{\mathcal{F}}(S) \to \tilde{\mathcal{F}}(T)\) is an injection, then the construction of \(\tilde{\mathcal{F}}\) (resp., of \(\tilde{\mathcal{F}}\)) is quite simple, and we have

\[
\tilde{\mathcal{F}}(T) \subset \tilde{\mathcal{F}}(T) \subset \tilde{\mathcal{F}}(T)
\]

for every \(T \in \text{SAlg}\); see [10, Remark 3.8]. This is the case if \(\mathcal{F}\) is the \(k\)-functor which assigns to every \(T\), the set \(X(T)/G(T)\) of \(G(T)\)-orbits in \(X(T)\), provided the action by \(G\) on \(X\) is free. In this case \(\tilde{\mathcal{F}}\) (resp., \(\tilde{\mathcal{F}}\)) is denoted \(X/\tilde{G}\) (resp., \(\tilde{X}/G\)).

For our purpose it is enough to work only with free actions.

**Definition 3.2** (cf. [13, Definition 8.1.1]). If the equivalent conditions (i)–(iv) in Part 1 of Theorem 3.1 are satisfied, we say that \(A \supset B\) is a \(D\)-Galois extension.

**Remark 3.3.** (1) H.-J. Schneider [15, Theorem I] proved the Oberst Satz cited above in the non-commutative, purely Hopf-algebraic situation, in which \(A, B, D\) as above may be non-commutative while neither \(G\) nor \(X\) is referred to. In [10] the theorem reproduced above was derived from Schneider’s Theorem, by using the bosonization technique developed therein.

(2) As is remarked by [10, Remark 7.2], \(\text{SMod}^D_A = (\text{SMod}^D_A, \otimes_A, A)\) is a tensor category with respect to the tensor product \(\otimes_A\) over \(A\) and the unit object \(A\). In particular, \(A\) is an algebra object in it. Moreover, the functor (3.1) is a tensor functor \(\text{SMod}_B = (\text{SMod}_B, \otimes_B, B) \to \text{SMod}^D_A\).

**3.3. The quotient superscheme \(X/G\).** Suppose that an affine supergroup \(G\) acts freely on an affine superscheme \(X\) from the right. The *quotient superscheme* \(X/G\) is a superscheme equipped with a morphism from \(X\), such that

\[
X \times G \rightrightarrows X \to X/G
\]

is a co-equalizer diagram of superschemes, where the paired arrows indicate the original \(G\)-action and the trivial \(G\)-action. If such a superscheme exists it is unique in the obvious sense. The morphism \(X \to X/G\) will not be referred to if it is obvious.

**Lemma 3.4.** *If the faisceau \(X/G\) happens to be a functorial superscheme, then the quotient superscheme \(X/G\) exists, and it necessarily represents \(X/G\).*

**Proof.** As is seen from the construction of \(X/G\), we have the co-equalizer diagram \(X \times G \rightrightarrows X \to X/G\) of faisceaux. This is a diagram of functorial superschemes under the assumption above. Now, Theorem 2.3 proves the lemma. \(\square\)
3.4. Tensor product decomposition of a Hopf superalgebra. Let $G = \text{Spec} \mathbb{C}$ be an affine supergroup. Thus $\mathbb{C} = \mathbb{C}_0 \oplus \mathbb{C}_1$ is a Hopf superalgebra. We assume that $G$ is an algebraic supergroup, or in other words, $\mathbb{C}$ is finitely generated. For later use we set this assumption, without which many of what follows, however, are known to be true. The coproduct, the counit and the antipode of this or any other Hopf superalgebra will be denoted so as

\[ \Delta_C : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, \quad \varepsilon_C : \mathbb{C} \to k, \quad S_C : \mathbb{C} \to \mathbb{C}, \]

respectively.

The super-ideal $(\mathbb{C}_1)$ generated by $\mathbb{C}_1$ is a Hopf super-ideal of $\mathbb{C}$, so that the algebra $\mathbb{C} = \mathbb{C}/(\mathbb{C}_1)$ associated with $\mathbb{C}$ (see (2.1)) is a quotient, ordinary Hopf algebra of $\mathbb{C}$, which is obviously finitely generated. For later use we set this assumption, without which many of what follows, however, are known to be true. The underlying topological space $|G|$ of $G$ is naturally identified with that space $|\mathbb{C}|$ of $\mathbb{C}$, and we have the closed embedding $G \supset \mathbb{C}$ which is identical on the underlying topological space.

Let $q : \mathbb{C} \to \mathbb{C}$ denote the quotient map. The composite

\[ \lambda_C : \mathbb{C} \xrightarrow{\Delta_C} \mathbb{C} \otimes \mathbb{C} \xrightarrow{\varepsilon_C \otimes \text{id}} \mathbb{C} \otimes \mathbb{C}, \]

makes $\mathbb{C}$ into a left $\mathbb{C}$-super-comodule superalgebra; this is equivalent to saying that $G$ is a left $G$-equivariant superscheme. Note that $\mathbb{C}$ includes $\mathbb{C}_0$ as a $\mathbb{C}$-comodule subalgebra. Analogously to (3.3), we let

\[ (3.9) \quad \co\mathbb{C} = \{ x \in \mathbb{C} | \lambda_C(x) = 1 \otimes x \} \]

denote the sub-superalgebra of $\mathbb{C}$ consisting of all left $\mathbb{C}$-coinvariants in $\mathbb{C}$.

**Lemma 3.5** ([8, Footnote 5]). There exists a left $\mathbb{C}$-comodule algebra map $\xi : \mathbb{C} \to \mathbb{C}_0$ such that $q \circ \xi = \text{id}_C$. It gives rise to an isomorphism,

\[ (3.10) \quad \mathbb{C} \otimes \co\mathbb{C} \xrightarrow{\cong} \mathbb{C}, \quad c \otimes x \mapsto \xi(c)x, \]

of left $\mathbb{C}$-super-comodule superalgebras.

For a proof of the lemma see Remark 3.7 (2) below.

Let $\mathbb{C}^+ = \text{Ker}(\varepsilon_C)$ denote the augmentation super-ideal of $\mathbb{C}$. Since $\mathbb{C}$ is finitely generated, it follows that for every $n > 0$, $(\mathbb{C}^+)^n$ is co-finite-dimensional, or $\dim(\mathbb{C}/(\mathbb{C}^+)^n) < \infty$. By definition

\[ T^*_x(G) = \mathbb{C}^+/(\mathbb{C}^+)^2 \]

is the cotangent super-vector space of $G$ at the identity element; it is finite-dimensional. The odd component $T^*_x(G)_1$ of this $T^*_x(G)$ is denoted by

\[ W_G = \mathbb{C}_1/\mathbb{C}_0^+ \mathbb{C}_1, \]

where $\mathbb{C}_0^+ = \mathbb{C}_0 \cap \mathbb{C}^+$. The even component $T^*_x(G)_0$ is seen to coincide with the cotangent space $T^*_x(G)$ of $G$ at the identity element. The right adjoint action

\[ G(T) \times G(T) \to G(T), \ (h, g) \mapsto g^{-1}hg; \quad T \in \text{SAlg} \]
induces a left $G$-supermodule structure, or equivalently, a right $C$-supercomodule structure on $C$, which in turn induces such a structure on $T^*_x(G)$, and hence on $W_G$ by restriction. The resulting $G$-action on $W_G$ is called the left co-adjoint action. The right $G$-action on $W_G$ analogously induced from the left $G$-adjoint action $(g, h) \mapsto ghg^{-1}$ on $G$ is called the right co-adjoint action.

