The Partition Function of 2D String Theory

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We derive a compact and explicit expression for the generating functional of all correlation functions of tachyon operators in 2D string theory. This expression makes manifest relations of the \(c = 1\) system to KP flow and \(W_{1+\infty}\) constraints. Moreover we derive a Kontsevich-Penner integral representation of this generating functional.

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1. Introduction

One of the beautiful aspects of the matrix-model formulation of $c < 1$ string theory is that it gives a natural and mathematically precise formulation of the partition function of strings moving in different backgrounds. This result began with Kazakov’s fundamental discovery of the appearance of matter fields in the one-matrix model \[1\] and culminated in the discovery of the generalized KdV flow equations and the associated $W_N$ constraints in the $c < 1$ matrix models coupled to gravity \[2–6\]. Recently these results have been further deepened through the use of a Kontsevich matrix model representation for the tau functions relevant to these flows \[7\], see also \[8,9\]. Analogous results in the $c = 1$ model have been strangely absent, and this paper is a first step in an attempt to change that situation. Using recently developed techniques for calculating tachyon correlators in the $c = 1$ model we derive a simple and compact expression (equation (3.10)) for the generating functional of tachyon correlators, or equivalently the string partition function in an arbitrary tachyon background, valid to all orders in string perturbation theory. In Euclidean space this quantity can be interpreted as the partition function of a nonlinear sigma model as a function of an infinite set of coupling constants $t_k, \bar{t}_k$ for a set of marginal operators. Upon appropriate analytic continuation to Minkowski space the partition function may be interpreted as the string $S$-matrix in a coherent state basis.

One immediate consequence of our result (3.10) is that the partition function is naturally represented as a tau function of the Toda hierarchy. From this result we obtain $W_\infty$ flow equations (equation 4.10) when the $c = 1$ coordinate $X$ is compactified at the self-dual radius. Moreover, this expression can be used to derive a Kontsevich-Penner representation of the partition function as a matrix integral, as described in section five below. In section six we discuss how time-independent changes in the matrix model background fit into our formalism, and in section seven we discuss some open problems and the relation of this work to other recent papers on $c = 1$ and $W_\infty$.

2. Defining correlation functions

In a particular background, string propagation in a two-dimensional spacetime is described on the string worldsheet by the conformal field theory of a massless scalar $X$ coupled to a $c = 25$ Liouville theory $\phi$ with worldsheet action (excluding ghosts)

$$\mathcal{A} = \int \frac{1}{2} \partial X \bar{\partial} X + \partial \phi \bar{\partial} \phi + \sqrt{2} R^{(2)} \phi + \mu e^{\sqrt{2} \phi}.$$ \hspace{1cm} (2.1)
Via its dual interpretation as the conformal gauge action for the coupling of \( X \) to two-dimensional gravity, (2.1) is expressible as the continuum limit of a sum over discretized surfaces. The discrete sum, as is by now well known, is generated by a matrix integral. In the double scaling limit which leads to the continuum theory this is in turn equivalent to a theory of free nonrelativistic fermions with action

\[
S = \int_{-\infty}^{\infty} dx d\lambda \hat{\psi}^\dagger \left( i \frac{d}{dx} + \frac{d^2}{d\lambda^2} - V(\lambda) \right) \hat{\psi} .
\] (2.2)

The potential \( V(\lambda) \) in (2.2) is required to approach \(-\frac{1}{4} \lambda^2\) for large \( \lambda \) in order to reproduce the topological expansion of string theory.\(^2\)

The theory contains one field theoretic degree of freedom, the massless “tachyon”. Tachyon correlators are calculated in the theory (2.1) by the insertion of vertex operators

\[
\mathcal{T}_q = \frac{\Gamma(|q|)}{\Gamma(-|q|)} \int_{\Sigma} e^{i q X/\sqrt{2}} e^{\sqrt{2}(1 - \frac{1}{2} |q|) \phi} .
\] (2.3)

In this section we show how these correlators are calculated in the double scaled matrix model (2.2). The presentation is a modification of the original derivation in [10], which stressed the spacetime interpretation via collective field theory, in that here we emphasize the relation to macroscopic loop amplitudes.

We recall that the objects calculated in a sum over continuous geometries on two-surfaces with boundary are “macroscopic loop amplitudes” defined by fixing the boundary values of the two-metric \( e^{\sqrt{2} \phi} \) so that the bounding circles \( C \) have lengths \( \ell = \oint_C e^{\phi/\sqrt{2}} \) [2,11–16]. In [14,16] tachyon correlators were defined as the coefficients of nonanalytic powers of \( \ell \) in the small-\( \ell \) expansion of macroscopic loop amplitudes. In particular, in this limit the macroscopic loop may be written as a sum of local operators

\[
W_{in}(\ell, p) = -\mathcal{T}_p \frac{\pi}{\sin \pi |p|} \mu^{-|p|/2} I_p(2\sqrt{\mu\ell}) - \sum_{r=1}^{\infty} \hat{B}_{r,p} \frac{2(-1)^r}{r^2 - \mu^2} \mu^{-r/2} I_r(2\sqrt{\mu\ell})
\] (2.4)

\(^2\) The behavior of \( V(\lambda) \) for negative \( \lambda \) is irrelevant to all orders of perturbation theory in \( 1/\mu \). Indeed, the results of this paper should be interpreted in this perturbative sense. Many results are true in the nonperturbative context and we will indicate this in the appropriate places. Where we mention nonperturbative results we will refer to potentials which grow sufficiently rapidly for large negative \( \lambda \). In [10] these were termed “type I” models.
where $\hat{B}_{r,p}$ are redundant operators for $p \notin \mathbb{Z}$. We may thus extract tachyon correlators from macroscopic loop amplitudes as

$$
\langle \prod_{i=1}^{n} W(\ell_i, q_i) \rangle = \prod_{i=1}^{n} \Gamma(-|q_i|) \ell_i^{|q_i|} \left( \prod_{i=1}^{n} \mathcal{T}_{q_i} \right) + \mathcal{O}(\ell_i^2) + \text{analytic in } \ell_i .
$$

(2.5)

The matrix model formulation of the theory leads to a simple computation of the appropriate limits of loop amplitudes. In the matrix model the macroscopic loop is related by a Laplace transform to the eigenvalue density $\hat{\rho}(\lambda, x) = \hat{\psi}^\dagger \hat{\psi}(\lambda, x)$:

$$
W(\ell, x) = \int_{0}^{\infty} e^{-\ell \lambda} \hat{\rho}(\lambda, x) d\lambda
$$

$$
W(\ell, q) = \int_{-\infty}^{\infty} e^{iqx} W(\ell, x) dx .
$$

(2.6)

Defining

$$
\hat{W}(z, x) = \int_{0}^{\infty} e^{-z\ell} W(\ell, x) d\ell
$$

we recover

$$
\hat{\rho}(\lambda, x) = -\frac{i}{\pi} \text{Disc} (\hat{W}(z, x)) \bigg|_{z=-\lambda} .
$$

(2.7)

(2.8)

Inserting (2.5) we find now

$$
\langle \prod_{i=1}^{n} \hat{\rho}(\lambda_i, q_i) \rangle = \prod_{i=1}^{n} \lambda_i^{-q_i - 1} \left( \prod_{i=1}^{n} \mathcal{T}_{q_i} \right) + \mathcal{O}(\lambda_i^{-2})
$$

(2.9)

for generic $q_i$. In the next section we show how this leads to a simple formula for the correlation functions.

