On the truncated matricial Stieltjes moment problem $M[[\alpha, \infty); (s_j)_j^{m}; \leq]$

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This paper gives via Stieltjes transform a complete description of the solution set of a matricial truncated Stieltjes-type power moment problem in the non-degenerate and degenerate cases. The approach is based on the Schur type algorithm which was worked out in the papers [9,10]. Furthermore, the subset of parameters is determined which corresponds to another truncated matricial Stieltjes-type moment problem.

**Keywords:** Stieltjes moment problem, Schur type algorithm, Stieltjes pairs.

1. Introduction

This paper is closely related to [9,10]. The main goal is to achieve a simultaneous treatment of the even and odd cases of a further truncated matricial Stieltjes moment problem, which is related but different from that truncated matricial Stieltjes moment problem which was studied in [9,10] on the basis of Schur analysis methods. We will demonstrate that appropriate modifications of our Schur analysis conceptions lead to a complete solution to the problem under consideration in the non-degenerate and degenerate cases. Our conception in [9,10] was based on working out two interrelated versions of Schur type algorithms, namely an algebraic one and a function-theoretic one, and then using the interplay between both of them. This strategy stands again in the center of our investigations.

In order to describe more concretely the central topics studied in this paper, we introduce some notation. Throughout this paper, let $p$ and $q$ be positive integers. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of all positive integers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers, respectively. For every choice of $\rho, \kappa \in \mathbb{R}\cup\{-\infty, \infty\}$, let $\mathbb{Z}_{\rho,\kappa} := \{k \in \mathbb{Z} : \rho \leq k \leq \kappa\}$. We will write $\mathbb{C}^{p\times q}$, $\mathbb{C}_H^{q\times q}$, $\mathbb{C}_\geq^{q\times q}$, and $\mathbb{C}_>^{q\times q}$ for the set of all complex $p \times q$ matrices, the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex $q \times q$ matrices, and the set of all positive Hermitian complex $q \times q$ matrices, respectively.

We will use $\mathcal{B}_\mathbb{R}$ to denote the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$. For each $\Omega \in \mathcal{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathcal{B}_\Omega := \mathcal{B}_\mathbb{R} \cap \Omega$. Furthermore, for each $\Omega \in \mathcal{B}_\mathbb{R} \setminus \{\emptyset\}$, we will write $\mathcal{M}_q^\geq(\Omega)$ to designate the set of all non-negative Hermitian $q \times q$ measures defined on $\mathcal{B}_\Omega$, i.e., the set of
all $\sigma$-additive mappings $\mu: \mathfrak{B}_\Omega \to \mathbb{C}^{q \times q}$. We will use the integration theory with respect to non-negative Hermitian $q \times q$ measures, which was worked out independently by I. S. Kats [17] and M. Rosenberg [22]. Some features of this theory are sketched in Appendix B. For every choice of $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$ and $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we will use $\mathcal{M}_{q,\kappa}^2(\Omega)$ to denote the set of all $\sigma \in \mathcal{M}_{q,\kappa}^2(\Omega)$ such that the integral
\[
\sigma_j^{(\kappa)} := \int_{\Omega} x^j \sigma(dx)
\]
exists for all $j \in \mathbb{Z}_{0,\kappa}$.

Remark 1.1. Let $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $\sigma \in \mathcal{M}_{q,\kappa}^2(\Omega)$. In view of (1.1), then one can easily check that $(\sigma_j^{(\kappa)})^* = \sigma_j^{(\kappa)}$ holds true for all $k \in \mathbb{Z}_{0,\kappa}$.

Remark 1.2. If $k, \ell \in \mathbb{N}_0$ with $k < \ell$, then $\mathcal{M}_{q,\ell}^\infty(\Omega) \subseteq \mathcal{M}_{q,k}^\infty(\Omega)$ holds true.

The central problem studied in this paper is formulated as follows:

Problem (M[$[\alpha, \infty)$; $(s_j)_{j=0}^m \leq \omega$]). Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \leq \omega]$ of all $\sigma \in \mathcal{M}_{q,m}^\omega([\alpha, \infty))$ for which the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m \geq 1$, moreover $s_j^{(\sigma)} = s_j$ is satisfied for each $j \in \mathbb{Z}_{0,m-1}$.

The papers [9][10] were concerned with the study of the following question:

Problem (M[$[\alpha, \infty)$; $(s_j)_{j=0}^m \omega$]). Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \omega]$ of all $\sigma \in \mathcal{M}_{q,m}^\omega([\alpha, \infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,m}$.

A closer look at the just formulated two problems leads to the following more or less obvious observations on interrelations between the two problems:

Remark 1.3. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \omega] \subseteq \mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \leq \omega]$.

Remark 1.4. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. For all $\ell \in \mathbb{Z}_{0,m-1}$, then $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \leq \omega] \subseteq \mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^\ell \omega]$.

In the case that a sequence $(s_j)_{j=0}^m$ of complex $q \times q$ matrices is given for which the set $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \omega]$ is non-empty, we obtained in [10] Theorem 13.1] a complete parametrization of this set via a linear fractional transformation of matrices the generating function of which is a $2q \times 2q$ matrix polynomial built from the sequence $(s_j)_{j=0}^m \omega$ of the given original data. The set of parameters is chosen as a particular class of $q \times q$ matrix-valued functions which are holomorphic in the domain $\mathbb{C} \setminus [\alpha, \infty)$. Thus, combining this with Remark 1.3 we were encouraged to adopt the approach from [14] in order to parametrize the set $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \leq \omega]$ via a linear fractional transformation with the same $2q \times 2q$ matrix polynomial as generating function. However, the class of parameter functions has to be appropriately extended.

The realization of this idea determines the basic strategy of this paper. In [9][10], we presented a Schur analysis approach to Problem $\mathcal{M}([\alpha, \infty); (s_j)_{j=0}^m \omega]$. In this paper, we will indicate that our method can be appropriately modified to produce a complete description of the set $\mathcal{M}_{q,\infty}^\omega([\alpha, \infty); (s_j)_{j=0}^m \leq \omega]$ in the non-degenerate and degenerate cases.
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In order to state a necessary and sufficient condition for the solvability of each of the above formulated moment problems, we have to recall the notion of two types of sequences of matrices.

If \( n \in \mathbb{N}_0 \) and if \( (s_j)_{j=0}^{2n} \) is a sequence of complex \( q \times q \) matrices, then \( (s_j)_{j=0}^{2n} \) is called Hankel non-negative definite (respectively, Hankel positive definite) if

\[
H_n := [s_{j+k}]_{j,k=0}^{n}
\]

is non-negative Hermitian (respectively, positive Hermitian). A sequence \( (s_j)_{j=0}^{\infty} \) of complex \( q \times q \) matrices is called Hankel non-negative definite (respectively, Hankel positive definite) if \( (s_j)_{j=0}^{2n} \) is Hankel non-negative definite (respectively, Hankel positive definite) for all \( n \in \mathbb{N}_0 \). For all \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), we will write \( \mathcal{H}_{q,2n}^{\geq,\kappa} \) (respectively, \( \mathcal{H}_{q,2n}^{>\kappa} \)) for the set of all Hankel non-negative definite (respectively, Hankel positive definite) sequences \( (s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices.

Furthermore, for all \( n \in \mathbb{N}_0 \), let \( \mathcal{H}_{q,2n}^{\geq,e} \) be the set of all sequences \( (s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices for which there exist complex \( q \times q \) matrices \( s_{2n+1} \) and \( s_{2n+2} \) such that \( (s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\geq,e} \), whereas \( \mathcal{H}_{q,2n+1}^{\geq,e} \) stands for the set of all sequences \( (s_j)_{j=0}^{2n+1} \) of complex \( q \times q \) matrices for which there exist some \( s_{2n+1} \in \mathbb{C}^{q \times q} \) such that \( (s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^{\geq,e} \). For each \( m \in \mathbb{N}_0 \), the elements of the set \( \mathcal{H}_{q,\infty}^{\geq,m} \) are called Hankel non-negative definite extendable sequences.

Besides the just introduced classes of sequences of complex \( q \times q \) matrices, we need analogous classes of sequences \( (s_j)_{j=0}^{n} \) of complex \( q \times q \) matrices, which take into account the influence of the prescribed number \( \alpha \in \mathbb{R} \). We will introduce several classes of finite or infinite sequences of complex \( q \times q \) matrices which are characterized by properties of the sequences \( (s_j)_{j=0}^{n} \) and \( (-\alpha s_j + s_{j+1})_{j=0}^{n} \).

Let \( (s_j)_{j=0}^{n} \) be a sequence of complex \( p \times q \) matrices. Then, for all \( n \in \mathbb{N}_0 \) with \( 2n+1 \leq \kappa \), we introduce the block Hankel matrix

\[
K_n := [s_{j+k+1}]_{j,k=0}^{n}.
\]

Let \( \alpha \in \mathbb{R} \). Let \( \mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq} \) and, for all \( n \in \mathbb{N} \), let \( \mathcal{K}_{q,2n,\alpha}^{\geq} \) be the set of all sequences \( (s_j)_{j=0}^{2n} \) of complex \( q \times q \) matrices for which the block Hankel matrices \( H_n \) and \( -\alpha H_{n-1} + K_{n-1} \) are both non-negative Hermitian, i.e.,

\[
\mathcal{K}_{q,2n,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} : (-\alpha s_j + s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}\right\}.
\] (1.2)

Furthermore, for all \( n \in \mathbb{N}_0 \), let \( \mathcal{K}_{q,2n+1,\alpha}^{\geq} \) be the set of all sequences \( (s_j)_{j=0}^{2n+1} \) of complex \( q \times q \) matrices for which the block Hankel matrices \( H_n \) and \( -\alpha H_{n-1} + K_{n} \) are both non-negative Hermitian, i.e., if \( \mathcal{F}_{q,2n+1} \) is the set of all sequences \( (s_j)_{j=0}^{2n+1} \) of complex \( q \times q \) matrices, then

\[
\mathcal{K}_{q,2n+1,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1} : \left( (s_j)_{j=0}^{2n}, (-\alpha s_j + s_{j+1})_{j=0}^{2n} \right) \in \mathcal{H}_{q,2n}^{\geq} \right\}.
\] (1.3)

Remark 1.5. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \( (s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq} \). Then it is easily checked that \( (s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell,\alpha}^{\geq} \) for all \( \ell \in \mathbb{Z}_{0,m} \).

Let \( \alpha \in \mathbb{R} \). In view of Remark 1.5, let \( \mathcal{K}_{q,\infty,\alpha}^{\geq} \) be the set of all sequences \( (s_j)_{j=0}^{\infty} \) of complex \( q \times q \) matrices such that \( (s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq} \) for all \( m \in \mathbb{N}_0 \). Formulas (1.2) and (1.3) show that the sets \( \mathcal{K}_{q,2n,\alpha}^{\geq} \) and \( \mathcal{K}_{q,2n+1,\alpha}^{\geq} \) are determined by two conditions. The condition \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \)
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ensures that a particular Hamburger moment problem associated with the sequence \((s_j)_{j=0}^{2n}\) is solvable (see, e.g., [6, Theorem 4.16]). The second condition \((-\alpha s_j + s_{j+1})_{j=0}^{2(n-1)} \in H_{q,2n}^2\) (respectively \((-\alpha s_j + s_{j+1})_{j=0}^{2n} \in H_{q,2n}^2\)) controls that the original sequences \((s_j)_{j=0}^{2n}\) and \((s_j)_{j=0}^{2n+1}\) are well adapted to the interval \([\alpha, \infty)\).

For each \(m \in \mathbb{N}_0\), let \(K_{q,m,\alpha}^e\) be the set of all sequences \((s_j)_{j=0}^m\) of complex \(q \times q\) matrices for which there exists an \(s_{m+1} \in \mathbb{C}^{q \times q}\) such that \((s_j)_{j=0}^{m+1}\) belongs to \(K_{q,m+1,\alpha}^e\).

**Remark 1.6.** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\). Then \((s_j)_{j=0}^\ell \in K_{q,\ell,\alpha}^e\) for all \(\ell \in \mathbb{Z}_{0,m}\).

**Remark 1.7.** Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), and let \((s_j)_{j=0}^\kappa \in K_{q,\kappa,\alpha}^{e,\ell}\). Then \((s_j)_{j=0}^\ell \in K_{q,\ell,\alpha}^{e,\ell}\) for all \(\ell \in \mathbb{Z}_{0,\kappa-1}\).

Let \(m \in \mathbb{N}_0\). Then we call a sequence \((s_j)_{j=0}^m\) of complex \(q \times q\) matrices right-sided \(\alpha\)-Stieltjes non-negative definite if it belongs to \(K_{q,m,\alpha}^e\) and right-sided \(\alpha\)-Stieltjes non-negative definite extendable if it belongs to \(K_{q,m,\alpha}^{e,\ell}\).

Now we can characterize the situations in which the problems formulated above have a solution:

**Theorem 1.8 ( [6, Theorems 1.3 and 1.4]).** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)_{j=0}^m\) be a sequence of complex \(q \times q\) matrices. Then:

(a) \(M_q^e[[\alpha, \infty); (s_j)_{j=0}^m, =] \neq \emptyset\) if and only if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\).
(b) \(M_q^e[[\alpha, \infty); (s_j)_{j=0}^m, \leq] \neq \emptyset\) if and only if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^{e,\ell}\).

**Corollary 1.9.** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \(\sigma \in M_{q,m}([\alpha, \infty))\). Then \((s_j)_{j=0}^{\sigma}\) belongs to \(K_{q,m,\alpha}^{e,\ell}\).

**Proof.** Apply Theorem 1.8.

Theorem 1.8 indicates the importance of the sets \(K_{q,m,\alpha}^e\) and \(K_{q,m,\alpha}^{e,\ell}\) for the above formulated truncated matricial Stieltjes-type moment problems. The following result reflects an interrelation between these two sets, which strongly influences our following considerations:

**Theorem 1.10 ( [6, Theorem 5.2]).** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\). Then there is a unique sequence \((\tilde{s}_j)_{j=0}^m \in K_{q,m,\alpha}^{e,\ell}\) such that

\[
M_q^e[[\alpha, \infty); (s_j)_{j=0}^m, \leq] = M_q^e[[\alpha, \infty); (\tilde{s}_j)_{j=0}^m, \leq].
\tag{1.4}
\]

**Theorem 1.10** leads us to the following notion:

**Definition 1.11.** If \(m \in \mathbb{N}_0\) and if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\), then the unique sequence \((\tilde{s}_j)_{j=0}^m\) belonging to \(K_{q,m,\alpha}^{e,\ell}\) for which (1.4) holds true is said to be the right-sided \(\alpha\)-Stieltjes non-negative definite extendable sequence equivalent to \((s_j)_{j=0}^m\).

**Lemma 1.12.** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}\), and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\). Denote by \((\tilde{s}_j)_{j=0}^m\) the right-sided \(\alpha\)-Stieltjes non-negative definite extendable sequence equivalent to \((s_j)_{j=0}^m\). Then \(s_m - \tilde{s}_m \in \mathbb{C}_{1}^{q \times q}\). If \(m \geq 1\), then \(s_j = \tilde{s}_j\) for all \(j \in \mathbb{Z}_{0, m-1}\). Moreover, \((s_j)_{j=0}^m = (\tilde{s}_j)_{j=0}^m\) if and only if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^{e,\ell}\).
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Proof. By construction, we have \((\tilde{s}_j)_{j=0}^m \in K_{q,m,q}^e\) and (1.4). Then we infer from Theorem 1.8 that \(M_q^e[\alpha,\infty];(\tilde{s}_j)_{j=0}^m;=\) \(\not=\emptyset\). Let \(\sigma \in M_q^e[\alpha,\infty];(\tilde{s}_j)_{j=0}^m;=\). This implies \(s_m^{(\sigma)} = \tilde{s}_m\). Consequently, from Remark 1.3 it follows \(\sigma \in M_q^e[\alpha,\infty];(\tilde{s}_j)_{j=0}^m;=\). Hence, (1.4) implies \(\sigma \in M_q^e[\alpha,\infty];(s_j)_{j=0}^m;=\). Thus, \(s_m - s_m^{(\sigma)} \in C_\geq^\times\). Because of \(s_m^{(\sigma)} = \tilde{s}_m\), we get \(s_m - \tilde{s}_m \in C_\geq^\times\). The remaining assertions are immediate consequences of Theorem 1.10.

Theorem 1.10 is essential for the realization of the above formulated basic strategy of our approach, because it is namely possible to restrict our considerations to the case that the given sequence \((s_j)_{j=0}^m\) belongs to the subclass \(K_{q,m,q}^e\) of \(K_{q,m,q}^e\). This provides the opportunity to use immediately the basics of the machinery developed in [9,10].

Similar as in [9,10], we reformulate the original truncated matricial moment problem via Stieltjes transform into an equivalent problem of prescribed asymptotic expansions for particular classes of matrix-valued functions, which are holomorphic in \(\mathbb{C}\setminus[\alpha,\infty)\). The key for the success of our approach is caused by the fact that the Schur-Stieltjes transform for \(q \times q\) matrix-valued holomorphic functions in \(\mathbb{C}\setminus[\alpha,\infty)\), which we worked out in [10], is also compatible with the problem under consideration in this paper.

This paper is organized as follows. In Section 2, we recall some basic facts on some classes of matrix-valued holomorphic functions, which are meromorphic in \(\mathbb{C}\setminus[\alpha,\infty)\). We also summarize some essential features of the Schur type algorithm for finite or infinite sequences of complex \(p \times q\) matrices, which was constructed in [9] Sections 8 and 9.

In Section 3, we recall some basic facts on some classes of matrix-valued holomorphic functions. The scalar versions of these classes had been proved to be essential tools for studying classical moment problems (see, e.g. [10]).

In Section 4, we summarize some basic facts on the classes \(S_{\kappa,q,[\alpha,\infty)\}}((s_j)_{j=0}^m;=\) of \([\alpha,\infty)\)-Stieltjes transforms of the measures belonging to \(M_q^e[\alpha,\infty);(s_j)_{j=0}^m;=\).

In Section 5, we translate the original moment problem \(M([\alpha,\infty);(s_j)_{j=0}^m;=\) via \([\alpha,\infty)\)-Stieltjes transformation into the language of a particular class of holomorphic matrix-valued functions, namely the class \(S_{q,[\alpha,\infty)\}}\).

Section 6 is written against a special background. Indeed, there arises a new phenomenon in comparison with our approach to the moment problem \(M([\alpha,\infty);(s_j)_{j=0}^m;=\), which was undertaken in [9,10]. As we will see later, in contrast to [10], Theorem 13.1, the set of all \([\alpha,\infty)\)-Stieltjes transforms of the solution set of the moment problem \(M([\alpha,\infty);(s_j)_{j=0}^m;=\) can not be parametrized by a linear fractional transformation, where the set of parameters is given by an appropriate subclass of \(S_{q,[\alpha,\infty)\}}\). Now we will be confronted with a particular class of ordered pairs of \(q \times q\) matrix-valued functions which are meromorphic in \(\mathbb{C}\setminus[\alpha,\infty)\). For this class of ordered pairs, essential features are given.

Sections 7, 8, and 10 lie in the heart of this paper. In the center of these sections stands the function-theoretic version of the Schur algorithm which was the basic tool of our approach to Problem \(M([\alpha,\infty);(s_j)_{j=0}^m;=\) chosen in [9,10]. Now we demonstrate that this algorithm
2. On further classes of sequences of complex $q \times q$ matrices

In this section, we introduce two classes of finite sequences of complex $q \times q$ matrices, which prove to be right-sided $\alpha$-Stieltjes non-negative definite extendable. We start with some notation. By $I_p$ and $0_{p \times q}$ we designate the unit matrix in $C^{p \times p}$ and the null matrix in $C^{p \times q}$. is also suitable to handle Problem $M[[\alpha, \infty]; (s_j)_{j=0}^m \leq]$. The main reason for this is that we already know how this Schur algorithm acts in the framework of Problem $M[[\alpha, \infty); (s_j)_{j=0}^m =]$. Then we can apply the corresponding results (see Theorems 9.4 and 10.1), which provided key instruments for the successful realization of our strategy chosen in 9.10. It should be mentioned that the proofs of Theorems 9.1 and 10.1, presented in 10, are based on matricial versions of the classical Hamburger-Nevanlinna theorem (see [10, Theorem 6.1]).
respectively. If the size of a unit matrix and a null matrix is obvious, then we will also omit
the indexes. For each \( A \in \mathbb{C}^{q \times q} \), let \( \text{tr} \, A \) be the trace of \( A \) and let \( \det A \) be the determinant of \( A \). If \( A \in \mathbb{C}^{p \times q} \), then we denote by \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \) the null space of \( A \) and the column space of \( A \), respectively, and we will use \( \text{rank} \, A \) and \( \|A\|_2 \) to denote the rank of \( A \) and the operator norm of \( A \), respectively. For every choice of \( x, y \in \mathbb{C}^q \), the notation \( \langle x, y \rangle_E \) stands for the (left) Euclidean inner product. For each \( A \in \mathbb{C}^{p \times q} \), let \( \|A\|_E := \sqrt{\text{tr}(A^*A)} \) be the Euclidean norm of \( A \). If \( \mathcal{M} \) is a non-empty subset of \( \mathbb{C}^q \), then \( \mathcal{M}^\perp \) stands for the (left) orthogonal complement of \( \mathcal{M} \). If \( \mathcal{U} \) is a linear subspace of \( \mathbb{C}^q \), then let \( P_{\mathcal{U}} \) be the orthogonal projection matrix onto \( \mathcal{U} \), i.e., \( P_{\mathcal{U}} \) is the unique complex \( q \times q \) matrix \( P \) that fulfills the three conditions \( P^2 = P \), \( P^* = P \), and \( \mathcal{R}(P) = \mathcal{U} \). We will often use the Moore-Penrose inverse of a complex \( p \times q \) matrix \( A \). This is the unique complex \( q \times p \) matrix \( X \) such that the four equations \( AXA = A \), \( XAX = X \), \((AX)^* = AX \), and \((XA)^* =XA \) hold true (see, e.g. [3] Proposition 1.1.1). As usual, we will write \( A^\dagger \) for this matrix \( X \). If \( n \in \mathbb{N} \), if \( (p_j)_{j=1}^n \) is a sequence of positive integers, and if \( A_j \in \mathbb{C}^{p_j \times q} \) for all \( j \in \mathbb{Z}_{1,n} \), then let

\[
\text{col}(A_j)_{j=1}^n := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.
\]

We use the Löwner semi-ordering in \( \mathbb{C}^{q \times q} \), i.e., we write \( A \succeq B \) or \( B \preceq A \) in order to indicate that \( A \) and \( B \) are Hermitian complex matrices such that the matrix \( A - B \) is non-negative Hermitian.

Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. We will associate with \( (s_j)_{j=0}^\kappa \) several matrices, which we will often need in our subsequent considerations: For all \( l,m \in \mathbb{N}_0 \) with \( l \leq m \leq \kappa \), let

\[
y_{l,m}^{(s)} := \text{col}(s_j)_{j=l}^m \quad \text{and} \quad z_{l,m}^{(s)} := [s_l, s_{l+1}, \ldots, s_m]. \tag{2.1}
\]

Let

\[
H_n^{(s)} := [s_j+k]_{j,k=0}^n \quad \text{for all } n \in \mathbb{N}_0 \text{ with } 2n \leq \kappa, \tag{2.2}
\]

\[
K_n^{(s)} := [s_j+k+1]_{j,k=0}^n \quad \text{for all } n \in \mathbb{N}_0 \text{ with } 2n + 1 \leq \kappa. \tag{2.3}
\]

Let

\[
L_0^{(s)} := s_0 \quad \text{and let} \quad L_n^{(s)} := s_{2n} - z_{n,2n-1}^{(s)}(H_n^{(s)})^+z_{n,2n-1}^{(s)} \tag{2.4}
\]

for all \( n \in \mathbb{N} \) with \( 2n \leq \kappa \). Let

\[
\Theta_0^{(s)} := 0_{p \times q} \quad \text{and let} \quad \Theta_n^{(s)} := z_{n,2n-1}^{(s)}(H_n^{(s)})^+z_{n,2n-1}^{(s)} \tag{2.5}
\]

for all \( n \in \mathbb{N} \) with \( 2n - 1 \leq \kappa \). In situations in which it is obvious which sequence \( (s_j)_{j=0}^\kappa \) of complex matrices is meant, we will also write \( y_{l,m}, z_{l,m}, H_n, K_n, L_n, \text{ and } \Theta_n \) instead of \( y_{j,k}^{(s)}, z_{j,k}^{(s)}, H_n^{(s)}, K_n^{(s)}, L_n^{(s)}, \text{ and } \Theta_n^{(s)} \), respectively.

Let \( \alpha \in \mathbb{C} \) and let \( \kappa \in \mathbb{N} \cup \{\infty\} \). Then the sequence \( (v_j)_{j=0}^{\kappa-1} \) given by

\[
v_j := s_{\alpha \omega j} \quad \text{and} \quad s_{\alpha \omega j} := -\alpha s_j + s_{j+1}. \tag{2.6}
\]
for all $j \in \mathbb{Z}_{0,k-1}$ plays a key role in our following considerations. We define

\begin{align}
\Theta_{\alpha n} &:= \Theta_n^{(v)} & &\text{for all } n \in \mathbb{N}_0 \text{ with } 2n \leq k, \\
H_{\alpha n} &:= H_n^{(v)} & &\text{and } L_{\alpha n} := L_n^{(v)} & &\text{for all } n \in \mathbb{N}_0 \text{ with } 2n + 1 \leq k, \\
K_{\alpha n} &:= K_n^{(v)} & &\text{for all } n \in \mathbb{N}_0 \text{ with } 2n + 2 \leq k,
\end{align}

and $y_{\alpha l,m} := y_l^{(v)}$ and $z_{\alpha l,m} := z_l^{(v)}$ for all $l, m \in \mathbb{N}_0$ with $l \leq m \leq k$. In view of (2.2), (2.3), (2.6), and (2.8), then $-\alpha H_n + K_n = H_{\alpha n}$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq k$.

**Definition 2.1** ([7 Definition 4.2]). Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)^\kappa_{j=0}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(Q_j)^\kappa_{j=0}$ given by $Q_{2k} := L_k$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and by $Q_{2k+1} := L_{a_{2k}}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$ is called the right $\alpha$-Stieltjes parametrization of $(s_j)^\kappa_{j=0}$. In the case $\alpha = 0$, the sequence $(Q_j)^\kappa_{j=0}$ is simply said to be the right Stieltjes parametrization of $(s_j)^\kappa_{j=0}$.

**Remark 2.2** ([7 Remark 4.3]). Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(Q_j)^\kappa_{j=0}$ be a sequence of complex $p \times q$ matrices. Then it can be immediately checked by induction that there is a unique sequence $(s_j)^\kappa_{j=0}$ of complex $p \times q$ matrices such that $(Q_j)^\kappa_{j=0}$ is the right $\alpha$-Stieltjes parametrization of $(s_j)^\kappa_{j=0}$, namely the sequence $(s_j)^\kappa_{j=0}$ recursively given by $s_{2k} = \Theta_k + Q_{2k}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and $s_{2k+1} = \alpha s_{2k} + \Theta_{a_{2k}} + Q_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$.

In [7] one can find characterizations of the membership of sequences of complex $q \times q$ matrices to the class $\mathcal{K}_{q,\kappa,\alpha}$ and to several of its subclasses, respectively. For our following considerations, we introduce some of these subclasses.

Let $\alpha \in \mathbb{R}$. Let $\mathcal{K}_{q,0,\alpha}^{\kappa} := \mathcal{H}_{q,0}^{\kappa}$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\kappa}$ be the set of all sequences $(s_j)^{2n}_{j=0}$ of complex $q \times q$ matrices for which the block Hankel matrices $H_n$ and $-\alpha H_{n-1} + K_{n-1}$ are positive Hermitian, i.e., $\mathcal{K}_{q,2n,\alpha}^{\kappa} := \{(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^{\kappa} : (s_{\alpha k})^{2(n-1)}_{j=0} \in \mathcal{H}_{q,2(n-1)}^{\kappa}\}$. Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^{\kappa}$ be the set of all sequences $(s_j)^{2n+1}_{j=0}$ of complex $q \times q$ matrices for which the block Hankel matrices $H_n$ and $-\alpha H_{n-1} + K_{n-1}$ are positive Hermitian, i.e., $\mathcal{K}_{q,2n+1,\alpha}^{\kappa} := \{(s_j)^{2n+1}_{j=0} \in (\mathbb{C}^{q \times q})_{0,2n+1} : \{(s_j)^{2n}_{j=0}, (s_{\alpha k})^{2n}_{j=0} \} \in \mathcal{H}_{q,2n}^{\kappa}\}$.

Let $\mathcal{K}_{q,\kappa,\alpha}^{\kappa}$ be the set of all sequences $(s_j)^{\kappa}_{j=0}$ of complex $q \times q$ matrices such that $(s_j)^{\kappa}_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\kappa}$ for all $m \in \mathbb{N}_0$.

**Proposition 2.3** ([7 Proposition 2.20]). Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then $\mathcal{K}_{q,m,\alpha}^{\kappa} \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\kappa}$.

For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\kappa,cd} := \{(s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2n}^{\kappa} : L_n^{(s)} = 0_{q \times q}\}$, where $L_n^{(s)}$ is given by (2.4). The elements of the set $\mathcal{H}_{q,2n}^{\kappa,cd}$ are called Hankel completely degenerate. For every choice of $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n,\alpha}^{\kappa,cd} := \mathcal{K}_{q,2n,\alpha}^{\kappa} \cap \mathcal{H}_{q,2n}^{\kappa,cd}$ and let $\mathcal{K}_{q,2n+1,\alpha}^{\kappa,cd} := \{(s_j)^{2n+1}_{j=0} \in \mathcal{K}_{q,2n+1,\alpha}^{\kappa} : (s_{\alpha k})^{2n+1}_{j=0} \in \mathcal{H}_{q,2n+1}^{\kappa,cd}\}$.

**Definition 2.4.** Let $\alpha \in \mathbb{R}$ and let $(s_j)^{\infty}_{j=0} \in \mathcal{K}_{q,\kappa,\alpha}^{\kappa}$.

(a) Let $m \in \mathbb{N}_0$. Then $(s_j)^{\infty}_{j=0}$ is called $[\alpha, \infty)$-Stieltjes completely degenerate of order $m$ if $(s_j)^{mn}_{j=0} \in \mathcal{K}_{q,m,\alpha}^{\kappa,cd}$.

(b) The sequence $(s_j)^{\infty}_{j=0}$ is called $[\alpha, \infty)$-Stieltjes completely degenerate if there exists an $m \in \mathbb{N}_0$ such that $(s_j)^{\infty}_{j=0} \in [\alpha, \infty)$-Stieltjes completely degenerate of order $m$.

**Proposition 2.5** ([7 Proposition 5.9]). Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then $\mathcal{K}_{q,m,\alpha}^{\kappa,cd} \subseteq \mathcal{K}_{q,m,\alpha}^{\kappa,cd, e}$. 8
3. A Schur type algorithm for sequences of complex matrices

For the convine of the reader, we add a technical result:

Lemma 2.6 ([7 Lemma 2.9]). Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \( (s_j)_{j=0}^\kappa \in \mathbb{K}^\kappa_{q,\alpha} \).

(a) \( s_j \in \mathbb{C}^{q \times q}_H \) for all \( j \in \mathbb{Z}_{0,\kappa} \) and \( s_{\alpha j} \in \mathbb{C}^{q \times q}_H \) for all \( j \in \mathbb{Z}_{0,\kappa-1} \).

(b) \( s_{2k} \in \mathbb{C}^{q \times q}_\geq \) for all \( k \in \mathbb{N}_0 \) with \( 2k \leq \kappa \) and \( s_{\alpha 2k} \in \mathbb{C}^{q \times q}_\geq \) for all \( k \in \mathbb{N}_0 \) with \( 2k + 1 \leq \kappa \).

3. A Schur type algorithm for sequences of complex matrices

3.1. Some observations on the \( \alpha \)-Schur-transform of sequences of complex matrices

The basic object of this section was introduced in [9]. We want to recall its definition. To do this we start with the reciprocal sequence corresponding to a given sequence of complex \( p \times q \) matrices.

Definition 3.1 ([13 Definition 4.13]). Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. The sequence \( (s_j^\kappa)_{j=0}^\kappa \) given by \( s_0^\kappa := s_0^\dagger \) and \( s_j^\kappa := -s_0^\dagger \sum_{l=1}^{j-1} s_{j-l}^\dagger s_l \) for all \( j \in \mathbb{Z}_{1,\kappa} \) is said to be the reciprocal sequence corresponding to \( (s_j)_{j=0}^\kappa \).

Remark 3.2 ([13 Remark 4.17]). Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices with reciprocal sequence \( (s_j^\kappa)_{j=0}^\kappa \). For all \( m \in \mathbb{Z}_{0,\kappa} \), then \( (s_j^m)_{j=0}^\kappa \) is the reciprocal sequence corresponding to \( (s_j)_{j=0}^\kappa \).

Definition 3.3 ([9 Definition 4.1]). Let \( \alpha \in \mathbb{C} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Then we call the sequence \( (s_j^{[+,\alpha]})_{j=0}^\kappa \) given by \( s_j^{[+,\alpha]} := -\alpha s_{j-1} + s_j \) for all \( j \in \mathbb{Z}_{0,\kappa} \), where \( s_{-1} := 0_{p \times q} \), the \([+,\alpha]\)-transform of \( (s_j)_{j=0}^\kappa \).

Obviously, the \([+,\alpha]\)-transform of \( (s_j)_{j=0}^\kappa \) is connected with the sequence \( (s_{\alpha j})_{j=0}^{\kappa-1} \) given in (2.6) via \( s_j^{[+,\alpha]} = s_{\alpha j} \) for all \( j \in \mathbb{Z}_{0,\kappa-1} \). Furthermore, we have \( s_0^{[+,\alpha]} = s_0 \).

Let \( \alpha \in \mathbb{C} \). In order to prepare the basic construction in Section 9, we study the reciprocal sequence corresponding to the \([+,\alpha]\)-transform of a sequence. Let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices with \([+,\alpha]\)-transform \( (u_j)_{j=0}^\kappa \). Then we define \( (s_j^{[\alpha]})_{j=0}^\kappa \) by \( s_j^{[\alpha]} := u_j^\dagger \) for all \( j \in \mathbb{Z}_{0,\kappa} \), i.e., the sequence \( (s_j^{[\alpha]})_{j=0}^\kappa \) is defined to be the reciprocal sequence corresponding to the \([+,\alpha]\)-transform of \( (s_j)_{j=0}^\kappa \).

Definition 3.4 ([9 Definition 7.1]). Let \( \alpha \in \mathbb{C} \), let \( \kappa \in \mathbb{N} \cup \{\infty\} \), and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Then the sequence \( (s_j^{[1,\alpha]})_{j=0}^\kappa \) defined by \( s_j^{[1,\alpha]} := -s_0 s_{j+1}^\dagger s_0 \) for all \( j \in \mathbb{Z}_{0,\kappa-1} \) is called the first \( \alpha \)-Schur transform (or short the first \( \alpha \)-S-transform) of \( (s_j)_{j=0}^\kappa \).

Theorem 3.5 ([9 Theorem 7.21(a) and (b)]). Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), and let \( (s_j)_{j=0}^m \) be a sequence of complex \( q \times q \) matrices with first \( \alpha \)-S-transform \( (s_j^{[1,\alpha]})_{j=0}^{m-1} \). Then:

(a) If \( (s_j)_{j=0}^m \in \mathbb{K}^m_{q,m,\alpha} \), then \( (s_j^{[1,\alpha]})_{j=0}^{m-1} \in \mathbb{K}^m_{q,m-1,\alpha} \).

(b) If \( (s_j)_{j=0}^m \in \mathbb{K}^m_{q,m,\alpha} \), then \( (s_j^{[1,\alpha]})_{j=0}^{m-1} \in \mathbb{K}^m_{q,m-1,\alpha} \).
The next result should be considered against to the background of Problem \( M[[\alpha, \infty); (s_j)^m_{j=0}, \leq] \) and indicates that the first \( \alpha-S \)-transform for finite sequences preserves a particular matrix inequality with respect to the Löwner semi-ordering for Hermitian matrices. This observation has far-reaching consequences for our further considerations.

**Lemma 3.6.** Let \( \alpha \in \mathbb{R} \) and \( m \in \mathbb{N} \). Furthermore, let \((s_j)^m_{j=0}\) and \((t_j)^m_{j=0}\) be sequences of Hermitian complex \( q \times q \) matrices such that

\[
t_j = s_j \quad \text{for all} \quad j \in \mathbb{Z}_{0,m-1} \quad \text{and} \quad t_m \leq s_m.
\]

Denote by \((s_j^{[1,\alpha]}), (t_j^{[1,\alpha]})_{j=0}^m\) the first \( \alpha-S \)-transforms of \((s_j)^m_{j=0}\) and \((t_j)^m_{j=0}\), respectively. Then:

(a) For each \( j \in \mathbb{Z}_{0,m-1} \), the matrices \( s_j^{[1,\alpha]} \) and \( t_j^{[1,\alpha]} \) are both Hermitian.

(b) The inequality \( t_j^{[1,\alpha]} \leq s_j^{[1,\alpha]} \) holds true. Furthermore, if \( m \geq 2 \), then \( t_j^{[1,\alpha]} = s_j^{[1,\alpha]} \) for all \( j \in \mathbb{Z}_{0,m-2} \).

**Proof.** Since \( \alpha \in \mathbb{R} \) is supposed, (a) follows from [9] Lemma 7.5(f).

Since \((s_j)^m_{j=0}\) is a sequence of Hermitian matrices, from Remark A.9 we have \((s_0^0s_0)^* = s_0^0s_0^\dagger\) and, in view of [9] Remark 8.7, furthermore \( \{s_j^{[1,\alpha]}, t_j^{[1,\alpha]}\}_{j=1}^m \subseteq \mathbb{H}^{q \times q} \).

