On capacity and torsional rigidity

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Abstract

We investigate extremal properties of shape functionals which are products of Newtonian capacity \( \text{cap} (\Omega) \), and powers of the torsional rigidity \( T(\Omega) \), for an open set \( \Omega \subset \mathbb{R}^d \) with compact closure \( \overline{\Omega} \), and prescribed Lebesgue measure. It is shown that if \( \Omega \) is convex, then \( \text{cap} (\Omega)T^q(\Omega) \) is (i) bounded from above if and only if \( q \geq 1 \), and (ii) bounded from below and away from 0 if and only if \( q \leq \frac{d-2}{2(d-1)} \). Moreover a convex maximiser for the product exists if either \( q > 1 \), or \( d = 3 \) and \( q = 1 \). A convex minimiser exists for \( q < \frac{d-2}{2(d-1)} \). If \( q \leq 0 \), then the product is minimised among all bounded sets by a ball of measure 1.

1. Introduction and main results

Several classical inequalities of mathematical physics are of the following form. Let \( F \) and \( H \) be strictly positive set functions defined on a suitable collection \( \mathcal{C} \) of open sets in \( \mathbb{R}^d \), and which satisfy scaling relations

\[
F(t\Omega) = t^{\beta_1} F(\Omega), \quad H(t\Omega) = t^{\beta_2} H(\Omega), \quad t > 0,
\]

where \( t\Omega \) is homothety of \( \Omega \), and \( \beta_1, \beta_2 \) are constants. Then the shape functional

\[
G(\Omega) = H(\Omega)F(\Omega)^{-\beta_2/\beta_1},
\]

is invariant under homotheties, and in some cases this quantity is minimal (respectively, maximal) for some open set \( \Omega^* \in \mathcal{C} \),

\[
G(\Omega) \geq G(\Omega^*) \quad (\text{respectively}, G(\Omega) \leq G(\Omega^*)), \quad \Omega \in \mathcal{C}.
\]

The Faber–Krahn, Krahn–Szegő, and Kohler–Jobin inequalities are of this form. See, for example, the seminal text [11]. In a recent paper [3], a more general set of inequalities was investigated. These are of the following form: let \( q \in \mathbb{R} \), and consider the shape functional

\[
G(\Omega) = H(\Omega)F(\Omega)^q.
\]

Then, unless \( q = -\beta_2/\beta_1 \), this product is not scaling invariant. However, denoting by \( |\Omega| \) the Lebesgue measure of \( \Omega \), the quantity

\[
\frac{H(\Omega)F(\Omega)^q}{|\Omega|^{(\beta_2+q\beta_1)/d}}
\]

is scaling invariant. The case where \( H \) is the principal Dirichlet eigenvalue, and \( F \) is the torsional rigidity was analysed in [3]. In the present paper, we investigate, in the spirit of [11],

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the case where \( H \) is the Newtonian capacity, and \( F \) is the torsional rigidity. Since the Newtonian capacity is most easily defined for compact subsets of \( \mathbb{R}^d, \ d \geq 3 \), we restrict ourselves to open sets \( \Omega \subset \mathbb{R}^d, \ d \geq 3 \) which are precompact. In that case, the Newtonian capacity scales as a power \( \beta_2 = d - 2 \) of the homothety.

Throughout this paper we let \( \Omega \) be a non-empty, open, bounded set in Euclidean space \( \mathbb{R}^d, \ d \geq 3 \). For a set \( A \subset \mathbb{R}^d \), we denote by \( \overline{A} \) its closure, \( \text{diam}(A) = \sup\{ |x - y| : x \in A, \ y \in A \} \) its diameter, and \( r(A) = \sup\{ r \geq 0 : (\exists x \in A, \ B_r(x) \subset A) \} \) its inradius, where \( B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \} \) is the ball of radius \( r \) centred at \( x \). Before we state the main results we recall some basic facts about the torsion function, torsional rigidity, and Newtonian capacity.

