Some Applications of the Mellin Transform to Branching Processes

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Abstract
We introduce a Mellin transform of functions which live on all of \( \mathbb{R} \) and discuss its applications to the limiting theory of Bellman-Harris processes, and specifically Luria-Delbrück processes. More precisely, we calculate the life-time distribution of particles in a Bellman-Harris process from their first-generation offspring and limiting distributions, and prove a formula for the Laplace transform of the distribution of types in a Luria-Delbrück process in the Mittag-Leffler limit. As a by-product, we show how to easily derive the (classical) Mellin transforms of certain stable probability distributions from their Fourier transform.

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1 Introduction

In this note, we shall concern ourselves with some applications of the Mellin transform to the theory of branching processes; see Sections 4 and 5 for further motivation. We start with a definition of the Mellin transform for functions that live on all of $\mathbb{R}$. Next, we give an overview of the classical definition of the Mellin transform and some of its properties. This is material which, except perhaps for the Plancherel formula, is well-known and might as well have fitted into an appendix. We prove our main theorem (which is actually nothing more than a formula to relate the Mellin transform of a function to its Laplace transform) in Section 3. The final two sections are applications of the main theorem.

To begin, let $X$ and $Y$ be two $\mathbb{R}$-valued random variables such that $P(X \leq x) =: F(x)$ and $P(Y \leq y) =: G(y)$. Consider their product $XY$. Then, by total probability,

$$P(XY \leq z) = \int_{-\infty}^{\infty} P(XY \leq z | Y \in dy) P(Y \in dy)$$

$$= \int_{-\infty}^{0} P(X \geq zY^{-1} | Y \in dy) dG(y)$$

$$+ \int_{0}^{\infty} P(X \leq zY^{-1} | Y \in dy) dG(y) + P(0 \leq z) \cdot g_0,$$

if $G$ charges $\{0\}$. It follows that if $X$ and $Y$ are independent,

$$P(XY \leq z) = \int_{-\infty}^{0} (1 - F(zy^{-1})) dG(y) + \int_{0}^{\infty} F(zy^{-1}) dG(y) + P(0 \leq z) \cdot g_0,$$

and if they both have a density, then

$$f \ast g(z) := \int_{0}^{\infty} \left[ f(zy^{-1})g(y) + f(-zy^{-1})g(-y) \right] \frac{dy}{y}$$

is the density of the product $XY$. $f \ast g$ is called the Mellin convolution of $f$ and $g$. We would like to calculate its Mellin transform (see the following section for the definition and a number of properties of the Mellin transform). To that end, we define

$$(f \ast g)^+(z) = f \ast g(z) \cdot I_{\mathbb{R}^+}(z) \tag{1}$$

and

$$(f \ast g)^-(z) = f \ast g(-z) \cdot I_{\mathbb{R}^-}(-z) = f \ast g(-z) \cdot I_{\mathbb{R}^+}(z) \tag{2}.$$ 

Then both $(f \ast g)^+$ and $(f \ast g)^-$ live on $\mathbb{R}^+$. Now we define the Mellin transform of $f \ast g$ as the pair

$$\mathcal{M}[f \ast g(x); s] := \left( \mathcal{M}(f \ast g)^+(s), \mathcal{M}(f \ast g)^-(s) \right),$$

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of Mellin transforms of $f \otimes g^+$ and $f \otimes g^-$, and similarly for any other function that lives on $\mathbb{R}$. (We reserve the symbol $\mathcal{M}$ for the Mellin transform of functions $f$ that live on $\mathbb{R}^+$. ) Now

$$\mathcal{M}(f \otimes g)^+(s) = \int_0^\infty (f \otimes g)^+(z) z^{s-1} \, dz = \int_0^\infty f(x) g(z) z^{s-1} \, dz$$

$$= \int_0^\infty \int_0^\infty \left[ f(zy^{-1})g(y) + f(-zy^{-1})g(-y) \right] \frac{dy}{y} z^{s-1} \, dz$$

$$= \int_0^\infty y^s g(y) \int_0^\infty f(zy^{-1}) \left( \frac{z}{y} \right)^{s-1} d \left( \frac{z}{y} \right) \frac{dy}{y}$$

$$+ \int_0^\infty y^s g^-(y) \int_0^\infty f(-zy^{-1}) \left( \frac{z}{y} \right)^{s-1} d \left( \frac{z}{y} \right) \frac{dy}{y},$$

so that

$$\mathcal{M}(f \otimes g)^+(s) = \mathcal{M}f^+(s) \mathcal{M}g^+(s) + \mathcal{M}f^-(s) \mathcal{M}g^-(s).$$

Similarly, one finds

$$\mathcal{M}(f \otimes g)^-(s) = \mathcal{M}f^+(s) \mathcal{M}g^-(s) + \mathcal{M}f^-(s) \mathcal{M}g^+(s).$$

Define a product $\odot$ of two Mellin transforms according to

$$\mathcal{M}[f(x); s] \odot \mathcal{M}[g(x); s] := (\mathcal{M}f^+(s) \mathcal{M}g^+(s) + \mathcal{M}f^-(s) \mathcal{M}g^-(s), \mathcal{M}f^+(s) \mathcal{M}g^-(s) + \mathcal{M}f^-(s) \mathcal{M}g^+(s)) .$$

We have proved

**Theorem 1** Let $X$ and $Y$ be random variables with densities $f$ and $g$, respectively. Then

$$\mathcal{M}[f \otimes g(x); s] = \mathcal{M}[f(x); s] \odot \mathcal{M}[g(x); s].$$

This is well-known for functions that live on $\mathbb{R}^+$. The multiplication $\odot$ is the same as the multiplication rule for hyperbolic numbers [9]; it remains to see whether this observation has more than curiosity value.

