In this paper, we study a frictional contact model which takes into account the damage and the memory. The deformable body consists of a viscoelastic material and the process is assumed to be quasistatic. The mechanical damage of the material which caused by the tension or the compression is included in the constitutive law and the damage function is modelled by a nonlinear parabolic inclusion. Then the variational formulation of the model is governed by a coupled system consisting of a history-dependent hemivariational inequality and a nonlinear parabolic variational inequality. We introduce and study a fully discrete scheme of the problem and derive error estimates for numerical solutions. Under appropriate solution regularity assumptions, an optimal order error estimate is derived for the linear finite element method. Several numerical experiments for the contact problem are given for providing numerical evidence of the theoretical results.

1. Introduction. In this paper, we study a frictional contact problem between a viscoelastic body and a foundation. The effect due to the damage of the material is also considered. In many materials, such as concrete, because of development and growth of internal cracks, there is an observed decrease over time in the load-bearing capacity. There exists a large number of engineering work on it since it affects the useful life-span of the designed structure or component, we refer to ([1, 10, 14]). However, only a few models considering the effect of the internal damage on the contact process have been studied mathematically, especially for the numerical analysis, such references include ([5, 6, 11]).

This paper is dedicated to the study on numerical approximation of a system coupled by a general evolutional hemivariational inequality involving history-dependent operators and a nonlinear parabolic variational inequality which models a quasistatic frictional contact problem with damage and long memory. We give the existence and uniqueness result for the system and consider the numerical method to solve it. Optimal error estimates for the scheme are derived.

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* Corresponding author: Hailing Xuan.
We first introduce the following frictional contact problem. Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^d \) which is occupied by a viscoelastic body with \( d = 2,3 \). The boundary \( \Gamma \) of \( \Omega \) is supposed to be Lipschitz continuous and partitioned into three mutually disjoint parts \( \Gamma_D, \Gamma_N \) and \( \Gamma_C \) and the measure of \( \Gamma_D \), denoted \( m(\Gamma_D) \), is positive. The body is held fixed on \( \Gamma_D \), and then the displacement field vanishes there. Time-dependent surface tractions of density \( f_N \) act on \( \Gamma_N \) and time-dependent volume forces of density \( f_0 \) act in \( \Omega \). We pay attention to the evolutionary process of the mechanical state of the body in the time interval \( (0,T) \) with \( T > 0 \).

We denote by \( u = (u_i) \), \( \sigma = (\sigma_{ij}) \) and \( \epsilon(u) = (\epsilon_{ij}(u)) \) the displacement vector, the stress tensor, and linearized strain tensor, respectively. For simplicity, we do not indicate explicitly the dependence of the variables on the spatial variable \( x \). Recall that the components of the linearized strain tensor \( \epsilon(u) \) are \( \epsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \), where \( u_{i,j} = \partial u_i / \partial x_j \). The indices \( i,j,k,l \) run between 1 and \( d \) and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable \( x \). A superscript prime of a variable stands for the time derivative of the variable. The outward unit normal on \( \partial \Omega \) is denoted by \( \nu \) and the notation \( v_\nu \) and \( v_\tau \) are used for the normal and tangential components of \( v \) on \( \partial \Omega \) given by \( v_\nu = v \cdot \nu \) and \( v_\tau = v - v_\nu \nu \). The normal and tangential components of the stress field \( \sigma \) on the boundary are defined by \( \sigma_\nu = (\sigma \nu) \cdot \nu \) and \( \sigma_\tau = \sigma \nu - \sigma_\nu \nu \), respectively. The symbol \( S^d \) represents the space of second order symmetric tensors on \( \mathbb{R}^d \).

The mathematical model of the contact problem is stated as follows.

**Problem 1.1.** Find a displacement field \( u : \Omega \times (0,T) \rightarrow \mathbb{R}^d \), a stress field \( \sigma : \Omega \times (0,T) \rightarrow S^d \) and a damage field \( \zeta : \Omega \times (0,T) \rightarrow \mathbb{R} \) such that for all \( t \in (0,T) \),

\[
\begin{align*}
\sigma(t) &= \mathcal{A} \epsilon(u'(t)) + \mathcal{B} (\epsilon(u(t)), \zeta(t)) & \text{in } \Omega, & (1.1) \\
\text{Div } \sigma(t) + f_0(t) &= 0 & \text{in } \Omega, & (1.2) \\
\zeta'(t) - \kappa \Delta \zeta(t) + D \psi[0,1](\zeta(t)) & \geq \phi(\epsilon(u(t)), \zeta(t)) & \text{in } \Omega, & (1.3) \\
\frac{\partial \zeta(t)}{\partial \nu} &= 0 & \text{on } \Gamma, & (1.4) \\
u(t) &= 0 & \text{on } \Gamma_D, & (1.5) \\
\sigma(t) \nu &= f_N(t) & \text{on } \Gamma_N, & (1.6) \\
- \sigma_\nu(t) &= p(u_\nu(t)) + \int_0^t b(t - s) u_\nu(s) \, ds & \text{on } \Gamma_C, & (1.7) \\
- \sigma_\tau(t) &= \partial j_\tau(u'_\xi(t)) & \text{on } \Gamma_C, & (1.8) \\
u(0) &= u_0, & \zeta(0) = \zeta_0 & \text{in } \Omega. & (1.9)
\end{align*}
\]

Eq. (1.1) represents the constitutive law for viscoelastic materials with damage in which \( \mathcal{A} \) represents the viscosity operator, \( \mathcal{B} \) represents the elasticity operator and the damage \( \zeta \) only affects the elastic behavior of the material and does not affect its viscosity. The damage of the material measures the density of the microcracks: when \( \zeta = 1 \), the material is undamaged, when \( \zeta = 0 \), the material is completely damaged and when \( 0 < \zeta < 1 \), there is a partial damage. Eq. (1.2) is the equation of equilibrium. The evolution of the damage is governed by the parabolic nonlinear differential inclusion (1.3). Here \( \Delta \) is the Laplacian, \( k > 0 \) is the damage diffusion constant, \( \phi \) is the damage source function and \( \partial \psi[0,1] \) denotes the subdifferential
of the indicator function $\psi_{[0,1]}$ of the interval $[0,1]$. We assume that there is no damage influx throughout the boundary $\Gamma$ and then Eq. (1.4) is used. We use the clamped boundary condition (1.5) on $\Gamma_D$ and (1.6) is the surface traction boundary condition.

Relation (1.7) is the contact condition in which $p$, $b$ are given functions describing the instantaneous and the memory reaction of the obstacle, respectively. It can be seen from (1.7) that the reaction of the obstacle depends both on the current value of the normal displacement (expressed by the term $p(u_\nu(t)))$ as well as on the history of the normal displacement (expressed by the integral term in (1.7)). Condition (1.8) describes a friction law where $j_\tau$ is a given function and $\partial j_\tau$ denotes the Clarke subdifferential of $j_\tau$. $u_0$ and $\zeta_0$ represent the initial values of the displacement and damage fields, respectively.

