Research Article

Three Nontrivial Solutions for Second-Order Partial Difference Equation via Morse Theory

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In the present paper, we consider a second-order nonlinear partial difference equation with Dirichlet boundary conditions. Applying variational method together with the Morse theory, we establish a criterion to obtain at least three nontrivial solutions. An example is also elaborated to demonstrate our main result.

1. Introduction

Let \( \mathbb{N} \) and \( \mathbb{Z} \) are natural number set and integer set, respectively. Denote \( \mathbb{Z}(a,b) = \{a,a+1, \cdots ,b\} \) for any \( a, b \in \mathbb{Z} \) and \( a \leq b \). \( \Delta_1 \) and \( \Delta_2 \) are the forward difference operators defined by \( \Delta_1 u(i,j) = u(i+1,j) - u(i,j) \) and \( \Delta_2 u(i,j) = u(i,j+1) - u(i,j) \). Given integers \( T_1, T_2 \geq 2 \), write \( \Omega = \mathbb{Z}(1,T_1) \times \mathbb{Z}(1,T_2) \), we deal with self-adjoint partial difference equation of the form

\[
\begin{align*}
\Delta_1 [p(i-1,j)\Delta_1 u(i-1,j)] + \Delta_2 [r(i,j-1)\Delta_2 u(i,j-1)] \\
+ q(i,j)u(i,j) = -f((i,j),u(i,j)), \ (i,j) \in \Omega,
\end{align*}
\]

subject to Dirichlet boundary conditions

\[
u(i,0) = u(i, T_2 + 1) = 0 \quad i \in \mathbb{Z}(1,T_1), \ u(0,j) = u(T_1 + 1,j) = 0 \quad j \in \mathbb{Z}(1,T_2). \tag{2}
\]

Here, \( f((i,j),u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is differentiable in \( u \) and there exists a function \( F((i,j),u) \) such that \( F((i,j),u) = \int_0^u f((i,j),r)dr \) for each \( (i,j) \in \Omega \). Further, for all \( (i,j) \in \Omega \), throughout this paper, we assume that

\[
\begin{align*}
(f_0)(f((i,j),0)) &= 0, \\
(p)p(i,j) > 0, \ r(i,j) > 0, \ q(i,j) \leq 0 \text{ and } q(i,j)\equiv 0.
\end{align*}
\]

As usual, if \( u \) satisfies (1.1)-(1.2), we say \( u \) is a solution of (1) and (2). According to \( f_0 \), it is easy to see that (1) and (2) admit a trivial solution \( u = 0 \). Meanwhile, what we care about is the existence and multiplicity of nontrivial solutions of (1)-(2).

Consider (1), a partial difference equation, involving functions with two discrete variables, it can be used to many investigations related to image processing, population models, and digital control systems [1]. Due to the rapid development of modern digital computing devices, more and more important information about the behavior of complex systems can be revealed by simulations by modern digital computing devices in a simple way, which contributes greatly to the increasing interest in discrete problems and they are investigated in many literatures, for example, [2–8].

Let \( r(i,j-1) \equiv 0 \) in (1), it becomes an ordinary difference equation, namely

\[
\Delta [p(t)\Delta u(t-1)] + q(t)u(t) = f(t,u(t)), \ t \in \mathbb{Z}. \tag{3}
\]

(3) has captured many interests and has been studied extensively. Here, mention a few and [9] considered the existence of periodic solutions via critical theory. Ma and Guo [10] discussed homoclinic orbits. [11] got sign-changing solutions, whereas, when \( r(i,j-1) \neq 0 \), it seems that there are rare literatures.
Moreover, (1) can be regarded as a discrete analog of a partial differential equation. It is well known that, with the rapid development of critical point theory, it becomes a more and more powerful to deal with the existence and multiplicity solutions of both partial differential equations and partial difference equations [12–14]. As mentioned, the Morse theory is a very useful tool to study the existence of multiple solutions of differential equations having variational structure, and it has been applied successfully to study differential equations [15–18]. At the same time, difference equations, regarded as discretizations of differential equations, are considered and multiple solutions are achieved via the Morse theory in some literatures [19, 20]. However, there are few literatures using the Morse theory to study partial difference equations. Due to abovementioned reasons, we devote to studying the Dirichlet boundary value problem of second-order partial difference equations (equations (1) and (2)) by the Morse theory.

The organization of the rest of this paper reads as follows. In Section 2, we construct a suitable variational framework corresponding to (1) and (2) and reduce the existence of solutions of (1) and (2) to the existence of critical points of the associated functional. With preparation of Section 2, Section 3 not only displays the main result of this paper but also provides detailed proof of the main result. Finally, an example is exhibited to demonstrate our main result in Section 4.

2. Variational Structure and Some Auxiliary Results

In this section, we construct a variational functional corresponding to (1) and (2) on a suitable function space and state some basic facts.

Let

\[ S = \{ u : \mathbb{Z}(0, T_1 + 1) \times \mathbb{Z}(0, T_2 + 1) \rightarrow \mathbb{R} \text{ such that } u(i, 0) = u(i, T_2 + 1) = 0, \forall i \in \mathbb{Z}(0, T_1 + 1) \text{ and } u(0, j) = u(T_1 + 1, j) = 0, j \in \mathbb{Z}(0, T_2 + 1) \} \].