The dual super-vector spaces $(C/((C^+)^n)^* \text{ of } C/((C^+)^n)$, $n > 0$, amount to the hyper-superalgebra of $G$

$$\text{hy}(G) = \bigcup_{n>0} (C^*/((C^+)^n)^*$$

in $C^*$. This is a super-cocommutative Hopf superalgebra with trivial coradical, which is not necessarily super-commutative; see [7, Sections 2.5, 4.3], for example. The Lie superalgebra consisting of all primitive elements in $\text{hy}(G)$ is the Lie superalgebra $\text{Lie}(G)$ of $G$. Its odd component is dual to $W_G$, or in notation,

$$(3.11) \quad \text{Lie}(G)_1 = W_G^c.$$

The left (or right) $G$-action on $\text{Lie}(G)_1$ dual to the right (or left) co-adjoint $G$-action on $W_G$ is called the adjoint action. We remark that the even component $\text{Lie}(G)_0$ of $\text{Lie}(G)$ coincides with the Lie algebra $\text{Lie}(G)$ of the affine algebraic group $G$.

The exterior algebra $\wedge(W_G)$ on $W_G$ has the natural Hopf-superalgebra structure with every element in $W_G$ being primitive; thus, the counit is such that $\varepsilon_{\wedge(W_G)}(w) = 0$ for every $w \in W_G$.

**Theorem 3.6 ([6] Theorem 4.5]).** There exists an isomorphism of left $C$-super-comodule superalgebras

$$(3.12) \quad \psi : C \xrightarrow{\cong} C \otimes \wedge(W_G)$$

such that $(\varepsilon_C \otimes \text{id}_{\wedge(W_G)}) \circ \psi : C \to \wedge(W_G)$, composed with the projection $\wedge(W_G) \to \wedge^0(W_G) \oplus \wedge^1(W_G) = k \oplus W_G$, coincides with the natural map $C \to C_0/C_0^+ \oplus C_1/C_1^+ C_1 = k \oplus W_G$.

**Remark 3.7.** (1) The properties of $\psi$ above implies that $(\varepsilon_C \otimes \text{id}_{\wedge(W_G)}) \circ \psi : C \to \wedge(W_G)$, composed with the projection $\wedge(W_G) \to \wedge^0(W_G) = k$, coincides with the canonical $C \to C_0/C_0^+ = k$. This is the same as saying that $\psi$ is counit-preserving, or explicitly, $\varepsilon_C = (\varepsilon_C \otimes \varepsilon_{\wedge(W_G)}) \circ \psi$.

(2) The cited [6] Theorem 4.5] states that there exists a counit-preserving isomorphism $\tilde{\psi} : C \xrightarrow{\cong} C \otimes \wedge(W_G)$ of left $C$-super-comodule superalgebras. The isomorphism constructed in the proof (see [6, p.301, line 9]) is seen to have the stronger property above. One sees that $C \to C_0$, $c \mapsto \psi^{-1}(c \otimes 1)$ is such a $\xi$ as in Lemma 3.5; it indeed gives rise to the isomorphism $C \xrightarrow{\cong} \wedge(W_G)$ of superalgebras.

Note that consequently, we have an isomorphism $\text{coC}(C \xrightarrow{\cong} \wedge(W_G)$ of superalgebras.
3.5. The associated graded Hopf superalgebra. Retain $G = \text{Spec } C$ as above. Note that the construction of $\text{gr } A$ in \cite{22} gives rise to an endo-functor $A \mapsto \text{gr } A$ on SAlg which preserves the tensor product. It then follows that

$$\text{gr } C = (\text{gr } C, \text{gr}(\Delta_C), \text{gr}(\varepsilon_C), \text{gr}(S_C))$$

is a Hopf superalgebra. Notice from Theorem 3.6 that $C \simeq \text{gr } C$ as superalgebras; but they are not necessarily isomorphic as Hopf superalgebras, see Lemma-Definition 3.9 below. Let

$$\text{gr } G = \text{Spec}(\text{gr } C)$$

denote the affine algebraic supergroup represented by $\text{gr } C$. One sees that $\text{gr } C$ includes $C = (\text{gr } C)(0)$ as a Hopf sub-superalgebra, and the associated, quotient Hopf superalgebra $\text{gr } C/(C^+)$ is $\wedge(W_G)$; see \cite{6}, Proposition 4.9 (2)]. Let

$$q_0 : \text{gr } C \to (\text{gr } C)(0) = C, \quad q_1 : \text{gr } C \to \text{gr } C/(C^+) = \wedge(W_G)$$

denote the quotient maps. The right $C$-comodule structure $W_G \to W_G \otimes C$ on $W_G$ which corresponds to the left co-adjoint action by $G$ uniquely extends to a right $C$-super-comodule superalgebra and super-coalgebra structure

$$\wedge(W_G) \to \wedge(W_G) \otimes C.$$ 

The associated super-coalgebra $C \triangleleft \wedge(W_G)$ of smash coproduct \cite{13} p.207, being the tensor product $C \otimes \wedge(W_G)$ as superalgebra, is a Hopf superalgebra which is canonically isomorphic to $\text{gr } C$ through

$$q_0 \otimes q_1 \circ \text{gr}(\Delta_C) : \text{gr } C \xrightarrow{\simeq} G \triangleleft \wedge(W_G);$$

see \cite{6} Proposition 4.9 (2)], again. In terms of supergroups, the affine supergroup $\text{gr } G$ include $G$ and $\text{Spec}(\wedge(W_G))$ as closed sub-supergroups, so that $\text{Spec}(\wedge(W_G))$ is normal, and the product morphism give a canonical isomorphism $G \ltimes \text{Spec}(\wedge(W_G)) \simeq \text{gr } G$.

**Lemma 3.8.** Choose such an isomorphism $\psi : \text{Spec}(\wedge(W_G)) \xrightarrow{\simeq} \text{gr } C$ as in Theorem 3.6. Then the associated isomorphism $\text{gr } \psi$ of graded superalgebras coincides with the canonical isomorphism \cite{3}.14.

**Proof.** This follows since the chosen $\psi$ is such that the graded superalgebra map $\text{gr } C \to \wedge(W_G)$ associated with $(\varepsilon_C \otimes \text{id}_{\wedge(W_G)}) \circ \psi : C \to \wedge(W_G)$ is the quotient map $q_1$ given in \cite{3}.13. $\square$

3.6. Harish-Chandra pairs. A Harish-Chandra pair \cite{13} Section 6.1] is a pair $(G, V)$ of an affine algebraic group $G$ and a right $G$-module $V$, which is equipped with a $G$-equivariant, symmetric bilinear map $[ \cdot, \cdot : V \times V \to \text{Lie}(G)$ such that $v \triangleleft [v, v] = 0$ for all $v \in V$. Here, $\text{Lie}(G)$ is supposed to be a right $G$-module by the adjoint action, and $\triangleleft$ indicates the $\text{Lie}(G)$-action on $V$ induced from the original $G$-action.

To every affine algebraic supergroup $G$, a Harish-Chandra pair $(G, V)$ is naturally assigned, where $G$ is the associated affine algebraic group, $V$ is the $\text{Lie}(G)_1 = W_G$ given the right adjoint $G$-action, and $\cdot, \cdot$ is the bracket of $\text{Lie}(G)$ restricted to $V = \text{Lie}(G)_1$. It is proved by \cite{7} Theorem 3.2] (see also \cite{9} Theorem 6.1]) that the assignment above gives rise to an equivalence
from the category of affine algebraic supergroups to the category of Harish-Chandra pairs. This is a very useful result, but it is used in this paper only at the following proof. We remark that the category-equivalence above is extended to those algebraic supergroups which are not necessarily affine, as will be proved in the forthcoming \[12\].

