Breakdown of the linear approximation in the perturbative analysis of heat conduction in relativistic systems

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Abstract

We analyze the effects of thermal conduction in a relativistic fluid just after its departure from spherical symmetry, on a time scale of the order of relaxation time. Using first order perturbation theory, it is shown that, as in spherical systems, at a critical point the effective inertial mass density of a fluid element vanishes and becomes negative beyond that point. The impact of this effect on the reliability of causality conditions is discussed.

1 Introduction

The behaviour of dissipative systems at the very moment when they depart from hydrostatic equilibrium has been recently studied [1, 2, 3]. As result of these works, it appears that a parameter formed by a specific combination of relaxation time, temperature, proper energy density and pressure, may critically affect the evolution of the object.

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Specifically, by means of first order perturbation theory, it was shown that in the equation of motion of any fluid element, the inertial mass density term is multiplied by a factor which vanish for a given value of the parameter \( \alpha \) defined below in equation (50) (critical point) and changes of sign beyond that value. Nevertheless, even though self-gravitating systems seem to become more unstable as the above mentioned parameter grows \( \text{(4)} \), an exact description of the evolution \( \text{(4)} \) does not indicate any anomalous behaviour of the system at or beyond the critical point. Thus, first order perturbation theory seems to be not reliable close or beyond the critical point.

Although causality conditions prevent, in some cases (pure shear or bulk viscosity \( \text{(3)} \)), reaching the critical point. This may not be the case in the most general situation (heat conduction plus viscosity \( \text{(3)} \) or even in the non-viscous case \( \text{(3)} \)). It is worth mentioning that causality conditions were found using a first order perturbation method \( \text{(6)} \), thus, in some cases these conditions must be revised.

However in all these works it has been assumed that spherical symmetry is preserved. It is therefore pertinent to ask if the appearance of the critical point is closely related to that kind of symmetry (spherical) or if it represents a general feature of dissipative systems.

It is the purpose of this work to provide an answer to the question above by considering a system which, although initially spherically symmetric, is submitted to perturbations deviating it from spherical symmetry.

By examining the equation of motion of an arbitrary fluid element along the “meridional” direction, it is shown that a critical point also appears in this case, indicating thereby the independence of this effect from the spherical symmetry.

The paper is organized as follows. In the next section the field equations and conventions are presented. In section 3 we briefly present the equation for the heat conduction and write down explicitly the \( \theta \)-component of this equation. In section 4 the full system of equations is evaluated at the time when the object starts to depart from spherical symmetry. Finally, a discussion of results is given in the last section.

In the whole text, a bar over a quantity denote that this one has been measured by a Minkowskian observer. If the symbol over a given quantity is a tilde, then this one is evaluated after the perturbation.

## 2 The field equations and conventions

We shall consider axially and reflection symmetric distributions of fluid dissipating energy through a heat flux vector. In null coordinates the metric is given by the standard Bondi expression \( \text{(6)} \)

\[
ds^2 = \left( \frac{V}{c^2} e^{2\beta} - U^2 r^2 e^{2\gamma} \right) du^2 + 2e^{2\beta} du \, dr + 2U r^2 e^{2\gamma} du \, d\theta - r^2 \left( e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta \, d\phi^2 \right),
\]  

(1)
where \( V, \beta, U \) and \( \gamma \) are functions of \( u, r \) and \( \theta \).

We number the coordinates \( x^{0,1,2,3} = u, r, \theta, \phi \), respectively. \( u \) is a time-like coordinate such that \( u = \text{constant} \) defines a null surface. In flat space-time this surface coincides with the null light cone open to the future. \( r \) is a null coordinate \((g_{rr} = 0)\) and \( \theta \) and \( \phi \) are two angle coordinates (see \([7]\) for details).

In order to guarantee the regularity of the metric within the fluid distribution at \( r = 0 \), we impose the conditions\([8]\)

\[
V = r + O(r^3); \quad \beta = O(r^2); \quad U = O(r); \quad \gamma = O(r^2).
\]

In this case, the energy-momentum tensor takes the usual form

\[
T_{\mu\nu} = (\rho + p)v_\mu v_\nu - pg_{\mu\nu} + q_\mu v_\nu + q_\nu v_\mu
\]

where \( \rho, p \) and \( q_\mu \) denote proper energy density, pressure and the heat flow vector, respectively, and the four-velocity \( v^\mu \) satisfies the conditions

\[
v^\mu v_\mu = 1; \quad q^\mu v_\mu = 0
\]

Since reflection symmetry is preserved, we have

\[
v^3 = v_3 = 0; \quad q^3 = q_3 = 0
\]

