Detection of entanglement and Bell’s inequality violation

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(Dated: January 7, 2022)

We propose a new method for detecting entanglement of two qubits and discuss its relation with the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality. Without the need for full quantum tomography for the density matrix we can experimentally detect the entanglement by measuring less than 9 local observables for any given state. We show that this test is stronger than the CHSH-Bell inequality and also gives an estimation for the degree of entanglement. If prior knowledge is available we can further greatly reduce the number of required local observables. The test is convenient and feasible with present experimental technology.

PACS numbers: 03.67.-a, 03.65.Ud, 03.65.Ta

Since the well-known debate of Einstein, Podolsky and Rosen [1] with Schrödinger [2] about the completeness of quantum mechanics, entangled states have intrigued physicists for decades. In particular, in recent years entangled states have become the key ingredient in the rapidly expanding field of quantum information science, with remarkable prospective applications such as quantum teleportation, quantum cryptography, quantum dense coding and parallel computation [3, 4, 5, 6]. However, the intrinsic nature of entanglement is by no means fully understood and the theory is far from complete. Moreover, from a practical point of view, even if a perfect entangled state has been generated in a laboratory we cannot guarantee its entangled character after interaction with the environment, due to unavoidable quantum noise (e.g. in long distance quantum communication). Thus efficient detection of entanglement is crucial for various quantum information tasks.

In 1964, John Bell showed that no local hidden-variable theory can reproduce all of the statistical predictions of quantum mechanics [6]. This was developed further in the form of the Clauser-Horne-Shimony-Holt inequality (CHSH-Bell inequality) for experimentally testing nonlocal quantum correlation between two separated entangled particles [7], and was first demonstrated experimentally by Aspect et al. [8]. In general, however, we have to take into account all possible settings for all the local observables that appear in the Bell inequality to test its violation. This is not efficient experimentally if we have no prior knowledge of a given state. Also, there are some entangled states which do not violate the Bell inequality, such as some of the Werner states [9]. In this case an alternative method is to use quantum state tomography [10, 11] to obtain the complete density matrix for a quantum state and then apply certain known sufficient or necessary entanglement criteria. (For recent good reviews we refer to [12, 13, 14] and references therein.) Among these, the Peres-Horodecki criterion [15, 16], the recent realignment criterion [17, 18, 19] and its multipartite generalization “the generalized partial transposition criterion” [19] are three strong operationally-friendly entanglement criteria which can fully recognize entanglement in $2 \times 2$ and $2 \times 3$ systems as well as distinguish most bound entangled states (which are not distillable) in higher dimensions. Moreover, we can use Wootters's elegant formula to calculate the entanglement of formation for two qubits [20]. The big disadvantage of these methods is that we have to make a large number of measurements ($4^2 - 1 = 15$ parameters for two qubits) to determine the complete density matrix.

Recently, much effort has been devoted to finding ways to detect the entanglement directly without having to measure the whole density matrix. For pure states, a possible optimal strategy is given in [21]. Horodecki and Ekert proposed a method based on structural physical approximations and collective measurements that can be applied to mixed states [22]. For 2 qubits this only needs 4 parameters to determine the degree of entanglement [23], but in practice it requires the construction of quantum gates and networks, which is not easy to implement with present experimental technology. If, however, we are given some prior knowledge of a quantum state, a few local measurements are enough to detect entanglement for two or three qubits and certain bound entangled states [24]. This has been generalized [25] to higher dimensions and some families of $n$ qubits by making use of the geometrical character of entanglement witnesses [26, 27, 28, 29]. For depolarized states of bipartite systems in arbitrary dimensions there is another scheme which only requires three local measurements [30]. However, these procedures all require some prior knowledge of the quantum state and are only efficient for special classes of states, so they are of limited use in realistic quantum information processing.

In this Letter we develop a new method of entanglement detection for two qubits which requires only a few local measurements and no prior knowledge of the quantum state. We also show its relationship with the CHSH-Bell inequality and the entanglement measure in terms of the concurrence [20]. With the experimental technol-
ogy currently available it should not be too difficult to implement the test.

In Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ it is well known that we can represent the density matrix of any two qubits by the bases in terms of the Kronecker product of the Pauli matrices, as follows:

$$\rho = \frac{1}{4} \sum_{i,j=0}^{3} R_{ij} \sigma_i \otimes \sigma_j,$$

where $\sigma_0$ is the identity operator and $\sigma_{1,2,3}$ are the standard Pauli matrices. Here $R_{ij}$ is real and can be calculated as $R_{ij} = Tr(\rho \sigma_i \otimes \sigma_j)$ since $Tr(\sigma_i \sigma_j) = 2\delta_{ij}$ and $\delta_{ij}$ is the Kronecker delta symbol. Thus $R$ is a $4 \times 4$ real matrix and a representation for the original density matrix $\rho$. For convenience, we denote the $3 \times 3$ sub-matrix $[R_{ij}]$ ($i,j = 1,2,3$) by $T_\rho$.

We shall now derive a practical detection method by using only 9 expectation values of the local observables $\sigma_i \otimes \sigma_j$.

**Theorem 1:** For any separable state of two qubits, the trace norm $\|T_\rho\|$ of $T_\rho$, which is the sum of all the singular values $s_i$ of $T_\rho$, is less than or equal to 1, that is, $\|T_\rho\| = \sum_{i=1}^{3} s_i (T_\rho) \leq 1$, while the state is entangled if $\|T_\rho\| > 1$.

**Proof:** A separable quantum state is a state which cannot be prepared locally and in which there is no quantum correlation. Mathematically, this means that the density matrix $\rho$ can be decomposed into an ensemble of product states:

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

where $\rho_i^A = |\psi_i\rangle_A \langle \psi_i|$, $\rho_i^B = |\phi_i\rangle_B \langle \phi_i|$, $\sum_i p_i = 1$ and $|\psi_i\rangle_A$, $|\phi_i\rangle_B$ are normalized pure states of the subsystems $A$ and $B$, respectively. It should be noted that $\rho_i^{A,B}$ can be characterized by 3-dimensional real vectors (Bloch vectors) $(\lambda_1^i, \lambda_2^i, \lambda_3^i)$ and $(\eta_1^i, \eta_2^i, \eta_3^i)$, respectively, as

$$\rho_i^A = \frac{1}{2} \sum_{k=0}^{3} \lambda_k^i \sigma_k, \rho_i^B = \frac{1}{2} \sum_{k=0}^{3} \eta_k^i \sigma_k,$$

where $\lambda_k^i = \eta_k^i = 1$. The conditions $(\lambda_1^i)^2 + (\lambda_2^i)^2 + (\lambda_3^i)^2 = (\eta_1^i)^2 + (\eta_2^i)^2 + (\eta_3^i)^2 = 1$ should be satisfied because the set of pure states corresponds to the surface of the Bloch sphere. Thus we have

$$\rho_i^A \otimes \rho_i^B = \frac{1}{4} \sum_{k=0}^{3} \sum_{l=0}^{3} \lambda_k^i \sigma_k \otimes \lambda_l^i \sigma_l,$$

and $R_{kl} = Tr((\rho_i^A \otimes \rho_i^B) \sigma_k \otimes \sigma_l) = \lambda_k^i \eta_l^i$. It is obvious then that $T_{\rho_i^A \otimes \rho_i^B} = (\lambda_1^i, \lambda_2^i, \lambda_3^i \eta_1^i, \eta_2^i, \eta_3^i)$ where $t$ denotes the standard transposition. Furthermore, it is clear that

$$\|T_{\rho_i^A \otimes \rho_i^B}\| = \|\lambda_1^i, \lambda_2^i, \lambda_3^i\| \times \|\eta_1^i, \eta_2^i, \eta_3^i\| = 1.$$ 

Hence, $\|T_\rho\| \leq \sum_i p_i \|T_{\rho_i^A \otimes \rho_i^B}\| = 1$ due to the convex property of the trace norm.