**Lemma-Definition 3.9.** For an affine algebraic supergroup \( G = \text{Spec} \mathbb{C} \), the following are equivalent:

1. \( G \cong \text{gr} G \) as affine supergroups;
2. \( \mathbb{C} \cong \text{gr} \mathbb{C} \) as Hopf superalgebras;
3. The Hopf superalgebra map \( q : \mathbb{C} \to \mathbb{C} \) splits;
4. The bracket on \( \text{Lie}(G) \), restricted to \( \text{Lie}(G)_1 \times \text{Lie}(G)_1 \), vanishes, so that \([\text{Lie}(G)_1, \text{Lie}(G)_1] = 0\).

If these equivalent conditions are satisfied, we say that \( G \) is \textit{graded}.

**Proof.** (i) \( \Leftrightarrow \) (ii). This is obvious.

(iv) \( \Rightarrow \) (ii). This follows from the category-equivalence mentioned above, since the Harish-Chandra pair corresponding to \( \text{gr} G \) is obtained from that pair of \( G \) just by replacing the associated \([ , ] : \text{Lie}(G)_1 \times \text{Lie}(G)_1 \to \text{Lie}(G)_0 \) with the zero map; see \[7\] Section 4.6, \[9\] Section 4.2.

(ii) \( \Rightarrow \) (iii). Assume (ii). Then the isomorphism \( \mathbb{C} \cong \text{gr} \mathbb{C} \), composed with the the natural Hopf superalgebra map \( \text{gr}(\mathbb{C}) \to \text{gr}(\mathbb{C})/(\text{gr}(\mathbb{C})_1) = C \) which obviously splits, coincides with the composite of \( q : \mathbb{C} \to \mathbb{C} \) with some automorphism of \( C \). This shows (iii).

(iii) \( \Rightarrow \) (iv). Assume (iii). Since we then have the split exact sequence \( 0 \to \text{Lie}(G)_0 \to \text{Lie}(G) \to \text{Lie}(G)_1 \to 0 \) of Lie superalgebras, (iv) follows. \( \square \)

4. The main theorem and its consequences

This section is the main body of the paper. Throughout, \( G = \text{Spec} \mathbb{C} \) denotes an affine algebraic supergroup which includes a closed sub-supergroup \( H = \text{Spec} \mathbb{D} \).

4.1. The key construction of open embeddings. We have the closed embedding and the associated surjection of Hopf superalgebras

\[
G \supset H, \quad \mathbb{C} \to \mathbb{D}.
\]

Since \( H \) acts freely on \( G \) by the right multiplication we can and we will discuss the quotient superscheme \( G/H \) and the faisceau \( G/\mathbb{H} \). The results which we are going to obtain for these have the obvious, side-switched analogues for \( H \setminus G \) or \( H \setminus G \), which hold true, indeed.

The second map in (4.1) induces a linear surjection

\[ W_G \to \mathbb{W}. \]

The kernel is denoted by

\[
Z = Z_{G/\mathbb{H}} := \text{Ker}(W_G \to \mathbb{W}).
\]

Let \( G = \text{Spec} C \) and \( H = \text{Spec} D \) denote the affine algebraic groups associated with \( G, H \), respectively. We thus have

\[ C = \mathbb{C}/(\mathbb{C}_1), \quad D = \mathbb{D}/(\mathbb{D}_1). \]
The embedding and the surjection in (4.1) induce a closed embedding of affine algebraic groups and a Hopf-algebra surjection

\[ G \supset H, \quad C \to D. \]

Here is a classical result; see [4, Part I, Sections 5.6–5.7], for example. There exists a (necessarily, unique) quotient scheme \( G/H \), which is Noetherian, and represents the faisceau \( G/\tilde{H} \). The canonical morphism of schemes

\[ \pi : G \to G/H \]

is affine, faithfully flat and finitely presented [4, Part I, Section 5.7, (1)]. Choose arbitrarily a non-empty affine open subscheme \( U \subset G/H \). Then \( \pi^{-1}(U) \) is an \( H \)-stable affine open subscheme of \( G \) such that

\[ \pi^{-1}(U)/H = U. \]

Suppose that \( \pi^{-1}(U) = \text{Spec} \, A \) and \( U = \text{Spec} \, B \). Then \( A \) is fpf over \( B \), and \( A \supset B \) is a \( D \)-Galois extension; see Definition 3.2.

By the definition (4.2) we have the short exact sequence

\[ (4.3) \quad 0 \to Z \to W_C \to W_H \to 0 \]

of right \( D \)-comodules.

**Lemma 4.1.** The tensor product \( A \otimes Z \) of right \( D \)-comodules, given the obvious multiplication by \( A \), turns into an object of \( \text{SMod}_{\hat{D}}^D \). This naturally gives rise to the right \( D \)-super-comodule superalgebra

\[ A \otimes \wedge(Z) \quad (= \wedge_A(A \otimes Z)) \]

over \( A \); notice from Remark 3.3 (2) that such a superalgebra is precisely an algebra object of the tensor category \( \text{SMod}_{\hat{D}}^D \). These \( A \otimes Z \) and \( A \otimes \wedge(Z) \) are injective as right \( D \)-comodules.

**Proof.** The assertions are obvious except the last. The last assertion follows since the right \( D \)-comodule \( A \) is injective. Indeed, an injective \( D \)-comodule tensored with any \( D \)-comodule is injective; see [4, Part I, Section 3.10, Proposition c)]. \( \square \)

Define superalgebras by

\[ (4.4) \quad A = (A \otimes \wedge(Z)) \square_D \mathbb{D}, \quad \mathbb{B} = (A \otimes \wedge(Z))^{coD}. \]

**Proposition 4.2.** We have the following.

1. \( A \) is naturally a right \( \mathbb{D} \)-super-comodule superalgebra such that

\[ (4.5) \quad A^{coD} = \mathbb{B}. \]

Moreover, \( A \) is finitely generated as a superalgebra, and is injective as a \( \mathbb{D} \)-super-comodule.

2. \( \mathbb{B} \) is a graded subalgebra of \( A \otimes \wedge(Z) \), whose 0-th component \( \mathbb{B}(0) \) is \( B \). The first component

\[ \mathbb{B}(1) = (A \otimes Z)^{coD} \]

is a finitely generated projective \( B \)-module of constant rank \( \dim \, Z \). The graded \( B \)-algebra map

\[ (4.6) \quad \wedge_B (\mathbb{B}(1)) \to \mathbb{B} \]
induced from the inclusion $B(1) \subset B$ is an isomorphism. Moreover, $B$ is Noetherian.

Proof. (1) The first assertion easily follows once one sees

$$A^{\text{co}D} = (A \otimes \wedge(Z)) \square_D D \square D k = (A \otimes \wedge(Z)) \square_D D = B.$$  

By Lemma 3.5 applied to $D$ we see that the inclusion $A \otimes \wedge(Z) \otimes^{\text{co}D} D \hookrightarrow A \otimes \wedge(Z) \otimes D$ gives the canonical isomorphism

$$(4.7) \quad A \otimes \wedge(Z) \otimes^{\text{co}D} D = A.$$  

Since $^{\text{co}D} D (\simeq \wedge(W_H))$ is finite-dimensional, $A$ is finitely generated.

Note that a $D$-(super-)comodule is injective if and only if it is a direct summand of the direct sum of some copies of $D$; see Section 2.1. One then sees that $A$ is $D$-injective since $A \otimes \wedge(Z)$ is $D$-injective by the previous lemma.

