SPLITTING OF THE VIRTUAL CLASS FOR GENUS ONE STABLE QUASIMAPS

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ABSTRACT. We analyse the local structure of moduli space of genus one stable quasimaps. Combining it with the p-fields theory developed in [9], we prove the splitting formula for the virtual cycle of stable quasimaps to complete intersections in \( \mathbb{P}^n \).

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1. INTRODUCTION

The moduli space of stable quasimaps to arbitrary GIT quotient is a generalization of the moduli space of stable quotient defined by Marian, Oprea and Pandharipande [33], which was constructed and studied by Ciocan-Fontanine, Kim and Maulik [13]. When the target is a projective complete intersection, Ciocan-Fontanine and Kim [14] proved that the invariants of stable quasimaps can be related to the Gromov-Witten invariants by mirror map for all genus (see also [11], [12], [35] for different cases, and [15], [16] for different proofs). The genus zero stable quasimap (stable quotient) invariants of complete intersections are computed by Cooper and Zinger [18], and Ciocan-Fontanine and Kim [11]. Kim and Lho [26] calculate the genus one invariants of complete intersection without markings by using infinitesimal marked points.

Let \( X = (q_1(x) = \cdots = q_m(x) = 0) \subset \mathbb{P}^n \) be a smooth complete intersection. Let \( Q_{g,k}(X,d) \) be the moduli stack of genus \( g \) stable quasimaps to \( X \) with degree \( d \) and \( k \) markings. It is a proper Deligne Mumford (DM for short)-stack, and carries a canonical virtual cycle \([Q_{g,k}(X,d)]^{\text{vir}}\).

Especially, for \( k = 1 \), \( X = \mathbb{P}^n \) case, \( \mathcal{X} := Q_{1,1}(\mathbb{P}^n,d) \) has two smooth components by Theorem 2.11. One is the main component \( \mathcal{X}_{\text{red}} \), and the other component is the ghost component \( \mathcal{X}_{\text{gst}} \). Let \( \pi_{\mathcal{X}} : \mathcal{C}_X \to \mathcal{X} \) be the universal family, and \( \mathcal{L}_X \) be the universal line bundle over \( \mathcal{C}_X \). Then the restriction \( \pi_{\mathcal{X}}^* \mathcal{L}_X^{(r)}|_{\mathcal{X}_{\text{red}}} \) is locally free for all positive integers \( r \). In [13], we define the reduced virtual cycle \( A_{1,d}^{\text{red}} \) by the refined euler class of the bundle \( \pi_{\mathcal{X}}^* \mathcal{L}_X^{(r)}|_{\mathcal{X}_{\text{red}}} \). Then we have the following splitting formula for virtual cycle.
Theorem 1.1. Let $X = (q_1(x) = \cdots = q_m(x) = 0) \subset \mathbb{P}^n$ be a smooth complete intersection, then
\[
[Q_{1,1}(X,d)]^{vir} = A_{1,d}^{red} + \langle (-1)^{\sum \deg q_i} t^d \left( \frac{c(H^\vee \otimes ev_1^* TX)}{c(H^\vee \otimes L_2)} \right) \rangle_{n-m-1} \cap ([\overline{M}_{1,1}] \times [Q_{0,2}(X,d)]^{vir})
\]
where $\iota : \overline{M}_{1,1} \times Q_{0,2}(X,d) \to Q_{1,1}(X,d)$ is the node-identifying morphism, $H$ is the Hodge bundle over $\overline{M}_{1,1}$, $L_2$ is the universal tangent bundle over $Q_{0,2}(\mathbb{P}^n, d)$ at the second marked point, which comes from splitting of the node and $A_{1,d}^{red}$ is the reduced virtual cycle defined by (4.3).

Let $\gamma$ be the psi-class of $Q_{1,1}(X,d)$ at the marked point. For $\gamma \in H^{2k}(X, \mathbb{Q})$, $k \leq 1$, we can define the following stable quasimap invariants
\[
\langle \gamma^a ev^* \gamma \rangle_{1,1,d} := \int [Q_{1,1}(X,d)]^{vir} \gamma^a ev^* \gamma,
\]
when $a + k = \text{vdim} Q_{1,1}(X,d)$.

The reduced genus one invariants of stable quasimaps to smooth complete intersection $X \subset \mathbb{P}^n$ is defined as follows

**Definition 1.2.**
\[
(1.1) \quad \langle \gamma^a ev^* \gamma \rangle_{1,1,d}^{red} := \int A_{1,d}^{red} \gamma^a ev^* \gamma.
\]

Then we prove the following equality as formula (1.2) in the paper,
\[
(1.2) \quad \langle \gamma^a ev^* \gamma \rangle_{1,1,d}^{red} = \int_{\kappa_{red}} \gamma^a ev^* \gamma \cup e^{\text{ref}} \left( \sum_{i=1}^m \pi_{X*} L_{\mathcal{A}} \otimes \deg q_i |_{\kappa_{red}} \right).
\]

This reduced invariants can be calculated by using the localization formula similarly as Zinger [34] did in genus one Gromov-Witten invariants, and as the second author [32] did in genus one stable quasimap invariants without marking. We have the following formula which connect the reduced and standard stable quasimap invariants for complete intersections

**Corollary 1.3.** Let $X = (q_1(x) = \cdots = q_m(x) = 0) \subset \mathbb{P}^n$ be a smooth complete intersection. For $\gamma \in H^{2k}(X, \mathbb{Q})$ where $k \leq 1$, we have
\[
\langle \gamma^a ev^* \gamma \rangle_{1,1,d} = \langle \gamma^a ev^* \gamma \rangle_{1,1,d}^{red} - \frac{1}{24} \left( \int_{Q_{0,2}(X,d)} \gamma^a ev^*_1 \gamma \cup c_{n-m-2}(ev_2^* TX) \right. \\
- (n-m-1) \left. \int_{Q_{0,2}(X,d)} \gamma^a ev^*_2 \gamma \right),
\]
where $a + k = \text{vdim} Q_{1,1}(X,d)$. Furthermore, if $X$ is a Calabi-Yau threefold, then $c_1(T_X) = 0$, and
\[
\langle \gamma^a ev^* \gamma \rangle_{1,1,d}^{red} = \langle \gamma^a ev^* \gamma \rangle_{1,1,d} + \frac{1}{12} \int_{Q_{0,2}(X,d)} \gamma^a ev^*_2 \gamma.
\]

The term $\langle \gamma^a \rangle_{1,1,d}^{red}$ plays an important role in Oh and the authors’ splitting formula [30] for genus two stable quasimap invariant of complete intersection Calabi-Yau threefolds in $\mathbb{P}^n$. Thus this paper can be seen as the first step in our approach to the calculation of genus two stable quasimap invariants.

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2. Local charts and local equations

2.1. Relative obstruction theories of quasi-map spaces. Here we introduce relative perfect obstruction theories of the quasi-map space $Q_{1,k}(P^n,d)$ and the quasi-map space with fields $Q_{1,k}(P^n,d)^P$. We introduce some Artin stacks, which will be used as bases of the relative perfect obstruction theories. Let $\mathcal{M}_{1,k}$ be the Artin stack of nodal curves of genus one with $k$-markings.

Definition 2.1. Let $\mathcal{M}_{1,k,d}^\text{wit}$ be the groupoid associating each scheme $S$ to the set $\mathcal{M}_{1,k,d}^\text{wit}(S) = (c_{S}, \{ p_j : S \to c_{S} \}_{j=1}^{k})$ where $(\pi : c_{S} \to S, \chi)$ is a flat family of prestable genus one weighted nodal curves with $k$ marked points. We will usually abbreviate it by $\mathcal{M}_{1,k,d}^\text{wit}$.

Definition 2.2. Let $\mathcal{M}_{1,k}^\text{line}$ be the groupoid associating each scheme $S$ to the set $\mathcal{M}_{1,k}^\text{line}(S) = (c_{S}, \{ p_j : S \to c_{S} \}_{j=1}^{k}, L)$, where $\pi : c_{S} \to S$ is a flat family of connected genus one nodal curves and $\{ L \}$ is a line bundle on $c_{S}$ of degree $d$ along fibers of $c_{S}/S$. An arrow from $(c_{S}, \{ p_j : S \to c_{S} \}_{j=1}^{k}, L)$ to $(c'_{S}, \{ p'_j : S \to c'_{S} \}_{j=1}^{k}, L')$ consists of $f : c_{S} \to c'_{S}$ and an isomorphism $\theta_{f} : f^{*}L' \to L$, which preserve the markings and the sections.

Let $(C, (p_j)_{j=1}^{k}, D)$ be the $k$-pointed (connected) nodal elliptic curves $C$ with effective divisors $D \subset C$ supported on the smooth loci of $C$. Then $(C, (p_j)_{j=1}^{k}, D)$ is stable if the induced weighted nodal curve $(C, (p_j)_{j=1}^{k}, \deg D)$ is stable.

Definition 2.3. Let $\mathcal{M}_{1,k,d}^\text{div}$ be the groupoid associating each scheme $S$ to the set $\mathcal{M}_{1,k,d}^\text{div}(S) = (c_{S}, \{ p_j : S \to c_{S} \}_{j=1}^{k}, L)$, where $\pi : c_{S} \to S$ is a flat family of connected stable genus one nodal curves and $\{ L \}$ is an effective divisor on $c_{S}$ whose degree is $d$ on each fiber.

Note that $\mathcal{M}_{1,k}, \mathcal{M}_{1,k,d}^\text{wit}, \mathcal{M}_{1,k}^\text{line}$ and $\mathcal{M}_{1,k,d}^\text{div}$ are smooth Artin stacks. The morphism $\mathcal{M}_{1,k,d}^\text{div} \to \mathcal{M}_{1,k,d}^\text{wit}$ is smooth and proper with connected fibers, and the morphism $\mathcal{M}_{1,k} \to \mathcal{M}_{1,k}^\text{line}$ is étale. The natural (dual) relative obstruction theory of $Q_{1,k}(P^n,d)$ over $\mathcal{M}_{1,k}^\text{line}$ is defined by

\[
\mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}}^\vee := R\pi_{*}L_{C}^{\text{div}},
\]

where $\pi : C \to Q_{1,k}(P^n,d)$ is the universal curve and $L_{C}$ is the universal bundle over $C$, which coincides with the pull-back of the universal bundle $L$ over $\mathcal{M}_{1,k}^\text{line}$ via the forgetful morphism $f : Q_{1,k}(P^n,d) \to \mathcal{M}_{1,k}^\text{line}$.

