Covering three-tori with cubes

Ilya I. Bogdanov* Oleg Grigoryan† Maksim Zhukovskii†

Abstract

Let $\mu(\varepsilon)$ be the minimum number of cubes of side $\varepsilon$ needed to cover the unit three-torus $[R/Z]^3$. We prove new lower and upper bounds for $\mu(\varepsilon)$ and find the exact value for all $\varepsilon \geq \frac{7}{15}$ and all $\varepsilon \in \left[\frac{1}{r+1}/(r^2+1), \frac{1}{r-1}/(r^2-1)\right]$ for any integer $r \geq 3$.

1 Introduction

Let $d$ be a positive integer, and let $\varepsilon \in (0, 1)$. Consider the torus $T^d := [R/Z]^d$ and the set $\mathcal{J}_\varepsilon$ of 'sub-cubes' of the form $\{(x_1, \ldots, x_d) : x_i \in [x^0_i, x^0_i+\varepsilon]\}$. The question is, what is the minimum number $\mu := \mu(d; \varepsilon)$ of sets $A_1, \ldots, A_\mu$ from $\mathcal{J}_\varepsilon$ needed to cover $T^d$ (i.e., $T^d = A_1 \cup \ldots \cup A_\mu$)?

In [6], it is proven that

$$\mu \geq \lceil \frac{1}{\varepsilon} \rceil^d, \quad (1)$$

where $[x]^{(i)} = [x]^{([x]^{(i-1)})}$ and $[x]^{(1)} = [x]$. Moreover, it is shown that, for $d = 2$, this lower bound is sharp, i.e. $\mu(2; \varepsilon) = \lceil \frac{1}{\varepsilon} \rceil^2$.

In our paper, we consider $d = 3$. Since $\mu(1; \varepsilon) = \lceil \frac{1}{\varepsilon} \rceil$ and $\mu(2; \varepsilon) = \lceil \frac{1}{\varepsilon} \rceil^2$, we get that

$$\lceil \frac{1}{\varepsilon} \rceil \cdot \lceil \frac{1}{\varepsilon} \rceil \leq \mu(3; \varepsilon) \leq \lceil \frac{1}{\varepsilon} \rceil \cdot \lceil \frac{1}{\varepsilon} \rceil. \quad (2)$$

In [6], the authors also noticed that the lower bound in [2] for $d = 3$ is not sharp. For example, $\mu(3; \frac{2}{3}) > \lceil \frac{1}{3} \rceil^2$. Unfortunately, no general bound better than [2] is known.

The case of large $d$ was studied more extensively. Note that (1) implies $\mu(d; \varepsilon) \geq (1/\varepsilon + o(1))^d$. From a general result by Erdős and Rogers [3] it follows that $1/\varepsilon$ is the correct base of the exponent, i.e. $\mu(d; \varepsilon) = (1/\varepsilon + o(1))^d$. More precisely, they proved that $\mu(d; \varepsilon) = O(d \log d(1/\varepsilon)^d)$. In the discrete case (i.e. $[Z/tZ]^d$ is covered by cubes with side $s \in [t] := \{1, 2, \ldots, t\}$, $\varepsilon = s/t$) a direct application of the probabilistic method allows to put away the $\log d$ factor (see [1]), and it is easy to see that it implies

*Moscow Institute of Physics and Technology, Laboratory of Combinatorial and Geometric Structures, Moscow, Russia
†Yandex School of Data Analysis, Moscow, Russia
the same bound for any $\varepsilon$ (both rational and irrational): $\mu(d; \varepsilon) = O(d(1/\varepsilon)^d)$ while \(1\) gives $\mu(d; \varepsilon) = \Omega((1/\varepsilon)^d)$. Note that, as we show in Section 3, the discrete and continuous problems are in some sense equivalent. Let us also mention that, for $\varepsilon = 2^r$, $r \in \mathbb{N}$, the corresponding packing problem (finding $\nu(d; \varepsilon)$ — the maximum number of non-overlapping sub-cubes with side $\varepsilon$ inside $T^d$) is related to the problem of finding Shannon capacity $c(C_r)$ of a simple cycle on $r$ vertices [2]: $c(C_r) = \sup_{d \geq 1} (\nu(d; 2/r))^{1/d}$ (the same connection works for other rational $\varepsilon$ but the respective graphs are not so foreseeable). For even $r$, $c(C_r) = r/2$ is straightforward. It was shown byLovász [5] that $c(C_5) = \sqrt{5}$. For larger odd $r$, finding $c(C_r)$ is still open.

In this paper, we have found the exact value of $\mu(3; \varepsilon)$ for $\varepsilon \geq 7/15$. We have also found exact values of $\mu(3; \varepsilon)$ for $\varepsilon$ close to $1/r$, $r \in \mathbb{N}$. In Section 2 we state new results.

For convenience, for any $a, b \in \mathbb{R}/\mathbb{Z}$, we denote by $|a - b|$ the smallest $\nu \in [0, 1)$ such that $a = b \pm \nu$. Similarly, for every $a, b \in \mathbb{Z}/t\mathbb{Z}$, we denote by $|a - b|$ the smallest $\nu \in \{0, 1, \ldots, t-1\}$ such that $a = b \pm \nu$.