## 2 The Mellin Transform

Let $f$ be a complex-valued function on $\mathbb{R}^+ \cup \{0\}$ such that the integral

$$\mathcal{M}[f(x); s] := \int_0^\infty f(x) x^{s-1} \, dx$$

exists for all complex $s$ in the fundamental strip $\langle \alpha, \beta \rangle := \{ s \in \mathbb{C} : \alpha < \Re(s) < \beta \}$. Then $\mathcal{M}[f(x); s]$ is called the Mellin transform of $f$ with respect to $s$. In what follows, we shall only be concerned with the Mellin transform of functions which live on $\mathbb{R}^+$, and consequently write $\mathcal{M}f$ or $\mathcal{M}f(s)$ instead of $\mathcal{M}[f(x); s]$ when this is feasible. The Mellin transform has a number of interesting properties. We start with...
Lemma 1
\[ M[f(\lambda x); s] = \lambda^{-s} M[f(x); s] \] (5)
and
\[ M[f(x^\mu); s] = \mu^{-1} M[f(x); s/\mu] . \] (6)

Proof. Consider
\[ M[f(\lambda x^\mu); s] = \int_0^\infty f(\lambda x^\mu) x^{s-1} \, dx \]
\[ = \frac{1}{\mu} \int_0^\infty f(\lambda y) y^{s/\mu-1} y^{1/\mu-1} \, dy = \frac{\lambda^{-s/\mu}}{\mu} \int_0^\infty f(x) x^{s/\mu-1} \, dx . \]

Now set \( \lambda = 1 \) or \( \mu = 1 \). \( \Box \)

This was easy. The following is a little harder:

Theorem (Mellin Inversion) 2 Suppose \( \gamma \) belongs to the fundamental strip of \( f \). Then
\[ f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} Mf(s) x^{-s} \, ds . \]

We first prove the following

Lemma 2
\[ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} Mf(s) \, ds = f(1) . \]

Proof of the lemma. Fix \( \varepsilon > 0 \). Then, because \( e^{s^2} \) is analytic in \( s \) everywhere in the complex plane, we have
\[ \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\pi\varepsilon^2 s^2} \, ds = \int_{-\infty}^{\infty} e^{\pi\varepsilon^2 s^2} \, ds = \frac{i}{\varepsilon} , \]
and so
\[ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\pi\varepsilon^2 s^2} Mf(s) \, ds = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\pi\varepsilon^2 s^2} \int_0^\infty f(z) z^{s-1} \, dz \, ds \]
\[ = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\pi\varepsilon^2 s^2} \int_{-\infty}^\infty f(e^{-y}) e^{-ys} \, dy \, ds \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(e^{-y}) \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\pi\varepsilon^2 s^2} e^{-2\pi\varepsilon y(2\pi\varepsilon)^{-1}} \, ds \, dy \]
\[ = \frac{1}{2\pi \varepsilon} \int_{-\infty}^{\infty} f(e^{-y}) e^{-\pi y^2(2\pi\varepsilon)^{-2}} \, dy = \int_{-\infty}^{\infty} f(e^{-2\pi\varepsilon y}) e^{-\pi y^2} \, dy . \]

Now let \( \varepsilon \to 0 \). \( \Box \)
Proof of the theorem. By the lemma and the scaling property (5), we have

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Mf(s) x^{-s} \, ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} M[f(xy); s] \, ds = f(xy) \bigg|_{y=1} = f(x),
\]

which concludes the proof. □

The following theorem is in a similar spirit.

Theorem (Plancherel Formula) 3 Suppose \( \gamma \) belongs to the fundamental strip of both \( f \) and \( g \). Then

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Mf(s) \overline{Mg(s)} \, ds = \int_0^\infty f(x) g(x) x^{2\gamma-1} \, dx.
\]

Proof. As for the proof of Lemma (2), one checks that

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Mf(s) \, ds = \overline{f(1)}
\]

and

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Mf(s) x^{-s} \, ds = \frac{x^{-2\gamma}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Mf(s) \left(\frac{1}{x}\right)^{-s} \, ds = x^{-2\gamma} f(1/x).
\]

Then

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\pi s^2} Mf(s) \overline{Mg(s)} \, ds
\]

\[
= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\pi s^2} \int_0^\infty f(x) x^{s-1} \, dx \int_0^\infty y^{-2\gamma} g(1/y) y^{s-1} \, dy \, ds
\]

\[
= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\pi s^2} \int_{-\infty}^{\infty} f(e^{-u}) e^{-us} \, du \int_{-\infty}^{\infty} e^{-2\gamma v} g(e^{-v}) e^{vs} \, dv \, ds
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(e^{-u}) \int_{-\infty}^{\infty} e^{-2\gamma v} g(e^{-v}) \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\pi s^2} e^{-2\pi \varepsilon(u-v)(2\pi \varepsilon)^{-1}} \, ds \, dv \, du
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-2\gamma v} g(e^{-v}) \int_{-\infty}^{\infty} f(e^{-u-v}) e^{-\pi(u-v)^2(2\pi \varepsilon)^{-2}} \, du \, dv
\]