(2) The first assertion is easy to see. Since $A \supset B$ is a $D$-Galois extension (Definition 3.2), we have by Theorem 3.1 (1) the category-equivalence $S\text{Mod}_D A \approx S\text{Mod}_B$. This shows that the $A$-actions on $A \otimes \wedge(Z)$, and on its first component $A \otimes Z$ give isomorphisms

$$A \otimes_B B \simeq A \otimes \wedge(Z), \quad A \otimes_B B(1) \simeq A \otimes Z$$

in $S\text{Mod}_D A$. Since $B \to A$ is faithfully flat, the second isomorphism shows that $A$ is such a $B$-module as claimed above. The result implies that the first isomorphism above, composed with the base extension of (4.6) along $B \to A$, is an isomorphism. Again by the faithful flatness, the map (4.6) is an isomorphism. It follows that $B$ is Noetherian since $B$ is. \hfill $\square$

We will see in the proof of Corollary 4.10 that $A \supset B$ is a $D$-Galois extension.

Remark 4.3. Here are two alternative ways of describing $B(1)$.

(1) Given a right $D$-comodule, one has a left $D$-comodule, twisting the side of the coaction through the antipode of $D$. Applied to $Z$, the resulting left $D$-coaction on $Z$ is what corresponds to the right coadjoint action by $H$; see Section 3.4. Regrading $Z$ thus as a left $D$-comodule, we have the alternative description

$$(4.8) \quad B(1) = A \square_D Z,$$

which will be often used. By Proposition 4.4 (2) we have

$$(4.9) \quad B \simeq \wedge_B(A \square_D Z).$$

(2) By (3.11) the dual $Z^*$ of $Z$ is the quotient vector space $\text{Lie}(G)_1 / \text{Lie}(H)_1$, on which $H$ acts by adjoint from the left. We see that $B(1)$ is identified so as

$$(4.10) \quad B(1) = \text{Comod}^D(Z^*, A) (= \text{Mod}_H(Z^*, A))$$

with the vector space of right $D$-comodule (or left $H$-module) maps.

We regard the tensor product $A \otimes W_G$ of right $D$-comodules as a purely odd object of $S\text{Mod}_D A$, with respect to the obvious multiplication by $A$; this then includes $A \otimes Z$ as a sub-object.
Lemma 4.4. The inclusion $A \otimes Z \hookrightarrow A \otimes W_G$ splits in $\text{SMod}_A^D$.

Proof. Since $A$ is $D$-injective, the unit map $\kappa \rightarrow A$, which is $D$-colinear, extends to a $D$-comodule map

$$\eta : D \rightarrow A,$$

which thus satisfies $\eta(1) = 1$. It follows by \cite{3} Theorem 1] that a short exact sequence in $\text{SMod}_A^D$ splits if it splits $A$-linearly. Since obviously, the inclusion in question splits $A$-linearly, it splits in $\text{SMod}_A^D$, as desired. See the following remark for the explicit retraction constructed from $\eta$. \hfill $\square$

Remark 4.5. (1) Let $\eta : D \rightarrow A$ be as in (1). Given an object $M$ of $\text{SMod}_A^D$, let $M \rightarrow M \otimes D$, $m \mapsto m(0) \otimes m(1)$ denote the $D$-comodule structure map. This is a monomorphism in $\text{SMod}_A^D$, and its retraction is given by

$$\sigma_M : M \otimes D \rightarrow M, \quad \sigma_M(m \otimes d) = m(0)\eta(S_D(m(1))d),$$

where $S_D$ denotes the antipode as in \cite{3}; see \cite{3} Page 100, line -1]. One sees easily that a retraction of the inclusion $A \otimes Z \hookrightarrow A \otimes W_G$ above is given by the composite

$$A \otimes W_G \rightarrow (A \otimes W_G) \otimes D \xrightarrow{id \otimes r \otimes id} (A \otimes Z) \otimes D \xrightarrow{\kappa \otimes \eta} A \otimes Z,$$

where the first arrow is the $D$-comodule structure map on $A \otimes W_G$, and $r : W_G \rightarrow Z$ is an arbitrarily chosen, linear retraction of the inclusion $Z \hookrightarrow W_G$.

(2) Let $(O_G(\pi^{-1}(U)) =) A \rightarrow A' = O_G(\pi^{-1}(U'))$ be the restriction map associated with $\pi^{-1}(U) \supset \pi^{-1}(U')$, where $U'$ is any non-empty affine open subscheme of $G/H$ included in $U$. It then follows that retractions such as above can be chosen so as to be compatible with $A \rightarrow A'$, since $\eta$ can be so chosen.

Let us choose a retraction in $\text{SMod}_A^D$

$$\theta : A \otimes W_G \rightarrow A \otimes Z$$

of the inclusion $A \otimes Z \hookrightarrow A \otimes W_G$; it may not be such as above that was constructed from some $\eta$.

Recall that $A$ is an (algebra) object of $\text{SMod}_A^D$. We regard $W_G \otimes A$ as such an object with respect to the structure possessed by the tensor factor $A$. Recall that $\pi^{-1}(U) = \text{Spec} A$ is an affine open subset of $G = \text{Spec} C$. Let $\iota : C \rightarrow A$ is the algebra map associated with $G \supset U$.

Lemma 4.6. The map

$$\kappa = \kappa_A : A \otimes W_G \rightarrow W_G \otimes A, \quad \kappa(a \otimes w) = w(0) \otimes a \iota(w(1))$$

is an isomorphism in $\text{SMod}_A^D$, where $w \mapsto w(0) \otimes w(1)$ indicates the right $C$-comodule structure map $W_G \rightarrow W_G \otimes C$.

Proof. Indeed, $w \otimes a \rightarrow a \iota(S_D(w(1))) \otimes w(0)$ gives an inverse. \hfill $\square$

Remark 4.7. Assume that $Z$ is a $C$-subcomodule (or equivalently, a $G$-submodule) of $W_G$; by \cite{11} Lemma 3.5] this is satisfied, if $\mathbb{H}$ is normal in $G$, or namely, if for every $T \in \text{SAlg}$, $\mathbb{H}(T)$ is normal in $G(T)$. Since the restriction $\kappa|_{A \otimes Z}$ of $\kappa$ to $A \otimes Z$ then maps isomorphically onto $Z \otimes A$, we have

$$\mathbb{H}(1) \simeq Z \otimes A^{coD} = Z \otimes B,$$
so that $\mathcal{B}(1)$ is a free $B$-module of rank $\dim Z$; cf. (4.17).

Let

$$
\theta' : W_G \otimes A \xrightarrow{\kappa^{-1}} A \otimes W_G \xrightarrow{\theta} A \otimes Z
$$

be the composite of $\kappa^{-1}$ with the $\theta$ chosen before. This is thus a retraction of $\kappa|_{A \otimes Z} : A \otimes Z \to W_G \otimes A$ in $\text{SMod}_A^D$. There arises the $A$-algebra morphism

$$\wedge(\theta') : \wedge(W_G) \otimes A \to A \otimes \wedge(Z)$$

in the tensor category $\text{SMod}_A^D$ (see Remark 3.3 (2)), which is a retraction of $\wedge(\kappa|_{A \otimes Z})$. Essentially by Theorem 3.6 we can choose an isomorphism

$$\psi' : C \xrightarrow{\sim} \wedge(W_G) \otimes C$$

with the analogous, opposite-sided properties to those one s which $\psi : C \xrightarrow{\sim} C \otimes \wedge(W_G)$ such as in (3.12) has. We define

$$\omega_\theta : C \to A = (A \otimes \wedge(Z)) \square_D \mathbb{D}$$

to be the composite

$$
\begin{align*}
C & \xrightarrow{\Delta} C \square_D C \longrightarrow C \square_D \mathbb{D} \\
& \xrightarrow{(\text{id}_C \otimes \text{id}) (\wedge(W_G) \otimes A) \square_D \mathbb{D}} (A \otimes \wedge(Z)) \square_D \mathbb{D}
\end{align*}
$$

where the second arrow is the Hopf algebra quotient $C \to D$ co-tensored with $\text{id}_C$. As for the first arrow note that the coproduct $\Delta_C$ goes into the co-tensor product $C \square_D C$. As for the third, $\psi'$, being $C$-colinear, is $D$-colinear.