### 2 New results

Since, for an integer $r \geq 2$ and $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1/r^2}\right)$, the lower bound and the upper bound in \(2\) are equal, the value of $\mu(3; \varepsilon)$ is straightforward and equals $r^3$. We have also found left-neighborhoods of all $1/r$ where the lower bound in \(2\) is tight (notice that for such $\varepsilon$ the difference between the upper and the lower bounds is, conversely, large).

**Theorem 1.** Let $r \in \mathbb{N}$. If $\varepsilon \in \left[\frac{1}{r+1/(r^2+r+1)}, \frac{1}{r}\right)$, then the lower bound is tight, i.e. $\mu(3; \varepsilon) = r^3 + r^2 + r + 1$.

Moreover, we have proved that the trivial right-neighborhoods of any number of the form $1/r$ where the upper bound is tight can be extended in the following way.

**Theorem 2.** Let $r \geq 2$ be an integer. If $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1/(r^2-1)}\right)$, then the upper bound is tight, i.e. $\mu(3; \varepsilon) = r^3$.

We have also improved the lower bound from \(2\) in some special cases.

**Theorem 3.** Let $r \geq 2$ be an integer, $\xi \in [r]$ be such that

$$\xi^2 \leq \xi + (r + 1) \left\lfloor \frac{\xi^2}{r+1} \right\rfloor.$$  \hspace{1cm} (3)

Let

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + 2\xi r + \left\lfloor \frac{\xi^2}{r+1} \right\rfloor$. 

\(2\)
Assume that $t$ and $s$ are coprime. Then $\mu\left(3; \frac{s}{t}\right) > t$, i.e. bigger than the lower bound.

Condition (3) implies that either $\xi \geq \sqrt{r + 1}$ or $\xi = 1$. In the interval $[1/3, 1/2)$, there are two such values of $\frac{\xi}{t}$, namely $\frac{\xi}{t} \in \{\frac{7}{16}, \frac{8}{21}\}$.

Finally, we have improved the upper bound from (2) in some special cases.

**Theorem 4.** Let $r \geq 2$ be an integer, $\xi \in [r]$. Let

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + \xi(r + 1)$.

Then $\mu\left(3; \frac{s}{t}\right) \leq t$, i.e. smaller than the upper bound.

Notice that Theorem 1 follows from Theorem 4 for $\xi = 1$. In Section 6, we give a complete proof of Theorem 4 for all values of $\xi$.

Both Theorem 3 and Theorem 4 imply improvements of the bounds in (2) for values of $\varepsilon$ in certain right-interval of the respective $s/t$ (see Lemma 1 from Section 3).

The results above allow to find the values of $\mu(3; \varepsilon)$ for all $\varepsilon \in [1/2, 1)$ as follows.

**Theorem 5.** We have

$$
\mu(3; \varepsilon) = \begin{cases}
4, & \varepsilon \in [3/4, 1); \\
5, & \varepsilon \in [2/3, 3/4); \\
7, & \varepsilon \in [3/5, 2/3); \\
8, & \varepsilon \in [1/2, 3/5).
\end{cases}
$$

Notice that, in contrast to $d = 2$, when $\varepsilon \in [1/2, 1)$, the value of $\mu(d = 3; \varepsilon)$ achieves the lower bound in (2) if and only if $\varepsilon \in [1/2, 4/7) \cup [3/5, 1)$.

For $\varepsilon \in [1/3, 1/2)$, Theorems 1, 2, 3, 4 and Lemma 1 from Section 3 imply that

- for $\varepsilon \in \left[\frac{1}{3}, \frac{8}{27}\right)$, $\mu(3; \varepsilon) = 27$ (by Theorem 2);
- for $\varepsilon \in \left[\frac{8}{27}, \frac{5}{13}\right)$, $\mu(3; \varepsilon)$ is constant and lies on $[22, 24]$ (by Theorem 3, Lemma 1 and (2));
- for $\varepsilon \in \left[\frac{7}{16}, \frac{4}{9}\right)$, $\mu(3; \varepsilon)$ is constant and lies on $[17, 21]$ (by Theorem 3, Lemma 1 and (2));
- for $\varepsilon \in \left[\frac{4}{9}, \frac{7}{15}\right)$, $\mu(3; \varepsilon) \in [16, 18]$ (by Theorem 4 and (2));
- for $\varepsilon \in \left[\frac{7}{15}, \frac{1}{2}\right)$, $\mu(3; \varepsilon) = 15$ (by Theorem 1 and Lemma 1).
3 Integer lattices

On one hand, there are continuously many \( \varepsilon \) left for which the answer is not known. On the other hand, the lemma below implies that the problem reduces to a countable set.

Let \( d \geq 2 \) be an integer.

**Lemma 1.** There exists an infinite sequence of rational numbers 1 > \( \frac{\mu_1}{t_1} > \frac{\mu_2}{t_2} > \ldots \) > 0 such that, for every \( i \in \mathbb{N}, t_i \leq \mu(d; s_i/t_i) \) and \( \mu(d; \varepsilon) = \mu(d; s_i/t_i) \) for all \( \varepsilon \in [s_i/t_i, s_{i-1}/t_{i-1}) \), where \( s_0 = t_0 = 1 \).