The torsion function for an open set \( \Omega \) with finite measure is the solution of
\[
-\Delta u = 1, \quad u \in H^1_0(\Omega),
\]
and is denoted by \( u_\Omega \). It is convenient to extend \( u_\Omega \) to all of \( \mathbb{R}^d \) by defining \( u_\Omega = 0 \) on \( \mathbb{R}^d \setminus \Omega \). It is well known that \( u_\Omega \) is non-negative, bounded \((2, 4, 6, 12)\), and monotone increasing with respect to \( \Omega \), that is
\[
\Omega_1 \subset \Omega_2 \Rightarrow u_{\Omega_1} \leq u_{\Omega_2}.
\]
The torsional rigidity of \( \Omega \), or torsion for short, is denoted by
\[
T(\Omega) = \| u_\Omega \|_1,
\]
where \( \| \cdot \|_p, \ 1 \leq p \leq \infty \) denotes the usual \( L^p \) norm. It follows that
\[
\Omega_1 \subset \Omega_2 \Rightarrow T(\Omega_1) \leq T(\Omega_2), \tag{1}
\]
and that the torsion satisfies the scaling property
\[
T(t\Omega) = t^{d+2}T(\Omega), \quad t > 0. \tag{2}
\]
Moreover \( T \) is additive on unions of disjoint families of open sets:
\[
T(\bigcup_{i \in I} \Omega_i) = \sum_{i \in I} T(\Omega_i).
\]
It is straightforward to verify that if \( E(a) \), with \( a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d_+ \), is the ellipsoid
\[
E(a) = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1 \right\},
\]
then
\[
u_{E(a)}(x) = \frac{1}{2} \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1} \left( 1 - \sum_{i=1}^d \frac{x_i^2}{a_i^2} \right),
\]
and
\[
T(E(a)) = \frac{\omega_d}{d + 2} \left( \prod_{i=1}^d a_i \right) \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1}, \tag{3}
\]
where
\[
\omega_d = \frac{\pi^{d/2}}{\Gamma((d + 2)/2)}
\]
is the Lebesgue measure of a ball \( B_1 \) with radius 1 in \( \mathbb{R}^d \). We put
\[
\tau_d = T(B_1) = \frac{\omega_d}{d(d + 2)}.
\]
The de Saint-Venant inequality (see, for instance, [11, Chapter V]) asserts that
\[ T(\Omega) \leq T(\Omega^*), \]  
where \( \Omega^* \) is any ball with \( |\Omega| = |\Omega^*| \). It follows by scaling that
\[ \frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \leq \frac{\tau_d}{\omega_d^{(d+2)/d}} = \frac{1}{d(d + 2)\omega_d^{2/d}}. \]  

Below we recall some basic facts about the Newtonian capacity \( \text{cap}(K) \) of a compact set \( K \subset \mathbb{R}^d, d \geq 3 \). There are several equivalent definitions of \( \text{cap}(K) \) of which we choose
\[ \text{cap}(K) = \inf \left\{ \int_{\mathbb{R}^d} |D\varphi|^2 \, dx : \varphi_K \geq 1, \varphi \in C_0^1(\mathbb{R}^d), \varepsilon > 0 \right\}, \]
where \( \varphi_K \) is the restriction of \( \varphi \) to \( K \). It follows that \( K_1 \subset K_2 \Rightarrow \text{cap}(K_1) \leq \text{cap}(K_2) \), and that the capacity satisfies the scaling property
\[ \text{cap}(tK) = t^{d-2} \text{cap}(K), \quad t > 0. \]  
Moreover if \( \{K_i, i \in I\} \) is a countable family of compact sets such that \( \bigcup_{i \in I} K_i \) is compact, then
\[ \text{cap}(\bigcup_{i \in I} K_i) \leq \sum_{i \in I} \text{cap}(K_i). \]

It was reported in [8, pp. 260] that the Newtonian capacity of an ellipsoid was computed in volume 8, [5, pp. 103–104]. The formula is given in terms of an elliptic integral, and reads
\[ \text{cap}\left(\frac{E(a)}{\pi}\right) = \frac{4\pi^{d/2}}{\Gamma(d/2)} \left( e(a) - 1 \right), \]  
where
\[ e(a) = \int_0^\infty \left( \prod_{i=1}^d \left( a_i^2 + t \right) \right)^{-1/2} dt. \]  

We put
\[ \kappa_d = \text{cap}\left(\frac{B_1}{\pi}\right) = \frac{4\pi^{d/2}}{\Gamma((d-2)/2)}, \]
so that
\[ \text{cap}\left(\frac{E(a)}{\pi}\right) = \frac{2\kappa_d}{d-2} e(a)^{-1}. \]  

The isoperimetric inequality for Newtonian capacity (see [11]) asserts that for all compact sets \( K \subset \mathbb{R}^d, d \geq 3 \),
\[ \text{cap}(K) \geq \text{cap}(K^*), \]
where \( K^* \) is a closed ball with \( |K| = |K^*| \). It follows by scaling that
\[ \frac{\text{cap}(K)}{|K|^{(d-2)/d}} \geq \frac{\kappa_d}{\omega_d^{(d-2)/d}}. \]  

The shape functional we consider in the present paper is
\[ G_q(\Omega) = \frac{\text{cap}(\Omega)T(\Omega)^q}{|\Omega|^{1+q+2(q-1)/d}}, \]  
where \( \text{cap}(\Omega) \) and \( T(\Omega) \) are defined as above.
defined for a bounded open set $\Omega \subset \mathbb{R}^d$, $d \geq 3$. By (2) and (7), we obtain that $G_q$ is scaling invariant. With the definitions above, we have

$$G_q(B_1) = \frac{\kappa_d \tau^q_d}{\omega_d^{1+q+2(q-1)/d}}.$$

Since the ball $\Omega^*$ with measure $|\Omega^*| = |\Omega|$ maximises the torsional rigidity $T(\Omega)$ (de Saint-Venant), and its closure minimises the Newtonian capacity $\text{cap}(\Omega)$, competition enters in the minimisation or maximisation problems for the functional in (12).