First we assume \( m = 1 \). Taking into account [9] Lemma 7.5(f) and \((s_0^0s_0)^* = s_0^0s_0^\dagger\), we obtain

\[
s_j^{[1,\alpha]} - t_j^{[1,\alpha]} = \begin{cases} s_0^{[1,\alpha]} - t_0^{[1,\alpha]} & \text{if } j = 0 \\ s_0^{[1,\alpha]} &= 0 \\ s_j^{[1,\alpha]} &= 0 \\ s_{j-1}^{[1,\alpha]} &= s_0^{[1,\alpha]} + \sum_{l=0}^{m-2} \sigma_{l+1}^{[1,\alpha]} t_{l}^{[1,\alpha]} \end{cases}
\]

\[
s_{m-1}^{[1,\alpha]} = s_0^{[1,\alpha]} + \sum_{l=0}^{m-2} \sigma_{l+1}^{[1,\alpha]} t_{l}^{[1,\alpha]} + \sum_{l=0}^{m-2} \sigma_{l}^{[1,\alpha]} t_{l}^{[1,\alpha]}.
\]

Now we consider the case \( m \geq 2 \). From (3.1) and [9] Remark 7.3 we get \( t_j^{[1,\alpha]} = s_j^{[1,\alpha]} \) for all \( j \in \mathbb{Z}_{0,m-2} \). Because of [9] Lemma 7.8, (3.1), and \( (s_0^0s_0)^* = s_0^0s_0^\dagger \), we conclude

\[
s_j^{[1,\alpha]} = s_0^{[1,\alpha]} + \sum_{l=0}^{m-2} \sigma_{l+1}^{[1,\alpha]} t_{l}^{[1,\alpha]} + \sum_{l=0}^{m-2} \sigma_{l}^{[1,\alpha]} t_{l}^{[1,\alpha]}.
\]

In view of \( t_m \leq s_m \), we see from (3.2) and (3.3) that \( t_j^{[1,\alpha]} \leq s_j^{[1,\alpha]} \).
3. A Schur type algorithm for sequences of complex matrices

3.2. The algorithm

The basic object of this section was introduced in \[9\] Section 10. For the convenience of the reader, we want to recall our motivation: Let \(\alpha \in \mathbb{C}\) and \(\kappa \in \mathbb{N} \cup \{\infty\}\). Let \((s_j)_{j=0}^\infty\) be a sequence from \(\mathbb{C}^{q \times q}\) and let \((s_j^{[1,\alpha]})_{j=0}^{\kappa-1}\) be its first \(\alpha\)-S-transform (see Definition 3.1). Then we want to recover the sequence \((s_j)_{j=0}^\infty\) on the basis of the sequence \((s_j^{[1,\alpha]})_{j=0}^{\kappa-1}\) and the matrix \(s_0\). Against this background we recall the following notion.

**Definition 3.7** (\[9\] Definition 10.1). Let \(\alpha \in \mathbb{C}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), let \((t_j)_{j=0}^\infty\) be a sequence of complex \(p \times q\) matrices, and let \(A\) be a complex \(p \times q\) matrix. The sequence \((t_j^{[1,\alpha,A]})_{j=0}^{\kappa-1}\) recursively defined by

\[
t_j^{[1,\alpha,A]} := A + \sum_{l=1}^j \alpha^{j-l} A \sum_{k=0}^{l-1} t_{l-k-1} A^\dagger (t_{k}^{[1,\alpha,A]}+[\alpha])
\]

for all \(j \in \mathbb{Z}_{1,\kappa+1}\) is called the first inverse \(\alpha\)-S-transform corresponding to \([(t_j)_{j=0}^m,A]\).

**Remark 3.8** (\[9\] Remark 10.2). Let \(\alpha \in \mathbb{C}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), let \((t_j)_{j=0}^\infty\) be a sequence from \(\mathbb{C}^{p \times q}\), and let \(A \in \mathbb{C}^{p \times q}\). Denote by \((s_j)_{j=0}^\infty\) the first inverse \(\alpha\)-S-transform corresponding to \([(t_j)_{j=0}^\infty,A]\). In view of Definition 3.7, one can easily see that, for all \(m \in \mathbb{Z}_{0,\kappa}\), the sequence \((s_j)_{j=0}^m\) depends only on the matrices \(A\) and \(t_0, t_1, \ldots, t_m\) and it is hence exactly the first inverse \(\alpha\)-S-transform corresponding to \([(t_j)_{j=0}^m,A]\).

**Definition 3.9** (\[13\] Definition 4.3). Let \(\kappa \in \mathbb{N} \cup \{\infty\}\) and let \((s_j)_{j=0}^\infty\) be a sequence of complex \(p \times q\) matrices. We then say that \((s_j)_{j=0}^\infty\) is first term dominant if \(\bigcup_{l=0}^{\infty} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)\) and \(\mathcal{N}(s_0) \subseteq \bigcap_{l=0}^{\infty} \mathcal{N}(s_j)\). The set of all first term dominant sequences \((s_j)_{j=0}^\infty\) of complex \(p \times q\) matrices will be denoted by \(D_{p \times q,\kappa}\).

**Remark 3.10** (\[9\] Remark 10.3). Let \(\alpha \in \mathbb{C}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), let \((t_j)_{j=0}^\infty\) be a sequence from \(\mathbb{C}^{p \times q}\), and let \(A \in \mathbb{C}^{p \times q}\). Denote by \((s_j)_{j=0}^{\kappa+1}\) the first inverse \(\alpha\)-S-transform corresponding to \([(t_j)_{j=0}^\infty,A]\). From Definition 3.7, we easily see then that \((s_j)_{j=0}^{\kappa+1} \in D_{p \times q,\kappa+1}\).

**Lemma 3.11** (\[9\] Lemma 10.4). Let \(\alpha \in \mathbb{C}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), let \((t_j)_{j=0}^\infty\) be a sequence from \(\mathbb{C}^{p \times q}\), and let \(A \in \mathbb{C}^{p \times q}\). Denote by \((s_j)_{j=0}^{\kappa+1}\) the first inverse \(\alpha\)-S-transform corresponding to \([(t_j)_{j=0}^\infty,A]\) and by \((s_j^{[1,\alpha,A]})_{j=0}^{\kappa+1}\) the \([+,\alpha]\)-transform of \((s_j)_{j=0}^{\kappa+1}\). Then \(s_0 = A\) and \(s_j = \alpha s_j-1 + AA^\dagger \sum_{k=0}^{j-1} t_{j-k-1} A^\dagger s_k^{[+,\alpha]}\) for all \(j \in \mathbb{Z}_{1,\kappa+1}\).

**Lemma 3.12** (\[9\] Lemma 10.13). Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N} \cup \{\infty\}\), let \((t_j)_{j=0}^\infty\) be a sequence of Hermitian complex \(q \times q\) matrices, and let \(A\) be a Hermitian complex \(q \times q\) matrix. Then the first inverse \(\alpha\)-S-transform \((t_j^{[1,\alpha,A]})_{j=0}^{\kappa+1}\) corresponding to \([(t_j)_{j=0}^\infty,A]\) is a sequence of Hermitian complex \(q \times q\) matrices. The following result can be considered as an analogue of Lemma 3.6 for the first inverse \(\alpha\)-S-transform in the case of given sequences of Hermitian complex \(q \times q\) matrices.

**Lemma 3.13.** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \(A \in \mathbb{C}_H^{q \times q}\). Further, let \((s_j)_{j=0}^m\) and \((t_j)_{j=0}^m\) be sequences of Hermitian complex \(q \times q\) matrices such that \(t_m \leq s_m\) and, in the case \(m \geq 1\), furthermore, \(t_j = s_j\) for all \(j \in \mathbb{Z}_{0,m-1}\) hold true. Denote by \((s_j^{[1,\alpha,A]})_{j=0}^{m+1}\) and \((t_j^{[1,\alpha,A]})_{j=0}^{m+1}\) the first inverse \(\alpha\)-S-transform corresponding to \([(s_j)_{j=0}^m,A]\) and \([(t_j)_{j=0}^m,A]\), respectively. Then \(t_j^{[1,\alpha,A]} = s_j^{[1,\alpha,A]}\) for each \(j \in \mathbb{Z}_{0,m}\) and \(t_j^{[1,\alpha,A]} \leq s_j^{[-1,\alpha,A]}\).
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Proof. For each $j \in \mathbb{Z}_{0,m+1}$ let $r_j := s_j^{-1,A}$ and $u_j := t_j^{-1,A}$. From (3.11) and Remark 5.8 we obtain immediately $u_j = r_j$ for each $j \in \mathbb{Z}_{0,m}$. It remains to check $u_{m+1} \leq r_{m+1}$. Because of $A^* = A$ and Remark 3.6 we have $(A^*A)^* = AA^\dagger$. Since $(s_j)_{j=0}^m$ and $(t_j)_{j=0}^m$ are sequences of Hermitian complex matrices, Lemma 3.11 yields \( \{ r_{m+1}, u_{m+1} \} \subseteq \mathbb{C}^{q \times q} \). If $m = 0$, then Lemma 3.11 \((A^\dagger A)^* = AA^\dagger, \) and (3.1) provide us

\[
\begin{align*}
    r_{m+1} - u_{m+1} &= r_1 - u_1 = \alpha r_0 + AA^\dagger s_0 A^\dagger r_0^+ - (\alpha u_0 + AA^\dagger t_0 A^\dagger u_0) \\
    &= \alpha A + AA^\dagger s_0 A^\dagger (\alpha + AA^\dagger t_0 A^\dagger) \\
    &= AA^\dagger (s_0 - t_0) A^\dagger A = (A^\dagger A)^*(s_m - t_m) A^\dagger A.
\end{align*}
\]

Now we consider the case $m \geq 1$. Using Lemma 3.11, \((A^\dagger A)^* = AA^\dagger, \) and again (3.1), we get then

\[
\begin{align*}
    r_{m+1} - u_{m+1} &= \alpha r_m + AA^\dagger \sum_{k=0}^m s_{m-k} A^\dagger r_k^+ - (\alpha u_m + AA^\dagger \sum_{k=0}^m t_{m-k} A^\dagger u_k^+) \\
    &= \alpha A + AA^\dagger \sum_{k=1}^m s_{m-k} A^\dagger r_k^+ \\
    &\quad - \left( \alpha u_m + AA^\dagger t_m A^\dagger u_0^+ + AA^\dagger \sum_{k=1}^m t_{m-k} A^\dagger u_k^+ \right) \\
    &= \alpha A + AA^\dagger s_m A^\dagger r_0 + AA^\dagger \sum_{k=1}^m s_{m-k} A^\dagger (-\alpha r_{k-1} + r_k) \\
    &\quad - \left[ \alpha u_m + AA^\dagger t_m A^\dagger u_0 + AA^\dagger \sum_{k=1}^m t_{m-k} A^\dagger (-\alpha u_{k-1} + u_k) \right] \\
    &= \alpha A + AA^\dagger s_m A^\dagger A + AA^\dagger \sum_{k=1}^m s_{m-k} A^\dagger (-\alpha r_{k-1} + r_k) \\
    &\quad - \left[ \alpha A + AA^\dagger t_m A^\dagger A + AA^\dagger \sum_{k=1}^m s_{m-k} A^\dagger (-\alpha r_{k-1} + r_k) \right] \\
    &= AA^\dagger (s_m - t_m) A^\dagger A = (A^\dagger A)^*(s_m - t_m) A^\dagger A.
\end{align*}
\]

In view of $t_m \leq s_m$, we see from (3.4) and (3.5) that $u_{m+1} \leq r_{m+1}$. \(\square\)

The $\alpha$-Schur transform for sequences of complex $p \times q$ matrices introduced in Section 3.1 generates in a natural way a corresponding algorithm for (finite and infinite) sequences of complex $q \times q$ matrices. In generalization of Definition 3.4, we introduced the following:

Definition 3.14 ([9, Definition 8.1]). Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{ \infty \}$, and let \((s_j)_{j=0}^\kappa \) be a sequence of complex $p \times q$ matrices. The sequence \((s_j^{[0,\alpha])_{j=0}^\kappa \) given by \(s_j^{[0,\alpha]} := s_j\) for all \(j \in \mathbb{Z}_{0,\kappa}\) is called the 0-th $\alpha$-S-transform of \((s_j)_{j=0}^\kappa \). In the case $\kappa \geq 1$, for all \(k \in \mathbb{Z}_{1,\kappa}\), the $k$-th $\alpha$-S-transform \((s_j^{[k,\alpha])_{j=0}^{\kappa-k} \) of \((s_j)_{j=0}^\kappa \) is recursively defined by $s_j^{[k,\alpha]} := t_j^{[1,\alpha]}$ for all $j \in \mathbb{Z}_{0,\kappa-k}$, where \((t_j)_{j=0}^{\kappa-(k-1)} \)$ denotes the $(k-1)$-th $\alpha$-S-transform of \((s_j)_{j=0}^\kappa \).

A comprehensive investigation of this algorithm was carried out in [9].
4. On some classes of holomorphic matrix-valued functions

4.1. The class $S_{q,\alpha,\infty}$

The use of several classes of holomorphic matrix-valued functions is one of the special features of this paper. In this section, we summarize some basic facts about the class of $[\alpha, \infty)$-Stieltjes functions of order $q$, which are mostly taken from our former paper [11]. If $A \in \mathbb{C}^{q \times q}$, then let $\Re A := \frac{1}{2}(A + A^*)$ and $\Im A := \frac{1}{2i}(A - A^*)$ be the real part and the imaginary part of $A$, respectively. Let $\Pi_+ := \{z \in \mathbb{C}: \Re z > 0\}$ be the open upper half plane of $\mathbb{C}$. The first class of functions, which plays an essential role in this paper, is the following.

**Definition 4.1.** Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$. Then $F$ is called a $[\alpha, \infty)$-Stieltjes function of order $q$ if $F$ satisfies the following three conditions:

(I) $F$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$.

(II) For all $w \in \Pi_+$, the matrix $\Im [F(w)]$ is non-negative Hermitian.

(III) For all $w \in (-\infty, \alpha)$, the matrix $F(w)$ is non-negative Hermitian.

We denote by $S_{q,\alpha,\infty}$ the set of all $[\alpha, \infty)$-Stieltjes functions of order $q$.

**Example 4.2.** Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}_z^{q \times q}$. Then $F: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ defined by $F(z) := A$ belongs to $F \in S_{q,\alpha,\infty}$.

For a comprehensive survey on the class $S_{q,\alpha,\infty}$, we refer the reader to [11]. We give now a useful characterization of the membership of a function to the class $S_{q,\alpha,\infty}$. Let $\Pi_- := \{z \in \mathbb{C}: \Im z < 0\}$ and let $\mathbb{C}_{\alpha,-} := \{z \in \mathbb{C}: \Re z \in (-\infty, \alpha)\}$.

**Proposition 4.3** ([11] Proposition 4.4). Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Then $F$ belongs to $S_{q,\alpha,\infty}$ if and only if the following four conditions are fulfilled:

(I) $F$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$.

(II) For each $z \in \Pi_+$, the matrix $\Im F(z)$ is non-negative Hermitian.

(III) For each $z \in \Pi_-$, the matrix $-\Im F(z)$ is non-negative Hermitian.

(IV) For each $z \in \mathbb{C}_{\alpha,-}$, the matrix $\Re F(z)$ is non-negative Hermitian.
Corollary 4.4. Let $\alpha \in \mathbb{R}$ and let $F \in S_{q,[\alpha,\infty)}$. For each $z \in \mathbb{C} \setminus \mathbb{R}$, then $(\text{Im } z)^{-1} \text{Im } F(z) \in \mathbb{C}^q_{\geq q}$.

Proof. In view of $\mathbb{C} \setminus \mathbb{R} = \Pi_+ \cup \Pi_-$ and $F \in S_{q,[\alpha,\infty)}$, this follows from conditions (II) and (III) of Proposition 4.3. \hfill \Box

The functions belonging to the class $S_{q,[\alpha,\infty)}$ admit an important integral representation. To state this, we introduce some terminology: If $\mu$ is a non-negative Hermitian measure on a measurable space $(\Omega, \mathcal{A})$ and if $K \in \{\mathbb{R}, \mathbb{C}\}$, then we will use $L^1(\Omega, \mathcal{A}, \mu; K)$ to denote the space of all Borel-measurable functions $f: \Omega \to K$ for which the integral $\int_{\Omega} f d\mu$ exists. In preparing the desired integral representation, we observe that, for all $\mu \in M^+_q([\alpha, \infty))$ and each $z \in \mathbb{C} \setminus [\alpha, \infty)$, the function $h_{\alpha,z}: [\alpha, \infty) \to \mathbb{C}$ defined by $h_{\alpha,z}(t) := (1 + t - \alpha)/(t - z)$ belongs to $L^1([\alpha, \infty), \mathcal{B}_{[\alpha,\infty)}, \mu; \mathbb{C})$.

Theorem 4.5 (cf. [11, Theorem 3.6]). Let $\alpha \in \mathbb{R}$ and let $F: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$. Then:

(a) If $F \in S_{q,[\alpha,\infty)}$, then there is a unique matrix $\gamma \in \mathbb{C}^{q \times q}$ and a unique non-negative Hermitian measure $\mu \in M^+_q([\alpha, \infty))$ such that

\[ F(z) = \gamma + \int_{[\alpha, \infty)} \frac{1 + t - \alpha}{t - z} \mu(dt) \quad (4.1) \]

holds true for each $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(b) If there are a matrix $\gamma \in \mathbb{C}^{q \times q}$ and a non-negative Hermitian measure $\mu \in M^+_q([\alpha, \infty))$ such that $F$ can be represented via (4.1) for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $F$ belongs to the class $S_{q,[\alpha,\infty)}$.

In the special case that $q = 1$ and $\alpha = 0$ hold true, Theorem 4.5 can be found in Krein/Nudelman [19, Appendix].

Notation 4.6. For all $F \in S_{q,[\alpha,\infty)}$, we will write $(\gamma_F, \mu_F)$ for the unique pair $(\gamma, \mu) \in \mathbb{C}^{q \times q} \times M^+_q([\alpha, \infty))$ for which the representation (4.1) holds true for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

We are interested in the structure of the values of functions belonging to $S_{q,[\alpha,\infty)}$.

Proposition 4.7 (cf. [11, Proposition 3.15]). Let $\alpha \in \mathbb{R}$ and let $F \in S_{q,[\alpha,\infty)}$. For all $z \in \mathbb{C} \setminus [\alpha, \infty)$, then $\mathcal{R}(F(z)) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, \infty)))$, $N(F(z)) = N(\gamma_F) \cap N(\mu_F([\alpha, \infty)))$, and $\mathcal{R}([F(z)]^*) = \mathcal{R}(F(z))$.

In the sequel, we will sometimes meet situations where interrelations between the null space (respectively, column space) of a function $F \in S_{q,[\alpha,\infty)}$ and the null space (respectively, column space) of a given matrix $A \in \mathbb{C}^{p \times q}$ are of interest.

Notation 4.8. Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q \times p}$. We denote by $S_{q,[\alpha,\infty)}[A]$ the set of all $F \in S_{q,[\alpha,\infty)}$ which satisfy $\mathcal{R}(F(z)) \subseteq \mathcal{R}(A)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Observe that the fact that a matrix-valued function $F \in S_{q,[\alpha,\infty)}$ belongs to the subclass $S_{q,[\alpha,\infty)}[A]$ of the class $S_{q,[\alpha,\infty]}$ can be characterized by several conditions (see [11, Lemma 3.18]). In particular, we are led to the following useful characterization of the elements of the class $S_{q,[\alpha,\infty)}[A]$.

Lemma 4.9. Let $\alpha \in \mathbb{R}$, let $F \in S_{q,[\alpha,\infty)}$, and let $A \in \mathbb{C}^{q \times p}$. Then $F \in S_{q,[\alpha,\infty)}[A]$ if and only if $\mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, \infty))) \subseteq \mathcal{R}(A)$.

Proof. Combine Proposition 4.7 with Remark A.6. \hfill \Box
4. On some classes of holomorphic matrix-valued functions

4.2. On some subclasses of $S_{q,[α,∞)}$

An essential feature of our subsequent considerations is the use of several subclasses of $S_{q,[α,∞)}$. In this section, we summarize some basic facts about these subclasses, which are characterized by growth properties on the positive imaginary axis. It should be mentioned that scalar versions of the function classes were introduced and studied by Kats/Krein [16]. Furthermore, we recognized in [10] that a detailed analysis of the behavior on the positive imaginary axis of the concrete functions of $F \in S_{q,[α,∞)}$ under study is very useful. For this reason, we turn now our attention to some subclasses of $S_{q,[α,∞)}$, which are described in terms of their growth on the positive imaginary axis.

For each $α \in \mathbb{R}$, first we consider the set

$$S_{q,[α,∞)}^α := \left\{ F \in S_{q,[α,∞)} : \lim_{y \to ∞} \|F(iy)\|_S = 0 \right\}$$

and we observe that $S_{q,[α,∞)}^α = \{ F \in S_{q,[α,∞)} : γ_F = 0 \}$ (see [11, Corollary 3.14]). In the following considerations, we associate with a function $F \in S_{q,[α,∞)}$ often the unique ordered pair $(γ_F, μ_F)$ given via Notation 4.6.

In [10, Section 4], we considered a particular subclass of the class $S_{q,[α,∞)}$, namely $S_{q,[α,∞)}^α$. We have seen there that, for an arbitrary function $F \in S_{q,[α,∞)}$, the null space of $F(z)$ is independent from the concrete choice of $z \in C \setminus [α, ∞)$. For the case that $F$ belongs to $S_{q,[α,∞)}^α$, a complete description of this constant null space was given in [10, Proposition 3.7]. Against to this background, we single out now a special subclass of $S_{q,[α,∞)}^α$, which is characterized by the interrelation of this constant null space to the null space of a prescribed matrix $A \in \mathbb{C}^{p \times q}$.

More precisely, in view of Notation 4.6 for all $A \in \mathbb{C}^{p \times q}$, let

$$S_{q,[α,∞),A}^α := \left\{ F \in S_{q,[α,∞)} : \mathcal{N}(A) \subseteq \mathcal{N}(μ_F([α, ∞))) \right\}.$$  \hspace{1cm} (4.3)

In our investigations in [10], the role of the matrix $A$ was taken by a matrix which is generated from the sequence of the given data of the moment problem via a Schur type algorithm. In [10, Remark 4.4], some characterizations of the set $S_{q,[α,∞),A}^α$ are proved. Furthermore, note that [10, Proposition 4.7] contains essential information on the structure of the set $S_{q,[α,∞),A}^α$, where $A \in \mathbb{C}^{p \times q}$ is arbitrarily given. In our considerations, the case $A \in \mathbb{C}^{p \times q}$ is of particular interest.

**Remark 4.10.** Let $α \in \mathbb{R}$ and let $A \in \mathbb{C}^{q \times q}_H$. Then [10, Remark 4.4] yields $S_{q,[α,∞),A}^α = \{ F \in S_{q,[α,∞)} : \mathcal{R}(μ_F([α, ∞))) \subseteq \mathcal{R}(A) \}$. Furthermore, $S_{q,[α,∞),A}^α[A] = S_{q,[α,∞)}^α \cap S_{q,[α,∞),A}^α[A]$, where $A \in \mathbb{C}^{p \times q}$ is arbitrarily given. In our considerations, the case $A \in \mathbb{C}^{q \times q}$ is of particular interest.

A further important subclass of the class $S_{q,[α,∞)}$ is the set

$$S_{q,[α,∞)} := \left\{ F \in S_{q,[α,∞)} : \sup_{y \in [1,∞)} y \|F(iy)\|_S < ∞ \right\}. \hspace{1cm} (4.4)$$

Let $Ω$ be a non-empty closed subset of $\mathbb{R}$ and let $σ \in \mathcal{M}_q^π(Ω)$. Then, in view of [11, Lemma A.4], for each $z \in C \setminus Ω$, the function $f_z : Ω \to \mathbb{C}$ defined by $f_z(t) := (t \cdot z)^{-1}$ belongs to $\mathcal{L}^1(Ω, \mathbb{B}_Ω, σ; \mathbb{C})$. In particular, for each $α \in \mathbb{R}$ and each $σ \in \mathcal{M}_q^π([α, ∞))$, the matrix-valued function $S_σ : C \setminus [α, ∞) \to \mathbb{C}^{q \times q}$ given by

$$S_σ(z) := \int_{[α, ∞)} \frac{1}{t \cdot z} σ(dt). \hspace{1cm} (4.5)$$
is well defined and it is called \([\alpha, \infty)\)-Stieltjes transform of \(\sigma\). It is readily checked that, for each \(z \in \mathbb{C} \setminus [\alpha, \infty)\), one has \(\tau \in \mathbb{C} \setminus [\alpha, \infty)\) and \(S_\sigma(\tau) = [S_\sigma(z)]^*\).

There is an important characterization of the set of all \([\alpha, \infty)\)-Stieltjes transforms of measures belonging to \(\mathcal{M}_q^\infty([\alpha, \infty))\):

**Theorem 4.11** (\cite{10} Theorem 3.2). Let \(\alpha \in \mathbb{R}\). The mapping \(\sigma \mapsto S_\sigma\) given by \(4.3\) is a bijective correspondence between \(\mathcal{M}_q^\infty([\alpha, \infty))\) and \(S_{0,q,[\alpha, \infty)}\). In particular, \(S_{0,q,[\alpha, \infty)} = \{S_\sigma : \sigma \in \mathcal{M}_q^\infty([\alpha, \infty))\}\).

For each \(F \in S_{0,q,[\alpha, \infty)}\), the unique measure \(\sigma \in \mathcal{M}_q^\infty([\alpha, \infty))\) satisfying \(S_\sigma = F\) is called the \([\alpha, \infty)\)-Stieltjes measure of \(\sigma\) and we will also write \(\sigma_F\) for \(\sigma\). Theorem 4.11 and (4.4) indicate that the \([\alpha, \infty)\)-Stieltjes transform \(S_\sigma\) of a measure \(\sigma \in \mathcal{M}_q^\infty([\alpha, \infty))\) is characterized by a particular mild growth on the positive imaginary axis.

In view of Theorem 4.11, the Problems \(M[[\alpha, \infty); (s_j)^m_{j=0}, =]\) and \(M[[\alpha, \infty); (s_j)^m_{j=0}, \leq]\) can be reformulated as an equivalent problem in the class \(S_{0,q,[\alpha, \infty)}\) as follows:

**Problem** \((S[[\alpha, \infty); (s_j)^m_{j=0}, =])\). Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)^m_{j=0}\) be a sequence of complex \(q \times q\) matrices. Parametrize the set \(\mathcal{S}_{\ell,q,[\alpha, \infty)}[[s_j]^m_{j=0}, =]\) of all \(F \in S_{\ell,q,[\alpha, \infty)}\) the \([\alpha, \infty)\)-Stieltjes measure of which belongs to \(\mathcal{M}_q^\infty([\alpha, \infty)); (s_j)^m_{j=0}, =\).

**Problem** \((S[[\alpha, \infty); (s_j)^m_{j=0}, \leq])\). Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)^m_{j=0}\) be a sequence of complex \(q \times q\) matrices. Parametrize the set \(\mathcal{S}_{\ell,q,[\alpha, \infty)}[[s_j]^m_{j=0}, \leq]\) of all \(F \in S_{\ell,q,[\alpha, \infty)}\) the \([\alpha, \infty)\)-Stieltjes measure of which belongs to \(\mathcal{M}_q^\infty([\alpha, \infty)); (s_j)^m_{j=0}, \leq\).

In \cite{10} Section 6, we stated a reformulation of the original power moment problem \(M[[\alpha, \infty); (s_j)^m_{j=0}, =]\) as an equivalent problem of finding a prescribed asymptotic expansion in a sector of the open upper half plane \(\Pi_+\).

For all \(\alpha \in \mathbb{R}\) and all \(\kappa \in \mathbb{N} \cup \{\infty\}\), we now consider the class

\[
\mathcal{S}_{\kappa,q,[\alpha, \infty)} := \left\{ F \in S_{\kappa,q,[\alpha, \infty)} : \sigma_F \in \mathcal{M}_q^\infty([\alpha, \infty)) \right\}. \tag{4.6}
\]

**Lemma 4.12** (\cite{10} Lemma 3.9). Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N}_0 \cup \{\infty\}\), let \(F \in \mathcal{S}_{\kappa,q,[\alpha, \infty)}\), and let \(\sigma_F\) be the \([\alpha, \infty)\)-Stieltjes measure of \(F\). For each \(z \in \mathbb{C} \setminus [\alpha, \infty)\), then \(\mathcal{R}(F(z)) = \mathcal{R}(\sigma_F([\alpha, \infty)))\) and \(\mathcal{N}(F(z)) = \mathcal{N}(\sigma_F([\alpha, \infty)))\).

**Remark 4.13.** Let \(\alpha \in \mathbb{R}\), let \(\kappa \in \mathbb{N}_0 \cup \{\infty\}\), and let \(F \in \mathcal{S}_{\kappa,q,[\alpha, \infty)}\). From (4.6) and (4.4) we see that \(\lim_{y \to \infty} F(iy) = 0_{q \times q}\).

**Remark 4.14.** Let \(\alpha \in \mathbb{R}\). In view of the (4.6), Remark 1.2 (4.3), Remark 1.13 and (4.2), then \(\mathcal{S}_{\infty,q,[\alpha, \infty)} \subseteq \mathcal{S}_{\ell,q,[\alpha, \infty)} \subseteq \mathcal{S}_{m,q,[\alpha, \infty)} \subseteq \mathcal{S}_{0,q,[\alpha, \infty)} \subseteq \mathcal{S}_{q^*,[\alpha, \infty)} \subseteq \mathcal{S}_{q,[\alpha, \infty)}\) for all \(\ell, m \in \mathbb{N}\) with \(\ell \geq m\).

### 4.3. On the class \(\mathcal{R}_{0,q}(\Pi_+)\)

In this section, we consider a class of holomorphic \(q \times q\) matrix-valued functions in the open upper half plane \(\Pi_+ := \{z \in \mathbb{C} : \text{Im} z \in (0, \infty)\}\). A function \(F : \Pi_+ \to \mathbb{C}^{q \times q}\) is called a Herglotz-Nevanlinna function in \(\Pi_+\) if \(F\) is holomorphic in \(\Pi_+\) and satisfies \(\text{Im}[F(z)] \in \mathbb{C}^{q \times q}\) for all \(z \in \Pi_+\). We denote by \(\mathcal{R}_q(\Pi_+)\) the set of all Herglotz-Nevanlinna functions in \(\Pi_+\). Furthermore, let

\[
\mathcal{R}_{0,q}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) : \sup_{y \in [1, \infty)} y \| F(iy) \|_S < \infty \right\}. \tag{4.7}
\]
The functions belonging to \( \mathcal{R}_{0,q}(\Pi_+) \) admit an important integral representation.

**Theorem 4.15** (see, e.g., [2 Theorem 8.7]).  
(a) For each \( F \in \mathcal{R}_{0,q}(\Pi_+) \), there is a unique non-negative Hermitian \( q \times q \) measure \( \mu \in \mathcal{M}_q^\geq(\mathbb{R}) \) such that \( F \) admits the representation

\[
F(z) = \int_{\mathbb{R}} \frac{1}{t - z} \mu(dt) \tag{4.8}
\]

for all \( z \in \Pi_+ \).

(b) If \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) is a matrix-valued function for which there exists a \( \mu \in \mathcal{M}_q^\geq(\mathbb{R}) \) such that (4.8) is satisfied for all \( z \in \Pi_+ \), then \( F \in \mathcal{R}_{0,q}(\Pi_+) \).

For each \( F \in \mathcal{R}_{0,q}(\Pi_+) \), the unique \( \mu \in \mathcal{M}_q^\geq(\mathbb{R}) \) satisfying (4.8) is called the Stieltjes measure of \( F \).

The following result on the class \( \mathcal{R}_{0,q}(\Pi_+) \), which is stated, e.g., in [2 Lemma 8.9], plays a key role in our subsequent considerations.

**Lemma 4.16.** Let \( M \in \mathbb{C}^{q \times q} \) and let \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) be a matrix-valued function which is holomorphic in \( \Pi_+ \) and which fulfills

\[
\begin{bmatrix}
M \\
F^*(w)
\end{bmatrix} \in \mathbb{C}^{2q \times 2q}
\]

for all \( w \in \Pi_+ \).

Then \( F \in \mathcal{R}_{0,q}(\Pi_+) \) and \( \text{sup}_{y \in (0,\infty)} \| F(iy) \| \leq \| M \|_S \). Furthermore, the Stieltjes measure \( \mu \) of \( F \) satisfies \( M - \mu(\mathbb{R}) \in \mathbb{C}^{q \times q}_{\geq} \).

**Proposition 4.17.** Let \( \alpha \in \mathbb{R} \), let \( G \in \mathcal{S}_{0,q}[\alpha,\infty) \), and let \( F := \text{Rstr}_{\Pi_+} G \). Then \( F \in \mathcal{R}_{0,q}(\Pi_+) \).

If \( \sigma_G \) be the \( [\alpha,\infty) \)-Stieltjes measure of \( G \) and if \( \mu_F \) is the Stieltjes measure of \( F \), then \( \mu_F(\mathbb{R} \setminus [\alpha, \infty)) = 0_{q \times q} \) and \( \sigma_G = \text{Rstr}_{\alpha,\infty} \mu_F \).

**Proof.** From Definition 4.1, (4.3), and (4.7) we see that \( F \in \mathcal{R}_{0,q}(\Pi_+) \). The rest was already shown in the proof of [11 Theorem 5.1].

5. The class \( \mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] \)

In [10] Section 4, we considered particular subclasses of the class \( \mathcal{S}_{\kappa, q, \{\alpha, \infty\}} \), which was introduced in (1.4) for \( \kappa = 0 \) and in (4.6) for all \( \kappa \in \mathbb{N} \cup \{ \infty \} \). In view of Theorem 4.11 for each function \( F \) belonging to one of the classes \( \mathcal{S}_{\kappa, q, \{\alpha, \infty\}} \) with some \( \kappa \in \mathbb{N} \cup \{ \infty \} \), we can consider the \( [\alpha, \infty) \)-Stieltjes measure \( \sigma_F \) of \( F \). Now, taking Remark 1.2 into account, we turn our attention to subclasses of functions \( F \in \mathcal{S}_{\kappa, q, \{\alpha, \infty\}} \) with prescribed first \( \kappa + 1 \) power moments of the \( [\alpha, \infty) \)-Stieltjes measure \( \sigma_F \).

For all \( \alpha \in \mathbb{R} \), all \( \kappa \in \mathbb{N} \cup \{ \infty \} \), and each sequence \( (s_j)_{j=0}^\kappa \) of complex \( q \times q \) matrices, we consider the class

\[
\mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] := \left\{ F \in \mathcal{S}_{\kappa, q, \{\alpha, \infty\}} : \sigma_F \in \mathcal{M}_q^\geq([\alpha, \infty); (s_j)_{j=0}^\kappa, =) \right\}. \tag{5.1}
\]

**Lemma 5.1** ([10 Lemma 5.6]). Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N} \cup \{ \infty \} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] \subseteq \mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] \). If \( \iota \in \mathbb{N} \cup \{ \infty \} \), with \( \iota \leq \kappa \) then

\[
\mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] = \bigcap_{m=0}^\kappa \mathcal{S}_{m, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =] \subseteq \mathcal{S}_{\kappa, q, \{\alpha, \infty\}}[(s_j)_{j=0}^\kappa, =].
\]
The following result shows the relevance of the set $K_{q,k,\alpha}^{\geq e}$.

**Theorem 5.4 (17 Theorem 5.3).** Let $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then $S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=] \neq \emptyset$ if and only if $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$.

**Remark 5.3.** Let $\alpha \in \mathbb{R}$ and let $F : \mathbb{C} \setminus [\alpha, \infty) \rightarrow \mathbb{C}^{q \times q}$ be a function. Then:

(a) If $F \in S_{0,q,[\alpha,\infty]}$, then $F \in S_{0,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$ with $s_0 := \sigma_F([\alpha, \infty))$.

(b) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$. If $F \in S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$, then $F \in S_{0,q,[\alpha,\infty]}$ and $\sigma_F([\alpha, \infty)) = s_0$.

Now we state a useful characterization of the set of functions given in (5.1).

**Theorem 5.4 (17 Theorem 5.4).** Let $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. In view of (1.5), then

$$S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=] = \{ \sigma \in \mathcal{M}_q^\infty [\alpha, \infty); (s_j)_{j=0}^\kappa,= \}$$

Theorem 5.4 shows that $S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$ coincides with the solution set of Problem $S[\alpha, \infty); (s_j)_{j=0}^\kappa,=]$, which is via $\alpha, \infty)$-Stieltjes transform equivalent to the original Problem $M[\alpha, \infty); (s_j)_{j=0}^\kappa,=]$. Thus, the investigation of the set $S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$ is a central theme of our further considerations. At the end of this section, we give a useful technical result. It is based on the following:

**Proposition 5.5 (17 Proposition 5.5).** Let $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$, and let $F \in S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$. Then:

(a) $\mathcal{R}(F(z)) = \mathcal{R}(s_0)$ and $\mathcal{N}(F(z)) = \mathcal{N}(s_0)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(b) $[F(z)][F(z)]^\dagger = s_0 s_0^\dagger$ and $[F(z)]^\dagger[F(z)] = s_0^\dagger s_0$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

(c) The function $F$ belongs to the class $S_{q,[\alpha,\infty]}$ with $\mathcal{R}(s_0) = \mathcal{R}(\gamma_F) + \mathcal{R}(\mu_F([\alpha, \infty)))$ and $\mathcal{N}(s_0) = \mathcal{N}(\gamma_F) \cap \mathcal{N}(\mu_F([\alpha, \infty)))$.

**Proposition 5.6.** Let $\alpha \in \mathbb{R}$, $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa \in K_{q,k,\alpha}^{\geq e}$. Then $S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=] \subseteq S_{q,[\alpha,\infty]}[s_0]$.

**Proof.** Let $F \in S_{\kappa,q,[\alpha,\infty]}[(s_j)_{j=0}^\kappa,=]$. In view of (5.1), (1.4), and (4.6), then $F \in S_{q,[\alpha,\infty]}$. Proposition 5.5 shows that $F F^\dagger = s_0 s_0^\dagger$. Thus, $s_0 s_0^\dagger F = F$. Hence, $F \in S_{q,[\alpha,\infty]}[s_0]$.

## 6. The class $S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m,\leq]$
Remark 6.2. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), and let \((s_j)_{j=0}^m\) be a sequence from \( \mathbb{C}^{q \times q} \). Then Remark 1.4 yields \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \subseteq S_{\ell,q,[\alpha,\infty]}[(s_j)_{j=0}^{m-1}, =] \) for all \( \ell \in \mathbb{Z}_{0,m-1} \).

Remark 6.3. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), and let \((s_j)_{j=0}^m\) be a sequence from \( \mathbb{C}^{q \times q} \). Let \( F \in S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \) with \( [\alpha,\infty) \)-Stieltjes measure \( \sigma_F \). Combining Remark 6.2 and Remark 5.3[b], we get then \( s_0 = s_0^{(\sigma_F)} = \sigma_F([\alpha,\infty)) \).

Now we characterize those sequences for which the sets defined in (6.1) are non-empty. (It is a reformulation of Theorem 1.8[b].)

Theorem 6.4. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \((s_j)_{j=0}^m\) be a sequence from \( \mathbb{C}^{q \times q} \). Then \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \neq \emptyset \) if and only if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^{\geq} \). Furthermore, in view of (5.5), if \((s_j)_{j=0}^m \in K_{q,m,\alpha}^{\geq} \), then \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] = \{ \sigma : \sigma \in M_{q,m}^{\geq}([\alpha,\infty)); (s_j)_{j=0}^m, \leq \} \).

Proof. Combine (4.4) and (6.1) with Theorem 1.8(b) and Theorem 4.11.

Proposition 6.5. Let \( \alpha \in \mathbb{R} \), \( m \in \mathbb{N} \), and \( (s_j)_{j=0}^m \in K_{q,m,\alpha}^{\geq} \). Combining Remark 6.2 and Remark 5.3[b], we get then \( s_0 = s_0^{(\sigma_F)} = \sigma_F([\alpha,\infty)) \).

Now we modify Lemma 5.1.

Proposition 6.6. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^{\geq} \). Then \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \subseteq S_{q,[\alpha,\infty]}[s_0] \).

Proof. Let \( F \in S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \). First we consider the case \( m \in \mathbb{N} \). In view of Remark 1.7, we have \((s_j)_{j=0}^{m-1} \in K_{q,m-1,\alpha}^{\geq,e} \), whereas Remark 6.2 yields \( F \in S_{m-1,q,[\alpha,\infty]}[(s_j)_{j=0}^{m-1}, =] \). Thus, applying Proposition 5.5 completes the proof.

