Triple product $p$-adic $L$-functions for balanced weights

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Abstract
We construct $p$-adic triple product $L$-functions that interpolate (square roots of) central critical $L$-values in the balanced region. Thus, our construction complements that of Harris and Tilouine. There are four central critical regions for the triple product $L$-functions and two opposite settings, according to the sign of the functional equation. In the first case, three of these regions are of interpolation, having positive sign; they are called the unbalanced regions and one gets three $p$-adic $L$-functions, one for each region of interpolation (this is the Harris-Tilouine setting). In the other setting there is only one region of interpolation, called the balanced region: we produce the corresponding $p$-adic $L$-function. Our triple product $p$-adic $L$-function arises as $p$-adic period integrals interpolating normalizations of the local archimedean period integrals. The latter encode information about classical representation theoretic branching laws. The main step in our construction of $p$-adic period integrals is showing that these branching laws vary in a $p$-adic analytic fashion. This relies crucially on the Ash-Stevens theory of highest weight representations over affinoid algebras.

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1 Introduction

Consider three finite slope cuspidal $p$-adic Coleman eigenfamilies $\mathbf{f} = (f_1, f_2, f_3)$ new of tame levels $(N_1, N_2, N_3)$, nebetypes $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and eigenvalues $(\alpha_1, \alpha_2, \alpha_3)$ for the $U_p$-operator of slope $(h_1, h_2, h_3)$, parametrized by the product of three connected affinoid subdomains $U = U_1 \times U_2 \times U_3 \subset X^3$, where $X$ denotes the weight space and we suppose these rigid analytic objects to be defined over a $p$-adic field $F$. By an integer point we mean $k \in \mathbb{N}^3 \cap U$, say $k = (k_1, k_2, k_3)$, such that $k_i > h_i - 1$. We also call $k_i$ an integer point. If $k$ is an integer point, we write $f_k = (f_{k_1}, f_{k_2}, f_{k_3})$ for the specialization of $\mathbf{f}$, which is a triple of classical modular forms with $f_{k_i}$ of weight $k_i + 2$. Because $\alpha_{i,k_i} = \pm p^{k_i/2}$ if $f_{i,k_i}$ is $p$-new, for every $k_i \neq 2h_i$ we know that $f_{i,k_i} = f_{i,k_i}$ is old at $p$ and it is the $p$-stabilization of some newform

$$f_{i,k_i} = f_{i,k_i}^{\#} \in S_{k_i+2} (\Gamma_0 (N_i), \varepsilon_i)_{\text{new}}^N.$$

We say that $k_i$ is a generic integer point in this case and we say that $k$ is a generic integer point if $k_i$ is such a point for $i = 1, 2, 3$. We refer the reader to Remark 8.1 below for more details.

The problem we are interested in is about interpolating the function

$$k := (k_1, k_2, k_3) \mapsto L \left( f_{1,k_1}^{\#} \times f_{2,k_2}^{\#} \times f_{3,k_3}^{\#}, c_k \right)$$

for the central critical value $c_k := \frac{k_1 + k_2 + k_3 + 4}{2}$. Here $L \left( f_{1,k_1}^{\#} \times f_{2,k_2}^{\#} \times f_{3,k_3}^{\#}, s \right)$ is the triple product complex $L$-function (see for example [24, §1] for its definition). Let

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us write \( \pi_i := \pi_{f_i, k_i} \) for the automorphic representation attached to \( f_i := f_{i, k_i}^\# \). If we want \( \Pi_k := \pi_1 \otimes \pi_2 \otimes \pi_3 \) to be selfdual, the following condition \((CC)_{k_1, k_2, k_3}\) needs to be imposed:

\[
(CC)_{k_1, k_2, k_3} : \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1, \quad \text{so that} \quad k_1 + k_2 + k_3 \in 2\mathbb{N}.
\]

There are four central critical regions, namely

\[
\begin{align*}
\Sigma_1 & := \{(k_1, k_2, k_3) : k_1 > k_2 + k_3 \text{ and } (CC)_{k_1, k_2, k_3} \text{ holds} \}, \\
\Sigma_2 & := \{(k_1, k_2, k_3) : k_2 > k_1 + k_3 \text{ and } (CC)_{k_1, k_2, k_3} \text{ holds} \}, \\
\Sigma_3 & := \{(k_1, k_2, k_3) : k_3 > k_1 + k_2 \text{ and } (CC)_{k_1, k_2, k_3} \text{ holds} \}, \\
\Sigma_{123} & := \{(k_1, k_2, k_3) : k_1 \leq k_2 + k_3, k_2 \leq k_1 + k_3, k_3 \leq k_1 + k_2 \text{ and } (CC)_{k_1, k_2, k_3} \text{ holds} \}
\end{align*}
\]

The transcendental nature of the Deligne’s period \( \Omega \) depends on the critical region. We have, up to powers of \( \pi \),

\[
\Omega = (f_i, f_j)^2 \quad \text{on} \quad \Sigma_i \quad \text{and} \quad \Omega = (f_1, f_1) (f_2, f_2) (f_3, f_3) \quad \text{on} \quad \Sigma_{123}.
\]

Here \((f, \pi) = (f, \pi)_k\) is the Petersson inner product, that we normalized by the volume of the corresponding modular curve:

\[
(f, \pi)_k := \frac{1}{\mu(\mathcal{H}/\Gamma_0(N))} \int_{\mathcal{H}/\Gamma_0(N)} f(z) \overline{\pi(z)} \mu(z), \mu = \frac{dx dy}{y^2}. \quad (1)
\]

Let \( S_i \) be the set of places such that \( \pi_{i,v} \) admits a Jacquet–Langlands lift \( \pi_{i,v}^D \) to the group of units of the division quaternion \( \mathbb{Q}_v \)-algebra. Set \( S := S_1 \cap S_2 \cap S_3 \), and, for every \( v \in S \), let \( d_v \) (resp. \( d_v^D \)) be the dimension of the space of trilinear forms on \( \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v} \) (resp. \( \pi_{1,v}^D \otimes \pi_{2,v}^D \otimes \pi_{3,v}^D \)). Define, for every \( v \in S \),

\[
\varepsilon_v (f_1 \times f_2 \times f_3) = \begin{cases} 
1 & \text{if } d_v = 1 \text{ and } d_v^D = 0 \\
-1 & \text{if } d_v = 0 \text{ and } d_v^D = 1.
\end{cases}
\]

It is a theorem of Prasad (see [34]) that the above function is indeed well defined, i.e. only one of the above two possibilities occurs. Write \( S = S^+ \cup S^- \), where \( S^\pm := \{ v : \varepsilon_v (f_1 \times f_2 \times f_3) = \pm 1 \} \), set \( D = D_{D, \Pi} := \prod_{l \in S_- - \{\infty\}} l \) and let \( B_D = B_{D, \Pi} \) be the quaternion algebra ramified at the finite primes dividing \( D \). If \( \Pi \) is an irreducible cuspidal automorphic representation of \( \text{GL}_3 \), we let \( B_{D, \Pi} \) be the quaternion algebra obtained by the above recipe and say that it is the one predicted by [34]. Recalling the dependence of these considerations from the weight, so that \( S^- = S^-_{D, \Pi} \) (resp. \( D = D_{D, \Pi} \)), the sign of the function at \( k \) is given by the formula

\[
\varepsilon(f_k^\#) = \prod_{v \in S} \varepsilon_v \left( f_k^\# \right) := \prod_{v \in S} \varepsilon_v (f_1 \times f_2 \times f_3) = (-1)^{\#S^-_{D, \Pi}}.
\]
Let $\varepsilon_{\mathfrak{f}} \left( f\# _k \right)$ be the product of the finite local signs, so that $\varepsilon \left( f\# _k \right) = \varepsilon_{\mathfrak{f}} \left( f\# _k \right) \varepsilon_{\infty} \left( f\# _k \right)$. We remark that the nature of the local sign at infinity depends on the critical region: we have $\varepsilon_{\infty} \left( f\# _k \right) = 1$ if $k \in \Sigma_i$ for $i = 1, 2$ or $3$, while $\varepsilon_{\infty} \left( f\# _k \right) = -1$ if $k \in \Sigma_{123}$.

Let us assume that $N_i$ is squarefree for $i = 1, 2, 3$. It is easy to see that $\varepsilon_{\mathfrak{f}} \left( f\# _k \right) \equiv 1$ or $\varepsilon_{\mathfrak{f}} \left( f\# _k \right) \equiv -1$ for every generic integer $k$. Indeed, under this assumption we have, for every such $k$ and every finite $v = l \in S_{f\# _k}$,

$$\varepsilon_l \left( f\# _k \right) = -a_l \left( f_{1,k_1} \right) a_l \left( f_{2,k_2} \right) a_l \left( f_{3,k_3} \right) l^{-\frac{k_1+k_2+k_3}{2}},$$

a function which can be $p$-adically interpolated and then needs to be constant for all weights (by connectedness of the $U_i$’s). Hence, having fixed $\mathfrak{f}$ there is a well defined finite “generic sign” $\varepsilon_{\mathfrak{f}} \left( f\right)$ of the family, $B_D = B_{\Pi \mathfrak{f}}$ does not depend on the generic integer point $k$ and we have a well-posed interpolation problem. Of course, we expect $\varepsilon_{\mathfrak{f}} \left( f\right)$ and $B_D = B_{\Pi \mathfrak{f}}$ to be defined in general, i.e. independent of the generic integer point $k$ (as explained below, we can give evidences). At this point the consideration splits in two cases.

If $\varepsilon_{\mathfrak{f}} \left( f\right) = 1$ (hence $D$ is the product of an even number of primes), then $\varepsilon \left( f\# _k \right) = 1$ for every generic $k \in \Sigma_1, \Sigma_2$ or $\Sigma_3$. One gets three (square root) $p$-adic $L$-function $\mathcal{L}_{\Sigma_i} (f\# )$, one for every region $\Sigma_i$, with the property that

$$\mathcal{L}_{\Sigma_i} (f\# ) (k)^2 = L \left( f_{1,k_1} \times f_{2,k_2} \times f_{3,k_3}, c_k \right) \text{ for } k \in \Sigma_i \text{ generic}.$$ 

Here we write $\equiv$ to mean equality up to Euler factors, periods and local constants. On the other hand, $L \left( f_{k_1} \times f_{k_2} \times f_{k_3}, c_k \right) = 0$ when $k \in \Sigma_{123}$ is generic because of the sign of the functional equation and the interpolation problem on $\Sigma_{123}$ is trivial. This is the Harris-Tilouine setting studied in [25], under some ordinarity assumption and supposing $N_1 = N_2 = N_3 = D = 1$. These $p$-adic $L$-functions have recently found interesting applications in [15,16]. See also [1] for an extension of Harris-Tilouine construction to Coleman families and [14] for related constructions.

When $\varepsilon_{\mathfrak{f}} \left( f\right) = -1$ (hence $D$ is the product of an odd number of primes), then $\varepsilon \left( f\# _k \right) = 1$ for every generic $k \in \Sigma_{123}$. The interpolation problem is therefore non-trivial only in the balanced region. We get a (square root) $p$-adic $L$-functions $\mathcal{L}_{\Sigma_{123}} (f\# )$ which interpolates in the region $\Sigma_{123}$:

$$\mathcal{L}_{\Sigma_{123}} (f\# ) (k)^2 = L \left( f_{1,k_1} \times f_{2,k_2} \times f_{3,k_3}, c_k \right) \text{ for } k \in \Sigma_{123} \text{ generic}.$$ 

In order to formulate the result in a simpler form, let us assume that $M = N_1 = N_2 = N_3$ and that the nebetypes $\varepsilon_l$ are trivial (but we will keep track of the characters in order to state (5) below with the correct Euler factors). Then we get a formula

$$\mathcal{L}_{\Sigma_{123}} (f\# ) (k)^2 = \mathcal{E}_p (\alpha, k)^2 \frac{\left( \varphi_{1,k}^\# \cdot \varphi_{2,k}^\# \right)_k}{2L \left( 1, \Pi \mathfrak{f}, \text{Ad} \right)} L \left( 1/2, \Pi \mathfrak{f} \right) \prod_{l \mid M} \frac{2}{l} \left( 1 + \frac{1}{l} \right),$$

(2) Springer
where the quantities appearing in the right hand side are the following. Define $k_1^* := -k_1 + k_2 + k_3$, $k_2^* := k_1 - k_2 + k_3$, $k_3^* := k_1 + k_2 - k_3$ and $k^* := k_1 + k_2 + k_3$, which are integers in view of the definition of $\Sigma_{123}$. We have the Euler factor

$$
E_p (\alpha, k) = E_{p,1} (\alpha, k) E_{p,2} (\alpha, k) E_{p,3} (\alpha, k) \left( 1 - \frac{1}{\alpha_1 \alpha_2 \alpha_3} p^{k_{\alpha} + 1} \right),
$$

where $E_{p,1} (\alpha, k) := 1 - \varepsilon_1 (p)^{-1} \frac{\alpha_1}{\alpha_2 \alpha_3} p^{k_1^*}$ and $E_{p,2} (\alpha, k)$ and $E_{p,3} (\alpha, k)$ are defined in a similar way. By the $p$-adic Jacquet–Langlands correspondence (see [10]), the three Coleman families $\tilde{\mathfrak{G}}_f$ lift to three eigenfamilies $\varphi_{f,i} = (\varphi_{f_1}, \varphi_{f_2}, \varphi_{f_3})$ on $B_D$, whose specialization $\varphi_{f,k} = (\varphi_{f_1,k}, \varphi_{f_2,k}, \varphi_{f_3,k})$ at a generic $k$ is the $p$-stabilization of three newforms $\varphi_{f,k}^\# = (\varphi_{f_1,k}^\#, \varphi_{f_2,k}^\#, \varphi_{f_3,k}^\#)$ on $B_D$ of level $M/D$; then $\left( \varphi_{f,k}^\#, \varphi_{f,k}^\#_{\text{L}} \right)_{k_i}$ is the product of quaternionic Petersson products $(\varphi_{f,i,k}^\#, \varphi_{f,i,k}^\#_{\text{L}})_{k_i}$ (see (19) and the definition before Lemma 3.2). Finally, setting $\Gamma_{\mathbb{C}} (s) := 2 (2 \pi)^{-s} \Gamma (s)$, we have

$$
L \left( 1/2, \Pi_{\mathbb{k}} \right) = L_{\infty} \left( 1/2, \Pi_{\mathbb{k}} \right) L \left( f_{1,k_1}^\#, f_{2,k_2}^\#, f_{3,k_3}^\#, c_{\mathbb{k}} \right),
$$

where

$$
L_{\infty} \left( 1/2, \Pi_{\mathbb{k}} \right) = \Gamma_{\mathbb{C}} \left( k^* + 2 \right) \Gamma_{\mathbb{C}} (k_1^* + 1) \Gamma_{\mathbb{C}} (k_2^* + 1) \Gamma_{\mathbb{C}} (k_3^* + 1)
$$

$$
= 2^4 \left( 2 \pi \right)^{-2k^* - 5} (k^* + 1)! k_1^*! k_2^*! k_3^*!
$$

and $L (s, \Pi_{\mathbb{k}}, \text{Ad})$ is a adjoint $L$-function.

**Remark 1.1** It follows from Deligne’s proof of the generalized Ramanujan conjecture that $E_p (\alpha, k) \neq 0$ for every generic integer point $k$.

Before explaining the idea of the proof, let us remark that the ratio $\frac{(\varphi_{f,k}^\#, \varphi_{f,k}^\#_{\text{L}})}{L (1/2, \Pi_{\mathbb{k}}, \text{Ad})}$ can be given a different arrangement as follows [see (73)]. First, a result of Shimura and Hida relates $L (1, \Pi_{\mathbb{k}}, \text{Ad})$ to the period $\left( \begin{smallmatrix} k_1^* \ k_2^* \ k_3^* \end{smallmatrix} \right)$ of

$$
\frac{\left( f_{1,k_1}^\#, f_{1,k_1}^\#_{\text{L}} \right)_{k_1} \left( f_{2,k_2}^\#, f_{2,k_2}^\#_{\text{L}} \right)_{k_2} \left( f_{3,k_3}^\#, f_{3,k_3}^\#_{\text{L}} \right)_{k_3}}{L (1, \Pi_{\mathbb{k}}, \text{Ad})}.
$$

Second we define, using Proposition 7.8, a rigid analytic function $\left( \varphi_{f}, \varphi_{f}^\# \right)_{p}$ uniquely characterized by the interpolation formula

$$
\left( \varphi_{f}, \varphi_{f}^\# \right)_{p} (k) = E_{p} (\alpha, k) \left( \varphi_{f,k}^\#, \varphi_{f,k}^\#_{\text{L}} \right)_{k},
$$

where $E_{p} (\alpha, k) = \prod_{i=1,2,3} E_{p} (\alpha_i, k_i)$ and $E_{p} (\alpha_i, k_i) := \left( 1 - \varepsilon_i (p) \alpha_i^{-2} p^{k_i} \right) \left( 1 - \varepsilon_i (p) \alpha_i^{-2} p^{k_i + 1} \right)$.
Remark 1.2 It follows from Deligne’s proof of the generalized Ramanujan conjecture that, for every generic integer point \( k, 1 - \varepsilon_i (p) \alpha_i^{-2} p^k_i \neq 0 \). Hence, we have \( \mathcal{E}_p^\Omega (\alpha_i, k_i) = 0 \) if and only if \( \alpha_i = \pm \sqrt[ki+1]{\varepsilon_i (p) p^{-2}} \). In particular, assuming that the slope of \( \alpha_i \) is \( \leq h_i \), we see that the condition \( \mathcal{E}_p^\Omega (\alpha_i, k_i) = 0 \) implies \( k_i \leq 2h_i - 1 \). Consequently, for all but finitely many generic integer point \( k \), we have \( \mathcal{E}_p^\Omega (\alpha, k) \neq 0 \).

Then one finds, assuming that \( \mathcal{E}_p^\Omega (\alpha, k) \neq 0 \),

\[
\frac{\left( \varphi_{\ell, k}, \varphi_{\ell, \#} \right)_k}{L (1, \Pi_k, \text{Ad})} = \frac{27}{4 \pi^3} \sum_{\ell | M} \left( 1 + \frac{1}{l} \right)^3 \mathcal{E}_p^\Omega (\alpha, k) \frac{\left( \varphi_{\ell, \#}, \varphi_{\ell} \right)_p (k)}{\left( \varphi_{\ell, k}, \varphi_{\ell, \#} \right)_k}
\]

[see (74)]. Hence (2) becomes

\[
L_{\Sigma^{123}}^\Lambda (\ell) (k)^2 = \frac{27}{4 \pi^3} \mathcal{E}_p^\Omega (\alpha, k)^2 \mathcal{E}_p^\Omega (\alpha, k) \frac{\left( \varphi_{\ell, \#}, \varphi_{\ell} \right)_p (k)}{\left( \varphi_{\ell, k}, \varphi_{\ell, \#} \right)_k} L (1/2, \Pi_k) \prod_{\ell | M} \left( 1 + \frac{1}{l} \right)^4.
\]

Remark 1.3 The \( p \)-adic period function \( \left( \varphi_{\ell, \#}, \varphi_{\ell} \right)_p \) is the tensor product of the three functions \( \left( \varphi_{\ell, \#}, \varphi_{\ell} \right)_p \equiv \frac{p+1}{\alpha_i} \left( \varphi_{\ell, \#}, \varphi_{\ell} \right) \) characterized by the interpolation property \( \left( \varphi_{\ell, \#}, \varphi_{\ell} \right)_p (k) = \mathcal{E}_p^\Omega (\alpha_i, k) \left( \varphi_{\ell, k}, \varphi_{\ell, \#} \right)_k \) (see Proposition 7.8). In the ordinary case, one can normalize the families \( \varphi_i \) and relate them to Hida’s canonical periods and congruence ideals (see [35, §2.1 and §2.2 and Proposition 6.4], [28, Theorem 0.1, Conjecture 0.2 (iii)], [27, Corollary 10.6], [26, Theorem 5.1] and [32, Proposition 10.1.1]).

Let us now explain how (2) is proved and the relevance of the assumption that we have done on the Coleman family. First, as explained, one problem is that there a priori no well defined interpolation problem because of the lack of a generic sign fixing a region of interpolation; second, as we will see, the special value formula requires test vectors and it is not clear that they move \( p \)-adically in general. In order to circumvent this issue, we are inspired by Ichino’s special value formula. Let us fix a (definite for our purposes) quaternion algebra \( B \), let \( B^\times \) be the algebraic group associated to its invertible elements with center \( B^\times \) and set \([B^\times (\Lambda)] \setminus B^\times (\Lambda) / B^\times \). Then Ichino’s formula takes the form

\[
I_B (\psi)^2 = \frac{C \zeta^2 (2) L (1/2, \Pi' (\psi))}{23 L (1, \Pi' (\psi), \text{Ad})} \prod_v I_v (\psi_v) = \frac{\left( \psi, \psi^\vee \right)_L^2 \zeta^2 (2) L (1/2, \Pi' (\psi))}{23 L (1, \Pi' (\psi), \text{Ad})} \prod_v C_v \psi^\vee \psi^\vee (\psi_v),
\]

where the notations are as follows. In the first equality we have that \( \psi = \otimes_v \psi_v \in \Pi (\psi) \) is an \( L^2 \)-automorphic form on \( B^\times ^3 \), that we assume to be in an irreducible
representation $\Pi (\psi)$ of $B^{\times 3}$, $\Pi' (\psi)$ is the automorphic representation of $GL_2^3$ which corresponds to $\Pi (\psi)$ via the Jacquet–Langlands correspondence, $I_B (\psi) = \int_{B^{\times (\Lambda)}} I_{B^{\times (\Lambda)}} (x) d \mu (B^{\times (\Lambda)}) \psi (x)$, $C$ is a non-zero constant defined in (22) below (which depends on the choice of local pairings) and $I_v (\psi_v)$ is defined in (25) [see (17) and the lines after (18) for the definition of $\psi \in \Pi (\psi) \vee$ appearing in (25)]. In the second equality we have determined $C = \frac{\langle \psi^\vee, \psi^\vee \rangle_{L^2}^v}{\Pi_v (\psi^\vee, \psi^\vee)}$, and defined $C_v^{\psi^\vee, \psi^\vee} (\psi_v) := \frac{I_v (\psi)}{\langle \psi_v^\vee, \psi_v^\vee \rangle_v^v}$ using auxiliary $\psi^\vee \in \Pi (\psi)$ and $\psi^\vee \in \Pi (\psi) \vee$ such that $\{\psi^\vee, \psi^\vee \rangle_{L^2} \neq 0$. Let us remark that, as explained in the proof of Theorem 3.4, formula (3) is a special case of a more general Ichino’s formula obtained choosing the dual vector $\psi^\vee = \tilde{\psi}$ in order to get the square $I_B (\tilde{\psi})^2$. Though qualitatively equivalent, the local constants $C_v^{\psi^\vee, \psi^\vee} (\psi_v)$ that appear in the second expression are more convenient to work with (see Remark 3.5). Although by a result of Prasad (see [34]) $I_B = 0$ on $\Pi (\psi)$ except in case $B = B_{\Pi (\psi)}$, a problem which is always meaningful is to try to interpolate the function $k \mapsto I_B (\psi_k)$ if $\psi_k$ is an $L^2$-automorphic form canonically attached to a vector valued modular form $\varphi_k = (\varphi_{1,k_1}, \varphi_{2,k_2}, \varphi_{3,k_3})$ on $B^{\times 3}$ (as explained below) which comes as the specialization of a $p$-adic family $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ on $B^{\times 3}$. Also, the Jacquet conjecture proved by Harris and Kudla (see [24] and Theorem 3.4 (2) below) tells us that, when $B = B_{\Pi (\psi_k)}$, there exists $\psi_k^0 \in \Pi' (\psi_k)$ such that the associated local constants $I_v (\psi_k^0)$ are non-zero and, hence

$$I_B (\psi_k^0) \neq 0 \iff L (1/2, \Pi' (\psi_k^0)) \neq 0. \quad (4)$$

In this case, we say that $\psi_k^0$ is a test vector. Formulated in this way our interpolation problem, we can remove all the assumptions that was done on the three Coleman families $I$ and give an unconditional $p$-adic Ichino’s formula analogous to (3). In order to state our main result, let us recall that $\varphi_i \in M^p (D_{k_i} (W), \omega_{0,p}^{k_i})$, where the notation is the following. Let $\Sigma_0 (p\mathbb{Z}_p) \subset M_2 (\mathbb{Z}_p)$ be the semigroup of matrices having non-zero determinant, upper left entry $a \in \mathbb{Z}_p^\times$ and lower left entry $c \in p\mathbb{Z}_p$ and set $\Gamma_0 (p\mathbb{Z}_p) := \Sigma_0 (p\mathbb{Z}_p) \cap GL_2 (\mathbb{Z}_p)$. The inclusion $U_i \subset X$ corresponds to a continuous character $k_i : \mathbb{Z}_p^\times \rightarrow \mathcal{O} (U_i)$ and one may consider the space $D_{k_i} (W)$ of locally analytic distributions on $W := \mathbb{Z}_p^\times \times \mathbb{Z}_p$ that are “homogeneous of weight $k_i$” (see Sect. 4.1 for the precise definition). Also, $\omega_{0,p}^{k_i} : Z_{B^{\times}} (A_\mathbb{F}) \rightarrow \mathcal{O} (U_i)$ is the character defined by the formula $\omega_{0,p}^{k_i} (z) := \omega_{l,i} (z) \left( \frac{z}{N_l (z)} \right)^{-k_i}_p$, where $N_l (z)$ is defined by the formula $z = N_l (z)$ is defined by the formula $z = N_l (z)$ with $\mathcal{O}_p^{N_l (z)}$ and $\mathfrak{z}^{N_l (z)} \in \mathbb{Z}_p$ (see also Sect. 2.1), $\omega_{l,i} (z)$ is the inverse of the adelization $\omega_{l,i}^{-1} (z) := \varepsilon_i \left( \frac{z}{N_l (z)} \right)$ of $\varepsilon_i$ and $(-)_p$ means that we take the $p$-component (which is indeed an element of $\mathbb{Z}_p^\times$).

1 The fact that the central characters are the inverse of the usual ones is due to the fact that the $L^2$-automorphic forms on $B$ that appear in Sect. 3.2 enjoy the equivariance property $f (xg) = f (x)$, as opposite.
Because $\Sigma_0(p\mathbb{Z}_p)$ acts on $\mathcal{D}_{k_i}(W)$ from the right (by right multiplication on the row vectors in $W$) one may form the space $M^\circ_p(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})$, which is a subspace of the space of those functions $\varphi_i : \mathcal{B}^\times (\mathcal{A}_f) \rightarrow \mathcal{D}_{k_i}(W)$ with the property that there exists an open and compact subgroup $K^p \subset \mathcal{B}^\times (\mathcal{A}_f)$ (here $\mathcal{A}_f$ is the prime to $p$ part of $\mathcal{A}_f$) such that $\varphi_i (zuxg) = \omega_0^{k_i} (z) \varphi_i (x) u_p^{-1}$ for every $z \in \mathbb{Z}_p \times (\mathcal{A}_f)$, $u \in K^p \Gamma_0\left(p\mathbb{Z}_p\right)$, $x \in \mathcal{B}^\times (\mathcal{A}_f)$ and $g \in \mathcal{B}^\times$. These spaces are naturally $\mathcal{B}^\times$ -modules and they are further endowed with the action of a $U_p$-operator (see the lines before Proposition 5.4): the superscript $(-)^\alpha$ refers to the $\alpha_i$-eigenspace for the $U_p$-operator. Also, if $k_i \in U_i$ is an irreducible representation $\pi_i \in \Pi_1$ (see (41)) satisfying the following properties (easily proved by means of the Ash-Stevens machinery [4], see also [9]): it respects the above actions, both $M^\circ_p(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})$ and $M^\circ(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})$ admit slope $\leq h$ decompositions for the $U_p$-operator (as defined in [4]) for every open and compact subgroup $K^p$ as above and the slope $\leq h$ parts $(-)^\alpha$ satisfy the control theorem $I_{k_i} \overset{M^\circ_p(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})}{\sim} M^\circ(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_i})$ for every $k + 1 > h$ (here $I_{k_i} \subset \mathcal{O}(U_i)$ is the ideal of functions vanishing at $k_i$). Let us write

$$M^\circ_p(U) := M^\circ_p(\mathcal{D}_{k_i}(W), \omega_{0,p}^{k_1}) \otimes M^\circ_p(\mathcal{D}_{k_2}(W), \omega_{0,p}^{k_2}) \otimes M^\circ_p(\mathcal{D}_{k_3}(W), \omega_{0,p}^{k_3}),$$

so that we focus on elements $\varphi \in M^\circ_p(U)$. Our main result is that there is a (unique up to sign) $\mathcal{O}(U)$ -valued $\mathcal{O}(U)$ -linear functional $L^\alpha_p(\varphi) : M^\circ_p(U) \rightarrow \mathcal{O}(U)$ such that, for every $\varphi \in M^\circ_p(U)$ which is the tensor product of three families, if $\varphi^\#_k$ belongs to an irreducible representation $\Pi(\varphi^\#_k)$, then

$$L^\alpha_p(\varphi)(k)^2 = E_p(a, k)^2 \frac{\gamma^2_\varphi(2)}{2^{3/2}} \frac{L\left(1/2, \Pi^\ldots(\varphi^\#_k)\right)}{L\left(1, \Pi^\ldots(\varphi^\#_k), \text{Ad}\right)} \prod_v I_v(\varphi^\#_k).$$

Footnote 1 continued to the usual convention $f(ux) = f(x)$. Thus we consider right $\mathcal{B}^\times$ action $(fu)(x) := f(ux)$ on them, rather than the usual left action $(uf)(x) := f(xu^{-1})$. The rule $f^*(x) := f(x^{-1})$, which satisfies $(fu)^* = uf^*$, exchange the two spaces, but the central characters of the corresponding spaces are reversed.
\[
\mathcal{E}_p (\omega, k)^2 \frac{\left( \varphi_k^\# , \varphi_k^{\# \#} \right)_k}{2L \left( 1, \Pi' \left( \varphi_k^\# \right), \text{Ad} \right)} L \left( 1/2, \Pi' \left( \varphi_k^\# \right) \right) \prod_{v \neq \infty, p} C_v \varphi_k^\# \varphi_k^{\# \#} \left( \varphi_k^\# \right),
\]

(5)

where \( \varphi_k^\# = (\varphi_{k,1}^\#, \varphi_{k,2}^\#, \varphi_{k,3}^\#) \) is the unique triple which componentwisely has \( p \)-stabilization \( \varphi_k \), \( \Pi' \left( \varphi_k^\# \right) = \Pi' \left( \psi_k \right) \), \( C_k = C \) is defined in (22) below, \( I_v (\varphi_k^\#) \) and \( C_v \varphi_k^\# \varphi_k^{\# \#} \left( \varphi_k^\# \right) \) are again defined in (25) and, as in (3) above, the second equality holds with the auxiliary choice of vectors \( \varphi_k^\# \) and \( \varphi_k^{\# \#} \) such that \( \left( \varphi_k^\# , \varphi_k^{\# \#} \right)_k \neq 0 \) and satisfying a local condition at \( p \) [see the lines before Theorem 7.3]. This is proved in Theorems 7.3 and 8.3, from which (2) is deduced as a special case [see also Sect. 8.3], is obtained by providing conditions on the Coleman families under which one knows a priori that \( B = B_{D_k} \) for every generic integer \( k \), an explicit test vector \( \varphi \) moving in families can be written down for which the corresponding local constants \( C_v \varphi_k^\# \varphi_k^{\# \#} \left( \varphi_k^\# \right) \) has been computed (that is, we take \( \varphi_k^\# = \varphi_k^{\# \#} = \varphi_k^{(\# \#)} \)) and then relating \( \left( \varphi_k^\# , \varphi_k^{\# \#} \right)_k \) to \( \left( \varphi_k^\# , \varphi_k^{(\# \#)} \right)_k \) [see (76)].

Let us write \( M_p^\circ \left( \mathcal{O}, \varphi_{\mathcal{O}}^\#: \left( \mathfrak{v}_k \right) \right) \subset M_p^{\alpha} \left( \mathcal{U} \right) \) for the \( \mathbf{B}^\times \left( \mathfrak{k}_P \right) \)-representation generated by \( \varphi_{\mathcal{O}} \) over \( \mathcal{O} \left( \mathcal{U} \right) \) and suppose that there is some generic integer point \( k^0 \) such that \( L \left( f_{1,k^0}, f_{2,k^0}^\#, f_{3,k^0}^\#, c_{\mathcal{O}}^\# \right) \neq 0 \) for \( B = B_{D_{k^0}} \). Then we see from (5), (4) and Remark 1.1 that \( L_p^{\alpha} \neq 0 \) as a functional on \( M_p^\circ \left( \mathcal{U}, \varphi_{\mathcal{O}}^\#: \left( \mathfrak{v}_k \right) \right) \) and, hence, there is some \( \varphi \in M_p^\circ \left( \mathcal{U}, \varphi_{\mathcal{O}}^\#: \left( \mathfrak{v}_k \right) \right) \) such that \( L_p^{\alpha} (\varphi) \neq 0 \). In particular, we see that \( B = B_{D_k} \) and that \( \varphi_k \) is a test vector for every generic integer point \( k \) in a Zariski open subset of \( \mathcal{U} \), by (5) and Remark 1.1.

Let us briefly explain how Theorem 7.3 is proved. First, the \( L^2 \)-automorphic forms to which \( \psi \) belongs are related to the vector valued modular forms to which \( \varphi_k \) described above belongs; via this identification, the integral \( I_B \left( \psi \right) \) that appear in the left hand side of the Ichino’s formula is related to a functional \( t_k \) on vector valued modular forms (see Sect. 3 and Theorem 3.4). This is done by appealing to the results of [22], which set up a general formalism for getting such a kind of results in the setting of Gan-Gross-Prasad conjectures when the real points of the algebraic group are compact modulo the center. This linear functional \( t_k \) is obtained by evaluating the vector valued forms at the product \( \Delta_{E/k_E} \in \mathcal{P}_{k_1,F} \otimes \mathcal{P}_{k_2,F} \otimes \mathcal{P}_{k_3,F} \) of certain powers of determinants [see (13)]: the resulting formula could be viewed as an analogous of the Hatcher’s formula in our setting (which indeed can be deduced from Waldspurger’s formula via the method of [22]). It turns out that \( t_k \) can not be deformed \( p \)-adically, but it is closely related to three linear forms \( t_{k,j}^\# \) (only defined on distribution valued modular forms) which can be easily moved \( p \)-adically. The relationship between \( t_k \)
and $t_{i,k}^{\alpha}$ is that they differ by the Euler factor $E_{p,i}(\alpha, k)$ and the action of an operator $W_p$ (closely related to the Atkin-Lehner operator). For example (see Corollary 5.6 and (46)):

$$t_{3,k}^{\alpha}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mid W_p) = E_{p,3}(\alpha, k) t_k(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mid W_p).$$

Expressing $t_k(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mid W_p)$ in terms of $t_k(\varphi_k^\#)$ gives rise to another Euler factor $\alpha_i E_{p,i}(\alpha, k)$ such that $E_{p,i}(\alpha, k) E_{p,i}(\alpha, k) = E_p(\alpha, k)$ does not depend on the choice of $i = 1, 2, 3$ (see Proposition 6.5). The result is that, writing $L_p^{\alpha}(\varphi)(k)$ for any one of $\frac{p+1}{\alpha_1} t_{1,k}^{\alpha}(\varphi_1 \mid W_3 \otimes \varphi_2 \otimes \varphi_3)$, $\frac{p+1}{\alpha_2} t_{2,k}^{\alpha}(\varphi_1 \otimes \varphi_2 \mid W_3 \otimes \varphi_3)$ or $\frac{p+1}{\alpha_3} t_{2,k}^{\alpha}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mid W_3)$, all of them satisfy the same interpolation property

$$L_p^{\alpha}(\varphi)(k) := E_p(\alpha, k) t_k(\varphi_k^\#);$$

by Zariski density of the integer points, these $p$-adically moving quantities needs to be the same and this interpolation formula uniquely characterize $L_p^{\alpha}(\varphi)$ (thus fixing the sign). Applying our vector valued version of Ichino’s formula (Theorem 3.4), (5) is deduced from (6). Let us remark that one feature of our asymmetric construction is that one gets in a natural way improved $p$-adic $L$-functions defined on appropriate improving planes (see Proposition 7.7).