All of our main results are for $d \geq 3$, and are as follows.

**Theorem 1.** (i) If $q \in \mathbb{R}$, then

$$\sup \{ G_q(\Omega) : \Omega \text{ open and bounded} \} = +\infty.$$  

(ii) If $q \leq 0$, then

$$\min \{ G_q(\Omega) : \Omega \text{ open and bounded} \} = G_q(B_1),$$  

with equality if and only if $\Omega$ is (up to sets of capacity 0) a ball in $\mathbb{R}^d$.

(iii) If $q > 0$, then

$$\inf \{ G_q(\Omega) : \Omega \text{ open and bounded} \} = 0.$$

**Theorem 2.** (i) If $q < 1$, then

$$\sup \{ G_q(\Omega) : \Omega \text{ open, bounded and convex} \} = +\infty.$$  

(ii) If $q \geq 1$, then

$$\sup \{ G_q(\Omega) : \Omega \text{ open, bounded and convex} \} \leq \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} G_q(B_1).$$  

If $q > 1$, then the variational problem in the left-hand side of (14) has a maximiser, say $\Omega^+$. For any such maximiser,

$$\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq 2d \left( \frac{2^{(d+2)/2} d^{3q-2+d(q+1)}}{d-2} \right)^{d/(2(q-1))}.$$  

(iii) If $q = 1$ and $d = 3$, then the variational problem in the left-hand side of (14) has a maximiser, say $\Omega^+$. For any such maximiser,

$$\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq 2 \cdot 3^8 e^{37}.$$  

**Theorem 3.** (i) If $q > (d-2)/(2(d-1))$, then

$$\inf \{ G_q(\Omega) : \Omega \text{ open, bounded and convex} \} = 0.$$  

(ii) If $0 < q \leq (d-2)/(2(d-1))$, then

$$\inf \{ G_q(\Omega) : \Omega \text{ open, bounded and convex} \} \geq \frac{1}{2d(d+2)q} G_q(B_1).$$  

If $0 < q < (d-2)/(2(d-1))$, then the variational problem in the left-hand side of (18) has a convex minimiser, say $\Omega^-$. For any such minimiser,

$$\frac{\text{diam}(\Omega^-)}{r(\Omega^-)} \leq 2d \left( \frac{2(d+2)q}{d-2-2q(d-1)} \right)^{d/(d-1)}.$$
We were unable to prove the existence or non-existence of a maximiser for the left-hand side of (14) for \( q = 1 \) and \( d > 3 \). In these higher dimensional cases, there is a lack of compactness. For example, if

\[
\alpha_\varepsilon = \left(1, \ldots, 1, \varepsilon, \ldots, \varepsilon\right),
\]

and if \( k \geq 3 \), then \( \lim_{\varepsilon \to 0} G_1(E(\alpha_\varepsilon)) \) exists and is strictly positive. Similarly we were unable to prove the existence of a minimiser of the left-hand side of (18) at the critical point \( q = (d - 2)/(2d - 2) \).

The proofs of Theorems 1–3 are deferred to Sections 2–4, respectively. A key ingredient in these proofs is John’s ellipsoid theorem \[9\]. This theorem asserts that for any open, bounded convex set \( \Omega \) in \( \mathbb{R}^d \) there exists a translation and rotation of \( \Omega \), again denoted by \( \Omega \), and an open ellipsoid \( E(a) \) such that

\[
E(a/d) \subset \Omega \subset E(a).
\]

Moreover, among all ellipsoids in \( \Omega \), \( E(a/d) \) has maximal measure.

Finally in Section 5 we discuss the optimisation of a functional over all open bounded planar convex sets with fixed measure, and which involves the logarithmic capacity and torsional rigidity.

2. Proof of Theorem 1

Proof. To prove the assertion under (i), we let \( \Omega \) be the disjoint union of an open ball \( B' \) of measure 1/2 and an open ellipsoid \( E(b_\varepsilon) \), with \( b_\varepsilon = (L_\varepsilon, \ldots, L_\varepsilon, \varepsilon) \), of measure 1/2, where

\[
L_\varepsilon = (2\omega_d \varepsilon)^{1/(1-d)}.
\]

