ON ORDER AUTOMORPHISMS OF THE EFFECT ALGEBRA

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Abstract. We give short proofs of two Šemrl’s descriptions of order automorphisms of the effect algebra. This sheds new light on both formulas that look quite complicated. Our proofs rely on Molnár’s characterization of order automorphisms of the cone of all positive operators.

Key words: self-adjoint operator, operator interval, effect algebra, order isomorphism, operator monotone function
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1. Introduction

Throughout the paper, let $H$ be a complex Hilbert space of dim $\dim H \geq 2$ with the inner product $\langle \cdot, \cdot \rangle$. By $S(H)$ we denote the set of all bounded linear selfadjoint operators on $H$. An operator $A \in S(H)$ is said to be positive, $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in H$, and $A$ is called strictly positive, $A > 0$, if $A$ is positive and invertible. The set $S(H)$ is partially ordered by the relation $\leq$ defined by $A \leq B \iff B - A \geq 0$. A real function $f$ defined on an interval $J \subseteq [0, \infty)$ is said to be operator monotone on $J$ if, for every operators $A$ and $B$ in $S(H)$ with spectra contained in $J$, the inequality $A \leq B$ implies that $f(A) \leq f(B)$.

An additive map $T : H \to H$ is conjugate-linear if $T(\lambda x) = \overline{\lambda} Tx$ for every $x \in H$ and $\lambda \in \mathbb{C}$. For a bounded conjugate-linear operator $T : H \to H$ we define the adjoint $T^*$ to be the unique bounded conjugate-linear map $T^* : H \to H$ satisfying $\langle Tx, y \rangle = \overline{\langle x, T^* y \rangle}$ for all pairs $x, y \in H$.

Let us recall the definition of the group $G_1$ from [5]. By $GL(H)$ we denote the general linear group on $H$, that is, the multiplicative group of all invertible linear bounded operators on $H$. Furthermore, let $CGL(H)$ denote the group of all invertible bounded either linear or conjugate-linear operators on $H$. The multiplicative group $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ can be naturally embedded into $CGL(H)$ by $z \mapsto zI$, where $I$ denotes the identity operator on $H$. By $G_1$ we denote the quotient group $G_1 = CGL(H)/S^1$. Note that each element of $G_1$ has the form $[T] = \{ zT : z \in S^1 \}$, where $T : H \to H$ is a bounded invertible either linear or conjugate-linear operator on $H$. 

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Assume that $A$ and $B$ are operators in $S(\mathcal{H})$ such that $B - A > 0$. Then operator intervals are defined by

\[ [A, B] = \{ C \in S(\mathcal{H}) : A \leq C \leq B \}, \]

\[ (A, B) = \{ C \in S(\mathcal{H}) : A < C \leq B \}, \]

and

\[ (A, B) = \{ C \in S(\mathcal{H}) : A < C < B \}. \]

Similarly, we define

\[ (A, \infty) = \{ C \in S(\mathcal{H}) : A < C \}, \]

\[ [A, \infty) = \{ C \in S(\mathcal{H}) : A \leq C \}, \]

and $(-\infty, \infty) = S(\mathcal{H})$. The notations $[A, B)$, $(-\infty, A)$, $(-\infty, A]$ are now self-explanatory. The operator interval $[0, I]$ is also called the effect algebra on $\mathcal{H}$.

Let $J$ and $K$ be operator intervals. A bijective map $\phi : J \to K$ is called an order isomorphism if for every pair of operators $A, B$ in $J$ we have

\[ A \leq B \iff \phi(A) \leq \phi(B), \]

and it is called an order anti-isomorphism if for every pair of operators $A, B$ in $J$,

\[ A \leq B \iff \phi(A) \geq \phi(B). \]

If, in addition, $J = K$, the map $\phi$ is called an order automorphism in the first case, and an order anti-automorphism in the second one.

Order isomorphisms of operator intervals have been studied systematically in [4]. Their study was motivated by problems in mathematical physics; see [1], [2] and the references therein. In particular, it was shown in [4] that every operator interval is order isomorphic or order anti-isomorphic to one of the following operator intervals: $(-\infty, \infty)$, $[0, \infty)$, $(0, \infty)$, and $[0, I]$. In the first three cases order automorphisms have simple forms. For example, order automorphisms of the operator interval $[0, \infty)$ are only congruence transformations $A \mapsto SAS^*$ for some invertible bounded either linear or conjugate-linear operator $S : \mathcal{H} \to \mathcal{H}$ (see Theorem 2.3). The form of order automorphisms looks quite complicated only in the case of the effect algebra $[0, I]$. Namely, it was proved in [4] that each order automorphism $\phi$ of the effect algebra $[0, I]$ has the form

\[ \phi(A) = f_p \left( (I + (TT^*)^{-1})^{1/2} (I - (I + TAT^*)^{-1}) (I + (TT^*)^{-1})^{1/2} \right), \quad A \in [0, I], \]

where $T : \mathcal{H} \to \mathcal{H}$ an invertible bounded either linear or conjugate-linear operator, $p$ is a negative real number, and $f_p$ is the operator monotone function defined by (1) below. The aim of the present paper is to give a short proof that better explains this formula.
2. Preliminaries