Recently, more and more researchers focus on deriving error estimates for numerical solutions, such references include [8, 11, 12]. The reference [8] provides an optimal error estimate for the numerical solution of a quasistatic viscoelastic problem with long memory terms. In [11], the quasistatic contact with friction and the resulting damage caused by mechanical strain was studied, the finite element method is used to discretize the domain and a backward Euler finite difference is used to discretize the time derivative and an optimal-order error estimate is obtained under appropriate regularity assumptions. The numerical solution of a family of stationary variational-hemivariational inequality was considered in [12] and the numerical simulation on the contact model between an elastic body and a foundation was carried out to illustrate their theoretically predicted optimal first order convergence. More recently, a history-dependent variational-hemivariational inequality is considered in [16] and the corresponding Céa’s type inequality is derived for error estimation. The paper [17] considers the numerical analysis of a dynamic contact problem with long memory and the contact involves a nonmonotone Clarke subdifferential boundary condition and the friction is modeled by a version of the Coulomb’s law of dry friction with the friction bound depending on the total slip. In [7], a spatially semidiscrete and fully discrete schemes for a variational-hemivariational inequality which describes the adhesive contact between a deformable body and a foundation are considered and the optimal error estimates are obtained for linear element method.

The rest of the paper is organized as follows. In Section 2, we first introduce preliminary materials and list some assumptions on the data. And then, we establish a history-dependent hemivariational inequality and a nonlinear parabolic variational inequality corresponding to the contact model. In Section 3, we first introduce a discrete problem and give an optimal order error estimate for finite element method. In the last section, we give the numerical simulation results of a two-dimensional contact problem and provide numerical evidence of optimal order convergence for the linear element solutions.

2. Notation and assumptions. In this section, we recall notation, basic definitions and unique solvability result of a history-dependent hemivariational inequality and a nonlinear parabolic variational inequality. We start with the definitions of Clarke’s directional derivative and Clarke’s subdifferential. Let $X$ be a Banach space, and $X^*$ its dual. Denote by $(\cdot, \cdot)_{X^* \times X}$ the duality pairing between $X^*$ and $X$. 
Definition 2.1. Let \( \Phi : X \to \mathbb{R} \) be a locally Lipschitz function. The generalized directional derivative, in the sense of Clarke’s, of \( \Phi \) at \( x \in X \) in the direction \( v \in X \), denoted by \( \Phi^0(x; v) \), is defined by

\[
\Phi^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{\Phi(y + \lambda v) - \Phi(y)}{\lambda}
\]

and the Clarke’s subdifferential of \( \Phi \) at \( x \), denoted by \( \partial \Phi(x) \), is a subset of a dual space \( X^* \) given by

\[
\partial \Phi(x) = \{ \zeta \in X^* \mid \Phi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \quad \forall \ v \in X \}.
\]

We use the standard notation for Lebesgue and Sobolev spaces. For \( v \in H^1(\Omega; \mathbb{R}^d) \), we use the same symbol \( v \) for the trace of \( v \) on \( \Gamma \) and we use the notation \( v_\nu \) and \( v_\tau \) for its normal and tangential traces. In addition, we introduce spaces \( V \) and \( H \) as follows:

\[
V = \{ v = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D \},
\]

\[
H = L^2(\Omega; \mathbb{R}^d),
\]

\[
H = L^2(\Omega; \mathbb{R}^d).
\]

These are real Hilbert spaces with the canonical inner products in \( H \) and \( H \), and the inner product

\[
\langle u, v \rangle_V = (\varepsilon(u), \varepsilon(v))_H
\]

in \( V \). The associated norms are \( \| \cdot \|_V \), \( \| \cdot \|_H \) and \( \| \cdot \|_H \). By the Sobolev trace theorem,

\[
\|v\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq \|\gamma\|_V \quad \forall \ v \in V,
\]

(2.1)

where \( \|\gamma\| \) represents the norm of the trace operator \( \gamma : V \to L^2(\Gamma_C; \mathbb{R}^d) \). Moreover, we employ \( Z_0 \) and \( Z_1 \) for the spaces \( L^2(\Omega) \) and \( H^1(\Omega) \) whenever they are used for the damage function \( \zeta(t) \) or its approximations. We also use the \( L^2(\Omega) \) norm and inner product

\[
\| \cdot \|_{Z_0} = \| \cdot \|_{L^2(\Omega)}, \quad (\cdot, \cdot)_{Z_0} = (\cdot, \cdot)_{L^2(\Omega)};
\]

(2.2)

respectively as well as the \( H^1(\Omega) \) seminorm

\[
| \cdot |_{Z_1} = | \cdot |_{H^1(\Omega)}.
\]

(2.3)

The damage function \( \zeta(t) \) will be sought in the subset \( \mathcal{K} \subset Z_1 \), defined as

\[
\mathcal{K} = \{ \xi \in Z_1 : \xi \in [0, 1] \ a.e. \ in \ \Omega \}.
\]

(2.4)

Note that \( V \subset H \subset V^* \) form an evolution triple of function spaces. Given \( 0 < T < +\infty \), we introduce spaces \( V = L^2(0, T; V) \) and the dual of \( V \) is \( V^* = L^2(0, T; V^*) \). The duality pairing between \( V^* \) and \( V \) is

\[
\langle w, v \rangle_{V^* \times V} = \int_0^T \langle w(t), v(t) \rangle_{V^* \times V} \, dt, \quad w \in V^*, v \in V.
\]

Define a space of fourth order tensor fields,

\[
Q_\infty = \{ \varepsilon = (\varepsilon_{ijkl}) \mid \varepsilon_{ijkl} = \varepsilon_{ijkl} = \varepsilon_{klij} \in L^\infty(\Omega), \ 1 \leq i, j, k, l \leq d \}.
\]

This is a real Banach space with the norm

\[
\| \varepsilon \|_{Q_\infty} = \sum_{0 \leq i, j, k, l \leq d} \| \varepsilon_{ijkl} \|_{L^\infty(\Omega)}.
\]
Now we introduce assumptions on the data in the study of Problem 1.1. For the viscosity operator $A : \Omega \times S^d \to S^d$, we assume

$$\begin{align*}
(a) & \text{ there exists } L_A > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in S^d \text{ and a.e. } x \in \Omega, \\
& \|A(x, \varepsilon_1) - A(x, \varepsilon_2)\| \leq L_A \|\varepsilon_1 - \varepsilon_2\|;
(b) & \text{ there exists } m_A > 0 \text{ such that for all } \varepsilon_1, \varepsilon_2 \in S^d \text{ and a.e. } x \in \Omega, \\
& (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A \|\varepsilon_1 - \varepsilon_2\|^2; \\
(c) & \text{ the mapping } x \mapsto A(x, \varepsilon) \text{ is measurable on } \Omega, \text{ for all } \varepsilon \in S^d; \\
(d) & A(x, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*}$$

For the elasticity operator $B : \Omega \times S^d \times \mathbb{R} \to S^d$, we assume

$$\begin{align*}
(a) & \text{ there exists } L_B > 0 \text{ such that } \\
& \|B(x, \varepsilon_1, \zeta_1) - B(x, \varepsilon_2, \zeta_2)\| \leq L_B (\|\varepsilon_1 - \varepsilon_2\| + |\zeta_1 - \zeta_2|) \\
& \text{ for all } \varepsilon_1, \varepsilon_2 \in S^d, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega; \\
(b) & \text{ the mapping } x \mapsto B(x, \varepsilon, \zeta) \text{ is measurable on } \Omega, \\
& \text{ for all } \varepsilon \in S^d \text{ and } \zeta \in \mathbb{R}; \\
(c) & B(x, 0, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*}$$