Next we consider a relative obstruction theory of $Q_{1,k}(P^n,d)$ over $\mathcal{M}_{1,k}$. The morphism $\mathcal{M}_{1,k}^\text{line} \to \mathcal{M}_{1,k}^\text{wit}$ is given by associating $(C, L)$ to the weight on $C$, given by the degree of the line bundle $L$ restricted on each irreducible component of $C$. Note that this morphism is smooth. Hence the morphism $\mathcal{M}_{1,k}^\text{line} \to \mathcal{M}_{1,k}$ is smooth. Hence there is a natural relative obstruction theory $\mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}}$ to $L_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}}$, which is induced from the relative obstruction theory $\mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}} \to L_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}}$ [1 Proposition 7.2].

From the definition of relative obstruction theories and octahedral axiom of derived categories, there is a natural distinguished triangle:

\[
f^{*}T_{\mathcal{M}_{1,k}^\text{line}/\mathcal{M}_{1,k}}[-1] \to \mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}}^\vee \to \mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}}^\vee +1
\]

which fits in to the commutative diagram of distinguished triangles:

\[
\begin{array}{ccc}
\mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}}^\vee & \to & \mathcal{E}_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}}^\vee +1 \\
\downarrow & & \downarrow \\
T_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}^\text{line}} & \to & T_{Q_{1,k}(P^n,d)/\mathcal{M}_{1,k}} +1
\end{array}
\]

On the other hand, in a similar manner as in [2 Lemma 2.8] we have the following commutative diagram of distinguished triangles:
where $\pi : C_{Q_1,k}(\mathbb{P}^n, d) \to Q_1,k(\mathbb{P}^n, d)$ is the universal curve and $L_C$ is the universal bundle over $C_{Q_1,k}(\mathbb{P}^n, d)$, $f : C_{Q_1,k}(\mathbb{P}^n, d) \to [\mathbb{C}^{n+1}/\mathbb{C}^*]$ is a universal morphism induced from the universal section $(u_0, \ldots, u_n)$ of $L_C^{\oplus n+1}$. Note that the (pull-back of) the tangent complex $T_{\mathbb{C}^{n+1}/\mathbb{C}^*}$ of the quotient stack is the complex

$$O_{\mathbb{C}^{n+1}}(x_0, \ldots, x_n) \to O_{\mathbb{C}^{n+1}}$$

where $x_0, \ldots, x_n$ is the coordinate functions of $\mathbb{C}^{n+1}$. Note that the distinguished triangle on the first horizontal arrow is obtained from the exact sequence

$$0 \to O_{Q_1,k}(\mathbb{P}^n, d) \to L_C^{\oplus n+1} \cong L_C \times \mathbb{C}^{n+1} \to f^* T_{[\mathbb{C}^{n+1}/\mathbb{C}^*]} \to 0$$

by taking the pull-back and the pushforward. Then we have

$$E^\vee_{Q_1,k}(\mathbb{P}^n, d) \cong \text{cone} \left( \begin{array}{c} \pi^* T^\vee_{M_{1,k}^{\text{div}} / \mathbb{P}^n, d} \to \pi^* T_{Q_1,k(\mathbb{P}^n, d) / M_{1,k}^{\text{div}}} \to \pi^* T_{Q_1,k(\mathbb{P}^n, d) / \mathbb{P}^n} \rightarrow \mathbb{C}^{n+1} \end{array} \right)$$

Remark 2.4. By the above argument, we can replace $\varphi$ by $\varphi'$, which is the morphism induced from the section $(u_0, \ldots, u_n) : O_{Q_1,k}(\mathbb{P}^n, d) \to L_C^{\oplus n+1}$ by taking the derived pushforward.

Next we define a local relative obstruction theory of $Q_1,k(\mathbb{P}^n, d)$ over $M_{1,k,d}^{\text{div}}$. Although there is no natural morphism from $Q_1,k(\mathbb{P}^n, d)$ to $M_{1,k,d}^{\text{div}}$, we can consider the morphism locally as follows. Consider a point $x = [(C, p_1, \ldots, p_k, L, \{u_i\}_{i=0}^n)] \in Q_1,k(\mathbb{P}^n, d)$ and an open subset $U_0 \subset Q_1,k(\mathbb{P}^n, d)$ defined by the condition $u_0 \neq 0$ containing $x$. Then there is a morphism $p : U_0 \to M_{1,k,d}^{\text{div}}$ defined by

$$p : U_0 \to M_{1,k,d}^{\text{div}},$$

$$[(C, p_1, \ldots, p_k, L, \{u_i\}_{i=0}^n)] \mapsto [(C, p_1, \ldots, p_k, u_0^{-1}(0))].$$

Over this local chart $U_0$ of $Q_1,k(\mathbb{P}^n, d)$, a (dual) relative obstruction theory $E^\vee_{U_0 / M_{1,k,d}^{\text{div}}}$ is defined by the following in [2]:

$$(2.4) \quad E^\vee_{U_0 / M_{1,k,d}^{\text{div}}} := R\pi_* O_C(D)^{\oplus n}$$

where $\pi : C \to U_0$ is the universal curve and $D \subset C$ is the universal divisor defined by the universal section $s_0$ of the universal bundle $L_C$ on $C$.

2.2. Local charts and local equations. In this section, we will study the local structure of $Q_1,k(\mathbb{P}^n, d)$, parallel to [22] which studied local structure of the stable map space $M_{1,k}(\mathbb{P}^n, d)$.

Recall the the morphism from the open neighbourhood $U_0 \subset Q_1,k(\mathbb{P}^n, d)$ to the Artin stack $M_{1,k,d}^{\text{div}}$ defined in Section 2.1. We also consider a closed point

$$x = [(C, p_1, \ldots, p_k, L, \{u_i\}_{i=0}^n)] \in U_0.$$
Let us denote the divisor $u_0^{-1}(0)$ by $D$ and let $\mathcal{V} \to \mathfrak{M}_{1,k,d}^{\text{div}}$ be a smooth affine chart with

$$[(\mathcal{C}_V)_0, p_1(0), \ldots, p_j(0), D_0] = [(C, p_1, \ldots, p_k, D)] = q(x).$$

Here, $\mathcal{C}_V$ is a canonical curve over $\mathcal{V}$, $p_i : \mathcal{V} \to \mathcal{C}_V$ are universal sections and $D$ is a universal divisor on $\mathcal{C}_V$. In fact, $U_0$ will be turned out as an open set of a total space of $\rho_* \mathcal{O}_{\mathcal{C}_V}(D)$ where $D$ is a universal divisor on the universal curve $\rho : \mathcal{C}_V \to \mathcal{V}$. So we need to find a resolution of $\rho_* \mathcal{O}_{\mathcal{C}_V}(D)$. For this, we first show the following lemma.

**Lemma 2.5.** By taking $\mathcal{V}$ small enough, there is an equivalence of line bundles:

$$\mathcal{O}_{\mathcal{C}_V}(rD) \cong \mathcal{O}_{\mathcal{C}_V}(D_1 + \cdots + D_{rd})$$

where $r \geq 1$ is an integer, $D_1, \ldots, D_{rd}$ are sections $\mathcal{V} \to D$ disjoint to each others.

**(Sketch of the proof).** Basically the proof can be obtained similarly as [22] Lemma 2.1. Case 1) $d = 1$. It is clear that there is nothing to proof. So we will just sketch the proof.

Case 2) $d \geq 2$. Take the neighbourhood $\mathcal{V}$ small enough. Then, from the degree condition,

$$\text{degree} \geq 2,$$

we can find two sections $s_1, s_2$ of $\mathcal{O}_{\mathcal{C}_V}(rD)$ which gives a family of degree $r \cdot d$ morphisms to $\mathbb{P}^1$. Since $\mathcal{V}$ is small enough, we can find a linear combination $as_1 + bs_2$ whose zero is $D_1 + \cdots + D_{rd}$ where $D_i$ are family of degree 1 effective divisors disjoint to each others. □

Same as the stable map spaces case [22]. We can choose sections $A, B : \mathcal{V} \to \mathcal{C}_V$ lies in core subcurves for each fiber, and disjoint with each others. Moreover we may assume that $A, B$ are disjoint to the divisors $D_1, \ldots, D_{rd}$. Here, we define core subcurve of a genus $g$ curve $X$ by a minimal genus $g$ subcurve of $X$.

Let $\mathcal{L} := \mathcal{O}_{\mathcal{C}_V}(D)$. By the above lemma, we have $\mathcal{L}^\otimes r \cong \mathcal{O}_{\mathcal{C}_V}(D_1 + \cdots D_{rd})$. We consider the inclusion of sheaves

$$\mathcal{M}_i := \mathcal{O}_{\mathcal{C}_V}(D_i + A - B) \subset \mathcal{M} := \mathcal{O}_{\mathcal{C}_V}\left(\sum_{i=1}^{rd} D_i + A - B\right)$$

and the induced inclusions

$$\eta_i : \rho_* \mathcal{M}_i \hookrightarrow \rho_* \mathcal{M}.$$

Both are locally free since $R^1 \rho_* \mathcal{M}_i$ and $R^1 \rho_* \mathcal{M} = 0$. By Riemann-Roch, $\rho_* \mathcal{M}_i$ is invertible and the rank of $\rho_* \mathcal{M}$ is $d$. We then let

$$\varphi : \rho_* \mathcal{M} \longrightarrow \rho_* \left(\mathcal{O}_{\mathcal{C}_V}\left(\sum_{i=1}^{rd} D_i + A - B\right)\right)_{|A} = \rho_* (\mathcal{O}_A(A))$$

and

$$\varphi_i : \rho_* \mathcal{M}_i \longrightarrow \rho_* (\mathcal{O}_{\mathcal{C}_V}\left(\sum_{i=1}^{rd} D_i + A - B\right)_{|A}) = \rho_* (\mathcal{O}_A(A))$$

be the evaluation homomorphisms. Obviously, $\varphi = \varphi \circ \eta_i$. Since we assumed that $\mathcal{V}$ is affine, the sheaf $\rho_* (\mathcal{O}_A(A))$ is isomorphic to $\mathcal{O}_V$.

