Quantum dice rolling

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A coin is just a two sided dice. Recently, Mochon proved that quantum weak coin flipping with an arbitrarily small bias is possible. However, the use of quantum resources to allow $N$ remote distrustful parties to roll an $N$-sided dice has yet to be addressed. In this paper we show that contrary to the classical case, $N$-sided dice rolling with arbitrarily small bias is possible for any $N$. In addition, we present a six-round three-sided dice rolling protocol, achieving a bias of 0.181, which incorporates weak imbalanced coin flipping.

II. QUANTUM DICE ROLLING WITH ARBITRARILY SMALL BIAS

It is straightforward to generalize the problem of coin flipping to multi-party settings. In multi-party coin flipping $N$ remote distrustful parties must decide on a bit. An analysis of multi-party SCF in a quantum setting...
was carried out in \cite{16}, and similarly to the two-party case, it was shown that the use of quantum resources is advantageous. Another possible generalization is the problem of \(N\) remote distrustful parties having to agree on a number from 1 to \(N\), with party \(i\) preferring the \(i\)-th outcome. To the best of our knowledge this problem has never been considered in classical settings. We shall term this problem “dice rolling”. For \(N = 2\) the problem reduces to WCF, while for \(N > 2\), there are now many different cheating scenarios, as any number of parties \(n < N\) may be dishonest. We will be interested in the \(N\) “worst case” scenarios where all but one of the parties are dishonest and, moreover, are cooperating with one another. That is, the dishonest parties share classical and quantum communication channels. In addition, we will require that the protocol be “fair” in the sense that the honest party’s maximum losing probability be the same in each of these \(N\) scenarios. Of course, the security of the protocol can be evaluated with respect to any other cheating scenario, but as we will consider only fair protocols, the security of any cheating scenarios is never poorer than that provided by the aforementioned \(N\) scenarios.

We begin by observing that in coin flipping protocols the bias has a complementary definition. We could just as well define it as

\[
\epsilon_i = P_i^* - 1/2, \quad i = A, B
\]

where \(P_i^* = P_{j \neq i}\) is the maximum probability that party \(i\) loses. According to this definition the bias tells us to what extent party \(j \neq i\) can increase the other party \(i\)'s chances of losing beyond one half. In the case of \(N\) parties, the bias \(\epsilon_i\) then tells us to what extent the \(N - 1\) dishonest parties can increase party \(i\)'s chances of losing beyond \(1 - 1/N\), rather than to what extent a sole dishonest party can increase its chances of winning beyond \(1/N\). We shall always use this redefinition of the bias when considering dice rolling. The computation of biases in dice rolling is therefore equivalent to the computation of biases in a weak imbalanced coin flipping protocol.

We shall now prove that that quantum weak \(N\)-sided dice rolling with arbitrarily small bias is possible for any \(N\). The proof is by construction. Consider the following \(N\)-party protocol. Each party is uniquely identified according to a number from 1 to \(N\). The protocol consists of \(N - 1\) stages. In stage one parties 1 and 2 “weakly flip” a balanced quantum coin. The winner and party 3 then weakly flip an imbalanced quantum coin in stage two, where if both parties are honest 3’s winning probability equals 1/3. And so on, the rule being that in stage \(n \geq 2\) the winner of stage \(n - 1\) and party \(n\) “weakly flip” an imbalanced quantum coin, where if both parties are honest, \(n\)’s winning probability equals 1/(\(n + 1\)). Thus, if all parties are honest each has the same overall winning probability of 1/\(N\). Using Mochon’s formalism \cite{14}, Chailloux and Kerenidis have recently proved that weak imbalanced coin flipping with arbitrarily small bias is possible \cite{13}. It follows that in the limit where each of the weak imbalanced coin flipping protocols, used to implement our dice rolling protocol, admits a vanishing bias (and \(N\) is finite), any honest party’s winning probability tends to 1/\(N\); for a formal proof see the appendix. Moreover, since we have considered the worst case cheating scenario, this result holds for any other cheating scenario.