We have, by (8) and (9),

\[
\text{cap}\left(\overline{E(b_\varepsilon)}\right) = \frac{4\pi^{d/2}}{\Gamma(d/2)} \left(\int_0^\infty dt \left( L_\varepsilon^2 + t \right)^{(1-d)/2} \left( \varepsilon^2 + t \right)^{-1/2}\right)^{-1} \\
\geq \frac{4\pi^{d/2}}{\Gamma(d/2)} \left(\int_0^\infty dt \left( L_\varepsilon^2 + t \right)^{(1-d)/2} t^{-1/2}\right)^{-1} \\
= \frac{4\pi^{(d-1)/2} \Gamma((d-1)/2)}{\Gamma(d/2) \Gamma((d-2)/2)} L_\varepsilon^{d-2},
\]

where we have used formulae \[7, 8.380.3 \text{ and } 8.384.1\]. Hence

\[
G_q(\Omega) = \text{cap}\left(\overline{B'} \cup \overline{E(b_\varepsilon)}\right) T(B' \cup E(b_\varepsilon))^q \\
\geq \text{cap}\left(\overline{E(b_\varepsilon)}\right) T(B')^q \\
\geq \frac{4\pi^{(d-1)/2} \Gamma((d-1)/2)}{\Gamma(d/2) \Gamma((d-2)/2)} T(B')^q (2\omega_d \varepsilon)^{(d-2)/(1-d)},
\]

which tends to \( +\infty \) as \( \varepsilon \downarrow 0 \).

To prove the assertion under (ii), we recall (4), and infer that \( T^q(\Omega) \geq T^q(\Omega^*) \) for \( q \leq 0 \). This implies (13) by (5) and (11).

To prove (iii), we let \( Q \subset \mathbb{R}^d \) be a cube with \( |Q| = 1 \). Let \( N \in \mathbb{N} \) be arbitrary. The cube \( Q \) contains \( N^d \) open disjoint cubes each of measure \( N^{-d} \). Each open cube contains an open
ball with radius $1/(2N)$. Let $Q_N$ be the union of these $N^d$ open balls. Since $Q_N \subset Q$ we have $\text{cap}(Q_N) \leq \text{cap}(Q)$. On the other hand, additivity and scaling properties of the torsion give
\[ T(Q_N) = N^d(2N)^{-(d+2)}T(B_1) = 2^{-d-2}N^{-2}T(B_1). \]
Furthermore,
\[ |Q_N| = \frac{\omega_d}{2^d}. \]
Hence
\[ \inf \{ G_q(\Omega) : \Omega \text{ open and bounded} \} \leq \frac{\text{cap}(Q)T^q(Q_N)}{|Q_N|^{1+q+2(q-1)/d}} = \frac{2^{d-2}}{\omega_d^{1+q+2(q-1)/d}} \text{cap}(Q)T^q(B_1)N^{-2q}. \]
This implies (17) since $q > 0$, and $N \in \mathbb{N}$ was arbitrary. □

3. Proof of Theorem 2

Proof. To prove (i) we consider the open ellipsoid $E(a_{\varepsilon})$ with $a_{\varepsilon} = (1, \varepsilon, \ldots, \varepsilon)$. We have
\[ |E(a_{\varepsilon})| = \omega_d \varepsilon^{d-1}, \]
\[ T(E(a_{\varepsilon})) = \frac{\omega_d}{d+2} \varepsilon^{d+1}, \]
\[ \text{cap}(E(a_{\varepsilon})) = \begin{cases} 4\pi \varepsilon(\log(\varepsilon^{-1}))^{-1}(1 + o(1)), & d = 3, \varepsilon \downarrow 0, \\ 2\pi d/(d-3) \varepsilon^{d-3}(1 + o(1)), & d > 3, \varepsilon \downarrow 0, \end{cases} \]
where we have used the formulae on [8, p. 260]. Hence
\[ G_q(E(a_{\varepsilon})) = \begin{cases} C_d \varepsilon^{2(q-1)/3}(\log \varepsilon^{-1})^{-1}(1 + o(1)), & d = 3, \varepsilon \downarrow 0, \\ C_d \varepsilon^{2(q-1)/d}(1 + o(1)), & d > 3, \varepsilon \downarrow 0, \end{cases} \]
where $C_d$ is a positive constant depending only on $d$. Since $q < 1$, we obtain the desired result by letting $\varepsilon \downarrow 0$.

To prove (ii), we first observe that the formulae for $|E(a)|$, $\text{cap}(E(a))$, and $T(E(a))$ are symmetric in $a_1, \ldots, a_d$. Without loss of generality we may therefore assume here, and throughout this paper, that $a_1 \geq a_2 \geq \ldots \geq a_d$. By inclusion, and (20) we have
\[ d^{-d} \omega_d \prod_{i=1}^d a_i = |E(a/d)| \leq |\Omega| \leq |E(a)| = \omega_d \prod_{i=1}^d a_i, \quad (21) \]
and
\[ T(\Omega) \geq T(E(a/d)) = \frac{\omega_d}{d^{d+2}(d+2)} \left( \prod_{i=1}^d a_i \right) \left( \sum_{i=1}^d \frac{1}{a_i^2} \right)^{-1} \geq \frac{\tau_d}{d^{d+2}} \left( \prod_{i=1}^d a_i \right) a_d^2. \quad (22) \]
We have by (8),
\[ \varepsilon(a) \geq \int_0^{a_d^2} dt \left( \prod_{i=1}^d (a_i^2 + t) \right)^{-1/2} \]
\[ \geq a_d^2 \left( \prod_{i=1}^{d} \left( a_i^2 + a_d^2 \right) \right)^{-1/2} \]
\[ \geq 2^{-d/2} a_d^2 \left( \prod_{i=1}^{d} a_i^2 \right)^{-1/2} \]
\[ = 2^{-d/2} \left( \prod_{i=1}^{d} a_i \right)^{-1} a_d^2. \]  
(23)