Now let \( \varepsilon \to 0 \) and change variables. □

We now continue with the derivation of
3 The Main Theorem

The gist of the paper is the following

**Theorem 4** Let \( f \) be a function on \( \mathbb{R} \) with bilateral Laplace transform

\[
\varphi(u) := \int_{-\infty}^{\infty} e^{-ux} f(x) \, dx
\]

and Mellin transform \( \mathcal{M}[f(x); s] = (\mathcal{M}f^+(s), \mathcal{M}f^-(s)) \). Define

\[
\alpha^+ := \sup \{ \alpha \in \mathbb{R} : \lim_{x \to \infty} e^{\alpha x} f^+(x) = 0 \}
\]

and

\[
\alpha^- := \sup \{ \beta \in \mathbb{R} : \lim_{x \to \infty} e^{\alpha x} f^-(x) = 0 \}.
\]

Suppose that \( \alpha^+ \) and \( \alpha^- \) are strictly larger than zero, and choose for \( \beta \) any number in \( \mathbb{R}^- \) such that \( 0 < -\beta < \min(\alpha^+, \alpha^-) \). Then we have, for \( s > 0 \) and in the fundamental strip of \( f^+ \) and \( f^- \),

\[
\mathcal{M}f^+(s) = \frac{\Gamma(s)}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \varphi(u) (-u)^{-s} \, du
\]

and

\[
\mathcal{M}f^-(s) = \frac{\Gamma(s)}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \varphi(-u) (-u)^{-s} \, du.
\]

Furthermore,

\[
\varphi(u) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}f^+(s) \Gamma(1-s) \, u^{s-1} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}f^-(s) \Gamma(1-s) \, (-u)^{s-1} \, ds
\]

for any \( \gamma \) that belongs to the fundamental strips of both \( f^+ \) and \( f^- \).

**Proof.** We first prove the theorem under the assumption that \( f \) is concentrated on \( \mathbb{R}^+ \), such that \( f = f^+ \), and we may write \( \mathcal{M}f \) for its Mellin transform. Choose \( \beta := \beta^+ \) as required. By Laplace inversion, we have

\[
f(x) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \varphi(u) e^{ux} \, du
\]

and so

\[
2\pi i \mathcal{M}f(s) = \int_{0}^{\infty} \int_{\beta-i\infty}^{\beta+i\infty} \varphi(u) e^{ux} \, du \, x^{s-1} \, dx.
\]
The standard trick here is often to first convolve $f$ with some $(\pi \varepsilon)^{-1/2} e^{-x^2/\varepsilon}$. The Laplace transform of the latter is $e^{\varepsilon u^2/4}$, and

\[
\int_0^\infty \int_{\beta-i\infty}^{\beta+i\infty} e^{\varepsilon u^2/4} \varphi(u) e^{ux} du \, dx \\
= \int_{-\beta-i\infty}^{\beta+i\infty} e^{\varepsilon u^2/4} \varphi(-u) \frac{1}{u^s} \int_0^\infty e^{-x} \, x^{s-1} \, dx \, du \\
= \Gamma(s) \int_{\beta-i\infty}^{\beta+i\infty} e^{\varepsilon u^2/4} \varphi(u) (-u)^s \, du,
\]

where interchanging the order of integration is permissible because the integrand becomes exponentially small if either $u$ or $x$ are of the order $\sim \varepsilon^{-1}$. One can then undo the convolution by letting $\varepsilon \to 0$. As for the converse, choose $\gamma$ in the fundamental strip of $f$. Then

\[
f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M} f(s) \, x^{-s} \, ds,
\]

and

\[
\int_0^\infty \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M} f(s) \, x^{-s} \, ds \, e^{-ux} \, dx
\]

\[
= \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M} f(s) u^{s-1} \int_0^\infty \, e^{-x} \, dx \, ds = \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M} f(s) \Gamma(1-s) \, u^{s-1} \, ds.
\]

To justify the interchange of the order of integration, one can Mellin convolve $f$ with $(\pi \varepsilon)^{-1/2} e^{-(\log x)^2/\varepsilon}$, whose Mellin transform is $e^{\varepsilon u^2/4}$, and then again undo the convolution by letting $\varepsilon \to 0$. This proves the theorem in case $f$ is concentrated on $\mathbb{R}^+$. For the general case, let $\varphi^+$ and $\varphi^-$ be the Laplace transforms of $f^+$ and $f^-$, respectively. Then

\[
\varphi(u) = \int_0^\infty f^+(x) e^{-ux} \, dx + \int_0^\infty f^-(x) e^{ux} \, dx = \varphi^+(u) + \varphi^-(u),
\]

and

\[
\mathcal{M} f^+(s) = \frac{\Gamma(s)}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \varphi(u) \, (-u)^{-s} \, du - \frac{\Gamma(s)}{2\pi i} \int_{-\beta-i\infty}^{\beta+i\infty} \varphi^-(u) \, (-u)^{-s} \, du,
\]

by what we have proved already. Now $\varphi^-$ cannot have any singularities in the half-plane $\Re(s) > -\beta$. We therefore can close the contour of integration in the second integral along a semi-circle of radius $r$ and find, by analyticity of $\varphi^-$,

\[
\int_{\beta-i\infty}^{\beta+i\infty} \varphi^-(u) \, (-u)^{-s} \, du = -r \int_{-\pi/2}^{\pi/2} \varphi^-(re^{i\theta}) \, (re^{i\theta})^{-s} \, d\theta,
\]

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which is of order
\[
r^{1-s} \int_{-\pi/2}^{\pi/2} e^{-r \cos \theta} \, d\theta \leq 2r^{1-s} \int_{0}^{\pi/2} e^{-r(1-2\theta/\pi)} \, d\theta = \pi r^{-s} (1 - e^{-r})
\]
in absolute value, and therefore tends to 0 if \( s > 0 \). This proves (9). (10) is proved in a similar manner, and (11) is an easy consequence of (12) and (13). □

