On the Verification of Logically Decorated Graph Transformations

Jon Haël Brenas\textsuperscript{1}, Rachid Echahed\textsuperscript{2}, and Martin Strecker\textsuperscript{3}

\textsuperscript{1} UTHSC - ORNL, Memphis, Tennessee, USA
\texttt{jhael@uthsc.edu}
\textsuperscript{2} CNRS and University Grenoble-Alpes, Grenoble, France
\texttt{rachid.echahed@imag.fr}
\textsuperscript{3} Université de Toulouse, IRIT Institute, Toulouse, France,
\texttt{martin.strecker@irit.fr}

Abstract. We address the problem of reasoning on graph transformations featuring actions such as addition and deletion of nodes and edges, node merging and cloning, node or edge labelling and edge redirection. First, we introduce the considered graph rewrite systems which are parameterized by a given logic $\mathcal{L}$. Formulas of $\mathcal{L}$ are used to label graph nodes and edges. In a second step, we tackle the problem of formal verification of the considered rewrite systems by using a Hoare-like weakest precondition calculus. It acts on triples of the form $\{\text{Pre}\} (R, \text{strategy}) \{\text{Post}\}$ where $\text{Pre}$ and $\text{Post}$ are conditions specified in the given logic $\mathcal{L}$, $R$ is a graph rewrite system and $\text{strategy}$ is an expression stating how rules in $R$ are to be performed. We prove that the calculus we introduce is sound. Moreover, we show how the proposed framework can be instantiated successfully with different logics. We investigate first-order logic and several of its decidable fragments with a particular focus on different dialects of description logic (DL). We also show, by using bisimulation relations, that some DL fragments cannot be used due to their lack of expressive power.

1 Introduction

Graphs, as well as their transformations, play a central role in modeling data in various areas such as chemistry, civil engineering or computer science. In many such applications, it may be desirable to be able to prove that the transformations are correct, i.e., from any graph (or state) satisfying a given set of conditions, only graphs satisfying another set of conditions can be obtained.

In this paper, we address the problem of correctness of programs defined as graph rewrite rules. The correctness properties are stated as logical formulas obtained using a Hoare-like calculus. The considered graph structures are attributed with logical formulas which label both nodes and edges. Definitions of the structures as well as their transformation are provided in a generic framework parameterized by a given logic $\mathcal{L}$. Rewrites rules follow an algorithmic approach where the left-hand sides are attributed graphs and the right-hand
sides are sequences of elementary actions. Among the considered actions, we quote node and edge addition or deletion, node and edge labelling and edge redirection, in addition to node merging and cloning. To our knowledge, the present work is the first to consider the verification of graph transformations including the last two actions, namely node merging and node cloning. We propose a sound Hoare calculus for the considered specifications defined as triples of the form \( \{\text{Pre}\}\{R, \text{strategy}\}\{\text{Post}\} \) where \( \text{Pre} \) and \( \text{Post} \) are conditions specified in a given logic \( L \), \( R \) is a graph rewrite system and \( \text{strategy} \) is an expression stating how rules in \( R \) are to be performed. Different instances of the logic \( L \) are provided in this paper in order to illustrate the effectiveness of the proposed method.

The correctness of graph transformations has attracted some attention in recent years. One prominent approach is model checking such as the Groove tool. The idea is to carry out a symbolic exploration of the state space, starting from a given graph, in order to find out whether certain invariants are maintained or certain states are reachable. The Viatra tool has similar model checking capabilities and in addition allows the verification of elaborate well-formedness constraints imposed on graphs. Well-formedness is within the realm of our approach (and amounts to checking the consistency of a formula), but it is not the primary goal of this paper which is on the dynamics of graphs. The Alloy analyser uses bounded model checking for exploring relational designs and transformations. Counter-examples are presented in graphical form. The aforementioned techniques are sometimes combined with powerful SAT- or SMT-solvers, but do not carry out a complete deductive verification, though.

Hoare-like calculi for the verification of graph transformations have already been proposed with different logics to express the pre- and post-conditions. Among the most prominent approaches figure nested conditions that are explicitly created to describe graph properties. The considered graph rewrite transformations are based on the double pushout approach with linear spans which forbid actions such as node merging and node cloning.

Other logics might be good candidates to express graph properties which go beyond first-order definable properties such as monadic second-order logic or the dynamic logic defined in which allows one to express both rich graph properties as well as the graph transformations at the same time. These approaches are undecidable in general and thus either cannot be used to prove correctness of graph transformations in an automated way or only work on limited classes of graphs.

Starting from the other side of the logical spectrum, one could consider the use of decidable logics such as fragments of Description Logics to specify graph properties. Decidable fragments of first-order logics such as two-variable logic with counting and logics with exists-forall-prefix, among others, can be of practical use as well in the verification of graph transformation.

The paper is organized as follows. Formal preliminary definitions of the considered graph structures and the elementary transformation actions are intro-
duced in the next section. In Section 3 we define the investigated class of graph rewrite systems and the used notion of rewrite strategies. The proposed Hoare-calculus for the verification of the correctness of graph transformations is presented Section 4. In Section 5 some logics that can be used for the considered verification problems are presented. We also point out some fragments of Description Logic whose expressive power is not sufficient enough to be useful in reasoning on graph dynamics. Concluding remarks are given in Section 6. The missing proofs can be found in the appendix.

2 Preliminaries

We start by introducing the notion of logically decorated graphs. Nodes and edges of such graph structures are labeled by logic formulas. The definition below is parameterized by a given logic $\mathcal{L}$ seen as a set of formulas. Section 5 provides some examples of possible candidates for such a logic $\mathcal{L}$.

**Definition 1 (Logically Decorated Graph).** Let $\mathcal{L}$ be a logic (set of formulas). A graph alphabet is a pair $(\mathcal{C}, \mathcal{R})$ of sets of elements of $\mathcal{L}$, that is $\mathcal{C} \subseteq \mathcal{L}$ and $\mathcal{R} \subseteq \mathcal{L}$. $\mathcal{C}$ is the set of node formulas or concepts and $\mathcal{R}$ is the set of edge formulas or roles. Subsets of $\mathcal{C}$ and $\mathcal{R}$, respectively named $\mathcal{C}_0$ and $\mathcal{R}_0$, contain basic (propositional) concepts and roles respectively. A logically decorated graph $G$ over a graph alphabet $(\mathcal{C}, \mathcal{R})$ is a tuple $(N, E, \Phi_N, \Phi_E, s, t)$ where $N$ is a set of nodes, $E$ is a set of edges, $\Phi_N$ is the node labeling function, $\Phi_N : N \rightarrow \mathcal{P}(\mathcal{C})$, $\Phi_E$ is the edge labeling function, $\Phi_E : E \rightarrow \mathcal{R}$, $s$ is the source function $s : E \rightarrow N$ and $t$ is the target function $t : E \rightarrow N$.

Transformation of logically decorated graphs, considered in the next section, will be defined following an algorithmic approach based on the notion of elementary actions as introduced below. These actions constitute a set of elementary graph transformations such as the addition/deletion of nodes, concepts or edges; redirection of edges; merge or clone of nodes. Formal definitions of the considered elementary actions are given in Figure 1.

**Definition 2 (Elementary action, action).** An elementary action, say $a$, may be of the following forms:

- a node addition $\text{add}_N(i)$ (resp. node deletion $\text{del}_N(i)$) where $i$ is a new node (resp. an existing node). It creates the node $i$. $i$ has no incoming nor outgoing edge and it is not labeled with any concept (resp. it deletes $i$ and all its incoming and outgoing edges).
- a concept addition $\text{add}_C(i, c)$ (resp. concept deletion $\text{del}_C(i, c)$) where $i$ is a node and $c$ is a basic concept (a proposition name) in $\mathcal{C}_0$. It adds the label $c$ to (resp. removes the label $c$ from) the labeling of node $i$.

---

4 The names concept and role are borrowed from Description Logics’ vocabulary [2].
an edge addition $\text{add}_E(e, i, j, r)$ (resp. edge deletion $\text{del}_E(e, i, j, r)$) where $e$ is an edge, $i$ and $j$ are nodes and $r$ is a basic role (edge label) in $R_0$. It adds the edge $e$ with label $r$ between nodes $i$ and $j$ (resp. removes the edge $e$). When the edge that is affected is clear from the context, we will usually simply write $\text{add}_E(i, j, r)$ (resp. $\text{del}_E(i, j, r)$).

- a global edge redirection $i \gg j$ where $i$ and $j$ are nodes. It redirects all incoming edges of $i$ towards $j$.

- a merge action $\text{mrng}(i, j)$ where $i$ and $j$ are nodes. This action merges the two nodes. It yields a new graph in which the first node $i$ is labeled with the union of the labels of $i$ and $j$ and such that all incoming or outgoing edges of any of the two nodes are gathered.

- a clone action $\text{cl}(i, j, L_{in}, L_{out}, L_{i,j}, L_{i,j}, L_{loop})$ where $i$ and $j$ are nodes and $L_{in}, L_{out}, L_{i,j}$ and $L_{j,i}$ are sets of basic roles. It clones a node $i$ by creating a new node $j$ and connect $j$ to the rest of a host graph according to different information given in the parameters $L_{in}, L_{out}, L_{i,j}, L_{i,j}, L_{loop}$.

The result of performing an elementary action $a$ on a graph $G = (N^G, E^G, C^G, R^G, \Phi^G_N, \Phi^G_E, s^G, t^G)$, written $G[\alpha]$, produces the graph $G' = (N^{G'}, E^{G'}, C^{G'}, R^{G'}, \Phi^{G'}_N, \Phi^{G'}_E, s^{G'}, t^{G'})$ as defined in Figure 7. An action, say $\alpha$, is a sequence of elementary actions of the form $\alpha = a_1; a_2; \ldots; a_n$. The result of performing $\alpha$ on a graph $G$ is written $G[\alpha]$. $G[a; \alpha] = (G[a])[\alpha]$ and $G[\epsilon] = G$ where $\epsilon$ is the empty sequence.

The elementary action $\text{cl}(i, j, L_{in}, L_{out}, L_{i,j}, L_{i,j}, L_{loop})$ might be not easy to grasp at first sight. It thus deserves some explanations. Let node $j$ be a clone of node $i$. What would be the incident edges of the clone $j$? answering this question is not straightforward. There are indeed different possibilities to connect $j$ to the neighborhood of $i$. Figure 2 illustrates such a problem: there are indeed different possibilities to connect node $q_1$, a clone of node $q_1$, to the other nodes. In order to provide flexible clone action, the user may tune the way the edges connecting a clone are treated through the five parameters $L_{in}, L_{out}, L_{i,j}, L_{i,j}, L_{loop}$. All these parameters are subsets of the set of basic roles $R_0$ and are explained informally below:

- $L_{in}$ indicates that every incoming edge $e$ of $i$ which is not a loop and whose label is in $L_{in}$ is cloned as a new edge $e'$ such that $s(e') = s(e)$ and $t(e') = j$.
- $L_{out}$ indicates that every outgoing edge $e$ from $i$ which not a loop and whose label is in $L_{out}$ is cloned as a new edge $e'$ such that $s(e') = j$ and $t(e') = t(e)$.
- $L_{i,j}$ indicates that every self-loop $e$ over $i$ whose label is in $L_{i,j}$ is cloned as a new edge $e'$ such that $s(e') = i$ and $t(e') = j$. (e.g., see the blue arrow in Figure 2)
- $L_{j,i}$ indicates that every self-loop $e$ over $i$ whose label is in $L_{j,i}$ is cloned as a new edge $e'$ such that $s(e') = j$ and $t(e') = i$. (e.g., see the red arrow in Figure 2)
- $L_{loop}$ indicates that every self-loop $e$ over $i$ whose label is in $L_{loop}$ is cloned as a new edge $e'$ which is a self-loop over $j$, i.e., $s(e') = j$ and $t(e') = j$. (e.g., see the selfloop over node $q_1'$ in Figure 2)
Additionally, the semantics of the cloning actions as defined in Figure 1 use several sets of edges, representing the edges that are created depending on how they should be connected to i and j, permitting to associate to each new edge the old one of which it is a copy. The sets $E_{in}', E_{out}', E_{j, in}', E_{j, out}'$ and $E_{j, loop}'$, used in Figure 1, are pairwise disjoint sets of new (fresh) edges, and the functions $\text{in}, \text{out}, L_{in}, L_{out}$ and $L_{loop}$ are bijections defined such that:

1. $E_{in}'$ is in bijection through function $\text{in}$ with the set $\{e \in E^G | t^G(e) = i \land s^G(e) \neq i \land \Phi^G_E(e) \in L_{in}\}$.
2. $E_{out}'$ is in bijection through function $\text{out}$ with the set $\{e \in E^G | s^G(e) = i \land t^G(e) \neq i \land \Phi^G_E(e) \in L_{out}\}$.
3. $E_{j, in}'$ is in bijection through function $L_{in}$ with the set $\{e \in E^G | s^G(e) = t^G(e) = i \land \Phi^G_E(e) \in L_{j, in}\}$.
4. $E_{j, out}'$ is in bijection through function $L_{out}$ with the set $\{e \in E^G | s^G(e) = t^G(e) = i \land \Phi^G_E(e) \in L_{j, out}\}$.
5. $E_{j, loop}'$ is in bijection through function $L_{loop}$ with the set $\{e \in E^G | s^G(e) = t^G(e) = i \land \Phi^G_E(e) \in L_{j, loop}\}$.

Informally, the set $E_{in}'$ contains a copy of every incoming edge $e$ of node $i$, i.e. such that $t^G(e) = i$, which is not a self-loop, i.e. $s^G(e) \neq i$, and having a label in $L_{in}$, i.e. $\Phi^G_E(e) \in L_{in}$. $L_{in}$ is thus used to select which incoming edges are cloned. The other sets ($E_{out}', E_{j, in}', E_{j, out}'$ and $E_{j, loop}'$) are defined similarly.

Notice that a node and its clone have the same basic (propositional) labels (see Figure 1). It is possible to define alternate versions of cloning regarding node labels (clone none of them, add a parameter stating which ones to clone, etc.). The results presented in this paper can be extended to these other definitions of cloning in a straightforward manner.

