INTEGRAL OPERATORS IN GRAND MORREY SPACES

Alexander Meskhi

Abstract. We introduce grand Morrey spaces and establish the boundedness of Hardy–Littlewood maximal, Calderón–Zygmund and potential operators in these spaces. In our case the operators and grand Morrey spaces are defined on quasi-metric measure spaces with doubling measure. The results are new even for Euclidean spaces.

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Introduction

In the paper we introduce the grand Morrey spaces $L^{p,\theta,\lambda}$ and derive the boundedness of a class of integral operators (Hardy–Littlewood maximal functions, Calderón–Zygmund singular integrals and potentials) in these spaces. We study the boundedness problem in the frame of quasi-metric measure spaces with doubling measure but the results are new even for Euclidean spaces.

The classical grand Lebesgue spaces $L^{p,\theta}$ were introduced in the paper by T. Iwaniec and C. Sbordone [19] when they studied the problem of the integrability of the Jacobian $J(f, x)$ of the order preserving mapping $f = (f_1, \cdots, f_n) : \Omega \rightarrow \mathbb{R}^n$ under minimal hypothesis, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $n \geq 2$.

Later the generalized grand Lebesgue spaces $L^{p,\theta}$ appeared in the paper by L. Greco, T. Iwaniec and C. Sbordone [17], where the existence and uniqueness of the nonhomogeneous $n$-harmonic equation $\text{div} A(x, \nabla u) = \mu$ were established.

Structural properties of these spaces were investigated in the papers [11], [13], [4] etc.

A. Fiorenza, B. Gupta and P. Jain [12] proved the boundedness of the Hardy–Littlewood maximal operator defined on an interval in weighted $L^{p,\theta}$ space, while the same problem for the Hilbert transform and other singular integrals were studied in the papers [22], [20].

The Morrey spaces $L^{p,\lambda}$, which were introduced by C. Morrey in 1938 (see [26]) in order to study regularity questions which appear in the calculus of variations, describe local regularity more precisely than Lebesgue spaces and widely use not only harmonic analysis but also partial differential equations (c.f. [15], [16]).

For essential properties of $L^{p,\lambda}$ spaces and the boundedness of maximal, fractional and singular operators in these spaces we refer to the papers [1], [27], [3], [7], [28], [15], etc.

Finally we mention that necessary and sufficient conditions for the boundedness of maximal operators and Riesz potentials in the local Morrey–type spaces were derived in [2], [3].
1 Preliminaries

Let $X := (X, \rho, \mu)$ be a topological space with a complete measure $\mu$ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a non-negative real-valued function (quasi-metric) $d$ on $X \times X$ satisfying the conditions:

(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) there exists a constant $a_1 > 0$, such that $\rho(x, y) \leq a_1(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$;
(iii) there exists a constant $a_0 > 0$, such that $\rho(x, y) \leq a_0 \rho(y, x)$ for all $x, y, \in X$.

We assume that the balls $B(x, r) := \{y \in X : \rho(x, y) < r\}$ are measurable and $0 \leq \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$; for every neighborhood $V$ of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. Throughout the paper we also suppose that $\mu\{x\} = 0$ and that

$$B(x, R) \setminus B(x, r) \neq \emptyset$$

for all $x \in X$, positive $r$ and $R$ with $0 < r < R < d$, where

$$d := \text{diam} (X) = \sup \{\rho(x, y) : x, y \in X\}.$$

Throughout the paper we suppose that $d < \infty$ and that the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r))$$

for $\mu$ is satisfied, where the positive constant $c$ does not depend on $x \in X$ and $r > 0$. In this case the triple $(X, d, \mu)$ is called a space of homogeneous type (SHT). For the definition, examples and some properties of an SHT see, e.g., monographs [29], [6].

A quasi-metric measure space, where the doubling condition is not assumed is called a non-homogeneous space.

Notice that the condition $d < \infty$ implies that $\mu(X) < \infty$ because every ball in $X$ has a finite measure.

We say that the measure $\mu$ is upper Ahlfors $Q$-regular if there is a positive constant $c_1$ such that $\mu(B(x, r)) \leq c_1 r^Q$ for for all $x \in X$ and $r > 0$. Further, $\mu$ is lower Ahlfors $q$-regular if there is a positive constant $c_2$ such that $\mu(B(x, r)) \geq c_2 r^q$ for for all $x \in X$ and $r > 0$.

Let $1 < p < \infty$, $\theta > 0$ and $0 \leq \lambda < 1$. We denote by $L^{p, \theta, \lambda}(X, \mu)$ the class of those $f : X \to \mathbb{R}$ for which the norm

$$\|f\|_{L^{p, \theta, \lambda}(X, \mu)} = \sup_{0 < \varepsilon < p - 1} \sup_{x \in X} \sup_{0 < r < d} \left[ \int_{B(x, r)} |f(y)|^{p - \varepsilon} d\mu(y) \right]^{\frac{1}{p - \varepsilon}}$$

is finite.

If $\lambda = 0$, then $L^{p, \theta, \lambda}(X, \mu)$ is the grand lebesgue space defined on $X$ and denoted by $L^{p, \theta}(X, \mu)$. Further, if $\theta = 1$, then we use the symbol $L^{p, \lambda}(X, \mu)$ instead of $L^{p, \theta, \lambda}(X, \mu)$.

Using Hölder’s inequality it is easy to see that the following embeddings hold for $L^{p, \theta}$ spaces (see also [12], [17]):

$$L^{p, \theta_1}_w(X, \mu) \subset L^{p, \theta_2}_w(X, \mu) \subset L^{p, \lambda}_w(X, \mu) \subset L^{p - \varepsilon}_w(X, \mu),$$

where $0 < \varepsilon < p - 1$ and $\theta_1 < \theta_2$.

The classical Morrey space, denoted by $L^{p, \lambda}(X, \mu)$, is defined by the norm

$$\|f\|_{L^{p, \lambda}(X, \mu)} = \sup_{x \in X} \sup_{0 < r < d} \left[ \frac{1}{(\mu(B(x, r)))^\lambda} \int_{B(x, r)} |f(y)|^p d\mu(y) \right]^{\frac{1}{p}}.$$
Finally we mention that constants (often different constants in the same series of inequalities) will generally be denoted by $c$ or $C$. By the symbol $p'$ we denote the conjugate number of $p$, i.e. $p' := \frac{p}{p-1}$, $1 < p < \infty$.

# 2 Maximal Operator in Grand Morrey Spaces

In this section we prove the boundedness of the Hardy–Littlewood maximal operator

$$(Mf)(x) = \sup_{x \in X} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|d\mu(y), \ x \in X,$$

in $L^{p,\theta,\lambda}(X,\mu)$.

Our main theorem in this section is the following statement:

**Theorem 2.1.** Let $1 < p < \infty$, $\theta > 0$ and let $0 \leq \lambda < 1$. Suppose that $d < \infty$. Then the Hardy-Littlewood maximal operator $M$ is bounded in $L^{p,\theta,\lambda}(X,\mu)$.

To prove Theorem 2.1 we need some auxiliary statements.

**Proposition 2.1.** Let $1 < p < \infty$. Then there is a positive constant $c_0$ non-depending on $p$ such that

$$\|Mf\|_{L^{p}(X,\mu)} \leq c_0 (p')^{\frac{1}{p}} \|f\|_{L^{p}(X,\mu)}$$

(2.1)

**Proof.** The proof follows directly from the Marcinkiewicz interpolation theorem. The constant $c_0$ arises from the appropriate covering lemma (see, e. g., [8], p. 29).

