Lectures on Instanton Counting

Hiraku Nakajima and Kōta Yoshioka

Dedicated to Professor Akhiro Tsuchiya on his sixtieth birthday

Abstract. These notes have two parts. The first is a study of Nekrasov’s deformed partition functions $Z(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau})$ of $N = 2$ SUSY Yang-Mills theories, which are generating functions of the integration in the equivariant cohomology over the moduli spaces of instantons on $\mathbb{R}^4$. The second is review of geometry of the Seiberg-Witten curves and the geometric engineering of the gauge theory, which are physical backgrounds of Nekrasov’s partition functions.

The first part is continuation of our previous paper [61], where we identified the Seiberg-Witten prepotential with $Z(0, 0, \vec{a}; q, 0)$. We put higher Casimir operators to the partition function and clarify their relation to the Seiberg-Witten $u$-plane. We also determine the coefficients of $\varepsilon_1 \varepsilon_2$ and $(\varepsilon_1^2 + \varepsilon_2^2)/3$ (the genus 1 part) of the partition function, which coincide with two measure factors $A, B$ appeared in the $u$-plane integral. The proof is based on the blowup equation which we derived in [61].

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Contents

1. Introduction 2
2. Seiberg-Witten curves 7
3. Instanton moduli spaces 17
4. Nekrasov’s deformed partition function 26
5. The blowup equation and Nekrasov’s conjecture 33
6. Fintushel-Stern’s blowup formula 37
7. Gravitational corrections 41
Appendix A. The root system of type $A_{n-1}$ 48
Appendix B. Theta functions 48
Appendix C. Equivariant Borel-Moore homology 52
Appendix D. The proof of (3.20) by Hiroyuki Ochiai 53
Appendix E. Perturbation term 54

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1. Introduction

In this long introduction, we review a history of Donaldson invariants and Seiberg-Witten geometry, which leads to the Nekrasov’s deformed partition function. This section contains no mathematically rigorous results, but provides the motivation for our study in later sections.

1.1. Donaldson invariants: a mathematical definition. Let $X$ be a smooth, compact, oriented, 4-manifold with a Riemannian metric $g$ with $b^+ \geq 1$ and odd. We also assume $\pi_1(X) = 1$ for brevity. Let $P \to X$ be an SO(3)-bundle over $X$. Let $M(P)$ be the moduli space of irreducible anti-self-dual connections on $P$. This is a manifold with dimension $-\frac{2}{3}p_1(P) - 3(1 + b^+)$ for a generic metric $g$.

Let $P \to X \times M(P)$ be the universal bundle. Then the Donaldson invariant is a polynomial on $H_0(X) \oplus H_2(X)$ defined by

$$D_P(p^a S^b) = \int_{M(P)} \mu(p)^a \mu(S)^b, \quad p \in H_0(X), S \in H_2(X),$$

where $\mu: H^i(X) \to H^{4-i}(M(P))$ is given by the slant product $\mu(\bullet) = -\frac{1}{2}p_1(P)/\bullet$. Since $M(P)$ is not compact, we must justify the definition of the integration, and this can be done by using the Uhlenbeck compactification, as one can find in textbooks on the Donaldson theory [14, 25].

We then formulate a generating function

$$D_\xi(p, S) = \sum_P \sum_{m, n \geq 0} \frac{D_p(S^m p^n)}{m! n!},$$

where $\xi = w_2(P)$ is fixed. When $b^+ > 1$, $D_\xi$ is independent of the metric and defines invariants of the differentiable structure of $X$. When $b^+ = 1$, it is piecewise constant as a function of $g$.

Although the invariants $D_\xi$ can be defined, their calculation was difficult in general. This was because it is difficult to describe the moduli spaces $M(P)$ explicitly. The situation was changed when Kronheimer-Mrowka [37] proved a structure theorem for $D_\xi$ in 1994: Although $D_\xi$ involves infinitely many moduli spaces, it is determined by finite data, if $D_\xi$ satisfies a so-called simple type condition.

Soon afterward, Fintushel-Stern obtained the ‘blowup formula’ which describe the relation between $D_\xi$ on $X$ and that on the blowup $\hat{X}$ [24]. The formula involves an elliptic function. The underlying elliptic curve is related to the structure theorem so that the simple type condition means that it degenerates to a rational curve. The blowup formula will play a fundamental role in this paper.
1.2. Seiberg-Witten geometry. In 1988, Witten described Donaldson invariants as correlation functions of certain operators in a twisted version of $\mathcal{N} = 2$ SUSY (supersymmetric) Yang-Mills theory [70]. We do not explain what this statement means here, but mention that it is an infinite dimensional analogue of the Chern-Weil formula [5].

Shortly after [37] was appeared, Seiberg-Witten analyzed the original $\mathcal{N} = 2$ SUSY Yang-Mills theory with gauge group SU(2) [66]. The original theory is formulated on $\mathbb{R}^{4}$, and was no mathematically rigorous definition of the ‘prepotential’, which they calculated, at that time. Giving such a definition is one of the main purpose of these notes. (See Theorem 5.7.) But we present an ‘informal’ definition here.

Let $H^{*}_{\text{SU}(2)}(\text{pt})$ be the SU(2)-equivariant cohomology of a point with complex coefficients. It is naturally identified with the Weyl group (in this case $\{ \pm 1 \}$) invariant part of the symmetric product of the dual of the (complexified) Cartan subalgebra $\mathfrak{h}$ (in this case $\mathbb{C}$). It is the coordinate ring of $\mathfrak{h}/W$. This space $\mathfrak{h}/W$ is the classical limit of the so-called $u$-plane, a family of ‘vacuum states’, which plays the most important role in the Seiberg-Witten geometry.

The coordinate ring $A(\mathfrak{h}/W) = H^{*}_{\text{SU}(2)}(\text{pt})$ has a generator $- \frac{1}{2} \text{tr} \left( \begin{array}{cc} -a & 0 \\ 0 & a \end{array} \right)^{2} = -a^{2}$, where $a$ is considered as a coordinate on $\mathfrak{h}$. Let us denote it by $u_{\text{cl}}$ since it is a coordinate of the classical limit of the $u$-plane. We make a ‘quantum correction’ $u$ of the function $u_{\text{cl}}$ by using the framed moduli space $M(2, n)$ of instantons on $S^{4}$. The precise definition will be given below, but it is roughly given by

$$u = - \sum_{n \geq 0} \Lambda^{4n} \int_{M(2, n)} \mu(p) \left/ \sum_{n \geq 0} \Lambda^{4n} \int_{M(2, n)} 1 \right..$$

Here $\Lambda$ is a formal variable, the integration is done in the equivariant homology group, and $\mu$ is defined by the same formula as in Donaldson invariants. The moduli space $M(2, n)$ has an SU(2)-action given by the change of the framing. The classical part is the term $n = 0$, then $M(2, 0)$ is a single point, so the integration is just an identity operator. In this case, $\mu(p) \in H^{*}_{\text{SU}(2)}(\text{pt})$ is nothing but the generator $-u_{\text{cl}}$. Thus $u_{\text{cl}}$ is the classical limit of $u$ as we explained.

When $n > 0$, the moduli space $M(2, n)$ is noncompact and we need to justify the integration. Here the problem is not a technical one, and has a very different nature from the noncompactness appeared in the definition of Donaldson invariants, which was overcome by Uhlenbeck compactification. In fact, if $M(2, n)$ had a suitable compactification, the integration of 1 would be 0 by the degree reason. The integration will be defined via the localization theorem in the equivariant homology group. The precise formulation will be given in [41]. As the upshot, the integral does not have the value $H^{*}_{\text{SU}(2)}(\text{pt})$, but in its fractional field. (In fact, we need to consider extra two dimensional torus as below. Or we should consider $u$ as an operator as in [52].) Thus $u$ is a rational function on $\mathfrak{h}/W$. In the Seiberg-Witten geometry, the role of $u$ and $u_{\text{cl}}$ is reversed. We define the $u$-plane as the parameter space for $u$, i.e., $u$ is the coordinate of the $u$-plane. Then we consider $u_{\text{cl}}$ (and $a$) as a rational function on the $u$-plane.

Other than the function $u$, there are several important geometric objects on the $u$-plane. They are defined via the integration over the instanton moduli spaces.
One of the most important objects is the prepotential, which has a form:

\[ F_0 = F_0^{\text{pert}} + F_0^{\text{inst}}, \]

where \( F_0^{\text{pert}} \) is the perturbative part of the prepotential, which is an explicit rational function on \( h/W \). The part \( F_0^{\text{inst}} \) is the instanton part, and is a power series in \( \Lambda^4 \). The coefficient of \( \Lambda^{4n} \) is given by integration over \( M(2,n) \). The \( u \)-plane is a special Kähler manifold, where the prepotential is included in its definition. For example, the Kähler metric is the imaginary part of the second derivative of the prepotential. See [26] for more detail.

The main result of [66] is the determination of the \( u \)-plane and the prepotential \( F_0 \). As a result, the \( u \)-plane is the parameter space for elliptic curves:

\[ y^2 = (z^2 + u - 2\Lambda^2)(z^2 + u + 2\Lambda^2). \]

The prepotential \( F_0 \) is given by using certain elliptic integrals. The original method used for the determination was a highly nontrivial physical argument. One of the most essential ingredients is understanding of its behavior under the ‘duality’ transformation \( \tau \mapsto -1/\tau \), where \( \tau \) is the period of the above elliptic curve, which is given by the second derivative of \( F_0 \) with respect to the coordinate \( a \). This is rather mysterious transformation in view of the definition (1.2). In our approach, we will see theta functions quite naturally. So the duality will come from the Poisson summation formula, but we do not really understand its geometric origin.

Note that this picture is very similar to that of the mirror symmetry. The prepotential above is a counterpart of the Gromov-Witten invariants and is on the ‘symplectic’ side. The elliptic curves (Seiberg-Witten curves) are on the ‘complex’ side. In fact, this is not just analogy. The geometric engineering which will be reviewed in §7.5 explains the result as a special case of the mirror symmetry.

For a later purpose, we give some functions explicitly. Let \( \tau \) be the period of the Seiberg-Witten elliptic curve. Then

\begin{align*}
    u &= -\frac{\theta_{10}^4 + \theta_{10}^4}{\theta_{00}^4 \theta_{10}^4} \Lambda^2, \\
    \frac{du}{da} &= \frac{2 \sqrt{-1}}{\theta_{00} \theta_{10}} \Lambda, \\
    a &= \sqrt{-1} \frac{2 E_2 + \theta_{00}^4 + \theta_{10}^4}{3 \theta_{00} \theta_{10}} \Lambda.
\end{align*}

Here \( \theta_\ast = \theta_\ast(0|\tau) \) is the theta function and \( E_2 = E_2(\tau) \) is the (normalized) second Eisenstein series. The reader should be careful when he/she compares these with the formulas in [54]. Our \( u \) (resp. \( a \)) is multiplied by \(-2\) (resp. \( 2 \sqrt{-1} \)).

Finally note that the elliptic curve becomes singular at \( u = \pm 2\Lambda^2 \). In the classical limit \( \Lambda \to 0 \), these fall into a single point 0, which is the singular point in the classical \( u \)-plane \( h/W \).

1.3. The \( u \)-plane integral. We return back to Donaldson invariants. Witten [71] explained that \( D_\xi \) has three contributions:

\[ D_\xi(p, S) = Z_u(p, S) + Z_+(p, S) + Z_-(p, S). \]

The parts \( Z_\pm(p, S) \) come from the measure supported on the singularity \( \pm 2\Lambda^2 \) of the \( u \)-plane. These are given by invariants defined via the moduli spaces of monopoles, called Seiberg-Witten invariants. As for application to topology, \( Z_u \) is irrelevant as
it depends only on $H^2(X, \mathbb{Z})$. Furthermore, $Z_u$ vanishes when $b_+ > 1$. But we are interested in structures of instanton moduli spaces which are reflected in $Z_u$.

When $b_+ = 1$, more precise description of $D_k$ was given by Moore-Witten [54]. (See also [45], [46] for similar results.) We briefly recall their description, since some parts are closely related to our study. The parts $Z_\pm(p, S)$ are written by the Seiberg-Witten invariants summed over various choices of Spin$^c$ structures. See [54] [7] for the explicit expression. The remaining part $Z_u$ is the integration with respect to a smooth volume form. It is called the $u$-plane integral. We choose and fix a harmonic self-dual two form $\omega$ with $\int_X \omega \wedge \omega = 1$. This is unique up to sign, and the choice of $\omega$ is related to the orientation of the moduli space. We also put $\Lambda = 1$. Then

$$Z_u(p, S) = \int_{\text{u-plane}} da d\sigma A(u)^{\lambda} B(u)^{\tau} e^{\rho u + S^2 T} \Psi,$$

with

$$A(u) = \alpha \left( \frac{du}{da} \right)^{1/2}, \quad B(u) = \beta (u^2 - 4)^{1/8},$$

$$T = \frac{1}{24} \left( \frac{du}{da} \right)^2 E_2(\tau) - \frac{1}{6} u,$$

$$\Psi = -\sqrt{-2} \frac{d\tau}{4y^{1/2} \frac{d\sigma}{d\alpha}} \exp \left[ \frac{1}{8\pi y} \left( \frac{du}{da} \right)^2 S^2_+ \right] e^{2\pi \sqrt{-1} \lambda^2_+} \sum_{\lambda \in H^2 + \frac{i}{2} \xi} (-1)^{(\lambda - \lambda_0) \cdot w_2(X)}$$

$$\times \left[ (\lambda, \omega) - \frac{1}{4\pi y} \frac{du}{da} (S, \omega) \right] \exp \left[ -\sqrt{-1} \pi \lambda^2_+ - \sqrt{-1} \pi \tau \lambda^2_+ + \frac{du}{da} (S, \lambda_-) \right].$$

Here $\chi$ (resp. $\sigma$) is the Euler number (resp. signature) of $X$, $\alpha, \beta$ are universal constants independent of $X$, $\tau = x + iy$, $\lambda_0$ is a fixed element in $\frac{1}{2} \xi + H^2(X, \mathbb{Z})$, and $(\bullet)_+$ denotes the self-dual and anti-self-dual part of $\bullet$ respectively.

Since this is a divergent integral, and we must regularize it. See the original paper [54] how it is done.

The term $T$ is called a contact term. Its determination can be done by several ways. In [45] by equating the answers given by various ways, a nontrivial equation was derived. This is the contact term equation, which will be important for our study. See Theorem 2.11 and Theorem 5.7. The terms $A, B$ come from a Riemannian metric $g$.

Let us analyze the effect of the blowup $\hat{X} \to X$ on the $u$-plane integral since it is closely related to our study. Let $C$ be the exceptional curve. We want to evaluate $Z_u(p, S + tC)$, where $S \in H_2(X)$ is considered as a class of $H_2(\hat{X})$ via the projection.

Since $\chi(\hat{X}) = \chi(X) + 1$, $\sigma(\hat{X}) = \sigma(X) - 1$, the factor $A(u)^{\lambda} B(u)^{\tau}$ is multiplied by

$$\frac{A(u)}{B(u)} = \frac{\alpha}{\beta} (u^2 - 4)^{-1/8} \left( \frac{du}{da} \right)^{1/2} = \frac{1}{\delta_0} \text{(up to constant)}.$$
We work in a chamber $C_+ = 0$, so we have
\[
\frac{\Psi_X}{\Psi} = \sum_{n \in \mathbb{Z} + \frac{1}{2}w_2(\tilde{P}) \cdot C} (-1)^n \exp \left( \sqrt{-1} \pi \tau n^2 - nt \frac{du}{d\tau} \right) = \theta^*(t \sqrt{-\frac{1}{2}} \frac{du}{d\tau}),
\]
where $* = 01$ or $11$ according to $w_2(\tilde{P}) \cdot C = 0$ or $1$. Therefore we get
\[
\frac{Z_{u}(p,S + tC)}{Z_{u}(p,S)} = \exp \left( -t \frac{\Delta}{2\pi} \frac{du}{d\tau} \right) \theta^*(0 \mid \tau),
\]
up to a constant multiple. The constant turns out to be 1 as left hand side is 1 at $t = 0$ when $* = 01$.

**1.4. Nekrasov’s deformed partition function.** As we explained, the pre-potential $F_0$ was given as integration over instanton moduli spaces. Before Nekrasov gave an explicit expression [62], it was written in terms of differential forms on moduli spaces. So it was difficult to calculate, understand its meaning... (See [15].) Nekrasov’s idea was to use an extra 2-dimensional torus action and apply the localization theorem in the equivariant homology. Technically it was also important that the Uhlenbeck (partial) compactification of the moduli space has a nice resolution of singularities introduced by the first author [59]. (The latter space will be denoted by $M(2,n)$ in the main body of the paper.) Let $\varepsilon_1, \varepsilon_2$ be two generators of $H^*_T(pt)$, then we define
\[
(1.5) \quad F = \varepsilon_1 \varepsilon_2 F_{\text{pert}} + \varepsilon_1 \varepsilon_2 \log \left( \sum_{n \geq 0} \Lambda^{4n} \int_{M(2,n)} 1 \right),
\]
where $F_{\text{pert}}$ is a certain two parameter deformation of $F^{\text{pert}}_0$. Each coefficient of $\Lambda^{4n}$ is a rational function in $\varepsilon_1, \varepsilon_2$, and is a mathematically rigorously defined. Nekrasov conjectured $F_{\mid \varepsilon_1, \varepsilon_2 = 0}$ is equal to $F_0$, given by the Seiberg-Witten curve. This is mathematically meaning full statement. This conjecture was proved by [61] and [63] by totally different methods.

The method used in [63] was geometric and a standard technique in the study of Donaldson invariants. We consider the instanton moduli spaces $\hat{M}(2,c_1,n)$ on the blowup, introduce an operator $\mu(C)$ in this equivariant setting, and compute this equivariant analog of Donaldson invariants. From the explicit expression given by the localization theorem, it is very easy to derive the blowup formula in a combinatorial form. On the other hand, by a simple dimension counting argument shows that $\int_{\hat{M}(2,0,n)} \mu(C)^2 = 0$. This vanishing give a differential equation satisfied by the original $F$. We call it the blowup equation. (See [62].) It characterizes $F$. When we put $\varepsilon_1 = \varepsilon_2 = 0$, this equation turns out to be the contact term equation, which we mentioned. Since the contact term equation can be derived from the Seiberg-Witten curve in a mathematically rigorous way (see [2]), this gives a proof of Nekrasov’s conjecture.

**1.5. Gravitational corrections.** After identifying $F_{\mid \varepsilon_1, \varepsilon_2 = 0}$ with the Seiberg-Witten prepotential [24], it becomes natural to ask the meaning of higher order
In the expansion

\[ F = F_0 + (\varepsilon_1 + \varepsilon_2)H + \varepsilon_1 \varepsilon_2 A + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B + \cdots . \]

Nekrasov asserted that these are gravitational corrections to the gauge theory \[62\] §4. Using the differential equation mentioned above, we prove these \( A, B \) coincides with those \( A, B \) appeared in the \( u \)-plane integrand. (\( H \) turns out to be a simple function.) (The calculation was done jointly with N. Nekrasov.)

Moreover, by the geometric engineering \[36\] (see §7.5), we can expect these terms are certain limits of higher genus Gromov-Witten invariants for a noncompact Calabi-Yau 3-fold, in this case the canonical bundle of \( \mathbb{P}^1 \times \mathbb{P}^1 \). More precisely, we put \( \varepsilon_1 = -\varepsilon_2 = \hbar \) and consider

\[ F = F_0 + F_1 \hbar^2 + F_2 \hbar^4 + \cdots . \]

Then \( F_g \) is a limit of the genus \( g \) Gromov-Witten invariants. Since \( 1/\varepsilon_1 \varepsilon_2 F \) is more fundamental (see (1.5)), we should write this as

\[ \frac{1}{\hbar^2} F = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g. \]

This is more natural as \( 2 - 2g \) is the Euler number of a genus \( g \) Riemann surface. It probably explains the singularity \( 1/\hbar^2 \).

Recently many Gromov-Witten invariants for noncompact toric Calabi-Yau have been calculated (see \[1\] and the references therein). These are identified with the Jones-Witten invariants via the geometric transition (called ‘large \( N \) duality’), as first proposed by Gopakumar-Vafa \[28\]. A first of such examples is the identification of Gromov-Witten for the resolved conifold and the SU(\( N \)) Jones-Witten invariant for \( S^3 \). These identifications have been proved in a mathematical rigorous way in a number of examples (see \[65, 73\]).

In the case of \( K_{\mathbb{P}^1 \times \mathbb{P}^1} \), the Jones-Witten side is SU(\( N \))-invariants for the Hopf link. Using the calculation by Morton-Lukac \[56\], Iqbal+Kashani-Poor show that the invariants of the Hopf link has the same combinatorial expression as that of \( F \) given by the localization formula \[35\]. (See also \[19\].)

Note that these results identify the \( n \)-instanton correction with the Gromov-Witten invariants of degree \( n \) (with respect to one of the factors of \( \mathbb{P}^1 \times \mathbb{P}^1 \)) for each \( n \). Thus they do not say much about the structure of the generating function \( F \), which is studied in this paper. Therefore it is interesting to understand the blowup equation from the Gromov-Witten side.

### 2. Seiberg-Witten curves

In this section we introduce the Seiberg-Witten curves, give the definition of the prepotential, and derive the renormalization equation and the contact term equation, which will characterize the prepotential.

We give some details, though one can find most of them in physics literature. The reason is that we must carefully choose cycles on the Seiberg-Witten curve to determine a characteristic of the theta function in a mathematically rigorous way. It is a standard exercise but we cannot find the argument in the literature.

The material discussed here is a minimum of the Seiberg-Witten geometry. We omit many things, such as monodromies, Picard-Fuchs equations, relations to
integrable systems, etc. Even for the differential equations satisfied by the prepotential, our treatment is a minimum. The Whitham hierarchy underlying these equations will not be discussed. The reader may wonder where these equations come from, though the authors’ approach through the instanton moduli spaces will be explained in [42]. For the original approaches, see [49, 51] and the references therein.

There is a nice survey article [12] for mathematicians which describes relation between integrable systems and the Seiberg-Witten geometry, as well as background on physics. We recommend it to our reader since it has no overlaps with this paper.

Note that we multiply $a_o$ by $-\sqrt{-1}$ from the conventional one in order to match with one in the instanton counting.

### 2.1. Definition of the Seiberg-Witten prepotential.

We consider a family of curves (Riemann surfaces) parametrized by $\vec{u} = (u_2, \ldots, u_r)$:

$$C_{\vec{u}} : \Lambda^r \left( w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + u_3 z^{r-3} + \cdots + u_r.$$  

We call them Seiberg-Witten curves. The projection $C_{\vec{u}} \ni (w, z) \rightarrow z \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution $\iota$ is given by $\iota(w) = 1/w$.

If we introduce $y = \Lambda^r (w - \frac{1}{w})$, we have

$$y^2 = P(z)^2 - 4\Lambda^{2r} = (P(z) - 2\Lambda^r)(P(z) + 2\Lambda^r).$$

This special form of the right hand side will play a crucial role later.

The parameter space $\{ \vec{u} \in \mathbb{C}^{r-1} \}$ is called the $u$-plane. Here $\Lambda$ is also a parameter, but we treat it separately from $\vec{u}$. The parameter $\Lambda$ is called the renormalization scale in physics. When $\Lambda = 0$, the theory goes to the classical limit. We consider $\vec{u} = (u_2, \ldots, u_r)$ as a coordinate system on the $u$-plane. This is a global coordinate.

Let $z_1, \ldots, z_r$ be the solutions of $P(z) = 0$. We will work on a region of the $u$-plane where $|z_\alpha - z_\beta|, |z_\alpha|$ are much larger than $|\Lambda|$, and then analytically continue. In particular $z_\alpha$’s are distinct. The vector $\vec{z} = (z_1, \ldots, z_r)$ ($\sum_\alpha z_\alpha = 0$) is a local coordinate on the $u$-plane. The relation between $\vec{z}$ and $\vec{u}$ is very simple. The former is a coordinate on $\mathbb{C}^{r-1}$ while the latter is on $\mathbb{C}^r/S_r \approx \mathbb{C}^{r-1}$, where $S_r$ is the symmetric group of $r$ letters. In other words, $(-1)^p u_p$ is the $p$th elementary symmetric function in $z_1, \ldots, z_r$. It is better to keep this simple relation in mind, since this coordinate system $\vec{z}$ is a quantum correction of another coordinate system $\vec{a}$ introduced below.

We can find $z_\alpha^\pm$ near $z_\alpha$ such that $P(z_\alpha^\pm) = \pm 2\Lambda^r$ when $|u| \gg |\Lambda|$. These are the $2r$-branched points of the projection $C_{\vec{u}} \rightarrow \mathbb{P}^1$. The infinity is not a branched point, and its inverse image consists of $\infty_+ (w = \infty)$ and $\infty_- (w = 0)$. The genus of $C_{\vec{u}}$ is $r - 1$. In the classical limit $\Lambda \rightarrow 0$, both $z_\alpha^\pm$ go to $z_\alpha$, and the curves develop singularities.

Let us define the quantum discriminant by

$$\Delta = (4\Lambda^r)^{2r} \prod_{\alpha < \beta} (z^+_{\alpha} - z^+_{\beta})^2 (z^-_{\alpha} - z^-_{\beta})^2.$$  

On the locus $\Delta = 0$, the Seiberg-Witten curves develop singularities. As we mentioned, we study a region away from this locus.

The hyperelliptic curve $C_{\vec{u}}$ is made of two copies of the Riemann sphere, glued along the $r$-cuts between $z^-_{\alpha}$ and $z^+_{\alpha}$ ($\alpha = 1, \ldots, r$), as usual. Let $A_\alpha$ be the
cycle encircling the cut between $z_\alpha^-$ and $z_\alpha^+$. We have $\sum \alpha A_\alpha = 0$. We draw $C_u$ as in Figure 1. The hyperelliptic involution $\iota$ is the rotation by $\pi$ about the axis passing through the branched points $z_\alpha^\pm$. Then we choose cycles $B_\alpha$ ($\alpha = 2, \ldots, r$) as in Figure 1 so that \{ $A_\alpha$, $B_\beta$ | $\alpha = 2, \ldots, r$ \} form a symplectic basis of $H_1(C_u, \mathbb{Z})$, i.e., $A_\alpha \cdot B_\beta = 0 = B_\alpha \cdot B_\beta$, $A_\alpha \cdot B_\beta = \delta_{\alpha\beta}$ for $\alpha, \beta = 2, \ldots, r$. (The cycle $A_1$ is omitted.)