**Example 1.** Let $A = \{Q, \Sigma, \delta, q_0, F\}$ be the automaton of Figure 2A. Performing the action $\text{cl}(q_1, q'_1, \Sigma, \Sigma, X, Y, Z)$ gives the automaton presented in B where the blue - plain - (resp. red - dashed -, purple - dotted) transition exists iff $X$ (resp. $Y$, $Z$) contains the label $\{a\}$.

**Example 2.** Let $(V, \geq)$ be a (weak) ordering on a finite set $V$. The relation $\geq$ is reflexive, asymmetric and transitive. Let us consider the representation of this ordering as a graph having $V$ as a set of nodes and there is an edge $e$ labeled with $\geq$ between every two nodes (elements) $i$ and $j$ of $V$ iff $i \geq j$. Then, creating an clone of a node does not make sense in general if one wishes to keep the correspondence between the edges of the graph and the property of ordering. However cloning can be used to create an element that is the next smaller element (if performing $\text{cl}(n, n', \{\leq\}, \{\leq\}, \{\leq\}, \emptyset, \{\leq\})$) or the next greater element (if performing $\text{cl}(n, n', \{\leq\}, \{\leq\}, \emptyset, \{\leq\}, \emptyset)$).

Readers familiar with algebraic approaches to graph transformation may recognize the cloning flexibility provided by the recent PBPO (pullback-pushout) approach of [10]. The parameters of the clone action reflect somehow the typing morphisms of [10]. Cloning a node according to the approach of Sesquipushout
If $\alpha = addc(i, c)$ then:

$N^G' = N^G, E^G' = E^G$,

$\Phi^G_n'(n) = \begin{cases} \Phi^G_n(n) \cup \{c\} & \text{if } n = i \\ \Phi^G_n(n) & \text{if } n \neq i \end{cases}$

$\Phi^E_{\Phi'} = \Phi^E_{\Phi'}, s^G = s^G, t^G = t^G$

If $\alpha = addc(e, i, j, r)$ then:

$N^G' = N^G, \Phi^G_N' = \Phi^G_N$

$E^G' = E^G \cup \{e\}$

$\Phi^G_{\Phi'}(e') = \begin{cases} r & \text{if } e' = e \\ \Phi^G_{\Phi'}(e') & \text{if } e' \neq e \end{cases}$

$s^G(e') = s^G(e')$ if $e' \neq e, s^G(e') = i$

$t^G(e') = t^G(e')$ if $e' \neq e, t^G(e') = j$

If $\alpha = addx(i)$ then:

$N^G' = N^G \cup \{i\}$ where $i$ is a new node

$E^G' = E^G \cup \{e\}$, $\Phi^G_{\Phi'} = \Phi^G_{\Phi'}, s^G = s^G, t^G = t^G$

$\Phi^G_N(n) = \begin{cases} \emptyset & \text{if } n = i \\ \Phi^G_N(n) & \text{if } n \neq i \end{cases}$

If $\alpha = delx(i)$ then:

$N^G' = N^G \setminus \{i\}$

$E^G' = E^G \setminus \{e\} s^G(e) = i \lor t^G(e) = i$

$\Phi^G_N' = \text{the restriction of } \Phi^G_N \text{ to } N^G'$

$\Phi^G_{\Phi'} = \text{the restriction of } \Phi^G_{\Phi'} \text{ to } E^G'$

$s^G = \text{the restriction of } s^G \text{ to } E^G'$

$t^G = \text{the restriction of } t^G \text{ to } E^G'$

If $\alpha = i \gg j$ then:

$N^G' = N^G, E^G' = E^G$

$\Phi^G_N' = \Phi^G_N, \Phi^G_{\Phi'} = \Phi^G_{\Phi'}, s^G = s^G$

$t^G(e) = \begin{cases} j & \text{if } t^G(e) = i \\ t^G(e) & \text{if } t^G(e) \neq i \end{cases}$

If $\alpha = mrg(i, j)$ then:

$N^G' = N^G \setminus \{i\}, E^G' = E^G, \Phi^G_{\Phi'}(e) = \Phi^G_{\Phi'}(e)$

$\Phi^G_N'(n) = \begin{cases} \Phi^G_N(i) \cup \Phi^G_N(j) & \text{if } n = i \\ \Phi^G_N(n) & \text{otherwise} \end{cases}$

$s^G(e) = \begin{cases} i & \text{if } s^G(e) = j \\ s^G(e) & \text{otherwise} \end{cases}$

$t^G(e) = \begin{cases} j & \text{if } t^G(e) = j \\ t^G(e) & \text{otherwise} \end{cases}$

\[ s^G'(e) = \begin{cases} s^G'(in(e)) & \text{if } e \in E_{in} \\ s^G'(out(e)) & \text{if } e \in E_{out} \end{cases} \]

\[ s^G'(j) = \begin{cases} j & \text{if } e \in E_{\text{in}} \\ j & \text{if } e \in E_{\text{out}} \end{cases} \]

\[ t^G'(e) = \begin{cases} t^G'(in(e)) & \text{if } e \in E_{\text{in}} \\ t^G'(out(e)) & \text{if } e \in E_{\text{out}} \end{cases} \]

Fig. 1. $G' = G[\alpha]$, summary of the effects of the elementary actions: $addx(i)$, $delx(i)$, $addc(i, c)$, $delc(i, c)$, $addc(e, i, j, r)$, $delc(e)$, $i \gg j$, $mrg(i, j)$ and $cl(i, j, L_{\text{in}}, L_{\text{out}}, L_{\text{in}}', L_{\text{out}}', L_{\text{loop}})$. $C$ and $R$ are never modified.
A) An automaton and B) the possible results of cloning node $q_1$ as node $q'_1$

[11] could be easily simulated by instantiating all the parameters by the full set of basic roles $cl(i, j, R_0, R_0, R_0, R_0, R_0)$.

Another action which may affect several edges in a row is the merge action. Figure 3 illustrates an example of node merging. To be more precise, node $j$ of the left graph is merged with node $i$. Notice that, except for the name of the resulting node ($i$ in this case), $\text{mrg}(i, j)$ and $\text{mrg}(j, i)$ are the same. After the action is performed, the edges between nodes $i$, $l$ and $k$ already present before the merge action remain unchanged and a new edge is added between $i$ and $k$ inherited from the link between $j$ and $k$. A loop over $i$ is also added representing the edge between $i$ and $j$ in the initial graph.

3 Graph Rewriting Systems and Strategies

In this section, we introduce the notion of logically decorated graph rewriting systems, or LDGRS. These are extensions of the graph rewriting systems defined in [14] where graphs are attributed with formulas from a given logic. The left-hand sides of the rules are thus logically decorated graphs whereas the right-hand sides are defined as sequences of elementary actions.

**Definition 3 (Rule, LDGRS).** A rule $\rho$ is a pair $(L, \alpha)$ where $L$, called the left-hand side, is a logically decorated graph and $\alpha$, called the right-hand side, is
an action. Rules are usually written $L \rightarrow \alpha$. A logically decorated graph rewriting system, LDGRS, is a set of rules.

Let us point out that the left-hand side of a rule is an attributed graph, that is it can contain nodes labeled with formulas. This is not insignificant as these formulas express additional conditions to be satisfied during the matching process, e.g. reachability (graph accessibility) condition, constraints on the number of neighbors (counting quantifiers), etc. depending on the underlying logic. In the sequel, we will use the symbol $\models$ to indicate the satisfiability relation between items of a graph (nodes or edges) and logical formulas.

**Definition 4 (Match).** A match $h$ between a left-hand side $L$ and a graph $G$ is a pair of functions $h = (h^N, h^E)$, with $h^N : N^L \rightarrow N^G$ and $h^E : E^L \rightarrow E^G$ such that:

1. $\forall n \in N^L, \forall c \in \Phi^L_N(n), h^N(n) \models c$
2. $\forall e \in E^L, \Phi^G_E(h^E(e)) = \Phi^L_E(e)$
3. $\forall e \in E^L, s^G(h^E(e)) = h^N(s^L(e))$
4. $\forall e \in E^L, t^G(h^E(e)) = h^N(t^L(e))$

The third and the fourth conditions are classical and say that the source and target functions and the match have to agree. The first condition says that for every node $n$ of the left-hand side, the node to which it is associated, $h(n)$, in $G$ has to satisfy every concept in $\Phi^L_N(n)$. This condition clearly expresses additional negative and positive conditions which are added to the “structural” pattern matching. The second one ensures that the match respects edge labeling.

**Definition 5 (Rule application).** A graph $G$ rewrites to graph $G'$ using a rule $\rho = (L, \alpha)$ iff there exists a match $h$ from $L$ to $G$. $G'$ is obtained from $G$ by performing actions in $h(\alpha)$. Formally, $G' = G[h(\alpha)]$. We write $G \rightarrow_\rho G'$.

Confluence of graph rewrite systems is not easy to establish. For instance, orthogonal graph rewrite systems are not always confluent, see e.g. [14]. We use the notion of rewrite strategies to control the use of possible rules. Informally, a strategy specifies the application order of different rules. It does not point to where the matches are to be found nor does it ensure unique normal forms.

**Definition 6 (Strategy).** Given a graph rewriting system $R$, a strategy is a word of the following language defined by $s$, where $\rho$ is any rule in $R$:

$s := \epsilon$ (Empty strategy) \hspace{1cm} $\rho$ (Rule)
$s \cdot s$ (Composition) \hspace{1cm} $s \oplus s$ (Choice)
$\rho !$ (Mandatory Rule) \hspace{1cm} $s^*$ (Closure) \hspace{1cm} $\rho ?$ (Rule trial)

Informally, the strategy ”$s_1; s_2$” means that strategy $s_1$ should be applied first, followed by the application of strategy $s_2$. On the other hand, $s_1 \oplus s_2$ means that either the strategy $s_1$ or the strategy $s_2$ is applied. The strategy $\rho^*$ means that rule $\rho$ is applied as many times as possible. Notice that the closure is the standard “while” construct: if the strategy we use is $s^*$, the strategy $s$ is used as long as it is possible and not an undefined number of times. The

$h(\alpha)$ is obtained from $\alpha$ by replacing every node name, $n$, of $L$ by $h(n)$.
strategies $ρ$, $ρ?$ and $ρ!$ try to apply the rule $ρ$. They behave in the same way when the rule $ρ$ matches the host graph. However, when rule $ρ$ does not match the host graph, the strategy written $ρ$ ends the rewriting process successfully. The strategy $ρ?$, called Rule Trial, simply skips the application of the rule and the rewriting process proceeds to the following strategy. The strategy $ρ!$, named Mandatory Rule, stops and the rewriting process fails.

We write $G \Rightarrow_s G'$ to denote that graph $G'$ is obtained from $G$ by applying the strategy $s$. In Figure 4 we provide the rules that specify how strategies are used to rewrite a graph. For that we use the following atomic formula $\text{App}(s)$ such that for all graphs $G$, $G \models \text{App}(s)$ iff the strategy $s$ can perform at least one step over $G$. This atomic formula is defined below.

- $G \models \text{App}(ρ)$ iff there exists a match $h$ from the left-hand side of $ρ$ to $G$
- $G \models \text{App}(ρ!)$ iff there exists a match $h$ from the left-hand side of $ρ$ to $G$
- $G \models \text{App}(ǫ)$
- $G \models \text{App}(s_0; s_1)$ iff $G \models \text{App}(s_0)$ or $G \models \text{App}(s_1)$
- $G \models \text{App}(ρ?)$

Notice that $G \models \text{App}(s)$ does not mean that the whole strategy can be applied on $G$, but just its first step can be applied. Indeed, let us assume the strategy $s = (s_0; s_1)$ where $s_0$ can be applied but may yield a state where $s_1$ cannot. In this case, the strategy $s$ can be applied on $G$ ($G \models \text{App}(s)$) but the execution may stop after performing one step of $s_0$.

Fig. 4. Strategy application rules
The three strategies using rules (i.e. \( \rho, \rho! \) and \( \rho? \)) behave the same way when \( G \models \text{App}(\rho) \) holds, as shown in Figure 4, but they do differ when \( G \not\models \text{App}(\rho) \). In such a case, \( \rho \) can yield any graph, denoted by \( \perp \), (i.e. the process stops without an error), \( \rho! \) stops the rewriting process with failure and \( \rho? \) ignores the rule application and moves to the next step of the execution of the strategy.

**Example 3.** Let us assume that we are managing a set of servers. Clients can connect to proxy servers that are themselves connected to mail servers, print servers, web servers, etc. We use graph transformations to generate new proxy servers to avoid over- or under-use of proxy servers. The rules that are used are shown in Figure 5. They use the description logic \( \mathbf{ALCQUOI} \) introduced in Section 5. For this example, actions that affect an edge, e.g. \( \text{add}_E(e, i, j, r) \), identify an edge from its extremities, e.g. we will write \( \text{add}_E(i, j, r) \) instead.

Both rules select a Client that Requested a connection to a Proxy. If the proxy has less than \( N \) currently established client-to-proxy (C2P) connections (rule \( \rho_0 \)), the label Request is removed and the label C2P is added to the edge between the Client and the Proxy. If the Proxy already has more than \( N \) client-to-proxy connections (rule \( \rho_1 \)), the Proxy is cloned. All its incoming edges, except for those labeled with Request or C2P, are cloned as well as all outgoing edges. Self-loops are not cloned. The label Request is then dropped from the edge between the Client and the new Proxy and the edge from the Client to the new Proxy is labeled with C2P.

The application condition for the first rule, \( \text{App}(\rho_0) \), is \( \exists U. (i \land \text{Client} \land \exists \text{Request}. (j \land \text{Proxy} \land (< N \text{C2P} - \top)) \). This condition can be understood as "there exists a node named \( i \) labeled with Client that is the source of an edge labeled with Request whose target is a node named \( j \) labeled with Proxy and such that there are strictly less than \( N \) different nodes connected through C2P to \( j \)." The used strategy is \( s = \rho_0 \oplus \rho_1 \) i.e. either \( \rho_0 \) or \( \rho_1 \) is applied but not both.

4 Verification

Reasoning on graph transformations does not benefit yet from standard proof techniques as it is the case for term rewriting. For instance, generalization of equational reasoning to graph rewriting systems is not even complete \[9\]. In this section, we follow a Hoare style to specify properties of LDGRS’s for which we establish a proof procedure.