**Remark 1.4** The local constants $C_v^{\psi, \psi, \psi, \psi}(\varphi_v, \psi_v)$ that appear in (24) below have been quite largely studied in the literature about subconvexity problems when $\psi_v = \psi_v$ and $\psi_v = \psi_v$ (see [30,43]). They appear when one specializes Ichino’s formula (27) to the case $\psi_v = \psi$: then $I_B(\psi)^2$ (of (3)) is replaced by $|I_B(\psi)|^2$ and $\langle \psi, \psi \rangle_{L^2}$ by $\langle \psi, \psi \rangle_{L^2}$, which is always non-zero. Our assumption that the nebetype are trivial in Theorem 8.3 allows us to appeal to this existing literature. Time after our works was completed, Hsieh has successfully computed the completion of the local constants $C_v^{\psi, \psi}(\varphi_v, \psi_v) \neq 0$ taking $\psi_v$ and $\psi_v$ new vectors as defined in Example 3.3 and explicit vectors $\psi_v$ linearly depending on $\psi_v$ (see [29, §6.1]). Rather than appealing to his results, we illustrate the method for specifying the local constants in (5) in order to get Theorem 8.3 in a simplified setting, allowing us to appeal to the previously existing literature and make an easier choice of test vectors. In the ordinary case, Hsieh has (also) given a construction of the balanced triple product $p$-adic $L$-functions based on our vector valued version of Ichino’s formula (Theorem 3.4) involving the trilinear form $t_k$ (see [29, Proposition 4.10]). He was able, in this case, to give a very nice Gross’ style interpretation of our $p$-adically moving trilinear form $t_{3,k}^{\alpha}$ (cfr. the proof of [29, Proposition 4.9] with our Proposition 5.4 and Corollary 5.6).

**Remark 1.5** Suppose that $k : \mathbb{Z}_p^\times \to \mathcal{O}_p^\times$ and $k' : \mathbb{Z}_p^\times \to \mathcal{O}_p^\times$ valued in (the invertible elements of) $F$-Banach algebras and are two continuous homomorphisms and that $\phi : \mathcal{O} \to \mathcal{O}'$ is a continuous homomorphism of $F$-algebras with the property that $k' = \phi \circ k$. Then it is easy to see that the canonical map $M_p(\mathcal{D}_k(W), \phi_{0,p}^k) \otimes_{\mathcal{O}_p} \phi \mathcal{O}' \to \cdots$ Springer
$M^\otimes_p(D_k(W), \omega^k_{0,p})$ is an isomorphism. In particular, the definition of $M^\otimes_p(U)$ can be uniquely extended to arbitrary admissible open subsets of $\mathcal{X}$. Correspondingly, it follows from the fact that our construction of $L^\otimes_{p,U}$ behaves well with respect to base changes that we can uniquely extend its definition to arbitrary admissible open subsets of $\mathcal{X}$ and that the resulting functional is uniquely characterized by the interpolation property (6) and, up to sign, by (5).

Suppose that $k$ is an integer point which is not generic: then at least one of the forms $\varphi_i, k_i$ is new at $p$. We compute the Euler factors in this case in Proposition 7.5. Our interest is motivated by the forthcoming work [6] in which exceptional zero phenomena are investigated, an analogue of those discovered in [33] and studied in [23] (see Remark 7.6). We will give an algebraic interpretation of these results in the framework of Nekovar-style weight pairings as defined in [39] (see also [40] and [41]). Particularly interesting is the case where a local change of sign at $p$ produces an extra vanishing due to the complex $L$-function (see Remark 7.6): we relate the derivatives of our $p$-adic $L$-function to the Abel-Jacobi image of diagonal cycles.

2 Modular forms and $p$-adic modular forms

Let $B$ be a definite quaternion division $\mathbb{Q}$-algebra which is split at the prime $p$ and let $B$ (resp. $B\times$) be the associated ring scheme (resp. algebraic group). We write $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ for the adele ring of $\mathbb{Q}$ and define $\mathbb{A}_f^p$ by the rule $\mathbb{A}_f = \mathbb{A}_f^p \times \mathbb{Q}_p$. We set $B_f := B (\mathbb{A}_f)$ (resp. $B_f^\times := B\times (\mathbb{A}_f)$), $B_f^{\times,p} := B (\mathbb{A}_f^p)$ and $B_v := B (\mathbb{Q}_v)$ (resp. $B_v^\times := B\times (\mathbb{Q}_v)$) if $v$ is either a finite place or $v = \infty$, so that $B_f^\times = B_f^{\times,p} \times B_p^\times$. We write $b \mapsto b^i$ for the main involution and $\text{ndr} : B\times \to G_m$ for the reduced norm.

If $Z \subset Z_{B\times} = G_m$ is a closed subgroup (such as the trivial subgroup or the whole center), we define $Z_f := Z (\mathbb{A}_f)$, $Z_v := Z (\mathbb{Q}_v)$ and $Z_f^p := Z (\mathbb{A}_f^p)$, so that $Z_f = Z_f^p \times Z_p$. We will need to consider double cosets of the form

$$[B\times (\mathbb{A})]_Z := Z (\mathbb{A}) \setminus B\times (\mathbb{A}) / B\times \quad \text{and} \quad [B_f^\times]_Z := Z_f \setminus B_f^\times / B_f^\times.$$ 

In order to later apply the results from [22], we fix measures as follows. We take the Tamagawa measure $\mu_{Z(\mathbb{A}) \setminus B\times (\mathbb{A})}$ on $Z (\mathbb{A}) \setminus B\times (\mathbb{A})$ and write $\mu_{[B\times (\mathbb{A})]_Z}$ for the quotient measure (normalized in the usual way). Next we choose $\mu_{Z(\mathbb{R}) \setminus B\times (\mathbb{R})}$ and $\mu := \mu_{B_f^\times}$ on $B_f^\times$ such that $\mu (K) \in \mathbb{Q}$ for some (and hence every) open and compact subgroup $K \subset B_f^\times$ and such that, writing $\mu_{B_f^\times / B\times}$ for the induced quotient measure on $B_f^\times / B\times$ (normalized in the usual way), which restricts to an invariant measure $\mu_{[B_f^\times]_Z}$ on $C (B_f \setminus B_f^\times / B\times) \subset C (B_f^\times / B\times)$,

$$\int_{[B\times (\mathbb{A})]_Z} f (x) d\mu_{[B\times (\mathbb{A})]_Z} (x) = \int_{[B_f^\times]_Z} \left( \int_{Z(\mathbb{R}) \setminus B\times (\mathbb{R})} f (x, x_\infty) d\mu_{Z(\mathbb{R}) \setminus B\times (\mathbb{R})} (x_\infty) \right) d\mu_{[B_f^\times]_Z} (x_f) \quad (7)$$
is satisfied. We let \( m_{\mathbb{Z}^{\mathbb{R}},\infty} \) be the total measure of \( \mathbb{Z} (\mathbb{R}) \backslash \mathbb{B}^{\infty} (\mathbb{R}) \).

Let \( \Sigma_0 (p\mathbb{Z}_p) \subset M_2 (\mathbb{Z}_p) \) be the subsemigroup of matrices having non-zero determinant, upper left entry \( a \in \mathbb{Z}_p^{\times} \) and lower left entry \( c \in p\mathbb{Z}_p \) and set \( \Gamma_0 (p\mathbb{Z}_p) := \Sigma_0 (p\mathbb{Z}_p) \cap \text{GL}_2 (\mathbb{Z}_p) \). Consider an open and compact subgroup \( K_p^{\infty} \subset B_p^{\infty} \) (it will be \( \Gamma_0 (p\mathbb{Z}_p) \) in our applications). We will also need to consider a subsemigroup \( K_p^{\infty} \subset \Sigma_p \subset B_p^{\infty} \) and to define \( \Sigma_p \left( B_f^{\infty} \right) := B_f^{\infty:p} \times \Sigma_p \) (we will take \( \Sigma_p = K_p^{\infty}, \Sigma_0 (p\mathbb{Z}_p) \) or \( B_p^{\infty} \)).

Let \( K := K \left( B_f^{\infty} \right) \) (resp. \( K^{\infty} := K \left( B_f^{\infty}, K_p^{\infty} \right) \)) be the set of open and compact subgroups \( K \subset B_f^{\infty} \) (resp. \( K = K^p \times K_p \) with \( K^p \subset B_f^{\infty:p} \) and \( K_p \subset K_p^{\infty} \) open and compact). If \( S \) is a \( B_f^{\infty} \)-module (resp. a \( \Sigma_p \left( B_f^{\infty} \right) \)-module), then we define

\[
S^K := \bigcup_{K \in K} S^K \quad \text{(resp. } S^{K^{\infty}} := \bigcup_{K \in K^{\infty}} S^K).
\]

We note that the Hecke operators \( \mathcal{H} \left( B_f^{\infty} \right) \) (resp. \( \mathcal{H} \left( \Sigma_p \left( B_f^{\infty} \right) \right) \)) act on \( S^K \) (resp. \( S^{K^{\infty}} \)) by double cosets of elements of \( B_f^{\infty} \) (resp. \( \Sigma_p \left( B_f^{\infty} \right) \)). We describe the action on \( S^{K^{\infty}} \) for a \( \Sigma_p \left( B_f^{\infty} \right) \)-module \( S \) (the action of \( S^K \) is similar). If \( K_1, K_2 \in K^{\infty} \) and \( \pi \in \Sigma_p \left( B_f^{\infty} \right) \), the space \( K_1 \backslash K_1 \pi K_2 \) is finite\(^2\) and we may write

\[
K_1 \pi K_2 = \bigsqcup_{x \in K_1 \backslash K_1 \pi K_2} K_1 x.
\]

As usual, we may define

\[
\cdot | K_1 \pi K_2 : S^{K_1} \to S^{K_2}
\]

by the rule

\[
v | K_1 \pi K_2 = \sum_{x \in K_1 \backslash K_1 \pi K_2} v x.
\]

We can define in this way an action of the Hecke algebra \( \mathcal{H} \left( \Sigma_p \left( B_f^{\infty} \right) \right) \) of elements of \( \Sigma_p \left( B_f^{\infty} \right) \). When \( K^p = B_p^{\infty} \), we have \( V^{K^{\infty}} = V^K \) and we have an action of \( \mathcal{H} \left( \Sigma_p \left( B_f^{\infty} \right) \right) = \mathcal{H} \left( B_f^{\infty} \right) \). Let \( K^{\infty : p} \subset K^{\infty} \) be the subset of those groups such that \( K_p = K^p \) and write \( \mathcal{H} \left( \Sigma_p \right) \) for the Hecke algebra of double cosets \( K \pi K \) with \( \pi \) concentrated in \( \pi_p \in \Sigma_p \) and \( K \in K^{\infty : p} \). Then (8) defines an operator on \( V^{K^{\infty : p}} = \left( V^K \right)^{K^p} \) by means of the formula \( vU^\pi := v | K \pi K \) if \( v \in V^K \) where \( K \in K^{\infty : p} \), i.e. it does not depend on \( K \in K^{\infty : p} \). It follows that \( V^{K^{\infty : p}} \) is endowed with an action of \( B_f^{\infty:p} \times \mathcal{H} \left( \Sigma_p \right) \):

\(^2\) Indeed note that \( K_1 \pi K_2 \) is compact, being the image of \( K_1 \times K_2 \) by means of the continuous map given by \( (x, y) \mapsto x \pi y \). Since \( K_1 \) is open, \( K_1 \pi K_2 = \bigsqcup_i K_1 \pi_i \) is an open covering which, by compactness, admits a finite refinement.
write $\widehat{\pi}_p$ for the idèle concentrated at $p$, where we have $(\widehat{\pi}_p)_p = \pi_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$; then $V^{K^{\infty}}$ is endowed with an action of the operator $U_p := U\widehat{\pi}_p$.

Let $(V, \rho)$ be a right representation of $G_{\infty} \in \{ B^{\times}, B^{\times} \}$ (resp. $\Sigma_p$) with coefficients in some commutative unitary ring $R$. If $g \in B^{\times} (\AA)$, we will write $g_v \in B^{\times}_v$ for its $v$-component. When $\rho$ is understood, we simply write $v\rho (g_\infty)$ (resp. $v\rho (g_p)$). Fix a character $\omega_0 : \mathbb{Z}_f \rightarrow \mathbb{R}^\times$ (resp. $\omega_{0,p} : \mathbb{Z}_f \rightarrow \mathbb{R}^\times$). Define $S (B^{\times}_f, \rho)$ (resp. $S_p (B^{\times}_f, \rho, \omega)$) to be the space of maps $\phi : B^{\times}_f \rightarrow V$ endowed with the $(B^{\times}_f, B^{\times}_f)$-action (resp. $(B^{\times}, \Sigma_p (B^{\times}_f))$-action) given by

$$(g\phi u) (x) := \phi (uxg_f) \rho \left( g^{-1}_\infty \right), \text{ where } g \in B^{\times} \text{ and } u \in B^{\times}_f$$

$$(\text{resp. } (g\phi u) (x)) := \phi (uxg_f) \rho \left( u_p \right), \text{ where } g \in B^{\times} \text{ and } u \in \Sigma_p (B^{\times}_f).$$

Then

$$S (B^{\times}_f, \rho, \omega_0) := \{ \phi \in S (B^{\times}_f, \rho) : \phi (zx) = \omega_0 (z) \phi (z) \text{ for all } z \in \mathbb{Z}_f \}$$

(resp. $S_p (B^{\times}_f, \rho, \omega_{0,p}) := \{ \phi \in S_p (B^{\times}_f, \rho) : \phi (zx) = \omega_{0,p} (z) \phi (x) \text{ for all } z \in \mathbb{Z}_f \}$)

is a sub $(B^{\times}, B^{\times}_f)$-module (resp. sub $(B^{\times}, \Sigma_p (B^{\times}_f))$-module). We also write

$$S (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_0) := S (B^{\times}_f, \rho, \omega_0) \left( B^{\times}, 1 \right) \left( \text{resp. } S_p (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_{0,p}) \right)$$

$$:= S_p (B^{\times}_f, \rho, \omega_{0,p}) \left( B^{\times}, 1 \right)$$

and

$$M (B^{\times}_f, \rho, \omega_0) := S (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_0) \left( 1, K \right) \left( \text{resp. } M_p (B^{\times}_f, \rho, \omega_0) \right)$$

$$:= S_p (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_{0,p}) \left( 1, K^{\infty} \right).$$

The former is called the space of $\rho$-valued modular forms and the latter the space of $\rho$-valued $p$-adic modular forms; they are Hecke modules as explained above. Also, setting

$$M^\omega (B^{\times}_f, \rho, \omega_{0,p}) := S_p (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_{0,p}) \left( 1, K^{\infty} \right) = M_p (B^{\times}_f, \rho, \omega_{0,p}) \left( K^{\infty}_p \right),$$

$$M^\omega (B^{\times}_f, \rho, \omega_0) := S (B^{\times}_f / B^{\times}, \rho / \rho_{/B^{\times}}, \omega_0) \left( 1, K^{\infty} \right) = M (B^{\times}_f, \rho, \omega_0) \left( K^{\infty}_p \right).$$

we get a $B^{\times} \times \mathcal{H} (\Sigma_p)$-module, as explained above. We omit $\omega_0$ from the notation when $\mathbb{Z}_f = 1$ and write $M (\mathbb{Z}_f \backslash B^{\times}_f, \rho) := M (B^{\times}_f, \rho, \omega_{0,p})$ when $\omega_0$ is the trivial character of $\mathbb{Z}_f$. Sometimes we will abusively replace $\rho$ with the underlying subspace $V$ in the notation. The same shorthands apply in the $p$-adic case.

The following remarks are easily checked.
Remark 2.1 Suppose that $\chi_0 : B_f^\times \to R^\times$ is a character with the property that $\chi_0(K) = 1$ for some $K \subseteq K$ and that $\chi_\infty : G_\infty \to R^\times$ is a character with the property that $\chi_{0|B^\times} = \chi_\infty|B^\times$.

1. If $\varphi \in M \left(B_f^\times, \rho, \omega_0\right)$, then the rule $(\chi_0 \varphi)(x) := \chi_0(x) \varphi(x)$ defines an element $\chi_0 \varphi \in M \left(B_f^\times, \rho(\chi_\infty), \chi_0|Z\omega_0\right)$.
2. We have $\chi_0 \in M \left(B_f^\times, R(\chi_\infty), \chi_0|Z\right)$.

Remark 2.2 Suppose that $\chi_{0,p} : B_f^\times \to R_p^\times$ and $\chi_p : \Sigma_p \to R_p^\times$ are characters such that there is some $K \subseteq K^\circ$ such that $\chi_{0,p}(u) = \chi_p(u_p)^{-1}$ every $u \in K$ and $\chi_{0,p|B^\times} = 1$.

1. If $\varphi \in M_p \left(B_f^\times, \rho, \omega_{0,p}\right)$, then the rule $(\chi_{0,p} \varphi)(x) := \chi_{0,p}(x) \varphi(x)$ defines an element $\chi_{0,p} \varphi \in M_p \left(B_f^\times, \chi_p \rho, \chi_{0,p}|Z\omega_{0,p}\right)$.
2. We have $\chi_{0,p} \in M \left(B_f^\times, R(\chi_p), \chi_{0,p}|Z\right)$.

The connection between modular forms and $p$-adic modular forms is the content of the following proposition. We suppose that we are given a representation $\rho \in R$ and the coefficient rings $i_\infty : R \subseteq R_\infty$ and $i_p : R \subseteq R_p$. For a character $\chi$ of some group with values in $R^\times$, we let $i_p(\chi) := i_p \circ \chi$ and $i_\infty(\chi) := i_\infty \circ \chi$. We also assume that we are given a representation $\rho_p$ (resp. $\rho_\infty$) of $B_p^\times$ (resp. $B_\infty^\times$) with coefficients in $R_p$ (resp. $R_\infty$) with the property that

$$\rho := \rho_p|B^\times = \rho_\infty|B^\times \subseteq \rho_p, \rho_\infty$$

takes coefficients in $R$: we distinguish between the $R$-valued representation $\rho$ of $B^\times$ and the $R_p$-valued representation $\rho_p|B^\times$ of $B^\times$.

Lemma 2.3 The rules

$$M \left(\chi_f^\times, \rho_p|B^\times\right) \to M_p \left(\chi_f^\times, \rho_p\right) \quad M_p \left(\chi_f^\times, \rho_p|B^\times\right) \to M \left(\chi_f^\times, \rho_p\right)$$

set up a right $\Sigma_p \left(\chi_f^\times\right)$-equivariant bijection and $M \left(\chi_f^\times, \rho\right) \subseteq M \left(\chi_f^\times, \rho_p|B^\times\right)$ is identified with the submodule of those $\psi \in M_p \left(\chi_f^\times, \rho_p\right)$ such that $\psi(x) \in \rho \subseteq \rho_p$ for every $x \in \chi_f^\times$. Furthermore, if $\rho_p$ has central character $\omega_{p,0}$ and $\left(-\right) : Z_f \to Z_p$ is the projection induced by $\chi_f^\times \to \chi_p^\times$, then the bijection induces

$$M \left(\chi_f^\times, \rho, \omega_0\right) \subseteq M \left(\chi_f^\times, \rho_p|B^\times, i_{p*}(\omega_0)\right) \cong M_p \left(\chi_f^\times, \rho_p, \omega_{0,p}\right)$$

with $\omega_{0,p} := i_{p*}(\omega_0) \omega_p^{-1}(\left(-\right)\rho)$. These identifications and inclusions are $\mathcal{H}(\Sigma_p \left(G_f\right))$-equivariant.

Proof Indeed the above rules induce a $\left(B^\times, \Sigma_p \left(\chi_f^\times\right)\right)$-equivariant identification $S \left(\chi_f^\times, \rho_p|B^\times\right) \cong S_p \left(\chi_f^\times, \rho_p\right)$. Since $B_f = B_f^\times \times B_f^\times$ topologically, $K^\circ \subseteq K$ is a cofinal family and we have $S^K = S^{K^\circ}$ for every $B_f^\times$-module. Hence, taking
Example 2.4 The above lemma notably applies in the following setting: let $V$ be an algebraic representation of $B^\times$ over $R = \mathbb{Q} \subset \mathbb{C}, \mathbb{Q}_p$ (or a quadratic field $R = \mathbb{K} \subset \mathbb{C}, \mathbb{Q}_p$ which splits $B$) and set $(V, \rho) := V(\mathbb{Q})$ (or $V(\mathbb{K}), (V_\infty, \rho_\infty) := V(\mathbb{C})$ (with the action restricted to $B_\infty^\times \subset B^\times (\mathbb{C})$) and $(V_\infty, \rho_p) := V(\mathbb{Q}_p)$. We can also take $R$ large enough for the values of the characters $\omega_\rho$ to take values in it (and replace $\mathbb{Q}_p$ by a finite extension $F$ and consider the action restricted to $B_p^\times \subset B^\times (F)$).

Suppose that we are given $\omega_0 : Z_f \to R^\times$ (resp. $\omega_{0,p} : Z_f \to R_p^\times$) and write $X(B^\times, \omega_0)$ (resp. $X_p (B^\times, \omega_{0,p})$) to denote the set of couples $(\chi_0, \chi_\infty)$ as in Remark 2.1 (resp. Remark 2.2) such that $\chi_0|_Z = \omega_0$ (resp. $\chi_{0,p}|_Z = \omega_{0,p}$). We also suppose, in the following remark, that we are given characters $\chi_0 : B_1^\times \to R^\times, \chi_p : B_p^\times \to R_p^\times$ and $\chi_\infty : G_\infty \to R_\infty^\times$ such that $\chi := \chi_{p|B^\times} = \chi_\infty|_{B^\times : B^\times \to R^\times}$ and that $\chi_p$ and $\chi_\infty$ are continuous with respect to topologies on $R_p$ and, respectively, $R_\infty$. Then the condition $\chi_{p|B^\times} = \chi_\infty|_{B^\times}$ implies that $\chi_p$ and $\chi_\infty$ determine each other.

Remark 2.5 We have $(\chi_0, \chi) \in X(B^\times, \omega_0)$ (equivalently, $(i_{\infty,*}(\chi_0), \chi_\infty) \in X(B^\times, i_{\infty,*}(\omega_0))$) if and only if $(\chi_{0,p}, \chi_p) \in X_p(B^\times, \omega_{0,p})$, where $\chi_{0,p}(x) := i_{p*}(\chi_0)(x) \chi_{p}^{-1}(x_p)$ and $\omega_{0,p}(z) := i_{p*}(\omega_0)(z) \chi_{p}^{-1}(z_p)$. In this case, regarding $\chi_0$ (resp. $\chi_{0,p}$) as modular forms via Remark 2.1 (2) (resp. Remark 2.2 (1)), we have that $\chi_0$ corresponds to $i_{p*}(\chi_p) \simeq \chi_{0,p}$ via the inclusion/identifications $\chi_0 \in M(G_0, R(\chi), \omega_0) \subset M(G_0, R_p(\chi_p), i_{p*}(\omega_0)) \simeq M_p(G_0, R_p(\chi_p), \omega_{0,p})$ provided by Lemma 2.3. Furthermore, via the inclusions/identifications provided by Lemma 2.3, twisting by $\chi_0$ (or $i_{p*}(\chi_p)$) as in Remark 2.1 (1) corresponds to twisting by $\chi_{0,p}$ as in Remark 2.2 (2).

2.1 The norm forms

Here is a key example of modular form. Consider the (normalized) absolute value functions $|-|_V : \mathbb{Q}_v^\times \to \mathbb{R}^\times_+, |-|_{A_f} : A_f^\times \to \mathbb{Q}_v^\times_+$ and $|-|_A : A^\times \to \mathbb{R}_+^\times$. Setting

$N := |-|_{A_f}^{-1}|-|_A : A^\times = G_m(A) \to \mathbb{C}^*$

gives a function such that $N_f N_\infty^{-1} = |-|_A^{-1}$ is trivial on $\mathbb{Q}^\times = G_m(\mathbb{Q})$ by the product formula. Suppose that $\chi : B^\times \to G_m$ is an algebraic character and that $\tau : R^\times \to G$ is a character. Then we define $\tau_\chi : B^\times (R) \xrightarrow{\chi_R} R^\times \xrightarrow{\tau} G$. In particular, we have the continuous character

$N_\chi : B^\times (A) \xrightarrow{\chi_A} A^\times \xrightarrow{N} \mathbb{R}_+^\times$

and, recalling that $N_f = |-|_{A_f}^{-1}$ and $N_\infty = |-|_\infty$,\n
\[\square\ Springer\]
\[ N_{\chi,f} : B^\times (\mathbb{A}_f) \xrightarrow{\chi_{A_f}} \mathbb{A}_f^\times \xrightarrow{|-|_{A_f}} Q_+^\times \text{ and } N_{\chi,\infty} : B^\times (\mathbb{R}) \xrightarrow{\chi_{\infty}} \mathbb{R}^\times \xrightarrow{|-|_\infty} R_+^\times. \]

Of course \( N_{\chi,f} \) (resp. \( N_{\chi,\infty} \)) is the finite adele (resp. \( \infty \)) component of \( N_{\chi} \), as suggested by the notation. If \( \kappa : Q_+^\times \rightarrow R^\times \) is a character (that we usually write exponentially \( r \mapsto r^\kappa \)), we can also define

\[ N_{\kappa \chi,f} : G(\mathbb{A}_f) \xrightarrow{N_{\chi,f}} Q_+^\times \xrightarrow{\kappa} R^\times. \]

Note that \( \chi_\infty (B^\times (\mathbb{R})) = \chi_\infty (B^\times (\mathbb{R})^\circ) \subset \mathbb{R}_+^\times \) (because \( B \) is definite), implying that \( \chi_\mathbb{Q} (B^\times (\mathbb{Q})) \subset Q_+^\times \) and we may consider \( \kappa_{\chi} := \kappa \circ \chi_{\mathbb{Q}} \). If \( V = (V, \rho) \) is a representation of \( G(\mathbb{R}) \) with coefficients in \( R \), we write \( V(\kappa_{\chi}) = (V, \rho (\kappa_{\chi})) \) for the representation \( \rho (\kappa_{\chi}) (g) (v) := \kappa_{\chi} (g) \rho (g) v. \)

**Remark 2.6** The continuous character \( N_{\chi} \) is such that \( N_{\chi,f} N_{\chi,\infty}^{-1} \) is trivial on \( B^\times (\mathbb{Q}) \) and we have

\[ N_{\kappa \chi,f} \in M \left( B^\times (\mathbb{A}_f), R (\kappa_{\chi}), N_{\chi,f,Z_f}^{\kappa} \right)_K \]

for every open and compact \( K \in \mathcal{K} \).

**Proof** This is an application of the product formula and the fact that \( \chi_\mathbb{Q} (B^\times (\mathbb{Q})) \subset Q_+^\times \), implying that Remark 2.1 (2) applies with \( (\chi_0, \chi_\infty) = (N_{\chi,f}^{\kappa}, N_{\chi,\infty}^{\kappa}) \) and \( N_{\chi,\infty}^{\kappa} := \kappa \circ N_{\chi,\infty} = \kappa \circ \chi_{\mathbb{Q}}. \)

Taking \( \chi = \text{nr} d : B^\times \rightarrow G_m \)

yields, for every \( \kappa = k \in \mathbb{Z} \) (viewed as the character \( k : \mathbb{Q}^\times \rightarrow R \) via \( r \mapsto r^k \)), the norm form

\[ \text{Nrd}_{\chi,f}^k := N_{\chi,f}^k \in M \left( G(\mathbb{A}_f), \mathbb{Q} (k), N_{\ell,Z_f}^{2k} \right)_K, \text{ for every } K \in \mathcal{K}. \]

We also write \( \text{Nrd}_{\chi,\infty}^k := N_{\chi,\infty}^k \) in this case. Applying Lemma 2.3 with \( \rho = \mathbb{Q} (k), \rho_p = \mathbb{Q}_p (k) \) and \( \varphi = \text{Nrd}_{\chi}^k (\varphi)_{\rho} \in M \left( G(\mathbb{A}_f), \mathbb{Q}_p (k), 2k \right)_K \) yields the \( p \)-adic modular form

\[ \text{Nrd}_p^k := \psi_\varphi \in M_p \left( G(\mathbb{A}_f), \mathbb{Q}_p (k), N_{\ell,Z_f}^{2k} \right)_K, \text{ for every } K \in \mathcal{K}. \]

We have, explicitly, writing \( (\varphi)_{\rho} \) for the \( p \)-component of an adelic element (and viewing the rational numbers diagonally in \( \mathbb{A}_f \)).
\[ \text{Nrd}_p^k(x) = \text{Nrd}_f^k(x)_p x^{-1} = \left( \frac{\text{Nrd}_f(x)_p}{\text{nrd}_p(x)_p} \right)^k = \left( \frac{\text{Nrd}_f(x)}{\text{nrd}_A(x)} \right)_p^k \quad \text{and} \]

\[ \text{N}_p(z) = \left( \frac{\text{N}_f(z)}{z} \right)_p. \]

We now remark that \( \text{Nrd}_f^k(x)_p \in \mathbb{Z}_p^\times \) for any \( x \in B_1^\times \). Suppose now that we are given \( k : \mathbb{Z}_p \to \mathbb{R}_\times \) which is a continuous group homomorphism, with \( \mathcal{O} \) a locally convex \( \mathbb{Q}_p \)-algebra. Since \( \mathcal{K}_p \subset B_1^\times \) is a compact subgroup, \( \text{nrd}_p \) maps it into the maximal open compact subgroup \( \mathbb{Z}_p^\times \subset \mathcal{O}_\times^\times \):

\[ \text{nrd}_p : \mathcal{K}_p \to \mathbb{Z}_p^\times. \]

If \( D \) is a \( \mathcal{K}_p^\times \)-module with coefficients in \( \mathcal{O} \), it makes sense to consider \( D^k \) the same representation with action \( v \cdot_k g := \text{nrd}_p^k(g) v g \). With this notation, we have

\[ \text{Nrd}_p^k \in M_p \left( \mathcal{G}(\Lambda), \mathcal{O}(k), \text{N}_p^{2k} \right)^K_{nrd} \quad \text{for every} \quad K \in \mathcal{K}_p^\times \]

which interpolates the norm forms \( \text{Nrd}_f^k \simeq \text{Nrd}_p^k \) with \( k \in \mathbb{Z} \).

### 2.2 Multilinear forms

For \( x \in B_1^\times \) and \( K \in \mathcal{K}_p \), define \( \Gamma_K(x) = B^\times \cap x^{-1} K x \). Being discrete (as \( B^\times \) is) and compact (as \( K \) is), the set \( \Gamma_K(x) \) is finite. For each \( K \in \mathcal{K}_p \) and each set \( R_K \subset B_1^\times \) of representatives of \( K \backslash B_1 \backslash B^\times \), define

\[ T_{R_K} : M \left( B_1^\times, R \right)^K \to R \quad \text{by} \quad T_{R_K}(f) := \mu(K) \sum_{x \in R_K} \frac{f(x)}{\left| \Gamma_K(x) \right|}. \]

It is easy to see that this is a well defined quantity which is independent from the choice of \( K \) (see [22, §3.1.1] for details), implying that this family defines

\[ T_{B_1^\times / B^\times} : M \left( B_1^\times, R \right) \to R \quad \text{and} \quad T_{Z_1 \backslash B_1^\times / B^\times} : M \left( Z_1 \backslash B_1^\times, R \right) \to R \]

where \( T_{B_1 / B^\times} = T_K \) on \( M \left( B_1^\times, R \right)^K \) and 

\[ T_{Z_1 \backslash B_1^\times / B^\times} := T_{B_1^\times / B^\times \mid Z_1 \backslash B_1^\times, R}. \]

Suppose that we are given a right representation \( (V, \rho) \) of \( G_\infty \in \left\{ B_1^\times, B_\infty^\times \right\} \) (resp. \( \Sigma_p \))) with coefficients in some commutative unitary ring \( R \) and group homomorphisms \( k : \mathbb{Q}_p^\times \to R^\times \) (resp. \( k : \mathbb{Z}_p \to R^\times \)). If

\[ \Lambda \in \text{Hom}_{R[B_\infty]}(\rho, R(k)) \quad \text{(resp.} \quad \in \text{Hom}_{R[K_p^\times]}(\rho, R(k)), \]

\[ \text{we remark that} \quad \text{Nrd}_f^k(x)_p \in \mathbb{Z}_p^\times. \]

\[ \text{Suppose now that we are given} \quad k : \mathbb{Z}_p \to \mathbb{R}_\times \quad \text{which is a continuous group homomorphism, with} \quad \mathcal{O} \quad \text{a locally convex} \quad \mathbb{Q}_p \quad \text{-algebra. Since} \quad \mathcal{K}_p \subset B_1^\times \quad \text{is a compact subgroup,} \quad \text{nrd}_p \quad \text{maps it into the maximal open compact subgroup} \quad \mathbb{Z}_p^\times \subset \mathcal{O}_\times^\times: \]

\[ \text{nrd}_p : \mathcal{K}_p \to \mathbb{Z}_p^\times. \]

\[ \text{If} \quad D \quad \text{is a} \quad \mathcal{K}_p^\times \text{-module with coefficients in} \quad \mathcal{O}, \quad \text{it makes sense to consider} \quad D^k \text{ the same representation with action} \quad v \cdot_k g := \text{nrd}_p^k(g) v g. \quad \text{With this notation, we have} \]

\[ \text{Nrd}_p^k \in M_p \left( \mathcal{G}(\Lambda), \mathcal{O}(k), \text{N}_p^{2k} \right)^K_{nrd} \quad \text{for every} \quad K \in \mathcal{K}_p^\times \]

which \textbf{interpolates} the norm forms \( \text{Nrd}_f^k \simeq \text{Nrd}_p^k \) with \( k \in \mathbb{Z} \).
Then we may define the $R$-linear morphisms

$$M (\Lambda) : M \left( B_f^\times, \rho, N_{t[Z]}^{2k} \right) \to R \text{ (resp. } M_p (\Lambda) : M_p \left( B_f^\times, \rho, N_p^{2k[Z]} \right) \to R \text{)}$$

by the rule

$$M (\Lambda) (\varphi) := \mu (K) \sum_{x \in K \setminus B^*/B^*} \frac{\Lambda (\varphi (x))}{|\Gamma_K (x)| Nrd_{K} (x)} \text{ if } \varphi \in M \left( B_f^\times, \rho, N_{t[Z]}^{2k} \right)^K$$

(resp. $M (\Lambda) (\varphi) := \mu (K) \sum_{x \in K \setminus B^*/B^*} \frac{\Lambda (\varphi (x))}{|\Gamma_K (x)| Nrd_{p} (x)} \text{ if } \varphi \in M_p \left( B_f^\times, \rho, N_p^{2k[Z]} \right)^K$)

Alternatively, we have

$$M (\Lambda) : M \left( B_f^\times, \rho, N_{t[Z]}^{2k} \right) \xrightarrow{\Lambda_{\ast}} M \left( B_f^\times, R (k), N_{t[Z]}^{2k} \right) \xrightarrow{(\cdot, Nrd_{t[Z]}^{-k})} R$$

(resp. $M_p (\Lambda) : M_p \left( B_f^\times, \rho, N_p^{2k[Z]} \right) \xrightarrow{\Lambda_{\ast}} M_p \left( B_f^\times, R (k), N_p^{2k[Z]} \right) \xrightarrow{(\cdot, Nrd_{p[Z]}^{-k})} R$),

where:

- $\Lambda_{\ast}$ is the morphism induced by functoriality and $\Lambda$, i.e. $\Lambda_{\ast} (\varphi (x)) := \Lambda (\varphi (x))$;
- $\langle \cdot, \cdot \rangle$ is the natural pairing

$$\langle \cdot, \cdot \rangle : M \left( B_f^\times, R (k), N_{t[Z]}^{2k} \right) \otimes R M \left( B_f^\times, R (-k), N_{t[Z]}^{-2k} \right) \xrightarrow{T_{Z_f[B]^*/B}} R$$

(resp. $\langle \cdot, \cdot \rangle : M_p \left( B_f^\times, R (k), N_p^{2k[Z]} \right) \otimes R M_p \left( B_f^\times, R (-k), N_p^{-2k[Z]} \right) \xrightarrow{T_{Z_f[B]^*/B}} R$),

with $(\varphi_1 \otimes \varphi_2) (x) := \varphi_1 (x) \varphi_2 (x)$.

It follows from this description that the quantity is well defined. Finally, when $\rho = \rho_1 \otimes_R \cdots \otimes_R \rho_n$ and $\omega_{0,i}$ (resp. $\omega_{0,p,i}$) are such that

$$\omega_{0,1 \ldots \omega_{0,n}} = N_{t[Z]}^{2k} \text{ (resp. } \omega_{0,p,1 \ldots \omega_{0,p,n}} = N_{p[Z]}^{2k}, \text{)}$$

(10)
we can define the $R$-linear morphism
\[
J (\Lambda) : M \left( B_f^\times, \rho_1, \omega_{0,1} \right) \otimes_R \cdots \otimes_R M \left( B_f^\times, \rho_n, \omega_{0,n} \right)
\]
\[
\otimes M \left( B_f^\times, \rho, N_{T_{[\mathbb{Z}_f]}^{2k}} \right)^{M(\Lambda)} \rightarrow R
\]
(resp. $J_p (\Lambda) : M_p \left( B_f^\times, \rho_1, \omega_{0,1} \right) \otimes_R \cdots \otimes_R M_p \left( B_f^\times, \rho_n, \omega_{0,n} \right)$
\[
\otimes M_p \left( B_f^\times, \rho, N_{T_{[\mathbb{Z}_f]}^{2k}} \right)^{M_p(\Lambda)} \rightarrow R
\] (11)
where $\otimes$ is obtained by iteration of $(\varphi_1 \otimes \varphi_2) (x) := \varphi_1 (x) \otimes_R \varphi_2 (x)$ in case $n = 2$.

