Dirichlet-Ford domains and Double Dirichlet domains*

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Abstract

We continue the investigations of Lakeland on Fuchsian and Kleinian groups which have a Dirichlet fundamental domain that also is a Ford domain in hyperbolic two- or three-space; such a domain is called a DF-domain. Making use of earlier obtained concrete formulas for the bisectors in the upper half-space model, we obtain an easy algebraic criterion on the side-pairing transformations of the fundamental domain of a Fuchsian or Kleinian group to be a DF-domain. Using the same methods, we also complement some of the results of Lakeland on Fuchsian groups having a Dirichlet domain with multiple centers.

1 Introduction

Describing generators and relations of discrete subgroups of $\text{PSL}_2(\mathbb{R})$ and $\text{PSL}_2(\mathbb{C})$ was started in the nineteenth century. As these are groups acting discontinuously on hyperbolic spaces, Poincaré’s Polyhedron Theorem gives a method to determine a presentation of such groups. For this one needs to describe a fundamental domain. However, a major difficulty one encounters is the effective construction of such a domain. Theoretically, one has the nicely described Ford and Dirichlet fundamental domains. Nevertheless, in practice, constructing these domains proves to be more difficult. This problem was considered by Ford, Poincaré, Serre, Swan and many others. Today, computer aided methods also exist; we refer to [8, 9] for Fuchsian groups, to [17] and [12] for Kleinian groups and to [4] for cocompact groups. In all these cases, the problem of constructing an explicit fundamental domain remains a challenge. As the Ford domain is based on isometric spheres, the concrete formulas for its construction are easy. However, the Ford domain is not always the most convenient fundamental domain to work with. Therefore, in [7], the authors develop explicit formulas for the bisectors determining the Dirichlet domain (see [15]) in hyperbolic two-space $\mathbb{H}^2$ and hyperbolic three-space $\mathbb{H}^3$. These are then used to tackle the non-trivial problem of describing units in an order of a non-commutative non-split division algebra. As the unit groups of some of these orders may be considered as discrete subgroups of $\text{SL}_2(\mathbb{C})$, fundamental domains and Poincaré’s Polyhedron Theorem are of potential use to determine these unit groups. First attempts to this were done in [13, 14], where the authors use Ford domains to get presentations of some small subgroups of congruence subgroups of Bianchi groups and in [4], where a presentation for the

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unit group of a small non-commutative division algebra is given using Dirichlet domains. As an application, one obtains a description of subgroups of finite index in the unit group of an integral group ring of some finite groups. Making use of concrete formulas, the authors obtain in [7] a more general approach than the isolated cases described above.

Motivated by all the above and making use of these concrete formulas constructed in [7], in this paper we continue the investigations initiated by Lakeland in [10] on DF-domains, i.e. fundamental domains in $\mathbb{H}^2$ or $\mathbb{H}^3$ that are Dirichlet and Ford domains at the same time, and on double Dirichlet domains, i.e. Dirichlet domains with multiple centers. The motivation in [10] is answering a question raised in [1]: the construction of a maximal arithmetic hyperbolic reflection group which is not congruence.

In [10, Theorem 5.3] it is proved that a finitely generated, finite coarea Fuchsian group $\Gamma$ admits a DF-domain if and only if $\Gamma$ is an index 2 subgroup of a reflection group. It also is proved that a Kleinian group $\Gamma$ has a generating set consisting of elements whose traces are real ( [10, Theorem 6.3].) We give a new and independent criterion for the result of [10] that also applies to Kleinian groups. Our criterion is of algebraic nature and easily can be checked once a set of side-pairing transformations is given:

**Theorem 1.1** Let $\Gamma$ be a geometrically finite discrete subgroup of $\text{PSL}_2(\mathbb{C})$, acting on $\mathbb{H}^2$, respectively $\mathbb{H}^3$, and $P_0 = i$, respectively $P_0 = j$. Suppose that the stabilizer of $P_0$ in $\Gamma$ is trivial. Then, $\Gamma$ admits a DF domain $\mathcal{F}$ with center $P_0$ if and only if for every side-pairing transformation $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$ of $\mathcal{F}$ we have that $d = \bar{a}$. Moreover, if $\Gamma$ is a cofinite Fuchsian group, then $\check{\Gamma} = \langle \sigma, \Gamma \rangle$ is a reflection group and $\hat{\Gamma} = \langle \tau, \Gamma \rangle$ is a Coxeter Kleinian group, where $\sigma$ is the reflection in the imaginary axis and $\tau$ is the linear operator represented by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Also, both groups contain $\Gamma$ as a subgroup of index two.

As an immediate consequence, one obtains, in the Kleinian case, that the traces of the generating elements are real (see Corollary 4.4). Moreover, as an application, we get most of the results on DF and double Dirichlet domains obtained in [10]. Also, some of the proofs of [10] can be simplified using Theorem 1.1.