By (9), (10) and (23), taking into account that \( \Gamma(z + 1) = z \Gamma(z), z > 0, \)
\[ \text{cap} (\Omega) \leq \text{cap} (E(a)) \leq 2 \left( d + 2 \right) / 2 \kappa_d d \left( \prod_{i=1}^{d} a_i \right) a_d^{-2}. \]  
(24)

By (1) and (3),
\[ T(\Omega) \leq T(E(a)) \leq d \tau_d \left( \prod_{i=1}^{d} a_i \right) a_d^2. \]  
(25)

By (21), (24), (25), \( a_1 \geq a_2 \geq \ldots \geq a_d, \) and \( q > 1, \) we obtain,
\[ G_q(\Omega) \leq \frac{\text{cap}(E(a)) T(E(a))^q}{|E(a/d)|^{1+q+2(q-1)/d}} \]
\[ \leq \frac{2^{(d+2)/2} \kappa_d (d \tau_d)^q \left( d^{-d} \omega_d \right)^{-1+q+2(q-1)/d}}{d-2} \left( \prod_{i=1}^{d} a_i \right)^{(2(1-q)/d) - 2} a_d^{2q-2} \]
\[ = \frac{2^{(d+2)/2} d^{q-2+d(q+1)}}{d-2} G_q(B_1) \left( \prod_{i=1}^{d} a_i \right)^{(2(1-q)/d) - 2} a_d^{2q-2} \]
\[ \leq \frac{2^{(d+2)/2} d^{q-2+d(q+1)}}{d-2} G_q(B_1). \]  
(26)

This proves (14).

To prove the existence of a maximiser, we observe that if the left-hand side of (14) equals \( G_q(B_1), \) then \( B_1 \) is a maximiser which satisfies (15). If the left-hand side of (14) is strictly greater than \( G_q(B_1), \) we let \( \Omega \) be bounded, open, and convex, and such that
\[ G_q(\Omega) > G_q(B_1). \]  
(27)

By the third inequality in (26), (27), \( q > 1, \) and \( a_1 \geq a_2 \geq \ldots \geq a_d, \) we find that
\[ a_1 \leq \beta_d^{d/(2(q-1))} a_d, \]  
(28)

where \( \beta_d \) is the coefficient of \( G_q(B_1) \) in the right-hand side of (26). Since
\[ \text{diam}(\Omega) \leq \text{diam}(E(a)) \leq 2a_1, \]  
(29)

and
\[ r(\Omega) \geq r(E(a/d)) = \frac{a_d}{d}, \]  
(30)
we obtain by (28)–(30),
\[
\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2d \left( \frac{2^{(d+2)/2}d\beta_{q-2+d(q+1)}^{d/(2(q-1))}}{d-2} \right)^{d/(2(q-1))}.
\]

(31)

Let \((\Omega_n)\) be a maximising sequence for the left-hand side of (14). Since this supremum is scaling invariant, we fix \(r(\Omega_n) = 1\). By (31), \(\text{diam}(\Omega_n) \leq L\) for some \(L < \infty\), and for all \(n\). By taking translations of \((\Omega_n)\), these translates are contained in a closed ball \(B_L\) of radius \(L\). Since the Hausdorff metric is compact on the space of convex, compact sets in \(B_L\), there exists a subsequence of \((\Omega_n)\), again denoted by \((\Omega_n)\), which converges in the Hausdorff (and in the complementary Hausdorff) metric to an element say \(\bar{\Omega}^+\). Set \(\Omega^+ = \text{int}(\bar{\Omega}^+)\). Note that \(\Omega^+\) is an open, bounded, convex set which is non-empty since \(\bar{\Omega}^+\) has inradius 1. Furthermore measure, torsion, capacity, and diameter are all continuous with respect to this metric. Hence
\[
G_q(\Omega^+) = \lim_{n \to \infty} G_q(\Omega_n),
\]
and \(\Omega^+\) is a maximiser which satisfies (15).