**Lemma 2.6.** [22] Lemma 4.10 We have

1. $\rho_* \mathcal{L}^\otimes r \cong \mathcal{O}_V \oplus \rho_* \mathcal{O}_{\mathcal{C}_V}\left(\sum_{i=1}^{rd} D_i - B\right)$;
2. $\rho_* \mathcal{O}_{\mathcal{C}_V}\left(\sum_{i=1}^{rd} D_i - B\right) \cong \ker \varphi$;
3. $\oplus_{i=1}^{rd} \eta_i : \bigoplus_{i=1}^{rd} \rho_* \mathcal{M}_i \longrightarrow \rho_* \mathcal{M}$ is an isomorphism, and $\oplus_{i=1}^{rd} \varphi_i = \varphi \circ \oplus_{i=1}^{rd} \eta_i$.

Note that $\rho_* \mathcal{M}_i \cong \mathcal{O}_V$ and $\rho_* (\mathcal{O}_A(A)) \cong \mathcal{O}_V$ since we may assume $\mathcal{V}$ sufficiently small. Then $\varphi_i$ is a morphism between trivial bundles. To describe each morphism $\varphi$ explicitly, we review arguments in [22] Section 4.

For a weighted genus one nodal curve $C$, Let $\gamma^0$ be the associated dual graph. Then we contract a subgraph of $\gamma^0$ comes from the core subcurve, making the new graph $\gamma^1$. We denote the contracted vertex by ‘o’. o is also called the root of the graph. Using the following four operations on the rooted tree $\gamma^1$, pruning, collapsing, specialization, and
advancing, we obtain a terminally weighted tree $\gamma$. See [22 Section 3.2] for details. Here, ‘terminally weighted’ means weights are concentrated on the terminal (=maximal order) vertices. Note that the vertex set of every rooted tree has natural order having the root vertex as a minimal element.

Let $\gamma$ be the terminally weighted tree associated to $(C, p_1, \cdots, p_k, L)$. The weight is given by degrees of $L$ on each components of $C$. For each vertex $v \in \gamma$ we define

$$\zeta_v = \zeta_q \in \Gamma(\mathcal{O}_\gamma),$$

where $q$ is the associated node of $v$, and $\Sigma_q = \{\zeta_q = 0\}$ is the locus such that the node $q$ is not smoothed. For any terminal vertex $i \in \text{Ver}(\gamma)^t$, we let

$$\zeta_{i,o} = \prod_{i \geq v > o} \zeta_v.$$

We have the following theorem,

**Theorem 2.7.** [22 Lemma 4.16] The direct image sheaf $\rho_*\mathcal{L}^\oplus$ is a direct sum of $\mathcal{O}_\mathcal{V}^{(rd-\ell+1)}$ with the kernel sheaf of the homomorphism

$$\varphi_i: \mathcal{O}_\mathcal{V}^\oplus \rightarrow \mathcal{O}_\gamma, \quad \varphi_i = c_i \cdot \zeta_{i,o}, \quad c_i \in \mathbb{C}$$

where $\ell$ is the number of terminals vertices of $\gamma$.

For a point in $Q_{1,1}(\mathbb{P}^n, d)$, let $U$ be a small neighborhood of it. We pick a smooth chart $\mathcal{V} \rightarrow \mathfrak{M}^{\text{div}}_{1,1,k,d}$, which contains the image of $U \rightarrow \mathfrak{M}^{\text{div}}_{1,1,k,d}$. Let $U = \mathcal{V} \times_{\mathfrak{M}^{\text{div}}_{1,1,k,d}} \mathcal{E}_\gamma$ be the total space of the vector bundle $\rho_*\mathcal{L}(A)^\oplus_n$. Let $p: \mathcal{E}_\gamma \rightarrow \mathcal{V}$ be the projection. Then the tautological restriction homomorphism

$$\text{rest}: \rho_*\mathcal{L}(A)^{\oplus n} \rightarrow \rho_*\mathcal{L}(A)^{\oplus n}{|_A}$$

lifts to a section

$$F \in \Gamma(\mathcal{E}_\gamma, p^*\rho_*\mathcal{L}(A)^{\oplus n}{|_A}).$$

Then there is a canonical open immersion $U \rightarrow (F = 0) \subset \mathcal{E}_\gamma$. To a terminal vertex $b \in \text{Ver}(\gamma)^t$, we associate $n$ coordinate functions $w_{b,1}, \cdots, w_{b,n} \in \Gamma(\mathcal{O}_{\mathcal{E}_\gamma})$. We then set

$$\Phi_\gamma = (\Phi_{\gamma,1}, \cdots, \Phi_{\gamma,n}), \quad \Phi_{\gamma,e} = \sum_{b \in \text{Ver}(\gamma)^t} \zeta_{b,o} w_{b,e}.$$

Similar to Hu and Li’s [22 Theorem 2.19], we have the following theorem

**Theorem 2.8.** For a point in $Q_{1,1}(\mathbb{P}^n, d)$, let $\gamma$ be the associated weighted tree, choosing $\mathcal{V}$ as above and shrinking it if necessary and fix an isomorphism $p^*\rho_*\mathcal{L}(A)^{\oplus n}{|_A} \cong \mathcal{O}_{\mathcal{E}_\gamma}^{\oplus n}$. Then we can find regular functions over $\mathcal{E}_\gamma, w_{b,1}, \cdots, w_{b,n}$, from coordinate functions of $\mathcal{O}_{\mathcal{E}_\gamma}^{\oplus n}$ and node-smoothing parameter functions $\zeta_i$ such that

$$F = (\Phi_{\gamma,1}, \cdots, \Phi_{\gamma,n}).$$

When $k = 1$, let $\gamma$ be a stable terminally weighted rooted trees of total weight $d$. We can easily check that $\gamma$ is a one path trees. Therefore $\gamma$ has only one terminal vertex, so that we have

$$\Phi_{\gamma,e} = \zeta_1 w_e, \quad \Phi_\gamma = (\zeta_1 w_1, \cdots, \zeta_1 w_n)$$

where $\zeta_1$ is a node-smoothing parameter correspond to the unique terminal vertex of $\gamma$. Let us denote $\zeta_1$ by $\zeta$. The local equation for $Q_{1,1}(\mathbb{P}^n, d)$ can be easily described as the following.

**Corollary 2.9.** For a point in $Q_{1,1}(\mathbb{P}^n, d)$, choosing $\mathcal{V}$ as above and shrinking it if necessary and fix an isomorphism $p^*\rho_*\mathcal{L}(A)^{\oplus n}{|_A} \cong \mathcal{O}_{\mathcal{E}_\gamma}^{\oplus n}$, we can find $n+1$ regular functions $w_1, \cdots, w_n, \zeta$ over $\mathcal{E}_\gamma$ such that

$$F = (w_1\zeta, \cdots, w_n\zeta).$$

Furthermore, each $w_i$ and $\zeta$ has smooth vanishing locus, which intersect transversally to each others.
When \( k > 1 \), as in \[22\], let \( \Theta_s \) be the closure in \( \mathcal{M}^{\text{inst}}_{1,k} \) of the locus where the weight is zero on the genus one core component, and has \( s \) rational components attach to the genus one component. Let \( \mathcal{M}_{1,k}^{\text{red}} \) be the successive blow up \( \mathcal{M}_{1,k}^{\text{inst}} \) along \( \Theta_1, \ldots, \Theta_d \). Then irreducible components of \( \mathcal{Q}_{1,k}(\mathbb{P}, d) := \mathcal{Q}_{1,k}(\mathbb{P}, d)_{\mathcal{M}_{1,k}^{\text{inst}}} \times \mathcal{M}_{1,k}^{\text{red}} \) are smooth and intersect transversally, and we also have the following local equations. The following is a direct analogue of \[22\] Theorem 2.19 and \[28\] Proposition 2.1 in stable quasi-map spaces.

**Theorem 2.10.** For a point in \( \mathcal{Q}_{1,k}(\mathbb{P}, d) \) choosing an smooth affine chart \( \hat{Y} \) of \( \mathcal{M}_{1,k} \), shrinking it if necessary and fixed \( \phi^* \rho_*(\mathcal{L}(\mathcal{A})^{\oplus n}|_{\mathcal{A}}) \cong \mathcal{E}_{\mathcal{V}}^{\oplus n} \), we can find \( n + d' \) regular functions \( w_1, \ldots, w_n \) and \( \zeta_1, \ldots, \zeta_{d'} \) over \( \mathcal{E}_{\mathcal{V}} \) where \( d' = \min\{k, d\} \), such that
\[
F = (w_1 \tau, \ldots, w_n \tau), \quad \tau := \zeta_1 \cdots \zeta_{d'}.
\]
Furthermore, each \( w_i \) and \( \zeta_j \) has smooth vanishing locus, and they intersect transversally to each others.

Set \( \mathcal{X} = Q_{1,1}(\mathbb{P}^n, d) \), let \( \pi_X : \mathcal{C}_X \to \mathcal{X} \) be the universal family and \( \mathcal{L}_X \) be the universal line bundle over \( \mathcal{C}_X \). By the stability conditions, we know that \( \mathcal{X} \) has two different irreducible components, the main component \( \mathcal{X}_{\text{red}} \) (where the underlying curves of the generic points are smooth elliptic curves), and the other is the so called ghost component \( \mathcal{X}_{\text{gst}} \). Locally, \( \mathcal{X}_{\text{red}} = \{ w_1 = \cdots = w_n = 0 \} \) and \( \mathcal{X}_{\text{gst}} = \{ \tau = 0 \} \). Then by the proof of \[22\] Theorem 2.11, we have

**Theorem 2.11.** The direct image sheaf \( \pi_{\mathcal{X}_{\text{red}}}^* \left( \mathcal{L}_X^\otimes r|_{\mathcal{X}_{\text{red}}} \right) \) is locally free of rank \( rd \), and the direct image sheaf \( \pi_{\mathcal{X}_{\text{gst}}}^* \left( \mathcal{L}_X^\otimes r|_{\mathcal{X}_{\text{gst}}} \right) \) is locally free of rank \( rd + 1 \).

**Remark 2.12.** For \( k > 1 \), we can obtain similar result as Theorem 2.11. In this case, ghost component is not irreducible. For each irreducible component of \( Q_{1,k}(\mathbb{P}^n, d) \), denoted by \( \tilde{Q}_\gamma \), the direct image sheaves \( \pi_{\tilde{Q}_{\gamma}}^* \left( \mathcal{L}_{Q_{1,k}(\mathbb{P}^n, d)}^\otimes r \right) \) is locally free of rank \( rd + 1 \). Also, the direct image sheaf \( \pi_{\tilde{Q}_{\text{red}}}^* \left( \mathcal{L}_{Q_{1,k}(\mathbb{P}^n, d)}^\otimes r |_{\tilde{Q}_{\text{red}}} \right) \) is locally free of rank \( rd \), where \( \tilde{Q}_{\text{red}} \) denotes the main component.