Let us now assume that we are in the setting of Lemma 2.3, with representations $\rho_i, p$ (resp. $\rho_i, \infty$) of $B_f^\times$ (resp. $B_\infty^\times$) having coefficients in $R_p$ (resp. $R_\infty$), the property that $\rho_i := \rho_{i,p}|_{B^\times} = \rho_{i,\infty}|_{B^\times} \subset \rho_i, p, \rho_i, \infty$ has coefficients in $R$ and suppose that (10) satisfied. Furthermore, suppose that $(\Lambda_p, \Lambda_\infty)$ is a couple of elements $\Lambda_p \in Hom_{R_p[B_p^\times]} (\rho_p, R_p (k))$ and $\Lambda_\infty \in Hom_{R_\infty[B_\infty^\times]} (\rho_\infty, R_\infty (k))$ with the property that
\[
\Lambda := \Lambda_p|_\rho = \Lambda_\infty|_\rho \in Hom_{R[B^\times]} (\rho, R (k))
\]
(Here we assume that $k : \mathbb{Q}^\times \rightarrow R$ and identify it with $i_p (\chi) := i_p \circ \chi$ and $i_\infty (\chi) := i_\infty \circ \chi$).

**Proposition 2.7** Via the inclusions/identifications provided by Lemma 2.3, we have
\[
J_p (\Lambda_p)|_{\otimes_{i=1}^n M (B_f^\times, \rho_i, \omega_{0,i})} = J (\Lambda)|_{\otimes_{i=1}^n M (B_f^\times, \rho_i, \omega_{0,i})}
\]
on
\[
\otimes_{i=1}^n M (B_f^\times, \rho_i, \omega_{0,i}) \subset \otimes_{i=1}^n M_p (B_f^\times, \rho_i, p, \omega_{0,p,i})
\]
\[
\otimes_{i=1}^n M (B_f^\times, \rho_i, \infty, i_\infty (\omega_{0,i}))
\]

**Proof** It is easily checked that all the canonical morphisms involved in the definition of $J_p (\Lambda_p)$ and $J (\Lambda_\infty)$ match: the non canonical ones, namely $\langle \cdot, \text{Nrd}_{f}^{-k} \rangle$ and $\langle \cdot, \text{Nrd}_{p}^{-k} \rangle$ match because $\text{Nrd}_{f}^{-k}$ corresponds to $\text{Nrd}_{p}^{-k}$ via Lemma 2.3.  

\[\square\]

### 2.3 Pairings and adjointness

Suppose that $D$ (resp. $E$) is a $\Sigma_D$ (resp. $\Sigma_E$) module, where $\Sigma_D$ (resp. $\Sigma_E$) satisfies the assumption that was done on $\Sigma_p$, and we let $\omega_{0,p,D}, \omega_{0,p,E} : \mathbb{Z}_f \rightarrow \mathbb{R}^\times$ be characters such that $\omega_{0,p,D} \omega_{0,p,E} = \omega_{0,p}$. We assume that we are given a group homomorphism $k : \mathbb{Z}_p^\times \rightarrow \mathbb{R}^\times$ and a pairing
\[
\langle - , - \rangle \in Hom_{R[K^\times_p]} (D \otimes_R E, R (k))
\]
Then (11) gives
\[
\langle - , - \rangle_{M_p} : M_p (B_f^\times, D, \omega_{0,p,D}) \otimes_R M_p (B_f^\times, E, \omega_{0,p,E}) \rightarrow R.
\]
We suppose $\Sigma_{D} = \Sigma_{p}$, $\Sigma_{D} = \Sigma'_{p}$ and $(K_{p}^{\circ})' = K_{p}^{\circ} \subset \Sigma_{p} \cap \Sigma'_{p}$ (as in case $K_{p}^{\circ} = \Gamma_{0}(p\mathbb{Z}_{p})$ and $\Sigma_{p} = \Sigma_{0}(p\mathbb{Z}_{p})$) and that $Z = \mathbb{Z}_{B_{p}} = \mathbb{G}_{m}$. Assuming that $E$ has central character $\kappa_{E} : \mathbb{Z}_{p}^{\times} \to R^{\times}$, we can consider the second of the following compositions:

$$\text{nrdf}_{\alpha^{0},E} : B_{f}^{\times} \to \kappa_{E}^{\times} = Z_{f}^{\alpha^{0},E} \to R^{\times} \quad \text{and} \quad \text{nrdf}_{p}^{\kappa_{E}} : K_{p}^{\circ} \to \mathbb{Z}_{p}^{\times} \to R^{\times}.$$ 

Suppose that $k, \kappa_{E} : \mathbb{Z}_{p}^{\times} \to R^{\times}$ extends to a character $\tilde{k}, \tilde{\kappa}_{E} : \mathbb{Q}_{p}^{\times} \to R^{\times}$. Then

$$\text{nrdf}_{p}^{\tilde{\kappa}_{E}} : \Sigma_{p} \to \mathbb{Q}_{p}^{\times} \to \tilde{\kappa}_{E}^{\times} \to R^{\times}$$

is an extension of $\text{nrdf}_{p}^{\kappa_{E}}$ to $\Sigma_{p}$ and we let $\text{Hom}_{R[\Sigma_{p}, \Sigma_{p}]}(D \otimes E, R(\tilde{k}))$ be the set of those pairings such that

$$\langle v\sigma, w \rangle = \text{nrdf}_{p}^{\tilde{\kappa}_{E}}(\sigma) \langle v, w^{\sigma} \rangle$$

for every $\sigma \in \Sigma_{p}$.

We remark that, for every element $u \in K_{p}^{\circ}$,

$$\langle vu, wu \rangle = \text{nrdf}_{p}^{\kappa_{E}}(u) \langle v, wuu^{\sigma} \rangle = \text{nrdf}_{p}^{k}(u) \langle v, w \rangle,$$

so that $\text{Hom}_{R[\Sigma_{p}, \Sigma_{p}]}(D \otimes E, R(\tilde{k})) \subset \text{Hom}_{R[\kappa_{E}^{\circ}]}(D \otimes E, R(k))$.

**Remark 2.8** Suppose now that $D \subset \tilde{D}$ and $E \subset \tilde{E}$, where $\tilde{D}$ and $\tilde{E}$ are $B_{p}^{\times}$-modules, the above inclusions are $\Sigma_{p}$ and, respectively, $\Sigma'_{p}$-equivariant and that $\tilde{E}$ has central character $\kappa_{\tilde{E}} = \tilde{\kappa}_{E}$ extending $\kappa_{E}$. If $\langle \cdot, \cdot \rangle \in \text{Hom}_{O[\kappa_{E}]}(D \otimes E, R(k))$ extends to $\langle \cdot, \cdot \rangle^\sim \in \text{Hom}_{O[B_{p}^{\times}])(\tilde{D} \otimes \tilde{E}, R(\tilde{k}))$ then $\langle \cdot, \cdot \rangle \in \text{Hom}_{O[\Sigma_{p}, \Sigma_{p}]}(D \otimes E, R(\tilde{k}))$:

$$\langle v\sigma, w \rangle = \langle v, w^{-1} \sigma \rangle = \text{nrdf}_{p}^{\tilde{\kappa}_{E}}(\sigma) \langle v, w^{-1} \sigma \rangle = \text{nrdf}_{p}^{\tilde{\kappa}_{E}}(\sigma) \langle v, w^{\sigma} \rangle.$$ 

In the following proposition, we suppose that $f \in M_{p}(B_{f}^{\times}, D, \alpha_{0}, p, D)^{K_{1}}$ and $g \in M_{p}(B_{f}^{\times}, E, \omega_{0}, p, E)^{K_{2}}$ (and make a similar assumption for classical, i.e. non $p$-adic, modular forms in the $M$‘s spaces). Finally, we assume that

$$\langle - , - \rangle \in \text{Hom}_{R[\Sigma_{p}, \Sigma_{p}]}(D \otimes E, R(\tilde{k}))$$

(but for classical modular forms, we suppose $\langle - , - \rangle \in \text{Hom}_{R[B^{\times}]}(D \otimes R E, R(\tilde{k}))$ where $\tilde{k} : \mathbb{Q}^{\times} \to R^{\times}$ and does not require $E$ to have central character $\kappa_{E} : \mathbb{Q}^{\times} \to R^{\times}$). We write $T_{\pi} := K_{1}\pi K_{2}, T_{\pi^{\prime}} := K_{2}\pi^{\prime} K_{1}$ and $T_{\pi^{\prime}} := K_{2}\pi^{\prime} K_{1}$.
Proposition 2.9 We have the following formulas, in the $p$-adic case:

$$\mu(K_2)^{-1}(f \mid T_\pi, g) = \text{Nrd}_p^k(\pi) \text{Nrd}_p^{\kappa E}(\pi) \text{Nrd}_p^{\omega_0, p, E}(\pi) \mu(K_1)^{-1}(f, g \mid T_{\pi^i}).$$

For classical modular forms, $\mu(K_2)^{-1}(f \mid T_\pi, g) = \text{Nrd}_p^k(\pi) \mu(K_1)^{-1}(f, g \mid T_{\pi^{-1}})$ and, whenever $E$ has central character $\kappa_E$, $\mu(K_2)^{-1}(f \mid T_\pi, g) = \text{Nrd}_p^k(\pi) \text{Nrd}_p^{\kappa E}(\pi) \mu(K_1)^{-1}(f, g \mid T_{\pi^i}).$

Proof Note that $K_i$ always contains a decomposable open and compact subgroup $K_i'$ and, because $\pi \in B_i^\times$, we see that $\pi_1 \in K_{i,1} \subset K_i$ for all but finitely may $l$’s (the inclusion viewing $B_i^\times \subset B_i^\times$ as the $l$-component of $B_i^\times$). It follows that we may assume that $\pi$ is concentrated at a finite number of components; then, we leave to the reader to check that we may assume that $\vert \Gamma_{K_i}(x) \vert = 1$ for all $x \in B_i^\times$ and $i = 1, 2$. Having made this reduction, we compute, for $p$-adic modular forms,

$$\mu(K_2)^{-1}(f \mid K_1 \pi K_2, g) = \sum_{x \in K_2 \setminus B_i^*/B^\times} \frac{((f \mid K_1 \pi K_2)(x), g(x))}{\text{Nrd}_p^k(x)}$$

$$= \sum_{u \in (K_2 \cap \pi^{-1} K_1) \setminus K_2 \times B_i^*/B^\times} \frac{f((\pi u x) \pi_p u_p, g(x))}{\text{Nrd}_p^k(x)}$$

$$= \sum_{u \in (K_2 \cap \pi^{-1} K_1) \setminus K_2 \times B_i^*/B^\times} \frac{f((\pi u x) \pi_p u_p, g(u x) u_p)}{\text{Nrd}_p^k(u) \text{Nrd}_p^k(u)}$$

$$= \sum_{u \in (K_2 \cap \pi^{-1} K_1) \setminus K_2 \times B_i^*/B^\times} \frac{f((\pi u x) \pi_p, g(u x))}{\text{Nrd}_p^k(u)}$$

$$= \sum_{y \in (K_2 \cap \pi^{-1} K_1) \setminus B_i^*/B^\times} \frac{f((\pi y) \pi_p, g(y))}{\text{Nrd}_p^k(y)}.$$

Here we have employed (9) in the second equality, the $K_2$-invariance of $g$ and $\text{Nrd}_p^k$ in the third equality and the $K_2$-equivariance of $\langle - , - \rangle$ in the fourth equality. Letting $g, f, K_2, K_1$ and $\pi^i$ play the roles of $f, g, K_1, K_2$ and $\pi$ respectively, we also see that

$$\mu(K_1)^{-1}\langle f, g \mid K_2 \pi^i K_1 \rangle = \sum_{z \in (K_1 \cap \pi^{-i} K_2 \pi^i) \setminus B_i^*/B^\times} \frac{f(z), g((\pi z) \pi^i)}{\text{Nrd}_p^k(z)}.$$

Note, however, that $y \mapsto \pi y$ induces a well defined map $H \times B_i^*/B^\times \rightarrow \pi H \times B_i^*/B^\times$ for any subgroup $H$ and we have $\pi^i H \pi^{-i} = \pi^{-1} H \pi$. Taking $H = K_2 \cap \pi^{-1} K_1$ we see that $\pi H \pi^{-1} = K_1 \cap \pi^{-i} K_2 \pi^i$. Making the change of variables $z = \pi y$, we have
\[ \mu (K_1)^{-1} \left\{ f, g \mid K_2 \pi^i K_1 \right\} = \sum_{y \in (K_2 \cap \pi^{-1} K_1 \pi)} B^\times_{\pi} \left( \frac{f (\pi y), g (\pi^i \pi y) \pi^i_p}{\text{Nrd}_{\pi}^k (\pi y)} \right) \]
\[ = \text{nrd}_{\pi}^{\tilde{\kappa} E} (\pi_p) \text{nrd}_{\pi}^{\kappa} (\pi) \times \sum_{y \in (K_2 \cap \pi^{-1} K_1 \pi)} B^\times_{\pi} \left( \frac{f (\pi y) \pi^i, g (\pi^i \pi y)}{\text{Nrd}_{\pi}^k (y)} \right). \]

Here we have used \( \left\{ v, w \pi^i \right\} = \text{nrd}_{\pi}^{\tilde{\kappa} E} (\pi_p) \left\{ v \pi^i, w \right\}. \) We now remark that \( \pi^i \pi = \text{nrd} (\pi) \in Z_f, \) so that \( g (\pi^i \pi y) = \text{nrd}_{\pi}^{\kappa_0, p, E} (\pi) g (y). \) It follows that
\[ \mu (K_1)^{-1} \left\{ f, g \mid K_2 \pi^i K_1 \right\} = \text{nrd}_{\pi}^{\tilde{\kappa} E} (\pi_p) \text{nrd}_{\pi}^{\kappa} (\pi) \text{nrd}_{\pi}^{\kappa_0, p, E} (\pi) \mu (K_2)^{-1} \left\{ f \mid K_1 \pi K_2, g \right\}. \]

The relation \( \text{Nrd}_{\pi}^f (x) = \left( \frac{\text{Nrd}_{\pi} (x)_{p}}{\text{Nrd}_{\pi} (x_p)} \right)^k \) gives the claim:
\[ \text{nrd}_{\pi}^{\tilde{\kappa} E} (\pi_p) \text{nrd}_{\pi}^{\kappa} (\pi) \text{nrd}_{\pi}^{\kappa_0, p, E} (\pi) \text{nrd}_{\pi}^{\kappa} (\pi_p) = \text{nrd}_{\pi}^{\tilde{\kappa} E} (\pi_p) \text{nrd}_{\pi}^{\kappa} (\pi_p). \]

For modular forms one finds, by a similar computation,
\[ \mu (K_2)^{-1} \left\{ f \mid K_1 \pi K_2, g \right\} = \sum_{y \in (K_2 \cap \pi^{-1} K_1 \pi)} B^\times_{\pi} \left( \frac{f (\pi y), g (y)}{\text{Nrd}_{\pi}^k (y)} \right), \]
\[ \mu (K_1)^{-1} \left\{ f, g \mid K_2 \pi^{-1} K_1 \right\} = \sum_{z \in (K_1 \cap \pi^{-1} K_2 \pi^{-1})} B^\times_{\pi} \left( \frac{f (\pi z), g (\pi z)}{\text{Nrd}_{\pi}^k (z)} \right). \]

The first equality in this case follows and the second, which can also be proved by a similar computation as above, is actually a consequence of the first in this setting, since one checks \( g \mid K_2 \pi^{-1} K_1 = \text{nrd}_{\pi}^{\kappa_0, E} (\pi) \cdot g \mid K_2 \pi^i \pi K_1 \) (because \( \text{nrd} (\pi) \in Z_f = Z_{B^\times_i}). \)

\[ \square \]

3 The special value formula and its \( p \)-adic avatar

We are now going to recall the special value formula proved in [22], specialized to the triple product case, which can be regarded as a vector valued version of Ichino’s formula [31] and a generalization of [7].

Let \( E/\mathbb{Q} \) be a Galois splitting field for \( B \) and fix \( B / E \simeq \mathbf{M}_2 / E \) inducing \( B^\times_{/E} \simeq \mathbf{GL}_2 / E. \) If \( k \in \mathbb{N}, \) we let \( \mathbf{P}_{k/ E} \) be the left \( \mathbf{GL}_2 / E \)-representation on two variables polynomials of degree \( k, \) the action being defined by the rule \( (g P) (X, Y) = P ((X, Y) g). \) We write \( \mathbf{V}_k \) for the dual right representation. If \( \mathbf{k} := (k_1, \ldots, k_r) \in \mathbb{N}^r, \) we may identify \( \mathbf{P}_{k_1/ E} \otimes \cdots \otimes \mathbf{P}_{k_r/ E} \) with the space of \( 2r \)-variable polynomials \( \mathbf{P}_{k/ E} \) which are homogeneous of degree \( k_i \) in the \( i \)-th couple of variables \( W_i := (X_i, Y_i). \)
Then \( V_{k_1/E} \otimes \cdots \otimes V_{k_r/E} \) is identified with the dual \( V_{k/E} \) of \( P_{k/E} \) and any \( P \in P_{k/E} (-r)^{GL_2/E} \), i.e. such that \( g P = \det (g)^r P \), induces

\[
\Lambda_P \in Hom_{GL_2/E} \left( V_{k/E}, 1_E (r) \right)
\]

by the rule \( \Lambda_P (l) := l (P) \). Note also that, if \( P \neq 0 \) then there is \( l \) such that \( l (P) = 1 \) and we see that \( \Lambda_P \neq 0 \). Setting \( 0 \neq \delta^k (X_1, Y_1, X_2, Y_2) := \left[ \begin{array}{cc} X_1 Y_1^k \\ X_2 Y_2 \end{array} \right] \), we have \( \delta^1 (W_1 g, W_2 g) = \det (g) \delta^1 (W_1, W_2) \), from which it follows that \( \delta_k \in P_{k,k/E} \) and \( g \delta^k = \det (g)^k \delta^k \). We deduce that \( \langle -, - \rangle_{k/E} := \Lambda_{\delta^k} \neq 0 \) satisfies the above requirement and, hence, defines

\[
\langle -, - \rangle_{k/E} \in Hom_{GL_2/E} \left( V_{k/E} \otimes V_{k/E}, 1_E (k) \right) ;
\]

then the irreducibility of the \( GL_2/E \) representation \( V_{k/E} \) implies that this non-zero pairing is perfect and symmetric. Next, if \( k := (k_1, k_2, k_3) \in \mathbb{N}^3 \), we define

\[
\langle -, - \rangle_{k/E} := \langle -, - \rangle_{k_1/E} \otimes \langle -, - \rangle_{k_2/E} \otimes \langle -, - \rangle_{k_3/E}
\]

\[
\in Hom_{GL_2/E} \left( V_{k/E} \otimes V_{k/E}, 1_E (k) \right).
\]

We remark that, viewing a left representation as a right representation by means of the inversion we have \( P_{k/E} = V^\vee_{k/E} \) (resp. \( P_{k/E} = V^\vee_{k/E} \)) and, hence, \( \langle -, - \rangle_{k/E} \) (resp. \( \langle -, - \rangle_{k/E} \)) induces \( P_{k/E} = V^\vee_{k/E} \simeq V_{k/E} (-k) \) (resp. \( P_{k/E} = V^\vee_{k/E} \simeq V_{k/E} (-k^* \rangle \), which in turn induces

\[
\langle -, - \rangle_{k/E} \in Hom_{GL_2/E} \left( P_{k/E} \otimes P_{k/E}, 1_E (-k) \right)
\]

\[
\in Hom_{GL_2/E} \left( P_{k/E} \otimes P_{k/E}, 1_E (-k) \right)
\]

(and the latter is the tensor product of the former pairings).

If \( k := (k_1, k_2, k_3) \in \mathbb{N}^3 \), we define the quantities \( k^* := \frac{k_1 + k_2 + k_3}{2} \), \( k^*_1 := -\frac{k_1 + k_2 + k_3}{2} \), \( k^*_2 := -\frac{k_1 - k_2 + k_3}{2} \) and \( k^*_3 := -\frac{k_1 + k_2 - k_3}{2} \). With a slight abuse of notation, we write \( P_{k/E} \) and \( V_{k/E} \) to denote the external tensor product, which is a representation of \( GL_3_{2/E} \). When \( k \) is balanced, we can also define

\[
\Lambda_{k/E} \in Hom_{GL_2/E} \left( V_{k/E}, 1_E (k^*) \right)
\]

as follows. The balanced condition precisely means that \( k_i^* \in \mathbb{N} \) for \( i = 1, 2, 3 \), so that we can consider

\[
0 \neq \Delta_{k/E} := \delta_{k_1^*} (W_2, W_3) \delta_{k_2^*} (W_1, W_3) \delta_{k_3^*} (W_1, W_2) \in P_{k/E}.
\]

We have \( g \Delta_{k/E} = \det (g)^{k^*} \Delta_{k/E} \). Hence \( \Delta_{k/E} \in P_{k/E} (-k^*)^{GL_2/E} \) and we may set

\[
\Lambda_{k/E} := \Lambda_{\Delta_{k/E}} \neq 0.
\]
The following result is an application of the Clebsch-Gordan decomposition that we leave to the reader.

**Lemma 3.1** Suppose that $2k^* = k_1 + k_2 + k_3 \in 2\mathbb{N}$ and $k$ is balanced.

1. There is a representation $V_k$ of $B^\times$ (with the diagonal action) such that $E \otimes V_k \simeq V_{k/E}$ via $B_{k/E}^\times \simeq \text{GL}_{2/E}$.

2. We have, setting $B_1^\times := \ker(\text{nr})$,

$$\dim \left( \text{Hom}_{B_1^\times} \left( V_k, 1 \right) \right) = \dim \left( \text{Hom}_{\text{SL}_{2/E}} \left( V_{k/E}, 1/E \right) \right) = 1.$$ 

For $i = 1, 2, 3$, let $\omega_i$ be an unitary Hecke character of the form $\omega_i = \omega_{i,f} \otimes \text{sgn} (-)^{k_i}$ and set $\omega_{0,i} := \omega_{i,f} N_{i,f}^{k_i}$. Assuming that $\omega_1 \omega_2 \omega_3 = 1$, we see that

$$k^*_i \in \mathbb{N} \text{ and } N_{i,f}^{2k_i} = \omega_{1,0} \omega_{2,0} \omega_{3,0}. \quad (14)$$

It follows from an adelic version of the Peter-Weyl Theorem (see [22, Proposition 6.1]) that, if $\pi_i = \pi_{i,f} \otimes V_{k_i,\mathbb{C}}^u$ is an irreducible unitary automorphic form with central character $\omega_i$ (and $V_{k_i,\mathbb{C}}^u$ the unitary twist of $V_{k_i,\mathbb{C}}$), the rule $f_1 (\Lambda \otimes \phi) (x) := \text{nr}_{f}^{k_i/2} (x_f) \text{nr}_{f}^{k_i/2} \left( x_{\infty} \right) \Lambda \left( \phi \left( x_f \right) x_{\infty} \right)$ defines a canonical $B^\times \left( \mathbb{A} \right)$-equivariant identification:

$$f_i : V_{k_i,\mathbb{C}}^u \otimes_{\mathbb{C}} M \left( B_{k_i,\mathbb{C}}^\times, V_{k_i,\mathbb{C}}, \omega_{0,i} \right) \left[ \text{nr}_{f}^{-k_i/2} \pi_{i,f} \right] \simeq A \left( B^\times \left( \mathbb{A} \right), \omega_i \right) \left[ \pi_i \right], \quad (15)$$

where $(-) \left[ \theta \right]$ means taking the $\theta$-component and $A \left( B^\times \left( \mathbb{A} \right), \omega_i \right)$ is the space of $K$-finite automorphic forms. We remark the we could have considered automorphic forms for the algebraic group $B_{3,\mathbb{C}}^\times$ and, with $\Pi := \pi_1 \otimes \pi_2 \otimes \pi_3$, so that $\Pi = \Pi_f \otimes V_{k,\mathbb{C}}^u$, we have the canonical $B^\times \left( \mathbb{A} \right)$-equivariant identification:

$$f : V_{k,\mathbb{C}}^u \otimes_{\mathbb{C}} M \left( B_{3,\mathbb{C}}^\times, V_{k,\mathbb{C}}, \omega_0 \right) \left[ \text{nr}_{f}^{-k/2} \Pi_f \right] \simeq A \left( B^\times \left( \mathbb{A} \right), \omega \right) \left[ \Pi \right], \quad (16)$$

where $\omega = (\omega_1, \omega_2, \omega_3)$, $\text{nr}_{f}^{-k/2} := \left( \text{nr}_{f}^{-k_1/2}, \text{nr}_{f}^{-k_2/2}, \text{nr}_{f}^{-k_3/2} \right)$, $N_{i,f}^{k/2}$ and $N_{i,f}^{k_1} = \left( N_{i,f}^{k_1}, N_{i,f}^{k_2}, N_{i,f}^{k_3} \right)$ are defined in a similar way, $\omega_0 := \omega_f N_{f}^{k}$ and $f (\Lambda \otimes \phi) \left( x \right) := \text{nr}_{f}^{-k/2} (x_f) \text{nr}_{f}^{k/2} \left( x_{\infty} \right) \Lambda \left( \phi \left( x_f \right) x_{\infty} \right)$.

### 3.1 Periods

From now on, we will need to fix an embedding $E \subset \mathbb{C}$, which allows us to regard $V_{k_i,E}$-valued (resp. $V_{k,E}$-valued) modular forms as $V_{k_i,C}$-valued (resp. $V_{k,C}$-valued) and suppose that $E$ is large enough to contain the values of the characters $\omega_{i,0}$. Let us remark that we have a morphism

$$M \left( B_{k_i,E}^\times, V_{k_i,E}, \omega_{0,i} \right) \longrightarrow M \left( B_{f,E}^\times, V_{k_i,E}, N_{i,f}^{2k_i} \omega_{0,i}^{-1} \right) \quad (17)$$
defined by the rule \( \varphi \mapsto \tilde{\varphi} \), where \( \tilde{\varphi}(x) := \text{rd}_f^{-\omega_i}(x)\varphi(x) = N_f^{k_i}(x)\text{rd}_f^{-\omega_0}(x)\varphi(x) \) with our usual shorthand \( \text{rd}_f^{-\omega} := \chi^{-1} \circ \text{rd}_f \) (the equality because \( \omega_i, f = \omega_0 N_f^{-k_i} \)). Indeed, Remark 2.1 applies as follows. Since \( \omega_i \) is a Hecke character, we have \( \text{rd}_f^{-\omega_i} = \text{rd}_f^{-\omega_i, \infty} \) and we see that \( \left( \text{rd}_f^{-\omega_i}, \text{rd}_f^{-\omega_0, \infty} \right) \in X \left( B^\times, \text{rd}_f^{-\omega_i, i} \right) \). Finally, because \( B \) is definite, \( \text{rd}_\infty(B^\times(\mathbb{R})) \subset \mathbb{R}_+^* \), so that \( \text{rd}_\infty^{-\omega, i} = 1 \) (because \( \omega_\infty, i = \text{sgn}(\omega_i) \)), and \( \text{rd}_f^{-\omega_i}(x) = \omega_i^{-1} \), so that \( \text{rd}_f^{-\omega_0, i}(x) = N_f^{2k_i} \omega_0^{-1} \); we have checked that
\[
\left( \text{rd}_f^{-\omega_0}, 1 \right) \in X \left( B^\times, \omega_i^{-2} \right). \tag{18}
\]
A similar twist works with the modular forms on \( B^X \), with \( \text{rd}_f^{-\omega_i} \) replaced by \( \text{rd}_f^{-\omega_i} := (\text{rd}_f^{-\omega_i, 1}, \text{rd}_f^{-\omega_i, 2}, \text{rd}_f^{-\omega_i, 3}) \) and \( N_f^{2k_i} \omega_0^{-1} \) replaced by \( N_f^{2k_i} \omega_0^{-1} \), where \( N_f^{2k_i} = (N_f^{2k_1}, N_f^{2k_2}, N_f^{2k_3}) \). Then, of course, \( \varphi \mapsto \tilde{\varphi} \) commutes with the tensor product.

It follows that we can consider
\[
(-, -)_{k_i} : M \left( B^X_f, V_{k_i/E, \omega_0, i} \right) \otimes \mathbb{C} M \left( B^X_f, V_{k_i/E, \omega_0, i} \right) \rightarrow E \tag{19}
\]
and
\[
(-, -)_{k} : M \left( B^X_f, V_{k/E, \omega_0} \right) \otimes \mathbb{C} M \left( B^X_f, V_{k/E, \omega_0} \right) \rightarrow E
\]
defined as follows. Let us write again
\[
(-, -)_{k/E} : M \left( B^X_f, V_{k/E, \omega_0, i} \right) \otimes_E M \left( B^X_f, V_{k/E, N_f^{2k_i} \omega_0^{-1}} \right) \rightarrow E
\]
for the morphism (11) induced by \( (-, -)_{k_i/E} \) [see (12)] on modular forms (defined because \( \omega_0, i N_f^{2k_i} \omega_0^{-1} = N_f^{2k_i} \)). Then we define
\[
(\varphi_1, \varphi_2)_{k_i} := \langle \varphi_1, \tilde{\varphi_2} \rangle_{k_i/E} = \mu(K) \sum_{x \in K \backslash B^\times / B^\times} \frac{\langle \varphi_1(x), \varphi_2(x) \rangle_{k_i/E}}{|\Gamma_K(x)| \text{nd}_f^{-\omega_0, i}(x)},
\]
if \( \varphi_1 \) and \( \tilde{\varphi_2} \) are \( K \)-invariant. Working in a similar way with the algebraic group \( B^X \) we get \( (-, -)_{k_i} \). Alternatively, the right hand side of the equality
\[
(-, -)_{k} = (-, -)_{k_1} \otimes_E (-, -)_{k_2} \otimes_E (-, -)_{k_3}
\]
gives an alternative definition.

**Lemma 3.2** We have that \((-,-)_k\) is a perfect pairing.

**Proof** Let us first remark that, from the definition of the adelic Peter-Weyl theorem [22, Proposition 6.1], the fact that \( \tilde{\varphi}(x) := \text{nd}_f^{-\omega_0, i}(x)\varphi(x) \) and the fact that \( \text{nd}_\infty^{-\omega_0, i} = 1 \) there is a commutative diagram
\[
\begin{array}{ccc}
V^\mu_{k_i, \mathbb{C}} \otimes \mathbb{C} M \left( B^X_f, V_{k_i, \mathbb{C}, \omega_0, i} \right) & \rightarrow & V^\mu_{k_i, \mathbb{C}} \otimes \mathbb{C} M \left( B^X_f, V_{k_i, \mathbb{C}, N_f^{2k_i} \omega_0^{-1}} \right) \\
\text{nd}_f^{-k_i/2} \pi_{1, i} & \downarrow & \text{nd}_f^{-k_i/2} \pi_{1, i} \\
A \left( B^\times(\hat{\mathbb{A}}, \omega_0) \right) [\pi_i] & \rightarrow & A \left( B^\times(\hat{\mathbb{A}}, \omega_0) \right) [\pi_i]
\end{array}
\]
if we define the lower arrow via \( \psi \mapsto \tilde{\psi} \), where \( \tilde{\psi}(x) := \text{nd}_\mathbb{A}^\alpha(x)\psi(x) \). There is also a similar commutative diagram for automorphic forms on \( B^{\times 3} \), which is the tensor product of the above commutative diagrams. Then, arguing as in the proof of [22, Proposition 6.1 (2)] (based on the Schur orthogonality relations, but for linear pairings, to which [22, Lemma 3.7] and (7) are applied), we see that

\[
\left\langle \psi_1, \tilde{\psi}_2 \right\rangle_{L_2} := \int_{[B^{\times 3}(\mathbb{A})]_{B^{\times 3}}} \psi_1(x)\tilde{\psi}_2(x)\mu(B^{\times 3}(\mathbb{A}))_{ZB^{\times 3}}(x) = \frac{\langle \Delta_1, \Delta_2 \rangle_k}{d_k} \left\langle \varphi_1, \tilde{\varphi}_2 \right\rangle_k = \frac{\langle \Delta_1, \Delta_2 \rangle_k}{d_k} (\varphi_1, \varphi_2)_k.
\]

if \( \psi_h = f(\Delta_h \otimes \rho \varphi_h) \) is in \( A(B^{\times 3}(\mathbb{A}), \omega)[\Pi] \) and \( [B^{\times 3}(\mathbb{A})]_{ZB^{\times 3}} = ZB^{\times 3}(\mathbb{A}) \setminus B^{\times 3}(\mathbb{Q}) \) with the product measure. Here \( d_k \) is the formal degree of \( V_{k,\mathbb{C}} \) (which only depends on the Haar measure, once \( (-,-)_E \) on \( P_{k/E} = V_{k/E} \) is obtained from \( (-,-)_k \) on \( V_{k/E} \) tautologically as above). Recalling that the formal degree of a representation of \( B^{\times 3}(\mathbb{R}) \) equals the dimension times the inverse of the total measure \( m^3_{ZB^{\times 3}(\mathbb{R})} \) of \( ZB^{\times 3}(\mathbb{R}) \setminus B^{\times 3}(\mathbb{R}) \) (by compactness of \( ZB^{\times 3}(\mathbb{R}) \setminus B^{\times 3}(\mathbb{R}) \)), we find

\[
\left\langle \psi_1, \tilde{\psi}_2 \right\rangle_{L^2} = \frac{m^3_{ZB^{\times 3}(\mathbb{R})}}{(k_1 + 1)(k_2 + 1)(k_3 + 1)} (\varphi_1, \varphi_2)_k. \tag{20}
\]

Because the left hand side is a perfect pairing, our claim follows. (It is not difficult to see that \( (-,-)_k \) on modular forms is perfect because it is induced by the perfect pairing \( (-,-)_k \) on the finite dimensional vector spaces \( V_{k,\mathbb{C}} \); next, noticing that \( \varphi \mapsto \tilde{\varphi} \) is a linear bijection, gives a direct proof of the perfectness of \( (-,-)_k \). However, we will need (20) below).

Fix an identification \( B\left(\mathbb{A}_f^{\text{Disc}(B)}\right) \simeq M_2\left(\mathbb{A}_f^{\text{Disc}(B)}\right) \) and, for an integer \( N \) such that \( (N, \text{Disc}(B)) = 1 \), write \( K_0^{\text{Disc}(B)}(N) \subset B^{\times}(\mathbb{A}_f^{\text{Disc}(B)}) \) (resp. \( K_1^{\text{Disc}(B)}(N) \subset R^{\text{Disc}(B)}(N) \)) for the subgroup which corresponds to matrices with integral coefficients having lower left entry \( c \equiv 0 \bmod (N) \) (resp. \( c \equiv 0 \bmod (N) \) and upper left entry \( a = 1 \)). Setting \( O_{\text{Disc}(B)} := \prod_{l|\text{Disc}(B)} O_{B_l}^\times \) we can define

\[
K_0(N) := K_0^{\text{Disc}(B)}(N) \times O_{\text{Disc}(B)}^\times \quad \text{and} \quad K_1(N) := K_1^{\text{Disc}(B)}(N) \times O_{\text{Disc}(B)}^\times.
\]

Assuming that \( \mu \varphi(N) \subset E \), we can decompose

\[
M\left(B_f^{\times}, V_{k,E}\right)^{K_1(N)} = \bigoplus_{\epsilon \in \left(\frac{\mathbb{Q}}{N\mathbb{Z}}\right)^\times} O^\times M\left(B_f^{\times}, V_{k,E}\right)^{K_1(N)}(\epsilon),
\]
where $M(\varepsilon)$ is the submodule of elements $x \in M$ such that $xu = \varepsilon(u)x$ if we define $\varepsilon(u) := \varepsilon(a_u)$ for $a_u$ the upper left entry of $u \in K_0(N)$. We also have

$$M\left( B_1^\times, V_{k,E} \right)^{K_1(N)}(\varepsilon) = M\left( B_1^\times, V_{k,E}, \omega_0^{e,k} \right)^{K_0(N)}.$$ 

**Example 3.3** Suppose that $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ and that

$$\varphi_i \in M\left( B_1^\times, V_{k_i,E} \right)^{K_1(N_i)}(\varepsilon_i) = M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i,k_i} \right)^{K_0(N_i)}.$$ 

Then we see that $\varphi \mapsto \tilde{\varphi}$ is a map

$$M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i,k_i} \right)^{K_0(N_i)} \rightarrow M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i^{-1},k_i} \right)^{K_0(N_i)}.$$

Indeed, $\omega_0^{e_i,k_i} \omega_0^{e_i^{-1},k_i} = N_l^{2k_i}$, so that we have

$$(-,-)_{k_i/E} : M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i,k_i} \right) \otimes_E M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i^{-1},k_i} \right) \rightarrow E$$

and $(\varphi_i, \psi_i)_{k_i} = \left( \varphi_i, \tilde{\psi}_i \right)_{k_i/E}$ by definition. Write $\tilde{\omega}_{N_i} \in B_1^\times$ for the matrix concentrated at the primes $l$ such that $l^{e_i} \mid N_i$, where we have $(\tilde{\omega}_{N_i})_l = \omega_l^{e_i} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and write $W_{N_i}$ for the Hecke operator it induces. If $\varphi_i^b \in M\left( B_1^\times, V_{k_i,E}, \omega_0^{e_i,k_i} \right)^{K_0(N_i)}$ is a newvector, then setting $\tilde{\varphi}_i^{bb} := \varphi_i^b \mid W_{N_i}$ it is easily checked that $(\tilde{\varphi}_i^{bb})$ is a newvector in the dual representation, hence a scalar multiple of $\varphi_i^b$ and then the proof of Lemma 3.2 shows that

$$\Omega(\varphi_i^b) := \left( \varphi_i^b, \tilde{\varphi}_i^{bb} \right)_{k_i} \neq 0;$$

in particular, setting $\varphi^b := \varphi_1^b \otimes \varphi_2^b \otimes \varphi_3^b$ and $\varphi^{bb} := \varphi_1^{bb} \otimes \varphi_2^{bb} \otimes \varphi_3^{bb}$ we see that

$$\Omega(\varphi^b) := \left( \varphi^b, \varphi^{bb} \right)_{k} \neq 0. \quad (21)$$

**3.2 The special value formula**

It follows from (14), that we can consider the quantity $t_\lambda := J(\Lambda_{k/E})$ defined by (11):

$$t_\lambda : M\left( B_1^\times, V_{k_1,E}, \omega_0 \right) = M\left( B_1^\times, V_{k_1,E}, \omega_0,1 \right) \otimes_E M\left( B_1^\times, V_{k_2,E}, \omega_0,2 \right) \otimes_E M\left( B_1^\times, V_{k_3,E}, \omega_0,3 \right) \rightarrow E.$$
The choice of \( \Lambda_{k/E} \in \mathcal{V}^\vee_{k,E} \) (and \( E \in \mathbb{C} \)) yields, via (16), the embedding

\[
f_{\Lambda_{k/E}} : M \left( B_t^{x^3}, V_{k,E}, \omega_0 \right) \left[ \text{Nrd}_{t}^{-k/2} \Pi_f \right] \hookrightarrow A \left( B^{x^3} (\mathbb{A}), \omega \right) [\Pi]
\]

for every irreducible automorphic representation \( \Pi \) of \( B^{x^3} \). Let us write \( \Pi' \) for the automorphic representation of \( \text{GL}^3_2 \) which corresponds to \( \Pi \) under the Jacquet–Langlands correspondence.