Now let \( m = 0 \). Because of the choice of \( F \) and (6.1), we obtain then \( F \in S_{0,q,[\alpha,\infty]} \) and \( s_0 - \sigma_F([\alpha,\infty)) \in \mathbb{C}^{q \times q} \). Hence, we get from Lemma 4.12 that \( R(F(z)) = R(\sigma_F([\alpha,\infty])) \) for all \( z \in \mathbb{C} \setminus [\alpha,\infty) \) and from \( \sigma_F([\alpha,\infty)) \in \mathbb{C}^{q \times q} \) and Remark 4.3 that \( R(\sigma_F([\alpha,\infty)) \subseteq R(s_0) \). Consequently, \( R(F(z)) \subseteq R(s_0) \) for all \( z \in \mathbb{C} \setminus [\alpha,\infty) \). Hence, Notation 4.8 shows that \( F \in S_{q,[\alpha,\infty]}[s_0] \).

Remark 6.7. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), and let \((s_j)_{j=0}^m\) be a sequence from \( \mathbb{C}^{q \times q} \). In view of Remark 6.2, then \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \subseteq S_{m-1,q,[\alpha,\infty]}[(s_j)_{j=0}^{m-1}, =] \). Thus, Lemma 5.1 shows that \( S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] \subseteq S_{q,[\alpha,\infty]}[s_0] \).

7. Stieltjes pairs of meromorphic \( q \times q \) matrix-valued functions in \( \mathbb{C} \setminus [\alpha,\infty) \)

Let \( \alpha \in \mathbb{R} \). Then the set \( \mathbb{C} \setminus [\alpha,\infty) \) is clearly a region in \( \mathbb{C} \). We consider a class of ordered pairs of \( q \times q \) matrix-valued meromorphic functions in \( \mathbb{C} \setminus [\alpha,\infty) \), which turns out to be closely related to the class \( S_{q,[\alpha,\infty]} \) introduced in Definition 4.1. This set of ordered pairs of \( q \times q \) matrix-valued meromorphic functions in \( \mathbb{C} \setminus [\alpha,\infty) \) plays an important role in our subsequent considerations.
7. Stieltjes pairs of meromorphic $q \times q$ matrix-valued functions in $\mathbb{C} \setminus [\alpha, \infty)$

Indeed, this set acts as the set of parameters in our description of the set of Stieltjes transforms of the solutions of our original truncated matricial moment problem at the interval $[\alpha, \infty)$.

Now we introduce the central object of this section. In doing this, we use the notation $[\mathcal{M}(\mathbb{C} \setminus [\alpha, \infty))]^{q \times q}$ introduced in Appendix C and the particular signature matrix $\tilde{J}_q$ defined by (A.3). We call a subset $\mathcal{D}$ of $\mathbb{C}$ discrete if for every bounded subset $\mathcal{B}$ of $\mathbb{C}$ the intersection $\mathcal{D} \cap \mathcal{B}$ contains only a finite number of points.

**Definition 7.1.** Let $\alpha \in \mathbb{R}$. Let $\phi, \psi \in [\mathcal{M}(\mathbb{C} \setminus [\alpha, \infty))]^{q \times q}$. Then $(\phi, \psi)$ is called a $q \times q$ Stieltjes pair in $\mathbb{C} \setminus [\alpha, \infty)$ if there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the following three conditions are fulfilled:

(i) $\phi$ are $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(ii) $\text{rank}[\phi(z) \psi(z)] = q$ for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(iii) For each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$,

\[
\begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix}^* \begin{bmatrix}
-\tilde{J}_q \\
2\text{Im} z
\end{bmatrix} \begin{bmatrix}
\phi(z) \\
\psi(z)
\end{bmatrix} \in \mathbb{C}^{q \times q}_\geq \quad (7.1)
\]

and

\[
\begin{bmatrix}
(z - \alpha)\phi(z) \\
\psi(z)
\end{bmatrix}^* \begin{bmatrix}
-\tilde{J}_q \\
2\text{Im} z
\end{bmatrix} \begin{bmatrix}
(z - \alpha)\phi(z) \\
\psi(z)
\end{bmatrix} \in \mathbb{C}^{q \times q}_\geq. \quad (7.2)
\]

The set of all $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha, \infty)$ will be denoted by $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

**Definition 7.2.** Let $\alpha \in \mathbb{R}$. A pair $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$ is said to be a proper $q \times q$ Stieltjes pair in $\mathbb{C} \setminus [\alpha, \infty)$ if $\det g$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \infty)$. The set of all proper $q \times q$ Stieltjes pairs in $\mathbb{C} \setminus [\alpha, \infty)$ will be denoted by $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

**Remark 7.3.** Let $\alpha \in \mathbb{R}$, let $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$, and let $g$ be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that $\det g$ does not vanish identically. Then it is readily checked that $(\phi g, \psi g) \in \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

**Remark 7.4.** Let $\alpha \in \mathbb{R}$ and let $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$. We will call the Stieltjes pairs $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ equivalent if there is a function $\theta \in [\mathcal{M}(\mathbb{C} \setminus [\alpha, \infty))]^{q \times q}$ such that $\det \theta$ does not identically vanish in $\mathbb{C} \setminus [\alpha, \infty)$ and that

\[
\begin{bmatrix}
\phi_2 \\
\psi_2
\end{bmatrix} = \begin{bmatrix}
\phi_1 \theta \\
\psi_1 \theta
\end{bmatrix} \quad (7.3)
\]

is satisfied.

**Remark 7.5.** It is easily checked that the relation introduced in Definition 7.4 is an equivalence relation on the set $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$. For each $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$, we denote by $\langle (\phi, \psi) \rangle$ the equivalence class generated by $(\phi, \psi)$. Furthermore, we write $\langle \mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty)) \rangle$ for the set of all these equivalence classes.
7. Stieltjes pairs of meromorphic \( q \times q \) matrix-valued functions in \( \mathbb{C} \setminus [\alpha, \infty) \)

The following result contains an essential property of \( q \times q \) Stieltjes pairs.

**Proposition 7.6.** Let \( \alpha \in \mathbb{R} \) and let \( (\phi, \psi) \in \mathcal{P}^{(q,q)}_{\mathcal{D}}(\mathbb{C} \setminus [\alpha, \infty)) \). Let \( \mathcal{D} \) be a discrete subset of \( \mathbb{C} \setminus [\alpha, \infty) \) such that the conditions in Definition 7.1 are satisfied. For each \( z \in \mathbb{C}_{\alpha,-} \setminus \mathcal{D} \), then \( \text{Re}[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}_{\geq} \).

**Proof.** From the definition of \( \mathbb{C}_{\alpha,-} \) we see that

\[
\mathbb{C}_{\alpha,-} \setminus \mathcal{D} = \left( (\mathbb{C}_{\alpha,-} \setminus \mathbb{R}) \cup (-\infty, \alpha) \right) \setminus \mathcal{D} = \left( \mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D}) \right) \cup \left( (-\infty, \alpha) \setminus \mathcal{D} \right). \tag{7.4}
\]

(I) First we consider an arbitrary \( z \in \mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D}) \). In view of condition (iii) of Definition 7.1, then (7.1) and (7.2) hold true. Hence, Remark \( \text{A.1} \) implies

\[
\frac{\text{Im}[\psi^*(z)\phi(z)]}{\text{Im}z} \in \mathbb{C}^{q \times q}_{\geq} \quad \text{and} \quad \frac{\text{Im}[(z - \alpha)\psi^*(z)\phi(z)]}{\text{Im}z} \in \mathbb{C}^{q \times q}_{\geq} \tag{7.5}
\]

follow. Thus, taking into account \( \alpha - \text{Re}z \in (0, \infty) \) and (7.4), we get

\[
\frac{\alpha - \text{Re}z}{\text{Im}z} \text{Im}[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}_{\geq}. \tag{7.6}
\]

Remark \( \text{A.1} \) implies \( \text{Im}[z\psi^*(z)\phi(z)] = \text{Re}(z) \text{Im}[\psi^*(z)\phi(z)] + \text{Im}(z) \text{Re}[\psi^*(z)\phi(z)] \). Consequently, because of \( \text{Im}z \neq 0 \), we infer

\[
\text{Re}[\psi^*(z)\phi(z)] = \frac{1}{\text{Im}z} \text{Im}[z\psi^*(z)\phi(z)] + \frac{\alpha - \text{Re}z}{\text{Im}z} \text{Im}[\psi^*(z)\phi(z)] - \frac{\alpha}{\text{Im}z} \text{Im}[\psi^*(z)\phi(z)]. \tag{7.7}
\]

In view of \( \alpha \in \mathbb{R} \), then Remark \( \text{A.1} \) yields \( \text{Im}[\alpha\psi^*(z)\phi(z)] = \alpha \text{Im}[\psi^*(z)\phi(z)] \) and, hence, \( \text{Im}[z\psi^*(z)\phi(z)] = \text{Im}[(z - \alpha)\psi^*(z)\phi(z)] \). In view of, (7.7), this implies

\[
\text{Re}[\psi^*(z)\phi(z)] = \frac{\alpha - \text{Re}z}{\text{Im}z} \text{Im}[\psi^*(z)\phi(z)] + \frac{\text{Im}[(z - \alpha)\psi^*(z)\phi(z)]}{\text{Im}z}. \tag{7.8}
\]

Combining (7.6), (7.7), and (7.8) yields

\[
\text{Re}[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}_{\geq}. \tag{7.9}
\]

(II) Now we consider an arbitrary \( z \in (-\infty, \alpha) \setminus \mathcal{D} \). Since \( \mathcal{D} \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \infty) \), there is a sequence \( (z_n)_{n=1}^\infty \) from \( \mathbb{C}_{\alpha,-} \setminus (\mathbb{R} \cup \mathcal{D}) \) such that \( \lim_{n \to \infty} z_n = z \). By continuity of the functions \( \phi \) and \( \psi \) in \( \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \), we have then

\[
\lim_{n \to \infty} \text{Re}[\psi^*(z_n)\phi(z_n)] = \text{Re}[\psi^*(z)\phi(z)]. \tag{7.10}
\]

Since part (I) of the proof shows that \( \text{Re}[\psi^*(z_n)\phi(z_n)] \in \mathbb{C}^{q \times q}_{\geq} \) holds true for all \( n \in \mathbb{N} \), equation (7.10) implies \( \text{Re}[\psi^*(z)\phi(z)] \in \mathbb{C}^{q \times q}_{\geq} \).

(III) In view of (7.10), part (I), and part (II), the proof is complete. \( \square \)

Our next considerations show that there are intimate relations between the class \( \mathcal{S}_{q,[\alpha,\infty)} \) of \( [\alpha, \infty) \)-Stieltjes functions of order \( q \) (see Definition 4.1) and the set of all \( q \times q \) Stieltjes pairs in \( \mathbb{C} \setminus [\alpha, \infty) \).

Denote by \( \mathcal{D}_q \) and \( \mathcal{J}_q \) the constant \( \mathbb{C}^{q \times q} \)-valued functions in \( \mathbb{C} \setminus [\alpha, \infty) \) with values \( 0_{q \times q} \) and \( I_q \), respectively.
Proposition 7.7. Let \( \alpha \in \mathbb{R} \) and let \( f \in S_{q,|\alpha,\infty)} \). Then:

(a) The pair \( (f, \mathcal{J}_q) \) belongs to \( \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \).

(b) Let \( g \in S_{q,|\alpha,\infty)} \). Then \( \langle (f, \mathcal{J}_q) \rangle = \langle (g, \mathcal{J}_q) \rangle \) if and only if \( f = g \).

Proof. \( \square \) Because of \( f \in S_{q,|\alpha,\infty)} \) and the choice of \( \mathcal{J}_q \), the functions \( f \) and \( \mathcal{J}_q \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \). Obviously, we have \( \text{rank} \{ f(z) \} = q \) for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). We consider arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). From Corollary 4.4 and Lemma 4.2 we conclude then \( \text{rank} \{ f(z) \} = q \) for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). We consider arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). From Corollary 4.4 and Lemma 4.2 we conclude then \( \text{rank} \{ f(z) \} = q \) for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). Thus, the pair \( (f, \mathcal{J}_q) \) belongs to \( \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \).

In view of \( \langle f, \mathcal{J}_q \rangle = \langle g, \mathcal{J}_q \rangle \) there exists a \( q \times q \) matrix-valued function \( h \) which is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) such that \( h \neq 0 \) and, consequently, \( f = g \).

Proposition 7.7 shows that the class \( \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \) can be considered as a projective extension of the class \( S_{q,|\alpha,\infty)} \).

Example 7.8. Let \( \alpha \in \mathbb{R} \). Then Proposition 7.7 shows that \( (\mathcal{D}_q, \mathcal{J}_q) \) belongs to \( \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Furthermore, Remark A.13 yields \( (\mathcal{J}_q, \mathcal{D}_q) \in \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \).

Remark 7.9. Let \( \alpha \in \mathbb{R} \). Then Example 7.8 shows that the set \( \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \) is non-empty.

Proposition 7.10. Let \( \alpha \in \mathbb{R} \). Further, let \( \phi \) be a \( q \times q \) matrix-valued function which is meromorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) such that the condition \( (\phi, \mathcal{J}_q) \in \mathcal{P}_{J_q}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \) is satisfied. Then \( \phi \in S_{q,|\alpha,\infty)} \).

Proof. Our strategy of proof is based on applying Proposition 4.3. In view of Definition 7.7 and the choice of \( \phi \), we choose a discrete subset \( \mathcal{D} \) of \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \) such that the following conditions are satisfied:

(i) The function \( \phi \) is holomorphic in \( \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) \).

(ii) For each \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \),

\[
\left\{ \begin{array}{c}
\left( \begin{array}{c}
\phi(z) \\
I_q
\end{array} \right)^* \begin{pmatrix}
-\bar{\mathcal{J}}_q \\
2\text{Im } z
\end{pmatrix} \left( \begin{array}{c}
\phi(z) \\
I_q
\end{array} \right), \\
\left( z - \alpha \right) \phi(z) \\
I_q
\end{array} \right)^* \begin{pmatrix}
-\bar{\mathcal{J}}_q \\
2\text{Im } z
\end{pmatrix} \left( \begin{array}{c}
\left( z - \alpha \right) \phi(z) \\
I_q
\end{array} \right) \right\} \subseteq \mathbb{C}^{q\times q}. 
\]

Obviously, we have

\[
\mathbb{C} \setminus [\alpha, \infty) = \mathbb{C}_{\alpha,-} \cup \Pi_+ \cup \Pi_-.
\]

From (7.11) we get

\[
\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) = (\mathbb{C}_{\alpha,-} \setminus \mathcal{D}) \cup (\Pi_+ \setminus \mathcal{D}) \cup (\Pi_- \setminus \mathcal{D}).
\]

In particular, from (i) and (7.12) we conclude that \( \phi \) is holomorphic in each of the regions \( \mathbb{C}_{\alpha,-} \setminus \mathcal{D}, \Pi_+ \setminus \mathcal{D}, \) and \( \Pi_- \setminus \mathcal{D} \). In view of (i) Proposition 7.6 shows that \( \text{Re}[\phi(z)] \in \mathbb{C}^{q\times q} \) for each \( z \in \mathbb{C}_{\alpha,-} \setminus \mathcal{D} \). Thus, Lemma C.2(a) yields that \( \phi \) is holomorphic in \( \mathbb{C}_{\alpha,-} \) and satisfies
Re[φ(z)] ∈ C≥0 for all z ∈ Cα. For all z ∈ C \ (R ∪ D), we infer from Remark A.13 and (ii) that \( \text{Im}[φ(z)] / \text{Im}(z) \in \mathbb{C}_{≥0} \) holds true. In view of \( C \subset (R ∪ D) = (Π_+ \setminus D) \cup (Π_- \setminus D) \), we obtain then \( \text{Im}[φ(z)] \in \mathbb{C}_{≥0} \) for all \( z ∈ Π_+ \setminus D \) and \( -\text{Im}[φ(z)] \in \mathbb{C}_{≥0} \) for all \( z ∈ Π_- \setminus D \). Consequently, since \( φ \) is holomorphic in \( Π_+ \setminus D \) and \( Π_- \setminus D \), the application of Lemma C.2 yields that the function \( φ \) is holomorphic in \( Π_+ \setminus D \) and satisfies \( \text{Im}[φ(z)] \in \mathbb{C}_{≥0} \) for all \( z ∈ Π_+ \) and that the function \( φ \) is holomorphic in \( Π_- \setminus D \) and satisfies \( -\text{Im}[φ(z)] \in \mathbb{C}_{≥0} \) for all \( z ∈ Π_- \). Taking into account (1111), we in particular see that \( φ \) is holomorphic in \( C \setminus [α, ∞) \). The application of Proposition 4.3 brings \( φ ∈ S_q,α,∞) \).

The following results complement the statements of Propositions 7.7 and 7.10. We see now that the equivalence class of a proper element of \( P^{(q, q)}_J(C \setminus [α, ∞)) \) is always represented by a function belonging to \( S_q,α,∞) \).

**Proposition 7.11.** Let \( α ∈ R \) and let \( (φ, ψ) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \). Then:

(a) The function \( S := φψ_1 \) belongs to \( S_q,α,∞) \).

(b) \( (S, J_q) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \) and \( (⟨φ, ψ⟩) = (⟨S, J_q⟩) \).

**Proof.** By the choice of \( (φ, ψ) \), we know that the function \( \text{det} ψ \) does not identically vanish in \( C \setminus [α, ∞) \). Thus, (b) follows from Remark 7.3 and Definition 7.4. In view of (b), Proposition 7.10 yields \( S ∈ S_q,α,∞) \).

**Corollary 7.12.** Let \( α ∈ R \) and let \( ρ_α : (P^{(q, q)}_J(C \setminus [α, ∞)) \to S_q,α,∞) \) be defined by

\[
ρ_α(⟨φ, ψ⟩) := φψ_1^{-1}.
\]

Then \( ρ_α \) is well defined and bijective with inverse \( \iota_α : S_q,α,∞) \to (P^{(q, q)}_J(C \setminus [α, ∞)) \) given by

\[
\iota_α(F) := (⟨F, J_q⟩).
\]

**Proof.** Consider two pairs \( (φ_1, ψ_1), (φ_2, ψ_2) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \) with \( ⟨φ_1, ψ_1⟩ = ⟨φ_2, ψ_2⟩ \). In view of Definition 7.4, there is a function \( θ ∈ M(C \setminus [α, ∞))^{q×q} \) such that \( \text{det} θ \) does not identically vanish in \( C \setminus [α, ∞) \) satisfying (7.3), i.e., \( θ \circ φ_1 = φ_2 \). Consequently, \( φ_1ψ_2 = φ_2ψ_1 \). Thus, since Proposition 7.11(b) shows that \( φψ_1^{-1} ∈ S_q,α,∞) \) for all \( (φ, ψ) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \), the mapping \( ρ_α \) is well defined. From Proposition 7.4.13 we can conclude that \( \iota_α \) is well defined, too. For all \( F ∈ S_q,α,∞) \), we obviously \( (ρ_α ∘ \iota_α)(F) = ρ_α(⟨F, J_q⟩) = F \). Using Proposition 7.4.13, we conclude \( (ι_α ∘ ρ_α)(⟨φ, ψ⟩) = ρ_α(φψ_1^{-1}) = (⟨φψ_1^{-1}, J_q⟩) = (⟨φ, ψ⟩) \) for all \( (φ, ψ) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \).

Now we turn our attention to a particular subclass of \( P^{(q, q)}_J(C \setminus [α, ∞)) \), which occupies an essential role in our subsequent considerations.

**Notation 7.13.** Let \( α ∈ R \) and let \( A ∈ C^{q×q} \). We denote by \( P_q,α[A] \) the set of all \( (φ, ψ) ∈ P^{(q, q)}_J(C \setminus [α, ∞)) \) such that \( R(φ(z)) ⊆ R(A) \) for all points \( z ∈ C \setminus [α, ∞) \) which are points of holomorphicity of \( φ \). Further, let \( \tilde{P}_q,α[A] := P_q,α[A] ∩ P^{(q, q)}_J(C \setminus [α, ∞)) \).
Example 7.14. Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q \times q}$. Then the pair $(\mathcal{D}_q, \mathcal{J}_q)$ introduced in Example 7.8 belongs to $\mathcal{P}_{q,\alpha}[A]$.

Remark 7.15. Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{q \times q}$ be such that $\det A \neq 0$. Then $A^\dagger = A^{-1}$ and, consequently, $\mathcal{P}_{q,\alpha}[A] = \mathcal{P}_{q,q}^{-\mathcal{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$.

Remark 7.16. Let $\alpha \in \mathbb{R}$. In view of Definition 7.1, Notation 7.13 and Example 7.14 one can easily check that $\langle \mathcal{P}_{q,\alpha}[0_{q \times q}] \rangle = \{(\mathcal{D}_q, \mathcal{J}_q)\}$.

Now we will see that the procedure of constructing subclasses of $\mathcal{P}_{q,q}^{-\mathcal{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$ via Notation 7.13 stands in full harmony with the equivalence relation in $\mathcal{P}_{q,q}^{-\mathcal{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$ introduced in Definition 7.4.

Lemma 7.17. Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{q \times q}$, and let $(\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[A]$. Let $(\phi_2, \psi_2) \in \mathcal{P}_{q,q}^{-\mathcal{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$ be such that $((\phi_1, \psi_1)) = ((\phi_2, \psi_2))$. Then $(\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[A]$.

Proof. By assumption and Remark 7.1, we have $AA^\dagger \phi_1 = \phi_1$ and there exists a meromorphic $q \times q$ matrix-valued function $g$ in $\mathbb{C} \setminus [\alpha, \infty)$ with non-identically vanishing determinant such that $\frac{\phi_2}{\psi_2} = \frac{\phi_1}{\psi_1}g$. Thus, $AA^\dagger \phi_2 = AA^\dagger (\phi_1g) = (AA^\dagger \phi_1)g = \phi_1g = \phi_2$. Hence, $(\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[A]$.

Now we turn our attention again to the topic opened by Proposition 7.6. The following result shows that the class $\mathcal{P}_{q,\alpha}[A]$ can be considered as a projective extension of the class $\mathcal{S}_{q,\alpha}[A]$ introduced in Notation 4.8.

Remark 7.18. Let $\alpha \in \mathbb{R}$ and let $f \in \mathcal{S}_{q,\alpha}[\alpha, \infty)$. According to Proposition 7.7 and Proposition 7.13, then:

(a) $(f, \mathcal{J}_q)$ belongs to $\mathcal{P}_{q,q}^{-\mathcal{J}_q}(\mathbb{C} \setminus [\alpha, \infty))$.

(b) Let $A \in \mathbb{C}^{q \times q}$. Then $f \in \mathcal{S}_{q,\alpha}[\alpha, \infty)[A]$ if and only if $(f, \mathcal{J}_q) \in \mathcal{P}_{q,\alpha}[A]$.

8. On a coupled pair of Schur-Stieltjes-type transforms

The main goal of this section is to recall some basic facts which were necessary for the preparation of the elementary step of our Schur type algorithm for the class $\mathcal{S}_{q,\alpha}[\alpha, \infty)$. The material is mostly taken from [10] Section 9. We will be led to a situation which, roughly speaking, looks as follows: Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{p \times q}$ and let $F: \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p \times q}$. Then the matrix-valued functions $F^{[-,\alpha]; A}; \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p \times q}$ and $F^{-\mathcal{J}_q}; \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{p \times q}$ which are defined by

$$ F^{[-,\alpha]; A}(z) := -(z - \alpha)^{-1}A \left[ I_q + (z - \alpha)^{-1}A \right]^\dagger,$$

and

$$ F^{[-,\alpha]; A}(z) := -(z - \alpha)^{-1}A \left[ I_q + A^\dagger F(z) \right]^\dagger, $$

respectively, will be central objects in our further considerations. Against to the background of our later considerations, the matrix-valued functions $F^{[-,\alpha]; A}$ and $F^{[-,\alpha]; A}$ are called the $(\alpha, A)$-Schur-Stieltjes transform of $F$ and the inverse $(\alpha, A)$-Schur-Stieltjes transform of $F$.

The generic case studied here concerns the situation where $p = q$, where $A$ is a complex $q \times q$ matrix with later specified properties and where $F \in \mathcal{S}_{q,\alpha}[\alpha, \infty)$. 

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An important theme of [10] was to choose, for a given function \( F \in S_{q,[a,\infty]} \), special matrices \( A \in \mathbb{C}^{q \times q} \) such that the function \( F^{[+,\alpha,A]} \) and \( F^{[\alpha,A]} \), respectively, belong to \( S_{q,[a,\infty]} \) (see [10] Propositions 9.5 and 9.10). Furthermore, from [10] Propositions 9.11 and 9.13 we know that under appropriate conditions the equations \( (F^{[\alpha,A]})^{[+,\alpha,A]} = F \) and \( (F^{[+,\alpha,A]})^{[\alpha,A]} = F \) hold true.

Now we verify that in essential cases formulas (8.1) and (8.2) can be rewritten as linear fractional transformations with appropriately chosen generating matrix-valued functions. The role of these generating functions will be played by the matrix polynomials \( W_{\alpha,A} \) and \( V_{\alpha,A} \), which are studied in Appendix E. In the sequel, we use the terminology for linear fractional transformations of matrices, which are introduced in Appendix D.

**Lemma 8.1** ([10] Lemma 9.6]). Let \( \alpha \in \mathbb{R} \), let \( F : \mathbb{C} \setminus [a,\infty) \to \mathbb{C}^{q \times q} \) be a matrix-valued function, and let \( A \in \mathbb{C}^{q \times q} \) be such that \( R(F(z)) \subseteq R(A) \) and \( N(F(z)) \subseteq N(A) \) for all \( z \in \mathbb{C} \setminus [a,\infty) \). Then \( F(z) \in Q_{[-(z-a)A^t,I_q-A^tA]} \) and \( F^{[+,\alpha,A]}(z) = F_{[\alpha,A]}(z) \) for all \( z \in \mathbb{C} \setminus [a,\infty) \).

The following application of Lemma 8.1 prepares our considerations in Section 12.

**Proposition 8.2** ([10] Proposition 9.7]). Let \( \alpha \in \mathbb{R} \), let \( \kappa \in \mathbb{N}_0 \cup \{\infty\} \), and let \( (s_j)^\kappa_{j=0} \in K_{q,\kappa,\alpha}^{e} \) and \( A \in \mathbb{C}^{q \times q} \). Then \( F(z) \in Q_{[-(z-a)I_q,A-I_qA^t]} \) and \( F^{[+,\alpha,A]}(z) = F^{[\alpha,A]}(z) \) for all \( z \in \mathbb{C} \setminus [a,\infty) \).

At the end of this section we are going to consider the situation which will turn out to be typical for larger parts of our following considerations. Let \( A \in \mathbb{C}^{q \times q} \) and let \( G \) belong to the class \( S_{q,[a,\infty]}[A] \) introduced in Notation 4.8. Our aim is then to investigate the function \( G^{[\alpha,A]} \) given by (8.2). We begin by rewriting formula (8.2) as linear fractional transformation. In the sequel, we will often use the fact that, for each \( G \in S_{q,[a,\infty]} \), the matrix \( \gamma_G \) given via Notation 4.6 is non-negative Hermitian.

**Proposition 8.3** ([10] Proposition 9.9]). Let \( \alpha \in \mathbb{R} \), let \( A \in \mathbb{C}_{\geq 0}^{q \times q} \), and let \( G \in S_{q,[a,\infty]}^{\gamma}[A] \). Then \( G(z) \in Q_{[(z-a)A^t],(z-a)I_q} \) and \( G^{[\alpha,A]}(z) = G_{[\alpha,A]}(z) \) for all \( z \in \mathbb{C} \setminus [a,\infty) \).

9. On the transform \( F^{[+,\alpha,s_0]} \) for functions \( F \) belonging to the class \( S_{m,q,[a,\infty]}[(s_j)^m_{j=0}, \leq] \)

In [10] Section 10, we considered the following situation: Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \( (s_j)^m_{j=0} \in K_{q,m,\alpha}^{e} \). Then Theorem 5.2 yields that the class \( S_{m,q,[a,\infty]}[(s_j)^m_{j=0}, \leq] \) is non-empty. If \( F \in S_{m,q,[a,\infty]}[(s_j)^m_{j=0}, \leq] \), then our interest in [10] Section 10 was concentrated on the \( (\alpha,s_0) \)-Schur-Stieltjes transform \( F^{[+,\alpha,s_0]} \) of \( F \). The following result on this theme is of fundamental importance.

**Theorem 9.1** ([10] Theorem 10.3]). Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), let \( (s_j)^m_{j=0} \in K_{q,m,\alpha}^{e} \) with \( \alpha \)-S-transform \( (s_j^{[1,\alpha]})^{m-1}_{j=0} \), and let \( F \in S_{m,q,[a,\infty]}[(s_j)^m_{j=0}, \leq] \). Then \( F^{[+,\alpha,s_0]} \) is in \( S_{m-1,q,[a,\infty]}[(s_j^{[1,\alpha]})^{m-1}_{j=0}, \leq] \).

Now we will consider a function \( F \in S_{m,q,[a,\infty]}[(s_j)^m_{j=0}, \leq] \). We are interested in its \( (\alpha,s_0) \)-Schur-Stieltjes transform.
Theorem 9.2. Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e \) with \( \alpha \)-S-transform \((s_j^{[1,\alpha]})_{j=0}^{m-1} \), and let \( F \in S_{m,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \leq] \). Then \( F^{[+,\alpha,s_0]} \) belongs to \( S_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \leq] \).

Proof. Because of (6.1), we see that \( F \in \mathcal{S}_{m,q,[\alpha,\infty)} \) and \( \sigma_F \in \mathcal{M}_{q}^{\geq}[\alpha,\infty);(s_j)_{j=0}^{m}, \leq \). In particular, this implies \( \sigma_F \in \mathcal{M}_{q}^{\geq}([\alpha,\infty)). \) We set
\[
t_j := s_j^{(\sigma_F)} \quad \text{for all } j \in \mathbb{Z}_{0,m}.
\]
Because of (9.1), the application of Corollary 1.9 yields
\[
(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^e.
\]
Furthermore, from (9.1) and (5.1) we conclude
\[
F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(t_j)_{j=0}^m, =].
\]
Using (9.2) and Remark 1.6 we get \((t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^e \). Because of this and \((s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^e \), the application of Lemma 2.6(a) yields \( s_j^* = s_j \) and \( t_j^* = t_j \) for all \( j \in \mathbb{Z}_{0,m} \). From (9.1) and the choice of \( F \) we infer
\[
s_j = t_j \quad \text{for all } j \in \mathbb{Z}_{0,m-1} \quad \text{and} \quad t_m \leq s_m.
\]
We denote by \((s_j^{[1,\alpha]})_{j=0}^{m-1}\) and \((t_j^{[1,\alpha]})_{j=0}^{m-1}\) the \( \alpha \)-S-transforms of \((s_j)_{j=0}^m\) and \((t_j)_{j=0}^m\), respectively. Since \( s_j = t_j \) and \( t_j^* = t_j \) are true for each \( j \in \mathbb{Z}_{0,m} \) and (9.1) is valid, Lemma 3.6 yields that \((s_j^{[1,\alpha]})_{j=0}^{m-1}\) and \((t_j^{[1,\alpha]})_{j=0}^{m-1}\) are sequences of Hermitian matrices which satisfy
\[
t_j^{[1,\alpha]} \leq s_j^{[1,\alpha]}\quad \text{for all } j \in \mathbb{Z}_{0,m-2}.
\]
In view of (9.2) and (9.3), the application of Theorem 9.1 yields \( F^{[+,\alpha,s_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j^{[1,\alpha]})_{j=0}^{m-1}, =] \). Combining this with the identity \( s_0 = t_0 \), which follows from (9.1), we get \( F^{[+,\alpha,s_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(t_j^{[1,\alpha]})_{j=0}^{m-1}, =] \). Because of (9.5) and (9.6), this implies that \( F^{[+,\alpha,s_0]} \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \leq] \). 

10. On the transform \( F^{[-,\alpha,s_0]} \) for functions

\[
F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, \leq]
\]

Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N} \), let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e \) with first \( \alpha \)-S-transform \((s_j^{[1,\alpha]})_{j=0}^{m-1}\), and let \( F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[(s_j^{[1,\alpha]})_{j=0}^{m-1}, =] \). Then our interest in Section 11 was concentrated on the inverse \((\alpha,s_0)\)-Schur-Stieltjes transform \( F^{[-,\alpha,s_0]} \) of \( F \). The following result on this theme is of fundamental importance for our considerations.
10. On the transform $F[-,\alpha,s_0]$ for functions $F \in \mathcal{S}_{m-1,q,\alpha,\infty}[[s_j^{1,\alpha}]_{j=0}^{m-1}, \leq]$

**Theorem 10.1** ([10] Theorem 11.3]). Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j^{m})_{j=0}^{m-1} \in \mathcal{K}_{\alpha,q,e,m}$ with first $\alpha$-S-transform $(s_j^{1,\alpha})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m-1,q,\alpha,\infty}[[s_j^{1,\alpha}]_{j=0}^{m-1}, \leq]$. Then $F[-,\alpha,s_0]$ belongs to $\mathcal{S}_{m,q,\alpha,\infty}[[s_j^{m}]_{j=0}^{m-1}, \leq]$.

Against to the background of Theorems 10.1 and 0.4 we are lead to the investigation of the following question: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j^{m})_{j=0}^{m-1} \in \mathcal{K}_{\alpha,m,\alpha}$ with first $\alpha$-S-transform $(s_j^{1,\alpha})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m-1,q,\alpha,\infty}[[s_j^{1,\alpha}]_{j=0}^{m-1}, \leq]$. Does then $F[-,\alpha,s_0]$ belong to $\mathcal{S}_{m,q,\alpha,\infty}[[s_j^{m}]_{j=0}^{m-1}, \leq]$? The following example shows that the answer to this question is negative.

**Example 10.2.** Let $s_0 := [1,0]$, let $s_1 := [1,1]$, and let $F: \mathbb{C} \setminus [0,\infty) \to \mathbb{C}^{2 \times 2}$ be given by $F(z) := -\frac{1}{2}s_0$. Then $(s_j^1)_{j=0}^{1} \in \mathcal{K}_{\alpha,m,\alpha}$. However, because of $\mathcal{R}(s_1) \not\subseteq \mathcal{R}(s_0)$, the sequence $(s_j^1)_{j=0}^{1}$ does not belong to $\mathcal{D}_{2,\times 2}$. Thus, $(s_j^1)_{j=0}^{1} \in \mathcal{K}_{\alpha,m,\alpha} \setminus \mathcal{D}_{2,\times 2}$. Obviously, because of $s_0^1 = s_0$ and $s_0^1 = s_0$, we have

$$s_0^{[1,0]} = -s_0^1 = s_0 = -s_0^1 \left[(-[z,0]^j)^{1,0} \right] = s_0^1(s_0^1 - s_0 + 1) = s_0.$$

Let $\delta_0$ be the Dirac measure defined on $\mathcal{B}_{[0,\infty)}$ with unit mass at the point 0 and let $\sigma := s_0\delta_0$. Then $\sigma \in \mathcal{M}_{\alpha}^{\geq}([0,\infty); (s_j^{[1,0]})_{j=0}^{1}, \leq]$. One can easily see that $\int_{[0,\infty)} (t - z)^{-1} \sigma(dt) = F(z)$ for each $z \in \mathbb{C} \setminus [0,\infty)$. In view of Theorem 10.1, consequently, $F \in \mathcal{S}_{0,2,[0,\infty)}[[s_j^{1,0}]_{j=0}^{1}, \leq]$. Obviously, if $\delta_1$ is the Dirac measure defined on $\mathcal{B}_{[0,\infty)}$ with unit mass at the point 1, then $\mu := s_0 \delta_1 \in \mathcal{M}_{\alpha}^{\geq}([0,\infty))$. In view of Remark 5.3 and $(s_j^1)_{j=0}^{1} = s_0 s_0^1 = s_0$, we get then

$$F[-,0,s_0](z) = -(z - 0)^{-1} s_0 \left[ I_2 + s_0^1 F(z) \right] = -\frac{1}{z} \left[ I_2 + \frac{1}{z} s_0 s_0^1 \right]^\dagger = \frac{1}{z} \left[ I_2 + s_0 \right] = \frac{z}{z - s_0} \int_{[0,\infty)} t - z \mu(dt).$$

Furthermore, we obtain $s_0 = \int_{[0,\infty)} t \mu(dt) = s_0$ and, hence, $s_1 - s_0^1 = [1,1] \not\subseteq \mathbb{C}^{2 \times 2}$. This implies $\mu \notin \mathcal{M}_{\alpha}^{\geq}([0,\infty); (s_j^1)_{j=0}^{1}, \leq]$. Because of 10.1, then $F[-,0,s_0] \notin \mathcal{S}_{0,2,[0,\infty)}[[s_j^{1,0}]_{j=0}^{1}, \leq]$ follows.

As a consequence of Example 10.2 we have to look for a “large” proper subclass $\mathcal{K}_{\alpha,m,\alpha}$ of $\mathcal{K}_{\alpha,q,e,m}$ with the following property:

Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j^m)_{j=0}^{m-1} \in \mathcal{K}_{\alpha,m,\alpha}$ with first $\alpha$-S-transform $(s_j^{1,\alpha})_{j=0}^{m-1}$, and let $F \in \mathcal{S}_{m-1,q,\alpha,\infty}[[s_j^{1,\alpha}]_{j=0}^{m-1}, \leq]$. Then $F[-,\alpha,s_0]$ belongs to $\mathcal{S}_{m,q,\alpha,\infty}[[s_j^{m}]_{j=0}^{m-1}, \leq]$.

The search for such a class $\mathcal{K}_{\alpha,m,\alpha}$ determines the direction of our next considerations. We will see that the class $\mathcal{D}_{q,\times q,m}$ of first term dominant sequences $(s_j^m)_{j=0}^{m}$ of complex $q \times q$ matrices (see Definition 3.3) will turn out to provide the key for finding the desired class $\mathcal{K}_{\alpha,m,\alpha}$. In order to prepare the proof of the main result of this section, we note that some technical results on the class $\mathcal{D}_{q,\times q,m}$ are given in [9]. In particular, we have the following:

**Proposition 10.3** ([9] Proposition 3.8(a)). Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then $\mathcal{K}_{\alpha,m,\alpha} \subseteq \mathcal{H}_{q,m}^{\geq,e} \subseteq \mathcal{D}_{q,\times q,m}$.

Now we come to the central result of this section.
10. On the transform $F[-\alpha,s_0]$ for functions $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[\{s_j^{[1,\alpha]}\}_{j=0}^{m-1}]$, where $s_j^{[1,\alpha]}$ is the application of \[9, \text{Proposition 10.8}\] yields $s_j^{[1,\alpha]} \in K_{q,m-1,\alpha}$ and $\mathcal{S}_{m-1,q,[\alpha,\infty)}[\{t_j^{[1,\alpha]}\}_{j=0}^{m-1}] \neq \emptyset$.

(a) If $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[\{s_j^{[1,\alpha]}\}_{j=0}^{m-1}]$, then $F[-\alpha,s_0] \in \mathcal{S}_{m,q,[\alpha,\infty)}[\{s_j^{m}\}]$.