For every real number \( p < 1 \) the function \( f_p \) is defined by

\[
 f_p(x) = \frac{x}{px + 1 - p}.
\]

For \( p \in [0, 1) \), let the interval \([0, \infty)\) be the domain of \( f_p \), while for \( p < 0 \) its domain is the interval \([0, 1 - \frac{1}{p})\). Clearly, the common domain for all members of the set \( \mathcal{G} = \{ f_p : p < 1 \} \) is the interval \([0, 1]\), and each function \( f_p \in \mathcal{G} \) is a bijective increasing function on the unit interval \([0,1]\) onto itself. By \( \mathcal{P} \) we denote the multiplicative group of positive real numbers. We begin with a lemma that slightly improves [4, Lemma 3.10].

**Lemma 2.1.** For every \( p \in [0, 1) \) the function \( f_p \) is operator monotone on the interval \([0, \infty)\), and for every real number \( p < 0 \) it is operator monotone on the interval \([0, 1 - \frac{1}{p})\). The set \( \mathcal{G} \) with the operation of functional composition is a group of functions from \([0, 1]\) onto itself, and we have

\[
 f_p \circ f_q = f_{p+q-pq}
\]

and

\[
 f_p^{-1} = f_{\frac{1}{p}}^{-1}.
\]

Moreover, the map \( p \mapsto f_{1-p} \) is a group isomorphism between the groups \( \mathcal{P} \) and \( \mathcal{G} \).

**Proof.** Since the function \( f_0(x) = x \) is clearly operator monotone on the interval \([0, \infty)\), we can assume that \( p \neq 0 \). We will use the well-known fact that if \( A \) and \( B \) in \( \mathcal{S}(\mathcal{H}) \) are strictly positive operators, then \( A \leq B \iff A^{-1} \geq B^{-1} \). If \( p \in (0, 1) \), then for any positive operator \( A \),

\[
 f_p(A) = \frac{1}{p} I - \frac{1-p}{p^2} \left( A + \left( \frac{1}{p} - 1 \right) I \right)^{-1},
\]

and so the function \( f_p \) is operator monotone on the interval \([0, \infty)\). If \( p < 0 \), then for any operator \( A \in [0, (1 - 1/p)I) \),

\[
 f_p(A) = \frac{1}{p} I + \frac{1-p}{p^2} \left( \left( 1 - \frac{1}{p} \right) I - A \right)^{-1},
\]

implying that the function \( f_p \) is operator monotone on the interval \([0, 1 - \frac{1}{p})\).

It is straightforward to verify the remaining assertions. \( \square \)

The following assertion is actually shown in the proof of [4, Corollary 5.2].

**Lemma 2.2.** If \( \phi : [0, I] \to [0, I] \) is an order automorphism, then

\[
 \phi([0, I]) = (0, 1].
\]

The following description of order automorphisms of the operator interval \([0, \infty)\) was proved in [1]; see also [2, Theorem 2.5.1].
Theorem 2.3. Assume that \( \phi : [0, \infty) \to [0, \infty) \) is an order automorphism. Then there exists an invertible bounded either linear or conjugate-linear operator \( S : \mathcal{H} \to \mathcal{H} \) such that
\[
\phi(A) = SAS^*
\]
for every \( A \in [0, \infty) \).

3. Results

The functions from the group \( \mathcal{G} \) induce order automorphisms of the effect algebra \([0, I]\) via functional calculus.

Lemma 3.1. For every real number \( p < 1 \), the map \( A \mapsto f_p(A) \), \( A \in [0, I] \), is an order automorphism and a homeomorphism of the effect algebra \([0, I]\).

Proof. Since the functions \( f_p \) and \( f_p^{-1} \) are operator monotone on the interval \([0, 1]\) by Lemma 2.1, the map \( A \mapsto f_p(A) \) is an order automorphism of the effect algebra \([0, I]\). Furthermore, the two formulas for \( f_p(A) \) given in the proof of Lemma 2.1 ensure that the map \( A \mapsto f_p(A) \) is a composition of translations, multiplication by a real constant and the map \( A \mapsto A^{-1} \) (defined on the set of all invertible operators). Since all of these maps are continuous, we conclude together with (2) that the map \( A \mapsto f_p(A) \) is continuous in both directions. This completes the proof. \( \square \)

The following order automorphisms of the effect algebra \([0, I]\) were introduced in [4].