To prove (iii), we let \(q = 1\) and \(d = 3\). Let \(\Omega\) be an element of a maximising sequence. We may assume that
\[
G_1(B_1) \leq G_1(\Omega) \leq \frac{\text{cap} \left( E(a) \right) T(E(a))}{|E(a/3)|^2}.
\]
We obtain an upper bound on \(\text{cap} \left( E(a) \right)\) by obtaining a lower bound on \(e(a)\). By (9), we have
\[
e(a_1, a_2, a_3) \geq \int_0^\infty (a_1^2 + t)^{-1/2} (a_2^2 + t)^{-1} dt
\]
\[
= \frac{2}{(a_1^2 - a_2^2)^{1/2}} \log \left( \frac{a_1}{a_2} + \left( \frac{a_1^2}{a_2^2} - 1 \right)^{1/2} \right)
\]
\[
\geq \frac{2}{a_1} \log \left( a_1a_2^{-1} \right).
\]
By (8) for \(d = 3\), and (33),
\[
\text{cap} \left( E(a) \right) \leq \kappa_3 a_1 \left( \log \left( a_1a_2^{-1} \right) \right)^{-1}.
\]
Since \(\Omega \subset B_{a_1}\), we also have
\[
\text{cap} \left( \Omega \right) \leq \text{cap} \left( B_{a_1} \right) = \kappa_3 a_1.
\]
Hence
\[
\text{cap} \left( \Omega \right) \leq \kappa_3 a_1 \min \left\{ 1, \left( \log \left( a_1a_2^{-1} \right) \right)^{-1} \right\}.
\]
In addition,
\[
T(E(a)) \leq 3\tau_3 a_1a_2a_3^3, \quad |E(a/3)| = 3^{-3}\omega_3 a_1a_2a_3.
\]
Summarising, from (32) we obtain
\[
G_1(\Omega) \leq 3^7 G_1(B_1) \cdot \frac{a_3}{a_2} \cdot \min \left\{ 1, \left( \log \left( a_1a_2^{-1} \right) \right)^{-1} \right\}.
\]
(34)

If the supremum in the left-hand side of (14) equals \(G_1(B_1)\), then \(B_1\) is a maximiser which satisfies (16). If not then we may assume that \(G_1(\Omega) > G_1(B_1)\). This, together with (34),
yields $a_3 \geq 3^{-7} a_2$, $a_1 \leq a_2 e^{3^7}$. These inequalities imply that $a_1/a_3 \leq 3^7 e^{3^7}$. Hence (29) and (30) yield
\[
\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2 \cdot 3^8 e^{3^7}.
\]
The remaining part of the proof follows similar lines as those in the proof of (ii). \qed

4. Proof of Theorem 3

Proof. To prove (i), we consider as $\Omega$ the ellipsoid $E(a)$ with
\[
a = (a_1, \ldots, a_d), \quad a_1 \geq a_d, \quad a_1^{d-1} a_d = 1,
\]
where $a_d \in (0, 1)$ is arbitrary. Since $E(a) \subset B_{a_1}$, we have by (6), (7) and (10),
\[
\text{cap}(E(a)) \leq \kappa_d a_1^{-d/2}.
\]
(35)

By (3) and (35),
\[
G_q(E(a)) \leq \frac{\kappa_d a_1^{-d/2}}{(d+2)^{1+2(q-1)/d}} \left( \frac{a_1^2 a_2^2}{d+2 (d-1) a_2^2 + a_1^2} \right)^q \leq \frac{\kappa_d a_1^{-d/2} a_d^{2q}}{(d+2)^{1+2(q-1)/d}}.
\]

Since $a_1 = a_d^{-1/(d-1)}$, we have
\[
G_q(E(a)) \leq \frac{\kappa_d}{(d+2)^{1+2(q-1)/d}} a_d^{-(d-2)/(d-1)+2q},
\]
and since $a_d \in (0, 1)$ was arbitrary, we obtain (17).

To prove (ii), we let $a_1 \geq a_2 \geq \ldots \geq a_d$. We have
\[
\text{cap}(\Omega) \geq \text{cap}(E(a/d)) = d^{-d} \text{cap}(E(a)) = \frac{2 \kappa_d}{d^{-d} (d-2)} (\epsilon(a))^{-1}.
\]
(36)