For \( d = 3 \), due to (2), the denominator of the critical point \( \frac{\varepsilon}{t} \) is at most \( \left\lfloor \frac{\varepsilon}{s_i} \right\rfloor \cdot \left\lfloor \frac{\varepsilon}{s_i} \right\rfloor \cdot \left\lfloor \frac{\varepsilon}{s_i} \right\rfloor \). Therefore, for every integer \( r \geq 2 \), on \( \left[ \frac{1}{r}, \frac{1}{r-1} \right) \) there are at most \( \frac{r^3(r^3+1)}{2(r-1)} + r^3 \) candidates for the role of a critical point.

We give the proof of Lemma 1 in Section 4. The proof also yields that the problem can be equivalently reformulated for integer lattices (as stated below in Lemma 2).

Let \( s \leq t \) be positive integers. Consider the torus \( \mathbb{Z}/t\mathbb{Z} \) and the set of its ‘sub-cubes’ with edges of size \( s \): \( \{(x_1, \ldots, x_d) : x_i^0 \leq x_i \leq x_i^0 + s - 1 \mod t \}, x_i^0 \in \mathbb{Z}/t\mathbb{Z} \). Throughout the paper, for a ‘sub-cube’ \( \{(x_1, \ldots, x_d) : x_i^0 \leq x_i \leq x_i^0 + s - 1 \mod t \} \), we call its node \( (x_1^0, \ldots, x_d^0) \) the base vertex of the cube. Let \( \mu_0(d; s, t) \) be the minimum number of such ‘sub-cubes’ needed to cover the torus.

**Lemma 2.** Let \( r \geq 2 \) be an integer, \( \varepsilon \in \left[ \frac{1}{r}, \frac{1}{r-1} \right) \). Let \( \frac{\varepsilon}{t} \leq \varepsilon \) be the (leftwards) closest rational number to \( \varepsilon \) with \( t \leq r^d \). Then \( \mu(d; \varepsilon) = \mu_0(d; s, t) \). Moreover, for any rational \( \varepsilon = s/t \), we have \( \mu(d; \varepsilon) = \mu_0(d; s, t) \).

Since \( \mu(d; s/t) = \mu_0(d; s, t) \), we get that \( \mu_0(d; s, t) \) depends only on \( s/t \).

4 Proofs of Lemmas 1 and 2

In this section, we prove the following.

**Claim 1.** Let \( \mu \) be a positive integer. Choose a minimal \( \varepsilon_0 \) such that there is a covering \( \mathcal{A} \) of the torus \( T^d \) by \( \varepsilon_0 \)-sub-cubes

\[
A_j := \{(x_1, \ldots, x_d) : x_i \in [x_i^j, x_i^j + \varepsilon_0] \} , \quad j \in [\mu].
\]

Then \( \varepsilon_0 = s_0/t_0 \) where \( s_0 \) and \( t_0 \) are coprime integers, with \( t_0 \leq \mu \).

Moreover, for every rational \( \varepsilon = s/t \geq \varepsilon_0 \), there exists a covering of \( T^d \) with \( \mu \) ‘\( \varepsilon \)-sub-cubes’ such that all their base vertices are multiples of \( 1/t \).

Lemma 1 is a direct consequence of the first part of Claim 1. To show that Claim 1 also yields Lemma 2, take an \( \varepsilon \in \left[ \frac{1}{r}, \frac{1}{r-1} \right) \) and put \( \mu = \mu(d; \varepsilon) \); notice that \( \mu \leq r^d \) and choose the fraction \( s/t \) as in the statement of Lemma 2. By Claim 1, the minimal number \( \varepsilon_0 \) such that there is a covering of the torus \( T^d \) by \( \varepsilon_0 \)-sub-cubes has the form \( \varepsilon_0 = s_0/t_0 \) with \( t_0 \leq \mu \leq r^d \); therefore, \( \varepsilon_0 \leq s/t \leq \varepsilon \) and hence \( \mu(d; \varepsilon_0) = \mu(d; s/t) = \mu \).
Moreover, assuming that \( \varepsilon = s'/t' \) is rational, the second part of Claim \( \text{[1]} \) implies that there exists a covering of \( T^d \) with \( \mu 's'/t' \)-sub-cubes' such that all their base vertices are integer multiples of \( 1/t \). This covering induces a covering of \( (\mathbb{Z}/t\mathbb{Z})^d \) by \( \mu 's'/t' \)-sub-cubes' of side length \( s \), thus showing that \( \mu(d; s', t') \leq \mu(d; s'/t') \). This yields Lemma \( \text{[2]} \) as the converse inequality \( \mu_0(d; s', t') \geq \mu(d; s'/t') \) is trivial.

The remaining part of the section is devoted to the proof of Claim \( \text{[1]} \).

Notice that the minimal \( \varepsilon_0 \) chosen in the statement of Claim \( \text{[1]} \) exists by a standard compactness argument. Consider now any \( \varepsilon \) such that there is a covering \( A \) of the torus \( T^d \) by \( \mu '\varepsilon\)-sub-cubes'

\[
A_j := \{(x_1, \ldots, x_d) : x_i \in [x_i^j, x_i^j + \varepsilon]\}, \quad j \in [\mu].
\]

We will modify this covering in two steps. The result of Step 1 will, in particular, establish the second part of Claim \( \text{[1]} \). In Step 2, we will see that, if \( \varepsilon \) is not of the form \( \varepsilon = s/t \) with \( t \leq \mu \), then there is a covering with \( \mu '\varepsilon\)-sub-cubes' for some \( \varepsilon' < \varepsilon \), thus establishing the first part of Claim \( \text{[1]} \).