The outline of the paper is as follows. For the sake of completeness, we record in Section 2 some fundamentals on hyperbolic geometry and on Fuchsian and Kleinian groups. In Section 3, we recall some results from [7] and develop some propositions that are necessary to prove our main result and its corollaries. In particular, we give conditions for an isometric sphere, in the upper half-space (-plane) model, to be a bisector. In the last section, we consider DF domains and double Dirichlet domains, prove Theorem 1.1 and show some corollaries.

2 Background

We begin by recalling basic facts on hyperbolic spaces and we fix notation. Standard references are [2, 3, 5, 6, 11, 15]. By $\mathbb{H}^3$ (respectively $\mathbb{B}^3$) we denote the upper half-space (respectively the ball) model of hyperbolic 3-space.

So, $\mathbb{H}^3 = \mathbb{C} \times [0, \infty[$. As is common, we often shall identify $\mathbb{H}^3$ with the subset $\{z + rj \in \mathcal{H} \mid z \in \mathbb{C}, r \in \mathbb{R}^+\}$ of the classical (real) quaternion algebra $\mathcal{H} = \mathcal{H}(-1, -1, \mathbb{R})$. The ball model $\mathbb{B}^3$ may be identified with $\{z + rj \in \mathbb{C} + \mathbb{R}j \mid |z|^2 + r^2 < 1\}$ in $\mathcal{H}$. Denote by $\text{Iso}(\mathbb{H}^3)$ (respectively $\text{Iso}(\mathbb{B}^3)$)
the group of isometries of \( \mathbb{H}^3 \) (respectively \( \mathbb{B}^3 \)). The respective groups of orientation-preserving isometries are denoted by \( \text{Iso}^+(\mathbb{H}^3) \) and \( \text{Iso}^+(\mathbb{B}^3) \). It is well known that \( \text{Iso}^+(\mathbb{H}^3) \) and \( \text{Iso}^+(\mathbb{B}^3) \) are isomorphic with \( \text{PSL}_2(\mathbb{C}) \) and that \( \text{Iso}(\mathbb{H}^3) \) and \( \text{Iso}(\mathbb{B}^3) \) are isomorphic with \( \text{PSL}_2(\mathbb{C}) \times C_2 \).

Throughout, we will use the notation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) both for an element of \( \text{SL}_2(\mathbb{C}) \) as well as for its natural image in \( \text{PSL}_2(\mathbb{C}) \). Moreover, abusing notation, we use the same letter for both the matrix in \( \text{SL}_2(\mathbb{C}) \) and the M"obius transformation acting on \( \mathbb{H}^3 \). For \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \), we write \( a = a(\gamma) \), \( b = b(\gamma) \), \( c = c(\gamma) \) and \( d = d(\gamma) \) when it is necessary to stress the dependence of the entries on the matrix \( \gamma \).

The action of \( \text{PSL}_2(\mathbb{C}) \) on \( \mathbb{H}^3 \) is given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (P) = (aP + b)(cP + d)^{-1},
\]

where \( (aP + b)(cP + d)^{-1} \) is calculated in \( \mathcal{H} \). Explicitly, if \( P = z + rj \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then

\[
\gamma(P) = \frac{(az + b)(\overline{c}z + d) + a\overline{c}r^2}{|cz + d|^2 + |c|^2r^2} + \frac{r}{|cz + d|^2 + |c|^2r^2}j. \tag{1}
\]

Let \( u = u_0 + u_1 i + u_2 j + u_3 k \in \mathcal{H} \) and define \( \overline{u} \) to be \( u_0 - u_1 i - u_2 j - u_3 k \), the conjugate of \( u \). Moreover let \( u' = u_0 - u_1 i - u_2 j + u_3 k \) and \( u^* = u_0 + u_1 i + u_2 j - u_3 k \). Let

\[
\text{SB}_2(\mathcal{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{H}) \mid d = a', \ b = c', \ a\overline{c} - c\overline{a} = 1, \ a\overline{c} \in \mathbb{C} + RJ \right\}.
\]

Let \( g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \in M_2(\mathcal{H}) \). The embedding \( \Psi : \text{SL}_2(\mathbb{C}) \to \text{SB}_2(\mathcal{H}) \) defined by

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \overline{g}\gamma g = \frac{1}{2} \begin{pmatrix} a + \overline{d} + (b - \overline{c})j & b + \overline{c} + (a - \overline{d})j \\ c + \overline{b} + (d - \overline{a})j & \overline{d} + (c - \overline{b})j \end{pmatrix}
\]

is a group embedding.