**Figure 1.** Seiberg-Witten curve and cycles ($r = 3$)

Note that we cannot take $A, B$-cycles *globally* on the $u$-plane. The cycles are transformed by monodromies around the locus $\Delta = 0$. In fact, the study of monodromies is important as it has been used for constancy checks of the Seiberg-Witten curves to some physically expected properties of the prepotential (introduced below). However we do not study monodromy behavior here except that around $\Lambda = 0$. We first fix a small region in the $u$-plane and then analytically continue. We choose a region containing the part that $z_\alpha$'s are real and satisfy $z_1 > z_2 > \cdots > z_r$. We also assume $\Lambda$ is a positive real number. Since we assume $\Lambda$ small, we have $z_1^+ > z_1^- > z_2^- > z_2^+ > \cdots$. This choice determines $A, B$-cycles as in Figure 1. Note the branched points are lined from the right by the order in Figure 1. Thus the choice is natural in this region. Note also that we choose the inverse image of the region with respect to the projection $C^r - 1 \to C^r - 1/S_r$.

The permutation ambiguity is less important than the monodromies, but we use the choice as we want to specify what is $a_\alpha$.

Let us define the *Seiberg-Witten differential* by

$$dS = \frac{1}{2\pi} \frac{dw}{w} = \frac{1}{2\pi} \frac{zP'(z)dz}{\sqrt{P(z)^2 - 4\Lambda^2 r}} = -\frac{1}{2\pi} \frac{zP'(z)dz}{y}. $$

It is a meromorphic differential having poles at $\infty_\pm$. We define functions $a_\alpha, a_\beta^\alpha$ on the $u$-plane ($|u| \gg |\Lambda|$) by

$$ a_\alpha = \int_{A_\alpha} dS, \quad a_\beta^\alpha = 2\pi \sqrt{-1} \int_{B_\beta} dS, \quad \alpha = 1, \ldots, r, \ \beta = 2, \ldots, r. $$

Let us study the behavior of the function $a_\alpha$ around $\Lambda = 0$. We move the cycle $A_\alpha$ so that $P(z)$ and $1/P(z)$ are bounded there. In particular, we are in a sheet where $\sqrt{P(z)^2 - 4\Lambda^2 r}$ is single-valued. We choose the sheet so that it is approximated by $P(z)$ on the $A_\alpha$-cycle. We suppose $A_\alpha$ has the counterclockwise
rotation in the sheet. In Figure 1 the sheet is the part lower than the plane containing \( A_\alpha \)'s. (See also the proof of Proposition 2.7 below.) Then we have the following expansion:

\[
a_\alpha = -\frac{1}{2\pi} \int_{A_\alpha} dz \frac{P'(z)}{P(z)} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n (4\Lambda^2 r \rho(z))^{n} = -\sqrt{-1}z_\alpha + O(\Lambda^{2r}),
\]

where \( \binom{n}{a} \) is the binomial coefficient. In particular, \( \vec{a} = (a_1, \ldots, a_r) \) is a local coordinate system for small \( \Lambda \). As we mentioned before, the coordinate \( \vec{z} \) is the quantum correction of \( \vec{a} \). The \( a_\alpha \) is a function in \( u_p \), but conversely we consider \( u_p \) as a function in \( a_\alpha \) (and also in \( \Lambda \)).

We differentiate the Seiberg-Witten differential \( dS \) by setting \( w \) to be constant:

\[
\frac{\partial}{\partial u_p} dS \bigg|_{w=\text{const}} = \frac{1}{2\pi} \frac{z^{r-p}}{P'(z)} \frac{dw}{w} = \frac{1}{2\pi} \frac{z^{r-p} dz}{w}.
\]

It is well-known that these form a basis of holomorphic differentials on \( C_{\vec{a}} \) for \( p = 2, \ldots, r \) (see e.g., [33 §2.3]). In other words, the Seiberg-Witten differential is a 'potential' for holomorphic differentials. Let \( (\sigma_{\alpha p}) \) be the matrix given by

\[
\sigma_{\alpha p} = \frac{\partial a_\alpha}{\partial u_p} = \frac{1}{2\pi} \int_{A_\alpha} z^{r-p} dw \frac{P'(z)}{P(z)}, \quad \alpha, p = 2, \ldots, r.
\]

If \( (\sigma^{p\alpha}) \) is the inverse matrix, the normalized holomorphic 1-forms

\[
\omega_\beta = \frac{1}{2\pi} \sum_p \sigma^{p\beta} \frac{z^{r-p}}{P'(z)} \frac{dw}{w} = \left. \frac{\partial}{\partial a_\beta} dS \right|_{w=\text{const}}
\]

satisfies \( \int_{A_\alpha} \omega_\beta = \delta_{\alpha\beta} \). Therefore the period matrix \( \tau = (\tau_{\alpha\beta}) \) of the curve \( C_{\vec{a}} \) is given by

\[
\tau_{\alpha\beta} = \int_{B_\alpha} \omega_\beta = \frac{1}{2\pi \sqrt{-1}} \frac{\partial a_\alpha^D}{\partial a_\beta}.
\]

Since \( (\tau_{\alpha\beta}) \) is symmetric (see e.g., [33 §2.2]), there exists a locally defined function \( F_0 \) on the \( u \)-plane such that

\[
a_\alpha^D = -\frac{\partial F_0}{\partial a_\alpha}.
\]

It is unique up to constant. We fix the constant so that \( F_0 \) is homogeneous of degree 2:

\[
(\sum a_\alpha \frac{\partial}{\partial a_\alpha} + \Lambda \frac{\partial}{\partial \Lambda}) F_0 = 2F_0.
\]

This function \( F_0 \) is called the Seiberg-Witten prepotential. We may also write \( F_0(\vec{a}) \) or \( F_0(\vec{a}; \Lambda) \). From the definition we have

\[
\tau_{\alpha\beta} = -\frac{1}{2\pi \sqrt{-1}} \frac{\partial^2 F_0}{\partial a_\alpha \partial a_\beta}.
\]

We put the subscript 0 because this will be identified with the genus 0 part of the Nekrasov’s deformed partition function.
2.2. The logarithmic singularities of the Seiberg-Witten prepotential. Let us study the behavior of $F_0$ as a function in $\Lambda$, following [11]. Our aim is to show

**Proposition 2.7.** We have

$$F_0 = \sum_{\alpha \neq \beta} \gamma_0(a_\alpha - a_\beta; \Lambda) + O(\Lambda^{2r})$$

$$= \sum_{\alpha \neq \beta} \left[ \frac{1}{2}(a_\alpha - a_\beta)^2 \log \left( \frac{a_\alpha - a_\beta}{\Lambda} \right) - \frac{3}{4}(a_\alpha - a_\beta)^2 \right] + O(\Lambda^{2r}),$$

where $\gamma_0(x; \Lambda) = \frac{1}{2} x^2 \log \left( \frac{x}{\Lambda} \right) - \frac{3}{4} x^2$ is the coefficient of $1/\varepsilon_1 \varepsilon_2$ in $-\gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda)$ as in (12).

The part $\sum_{\alpha \neq \beta} \gamma_0(a_\alpha - a_\beta; \Lambda)$ is called the *perturbative part* of the prepotential $F_0$, and denoted by $F_0^{\text{pert}}$. The remaining part is called the *instanton part*, and denoted by $F_0^{\text{inst}}$. It is a power series in $\Lambda^{2r}$: $F_0^{\text{inst}} = f_1 \Lambda^{2r} + f_2 \Lambda^{4r} + \cdots + f_n \Lambda^{2rn} + \cdots$. The coefficient $f_n$ is called the *$n$th instanton correction* to the prepotential. This is because we will identify $f_n$ something defined via the $n$-instanton moduli space.

The choice of the branch of log is as follows. Suppose that $z_\alpha, z_\alpha^\pm \in \mathbb{R}$, $\Lambda \in \mathbb{R}_{>0}$ and $z_1 > z_2 > \cdots > z_r$ as above. We choose a path $A_\alpha$ encircling $z_\alpha^+$ and $z_\alpha^-$ so that it is invariant under the complex conjugation $z \mapsto \bar{z}$. Then $a_\alpha$ is purely imaginary. We have $\sqrt{-1}a_1 > \sqrt{-1}a_2 > \cdots > \sqrt{-1}a_r$. We choose the branch of log so that

$$\frac{1}{2} \log \left( \frac{a_\alpha - a_\beta}{\Lambda} \right) + \frac{1}{2} \log \left( \frac{a_\beta - a_\alpha}{\Lambda} \right) = \log \left( \sqrt{-1}(a_\alpha - a_\beta) \right)$$

is real for $\alpha < \beta$. In what follows, we assume this choice of $z_\alpha$, etc. It is enough to consider this case by analytic continuation.

**Proof of Proposition 2.7.** First we study

$$a_\alpha^D = 2\pi \sqrt{-1} \int_{B_\alpha} dS.$$

Locally, this is a function in $\Lambda^{2r}$. But it is multi-valued, as the cycle $B_\alpha$ transforms to $B_\alpha + A_\alpha - A_1$ when we analytically continue from $\Lambda^{2r}$ to $e^{2\pi \sqrt{-1}} \Lambda^{2r}$. Therefore $a_\alpha^D + \sqrt{-1}(z_\alpha - z_1) \log \Lambda^{2r}$ is a single-valued function in $\Lambda$. This kind of the monodromy behavior is quite important in the conventional arguments.

For $\beta = 2, \ldots, r$, let $C_\beta^0$ be the straight line from $z_{\beta-1}^\pm$ to $z_\beta^\pm$ (+ for $\beta$ odd, − for $\beta$ even) in one sheet and define the cycle $C_\beta$ as $C_\beta^0$ followed by $-i C_\beta^0$. This is a cycle rounding the hole in Figure 11. We have

$$B_\alpha = \sum_{\beta=2}^n C_\beta,$$

and

$$-2\pi \int_{C_\beta} dS = 2 \int_{z_\beta^\pm}^{z_{\beta-1}^\pm} \frac{z P'(z) dz}{\sqrt{P(z)^2 - 4 \Lambda^{2r}}}.$$

In the last expression, $z$ is real. But we should be careful for the choice of the branch of $\sqrt{P(z)^2 - 4 \Lambda^{2r}}$. This is not necessarily $\geq 0$ contrary to the usual convention for the real function. It is determined by the analytic continuation. We choose the
sheet so that \( \sqrt{P(z)^2 - 4\Lambda^2 r} \) has the same sign as \( P(z) \) on each interval \([z_{\beta-1}^\pm, z_{\beta}^\pm]\). This is the same sheet used in (2.3) (i.e., the lower half) and we have the right orientation so that \( A_\alpha \cdot B_\alpha = 1 \). Note also that \( a_\alpha^D \) is pure imaginary as the integrals are real.

Fix \( \delta > 0 \) small with \(|\Lambda| \ll |\delta|\) and rewrite the integral as

\[
(2.8) \quad -\pi \int_{C_\beta} dS = \left( \int_{z_{\beta-1} + \delta}^{z_{\beta}^+} + \int_{z_{\beta}^+}^{z_{\beta}^-} - \int_{z_{\beta-1}^-}^{z_{\beta-1} + \delta} \right) \frac{z P'(z) dz}{\sqrt{P(z)^2 - 4\Lambda^2 r}}
\]

The first integral is regular at \( \Lambda = 0 \):

\[
\int_{z_{\beta-1} + \delta}^{z_{\beta}^+} \frac{z P'(z) dz}{P(z)} + O(\delta) = \int_{z_{\beta-1} + \delta}^{z_{\beta}^+} \sum_{\gamma = 1}^{r} \log \left( \frac{z - z_\gamma}{z - z_{\beta-1}} \right) + O(\delta)
\]

\[
= z_\beta (r + \log \delta) - z_{\beta-1} (r + \log \delta) + \sum_{\gamma \neq \beta} z_\gamma \log |z_\beta - z_\gamma| - \sum_{\gamma \neq \beta-1} z_\gamma \log |z_{\beta-1} - z_\gamma| + O(\delta).
\]

Here we choose the branch of \( \log \) so that all the above expressions are real.

The second integral of (2.8) is equal to

\[
(2.9) \quad z_\beta \int_{z_{\beta}^-}^{z_{\beta}^+} \frac{P'(z) dz}{\sqrt{P(z)^2 - 4\Lambda^2 r}} + \int_{z_{\beta^-}}^{z_{\beta}^+} \frac{(z - z_\beta) P'(z) dz}{\sqrt{P(z)^2 - 4\Lambda^2 r}}
\]

The first term is

\[
z_\beta \int_{w = w_\beta(\delta)}^{w = w_\beta(\delta) + \pm 1} \frac{dw}{w} = -z_\beta \log |w_\beta(\delta)|,
\]

where

\[
w_\beta(\delta) = \frac{1}{2\Lambda^r} \left( P(z_\beta - \delta) + \sqrt{P(z_\beta - \delta)^2 - 4\Lambda^2 r} \right).
\]

We have

\[
\log |w_\beta(\delta)| = \log \left( \frac{|P(z_\beta - \delta)|}{2\Lambda^r} + \log \left( 1 + \frac{1}{\sqrt{1 - \frac{4\Lambda^2 r}{P(z_\beta - \delta)^2}}} \right) \right) \]

\[
= \log \left( \frac{\delta \prod_{\gamma \neq \beta} (z_\beta - z_\gamma)}{\Lambda^r} \right) + O(\delta).
\]

Let us consider the second term of (2.9). Let \( z - z_\beta = \prod_{\gamma \neq \beta} (z_\beta - z_\gamma)^{-1} P(z) + E(z) \). We have \( E(z) = O(\delta^2) \) in the range of the integration. But the above calculation of the first part shows that the integral of \( E(z) \) yields \( O(\delta^2) O(\log \delta) = O(\delta) \). Therefore the second term is

\[
\prod_{\gamma \neq \beta} \left( \int_{z_{\beta}^-}^{z_{\beta}^+} \frac{P(z) P'(z) dz}{\sqrt{P(z)^2 - 4\Lambda^2 r}} \right) = \prod_{\gamma \neq \beta} \left( \sqrt{P(z)^2 - 4\Lambda^2 r} \right)_{z_{\beta}^-}^{z_{\beta}^+} + O(\delta).
\]

But as \( P(z_{\beta}^\pm) = 2\Lambda^r, P(z_{\beta} - \delta) = O(\delta) \), the contribution is \( O(\delta) \).
The third integral has a similar expression with $z_\beta$, $\delta$ replaced by $z_{\beta-1}$, $-\delta$ respectively. Altogether we get

$$- \pi \int_{C_\beta} dS - r(z_\beta - z_{\beta-1})(1 + \log \Lambda)$$

$$+ \sum_{\gamma \neq \beta} (z_\beta - z_\gamma) \log |z_\beta - z_\gamma| - \sum_{\gamma \neq \beta-1} (z_{\beta-1} - z_\gamma) \log |z_{\beta-1} - z_\gamma| = O(\delta).$$

But the left hand side is independent of $\delta$. This means that the left hand side is, in fact, $O(\Lambda^{2r})$. Combining with (2.3), we have

$$\frac{1}{2} a_\alpha^D = r(a_\alpha - a_1) (1 + \log \Lambda) - \sum_{\beta \neq \alpha} (a_\alpha - a_\beta) \log |\sqrt{-1}(a_\alpha - a_\beta)|$$

$$+ \sum_{\beta \neq 1} (a_1 - a_\beta) \log \sqrt{-1}(a_1 - a_\beta) + O(\Lambda^{2r}).$$

In the last part, we do not take the absolute value of $\sqrt{-1}(a_1 - a_\beta)$ since $\sqrt{-1}a_1 > \sqrt{-1}a_\beta$.

Now let us differentiate $F_0^{\text{pert}}$ in the statement. Let $\gamma_0(x; \Lambda) = x^2 \log \left( \frac{\sqrt{-1}x}{\Lambda} \right) - \frac{\lambda}{2} x^2$. (Remember our choice of the branch of log.) We have the following

$$- \frac{\partial F_0^{\text{pert}}}{\partial a_\alpha} = - \sum_{\beta < \gamma} \frac{\partial}{\partial a_\alpha} \gamma_0(a_\beta - a_\gamma; \Lambda)$$

$$= - \sum_{\alpha < \beta} \gamma_0(a_\alpha - a_\beta; \Lambda) + \sum_{\beta < \alpha} \gamma_0(a_\beta - a_\alpha; \Lambda) + \sum_{\beta \neq 1} \gamma_0(a_1 - a_\beta; \Lambda).$$

We substitute $\gamma_0(x) = 2x \log \frac{\sqrt{-1}x}{\Lambda} - 2x$ to get

$$- \frac{1}{2} \frac{\partial F_0^{\text{pert}}}{\partial a_\alpha} = r(a_\alpha - a_1) - \sum_{\beta \neq \alpha} (a_\alpha - a_\beta) \log \left( \frac{\sqrt{-1}(a_\alpha - a_\beta)}{\Lambda} \right)$$

$$+ \sum_{\beta \neq 1} (a_1 - a_\beta) \log \left( \frac{\sqrt{-1}(a_1 - a_\beta)}{\Lambda} \right).$$

This coincides with the above expression. The proof of Proposition 2.7 is completed.

\[\Box\]

### 2.3. A renormalization group equation.

We prove the so-called ‘renormalization group equation’ following (68) in this subsection:

**PROPOSITION 2.10.**

$$\Lambda \frac{\partial}{\partial \Lambda} F_0 = -2ru_2.$$

This equation was found earlier by (68) for SU(2), and independently by (20).

See also (10).

**PROOF.** We differentiate the Euler equation (64):

$$\frac{\partial}{\partial u_p} \left( \Lambda \frac{\partial}{\partial \Lambda} F_0 \right) = 2 \frac{\partial F_0}{\partial u_p} - \sum_{\alpha} \frac{\partial}{\partial u_p} \left( a_\alpha \frac{\partial F_0}{\partial a_\alpha} \right) = - \sum_{\alpha} \left( \frac{\partial a_\alpha}{\partial u_p} a_\alpha^D - a_\alpha \frac{\partial a_\alpha^D}{\partial u_p} \right)$$

$$= - 2\pi \sqrt{-1} \sum_{\alpha} \left[ \int_{A_\alpha} \frac{\partial}{\partial u_p} dS \int_{B_\alpha} dS - \int_{A_\alpha} dS \int_{B_\alpha} \frac{\partial}{\partial u_p} dS \right].$$
Let us make a change of variable $x = 1/z$. We expand the Seiberg-Witten differential and its differential around $x = 0$:
\[ dS = \frac{1}{2\pi} \frac{zP'(z)dz}{\sqrt{P(z)^2 - 4\Lambda^2}} = (s_{-2}x^{-2} + s_0 + s_1x + \cdots) \, dx, \]
\[ \frac{\partial}{\partial u_p} dS = \frac{1}{2\pi} \frac{z^{r-p}dz}{\sqrt{P(z)^2 - 4\Lambda^2}} = (\omega_0^p + \omega_1^px + \cdots) \, dx. \]
(Recall that $dS$ is a meromorphic differential having poles only at $\infty_{\pm}$.)

By the Riemann bilinear relation (see e.g., [33, §2.3]), we have
\[ \frac{\partial}{\partial u_p} \left( \Lambda \frac{\partial}{\partial \Lambda} F_0 \right) = 8\pi^2 \sum_{n} \frac{s_{-n}\omega_n^p}{n-1} = 8\pi^2 s_{-2}\omega_0^p, \]

Since $s_{-2} = \frac{r}{2\pi}$, $\omega_0^p = -\frac{1}{2\pi}\delta_{p2}$, we get
\[ \frac{\partial}{\partial u_p} \left( \Lambda \frac{\partial}{\partial \Lambda} F_0 \right) = -2r\delta_{p2}. \]

Integrating out, we get the assertion. Here the integration constant is zero thanks to the homogeneity of $F_0$. \qed

2.4. The contact term equation. In this subsection we show the following partial differential equation:

**Theorem 2.11.** We have
\[ \Lambda \frac{\partial}{\partial \Lambda} u_p = \frac{2r}{\pi \sqrt{-1}} \sum_{\alpha, \beta = 2}^{r} \frac{\partial u_p}{\partial a_\alpha} \frac{\partial u_2}{\partial a_\beta} \frac{\partial}{\partial \tau_{\alpha\beta}} \log \Theta_E(0|\tau) \]

for $p = 2, 3, \ldots, r$. Here $E$ is the even half-integer characteristic given by $\left[ \frac{\delta}{\Lambda} \right]$ in [33].

This equation was first derived by Losev-Nekrasov-Shatashvili [45, 46] during their study of the topologically twisted version of $\mathcal{N} = 2$ SUSY Yang-Mills theory, i.e., the physical counterpart of the Donaldson theory. More precisely, they derived the equation by studying the effect of the blowup on the so-called ‘contact terms’. So we call the equation the contact term equation. Later Gorsky, Marshakov, Mironov and Morozov [31] derived the contact term equation in the framework of the Seiberg-Witten curve. We give the proof following their approach in this subsection. In fact, they did not determine the characteristic. It was determined in [45, 46], but the argument involves a physical intuition. Here we can give a mathematically rigorous proof thanks to our precise definition of the $B$-cycles used in the definition of the prepotential.

For a later purpose, we give a remark. Recall that $(-1)^pu_p$ is the $p$th elementary symmetric function in variables $z_1, \cdots, z_r$. Let $c_p$ be the $p$th power sum multiplied by $\left( -\sqrt{-1} \right)^p$:
\[ c_p = \left( -\sqrt{-1} \right)^p \sum_{\alpha=1}^{r} z_\alpha^p, \quad c_1 = 0, c_2 = u_2, \cdots, \text{etc.} \]
Then \((c_2, c_3, \ldots, c_r)\) is another coordinate system on the \(\vec{u}\)-plane. Since \(c_p\) is a polynomial in \(u_q\)'s, it is also a solution of the contact term equation:

\[
\Lambda \frac{\partial}{\partial \Lambda} c_p = \frac{2r}{\pi \sqrt{-1}} \sum_{\alpha, \beta = 2}^{r} \frac{\partial c_p}{\partial a_\alpha} \frac{\partial u_2}{\partial a_\beta} \frac{\partial}{\partial \tau_{\alpha \beta}} \log \Theta_E(0|\tau).
\]

Before giving the proof of Theorem 2.11, we give a corollary which will play an important role later.

**Corollary 2.13.**

\[
\left(\Lambda \frac{\partial}{\partial \Lambda}\right)^2 F_0 = \frac{-1}{\pi \sqrt{-1}} \sum_{\alpha, \beta = 2}^{r} \frac{\partial}{\partial a_\alpha} \left(\Lambda \frac{\partial}{\partial \Lambda} F_0\right) \frac{\partial}{\partial a_\beta} \left(\Lambda \frac{\partial}{\partial \Lambda} F_0\right) \frac{\partial}{\partial \tau_{\alpha \beta}} \log \Theta_E(0|\tau).
\]

This equation together with the description of the perturbative part (Proposition 2.7) completely determines the prepotential \(F_0\). See the proof of Theorem 5.7 and §5.2 below. This observation was due to [17]. (See also [53] for an earlier result for \(SU(2)\).)

**Proof of Theorem 2.11.**

Recall that we consider \(u_p\) as functions of \(a_\alpha, \Lambda\). We differentiate (2.2) by \(\log \Lambda\) to get

\[
\sum_p \frac{\partial u_p}{\partial \log \Lambda} \int_{A_\alpha} \frac{\partial}{\partial u_p} dS + \int_{A_\alpha} \frac{\partial}{\partial \log \Lambda} dS = 0.
\]

Therefore

\[
\sum_p \frac{\partial u_p}{\partial \log \Lambda} \frac{\partial a_\alpha}{\partial u_p} = - \int_{A_\alpha} \frac{\partial}{\partial \log \Lambda} dS = \frac{r}{2\pi} \int_{A_\alpha} \frac{P(z)}{P'(z)} dw = \frac{r}{2\pi} \int_{A_\alpha} \frac{P(z) dz}{y}.
\]

The last expression can be given by the Szegö kernel (see (B.7)) as

\[
\Psi_E^\pm(z_1, \infty_\pm) = \frac{-P(z_1) \pm y(z_1)}{2y(z_1)} d \left(\frac{1}{z_2}\right) \bigg|_{z_2 = \infty_\pm}.
\]

Here we choose the leftmost point \(z_1^+\) as the base point for the Abel-Jacobi map. And the even half-integer characteristic \(E\) corresponds to the partition of the branched points into

\[
\{z_\alpha^+ | \alpha = 1, \ldots, r\} \sqcup \{z_\alpha^- | \alpha = 1, \ldots, r\}.
\]

On the other hand, we have

\[
\omega_\beta|_{z = \infty_\pm} = - \frac{1}{2\pi} \sum_p \frac{\partial u_p}{\partial a_\beta} \frac{z^r - r dz}{y} \bigg|_{z = \infty_\pm} = \frac{1}{2\pi} \frac{\partial u_2}{\partial a_\beta} d \left(\frac{1}{z}\right) \bigg|_{z = \infty_\pm},
\]

where we have used \(y \sim z^r\) at \(z = \infty_\pm\) in the second equality. Therefore by Fay's identity [136] we have

\[
\frac{r}{2\pi} \int_{A_\alpha} \frac{(P(z) + y) dz}{y} = \frac{r}{2\pi^2} \frac{\partial u_2}{\partial a_\beta} \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} \log \Theta_E(0|\tau)
\]

\[
= \frac{2r}{\pi \sqrt{-1}} \frac{\partial u_2}{\partial a_\beta} \frac{\partial}{\partial \tau_{\alpha \beta}} \log \Theta_E(0|\tau).
\]
In the second equality we have used the heat equation and the fact that $E$ is an even half-integer characteristic, and hence the derivative of $\Theta_E(\xi \tau)$ at $\xi = 0$ vanishes. Multiplying the matrix $\left( \frac{\partial u_+}{\partial a_+} \right)$ to both hand sides, we get the differential equation as in the assertion.

Finally we determine the even half-integer characteristic $E$ explicitly. Looking at Figure 11 we find that the partition corresponding to characteristic 0 is

\[ \{ z_1^+, z_2^-, z_3^+, \ldots \} \cup \{ z_1^-, z_2^+, z_3^-, \ldots \} . \]

Namely $z_\alpha^+$ and $z_\alpha^-$ for $\alpha$ even are interchanged from $E$. Since $\int_{z_\alpha}^{z_\alpha^+} \omega_\beta = \frac{1}{2} \int_{A_\alpha} \omega_\beta = \frac{1}{2} \delta_{\alpha \beta}$, we find that the characteristic $E$ is $\left[ \begin{array}{c} 0 \\ \Lambda \end{array} \right]$ in (B.1).

