Pursuit Game for an Infinite System of First-Order Differential Equations with Negative Coefficients

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Abstract. In this paper we study a linear pursuit differential game described by an infinite system of first-order differential equations in Hilbert space. The control functions of players are subject to geometric constraints. The pursuer attempts to bring the system from a given initial state to the origin for a finite time and the evader’s purpose is opposite. We obtain a guaranteed pursuit time and construct a strategy for pursuer.

1. Introduction

A game has players, strategies and scores to explain why the players win or lose. Differential games are games which are modeled with differential equations and were initiated by Isaacs [14]. Pursuit–evasion games are common examples of games that are abstract models for pursuers who try to catch evaders who are running away. Differential games and pursuit–evasion problems are investigated by many authors. For example see, Blagodatskih and Petrov [2], Krasovskii [15], Pashkov and Terekhov [16], Petrov [18, 19], Petrosyan [20], Pontryagin [21], Pshenichii [22], Rikhsiev [23] and Rzymowski [24]. Finding the value of the game and identifying optimal strategies of players are two interesting subjects in the study of pursuit evasion problems. Such issues for many players with a variety of constraints have been studied in [6], [8], [9], [10], [12], [13], [25], [26] and [27].

In [9] Ibragimov investigates a pursuit–evasion game of optimal approach of countably many pursuers to one evader in a Hilbert space with geometric constraints on the controls of the players. Ibragimov and Salimi [12] worked on a similar game for inertial players with integral constraints with the assumption that the control resource of the evader is less than that of each pursuer. Evasion of evader from many pursuers in simple motion differential games was developed by Ibragimov et al. in [13] as well. This class of differential games occurs naturally as a reformulation for solving controlled distributed systems described by

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a parabolic equation as in [28] (see also [4, 10] for more details). In [29] Satimov and Tukhtasinov study pursuit and evasion problems for controlled equations of the parabolic type. The control parameters occur on the right-hand side of the equations as additive terms. They consider various cases of control constraints. For some cases, they single out pairs of sets of initial states such that capture is guaranteed if the initial point belongs to the first set and evasion of the terminal set is guaranteed if the initial point belongs to the second set.

In [11] Ibragimov et al. study a differential game of optimal approach of finite or countable number of pursuers with one evader in the Hilbert space $l_2$. In their research, they found a formula for the value of the game and constructed explicitly optimal strategies of the players. Important point to note was that the energy resource of any pursuer needs not be greater than that of the evader.

In the work of Huang et al. [7], a game of multiple pursuers cooperating to capture a single evader in a bounded, convex polygon was studied in the plane. The main result of that paper is construction of a decentralized, guaranteed pursuit strategy where the pursuers cooperatively minimize the area of the evaders Voronoi cell by independently controlling each pursuers shared Voronoi boundary with the evader. In the differential game of many pursuers in a planar domain studied by Zhengyuan et al. [30] pursuit strategies are constructed based on Voronoi partition as well.

In the paper of Salimi and Ferrara [26], authors consider a finite time pursuit–evasion game in which a finite or countable number of pursuers pursue a single evader. The control functions of players are subjected to integral constraints. They introduce the value of the game and identify optimal strategies of the pursuers. Azamov and Ruziboev [1] considered the time-optimal problem for a controlled system with evolution-type distributed parameters. They obtained an upper estimate for the optimal transition time into the zero state.

In order to motivate the study of countably many pursuers in contrast to the classical problem of finitely many pursuers, we would like to draw a comparison with optimal control theory [17]. In order to design a controller for an infinite-dimensional problem, e.g. a partial differential equation, one often follows the ‘approximate-then-design’ method, i.e. first the infinite-dimensional problem is approximated, e.g. by a Galerkin approximation, and then a controller is designed for the resulting finite-dimensional approximation. Finally the finite-dimensional controller or strategy is shown to work also for the original problem [5]. A dual approach which is still in its infancy is the ‘design-then-approximate’ method, i.e. first an
infinite-dimensional controller or strategy for the infinite-dimensional problem is designed, and then approximated by a finite-dimensional controller which is then proved to work also for the original problem [3]. The results in this paper are in the spirit of the 'design-then-approximate' method and in case one is interested in finite-dimensional approximations or implementations of our methods, further investigations are required.