To detect entanglement we only need to measure the expectation values of the 9 local observables $\sigma_i \otimes \sigma_j$ to obtain $(T_\rho)_{i,j}$, then compare the trace norm of $T_\rho$ and 1. One question is immediate: is this test stronger or weaker than the standard Bell inequality test? According to [[3], [31], [32]], the CHSH-Bell test can be formulated using the expectation value $\langle B \rangle$ of the Bell operator

$$\mathcal{B} = \sum_{ij=1}^{3} (a_i(c_j + d_j) + b_i(c_j - d_j)) \sigma_i \otimes \sigma_j,$$

with $(a, b, c, d)$ being any real unit vectors and $\sigma_i$ the Pauli matrices, so that $\langle B \rangle = Tr(\rho \mathcal{B}) \leq 2$ should be satisfied by any local classical model. An important advance made by the Horodecki family [[32]] gave a necessary and sufficient condition for two qubits to violate the CHSH-Bell test: the inequality is violated iff $s_1^2 + s_2^2 > 1$, where $s_1$ and $s_2$ are two of the maximal singular values of $T_\rho$. Applying this result, we derive a close relationship between Theorem 1 and the Bell inequality test:

**Theorem 2:** Detection of Theorem 1 is stronger than the Bell inequality, i.e. any entanglement which can be detected by the Bell inequality can also be detected by the condition $\|T_\rho\| > 1$.

**Proof:** For any state satisfying $\|T_\rho\| \leq 1$ we have $s_1 + s_2 + s_3 \leq 1$, so $s_1^2 + s_2^2 \leq s_1^2 + s_2^2 + s_3^2 \leq s_1 + s_2 + s_3 \leq 1$ since $s_i \geq 0$ and $s_2^2 \leq s_i$. Thus any state violating the Bell inequality satisfies $s_1^2 + s_2^2 > 1$ and gives $s_1 + s_2 + s_3 > 1$, which cannot escape detection by Theorem 1.

Now we would like to know how entangled is a given quantum state. For two qubits the degree of entanglement, in terms of the entanglement of formation [[33]], can be calculated by the elegant formula of Wootters [[20]]:

$$E_f(\rho) = h \left( 1 + \sqrt{1 - C^2} \right),$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ and the concurrence $C = \max \{0, \tau_1 - (\tau_2 + \tau_3 + \tau_4)\}$ with $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ being the decreasingly ordered eigenvalues of $\rho(\sigma_2 \otimes \sigma_2)\rho^T(\sigma_2 \otimes \sigma_2)$. Noting that $E_f(\rho)$ is in fact a convex and monotone function with respect to the concurrence $C$, we can use the concurrence for convenience in the following. For any pure state in the standard basis:

$$|\psi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle,$$

where $a, b, c, d$ are complex numbers satisfying $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$, we have a simpler expression for the concurrence $C(|\psi\rangle) = 2|ad - bc|$. Now we can see that Theorem 1 gives further an estimation for the amount of
entanglement in terms of the concurrence:

**Theorem 3:** For any pure state $|\psi\rangle$, $\frac{\|T_{\psi}\|}{2}$ is equal to the concurrence $C(|\psi\rangle)$. For any mixed state it gives a lower bound for the concurrence $C(\rho)$, i.e. $\frac{\|T_{\psi}\|}{2} \leq C(\rho)$.

**Proof:** For the pure state $|\psi\rangle$ of Eq. (7) we have $\{T_{\psi}\}_{ij} = Tr((a, b, c, d)T_{|ab\rangle})(a^*, b^*, c^*, d^*)\sigma_i \otimes \sigma_j$. It is straightforward to calculate the eigenvalues $s_1^2, s_2^2, s_3^2$ of $T_{|\psi\rangle}T_{|\psi\rangle}^\dagger$ and to obtain $\|T_{|\psi\rangle}\| = s_1 + s_2 + s_3 = 4|ad - bc| + 1 = 2C(|\psi\rangle) + 1$. Thus we have $C(|\psi\rangle) = \frac{\|T_{|\psi\rangle}\|}{2} - 1$. Suppose that for the mixed state $\rho$ we have the decomposition which gives the concurrence $C(\rho)$. That is,