Although we shall not use it as heavily, we note the following variant of Theorem 4:

**Theorem 5** Let \( f \) be a function on \( \mathbb{R} \) with Mellin transform \((\mathcal{M}f^+, \mathcal{M}f^-)\) and Fourier transform
\[
f^*(y) := \int_{-\infty}^{\infty} e^{iyx} f(x) \, dx = \int_{0}^{\infty} e^{-iyx} f^-(x) \, dx + \int_{0}^{\infty} e^{iyx} f^+(x) \, dx.
\]
Then, for \( 0 < s < 1 \),
\[
\mathcal{M}f^+(s) = \frac{\Gamma(s)}{\pi} \Re \left( \exp \left( -i \frac{s\pi}{2} \right) \int_{0}^{\infty} f^*(y) \, y^{-s} \, dy \right)
\]
and
\[
\mathcal{M}f^-(s) = \frac{\Gamma(s)}{\pi} \Re \left( \exp \left( i \frac{s\pi}{2} \right) \int_{0}^{\infty} f^*(y) \, y^{-s} \, dy \right).
\]
Moreover,
\[
f^*(y) = i \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}f^+(s) \Gamma(1-s) \, y^{s-1} \, ds
\]
\[- i \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{M}f^-(s) \Gamma(1-s) \, y^{s-1} \, ds
\]
for any \( \gamma \) that belongs to the fundamental strips of \( f^+ \) and \( f^- \).

**Proof.** We need the following

**Lemma 3** For \( 0 < s < 1 \),
\[
\int_{0}^{\infty} e^{\pm ix} x^{s-1} \, dx = \Gamma(s) \exp \left( \pm \frac{s\pi i}{2} \right).
\]

**Proof of the lemma.** We have
\[
\int_{0}^{\infty} e^{\pm ix} e^{-xu} \, dx = \frac{1}{u \mp i} = \frac{u}{u^2 + 1} \pm \frac{i}{u^2 + 1},
\]
and then, by Theorem 3
\[
\int_{0}^{\infty} \cos x e^{-\lambda x} x^{s-1} \, dx = \frac{\Gamma(s)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{u + \lambda}{(u + \lambda)^2 + 1} (-u)^s \, du
\]
for arbitrary $\lambda > 0$. It then follows by the Residue theorem that
\[
\int_0^\infty \cos x e^{-\lambda x} x^{s-1} \, dx = \Gamma(s) ((\lambda - i)^{-s} + (\lambda + i)^{-s}) = \Gamma(s) \frac{\cos \left( -s \operatorname{arg}(\lambda - i) \right)}{\left( \lambda^2 + 1 \right)^{s/2}},
\]
which implies
\[
\int_0^\infty \cos x x^{s-1} \, dx = \Gamma(s) \cos \frac{s\pi}{2},
\]
by arbitrariness of $\lambda$. Similarly, one proves
\[
\int_0^\infty \sin x x^{s-1} \, dx = \Gamma(s) \sin \frac{s\pi}{2},
\]
and this already implies the lemma. $\square$

Proof of the theorem. Inverting the Fourier transform, we find
\[
2\pi f(x) = \int_{-\infty}^{\infty} f^*(y) e^{-ixy} \, dy,
\]
and so
\[
2\pi M f^+(s) = \int_0^\infty \int_{-\infty}^{\infty} f^*(y) e^{-ixy} y^{s-1} \, dx \, dy
= \int_0^\infty f^*(-y) \int_{-\infty}^{\infty} e^{i\pi y} x^{s-1} \, dx \, dy + \int_0^\infty f^*(y) \int_{-\infty}^{\infty} e^{-i\pi y} x^{s-1} \, dx \, dy
= \Gamma(s) \exp \left( i \frac{s\pi}{2} \right) \int_0^\infty f^*(-y) y^{-s} \, dy + \Gamma(s) \exp \left( -i \frac{s\pi}{2} \right) \int_0^\infty f^*(y) y^{-s} \, dy,
\]
by Lemma 3. But this already implies the first of Equations (15) - (17), because $f^*(-y) = f^*(y)$. (One again checks that interchanging the order of integration is permissible at worst after convoluting $f$ with some $(\pi \varepsilon)^{-1/2} e^{-x^2/\varepsilon}$.) The rest are proved in a similar manner. $\square$

As an application, we now calculate the Mellin transforms of those stable probability distributions with Fourier transform
\[
\psi^*(y) := \psi^*_{\alpha,\theta}(y) := \exp \left( -|y|^{\alpha} e^{i\pi \theta \text{sgn} y/2} \right).
\]
Writing $e^{i\pi \theta/2} =: \zeta$, we find
\[
\int_0^\infty \psi^*(y) y^{-s} \, dy = \int_0^\infty \exp (-|y|^{\alpha} \zeta) y^{-s} \, dy
= \frac{1}{\alpha \zeta^{(1-s)/\alpha}} \int_0^\infty e^{-y} y^{(1-s)/\alpha-1} \, dy = \Gamma\left((1-s)/\alpha\right) \frac{\zeta^{(1-s)/\alpha}}{\alpha \zeta^{(1-s)/\alpha}}
= \frac{1}{\alpha} \Gamma\left((1-s)/\alpha\right) \left( \cos \frac{(1-s)\pi \theta}{2\alpha} + i \sin \frac{(1-s)\pi \theta}{2\alpha} \right),
\]
because $|\zeta| = 1$. Multiplication by $e^{-is\pi/2}$ and taking the real part then gives

$$
\mathcal{M}\psi^+(s) = \frac{\Gamma(s)\Gamma((1-s)/\alpha)}{\alpha\pi} \cos \left(\frac{(1-s)\pi\theta + s\pi\alpha}{2\alpha}\right)
= \frac{\Gamma(s)\Gamma((1-s)/\alpha)}{\alpha\pi} \sin \left(\frac{(1-s)\pi(\alpha - \theta)}{2\alpha}\right)
= \frac{\rho^+ \Gamma(s)\Gamma((1-s)/\alpha)}{\Gamma((1+\rho^+(1-s))\Gamma(1-\rho^+(1-s))},
$$