Lemma 3.2. Let \( p \) be a negative real number and \( T : \mathcal{H} \to \mathcal{H} \) an invertible bounded either linear or conjugate-linear operator. Then the map \( \phi_{p,T} : [0, I] \to [0, I] \) given by
\[
\phi_{p,T}(A) = f_p \left( (I + (TT^*)^{-1})^{1/2}(I - (I + TAT^*)^{-1})(I + (TT^*)^{-1})^{1/2} \right), \ A \in [0, I],
\]
is an order automorphism and a homeomorphism of the effect algebra \([0, I]\).

Proof. Since the map \( A \mapsto I - (I + TAT^*)^{-1}, \ A \in [0, I], \) is an order isomorphism of the effect algebra \([0, I]\) onto the operator interval \([0, (I + (TT^*)^{-1})^{-1}]\), the map
\[
A \mapsto (I + (TT^*)^{-1})^{1/2}(I - (I + TAT^*)^{-1})(I + (TT^*)^{-1})^{1/2} = f_p^{-1}(\phi_{p,T}(A)), \ A \in [0, I],
\]
is an order automorphism of the effect algebra \([0, I]\). Now, we apply Lemma 3.1 to conclude that \( \phi_{p,T} \) is also an order automorphism of the effect algebra \([0, I]\).

To prove the continuity of \( \phi_{p,T} \), we note that
\[
\phi_{p,T}(A) = f_p \left( \frac{1}{2} (I + (TT^*)^{-1})^{1/2} f_{1/2}(TAT^*)(I + (TT^*)^{-1})^{1/2} \right), \ A \in [0, I].
\]
Hence, $\phi_{p,T}$ is a composition of two congruence transformations and two maps from the Lemma 3.1. Since all of them are continuous, it is continuous as well. The same conclusion holds for the inverse of $\phi_{p,T}$, completing the proof. □

As the main contribution of this paper, we give a short proof of [4, Theorem 2.3]. Our proof relies on Theorem 2.3, and it gives us a better insight into the structure of order automorphisms of the effect algebra $[0, I]$.

**Theorem 3.3.** Assume that $\phi : [0, I] \to [0, I]$ is an order automorphism. Then there exist a negative real number $p$ and an invertible bounded either linear or conjugate-linear operator $T : \mathcal{H} \to \mathcal{H}$ such that $\phi(A) = \phi_{p,T}(A)$ for all $A \in [0, I]$.

**Proof.** By Lemma 2.2, the restriction $\phi|_{[0, I]}$ is an order automorphism on the operator interval $(0, I]$. The map $A \mapsto A^{-1} - I$ is an order anti-isomorphism of $(0, I]$ onto $[0, \infty)$ and its inverse $A \mapsto (I + A)^{-1}$ is an order anti-isomorphism of $[0, \infty)$ onto $(0, I]$. These observations together with Theorem 2.3 imply that there exists an invertible bounded either linear or conjugate-linear operator $S : \mathcal{H} \to \mathcal{H}$ such that

$$\phi(A) = (I + S(A^{-1} - I)S^*)^{-1}$$

for all $A \in (0, I]$. Choose a real number $\lambda \in (1, \infty)$ such that $\lambda > \|SS^*\| = \|S\|^2$. Let $A \in (0, I]$. Then

$$\phi(A)^{-1} + (\lambda - 1)I = \lambda I - SS^* + SA^{-1}S^* = (\lambda I - SS^*)^{1/2}(I + RA^{-1}R^*)(\lambda I - SS^*)^{1/2},$$

where an invertible bounded either linear or conjugate-linear operator $R : \mathcal{H} \to \mathcal{H}$ is given by

$$R = (\lambda I - SS^*)^{-1/2} S.$$ 

The equality $R^* = T^{-1}$ defines an invertible bounded either linear or conjugate-linear operator $T : \mathcal{H} \to \mathcal{H}$. Clearly, we have

$$(TT^*)^{-1} = RR^* = (\lambda I - SS^*)^{-1} SS^*,$$

so that

$$I + (TT^*)^{-1} = \lambda (\lambda I - SS^*)^{-1}$$

and

$$\lambda I - SS^* = \lambda (I + (TT^*)^{-1})^{-1}.$$ 

Therefore,

$$\phi(A)^{-1} + (\lambda - 1)I = (\lambda I - SS^*)^{1/2}(I + (TAT^*)^{-1})(\lambda I - SS^*)^{1/2} =$$

$$= \lambda (I + (TT^*)^{-1})^{-1/2}(I + TAT^*)(TAT^*)^{-1}(I + (TT^*)^{-1})^{-1/2}. $$

It follows that

$$\lambda(\phi(A)^{-1} + (\lambda - 1)I)^{-1} = (I + (TT^*)^{-1})^{1/2}(TAT^*)(I + TAT^*)^{-1}(I + (TT^*)^{-1})^{1/2},$$
and so
\[ f_q(\phi(A)) = (I + (TT^*)^{-1})^{1/2}(I - (I + TAT^*)^{-1})(I + (TT^*)^{-1})^{1/2}, \]
where \( q = 1 - 1/\lambda \in (0, 1) \). Letting \( p = q/(q - 1) \in (-\infty, 0) \) we have \( f_q = f_p^{-1} \) by (2), and so we obtain that \( \phi(A) = \phi_{p,T}(A) \) for every \( A \in [0, I] \).