For the damage source function $\phi : \Omega \times S^d \times \mathbb{R} \to \mathbb{R}$, we assume

$$\begin{align*}
(a) & \text{ there exists } L_\phi > 0 \text{ such that } \\
& |\phi(x, \varepsilon_1, \zeta_1) - \phi(x, \varepsilon_2, \zeta_2)| \leq L_\phi (\|\varepsilon_1 - \varepsilon_2\| + |\zeta_1 - \zeta_2|) \\
& \text{ for all } \varepsilon_1, \varepsilon_2 \in S^d, \zeta_1, \zeta_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega; \\
(b) & \text{ the mapping } x \mapsto \phi(x, \varepsilon, \zeta) \text{ is measurable on } \Omega, \\
& \text{ for all } \varepsilon \in S^d \text{ and } \zeta \in \mathbb{R}; \\
(c) & \phi(x, 0, 0) \in L^2(\Omega).
\end{align*}$$

For the function $j_\tau : \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$, we assume

$$\begin{align*}
(a) & \text{ } j_\tau(\cdot, \xi) \text{ is measurable on } \Gamma_C \text{ for all } \xi \in \mathbb{R}^d \text{ and there exists } e \in L^2(\Gamma_C; \mathbb{R}^d) \text{ such that } j_\tau(\cdot, e(\cdot)) \in L^2(\Gamma_C); \\
(b) & \text{ } j_\tau(x, \cdot) \text{ is Lipschitz continuous on } \mathbb{R}^d \text{ for a.e. } x \in \Gamma_C; \\
(c) & \|\partial j_\tau(x, \xi)\| \leq c_0 \text{ for a.e. } x \in \Gamma_C, \text{ all } \xi \in \mathbb{R}^d \text{ with } c_0 > 0; \\
(d) & \|j_{\tau_1}(x, \xi_1) - j_{\tau_2}(x, \xi_2)\| \leq \bar{\beta} \|\xi_1 - \xi_2\|_{\mathbb{R}^d} \text{ for a.e. } x \in \Gamma_C, \text{ all } \xi_i \in \mathbb{R}^d, \ i = 1, 2 \text{ with } \bar{\beta} \geq 0.
\end{align*}$$

For the memory function $b : \Gamma_C \to \mathbb{R}$, we assume

$$\begin{align*}
(a) & b \in L^1(0, T; L^\infty(\Gamma_C)); \\
(b) & b(x, t) \geq 0 \text{ for a.e. } x \in \Gamma_C, t \in (0, T).
\end{align*}$$

For the contact function $p : \Gamma_C \times \mathbb{R} \to \mathbb{R}$, we assume

$$\begin{align*}
(a) & p(\cdot, r) \text{ is measurable on } \Gamma_C, \text{ for any } r \in \mathbb{R}; \\
(b) & p(\cdot, 0) \in L^2(\Gamma_C); \\
(c) & \text{ there exists } L_p > 0 \text{ such that for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \text{ } \\
& |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2|.
\end{align*}$$

Moreover, we assume that the densities of body forces, surface tractions have the regularity

$$\begin{align*}
f_0 & \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad f_N \in C(0, T; L^2(\Gamma_N; \mathbb{R}^d)),
\end{align*}$$

where $\Omega$ is the domain of the problem and $\Gamma_N$ are the boundaries of the contact surfaces.
and the microcrack diffusion coefficient verifies
\[ \kappa > 0. \]  
(2.12)

Finally, the initial data satisfy
\[ u_0 \in V, \quad \zeta_0 \in K. \]  
(2.13)

Define a function \( f : (0, T) \to V^* \) by
\[ \langle f(t), v \rangle_{V^* \times V} = (f_0(t), v)_H + (f_N(t), v)_{L^2(\Gamma_N; \mathbb{R}^d)}, \quad v \in V, \ a.e. \ t \in (0, T). \]  
(2.14)

Then conditions (2.11) imply
\[ f \in C([0, T]; V). \]  
(2.15)

Let \( a : Z_1 \times Z_1 \to \mathbb{R} \) be the bilinear form
\[ a(\xi, \eta) = \kappa \int_{\Omega} \nabla \xi \cdot \nabla \eta \, dx. \]  
(2.16)

Through a list of standard derivation, we obtain the following variational formulation of Problem 1.1.

**Problem 2.1.** Find a displacement field \( u : (0, T) \to V \) and a damage field \( \zeta : (0, T) \to Z_1 \) such that for a.e. \( t \in (0, T) \),
\[ (A \varepsilon(u'(t)), \varepsilon(v))_H + (B(\varepsilon(u(t)), \zeta(t)), \varepsilon(v))_H + \int_{\Gamma_C} p(u_\nu(t)) v_\nu \, d\Gamma + \int_{\Gamma_C} b(t-s) u_\nu(s) \, ds + \int_{\Gamma_C} j_0^\nu(u_\tau'; t; v_\tau) \, d\Gamma \geq \langle f(t), v \rangle_{V^* \times V} \forall v \in V, \]
\[ \zeta(t) \in K, \quad (\zeta'(t), \xi - \zeta(t))_{Z_0} + a(\zeta(t), \xi - \zeta(t)) \geq (\phi(\varepsilon(u(t)), \zeta(t)), \xi - \zeta(t))_{Z_0} \forall \xi \in K \]  
(2.17)

and
\[ u(0) = u_0, \quad \zeta(0) = \zeta_0. \]  
(2.18)

The unique solvability of Problem 2.1 is provided in the following result.

**Theorem 2.2.** Assume (2.5)–(2.13). If
\[ m_A > \bar{\beta} \| \gamma \|^2, \]  
(2.20)

then Problem 2.1 has a unique solution \((u, \zeta)\) with regularity
\[ u \in C^1([0, T]; V), \quad \zeta \in H^1(0, T; Z_0) \cap L^2(0, T; Z_1). \]  
(2.21)

We refer to [15] for a standard argument to show the existence of a unique solution \((u, \zeta)\).

For the convenience of the numerical approximation, we reformulate the history-dependent hemivariational inequality in terms of the velocity variable
\[ w = u'. \]  
(2.22)

By using the initial value condition (2.19), we can recover \( u \) from \( w \) as follows:
\[ u(t) = u_0 + (I w)(t), \]  
(2.23)

where
\[ (I w)(t) = \int_0^t w(s) \, ds. \]  
(2.24)

Then Problem 2.1 can be equivalently stated as follows.
Problem 2.3. Find a velocity field $w \in V$ and a damage field $\zeta : (0, T) \to Z_1$ such that for a.e. $t \in (0, T)$,

\[
(A\varepsilon(w(t)), \varepsilon(v))_H + (B(\varepsilon(u_0 + (Iw)(t)), \zeta(t)), \varepsilon(v))_H \\
+ \int_{\Gamma_C} p(u_{0,\nu} + (Iw)(t,\nu))v_\nu d\Gamma + \int_{\Gamma_C} \left( \int_0^1 b(t-s)u_{0,\nu} + (Iw)(s,\nu) \right) ds v_\nu d\Gamma \\
+ \int_{\Gamma_C} j_s^0(w(t); v_\tau) d\Gamma \geq (f(t), v)_{V^* \times V} \quad \forall v \in V,
\]

where the constant $c > 0$ depends on $c$ independent of $N$ or $k$.