(19)

if we define

$$
\rho^+ = \frac{\alpha - \theta}{2\alpha},
$$

(20)

and make use of $\Gamma(s)\Gamma(1-s) = \pi\sin(\pi s)^{-1}$. (19) is equivalent to Formula 17 in [5] or Formula 6.8 in [6]. Similarly, one finds

$$
\mathcal{M}\psi^-(s) = \frac{\rho^- \Gamma(s)\Gamma((1-s)/\alpha)}{\Gamma((1+\rho^-(1-s))\Gamma(1-\rho^-(1-s))},
$$

(21)

if we define

$$
\rho^- = \frac{\alpha + \theta}{2\alpha}.
$$

(22)

In particular, $\mathcal{M}\psi^+(1) + \mathcal{M}\psi^-(1) = \rho^+ + \rho^- = 1$, as required.

### 4 An Application to Bellman-Harris Processes

Let $\{Z_t\}_{t\geq 0}$ be a supercritical Bellman-Harris process with offspring distribution $\{\pi_k\}_{k=0}^\infty$ and life-time distribution $G$. Thus, $G(t)$ is the probability that a newborn individual survives at least until time $t$, and $\pi_k$ is the probability that once it splits into a number $Z_+$ of progeny, it will split into exactly $k$ of these. We denote by $f(s) := E(e^{sZ_+}) = \sum_{k=0}^\infty \pi_k s^k$ the corresponding generating function. By supercriticality, $\mu := f'(1) > 1$, and there exists $q \in [0,1)$ such that $f(q) = q$. We assume that $G$ is non-lattice, that $G(0^+) = G(0) = 0$, and that $\mu > \infty$. Then there exist ‘constants’ $\chi_t$ (the Seneta constants) such that, on the set of non-extinction, $\chi_t Z_t$ converges almost surely to a non-degenerate random variable $Z$ whose Laplace transform $\psi(u) = \mathbb{E}(e^{-uZ})$ satisfies

$$
\psi(u) = \int_0^\infty f \circ \psi(ue^{-\beta t}) dG(t),
$$

(23)

where $\beta$ is the Malthusian parameter, that is, the unique root of

$$
\mu \int_0^\infty e^{-yt} dG(t) = 1,
$$

$y \in (0, \infty)$. There are only a few instances where the solution of (23) is known for given $f$ and $G$. We turn the problem on its head: Suppose $\psi$ were (the Laplace transform of some function $g$) such that it fulfills the conditions of Theorem 4. Then

$$
\mathcal{M}g(s) = \frac{\Gamma(s)}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \psi(u) (-u)^{-s} du
$$
for some suitably chosen value of $\beta$, and it follows by Equation (23) that
\[
\frac{2\pi i}{\Gamma(s)} Mg(s) = \int_{\beta - i\infty}^{\beta + i\infty} f \circ \psi(ue^{-t}) dG(t) \left( -u \right)^{-s} du = \int_{\beta - i\infty}^{\beta + i\infty} f \circ \psi(u) (-u)^{-s} du \int_{0}^{\infty} e^{-t(s-1)} dG(t).
\]

In other words, we have for the Laplace transform of $G$ at $s$,
\[
\int_{0}^{\infty} e^{-ts} dG(t) = \frac{\int_{\beta - i\infty}^{\beta + i\infty} \psi(u) (-u)^{-s-1} du}{\int_{\beta - i\infty}^{\beta + i\infty} f \circ \psi(u) (-u)^{-s-1} du}.
\]

Example. Suppose that $g(x) \propto e^{-x^{\kappa-1}}$ for some $\kappa > 0$ and $f(s) = s^m$. Then
\[
\int_{\beta - i\infty}^{\beta + i\infty} \psi(u) (-u)^{-s-1} du = 2\pi i \frac{Mg(s+1)}{\Gamma(s+1)} = \frac{2\pi i}{\Gamma(s+1) \Gamma(\kappa)} \int_{0}^{\infty} e^{-x^{\kappa-1} x^s} dx = 2\pi i \frac{\Gamma(s+\kappa)}{\Gamma(s+1) \Gamma(\kappa)},
\]
which eventually implies that
\[
\int_{0}^{\infty} e^{-ts} dG(t) = \frac{\Gamma(s+\kappa) \Gamma(m\kappa)}{\Gamma(s+m\kappa) \Gamma(\kappa)}.
\]

and that $G$ has a density:
\[
\frac{dG(t)}{dt} = \frac{\Gamma(m\kappa)}{\Gamma(\kappa) \Gamma(m\kappa-\kappa)} e^{-\kappa t} (1 - e^{-t})^{(m-1)\kappa-1}. \tag{24}
\]

In case $f$ is a polynomial $f(s) = \sum_{j=1}^{m} \pi_j s^j$, we similarly obtain
\[
\int_{0}^{\infty} e^{-ts} dG(t) = \frac{\Gamma(s+\kappa)}{\Gamma(\kappa) \sum_{i=1}^{m} \frac{\Gamma(s+j\kappa)}{\Gamma(j\kappa)}}. \tag{25}
\]

It is not obvious that the function on the right-hand side of (25) has all the properties of a Laplace transform (most of all, complete monotonicity), but this is what we have shown. So the question is, When does the Laplace transform $\psi$ of the random variable $Z$ have an exponential tail? The answer is given by the following
**Theorem 6** Let $F_t$ be the PGF of particle numbers in a Bellman-Harris process at time $t$, and let $f$ be the PGF of the corresponding first-generation offspring distribution. Say that $f$ has exponential moments up to order $r > 0$ if $f(e^u) < \infty$ for $u < r$, and let $M_t := F_t'(1)$. Then $F_t$ has exponential moments up to order $O(r/M_t)$. In particular, there exists $r' > 0$ such that Laplace transform

$$\tilde{\psi}(u) := \lim_{t \to \infty} F_t(e^{-u/M_t})$$

is analytic for $u > -r'$.