The above result is in stark contrast to the classical case where if the number of honest parties is not strictly greater than \(N/2\), then the dishonest parties can force any outcome they desire. To see why this is so, let us consider a classical \(N\)-sided dice rolling protocol and partition the parties into two groups of \(m \leq \lfloor N/2 \rfloor\) and \(n = N - m\) parties. If both groups are honest, the probability that a party in the first (second) group wins is \(m/N\) (1 \(- m/N\)). Therefore, any dice rolling protocol can serve as a weak imbalanced coin flipping protocol. Suppose now that all of the parties in the second group are dishonest, and are nevertheless unable to force with certainty the outcome they choose. Clearly, this would still be the case even if they were the smaller group, i.e. \(n < m\) (\(m > \lfloor N/2 \rfloor\)), and we get a contradiction, since in classical weak imbalanced coin flipping (as in weak balanced coin flipping) at least one of the parties should always be able to force whichever outcome they desire \cite{17}.

### III. A THREE-ROUND WEAK IMBALANCED COIN FLIPPING PROTOCOL

In this section we analyze a three-round weak imbalanced coin flipping protocol based on quantum gambling. It is constructed such that if both parties are honest Alice’s winning probability equals \(1 - p\). Interestingly, it turns out that this protocol coincides with the generalization of Spekkens and Rudolph’s work to the imbalanced case. The protocol will be used in the subsequent section to implement a six-round three-sided dice rolling protocol.

The protocol

1. Alice prepares a superposition of two qubits

\[
|\psi_0\rangle = \sqrt{1 - p - \eta} |\uparrow_1 \downarrow_2\rangle + \sqrt{p + \eta} |\downarrow_1 \uparrow_2\rangle, \quad 0 \leq \eta \leq 1 - p,
\]
where the subscripts serve to distinguish between the first and second qubit and will be omitted when the distinction is clear. She then sends the second qubit to Bob.

2. Bob carries out a unitary transformation $U_\eta$ on the qubit he received and another qubit (labelled by the subscript 3) prepared in the state $|\downarrow\rangle$ such that

$$|\uparrow_2\downarrow_3\rangle \rightarrow U_\eta |\uparrow_2\downarrow_3\rangle = \sqrt{\frac{p}{p+\eta}} |\uparrow_2\downarrow_3\rangle + \sqrt{\frac{\eta}{p+\eta}} |\downarrow_2\uparrow_3\rangle,$$

and

$$|\downarrow_2\uparrow_3\rangle \rightarrow U_\eta |\downarrow_2\uparrow_3\rangle = \sqrt{\frac{\eta}{p+\eta}} |\downarrow_2\uparrow_3\rangle - \sqrt{\frac{p}{p+\eta}} |\downarrow_2\uparrow_3\rangle,$$

with $U_\eta$ acting trivially on all other states. The resulting state is then

$$|\psi_1\rangle = U_\eta |\psi_0\rangle = \sqrt{1-p-\eta} |\uparrow_1\downarrow_2\downarrow_3\rangle + \sqrt{p} |\downarrow_1\uparrow_2\downarrow_3\rangle + \sqrt{\eta} |\downarrow_1\downarrow_2\uparrow_3\rangle,$$

(6)

Following this, he checks whether the second and third qubits are in the state $|\uparrow_2\downarrow_3\rangle$.

3. Bob wins if he finds the qubits in the state $|\uparrow_2\downarrow_3\rangle$. Alice then checks whether the first qubit is in the state $|\downarrow\rangle$, in which case Bob passes the test. If Bob does not find the qubits in the state $|\uparrow_2\downarrow_3\rangle$, he asks Alice for the first qubit and checks whether all three qubits are in the state

$$|\xi\rangle = \frac{1-p-\eta}{1-p} |\uparrow_1\downarrow_2\downarrow_3\rangle + \frac{\eta}{1-p} |\downarrow_1\downarrow_2\uparrow_3\rangle,$$

(7)

in which case she passes the test.