Hence, \( \|\Psi(\gamma)\|^2 = \|\gamma\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 \). If \( \Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix} \), then the action of \( \Psi(\gamma) \) on \( \mathbb{B}^3 \) is given by

\[
\Psi(\gamma)(P) = (AP + C')(CP + A')^{-1},
\]

where the latter is calculated in the classical quaternion algebra \( \mathcal{H}(-1, -1, \mathbb{R}) \). Further details on this may be found in [5, Section 1.2].

The map

\[
\eta_0 : \mathbb{H}^3 \to \mathbb{B}^3 : P \mapsto (P - j)(-jP + 1)^{-1}
\]

is an \( \Psi \)-equivariant isometry between the hyperbolic models \( \mathbb{H}^3 \) and \( \mathbb{B}^3 \), that is, \( \eta_0(MP) = \psi(M)\eta_0(P) \), for \( P \in \mathbb{H}^3 \) and \( M \in \text{PSL}_2(\mathbb{C}) \) (see [5, Proposition, I.2.3]).

The hyperbolic distance \( \rho \) in \( \mathbb{H}^3 \) is determined by \( \cosh \rho(P, P') = 1 + \frac{d(P, P')^2}{2rr'} \), where \( d \) is the Euclidean distance and \( P = z + rj \) and \( P' = z' + r'j \) are two elements of \( \mathbb{H}^3 \).

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Recall that a discrete subgroup of $\text{PSL}_2(\mathbb{R})$ is called Fuchsian and a discrete subgroup of $\text{PSL}_2(\mathbb{C})$ is called Kleinian. We now define the concepts of Dirichlet and Ford fundamental domain and recall how these can be used to give a presentation for the considered groups, the so called Poincaré method (for details see for example [15]). Let $\Gamma$ be a discrete subgroup of $\text{Iso}^+(\mathbb{H}^3)$. Let $\gamma_j$ be the stabilizer in $\Gamma$ of $j \in \mathbb{H}^3$ and let $\mathcal{F}_0$ be a fundamental domain of $\gamma_j$. Put $D_\gamma(j) = \{ u \in \mathbb{H}^3 : \rho(j, u) \leq \rho(u, \gamma(j)) \}$. The border $\partial D_\gamma(j) = \{ u \in \mathbb{H}^3 : \rho(j, u) = \rho(u, \gamma(j)) \}$ is the hyperbolic bisector of the geodesic linking $j$ to $\gamma(j)$. This is called a Poincaré bisector. The set

$$
\mathcal{F} = \mathcal{F}_j \cap \left( \bigcap_{\gamma \in \Gamma \setminus \Gamma_j} D_\gamma(j) \right)
$$

is the Dirichlet fundamental domain of $\Gamma$ with center $j$. Moreover, it may be shown that $\mathcal{F}$ is a polyhedron and if $\Gamma$ is geometrically finite then a set of generators for $\Gamma$ consists of the elements $\gamma \in \Gamma$ such that $\mathcal{F} \cap \gamma(\mathcal{F})$ is a side of the polyhedron together with $\gamma_j$, i.e. $\Gamma = \langle \gamma_j, \gamma \mid \gamma(\mathcal{F}) \cap \mathcal{F} \text{ is a side } \rangle$ (see [15, Theorem 6.8.3]). Note that the same construction can be done for $\Gamma \leq \text{Iso}^+(\mathbb{H}^2)$, by replacing the point $j$ by the point $i$.

Let $\Gamma$ be a discrete subgroup of $\text{PSL}_2(\mathbb{C})$ and denote by $\Gamma_\infty$ the stabilizer in $\Gamma$ of the point $\infty$. Denote a fundamental domain of $\Gamma_\infty$ by $\mathcal{F}_\infty$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_\infty$, denote the isometric sphere of $\gamma$ by $\text{ISO}^- \gamma$. Note that these are the points $P \in \mathbb{H}^3$ such that $\|CP + d\| = 1$. Denote the set $\{ P \in \mathbb{H}^3 : \|CP + d\| \geq 1 \}$ by $\text{ISO}^2 \gamma$. It is known that if $\Gamma$ contains a parabolic element then

$$
\mathcal{F} = \mathcal{F}_\infty \cap \left( \bigcap_{\gamma \in \Gamma \setminus \Gamma_\infty} \text{ISO}^2 \gamma \right)
$$

is a fundamental domain of $\Gamma$ called the Ford fundamental domain of $\Gamma$.

**Remark 2.1** If we talk about Ford domains, we implicitly assume the discrete subgroup $\Gamma$ to have a parabolic element.

### 3 Poincaré bisectors and isometric spheres

For completeness’ sake, we recall in this section the results from [7] that are needed to prove our main result. Based on these results we also develop two more propositions that will be used later.