Let us denote by $L^{p,\lambda}(X,\mu)$ the classical Morrey space, where $1 < p < \infty$ and $0 \leq \lambda < 1$, which is the class of all $\mu$-measurable functions $f$ for which the norm

$$\|f\|_{L^{p,\lambda}(X,\mu)} = \sup_{x \in X} \left[ \frac{1}{(\mu B(x,r))^\lambda} \int_{B(x,r)} |f(y)|^p d\mu(y) \right]^{\frac{1}{p}}$$

is finite. If $\lambda = 0$, then $L^{p,\lambda}(X,\mu) = L^p(X,\mu)$.

**Proposition 2.2.** Let $1 < p < \infty$ and let $0 \leq \lambda < 1$. Then

$$\|Mf\|_{L^{p,\lambda}(X,\mu)} \leq \left( \frac{1}{(\mu B(x,r))^\lambda} \int_{B(x,r)} (Mf)^p d\mu(y) \right)^{\frac{1}{p}}$$

holds, where the positive constant $b$ arises in the doubling condition for $\mu$ and $c_0$ is the constant from (2.1).

**Proof.** Let $r$ be a small positive number and let us represent $f$ as follows:

$$f = f_1 + f_2,$$

where $f_1 = f \cdot \chi_{B(x,\pi r)}$, $f_2 = f - f_1$ and $a$ is the positive constant given by $\overline{a} = a_1(a_1(a_0 + 1) + 1)$ (here $a_0$ and $a_1$ are constants arisen in the triangle inequality for the quasi-metric $\rho$).

We have

$$\left[ \frac{1}{(\mu B(x,r))^\lambda} \int_{B(x,r)} (Mf)^p d\mu(y) \right]^{\frac{1}{p}} \leq \left( \frac{1}{(\mu B(x,r))^\lambda} \int_{B(x,r)} (Mf_1)^p d\mu(y) \right)^{\frac{1}{p}}$$
\[
\left( \frac{1}{(\mu_B(x,r))^\lambda} \int_{B(x,r)} (Mf_2)^p(y) d\mu(y) \right)^{1/p} + J_1(x,r) + J_2(x,r).
\]

By applying Proposition 2.1 we have that
\[
J_1(x,r) \leq \frac{1}{(\mu_B(x,r))^{\lambda/p}} \left( \int_X (Mf_1(y))^p d\mu(y) \right)^{1/p} \leq c_0 (p')^{\frac{1}{p}} \mu_B(x,r)^{-\lambda/p} \| f \|_{L^{p,\lambda}(X,\mu)},
\]
where \( c_0 \) is the constant from (2.1) and \( b \) arises from the inequality
\[
\mu_B(x,ar) \leq b\mu_B(x,r)
\]
which is a consequence of the doubling condition. Further, observe that (see also [23], p. 23) if \( y \in B(x,r) \), then \( B(x,r) \subset B(y,a_1(a_0+1)r) \subset B(x,\overline{a},r) \). Hence, if \( y \in B(x,r) \), then
\[
Mf_2(y) \leq \sup_{B(x,r) \subset B} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).
\]
Consequently,
\[
J_2(x,r) \leq \mu(B(x,r))^{1/p} \sup_{B(x,r) \subset B} \left( \frac{1}{\mu(B)} \int_B |f(y)|^p d\mu(y) \right)^{1/p} \leq \sup_B (\mu B)^{-\lambda/p} \left( \int_B |f(y)|^p d\mu(y) \right)^{1/p} = \| f \|_{L^{p,\lambda}(X,\mu)}.
\]
Taking into account the estimates for \( J_1(x,r) \) and \( J_2(x,r) \) we conclude that
\[
\left( \frac{1}{\mu(B(x,r))^{\lambda}} \int_{B(x,r)} (Mf(y))^p d\mu(y) \right)^{1/p} \leq (c_0 b^{\lambda/p} (p')^{1/p} + 1) \| f \|_{L^{p,\lambda}(X,\mu)}.
\]

\textbf{Proof of Theorem 2.1.} It is obvious that
\[
\| M f \|_{L^{p,\theta,\lambda}(X,\mu)} = \max \left\{ \sup_{0 < \epsilon \leq \sigma} \sup_{x \in X} \left( \frac{\epsilon^\theta}{(\mu_B(x,r))^\lambda} \int_{B(x,r)} (Mf(y))^{p-\epsilon} d\mu(y) \right)^{\frac{1}{p-\epsilon}} \right\};
\]

\[
= \max \left\{ \sup_{\sigma < \epsilon < p-1} \sup_{x \in X} \left( \frac{\epsilon^\theta}{(\mu_B(x,r))^\lambda} \int_{B(x,r)} (Mf(y))^{p-\epsilon} d\mu(y) \right)^{\frac{1}{p-\epsilon}} \right\} =: \max \{ A_1, A_2 \}.
\]
We begin to estimate $A_2$. Using the facts that $\sup_{\sigma \leq \varepsilon < p-1} \varepsilon^{\frac{1}{p-\varepsilon}} = p - 1$, $\frac{1}{p-\varepsilon} > \frac{1}{p-\sigma}$ (when $\sigma < \varepsilon < p - 1$) and Hölder’s inequality we have that

$$A_2 = \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| Mf \|_{L^{p-\varepsilon,\lambda}(X,\mu)}$$

$$= \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left( \sup_{x \in X} \left( \frac{1}{\mu B(x,r)} \int_{B(x,r)} (Mf(y))^{p-\varepsilon} d\mu(y) \right) \right)^{\frac{1}{p-\varepsilon}}$$

$$\leq \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left( \sup_{x \in X} \left( \frac{1}{\mu B(x,r)} \int_{B(x,r)} (Mf(y))^{p-\sigma} d\mu(y) \right) \right)^{\frac{1}{p-\sigma}}$$

$$\leq (p-1)^{\theta} \sup_{x \in X} \left( \frac{1}{\mu B(x,r)} \int_{B(x,r)} (Mf(y))^{p-\sigma} d\mu(y) \right)^{\frac{1}{p-\sigma}}$$

$$= (p-1)^{\theta} \sup_{x \in X} \left( \frac{1}{\mu B(x,r)} \int_{B(x,r)} (Mf(y))^{p-\sigma} d\mu(y) \right)^{\frac{1}{p-\sigma}}$$

Hence, by using Proposition 2.2 we find that

$$\| Mf \|_{L^{p,\lambda}(X,\mu)} \leq p \sigma^{\frac{\theta}{p-\sigma}} \sup_{0 < \varepsilon \leq \sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| Mf \|_{L^{p-\varepsilon,\lambda}(X,\mu)}$$

$$\leq c_0 p \cdot \sigma^{\frac{\theta}{p-\sigma}} \left( \sup_{0 < \varepsilon \leq \sigma} b_{\varepsilon}^{\lambda} \left( \frac{1}{p-\varepsilon} + 1 \right) \right) \varepsilon^{\frac{\theta}{p-\varepsilon}} \| f \|_{L^{p-\varepsilon,\lambda}(X,\mu)}$$

$$\leq c_0 p \cdot \sigma^{\frac{\theta}{p-\sigma}} \left( \sup_{0 < \varepsilon \leq \sigma} b_{\varepsilon}^{\lambda} \left( \frac{1}{p-\varepsilon} + 1 \right) \right) \| f \|_{L^{p,\lambda}(X,\mu)}$$

Since $\sigma$ is sufficiently small, we have that the expression

$$S_{p,\sigma} := c_0 p \sigma^{\frac{\theta}{p-\sigma}} \sup_{0 < \varepsilon \leq \sigma} b_{\varepsilon}^{\lambda} \left[ (p - \varepsilon) \frac{1}{p-\varepsilon} + 1 \right]$$

is finite.