**Proof.** We can assume $f(0) = 0$, so $Z_t \to \infty$ almost surely. Because

$$e^u = F_t \circ \exp \left( \frac{M_t \log F_{-t}(e^u)}{M_t} \right)$$

(we write $F_{-t}$ to denote the inverse of $F_t$), one readily checks that Theorem 6 holds true iff

$$\frac{d}{du} M_t \log F_{-t}(e^u) = \frac{e^{-\beta t} M_t e^u}{F_{-t}(e^u) e^{-\beta t} F_t' \circ F_{-t}(e^u)} < e^u$$

(supposing $u > 0$) remains bounded away from zero. Now $e^{-\beta t} M_t$ converges to something non-zero by the Kesten-Stigum theorem, so $F_t(e^u) \to \infty$ for $u > 0$ by convexity of $F_t$. But then $F_t \circ F_{-t}(e^u) = e^u$ implies $F_{-t}(e^u) \to 1$. All that is left show is that $e^{-\beta t} F_t' \circ F_{-t}(e^u)$ remains bounded. Write $s$ instead of $e^u$. We have

$$F_t(s) = (1 - G(t))s + \int_0^t f \circ F_{t-u}(s) \, dG(u) ,$$

so

$$X_t(s) := e^{-\beta t} F_t'(s)$$

\[= e^{-\beta t} (1 - G(t)) + \int_0^t f \circ F_{t-u}(s) e^{-\beta u} F_u'(s) e^{-\beta (t-u)} dG(t-u)\]

\[= e^{-\beta t} (1 - G(t)) + \int_0^t \left( \frac{f' \circ F_u(s)}{\mu} - 1 \right) X_u(s) \, dG_{\beta}(t-u) + \int_0^t X_u(s) \, dG_{\beta}(t-u) ,\]

where we have introduced the measure

$$G_{\beta}(t) = \mu \int_0^t e^{-\beta u} \, dG(u) .$$
We now use the final expression in \( \text{(26)} \) to obtain

\[
\begin{align*}
\int_0^t X_u(s) \, dG_t(u) - \int_0^t e^{-\beta u} (1 - G(u)) \, dG_t(u) &= \\
= & \int_{u=0}^{t} \int_{v=0}^{u} \frac{f' \circ F_u(s)}{\mu} X_v(s) \, dG_t(u-v) \, dG_t(u) - \\
& \int_{v=0}^{t} \frac{f' \circ F_u(s)}{\mu} X_v(s) \int_{u=0}^{t} dG_t(u-v) \, dG_t(u) - \\
& \int_{v=0}^{t} \frac{f' \circ F_u(s)}{\mu} X_v(s) \, dG^\ast_\beta(t-v),
\end{align*}
\]

and find by induction and Fubini’s theorem,

\[
X_t(s) = e^{-\beta t} (1 - G(t)) + \int_0^t e^{-\beta u} (1 - G(u)) \, dU_t(u) + \\
\int_0^t \left( \frac{f' \circ F_u(s)}{\mu} - 1 \right) X_u(s) \, dU_t(u),
\]

(27)

or

\[
X_t(s) - X_t(1) = \int_0^t \left( \frac{f' \circ F_{t-u}(s)}{\mu} - 1 \right) X_{t-u}(s) \, dU_t(u),
\]

(28)

where

\[
U_\beta(t) = \sum_{i=1}^{\infty} G^2_{\beta,i}(t)
\]

is essentially the renewal measure for \( G_\beta \). Suppose \( X_t \circ F_t(s) \) does not remain bounded. Then we can find a sequence of values \( t_1, t_2, \ldots \) tending to infinity such that

\[
1 \leq \lim_{k \to \infty} \int_0^{t_k} \left( \frac{f' \circ F_{tk-u} \circ F_{-tk}(s)}{\mu} - 1 \right) \, dU_t(u).
\]

But \( Z_{tk}/Z_t \to e^{-\beta u} \) almost surely \( \text{[8]} \), hence

\[
\lim_{k \to \infty} F_{tk-u} \circ F_{-tk}(s) = \lim_{k \to \infty} \mathbb{E} \left( F_{-tk}(s) Z_{tk} Z_{tk-u}/Z_{tk} \right) \leq s^{-\beta u},
\]

by Jensen’s inequality. Therefore,

\[
1 \leq \int_0^{\infty} \left( \frac{f' \circ F_{s^{-\beta u}}(s)}{\mu} - 1 \right) \, dU_t(u) \simeq \frac{f''(1)}{\mu} \log s
\]

if \( s < e^r \) (recall that \( f \) has exponential moments up to order \( r \)), because \( dU_t(u) \) is essentially Lebesgue measure plus a term of order \( e^{-\alpha u} du \) for some \( \alpha \in (0, \beta] \) \( \text{[10]} \). But \( 1 \leq O(\log s) \) is a contradiction for \( s \) sufficiently close to 1. The theorem follows.

