Uphill currents in coupled, driven zero–range processes

Emilio N.M. Cirillo,† Matteo Colangeli,‡ and Ronald Dickman

1 Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, via A. Scarpa 16, I–00161, Roma, Italy.
2 Dipartimento di Ingegneria e Scienze dell’Informazione e Matematica, Università degli Studi dell’Aquila, via Vetoio, 67100 L’Aquila, Italy.
3 Departamento de Física and National Institute of Science and Technology for Complex Systems, ICEx, Universidade Federal de Minas Gerais, C.P. 702, 30123–970, Belo Horizonte, Minas Gerais, Brazil.

In particle systems subject to a nonuniform drive, particle migration is observed from the driven to the non–driven region and vice–versa, depending on details of the hopping dynamics, leading to apparent violations of Fick’s law and of steady–state thermodynamics. We propose and discuss a very basic model in the framework of the zero–range process, in which this phenomenon is observed and fully explained.

I. INTRODUCTION

Particle transport in far–from–equilibrium systems often exhibits features that run counter to intuition based on equilibrium thermodynamics. One class of paradoxical behavior is that of “uphill currents,” that is, migration of particles from regions of lower to higher density in the absence of an interaction or external potential, in apparent violation of Fick’s law. Examples are found in driven lattice gases employed in the study of steady–state thermodynamics. Since these observations have eluded explanation, it is of interest to have simple, analytically solvable examples in which uphill currents arise. The purpose of this work is to analyze such a system.

A motivation for the present study is Ref. [1] in which a lattice gas with nearest–neighbor exclusion (NNE) and a driven hopping dynamics is considered on a two–dimensional (2D) square torus; the system is observed to relax to a nonequilibrium steady state. The NNE rule means that if a site is occupied, all of its nearest neighbors must be empty. Thus a particle can only jump to sites which are vacant, and with all their nearest neighbors vacant (other than the site occupied by the particle just before it jumps).

The stochastic dynamics is driven along one direction ($x$, say) of the square lattice: in the case of nearest–neighbor (NN) hopping, each particle attempts to jump along the $+x$ direction at a rate $(1 + p)/4$ while the rate for hops in the opposite direction is $(1 − p)/4$. Particles attempt to hop along the $±y$ directions with rate $1/4$. For nearest– and next–nearest neighbor (NNN) hopping, the rates are $((1 ± p))/8$ for displacements with a component along the $±x$ direction, and $1/8$ for hopping along the $±y$ directions. Thus $p = 0$ corresponds to equilibrium, and $p = 1$ to maximum drive (displacements with a projection along the $−x$ direction forbidden). (Given the periodic boundaries, the bias in the hopping rates cannot be associated with a single–valued potential energy function, and the system cannot be described via equilibrium statistical mechanics.)

Monte Carlo simulations of the driven NNE lattice gas show that in a uniformly driven system, at low densities, the system attains a spatially uniform nonequilibrium stationary state (NESS). As shown in [1], it is possible to define a chemical potential $\mu(\rho, p)$ in this NESS, such that when two such systems are allowed to exchange particles at a vanishingly small rate, the densities at coexistence (i.e., at long times, when the net particle flux is zero) is predicted by equating the chemical potentials that describe the two systems in isolation. Here it is important to note that particle transfer occurs directly between the two uniform steady states, i.e., two tori free of boundaries. The vanishingly small exchange rate guarantees that (subject to the constraint of fixed total particle number) the two systems are statistically independent.

Another, in principle more realistic, manner of placing two systems in contact is through particle exchange across a boundary. In equilibrium, this form of contact leads to the same steady–state behavior as exchange between uniform systems. To study this issue in the driven lattice gas, the lattice is partitioned into two equal parts,
with a boundary parallel to the drive (that is, along the \(x\) direction). In the upper region particles experience a rightward drift, while the dynamics is unbiased in the lower region \([1]\). The usual dynamics allows particles to hop between the driven and undriven regions. (Note that there is no bias in the hopping rates along this direction, perpendicular to the drive.) If particles are allowed to jump only to nearest neighbor sites, in the stationary state mass migration towards the driven region is observed. On the other hand, if jumps to next to the nearest neighbor sites are also allowed, the mass displacement is observed in the opposite direction, from the non-driven to the driven region. In both cases, the stationary densities in the two regions cannot be predicted by equating the chemical potentials determined from the uniform systems. This is consistent with the work of Guiotto and Bertin \([2]\), who show that intensive parameters such as chemical potential can be defined if the transition rates satisfy a factorization condition. The nonuniform density profile that arises in the vicinity of the boundaries between driven and undriven regions violates this condition.

