Abstract

The framed $n$-discs operad $fD_n$ is studied as semidirect product of $SO(n)$ and the little $n$-discs operad. Our equivariant recognition principle says that a grouplike space acted on by $fD_n$ is equivalent to the $n$-fold loop space on a $SO(n)$-space. Examples of $fD_2$-spaces are nerves of ribbon braided monoidal categories. We compute the rational homology of $fD_n$. Koszul duality for semidirect product operads of chain complexes is defined and applied to compute the double loop space homology as BV-algebra.

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1 Introduction

The topology of iterated loop spaces was thoroughly investigated in the seventies [18, 16, 24]. These spaces have a wealth of homology operations parametrised by the famous operads of little discs. The notion of operad was...
introduced for the first time for this purpose \[1, 18\]. Such machinery allows for example to reconstruct an iterated delooping if one has full knowledge of the operad action on an iterated loop space. Moreover any connected space acted on by the little discs is homotopy equivalent to an iterated loop space. This fact is the celebrated recognition principle.

Our main objective is to extend this theory by adding the operations rotating the discs. The operad generated by the little \(n\)-discs \(D_n\) and the rotations in \(SO(n)\) is called the framed \(n\)-discs operad, and was introduced in \[7\]. Our equivariant recognition principle (theorem 3.1) says that a connected (or grouplike) space acted on by the framed \(n\)-discs operad is weakly homotopic to the \(n\)-fold loop space on an \(SO(n)\)-space.

Thus the looping and delooping functors induce a categorical equivalence between \(SO(n)\)-spaces and spaces acted on by the framed \(n\)-discs operad, under the correct connectivity assumptions. The main technique consists in presenting the framed little discs as a semidirect product of the little discs and the special orthogonal group, so that an algebra over the framed \(n\)-discs is nothing else than an algebra over the original little \(n\)-discs in the category of \(SO(n)\)-spaces.

The principle generalises to any representation \(G \rightarrow O(n)\) of a topological group. If \(G\) is trivial we recover the original recognition principle by May.

Is there a way of producing spaces acted on by the framed discs operad from category theory? Fiedorowicz showed that the nerve of a braided monoidal category is equivalent to an algebra over the little 2-discs \[5\]. We extend this fact to ribbon braided monoidal categories, which are braided categories equipped with a “twist”, and the framed little 2-discs operad (theorem 4.11).

Is there a procedure to detect framed little 2-discs operad up to homotopy? Fiedorowicz \[4\] has a procedure to recognise topological operads weakly equivalent to the little 2-discs. We extend his work to the twisted case giving a procedure to detect whether an operad is weakly equivalent to the framed 2-discs (Theorem 4.3).

We investigate next the homology of spaces acted on by the framed discs operad. In the category of chain complexes, in analogy with the topological situation, we define the semidirect product operad of a Hopf algebra \(H\) and an operad acted on by \(H\). An algebra over the semidirect product will be exactly an algebra over the original operad in the monoidal category of \(H\)-modules. The homology functor with coefficients in a field commutes with the semidirect product construction. This approach yields a conceptual proof that the rational homology of an algebra over the framed 2-discs is a Batalin-Vilkovisky algebra \[7\], and computes more generally the rational homology.
of the framed \(n\)-discs operad for any \(n\) (Theorem 5.3).

What is the dual of a semidirect product operad of chain complexes? We redefine the concept of Koszul duality for semidirect products, in contrast to [9], so that the dual is still naturally a semidirect product. This notion induces equivalence of derived categories over dual operads. Moreover a Koszul operad remains such after taking semidirect products. For example the BV-operad is Koszul self-dual up to a shift. This is based upon the analogous result for Gerstenhaber algebras [8].

As application we explain how to compute the rational homology of a double loop space on a \(S^1\)-space \(X\) as a BV-algebra, starting from a minimal model of the \(S^1\)-space together with the derivation induced by the action (Theorem 7.6). This extends results in [7], where \(X\) is a double suspension, and [8], where only the Gerstenhaber algebra structure is considered.

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\section{Semidirect product of topological operads}

We work in the category \(\mathbf{Top}\) of compactly generated weak Hausdorff topological spaces. Let \(G\) be a topological group. The category of left \(G\)-spaces, denoted \(\mathbf{G-Top}\), is a symmetric monoidal category by the cartesian product. We can thus consider operads in this category, which we call \(G\)-operads.

Let \(\mathcal{A}\) be a \(G\)-operad. So \(\mathcal{A}\) consists of a sequence of \(G\)-spaces \(\mathcal{A}(k)\) for \(k \in \mathbb{N}\), with \(G\)-equivariant operad structure maps and symmetric group actions. Note that the unit \(1 \in \mathcal{A}(1)\) is also preserved by the \(G\)-action.

We will denote the action of an element \(g \in G\) on an element \(a \in \mathcal{A}(k)\) by \(ga\).

\textbf{Definition 2.1.} Let \(\mathcal{A}\) be a \(G\)-operad. Define \(\mathcal{A} \times G\), the semidirect product of \(\mathcal{A}\) and \(G\), to be the following operad in \(\mathbf{Top}\) : for \(k \in \mathbb{N}\),

\[(\mathcal{A} \times G)(k) = \mathcal{A}(k) \times G^k\]

with \(\Sigma_k\) acting diagonally on the right, permuting the components of \(G^k\) and acting on \(\mathcal{A}(k)\), and the map

\[
\gamma : (\mathcal{A} \times G)(k) \times (\mathcal{A} \times G)(n_1) \times \cdots \times (\mathcal{A} \times G)(n_k) \longrightarrow (\mathcal{A} \times G)(n_1 + \cdots + n_k)
\]

given by

\[
\gamma((a, g), (b_1, h_1^1), \ldots, (b_k, h_k^k)) = (\gamma_\mathcal{A}(a, g_1 b_1, \ldots, g_k b_k), g_1 h_1^1, \ldots, g_k h_k^k),
\]

where \(h_i^j = (h_{i1}^j, \ldots, h_{it_i}^j)\) and \(g_i h_i^j = (g_i h_{i1}^j, \ldots, g_i h_{it_i}^j)\). The unit in \(\mathcal{A} \times G(1)\) is \((1, e)\), formed of the units of \(\mathcal{A}\) and \(G\).
The $G$-equivariance of $\gamma_A$ is necessary for the associativity of the structure map of the semidirect product operad.

Example 2.2. The example we have in mind is the framed discs operad $f\mathcal{D}_n$. Let $\mathcal{D}_n$ be the little $n$-discs operad of Boardman and Vogt. Hence $\mathcal{D}_n(k)$ is the space of embeddings $\coprod_k D^n \to D^n$ of $k$ copies of the unit $n$-disc to itself such that the maps are compositions of positive dilations and translations, and the images are disjoint. The framed discs, $f\mathcal{D}_n$, is defined similarly but one allows rotations for the embeddings. As spaces, $f\mathcal{D}_n(k) = \mathcal{D}_n(k) \times (SO(n))^k$, the $i$-th element of $SO(n)$ encoding the rotation of the $i$-th disc. In fact, $\mathcal{D}_n$ is an $SO(n)$-operad and $f\mathcal{D}_n$ is a semidirect product in the above sense:

$$f\mathcal{D}_n = \mathcal{D}_n \rtimes SO(n).$$

The action of $SO(n)$ on $\mathcal{D}_n(k)$ rotates the little discs around their center. Note that the whole orthogonal group $O(n)$ acts on $\mathcal{D}_n(k)$ in such a way that $\mathcal{D}_n \rtimes O(n)$ is well-defined. We will consider semidirect products of $\mathcal{D}_n$ with any topological group $G$ equipped with a continuous homomorphism $\phi : G \to O(n)$. We will suppress $\phi$ from the notation and denote the resulting semidirect product by $\mathcal{D}_n \rtimes G$.