Lemma 3.1. Let $T > 0$ be given. For a positive integer $N$, define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying

\[
e_n \leq \bar{c}g_n + \bar{c} \sum_{j=1}^n ke_j, \quad n = 1, \ldots, N
\]

for a positive constant $\bar{c}$ independent of $N$ or $k$. Then, there exists a positive constant $c$, independent of $N$ or $k$, such that

\[
\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.
\]

We will also make use of the following modified Cauchy-Schwarz inequality with an arbitrary $\epsilon > 0:

\[
a b \leq \epsilon a^2 + c b^2, \quad a, b \in \mathbb{R},
\]

where the constant $c > 0$ depends on $\epsilon$; Here, we may simply take $c = 1/(4 \epsilon)$.

Let $V^h$ be a finite dimensional subspace of $V$, $Z_1^h$ be a finite dimensional subspace of $Z_1$ where $h > 0$ denotes a spatial discretization parameter. Let $K^h = K \cap Z_1^h$. In addition, we consider an equidistant time grid with abscissae $t_n = nk$, where $n = 0, 1, \cdots, N$, $N \in \mathbb{N}$, and the constant step-size $k = T/N$. For a time continuous function $z = z(t)$, we write $z_n = z(t_n)$ for $n = 0, 1, \cdots, N$. For a sequence $\{z_n\}_{n=0}^N$, we let $\delta z_n = \frac{z_n - z_{n-1}}{k}$ be its corresponding divided differences.

Let $u_0^h$ and $\xi_0^h$ be the appropriate approximation of the initial condition $u_0$ and $\xi_0$. The integration operator $I$ of (2.24) will be approximated by the discrete operator $I^k$ by the form

\[
(I^k w)_n = k \sum_{j=1}^n w_j
\]
and the discrete displacement field and discrete velocity field are related by the relation
\[ u_{n}^{hk} = u_{0}^{h} + (I^{k}w_{n}^{hk})_{n}, \]  
(3.3)
where \((I^{k}w_{n}^{hk})_{n} = k \sum_{j=1}^{n} w_{j}^{hk}\).

**Problem 3.2.** Find a discrete velocity field \(w_{n}^{hk} = \{w_{n}^{hk}\}_{n=0}^{N} \subset V^{h}\), a discrete damage field \(c_{n} = \{c_{n}^{hk}\}_{n=0}^{N} \subset K^{h}\) such that for \(1 \leq n \leq N\),

\[
(A \varepsilon(w_{n}^{hk}), \varepsilon(v_{h}))_{V} + (B(\varepsilon(u_{0}^{h} + (I^{k}w_{n}^{hk})_{n-1}), c_{n}^{hk}), \varepsilon(v_{h}))_{V} + \int_{C} \int_{0}^{t_{n}} b(t_{n} - t_{j-1}) (u_{0}^{h} + (I^{k}w_{n}^{hk})_{j-1,\nu}) v_{h}^{n,\nu} dt_{j} d\Gamma + I_{C} \int_{0}^{t_{n}} b(t_{n} - s) (u_{0}^{h} + (I^{k}w_{n}^{hk})_{j-1,\nu}) (v_{h}^{n,\nu} - v_{n,\nu}) d\Gamma \geq (f, v_{h})_{V_{\nu \times V}} \forall v_{h} \in V^{h},
\]  
(3.4)

where \(c^{hk} = \{c_{n}^{hk}\}_{n=0}^{N} \subset K^{h}\). We first set \(t = t_{n}\) and \(v = w_{n}^{hk} - u_{n}^{h}\) in (2.25) and we also replace \(v_{h}^{n} - w_{n}^{hk}\) in (3.4). By adding the resulting inequalities, we obtain that

\[
(A \varepsilon(w_{n}^{hk}), \varepsilon(w_{n}^{hk} - w_{n}^{hk}))_{V} \leq I_{1} + I_{2} + I_{3},
\]  
(3.9)

where

\[
I_{1} = (B(\varepsilon(u_{0} + (I^{k}w_{n})), \zeta_{n}) - B(\varepsilon(u_{0}^{h} + (I^{k}w_{n}^{hk})_{n-1}), \zeta_{n}^{hk}), \varepsilon(w_{n}^{hk} - w_{n}^{hk}))_{V},
\]

\[
I_{2} = \int_{C} \int_{0}^{t_{n}} (p(u_{0,\nu} + (I^{k}w_{n,\nu})_{n-1,\nu}) - p(u_{0,\nu}^{h} + (I^{k}w_{n}^{hk})_{n-1,\nu})) (w_{n,\nu}^{h} - w_{n,\nu}) dt_{j} d\Gamma + I_{C} \int_{0}^{t_{n}} b(t_{n} - s) (u_{0}^{h} + (I^{k}w_{n}^{hk})_{j-1,\nu}) (v_{h}^{n,\nu} - v_{n,\nu}) d\Gamma,
\]

\[
I_{3} = \int_{C} j_{\tau}^{0}(w_{n,\tau}^{hk} - v_{h}^{n,\tau}) d\Gamma + \int_{C} j_{\tau}^{0}(w_{n,\tau}^{hk} + v_{h}^{n,\tau} - w_{n,\tau}^{hk}) d\Gamma.
\]
Lemma 3.3. Let $w$ and $w^{hk}$ be solutions to Problems 2.3 and 3.2, respectively. Under the regularity assumption $w \in H^1(0,T;V)$, we have the following bound for $n = 1, \ldots, N$:

$$
\|(Iw)_n - (Ikw^{hk})_{n-1}\|_V \leq \epsilon_k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V. \tag{3.10}
$$

Proof. From the definitions of $(Iw)_n$ and $(Ikw^{hk})_n$, we have

$$
\|(Iw)_n - (Ikw^{hk})_{n-1}\|_V \leq \left\| \int_0^{t_n} w(s) \, ds - \int_0^{t_{n-1}} w(s) \, ds \right\|_V
$$

$$
+ \| \int_0^{t_n} w(s) \, ds - k \sum_{j=1}^{n-1} w_j \|_V + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V
$$

$$
= \int_{t_{n-1}}^{t_n} \|w(s)\|_V \, ds + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left( w(s) - w_j \right) \, ds \|_V + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V
$$

$$
= \int_{t_{n-1}}^{t_n} \|w(s)\|_V \, ds + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \frac{d}{d\tau}(w(\tau)) \, d\tau \|_V + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V
$$

$$
\leq k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w'(\tau)\|_V \, d\tau + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V,
$$

i.e., (3.10) holds.

The next lemma provides a result on the estimate for history-dependent terms in (3.9).