Let \( i \in [d] \). Consider the segments \( [x_i^j, x_i^j + \varepsilon], j \in [\mu] \). Let us introduce a graph \( G_i(A) \) with vertex set \([\mu]\). Let vertices \( j_1, j_2 \in [\mu] \) be adjacent in \( G_i(A) \) if and only if the sets \( \{x_i^{j_1}, x_i^{j_1} + \varepsilon\}, \{x_i^{j_2}, x_i^{j_2} + \varepsilon\} \) are not disjoint (i.e., the respective segments have at least one common endpoint).

**Step 1.** We show that there exists a covering \( \tilde{A} \) of \( T^d \) by \( \mu '\varepsilon\)-sub-cubes' such that \( G_i(\tilde{A}) \) is connected, for every \( i \in [d] \). For that purpose, we shift some 'sub-cubes' as follows.

Assume that, in \( G_i(A) \), there are several connected components \( H_1, \ldots, H_\ell, \ell \geq 2 \). Choose an endpoint \( a \) of segment \( j \) with \( j \in H_1 \) and an endpoint \( b \) of segment \( j' \in [\mu] \setminus H_1 \) (i.e., \( a \in \{x_i^j, x_i^j + \varepsilon\} \) and \( b \in \{x_i^{j'}, x_i^{j'} + \varepsilon\} \)) such that their distance \( \rho = |a - b| \) is minimal over the choice of the components, of the segments inside them, and their endpoints; since \( j \) and \( j' \) are in different components, we have \( \rho > 0 \).

Without loss of generality, \( a > b \). Now let us shift all segments labeled by \( H_1 \) leftwards by distance \( \rho \). By the choice of \( \rho \), no point may remain uncovered, so we get a covering \( A_1 \) of \( T^d \), where the graph \( G_i(A_1) \) consists of at most \( \ell - 1 \) components. If \( G_i(A_1) \) is not connected, we perform the same procedure with \( A_1 \) and obtain a covering \( A_2 \) with \( G_i(A_2) \) having at most \( \ell - 2 \) components. Proceeding in this way, we reach a covering \( \tilde{A} \) where \( G_i(\tilde{A}) \) is connected.

Applying the same procedure for every \( i \in [d] \), we get a desired covering \( \tilde{A} \).

Now, if \( \varepsilon = s/t \) is rational, we may assume that that \( x_1 = (0, 0, \ldots, 0) \). By connectedness of all graphs \( G_i(\tilde{A}) \), all coordinates of the vertices of the 'sub-cubes' are multiples of \( 1/t \); thus the second part of Claim \( \text{[1]} \) is established.

**Step 2.** Now we may assume that for every \( i \in [d] \), the graph \( G_i(A) \) is connected. Suppose that there is no integer \( t \in [\mu] \) such that \( t\varepsilon \in \mathbb{N} \); we will show that there exists an \( \varepsilon' < \varepsilon \) such that \( T^d \) can be covered by \( \mu '\varepsilon\)-sub-cubes', thus proving the first part of Claim \( \text{[1]} \).

Fix \( i \in [d] \) and consider the following relation \( <_i \) on the set of segments \( [x_i^j, x_i^j + \varepsilon] \):
if \( x_i^{j_1} + \varepsilon = x_i^{j_2} \mod 1 \), then \( [x_i^{j_1}, x_i^{j_1} + \varepsilon] <_i [x_i^{j_2}, x_i^{j_2} + \varepsilon] \).

Since \( q \varepsilon \not\in \mathbb{N} \) for any \( q \in [\mu] \), we get that \( <_i \) is a partial order (recall that some numbers of the form \( x_i^j \) may coincide).

Choose now a minimal segment \( [x_i^{j_1}, x_i^{j_1} + \varepsilon] \) with respect to \( <_i \). Shifting all \( i \)th coordinates by \( x_i^{j_1} \), we may assume that \( x_i^{j_1} = 0 \). Performing such shifts along all coordinates, we arrive at the situation where all coordinates of all vertices of the 'sub-cubes' lie in the set

\[
I_\varepsilon = \{ k \varepsilon \mod 1 : k = 0, 1, \ldots, \mu \}.
\]

Notice that \( I_\varepsilon \) consists of \( \mu + 1 \) distinct numbers; let \( \gamma \) be the minimal distance between elements of \( I_\varepsilon \) (modulo 1).

Choose a positive \( \delta < \gamma/\mu \) and \( \varepsilon' = \varepsilon - \delta \). Then, for every \( k_1, k_2 \in \{0, 1, \ldots, \mu\} \) we have

\[
\{k_1 \varepsilon\} < \{k_2 \varepsilon\} \quad \text{if and only if} \quad \{k_1 \varepsilon'\} < \{k_2 \varepsilon'\}.
\]

Informally speaking, the sets \( I_\varepsilon \) and \( I_{\varepsilon'} \) have the same combinatorial structure.