3. Calculating correlators

3.1. Graphical rules for tachyon correlators

The calculation of tachyon correlation functions now reduces to the study of asymptotics of correlation functions of the eigenvalue density. An explicit formula for these was found in [10]; we will use a generalization of this to a compactification radius $\beta$ for the Euclidean $X$ field. Momenta for the tachyon field are therefore always of the form $q = n/\beta$ for $n \in \mathbb{Z}$. The required modification follows by interpreting the compactified Euclidean time as a finite temperature for the free fermion system [17]. Allowed fermionic momenta are of the form $p_m = (m + \frac{1}{2})/\beta$ with $m \in \mathbb{Z}$. 

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The eigenvalue correlator can be written as:

\[
\langle \prod \hat{\rho}(\lambda_i, q_i) \rangle = \sum_{m=-\infty}^{\infty} \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} I(Q_k^\sigma, \lambda_{\sigma(k)}, \lambda_{\sigma(k+1)})
\]

(3.1)

where \( \Sigma_n \) is the set of permutations of \( n \) objects, \( Q_k^\sigma \equiv p_m + q_{\sigma(1)} + \cdots + q_{\sigma(k)} \). The \( \lambda \) dependence of the correlator is determined by the function

\[
I(q, \lambda_1, \lambda_2) = (I(-q, \lambda_1, \lambda_2))^* = \langle \lambda_1 | \frac{1}{\lambda - \mu - iq} | \lambda_2 \rangle
\]

where \( H \) is the one-body Hamiltonian for the free fermion system. This has large \( \lambda \) asymptotics given by a direct and a reflected contribution

\[
I(q, \lambda_1, \lambda_2) \xrightarrow{\lambda \to \infty} -i \sqrt{\frac{\lambda_1}{\lambda_2}}\left[ D(q, \lambda_1, \lambda_2) + R(q, \lambda_1, \lambda_2) \right]
\]

where

\[
D(q, \lambda_1, \lambda_2) = e^{\pi \mu + iq} \log (\lambda_1/\lambda_2) - \frac{i}{4} (\lambda_1^2 - \lambda_2^2)
\]

\[
R(q, \lambda_1, \lambda_2) = R_q \exp \left[ i(\mu + iq) \log (\lambda_1 \lambda_2) - \frac{i}{4} (\lambda_1^2 + \lambda_2^2) \right]
\]

(3.2)

for \( q > 0 \). The function \( R_q \) is the reflection coefficient for the nonrelativistic free fermions in the double scaled potential \( V \). For the “standard” case \( V = -\frac{1}{4} \lambda^2 \) with an infinite wall at \( \lambda = 0 \) it is given by

\[
R_q = i \sqrt{\frac{1 + i e^{-\pi (\mu + iq)}}{1 - i e^{-\pi (\mu + iq)}}} \frac{\Gamma(\frac{1}{2} - i\mu + q)}{\Gamma(\frac{1}{2} + i\mu - q)}
\]

(3.3)

Inserting (3.2) in the expression (3.1) leads to a sum of terms. The calculation of tachyon correlators requires the extraction of those terms in the sum with the correct asymptotic dependence on \( \lambda_i \). For each permutation \( \sigma \), at most a finite number of terms in the sum over the loop momentum \( p_m \) contribute to the result.

A graphical procedure for performing this extraction was developed in [10] and used to derive an explicit expression for arbitrary tachyon correlators. We divide the tachyon insertions into “incoming” \( (q < 0) \) and ‘outgoing’ \( (q > 0) \) particles. As in a Feynman diagram there is a vertex in the \((x, \lambda)\) half-space corresponding to each operator \( \hat{\rho}(x, \lambda) \). While the final result will of course be independent of the order in which the \( \lambda_i \) are increased to infinity, in intermediate steps we will choose some order and locate the vertices accordingly. Points are connected by line segments, representing the integral \( I \), to form a one-loop graph. Since the expression for \( I \) in (3.2) has two terms we have both direct and reflected propagators as in fig. 1. Each line segment carries a momentum and an arrow. Note that in fig. 1 the reflected propagator, which we call simply a “bounce,” is composed
of two segments with opposite arrows and momenta. These line segments are joined to form a one-loop graph according to the following rules:

RH1. Lines with positive (negative) momenta slope upwards to the right (left).

RH2. At any vertex arrows are conserved and momentum is conserved as time flows upwards. In particular momentum $q_i$ is inserted at the vertex in fig. 2.

RH3. Outgoing vertices at $(x_{out}, \lambda_{out})$ all have later times than incoming vertices $(x_{in}, \lambda_{in})$: $x_{out} > x_{in}$.

Diagrams drawn according to these rules correspond to possible physical processes in real time and were hence termed "real histories". The connected tachyon correlation function is found by summing the terms in (3.1) corresponding to all real histories, and reads schematically

$$\langle \prod_{i=1}^{n} T_{q_i} \rangle = (-i)^n \sum_{\text{RH}} \pm \sum_{m} \prod_{\text{bounces}} R_Q(-R_Q)^*.$$ (3.4)

The graphical rules allow one to convert (3.4) into an explicit formula for the amplitude [10]. In the next subsection we will show that this result may be written quite simply in terms of free fermionic fields, representing a fermionized version of the free relativistic bosonic field which describes the asymptotic behavior of the tachyon.

3.2. Free Energy in terms of free oscillators

One of the central results of [10] is that the graphical rules described above are equivalent to the composition of three transformations on the scattering states: fermionization, free fermion scattering, and bosonization: $i_{f \rightarrow b} \circ S_{ff} \circ i_{b \rightarrow f}$ as in fig. 3. The various real histories correspond to the possible contractions among the incoming and outgoing fermions, and the fermion scattering matrix describes a simple one-body process, given essentially by the phase shift in the nonrelativistic problem. It should be noted that this does not imply the (false) statement that bosonization is exact for the nonrelativistic fermion problem. Rather, it is a statement about the asymptotics of certain correlators in the theory for a particular class of potentials. Here we will rewrite the tachyon amplitude using this formulation as a matrix element of a certain operator in the conformal field theory of a free Weyl fermion.
It is convenient to define rescaled tachyon vertex operators $V_q = \mu^{1-|q|/2} T_q$ and two free scalar fields $\partial \phi^{in/out} = \sum_n \alpha^{in/out}_n z^{-n-1}$, such that

$$\langle \prod_{i=1}^n V_{n_i} / \beta \prod_{j=1}^m V_{-n'_j} / \beta \rangle = -\frac{(i\mu)^n}{\beta} \langle 0 \mid \prod \alpha^{out}_{n_i} \prod \alpha^{in}_{-n'_j} \mid 0 \rangle$$

(3.5)

where $|0\rangle$ is the standard $SL(2, \mathbb{R})$ invariant vacuum. The equivalence of the graphical rules to bosonization then implies that while the relation between the two bosonic fields is complicated and nonlinear it may in fact be expressed as a simple linear transformation in the fermionized version. Thus we write $\partial \phi = \psi(z) \bar{\psi}(z)$ where $\psi, \bar{\psi}$ are Weyl fermions of weight $\frac{1}{2}$ with expansions

$$\psi(z) = \sum_{m \in \mathbb{Z}} \psi_m z^{-m-1}$$

$$\bar{\psi}(z) = \sum_{m \in \mathbb{Z}} \bar{\psi}_m z^{-m-1}$$

$$\{ \psi_r, \bar{\psi}_s \} = \delta_{r+s,0}.$$  
(3.6)