In fact,

$$S_{p,\sigma} \leq c_0 p \sigma^{\frac{\theta}{p-\sigma}} b_{\varepsilon}^{\lambda} \left[ (p - \sigma) + 1 \right].$$
Finally,

\[ \|Mf\|_{L^p,\theta,\lambda(X,\mu)} \leq \left( \inf_{0<\sigma<p-1} S_{p,\sigma} \right) \|f\|_{L^p,\theta,\lambda(X,\mu)}. \]

\[ \square \]

3 Calderón-Zygmund Operators in Grand Morrey Spaces

Let

\[ Tf(x) = \text{p.v.} \int_X k(x,y)f(y)d\mu(y), \]

where \( k : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{R} \) be a measurable function satisfying the conditions:

\[ |k(x,y)| \leq \frac{c}{\mu B(x,\rho(x,y))}, \quad x,y \in X, \ x \neq y; \]

\[ |k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \leq c\omega \left( \frac{\rho(x_2,x_1)}{\rho(x_2,y)} \right) \frac{1}{\mu B(x_2,\rho(x_2,y))} \]

for all \( x_1, x_2 \) and \( y \) with \( \rho(x_2,y) > c\rho(x,x_2) \), where \( \omega \) is a positive non-decreasing function on \((0,\infty)\) which satisfies the \( \Delta_2 \) condition: \( \omega(2t) \leq c\omega(t) \) \((t > 0)\); and the Dini condition:

\[ \int_0^1 \left( \frac{\omega(t)}{t} \right) dt < \infty. \]

We also assume that for some constant \( p_0, 1 < p_0 < \infty \), and all \( f \in L^{p_0}(X,\mu) \) the limit \( Tf(x) \) exists almost everywhere on \( X \) and that \( T \) is bounded in \( L^{p_0}(X,\mu) \).

For simplicity we will assume that \( p_0 = 2 \). We call \( T \) the Calderón-Zygmund operator.

**Lemma 3.1.** Let \( T \) be the Calderón-Zygmund operator. Then there is a positive constant \( c \) non-depending on \( p \) such that the following estimates hold:

\[ \|T\|_{L^p(X,\mu) \to L^p(X,\mu)} \leq c \left( \frac{p}{p-1} + \frac{p}{2-p} \right), \quad 1 < p < 2, \]

\[ \|T\|_{L^p(X,\mu) \to L^p(X,\mu)} \leq c \left( p + \frac{p}{p-2} \right), \quad p > 2. \]

**Proof.** Since \( T \) has weak \((1,1)\) and strong \((2,2)\) types the Marcinkiewicz interpolation theorem (see, e. g., [8], p. 29) we have that

\[ \|T\|_{L^p(X,\mu) \to L^p(X,\mu)} \leq \left( \frac{2p}{p-1} A_0 \right)^{\frac{1}{p}} + \left( \frac{4p}{2-p} A_1^2 \right)^{\frac{1}{p}} \|f\|_{L^p(X,\mu)}, \quad 1 < p < 2, \]

where \( A_0 \) is the constant arisen in the weak \((1,1)\) type inequality for \( T \) and \( A_1 \) is the constant from the strong \((2,2)\) type inequality for \( T \). Observe now that

\[ \left[ \frac{2p}{p-1} A_0 + \frac{4p}{2-p} A_1^2 \right]^{1/p} \leq 2^{1/p} \left( \frac{p}{p-1} \right)^{1/p} A_0^{1/p} + 4^{1/p} \left( \frac{p}{2-p} \right)^{1/p} A_1^{2/p} \]

\[ \leq c \left( \frac{p}{p-1} + \frac{p}{2-p} \right), \]
where the positive constant $c$ does not depend on $p$.

Let now $p > 2$. By using the above-mentioned arguments we have that

\[
\|T\|_{L^p \to L^p} = \|T\|_{L^{p'} \to L^{p'}} \leq \left( \frac{p'}{p' - 1} + \frac{p'}{2 - p'} \right) = c \left( p + \frac{p}{p - 2} \right).
\]

\[\square\]

**Proposition 3.1.** Let $1 < p < \infty$, $0 \leq \lambda < 1$. Then

\[
\|T\|_{L^p,\lambda(X,\mu)} \leq c \left[ \frac{p}{p - 1} + \frac{p - \lambda + 1}{2 - p} \right], \quad 1 < p < 2,
\]

\[
\|T\|_{L^p,\lambda(X,\mu)} \leq c \left[ p + \frac{p - \lambda + 1}{1 - \lambda} \right], \quad p > 2.
\]

**Proof.** Let us take small $r > 0$ and $x \in X$. Represent $f$ as follows: $f = f_1 + f_2$, where $f_1 = f \cdot \chi_{B(x,2a_1r)}$, $f_2 = f - f_1$, where $a_1$ is the constant from the triangle inequality for the quasi-metric $\rho$. Observe that if $y \in B(x,r)$ and $z \in X \setminus B(x,2a_1r)$, then

\[
\mu B(x, \rho(x,z)) \leq c \mu B(y, \rho(y,z)). \tag{3.1}
\]

Inequality (3.1) follows from the estimates

\[
\mu B(x, \rho(x,z)) \leq c_1 \mu B(x, \rho(y,z)) \leq c_2 \mu B(y, \rho(y,z)). \tag{3.2}
\]

To show the first part of (3.2) observe that

\[
\rho(x,z) \leq a_1 \rho(x,y) + a_1 \rho(y,z) \leq a_1 r + a_1 \rho(y,z) \leq \frac{\rho(x,z)}{2} + a_1 \rho(y,z)
\]

Hence, $\frac{\rho(x,z)}{2a_1} \leq \rho(y,z)$. Now by the doubling condition we have the first part of (3.2).

The second part of (3.2) follows easily.

Recall now that the doubling condition for $\mu$ implies the reverse doubling condition for $\mu$: there one constants $0 < \alpha, \beta < 1$ such that for all $x \in X$ and small positive $r$,

\[
\mu B(x, \alpha r) \leq \beta \mu B(x, r). \tag{3.3}
\]

Let us take an integer $m_0$ so that $\alpha^{m_0} d$ is sufficiently small, where $d$ is the diameter of $X$.