**Remark 2.14.** In [31] ‘time variables’ $T_1$, $T_2$, ..., $T_{r-1}$ are introduced in the framework of Whitham hierarchy. Then the contact term equations are the specialization of the equations at $T_2 = T_3 = \cdots = 0$ ($T_1$ is essentially $\log \Lambda$). On the other hand, we will introduce infinitely many variables $\tau_1, \tau_2, \ldots$ in [4]. We will show $\frac{\partial}{\partial \tau_p} = \frac{\partial}{\partial \tau_p}$ for $p = 1, 2, \ldots, r - 1$ when it is restricted to $\tau_1 = \tau_2 = \cdots = 0$ in Theorem 2.11(2). However the equations for the $\tau_p$-derivatives are not explicitly written down, and are different from the equations for $T_p$-derivatives outside this subspace.

**2.5. Rank 2 case.** When $r = 2$, i.e., the Seiberg-Witten curve is an elliptic curve, we have the expressions [43] for $u, a$ in terms of theta functions and Eisenstein series. Here we write $u = u_2$, $a = \alpha_2$. The derivation of the expressions are left to the reader as an exercise. One can prove Theorem 2.11 using the expressions. See [31] Appendix.

Let us record the following formula for the later purpose.

\[
\Lambda \frac{\partial u}{\partial \Lambda} = 2u - a \frac{du}{da} = 2u + \sqrt{1 - \frac{2E_2}{3\theta_0^2\theta_{10}} \Lambda} \frac{du}{da} \\
= 2u - \frac{1}{3} E_2 \left( \frac{du}{da} \right)^2 + \frac{2}{3} \frac{\theta_{10}^2 + \theta_4^1}{\theta_0^2 \theta_{10}} \Lambda^2 = -\frac{1}{3} E_2 \left( \frac{du}{da} \right)^2 + \frac{4}{3} u,
\]

where the first equality follows from the homogeneity of $u$.

For the reader who wants to compare the formulas with ones in the literature, we record how parameters differ. Let us make a change of variable by

\[ w - \frac{u}{3} = -\Lambda^2 \frac{z - \sqrt{-u + 2\Lambda^2}}{z + \sqrt{-u + 2\Lambda^2}}. \]

The branched points $z = \sqrt{-u + 2\Lambda^2}, -\sqrt{-u + 2\Lambda^2}, \sqrt{-u - 2\Lambda^2}, -\sqrt{-u - 2\Lambda^2}$ are mapped to $w = u/3, \infty, (-u - 3\sqrt{u^2 - 4\Lambda^4})/6, (-u + 3\sqrt{u^2 - 4\Lambda^4})/6$ respectively. And the curve $y^2 = (z^2 + u)^2 - 4\Lambda^4$ is isomorphic to a Weierstrass form $y^2 = 4w^3 - g_2w - g_3$ with

\[ g_2 = 4 \left( \frac{1}{3} u^2 - \Lambda^4 \right), \quad g_3 = -\frac{1}{27} u \left( 8u^2 - 36\Lambda^4 \right). \]

This is the form of curves appeared in [24] with $u$ replaced by $-x$ therein. If we make a further change of variable as $w = 2x + u/3$, we get

\[ \frac{1}{32} y^2 = x \left( x + \frac{u}{2} x + \frac{\Lambda^4}{4} \right). \]
This is the form of the curves in $\mathbb{S}_4^6$ after the replacement $u \mapsto -2u$, $y \mapsto 4\sqrt{2}y$.

### 3. Instanton moduli spaces

#### 3.1. Basic definitions.

In this and next subsections we briefly recall properties of framed moduli spaces of instantons (resp. torsion-free sheaves) on $S^4$ (resp. $\mathbb{P}^2$) and the corresponding moduli spaces on blowup. For more detail, see [61 §1] and [60 Chapters 2,3] and the references therein.

Let $M(r,n)$ be the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank $r$ and $c_2 = n$, which parametrizes isomorphism classes of $(E, \Phi)$ such that

1. $E$ is a torsion free sheaf of rank $E = r$, $\langle c_2(E), [\mathbb{P}^2] \rangle = n$ which is locally free in a neighbourhood of $\ell_\infty$,

2. $\Phi: E|_{\ell_\infty} \cong O_{\ell_\infty}^{2r}$ is an isomorphism called ‘framing at infinity’.

Here $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2 \subset \mathbb{P}^2$ is the line at infinity. Notice that the existence of a framing $\Phi$ implies $c_1(E) = 0$.

This is known to be nonsingular of dimension $2nr$.

Let $M^\text{reg}_0(r,n)$ be the open subset consisting of locally free sheaves. By a result of Donaldson [15] it can be identified with the framed moduli space of instantons on $S^4$ which parametrizes anti-self-dual connections $A$ on a principal $SU(r)$-bundle $P$ with $\langle c_2(P), [S^4] \rangle = n$ modulo gauge transformations $\gamma$ with $\gamma_\infty = id$.

Let $M_0(r,n)$ be the Uhlenbeck (partial) compactification of $M^\text{reg}_0(r,n)$. Set theoretically it is defined by

$$M_0(r,n) = \bigsqcup_{k=0}^n M^\text{reg}_0(r,n-k) \times S^k \mathbb{C}^2,$$

where $S^k \mathbb{C}^2$ is the $k$th symmetric product of $\mathbb{C}^2$. We can endow a structure of an affine algebraic variety to $M_0(r,n)$ so that there is a projective morphism

$$\pi: M(r,n) \to M_0(r,n).$$

The corresponding map between closed points can be identified with

$$(E, \Phi) \mapsto ((E^{\vee\vee}, \Phi), \text{Supp}(E^{\vee\vee}/E)) \in M^\text{reg}_0(r,n') \times S^{n-n'} \mathbb{C}^2.$$

where $E^{\vee\vee}$ is the double dual of $E$ and $\text{Supp}(E^{\vee\vee}/E)$ is the support of $E^{\vee\vee}/E$ counted with multiplicities. Note that $E^{\vee\vee}$ is a locally free sheaf. For moduli spaces on general projective surfaces, such morphisms from moduli spaces of sheaves to Uhlenbeck compactifications were constructed by J. Li [39] and Morgan [55].

Let $T$ be the maximal torus of $\text{GL}_r(\mathbb{C})$ consisting of diagonal matrices and let $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T$. We define an action of $\widetilde{T}$ on $M(r,n)$ as follows: For $(t_1,t_2) \in \mathbb{C}^* \times \mathbb{C}^*$, let $F_{t_1,t_2}$ be an automorphism of $\mathbb{P}^2$ defined by

$$F_{t_1,t_2}([z_0 : z_1 : z_2]) = [z_0 : t_1z_1 : t_2z_2].$$

For $\text{diag}(e_1,\ldots,e_r) \in T$ let $G_{e_1,\ldots,e_r}$ denotes the isomorphism of $O^\boxtimes 2r$ given by

$$O_{\ell_\infty}^\boxtimes (s_1,\ldots,s_r) \mapsto (e_1s_1,\ldots,e_rs_r).$$

Then for $(E, \Phi) \in M(r,n)$, we define

$$\Phi' = (F_{t_1,t_2})^* \Phi,$$

where $\Phi'$ is the composite of homomorphisms

$$(F_{t_1,t_2})^*: E|_{\ell_\infty} \xrightarrow{(F_{t_1,t_2})^* \Phi} (F_{t_1,t_2})^* O_{\ell_\infty}^\boxtimes \xrightarrow{G_{e_1,\ldots,e_r}} O_{\ell_\infty}^\boxtimes.$$
Here the middle arrow is the homomorphism given by the action.

In a similar way, we have a $\tilde{T}$-action on $M_0(r, n)$. The map $\pi: M(r, n) \to M_0(r, n)$ is equivariant.

The fixed points $M(r, n)^{\tilde{T}}$ consist of $(E, \Phi) = (I_1, \Phi_1) \oplus \cdots \oplus (I_r, \Phi_r)$ such that

a) $I_\alpha$ is an ideal sheaf of 0-dimensional subscheme $Z_\alpha$ contained in $\mathbb{P}^2 \setminus \mathcal{E}_\infty$.

b) $\Phi_\alpha$ is an isomorphism from $(I_\alpha)_{\mathcal{E}_\infty}$ to the $\alpha$th factor of $O_{\mathbb{P}^2}^{\oplus r}$.

c) $I_\alpha$ is fixed by the action of $\mathbb{C}^* \times \mathbb{C}^*$ coming from that on $\mathbb{P}^2$.

On the other hand, the fixed points $M_0(r, n)^{\tilde{T}}$ consist of the single point $n[0] \in S^n \mathbb{C}^2 \subset M_0(r, n)$.

We parametrize the fixed point set $M(r, n)^{\tilde{T}}$ by a $r$-tuple of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_r)$ so that the ideal $I_\alpha$ is spanned by monomials $x^i y^j$ placed at $(i - 1, j - 1)$ outside $Y_\alpha$ as illustrated in Figure 2. The constraint is that the total number of boxes $|\vec{Y}| \overset{\text{def}}{=} \sum_{\alpha} |Y_\alpha|$ is equal to $n$.

Let $Y = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a Young diagram, where $\lambda_i$ is the length of the $i$th column. Let $Y' = (\lambda'_1 \geq \lambda'_2 \geq \cdots)$ be the transpose of $Y$. Thus $\lambda'_j$ is the length of the $j$th row of $Y$. Let $l(Y)$ denote the number of columns of $Y$, i.e., $l(Y) = \lambda'_1$. Let

\begin{align*}
a_Y(i, j) &= \lambda_i - j, \quad a'_j(i, j) = j - 1 \\
l_Y(i, j) &= \lambda'_j - i, \quad l'_j(i, j) = i - 1.
\end{align*}

Here we set $\lambda_i = 0$ when $i > l(Y)$. Similarly $\lambda'_j = 0$ when $j > l(Y')$. When the square $s = (i, j)$ lies in $Y$, these are called arm-length, arm-colength, leg-length, leg-colength respectively, and we usually consider in this case. But our formula below involves these also for squares outside $Y$. So these take negative values in general. Note that $a'$ and $l'$ does not depend on the diagram, and we do not write the subscript $Y$.

**Theorem 3.2.** Let $(E, \Phi)$ be a fixed point of the $T$-action corresponding to $\vec{Y} = (Y_1, \ldots, Y_r)$. Then the $T$-module structure of $T_{(E, \Phi)} M(r, n)$ is given by

\[
\sum_{\alpha, \beta = 1}^r N_{\alpha, \beta}^\vec{Y}(t_1, t_2),
\]
where
\[ N^T_{\alpha, \beta}(t_1, t_2) = e_\beta e_\alpha^{-1} \times \left\{ \sum_{s \in Y_\alpha} \left( t_1^{-y_\beta(s)} t_2^{a_y(s)+1} \right) + \sum_{t \in Y_\beta} \left( t_1^{y_\alpha(t)+1} t_2^{-\alpha y_\beta(t)} \right) \right\}. \]

Here we have used the following notation.

**Notation 3.3.** We denote by \( e_\alpha \) (\( \alpha = 1, \ldots, r \)) the one dimensional \( T \)-module given by
\[
\tilde{T} \supset (t_1, t_2, e_1, \ldots, e_r) \mapsto e_\alpha.
\]

Similarly, \( t_1, t_2 \) denote one-dimensional \( T \)-modules. Thus the representation ring \( R(\tilde{T}) \) is isomorphic to \( \mathbb{Z}[t_1^\pm, t_2^\pm, \ldots, e^\pm_\alpha] \), where \( e_\alpha^{-1} \) is the dual of \( e_\alpha \).

### 3.2. Moduli spaces on the blowup.

Let \( \hat{\mathbb{P}}^2 \) be the blowup of \( \mathbb{P}^2 \) at \([1 : 0 : 0] \). Let \( p: \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \) denote the projection. The manifold \( \hat{\mathbb{P}}^2 \) is the closed subvariety of \( \mathbb{P}^2 \times \mathbb{P}^1 \) defined by
\[
\{ ([z_0 : z_1 : z_2], [z : w] \in \mathbb{P}^2 \times \mathbb{P}^1 | z_1w = z_2z \},
\]
where the map \( p: \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \) is the projection to the first factor. Let us denote the inverse image of \( \ell_\infty \) under \( \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \) also by \( \ell_\infty \) for brevity. It is given by the equation \( z_0 = 0 \). The complement \( \hat{\mathbb{P}}^2 \setminus \ell_\infty \) is the blowup \( \hat{\mathbb{C}}^2 \) of \( \mathbb{C}^2 \) at the origin. Let \( C \) denote the exceptional set. It is given by \( z_1 = z_2 = 0 \).

In this subsection, \( \mathcal{O} \) denotes the structure sheaf of \( \hat{\mathbb{P}}^2 \), \( \mathcal{O}(C) \) the line bundle associated with the divisor \( C \), \( \mathcal{O}(mC) \) its \( m \)th tensor product.

Let \( \hat{M}(r, k, n) \) be the framed moduli space of torsion free sheaves \((E, \Phi)\) on \( \hat{\mathbb{P}}^2 \) with rank \( r \), \( \langle c_1(E), [C] \rangle = -k \) and \( \langle c_2(E) - \frac{1}{2r}c_1(E)^2, [\mathbb{P}^2] \rangle = n \). This is also nonsingular of dimension \( 2nr \). (Remark that \( n \) may not be integer in general.)

**Theorem 3.4.** There is a projective morphism \( \hat{\pi}: \hat{M}(r, k, n) \to M_0(r, n - \frac{1}{r}k(r - k)) \) (\( 0 \leq k < r \)) defined by
\[
(E, \Phi) \mapsto \left( ((p_*E)^{\vee \vee}, \Phi), \text{Supp}(p_*E^{\vee \vee}/p_*E) + \text{Supp}(R^1p_*E) \right).
\]

Let us define an action of the \((r + 2)\)-dimensional torus \( \hat{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \) on \( \hat{M}(r, k, n) \) by modifying the action on \( M(r, n) \) as follows. For \((t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^* \), let \( F^T_{t_1, t_2} \) be an automorphism of \( \hat{\mathbb{P}}^2 \) defined by
\[
F^T_{t_1, t_2}([z_0 : z_1 : z_2], [z : w]) = ([z_0 : t_1z_1 : t_2z_2], [t_1z : t_2w]).
\]

Then we define the action by replacing \( F_{t_1, t_2} \) by \( F^T_{t_1, t_2} \) in (3.2). The action of the latter \( T \) is exactly the same as before. The morphism \( \hat{\pi} \) is equivariant.

Note that the fixed point set of \( \mathbb{C}^* \times \mathbb{C}^* \) in \( \hat{\mathbb{C}}^2 = \mathbb{C}^2 \setminus \ell_\infty \) consists of two points \([1 : 0 : 0], [1 : 0] \). Let us denote them \( p_1 \) and \( p_2 \).

Let us define an action of the \((r + 2)\)-dimensional torus \( \tilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \) on \( \hat{M}(r, k, n) \) by modifying the action on \( M(r, n) \) as follows. For \((t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^* \), let \( F^T_{t_1, t_2} \) be an automorphism of \( \hat{\mathbb{P}}^2 \) defined by
\[
F^T_{t_1, t_2}([z_0 : z_1 : z_2], [z : w]) = ([z_0 : t_1z_1 : t_2z_2], [t_1z : t_2w]).
\]

Then we define the action by replacing \( F_{t_1, t_2} \) by \( F^T_{t_1, t_2} \) in (3.1). The action of the latter \( T \) is exactly the same as before. The morphism \( \hat{\pi} \) is equivariant.

The fixed points \( \hat{M}(r, k, n)^T \) consist of \((E, \Phi) = (I_1(k_1C), \Phi_1) \oplus \cdots \oplus (I_r(k_rC), \Phi_r)\) such that
a) \( I_\alpha(k_\alpha C) \) is the tensor product \( I_\alpha \otimes O(k_\alpha C) \), where \( k_\alpha \in \mathbb{Z} \) and \( I_\alpha \) is an ideal sheaf of 0-dimensional subscheme \( Z_\alpha \) contained in \( \overline{\mathbb{P}^2} \setminus \ell_{\infty} \).

b) \( \Phi_\alpha \) is an isomorphism from \((I_\alpha)_{\ell_{\infty}}\) to the \( \alpha \)th factor of \( O_{\ell_{\infty}}^{\oplus r} \).

c) \( I_\alpha \) is fixed by the action of \( \mathbb{C}^* \times \mathbb{C}^* \), coming from that on \( \overline{\mathbb{P}^2} \).

The support of \( Z_\alpha \) must be contained in the fixed point set in \( \overline{\mathbb{P}^2} \), i.e., \{\( p_1, p_2 \}\). Thus \( Z_\alpha \) is a union of \( Z_\alpha^1 \) and \( Z_\alpha^2 \), subschemes supported at \( p_1 \) and \( p_2 \) respectively. If we take a coordinate system \((x, y) = (z_1/z_0, w/z)\) (resp. \((z/w, z_2/z_0)\)) around \( p_1 \) (resp. \( p_2 \)), then \( Z_\alpha^1 \) (resp. \( Z_\alpha^2 \)) is generated by monomials \( x^i y^j \). (See Figure 3.) Then \( Z_\alpha^1 \) (resp. \( Z_\alpha^2 \)) corresponds to a Young diagram \( Y_\alpha^1 \) (resp. \( Y_\alpha^2 \)) as before. Therefore the fixed point set is parametrized by \( r \)-tuples \((\vec{k}, Y^1, Y^2) = ((k_1, Y_1^1, Y_1^2), \ldots, (k_r, Y_r^1, Y_r^2))\), where \( k_\alpha \in \mathbb{Z} \) and \( Y_\alpha^1, Y_\alpha^2 \) are Young diagrams. The constraint is

\[
\sum_{\alpha} k_\alpha = k, \quad |Y^1| + |Y^2| + \frac{1}{2r} \sum_{\alpha < \beta} |k_\alpha - k_\beta|^2 = n.
\]

We will use the convention for \( \vec{k} \) in [41].

Note that the fixed point data have three parts, \( \vec{k}, Y^1 \) and \( Y^2 \). This will be reflected in the blowup formula (4.6) below. Also, the appearance of \( \vec{k} \in \mathbb{Z}^r \) explains why many formulas below contain the theta function.

![Figure 3. Blowup and fixed points](image)

**Theorem 3.6.** Let \((E, \Phi)\) be a fixed point of \( \vec{T} \)-action corresponding to \( (\vec{k}, Y^1, Y^2) \). Then the \( \vec{T} \)-module structure of \( T_{(E, \Phi)} \vec{M}(r, k, n) \) is given by

\[
\sum_{\alpha, \beta = 1}^r L_{\alpha, \beta}^E(t_1, t_2) + t_1^{k_\alpha - k_\beta} N_{\alpha, \beta}^1(t_1, t_2/t_1) + t_2^{k_\beta - k_\alpha} N_{\alpha, \beta}^2(t_1/t_2, t_2),
\]

where

\[
L_{\alpha, \beta}^E(t_1, t_2) = e_\beta e_\alpha^{-1} \times \begin{cases} 
   t_1^{-i} t_2^{-j} & \text{if } k_\alpha > k_\beta, \\
   \sum_{i+j \leq k_\alpha - k_\beta - 1} t_1^{i+1} t_2^{j+1} & \text{if } k_\alpha + 1 < k_\beta, \\
   0 & \text{otherwise.}
\end{cases}
\]
The reason for the change of weights \((t_1, t_2/t_1), (t_1/t_2, t_2)\) is clear. It comes from the action on the coordinate system around \(p_1\) and \(p_2\).

### 3.3. Topology of moduli spaces.

Thanks to the existence of the torus action, the homology groups of \(M(r, n)\), \(\tilde{M}(r, k, n)\) enjoy nice properties. In particular, we can calculate their Betti numbers whose generating functions have beautiful formulas. The results of this and next subsections will not be used in the other parts of this paper. The reader in a hurry may skip this and next subsections.

**Theorem 3.7.** (1) \(\pi^{-1}(n[0])\) is isomorphic to the punctual quot-scheme parameterizing zero dimensional quotients \(O^\Sigma_{\mathbb{P}^2} \to Q\) with \(\text{Supp}(Q) = n[0]\).

(2) \(M(r, n)\) is homotopy equivalent to \(\pi^{-1}(n[0])\).

(3) Both \(M(r, n)\) and \(\pi^{-1}(n[0])\) have \(\alpha\)-partitions into affine spaces.

(4) \(H_{\text{odd}}(M(r, n), \mathbb{Z}) = 0\) and \(H_{\text{even}}(M(r, n), \mathbb{Z})\) is a free abelian group. The cycle map \(A_\ast(M(r, n)) \to H_{\text{even}}(M(r, n), \mathbb{Z})\) is an isomorphism. The same assertions hold for \(\pi^{-1}(n[0])\).

Recall that a finite partition of a variety \(X\) into locally closed subvarieties is said to be an \(\alpha\)-partition if the subvarieties in the partition can be indexed \(X_1, \ldots, X_n\) in such a way that \(X_1 \cup X_2 \cup \cdots \cup X_i\) is closed in \(X\) for \(i = 1, \ldots, n\).

Here \(H_\ast(\bullet, \mathbb{Z})\), \(A_\ast(\bullet)\) denote the Borel-Moore homology group and the Chow group. See [C].

**Proof.** (1) By the geometric description \(5.1\) of the map \(\pi\), the fiber \(\pi^{-1}(n[0])\) consists of \((E, \Phi)\) such that \(E \to \phi\) and \(\text{Supp}(E) = n[0]\). Thus the quotient \(O^\Sigma_{\mathbb{P}^2}/E\) is a point in the punctual quot-scheme.

(2) A similar result was proved in [58] 5.5 for quiver varieties by using a \(\mathbb{C}^\ast\)-action. The proof can be adapted to our situation as follows.

Let us consider a one parameter subgroup \(\mathbb{C}^\ast \ni t \mapsto \lambda(t) = (t, t, 1, \ldots, 1) \in \mathbb{T}\). When \(t \to 0\), \(\lambda(t) \cdot x\) goes to 0 for any \(x \in M_0(r, n)\). Now apply an argument of Slodowy [67] 4.3 to \(\pi: M(r, n) \to M_0(r, n)\).

(3) Choose a generic one parameter subgroup \(\lambda: \mathbb{C}^\ast \to \mathbb{T}\) so that the fixed point set is unchanged: \(M(r, n)\lambda(\mathbb{C}^\ast) = M(r, n)\mathbb{T}\). Moreover we take so that \(\lim_{t \to 0} \lambda(t) = 0\). Then points fixed by \(\lambda(\mathbb{C}^\ast)\) are given as above. In particular, they form a finite set. For each fixed point \(w\), we consider \((\pm)\)-attracting set:

\[
S_w = \left\{ x \in M(r, n) \left| \lim_{t \to 0} \lambda(t) \cdot x = w \right\} \right., \quad U_w = \left\{ x \in M(r, n) \left| \lim_{t \to \infty} \lambda(t) \cdot x = w \right\} \right..
\]

These are affine spaces by [7]. Moreover, there exists an order on fixed points so that \(\bigcup_{y \leq w} U_y\) is closed in \(\bigcup_{y \leq w} S_y\) for each \(w\). (See e.g., [3] §1.) We claim that \(\bigcup S_w = M(r, n), \bigcup U_w = \pi^{-1}(n[0])\). Consider the corresponding action on \(M_0(r, n)\). For this purpose, we recall the ADHM description: \(M_0(r, n)\) is an affine albro-geometric quotient

\[
\{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\}/\text{GL}_n(\mathbb{C}),
\]

where \(B_1, B_2\) are \(n \times n\) complex matrices, and \(i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n), j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)\).

(See [50] Chapter 2 and [51],) The action of \(\text{GL}_n(\mathbb{C})\) is given by \(g \cdot (B_1, B_2, i, j) = (g B_1 g^{-1}, g B_2 g^{-1}, g i, g j^{-1})\). The \(T\)-action is given by

\[
(B_1, B_2, i, j) \mapsto (t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j),
\]

for \(t_1, t_2 \in \mathbb{C}^\ast, e = \text{diag}(e_1, \ldots, e_r) \in (\mathbb{C}^\ast)^r\).
By [48] the coordinate ring of $M_0(r, n)$ is generated by the following two types of functions:

a) $\text{tr}(B_{\alpha N} \cdots B_{\alpha_1} : C^n \to C^n)$, where $\alpha_i = 1$ or 2.

b) $\chi(jB_{\alpha N} \cdots B_{\alpha_1}i)$, where $\alpha_i = 1$ or 2, and $\chi$ is a linear form on $\text{End}(C^n)$.

Both types of functions have positive weights with respect to $\lambda$. Therefore every point in $M_0(r, n)$ converges to 0 as $t \to 0$, and every point except 0 goes to infinity as $t \to \infty$. Since $\pi$ is proper, we get the claim. (We use the fact that the orbit of a one-parameter subgroup has limit if it is contained in a compact set.) Thus $\bigcup S_w$ (resp. $\bigcup U_w$) gives us an $\alpha$-partition of $M(r, n)$ (resp. $\pi^{-1}(0)$) into affine spaces.

(4) is a consequence of (3) by [9], Lemma 1.8.

Theorem 3.8. The Poincaré polynomials of the punctual quot-scheme $\pi^{-1}(n[0])$ is given by

$$P_t(\pi^{-1}(n[0])) = \sum_{(Y_1, \ldots, Y_r)} \prod_{\alpha=1}^r t^{2(r|\alpha| - \alpha_l[Y_\alpha])},$$

where the summation runs over the set of $r$-tuple of Young diagrams $Y = (Y_1, \ldots, Y_r)$ with $|Y| = n$.

As in the calculation in [60], the end of Chapter 6, we get the following nice expression for the generating function.