Now the result of [10] states that (3.5) is equivalent to

$$\psi^{in}_{-(m+\frac{1}{2})} = R_{p_m} \psi^{out}_{-(m+\frac{1}{2})}$$

$$\bar{\psi}^{in}_{-(m+\frac{1}{2})} = R^*_{p_m} \bar{\psi}^{out}_{-(m+\frac{1}{2})}.$$  
(3.7)

Unitarity of the tachyon $S$-matrix is equivalent to the identity

$$R_q R^*_{-q} = 1$$

(3.8)

on the reflection factors. Using this, we can rewrite (3.7) as a unitary transformation

$$\psi^{in}(z) = S \psi^{out}(z) S^{-1}$$

$$\bar{\psi}^{in}(z) = S \bar{\psi}^{out}(z) S^{-1}$$

(3.9)

$$S =: \exp \left[ \sum_{m \in \mathbb{Z}} \log R_{p_m} \psi^{out}_{-(m+\frac{1}{2})} \bar{\psi}^{out}_{m+\frac{1}{2}} \right].$$

\[3\] This holds to all orders in perturbation theory for any of the potentials we consider. The question of its nonperturbative validity was discussed in [10]. Essentially, this requires that $V(\lambda)$ grow sufficiently rapidly for large negative $\lambda$.  

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Thus we may write the full generating functional for connected Green’s functions in terms of a single free boson with modes $\alpha_n$:

$$\mu^2 F \equiv \langle e^{\sum_{n \geq 1} t_n V_{n/\beta} + \sum_{n \geq 1} \bar{t}_n V_{-n/\beta}} \rangle_c$$

$$= -\frac{1}{\beta} \langle 0 | e^{i\mu \sum_{n \geq 1} t_n \alpha_n} S e^{i\mu \sum_{n \geq 1} \bar{t}_n \alpha_{-n}} | 0 \rangle_c .$$

(3.10)

With this definition $F$ has a genus expansion $F = F_0 + \frac{1}{\mu^2} F_1 + \cdots$. This formula is an enormous simplification over previous expressions for $c = 1$ amplitudes. The generating function for all amplitudes is

$$Z = e^{\mu^2 F} .$$

(3.11)

4. $W_{1+\infty}$ constraints

In correlation functions of tachyons with integer (Euclidean) momentum, the bounce factors $R_q$ of (3.3) simplify due to the following identity

$$R^*_\xi R_{n-\xi} = (-i\mu)^{-n} \left( \frac{1}{2} - i\mu - \xi \right)_n .$$

(4.1)

This is valid to all orders in perturbation theory. (In the “standard” potential it also holds nonperturbatively if $n \in 2\mathbb{Z}$. ) Note that equation (2.9) holds for generic momenta; the results for integer momenta are defined by continuity. Working with the generating functional of all amplitudes (3.11) we have:

$$\frac{i\beta}{\mu} \frac{\partial Z}{\partial t_n} = \langle 0 | e^{i\mu \sum_{n \geq 1} t_n \alpha_n} (S \alpha_{-n} S^{-1}) S e^{i\mu \sum_{n \geq 1} \bar{t}_n \alpha_{-n}} | 0 \rangle$$

$$= \oint dw \int dz \left[ \sum_{m \in \mathbb{Z}} R_{p_m} R^*_{n/\beta - p_m} \left( \frac{w}{z} \right)^m \right]$$

$$\langle 0 | e^{i\mu \sum_{n \geq 1} t_n \alpha_n} \psi(z) \bar{\psi}(w) S e^{i\mu \sum_{n \geq 1} \bar{t}_n \alpha_{-n}} | 0 \rangle$$

(4.2)

At the self-dual radius $\beta = 1$, where all tachyon momenta are integral, we may simplify the sum on $m$ using (4.1)

$$(-i\mu)^{-n} (-i\mu + z \frac{\partial}{\partial z})_n \sum_{m \in \mathbb{Z}} \left( \frac{w}{z} \right)^m$$

(4.3)

the latter sum acting like a delta function. Now integrate by parts and use the identity

$$(-i\mu - z \frac{\partial}{\partial z})_n = (-z)^n z^{-i\mu} \left( \frac{\partial}{\partial z} \right)^n z^{i\mu} .$$

(4.4)
It is convenient to bosonize $\psi(z) = e^{\phi(z)}$ and shift the zero mode:

$$\tilde{\phi}(z) = \phi(z) + i \mu \log z . \quad (4.5)$$

Taking the operator product of the two exponentials in $\phi$, and using the delta function and charge conservation we find the operator:

$$\oint dw(i\mu)^{-n} \frac{1}{n+1} : e^{-\tilde{\phi}(w)} \partial_w^{n+1} e^{\tilde{\phi}(w)} : . \quad (4.6)$$

Now go to the coherent state basis in the $t_n$'s, and redefine the scalar field by a factor of $i\mu$ to obtain the final result:

$$\frac{\partial F}{\partial t_n} = Z^{-1} \oint dw \frac{(i\mu)^{-(n+1)}}{n+1} : e^{-i\mu \varphi(w)} \partial_w^{n+1} e^{i\mu \varphi(w)} : Z \quad (4.7)$$

where

$$\varphi(w) = \frac{1}{w} + \sum_{n>0} n t_n w^{n-1} - \frac{1}{\mu^2} \sum_{n>0} \frac{\partial}{\partial t_n} w^{-n-1} . \quad (4.8)$$

The genus zero result of [18] is easily obtained from this as the leading term at large $\mu$. (Note that this was obtained at $\beta = \infty$ but genus zero correlators are independent of $\beta$ [17].)

The operators $P^{(n)}(z) = : e^{-\tilde{\phi}(z)} \partial^n e^{\tilde{\phi}(z)} :$ and their derivatives generate the algebra $W_{1+\infty}$ [19]. The standard generators are related to these by

$$W^{(n)}(z) = \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)!} \frac{((n-l)_l)}{(2n-2)_l} \partial^l P^{(n-l)}(z) . \quad (4.9)$$

The rescaling of the scalar field required to obtain (4.8) is simply a change of basis effected by the operator: $e^{\log(i\mu) \pi_{\tilde{\phi}}} :$ where $\pi_{\tilde{\phi}}$ is the momentum conjugate to $\tilde{\phi}$. Inserting this we can rewrite (4.7) as

$$\frac{\partial Z}{\partial t_n} = W^{(n+1)}_{-n} Z$$

$$W^{(n+1)}_{-n} \equiv \oint dz W^{(n+1)}(z) . \quad (4.10)$$

Generalizations of (4.10) to other radii $\beta \neq 1$ follow from (4.2).

In the conclusions we comment briefly on the relation of this result to the many other occurrences of $W_{\infty}$ in this subject.
5. Tau-functions and the Kontsevich-Penner matrix integral

In this section we will point out that the above reformulation of the generating functional of the $c = 1$ string represents mathematically a $\tau$-function of the Toda Lattice hierarchy. The Toda Lattice naturally contains the KP and KdV hierarchies, and thus the $c = 1$ results are closely related to the expressions obtained for $c < 1$. We will also show how to rewrite the partition function (at the self-dual radius) as a matrix integral, generalizing expressions previously considered by Kontsevich [7] and Penner [20].