3. Moduli of stable quasimaps with fields

3.1. **Stable quasimaps with fields.** First we recall the moduli stack of stable quasimaps with fields introduced in \[9\]. To simplify the notation, we will focus on the genus one case. Let us abbreviate \( Q := Q_{1,k}(\mathbb{P}^n, d) \). Let
\[
\pi_Q : \mathcal{C}_Q \to Q, \quad \mathcal{P}_Q^i = \mathcal{L}_Q^\otimes \deg q_i \otimes \omega_{\mathcal{C}_Q/Q}, \quad 1 \leq i \leq m.
\]
As in \[9\], let \( \mathcal{P} = \mathcal{P}_{1,k} = C(\oplus_{i=1}^m \pi_{Q*} \mathcal{P}_Q) \) be the cone stack over \( Q \). The relative perfect obstruction theory over \( \mathcal{P} \to \mathcal{M}^{\text{inst}}_{1,k} \) is given by
\[
(3.1) \quad \phi_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} : T_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} \to E_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}}^\vee, \quad E_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}}^\vee := R^* \pi_{\mathcal{P}*}(\mathcal{L}_\mathcal{P}^\otimes (n+1) \oplus_i \mathcal{P}_Q^i),
\]
where
\[
\pi_P : \mathcal{C}_P \to \mathcal{P}, \quad \mathcal{P}_P^i = \mathcal{L}_P^\otimes \deg q_i \otimes \omega_{\mathcal{C}_P/P}, \quad 1 \leq i \leq m
\]
is the universal curve and \( T_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} \) denotes the relative tangent complex.

According to the convention, we call the cohomology sheaf
\[
\mathcal{O}b_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} := H^1(E_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}}^\vee) = R^1 \pi_{\mathcal{P}*}(\mathcal{L}_\mathcal{P}^\otimes (n+1) \oplus_i \mathcal{P}_P^i)
\]
the relative obstruction sheaf of \( \phi_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} \).

The authors \[9\] constructed a cosection of \( \mathcal{O}b_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} \) by using the defining polynomials \( q_1(x) = \cdots = q_m(x) = 0 \) of \( X \). Namely a homomorphism
\[
(3.2) \quad \sigma' : \mathcal{O}b_{\mathcal{P}/\mathcal{M}^{\text{inst}}_{1,k}} \to \mathcal{O}_P.
\]
This cosection can be lifted to a cosection $\tilde{\sigma} : \mathcal{O}_P \to \mathcal{O}_P$ of the obstruction sheaf $\mathcal{O}_P$. Note that the obstruction sheaf $\mathcal{O}_P$ fits into the exact sequence

$$f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}} \longrightarrow \mathcal{O}_{P/\mathfrak{m}_{1,k}^{\text{line}}} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$  

The degeneracy locus $D(\sigma')$ of $\sigma'$, where $\sigma$ is not surjective, is the closed subset

$$D(\sigma') = Q_{1,k}(X, d) \subset \mathcal{P}.$$  

Moreover we have $A_\ast D(\sigma') = A_\ast Q_{1,k}(X, d)$ by the result in [2]. Furthermore, in [2] the authors defined the (localized) virtual cycle for $\mathcal{P}$ as

$$[\mathcal{P}]_{\text{loc}}^{\text{vir}} := [0]_{\sigma'_{\text{loc}}}[\mathcal{C}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}} \in A_\ast D(\sigma') = A_\ast Q_{1,k}(X, d)$$

where $0_{\sigma'_{\text{loc}}}$ is the the localized Gysin map defined in [25] for the cosection $\sigma'$, and $\mathcal{C}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}$ is the relative intrinsic normal cone.

**Theorem 3.1 ([9, 27]).** We have

$$[\mathcal{P}]_{\text{loc}}^{\text{vir}} = (-1)^{\sum_i \deg g_i} [Q_{1,k}(X, d)]^{\text{vir}}.$$  

We remark that this Theorem holds for all genus $g$ and $k$. For our purpose here, we only state in the case $g = 1$. Set $\phi : \mathfrak{m}_{1,k}^{\text{line}} \to \mathfrak{m}_{1,k}$. Then we have the following distinguished triangles

$$f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1] \longrightarrow T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}} \longrightarrow T_P \longrightarrow f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}.$$  

By [2] Lemma 3.6, the composing with $\sigma' \circ H^1(\phi_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) : T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}} \longrightarrow \mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}} \longrightarrow \mathcal{O}_P$ is zero. From the following commutative diagram, the cosection $\sigma$ induces a cosection $\sigma : H^1(\mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \to \mathcal{O}_P$.

$$H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1]) \longrightarrow H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1])$$

$$\downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(E_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(\mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

$$\phi_{\text{int}} \downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(E_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(\mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

$$\sigma' \downarrow$$

$$\sigma' \downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(E_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(\mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

By [2] Proposition 3.5, the following morphism

$$\eta : H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1]) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(E_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow \mathcal{O}_P$$

is zero. Let $g_P := \phi \circ f_P : \mathcal{P} \to \mathfrak{m}_{1,k}$. By the commutative diagram below

$$H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1]) \longrightarrow H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1]) \longrightarrow H^1(f_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1])$$

$$\downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

$$\eta_{\text{int}} \downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

$$\phi_{\text{int}} \downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

$$\sigma' \downarrow$$

$$\sigma' \downarrow$$

$$H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}})$$

the morphism

$$\eta' : H^1(g_P^* T_{\mathfrak{m}_{1,k}^{\text{line}}}[-1]) \longrightarrow H^1(T_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow H^1(\mathcal{E}_{\mathcal{P}/\mathfrak{m}_{1,k}^{\text{line}}}) \longrightarrow \mathcal{O}_P$$
obtained by the composition is zero. Thus the cosection \( \sigma : H^1(\mathcal{E}_P^\vee / \mathfrak{M}_{1,k}) \to \mathcal{O}_P \) can be lifted to the cosection \( \mathcal{O}_P \to \mathcal{O}_P \). Therefore we can define the following virtual cycle

\[
0^l_{\sigma, \text{loc}}(\mathcal{E}_P / \mathfrak{M}_{1,k})
\]

Since \( \phi : \mathfrak{M}_{1,k} \to \mathfrak{M}_{1,k} \) is smooth, we have the following commutative diagram:

\[
\begin{array}{ccc}
\quad & \quad & \\
\scriptstyle h^1/h^0(\mathcal{E}_P^\vee / \mathfrak{M}_{1,k}) & \xrightarrow{f} & h^1/h^0(\mathcal{E}_P^\vee / \mathfrak{M}_{1,k}) \\
\quad & \quad & \\
\scriptstyle h^1/h^0(\mathcal{E}_P^\vee / \mathfrak{M}_{1,k}) & \xrightarrow{\phi} & h^1/h^0(\mathcal{E}_P^\vee / \mathfrak{M}_{1,k}) \\
\end{array}
\]

\[
(3.10)
\]

Hence we have

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From the cotangent complexes associated to the triples \( Y \to \mathcal{M}_{1,1}^{\text{line}} \to \mathcal{M}_{1,1} \) and \( X \to \mathcal{M}_{1,1}^{\text{line}} \to \mathcal{M}_{1,1} \), we obtain the diagram

\[
H^1(f_Y^* T_{\mathcal{M}_{1,1}^{\text{line}}/\mathcal{M}_{1,1}}[-1]) \longrightarrow H^1(f_Y^* T_{\mathcal{M}_{1,1}^{\text{line}}/\mathcal{M}_{1,1}}[-1])
\]

Furthermore, we have

\[
H^1(T_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \longrightarrow p^* H^1(T_{X/\mathcal{M}_{1,1}^{\text{line}}})
\]

\[
H^1(T_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \longrightarrow p^* H^1(T_{X/\mathcal{M}_{1,1}^{\text{line}}}).
\]

Furthermore, we have

\[
H^1(f_Y^* T_{\mathcal{M}_{1,1}^{\text{line}}/\mathcal{M}_{1,1}}[-1]) \longrightarrow H^1(f_Y^* T_{\mathcal{M}_{1,1}^{\text{line}}/\mathcal{M}_{1,1}}[-1])
\]

\[
H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \longrightarrow j^* H^1(E_{X/\mathcal{M}_{1,1}^{\text{line}}}).
\]

\[
H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \longrightarrow p^* H^1(E_{X/\mathcal{M}_{1,1}^{\text{line}}}).
\]

Here \( j \) is a morphism which gives the splitting (3.14) of \( H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \). Note that the vertical arrows \( H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \to H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) \) and \( p^* H^1(E_{X/\mathcal{M}_{1,1}^{\text{line}}}) \to p^* H^1(E_{X/\mathcal{M}_{1,1}^{\text{line}}}) \) are surjective. Then, by chasing the diagram we can show that \( j \) induce the morphism \( \bar{j} \), which gives the splitting. So we obtain the decomposition

\[
H^1(E_{Y/\mathcal{M}_{1,1}^{\text{line}}}) = p^* H^1(E_{X/\mathcal{M}_{1,1}^{\text{line}}}) \bigoplus_i p^* H^1(R^i \pi_{X*}(\mathcal{P}_X)).
\]

Parallel to [29, Lemma 2.4], we will prove the following lemma.