$$C(\rho) = \sum_i p_i C(|\psi_i\rangle),$$

(8)

where $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ and $\sum_i p_i = 1$. It is natural that we have $\|T_{\rho}\| = \|\sum_i p_i T_{|\psi_i\rangle}\| \leq \sum_i p_i \|T_{|\psi_i\rangle}\| = \sum_i p_i (2C(|\psi_i\rangle) + 1) = 2C(\rho) + 1$ due to the convexity of the trace norm. Therefore $\frac{\|T_{\rho}\|}{2} - 1$ leads to a lower bound for the concurrence $C(\rho)$.

In most practical applications we do have some prior knowledge of the quantum state for a given system. For example, we can generate a known perfectly entangled pure state $|\psi\rangle$ in one place and send it by some classical or quantum channel to another place where we wish to know its final state on arrival. Due to interaction with noise in the environment, the state $|\psi\rangle$ will evolve to a mixed state. One typical example is after going through a depolarizing channel, the state will transform to:

$$\rho = p |\psi\rangle \langle \psi| + (1-p)I/4. \quad (9)$$

where $I/4$ is the maximally mixed state and $p$ is a constant $0 \leq p \leq 1$ representing the degree of depolarization. This class of states can now be determined by Theorem 1. We know that any pure state $|\psi\rangle$ of Eq. (7) can evolve through a Schmidt decomposition to $|\psi\rangle = \lambda_1 |00\rangle + \lambda_2 |11\rangle$ after a unitary transformation of the local bases. Without loss of generality, we suppose that $|\psi\rangle = a |00\rangle + b |11\rangle$ where $a, b > 0$ and $a^2 + b^2 = 1$, from which we obtain

$$T_{\rho} = \begin{pmatrix} 2abp & 0 & 0 \\ 0 & 2abp & 0 \\ 0 & 0 & -p \end{pmatrix}. \quad (10)$$

Thus $\|T_{\rho}\| = 2|2abp| + | -p | = 4abp + p$. Theorem 1 says that $4abp + p > 1$ implies entanglement, i.e. $1 - p - 4abp < 0$. Noticing that $(1 - p)/4 - abp$ gives the minimal eigenvalue for partial transposition of $\rho$ with respect to the first subsystem, our test is surprisingly equivalent to the Peres-Horodecki criterion and can completely identify this class of states. We only need to make three measurements of the local observables $\sigma_i \otimes \sigma_i (i = 1, 2, 3)$ (or even just two since $(T_{\rho})_{11} = (T_{\rho})_{22}$).

From the above-mentioned operations we notice that $1 - \|T_{\rho}\| = 1 - (T_{\rho})_{11} - (T_{\rho})_{22} + (T_{\rho})_{33} = Tr(\rho(I - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3))$. This suggests that $W = I - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3$ may act as a witness operator to detect entanglement. An entanglement witness $W$ is a Hermitian operator (an observable) which satisfies $Tr(W\rho) > 0$ for all separable states. Thus a state $\rho$ is entangled if we have $Tr(W\rho) < 0 \Rightarrow 26, 27, 28, 29$. In fact, $W$ can here be expressed as

$$W = 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

and is indeed an optimal witness, the same as the one given in [24] up to a constant factor $4$. Thus Theorem 1 is strong enough to detect any degree of entanglement in the class of states represented by Eq. (7). It recognizes entanglement in a subtle way while only requiring less than 3 measurements of local observables. We can also derive an optimal witness operator from the construction of our test, as well as recover the result of [30] which involves three local observable measurements. However, our test is even better in that we need only use two local measurements when we consider the structural character of expression (10).