**Definition 7 (Specification).** A specification \( SP \) is a triple \( \{ \text{Pre}\}(R, s)\{ \text{Post}\} \) where \( \text{Pre} \) and \( \text{Post} \) are formulas (of a given logic), \( R \) is a graph rewriting system and \( s \) is a strategy.

**Definition 8 (Correctness).** A specification \( SP \) is said to be correct iff for all graphs \( G, G' \) such that \( G \Rightarrow_s G' \) and \( G \models \text{Pre} \), then \( G' \models \text{Post} \).

\[6\] This formula is actually not as expressive as \( \text{App}(\rho_0) \). This problem is discussed more in Section 5.
In order to show the correctness of a specification, we follow a Hoare-calculus style \cite{18} and compute the weakest precondition $wp(S, Post)$. For that, we give in Figure 6 (resp. in Figure 7) the definition of the function $wp$ which yields the weakest precondition of a formula $Q$ w.r.t. an action (resp. a strategy).

$$wp(a, Q) = Q[a] \quad wp(a; \alpha, Q) = wp(a, wp(\alpha, Q))$$

Fig. 6. Weakest preconditions w.r.t. actions where $a$ (resp. $\alpha$) stands for an elementary action (resp. action) and $Q$ is a formula.

The weakest precondition of an elementary action, say $a$, and a postcondition $Q$ is defined as $wp(a, Q) = Q[a]$ where $Q[a]$ stands for the precondition consisting of $Q$ to which is applied a substitution induced by the action $a$ that we denote by $[a]$. The notion of substitution used here is the one coming from Hoare-calculi.

**Definition 9 (Substitutions).** To each elementary action $a$ is associated a substitution, written $[a]$, such that for all graphs $G$ and formula $\phi$, $(G \models \phi[a]) \iff (G[a] \models \phi)$.

Notice that, in general, substitutions are not defined as formulas of a given logic $\mathcal{L}$. They are defined as a new formula constructor whose meaning is that the weakest preconditions for elementary actions, as defined above, are correct. In general, the addition of a constructor for substitutions is not harmless. That is to say, if $\phi$ is a formula of a logic $\mathcal{L}$, $\phi[a]$ is not necessarily a formula of...
It is a very interesting problem to figure out which logics are closed under the considered substitutions. Some positive and negative answers are given in Section 5.

\[wp(\varepsilon, Q) = Q\]  
\[wp(s_0; s_1, Q) = wp(s_0, wp(s_1, Q))\]  
\[wp(s_0 \oplus s_1, Q) = wp(s_0, Q) \land wp(s_1, Q)\]  
\[wp(\rho, Q) = \text{App}(\rho) \Rightarrow wp(\alpha_\rho, Q)\]  
\[wp(\rho!, Q) = \text{App}(\rho) \land wp(\alpha_\rho, Q)\]  
\[wp(\rho?, Q) = (\text{App}(\rho) \Rightarrow wp(\alpha_\rho, Q)) \land (\lnot \text{App}(\rho) \Rightarrow Q)\]

Fig. 7. Weakest preconditions for strategies. \(\alpha_\rho\) denotes the right-hand side of rule \(\rho\).

The definition of \(wp(s, Q)\) for the empty strategy, the composition and the choice are quite direct. The definitions for the rule, mandatory rule and trial differ on what happens if the rule cannot be applied. When the rule \(\rho\) can be applied, then applying it should lead to a graph satisfying \(Q\). When the rule \(\rho\) cannot be applied, \(wp(\rho, Q)\) indicates that the considered specification is correct; while \(wp(\rho!, Q)\) indicates that the specification is not correct and \(wp(\rho?, Q)\) leaves the postcondition unchanged and thus transformations can move to possible next steps.

The weakest precondition for the closure is close to the while imperative instruction. It requires an invariant \(\text{inv}_s\) to be defined. \(wp(s^*, Q) = \text{inv}_s\) which means that the invariant has to be true when entering the iteration for the first time. On the other hand, it is obviously not enough to be sure that \(Q\) will be satisfied when exiting the iteration or that the invariant will be maintained throughout execution. To make sure that iterations behave correctly, we need to introduce some additional verification conditions computed by means of a function \(vc\), defined in Figure 8.

\[vc(\varepsilon, Q) = \top (\text{true})\]  
\[vc(s_0; s_1, Q) = vc(s_0, wp(s_1, Q)) \land vc(s_1, Q)\]  
\[vc(s_0 \oplus s_1, Q) = vc(s_0, Q) \land vc(s_1, Q)\]  
\[vc(s^*, Q) = vc(s, Q) \land (\text{inv}_s \land \text{App}(s) \Rightarrow wp(s, \text{inv}_s)) \land (\text{inv}_s \land \lnot \text{App}(s) \Rightarrow Q)\]  
\[vc(\rho, Q) = \top\]  
\[vc(\rho!, Q) = \top\]  
\[vc(\rho?, Q) = \top\]

Fig. 8. Verification conditions for strategies.
As the computation of \( wp \) and \( vc \) requires the user to provide invariants, we now introduce the notion of annotated strategies and specification.

**Definition 10 (Annotated strategy, Annotated specification).** An annotated strategy is a strategy in which every iteration \( s^* \) is annotated with an invariant \( inv_s \). It is written \( s^*\{ inv_s \} \). An annotated specification is a specification whose strategy is an annotated strategy.

**Definition 11 (Correctness formula).** We call correctness formula of an annotated specification \( SP = \{ Pre\}(\mathcal{R}, s)\{ Post \} \), the formula:

\[
\text{correct}(SP) = (Pre \Rightarrow wp(s, Post)) \land vc(s, Post).
\]

Before stating the soundness of the proposed verification method, we state a first simple lemma.

**Lemma 1.** Let \( Q \) be a formula and \( \alpha \) be an action. For all graphs \( G, G' \) such that \( G \Rightarrow_\alpha G', G \models wp(\alpha, Q) \) implies \( G' \models Q \).

**Proof.**
- Let us assume \( \alpha = a \), an elementary action. Then \( wp(a, Q) = Q[a] \).
  
  Let \( G \) be a graph such that \( G \models Q[a] \). By definition of the substitutions, \( G \models Q[a] \) implies that for any graph \( G' \) such that \( G \Rightarrow_\alpha G', G' \models Q \). Thus \( G \models wp(\alpha, Q) \Rightarrow G' \models Q \).

- Let us assume \( \alpha = a; \alpha' \) where \( a \) is an elementary action and \( \alpha' \) is an action. Then \( wp(a; \alpha', Q) = wp(a, wp(\alpha', Q)) \). Let \( G \) be a graph such that \( G \models wp(a, wp(\alpha', Q)) \) and let \( G' \) be a state such that \( G \Rightarrow_{a;\alpha'} G' \). Then there exists \( G'' \) with \( G \Rightarrow_\alpha G'' \) and \( G'' \Rightarrow_{a'} G' \). As \( G \models wp(a, wp(\alpha', Q)) \), by induction, \( G'' \models wp(\alpha', Q) \). Then, by an additional induction, \( G' \models Q \). Thus \( G \models wp(\alpha, Q) \Rightarrow G' \models Q \).

**Theorem 1 (Soundness).** Let \( SP = \{ Pre\}(\mathcal{R}, s)\{ Post \} \) be an annotated specification. If \( \text{correct}(SP) \) is valid, then for all graphs \( G, G' \) such that \( G \Rightarrow_s G', G \models Pre \) implies \( G' \models Post \).

**Proof.** This proof is done by induction on the semantic of the programming language.

- Let us assume \( s = \rho \). Then \( \text{correct}(SP) = Pre \Rightarrow wp(\rho, Post) \). Let \( G, G' \) be graphs such that \( G \models Pre \) and \( G \Rightarrow_\rho G' \) then, as \( \text{correct}(SP) \) is valid, \( G \models Pre \Rightarrow wp(\rho, Post) \). Thus, by modus ponens, \( G \models wp(\rho, Post) \).
  
  As \( wp(\rho, Post) = App(\rho) \Rightarrow wp(\alpha, Post) \) and, by definition of \( \Rightarrow_\rho \), \( G \models App(\rho) \) and thus, by modus ponens, \( G \models wp(\alpha, Post) \). Then, by applying the lemma, \( G' \models Post \).

- Let us assume \( s = \rho! \). Then \( \text{correct}(SP) = Pre \Rightarrow wp(\rho!, Post) \). Let \( G, G' \) be graphs such that \( G \models Pre \) and \( G \Rightarrow_{\rho!} G' \) then, as \( \text{correct}(SP) \) is valid, \( G \models Pre \Rightarrow wp(\rho!, Post) \). Thus, by modus ponens, \( G \models wp(\rho!, Post) \).
  
  As \( wp(\rho!, Post) = App(\rho) \land wp(\alpha, Post) \), \( G \models wp(\alpha, Post) \). Then, by applying the lemma, \( G' \models Post \).
Let us assume \( s = \rho ? \). Then correct\((SP) = Pre \Rightarrow wp(\rho ?), Post\). Let \( G, G' \) be graphs such that \( G \models Pre \) and \( G \Rightarrow \rho ? G' \) then, as correct\((SP) \) is valid, \( G \models Pre \Rightarrow wp(\rho ?), Post \). Thus, by modus ponens, \( G \models wp(\rho ?), Post \). As \( wp(\rho ?), Post \) = \((App(\rho ) \Rightarrow wp(\alpha_\rho , Post)) \land (\neg App(\rho ) \Rightarrow Post)\), we have to treat two different cases:

- if \( G \models App(\rho ) \) then, by modus ponens, \( G \models wp(\alpha_\rho , Post) \) and then, as, by definition of \( \Rightarrow \rho ? \), \( G \Rightarrow \rho \ G' \), using the lemma, \( G' \models Post \).
- otherwise, \( G \models \neg App(\rho ) \) and thus, by modus ponens, \( G \models Post \). But, by definition of \( \Rightarrow \rho ? \), \( G = G' \) and thus \( G' \models Post \).

Thus \( G \models Pre \Rightarrow G' \models Post \).

Let us assume \( s = s_0 ; s_1 \). Then correct\((SP) = vc(s_0 ; s_1 , Post) \land (Pre \Rightarrow wp(s_0 ; s_1 , Post)). As \( vc(s_0 ; s_1 , Post) = vc(s_0 , wp(s_1 , Post)) \land vc(s_1 , Post) \) and \( wp(s_0 ; s_1 , Post) = wp(s_0 , wp(s_1 , Post)) \), correct\((SP) = vc(s_0 , wp(s_1 , Post)) \land (Pre \Rightarrow wp(s_0 , wp(s_1 , Post))). Let \( G \) be a graph such that \( G \models Pre \). As correct\((SP) \) is valid, \( G \models correct(SP) \). Let \( G' \) be a graph such that \( G \Rightarrow s_0 ; s_1 G' \). Then there exists \( G'' \) with \( G \Rightarrow s_0 G'' \) and \( G'' \Rightarrow s_1 G' \). As \( G \models Pre \) and \( G \models vc(s_0 , wp(s_1 , Post) \land (Pre \Rightarrow wp(s_0 , wp(s_1 , Post)) \), by induction with \( s_0 = (Pre , wp(s_1 , Post))\), \( s_1 \), \( G'' \models wp(s_1 , Post) \). As correct\((SP) \) is valid, so is \( vc(s_1 , Post) \) and thus also \( vc(s_1 , Post) \land (wp(s_1 , Post) \Rightarrow wp(s_1 , Post)) \) and then, by induction with \( s_1 = (wp(s_1 , Post)) \), \( G' \models Post \). Thus \( G \models Pre \Rightarrow G' \models Post \).

Let us assume that \( s = \epsilon \). Then correct\((SP) = Pre \Rightarrow Post \). Let \( G \) and \( G' \) be graphs such that \( G \models G' \) \( G \models Pre \). By definition, \( G = G' \) and thus, by modus ponens, \( G' \models Post \). Thus \( G \models Pre \Rightarrow G' \models Post \).

Let us assume \( s = s_0 \oplus s_1 \). Then correct\((SP) = vc(s_0 , Post) \land vc(s_1 , Post) \land (Pre \Rightarrow wp(s_0 , Post) \land wp(s_1 , Post)) \). Let \( G \) and \( G' \) be graphs such that \( G \Rightarrow s_0 \oplus s_1 G' \) and \( G \models Pre \). By definition of \( \Rightarrow s_0 \oplus s_1 \), there are two possible cases:

- If \( G \Rightarrow s_0 G' \) then, as correct\((SP) \) is valid, so is \( vc(s_0 , Post) \). As \( G \models Pre \), by induction, \( G' \models Post \).
- otherwise, \( G \models s_1 G' \) and then, as correct\((SP) \) is valid, \( vc(s_1 , Post) \land (Pre \Rightarrow wp(s_1 , Post)) \). As \( G \models Pre \), by induction, \( G' \models Post \).

Thus \( G \models Pre \Rightarrow G' \models Post \).

Let us assume \( s = s_0' \{ inv \} \). Then correct\((SP) = vc(s_0 , inv) \land (inv \land App(s_0) \Rightarrow wp(s_0 , inv)) \land (Post \land \neg App(s_0) \Rightarrow Post) \land (Pre \Rightarrow inv) \). Let \( G \) and \( G' \) be graphs such that \( G \models Pre \) and \( G \Rightarrow s_0' G' \). There are two possible cases:

- Let us assume that \( G \models App(s_0) \). By definition of \( \Rightarrow s_0' \), there exist \( G'' \) such that \( G \Rightarrow s_0 G'' \) and \( G'' \Rightarrow s_0' G' \). As correct\((SP) \) is valid, so is \( vc(s_0 , inv) \land (Pre \Rightarrow (App(s_0) \land inv \Rightarrow wp(s_0 , inv))) \). By induction with \( S' = (Pre \land App(s_0) \land inv , R , s_0) \), \( G \models Pre \land App(s_0) \land inv , G'' \land inv \). Similarly, with \( S'' = (inv , Post , R , s_0' \{ inv \}) \), by induction, \( G' \models Post \).
- otherwise, \( G \models \neg App(s_0) \) and thus, by modus ponens, \( G \models Post \). But, by definition of \( \Rightarrow s_0' \), \( G = G' \) and thus \( G' \models Post \).