Corollary 3.10. The generating function of the Poincaré polynomials of $\pi^{-1}(n[0])$ is given by

$$\sum_n P_t(\pi^{-1}(n[0]))q^n = \prod_{n=1}^r \prod_{d=1}^{\infty} \frac{1}{1 - t^{2(rd - \alpha)}q^d}.$$
negative weight spaces for $\mathbb{C}^*$-action are direct sum of weight spaces for $\bar{T}$-action such that the one of the followings holds

1. weight of $t_2$ is negative,
2. weight of $t_2$ is zero and weight of $e_1$ is negative,
3. weight of $t_2$, $e_1$, $e_2$ are zero and weight of $e_3$ is negative,

\[ \ldots \]

\[(r + 1)\text{ weight of } t_2, e_1, e_2, \ldots, e_{r-1} \text{ are zero and weight of } e_r \text{ is negative},\]

\[(r + 2)\text{ weight of } t_2, e_1, e_2, \ldots, e_r \text{ are zero and weight of } t_1 \text{ is negative}.\]

Recall that we decompose the tangent space into $\sum_{\alpha, \beta} \mathcal{N}^{\alpha, \beta}(t_1, t_2)$ in Theorem 3.2. We calculate the sum of dimensions of weight spaces with the above condition in each summand separately, and then sum up the contribution from each summand. In the summand $\alpha = \beta$, the contribution is

$$|Y_\alpha| - l(Y_\alpha).$$

If $\alpha < \beta$, the above condition is equivalent to that weight of $t_2$ is nonpositive. Hence the contribution is equal to the number of terms in [61, (1.18)], i.e.,

$$|Y_\beta|.$$

If $\alpha > \beta$, the above condition is equivalent to that weight of $t_2$ is negative. If we look at [61, (1.18)], we find that the contribution is

$$|Y_\beta| - l(Y_\beta).$$

Thus the total contribution is

$$\sum_{\beta=1}^r r|Y_\beta| - (r - \beta + 1)l(Y_\beta).$$

Changing the variable as $\alpha = r - \beta + 1$, we get the formula 3.9.

We now turn to the moduli spaces on blowup. The proof of the following is the same as that of Theorem 3.7 and hence omitted.

**Theorem 3.13.** (1) $\bar{M}(r, k, n)$ is homotopy equivalent to $\pi^{-1}(n[0])$.
(2) Both $\bar{M}(r, k, n)$ and $\pi^{-1}(n[0])$ have $\alpha$-partitions into affine spaces.
(3) $H_{\text{odd}}(\bar{M}(r, k, n), \mathbb{Z}) = 0$ and $H_{\text{even}}(\bar{M}(r, k, n), \mathbb{Z})$ is a free abelian group.
The cycle map $A_*(\bar{M}(r, k, n)) \rightarrow H_{\text{even}}(\bar{M}(r, k, n), \mathbb{Z})$ is an isomorphism. The same holds for $\pi^{-1}(n[0])$.

The following theorem is proved in a similar way as Theorem 3.8. The detail is left to the reader.

**Theorem 3.14.** The Poincaré polynomial of $\pi^{-1}(n[0])$ is given by

$$P_t(\pi^{-1}(n[0])) = \sum \prod_{\alpha=1}^r t^{2(r|Y_\alpha^1|+r|Y_\alpha^2|-\alpha|Y_\alpha^2|)} \prod_{\alpha<\beta} t^{(k_\alpha-k_\beta)(k_\alpha-k_\beta+1)},$$

where the summation runs over the set $((k_1,Y_1^1,Y_1^2),\ldots,(k_r,Y_r^1,Y_r^2))$ with 3.9.
Corollary 3.15. The generating function of the Poincaré polynomials of \( \pi^{-1}(n[0]) \) is given by
\[
\sum_n P_t(\pi^{-1}(n[0])) q^n = \left( \prod_{\alpha=1}^\ell \prod_{d=1}^r \frac{1}{1 - t^{2(\ell d - \alpha)} q^d} \right) \left( \prod_{d=1}^r \frac{1}{1 - t^{2rd} q^d} \right)^r \times \sum_{\{\hat{k}\} = \frac{1}{r}} t^{2(\hat{k}, q)} (t^{2r} q)^{(\hat{k}, \hat{k})/2}.
\]

3.4. A different choice of the one parameter subgroup. This subsection is an interesting detour. We compute Betti numbers of \( \hat{\pi}^{-1}(n[0]) \) in a different way. A comparison with the formula in the previous subsection gives us a nontrivial combinatorial identity.

Let us choose weights for the one-parameter subgroup \( \lambda \in \mathfrak{t} \) so that
\[
m_1 = m_2 \gg n_1 > n_2 > \cdots > n_r > 0
\]
and \( m_1, n_\alpha \) are generic.

Since this \( \lambda \) is not generic, the fixed points are different from those for \( \hat{T} \). But they are described similarly as \( (E, \Phi) = (I_1(k_1 C), \Phi_1) \oplus \cdots \oplus (I_r(k_r C), \Phi_r) \) such that
\begin{enumerate}
    \item \( I_\alpha(k_\alpha C) \) is the tensor product \( I_\alpha \otimes \mathcal{O}(k_\alpha C) \), where \( k_\alpha \in \mathbb{Z} \) and \( I_\alpha \) is an ideal sheaf of 0-dimensional subscheme \( Z_\alpha \) contained in \( \mathbb{C}^2 = \mathbb{P}^2 \setminus \ell_\infty \).
    \item \( \Phi_\alpha \) is an isomorphism from \( (I_\alpha)_{\ell_\infty} \) to the \( \alpha \)th factor of \( \mathcal{O}_{\ell_\infty} \).
    \item \( I_\alpha \) is fixed by the diagonal subgroup \( \Delta \mathbb{C}^* \) of \( \mathbb{C}^* \times \mathbb{C}^* \), coming from that on \( \mathbb{P}^2 \).
\end{enumerate}

Furthermore, we can parametrize the components of the fixed point set by \( (\hat{k}, \hat{Y}) = ((k_1, Y_1), \ldots, (k_r, Y_r)) \) with \( k_\alpha \) as above and \( Y_\alpha \) is a Young diagram. Here the constraint is
\[
\sum_\alpha k_\alpha = k, \quad |\hat{Y}| + \frac{1}{2r} \sum_{\alpha < \beta} |k_\alpha - k_\beta|^2 = n.
\]

Since this can be proved by the method in [60 Chapter 7], we explain it only briefly. A general point in the component \( (\hat{k}, \hat{Y}) \) is \( (E, \Phi) = (I_1(k_1 C), \Phi_1) \oplus \cdots \oplus (I_r(k_r C), \Phi_r) \) such that
\begin{enumerate}
    \item the support of \( I_\alpha \) consists of \( P_1, P_2, \ldots, P_{l_\alpha} \), contained in the exceptional curve \( C \),
    \item if \( \xi \) is the inhomogeneous coordinate of \( C = \mathbb{P}^1 \) and \( \eta \) is the coordinate of the fiber \( \mathbb{C}^2 \cong \mathcal{O}(-1) \to C \),
    \item \[ I_\alpha = (\xi - \xi_1, \eta^\lambda_i) \cap (\xi - \xi_2, \eta^\lambda_i) \cap \cdots \]
    \item with \( \xi_\ell = \xi(P_\ell) \).
\end{enumerate}
See [60 Figure 7.4]. The points \( P_1, P_2, \ldots \) move in \( \mathbb{P}^1 \), but their order is irrelevant when the values \( \lambda_i \) are the same. Therefore the component is isomorphic to
\[ S^Y \mathbb{P}^1 \times \cdots \times S^Y \mathbb{P}^1, \]
with the following notation: For a Young diagram \( Y = (\lambda_1 \geq \lambda_2 \geq \cdots) \), we define \( m_i = \# \{ l \mid \lambda_l = i \} \). We denote \( Y = (1^{m_1} 2^{m_2} \cdots) \) in this case. We set
\[ S^Y \mathbb{P}^1 = S^{m_1} \mathbb{P}^1 \times S^{m_2} \mathbb{P}^1 \times \cdots = \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots, \]
where \( S^m \mathbb{P}^1 \) is the \( m \)th symmetric product of \( \mathbb{P}^1 \).

Let \( (E, \Phi) \) be a fixed point in the component corresponding to \( ((a_1, Y_1), \ldots, (a_r, Y_r)) \). Then the tangent space \( T_{(E, \Phi)} \tilde{M}(r, k, n) \) is a \( \Delta \mathbb{C}^* \times T^r \)-module. The \( \Delta \mathbb{C}^* \times T^r \)-module structure is independent of the choice of a point, we take the \( \tilde{T} \)-fixed point corresponding to \( ((k_1, 0, Y_1), \ldots, (k_r, 0, Y_r)) \). By the formula in Theorem 3.6 we have

\[
T_{(E, \Phi)} \tilde{M}(r, k, n) = \sum_{\alpha, \beta} (L_{\alpha, \beta}^k(t_1, t_1) + \epsilon_{\alpha}^\alpha - a_{\alpha}^\alpha) N_{\alpha, \beta}^Y(t_1),
\]

where \( N_{\alpha, \beta}^Y(t_1) = N_{\alpha, \beta}^Y(1, t_1) \). By Theorem 3.2 we have

\[
(3.17) \quad N_{\alpha, \beta}^Y(t_1) = \left( \sum_{s \in Y_{\alpha}} t_{1}^{a_{\alpha}(s)+1} + \sum_{t \in Y_{\beta}} t_{1}^{-a_{\beta}(t)} \right) e_{\beta} e_{\alpha}^{-1}.
\]

The following theorem is proved in a similar way as Theorem 3.8. The detail is left to the reader.

**Theorem 3.18.** The Poincaré polynomial of \( \tilde{\pi}^{-1}(n[0]) \) is given by

\[
P_\ell(\tilde{\pi}^{-1}(n[0])) = \sum_{\alpha, \beta} \prod_{r=1}^{\infty} \left( 1 - \frac{1}{1 - t^{2d} q^d} \right) \frac{1}{(1 - t^{4d} q^d) (1 - t^{4d-2} q^d)^2 (1 - t^{4d-4} q^d)}
\]

where the summation runs over the set \( (\tilde{k}, Y) \) with \( \tilde{k} = (k_1, \ldots, k_r) \), and

\[
l'_{\alpha, \beta} = \begin{cases} \frac{1}{2} (k_{\alpha} - k_{\beta} + 1) (k_{\alpha} - k_{\beta}) & \text{if } k_{\alpha} \geq k_{\beta}, \\ \frac{1}{2} (k_{\beta} - k_{\alpha} + 1) (k_{\beta} - k_{\alpha}) - 1 & \text{otherwise}, \end{cases}
\]

\[
n'_{\alpha, \beta} = \begin{cases} \text{(# of columns of } Y_{\alpha} \text{ which are longer than } k_{\alpha} - k_{\beta}) & \text{if } k_{\alpha} \geq k_{\beta}, \\ \text{(# of columns of } Y_{\beta} \text{ which are longer than } k_{\beta} - k_{\alpha} - 1) & \text{otherwise}. \end{cases}
\]

Let us consider the generating function of Poincaré polynomials. We consider the simplest case.

**Corollary 3.19.** Assume \( r = 2 \) and \( c_1 = 0 \). The generating functions of Poincaré polynomial is

\[
\left( \prod_{d=1}^{\infty} \frac{1}{(1 - t^{4d} q^d) (1 - t^{4d-2} q^d)^2 (1 - t^{4d-4} q^d)} \right) \times \left[ \sum_{k \geq 0} \prod_{d=1}^{2k} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k(2k+1)} q^{k^2} + \sum_{k \geq 0} \prod_{d=1}^{2k-1} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k(2k+1)-2} q^{k^2} \right].
\]

Comparing with the formula in Corollary 3.8 we get the following identity

\[
(3.20) \quad \sum_{k \geq 0} \prod_{d=1}^{2k} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k(2k+1)} q^{k^2} + \sum_{k \geq 0} \prod_{d=1}^{2k-1} \frac{1 - t^{4d-4} q^d}{1 - t^{4d} q^d} t^{2k(2k+1)-2} q^{k^2}
\]

\[
= \prod_{d=1}^{\infty} \frac{1 - t^{4d-2} q^d}{1 - t^{4d} q^d} \sum_{k=-\infty}^{\infty} t^{2k(2k+1)} q^{k^2}.
\]

Since this identity does not involve any geometric information, it is natural to expect to have a direct proof. Such a proof was provided for us by Hiroyuki Ochiai. See [4D]
Finally let us remark that the results of this and the previous subsections give the blowup formula for the virtual Hodge polynomials of moduli spaces for an arbitrary projective surface $X$. Let $H$ be an ample line bundle over $X$. For $c_1 \in H^2(X, \mathbb{Z})$, $n \in \mathbb{Q}$, let $M_H(r, c_1, n)$ be the moduli space of $H$-stable sheaves $E$ on $X$ with $c_1(E) = c_1$, $c_2(E) - \frac{r-1}{2}c_1(E)^2 = n$. We assume $\text{GCD}(r, (c_1, H)) = 1$.

Let $\tilde{M}_H(r, c_1 + kC, n)$ be the moduli space of $(H - \epsilon C)$-stable sheaves $E$ on $\tilde{X}$ with $c_1(E) = p^*c_1 + kC$, $c_2(E) - \frac{r-1}{2}c_1(E)^2 = \Delta$, where $c_1, n$ is as above, $k \in \mathbb{Z}$, and $\epsilon > 0$ is sufficiently small.

Let $e(Y; x, y)$ denote the virtual Hodge polynomial of $Y$ introduced in \cite{3}.

**Theorem 3.21.** The ratio

$$\frac{\sum_n e(\tilde{M}_H(r, c_1 + kC, n); x, y)q^n}{\sum_n e(M_H(r, c_1, n); x, y)q^n}$$

is independent of the surface $X$ and is given by

$$\left(\prod_{d=1}^r \left(1 - (xy)^d q^d\right)\right)^r \sum_{\{\tilde{k}\} = \frac{r}{2}} (xy)^{\langle\tilde{k}, \rho\rangle} ((xy)^r q)^{\langle\tilde{k}, \tilde{k}\rangle/2}.$$ 

This result is proved as follows. From the proof of Theorem 5.1 for arbitrary surface $X$, we have a stratification of $M_H(r, c_1, n)$ such that $\tilde{\pi}$ is a fibration over each stratum. The fiber is independent of $X$, and isomorphic to our $\tilde{\pi}^{-1}(0)$ defined for the framed moduli spaces. Then properties of virtual Hodge polynomials give the above assertion.

Finally remark that the above holds in the Grothendieck group of varieties, if we replace $(xy)^n$ by $[\mathbb{C}^n]$. This generalizes \cite{72} from rank 2 to higher ranks.

**Remark 3.22.** This result was obtained by the second author \cite{72}. In fact, he assumed that the moduli spaces are nonsingular and used the Weil conjecture to count numbers of rational points over finite fields. He did not use the framed moduli spaces nor the morphism $\tilde{\pi}$ as did in here. The above proof, under the same assumption, was obtained in July, 1997. The authors then noticed that W-P. Li and Z. Qin \cite{20, 11, 42} obtained the above result for rank 2 case, where the universal function is given in the form corresponding to Corollary 8.19. The authors then learned the virtual Hodge polynomials are natural language here.

**4. Nekrasov’s deformed partition function**

Nekrasov’s deformed partition function \cite{62}, more precisely, the one with higher order Casimir operators turned on, can be considered as the generating function of the equivariant homology version of Donaldson invariants on $\mathbb{C}^2$. We give its definition and also the one for the blowup $\tilde{\mathbb{C}}^2$ at the origin in this section. We then study their relation by using the localization theorem in the equivariant homology groups. (See \cite{14})

As for the calculation of original Donaldson invariants, the localization technique was not so useful even if we assume the base manifold has large symmetry (say $X = \mathbb{P}^2$). This was because the fixed point sets are not isolated in general, and are still difficult to study. The crucial difference here is the existence of the framing: A point in $M(r, n)$ is fixed by the action given by the change of the framing if and only if it is a direct sum of rank 1 sheaves. The rank 1 sheaves are easy to study.
4.1. Equivariant integration. Before giving the definition, we explain a general setting for the ‘equivariant integration’ via the localization theorem. Let $T$ be a torus acting on an algebraic variety $N$. Suppose that the fixed point set $N^T$ consists of a single point $o$. Then the push-forward homomorphism for the inclusion $i_o: \{o\} \to N$ induces an isomorphism between localized equivariant homology groups

\[(i_o)_*: S \cong H^*_T(o) \otimes_S S \cong H^*_T(N) \otimes_S S,\]

where $S = H^*_T(pt)$ and $S$ is its quotient field. Furthermore, if $f: M \to N$ is a $T$-equivariant proper morphism, we can define $H^*_T(M) \to S$ by

\[\alpha \mapsto (i_o)_*^{-1} f_* \alpha.\]

We denote this by $\int_M$. This makes sense even when $M$ is not necessarily compact. But it takes a value in the rational function field $S$. When $M$ is compact, it coincides with the usual integration and has values in $S$.

For equivariant $K$-homology groups, we have a similar homomorphism defined by the same formula:

\[K^G(N) \ni \alpha \mapsto (i_o)_*^{-1} f_* \alpha \in \mathcal{R},\]

where $\mathcal{R}$ is the quotient field of the representation ring of $G$. This has a relation with equivariant Hilbert polynomials. See [61] §3.

Suppose $M$ is nonsingular. Let $M^T = \bigsqcup_i F_i$ be the decomposition of the fixed point set $M^T$ to irreducible components. Let $N_i$ be the normal bundle. Note that we only have finitely many components, and each $F_i$ is compact, as $f$ is proper and $N^T = \{o\}$. By the functoriality of the push-forward homomorphism, we have

\[\int_M \alpha = \sum_i \int_{F_i} \frac{1}{e_T(F_i)} t_i^* \alpha,\]

where $e_T(N_i)$ is the equivariant Euler class and $t_i^*$ is the pull-back homomorphism for the inclusion $i: F_i \to M$ defined via the Poincaré duality homomorphism. Here $\int_{F_i}$ is the usual integration as $F_i$ is compact. When $M$ is compact, the fractional parts of each summand of the right hand side cancel out, and the final answer is in $S$. But this does not happen when $M$ is noncompact in general.

4.2. Universal sheaves. Since we need higher rank generalization of Donaldson’s $\mu$-map, we begin with the description of universal sheaves on the moduli spaces.

Over the moduli space $M(r,n)$, we have a natural vector bundle $V$, whose fiber at $(E, \varphi)$ is $H^1(E(-\ell_\infty))$. In the ADHM description in [60] Chapter 2, this is the bundle associated with the natural principal $GL_r(\mathbb{C})$-bundle, coming from the construction of $M(r,n)$ as a quotient space. If $\mathcal{E}$ denotes a universal sheaf on $\mathbb{P}^2 \times M(r,n)$, we have $V = R^1 p_{2*}(\mathcal{E} \otimes p_1^* (\mathcal{O}(\ell_\infty)))$.

We also have another natural vector bundle $W$, given by the fiber at infinity: $W = H^0(\mathcal{E}|_{\ell_\infty})$. This is a trivial bundle, but nontrivial as an equivariant bundle. We also consider bundles $S^+ = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ and $S^- = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ over $\mathbb{P}^2$ with the $T^\mathbb{C}$-action such that $\text{ch} S^+_0 = 1 + t_1^{-1} t_2^{-1}$, $\text{ch} S^- = t_1^{-1} + t_2^{-1}$ on the fibers at the origin. (They are positive and negative spinor bundles $S^+, S^-$ when restricted on $\mathbb{R}^4$.) We then form a virtual equivariant vector bundle on $\mathbb{P}^2 \times M(r,n)$ by $\mathcal{O}_{\mathbb{P}^2} \boxtimes W + (S^- - S^+) \boxtimes V$. 

Lectures on Instanton Counting 27
By \cite{61} Lemma 1.8, this virtual equivariant bundle is isomorphic to the universal sheaf $E$ in the equivariant $K$-cohomology group $K_T(\mathbb{P}^2 \times M(r,n))$. We denote the virtual bundle by $E$ hereafter.

The character of the fiber of $E$ at the fixed point $(0, \bar{Y})$ is given by

\begin{equation}
\chi^*_{(0, \bar{Y})} \chi(E) = \sum_{a=1}^{r} e^{\alpha_a} \left( 1 - (1 - e^{\varepsilon_1})(1 - e^{\varepsilon_2}) \sum_{s \in Y} e^{l'(s)\varepsilon_1 + a'(s)\varepsilon_2} \right),
\end{equation}

where we set $e_{\alpha} = e^{-\alpha_{\alpha}}$, $t_1 = e^{-\varepsilon_1}$, $t_2 = e^{-\varepsilon_2}$ as usual. Let $q_1 = [0 : 1 : 0]$, $q_2 = [0 : 0 : 1]$ be two other fixed points in $\mathbb{P}^2$. We have

$$\chi^*_{(q_1, \bar{Y})} \chi(E) = \chi^*_{(q_2, \bar{Y})} \chi(E) = \sum_{a=1}^{r} e^{\alpha_a}.$$

We define

$$\chi(E)/[C^2] = \frac{1}{\varepsilon_1 \varepsilon_2} \chi^*_{(0, \bar{Y})} \chi(E).$$

Since $C^2$ is noncompact, the slant product $/[C^2]$ is not defined in the usual sense. So we define it by formally applying Bott’s formula. The homogeneous degree part of this is an element of the localized equivariant cohomology group $H^*_T(M(r,n)) \otimes_S S$, but its fractional part is a constant in the following sense:

$$\chi^*_{\bar{Y}} (\chi(E)/[C^2])$$

$$= \chi^*_{\bar{Y}} (\chi(E)/[\mathbb{P}^2]) + \frac{1}{\varepsilon_1 (\varepsilon_2 - \varepsilon_1)} \chi^*_{(q_1, \bar{Y})} \chi(E) + \frac{1}{(\varepsilon_1 - \varepsilon_2) \varepsilon_2} \chi^*_{(q_2, \bar{Y})} \chi(E)$$

$$= \chi^*_{\bar{Y}} (\chi(E)/[\mathbb{P}^2]) + \frac{1}{\varepsilon_1 \varepsilon_2} \sum_{a=1}^{r} e^{\alpha_a},$$

and each degree part of

$$\chi^*_{\bar{Y}} (\chi(E)/[\mathbb{P}^2])$$

is a polynomial. We remark

$$\chi(E)/[\mathbb{P}^2] = \chi((S^- - S^+) \boxtimes V)/[\mathbb{P}^2]$$

since $\chi(W \boxtimes \mathcal{O}_{\mathbb{P}^2})/[\mathbb{P}^2] = 0$. Now the slant product $/[\mathbb{P}^2]$ in the right hand side can be replaced by $/[C^2]$ since $S^- - S^+$ is zero at $t_\infty = \mathbb{P}^2 \setminus C^2$. This observation also matches with the above formula of $\chi^*_{\bar{Y}} (\chi(E)/[\mathbb{P}^2])$.

Let us define the instanton part of the partition function by

$$Z^{\text{inst}}(\varepsilon_1, \varepsilon_2; q, \bar{Y}) = \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \exp \left( \sum_{p=1}^{\infty} \tau_p \chi_{p+1}(E)/[C^2] \right)$$

$$= \sum_{\bar{Y}} \prod_{\alpha, \beta} \sqrt{t_{\alpha, \beta}(\varepsilon_1, \varepsilon_2, \bar{Y})}$$

$$\times \exp \left( \sum_{p=1}^{\infty} \sum_{\alpha=1}^{r} \tau_p \left[ e^{\alpha_{\alpha}} \left( 1 - (1 - e^{\varepsilon_1})(1 - e^{\varepsilon_2}) \sum_{s \in Y} e^{l'(s)\varepsilon_1 + a'(s)\varepsilon_2} \right) \right]_{p=1} \right),$$
where \( \text{ch}_{p+1} \) is the degree \((p + 1)\)-part of the Chern character, \(|\nabla|_{p-1} \) denotes the degree \((p - 1)\)-part of \( \nabla \), and \( \int_{M(r,n)} \) means \((\iota_0 \ast)^{-1} \pi_* (\bullet \cap [M(r,n)]) \) with \( \iota_0 : \{0\} \to M_0(r,n) \) is the inclusion of the unique fixed point \( 0 \in M_0(r,n) \). Furthermore, \( n^Y_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a}) \) is the equivariant Euler class of the tangent space at the fixed point \( \vec{Y} \). It is given by the explicit formula:

\[
n^Y_{\alpha,\beta}(\varepsilon_1, \varepsilon_2, \vec{a}) = \prod_{s \in Y_\alpha} \left( -l_{Y_\alpha}(s)\varepsilon_1 + (a_{Y_\alpha}(s) + 1)\varepsilon_2 + a_\beta - a_\alpha \right) \times \prod_{t \in Y_\beta} \left( (l_{Y_\beta}(t) + 1)\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha \right).
\]

The indeterminate \( q \) should be distinguished from \( q = e^{\pi \sqrt{-1} \tau} \) which will appear later.

Note that each coefficient of \( q^n \) has a complicated, but explicit expression. For small \( n \), it is easy to compute. (The authors wrote a MAPLE program, which was very useful when we found our main result.) But when \( n \) increases, the number of Young diagrams becomes large, and it becomes difficult to compute. We are interested not in individual coefficients, but in the generating function.

By (4.1) we have

\[
\iota_Y^* \left( \text{ch}(\mathcal{E})/|\mathbb{C}^2| \right) = \frac{1}{\varepsilon_1\varepsilon_2} \sum_{\alpha} a_\alpha = 0, \quad \iota_Y^* \left( \text{ch}_2(\mathcal{E})/|\mathbb{C}^2| \right) = \frac{1}{2\varepsilon_1\varepsilon_2} \sum_{\alpha} a_\alpha^2 - n.
\]

By the first equality, we did not include \( \tau_0 \) in \( \vec{\tau} \). Also, the second equation means that \( \tau_1 \) is essentially equal to \( -\log q \) (see also 3.1), but we use both of them to simplify the blowup formula below.

If we set \( \vec{\tau} = 0 \), we get the partition function studied in [61], which was denoted by \( Z(\varepsilon_1, \varepsilon_2, \vec{a}; q) \) there. But we emphasize that this is only the instanton part of the partition function. It is more natural to include the perturbative part also. This will be done in 4.3. This is the reason why we change the notation.

If we expand the exponential, \( Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}) \) becomes

\[
\sum_{n=0}^{\infty} q^n \sum_{\mu} \prod_{i} \tau_{\mu_i} \int_{M(r,n)} \prod_{i} \text{ch}_{\mu_i + 1}(\mathcal{E})/|\mathbb{C}^2|,
\]

where \( \mu = (\mu_1 \geq \mu_2 \geq \cdots) \) is a partition. Therefore \( Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}) \) is the generating function of all intersection numbers of Chern classes of universal sheaves slanted by the fundamental cycle \([\mathbb{C}^2]\).