(4)

For any \( u, v \in S \), define an inner product \( \langle \cdot, \cdot \rangle \) by

\[ \langle u, v \rangle = \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} u(i, j)v(i, j) \].

(5)

then the induced norm is

\[ ||u|| = \sqrt{\langle u, u \rangle} = \left( \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} |u(i, j)|^2 \right)^{1/2}, \forall u \in S. \]

(6)

Hence, \((S, \langle \cdot, \cdot \rangle)\) is a \( T_1T_2 \)-dimensional Hilbert space.

Consider the functional \( I : S \rightarrow \mathbb{R} \) as the following form:

\[ I(u) = \frac{1}{2} \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} p(i-1, j)\Delta_i u(i-1, j)^2 \]

\[ + \frac{1}{2} \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} r(i, j-1)\Delta_j u(i, j-1)^2 \]

\[ - \frac{1}{2} \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} q(i, j)|u(i, j)|^2 \]

\[ - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} F((i, j), u(i, j)), \forall u \in S. \]

(7)

Note that \( f((i, j), u(i, j)) \) is differentiable in \( u \), which ensures \( I(u) \) is twice differentiable. What is more, for any \( u, v \in S \), make use of the boundary conditions (2), we have

\[ \langle I'(u), v \rangle = -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \{ \Delta_i [p(i-1, j)\Delta_i u(i-1, j)] \]

\[ + \Delta_j [r(i, j-1)\Delta_j u(i, j-1)] + q(i, j)u(i, j) \]

\[ + f((i, j), u(i, j))v(i, j), \]  

(8)

and we transfer the existence of nontrivial solutions of (1) and (2) into the existence of critical points of \( I \) on \( S \).

In the following, we introduce some basic facts.

Definition 1 [21]. The functional \( I \) satisfies the weaker Cerami condition ((C)\(_c\) condition for short) at the level \( c \in \mathbb{R} \) if any sequence \( \{ u_n \} \subseteq S \) satisfying \( I(u_n) \longrightarrow c, (1 + ||u_n||)||I'(u_n)|| \longrightarrow 0 \) as \( n \rightarrow \infty \) has a convergent subsequence. \( I \) satisfies (C) condition if \( I \) satisfies (C)\(_c\) condition at any \( c \in \mathbb{R} \).

Definition 2 [16, 22]. Let \( u_0 \) be an isolated critical group of \( I \) with \( I(u_0) = c \in \mathbb{R} \) and \( U \) be a neighborhood of \( u_0 \), the group

\[ C_q(I, u_0) = \text{H}_q(I' \cap U, I' \cap U \setminus u_0), q \in \mathbb{Z}. \]

(9)

is called the q-th critical group of \( I \) at \( u_0 \). If \( I(G) \) is bounded from below by \( a \in \mathbb{R} \) and \( I \) satisfies (D)\(_c\) condition for all \( c \leq a \). Then, the group

\[ C_q(I, \infty) = \text{H}_q(S, I'), q \in \mathbb{Z}. \]

(10)

is called the q-th critical group of \( I \) at infinity.

In applications of Morse theory, it is necessary to make the functional satisfy the deformation condition (D), which is introduced by [23]. And [24] proves that once the functional \( I \) satisfies the (C) condition, it must satisfy the deformation condition (D). Let \( S \) be a real Hilbert space and \( I \in C^2(S, \mathbb{R}) \). Denote Morse index and zero dimension of
\[ u_0 \text{ by } \mu(u_0) \text{ and } \nu(u_0), \text{ respectively. The following propositions are essential tools to verify our main result.} \]

**Proposition 3** [12]. Let I satisfy the (D) condition. We have 
\[ (I_1) \text{ if } C_q(I, \infty) \neq 0 \text{ holds for some } q; \text{ then, I must have a critical point } x \text{ such that } C_q(I, x) \neq 0; \]
\[ (I_2) \text{ if } 0 \text{ is the isolated critical point of } I \text{ and } C_q(I, \infty) \neq C_q(I, 0) \text{ holds for some } q; \text{ then, I has a nonzero critical point.} \]

**Proposition 4** [18]. Suppose \( u_0 \) is the isolated critical point of \( I \) and \( I'(u_0) \) is a Fredholm operator. Further, if \( \mu(u_0) \) and \( \nu(u_0) \) are finite, there holds
\[ (I_3) \text{ if } u_0 \text{ is the local minimum point of } I, \text{ then} \]
\[ C_q(I, u_0) \supseteq \delta_{k_0}Z, \quad q = 0, 1, 2, \ldots. \]

**Proposition 5** [18]. Let \( A : S \to S \) be a self-adjoint linear operator with the isolated spectral point 0 and write I in the form
\[ I(u) = \frac{1}{2} (Au, u) + Q(u), \]
where \( Q \in C^1(S, \mathbb{R}) \) such that \( \lim_{|u| \to \infty} \|Q'(u)\| = 0. \)

Define \( V = \ker A, W = V^\perp = W^* \oplus W^\perp, \) and \( \mu = \dim V^\perp \) and \( \nu = \dim V \neq 0 \) are finite numbers. Suppose \( f \) satisfies the (D) condition and \( I \) satisfies the angle condition at infinity; \( (AC^\infty) \) there exist constants \( M > 0 \) and \( \alpha \in (0, 1) \) such that
\[ \pm \langle I'(u), \varphi \rangle \geq 0, \forall u = v + w, \|u\| \geq M, \quad \|w\| \leq \alpha\|u\|. \]

Then,
\[ C_q(I, \infty) \supseteq \delta_{k_k}Z, \]
where \( k_k = \mu, k_k = \mu + \nu, v \in V, w \in W. \)

In our proofs, we also need the following Mountain Pass Lemma.