Thus \( G \models Pre \Rightarrow G' \models Post \).
Example 4. Let us consider Example 3. We want to prove that the specification \( \{ \text{Pre}\}{(R, \rho_0 \oplus \rho_1)}\{\text{Post} \} \), where \( \text{Pre} \equiv \exists U. (\text{Client} \land \exists \text{Request. Proxy}) \land \forall U. (\text{Proxy} \Rightarrow (\leq N C2P \top)) \) and \( \text{Post} = \forall U. (\text{Proxy} \Rightarrow (\leq N C2P \top)) \), is correct. \( \text{Pre} \) means that there exist a \text{Client} that \text{Requested} a connection to a \text{Proxy} and that no \text{Proxy} has more than \( N \) different C2P incoming connections. \( \text{Post} \) means that no \text{Proxy} has more than \( N \) different C2P incoming connections. The correctness formula is then \( (\text{Pre} \Rightarrow \text{wp}(\rho_0 \oplus \rho_1, \text{Post})) \land \text{vc}(\rho_0 \oplus \rho_1, \text{Post}) \). It can be simplified, however, as \( \text{vc}(\rho_0 \oplus \rho_1, \text{Post}) = \text{vc}(\rho_0, \text{Post}) \land \text{vc}(\rho_1, \text{Post}) \) and both \( \text{vc}(\rho_0, \text{Post}) \) and \( \text{vc}(\rho_1, \text{Post}) \) are, by definition, true. The correctness formula is thus \( \text{Pre} \Rightarrow (((\text{App}(\rho_0) \Rightarrow \text{wp}(\alpha_{\rho_0}, \text{Post}))) \land ((\text{App}(\rho_1) \Rightarrow \text{wp}(\alpha_{\rho_1}, \text{Post}))) \)

5 Assertion Logics

The framework presented so far regarding the considered rewrite systems (LDRSs) and specifications is parameterized by a given logic \( \mathcal{L} \). In this section, we present some logics that could possibly be used to instantiate this general framework. The logics that are used should be closed under the substitutions generated by the elementary actions. Otherwise, the computation of weakest preconditions may be outside the considered logic. We start by considering first-order logic as well as some of its decidable fragments. We focus more particularly on description logics (DL) in a second time. We show that some of them are closed under substitutions for all the actions that we have presented in this paper. We also provide a negative result by proving that some DL fragments are not closed under substitutions generated by the elementary action \text{merge (mrg(i, j))}. The results presented in this section are new and complete those already given in [6].

For all the logics we consider, we discuss the closure under substitutions and the expression of the literal \( \text{App}(\rho) \).

5.1 First-order logic

We start by recalling briefly the first-order formulas useful for our purpose as well as the notions of interpretations and models.

Definition 12 (First-order formula). Let \( \mathcal{A} = (\mathcal{V}, \mathcal{C}, \mathcal{R}) \) where \( \mathcal{V} \) is a set of variables, \( \mathcal{C} \) is a set of unary predicates, and \( \mathcal{R} \) is a set of binary predicates including equality (=). Given \( x, y \in \mathcal{V}, C \in \mathcal{C} \) and \( R \in \mathcal{R} \), the set of first-order formulas \( \phi \) we consider is defined by:

\[
\phi := T \mid C(x) \mid R(x, y) \mid x = y \mid \neg \phi \mid \phi \lor \phi \mid \exists x. \phi
\]

For the sake of conciseness, we define \( \bot \equiv \neg T, \phi \land \psi \equiv \neg (\neg \phi \lor \neg \psi), \forall x. \phi \equiv \neg (\exists x. \neg \phi) \).

A variable \( x \) is free in \( \phi \) iff \( \phi = C(t_0), \phi = R(t_0, t_1) \) or \( \phi = "t_0 = t_1" \) and \( x \) occurs in \( t_0 \) or \( t_1 \), or \( \phi = \neg \psi \) or \( \phi = \psi \lor \psi' \) and \( x \) is free in \( \psi \) and \( \psi' \), or \( \phi = \exists y. \psi \) and \( x \) is free in \( \psi \) and \( x \) is different from \( y \). A formula with no free variable is a sentence. We only consider sentences hereafter.
Definition 13 (Model). Let $G = (N, E, \Phi_N, \Phi_E, s, t)$ be a graph over the alphabet $(C, \mathcal{R})$, an interpretation over the alphabet $(\mathcal{V}, C, \mathcal{R})$ is a tuple $(\Delta, \mathcal{I})$ such that $N \subseteq \Delta$ and $\mathcal{I}$ is a function over formulas defined by:

- $\top^\mathcal{I}$ is true
- $C(x)^\mathcal{I}$ is true if and only if $C \in \Phi_N(x)$
- $R(x, y)^\mathcal{I}$ is true if and only if $\exists e \in E.s(e) = x$ and $t(e) = y$ and $R \in \Phi_E(e)$
- $x = y^\mathcal{I}$ is true if and only if $x = y$
- $(\exists x.\phi)^\mathcal{I}$ is true if and only if $\exists n \in N.\phi[x \rightarrow n]^\mathcal{I}$ where $\phi[x \rightarrow n]$ is $\phi$ where each occurrence of $x$ is replaced with $n$
- $(\neg \phi)^\mathcal{I}$ is true if and only if not $\phi^\mathcal{I}$
- $(\phi \lor \psi)^\mathcal{I}$ is true if and only if $\phi^\mathcal{I}$ or $\psi^\mathcal{I}$

We say that a graph $G$ models a first-order formula $\phi$, written $G \models \phi$ if there exists an interpretation $(\Delta, \mathcal{I})$ such that $\phi^\mathcal{I}$ is true.

One may remark that $N \subseteq \Delta$ and not $\Delta = \Delta$. This is because some actions (e.g. node addition, deletion, merging ...) may modify the set of nodes currently existing (i.e. nodes of the current graph). To keep track of $N$, we follow $\mathcal{B}$ and introduce a special unary predicate $\text{Active}$ that denotes the existing nodes. We transform formulas so that all $\exists x.\phi$ become $\exists x.\text{Active}(x) \land \phi$ and add to the definition of $\mathcal{I}$ the fact that $\text{Active}(x)^\mathcal{I}$ is true if and only if $x \in N$ and $\exists x.\phi$ if and only if $\exists n \in \Delta.\phi[x \rightarrow n]$ where $\phi[x \rightarrow n]$ is $\phi$ where each occurrence of $x$ is replaced with $n$.

Theorem 2. First-order logic is closed under substitutions.

Proof. The proof is done by induction on the formula constructors. We focus here on the substitutions generated by the elementary actions $\text{mrg}$ and $\text{cl}$. The full proof is reported in the appendix.

We start by giving formulas without substitutions that are equivalent to those with substitutions.

- $\top[\sigma] \leadsto \top$
- $\text{Active}(x)[\sigma] \leadsto \text{Active}(x)$ if $\sigma \neq \text{add}_N(i)$ and $\sigma \neq \text{cl}(i, j, \ldots)$ and $\sigma \neq \text{mrg}(i, j)$
- $\text{Active}(x)[\text{add}_N(i)] \leadsto \text{Active}(x) \lor i = x$
- $\text{Active}(x)[\text{del}_N(i)] \leadsto \text{Active}(x) \land i \neq x$
- $\text{Active}(x)[\text{cl}(i, j, \ldots)] \leadsto \text{Active}(x) \land x = j$
- $\text{Active}(x)[\text{mrg}(i, j)] \leadsto \text{Active}(x) \land x \neq j$
- $C'(x)[\text{add}_C(i, C)] \leadsto C'(x)$
- $C(x)[\text{add}_C(i, C)] \leadsto C(x) \lor i = x$
- $C'(x)[\text{del}_C(i, C)] \leadsto C'(x)$
- $C(x)[\text{del}_C(i, C)] \leadsto C(x) \land \neg i = x$
- $C(x)[\text{add}_R(i, j, R)] \leadsto C(x)$
- $C(x)[\text{del}_R(i, j, R)] \leadsto C(x)$
- $C(x)[\text{add}_N(i)] \leadsto C(x)$ for $C \neq \text{Active}$
- $C(x)[\text{del}_N(i)] \leadsto C(x) \land \neg i = x$ for $C \neq \text{Active}$
- $C(x)[i \gg j] \rightarrow C(x)$
- $C(x)[cl(i,j,\ldots)] \rightarrow C(x) \lor (x = j \land C(i))$ if $C \neq \text{Active}$
- $C(x)[mrg(i,j)] \rightarrow x \neq j \land (C(x) \lor (x = i \land C(j)))$ if $C \neq \text{Active}$
- $R(x,y)[\text{add}_C(i,C)] \rightarrow R(x,y)$
- $R(x,y)[\text{del}_C(i,C)] \rightarrow R(x,y)$
- $R'(x,y)[\text{add}_R(i,j,R)] \rightarrow R'(x,y)$
- $R(x,y)[\text{add}_R(i,j,R)] \rightarrow R(x,y) \lor (i = x \land j = y)$
- $R'(x,y)[\text{del}_R(i,j,R)] \rightarrow R'(x,y)$
- $R(x,y)[\text{del}_R(i,j,R)] \rightarrow R(x,y) \land (\neg i = x \lor \neg j = y)$
- $R(x,y)[\text{add}_N(i)] \rightarrow R(x,y)$
- $R(x,y)[\text{del}_N(i)] \rightarrow R(x,y) \land \neg i = x \land \neg i = y$
- $R(x,y)[i \gg j] \rightarrow (R(x,y) \land \neg i = y) \lor (R(x,i) \land j = y))$
- $R(x,y)[cl(i,j,\ldots)] \rightarrow R(x,y) \lor \phi_{in} \lor \phi_{out} \lor \phi_{jn} \lor \phi_{inout} \lor \phi_{jloop}$ where:
  - $\phi_{in} = \begin{cases} R(x,i) \land y = j \land \neg(x = i) & \text{if } R \in L_{in} \\ \bot & \text{otherwise} \end{cases}$
  - $\phi_{out} = \begin{cases} R(i,y) \land \neg(y = i) & \text{if } R \in L_{out} \\ \bot & \text{otherwise} \end{cases}$
  - $\phi_{jn} = \begin{cases} R(i,i) \land x = i \land y = j & \text{if } R \in L_{jn} \\ \bot & \text{otherwise} \end{cases}$
  - $\phi_{inout} = \begin{cases} R(i,i) \land x = j \land y & \text{if } R \in L_{inout} \\ \bot & \text{otherwise} \end{cases}$
  - $\phi_{jloop} = \begin{cases} R(i,i) \land x = j \land y & \text{if } R \in L_{jloop} \\ \bot & \text{otherwise} \end{cases}$
- $N^G \rightarrow \phi^G \lor (\exists x,y)[\phi^G(x) \land \phi^G(y)]$
- $\phi^G \lor (\exists x)[\phi^G(x)] \rightarrow (\exists x)[\phi^G(x)]$
- $(\phi \land \psi)^G \rightarrow \phi^G \land \psi^G$
- $(\neg \phi)^G \rightarrow \neg(\phi^G)$

Let us now prove that the proposed formulas without substitutions are indeed equivalent to the ones with substitutions. For lack of space, we will illustrate these equivalences only for some of them. To do that, we introduce the interpretations $(\Delta^G, \Gamma^G)$ and $(\Delta^G, \Gamma^G')$ that results from the cloning or merging action.

$\top[\sigma]$: No matter what action is performed, $\top$ is satisfied.

$\text{Active}(x)[\sigma]$: If $\sigma$ is not a node creation, deletion, cloning or merging all nodes that were active stay so and vice-versa.

$\text{Active}(x)[\text{add}_N(i)]$: The valuation of $\text{Active}$ becomes $\text{Active}^G \cup \{i^G\}$.

$\text{Active}(x)[\text{del}_N(i)]$: The valuation of $\text{Active}$ becomes $\text{Active}^G \setminus \{i^G\}$.

$\text{Active}(x)[cl(i,j,\ldots)]$: As $N^G' = N^G \cup j$, $\text{Active}(x) \in \phi^G_N(n)$ if and only if $(\text{Active}(x) \lor x = j) \in \phi^G_N(n)$.

$\text{Active}(x)[mrg(i,j)]$: As $N^G' = N^G \setminus j$, $\text{Active}(x)^G$ if and only if $(\text{Active}(x) \land x \neq j)^G$.

$C'(x)[\text{add}_C(i,C)]$: The valuation of $C'$ is left untouched.

$C(x)[\text{add}_C(i,C)]$: $C^G$ after performing $\text{add}_C(i,C)$ is $C^G \cup \{i^G\}$.

$C'(x)[\text{del}_C(i,C)]$: The valuation of $C'$ is left untouched.
\(C(x)[\text{add}_R(i, j, R)]\): The valuation of \(C\) is left untouched.

\(C(x)[\text{del}_R(i, j, R)]\): The valuation of \(C\) is left untouched.

\(C(x)[\text{del}_C(i, C)]\): \(C'\) after performing \(\text{del}_C(i, C)\) is \(C\setminus \{i^G\}\).

\(C(x)[\text{add}_N(i)]\): The valuation of \(C\) is left untouched.

\(C(x)[\text{del}_N(i)]\): \(C'\) = \(C\setminus \{i^G\}\).

\(C(x)[i \gg j]\): \(C' = C\).

\(C(x)[cl(i, j, \ldots)]\): If \(\phi^G_N(j) = \phi^G_N(i)\) and \(\forall n \neq j, \phi^G_N(n) = \phi^G_N(n), C(x)^G\) if and only if \((C(x) \lor (x = j \land C(i))^G)\).

\(C(x)[mrg(i, j)]\): If \(\phi^G_N(i) = \phi^G_N(i) \lor \phi^G_N(j)\) and \(\forall n \neq j, \phi^G_N(n) = \phi^G_N(n), C(x)^G\) if and only if \((x \neq j \land (C(x) \lor (x = i \land C(j))^G))\).

\(R(x, y)[\text{add}_C(i, C)]\): The valuation of \(R\) is left untouched.

\(R(x, y)[\text{del}_C(i, C)]\): The valuation of \(R\) is left untouched.

\(R'(x, y)[\text{add}_R(i, j, R)]\): The valuation of \(R'\) is left untouched.

\(R'(x, y)[\text{del}_R(i, j, R)]\): The valuation of \(R'\) is left untouched.

\(R(x, y)[\text{add}_N(i)]\): The valuation of \(R\) is left untouched.

