An algorithm for detecting absorbable constants in Riemannian spaces

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Dedication. The author dedicates this work to Elena and Katerina.

Abstract. The present lecture outlines the problem of distinguishing between essential and spurious (i.e., absorbable) constants contained in a metric tensor field in a Riemannian geometry, as well as a proposed solution to it. The solution consists of a sufficient and necessary criterion, given in a covariant form. The result is that the problem of characterization is reduced to that of solving a system of partial differential equations of the first order. Except the fact that the study is purely of local character, the only assumption is the smoothness of the metric tensor field with respect to the constant to be tested.

1. Introduction
In studying Riemannian spaces, especially in the level of a local description and coordinate approach, one frequently confronts the problem of finding the nature of a constant (or a parameter), which may enter a metric tensor field. Generally, there are two options: a constant may be either global (i.e., topological) or local. In the first case, the constant is removable from local coordinate patches but it appears in the transforms between them (e.g., in the coordinate range). In the second case, the constant may or may not be absorbable through coordinate transformations. So, local constants can be divided into two subcategories, each for the two possibilities of the second case: spurious (or absorbable) and essential. Obviously, these two notions are complementary.

The problem under discussion is of great interest when general relativity and its applications (e.g., the problem of limits of space-times, ref. [1]) are taken into account (where the metric tensor field is solution to the Einstein equations, and the constants emerge from the integration procedure), though this observation should not limit the spirit of the study.

In order to solve the problem of attributing the proper characterization to a local constant, one can find two main approaches in the relevant literature:

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One tries to find the desired coordinate transformation. In the case of a positive result, the constant is incorporated in the very definition of the new coordinate system. But, in general, there is not a systematic way towards this scope and failure to find such a transformation does not necessarily imply the essentiality.

The second main approach is divided into two subcategories: one can either use the invariant classification methods for a single Riemannian space or implement the methods of the equivalence problem (ref. [2]).

The second approach can be extremely laborious because the maximum order of the covariant derivative of the Riemann tensor entering the calculations increases in a way disproportionate to the dimensions of the Riemannian space (especially when the dimensions are greater than four). Besides, in that approach one must make use of a great series of complex (in structure) quantities such as: syzygies, polynomial invariants, mixed invariants, and the Cartan invariants –see ref. [2] and the references therein for details—, or the “ratios” between two tensors (obtained from the Riemann tensor and its covariant derivatives) which differ only by a factor—these quantities are described in ref. [3] (especially the last two references therein).

2. Criterion
Having seen the vagueness of the first approach and the (practical) difficulties of the second, naturally, one could pose the question of whether a necessary, sufficient and covariant criterion on the characterization of a local constant can be found or not. It turns out that a positive answer exists, but it is more practical, though equivalent, to be given in terms of spurious constants –instead of essential constants.

In the following lines, only a sketch of the basic ideas and steps towards the establishing of the criterion is exhibited.

The solution to the posed question consists in the following steps:

$I_1$ One treats the “suspect” local constant as one extra coordinate (while smoothness of the metric tensor field with respect to the constant is assumed).

$I_2$ A local isometric embedding (see e.g., ref. [4]) of the initial $n$-dimensional Riemannian space into another of $(n+1)$-dimensions is considered.

$I_3$ If the constant is spurious one can, in the coordinates in which the constant is removed, take a product metric tensor field on $S \times I$ (where $S$ is the initial $n$-dimensional manifold, and $I$ the domain of definition of the spurious constant), and then conclude that the only non-zero components of curvature in $n+1$ dimensions are those which correspond to the curvature tensor of the initial $n$-dimensional metric tensor field. In these coordinates the normals to $S$ form a symmetry. Alternatively, one can also consider the $(n+1)$-dimensional manifold using the original coordinates, with the spurious constant as the extra coordinate, and use the constant to label the $n$-dimensional slices.

Due to the difference of dimensions between the initial and the final space, which is 1, there is no torsion, and the Mainardi-Codazzi conditions are identically satisfied. On the other hand, the Weingarten-Gauss conditions as well as the observation that the steps $I_1$, $I_2$ and $I_3$ are independent of the local isometric embedding, give rise to the:

**Criterion:** A local constant is spurious if and only if the Lie derivative of the metric tensor field of the embedding space with respect to the normal (to the subspace) vector vanishes.