We shall now write the field equations according to the scheme presented by Isaacson et al.\([8]\) (see also \([9]\)). Thus, there are three main equations which only contain derivatives with respect to \( r \) and \( \theta \). In our case the corresponding expressions are given by

\[
2\pi r [(\rho + p) v_1 + 2q_1] v_1 = \beta,1 - \frac{1}{2} r (\gamma,1)^2
\]

\[
16\pi r^2 [(\rho + p) v_1 v_2 + q_1 v_2 + q_2 v_1] = \left[r^4 e^{2(\gamma - \beta)} U,1\right],1 - 2\pi r^2 \left[r^2 (r^{-2} \beta),12 - \sin^{-2} \theta (\gamma \sin^2 \theta),12 + 2\gamma,1 \gamma,2\right]
\]

\[
-8\pi r^2 e^{2\beta} [\rho - p + r^{-2} e^{-2\gamma} \left(|\rho + p| (v_2)^2 + 2q_2 v_2\right)] = 2V,1 + \frac{r^4}{4} e^{2(\gamma - \beta)} (U,1)^2 - (r^2 \sin \theta)^{-1} (r^4 U \sin \theta),12 + 2e^{2(\beta - \gamma)} \times \\
\left[-1 + (\sin \theta)^{-1} (\beta,2 \sin \theta),2 - \gamma,22 - 3\gamma,2 \cot \theta + (\beta,2)^2 + 2\gamma,2 (\gamma,2 - \beta,2)\right]
\]

where a comma denotes partial derivative. Following the Bondi-Isaacson scheme we shall assume that variables \( \rho, p, v_1, v_2, q_1, q_2 \) and \( \gamma \) are known on
a given initial null hypersurface \( u = u_0 = \text{constant} \), which we shall call the “initial cone”.

At \( u = u_0 \) perturbations are introduced into the system, forcing it to deviate from spherical symmetry. We shall evaluate the system immediately after perturbing it, on a null surface \( u = \tilde{u} = \text{constant} \) such that \( \tilde{u} - u_0 \) is much smaller than the typical time required for \( v_1, v_2, q_1, q_2 \) and \( \gamma \) to change significantly (where “significantly” refers to terms linear in the perturbation, or larger). In other words, on the surface \( u = \tilde{u} \), which we shall call the “evaluation cone”, the components \( v_1 \) and \( v_2 \) of the four-velocity, \( q_1 \) and \( q_2 \) of the heat flow vector and the metric function \( \gamma \) have the same magnitude as on the initial cone (up to quadratic and higher terms in the perturbation parameter).

The “standard equation” which provides an expression for \( \gamma_0 \) reads

\[
4r (r\gamma)_{,01} = \left[ 2r\gamma_1 V - r^2 (2\gamma_2 U + U_2 - U \cot \theta) \right]_{,1} - 2r^2 \sin^{-1} \theta (\gamma_1 U \sin \theta)_{,2} + \frac{r^4}{2} e^{2(\gamma-\beta)} (U_1)^2 + 2 e^{2(\beta-\gamma)} \times \left[ (\beta_{,2})^2 + \beta_{,22} - \beta_{,2} \cot \theta + 4\pi ([\rho + p] (v_2)^2 + 2q_2v_2) \right]
\]

(9)

Next, in order to give physical meaning to quantities appearing in (3), taking into account the axial symmetry of the case considered, we shall develop a procedure similar to that used by Bondi \[10\] in his study of non-static spherically symmetric sources. Thus, let us introduce purely local Minkowski coordinates \((\not{x}^0, \not{x}^1, \not{x}^2, \not{x}^3)\), defined by \[11\]

\[
d\not{x}^0 = Adu + Bdr + Cd\theta
\]

(10)

\[
d\not{x}^1 = Bdr + Cd\theta
\]

(11)

\[
d\not{x}^2 = Fd\theta
\]

(12)

\[
d\not{x}^3 = Gd\phi
\]

(13)

where coefficients \( A, B, C, F \) and \( G \) are given by

\[
A = \left( \frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right)^{\frac{1}{2}}
\]

(14)

\[
B = e^{2\beta} \left( \frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right)^{-\frac{1}{2}}
\]

(15)

\[
C = U r^2 e^{2\gamma} \left( \frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right)^{-\frac{1}{2}}
\]

(16)
\[ F = re^\gamma \]  

\[ G = re^{-\gamma} \sin \theta \]  

so that,

\[ ds^2 = (d\tau^0)^2 - (d\tau^1)^2 - (d\tau^2)^2 - (d\tau^3)^2 \]  

Now, for the Minkowski observer defined above the fluid moves in such a way that the four-velocity vector of the fluid is given by

\[ \tau^\mu = \lambda (1, \omega_1, \omega_2, 0), \]  

where \( \omega_1 \) and \( \omega_2 \) are the components of the velocity along the \( \tau^1 \) and \( \tau^2 \) axes, respectively (as measured by our locally Minkowski observer), and

\[ \lambda = \frac{1}{\sqrt{1 - \omega_1^2 - \omega_2^2}} \]

Thus, a locally Minkowski observer comoving with the fluid is one for which \( \omega_1 = \omega_2 = 0 \).