Now choose the '\( \varepsilon' \)-sub-cubes’

\[
A'_j = \{ (x_1, \ldots, x_d) : x_i \in [(x'_1)_i, (x'_1)_i + \varepsilon'] \}, \quad j \in [\mu],
\]

as follows: if \( x_i^j = \{k \varepsilon\} \), then \( (x'_1)_i^j = \{k \varepsilon'\} \). The above relation shows that for every \( a' \in [0, 1) \) there exists \( a \in [0, 1) \) such that

\[
a' \in [(x'_1)_i^j, (x'_1)_i^j + \varepsilon'] \quad \text{if and only if} \quad a \in [x_i^j, x_i^j + \varepsilon].
\]

Therefore, for any point \( x' = (x'_1, \ldots, x'_d) \in T^d \) there exists a point \( x = (x_1, \ldots, x_d) \in T^d \) such that \( x' \) is covered by \( A'_j \) if and only if \( x \) is covered by \( A_j \). Thus, since the \( A_j \) form a covering of \( T^d \), so do the \( A'_j \), and we have constructed a covering by \( \mu '\varepsilon'-\text{sub-cubes} \), as desired.

## 5 Proofs of Theorems 2 and 3

Introduce the following conditions on positive integers \( s \) and \( t \):

(i). \( \lfloor \frac{s}{r} \rfloor = r \geq 2 \) and \( \lceil \frac{s}{s} \rceil = r + 1 \),

(ii). \( \lfloor \frac{s}{s} \lfloor \frac{s}{s} \rfloor \rfloor = s \) and, consequently, \( \lfloor \frac{s}{s} \lfloor \frac{s}{s} \rfloor \rfloor = t \),

(iii). \( (s^2 - t(r + 1))r \leq s - (s^2 - t(r + 1)) \).

(iv). \( r(s^2 + s - t(r + 1)) \leq t \);

(v). at least one of the inequalities in (iii) and (iv) is strict.

In fact, Theorems 2 and 3 are particular cases of the following lemma. We start with proving the Lemma, and then we derive both theorems from it.
Lemma 3. If \(t, s\) are coprime positive integers satisfying the conditions \([i] [v]\), then \(\mu(3; s/t) > t\).

Proof. Due to Lemma 2, it suffices to show \(\mu_0(3; s, t) > t\). By the indirect assumption, there is a covering of \((\mathbb{Z}/t\mathbb{Z})^3\) with \(t\) ‘sub-cubes’ \(C_1, C_2, \ldots, C_t\) of side \(s\).

For \(i \in [t]\), let \(x^0(i) = (x^0_1(i), x^0_2(i), x^0_3(i))\) be the base vertex of cube \(C_i\). For every \(\alpha \in \mathbb{Z}/t\mathbb{Z}\), define the \(\alpha\)th layer \(S_\alpha\) as the intersection of the torus with the hyperplane \(x_3 = \alpha\), i.e., \(S_\alpha = (\mathbb{Z}/t\mathbb{Z})^3_{|_{x_3=\alpha}}\). Each such intersection is a 2-torus covered with the squares, i.e., the (nonempty) intersections of the ‘sub-cubes’ with \(S_\alpha\).

Let us denote the number of squares covering layer \(S_\alpha\) by \(f(\alpha)\). Since \(\mu(2; z) = \lceil \frac{1}{z} \rceil\) (see [6]), we get \(f(\alpha) = s\) for every \(\alpha \in \mathbb{Z}/t\mathbb{Z}\). In particular, for every \(i \in [t]\) we have \(f(x^0_3(i) + s - 1) = f(x^0_3(i) + s)\). In other words, the last layer of the cube \(C_i\) is covered by the same number of squares as the next layer. It is only possible when there is \(j \in [t]\) such that \(x^0_3(j) = x^0_3(i) + s\); informally speaking, cube \(C_j\) ‘starts’ exactly when \(C_i\) ‘finishes’. In this situation, say that cube \(C_j\) is a successor of cube \(C_i\).

Since \(s\) and \(t\) are coprime, for every \(\alpha \in \mathbb{Z}/t\mathbb{Z}\), there is a cube \(C_i\) with \(x^0_3(i) = \alpha\). Renumbering the cubes \(C_i\), we assume that \(x^0_3(i) = i\) for all \(i \in [t]\); we assume that the numeration of the cubes is cyclic, i.e., \(C_{i+t} = C_i\); so, further we assume that the indices \(i\) run over \(\mathbb{Z}/t\mathbb{Z}\). Now, each cube \(C_i\) has a unique successor \(C_{i+s}\).

For \(\gamma \in \{1, 2, 3\}\), we say that a \(\gamma\)-column is a set of \(t\) elements in \((\mathbb{Z}/t\mathbb{Z})^3\) differing only in the \(\gamma\)th coordinate. Due to [i] each \(\gamma\)-column meets at least \(r + 1\) ‘sub-cubes’.

Claim 2. Assume that a layer \(S\) is covered with squares \(A_1, A_2, \ldots, A_s\). Fix \(\gamma \in \{1, 2\}\). Then there are no more than \(s^2 - t(r + 1)\) \(\gamma\)-columns crossing at least \(r + 2\) squares in the covering.