Proof. Part (a) follows immediately from Theorem 3.3\(\alpha\) and Theorem 1.5\(\alpha\). Let $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[\{s_j^{[1,\alpha]}\}_{j=0}^{m-1}]$. In view of (a), the application of Lemma 2.6\(\alpha\) yields $s_j^{[1,\alpha]} \in \mathbb{C}^q_{\mathcal{H}}$ for all $j \in \mathbb{Z}_{0,m-1}$. Because of (6.1), we have $F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}$ and $\sigma_F \in \mathcal{M}_{q,m-1}^\geq[\alpha,\infty)$; $(r_j^{[1,\alpha]} \in \mathbb{R})$. In particular, we infer $\sigma_F \in \mathcal{M}_{q,m-1}^\geq[(\alpha,\infty)]$. We set

$$t_j := s_j^{(\sigma_F)}$$

for all $j \in \mathbb{Z}_{0,m-1}$. (10.2)

Taking into account (10.2), the application of Corollary 1.9 yields $(t_j^{m-1})_{j=0}^{m-1} \in K_{q,m-1,\alpha}^\geq$. By virtue of Remark 1.6, we get $(t_j^{m-1})_{j=0}^{m-1} \in K_{q,m-1,\alpha}^\geq$. Thus, Lemma 2.6\(\alpha\) yields

$$t_j \in \mathbb{C}^q_{\mathcal{H}}$$

for all $j \in \mathbb{Z}_{0,m-1}$, (10.3)

whereas Lemma 2.6\(\beta\) gives $t_0 \in \mathbb{C}^q_{\mathcal{H}}$. From (10.2) and the choice of $F$ we infer

$$t_{m-1} \leq s_{m-1}^{[1,\alpha]} \quad \text{and, in the case, } m \geq 2 \quad \text{moreover } t_j = s_j^{[1,\alpha]} \quad \text{for all } j \in \mathbb{Z}_{0,m-2}. \quad (10.4)$$

In particular, $t_0 \leq s_0^{[1,\alpha]}$. Combining $s_0^{[1,\alpha]} \in \mathbb{C}^q_{\mathcal{H}}$ with $t_0 \in \mathbb{C}^q_{\mathcal{H}}$ and $t_0 \leq s_0^{[1,\alpha]}$, then Remark A.6 and Definition 3.3 give

$$\mathcal{R}(t_0) \subseteq \mathcal{R}(s_0^{[1,\alpha]}) \subseteq \mathcal{R}(s_0) \quad \text{and} \quad \mathcal{N}(s_0) \subseteq \mathcal{N}(s_0^{[1,\alpha]}) \subseteq \mathcal{N}(t_0). \quad (10.5)$$

In view of $(s_j)^m_{j=0} \in K_{q,m,\alpha}^\geq$, Lemma 2.6\(\alpha\) gives $s_0 \in \mathbb{C}^q_{\mathcal{H}}$. Denote by $(r_j)^m_{j=0}$ the first inverse $\alpha$-S-transform corresponding to $[(t_j)^m_{j=0}, s_0]$. Taking into account $(t_j)^m_{j=0} \in K_{q,m-1,\alpha}^\geq$ and $s_0 \in \mathbb{C}^q_{\mathcal{H}}$, then the application of [9] Proposition 10.15 yields $(r_j)^m_{j=0} \in K_{q,m-1,\alpha}^\geq$. In view of $(t_j)^m_{j=0} \in K_{q,m-1,\alpha}^\geq$, we infer from Proposition 10.3 that $(t_j)^m_{j=0} \in \mathcal{D}_{q,m,m-1}$. Denote by $(r_j^{[1,\alpha]})_{j=0}^{m-1}$ the first $\alpha$-S-transform of $(r_j)^m_{j=0}$. Then, because of $(t_j)^m_{j=0} \in \mathcal{D}_{q,m,m-1}$ and (10.5), the application of [9] Proposition 10.8 yields $r_j^{[1,\alpha]} = t_j$ for all $j \in \mathbb{Z}_{0,m-1}$. Combining this with (10.2), we get $\sigma_F \in \mathcal{M}_{q,m-1}^\geq[\alpha,\infty)$; $(r_j^{[1,\alpha]} \in \mathbb{R})$ and hence, in view of (5.1), then

$$F \in \mathcal{S}_{m-1,q,[\alpha,\infty)}[\{r_j^{[1,\alpha]}\}_{j=0}^{m-1}], \quad (10.6)$$

From $(t_j)^m_{j=0} \in K_{q,m-1,\alpha}^\geq$ and $r_j^{[1,\alpha]} = t_j$ for all $j \in \mathbb{Z}_{0,m-1}$ we infer $(r_j^{[1,\alpha]} \in \mathbb{R})$. Thus, in view of (10.6), the application of Theorem 10.1 implies

$$F[-\alpha,s_0] \in \mathcal{S}_{m,q,\alpha}^\geq[\{r_j\}_{j=0}^{m}], \quad (10.7)$$

Denote by $(u_j)^m_{j=0}$ the first inverse $\alpha$-S-transform corresponding to $[(s_j^{[1,\alpha]} \in \mathbb{R})]$. Because of (10.5)–(10.7), we infer from Lemma 3.13 that $r_j = u_j$ for all $j \in \mathbb{Z}_{0,m-1}$ and that $r_m \leq u_m$. Because of the assumption $(s_j)^m_{j=0} \in \mathcal{D}_{q,m,m}$, then [9] Proposition 10.10 yields $s_j = u_j$ for all $j \in \mathbb{Z}_{0,m}$. Consequently, from (10.7) we get $F[-\alpha,s_0] \in \mathcal{S}_{m,q,\alpha}^\geq[\{s_j\}_{j=0}^{m}]$. □
11. On the set $S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}, \leq]$

Finally, we turn our attention to an important consequence of Theorem 10.4.

**Theorem 10.5.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, let $(s_j)^m_{j=0} \in K^{\geq e}_{q,m,\alpha}$ with first $\alpha$-$S$-transform $(s_j^{|\alpha|})_{j=0}^{m-1}$. Then $(s_j^{|\alpha|})_{j=0}^{m-1} \in K^{\geq e}_{q,m,1-\alpha}$ and $S_{m-1,q,[\alpha,\infty)}[(s_j^{|\alpha|})_{j=0}^{m-1}, \leq] \neq \emptyset$. Furthermore, if $F \in S_{m-1,q,[\alpha,\infty)}[(s_j^{|\alpha|})_{j=0}^{m-1}, \leq]$, then $F[-\alpha,\infty) \in S_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0}, \leq]$. 

**Proof.** In view of Remark 1.6 and Proposition 10.3, the inclusion $K^{\geq e}_{q,m,\alpha} \subset K^{\geq e}_{q,m} \cap D_{q \times q,m}$ holds true. Hence, the application of Theorem 10.4 completes the proof. □

Taking into account Theorems 1.8 and 1.10, we recognize the particular importance of Theorem 10.5 for the treatment of Problem M\([\alpha,\infty); (s_j)^m_{j=0}, \leq\].

11. On the set $S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}, \leq]$ 

In this section, for arbitrarily given $\alpha \in \mathbb{R}$ and $s_0 \in \mathbb{C}^{q \times q}$, we study the set $S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}, \leq]$ introduced in (6.1). We will see that this set can be parametrized by a linear fractional transformation with the linear $2q \times 2q$ matrix polynomial $V_{\alpha,s_0}$ given via (E.1) as generating matrix-valued functions. The role of parameters will be taken by the set $P_{q,\alpha}[s_0]$ of pairs $(\phi, \psi)$ of meromorphic $q \times q$ matrix-valued functions, which were introduced in Notation 7.13.

**Lemma 11.1.** Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}^{q \times q}$, and let $F \in S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}, \leq]$. For all $w \in \mathbb{C} \setminus \mathbb{R}$, then 

$$
\frac{1}{\text{Im } w} \text{ Im } F(w) \geq [F(w)]^* s_0^t [F(w)].
$$

(11.1)

**Proof.** Let $w \in \mathbb{C} \setminus \mathbb{R}$. Because of $F \in S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}, \leq]$, we have $F \in S_{0,q,[\alpha,\infty)}$. Let $\sigma_F$ be the $[\alpha,\infty)$-Stieltjes measure of $F$ and let $g_w : [\alpha, \infty) \to \mathbb{C}$ be defined by $g_w(t) = (t - w)^{-1}$. Then $g_w \in L^1([\alpha, \infty), \mathcal{B}_{[\alpha,\infty)}, \sigma_F; \mathbb{C})$ and

$$
F(w) = \int_{[\alpha, \infty)} g_w d\sigma_F.
$$

(11.2)

Hence, $\text{Im } g_w \in L^1([\alpha, \infty), \mathcal{B}_{[\alpha,\infty)}, \sigma_F; \mathbb{C})$ and

$$
\text{Im } F(w) = \int_{[\alpha, \infty)} \text{Im } (g_w) d\sigma_F.
$$

(11.3)

For all $t \in [\alpha, \infty)$, we have $\text{Im } g_w(t) = |\text{Im } (w)| |t - w|^{-2} = |\text{Im } (w)| [g_w(t)]^2$ and, consequently, $|g_w(t)|^2 = \text{Im } g_w(t)/|\text{Im } (w)|$. This implies

$$
|g_w|^2 \in L^1([\alpha, \infty), \mathcal{B}_{[\alpha,\infty)}, \sigma_F; \mathbb{C}) \quad \text{and} \quad \int_{[\alpha, \infty)} |g_w|^2 d\sigma_F = \frac{1}{\text{Im } (w)} \int_{[\alpha, \infty)} \text{Im } (g_w) d\sigma_F.
$$

(11.4)

Comparing (11.3) and (11.4), we conclude

$$
\text{Im } F(w) = |\text{Im } (w)| \int_{[\alpha, \infty)} |g_w|^2 d\sigma_F.
$$

(11.5)

From (11.4) and Corollary B.6, we obtain

$$
\left( \int_{[\alpha, \infty)} g_w d\sigma_F \right)^* [\sigma_F([\alpha, \infty)) ]^t \left( \int_{[\alpha, \infty)} g_w d\sigma_F \right) \leq \int_{[\alpha, \infty)} |g_w|^2 d\sigma_F.
$$

(11.6)
11. On the set $S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}=0, \leq$]

The matrices $B := \sigma_F([\alpha, \infty))$ and $s_0$ are both non-negative Hermitian. In particular, using Remark A.6[1], we get $B = B^*$ and $s_0 = s_0^*$. By virtue of $F \in S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}=0, \leq$], we see that $\sigma_F \in \mathcal{M}_q^{[\alpha, \infty)}(\mathbb{C})$; $(s_j)^0_{j=0}=0, \leq$] and, consequently, $s_0 \geq s_0^{(\sigma_F)} = \sigma_F([\alpha, \infty)) = B \geq 0_q \otimes q$ hold true. Thus, taking additionally into account $B^* = B$, $s_0^* = s_0$, and Lemma A.12 we infer

$$B^* \geq B^* B s_0^* B B^*. \quad (11.7)$$

Since $F$ belongs to $S_{0,q,[\alpha,\infty)}$, in view of Lemma A.12 and $B = \sigma_F([\alpha, \infty))$ we obtain $\mathcal{R}(F(z)) = \mathcal{R}(B)$ for all $z \in \mathbb{C} \setminus [\alpha, \infty)$. Thus, Remark A.6[1] implies

$$BB^* F = F. \quad (11.8)$$

Because of $B^* = B$ and Remark A.6[1], we conclude $B^* B = (BB^*)^*$. Consequently, using additionally (11.5), (11.6), (11.2), $B = \sigma_F([\alpha, \infty))$, (11.7), and (11.8), we get then

$$\frac{1}{\text{Im } w} \text{ Im } F(w) = \int_{[\alpha, \infty)} |g_w|^2 d\sigma_F \geq \left(\int_{[\alpha, \infty)} g_w d\sigma_F\right)^* \left(\int_{[\alpha, \infty)} g_w d\sigma_F\right)$$

$$= [F(w)]^* B^* [F(w)] \geq [F(w)]^* B^* B s_0^* B B^* B F(w) = [F(w)]^* (BB^*)^* s_0^* B B^* F(w)$$

$$= \left[BB^* F(w)\right]^* s_0^* \left[BB^* F(w)\right] = [F(w)]^* s_0^* F(w). \quad \square$$

**Proposition 11.2.** Let $\alpha \in \mathbb{R}$, let $s_0 \in \mathbb{C}^{q \times q}$, and let $F \in S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}=0, \leq]$. Further, let $W_{\alpha,s_0}$ be given by (E.1) and let

$$W_{\alpha,s_0} \cdot \begin{bmatrix} F \\ I_q \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad (11.9)$$

be the $q \times q$ block representation of $W_{\alpha,s_0} \cdot \begin{bmatrix} F \\ I_q \end{bmatrix}$. Then:

(a) The functions $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$.

(b) The pair $(\phi, \psi)$ belongs to $\mathcal{P}_{q,\alpha}[s_0]$.

(c) The inequality $\det[(z - \alpha)s_0^+ \phi(z) + (z - \alpha)I_q \cdot \psi(z)] \neq 0$ and the equation

$$F(z) = [0_q \otimes q \cdot \phi(z) - s_0 \psi(z)] [(z - \alpha)s_0^+ \phi(z) + (z - \alpha)I_q \cdot \psi(z)]^{-1}$$

hold true for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

**Proof.** Since $F$ belongs to $S_{0,q,[\alpha,\infty)}[(s_j)^0_{j=0}=0, \leq]$, we have $F \in S_{0,q,[\alpha,\infty)}$ and $F \in S_{q,[\alpha,\infty)}$. In particular, $F$ is holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$. Thus, since $W_{\alpha,s_0}$ is a $2q \times 2q$ matrix polynomial, we see from (11.9) that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ as well.

(a) For each $z \in \mathbb{C} \setminus [\alpha, \infty)$, using (E.1), we obtain

$$[s_0^+, I_q] W_{\alpha,s_0}(z) = [s_0^+, I_q] \begin{bmatrix} (z - \alpha)I_q \\ -(z - \alpha)s_0^+ \\ I_q - s_0^+ s_0 \end{bmatrix} = [0_q \otimes q, I_q]$$

and, in view of (11.9), then

$$q = \text{rank } I_q = \text{rank } \begin{bmatrix} [0_q \otimes q, I_q] \begin{bmatrix} F(z) \\ I_q \end{bmatrix} \end{bmatrix} = \text{rank } \begin{bmatrix} [s_0^+, I_q] W_{\alpha,s_0}(z) \begin{bmatrix} F(z) \\ I_q \end{bmatrix} \end{bmatrix}$$

$$= \text{rank } \begin{bmatrix} [s_0^+, I_q] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \end{bmatrix} \leq \text{rank } \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \leq q.$$
This implies
\[
\text{rank } \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q \quad \text{for all } z \in \mathbb{C} \setminus [\alpha, \infty). \tag{11.10}
\]

Now consider an arbitrary \( w \in \mathbb{C} \setminus \mathbb{R}. \) In view of \( F \in \mathcal{S}_{\mathbb{Q},[\alpha, \infty)}, \) then \([11] \text{ Lemma 4.2} \) yields
\[
\frac{\text{Im}[(w - \alpha)F(w)]}{\text{Im} w} \in \mathbb{C}_{\geq}^{q \times q}. \tag{11.11}
\]
By virtue of \( s_0 \in \mathbb{C}^{\geq q}, \) we have \( s_0^* = s_0. \) Thus, Proposition \([E.6] \) implies
\[
[W_{\alpha,s_0}(w)]^* (-\bar{J}_q) W_{\alpha,s_0}(w) = [\text{diag}((w - \alpha)I_q, I_q)]^* (-\bar{J}_q) [\text{diag}((w - \alpha)I_q, I_q)]. \tag{11.12}
\]
Because of \((11.9), \ (11.12), \) Remark \( \text{A.13} \) and \((11.11), \) we infer
\[
\begin{align*}
\begin{bmatrix} \phi(w) \\ \psi(w) \end{bmatrix}^* \begin{pmatrix} -\bar{J}_q \\ 2 \text{Im} w \end{pmatrix} \begin{bmatrix} \phi(w) \\ \psi(w) \end{bmatrix} & = \frac{1}{2 \text{Im} w} \begin{bmatrix} F(w) \\ I_q \end{bmatrix}^* [W_{\alpha,s_0}(w)]^* (-\bar{J}_q) [W_{\alpha,s_0}(w)] \begin{bmatrix} F(w) \\ I_q \end{bmatrix} \\
& = \frac{1}{2 \text{Im} w} \begin{bmatrix} F(w) \\ I_q \end{bmatrix}^* [\text{diag}((w - \alpha)I_q, I_q)]^* (-\bar{J}_q) [\text{diag}((w - \alpha)I_q, I_q)] \begin{bmatrix} F(w) \\ I_q \end{bmatrix} \\
& = \frac{1}{2 \text{Im} w} \begin{bmatrix} (w - \alpha)F(w) \\ I_q \end{bmatrix}^* (-\bar{J}_q) \begin{bmatrix} (w - \alpha)F(w) \\ I_q \end{bmatrix} = \frac{\text{Im}[(w - \alpha)F(w)]}{\text{Im} w} \in \mathbb{C}_{\geq}^{q \times q}. \tag{11.13}
\end{align*}
\]
From \( F \in \mathcal{S}_{\mathbb{Q},[\alpha, \infty)} \) we see that the \([\alpha, \infty)\)-Stieltjes measure \( \sigma_F \) of \( S \) satisfies \( \sigma_F \in \mathcal{M}_{\geq}^{\infty}([\alpha, \infty); (s_j)_{j=0}^0, \leq), \) which implies \( 0_{q \times q} \leq \sigma_F([\alpha, \infty)) = s_0^{(\sigma_F)} \leq s_0. \) Thus, Remark \( \text{A.5} \) yields \( \mathcal{R}(\sigma_F([\alpha, \infty))) \subset \mathcal{R}(s_0). \) Hence, in view of \( F \in \mathcal{S}_{\mathbb{Q},[\alpha, \infty)} \) and Lemma \( \text{4.12} \) we get that \( \mathcal{R}(F(z)) = \mathcal{R}(\sigma_F([\alpha, \infty))) \subset \mathcal{R}(s_0) \) for each \( z \in \mathbb{C} \setminus [\alpha, \infty). \) Consequently, by virtue of Remark \( \text{A.6} \), we obtain then
\[
s_0 s_0^* F = F. \tag{11.14}
\]
In view of \( s_0^* = s_0, \) the application of \( \text{E.5} \) provides
\[
\begin{align*}
\begin{bmatrix} F(w) \\ I_q \end{bmatrix}^* (-\bar{J}_q) \begin{bmatrix} F(w) \\ I_q \end{bmatrix} = \text{Im}[F(w)] \quad \text{and} \quad \begin{bmatrix} 0_{q \times q} \\ I_q \end{bmatrix}^* (-\bar{J}_q) \begin{bmatrix} 0_{q \times q} \\ I_q \end{bmatrix} = 0_{q \times q}. \tag{11.16}
\end{align*}
\]
By virtue of \( s_0 \in \mathbb{C}_{\geq}^{q \times q} \) and \( F \in \mathcal{S}_{\mathbb{Q},[\alpha, \infty)} \) \((s_j)_{j=0}^0, \leq), \) Lemma \( \text{11.1} \) yields \( \text{11.1}. \) Using \( \text{11.9}, \)
11. On the set $S_{0,q,[\alpha,\infty)}[s_0]_{j=0}^\infty$, we conclude

\[ (w - \alpha) \phi(w) \psi(w) = \frac{1}{2 \text{Im } w} \left[ F(w) \right] \left[ 2 \text{Im } w \right] \]

Thus, in view of Remark A.6(a), in order to complete the proof of part (b), it remains to check that

\[ (\phi, \psi) \]

From (a), (11.10), (11.13), and (11.17) we see that \((\phi, \psi) = \phi \in \mathbb{C}\) holds true. Thus, in view of Remark A.6(a), in order to complete the proof of part (b), it remains to check that \(s_0 s_0^\dagger \phi = \phi\). Since \(\phi\) and \(F\) are holomorphic in \(\mathbb{C} \setminus [\alpha, \infty)\) and, because of (11.9) and (11.10), for all \(z \in \mathbb{C} \setminus [\alpha, \infty)\), we get

\[ \phi(z) = [I_q, 0_{q \times q}] W_{\alpha,s_0}(z) \left[ F(z) \right]_{I_q} = [I_q, 0_{q \times q}] \left[ (z - \alpha) I_q, s_0 \right] \left[ F(z) \right]_{I_q} \]

and, according to (11.10), consequently \(s_0 s_0^\dagger \phi(z) = (z - \alpha) F(z) + s_0 = \phi(z)\).

Let \(z \in \mathbb{C} \setminus [\alpha, \infty)\). Our proof is based on an application of Proposition D.7. The roles of the matrices \(E_1\) and \(E_2\) in Proposition D.7 will be played by the matrices

\[ W_{\alpha,s_0}(z) = \left[ \begin{array}{cc} (z - \alpha) I_q & s_0 \\ -(z - \alpha) s_0^\dagger & I_q - s_0^\dagger s_0 \end{array} \right] \quad \text{and} \quad V_{\alpha,s_0}(z) = \left[ \begin{array}{cc} 0_{q \times q} & -s_0 \\ (z - \alpha) s_0^\dagger & (z - \alpha) I_q \end{array} \right]. \]
11. On the set $S_{0,q, [\alpha, \infty)](s_j)_{j=0}^{\infty} : \leq}$

Taking into account Remark [E.4] we have

$$\text{rank} \left[ -(z - \alpha)s_0, I_q - s_0^t s_0 \right] = q \quad \text{and} \quad \text{rank} \left[ (z - \alpha)s_0^t, (z - \alpha)I_q \right] = q. \quad (11.19)$$

From Remark [E.2] we get

$$[V_{\alpha, s_0}(z)][W_{\alpha, s_0}(z)] = (z - \alpha) \cdot \text{diag}(s_0s_0^t, I_q). \quad (11.20)$$

Because of $z \in \mathbb{C} \setminus [\alpha, \infty)$, we obtain $\text{det}[(z - \alpha) \cdot 0_{q \times q} \cdot F(z) + (z - \alpha)I_q \cdot I_q] = (z - \alpha)^q \neq 0$ and, in view of (11.14), then

$$[z - \alpha)s_0s_0^tF(z) + 0_{q \times q} \cdot I_q]\left[(z - \alpha)0_{q \times q} \cdot F(z) + (z - \alpha)I_q \cdot I_q\right]^{-1}
= (z - \alpha)s_0s_0^t[0_{q \times q} \cdot F(z)]\left[(z - \alpha)I_q\right]^{-1} = s_0s_0^tF(z) = F(z). \quad (11.21)$$

By virtue of (11.18)–(11.21), the application of Proposition D.7 yields the inequality $\text{det}[(z - \alpha)s_0\phi(z) + (z - \alpha)I_q \cdot \psi(z)] \neq 0$ and the equation

$$0_{q \times q} \cdot \phi(z) - s_0\psi(z)\left[(z - \alpha)s_0^t\phi(z) + (z - \alpha)I_q \cdot \psi(z)\right]^{-1} = F(z). \quad \square$$

**Proposition 11.3.** Let $\alpha \in \mathbb{R}$ and let $s_0 \in \mathbb{C}^{q \times q}_{\geq}$. Let $V_{\alpha, s_0}$ be given by (11.1). Further, let $(\phi, \psi) \in \mathcal{P}_{q, \alpha}[s_0]$, and let

$$V_{\alpha, s_0} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \quad (11.22)$$

be the $q \times q$ block representation of $V_{\alpha, s_0}[\phi]$. Let $\mathcal{D}$ be a discrete subset of $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$ such that conditions (i) and (ii) in Definition 7.1 are fulfilled. Then:

(a) $\det Y(z) \neq 0$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(b) The functions $X$ and $Y$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.

(c) $(X, Y)$ belongs to $\mathcal{P}_{q, q}^{-J_q, \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

(d) The function $\det Y$ does not identically vanish in $\mathbb{C} \setminus [\alpha, \infty)$ and $F := XY^{-1}$ belongs to $S_{0,q, [\alpha, \infty)](s_j)_{j=0}^{\infty} : \leq]$.

**Proof.** (i) From the assumption $(\phi, \psi) \in \mathcal{P}_{q, \alpha}[s_0]$ we see that $(\phi, \psi) \in \mathcal{P}_{-J_q, \geq}^{-J_q, \geq}(\mathbb{C} \setminus [\alpha, \infty))$ is valid and, in view of Remark [A.6][E], that

$$s_0s_0^t\phi = \phi \quad (11.23)$$

holds true. Because of Remark [E.1] the function $V_{\alpha, s_0}$ is holomorphic in $\mathbb{C}$. Thus, taking into account that, by condition (i) of Definition 7.1 the functions $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$, we conclude from (11.22) then:

(iv) The functions $X$ and $Y$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. 

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11. On the set $S_{0,q,\{\alpha, \infty\}}[(s_j)_{j=0}^\infty, \leq]$

Let $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. Using (iv) and (E.1), we obtain

$$\begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} = V_{\alpha,s_0}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \begin{bmatrix} 0_{q \times q} & -s_0 \\ (z - \alpha)s_0^\dagger & (z - \alpha)I_q \end{bmatrix} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \begin{bmatrix} -s_0 \psi(z) \\ (z - \alpha)[s_0^\dagger \phi(z) + \psi(z)] \end{bmatrix}.$$  

Hence,

$$X(z) = -s_0 \psi(z) \quad \text{and} \quad Y(z) = (z - \alpha)[s_0^\dagger \phi(z) + \psi(z)]. \quad (11.24)$$

Now we are going to verify that $\det Y(z) \neq 0$. Let

$$v \in \mathcal{N}(Y(z)). \quad (11.25)$$

Applying (11.24) and (11.25), we infer $(z - \alpha)[s_0^\dagger \phi(z) + \psi(z)]v = [Y(z)]v = 0_{q \times 1}$ and, consequently, $\psi(z)v = -s_0^\dagger \phi(z)v$. Thus, (11.23) implies

$$-s_0 \psi(z)v = s_0 s_0^\dagger \phi(z)v = \phi(z)v. \quad (11.26)$$

Taking into account (11.26) and $s_0 \in \mathbb{C}^q_{\geq 0}$, we get

$$-v^*[\psi(z)]^*[\phi(z)]v = v^*[\psi(z)]^*s_0[\psi(z)]v = [\sqrt{s_0} \psi(z)v]^*[\sqrt{s_0} \psi(z)v] = \|\sqrt{s_0} \psi(z)v\|_E^2. \quad (11.27)$$

Obviously,

$$\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}) = (\mathbb{C}_{\alpha, -} \setminus \mathcal{D}) \cup [\mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})]. \quad (11.28)$$

(II) Now we consider the particular case that

$$z \in \mathbb{C}_{\alpha, -} \setminus \mathcal{D}. \quad (11.29)$$

Because of (11.29), (11.25), and (11.24), we have then

$$-v^* \psi^*(z) \phi(z)v = \|\sqrt{s_0} \psi(z)v\|_E^2. \quad (11.30)$$

In view of (11.28), the choice of $\mathcal{D}$, and (11.29), the application of Proposition 7.6 yields $\text{Re}[^* \psi^*(z) \phi(z)] \in \mathbb{C}_{\geq 0}^{q \times q}$. Using (11.30), we obtain then

$$0 \leq \|\sqrt{s_0} \psi(z)v\|_E^2 = \text{Re}[\|\sqrt{s_0} \psi(z)v\|_E^2] = \text{Re}[-v^* \psi^*(z) \phi(z)v] = -v^* \text{Re}[\psi^*(z) \phi(z)]v \leq 0.$$ 

Thus, $\|\sqrt{s_0} \psi(z)v\|_E = 0$ and, consequently,

$$\sqrt{s_0} \psi(z)v = 0_{q \times 1}. \quad (11.31)$$

(III) Now we consider the further particular case that $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Because of (11.25) and (11.27), we have then

$$-v^* \psi^*(z) \phi(z)v = \|\sqrt{s_0} \psi(z)v\|_E^2. \quad (11.32)$$
11. On the set $S_{0,q,[\alpha,\infty)}[(s_j)_j^0=0,\leq]$

In view of (11.28) and the choice of $D$, we infer from condition (iii) in Definition 7.1 that (7.2) holds true. Hence, Remark A.13 yields

$$\frac{\text{Im}[(z-\alpha)\psi^*(z)\phi(z)]}{\text{Im} z} \in \mathbb{C}^{q\times q}. \quad (11.33)$$

Using $\alpha \in \mathbb{R}$ and applying (11.32), we get

$$-v^*\left(\frac{1}{\text{Im} z} \text{Im}[(z-\alpha)\psi^*(z)\phi(z)]\right)v = \frac{\text{Im}[(z-\alpha)[-v^*\psi^*(z)\phi(z)v]]}{\text{Im} z} = \frac{\text{Im}[(z-\alpha)||\sqrt{s_0}\psi(z)v||^2_E]}{\text{Im} z} = \frac{\text{Im}[(z-\alpha)||\sqrt{s_0}\psi(z)v||^2_E]}{\text{Im} z} = ||\sqrt{s_0}\psi(z)v||^2_E. \quad (11.34)$$

Combining (11.34) and (11.33), we obtain

$$0 \leq ||\sqrt{s_0}\psi(z)v||^2_E = -v^*\left(\frac{1}{\text{Im} z} \text{Im}[(z-\alpha)\psi^*(z)\phi(z)]\right)v \leq 0.$$  

Thus, $||\sqrt{s_0}\psi(z)v||^2_E = 0$. Consequently, (11.31) holds true.

(IV) Now we consider again the general case $z \in \mathbb{C} \setminus (\{\alpha, \infty\} \cup D)$. In view of (11.28), part (II), and part (III), then (11.31) is proved. Using (11.26) and (11.31), it follows $\phi(z)v = s_0\psi(z)v = \sqrt{s_0}\sqrt{s_0}\psi(z)v = 0_{q\times 1}$. The last equations and (11.25) imply $[\phi(z)]v = 0_{q\times 1}$. Because of $(\phi, \psi) \in \mathcal{P}_{Q,q}^q_{\mathbb{C}}(\mathbb{C} \setminus [\alpha, \infty))$ and the choice of $D$, then condition (ii) in Definition 7.1 yields $v = 0_{q\times 1}$. Thus, taking into account (11.25), we have $Y(z) \neq 0$. The proof of part (a) is complete.

(b) This follows from (iv).

(c) Part (a) provides rank $X(z) = q$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty] \cup D)$. Now we consider an arbitrary $z \in \mathbb{C} \setminus (\mathbb{R} \cup D)$. From condition (iii) in Definition 7.1 we infer that (7.1) holds true. In view of $s_0 \in \mathbb{C}^{q\times q}, \subseteq \mathbb{C}_{\mathbb{H}}^{q\times q}$, the application of Corollary A.13 yields

$$[V_{a,s_0}(z)]^*(-J_q)[V_{a,s_0}(z)] = \left[\text{diag}\left((z-\alpha)s_0, s^*_0\right)\right]^*(-J_q)\left[\text{diag}\left((z-\alpha)s_0, s^*_0\right)\right] + 2\text{Im}(z)\text{diag}(0_{q\times q}, s_0) \quad (11.35)$$

and

$$[\text{diag}((z-\alpha)I_q, I_q)V_{a,s_0}(z)]^*(-J_q)[\text{diag}((z-\alpha)I_q, I_q)V_{a,s_0}(z)] = |z-\alpha|^2\left[\text{diag}(s_0, s^*_0)\right]^*(-J_q)\left[\text{diag}(s_0, s^*_0)\right]. \quad (11.36)$$

In view of $s^*_0 = s_0$, Lemma A.17 yields

$$[\text{diag}(s_0, s^*_0)]^*(-J_q)\left[\text{diag}(s_0, s^*_0)\right] = \left[\text{diag}(s_0s^*_0, I_q)\right]^*(-J_q)\left[\text{diag}(s_0s^*_0, I_q)\right]. \quad (11.37)$$

and

$$[\text{diag}\left((z-\alpha)s_0, s^*_0\right)]^*(-J_q)\left[\text{diag}\left((z-\alpha)s_0, s^*_0\right)\right] = \left[\text{diag}\left((z-\alpha)s_0s^*_0, I_q\right)\right]^*(-J_q)\left[\text{diag}\left((z-\alpha)s_0s^*_0, I_q\right)\right]. \quad (11.38)$$

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Using (11.22) and (11.35), we get

\[
\begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} \begin{bmatrix} -\tilde{J}_q \\ \frac{2}{2 \text{Im } z} \end{bmatrix} \begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} = \left(V_{\alpha, s_0}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right)^* \left(-\tilde{J}_q \right) \left(V_{\alpha, s_0}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right)
\]

\[
= \left[ \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right]^* \left[V_{\alpha, s_0}(z)^* \begin{bmatrix} -\tilde{J}_q \\ \frac{2}{2 \text{Im } z} \end{bmatrix} \left[V_{\alpha, s_0}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right] \right]^* 
\]

\[
= \left[ \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right]^* \left(\text{diag}\left(\alpha, s_0, s_0^\dagger\right)^* \frac{\tilde{J}_q}{\text{Im } z} \right) \left(\text{diag}\left(\alpha, s_0, s_0^\dagger\right) \right) + \text{diag}(0_{q \times q}, s_0) \right] \left[ \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right]
\]

\[(11.39)\]

Applying (11.33) and (11.23), it follows

\[
\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \left[ \text{diag}\left(\alpha, s_0, s_0^\dagger\right)^* \frac{\tilde{J}_q}{\text{Im } z} \right] \left[ \text{diag}\left(\alpha, s_0, s_0^\dagger\right) \right] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}
\]

\[
= \left[ \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right]^* \left[ \text{diag}\left(\alpha, s_0, s_0^\dagger\right)^* \frac{\tilde{J}_q}{\text{Im } z} \right] \left[ \text{diag}\left(\alpha, s_0, s_0^\dagger\right) \right] \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}
\]

\[
= \left( \text{diag}\left(\alpha, s_0, s_0^\dagger\right) \right)^* \frac{\tilde{J}_q}{\text{Im } z} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}
\]

\[(11.40)\]

Because of (11.39) and (11.40), we have

\[
\begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} \begin{bmatrix} \frac{\tilde{J}_q}{2 \text{Im } z} \\ \frac{-\tilde{J}_q}{2 \text{Im } z} \end{bmatrix} \begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} = \left(\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right)^* \left(\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \right) + |\psi(z)|^2 s_0 |\psi(z)|
\]

\[(11.41)\]

Since $s_0$ is non-negative Hermitian, we know that $|\psi(z)|^2 s_0 |\psi(z)| \in \mathbb{C}^{T \times q}_z$. Consequently, (11.41) and (11.22) imply

\[
\begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} \begin{bmatrix} \frac{\tilde{J}_q}{2 \text{Im } z} \\ \frac{-\tilde{J}_q}{2 \text{Im } z} \end{bmatrix} \begin{bmatrix} X(z) \\ Y(z) \end{bmatrix} \in \mathbb{C}^{T \times q}_z.
\]

\[(11.42)\]
Using (11.22), (11.36), (11.37), and (11.23), we obtain

\[
\begin{align*}
\left[ (z - \alpha)X(z) \right]^* & \left( -\bar{J}_q \right) \left[ (z - \alpha)X(z) \right] \\
& = \left( \text{diag}((z - \alpha)I_q, I_q) \left[ X(z) \right] \right)^* \left( -\bar{J}_q \right) \left( \text{diag}((z - \alpha)I_q, I_q) \left[ X(z) \right] \right) \\
& = \left( \text{diag}((z - \alpha)I_q, I_q) V_{\alpha,s_0}(z) \left[ \phi(z) \right] \left[ \psi(z) \right] \right)^* \left( -\bar{J}_q \right) \left( \text{diag}((z - \alpha)I_q, I_q) V_{\alpha,s_0}(z) \left[ \phi(z) \right] \left[ \psi(z) \right] \right) \\
& = \left[ \phi(z) \right] \left[ \psi(z) \right] \left[ \text{diag}((z - \alpha)I_q, I_q) V_{\alpha,s_0}(z) \right]^* \left( -\bar{J}_q \right) \left[ \text{diag}((z - \alpha)I_q, I_q) V_{\alpha,s_0}(z) \right] \\
& = \left[ \phi(z) \right] \left[ \psi(z) \right] \left[ \text{diag}(s_0^0, s_0^1, I_q)^* \left( -\bar{J}_q \right) \text{diag}(s_0^0, s_0^1, I_q) \right] \\
& = |z - \alpha|^2 \left( \text{diag}(s_0^0, s_0^1, I_q) \left[ \phi(z) \right] \left[ \psi(z) \right] \right)^* \left( -\bar{J}_q \right) \left[ \text{diag}(s_0^0, s_0^1, I_q) \left[ \phi(z) \right] \left[ \psi(z) \right] \right) \\
& = |z - \alpha|^2 \left[ s_0^0, s_0^1, I_q \right] \left[ \phi(z) \right] \left[ \psi(z) \right] \left( -\bar{J}_q \right) \left[ \phi(z) \right] \left[ \psi(z) \right]. \\
\end{align*}
\]

(11.43)

From (11.43) and (7.2) we conclude now \[ \frac{(z - \alpha)X(z)}{Y(z)} \in \mathbb{C}^{q \times q}. \] In view of parts (a) and (b) and Definition (L.4) the proof of (c) is complete.