Proposition 2.3. Let $A$ and $G$ be as in definition 2.1. A space $X$ is an $(A \rtimes G)$-algebra if and only if $X$ is an $A$-algebra in the category of $G$-spaces, i.e. $X$ admits a $G$-action and $A$-algebra structure maps $\theta_A : A(k) \times X^k \to X$ satisfying $g(\theta_A(a, x_1, \ldots, x_k)) = \theta_A(ga, gx_1, \ldots, gx_k)$.

In this case $\theta_{A \rtimes G}((a, (g_1, \ldots, g_k)), x_1, \ldots, x_k) = \theta_A(a, g_1x_1, \ldots, g_kx_k)$.

As an immediate consequence we have

Corollary 2.4. Let $X, Y$ be two $(A \rtimes G)$-algebras. A map $f : X \to Y$ is a map of $(A \rtimes G)$-algebras if and only if it is an $A$-algebra map and a $G$-map.
We will use the following examples of framed algebras:

**Example 2.5.** Let $Y$ be a pointed $G$-space and let $D_n$ denote the monad associated to the operad $\mathcal{D}_n$ ([18] construction 2.4). $D_n Y$ is the free $\mathcal{D}_n$-algebra on the pointed space $Y$. Let $\Omega^n Y$ denote the based $n$-fold loop space on $Y$, seen as the space of maps from the unit $n$-disc $D^n$ to $Y$ sending the boundary to the base point. The space $\Omega^n Y$ carries a natural $\mathcal{D}_n$-algebra structure [18].

Let $\phi: G \to O(n)$ be a continuous group homomorphism. The spaces $D_n Y$ and $\Omega^n Y$ are $\mathcal{D}_n \rtimes G$-algebras, with the action of $g \in G$

on $[c; y_1, \ldots, y_k] \in D_n Y$, where $c \in D_n(k)$, $y_i \in Y$, given by

$$g[c; y_1, \ldots, y_k] = [\phi(g)c; gy_1, \ldots, gy_k],$$

and on $[y(t)] \in \Omega^n Y$, where $t \in D^n$ and $[y(t)]$ denotes the n-fold loop $t \mapsto y(t)$, given by

$$g[y(t)] = [gy(\phi(g)^{-1}(t))].$$

3 EQUIVARIANT RECOGNITION PRINCIPLE

Let $\phi: G \to O(n)$ be as above and let $X$ be a grouplike $\mathcal{D}_n \rtimes G$-algebra, i.e. the components of $X$ form a group by the product induced by any element in $\mathcal{D}_n(2)$. As $X$ is a $\mathcal{D}_n \rtimes G$-algebra, it is in particular a $\mathcal{D}_n$-algebra.

May introduced a deloop functor $B_n$ from $\mathcal{D}_n$-algebras to pointed spaces defined by $B_n X := B(\Sigma^n, D_n, X)$, where $B$ is the double bar construction ([18] construction 9.6), and $\Sigma$ the (reduced) suspension. May’s recognition principle [18, 3] says that $X$ is weakly equivalent to $\Omega^n B_n X$ as $\mathcal{D}_n$-algebra. May also showed that, conversely, $B_n$ applied to an n-fold loop space $\Omega^n Y$ produces a space weakly homotopy equivalent to $Y$.

In what follows, we consider the behaviour of $\Omega^n$ and $B_n$ with respect to $G$-actions and provide a recognition principle for algebras over $\mathcal{D}_n \rtimes G$.

Let $\mathcal{D}_n \rtimes G - \text{Top}_{gl}$, $\mathcal{D}_n \rtimes G - \text{Top}_0$ and $G - \text{Top}^*_n$ be the categories of grouplike, connected $\mathcal{D}_n \rtimes G$-algebras and $n$-connected pointed $G$-spaces respectively. Those three categories are closed model categories with weak homotopy equivalences as weak equivalences [22]. For a model category $\mathcal{C}$, we will denote $\text{Ho}(\mathcal{C})$ its associated homotopy category, obtained by inverting the weak equivalences.

For any $G$-space $Y$, we have seen in example 2.3 that $\Omega^n Y$ has a $\mathcal{D}_n \rtimes G$-algebra structure induced by the diagonal action of $G$. On the other hand,
we will define a $G$-action on $B_nX$ for any $D_n \rtimes G$-algebra $X$. Hence, $\Omega^n$ and $B_n$ will be functors between the categories of pointed $G$-spaces and of $D_n \rtimes G$-algebras.

**Theorem 3.1.** For each continuous homomorphism $\phi : G \rightarrow O(n)$, we have functors

$$\Omega^n_\phi = \Omega^n : G - \text{Top}_{n-1}^* \longrightarrow D_n \rtimes G - \text{Top}_{gl}$$

$$B^n_\phi = B_n : D_n \rtimes G - \text{Top}_{gl} \longrightarrow G - \text{Top}_{n-1}^*$$

which induce an equivalence of homotopy categories

$$Ho(G - \text{Top}_{n-1}^*) \simeq Ho(D_n \rtimes G - \text{Top}_{gl}).$$

This equivalence restricts to

$$Ho(G - \text{Top}_n^*) \simeq Ho(D_n \rtimes G - \text{Top}_0).$$

**Proof.** May’s recognition principle ([18] Theorem 13.1) is obtained through the following maps:

$$X \leftarrow B(D_n, D_n, X) \xrightarrow{\alpha} B(\Omega^n \Sigma^n, D_n, X) \longrightarrow \Omega^n B(\Sigma^n, D_n, X) = \Omega^n B_n X,$$

where all maps are $D_n$-maps between $D_n$-spaces. When $X$ is a $D_n \rtimes G$-algebra, we want to define $G$-actions on the spaces involved which induce $D_n \rtimes G$-algebra structures and such that all maps are $G$-maps.

The functors $D_n$, $\Sigma^n$ and $\Omega^n$ restrict to functors in the category of $G$-spaces, where, for any $G$-space $Y$, we define the action on $D_n Y$, $\Sigma^n Y$ and $\Omega^n Y$ diagonally as in example 2.5. Hence for any $G$-space $Y$ the $G$-action on $\Omega^n \Sigma^n Y$ is given by

$$g[\sigma(t), y(t)] = [\phi(g)\sigma(\phi(g)^{-1}t), gy(\phi(g)^{-1}t)],$$

where $g \in G$, $t, \sigma(t) \in D^n$ and $y(t) \in Y$. This produces a $D_n \rtimes G$-algebra structure on $\Omega^n \Sigma^n Y$ such that May’s map $\alpha : D_n Y \longrightarrow \Omega^n \Sigma^n Y$ is a $G$-map, and thus a $D_n \rtimes G$-map.

We extend now these actions on the simplicial spaces $B(D_n, D_n, X)$, $B(\Omega^n \Sigma^n, D_n, X)$ and $\Omega^n B(\Sigma^n, D_n, X)$.

Recall that the double bar construction $B(F, C, X)$ is defined simplicially, for a monad $C$, a left $C$-functor $F$ and a $C$-algebra $X$ by $B(F, C, X) = |B_*(F, C, X)|$, where $B_p(F, C, X) = FC^pX$, with boundary and degeneracy maps using the left functor, monad and algebra structure maps. The group
acts then on $B_p(F, C, X)$ through its action on the functors $F$ and $C$, which comes to “rotate everything”. For example, the action of $g \in G$ on a 1-simplex of $B(\Omega^n \Sigma^n, D_n, X)$ is given by $g[\sigma(t), c(t), x_1(t), \ldots, x_k(t)]$

\[= \left[\phi(g)\sigma(\phi(g)^{-1}t), \phi(g)c(\phi(g)^{-1}t), gx_1(\phi(g)^{-1}t), \ldots, gx_k(\phi(g)^{-1}t)\right].\]

With these actions, all maps above are $G$-maps between $D_n \rtimes G$-spaces and $B_n X$ is equipped with an explicit $G$-action.