Before stating our next result, we need to recall the definitions of some relevant quantities. Let \( \langle -, - \rangle_{L^2} \) be the pairing defined before (20) and fix non-zero \( B^{x^3}(\mathbb{Q}_v) \)-invariant pairings \( \langle -, - \rangle_v \) between \( \Pi_v \) and its dual representation. The irreducibility of \( \Pi_v \) implies that there is a non-zero constant \( C \) such that

\[
\langle \psi^b, \psi^{b^\vee} \rangle_{L^2} = C \prod_v \langle \psi_v^b, \psi_v^{b^\vee} \rangle_v.
\]

Then one defines the bilinear form on \( \Pi_v \times \Pi_v^\vee \) via the formula

\[
I_v(\psi_v \otimes \psi_v^\vee) := \frac{L_v \left( 1, \Pi_v', \text{Ad} \right)}{\zeta_{\mathbb{Q}_v}(2) L_v \left( 1/2, \Pi_v' \right)} \int_{B(\mathbb{Q}_v) \backslash B^x(\mathbb{Q}_v)} (\psi_v \Pi_v(x)^{-1}, \psi_v^{\vee})_v \mu_{B(\mathbb{Q}_v) \backslash B^x(\mathbb{Q}_v)}(x),
\]

depending on the choice of the local measure, and set

\[
C_{\psi_v^b,\psi_v^{b^\vee}}(\psi_v, \psi_v^{\vee}) := \frac{I_v(\psi_v \otimes \psi_v^\vee)}{\langle \psi_v^b, \psi_v^{b^\vee} \rangle_v}.
\]

(24)

Note also that (16) shows that

\[
f_{\Lambda_{k/E}} : \mathbb{C} \Lambda_{k/E} \otimes M \left( B_t^{x^3}, V_{k,E}, \omega_0 \right) \left[ \text{Nrd}_{t}^{-k/2} \Pi_f \right] \sim \Pi_f.
\]

Suppose that \( \varphi, \varphi^b \) and \( \varphi^{b^b} \) in \( M \left( B_t^{x^3}, V_{k,E}, \omega_0 \right) \left[ \text{Nrd}_{t}^{-k/2} \Pi_f \right] \) corresponds to pure tensors \( \psi := f_{\Lambda_{k/E}}(\varphi) = \otimes_v \psi_v, \psi^b := f_{\Lambda_{k/E}}(\varphi^b) = \otimes_v \psi_v^b \) and \( \psi^{b^b} := f_{\Lambda_{k/E}}(\varphi^{b^b}) = \otimes_v \psi_v^{b^b} \). Setting \( \psi^{b^\vee} := (\tilde{\psi}^{b^b}) \) (where \( \psi^{b^b} \mapsto (\tilde{\psi}^{b^b}) \) is the operation in the lower horizontal row in the proof of Lemma 3.2), we define

\[
I_v(\varphi) = I_v(\psi_v) := I_v(\psi_v, \tilde{\psi}_v) \quad \text{and} \quad C_{\psi_v^b,\psi_v^b^\vee}(\varphi) = C_{\psi_v^b,\psi_v^b^\vee}(\psi_v) := C_{\psi_v^b,\psi_v^b^\vee}(\psi_v, \tilde{\psi}_v).
\]

(Note that \( \tilde{\psi}^{b^b} \) is again a pure tensor, being the product of \( \psi^{b^b} \) by a pure tensor).

The following result is deduced in [22, Theorem 8.2] from [24,31] and the Jacquet conjecture proved in [24].
Theorem 3.4 Suppose that \( k \) is balanced and that \( \omega_i = \omega_{i,f} \otimes \text{sgn} (-)^{k_i} \) are unitary Hecke characters such that \( \omega_1 \omega_2 \omega_3 = 1 \), implying \( k^* \in \mathbb{N} \). Consider the quantity

\[
t_k(\varphi) = \mu(K_\varphi) \sum_{x \in K_\varphi \backslash B_k^* / B_k^*} \frac{\Lambda_k(\varphi_1(x) \otimes \varphi_2(x) \otimes \varphi_3(x))}{|\Gamma_{K_\varphi}(x)| \text{Nrd}_{t_k}^*(x)},
\]

where \( K_\varphi \in \mathcal{K} \) is such that \( K_\varphi \subset K_{\varphi_1} \cap K_{\varphi_2} \cap K_{\varphi_3} \) and

\[
\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \in \bigotimes_{i=1}^3 M(B_f^x, V_{k_i}, \omega_i, 0)^{K_{\varphi_i}} = M \left( B_f^x, V_{k,E}, \omega_0 \right)^{K_{\varphi_1} \times K_{\varphi_2} \times K_{\varphi_3}}.
\]

(1) We have the equality

\[
t_k^2(\varphi) = \frac{C}{2^{19/32} L(1/2, \Pi')} \left( \prod_v I_v(\varphi) \right) \frac{(\varphi^b, \varphi^{b,2})_k}{2 L(1, \Pi, \text{Ad})} L(1/2, \Pi') \prod_{v \neq \infty} C_v^{\varphi^b, \varphi^{b,2}}(\varphi)
\]

as quadratic forms on

\[
f_{\Lambda_k/E} : M \left( B_f^x, V_{k,E}, \omega_0 \right) \left[ \text{Nrd}_{t_f}^{-k/2} \Pi_f \right] \hookrightarrow A \left( B^x, (\mathbb{A}), \omega \right)[\Pi],
\]

where \( C \neq 0 \) is the constant defined in (22) below, \( I_v(\varphi) \) and \( C_v^{\varphi^b, \varphi^{b,2}}(\varphi) \) are defined in (25) and the second equality depends on the choice of vectors \( \varphi^b, \varphi^{b,2} \in \text{Nrd}_{t_f}^{-k/2} \Pi_f \subset M \left( B_f^x, V_{k,E}, \omega_0 \right) \) such that \( (\varphi^b, \varphi^{b,2})_k \neq 0 \) (see Lemma 3.2 for their existence and (21) for a specific choice).

(2) Suppose that \( B = B_{\Pi'} \) is the quaternion algebra predicted by [34]. Then there exists \( \varphi \) whose associated local constants \( I_v \) are all non-zero and, hence, \( L \left( \Pi', 1/2 \right) \neq 0 \) if and only if \( t_k \neq 0 \) on \( M \left( B_f^x, V_{k,E}, \omega_0 \right) \left[ \text{Nrd}_{t_f}^{-k/2} \Pi_f \right] \).

Proof (1) Let us explain how to deduce the result from [22, Theorem 8.2] in the form we need here. There, it is taken the normalization from [31, Theorem 1.1], which requires [31, (1.3) and (1.4)]. We will explicitly fix our local measures in Sect. 3.2.1 below in such a way that [31, (1.4)] is in force and (7) and the conditions before it are satisfied (so that [22, Theorem 7.2] is in force).

If \( \varphi \) (resp. \( \varphi^\vee \) in the dual representation) correspond to a pure tensor \( \psi := f_{\Lambda_k/E}(\varphi) = \otimes_v \psi_v \) (resp. \( \psi^\vee := f_{\Lambda_k/E}(\varphi^\vee) = \otimes_v \psi_v^\vee \)), then [31, Theorem 1.1] (but where we allow \( C \) to be arbitrary, i.e. [31, (1.4)] is in force but [31, (1.3)] may be not) gives, thanks to [22, Theorem 7.2 and §8.1]:

\[\otimes \text{ Springer}\]
\[ t_\hbar^\infty (\varphi) = \frac{C}{2^3 m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^2} \frac{\zeta_2^2 (2) \, L (1/2, \Pi')}{L (1, \Pi', \text{Ad})} \prod_v c_v (\psi_v \otimes \psi_v^\infty). \]

where the latter equality holds when \( \langle \psi^b, \psi^{b\varphi} \rangle_{L^2} \neq 0 \). If we specialize to the case where \( \varphi^\infty := \hat{\varphi} \), then we see that \( t_\hbar (\varphi) = t_\hbar (\hat{\varphi}) \) because we are twisting by \( nrd_{-1} \), which restricts on the diagonally embedded center of \( \mathcal{B}_x \) to 1 and \( t_\hbar \) only depend on its diagonal restriction. Define \( \psi^b := f_{\Lambda_{k/E}} (\varphi^b) \), \( \psi^{bb} := f_{\Lambda_{k/E}} (\varphi^{bb}) \) and \( \psi^{b\varphi} := (\psi^{bb}) \). The first formula is obtained from the definition (25), once we recall that \( \tilde{\varphi} = f_{\Lambda_{k/E}} (\hat{\varphi}) = \otimes_v \tilde{\varphi}_v \) (from the commutative diagram for automorphic forms on \( \mathcal{B}^3 \) analogous to those displayed in the proof of Lemma 3.2) and we will fix the measures in such a way that \( m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^{\infty} = 24 \).

It follows from (16) and the definition of \( f_{\Lambda_{k/E}} \) that we have \( \psi_\infty = \psi^b_\infty = \psi^{bb}_\infty = \Lambda_k = \Lambda_{k} \) (via \( \mathcal{V}_{k,C} = \mathcal{P}_{k,C} \) provided by the tautological evaluation pairing) and \( \tilde{\varphi}_\infty = (\psi^{bb})_\infty = \psi_\infty \) because there is no twist at infinity in the definition of \( \tilde{\varphi} \) (as remarked in the proof of Lemma 3.2, \( nrd_{-1}^{\infty, i} = 1 \)). Then, the invariance property of \( \Delta_k \) under the (unitarized) diagonal action implies that the integral that appears in (24) for \( v = \infty \) is \( m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^{\infty} (\Delta_k, \Delta_k)_{\infty} \). Hence we find

\[
C_{\infty}^{\psi^b, \psi^{bb}} (\psi_\infty) = m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^{\infty} \frac{L_\infty (1, \Pi', \text{Ad})}{\zeta_2^2 (2)} \frac{L_\infty (1/2, \Pi')}{L (1, \Pi', \text{Ad})}. \]

Also, (20) specializes to

\[
\langle \psi^b, \psi^{b\varphi} \rangle_{L^2} = \frac{m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^3}{(k_1 + 1) (k_2 + 1) (k_3 + 1)} \frac{\Delta_k, \Delta_k_k}{(\varphi^b, \varphi^{bb})_k} \frac{L_\infty (1/2, \Pi')}{L (1, \Pi', \text{Ad})}. \]

Inserting these last two equations in (27) gives the formula:

\[
t_\hbar^\infty (\varphi) = \gamma_{\infty, k} \frac{\zeta_2^2 (2)}{(\varphi^b, \varphi^{bb})_k} \frac{L_\infty (1/2, \Pi')}{L (1, \Pi', \text{Ad})} \prod_v c_v (\psi_v \otimes \psi_v^\infty). \]

where

\[
\gamma_{\infty, k} = \frac{1}{2^3 m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^2} \frac{\langle \psi, \tilde{\varphi} \rangle_{L^2}^2}{\langle \varphi^b, \varphi^{bb} \rangle_\hbar^2} \frac{C_{\infty}^{\psi^b, \psi^{bb}} (\psi_\infty)}{\zeta_2^2 (2)} = \frac{m_{\mathcal{Z}_B \backslash \mathcal{B}_x}^2}{2^3} \frac{(\Delta_k, \Delta_k k)}{(k_1 + 1) (k_2 + 1) (k_3 + 1)} \frac{L_\infty (1/2, \Pi')}{L (1, \Pi', \text{Ad})}. \]
The claim is proved, once we fix local measures which in turn fix the local constants \( C_v^b (\ldots) \) in such a way that \( m_{\mathbb{Z}_p} \mathbb{B}^{\times, \infty} = 24 \), make explicit the ratio 

\[
\frac{\langle \Delta_{\infty}, \Delta_{\infty} \rangle_{L_1} (1, \Pi_v, \text{Ad})}{(k_1+1)(k_2+1)(k_3+1)L_0 (1/2, \Pi_v)} = \frac{1}{4\pi^3}
\]

and insert the value \( \xi, f \) \( (2) = \frac{\pi^2}{6} \); this is detailed in 3.2.1 below.

(2) This is proved in [22, Theorem 8.1 (3)], as a consequence of the Jacquet’s conjecture proved in [24], using [22, Theorem 8.1 (2)] and (2).

**Remark 3.5** If one wants to make (26) explicit, the first equality is poorly useful because \( I_v \) depends on the choice of \( \langle -,- \rangle_v \) and, hence, on the fixed local model. On the other hand, \( C_v^b,\psi^b \) is much more canonical: in fact, we see from (23) and (24) that \( C_v^{\psi^b,\psi^b} (\psi_v, \psi_v^\vee) \) does not depend on the choice of the local pairing \( \langle -,- \rangle_v \); also, if \( \psi_v \) and \( \psi_v^\vee \) linearly depends on \( \psi_v^b \) and, respectively, \( \psi_v^{\vee} \), we see that \( C_v^{\psi^b,\psi^b} (\psi_v, \psi_v^\vee) \) does not depend on the line spanned by either \( \psi_v \) and \( \psi_v^b \) or \( \psi_v^\vee \) and \( \psi_v^{\vee} \). This is very convenient in order to “transport” calculations from abstract local models.

Regarding \( t_k^2 \) as the algebraic part of \( L (1/2, \Pi) \) (see [22] for a justification), it follows from Theorem 3.4 that the relevant part to be interpolated is \( t_k \). Applying Proposition 2.7, we place \( t_k \) in a \( p \)-adic setting, making it correspond to \( t_k := J_p (\Lambda_{k/Q_p}) \)

\[
t_k (\varphi_1, \varphi_2, \varphi_3) = \mu (K_p) \sum_{x \in K_p \backslash \mathbb{B}^{\times, \infty} / \mathbb{B}^{\times, \infty}} \frac{\Lambda_{k/Q_p} (\varphi_1 (x) \otimes \varphi_2 (x) \otimes \varphi_3 (x))}{\vert \Gamma_{K_p} (x) \vert \text{Nrd}^k_{p} (x)}.
\]

We have already interpolated the association \( k \mapsto \text{Nrd}^k_p (x) \) in Sect. 2.1 and we will now proceed to interpolate the association \( k \mapsto \Lambda_{k/Q_p} \). To this end, we first review and prove some facts on distribution modules, by means of which \( p \)-adic families of modular forms are defined.

### 3.2.1 Choice of measures and further computations

Let us fix local measures in a such a way that the condition \( \mu_{B^{\times, \infty}_1} (K) \in \mathbb{Q} \) for some (and hence every) open and compact subgroup \( K \subset B_1^{\times, \infty} \) and the integration formula (7) are satisfied: at the same time this fix the constant \( m_{\mathbb{Z}_p} \mathbb{B}^{\times, \infty} \), that we have to determine, and show that [31, (1.4)] is in force. To this end, we first record the following lemma, whose proof is easy and left to the reader. Suppose that \( \Gamma \subset G = G_0 \times G_{1\infty} \) is a discrete subgroup of a product of Hausdorff and locally compact topological groups \( G \) and that composition with the projection makes \( \Gamma \subset G \rightarrow G_0 \) a discrete subgroup such that \( G_0 / \Gamma \) is compact. Let us fix Haar measures \( \mu_{G} \) (resp. \( \mu_{G_0} \)) on \( G \) (resp. \( G_0 \)) and let \( \mu_{G/\Gamma} \) (resp. \( \mu_{G_0/\Gamma} \)) be the \( G \)-invariant (resp. \( G_0 \)-invariant) quotient measures on \( G/\Gamma \) (resp. \( G_0/\Gamma \)), normalized as usual. Let \( Z = Z_0 \times Z_{1\infty} \subset G_0 \times G_{1\infty} = G \) be a fixed closed subgroup in the center. Using the compactness of \( G_0 / \Gamma \), it is not difficult to see that there is a (unique up to non-zero scalar) non-zero left \( G_0 \)-invariant Radon measure \( \mu_{Z_0} G_0 / \Gamma \) on \( Z_0 \backslash G_0 / \Gamma \) with the property that \( \mu_{Z_0} G_0 / \Gamma = \mu_{G_0/\Gamma} |C_c (Z_0 \backslash G_0 / \Gamma) \) if \( C_c (Z_0 \backslash G_0 / \Gamma) \subset C_c (G_0 / \Gamma) \) by means of the
pull-back induced by $\pi : G_0/\Gamma \to Z/G_0/\Gamma$. On the other hand, we set $\mu_{Z/G_0/\Gamma} := \mu_{\mathcal{Z}/\mathcal{Z}}^\mathbf{Z}$, where once again the subquotient measures on the right hand side are obtained from $\mu_G$ with the usual normalizations.

We normalize the Haar measure $\mu_{G_\infty}$ on $G_\infty$ so that $\mu_G = \mu_{G_0} \times \mu_{G_\infty}$ is satisfied:

$$\int_G f (g) \, d\mu_G (g) = \int_{G_\infty} \left( \int_{G_\infty} f (g_0 g_\infty) \, d\mu_{G_\infty} (g_\infty) \right) \, d\mu_{G_0} (g_0).$$

Assuming that $Z_\infty \setminus G_\infty$ is compact, we get the formula

$$\int_{Z\setminus G/\Gamma} f (x) \, d\mu_{Z\setminus G/\Gamma} (x) = c \int_{Z_0 \setminus G_0/\Gamma} \left( \int_{Z_\infty \setminus G_\infty} f (x_0 x_\infty) \, d\mu_{Z_\infty/G_\infty} (x_\infty) \right) \, d\mu_{Z_0 \setminus G_0/\Gamma} (x_0)$$

for some $c \in \mathbb{R}_+^\times$.

**Lemma 3.6** Suppose that $G_0$ is locally profinite and let $\mathcal{K} = \mathcal{K} (G_0)$ be the set of its open and compact subgroups.

1. We have $c = 1$.
2. $\mu_{Z\setminus G/\Gamma} (Z\setminus G/\Gamma) = \mu_{Z_0 \setminus G_0/\Gamma} (Z_0 \setminus G_0/\Gamma) \mu_{Z_\infty/G_\infty} (Z_\infty/G_\infty)$.
3. If $\mu_{G_0} (K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}$, we have $\mu_{Z_0 \setminus G_0/\Gamma} (Z_0 \setminus G_0/\Gamma) \in \mathbb{Q}$.

Let us apply this lemma with $\Gamma = B_\times$, $G_0 = B_1^\times$ and $G_\infty = B_\infty (\mathbb{R})$. Let us write $D$ for the discriminant of our quaternion algebras and fix an identification $\mathbf{B} (\mathbb{A}_F^D) \simeq \mathbf{M}_2 (\mathbb{A}_F^D)$. Fix local measures $\mu_{B_\times, l}$ (resp. $\mu_{Z_B \setminus B_\times, l}$) of $B_\times (\mathbb{Q}_l)$ (resp. $Z_B (\mathbb{Q}_l) \setminus B_\times (\mathbb{Q}_l)$) at the finite primes $l$ as in [43, 2.2]: if $l \nmid D$ (resp. $l \mid D$), $\mu_{Z_B \setminus B_\times, l}$ is the Haar measure such that $\mu_{B_\times, l} (\text{GL}_2 (\mathbb{Z}_l)) = 1$ (resp. $\mu_{B_\times, l} (\mathcal{O}_{B_l}^\times) = (p - 1)^{-1}$, where $\mathcal{O}_{B_l}$ is a maximal order in $B_l$) and $\mu_{Z_B \setminus B_\times, l}$ the quotient measure, normalized in the usual way. Then $\mu_{B_1^\times}$ satisfies our first requirement. Next, we take $\mu_{B_\times, l}$ on $B_\times (\mathbb{A}_F)$ in such a way that the induced quotient measure $\mu_{Z_B \setminus B_\times, l}$ on $Z_B (\mathbb{A}_F) \setminus B_\times (\mathbb{A}_F)$ is the Tamagawa measure, so that [31, (1.4)] is in force. Finally, we fix $\mu_{B_\times, \infty}$ via the formula $\mu_{B_\times, l} = \mu_{B_1^\times} \times \mu_{B_\times, \infty}$. It follows from Lemma 3.6 (1) that (refMeasuresF1) is in force and from Lemma 3.6 (2) that we have:

$$\mu_{[B_\times, l]_{Z_B}} ([B_\times (\mathbb{A}_F)]_{Z_B}) = m_{Z_B \setminus B_\times, \infty} \mu_{[B_1^\times]_{Z_B}} ([B_1^\times]_{Z_B}).$$

The left hand side is the Tamagawa number of $Z_B \setminus B_\times$, which is known to be 2 (see [42, Theorem 3.2.1]). On the other hand, choosing the local measures at the finite primes $l \mid D$ in such a way that $\mu_{B_\times, l} (\mathcal{O}_{B_l}^\times) = 1$, the total measure of $[B_1^\times]_{Z_B}$ is known to be $\frac{1}{12} \prod_{l \mid D} (l - 1)$ by the Eichler’s mass formula (see [44, Lemma 2.2]...
and Theorem 3.6, (3.17)], for example, where \( \zeta_{Q,f}(-1) = -1/12 \). We deduce that

\[
\mu_{[B^\times_f]}(\mathbb{Z}_p) = \frac{1}{12} \quad \text{and, hence,}
\]

\[
m_{\mathbb{Z}_p}[B^\times, \infty] = 24.
\]

Next, we recall that we have, setting \( \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2) \) and \( \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s/2}\Gamma(s) \),

\[
L_\infty(s, \Pi', \text{Ad}) = \pi^{-3}\Gamma_{\mathbb{C}}(s + k_1 + 2)\Gamma_{\mathbb{C}}(s + k_2 + 2)\Gamma_{\mathbb{C}}(s + k_3 + 2),
\]

\[
L_\infty(s, \Pi') = \Gamma_{\mathbb{C}}(s + k_1^* + \frac{3}{2})\Gamma_{\mathbb{C}}(s + k_2^* + \frac{1}{2}) \times \Gamma_{\mathbb{C}}(s + k_3^* + \frac{1}{2}).
\]

Then we see that

\[
\frac{L_\infty(1, \Pi', \text{Ad})}{L_\infty(1/2, \Pi')} = \frac{\Gamma(k_1 + 2)\Gamma(k_2^* + 2)\Gamma(k_3^* + 2)}{4\pi^4\Gamma(k_1^* + 2)\Gamma(k_2^* + 1)\Gamma(k_3^* + 1)}.
\]

We claim that

\[
\langle \Delta_{k}, \Delta_{\tilde{k}} \rangle = \frac{(k_1^* + 1)k_1^{*+1}k_2^{*+1}k_3^{*+1}}{k_1!k_2!k_3!} = \frac{\Gamma(k_1^* + 2)\Gamma(k_2^* + 1)\Gamma(k_3^* + 1)}{\Gamma(k_1 + 1)\Gamma(k_2 + 1)\Gamma(k_3 + 1)},
\]

from which the equality \( \langle \Delta_{k}, \Delta_{\tilde{k}} \rangle L_\infty(1, \Pi', \text{Ad}) = \frac{1}{4\pi^4} \) follows. The pairing \( \langle - , - \rangle_{k/E} \) on the polynomials is explicitly given by

\[
\left< X^r_i y^k_i = r , X^s_i y^k_i = s \right>_{k/E} = \begin{cases} (-1)^{k_i - r} (k_r)^{-1} & \text{if } r + s = k_i, \\ 0 & \text{if } r + s \neq k_i. \end{cases}
\]

Let us write \( \delta_3 : \mathbf{P}_{k_1,k_2,k_3} \rightarrow \mathbf{P}_{k_1+1,k_2+1,k_3} \) for the multiplication by \( \delta^1 \) \( (W_1, W_2) \) map and let \( \delta_3^* : \mathbf{P}_{k_1+1,k_2+1,k_3} \rightarrow \mathbf{P}_{k_1,k_2,k_3} \) be its adjoint with respect to the perfect pairings \( \langle - , - \rangle_{(k_1,k_2,k_3)} \) and \( \langle - , - \rangle_{(k_1+1,k_2+1,k_3)} \) (we do not write the subscript /E for brevity). Then one checks that \( \delta_3(\Delta_{k_1,k_2,k_3}) = \Delta_{k_1+1,k_2+1,k_3} \) and \( \delta_3^*(\Delta_{k_1+1,k_2+1,k_3}) = \frac{(k_1^* + 2)(k_2^* + 1)}{(k_1 + 1)(k_2 + 1)}\Delta_{k_1,k_2,k_3} \) (note that \( \delta_3 \) and hence \( \delta_3^* \) are \( \text{SL}_2 \)-equivariant, from which the equality \( \delta_3^*(\Delta_{k_1+1,k_2+1,k_3}) = \lambda \Delta_{k_1,k_2,k_3} \) is known a priori taking the \( \text{SL}_2 \)-invariants for a scalar factor \( \lambda \)).

Assuming by induction that we have proved our claim for \( k = (k_1, k_2, k_3) \), we find the claim for \( (k_1 + 1, k_2 + 1, k_3) \):

\[
\lambda \langle P, \Delta_{k_1,k_2,k_3} \rangle_{k_1,k_2,k_3} = \langle P, \delta_3^*(\Delta_{k_1+1,k_2+1,k_3}) \rangle_{k_1+1,k_2+1,k_3} = \langle \delta_3 P, \Delta_{k_1+1,k_2+1,k_3} \rangle_{k_1+1,k_2+1,k_3}.
\]

A good choice is to take \( P = X_1^{k_1} \otimes X_2^{k_2} \otimes X_3^{k_3} \).

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\[ \begin{align*}
\delta_3 \left( \Delta_{k_1,k_2,k_3}, \Delta_{k_1+1,k_2+1,k_3} \right)_{k_1+1,k_2+1,k_3} &= \delta_3 \left( \Delta_{k_1,k_2,k_3}, \Delta_{k_1+1,k_2+1,k_3} \right)_{k_1,k_2,k_3} \\
&= \left( \frac{k^* + 2}{k!} \right) \left( \frac{k^* + 1}{k!} \right) \left( \frac{k^* + 1}{k!} \right) \left( \frac{k^* + 1}{k!} \right) \left( \frac{k^* + 1}{k!} \right) \left( \frac{k^* + 1}{k!} \right) = \frac{(k^* + 2)!k^*!k^*!k^*!k^*!}{(k!)(k!)(k!)(k!)}.
\end{align*} \]

4 Spaces of homogeneous $p$-adic distribution spaces

4.1 Locally analytic homogeneous distributions

By a $p$-adic manifold $X$ we always mean a locally compact and paracompact manifold over a fixed spherically complete non-archimedean $p$-adic field. For a Banach algebra $O$, let $A(X, O)$ be the space of $O$-valued locally analytic functions on $X$ and set $D(X, O) := \mathcal{L}_O(A(X, O), O) \subset \mathcal{L}(A(X, O), O)$, the strong $O$-dual of $A(X, O)$. If $f : X \to Y$ is a morphism of $p$-adic manifolds, we have

\[ f^*_O : A(Y, O) \to A(X, O) \quad \text{and} \quad f^*_O : D(X, O) \to D(Y, O), \]

the first being the pull-back of functions $f^*_O(F) := F \circ f$ and the second operation being the strong $O$-dual of the first. We note that

\[ f^*_x \left( \delta^O_x \right) = \delta^O_{f(x)}, \quad \text{for every } x \in X \quad (29) \]

if $\delta^O : X \to D(X, O)$ denotes the Dirac distribution map. It is useful to remark that the $O$-linear span of $\{ \delta^O_x : x \in X \}$ is dense in $D(X, O)$ (see [21, (52)]): we refer to this fact using the set phrase “by density of Dirac distributions”. It can be shown that there are topological identifications\(^4\)

\[ T^O_{\mathcal{D}(X)} : O \widehat{\otimes} D(X) \sim D(X, O) \]

and

\[ P^O_{D(X_1), D(X_2)} : D(X_1, O_1) \otimes D(X_2, O_2) \sim D(X_1 \times X_2, O_1 \widehat{\otimes} O_2). \]

They are characterized by the equalities:

\[ T^O_{\mathcal{D}(X)} \left( 1 \widehat{\otimes} \delta^O_X \right) = \delta^O_X \quad \text{and} \quad P^O_{D(X_1), D(X_2)} \left( \delta^O_{X_1} \widehat{\otimes} \delta^O_{X_2} \right) = \delta^O_{X_1 \times X_2}. \quad (30) \]

\(^4\) We write $V \otimes_1 W$ (resp. $V \otimes W$) to denote $V \otimes W$ with the inductive (resp. projective) tensor topology.
We will usually suppress the reference to the Banach algebra when this is the fixed $p$-adic field.

Suppose from now on that $X$ is endowed with the action of a $p$-adic Lie group $T$, meaning that the action is given by a locally analytic map $a : T \times X \to X$. Then $T$ naturally acts from the right on $\mathcal{A}(X, \mathcal{O})$ and from the left on $\mathcal{D}(X, \mathcal{O})$. The left action of $T$ on $\mathcal{D}(X)$ can be extended, with respect to $\delta : T \to \mathcal{D}(T)$, to a left action of $\mathcal{D}(T)$ making $\mathcal{D}(X)$ a $\mathcal{D}(T)$-module by the convolution product:

$$\mathcal{D}(T) \otimes_{\mathcal{O}} \mathcal{D}(X) \xrightarrow{p_{\mathcal{D}(T), \mathcal{D}(X)}} \mathcal{D}(T \times X) \xrightarrow{a_{\mathcal{O}}} \mathcal{D}(X).$$ (31)

We note the formula

$$\delta_t \cdot \delta_x = \delta_{tx} \quad \text{for} \quad t \in T \quad \text{and} \quad x \in X,$$ (32)

which indeed characterizes the multiplication law by density of the Dirac distributions.

Also, we remark that the multiplication map is in general separately continuous, while it is continuous if we assume that $T$ and $X$ are compact. In particular, one checks that $\mathcal{D}(T)$ becomes an algebra in this way. We write $\text{Hom}_{\mathcal{A}}(T, \mathcal{O}^\times)$ to denote the group of those group homomorphisms such that their composition with the inclusion $\mathcal{O}^\times \subset \mathcal{O}$ belongs to $\mathcal{A}(T, \mathcal{O})$. We also write $\text{Hom}_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O})$ to denote the space of those morphisms of locally convex spaces that are morphisms of algebras. Then there is a bijection (see [21, Lemma 7.2])

$$C^\mathcal{O} : \text{Hom}_{\mathcal{L}}(\mathcal{D}(T), \mathcal{O}) \sim \text{Hom}_{\mathcal{A}}(T, \mathcal{O}^\times), \text{ via } C^\mathcal{O}(k)(t) := k(\delta_t).$$ (33)

We will abuse of notations, when there will be no risk of confusion, and identify these two sets, deserving the exponential notation to the group homomorphisms and calling the elements of these sets weights.

If $k$ is a weight, we may consider the space of locally analytic homogeneous functions:

$$\mathcal{A}_k(X) = \mathcal{A}(X, k) = \left\{ F \in \mathcal{A}(X, \mathcal{O}) : F(tx) = t^k F(x) \right\}.$$

It is indeed a closed $\mathcal{O}$-submodule of $\mathcal{A}(X, \mathcal{O})$. Viewing both $\mathcal{O}$ and $\mathcal{D}(X)$ as $\mathcal{D}(T)$-modules by means of $k$ and, respectively, the convolution product, we may define

$$\mathcal{D}_k(X) := \mathcal{O} \widehat{\otimes}_k \mathcal{D}(X) \quad \text{and} \quad \mathcal{D}(X, k) := \mathcal{L}_\mathcal{O} (\mathcal{A}_k(X), \mathcal{O}).$$

We also assume from now on that $X$ is endowed with a right action by a semigroup $\Sigma$ such that $\sigma : \Sigma \to \Sigma$ is locally analytic for every $\sigma \in \Sigma$, which is compatible with the left $T$-action in the sense that $t(x\sigma) = (tx)\sigma$ for all $t \in T$, $x \in X$, and $\sigma \in \Sigma$. It follows that $\sigma$ induces a well defined action on $\mathcal{A}_k(X)$, $\mathcal{D}_k(X)$ and $\mathcal{D}(X, k)$. The relation between the space $\mathcal{D}_k(X)$ and $\mathcal{D}(X, k)$ is expressed by means of an $(\mathcal{O}, \Sigma)$-equivariant morphism of locally convex spaces

$$\mathcal{T}_{\mathcal{D}(X)}^k : \mathcal{D}_k(X) \to \mathcal{D}(X, k)$$ (34)
which is an isomorphism when \( X \) is a trivial (equivalently locally trivial) \( T \)-bundle. It is characterized by the property that

\[
T^k_{\mathcal{D}(X)} \left( 1 \otimes_k \delta_x \right) = \delta^k_x \quad \text{for every } x \in X,
\]

if \( \delta^k_x \) is the image of \( \delta^O_x \). We refer the reader to [21, Lemma 7.3 and Proposition 7.6] for details.

It follows from (34) that the elements of \( \mathcal{D}_k (X) \) naturally integrates functions in \( \mathcal{A}_k (X) \). Furthermore they are endowed with natural specialization maps, not possessed by the spaces \( \mathcal{D} (X, k) \), defined as follows. If we are given \( k_i \in Hom_L (\mathcal{D} (T_i), \mathcal{O}_i) \), we say that \( k_1 \) specializes via \( \phi \) to \( k_2 \), and we write \( k_1 \rightarrow \phi k_2 \), if \( \phi \in Hom_L (\mathcal{O}_1, \mathcal{O}_2) \) and \( k_2 = \phi \circ k_1 \). Then we have an induced specialization map

\[
\phi_* : \mathcal{D}_{k_1} (X) \rightarrow \mathcal{D}_{k_2} (X) \quad \text{via} \quad \phi_* (\alpha \otimes_{k_1} \mu) := \phi (\alpha) \otimes_{k_2} \mu.
\]

(35)

### 4.2 Multiplying locally analytic homogeneous distributions

Now suppose that we are given two \( p \)-adic locally compact and paracompact manifolds \( X_i \) endowed with analytic actions of \( T_i \) for \( i = 1, 2 \), so that \( T_1 \times T_2 \) act on \( X_1 \times X_2 \) in the obvious way. Let us be given \( k_i \in Hom_L (\mathcal{D} (T_i), \mathcal{O}_i) \). We define the continuous morphism of locally convex spaces

\[
k_1 \boxplus k_2 : \mathcal{D} (T_1 \times T_2) \overset{P^{-1}_{\mathcal{D}(T_1), \mathcal{D}(T_2)}}{\longrightarrow} \mathcal{D} (T_1) \otimes_i \mathcal{D} (T_2) \overset{k_1 \otimes \mathcal{O}_1 \otimes k_2}{\longrightarrow} \mathcal{O}_1 \otimes \mathcal{O}_2 \overset{\mathcal{O}_1 \otimes \mathcal{O}_2}{\longrightarrow} \mathcal{O}_1 \otimes \mathcal{O}_2.
\]

(36)

Exploiting the effect on Dirac distributions and noticing that the multiplications laws are separately continuous by (31), it is not difficult to deduce from the density of Dirac distributions that \( k_1 \boxplus k_2 \) is a morphism of algebras, hence

\[
k_1 \boxplus k_2 \in Hom_L (\mathcal{D} (T_1 \times T_2), \mathcal{O}_1 \otimes \mathcal{O}_2).
\]

We assume that \( X_i \) is further endowed with a right action by a semigroup \( \Sigma_i \) having the same properties of the \( \Sigma \)-action considered above.