(d) In view of (c), the application of Proposition (L.11) yields \( F \in S_{q,[\alpha, \infty)} \). In order to complete the proof of (d) we are going to apply Lemma (L.10) from \( F \in S_{q,[\alpha, \infty)} \) and Proposition (4.17) we get \( \text{Rstr}_{\mathbb{H}} \in \mathcal{R}_q(\Pi_\infty) \). Now let \( z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}) \). Then we have \( \text{Im} z \neq 0 \) and \( \frac{1}{\text{Im} z} \text{Im} F(z) = F(z) - \frac{F^*(z)}{z - \bar{z}} \). Consequently, Remark (A.13) and (11.42) provide us

\[
\begin{align*}
\frac{F(z) - F^*(z)}{z - \bar{z}} &= \frac{1}{\text{Im} z} \text{Im} F(z) = \frac{1}{\text{Im} z} \text{Im} \left( [X(z)][Y(z)]^{-1} \right) \\
& = \frac{1}{2\text{Im} z} \left[ Y(z) \right]^{-*} \left[ X(z) \right]^* \left( -\bar{J}_q \right) \left[ X(z) \right] \left[ Y(z) \right]^{-1} \\
& = \left[ Y(z) \right]^{-*} \left[ \left( z - \alpha \right) \phi(z) \right]^* \left( -\bar{J}_q \right) \left[ \left( z - \alpha \right) \phi(z) \right] \left[ \psi(z) \right] + [\psi(z)]^* s_0[\psi(z)] \left[ Y(z) \right]^{-1} \\
& = \left[ Y(z) \right]^{-*} \left[ \left( z - \alpha \right) \phi(z) \right]^* \left( -\bar{J}_q \right) \left[ \left( z - \alpha \right) \phi(z) \right] \left[ Y(z) \right]^{-1} + [\psi(z)]^* s_0[\psi(z)] \left[ Y(z) \right]^{-1}.
\end{align*}
\]

(11.44)
Applying (7.2) and (11.46), we conclude

\[ \begin{align*}
[Y(z)]^{-*}[\psi(z)]^*s_0[\psi(z)]&= [Y(z)]^{-*}[\psi(z)]^*s_0 s_0^* s_0 [\psi(z)] [Y(z)]^{-1} \\
&= [Y(z)]^{-*}[\psi(z)]^* s_0^* s_0 [\psi(z)] [Y(z)]^{-1} \\
&= [Y(z)]^{-*} [-X(z)]^* s_0^* [-X(z)] [Y(z)]^{-1} = [Y(z)]^{-*} [X(z)]^* s_0^* [X(z)] [Y(z)]^{-1} \\
&= [F(z)]^* s_0^* [F(z)].
\end{align*} \]

Combining (11.44) and (11.45), it follows

\[ \begin{align*}
\frac{F(z) - F^*(z)}{z - \overline{z}} - [F(z)]^* s_0^* [F(z)]
&= [Y(z)]^{-*} \left[ (z - \alpha) \phi(z) \right]^* \left( \frac{-i}{2 \mathrm{Im} \, z} \right) \phi(z) [Y(z)]^{-1}.
\end{align*} \]

Applying (7.2) and (11.46), we conclude

\[ \begin{align*}
\frac{F(z) - F^*(z)}{z - \overline{z}} - [F(z)]^* s_0^* [F(z)] 
&\in \mathbb{C}^{q \times q}. \tag{11.47}
\end{align*} \]

Furthermore, using again the first equation in (11.24), we get

\[ \mathcal{R}(F(z)) = \mathcal{R} \left( X(z) [Y(z)]^{-1} \right) \subseteq \mathcal{R}(X(z)) = \mathcal{R}(-s_0 \psi(z)) \subseteq \mathcal{R}(s_0) \]

and, therefore, \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(s_0) \). Consequently, taking additionally into account \( s_0 \in \mathbb{C}^{q \times q}_\geq \) and (11.47), we conclude with the aid of Lemma A.10 then

\[ \begin{bmatrix} s_0 & F(z) - F^*(z) \\ F^*(z) & \overline{z} \end{bmatrix} \in \mathbb{C}^{2q \times 2q}. \tag{11.48} \]

Since (11.48) is fulfilled for all \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathbb{D}) \) and since \( F \) is holomorphic and in particular continuous in \( \mathbb{C} \setminus [\alpha, \infty) \), we obtain that (11.48) holds true for all \( z \in \mathbb{C} \setminus \mathbb{R} \). Thus, since \( F \) is holomorphic in \( \Pi_+ \), the application of Lemma 11.16 yields that

\[ \begin{align*}
\text{Rstr}_{\Pi_+} F &\in \mathcal{R}_{0, q}(\Pi_+) \tag{11.49}
\end{align*} \]

and that the Stieltjes measure \( \mu \) of \( \text{Rstr}_{\Pi_+} F \) satisfies

\[ s_0 - \mu(\mathbb{R}) \in \mathbb{C}^{q \times q}_\geq. \tag{11.50} \]

From \( F \in \mathcal{S}_{q,[\alpha, \infty)} \) and (11.49) we get \( F \in \mathcal{S}_{0,q,[\alpha, \infty)} \). Moreover, in view of Proposition 11.17 the \([\alpha, \infty)\)-Stieltjes measure \( \sigma_F \) of \( F \) satisfies \( \sigma_F([\alpha, \infty)) = \mu(\mathbb{R}) \). Thus, (11.50) implies \( s_0 - \sigma_F([\alpha, \infty)) \in \mathbb{C}^{q \times q}_\geq \). Consequently, (11.49) shows that \( F \in \mathcal{S}_{0,q,[\alpha, \infty)} ([s_j]_{j=0}^m, \leq). \)

12. **A first description of the set \( \mathcal{S}_{m,q,[\alpha, \infty)} ([s_j]_{j=0}^m, \leq) \)**

In this section, we give a Schur type algorithm for functions which belong to the class \( \mathcal{S}_{0,q,[\alpha, \infty)} \). This enables us to construct an explicit bijective mapping between \( \{ \mathcal{P}_{q, \alpha}([s_0]) \} \) and \( \mathcal{S}_{m,q,[\alpha, \infty)} ([s_j]_{j=0}^m, \leq) \).
12. A first description of the set $\mathcal{S}_{m,q,\alpha,\infty}[(s_j)_{j=0}^m, \leq]$

**Proposition 12.1.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in K_{q,m,\alpha}^\geq$, let $(s_j^{[m,\alpha]})_{j=0}^0$ be the $m$-th $\alpha$-$S$-transform of $(s_j)^{[m,\alpha]}_{j=0} = 0$ such that conditions (i) and (ii) in Definition 7.1 are fulfilled. Let $v_{jk}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}]}$ be given for every choice of $j,k \in \{1,2\}$ via (E.26) and (E.27). Then

\[
\det \left[ v_{21}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}}(z) \right] + v_{22}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}}(z) \psi(z) \right] \neq 0 \quad \text{for each } z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \quad (12.1)
\]

and

\[
\left[ v_{11}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}} + v_{12}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}}(z) \psi(z) \right] \left[ v_{21}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}} + v_{22}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}}(z) \psi(z) \right]^{-1} \in \mathcal{S}_{m,q,\alpha,\infty}[(s_j)_{j=0}^m, \leq]. \quad (12.2)
\]

**Proof.** We will prove Proposition 12.1 by induction.

First we consider the case $m = 0$. Because of Remark 1.6 and Lemma 2.9, we have $s_0 \in \mathbb{C}^{q \times q}$. From Definition 3.11 we get $s_0^{[0,\alpha]} = s_0$ and, hence, $(\phi, \psi) \in P_{q,\alpha}[s_0]$. In view of (E.26), we obtain $\mathcal{G}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}} = V_{\alpha,s_0}$, and, consequently,

\[
V_{\alpha,s_0} \left[ \begin{array}{c} \phi \\ \psi \end{array} \right] = \mathcal{G}^{[\alpha,(s_j)^{[m,\alpha]}_{j=0}} \left[ \begin{array}{c} \phi \\ \psi \end{array} \right] = \left[ \begin{array}{c} \alpha,(s_j)^{[m,\alpha]}_{j=0} \\ \alpha,(s_j)^{[m,\alpha]}_{j=0} \end{array} \right] \left[ \begin{array}{c} \phi \\ \psi \end{array} \right].
\]

Thus, Proposition 11.3(31) shows that (12.1) holds true for $m = 0$. Proposition 11.3(31) yields (12.2) for $m = 0$. Thus, Proposition 11.3(31) is proved for $m = 0$.

Now we assume that there is an $n \in \mathbb{N}$ such that Proposition 12.1 is checked for each $m \in \mathbb{Z}_{0,n-1}$. We consider the case $m = n$. Let $t_j := s_j^{[1,\alpha]}$ for each $j \in \mathbb{Z}_{0,n-1}$. From Theorem 8.1(13) we see then that $(t_j)^{n-1}_{j=0} \in K_{q,n-1,\alpha}^\geq$. Because of Remark 8.13 we also have $t_0^{[n-1,\alpha]} = s_0^{[n,\alpha]}$. Thus, $(\phi, \psi) \in P_{q,\alpha}[t_0^{[n-1,\alpha]}]$. Since Proposition 12.1 is assumed to be true for $m = n - 1$, we get

\[
\det \left[ v_{21}^{[\alpha,(t_j)^{n-1}_{j=0}}(z) \phi(z) + v_{22}^{[\alpha,(t_j)^{n-1}_{j=0}}(z) \psi(z) \right] \neq 0 \quad \text{for each } z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \quad (12.3)
\]

and that $G := \mathcal{G}^{[\alpha,(t_j)^{n-1}_{j=0}}((\phi, \psi))$ is a well-defined matrix-valued function for which

\[
G = \left[ v_{11}^{[\alpha,(t_j)^{n-1}_{j=0}} + v_{12}^{[\alpha,(t_j)^{n-1}_{j=0}}(z) \psi(z) \right] \left[ v_{21}^{[\alpha,(t_j)^{n-1}_{j=0}} + v_{22}^{[\alpha,(t_j)^{n-1}_{j=0}}(z) \psi(z) \right]^{-1} 
\in \mathcal{S}_{n-1,q,\alpha,\infty}[(t_j)^{n-1}_{j=0}, \leq]. \quad (12.4)
\]

is true. Because of Definition 8.3 we have $\mathcal{N}(s_0) \subseteq \mathcal{N}(s_0^{[1,\alpha]})$ and, in view of 8.3, then $\mathcal{S}_{q,\alpha,\infty}[s_0^{[1,\alpha]}] \subseteq \mathcal{S}_{q,\alpha,\infty}[s_0]$. Combining this with Remark 6.7, we obtain

\[
\mathcal{S}_{n-1,q,\alpha,\infty}[(t_j)^{n-1}_{j=0}, \leq] \subseteq \mathcal{S}_{q,\alpha,\infty}[t_0] = \mathcal{S}_{q,\alpha,\infty}[s_0^{[1,\alpha]}] \subseteq \mathcal{S}_{q,\alpha,\infty}[s_0].
\]

Hence, (12.4) implies $G \in \mathcal{S}_{q,\alpha,\infty}[s_0]$. Because of Proposition 8.3, we conclude then

\[
\det \left[ (z - \alpha)s_0^\dagger G(z) + (z - \alpha)I_q \right] \neq 0 \quad (12.5)
\]

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Remark E.5 and Lemma D.2 show that Proposition 12.2. Let

$$G_{[-\alpha, s_0]}(z) = [S_{V_0, s_0}^{(q, q)}(G)](z)$$

for each $z \in \mathbb{C} \setminus [\alpha, \infty)$. (12.6)

By virtue of Remark E.5, we get $
\text{rank}[(z - \alpha)s_0^1, (z - \alpha)I_q] = q$ for each $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Lemma D.2 and (12.3) yield $
\text{rank}[v_{21}^{[\alpha, (t_j)_{j=0}^{n-1}], (z)}v_{22}^{[\alpha, (t_j)_{j=0}^{n-1}]}(z)] = q$

for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$. Hence, (12.5), (12.3), and Proposition D.4 show that the inequality $
\text{det}[(\phi(z)) + v_{22}^{[\alpha, (s_j)_{j=n}]}(z)\psi(z)] \neq 0$

and the equations

$$\left(\alpha^{(t_j)_{j=0}^{n-1}}_0 \phi + v_{12}^{[\alpha, (s_j)_{j=n}]_0} \psi\right)\left(\alpha^{[\alpha, (s_j)_{j=n}]_0}_{V_0, s_0} \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right)^{-1} = \left(\alpha^{(t_j)_{j=0}^{n-1}}_0 \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right) = S_{V_0, s_0}^{(q, q)}(G) = G_{[-\alpha, s_0]}(z)$$

are valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$. Because of $(s_j)_{j=0}^{n} \in K_{q, n, \alpha}^{\geq e}$ and Proposition 10.3 we have $(s_j)_{j=0}^{n} \in K_{q, n, \alpha}^{\geq e} \cap D_{q, n, a}^{\geq e}$. Hence, from (12.4) and Theorem 10.4(b) we obtain $G_{[-\alpha, s_0]} \in S_{n, q, [\alpha, \infty]}(s_j)_{j=0}^{n} \leq \cdot$. Combining this with Remark E.5, (12.7), and (12.6), we get

$$\left[\alpha^{[\alpha, (s_j)_{j=n}]_0}_{V_0, s_0} \phi + v_{12}^{[\alpha, (s_j)_{j=n}]_0} \psi\right] \left[\alpha^{[\alpha, (s_j)_{j=n}]_0}_{V_0, s_0} \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right]^{-1} = \left(\alpha^{(t_j)_{j=0}^{n-1}}_0 \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right) = S_{V_0, s_0}^{(q, q)}(G) = G_{[-\alpha, s_0]}(z) \in S_{n, q, [\alpha, \infty]}(s_j)_{j=0}^{n} \leq \cdot$$

Consequently, Proposition 12.1 is proved for $m = n$ as well. The proof is complete. □

**Proposition 12.2.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{n} \in K_{q, m, \alpha}^{\geq e}$, let $(s_j)_{j=0}^{m} \in K_{q, m, \alpha}^{\geq e}$, and let $F \in S_{m, q, [\alpha, \infty]](s_j)_{j=0}^{n} \leq \cdot]$. Then there exists a $q \times q$ Stieltjes pair $(\phi, \psi) \in \mathcal{P}_{q, a, [s_0]_{m, [\alpha, \infty]}^{[\alpha, \infty]}\] such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and that, for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, the inequality $\text{det}[v_{21}^{[\alpha, (s_j)_{j=n}]_0}(z)\phi(z) + v_{22}^{[\alpha, (s_j)_{j=n}]_0}(z)\psi(z)] \neq 0$ and the representation

$$F(z) = \left[\alpha^{[\alpha, (s_j)_{j=n}]_0}_{V_1, s_0} \phi + v_{12}^{[\alpha, (s_j)_{j=n}]_0} \psi\right] \left[\alpha^{[\alpha, (s_j)_{j=n}]_0}_{V_2, s_0} \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right]^{-1} \times \left[\alpha^{(t_j)_{j=0}^{n-1}}_0 \phi + v_{22}^{[\alpha, (s_j)_{j=n}]_0} \psi\right]$$

of $F$ hold true.

**Proof.** (1) Since $(s_j)_{j=0}^{n}$ belongs to $K_{q, m, \alpha}^{\geq e}$, from Remark E.6 and Lemma 2.6(b) we obtain $s_0 \in \mathbb{C}^{q 	imes q}$. Obviously,

$$\text{rank}[(z - \alpha)s_0^1, (z - \alpha)I_q] = q$$

for all $z \in \mathbb{C} \setminus \{\alpha\}$. (12.8)

Remark E.5 and Lemma D.2 show that

$$\text{rank}[-(z - \alpha)s_0^1, I_q - s_0^1s_0] = q$$

for all $z \in \mathbb{C} \setminus \{\alpha\}$. (12.9)

(II) Now our proof works inductively. In the case $m = 0$, the assertion follows immediately applying (E.26), (E.7), Definition 3.13, and Proposition 11.2.
12. A first description of the set $S_{m,q,[\alpha,\infty]}((s_j)_{j=0}^{n-1}, \leq)$

(III) Because of part (II) of the proof, we can assume that there is a positive integer $n$ such that Proposition 12.2 is already proved for each $m \in \mathbb{Z}_{0,n-1}$.

(IV) We consider now the case $m = n$. Because of $F \in S_{n,q,[\alpha,\infty]}((s_j)_{j=0}^{n-1}, \leq)$, the $[\alpha,\infty)$-Stieltjes measure $\sigma_F$ of $F$ belongs to $\mathcal{M}_{\geq}^{\infty}([\alpha,\infty); (s_j)_{j=0}^{n-1}, \leq]$. Hence, $s_j = s_j^{(\sigma_F)}$ for each $j \in \mathbb{Z}_{0,n-1}$. In particular, $s_0 = s_0^{(\sigma_F)} = s_0([\alpha,\infty))$. Therefore, and in view of Lemma E.2 for each $z \in \mathbb{C} \setminus [\alpha,\infty)$, we get $R(s_0) = R(\sigma_F([\alpha,\infty))) = R(F(z))$. Thus, Remark A.6 shows

$$s_0 s_0^{\dagger} F(z) = F(z)$$

for each $z \in \mathbb{C} \setminus [\alpha,\infty)$. (12.10)

Lemma E.3(b) yields $F(z) \in \mathcal{Q}_{[-(z-\alpha)s_0, l_q-s_0]}$ for each $z \in \mathbb{C} \setminus \{\alpha\}$. Consequently,

$$\det \left[ -(z - \alpha)s_0^{\dagger} F(z) + (I_q - s_0 s_0^{\dagger}) \right] \neq 0$$

for all $z \in \mathbb{C} \setminus [\alpha,\infty)$. (12.11)

Moreover, from Proposition 5.2 we get

$$\det \left[ (z - \alpha)s_0^{\dagger} \left( S_{W_{\alpha,q_0}}^{(q,q)} (F) \right) (z) + (z - \alpha)I_q \right] \neq 0$$

for all $z \in \mathbb{C} \setminus [\alpha,\infty)$. (12.12)

Combining (E.1), (E.11), (12.9), (12.8), (12.11), (12.12), Corollary D.5, Remark E.2, and (12.10), we infer

$$S_{V_{\alpha,q_0}(z)}^{(q,q)} \left( S_{W_{\alpha,q_0}(z)}^{(q,q)} (F(z)) \right) = \left( S_{V_{\alpha,q_0}(z)-\alpha,q_0}^{(q,q)} \right)_{z=\alpha} \left( S_{W_{\alpha,q_0}(z)-\alpha,q_0}^{(q,q)} (F(z)) \right) = \left( (z - \alpha)s_0 s_0^{\dagger} F(z) \right) [z - (z - \alpha)I_q]^{-1} = s_0 s_0^{\dagger} F(z) = F(z)$$

for each $z \in \mathbb{C} \setminus [\alpha,\infty)$. Let $t_j := s_j^{[1,\alpha]}$ for each $j \in \mathbb{Z}_{0,n-1}$. From Theorem 3.5(b) we see then that $(t_j)_{j=0}^{n-1} \in K_{\geq}^{\infty}$. Let $R$ be the $(\alpha, s_0)$-Schur-Stieltjes transform of $F$. By virtue of (8.1), we obtain then $R = F^{[\alpha,\alpha]}$. Consequently, Theorem 9.2 yields $R \in S_{(0,q,[\alpha,\infty)]}((t_j)_{j=0}^{n-1}, \leq)$. Because of part (III) of the proof, then there exists a pair $(\phi, \psi) \in \mathcal{P}_{q,\alpha}[l_0]$ such that

$$\det \left[ \phi(z) + \psi(z) \right] \neq 0$$

and

$$R(z) = \left[ \phi(z) + \psi(z) \right]^{-1}$$

hold true for each $z \in \mathbb{C} \setminus [\alpha,\infty)$. Hence, from (E.27) and Notation D.1 we get

$$R(z) = \left[ \begin{array}{c} \phi(z) + \psi(z) \\ \phi(z) + \psi(z) \end{array} \right]$$

for each $z \in \mathbb{C} \setminus [\alpha,\infty)$. Regarding (12.14), Lemma D.2 shows that

$$\text{rank} \left[ \begin{array}{c} \phi(z) + \psi(z) \\ \phi(z) + \psi(z) \end{array} \right] = q$$

(12.16)
for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). In view of (12.12), equation (12.15) and Proposition 5.2 yield \( F(z) \in \mathcal{Q}_{[-(z-\alpha)s_0^q, I_q - s_0^q]} \) and

\[
\begin{align*}
\tilde{S}^{(q,q)}_{g\alpha,(t_j^q)^{n-1}}(z) (\phi(z), \psi(z)) &= \left[ \tilde{S}^{(q,q)}_{g\alpha,(t_j^q)^{n-1}}(z) (\phi, \psi) \right](z) = R(z) = F^{+[\alpha,s_0]}(z) \\
&= \left[ S_{W_{\alpha,s_0}}(F) (z) = S_{W_{\alpha,s_0}}(F)(z) \right] \\
(12.17)
\end{align*}
\]

for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). Hence, (12.17) and (12.12) provide us

\[
\det \left( (z - \alpha)s_0^q \left[ \tilde{S}^{(q,q)}_{g\alpha,(t_j^q)^{n-1}}((\phi(z), \psi(z))) \right](z) + (z - \alpha)I_q \right) \neq 0 \quad (12.18)
\]

for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \). Taking into account \( \text{E.1}, \text{12.11}, \text{12.12}, \text{12.13}, \text{12.17}, \text{E.27}, \text{12.16} \), again \( \text{E.1}, \text{12.28}, \text{12.11}, \text{12.18} \), Proposition D.1 \( t_j := s_j^{[1,a]} \) for all \( j \in \mathbb{Z}_{0,n-1} \), Remark E.0 again \( \text{E.27} \), and Notation D.1 for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \), we infer

\[
F(z) = S^{(q,q)}_{\alpha,s_0}(z) \left( S^{(q,q)}_{W_{\alpha,s_0}}(F(z)) \right) = S^{(q,q)}_{\alpha,s_0}(z) \left( S^{(q,q)}_{\alpha,s_0}(z) \left( (\phi(z), \psi(z)) \right) \right) \\
= \left[ S^{(q,q)}_{\alpha,s_0}(z) \right]^{-1} (\phi(z), \psi(z)) \\
\text{[11,12]} \left( \phi(z) + v_{12}^{[\alpha,(s_j)^{n-0}])(z) \psi(z) \right) v_{21}^{[\alpha,(s_j)^{n-0}])(z) \phi(z) + v_{22}^{[\alpha,(s_j)^{n-0}])(z) \psi(z) \right]^{-1}.
\]

By induction, the proof is complete.

If \( M \in \mathbb{C}^{p \times q} \), if \( (\phi_1, \psi_1) \in \mathcal{P}_{[a,m]}[M] \), and if \( (\phi_2, \psi_2) \in \mathcal{P}_{[a,m]}[\mathbb{C} \setminus [\alpha, \infty)) \) fulfills \( ((\phi_1, \psi_1)) = ((\phi_2, \psi_2)) \), then it is easily checked that \( (\phi_2, \psi_2) \in \mathcal{P}_{[a,m]}[M] \).

We will now collect the results of this section. In this way, we obtain a first parametrization of Problem M\([\alpha, \infty); (s_j)^{m-0} \leq \] formulated in Section I

**Theorem 12.3.** Let \( \alpha \in \mathbb{R} \), let \( m \in \mathbb{N}_0 \), and let \( (s_j)^{m-0} \in \mathcal{K}_{q,m,a}^\geq \). Let \( (s_j^{[m,a]})^{j=0}_0 \) be the \( m \)-th \( \alpha \)-S-transform of \( (s_j)^{m-0} \). Let \( \mathcal{Q}_{[\alpha,(s_j)^{n-0}] \in \} \) be defined via \( \text{E.26} \) and \( \text{E.1} \). Furthermore, let \( \mathcal{Q}_{[\alpha,(s_j)^{n-0}] \in \} \) be the \( q \times q \) block representation of \( \mathcal{Q}_{[\alpha,(s_j)^{n-0}] \in \} \). Then the following statements hold true:

(a) For each \( (\phi, \psi) \in \mathcal{P}_{[\alpha, \infty]}[s_0^{[m,a]}], \) the function \( \det(v_{11}^{[\alpha,(s_j)^{n-0}] \phi} + v_{12}^{[\alpha,(s_j)^{n-0}] \psi})(v_{21}^{[\alpha,(s_j)^{n-0}] \phi} + v_{22}^{[\alpha,(s_j)^{n-0}] \psi})^{-1} \) (12.19) belongs to \( S_{m,q,[\alpha, \infty]}[(s_j)^{m-0}; \leq] \).

(b) For every choice of \( F \in S_{m,q,[\alpha, \infty]}[(s_j)^{m-0}; \leq] \), there exists a pair \( (\phi, \psi) \in \mathcal{P}_{[\alpha, \infty]}[s_0^{[m,a]}] \) of \( q \times q \) matrix-valued functions \( \phi \) and \( \psi \) which are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \) such that, for each \( z \in \mathbb{C} \setminus [\alpha, \infty) \), the inequality \( \det(v_{11}^{[\alpha,(s_j)^{n-0}] \phi} + v_{12}^{[\alpha,(s_j)^{n-0}] \psi})(v_{21}^{[\alpha,(s_j)^{n-0}] \phi} + v_{22}^{[\alpha,(s_j)^{n-0}] \psi}) \neq 0 \) and
the representation

\[ F(z) = \begin{bmatrix} v_1^{[\alpha(s)_j^m]}(z)\phi(z) + v_{12}^{[\alpha(s)_j^m]}(z)\psi(z) \\ v_2^{[\alpha(s)_j^m]}(z)\phi(z) + v_{22}^{[\alpha(s)_j^m]}(z)\psi(z) \end{bmatrix}^{-1} \]

of \( F \) hold true.

(c) Let \((\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}].\) Then the following statements are equivalent:

\begin{enumerate}[(i)]
  \item \((v_1^{[\alpha(s)_j^m]} \phi_1 + v_{12}^{[\alpha(s)_j^m]} \psi_1)(v_{21}^{[\alpha(s)_j^m]} \phi_1 + v_{22}^{[\alpha(s)_j^m]} \psi_1) = 0 = (v_1^{[\alpha(s)_j^m]} \phi_2 + v_{12}^{[\alpha(s)_j^m]} \psi_2)(v_{21}^{[\alpha(s)_j^m]} \phi_2 + v_{22}^{[\alpha(s)_j^m]} \psi_2).\)
  \item \(\langle (\phi_1, \psi_1) \rangle = \langle (\phi_2, \psi_2) \rangle.\)
\end{enumerate}

Proof. \(\square\) Combining Definition 7.1 and Proposition 12.1\(\square\)

(i) This follows from Proposition 12.2\(\square\)

(ii) Let

\[ F := (v_1^{[\alpha(s)_j^m]} \phi_1 + v_{12}^{[\alpha(s)_j^m]} \psi_1)(v_{21}^{[\alpha(s)_j^m]} \phi_1 + v_{22}^{[\alpha(s)_j^m]} \psi_1)^{-1}. \]  

Then

\[ \begin{bmatrix} F \\ \mathcal{J}_q \end{bmatrix} = \begin{bmatrix} v_1^{[\alpha(s)_j^m]} \phi_1 + v_{12}^{[\alpha(s)_j^m]} \psi_1 \\ v_{21}^{[\alpha(s)_j^m]} \phi_1 + v_{22}^{[\alpha(s)_j^m]} \psi_1 \\ v_1^{[\alpha(s)_j^m]} \phi_1 + v_{12}^{[\alpha(s)_j^m]} \psi_1 \end{bmatrix}^{-1} \]

\[ = \mathcal{W}^{[\alpha(s)_j^m]} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \]

Let \(g_\alpha : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C} \) be defined via \(g_\alpha(z) := (z - \alpha)^{m+1}.\) From Lemma 12.10 we obtain then

\[ \mathcal{W}^{[\alpha(s)_j^m]} \mathcal{W}^{[\alpha(s)_j^m]} = g_\alpha \cdot \text{diag}(s_0^{[m,\alpha]}(s_0^{[m,\alpha]})^\dagger, I_q). \]

Because of \((\phi_1, \psi_1) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}] \) and Remark A.6(b), we have \(s_0^{[m,\alpha]}(s_0^{[m,\alpha]})^\dagger \phi_1 = \phi_1.\) Consequently, combining 12.22 and 12.23 gives us

\[ \mathcal{W}^{[\alpha(s)_j^m]} \mathcal{W}^{[\alpha(s)_j^m]} \begin{bmatrix} F \\ \mathcal{J}_q \end{bmatrix} = g_\alpha \cdot \text{diag}(s_0^{[m,\alpha]}(s_0^{[m,\alpha]})^\dagger, I_q) \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \]

\[ = g_\alpha \begin{bmatrix} s_0^{[m,\alpha]}(s_0^{[m,\alpha]})^\dagger \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1} \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix}^{-1}. \]
13. Parametrization of the matricial Stieltjes moment problem in the general case

From (12.21) and (3) we get

\[ F = (v_{11}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{12}^{[\alpha,(s_j)_{j=0}^m]} \psi_2) (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_2)^{-1}. \] (12.25)

Then, because of (12.25) and \((\phi_2, \psi_2) \in \mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}],\) the above computations (see (12.21) and (12.24)) show that

\[ \mathcal{M}[\alpha,(s_j)_{j=0}^m] \mathcal{G}_{q,\alpha}[\alpha,(s_j)_{j=0}^m] \begin{bmatrix} F \\ \mathcal{G}_q \end{bmatrix} = g_{\alpha} \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_2)^{-1}. \] (12.26)

From (12.21), (12.26), and the definition of the function \(g_{\alpha}\), we get

\[ \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_1 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_1)^{-1} = \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_2)^{-1} \]

and, consequently,

\[ \begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_1 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_1)^{-1} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_2). \] (12.27)

In view of

\[ \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_1 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_1)^{-1} (v_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi_2 + v_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi_2) \in \mathcal{M}(\mathbb{C} \setminus [\alpha, \infty])^{q \times q} \]

and the fact that the determinant of this function does not identically vanish in \(\mathbb{C} \setminus [\alpha, \infty)\), we see from (12.27) that \(\langle (\phi_1, \psi_1) \rangle = \langle (\phi_2, \psi_2) \rangle\). Thus, (ii) is satisfied.

(ii) \(\Rightarrow\) (i): This implication holds trivially.

Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)_{j=0}^m\) be a sequence of complex \(q \times q\) matrices. Denote by \((s_j)_{j=0}^m\) the \(m\)-th \(\alpha\)-S-transform of \((s_j)_{j=0}^m\). According to Lemma 12.17, we write \((\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}])\) for the set of the equivalence classes of which belong to \(\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]\).

**Corollary 12.4.** Let \(\alpha \in \mathbb{R}\), let \(m \in \mathbb{N}_0\), and let \((s_j)_{j=0}^m\) be the \(m\)-th \(\alpha\)-S-transform of \((s_j)_{j=0}^m\). Then the mapping \(\Sigma: (\mathcal{P}_{q,\alpha}[s_0^{[m,\alpha]}]) \to S_{m,q,[\alpha,\infty]}(s_j)_{j=0}^m, \leq \) defined by

\[ \Sigma((\phi, \psi)) := \begin{bmatrix} \phi_{11}^{[\alpha,(s_j)_{j=0}^m]} \phi + \phi_{12}^{[\alpha,(s_j)_{j=0}^m]} \psi \end{bmatrix} \begin{bmatrix} \phi_{21}^{[\alpha,(s_j)_{j=0}^m]} \phi + \phi_{22}^{[\alpha,(s_j)_{j=0}^m]} \psi \end{bmatrix}^{-1} \]

is well defined and bijective.

**Proof.** Apply Theorem 12.3.

13. Parametrization of the matricial Stieltjes moment problem in the general case

Theorem 12.3 gives a parametrization of the moment problem in question with a parameter set which depends on the given data indicated by the matrix \(s_0^{[m,\alpha]}\). In this section, we prove a parametrization with a parameter set which is independent of the given data. We distinguish the following three cases:
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(I) The non-degenerate case $\text{rank } s_0^{[m,\alpha]} = q$. In view of [9, Proposition 9.19] this is equivalent to $(s_j)_{j=0}^m \in K_{q,m,\alpha}^>$. 

(II) The completely degenerate case $s_0^{[m,\alpha]} = 0$. In view of [9, Proposition 9.22] this is equivalent to $(s_j)_{j=0}^m \in K_{q,m,\alpha}^{cd}$. 

(III) The degenerate, but not completely degenerate case $1 \leq \text{rank } s_0^{[m,\alpha]} \leq q - 1$. 

13.1. The non-degenerate case

First we want to turn our attention to the particular cases in Theorem 12.3, where the rank of the matrix $s_0^{[m,\alpha]}$ attains the extremal values $q$ and 0, respectively. We start with the full rank case.

**Theorem 13.1.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in K_{q,m,\alpha}^>$. Let $(s_j)_{j=0}^m$ be the $m$-th $\alpha$-S-transform of $(s_j)_{j=0}^m$. Then:

(a) $(s_j)_{j=0}^m \in K_{q,m,\alpha}^e$ and $\text{rank } s_0^{[m,\alpha]} = q$. 

(b) Let $\mathfrak{F}^{[\alpha,(s_j)_{j=0}^m]}$ be defined via (E.26) and (E.1). Furthermore, let (E.27) be the $q \times q$ block representation of $\mathfrak{F}^{[\alpha,(s_j)_{j=0}^m]}$. Then:

(b1) For each $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-J_{q,\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$, the function $\det[\phi^{[\alpha,(s_j)_{j=0}^m]} + \psi^{[\alpha,(s_j)_{j=0}^m]}]$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and does not vanish identically. Furthermore,

$$\left(\psi^{[\alpha,(s_j)_{j=0}^m]} \phi + \phi^{[\alpha,(s_j)_{j=0}^m]} \psi^T \right) (\psi^{[\alpha,(s_j)_{j=0}^m]} \phi + \phi^{[\alpha,(s_j)_{j=0}^m]} \psi^T)^{-1}$$

belongs to $\mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq)$. 

(b2) For every choice of $F \in \mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq)$, there exists a pair $(\phi, \psi) \in \mathcal{P}^{(q,q)}_{-J_{q,\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$ of $q \times q$ matrix-valued functions $\phi$ and $\psi$ which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that, for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, the inequality $\det[\psi^{[\alpha,(s_j)_{j=0}^m]}(z) \phi(z) + \phi^{[\alpha,(s_j)_{j=0}^m]}(z) \psi(z)] \neq 0$ and the representation (12.20) of $F$ hold true.

(b3) Let $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}^{(q,q)}_{-J_{q,\alpha}}(\mathbb{C} \setminus [\alpha, \infty))$. Then conditions (a) and (b) stated in Theorem 12.3 are equivalent.

**Proof.** [7, Proposition 2.20] implies that $(s_j)_{j=0}^m \in K_{q,m,\alpha}^>$, whereas [9, Proposition 9.19] yields that $\det s_0^{[m,\alpha]} \neq 0$. Thus, part (a) is proved. Because of (a), Theorem 12.3 and Remark 7.15 part (b) follows. 

It should be mentioned that in the situation of Theorem 13.1 an alternate approach to the determination of the set $\mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq]$ was presented in the recent PhD thesis [7,15] of B. Jeschke. His approach was inspired by the technique used by Yu. M. Dyukarev [4] in the case $\alpha = 0$. Moreover, B. Jeschke extended results of A. E. Choque Rivero [3] on various kinds of matrix polynomials which are connected to the case $\alpha = 0$ to the case of arbitrary $\alpha \in \mathbb{R}$. 

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13.2. The completely degenerate case

Now we see in particular that in the completely degenerate case the moment problem $M[[\alpha, \infty); (s_j)]_{j=0}^m \leq$ has a unique solution.

**Theorem 13.2.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}^{\geq}_{\mathcal{R}, m, \alpha}$. Let $(s_j^{[m,\alpha]})_{j=0}^0$ be the $m$-th $\alpha$-$S$-transform of $(s_j)_{j=0}^m$. Then:

(a) The relations $(s_j)_{j=0}^m \in \mathcal{K}^{\geq}_{\mathcal{R}, m, \alpha}$ and $s_0^{[m,\alpha]} = 0_{q \times q}$ hold true.

(b) Let $\mathcal{V}^{[\alpha,(s_j)]}_{j=0} m$ be defined via \([E.26]\) and \([E.1]\). Furthermore, let \([E.27]\) be the $q \times q$ block representation of $\mathcal{V}^{[\alpha,(s_j)]}_{j=0} m$. Then

$$
S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] = \left\{ v_{12}^{[\alpha,(s_j)]_{j=0}^m} (v_{22}^{[\alpha,(s_j)]_{j=0}^m})^{-1} \right\}.
$$

(c) $S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq] = S_{m,q,[\alpha,\infty]}[(s_j)_{j=0}^m, \leq]$. 

**Proof.** (a) From Proposition 2.5 we get $(s_j)_{j=0}^m \in \mathcal{K}^{\geq}_{\mathcal{R}, m, \alpha}$, whereas \([E.26]\) Proposition 9.20 yields $s_0^{[m,\alpha]} = 0_{q \times q}$.

(b) Combine (a), Theorem 12.3, and Remark 12.10.

(c) In view of part (b) and Remark 12.10, part (c) follows from Theorem 12.5(a). \(\square\)

Observe that the unique solution of Problem $M[[\alpha, \infty); (s_j)_{j=0}^m \leq]$ which occurs in the situation of Theorem 13.2 is a non-negative Hermitian measure concentrated on a finite set of points (see \([E.26]\)).

**Corollary 13.3.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}^{\geq}_{\mathcal{R}, m, \alpha}$. Then $M^\infty Y_{\alpha, \infty} ((s_j)_{j=0}^m, \leq) = M^\infty Y_{\alpha} ((s_j)_{j=0}^m, \leq)$.

**Proof.** Use Theorems 1.11 and 13.2 \(\square\)

13.3. The degenerate, but not completely degenerate case

In this section, we prove a parametrization of the set of $[\alpha, \infty)$-Stieltjes transforms of the solutions of the Stieltjes-type power moment problem $M[[\alpha, \infty); (s_j)_{j=0}^m \leq]$ with free parameters in the case $1 \leq \text{rank} s_0^{[m,\alpha]} \leq q - 1$.

**Lemma 13.4.** Let $M \in \mathbb{C}^{q \times p}$ be such that $r := \text{rank} M$ fulfills $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(M)$, let $U := [u_1, u_2, \ldots, u_r]$, and let $Q := P_{\mathcal{N}(M^*)}$. Then the mapping $\gamma: \mathcal{P}^{(r,r)}_{\mathcal{N}(M^*)}(\mathbb{C} \setminus [\alpha, \infty)) \rightarrow \mathcal{P}_{\mathcal{N}(M^*)}[M]$ given by $\gamma((\phi, \psi)) := (U\phi U^*, U\psi U^* + Q)$ is well defined and injective.