Lemma 3.4. Let $w$ and $w^{hk}$ be solutions to Problems 2.3 and 3.2, respectively. Under the regularity assumption $b \in H^1(0,T;L^\infty(\mathcal{C}))$ and $w \in H^1(0,T;V)$, we have the following inequality:

$$
I_1 + I_2 + I_3 \leq c \|w^{hk}_n - v^0_n\|_V \left( k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V \right) \tag{3.11}
$$

$$
+ \|u_0 - u^0_0\|_V + k \|\zeta'\|_{C([0,T];Z_0)} + \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Z_0 + \beta \|\gamma\|_V \|w_n - w^{hk}_n\|_V^2
$$

$$
+ c \|w_n - v^0_n\|_{L^2(\mathcal{C} \times \mathbb{R}^d)} \text{ for all } v^0_n \in V^h, \ n = 1, \ldots, N.
$$

Proof. We first derive an upper bound on $I_1$. By using (2.6)(a), we obtain

$$
\left\| B(\varepsilon(u_0 + (Iw)_n), \zeta_n) - B(\varepsilon(u^0_0 + (Ikw^{hk})_{n-1}), \zeta^{hk}_{n-1}) \right\| \leq L_B(\|u_0 - u^0_0\|_V + \|(Iw)_n - (Ikw^{hk})_{n-1}\|_V + \|\zeta_n - \zeta^{hk}_{n-1}\|_Z_0).
$$

Using (3.10) we conclude that there is a constant $c$ such that

$$
I_1 \leq c \|w^{hk}_n - v^0_n\|_V \left( k \|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w^{hk}_j\|_V + \|u_0 - u^0_0\|_V + \|\zeta_n - \zeta^{hk}_{n-1}\|_Z_0 \right). \tag{3.12}
$$

Next, we bound $I_2$. For the term containing function $p$, by using 2.10(c), we have

$$
\|p(u_{0,\nu} + (Iw)_{n,\nu}) - p(u^0_{0,\nu} + (Ikw^{hk})_{n-1,\nu})\|_{L^2(\mathcal{C})}
$$

$$
\leq L_p \|u_{0,\nu} - u^0_{0,\nu} + (Iw)_{n,\nu} - (Ikw^{hk})_{n-1,\nu}\|_{L^2(\mathcal{C})}
$$

$$
\leq L_p \|\gamma\| \|u_0 - u^0_0 + (Iw)_n - (Ikw^{hk})_{n-1}\|_V.
$$
For the term containing $b$, we obtain
\[
\left\| \int_0^{t_n} b(t_n - s) (u_{0,\nu} + (Iw)(s)_{\nu}) \, ds - k \sum_{j=1}^n b(t_n - t_j) (u_{0,\nu}^{(j)} + (I^k w)_{j-1,\nu}) \right\|_{L^2(G_C)} \\
\leq \left\| \int_0^{t_n} b(t_n - s) (u_{0,\nu} + (Iw)(s)_{\nu}) \, ds - \int_0^{t_n} b(t_n - s) (u_{0,\nu} + (Iw)(s)_{\nu}) \, ds \right\|_{L^2(G_C)} \\
+ \left\| \int_0^{t_n-1} b(t_n - s) (u_{0,\nu} + (Iw)(s)_{\nu}) \, ds - k \sum_{j=1}^n b(t_n - t_j) (u_{0,\nu}^{(j)} + (I^k w)_{j-1,\nu}) \right\|_{L^2(G_C)} \\
+ \left\| k \sum_{j=1}^n b(t_n - t_j) (u_{0,\nu}^{(j)} + (I^k w)_{j-1,\nu}) - k \sum_{j=1}^n b(t_n - t_j) (u_{0,\nu}^{(j)} + (I^k w)_{j-1,\nu}) \right\|_{L^2(G_C)} \\
\leq c(k\|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w_h^k\|_V + \|u_0 - u_0^h\|_V).
\]

Consequently, there is a constant $c$ such that
\[
I_2 \leq c\|w_n^h - v_n^h\|_V \left( k\|w\|_{H^1(0,T;V)} + k \sum_{j=1}^{n-1} \|w_j - w_h^k\|_V + \|u_0 - u_0^h\|_V \right). \tag{3.13}
\]

Finally, we bound $I_3$, which is expressed as
\[
I_3 = \int_{\Gamma_C} j^0_{r}(w_{n,\tau}; w_{n,\tau} - v_{n,\tau}) \, d\Gamma + \int_{\Gamma_C} j^0_{r}(w_{n,\tau}; v_{n,\tau} - w_{n,\tau}) \, d\Gamma.
\]

By (2.8) (c), $j^0_{r}(x, \xi; \eta) \leq \tau_0 |\eta|$. Using the subadditivity of the generalized directional derivative, (2.8) (d) and Cauchy-Schwarz inequality, it follows that
\[
\begin{align*}
\int_{\Gamma_C} j^0_{r}(w_{n,\tau}; w_{n,\tau} - v_{n,\tau}) \, d\Gamma &+ \int_{\Gamma_C} j^0_{r}(w_{n,\tau}; v_{n,\tau} - w_{n,\tau}) \, d\Gamma \\
&\leq \int_{\Gamma_C} [j^0_{r}(w_{n,\tau}; w_{n,\tau} - w_{n,\tau}) + j^0_{r}(w_{n,\tau}; w_{n,\tau} - v_{n,\tau})] \, d\Gamma \\
&+ \int_{\Gamma_C} [j^0_{r}(w_{n,\tau}; v_{n,\tau} - w_{n,\tau}) + j^0_{r}(w_{n,\tau}; v_{n,\tau} - v_{n,\tau})] \, d\Gamma \\
&\leq \tilde{\beta}\gamma \|w_n - w_n^h\|_V^2 + 2\tau_0 \sqrt{m(\Gamma_C)} \|w_n - v_n^h\|_{L^2(\Gamma_C; \mathbb{R}^d)}.
\end{align*}
\tag{3.14}
\]

Moreover, we use the inequality
\[
\|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0} \leq \|\zeta_n - \zeta_{n-1}\|_{Z_0} + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}
\]
and obtain
\[
\|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0} \leq \|\zeta_n - \zeta_{n-1}\|_{Z_0} + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0} + k\|\zeta^\prime\|_{C(0,T;Z_0)}.
\tag{3.15}
\]

From (3.12)–(3.15), we have (3.11), which finishes the proof.