**Proposition 6** [22]. Let \( S \) be a real Banach space and \( I \in C^1(S, \mathbb{R}) \) satisfy the Palais-Smale (PS in short) condition. Further, if \( I(0) = 0 \) and
\[ (I_1) \text{ there exist constants } \rho, \alpha > 0 \text{ such that } I_{|B_\rho} \geq \alpha, \]
\[ (I_2) \text{ there is } e \in S \setminus B_\rho \text{ such that } I(e) \leq 0. \]

Then, \( I \) possesses a critical value \( c \geq 0 \) given by
\[ c = \inf_{h \in \Gamma} \sup_{x \in [0, 1]} f(h(x)), \]
where
\[ \Gamma = \{ h \in C([0, 1], S) \mid h(0) = 0, h(1) = e \}. \]

Denoted by \( \lambda_k \) be the k-th eigenvalue corresponding to linear eigenvalue problem of the equation (1), namely,
\[ \Delta_I [p(i - 1, j)\Delta_I u(i - 1, j)] + \Delta_I [r(i, j - 1)\Delta_I u(i - 1, j - 1)] + q(i, j)u(i, j) + \lambda u(i, j) = 0. \]

We claim

**Lemma 7.** For \( 1 \leq k \leq T_1T_2 \), the eigenvalue \( \lambda_k \) of (17)-(2) is positive.

**Proof.** Rewrite \( u \) as
\[ u = (u(0, 1), \cdots, u(T_1, 1), \cdots, u(T_1 + 1, T_2))^T, \]
where \( ^T \) denotes the transpose of vector. Then, the functional \( I \), defined by (7), can be expressed by
\[ I(u) = \frac{1}{2} (Au, u) - \mathfrak{F}(u), \forall u \in S, \]
where \( \mathfrak{F}(u) = \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} F(i, j, u(i, j)), \) \( A \) is the \( T_1T_2 \times T_1T_2 \) matrix corresponding to the quadratic form
\[ \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} [p(i - 1, j) \Delta_I u(i - 1, j)]^2 + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} [r(i, j - 1) \Delta_I u(i, j - 1)]^2 - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} q(i, j) |u(i, j)|^2. \]

\[ \square \]

At first, it is easy to get that \( A \) possesses at most \( T_1T_2 \) eigenvalues. Subsequently, we need to prove that \( A \) is a positive definite matrix.

In fact, for all \( u \in \mathbb{R} \) and \( u \neq 0 \), we have
\[ u^T Au = \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} [p(i - 1, j) \Delta_I u(i - 1, j)]^2 + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} [r(i, j - 1) \Delta_I u(i, j - 1)]^2 - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} q(i, j) |u(i, j)|^2 \geq 0. \]

If \( u^T Au = 0 \), according to \( (p) \), there hold
\[ p(i - 1, j) \Delta_I u(i - 1, j)]^2 = 0, \quad \forall (i, j) \in \mathbb{Z}(1, T_1 + 1) \times \mathbb{Z}(1, T_2), \]
\[ r(i, j - 1) \Delta_I u(i, j - 1)]^2 = 0, \quad \forall (i, j) \in \mathbb{Z}(1, T_1) \times \mathbb{Z}(1, T_2 + 1), \]
\[ q(i, j) |u(i, j)|^2 = 0, \quad \forall (i, j) \in \Omega. \]
which lead to that $u(i, j) = u(i + 1, j) = u(i + 1, j + 1)$ for all $(i, j) \in \Omega$ and there exists some $(i_0, j_0) \in \Omega$ such that $u(i_0, j_0) = 0$. Together with Dirichlet boundary conditions (2), we can deduce that $u(1, 1) = \cdots = u(T_1, T_2) = 0$, which show a contradiction with respect to $u \neq 0$. Therefore, $A$ is positive definite, which ensures that $\lambda_k > 0$ for all $k \in \{1, T_1, T_2\}$. Without loss of generality, we can rearrange all eigenvalues of $A$ as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{T_1 T_2}$.

### 3. Main Result and Its Proof

Thanks to above preparations, we are ready to establish our main result and state the detailed proof of it.

First, we give some notations.

Let $\phi_k = (\phi_k(1), \phi_k(2), \cdots, \phi_k(T_1 T_2))^T$ and $1 \leq k \leq T_1 T_2$, represent an eigenvector corresponding to the eigenvalue $\lambda_k$. Write $W^* = \text{span}\{\phi_1, \cdots, \phi_{T_1 T_2}\}$, $W^0 = \text{span}\{\phi_k\}$, and $W^+ = (W^- \oplus W^0)^*$; then, $S$ can be split into

$$S = W^- \oplus W^+ \oplus W^0.$$  \hfill (23)

Denote

$$\lim_{|u| \to \infty} f((i, j), u) = \lambda_k, \quad \forall (i, j) \in \Omega,$$  \hfill (24)

and $g((i, j), u) = f((i, j), u) - \lambda_k u$. Then, it yields that

$$\lim_{|u| \to \infty} \frac{g((i, j), u)}{u} = 0, \quad \forall (i, j) \in \Omega.$$  \hfill (25)

We also need the following denotation.