5.1. Grassmannians and tau-functions

Let us first briefly explain the notion of a tau-function and its relation with the universal Grassmannian. For more details see e.g. [21] and [22]. We will focus here on the relation with conformal field theory instead of the Lax pair formulation.

Consider a two-dimensional free chiral scalar field $\varphi(z)$, with the usual mode expansion

$$\partial \varphi(z) = \sum_{n} \alpha_n z^{-n-1}.$$  \hspace{1cm} (5.1)

The reader is encouraged to think about this scalar field as the target space tachyon field at spatial infinity with a periodic Euclidean time coordinate. We have a Hilbert space $\mathcal{H}$ built on the vacuum $|0\rangle$, and as in the case of a harmonic oscillator one can consider coherent states,

$$|t\rangle = \exp \sum_{n=1}^{\infty} it_n \alpha_{-n}$$ \hspace{1cm} (5.2)

and their Hermitian conjugates

$$\langle t| = \langle 0| \exp \sum_{n=1}^{\infty} -it_n \alpha_n$$ \hspace{1cm} (5.3)

(The parameters $t_n$ are considered to be real here.) Now to any state $|W\rangle$ in the Hilbert space $\mathcal{H}$ we can associate a coherent state wavefunction $\tau_W(t)$ by considering the inner product

$$\tau_W(t) = \langle t|W\rangle.$$ \hspace{1cm} (5.4)

This function is a tau-function of the KP hierarchy if and only if the state $|W\rangle$ lies in the so-called Grassmannian.
To explain the concept of the Grassmannian we have to turn to the alternative description of this chiral conformal field theory in terms of chiral Weyl fermions $\psi(z), \bar{\psi}(z)$ by means of the well-known bosonization formulas:

$$i\partial \varphi = \bar{\psi}\psi, \quad \psi = e^{i\varphi}, \quad \bar{\psi} = e^{-i\varphi}.$$ (5.5)

Loosely speaking, the Grassmannian can be defined as the collection of all fermionic Bogoliubov transforms of the vacuum $|0\rangle$. That is, the state $|W\rangle$ belongs to the Grassmannian if it is annihilated by particular linear combinations of the fermionic creation and annihilation operators.

$$(\psi_{n+\frac{1}{2}} - \sum_{m=1}^{\infty} A_{nm} \psi_{-m+\frac{1}{2}})|W\rangle = 0, \quad n \geq 0,$$ (5.6)

or equivalently,

$$|W\rangle = S \cdot |0\rangle, \quad S = \exp \sum_{n,m} A_{nm} \bar{\psi}_{-n-\frac{1}{2}} \psi_{-m+\frac{1}{2}}.$$ (5.7)

Note that the operator $S$ can be considered as an element of the infinite-dimensional linear group, $S \in GL(\infty, \mathbb{C})$.

By replacing the vacuum $|0\rangle$ by the state $|W\rangle$, we simply made another decomposition into positive and negative energy states, and filled these new negative energy states. The positive energy wave-functions are no longer of the form $z^n \ (n \geq 0)$ but are now given by the functions

$$v_n(z) = z^n - \sum_{m=1}^{\infty} A_{nm} z^{-m}.$$ (5.8)

If one prefers the language of semi-infinite differential forms, we have a formula of the form

$$|W\rangle = v_0 \wedge v_1 \wedge \ldots$$ (5.9)

In the above fashion one generates solutions to the KP hierarchy. This construction can be extended to give a tau-function for the Toda Lattice hierarchy by considering a second set of times $\bar{t}_k$, as discussed in detail in [23]. In terms of our conformal field theory, the Toda tau-function is simply obtained as

$$\tau(t, \bar{t}) = \langle t | S | \bar{t} \rangle,$$ (5.10)

$^4$ Since we do not wish to flaunt tradition we change conventions for bosonization in this section relative to the previous sections.
with \( |\bar{t}\rangle \) and \( \langle t| \) the coherent states (5.2) and (5.3) and \( S \) a general \( GL(\infty,\mathbb{C}) \) element, i.e. an exponentiated fermion bilinear of type (5.7). The \( S \)-matrix (3.9) of the \( c=1 \) string is definitely of this form, which allows us to conclude that the string partition function can indeed (after a rescaling \( t_n \to \mu \cdot t_n \)) be identified as a Toda Lattice tau-function.

Instead of taking the inner product of the state \( |W\rangle \) with a coherent bosonic state, one can also consider fermionic \( N \)-point functions (see e.g. [9]). In fact, one finds in this way a simple expression in terms of an \( N \times N \) determinant of the wave-functions (5.8)

\[
\langle N|\psi(z_1)\ldots\psi(z_N)|W\rangle = \det v_{j-1}(z_i).
\] (5.11)

Using the bosonization formulas, one recognizes this correlation function as a special coherent state where the parameters \( t_n \) are given by

\[
t_n = \sum_{i=1}^{N} \frac{1}{n} z_i^{-n}.
\] (5.12)

With this choice of parameterization, and after taking into account a normal ordering contribution, the tau-function can be written as

\[
\tau(t) = \frac{\det v_{j-1}(z_i)}{\Delta(z)},
\] (5.13)

with \( \Delta(z) \) the Vandermonde determinant \( \Delta(z) = \det z_i^{j-1} = \prod_{i>j} (z_i - z_j) \). We apply this result in the next section.

5.2. Kontsevich integrals and the \( c<1 \) models.

Since our expression for the \( c=1 \) partition function is very analogous to the result found for the \( c<1 \) string theories, we will briefly summarize the latter (see [2,3,4]). Recall that the \( c<1 \) matrix models naturally give rise to a universal set of observables \( O_n \) \((n = 1, 2, \ldots)\) whose correlation functions

\[
\langle O_{n_1} \ldots O_{n_s} \rangle_g
\] (5.14)

at a specific genus \( g \) are unambiguously determined. The generating functional \( \tau(t) \) of these correlators has an asymptotic expansion in the string coupling constant \( \lambda \)

\[
\log \tau(t) = \sum_{g=0}^{\infty} \lambda^{2-2g} \langle \exp \sum_{n=1}^{\infty} t_n O_n \rangle_g.
\] (5.15)
The techniques of the double-scaled matrix models leads to two important results. First, the partition function $\tau(t)$ is a tau-function of the KP hierarchy, that is, it can be written as

$$\tau(t) = \langle t | W \rangle = \langle t | S | 0 \rangle ,$$

for some state $|W\rangle$ and matrix $S \in GL(\infty, \mathbb{C})$. Secondly, all minimal models of type $(p,q)$ with fixed $p$ belong to one KP orbit. More precisely, relative to a convenient choice of origin, the $(p,q)$ model is obtained at the value $t_k = \delta_{k,p+q}$. Furthermore, the KP hierarchy reduces to the $p^{th}$ KdV hierarchy, which implies that all correlation functions of the operators $O_n$ with $n \equiv 0 \pmod{p}$ vanish.