Now, we are ready to bound the other term on the right side of (3.9). From (2.5)(a) and applying the modified Cauchy-Schwarz inequality (3.1), we have
\[
(\mathbb{A}e(w_n) - \mathbb{A}e(w_n^h), e(w_n - v_n^h))_H \leq L_A \|w_n - w_n^h\|_V \|w_n - v_n^h\|_V \\
\leq \epsilon \|w_n - w_n^h\|_V^2 + c \|w_n - v_n^h\|_V^2. \tag{3.16}
\]

Then, by using inequalities (3.11), (3.16) on the right side of the inequality (3.9) and taking $\epsilon > 0$ sufficiently small, under assumption $m_A > \tilde{\beta}\gamma$, we obtain the following result.
\[
\|w_n - w_n^{hk}\|_V^2 \leq c(\|w_n - v_n^h\|_V^2 + \|w_n - v_n^h\|_{L^2(\Gamma_C; \mathbb{R}^d)})
+ c\|w_n^{hk} - v_n^h\|_V(k\|w\|_{H^1(0, T; V)} + k\sum_{j=1}^{n-1} \|w_j - w_j^{hk}\|_V + \|u_0 - u_0^h\|_V
+ k\|\zeta\|_{C(0, T; Z_0)} + \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_{Z_0}).
\]

Since \(\|w_n^{hk} - v_n^h\|_V^2 \leq 2(\|w_n^{hk} - w_n\|_V^2 + \|w_n - v_n^h\|_V^2)\), it follows that
\[
\|w_n - w_n^{hk}\|_V^2 \leq c(\|w_n - v_n^h\|_V^2 + \|w_n - v_n^h\|_{L^2(\Gamma_C; \mathbb{R}^d)})
+ c(k\|w\|_{H^1(0, T; V)} + k\sum_{j=1}^{n-1} \|w_j - w_j^{hk}\|_V + \|u_0 - u_0^h\|_V
+ k\|\zeta\|_{C(0, T; Z_0)} + \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_{Z_0})^2. \tag{3.17}
\]

The next, we estimate \(\|\zeta_n - \zeta_n^{hk}\|_{Z_0}\). Now, we choose \(\xi = \zeta_n^{hk}\) in (2.26) at \(t = t_n\) and find
\[
(\zeta_n', \zeta_n^{hk} - \zeta_n)_{Z_0} + a(\zeta_n, \zeta_n^{hk} - \zeta_n) \geq \langle \phi(\mathbf{u}_0 + \int_0^{t_n} \mathbf{w}(s) \, ds), \zeta_n, \zeta_n^{hk} - \zeta_n \rangle_{Z_0}.
\tag{3.18}
\]
Adding (3.18) and (3.5), with \(\xi = \zeta_n^{hk} \in K^h\), yields
\[
(\delta(\zeta_n - \zeta_n^{hk}), \zeta_n - \zeta_n^{hk})_{Z_0} + a(\zeta_n - \zeta_n^{hk}, \zeta_n - \zeta_n^{hk}) \leq (\delta\zeta_n', \zeta_n - \zeta_n^{hk})_{Z_0} + (\delta(\zeta_n - \zeta_n^{hk}), \zeta_n - \zeta_n^{hk})_{Z_0}
+ a(\zeta_n - \zeta_n^{hk}, \zeta_n - \zeta_n^{hk}) - (\delta\zeta_n, \zeta_n - \zeta_n^{hk})_{Z_0}
- a(\zeta_n, \zeta_n - \zeta_n^{hk}) + \langle \phi(\mathbf{u}_0 + \int_0^{t_n} \mathbf{w}(s) \, ds), \zeta_n, \zeta_n - \zeta_n^{hk} \rangle_{Z_0}
+ \langle \phi(\mathbf{u}_0 + \int_0^{t_n} \mathbf{w}(s) \, ds), \zeta_n - \phi(\mathbf{u}_0 + (I^k w_n^{hk})_{n-1}, \zeta_n^{hk}), \zeta_n^{hk} - \zeta_n^{hk} \rangle_{Z_0}. \tag{3.19}
\]
Next, by the assumption on the bilinear form \(a(\cdot, \cdot)\), we have
\[
a(\zeta_n - \zeta_n^{hk}, \zeta_n - \zeta_n^{hk}) \geq c_0|\zeta_n - \zeta_n^{hk}|_{Z_0}^2, \tag{3.20}
\]
The first term on the left-hand of (3.19) is bounded from below
\[
(\delta(\zeta_n - \zeta_n^{hk}), \zeta_n - \zeta_n^{hk})_{Z_0} \geq \frac{1}{2k}(\|\zeta_n - \zeta_n^{hk}\|_{Z_0}^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_{Z_0}^2). \tag{3.21}
\]
Now we use (3.20) and (3.21) in (3.19) and after some simple estimates just like [15], we obtain

\[
\|\zeta_n - \zeta_n^{h_k}\|_{Z_0}^2 + k^2 \sum_{j=1}^{n} |\zeta_j - \zeta_j^{h_k}|_{Z_1}^2 \geq c \left\{ \|\zeta_0 - \zeta_0^{h_k}\|_{Z_0}^2 + \|\zeta_1 - \zeta_1^{h_k}\|_{Z_0}^2 + \|u_0 - u_0^{h_k}\|_{V}^2 \right\} + k^2 (\|w_{T;0,V}\| + \|\zeta^{h_k}'\|_{C([0,T];Z_0)}) + k \sum_{j=1}^{n} \|\delta\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 \\
+ \|\zeta_n - \zeta_n^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n-1} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 \\
+ k \sum_{j=1}^{n} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n} \|w_j - w_j^{h_k}\|_{V}^2 \right\}.
\]

Combining this inequality with (3.17), we obtain

\[
\|w_n - w_n^{h_k}\|_{V}^2 + \|\zeta_n - \zeta_n^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n} |\zeta_j - \zeta_j^{h_k}|_{Z_1}^2 \geq c \left\{ \|w_n - w_n^{h_k}\|_{V}^2 + \|w_n - w_n^{h_k}\|_{L^2([\Gamma_c;\mathbb{R}^d])} \right\} + k^2 (\|w_{T;0,V}\| + \|\zeta^{h_k}'\|_{C([0,T];Z_0)}) + k \sum_{j=1}^{n} \|\delta\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 \\
+ \|\zeta_n - \zeta_n^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n-1} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 \\
+ k \sum_{j=1}^{n} \|\zeta_j - \zeta_j^{h_k}\|_{Z_0}^2 + k \sum_{j=1}^{n} \|w_j - w_j^{h_k}\|_{V}^2 \right\}.
\]

Applying the Gronwall inequality, we have

\[
\max_{1 \leq n \leq N} \left( \|w_n - w_n^{h_k}\|_{V}^2 + \|\zeta_n - \zeta_n^{h_k}\|_{Z_0}^2 \right) \leq c \left( \|u_0 - u_0^{h_k}\|_{V}^2 + \|\zeta_0 - \zeta_0^{h_k}\|_{Z_0}^2 \right) + c k^2 (\|w_{T;0,V}\| + \|\zeta^{h_k}'\|_{C([0,T];Z_0)}) + c \max_{1 \leq n \leq N} \hat{R}_n
\]
where
\[
\tilde{R}_n = \|w_n - u_n^h\|_V^2 + \|w_n - u_n^h\|_{L^2(\Gamma; \mathbb{R}^d)} \tag{3.25}
\]
\[
+ \|\zeta_1 - \xi_1^h\|_Z^2 + k \sum_{j=1}^n \|\delta \xi_j - \zeta_j^h\|_Z^2 + \|\zeta_n - \xi_n^h\|_Z^2
\]
\[
+ \sum_{j=1}^n k \|\phi(\xi(\int_0^{t_j} w(s) ds), \zeta_j) - \delta \zeta_j + k \Delta \zeta_j\|_Z \|\zeta_j - \xi_j^h\|_Z
\]
\[
+ k \sum_{j=1}^n \|\zeta_j - \xi_j^h\|_Z^2 + k^{-1} \sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Z^2.
\]

Summarizing the above arguments, we have the following theorem.