**Lemma 4.1** There is a unique morphism of locally convex spaces \( P^{k_1,k_2}_{\mathcal{D}(X_1), \mathcal{D}(X_2)} \) making the following diagram commutative, which is \( (\mathcal{O}_1 \otimes \mathcal{O}_2, \Sigma_1 \times \Sigma_2) \)-equivariant:

\[
\begin{array}{ccc}
\mathcal{O}_1 \otimes \mathcal{D} (X_1) \otimes \mathcal{O}_2 \otimes \mathcal{D} (X_2) & \overset{1 \otimes \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}}{\longrightarrow} & \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{D} (X_1 \times X_2) \\
\downarrow & & \downarrow \\
\mathcal{D}_{k_1} (X_1) \otimes_i \mathcal{D}_{k_2} (X_2) & \overset{P^{k_1,k_2}_{\mathcal{D}(X_1), \mathcal{D}(X_2)}}{\longrightarrow} & \mathcal{D}_{k_1 \boxplus k_2} (X_1 \times X_2).
\end{array}
\]

(37)

**Proof** Let \( B \) be the composition of \( 1 \otimes \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes P_{\mathcal{D}(X_1), \mathcal{D}(X_2)} \) with the right vertical morphism. Since we know that \( B \) is continuous and \( \mathcal{D}_{k_1 \boxplus k_2} (X_1 \times X_2) \) is Hausdorff.
and complete, we first need to show that, for every \( \alpha_i \in \mathcal{O}_i \), \( \mu_i \in \mathcal{D}(X_i) \) and \( \nu_i \in \mathcal{D}(T_i) \)

\[
B \left( \alpha_1 k_1 (v_1) \otimes \mu_1 \otimes \alpha_2 k_1 (v_2) \otimes \mu_2 \right) = B \left( \alpha_1 \otimes (v_1 \cdot \mu_1) \otimes \alpha_2 \otimes (v_2 \cdot \mu_2) \right).
\]

It turns out that this is equivalent to checking the equalities

\[
P_{\mathcal{D}(T_1), \mathcal{D}(T_2)}(v_1 \otimes v_2) \cdot P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}(\mu_1 \otimes \mu_2) = P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}((v_1 \cdot \mu_1) \otimes (v_2 \cdot \mu_2)) \in \mathcal{D}(X_1 \times X_2).
\]

When \( \mu_i = \delta_{x_i} \) and \( \nu_i = \delta_{t_i} \) we have indeed, by (30) and (32)

\[
P_{\mathcal{D}(T_1), \mathcal{D}(T_2)}(v_1 \otimes v_2) \cdot P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}(\mu_1 \otimes \mu_2) = \delta_{(t_1, t_2)} \cdot \delta_{(x_1, x_2)} = \delta_{(t_1, x_1, t_2, x_2)},
\]

\[
P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}((v_1 \cdot \mu_1) \otimes (v_2 \cdot \mu_2)) = P_{\mathcal{D}(X_1), \mathcal{D}(X_2)}(\delta_{t_1 x_1} \otimes \delta_{t_2 x_2}) = \delta_{(t_1, x_1, t_2, x_2)}.
\]

We note that both the left and the right hand sides of (38) are linear in the variables \( \mu_i \) and \( \nu_i \). Furthermore, if we fix three of these variables, the two resulting functions are continuous in the remaining variable thanks to (31) showing that the multiplication laws are separately continuous. Hence the claimed equality (38) follows from the density of Dirac distributions. The existence and uniqueness of \( P_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \) follows and, since \( \mathcal{O}_1 \otimes \mathcal{D}(X_1) \otimes \mathcal{O}_2 \otimes \mathcal{D}(X_2) \rightarrow \mathcal{D}(X_1) \otimes \mathcal{D}(X_2) \) is surjective and all the arrows other than \( P_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \) in (37) are \( (\mathcal{O}_1 \otimes \mathcal{O}_2, \Sigma_1 \times \Sigma_2) \)-equivariant (by (30)), implying that \( P_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \) is equivariant as well.

In particular we may define

\[
\mathcal{P}_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) : \mathcal{D}(X_1) \otimes \mathcal{D}(X_2) \rightarrow \mathcal{D}(X_1 \times X_2),
\]

\[
\mathcal{P}_{k_1 \oplus k_2}^{k_1 \oplus k_2} \mathcal{D}(X_1 \times X_2) \rightarrow \mathcal{D}(X_1 \times X_2, k_1 \oplus k_2).
\]

If \( \mu_i \in \mathcal{D}(X_i) \) for \( i = 1, 2 \), we set

\[
\mu_1 \otimes \mu_2 := \mathcal{P}_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2)(\mu_1 \otimes \mu_2) \in \mathcal{D}(X_1 \times X_2, k_1 \oplus k_2).
\]

Of course, the formation of \( k_1 \oplus k_2, \mathcal{P}_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \) and \( \mathcal{P}_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \) extends to a finite number of indices and the usual associativity constraints are satisfied, as well as the compatibility with the commutativity constraints in the sources and the targets of these maps. We finally remark that the equations

\[
P_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \left( 1 \otimes \delta_{x_1} \otimes \delta_{x_2} \right) = 1 \otimes \delta_{x_1} \otimes \delta_{x_2},
\]

\[
P_{k_1, k_2}^{k_1, k_2} \mathcal{D}(X_1), \mathcal{D}(X_2) \left( 1 \otimes \delta_{x_1} \otimes \delta_{x_2} \right) = \delta_{(x_1, x_2)}.
\]

characterize these maps.
4.3 Algebraic operations on weights

Setting \( \mathcal{X}_T (\mathcal{O}) := \text{Hom}_A (T, \mathcal{O}^\times) \) defines a group functor on Banach algebras, so that we have

\[
+ : \mathcal{X}_T (\mathcal{O}) \times \mathcal{X}_T (\mathcal{O}) \to \mathcal{X}_T (\mathcal{O}) \quad \text{and} \quad - : \mathcal{X}_T (\mathcal{O}) \to \mathcal{X}_T (\mathcal{O}) .
\]

It follows from (33) that we may transport these operations getting

\[
+ : \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i) \times \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i) \to \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i)
\]

and

\[
- : \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i) \to \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i) .
\]

Our next task it to interpolate these operations.

If we are given \( k_i \in \text{Hom}_L (\mathcal{D} (T), \mathcal{O}_i) \), then we define

\[
k_1 \oplus k_2 : \mathcal{D} (T) \xrightarrow{\Delta} \mathcal{D} (T \times T) \xrightarrow{k_1 \boxplus k_2} \mathcal{O}_1 \boxtimes \mathcal{O}_2 ,
\]

where \( \Delta : T \to T \times T \) is the diagonal map and \( k_1 \boxplus k_2 \) is given by (36).

If \( k \in \text{Hom}_L (\mathcal{D} (T), \mathcal{O}) \), then we define

\[
\ominus k : \mathcal{D} (T) \xrightarrow{i_s} \mathcal{D} (T) \xrightarrow{k} \mathcal{O} ,
\]

where \( i : T \to T \) is the inversion, and set

\[
k_1 \ominus k_2 := k_1 \oplus (\ominus k_2)
\]

We note that these operations are obviously functorial and compatible with specialization.

Exploiting the definitions and making (36) explicit it is easy to check the following result.

**Lemma 4.2** Suppose that \( k, k_i \in \text{Hom}_L (\mathcal{D} (T), \mathcal{O}) \) and write

\[
m_\mathcal{O} : \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}
\]

for the multiplication map. Then \( -k = \ominus k \) and

\[
k_1 + k_2 : \mathcal{D} (T) \xrightarrow{k_1 \oplus k_2} \mathcal{O} \boxtimes \mathcal{O} \xrightarrow{m_\mathcal{O}} \mathcal{O} .
\]

We now illustrate why \( k_1 \oplus k_2 \) interpolates the + operation. Suppose that \( F \) is our \( p \)-adic working field and that \( k_i \in \text{Hom}_L (\mathcal{D} (T), F) \) are such that \( \phi_i \to k_i \). Then

\[
k_1 \oplus k_2 \xrightarrow{\phi_1 \boxplus \phi_2} k_1 \oplus k_2
\]
by the compatibility of the $\oplus$-operation with specializations. But we have $F \otimes F = F$ canonically and the identification is given by $m_F$. Hence $k_1 \oplus k_2$ specializes via $\phi_1 \otimes \phi_2$ to $k_1 + k_2$, thanks to Lemma 4.2. In particular, suppose that $\mathcal{X}_T$ is representable by a rigid analytic space (for example because $T$ is compact) and that $k_i \to k_i$ corresponds to $k_i \in U_i$, with $U_i \subset \mathcal{X}_T$ an affinoid neighbourhood of $k$. Then $k_1 \oplus k_2$ corresponds to

$$U_1 \times U_2 \subset \mathcal{X}_T \times \mathcal{X}_T \overset{\phi}{\to} \mathcal{X}_T.$$

We finally remark that, as a consequence of the associativity of the operation in $T$, we have

$$(k_1 \oplus k_2) \oplus k_3 \simeq k_1 \oplus (k_2 \oplus k_3)$$

up to

$$(\mathcal{O}_1 \otimes \mathcal{O}_2) \otimes \mathcal{O}_3 \simeq \mathcal{O}_1 \otimes (\mathcal{O}_2 \otimes \mathcal{O}_3).$$

A similar compatibility holds true for the commutativity, when $T$ is commutative as in our applications.

Suppose now $T \simeq \Delta \times (1 + p\mathbb{Z}_p)^r$, where $\Delta$ is the torsion part of $T$, and consider the multiplication by 2 map $t \mapsto t^2$ (we write $T$ multiplicatively). We say that $k \in \mathcal{X}_T(\mathcal{O})$ is even if it is in the image of $2^* : \mathcal{X}_T(\mathcal{O}) \to \mathcal{X}_T(\mathcal{O})$ and set $\frac{k}{2}$ for an element in the inverse image of $k$. For example, suppose that $p \neq 2$ and $T = \mathbb{Z}_p^\times \simeq \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$. We can decompose every $k \in \mathcal{X}_T(\mathcal{O})$ in the form $k = ([k], (k))$, where $[k] \in \mathbb{F}_p^\times$ and $(k) \in \mathcal{X}_{1 + p\mathbb{Z}_p}(\mathcal{O})$. Since $t \mapsto t^2$ is invertible on $1 + p\mathbb{Z}_p$, $k = ([k], (k))$ is even if and only if $[k] \in \mathbb{F}_p^\times$ and then $\frac{k}{2} \in \{ \left( \left[ \frac{k}{2} \right], \left[ \frac{k}{2} \right] \right), \left( -\left[ \frac{k}{2} \right], \left[ \frac{k}{2} \right] \right) \}$; if $[k] = [k_0]$ for some integer $k_0$ our convention is to choose $\frac{k}{2} = \left( \left[ \frac{k_0}{2} \right], \left[ \frac{k_0}{2} \right] \right)$. Then $k \to \frac{k}{2} \in \mathbb{Z}_p$ implies $k \in 2\mathbb{N}$ and $\frac{k}{2} \to \frac{k}{2}$.

The elements of $Hom_L(D(T), \mathcal{O}) \simeq \mathcal{X}_T(\mathcal{O})$ are called $\mathcal{O}$-weights; we will freely identify $k_1 \boxtimes k_2 \simeq (k_1, k_2)$. We will write $\mathcal{X} := \mathcal{X}_{\mathbb{Z}_p^\times}$.

5 The $p$-adic trilinear form

The semigroup $\Sigma_0(p\mathbb{Z}_p) \subset M_2(\mathbb{Z}_p)$ acts from the right on the set $W := \mathbb{Z}_p^\times \times \mathbb{Z}_p$. Setting $\omega_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, we have $\tilde{W} := W \omega_p = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$, on which $\Sigma_0(p\mathbb{Z}_p)^t$ acts from the right. Hence, for an $\mathcal{O}$-weight $k$, we may form the right $\Sigma_0(p\mathbb{Z}_p)$-module (resp. $\Sigma_0(p\mathbb{Z}_p)^t$-module) $D_k(W)$ (resp. $D_k(\tilde{W})$). Taking $K_\mathcal{O}^\circ = \Gamma_0(p\mathbb{Z}_p) := \Sigma_0(p\mathbb{Z}_p) \cap GL_2(\mathbb{Z}_p)$, we may form the spaces of $p$-adic families of modular forms on $B^\times$:

$$M_p^\circ(D_k(W), \omega_0, p) := M_p^\circ(B_1^\times, D_k(W), \omega_0, p).$$

Recall we work over a $p$-adic field $F$ and consider Banach $F$-algebras: we set $M_p^\circ(V_{k,F}, \omega_0, p) := M_p^\circ(B_1^\times, V_{k,F}, \omega_0, p)$ and $M_p^\circ(V_{k,F}, \omega_0) := M^\circ(B_1^\times, V_{k,F}, \omega_0)$. 

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As explained after (9), they are naturally \( B_f^{\infty, p} \times \mathcal{H}(\Sigma_0(p\mathbb{Z}_p)) \)-modules and, in particular, they are endowed with a \( U_p \)-operator.

**Example 5.1** Recall the open and compact subgroups \( K_0(N), K_1(N) \subset B_f^{\infty} \) defined before Example 3.3. Assuming that \( \mu_{\psi(N)} \subset F \), we can decompose

\[
M_p^{\infty}(D_k(W))^{K_1(N)}(\varepsilon) = \bigoplus_{\varepsilon \in (\mathbb{Z}/p\mathbb{Z})^{\times}} M_p^{\infty}(D_k(W))^{K_1(N)}(\varepsilon),
\]

where \( M(\varepsilon) \) is the submodule of elements \( x \in M \) such that \( xu = \varepsilon(u)x \) if we define \( \varepsilon(u) := \varepsilon(a_u) \) for \( a_u \) the upper left entry of \( u \in K_0(N) \). Setting \( \omega_{0,p}^{\varepsilon,k}(z) := \varepsilon\left(\frac{z}{N_f(z)}\right)N^k_p(z) = \varepsilon\left(\frac{z}{N_f(z)}\right)\left(\frac{z}{N_f(z)}\right)^{-k}_p \), we have

\[
M_p^{\infty}(D_k(W))^{K_1(N)}(\varepsilon) = M_p^{\infty}(D_k(W), \omega_{0,p}^{\varepsilon,k})^{K_0(N)} \subset M_p^{\infty}(D_k(W), \omega_{0,p}^{\varepsilon,k}).
\]

If \( k \rightarrow k \in \mathbb{N} \), there is a specialization map

\[
\phi^\text{alg}_k : M_p^{\infty}(D_k(W)) \xrightarrow{\phi_k} M_p^{\infty}(D_k(W)) \xrightarrow{\nu_k} M_p^{\infty}(V_{k,F}) \tag{40}
\]

where the first arrow is induced by (35) and the second is the restriction to polynomials map (regarded as functions on \( W \)). We will usually write \( \varphi_k := \phi^\text{alg}_k(\varphi) \) when \( \varphi \in M_p^{\infty}(D_k(W)) \).

**Example 5.2** Suppose that we are in the setting of Example 5.1, so that \( \mu_{\psi(N)} \subset F \). Then we can decompose \( M^{\infty}(V_{k,F})^{K_1(N)} \) as we did for \( p \)-adic forms. Setting \( \omega_{0}^{\varepsilon,k}(z) := \varepsilon\left(\frac{z}{N_f(z)}\right)N^k_f(z) \), we have

\[
M^{\infty}(V_{k,F})^{K_1(N)}(\varepsilon) = M^{\infty}(V_{k,F}, \omega_{0}^{\varepsilon,k})^{K_0(N)} \subset M^{\infty}(V_{k,F}, \omega_{0}^{\varepsilon,k}).
\]

Furthermore, when \( \varepsilon : \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^{\times} \rightarrow \mu_{\psi(N)} \subset F^{\times} \), the specialization map induces (see Lemma 2.3 for the isomorphism)

\[
\phi^\text{alg}_k : M_p^{\infty}(D_k(W), \omega_{0}^{\varepsilon,k}) \rightarrow M_p^{\infty}(V_{k,F}, \omega_{0}^{\varepsilon,k}) \simeq M^{\infty}(V_{k,F}, \omega_{0}^{\varepsilon,k}).
\]

Justified by the above example and changing a bit the notation to make it consistent with that of Sect. 3, we will consider characters of the form \( \omega_{0,k}^{p}(z) = \omega_f(z)N^k_p(z) = \omega_f(z)\left(\frac{z}{N_f(z)}\right)^{-k}_p \), where \( \omega_f \) is the finite part of a unitary Hecke character which is unramified outside \( p \). As explained above, setting \( \omega_0^k(z) := \omega_f(z)N^k_f(z) \) we see that (40) induces

\[
M^{\infty}(D_k(W), \omega_{0,k}^{p}) \rightarrow M^{\infty}(V_{k,F}, \omega_{0,k}^{p}) \simeq M^{\infty}(V_{k,F}, \omega_{0}^{k}). \tag{41}
\]
We identify \( \psi : \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \sim \mathbf{M}_2 (\mathbb{Q}_p) \) by the rule \( \psi (x_1, y_1, x_2, y_2) := \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \) and, for a subset \( S \subset \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \), we define

\[
\delta_S : S \to \mathbb{Q}_p, \quad \delta_S (s) := \det (\psi (s)) \quad \text{and} \quad S_n := \delta_S^{-1} \left( p^n \mathbb{Z}_p^\times \right).
\]

For a continuous group homomorphism \( k : \mathbb{Z}_p^\times \to \mathcal{O}_\mathbb{Z}^\times \) valued in a \( \mathbb{Q}_p \)-Banach algebra \( \mathcal{O} \), we may consider

\[
\delta^k_{S_0} : S_0 \to \mathbb{Z}_p^\times \to \mathcal{O}_\mathbb{Z}^\times, \quad \delta^k_S (s) := \delta_S (s)^k.
\]

It is a locally analytic function, when \( S \subset \mathbb{Q}_p^2 \times \mathbb{Q}_p^2 \) is a submanifold, because \( k \) is locally analytic and \( S_0 \subset S \) is a submanifold.

Since \( \psi (w_1 g, w_2 g) = \psi (w_1, w_2) g \) for any \( w_i = (x_i, y_i) \) with \( i = 1, 2 \) and \( g \in \text{GL}_2 (\mathbb{Q}_p) \), we have \( \delta_S (g) = \delta_S (s) \det (g) \) for any \( s \in S \) and \( (Sg)_n = S_n - \nu_p (s) g \) where \( \nu_p := \text{ord}_p \circ \det \). In particular, if \( \Sigma \subset \text{GL}_2 (\mathbb{Q}_p) \) is a subsemigroup acting on \( S \), then \( \Gamma_\Sigma := \Sigma \cap \text{det}^{-1} \left( \mathbb{Z}_p^\times \right) \) acts on \( S_n \) for every \( n \). We have

\[
\delta^k_{S_0} (sg) = \det (g)^k \delta^k_{S_0} (s) \tag{42}
\]

and, in particular,

\[
\delta^k_{S_0} (t_1 s_1, t_2 s_2) = t_1^k t_2^k \delta^k_{S_0} (s) \tag{43}
\]

Noticing that \( (W \times \hat{W})_0 = W \times \hat{W} \) and \( (\hat{W} \times W)_0 = \hat{W} \times W \), we may consider the locally analytic functions

\[
\delta^k_{W \times \hat{W}} : W \times \hat{W} \to \mathcal{O}_\mathbb{Z}^\times, \quad \delta^k_{W \times W} : \hat{W} \times W \to \mathcal{O}_\mathbb{Z}^\times \quad \text{and} \quad \delta^k_0 (W \times W)_0 : W \times W \to \mathcal{O}_\mathbb{Z}^\times.
\]

Recall our notation for the twists by the norm. Since \( \Gamma_0 (p \mathbb{Z}_p) \subset \text{GL}_2 (\mathbb{Q}_p) \) is compact, \( \text{nr}_p \) maps it into the maximal open compact subgroup \( \mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times \). Hence, if \( D \) is a \( \Gamma_0 (p \mathbb{Z}_p) \)-module with coefficients in \( \mathcal{O} \), it makes sense to consider \( D (k) := D \left( \text{nr}_p^k \right) \), the same representation with action \( v \cdot k \ g := \text{nr}_p^k (g) \ v g \). Also, recall we have \( \text{Nr}_p^k \in M_p^\circ (\mathcal{O} (k), N^2 p_k) \) for every \( K \in \mathbb{K}^\circ \) (indeed \( N^2 p_k = \text{Nr}_p^k \) on \( \mathbb{Z}_f = \mathbb{A}_f^\times \)).

If \( k = (k_1, k_2, k_3) \) where \( k_i : \mathbb{Z}_p^\times \to \mathcal{O}_i^\times \) are \( \mathcal{O}_i \)-valued weights such that \( k_1 \oplus k_2 \oplus k_3 \) is even, set \( k_* := k_1 \oplus k_2 \oplus k_3 \), \( k_1^* := \frac{k_1 \oplus k_2 \oplus k_3}{2} \), \( k_2^* := \frac{k_1 \oplus k_2 \oplus k_3}{2} \) and \( k_3^* := \frac{k_1 \oplus k_2 \oplus k_3}{2} \), so that \( k_* : \mathbb{Z}_p^\times \to \mathcal{O}_k^\times \) for \( \mathcal{O}_k := \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 \). We define \( W := W \times W \times W, W_1 := \hat{W} \times W \times W, W_2 := W \times \hat{W} \times W \) and \( W_3 := W \times W \times \hat{W} \). Also, if \( p_i : W_i \to W \times W \) denotes the projection onto the components which are different from \( i \), we define \( \overline{W}_i := p_i^{-1} ((W \times W)_0) \) (for example, \( \overline{W}_3 := (W \times W)_0 \times \hat{W} \)).
Then we define the locally analytic functions

$$\Delta^\circ_{i,k} : W_i^\circ \to \mathcal{O}_k^\times$$

by the rule

$$\Delta^\circ_{1,k} (w_1, w_2, w_3) := \delta_{W(W \times W)_0}^k (w_2, w_3) \delta_{W(W \times W)_0}^k (w_1, w_3) \delta_{W(W \times W)_0}^k (w_1, w_2),$$

$$\Delta^\circ_{2,k} (w_1, w_2, w_3) := \delta_{W(W \times W)_0}^k (w_2, w_3) \delta_{W(W \times W)_0}^k (w_1, w_3) \delta_{W(W \times W)_0}^k (w_1, w_2),$$

$$\Delta^\circ_{3,k} (w_1, w_2, w_3) := \delta_{W(W \times W)_0}^k (w_2, w_3) \delta_{W(W \times W)_0}^k (w_1, w_3) \delta_{W(W \times W)_0}^k (w_1, w_2).$$

We remark that $\Gamma_0 (p \mathbb{Z}_p) = \Gamma_0 (p \mathbb{Z}_p)^i = \Sigma_0 (p \mathbb{Z}_p) \cap \Sigma_0 (p \mathbb{Z}_p)^i$ acts diagonally on $W_0^\circ$. The following lemma is an application of (42), (43) and the definitions of Sect. 4.3.

Lemma 5.3 We have $\Delta^\circ_{i,k} \in \mathcal{A}_{k_1 \boxtimes k_2 \boxtimes k_3} (W_i^\circ) (-k^*) \Gamma_0 (p \mathbb{Z}_p)$.

We will now focus on the $i = 3$ index, the other cases being similar. Since $W_3 = W_3^0 \cup (W_3 - W_3^0)$ (resp. $W_2 = (W_2)^0 \cup (W_2 - (W_2)^0)$) is a disjoint decomposition in open subsets, we have an extension by zero map $\cdot : \mathcal{A}_k (W_3^0) \to \mathcal{A}_k (W_3)$ (resp. $\cdot : \mathcal{A}_k (W_2^0) \to \mathcal{A}_k (W_2)$). By duality, we obtain a map

$$\cdot : \mathcal{D} (W_3^0, k) \to \mathcal{D} (W_3^0, k)$$

(resp. $\cdot : \mathcal{D} (W_2^0, k) \to \mathcal{D} (W_2^0, k)$). It follows from Lemma 5.3 that we may consider

$$\Lambda^\circ_{3,k} (\mu_1, \mu_2, \mu_3) := (\mu_1 \boxtimes \mu_2 \boxtimes \mu_3)^\circ (\Delta^\circ_{3,k}) \in \mathcal{O}_k (\mu_i \in \mathcal{D}_{k_i} (W) \text{ for } i = 1, 2 \text{ and } \mu_3 \in \mathcal{D}_{k_3} (\widehat{W}))$$

and that we have

$$\Lambda^\circ_{3,k} \in \text{Hom}_{\mathcal{O}[\Gamma_0 (p \mathbb{Z}_p)]} (\mathcal{D}_{k_1} (W) \otimes \mathcal{O} \mathcal{D}_{k_2} (W) \otimes \mathcal{O} \mathcal{D}_{k_3} (\widehat{W}), \mathcal{O}_k (k^*)).$$

Suppose that we are given characters $\omega_i^k_{0,p}$ for $i = 1, 2, 3$ such that $N_{i,p}^{2k^*} = \omega_i^k_{0,p} \omega_i^k_{0,p} \omega_i^k_{0,p}$. Taking $\Lambda = \Lambda^\circ_{3,k}$ in (11)) gives the trilinear form

$$\iota^\circ_{3,k} : M_p^\circ (\mathcal{D}_{k_1} (W), \omega_i^k_{0,p}) \otimes M_p^\circ (\mathcal{D}_{k_2} (W), \omega_i^k_{0,p}) \otimes M_p^\circ (\mathcal{D}_{k_3} (\widehat{W}), \omega_i^k_{0,p}) \to \mathcal{O}_k.$$

Suppose, for example, that we may write $\omega_i^k_{0,p} (z) = \omega_{i,i} (z) N_{p}^{k_i} (z) = \omega_{i,i} (z) \left( \frac{z}{\lambda_i (z)} \right)^{-k_i}$ with $\omega_{i,i}$ taking values in $F$ with $\omega_{i,1} \omega_{i,2} \omega_{i,3} = 1$; then we see that $\omega_i^k_{0,p} \omega_i^k_{0,p} \omega_i^k_{0,p} = N_{i,p}^{2k^*}$ and the above definition applies. Let us remark that, when $k$ is such that $k^* = c \in \mathbb{N}$, the expression (13) defines an element $\Delta_{3,k} \in \mathcal{A}_{k_1 \boxtimes k_2 \boxtimes k_3} (W_3^0)$.
$A_{k_1 \boxtimes k_2 \boxtimes k_3}^\circ (W_3) (\omega^*) \Gamma_0 (p \mathbb{Z})$. We can therefore integrate this function without first applying to the measures involved. The result is a trilinear form

$$t_{3,k} : M_p^\circ \left( D_{k_1} (W), \omega_{0,p}^{k_1} \right) \otimes M_p^\circ \left( D_{k_2} (W), \omega_{0,p}^{k_2} \right) \otimes M_p^\circ \left( D_{k_3} (\hat{W}), \omega_{0,p}^{k_3} \right) \rightarrow \mathcal{O}_K.$$  

(44)

Let $\hat{\omega}_p$ be the idele concentrated at $p$, where we have $(\hat{\omega}_p)_p = \omega_p$. Because $\omega_p : D_{k_3} (W) \rightarrow D_{k_3} (\hat{W})$, the formula $(\varphi_3 | W_p) (x) := (\varphi_3 \hat{\omega}_p) (x) = \varphi_3 (\hat{\omega}_p x) \omega_p$ defines

$$W_p : M_p^\circ \left( D_{k_3} (W), \omega_{0,p}^{k_3} \right) \rightarrow M_p^\circ \left( D_{k_3} (\hat{W}), \omega_{0,p}^{k_3} \right)$$

(45)

It follows from (39) that, if $k_i^* = c \in \mathbb{N}$ and $\varphi_{i,k_i} = \phi_{i,*}^\text{alg} (\varphi_i)$ where $\phi : k_i \rightarrow k_i$, then

$$\phi \left( t_{3,k} (\varphi_1 \otimes \varphi_2 \otimes \varphi_3 | W_3) \right) = t_{k} (\varphi_{1,k_1}, \varphi_{2,k_2}, \varphi_{3,k_3} | W_p).$$

(46)

This applies, in particular, when $\phi$ is the identity and $k = k \in \mathbb{N}^3$ is balanced: then $\varphi_{i,k_i} = \varphi_{i,*}^\text{alg} (\varphi_i) = v_{k_i} (\varphi_i)$ is just the restriction to polynomials map [see (40)] and $\phi$ that appears in (46) is the identity.

The following key calculation relates the trilinear form $t_{3,k}^\circ$ to $t_{3,k}$ under the running assumption that $k = k \in \mathbb{N}^3$ is balanced (to be in force from now until the end of Corollary 5.6 below). Write $\hat{\pi}_p$ for the idele concentrated at $p$, where we have $(\hat{\pi}_p)_p = \pi_p := \left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right)$. If $\varphi \in M_p^\circ (D_k (W))$, recall the $U_p$-operator defined by the double coset $K \hat{\pi}_p K$, where $K = K^p \Gamma_0 (p \mathbb{Z})$ and $\varphi \in M_p^\circ (D_k (W))^K$ (see the discussion after (9)).

**Proposition 5.4** For $i = 1, 2$, suppose $\varphi_i \in M_p^\circ (D_{k_i} (W))$ is a $U_p$-eigenvector with $\varphi_i | U_p = a_i \varphi_i$, and view $\varphi_1 \otimes \varphi_2$ as an element of $M_p^\circ (D_{k_1 \boxtimes k_2} (W \times W))$. Then

$$(\varphi_1 \otimes \varphi_2) | U_p = a_1 a_2 (\varphi_1 \otimes \varphi_2 - i_*(\varphi_1 \otimes \varphi_2)^0),$$

where $i_* : M_p^\circ (D_{k_1 \boxtimes k_2} ((W \times W)_0)) \rightarrow M_p^\circ (D_{k_1 \boxtimes k_2} (W \times W))$ is induced by the map $i_* : D_{k_1 \boxtimes k_2} ((W \times W)_0) \rightarrow D_{k_1 \boxtimes k_2} (W \times W)$ obtained from the inclusion $i : (W \times W)_0 \hookrightarrow W \times W$.

**Proof** Consider the decomposition

$$W = \bigsqcup_{i=0}^{p-1} W_i, \quad \text{where} \quad W_i = \hat{W} \pi_i = \{ w = (x, y) \in W : y \equiv ix \pmod{p} \}.$$  

Then $K \hat{\pi}_p K = \bigsqcup_{i=0}^{p-1} K \pi_i$ and we can compute:

$$a_1 a_2 (\varphi_1 \otimes \varphi_2) (x) = (\varphi_1 | U_p \otimes \varphi_2 | U_p)(x)$$

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\[
\begin{align*}
&= \sum_{i, j=0}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j \\
&= \sum_{i=0}^{p-1} \varphi_1(\pi x)\pi_i \otimes \varphi_3(\pi_i x)\pi_i + \sum_{i, j=0}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j \\
&= ((\varphi_1 \otimes \varphi_2) | U_p)(x) + A.
\end{align*}
\]

It remains to show that \( A = a_1a_2i_+(\varphi_1 \otimes \varphi_2)^0(x) \). To this end, note that we may write
\[
W^2 = \bigoplus_{i, j=0}^{p-1} W_i \times W_j = \bigoplus_{i=0}^{p-1} W_i \times W_i \sqcup \bigoplus_{i, j=0}^{p-1} W_i \times W_j.
\]

Subordinate to this decomposition of spaces, we have a corresponding decomposition of \( \mathcal{D}_{k_1 \boxplus k_2}(W^2) \):
\[
\mathcal{D}_{k_1 \boxplus k_2}(W^2) = \bigoplus_{i=0}^{p-1} \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_i) \oplus \bigoplus_{i, j=0}^{p-1} \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j).
\]

Note that the spaces \( W_i \) are all \( \mathbb{Z}^\times \)-stable, so that these spaces of distributions are defined. Writing \( \text{proj}_{i, j} : \mathcal{D}_{k_1 \boxplus k_2}(W^2) \rightarrow \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j) \) for the associated projections, we have
\[
a_1a_2 \sum_{i, j=0}^{p-1} \text{proj}_{i, j}(\varphi_1(x) \otimes \varphi_2(x))
\]
\[
= a_1a_2(\varphi_1(x) \otimes \varphi_2(x)) = \sum_{i, j=0}^{p-1} \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j.
\]

Since
\[
\mu_1\pi_i \otimes \mu_2\pi_j \in \mathcal{D}_{k_1 \boxplus k_2}(W_i \times W_j),
\]
for every \( \mu_1 \in \mathcal{D}_{k_1}(W) \) and \( \mu_2 \in \mathcal{D}_{k_2}(W) \) (as it can be checked on Dirac distributions), taking \( \mu_1 = \varphi_1(\pi_i x) \) and \( \mu_2 = \varphi_2(\pi_j x) \) it follows that
\[
a_1a_2 \text{proj}_{i, j}(\varphi_1(x) \otimes \varphi_2(x)) = \varphi_1(\pi_i x)\pi_i \otimes \varphi_2(\pi_j x)\pi_j.
\]
for all \( i, j \). Therefore,

\[
A = a_1 a_2 \sum_{i,j=0}^{p-1} \text{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)).
\]

One easily verifies the equality

\[
(W \times W)_0 = \bigcup_{i,j=0}^{p-1} W_i \times W_j,
\]

implying that

\[
\sum_{i,j=0}^{p-1} \text{proj}_{i,j}(\varphi_1(x) \otimes \varphi_2(x)) = i_*(\varphi_1 \otimes \varphi_2)^0(x).
\]

Now substitute (48) into (47).

We are going to apply the results of Sect. 2.3. We take \( \Sigma_p = \Sigma_0(p\mathbb{Z}_p) \), \( D = \mathcal{D}_{k_1 \boxplus k_2}(W \times W) \) and \( E = \mathcal{D}_{k_3}(\hat{W}) \). Then \( k = k^* \) (resp. the central character \( \kappa_E = k_3 \) of \( E = \mathcal{D}_{k_3}(\hat{W}) \) extends to the character \( \tilde{k} = k^* \) of \( \mathbb{Q}^\times_p \) (resp. the character \( \tilde{\kappa}_E = k_3 \) of \( \mathbb{Q}^\times_{\mathbb{F}_p} \)). Finally, we suppose that we may write \( \omega_{\tilde{k}} \otimes \omega_{\tilde{k}} \otimes \omega_{\tilde{k}} = 1 \). Then

\[
\omega_{\tilde{k}} \otimes \omega_{\tilde{k}} \otimes \omega_{\tilde{k}}(z) = \omega_{\tilde{k}}(z) \omega_{\tilde{k}}(z) \omega_{\tilde{k}}(z)
\]

and we have \( \omega_{\tilde{k}} \otimes \omega_{\tilde{k}} \otimes \omega_{\tilde{k}} = \omega_{\tilde{k}} \otimes \omega_{\tilde{k}} \otimes \omega_{\tilde{k}}(z) = \text{Nrd}_{\tilde{k}}(\kappa_E)(\pi_p) \).

**Lemma 5.5** With these notations the trilinear form \( t_{3,k} \) defines an element of \( \text{Hom}_{\mathcal{D}_p}(\Sigma_p, \Sigma_p, \Sigma_p) \). Furthermore, we have \( \text{Nrd}_{\tilde{k}}(\pi_p) \text{Nrd}_{\tilde{k}}(\kappa_E)(\pi_p) \).

**Proof** Note that \( \Delta_{i,k} \) defines indeed \( \tilde{\Delta}_k \) \( \in \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}^2_p \otimes \mathbb{Q}^2_p \otimes \mathbb{Q}^2_p) \) such that \( \tilde{\Delta}_k|_{W_3} = \Delta_{i,k} \). We take \( \hat{D} = \mathcal{D}_{k_1 \boxplus k_2}(\mathbb{Q}^2_p \otimes \mathbb{Q}^2_p) \) and \( \hat{E} = \mathcal{D}_{k_3}(\mathbb{Q}^2_p) \), so that \( \tilde{k} = k_3 \). The pairing associated to \( t_{3,k} \) is given by \( (\mu_{12}, \mu_{3}) := (\mu_{12} \otimes \mu_{3})(\Delta_k) \) and we define \( (\mu_{12}, \mu_{3}) := (\mu_{12} \otimes \mu_{3})(\Delta_k) \). Since \( \mathbb{Q}^2_p \otimes \mathbb{Q}^2_p \otimes \mathbb{Q}^2_p = (W^2 \times \hat{W}) \), with \( \mathbb{Z} \) an open subset,

\[
\mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}^2_p \otimes \mathbb{Q}^2_p \otimes \mathbb{Q}^2_p) = \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(W^2 \times \hat{W}) \oplus \mathcal{A}_{k_1 \boxplus k_2 \boxplus k_3}(Z)
\]

and

\[
\mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(\mathbb{Q}^2_p \otimes \mathbb{Q}^2_p \otimes \mathbb{Q}^2_p) = \mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(W^2 \times \hat{W}) \oplus \mathcal{D}_{k_1 \boxplus k_2 \boxplus k_3}(Z).
\]
For elements $\mu_{12} \in \mathcal{D}_{k_1 \boxtimes k_2}(W^2)$ and $\mu_3 \in \mathcal{D}_{k_3}(\hat{W})$, the distribution $\mu_{12} \hat{\otimes} \mu_3$ is supported on $\mathcal{D}_{k_1 \boxtimes k_2 \boxtimes k_3}(W^2 \times \hat{W})$ and we see that $(\mu_{12}, \mu_3)^\sim = \langle \mu_{12}, \mu_3 \rangle$. We note that the relation $\sigma \Delta_k = \det(\sigma) k^* \Delta_k$ for every $\sigma \in \Sigma_p$ implies

$$\langle \mu_{12}, \mu_3 \rangle^\sim := (\mu_{12} \sigma \otimes \mu_3) (\Delta_k^\sigma) = \det(\sigma) k^* (\mu_{12} \otimes \mu_3) (\Delta_k^\sigma) = \det(\sigma) k^* \langle \mu_{12}, \mu_3 \rangle^\sim.$$

Now apply Remark 2.8 in order to get the first statement. Finally, the second statement follows by a simple computation. \hfill \square

Write $p' \in \mathbb{A}_f^\times$ for the finite idele $(p')_v = p$ for every $v \neq p$ and $(p')_p = 1$.