The state $|W\rangle$ corresponding to this orbit is most simply described at the $(p,1)$ point, where a description in terms of topological field theory can be given. For its basis one can take the wave-functions

$$v_n(z) = \sqrt{\frac{ipz^{p-1}}{2\pi \lambda}} e^{\frac{ipz^{p+1}}{p+1}/\lambda} \cdot \int_{-\infty}^{\infty} dy \cdot \frac{y^n \cdot e^{i(z^p y - \frac{y^{p+1}}{p+1})/\lambda}}{\Delta(z)} ,$$

where the normalization is chosen such that we have the appropriate asymptotic expansion

$$v_n(z) = z^n (1 + O(z^{-1})) .$$

Since the wave-functions are moments in a Fourier transform, the fermionic formula (5.13) can be explicitly evaluated, and gives rise to a so-called Kontsevich integral [7] (see also [8,9,24])

$$\tau(t) = c \cdot \int DY \cdot e^{i Tr(Z^p Y - \frac{Y^{p+1}}{p+1})/\lambda} .$$

Here $Y$ and $Z$ are both $N \times N$ Hermitian matrices, and the parameterization of the KP times $t_k$ in terms of the matrix $Z$ is

$$t_k = \frac{\lambda}{k} Tr Z^{-k} .$$

This result can be generalized to the ‘generalized Kontsevich model’ [8] which features an arbitrary potential $V(z)$

$$\tau(t) = c(Z) \cdot \int DY \cdot e^{i Tr(V'(Z) Y - V(Y))/\lambda} .$$

with

$$c(Z) = (2\pi i/\lambda)^{-N^2/2} \cdot \det V''(Z) \cdot \frac{\Delta(V'(Z))}{\Delta(Z)} \cdot e^{i Tr(V(Z) - V'(Z) Z)/\lambda} .$$

It has been noticed by many authors that the case $p = -1$ (i.e. a logarithmic potential $V(z) = \log z$) is likely associated with the $c = 1$ model [25]. We will now proceed to show that this is indeed the case.
5.3. The Kontsevich-Penner integral

We have seen that the $c = 1$ partition function can be succinctly written as a tau-function of the Toda Lattice hierarchy

$$\tau(t, \bar{t}) = \langle t | S | \bar{t} \rangle . \quad (5.23)$$

For fixed $\bar{t}_k$ we recover a tau-function of the KP hierarchy, which we can study with the techniques of the previous subsection. Indeed the operators $O_n$ of the minimal models should now be compared to the outgoing tachyons of the $c = 1$ model.

We want to determine in more detail the element $W(\bar{t})$ in the Grassmannian that parametrizes this particular orbit of the KP flows. To this end we have to consider the state

$$|W(\bar{t})\rangle = S \cdot U(\bar{t}) \cdot |0\rangle, \quad U(\bar{t}) = \exp \sum_{n=1}^{\infty} i\mu \bar{t}_n \alpha_{-n} . \quad (5.24)$$

We will describe $|W(\bar{t})\rangle$ by giving a basis $v_k(z; \bar{t})$, $k \geq 0$, of one-particle wave-functions. First we observe that the operator $U(\bar{t})$ acts on the wave-functions $z^n$ by simple multiplication

$$U(\bar{t}) : z^n \to \exp \left( \sum i\mu \bar{t}_k z^{-k} \right) \cdot z^n . \quad (5.25)$$

Similarly we have for the action of $S$ a multiplication

$$S : z^n \to R_{pn} \cdot z^n . \quad (5.26)$$

We have already seen that the reflection factors $R_{pn}$ contain all the relevant information of the $c = 1$ matrix model. At radius $\beta$ they can be chosen to be

$$R_{pn} = (-i\mu)^{-n+\frac{1}{\beta}} \frac{\Gamma(\frac{1}{2}-i\mu + \frac{n+\frac{1}{\beta}}{\beta})}{\Gamma(\frac{1}{2}-i\mu)} . \quad (5.27)$$

(Recall, we are only interested in the perturbative part in $\mu^{-1}$ of this expression.) The usual vacuum $|0\rangle$ is spanned by the non-negative powers $z^k$. Therefore the basis elements $v_k(z; \bar{t})$ of $W$ are simply determined as

$$v_k(z; \bar{t}) = c_k \cdot S \circ U(\bar{t}) z^k , \quad (5.28)$$
with a normalization constant $c_k$ such that $v_k(z; 0) = z^k$. (This corresponds to the normal ordering of the $S$-matrix in (3.9).) Since the reflection factor is basically a gamma function, the result can be expressed as a Laplace transform

$$v_k(z; \bar{t}) = c'(z) \cdot \int_0^\infty dy \cdot y^k \cdot y^{-i\mu \beta + (\beta - 1)/2} e^{i\mu (y/z)^\beta} \exp \left( \sum_i i\mu \bar{t}_k y^{-k} \right)$$  \tag{5.29}$$

Here the constant $c'(z)$ is given by

$$c'(z) = \beta \frac{(-i\mu/z^\beta)^{1/2} - i\mu}{\sqrt{\Gamma(1/2 - i\mu)}}.$$  \tag{5.30}$$

These integral representations are of Kontsevich type if and only if $\beta = 1$, that is, only at the self-dual radius. Indeed in that case we have

$$v_k(z; \bar{t}) = c'(z) \cdot \int_0^\infty dy \cdot y^k \cdot \exp i\mu \left( y/z - \log y + \sum \bar{t}_k y^{-k} \right)$$  \tag{5.31}$$

Therefore, following the procedure in [8,9], we can write the following matrix integral representation for the generating functional. Define the integral

$$\sigma(Z, \bar{t}) = \int dY e^{i\mu Tr[YZ^{-1} + V(Y)]},$$  \tag{5.32}$$

where

$$V(Y) = -\log Y + \sum \bar{t}_k Y^{-k},$$  \tag{5.33}$$

and we integrate over positive definite matrices $Y$. Then we have

$$\tau(t, \bar{t}) = \frac{\sigma(Z, \bar{t})}{\sigma(Z, 0)},$$  \tag{5.34}$$

with the parameterization

$$t_n = \mu^{-1} \cdot \frac{1}{n} Tr Z^{-n}.$$  \tag{5.35}$$

Note that with this normalization $\tau(t, 0) = 1$, which is appropriate since we consider normalized correlation functions. In order to write down the result (5.34) we had to treat the incoming and outgoing tachyons very differently, parametrizing the outgoing states through (5.35), whereas the coupling coefficients to the incoming states enter the matrix integral in a much more straightforward fashion. Equation (5.34) should be considered as an asymptotic expansion in $\mu^{-1}$, but, for small enough $t_k, \bar{t}_k$ the expansion in these variables will be convergent. In some cases, (e.g. the sine-Gordon case considered in [26]) the expansion has a finite radius of convergence, and as we increase $|t_k|$ beyond the radius of convergence we can have phase transitions.
5.4. The partition function

Matrix integrals of the above type have appeared in the work of the mathematicians Harer and Zagier [27] and Penner [20] in their investigations of the Euler characteristic of the moduli space $\mathcal{M}_{g,s}$ of Riemann surfaces with $g$ handles and $s$ punctures. (See [28] for more details on these wonderful calculations.) The double scaling limit of this so-called Penner integral was considered by Distler and Vafa [29] who also speculated on the relation with $c = 1$ string theory. Their work has been followed by a number of papers concerned with double scaling limits and multi-critical behaviour of matrix models with logarithmic potentials [25]. All these papers considered essentially the case $Z = 1$ and $\bar{t}_n = 0$, in the notation of (5.32).