The following formula will be useful later:

\[
(4.2) \quad \left( \prod_{i} \frac{\partial}{\partial \tau_{\mu_i}} \right) Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau})
\]

\[
= \sum_{n=0}^{\infty} q^n \int_{M(r,n)} \left( \prod_i \text{ch}_{\mu_i + 1}(\mathcal{E})/|\mathbb{C}^2| \right) \cap \exp \left( \sum_{p=1}^{\infty} \tau_p \text{ch}_{p+1}(\mathcal{E})/|\mathbb{C}^2| \right)
\]

for a partition \( \mu = (\mu_1 \geq \mu_2 \geq \cdots) \).
4.3. Partition function on the blowup. We consider similar partition functions on the blowup:

\[
\hat{Z}_{\text{inst}}^{r}(\varepsilon_1, \varepsilon_2, \tilde{\alpha}; q, \tau, \tilde{t}) = \sum_{n} q^{n} \int_{\hat{M}(r, k, n)} \exp \left( \sum_{p=1}^{\infty} \left\{ t_p \left( \text{ch}_{p+1}(\hat{E})/|C| \right) + \tau_p \left( \text{ch}_{p+1}(\hat{E})/|\hat{C}^2| \right) \right\} \right),
\]

where \( \hat{E} \) is a universal sheaf over \( \hat{\mathbb{P}}^2 \times \hat{M}(r, k, n) \) and \( \text{ch}(\hat{E})/|\hat{C}^2| \) is defined as above via the localization (see below). And the summation runs over \( n \in \mathbb{Z}_{\geq 0} + \frac{1}{2p} k(r-k) \). Here we do not include \( t_0 \) though \( \text{ch}_1(\hat{E})/|C| \) is not 0 in general. In fact, it is constant \(-k\).

We calculate this by using the localization formula. Recall \( \{(z_0 : z_1 : z_2), [z : w] \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_{11} = z_{22} \} \). Let \( p_1 = \{(1 : 0 : 0), [1 : 0] \}, p_2 = \{(1 : 0 : 0), [0 : 1] \}, q_1 = \{(0 : 1 : 0), [1 : 0] \}, q_2 = \{(0 : 1 : 0), [0 : 1] \} \) be the fixed points in \( \mathbb{P}^2 \). We use the same notation for the latter two points as fixed points in \( \mathbb{P}^2 \), since they are mapped to corresponding points under the projection \( p: \hat{\mathbb{P}}^2 \to \mathbb{P}^2 \). The characters of the fibers over the fixed points are given by

\[
\begin{align*}
\tilde{t}_{(p_1, \tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})) &= \tilde{t}_{(0, \tilde{Y}^1)}^*(\text{ch}(\hat{E})) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_1, \tilde{a} \to \tilde{a} + \varepsilon_1 \tilde{k}}, \\
\tilde{t}_{(p_2, \tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})) &= \tilde{t}_{(0, \tilde{Y}^2)}^*(\text{ch}(\hat{E})) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_2, \tilde{a} \to \tilde{a} + \varepsilon_2 \tilde{k}}, \\
\tilde{t}_{(q_1, \tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})) &= \tilde{t}_{(q_2, \tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})) = \sum_{\alpha=1}^{r} e^{-a_{\alpha}}.
\end{align*}
\]

For the first two equalities, see the proof of [51] 2.4. The last two equalities are obvious since \( p \) is isomorphism outside the exceptional set. Therefore \( \text{ch}(\hat{E})/|\hat{C}^2| \) has the same constant fractional part as \( \text{ch}(\hat{E})/|C^2| \).

Therefore we have

\[
\begin{align*}
\tilde{t}_{(\tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})/|C|) &= \sum_{\alpha=1}^{r} \left[ \frac{e^{a_{\alpha} + \varepsilon_1 k + \varepsilon_1} \varepsilon_1}{\varepsilon_1 (\varepsilon_2 - \varepsilon_1)} \left( 1 - (1 - e^{\varepsilon_1}) (1 - e^{\varepsilon_2 - \varepsilon_1}) \sum_{\alpha \in Y_2^1} e^{(s + a') \varepsilon_1 + a' (s + \varepsilon_2)} \right) \\
&\quad + \frac{e^{a_{\alpha} + \varepsilon_2 k} \varepsilon_2}{(\varepsilon_1 - \varepsilon_2) \varepsilon_2} \left( 1 - (1 - e^{\varepsilon_1 - \varepsilon_2}) (1 - e^{\varepsilon_2}) \sum_{\alpha \in Y_2^2} e^{(s + a') (s + \varepsilon_1 - \varepsilon_2 + a' (s + \varepsilon_1 - \varepsilon_2))} \right) \right] \\
&= \varepsilon_1 \tilde{t}_{\tilde{Y}^1}^* (\text{ch}(\hat{E})/|C^2|) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_1, \tilde{a} \to \tilde{a} + \varepsilon_1 \tilde{k}} + \varepsilon_2 \tilde{t}_{\tilde{Y}^2}^* (\text{ch}(\hat{E})/|C^2|) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_2, \tilde{a} \to \tilde{a} + \varepsilon_2 \tilde{k}}
\end{align*}
\]

and

\[
\begin{align*}
\tilde{t}_{(\tilde{k}, \tilde{Y}^1, \tilde{Y}^2)}^*(\text{ch}(\hat{E})/|\hat{C}^2|) &= \tilde{t}_{\tilde{Y}^1}^* (\text{ch}(\hat{E})/|C^2|) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_1, \tilde{a} \to \tilde{a} + \varepsilon_1 \tilde{k}} + \tilde{t}_{\tilde{Y}^2}^* (\text{ch}(\hat{E})/|\hat{C}^2|) \bigg|_{\varepsilon_1 \to \varepsilon_1, \varepsilon_2 \to \varepsilon_2 - \varepsilon_2, \tilde{a} \to \tilde{a} + \varepsilon_2 \tilde{k}}.
\end{align*}
\]
Note also that the equivariant Euler class of the tangent space \((\vec{k}, Y^1, Y^2)\) is given by
\[
\prod_{\alpha, \beta} \eta^k_{\alpha, \beta}(\varepsilon_1, \varepsilon_2, \vec{a}) \eta_{\alpha, \beta}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}) \eta_{\alpha, \beta}(\varepsilon_1 - \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}),
\]
where
\[
\eta^k_{\alpha, \beta}(\varepsilon_1, \varepsilon_2, \vec{a}) = s^{k_{\alpha} - k_{\beta}}(\varepsilon_1, \varepsilon_2, a_{\beta} - a_{\alpha}),
\]
\[
(4.3)
s^k(\varepsilon_1, \varepsilon_2, x) = \begin{cases} 
\prod_{i, j \geq 0} \left(-i \varepsilon_1 - j \varepsilon_2 + x\right) & \text{if } k > 0, \\
\prod_{i, j \geq 0} \left((i + 1) \varepsilon_1 + (j + 1) \varepsilon_2 + x\right) & \text{if } k < -1, \\
1 & \text{if } k = 0 \text{ or } -1.
\end{cases}
\]
Therefore
\[
\tilde{Z}_{\varepsilon_1 = k}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{r}, \vec{t})
= \sum_{\{\vec{k}\} = -\frac{1}{k}} \frac{q^{\vec{k}}(\vec{r}, \vec{t})}{\prod_{\alpha, \beta} \eta^k_{\alpha, \beta}(\varepsilon_1, \varepsilon_2, \vec{a})} Z^{\text{inst}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}; q, \vec{r} + \varepsilon_1 \vec{t}) \times Z^{\text{inst}}(\varepsilon_1 - \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}; q, \vec{r} + \varepsilon_2 \vec{t}),
\]
where \(\vec{r} + \varepsilon_1 \vec{t}\) means \((\tau_1 + \varepsilon_1 \tau_1, \tau_2 + \varepsilon_1 \tau_2, \ldots)\), and This is a generalization of the blowup formula. (In our previous paper \cite{61}, we only consider the case \(\vec{r} = (t, 0, 0, \ldots)\).)

### 4.4. Adding perturbation term.

We define the full partition function by
\[
Z(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{r}) = \exp \left[ -\sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2}(a_{\alpha} - a_{\beta}; \Lambda) \right] Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{r}),
\]
where \(\gamma_{\varepsilon_1, \varepsilon_2}\) is as in \cite{42} and \(\Lambda = q^{\frac{1}{2}}\) as usual. The first term is the perturbation term of the partition function.

We choose the branch of log in the perturbative term as explained in the paragraph after Proposition 2.7. See \cite{E.5}.

We have
\[
\sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2 - \varepsilon_1}(a_{\alpha} - a_{\beta} + \varepsilon_1(k_{\alpha} - k_{\beta}); \Lambda) + \gamma_{\varepsilon_1 - \varepsilon_2, \varepsilon_2}(a_{\alpha} - a_{\beta} + \varepsilon_2(k_{\alpha} - k_{\beta}); \Lambda)
= \sum_{\alpha \neq \beta} \left[ \gamma_{\varepsilon_1, \varepsilon_2}(a_{\alpha} - a_{\beta}; \Lambda) + \log s^{k_{\beta} - k_{\alpha}}(\varepsilon_1, \varepsilon_2, a_{\alpha} - a_{\beta}) - \frac{(k_{\alpha} - k_{\beta})^2}{2} \log \Lambda \right]
\]
Therefore
\[
Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} + \varepsilon_1 \bar{k}; q, \bar{\tau} + \varepsilon_1 \bar{l}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \bar{a} + \varepsilon_2 \bar{k}; q, \bar{\tau} + \varepsilon_2 \bar{l})
\]
\[
= \exp \left[ - \sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2} (a_{\alpha} - a_{\beta}; \Lambda) \right] \prod_{\alpha, \beta \neq \alpha} \frac{\Lambda(k_{\beta} - k_{\alpha})^2/2}{k_{\beta, \alpha}(\varepsilon_1, \varepsilon_2, \bar{a})} \times Z_{\text{inst}}(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} + \varepsilon_1 \bar{k}; q, \bar{\tau} + \varepsilon_1 \bar{l}) Z_{\text{inst}}(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \bar{a} + \varepsilon_2 \bar{k}; q, \bar{\tau} + \varepsilon_2 \bar{l}).
\]
Then the blowup formula \(4.5\) is simplified as
\[
Z_{c_{1}=k}(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}, \bar{l}) = \exp \left[ - \sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2} (a_{\alpha} - a_{\beta}; \Lambda) \right] \hat{Z}_{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}, \bar{l}).
\]

Remark 4.7. As we saw, the blowup equations are simplified if we include the perturbation part. This was pointed out to us by N. Nekrasov. Probably this is already enough for the reason for the perturbation term. But it has a geometric meaning as a regularization of the Euler class of an ‘infinite rank’ vector bundle. (See [32, §3.10] and §7.2 below.)

4.5. \(\tau_1\) versus \(\log q\). Let \(\bar{r}_1 = (\tau_1, 0, 0, \cdots)\) be a vector with the first entry only. We have
\[
Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau} + \bar{r}_1) = \exp \left[ - \frac{\tau_1}{2 \varepsilon_1 \varepsilon_2} \sum_{\alpha} a_{\alpha}^2 \right] Z_{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q e^{-\tau_1}, \bar{\tau}).
\]
On the other hand, we have
\[
\exp \left[ - \frac{\tau_1}{2 \varepsilon_1 \varepsilon_2} \sum_{\alpha} a_{\alpha}^2 \right] \exp \left[ - \sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2} (a_{\alpha} - a_{\beta}; \Lambda) \right] \exp \left[ - \frac{\tau_1}{2 \varepsilon_1 \varepsilon_2} \left( \frac{(a_{\alpha} - a_{\beta})^2}{2 \varepsilon_1 \varepsilon_2} + \frac{(a_{\alpha} - a_{\beta})(\varepsilon_1 + \varepsilon_2)}{2 \varepsilon_1 \varepsilon_2} + \frac{\varepsilon_1^2 + \varepsilon_2^2 + 3 \varepsilon_1 \varepsilon_2}{12 \varepsilon_1 \varepsilon_2} \right) \right] = \exp \left[ - \frac{\tau_1}{2 \varepsilon_1 \varepsilon_2} \sum_{\alpha} a_{\alpha}^2 \right] \exp \left[ - \frac{\tau_1 (r - 1)(\varepsilon_1^2 + \varepsilon_2^2 + 3 \varepsilon_1 \varepsilon_2)}{24 \varepsilon_1 \varepsilon_2} \right].
\]
by \(1.7.4\). Therefore
\[
Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau} + \bar{r}_1) = \exp \left[ - \frac{\tau_1 (r - 1)(\varepsilon_1^2 + \varepsilon_2^2 + 3 \varepsilon_1 \varepsilon_2)}{24 \varepsilon_1 \varepsilon_2} \right] Z(\varepsilon_1, \varepsilon_2, \bar{a}; q e^{-\tau_1}, \bar{\tau}).
\]
In particular, we have
\[
\left( \frac{\partial}{\partial \tau_1} \right)^N Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) = \left( - \frac{(r - 1)(\varepsilon_1^2 + \varepsilon_2^2 + 3 \varepsilon_1 \varepsilon_2)}{24 \varepsilon_1 \varepsilon_2} - q \frac{\partial}{\partial q} \right)^N Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau})
\]
for $N \in \mathbb{Z}_{\geq 0}$.

5. The blowup equation and Nekrasov’s conjecture

The blowup formula (4.6) equates the unknown function $Z_{c_1=k}$ to the unknown $Z$. It is useless unless we know either or their independent relation. We do not have such knowledge so far in general, but we do know something when we restrict to the subspace $\tau = 0$. This will be given in this section. An application is a solution of Nekrasov’s conjecture: $\varepsilon_1 \varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; q, 0)|_{\varepsilon_1, \varepsilon_2=0}$ is equal to the Seiberg-Witten prepotential $\mathcal{F}_0(\vec{a}; \Lambda)$ introduced in [2].

Let

$$F(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}) = \varepsilon_1 \varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}).$$

Here the logarithm is defined as follows: We first separate this into the perturbative part and the instanton part

$$F(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}) = -\varepsilon_1 \varepsilon_2 \sum_{\alpha \neq \beta} \gamma_{\varepsilon_1, \varepsilon_2}(a_{\alpha} - a_{\beta}; \Lambda) + \varepsilon_1 \varepsilon_2 \log Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}).$$

We denote the second part by $F^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau})$. It has the form

$$\sum_{p=1}^r \sum_{\alpha=1}^r \tau_p [e^{\alpha \vec{a}}]_p + \varepsilon_1 \varepsilon_2 \log \left[ \sum_{n=0}^\infty q^n \int_{M(r, n)} \exp \left( \sum_{p=1}^r \tau_p \text{ch}_{p+1}(E)/[\mathbb{C}^2] \right) \right].$$

Since the summation in the last part starts with 1, its logarithm makes sense as a formal power series in $q$.

5.1. Gap from the dimension counting. In this subsection we derive a differential equation from a simple geometric consideration, which is well-known in the context of Donaldson invariants.

**Lemma 5.1.** Let $0 < k < r$. Then

$$\tilde{Z}_{c_1=k}(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau}, \vec{\tau}) = O((\tau^* \vec{\tau})^{k(r-k)}).$$

Here we put $\deg \tau_p = p - 1$, $\deg t_p = p$ and $O((\tau^* \vec{\tau})^N)$ means that it is a sum of monomials of degree greater than or equal to $N$. When it is a function only in $\vec{\tau}$ (or $\vec{\tau}$), we simply denote by $O(\vec{\tau}^N)$ (or $O(\vec{\tau}^N)$). This convention will be used in what follows.

**Proof.** Consider the projective morphism $\hat{\pi}: \hat{M}(r, k, n) \rightarrow M_0(r, n - \frac{1}{2r} k(r - k))$. If $x \in H^{2d}_{2r}(\hat{M}(r, k, n))$, we have

$$\hat{\pi}_* \left( x \cap [\hat{M}(r, k, n)] \right) \in H^{2d}_{2r} \hat{M}(r, k, n) - 2d(M_0(r, n - \frac{1}{2r} k(r - k))),$$

and this is 0 if

$$\dim \hat{M}(r, k, n) - d > \dim M_0(r, n - \frac{1}{2r} k(r - k)) \iff k(r - k) > d.$$

In the definition of the partition function on the blowup, we have $\text{ch}_{p+1}(E)/[C] \in H^{2p}_{2r}, \text{ch}_{p+1}(\hat{E})/[\mathbb{C}^2] \in H^{2(p-1)}_{2r}$. The degrees exactly match with the above definition of the degrees of $t_p$ and $\tau_q$. \qed
Similarly we consider the \( k = 0 \) case. The morphism \( \bar{\pi} : \bar{M}(r, 0, n) \to M_0(r, n) \) is an isomorphism outside the inverse image of the closure of \( \{ 0 \} \times M_0^{\text{reg}}(r, n - 1) \). Furthermore we have \( \bar{\pi}^*(\mathcal{E}) \cong \bar{\mathcal{E}} \) there. Since \( \text{codim}(\{ 0 \} \times M_0^{\text{reg}}(r, n - 1)) = 2r \), the same argument as above shows

\[
\bar{Z}_{\varepsilon_1 = 0}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}, \bar{\iota}) = Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) + O((\bar{\tau}, \bar{\iota})^{2r}).
\]

Combined with the blowup formula in the previous subsection, we get

\[
(5.2) \quad \sum_{(\bar{k}) = 0} Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} + \varepsilon_1 \bar{k}; q, \bar{\tau} + \varepsilon_1 \bar{t}) Z(\varepsilon_1 - \varepsilon_2, \bar{a} + \varepsilon_2 \bar{k}; q, \bar{\tau} + \varepsilon_2 \bar{t}) = Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) + O((\bar{\tau}, \bar{\iota})^{2r}),
\]

\[
(5.3) \quad \sum_{(\bar{k}) = -k} Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} + \varepsilon_1 \bar{k}; q, \bar{\tau} + \varepsilon_1 \bar{t}) Z(\varepsilon_1 - \varepsilon_2, \bar{a} + \varepsilon_2 \bar{k}; q, \bar{\tau} + \varepsilon_2 \bar{t}) = O((\bar{\tau}, \bar{\iota})^{k(r-k)}) \quad (0 < k < r).
\]

We call these blowup equations.

### 5.2. Recursive structure

In this subsection we illustrate the power of the blowup equations: They determine \( Z(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) \) up to \( O(\bar{\tau}^{2r-3}) \).

We introduce two auxiliary functions:

\[
F_a(\bar{a}) = F(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a}; q, \bar{\tau}), \quad F_b(\bar{a}) = F(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \bar{a}; q, \bar{\tau}).
\]

We suppress the \( \varepsilon_1, \varepsilon_2 \)-dependence in the notation.

We divide \( \text{Eq.} \) for \( c_1 = k = 0 \) by \( Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a}; q, \bar{\tau})Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \bar{a}; q, \bar{\tau}) \), expand with respect to the variables \( \bar{t} \), and take the coefficients of \( t_1 \) and \( t_1^2 \):

\[
(5.4) \quad \sum_{(\bar{k}) = 0} \left[ \left\{ \frac{1}{\varepsilon_2 - \varepsilon_1} \left( \frac{\partial}{\partial \bar{\tau}_1} F_a(\bar{a} + \varepsilon_1 \bar{k}) - \frac{\partial}{\partial \bar{\tau}_1} F_b(\bar{a} + \varepsilon_2 \bar{k}) \right) \right\}^2 \right.
\]

\[
+ \frac{1}{\varepsilon_2 - \varepsilon_1} \left( \frac{\partial^2}{\partial \bar{\tau}_1^2} F_a(\bar{a} + \varepsilon_1 \bar{k}) - \varepsilon_2 \frac{\partial^2}{\partial \bar{\tau}_1^2} F_b(\bar{a} + \varepsilon_2 \bar{k}) \right)
\]

\[
\times \exp \left[ \frac{1}{\varepsilon_2 - \varepsilon_1} \left( \frac{F_a(\bar{a} + \varepsilon_1 \bar{k}) - F_a(\bar{a})}{\varepsilon_1} - \frac{F_b(\bar{a} + \varepsilon_2 \bar{k}) - F_b(\bar{a})}{\varepsilon_2} \right) \right]
\]

\[
= O(\bar{\tau}^{2r-2}).
\]

**Theorem 5.5.** (1) The solution \( F(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) \) of the equations \( (5.3) \) is unique up to \( O(\bar{\tau}^{2r-3}) \). In fact, determine the coefficients of \( q^n \) in \( F^{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) \) recursively up to \( O(\bar{\tau}^{2r-4}) \).

(2) The coefficients of monomials in \( \bar{\tau} \) of degree \( < 2r - 2 \) in \( F(\varepsilon_1, \varepsilon_2, \bar{a}; q, \bar{\tau}) \) is regular at \( \varepsilon_1 = \varepsilon_2 = 0 \).
PROOF. We prove the assertions by the induction on the power of \( q \). Suppose that the coefficient of \( q^m \) in \( F(\varepsilon_1, \varepsilon_2; \vec{a}; q, \vec{\tau}) \) are determined for \( m < n \). We show that the coefficients of \( q^n \) in \( F_a, F_b \), and hence in \( F(\varepsilon_1, \varepsilon_2; \vec{a}; q, \vec{\tau}) \) are determined from (5.3).

Let us separate the terms with \( k = 0 \). The remaining terms with \( k \neq 0 \) are divisible by \( q \) by (4.5) (recall \( \prod_{\alpha, \beta} \Lambda^{(k_\alpha - k_\beta)^2/2} = q^{k_\vec{k}}(k, \vec{k}) \)). Then the equations are written as

\[
\frac{\partial}{\partial \tau_1} F_a(\vec{a}) - \frac{\partial}{\partial \tau_1} F_b(\vec{a}) = q \times \text{known up to order } n - 1,
\]

\[
\varepsilon_1 \frac{\partial^2}{\partial \tau_1^2} F_a(\vec{a}) - \varepsilon_2 \frac{\partial^2}{\partial \tau_1^2} F_b(\vec{a}) = q \times \text{known up to order } n - 1.
\]

After noticing that \( \frac{\partial}{\partial \tau} \) is essentially equal to \( q \frac{\partial}{\partial q} \) as \( \varepsilon_0 \) the above equations gives a system of linear equations on the coefficients of \( q^n \) in \( F_0(\vec{a}, \varepsilon_1 \vec{\tau}), F_b(\vec{a}, \varepsilon_2 \vec{\tau}) \). This system is uniquely solvable since the determinant of \( \begin{pmatrix} n & -n \\ \varepsilon_1 n^2 & -\varepsilon_2 n^2 \end{pmatrix} \) is nonzero.

Furthermore, the right hand sides divided by \( \varepsilon_1 - \varepsilon_2 \) are regular at \( \varepsilon_1, \varepsilon_2 = 0 \) if \( F(\varepsilon_1, \varepsilon_2; \vec{a}; q, \vec{\tau}) \) is regular. Again by the induction, we get the second assertion. \( \square \)

### 5.3. Contact term equations as limit of blowup equations

In this subsection we study the specialization of the differential equation (5.2) at \( \varepsilon_1 = \varepsilon_2 = 0 \).

Let

\[
F(\varepsilon_1, \varepsilon_2; \vec{a}; q, \vec{\tau}) = F_0(\vec{a}; q, \vec{\tau}) + (\varepsilon_1 + \varepsilon_2)H(\vec{a}; q, \vec{\tau}) + \varepsilon_1 \varepsilon_2 F_1(\vec{a}; q, \vec{\tau}) + \cdots,
\]

where we consider terms up to \( O(\varepsilon_1 \varepsilon_2^{-r - 2}) \).

By the exactly same argument as in \( \[61\] 6.1 \), we have \( Z_{\text{inst}}(\varepsilon_1, -2\varepsilon_1, \vec{a}; q, \vec{\tau}) = Z_{\text{inst}}(\varepsilon_1, -\varepsilon_1, \vec{a}; q, \vec{\tau}) \) up to \( O(\varepsilon_1 \varepsilon_2^{-r - 2}) \). In particular, \( H(\vec{a}; q, \vec{\tau}) \) up to \( O(\varepsilon_1 \varepsilon_2^{-r - 2}) \) comes only from the perturbative term:

\[
H(\vec{a}; q, \vec{\tau}) = \frac{1}{2} \sum_{\alpha < \beta} (a_\alpha - a_\beta) \log(-1) = \pi \sqrt{-1}(\vec{a}, \rho).
\]

See \( [156] \).

The first part of the following was proved by \( [61] \) and independently by \( [63] \).

**Theorem 5.7.** (1) \( F_0(\vec{a}; q, 0) \) is equal to the Seiberg-Witten prepotential \( F_0(\vec{a}; \Lambda) \) with \( q = \Lambda^{2r} \).

(2) For \( p = 2, \ldots, r \), let \( c_p \) be the \( p \)th power sum in \( z_1, \ldots, z_r \) multiplied by \( \left( -\frac{1}{p!} \right)^p \) given in \( \[2.12\] \). We have

\[
\left. \frac{\partial F_0}{\partial \tau_{r-1}} \right|_{\vec{\tau} = 0} = c_p.
\]

**Proof.** As the name of this subsection suggests, we prove the assertion by studying limit of blowup equations.
We have
\[ F_0(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t}) \]
\[ = \frac{1}{\varepsilon_1 \varepsilon_2} F_0 + \frac{\partial^2 F_0}{\partial \tau_p \partial \tau_q} t_p t_q \frac{1}{2} + \frac{\partial^2 F_0}{\partial \tau_p \partial a^l} t_p k^l + \frac{\partial^2 F_0}{\partial a^l \partial a^m} \frac{k^l k^m}{2} + \cdots, \]
\[ = \frac{\varepsilon_2 H(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t})}{\varepsilon_1 \varepsilon_2} + \frac{\varepsilon_1 H(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t})}{\varepsilon_1 \varepsilon_2} \]
\[ = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 \varepsilon_2} H + \frac{\partial H}{\partial \tau_p} t_p + \frac{\partial H}{\partial a^l} k^l \]
\[ = (\varepsilon_1 + \varepsilon_2) \frac{H + \pi \sqrt{-1} \langle \vec{k}, \rho \rangle}{\varepsilon_1 \varepsilon_2} + \cdots, \]
\[ \frac{\varepsilon_2^2 G(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t})}{\varepsilon_1 \varepsilon_2} + \frac{\varepsilon_1^2 G(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t})}{\varepsilon_1 \varepsilon_2} = (\varepsilon_1 + \varepsilon_2)^2 - \varepsilon_1 \varepsilon_2 G + \cdots, \]
\[ F_1(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t}) + F_1(\vec{a} + \varepsilon \vec{K}; \vec{q}, \varepsilon \vec{t}) = 2F_1 + \cdots. \]

Here and throughout the proof, \( F_0, H, G, F_1 \) and their derivatives in the right hand side are all restriction to \( \vec{t} = 0 \).