\(R(x, y)[\text{del}_N(i)]\): \(R' = R\setminus \{(i^G, j^G)\}\).

\(R(x, y)[\text{add}_C(i)]\): \(R' = R\setminus \{(i, j)^G\}\).

\(R(x, y)[\text{del}_C(i)]\): \(R' = R\setminus \{(i^G, j^G)\}\).

\(R(x, y)[cl(i, j, \ldots)]\): \(R(x, y)[cl(i, j, \ldots)] \sim R(x, y) \lor \phi_{in} \lor \phi_{out} \lor \phi_{\ell_{in}} \lor \phi_{\ell_{out}} \lor \phi_{\ell_{oop}}\).

If \(R \in \phi^G_E(e')\) then either:
- \(e' \in E^G_1\), and then \(x = s^G(in(e'))\) and \(y = j\), that is there exists \(e\) such that \(R \in \phi^G_E(e)\) and \(s^G(e) = x\) and \(t^G(e) = i\). Thus \((R(x, i) \land y = j \land (x = i)^G)\).
- \(e' \in E^G_2\), and then \(x = j\) and \(y = t^G(out(e'))\), that is there exists \(e\) such that \(R \in \phi^G_E(e)\) and \(s^G(e) = i\) and \(t^G(e) = y\). Thus \((R(i, y) \land x = j \land (y = i)^G)\).
- \(e' \in E^G_3\), and then \(x = i\) and \(y = j\), and there exists \(e\) such that \(R \in \phi^G_E(e)\) and \(s^G(e) = i\) and \(t^G(e) = y = j\). Thus \((R(i, y) \land x = i \land (y = j)^G)\).
- \(e' \in E^G_4\), and then \(x = y\) and \(y = i\) and there exists \(e\) such that \(R \in \phi^G_E(e)\) and \(s^G(e) = i\) and \(t^G(e) = x\). Thus \((R(i, y) \land x = i \land (y = j)^G)\).
- otherwise, \(e' \in E^G_5\), and thus \((R(x, y)^G)\).

\(R(x, y)[mrg(i, j)]\): \(R^G = \{(x, y)\mid x \neq j \land y \neq i \land (R(x, y) \lor (R(x, j) \land y = i) \lor (R(j, y) \land x = i)\}\}

\((\exists x)\phi\)[\(\sigma\)]\): The substitutions do not modify the existence or not of a node.

\((\phi \land \psi)[\sigma]\): If \(\phi \land \psi\) is satisfied after performing \(\sigma\), so must be \(\phi\) and \(\psi\) and the other way round.

\((\neg \phi)[\sigma]\): If \(\phi\) is not satisfied after performing \(\sigma\), it is not possible that \(\phi\) be satisfied after performing \(\sigma\).

The proof of the previous theorem shows that the shape of the used formulas is conserved. If the formulas (ignoring the substitutions) belonged to less expressive and decidable fragments of first-order logic, namely the two-variable
fragment with counting C2[10] and $\forall^* \exists^*$, the fragment containing only formulas that, in prenex normal form, can be written as $\forall x_0 \ldots \forall x_n \exists y_0 \ldots \exists y_m \phi$ with $\phi$ quantifier free[8], so do the equivalent formulas without substitution.

**Corollary 1.** $\forall^* \exists^*$ and C2 are closed under substitutions.

The correctness formula includes substitutions as well as literals of the form $App(\rho)$. We proved previously that it is possible to remove the substitutions. Below, we show that $App(\rho)$ can be expressed in first-order logic.

**Proposition 1.** Let us assume that $\rho$ is a rule such that the labels of its left-hand side are in first-order logic. It is possible to express $App(\rho)$ in first order logic.

**Proof.** Let $L = (N^L, E^L, \Phi^L_N, \Phi^L_E, s^L, t^L)$ be the left-hand side of $\rho$. Let $A = \exists_{n \in N^L} x_n. \bigwedge_{n \in N^L} \psi_n \land \bigwedge_{e \in E^L} \psi_e$ where $\psi_n = \bigwedge_{c \in \phi^L(n)} c(x_n)$ and $\psi_e = \bigwedge_{t \in \phi^L_E(e)} r(x_{t(e)}, x_{tL(e)})$.

Let us assume that $G = (N^G, E^G, \Phi^G_N, \Phi^G_E, s^G, t^G)$ is a graph.

Let us assume that $G \models n^G(n) = x_n$, for $n \in N^L$, and $h^G(e) = \xi_e$ where $\xi_e \in \{e' \in E^G | s^G(e') = x_{t(e)} \land t^G(e') = x_{tL(e)}\}$, for $e \in E^L$.

1. For all $n \in N^L$, for all $c \in \phi^L_N(n)$, $x_n \models c$
2. For all $e \in E^L$, for all $r \in \phi^L_E(e)$, $\xi_e \models r$
3. For all $e \in E^L$, $s^G(\xi_e) = x_{t(e)}$
4. For all $e \in E^L$, $t^G(\xi_e) = x_{tL(e)}$

$(h^N, h^E)$ is thus a match. Hence, there exist at least one.

Let us now assume that there exists a match $(h^N, h^E)$ from $L$ to $G$. Then, by definition, the $x_n = h^N(n)$’s (and $\xi_e = h^E(e)$’s) of $A$ exist. Additionally, due to the first condition, $x_n$ is a model of $\psi_n$ and, thanks to the other conditions, $\xi_e$ is a model of $\psi_e$. Thus $G \models A$.

Thus, $A \leftrightarrow App(\rho)$.

**Example 5.** Let us consider the rule $\rho$ of Figure[9] The corresponding $App(\rho)$ in first-order logic is $\exists j, k, C(j) \land (C(k) \lor D(k)) \land R(i, j) \land R(j, k)$. This formula is in $\forall^* \exists^*$ but not in C2. However, it is equivalent to the C2 formula $\exists x, y, (R(x, y) \land C(y) \land \exists x, (C(x) \lor D(x)) \land R(y, x))$.

### 5.2 Description logics

Description Logics are also fragments of first-order logic but not all description logics are closed under substitutions. We mainly focus in this subsection on the substitutions generated by the cloning and merging elementary actions. Closure under classical substitutions have been considered in [6]. We prove that with the

---

[8] $\exists_{n \in N} x_n$ is used as a shorthand for $\exists x_{n_0} \ldots \exists x_{n_k}$ where $N = \{n_0, \ldots, n_k\}$. 

[9]
addition of merge and global edge redirection, some logics are still closed while others no longer are.

We assume that the reader is familiar with Description Logics (see [2] for extended definitions). We only focus on extensions of \( \mathcal{ALC} \). We recall that these extensions are named by appending a letter representing additional constructors to the logic name. We focus on nominals (represented by \( \mathcal{O} \)), counting quantifiers (\( \mathcal{Q} \)), self-loops (\( \mathcal{S}elf \)), inverse roles (\( \mathcal{I} \)) and the universal role (\( \mathcal{U} \)). For instance, the logic \( \mathcal{ALCUO} \) extends \( \mathcal{ALC} \) with the universal role and nominals. Below, we recall the definition of \( \mathcal{ALC} \) and the possible additional constructors.

**Definition 14 (Concept, Role, \( \mathcal{ALC} \)).** Let \( \mathcal{A} = (\mathcal{O}, \mathcal{C}_0, \mathcal{R}_0) \) be an alphabet where \( \mathcal{O} \) (resp. \( \mathcal{C}_0, \mathcal{R}_0 \)) is the set of nominals (resp. atomic concepts, atomic roles), given \( o \in \mathcal{O}, c_0 \in \mathcal{C}_0, r_0 \in \mathcal{R}_0 \) and \( n \) and integer, \( \mathcal{ALC} \) concepts \( C \) and roles \( R \) are defined by:

\[
\begin{align*}
\mathcal{C} & := \top | \mathcal{C}_0 | \exists R.C | \neg \mathcal{C} | \mathcal{C} \lor \mathcal{D} \\
\mathcal{R} & := r_0 | \exists R.Self | (\geq n R C) \quad \text{for} \quad n \geq 0
\end{align*}
\]

\( \mathcal{ALC} \) can be extended by adding some of the following concept and role constructors:

\[
\begin{align*}
\mathcal{C} & := o \quad \text{(nominals)} \quad | \quad \exists R.Self \quad \text{(self loops)} \quad | \quad (\geq n R C) \quad \text{(counting quantifiers)} \\
\mathcal{R} & := U \quad \text{(universal role)} \quad | \quad R^{-} \quad \text{(inverse role)}
\end{align*}
\]

For the sake of conciseness, we define \( \bot \equiv \neg \top, C \land C' \equiv \neg (\neg C \lor \neg C'), \forall R.C \equiv \neg (\exists R.\neg C) \) and \( (\geq n R C) \equiv \neg (< n R C) \).

**Definition 15 (Interpretation).** An interpretation over an alphabet \( (\mathcal{C}_0, \mathcal{R}_0, \mathcal{O},) \) is a tuple \( (\mathcal{T}, \mathcal{I}) \) where \( \mathcal{I} \) is a function such that \( c^I_0 \subseteq \Delta^I \), for every atomic concept \( c_0 \in \mathcal{C}_0, r^n_0 \subseteq \Delta^I \times \Delta^I \), for every atomic role \( r_0 \in \mathcal{R}_0, \sigma^I \subseteq \Delta^I \), for every nominal \( o \in \mathcal{O} \). The interpretation function is extended to concept and role descriptions by the following inductive definitions:

\[
\begin{align*}
\top^I & = \Delta^I \\
(\neg C)^I & = \Delta^I \setminus C^I \\
(C \lor D)^I & = C^I \cup D^I \\
(\exists R.C)^I & = \{ n \in \Delta^I | \exists m, (n, m) \in R^I \land m \in C^I \} \\
(\exists R.Self)^I & = \{ n \in \Delta^I | (n, n) \in R^I \} \\
(< n R C)^I & = \{ \delta \in \Delta^I | \#(\{ m \in \Delta^I | (\delta, m) \in R^I \land m \in C^I \}) < n \}
\end{align*}
\]
Definition 16 (Interpretation induced by a decorated graph). Let $G = (N, E, \Phi_N, \Phi_E, s, t)$ be a graph over an alphabet $(C, R)$ such that $C_0 \cup O \subseteq C$ and $R_0 \subseteq R$. The interpretation induced by the graph $G$, denoted $(\Delta^G, \iota^G)$ such that $\Delta^G = N$, $\iota^G = \{(n, m) \in N \times N|C_0 \in \Phi_N(n)\}$, for every atomic concept $c_0 \in C_0$, $\omega^G = \{(n, m) \in N \times N|\exists e \in E. s(e) = n \text{ and } t(e) = m \text{ and } r_0 \in \Phi_E(e)\}$, for every atomic role $r_0 \in R_0$, $\iota^G = \{n \in N|o \in \Phi_N(n)\}$ for every nominal $o \in O$.

We say that a node $n$ of a graph $G$ satisfies a concept $c$, written $n \models c$ if $n \in c^G$. We say that a graph $G$ satisfies a concept $c$, written $G \models c$ if $c^G = N$ that is every node of $G$ belongs to the interpretation of $c$ induced by $G$.

We first consider the possibility to express $\text{App}(\rho)$ in a Description Logic $\mathcal{L}$ for a given rule $\rho$. The definition of $\text{App}(\rho)$ depends of the shape of the left-hand side of $\rho$ on one side and on the expressive power of the considered logic $\mathcal{L}$. Below, we give a general expression for $\text{App}(\rho)$ for a particular class of left-hand sides and logics including $\mathcal{ALCU}$.

Proposition 2. Let $\mathcal{L}$ be a logic extending $\mathcal{ALCU}$. Let us assume that $\rho$ is a rule whose left-hand side is a tree labeled with $\mathcal{L}$ such that its edges have only one label. $\text{App}(\rho)$ can be expressed in $\mathcal{L}$.

Proof. Let $L = (N^L, E^L, \Phi_N^L, \Phi_E^L, s^L, t^L)$ be the left-hand side of $\rho$. Let $r$ be the root of $L$. Let $A = \exists U.\psi_u(r) \land \bigwedge_{e \in \xi(e) \subseteq \mathcal{L}} \psi_e(e)$ where $\mathcal{L}(n) = \{e \in E^L|s^L(e) = n\}, \psi_u(n) = \bigwedge_{c \in \Phi_N^L(n)} c$ and

$$\psi_e(e) = \exists \Phi_E^L(e). \psi_n(t(e)).$$

Let us assume that $G = (N^G, E^G, \Phi_N^G, \Phi_E^G, s^G, t^G)$ is a graph.

Let us assume that $G \models A$. Then, $\{x_r \in N^G|\exists x_n \in N^G \text{ for } n \in N^L, \exists \xi_e \in e^G \text{ for } e \in E^L \text{ such that } \psi_u(n) \in \Phi_N^G(x_n), n = s^L(e) \Rightarrow x_n = s^G(\xi_e), n = t^L(e) \Rightarrow x_n = t^G(\xi_e) \text{ and } \Phi_E^G(e) \in \Phi_E^G(\xi_e)\}$ is not empty. Then, let us define $h^N(n) = x_n$ and $h^E(e) = \xi_e$.

1. For all $n \in N^L$, for all $e \in N^G(n), x_n \models e$ by induction.
2. For all $e \in E^L$, for all $r \in \Phi_E^L(e), \xi_e \models r$
3. For all $e \in E^L, s^G(\xi_e) = x_r$
4. For all $e \in E^L, t^G(\xi_e) = x_l$

$(h^N, h^E)$ is thus a match. Hence, there exist at least one.

Let us now assume that there exists a match $(h^N, h^E)$ from $L$ to $G$. Then, by definition, the $x_n = h^N(n)$’s and $\xi_e = h^E(e)$’s defined previously exist. Additionally, due to the first condition, $x_n$ is a model of $\psi_n$ and, thanks to the other conditions, $\xi_e$ is a model of $\psi_e$. Thus $G \models A$.

Thus, $A \leftrightarrow \text{App}(\rho)$.

Example 6. Let us consider the rule $\rho$ of Figure 9. $\text{App}(\rho)$ can be expressed in $\mathcal{ALCU}$ as $\exists U. (\exists R. (C \lor \exists R. (C \lor D)))$. 
We now discuss the closure under substitution of various Description Logics.