**Corollary 5.6** For $i = 1, 2$, let $\varphi_i \in M^0_p(\mathcal{D}_{k_i}(W), \omega^k_{0, p})$ be a $U_p$-eigenvector with $\varphi_i \mid U_p = \alpha_i \varphi_i$ and let $\varphi_3 \in M^0_p(\mathcal{D}_{k_3}(\hat{W}), \omega^k_{0, p})$ be a $U_p$-eigenvector with $\varphi_3 \mid U_p = \alpha_3 \varphi_3$. Then

$$t^3_{3, k}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) = \left(1 - \omega_{1, 3} (p') \frac{\alpha_3}{\alpha_1 \alpha_2} \right) t^3_{3, k}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3).$$

**Proof** Recall the morphism $^0 : \mathcal{D}_{k_1 \boxtimes k_2}(W \times W) \to \mathcal{D}_{k_1 \boxtimes k_2}((W \times W)_0).$ Given $\mu_i \in \mathcal{D}_{k_i}(W)$ for $i = 1, 2$ and $\mu_3 \in \mathcal{D}_{k_3}(\hat{W})$, we may therefore consider

$$\langle \mu_{12}, \mu_3 \rangle^\circ := \left(\mu_{12} \boxtimes \mu_3 \right) \left(\Delta^\circ_k \right).$$

This is granted by Lemma 5.3, which also implies $\langle - , - \rangle^\circ \in \text{Hom}_O[\Gamma_0(p\mathbb{Z}_p)] (\mathcal{D}_{k_1 \boxtimes k_2}((W^2)_0) \otimes \mathcal{D}_{k_3}(\hat{W}), O(k^*)).$ Taking $\Lambda = (- , -)^\circ$ in (11) gives the bilinear form

$$\langle - , - \rangle^\circ : M^0_p(\mathcal{D}_{k_1 \boxtimes k_2}((W^2)_0), \omega^{0, p, D}) \otimes M^0_p(\mathcal{D}_{k_3}(\hat{W}), \omega^{0, p, E}) \to O.$$
By Proposition 5.4, we have
\[ \langle \varphi_1 \otimes \varphi_2, \varphi_3 \rangle_t = \frac{1}{\alpha_1 \alpha_2} \langle (\varphi_1 \otimes \varphi_2) | U_p, \varphi_3 \rangle_t + \langle \iota_* (\varphi_1 \otimes \varphi_2)^0, \varphi_3 \rangle_t \] (49)

Proposition 2.9, which applies thanks to Lemma 5.5, implies that
\[ \langle (\varphi_1 \otimes \varphi_2) | U_p, \varphi_3 \rangle_t = \omega_{t,3} (p^t) \ p^{k_3} \langle \varphi_1 \otimes \varphi_2, \varphi_3 | U_p^t \rangle_t = \omega_{t,3} (p^t) \ p^{k_3} \alpha_3 \langle \varphi_1 \otimes \varphi_2, \varphi_3 \rangle_t. \] (50)

Finally, it is easy to see that \( \left\langle \iota_* (\mu_1 \otimes \mu_2)^0, \mu_3 \right\rangle_t = \left\langle (\mu_1 \otimes \mu_2)^0, \mu_3^\circ \right\rangle_t \) (once again checking the equality on Dirac distributions), from which we see that
\[ \left\langle \iota_* (\varphi_1 \otimes \varphi_2)^0, \varphi_3 \right\rangle_t = \left\langle (\varphi_1 \otimes \varphi_2)^0, \varphi_3 \right\rangle^\circ. \] (51)

The result follows by combining (49), (50), and (51). \qed

5.1 \textit{p}-adic periods

Writing \( k = k_t \) and \( O = O_t \), as remarked after (43), we have \( \delta^k_{W \times \hat{W}} : W \times \hat{W} \to O^\times \); it follows from (42) that we have \( \delta^k_{W \times \hat{W}} \in \mathcal{A}_{k \otimes k} (W \times \hat{W}) (-k)^\Gamma_0 (p \mathbb{Z}_p) \) and we can consider
\[ B_k \in \text{Hom}_{O[[\Gamma_0 (p \mathbb{Z}_p)]]} \left( \mathcal{D}_k (W) \otimes_b \mathcal{D}_k (\hat{W}), O (k) \right) \]
defined by \( B_k (\mu_1 \otimes \mu_2) := (\mu_1 \otimes \mu_2) (\delta^k_{W \times \hat{W}}) \). Taking \( \Lambda = B_k \) in (11) gives the bilinear form
\[ \langle - , - \rangle_k : M^\circ_p \left( \mathcal{D}_k (W), \omega^k_{0,p} \right) \otimes M^\circ_p \left( \mathcal{D}_k (\hat{W}), N^{2k}_p \left( \omega^k_{0,p} \right)^{-1} \right) \to O. \]

Suppose now that \( k \xrightarrow{\phi} k \in \mathbb{N} \) and let us remark that there are specialization maps
\[ \phi^\text{alg}_* : M^\circ_p \left( \mathcal{D}_k (\hat{W}), \omega^k_{0,p} \right) \to M^\circ_p \left( \mathcal{V}_{k,E}, \omega^k_{0,p} \right) \cong M^\circ \left( \mathcal{V}_{k,E}, \omega^k_0 \right) \] and
\[ \phi^\text{alg}_* : M^\circ_p \left( \mathcal{D}_k (\hat{W}), N^{2k}_p \left( \omega^k_{0,p} \right)^{-1} \right) \to M^\circ_p \left( \mathcal{V}_{k,E}, N^{2k}_p \left( \omega^k_{0,p} \right)^{-1} \right), \] (52)
defined in the same way as (41) was defined, i.e. via the morphism induced by (35) and the restriction to polynomials (now regarded as functions on \( \hat{W} \)). Let us apply Remark 2.5 to (18). First, it tells us that \( \left( \nrd_{\hat{f}^\text{ord}}^{-1}, 1 \right) \in X \left( B^\times, \omega^2_t \right) \) corresponds to \( \left( \nrd_{\hat{f}^\text{ord}}^{-1}, 1 \right) \in X_p \left( B^\times, \omega^{-2}_t \right) \) (taking \( \chi_0 = \nrd_{\hat{f}^\text{ord}}^{-1} \) and \( \chi_\infty = \chi_p = 1 \) in loc.cit. we see that \( \chi_{0,p} = \nrd_{\hat{f}^\text{ord}}^{-1} \)). Second, it tells us that (17) (for the central character

\[ \xi \] Springer
\( \omega_0^k (z) = \omega_f (z) N_1^k (z) \) corresponds to

\[
M^\circ_p \left( \mathbf{V}_{k, E}, \omega_{0, p}^k \right) \longrightarrow M^\circ_p \left( \mathbf{V}_{k, E}, N_{2, p}^{2k} \left( \omega_{0, p}^k \right)^{-1} \right)
\]  

(53)

defined via Remark 2.2 (1), i.e. given by \( \varphi \mapsto \tilde{\varphi} \), where \( \tilde{\varphi}(x) := \operatorname{nd}_{f}^{-\omega_{0, i}}(x) \varphi(x) = \operatorname{nd}_{f}^{-\omega_{0, p}}(x) \varphi(x) \) (the equality because \( \omega_{0, p}^k (z) = \omega_f (z) N_1^k (z) \), implying that \( \omega_f = \omega_{0, p}^k N_1^{-k} \)). Hence, the same formula \( \varphi \mapsto \tilde{\varphi} \), where \( \tilde{\varphi}(x) := \operatorname{nd}_{f}^{-\omega_{0, i}}(x) \varphi(x) \), defines

\[
M^\circ_p \left( D_k (\hat{W}), \omega_{0, p}^k \right) \longrightarrow M^\circ_p \left( D_k (\hat{W}), N_{2, p}^{2k} \left( \omega_{0, p}^k \right)^{-1} \right)
\]  

(54)

which interpolates (53) \( \simeq (17) \) via (52). Recall the \( W_p \)-operator \( M^\circ_p \left( D_k (W), \omega_{0, p}^k \right) \rightarrow M^\circ_p \left( D_k (\hat{W}), \omega_{0, p}^k \right) \) [see (45)] which interpolates the \( W_p \)-operator \( M^\circ_p \left( V_{k, E}, \omega_{0, p}^k \right) \rightarrow M^\circ_p \left( V_{k, E}, N_{2, p}^{2k} \right) \) (defined by the same formula) via the specialization maps (41) and (52). Finally, it is clear from its definition that \( (-, -)_k \) interpolates \( (-, -)_k \) via the specialization maps (41) and (52). Hence, setting \( (\varphi, \psi) := \left( \varphi, (\psi \mid W_p) \right) \), we have proved the following result (the uniqueness follows from the fact that the weight space is reduced and \( \mathbb{N} \) is Zariski dense in the open affinoid subdomain \( U \subset \mathcal{X} \)).

**Lemma 5.7** Suppose that \( k : \mathbb{Z}_p^\times \rightarrow \mathcal{O} \) corresponds to an open affinoid subdomain \( U \subset \mathcal{X} \) and that \( \varphi, \psi \in M^\circ_p \left( D_k (W), \omega_{0, p}^k \right) \). There is a unique \( (\varphi, \psi) \in \mathcal{O} \) such that for every \( k \in U \cap \mathbb{N} \) which corresponds to \( k \in \mathbb{N} \) we have, setting \( \varphi_k := \varphi_{k}^{\operatorname{alg}}(\varphi) \) and \( \psi_k := \psi_{k}^{\operatorname{alg}}(\psi) \):

\[
(\varphi, \psi) (k) := \phi ((\varphi, \psi)) = (\varphi_k, \psi_k \mid W_p)_k.
\]

**6 Degeneracy maps and \( p \)-stabilizations**

If \( g \in \text{GL}_2 (\mathbb{Q}_p) \), we let \( \hat{g} \) be the idele concentrated in \( p \), where we have \( \hat{g}_p = g \). In particular, we write \( \hat{\pi}_p \) (resp. \( \hat{\omega}_p \)) for the idele concentrated at \( p \), where we have

\[
(\hat{\pi}_p) = \pi_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad (\hat{\omega}_p) = \omega_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.
\]

We fix levels \( K \subset K^\# \) of the form \( K = K^p \Gamma_0 (p \mathbb{Z}_p) \) and \( K^\# = K^p \text{GL}_2 (\mathbb{Z}_p) \). Let us record the following fact.
Lemma 6.1  We have $K^\# = \bigsqcup_{i=0,...,p-1,\infty} K\hat{\gamma}_i$ with $\gamma_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$ for $i = 0, \ldots, p-1$ and $\gamma_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $K\hat{\pi}_p K^\# = \bigsqcup_{i=0}^{p-1} K\hat{\pi}_i \sqcup K\hat{\omega}_p$ with $\pi_i = \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}$. Also,

$$K^\# \hat{\pi}_p K^\# = \bigsqcup_{i=0,...,p-1,\infty} K^\# \pi_i K^\# = \bigsqcup_{i=0,...,p-1} K^\# \pi_i \sqcup K^\# \pi_p^i .$$

Proof  Indeed, a direct computation shows that $\bigsqcup_i K\hat{\gamma}_i \subset K^\#$ (resp. $\bigsqcup_i^{p-1} K\hat{\pi}_i \sqcup K\hat{\omega}_p \subset K\hat{\pi}_p K^\#$), $\bigsqcup_i K^\# \pi^i \gamma_i \subset K^\# \pi_p K^\#$ and $\bigsqcup_{i=0,...,p-1}^{p-1} K^\# \pi_i \sqcup K^\# \pi_p^i \subset K^\# \pi_p K^\#$. The first equality is then easily checked (and equivalent to $[K^\# : K_p] = p + 1$ or the fact that there are $p + 1$ index $p \mathbb{Z}_p$-sublattices in $\mathbb{Z}_p^2$). To see the other equalities, we may fix a left and right invariant Haar measure $\mu$ and check that both sides have the same measure as follows. First we remark that, because $\hat{\omega}_p \hat{\gamma}_\infty = \hat{\pi}_p$,

$$K\hat{\pi}_p K^\# = K\hat{\omega}_p \hat{\gamma}_\infty K^\# = K\hat{\omega}_p K^\# = \hat{\omega}_p K K^\# = \hat{\omega}_p K^\#. \quad (55)$$

Then we see that

$$\mu \left( K\hat{\pi}_p K^\# \right) = \mu \left( \hat{\omega}_p K^\# \right) = \mu \left( K^\# \right) = (p + 1) \mu \left( K \right) ,$$

proving the second equality because

$$\mu \left( \bigsqcup_{i=0}^{p-1} K\hat{\pi}_i \sqcup K\hat{\omega}_p \right) = (p + 1) \mu \left( K \right) .$$

Using the first equality, one checks that

$$K^\# \pi_p K^\# = \bigsqcup_{i=0,...,p-1,\infty} \hat{\gamma}_i^{-1} K\hat{\pi}_p K^\# .$$

Then we see that

$$\mu \left( K^\# \pi_p K^\# \right) \leq (p + 1) \mu \left( K\hat{\pi}_p K^\# \right) = (p + 1) \mu \left( K^\# \right) ,$$

proving the third and the fourth equalities because

$$\mu \left( \bigsqcup_{i=0,...,p-1,\infty} K^\# \pi_p^i \gamma_i \right) = \mu \left( \bigsqcup_{i=0,...,p-1} K^\# \pi_i \sqcup K^\# \pi_p^i \right) = (p + 1) \mu \left( K^\# \right) .$$

$\Box$

We suppose in this Sect. 6 that $\hat{\mathbb{Z}}^\times \subset K$ and that we may write $\omega_0 (z) = \omega_l (z) N_l^K (z)$. We have two degeneracy maps

$$K^\# 1K = K^\# 1, K^\# \pi_p^i K : M \left( V_{k,F}, \omega_0 \right)^K \rightarrow M \left( V_{k,F}, \omega_0 \right)^K .$$
Define

$$\varphi^{(p)} := \varphi \mid K^\# \widehat{\pi}_p K \in M \left( V_{k,F}, \omega_0 \right)^K.$$

Let us now fix $0 \neq \varphi \in M \left( V_{k,F}, \omega_0 \right)^K$ such that $\varphi \mid T_p = a_p \left( \varphi \right)$ and define $M \left( V_{k,F}, \omega_0 \right)^{K, \varphi\text{-old}} \subset M \left( V_{k,F}, \omega_0 \right)^K$ to be the span of $\left\{ \varphi, \varphi^{(p)} \right\}$. We define the Hecke polynomial at $p$ and the quantities $\alpha_p \left( \varphi \right)$ and $\beta_p \left( \varphi \right)$ to via the formula

$$X^2 - a_p \left( \varphi \right) X + \omega_f \left( p' \right)^{-1} p^{k+1} = (X - \alpha_p \left( \varphi \right)) \left( X - \beta_p \left( \varphi \right) \right). \quad (56)$$

Next, let us set

$$\varphi^\alpha = \varphi^{\alpha_p \left( \varphi \right)} := \varphi - \alpha_p \left( \varphi \right)^{-1} \varphi^{(p)} \in M \left( V_{k,F}, \omega_0 \right)^{K, \varphi\text{-old}},$$

$$\varphi^\beta = \varphi^{\beta_p \left( \varphi \right)} := \varphi - \beta_p \left( \varphi \right)^{-1} \varphi^{(p)} \in M \left( V_{k,F}, \omega_0 \right)^{K, \varphi\text{-old}}.$$

We say that $\varphi$ is semisimple (resp. not semisimple) if $\alpha_p \left( \varphi \right) \neq \beta_p \left( \varphi \right)$ (resp. $\alpha_p \left( \varphi \right) = \beta_p \left( \varphi \right)$). Conjecturally, $\varphi$ is always semisimple, as shown in [12, Corollary 3.2 and Remark 3.3].

**Remark 6.2** Suppose that $\widehat{\mathbb{Z}}^\times \subset K'$ and that $\psi \in M \left( V_{k,F}, \omega_0 \right)^{K'}$. Write $p := \widehat{p}$ (resp. $p' \in \widehat{\mathbb{Z}}^\times \subset K'$) for the idele concentrated at $p$, where we have $p_p = p$ (resp. the finite idele defined by the conditions $(p'_v) = p$ for every $v \neq p$ and $(p'_p) = 1$).

Then $\varphi \mid K' \psi = \omega_f \left( p' \right)^{-1} p^k \psi$.

**Proof** Indeed, we may write $p = p p'$ and we see that:

$$\left( \varphi \mid K' \psi \right) (x) = \varphi \left( p x \right) \varphi \left( p' x p \right) = \omega_f \left( p' \right)^{-1} \varphi \left( x \right) = \omega_f \left( p' \right)^{-1} p^k \varphi \left( x \right).$$

$$\Box$$

Set $T_p := K^\# \widehat{\pi}_p K^\#$ (acting on $M \left( V_{k,F}, \omega_0 \right)^{K^\#}$), $U_p := K \widehat{\pi}_p K$ and $W_p := K \widehat{\omega}_p$ (both acting on $M \left( V_{k,F}, \omega_0 \right)^{K}$).

**Corollary 6.3** If $\varphi \in M \left( V_{k,F}, \omega_0 \right)^{K^\#}$ we have

$$\varphi \mid U_p = \varphi \mid T_p - \varphi^{(p)}, \varphi^{(p)} \mid U_p = \omega_f \left( p' \right)^{-1} p^{k+1} \varphi$$

$$\varphi \mid W_p = \varphi^{(p)} \text{ and } \varphi^{(p)} \mid W_p = \omega_f \left( p' \right)^{-1} p^k \varphi.$$

**Proof** Noticing that $K^\# \widehat{\pi}_p K = K^\# \widehat{\pi}_p$ (because $\widehat{\gamma} \widehat{\omega}_p = \widehat{\pi}_p$, arguing as in (55)), the first equality is a direct consequence of the last decomposition of Lemma 6.1 and the definition $\varphi^{(p)} := \varphi \mid K^\# \widehat{\pi}_p K$. Since $\varphi^{(p)} = \varphi \mid K^\# \widehat{\pi}_p$ we find

$$\varphi^{(p)} \mid U_p = \sum_{i=0}^{p-1} \left( \varphi \mid K^\# \widehat{\pi}_p \right) \widehat{\pi}_i = \sum_{i=0}^{p-1} \varphi \widehat{\pi}_i \widehat{\pi}_i.$$
We now remark that
\[ \pi_p \pi_i = \left( \begin{array}{cc} p & ip \\ 0 & p \end{array} \right) = \left( \begin{array}{cc} 1 & i \\ 0 & 1 \end{array} \right) p, \]
so that \( \hat{\pi}_p \hat{\pi}_i \in Kp \). It follows from Remark 6.2 and the \( K \)-invariance of \( \varphi \) that we have \( \varphi \hat{\pi}_p \hat{\pi}_i = \varphi p = \varphi | Kp = \omega_f (p')^{-1} p^k \varphi \). Hence we find
\[
\varphi^{(p)} | U_p = \omega_f (p')^{-1} p^k \sum_{i=0}^{p-1} \varphi = \omega_f (p')^{-1} p^{k+1} \varphi.
\]

The equality \( \varphi | W_p = \varphi^{(p)} \) follows from \( K^# \hat{\pi}_p K = K^# \hat{\omega}_p \) (because we remarked that \( K^# \hat{\pi}_p K = K^# \hat{\pi}_i \) and \( K^# \hat{\pi}_p = K^# \hat{\omega}_p \) in view of \( \hat{\pi}_p = \gamma_{\infty} \hat{\omega}_p \)) and the fact that \( \varphi | K \hat{\omega}_p = \varphi | K^# \hat{\omega}_p \) because \( \varphi \in M \left( V_{k,F}, \omega_0 \right)^{K^#} \). Finally, since \( \omega_2^2 = -p \), once again noticing that \( K^# \hat{\pi}_p K = K^# \hat{\omega}_p \), we find
\[
\varphi^{(p)} | W_p = \left( \varphi | K^# \hat{\omega}_p \right) \hat{\omega}_p = \varphi \hat{\omega}_p = \varphi | K^# p = \omega_f (p')^{-1} p^k \varphi.
\]

The following result, whose proof is left to the reader, can now be deduced from Corollary 6.3 by standard linear algebra and the well-known fact that \( \text{Im}(K^# 1 K) \cap \text{Im}(K^# \hat{\pi}_p K) = 0 \).

**Proposition 6.4** The following facts are true, assuming that \( F \) is a field such that \( \alpha_p (\varphi), \beta_p (\varphi) \in F \) for the statements (2)–(5).

1. The space \( M \left( V_{k,F}, \omega_0 \right)^{K,\varphi-\text{old}} \) is two dimensional with basis \( \{ \varphi, \varphi^{(p)} \} \), stable under the action of the \( U_p \) and \( W_p \) operators.
2. We have \( \varphi^\alpha | U_p = \alpha_p (\varphi) \varphi, \varphi^\beta | U_p = \beta_p (\varphi) \varphi \) and, if \( \psi \in M \left( V_{k,F}, \omega_0 \right)^{K,\varphi-\text{old}} \) is such that \( \psi | U_p = \rho \psi \), then \( \psi = \varphi^\alpha \) or \( \psi = \varphi^\beta \) up to a scalar factor.
3. We have
   \[
   F \left( \varphi^\alpha | W_p \right) \cap F \varphi^\alpha = F \left( \varphi^\alpha | W_p \right) \cap F \varphi^\beta
   = 0 \text{ (resp. } F \left( \varphi^\beta | W_p \right) \cap F \varphi^\alpha = F \left( \varphi^\beta | W_p \right) \cap F \varphi^\beta = 0 \text{)}
   \]
   unless \( \alpha_p (\varphi)^2 = \omega_f (p')^{-1} p^k \) (resp. \( \beta_p (\varphi)^2 = \omega_f (p')^{-1} p^k \)) and, in this case,
   \( \varphi^\alpha | W_p = -\alpha_p (\varphi) \varphi^\alpha \) (resp. \( \varphi^\beta | W_p = -\beta_p (\varphi) \varphi^\beta \)). In general,
   \[
   \varphi^\alpha | W_p = \varphi^{(p)} - \alpha_p (\varphi)^{-1} \omega_f (p')^{-1} p^k \varphi \text{ (resp. } \varphi^\beta | W_p = \varphi^{(p)} - \beta_p (\varphi)^{-1} \omega_f (p')^{-1} p^k \varphi \text{)}.
   \]
4. If \( \varphi \) is semisimple, then \( U_p \) is diagonalizable on \( M \left( V_{k,F}, \omega_0 \right)^{K,\varphi-\text{old}} \) and we have
   \[
   M \left( V_{k,F}, \omega_0 \right)^{K,\varphi-\text{old}} = F \varphi^\alpha \oplus F \varphi^\beta.
   \]
(5) If $\varphi$ is not semisimple, then

$$0 \neq F\varphi^\alpha = F\varphi^\beta \subset M(V_{k,F}, \omega_0)^{K,\varphi\text{-old}}$$

is a one dimensional subspace of $M(V_{k,F}, \omega_0)^{K,\varphi\text{-old}}$ and $U_p$ is not diagonalizable on $M(V_{k,F}, \omega_0)^{K,\varphi\text{-old}}$.

6.1 The case of three $p$-old forms

Let us assume that $\varphi_i \in M(V_{k,F}, \omega_0)_{AK}$ are such that $\varphi_i \mid T_p = a_p(\varphi_i)\varphi_i$ for $i = 1, 2, 3$ and let us write $\alpha_i := \alpha(\varphi_i)$ and $\beta_i := \beta(\varphi_i)$. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $k = (k_1, k_2, k_3)$, we define

$$E_{p,1}(\alpha, k) := 1 - \omega_{1,1}(p')\frac{\alpha_1}{\alpha_2}\alpha_3 p^{k_1}, E_{p,2}(\alpha, k)$$

$$:= 1 - \omega_{1,2}(p')\frac{\alpha_2}{\alpha_1}\alpha_3 p^{k_2}, E_{p,3}(\alpha, k) := 1 - \omega_{1,3}(p')\frac{\alpha_3}{\alpha_1\alpha_2} p^{k_3}$$

$$E_p(\alpha, k) := E_{p,1}(\alpha, k) E_{p,2}(\alpha, k) E_{p,3}(\alpha, k) \left(1 - \frac{1}{\alpha_1\alpha_2\alpha_3} p^{k_1+k_2+k_3+1}\right)$$

and then (the last equality with the convention that it simply means “formally remove the $E_{p,i}(\alpha, k)$ factor”):

$$\widehat{E}_{p,i}(\alpha, k) := \prod_{j \neq i} E_{p,j}(\alpha, k) \left(1 - \frac{1}{\alpha_1\alpha_2\alpha_3} p^{k_1+k_2+k_3+1}\right) = \frac{E_p(\alpha, k)}{E_{p,i}(\alpha, k)}.$$  

Proposition 6.5 With the above notations, the following formulas hold:

$$t_k\left(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}\right) = \frac{\alpha_1}{p+1} \widehat{E}_{p,1}(\alpha, k) t_k(\varphi_1, \varphi_2, \varphi_3).$$

$$t_k\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)}\right) = \frac{\alpha_2}{p+1} \widehat{E}_{p,2}(\alpha, k) t_k(\varphi_1, \varphi_2, \varphi_3).$$

$$t_k\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p\right) = \frac{\alpha_3}{p+1} \widehat{E}_{p,3}(\alpha, k) t_k(\varphi_1, \varphi_2, \varphi_3).$$

Proof We have, by definition and Proposition 6.4 (3),

$$t_k\left(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p\right) = -t_k\left(\varphi_1 - \alpha_1^{-1}\varphi_1^{(p)}, \varphi_2 - \alpha_2^{-1}\varphi_2^{(p)}, \varphi_3 - \alpha_3^{-1}\varphi_3^{(p)}\mid p^k \omega_{1,3}(p')^{-1}\varphi_3 - \varphi_3^{(p)}\right)$$

$$= - \left(A^{(3)} - B^{(3)} + C^{(3)} - D^{(3)}\right).$$
where

\[
A^{(3)} = \omega_{r,3} \left( p' \right)^{-1} \alpha_3^{-1} p^{k_3} t_k (\varphi_1, \varphi_2, \varphi_3),
\]

\[
B^{(3)} = \omega_{r,3} \left( p' \right)^{-1} \alpha_1^{-1} \alpha_3^{-1} p^{k_3} t_k \left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right) + \omega_{r,3} \left( p' \right)^{-1} \alpha_2^{-1} \alpha_3^{-1} p^{k_3} t_k \left( \varphi_1, \varphi_2^{(p)}, \varphi_3 \right) + t_k \left( \varphi_1, \varphi_2, \varphi_3^{(p)} \right)
\]

\[
C^{(3)} = \omega_{r,3} \left( p' \right)^{-1} \alpha_1^{-1} \alpha_2^{-1} \alpha_3^{-1} p^{k_3} t_k \left( \varphi_1^{(p)}, \varphi_2^{(p)}, \varphi_3^{(p)} \right)
\]

\[
D^{(3)} = \alpha_1^{-1} \alpha_2^{-1} t_k \left( \varphi_1^{(p)}, \varphi_2^{(p)}, \varphi_3^{(p)} \right).
\]

Regarding \( t_k \) as a pairing as we did in the proof of Corollary 5.6 (for \( t_{3,k} \)), we compute

\[
t_k \left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right)
= \left\{ \varphi_1 | K^{\#} \pi^i_p K, \varphi_2 \otimes \varphi_3 \right\} \tag{57}
\]

(57)

(57)

(57)

(57)

It follows from Lemma 6.1 that, if \( K^{\#} = \bigcup_{l} K \gamma_l \), then \( K^{\#} \pi^i_p K^{\#} = \bigcup_{l} K^{\#} \pi^i_p \gamma_l \). Therefore,

\[
\sum_{\gamma_l \in K \setminus K^{\#}} \left\{ \varphi_1 | T_p, \varphi_2 \otimes \varphi_3 \right\} \tag{58}
\]

(58)

(58)

(58)

We have proved that

\[
t_k \left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right) = (p + 1)^{-1} a_p \left( \varphi_1 \right) t_k \left( \varphi_1, \varphi_2, \varphi_3 \right). \tag{58}
\]

(58)

(58)

(58)

Working in a similar way for the other two terms of \( B^{(3)} \) we deduce (recall \( \alpha_3 p^{-k_3} = \omega_{r,3} \left( p' \right) \beta_3^{-1} t_{k} \)):

\[
(p + 1) B^{(3)} = \left\{ \omega_{r,3} \left( p' \right)^{-1} \alpha_1^{-1} a_p \left( \varphi_1 \right) \alpha_3^{-1} p^{k_3} \right. \right. \\
+ \left. \omega_{r,3} \left( p' \right)^{-1} \alpha_2^{-1} a_p \left( \varphi_2 \right) \alpha_3^{-1} p^{k_3} + a_p \left( \varphi_3 \right) \right\} t_k \left( \varphi_1, \varphi_2, \varphi_3 \right)
\]
\[
\begin{align*}
= \{ \omega_{r,3} (p')^{-1} \alpha_3^{-1} p^{k_3} + \omega_{r,3} (p')^{-1} \alpha_1^{-1} \beta_1 p^{k_3} + \omega_{r,3} (p')^{-1} \alpha_3^{-1} p^{k_3} \\
+ \omega_{r,3} (p')^{-1} \alpha_2^{-1} \alpha_3^{-1} \beta_2 p^{k_3} + \alpha_3 + \beta_3 \} t_k (\varphi_1, \varphi_2, \varphi_3).
\end{align*}
\]

Noticing that we have \( K^\# \tilde{\pi}_t p K = K^\# \tilde{\pi}_t p \), we find
\[
\varphi_1^{(p)} \otimes \varphi_2^{(p)} = (\varphi_1 \otimes \varphi_2) | K^\# \tilde{\pi}_t p K.
\]

Hence we find, using the adjointness property of Proposition 2.9,
\[
t_k \left( \varphi_1^{(p)}, \varphi_2^{(p)}, \varphi_3 \right) = \left( (\varphi_1 \otimes \varphi_2) | K^\# \tilde{\pi}_t p K, \varphi_3 | K^\# 1 K \right)_t
= (p + 1)^{-1} \omega_{r,3} (p') p^{k_2} \left( \varphi_1 \otimes \varphi_2, \varphi_3 | K^\# 1 K \right) | K^\# p K^\#_t
= (p + 1)^{-1} \omega_{r,3} (p') p^{k_2} \alpha p (\varphi_3) (\varphi_1 \otimes \varphi_2, \varphi_3)_t
= (p + 1)^{-1} \omega_{r,3} (p') p^{k_2} \alpha p (\varphi_3) t_k (\varphi_1, \varphi_2, \varphi_3).
\]

Working in a similar way for the other two terms of \( C^{(3)} \) we deduce (recall \( \alpha_3 p^{-k_3} = \omega_{r,3} (p') \beta_3^{-1} p \))
\[
(p + 1) C^{(3)} = \left\{ \alpha_1^{-1} \alpha_2^{-1} \alpha p (\varphi_3) \alpha_3^{-1} p^{k_3 + k_2} + \omega_{r,2} (p') \alpha_1^{-1} \alpha p (\varphi_2) p^{k_2} \right. \\
+ \omega_{r,1} (p') \alpha_2^{-1} \alpha p (\varphi_1) p^{k_2} \right\} t_k (\varphi_1, \varphi_2, \varphi_3)
= \left\{ \alpha_1^{-1} \alpha_2^{-1} p^{k_3 + k_2} + \alpha_1^{-1} \alpha_2^{-1} \alpha_3^{-1} \beta_3 p^{k_3 + k_2} \\
+ \omega_{r,2} (p') \alpha_1^{-1} \alpha_2 p^{k_2} + \omega_{r,2} (p') \alpha_1^{-1} \beta_2 p^{k_2} \\
+ \omega_{r,1} (p') \alpha_2^{-1} \alpha_1 p^{k_2} + \omega_{r,1} (p') \alpha_2^{-1} \beta_1 p^{k_2} \right\} t_k (\varphi_1, \varphi_2, \varphi_3).
\]

Finally, once again using \( K^\# \tilde{\pi}_t p K = K^\# \tilde{\pi}_t p \), we find
\[
\varphi_1^{(p)} \otimes \varphi_2^{(p)} \otimes \varphi_3^{(p)} = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 | K^\# \tilde{\pi}_t p
\]
and \( t_k \left( \varphi_1^{(p)}, \varphi_2^{(p)}, \varphi_3^{(p)} \right) = p^{k^*} t_k (\varphi_1, \varphi_2, \varphi_3) \). Hence we find
\[
D^{(3)} = \alpha_1^{-1} \alpha_2^{-1} p^{k^*} t_k (\varphi_1, \varphi_2, \varphi_3).
\]

Putting everything together, we have computed that
\[
t_k \left( \varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} | W_p \right) = - (p + 1)^{-1} E \cdot t_k (\varphi_1, \varphi_2, \varphi_3),
\]
where (using \( \beta_i = \omega_{r,i} (p')^{-1} \alpha_i^{-1} p^{k_{i+1}} \)):
\[
E = \omega_{r,3} (p')^{-1} \alpha_3^{-1} p^{k_3} + \omega_{r,3} (p')^{-1} \alpha_3^{-1} p^{k_3+1}
\]
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Suppose that the triple product simplifications:

Let us now remark that, writing \( \omega \equiv \omega_1, \omega_2, \omega_3 \) for the j-term of the i-line, we have the following simplifications: \((1, 1)\) with \((2, 1)\), \((1, 2)\) with \((3, 3)\), \((4, 1)\) with \((7, 1)\) (because \(k_3 + k_3^* = k\)) and \((6, 2)\) with \((7, 2)\) (because \(k_1 + k_1^* = k\)). Hence, we find (recalling that \( \omega_{i,1} \omega_{i,2} \omega_{i,3} = 1 \) on \( \mathbb{Z}_p \times \) in the first equality and noticing that \( k_1 + k_3 + 1 = k_2 + k_3 + 1 = k_1^* + k_3^* + 1 \), \( k_3 = k_2^* + k_2^*, k_2 + k_3 + 1 = k_1^* + k_3^* + 1, 2k_3 + k_3^* + 1 = k_1^* + k_2^* + k_3^* + 1 \) and \( k_2 + k_3^* + 1 = k_2^* + 1 \) to get the factorization):

\[
E = -\alpha_3 (\omega_{i,2} (p') \alpha_1^2 \alpha_3^2 p^{k_1+k_3+1} + \omega_{i,1} (p') \omega_{i,2} (p') \alpha_3^2 p^{k_3})
\]

\[
\omega_{i,1} (p') \alpha_2^2 \alpha_3^2 p^{k_3} + \omega_{i,2} (p') \alpha_2^2 \alpha_3^2 p^{k_3} + 1 - \omega_{i,3} (p')^{-1} \alpha_1^{-1} \alpha_2^{-1} \alpha_3^{-1} p^{2k_3+k_3^*+1}
\]

\[
- \omega_{i,2} (p') \alpha_2^{-1} \alpha_3^{-1} p^{k_3^*} - \omega_{i,1} (p') \alpha_2^{-1} \alpha_3^{-1} p^{k_3} - \omega_{i,1} (p') \alpha_2^{-1} \alpha_3^{-1} p^{k_3^*}
\]

\[
= -\alpha_3 \left( 1 - \omega_{i,1} (p') \frac{\alpha_1}{\alpha_2 \alpha_3} p^{k_3^*} \right) \left( 1 - \omega_{i,2} (p') \frac{\alpha_2}{\alpha_1 \alpha_3} p^{k_3^*} \right) \left( 1 - \frac{1}{\alpha_1 \alpha_2 \alpha_3} p^{k_3^*+1} \right).
\]

Inserting this computation of \( E \) in (59) gives the third equation. The first two equations are proved in a similar way.

Let us discuss the \( p \)-adic periods.