Distler and Vafa noticed that — after a double scaling limit and an analytic continuation — the Penner matrix integral could reproduce the $c = 1$ partition function at the self-dual radius $\beta = 1$. Recall that the free energy at that radius is given by [30]

$$\frac{\partial^2 F}{\partial \mu^2} = \text{Re} \int_0^\infty dx \frac{e^{-i\mu x}}{x} \left( \frac{x/2}{\sinh x/2} \right)^2,$$

(5.36)

and has an expansion

$$F = \frac{1}{2} \mu^2 \log \mu - \frac{1}{12} \log \mu + \sum_{g = 2}^\infty (-1)^g \frac{B_{2g}}{2g(2g - 2)} \mu^{2 - 2g}.$$  

(5.37)

(Up to analytic terms in $\mu$.) This makes one wonder whether our result (5.34) can be sharpened to give the unnormalized correlation functions.

To this end let us put the incoming coupling constants $\bar{t}_k$ to zero (and thereby also $t_k = 0$) and take a closer look at the integral

$$\sigma(Z) = \int dY e^{i\mu Tr[Y Z^{-1} - \log Y]}.$$  

(5.38)

First of all it has a trivial $Z$-dependence

$$\sigma(Z) = (\det Z)^{N - i\mu} \cdot \sigma(1).$$  

(5.39)

Actually, it is convenient to work with the quantity $\tilde{F}$ defined by

$$e^{\tilde{F}} = (\pi i/\mu)^{-N^2/2} e^{-i\mu N^2/2} \cdot \sigma(1).$$  

(5.40)
As an asymptotic expansion in $1/\mu$ it has the representation
\[
e^{\tilde{F}} = \frac{\int dY \cdot e^{i\mu \sum_{k=2}^{\infty} \frac{1}{k} \text{Tr} Y^k}}{\int dY \cdot e^{i\mu \frac{1}{2} \text{Tr} Y^2}}.
\]
(5.41)

This is known as the Penner integral [20] and is usually considered in ‘Euclidean signature’, i.e. after analytic continuation $\mu = i\nu$, $\nu$ real and positive.

The quantity $\tilde{F}$ has a beautiful geometrical interpretation, calculating the virtual Euler characteristic of moduli space in the open string field theory cell decomposition of moduli space. This is essentially the same description of moduli space used by Kontsevich [7]. The expansion of $\tilde{F}$ reads
\[
\tilde{F} = \sum_{g=0}^{\infty} \sum_{s=1}^{\infty} N^s (-i\mu)^{2-2g-s} \tilde{F}_{g,s}.
\]
(5.42)

(Here $s \geq 3$ in the case $g = 0$.) The coefficients are directly related to the Euler numbers
\[
\tilde{F}_{g,s} = \chi(M_{g,s}).
\]
(5.43)

The $1/\mu$ asymptotics of the integral (5.41) can be evaluated using the methods described in [28] to give
\[
e^{\tilde{F}} = e^{-i\mu N \left( \frac{2\pi i}{\mu} \right)^{-N/2} \left( (-i\mu)^{i\mu} \Gamma(-i\mu) \right)^N \prod_{p=1}^{N-1} (1 - p/i\mu)^{N-p}}
\]
(5.44)

from which one may obtain the formulae:
\[
\tilde{F}_{g,s} = \frac{(-1)^s B_{2g}}{2g(2g - 2 + s)} \binom{2g - 2 + s}{s},
\]
(5.45)

It is important that the terms with $s = 0$, that is, the surfaces without punctures, are absent.

We can explicitly do the summation over $s$ in (5.42) to obtain
\[
\tilde{F} = \sum_{g=0}^{\infty} \mu^{2-2g} \tilde{F}_g(N/i\mu).
\]
(5.46)

with
\[
\tilde{F}_g(x) = \frac{(-1)^g B_{2g}}{2g(2g - 2)} [1 - (1 - x)^{2-2g}], \quad g \geq 2,
\]
\[
\tilde{F}_1(x) = -\frac{1}{12} \log(1 - x),
\]
\[
\tilde{F}_0(x) = -\frac{1}{2} (1 - x)^2 \log(1 - x) + \frac{3}{4} x^2 - \frac{1}{2} x.
\]
(5.47)
The double-scaling limit considered by Distler and Vafa in [29] keeps \( N - i\mu \) fixed, while sending \( N, \mu \to \infty \) (and \( x \to 1 \) in (5.47)). This is clearly only possible for imaginary \( \mu \), which is precisely the case they study. However, here we want to consider a simpler limit in which \( \mu \) is kept fixed, but \( N \) tends to infinity. We already mentioned that the parameterization (5.35) only makes sense in this limit. Indeed, the absence of a double scaling limit is very much in the spirit of Kontsevich integrals. The contribution for genus 2 or higher have a smooth limit, as is evident from (5.47). (Recall, we send \( x \to \infty \).) However, we have to worry about the genus zero and one pieces, which have to be corrected by hand. (This is by the way also true for the double scaling limit.)

Combining all ingredients we obtain the following final result for the unnormalized generating functional for the \( c = 1 \) string theory

\[
\tau(t, \bar{t}) = c(Z) \cdot \int dY \exp i\mu \text{Tr} \left[ YZ^{-1} - \log Y + \sum \bar{t}_k Y^{-k} \right].
\] (5.48)

where the normalization constant is given by

\[
c(Z) = e^{-i\mu N} (2\pi i/\mu)^{N^2/2} (\det Z)^{i\mu - N} \\
(1 + iN/\mu)^{\frac{1}{2}(\mu + iN)^2 + \frac{1}{12} \mu^2 - \frac{1}{12} e^{\frac{3}{4} N^2/\mu^2} - \frac{1}{2} N/\mu). \] (5.49)

The expression (5.48) has a smooth large \( N \) limit.

6. Other Backgrounds

The results of the previous sections comprise in principle a calculation of the partition function in arbitrary tachyon backgrounds (subject to the equations of motion). The full space of classical backgrounds in the theory includes in addition to these excitations of the “discrete states” corresponding to global modes like the radius of the 1D universe and generalizations thereof. Of these, the ones best understood in terms of the matrix model are the zero-momentum excitations which are thought to be represented by variations in the double-scaled potential. In this section we study the dependence of the amplitudes on these extra parameters. We note that in principle the formulation of section three applies in arbitrary potentials. What we add here is a study of the variation of the reflection factor \( R_q \), hence of the partition function, under variations of the potential.
6.1. Dependence on $\beta$

The most obvious parameter is $\beta$, the radius at which we compactify the scalar field $X$. The formulas of section four are valid for arbitrary $\beta$, however as pointed out in [17], correlation functions at different radii are related. The relation is most simply written in terms of rescaled couplings $t_n$. Defining

$$\hat{F}[t_n, i_n; \beta; \mu] \equiv \mu^2 F[\mu^{2\beta} t_n, \mu^{2\beta} i_n; \beta; \mu] \quad (6.1)$$

so that derivatives of $\hat{F}$ yield correlation functions of $T_q$, we have

$$F[t_n, i_n; \beta; \mu] = \frac{1}{2 \beta \partial \mu} \hat{F}[\mu^{2\beta} t(n/\beta), \mu^{2\beta} i(n/\beta); \infty; \mu] \quad (6.2)$$

Comparing this with the $\beta \to \infty$ limit of the previous calculations is a pretty consistency check. As an example, set $\beta = 1$ and consider the two-point function. Computing the one-loop graph we find

$$\frac{\partial \hat{F}[t_n, i_n; 1; \mu]}{\partial t_n \partial i_n} = \mu^n \sum_{m=0}^{n-1} R^*_p R_{n-p} = i^n \sum_{m=0}^{n-1} (-i\mu - m)_n \quad (6.3)$$

Inverting the operator in (6.2) as

$$\sin \left(\frac{\partial}{2 \partial \mu}\right) \hat{F} = \int_{-1/2}^{1/2} ds \hat{F}[\mu \to \mu + is] \quad (6.4)$$

we obtain

$$\langle T_n T_{-n} \rangle_{\beta=\infty} = i^n \int_0^{1} dx \left(\frac{1}{2} - i\mu - x\right)_n$$

in agreement with the result of [10].