On the other hand, we have a weak homotopy equivalence [18, 3]

\[B_n \Omega^n Y = B(\Sigma^n, D_n, \Omega^n Y) \xrightarrow{[\delta_n^0]} \Sigma^n \Omega^n Y \xrightarrow{e} Y\]

for any $(n - 1)$-connected space $Y$. If $Y$ is a $G$-space, then this composite is a $G$-map with the actions on $B_n \Omega^n Y$ and $\Sigma^n \Omega^n Y$ defined as above.

4 Ribbon braid categories and operads

We explain in this section how braid groups and ribbon braid groups are related to the little 2-discs and framed 2-discs operads respectively. The braid case was done by Fiedorowicz [5, 6]. Details about the ribbon case can be found in [28, 29].

We will denote by $\beta_k$ the braid group on $k$ strings, the fundamental group of the configuration space of $k$ unordered particles in $\mathbb{R}^2$. There is a natural surjection $\beta_k \twoheadrightarrow \Sigma_k$, sending a braid to the induced permutation of the ends. The pure braid group $P\beta_k$ is the kernel of this surjection.

We will denote by $R\beta_k$ the ribbon braid group on $k$ elements, the fundamental group of the configuration space of $k$ unordered particles in $\mathbb{R}^2$ with label in $S^1$. One can think of an element of $R\beta_k$ as a braid on $k$ ribbons, where full twists of the ribbons are allowed.

The pure ribbon braid group $PR\beta_k$ is the kernel of the surjection $R\beta_k \twoheadrightarrow \Sigma_k$.

The groups $R\beta_k$ and $PR\beta_k$ are isomorphic to $\beta_k \rtimes \mathbb{Z}^k$ and $P\beta_k \rtimes \mathbb{Z}^k$ respectively, where $\mathbb{Z}^k$ encodes the number of twists on each ribbon.

We want to characterise operads equivalent to the framed 2-discs. We consider the following notion of equivalence:

**Definition 4.1.** An operad map $\mathcal{A} \rightarrow \mathcal{B}$ is an equivalence if each map $\mathcal{A}(k) \rightarrow \mathcal{B}(k)$ is a $\Sigma_k$-equivariant homotopy equivalence.

An operad $\mathcal{A}$ is a $E_n$-operad (resp. $fE_n$-operad) if there is a chain of equivalences connecting $\mathcal{A}$ to $D_n$ (resp. $fD_n$).

Z. Fiedorowicz gave a recognition principle for $E_2$-operads. It requires the introduction of “braid operads”, which resemble operads except that the
symmetric group actions are replaced by braid group actions in a natural way. More precisely, a collection of spaces \( \mathcal{A} = \{ \mathcal{A}(k) \} \) is a braid operad if \( \beta_k \) acts on \( \mathcal{A}(k) \) for each \( k \) and if there are associative structure maps

\[
\gamma : \mathcal{A}(k) \times \mathcal{A}(n_1) \times \cdots \times \mathcal{A}(n_k) \to \mathcal{A}(n_1 + \cdots + n_k)
\]

with two-sided unit \( e \in \mathcal{A}(1) \), satisfying the equivariance conditions

\[
\gamma(a^\sigma, b_1, \ldots, b_k) = \gamma(a, b_{[\sigma]^{-1}(1)}, \ldots, b_{[\sigma]^{-1}(k)})^{\sigma(n_1, \ldots, n_k)}
\]

and

\[
\gamma(a, b_1^{\tau_1}, \ldots, b_k^{\tau_k}) = \gamma(a, b_1, \ldots, b_k)^{([\tau_1] \oplus \cdots \oplus \tau_k)}
\]

for all \( a \in \mathcal{A}(k), b_i \in \mathcal{A}(n_i), \sigma \in \beta_k, \tau_i \in \beta_{n_i} \), where \([\sigma]\) is the permutation induced by \( \sigma \), the braid \( \sigma(n_1, \ldots, n_k) \) on \( n_1 + \cdots + n_k \) strings is obtained from \( \sigma \) by replacing the \( i \)th string by \( n_i \) strings, and \( (\tau_1 \oplus \cdots \oplus \tau_k) \) is the block sum of the braids \( \tau_1, \ldots, \tau_k \).

A braid operad \( \mathcal{A} \) is called a \( B_\infty \) operad if each \( \mathcal{A}(k) \) is contractible and is acted on freely by the braid group \( \beta_k \).

**Theorem 4.2.** [6] An operad \( \mathcal{A} \) is an \( E_2 \) operad if and only if its operad structure lifts to a \( B_\infty \) operad structure on its universal cover \( \tilde{\mathcal{A}} \).

**Sketch of the proof.** Z. Fiedorowicz constructed in [4] a lift of the operad structure of \( \mathcal{D}_2 \) to a \( B_\infty \) structure on its universal cover \( \tilde{\mathcal{D}}_2 \). The difficulty is that there is no consistent choice of base-points to lift the map \( \gamma \). Fiedorowicz uses the inclusion of the unordered little intervals in the little discs \( \mathcal{D}^u_1 \), seeing \( \mathcal{D}^u_1(k) \) as the component of \( \mathcal{D}_1(k) \) with the intervals ordered in the canonical way, from left to right. This provides contractible subspaces \( \mathcal{D}^u_1(k) \subset \mathcal{D}_2(k) \), which will play the role of base-points for the lifting.

To lift the operad maps for any \( E_2 \) space, one uses the cofibrant resolution of \( \mathcal{D}_2 \), the operad \( W\mathcal{D}_2 \) constructed by Boardman and Vogt in [4] (see also [12]). For any \( E_2 \) operad \( \mathcal{A} \), they show that there is an equivalence of operads \( W\mathcal{D}_2 \cong \mathcal{A} \). We can then use the inclusion \( W\mathcal{D}_1 \hookrightarrow W\mathcal{D}_2 \) to produce a \( B_\infty \) structure on \( \tilde{\mathcal{A}} \).

For the converse, one first notes that the product of two \( B_\infty \) operads is again a \( B_\infty \) operad. Now, using the fact that, as for \( E_\infty \) operads, a braid operad map between two \( B_\infty \) operads is always an equivalence, the theorem is proved using the maps

\[
\begin{array}{c}
\tilde{\mathcal{A}} \cong \tilde{\mathcal{A}} \times \mathcal{D}_2 \cong \mathcal{D}_2 \\
A \cong (\tilde{\mathcal{A}} \times \mathcal{D}_2)/P\beta_e \cong \mathcal{D}_2.
\end{array}
\]
One can define ribbon operads and $R_\infty$ operads simply by replacing the braid groups by ribbon braid groups in the definitions of braid and $B_\infty$ operads. As in the braid case, we have that all $R_\infty$ operads are equivalent.

As we have $D_1 \hookrightarrow D_2 \hookrightarrow fD_2$, one can again use the little intervals to lift the operad maps of $fD_2$ to its universal cover $\tilde{fD}_2$, obtaining then an $R_\infty$ operad structure, and similarly, for any $fE_2$-operad, using the $W$ construction.

Adapting the proof of theorem 4.2 we obtain:

**Theorem 4.3.** An operad $A$ is an $fE_2$ operad if and only if its operad structure lifts to an $R_\infty$ operad structure on its universal cover $\tilde{A}$.

Here are our main examples. There is a general method to construct categorical operads from certain families of groups [26, 28].