It remains to prove that \( \phi(A) = \phi_{p,T}(A) \) for every \( A \in [0, I] \). In other words, we need to show that \( \phi_{p,T} \) is the only order automorphism on the effect algebra \([0, I]\) that extends the restriction \( \phi_{[0,I]} \). Given \( A \in [0, I] \), define a decreasing sequence \( \{A_n\}_{n \in \mathbb{N}} \) in the order interval \((0, I]\) by \( A_n = (1 - \frac{1}{n})A + \frac{1}{n}I \). Clearly, it converges to \( A \), and so Lemma \( \text{3.2} \) implies that \( \{\phi_{p,T}(A_n)\} = \{\phi(A_n)\} \) converges to \( B = \phi_{p,T}(A) \). Since the sequence \( \{\phi(A_n)\} \) is decreasing, it is easily seen that \( \phi(A_n) \geq B \) for all \( n \). From \( A_n \geq A \) it follows that \( \phi(A_n) \geq \phi(A) \) for all \( n \), and so \( B \geq \phi(A) \). Since \( \phi \) is bijective, there is an operator \( A' \in [0, I] \) such that \( \phi(A') = B \). Hence, \( A' \geq A \), as \( \phi(A') \geq \phi(A) \). Since \( \phi(A_n) \geq \phi(A') \), we have \( A_n \geq A' \) for all \( n \), and so \( A \geq A' \). Thus, \( A = A' \) and \( \phi(A) = B \).
So, \( \phi(A) = \phi_{p,T}(A) \) for every \( A \in [0, I] \).

The preceding proof also reveals that the group of order automorphisms of the effect algebra \([0, I]\) is isomorphic to the group \( G_1 \), as it has been shown in [5].

In [3] another description of order automorphisms of the effect algebra \([0, I]\) was given. We now show that it is closely related to the description from Theorem 3.3.

**Theorem 3.4.** Assume that \( \phi : [0, I] \to [0, I] \) is an order automorphism. Then there exist a negative real number \( p \), a real number \( r \in (0, 1) \) and an invertible bounded either linear or conjugate-linear operator \( S : \mathcal{H} \to \mathcal{H} \) with \( \|S\| \leq 1 \) such that
\[ \phi(A) = f_p \left( (f_r(SS^*))^{-1/2} f_r(SAS^*)(f_r(SS^*))^{-1/2} \right) \]
for all \( A \in [0, I] \).

**Proof.** By Theorem 3.3 there exist a negative real number \( p \) and an invertible bounded either linear or conjugate-linear operator \( T : \mathcal{H} \to \mathcal{H} \) such that \( \phi(A) = \phi_{p,T}(A) \) for all \( A \in [0, I] \). If \( \|T\| \leq 1 \) then take \( S = T \) and observe that
\[ f_p^{-1}(\phi_{p,T}(A)) = (f_{1/2}(SS^*))^{-1/2} f_{1/2}(SAS^*)(f_{1/2}(SS^*))^{-1/2}, \]
so that the desired formula with \( r = 1/2 \) follows.

Assume therefore that \( \|T\| > 1 \). Let
\[ S = \frac{1}{\|T\|} T \quad \text{and} \quad r = \frac{\|T\|^2}{1 + \|T\|^2}, \]
so that
\[ \|S\| = 1 \quad \text{and} \quad \frac{r}{1 - r} = \|T\|^2. \]
Then
\[ (f_r(SS^*))^{-1} = r + (1 - r)(SS^*)^{-1} = r(I + (TT^*))^{-1}, \]
and 
\[ r f_r(SAS^*) = I - (1 - r)(rSAS^* + (1 - r)I)^{-1} = I - (I + TAT^*)^{-1}, \]
and so 
\[ (f_r(SS^*))^{-1/2} f_r(SAS^*) (f_r(SS^*))^{-1/2} = (I + (TT^*)^{-1})^{1/2}(I - (I + TAT^*)^{-1})(I + (TT^*)^{-1})^{1/2} = f_r^{-1}(\phi_{p,T}(A)), \]
completing the proof. \(\Box\)

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