Now, from (24) and (10)–(13) we can obtain expressions for the four-velocity components in the original Bondi coordinates. In terms of \( \omega_1 \) and \( \omega_2 \), they are

\[ v^\mu = \lambda \left[ \frac{1 - \omega_1}{A}, \frac{1}{B} \left( \omega_1 - \frac{C\omega_2}{F} \right), \frac{\omega_2}{F}, 0 \right] \]

or, for the covariant components,

\[ v_0 = \lambda \left( \frac{V e^{2\beta} - U^2 e^{2\gamma}}{r} \right)^{1/2} \]
\[ v_1 = \frac{\lambda e^{2\beta} (1 - \omega_1)}{\left( \frac{V e^{2\beta} - U^2 e^{2\gamma}}{r} \right)^{1/2}} \]
\[ v_2 = \lambda \left[ \frac{U r^2 e^{2\gamma} (1 - \omega_1)}{\left( \frac{V e^{2\beta} - U^2 e^{2\gamma}}{r} \right)^{1/2}} - re^\gamma \omega_2 \right] \]
\[ v_3 = 0 \]

It is worth noticing that the Minkowski observer will say that the fluid is at rest when \( \omega_1 = \omega_2 = 0 \). Also, observe that \( \omega_2 \), as it follows from (12), measures the velocity of a fluid element along the \( \theta \) (meridional) coordinate line.

Finally, defining \( q \) as

\[ q = \sqrt{-q^\mu q_\mu} \]
we obtain

\[ q_0 = q \omega_1 \sqrt{\frac{V e^{2\beta}}{r} - U^2 r^2 e^{2\gamma}} \]  

\[ q_1 = -q e^{2\beta} \left( \frac{V e^{2\beta}}{r} - U^2 r^2 e^{2\gamma} \right)^{-1/2} \left( \frac{\omega_1^2 + \omega_2^2}{\omega_1^2 + \omega_2^2} - \lambda \omega_1 \right) \]  

\[ q_2 = -q e^{2\gamma} \times \left[ U e^{2\gamma} \left( \frac{V e^{2\beta}}{r} - U^2 r^2 e^{2\gamma} \right)^{-1/2} \left( \frac{\omega_1^2 + \omega_2^2}{\omega_1^2 + \omega_2^2} - \lambda \omega_1 \right) + \frac{\omega_1 \omega_2}{\omega_1^2 + \omega_2^2} (\lambda - 1) \right] \]  

\[ q_3 = 0 \]

3 Heat Conduction Equation.

In the study of star interiors it is usually assumed that the energy flux of radiation (and thermal conduction) is proportional to the gradient of temperature (Maxwell-Fourier law or Eckart-Landau in general relativity).

However, it is well known that the Maxwell-Fourier law for the radiation flux leads to a parabolic equation (diffusion equation) which predicts propagation of perturbation with infinite speed (see \[12\]–\[14\] and references therein). This simple fact is at the origin of the pathologies \[6\] found in the approaches of Eckart \[15\] and Landau \[16\] for relativistic dissipative processes.

To overcome such difficulties, different relativistic theories with non-vanishing relaxation times have been proposed in the past \[17\]–\[20\]. The important point is that all these theories provide a heat transport equation which is not of Maxwell-Fourier type but of Cattaneo type \[21\], leading thereby to a hyperbolic equation for the propagation of thermal perturbation.

Accordingly we shall describe the heat transport by means of the relativistic Israel-Stewart equation \[14\], which reads

\[ \tau \frac{Dq^\alpha}{Ds} + q^\alpha = \kappa P^{\alpha \beta} (T_{\beta \gamma} - T a_{\beta \gamma}) - \tau v^\alpha q_\beta a^{\beta} - \frac{1}{2} \kappa T^2 \left( \frac{\tau}{\kappa T^2} v^\beta \right) \cdot q^\alpha \]  

where \( \kappa, \tau, T \) and \( a^\beta \) denote thermal conductivity, thermal relaxation time, temperature and the components of the four-acceleration, respectively. Also, \( P^{\alpha \beta} \) is the projector onto the hypersurface orthogonal to the four velocity \( v^\alpha \) and

\[ Dq^\alpha \]

\[ Ds \equiv v^\beta q^\beta_{\beta} \equiv q_\alpha \]
For the purpose of this work we only need the components of (32) containing
$u$-derivatives of $q_2$ and $v_2$. Such terms only appear in the $\theta$-component of (32).
Thus we have

$$\tau \dot{q}_\nu P^{\nu 2} + q^2 = \kappa P^{\alpha 2} (T_\alpha - T a_\alpha) - \frac{1}{2} \kappa T^2 \left( \frac{\tau}{\kappa T^2} v^\beta \right) \dot{q}^2$$