**Alice’s maximal bias**

Most generally Alice can prepare any state of the form

$$|\psi'_0\rangle = \sum_{i,j=\uparrow,\downarrow} \alpha_{ij} |ij\rangle \otimes |\Phi_{ij}\rangle,$$

(8)

where the $|\Phi_{ij}\rangle$ are states of some ancillary system at her possession. After Bob applies $U_\eta$ the resulting composite state is given by

$$|\psi'_1\rangle = U_\eta |\psi'_0\rangle \otimes |\downarrow\rangle = \alpha_{\uparrow\uparrow} \left( \sqrt{\frac{p}{p+\eta}} |\uparrow\uparrow\rangle + \sqrt{\frac{\eta}{p+\eta}} |\uparrow\downarrow\rangle \right) \otimes |\Phi_{\uparrow\uparrow}\rangle$$

$$+ \alpha_{\uparrow\downarrow} |\uparrow\downarrow\rangle \otimes |\Phi_{\uparrow\downarrow}\rangle + \alpha_{\downarrow\uparrow} \left( \sqrt{\frac{\eta}{p+\eta}} |\downarrow\uparrow\rangle + \sqrt{\frac{p}{p+\eta}} |\downarrow\downarrow\rangle \right) \otimes |\Phi_{\downarrow\uparrow}\rangle$$

$$+ \alpha_{\downarrow\downarrow} |\downarrow\downarrow\rangle \otimes |\Phi_{\downarrow\downarrow}\rangle,$$

(9)

The probability that Bob does not find find the second and third qubits in the state $|\downarrow_2\downarrow_3\rangle$ is

$$P_{\uparrow\downarrow} = 1 - P_{\uparrow\downarrow} = 1 - \frac{|\alpha_{\uparrow\downarrow}|^2 p + |\alpha_{\downarrow\uparrow}|^2 p}{p + \eta},$$

and the resulting composite state then is

$$|\psi'_2\rangle = \mathcal{N} \left( \alpha_{\uparrow\uparrow} \sqrt{\frac{\eta}{p+\eta}} |\uparrow\downarrow\rangle \otimes |\Phi_{\uparrow\downarrow}\rangle + \alpha_{\uparrow\downarrow} |\uparrow\downarrow\rangle \otimes |\Phi_{\uparrow\downarrow}\rangle$$

$$+ \alpha_{\downarrow\uparrow} \sqrt{\frac{p}{p+\eta}} |\downarrow\uparrow\rangle \otimes |\Phi_{\downarrow\uparrow}\rangle + \alpha_{\downarrow\downarrow} |\downarrow\downarrow\rangle \otimes |\Phi_{\downarrow\downarrow}\rangle \right),$$

(11)

where $\mathcal{N}$, the normalization, is

$$\frac{1}{\mathcal{N}^2} = 1 - \frac{p}{p + \eta} \left( |\alpha_{\uparrow\downarrow}|^2 + |\alpha_{\downarrow\uparrow}|^2 \right).$$

(12)
The probability that Alice passes the test is therefore given by

\[ P_{\text{test}} = \| \langle \xi | \psi' \rangle \|^2 = N^2 \left| \alpha_{1\dagger} \sqrt{\frac{1 - p - \eta}{1 - p}} |\Phi_{1\dagger}\rangle + \alpha_{1} \sqrt{\frac{\eta^2}{(1 - p)(p + \eta)}} |\Phi_{1}\rangle \right|^2. \]  

(13)

The maximum obtains for \( |\Phi_{1\dagger}\rangle = |\Phi_{1}\rangle \). This choice of the ancillary states does not affect the maximum of \( P_{1\dagger} \). Hence, Alice obtains no advantage by using ancillary systems and we can do away with them. Alice’s maximum cheating probability is then

\[ P_A^* = \max_{\alpha_{ij}} P_{1\dagger} \cdot P_{\text{test}}, \]  

where now

\[ P_{1\dagger} \cdot P_{\text{test}} = \left| \alpha_{1\dagger} \sqrt{\frac{1 - p - \eta}{1 - p}} + \alpha_{1} \sqrt{\frac{\eta^2}{(1 - p)(p + \eta)}} \right|^2. \]  

(15)

(\( P_{1\dagger} = 1/N^2 \)). Clearly, this expression is maximum when \( \alpha_{1\dagger} = \alpha_{1} = 0 \). Therefore, to maximize her chance of successfully cheating Alice will prepare a state of the form

\[ |\psi'_0\rangle = \sqrt{1 - \delta} |\uparrow\downarrow\rangle + \sqrt{\delta} |\downarrow\uparrow\rangle, \]  

where with no loss of generality we have set \( \alpha_{1\dagger} = \sqrt{1 - \delta} \) and \( \alpha_{1} = \sqrt{\delta} \). So that

\[ P_A^* = \max_{\delta} \left( \sqrt{\frac{(1 - p - \eta)(1 - \delta)}{1 - p}} + \sqrt{\frac{\eta^2 \delta}{(1 - p)(p + \eta)}} \right)^2. \]  