**Lemma 3.4.** (1) For a sufficiently small open neighbourhood \( U \subset X \), and \( U_{\text{gst}} := U \times_X \mathcal{X}_{\text{gst}} \) we have

\[
H^1(E_{U/\mathcal{M}_{1,1}^{\text{line}}} |_{U_{\text{gst}}}) \cong H^1(E_{U'/\mathcal{M}_{1,1}^{\text{line}}} |_{U'_{\text{gst}}}).
\]

(2) Also, for a sufficiently small open neighbourhood \( U' \subset Y \), and \( (U')_{\text{gst}} := U' \times_X \mathcal{X}_{\text{gst}} \) we have

\[
H^1(E_{U'/\mathcal{M}_{1,1}^{\text{line}}} |_{(U')_{\text{gst}}}) \cong H^1(E_{U'/\mathcal{M}_{1,1}^{\text{line}}} |_{U'_{\text{gst}}}).
\]

**Proof.** Since the proof of (2) is parallel to (1), we will only prove (1) here. We first consider the neighbourhood \( U \subset X \). We may assume that \( U \subset \mathcal{U}_0 \). Note that \( E_{X/\mathcal{M}_{1,1}^{\text{line}}} | U \cong R^i \pi_{X*} \mathcal{O}_{\mathcal{U}_t}(\mathcal{D}_t)^{\boxplus n+1} \) on the neighbourhood \( U \). Recall the remark [24] which says that the horizontal arrow \( \phi : R^i \pi_{X*} \mathcal{O}_{\mathcal{U}_t} | U \to E_{X/\mathcal{M}_{1,1}^{\text{line}}} | U \) in (2.2) is induced from the arrow

\[
\mathcal{O}_{\mathcal{U}_t} \xrightarrow{\phi} \mathcal{O}_{\mathcal{U}_t}(\mathcal{D}_t)^{\boxplus n+1}
\]

by taking \( R^i \pi_{X*}(-) \). Consider the exact sequence of complexes

\[
0 \to [0 \to \mathcal{O}_{\mathcal{U}_t}(\mathcal{D}_t)^{\boxplus n}] \to \mathcal{O}_{\mathcal{U}_t} \xrightarrow{\phi} \mathcal{O}_{\mathcal{U}_t}(\mathcal{D}_t)^{\boxplus n+1} \to \mathcal{O}_{\mathcal{U}_t} \to 0.
\]

Since \( E_{X/\mathcal{M}_{1,1}^{\text{line}}} | U \) is equivalent to the mapping cone \( \text{cone}(\phi) \), and \( [\mathcal{O}_{\mathcal{U}_t} \xrightarrow{\phi} \mathcal{O}_{\mathcal{U}_t}(\mathcal{D}_t)] \cong \mathcal{O}_{\mathcal{U}_t} \), we have the distinguished triangle

\[
E_{X/\mathcal{M}_{1,1}^{\text{line}}} | U \to E_{X/\mathcal{M}_{1,1}^{\text{line}}} | U \to R^i \pi_{X*} \mathcal{O}_{\mathcal{U}_t}^{\boxplus 1}
\]
by taking $R^*\pi_{X*}$ to the sequence (3.18). Then, by taking the long exact sequence of this distinguished triangle, we obtain the exact sequence:

$$H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) \rightarrow H^1(E^\vee_{\mathcal{O}m}^{|x|}) \rightarrow 0$$

for any closed point $x \in \mathcal{U}$.

On the other hand, we can consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{U}} \xrightarrow{s_0} \mathcal{O}_{\mathcal{U}}(\mathcal{D}) \rightarrow \mathcal{O}_D \rightarrow 0$$

where $\mathcal{D} = s_0^{-1}(0)$ is the family of degree $d$ divisors on the universal curve $\mathcal{C}_U \rightarrow \mathcal{U}$. Therefore we have the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{U}} \xrightarrow{s_0} \mathcal{O}_{\mathcal{U}}(\mathcal{D})^\oplus \rightarrow \mathcal{O}_{\mathcal{U}}(\mathcal{D})^\oplus \oplus \mathcal{O}_D \rightarrow 0.$$ 

From the isomorphisms

$$E^\vee_{X/\mathcal{O}m}^{|x|} \simeq \text{Cone}[R^*\pi_{X*}\mathcal{O}_{\mathcal{U}} \rightarrow E^\vee_{\mathcal{O}m}^{|x|}],$$

and $E^\vee_{\mathcal{O}m}^{|x|} \simeq R^*\pi_{X*}\mathcal{O}_{\mathcal{U}}(\mathcal{D})^\oplus + \mathcal{O}_D$ we obtain

$$\dim H^0(E^\vee_{X/\mathcal{O}m}^{|x|}) = h^0(C_x, \mathcal{O}_{C_x}(D_x))^\oplus + h^0(C_x, \mathcal{O}_{D_x}) = n \cdot h^0(C_x, \mathcal{O}_{C_x}(D_x)) + d,$$

$$\dim H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) = h^0(C_x, \mathcal{O}_{C_x}(D_x))^\oplus + h^1(C_x, \mathcal{O}_{D_x}) = n \cdot h^1(C_x, \mathcal{O}_{C_x}(D_x))$$

for each closed point $x \in \mathcal{U}$. The fiber $C_x = C_U|_x$ of the universal curve over $x$ and the degree $d$ divisor $D_x = D|_x$ on $C_x$, which is the fiber of the universal divisor $\mathcal{D}$ over $x$. If $x \in \mathcal{U}_{\text{gst}}$, we observe that

$$\dim H^0(E^\vee_{X/\mathcal{O}m}^{|x|}) = n(d + 1) + d, \quad \dim H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) = n$$

from (3.20). Also it is trivial that $\dim H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) = n \cdot h^1(C_x, \mathcal{O}_{C_x}(D_x)) = 1$ for $x \in \mathcal{U}_{\text{gst}}$. Therefore, for an arbitrary closed points $x \in \mathcal{U}_{\text{gst}}$, the morphism

$$H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) \rightarrow H^1(E^\vee_{X/\mathcal{O}m}^{|x|})$$

from (3.19) is an isomorphism since it is surjective and both vector spaces have same dimension $n$. Since $\mathcal{U}_{\text{gst}}$ is a reduced scheme, we have an isomorphism

$$H^1(E^\vee_{X/\mathcal{O}m}^{|x|}) \xrightarrow{\cong} H^1(E^\vee_{X/\mathcal{O}m}^{|x|}).$$

Because the sheaf $H^1(E^\vee_{X/\mathcal{O}m}^{|x|})$ is locally free by Remark 3.3 we have the following.

**Proposition 3.5.** The obstruction sheaf $H^1(E^\vee_{X/\mathcal{O}m}^{|x|})$ is locally free.

### 3.3. Decomposition of the intrinsic normal cone

Let $R = \text{Spec}(B)$ be a smooth affine variety. Let $\tilde{R} := R \times \mathbb{C}^{n+m}$, and $F$ be the section of $\mathcal{O}^{n+m}$ with $F = (w_1z, \ldots, w_{n+m}z)$, where $w_i$ are coordinates of $\mathbb{C}^{n+m}$, and $z \in B$ is a regular function. Denote by $Z = F^{-1}(0)$ the zero loci of $F$. Then $Z$ has two different components, where $Z = Z_1 \cup Z_2$ with $Z_1 = \{w_1 = \cdots = w_{n+m} = 0\}$ and $Z_2 = \{z = 0\}$.

**Lemma 3.6.** Let $Z_1/\tilde{R}$ be the normal cone of $Z$ in $\tilde{R}$, then $C_{Z_1/\tilde{R}} = C_1 \cup C_2$ has two different irreducible components $C_1$ and $C_2$ support on $Z_1$ and $Z_2$ respectively, and there is a canonical dominant morphism

$$C_{Z_1/\tilde{R}} \rightarrow C_{Z_1/\tilde{R}}/Z_2.$$

**Proof.** Let $\mathfrak{R} := B[w_1, \ldots, w_{n+m}]/(w_1z_1, \ldots, w_{n+m}z_1)$ be the coordinate ring of $Z$. Consider the following surjective morphism

$$\mathfrak{R}[A_1 \cdots, A_{n+m}] \rightarrow \bigoplus_{k \geq 0} \mathfrak{R}[Z_1/\tilde{R}]/Z_1/\tilde{R}^{k+1} \quad A_i \mapsto w_i z_1.$$
Then $C_{Z/\hat{R}} = \text{Spec}\left( \mathcal{R}[A_1 \cdots , A_{n+m}]/(w_iA_j - w_jA_i) \right)$, which supports on $Z_1$ and $Z_2$. We have
\[
C_{Z/\hat{R}}|_{Z_1} = \text{Spec}\left( \mathcal{R}[A_1 \cdots , A_{n+m}]/(w_iA_j - w_jA_i) \otimes \mathcal{R}/(w_1, \cdots , w_{n+m}) \right)
\]
and
\[
C_{Z/\hat{R}}|_{Z_2} = \text{Spec}\left( B/[A_1 \cdots , A_{n+m}] \right),
\]
Thus the fiber over $C_{Z/\hat{R}}|_{Z_2}$ over $Z_2$ is the affine cone of the blowing up $\text{Bl}_0 \mathbb{C}^{n+m}$, and $C_{Z/\hat{R}}|_{Z_1}$ is a vector bundle over $Z_1$. They are all irreducible. Hence $C_{Z/\hat{R}}|_{Z_2}$ and $C_{Z/\hat{R}}|_{Z_1}$ are irreducible.

Because $Z_2 \subset Z \subset \hat{R}$, there is a canonical morphism
\[
C_{Z/\hat{R}} \rightarrow C_{Z/\hat{R}}|_{Z_2}.
\]
The ideal $I_{Z_2/\hat{R}}$ is equal to $(\zeta)$, the cone $C_{Z_2/\hat{R}}$ is isomorphic to $N_{Z_2/\hat{R}}$ which is a line bundle. Since $I_{Z_2/\hat{R}} = (w_1\zeta, \ldots , w_{n+m}\zeta)$, the composition of the morphism (3.24) with the inclusion $C_{Z/\hat{R}}|_{Z_2} \hookrightarrow Z_2 \times \mathbb{C}^{n+m}$ is given by
\[
N_{Z_2/\hat{R}} = C_{Z_2/\hat{R}} \rightarrow C_{Z/\hat{R}}|_{Z_2} \hookrightarrow Z_2 \times \mathbb{C}^{n+m}
\]
where 1 is a local generator of the line bundle. From the local description (3.23) of $C_{Z/\hat{R}}|_{Z_2}$, we can check that $C_{Z/\hat{R}}|_{Z_2}$ is a closure of the image of the above morphism. Hence (3.24) is dominant.