(33)
or simbolically

$$\tau g^{01} v_1 q_{2,0} P^{22} + \kappa T P^{22} g^{01} v_1 v_{2,0} = \mathcal{H}$$

(34)
where $\mathcal{H}$ is a combination of terms not containing $q_{2,0}$ or $v_{2,0}$.
Alternatively, we may write from (34)

$$q_{2,0} = \frac{\kappa T}{\tau} v_{2,0} + \mathcal{I},$$

(35)
where $\mathcal{I}$, as $\mathcal{H}$, represents a combination of terms not containing $q_{2,0}$ or $v_{2,0}$.
We shall now get into the central problem of this work.

4 Thermal conduction and departure from hydrostatic equilibrium

Let us now consider a fluid distribution which on the initial cone ($u = u_0$) is
spherically symmetric. We shall not impose any further restrictions on it, so
that in principle the system may not be in hydrostatic equilibrium along the
radial direction at $u = u_0$.

Next, let us assume that the system is perturbed, as a result of which it
departs from spherical symmetry.

Since the system is initially spherically symmetric it is clear that on the
initial cone the fluid is in equilibrium along the $\theta$-line of coordinates (i.e. at
$u = u_0, \omega_2 = q_2 = 0$ and $\omega_{2,0} = q_{2,0} = 0$).

For the purpose of this paper (i.e. in order to put in evidence the existence
of a critial point along a non-radial direction) it will suffice to consider those
equations containing $q_{2,0}$ and $\omega_{2,0}$.

Let us now evaluate our system on the evaluation cone ($u = \bar{u}$).

As mentioned in the precedent section, the only component of (32) containing
terms with $u$-derivatives of $q_2$ or $\omega_2$ (the $\theta$ -component) leads to eq.(35).

This last equation after perturbation keeps the same form as (34), where all
the quantities are evaluated on the $u = \bar{u}$ cone, i.e.

$$\bar{q}_{2,0} = -\frac{\kappa T}{\tau} \bar{v}_{2,0} + \bar{\mathcal{I}}.$$  

(36)

Next, instead of dealing with the field equations (31)–(34), it will be more
useful to consider the “conservation” equations
Thus, the $u$-component of (37) gives an expression of the form

$$[(\rho + p)v_0 + q_0]v_{1,0} + [(\rho + p)v_1 + q_1]v_{0,0} + q_{1,0}v_0 + q_{0,0}v_1 = J$$

(38)

where $J$ denotes a combination of terms without $u$-derivatives of matter variables. Observe that (38) does not contain terms with $u$-derivatives of $q_2$ or $\omega_2$. Also note that all terms containing $\omega_{2,0}$, appearing from the $u$-derivatives of $\lambda$ in $v_{1,0}$, $v_{0,0}$, $q_{1,0}$ and $q_{0,0}$ are multiplied by $\omega_2$ and therefore vanish on the evaluation cone (see eqs. (22)–(31)).

Then, (38) may be written alternatively as

$$f(\omega_{1,0}; q, 0) = K,$$

(39)

where $K$, as $J$, is a combination of terms without $u$-derivatives of matter variables.

The $r$-component of (37) yields an expression of the form

$$[(\rho + p)v_1 + q_1]v_{1,0} + q_{1,0}v_1 = L,$$

(40)

where $L$ is again a combination of terms without $u$-derivatives of matter variables.

As in the precedent case, when evaluating (40) at $u = \tilde{u}$, all terms with $\omega_{2,0}$ will vanish since they appear multiplied by $\omega_2$, and therefore this later equation may be written as

$$g(\omega_{1,0}; q, 0) = M,$$

(41)

$M$ denoting another combination of terms without $u$-derivatives of matter variables.

Finally, let us consider the $\theta$-component of (37), it leads to an expression of the form

$$[(\rho + p)v_2 + q_2]v_{1,0} + [(\rho + p)v_1 + q_1]v_{2,0} + q_{2,0}v_2 + q_{2,0}v_1 = N,$$

(42)

or, using (38) and (41)

$$[(\rho + p)v_1 + q_1]v_{2,0} + q_{2,0}v_1 = N + h(K, M),$$

(43)

Since $N$ does not contain terms with $u$-derivatives of matter variables, these later terms are absent on the right hand side of (43).