Let $y \in B(x,r)$. Then by (3.1) and Fubini’s theorem we have that

\[
|T f_2(y)| \leq c \int_{X \setminus B(x,2a_1r)} |f(z)| (\mu B(x,z))^{-1} d\mu(z)
\]

\[
\leq c_\beta \int_{X \setminus B(x,2a_1r)} |f(z)| \left( \int_{B(x,\alpha^{m_0} \rho(x,z)) \setminus B(x,\alpha^{m_0} - 1 \rho(x,z))} (\mu B(x, \rho(x,t))^{-2} d\mu(t)) \right) d\mu(z)
\]

\[
\leq c_\beta \int_{X \setminus B(x,2\alpha^{m_0} - 1 a_1r)} (\mu B(x, \rho(x,t)))^{-2} \left( \int_{B(x,\alpha^{1-m_0} \rho(x,t))} |f(z)| d\mu(z) \right) d\mu(t)
\]
where
\[
\overline{f}(x,t) := \left[\mu B(x, \alpha^{1-m_0})\right]^{-1} \int_{B(x, \alpha^{1-m_0} \rho(x,t))} |f(z)| \, d\mu(z).
\]

Observe that by Hölder’s inequality the following estimates hold:
\[
\overline{f}(x,t) \leq \mu B(x, \alpha^{1-m_0} \rho(x,t))^{-1} \|f\|_{L^p(B(x, \alpha^{1-m_0} \rho(x,t)))} \chi_{B(x, \alpha^{1-m_0} \rho(x,t))} \|\\cdot\|_{L^p(X)}
\]
\[
\leq c \mu B(x, \alpha^{1-m_0} \rho(x,t))^{-1} \|f\|_{L^p(B(x, \alpha^{1-m_0} \rho(x,t)))}
\]
\[
\leq c \mu B(x, \alpha^{1-m_0} \rho(x,z))^{\lambda/p} \|f\|_{L^{p,\lambda}(X,\mu)}.
\]

By applying now Lemma 1.2 of [23] (see also the monograph [9], p. 372) we find that for
\[y \in \overline{B(x,r)},\]
\[
|Tf_2(y)| \leq c \|f\|_{L^{p,\lambda}(X,\mu)} \int_{X \setminus \overline{B(x,2\alpha^{1-m_0}a_1r)}} \left[\mu B(x, \alpha^{1-m_0} \rho(x,t))\right]^{\lambda-1/p} \, d\mu(t)
\]
\[
\leq c \mu B(x, 2\alpha^{1-m_0}a_1r)^{\lambda-1/p} \cdot \mu B(x, 2\alpha^{1-m_0}a_1r)^{\lambda-1/p} \|f\|_{L^{p,\lambda}(X,\mu)}
\]
where the positive constant \(c\) does not depend on \(\lambda\) and \(p\).

Consequently, by Lemma 3.1 we find that
\[
\left[\left(\mu B(x,r)\right)^{-\lambda} \int_{B(x,r)} |Tf(y)|^p \, d\mu(y)\right]^{1/p} \leq (\mu B(x,r))^{-\lambda/p} \left(\int_{B(x,r)} |Tf_1(y)|^p \, d\mu(y)\right)^{1/p}
\]
\[
+ (\mu B(x,r))^{-\lambda/p} \left(\int_{B(x,r)} |Tf_2(y)|^p \, d\mu(y)\right)^{1/p}
\]
\[
\leq c \cdot c_p (\mu B(x,r))^{-\lambda/p} \left(\int_{B(x,2a_1r)} |f(y)|^p \, d\mu(y)\right)^{1/p}
\]
\[
+ c \left(\mu B(x, 2\alpha^{1-m_0}a_1r)^{\lambda-1/p} (\mu B(x, 2a_1r))^{\lambda/p}\right)
\]
\[
\times (\mu B(x,r))^{-\lambda/p} \|f\|_{L^{p,\lambda}(X,\mu)}.
\]
Proof. We have \( \| f \|_{L^p(X, \mu)} = c \left( c_p + \frac{p - \lambda + 1}{1 - \lambda} \right) \| f \|_{L^p(X, \mu)} \), where the constant \( c > 0 \) does not depend on \( p \) and \( \lambda \), and
\[
 c_p = \begin{cases} 
 \frac{p}{p-1} + \frac{p}{2-p}, & 1 < p < 2, \\
 p + \frac{p}{p-2}, & p > 2.
\end{cases}
\]
Observe that the constant \( c \) does not depend on \( \lambda \) and \( p \).

\[\square\]

**Theorem 3.1.** Let \( 1 < p < \infty, \theta > 0 \) and let \( 0 < \lambda < 1 \). Then the operator \( T \) is bounded in \( L^p, \theta, \lambda(X, \mu) \).

**Proof.** We have
\[
\| T f \|_{L^p, \theta, \lambda(X, \mu)} = \max \left\{ \sup_{0 < \sigma \leq \theta} \sup_{x \in X} \left( \frac{\varepsilon^\theta}{(\mu B(x, r))^\lambda} \int_{B(x, r)} |T f|^{p-\varepsilon} \right)^{\frac{1}{p-\varepsilon}} \right\}
\]
where \( \sigma \) is the number satisfying the condition \( 0 < \sigma < p - 1 \).

Now we estimate \( A_2 \). By applying Hölder’s inequality we find that
\[
A_2 = \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| T f \|_{L^{p-\varepsilon, \lambda}(X, \mu)} = \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \mu(B(x, r))^{-\frac{1}{p-\varepsilon}} \left( \mu(B(x, r)) \right)^{\frac{\theta}{p-\varepsilon}}
\]
\[
\times \left( \frac{1}{\mu B(x, r)} \int_{B(x, r)} |T f|^{p-\varepsilon} \right)^{\frac{1}{p-\varepsilon}}
\]
\[
\leq \sup_{\sigma < \varepsilon < p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} (\mu B(x, r))^{-\frac{1}{p-\varepsilon}} \left( \mu B(x, r) \right)^{-1} \int_{B(x, r)} |T f|^{p-\sigma} \right)^{\frac{1}{p-\sigma}}.
\]

Further, without loss of generality we can assume that \( \mu(X) = 1 \) and, consequently, \( (\mu B(x, r))^{-\frac{1}{p-\varepsilon}} \leq (\mu B(x, r))^{-\frac{1}{p-\varepsilon}} \). Hence,
\[
A_2 \leq (p - 1)\theta \sigma^{-\frac{\theta}{p-\varepsilon}} \sup_{0 < \varepsilon \leq \sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| T f \|_{L^{p-\varepsilon, \lambda}(X, \mu)}.
\]

Hence, by Proposition 3.1 we conclude that
\[
\| T f \|_{L^p, \theta, \lambda(X, \mu)} \leq \left( (p - 1)\theta \sigma^{-\frac{\theta}{p-\varepsilon}} + 1 \right) \sup_{0 < \varepsilon \leq \sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| T f \|_{L^{p-\varepsilon, \lambda}(X, \mu)}
\]
\[
\leq \left( (p - 1)\theta \sigma^{-\frac{\theta}{p-\varepsilon}} + 1 \right) \sup_{0 < \varepsilon \leq \sigma} C_{p, \lambda, \theta} \varepsilon^{\frac{\theta}{p-\varepsilon}} \| f \|_{L^{p-\varepsilon, \lambda}(X, \mu)}
\]
\[
(p - 1)^\theta \sigma^{-\frac{\theta}{p - \sigma}} + 1 \sup_{0 < \varepsilon \leq \sigma} C_{p, \lambda, \varepsilon} \sup_{x \in X, 0 \leq r < d} \left[ \frac{\varepsilon^\theta}{(\mu B(x, r))^\lambda} \int_{B(x, r)} |f(y)|^{p - \varepsilon} d\mu(y) \right]
\]

\[
\leq \left[ (p - 1)^\theta \sigma^{-\frac{\theta}{p - \sigma}} + 1 \right] \|f\|_{L^{p, \sigma}(X, \mu)} \sup_{0 < \varepsilon \leq \sigma} C_{p, \lambda, \varepsilon},
\]

where

\[
C_{p, \lambda, \varepsilon} = \begin{cases} 
\frac{p - \varepsilon - \lambda + 1}{1 - \lambda} + \frac{p - \varepsilon}{p - \varepsilon - 1} + \frac{p - \varepsilon}{2 - p + \varepsilon}, & 1 < p < 2 \\
\frac{p - \varepsilon - \lambda + 1}{1 - \lambda} + \frac{p - \varepsilon}{1 - \lambda} + p - \varepsilon + \frac{p - \varepsilon}{p - \varepsilon - 2}, & p > 2.
\end{cases}
\]