The braid groups give rise this way to an operad $B$, where $B(k)$ is the category with set of objects $\beta_k/P\beta_k = \Sigma_k$ and morphisms $\text{Hom}_B(\sigma P\beta_k, \tau P\beta_k) \cong P\beta_k$ are by left multiplication

$$\tau P\beta_k \xleftarrow{\tau h\sigma^{-1}} \sigma P\beta_k,$$

where $h \in P\beta_k$ and $\sigma, \tau \in \beta_k$. The operad structure maps are defined on object by the usual maps on symmetric groups (from the associative operad) and on morphisms by

$$\gamma(\sigma_1 \xleftarrow{\tau} \sigma_0, \rho_1, \ldots, \rho_k) = \tau(n_{\rho_1^{-1}(1)}, \ldots, n_{\rho_k^{-1}(k)})((\rho_{\rho_1^{-1}(1)} \oplus \cdots \oplus \rho_{\rho_k^{-1}(k)})\sigma_0),$$

where the right hand side is defined as in the definition of braid operads above.

**Proposition 4.4.** The nerve construction gives an $E_2$-operad $|B|$.

**Proof.** Let $\tilde{B}(k)$ be the translation category of $\beta_k$, having $\beta_k$ as set of objects and $\text{Mor}_{\tilde{B}(k)}(\sigma, \tau) = \{ \tau\sigma^{-1} \}$. There is then an obvious “projection” functor $\tilde{B} \to B$ which induces a covering map on the nerves. As $|\tilde{B}|$ is contractible, it is in fact the universal cover of $|B|$. Also, the operad structure of $|B|$ lifts to a natural braid operad structure on $|\tilde{B}|$, which is a $B_\infty$ operad. We conclude by theorem 4.2.

Similarly, the ribbon braid groups give rise to a categorical operad $R$, where the category $R(k)$ has set of objects $\Sigma_k$ and morphisms sets equivalent to $\text{PR}\beta_k$.  

\[\square\]
Proposition 4.5. The nerve of $R$ yields an $fE_2$-operad $|R|$.  

The proof is similar and uses theorem 4.3. Here the universal cover is the nerve of the translation categories of the groups $R\beta_k$.  

We note that $|R|$ is the semidirect product $|B| \rtimes B\mathbb{Z}$.  

We will describe $B$- and $R$-algebras.  

Definition 4.6. A braided monoidal category is a monoidal category $(A, \otimes)$ equipped with a braiding, i.e. a natural family of isomorphisms  

$$c = c_{A,B} : A \otimes B \to B \otimes A$$  

satisfying the “braid relations” (see figure 2):  

$$(id \otimes c_{A,C}) \circ a \circ (c_{A,B} \otimes id) = a \circ c \circ a : (A \otimes B) \otimes C \to B \otimes (C \otimes A),$$  

and  

$$(c_{A,C} \otimes id) \circ a^{-1} \circ (id \otimes c_{B,C}) = a^{-1} \circ c \circ a^{-1} : A \otimes (B \otimes C) \to (C \otimes A) \otimes B,$$  

where $a$ denotes the associativity isomorphism.  

It is called a braided strict monoidal category if the monoidal structure is strict, i.e. if the associativity and unit isomorphisms are given by the identity.  

![Figure 2: braiding and braid relations](image)

Braided monoidal categories arise in the theory of quantum groups and their associated link invariants.  

Proposition 4.7. A braided strict monoidal category is exactly a $B$-algebra.  

If $A$ is a $B$-algebra, there are functors $\theta_k : B(k) \times A^k \to A$. The product in $A$ is defined on objects by $A \otimes B = \theta_2(id_{\Sigma_2}, A, B)$ and on morphisms by $f \otimes g = \theta_2(id_{\beta_2}, f, g)$, while the braiding is given by $c_{A,B} = \theta_2(b, id_A, id_B)$, where $b$ is the generator of $\beta_2$ (see figure 2).  

The recognition principles of Fiedorowicz and May lead to the following theorem:
Theorem 4.8. [5] After group completion the nerve of a braided monoidal category is weakly homotopy equivalent to a double loop space.

Sketch of the proof. Let $\mathcal{C}$ be a braided monoidal category. Then $\mathcal{C}$ is equivalent to a braided strict monoidal category $\mathcal{C}'$. Now $|\mathcal{C}'|$ is a $|\mathcal{B}|$-algebra. Define $X$ to be the double bar construction $B(D_2, \tilde{D}_2 \times_{\mathcal{B}} \tilde{B}, |\mathcal{C}'|)$. The space $X$ is a $D_2$-algebra weakly equivalent to $|\mathcal{C}'|$. Finally, the recognition principle tells us that the group completion of $X$ is equivalent to a double loop space. □

Definition 4.9. [11] A ribbon braided (strict) monoidal category $(\mathcal{A}, \otimes, c, \tau)$ is a braided (strict) monoidal category $(\mathcal{A}, \otimes, c)$ equipped with a twist, i.e. a natural family of isomorphisms

$$\tau = \tau_A : A \to A$$

such that $\tau_1 = \text{id}_1$, where $1$ is the unit object of $\mathcal{A}$, and satisfying the following compatibility with the braiding: $\tau_{A \otimes B} = c_{B,A} \circ \tau_B \otimes \tau_A \circ c_{A,B} : A \otimes B \to A \otimes B$ (see figure 3).

![Figure 3: compatibility between the twist and the braiding](image)

Proposition 4.10. [28] An $R$-algebra is exactly a ribbon braided strict monoidal category.

For an $R$-algebra $\mathcal{A}$, the braided monoidal structure is defined as for $B$-algebras. The twist in $\mathcal{A}$ is defined by $\tau_A = \theta_1(t, \text{id}_A)$, where $t$ is the generator of $R\beta_1$. 

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Consider the monoid $\mathbb{R} \times \mathbb{Z} E \mathbb{Z} \subset f \mathcal{D}_2(1) \times \mathcal{P}_R | \mathcal{R} |(1)$. There are monoid maps

$$S^1 \cong (\mathbb{R} \times \mathbb{Z} \ast) \xrightarrow{\sim} \mathbb{R} \times \mathbb{Z} E \mathbb{Z} \xrightarrow{\sim} (\ast \times \mathbb{Z} E \mathbb{Z}) \cong B \mathbb{Z}.$$ 

So any $S^1$-space or $B \mathbb{Z}$-space is canonically an $\mathbb{R} \times \mathbb{Z} E \mathbb{Z}$-space. The above maps are restrictions of the operad maps $\mathcal{F}_2 \leftrightarrow \mathcal{F}_2 \times \mathcal{P}_R | \mathcal{R} | \xrightarrow{\sim} | \mathcal{R} |$ in arity 1. Using our recognition principle (theorem 3.1) and theorem 4.3, we obtain the following:

**Theorem 4.11.** The nerve of a ribbon braided monoidal category $\mathcal{C}$, after group completion, is weakly homotopy equivalent to a double loop space $\Omega^2 Y$. Moreover, the twist on $\mathcal{C}$ induces an $S^1$-action on $Y$ in such a way that the equivalence is $\mathbb{R} \times \mathbb{Z} E \mathbb{Z}$-equivariant.

**Proof.** Let $\mathcal{C}$ be a ribbon braided monoidal category and let $\mathcal{C}'$ be the strictification of $\mathcal{C}$ as a monoidal category. The category $\mathcal{C}'$ then inherits a ribbon braided structure from the one existing on $\mathcal{C}$. Its nerve $|\mathcal{C}'|$ is an $|\mathcal{R}|$-algebra. The space $|\mathcal{C}|$ is not necessarily an $|\mathcal{R}|$-algebra, but it admits a $B \mathbb{Z}$-action induced by the twist on $\mathcal{C}$, and the equivalence $|\mathcal{C}| \xrightarrow{\sim} |\mathcal{C}'|$ is $B \mathbb{Z}$-equivariant.