2. Statement of problem

Consider the Hilbert space

\[ l^2 = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) \mid \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\} \]

with inner product and norm defined by

\[ \langle \alpha, \beta \rangle = \sum_{k=1}^{\infty} |\alpha_k\beta_k|, \quad \alpha, \beta \in l^2, \quad ||\alpha|| = \left( \sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2} \]

and the following differential game described by the equations

\[ \dot{z}_i = -\lambda_i z_i + u_i - v_i, \quad i = 1, 2, \ldots; \quad z_i(0) = z_{i0}, \quad \text{(2.1)} \]

where \( z_i, u_i, v_i \in \mathbb{R} \), \( u = (u_1, u_2, \ldots) \) and \( v = (v_1, v_2, \ldots) \) are control parameters of pursuer and evader respectively, and \( \lambda_k, k = 1, 2, \ldots \), are positive numbers. It is assumed that \( z_0 = (z_{10}, z_{20}, \ldots) \neq 0 \).

**Definition 1.** Let \( T \) be an arbitrary number. A vector function \( u(t) = (u_1(t), u_2(t), \ldots), ||u(t)|| \leq \rho, 0 \leq t \leq T \), is called admissible control of the pursuer, where \( \rho \) is a given positive number.

**Definition 2.** A vector function \( v(t) = (v_1(t), v_2(t), \ldots), ||v(t)|| < \sigma, 0 \leq t \leq T \) is called admissible control of the evader, where \( \sigma \) is a given positive number.

**Definition 3.** A function of the form \( u(v) = (u_1(v_1), u_2(v_2), \ldots) \) with measurable coordinates of \( u_i(\cdot) \in \mathbb{R} \), that satisfies the condition \( ||u(v)|| \leq \rho \) for any admissible control of evader \( v = v(t), 0 \leq t \leq T \), is called strategy of pursuer.

Pursuit is started from the initial positions \( z_0 = (z_{10}, z_{20}, \ldots) \) at time \( t = 0 \) where \( z_{i0} \in \mathbb{R}, i = 1, 2, \ldots. \)
If we replace the parameters \( u_i, v_i \) in the equation (2.1) by some measurable functions \( u_i(t), v_i(t), 0 \leq t \leq T \), then it follows from the theory of differential equations that the initial value problem (2.1), \( z_0 = (z_{10}, z_{20}, \ldots) \) has a unique solution on the time interval \([0, T]\).

The solution

\[
z(t) = (z_1(t), z_2(t), \ldots), \quad 0 \leq t \leq T
\]

of infinite system of differential equations (2.1) is considered in the space of functions \( f(t) = (f_1(t), f_2(t), \ldots) \in l^2 \) with absolutely continuous coordinates \( f_i(t) \) defined on the interval \(0 \leq t \leq T\).

**Definition 4.** A number \( T_0, T_0 \leq T \), is called a guaranteed pursuit time if there exists a strategy of pursuer such that for any control of the evader, the solution of the initial value problem (2.1), \( z_0 = (z_{10}, z_{20}, \ldots), z(t) = (z_1(t), z_2(t), \ldots) \) equals zero, at some \( T, 0 \leq T \leq T_0 \), i.e. \( z_i(T) = 0 \) for all \( i = 1, 2, \ldots \).

**Problem:** Find a guaranteed pursuit time in the game (2.1)-(\( z_0 = (z_{10}, z_{20}, \ldots) \)).

### 3. Main Result

We assume that \( \rho > \sigma \) and define the strategy for the pursuer as following:

\[
u_i(t) = \begin{cases} 
- \frac{z_{10}}{||z_0||}(\rho - \sigma) + v_i(t), & i = 1, 2, \ldots, \quad 0 \leq t \leq T_i, \\
v_i(t), & t > T_i
\end{cases}
\]

where

\[
T_i = \frac{1}{\lambda_i} \ln \left( \frac{\lambda_i ||z_0||}{\rho - \sigma} \right) + 1.
\]

The above strategy is admissible, indeed using the Minkowskii inequality we get,

\[
||u|| = \left( \sum_{i=1}^{\infty} \left( - \frac{z_{10}}{||z_0||}(\rho - \sigma) + v_i \right)^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{i=1}^{\infty} \frac{z_{10}^2(\rho - \sigma)^2}{||z_0||^2} \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{\infty} v_i^2 \right)^{\frac{1}{2}} \\
\leq \rho - \sigma + \sigma = \rho.
\]

Thus, \( ||u|| \leq \rho \) and the strategy is admissible.