**Proof.** In view of Remarks A.3 and A.4 we have obviously

$$
Q^* = Q, \quad U^* U = I_r, \quad U U^* = P_{\mathcal{R}(M)} = P_{\mathcal{N}(M^*)}^* = I_q - Q.
$$

From (13.2) and (13.1) we obtain

$$
U^* Q = U^* U U^* Q = U^* (I_q - Q) Q = 0_{r \times q}.
$$

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Let \((\phi, \psi) \in \mathcal{P}^{(r,r)}_{-J_{r} \geq}(\mathbb{C} \setminus [\alpha, \infty))\). By Definition 7.1, there is a discrete subset \(D\) of \(\mathbb{C} \setminus [\alpha, \infty)\) such that the conditions \((i) \quad (iii)\) of Definition 7.1 are fulfilled. Because of Definition 7.1(ii), the matrix-valued functions \(F \coloneqq U\phi U^*\) and \(G \coloneqq U\psi U^* + Q\) are meromorphic and holomorphic in \(\mathbb{C} \setminus ([\alpha, \infty) \cup D)\). Using (13.2), we get then

\[
G^* F = (I_q - UU^*) (U\phi U^*) = U\phi^* U^*, \quad (13.4)
\]

\[
F^* F = U\phi^* U^*, \quad (13.5)
\]

and, in view of (13.3) and (13.1), furthermore

\[
G^* G = U\psi^* U^* U\psi U^* + Q^* U\psi U^* + U\psi^* U^* Q + Q^* Q = U\psi^* U^* + Q. \quad (13.6)
\]

By virtue of (13.5) and (13.6), we have

\[
F^* F + G^* G = U(\phi^* \phi + \psi^* \psi) U^* + Q. \quad (13.7)
\]

We consider now an arbitrary \(z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)\) and an arbitrary \(v \in \mathcal{N}([F(z)]^*)\). Then (13.7) implies

\[
0 = v^* ([F(z)]^*[F(z)] + [G(z)]^*[G(z)]) v = (U^* v)^* ([\phi(z)]^*[\phi(z)] + [\psi(z)]^*[\psi(z)]) v + v^* Q v. \quad (13.8)
\]

Taking into account Definition 7.1(ii) and (ii) we get \(\text{rank}([\phi(z)]^*[\phi(z)] + [\psi(z)]^*[\psi(z)]) = \text{rank}[\phi(z)] = r\) and, consequently,

\[
[\phi(z)]^*[\phi(z)] + [\psi(z)]^*[\psi(z)] \in \mathbb{C}^{r \times r}. \quad (13.9)
\]

Obviously, \(Q = QQ^* \in \mathbb{C}^{q \times q}\). Thus, from (13.8) and (13.9), we conclude \(U^* v = 0_{r \times 1}\) and \(v^* Q v = 0\). Since (13.2) shows then that

\[
0 = v^* UU^* v = v^* (I_q - Q)v = v^* v - v^* Qv = v^* v = \|v\|_{F}^2
\]

holds true, we infer \(v = 0_{q \times 1}\). Hence, \(\mathcal{N}([F(z)]^*) = \{0_{q \times 1}\}\) for each \(z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)\). For every choice of \(\eta \in \{1, z - \alpha\}\) and \(z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)\), from (13.4) we obtain

\[
\frac{1}{\text{Im} z} \text{Im} [\eta G^* (z) F(z)] = U \left( \frac{1}{\text{Im} z} \text{Im} [\eta \psi^* (z) \phi(z)] \right)^* U^*
\]

and, in view of Remark A.13 and Definition 7.1(ii), consequently,

\[
\left[ \frac{\eta F(z)}{G(z)} \right]^* \left( \frac{-J_q}{2 \text{Im} z} \right) \left[ \frac{\eta F(z)}{G(z)} \right] = \frac{1}{\text{Im} z} \text{Im} [\eta G^* (z) F(z)] = U \left( \frac{1}{\text{Im} z} \text{Im} [\eta \psi^* (z) \phi(z)] \right)^* U^*
\]

and

\[
= U \left( \frac{\eta \phi(z)}{\psi(z)} \right)^* \left( \frac{-J_q}{2 \text{Im} z} \right) \left[ \frac{\eta \phi(z)}{\psi(z)} \right] U^* \in \mathbb{C}^{q \times q}.
\]

Thus, \((F,G)\) belongs to \(\mathcal{P}^{(q,q)}_{-J_{r} \geq}(\mathbb{C} \setminus [\alpha, \infty))\). Since \(\mathcal{R}(F) = \mathcal{R}(U\phi U^*) \subseteq \mathcal{R}(U) = \mathcal{R}(M)\) and Remark A.6 imply \(M M^* F = F\), we see that \(F\) belongs to \(\mathcal{P}^{(q,q)}_{-J_{r} \geq}(\mathbb{C} \setminus [\alpha, \infty))\). Consequently, the mapping \(\gamma\) is well defined. Now we check that \(\gamma\) is injective. For this reason, we consider arbitrary \((\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}^{(r,r)}_{-J_{r} \geq}(\mathbb{C} \setminus [\alpha, \infty))\) such that \(\gamma((\phi_1, \psi_1)) = \gamma((\phi_2, \psi_2))\). Then

\[
U\phi_1 U^* = U\phi_2 U^* \quad \text{and} \quad U\psi_1 U^* + Q = U\psi_2 U^* + Q.
\]

In view of (13.2), this implies \(\phi_1 = \phi_2\) and \(\psi_1 = \psi_2\). Thus, \(\gamma\) is injective.
Remark 13.5 ([29] Lemma 11.3)). Let $\alpha \in \mathbb{R}$, let $W$ be a complex $q \times q$ matrix with $W^* V = I_q$. Let $(\phi_1, \psi_1) \in \mathcal{P}_{-J_{\ell r}, \geq}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty))$. Then $\phi$ and $\psi$ are functions meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and there exists a discrete subset $D$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that the conditions (i) and (iii) of Definition 7.1 are fulfilled. Hence, $(\phi_2, \psi_2) := V \phi_1$ and $(\phi_2, \psi_2) := W \psi_1$ are functions meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ which are holomorphic in $\mathbb{C} \setminus ((\alpha, \infty) \cup D)$ such that $[\phi_2 \psi_2] = \text{diag}(V, W) \cdot [\phi_1 \psi_1]$. Thus, rank$[\phi_2(z)] = \text{rank}[\phi_1(z)] = q$ for all $z \in \mathbb{C} \setminus ((\alpha, \infty) \cup D)$ and

$$
\begin{bmatrix}
(z - \alpha)^{\ell} \phi_2(z) \\
\psi_2(z)
\end{bmatrix}^* \begin{pmatrix}
-\tilde{J}_q \\
2 \text{Im } z
\end{pmatrix} \begin{bmatrix}
(z - \alpha)^{\ell} \phi_2(z) \\
\psi_2(z)
\end{bmatrix} = \begin{bmatrix}
(z - \alpha)^{\ell} \phi_1(z) \\
\psi_1(z)
\end{bmatrix}^* \begin{pmatrix}
-\tilde{J}_q \\
2 \text{Im } z
\end{pmatrix} \begin{bmatrix}
(z - \alpha)^{\ell} \phi_1(z) \\
\psi_1(z)
\end{bmatrix} \in \mathbb{C}^{q \times q}
$$

for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup D)$ and all $\ell \in \{1, 2\}$. Consequently, $(\phi_2, \psi_2) \in \mathcal{P}_{-J_{\ell r}, \geq}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty))$.

Lemma 13.6. Let $M \in \mathbb{C}^{q \times p}$ be such that $r := \text{rank } M$ fulfills $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(M)$, let $U := [u_1, u_2, \ldots, u_r]$, and let $Q := P_{N(M^*)}$. Furthermore, let $(F, G) \in \mathcal{P}_{q, \alpha}[M]$. Then:

(a) The matrix-valued function $B := G - iF$ is meromorphic and the function det $B$ does not vanish identically.

(b) Let $\phi := U^* F B^{-1} U$ and let $\psi := U^* G B^{-1} U$. Then $(\phi, \psi) \in \mathcal{P}_{r, r}^{(q, q)}(\mathbb{C} \setminus [\alpha, \infty))$ and

$$
\begin{bmatrix}
F \\
G
\end{bmatrix} B^{-1} = \begin{bmatrix}
U \phi U^* \\
U \psi U^* + Q
\end{bmatrix}.
$$

(13.10)

Proof. Obviously, (13.2) holds true. In view of Remarks 4.4 and 4.8 we have then $UU^* = I_q - Q = I_q - P_{R(M)^+} = MM^\dagger$. Thus, because of $(F, G) \in \mathcal{P}_{q, \alpha}[M]$, we get

$$
UU^* F = MM^\dagger F = F.
$$

(13.11)

For parts of the following, we adopt the method used in the proof of [29] Lemma 1.6(a). Since $(F, G)$ belongs to $\mathcal{P}_{q, \alpha}[M]$, we see that $F$ and $G$ are in $\mathbb{C} \setminus [\alpha, \infty)$ meromorphic $\mathbb{C}^{q \times q}$-valued functions and that there exists a discrete subset $D$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that conditions (i) and (iii) of Definition 7.1 hold true. Hence, $B$ is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ fulfilling

$$
[B(z)]^*[B(z)] = ([G(z)]^* + i[F(z)]^*[G(z)] - iF(z))
$$

$$
= [G(z)]^*[G(z)] + [F(z)]^*[F(z)] + 2 \text{Im } ([G(z)]^*[F(z)]) \geq F(z),
$$

$$
\begin{bmatrix}
F(z) \\
G(z)
\end{bmatrix}^* \begin{bmatrix}
F(z) \\
G(z)
\end{bmatrix} \in \mathbb{C}^{q \times q}
$$

for all $z \in \Pi_+ \setminus D$. In particular, the function det $B$ does not vanish in $\Pi_+ \setminus D$. Thus, $S := (G + iF)B^{-1}$ is an in $\mathbb{C} \setminus [\alpha, \infty)$ meromorphic $\mathbb{C}^{q \times q}$-valued function which fulfills

$$
FB^{-1} = \frac{i}{2}(I_q - S) \quad \text{and} \quad GB^{-1} = \frac{1}{2}(I_q + S).
$$

(13.12)
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Moreover, direct calculation shows

\[ I_q - [S(z)]^* [S(z)] = 4B(z)^{-1} \text{Im}([G(z)]^*[F(z)][B(z)])^{-1} \in C_{\geq}^{q \times q} \]

for all \( z \in \Pi_+ \setminus \mathcal{D} \). Using Riemann’s theorem on removable singularities, we can conclude that \( \hat{S} := \text{Rstr}_{\Pi_+} S \) belongs to the Schur class \( S_{q \times q}(\Pi_+) \).

First we consider the case \( 1 \leq r \leq q - 1 \). We choose then \( u_{r+1}, u_{r+2}, \ldots, u_q \in C^q \) such that \( u_1, u_2, \ldots, u_q \) is an orthonormal basis of \( C^q \). Let \( V := [u_{r+1}, u_{r+2}, \ldots, u_q] \) and let \( W := [U, V] \). Then

\[ V^*U = 0_{(q-r) \times r}, \quad V^*V = I_q-r, \quad (13.13) \]

and \( W^*W = I_q \) hold true. This implies \( U^*V = 0_{r \times (q-r)}, \quad WW^* = I_q \), and, because of \( (13.2) \), hence

\[ I_q = [U, V][U, V]^* = UU^* + VV^* = I_q - Q + VV^*. \quad (13.14) \]

This shows that

\[ Q = VV^*. \quad (13.15) \]

From \( (13.11) \) and \( (13.13) \) we see that

\[ V^*F = V^*U^*F = 0_{(q-r) \times q}. \quad (13.16) \]

Using \( (13.12) \) and \( (13.16) \), we get

\[ V^* - V^*S = V^*(I_q - S) = -2iV^*FB^{-1} = 0_{(q-r) \times q} \]

and, consequently, \( V^*S = V^* \). Thus, in view of \( (13.13) \), we obtain \( V^*SU = V^*U = 0_{(q-r) \times r} \) and \( V^*SV = V^*V = I_q-r \). Hence,

\[ W^*SW = \begin{bmatrix} U^*SU & U^*SV \\ V^*SU & V^*SV \end{bmatrix} = \begin{bmatrix} U^*SU & U^*SV \\ 0_{(q-r) \times r} & I_{q-r} \end{bmatrix}. \quad (13.17) \]

Since \( S(z) \) is contractive for each \( z \in \Pi_+ \), the matrix \( W^*S(z)W \) is contractive for each \( z \in \Pi_+ \). Thus, \( (13.17) \) and Remark A.11 yield that \( U^*S(z)V = 0_{r \times (q-r)} \) for all \( z \in \Pi_+ \). Taking into account that \( S \) is meromorphic in \( \mathbb{C} \setminus [0, \infty) \), we obtain \( U^*SV = 0_{r \times (q-r)} \). Because of \( (13.17) \), this implies \( W^*SW = \text{diag}(U^*S, I_{q-r}) \). By virtue of \( WW^* = I_q \), then \( S = W \cdot \text{diag}(U^*S, I_{q-r}) \cdot W^* \) follows. Furthermore, since \( (13.2) \) holds true, for all \( \zeta \in \mathbb{C} \), we have then

\[ I_q + \zeta S = W[I_q + \text{diag}(U^*S, I_{q-r})]W^* = W \cdot \text{diag}(U^*S, I_{q-r}) \cdot W^*. \quad (13.18) \]

According to \( (13.12) \) and \( (13.18) \), we conclude

\[ FB^{-1} = \frac{1}{2} W \cdot \text{diag}(U^*(I_q - S)U, 0_{(q-r) \times (q-r)}) \cdot W^* = W \cdot \text{diag}(U^*FB^{-1}U, 0_{(q-r) \times (q-r)}) \cdot W^* = [U, V] \cdot \text{diag}(\phi, 0_{(q-r) \times (q-r)}) \cdot \begin{bmatrix} U^* \\ V^* \end{bmatrix} = U\phi U^* \quad (13.19) \]
and
\[ GB^{-1} = \frac{1}{2} W \cdot \text{diag}(U^*(I_q + S)U, 2I_{q-r}) \cdot W^* \]
\[ = W \cdot \text{diag}(U^*GB^{-1}U, I_{q-r}) \cdot W^* = [U, V] \cdot \text{diag}(\psi, I_{q-r}) \cdot \begin{bmatrix} U^* \\ V^* \end{bmatrix} = U\psi U^* + VV^*. \]
\[ (13.20) \]
By virtue of (13.20) and (13.15), we get \( GB^{-1} = U\psi U^* + Q \). Combining this with (13.19), we infer (13.10). Furthermore, (13.19) and (13.20) yield
\[ \begin{bmatrix} F \\ G \end{bmatrix} B^{-1}W = \begin{bmatrix} U\phi U^*W \\ U\psi U^*W + VV^*W \end{bmatrix}. \]
(13.21)
Using \( U^*W = U^*[U, V] = [U^*U, U^*V] = [I_r, 0_{r \times (q-r)}] \) and \( V^*W = V^*[U, V] = [V^*U, V^*V] = [0_{(q-r) \times r}, I_{q-r}] \), then
\[ \begin{bmatrix} FB^{-1}W \\ GB^{-1}W \end{bmatrix} = \begin{bmatrix} [U\phi, 0_{q \times (q-r)}] \\ [U\psi, V] \end{bmatrix}. \]
(13.22)
follows. Since \((F, G)\) belongs to \( P_{-J_q \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \) and since \( \det(B^{-1}W) \) does not vanish identically, Remark 7.3 yields that \((FB^{-1}W, GB^{-1}W)\) belongs to \( P_{-J_q \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Consequently, (13.22) provides \([U\phi, 0_{q \times (q-r)}] \in P_{-J_q \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \) and, in view of \((W^*)^*W^* = I_q \) and Remark 13.5, we have \([W^*[U\phi, 0_{q \times (q-r)}]] \in P_{-J_q \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Because of \( U^*U = I_r \) and \( V^*U = 0_{(q-r) \times r} \), we have
\[ W^*[U\phi, 0_{q \times (q-r)}] = \begin{bmatrix} U^*U\phi \\ V^*U\phi \end{bmatrix} = \begin{bmatrix} 0_{r \times (q-r)} \\ 0_{(q-r) \times (q-r)} \end{bmatrix} = \text{diag}(\phi, 0_{(q-r) \times (q-r)}) \]
and
\[ W^*[U\psi, V] = \begin{bmatrix} U^*U\psi \\ V^*U\psi \end{bmatrix} = \text{diag}(\psi, I_{q-r}). \]
Hence, \( \begin{bmatrix} \text{diag}(\phi, 0_{(q-r) \times (q-r)}) \\ \text{diag}(\psi, I_{q-r}) \end{bmatrix} \in P_{-J_q \geq}^{(q,q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Definition 7.1 shows then that there is a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \infty) \) such that \( \phi \) and \( \psi \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \infty) \), that \( \text{rank}[\phi(z)] = q - \text{rank}I_{q-r} = r \) for all \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \), and, in view of Remark A.13, that
\[ \text{diag}\left( \begin{bmatrix} (z - \alpha)^k\phi(z) \\ (z - \alpha)^k\psi(z) \end{bmatrix} \begin{bmatrix} 0_{q \times (q-r)} \end{bmatrix}, \right) \]
\[ = \begin{bmatrix} (z - \alpha)^k \cdot \text{diag}(\phi(z), 0_{(q-r) \times (q-r)}) \\ \text{diag}(\psi(z), I_{q-r}) \end{bmatrix} \]
\[ \in \mathbb{C}_{\geq}^{q \times q} \]
for every choice of \( k \in \mathbb{Z}_{\geq 0} \) and \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \). In particular, (7.1) and (7.2) hold true for all \( z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D) \). Thus, \( (\phi, \psi) \in P_{-J_r \geq}^{(r,r)}(\mathbb{C} \setminus [\alpha, \infty)) \).
It remains to consider the case $r = q$. Then $U$ is unitary. Consequently, $FB^{-1} = U\phi U^*$ and $GB^{-1} = U\psi U^*$, which shows that

\[
\begin{bmatrix}
FB^{-1} \\
GB^{-1}
\end{bmatrix} = \begin{bmatrix}
U\phi U^* \\
U\psi U^*
\end{bmatrix}.
\]

(13.23)

From [13.22] we conclude $I_q = U^*U = I_q - Q$ and, therefore, $Q = 0_{q \times q}$. This implies $GB^{-1} = U\psi U^* + Q$. Hence, [13.10] holds true. Since $(F, G)$ belongs to $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$ from [13.23], part (ii), and Remark 7.3 we see that $(U\phi U^*, U\psi U^*)$ belongs to $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$ as well.

Since $U$ is unitary, from Definition 7.1 we get then that there is a discrete subset $D$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi$ and $\psi$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup D)$, that rank $[\phi(z)] = q$ holds true for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$, and, in view of Remark [A.13] and $U^*U = I_q$, that

\[
U \begin{bmatrix}
(z - \alpha)^k \phi(z) \\
\psi(z)
\end{bmatrix} \begin{bmatrix}
\frac{-J_q}{2 \text{Im } z} \\
\psi(z)
\end{bmatrix} U^* = U \frac{1}{\text{Im } z} \text{Im} \left( [z - \alpha]^k U\psi(z)] \psi(z) \right) U^* = \frac{1}{\text{Im } z} \text{Im} \left( [z - \alpha]^k U[\psi(z)] \psi(z) \right) U^*
\]

is fulfilled for every choice of $k \in \mathbb{Z}_{0,1}$ and $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$. Using $U^*U = I_q$ again, then, for each $k \in \mathbb{Z}_{0,1}$ and each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$, we get $[13.1]$ and $[13.2]$. Consequently, $(\phi, \psi)$ belongs to $\mathcal{P}^{(q,q)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$.

Lemma 13.7. Let $\alpha \in \mathbb{R}$ and let $M \in \mathbb{C}^{n \times n}$ be such that $r := \text{rank } M$ fulfills $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathcal{R}(M)$, let $U := [u_1, u_2, \ldots, u_r]$, and let $Q := P_{\mathcal{N}(M^*)}$. Then $\Gamma_U : (\mathcal{P}^{(r,r)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))) \to (\mathcal{P}_{q,\alpha}[M])$ given by

\[
\Gamma_U((\phi, \psi)) := \langle U\phi U^*, U\psi U^* + Q \rangle
\]

(13.24)

is well defined and bijective.

Proof. Obviously, [13.11], [13.2], and [13.3] are valid. Let $(\phi, \psi) \in \mathcal{P}^{(r,r)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$. According to Lemma 13.3 then $\Gamma_U((\phi, \psi))$ belongs to $(\mathcal{P}_{q,\alpha}[M])$. We consider now arbitrary $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}^{(r,r)}_{-J_q \geq}(\mathbb{C} \setminus [\alpha, \infty))$ which fulfill $\langle (\phi_1, \psi_1) \rangle = \langle (\phi_2, \psi_2) \rangle$. By virtue of Definition 13.4 then there are a meromorphic $r \times r$ matrix-valued function $\theta$ and a discrete subset $D$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi_1, \phi_2, \psi_1, \psi_2$, and $\theta$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup D)$ and that $\det \theta(z) \neq 0$. Then $\phi_2(z) = \phi_1(z)\theta(z)$, and $\psi_2(z) = \psi_1(z)\theta(z)$ hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup D)$.

Then $F_1 := U\phi_1 U^*, F_2 := U\phi_2 U^*, G_1 := U\psi_1 U^* + Q, G_2 := U\psi_2 U^* + Q$, and $T := U\theta U^* + Q$ are meromorphic matrix-valued functions which, in view of [13.2], [13.3], and [13.11], fulfill

\[
F_1(z)T(z) = U\phi_1(z)U^*U\theta(z)U^* + U\phi_1(z)U^*(I_q - UU^*)
= U\phi_1(z)\theta(z)U^* = U\phi_2(z)U^* = F_2(z)
\]

and

\[
G_1(z)T(z) = U\psi_1(z)U^*U\theta(z)U^* + U\psi_1(z)U^*Q + QU\theta(z)U^* + Q^2
= U\psi_1(z)\theta(z)U^* + Q = G_2(z)
\]

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for each $z \in \mathbb{C} \setminus [\alpha, \infty)$. Now let $z \in \mathbb{C} \setminus [\alpha, \infty)$. We are going to check that $\det T(z) \neq 0$. For this reason, let $v \in \mathcal{N}(T(z))$. Then

$$0_{q \times 1} = T(z)v = U\theta(z)U^*v + Qv. \quad (13.25)$$

According to (13.2) and (13.25), we get $0_{r \times 1} = \theta(z)U^*v + U^*(I_q - UU^*)v = \theta(z)U^*v$. Because of $\det \theta(z) \neq 0$, then $U^*v = 0_{r \times 1}$ follows. Thus, taking into account (13.25) and (13.2) again, this implies $0_{q \times 1} = Qv = (I_q - UU^*)v = v - UU^*v = v$. Consequently, $\mathcal{N}(T(z)) \subseteq \{0_{q \times 1}\}$ and, hence, $\det T(z) \neq 0$. Therefore, the pairs $(F_1, G_1)$ and $(F_2, G_2)$ are equivalent. In other words, the mapping $\Gamma_U$ is well defined.

In order to prove that $\Gamma_U$ is injective, we consider arbitrary $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{P}_{q,q}^{(r,r)}(\mathbb{C} \setminus [\alpha, \infty))$ with $\Gamma_U(((\phi_1, \psi_1))) = \Gamma_U(((\phi_2, \psi_2)))$. Then there are a meromorphic $q \times q$ matrix-valued function $\theta$ and a discrete subset $\hat{D}$ of $\mathbb{C} \setminus [\alpha, \infty)$ such that $\phi_1, \phi_2, \psi_1, \psi_2$, and $\hat{\theta}$ are holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \hat{D})$ and that $\det \theta(z) \neq 0$, $U\phi_2(z)U^* = U\phi_1(z)U^*\hat{\theta}(z)$, and

$$U\psi_2(z)U^* + Q = [U\psi_1(z)U^* + Q]\hat{\theta}(z). \quad (13.26)$$

Thus, setting $\Lambda := U^*\hat{\theta}U$, from (13.2) and (13.3), we have

$$\phi_1\Lambda = U^*\phi_1\Lambda = U^*U\phi_1U^*\hat{\theta}U = U^*U\phi_2U^*U = \phi_2$$

and

$$\psi_1\Lambda = U^*U\psi_1\Lambda = U^*U\psi_1U^*\hat{\theta}U = U^*(U\psi_2U^* + Q - Q\hat{\theta})U = \psi_2 + U^*Q(I_q - \hat{\theta})U = \psi_2.$$ 

If $r = q$, then $UU^* = I_q$ and, consequently, $\det \Lambda(z) = \det \hat{\theta}(z) \neq 0$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \hat{D})$. Now we assume $r < q - 1$. Then we choose vectors $u_{r+1}, u_{r+2}, \ldots, u_q \in \mathbb{C}^q$ such that $u_1, u_2, \ldots, u_q$ is an orthonormal basis of $\mathbb{C}^q$. Put $V := [u_{r+1}, u_{r+2}, \ldots, u_q]$. Obviously, (13.13) holds true. Multiplying equation (13.26) by $V^*$ from the left-hand side and using (13.13), we get $V^*Q = V^*Q\hat{\theta}$. By virtue of (13.13) and (13.2), this implies $V^* = V^*(I_q - UU^*) = V^*(I_q - UU^*)\hat{\theta} = V^*\hat{\theta}$. Thus, $V^*\hat{\theta}U = V^*U = 0_{(q-r)\times r}$ and $V^*\hat{\theta}V = V^*V = I_{q-r}$. Hence, $W := [U, V]$ fulfills $W^*W = I_q$ and

$$W^*\hat{\theta}W = \begin{bmatrix} U^*\hat{\theta}U & U^*\hat{\theta}V \\ V^*\hat{\theta}U & V^*\hat{\theta}V \end{bmatrix} = \begin{bmatrix} \Lambda & U^*\hat{\theta}V \\ V^*\hat{\theta}U & V^*\hat{\theta}V \end{bmatrix}.$$ 

In particular, $\det \Lambda(z) = \det |W^*\hat{\theta}(z)W| = \det \hat{\theta}(z) \neq 0$ for all $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \hat{D})$. Hence, in the case $r = q$ as well as in the case $r \leq q - 1$, the function $\det \Lambda$ does not vanish identically. Consequently, $((\phi_1, \psi_1)) = ((\phi_2, \psi_2))$. Therefore, the mapping $\Gamma_U$ is injective. It remains to prove that $\Gamma_U$ is surjective. Let $(F, G) \in \mathcal{P}_{q,q}[M]$. Then we know from Lemma 13.5 that $B := G - 1F$ is a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ for which $\det B$ does not vanish identically. Furthermore, Lemma 13.6 shows that $\phi := U^*FB^{-1}U$ and $\psi := U^*GB^{-1}U$ are meromorphic matrix-valued functions such that $(\phi, \psi) \in \mathcal{P}_{q,q}^{(r,r)}(\mathbb{C} \setminus [\alpha, \infty))$ and (13.10) hold true. Consequently, $\Gamma_U(((\phi, \psi))) = ((F, G))$. Hence, $\Gamma_U$ is surjective as well. \hfill \Box

Remark 13.8. Let $M \in \mathbb{C}^{q \times p}$ be such that $r := \text{rank } M$ fulfills $1 \leq r \leq q - 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_r$ is an orthonormal basis of $\mathcal{R}(M)$. Let
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$W := [u_1, u_2, \ldots, u_q]$, let $U := [v_1, v_2, \ldots, v_r]$, and let $V := [u_{r+1}, u_{r+2}, \ldots, u_q]$. Then $W = [U, V]$ and (13.2) hold true. Because $WW^* = I_q$, we see that (13.14) and (13.15) are true. Thus, the mapping $\Gamma_U : \mathcal{P}^{(r,r)}_{-J_r, \geq} (\mathbb{C} \setminus [\alpha, \infty)) \to \mathcal{P}_{q, \alpha} [M]$ defined by (13.21) fulfills

$$\Gamma_U : \langle (\phi, \psi) \rangle = \langle (W \cdot \text{diag}(\phi, 0_{(q-r) \times (q-r)}) \cdot W^*, W \cdot \text{diag}(\psi, I_{q-r}) \cdot W^*) \rangle.$$

Now we obtain the announced description of the set $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0}, \leq]$ in the degenerate, but not completely degenerate case as well:

**Theorem 13.9.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)^m_{j=0} \in \mathbb{K}^{\geq, e}_{q,m,\alpha}$ be such that $r := \text{rank} s_0^{[m,\alpha]}$ fulfills $1 \leq r \leq q - 1$. Let $\mathcal{Q}^{[\alpha, (s_j)^m_{j=0}]}$ be defined via (12.2), and Remark 12.1. Furthermore, let $(\mathcal{Q}^{[\alpha, (s_j)^m_{j=0}]}_{\infty})$ be the $q \times q$ block representation of $\mathcal{Q}^{[\alpha, (s_j)^m_{j=0}]}$. Let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathcal{C}$ such that $u_1, u_2, \ldots, u_r$ is an orthonormal basis of $\mathcal{R}(s_0^{[m,\alpha]})$ and let $W := [u_1, u_2, \ldots, u_q]$. Then:

(a) For each pair $(\phi, \psi) \in \mathcal{P}^{(r,r)}_{-J_r, \geq} (\mathbb{C} \setminus [\alpha, \infty))$, the function

$$\det \left[ v_{11}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\phi, 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\psi, I_{q-r}) \right]$$

is meromorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and does not vanish identically. Furthermore,

$$F := \left[ v_{11}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\phi, 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\psi, I_{q-r}) \right]$$

$$\times \left[ v_{21}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\phi, 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha, (s_j)^m_{j=0}]} W \cdot \text{diag}(\psi, I_{q-r}) \right]^{-1}$$

belongs to $\mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0}, \leq]$.

(b) For each $F \in \mathcal{S}_{m,q,[\alpha,\infty)}[(s_j)^m_{j=0}, \leq]$, there exists a pair $(\phi, \psi) \in \mathcal{P}^{(r,r)}_{-J_r, \geq} (\mathbb{C} \setminus [\alpha, \infty))$ of $q \times q$ matrix-valued functions $\phi$ and $\psi$ which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ such that, for each $z \in \mathbb{C} \setminus [\alpha, \infty)$, the inequality

$$\det \left[ v_{11}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\phi(z), 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\psi(z), I_{q-r}) \right] \neq 0$$

and the representation

$$F(z) = \left[ v_{11}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\phi(z), 0_{(q-r) \times (q-r)}) + v_{12}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\psi(z), I_{q-r}) \right]$$

$$\times \left[ v_{21}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\phi(z), 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha, (s_j)^m_{j=0}]} (z) W \cdot \text{diag}(\psi(z), I_{q-r}) \right]^{-1}$$

hold true.
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(c) Let \((\phi_1, \psi_1), (\phi_2, \psi_2) \in P^{(r,r)}_{-J_r, \geq} (C \setminus [\alpha, \infty)).\) Then

\[
\begin{align*}
\left[ v_{11}^{[\alpha,(s_j)_{j=0}^m]} W \cdot \text{diag}(\phi_1, 0_{(q-r) \times (q-r)}) + v_{12}^{[\alpha,(s_j)_{j=0}^m]} W \cdot \text{diag}(\psi_1, I_{q-r}) \right] \\
\times \left[ v_{21}^{[\alpha,(s_j)_{j=0}^m]} W \cdot \text{diag}(\phi_2, 0_{(q-r) \times (q-r)}) + v_{22}^{[\alpha,(s_j)_{j=0}^m]} W \cdot \text{diag}(\psi_2, I_{q-r}) \right]^{-1}
\end{align*}
\]

if and only if \(((\phi_1, \psi_1)) = ((\phi_2, \psi_2)).\)

Proof. In view of \(WW^* = I_p\), the assertion follows immediately by application of Theorem 12.3 and Remark 13.7.\qed

Remark 13.10. Let \(\alpha \in \mathbb{R}, \) let \(m \in \mathbb{N}_0, \) and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e\) be such that \(r := \text{rank} s_0^{[m,\alpha]} \) fulfills \(r \geq 1.\) Let \(u_1, u_2, \ldots, u_r\) be an orthonormal basis of \(\mathcal{R}(s_0^{[m,\alpha]}),\) and let \(U := [u_1, u_2, \ldots, u_r].\) Let \(\Gamma_U: P^{(r,r)}_{-J_r, \geq} (C \setminus [\alpha, \infty)) \to \langle P_{q,\alpha}[s_0^{[m,\alpha]}] \rangle\) be defined by \((13.24)\) and let \(\Sigma: \langle P_{q,\alpha}[s_0^{[m,\alpha]}] \rangle \to \mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq)\) be given by \((12.28)\). Then one can see from Lemma 13.7 and Corollary 12.4 that \(\Sigma \circ \Gamma_U\) realizes a bijection between \(\langle P^{(r,r)}_{-J_r, \geq} (C \setminus [\alpha, \infty)) \rangle\) and \(\mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq).\)

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Let \(\alpha \in \mathbb{R}, \) let \(m \in \mathbb{N}_0, \) and let \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e.\) In [10] Theorem 13.1] a complete description of the set of solutions of the moment problem \(M[[\alpha, \infty); (s_j)_{j=0}^m, \leq]\) was given in terms of the Stieltjes transforms of the solutions. Because of Remark 13.3 this set is a subset of the set of all Stieltjes transforms of solutions of the moment problem \(M[[\alpha, \infty); (s_j)_{j=0}^m, \leq].\) The latter set was determined in Theorem 12.3 and Section 13. Now we will demonstrate where the Stieltjes transforms of the solutions of the moment problem \(M[[\alpha, \infty); (s_j)_{j=0}^m, \leq]\) can be found amongst the Stieltjes transforms of the solutions of the moment problem \(M[[\alpha, \infty); (s_j)_{j=0}^m, \leq].\) In the completely degenerate case, an answer to this question was already given in Theorem 13.2 and Corollary 13.3.

Let \(\alpha \in \mathbb{R}.\) We set

\[
\hat{P}_{q,\alpha}^\circ := \{ (\phi, \psi) \in \hat{P}_{-J_r, \geq} (C \setminus [\alpha, \infty)): \lim_{y \to \infty} \| \phi \psi^{-1} (iy) \|_S = 0 \}. \quad (14.1)
\]

If \(A \in \mathbb{C}^{q \times q},\) then let

\[
\hat{P}_{q,\alpha}^\circ [A] := \hat{P}_{q,\alpha}^\circ \cap P_{q,\alpha}[A]. \quad (14.2)
\]

If \(m \in \mathbb{N}_0\) and \((s_j)_{j=0}^m \in K_{q,m,\alpha}^e,\) then we use the notation

\[
\mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_{j=0}^m, \leq) := \{ F \in \mathcal{S}_{m,q,[\alpha,\infty]}: \sigma_F \in M_{q}^{\geq} ([\alpha, \infty); (s_j)_{j=0}^m, \leq) \}. \quad (14.3)
\]

Lemma 14.1. Let \(\alpha \in \mathbb{R}\) and let \(\rho_\alpha^\circ: \hat{P}_{q,\alpha}^\circ \to \mathcal{S}_{q,[\alpha,\infty]}^\circ\) be defined by

\[
\rho_\alpha^\circ((\phi, \psi)) := \phi \psi^{-1}. \quad (14.4)
\]
Then $\rho^\circ_\alpha$ is well defined and bijective with inverse $\iota^\circ_\alpha : S^\circ_{\eta,[\alpha,\infty)} \to \langle \tilde{P}^\circ_{q,\alpha} \rangle$ given by

$$\iota^\circ_\alpha(F) := \langle (F, J_q) \rangle. \quad (14.5)$$

**Proof.** First observe that $\tilde{P}^\circ_{q,\alpha} \subseteq \tilde{P}^\circ_{\eta,[\alpha,\infty)} \subseteq S^\circ_{\eta,[\alpha,\infty)}$. Because of Corollary 7.12 it is thus sufficient to show $\rho_\alpha(\langle \tilde{P}^\circ_{q,\alpha} \rangle) \subseteq S^\circ_{\eta,[\alpha,\infty)}$ and $\iota_\alpha(S^\circ_{\eta,[\alpha,\infty)}) \subseteq \langle \tilde{P}^\circ_{q,\alpha} \rangle$, where $\rho_\alpha: \langle \tilde{P}^\circ_{q,\alpha} \rangle \to S^\circ_{\eta,[\alpha,\infty)}$ and $\iota_\alpha: S^\circ_{\eta,[\alpha,\infty)} \to \langle \tilde{P}^\circ_{q,\alpha} \rangle (C \setminus [\alpha, \infty))$ are given by (14.3) and (14.4), respectively. Consider an arbitrary pair $(\phi, \psi) \in \tilde{P}^\circ_{q,\alpha}$. According to Corollary 7.12 then $F := \phi \psi^{-1} = \rho_\alpha(\langle \phi, \psi \rangle)$ belongs to $S^\circ_{\eta,[\alpha,\infty)}$. Because of (14.1), furthermore $\lim_{y \to \infty} \| F_{iy} \|_S = 0$. Thus, $F \in S^\circ_{\eta,[\alpha,\infty)}$ by virtue of (14.2). Conversely, now we consider a function $F \in S^\circ_{\eta,[\alpha,\infty)}$. Using Corollary 7.12 then we see that $\langle (F, J_q) \rangle = \iota_\alpha(F)$ belongs to $\langle \tilde{P}^\circ_{q,\alpha} \rangle$. In particular, $(F, J_q) \in \langle \tilde{P}^\circ_{q,\alpha} \rangle (C \setminus [\alpha, \infty))$. Furthermore, we have $\lim_{y \to \infty} \| F_{iy} \|_S = 0$ by virtue of (14.2). In view of (14.1), thus $\langle (F, J_q) \rangle \in \langle \tilde{P}^\circ_{q,\alpha} \rangle$ follows. \qed

**Lemma 14.2.** Let $\alpha \in \mathbb{R}$, let $A \in C^{\eta \times \eta}$, and let $\rho^\circ_{\alpha,A} : \langle \tilde{P}^\circ_{q,\alpha} [A] \rangle \to S^\circ_{\eta,[\alpha,\infty)} [A]$ be defined by

$$\rho^\circ_{\alpha,A}(\langle \phi, \psi \rangle) := \phi \psi^{-1}. \quad (14.6)$$

Then $\rho^\circ_{\alpha,A}$ is well defined and bijective with inverse $\iota^\circ_{\alpha,A} : S^\circ_{\eta,[\alpha,\infty)} [A] \to \langle \tilde{P}^\circ_{q,\alpha} [A] \rangle$ given by

$$\iota^\circ_{\alpha,A}(F) := \langle (F, J_q) \rangle. \quad (14.7)$$

**Proof.** Taking into account (14.2) and (14.3), we have $\tilde{P}^\circ_{q,\alpha} [A] \subseteq \tilde{P}^\circ_{q,\alpha}$ and $S^\circ_{\eta,[\alpha,\infty)} [A] \subseteq S^\circ_{\eta,[\alpha,\infty)}$. In view of Lemma 14.1 it is thus sufficient to prove $\rho^\circ_{\alpha,A}(\langle \tilde{P}^\circ_{q,\alpha} [A] \rangle) \subseteq S^\circ_{\eta,[\alpha,\infty)} [A]$ and $\iota^\circ_{\alpha,A}(S^\circ_{\eta,[\alpha,\infty)} [A]) \subseteq \langle \tilde{P}^\circ_{q,\alpha} [A] \rangle$, where $\rho^\circ_{\alpha} : \langle \tilde{P}^\circ_{q,\alpha} \rangle \to S^\circ_{\eta,[\alpha,\infty)}$ and $\iota^\circ_{\alpha} : S^\circ_{\eta,[\alpha,\infty)} \to \langle \tilde{P}^\circ_{q,\alpha} \rangle$ are given by (14.3) and (14.4), respectively. First we consider an arbitrary pair $(\phi, \psi) \in \tilde{P}^\circ_{q,\alpha} [A]$. According to Lemma 14.1 then $F := \phi \psi^{-1} = \rho^\circ_{\alpha}(\langle \phi, \psi \rangle)$ belongs to $S^\circ_{\eta,[\alpha,\infty)}$. Because of (14.2), the pair $(\phi, \psi)$ belongs to $P^\circ_{q,\alpha} [A]$. Notation 7.13 and Remark A.6.16 hence imply $AA^\dagger \phi = \phi$. Since $F$ is holomorphic in $C \setminus [\alpha, \infty)$, a continuity argument provides $AA^\dagger F = F$. Thus, $F \in S^\circ_{\eta,[\alpha,\infty)} [A]$ according to Remark A.6.17 and Notation 1.8. Consequently, Remark 1.10 yields $F \in S^\circ_{\eta,[\alpha,\infty)} [A]$. Conversely, now we consider an arbitrary function $F \in S^\circ_{\eta,[\alpha,\infty)} [A]$. Because of Lemma 14.1 then $\langle (F, J_q) \rangle = \iota_\alpha(F)$ belongs to $\langle \tilde{P}^\circ_{q,\alpha} \rangle$. In particular, $(F, J_q) \in \tilde{P}^\circ_{q,\alpha}$. Since Remark 1.10 yields $F \in S^\circ_{\eta,[\alpha,\infty)} [A]$, we have $(F, J_q) \in \tilde{P}^\circ_{q,\alpha} [A]$ by virtue of (14.1) and Notations 1.8 and 11.3 in view of (14.2), thus $\langle (F, J_q) \rangle \in \langle \tilde{P}^\circ_{q,\alpha} [A] \rangle$. \qed

**Theorem 14.3.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in K_{\eta,[\alpha,\infty)}$ with right $\alpha$-Stieltjes parametrization $(Q_j)_{j=0}^m$. Then $\Sigma^\circ : \langle \tilde{P}^\circ_{q,\alpha} [Q_m] \rangle \to S^\circ_{m,q,[\alpha,\infty)} (s_j)_{j=0}^m$ is given by

$$\Sigma^\circ(\langle \phi, \psi \rangle) := (\nu_{11}^{[\alpha,(s_j)^{m}]_{j=0}} \phi + \nu_{12}^{[\alpha,(s_j)^{m}]_{j=0}} \psi)(\nu_{21}^{[\alpha,(s_j)^{m}]_{j=0}} \psi + \nu_{22}^{[\alpha,(s_j)^{m}]_{j=0}} \psi)^{-1}. \quad (14.8)$$

is well defined and bijective.