**Theorem 3.** The logics $\text{ALC\text{UO}}, \text{ALC\text{UIO}}, \text{ALC\text{UO}Self}, \text{ALC\text{UIO}Self}, \text{ALC\text{QUO}}$ and $\text{ALC\text{QUO}Self}$ are closed under substitutions.

**Proof.** We proved in [6] that $\text{ALC\text{QUO}Self}$ is closed under substitution for every action but clone and merge. The proof uses a rewriting system that replaces formulas with substitutions with equivalent formulas without substitutions. It is not possible to remove all the substitutions in one step. Some rules are used to move the substitutions closer to atomic formulas.

---

**Fig. 10.** Illustrations of the various ways for a node to satisfy $\exists R.C[i, j, \ldots]$ by gaining a new $R$-neighbor satisfying $C[i, j, \ldots]$. The node where the concept is evaluated is in red; created edges are dashed and red. Squares are nodes that satisfy $C[i, j, \ldots]$. A) $i$ will have a new neighbor $j$ after $[cl(i, j, \ldots)]$ if $R \in L_{\text{lin}}$ and it has a self-loop; B) A node that is neither $i$ nor $j$ will have a new neighbor $j$ if $i$ was its neighbor and $R \in L_{\text{in}}$; C) $j$ will have new neighbours after $[cl(i, j, \ldots)]$ if $i$ has a self-loop and $R \in L_{\text{Out}}$ ($C_1$), $i$ has a self-loop and $R \in L_{\text{Loop}}$ ($C_2$) or if $R \in L_{\text{Out}}$ ($C_3$).

---

- $\top \sigma \rightsquigarrow \top$
- $\sigma \rightsquigarrow \sigma$
- $C_0[\text{add}_C(i, C')] \rightsquigarrow C_0$
- $C_0[\text{del}_C(i, C')] \rightsquigarrow C_0$
- $C_0[\text{add}_C(i, C_0)] \rightsquigarrow C_0 \lor i$
- $C_0[\text{del}_C(i, C_0)] \rightsquigarrow C_0 \land \neg i$
- $C_0[\text{add}_R(i, j, R)] \rightsquigarrow C_0$
- $C_0[\text{del}_R(i, j, R)] \rightsquigarrow C_0$
- $C_0[\text{add}_N(i)] \rightsquigarrow C_0$ if $C_0 \neq \text{Active}$
- $C_0[\text{del}_N(i)] \rightsquigarrow C_0 \land \neg j$
- $C_0[i \gg j] \rightsquigarrow C_0$

---

8 Description Logic names are such that each letter represents a (groups of) constructor(s). More information can be found in [2].
\[C_0[mrg(i,j)] \rightarrow \neg j \land (C_0 \lor (i \land \exists U.(j \land C_0))) \] where \(C_0\) is an atomic formula different from \(Active\).

\[C_0[cl(i,j,\ldots)] \rightarrow C_0 \lor (j \land \exists U.(i \land C_0)) \] where \(C_0\) is an atomic formula different from \(Active\).

\[Active[add_N(i)] \rightarrow C \lor i \]

\[Active[mrg(i,j)] \rightarrow \text{Active} \land \neg j \]

\[Active[cl(i,j,\ldots)] \rightarrow \text{Active} \lor j \]

\[\phi[\sigma] \rightarrow o \]

\[\neg(C) \rightarrow \neg(C\sigma) \]

\[(C \lor D)\sigma \rightarrow C\sigma \lor D\sigma \]

\[\exists R.Self[add_C(i,C_0)] \rightarrow \exists R.Self \]

\[\exists R.Self[del_C(i,C_0)] \rightarrow \exists R.Self \]

\[\exists R.Self[add_R(i,j,R')] \rightarrow \exists R.Self \]

\[\exists R.Self[mrg(i,j)] \rightarrow \neg j \land (\exists R.Self \lor ((i \lor \{j\}) \lor \exists R.Self \lor \exists R.\{i\}) \lor \exists U.((i \lor \exists R.\{j\}) \lor \exists U.((i \lor \exists R.\{i\}) \lor \exists U.((i \lor \exists R.Self)))) \]

\[\exists R.Self[cl(i,j,\ldots)] \rightarrow \exists R.Self \lor C_S \] where \(C_S = \{j\} \) if \(R \in L_1\) and \(C_S = \bot\) otherwise.

\[\exists R.\phi[add_C(i,C_0)] \rightarrow \exists R.(\phi[add_C(i,C_0)]) \]

\[\exists R.\phi[del_C(i,C_0)] \rightarrow \exists R.(\phi[del_C(i,C_0)]) \]

\[\exists R.\phi[add_R(i,j,R')] \rightarrow \exists R.(\phi[add_R(i,j,R')]) \]

\[\exists R.\phi[del_R(i,j,R')] \rightarrow \exists R.(\phi[del_R(i,j,R')]) \]

\[\exists R.\phi[add_R(i,j,R)] \rightarrow \{i\} \land \exists U.((i \lor \phi[add_R(i,j,R)]) \lor \exists R.\phi[add_R(i,j,R)]) \]

\[\exists R.\phi[del_R(i,j,R)] \rightarrow \{i\} \lor \exists R.(\phi[del_R(i,j,R)]) \]

\[\exists R.\phi[add_N(i)] \rightarrow \exists R.(\phi[add_N(i)]) \]

\[\exists R.\phi[del_N(i)] \rightarrow \neg \{i\} \land \exists R.(\phi[del_N(i)] \land \neg \{i\}) \]

\[\exists R.\phi[i \gg j] \rightarrow (\exists U.\{(i) \land \{j\}) \Rightarrow \exists R.\phi[i \gg j]) \]

\[\land(\exists U.\{(i) \land \neg \{j\}) \Rightarrow (\exists R.\{(i) \land \phi[i \gg j]) \land \forall R.\neg \{j\} \land \exists U.\{(j) \land \neg \phi[i \gg j]) \Rightarrow \exists R.\phi[i \gg j] \land \neg \{i\}) \land (\exists R.\{(i) \land \neg \{j\}) \land \exists U.\{(j) \land \phi[i \gg j]) \land \forall R.\neg \{j\} \land \exists R.\{(j) \Rightarrow \exists R.\phi[i \gg j] \land \neg \{i\}) \land (\forall R.\neg \{i\}) \land (\exists R.\{(i) \land \neg \phi[i \gg j]) \land \forall R.\{(j) \]
\[ \forall (\exists R. (\{i\} \land \neg \phi[i \gg j]) \land \forall R. \neg \{j\} \land \exists U. (\{j\} \land \neg \phi[i \gg j]) \Rightarrow \\
\exists R. \phi[i \gg j]) \]

\[ - (\exists R. \phi)[mrg(i, j)] \rightarrow \neg \{j\} \land (\exists R. (-\{j\} \land \phi[mrg(i, j)])) \lor R. (-\{j\} \land \phi[mrg(i, j)]) \lor \\
\{i\} \land \exists U. ((\{i\} \lor \{j\}) \lor \exists R. \phi[mrg(i, j)]) \]

\[ - (\exists R. \phi)[cl(i, j, \ldots)] \rightarrow \\
\exists R. (\phi[cl(i, j, \ldots)]) \lor c_{in} \lor c_{out} \lor c_{in} \lor c_{out} \lor c_{loop} \text{ where:} \]

\[ c_{in} = \begin{cases} 
\neg \{i\} \land \neg \{j\} \land \exists R. \{i\} \land \\
(\exists U. (\{j\} \land \phi[cl(i, j, \ldots)]) & \text{if } R \in L_{in} \\
\bot & \text{otherwise} 
\end{cases} \]

\[ c_{out} = \begin{cases} 
\{j\} \land (\exists U. (\{i\} \land \exists R. (-\{i\} \land \phi[cl(i, j, \ldots)])) & \text{if } R \in L_{out} \\
\bot & \text{otherwise} 
\end{cases} \]

\[ c_{in} = \begin{cases} 
\{i\} \land \exists R. \{i\} \land \\
(\exists U. (\{j\} \land \phi[cl(i, j, \ldots)]) & \text{if } R \in L_{in} \\
\bot & \text{otherwise} 
\end{cases} \]

\[ c_{out} = \begin{cases} 
\{j\} \land \exists U. (\{i\} \land \exists R. \{i\} \land \phi[cl(i, j, \ldots)]) & \text{if } R \in L_{out} \\
\bot & \text{otherwise} 
\end{cases} \]

\[ c_{loop} = \begin{cases} 
\{j\} \land \phi[cl(i, j, \ldots)] \land \exists U. (\{i\} \land \exists R. \{i\}) & \text{if } R \in L_{loop} \\
\bot & \text{otherwise} 
\end{cases} \]

\[ - (n \land R \phi)[add_{C}(i, C_{0})] \rightarrow (n \land R \phi)[add_{C}(i, C_{0})] \]

\[ - (n \land R \phi)[del_{C}(i, C_{0})] \rightarrow (n \land R \phi)[del_{C}(i, C_{0})] \]

\[ - (n \land R \phi)[add_{R}(i, j, R')] \rightarrow (n \land R \phi)[add_{R}(i, j, R')] \]

\[ - (n \land R \phi)[del_{R}(i, j, R')] \rightarrow (n \land R \phi)[del_{R}(i, j, R')] \]

\[ - (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ - (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]

\[ \forall (n \land R \phi)[add_{N}(i)] \rightarrow (n \land R \phi)[add_{N}(i)] \]

\[ \forall (n \land R \phi)[del_{N}(i)] \rightarrow (n \land R \phi)[del_{N}(i)] \]
\[
- (< n R^- \phi)[i \gg j] \rightarrow ([i] \land \neg\{j\}) \lor \\
(\neg\{i\} \land \{j\} \Rightarrow \\
\bigcup_{k \in [0, n]} (< k R^- \phi[i \gg j]) \land \\
\exists U.([i] \land (< (n - k) R^- (\phi[i \gg j] \land \neg\exists R^- \cdot \{j\})))) \\
\lor ((\{i\} \leftrightarrow \{j\}) \Rightarrow (< n R^- \phi[i \gg j])) \\
- (< n R \phi)[mrg(i, j)] \sim \{i\} \lor \\
(\{i\} \land \forall_{k=1} \langle < k R (\phi[mrg(i, j)] \land \forall R^- \cdot \{j\}) \rangle \lor \exists U.([i] \land (< (n - k) R \phi[mrg(i, j)])) \lor \neg\{i\} \land \neg\{j\} \land \\
(\exists R.([j] \land \neg\phi[mrg(i, j)])) \land \forall R^- \cdot \{i\} \land \exists U.([i] \land \phi[mrg(i, j)])) \land (< n - 1 R \phi[mrg(i, j)])) \lor \\
(\exists R.([\{i\} \land \neg\phi[mrg(i, j)]]) \land \forall R^- \cdot \{j\} \land \exists U.([\{j\} \land \phi[mrg(i, j)])) \land (< n - 1 R \phi[mrg(i, j)])) \lor \\
(\exists R.([i] \land \forall R^- \cdot \{i\} \land \exists U.([\{j\} \land \phi[mrg(i, j)])) \land (< n - n + 1 R \phi[mrg(i, j)])) \lor \\
(\forall R^- \cdot \{i\} \lor \exists U.([\{j\} \land \neg\phi[mrg(i, j)])) \lor (< n - n R \phi[c(d(i, j), \ldots)]) \lor \\
C_j = \top \text{ if } R \notin L_{out} \text{ and either:} \\
* R \notin L_{out} \cup L_{loop}, \text{ or} \\
* R \notin L_{out} \cap L_{loop} \text{ and } n > 1, \text{ or} \\
* n > 2 \\
, \text{ and} \\
C_j = (\exists U.([\{i\} \land \exists R.([\{i\} \land \phi[c(d(i, j), \ldots)]) \Rightarrow \\
\bot) \land \\
(\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\top) \text{ if } R \notin L_{out} \cup L_{loop} \text{ and } R \in L_{out} \text{ and } n = 1, \text{ and} \\
C_j = (\exists U.([\{i\} \land \exists R.([\{i\} \land \phi[c(d(i, j), \ldots)]) \Rightarrow \\
\bot) \land \\
(\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\top) \text{ if } R \notin L_{out} \cup L_{loop} \text{ and } R \in L_{loop} \text{ and } n = 1, \text{ and} \\
C_j = (\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \lor \exists U.([\{i\} \land \phi[c(d(i, j), \ldots)]) \Rightarrow \\
\bot) \land \\
(\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \lor \exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\top) \text{ if } R \notin L_{out} \text{ and } R \in L_{out} \cap L_{loop} \text{ and } n = 2, \text{ and} \\
C_j = (\exists U.([\{i\} \land \exists R.([\{i\} \land \phi[c(d(i, j), \ldots)]) \lor \exists U.([\{i\} \land \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\bot) \land \\
(\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \lor \exists U.([\{i\} \land \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\top) \text{ if } R \notin L_{out} \text{ and } R \in L_{out} \cap L_{loop} \text{ and } n = 1, \text{ and} \\
C_j = (\exists U.([\{i\} \land \exists R.([\{i\} \land \phi[c(d(i, j), \ldots)]) \Rightarrow \\
\exists U.([\{i\} \land (< n - 1 R^- (\neg i \land \phi[c(d(i, j), \ldots)]))) \Rightarrow \\
\top) \land \\
(\exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \lor \exists U.([\{i\} \land (\forall R^- \cdot \{i\} \lor \neg\phi[c(d(i, j), \ldots)]) \Rightarrow \\
\top) \text{ if } R \notin L_{out} \text{ and } R \in L_{out} \cap L_{loop} \text{ and } n = 1, \text{ and} \\
C_j = (\exists U.([\{i\} \land \exists R.([\{i\} \land \phi[c(d(i, j), \ldots)]) \Rightarrow \\
\exists U.([\{i\} \land (< n - 1 R^- (\neg i \land \phi[c(d(i, j), \ldots)]))) \Rightarrow)
\[
(\exists U.\{i\} \land (\forall R.\neg\{i\} \lor \neg\phi(d(i, j, \ldots)))) \Rightarrow \\
\exists U.\{i\} \land (< n R (\neg\{i\} \land \phi[d(i, j, \ldots)])) \text{ if } R \in L_{out} \cup L_{Lout}
\]
and \( R \notin L_{Lloop} \), and