We divide both hand sides of (5.2) by \( Z(\varepsilon_1, \varepsilon_2; \vec{a}; \vec{t}) \), set \( \vec{t} = 0 \), and take limit \( \varepsilon_1, \varepsilon_2 \to 0 \):
\[ \sum_{\{k\}} \exp \left[ - \frac{\partial^2 F_0}{\partial \tau_p \partial \tau_q} t_p t_q \frac{1}{2} - \frac{\partial^2 F_0}{\partial \tau_p \partial a^l} t_p k^l - \frac{\partial^2 F_0}{\partial a^l \partial a^m} \frac{k^l k^m}{2} \right] \]
\[ \times (-1)^{\langle \vec{k}, \rho \rangle} \exp (F_1 - G) = 1 + O(\vec{t}^r), \]
where the summation symbol over \( p, q, l, m \) are omitted. Logically speaking, we only show \( F \) is regular up to \( O(\vec{t}^{2r-2}) \) at this moment. Thus the higher order terms may diverge in the limit \( \varepsilon_1 = \varepsilon_2 = 0 \). Therefore this equation should be understood that the left hand side is equal to 1 if we set all higher order terms to be zero.

Let
\[ \tau_{kl} = - \frac{1}{2 \pi \sqrt{-1}} \frac{\partial^2 F_0}{\partial a^k \partial a^l}. \]
This is symmetric and positive definite for \( q \) small. So we consider the corresponding theta function \( \Theta_E \) with the characteristic \( E = t^l \left( \frac{1}{2}, \frac{1}{2}, \cdots \right) \) in the notation for the root system of type \( A_{r-1} \).

Comparing the constant term of (5.8), we get
\[ \exp(G - F_1) = \Theta_E(0|\tau). \]
Comparing the coefficients of \( t_p t_q \) with \( p + q \leq 2r - 1 \), we get
\[ 0 = - \frac{\partial^2 F_0}{\partial \tau_p \partial \tau_q} + \frac{1}{2 \pi \sqrt{-1}} \sum_{l,m} \frac{\partial^2 F_0}{\partial \tau_p \partial a^l} \frac{\partial}{\partial \tau^l} \frac{\partial^2 F_0}{\partial a^m \partial \tau} \log \Theta_E(0|\tau). \]

By (5.8) we have \( \frac{\partial}{\partial \tau^l} F_0 = -q \frac{\partial}{\partial \tau} F_0 = -\frac{1}{2} \Lambda \frac{\partial}{\partial \Lambda} F_0 \) and \( \frac{\partial^2}{\partial \tau^l} F_0 = \frac{1}{2 \pi} \left( \Lambda \frac{\partial}{\partial \Lambda} \right)^2 F_0 \).
Therefore the equation with \( p = q = 1 \) is nothing but the contact term equation in Corollary 2.13.

When we consider the contact term equation as the differential equation for \( F_0 \), it has similar recursive structure as the blowup equation studied in (5.2). The coefficients of \( q^n \) in the instanton part of \( F_0 \) are determined from lower coefficients.
In particular, the solution is unique if the perturbative part is given. Since \( F_0 \) and
the Seiberg-Witten prepotential $F_0$ have the same perturbative part, we have (1) of Theorem 5.7.

Let us prove the second assertion. The case $p = 2$ is nothing but the renormalization group equation in Proposition 2.10. We substitute $p = p - 1, q = 2$ in (5.11):

$$\Lambda \frac{\partial}{\partial \Lambda} \left( \frac{\partial F_0}{\partial \tau_{p-1}} \right) = \frac{2r}{\pi \sqrt{-1}} \sum_{l,m} \frac{\partial u_2}{\partial a^m} \frac{\partial}{\partial \tau_l} \left( \frac{\partial F_0}{\partial \tau_{p-1}} \right) \frac{\partial}{\partial \tau_m} \log \Theta_E(0|\tau).$$

If we expand the both hand sides into the power series in $q$ (plus the perturbative part), the equation determines the coefficients recursively. The point here is the observation that $\frac{\partial}{\partial \tau_m} \log \Theta_E(0|\tau)$ is divisible by $q$. In particular, the solution is unique if the perturbative part is given. The perturbative part of $\frac{\partial F_0}{\partial \tau_{p-1}}$ is given by

$$[e^{a(a)}]_p = \sum_{\alpha} \frac{a^p}{p!}.$$ 

This is equal to the perturbative part of $c_p$. This shows our assertion. □

6. Fintushel-Stern's blowup formula

6.1. In this and next subsections we assume that derivatives of $F(\varepsilon_1, \varepsilon_2, \vec{a}; q, \vec{\tau})$ up to the second order are regular at $\varepsilon_1 = \varepsilon_2 = 0$, when restricted to $\vec{\tau} = 0$.

As we did in (5.8) we can derive the limit of the blowup formula (4.6) as

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \hat{Z}^{c_1=k}(\varepsilon_1, \varepsilon_2, \vec{a}; q, 0, \vec{t})$$

$$= \sum_{(\vec{k})=-k} \exp \left[ - \frac{\partial^2 F_0}{\partial \tau_p \partial \tau_q} t_p t_q - \frac{\partial^2 F_0}{\partial \tau_p \partial a^l} t_p k^l - \frac{\partial^2 F_0}{\partial a^l \partial a^m} k^l k^m \right]$$

$$\times (\sqrt{-1})^{2(\vec{k}, \vec{\rho})} \exp \left( F_1 - G \right),$$

where the summation symbol is omitted as before. And we restrict functions to the subspace $\vec{\tau} = 0$ also as before.

We define the contact term by

$$T_{p,q}(\vec{a}; q) = \frac{1}{2} \frac{\partial^2 F_0}{\partial \tau_p \partial \tau_q} (\vec{a}; q, 0).$$

We also set $t_r = t_{r+1} = \cdots = 0$. Then the right hand side of (6.1) is rewritten as

$$\exp \left( - \sum_{p,q=1}^{r-1} T_{p,q} t_p t_q \right) \frac{\Theta_{E_k} \left( \frac{\sqrt{-1}}{2\pi} \sum_{p=1}^{r-1} \frac{dc_{p+1}}{d\vec{a}} \bigg| t_p \bigg| \tau \right)}{\Theta_E(0|\tau)}$$

where

$$\frac{dc_{p+1}}{d\vec{a}} = t \left( \frac{\partial c_{p+1}}{\partial a^1} \ldots \frac{\partial c_{p+1}}{\partial a^{r-1}} \right),$$

and $E_k$ is the characteristic given in §B.1. We have used Theorem 5.7(2) and (5.10).

This is a generalization of (7.1). Note that this expression for $r = 2$ coincides with the blowup formula derived from the $u$-plane integral in (1.8). The identification of the contact term follows from (2.15).
For a physical derivation of a higher rank generalization, see [45].

6.2. A reformulation. We reformulate the blowup formula in the previous subsection in a form which does not involve the limit. It also provides an interpretation of Theorem 5.7(2) which does not involve the limit. The goal is to express the formula in $H^T_* \hat{\lambda}$ instead of $H^T_*$. (Recall $\hat{T} = \mathbb{C}^* \times \mathbb{C}^* \times T$.)

Let $j': H^T_* (M_0(r, n)) \to H^T_* (M_0(r, n))$ be the homomorphism given by the restriction of the action. This can be defined via the pull-back homomorphism with respect to a locally trivial fibration with fiber $\hat{T}/T$ $j: M_0(r, n) \times_T U \to M_0(r, n) \times_\hat{T} U$, where $U$ is a $\hat{T}$-variety as in [45].

Recall that we have made an identification

$$\prod_n \left( H^T_* (M_0(r, n)) \otimes S \right) \Lambda^{2r n} \cong \prod_n S \Lambda^{2r n}$$

via $(\iota_0)_*$. The multiplication of $\Lambda$ in the right hand side is identified with the push-forward homomorphism $i_{n, n+1}$ of the natural embedding $i_{n, n+1}: M_0(r, n) \to M_0(r, n + 1)$. The identification follows from the commutativity of the diagram

$$H^T_* (pt) \xrightarrow{\iota_{0*}} H^T_* (M_0(r, n)) \xrightarrow{i_n} H^T_* (M_0(r, n + 1)).$$

In particular, the multiplication of $\Lambda^{2r}$ makes sense as operators on $\prod_n H^T_* (M_0(r, n)) \Lambda^{2r n}$. It does makes sense also on $\prod_n H^T_* (M_0(r, n)) \Lambda^{2r n}$, and two homomorphisms commute with $j^!$.

We consider

$$Z_{\text{inst}} (\vec{a}; \Lambda) = \sum_n \Lambda^{2r n} \pi_* [M(r, n)],$$

$$\hat{Z}_{c_1 = k} (\vec{\alpha}; \Lambda, \vec{t}) = \sum_n \Lambda^{2r n} \pi_* \left[ \exp \left( \sum_{p=1}^{\infty} t_p \left( \text{ch}_{p+1} (\vec{E}) / [C] \right) \right) \cap [\bar{M}(r, k, n)] \right].$$

These are the formal sums of elements in $H^T_* (M_0(r, n))$. They are the pull-backs of the corresponding elements in $H^T_* (M_0(r, n))$ via $j^!$.

**Lemma 6.3.** Let $R = \{ f(\vec{a}, \varepsilon_1, \varepsilon_2) \mid f(\vec{a}, 0, 0) \neq 0 \}$. Then $H^T_* (M_0(r, n))_R := H^T_* (M_0(r, n)) \otimes_{S(T)} S(T)_R$ is a torsion free $S(\hat{T})_R$-module.

**Proof.** By the localization theorem, we have

$$H^T_* (M_0(r, n))_R \cong H^T_* (M_0(r, n))^T_R \cong H^C_* \times C^* (M_0(r, n)^T) \otimes_{\mathbb{C}} S(T)_R.$$ 

Since $M_0(r, n)^T = S^n \mathbb{C}^2, H^C_* \times C^* (M_0(r, n)^T)$ is a torsion free $\mathbb{C} [\varepsilon_1, \varepsilon_2]$-module. \qed

Therefore the blowup formula in the previous section can be restricted:

$$\hat{Z}_{c_1 = k} (\vec{a}; \Lambda, \vec{t}) = [E^2] \times Z_{\text{inst}} (\vec{a}; \Lambda).$$

This is an equality in the formal power series in $\Lambda^{2r}$ and $\vec{t}$ with values in $H^T_* (M_0(r, n)) \otimes_{S(T)} S(T)$. Let us emphasize again that the multiplication of $\Lambda^{2r}$ is $i_{n, n+1*}$. 


Recall $Z^\text{inst}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{r}) = \exp(F^\text{inst}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{r})/\varepsilon_1 \varepsilon_2)$. By $(6.4)$ we have

$$
\sum_{n=0}^\infty \Lambda^{2rn} \int_{M(r,n)} \prod_i (\text{ch}_{\mu_i+1}(\mathcal{E})/\{0\}) = \left( \prod_i \varepsilon_1 \varepsilon_2 \frac{\partial}{\partial \tau_{\mu_i}} \right) Z^\text{inst}(\varepsilon_1, \varepsilon_2, \bar{a}; \Lambda^{2r}, \bar{r})
$$

$$
= \left( \prod_i \frac{\partial F^\text{inst}}{\partial \tau_{\mu_i}}(\varepsilon_1, \varepsilon_2, \bar{a}; \Lambda^{2r}, 0) \right) Z^\text{inst}(\varepsilon_1, \varepsilon_2, \bar{a}; \Lambda^{2r}, 0)
$$

where $\{0\}$ is the fundamental class of the origin. We can make a restriction:

$$(6.5) \sum_{n=0}^\infty \Lambda^{2rn} \pi_* \left( \prod_i (\text{ch}_{\mu_i+1}(\mathcal{E})/\{0\}) \cap [M(r,n)] \right) = \left( \prod_i c_{\mu_i+1} \right) Z^\text{inst}(\bar{a}; \Lambda^{2r})$$

thanks to Theorem $(5.74)$, where we assume all $\mu_i \leq r-1$.

Note that this formula explains the meaning of $c_p = \partial F_0/\partial \tau_{p-1}$ without taking limit. It is just multiplication of $\text{ch}_p(\mathcal{E})/\{0\}$. Also it formally looks like $(1.1)$ when $r = 2, \mu_i = 1$.

We suppose

(1) The factor $(6.2)$ is in $\mathbb{C}[c_2, \ldots, c_r, \Lambda^{2r}]\{[t_1, \ldots, t_{r-1}]\}$.

Let us denote it by $B^{c_1=k}(\bar{c}, \Lambda, \bar{t})$. Note that this is a conjecture on theta functions. It seems that this was proved in $(16)$. But we do not quite check the detail. Then $(6.4)$ becomes

$$(6.6) \sum_{n=0}^\infty \Lambda^{2rn} \pi_* \left[ \exp \left( \sum_{p=1}^\infty t_p \left( \text{ch}_{p+1}(\mathcal{E})/\{C\} \right) \right) \cap [M(r,k,n)] \right]
$$

$$
= \sum_{n=0}^\infty \Lambda^{2rn} \pi_* \left( B^{c_1=k}(\text{ch}(\mathcal{E})/\{0\}, \Lambda, \bar{t}) \cap [M(r,n)] \right).
$$

We conjecture

(2) $(6.6)$ holds for moduli spaces $M_H(r,c_1,n)$, $\widetilde{M}_H(r,c_1+kC,n)$ for an arbitrary projective surface $X$.

Note also that $(6.5)$ explains the meaning Kronheimer-Mrowka’s simple type condition if we have the same formula for an arbitrary surface:

$$
\sum_{n=0}^\infty \Lambda^{4n} \pi_* \left( \left( \text{ch}_2(\mathcal{E})/\{0\} \right)^2 \cap [M_H(2,c_1,n)] \right) = 4\Lambda^2 \sum_{n=0}^\infty \Lambda^{4n} \pi_* [M_H(2,c_1,n)].
$$

It is equivalent to $c_2^2 = u_2^2 = 4\Lambda^2$. It means that the Seiberg-Witten curve is singular.

6.3. Rank 2 case. We assume $r = 2$. In this subsection we give an explicit expression for $B^{c_1=k}(u, \Lambda, t)$ in terms of Weierstrass $\sigma$-functions as in $(24)$. This is an exercise in elliptic functions and involves no geometry.

The regularity assumption made in the previous subsections is true as we consider the derivative with respect to $\tau_1$.

6.3.1. Weierstrass functions. Let $\wp(z)$ be the Weierstrass $\wp$-function with the period $\omega + \omega'$, where $\omega'/\omega = \tau, 3\tau > 0$. Then the associated elliptic curve is given by

$$
y^2 = 4(x-e_1)(x-e_2)(x-e_3)
$$

$$
= 4x^3 - g_2x - g_3.
$$
where \(e_1 = \varphi(\omega/2), e_2 = \varphi(-\omega/2 - \omega'/2)\) and \(e_3 = \varphi(\omega'/2)\).

Let \(\sigma(z)\) be the Weierstrass \(\sigma\)-function. We have the following expansion:
\[
\varphi(z) = \sum_{n \geq 0} c_n(g_2, g_3)z^{2n-2} = \frac{1}{z^2} + \frac{g_2}{2^2 \cdot 5} z^2 + \frac{g_3}{2^2 \cdot 7} z^4 + \cdots
\]
\[
\sigma(z) = \sum_{n \geq 0} c_n'(g_2, g_3)z^{2n+1} = z - \frac{g_2}{2^4 \cdot 3 \cdot 5} z^5 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} z^7 - \cdots
\]
where \(c_n, c'_n\) are weighted homogeneous polynomials of degree \(n\) with \(\text{deg} \ g_2 = 2\) and \(\text{deg} \ g_3 = 3\). Let \(\sigma_i(z), i = 1, 2, 3\) be three more sigma functions associated to \(e_i\).

We assign \(\text{deg} \ e_i = 1\). Since \(\sigma_i(z)^2 = \sigma(z)^2(\varphi(z) - e_i)\), \(\sigma_i(z)\) also has an expansion
\[
(6.7) \quad \sigma_i(z) = 1 - \frac{e_i}{2} z^2 + \sum_{n \geq 2} c''_n(e_i, g_2, g_3) z^{2n}
\]
where \(c''_n\) are weighted homogeneous polynomials of degree \(n\).

In terms of modular forms, we have the following expressions:
\[
\sigma(\omega z) = \omega e^{\omega \eta z^2} \frac{\theta_{11}(z|\tau)}{\theta_{11}(0|\tau)} = -\frac{\omega}{\pi \theta_{10}(0|\tau) \theta_{00}(0|\tau)} \frac{e^{\omega \eta z^2} \theta_{11}(z|\tau)}{\theta_{01}(0|\tau)}
\]
\[
(6.8) \quad \sigma_3(\omega z) = e^{\omega \eta z^2} \frac{\theta_{01}(z|\tau)}{\theta_{01}(0|\tau)},
\]
where \(\eta = \zeta(\omega/2)\) is given by
\[
\eta \omega = \frac{\pi^2}{6} E_2(\tau).
\]

We also have
\[
(6.9) \quad e_1 = \frac{1}{3} \left(\frac{\pi}{\omega}\right)^2 (\theta_{00}^1 + \theta_{01}^1),
\]
\[
e_2 = \frac{1}{3} \left(\frac{\pi}{\omega}\right)^2 (\theta_{10}^1 - \theta_{01}^1),
\]
\[
e_3 = -\frac{1}{3} \left(\frac{\pi}{\omega}\right)^2 (\theta_{10}^4 + \theta_{00}^4).
\]

\[
(6.10) \quad g_2 = -4 \left(\frac{\pi}{\omega}\right)^4 \left(-\frac{1}{3} (\theta_{10}^1 + \theta_{00}^1)^2 + (\theta_{10}^4 \theta_{00}^4)\right),
\]
\[
g_3 = \frac{1}{27} \left(\frac{\pi}{\omega}\right)^6 (\theta_{10}^1 + \theta_{00}^1)^4 (8(\theta_{10}^4 + \theta_{00}^4)^2 - 36(\theta_{10}^4 \theta_{00}^4)).
\]

6.3.2. We write \(u = u_2, a = a_2\) as in \(\text{III.A.}\) We set \(\omega = \pi \theta_{00} \theta_{10} / \Lambda\). Then
\[
g_2 = 4 \left(\frac{1}{3} u^2 - \Lambda^4\right)
\]
\[
g_3 = -\frac{1}{27} u (8u^2 - 36\Lambda^4)
\]
\[
e_3 = \frac{u}{3}.
\]
Hence we get
\[ e^{-T_{1,1} t^2} \frac{\theta_{01}(\frac{\sqrt{-1}}{2\pi} \frac{dt}{dz} | \tau)}{\theta_{01}(0 | \tau)} = e^{\frac{\pi}{4} i \sigma_3} c(t), \]
\[ e^{-T_{1,1} t^2} \frac{\theta_{11}(\frac{\sqrt{-1}}{2\pi} \frac{dt}{dz} | \tau)}{\theta_{01}(0 | \tau)} = e^{\frac{\pi}{4} i \sigma_3} (t) \Lambda. \]

This checks the conjecture (1) in the previous subsection.

Since \( \text{ch}_2(E) = -c_2(E) \), we put \( x = -u \). Then \( x \) corresponds to the insertion of the point class \( \mu(p) = -c_2(E)/[0] \) in (6.6). The above (6.11) exactly coincides with the functions \( B(x, t), S(x, t) \) appeared in Fintushel-Stern's blowup formula for Donaldson invariants [24].

This checks the conjecture (2) in a weak sense, i.e., if we multiply products of \( \mu(S) \) and integrate, then the equality holds. Conversely if we can prove (2) directly, it gives a new proof of Fintushel-Stern's blowup formula. The proof of (2) probably requires more detailed study of the map \( \tilde{\pi} \).

### 7. Gravitational corrections

As we mentioned in Introduction, Nekrasov asserts that higher order terms in (5.6) are gravitational corrections to the gauge theory [62 §4]. In some cases these are some known quantities, which are really related genus \( g \) curves, e.g., Gromov-Witten invariants with domain genus \( g \). We are still far away from verifying this conjecture in full generality. But we have some nontrivial examples, which we review in this section.

#### 7.1. Genus 1 part.

The result of this subsection is based on discussions with N. Nekrasov.

We determine the coefficients \( G, F_1 \) in the expansion (5.6) for \( \tilde{\tau} = 0 \). These terms are considered as genus 1 gravitational corrections as we said. Since we are only interested in \( \tilde{\tau} = 0 \) case, we omit \( \tilde{\tau} \) from the notation.

We consider the blowup equation (5.3) for \( c_1 = kC \ (k \neq 0) \) with \( \tilde{\tau} = \tilde{t} = 0 \):
\[ 0 = \sum_{\{\tilde{k}\} = -\frac{1}{2}} Z(\varepsilon_1, \varepsilon_2, \ddbar{a} + \varepsilon_1 \ddbar{k}; q) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \ddbar{a} + \varepsilon_2 \ddbar{k}; q). \]

As in the derivation of (5.8) we have
\[ 0 = \sum_{\{\tilde{k}\} = -\frac{1}{2}} \exp \left[ - \frac{\partial^2 F_0}{\partial a^l \partial a^m} k^l k^m}{2} + \partial H \frac{\partial a^l}{\partial a^i} k^l + (\varepsilon_1 + \varepsilon_2) \left\{ - \frac{\partial^3 F_0}{\partial a^l \partial a^m \partial a^n} \frac{k^l k^m k^n}{3!} + \frac{\partial (G + F_1)}{\partial a^l} k^l \right\} + \cdots \right] \]
\[ = \sum_{\{\tilde{k}\} = -\frac{1}{2}} \sqrt{-1}^{2(k, \psi)} \exp \left[ \pi \sqrt{-1} \tau_{lm} k^l k^m \right. \]
\[ + (\varepsilon_1 + \varepsilon_2) \left\{ - \frac{\partial^3 F_0}{\partial a^l \partial a^m \partial a^n} \frac{k^l k^m k^n}{3!} + \frac{\partial (G + F_1)}{\partial a^l} k^l \right\} + \cdots \],

where \( \tau_{lm} \) is the period of the Seiberg-Witten curve as before (4.6).
Setting \( \varepsilon_1 = \varepsilon_2 = 0 \), we get
\[
0 = \sum_{\{k\}=-\frac{1}{r}} \sqrt{-1}^{2\langle \vec{k}, \rho \rangle} \exp \left( \pi \sqrt{-1} \tau_{lm} k^l k^m \right) \Theta_{E_{k}}(\vec{0}|\tau).
\]

Next we take the coefficient of \( \varepsilon_1 + \varepsilon_2 \) in the above to get
\[
0 = \sum_{\{k\}=-\frac{1}{r}} \sqrt{-1}^{2\langle \vec{k}, \rho \rangle} \exp \left[ \pi \sqrt{-1} \tau_{lm} k^l k^m \right] \times \left\{ 2\pi \sqrt{-1} \frac{\partial \tau_{mn}}{\partial a^l} \frac{\partial^2 (G + F_1)}{\partial a^l \partial \xi^l} \right\},
\]
i.e.,
\[
\sum_{l} \frac{\partial (G + F_1)}{\partial a^l} \frac{\partial}{\partial \xi^l} \Theta_{E_{k}}(\vec{0}|\tau) + \frac{1}{3} \frac{\partial^2 (G + F_1)}{\partial a^l \partial \xi^l} \Theta_{E_{k}}(\vec{0}|\tau) = 0.
\]

We believe this equation with \( k = 1, \ldots, r - 1 \) determine \( \frac{\partial (G + F_1)}{\partial a^l} \) for \( l = 2, \ldots, r \).

But we do not know the required identities for the theta functions, as far as the authors are concerned. So we assume \( r = 2 \) from now. Then the equation is
\[
(7.1) \quad \frac{\partial (G + F_1)}{\partial a} = -\frac{1}{3} \frac{\partial}{\partial a} \log \left( \frac{\partial}{\partial \xi} \Theta_{E_{1}}(\vec{0}|\tau) \right).
\]

We now switch to the notation in \[ 13, \text{§2} \] From \[ 10, \text{§1} \], we get
\[
\exp (G - F_1) = \theta_{01}(0, \tau), \quad \exp (G + F_1) = C \theta'_{11}(0, \tau)^{-1/3}
\]
for some constant \( C \) independent of \( a \).

Therefore
\[
\exp (2F_1) = C' \theta'_{11}(0, \tau)^{-1/3} \theta_{01}(0, \tau)^{-1}.
\]

By Jacobi’s triple product identity (see e.g., \[ 57 \] Chap. I, §14] we have
\[
\theta_{01}(0, \tau) = \prod_{d=1}^{\infty} \left[ (1 - q^{2d})(1 - q^{2d-1})^2 \right].
\]

By Jacobi’s derivative formula (see \[ 57 \] Chap. I, §13] we have
\[
\theta'_{11}(0, \tau) = -2\pi q^{\frac{1}{2}} \prod_{d=1}^{\infty} (1 - q^{2d})^3.
\]

Therefore we get
\[
\exp F_1 = C' q^{-\frac{1}{2}} \prod_{d=1}^{\infty} (1 - q^d)^{-1} = \frac{C'}{\eta(\frac{\tau}{2})}
\]
for some constant \( C' \) independent of \( a \). A priori, \( C' \) may depend on \( q \) (or \( \Lambda \)), but in fact, it does not as follows. Let us define degrees of variables by
\[
\deg \varepsilon_1 = \deg \varepsilon_2 = \deg a_\alpha = 1, \quad \deg q = 2r \quad (\deg \Lambda = 1).
\]

(This definition applies for arbitrary \( r \), not necessarily 2.) By the definition, \( Z \) has degree 0. Therefore \( F_0 \) has degree 2, while \( G \) and \( F_1 \) have degree 0. Then \( \tau \) has degree 0, and hence so is \( \eta(\frac{\tau}{2}) \). Therefore \( C' \) has degree 0. Since it is independent
of $a$, it means that it is also independent of $q$ (or $\Lambda$). Therefore $C'$ can be computed by studying the expansions of $F_1$ and $\eta$ in $a/\Lambda$:

$$F_1 = \frac{1}{12} \left\{ \log \left( \frac{2a}{\Lambda} \right) + \log \left( -\frac{2a}{\Lambda} \right) \right\} + \cdots = \frac{1}{6} \log \left( \frac{2\sqrt{-1}a}{\Lambda} \right) + \cdots,$$

$$\log \frac{1}{\eta(\tau)} = -\frac{1}{24} \log q - \sum_{d=1}^{\infty} \log(1 - q^d) = -\frac{1}{24} \pi \sqrt{-1} \tau + \cdots.$$

Furthermore, we have

$$\tau = \sqrt{-1} \pi \log \left( \frac{2\sqrt{-1}a}{\Lambda} \right) + \cdots.$$

Therefore we get

$$C' = 1,$$

$$F_1 = -\log \eta(\tau), \quad G = \log \left[ q^{-\frac{1}{24}} \prod_{d=1}^{\infty} (1 - q^{2d-1}) \right].$$

It is better to make the following combination:

$$F_1 = A - \frac{2}{3} B, \quad G = \frac{1}{3} B.$$

Then

$$\exp A = \exp (F_1 + 2G) = q^{-\frac{1}{24}} \prod_{d=1}^{\infty} \frac{1 - q^{2d-1}}{1 - q^{2d}} = \left( -\frac{2\pi \theta_{01}}{\theta'_{11}} \right)^{\frac{1}{2}} = \left( \frac{2}{\theta_{00} \theta_{10}} \right)^{\frac{1}{2}},$$

$$\exp B = \exp (3G) = q^{-\frac{1}{24}} \prod_{d=1}^{\infty} (1 - q^{2d-1})^3 = \left( -\frac{2\pi \theta_{01}^3}{\theta'_{11}^3} \right)^{\frac{1}{2}} = \left( \frac{2\theta^2_{01}}{\theta_{00} \theta_{10}} \right)^{\frac{1}{2}}.$$

Comparing with (1.3), we find

$$\exp A = \left( \sqrt{-1} \frac{du}{da} \right)^{\frac{1}{2}}, \quad \exp B = \left( \frac{4(u^2 - 4\Lambda^4)}{\Lambda^4} \right)^{\frac{1}{2}}.$$

Note that the last expression is given by the quantum discriminant (2.1):

$$\Delta = 2^{12} \Lambda^8 (u^2 - 4\Lambda^4).$$

Therefore

$$\epsilon_1 \epsilon_2 F_1 + (\epsilon_1 + \epsilon_2)^2 G = \epsilon_1 \epsilon_2 \log \left( \sqrt{-1} \frac{du}{da} \right)^{\frac{1}{2}} + \frac{\epsilon_1^2 + \epsilon_2^2}{3} \log \left( \frac{\Delta}{2^{10} \Lambda^4} \right)^{\frac{1}{2}}.$$

Comparing with (1.4), this suggests the following formula for the equivariant Euler number and signature for $\mathbb{C}^2$:

$$\chi(\mathbb{C}^2) = \epsilon_1 \epsilon_2, \quad \sigma(\mathbb{C}^2) = \frac{\epsilon_1^2 + \epsilon_2^2}{3}.$$

This is natural from the following formal computation:

$$\chi(\mathbb{C}^2) = c_2(\mathbb{C}^2), \quad \sigma(\mathbb{C}^2) = \frac{1}{3} \left( c_1(\mathbb{C}^2)^2 - 2c_2(\mathbb{C}^2) \right),$$

$$c_1(\mathbb{C}^2) = \epsilon_1 + \epsilon_2, \quad c_2(\mathbb{C}^2) = \epsilon_1 \epsilon_2.$$
Nekrasov conjectures that (7.2) holds higher rank case also if we replace \( \frac{\partial}{\partial a} \) by \( \det(\frac{\partial}{\partial a}) \).