Now the space $X = B(f \mathcal{D}_2, f \mathcal{D}_2 \times \mathcal{P}_R | \mathcal{R} |, |\mathcal{C}'|)$ is weakly homotopy equivalent to $|\mathcal{C}'|$ and is an $f \mathcal{D}_2$-algebra. The equivalence is obtained through the following diagram of weak equivalences in $\mathbb{R} \times \mathbb{Z} E \mathbb{Z}^{-} \text{Top}$.

$$B(f \mathcal{D}_2, f \mathcal{D}_2 \times \mathcal{P}_R | \mathcal{R} |, |\mathcal{C}'|) \xleftarrow{\sim} B(f \mathcal{D}_2 \times \mathcal{P}_R | \mathcal{R} |, f \mathcal{D}_2 \times \mathcal{P}_R | \mathcal{R} |, |\mathcal{C}'|)$$

The group completion of $X$ is then equivalent to a double loop space $\Omega^2 Y$, where $Y = B(\Sigma^2, D_2, X)$ and the $S^1$-action on $X$ now induces one on $Y$, as explained in theorem 3.1.

5 Semidirect products of algebraic operads

We work from now on in the category of chain complexes over a field $k$ (possibly with trivial differential). For an element $x$ of a chain complex, we denote by $|x|$ its degree. We call operads in this category differential graded operads, or $dg$-operads.

Let $H$ be a graded associative cocommutative Hopf algebra over a field $k$. The tensor product of two $H$-modules inherits an $H$-structure which is
induced by the coproduct of $H$. As $H$ is cocommutative, the category of differential graded $H$-modules, denoted $H$-$\text{Mod}$, is a symmetric monoidal category with product the ordinary tensor product. Hence it makes sense to consider operads and their algebras in this category. We call such operads $dg$-operads of $H$-modules.

As in the topological case, we can construct semidirect products for these operads.

**Proposition 5.1.** Let $P$ be an operad of $H$-modules. There exists a differential graded operad, the semidirect product $P \rtimes H$, such that algebras over $P$ in the category of $H$-modules are exactly $P \rtimes H$-algebras.

The operad is defined by $(P \rtimes H)(n) = P(n) \otimes H^\otimes n$. The structure maps are defined similarly to the topological case, using the comultiplication $c$ of $H$ and using interchanging homomorphisms with appropriate signs.

Homology provides a bridge from the topological to the algebraic setting:

**Proposition 5.2.** Let $G$ be a topological group acting on a topological operad $\mathcal{A}$.

There is a natural isomorphism of operads $H(\mathcal{A} \rtimes G) \cong H(\mathcal{A}) \rtimes H(G)$.

Suppose now that $P$ is a quadratic dg-operad, namely $P$ has binary generators and 3-ary relations \[.\] We will restrict ourselves to the case where $P(1) = k$, concentrated in dimension 0. Explicitly $P = F(V)/(R)$, where $F(V)$ is the free operad generated by a $k[\Sigma_2]$-module of binary operations $V$ and $(R)$ is the ideal generated by a $k[\Sigma_3]$-submodule $R \subset F(V)(3)$.

**Proposition 5.3.** Let $H$ be a cocommutative Hopf algebra and $P = F(V)/(R)$ a quadratic operad. Then $P$ is an operad of $H$-modules if and only if

(i) $V$ is an $(H, k[\Sigma_2])$-bimodule;

(ii) $R \subseteq F(V)(3)$ is an $(H, k[\Sigma_3])$-sub-bimodule.

In this case, we will call $P$ a quadratic operad of $H$-modules.

**Proof.** An element of the free operad on $V$ is described by a tree with vertices labelled by $V$. We define the action of $H$ on such element by acting on the labels of the vertices, using the comultiplication of $H$. This is well defined as $H$ is cocommutative. It induces an $H$-module structure on $F(V)$ which induces one on $P(n)$ for all $n$ by condition (ii). The operad structure maps are then $H$-equivariant by construction.
Let \( c : H \to H \otimes H \) be the comultiplication. For \( g \in H \) we write informally \((c \otimes id)(c(g)) = \sum_i g'_i \otimes g''_i \otimes g''''_i\).

**Proposition 5.4.** Let \( P = F(V)/(R) \) be a quadratic operad of \( H \)-modules as above. A chain complex \( X \) is an algebra over \( P \rtimes H \) if and only if

(i) \( X \) is an \( H \)-module

(ii) \( X \) is a \( P \)-algebra

(iii) for each \( g \in H, v \in V \) and \( x, y \in X \),

\[
g(v(x, y)) = \sum_i (-1)^{|g''_i|+|g''''_i|(|v|+|x|)} g'_i(v)(g''_i(x), g''''_i(y)).
\]

**Proof.** The \( H \)-equivariance of the algebra map \( \theta_2 : P(2) \otimes X \otimes X \to X \) is given by the commutativity of the following diagram:

\[
\begin{array}{ccc}
H \otimes P(2) \otimes X \otimes X & \xrightarrow{H \otimes \theta_2} & H \otimes X \\
\downarrow \psi \otimes \phi & & \downarrow \phi \\
H \otimes P(2) \otimes H \otimes X \otimes H \otimes X & \xrightarrow{T \circ (c \otimes id) \circ c} & H \otimes H \otimes X \otimes X \\
\downarrow \phi & & \downarrow \psi \otimes \phi \\
P(2) \otimes X \otimes X & \xrightarrow{\theta_2} & X,
\end{array}
\]

where \( \phi \) and \( \psi \) give the action of \( H \) on \( X \) and \( P(2) \) respectively and \( T \) is the interchange. This diagram translates, for the generators of \( P(2) \), into condition (iii) of the proposition. The \( H \)-equivariance of the structure maps \( \theta_k \) for \( k > 2 \) is a consequence of the fact that \( V \) generates \( P(k) \), that the operadic composition is \( H \)-equivariant and that the structure maps \( \theta \) satisfy the associativity axiom. \( \Box \)

## 6 Batalin-Vilkovisky algebras

From now on we work over a field \( k \) of characteristic 0. As first application we give a conceptual proof of a theorem of Getzler [7]. Recall that a Batalin-Vilkovisky algebra \( X \) is a graded commutative algebra with a linear endomorphism \( \Delta : X \to X \) of degree 1 such that \( \Delta^2 = 0 \) and for each \( x, y, z \in X \) the following BV-axiom holds:

\[
\Delta(xyz) = \Delta(xyz) + (-1)^{|x|}x\Delta(yz) + (-1)^{(|x|+1)|y|}y\Delta(xz) - \Delta(xyz) - (-1)^{|x|}x\Delta(y)z - (-1)^{|x|+|y|}xy\Delta(z).
\]

(1)
Theorem 6.1. Let $H(fD_2) = H(D_2) \rtimes H(SO(2))$ be the homology of the framed little 2-discs operad. An $H(fD_2)$-algebra is exactly a Batalin-Vilkovisky algebra.

Proof. Let $X$ be an algebra over $H(fD_2)$. By proposition 5.4 (condition (i)), $X$ is an $H(SO(2))$-module. As an algebra, $H(SO(2)) = k[\Delta] / \Delta^2$, where $\Delta \in H_1(SO(2))$ is the fundamental class. This provides $X$ with an operator $\Delta$ of degree 1 satisfying $\Delta^2 = 0$. Condition (ii) of proposition 5.4 tells us that $X$ is an algebra over $H(D_2)$. The operad $H(D_2)$, called the Gerstenhaber operad, was identified by F. Cohen. This operad is quadratic, generated by the operations $* \in H_0(D_2(2))$ and $b \in H_1(D_2(2))$, corresponding to the class of a point and the fundamental class under the $SO(2)$-equivariant homotopy equivalence $D_2(2) \simeq S^1$. The class $*$ induces a graded commutative product on $X$, while $b$ induces a Lie bracket of degree 1, i.e. a Lie algebra structure on $\Sigma X$, the suspension of $X$, where $(\Sigma X)_i = X_{i-1}$. The bracket is defined on $X$ by $[x, y] = (-1)^{|x|}b(x, y)$. Cohen proved that the product and the bracket satisfy the following Poisson relation:

$$[x, y * z] = [x, y] * z + (-1)^{|y|(|x|+1)}y * [x, z].$$

In order to unravel condition (iii) of proposition 5.4, we must understand the effect in homology of the $SO(2)$-action on $D_2(2) \simeq S^1$. Clearly $\Delta(*) = b$ because the rotation of the generator in degree 0 gives precisely the fundamental 1-cycle. Moreover $\Delta(b) = 0$ for dimensional reasons. As $\Delta$ is primitive, condition (iii) applied respectively to $g = \Delta$, $v = *$ and $g = \Delta$, $v = b$ provides the following relations:

$$\Delta(x * y) = \Delta(*) (x, y) + \Delta(x) * y + (-1)^{|x|}x * \Delta(y);$$

$$\Delta(b(x, y)) = \Delta(b)(x, y) - b(\Delta(x), y) + (-1)^{|x|+1}b(x, \Delta(y)).$$