Let $\mathcal{V}$ and $\mathcal{U}$ be smooth affine charts of $\mathfrak{M}_{1,1}$ and $\mathcal{V}$ as in Proposition 3.2. Denote by $\overline{\mathcal{U}} := \mathcal{V} \times \mathbb{C}^{dn} \times \mathbb{C}^{n+m}$. Then the cone $\mathcal{C}_{\mathcal{V}/\mathfrak{M}_{1,1}}|\mathcal{U} = [\mathcal{C}_{\mathcal{U}/\overline{\mathcal{U}}}/T_{\mathcal{U}/\overline{\mathcal{U}}}]$ has two different components by Proposition 3.2 and Lemma 3.4. Hence $\mathcal{C}_{\mathcal{V}/\mathfrak{M}_{1,1}}$ has two different components. Denote them by
\[
\mathcal{C}_{\mathcal{V}/\mathfrak{M}_{1,1}} = \mathcal{C}_{\text{red}} \cup \mathcal{C}_{\text{gst}},
\]
which are supported on $\mathcal{V}_{\text{red}}$ and $\mathcal{V}_{\text{gst}}$ respectively. Consequently,
\[
[\mathcal{V}]^\text{vir} = 0^\text{vir}_{\text{loc}}[\mathcal{C}_{\text{red}}] + 0^\text{vir}_{\text{loc}}[\mathcal{C}_{\text{gst}}].
\]
Let $\mathcal{C}_{\text{gst}}$ be the coarse moduli space of $\mathcal{C}_{\text{gst}}$, then $\mathcal{C}_{\text{gst}} \subset H^1(\mathbb{E}_{\mathcal{V}/\mathfrak{M}_{1,1}})|_{\mathcal{V}_{\text{gst}}}.

Let us define $\mathfrak{M}_{\text{gst}} := \iota(\mathfrak{M}_{1,1} \times \mathfrak{M}_{0,2}) \subset \mathfrak{M}_{1,1}$ where $\iota$ is the node-identifying morphism. It is a substack whose general points are stable genus one curves attached by rational tails. Moreover let $\mathfrak{g}_{\mathcal{V}} : \mathcal{V} \rightarrow \mathfrak{M}_{1,1}$ be the forgetful morphism and $\mathfrak{g}_{\mathcal{V}_{\text{gst}}} : \mathcal{V}_{\text{gst}} \rightarrow \mathfrak{M}_{1,1}$ be the restriction of $\mathfrak{g}_{\mathcal{V}}$ on $\mathcal{V}_{\text{gst}}$. Consider the coarse moduli space $\mathcal{C}_{\mathcal{V}_{\text{gst}}/\mathfrak{M}_{1,1}}$ of the intrinsic normal cone $\mathcal{C}_{\mathcal{V}_{\text{gst}}/\mathfrak{M}_{1,1}}$. Note that we have
\[
\mathcal{C}_{\mathcal{V}_{\text{gst}}/\mathfrak{M}_{1,1}} = \mathfrak{g}_{\mathcal{V}_{\text{gst}}}^{*}N_{\mathfrak{M}_{\text{gst}}/\mathfrak{M}_{1,1}},
\]
where $N_{\mathfrak{M}_{\text{gst}}/\mathfrak{M}_{1,1}}$ is the normal bundle of $\mathfrak{M}_{\text{gst}} \subset \mathfrak{M}_{1,1}$. Since $\mathcal{V}_{\text{gst}} \subset \mathcal{V}$, there is a nature morphism
\[
\phi : \mathcal{C}_{\mathcal{V}_{\text{gst}}/\mathfrak{M}_{1,1}} \rightarrow \mathcal{C}_{\mathcal{V}/\mathfrak{M}_{1,1}}|_{\mathcal{V}_{\text{gst}}} = \mathcal{C}_{\text{gst}} \subset H^1(\mathbb{E}_{\mathcal{V}/\mathfrak{M}_{1,1}})|_{\mathcal{V}_{\text{gst}}}.
\]
By (3.26), $\phi$ is locally expressed by
\[
\phi|_{\mathcal{U}} : 1 \rightarrow (w_1, \ldots , w_{n+m}, 0, \cdots , 0).
\]
Moreover, from the above local computation for the normal cone, we observe that $\phi$ is a birational morphism. Hence $C_{\text{gst}}$ is birational to the line bundle $g_{\text{gst}}^* N_{M_{\text{gst}}/\mathfrak{M}_{1,1}}$. We will use this to describe $\theta_{1,\text{loc}}(c_{\text{gst}})$ in the next section.

4. Calculations

4.1. Proof of the Theorem 1.1. Basically our proof follows contents in [29] Section 4, which proved a similar statement to our Theorem 1.1 in the case of stable map spaces.

Let $M := \mathcal{X}_{\text{gst}}$, and $\pi_M : \mathcal{C}_M \to M$ be the universal family. Let $\mathcal{L}_M$ be the universal bundle over $\mathcal{C}_M$, and $\mathcal{P}^i_M = \mathcal{L}_{\mathcal{M}^i_{\text{gst}}/\mathcal{M}} \otimes \omega_{\mathcal{C}_M/M}$. By definition the component $\mathcal{Y}_{\text{gst}}$ is the total space of a vector bundle $\mathcal{L}$ on $M$, where

$$\mathcal{L} = \oplus_{i=1}^m \pi_M \mathcal{P}^i_M.$$ 

Furthermore, let

$$\gamma : W := \mathcal{Y}_{\text{gst}} = \text{Tot}(\mathcal{L}) \to M$$

be the induced (tautological) projection. Here $\text{Tot}(-)$ denote the total space of the bundle. We denote the bundles

$$V'_1 = R^1 \pi_{M *} \mathcal{L}^{\oplus (n+1)}_M, \quad V'_2 = \oplus_{i=1}^m R^1 \pi_M \mathcal{P}^i_M,$$

and $V' = V'_1 \oplus V'_2$.

By [9] Proposition 2.8], we have $H^1(\mathcal{T}_{M/\mathfrak{M}_{1,1}}) \cong V'_1$. For any point $x = (C, p_1, \{u_i\}) \in M$, we define

$$\xi'_1 : (V'_1 \otimes \mathcal{L})|_x \to \mathbb{C}, \quad \xi'(x)(\hat{u_i} \otimes \chi) = \sum_{i=1}^m \chi_i \partial q(u) \hat{u}_i,$$

$$\xi'_2 : V'_2|_x \to \mathbb{C}, \quad \xi'(x)(\hat{u_i}) = \sum_{i=1}^m \chi_i \partial q(u), \quad \chi = (\chi_1, \cdots, \chi_m) \in \Gamma(M, \mathcal{L})$$

On the other hand, let $\gamma_W : \mathcal{C}_W \to \mathcal{Y}$ be the universal family, and $\mathcal{L}_W$ be the universal line bundle over $\mathcal{C}_W$. Denote $\pi_W : \mathcal{C}_W \to W$, $\mathcal{P}^i_W = \mathcal{P}^i_M|_W$ and $\mathcal{L}_W := \mathcal{L}_W|_{\mathcal{C}_W}$. Recall that the dual perfect obstruction theory of $\mathcal{Y}/\mathfrak{M}_{1,1}$ is $E_{\mathcal{Y}/\mathfrak{M}_{1,1}} = R^* \pi_{\mathcal{Y}*}(\mathcal{L}^{\oplus (n+1)}_W \oplus \oplus_{i=1}^m \mathcal{P}^i_W)$.

We let

$$\tilde{V}'_1 = H^1(R^* \pi_{W*} \mathcal{L}^{\oplus (n+1)}_{\mathcal{M}}) \cong \gamma^* V'_1, \quad \tilde{V}'_2 = H^1(\oplus_{i=1}^m R^* \pi_{W*} \mathcal{P}^i_W) \cong \gamma^* V'_2,$$

and $\tilde{V}' = \tilde{V}'_1 \oplus \tilde{V}'_2$. Both $\tilde{V}'_1$ and $\tilde{V}'_2$ are locally free on $W$.

Denote $\tilde{\xi}'_i = (\tilde{\xi}'_i, \tilde{\xi}'_2)$, where $\tilde{\xi}'_1 := \gamma^* (\xi'_1)(\cdot \otimes \epsilon), \epsilon \in \Gamma(W, \gamma^* \mathcal{L})$ is the tautological section and $\tilde{\xi}'_2 := \gamma^* (\xi'_2)$. Then we have

$$\tilde{\xi}' = \sigma'|_{\mathcal{Y}_{\text{gst}}} : \tilde{V}' \to \mathcal{O}_W,$$

where $\sigma'$ is the cosection defined in (3.2). Next we consider the obstruction theory over the Artin stack $\mathfrak{M}_{1,1}$. Moreover we denote

$$V'_1 = R^1 \pi_{E*} f^*_E T_p \cong H^1(E'_{\mathcal{X}/\mathfrak{M}_{1,1}}|_{\mathcal{Y}_{\text{gst}}}), \quad V'_2 = \oplus_{i=1}^m R^1 \pi_M \mathcal{P}^i_M,$$

and $V = V'_1 \oplus V'_2$ where $\pi : \mathcal{C} \to M$ is the universal curve, $\mathcal{C}_E \subset \mathcal{C}$ is the universal family of minimal genus 1 subcurves, $\pi_E : \mathcal{C}_E \to M$ is the projection morphism, and $f_E : \mathcal{C}_E \to \mathcal{M}$ is the universal morphism. They are vector bundles (locally free sheaves) on $M$ (c.f. Proposition 3.4). Let $\tilde{V}_1 := \gamma^* V_1$ and $\tilde{V}_2 := \gamma^* V_2$. Then $H^1(E'_{\mathcal{Y}/\mathfrak{M}_{1,1}}|_{\mathcal{Y}_{\text{gst}}}) = \tilde{V} := \tilde{V}_1 \oplus \tilde{V}_2$. Then the cosection $\tilde{\xi}' = (\tilde{\xi}'_1, \tilde{\xi}'_2)$ induces the cosection $\xi' = (\tilde{\xi}'_1, \tilde{\xi}'_2) : V = \tilde{V}_1 \oplus \tilde{V}_2 \to \mathcal{O}_W$.

Following [9] Proposition 3.2], the non-surjective locus $D(\xi)$ of $\xi = \sigma'|_{\mathcal{Y}_{\text{gst}}}$ is

$$D(\sigma) \times_X M = Q_{1,1}(X, d) \times_{Q_{1,1}(\mathbb{P}^n, d)} M,$$

which is proper. Let

$$\tilde{\nu}_1 = h^1/h^0(E'_{\mathcal{Y}/\mathfrak{M}_{1,1}}|_{\mathcal{Y}_{\text{gst}}}), \quad \tilde{\nu}_2 = h^1/h^0(\oplus_{i=1}^m R^\sigma \pi_{W*} \mathcal{P}^i_W), \quad \tilde{\nu} = \tilde{\nu}_1 \times_W \tilde{\nu}_2.$$

be the vector bundle stacks. Then there is a canonical morphism $\rho_j : \tilde{V}_j \to \tilde{V}_j$ from the bundle stack to its coarse moduli space, for $j = 1, 2$. Note that both $\rho_j$ are proper morphisms.