6.2. Other zero-momentum modes

The matrix model naturally suggests candidate representatives of the special states at zero $X$ momentum. Operators with the appropriate quantum numbers may be introduced as generating variations in the double-scaled potential $V(\lambda)$. Their correlators may thus be studied by analysis of the variation of the partition function $Z$ computed above under these changes in $V$. From the definition of $I(q, \lambda_1, \lambda_2)$ we can obtain directly constraints on the
variation of $R_q$. Essentially these follow upon integration by parts from the linear Gelfand-Dikii equation satisfied by a product of Sturm-Liouville eigenfunctions [31]. Explicitly, we have

$$L_{q,k} R_q = 0 \quad k \geq -1$$

$$L_{q,k} = -k(k^2 - 1) \frac{\partial}{\partial s_{k-2}} + 4iq(k+1) \frac{\partial}{\partial s_k} + 2 \sum_{p \geq 0} s_p (2k + p + 2) \frac{\partial}{\partial s_{p+k}}, \quad (6.6)$$

where the space of potentials is parametrized by the formal expansion $V = \sum_{n \geq 0} s_n \lambda^n$. The operators $L_{q,k}$ for any fixed $q$ are seen to satisfy the commutation relations of (one half of) the Virasoro algebra. These were derived from related considerations in [32]; the details of the derivation in the present context as well as the relation to this work appear in the appendix.

Via (3.10) these imply constraints on the $V(\lambda)$-dependence of the partition function, since this arises only through $R_q$. These however are nontrivial to write down explicitly. In particular, we note that they do not seem to fit into the $W_{1+\infty}$ algebra discussed in section four. Furthermore, it is easy to see that away from $s_k = 0$ the identity (4.1) ceases to hold. Thus perturbing away from the standard background may break the symmetry of section four. Further work is required to clarify the relation of the various symmetry algebras which appear in this model.

7. Discussion

Some remarks on the $W$-constraints (4.10) are in order. First, it cannot have escaped the reader that these constraints are strongly reminiscent of the famous $W$-constraints of $c < 1$ models coupled to gravity [33]. In these latter models there is only one continuous spacetime coordinate and there is only one set of couplings $t_j$ rather than the $t, \bar{t}$ of the $c = 1$ model. Moreover, closer comparison of the identities reveals some important differences. For example, at $c < 1$ the $t_n$ couple to an infinite set of gravitational descendents, while at $c = 1$ the $t, \bar{t}$ couple to gravitational primaries. Nevertheless, a clearer spacetime interpretation of the $c < 1$ models will probably emerge from a comparison of these identities.

5 In [33] proposals for $c = 1$ flow equations were made by taking the $N \to \infty$ limit of the $W_N$ constraints of the $c < 1$ models. It should be noted that, although our equations have some similarities to the proposals of [33], they are not equivalent.
From the relation of these results to a Kontsevich-type matrix model it appears that
we have taken a step closer to a unified description of all the \( c \leq 1 \) models along the lines
proposed by \cite{8,9}. Moreover, the description (5.48) of the partition function is a strong
hint that the \( c=1 \) correlators have a description in terms of a topological field theory. If
this is so then the present results provide a direct bridge between a topological field theory
at the self-dual radius and the local physics of the \( c = 1 \) tachyon in the uncompactified
theory.

There have been many discussions of \( W_\infty \) symmetry in the \( c = 1 \) system. Our con-
straints are related to the results of \cite{15,34–36}. The other modes of the \( W_\infty \) currents
appearing in equation (4.10) define a set of operators \( \sigma_n(T_q) \) whose correlation functions
are determined by the subleading terms proportional to \( \lambda^{-|q|-2n} \) in the large \( \lambda \) asymp-
totics of the eigenvalue correlators.\(^6\) These “operators” exist at any radius for \( X \) and have
free fermion representations as fermion bilinears. Their correlators are also given by a
Toda tau function generalizing that in (3.10). Note that these operators appear at any
momentum \( q \) and are related to fractional powers of \( \ell \) (or, equivalently, of \( \lambda \)). Therefore,
at generic \( q \) they cannot be the special state operators but rather are related to contact
terms associated to singular geometries created by intersecting macroscopic loops \cite{15}. At
integer \( q \) the distinction between special states and the \( \sigma_n(T_q) \) is less clear. We hope to
return to the subtleties of these contact terms in a future publication.

The \( W_\infty \) symmetry we have discussed might also be related to the \( W_\infty \) Ward identities
of \cite{37,38,39}. In these references the Liouville field is treated as a free field, in other words,
one works at \( \mu = 0 \). One should be cautious about identifying these \( W_\infty \) symmetries with
those of the matrix model. As we have emphasized, the \( W_\infty \) modes of the matrix model
\( \sigma_n(T_q) \) are constructed from the tachyon degrees of freedom in distinction to the \( W_\infty \)
currents of \cite{37,43}. Moreover, our Ward identities are highly nonlinear when expressed
in terms of the correlation functions\(^7\) in contrast to the quadratic identities of \cite{39,43}. Finally the ghost sector of the theory is crucial in \cite{39,43}, leading to many more “special
state operators” at given \( X, \phi \) momenta than are considered in \cite{15,34–36}. Clearly there
is a certain amount of tension between these two approaches and further work is needed
to see if these differences are superficial or essential.

\(^6\) In \cite{16} these operators were denoted \( \sigma_{2n}(O_q) \). In \cite{15} it was pointed out that they only have
contact term interactions.

\(^7\) This is already true at genus zero \cite{18}.
We must emphasize that at $c = 1$ the $W_\infty$-constraints are actually somewhat secondary, since we have an explicit solution of the appropriate Toda tau function given by (3.10). Analogous representations for the $c < 1$ tau functions (at nontopological points) replace the simple operator $S$ by complicated and uncomputable objects like the “star operators” of [14]. This is why the Virasoro constraints at $c < 1$ are essential to the actual computation of amplitudes.

It would be interesting to investigate further the physical properties of these different time-dependent backgrounds. In [18, 26] some results along these lines were discussed. Our result (3.10) should allow a much more complete analysis of the space of time-dependent backgrounds in 2D string theory and the various phase transitions occurring as one increases the coordinates $t_k$. What is needed for further progress is a more effective way to compute the tau function (perhaps from the Kontsevich representation) or a deeper understanding of the infinite dimensional geometry of the associated Grassmannian.

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Appendix A. Dependence on the potential

In this appendix we will derive constraints on the dependence of the free energy upon the double-scaled matrix model potential $V(\lambda)$. We restrict attention to variations of the potential which preserve the asymptotics $V(\lambda) \sim -\frac{1}{4} \lambda^2$ for large $\lambda$. For convenience, we will assume that the asymptotic behavior of $V$ at $\lambda \to -\infty$ is such that the eigenfunctions decay rapidly enough to render all integrals convergent and all boundary terms negligible.

To all orders of perturbation theory, of course, these details are irrelevant.