As $\Delta(*) = b$, equation 3 expresses the bracket in terms of the product and $\Delta$:

$$[x, y] = (-1)^{|x|}\Delta(x * y) - (-1)^{|x|}\Delta(x) * y - x * \Delta(y).$$

If we substitute this expression into the Poisson relation 2 we get exactly the BV-axiom 1. We can re-write equation 3 as

$$\Delta[x, y] = [\Delta(x), y] + (-1)^{|x|+1}[x, \Delta(y)]$$

which says that $\Delta$ is a derivation with respect to the bracket. To conclude we must show that 4 and the Lie algebra axioms follow from the BV-axiom. This is shown in Proposition 1.2 of [7].
Remark 6.2. The lantern relation, introduced by Johnson for its relevance to the mapping class group of surfaces \([10]\) is defined by the following equation:

\[ T_{E_1} = T_{E_2}T_{E_3}T_{C_1}T_{C_2}T_{C_3}, \]

where \(T_C\) denotes the Dehn twist along the curve \(C\). See figure 4 for the relevant curves on a sphere with four holes, or equivalently on a disc with three holes. The mapping class group of a sphere with 4 ordered holes, is the group of path components of orientation preserving diffeomorphisms which fix the boundary pointwise. The group is isomorphic to the pure ribbon braid group \(PR\beta_3\). The lantern relation is thus a relation in \(PR\beta_3\) and gives rise to a relation in \(H_1(f\mathcal{D}_2(3))\) which is the abelianisation of \(PR\beta_3\). It was noted by Tillmann that, with this interpretation, one gets precisely the BV-axiom. Indeed, up to signs, the curve \(E_1\) represents the operation \((x, y, z) \mapsto \Delta x \ast y \ast z\). Moreover \(E_2\) corresponds to \(x \ast \Delta y \ast z\), \(E_3\) to \(x \ast y \ast \Delta z\), \(E_4\) to \(\Delta(x \ast y \ast z)\), \(C_1\) to \(\Delta(x \ast y) \ast z\), \(C_2\) to \(x \ast \Delta(y \ast z)\) and \(C_3\) to \(y \ast \Delta(x \ast z)\).

This geometric interpretation shows that any \(H(f\mathcal{D}_2)\)-algebra is a BV-algebra.

We will use alternatively the notations \(e_n\) and \(H(D_n)\) for the homology of the little \(n\)-discs operad, by which we mean

\[
\begin{align*}
  e_n(k) & := H(D_n(k)) \quad k \geq 1 \\
  e_n(0) & = 0.
\end{align*}
\]

Algebras over the operad \(e_n\), \(n \geq 2\), are called \(n\)-algebras. By assumption they have no units. F. Cohen’s study of \(H(D_n)\) in \([3]\) implies that an \(n\)-algebra \(X\) is a differential graded commutative algebra with a Lie bracket of degree \(n - 1\), i.e.
\[(L1) \ [x, y] + (-1)^{|x|+n-1}(|y|+n-1)y = 0,\]
\[(L2) \ \partial [x, y] = [\partial x, y] + (-1)^{|x|+n-1}[x, \partial y],\]
\[(L3) \ [x, [y, z]] = [[x, y], z] + (-1)^{|x|+n-1}(|y|+n-1)y, [x, z]],\]
satisfying the Poisson relation
\[(P1) \ [x, y * z] = [x, y] * z + (-1)^{|y|(|x|+n-1)y * [x, z].}\]

Gerstenhaber algebras correspond to the case \(n = 2\).

Note that \(D_n(2)\) is \(SO(n)\)-equivariantly homotopic to \(S^{n-1}\). In an \(n\)-algebra, the product comes from the generating class \(* \in H_0(D_n(2)) \cong k\) and the bracket from the fundamental class \(b \in H_{n-1}(D_n(2)) \cong k\), if we define \([x, y] = (-1)^{(n-1)|x|}b(x, y)\). The operad \(e_n\) is quadratic \([8]\).

In order to determine the homology operad \(H(fD_n)\), we need to know the Hopf algebra structure of \(H(SO(n))\) and the effect in homology of the action of \(SO(n)\) on \(D_n(2)\). For dimensional reasons, one always has \(\delta(b) = 0\) for each \(\delta \in \tilde{H}(SO(n))\). On the other hand \(\delta(\ast) = \pi_\ast(\delta)\), where \(\pi_\ast\) is induced in homology by the evaluation map \(\pi: SO(n) \to S^{n-1}\), via \(D_n(2) \cong S^{n-1}\).

Let us give two further examples:

**Example 6.3.** (i) An \(H(fD_3)\)-algebra is a 3-algebra together with an endomorphism \(\delta\) of degree 3 such that \(\delta^2 = 0\) and \(\delta\) is a derivation both with respect to the product and the bracket.

(ii) An \(H(fD_4)\)-algebra is a commutative dg-algebra together with two linear endomorphisms \(\alpha, \beta\) of degree 3, such that \(\alpha^2 = 0, \beta^2 = 0, \alpha\beta = -\beta\alpha\), \(\alpha\) and the product satisfy the BV-axiom \([4]\) and \(\beta\) is a derivation with respect to the product.

**Proof.** (i) \(H(SO(3)) = \wedge(\delta)\) is the free exterior algebra generated by the fundamental class \(\delta \in H_3(SO(3))\), and \(\pi_\ast(\delta) = 0\) by dimension. By proposition \([5,4]\), \(X\) is an \(H(fD_3)\)-algebra if and only if \(X\) is a 3-algebra admitting an \(H(SO(3))\)-module structure, i.e. an operator \(\delta\) of degree 3 with \(\delta^2 = 0\), such that the following relations hold:

\[
\delta(x * y) = \delta x * y + (-1)^{|x|}x * \delta y \tag{7}
\]
\[
\delta[x, y] = [\delta x, y] + (-1)^{|x|}[x, \delta y], \tag{8}
\]

where those equations are obtained by setting \(g = \delta, v = \ast\) and \(g = \delta, v = b\) in turn in condition (iii) of proposition \([5,4]\). In this case the operator \(b\) is equal to the bracket and lies in degree 2 and \(\delta(b) = 0\).
(ii) The evaluation fibration \( SO(3) \to SO(4) \to S^3 \) splits as a product. So \( H(SO(4)) = \wedge(\alpha, \beta) \), with both generators in degree 3. A class \( \alpha \) comes from the basis, so \( \pi_*(\alpha) = b \), and another class \( \beta \) comes from the fibre, so \( \pi_*(\beta) = 0 \).