Observe now that

\[
\sup_{0 < \varepsilon \leq \sigma} C_{p, \lambda, \varepsilon} \leq \begin{cases} 
\frac{p - \lambda + 1}{1 - \lambda} + \frac{p - \sigma}{p - \sigma - 1} + \frac{p}{2 - p}, & 1 < p < 2, \\
\frac{p - \lambda + 1}{1 - \lambda} + \frac{p - \sigma}{p - \sigma - 1} + \frac{p}{p - \sigma - 2}, & p > 2,
\end{cases}
\]

where \(\sigma\) is sufficiently small. \(\square\)

4 Fractional integrals in grand Morrey spaces

4.1 Potentials \((I_\alpha f)(x) = \int_X \frac{f(y)}{\rho(x, y)^{\gamma - \alpha}} d\mu(y)\)

Let an SHT \((X, \rho, \mu)\) satisfy the condition: there are positive constants \(b\) and \(\gamma\) such that

\[
\mu B(x, r) \leq br^\gamma,
\]

for all \(x \in X\) and \(r, 0 < r < d\), i.e. \(\mu\) is upper \(\gamma\)-Ahlfors regular. As before we assume that \(d = diam(X) < \infty\).

Let

\[
(I_\alpha f)(x) = \int_X \frac{f(y)}{\rho(x, y)^{\gamma - \alpha}} d\mu(y), \quad x \in X,
\]

where \(0 < \alpha < \gamma\).

In this section we study the boundedness of \(I_\alpha\) in grand Morrey spaces. For this we define the classical Morrey space as follows: \(f \in \mathcal{L}^{p, \lambda}(X, \mu)\) \((1 < p < \infty, 0 \leq \lambda < \frac{1}{\gamma})\) if

\[
\|f\|_{\mathcal{L}^{p, \lambda}(X, \mu)} := \sup_{x \in X} \left( \frac{1}{r^{\gamma \lambda}} \int_{B(x, r)} |f(y)|^p d\mu(y) \right)^{1/p} < \infty.
\]

Further, let \(\varphi\) be a positive function in \((0, p - 1)\) which is increasing near 0 and satisfies the condition \(\varphi(0+) = 0\). We say that \(f \in \mathcal{L}^{p, \varphi(\cdot), \lambda}(X, \mu)\) if

\[
\|f\|_{\mathcal{L}^{p, \varphi(\cdot), \lambda}(X, \mu)} = \sup_{0 < \varepsilon < p - 1} \sup_{x \in X} \left( \frac{\varphi(\varepsilon)}{r^{\gamma \lambda}} \int_{B(x, r)} |f|^p d\mu \right)^\frac{1}{p} < \infty.
\]
Let \( \theta \) be a positive number. If \( \varphi(\varepsilon) \equiv \varepsilon^\theta \), then we denote \( \mathcal{L}^{p, \varphi(\cdot), \lambda}(X, \mu) =: \mathcal{L}^{p, \theta, \lambda}(X, \mu) \). For \( \theta = 1 \) we have the grand Morrey space \( \mathcal{L}^{p, \lambda}(X, \mu) \).

Let
\[
(\mathcal{M}f)(x) = \sup_{x \in X} \frac{1}{r^r} \int_{B(x, r)} |f(y)| \, d\mu(y), \quad x \in X.
\]
We begin with the following statement:

**Proposition 4.1.** Let \( 1 < p < \infty \) and let \( 0 \leq \lambda < 1 \). Then
\[
\| \mathcal{M}f \|_{L^{p, \lambda}(X, \mu)} \leq \left( (\overline{\alpha})^{\frac{\lambda}{p'}} c_0 (p')^{\frac{\lambda}{p}} + 1 \right) \| f \|_{L^{p, \lambda}(X, \mu)}
\]
holds, where \( c_0 \) is the constant from (2.1) and \( \overline{\alpha} = a_1(a_1(a_0 + 1) + 1) \).

**Proof.** Since \( \mathcal{M}f(x) \leq Mf(x) \), by Proposition 2.1 we have that
\[
\| \mathcal{M} \|_{L^{p}(X, \mu) \to L^{p}(X, \mu)} \leq c_0(p')^{1/p}.
\]
Repeating the proof of Proposition 2.2 we have the desired result. \( \square \)

**Lemma 4.1.** Let \( 1 < p < \infty \), \( 0 < \alpha < \frac{(1-\lambda)\gamma}{p} \), \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-\lambda)\gamma} \), where \( 0 \leq \lambda < 1/\gamma \). Then the inequality
\[
\| I_\alpha f \|_{L^{q, \lambda}(X, \mu)} \leq c(p, \alpha, \lambda, \gamma) \| f \|_{L^{p, \lambda}(X, \mu)}
\]
holds, where the positive constant \( c(p, \alpha, \lambda, \gamma) \) is given by
\[
c(p, \alpha, \lambda, \gamma) = c \frac{(1-\lambda)\gamma}{\alpha((1-\lambda)\gamma - \alpha p)} [(p')^{1/q} + 1],
\]
and the positive constant \( c \) does not depend on \( p \) and \( \alpha \).