**Proposition 4.8.** $\omega_\theta : C \to A$ gives rise to a right $\mathbb{H}$-equivariant embedding

$$\text{Spec}(\omega_\theta) : \text{Spec} A \to \text{Spec} \mathbb{C} = \mathbb{G}.$$

of superschemes onto the open subset $\pi^{-1}(U)$ of $|\mathbb{H}| (= |\mathbb{G}|)$.

**Proof.** For simplicity let us write $\mathcal{X}$ for $\text{Spec} A$. As is seen from (4.17), the underlying topological space $|\mathcal{X}| = \text{Spec}(A_0)$ of $\mathcal{X}$ is naturally identified with $\pi^{-1}(U) = \text{Spec} A$. Let $\mathcal{O}_\mathcal{G}$ denote the structure sheaf of $\mathcal{G}$. It remains to show that the restricted sheaf $\mathcal{O}_\mathcal{G}|_{|\mathcal{X}|}$ coincides with $\mathcal{O}_\mathcal{X}$. We should prove the following two:

1. The algebra map associated with $\omega_\theta$ coincides with $\iota : C \to A$, so that $\text{Spec}(\omega_\theta)$ gives the open embedding $\pi^{-1}(U) \hookrightarrow |\mathbb{G}| (= |\mathbb{G}|)$ of the underlying topological spaces;

2. The restricted sheaf $\mathcal{O}_\mathcal{G}|_{|\mathcal{X}|}$ coincides with $\mathcal{O}_\mathcal{X}$.

We wish to see what the graded superalgebra map $\text{gr}(\omega_\theta)$ associated with $\omega_\theta$ is. By the analogous, opposite-sided result to Lemma 3.5, $\text{gr} \psi'$ is the canonical isomorphism $\text{gr} \mathbb{C} = \wedge(W_G) \hookrightarrow \wedge(W_G) \otimes C$. Therefore, the graded algebra map associated with the first row (4.17) of the composite defining $\omega_\theta$ is

$$\text{gr} \mathbb{C} = C \otimes \wedge(W_G) \to C \otimes \wedge(W_G) \otimes \wedge(W_\mathbb{H}) \xrightarrow{\wedge(\kappa_C) \otimes \text{id}} \wedge(W_G) \otimes C \otimes \wedge(W_\mathbb{H}),$$

where the first arrow is the natural right $\wedge(W_\mathbb{H})$-super-comodule structure map, and the $\wedge(\kappa_C)$ in the second arrow is the graded-algebra isomorphism

$$\wedge(\kappa_C) : \wedge(W_G) \otimes C \otimes \wedge(W_\mathbb{H}) \xrightarrow{\sim} \wedge(W_G) \otimes C \otimes \wedge(W_\mathbb{H}).$$
arising from the isomorphism $\kappa_C$ as defined by (4.14). By using $\kappa_A \circ (\iota \otimes \text{id}_{W_G}) = (\text{id}_{W_G} \otimes \iota) \circ \kappa_C$, it follows that $\text{gr}(\omega_B)$ is the composite

$$
\text{gr} C = C \otimes \wedge(W_G) \to C \otimes \wedge(W_G) \otimes \wedge(W_H)
$$

$$
\iota \otimes \text{id} \otimes \text{id} : A \otimes \wedge(W_G) \otimes \wedge(W_H) \otimes \wedge(W_H)
$$

This is seen to be $\iota : C \to A$ in degree zero. This proves (1).

To prove (2), let $P \in \text{Spec}(A_0)$, and set $Q = \omega_B^{-1}(P)$. We should prove that the local superalgebra map of stalks

$$
(\omega_B)_P : C_Q \to A_P,
$$

at $P$ is an isomorphism. It suffices to prove that the associated graded algebra map $\text{gr}(\omega_B)_P$ is an isomorphism. As was seen in the proof of Lemma 2.1 we have $\text{gr}(\omega_B)_P = \text{gr}(\omega_B)_{\iota P}$. As for the latter, we may suppose $P \in \text{Spec} A$, $Q \in \text{Spec} C$ and that the relevant localizations $(\text{gr} A)_P$ and $(\text{gr} C)_Q$ are by those. In the same situation, $\iota_P : C_Q \to A_P$ is an isomorphism since $\pi^{-1}(U) \subset G$ is open. From the result obtained in the preceding paragraph we see that

$$
\text{gr}(\omega_B)_P : C_Q \otimes \wedge(W_G) \to A_P \otimes \wedge(Z) \otimes \wedge(W_H).
$$

is a right $\wedge(W_H)$-super-comodule superalgebra map, which, restricted to the $\wedge(W_H)$-coinvariants, coincides with $\iota_P \otimes \text{id} : C_Q \otimes \wedge(Z) \to A_P \otimes \wedge(Z)$. This last property shows that $\text{gr}(\omega_B)_P$ is an isomorphism, as desired. Indeed, $\text{gr}(\omega_B)_P$ is a morphism in that category $\text{SMod}_{\wedge(W_H)}$ which arises from the right $\wedge(W_H)$-super-comodule superalgebra $\wedge(W_G)$; the map is, moreover, a $\wedge(W_G)$-algebra morphism in the category. Since $\wedge(W_G) \supset \wedge(Z)$ is obviously a $\wedge(W_H)$-Galois extension, satisfying Condition (iii) of Theorem 3.1 (1), the resulting category-equivalence $\text{SMod}_{\wedge(W_G)} \approx \text{SMod}_{\wedge(Z)}$ can apply to see the result.

Remark 4.9. The argument of the last proof (see also the proof of Lemma 2.1) shows: given a superalgebra map $\omega : T \to S$ between Noetherian superalgebras, the associated morphism $\text{Spec}(\omega) : \text{Spec}S \to \text{Spec}T$ of affine superschemes is an open embedding if and only if the morphism $\text{Spec}(\text{gr}(\omega)) : \text{Spec}(\text{gr} S) \to \text{Spec}(\text{gr} T)$ associated with $\text{gr}(\omega) : \text{gr} T \to \text{gr} S$ is an open embedding. Note that the two associated continuous maps between the underlying topological spaces are naturally identified. The result is easily generalized in the obvious manner to morphisms of Noetherian superschemes.

4.2. The main theorem. Retaining the situation as above we have the following corollary to the previous proposition.

Corollary 4.10. The open subset $\pi^{-1}(U) \subset |G|$, regarded as an open sub-superscheme of $G$, is an $\mathbb{H}$-equivariant affine superscheme $\text{Spec}(O_G(\pi^{-1}(U)))$ such that the faisceau $\pi^{-1}(U)/\mathbb{H}$ is the affine superscheme

$$
\text{Spec} \left( O_G(\pi^{-1}(U))^\text{coD} \right),
$$

which is Noetherian.
Proof. By Proposition 3.1 (2), $\mathcal{O}_G(\pi^{-1}(U))$ is $\mathbb{H}$-stable and affine in $G$; in particular, $\mathcal{O}_G(\pi^{-1}(U))$ is a right $\mathbb{D}$-super-comodule superalgebra. Moreover, $\omega_B$ naturally factors through an isomorphism $\mathcal{O}_G(\pi^{-1}(U)) \xrightarrow{\sim} \mathbb{A}$ of $\mathbb{D}$-supercomodule superalgebras, which obviously restricts to $\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D} \xrightarrow{\sim} \mathbb{B}$. Recall from Proposition 4.12 that $\mathbb{A}$ is finitely generated, and $\mathbb{B}$ is Noetherian.