### 7.2. Coordinate rings of symmetric products

The next example is related to the perturbative part of the \( K \)-theory version of the partition function in \( \{1\} \) (See [63 A.0.3].) It fits with geometric engineering quite well (see below), but it looks like an accident if we understand it as a purely mathematical statement. The authors learned the result from [63 §A.0.3].

Let us consider the \( n \)th symmetric product \( S^n(\mathbb{C}^2) \) of the affine plane \( \mathbb{C}^2 \). We define an action of the two torus \( T^2 = \mathbb{C}^* \times \mathbb{C}^* \) on \( \mathbb{C}^2 \) by

\[
(x, y) \mapsto (t_1 x, t_2 y), \quad (t_1, t_2) \in T^2.
\]

We also have an induced action on \( S^n(\mathbb{C}^2) \).

The coordinate ring \( H^0(S^n(\mathbb{C}^2), \mathcal{O}) \), that is the ring of polynomial functions on \( S^n(\mathbb{C}^2) \), is a \( T^2 \)-module. We consider its character

\[
\mathrm{ch} H^0(S^n(\mathbb{C}^2), \mathcal{O}) = \sum_{m,n \geq 0} t_1^m t_2^n \dim H^0(S^n(\mathbb{C}^2), \mathcal{O})_{m,n},
\]

where

\[
H^0(S^n(\mathbb{C}^2), \mathcal{O})_{m,n} = \{ f \in H^0(S^n(\mathbb{C}^2), \mathcal{O}) \mid (t_1, t_2) \cdot f = t_1^m t_2^n f \}
\]

is a simultaneous eigenspace, i.e., a weight space. The character is called Hilbert series sometimes. It is standard in algebraic geometry to show that

1. each weight space is finite-dimensional, and hence the character is well-defined as a formal sum,
2. the character is a rational function in \( t_1^\pm, t_2^\pm \).

In fact, in this case, an explicit answer can be written down:

**Proposition 7.3.** The generating function of the character is given by

\[
(7.4) \quad \sum_{n=0}^{\infty} q^n \mathrm{ch} H^0(S^n(\mathbb{C}^2), \mathcal{O}) = \exp \left( \sum_{d=1}^{\infty} \frac{q^d}{d(1-t_1^d)(1-t_2^d)} \right).
\]

We refer [61 §3] for the proof. But we recommend the reader to write down the proof by himself/herself since it is a nice exercise on a treatment of generating functions.

The generating function (7.4) is the rank 1 version of Nekrasov’s deformed partition function (for the \( K \)-theory version). Let us expand it into a formal power series in \( \hbar \), after putting \( t_1 = \exp \hbar, t_2 = \exp(-\hbar) \):

\[
(7.5) \quad \log \left( \sum_{n=0}^{\infty} q^n \mathrm{ch} H^0(S^n(\mathbb{C}^2), \mathcal{O}) \right) \bigg|_{t_1=e^\hbar, t_2=e^{-\hbar}} = \sum_{d=1}^{\infty} \frac{q^d}{d(1-e^{\hbar d})(1-e^{-\hbar d})}
\]

\[
= -\sum_{d=1}^{\infty} \frac{q^d}{d^3 \hbar^{-2}} - \frac{1}{12} \log (1-q) \hbar^0 + \sum_{g \geq 2} \frac{B_{2g}}{2g(2g-2)!} \sum_{d=1}^{\infty} d^{2g-3} q^d \hbar^{2g-2},
\]

where \( B_{2g} \) is the \( 2g \)th Bernoulli number as in [12]. The series start with \( \hbar^{-2} \) and have only even powers of \( \hbar \). In the next subsection we will see that (7.5) is equal to the generating function of certain Gromov-Witten invariants. Then \( q \) will be identified with the genus of the domain curve, and hence \( 2 - 2g \) with the Euler number.
7.3. Gromov-Witten invariants of the resolved conifold. Let $X$ be the total space of the rank 2 vector bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. This space is called the resolved conifold in physics. The local Gromov-Witten invariants for target $X$ is defined as follows: Let $M_{g,n}(\mathbb{P}^1, d)$ be the moduli space of stable maps for target $\mathbb{P}^1$ from a genus $g$ curve with $n$ marked points with degree $d$. We consider $d > 0$ case only, that is the stable map is not constant. We have the diagram

$$
M_{g,0}(\mathbb{P}^1, d) \xleftarrow{\text{forget}} M_{g,1}(\mathbb{P}^1, d) \xrightarrow{\text{eval}} \mathbb{P}^1,
$$

where forget is the map given by forgetting the marked point, and eval is the map given by taking the image of the marked point under the stable map. We consider a vector bundle $E = R^1\text{forget}^*\text{eval}^*E$. Let $c_{\text{top}}(E)$ be its top Chern class. Then the local Gromov-Witten invariant is defined by

$$
C(g,d) = \int_{M_{g,0}(\mathbb{P}^1, d)^{\text{vir}}} c_{\text{top}}(E),
$$

where $M_{g,0}(\mathbb{P}^1, d)^{\text{vir}}$ is the virtual fundamental class. This is a rational number. This is a local contribution to the global Gromov-Witten invariant of a Calabi-Yau 3-fold of multiple covers of a fixed rational curve with the normal bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Thus the formula for $C(g,d)$ is important in the Gromov-Witten theory. The genus $g = 0$ case is known as the Aspinwall-Morrison formula. The complete answer is given by

**Theorem 7.6** ([2, 52, 69] for $g = 0$, [32] for $g = 1$, [21] for $g \geq 2$).

$$
C(0, d) = \frac{1}{d^3}, \quad C(1, d) = \frac{1}{12d}, \quad C(g,d) = \frac{(-1)^{g-1}B_{2g}d^{2g-3}}{2g(2g-2)!}.
$$

Comparing with (7.5), we get

$$
\log \left( \sum_{n=0}^{\infty} q^n \text{ch} H^0(S^n(\mathbb{C}^2), \mathcal{O}) \right) \bigg|_{t_1=e^h, t_2=e^{-h}} = \sum_{d=1}^{\infty} \sum_{g=0}^{\infty} C(g,d)q^d(ih)^{2g-2}.
$$

This is our first example of the assertion that the gauge theory partition function is identified with generating functions of Gromov-Witten invariants.

It is worthwhile mentioning that the exponential of the right hand side is the generating function of Gromov-Witten invariants whose domain curves are not necessarily connected. This does not make sense in the gauge theory side, but somehow related to the recursive structure among the symmetric products $S^n\mathbb{C}^2$ for various $n$.

7.4. The $r = 1$ case and Gromov-Witten invariants for $\mathbb{P}^1$. In the main body of the paper, the case $r = 1$ was excluded as the Seiberg-Witten geometry does not make sense. However Nekrasov’s partition function does make sense $r = 1$ also. The $K$-theory version with $\vartheta = 0$ is what we already saw in (7.2). This is because the moduli space $M(1, n)$ is nothing but the Hilbert scheme of points, and it is known that the higher direct image sheaves for $M(1, n) \to S^n\mathbb{C}^2$ vanish and $\text{ch} H^0(S^n\mathbb{C}^2, \mathcal{O})$ can be given by Atiyah-Bott formula for $M(1, n)$. (See [61, §3] for more detail.) When $\vartheta \neq 0$, it was studied in [34]. (His main result is a positivity property, which is not studied in this paper. But it is natural to conjecture a similar property for higher rank cases also.)
The homology version of the partition function, i.e., our $Z(\varepsilon_1, \varepsilon_2; q, \vec{\tau})$, has the presentation by the Fock space when it is restricted to $\varepsilon_2 = -\varepsilon_1$. Then comparing with the presentation for the Gromov-Witten invariants for $\mathbb{P}^1$ [64], one gets

$$\log Z(\varepsilon_1, -\varepsilon_1; q, \vec{\tau}) = \text{the generating function of the Gromov-Witten invariants for } \mathbb{P}^1,$$

where $\varepsilon_1$ is mapped to an indeterminate for the domain genus, and $\vec{\tau}$ to those for gravitational descendants. This remarkable observation was done by [43] and [44] independently. We refer the precise statement and the proof to the original papers.

### 7.5. Geometric Engineering

The geometric engineering of Katz-Klemm-Vafa [36] realizes 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories as limits of type IIA string theory compactified on certain noncompact Calabi-Yau 3-folds. Mathematically it poses the following conjecture:

\begin{equation}
\text{(7.8)}
\end{equation}

Partition functions in 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are equal to limits of generating functions of local Gromov-Witten invariants for certain noncompact Calabi-Yau 3-folds.

The noncompact Calabi-Yau 3-fold is chosen according to the gauge theory to realize. Typically it is an ALE space fibration over $\mathbb{P}^1$. Recall that an ALE space is the minimal resolution of a simple singularity and contains a configuration of $\mathbb{P}^1$'s intersecting as one of ADE Dynkin diagrams. The group of the gauge theory is the corresponding ADE group.

This statement is rather striking since it seems difficult to compare two types of moduli spaces directly, i.e., moduli spaces of stable maps and instanton moduli spaces. Moreover, we must sum up over degrees for Gromov-Witten invariants as we will see below.

For simplicity, we restrict ourselves to rank 2 case. (In higher rank cases, the above naive definition of the local Gromov-Witten invariants must be modified as the base space become singular.) As we have already seen in [7.8], the $K$-theory version of the partition function is more natural here as we do not need to take a limit. We define

$$Z_K(h,a;\beta) = \exp \left[-\gamma^K_h(2a;\beta) - \gamma^K_h(-2a;\beta) \right] Z_{K}^{\text{inst}}(h,a;\beta),$$

$$Z_{K}^{\text{inst}}(h,a;\beta) = \sum_{n=0}^{\infty} \beta^{2rn} \sum_{i=1}^{i} (-1)^i \text{ch} H^i(M(r,n),\mathcal{O}),$$

$$t_1 = e^{\beta h}, \quad t_2 = e^{-\beta h}, \quad e_1 = e^{-\beta a}, \quad e_2 = e^{\beta a},$$

where $\text{ch}$ is the character of $\tilde{T}$-module and the one-dimensional $\tilde{T}$-modules (in the notation [33]) are replaced as indicated. The characters of the cohomologies have expressions in terms of Young diagrams $\hat{Y}$ by the localization formula for the $K$-theory:

$$Z_{K}^{\text{inst}}(h,a;\beta) = \sum_{\hat{Y}} \beta^{2r|\hat{Y}|}$$

(See [61] §3 for detail.) The perturbative term is given by

$$\gamma^K_h(x;\beta) = \sum_{d=1}^{\infty} \frac{e^{-\beta dx}}{d(1-e^{hd})(1-e^{-hd})}.$$ 

This is equal to the one in [63] §A.0.3] up to polynomials in $a$. 
Note that we do not include higher Casimir operators. This is not for the brevity. We (at least the authors) do not know what are counterparts in the Gromov-Witten theory.

In the rank 2 case, the noncompact Calabi-Yau 3-fold is supposed to be the canonical bundle $K_{F_n}$ of the Hirzebruch surface $F_n$. And it is conjectured that the Gromov-Witten invariants are essentially independent of $n$. So we further restrict to the case $n = 0$, i.e., the $K_{F_0}$ of $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$. (For the actual calculation, it is necessary to assume $F_n$ is a toric variety in order to apply the localization technique.)

We define the local Gromov-Witten invariants as in the case of the resolved conifold. The degree of maps is a pair of integers $(n,d)$ corresponding to the base and fiber respectively. (Although we have a symmetry exchanging two factors, we break it and consider one is base and the other is fiber. This is automatic for other $F_n$.) Let us introduce two parameters $q_b, q_f$ respectively.

Now the geometric engineering asserts that $\log Z_K(\hbar, a; \beta)$ is equal to the generating function of Gromov-Witten invariants, under a suitable identification of parameters. The parameter $\hbar$ should count the genus of domain curves as in §7.3.

Next we match the degree $n$ for the base with the instanton number $n$. Although this identification is quite natural, we do not have any mathematically rigorous justification of this statement. Anyway we should identify $q_b$ with $\beta^{2r}$. In fact, the analysis below gives the exact answer:

$$q_b = \left(\frac{\beta^2}{2}\right)^{2r}.$$

Let us consider the Gromov-Witten invariants for $n = 0$. In this case, we only have multiple covers of the fiber. It is given by \ref{eq:7.6} multiplied by $-2$. Since $n = 0$ means zero instanton number in the gauge theory side, it should be equal to the perturbation term of the gauge theory, namely:

$$-\gamma \left(2a|\beta\right) - \gamma \left(-2a|\beta\right) \equiv -2 \sum_{d=1}^{\infty} \frac{q_f^d}{d(1-e^{hd})(1-e^{-hd})}.$$

Since the left hand side is equal to $-2\gamma \left(2a|\beta\right)$ up to a polynomial in $a$, thanks to the inversion formula for the polylogarithms, this (up to a polynomial in $a$) follows from what we observed in \ref{eq:7.3} when we equate the parameters as

$$q_f = e^{-2\beta a}.$$

Note that for this identification, we must sum up the Gromov-Witten invariants for various degrees on fibers (and various genus). This is also true for $n$-instanton corrections.

For $n > 0$, the genus 0 Gromov-Witten invariants were calculated using the local mirror symmetry for $X = K_{F_0}$ in \ref{36}. And the limit of their generating function was identified with the Seiberg-Witten prepotential $F_0$. In fact, they identify the limit of the local mirror of $X$ with the Seiberg-Witten curve. Note that genus 0 case is enough to identify the parameters.

Recently Iqbal+Kashani-Poor identify $\log Z_K$ with the generating function of all genus Gromov-Witten invariants by using the large $N$ duality \ref{35}. In fact, they identify the expression of $\log Z_K$ via Young diagrams $\mathbf{Y}$ with Jones-Witten invariants for the Hopf link. (They assume certain combinatorial identities which are proved in more recent paper \ref{19}.)
Appendix A. The root system of type $A_{r-1}$

Let $Q$ be the coroot lattice of type $A_{r-1}$:

$$Q = \left\{ \vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r \mid \sum_\alpha k_\alpha = 0 \right\}.$$ 

We take simple coroots

$$\alpha_i^\vee = (0, \ldots, 0, 1, -1, 0, \ldots, 0), \quad (i = 1, \ldots, r - 1).$$

We can write

$$Q \ni \vec{k} = \sum_i k_i \alpha_i^\vee.$$ 

For a given $k \in \mathbb{Z}$, elements $\vec{k} \in \mathbb{Z}^r$ with $\sum_\alpha k_\alpha = k$ are identified

$$\{ \vec{l} = (l_1, \ldots, l_r) \in \mathbb{Q}^r \mid \sum_\alpha l_\alpha = 0, \forall \alpha \ l_\alpha \equiv \pm \frac{k}{r} \mod \mathbb{Z} \}.$$ 

This is a subset of the coweight lattice $P = \{ \vec{l} = (l_1, \ldots, l_r) \in \mathbb{Q}^r \mid \sum_\alpha l_\alpha = 0, \exists k \in \mathbb{Z} \ \forall \alpha \ l_\alpha \equiv \pm \frac{k}{r} \mod \mathbb{Z} \}$. There exists a homomorphism $P \rightarrow \mathbb{Z}/r\mathbb{Z}$ by taking the fractional part of $l_\alpha$. It can be identified with the natural quotient homomorphism $P \rightarrow P/Q$. We denote it by $\vec{l} \mapsto \{ \vec{l} \}$. Hereafter we identify $\vec{l}$ with $\vec{k}$ and denote both by $\vec{k}$. We write $\vec{k} = \sum k_i \alpha_i^\vee$ in either case $k = 0 \neq 0$. But $k^i$ may be rational in the latter case. Let $(\ , \ )$ be the standard inner product on $P$. The Killing form $B_{SU(r)}$ of $SU(r)$ satisfies $B_{SU(r)} = 2r(\ , \ )$. The following formulas are useful later:

$$\frac{1}{2r} \sum_{\alpha, \beta} (k_\alpha - k_\beta)(a_\alpha - a_\beta) = \langle \vec{k}, \vec{a} \rangle = \sum_{i,j} C_{ij} a^i k^j,$$

$$\frac{1}{2r} \sum_{\alpha, \beta} (k_\alpha - k_\beta)^2 = \langle \vec{k}, \vec{k} \rangle = \sum_{i,j} C_{ij} k^i k^j,$$

$$\sum_{\alpha < \beta} \frac{k_\alpha - k_\beta}{2} = \langle \vec{k}, \rho \rangle = \sum_i k^i.$$ 

Here $C_{ij}$ is the Cartan matrix, and $\rho$ is the half of the sum of positive roots, as usual.

Appendix B. Theta functions

We give definitions and some properties of Riemann theta functions.

B.1. Riemann Theta functions. Let $Q = \mathbb{Z}^2$. Let $\tau = (\tau_{\alpha \beta})$ be a symmetric $g \times g$ complex matrix whose imaginary part is positive definite. For $\vec{\mu}, \vec{\nu} \in \mathbb{C}^g$, we define the theta function with characteristic $[\vec{\mu}, \vec{\nu}]$ by

$$\Theta \left[ \vec{\mu} \bigg| \vec{\nu} \right] (\xi | \tau) = \sum_{\vec{k} \in Q} \exp \left( \pi \sqrt{-1} \sum_{\alpha, \beta} \tau_{\alpha \beta}(k_\alpha + \mu_\alpha)(k_\beta + \mu_\beta) + 2\pi \sqrt{-1} \sum_\alpha (k_\alpha + \mu_\alpha)(\xi_\alpha + \nu_\alpha) \right).$$

When $[\vec{\mu}, \vec{\nu}] = [\vec{\mu}, \vec{\nu}]$, we simply denote it by $\Theta$. We have

$$\Theta \left[ \vec{\mu} \bigg| \vec{\nu} \right] (\xi | \tau) = \exp \left( \pi \sqrt{-1} \vec{\mu} \tau \vec{\mu} + 2\pi \sqrt{-1} \vec{\mu} (\vec{\xi} + \vec{\nu}) \right) \Theta(\vec{\xi} + \tau \vec{\mu} + \vec{\nu} | \tau).$$
The theta function is quasi-periodic with respect to the lattice \( \mathbb{Z}^g \oplus \tau \mathbb{Z}^g \). It satisfies the heat equation:

\[
\frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} \Theta \left[ \vec{\mu}, \vec{\nu} \right] (\xi | \tau) = 4\pi \sqrt{-1} \frac{\partial}{\partial \tau_{\alpha \beta}} \Theta \left[ \vec{\mu}, \vec{\nu} \right] (\xi | \tau).
\]

When \( \vec{\mu}, \vec{\nu} \in \frac{1}{2} \mathbb{Z}^g \), \( \left[ \vec{\mu}, \vec{\nu} \right] \) is called a half-integer characteristic. The set of half-integer characteristics are divided into two, odd or even, according to whether \( \Theta \left[ \vec{\mu}, \vec{\nu} \right] (\xi | \tau) \) is an odd or even function.

In the main body of the paper, we use the \( A_{r-1} \)-lattice

\[
\left\{ \vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r \mid \sum_{\alpha} k_\alpha = 0 \right\}.
\]

(See \[A\]) This is identified with \( \mathbb{Z}^{r-1} \) by taking \( (k_2, \ldots, k_r) \). Then we apply the above convention, i.e., the suffix runs \( \alpha = 2, \ldots, r \).

A theta function with a particular half-integer even characteristic appears often in this paper:

\[
\Theta_E(\xi | \tau) = \Theta \left[ \vec{0}, \vec{\nu} \right] (\xi | \tau), \quad \nu_2 = \frac{1}{2}, \nu_3 = 0, \nu_4 = \frac{1}{2}, \nu_5 = 0, \ldots.
\]

We denote this characteristic by \( E \) and the corresponding theta function by \( \Theta_E \).

We also use the notation for the root system of Lie algebra of type \( A_{r-1} \). Then

\[
\frac{1}{2} k_2 + \frac{1}{2} k_4 + \cdots \equiv -\frac{1}{2} k_2 - \frac{2}{2} k_3 - \cdots - \frac{r-1}{2} k_r
\]

\[
= \frac{r-1}{4} k_1 + \frac{r-3}{4} k_2 + \cdots + \frac{1-r}{4} k_r = \sum_{\alpha < \beta} \frac{k_\alpha - k_\beta}{4} = \frac{1}{2} \sum_i k_i,
\]

where \( \equiv \) means the equality modulo \( \mathbb{Z}^{r-1} \). The last equality is \( (A.1) \). Therefore the characteristic is

\[
\left( \frac{1}{2}, \frac{1}{2}, \ldots \right)
\]

in this notation.

We also use the theta function where the summation range is replaced by

\[
\left\{ \vec{k} = (k_1, \ldots, k_r) \in \mathbb{Q}^r \mid \sum_{\alpha} k_\alpha = 0, \forall \alpha, l_\alpha \equiv -\frac{k_\alpha}{r} \text{ mod } \mathbb{Z} \right\}
\]

for a fixed \( k \in \mathbb{Z} \). It is \( \Theta \left[ \vec{\alpha}_k \right] \) with \( \vec{\alpha}_k = \frac{k}{r} \left( 1, 2, \cdots, r-1 \right) \). We denote by \( E_k \) this characteristic.

**B.2.** When \( g = 1 \), we use the following notation for the theta functions:

\[
\theta_{00}(z, \tau) = \sum_{n \in \mathbb{Z}} q^n w^{2n}, \quad \theta_{01}(z, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^n w^{2n},
\]

\[
\theta_{10}(z, \tau) = \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2} w^{2n+1}, \quad \theta_{11}(z, \tau) = \sqrt{-1} \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+\frac{1}{2})^2} w^{2n+1},
\]

where

\[
q = \exp(\pi \sqrt{-1} \tau), \quad w = \exp(\pi \sqrt{-1} z).
\]

This is the same as \[57\]. We have

\[
\Theta_E = \theta_{01}, \quad \Theta_{E_1} = \theta_{11}.
\]
B.3. Riemann surfaces and theta functions. Let $C$ be a compact Riemann surface of genus $g$. Let $K_C$ be its canonical bundle. We choose and fix a symplectic basis $A_1, \ldots, A_g, B_1, \ldots, B_g$ of $H_1(C, \mathbb{Z})$ so that $A_a \cdot A_b = B_a \cdot B_b$, $A_a \cdot B_\beta = \delta_{a\beta}$ for $\alpha, \beta = 1, \ldots, g$. We then have a basis $\omega_1, \ldots, \omega_g$ of holomorphic differentials $H^0(C, K_C)$ such that $\int_{A_a} \omega_\beta = \delta_{a\beta}$. The period matrix of $C$ is defined by

$$\tau_{\alpha\beta} = \int_{B_\beta} \omega_\alpha.$$ 

It is symmetric and its imaginary part is positive-definite.

Using the period matrix $(\tau_{\alpha\beta})$ of the Riemann surface $C$, we consider the associated theta function $\Theta [\vec{\beta}](\vec{\xi}|\tau)$ as in the previous section. We consider it as a multi-valued function (or a section of a line bundle) on the Jacobian variety $J(C) = H^0(C, K_C)^*/H_1(C, \mathbb{Z})$. Here $H^0(C, K_C)$ is identified with $\mathbb{C}^g$ by the basis $\omega_1, \ldots, \omega_g$.

We choose a base point $P_0$ in $C$. Then we have the Abel-Jacobi map

$$C \ni P \mapsto \int_{P_0}^P \omega \in J(C); \quad \omega \in H^0(C, K_C).$$

We denote it by $A$. It extends a map from $J_g(C)$ the Picard variety of divisor classes (linear equivalence classes) of degree $g$ divisors. When $g = 0$, it is independent of the base point and we have an isomorphism $J_g(C) \cong J(C)$.