**Theorem 3.5.** Let \( (w, \zeta) \) and \( (w^{hk}, \xi^{hk}) \) be solutions to Problems 2.3 and Problem 3.2, respectively. Assume (2.5)–(2.13), \( m_A > \beta \|\gamma\|_V^2 \). Then under the regularity assumption in Lemma 3.4, we have
\[
\max_{1 \leq n \leq N} (\|w_n - w_n^{hk}\|_V^2 + \|\zeta_n - \zeta_n^{hk}\|_Z^2)
\]
\[
\leq c(\|u_0 - u_0^h\|_V^2 + \|\zeta_0 - \zeta_0^h\|_Z^2) + c \max_{1 \leq n \leq N} \tilde{R}_n
\]
where
\[
\tilde{R}_n = \|w_n - u_n^h\|_V^2 + \|w_n - u_n^h\|_{L^2(\Gamma; \mathbb{R}^d)} \tag{3.26}
\]
\[
+ \|\zeta_1 - \xi_1^h\|_Z^2 + k \sum_{j=1}^n \|\delta \xi_j - \zeta_j^h\|_Z^2 + \|\zeta_n - \xi_n^h\|_Z^2
\]
\[
+ \sum_{j=1}^n k \|\phi(\xi(\int_0^{t_j} w(s) ds), \zeta_j) - \delta \zeta_j + k \Delta \zeta_j\|_Z \|\zeta_j - \xi_j^h\|_Z
\]
\[
+ k \sum_{j=1}^n \|\zeta_j - \xi_j^h\|_Z^2 + k^{-1} \sum_{j=1}^{n-1} \|(\zeta_{j+1} - \xi_{j+1}^h) - (\zeta_j - \xi_j^h)\|_Z^2.
\]

Theorem 3.5 is the basis for the analysis of the optimal error estimate. As an example, let \( \Omega \) be a polygonal or polyhedral domain and let \( \mathcal{T}^h \) be a regular triangulation of \( \Omega \) compatible with the partition of the boundary \( \Gamma = \partial \Omega \) into \( \Gamma_D \), \( \Gamma_N \) and \( \Gamma_C \). For an element \( T \in \mathcal{T}^h \), denote by \( P_1(T; \mathbb{R}^d) \) the space of polynomials of a total degree less than or equal to one in \( T \). Then we can use the linear element space of piecewise continuous affine functions
\[
V^h = \{ v^h \in C(\Omega; \mathbb{R}^d) : v^h|_T \in P_1(T; \mathbb{R}^d) \ \forall \ T \in \mathcal{T}^h, \ v^h = 0 \ \text{on} \ \Gamma_D \},
\]
\[
K^h = \{ \xi^h \in C(\Omega) : \xi^h|_T \in P_1(T), \ \forall \ T \in \mathcal{T}^h \}.
\]

**Corollary 3.6.** Under the assumptions stated in Theorem 3.5. Assume \( \Omega \) is a polygonal or polyhedral domain, and let \( \{ V^h \} \), \( \{ K^h \} \) be the family of linear element spaces made of continuous and piecewise affine functions defined by (3.28) and (3.29), corresponding to a regular family of finite element triangulations of \( \Omega \)
into triangles or tetrahedrons. Assume further that
\[ \mathbf{w} \in C([0, T]; H^2(\Omega; \mathbb{R}^d)), \quad \mathbf{w}_{IC} \in C([0, T]; H^2(\Gamma_C; \mathbb{R}^d)), \quad \zeta \in C([0, T]; H^2(\Omega)) \cap H^2(0, T; \mathbb{R}) \cap H^1(0, T; Z_1). \]
Then we have the following optimal order error estimate:
\[
\max_{1 \leq n \leq N} (\|\mathbf{w}_n - \mathbf{w}_h\|_V^2 + \|\zeta_n - \zeta_h\|_{Z_0}^2) \leq c(k^2 + h^2). \tag{3.30}
\]

**Proof.** We apply the standard finite element interpolation error estimates ([2, 4, 9]). Since \( \mathbf{w} \in C(0, T; H^2(\Omega; \mathbb{R}^d)) \), we have
\[
\max_{1 \leq n \leq N} \inf_{v_h \in V_h} \|\mathbf{w}_n - v_h\|_V \leq c h \|\mathbf{w}\|_{C([0, T]; H^2(\Omega; \mathbb{R}^d))}. \tag{3.31}
\]
By the finite element interpolation error estimates, if we take \( \xi_j^h \) to be the interpolant of \( \zeta_j \) for \( j = 1, \ldots, N \), then
\[
\|\zeta_j - \xi_j^h\|_{Z_0} \leq c h \|\zeta_j\|_{H^2(\Omega)}, \tag{3.32}
\]
\[
\|\zeta_j - \xi_j^h\|_{Z_1} \leq c h \|\zeta_j\|_{H^2(\Omega)}. \tag{3.33}
\]
Since \( \mathbf{u}_h^0 \) is the finite element interpolant of \( \mathbf{u}_0 \), \( \zeta_h^0 \) be the finite element orthogonal projection of \( \zeta_0 \) in \( L^2(\Omega) \). Then we have the error estimate for the discrete initial values,
\[
\|\mathbf{u}_0 - \mathbf{u}_h^0\|_V + \|\zeta_0 - \zeta_h^0\|_{Z_0} \leq c h. \tag{3.34}
\]
Since \( \zeta \in H^2(0, T; Z_0) \), it follows that
\[
k \sum_{j=1}^{n} \|\delta \zeta_j - \xi_j^h\|_{Z_0} \leq c k \|\zeta\|_{H^2(0, T; Z_0)}, \quad 1 \leq n \leq N. \tag{3.35}
\]
From [11], we have
\[
\frac{1}{k} \sum_{j=1}^{n-1} \|\zeta_j - \xi_j^h - (\zeta_{j+1} - \xi_{j+1}^h)\|_{Z_0} \leq c h^2 \|\zeta\|_{H^1(0, T; Z_1)}, \quad 1 \leq n \leq N. \tag{3.36}
\]
Finally, we have
\[
\max_{1 \leq n \leq N} \|\mathbf{w}_n - v_h\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq c h \|\mathbf{w}\|_{C([0, T]; H^2(\Gamma_C; \mathbb{R}^d))}. \tag{3.37}
\]
Combining (3.31)–(3.37) and (3.26), we obtain the optimal order error estimate
\[
\max_{1 \leq n \leq N} (\|\mathbf{w}_n - \mathbf{w}_h\|_V^2 + \|\zeta_n - \zeta_h\|_{Z_0}^2) \leq c(k^2 + h^2).
\]
This concludes the proof of Corollary 3.6. \( \square \)

Finally, we note that for the error in the displacement,
\[
\mathbf{u}_n - \mathbf{u}_h^{kk} = (\mathbf{u}_0 - \mathbf{u}_h^0) + [(I \mathbf{w})_n - (I^k \mathbf{w}^{kk})_n].
\]
So from Corollary 3.6 and Lemma 3.3, we have the next result.