In this part we are going to prove that,

\[
z_i(t) = 0 \quad \text{for all} \quad t \geq T_i.
\]
Indeed, by (3.1) we have
\[
Z_i(t) = e^{-\lambda_i t} \left[ z_{i0} + \int_0^t e^{\lambda_i s} (u_i(s) - v_i(s)) ds \right]
\]
\[
= e^{-\lambda_i t} \left[ z_{i0} - \int_0^{T_i} e^{\lambda_i s} \frac{z_{i0}}{||z_0||} (\rho - \sigma) ds \right] = 0.
\]

Since
\[
\int_0^{T_i} e^{\lambda_i s} ds = \frac{e^{\lambda_i T_i} - 1}{\lambda_i} = \frac{||z_0||}{\rho - \sigma},
\]

Thus, by (3.2) the number
\[
T = \sup_i T_i = \sup_i \frac{1}{\lambda_i} \ln \left( \frac{\lambda_i ||z_0||}{\rho - \sigma} + 1 \right)
\]
is a guaranteed pursuit time.

Now we show that
\[
T = \frac{1}{\lambda} \ln \left( \frac{\lambda ||z_0||}{\rho - \sigma} + 1 \right) \tag{3.3}
\]
where \( \lambda = \inf \lambda_i \). To this end we consider the function
\[
f(x) = \frac{\ln \left( \frac{||z_0||}{\rho - \sigma} x + 1 \right)}{x}, \quad x > 0.
\]

It is not difficult to see that
\[
f'(x) = \frac{1}{x^2} g(x), \quad g(x) = 1 - \frac{1}{\frac{||z_0||}{\rho - \sigma} x + 1} - \ln \left( \frac{||z_0||}{\rho - \sigma} x + 1 \right),
\]
and
\[
g'(x) = -\frac{\left( \frac{||z_0||}{\rho - \sigma} \right)^2 x}{\left( \frac{||z_0||}{\rho - \sigma} x + 1 \right)^2} < 0.
\]

Hence \( g(x) \) is decreasing. Since \( g(0) = 0 \), therefore \( g(x) < 0, \ x > 0 \). Therefore, \( f'(x) < 0, \ x > 0 \). Consequently \( f(x) \) is decreasing. Thus, \( T \) is defined by formula (3.3).
4. Discussion and Conclusion

We studied a linear pursuit differential game described by an infinite system of first-order differential equations in Hilbert space \( l^2 \). The control functions of players are subjected to geometric constraints. The pursuer attempts to bring the system from a given initial state to the origin for a finite time and the evader’s purpose is opposite. We obtained a guaranteed pursuit time and constructed a strategy for pursuer.

In the differential game studied by Satimov and Tukhtasinov [28] the numbers \( \lambda_1, \lambda_2, \ldots \), satisfy the inequality \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow \infty \) and guaranteed pursuit time is \( T_0 = \frac{|| z_0 ||}{\rho - \sigma} \). For that differential game guaranteed pursuit time \( T \) defined by the formula (3.3) is

\[
T = \frac{1}{\lambda_1} \ln \left( \frac{\lambda_1 || z_0 ||}{\rho - \sigma} + 1 \right) < T_0 = \frac{|| z_0 ||}{\rho - \sigma}.
\]

In summary, we stress the following two advantages of the present work

(1) Guaranteed pursuit time \( T \) defined by (4) improves that of the work of Satimov, Tukhtasinov [28].

(2) The numbers \( \lambda_1, \lambda_2, \ldots \) needn’t satisfy the relations \( \lambda_1 \leq \lambda_2 \leq \cdots \). They are assumed to be any positive numbers.

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