As in the previous case, we know that an \( H(fD_4) \)-algebra \( X \) is a 4-algebra with two operators \( \alpha \) and \( \beta \) both in degree 3, satisfying \( \alpha^2 = 0 = \beta^2 \) and \( \alpha\beta = -\beta\alpha \) and relations obtained by setting \( (g, v) = (\alpha, *) \), \( (g, v) = (\alpha, b) \), \( (g, v) = (\beta, *) \), and \( (g, v) = (\beta, b) \) in turn in condition (iii) of proposition 5.4. Using the identification \( b(x, y) = (-1)^{|x|}[x, y] \), this gives the following equations:

\[
\alpha(x \ast y) = (-1)^{|x|}[x, y] + \alpha x \ast y + (-1)^{|x|}x \ast \alpha y \quad (9)
\]
\[
\alpha[x, y] = [\alpha x, y] + (-1)^{|x|+1}[x, \alpha y] \quad (10)
\]
\[
\beta(x \ast y) = \beta x \ast y + (-1)^{|x|}x \ast \beta y \quad (11)
\]
\[
\beta[x, y] = [\beta x, y] + (-1)^{|x|+1}[x, \beta y] \quad (12)
\]

Note that equations (9) and (10) correspond precisely to the equations we had for \( \Delta \) and the bracket in theorem 6.1. So, by the same calculations, we know that \( \alpha \) and the product form a Batalin-Vilkovisky algebra of higher degree, i.e. \( \alpha \) and \( \ast \) satisfy equation (1) but the operator \( \alpha \) has now degree 3. There is an additional operator \( \beta \) of degree 3. Equation (11) says that \( \beta \) is a derivation with respect to the product. Using equation (9), one can rewrite equation (12) in terms of \( \alpha, \beta \) and the product. This shows that equation (12) is redundant.

We need a lemma in order to state the general case.

**Lemma 6.4.** For \( n \geq 1 \), over a field of characteristic 0, the Hopf algebra \( H(SO(2n)) = \wedge(\beta_1, \ldots, \beta_{n-1}, \alpha_{2n-1}) \) is the free exterior algebra on primitive generators \( \beta_i \in H_{4i-1}(SO(2n)) \) and \( \alpha_{2n-1} \in H_{2n-1}(SO(2n)) \). Moreover, \( \pi_*(\beta_i) = 0 \) for all \( i \) and \( \pi_*(\alpha_{2n-1}) = b \in H_{2n-1}(S^{2n-1}) \) is the fundamental class.

The Hopf algebra \( H(SO(2n+1)) = \wedge(\beta_1, \ldots, \beta_n) \) is the free exterior algebra on primitive generators \( \beta_i \in H_{4i-1}(SO(2n+1)) \), and \( \pi_*(\beta_i) = 0 \) for all \( i \).

**Proof.** The homology Serre spectral sequence of the principal fibration \( SO(n) \to SO(n+1) \to S^n \) collapses at the \( E_2 \) term if \( n \) is odd; if \( n \) is even then there is a non-trivial differential \( d(b) = \alpha_{n-1} \) [17].
If a Hopf algebra $H$ acts trivially, via the counit, on an operad $P$, we call the semidirect product just the *direct product* and denote it by $P \times H$. Note that a $P \times H$-algebra is an $H$-module $X$ with a $P$-algebra structure satisfying an $H$-equivariance condition which is trivial only if $H$ acts trivially on $X$. In particular, any $P$-algebra is a $P \times H$-algebra with the trivial $H$-module structure.

Let us denote by $BV_n$, for $n$ even, the Batalin-Vilkovisky operad with the operator $\Delta$ in degree $n - 1$. Hence a $BV_n$-algebra is a differential graded commutative algebra with an operator $\Delta$ of degree $n - 1$ such that $\Delta^2 = 0$ and the BV-equation (1) holds.

Note that there is no non-trivial $\Sigma_2$-equivariant map from $H_0(D_{2n+1}(2))$ to $H_{2n}(D_{2n+1}(2))$. So $(2n + 1)$-algebras do not give rise to “odd” $BV$-structures like in the even case.

**Theorem 6.5.** For $n \geq 1$ there are isomorphisms of operads

$$H(fD_{2n+1}) \cong H(D_{2n+1}) \times H(SO(2n + 1))$$

and

$$H(fD_{2n}) \cong BV_{2n} \times H(SO(2n - 1)).$$

Hence an $H(fD_{2n+1})$-algebra is a $(2n + 1)$-algebra together with endomorphisms $\beta_i$ of degree $4i - 1$ for $i = 1, \ldots, n$ such that $\beta_i^2 = 0$, $\beta_i \beta_j = -\beta_j \beta_i$ for each $i, j$, and each $\beta_i$ is a $(2n + 1)$-algebra derivation, i.e. a derivation both with respect to the product and the bracket.

On the other hand, an $H(fD_{2n})$-algebra is a $BV_{2n}$-algebra together with endomorphisms $\beta_i$ of degree $4i - 1$ for $i = 1, \ldots, n - 1$ squaring to 0 and anti-commuting as in the odd case, which moreover anti-commute with the $BV$ operator $\Delta$ and are derivations with respect to the product.

The proof is similar to the proof of example 6.3.

We already saw that iterated loop spaces are algebras over the framed discs operad. We deduce the following example:

**Example 6.6.** The homology of an $n$-fold loop space on a pointed $SO(n)$-space is an algebra over $H(fD_n)$.

Another interesting class of algebras over $H(fD_n)$ is given by the space $\Lambda^n(X)$ of unbased maps from $S^n$ to a space $X$. Chas and Sullivan showed that the homology of a free loop space $\Lambda M$ on an oriented manifold $M$ is a Batalin-Vilkovisky algebra. Sullivan and Voronov generalised it to higher dimension and have a geometrical proof involving the so-called cacti operad.

**Example 6.7.** [25] Let $M$ be a $d$-dimensional oriented manifold. Then the $d$-fold desuspended homology $\Sigma^{-d}H(\Lambda^n M)$ of the unbased mapping space from the $n$-sphere into $M$ is an algebra over $H(fD_{n+1})$. 

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7 Koszul duality for semidirect products

Recall that we work in the category of differential graded vector spaces, also called chain complexes, over a field \( k \) of characteristic 0. Let \( P \) be a quadratic operad of \( H \)-modules. We assume that \( P(0) = 0 \) and \( P(1) = k \) is concentrated in degree 0.

Recall that if \( P = F(V)/(R) \) is the quadratic operad generated by \( V \) with relations \( R \), then its quadratic dual, as defined in [9] and [15] is given by \( P^! := F(\hat{V})/(R^\perp) \), where \( \hat{V} = V^* \otimes sgn_2 \), \( V^* \) is the linear dual, \( sgn_2 \) is the sign representation of the symmetric group, and \( R^\perp \) is the annihilator of \( R \) in \( F(\hat{V})(3) \).

The dual \( (P \rtimes H)^! \) of \( P \rtimes H \) in this sense is not naturally a semidirect product operad. Thinking of \( P \rtimes H \) as the operad \( P \) in \( H \)-Mod, we consider instead the following duality.

**Definition 7.1.** The dual of a semidirect product \( P \rtimes H \) is the operad \( (P \rtimes H)^\dagger := P^! \rtimes H^{\text{op}} \), where \( P^! \) is the quadratic dual of \( P \) and \( H^{\text{op}} \) denotes \( H \) with the opposite multiplication.

This makes sense because \( P^! \) is an operad of \( H^{\text{op}} \)-modules by proposition 5.3.

The suspension of an operad \( P \) is the operad \( \Lambda P \) defined by \( \Lambda P(n) = \Sigma^{n-1}(P(n)) \otimes sgn_n \). A chain complex \( A \) is a \( \Lambda P \)-algebra if and only if the suspension \( \Sigma A \) is a \( P \)-algebra.

In the following example, we show that the operad \( BV \), as a semidirect product, is self-dual up to suspension.

**Example 7.2.**

1. \( BV_{2n}^\dagger := H(D_{2n})^\dagger \rtimes k < \alpha_{2n-1} > = \Lambda^{1-2n}BV_{2n} \); 

2. \( H(fD_{2n})^\dagger := H(D_{2n})^\dagger \times H(SO(2n))^{\text{op}} = \Lambda^{1-2n}BV_{2n} \times H(SO(2n-1)) \); 

3. \( H(fD_{2n+1})^\dagger = \Lambda^{-2n}H(D_{2n+1}) \times H(SO(2n+1)) \).