($g^+$) If $\|u_m\| \to \infty$ such that $\|v_n\|/\|u_m\| \to 1$ as $n \to \infty$, then there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i, j), u_n(i, j), v_n(i, j)) \geq \delta, \quad n \geq N.$$  \hfill (26)

where $u_n = v_n + w_n, \ v_n \in W^0, \ w_n \in W^+ \oplus W^-.$

Now we state our main result as the following.

### Theorem 8.

Let $(f_0, \gamma, p)$, and (24) hold. If, for all $(i, j) \in \Omega$, $(V_1)$ $F \in C^2(\Omega, \mathbb{R})$ and $\partial^2 F((i, j), u)/\partial u^2 > 0, \ u \in \mathbb{R}$, $(V_2)$ $\partial^2 F((i, j), u)/\partial u^2 = 0, \ u \in \mathbb{R}$, are satisfied. Then, (1) and (2) possess at least three nontrivial solutions, including a positive solution and a negative solution under either $(\tilde{1})(g^+)$ with $k \geq 2$ or $(\tilde{1})(g^-)$ with $k \geq 3$ is fulfilled.

According to Proposition 3, we are to verify the compactness conditions (the $(C)$ condition) of $I$ under the assumptions given in our theorem.

### Lemma 9.

Let $(f_0)$ and (24) hold. If $(g^+)$ holds, then $I$ satisfies the $(C)$ condition.

### Proof.

Suppose that there exists a sequence $\{u_m\} \subseteq S$ such that

$$I(u_m) \to c, \ (1 + \|u_m\|)\|I'(u_m)\| \to 0, \ m \to \infty.$$  \hfill (27)

Due to $(S, \langle \cdot, \cdot \rangle)$ is a $T_1 T_2$-dimensional real Hilbert space, it suffices to verify that $\{u_m\}$ is bounded. Arguing indirectly, suppose $\{u_m\}$ is unbounded, namely,

$$\|u_m\| \to \infty, \ as \ m \to \infty.$$  \hfill (28)

Denote $\bar{u}_m = u_m/\|u_m\|$, then $\|\bar{u}_m\| = 1$. As a result, there is a convergent subsequence for $\{\bar{u}_m\}$. It might as well be set as itself, and there exists $\bar{u} \in S, \|\bar{u}\| = 1$ such that $\bar{u}_m \to \bar{u}$.

Recall $g((i, j), u) = f((i, j), u) - \lambda_k u$, then for all $\varphi \in S$, we have

$$\left\langle I'(u_m), \varphi \right\rangle = - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \Delta_1 \left[ p(i - 1, j) \Delta_1 \bar{u}_m(i - 1, j) \right] + \Delta_2 \left[ r(i - 1, j) \Delta_2 \bar{u}_m(i - 1, j) \right] + \frac{g((i, j), u_m(i, j))}{\|u_m\|} + \lambda_k \bar{u}_m(i, j) \varphi(i, j).$$  \hfill (29)

Meanwhile, (25) implies that $g((i, j), u_m)/\|u_m\| \to 0$ as $m \to \infty$. Thus, (29) gives

$$\Delta_1 \left[ p(i - 1, j) \Delta_1 \bar{u}(i - 1, j) \right] + \Delta_2 \left[ r(i - 1, j) \Delta_2 \bar{u}(i - 1, j) \right] + \frac{g(i, j) \bar{u}(i, j) + \lambda_k \bar{u}(i, j)}{\|u_m\|} = 0.$$  \hfill (30)

Denote $u_m = v_m + w_m$, where $v_m \in W^0, \ w_m \in W^+$, and $w_m \in W^-$; then according to $I'(u) = Au - f(u) = Au - g(u) - \lambda_k u$ we get

$$\left\langle \lambda_{k+1} - \lambda_k \right\rangle \|u_m^+\|^2 = \left\langle \lambda_{k+1} w_m^+, w_m^+ \right\rangle - \left\langle \lambda_k w_m^+, w_m^+ \right\rangle \leq \left\langle A w_m^+, w_m^+ \right\rangle - \left\langle \lambda_k w_m^+, w_m^+ \right\rangle = \left\langle A w_m^+, w_m^+ \right\rangle - \left\langle \lambda_k w_m^+, w_m^+ \right\rangle = \left\langle g(u_m), w_m^+ \right\rangle + \left\langle I'(u_m), w_m^+ \right\rangle.$$  \hfill (31)

Hence,

$$\frac{\|w_m^+\|}{\|u_m\|} \to 0, \ \frac{\|w_m^-\|}{\|u_m\|} \to 0, \ m \to \infty.$$  \hfill (32)
Since \( \forall m \geq N \),
\[
\|I(u_m)\| \longrightarrow 1, m \longrightarrow \infty. \tag{33}
\]

According to \((g^*)\), there exist \( \delta > 0 \) and \( N \in \mathbb{N} \) such that
\[
\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i, j), u_m(i, j))), v_m(i, j)) \geq \delta, \forall m \geq N. \tag{34}
\]