(17)

Bob’s maximal bias

Bob wins and passes the test whenever Alice does not find the first qubit in the state \(|\uparrow\rangle\). The probability for this is just \( p + \eta \). This gives an upper bound on Bob’s maximal cheating probability, which is reached if Bob always announces that he has won. That is,

\[ P_B^* = p + \eta. \]  

(18)

Biases in the balanced case

In the balanced case a protocol is fair if \( P_A^* = P_B^* \). We can play with \( \eta \) to make \( P_A^* \) and \( P_B^* \) minimal under this constraint. It is easy to show that the minimum then obtains for \( \eta = (\sqrt{2} - 1)/2 \). It follows that \( \epsilon_A = \epsilon_B = (\sqrt{2} - 1)/2 \) and \( P_A^* = P_B^* = 1/\sqrt{2} \).

IV. A SIX-ROUND THREE-SIDED DICE ROLLING PROTOCOL WITH A BIAS OF 0.181

Apart from the inherent limitations on the security of a multi-party quantum cryptographic protocol, it is most interesting, both from a theoretical and a practical viewpoint, to determine what degree of security is afforded using the least amount of communication. In this section we introduce a six-round three-sided dice rolling protocol following the general construction presented in the second section. The protocol consists of two three-round stages. In the first stage, we have Alice and Bob flip a balanced quantum coin. Following this, in the second stage, the winner and Claire flip an imbalanced quantum coin, such that if both parties are honest Claire’s winning probability equals 1/3. The protocol is considered fair if \( P_A^* = P_B^* = P_C^* \). Due to the protocols’ symmetry with respect to the interchange of Alice and Bob there are only two nonequivalent worst case scenarios, i.e. either only Alice is honest or only Claire is honest.
Using the protocol in the previous section an honest Alice has a maximum chance of $1 - 1/\sqrt{2}$ of progressing to the second stage. Therefore, Alice’s maximum losing probability is given by

$$P_A^* = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)\bar{\Pi}_{2/3}^*,$$

while an honest Claire’s maximum losing probability is given by

$$P_C^* = \bar{\Pi}_{1/3}^*,$$

with $\bar{\Pi}_{1/3}^*$ ($\bar{\Pi}_{2/3}^*$) the maximum losing probability of the party with a winning probability of $1/3$ ($2/3$) when both parties are honest. Hence, we require that

$$\bar{\Pi}_{1/3}^* = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)\bar{\Pi}_{2/3}^*.$$  

If we use the WCF protocol of the previous section to implement the second stage, then $\bar{\Pi}_{1/3}^*$ and $\bar{\Pi}_{2/3}^*$, and hence the $P_i^*$, will depend on $\eta$. We then have to minimize the $\bar{P}_i^*$ with respect to $\eta$ under the constraint that they are all equal, or what is the same thing, minimize $\bar{\Pi}_{1/3}^*$ under the constraint eq. (20). However, there are two possible implementations. Either $1 - p = 2/3$ and the second stage begins with Alice preparing the state $\sqrt{2/3 - \eta} |1\rangle_2 + \sqrt{1/3 + \eta} |1\rangle_2$, or else $1 - p = 1/3$ and the second stage begins with Claire preparing the state $\sqrt{1/3 - \eta} |1\rangle_2 + \sqrt{2/3 + \eta} |1\rangle_2$. In the first case we have to compute

$$\min_{\eta} \max_\delta \frac{1}{2} \left(\sqrt{(2 - 3\eta)(1 - \delta)} + \sqrt{9\eta^2 \delta \over (1 + 3\eta)}\right)^2$$

under the constraint that

$$\max_\delta \frac{1}{2} \left(\sqrt{(2 - 3\eta)(1 - \delta)} + \sqrt{9\eta^2 \delta \over (1 + 3\eta)}\right)^2 = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{3} + \eta\right),$$

while in the second case we have to compute

$$\min_{\eta} \left(\frac{2}{3} + \eta\right)$$

under the constraint that

$$\left(\frac{2}{3} + \eta\right) = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)\max_\delta \left(\sqrt{(1 - 3\eta)(1 - \delta)} + \sqrt{9\eta^2 \delta \over (2 + 3\eta)}\right).$$

The first of these yields the lower bias $\bar{\epsilon}_A = \bar{\epsilon}_B = \bar{\epsilon}_C \simeq 0.181$ corresponding to $P_A^* \simeq P_B^* \simeq 0.848$. The second yields a bias of 0.199.