The origin of $\mathbb{H}^3$ is denoted $0$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \setminus \text{SU}_2(\mathbb{C})$ and $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix}$. Consider the Euclidean sphere $\Sigma = \Sigma_{\Psi(\gamma)} = \{ P = z + rj \in \mathbb{C} + \mathbb{R}j : \|CP + A'\| = 1 \}$. It has center $P_{\Psi(\gamma)} = -C^{-1}A'$ and radius $R_{\Psi(\gamma)} = \frac{1}{|C|}$. It is well known that this is the isometric sphere of $\Psi(\gamma)$ in $\mathbb{H}^3$ and that it equals the bisector of the geodesic segment linking $0$ and $\Psi(\gamma^{-1})(0)$, i.e. $\Sigma_{\Psi(\gamma)} = \{ u \in \mathbb{H}^3 : \rho(0, u) = \rho(u, \Psi(\gamma^{-1})(0)) \}$ (see for instance [2, Theorem 9.5.2] for a proof in dimension 2). In [7, Theorem 3.1], the authors give an independent proof of this in dimension 3 (which is of course adaptable to dimension 2).

In general, in $\mathbb{H}^3$, an isometric sphere is not necessarily a Poincaré bisector. For this reason we define $\Sigma_\gamma := \eta_0^{-1}(\Sigma_{\Psi(\gamma)})$, where $\eta_0$ was defined in (2). As $\eta_0$ is an isometry between the two models,
where \( \Sigma_{\gamma} \) is the bisector of the geodesic linking \( \eta^{-1}_0(0) = j \) and \( \eta^{-1}_0(\Psi^{-1}(\gamma^{-1}(0))) = \gamma^{-1}(j) \). This bisector may be a Euclidean sphere or a plane perpendicular to \( \partial \mathbb{H}^3 \). In case it is a Euclidean sphere, we denote its center by \( P_{\gamma} \) and its radius by \( R_{\gamma} \).

The following result gives concrete formulas for the Poincaré bisectors in the upper half-space model. Recall that \( \gamma \in \text{SU}_2(\mathbb{C}) \) if and only if \( \gamma(j) = j \) or, equivalently, \( \Psi(\gamma)(0) = 0 \) (see [2, 5]). As the Poincaré bisector can only exist if \( \gamma \not\in \Gamma_j \), the case \( \gamma \in \text{SU}_2(\mathbb{C}) \) is excluded in the following proposition.

**Proposition 3.1** [7, Proposition 3.2] Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}), \) with \( \gamma \not\in \text{SU}_2(\mathbb{C}) \).

1. \( \Sigma_{\gamma} \) is a Euclidean sphere if and only if \( |a|^2 + |c|^2 \neq 1 \). In this case, its center and its radius are respectively given by \( P_{\gamma} = \frac{-\langle m, n \rangle}{|a|^2 + |c|^2 - 1} \) and \( R_{\gamma} = \frac{1 + \|P_{\gamma}\|^2}{|a|^2 + |c|^2} \).

2. \( \Sigma_{\gamma} \) is a plane if and only if \( |a|^2 + |c|^2 = 1 \). In this case \( \text{Re}(\langle \eta, z \rangle) + \frac{|a|^2}{\lambda} = 0, \) \( z \in \mathbb{C} \) is a defining equation of \( \Sigma_{\gamma} \), where \( \nu = \frac{\lambda b + \lambda d}{\eta} \).

The next proposition gives some information on the relation between \( \text{ISO}_{\Sigma} \) and \( \Sigma_{\gamma} \), for some \( \gamma \in \text{SL}_2(\mathbb{C}) \setminus \text{SU}_2(\mathbb{C}) \) with \( c(\gamma) \neq 0 \). This will be useful in the study of DF domains and double Dirichlet domains.

**Proposition 3.2** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \setminus \text{SU}_2(\mathbb{C}) \).

1. If \( c \neq 0 \) and \( |a|^2 + |c|^2 \neq 1 \) then \( |P_{\gamma} - \bar{P}_{\gamma}| = \frac{|d - \bar{\eta}|}{|a|^2 + |c|^2 - 1} \). Moreover, \( \text{ISO}_{\gamma} = \Sigma_{\gamma} \) if and only if \( d = \bar{\eta} \). In this case we also have that \( c = \lambda \bar{\nu} \), with \( \lambda \in \mathbb{R} \).

2. If \( \text{ISO}_{\gamma} \) and \( \Sigma_{\gamma} \) exist and are equal, then \( \text{tr}(\gamma) \in \mathbb{R} \).

**Proof.** Since \( \det(\gamma) = 1 \) we have that \( |P_{\gamma} - \bar{P}_{\gamma}| = \frac{|d - \bar{\eta}|}{|a|^2 + |c|^2 - 1} \). Hence if \( \text{ISO}_{\gamma} = \Sigma_{\gamma} \), then \( d = \bar{\eta} \) and hence \( bc = |a|^2 - 1 \in \mathbb{R} \). With the formula of \( R_{\gamma} \) at hand, we readily find that \( R_{\gamma} = \frac{1}{|a|^2} \).