Thus, if (in a local coordinate system):

\[
\tilde{g}_{AB} = \begin{pmatrix} N_{\rho} N^{\rho} + \varepsilon N^2 & N_3 \\ N_\alpha & g_{\alpha \beta} \end{pmatrix}
\]
where:

- $g_{\alpha\beta}$ are the components of the initial metric tensor field (expressed in a local coordinate system),
- $\lambda$ is the local constant under consideration, the symbol $\varepsilon$ express the time-like or space-like character of the extra coordinate, taking the values $\pm 1$,
- the quantity $N$ is the lapse function, and
- the object $N^\alpha$ is the shift vector (obviously the last two quantities depend on the embedding),

then, the criterion implies:

$$N = N(\lambda)$$

and

$$N_{\alpha\beta} + N_{\beta\alpha} = \dot{g}_{\alpha\beta}$$

where:

- the bar ($\bar{\cdot}$) denotes covariant derivative with respect to the initial metric $g$ of the subspace, while
- the dot ($\cdot$) denotes differentiation with respect to the extra coordinate, i.e., $\lambda$, while:

$$n^A = \frac{1}{N(\lambda)}\{1, -N^\alpha(\lambda, x^\beta)\}$$

is the normal vector field under discussion. Its integral curves define an one-parametric family of transformations in the extended space, which bring the initial extended metric tensor field to a Gaussian form; in that coordinate system the $n$-dimensional part does not contain the spurious constant.

The criterion stated above, not only constitutes a necessary and sufficient condition but also has a covariant form since it involves the vanishing of the Lie derivative of a tensor field, which itself is a tensor field, in the final space (for all the technical details entering the proof of the criterion, see ref. [5]).

3. An application and a pedagogical example

An immediate application of the criterion is deduced when it is applied to the case where the “suspect” constant is an overall factor; i.e., in a local system of coordinates $\{x^\mu\}$:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma; \lambda) \equiv \lambda G_{\alpha\beta}(x^\gamma)$$

Then, the criterion, in its “solved form” 1, results in:

$$N_{\alpha\beta} + N_{\beta\alpha} = \dot{g}_{\alpha\beta} = G_{\alpha\beta} \Rightarrow N_{\alpha\beta} + N_{\beta\alpha} = \frac{1}{\lambda} g_{\alpha\beta}$$

which is nothing but the well-known homothety equations for the subspace – a classical result.

The lecture concludes with a pedagogical example.

Let a two-dimensional metric tensor field, which in a local coordinate system $\{x^\mu\} \equiv \{u, v\}$, has the form:

$$g_{\alpha\beta}(u, v; \lambda) = (1 + \lambda^2) \begin{pmatrix} 0 & 1 + u^2 + (1 + \lambda^2)^2 v^2 \\ 1 + u^2 + (1 + \lambda^2)^2 v^2 & 0 \end{pmatrix}$$

Solution to:

$$N_{\alpha\beta} + N_{\beta\alpha} = \dot{g}_{\alpha\beta}$$

results in:

$$N^\alpha = \left\{0, \frac{2\lambda v}{1 + \lambda^2}, \frac{2y^0 y^2}{1 + (y^0)^2}\right\}$$
and hence:

\[ n^A = \frac{1}{N(y^0)} \left\{ 1, 0, -\frac{2y^0 y^2}{1 + (y^0)^2} \right\} \quad (8) \]

The corresponding flow lines are described by:

\[
\frac{dy^0}{ds} = \frac{1}{N(y^0)} \tag{9}
\]

\[
\frac{dy^1}{ds} = 0 \tag{10}
\]

\[
\frac{dy^2}{ds} = -\frac{1}{N(y^0)(1 + (y^0)^2)} \frac{2y^0 y^2}{1 + (y^0)^2} \tag{11}
\]

and the integral curves:

\[
\int N(y^0)dy^0 = s + \bar{y}^0 \quad (12)
\]

\[
y^1 = \bar{y}^1 \quad (13)
\]

\[
y^2 = \bar{y}^2 (1 + (y^0)^2)^{-1} \quad (14)
\]

Then, as expected, it is:

\[ \bar{n}^A = \{1, 0, 0\} \quad (15) \]

leading to the transformed embedding metric:

\[ \bar{g}_{AB} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{g}_{\alpha\beta} \end{pmatrix} \quad (16) \]

with:

\[ \bar{g}_{\alpha\beta} = \begin{pmatrix} 0 & 1 + \bar{n}^2 + \bar{v}^2 \\ 1 + \bar{n}^2 + \bar{v}^2 & 0 \end{pmatrix} \quad (17) \]

Though the example may seem simple and trivial, its purpose is to exhibit not only the implementation of the criterion but also all the details connected to it.

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