We wish to compute the change in the correlators (3.1) under a variation $V(\lambda) \to V + \delta V(\lambda)$ preserving the asymptotics. The change in (3.1) is clearly computed from the variation of the resolvent $I$. Thus write

$$\delta I(q, \lambda_1, \lambda_2) = \langle \lambda_1 | \delta \left( \frac{1}{H - \mu - iq} \right) | \lambda_2 \rangle = - \int_{-\infty}^{\infty} d\lambda \delta V(\lambda) I(q, \lambda_1, \lambda) I(q, \lambda, \lambda_2) . \quad (A.1)$$
We now recall the calculation of $I$ from [10] (see appendix A of this work for a detailed calculation for particular potentials). For simplicity let $q > 0$. We will make use of the eigenfunctions of $H = \frac{d^2}{d\lambda^2} - V(\lambda)$ with eigenvalue $z = \mu + iq$. We have for any $V$ with the correct asymptotics two solutions

$$\chi^\pm(z, \lambda) \sim \lambda^{-\frac{1}{2} + i\lambda^2} e^{\pm i\lambda^2}. \quad (A.2)$$

In terms of these we can write the resolvent quite easily by imposing the boundary conditions and the defining property $(H - z)I(q, \lambda_1, \lambda_2) = \delta(\lambda_1 - \lambda_2)$ as in [10]

$$I(q, \lambda_1, \lambda_2) = -i\theta(\lambda_1 - \lambda_2) \left[ \chi^-(z, \lambda_1)\chi^+(z, \lambda_2) + R_q\chi^-(z, \lambda_1)\chi^-(z, \lambda_2) \right] + (\lambda_1 \leftrightarrow \lambda_2). \quad (A.3)$$

The reflection factor $R_q$ contains all the effects of the potential, and for the standard $V$ is given by (3.3). Inserting this into (A.1) and neglecting terms of order $\delta V(\lambda_1, \lambda_2)$ for large $\lambda_i$, we find that a variation of $V$ yields

$$\delta R_q = -i \int_{-\infty}^{\infty} d\lambda \delta V(\lambda) \psi(z, \lambda)^2 \quad (A.4)$$

where $\psi = \chi^+ + R_q\chi^-$ is the solution satisfying the boundary conditions at small $\lambda$. The integrand $F(z, \lambda) = \psi(z, \lambda)^2$ in (A.4) satisfies a differential equation [31] following from that satisfied by $\psi$

$$F''' - 4(V(\lambda) + z)F' - 2V'F = 0 \quad (A.5)$$

where primes denote $\lambda$ differentiation. Let us choose as a convenient set of variations of the potential $\delta V(\lambda) = \varepsilon e^{-\ell \lambda}$. Inserting this in (A.4) and integrating by parts we find

$$[\ell^3 - 4\ell \varepsilon - 4\varepsilon V(-\frac{d}{d\ell}) + 2\varepsilon V'(-\frac{d}{d\ell})] \delta R_q = 0. \quad (A.6)$$

The integration by parts is justified by the limiting conditions we have imposed upon $\psi$ and $\delta V$.

Formally expanding $V = \sum_{n \geq 0} s_n \lambda^n$ the bounce factor becomes a function of the $s_j$: $R_q = R_q[s_1, s_2, \ldots]$. Rewriting $\delta V$ as a motion in $s_j$ and inserting the resulting expression for $\delta R_q$ in (A.6) we obtain (after shifting $s_0$)

$$L_{q,k}R_q = 0 \quad k \geq -1$$

$$L_{q,k} = -k(k^2 - 1) \frac{\partial}{\partial s_{k-2}} + 4iq(k + 1) \frac{\partial}{\partial s_{k}} + 2 \sum_{p \geq 0} s_p(2k + p + 2) \frac{\partial}{\partial s_{p+k}}. \quad (A.7)$$

Footnote 8: The similarity of this to the WdW equation of [16] is no coincidence; setting $z = \mu$ and $\lambda_1 = \lambda_2$ we find that (A.1) is essentially the WdW wavefunction of the cosmological constant.
These constraints were obtained in [32] by different means. We will show that the two results are equivalent, but note here that the present derivation has the advantage of working with potentials with the correct asymptotics throughout, as well as demonstrating explicitly the justification for the various integrations by parts.

The Virasoro constraints in [32] were obtained as differential equations for the cosmological constant one-point function. We have obtained above identical constraints on $R_q$, from which one can derive constraints on the partition function. We will now relate the two quantities, demonstrating that in fact the two sets of constraints coincide. Begin with the formula for the two-point function [10]:

$$\langle T_q T_{-q} \rangle = \int_0^q dx R_x R^*_q - x$$  \hspace{1cm} (A.8)

or its differentiated version:

$$\frac{\partial}{\partial \mu} \langle T_q T_{-q} \rangle = 2 \Im [R_q R^*_0]$$  \hspace{1cm} (A.9)

For $q > 0$ we may write:

$$R_q = e^{i\Theta(\mu + iq; V)}$$  \hspace{1cm} (A.10)

where for $E$ real the pure phase $e^{i\Theta(E; V)}$ is the reflection coefficient for a free fermion of energy $E$ in the double-scaled matrix potential $V(\lambda)$.

We now obtain the specific heat from the limit as $q \to 0$. As explained in [45, 46, 47] the cosmological constant vertex operator in the $c = 1$ theory is given by

$$T_0 = \int_\Sigma \phi e^{\sqrt{2} \phi}$$  \hspace{1cm} (A.11)

and is therefore obtained by the limit

$$\lim_{q \to 0} \frac{1}{q} T_q = T_0.$$  \hspace{1cm} (A.12)

Taking the limit in (A.9) we find the leading order begins at $q^2$, as expected from (A.12). Indeed, quite generally, the low energy theorem of [10] shows that for $n > 2$-point functions if $k < n$ momenta $q_i$ approach zero the amplitude behaves like

$$\prod_{i=1}^k q_i \left( \frac{\partial}{\partial \mu} \right)^k \langle \prod_{k+1}^n V_{q_i} \rangle$$

in accordance with the expectations of Liouville theory. Thus, the apparent $\frac{1}{q}$ divergence in (A.12) does not appear. From the Liouville point of view this may be interpreted as the decoupling of the wrong branch dressing of the vertex operator [46, 47]. The low energy theorem is more subtle in the case $n = 2$. In this case the leading
order behavior is $q + \mathcal{O}(q^2)$. The first term is $\mu$-independent and physically sensible, being the inverse on-shell propagator at genus zero. The second term of order $\mathcal{O}(q^2)$ defines the correct zero-momentum two-point function.

Taking the limit of (A.8) and bearing in mind the above remarks we obtain the equation

$$\langle T_0 T_0 \rangle = -\Theta'(\mu; V) \quad (A.13)$$

and hence the nonperturbative one-point function and vacuum energy are

$$\langle T_0 \rangle = -\Theta(\mu; V) = i \log R(\mu; V) \quad . \quad (A.14)$$
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Figure Captions

Fig. 1. (a) A pictorial version of the integral $I$ for positive momentum. (b) A pictorial version of the integral $I$ for negative momentum

Fig. 2. Incoming and outgoing vertices. The dotted line carrying negative (positive) momentum $q_i$ should be thought of as an incoming (outgoing) boson with energy $|q_i|$. Momentum carried by lines is always conserved as time flows upwards.

Fig. 3. A real history as a composition of three maps