If \( \text{ISO}_{\gamma} = \Sigma_{\gamma} \) then \( d = \bar{\eta} \) and thus \( \text{tr}(\gamma) = a + \bar{\eta} \in \mathbb{R} \). This proves the second item. \( \square \)

**Remark 3.3** Notice that if \( c = 0 \) and \( \Sigma_{\gamma} \) is a plane, we have that \( ad = 1 \) and \( 1 = |a|^2 + |c|^2 = |a|^2 \). Hence \( d = \bar{\eta} \) and \( c = \lambda \bar{\nu} \), with \( \lambda \in \mathbb{R} \). So, if the isometric sphere does not exist and the Poincaré bisector is a Euclidean plane, we get the same result as in item 1. Similarly, if \( c = 0 \) and \( \Sigma_{\gamma} \) is a plane, one may also easily prove that \( \text{tr}(\gamma) \in \mathbb{R} \), as in item 2.

We need one more elementary proposition.

**Proposition 3.4** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \) and \( \Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix} \). Then \( \Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)} \) if and only if \( \gamma_1 = \gamma_0 \gamma \) for some \( \gamma_0 \in \text{SU}_2(\mathbb{C}) \).
Proof.

Suppose \( \Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)} \). As \( \Psi(\gamma) \) and \( \Psi(\gamma_1) \) are Möbius transformations, \( \Psi(\gamma) = A \circ \sigma \) and \( \Psi(\gamma_1) = A_1 \circ \sigma_1 \) for \( A \) and \( A_1 \) two orthogonal maps and \( \sigma \) and \( \sigma_1 \) reflections in the spheres \( \Sigma_{\Psi(\gamma)} \) and \( \Sigma_{\Psi(\gamma_1)} \) respectively. Thus \( \sigma = \sigma_1 \) and thus \( \Psi(\gamma_1) = A_1 \circ \sigma_1 = A_1 \circ \sigma = A_1 A^{-1} \circ \Psi(\gamma) \). Put \( \Psi(\gamma_0) = A_1 A^{-1} \) and one implication is proved. To prove the inverse implication suppose \( \gamma_1 = \gamma_0 \gamma \) for some \( \gamma_0 \in \text{SU}_2(\mathbb{C}) \). Then \( \Psi(\gamma) = A \circ \sigma \) and \( \Psi(\gamma_1) = A_1 \circ \sigma_1 \) for \( A \) and \( A_1 \) two orthogonal maps and \( \sigma \) and \( \sigma_1 \) reflections in some spheres. We have that \( \Psi(\gamma_1) = \Psi(\gamma_0) \Psi(\gamma) = \Psi(\gamma_0) \Psi(\gamma) A \sigma \) and as \( \Psi(\gamma_0) A \) is an orthogonal map, \( \Psi(\gamma_0) A = A_1 \) and \( \sigma = \sigma_1 \). Consequently \( \Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)} \). □

4 DF Domains and Double Dirichlet Domains

The goal of this section is to prove Theorem 1.1 and give some consequences that reprove and complement some results in [10].

The following definitions are taken from [10].

**Definition 4.1** A Dirichlet fundamental domain which is also a Ford domain in \( \mathbb{H}^n \) is called a DF-domain. A Dirichlet fundamental domain which has multiple centers is called a double Dirichlet Domain.

Throughout this section, we work in \( \mathbb{H}^2 \) and \( \mathbb{H}^3 \) and we assume, without loss of generality, that the stabilizer of \( i \), or \( j \) respectively, in \( \Gamma \), is trivial. The latter is possible by conjugating, if needed, the group \( \Gamma \) by an adequate affine subgroup of \( \text{PSL}_2(\mathbb{R}) \), or \( \text{PSL}_2(\mathbb{C}) \) respectively.

We first give two lemmas on Fuchsian groups.

**Lemma 4.2** The following properties are equivalent for \( \gamma \in \text{PSL}_2(\mathbb{R}) \).

1. \( a(\gamma) = d(\gamma) \).
2. \( \gamma = \sigma \circ \sigma_\gamma \), where \( \sigma \) denotes the reflection in the imaginary axes, i.e., \( \sigma(z) = -\overline{z} \) and \( \sigma_\gamma \) is the reflection in \( \Sigma_\gamma \).
3. \( \Sigma_\gamma \) is the bisector of the geodesic linking \( t \) and \( \gamma^{-1} (t) \), for all \( t > 0 \).
4. There exists \( t_0 \neq 1 \) such that \( \Sigma_\gamma \) is the bisector of the geodesic segment linking \( t_0 i \) and \( \gamma^{-1}(t_0 i) \).