Riemann’s theorem ([22] Theorem 1.1], [57] Chap. II, 3.1]) says that there exists a vector $\vec{K} \in \mathbb{C}^g$ such that for all $e \in \mathbb{C}^g$ the composition $\Theta \circ (A + e)$ either vanishes identically, or has $g$ zeroes $Q_1, \ldots, Q_g$ such that $\sum_{k=1}^g A(Q_k) + e \equiv \vec{K}$ mod $H_1(C, \mathbb{Z})$. The vector is called the Riemann constant. The vector $\vec{K}$ depends on the choice of the symplectic basis of $H_1(C, \mathbb{Z})$ and the base point $P_0$. However, if we denote it by $\vec{K}_{P_0}$, then $\Delta = (g-1)P_0 + \vec{K}_{P_0} \in J_g-1(C)$ is independent of $P_0$. See [22] VI.3.7], [57] Chap. II, 3.11, 3.18]. In fact, we have $\Theta = W_{g-1} - \Delta$, where $\Theta$ is considered as a divisor in $J(C)$ as a zero set $\Theta$ = 0, and $W_{g-1} = \{x_1 + \cdots + x_{g-1} | x_\alpha \in C\}$.

The set $\Sigma$ of divisor classes $D \in J_g-1(C)$ such that $2D = K_C$ is called the set of theta characteristics. The above $\Delta$ is an example. The set $\Sigma$ is bijective to $\frac{1}{2}H_1(C, \mathbb{Z})/H_1(C, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$ (considered as a subset in $J(C)$) via $D \mapsto D - \Delta$. We identify $\Sigma$ with a characteristic for the theta function by the further identification $J(C) \cong \mathbb{C}^g/\mathbb{Z}^g + \tau \mathbb{Z}^g$ given by the choice of cycles.

B.4. Green functions. Let $\phi: \tilde{C} \to C$ be the universal covering of $C$. We take a nonsingular odd theta characteristic $D \in J_{g-1}(C)$ (i.e., the theta divisor is smooth at $D$ [57] IIIb,Lem. 1]) and let $\delta = [\vec{\beta}]$ be the corresponding half integer characteristic. By Riemann’s theorem,

$$\zeta(x) = \sum_{\alpha=1}^g \frac{\partial \Theta_\delta}{\partial \xi_\alpha}(0) \omega^\alpha(x)$$

is a section of $K_C$ which vanishes on $D$. Since $D$ is a nonsingular odd characteristic, $H^0(C, K_C(-D)) \cong H^0(C, \mathcal{O}_C(D)) \cong \mathbb{C}$, and hence $\sqrt{\zeta(x)}$ is a section of $\mathcal{O}_C(D)$. 
It is called the Szegö kernel for a half integer characteristic \( \{ \tilde{\tau} \} \).

We define a prime form by

\[
E(x, y) := \frac{\Theta_y \left( \int x \big| \tau \right)}{\sqrt{\zeta(x) \zeta(y)}}
\]

This is a holomorphic differential form on \( \tilde{C} \times \tilde{C} \). It is also regarded as a holomorphic section of a line bundle on \( C \times C \): Let \( \pi_i : C \times C \to C, \ i = 1, 2 \) be projections and \( \mu : C \times C \to J(C) \) be the map \( (x, y) \mapsto y - x \). Then \( E(x, y) \) is a section of \( \pi_1^* \mathcal{O}_C(-\Delta) \otimes \pi_2^* \mathcal{O}_C(-\Delta) \otimes \mu^*(\mathcal{O}_{J(C)}(\Theta)) \), where \( \Theta \) is the theta divisor.

We pick up some properties of \( E(x, y) \).

1. \( E(x, y) = 0 \iff \phi(x) = \phi(y) \).
2. \( E(x, y) \) has a first order zero along the diagonal \( \Delta \subset C \times C \) and locally \( E(x, y) = \frac{x - y}{\sqrt{\zeta(x) \zeta(y)}} (1 + O((x - y)^2)) \).
3. \( E(x, y) = -E(y, x) \).

Let

\[
W(z_1, z_2) = \partial_z \partial_{z_2} \log E(z_1, z_2).
\]

This is a well-defined meromorphic 2-form on \( C \times C \) and it is used to construct differentials of the 2nd kind. For \( c \in \mathbb{C}^g \) with \( \Theta(c) \neq 0 \), we set

\[
\Psi_c(z_1, z_2) = \frac{\Theta_c \left( \int z_1 \big| \tau \right)}{\Theta_c(0) E(z_1, z_2)}.
\]

It is called the Szegö kernel.

By Fay’s trisecant identity \(\text{(23) p. 34, formula 45} \text{ or } \text{(57 IIIb,2)}\), we have

\[
\Psi_c^2(z_1, z_2) = W(z_1, z_2) + \sum_{\alpha, \beta} \omega^\alpha(z_1) \omega^\beta(z_2) \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} \log \Theta_c(0|\tau).
\]

for a half integer characteristic \( c \) \(\text{(23 Cor. 2.12 formula 38, 57 IIIb,3 (2))}\).

**B.5. Hyperelliptic curves.** Let \( Q(z) \) be a polynomial of degree \( 2g + 2 \). Let \( C = \{ y^2 = Q(z) \} \) be the corresponding hyperelliptic curve of genus \( g \). We denote by \( \iota : C \to C \) be the involution. Let \( \{Q_1, \ldots, Q_{2g+2}\} \) be the set of branched points, i.e., the roots of \( Q(z) = 0 \). We choose cycles \( A_\alpha, B_\alpha \) as in Figure\( \text{I} \) where we replace as

\[
z_1^+ \to Q_1, \ z_1^- \to Q_2, \ z_2^- \to Q_3, \ z_2^+ \to Q_4, \ \ldots.
\]

Then the choice of cycles is exactly the same as \(\text{23 p.12 Example} \) with shifting the numbering by 1, i.e., \( A_2 \to A_1, B_2 \to B_1 \), etc. If we choose \( Q_1 \) for the base point of the Abel-Jacobi map, we have \(\text{23 p.14, 57 Chap. IIIa.5}\)

\[
\tilde{K} = \tau^t \left( \frac{1}{2} \quad \frac{1}{2} \quad \cdots \quad \frac{1}{2} \right) + \tau^t \left( \frac{1}{2} \quad \frac{2}{2} \quad \cdots \quad \frac{g}{2} \right).
\]

Let \( L \) be the divisor class of degree 2 containing \( P + \iota P \) for \( P \in C \). Then the set of theta characteristics is bijective to the set of subsets \( T \subset \{Q_1, \ldots, Q_{2g+2}\} \) with \( \# T \equiv (g + 1) \mod 2 \) modulo the equivalence relation \( T \sim T' \) by

\[
T \mapsto \sum_{P \in T} P + \frac{g - 1 - \# T}{2} L.
\]

Under this correspondence, the vector \( \Delta = (g - 1)Q_1 + \tilde{K} \) is mapped to the \{\(Q_1, Q_3, \ldots, Q_{2g+1}\}\).
When $\# T = (g + 1)$, the corresponding Szegö kernel is given explicitly by

$\Psi_c(z_1, z_2) = \frac{1}{2} \left( \sqrt[4]{\frac{\psi(z_1)}{\psi(z_2)}} + \sqrt[4]{\frac{\psi(z_2)}{\psi(z_1)}} \right) \sqrt{\frac{dz_1 dz_2}{z_1 - z_2}},$  

where $\psi(z) = \prod_{Q \in T} (z - Q_\alpha) \times \prod_{Q \in T^c} (z - Q_\beta)^{-1}$. See [23] p.12 Example.

Appendix C. Equivariant Borel-Moore homology

We use equivariant Borel-Moore homology in this paper. For the usual Borel-Moore homology, see e.g., [27] §B.2. As we only use the Borel-Moore homology, we denote it by $H_*(\cdot)$. If we do not specify the coefficients, we mean the complex coefficients.

The following properties are crucial.

a) If $X$ is nonsingular, $H_i(X)$ is isomorphic to the ordinary cohomology group $H^{2 \dim X - i}(X)$.

b) For an irreducible algebraic variety $X$, its fundamental class $[X] \in H_{2 \dim X}(X)$ is defined.

c) For a proper continuous map $f : X \to Y$, the push-forward homomorphism $f_* : H_*(X) \to H_*(Y)$ is defined.

d) If $U \subset X$ is open with complement $Y = X \setminus U$, we have the long exact sequence

$$\cdots \to H_i(Y) \xrightarrow{i_*} H_i(X) \xrightarrow{j^*} H_i(U) \to H_{i-1}(Y) \to \cdots,$$

where $i : Y \to X$, $j : U \to X$ are inclusions, and $j^*$ is the restriction homomorphism.

For an equivariant Borel-Moore homology, we use the one given in [47], but we shift the degree so that the fundamental class $[X]$ has degree $2 \dim X$. This definition is the same as [18].

Let us recall the definition briefly. Let $G$ be a linear algebraic group acting on an algebraic variety $X$. (Everything is over $\mathbb{C}$.) We have a finite dimensional approximation of the classifying space $EG \to BG$, i.e., for any $n$, there exists a smooth irreducible variety $U$ with $G$-action such that

a) The quotient $U \to U/G$ exists and is a principal $G$-bundle.

b) $H^i(U) = 0$ for $i = 1, \ldots, n$.

We then define

$H^G_n(X) = H_{n-2 \dim G + 2 \dim U}(X \times_G U).$

Here $U$ is smooth, in particular $\dim U$ makes sense. One can show that this is independent of the choice of $U$, using the double fibration argument.

Note that $H^G_n(X) = 0$ if $n > 2 \dim X$, but $H^G_n(X)$ may be nonzero for $n < 0$. ($X$ is pure dimensional.)

On the other hand, we define the equivariant co-homology as

$H^n_G(X) = H^n(X \times_G U),$

where $H^n(\cdot)$ is the ordinary cohomology. This coincides with the usual definition. It is a graded ring. We have the Poincaré duality isomorphism

$H^n_G(X) \cong H_{2 \dim X - n}(X)$

when $X$ is nonsingular.
As a projection $X \times_G U \to U/G$ is flat, $H^G_*(X)$ has a structure of a $H^G_*(pt)$-module.

Suppose that $G$ is reductive. Then $H^G_*(pt)$ is isomorphic to $S^*(\mathfrak{h}^*)^W$, where $\mathfrak{h}$ is a Cartan subalgebra, $S^*(\mathfrak{h}^*)$ is the symmetric algebra of its dual, and $W$ is the Weyl group. We denote this by $S$ or $S(G)$.

Let $T$ be a torus acting on $X$. Let $X^T$ be the fixed point set and $\iota: X^T \to X$ be the inclusion. We have the push-forward homomorphism $\iota_*: H^*_T(X^T) \to H^*_T(X)$. Since $T$ acts trivially on $X^T$, we have $H^*_T(X^T) = H_*(X^T) \otimes_C S$. The localization theorem (see [4]) says that $\iota_*$ becomes an isomorphism after tensoring the quotient field $S$ of $S$.

When $X$ is nonsingular, the inverse of $\iota_*$ can be explicitly given. Let $X^T = \bigsqcup F_i$ be the decomposition to irreducible components. Each $F_i$ is nonsingular. Let $N_i$ be the normal bundle of $F_i$ in $X$. Then we have

$$(\iota_*)^{-1} = \sum_i \frac{1}{e_T(N_i)} \iota_i^*.$$ 

where $e_T(N_i)$ is the equivariant Euler class and $\iota_i^*$ is the pull-back homomorphism for the inclusion $\iota_i: F_i \to X$ defined via the Poincaré duality homomorphism.

Appendix D. The proof of (5.20) by Hiroyuki Ochiai

Let

$$(a)_k = (a, q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}), \quad (a)_\infty = \prod_{d=0}^\infty (1 - aq^d).$$

We start with the formula, [9] p.16,(7.2).

We substitute $d = aq/c$, then we have

$$\sum_{k=0}^\infty \frac{(a)_k (e)_k (f)_k (1 - aq^{2k})(-a/qef)^k q^{k(k+3)/2}}{(aq/e)_k (aq/f)_k (q)_k (1 - a)} = \frac{(aq)_\infty (aq/ef)_\infty}{(aq/e)_\infty (aq/f)_\infty}.$$ 

($q$-hypergeometric part vanishes since $(1)_k = 0$ for $k \geq 1$.) We put $e = -(aq)^{1/2}$, then we have

$$\sum_{k=0}^\infty \frac{(a)_k (f)_k (1 - aq^{2k})(a^{1/2}q^{-3/2}/f)^k q^{k(k+3)/2}}{(aq/f)_k (q)_k (1 - a)} = \frac{(aq)_\infty (-aq^{1/2}/f)_\infty}{(-aq^{1/2})_\infty (aq/f)_\infty}.$$ 

Finally, we put $f = aq^{-1/2}$. Then we have

$$\sum_{k=0}^\infty \frac{(a)_k (aq^{-1/2})_k (1 - aq^{2k})(a^{-1/2}q^{-1})_k q^{k(k+3)/2}}{(q^{1/2})_k (q)_k} = \frac{(aq)_\infty (-a^{-1/2}q)_\infty}{(-a^{-1/2}q)_{\infty} (q^{1/2})_{\infty}}.$$ 

Using Jacobi triple product identity $(q)_\infty (-a^{1/2}q)_\infty = \sum_{l=-\infty}^\infty (q/a)^{1/2}q^{l^2/2}$, the right hand side is

$$\frac{(aq)_\infty (-a^{1/2}q^{1/2})_{\infty} (q^{3/2})_{\infty}}{(-a^{1/2}q^{1/2})_{\infty} (q^{3/2})_{\infty}} \sum_{l=-\infty}^\infty (q/a)^{1/2}q^{l^2/2}.$$ 

Using $(b)_\infty (bq^{1/2})_\infty = (b, q^{1/2})_\infty$, and $(b^2)_\infty = (b, q^{1/2})_\infty (-b, q^{1/2})_\infty$, we get

$$\sum_{k=0}^\infty \frac{(aq^{-1/2}, q^{1/2})_{2k} (1 - aq^{2k})(-a^{-1/2}q^{-1})_k q^{k(k+3)/2}}{(q, q^{1/2})_{2k}} = \frac{(a^{1/2}, q^{1/2})_{\infty} (q, q^{1/2})_{\infty}}{(q, q^{1/2})_{\infty}} \sum_{l=-\infty}^\infty (q/a)^{1/2}q^{l^2/2}.$$
Also
\[
\sum_{k=0}^{\infty} \frac{(aq^{-1/2}, q^{1/2})_{2k}(1 - aq^{2k})(a^{-1/2}q^{-1})^{k}q^{k(k+3)/2}}{(q^{1/2}, q^{1/2})_{2k+1}} = (a^{1/2}, q^{1/2})_{\infty} \sum_{l=-\infty}^{\infty} (q/a)^{l/2}q^{l^2/2}.
\]
Now we substitute \(a \mapsto q^2t^2\) and \(q \mapsto q^4t^2\). Then
\[
\sum_{k=0}^{\infty} \frac{(t, q^2t)_{2k}(1 - q^{8k+2t4k^2+2})q^{k(k+1)}t^k}{(q^2t, q^2t)_{2k+1}} = \frac{(qt, q^2t)_{\infty}}{(q^2t, q^2t)_{\infty}} \sum_{l=-\infty}^{\infty} q^{l(2l+1)t}t^2.
\]
Using the identity
\[
1 - q^{8k+2t4k^2+2} = (1 - q^{4k}t^{2k+1})q^{4k+2t2k+1} + (1 - q^{4k+2t2k+1}),
\]
we see the left hand side is
\[
\sum_{k=0}^{\infty} \frac{(t, q^2t)_{2k}q^{k(2k+1)}t^k}{(q^2t, q^2t)_{2k}} + \frac{(t, q^2t)_{2k+1}}{(q^2t, q^2t)_{2k+1}}q^{k+1)(2k+3)-1}t(k+1)^2
\]
This is the end of the proof.

Appendix E. Perturbation term

E.1. One parameter version. Let
\[
\gamma_h(x; \Lambda) = \frac{d}{ds} \bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t} t^{-s} e^{-tx} (e^{ht} - 1)(e^{-ht} - 1),
\]
where \(\Gamma(s)\) is the Gamma function
\[
\Gamma(s) = \int_0^{\infty} \frac{dt}{t} t^{s-1} e^{-t}.
\]
The integral in the right hand side converges when \(\Re(s) > 2\). The analytic continuation can be done by the standard procedure using the Taylor expansion of the integrand. (See below.)

If we formally expand as
\[
\frac{1}{(e^{ht} - 1)(e^{-ht} - 1)} = \sum_{m,n \geq 0} e^{h(m-n)t},
\]
we get
\[
\gamma_h(x; \Lambda) \text{ formally } = \sum_{m,n \geq 0} \log \left( \frac{x - h(m-n)}{\Lambda} \right).
\]
Thus \(\gamma_h(x; \Lambda)\) is a regularization of the right hand side.

We introduce Bernoulli numbers by
\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
\]
We have \(B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_{2k+1} = 0\) for \(k \geq 1\). Note
\[
\frac{1}{(e^t - 1)(e^{-t} - 1)} = \frac{1}{t^2} \sum_{g=0}^{\infty} \frac{B_{2g}}{2g(2g-2)!} t^{2g-2}.
\]
Then \[ \gamma_h(x; \Lambda) \]
\[
= \frac{d}{ds}\bigg|_{s=0} \left[ -\left(\frac{x}{\Lambda}\right)^2 \left(\frac{1}{x}\right)^s \frac{\Gamma(s - 2)}{\Gamma(s)} + \frac{B_2}{2} \right] x^s \\
+ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)!} \left(\frac{\Lambda}{x}\right)^{2g-2} \frac{\Gamma(s + 2g - 2)}{\Gamma(s)} \right] \\
= h^{-2} \left\{ \frac{1}{2} x^2 \log \left(\frac{x}{\Lambda}\right) - \frac{3}{4} x^2 \right\} - \frac{1}{12} \log \left(\frac{x}{\Lambda}\right) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \left(\frac{\Lambda}{x}\right)^{2g-2}.
\]

We have the difference equation

\[ \gamma_h(x + h; \Lambda) + \gamma_h(x - h; \Lambda) - 2\gamma_h(x; \Lambda) = \log \left(\frac{x}{\Lambda}\right). \]

In fact, the left hand side is equal to

\[
\frac{d}{ds}\bigg|_{s=0} \left( \Lambda^s \int_0^\infty \frac{dt}{t^s} e^{-tx} e^{th} e^{-th} - \frac{1}{(e^{ht} - 1)(e^{-ht} - 1)} \right) = - \frac{d}{ds}\bigg|_{s=0} \left( \Lambda^s \right).
\]

We have

\[
\gamma_h \left( x + \frac{h}{2}; \Lambda \right) - \gamma_h \left( x - \frac{h}{2}; \Lambda \right) = \frac{d}{ds}\bigg|_{s=0} \left( \Lambda^s \int_0^\infty \frac{dt}{t^s} e^{-tx} e^{th} e^{-th} \right) = \frac{d}{ds}\bigg|_{s=0} \left( \Lambda^s \right) \zeta \left( s, \frac{x}{h} + \frac{1}{2} \right)
\]

\[
= \log \left( \frac{\Lambda}{h} \right) \zeta \left( 0, \frac{x}{h} + \frac{1}{2} \right) + \zeta' \left( 0, \frac{x}{h} + \frac{1}{2} \right) = \log \left( \frac{1}{\sqrt{2\pi}} \left( \frac{h}{\Lambda} \right)^{\frac{h}{\Lambda}} \right) \Gamma \left( \frac{x}{h} + \frac{1}{2} \right)
\]

where \( \zeta(s, a) \) is the Hurwitz zeta function. And at the final equality, we have used the Lerch formula (see [38 XVIII]).

We have

\[ \gamma_h(x; \Lambda) + \gamma_h(\sqrt{-1}x; \Lambda) = 2\gamma_h(\sqrt{-1}x; \Lambda). \]

This can be seen from the expansion [38].

**E.2. Two parameter version.** Let us introduce a generalization of \( \gamma_h(x; \Lambda) \):

\[
\gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) = \frac{d}{ds}\bigg|_{s=0} \left( \Lambda^s \int_0^\infty \frac{dt}{t^s} e^{-t \varepsilon_1 t} e^{-t \varepsilon_2 t} \right).
\]

This is formally equal to

\[
\sum_{m,n \geq 0} \log \left( \frac{x - m \varepsilon_1 - n \varepsilon_2}{\Lambda} \right).
\]

The difference equation is

\[ \gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_1; \Lambda) + \gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_2; \Lambda) - \gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) - \gamma_{\varepsilon_1, \varepsilon_2}(x - \varepsilon_1 - \varepsilon_2; \Lambda) = \log \left( \frac{x}{\Lambda} \right). \]
Let $k$ be an integer. We have
\[
\gamma_{\epsilon_1, \epsilon_2}(x + \epsilon_1 k; \Lambda) + \gamma_{\epsilon_1, -\epsilon_2}(x + \epsilon_2 k; \Lambda)
\]
\[
= \frac{d}{ds} \bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} e^{-tx} \left( \frac{e^{-t\epsilon_1 k}}{(e^\epsilon_1 t - 1)(e^{\epsilon_2-\epsilon_1} t - 1)} + \frac{e^{-t\epsilon_2 k}}{(e^\epsilon_2 t - 1)(e^{\epsilon_1-\epsilon_2} t - 1)} \right)
\]
\[
= \frac{d}{ds} \bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} e^{-tx} \left( (1 - e^{-\epsilon_2 t}) e^{-\epsilon_1 t} - (1 - e^{-\epsilon_1 t}) e^{-\epsilon_2 t} \right)
\]
\[
\bigg| e^{\epsilon_1 t} - 1 \bigg| e^{\epsilon_2 t} - 1 \bigg| e^{-\epsilon_1 t} - 1 \bigg| e^{-\epsilon_2 t} - 1 \bigg|
\]
We claim that this is equal to
\[
(E.2) \quad \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) + \log s^{-k}(\epsilon_1, \epsilon_2, x) - \frac{k(k-1)}{2} \log \Lambda,
\]
where $s^{-k}(\epsilon_1, \epsilon_2, x)$ is given by (E.1). If $k = 0$ or 1, this is obvious. Suppose that $k \geq 2$. Then the above is equal to
\[
\sum_{l,m \geq 0 \atop l+m=k-1} \gamma_{\epsilon_1, \epsilon_2}(x + l\epsilon_1 + m\epsilon_2; \Lambda) - \sum_{l,m \geq 1 \atop l+m=k} \gamma_{\epsilon_1, \epsilon_2}(x + l\epsilon_1 + m\epsilon_2; \Lambda).
\]
On the other hand, (E.1) is equal to
\[
\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) + \sum_{l,m \geq 0 \atop l+m=k-2} \log \left( \frac{x + (l+1)\epsilon_1 + (m+1)\epsilon_2}{\Lambda} \right).
\]
\[
= \left( \sum_{l=0, m=0 \atop l+m=k} + \sum_{l \geq 0, m \geq 1 \atop l+m \leq k-1} - \sum_{l \geq 1, m \geq 1 \atop l+m \leq k} - \sum_{l \geq 0, m \geq 0 \atop l+m \leq k-2} \right) \gamma_{\epsilon_1, \epsilon_2}(x + l\epsilon_1 + m\epsilon_2; \Lambda)
\]
\[
= \left( \sum_{l=0, m=0 \atop l+m=k} + \sum_{l \geq 0, m \geq 0 \atop l+m \leq k-1} - \sum_{l \geq 0, m \geq 1 \atop l+m \leq k} + \sum_{l \geq 1, m \geq 1 \atop l+m \leq k-1} \right) \gamma_{\epsilon_1, \epsilon_2}(x + l\epsilon_1 + m\epsilon_2; \Lambda)
\]
by the difference equation. Thus we get the assertion when $k \geq 2$.

Similarly we have
\[
\gamma_{\epsilon_1, -\epsilon_2}(x + \epsilon_1 k; \Lambda) + \gamma_{\epsilon_1, \epsilon_2}(x + \epsilon_2 k; \Lambda)
\]
\[
= \gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) + \sum_{l,m \geq 0 \atop l+m \leq -k-1} \log \left( \frac{x - l\epsilon_1 - m\epsilon_2}{\Lambda} \right)
\]
for $k \leq -1$. This is nothing but the assertion.

**E.3. Expansion.** Let us define $c_n$ ($n = 0, 1, 2, \ldots$) by
\[
\frac{1}{(e^\epsilon_1 t - 1)(e^\epsilon_2 t - 1)} = \sum_{n=0}^\infty \frac{c_n t^{2-n}}{n!}.
\]
We have
\[
c_0 = \frac{1}{\epsilon_1 \epsilon_2}, \quad c_1 = -\frac{\epsilon_1 + \epsilon_2}{2\epsilon_1 \epsilon_2}, \quad c_2 = \frac{\epsilon_1^2 + \epsilon_2^2 + 3 \epsilon_1 \epsilon_2}{6\epsilon_1 \epsilon_2}, \quad \ldots.
\]
Then

\[
\gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) = \frac{d}{ds} \bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \sum_{n=0}^{\infty} c_n \int_0^{\infty} \frac{dt}{t} t^{n+s-2} e^{-tx}
\]

\[
= \frac{\frac{1}{\varepsilon_1 \varepsilon_2}}{12} \left\{ -\frac{1}{2} \varepsilon^2 + \frac{3\varepsilon^2}{2} + 3 \varepsilon_1 \varepsilon_2 \right\} \log \left( \frac{x}{\Lambda} \right) + \sum_{n=3}^{\infty} \frac{c_n x^{2-n}}{n(n-1)(n-2)}.
\]

In particular, we have

\[
\gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda e^n) = \gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) + u \left( \frac{x^2}{2\varepsilon_1 \varepsilon_2} + \frac{x (\varepsilon_1 + \varepsilon_2)}{2\varepsilon_1 \varepsilon_2} + \frac{\varepsilon_1^2 + \varepsilon_2^2 + 3\varepsilon_1 \varepsilon_2}{12 \varepsilon_1 \varepsilon_2} \right),
\]

\[
\gamma_{\varepsilon_1, \varepsilon_2}(x; \Lambda) + \gamma_{\varepsilon_1, \varepsilon_2}(-x; \Lambda)
\]

\[
= \frac{2}{\varepsilon_1 \varepsilon_2} \left\{ -\frac{1}{2} \varepsilon^2 + \frac{3\varepsilon^2}{4} + \frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1 \varepsilon_2} \pi \sqrt{-1} x \right\} + \frac{\varepsilon_1^2 + \varepsilon_2^2 + 3\varepsilon_1 \varepsilon_2}{6 \varepsilon_1 \varepsilon_2} \log \left( \frac{\sqrt{-1} x}{\Lambda} \right) + \sum_{g=2}^{\infty} \frac{2c_{2g} x^{2-2g}}{2g(2g-1)(2g-2)}.
\]

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