**Proof.** First we show that the Hedberg’s [18] type inequality holds:
\[
|(I_\alpha f)(x)| \leq c_{p, \lambda, \gamma, \alpha} \| \mathcal{M}f \|_{L^{p, \lambda}(X, \mu)}^{1 - \frac{\alpha}{(1-\lambda)\gamma}} \| f \|_{L^{p, \lambda}(X, \mu)}^{\frac{\alpha}{(1-\lambda)\gamma}}, \quad (4.2)
\]
where \( c_{p, \lambda, \gamma, \alpha} = \frac{2(1-\lambda)\gamma}{\alpha((1-\lambda)\gamma - \alpha p)} \). To prove (4.2) we set
\[
f_r(x) := \frac{1}{r^r} \int_{B(x, r)} |f(y)| \, d\mu(y).
\]
The inequality
\[
\rho(x, y) \leq 2 \int_{\rho(x, y)} t^{\alpha - \gamma - 1} \, dt, \quad 0 < \rho(x, y) < l,
\]
is obvious. By using (4.3) we find that
\[
|(I_\alpha f)(x)| \leq 2 \int_{X} |f(y)| \left( \int_{\rho(x, y)} t^{\alpha - \gamma - 1} \, dt \right) \, d\mu(y)
\]
\[
= 2 \int_{0}^{2d} \int_{\rho(x, y) < t} |f(y)| \, d\mu(y) \, dt \leq 2 \int_{0}^{2d} t^{\alpha - \gamma} f_t(x) \, dt.
\]
Taking $\varepsilon > 0$ (which will be chosen later) we have that
\[
|(I_\alpha f)(x)| \leq 2 \left[ \int_0^\varepsilon t^{\alpha-\gamma} f_1(x) dt + \int_\varepsilon^{2d} t^{\alpha-\gamma} f_1(x) dt \right] = 2 \left[ J_1^{(\varepsilon)}(x) + J_2^{(\varepsilon)}(x) \right].
\]
It is obvious that
\[
J_1^{(\varepsilon)}(x) \leq (\mathcal{M}f)(x) \int_0^\varepsilon t^{\alpha-1} dt = \frac{(\mathcal{M}f)(x)}{\alpha} \varepsilon^{\alpha}.
\]
Further, by Hölder’s inequality and condition (4.1) it is clear that
\[
f_1(x) = \frac{1}{t^\gamma} \int_{B(x,t)} |f(y)|d\mu(y) \leq \left( \frac{1}{t^\gamma} \int_{B(x,t)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}
= t^{-\frac{\lambda-1}{p}\gamma + \alpha} \left( \frac{1}{t^{\gamma\lambda}} \int_{B(x,t)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \leq t^{-\frac{\lambda-1}{p}\gamma + \alpha} \|f\|_{L^{p,\lambda}(X,\mu)}.
\]
By applying the condition $\frac{\lambda-1}{p}\gamma + \alpha < 0$ we find that
\[
|(I_\alpha f)(x)| \leq 2 \left[ \frac{(\mathcal{M}f)(x)}{\alpha} \varepsilon^{\alpha} + \left( \int_\varepsilon^{2d} t^{\frac{(\lambda-1)\gamma}{p} + \alpha - 1} dt \right) \|f\|_{L^{p,\lambda}} \right] = 2 \left[ \frac{(\mathcal{M}f)(x)}{\alpha} \varepsilon^{\alpha} + \frac{\varepsilon^{\frac{\lambda-1}{p}\gamma + \alpha}}{\alpha + \frac{\lambda-1}{p}\gamma} \|f\|_{L^{p,\lambda}(X,\mu)} \right].
\]
Let $\varepsilon = \left[ \frac{\|f\|_{L^{p,\lambda}(X,\mu)}}{(\mathcal{M}f)(x)} \right]^{\frac{p}{\lambda-1}}$. Then
\[
|(I_\alpha f)(x)| \leq 2 \left[ \frac{(\mathcal{M}f)^{1-\frac{\alpha p}{(1-\lambda)\gamma}}(x)}{\alpha} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{(1-\lambda)\gamma}} \right. \\
- \left. \frac{1}{\alpha + \frac{(\lambda-1)\gamma}{p}} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{(1-\lambda)\gamma}} (\mathcal{M}f)^{1-\frac{\alpha p}{(1-\lambda)\gamma}}(x) \right]
= 2 \left[ \frac{1}{\alpha} - \frac{p}{\alpha p + (\lambda-1)\gamma} \right] \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{(1-\lambda)\gamma}} (\mathcal{M}f)^{1-\frac{\alpha p}{(1-\lambda)\gamma}}(x)
= 2 \frac{(1-\lambda)\gamma}{\alpha((1-\lambda)\gamma - \alpha p)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{(1-\lambda)\gamma}} (\mathcal{M}f)^{1-\frac{\alpha p}{(1-\lambda)\gamma}}(x).
\]
Consequently, by the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-\lambda)\gamma}$ and Proposition 4.1 we have that
\[
\left( \frac{1}{t^{\gamma\lambda}} \int_{B(x,t)} |(I_\alpha f)(y)|^q d\mu(y) \right)^{\frac{1}{q}}
\]
Suppose that \( \theta \) Let us introduce the function:

\[
\proof.
\]

\[
\Hence it is enough to prove that \( I \) is bounded from \( \mathcal{L}^p,\theta_1,\lambda(X,\mu) \) to \( \mathcal{L}^q,\theta_2,\lambda(X,\mu) \).
\]

\[
\text{Theorem 4.1.} \quad \text{Let} \ 1 < p < \infty, \ 0 < \alpha < \frac{(1-\lambda)\gamma}{p}, \ 0 \leq \lambda < 1/\gamma \ \text{and let} \ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-\lambda)\gamma}. \ \text{Suppose that} \ \theta_1 > 0. \ \text{We set}
\]

\[
\theta_2 = \left[ 1 + \frac{\alpha \gamma}{(1-\lambda)\gamma} \right] \theta_1.
\]

\[
\text{Then the operator} \ I_\alpha \ \text{is bounded from} \ \mathcal{L}^p,\theta_1,\lambda(X,\mu) \ \text{to} \ \mathcal{L}^q,\theta_2,\lambda(X,\mu).
\]

\[
\proo.
\]

\[
\varphi(u) := \left[ p + \frac{(1-\lambda)(u-q)\gamma}{(1-\lambda)\gamma} \right]^{\frac{1}{(1-\lambda)-\frac{\alpha \gamma}{(1-\lambda)\gamma}}}.
\]

Observe that

\[
\varphi(t) \sim t^{1+\frac{\alpha \gamma}{(1-\lambda)\gamma}}, \quad \text{as} \ t \to 0^+.
\]

Hence it is enough to prove that \( I_\alpha \) is bounded from \( \mathcal{L}^p,\theta_1,\lambda(X,\mu) \) to \( \mathcal{L}^q,\theta_1,\lambda(X,\mu) \), where

\[
\psi(t) := \varphi(t^{\theta_1}).
\]

Let \( \sigma \) be a small positive number. As in the proofs of the main theorems of previous sections we have

\[
\|I_\alpha f\|_{\mathcal{L}^q,\psi,\lambda(X,\mu)} = \max \left\{ \sup_{0 < c \leq \sigma} \sup_{x \in X} \left( \frac{\psi(c)}{c^{\lambda \gamma}} \right) \int_{B(x,t)} |f(x)|^{q^{-\frac{1}{\psi}}} d\mu(x) \right\}^\frac{1}{q^{-\frac{1}{\psi}}}.
\]
Further, applying Lemma 4.1, for 

\[ \sup_{\sigma < \varepsilon < q-1} \sup_{x \in \mathcal{X}} \left( \frac{\psi(\varepsilon)}{t^{\lambda \gamma}} \int_{B(x,r)} |I_\alpha f(y)|^{q-\varepsilon} d\mu(y) \right) \]

\[ \sup_{0 \leq r < d} \left( \frac{1}{\varepsilon^{\frac{1}{q-\varepsilon}}} \right) =: \max\{A_1, A_2\}. \]

For \( A_2 \), we observe that

\[ \psi(\varepsilon) \frac{1}{\varepsilon^{\frac{1}{q-\varepsilon}} t^{\frac{(1-\lambda)\gamma}{\lambda \gamma}}} \left( \frac{1}{t^{\lambda \gamma}} \int_{B(x,r)} |I_\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}} \]

\[ \leq \sup_{\sigma < \varepsilon < q-1} \left( \psi(\varepsilon) \frac{1}{\varepsilon^{\frac{1}{q-\varepsilon}} t^{\frac{(1-\lambda)\gamma}{\lambda \gamma}}} \left( \frac{1}{t^{\lambda \gamma}} \int_{B(x,t)} |I_\alpha f(y)|^{q-\sigma} d\mu(y) \right)^{\frac{1}{q-\sigma}} \right) \]

\[ = \left[ \sup_{\sigma < \varepsilon < q-1} \left( \psi(\varepsilon) \frac{1}{\varepsilon^{\frac{1}{q-\varepsilon}}} \right) \psi(\sigma)^{-\frac{1}{q-\sigma}} \frac{\varphi(\sigma)}{t^{\lambda \gamma}} \sup_{0 < \sigma \leq \varepsilon} \sup_{x \in \mathcal{X}} \left( \frac{1}{t^{\lambda \gamma}} \int_{B(x,t)} |I_\alpha f(y)|^{q-\sigma} d\mu(y) \right)^{\frac{1}{q-\sigma}} \right]. \]