\( C_j = (\exists U.\{i\} \land \exists R.\{i\} \land \phi[d(i, j, \ldots)] \Rightarrow \\
\exists U.\{i\} \land (< n - 1 R (\neg\{i\} \land \phi[d(i, j, \ldots)])) \land \\
(3U.\{i\} \land \forall R.\neg\{i\} \lor \neg\phi[d(i, j, \ldots)] \Rightarrow \\
\exists U.\{i\} \land (< n R (\neg\{i\} \land \phi[d(i, j, \ldots)])) \text{ if } R \in L_{out} \cup L_{Lloop}
\]
and \( R \notin L_{Lout} \), and

\( C_o = (< n R \phi[d(i, j, \ldots)] ) \text{ if } R \notin L_{in}, \text{ and } \\
C_o = (\exists R.\{i\} \land \exists U.\{j\} \land \phi[d(i, j, \ldots)] \Rightarrow \\
(< n - 1 R \phi[d(i, j, \ldots)] ) \land \\
(\forall R.\neg\{i\} \lor \exists U.\{j\} \lor \neg\phi[d(i, j, \ldots)] \Rightarrow \\
(< n R \phi[d(i, j, \ldots)] ) \text{ if } R \in L_{in}
\)

We gave an illustration of the various possible cases for \((\exists R.C)[d(i, j, \ldots)]\) in Figure 10. As illustrated by the equivalence given, there are 2 ways for a node to satisfy \((\exists R.C)[d(i, j, \ldots)]\): either it already had such a neighbor before cloning or it gained it during cloning. The 5 possible ways for the second scenario to happen are given in Figure 10. Using this picture, one can also see the various cases of \(< n R C)[d(i, j, \ldots)]\). As it is quite complex and depends on \( L_{in}, L_{out}, L_{Lout}, L_{Lloop}, L_{Lout} \), \( L_{Lloop} \), and \( n \), we do not report the exact equivalence. We give an idea of what it is, though: assuming \( j \) will be labeled with \( C \), we remark that in case A) \( i \) needs \( n - 1 \) neighbors that will be labeled with \( C \) and it needs \( n \) otherwise, and, in case B), the same can be said for other nodes. \( j \) is more problematic. If \( R \notin L_{out} \), it will have at most 2 neighbors, if \( R \in L_{out} \) it will have as many as \( i \) plus, possibly, \( i \) and \( j \).

We give an illustration of the counting quantifiers in the case of \( mrg(i, j) \) in Figure 11. \( j \) always satisfies \(< n R C)[mrg(i, j)] \) as it has no neighbors after merging. \( i \) has its neighbors plus those of \( j \) that were not already its neighbors. All other nodes can either gain a new one \( i \), lose one \( j \) or both.

Let us now prove that the two sides of each rules are indeed equivalent.

\( \top \sigma \): By definition, \( \top \) is always satisfied

\( \circ \sigma \): The interpretation of \( o \) is never modified

\( C_0[add_C(i, C')] \): The interpretation of \( C_0 \) does not depend on the interpretation \( C' \).

\( C_0[del_C(i, C')] \): The interpretation of \( C_0 \) does not depend on the interpretation \( C' \).
Fig. 11. Illustrations of the various ways for a node to satisfy \( (< n \text{ RC} | \text{mrg}(i,j)) \) when the merging action affects the number of neighbors of a node. The node where the concept is evaluated is in red, the nodes that will be labeled with \( C \) are squares. A) \( j \) has no remaining neighbor, it thus satisfies \( (< n \text{ RC} | \text{mrg}(i,j)) \). B) \( i \) will have as neighbors all its neighbors plus those of \( j \). It is important to count each one only once. C) If the node is neither \( i \) nor \( j \), it will gain a new neighbor that will be labeled with \( C \) - \( i \) - if \( i \) will be labeled with \( C \), it is not yet a neighbor and \( j \) is a neighbor that would not be labeled with \( C \) (\( C_1 \)); on the other hand, it will lose a neighbor that will be labeled with \( C \) - \( j \) - if \( j \) is a neighbor that will be labeled with \( C \) and either \( i \) is also a neighbor that will be labeled with \( C \) (\( C_2 \)) or \( i \) will not be labeled with \( C \) (\( C_3 \)); otherwise, the number of neighbors that will be labeled with \( C \) stays the same either because there is no new neighbor (\( C_4 \)), because neither \( i \) nor \( j \) will be labeled with \( C \) (\( C_5 \)) or because it loses one neighbor that will be labeled with \( C \) - \( j \) - and gains one - \( i \) (\( C_6 \)).
$C_0[\text{add}_C(i, C_0)]$: The interpretation of $C_0$ becomes $C_0^T \cup i^T = (C_0 \lor i)^T$.

$C_0[\text{del}_C(i, C_0)]$: The interpretation of $C_0$ becomes $C_0^T \setminus i^T = (C_0 \land \neg i)^T$.

$C_0[\text{add}_R(i, j, R)]$: The interpretation of $C_0$ does not depend on the interpretation $R$.

$C_0[\text{del}_R(i, j, R)]$: The interpretation of $C_0$ does not depend on the interpretation $R$.

$C_0[\text{add}_N(i)]$: The interpretation of $C_0$ does not depend on the interpretation $R$.

$C_0[\text{del}_N(i)]$: The interpretation of $C_0$ becomes $C_0^T \setminus i^T = (C_0 \land \neg i)^T$.

$C_0[i \gg j]$: The interpretation of $C_0$ does not depend on the interpretation of any role.

$C_0[\text{merge}(i, j)]$: The interpretation of $C_0$ becomes $C_0^T \cup \{n | n = i^T \land j \in C_0^T\} \setminus j^T = (\neg j \land (C_0 \lor (i \land \exists U(j \land C_0))))^T$.

$C_0[\text{set}(i, j, \ldots)]$: The interpretation of $C_0$ becomes $C_0^T \cup \{n | n = j^T \land i \in C_0\} \setminus j^T = (\neg j \land (\exists U(i \land C_0))))^T$.

Active$[\text{add}_N(i)]$: The interpretation of $\text{Active}$ becomes $N^G \lor i^T = (\text{Active} \lor i)^T$.

Active$[\text{merge}(i, j)]$: The interpretation of $\text{Active}$ becomes $N^G \land j^T = (\text{Active} \land j)^T$.

Active$[\text{set}(i, j)]$: The interpretation of $\text{Active}$ becomes $N^G \land j^T = (\text{Active} \lor j)^T$.

$\exists \text{Self}[\text{add}_C(i, C_0)]$: The interpretation of $R$ does not depend on the interpretation of $C_0$.

$\exists \text{Self}[\text{del}_C(i, C_0)]$: The interpretation of $R$ does not depend on the interpretation of $C_0$.

$\exists \text{Self}[\text{add}_R(i, j, R')]$: The interpretation of $R$ does not depend on the interpretation of $R'$.

$\exists \text{Self}[\text{del}_R(i, j, R')]$: The interpretation of $R$ does not depend on the interpretation of $R'$.

$\exists \text{Self}[\text{add}_R(i, j, R)]:$ The interpretation of $R$ becomes $R^T \cup i^T \times j^T$. Thus $(\exists \text{Self})^T = \{n \in \Delta \exists e \in R^T, s(e) = n \land t(e) = \{\} \cup \{n \in \Delta | n = i^T \land n = j^T\}$ that is $(\exists \text{Self})^T = (\exists \text{Self} \lor (i \land j))^T$.

$\exists \text{Self}[\text{del}_R(i, j, R)]:$ The interpretation of $R$ becomes $R^T \setminus i^T \times j^T$. Thus $(\exists \text{Self})^T = \{n \in \Delta \{n, n \in R^T\} \cup \{n \in \Delta | n = i^T \land n = j^T\}$ that is $(\exists \text{Self})^T = (\exists \text{Self} \land (\neg i \lor j))^T$.

$\exists \text{Self}[\text{add}_N(i)]:$ The interpretation of $R$ is not modified.

$\exists \text{Self}[\text{set}(i, j)]:$ The interpretation of $R$ becomes $R^T \setminus \{c \in T(e) = i^T \lor t^T(e) = i^T\}$. Thus $(\exists \text{Self})^T = (\exists \text{Self} \land (\neg i))^T$.

$\exists \text{Self}[i \gg j]$: Let us assume that $(n, n) \in R^T$. Then, either:

- $n \neq i^T$ and $n \neq j^T$ and thus $(n, n) \in R^T$
- $n = i^T = j^T$ and thus $(n, n) \in R^T$
- or $n = j^T \neq i^T$ and thus either $(j, i^T) \in R^T$ or $(j^T, i) \in R^T$
- or $n = i^T \neq j^T$ which is impossible as $R^T \cap \{(n, i^T)\} = \emptyset$

Thus $(\exists \text{Self}[i \gg j])^T = \{(((\{\} \leftrightarrow \{\})) \Rightarrow \exists \text{Self}(\text{add}_N(i, j))\}^T$.

$\exists \text{Self}[\text{merge}(i, j)]:$ Let us assume that $(n, n) \in R^T$. Then, either:
\[ n = j^T \] which is impossible

\[ n = i^T \] and thus one of \((i^T, i^T), (i^T, j^T), (j^T, i^T)\) or \((j^T, j^T)\) \in \mathbb{R}^T \]

\[ n \neq j^T \] and \(n \neq i^T\) and thus \((n, n) \in \mathbb{R}^T\)

Thus \(\exists R.\text{Self}[\text{mrg}(i, j)]\)^T = \((\neg \{j\} \land (\exists R.\text{Self} \lor \{i\} \land (\exists R.\{j\} \lor \exists U.((\{j\} \land \exists R.\{i\})) \lor \exists U.((\{i\} \land \exists R.\{j\})))))^T\)

\[ \exists R.\text{Self}[\text{el}(i, j, \ldots)]: \text{Let us assume that } (n, n) \in \mathbb{R}^T. \text{ Then, either:} \]

\[ n = j^T \text{ and } R \in L_{\text{loop}} \]

\[ n \neq j^T \text{ and thus } (n, n) \in \mathbb{R}^T \]

Thus \(\exists R.\text{Self}[\text{mrg}(i, j)]\)^T = \((\exists R.\text{Sel} \lor C_S)^T \) where \(C_S = \{j\} \) if \(R \in L_{\text{loop}}\) and \(\bot\) otherwise.

\[ (\exists R.\phi)[\text{addc}(i, C_0)]: \text{As the valuation of } R \text{ is not modified by the substitution,} \]

\[ ((\exists R.\phi)[\text{addc}(i, C_0)])^T = (\exists R.(\phi[\text{addc}(i, C_0)])^T. \]

\[ (\exists R.\phi)[\text{delc}(i, C_0)]: \text{As the valuation of } R \text{ is not modified by the substitution,} \]

\[ ((\exists R.\phi)[\text{delc}(i, C_0)])^T = (\exists R.(\phi[\text{delc}(i, C_0)])^T. \]

\[ (\exists R.\phi)[\text{addp}(i, j, R')]\): As the valuation of \( R \) is not modified by the substitution, \((\exists R.\phi)[\text{addp}(i, j, R')]\)^T = \((\exists R.(\phi[\text{addp}(i, j, R')])^T). \]

\[ (\exists R.\phi)[\text{delr}(i, j, R')]\): As the valuation of \( R \) becomes \( R^T \cup \{i, j\} \), \((\exists R.\phi)[\text{delr}(i, j, R')]\)^T = \((\exists R.(\phi[\text{delr}(i, j, R')])^T. \]

\[ (\exists R.\phi)[\text{addn}(i)]: \text{As the valuation of } R \text{ is not modified by the substitution,} \]

\[ ((\exists R.\phi)[\text{addn}(i)])^T = (\exists R.(\phi[\text{addn}(i)])^T. \]

\[ (\exists R.\phi)[\text{deln}(i)]: \text{As the valuation of } R \text{ becomes } R^T \setminus \{i, n'\} \text{ if } n' = i^T, \]

\[ ((\exists R.\phi)[\text{deln}(i)])^T = (\neg \{i\} \land (\exists R.(-i \land (\exists R.\text{deln}(i, j, R'))^T. \]

\[ \exists R.\phi[i \gg j]: \text{Let us assume that there exists } (n, n') \in R^T \text{ and } n' \neq j^T. \]

Then, either:

\[ n^T = j^T \] and thus \((n, n') \in \mathbb{R}^T \]

\[ n^T \neq j^T \] and then either:

\( (n, n^T) \notin \mathbb{R}^T \) and thus \((n, n') \in \mathbb{R}^T \)

\( (n, n^T) \in \mathbb{R}^T \text{ and } j^T \notin (\phi[i \gg j])^T \) and thus \(n' \neq j^T \) and thus \((n, n') \in \mathbb{R}^T \)

\( (n, n^T) \in \mathbb{R}^T \text{ and } j^T \in (\phi[i \gg j])^T \) and thus \(j^T \) is a witness.

Thus the rule is correct.

\[ \exists R.\phi[\text{mrg}(i, j)]: \text{Let us assume that there exists } (n, n') \in \mathbb{R}^T \text{ with } n' \in \phi^T \]

\[ n \neq j^T \] and \(n' \neq j^T\). If \( n \neq i^T \) and \(n' \neq i^T\), then \((n, n') \in \mathbb{R}^T \) thus \( n \in (\neg \{j\} \land \exists R.(-\{j\} \land \phi[\text{mrg}(i, j)])^T. \)

\[ \text{If } n' = i^T \text{ then either } (n, j^T) \in \mathbb{R}^T \]

\[ (n, i^T) \in \mathbb{R}^T \text{ and } \{i, j^T\} \cap \phi^T \neq \text{emptyset} \text{ thus } n \in (\neg \{j\} \land \exists R.((\{i\} \lor (j^T \land \exists U.((\{i\} \lor \{j\}) \land \phi[\text{mrg}(i, j)])^T. \)

\[ \text{If } n = i^T \text{ then either } (i^T, n') \text{ or } (j^T, n') \in \mathbb{R}^T \text{ and thus } n \in (\neg \{j\} \land \exists R.((\{i\} \lor \{j\}) \land \exists R.\phi[\text{mrg}(i, j)])^T. \]

Thus \(\exists R.\phi[i \gg j^T] = (\neg \{j\} \land (\exists R.(-\{j\} \land (\exists R.\phi[\text{mrg}(i, j)] \lor \exists R.(-\{j\} \land \phi[\text{mrg}(i, j)])^T. \)

\[ \exists R.\phi[\text{cl}(i, j, \ldots)]: \text{Let us assume that there exists } c' \in E^T \text{ such that } s^T(c') = n, \]

\[ i^T(c') = n' \text{ and } (n, n') \in \mathbb{R}^T \text{ then either:} \]
$e' \in E'_{in}$ and then $n = s'(in(e'))$ and $n' = j^2$, that is there exists $e$ such that $(s^2(e), t^2(e)) \in R^2$ and $s^2(e) = n \neq j^2$ and $t^2(e) = i^2$. Thus $n \in (\neg \{i\} \land \exists R.\{i\} \land \exists U.(\{j\} \land \phi[cl(i, j, \ldots)])\}^2$.