\( \square \)
5 Limit Laws of Luria-Delbrück Processes

The Luria-Delbrück (LD) distribution arises as the distribution of types in a two-type Bellman-Harris process or, in a narrower sense, as a limiting distribution of types in such a process. If the life-time distribution is exponential and branching is binary, the theory of the LD distribution is essentially complete; see [1]. The same can be said if the life-time distribution of cells is exponential, but cells always produce a fixed number $\kappa$ of mutant or non-mutant progeny [2, 7]. We shall refer to such a process as a $(1 - \rho, \kappa)$-Luria-Delbrück process, where $\rho$ is the probability that upon division, a non-mutant cell produces one non-mutant and $\kappa$ mutant daughter cells. Mutants only produce $\kappa$ mutant progeny. The following theorem has been proved by Leona Schild in her diploma thesis:

**Theorem [7] 7** Denote by $L_n$ the number of non-mutants in a $(1 - \rho, \kappa)$-Luria-Delbrück process when population size has reached $n\kappa + 1$, and the process has been started from a single non-mutant individual. Then

$$\frac{L_n}{n^{1-\rho}} \to L \quad (29)$$

almost surely, and

$$L \overset{D}{=} B^{1-\rho} \cdot L_\kappa \cdot \kappa \quad (30)$$

where $\kappa = \kappa^{-1}$, $B$ is $((1-\rho)\kappa, \rho \kappa)$-Beta distributed, and $L_\kappa$ is $\kappa$-biased $(1-\rho)$-Mittag-Leffler.

We explain our terms: By a $(1 - \rho)$-Mittag-Leffler distribution we mean a distribution whose Laplace transform is the Mittag-Leffler function

$$E_{1-\rho}(u) = \sum_{k=0}^{\infty} \frac{(-u)^k}{\Gamma((1-\rho)k + 1)} \quad (31)$$

Next, a $\kappa$-biasing of a random variable $X$ with Laplace transform $E(e^{-uX})$ is a random variable $X_\kappa$ whose Laplace transform is

$$\frac{E(X_\kappa e^{-uX})}{E(X_\kappa)} \quad (32)$$

We refer to [7] for further background. Our goal here is to deduce

**Theorem [2] 8** The Laplace transform of $L$ is

$$E(e^{-uL}) = \Gamma(\kappa) \sum_{i=0}^{\infty} \left( -\frac{\kappa}{i} \right) \frac{s^i}{\Gamma(i(1-\rho) + \kappa)} \quad (33)$$

(which has been proved in [2] in a rather indirect manner) directly from Theorem 7. We need the following
Lemma 4 Suppose \( Y \) is \((1 - \rho)\)-Mittag-Leffler distributed. Then its Mellin transform is
\[
\mathcal{M}Y(s) = \frac{\Gamma(s)}{\Gamma((1 - \rho)(s - 1) + 1)}.
\]

Proof. First observe that
\[
\frac{\Gamma((1 - \rho)(k + 1) + 1)}{\Gamma((1 - \rho)k + 1)} \sim k^{1-\rho},
\]
which implies that \( E_{1-\rho} \) is analytic on all of \( \mathbb{C} \). Also,
\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^{\zeta}}{\zeta^z} d\zeta,
\]
where \( \mathcal{H} \) is a Hankel contour encircling the negative axis in counterclockwise direction. We fix \( \varepsilon > 0 \), and choose for \( \mathcal{H} = \mathcal{H}(\varepsilon) \) a vaguely lollipop-shaped figure as follows: It runs along (in fact, just below) the negative axis from \(-\infty\) to \(-\varepsilon\), runs around the origin on a circle of radius \( \varepsilon \) in counterclockwise direction, and then returns to \(-\infty\) just above the negative axis. Then we have
\[
\int_{\mathcal{H}} \frac{e^{\zeta}}{\zeta^{(1-\rho)k+1}} d\zeta = \int_{-\infty}^{-\varepsilon} \frac{e^{r\zeta} e^{-i\pi}}{r^{(1-\rho)k+1} e^{-i\pi(1-\rho)k-i\pi}} dr
\]
\[
+ \int_{-\varepsilon}^{\pi} \frac{e^{r\zeta e^{i\theta}}} {e^{(1-\rho)k+1} e^{i\theta(1-\rho)k+i\theta}} d\theta + \int_{-\pi}^{-\infty} \frac{e^{r\zeta e^{i\theta}}} {r^{(1-\rho)k+1} e^{i\pi(1-\rho)k+i\pi}} dr,
\]
and the three integrands are \( \sim \varepsilon^{-(1-\rho)k} \) in order of magnitude. It follows that if \( |u| < \varepsilon^{1-\rho} \), the sequence of functions
\[
\sum_{k=0}^{n} (-u)^k \frac{e^{r\zeta e^{-i\pi}}} {r^{(1-\rho)k+1} e^{-i\pi(1-\rho)k-i\pi}}
\]
is uniformly convergent in \( r \), and that of
\[
\sum_{k=0}^{n} (-u)^k \frac{e^{r\zeta e^{i\theta}}} {e^{(1-\rho)k+1} e^{i\theta(1-\rho)k+i\theta}}
\]
is uniformly convergent in \( \theta \). It is therefore permissible to interchange the order of summation and integration in the following chain of equations, and we find
\[
E_{1-\rho}(u) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} (-u)^k \int_{\mathcal{H}} \frac{e^{\zeta}}{\zeta^{(1-\rho)k+1}} d\zeta
\]
\[
= \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^{\zeta}}{\zeta} \sum_{k=0}^{\infty} (-u)^k \zeta^{(1-\rho)k} d\zeta = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^{\zeta}}{\zeta + u\zeta^\rho} d\zeta. \quad (34)
\]
Now the right-hand side of Equation (34) defines an analytic function in \( u \) except possibly for those \( u \) which belong to the zero set \( \mathcal{H}_0(\varepsilon) := \{ u \in \mathbb{C} : \zeta + u\zeta^\rho = 0 \} \).
0 for some $\zeta \in \mathcal{H}(\varepsilon)$ of the denominator. But one readily checks that $\mathcal{H}_0(\varepsilon)$ is a Hankel contour around the positive axis that winds around the origin along a segment of a circle of radius $\varepsilon^{1-\rho}$, so that if we choose some $\overline{\varepsilon} > \varepsilon$, we have
\[
\int_{\mathcal{H}(\varepsilon)} \frac{e^\zeta}{\zeta + u\zeta^\rho} d\zeta = \int_{\mathcal{H}(\overline{\varepsilon})} \frac{e^\zeta}{\zeta + u\zeta^\rho} d\zeta
\]
on a (connected) segment of the open torus $\{u \in \mathbb{C} : \varepsilon^{1-\rho} < |u| < \overline{\varepsilon}^{1-\rho}\}$. By analytic continuation, then, (34) holds for all $u \in \mathbb{C} \setminus \mathcal{H}_0(\varepsilon)$, and in particular holds for all $u$ for which $\Re(u) < -\varepsilon^{1-\rho}$. By Theorem 4 therefore, we can write
\[
\mathcal{M}Y(s) = -\frac{\Gamma(s)}{4\pi^2} \int_{\beta-i\infty}^{\beta+i\infty} \int_{\mathcal{H}} \frac{e^\zeta}{\zeta + u\zeta^\rho} (-u)^{-s} du d\zeta
\]
if $\beta < -\varepsilon^{1-\rho}$. Now one may interchange the order of integration (if necessary again after convoluting with $\sqrt{\pi \varepsilon} e^{-x^2/\varepsilon}$) and then apply the Residue theorem to find
\[
\mathcal{M}Y(s) = \frac{\Gamma(s)}{4\pi^2} \int_{\beta-i\infty}^{\beta+i\infty} \int_{\mathcal{H}} \frac{e^\zeta}{\zeta + u\zeta^\rho} (-u)^{-s} du d\zeta
\]
\[
= \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{e^\zeta}{\zeta + u\zeta^\rho} d\zeta
\]
which again follows from Hankel's representation of the reciprocal of the Gamma function. The proof of the lemma is complete. □