Proof. Let \(h\) be the number of \(\gamma\)-columns crossing at least \(r + 2\) squares in the covering. Consider the pairs of the form \((U, A_i)\), where \(i \in [s]\) and \(U\) is a \(\gamma\)-column in \(S\) that meets \(A_i\). The number of such pairs is exactly \(s^2\) since every square \(A_i\) meets \(s\) columns. Since each \(\gamma\)-column crosses at least \(r + 1\) squares, and \(h\) of them cross at least \(r + 2\) squares, we obtain

\[
s^2 \geq h(r + 2) + (t - h)(r + 1) = t(r + 1) + h,
\]

so \(h \leq s^2 - t(r + 1)\). \(\Box\)

Claim 3. For every \(i \in \mathbb{Z}/t\mathbb{Z}\) and every \(\gamma \in \{1, 2\}\), the intersection of the sets of \(\gamma\)-coordinates of \(C_i\) and \(C_{i+s}\) has cardinality at least \(s - (s^2 - (r + 1)t)\), i.e.

\[
|\{x^0_\gamma(i), \ldots, x^0_\gamma(i) + s - 1\} \cap \{x^0_\gamma(i + s), \ldots, x^0_\gamma(i + s) + s - 1\}| \geq s - (s^2 - (r + 1)t).
\]

Proof. The layer \(S_{s+1}\) meets ‘sub-cubes’ \(C_1, C_2, \ldots, C_s\), while \(S_{s+1}\) meets ‘sub-cubes’ \(C_2, C_3, \ldots, C_{s+1}\). Let \(A_{\ell}\) be the projection of \(C_{\ell}\) onto \(S := S_s\), for \(\ell \in [s + 1]\)

Consider the covering of \(S\) by \(A_1, A_2, \ldots, A_s\). By Claim 2 among the \(\gamma\)-columns crossing \(A_1\), there are at least \(s - (s^2 - t(r + 1))\) ones crossing exactly \(r + 1\) squares
in the covering. This means that each of those columns is not covered completely by \( A_2, A_3, \ldots, A_s \). Hence each of those columns needs to cross \( A_{s+1} \), since \( S \) is covered by \( A_2, A_3, \ldots, A_{s+1} \) as well. This finishes the proof.

By Claim 3 for each \( i \in \mathbb{Z}/t\mathbb{Z} \) and \( \gamma \in \{1, 2\} \) we have \(|x_\gamma^0(i) - x_\gamma^0(i+s)| \leq s^2 - t(r+1)\). Therefore,

\[
|x_\gamma^0(i) - x_\gamma^0(i+rs)| \leq \sum_{j=1}^{r} |x_\gamma^0(i + (j-1)s) - x_\gamma^0(i+js)| \leq (s^2 - t(r+1))r.
\]

Now we distinguish two cases.

**Case 1.** The above inequality is sometimes strict, i.e., there exist \( \gamma \in \{1, 2\} \) and \( i \in \mathbb{Z}/t\mathbb{Z} \) such that

\[
|x_\gamma^0(i) - x_\gamma^0(i+rs)| < (s^2 - t(r+1))r.
\]

Without loss of generality, we assume \( i = 0 \). For \( \kappa = 1, 2 \), we denote

\[
N_\kappa := |\{x_\kappa^0(0), \ldots, x_\kappa^0(0) + s - 1\} \cap \{x_\kappa^0(rs), \ldots, x_\kappa^0(rs) + s - 1\}|;
\]

in other words, \( N_\kappa \) is the number of common \( \kappa \)-coordinates in \( C_0 \) and \( C_{rs} \). Then we have

\[
N_\gamma > s - (s^2 - t(r+1))r \quad \text{and} \quad N_{3-\gamma} \geq s - (s^2 - t(r+1))r. \quad (4)
\]

We show that there exists \( \kappa \in \{1, 2\} \) such that

\[
N_\kappa > s(r+1) - t \quad \text{and} \quad N_{3-\kappa} > s^2 - t(r+1). \quad (5)
\]

Indeed, notice that

\[
s - (s^2 - t(r+1))r = s(r+1) - (s + s^2 - t(r+1))r \geq s(r+1) - t \quad (6)
\]

by \((iv)\) moreover, if the inequality in \((iv)\) is strict, then so is the last inequality above. By \((v)\) one of the inequalities in \((iii)\) and \((iv)\) is strict. If the inequality in \((iii)\) is strict, then we can put \( \kappa = \gamma \), since \( N_{3-\kappa} = N_{3-\gamma} \geq s - (s^2 - t(r+1))r > s^2 - t(r+1) \) by \((iii)\) and \( N_\kappa = N_\gamma > s - (s^2 - t(r+1))r \geq s(r+1) - t \) by \((3)\). Otherwise, \((iv)\) is strict, and we can put \( \kappa = 3 - \gamma \), as \( N_\kappa = N_{3-\gamma} \geq s - (s^2 - t(r+1))r > s(t+1) - t \) by \((6)\) and \( N_{3-\kappa} = N_\gamma > s - (s^2 - t(r+1))r \geq s^2 - t(r+1) \) by \((iii)\).

Without loss of generality, we assume that \( \kappa = 1 \) satisfies \((3)\). Consider \( S := S_0 \), and let \( A_\ell \) be the projection of \( C_\ell \) onto \( S \), for \( \ell \in \{t+1-s, t+2-s, \ldots, t\} \). Notice that \( S \) is covered by the \( A_\ell \), where \( \ell \) runs through the same range.