We claim that $\mathbb{A} \supset \mathbb{B}$ is a $\mathbb{D}$-Galilean extension; this implies the corollary by Theorem 4.1 (2). Since $\mathbb{A}$ is $\mathbb{D}$-injective by Proposition 4.2 (1), it suffices by Theorem 3.1 (1) (see Condition (iii)) to prove that the $\mathcal{A}_G$-superalgebra map $\alpha_{\mathcal{A}}$ in (3.6) is surjective. Let $P$, $Q$ be as in the last proof. Then we have the following commutative diagram which contains the map $(\omega_B)_P$ in (4.18); it has been proved to be an isomorphism.

$$
\begin{array}{ccc}
n C_Q \otimes C & \xrightarrow{(\alpha_C)_Q} & C_Q \otimes D \\
(\omega_B)_P \otimes \omega_B & \cong & (\omega_B)_P \otimes \text{id}
\end{array}
\begin{array}{ccc}
A_P \otimes A & \xrightarrow{(\alpha_{\mathcal{A}})_P} & A_P \otimes D
\end{array}
$$

Here the horizontal arrows are localizations of the alpha maps. The upper $(\alpha_C)_Q$ is surjective since the map $\alpha_C$ factors through the canonical isomorphism $C \otimes C \xrightarrow{\sim} C \otimes C$, $x \otimes y \mapsto x\Delta_C(y)$, and is, therefore, surjective. It follows that the lower $(\alpha_{\mathcal{A}})_P$ is as well, proving the desired surjectivity. \qed

Remark 4.11. As is seen from the proof above, the map $\mathcal{O}_G(\pi^{-1}(U)) \rightarrow \mathcal{O}_G(\pi^{-1}(U)) = A$ of superalgebras of sections, associated with $G \supset G$, induces an isomorphism from the algebra associated with $\mathcal{O}_G(\pi^{-1}(U))$ onto $A$. The last map restricts to $\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D} \rightarrow A^{\text{co}D} = B$, which induces an isomorphism from the algebra associated with $\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D}$ onto $B$. The underlying topological space of $\text{Spec}(\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D})$ is thus naturally identified with $U$, the underlying topological space of $\text{Spec} B$.

Given a non-empty affine open subset $U$ of $|G/H|$, we thus have the Noetherian affine superscheme $\text{Spec}(\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D})$ with underlying topological space $U$.

Theorem 4.12. The Noetherian affine superschemes

$$
\text{Spec}(\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D}),
$$

where $U$ ranges over non-empty affine open subsets of $|G/H|$, are uniquely glued into a superscheme, which is Noetherian, with the underlying topological space $|G/H|$. This superscheme is the quotient superscheme $\mathbb{G}/\mathbb{H}$ of $\mathbb{G}$ by $\mathbb{H}$, and represents the faisceau $\mathbb{G}/\mathbb{H}$.

Proof. The theorem consists of two assertions.

Proof of the first assertion. Let $U = \text{Spec} B \supset U' = \text{Spec} B'$ be affine open subschemes of $G/H$. The restriction map $\mathcal{O}_G(\pi^{-1}(U)) \rightarrow \mathcal{O}_G(\pi^{-1}(U'))$ restricts to

$$
\mathcal{O}_G(\pi^{-1}(U))^{\text{co}D} \rightarrow \mathcal{O}_G(\pi^{-1}(U'))^{\text{co}D}.
$$

Suppose $\pi^{-1}(U) = \text{Spec} A$, $\pi^{-1}(U') = \text{Spec} A'$ in $G$. From these $A$ and $A'$, we construct superalgebras $\mathbb{A} \supset \mathbb{B}$, $\mathbb{A}' \supset \mathbb{B}'$, respectively, as in (4.4).
By choosing retractions $A(\cdot) \otimes W_G \to A(\cdot) \otimes Z$ as in (4.12), we construct $D$-super-comodule superalgebra maps

$$C \to A = (A \otimes \langle Z \rangle) \square_D D, \quad C \to A' = (A' \otimes \langle Z \rangle) \square_D D,$$

as in (4.16), which give open embeddings of $\text{Spec} A$ and of $\text{Spec} A'$ into $G = \text{Spec} C$ by Proposition 4.8. As is seen from Remark 4.5 (2) and the description (4.12) of the retractions, we may suppose that the two superalgebra maps above are compatible with the map $(A \otimes \langle Z \rangle) \square_D D \to (A' \otimes \langle Z \rangle) \square_D D$ which arises from the restriction map $A \to A'$; this compatibility is expressed by commutativity of the diagram:

$$\begin{tikzcd}
A & A' = (A' \otimes \langle Z \rangle) \square_D D \\
C \\
A = (A \otimes \langle Z \rangle) \square_D D \\
\end{tikzcd}$$

Consequently, the map (4.19) may be supposed to be the map

$$(4.20) \quad B = A \square_D \langle Z \rangle \to A' \square_D \langle Z \rangle = B'$$

which arises from $A \to A'$, again. Here for $B$ and $B'$, we have used description analogous to (4.8).

Let $P \in U'(= \text{Spec} B')$. Let $Q$ be the pullback of $P$ in $B$ along the algebra map $B \to B'$ associated with $U \supset U'$. The map induces an isomorphism, $B_Q \cong B'_P$, of stalks. We claim that the superalgebra map above induces an isomorphism, $B_Q \cong B'_P$, between the stalks. Here one should notice from (4.3) that $B_Q = B \otimes_B B_Q$ and $B'_P = B' \otimes_{B'} B'_P$ are indeed the stalks; see the proof of Lemma 2.1. Since $B(\cdot) = A(\cdot) \otimes_D D$, the exactness of localization shows that the localized $B_Q \to B'_P$ coincides with the localized $A_Q \to A'_P$ cotensored over $D$ with the identity map $\langle Z \rangle \to \langle Z \rangle$. The map $A_Q \to A'_P$ is a $D$-comodule algebra map, and it restricts to the isomorphism $B_Q \cong B'_P$. Since $A_Q \supset B_Q$ and $A'_P \supset B'_P$ are $D$-Galois, it follows that $A_Q \to A'_P$ is an isomorphism, proving the claim. Indeed, we have the commutative diagram:

$$\begin{tikzcd}
A'_P \otimes_{B_Q} A_Q & A'_P \otimes D \\
A'_P \otimes_{B'_P} A'_P \\
\end{tikzcd}$$

Here the second row is the canonical isomorphism $\beta$ for $A'_P \supset B'_P$ (see (5.7)), while the first is the base extension of the isomorphism for $A_Q \supset B_Q$, along $A_Q \to A'_P$. Since $(B_Q \cong) B'_P \to A'_P$ is faithfully flat, $A_Q \to A'_P$ is an isomorphism.

Let

$$\mathcal{Y}_U := \text{Spec} \left( \mathcal{O}_G(\pi^{-1}(U))^{\text{comod}} \right).$$

The claim just proven implies that the structure sheaf of this $\mathcal{Y}_U$, restricted to $U'$, coincides with that sheaf of $\mathcal{Y}_{U'}$. This proves the first assertion: the Noetherian affine superschemes are uniquely glued into a superscheme, say...
\[ \mathcal{Y} \text{ is Noetherian since } G/H, \text{ being Noetherian, is covered by finitely many } U\text{'s.} \]

**Proof of the second assertion.** By Corollary 4.10 \( \mathcal{Y}_U \) represents the faisceau \( \pi^{-1}(U)/\mathbb{H} \). By Lemma 3.4 \( \mathcal{Y}_U \) is the quotient superscheme \( \pi^{-1}(U)/\mathbb{H} \). We know that \( \mathcal{Y} \) is the union \( \bigcup_i \mathcal{Y}_{U_i} \) (= \( \bigcup_i \pi^{-1}(U_i)/\mathbb{H} \)), where \( |G/H| = \bigcup_i U_i \). As is easily seen, the quotient morphisms \( \pi^{-1}(U_i) \to \mathcal{Y}_i \) uniquely extend to a morphism \( G \to \mathcal{Y} \). It follows that the superscheme \( \mathcal{Y} \) equipped with the last morphism is the quotient superscheme \( G/\mathbb{H} \). The category-equivalence \( X \mapsto \mathcal{X}^o \) in Theorem 2.3 preserves open embeddings, and \( \mathcal{Y}^o = \bigcup \pi^{-1}(U_i)/\mathbb{H} \); see [10] Lemma 5.2. It follows that \( \mathcal{Y}^o = G/\mathbb{H} \), or \( \mathcal{Y} \) represents \( G/\mathbb{H} \).