**Proof.** First observe that (14.2) yields $\tilde{P}^\circ_{q,\alpha} [Q_m] \subseteq P^\circ_{q,\alpha} [Q_m]$. Theorem 5.10 shows that $Q_m = s_0^{[m,\alpha]}$. In view of Corollary 12.4 it is thus sufficient to show $\Sigma(\langle \tilde{P}^\circ_{q,\alpha} [Q_m] \rangle) = \langle \tilde{P}^\circ_{q,\alpha} \rangle$.
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\[ \mathcal{S}_{m,q,[a,\infty)}[[s_j]_{j=0}^m = 0, =] \], where \( \Sigma : (\mathcal{P}_{q,a}[Q_m]) \to \mathcal{S}_{m,q,[a,\infty)}[[s_j]_{j=0}^m \leq] \) is given by (12.28). First we consider an arbitrary pair \((\phi, \psi) \in \mathcal{P}_{q,a}[Q_m]\). According to Lemma [14.2] then \( F := \phi\psi^{-1} \) belongs to \( \mathcal{S}_{q,[a,\infty)}[Q_m] \), the pair \((F, \mathcal{J}_q) \) belongs to \( \mathcal{P}_{q,a}[Q_m] \), and \( \langle (\phi, \psi) \rangle = \langle (F, \mathcal{J}_q) \rangle \). From [10] Proposition 12.13 and Theorem 13.1(a) we can conclude that \( \Sigma((F, \mathcal{J}_q)) \) belongs to \( \mathcal{S}_{m,q,[a,\infty)}[[s_j]_{j=0}^m = 0, =] \). Hence, \( \det \mathcal{J}_q \) does not identically vanish in \( \mathcal{S}_{m,q,[a,\infty)}[[s_j]_{j=0}^m = 0, =] \). Conversely, now we consider an arbitrary \( \Sigma \in \mathcal{S}_{m,q,[a,\infty)}[[s_j]_{j=0}^m = 0, =] \). From [10] Theorem 13.1(a) we get that there is a function \( F \in \mathcal{S}_{q,[a,\infty)}[Q_m] \) with \( S = (v_{11}^{[a,(s_j)_{j=0}^m]} F + v_{12}^{[a,(s_j)_{j=0}^m]} (v_{21}^{[a,(s_j)_{j=0}^m]} F + v_{22}^{[a,(s_j)_{j=0}^m]} - 1). \) Consequently, since Lemma [14.2] implies \( (F, \mathcal{J}_q) \in \mathcal{P}_{q,a}[Q_m] \), from (12.28) we have \( \Sigma((F, \mathcal{J}_q)) = 0 \).

**Lemma 14.4.** Let \( \alpha \in \mathbb{R} \) and let \( A \subseteq \mathbb{C}^{q \times p} \) with rank \( r \geq 1 \). Let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( \mathcal{R}(A) \), let \( U := [u_1, u_2, \ldots, u_r] \), and let \( Q := P_N(A^*) \). Then \( \Gamma_U : \mathcal{P}_{r,\alpha}^g \rightarrow \mathcal{P}_{q,a}^g[A] \) given by

\[ \Gamma_U((\phi, \psi)) := \langle U\phi U^*, U\psi U^* + Q \rangle \quad (14.9) \]
is well defined a bijective.

**Proof.** First observe that \( \mathcal{P}_{r,\alpha}^g \subseteq \mathcal{P}_{r,\alpha}^{(r)}(\mathbb{C} \setminus [\alpha, \infty)) \) by virtue of (14.11). Furthermore, (14.2) shows that \( \mathcal{P}_{q,a}^g[A] \subseteq \mathcal{P}_{q,a}^g[A] \). In view of Lemma [13.7] it is thus sufficient to show \( \Gamma_U((\mathcal{P}_{r,\alpha}^{(r)}(\mathbb{C} \setminus [\alpha, \infty)) \rightarrow \mathcal{P}_{q,a}^g[A] \) is given by (13.21). Observe that the same reasoning as in the proof of Lemma [14.4] yields (13.1), (13.2), and (13.3). According to (13.1) and (13.3), we have

\[ QU = Q^*U = (U^*Q)^* = 0_{q \times r} \quad (14.10) \]

First we consider an arbitrary pair \((\phi, \psi) \in \mathcal{P}_{r,\alpha}^{(r)}(\mathbb{C} \setminus [\alpha, \infty)) \) by virtue of (14.11) and \( \lim_{y \rightarrow \infty} \| (\phi \psi^{-1})(iy) \|_s = 0 \). Because of Definition [7.2] then \( (\phi, \psi) \) belongs to \( \mathcal{P}_{r,\alpha}^{(r)}(\mathbb{C} \setminus [\alpha, \infty)) = \mathcal{P}_{q,a}^g[A] \), and \( \det \psi \) does not identically vanish in \( \mathcal{S}_{q,[a,\infty)}[[s_j]_{j=0}^m = 0 \). Let \( F := U\psi U^* \) and \( G := U\hat{\psi} U^* + Q \). Then (13.2) implies (13.4). In view of Lemma [13.7] the equivalence class \((F, G) \) belongs to \( \mathcal{P}_{q,a}^g[A] \). In particular, \( (F, G) \in \mathcal{P}_{q,a}^g[A] \). Notation [14.13] shows that \((F, G) \) belongs to \( \mathcal{P}_{q,a}^{(q)}(\mathbb{C} \setminus [\alpha, \infty)) \). Using (13.2), (13.3), (14.10), and (13.1), we obtain

\[ G(U\psi^{-1}U^* + Q) = U\psi U^* U\psi^{-1}U^* + U\psi U^* Q + QU\psi^{-1}U^* + Q^2 = U\psi^{-1}U^* + Q^2 = I_q. \]

Hence, \( \det G \) does not identically vanish in \( \mathcal{S}_{q,[a,\infty)}[[s_j]_{j=0}^m \leq] \) and \( G^{-1} = U\psi^{-1}U^* + Q \). In particular, \((F, G) \in \mathcal{P}_{q,a}^g[A] \). Using (13.2) and (13.3), we get then

\[ FG^{-1} = U\phi U^* (U\psi^{-1}U^* + Q) = U\phi U^* U\psi^{-1}U^* + U\phi U^* Q = U\phi U^* - 1. \]

The equation \( \lim_{y \rightarrow \infty} \| (\phi \psi^{-1})(iy) \|_s = 0 \) implies \( \lim_{y \rightarrow \infty} \| (FG^{-1})(iy) \|_s = 0 \). Thus, \( (F, G) \) belongs to \( \mathcal{P}_{q,a}^g[A] \) according to (14.11). In view of \( (F, G) \in \mathcal{P}_{q,a}^g[A] \), then \( (F, G) \in \mathcal{P}_{q,a}^g[A] \) by virtue of (14.12). Hence, \( \Gamma_U((\phi, \psi)) \in \mathcal{P}_{q,a}^g[A] \). Conversely, consider now an arbitrary pair \((F, G) \in \mathcal{P}_{q,a}^g[A] \). Then \((F, G) \) belongs to \( \mathcal{P}_{q,a}^g[A] \) because of (14.2). Taking into account (14.11) and Definition [7.2] we infer then \((F, G) \in \mathcal{P}_{q,a}^g[A] \).
orthonormal basis of is well defined and bijective.

Theorem 14.5. Let \( r = \text{rank } Q \geq 1 \), where \( (Q_j)_{j=0}^m \) is the right \( \alpha \)-Stieltjes parametrization of \( \langle s_j, s_j \rangle_{m=0} = 1 \). Let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( R(Q_m) \), let \( U := [u_1, u_2, \ldots, u_r] \), and let \( \Gamma_U: \langle \tilde{P}_{r,\alpha} \rangle \to \langle \tilde{P}_{r,\alpha}(Q_m) \rangle \) be given by (14.9). Then \( \Sigma^* \circ \Gamma_U \) is well defined and bijective.

Proof. The assertion is an immediate consequence of Lemma 14.4 and Theorem 14.5.

Theorem 14.6. Let \( r = \text{rank } Q \geq 1 \), where \( (Q_j)_{j=0}^m \) is the right \( \alpha \)-Stieltjes parametrization of \( \langle s_j, s_j \rangle_{m=0} = 1 \). Let \( \nu^*: S_{r,\alpha}^{\infty} \to \langle \tilde{P}_{r,\alpha} \rangle \) be given by (14.3). Let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of
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$\mathcal{R}(Q_m)$, let $U := [u_1, u_2, \ldots, u_r]$, and let $\hat{\Gamma}_U^0 : (\mathcal{P}^{\circ}_{r,\alpha}) \to (\mathcal{P}^{\circ}_{q,\alpha}[Q_m])$ be given by \((14.9)\). Let $\Sigma^0 : (\mathcal{P}^{\circ}_{q,\alpha}[Q_m]) \to \mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_j^{m=0} =)$ be given by \((14.8)\). Then $\Sigma^0 \circ \hat{\Gamma}_U^0 \circ v^{\alpha}_0$ is well defined and bijective.

**Proof.** The assertion is an immediate consequence of Lemma \((14.1)\) and Theorem \((14.5)\).

**Corollary 14.7.** Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_j^{m=0} \in \mathbb{K}^{\mathbb{C},e,m,\alpha}$ be such that $r := \text{rank } Q_m$ fulfills $1 \leq r \leq q - 1$, where $(Q_j)_j^{m=0}$ is the right $\alpha$-Stieltjes parametrization of $(s_j)_j^{m=0}$. Let $v \alpha : \mathbb{S}_{r,[\alpha,\infty]} \to (\mathcal{P}^{\circ}_{r,\alpha})$ be given by \((14.5)\). Let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $u_1, u_2, \ldots, u_r$ is an orthonormal basis of $\mathcal{R}(Q_m)$ and let $W := [u_1, u_2, \ldots, u_q]$. Let $U := [u_1, u_2, \ldots, u_r]$ and let $\hat{\Gamma}_U^0 : (\mathcal{P}^{\circ}_{r,\alpha}) \to (\mathcal{P}^{\circ}_{q,\alpha}[Q_m])$ be given by \((14.9)\). Let $\Sigma^0 : (\mathcal{P}^{\circ}_{q,\alpha}[Q_m]) \to \mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_j^{m=0} =)$ be given by \((14.8)\). Then the mapping $\Theta := \Sigma^0 \circ \hat{\Gamma}_U^0 \circ v \alpha : \mathbb{S}_{r,[\alpha,\infty]} \to \mathcal{S}_{m,q,[\alpha,\infty]}((s_j)_j^{m=0} =)$ admits, for each $f \in \mathbb{S}_{r,[\alpha,\infty]}$, the representation

$$[\Theta(f)](z) = \left[ \begin{array}{c} \iota_2 \left( \left( \begin{array}{c} f(z) \end{array} \right) \cdot 0_{(q-r) \times (q-r)} \right) + \iota_1 \left( \left( \begin{array}{c} f(z) \end{array} \right) \cdot 0_{(q-r) \times (q-r)} \right) \right] \right]^{-1}$$

for all $z \in \mathbb{C} \setminus [\alpha, \infty)$.

**Proof.** For all $(\phi, \psi) \in \mathbb{P}^{\circ}_{q,\alpha}$, in view of \((14.1)\), the comparison of \((13.24)\) and \((14.9)\) shows that $\hat{\Gamma}_U^0((\phi, \psi)) = \Gamma_U^0((\phi, \psi))$. Consider an arbitrary function $f \in \mathbb{S}_{r,[\alpha,\infty]}$. Then $(f, \mathcal{J}_r)$ belongs to $\mathbb{P}^{(r,\mathcal{J}_r)}_{-r}((\mathbb{C} \setminus [\alpha, \infty))$ and Remark \((13.8)\) yields $(UfU^*, U \cdot \mathcal{J}_r \cdot U^* + P_{\mathcal{N}(Q_m)^*}) = (W \cdot \text{diag}(f, \Sigma_{q-r}) \cdot W^*, \mathcal{J}_q)$. Consequently, using \((14.3)\), \((14.9)\), \((14.8)\), and a continuity argument completes the proof.

A. Some facts from matrix theory

**Remark A.1.** Let $A \in \mathbb{C}^{q \times q}$. Then straightforward computations show that $\text{Re}(zA) = \text{Re}(z) \text{Re}(A) - \text{Im}(z) \text{Im}(A)$ and $\text{Im}(zA) = \text{Re}(z) \text{Im}(A) + \text{Im}(z) \text{Re}(A)$ hold true for all $z \in \mathbb{C}$. In particular, $\text{Re}(iA) = -\text{Im} A$ and $\text{Im}(iA) = \text{Re} A$ are valid.

**Lemma A.2.** Let $A \in \mathbb{C}^{q \times q}$. Then $B^*(-\mathcal{J}_q)B = -\mathcal{J}_q$, where

$$B := \begin{bmatrix} I_q & A \\ -A^\dagger & I_q - A^\dagger A \end{bmatrix}.$$  \hspace{1cm}  \text{(A.1)}

**Proof.** In view of $A^* = A$, the application of Remark \((A.6)\) and Remark \((A.9)\) yield $(A^\dagger)^* = A^\dagger$ and $A^\dagger A = AA^\dagger$. A straightforward calculation brings $B^*(-\mathcal{J}_q)B = -\mathcal{J}_q$.

**Remark A.3.** Let $U$ be a linear subspace of $\mathbb{C}^d$ with dimension $d \geq 1$ and let $u_1, u_2, \ldots, u_d$ be an orthonormal basis of $U$. Then $U := [u_1, u_2, \ldots, u_d]$ fulfills $U^*U = I_d$ and $UU^* = P_U$.

**Remark A.4.** For each $A \in \mathbb{C}^{q \times q}$, we have $\mathcal{R}(A)^{\perp} = N(A^*)$ and $N(A)^{\perp} = \mathcal{R}(A^*)$.

**Remark A.5.** Let $A \in \mathbb{C}^{q \times q}$, and let $B \in \mathbb{C}^{H \times q}$, be such that $B - A \in \mathbb{C}^{q \times q}$. Then $B \in \mathbb{C}^{q \times q}$, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, and $N(B) \subseteq N(A)$.

**Remark A.6.** Let $A \in \mathbb{C}^{p \times q}$. Then the following statements hold true:

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(a) \((A^\dagger)^\dagger = A\), \((A^\dagger)^* = (A^*)^\dagger\), \(N(A^\dagger) = N(A^*)\), \(R(A^\dagger) = R(A^*)\), and \(\text{rank}(AA^\dagger) = \text{rank} A\).

(b) Let \(r \in \mathbb{N}\) and \(B \in \mathbb{C}^{p \times r}\). Then \(R(B) \subseteq R(A)\) if and only if \(AA^\dagger B = B\).

(c) Let \(s \in \mathbb{N}\) and \(B \in \mathbb{C}^{s \times q}\). Then \(N(A) \subseteq N(C)\) if and only if \(CA^\dagger A = C\).

**Proposition A.7.** Let \(A \in \mathbb{C}^{p \times q}\) and let \(G \in \mathbb{C}^{q \times p}\). Then \(G = A^\dagger\) if and only if \(AG = P_{R(A)}\) and \(GA = P_{R(G)}\) hold true.

A proof of Proposition A.7 is given, e.g., in [3 Theorem 1.1.1, p. 15].

**Remark A.8.** Let \(A \in \mathbb{C}^{p \times q}\). In view of Proposition A.7 and Remark A.6, then one can easily see that \(I_q - AA^\dagger = P_{R(A)^\perp}\).

**Remark A.9.** If \(A \in \mathbb{C}_H^{q \times q}\), then \(AA^\dagger = A^\dagger A\).

**Lemma A.10.** Let \(E \in \mathbb{C}^{(p+q) \times (p+q)}\) and let

\[
E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

be the block partition of \(E\) with \(p \times p\) block \(a\). Then the following statements are equivalent:

(i) The matrix \(E\) is non-negative Hermitian.

(ii) \(a \in \mathbb{C}_{\geq q \times q}\), \(R(b) \subseteq R(a)\), \(c = b^*\), and \(d - ca^\dagger b \in \mathbb{C}_{\geq q \times q}\).

(iii) \(d \in \mathbb{C}_{\geq q \times q}\), \(R(c) \subseteq R(d)\), \(b = c^*\), and \(a - bd^\dagger c \in \mathbb{C}_{\geq q \times q}\).

A proof of Lemma A.10 is given, e.g., in [3 Lemma 1.1.9].

**Remark A.11.** Let the matrix \(E \in \mathbb{C}^{(p+r) \times (q+r)}\) be contractive and let \(\begin{bmatrix} A & B \\ B & C \end{bmatrix}\) be the block partition of \(E\) with \(p \times q\) block \(a\). If \(d = I_r\), then it is readily checked that \(b = 0_{p \times r}\) and \(c = 0_{q \times r}\).

**Lemma A.12 ([8 Lemma A.7]).** Let \(A, B \in \mathbb{C}_H^{q \times q}\). Then the following statements are equivalent:

(i) \(0_{q \times q} \leq B \leq A\).

(ii) \(0_{q \times q} \leq B^\dagger BA^\dagger BB^\dagger \leq B^\dagger\) and \(N(A) \subseteq N(B)\).

(iii) \(0_{q \times q} \leq BA^\dagger B \leq B^\dagger\) and \(N(A) \subseteq N(B)\).

(iv) \(0_{2q \times 2q} \leq \begin{bmatrix} A & B \\ B & C \end{bmatrix}\).

If (iv) is fulfilled, then \(N(BA^\dagger B) = N(B)\) and \(R(BA^\dagger B) = R(B)\).

We will use the 2\(q\) × 2\(q\) signature matrices

\[
\tilde{J}_q := \begin{bmatrix} 0_{q \times q} & -I_q \\ I_q & 0_{q \times q} \end{bmatrix} \quad \text{and} \quad J_q := \begin{bmatrix} 0_{q \times q} & -I_q \\ -I_q & 0_{q \times q} \end{bmatrix}.
\]

**Remark A.13.** For all \(A, B \in \mathbb{C}^{q \times q}\), the equations \(\begin{bmatrix} A \\ B \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} A \\ B \end{bmatrix} = 2\text{Im}(B^*A)\) and \(\begin{bmatrix} B \\ A \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} B \\ A \end{bmatrix} = 2\text{Re}(B^*A)\) hold true. In particular, \(\begin{bmatrix} A_q \\ I_q \end{bmatrix}^* (-J_q) \begin{bmatrix} A_q \\ I_q \end{bmatrix} = 2\text{Im} A\) and \(\begin{bmatrix} I_q \\ A_q \end{bmatrix}^* (-J_q) \begin{bmatrix} I_q \\ A_q \end{bmatrix} = 2\text{Re} A\) are valid for each \(A \in \mathbb{C}^{q \times q}\).
Remark A.14. Let $A, B \in \mathbb{C}^{q \times q}$. Suppose $\det B \neq 0$. In view of Remark A.13 it is readily checked then that $\text{Im}(AB^{-1}) = B^{-1}\left([A]^*F\tilde{\delta}q[A]\right)B^{-1}$.

Remark A.15. Suppose $q \geq 2$. Let $r \in \mathbb{Z}_{1,q-1}$, let $A \in \mathbb{C}^{r \times r}$, and let $B \in \mathbb{C}^{r \times r}$. In view of Remark A.13 it is readily checked then that

\[
\left[\text{diag}(A, 0_{(q-r) \times (q-r)})\right]^*(-\tilde{\delta}q)\left[\text{diag}(A, 0_{(q-r) \times (q-r)})\right] = \text{diag}\left([A]^*(-\tilde{\delta}q), 0_{(q-r) \times (q-r)}\right).
\]

Remark A.16. Let $H \in \mathbb{C}^{q \times q}$. Then $[\text{diag}(H, I_q)]^*(-\tilde{\delta}q)[\text{diag}(H, I_q)] = [\begin{pmatrix} 0_{q \times q} & iH^* \\ -iH & 0_{q \times q} \end{pmatrix}]$.

Lemma A.17. Let $\alpha \in \mathbb{R}$ and $H \in \mathbb{C}_{H}^{q \times q}$. Then:

(a) $[\text{diag}(H, H^\dagger)]^*(-\tilde{\delta}q)[\text{diag}(H, H^\dagger)] = [\text{diag}(HH^\dagger, I_q)]^*(-\tilde{\delta}q)[\text{diag}(HH^\dagger, I_q)]$.

(b) $[\text{diag}((z-\alpha)H, H^\dagger)]^*(-\tilde{\delta}q)[\text{diag}((z-\alpha)H, H^\dagger)] = [\text{diag}((z-\alpha)HH^\dagger, I_q)]^*(-\tilde{\delta}q)[\text{diag}((z-\alpha)HH^\dagger, I_q)]$ for each $z \in \mathbb{C}$.

Proof. In view of $H^\dagger = H$, from Remark A.6(a) we infer $[\text{diag}(H, H^\dagger)]^* = [\text{diag}(H, H^\dagger)]$. Furthermore, because of $H^\dagger = H$ and Remark A.9 we have $H^\dagger H = HH^\dagger$. Using additionally Remark A.16 and $(HH^\dagger)^* = HH^\dagger$, we conclude then

\[
[\text{diag}(H, H^\dagger)]^*(-\tilde{\delta}q)[\text{diag}(H, H^\dagger)] = [\text{diag}(H, H^\dagger)]\begin{pmatrix} 0_{q \times q} & iI_q \\ -iI_q & 0_{q \times q} \end{pmatrix}[\text{diag}(H, H^\dagger)] = [\begin{pmatrix} 0_{q \times q} & iHH^\dagger \\ -iHH^\dagger & 0_{q \times q} \end{pmatrix}][\text{diag}(HH^\dagger, I_q)]^*(-\tilde{\delta}q)[\text{diag}(HH^\dagger, I_q)].
\]

Using (a), for each $z \in \mathbb{C}$, we obtain

\[
[\text{diag}((z-\alpha)H, H^\dagger)]^*(-\tilde{\delta}q)[\text{diag}((z-\alpha)H, H^\dagger)] = [\text{diag}((z-\alpha)HH^\dagger, I_q)]^*(-\tilde{\delta}q)[\text{diag}((z-\alpha)HH^\dagger, I_q)].
\]

B. Particular results of the integration theory with respect non-negative Hermitian measures

Let $\Omega$ be a non-empty set and let $\mathcal{A}$ be a $\sigma$-algebra on $\Omega$. If $\nu$ is a measure on the measurable space $(\Omega, \mathcal{A})$, for each $s \in (0, \infty)$, then we use $\mathcal{L}^s(\Omega, \mathcal{A}, \nu; \mathbb{C})$ to denote the set of all $\mathcal{A}$-$\mathcal{B}_c$-measurable functions $f : \Omega \to \mathbb{C}$ such that $\int_\Omega |f|^s d\nu < \infty$. A matrix-valued function $\mu$ whose domain is $\mathcal{A}$ and whose values belong to the set $\mathbb{C}_{2 \times 2}$ of all non-negative Hermitian
complex $q \times q$ matrices is called non-negative Hermitian $q \times q$ measure on $(Ω, ℳ)$ if it is countable additive, i.e., if $μ$ fulfills $μ(∪_{k=1}^{∞} A_k) = \sum_{k=1}^{∞} μ(A_k)$ for each sequence $(A_k)_{k=1}^{∞}$ of pairwise disjoint sets that belong to $ℳ$. By $ℳ_p^q(Ω, ℳ)$ we denote the set of all non-negative Hermitian $q \times q$ measures on $(Ω, ℳ)$. We use some basic facts from the integration theory with respect to non-negative Hermitian measures (see [17] and [22]).

Let $μ = [μ_{jk}]_{j,k=1}^{q} ∈ ℳ_p^q(Ω, ℳ)$. Then each entry function $μ_{jk}$ is a complex measure on $(Ω, ℳ)$ and, in particular, the measure $τ := tr μ$ is a finite (non-negative real-valued) measure, the so-called trace measure of $μ$. For every choice of $j$ and $k$ in $ℤ_{+}q$, the complex measure $μ_{jk}$ is absolutely continuous with respect to $τ$ and, consequently, the matrix-valued function $μ_{jk}′ = [dμ_{jk}/dτ]_{j,k=1}^{q}$ of the Radon-Nikodym derivatives is well defined up to sets of zero $τ$-measure. We will write $ℳ_{p×q}$ for the $σ$-algebra of all Borel subsets of $ℂ^{p×q}$. An ordered pair $(Φ, Ψ)$ consisting of an $ℳ_{p×q}$-measurable function $Φ : Ω → ℂ^{p×q}$ and an $ℳ_{r×p}$-measurable function $Ψ : Ω → ℂ^{r×p}$ is said to be left-integrable with respect to $μ$ if $Φμ_{jk}′Ψ^∗$ belongs to $ℒ^1(Ω, ℳ, τ; ℂ)^{p×r}$. In this case, for each $A ∈ ℳ$, the integral $∫_A ΦdμΨ^∗ := ∫_A Φ(ω)μ_{jk}′(ω)Ψ^∗(ω)τ(dω)$ is well defined. The class of all $ℳ_{p×q}$-measurable functions $Φ : Ω → ℂ^{p×q}$ for which $(Φ, Ψ)$ is left-integrable with respect to $μ$ is denoted by $p × q – L^2(q, Ω, ℳ)$ if and only if the pair $(f I_q, g I_q)$ is left-integrable with respect to $μ$. Note that one can easily check that the set $ℒ^1(Ω, ℳ, μ; ℂ)$, consisting of all Borel-measurable functions $f : Ω → ℂ$ for which the integral $∫_Ω f dμ$ exists, coincides with $ℒ^1(Ω, ℳ, τ; ℂ)$.

Remark B.1. The set $ℒ^1(Ω, ℳ, μ; ℂ)$ coincides with the set of all $ℳ_{p×q}$-measurable functions $f : Ω → ℂ$ for which the pair $(f I_q, g I_q)$ is left-integrable with respect to $μ$. If $f ∈ ℒ^1(Ω, ℳ, μ; ℂ)$, then it is readily checked that $∫_A f dμ = ∫_A f I_q dμ I_q^*$ for all $A ∈ ℳ$. Furthermore, if $f = f I_q$ and $g = g I_q$ are $ℳ_{p×q}$-measurable functions defined on $Ω$, then $∫_Ω f(dμ) I_q^*$ is integrable with respect to $μ$. In this case, one can easily see that $∫_A f I_q dμ I_q^*$ belongs to $ℒ^1(Ω, ℳ, τ; ℂ)$ as well and $∫_A f I_q dμ(I_q^∗)^{∗} = ∫_A f dμ I_q^*$ for all $A ∈ ℳ$.

Remark B.2. Let $(Ω, ℳ)$ be a measurable space, let $μ ∈ ℳ_p^q(Ω, ℳ)$, and let $f ∈ ℒ^1(Ω, ℳ, τ; ℂ)$. Then $∫_Ω f dμ$ belongs to $ℒ^1(Ω, ℳ, τ; ℂ)$ as well and $∫_Ω f dμ(I_q^∗)^{∗} = ∫_Ω f dμ I_q^*$ for all $A ∈ ℳ$.

Proposition B.3. Let $(Ω, ℳ)$ be a measurable space and let $μ ∈ ℳ_p^q(Ω, ℳ)$. Furthermore, let

$Φ ∈ p × q – L^2(q, Ω, ℳ)$ and $Ψ ∈ r × q – L^2(q, Ω, ℳ).$  \hspace{1cm} (B.1)

Then

$N(∫_Ω Φ dμ Φ^∗) \subseteq N(∫_Ω Ψ dμ Φ^∗), \hspace{1cm} R(∫_Ω Φ dμ Φ^∗) \subseteq R(∫_Ω Ψ dμ Φ^∗),$  \hspace{1cm} (B.2)

and

$∫_Ω Ψ dμ Φ^∗(∫_Ω Φ dμ Φ^∗) † (∫_Ω Φ dμ Φ^∗) ≤ ∫_Ω Ψ dμ Φ^∗.$  \hspace{1cm} (B.3)

Proof. Because of (B.1), we have $(Φ, Ψ) ∈ (p+r) × q – L^2(q, Ω, ℳ)$ and

$∫_Ω 1^p dμ Φ^∗ = ∫_Ω 1^p dμ Φ^∗,$  \hspace{1cm} (B.4)

and

$∫_Ω Ψ dμ Φ^∗ = ∫_Ω Ψ dμ Φ^∗, \hspace{1cm} (∫_Ω Φ dμ Φ^∗)^∗ = ∫_Ω Φ dμ Φ^∗.$

Thus, the first inclusion in (B.2) follows from the second one and Remark A.4.
**Corollary B.4.** Let \((\Omega, \mathcal{A})\) be a measurable space, let \(\mu \in \mathcal{M}_\mu^\infty(\Omega, \mathcal{A})\), and let \(f, g \in L^2(\Omega, \mathcal{A}, \text{tr} \mu; \mathbb{C})\). Then \{f, g, |f|^2, |g|^2, f g, f g^\dagger\} \subseteq L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})\),

\[
\mathcal{N}\left(\int_\Omega |f|^2 \, d\mu\right) \subseteq \mathcal{N}\left(\int_\Omega f \, d\mu\right) \cap \mathcal{N}\left(\int_\Omega f^\dagger \, d\mu\right)^* \tag{B.4}
\]

and

\[
\mathcal{R}\left(\int_\Omega |f|^2 \, d\mu\right) \subseteq \mathcal{R}\left(\int_\Omega f \, d\mu\right) \cap \mathcal{R}\left(\int_\Omega f^\dagger \, d\mu\right)^* \tag{B.5}
\]

**Proof.** Since \(f\) and \(g\) belong to \(L^2(\Omega, \mathcal{A}, \text{tr} \mu; \mathbb{C})\) and because of \(\mathcal{R}(\Omega) < \infty\), we have \{f, g, |f|^2, |g|^2, f g, f g^\dagger\} \subseteq L^1(\Omega, \mathcal{A}, \tr \mu; \mathbb{C})\). In view of \(L^1(\Omega, \mathcal{A}, \mu; \mathbb{C}) = L^1(\Omega, \mathcal{A}, \tau \mu; \mathbb{C})\), then \{f, g, |f|^2, |g|^2, f g, f g^\dagger\} \subseteq L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})\) follows. Setting \(\Phi := f I_q\) and \(\Psi := g I_q\), we get \([B.4]\) and \([B.5]\) from Proposition \([B.3]\) and Remark \([B.1]\). \(\square\)

**Corollary B.5.** Let \((\Omega, \mathcal{A})\) be a measurable space, let \(\mu \in \mathcal{M}_\mu^\infty(\Omega, \mathcal{A})\), and let \(f \in L^2(\Omega, \mathcal{A}, \text{tr} \mu; \mathbb{C})\). Then \{f, f, |f|^2\} \subseteq L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})\),

\[
\mathcal{N}\left(\int_\Omega |f|^2 \, d\mu\right) \subseteq \mathcal{N}\left(\int_\Omega f \, d\mu\right) \cap \mathcal{N}\left(\int_\Omega f \, d\mu\right)^* \tag{B.6}
\]

and

\[
\mathcal{R}\left(\int_\Omega |f|^2 \, d\mu\right) \subseteq \mathcal{R}\left(\int_\Omega f \, d\mu\right) \cap \mathcal{R}\left(\int_\Omega f \, d\mu\right)^* \tag{B.7}
\]

**Proof.** Setting \(g := 1_\Omega\) and applying Corollary \([B.4]\), we conclude \(\mathcal{N}(f_0 |f|^2 \, d\mu) \subseteq \mathcal{N}(f_0 f \, d\mu)\) and \(\mathcal{R}(f_0 f \, d\mu) \subseteq \mathcal{R}(f_0 |f|^2 \, d\mu)\). Substituting \(f\) by \(f\), we obtain \(\mathcal{N}(f_0 |f|^2 \, d\mu) \subseteq \mathcal{N}(f_0 f \, d\mu)\) and \(\mathcal{R}(f_0 f \, d\mu) \subseteq \mathcal{R}(f_0 |f|^2 \, d\mu)\). Because of Remark \([B.2]\), then \([B.6]\) and \([B.7]\). The inequality in \([B.8]\) is an immediate consequence of Corollary \([B.5]\) where \(g := 1_\Omega\) is chosen in \([B.8]\). Similarly, \([B.9]\) follows from Corollary \([B.5]\) where \(f\) is substituted by \(f\). \(\square\)

**Corollary B.6.** Let \((\Omega, \mathcal{A})\) be a measurable space, let \(\mu \in \mathcal{M}_\mu^\infty(\Omega, \mathcal{A})\), and let \(g \in L^2(\Omega, \mathcal{A}, \text{tr} \mu; \mathbb{C})\). Then \{g, g, |g|^2\} \subseteq L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})\),

\[
\mathcal{N}(\mu(\Omega)) \subseteq \mathcal{N}\left(\int_\Omega g \, d\mu\right) \cap \mathcal{N}\left(\int_\Omega g \, d\mu\right)^* \tag{B.10}
\]

and

\[
\mathcal{R}(\mu(\Omega)) \subseteq \mathcal{R}\left(\int_\Omega g \, d\mu\right) \cap \mathcal{R}\left(\int_\Omega g \, d\mu\right)^* \tag{B.11}
\]

**Proof.** Let \(f := 1_\Omega\). Applying Corollary \([B.4]\) we conclude \(\mathcal{N}(\mu(\Omega)) \subseteq \mathcal{N}(f_0 g \, d\mu)\) and \(\mathcal{R}(f_0 g \, d\mu) \subseteq \mathcal{R}(\mu(\Omega))\). Substituting \(g\) by \(g\) then \(\mathcal{N}(\mu(\Omega)) \subseteq \mathcal{N}(f_0 g \, d\mu)\) and \(\mathcal{R}(f_0 g \, d\mu) \subseteq \mathcal{R}(\mu(\Omega))\) follow. In view of Remark \([B.2]\) we get then \([B.10]\) and \([B.11]\). Similarly, \([B.12]\) and \([B.13]\) can be proved using Corollary \([B.4]\) and Remark \([B.2]\). \(\square\)
C. Meromorphic matrix-valued functions

By a region we mean an open connected subset of $\mathbb{C}$. Let $\mathcal{G}$ be a region in $\mathbb{C}$ and let $f$ be a complex function in $\mathcal{G}$. Then $f$ is called meromorphic in $\mathcal{G}$ if there exists a discrete subset $\mathbb{P}_f$ of $\mathcal{G}$ such that $f$ is holomorphic in $\mathbb{H}_f := \mathcal{G} \setminus \mathbb{P}_f$ whereas $f$ has a pole in each point of $\mathbb{P}_f$. We denote by $\mathcal{M}(\mathcal{G})$ the set of all meromorphic functions in $\mathcal{G}$. The notation $\mathcal{H}(\mathcal{G})$ stands for the set of all complex-valued holomorphic functions in $\mathcal{G}$.

Now we want to extend these notions to matrix-valued functions. Let $\mathcal{G}$ be a region in $\mathbb{C}$ and let $r, s \in \mathbb{N}$.

Let $f = [f_{jk}]_{j=1,...,r} \in [\mathcal{M}(\mathcal{G})]^{r \times s}$. Then the sets $\mathbb{H}_f := \bigcap_{j=1}^r \bigcap_{k=1}^s \mathbb{H}_{f_{jk}}$ and $\mathbb{P}_f := \bigcup_{j=1}^r \bigcup_{k=1}^s \mathbb{P}_{f_{jk}}$ are called the holomorphicity set of $f$ and the pole set of $f$, respectively. Then one can easily see that $\mathbb{H}_f \cup \mathbb{P}_f = \mathcal{G}$ and $\mathbb{H}_f \cap \mathbb{P}_f = \emptyset$ hold true. We consider an $f \in [\mathcal{M}(\mathcal{G})]^{r \times s}$ also as a mapping $f$ between the sets $\mathbb{H}_f$ and $\mathcal{C}^{r \times s}$.

Now we consider two classes of $q \times q$ matrix-valued holomorphic functions.

Let $\mathcal{G}$ be a region in $\mathbb{C}$. A function $f \in [\mathcal{H}(\mathcal{G})]^{q \times q}$ is called a $q \times q$ Carathéodory function (resp. $q \times q$ Herglotz-Nevanlinna function) in $\mathcal{G}$ if $\text{Re} f(\mathcal{G}) \subseteq \mathbb{C}^{q \times q}$ (resp. $\text{Im} f(\mathcal{G}) \subseteq \mathbb{C}^{q \times q}$). We denote by $\mathcal{C}_q(\mathcal{G})$ (resp. $\mathcal{R}_q(\mathcal{G})$) the set of all $q \times q$ Carathéodory functions (resp. $q \times q$ Herglotz-Nevanlinna functions) in $\mathcal{G}$.

**Remark C.1.** Let $\mathcal{G}$ be a region in $\mathbb{C}$. Then Remark [A.1] shows that $\mathcal{R}_q(\mathcal{G}) = \{ f : f \in \mathcal{C}_q(\mathcal{G}) \}$ and $\mathcal{C}_q(\mathcal{G}) = \{-ig : g \in \mathcal{R}_q(\mathcal{G})\}$.

**Lemma C.2.** Let $\mathcal{G}$ be a region in $\mathbb{C}$ and let $\mathcal{D}$ be a discrete subset of $\mathcal{G}$.

(a) Let $f \in \mathcal{C}_q(\mathcal{G} \setminus \mathcal{D})$. Then each $w \in \mathcal{D}$ is a removable singularity of $f$ and the extended function $\tilde{f}$ satisfies $\tilde{f} \in \mathcal{C}_q(\mathcal{G})$.

(b) Let $f \in \mathcal{R}_q(\mathcal{G} \setminus \mathcal{D})$. Then each $w \in \mathcal{D}$ is a removable singularity of $f$ and the extended function $\tilde{f}$ satisfies $\tilde{f} \in \mathcal{R}_q(\mathcal{G})$.

**Proof.** Part (a) follows from [3] Lemma 2.1.9. Part (b) is a consequence of (a) and Remark C.1.

D. On Linear Fractional Transformations of Matrices

In this appendix, we summarize some basic facts on linear fractional transformations of matrices. A systematic treatment of this topic was handled by V. P. Potapov in [21] (see also [3], Section 1.6). We slightly extend the concept developed by V. P. Potapov by studying the more general version of linear fractional transformations of pairs of complex matrices. It should be mentioned that V. P. Potapov [21] pp. 80–81 observed that sometimes there are situations where linear fractional transformations of pairs of complex matrices arise, but not treated this case. We did not succeed in finding a convenient hint in the public literature. That’s why we state the corresponding results with short proofs.