Making use of (30), we get
\[
\langle I'(u_m), v_m \rangle = -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \{ \Delta_i|p(i-1, j)\Delta_i u_m(i-1, j)] + q(i, j) u_m(i, j) + f((i, j), u_m(i, j)) v_m(i, j) \}
\]
\[
+ \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \{ \Delta_i|p(i-1, j)\Delta_i u_m(i-1, j)] + q(i, j) u_m(i, j) + g((i, j), u_m(i, j)) + \lambda_k u_m(i, j) v_m(i, j) \}
\]
\[
= -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i, j), u_m(i, j)), v_m(i, j)) \leq -\delta, \forall m \geq N. \tag{35}
\]

Furthermore, since \( u_m = v_m + w_m \), it follows that
\[
\|I'(u_m)\|\|u_m\| \geq \|I'(u_m)\|\|v_m\| \geq \langle I'(u_m), v_m \rangle \geq \delta, \forall m \geq N, \tag{36}
\]
which contradicts the hypothesis. Therefore, \( \{u_m\} \) is bounded.

Now, we will calculate its critical groups at infinity, \( \mathcal{C}_q(1, \infty) \), via Proposition 5.

**Lemma 10.** Let \((f_0)\) and \((24)\) hold, then

(V3) \( \mathcal{C}_q(1, \infty) \cong \delta \Delta_k \mathbb{Z} \), if \((g^*)\) holds;

(V4) \( \mathcal{C}_q(1, \infty) \cong \delta \Delta_k \mathbb{Z} \) if \((g^-)\) holds.

**Proof.** Since \( S \) is a \( T_1 \times T_2 \)-dimensional Hilbert space and \( A \) is a positive definite matrix, there exists a self-adjoint linear operator, which can be still represented by \( A \). Write
\[
\langle Ju, v \rangle = \lambda_k \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (u(i, j), v(i, j)). \tag{37}
\]

Then,
\[
I(u) = \frac{1}{2} \langle (A - J)u, u \rangle - G(u), \tag{38}
\]
which has the form (12) with \( Q(u) = -G(u) \). According to (25), \( Q(u) \) satisfies
\[
\lim_{\|u\| \to \infty} \frac{||Q'(u)||}{\|u\|} = -\lim_{\|u\| \to \infty} \frac{\|g(u)\|}{\|u\|} = 0. \tag{39}
\]

Moreover, \( I \) satisfies the \((C)\) condition guarantees that the \((D)\) condition is fulfilled and \( \ker (A - J) = \text{span}\{\phi_k\} \).

Subsequently, what we need to do is to show \((AC^-)\) is met with the condition \((g^*)\). Otherwise, for every natural number \( m \) and every \( x_m = 1/m \), there exists \( u_m = v_m + w_m \in W^0 \oplus (W^* \oplus W^-) \), where \( v_m \in W^0 \), \( w_m \in W^* \oplus W^- \) such that \( u_m \geq m \), \( \|w_m\| \leq (1/m)\|u_m\| \). Therefore, it yields that
\[
\|u_m\| \longrightarrow \infty, \|v_m\| \longrightarrow 1, m \longrightarrow \infty. \tag{40}
\]

By \((g^*)\), there exist \( \delta > 0 \) and \( N \in \mathbb{N} \) such that
\[
\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i, j), u_m(i, j)), v_m(i, j)) \geq \delta, \forall m \geq N. \tag{41}
\]

Therefore,
\[
\langle I'(u_m), v_m \rangle = -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i, j), u_m(i, j)), v_m(i, j)) \leq -\delta, \forall m \geq N. \tag{42}
\]

Meanwhile, the assumption \((AC^-)\) indicates that
\[
\langle I'(u_m), v_m \rangle > 0, \tag{43}
\]
which is inconsistent with (42). As a result, \( I \) satisfy \((AC^-)\) condition.

In order to gain mountain pass type critical points, we need the following Lemmas.

**Lemma 11.** Let
\[
f^+((i, j), u) = \begin{cases} f((i, j), u), & u \geq 0, \\ 0, & u < 0, \end{cases} \tag{44}
\]
where \( F^+ = ((i, j), u) = \int_0^u f^+((i, j), \tau) d\tau \). If
\[
\lim_{u \to +\infty} \frac{f^+((i, j), u)}{u} = \sigma > \lambda_j, \tag{45}
\]
then the functional
\[ I^*(u) = \sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \frac{1}{2} p(i - 1, j) \Delta_1 v(i - 1, j) + \frac{1}{2} r(i, j - 1) \Delta_2 v(i, j - 1) - \sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \frac{1}{2} q(i, j) v(i, j) + q(i, j) v(i, j) + \sigma v^*(i, j) \phi(i, j) = 0, \quad (i, j) \in \Omega, \]

satisfies the (PS) condition.

**Proof.** Suppose that there exists a sequence \( \{u_m\} \subseteq S \) satisfying

\[ I^*(u_m) \longrightarrow c, \quad I^+(u_m) \longrightarrow 0, \quad m \longrightarrow \infty. \]  

\[ \square \]

Since \( (S, \langle \cdot, \cdot \rangle) \) is a \( T_1 T_2 \)-dimensional real Hilbert space, we only need to verify that \( \{u_m\} \) is bounded. On the contrary, suppose \( \{u_m\} \) is unbounded, that is, \( |u_m(i, j)| \longrightarrow \infty \).