Recall that \( t - s + 1 \leq rs < t \) by \((1)\). In \( S \), consider any of \( \kappa \)-columns crossing both \( A_{rs} \) and \( A_0(= A_t) \); there are at least \( N_{3-\kappa} > s^2 - t(r+1) \) such \( \kappa \)-columns. By Claim 2, at least one of those is covered by exactly \( r + 1 \) squares in the covering. But the intersections of the column with two of those squares, namely \( A_t \) and \( A_{rs} \), have \( N_\kappa \) common elements, hence those \( r + 1 \) squares cover at most

\[
s(r+1) - N_\kappa < s(r+1) - (s(r+1) - t) = t
\]
elements in the column. Therefore, this column is not covered completely — a contradiction.

Case 2. Conversely, assume now that, for every $\gamma \in \{1, 2\}$ and $i \in \mathbb{Z}/t\mathbb{Z}$,

$$|x^0_\gamma(i) - x^0_\gamma(i + rs)| = \sum_{j=1}^{r} |x^0_\gamma(i + (j - 1)s) - x^0_\gamma(i + js)| = (s^2 - t(r + 1))r. \quad (7)$$

Fix $\gamma \in \{1, 2\}$ and $i \in \mathbb{Z}/t\mathbb{Z}$. By the triangle inequality, (7) is only possible when, for every $j \in [r]$, $|x^0_\gamma(i + (j - 1)s) - x^0_\gamma(i + js)| = s^2 - t(r + 1)$ and, moreover, all expressions of the form $x^0_\gamma(i + (j - 1)s) - x^0_\gamma(i + js)$ are equal, for $j \in [r]$. This in fact means that all expressions of the form $x^0_\gamma(i) - x^0_\gamma(i + s)$ are equal, and this common value is $\pm(s^2 - t(r + 1))$. Without loss of generality, we assume that, for every $i \in \mathbb{Z}/t\mathbb{Z}$,

$$x^0_2(i + s) - x^0_2(i) = x^0_1(i + s) - x^0_1(i) = s^2 - t(r + 1)$$

(the generality is not lost, as we can renumber each coordinate as $i \mapsto -i$).

So, the first and the second coordinates of each base vertex are equal; so those coordinates of any point in a ‘sub-cube’ differ by at most $s - 1$. Therefore, no point of the form $(0, s, x_3)$ is covered by any ‘sub-cube’ (recall here that $t > 2s$ by (i)). This is a contradiction.

Proof of Theorem 3. We show that the parameters in the theorem satisfy (i)–(v); the theorem follows then from Lemma 3.

To prove (i) write

$$0 < t - rs = \xi r + \left[\xi^2/(r + 1)\right] \leq \xi r + r - 1 \leq r^2 + r - 1 < s.$$

Notice that $s^2 - t(r + 1) = \xi^2 - \left[\frac{\xi^2}{r + 1}\right](r + 1) \in [0, \xi]$, \quad (8)

where the last inclusion follows from (3). Therefore,

$$\left[\begin{array}{c} t \\ \frac{t}{s} \end{array}\right] = \left[\begin{array}{c} t(r + 1) \\ \frac{t(r + 1)}{s} \end{array}\right] = \left[\begin{array}{c} s - \frac{s^2 - t(r + 1)}{s} \end{array}\right] \in \left[\begin{array}{c} s - \frac{\xi}{s} \\ s \end{array}\right].$$

which yields (iii).

Moreover, (8) implies that

$$(s^2 - t(r + 1))(r + 1) \leq \xi(r + 1) \leq r^2 + r < r^2 + r + \xi = s.$$

This verifies (iii) with a strict inequality (hence (v) as well).
Finally, (8) implies that
\[(s + s^2 - t(r + 1))r \leq (s + \xi)r = (r^2 + r + 2\xi)r = r^3 + r^2 + 2r\xi \leq t\]
which establishes (iv). \(\square\)

**Proof of Theorem 2**. For convenience, we replace \(r\) with \(r + 1\) (i.e., we prove that, for any \(r \in \mathbb{N}\) and any \(\varepsilon \in \left[\frac{1}{r+1}, \frac{(r+1)^2 - 1}{(r+1)^2 - r - 2}\right]\), \(\mu(3; \varepsilon) = (r + 1)^3\)).

Let \(s = (r + 1)^2, t = (r + 1)^3 - 1\). Notice that the fractions \(\frac{1}{r+1}\) and \(\frac{(r+1)^2 - 1}{(r+1)^2 - r - 2}\) are Farey neighbors (for properties of Farey sequence, see [4, Chapter III]). Therefore, the rational number between them with the smallest denominator is \(\frac{1 + (r+1)^2 - 1}{(r+1) + (r+1)^2 - r - 2} = \frac{(r+1)^2}{(r+1)^2 - 1}\), and the next smallest denominator is already greater than \((r + 1)^3\). By Lemma 2, in order to prove Theorem 2, it suffices to show that \(\mu(3; s/t) \geq (r + 1)^3\). For that purpose, we show that the numbers \(s\) and \(t\) satisfy the conditions of Lemma 3.