**Proof.** We give a proof of (1). It is known that the quadratic dual of the operad \( e_{2n} = H(D_{2n}) \) is its own \( (2n-1) \)-fold desuspension \( \Lambda^{1-2n}e_{2n} \), with product \( p' = b^* \) dual to original bracket \( b \) and bracket \( b' = p^* \) dual to the original product \( p \) [8, 13]. If \( \alpha_{2n-1}(p) = b \), then \( \alpha_{2n-1}(b^*) = p^* \). So the class \( \alpha_{2n-1} \) gives an operator \( \Delta \) of degree \( 2n-1 \) with \( \Delta(p') = b' \) and \( \Delta(b') = 0 \), which thus induces a \( BV \) structure as in theorem 6.1 but this time with product in degree \( 1-2n \).
From now on, we assume that $P$ is a quadratic operad of $H$-modules such that $P(n)$ is a finite dimensional $k$-vector spaces for each $n$. The operad $P$ is thus admissible in the sense of Ginzburg and Kapranov.

Let $P - \text{Alg}_{\geq n}$ ($P - \text{Alg}_{\leq n}$) denote the category of $P$-algebras of finite type concentrated in degree $\geq n$ ($\leq n$). Getzler and Jones ([8], see also [12]) constructed contravariant adjoint functors

$$C_P : P - \text{Alg}_{\geq 1} \rightleftarrows P^! - \text{Alg}_{\leq -2} : T_P$$

such that $C_P$ and $T_P$ preserve quasi-isomorphisms. Moreover, if $P$ is Koszul, the unit and counit of the adjunction are quasi-isomorphisms.

If $\mathcal{C}$ is a category of chain complexes, we denote by $Ho(\mathcal{C})$ the category obtained from $\mathcal{C}$ by inverting all quasi-isomorphisms. We thus have the following equivalences of categories:

$$Ho(P - \text{Alg}_{\geq 1}) \simeq Ho(P^! - \text{Alg}_{\leq -2})$$

We want to see that $C_P$ and $T_P$ restrict to functors between categories of $P \rtimes H$-algebras and $P^! \rtimes H^{op}$-algebras.

Let $A$ be a $P$-algebra in $H\text{-Mod}$. The complex $C_P(A)$ is defined as the free $P^!$-algebra on the dual of the suspension of $A$ with differential $d_1 + d_2$, where $d_1$ is induced by the differential of $A$ and $d_2$ by the $P$-algebra structure of $A$. This object has an $H$-module structure induced by the action of $H$ on $A$ and on $P$. The $H$-action commutes with $d_1$ as $A$ is an $H$-module and it commutes with $d_2$ as the $P$-algebra structure maps of $A$ are $H$-equivariant. The $P^!$-algebra structure map of $C_P(A)$ is $H$-equivariant because $P$ is an operad of $H$-modules.

Similarly, for a $P^!$-algebra $X$, $T_P(X)$ is the free $P$-algebra on the dual of the suspension of $X$ with twisted differential, and is a $P$-algebra in $H\text{-Mod}$. Moreover the natural transformations $T_P C_P(A) \rightarrow A$ and $C_P T_P(X) \rightarrow X$ are respectively $H$- and $H^{op}$-equivariant.

We thus have the following theorem:

**Theorem 7.3.** Let $P$ be a quadratic operad of $H$-modules. Then there is an equivalence of categories

$$Ho(P \rtimes H - \text{Alg}_{\geq 1}) \simeq Ho(P^! \rtimes H^{op} - \text{Alg}_{\leq -2}).$$

**Remark 7.4.** The theorem can be extended to objects not of finite type, by considering coalgebras over an operad.
Definition 7.5. Let $A$ be a $P \rtimes H$-algebra. The operadic homology of $A$ is the $H(P!) \rtimes H^{op}$-algebra $H(C_P(A))$.

So the homology of $A$ over $P \rtimes H$ is the homology of $A$ over $P$ with an additional $H^{op}$-module structure.

We apply this machinery in order to compute explicitly the $BV$-algebra structure of the homology of a double loop space. The Gerstenhaber algebra structure is computed in 6.1 of [8] for the double loop space on a manifold via operadic homology.

Let $X$ be a 2-connected pointed CW-space of finite type acted on by $S^1$. Let $M(X)$ be the minimal model of $X$, a non-negatively graded unital commutative algebra with a differential of degree 1. The $S^1$-action induces a map $f : M(X) \to M(X) \otimes \Lambda(e_1)$ of type $f(x) = x + \Delta(x) \otimes e_1$, thus $\Delta$ is a derivation of degree -1. Let $m(X) = M(X)/ \langle 1 \rangle$ be the quotient by the unit. By changing signs we regard $m(X)$ as a $BV$-algebra concentrated in degree $\leq -3$, with trivial Lie bracket. Clearly the suspension $\Sigma m(X)$ is a $BV$-algebra concentrated in negative degree.

Theorem 7.6. The operadic homology of the $BV$-algebra $\Sigma m(X)$ is isomorphic to $H(\Omega^2(X))$ as $BV$-algebra.

Proof. If we substitute the expression $BV$ by $G$, then the result follows from 6.9 of [8] by the same proof as 6.1 of [8], with the minimal model replacing the deRham complex.

Since $H_\ast(\Omega^2 X)$ is the free commutative algebra on $\pi_\ast(\Omega^2 X)$, by the Milnor-Moore theorem, the homology $BV$-algebra structure is uniquely determined by $\Delta$ on spherical classes.

We identify geometrically the $BV$-operator on spherical classes. If $Y$ is a pointed $S^1$-space and $x \in \pi_n(Y)$, consider the map $S^n \wedge S^1_+ = (S^n \times S^1)/(\ast \times S^1) \to Y$ induced by the action. Since $S^n \wedge S^1_+ \simeq S^n \vee S^{n+1}$ let $s_Y(x) \in \pi_{n+1}(Y)$ be the restriction to the second summand. Clearly $s$ is natural with respect to $S^1$-equivariant maps.

The operation $s$ represents $\Delta$ on spherical classes, i.e. the following diagram commutes for any $n > 1$.

\[
\begin{array}{ccc}
\pi_{n+2}(X) & \xrightarrow{s_X} & \pi_{n+3}(X) \\
\downarrow & & \downarrow \\
H_n(\Omega^2 X; k) & \xrightarrow{\Delta} & H_{n+1}(\Omega^2 X; k)
\end{array}
\]

The only subtlety is to show that $s_X$ can be replaced by $s_{\Omega^2 X}$. Consider the universal example $X = S^n \wedge S^1_+$. Clearly $\pi_n(X) \otimes k = H_n(X; k) = \ldots$
$k < e_n >$, and $\pi_{n+1}(X) \otimes k = H_{n+1}(X; k) = k < e_{n+1} >$, because $\text{char}(k) = 0$ and $n > 2$. Moreover $s_X(e_n) = e_{n+1}$ because the composite

$$S^1 \times S^n \hookrightarrow S^1 \times X \to X \to S^1 \wedge S^n = S^{n+1}$$

has degree 1, and $s_{\Omega^2 X}(e_n) = e_{n+1}$ for the same reason.

Recall now that spherical classes are dual to indecomposables in the minimal model. In the operadic homology the BV-operator $t$ on indecomposables in $m(X)/(m(X) \cdot m(X))$ is induced by the BV-operator on $m(X)$ by definition. By naturality it is sufficient to show that $s$ and $t$ coincide in the universal case $X = S^n \wedge S^1$. But $t(e_n) = e_{n+1}$, because, for homological reasons, the map $f$ induced by the $S^1$-action on $m(X) = \Lambda(x_n, x_{n+1}, \ldots)$ gives $f(x_{n+1}) = x_{n+1} + x_n \otimes e_1$.

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