Denote that \( v_m = u_m/|u_m| \), then \( \|v_m\| = 1 \) and exists \( v \in S \), \( \|v\| = 1 \) such that \( v_m \rightarrow v \). For any \( \varphi \in S \), we get

\[ \frac{\langle I^+(u_m), \varphi \rangle}{\|u_m\|} = -\sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \left[ \Delta_1 [p(i - 1, j) \Delta_1 v_m(i - 1, j)] + \Delta_2 [r(i, j - 1) \Delta_2 v_m(i, j - 1)] + q(i, j) v_m(i, j) + f^*(i, j) \|u_m\| \right] \phi(i, j). \]

\[ \text{Since } |u_m(i, j)| \longrightarrow \infty \text{ for every } (i, j) \in \Omega, \text{ we get} \]

\[ \lim_{m \rightarrow \infty} f^*(i, j, u_m(i, j)) = \lim_{m \rightarrow \infty} f^*(i, j, u_m(i, j)) v_m(i, j) = \sigma v^*(i, j). \]

where \( v^*(i, j) = \max \{v(i, j) , 0\}, (i, j) \in \Omega. \) Therefore,

\[ -\sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \left[ \Delta_1 [p(i - 1, j) \Delta_1 v(i - 1, j)] + \Delta_2 [r(i, j - 1) \Delta_2 v(i, j - 1)] + q(i, j) v(i, j) + \sigma v^*(i, j) \phi(i, j) = 0, \]

which implies that \( v \neq 0 \) satisfies

\[ \Delta_1 [p(i - 1, j) \Delta_1 v(i - 1, j)] + \Delta_2 [r(i, j - 1) \Delta_2 v(i, j - 1)] + q(i, j) v(i, j) + \sigma v^*(i, j) = 0, \quad (i, j) \in \Omega, \]

which is the (PS) condition.

**Lemma 12.** Let

\[ f^-((i, j), u) = \begin{cases} f((i, j), u), & u \leq 0, \\ 0, & u > 0, \end{cases} \]

where \( F^-=(i, j), u) = \int_A f^-((i, j), \tau)d\tau. \) If

\[ \lim_{u \rightarrow -\infty} \frac{f^-((i, j), u)}{u} = \sigma > \lambda, \]

then the functional

\[ I^*(u) = \sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \frac{1}{2} p(i - 1, j) \Delta_1 u(i - 1, j)^2 + \frac{1}{2} r(i, j - 1) \Delta_2 u(i, j - 1)^2 - \sum_{i=1}^{T_2} \sum_{j=1}^{T_1} \frac{1}{2} q(i, j) u(i, j)^2 - \sum_{i=1}^{T_2} \sum_{j=1}^{T_1} F^*((i, j), u(i, j)) \]

satisfies the (PS) condition.

**Lemma 13.** Under the condition of Theorem 8, the functional \( I^* \) possesses a positive critical point \( u^* \) and \( C_q(I^*, u^*) \equiv \delta_{q, \ast} \), the functional \( I^* \) possesses a negative critical point \( u^- \) and \( C_q(I^*, u^-) \equiv \delta_{q, \ast} \).

**Proof.** We only prove the case of \( I^+ \) and the proof of the case of \( I^- \) can be obtained similarly. First of all, we need to prove that \( I^+ \) satisfies the Proposition 6 so that it has a nonzero critical point \( u^* \). As a matter of fact, \( I^+(0) = 0 \) and according
to the conclusion of Lemma 11, $I^*$ satisfies the (PS) condition. Now, we must prove that $I^*$ satisfies $(J_4)$ and $(J_5)$. □

On one hand, since (V2) is satisfied, there exist $\rho > 0$ and $\rho_1 > 0$ satisfying $F^\prime \prime((i, j), 0) < \rho_1 < \lambda_1$ such that

$$F((i, j), u) \leq \frac{1}{2}\rho_1 ||u||^2, ||u|| \leq \rho.$$ (59)

Then,

$$I^*(u) = \frac{1}{2}(Au, u) - F(u) \geq \frac{1}{2}\lambda_1 ||u||^2 - \sum_{i=1}^{T_i} \sum_{j=1}^{T_j} F((i, j), u(i, j))$$

$$\geq \frac{1}{2}\lambda_1 ||u||^2 - \frac{1}{2}\rho_1 \sum_{i=1}^{T_i} \sum_{j=1}^{T_j} |u(i, j)|^2$$

$$= \frac{1}{2}\lambda_1 ||u||^2 - \frac{1}{2}\rho_1 ||u||^2 = \frac{1}{2}(\lambda_1 - \rho_1) ||u||^2 > 0.$$ (60)

Therefore, $(J_4)$ is satisfied. On the other hand, (24) means that there exists $\gamma > \lambda_{k-1}(>\lambda_1), \ b \in \mathbb{R}$ such that

$$F(t) \geq \frac{\gamma}{2}t^2 + b, \ \forall t \in \mathbb{R}.$$ (61)

Choose $e \in \text{span}\{\phi_1\},$ where $\phi_1$ is the eigenvector corresponding to $\lambda_1$, we have

$$I^*(te) = \frac{1}{2}(Ate, te) - F(te) \leq \frac{1}{2}\lambda_1 ||te||^2 - \frac{\gamma||te||^2}{2} - bT_1T_2$$

$$= \frac{t^2}{2}(\lambda_1 - \gamma)||e||^2 - bT_1T_2.$$ (62)

Therefore, there exists a sufficiently large $t > \rho$ such that $I^*(ue_0) \leq 0$ for $u_0 = te \in S$, which makes $(J_5)$ is fulfilled. By Proposition 6, $I^*$ possesses a critical point $u^* \neq 0$.