Clearly, \(s\) and \(t\) are coprime. The conditions (i)–(ii) are straightforward. Next, since \(s^2 - t(r + 1) = r + 1\), we have
\[(s^2 - t(r + 1))r = (r + 1)r = (r + 1)^2 - (r + 1) = s - (s^2 - t(r + 1))\]
which yields (iii) (with the equality sign). It remains to show (iv) with the strict sign:
\[r(s^2 + s - t(r + 1)) = r(r + 1)(r + 2) = (r + 1)((r + 1)^2 - 1) < t. \] \(\square\)

6 Proof of Theorem 2

Let \(r \geq 2\) be an integer, \(\xi \in [r]\), \(s = r^2 + r + \xi, t = r^3 + r^2 + \xi(r + 1)\). We construct a covering of \([\mathbb{Z}/t\mathbb{Z}]^3\) consisting of \(t\) ‘sub-cubes’ of side \(s\).

The main idea here is the same as in [6]: to take each next ‘sub-cube’ shifted relative to the previous one by a fixed integer vector \(v\). We take
\[v = (1, r, (r^2 + r + 1)).\]

Thus, we need to prove that the ‘sub-cubes’ \(C_i\) with base vertices \((i, ri, (r^2 + r + 1)i), i \in \mathbb{Z}/t\mathbb{Z}\), cover the torus \([\mathbb{Z}/t\mathbb{Z}]^3\). Since the points of the lattice with the same first coordinate \(j\) are covered by \(s\) consecutive cubes \(C_{j-i}\) with \(i \in \{0, 1, \ldots, s - 1\}\), the following claim finishes the proof of Theorem 2.

**Claim 4.** The squares of side \(s\) with base vertices \((ri, (r^2 + r + 1)i), i \in \{0, 1, \ldots, s - 1\}\), cover \([\mathbb{Z}/t\mathbb{Z}]^2\).

**Proof.** Denote by \(B_i, i \in \{0, 1, \ldots, s - 1\}\), the square with the base vertex \((ri, (r^2 + r + 1)i)\). In what follows, we assume that \(i\) runs over \(\mathbb{Z}/s\mathbb{Z}\), i.e., \(i + 1 = 0\) for \(i = s - 1\).

Fix \(y \in \{0, 1, \ldots, t - 1\}\). Let \(C\) be the column consisting of all points whose first coordinate is \(y\). We show that \(C\) is covered completely by the \(B_i\).
Notice that the relative shift of the first coordinate of $B_{i+1}$ with respect to $B_i$ is $x_1^0(i + 1) - x_1^0(i) = r$ for $i \neq s - 1$ and
\[ x_1^0(0) - x_1^0(s - 1) = t - r(s - 1) = r + \xi \]
for $i = s - 1$. Since $r + \xi < s$, column $C$ meets several consecutive squares $B_i, B_{i+1}, \ldots, B_{i+p}$. Since each of $B_{i-1}$ and $B_{i+p+1}$ does not meet $C$, we have $x_1^0(i+p+1) - x_1^0(i-1) > s$. Therefore,
\[ r^2 + r + \xi + 1 = s + 1 \leq x_1^0(i + p + 1) - x_1^0(i - 1) \leq (p + 1)r + (r + \xi), \]
which yields $p \geq r$. So, $C$ meets the squares $B_i, B_{i+1}, \ldots, B_{i+r}$.

Now, the relative shift of the second coordinate of $B_{i+1}$ with respect to $B_i$ is $x_2^0(i + 1) - x_2^0(i) = r^2 + r + 1$ for $i \neq s - 1$ and
\[ x_2^0(0) - x_2^0(s - 1) = t(r + 1) - (r^2 + r + 1)(s - 1) = \xi r + 1 < r^2 + r + 1 \]
for $i = s - 1$. This yields that the parts of $C$ covered with consecutive squares are either tangent or even overlapping. So the interval they cover consists of at least
\[ \min\{t, s + (r - 1)(r^2 + r + 1) + (r\xi + 1)\} = t \text{ points}, \]
and hence the whole column is covered.

\section{Proof of Theorem \ref{thm:5}}
We start with the lower bounds for $\mu(3; \varepsilon)$ agreeing with the statement of the theorem. For $\varepsilon \in [1/2, 4/7] \cup [3/5, 1)$, such lower bounds follow from \cite{2}. For $\varepsilon \in [4/7, 3/5)$ the lower bound follows from Theorem \cite{2}.

In order to show that those bounds are achieved, it suffices to show that $\mu_0(3; 3, 4) = 4, \mu_0(3; 2, 3) = 5$, and $\mu_0(3; 3, 5) = 7$. The corresponding examples are provided by the following sets of base vertices:

\begin{align*}
(0, 0, 0), & \ (1, 1, 1), \ (2, 2, 2), \ (3, 3, 3), \quad \text{for } s/t = 3/4; \\
(0, 0, 0), & \ (1, 1, 1), \ (1, 2, 2), \ (2, 1, 2), \ (2, 2, 1), \quad \text{for } s/t = 2/3; \\
(0, 0, 0), & \ (1, 1, 3), \ (1, 3, 1), \ (2, 4, 4), \ (3, 0, 2), \ (3, 2, 0), \ (4, 3, 3), \quad \text{for } s/t = 3/5.
\end{align*}

The theorem is proved.

\section{Acknowledgements}
The study was supported by the Russian Science Foundation (grant number 21-71-10092).
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