**Lemma 6.6** Suppose that \( 0 \neq \varphi_1, \varphi_2 \in M (V_{k,F}, \omega_0)^k \) are such that \( \varphi_i \mid T_p = a_p (\varphi_i) \varphi_i \) and let \( \alpha_i = a_p (\varphi_i) \) be a root of the Hecke polynomial at \( p \) of \( \varphi_i \) (see (56)). If \( a_p (\varphi_1) = a_p (\varphi_2) \) and \( \alpha := \alpha_1 = \alpha_2 \), then we have

\[
\left( \varphi^{(\alpha)}, \varphi^{(\alpha)} \mid W_p \right)_k = \frac{\alpha \left( 1 - \omega (p')^{-1} \alpha^{-2} p^k \right) \left( 1 - \omega (p')^{-1} \alpha^{-2} p^{k+1} \right)}{p + 1} (\varphi, \varphi)_k.
\]

**Proof** We have, by Proposition 6.4 (3),

\[
\left( \varphi^{(\alpha)}_1, \varphi^{(\alpha)}_2 \mid W_p \right)_k = \left( \varphi_1 - \alpha^{-1} \varphi_1^{(p)}, \varphi_2^{(p)} - \alpha^{-1} \omega (p')^{-1} p^k \varphi_2 \right)_k
\]

\[
= -\omega (p')^{-1} \alpha^{-1} p^k (\varphi_1, \varphi_2)_k + \omega (p')^{-1} \alpha^{-2} p^k \left( \varphi_1^{(p)}, \varphi_2 \right)_k + \left( \varphi_1, \varphi_2^{(p)} \right)_k - \alpha^{-1} \left( \varphi_1^{(p)}, \varphi_2^{(p)} \right)_k.
\]
If \( \varphi, \psi \in M \left( V_{k,F}, \omega_0 \right)^K \) and \( \psi^\vee \in M \left( V_{k,F}, N_{1}^{2k} (\omega_0^{-1})^K \right) \) are such that \( \varphi ^{\upharpoonright T_p} = a_p (\varphi) \varphi \), the adjointness property of Proposition 2.9 gives, arguing as in (57) and (58):

\[
(p + 1) \left( \varphi^{(p)}, \psi \right)_k = \left( \varphi \right. ^{\upharpoonright T_p} \left. , \psi \right)_k
= a_p (\varphi) \left( \varphi, \psi \right)_k \quad \Longleftrightarrow \quad (p + 1) \left( \varphi^{(p)}, \psi \right)_k
= a_p (\varphi) \left( \varphi, \psi \right)_k.
\]

(61)

In order to get a symmetrical relation, suppose now we have \( \psi^\vee ^{\upharpoonright T_p} = a_p \left( \psi^\vee \right) \psi^\vee \) and apply once again Proposition 2.9 arguing as in (57) and (58) in order to get

\[
(p + 1) \left( \varphi, \left( \psi^\vee \right)^{(p)} \right)_k = a_p \left( \psi^\vee \right) \left( \varphi, \psi^\vee \right)_k.
\]

Next, note that twisting does not exactly commutes with the right \( B^{1}_F \)-actions: rather we have \( \tilde{\psi} \tilde{g} = nrd^{-1}_{\psi}(g) (\tilde{\varphi} \tilde{g}) \). It follows that \( \tilde{\psi}^{(p)} = \omega_{\tilde{\varphi}}^{-1} (\tilde{\varphi})(\tilde{\varphi}^{(p)}) \) and \( a_{\tilde{\psi}} = \omega_{\tilde{\varphi}}^{-1}(\tilde{\varphi}) a_{\varphi} (\psi) \). Then, taking \( \psi^\vee = \tilde{\psi} \) in the above relation gives

\[
(p + 1) \omega_{\tilde{\varphi}}^{-1} (\tilde{\varphi}) \left( \varphi, (\tilde{\varphi}^{(p)}) \right)_k = (p + 1) \left( \varphi, \tilde{\varphi}^{(p)} \right)_k
= a \left( \tilde{\varphi} \right) \left( \varphi, \tilde{\varphi} \right)_k = \omega_{\tilde{\varphi}}^{-1} (\tilde{\varphi}) a_{\varphi} (\psi) \left( \varphi, \tilde{\varphi} \right)_k
\]

\[
\Longleftrightarrow (p + 1) \left( \varphi, (\tilde{\varphi}^{(p)}) \right)_k = a_{\varphi} (\psi) \left( \varphi, \tilde{\varphi} \right)_k \Longleftrightarrow (p + 1) \left( \varphi, \psi^{(p)} \right)_k
= a_{\varphi} (\psi) \left( \varphi, \psi \right)_k.
\]

(62)

Finally, the invariance property of \( (-,-)_k \) gives \( \left( \varphi^{(p)}, (\psi^\vee)^{(p)} \right)_k = p^k \left( \varphi, \psi^\vee \right)_k \) and then we find (because \( \omega_{\tilde{\varphi}}^{-1} (\tilde{\varphi}) = \omega_{\tilde{\varphi}} (p') \)):

\[
\omega_{\tilde{\varphi}}^{-1} (\tilde{\varphi}) \left( \varphi^{(p)}, (\psi^\vee)^{(p)} \right)_k = \left( \varphi^{(p)}, \tilde{\psi}^{(p)} \right)_k = \left( \varphi^{(p)}, \tilde{\psi}^{(p)} \right)_k \Longleftrightarrow \left( \varphi^{(p)}, \psi^{(p)} \right)_k
= \omega_{\tilde{\varphi}} (p')^{-1} p^k \left( \varphi, \psi \right)_k.
\]

(63)

Inserting (61), (62) and (63) in (60) yields

\[
\left( \varphi_1^{(a)}, \varphi_2^{(a)} \right)_{W_p} = (p + 1)^{-1} E \cdot (\varphi_1, \varphi_1)_k,
\]

where (using \( \beta = \omega_{\tilde{\varphi}} (p')^{-1} \alpha^{-1} p^{k+1} \)):

\[
E = -\omega_{\varphi} (p')^{-1} \alpha^{-1} p^{k+1} - \omega_{\varphi} (p')^{-1} \alpha^{-1} p^k + \omega_{\varphi} (p')^{-1} \alpha^{-2} p^k a_{\varphi} (\varphi)
+ a_{\varphi} (\psi) - \omega_{\varphi} (p')^{-1} \alpha^{-1} p^{k+1} - \omega_{\varphi} (p')^{-1} \alpha^{-1} p^k
\]

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\[
\begin{align*}
&= -\omega_I(p')^{-1} - \frac{1}{\alpha} p^{k+1} + \omega_I(p')^{-1} - \frac{1}{\alpha} p^k + \omega_I(p')^{-2} - \frac{3}{\alpha} p^{2k+1} \\
&\quad + \alpha + \omega_I(p')^{-1} - \frac{1}{\alpha} p^{k+1} - \omega_I(p')^{-1} - \frac{1}{\alpha} p^k
\end{align*}
\]

6.2 The case of two \( p \)-old forms

Let us assume that \( \varphi_i \in M(V_{k_i}, \omega_{0,i})^K \) are such that \( \varphi_i \mid T_p = a_p(\varphi_i) \varphi_i \) for \( i = 1, 2 \) and that \( \varphi_3 \in M(V_{k_3}, 1)^K \) is \( p \)-new, has even weight and trivial central character, i.e. it is such that \( \varphi_3 \mid U_p = -w_{p,3} p^{k_3/2} \varphi_3 \) and \( \varphi_3 \mid W_p = w_{p,3} p^{k_3/2} \varphi_3 \) with \( w_{p,3} \in \{\pm 1\} \). To make the notation uniform, we define \( \alpha_3 := -w_{p,3} p^{k_3/2} \) and \( \varphi^{(\alpha_3)}_3 := \varphi_3 \). Then

\[
E_{p,1}(\alpha, k) = 1 + w_{p,3} \omega_I(1)(p')^{p(k_2-k_1)/2} \alpha_1 \alpha_2, E_{p,2}(\alpha, k)
\]

\[
E_{p,3}(\alpha, k) = 1 + w_{p,3} \alpha_3 \alpha_2.
\]

Proposition 6.7 With the above notations, the following formulas hold:

\[
t_k(\varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}) = E_{p,3}(\alpha, k) t_k(\varphi_1^{(p)}, \varphi_2, \varphi_3)
\]

\[
= w_{p,3} \omega_I(1)(p')^{-1} p(k_1-k_2)/2 E_{p,3}(\alpha, k) t_k(\varphi_1, \varphi_2(p), \varphi_3),
\]

\[
t_k(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)}) = E_{p,3}(\alpha, k) t_k(\varphi_1, \varphi_2(p), \varphi_3)
\]

\[
= w_{p,3} \omega_I(2)(p')^{-1} p(k_2-k_1)/2 E_{p,3}(\alpha, k) t_k(\varphi_1(p), \varphi_2, \varphi_3),
\]

\[
t_k(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p) = \frac{\alpha_3}{\alpha_1} E_{p,2}(\alpha, k) t_k(\varphi_1, \varphi_2(p), \varphi_3)
\]

\[
= \frac{\alpha_3}{\alpha_1} E_{p,1}(\alpha, k) t_k(\varphi_1(p), \varphi_2, \varphi_3).
\]

Proof We begin with the following remark from [17, discussion after Theorem 2]: because \( t_k \) is \( K^\# \)-invariant (resp. \( \omega_p^{-1} K^\# \omega_p \)-invariant)\(^5\) and \( \varphi_i \in M(V_{k_i}, \omega_{0,i})^K \)

\(^5\) The trilinear form \( t_k \) satisfies the invariance formula

\[
t_k(\varphi_1 u, \varphi_2 u, \varphi_3 u) = \text{Nrd}_f(u)t_k(\varphi_1, \varphi_2, \varphi_3)
\]

and we have \( \text{Nrd}_f(u) = 1 \) for \( u \in K^\# \) or \( u \in \omega_p^{-1} K^\# \omega_p \).
(resp. $\varphi_i^{(p)} \in M(V_{k_i,F}, \omega_{0,i})^{\omega_p^{-1}K^\#\omega_p}$ for $i = 1, 2$, $\varphi \mapsto t_\xi(\varphi_1, \varphi_2, \varphi)$ (resp. $\varphi \mapsto t_\xi(\varphi_1^{(p)}, \varphi_2^{(p)}, \varphi)$) is $K^\#$-invariant (resp. $\omega_p^{-1}K^\#\omega_p$-invariant) and, hence, it is zero on the irreducible representation $V_{\varphi_3} \subset M(V_{k_3,F}, \omega_{0,3})$ generated by $\varphi_3$, whose dual representation does not have non-zero $K^\#$-invariant (resp. $\omega_p^{-1}K^\#\omega_p$-invariant) vectors. In particular,

$$t_\xi(\varphi_1, \varphi_2, \varphi) = t_\xi(\varphi_1^{(p)}, \varphi_2^{(p)}, \varphi) = 0. \quad (64)$$

The computations of $t_\xi(\varphi_1^{(\alpha_1)} | W_p, \varphi_2^{(\alpha_2)}, \varphi_3)$ and $t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} | W_p, \varphi_3)$ are the same, so that we can work with the second. We have, by Proposition 6.4 (3) and (64):

$$t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3 | W_p)$$

$$= t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} | W_p, \varphi_3)$$

$$= t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3) + \omega_{0,2}^{(\alpha_2)}(1) \omega_{0,2}^{(\alpha_3)}(1) t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3) \quad (65)$$

Similarly, because $\varphi_3 | W_p = w_{p,3}p^{k_3/2}\varphi_3$ and by (64):

$$t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3 | W_p)$$

$$= w_{p,3}p^{k_3/2} \left( \varphi_1 - \omega_{0,2}^{(\alpha_2)} \varphi_2^{(\alpha_2)} - \omega_{0,2}^{(\alpha_3)} \varphi_3 \right)$$

$$= -w_{p,3}p^{k_3/2} \left( \alpha_1^{-1} t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3) + \alpha_2^{-1} t_\xi(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3) \right). \quad (66)$$

Because $K^{\#\omega_p^{-1}} K = K^{\#\omega_p}$, we have $\varphi_i^{(p)} = \varphi_i | K^{\#\omega_p}$; also, noticing that $\tilde{\omega}_p^2 = -\tilde{p}$, we see that $\tilde{\omega}_p^{-1} = -\tilde{p}^{-1} \omega_p = -\tilde{p}^{-1} \omega_p$, implying that $K^{\#\omega_p^{-1}} = K^{\#\omega_p}$ for $K' \in \{ K^\#, K \}$. Applying Remark 6.2 gives $\varphi_i | K^{\#\omega_p^{-1}} = \omega_{0,1}(\tilde{p})^{-1} \varphi_i | K^{\#\omega_p}$.

Hence we find (using the invariance property of $t_\xi$ in the second equality\textsuperscript{6} and $\varphi_3 | W_p = w_{p,3}p^{k_3/2}\varphi_3$ in the last equality):

$$t_\xi(\varphi_1^{(p)}, \varphi_2, \varphi_3) = t_\xi(\varphi_1 \tilde{\omega}_p, \varphi_2, \varphi_3) = p^{k_2} t_\xi(\varphi_1, \varphi_2 \tilde{\omega}_p, \varphi_3)$$

$$= \omega_{0,1}(\tilde{p})^{-1} \omega_{0,2}^{(\alpha_2)}(1) \omega_{0,2}^{(\alpha_3)}(1) t_\xi(\varphi_1, \varphi_2 \tilde{\omega}_p, \varphi_3)$$

$$= \omega_{0,2}^{(\alpha_2)}(1) \omega_{0,2}^{(\alpha_3)}(1) t_\xi(\varphi_1, \varphi_2^{(p)}, \varphi_3). \quad (67)$$

Inserting (67) in (65) and (66) yields the claimed formulas. \hfill \Box

\textsuperscript{6} We have $\text{Nrd}_f(\tilde{\omega}_p) = |\text{Nrd}(\tilde{\omega}_p)|_{\tilde{h}_f}^{-1} = \tilde{p}^{-1} = p$. 

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6.3 The case of one $p$-old form

Let us assume that $\varphi_i \in M \left( \mathbf{V}_{k_i,f}, \omega_{0,i} \right)^K$ are such that $\varphi_i \mid T_p = a_p(\varphi_i) \varphi_i$ for $i = 1$ and that $\varphi_i \in M \left( \mathbf{V}_{k_i,f}, 1 \right)^K$ are $p$-new, have even weight and trivial central character for $i = 2, 3$ (implying $\omega_{f,1}(p') = 1$). As in the setting of Proposition 6.7, we write $\alpha_i := -w_{p,i}p^{k_i/2}$ and $\varphi_i^{(\alpha_i)} := \varphi_i$ for $i = 2, 3$. Then

$$E_{p,1}(\alpha, k) = 1 - w_{p,2}w_{p,3}\alpha_1p^{-k_1/2} \text{ and } E_{p,2}(\alpha, k) = E_{p,3}(\alpha, k) = 1 - w_{p,2}w_{p,3}\alpha_1^{-1}p^{k_1/2}.$$  

**Proposition 6.8** With the above notations, the following formulas hold:

$$t_k\left( \varphi_1^{(\alpha_1)} \mid W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \right) = -\alpha_1^{-1}p^{k_1}E_{p,1}(\alpha, k)t_k(\varphi_1, \varphi_2, \varphi_3),$$

$$t_k\left( \varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)} \mid W_p, \varphi_3^{(\alpha_3)} \right) = -\alpha_2^{-1}E_{p,2}(\alpha, k)t_k(\varphi_1, \varphi_2, \varphi_3),$$

$$t_k\left( \varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} \mid W_p \right) = -\alpha_3^{-1}E_{p,3}(\alpha, k)t_k(\varphi_1, \varphi_2, \varphi_3).$$

**Proof** The computations of $t_k\left( \varphi_1^{(\alpha_1)}, \varphi_2 \mid W_p, \varphi_3 \right)$ and $t_k\left( \varphi_1^{(\alpha_1)}, \varphi_2, \varphi_3 \mid W_p \right)$ are the same, so that we can work with the second. We have, by Proposition 6.4 (3):

$$t_k\left( \varphi_1^{(\alpha_1)} \mid W_p, \varphi_2, \varphi_3 \right) = t_k\left( \varphi_1^{(p)} - \alpha_1^{-1}p^{k_1}\omega_{f,1}(p')^{-1}\varphi_1, \varphi_2, \varphi_3 \right) = t_k\left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right) - \alpha_1^{-1}\omega_{f,1}(p')^{-1}p^{k_1}t_k(\varphi_1, \varphi_2, \varphi_3),$$  

(68)

Also, because $\varphi_3 \mid W_p = w_{p,3}p^{k_3/2}\varphi_3$:

$$t_k\left( \varphi_1^{(\alpha_1)}, \varphi_2, \varphi_3 \mid W_p \right) = w_{p,3}p^{k_3/2}t_k\left( \varphi_1 - \alpha_1^{-1}\varphi_1^{(p)}, \varphi_2, \varphi_3 \right) = w_{p,3}p^{k_3/2}t_k\left( \varphi_1, \varphi_2, \varphi_3 \right) - \alpha_1^{-1}t_k\left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right).$$  

(69)

Arguing similarly as we did in (67) we find

$$t_k\left( \varphi_1^{(p)}, \varphi_2, \varphi_3 \right) = w_{p,2}w_{p,3}p^{k_1/2}t_k(\varphi_1, \varphi_2, \varphi_3)$$  

(70)

Inserting (70) in (68) and (70) yields the claimed formulas (recall $\omega_{f,1}(p') = 1$).  

6.4 The case of three $p$-new forms

Let us assume that $\varphi_i \in M \left( \mathbf{V}_{k_i,f}, 1 \right)^K$ are $p$-new, have even weight and trivial central character for $i = 1, 2, 3$. As usual, we write $\alpha_i := -w_{p,i}p^{k_i/2}$ and $\varphi_i^{(\alpha_i)} := \varphi_i$ for $i = 1, 2, 3$. Then
\[ \mathcal{E}_{p,1}(\alpha, k) = \mathcal{E}_{p,2}(\alpha, k) = \mathcal{E}_{p,3}(\alpha, k) = 1 + w_{1,p}w_{2,p}w_{3,p}. \]

**Proposition 6.9** With the above notations, the following formulas hold:

\[
t_k(\varphi_1^{(\alpha_1)} | W_p, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)}) = -\alpha_1 t_k(\varphi_1, \varphi_2, \varphi_3), t_k(\varphi_1^{(\alpha_1)}, \varphi_2^{(\alpha_2)}, \varphi_3^{(\alpha_3)} | W_p) = -\alpha_2 t_k(\varphi_1, \varphi_2, \varphi_3).
\]

Furthermore, we have \( t_k(\varphi_1, \varphi_2, \varphi_3) = 0 \) when \( w_{1,p}w_{2,p}w_{3,p} = -1 \).

**Proof** Just use \( \varphi_i \ | W_p = w_{p,i}p^{k_i/2}\varphi_i \) to get the first formulas. The last assertion is a consequence of the invariance property of \( t_k \), which gives the first of the following equalities, and again the relation \( \varphi_i \mid W_p = w_{p,i}p^{k_i/2}\varphi_i \), which gives second equality below:

\[
p_k^{k_i}t_k(\varphi_1, \varphi_2, \varphi_3) = t_k(\varphi_1 \mid W_p, \varphi_2 \mid W_p, \varphi_3 \mid W_p) = w_{1,p}w_{2,p}w_{3,p}p^{k_i}t_k(\varphi_1, \varphi_2, \varphi_3).
\]

\( \square \)

## 7 Proof of the main result

### 7.1 Interpolation property of the \( p \)-adic trilinear form

Recall our given \( k = (k_1, k_2, k_3) \) such that \( k_1 \oplus k_2 \oplus k_3 \) is even, where \( k_i : \mathbb{Z} \rightarrow O_i \) and \( O_k := O_1 \otimes O_2 \otimes O_3 \). Consider the spaces \( M_p^{\omega}(D_{k_i}(W), \omega_{0,p}^{k_i}) \), where \( \omega_{0,p}(z) = \omega_{T,i}(z)N_{p}^{k_i}(z) \) with \( \omega_{T,i} \) the finite part of a unitary Hecke character taking values in \( F \) that are unramified outside \( p \) and such that \( \omega_{T,1}\omega_{T,2}\omega_{T,3} = 1 \). Recall that specialization maps attached to \( \phi_i : k_i \rightarrow \mathbb{N} \):

\[
\phi_{i,*}^{\text{alg}} : M_p^{\omega_i}(D_{k_i}(W), \omega_{0,p}^{k_i}) \rightarrow M_p^{\omega_i}(V_{k_i,F}, \omega_{0,p}^{k_i}) \cong M_p^{\omega_i}(V_{k_i,F}, \omega_{0,p}^{k_i}),
\]

where \( \omega_{0,p}(z) = \omega_{T,i}(z)N_{p}^{k_i}(z) \) and \( \omega_{0,p}(z) = \omega_{T,i}(z)N_{p}^{k_i}(z) \). Let set up the following notation in order to precisely give our statement. Let us fix \( \alpha := (\alpha_1, \alpha_2, \alpha_3) \), where \( \alpha_i \in O_i^{\times} \) and write \( M_p^{\omega_i}(D_{k_i}(W), \omega_{0,p}^{k_i})_{>\alpha_i} \) for the subspace of \( M_p^{\omega_i}(D_{k_i}(W), \omega_{0,p}^{k_i}) \) which is the kernel of the \( U_p - \alpha_i \).

**Remark 7.1** There are plenty of examples of non-zero eigenvectors with associated invertible eigenvalue because the \( U_p \)-operator acts on these spaces and the Ash-Stevens theory of [4] applies to show that they have slope \( \leq h \in \mathbb{R} \) decompositions (as defined in [4]): writing \( (\mathcal{E}_{p,1}^h \oplus \mathcal{E}_{p,2}^h) \), any eigenvector in \( M_p^{\omega_i}(D_{k_i}(W), \omega_{0,p}^{k_i})_{>\alpha_i} \) has eigenvalue in \( O_i^{\times} \). Furthermore, the Ash-Stevens theory of [4] applies to show that
we have the control theorem in our setting, from which one can easily deduce that the $U_p$-eigenvectors of slope $\leq h < k + 1$ on $M_p^\omega(D_{k_1}(W), \omega_{0,p})^{\leq h}$ lifts to eigenfamilies belonging to $M_p^\omega(D_{k_i}(W), \omega_{0,p})^{\leq h}$ in an essential unique way, when $\phi_i : k_i \rightarrow k_i$ is obtained from $k_i \in U_i \subset X$ (see also [9] and [38, Theorem 3.7] for the control theorem in our setting and [11, Corollary B5.7.1] and [20, Corollary 11.4] for these kind of applications of the control theorem). These lifts to eigenfamilies in $M_p^\omega(D_{k_i}(W), \omega_{0,p})^{\leq h}$ provide such a kind of examples.

Set

$$M_p^\alpha := M_p^\omega(D_{k_1}(W), \omega_{0,p})^{\alpha_1} \otimes M_p^\omega(D_{k_2}(W), \omega_{0,p})^{\alpha_2} \otimes M_p^\omega(D_{k_3}(W), \omega_{0,p})^{\alpha_3}$$

Then define the $O_k$-valued $O_k$-linear functionals

$$\mathcal{L}_{p,i}^\alpha : M_p^\alpha \rightarrow O_k$$

via the formula

$$\mathcal{L}_{p,i}^\alpha (\varphi_1 \otimes \varphi_2 \otimes \varphi_3) := \frac{p+1}{\alpha_1} f_{i,k_i}^\alpha (\varphi_1 | W_3 \otimes \varphi_2 \otimes \varphi_3),$$

and $\mathcal{L}_{p,2}^\alpha$ and $\mathcal{L}_{p,3}^\alpha$ are defined in a similar way. We will use the notation $\varphi$ to denote an element of $M_p^\alpha$, that we may and will assume to be a pure tensor product.

Let us assume, from now on, that $k_i$ corresponds to $U_i \subset X$ and, if $\phi_i : k_i \rightarrow k_i \in \mathbb{N} \cap U_i$ is obtained from $k_i \in U_i \subset X$, $F \in O_k = O(U_1 \times U_2 \times U_3)$ and $k = (k_1, k_2, k_3)$, let $F(k) := (\phi_{i,k_i} \otimes \varphi_2 \otimes \varphi_3)(F)$ be its evaluation at $k$. Also, we write $\varphi_{i,k_i} := \varphi_{i,k_i}^{\text{alg}}(\varphi_i)$. Because we assume that $\alpha_i$ is invertible in $O_i$, and, hence, it has finite slope, except for a finite number of points $\varphi_{i,k_i}$ is old at $p$ and, more precisely, there is a unique $\varphi_{i,k_i} : M^\omega(V_{k_i,F}, \omega_{0,p})^{\text{GL}_2(\mathbb{Z}_p)}$ such that $\varphi_{i,k_i} = \varphi_{i,k_i}^{\text{old}(\alpha_i)}$. Let us write $U := U_1 \times U_2 \times U_3$.

**Definition 7.2** We say that $k = (k_1, k_2, k_3) \in \mathbb{N}^3 \cap U \subset X^3$ (resp. $k_i \in \mathbb{N} \cap U_i \subset X$) is a generic integer point for $\varphi$ (resp. $\varphi_i$) if $\varphi_{i,k_i}$ is old at $p$ for $i = 1, 2, 3$ (resp. $\varphi_{i,k_i}$ is old at $p$).

If $k \in \mathbb{N}^3 \cap U$, we write $\varphi_k := \varphi_{1,k_1} \otimes \varphi_{2,k_2} \otimes \varphi_{3,k_3}$ and, when $k$ is a generic integer point, we also write $\varphi_k^{\#} := \varphi_{1,k_1}^{\#} \otimes \varphi_{2,k_2}^{\#} \otimes \varphi_{3,k_3}^{\#}$. When it happens that $\varphi_k^{\#}$ belongs to an irreducible representation, we denote it by $\Pi(\varphi_k)$ and let $\Pi'(\varphi_k)$ be its Jacquet–Langlands lifts to $\text{GL}_2$, so that $\Pi(\varphi_k) = \Pi(\varphi_k^{\#})$ and $\Pi'(\varphi_k) = \Pi'(\varphi_k^{\#})$ when $k$ is generic. Finally, we write $M_p^\omega(\varphi) \subset M_p^\alpha$ for the $B^{3 \times 3}(A_1^p)$-representation generated by $\varphi$ over $O_k$; note that, if $\varphi' \in M_p^\omega(\varphi)$, then $\Pi(\varphi') = \Pi(\varphi)$ for every integer point $k$. Finally, we choose vectors $\varphi_k^{\#}, \varphi_k^{\#} \in \Pi(\varphi_k)$ such that $(\varphi_k^{\#}, \varphi_k^{\#})_k \neq 0$ (see Lemma 3.2 for their existence and (21) for a specific choice); we further assume that they satisfy the property that the local components at $p$ equals the local component at
of $\varphi_k^\dagger$ (indeed, because $\varphi_k^\dagger$ is new at $p$, $\varphi_k^\dagger$ is the tensor product of its local component at $p$, which is defined, and a prime to $p$-component). Having setup our notations, we can state our main result, which is a combination of Theorem 3.4, (46), Corollary 5.6 and Proposition 6.5.

**Theorem 7.3** There is a unique $O_k$-valued $O_k$-linear functional $L_p^\alpha : M_p^\alpha \rightarrow O_k$ such that, for every $\varphi \in M_p^\alpha$ and every balanced generic integer point $k \in U$ for $\varphi$,

$$L_p^\alpha(\varphi)(k) := \mathcal{E}_p(\alpha, k) \, t_k(\varphi_k^\dagger). \quad (71)$$

We have, indeed, $L_p^\alpha = L_{p,i}$ for $i = 1, 2, 3$ and, furthermore, if $\varphi \in M_p^\alpha$ is a tensor product of three families and $\varphi_k^\dagger$ belongs to the irreducible representation $\Pi(\varphi_k^\dagger)$, then

$$L_p^\alpha(\varphi)(k)^2 = \mathcal{E}_p(\alpha, k)^2 \, \frac{C_k}{2^9 \, 3^2} \, \frac{\xi_2^2(2)}{L}\left(1/2, \Pi'(\varphi_k^\dagger)\right) \prod_{\nu} I_\nu(\varphi_k^\dagger) \, \frac{\left(\varphi_k^\dagger, \varphi_k^\dagger\right)_k}{L\left(1, \Pi'(\varphi_k^\dagger)\right)} L\left(1/2, \Pi'(\varphi_k^\dagger)\right) \prod_{\nu \neq \infty, p} C_v^{\varphi_k^\dagger, \varphi_k^\dagger}(\varphi_k^\dagger), \quad (72)$$

where $C_k \neq 0$ is defined in (22) and $I_\nu(\varphi_k^\dagger)$ and $C_v^{\varphi_k^\dagger, \varphi_k^\dagger}(\varphi_k^\dagger)$ are defined in (25).

Also, suppose that there is a balanced generic integer point $k_0$ for $\varphi$ such that $B = B_{\Pi'(\varphi_k^\dagger)}$ is the quaternion algebra predicted by [34] and $L\left(1/2, \Pi'(\varphi_k^\dagger)\right) \neq 0$.

Then, up to shrinking $U$ in a neighbourhood of $k_0$, there exists $\varphi' \in M_p^\alpha$ such that, for every balanced generic integer point $k \in U$, we know that $B = B_{\Pi'(\varphi_k^\dagger)} = B_{\Pi'(\varphi_k^\dagger)}$ is the quaternion algebra predicted by [34] and we have satisfied the equivalence

$$L_p^\alpha(\varphi')(k) \neq 0 \Leftrightarrow L\left(1/2, \Pi'(\varphi_k^\dagger)\right) = L\left(1/2, \Pi'(\varphi_k^\dagger)\right) \neq 0.$$

**Proof** Applying Corollary 5.6 and Proposition 6.5 to any one of the $L_{p,i}^\alpha$’s gives (71) with $L_{p,i}^\alpha := L_{p,i}^\alpha$ and the uniqueness follows from the Zariski density of the balanced generic integer points in the open affinoid subdomain $U_1 \times U_2 \times U_3$ of $X^3$. Then everything is clear from Theorem 3.4, except we have to explain why we have excluded $C_p^{\varphi_k^\dagger, \varphi_k^\dagger}(\varphi_k^\dagger)$ from our local constants. This is because $\varphi_k^\dagger$ is the tensor product of $p$-new vectors which gives rise, in the local representation at $p$ attached to $\Pi(\varphi_k^\dagger)$, to a vector $\psi_p$ which is the tensor product of the (unique up to a scalar factor)

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$p$-new vectors in the local representations attached to the $\varphi_{i,k_i}^{\#,(\alpha_i)}$’s. Then, because the local components of $\varphi_k^\#$ and $\varphi_k^{\#(p)}$ at $p$ equals the local component at $p$ of $\varphi_k^\#$, the equality $C_{p_k}^{\#} \otimes \varphi_k^{\#(p)} = (\varphi_k^\#) = 1$ follows from [31, Lemma 2.2] (see Sect. 8.2 below for more details). The last assertion follows from (72), Theorem 3.4 (2) and Remark 1.1.

\[ \square \]

**Remark 7.4** The equality (72) should be understood as an equality of quadratic forms although, with an eye to its applications (see Sect. 8), we have suggestively stated it as if $\varphi_k^\#$ were a pure tensor. In general, even in case $\varphi$ is a tensor product of three families, it may happen that $\varphi_k^\#$ belongs to a sum of irreducible automorphic representations: then the scalar factor relating the two sides of (72) depend of these irreducible components via the above $L$-values. Let us also remark that, in the applications, one usually start with a tensor product of three eigenfamilies $\varphi$ and then take linear combinations of them: in this case $\varphi_k^\#$ belongs to a single irreducible representation $\prod (\varphi_k^\#)$ for every balanced integer point.

Let us now discuss the value of $L_{p_k}^\alpha (\varphi)$ at some balanced integer point $k \in U$ where one, two or three of the Galois representations attached to $\varphi_{i,k_i}$ are semistable: more precisely, we suppose that, in this case, the $p$-new form $\varphi_{i,k_i}$ has even weight and trivial central character, thus forcing the corresponding $\omega_{l,i}$ of the family $\varphi_i$ to be 1. The proof is essentially the same of Theorem 7.3: one replaces 6.5 by either 6.7, 6.8 or 6.9.

**Proposition 7.5** Suppose that $k \in U$ is a balanced integer point such that (i) $\varphi_{3,k_3}$, resp. (ii) $\varphi_{2,k_2}$ and $\varphi_{3,k_3}$ or resp. (iii) $\varphi_{1,k_1}, \varphi_{2,k_2}$ and $\varphi_{3,k_3}$ are $p$-new with even weight and trivial central character. Then (72) holds with the following modified Euler factor $E$ replacing $E_p(\alpha, k)^2$, where we write $\xi_{k_i} := (p + 1) \omega_{l,i}(\mathbf{p})^{-1}p^{k_i}$:

\[ (i) \quad E = \xi_{k_1} \frac{E_p^1(\alpha, k)E_p^2(\alpha, k)}{E_p^1(\alpha, k)E_p^2(\alpha, k)} = \xi_{k_2} \frac{E_p^2(\alpha, k)E_p^3(\alpha, k)}{E_p^2(\alpha, k)E_p^3(\alpha, k)}; \]

\[ (ii) \quad E = \frac{1}{p} \xi_{k_1} \frac{E_p^1(\alpha, k)^2}{E_p^1(\alpha, k)^2} = \frac{1}{p} (p + 1)^2 E_p(\alpha, k)^2 = \frac{1}{p} (p + 1)^2 E_p(\alpha, k)^2; \]

\[ (iii) \quad E = \frac{2}{p} \left( 1 + \frac{1}{p^2} \right) \left( 1 + w_{1,p}w_{2,p}w_{3,p} \right)^2. \]

**Proof** Case (i). Applying Corollary 5.6 and Proposition 6.7 gives (71) with $t_k(\varphi_k^\#)$ replaced by $t_k(\varphi_1^{\#(p)}, \varphi_2^{\#}, \varphi_3^{\#})$ (resp.$t_k(\varphi_1^{\#(p)}, \varphi_2^{\#(p)}, \varphi_3^{\#})$) and $E_p(\alpha, k)$ replaced by the modified Euler factor $E' := \frac{p^{k_2-k_3}}{a_2^i} \frac{2}{a_1^i} \frac{E_p(\alpha, k)E_p(\alpha, k)}{E_p(\alpha, k)E_p(\alpha, k)}$. Then, $p^{(k_2-k_3)/2} E_p(\alpha, k) E_p(\alpha, k)$ or $\frac{p^{k_1-k_2}}{a_1^i} \frac{2}{a_2^i} \frac{E_p(\alpha, k)E_p(\alpha, k)}{E_p(\alpha, k)E_p(\alpha, k)}$, which can be checked to agree (resp. $\frac{p^{k_1-k_2}}{a_1^i} w_{p,2} \omega_{l,1}(\mathbf{p})^{-1} \frac{2}{a_2^i} \frac{E_p(\alpha, k)E_p(\alpha, k)}{E_p(\alpha, k)E_p(\alpha, k)}$, which again can be checked to agree). Next, as above one applies Theorem 3.4 now with $\varphi_k^\#$ replaced by $\varphi_k^{\#(p)} := \varphi_1^{\#(p)} \otimes \varphi_2^{\#(p)} \otimes \varphi_3^{\#}$ (resp. $\varphi_k^{\#(p)} := \varphi_1^{\#(p)} \otimes \varphi_2^{\#(p)} \otimes \varphi_3^{\#}$) and again with $\varphi_k^{\#}$ and $\varphi_k^{\#(p)}$, which fixes $\left( \varphi_k^{\#}, \varphi_k^{\#(p)} \right)$. 

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and the local constants \(C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p))\). On the other hand, with the notations introduced in the proof of Theorem 3.4, set \(\psi_k^{\#(p)} := f_{\Lambda_k/E}(\varphi_k^{\#(p)})\), \(\psi_k^{\#(p)} := f_{\Lambda_k/E}(\varphi_k^{\#(p)})\) and \(\psi_k^{\#(p),\varphi} := (\psi_k^{\#(p)})\) where \(\varphi_k^{\#(p)} := \varphi_k^{\#(p)} \otimes \varphi_k^{\#(p)} \otimes \varphi_k^{\#(p)}\) (resp. \(\varphi_k^{\#(p)} := \varphi_k^{\#(p)} \otimes \varphi_k^{\#(p)} \otimes \varphi_k^{\#(p)}\) and the same for \(\varphi_k^{\#(p)}\)). Similarly as in the global calculation (63), one checks that

\[
\left(\frac{\psi_k^{\#(p)} \cdot \varphi_k^{\#(p)}}{\omega_{f,1}(p)} p^{-k_1} \frac{(\psi_k^{\#(p)} \cdot \varphi_k^{\#(p)})}{\omega_{f,2}(p) p^{k_2}}\right)
\]

and, consequently,

\[
C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) = \omega_{f,1}(p)^{p^{-k_1}} C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) \left(\omega_{f,2}(p) p^{k_2}\right).\]

Because the local components of \(\varphi_k^{\#(p)}\) and \(\varphi_k^{\#(p)}\) at \(p\) equals the local component at \(p\) of \(\varphi_k^{\#(p)}\), the local components at \(p\) of \(\varphi_k^{\#(p)}\) and \(\varphi_k^{\#(p)}\) equals the local component at \(p\) of \(\varphi_k^{\#(p)}\) and \([43, \text{Corollary 4.2}]\) gives

\[
C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) = \frac{1}{p} \left(1 + \frac{1}{p}\right)^{-1} = \frac{1}{p+1}\]

(see Sect. 8.2 below for more details). Hence

\[
C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) = \frac{\omega_{f,1}(p)^{p^{-k_1}}}{p+1}
\]

and we see that \(C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) E^{2} = E\), as claimed. The proof of the cases (ii) and (iii) is similar, noticing that we already have everything expressed in term of \(\varphi_k^{\#}\) and \(C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) = p^{-1}\) (resp. \(C_p^{\varphi_k^{\#} \cdot \varphi_k^{\#}}(\varphi_k^{\#}(p)) = 2p^{-1}(1 + p^{-1})\) in case (ii) (resp. (iii)) thanks to \([43, \text{Proposition 4.3}]\) (resp. \([43, \text{Proposition 4.4}]\)).