Next we need

**Lemma 5** Suppose $B$ is $(1-\rho)\kappa, \rho \kappa)$-Beta distributed. Then the Mellin Transform of $B^{1-\rho}$ is
\[
\mathcal{M}B^{1-\rho}(s) = \frac{\Gamma(\kappa)}{\Gamma((1-\rho)\kappa) \Gamma((1-\rho)(s-1) + \kappa)}.
\]

**Proof.** We have
\[
P(B \leq x) = \frac{\Gamma(\kappa)}{\Gamma((1-\rho)\kappa) \Gamma(\rho \kappa)} \int_0^x y^{(1-\rho)\kappa-1}(1-y)^{\rho \kappa-1} dy.
\]
Then $P(B^{1-\rho} \leq x) = P(B \leq x^{(1-\rho)^{-1}})$, and
\[
\int_0^1 \frac{d}{dx} \int_0^{x^{(1-\rho)^{-1}}} y^{(1-\rho)\kappa-1}(1-y)^{\rho \kappa-1} dy x^{s-1} dx
\]
\[
= \int_0^1 \frac{d}{du} \int_0^u y^{(1-\rho)\kappa-1}(1-y)^{\rho \kappa-1} dy u^{(1-\rho)(s-1)} du
\]
\[
= \int_0^1 u^{(1-\rho)\kappa-1}(1-u)^{\rho \kappa-1} u^{(1-\rho)(s-1)} du = \frac{\Gamma(\rho \kappa) \Gamma((1-\rho)(s-1) + \kappa)}{\Gamma((1-\rho)(s-1)+\kappa)}.
\]
which already finishes the proof. □

It follows from Lemma 4 that the Mellin transform of a $\kappa$-biased Mittag-Leffler distribution is

$$\mathcal{M}Y_\kappa(s) = \frac{\Gamma(s + \kappa)}{\Gamma((1 - \rho)(s + \kappa - 1) + 1)} \frac{\Gamma(1 + (1 - \rho)\kappa)}{\Gamma(1 + \kappa)}.$$  

Together with Lemma 5 and Theorem 1 this implies that the Mellin transform of $L$ satisfies

$$\mathcal{M}L(s) = \frac{\Gamma(s + \kappa - 1)}{\Gamma((1 - \rho)(s - 1) + \kappa)}.$$  

Specialising now to $s = i + 1$ for arbitrary integer $i \geq 0$, one has for the Laplace transform of $L$,

$$\mathbb{E}(e^{-uL}) = \sum_{i=0}^{\infty} \frac{\Gamma(i + \kappa)}{\Gamma((1 - \rho)i + \kappa)} \frac{(-u)^i}{i!}$$

$$= \sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{\kappa + j - 1}{j} \frac{(-u)^i}{\Gamma((1 - \rho)i + \kappa)} = \sum_{i=0}^{\infty} \left( -\frac{\kappa}{i} \right) \frac{u^i}{\Gamma(i(1 - \rho) + \kappa)}.$$  

This was to be proved. □

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