By the base change property of the $h^1/h^0$-construction, and by the definition of $\mathcal{C}_{gst}$, we have

$$[\mathcal{C}_{gst}] \in Z_{*}\tilde{V}; \quad \tilde{V} = h^1/h^0(E_{\gamma}^{\gamma}/\mathcal{O}_{V})|_{V}.$$  

Let $C_{gst}$ be the coarse moduli of $\mathcal{C}_{gst}$ relative to $V$, thus $C_{gst} \subset \tilde{V}$ since $\tilde{V}$ is the coarse moduli of $V$. Further, since the projection $\rho := \rho_1 \times \rho_2 : \tilde{V} \to \tilde{V}$ is smooth, we have an identity of cycles $\rho^{*}[C_{gst}] = [\mathcal{C}_{gst}] \in Z_{*}\tilde{V}$. Finally, because $[\mathcal{C}_{gst}] \in Z_{*}\tilde{V}(\sigma)$, we have

$$[C_{gst}] \in Z_{*}\tilde{V}(\xi).$$

Therefore we have the following identity.

**Proposition 4.1.** [3 Proposition 6.3]

$$0^{1}_{loc}[\mathcal{C}_{gst}] = 0^{1}_{loc}[C_{gst}] \in A_{*}D(\xi).$$

Now we calculate the cycle $0^{1}_{loc}[C_{gst}]$. We first introduce the following notations.

- $\nabla := \mathbb{P}(L \oplus \mathcal{O}_{M})$ be a completion of $W = Y_{gst}$,
- $\gamma : \nabla \to \mathcal{X}_{gst}$ be the projection morphisms,
- $\gamma : \tilde{V} := \mathcal{V}_{1}(-D_{\infty}), \mathcal{V}_{2} := \mathcal{V}_{1} \oplus \mathcal{V}_{2}$,
- $\xi_{1} : \mathcal{V}_{1} \to \mathcal{O}_{\nabla}$ and $\xi_{2} : \mathcal{V}_{2} \to \mathcal{O}_{\nabla}$ are cosections induced from $\xi_{1}$ and $\xi_{2}$ respectively,
- $\xi := \xi_{1} + \xi_{2}$.

To calculate $0^{1}_{loc}[C_{gst}]$, we approximate the cone $C_{gst}$ as a subvector bundle of $\nabla_{1}$. To do this, we consider $R := C_{gst}/C_{gst}$ where $C_{gst} := C_{gst} \cap \text{Tot}(0 \oplus \mathcal{V}_{2})$. It is a deformation of $C_{gst}$.

We can easily check that $R$ is embedded in $\text{Tot}(\tilde{V})$ and $[C_{gst}] = [R]$ in $A_{*}(\tilde{V}(\xi))$. Next we investigate the cone $R$ and its completion $\bar{R}$ in $\text{Tot}(\tilde{V})$. Similar to [29 (4.5)], by using a local computation we can check

$$C_{gst} \subset 0_{gst} \cup \gamma^{*}F$$

where $0_{gst} \subset Y_{gst} = \text{Tot}(L)$ is the zero section of the bundle $L$, $\Delta_{X} := \mathcal{X}_{gst} \cap \mathcal{X}_{red}$ and $F$ is a rank $m$ subbundle of $V_{2}|_{\Delta_{X}}$ defined in the below.

Recall the quasi-isomorphism

$$\oplus_{i=1}^{m} \pi_{X, \ast} \mathcal{P}_{\mathcal{X}}^{i} \xrightarrow{\text{loc}} \bigg( \big[ \mathcal{O}_{\mathcal{X}} \xrightarrow{x_{i}} \mathcal{O}_{X} \big] \oplus \big[ 0 \to \mathcal{O}_{\mathcal{X}}^{\text{deg} q_{i}} \big] \bigg),$$

we observe that $H^{1}\left( \oplus_{i=1}^{m} \pi_{X, \ast} \mathcal{P}_{\mathcal{X}}^{i} \right)_{\text{tor}}|_{Y}$ is a rank $m$ subbundle of $V_{2}$. Then we define $F := H^{1}\left( \oplus_{i=1}^{m} \pi_{X, \ast} \mathcal{P}_{\mathcal{X}}^{i} \right)_{\text{tor}}|_{\Delta_{X}} \subset V_{2}|_{\Delta_{X}}$.

Since $R$ is a cone over $C_{gst} \subset \text{Tot}(\tilde{V}_{2})$, we can write

$$[R] = [R_{1}] + [R_{2}] \in A_{*}\left( \text{Tot}(\tilde{V}) \right)$$

where $R_{1} := R|_{0_{gst}}$ and $[R_{2}]$ is a cycle supported on $\text{Tot}(\gamma^{*}F)$. Parallel to [3 Lemma 8.1] and [28 p. 24], we can check that

$$0^{1}_{\xi, loc}[R_{2}] = 0$$

since $\text{dim} \text{Tot}(\gamma^{*}F)$ is smaller than the degree of $[R_{2}] \in A_{*}(\text{Tot}(\tilde{V}))$.

Hence we have

$$0^{1}_{\xi, loc}[C_{gst}] = 0^{1}_{\xi, loc}[R] = 0^{1}_{\xi, loc}[R_{1}]$$

(4.6)
Moreover, by [28, Proposition 5.3], we have
\begin{equation}
0^1_{\xi,\text{loc}}[R_1] = \tilde{\gamma}_*0^1_{\xi_2,\text{loc}} \cdot 0_{V_1}[\overline{R_1}]
\end{equation}
where $\overline{R_1}$ is the closure of $R_1$ in $\text{Tot}(\overline{V})$.

Next we investigate the cone $R_1$. Using a local computation of $R_1$ similar to [29, (4.8), (4.9)], we conclude that $R_1$ is of the form $R_1 = \gamma^*R_1$. Here $R_1^\prime$ is given as the closure of the image of the natural composition morphism
\begin{equation}
\varphi : \overline{\mathfrak{g}}_{\text{Yost}}^* N_{\text{Yost}/\mathbb{P}^1} \cong C_{\mathbb{P}^1} \rightarrow C_{\mathbb{P}^1}/\mathbb{P}^1 |_{\text{Yost}} \rightarrow V.
\end{equation}

Similar to the local description of $\varphi$ in (3.30), we can locally describe $\varphi$ as follows:
\begin{equation}
\varphi|_{\mathcal{U}'} : \{w_1, \ldots, w_n,0,\ldots,0\}
\end{equation}
over some sufficiently small neighbourhood $\mathcal{U}' \subset X$. From this, we observe the degeneracy locus of the morphism $\varphi$ is $\Delta_X$. To resolve this, we consider the blow-up
\begin{equation}
\hat{M} := \text{Bl}_{\Delta_X} M, \quad p : \hat{M} \rightarrow M.
\end{equation}

Let $E$ be the exceptional divisor. Then there is an induced morphism
\begin{equation}
\hat{\varphi} : \left(p^*\overline{\mathfrak{g}}_{\text{Yost}}^* N_{\text{Yost}/\mathbb{P}^1} \right)(E) \rightarrow p^*V
\end{equation}
which is an injective morphism of vector bundles. Thus its image $\text{Im}(\hat{\varphi})$ is a line subbundle of $p^*V$. Then we have
\begin{equation}
p(\text{Tot}(\text{Im}(\hat{\varphi}))) = R_1'.
\end{equation}

There is the following induced morphism
\begin{equation}
\hat{\varphi} : \left(\hat{\gamma}^* \left(p^*\overline{\mathfrak{g}}_{\text{Yost}}^* N_{\text{Yost}/\mathbb{P}^1} \right)(E) \right)(q^*D_\infty) \rightarrow q^*V
\end{equation}
where $q : \hat{W} := \overline{W} \times_M \hat{M} \rightarrow \overline{W}$ is the projection, $\hat{\gamma} : \hat{W} \rightarrow \hat{M}$ is the projection. Note that $\hat{\varphi}$ is an injective morphism of vector bundles. We have
\begin{equation}
q(\text{Tot}(\text{Im}(\hat{\varphi}))) = \overline{R_1}.
\end{equation}

Then we obtain
\begin{equation}
0^1_{V_1}[\overline{R_1}] = q_*0^1_{\hat{\varphi}}[\text{Tot}(\text{Im}(\hat{\varphi}))] = q_* \left(c_{\text{top}}(q^*V/\text{Im}(\hat{\varphi})) \cap [\hat{M}] \right).
\end{equation}

Hence, by combining the above computation with (4.6) and (4.7), we have
\begin{equation}
0^1_{\xi,\text{loc}}[C_{\text{Yost}}] = \tilde{\gamma}_*0^1_{\xi_2,\text{loc}} \cdot 0_{V_1}[\overline{R_1}] = 0^1_{\xi_2,\text{loc}} \left(\tilde{\gamma}_*0^1_{V_1}[\overline{R_1}]\right) = 0^1_{\xi_2,\text{loc}} \left(\tilde{\gamma}_* q_* \left(c_{\text{top}}(q^*V/\text{Im}(\hat{\varphi})) \cap [\hat{M}] \right)\right)
\end{equation}
where the second equality comes form the functorial property of localized Gysin homomorphisms [25].

By using [29, Lemma 4.2] and [29, (4.13)], we have
\begin{equation}
\tilde{\gamma}_*q_* \left(c_{\text{top}}(q^*V/\text{Im}(\hat{\varphi})) \cap [\hat{M}] \right) = \left(\frac{c(V_1)s(L')} {c(\overline{\mathfrak{g}}_{\text{Yost}}^* N_{\text{Yost}/\mathbb{P}^1})}\right) _{\text{rank} V_1-m-1}.\n\end{equation}

Therefore we have
\begin{equation}
0^1_{\xi,\text{loc}}[C_{\text{Yost}}] = \left(\frac{c(V_1)s(L')} {c(\overline{\mathfrak{g}}_{\text{Yost}}^* N_{\text{Yost}/\mathbb{P}^1})}\right) _{\text{rank} V_1-m-1} \cap 0^1_{\xi_2,\text{loc}}[M].
\end{equation}

Note that $M$ is considered as a substack of $\text{Tot}(V_2)$ embedded by the zero section.