For pure states, a possible optimal strategy to detect entanglement is to measure the reduced density matrix, as shown in [21]. However, measuring all the 9 local observables by means of Theorem 1 is a costly matter if we have no prior knowledge of the state. Here we propose a better strategy which requires few local operations and uses only 3 observables:

**Proposition 1:** For any pure state $|\psi\rangle$ of Eq. (7) we have $\|\begin{pmatrix} R_{01} & R_{02} & R_{03} \end{pmatrix}\| = \|\begin{pmatrix} R_{10} & R_{20} & R_{30} \end{pmatrix}\| = \sqrt{1 - C^2(|\psi\rangle)}$.

**Proof:** For the pure state $|\psi\rangle$ of Eq. (7) we have $R_{0i} = Tr((a, b, c, d)(a^*, b^*, c^*, d^*)\sigma_0 \otimes \sigma_i)$ and $R_{i0} = Tr((a, b, c, d)(a^*, b^*, c^*, d^*)\sigma_i \otimes \sigma_0)$. It is straightforward to verify that $\|\begin{pmatrix} R_{01} & R_{02} & R_{03} \end{pmatrix}\| = \|\begin{pmatrix} R_{10} & R_{20} & R_{30} \end{pmatrix}\| = \sqrt{1 - 4|ad - bc|^2} = \sqrt{1 - C^2(|\psi\rangle)}$.

Proposition 1 provides a better detection method than Theorem 1 for pure states and only involves 3 local observables of $\sigma_0 \otimes \sigma_i$ or $\sigma_i \otimes \sigma_0 (i = 1, 2, 3)$. After measurement, we can calculate the exact amount of entanglement such as the entanglement of formation in terms of the concurrence $C(|\psi\rangle)$. This is in some degree similar to the scheme to measure the whole reduced density matrix proposed in [21].

The above two examples (the depolarized state and the pure state) show that, with some prior knowledge,
we can greatly reduce the number of local observables required for detecting entanglement. Using Theorem 1 and Proposition 1 is much more efficient than reconstructing the density matrix through quantum tomography. In terms of the factor $f \equiv \text{"number of parameters $\times$ number of copies"}$ defined in 23, our method is parametrically superior ($f = 9 \times 9 = 81$) to the density matrix reconstruction schemes ($f = 15 \times 15 = 225$) and is comparable with the proposal in 23 ($f = 4 \times 20 = 80$). Our scheme is feasible with present mature experimental technology and needs no prior knowledge of the state.

In any case our results lead the way to a new form of entanglement detection with few local measurements. It is stronger than the Bell-CHSH inequality test. If no prior knowledge is available, there exists an infinite number of Bell-CHSH inequalities which must be tested before its violation can be strictly proved. This is not efficient for practical applications. Theorem 1 also suggests that a stronger Bell-like inequality exists if we further add the contribution of the third singular values $s_3$ of $T_p$ to the degree of entanglement in Theorem 1. We only need a certain many-setting Bell-type inequality which is maximally violated iff $s_1^2 + s_2^2 + s_3^2 > 1$ (compare the standard maximal violation iff $s_1^2 + s_2^2 > 1$) or even a weaker condition. This is in fact possible, as proved in Ref. [34], where a certain 3-setting Bell-like inequality is shown to be a sufficient and necessary criterion for separability.

We are now closer to solving the problem of finding the minimal measurement cost of detecting entanglement. Our method gives a better practical test of entanglement without whole state estimation. It recovers previous results for some special classes of states (the depolarized and the pure states) and can be implemented with feasible experimental technology. Though our test is not sufficient to detect all the entangled states, it is stronger than the Bell-CHSH inequality. It also gives a lower bound for the concurrence and thus an estimate of the amount of entanglement for a given state. We expect that a similar result should exist for the case of higher dimensions in a bi-partite system. We leave this as an interesting open problem for future study.

Acknowledgements: the authors would like to thank O. Gühne for introducing their work, and Shao-Ming Fei and Fan Heng for stimulating discussions. K.C. is grateful to Guozhen Yang for his continuous encouragement. This work was supported by the Chinese Academy of Sciences, the National Program for Fundamental Research, the National Natural Science Foundation of China and the China Postdoctoral Science Foundation.

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