In the following, we devote to verifying $u^* > 0$. Due to $I^*(u^*) = 0$, there exists a sequence $\{u_m^*\}$ such that $I^*(u_m^*) \longrightarrow 0 = I^*(u^*)$ as $m \longrightarrow \infty$. Then, for all $\varphi \in S$, there holds

$$\langle I^*(u_m^*), \varphi \rangle = -\sum_{i=1}^{T_i} \sum_{j=1}^{T_j} \left[\Delta_1[p(i - 1, j)\Delta_1u_m^*(i - 1, j)] + \Delta_2[r(i, j - 1)\Delta_2u_m^*(i, j - 1)] + q(i, j)u_m^*(i, j) + f^\prime((i, j), u_m^*(i, j)))\varphi(i, j).$$ (63)

Let $m \longrightarrow \infty$ and $u^*(i, j) = \max \{u(i, j), 0\}$, we have

$$\lim_{m \longrightarrow \infty} f^\prime((i, j), u_m^*(i, j)) = \lim_{m \longrightarrow \infty} \frac{f^\prime((i, j), u_m^*(i, j))}{u_m^*(i, j)}u_m^*(i, j) = \sigma u^*(i, j),$$

$$\Delta_1[p(i - 1, j)\Delta_1u^*(i - 1, j)] + \Delta_2[r(i, j - 1)\Delta_2u^*(i, j - 1)] + q(i, j)u^*(i, j) + \sigma u^*(i, j) = 0.$$ (64)

Denote $u(i_0, j_0) = \min \{u(i, j)\}$. Now, we need to prove $u(i_0, j_0) > 0$ to get $u(i, j) > 0$. Arguing indirectly, let $u(i_0, j_0) \leq 0$, then

$$\Delta_1[p(i_0 - 1, j_0)\Delta_1u^*(i_0 - 1, j_0)] + \Delta_2[r(i_0, j_0 - 1)\Delta_2u^*(i_0, j_0 - 1)] = 0.$$ (65)

Applying (p), we get

$$\Delta_1u^*(i_0 - 1, j_0) = \Delta_2u^*(i_0, j_0 - 1) = 0,$$ (66)

which means $u^*(i_0 - 1, j_0) = u^*(i_0, j_0) = u^*(i_0, j_0 - 1)$. Then,

$$u^* = 0, \ \forall (i, j) \in \Omega$$ (67)

contradicts with $u^* \neq 0$. Consequently, $u^* > 0$.

Now, we calculate $C_q(I^*, u^*)$. On account of

$$\left\langle I^\prime\prime(u)v, w \right\rangle = w^TAv - \sum_{i=1}^{T_i} \sum_{j=1}^{T_j} F^\prime\prime((i, j), u(i, j))v(i, j)w(i, j),$$ (68)

for any $v \in S$, we have $\langle I^\prime\prime(u^*)v, v \rangle \geq 0$ and there exists $v_0 \neq 0$ such that $\langle I^\prime\prime(u^*)v_0, v \rangle = 0$, which indicates that $v_0$ is a solution of

$$\Delta_1[p(i - 1, j)\Delta_1v_0(i - 1, j)] + \Delta_2[r(i, j - 1)\Delta_2v_0(i, j - 1)] + q(i, j)v_0(i, j) - \lambda F^\prime((i, j), u^*)v_0(i, j) = 0, \ \forall (i, j) \in \Omega,$$

$$v_0(i, j) = v_0(i, T_2 + 1) = v_0(0, 0)$$

$$= v_0(T_1 + 1, j) = 0, \ \ i \in \mathbb{Z}(1, T_1), \ j \in \mathbb{Z}(1, T_2).$$ (69)

Therefore, the corresponding Dirichlet eigenvalue problem

$$\Delta_1[p(i - 1, j)\Delta_1v(i - 1, j)] + \Delta_2[r(i, j - 1)\Delta_2v(i, j - 1)] + q(i, j)v(i, j) - \lambda F^\prime((i, j), u^*)v(i, j) = 0, \ \forall (i, j) \in \Omega,$$ (70)

admits an eigenvalue $\lambda = 1$. According to (V1), 1 is a single eigenvalue, which means that $\text{dimker}(I^*(u^*)) = 1$. Combining with $I^*(u^*)$: $S \longrightarrow \mathbb{R}$ is a surjection, $I^*(u^*)$ is a Fredholm operator, we conclude that
\[ C_q(I^*, u^*) = \delta_{q1} Z \]
\[ C_q(I, u^*) \equiv C_q(I^*, u^*) = \delta_{q1} Z. \]  (71)

With the aid of preceding preparations, now we are in the position to complete the verification of Theorem 8 via Proposition 3.