Next, consider the node-identifying morphism
\begin{equation}
\iota : \overline{M}_{1,1} \times Q_{0,2}(\mathbb{P}^n, d) \rightarrow Q_{1,1}(\mathbb{P}^n, d) = X
\end{equation}

Let $\mathcal{H}$ be the Hodge bundle over $\overline{M}_{1,1}$, $L_1$ be the universal tangent bundle over $\overline{M}_{1,1}$ at the marked point, $L_2$ be the universal tangent bundle over $Q_{0,2}(\mathbb{P}^n, d)^p$ at the second

\footnote{Caution: $\varphi$ is similarly defined as $\hat{\varphi} : C_{\text{Yost}/\mathbb{P}^1}$. But it is slightly different.}
marked point, which comes from splitting of the node. We have \( H^\vee \cong L_1 \). Moreover we have

\[
\begin{align*}
\circ \ i^* V_1 & \cong H \otimes ev_2^* T_{\mathbb{P}^n}, \\
\circ \ i^* C^\vee & \cong H \otimes (\oplus_1 ev_2^* O_{\mathbb{P}^n}(\deg q_i)), \\
\circ \ i^* g_{\text{gst}}^* N_{\text{gst}}/_{/\text{gst}} & \cong H^\vee \otimes L_2, \\
\circ \ i^{-1}[M] & = \overline{M}_{1,1} \times Q_{0,2}(\mathbb{P}^n, d), \\
\circ \ \partial_1^* [\xi_{\text{loc}}(\overline{M}_{1,1}) \times [Q_{0,2}(\mathbb{P}^n, d)]) & = \overline{M}_{1,1} \times [Q_{0,2}(X, d)]^\text{vir}.
\end{align*}
\]

Thus we have

\[
0_1^* [C_{\text{gst}}] = \left( \frac{c(V_1) s(C^\vee)}{c(g_{\text{gst}}^* N_{\text{gst}}/_{/\text{gst}})} \right) \cap 0_1^* [M] \oplus \text{rank} _{Y_1-m-1}^n
\]

\[
= (-1)^{(\sum_1 \deg q_i) d} s_{\ast} \left( \frac{c(H^\vee \otimes ev_2^* T_{\mathbb{P}^n}) s(H^\vee \otimes ev_2^* (\oplus_1 O_{\mathbb{P}^n}(\deg q_i)))}{c(H^\vee \otimes L_2)} \right) \cap \text{rank} _{Y_1-m-1}^n
\]

\[
(\overline{M}_{1,1} \times [Q_{0,2}(X, d)])^\text{vir}.
\]

where the last identity comes from the short exact sequence \( 0 \to T_X \to T_{\mathbb{P}^n}|_X \to \oplus_1 O_{\mathbb{P}^n}(\deg q_i)|_X \to 0 \). Let us define

\[
A_{1,d}^{\red} := (-1)^{(\sum_1 \deg q_i) d} 0_1^* [c_\text{red}],
\]

We will call it the virtual cycle for reduced quasi-map invariants. We set

\[
N_{\text{red}} := \pi_* (\oplus_1 \mathbb{L}^q_\nu_{\ast} |_{\mathbb{P}}^\text{reg}), \quad \pi_* (\oplus_1 \mathbb{L}^q_\nu_{\ast} |_{\mathbb{P}}^\text{reg})
\]

for the universal curve \( \mathbb{P} := \pi|_{\mathbb{P}}: \mathbb{P}|_{\mathbb{P}} \to Y_{\text{red}} \). Then by Theorem 2.7, \( N_{\text{red}} \) is a vector bundle.

In the same manner as in [29, Section 4.3] we can show that

\[
(4.8) \quad A_{1,d}^{\red} = (-1)^{(\sum_1 \deg q_i) d} 0_1^* [Y_{\text{red}}] \in \mathcal{A}_s(Q_{1,1}(X, d)).
\]

where \( s \) is the natural section \( s: \mathcal{O}_{\mathbb{P}^n} \to N_{\text{red}} \) which is induced from the defining equations \( q_1, \ldots, q_m \) of \( X \subset \mathbb{P}^n \). Let \( e_{\text{ref}}(N_{\text{red}}) \) be the refined euler class localized by the section \( s \). Note that we have

\[
0_1^* [Y_{\text{red}}] = (-1)^{\text{rank}(N_{\text{red}})} e_{\text{ref}}(N_{\text{red}})[Y_{\text{red}}] = (-1)^{(\sum_1 \deg q_i) d} e_{\text{ref}}(N_{\text{red}})[Y_{\text{red}}]
\]

By the proof in [3, Section 5], we have

\[
(4.9) \quad A_{1,d}^{\red} = (-1)^{(\sum_1 \deg q_i) d} 0_1^* [Y_{\text{red}}] + e_{\text{ref}}(\oplus_1 \mathbb{L}^q_{\nu, \ast}) \cap \mathcal{X}_{\text{red}}.
\]

In summary, we obtain the following

\[
(4.10) \quad [Q_{1,1}(X, d)]^\text{vir}
\]

\[
= (-1)^{(\sum_1 \deg q_i) d} [Y]^\text{vir}
\]

\[
= (-1)^{(\sum_1 \deg q_i) d} \left( 0_{\ast, \text{loc}}[c_{\text{pri}}] + 0_1^* [C_{\text{gst}}] \right)
\]

\[
= A_{1,d}^{\red} + i_* \left( \frac{c(H^\vee \otimes ev_2^* T_X)}{c(H^\vee \otimes L_2)} \right) \cap (\overline{M}_{1,1} \times [Q_{0,2}(X, d)]^\text{vir}
\]

where \( i: \overline{M}_{1,1} \times Q_{0,2}(X, d) \to Q_{1,1}(X, d) \) is the node-identifying morphism. It proves the main Theorem 1.1.
4.2. Proof of the Corollary 1.3. Let $X \subset \mathbb{P}^n$ be a complete intersection with dimension $n-m$, then the virtual dimension

\[ \text{vdim } Q_{g,k}(X,d) = \int_{d[\mathbb{P}^1]} c_1(T_X) + (1-g)(n-m-3)+k. \]

Let $p_1: \overline{M}_{1,1} \times Q_{0,2}(X,d) \to \overline{M}_{1,1}$ and $p_2: \overline{M}_{1,1} \times Q_{0,2}(X,d) \to Q_{0,2}(X,d)$ be the two projections.

\[ c(\mathcal{H}^\vee \boxtimes ev_2^* T_X) = \sum_{i=0}^{n-m} c_1(\mathcal{H}^\vee)^{r-i} p_2^* c_i(e_2^* T_X) \]
\[ = 1 + p_1^* c_1(\mathcal{H}^\vee) \left( \sum_{i=0}^{n-m-1} p_2^* c_i(e_2^* T_X) \right) + \cdots , \]

where \( \cdots \) are the terms such that they contain factor of \( c_1(\mathcal{H}^\vee) \) with \( i > 1 \) and

\[ \frac{1}{c(\mathcal{H}^\vee \boxtimes L_2)} = 1 + \sum_{i \geq 1} (-1)^i (p_1^* c_1(\mathcal{H}^\vee) + p_2^* c_1(L_2)) \]
\[ = 1 + \sum_{i \geq 1} (-1)^i \left( \frac{i}{i} \right)^{\binom{n-m-1}{i}} p_1^* c_1(\mathcal{H}^\vee) p_2^* c_1(L_2)^{i-1} + \cdots . \]

\[ \frac{c(\mathcal{H}^\vee \boxtimes ev_2^* T_X)}{c(\mathcal{H}^\vee \boxtimes L_2)} = \left( 1 + p_1^* c_1(\mathcal{H}^\vee) \left( \sum_{i=0}^{n-m-1} p_2^* c_i(e_2^* T_X) \right) + \cdots \right) \]
\[ \times \left( 1 + \sum_{i \geq 1} (-1)^i \left( \frac{i}{i} \right)^{\binom{n-m-1}{i}} p_1^* c_1(\mathcal{H}^\vee) p_2^* c_1(L_2)^{i-1} + \cdots \right) \]
\[ = 1 + p_1^* c_1(\mathcal{H}^\vee) \left( \sum_{i=0}^{n-m-1} p_2^* c_i(e_2^* T_X) + \sum_{i \geq 1} (-1)^i \left( \frac{i}{i} \right)^{\binom{n-m-1}{i}} p_2^* c_1(L_2)^{i-1} \right) + \cdots . \]

Let \( \psi_i \) be the psi class, which is the first Chern class of the universal cotangent line bundle for the \( i \)-th marking. Let \( \gamma \in H^{2k}(X, \mathbb{Q}) \) be a cohomology class such that \( k \leq 1 \), and let \( a \) be an integer satisfies \( a + k = \text{vdim } Q_{1,1}(X,d) \). By formula (1.10), we have the following formula for stable quasimap invariants

\[ \langle \psi^a ev^* \gamma \rangle_{1,1,d} = \int_{\overline{M}_{1,1}} \psi^a ev^* \gamma + \int_{\overline{M}_{1,1}} \psi^a ev^* \gamma \]
\[ = \langle \psi^a ev^* \gamma \rangle_{1,1,d} + \int_{\overline{M}_{1,1}} c_1(\mathcal{H}^\vee) \left( \int_{Q_{0,2}(X,d)} \psi^a ev^* \gamma c_{n-m-2}(e_2^* T_X) \right) \]
\[ + (-1)^{n-m-1} \binom{n-m-1}{1} \int_{Q_{0,2}(X,d)} \psi^a ev^* \gamma c_1(L_2)^{n-m-2} \]
\[ = \langle \psi^a ev^* \gamma \rangle_{1,1,d} - \frac{1}{24} \left( \int_{Q_{0,2}(X,d)} \psi^a ev^* \gamma c_{n-m-2}(e_2^* T_X) \right) \]
\[ - (n-m-1) \int_{Q_{0,2}(X,d)^{\rm{vir}}} \psi^a ev^* \gamma \psi_2^{n-m-2} \].

Here we denoted \( c_1(\mathcal{H}^\vee) = \psi \). If \( X \) is a Calabi-Yau threefold, then \( c_1(T_X) = 0 \), and \( n-m = 3 \). So we obtain

\[ \langle \psi^a ev^* \gamma \rangle_{1,1,d} = \langle \psi^a ev^* \gamma \rangle_{1,1,d} + \frac{1}{12} \int_{Q_{0,2}(X,d)^{\rm{vir}}} \psi^a ev^* \gamma \psi_2. \]
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