**Proof.** We first prove that (1) and (2) are equivalent. Suppose that \( a(\gamma) = d(\gamma) \). Then, by Proposition 3.2, we have that \( \Sigma_\gamma = \text{ISO}_\gamma \). Hence the reflection \( \sigma_\gamma \) in \( \Sigma_\gamma \) is given by \( \sigma_\gamma(z) = P_\gamma - (|c|^2 \sigma(z - P_\gamma))^{-1} = \sigma(\gamma(z)) \), if \( c(\gamma) \neq 0 \). If \( c(\gamma) = 0 \), then we may take \( a(\gamma) = d(\gamma) = 1 \) and thus \( \sigma_\gamma(z) = \sigma(z - b(\gamma)) = \sigma(\gamma(z)) \). Hence, in either case, we have that \( \gamma = \sigma \circ \sigma_\gamma \). Now suppose that \( \gamma = \sigma \circ \sigma_\gamma \). We first suppose that \( P_\gamma \) exists, i.e., \( \Sigma_\gamma \) is a Euclidean sphere. In this case we have that \( \gamma(z) = \sigma \circ \sigma_\gamma(z) = \frac{-P_\gamma z + P_\gamma^2 - R^2}{z - P_\gamma} \), from which it follows that \( a(\gamma) = d(\gamma) \). If \( \Sigma_\gamma \) is a vertical line, \( x = x_0 \) say, then \( \sigma_\gamma(z) = -\overline{z} + 2x_0 \). Hence \( \gamma(z) = \sigma \circ \sigma_\gamma(z) = z - 2x_0 \) and hence \( a(\gamma) = 1 = d(\gamma) \).

Suppose now that \( \gamma = \sigma \circ \sigma_\gamma \) and let \( u \in \Sigma_\gamma \). Then \( \rho(\gamma^{-1}(ti), u) = \rho(\sigma_\gamma \circ \sigma(t), u) = \rho(\sigma_\gamma(t), u) = \rho(t, \sigma_\gamma(u)) = \rho(t, u) \) and hence \( \Sigma_\gamma \) is the bisector of the geodesic linking \( ti \) and \( \gamma^{-1}(ti) \). This proves that (2) implies (3). Obviously (3) implies (4).
We now prove that (4) item implies (1). Let \( u \in \Sigma_\gamma \). Then we have that \( \rho(t_0i, u) = \rho(\gamma^{-1}(t_0i), u) \) and hence \( \rho(t_0i, u) = \rho(t_0i, \gamma(u)) \). Since \( \gamma \) is a Möbius transformation we have that \( \text{Im}(\gamma(z)) = |\gamma'(z)|\text{Im}(z) \). Using this and the explicit formula of the hyperbolic distance in the upper half-space model (see Section 2), we obtain that \( |\gamma'(u)||t_0i - u|^2 = |t_0i - \gamma(u)|^2 \). It follows that \( \text{Re}(u)^2|\gamma'(u)| - \text{Re}(\gamma(u))^2 = (|\gamma'(u)| - 1)t_0^2 + (1 - |\gamma'(u)|)|\gamma'(u)|\text{Im}(u)^2 \). We may write this as an equation of the type \( \alpha t^2 = \beta \) having \( t = t_0 \) as a solution. However as \( u \in \Sigma_\gamma \), by definition \( \rho(u, i) = \rho(u, \gamma^{-1}(i)) \) and hence also \( t = 1 \) is also solution of the given equation. Thus we have that \( \alpha = \beta \) and \( \alpha(t_0^2 - 1) = 0 \). It follows that \( \alpha = 0 \) and thus \( |\gamma'(u)| = 1 \), for all \( u \in \Sigma_\gamma \), i.e. \( \Sigma_\gamma = \text{ISO}_\gamma \). Applying Proposition 3.2, we obtain that \( a(\gamma) = d(\gamma) \).

Recall that one says that an angle \( \alpha \) is a submultiple of an angle \( \beta \) if either there is a positive integer \( n \) such that \( n\alpha = \beta \) or \( \alpha = 0 \).

**Lemma 4.3** Let \( \Gamma \) be a geometrically finite cofinite discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \). Suppose that \( i \in \mathbb{H}^2 \) has trivial stabilizer and let \( \mathcal{F} \) be its Dirichlet fundamental polyhedron with center \( i \). Let \( \gamma_i \) be the side-pairing transformations of \( \mathcal{F} \) for \( 1 \leq i \leq n \). If, for every \( 1 \leq i \leq n \), \( a(\gamma_i) = d(\gamma_i) \), then \( \Gamma \) is the subgroup of the orientation-preserving isometries of a discrete reflection group.