**Remark 7.6** It follows from Deligne’s proof of the generalized Ramanujan conjecture that, in the setting of the above Proposition 7.5, we may have the vanishing of the Euler factor \(E\) only in case (i) or (iii). In the first case, we have indeed \(E_{p,3}(\alpha, k) \neq 0\) and \(E_{p,1}(\alpha, k) = E_{p,2}(\alpha, k) = 0\) if and only if the equivalence

\[
\frac{\alpha_1}{\alpha_2} = -w_{p,3} p^{(k_2 - k_1)/2} \omega_{f,1}(p)^{-1} \iff \frac{\alpha_2}{\alpha_1} = -w_{p,3} p^{(k_2 - k_1)/2} \omega_{f,2}(p)^{-1}
\]

is satisfied (recall \(\omega_{f,1}(p) \neq \omega_{f,2}(p)\)). In the second case, we have \(E = 0\) if and only if \(w_{1,1} w_{2,2} w_{3,3} = -1\) and, in this case, we see from Proposition 6.9 that there is an extra vanishing due to the complex \(L\)-function.

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Finally, let us discuss improved $p$-adic $L$-functions. Suppose that $c \in \mathbb{N}$ and consider the plane

$$H^c_i := \{ \kappa \in U : \kappa_i^* = c \} \subset U.$$ 

Let us remark that the Euler factor $E_{p,i} (\alpha, k)$ extends to a rigid analytic function on $H^c_i$. Suppose that $k'$ is such that $k'^* = c$; geometrically, this means that $k' : \mathbb{Z}^3 \rightarrow \mathcal{O}_k$ factors through the morphism $\mathcal{O}_k = \mathcal{O} (U) \rightarrow \mathcal{O} (H^c_i)$ which corresponds to $H^c_i \subset U$. Then we can consider the $\mathcal{O}_k'$-valued $\mathcal{O}_k'$-linear functional $L_{p,1}^{\alpha,c} : M_p^\alpha \rightarrow \mathcal{O}_k'$ defined via the formula

$$L_{p,1}^{\alpha,c} (\varphi_1 \otimes \varphi_2 \otimes \varphi_3) := \frac{p + 1}{\alpha_1} t_{1,k} \left( \varphi_1 \mid W_p \otimes \varphi_2 \otimes \varphi_3 \right),$$

and $L_{p,2}^{\alpha,c}$ and $L_{p,3}^{\alpha,c}$ are defined in a similar way when $k_2'^* = c$ or, respectively, $k_3'^* = c$ (see (44) for the definition of $t_{1,k}$). Taking $k'$ to be the morphism which corresponds to $H^c_i \subset U$ and applying Corollary 5.6 yields the following result.

**Proposition 7.7** The above $\mathcal{O} (H^c_i)$-linear functional $L_{p,i}^{\alpha,c}$ is uniquely characterized by the property that, for every $\varphi \in M_p^\alpha$,

$$L_{p}^{\alpha} (\varphi) |_{H^c_i} = E_{p,i} (\alpha, -) L_{p,i}^{\alpha,c} (\varphi)$$

as rigid analytic functions on $H^c_i$.

**7.2 Variants**

Let us explain how one can rewrite the term $\frac{\Omega (\varphi_1^\#)}{L (1, \Pi (\varphi_2^\#), \text{Ad})}$ that appears in the interpolation formula (72) when $\Omega (\varphi_1^\#) \neq 0$. Suppose that $f \in S_k (N, \varepsilon)$ is a normalized newform with nebetype $\varepsilon$ having conductor $N_k$ and write $\pi (f) = \otimes_v \pi_v (f)$ for the corresponding automorphic representation: we recall that a formula of Shimura and Hida relates $L (\text{ad} (\pi (f)), 1)$ and the Petersson inner product $(f, f)_k$ that we normalized as in (1). Let us define $L (\text{ad} (\pi (f)), s) = \prod_v L_v (\text{ad} (\pi (f)), s)$ and $L^H (\text{ad} (f), s) = \prod_v L_v^H (\text{ad} (f), s)$, where

$$L_\infty (\text{ad} (\pi (f)), s) = 2 (2\pi)^{-s+k-1} \Gamma (s + k - 1) \pi^{-s+1/2} \Gamma \left( \frac{s + 1}{2} \right)$$

and the Euler factors are defined as follows:

$$L_1 (\text{ad} (\pi), s)^{-1} = \left\{ \begin{array}{ll}
(1 - l^{-s}) (1 - \chi_1 \chi_2 (l) l^{-s}) (1 - \chi_1 \chi_2 (l) l^{-s}) & \text{if } \pi_l = \pi (\chi_1, \chi_2) \text{ is principal} \\
1 - l^{-1-s}, & \text{if } \pi_l \text{ is special,} \\
1 + l^{-s}, & \text{if } \pi_l \text{ is supcercuspidal and } \pi_l \simeq \pi_l \otimes \eta_l, \\
1, & \text{if } \pi_l \text{ is supcercuspidal and } \pi_l \not\simeq \pi_l \otimes \eta_l.
\end{array} \right.$$
\[
L^H_I (\text{ad} (f), s)^{-1} = \begin{cases} 
(1 - l^{k-1-s}) (1 - \varepsilon (q) - \alpha^2 l^{k-1-s}) (1 - \varepsilon (q) - 2 l^{k-1-s}) & \text{if } l \nmid N, \\
(1 - l^{-1-s}) (1 + l^{-s}) & \text{if } l \parallel N \text{ and } l \nmid N_s, \\
1 - l^{-1-s} & \text{if } l \mid N \text{ and } l \parallel N/N_s, \\
1 + l^{-s} & \text{otherwise}; 
\end{cases}
\]

Then (see [26, Theorem 5.1] and [14, Theorem 2.2.3 and Corollary 2.2.4] for the notations employed):

\[
L (\text{ad} (\pi (f)), 1) = \frac{2k\pi}{3} (f, f_k) \prod_{l \mid N} L^I_L (\text{ad} (\pi (f)), 1).
\]

Let us now go back to our family \( \varphi \in M^p_p \) specializing to \( \varphi_k = \varphi^\# (\varphi_k) \otimes \varphi^\# (\varphi_k) \) and write the \( \text{GL}_2 \) representation \( \pi^\prime (\varphi_k) \) attached to \( \varphi_k \) as \( \pi^\prime (\varphi_k) = \pi (f^\#_k) \) where \( f^\#_k \in S_k (N_i, \varepsilon_i) \) is a normalized newform (in particular, \( \varepsilon_i (z) = \omega^{-1} (z) \) viewing \( z \in \mathbb{Z} \) diagonally embedded in \( \mathbb{A}_k \)) and set \( f^\#_k := (f^\#_1, f^\#_2, f^\#_3) \) and \( \Omega (f^\#_k) := (f^\#_1, f^\#_1, f^\#_2, f^\#_2, f^\#_3, f^\#_3) \). Then we have \( L_v (s, \pi^\prime (\varphi^\#_k), \text{Ad}) = \prod_{i=1, 2, 3} L^H_I (\text{ad} (\pi (f^\#_k)), s) \) and we define \( L^H_I (s, f^\#_k, \text{Ad}) := \prod_{i=1, 2, 3} L^H_I (\text{ad} (f^\#_k), s) \) and \( M := \text{lcm}(N_1, N_2, N_3) \). The following result is a direct consequence of Lemmas 5.7 and 6.6.

**Proposition 7.8** Suppose that \( \mathbf{k} : \mathbb{Z}_k^\times \to \mathcal{O} \) corresponds to an open affinoid subdomain and that \( \varphi^b, \varphi^{bb} \in M^p_p (D_k (W), \omega^k_{0, p})^a \) for the same \( s \in \mathcal{O}^\times \). Then \( \frac{p+1}{\alpha} (\varphi^b, \varphi^{bb}) \in \mathcal{O} \) is the unique rigid analytic function such that, for every generic integer \( k \in \mathbb{N} \cap U \) (for \( \varphi^b \) and \( \varphi^{bb} \)),

\[
\left( \frac{p+1}{\alpha} (\varphi^b, \varphi^{bb}) (k) = \left( 1 - \omega (p)^{-1} \alpha^{-2} p^k \right) \left( 1 - \omega (p)^{-1} \alpha^{-2} p^{k+1} \right) \left( \varphi^b_k, \varphi^{bb}_k \right)_k,
\]

where \( \varphi^b_k \) (resp. \( \varphi^{bb}_k \)) is the unique vector in \( M^\mathcal{O} (V_k, F, \omega^k_{0, p})^\text{GL}_2 (\mathbb{Z}_p) \) such that \( \varphi^b_k = \varphi^b_k \) (resp. \( \varphi^{bb}_k \)) is the specialization of \( \varphi^b \) (resp. \( \varphi^{bb} \)) at \( k \).

In particular, if \( \varphi^b_i, \varphi^{bb}_i \in M^\mathcal{O}_p (D_k (W), \omega^k_{0, p})^a, \varphi^b := \varphi^b \otimes \varphi^{bb} \in M^\mathcal{O}_p, \varphi^{bb} := \varphi^{bb} \otimes \varphi^{bb} \otimes \varphi^b \in M^\mathcal{O}_p \), we can define

\[
\left( \varphi^b, \varphi^{bb} \right)_p := (p + 1) \left( \frac{\varphi^b_1}{\alpha_1}, \varphi^{bb}_1 \right) \otimes \left( \frac{\varphi^b_2}{\alpha_2}, \varphi^{bb}_2 \right) \otimes \left( \varphi^b_3, \varphi^{bb}_3 \right) \in \mathcal{O}_k,
\]

which satisfies, at every generic integer point \( k \in \mathbb{N}^3 \cap U \) (for \( \varphi^b \) and \( \varphi^{bb} \)), the interpolation property

\[
\left( \varphi^b, \varphi^{bb} \right)_p (k) = \mathcal{E}^\mathcal{O} (\alpha, k) \left( \varphi^b_k, \varphi^{bb}_k \right)_k.
\]
where
\[
E_{\Omega}^\alpha (\gamma, k) := \prod_{i=1,2,3} \left( 1 - \omega_{\gamma,i} (p')^{-1} \alpha_i^{-2} p^{ki} \right) \left( 1 - \omega_{\gamma,i} (p')^{-1} \alpha_i^{-2} p^{ki+1} \right)
\]
and \( \varphi^b := \varphi^b_k \otimes \varphi^b_k \otimes \varphi^b_k \) (resp. \( \varphi^{b\#} := \varphi^{b\#}_k \otimes \varphi^{b\#}_k \otimes \varphi^{b\#}_k \)) if \( \varphi_k \) (resp. \( \varphi^{b\#}_k \)) is the unique vector in \( M^3 (V_{k_1, F}, \omega^0_k) \) \( GL_2 (\mathbb{Z}_p) \) such that \( \varphi^{b\#}_k(\alpha) = \varphi_k^{b\#}(\alpha) = \varphi_k^{b\#} \) and \( \varphi_k^b \) (resp. \( \varphi_k^{b\#} \)) is the specialization of \( \varphi_i^b \) (resp. \( \varphi_i^{b\#} \)) at \( k \). We may choose \( \varphi_i^b \) and \( \varphi_i^{b\#} \) in such a way that, for every generic integer point \( k \in \mathbb{N}^3 \subseteq \mathcal{U} \) for both \( \varphi^b \) and \( \varphi^{b\#} \), we have \( \varphi_i^b \otimes \varphi_i^{b\#} \neq 0 \) and so that the local components at \( p \) of the specialization of \( \varphi_i^b \) equals the local components at \( p \) of the specializations of \( \varphi_i^b \); indeed, for example, we may take \( \varphi_i^b \) a newvector of tame level \( N_i \) and \( \varphi_i^{b\#} := \varphi_i^b \mid W_{N_i} \) [see (21)].

Then, thanks to (73) and Proposition 7.8, we find for every generic integer point \( k \in \mathbb{N}^3 \subseteq \mathcal{U} \) for both \( \varphi^b \) and \( \varphi^{b\#} \) (the notation \( \Omega (\forall E) \) means that the equality holds when \( E_{\Omega}^\alpha (\gamma, k) \neq 0 \), see Remark 1.2):

\[
\frac{(\varphi_k^b \cdot \varphi_k^{b\#})_k}{L \left( 1, \frac{\Omega (\forall E)}{\Gamma \left( \frac{\Omega (\forall E)}{\Gamma} \right)} \right)} = \frac{27}{4 \pi^3} \left( \varphi_k^b \cdot \varphi_k^{b\#} \right)_k \prod_{l \mid M} \frac{L_l^H (s, \pi'_l, \text{Ad})}{L_l (s, \frac{\Omega (\forall E)}{\Gamma} \mid W_{N_i} \mid, \text{Ad})}.
\]

8 An explicit example

Recall the cuspidal finite slope \( h_i \) Coleman eigenfamilies \( f_i \) of tame level \( N_i \), trivial nebentype \( e_i = 1 \) and \( U_p \)-eigenvalue \( \omega_i \in \mathcal{O} \) defined in the connected affinoid subdomain \( U_i \) of the weight space \( \mathcal{X} \) that was considered in the introduction (we are going to relax the assumption on the level that we did there, which is no longer in force).

Remark 8.1 Let us fix a Dirichlet character \( \varepsilon : \left( \frac{\mathbb{Z}}{N \mathbb{Z}} \right) \rightarrow \mathbb{C} \) of level prime to \( p \). It follows from the work of Coleman that prime to \( p \) newforms vary in families: there is a sheaf of families of overconvergent modular forms \( f \) of finite slope on the weight space whose specializations at \( k \) belongs to \( S_{k+2} (\Gamma_0 (p N), \varepsilon)^{N-\text{new}} \) for almost every \( k \)'s (see [11, discussion after Corollary B5.7.1]). Working over connected affinoids \( U \subseteq \mathcal{X} \) there are plenty of sections of this sheaf. More precisely, let us fix a real number \( h \geq 0 \) and let us assume that \( f \in S_{k+2} (\Gamma_0 (p N), \varepsilon)^{N-\text{new}} \) is such that \( k_0 > h + 1 \) and has slope \( h \). Under a mild further assumption, it is shown in [11, Corollary B5.7.1] the existence of a family \( f \) such that \( \mathbf{f}_k = f \) and, for every integer \( k > h + 1 \) (an integer points in our notations), \( \mathbf{f}_k \in S_{k+2} (\Gamma_0 (p N), \varepsilon)^{N-\text{new}} \) and
has slope $h$. Let us also remark that, if $\mathbf{f}$ is a finite slope overconvergent modular forms as above defined on a connected affinoid, then its slope is constant, say $h$, and $\mathbf{f}_k \in S_{k+2}(\Gamma_0(pN), \varepsilon)^{N-\text{new}}$ for every integer $k > h + 1$, as it follows from the Coleman’s classicality result.

One can give the following geometric interpretation of this result. As explained in [20, §12.1 and Corollary 10.7], the work on Ash-Stevens on slope decompositions combined with standard techniques due to Coleman-Mazur and Buzzard yields the construction of a curve $w : C_{N,\varepsilon}^{<h} \to \mathcal{X}$ parametrizing cuspidal eigenforms of level $\Gamma_0(pN)$ and nebetype $\varepsilon$ which are $N$-new and of slope $\leq h$. The Coleman-Mazur eigencurve $w : C_{N,\varepsilon} \to \mathcal{X}$ parametrizing these objects with the relaxed finite slope condition admits $C_{N,\varepsilon}^{<h}$ as a closed curve and it is the union of them. The families $\mathbf{f}$ obtained from [11, Corollary B5.7.1] corresponds to sections $s : U \to C_{N,\varepsilon}^{<h} \subset C_{N,\varepsilon}$ of $w$ (see [20, Corollary 10.7 (iii)]). Moreover, if $s : U \to C_{N,\varepsilon}$ is a section of $w : C_{N,\varepsilon} \to \mathcal{X}$, because $U$ is a connected affinoid, $s(U) \subset C_{N,\varepsilon}$ for some $h$.

Recall that, for every generic integer point $k$, the specialization $\mathbf{f}_{i,k} = f_{i,k}$ is the $p$-stabilization of a level $N_i$ newform $f_{i,k}^\# = f_{i,k}^\#$ and we let $\pi_{i,k} = \bigotimes_v \pi_{i,k,v}$ be the associated automorphic representation. We write

$$D := gcd(N_1, N_2, N_3) \text{ (resp. } M := lcm(N_1, N_2, N_3))$$

for the greatest common multiple (resp. the least common multiple).

**Lemma 8.2** Suppose that $\mathbf{f}$ is a finite slope cuspidal $p$-adic Coleman eigenfamily new of tame level $N$ and nebetype $\varepsilon$ defined on some open affinoid subdomain $U$ as in Remark 8.1 above. Let us write $\pi_k = \bigotimes_v \pi_{k,v}$ for the automorphic representation attached to an integer point $k \in U$. If $\pi_{k_0,v}$ is a principal series, special or supercuspidal representation of conductor $c_{k_0,v}$ at an integer point $k_0 \in U$ and $v \neq p, \infty$, then $\pi_{k,v}$ has the same property for every integer point $k \in U$.

**Proof** According to the work of Coleman-Mazur [13] (see also [2, Theorem 5.1]), there is a pseudocharacter $T : G_\mathbb{Q} \to \mathcal{O}_{C_{N,\varepsilon}}$ such that the specialization $T_l$ of $T$ at a classical point $y \in C_{N,\varepsilon}$ is (the pseudocharacter of) the representation $\rho_y$ attached to the eigenform $y$, characterized by the fact that the trace of the geometric Frobenius at $l$ is the eigenvalue of $T_l$ acting on $y$ for almost every $l$. Let us write $s^\# : \mathcal{O}_{C_{N,\varepsilon}} \to \mathcal{O} := \mathcal{O}_\mathcal{X}(U)$ for the morphism induced by $s : U \to C_{N,\varepsilon}$, where $s$ corresponds to $\mathbf{f}$, as explained in Remark 8.1. Then $T_y := s^\# \circ T$ has the property that its specialization at an integer point $k \in U$ is (the pseudocharacter of) the representation $\rho_k := \rho_{k_0}$ attached to the eigenform $\mathbf{f}_k$. Because $\mathcal{O}$ is a $PID$, it follows from [13, Theorem 5.1.2 and Remark after it] that $T = \rho$ is indeed (the pseudocharacter of) a representation $\rho : G_\mathbb{Q} \to GL_2(\mathcal{O})$ which interpolates the representations $\rho_k$’s. We also refer the reader to [2, §6.2] for an alternative construction of this representation.

According to [5, Lemma 7.8.14], writing $W_v$ for the Weil-Deligne group of $\mathbb{Q}$ at $v$, we have that $\rho_{W_v}$ is monodromic in the sense of [36, Definition 2.2]. This means

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7 Via the morphism $\mathcal{O}_{C_{N,\varepsilon}} \to \mathcal{O}_{C_{N,\varepsilon}^{<h}}$ which sends a Hecke operator acting on finite slope overconvergent modular forms to the Hecke operator acting on overconvergent modular forms of slope $\leq h$.  

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that we can attach to it a Weil-Deligne representation $WD_v(\rho)$ with coefficients in $O$ via [5, Definition 7.8.13]: we remark that, writing $WD_v(\rho)_k$ for its specialization at $k$, we have $WD_v(\rho)_k = WD_v(\rho_k)$ by the uniqueness assertion of [5, Lemma 7.8.12] characterizing the monodromy and the definition of $r$ in loc.cit. Let us remark that the automorphic type of $\pi_{k,v}$ and its conductor are encoded in the Frobenius semisimplification $WD_v(\rho_k)^{Fr-ss}$ of $WD_v(\rho_k)$ thanks to the local-global compatibility, which is well known in this case and widely covered by the Hilbert case handled in [8]. We can now apply [36, Theorem 3.1 (1) and (4)] and [37, Theorem 3.1] to deduce that the automorphic type of $\pi_{k,v}$ and its conductor are constant if $WD_v(\rho_k)^{Fr-ss}$ is pure, i.e. it satisfies the monodromy conjecture (see [36, Definition 2.10]). Since the conjecture is well known in our setting and, again, widely covered by [8], the lemma is proved.

Let us also remark that the assertion about the conductor also follows from the fact that $f_k$ is new outside $p$ and the constancy of automorphic types can also be proved along the lines of [19, Lemma 2.14]. Indeed, the proof of [19, Lemma 2.14] needs as an input a representation interpolating the $\rho_k$’s as the $\rho$ above and then the arguments of loc.cit. apply. More precisely, case (a) of loc.cit. does not need any further explanation, while case (b) of loc.cit. take advantage of a base change argument and, hence, needs a theory of Coleman families attached to Hilbert modular forms, which has been developed in [2, Theorem 5.1], and also a base change theorem for Coleman families. This latter ingredient can be proved as in the case of Hida families, but we have not been able to provide a reference in our broader setting. Alternatively, we remark that case (b) of loc.cit. also follows from [18, Lemma 2.6.2] without the need to make a base change, but again this result is formulated in the setting of Hida families. Let us explain another approach based on [36, Theorem 3.1 (2)] which avoids the base change argument and applies to Coleman families. The main point of (b) is proving that, if $\pi_{k_0,v}$ is special, then the same is true for $\pi_{k,v}$. By the local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, this means that if the monodromy of $WD_v(\rho_{k_0})^{Fr-ss}$ is non-trivial, then the same is true for $WD_v(\rho_k)^{Fr-ss}$: but this follows from [36, Theorem 3.1 (2)]. More precisely, writing $\overline{\mathbb{Q}}$ for an algebraic closure of the fraction field of $O$, let us write $t$ (resp. $t_k$) for the smallest integer $s$ such that $N_s = 0$ on $(\overline{\mathbb{Q}} \otimes O WD_v(\rho))^{Fr-ss}$ (resp. $WD_v(\rho_k)^{Fr-ss}$). Then a priori $t_k \leq t$ and [36, Theorem 3.1 (2)] gives $t_k = t$ whenever $WD_v(\rho_k)^{Fr-ss}$ is pure, from which the required $t_k = t = t_{k_0} = 2$ follows. We note that [36, Theorem 3.1 (2)] is already proved in the first paragraph of pag. 890 of loc.cit. taking into account that the integer $t_k$ can be characterized by the fact that $2(t_k - 1)$ equals the difference between the larger and the smaller weight of $WD_v(\rho_k)^{Fr-ss}$ when $WD_v(\rho_k)^{Fr-ss}$ is pure (or when it is indecomposable).

According to Lemma 8.2, the conductor of $\pi_{t_i,k,1}$ at a finite prime $l \neq p$ is a well defined quantity which does not depend on the choice of the point $k \in \mathbb{N} \cap U_i$: we denote it by $c_l(f_i)$ and let $c_i := \max_{i=1,2,3} \{ c_l(f_i) \}$. We recall that, if $l$ is a finite prime, an irreducible admissible representation of $GL_2(\mathbb{Q}_l)$ admits a Jacquet–Langlands lift if and only if it is special or supercuspidal. Hence, in view of Lemma 8.2, it makes sense to consider
$S_{JL} := \{ l \neq p \text{ finite primes} : \pi_{i,k,l} \text{ admits a JL-lift, } \forall k \in \mathbb{N} \cap U_i \text{ and } i = 1, 2, 3 \}$

and $D_{JL} := \prod_{l \in S_{JL}} l^{c_l} | D$.

Next, we suppose that $D_{JL}$ is squarefree and define

$$S_{JL}^{-} := \{ l \mid D_{JL} : -a_l (f_{1,k_1}) a_l (f_{2,k_2}) a_l (f_{3,k_3}) l^{-\frac{k_1+k_2+k_3}{2}} = -1 \}$$

and

$$D_{JL}^{-} := \prod_{l \in S_{JL}^{-}} l,$$

where we remark that $D_{JL}^{-}$ is indeed independent of $k$: because $l \neq p$, the function of $k$ that appears in the definition of $D_{JL}^{-}$ is rigid analytic and, being $\{\pm 1\}$-valued on the Zariski dense subset of integer points of the connected affinoid subdomain $U_1 \times U_2 \times U_3$, is indeed constant. Because we assume that $D_{JL}$ is squarefree, we have the equality

$$\varepsilon_l (f_{k}) = -a_l (f_{1,k_1}) a_l (f_{2,k_2}) a_l (f_{3,k_3}) l^{-\frac{k_1+k_2+k_3}{2}},$$

and we see that there is a well defined finite “generic sign” $\varepsilon_{\text{fin}} (f)$. Next, we make the following further assumptions that will be in force until the end of this section.

(Bal) $\varepsilon_{\text{fin}} (f) = -1$, i.e. $D_{JL}^{-}$ is the product of an odd number of primes; (Con)

For every $l \mid M$, we have $c_l (f_{i_l}) \leq 1$ for $i = 1, 2, 3$ or there is an index $i_l$ such that $c_l (f_{i_l}) \geq 2 \max_{i \neq i_l} \{ c_l (f_{i}) \}$.

Let us write $\alpha_i$ for the $U_p$-eigenvalues attached to $f_i$, so that $\alpha_i \in \mathcal{O}$. Write $B$ for the definite quaternion algebra of discriminant $D_{JL}^{-}$: then $f_i$ lifts to an element of $M_p^\diamond (\mathcal{D}_{k_i} (W), \omega_{0,p}^{k_i})^{\alpha_i}$ (by the $p$-adic Jacquet–Langlands correspondence, see [10]) and it follows from (Bal) that $B$ is the quaternion algebra predicted by [34] at every integer and balanced point $k$.

Under these assumptions, we can prove the following result.

**Theorem 8.3** There exists

$$\varphi \in M_p^\diamond (\mathcal{D}_{k_1} (W), \omega_{0,p}^{k_1})^{\alpha_1} \otimes M_p^\diamond (\mathcal{D}_{k_2} (W), \omega_{0,p}^{k_2})^{\alpha_2} \otimes M_p^\diamond (\mathcal{D}_{k_3} (W), \omega_{0,p}^{k_3})^{\alpha_3}$$

such that, for every balanced generic integer point $k \in U$ for $\varphi$,

$$L_p^\alpha (\varphi) (k)^2 = E_p (\alpha, k)^2 \frac{\left( \varphi_{\kappa}^\#, \varphi_{\kappa}^\# \right)_k}{2L (1, \Pi_{\kappa}, \text{Ad})} L (1/2, \Pi_{\kappa}) \prod_{l \mid M} C_l,$$

where the constants $C_l$ are defined as follows.

- If $l \mid D_{JL}$, then $C_l = 2 \frac{1}{l} (1 - \frac{1}{l})$;
If \( l \mid M / D_{JL} \) and \( c_l(\mathbf{f}_i) \leq 1 \) for \( i = 1, 2, 3 \), then
\[
C_l = \begin{cases} 
\frac{1}{L} \frac{1}{1 + \frac{1}{L}}, & \text{if one of the representations is special unramified}, \\
\frac{1}{L}, & \text{if two of the representations are special unramified}, \\
\frac{2}{\ell} \left(1 + \frac{1}{\ell} \right), & \text{if three of the representations are special unramified}; 
\end{cases}
\]

If \( l \mid M / D_{JL} \) and \( c := c_l(\mathbf{f}_i) \geq 2 \max_{i \neq i_l} \{ c_l(\mathbf{f}_i), 1 \} \), then
\[
C'_l, \text{ where } C'_l = \frac{\prod_{i \neq i_l} (1 - A(\pi_i))}{\prod_{i \neq i_l} (1 + \frac{1}{\ell})} \text{ and } A(\pi_i)
\]

\[
: \begin{cases} 
\frac{1}{\ell}, & \text{if } \pi_i \text{ is supercuspidal or } \pi_i = \pi(\chi_1, \chi_2) \text{ with } \chi_k \text{ ramified for } k = 1, 2, \\
\ell, & \text{if } \pi_i = \pi(\chi_1, \chi_2) \text{ is principal unramified,} \\
1, & \text{if } \pi_i \text{ is special unramified,} \\
0, & \text{if } \pi_i = \pi(\chi_1, \chi_2) \text{ with one } \chi_k \text{ ramified and the other unramified.} 
\end{cases}
\]

**Remark 8.4** The contribution of the local \( L \)-factors in the third case depends on the nature of the representations. For example, we have \( C_l = C'_l \) when the two representations having the smaller conductor are unramified.

### 8.1 The \( p \)-adic Jacquet–Langlands correspondence and the choice of the test vector

By Chenevier’s \( p \)-adic Jacquet–Langlands correspondence (see [10]), the eigenvector \( \mathbf{f}_i \) corresponds to an eigenvector \( \varphi_{\mathbf{f}_i} \) in \( M_{p}^\omega(\mathcal{D}_k(W), \omega_{0,p}^{k_i}) \) with the property that its specializations are new at all the primes \( l \), the only possible exception being at \( l = p \).

If \( l \nmid D_{JL} \) is a finite prime, we write \( \widehat{\delta}_l \) for the element of \( \text{GL}_2(\hat{\mathbb{A}}_f) \) defined by the following local conditions:

\[
\widehat{\delta}_l = \begin{cases} 
1 & \text{if } v \neq l, \\
\delta_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v = l.
\end{cases}
\]

Set \( \varphi_{\mathbf{f}_j} := \varphi_{\mathbf{f}_1} \otimes \varphi_{\mathbf{f}_2} \otimes \varphi_{\mathbf{f}_3} \) and define \( \varphi \) in \( M_{p}^\omega(\mathcal{D}_k(W), \omega_{0,p}^{k_i}) \) as follows. For every finite prime \( l \), let \( i_l \) be the index which realizes the largest conductor \( c_l(\mathbf{f}_i) \) and let \( j_l \neq i_l \) be the index which realizes the second largest conductor \( c_l(\mathbf{f}_j) \). Suppose we have, for example, \( i_l = 3 \) and \( j_l = 2 \); then we define \( \delta_l := 1 \otimes \delta_{j_l}(t_2) \otimes (t_3) \otimes 1 \). Finally, we set \( \delta := \prod_{l \nmid M} \delta_l \) and define \( \varphi := \varphi_{\mathbf{f}_j} \delta_M \). Let us remark that \( \varphi \) is designed so that its specialization at an arithmetic point \( k \) is a pure tensor whose component at the finite primes different from \( p \) are either a tensor product of new vectors when \( l \nmid M \) or, assuming as above that \( i_l = 3 \) and \( j_l = 2 \) and taking into account the twist (15):

\[
\psi_l = l^{k_2/2} \psi_{1,l} \otimes \psi_{2,l} \delta_{j_l}(t_2) \otimes \psi_{3,l} =: l^{k_2/2} \psi_{1,l}^0,
\]

\(8\) By a special unramified representation, we mean the twist by an unramified character of the special representation. Of course, this is a ramified representation of conductor 1.
where \( \psi_{i,1} \) is a new vector in \( \pi_{i,k_1} \). In particular, the same property is enjoyed by \( \varphi_{\#} \), whose local component at the primes \( \ell \neq p \) agrees with the local component of \( \varphi \).

### 8.2 Proof of Theorem 8.3

We have to make explicit the local constants \( C_{\psi_v, \tilde{\psi}_v} :\hat{\psi}_v(\psi_v) := \frac{I_v(\psi_v \otimes \tilde{\psi}_v)}{(\psi_v, \tilde{\psi}_v)} \) associated to the local components of \( \varphi_{\#} \) whose \( p \)-stabilization \( \varphi_{\#} \) is the specialization of \( \varphi \) at the finite primes that appear in Theorem 7.3 where we choose \( \varphi_{\#} = \varphi_{\#} = \varphi_{\#} \) (at the same time we will see that \( (\varphi_{\#}, \varphi_{\#})_k = (\varphi_{\#}, \varphi_{\#})_k \neq 0 \)) from the local calculation and (22)). As noticed in Remark 3.5, \( C_{\psi_v, \psi_v}(\psi_v, \psi_v) \) does not depend on the non-zero vector in the lines spanned by either \( \psi_v \) or \( \psi_v \): thanks to our assumption on the central characters, we see that \( \tilde{\psi}_{i,1} = \omega_{i,1}^{-1}(\psi_{i,1}) \) (resp. \( \tilde{\psi}_{i,1} \)) is a new vector (resp. the translate by \( \delta_{\ell} \) of a new vector) as it is \( \tilde{\psi}_{i,v} \) (resp. \( \tilde{\psi}_{i,v} \delta_{\ell} \)); it follows that \( \tilde{\psi}_{i,l}(\psi_{i,l}) \neq 0 \) and, from (75),

that we have \( C_{\psi_v, \tilde{\psi}_v}(\psi_v) = C_{\psi_v, \tilde{\psi}_v}(\psi_v, \tilde{\psi}_v) = C_{\psi_v, \tilde{\psi}_v}(\psi_v, \tilde{\psi}_v) = C_{\psi_v, \tilde{\psi}_v}(\psi_v, \tilde{\psi}_v) \). Hence our claim follows from the following computations of the local constants \( C_{\psi_v, \tilde{\psi}_v}(\psi_v, \tilde{\psi}_v) \), which always uses (75) as a test vector. When \( l \) is prime to \( M \), \( C_{\psi_v}(\psi_v) = 1 \) by [31, Lemma 2.2]. When \( l \mid D_{\ell}JL \) this is done in [43, Proposition 4.5]. When \( l \mid M/D_{\ell}JL \) and \( c_1(\ell_l) \leq 1 \), then \( \pi_{1,k_1} \otimes \pi_{2,k_2} \otimes \pi_{3,k_3} \) is the tensor product of unramified principal series representations with either one, two or three special unramified representations: then we may apply [43, Corollary 4.2], [43, Proposition 4.3] or, respectively, [43, Proposition 4.4]. Finally, the computation of \( \tilde{\psi}_{i,v}(\psi_v) \) in the last case follows from [30, Theorem 4.1], once we remark that the roots \( \alpha_l \) and \( \beta_l \) of the Hecke polynomial of a weight \( k + 2 \) supercuspidal eigenform \( f \) at \( l \) satisfy \( \chi_1(l) = \frac{\alpha_l}{p^{(k+3)/2}} \) and \( \chi_2(l) = \frac{\beta_l}{p^{(k+1)/2}} \), when the associated automorphic representation at \( l \) is principal unramified of the form \( \pi(\chi_1, \chi_2) \), implying that we have

\[
\frac{\chi_1(l)}{\chi_2(l)} \frac{\chi_2(l)}{\chi_1(l)} = \frac{\chi_1^2(l) + \chi_2^2(l)}{\chi_1(l) \chi_2(l)} = \frac{\alpha_l^2 + \beta_l^2}{\alpha_l \beta_l} = \frac{\alpha_l + \beta_l}{\alpha_l \beta_l}
\]

in loc.cit.

### 8.3 Variants

Let us remark that we have \( (\tilde{\psi} g) = \omega(\nd(g)) \tilde{\psi} g = \omega_f(\nd(g)) \tilde{\psi} g \) and then we see from the invariance of the left hand side of (20) (or directly from the definition of \( (\cdot, \cdot)_{\ell} \)) that we have

\[ \text{[43, Proposition 4.4]: the quantity } (1 - \varepsilon) \text{ should be } (1 + \varepsilon), \text{ which is 2 in our case,} \]

in accordance with the Prasad’s results.

\[ \text{[43, Proposition 4.4]: the quantity } (1 - \varepsilon) \text{ should be } (1 + \varepsilon), \text{ which is 2 in our case,} \]

in accordance with the Prasad’s results.
\[(\varphi_1 g, \varphi_2 g)_\kappa = \omega_I(\text{nd}(g))^{-1}(\varphi_1, \varphi_2)_\kappa.\]

Write \(\varphi_\#_\kappa := \varphi_{\#_1, k_1} \otimes \varphi_{\#_2, k_2} \otimes \varphi_{\#_3, k_3}\), where \(\varphi_{\#, k_i}\) is the unique vector whose \(p\)-stabilization \(\varphi_{i, k_i}\) is the specialization of \(\varphi_{\#}\). Then it follows from (75) that we have an equality \(\left( \varphi_{\#_1, k}, \varphi_{\#_2, k} \right)_\kappa = \mu \left( \varphi_{\#_1, k}, \varphi_{\#_2, k} \right)_\kappa\) for a non-zero constant \(\mu\) which does not depend on \(k\): a product of values of the kind \(\omega^{-1}_I(\text{nd}(\delta_{lc}))\), indeed, which is therefore 1. Thus we can substitute

\[
\left( \varphi_{\#_1, k}, \varphi_{\#_2, k} \right)_\kappa = \left( \varphi_{\#_1, k}, \varphi_{\#_2, k} \right)_\kappa
\]

in Theorem 8.3 and also apply (74).

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