**Remark 4.13.** Let \( \mathcal{O}_{G/\mathbb{H}} \) (resp., \( \mathcal{O}_{G/H} \)) denote the structure sheaf of \( G/\mathbb{H} \) (resp., \( G/H \)). In view of (4.9) we see from the last proof that \( \mathcal{O}_{G/\mathbb{H}} \) is locally isomorphic to

\[ \wedge \mathcal{O}_{G/\mathbb{H}}(\pi_* \mathcal{O}_D Z); \]

to be more precise the two sheaves are isomorphic, restricted to every open subset that is affine in \( G/H \), or equivalently, in \( G/\mathbb{H} \); see Proposition 4.16 (2) below. Here note that for every open subset \( U \subset |G/H| (= |G/\mathbb{H}|) \), \( \pi^{-1}(U) \approx U \times_{G/H} G \) is \( H \)-stable in \( G \). Hence \( \mathcal{O}_G(\pi^{-1}(U)) \) is naturally a right \( D \)-comodule, and the co-tensor product \( \mathcal{O}_G(\pi^{-1}(U)) \square_D Z \) makes sense. Since \( \dim Z < \infty \), it follows that the presheaf \( \pi_* \mathcal{O}_D Z \), which assigns \( \mathcal{O}_G(\pi^{-1}(U)) \square_D Z \) to every open \( U \), is a sheaf. It is indeed an \( \mathcal{O}_{G/H} \)-module sheaf, which is locally free, as is seen from Proposition 4.12 (2).

4.3. **Consequences of the theorem.** The first half of the following has been obtained in [10] Corollary 8.15, while the second half is hopefully new.

**Corollary 4.14.** The superscheme \( \mathcal{G}/\mathbb{H} \) is affine if and only if the scheme \( G/H \) is affine. In this case,

\[ \mathcal{G}/\mathbb{H} = \text{Spec}(\wedge_B(C \square_D Z)), \]

where we let \( B = \mathcal{C}^{\text{co}D} \), and so \( G/H = \text{Spec} B \).

**Proof.** The “only if” follows since one sees from the functorial viewpoint that if \( \mathcal{G}/\mathbb{H} = \text{Sp} \mathcal{B} \) is affine, then \( G/H = \text{Sp}(\mathcal{B}/(\mathcal{B}_1)) \). The remaining follows from Corollary 4.10 and Remark 4.13.

**Remark 4.15.** It follows by [21 Theorem 5.2] or Theorem 3.1 that \( \mathcal{G}/\mathbb{H} \) is affine if and only if

(i) \( \mathcal{C} \) is injective (or equivalently, coflat) as a left or right \( D \)-comodule.

It is known (see [6 Theorem 5.9], [21 Theorem 6.2]) that if \( \mathbb{H} \) is normal in \( \mathcal{G} \), then the equivalent conditions are satisfied, and \( \mathcal{G}/\mathbb{H} \) is naturally an affine algebraic supergroup. The classical counterpart (see [18 Theorem 10], [14] Folgerung B) states that \( G/H \) is affine if and only if

(ii) \( \mathcal{C} \) is injective (or equivalently, coflat) as a left or right \( D \)-comodule.

If \( H \) is normal in \( G \), then the equivalent conditions are satisfied, and \( G/H \) is naturally an affine algebraic group.

Therefore, Corollary 4.14 tells us that Conditions (i) and (ii) are equivalent. If \( H \) is normal in \( G \), then \( \mathcal{G}/\mathbb{H} \) is an affine algebraic superscheme since the \( B \) is in the corollary is then finitely generated.
The first half of the last corollary (or [10, Corollary 8.15]) is generalized by Part 2 of the next Proposition, which would reveal a remarkable feature of $G/H$.

**Proposition 4.16.** We have the following.

1. $G/H$ is naturally isomorphic to the scheme associated with the superscheme $G/H$, see Definition 2.2.
2. Given an open subset $U \subset |G/H|$ (= $|G/H|$), the open sub-superscheme $(U, \mathcal{O}_{G/H}|U)$ of $G/H$ is affine if and only if the open subscheme $(U, \mathcal{O}_{G/H}|U)$ of $G/H$ is affine.
3. $G/H$ is smooth if and only if $G/H$ is smooth.
4. The quotient morphism $G \rightarrow G/H$ is affine, faithfully flat and finitely presented.

**Proof.** (1) The structure sheaf of the associated scheme is naturally isomorphic to $\mathcal{O}_{G/H}$, since it is so on all affine open subsets of $|G/H|$ by Remark 4.11.

(2) The “if” follows from Theorem 4.12, while the “only if” follows from Part 1 above.

(3) For every point $P \in |G/H|$, $\mathcal{O}_{G/H,P}$ is the exterior algebra over $\mathcal{O}_{G/H,P}$ on a finitely generated free $\mathcal{O}_{G/H,P}$-module, as is seen from Proposition 4.2. It follows from [11, Theorem A.2] that $\mathcal{O}_{G/H,P}$ is smooth if and only if $\mathcal{O}_{G/H,P}$ is. This proves the result.

(4) In addition to Part 2 above we have the result in a special case that an open sub-superscheme of $G$ is affine if and only if the associated, open sub-scheme of $G$ is affine. Hence the desired affinity follows from the fact that $G \rightarrow G/H$ is affine. The faithful flatness follows since with the notation (4.4), $A$ is faithfully flat over $B$. The remaining follows, since the $B$-superalgebra $A$ is finitely presented, being so after base extension along $B \rightarrow A; this last is seen from the isomorphism $\beta : A \otimes_B A \xrightarrow{\sim} A \otimes B$ as in (3.7).

**Remark 4.17.** Corollary 9.10 of [10] proves that the first two properties of Part 4 above are possessed, more generally, by the quotient superscheme $X/G$, if it exists and represents the faisceau $X/G$, where $X$ is an affine superscheme, and $G$ is an affine algebraic supergroup which freely acts on $X$.

Let $X$ be a superscheme. We say that $X$ is split, if there exists a scheme with the same underlying topological space $|X| = |X|$ as that space of $X$, together with a locally free $\mathcal{O}_X$-module sheaf $\mathcal{M}$, such that

$$\mathcal{O}_X \simeq \wedge\mathcal{O}_X(\mathcal{M}),$$

that is, $\mathcal{O}_X$ and $\wedge\mathcal{O}_X(\mathcal{M})$ are globally isomorphic, where $\mathcal{M}$ is supposed to be purely odd. One sees that $X$ is necessarily the scheme associated with $X$. If $X$ is split, then $X = \text{gr} X$.