**Proof.** First note that, as \( a(\gamma_i) = d(\gamma_i) \), we have by Lemma 3.2 that \( \Sigma_{\gamma_i} = \text{ISO}_{\gamma_i} \) and more precisely \( \Sigma_{\gamma_i}^{-1} \) has the same radius as \( \Sigma_{\gamma_i} \) and their centers are the same in absolute value, but have opposite sign. Hence, if \( \Sigma_{\gamma_i} \) intersects the imaginary axis \( \Sigma \), then so does \( \Sigma_{\gamma_i}^{-1} \) and the angle between \( \Sigma \) and \( \Sigma_{\gamma_i} \) is half the the angle between \( \Sigma_{\gamma_i} \) and \( \Sigma_{\gamma_i}^{-1} \). Consider the polyhedron \( P \), whose sides are \( \Sigma \) and the \( \Sigma_{\gamma_i} \)'s with \( P_{\gamma_i} \geq 0 \). As \( \mathcal{F} \) is a fundamental Dirichlet polyhedron, its dihedral angles are submultiples of \( \pi \). Hence the dihedral angles of \( P \) are submultiples of \( \pi \) and thus, by [15. Theorem 7.1.3], the group \( \tilde{\Gamma} = \langle \sigma, \sigma_{\gamma_i} \mid P_{\gamma_i} \geq 0 \rangle \), where \( \sigma_{\gamma_i} \) denotes the reflection in \( \Sigma_{\gamma_i} \), is a discrete reflection group with respect to \( P \). The result then follows by Lemma 4.2.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \mathcal{F} \) be a DF domain, in \( \mathbb{H}^3 \), for \( \Gamma \) with center \( P_0 = \{i, j\} \). Let \( \Phi_0 \) be a set of side-pairing transformations, i.e., \( \Phi_0 \) consist of those elements of \( \Gamma \) whose isometric circles (respectively isometric spheres) and vertical lines (respectively vertical planes) form the boundary of \( \mathcal{F} \).

Then there exists a bijection \( f : \Phi_0 \rightarrow \Phi_0 \) such that if \( \gamma \in \Phi_0 \), \( \gamma \notin \Gamma_{\infty} \) then \( \text{ISO}_\gamma = \Sigma_f(\gamma) \). Since \( \mathcal{F} \) is a Ford domain, we have that \( \mathcal{F} \cap \gamma^{-1}(\mathcal{F}) = \Sigma_\gamma \) and hence \( f(\gamma) \in \Phi_0 \). Because, \( \mathcal{F} \) is also a Dirichlet domain we have that \( \mathcal{F} \cap \gamma^{-1}(\mathcal{F}) = \Sigma_\gamma \). Consequently \( \text{ISO}_{\gamma} = \Sigma_{\gamma} \) and thus, by Proposition 3.2, \( d(\gamma) = a(\gamma) \) in case \( \Sigma_{\gamma} \) is a Euclidean sphere. It also follows that \( \Sigma_{\gamma} = \Sigma_{f(\gamma)} \) and hence, by Proposition 3.4, \( f(\gamma) = \gamma g \), with \( g \in \Gamma_{P_0} \).

If \( \Sigma_{\gamma} \) is a Euclidean line or plane then also \( \Sigma_{\gamma}^{-1} \) is a line or a plane. Indeed suppose that \( \Sigma_{\gamma}^{-1} \) is not a Euclidean line or plane. Then \( \Sigma_{\gamma}^{-1} = \text{ISO}_{\gamma} \) and \( \gamma^{-1} \notin \Gamma_{\infty} \), which implies that \( \gamma \notin \Gamma_{\infty} \) and hence \( \Sigma_{\gamma} = \text{ISO}_{\gamma} \) is a Euclidean sphere, a contradiction. From this it follows that \( c(\gamma) = 0 \) and hence, by Proposition 3.1, \( |a(\gamma)| = |d(\gamma)| = 1 = a(\gamma)d(\gamma) \). Thus \( d(\gamma) = a(\gamma) \).

To prove the converse, one just has to use Proposition 3.2 to obtain that \( \text{ISO}_{\gamma} = \Sigma_{\gamma} \), for all side-pairing transformations \( \gamma \) whose bisector is not a Euclidean line or plane. For those \( \gamma \) such that \( \Sigma_{\gamma} \) is a Euclidean line or plane, one uses Remark 3.3, and hence we have a DF domain.
By Lemma 4.3 we have that \( \tilde{\Gamma} = \langle \sigma, \Phi \rangle \) is a reflection group containing \( \Gamma \) as a subgroup of index 2. Since \( o(\tau \gamma) = 2 \), \( \text{tr}(\tau \gamma) = 0 \), for all \( \gamma \in \Phi \) it follows that \( \hat{\Gamma} := \langle \tau, \Phi \rangle \) is a Coxeter group with \( [\hat{\Gamma} : \Gamma] = 2 \).

Note that a presentation of \( \tilde{\Gamma} \) and \( \hat{\Gamma} \) can be obtained using [5, Theorem II.7.5]. Also this result simplifies a lot the proof of [10, Theorem 3.1], i.e. it easily follows that the orbifold of \( \Gamma \) is a punctured sphere in the Fuchsian case. Moreover, as is shown by the next corollary, [10, Theorem 7.3] follows easily from Theorem 1.1.