In order to obtain an upper bound on $\epsilon(a)$, we have by using the inequality $(x^2 + t)^{1/2} \geq 2^{-1/2} (x + t^{1/2})$, and the change of variables $t = \theta^2$,
\[
\epsilon(a) \leq \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2}^2 + t)(a_{d-1}^2 + t)(a_d^2 + t) \right)^{-1/2}
\]
\[
\leq \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2}^2 + t)(a_{d-1}^2 + t)t \right)^{-1/2}
\]
\[
\leq 2 \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty dt \left( (a_{d-2}^2 + t^{1/2})(a_{d-1}^2 + t^{1/2})t^{1/2} \right)^{-1}
\]
\[
= 4 \left( \prod_{i \leq d-3} a_i^{-1} \right) \int_0^\infty d\theta \left( (a_{d-2} + \theta)(a_{d-1} + \theta) \right)^{-1}
\]
\[
= 4 \left( \prod_{i \leq d-3} a_i^{-1} \right) \left( 1 - \frac{a_{d-1}}{a_{d-2}} \right)^{-1} \log \left( \frac{a_{d-2}}{a_{d-1}} \right),
\]
(37)
where the product over the empty set in the right-hand side of (37) is defined to be equal to 1, and where the case $a_{d-2} = a_{d-1}$ follows by taking the appropriate limit in the right-hand side of (37). It is elementary to verify that
\[(1 - x)^{-1} \log(x^{-1}) \leq \log(e/x), \quad 0 < x < 1,
\]
and
\[\lim_{x \to 1} (1 - x)^{-1} \log(x^{-1}) = 1.
\]
This gives by (37),
\[\varepsilon(a) \leq 4 \left( \prod_{i \leq d-2} a_i^{-1} \right) \log(\epsilon a_{d-2}/a_{d-1}). \tag{38}
\]
Hence by (36) and (38),
\[\text{cap}(\Omega) \geq \frac{\kappa_d}{2^{d^2+2(d+2)q(d-2)}} \left( \prod_{i \leq d-2} a_i \right) (\log(\epsilon a_{d-2}/a_{d-1}))^{-1}. \tag{39}
\]
By (21), (22), and (39),
\[G_q(\Omega) \geq \frac{1}{2^{d^2+2(d+2)q(d-2)}} G_q(B_1) a_d^{2q-1} \left( \prod_{i=1}^{d} a_i \right)^{2(1-q)/d} (\log(\epsilon a_{d-2}/a_{d-1}))^{-1}. \tag{40}
\]
The $a$-dependence in the right-hand side of (40) is scaling invariant. It is convenient to choose $\prod_{i=1}^{d} a_i = 1$. We then have
\[\frac{a_{d-2}}{a_{d-1}} = \left( \prod_{i \leq d-3} a_i^{-1} \right) a_{d-2} a_{d-1}^{-1} \leq a_{d-1} a_{d-1}^{-1}. \]
This gives with (40),
\[G_q(\Omega) \geq \frac{1}{2^{d^2+2(d+2)q(d-2)}} G_q(B_1) a_d^{2q-1} (\log(\epsilon/(a_{d-1} a_d)))^{-1}. \tag{41}
\]
Since
\[x^{1/(d-1)} \log(e/x) \leq d - 1, \quad 0 < x < 1,
\]
we have, with $x = a_{d-1}^{d-1} a_d$,
\[(\log(\epsilon/(a_{d-1} a_d)))^{-1} \geq a_{d-1} a_d^{1/(d-1)} (d - 1)^{-1}. \]
This, together with (41), gives
\[G_q(\Omega) \geq \frac{1}{2^{d^2+2(d+2)q(d-2)(d-1)}} G_q(B_1) a_d^{2q-\frac{d-2}{d-1}}
\[\geq \frac{1}{2^{d^2+(d+2)q}} G_q(B_1) a_d^{2q-\frac{d-2}{d-1}}. \tag{42}
\]
This proves (18) since $a_d \in (0, 1]$, and $q \leq (d - 2)/(2(d - 1))$.

To prove the existence of a minimiser, we observe that if the left-hand side of (18) equals $G_q(B_1)$, then $B_1$ is a minimiser which satisfies (19). If the left-hand side of (18) is strictly less than $G_q(B_1)$, we let $\Omega$ be bounded and convex such that
\[G_q(\Omega) < G_q(B_1). \tag{43}
\]
By (42) and (43), we infer
\[ a_d \geq \left( \frac{1}{2d^d(d+2)^q} \right)^{(d-1)/(d-2-2q(d-1))}. \] (44)

Since \( \prod_{i=1}^{d} a_i = 1 \) and \( a_1 \geq a_2 \geq \ldots \geq a_d \), we have \( a_1 \leq a_d^{1-d} \). By (29), (30) and (44), we obtain
\[ \frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2da_d^{-d} \leq 2d\left(2d^d(d+2)^q\right)^{\frac{d(d-1)}{d-2-2q(d-1)}}. \] (45)

The proof of the existence of a minimiser is similar to the proof of the existence of a maximiser in Theorem 1(ii), and has been omitted. If \( \Omega^- \) is a minimiser, then, by continuity of diameter and inradius, \( \Omega^- \) satisfies (45). This proves (19).