**Notation D.1.** Let $E \in \mathbb{C}^{(p+q) \times (p+q)}$ and let $[A.2]$ be the block partition of $E$ with $p \times p$ block $A$. If $\text{rank}[c, d] = q$, then the linear fractional transformations $S_E^{(p,q)} : \mathcal{Q}_{[c,d]} \to \mathbb{C}^{p \times q}$ and $\tilde{S}_E^{(p,q)} : \tilde{\mathcal{Q}}_{[c,d]} \to \mathbb{C}^{p \times q}$ are defined by

$$S_E^{(p,q)}(x) := (ax + b)(cx + d)^{-1} \quad \text{and} \quad \tilde{S}_E^{(p,q)}((x,y)) := (ax + by)(cx + dy)^{-1}.$$
Lemma D.2. Let $c \in \mathbb{C}^{q \times p}$ and $d \in \mathbb{C}^{q \times q}$. Then the following statements are equivalent:

(i) The set $Q_{[c,d]} := \{ x \in \mathbb{C}^{p \times q} : \det(cx + d) \neq 0 \}$ is non-empty.

(ii) The set $\tilde{Q}_{[c,d]} := \{(x,y) \in \mathbb{C}^{p \times q} \times \mathbb{C}^{q \times q} : \det(cx + dy) \neq 0 \}$ is non-empty.

(iii) $\operatorname{rank}[c,d] = q$.

Furthermore, $\tilde{Q}_{[c,d]}$ is a subset of the set $Q_{p \times q}$ of all pairs $(x,y) \in \mathbb{C}^{p \times q} \times \mathbb{C}^{q \times q}$ which fulfill $\operatorname{rank}[^{x}_{y}] = q$.

Proof. "(i) $\Rightarrow$ (ii)": Choose $y = I_q$.

"(ii) $\Rightarrow$ (iii)": If $(x,y) \in \tilde{Q}_{[c,d]}$, then

$$q = \operatorname{rank}(cx + dy) = \operatorname{rank}
\begin{bmatrix}
[c, d] \\
[x, y]
\end{bmatrix}
\leq \min \left\{ \operatorname{rank}[c,d], \operatorname{rank}
\begin{bmatrix}
x \\
y
\end{bmatrix}\right\} \leq q. \quad \text{(D.1)}$$

"(iii) $\Rightarrow$ (i)": This implication is proved, e.g., in [3] Lemma 1.6.1, p. 52].

The rest follows from (D.1). \qed

Proposition D.3 (see, e.g. [3] Proposition 1.6.3). Let $a_1, a_2 \in \mathbb{C}^{p \times p}$, let $b_1, b_2 \in \mathbb{C}^{p \times q}$, let $c_1, c_2 \in \mathbb{C}^{q \times p}$, and let $d_1, d_2 \in \mathbb{C}^{q \times q}$ be such that $\operatorname{rank}[c_1,d_1] = \operatorname{rank}[c_2,d_2] = q$. Furthermore, let

$$E_1 := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad E_2 := \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad \text{(D.2)}$$

let $E_2 := [a_2 \ b_2]$, let $E := E_2 E_1$, and let $[A.2]$ be the block representation of $E$ with $p \times p$ block $a$. Then $Q := \{ x \in Q_{[c_1,d_1]} : S^{(p,q)}_{E_1}(x) \in Q_{[c_2,d_2]} \}$ is a non-empty subset of the set $Q_{[c,d]}$ and $S^{(p,q)}_{E_2} \left(S^{(p,q)}_{E_1}(x)\right) = S^{(p,q)}_{E}(x)$ holds true for all $x \in Q$.

Proposition D.4. Let $E_1, E_2 \in \mathbb{C}^{(p+q) \times (p+q)}$ and let the block partitions of $E_1$ and $E_2$ with $p \times p$ blocks $a_1$ and $a_2$ be given by (D.2). Let $E := E_2 E_1$ and let $[A.2]$ be the block partition of $E$ with $p \times p$ block $a$. Suppose that $\operatorname{rank}[c_1,d_1] = q$ and $\operatorname{rank}[c_2,d_2] = q$ hold true. Let $\tilde{Q} := \{ (x,y) \in \tilde{Q}_{[c_1,d_1]} : S^{(p,q)}_{E_1}(x,y) \in \tilde{Q}_{[c_2,d_2]} \}$. Then $\tilde{Q}_{[c,d]} \cap \tilde{Q}_{[c_1,d_1]} = \tilde{Q}$. Furthermore, if $\tilde{Q}_{[c,d]} \cap \tilde{Q}_{[c_1,d_1]} \neq \emptyset$, then $S^{(p,q)}_{E_2} \left(S^{(p,q)}_{E_1}(x,y)\right) = S^{(p,q)}_{E}(x,y)$ for all $(x,y) \in \tilde{Q}_{[c,d]} \cap \tilde{Q}_{[c_1,d_1]}$.

Proof. It is readily checked that

$$a = a_2 a_1 + b_2 c_1, \quad b = a_2 b_1 + b_2 d_1, \quad c = c_2 a_1 + d_2 c_1, \quad \text{and} \quad d = c_2 b_1 + d_2 d_1. \quad \text{(D.3)}$$

Suppose $\tilde{Q} \neq \emptyset$. We consider an arbitrary $(x,y) \in \tilde{Q}$. Then $(x,y) \in \tilde{Q}_{[c_1,d_1]}$ and $z := S^{(p,q)}_{E_1}(x,y)$ belongs to $Q_{[c_2,d_2]}$, i.e., $\det(c_2 z + d_2) \neq 0$ is true. Since (D.3) implies

$$c_2 z + d_2 = c_2(a_1 x + b_1 y)(c_1 x + d_1 y)^{-1} + d_2(c_1 x + d_1 y)(c_1 x + d_1 y)^{-1}$$

$$= [c_2(a_1 x + b_1 y) + d_2(c_1 x + d_1 y)](c_1 x + d_1 y)^{-1} = (c x + dy)(c_1 x + d_1 y)^{-1}, \quad \text{(D.4)}$$

$$D. \text{ On Linear Fractional Transformations of Matrices}$$
we get \( (cx + dy) \neq 0 \), i.e., that \((x, y)\) belongs to \(Q_{c,d}\). Thus, \( \tilde{Q} \subseteq Q_{c,d} \cap \tilde{Q}_{c_1,d_1} \) is checked. Similar to (D.4), we conclude \( a_2z + b_2 = (ax + by)(c_1x + d_1y)^{-1} \). Consequently, using (D.4), then we also obtain
\[
S_{E_2}^{(p,q)}(S_{E_1}^{(p,q)}((x,y))) = (a_2z + b_2)(c_1x + d_1y)^{-1} = (ax + by)(c_1x + d_1y)^{-1}[(cx + dy)(c_1x + d_1y)^{-1}]^{-1} = S_{E_2}^{(p,q)}((x,y)).
\]
It remains to prove that \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \subseteq \tilde{Q} \) is fulfilled. For this reason, we assume that \((x, y)\) belongs to \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \). Setting \( z := S_{E_1}^{(p,q)}((x,y)) \) again, we infer that (D.4) holds true. Because of (D.4) and \((x, y) \in \tilde{Q}_{c,d} \), we get \( \det(c_2z + d_2) \neq 0 \), i.e., \( z \in Q_{c_2,d_2} \). Hence, \((x, y)\) belongs to \( \tilde{Q} \).

From Proposition [D.4] immediately the following result follows:

**Corollary D.5 ( [3] Proposition 1.6.1).** Let \( E_1, E_2 \in C^{(p+q) \times (p+q)} \), let \( E := E_2E_1 \), and let [D.2] and [A.2] be the block partitions of \( E_1, E_2 \), and \( E \) with \( p \times p \) blocks \( a_1, a_2 \), and \( a \), respectively. Suppose that \( \text{rank}[c_1, d_1] = q \) and \( \text{rank}[c_2, d_2] = q \) hold true. Then \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} = \{ x \in Q_{c_1,d_1} : S_{E_1}^{(p,q)}(x) \in Q_{c_2,d_2} \} \}. \) Furthermore, if \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \neq \emptyset, \) then \( S_{E_2}^{(p,q)}(S_{E_1}^{(p,q)}(x)) = S_{E_2}^{(p,q)}(x) \) for all \( x \in \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \).

We modify Notation [D.1] for matrix-valued functions: Let \( G \) be a non-empty subset of \( C \), let \( V : G \to C^{(p+q) \times (p+q)} \) be a matrix-valued function, and let \( V = \left[ \begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix} \right] \) be the block partition of \( V \) with \( p \times p \) block \( v_{11} \). Then we make the following conventions: If \( F : G \to C^{p \times q} \) fulfills \( \det(v_{21}(z)F(z) + v_{22}(z)) \neq 0 \) for all \( z \in G \), then we use \( S_{V}^{(p,q)}(F) \) to denote the function \( S_{V}^{(p,q)}(F) : G \to C^{p \times q} \) defined by \( [S_{V}^{(p,q)}(F)](z) := S_{V(z)}^{(p,q)}(F(z)) \). Furthermore, if \( F_1 : G \to C^{p \times q} \) and \( F_2 : G \to C^{p \times q} \) are matrix-valued functions with \( \det(v_{21}(z)F_1(z) + v_{22}(z)F_2(z)) \neq 0 \) for all \( z \in G \), then \( S_{V}^{(p,q)}((F_1, F_2)) : G \to C^{p \times q} \) is given by \( [S_{V}^{(p,q)}((F_1, F_2))](z) := S_{V(z)}^{(p,q)}((F_1(z), F_2(z))) \).

The following example shows that the conditions \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \neq \emptyset \) and \( \tilde{Q}_{c_2,d_2} \neq \emptyset \) do not imply \( \tilde{Q}_{c,d} \cap \tilde{Q}_{c_1,d_1} \neq \emptyset \).

**Example D.6.** With \( a_1 = b_1 = d_1 = a_2 = b_2 = d_2 = 0_{q \times q} \), and \( c_1 = c_2 = I_q \) we have \( E = 0_{2q \times 2q} \), \( (I_q, I_q) \in \tilde{Q}_{c_1,d_1} \), \( I_q \in Q_{c_2,d_2} \), and \( \tilde{Q}_{c,d} = \emptyset \).

**Proposition D.7.** Let \( E_1, E_2 \in C^{(p+q) \times (p+q)} \) and let [D.2] be the block partitions of \( E_1 \) and \( E_2 \) with \( p \times p \) blocks \( a_1 \) and \( a_2 \). Let \( E := E_2E_1 \) and let [A.2] be the block partition of \( E \) with \( p \times p \) block \( a \). Suppose \( \text{rank}[c_1, d_1] = q \) and \( \text{rank}[c_2, d_2] = q \). Furthermore, let \((x, y)\) and \((\tilde{x}, \tilde{y})\) be from \( C^{p \times q} \times C^{q \times q} \) such that \( E_1 \left[ \begin{smallmatrix} \tilde{y} \\ \tilde{y} \end{smallmatrix} \right] = \left[ \begin{smallmatrix} \frac{x}{2} \\ \frac{y}{2} \end{smallmatrix} \right] \). Then:

(a) \((x, y) \in \tilde{Q}_{c,d}\) if and only if \((\tilde{x}, \tilde{y}) \in \tilde{Q}_{c_2,d_2}\).

(b) If \((x, y) \in \tilde{Q}_{c,d}\), then \( S_{E_2}^{(p,q)}((x,y)) = S_{E_2}^{(p,q)}((\tilde{x}, \tilde{y})) \).

**Proof.** By assumption, we have \( \tilde{x} = a_1x + b_1y \) and \( \tilde{y} = c_1x + d_1y \). In view of (D.3), then
\[
a_2\tilde{x} + b_2\tilde{y} = a_2(a_1x + b_1y) + b_2(c_1x + d_1y) = (a_2a_1 + b_2c_1)x + (a_2b_1 + b_2d_1)y = ax + by
\]
and
\[
c_2\tilde{x} + d_2\tilde{y} = c_2(a_1x + b_1y) + d_2(c_1x + d_1y) = (c_2a_1 + d_2c_1)x + (c_2b_1 + d_2d_1)y = cx + dy.
\]
Now (a) and (b) immediately follow.
E. The Matrix Polynomials $V_{\alpha,A}$ and $W_{\alpha,A}$

Now we study special matrix polynomials, which were already intensively used in [10].

**Remark E.1.** Let $\alpha \in \mathbb{R}$ and let $A \in \mathbb{C}^{p \times q}$. Then $V_{\alpha,A}: \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)}$ and $W_{\alpha,A}: \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)}$ given by

$$ V_{\alpha,A}(z) := \begin{bmatrix} 0_{p \times p} & -A \\ (z - \alpha)A^\dagger & (z - \alpha)I_q \end{bmatrix} \quad \text{and} \quad W_{\alpha,A}(z) := \begin{bmatrix} (z - \alpha)I_p & A \\ -(z - \alpha)A^\dagger & I_q - A^\dagger A \end{bmatrix} \quad \text{(E.1)} $$

respectively, are linear $(p + q) \times (p + q)$ matrix polynomials and, in particular, holomorphic in $\mathbb{C}$. For all $z \in \mathbb{C}$, furthermore

$$ V_{\alpha,A}(z) = \text{diag}(I_p, (z - \alpha)I_q) \cdot \begin{bmatrix} 0_{p \times p} & -A \\ A^\dagger & I_q \end{bmatrix} \quad \text{(E.2)} $$

and

$$ W_{\alpha,A}(z) = \begin{bmatrix} I_p & A \\ -A^\dagger & I_q - A^\dagger A \end{bmatrix} \cdot \text{diag}((z - \alpha)I_p, I_q). \quad \text{(E.3)} $$

The use of the matrix polynomial $V_{\alpha,A}$ was inspired by some constructions in the paper [14]. In particular, we mention [14] formula (2.3). In their constructions, Hu and Chen used Drazin inverses instead of Moore-Penrose inverses of matrices. Since both types of generalized inverses coincide for Hermitian matrices (see, e.g. [12] Proposition A.2), we can conclude that in the generic case the matrix polynomials $V_{\alpha,A}$ coincide for $\alpha = 0$ with the functions used in [13].

**Remark E.2 ( [10] Remark D.1).** Let $A \in \mathbb{C}^{p \times q}$ and let $\alpha \in \mathbb{C}$. For each $z \in \mathbb{C}$, then $V_{\alpha,A}(z)W_{\alpha,A}(z) = (z - \alpha) \cdot \text{diag}(AA^\dagger, I_q)$ and $W_{\alpha,A}(z)V_{\alpha,A}(z) = (z - \alpha) \cdot \text{diag}(AA^\dagger, I_q)$.

Now we are going to study the linear fractional transformation generated by the matrix $W_{\alpha,A}(z)$ for arbitrarily given $z \in \mathbb{C} \setminus \{\alpha\}$.

**Lemma E.3 ( [10] Lemma D.2).** Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, and let $z \in \mathbb{C} \setminus \{\alpha\}$. Then:

(a) The matrix $-(z - \alpha)^{-1}A$ belongs to $\mathcal{Q}_{[-(z - \alpha)A^\dagger, I_q - A^\dagger A]}$. In particular, $\mathcal{Q}_{[-(z - \alpha)A^\dagger, I_q - A^\dagger A]} \neq \emptyset$.

(b) Let $X \in \mathbb{C}^{p \times q}$ be such that $\mathcal{R}(A) \subseteq \mathcal{R}(X)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(X)$. Then $X$ belongs to $\mathcal{Q}_{[-(z - \alpha)A^\dagger, I_q - A^\dagger A]}$ and the equation $[-(z - \alpha)A^\dagger X + I_q - A^\dagger A]^{-1} = -(z - \alpha)^{-1}X^\dagger A + I_q - A^\dagger A$ holds true.

**Remark E.4.** Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, and let $z \in \mathbb{C} \setminus \{\alpha\}$. In view of Lemmas E.3 and D.2 then $\mathcal{Q}_{[-(z - \alpha)A^\dagger, I_q - A^\dagger A]} \neq \emptyset$.

**Remark E.5.** Let $\alpha \in \mathbb{C}$, let $A \in \mathbb{C}^{p \times q}$, and let $z \in \mathbb{C} \setminus \{\alpha\}$. Then $\text{rank}[(z - \alpha)A^\dagger, (z - \alpha)I_q] = q$ and the matrix $0_{q \times q}$ obviously belongs to $\mathcal{Q}_{[(z - \alpha)A^\dagger, (z - \alpha)I_q]}$. In particular, $\mathcal{Q}_{[(z - \alpha)A^\dagger, (z - \alpha)I_q]} \neq \emptyset$ and, in view of Lemma D.2 furthermore $\mathcal{Q}_{[(z - \alpha)A^\dagger, (z - \alpha)I_q]} \neq \emptyset$.

**Proposition E.6.** Let $\alpha \in \mathbb{R}$, let $A \in \mathbb{C}^{2q \times q}$, and let the $2q \times 2q$ matrix polynomial $W_{\alpha,A}$ be given via (E.1). Furthermore, let $J_q$ be given via (A.3). Let $z \in \mathbb{C}$. Then

$$ [W_{\alpha,A}(z)]^\dagger(-\tilde{J}_q)W_{\alpha,A}(z) = \text{diag}((z - \alpha)I_q, I_q)]^\dagger(-\tilde{J}_q)[\text{diag}((z - \alpha)I_q, I_q)] \quad \text{(E.4)} $$
and

\[
[\text{diag}((z - \alpha)I_q, I_q) \cdot W_{\alpha,A}(z)]^* (-\tilde{J}_q) \cdot [\text{diag}((z - \alpha)I_q, I_q) \cdot W_{\alpha,A}(z) ]
\]

\[
= |z - \alpha|^2 \left[ \left[ \text{diag}(AA^\dagger, I_q) \right]^*(-\tilde{J}_q) \cdot [\text{diag}(AA^\dagger, I_q) - 2 \text{Im}(z) \cdot \text{diag}(A^\dagger, 0_{q\times q})] \right.
\]

\[
+ \left[ \text{diag}((z - \alpha)^2(I_q - AA^\dagger), I_q) \right]^*(-\tilde{J}_q) \left[ \text{diag}((z - \alpha)^2(I_q - AA^\dagger), I_q) \right].
\]  
(E.5)

**Proof.** In view of Remark A.1, we have (E.3). Setting (A.1) and using $A^* = A$, Lemma A.2 yields $B^*(-\tilde{J}_q)B = -\tilde{J}_q$. Consequently, (E.4) follows. Let

\[
E(z) := [\text{diag}((z - \alpha)I_q, I_q) W_{\alpha,A}(z)]^*(-\tilde{J}_q)[\text{diag}((z - \alpha)I_q, I_q) W_{\alpha,A}(z)].
\]  
(E.6)

and let

\[
E(z) = \begin{bmatrix}
E_{11}(z) & E_{12}(z) \\
E_{21}(z) & E_{22}(z)
\end{bmatrix}
\]  
(E.7)

be the $q \times q$ block decomposition of $E(z)$. Obviously, we have

\[
E(z) = \begin{bmatrix}
\frac{i(z - \alpha)}{2} A^\dagger & \frac{i(z - \alpha)^2}{2} I_q \\
-i(I_q - A^\dagger A) & -(z - \alpha)A
\end{bmatrix}
\]  
(E.8)

By virtue of $A^* = A$, Remarks A.6 and A.9 yield $(A^\dagger)^* = A^\dagger$ and $AA^\dagger = A^\dagger A$, which implies $(I_q - AA^\dagger)A = (I_q - A^\dagger A)A = A - AA^\dagger A = 0_{q\times q}$. From (E.7) and (E.8) we conclude

\[
E_{11}(z) = i(z - \alpha)(z - \alpha)^2 A^\dagger - i(z - \alpha)^2 (z - \alpha) A^\dagger = i|z - \alpha|^2 (z - \alpha) - (z - \alpha)^2 A^\dagger,
\]

\[
E_{12}(z) = i|z - \alpha|^2 (z - \alpha) A^\dagger + i(z - \alpha)^2 (I_q - A^\dagger A) = i|z - \alpha|^2 AA^\dagger + i(z - \alpha)^2 (I_q - AA^\dagger),
\]

\[
E_{21}(z) = -i(z - \alpha)^2 (I_q - A^\dagger A) - i|z - \alpha|^2 A^\dagger = -i(z - \alpha)^2 (I_q - AA^\dagger) - i|z - \alpha|^2 AA^\dagger,
\]

and

\[
E_{22}(z) = -i(z - \alpha)(I_q - A^\dagger A) + i(z - \alpha) A (I_q - A^\dagger A) = 0_{q\times q}.
\]

Consequently, in view of (E.7), we conclude

\[
E(z) = \begin{bmatrix}
-0_{q\times q} & i|z - \alpha|^2 AA^\dagger \\
-i|z - \alpha|^2 AA^\dagger & -\text{diag}(2|z - \alpha|^2 \text{Im}(z)A^\dagger, 0_{q\times q})
\end{bmatrix} + \begin{bmatrix}
0_{q\times q} & i(z - \alpha)^2 (I_q - AA^\dagger) \\
-i(z - \alpha)^2 (I_q - AA^\dagger) & 0_{q\times q}
\end{bmatrix}
\]  
(E.9)

From Remark A.16 we get

\[
\begin{bmatrix}
0_{q\times q} & i|z - \alpha|^2 AA^\dagger \\
-i|z - \alpha|^2 AA^\dagger & 0_{q\times q}
\end{bmatrix} = |z - \alpha|^2 \left[ \text{diag}(AA^\dagger, I_q) \right]^*(-\tilde{J}_q) \left[ \text{diag}(AA^\dagger, I_q) \right]
\]  
(E.10)

and

\[
\begin{bmatrix}
0_{q\times q} & i(z - \alpha)^2 (I_q - AA^\dagger) \\
-i(z - \alpha)^2 (I_q - AA^\dagger) & 0_{q\times q}
\end{bmatrix} = \left[ \text{diag}((z - \alpha)^2(I_q - AA^\dagger), I_q) \right]^*(-\tilde{J}_q) \left[ \text{diag}((z - \alpha)^2(I_q - AA^\dagger), I_q) \right].
\]  
(E.11)

Using (E.6), (E.9), (E.10), and (E.11), we obtain (E.5).
Proposition E.7. Let $\alpha \in \mathbb{R}$. Further, let $A$ and $B$ be Hermitian complex $q \times q$ matrices such that $\mathcal{N}(A) \subseteq \mathcal{N}(B)$. For each $z \in \mathbb{C}$, then

$$\left[\text{diag}(A, A^\dagger) V_{A,B}(z)\right]^* (-\bar{J}_q) \left[\text{diag}(A, A^\dagger) V_{A,B}(z)\right]$$

$$= \left[\text{diag}\left((z - \alpha)B, B^\dagger\right)\right]^* (-\bar{J}_q) \left[\text{diag}\left((z - \alpha)B, B^\dagger\right)\right] + 2 \text{Im}(z) \text{diag}(0_{q \times q}, B)$$

and

$$\left[\text{diag}\left((z - \alpha)A, A^\dagger\right) V_{A,B}(z)\right]^* (-\bar{J}_q) \left[\text{diag}\left((z - \alpha)A, A^\dagger\right) V_{A,B}(z)\right]$$

$$= |z - \alpha|^2 \left[\text{diag}(B, B^\dagger)\right]^* (-\bar{J}_q) \left[\text{diag}(B, B^\dagger)\right].$$

Proof. Because of the assumption that the matrices $A$ and $B$ are Hermitian, we see from Remarks A.6 and A.9 that

$$A^\dagger = (A^\dagger)^*, \quad B^\dagger = (B^\dagger)^*, \quad AA^\dagger = A^\dagger A \quad \text{and} \quad BB^\dagger = B^\dagger B. \tag{E.12}$$

Since $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ is supposed, Remark A.6(c) brings

$$BA^\dagger A = B. \tag{E.13}$$

The application of the assumption $A^* = A$ and $B^* = B$, (E.12), and (E.13) yields

$$AA^\dagger B = A^*(A^\dagger)^* B^* = (BA^\dagger A)^* = B^* = B, \tag{E.14}$$

$$BAA^\dagger = BA^\dagger A = B, \quad \text{and} \quad A^\dagger AB = AA^\dagger B = B. \tag{E.15}$$

From (E.15) and (E.12) it follows

$$B^\dagger A^\dagger AB = B^\dagger B = BB^\dagger. \tag{E.16}$$

Taking into account (E.11), we infer

$$\text{diag}(A, A^\dagger) \cdot V_{A,B}(z) = \begin{bmatrix} 0_{q \times q} & -AB \\ (z - \alpha) A^\dagger B^\dagger & (z - \alpha) A^\dagger \end{bmatrix}. \tag{E.17}$$

Using (E.15), the assumption that $A$ and $B$ are Hermitian, and (E.12), we get

$$\left[\text{diag}(A, A^\dagger) \cdot V_{A,B}(z)\right]^* = \begin{bmatrix} 0_{q \times q} & (z - \alpha) B^\dagger A^\dagger \\ -BA & (z - \alpha) A^\dagger \end{bmatrix}. \tag{E.18}$$

In view of (E.14) and (E.15), we conclude

$$i(z - \alpha) A^\dagger AB - i(z - \alpha) BAA^\dagger = -i(z - \overline{\alpha}) B = 2 \text{Im}(z) B. \tag{E.19}$$

Because of Remark A.16 and Lemma A.17(1), we get

$$\begin{bmatrix} 0_{q \times q} & i(z - \alpha) BB^\dagger \\ -i(z - \alpha) BB^\dagger & 0_{q \times q} \end{bmatrix}$$

$$= \begin{bmatrix} \text{diag}\left((z - \alpha)B B^\dagger, I_q\right) \end{bmatrix}^* (-\bar{J}_q) \cdot \text{diag}\left((z - \alpha)B B^\dagger, I_q\right)$$

$$= \begin{bmatrix} \text{diag}\left((z - \alpha)B, B^\dagger\right) \end{bmatrix}^* (-\bar{J}_q) \cdot \text{diag}\left((z - \alpha)B, B^\dagger\right). \tag{E.20}$$
Combining (E.18), (E.17), (E.15), (E.16), (E.14), (E.19), and (E.20), we obtain
\[\left[\operatorname{diag}(A, A^\dagger) \cdot V_{\alpha,B}(z)\right]^* \left(-\tilde{J}_q\right) \left[\operatorname{diag}(A, A^\dagger) \cdot V_{\alpha,B}(z)\right] = \left[\begin{array}{c|c}
0_{q \times q} & \frac{(z - \alpha)B^\dagger A^\dagger}{(z - \alpha)A^\dagger} \\
-(z - \alpha)BA & 0_{q \times q}
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & iI_q \\
0_{q \times q} & 0_{q \times q}
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)A^\dagger B^\dagger \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right]
= \left[\begin{array}{c|c}
-i(z - \alpha)B^\dagger A^\dagger & 0_{q \times q} \\
-i(z - \alpha)A^\dagger & iBA 
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)A^\dagger B^\dagger \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right]
= \left[\begin{array}{c|c}
0_{q \times q} & i(z - \alpha)B^\dagger A^\dagger \\
-i(z - \alpha)B^\dagger & 0_{q \times q}
\end{array}\right] + 2 \operatorname{Im}(z) \cdot \operatorname{diag}(0_{q \times q}, B)
= \left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)A^\dagger B^\dagger \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right].
\] (E.21)

Using (E.17), we get
\[\operatorname{diag}\left((z - \alpha)A, A^\dagger\right) \cdot V_{\alpha,B}(z) = \operatorname{diag}\left((z - \alpha)I_q, I_q\right) \cdot \operatorname{diag}(A, A^\dagger) \cdot V_{\alpha,B}(z)
= \left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)AB \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right].
\] (E.22)

From Remark A.6\(^3\) and the assumption that A and B are Hermitian, we obtain \((A^\dagger)^* = A^\dagger\) and \((B^\dagger)^* = B^\dagger\). Thus, (E.22) implies
\[\left[\operatorname{diag}\left((z - \alpha)A, A^\dagger\right)V_{\alpha,B}(z)\right]^* = \left[\begin{array}{c|c}
0_{q \times q} & \frac{(z - \alpha)B^\dagger A^\dagger}{(z - \alpha)A^\dagger} \\
-(z - \alpha)BA & 0_{q \times q}
\end{array}\right].
\] (E.23)

By virtue of (E.13) and (E.14), we have
\[i(z - \alpha)^2 A^\dagger AB - i(z - \alpha)^2 BAA^\dagger = i(z - \alpha)^2 B - i(z - \alpha)^2 B = 0_{q \times q}.
\] (E.24)

Taking into account Remark A.16\(^2\) and Lemma A.17\(^3\), it follows
\[\left[\begin{array}{c|c}
0_{q \times q} & iBB^\dagger \\
-iBB^\dagger & 0_{q \times q}
\end{array}\right] = \left[\begin{array}{c|c}
0_{q \times q} & (z - \alpha)B^\dagger I_q \\
-(z - \alpha)BA & 0_{q \times q}
\end{array}\right] \left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)A^\dagger B^\dagger \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right]
= \left[\begin{array}{c|c}
\operatorname{diag}(B, B^\dagger) & (z - \alpha)I_q \\
(z - \alpha)I_q & \operatorname{diag}(B, B^\dagger)
\end{array}\right].
\] (E.25)

Applying (E.23), (E.22), (E.16), (E.24), (E.12), and (E.25), we get
\[\left[\operatorname{diag}\left((z - \alpha)A, A^\dagger\right) \cdot V_{\alpha,B}(z)\right]^* \left(-\tilde{J}_q\right) \left[\operatorname{diag}\left((z - \alpha)A, A^\dagger\right) \cdot V_{\alpha,B}(z)\right]
= \left[\begin{array}{c|c}
0_{q \times q} & \frac{(z - \alpha)B^\dagger A^\dagger}{(z - \alpha)A^\dagger} \\
-(z - \alpha)BA & 0_{q \times q}
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & iI_q \\
0_{q \times q} & 0_{q \times q}
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & -(z - \alpha)A^\dagger B^\dagger \\
(z - \alpha)A^\dagger B^\dagger & (z - \alpha)A^\dagger
\end{array}\right]
= \left[\begin{array}{c|c}
0_{q \times q} & i(z - \alpha)^2 B^\dagger A^\dagger \\
-(z - \alpha)^2 BAA^\dagger & 0_{q \times q}
\end{array}\right]
\left[\begin{array}{c|c}
0_{q \times q} & i(z - \alpha)^2 B^\dagger B^\dagger \\
-(z - \alpha)^2 BB^\dagger & 0_{q \times q}
\end{array}\right]
= |z - \alpha|^2 \left[\begin{array}{c|c}
0_{q \times q} & iBB^\dagger \\
-iBB^\dagger & 0_{q \times q}
\end{array}\right]
= |z - \alpha|^2 \left[\begin{array}{c|c}
\operatorname{diag}(B, B^\dagger) & (z - \alpha)I_q \\
(z - \alpha)I_q & \operatorname{diag}(B, B^\dagger)
\end{array}\right].
\] \(\square\)
Corollary E.8. Let $\alpha \in \mathbb{R}$ and let $B \in \mathbb{C}^{q \times q}$. For each $z \in \mathbb{C}$, then
\[
[V_{\alpha,B}(z)]^*(-\tilde{J}_q)|V_{\alpha,B}(z)|
= \left[\text{diag}\left((z-\alpha)B, B^\dagger\right)\right]^*(-\tilde{J}_q) \cdot \text{diag}\left((z-\alpha)B, B^\dagger\right) + 2\text{Im}(z) \cdot \text{diag}(0_{q \times q}, B)
\]
and
\[
[\text{diag}((z-\alpha)I_q, I_q) \cdot V_{\alpha,B}(z)]^*(-\tilde{J}_q)[\text{diag}((z-\alpha)I_q, I_q) \cdot V_{\alpha,B}(z)]
= |z-\alpha|^2\left[\text{diag}(B, B^\dagger)\right]^*(-\tilde{J}_q) \cdot \text{diag}(B, B^\dagger).
\]

Proof. From $\mathcal{N}(I_q) = \{0_{q \times 1}\} \subseteq \mathcal{N}(B)$ the assertion follows from Proposition E.7. \qed

Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)^m_{j=0}$ be a sequence of complex $p \times q$ matrices, and let $m \in \mathbb{Z}_{0,\kappa}$. For all $l \in \mathbb{Z}_{0,m}$, let $(s_j^{[l,\alpha]})_{j=0}^{m-l}$ be the $l$-th $\alpha$-S-transform of $(s_j)^m_{j=0}$. Let the sequence $(V_{\alpha_{s_0}, \alpha_{s_0}^{[m]}})_{j=0}^{m}$ be given via (E.1), let
\[
\mathfrak{H}^{[\alpha, (s_j)^m_{j=0}]} := V_{\alpha_{s_0}^{[0]}, \alpha_{s_0}^{[1]}} \cdots V_{\alpha_{s_0}^{[m-1]}, \alpha_{s_0}^{[m]}}, \tag{E.26}
\]
and let
\[
\mathfrak{M}^{[\alpha, (s_j)^m_{j=0}]} = \begin{bmatrix} v_{11}^{[\alpha, (s_j)^m_{j=0}]} & v_{12}^{[\alpha, (s_j)^m_{j=0}]} \\ v_{21}^{[\alpha, (s_j)^m_{j=0}]} & v_{22}^{[\alpha, (s_j)^m_{j=0}]} \end{bmatrix}, \tag{E.27}
\]
be the $q \times q$ block representation of $\mathfrak{H}^{[\alpha, (s_j)^m_{j=0}]}$ with $p \times p$ block $v_{11}^{[\alpha, (s_j)^m_{j=0}]}$. Furthermore, let the sequence $(W_{\alpha_{s_0}, \alpha_{s_0}^{[m]}})_{j=0}^{m}$ be given via (E.1), let
\[
\mathfrak{H}^{[\alpha, (s_j)^m_{j=0}]} := W_{\alpha_{s_0}^{[0]}, \alpha_{s_0}^{[1]}} \cdots W_{\alpha_{s_0}^{[m-1]}, \alpha_{s_0}^{[m]}}, \tag{E.28}
\]
and let
\[
\mathfrak{M}^{[\alpha, (s_j)^m_{j=0}]} = \begin{bmatrix} w_{11}^{[\alpha, (s_j)^m_{j=0}]} & w_{12}^{[\alpha, (s_j)^m_{j=0}]} \\ w_{21}^{[\alpha, (s_j)^m_{j=0}]} & w_{22}^{[\alpha, (s_j)^m_{j=0}]} \end{bmatrix},
\]
be the $q \times q$ block representation of $\mathfrak{H}^{[\alpha, (s_j)^m_{j=0}]}$ with $p \times p$ block $w_{11}^{[\alpha, (s_j)^m_{j=0}]}$.

Remark E.9. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{\infty\}$, and let $(s_j)^m_{j=0}$ be a sequence of complex $p \times q$ matrices. For all $m \in \mathbb{Z}_{1,\kappa}$ and all $l \in \mathbb{Z}_{0,m-1}$, one can see then from (E.26), (E.28), and \textbf{9} Remark 8.3 that
\[
\mathfrak{H}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-l}]} = \mathfrak{H}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]} V_{\alpha_{s_0}^{[m-l]}}, \quad \mathfrak{H}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-l}]} = V_{\alpha_{s_0}^{[l]}, \alpha_{s_0}^{[m-l]}} \mathfrak{H}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]},
\]
and
\[
\mathfrak{M}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-l}]} = \mathfrak{M}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]} W_{\alpha_{s_0}^{[m-l]}}, \quad \mathfrak{M}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-l}]} = W_{\alpha_{s_0}^{[l]}, \alpha_{s_0}^{[m-l]}} \mathfrak{M}^{[\alpha, (s_j^{[l,\alpha]})_{j=0}^{m-(l+1)}]},
\]
hold true where $t_j := s_j^{[l+1,\alpha]}$ for all $j \in \mathbb{Z}_{0,m-(l+1)}$. 70
Lemma E.10. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $p \times q$ matrices. Let $\mathcal{M}^{[\alpha[s_j]_{j=0}^m]}$ and $\mathcal{N}^{[\alpha[s_j]_{j=0}^m]}$ be given by (E.28) and (E.26), respectively. For each $z \in \mathbb{C}$, then

$$\mathcal{M}^{[\alpha[s_j]_{j=0}^m]}(z)\mathcal{N}^{[\alpha[s_j]_{j=0}^m]}(z) = (z - \alpha)^{m+1} \cdot \operatorname{diag}(s_0^{[m,\alpha]}(s_0^{[m,\alpha]})^\dagger, I_q). \quad (E.29)$$

Proof. Because of (E.28), (E.26), and Remark E.2, we have

$$\mathcal{M}^{[\alpha[s_j]_{j=0}^m]}(z)\mathcal{N}^{[\alpha[s_j]_{j=0}^m]}(z) = W_{\alpha,s_0^{[0,\alpha]}}(z) \mathcal{N}^{[\alpha,s_0^{[0,\alpha]}]}(z) = (z - \alpha)^{m} \operatorname{diag}(s_0^{[0,\alpha]}(s_0^{[0,\alpha]})^\dagger, I_q). \quad (E.30)$$

Hence, there is an $n \in \mathbb{N}$ such that (E.29) is fulfilled for each $m \in \mathbb{Z}_{0,n-1}$. Now we are going to prove that (E.29) is also true for $m = n$. In view of [9, Remark 8.5], we have $\mathcal{R}(s_0^{[n,\alpha]}) \subseteq \mathcal{R}(s_0^{[n-1,\alpha]})$. Thus, Remark A.6 yields $s_0^{[n-1,\alpha]}(s_0^{[n-1,\alpha]})^\dagger s_0^{[n,\alpha]} = s_0^{[n,\alpha]}$. Using this and additionally Remark E.9, (E.1), and (E.30), we obtain then

$$\mathcal{M}^{[\alpha[s_j]_{j=0}^m]}(z)\mathcal{N}^{[\alpha[s_j]_{j=0}^m]}(z) = W_{\alpha,s_0^{[n,\alpha]}}(z) \mathcal{M}^{[\alpha[s_j]_{j=0}^{n-1}]}(z)\mathcal{N}^{[\alpha[s_j]_{j=0}^{n-1}]}(z) \mathcal{N}^{[\alpha,s_0^{[n,\alpha]}]}(z) \mathcal{N}^{[\alpha,s_0^{[n,\alpha]}]}(z)
= \begin{bmatrix} (z - \alpha)I_p & 0_{p \times q} \\ - (z - \alpha)(s_0^{[n,\alpha]})^\dagger I_q - (s_0^{[n,\alpha]})^\dagger s_0^{[n,\alpha]} \end{bmatrix} \left( (z - \alpha)^n \operatorname{diag}(s_0^{[n-1,\alpha]}(s_0^{[n-1,\alpha]})^\dagger, I_q) \right) \times \begin{bmatrix} I_p & 0_{p \times q} \\ - (z - \alpha)(s_0^{[n,\alpha]})^\dagger I_q - (s_0^{[n,\alpha]})^\dagger s_0^{[n,\alpha]} \end{bmatrix}
= (z - \alpha)^{n+1} \operatorname{diag}(s_0^{[n,\alpha]}(s_0^{[n,\alpha]})^\dagger, I_q).$$

Thus, the assertion is proved inductively. \qed

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