A possible interpretation of this remarkable mass migration phenomenon is that in the first case a particle typically spends more time in the upper region rather than in the lower one. The opposite should happen in the second case. Although a simplistic argument in terms of effective diagonal barriers is mentioned in \([1]\), a detailed explanation of this phenomenon in terms of the exclusion interaction and hopping dynamics is lacking. In this note we propose a simple model consisting of two zero-range processes on rings, in which mass migration between driven and nondriven parts is observed and fully explained in terms of the typical times spent by particles in the two regions. The observed mass transport across different regions of the domain shares some interesting features with the uphill diffusion phenomenon discussed in \([3, 5]\) in the context of lattice gas or spin models, and with particle migration on a half-driven ladder \([6]\).

The remained of this work is organized as follows. In Sec. \([II]\) we define the model and present its stationary solution. Our analytic and simulation results are discussed in Sec. \([III\,A]\) illustrating uphill currents. The connection between uphill currents and particle sojourn times, and a connection between the latter and the gambler’s ruin problem, are analyzed in Sec. \([III\,B]\) Some interesting aspects of the particle density profiles are considered in Sec. \([III\,C]\) followed, in Sec. \([IV]\) by a summary and discussion of our findings.

\[ W_{i \pm 1, i} = \frac{1}{2} \quad \text{and} \quad W_{L-1, L} = W_{1, L} = \frac{1}{2} \quad \text{(1)} \]

for \(i = 2, \ldots, \ell - 1, \ell + 1, \ldots, L - 1, \)

\[ W_{L, 1} = W_{2, 1} = \frac{1/2}{1 + \psi} \quad \text{and} \quad W_{L+1, 1} = \frac{\psi}{1 + \psi} \quad \text{(2)} \]

with \(\psi \geq 0, \)

\[ W_{\ell \pm 1, \ell} = \frac{1/2}{1 + \sigma} \quad \text{and} \quad W_{L+\ell, \ell} = \frac{\sigma}{1 + \sigma} \quad \text{(3)} \]

with \(\sigma \geq 0. \) The hopping probabilities starting from a site in the upper ring are chosen similarly, the main difference being the presence of the drift \(\varepsilon \in [0, 1/2]: \)

\[ W_{i \pm 1, i} = \frac{1}{2} \pm \varepsilon, \quad W_{2L-1, 2L} = \frac{1}{2} - \varepsilon, \quad W_{L+1, 2L} = \frac{1}{2} + \varepsilon \quad \text{(4)} \]
for $i = L + 2, \ldots, L + \ell - 1, L + \ell + 1, \ldots, 2L - 1$,

$$W_{2L, L+1} = \frac{1}{2} - \frac{\varepsilon}{1 + \sigma}, W_{L+2, L+1} = \frac{1}{2} + \frac{\varepsilon}{1 + \sigma}, W_{1, L+1} = \frac{\sigma}{1 + \sigma}$$

and

$$W_{L+\ell \pm 1, L+\ell} = \frac{1}{2} \pm \frac{\varepsilon}{1 + \psi} \quad \text{and} \quad W_{\ell, L+\ell} = \frac{\psi}{1 + \psi}.$$ 

In Fig. 1 where $n$ denotes the number of particles at the generic site, we show a schematic representation of the model with the particle hopping rates.

We note that, since all transfer rates are proportional to $n$, each particle moves independently. Each particle has a probability of $1/N$ to be selected as the next to move, and when selected, it leaves its current position with probability 1.

Following the recipe provided, for instance, in the nice review [2] Section 2] the stationary measure on the configuration space comprised of the states $n = (n_1, \ldots, n_{2L})$ such that $n_1 + \cdots + n_{2L} = N$ is

$$\nu(n) = \frac{1}{Z} \prod_{i=1}^{2L} s_{n_i}^{n_i} \quad \text{with} \quad Z = \sum_n \prod_{i=1}^{2L} s_{n_i}^{n_i}$$

(5)

where the sum defining the partition function $Z$ is taken over configuration space and the fugacities $s_i$ satisfy the equation

$$s_j = \sum_{i=1}^{2L} s_i W_{j,i},$$

(6)

see [7 equation (16)]. In the stationary state the mean number of particles at the generic site $i$ is

$$\rho_i = \frac{1}{Z} \sum_n n_i \prod_{i=1}^{2L} s_{n_i}^{n_i} = N \left[ \sum_{j=1}^{2L} s_j \right]^{-1} s_i,$$

(7)

see [7 equation (11)] and [8 equation (37)].

The main quantities of interest in this note are the lower and upper particle profiles $\rho_i$ with $i = 1, \ldots, L$ and $i = L + 1, \ldots, 2L$, respectively, and the average mass displacement between the lower and the upper ring

$$\chi = \frac{1}{N} \left[ \sum_{i=1}^{L} \rho_i - \sum_{i=L+1}^{2L} \rho_i \right] = \frac{1}{N} \sum_{i=1}^{L} (\rho_i - \rho_{L+i}).$$

III. RESULTS

In this paper we focus on the behavior of the profiles $\rho_i$ and the mass displacement $\chi$ as a function of the model parameters. For given parameters, we obtain the exact value by first solving equations (6) to compute the fugacities $s_i$, and then equation (7) and (8). In the simulations, the total number of particles is $N = 200$ and $L = 50$.