5. The logarithmic capacity

We briefly recall some basic properties of the logarithmic capacity of a compact set \( K \) in \( \mathbb{R}^2 \). Let \( \mu \) be a probability measure supported on \( K \), and let
\[ I(\mu) = \int \int_{K \times K} \log \left( \frac{1}{|x-y|} \right) \mu(dx)\mu(dy). \]

Furthermore let
\[ V(K) = \inf \{ I(\mu) : \mu \text{ a probability measure on } K \}. \]

The logarithmic capacity of \( K \) is denoted by \( \text{cap}(K) \), and is the non-negative real number
\[ \text{cap}(K) = e^{-V(K)}. \]

It shares some of the properties of the Newtonian capacity. In particular, if \( K_1 \) and \( K_2 \) are compact sets in \( \mathbb{R}^2 \) with \( K_1 \subset K_2 \), then \( \text{cap}(K_1) \leq \text{cap}(K_2) \). Moreover, \( \text{cap}(K) \) is invariant under translations and rotations of \( K \), and
\[ \text{cap}(K) \geq \text{cap}(K^*), \] (46)
where \( K^* \) is the disc with \( |K| = |K^*| \). See [1] for some refinements. Finally for a homothety,
\[ \text{cap}(tK) = t\text{cap}(K), \quad t > 0. \] (47)

The classic treatise [10] gives various planar domains for which the logarithmic capacity can be computed analytically. In particular, for the ellipse with semi axes \( a_1 \) and \( a_2 \),
\[ \text{cap}(E(a_1, a_2)) = \frac{1}{2}(a_1 + a_2). \] (48)

For an open, bounded, convex planar set \( \Omega \), we define the functional
\[ H_q(\Omega) = \frac{\text{cap}(\Omega)}{|\Omega|^{(1+4q)/2}}. \]

In particular, we have
\[ H_q(B_1) = \frac{\tau_q}{\omega_2^{(1+4q)/2}} = \frac{1}{8^q\pi^{q+1/2}}. \]

We immediately see that by (2), and (47) that \( H_q(t\Omega) = H_q(\Omega), t > 0 \). Our main result is the following.
Theorem 4. (i) If \( q \geq 1/2 \), then
\[
\sup \{ H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex} \} \leq 2^{1+5q} H_q(B_1). \tag{49}
\]

(ii) If \( q > 1/2 \), then the left-hand side of (49) has an open, bounded, planar, and convex maximiser. For any such maximiser, say \( \Omega^+ \),
\[
\frac{\text{diam}(\Omega^+)}{r(\Omega^+)} \leq \frac{2^{14q}}{2q-1}. \tag{50}
\]

(iii) If \( q < 1/2 \), then
\[
\sup \{ H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex} \} = +\infty. \tag{51}
\]

(iv) If \( q \leq 1/2 \), then
\[
\inf \{ H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex} \} \geq 2^{-2(1+2q)} H_q(B_1). \tag{52}
\]

(v) If \( q < 1/2 \), then the left-hand side of (52) has an open, bounded, planar, and convex minimiser. For any such minimiser, say \( \Omega^- \),
\[
\frac{\text{diam}(\Omega^-)}{r(\Omega^-)} \leq 2^{2(3+2q)/(1-2q)}. \tag{53}
\]

(vi) If \( q > 1/2 \), then
\[
\inf \{ H_q(\Omega) : \Omega \text{ open, bounded, planar, and convex} \} = 0. \tag{54}
\]

Proof. (i) If \( E(a) \) is the John’s ellipsoid for \( \Omega \), then \( E(a/2) \subset \Omega \subset E(a) \) with \( a_1 \geq a_2 \). Furthermore,
\[
\text{cap}(E(a)) \leq a_1, \quad T(E(a)) \leq 2\tau_2 a_1 a_2^3, \quad |\Omega| \geq |E(a/2)| = \omega_2 a_1 a_2/4,
\]
so that
\[
H_q(\Omega) \leq 2^{1+5q} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left( \frac{a_2}{a_1} \right)^{q-1/2}. \tag{54}
\]
This implies (49) since \( q \geq 1/2 \).

(ii) To prove (50), we have that either the supremum in the left-hand side of (49) is attained for a ball, in which case the maximiser exists and satisfies (50), or we may assume that \( H_q(\Omega) > H_q(B_1) \). This implies, by (29) and (30),
\[
\frac{\text{diam}(\Omega)}{r(\Omega)} \leq \frac{2^{14q}}{2q-1}.
\]
The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(iii) By (21), (22), and (48),
\[
H_q(\Omega) \geq 2^{-2(1+2q)} \frac{\tau_2^q}{\omega_2^{(1+4q)/2}} \left( \frac{a_2}{a_1} \right)^{q-1/2}. \tag{55}
\]
This implies (51) by letting \( a_2/a_1 \to 0 \) in (55).

(iv) This follows from (55) and \( a_1 \geq a_2 \).

(v) Either the infimum in the left-hand side of (52) is attained for a ball, in which case the minimiser exists and satisfies (53), or we may assume that \( H_q(\Omega) < H_q(B_1) \). By (55), (29), and (30),
\[
\frac{\text{diam}(\Omega)}{r(\Omega)} \leq 2^{2(3+2q)/(1-2q)}.
\]
The remaining part of the proof is similar to the corresponding parts in the proof of Theorem 2.

(vi) This follows by letting \( a_2/a_1 \to 0 \) in (54). \(\square\)
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