**Corollary 3.7.** Keep the assumptions stated in Corollary 3.6. Then we have the optimal order error estimate
\[
\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_h^{kk}\|_V + \|\zeta_n - \zeta_h\|_{Z_0}) \leq c(k + h). \tag{3.38}
\]
4. **Numerical example.** In this section, we present some numerical results to illustrate the behavior of the solution of the history-dependent frictional contact problem Problem 1.1. In the examples, the elasticity operator has the form

\[
\mathcal{B}(\varepsilon(u), \zeta) = \eta_*(\zeta) \mathcal{B}(\varepsilon(u))
\]

where the elasticity tensor \( \mathcal{B} \) is given by

\[
(\mathcal{B} \tau)_{\alpha\beta} = \frac{E}{1 + \kappa} \tau_{\alpha\beta} + \frac{E \kappa}{(1 - \kappa)(1 - 2\kappa)} (\tau_{11} + \tau_{22}) \delta_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
\]

and the viscosity tensor \( \mathcal{A} \) is given by

\[
(\mathcal{A} \tau)_{\alpha\beta} = \mu_1 (\tau_{11} + \tau_{22}) \delta_{\alpha\beta} + \mu_2 \tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2, \quad \forall \tau \in \mathbb{S}^2,
\]

where \( \mu_1 \) and \( \mu_2 \) are viscosity constants, \( E \) and \( \kappa \) are Young’s modulus and Poisson’s ratio of the material, and \( \delta_{\alpha\beta} \) denotes the Kronecker symbol. In the numerical example, we take \( \mu_1 = 25 \), \( \mu_2 = 50 \), \( E = 1000 \) N/m, \( \kappa = 0.3 \).

The damage source function used here has the form

\[
\phi(\varepsilon(u), \zeta) = -\lambda_1 \left( 1 - \frac{\eta_*(\zeta)}{\eta_*(\zeta)} \right) - \frac{1}{2} \lambda_2 \Phi_*(\varepsilon(u)) + \lambda_3
\]

where \( \Phi_*(\varepsilon(u)) = \min\{\varepsilon(u) \cdot \varepsilon(u), q^*\} \) for some constant \( q^* > 0 \) and

\[
\eta_*(\zeta) = \begin{cases} 
\zeta, & \zeta < \zeta_*, \\
\zeta_*, & \zeta_* \leq \zeta \leq 1, \\
1, & \zeta > 1.
\end{cases}
\]

**Figure 1.** Reference configuration of the two-dimensional example.

For the friction law (1.8), it can be reduced as follows:

\[
\| \sigma(t) \| \leq F_b,
\]

\[
- \sigma(t) = F_b \left( (1 - a) e^{-\| u'_r(t) \|} + a \right) \frac{u'_r(t)}{\| u'_r(t) \|} \quad \text{if} \quad u'_r(t) \neq 0.
\]

(4.1)

Then, we take

\[
F_b = 1.
\]

(4.2)
and

\[-\sigma = \begin{cases} 
((a - 1)e^{-\|u^\prime(t)\|} - a) & \text{if } u^\prime(t) < 0, \\
((a - 1)e^{-\|u^\prime(t)\|} - a, ((1 - a)e^{-\|u^\prime(t)\|} + a)) & \text{if } u^\prime(t) = 0, \\
((1 - a)e^{-\|u^\prime(t)\|} + a) & \text{if } u^\prime(t) > 0.
\end{cases}\] (4.3)

We use a Primal-Dual Active Set Strategy to solve the discrete problem. We refer to paper [3] for more details about this strategy.

4.1. The first example

We consider the physical setting shown in Figure 1. Here, the domain \(\Omega = (0, 2) \times (0, 1)\), and its boundary is split into:

\(\Gamma_D = \{0\} \times (0, 1),\ \Gamma_N = ((0, 2) \times \{1\} \cup (\{2\} \times (0, 1))\) and \(\Gamma_C = (0, 2) \times \{0\}\), \(T = 1s\).

\(f_0 = (0, 0)N/m^2\) in \(\Omega\), \(f_N = (0, -30t)N/m\) on \((0, 2) \times \{1\}\), \(b_{ijkl} = 1, \ \zeta^* = 0.01, \ \lambda_1 = \lambda_2 = 2, \ \lambda_3 = 0, \ u_0 = 0, \ \zeta_0(x) = 1, \ \forall x \in \Omega\).

\(p(r) = d_1 r, \ d_1 = 100N/m^2\).

In Figure 2, we plot the deformed configuration during the compression process for coefficient \(a = 0.1, \ \mu = 0.1\) and the damage field at time \(t = 1s\), respectively. We note that the damage is most concentrated on the left boundary due to the clamping condition. Moreover, the linear convergence of the algorithm with respect to \(h + k\) is clearly observed in Table 1 and Figure 5.

4.2. The second example

As the second example, the physical setting is similar to the above test. Now, we change \(f_N\) into \((-30t, 0)N/m\) on \((\{2\} \times (0, 1))\).
In Figure 3, we plot the damage field at time $t = 1s$ for coefficient $a = 1$, $\mu = 0$ and $a = 1$, $\mu = 1$, respectively. We observe that in both cases the damage is concentrated on the left boundary due to the clamping condition. When $a = 1$, $\mu = 0$, there is no friction, we observe that on the upper boundary and the lower boundary, the damage is nearly the same, when $a = 1$, $\mu = 1$, due to the friction, the damage is more serious on the lower boundary than the upper boundary.

4.3. The third example

In the last example, we let $\Gamma_D = \{(0) \times (0, 1)\} \cup \{(2) \times (0, 1)\}$, $\Gamma_N = (0, 2) \times \{1\}$ and $\Gamma_C = (0, 2) \times \{0\}$. Moreover, now $f_N$ is made to be $(0,0)N/m$ on $\Gamma_N$ and $f_0 = (0,-100)N/m^2$ in $\Omega$.

In figure 4, we plot the deformed configuration during the compression process for coefficient $a = 1$, $\mu = 1$ and the damage field at time $t = 1s$, respectively. We note that the damage is most concentrated on the left boundary and the right boundary due to the clamping condition. And on the lower boundary, the damage is due to the friction.

From the above examples, we conclude that the damage is more concentrate on the clamping boundaries, the contact boundaries, i.e. the most stressed fields.
In order to examine the numerical convergence orders, we use the uniform discretizations of the problem domain to compute a sequence of numerical solutions according to the spatial discretization parameter $h$ and time step $k$. For example, the deformed configuration plotted in Figure 2 correspond to the choices $h = 1/64$ and $k = 1/128$. The numerical error $\|u - u^{hk}\|_V + \|\zeta - \zeta^{hk}\|_{Z_0}$ is computed for several discretization parameters of $h$ and $k$. Here, the boundary $\Gamma_C$ of $\Omega$ is divided into $1/h$ equal parts. The numerical solution corresponding to $h = 1/64$ and $k = 1/128$ was taken as the ‘exact’ solution to compute the errors of the numerical solutions. The numerical results are presented in Table 1 and Figure 5. It is obvious that the error $\|u - u^{hk}\|_V + \|\zeta - \zeta^{hk}\|_{Z_0}$ displays a linear convergence order pattern clearly, this matches well the theoretical prediction from (3.38).

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E-mail address: hailingxuan@zju.edu.cn
E-mail address: xiaoliangcheng@zju.edu.cn