**Corollary 4.4** Let \( \Gamma < \text{PSL}_2(\mathbb{C}) \) be a finitely generated cofinite discrete group and suppose \( \Gamma \) admits a DF domain \( F \). Then, for every side-pairing transformation \( \gamma \), \( \text{tr}(\gamma) \in \mathbb{R} \) and the vertical planes bisecting \( \Sigma_\gamma \) and \( \Sigma_\gamma^{-1} \) (for \( \gamma \not\in \Gamma_{\infty} \)) all intersect in a vertical axis.

**Proof.** Without loss of generality, we may assume that \( \Gamma \) admits a DF domain with center \( j \) (see the beginning of the section). That \( \text{tr}(\gamma) \in \mathbb{R} \), for \( \gamma \) a side-pairing transformations, is a direct consequence of Theorem 1.1 or of Proposition 3.2.

If \( \gamma \not\in \Gamma_{\infty} \), \( \Sigma_\gamma \) and \( \Sigma_\gamma^{-1} \) are Euclidean spheres with center \( -\frac{a(\gamma)}{c(\gamma)} \) and \( \frac{a(\gamma)}{c(\gamma)} \) respectively. A simple computation then shows that the Euclidean bisector of these two points contains the point 0 and hence the vertical plane bisecting \( \Sigma_\gamma \) and \( \Sigma_\gamma^{-1} \) contains the point \( j \). Hence all these vertical planes intersect in a vertical line through \( j \).

We now consider when a fundamental domain is a double Dirichlet domain. The next two corollaries of our main Theorem give an alternative way to [10, Section 4] to treat such domains.

**Corollary 4.5** Let \( \Gamma \) be a finitely generated cofinite Fuchsian group with trivial stabilizer of \( i \in \mathbb{H}^2 \). Then the following properties are equivalent.

1. \( \Gamma \) is the subgroup of orientation-preserving isometries of a Fuchsian reflection group containing the reflection in the imaginary axis.

2. \( \Gamma \) has a DF domain with center \( i \).

3. \( \Gamma \) has a Dirichlet fundamental domain \( F \) with center \( i \) such that, for every side-pairing transformation \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \), \( a = d \).

4. \( \Gamma \) has a Dirichlet fundamental domain \( F \) with \( i \) and \( t_0 i \) as centers, for some \( 1 \neq t_0 > 0 \).

5. \( \Gamma \) has a Dirichlet fundamental domain \( F \) such that all the points of the geodesic through \( i \) and \( ti \), for \( t > 0 \), are centers of \( F \).

**Proof.** Theorem 1.1 shows that (2) and (3) are equivalent. The equivalence of (3), (4) and (5) is given by Lemma 4.2. Moreover, by Lemma 4.3, (3) implies (1). We show that (1) implies (4). Fix a polyhedron \( P \) for the reflection group such that one of the sides is the imaginary axis \( \Sigma \). Then \( \sigma \) is the reflection in \( \Sigma \). Let \( \Sigma_i \) be a side of \( P \). Denote by \( \sigma_{\gamma_i} \), the reflection in \( \Sigma_i \) and let \( \gamma_i = \sigma \circ \sigma_{\gamma_i} \). Then the result follows from Lemma 4.2.

The previous result can be generalized, by conjugating the group \( \Gamma \). We then get the following result, where the Dirichlet fundamental domain has arbitrary center \( P \in \mathbb{H}^2 \). However, in that case, the third item has to be dropped. We also regrouped parts (4) and (5).
Corollary 4.6 Let $\Gamma$ be a finitely generated cofinite Fuchsian group with trivial stabilizer of $P \in \mathbb{H}^2$. The following properties are equivalent.

1. $\Gamma$ is the subgroup of orientation-preserving isometries of a Fuchsian reflection group containing the reflection in the vertical line through $P$.
2. $\Gamma$ has a DF domain with center $P$.
3. $\Gamma$ has a Dirichlet fundamental domain $F$ such that all the points of the geodesic through $P$ and $P + i$ are centers of $F$.

It follows immediately from Corollary 4.5, that the figure-8 knot group [16] does not have a DF domain. Moreover, all examples of [5, Section VII.3] are groups whose Ford domain is also a Dirichlet domain. Note that this does not follow immediately from the results of [10].

An interesting question is to analyse when the Bianchi groups have a DF domain. The next corollary gives a starting point to this problem. It may be proven by analysing the Dirichlet fundamental domain $F$ of these groups given in [7, Section 4.2] and using Theorem 1.1.

Corollary 4.7 Let $d \in \{1, 2, 3, 5, 7, 11, 19\}$. Then the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in $\mathbb{Q}(\sqrt{-d})$, has a DF domain.

We do not know if these are the only values of $d$ for a DF domain to exist.

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