A. Mass displacement: uphill currents

In this section our principal focus is on the behavior of $\chi$ as a function of the model parameters. Figs. 2 and 3 compare exact results for $\chi$ with those of Monte Carlo simulations.

In Fig. 2 the mass displacement $\chi$ is plotted as a function of the ratio $\psi/\sigma$ for different choices of the other parameters. For zero drift, the model is symmetric for any choice of parameters and the mass displacement $\chi$ vanishes. For $\varepsilon > 0$, mass migration is observed ($\chi \neq 0$), except for the symmetric case, $\sigma/\psi = 1$. Moreover, in the symmetric case $\sigma/\psi = 1$, for any $\varepsilon \geq 0$, the upper and lower profiles $\rho_i$ are uniform, independent of $i$. The result for $\psi = \sigma$ is expected: in this case symmetry is ensured by the fact that in a state in which the mean number of particles is the same at each site, the rates at which particles move upward and downward are equal in each channel.

For various choices of the parameters we find $\chi > 0$, namely, we observe mass migration towards the non-driven, lower ring. This phenomenon is not trivial, since the model is designed in such a way that upward and
downward rates in the two channels should compensate each other even when $\psi \neq \sigma$.

In Figs. 3 and 4 the mass displacement is plotted as a function of the drift $\varepsilon$ and the intra–channel distance $\Delta$, respectively. These graphs show that both positive and negative mass displacements are possible. More precisely, given $\varepsilon$ not too large, if the intra–channel distance $\Delta$ is small enough then the mass displacement is negative, i.e., the mass in the driven ring is larger than that in the non–driven one. The fact that mass migration is observed in both directions is similar to the phenomenon described in [1], although in that work it is connected to the possible jumps that particles may perform, whereas here it depends on simple geometric features.

### B. Sojourn times and gambler’s ruin

Mass migration can be interpreted in terms of the average residence times, called upper and lower sojourn time and denoted respectively by $\tau_{\text{up}}$ and $\tau_{\text{down}}$, that a particle spends in the upper and in the lower ring. These times can be defined by attaching a label to each particle $i$, and following its position over time. Then $\tau_{\text{up}}$ is defined as the mean time (over the evolution and over particles) between entering and exiting the upper ring, in the stationary state; $\tau_{\text{down}}$ is defined analogously. Sojourn times are similar to the residence times that have been widely studied for the simple exclusion process [9] and for the simple symmetric random walk [10]. The main difference is that in those studies the geometry of the strip was considered and the residence time was defined conditioning the particle to exit the strip through the side opposite the one where it started the walk.

The idea is very simple: if the upper sojourn time is larger than the lower one, then particles will spend more time in the upper ring and, at stationarity, a negative mass displacement will be observed. This idea is confirmed by the simulation results plotted in Fig. 5 where...
The ratio of the upper and the lower sojourn times is plotted as a function of the drift. Indeed, comparing data in Fig. 5 to those in Fig. 3, a perfect mirror behavior is seen, i.e., $\tau_{\text{up}} > \tau_{\text{down}}$ corresponds to $\chi < 0$ and vice-versa. Note, in particular, that the curves for $\Delta = 25$ and $\Delta = 49$ intersect at the same value of the drift $\varepsilon$ in both graphs.

The interpretation in terms of sojourn times also provides a nice explanation of the dependence of the sign of $\chi$ on the interchannel distance. In Figs. 3, 5, we consider $\sigma = 10 \gg (1/2 \pm \varepsilon)$ and $\psi = 1 \approx (1/2 \pm \varepsilon)$. Under this assumption particles typically jump from the upper ring to the lower one at site $L + 1$, and from the lower one to the upper one at site $\ell$.

Hence, the upper sojourn time is quite close to the typical time that a particle needs to move from site $L + \ell$ to site $L + 1$ along the upper ring, whereas the lower sojourn time is quite close to the typical time that a particle needs to move from site 1 to site $\ell$ along the lower ring. In view of this remark, due to the presence of the drift, the upper sojourn time is, for most choices of the parameters, smaller than the lower one, explaining why, in general the mass migration is observed towards the lower, non-driven ring. The contrary is observed when $\Delta$ is small. Indeed, in this case the lower sojourn time is substantially smaller than the upper one. Although no drift is present, the lower sojourn time is smaller than the upper one because the distance to be covered is much smaller than in the upper, driven ring, cf. Fig. 6.