**Proof of Theorem 14.** Let \( \{i, \bar{i}\} \) holds. On one hand, through Lemma 10 we get
\[ C_q(I, 0) \equiv \delta_{q0} Z. \]  (72)

On the other hand, because of \( F^{''}(0) < \lambda_1, I'(0) = 0 \) and
\[ \langle I''(0)u, u \rangle \geq \left( \lambda_1 - F^{''}(0) \right) \|u\|^2, \]  (73)
we can refer that 0 is the local minimum of \( I \). Moreover, 0 is the isolated critical point of \( I \) which implies that \( I^{''}(0) \) is a Fredholm operator with finite Morse index and zero dimension. From Proposition 4, it gives that
\[ C_q(I, 0) \equiv \delta_{q0} Z. \]  (74)

For the reason that \( I \) satisfies the \( (D) \) condition from Lemma 9, we can deduce \( I \) possesses a critical point \( u_i \neq 0 \) such that
\[ C_k(I, u_i) \neq 0 \quad [C_{k-1}(I, u_i) \neq 0]. \]  (75)

As a result, we conclude that when \( k \geq 2 [k \geq 3], u^*, u^* \), and \( u_i \) are the nontrivial critical points of \( I \). Thus, the proof of Theorem 14 is finished.

**4. An Example**

As an application of our result, an example is elaborated here.

**Example 15.** Take \( T_1 = 3, T_2 = 2 \), then for \( (i, j) \in Z(1, 3) \times Z(1, 2), p(i, j) = 1, q(i, j) = -1, r(i, j) = 1. \) Consider
\[ \Delta^2 u(i-1, j) + \Delta^2 u(i, j-1) - u(i, j) + \frac{(\lambda_1/2 - \lambda_k)u(i, j)}{1 + |u(i, j)|^2} + \lambda_k u(i, j) = 0, \]  (76)
with the boundary value conditions (2).

According to the expression of \( f((i, j), u) \), it follows that
\[ F((i, j), u) = \frac{\lambda_1/2 - \lambda_k}{2} \ln (1 + u^2) + \frac{1}{2} \lambda_k u^2. \]
\[ F^{''}(i, j), u) = \frac{(3\lambda_k - \lambda_1/2)u^2 + \lambda_k u^4 + (\lambda_1/2)}{(1 + u^2)^2} > 0. \]  (77)

It is not difficult to verify that \( f((i, j), 0) = 0, F((i, j), 0) = 0, F^{''}(i, j), 0 = \lambda_1/2 < \lambda_k \) and
\[ \lim_{|u| \to \infty} \frac{f((i, j), u)}{u} = \lim_{|u| \to \infty} \left[ \frac{\lambda_1/2 - \lambda_k}{1 + |u(i, j)|^2} + \lambda_k \right] = \lambda_k. \]  (78)

Since \( p(i, j) = 1, q(i, j) = -1, r(i, j) = 1 \) for each \( (i, j) \in Z(1, 3) \times Z(1, 2) \), then the matrix
\[
A = \begin{pmatrix}
5 & -1 & 0 & -1 & 0 & 0 \\
-1 & 5 & -1 & 0 & -1 & 0 \\
0 & -1 & 5 & 0 & 0 & -1 \\
-1 & 0 & 0 & 5 & -1 & 0 \\
0 & -1 & 0 & -1 & 5 & -1 \\
0 & 0 & -1 & 0 & -1 & 5
\end{pmatrix}
\]  (79)
is positive definite, and the eigenvalues are given by
\[ \lambda_1 = 4 - \sqrt{2}, \lambda_2 = 4, \lambda_3 = 4 + \sqrt{2}, \lambda_4 = 6 - \sqrt{2}, \lambda_5 = 6, \lambda_6 = 6 + \sqrt{2}. \]  (80)

Next, we check \( (i, j) \). If \( |u_m| \to \infty \) such that \( |v_m|/|u_m| \to 1 \) with \( k \geq 3 \), then there exist \( \delta > 0 \) and \( M \in \mathbb{N} \) such that
\[ -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g(u_m(i, j)), v_m(i, j)) \geq \delta, \quad m \geq M. \]  (81)

In fact, we can choose \( k = 4, \delta = \lambda_4 - \lambda_1 = 2 > 0 \). Since \( g((i, j), u) = (\lambda_1/2 - \lambda_k)u^1 + u^2 \), then
\[ -\sum_{i=1}^{3} \sum_{j=1}^{2} (g(u_m(i, j)), v_m(i, j)) \]
\[ = -\sum_{i=1}^{3} \sum_{j=1}^{2} (g(v_m(i, j)), v_m(i, j)) \]
\[ = -\sum_{i=1}^{3} \sum_{j=1}^{2} \frac{(\lambda_1/2 - \lambda_k)v_m(i, j)}{1 + (v_m(i, j))^2} \cdot v_m(i, j) \]
\[ = -\sum_{i=1}^{3} \sum_{j=1}^{2} \frac{1}{1 + (v_m(i, j))^2} = 6 \left( \lambda_4 - \frac{\lambda_1}{2} \right) > \delta. \]  (82)
Therefore, all conditions of Theorem 8 are satisfied and (76)-(2) admits at least three nontrivial solutions, which include a positive solution and a negative solution.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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