V. SUMMARY AND DISCUSSION

We have defined and studied a novel mutli-party generalization of weak coin flipping, which we have termed dice rolling. In $N$-sided dice rolling $N$ remote distrustful parties must decide on a number between 1 and $N$, with party $i$ having the preference for outcome $i$. We have shown that for any bias $\bar{\epsilon} \ll 1$ there exists a quantum protocol such that an honest party is always guaranteed a $1/N - \bar{\epsilon}$ chance of winning. This is in contrast with classical protocols where for $n \leq [N/2]$ honest parties, the dishonest parties can force any outcome they desire. We have also presented a six-round three-sided dice rolling protocol in which the honest party’s maximum losing probability equals 0.848 as compared to the 2/3 that obtains when all parties are honest. Finally, this problem also admits a “strong” variant in which $M \geq 2$ remote distrustful parties must agree on a number between 1 and $N > 2$, without any party being aware of any other’s preference. Kitaev’s bound generalizes to these cases. In particular, for $M = 2$ it can be saturated, as will be shown in an upcoming publication [18].

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APPENDIX A

For any dice rolling protocol, based on weak imbalanced coin flipping according to the scheme presented in section II, party \( n \)'s maximum chance of losing is given by

\[
\hat{P}^n = \frac{N-1}{N} + \bar{\varepsilon}_n = \bar{\Pi}^n_{n-1} + \sum_{k=n}^{N-1} \bar{\Pi}^n_k \prod_{j=0}^{k-n} (1 - \bar{\Pi}^n_{n-1+j}) \tag{A1}
\]

where \( \bar{\Pi}^n_k \) is party \( n \)'s maximum chance of losing stage \( k \) conditional on having made it to that round and \( \bar{\delta}_k \) the corresponding bias. If we now let \( \bar{\delta}^{(n)}_{\text{max}} \doteq \max_k \bar{\delta}_k \) and \( \bar{\delta}^{(n)}_{\text{min}} \doteq \min_k \bar{\delta}_k \) \( (k = n-1, \ldots, N-1) \), then

\[
\beta_n \leq \bar{\delta}^{(n)}_{\text{max}} + \frac{\bar{\delta}^{(n)}_{\text{max}}}{\bar{\delta}^{(n)}_{\text{min}}} \sum_{k=n}^{N-1} \left( \frac{1}{k} + \bar{\delta}^{(n)}_{\text{min}} \right) \prod_{j=1}^{k-n-1} \left( \frac{n+j}{n+j+1} - \bar{\delta}^{(n)}_{\text{min}} \right)
\]

\[
-\bar{\delta}^{(n)}_{\text{min}} \sum_{k=n}^{N-1} \left( \frac{1}{k} + \bar{\delta}^{(n)}_{\text{max}} \right) \left( \frac{1}{k} - \bar{\delta}^{(n)}_{\text{min}} \right) \sum_{m=1}^{k-n-1} \prod_{j \neq m} \left( \frac{n+j+1}{n+j+2} - \bar{\delta}^{(n)}_{\text{min}} \right)
\]

\[
\leq \bar{\delta}^{(n)}_{\text{max}} + \bar{\delta}^{(n)}_{\text{max}} \sum_{k=n}^{N-1} \frac{1}{k} \prod_{j=1}^{k-n-1} \left( \frac{n+j+1}{n+j+2} - \bar{\delta}^{(n)}_{\text{min}} \right)
\]

\[
< N \bar{\delta}^{(n)}_{\text{max}} \tag{A2}
\]

Hence, if each of the weak imbalanced coin flipping protocols, used to implement the dice rolling protocol, are such that \( \bar{\delta}^{(n)}_{\text{max}} \ll 1/N \) for any \( n \), an honest party’s winning probability tends to \( 1/N \).

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