Before perturbation (on the initial cone) (43) is just an identity $0 = 0$, since the system is spherically symmetric. After perturbation (43) keeps the same form, with all quantities evaluated at $u = \tilde{u}$, i.e.
\[(\bar{\rho} + \bar{p}) \bar{v}_1 + \bar{q}_1) \bar{v}_{2,0} + \bar{q}_{2,0} \bar{v}_1 = \bar{N} + \bar{h} (K, M) \quad (44)\]

or replacing \(\bar{q}_{2,0}\) by (36)

\[
\left[ \left( \bar{\rho} + \bar{p} - \frac{\kappa T}{\tau} \right) \bar{v}_1 + \bar{q}_1 \right] \bar{v}_{2,0} + \bar{I} \bar{v}_1 = \bar{N} + \bar{h} (K, M) \quad (45)
\]

or, after re-arranging terms

\[
\left( \bar{\rho} + \bar{p} - \frac{\kappa T}{\tau} + \frac{\bar{q}_1}{\bar{v}_1} \right) \bar{v}_{2,0} = \frac{\bar{N} + \bar{h} (K, M)}{\bar{v}_1} - \bar{I} \quad (46)
\]

Next, taking into account that

\[
\frac{\bar{q}_1}{\bar{v}_1} = -\bar{q} \quad (47)
\]

as it follows from (24) and (29) imposing \(\omega_2 = 0\), and keeping terms linear in the perturbation, we may write (46) as

\[
\left( \rho + p - \frac{\kappa T}{\tau} - q \right) \bar{v}_{2,0} = \frac{\bar{N} + \bar{h} (K, M)}{\bar{v}_1} - \bar{I}, \quad (48)
\]

It remains to expand \(\bar{v}_{2,0}\) by using (25). One obtains terms with \(\omega_{2,0}\), terms with \(\omega_{1,0}\) and terms not containing \(u\)-derivatives of matter variables. The latter may be included in the right hand of (48). Terms with \(\omega_{1,0}\) may also be transformed into terms not containing \(u\)-derivatives of matter variables by virtue of (39) and (41). Thus, we obtain finally

\[
R_\theta = -\omega_{2,0} \times \left[ 1 - \frac{\kappa T}{\tau (\rho + p - q)} \right] (\rho + p - q) \quad (49)
\]

The physical meaning of \(R_\theta\) is quite simple. If the fluid is in equilibrium along any meridional line then \(R_\theta = 0\), thus \(R_\theta\) represents the total force on any fluid element in the \(\theta\)-direction (observe that \(p, q\) enters in \(R_\theta\)). Therefore (49) may be regarded as an equation of the “Newtonian” form

\[
\text{Force} = \text{mass} \times \text{acceleration}
\]

where by “acceleration” we mean the time-like derivative of \(\omega_2\). Thus, if

\[
\alpha \equiv \frac{\kappa T}{\tau (\rho + p - q)} = 1 \quad (50)
\]

the effective inertial mass vanishes and any fluid element is out of hydrostatic equilibrium \((\omega_{2,0} \neq 0)\) even though \(R_\theta\) is zero. Note that beyond the critical
point \((\alpha > 1)\), the system exhibits an anomalous behaviour: If the total force on any fluid element is acting in the positive \(\theta\)-direction, then the system tends to accelerate in the negative \(\theta\)-direction.

This is the same critical point already found in the spherically symmetric case \([1, 2, 3]\). The only difference with the later case being the appearance of \(q\) in \(\alpha\). This is due to the fact that in the spherically symmetric case the system is initially static \((q = 0)\) or slowly evolving \((q\) is small\). However in our case, we have considered the more general situation when the system initially is spherically symmetric but, although \(q_2\) vanishes on the initial cone, \(q_1\), and therefore \(q\), is not necessarily zero or a small quantity.

5 Conclusions

We have described the departure of a fluid distribution from hydrostatic equilibrium along the meridional direction, which in its turn implies departure from spherical symmetry. It has been shown that a critical point, for which the effective inertial mass density of a fluid element vanishes, exists. It has been therefore established, that the occurrence of such critical point is independent of spherical symmetry.

As in spherically symmetric systems \([1, 2, 3]\), condition \(\alpha = 1\) establishes the upper limit for which first order perturbation theory can be applied. Thus, close to (and beyond) this point, causality conditions obtained from the linear approximation should be taken with caution.

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