Further, applying Lemma 4.1, for \( \varepsilon \) satisfying the condition \( 0 < \varepsilon \leq \sigma \), we have

\[ \left( \frac{\psi(\varepsilon)}{t^{\lambda \gamma}} \int_{B(x,t)} |I_\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}} = \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \left( \frac{1}{t^{\lambda \gamma}} \int_{B(x,t)} |I_\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}} \]

\[ \leq \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \sup_{x \in \mathcal{X}} \left( \frac{1}{t^{\lambda \gamma}} \int_{B(x,t)} |I_\alpha f(y)|^{q-\varepsilon} d\mu(y) \right)^{\frac{1}{q-\varepsilon}} \]

\[ = \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|I_\alpha f\|_{L^{q-\varepsilon, \lambda}(X, \mu)} \leq \]

\[ \leq c(p - \eta, \alpha, \lambda) \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|f\|_{L^{q-\varepsilon, \lambda}(X, \mu)}, \]

\[ \left( \text{where} \frac{1}{p - \eta} - \frac{1}{q - \varepsilon} = \frac{\alpha}{(1 - \lambda) \gamma} \right) \]

\[ = c \left( \frac{1 - \lambda}{\alpha ((1 - \lambda) \gamma - \alpha (p - \eta))} \right) \left( \frac{p - \eta}{p - \eta - 1} + 1 \right) \psi(\varepsilon)^{\frac{1}{q-\varepsilon}} \|f\|_{L^{p-\eta, \lambda}(X, \mu)} \]
For example, (4.4) holds if observe that when \( \sigma \) is small, then \( \eta \) is also small positive number; recall also that

\[
\varphi(u) = \left[ p + \frac{(1 - \lambda)(u - q)\gamma}{(1 - \lambda)\gamma - \alpha(u - q)} \right]^{\frac{1}{(1 - \lambda)\gamma - \alpha(u - q)}} \sim u^{1 + \frac{\alpha q}{(1 - \lambda)\gamma}}, \quad u \to 0 +,
\]

and \( \psi(\varepsilon) = 1 \)

\[
\leq \left[ \sup_{0 < \eta \leq \sigma_1} c(p - \eta, \alpha, \lambda) \right] \|f\|_{\mathcal{L}^{p, \theta}, \lambda(X, \mu)}
\]

Hence,

\[
\|I_{\alpha}f\|_{\mathcal{L}^{p, \theta}, \lambda(X, \mu)} \leq \left[ \sup_{0 < \eta \leq \sigma_1} c(p - \eta, \alpha, \lambda) \right] \|f\|_{\mathcal{L}^{p, \theta}, \lambda(X, \mu)},
\]

where

\[
c(p - \eta, \alpha, \lambda) = \frac{(1 - \lambda)\gamma}{\alpha[(1 - \lambda)\gamma - \alpha(p - \eta)]} \left( \frac{p - \eta}{p - \eta - 1} \right)^{\frac{1}{p - \eta}} + 1,
\]

\( \sigma_1 \) is the constant independent of \( p, \eta \) and \( \alpha \); \( \sigma_1 \) is a small positive number. Observe that if \( \sigma_1 \) is sufficiently small, then \( (1 - \lambda)\gamma - \alpha(p - \eta) \geq \eta_0 > 0 \), \( \frac{p - \eta}{p - \eta - 1} \leq p' + 1 \) for some \( \eta_0 \) when \( 0 < \eta \leq \sigma_1 \).

### 4.2 Potential

\[(T_{\alpha}f)(x) = \int_X \frac{f(y)}{\mu B(x, \rho(x, y))^{1 - \alpha}} d\mu(y)\]

Let

\[(T_{\alpha}f)(x) = \int_X \frac{f(y)}{\mu B(x, \rho(x, y))^{1 - \alpha}} d\mu(y), \quad 0 < \alpha < 1.
\]

Suppose that instead of condition (4.1) of the previous section the following conditions hold:

(i) \( \mu \{x\} = 0, \quad \text{for all } x \in X; \)

(ii) \( \mu B(x, t) \) is continuous in \( t \) for every \( x \in X \).

For example, (4.4) holds if \( \mu \{y \in X : \rho(x, y) = t\} = 0 \) for arbitrary \( x \in X \) and \( t \in [0, d) \).

We say that \( f \in L^{p, \varphi(\cdot), \lambda}(X, \mu) \) if

\[
\|f\|_{L^{p, \varphi(\cdot), \lambda}(X, \mu)} := \left( \sup_{0 < \varepsilon < p - 1} \sup_{x \in X} \left( \frac{\varphi(\varepsilon)}{\mu B(x, r)^{\lambda}} \int_{B(x, r)} |f(y)|^p d\mu(y) \right) \right)^{1/p} < \infty, \quad 0 \leq \lambda < 1,
\]

where \( \varphi \) is a positive function in \( (0, p - 1) \) which is increasing near 0 and satisfies the condition \( \varphi(0+) = 0 \). If \( \varphi(\varepsilon) = \varepsilon^\theta \), where \( \theta \) is a positive number, then we denote \( L^{p, \varphi(\cdot), \lambda}(X, \mu) \) by \( L^{p, \theta, \lambda}(X, \mu) \).

Our aim in this section is to prove the next statement:
Theorem 5.1. Let $1 < p < \infty$, $0 < \alpha < \frac{1-\lambda}{p}$, $0 \leq \lambda < 1$ and let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$. Let $\theta_1$ be a positive number. We set

$$
\theta_2 = \theta_1 \left( 1 + \frac{\alpha q}{1-\lambda} \right).
$$

Then the operator $T_\alpha$ is bounded from $L^{p,\lambda}(X,\mu)$ to $L^{q,\lambda}(X,\mu)$.

To prove Theorem 5.1 we need the following lemma

Lemma 5.1. Let $1 < p < \infty$, $0 < \alpha < \frac{1-\lambda}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$, where $0 \leq \lambda < 1$. Then the inequality

$$
\| T_\alpha f \|_{L^{q,\lambda}(X,\mu)} \leq c(p,\alpha,\lambda) \| f \|_{L^{p,\lambda}(X,\mu)}
$$

holds, where

$$
c(p,\alpha,\lambda) = C_\alpha + \frac{p}{1-\lambda - \alpha p} \left[ (p')^{1/q} + 1 \right]
$$

and the positive constant $c$ does not depend on $p$ and $\alpha$.