This latter observation suggests a way of estimating the sojourn times that should be quite accurate for $\sigma \gg 1 \gg \psi$. Indeed, in this case one can use a classic result of probability theory, namely, the gambler’s ruin estimate of the duration of the game, see [11, Chapter XIV, Section 3] and [12], to obtain

$$\tau_{\text{down}} \approx (\ell - 1)(L - \ell + 1)$$

and

$$\tau_{\text{up}} \approx \frac{\ell - 1}{1 - 2\left(\frac{1}{2} + \varepsilon\right)} - \frac{L}{1 - 2\left(\frac{1}{2} + \varepsilon\right)} 1 - \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{\ell - 1}.$$  

The ratio $\tau_{\text{up}}/\tau_{\text{down}}$ computed using the above expressions is plotted in Fig. 5, where, as expected, the agreement is perfect for $\sigma = 10^3$ and $\psi = 0$. Indeed, in such an extreme case, particles move along the two rings closely following the gambler’s ruin rules. The sole difference with respect to the gambler’s ruin problem is in the tiny probability $1 - \sigma/(1 + \sigma)$ that a particle at site $L + 1$ does not jump to the site 1 and a particle at the site $\ell$ does not jump to $L + \ell$.

C. Density profiles

This section is devoted to a brief discussion of the typical stationary particle density profiles, which depend on both the drift $\varepsilon$ and on the ratio $\sigma/\psi$. As explained above, we perform a series of MC simulations to investigate the stationary density profiles in the two rings. The initial datum, in our simulations, is represented by uniform density profiles, with the same number of particles shared by the two rings. The numerical results are then compared to the exact ones obtained as explained at the beginning of Section III.

We have already mentioned that in the symmetric case $\sigma/\psi = 1$, for any $\varepsilon \geq 0$, the upper and lower profiles $\rho_i$ are uniform, namely, they do not depend on the site $i$, and $\chi = 0$. This is expected since, in such a case, in a state in which the average number of particles is the same at each site, the rates at which particles move upward and downward are equal in each channel.

Setting $\psi \neq \sigma$ yields nonuniform stationary profiles in the two rings. In particular, Fig. 7, for $\varepsilon = 0$, shows the onset of piecewise linear density profiles (i.e., stationary solutions to the discrete Laplace equation) in both rings, with kinks at the locations of the two channels. For reasons of symmetry, there is no net mass transfer between the rings, hence we again have $\chi = 0$.

The center and the left panel in Fig. 7 show how the profile shape is modified when the drift in the upper channel is different from zero, which is the paradigmatic case.
that leads to “uphill currents”, namely states with \( \chi \neq 0 \).
We notice, indeed, that, since \( \chi > 0 \) holds in the cases shown in the figure, the overall mass residing in the upper ring, in the stationary state, is less than that in the lower ring. Moreover, it also worth noting that the profiles are not linear in the upper ring due to the presence of the drift.

The plots in Fig. 8 refer to an inter–channel distance \( \Delta = 5 \), small enough to produce a nonmonotonic behavior of \( \chi \) as a function of \( \epsilon \), already highlighted in Fig. 3 for \( \Delta = 1 \). In particular, for small values of the drift, particles gather on the upper ring (\( \chi < 0 \)), whereas for larger values of the drift they tend to move downwards to the lower ring (\( \chi > 0 \)).

We have also learned that in our model there are two classes of stationary states showing no uphill diffusion, as also visible in Fig. 2. The first class is obtained by setting \( \psi = \sigma \) and/or \( \epsilon = 0 \): in particular, the case with \( \psi = \sigma \) and \( \epsilon = 0 \) corresponds to a state with vanishing current and whose density profiles are uniform (except for the coupling sites) over the two rings. The second class includes the states for which the (moderate) effect of the drift in the upper ring is exactly compensated by the (small) interchannel distance covered by particles in the lower ring, see the third panel from the left in Fig. 8. These are nonequilibrium steady states characterized by nonuniform density profiles and nonvanishing currents, which nevertheless produce no net mass transfer between the rings.

IV. CONCLUSIONS

We consider a simple, exactly solvable model of noninteracting particles exhibiting migration despite overall symmetry. The model consists of a pair of zero-range processes on rings, with particle exchange between rings allowed at specified channels. The combination of asymmetry between the channels and a drift in one (but not both) rings leads to particle migration between rings, as shown by the exact solution of the model and confirmed in Monte Carlo simulations. In the limit of strong channel asymmetry, particle migration can be understood on the basis of the classic gambler’s ruin problem.
Our results show that “uphill diffusion,” of particle migration in apparent violation of Fick’s law, may be significantly more general than has been appreciated until now, in that it does not depend on interparticle interactions. A promising arena for observation of this phenomenon is the movement of passive tracers in cyclic geophysical flows.

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