**Proposition 4.18.** The graded superscheme $\text{gr}(G/H)$ associated with $G/H$ (see Definition 2.1) is split with the structure sheaf

$$\wedge\mathcal{O}_{G/H} (\pi_* \mathcal{O}_G \square_D Z);$$
see Remark 4.13. Moreover, the morphism \( \text{gr} \, G \to \text{gr}(G/\mathbb{H}) \) associated with the quotient morphism \( G \to G/\mathbb{H} \) induces an isomorphism

\[
\text{gr} \, G/ \text{gr} \, \mathbb{H} \simeq \text{gr}(G/\mathbb{H}).
\]

**Proof.** Recall that the structure sheaf \( \mathcal{O}_{G}(G/\mathbb{H}) \) of \( \text{gr}(G/\mathbb{H}) \) is a sheaf of graded superalgebras. The 0-th component is \( \mathcal{O}_{G/H} \) by Proposition 4.16 (1). The first component coincides with \( \pi_{*}\mathcal{O}_{G/H} \boxtimes_{D} Z \), since it does on all affine open subsets of \( [G/H] \); see Remark 4.14. Therefore, we have a natural morphism \( \wedge_{G/H}(\pi_{*}\mathcal{O}_{G} \boxtimes_{D} Z) \to \mathcal{O}_{G}(G/\mathbb{H}) \) of sheaves, which is identical in degree 0, 1; this is isomorphic since it is so on all affine open sets. The result just proven, combined with Proposition 4.19 below in Case (c), proves (4.22). □

**Proposition 4.19.** The superscheme \( G/\mathbb{H} \) is split with the structure sheaf as in (4.21), either if

1. \( Z \) is a \( C \)-subcomodule of \( W_{G} \),
2. \( Z \) is a \( D \)-comodule direct summand of \( W_{G} \), or
3. \( G \) is graded in the sense as defined by Definition 3.3.

To prove this we wish to show that the sheaf (4.21) and \( \mathcal{O}_{G/\mathbb{H}} \) are naturally isomorphic on all affine open subsets of \( [G/H] \), in Cases (a), (b) and in Case (c), separately.

**Proof in Cases (a), (b).** In these cases we construct retractions \( \vartheta : A \otimes W_{G} \to A \otimes Z \) as in (4.13), which do not depend on the \( \eta \) in (4.11). In Case (a), choose a linear retraction \( r : W_{G} \to Z \) of the inclusion \( Z \hookrightarrow W_{G} \), and define \( \vartheta \) to be the composite

\[
A \otimes W_{G} \xrightarrow{\xi^{-1}} W_{G} \otimes A \xrightarrow{r \otimes \text{id}} Z \otimes A \xrightarrow{(\alpha \otimes Z)^{-1}} A \otimes Z.
\]

This is indeed possible as is seen from Remark 4.14. In Case (b), choose a \( D \)-collinear retraction \( s : W_{G} \to Z \), and let \( \vartheta = \text{id} \otimes s : A \otimes W_{G} \to A \otimes Z \).

Using these \( \vartheta \), define \( \mathbb{D} \)-super-comodule superalgebra maps \( \omega_{\theta} : C \to (A \otimes (A \otimes Z)) \boxtimes_{\mathbb{D}} \mathbb{D} \) as in (4.14) for all \( A = \mathcal{O}_{G}(\pi^{-1}(U)) \), where \( U \subset [G/H] \) are affine open. Then the maps are seen to be compatible in the same sense as in Remark 4.15 (2), with respect to all pairs \( U \supset U' \) in \( [G/H] \). It follows that all superalgebra maps \( \mathcal{O}_{G}(\pi^{-1}(U)) \boxtimes_{\mathbb{D}} \mathbb{D} \to \mathcal{O}_{G}(\pi^{-1}(U')) \boxtimes_{\mathbb{D}} \mathbb{D} \) restricted from the restriction maps \( \mathcal{O}_{G}(\pi^{-1}(U)) \to \mathcal{O}_{G}(\pi^{-1}(U')) \) may be identified with those maps \( A \boxtimes_{D} (A \otimes Z) \to A' \boxtimes_{D} (Z) \) which arise from the restriction maps \( A = \mathcal{O}_{G}(\pi^{-1}(U)) \to \mathcal{O}_{G}(\pi^{-1}(U')) = A' \). In view of (4.9) this proves the desired result. □

**Proof in Case (c).** Assume (c). Then \( C = C \otimes \Lambda(W_{G}) \). One sees from Lemma 3.9 (see Condition (iv)) that \( \mathbb{H} \) as well is graded, so that \( \mathbb{D} = D \otimes \Lambda(W_{G}) \).

Let \( U \subset [G/H] \) be non-empty affine open. Set \( A = \mathcal{O}_{G}(\pi^{-1}(U)) \), and let \( \iota : C \to A \) be the (right \( D \)-comodule) algebra map associated with \( G \supset U \).

Compose the now canonical isomorphism

\[
\psi = \text{gr} \, \psi : C \xrightarrow{\sim} C \otimes \Lambda(W_{G}) (= C \otimes \Lambda(W_{G}))
\]

as in (3.12) with \( \iota \otimes \text{id} : C \otimes \Lambda(W_{G}) \to A \otimes \Lambda(W_{G}) \). The resulting \( C \to A \otimes \Lambda(W_{G}) \) is seen to be a right \( \mathbb{D} \)-super-comodule superalgebra map. Moreover,
it gives rise to a right $\mathbb{H}$-equivariant embedding $\text{Spec}(A \otimes \wedge(W_G)) \to G$ of superschemes onto the open subset $\pi^{-1}(U)$ of $|G| = |G|$; see the proof of Proposition 4.18. Here, the $\mathbb{D}$-super-comodule structure on $A \otimes \wedge(W_G)$ is such that $\wedge(W_H)$ co-acts naturally on the tensor factor $\wedge(W_G)$, and $D$ co-acts co-diagonally on the tensor product. Therefore, the $\wedge(W_H)$-coinvariants in $A \otimes \wedge(W_G)$ are given by

$$(A \otimes \wedge(W_G))^{co\wedge(W_H)} = A \otimes \wedge(Z).$$

This last is the tensor product of two right $D$-comodules. Its $D$-coinvariants coincide with the $D$-coinvariants in the original $A \otimes \wedge(W_G)$, and are given by

$$(A \otimes \wedge(Z))^{coD} = A \Box D \wedge(Z).$$

We thus have $O_G(\pi^{-1}(U)) \simeq A \Box D \wedge(Z) = \wedge_B(A \Box D Z)$; see (4.9). Since the isomorphism is natural in $U$, the desired result follows. □

**Remark 4.20.** (1) The super-Grassmanians $\text{Gr}(s|r,m|n)$, super-analogues of Grassmanians, are presented in the form $G/\mathbb{H}$ as in [10, Section 6]; $\text{Gr}(s|r,m|n)$ is a smooth algebraic superscheme which has the product $\text{Gr}(s,m) \times \text{Gr}(r,n)$ of Grassmanians as its associated scheme. It was early proved by Manin [5, Chapter 4, Section 3, 16. Example, p.200] that the super-Grassmanian $\text{Gr}(1|1,2|2)$ over the field of complex numbers is not split.

(2) E. G. Vishnyakova [19, 20] studies the splitting property of quotients $G/\mathbb{H}$ in the analytic situation for complex super Lie groups. Theorem 2 of [19] proves our Proposition 4.19 in Case (c) in the analytic situation. Example 3 of [20] tells us that the super-Grassmanian $\text{Gr}(s|r,m|n)$, constructed as a complex super-manifold, is not split if and only if $0 < s < m$ and $0 < r < n$. The "if" holds as well for our algebraic $\text{Gr}(s|r,m|n)$ (over the field of complex numbers), since one can prove the following: (i) The analytic $\text{Gr}(s|r,m|n)$ is the analytification of ours; (ii) If a smooth locally-algebraic superscheme is split, then its analytification is, as well.

**ACKNOWLEDGMENTS**

The first-named author was supported by JSPS Grant-in-Aid for Scientific Research (C) 17K05189. The authors thank Alexandr Zubkov for his helpful comments on an earlier version of this paper.

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