Proof. Following the idea of Hedberg [18] and taking into account the proof of (4.2) we have that

$$
|T_\alpha f(x)| \leq c(p,\lambda,\alpha)(Mf)^{1-\frac{\alpha p}{1-\lambda}}(x) \| f \|_{L^{p,\lambda}(X,\mu)}^{\alpha p}, \quad (4.5)
$$

where

$$
c(p,\lambda,\alpha) = C_\alpha + \frac{p}{1-\lambda - \alpha p}
$$

and

$$
(Mf)(x) = \sup_{x \in X} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y)|d\mu(y).
$$

We set

$$
\overline{f}(x,t) := \frac{1}{\mu B\left(x, \frac{\rho(x,t)}{\alpha}\right)} \int_{B\left(x, \frac{\rho(x,t)}{\alpha}\right)} |f(y)|d\mu(y),
$$

where $\alpha$ is the constant between 0 and 1. In fact $\alpha$ is the constant from the reverse doubling condition (3.3) (we use the symbol $\alpha$ instead of $\lambda$).

Observe that

$$
|I_\alpha f(x)| \leq b \int_X |f(y)| \left[ \int_{\rho(x,y) < \rho(x,t) < \rho(x,y)} \mu B(x, \rho(x,t))^{\alpha-2}d\mu(t) \right] d\mu(y),
$$

where $b$ is the constant depending on $\beta$ from (3.3).

Hence,

$$
|I_\alpha f(x)| \leq b \int_X \mu B(x, \rho(x,t))^{\alpha-2} \left( \int_{B\left(x, \frac{\rho(x,t)}{\alpha}\right)} |f(y)|d\mu(y) \right) d\mu(t)
$$

$$
\leq b_0 \int_X \mu B(x, \rho(x,t))^{\alpha-1} \overline{f}(t,x)d\mu(t),
$$

where $b_0$ is the positive constant which does not depend on $p$, $\alpha$ and $\lambda$. 
We take \( \varepsilon > 0 \) which will be chosen later. Then

\[
|I_\alpha f(x)| \leq b_0 \left[ \int_{B(x, \varepsilon)} \mu B(x, \rho(x,t))^{\alpha - 1} \overline{f}(t, x) d\mu(t) + \int_{X \setminus B(x, \varepsilon)} \mu B(x, \rho(x,t))^{\alpha - 1} \overline{f}(t, x) d\mu(t) \right] =: b_0 \left[ J^{(1)}(x, t) + J^{(2)}(x, t) \right].
\]

It is easy to see that (see also [9], p. 348)

\[
J^{(1)}(x, t) \leq M f(x) \int_{B(x, \varepsilon)} \mu B(x, \rho(x,t))^{\alpha - 1} d\mu(t) \leq c_\alpha M f(x) \mu B(x, \varepsilon)^\alpha,
\]

where the positive constant \( c_\alpha \) depends only on \( \alpha \).

Further, by Hölder’s inequality we find that

\[
f(t, x) \leq \frac{1}{\mu B \left( x, \frac{\rho(x,t)}{\overline{\rho}} \right)} \left( \int_{B(x, \rho(x,t))} |f(t)|^p d\mu(t) \right)^{1/p} \mu B \left( x, \frac{\rho(x,t)}{\alpha} \right)^{\frac{1}{p} - \frac{1}{\overline{\rho}}}
\]

Besides this, by the inequality

\[
\int_{X \setminus B(x, \varepsilon)} \mu B(x, \rho(x,t))^{\alpha - 1 + \frac{\lambda - 1}{p}} d\mu(t) \leq c \mu B(x, \varepsilon)^{\alpha + \frac{\lambda - 1}{p}}
\]

(see Proposition 6.1.2 of [9]) we obtain that

\[
|I_\alpha f(x)| \leq b_0 \left[ c_\alpha M f(x) \mu B(x, \varepsilon)^\alpha + \left( \int_{X \setminus B(x, \varepsilon)} \mu B(x, \rho(x,t))^{\alpha - 1 + \frac{\lambda - 1}{p}} d\mu(t) \right) \right]
\]

\[
\|f\|_{L^p, \lambda(X, \mu)} = b_0 \left[ c_\alpha (M f(x)) \mu B(x, \varepsilon)^\alpha + \overline{c}_{p, \lambda, \alpha} \mu B(x, \varepsilon)^\alpha + \frac{\lambda - 1}{p} \|f\|_{L^p, \lambda(X, \mu)} \right],
\]

where the positive constant \( c_\alpha \) depends only on \( \alpha \) and the positive constant \( \overline{c}_{p, \lambda, \alpha} \) is given by

\[
\overline{c}_{p, \lambda, \alpha} = \frac{p}{1 - \lambda - \alpha p}; \quad b_0 \text{ does not depend on } p, \alpha \text{ and } \lambda.
\]

Now we take (recall that \( \mu B(x, \varepsilon) \) is continuous in \( \varepsilon \))

\[
\mu B(x, \varepsilon) = \left[ \frac{\|f\|_{L^p, \lambda(X, \mu)}}{M f(x)} \right]^{\frac{1}{1 - \lambda}}.
\]

Consequently,

\[
|I_\alpha f(x)| \leq \left( c_\alpha - \frac{p}{\alpha p - 1 + \lambda} \right) \|f\|_{L^p, \lambda(X, \mu)}^{\frac{\alpha p}{1 - \lambda}} (M f)^{1 - \frac{\alpha p}{1 - \lambda}}(x).
\]

Using the condition \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1 - \lambda} \) and Proposition 2.2 we find that

\[
\left( \frac{1}{\mu B(x, r)^{\lambda}} \int_{B(x, r)} |I_\alpha f(x)|^q d\mu(x) \right)^{1/q}
\]
\[
\leq \mu B(x, r)^{-\lambda/q} \left( c_{\alpha} - \frac{p}{\alpha p - 1 + \lambda} \right) \left[ \int_{B(x, r)} (Mf(y))^q[1 - \frac{q}{q'}]d\mu(y) \right]^{\frac{1}{q'}} \|f\|^{\frac{\alpha p}{1 - \lambda}}_{L^{p,\lambda}(X, \mu)}
\]

\[
= \left( c_{\alpha} - \frac{p}{\alpha p - 1 + \lambda} \right) \left[ \frac{1}{\mu B(x, r)^{\lambda/q}} \int_{B(x, r)} (Mf(y))^p d\mu(y) \right]^{\frac{1}{q'}} \|f\|^{\frac{\alpha p}{1 - \lambda}}_{L^{p,\lambda}(X, \mu)}
\]

\[
\leq \left( c_{\alpha} - \frac{p}{\alpha p - 1 + \lambda} \right) \|Mf\|^{\frac{p}{q}}_{L^{p,\lambda}(X, \mu)} \|f\|^{\frac{\alpha p}{1 - \lambda}}_{L^{p,\lambda}(X, \mu)}
\]

\[
\leq \left( c_{\alpha} - \frac{p}{\alpha p - 1 + \lambda} \right) c_0 b^\lambda \left[ (p')^{1/q} + 1 \right] \|f\|_{L^{p,\lambda}(X, \mu)}.
\]

\[\square\]

**Proof of Theorem 5.1.** By using Lemma 5.1 and repeating the arguments of the proof of Theorem 4.1 we conclude that Theorem 5.1 holds. Details are omitted. \[\square\]

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Author’s Address:
A. Meskhi:
A. Razmadze Mathematical Institute, M. Aleksidze St., Tbilisi 0193, Georgia
Second Address: Department of Mathematics, Faculty of Informatics and Control Systems, Georgian Technical University, 77, Kostava St., Tbilisi, Georgia.
e-mail: meskhi@@rmi.acnet.ge