$e' \in E'_{out}$ and then $n = j^2$ and $n' = t^2(out(e'))$, that is there exists $e$ such that $(s^2(e), t^2(e)) \in R^2$ and $s^2(e) = i^2$ and $t^2(e) = i^2$. Thus $n \in (\{i\} \land \exists R.\{i\} \land \exists U.(\{j\} \land \phi[cl(i, j, \ldots)])\}^2$.

$e' \in E'_{in}$ and then $n = i^2$ and $n' = j^2$ and there exists $e$ such that $(s^2(e), t^2(e)) \in R^2$ and $s^2(e) = i^2$ and $t^2(e) = i^2$. Thus $n \in (\{i\} \land \exists R.\{i\} \land \exists U.(\{i\} \land \exists R.\{i\} \land \phi[cl(i, j, \ldots)])\}^2$.

$e' \in E'_{out}$ and then $n = j, n' = i$ and there exists $e$ such that $(s^2(e), t^2(e)) \in R^2$ and $s^2(e) = i^2$ and $t^2(e) = i^2$. Thus $n \in (\{j\} \land \exists R.\{i\} \land \exists U.(\{i\} \land \exists R.\{i\} \land \phi[cl(i, j, \ldots)])\}^2$.

$e' \in E'_{loop}$ and then $n = j, n' = j$ and there exists $e$ such that $(s^2(e), t^2(e)) \in R^2$ and $s^2(e) = i^2$ and $t^2(e) = i^2$. Thus $n \in (\{j\} \land \exists R.\{i\} \land \exists U.(\{i\} \land \exists R.\{i\} \land \phi[cl(i, j, \ldots)])\}^2$.

otherwise, $e' \in E'_{loop}$ and then $n \in (\{n\} \land \exists R.\{i\} \land \exists U.(\{i\} \land \exists R.\{i\} \land \phi[cl(i, j, \ldots)])\}^2$.

($\langle n R \phi[addC(i, C_0)]\rangle$: As the valuation of $R$ is not modified by the substitution, $(\langle n R \phi[addC(i, C_0)]\rangle)^2 = (\langle n R \phi[addC(i, C_0)]\rangle)^2$.

($\langle n R \phi[delC(i, C_0)]\rangle$: As the valuation of $R$ is not modified by the substitution, $(\langle n R \phi[delC(i, C_0)]\rangle)^2 = (\langle n R \phi[delC(i, C_0)]\rangle)^2$.

($\langle n R \phi[addR(i, j, R')]\rangle$: As the valuation of $R$ is not modified by the substitution, $(\langle n R \phi[addR(i, j, R')]\rangle)^2 = (\langle n R \phi[addR(i, j, R')]\rangle)^2$.

($\langle n R \phi[addR(i, j, R)]\rangle$: Let us assume that $n \in (\langle n R \phi[addR(i, j, R)]\rangle)^2$ then either:

\[ n = i^2, j^2 \in \phi[addR(i, j, R)]^2 \quad \text{and} \quad (n, j^2) \notin R^2 \quad \text{and} \quad n \in (\langle n + 1 \rangle R \phi[addR(i, j, R)]^2 \quad \text{and} \quad \text{the number of neighbors of } n \text{ is left unchanged and thus } n \in (\langle n R \phi[addR(i, j, R)]\rangle)^2 \]

Thus the rule is correct.

($\langle n R \phi[delR(i, j, R)]\rangle$: Let us assume that $n \in (\langle n R \phi[delR(i, j, R)]\rangle)^2$ then either:

\[ n = i^2, j^2 \in \phi[delR(i, j, R)]^2 \quad \text{and} \quad (n, j^2) \in R^2 \quad \text{and} \quad n \in (\langle n + 1 \rangle R \phi[delR(i, j, R)]^2 \quad \text{and} \quad \text{the number of neighbors of } n \text{ is left unchanged and thus } n \in (\langle n R \phi[delR(i, j, R)]\rangle)^2 \]

Thus the rule is correct.

($\langle n R \phi[addN(i)]\rangle$: As the valuation of $R$ is not modified by the substitution, $(\langle n R \phi[addN(i)]\rangle)^2 = (\langle n R \phi[addN(i)]\rangle)^2$.

($\langle n R \phi[delR(i, j, R')]\rangle$: As the valuation of $R$ becomes $R^2 \setminus \langle (n, n') | n = i^2 \text{ or } n' = j^2 \rangle$, $(\langle n R \phi[delR(i, j, R')]\rangle)^2 = (\langle i \rangle \lor (\langle n R \phi[delR(i, j, R')]\rangle \land \neg \{i\})\}^2$.

($\langle n R \phi[i \Rightarrow j]\rangle$: Let us consider whether the node $m \in (\langle n R \phi[i] \Rightarrow j\rangle)$ gains or loses neighbors in $\phi[\ldots]$:

- if $i^2 = j^2$, the transformation didn’t change anything.
– otherwise:
  • if \((m, i) \in R^2, (m, j) \notin R^2\) and \(j \in \phi^I\), \(m\) lost one and thus had less than \(n + 1\),
  • if \((m, i) \in R^2\) and \((m, j) \in R^2\), \(m\) lost one and thus had less than \(n + 1\),
  • if \((m, i) \notin R^2, (m, j) \in R^2\) and \(j \in \phi^I\), \(m\) gained one and thus had less than \(n - 1\),
  • otherwise, its number of neighbors in \(\phi^T\) does not change.

Thus the rule is correct.

\(< n R^- \phi)[i \gg j]\): Let us consider whether the node \(m \in (< n R \phi)^I\) gains or loses neighbors in \(\phi^I\):
– if \(m = i \gg j\), the transformation didn’t change anything,
– if \(m = i \gg j\), \(m\) lost all its neighbors and thus has less than \(n\),
– if \(m = j \gg j\), \(m\) gained all of \(i \gg j\)’s neighbors and thus the sum of its neighbors and those of \(i \gg j\) had to be less than \(n\),
– otherwise, its number of neighbors in \(\phi^I\) does not change.

Thus the rule is correct.

\(< n R C)[mrg(i, j)]\) Let us consider whether the node \(m \in (< n R \phi)^I\) gains or loses neighbors in \(\phi^I\):
– If \(m = j \gg j\), it has no remaining neighbor,
– If \(m = i \gg j\), it gains all neighbors of \(j \gg j\) and thus the sum of its neighbors and those of \(j \gg j\) had to be less than \(n\),
– otherwise:
  • if \((m, i) \in R^2, (m, j) \notin R^2, i \notin \phi^I\) and \(j \in \phi^I\), \(m\) gained one and thus had less than \(n - 1\),
  • if \((m, j) \in R^2, (m, i) \notin R^2, j \notin \phi^I\) and \(i \in \phi^I\), \(m\) gained one and thus had less than \(n - 1\),
  • if \((m, i) \in R^2, (m, j) \in R^2, j \in \phi^I\) and \(i \notin \phi^I\), \(m\) lost one and thus had less than \(n + 1\),
  • otherwise, they stay the same.

The rule is thus correct.

\(< n R C)[cl(i, j, . . .)]\) Let us consider whether the node \(m \in (< n R \phi)^I\) gains or loses neighbors in \(\phi^I\):
– If \(m = i \gg j\), it can only gain one possible neighbor \((j)\) if \((m, m) \in R^2\), \(j = \phi^I\) and \(R \in L_{\text{in}}\). In that case, it needs have one less neighbor.
– If \(m = j \gg j\), then either:
  • \(R \notin L_{\text{out}}\) and thus the only possible neighbors are \(i\) and \(j\). Then \(m \in (< n R \phi)^I\) if and only if one of the following is true:
    * \(n > 2\),
    * \(n > 1\) and \(R \notin L_{\text{out}} \cap L_{\text{loop}}\),
    * \(R \notin L_{\text{out}} \cup L_{\text{loop}}\),
    * \(n = 1\) and \(R \in L_{\text{out}} \setminus L_{\text{loop}}\) and \((i \gg j) \notin R^2\) or \(i \notin \phi^I\),
    * \(n = 1\) and \(R \in L_{\text{loop}} \setminus L_{\text{out}}\) and \((i \gg j) \notin R^2\) or \(j \notin \phi^I\),
    * \(n = 1\) and \(R \in L_{\text{out}} \cap L_{\text{loop}}\) and \((i \gg j) \notin R^2\) or both \(\{i \gg j, j \gg j\}\) and \(\phi^I = \emptyset\),
Theorem 4. The logics \texttt{ALCQUO} and \texttt{ALCQUOSelf} are not closed under substitutions.

In order to prove this theorem, we use the notion of bisimulation \cite{10}.

Definition 17 (\texttt{ALCQUO}-Bisimulation). Given a signature \((\mathbb{C}, R, I)\) and two interpretations \(I\) and \(J\), a non-empty binary relation \(Z \subseteq (\Delta^I \times \Delta^J)\) is an
ALCQUO- bisimulation if it satisfies:

\((ALC_1)\): \(d_1 Z d_2 \implies \forall A \in C, (d_1 \in A^I \iff d_2 \in A^J)\)

\((ALC_2)\): \(\forall R \in R, (d_1 Z d_2 \land (d_1, e_1) \in R^I \implies \exists e_2, (d_2, e_2) \in R^J \land e_1 Z e_2)\)

\((ALC_3)\): \(\forall R \in R, (d_1 Z d_2 \land (d_2, e_2) \in R^J \implies \exists e_1, (d_1, e_1) \in R^I \land e_1 Z e_2)\)

\((ALC_4)\): \(\forall i \in I, i \in J\)

\((U_1)\): \(\forall d \in \Delta^I, \exists d' \in \Delta^J. d Z d'\)

\((U_2)\): \(\forall d' \in \Delta^J, \exists d \in \Delta^I. d Z d'\)

\((Q)\): \(\forall R : (d_1 Z d_2 \implies Z \text{ is a bijection between the } R\text{-successors of } d_1 \text{ and those of } d_2)\)

**Fig. 12.** \(d_1\) is a model of \((\geq 2 \ R \ C)[mrg(i, j)]\) and \((\geq 2 \ R \ C)[i \gg j]\), \(d_1'\) is not. Nodes satisfying \(C\) are drawn in red.

**Theorem 5.** \([13]\) Let \((\Delta_1, ^I_1)\) and \((\Delta_2, ^I_2)\) be two interpretations and \(Z\) a \(ALCQUO\)-bisimulation relation between \(I_1\) and \(I_2\). Let \(C\) be an \(ALCQUO\) concept, then for all \(x_1 \in \Delta_1\) and \(x_2 \in \Delta_2\), \(x_1 Z x_2 \implies (x_1 \in C^I \iff x_2 \in C^J)\).

The notion of bisimulation can be extended to \(ALCQUOSelf\) as follows.

**Definition 18 (ALCQUOSelf-Bisimulation).** Given a signature \((C, R, I)\) and two interpretations \(I\) and \(J\), a non-empty binary relation \(Z \subseteq (\Delta^I \times \Delta^J)\) is an \(ALCQUOSelf\)-bisimulation if it is an \(ALCQUO\)-bisimulation and it satisfies:

\((Self)\): \(\forall R \in R, d_1 Z d_2 \implies ((d_1, d_1) \in R^I \iff (d_2, d_2) \in R^J)\).

**Proof (Theorem 4).** We use the interpretations from Figure 12 to show that some concept with substitution is not in \(ALCQUO\) or \(ALCQUOSelf\). Let us start by proving that two interpretations are indeed bisimilar:

\((ALC_1)\):
- \(d_1 Z d_1' \rightarrow (d_1 \in C^I \iff d_1' \in C^J)\)
\(- d_2 Zd'_2 \rightarrow (d_2 \in \mathcal{C}) \iff d'_2 \in \mathcal{C}^{\mathcal{J}} \checkmark \)
\(- d_3 Zd'_3 \rightarrow (d_3 \in \mathcal{C}) \iff d'_3 \in \mathcal{C}^{\mathcal{J}} \checkmark \)
\(- d_4 Zd'_4 \rightarrow (d_4 \in \mathcal{C}) \iff d'_4 \in \mathcal{C}^{\mathcal{J}} \checkmark \)

\((\mathcal{ALC}_2)\) :
\(- d_1 Zd'_1 \land (d_1, d_3) \in \mathcal{R}^{\mathcal{I}} \rightarrow (d'_1, d'_3) \in \mathcal{R}^{\mathcal{J}} \land d_3 Zd'_3 \checkmark \)
\(- d_2 Zd'_2 \land (d_2, d_4) \in \mathcal{R}^{\mathcal{I}} \rightarrow (d'_2, d'_4) \in \mathcal{R}^{\mathcal{J}} \land d_2 Zd'_2 \checkmark \)

\((\mathcal{ALC}_3)\) :
\(- d_1 Zd'_1 \land (d'_1, d'_3) \in \mathcal{R}^{\mathcal{I}} \rightarrow (d_1, d_3) \in \mathcal{R}^{\mathcal{J}} \land d_3 Zd'_3 \checkmark \)
\(- d_2 Zd'_2 \land (d'_2, d'_4) \in \mathcal{R}^{\mathcal{I}} \rightarrow (d_2, d_4) \in \mathcal{R}^{\mathcal{J}} \land d_2 Zd'_2 \checkmark \)

\((\mathcal{ALC}_4)\) :
\(- d^i Zd^j \checkmark \)
\(- J^i Z d^j \checkmark \)

\((\mathcal{O})\) :
\(- d_1 Zd'_1 \rightarrow (d_1 = i^{\mathcal{I}} \iff d'_1 = i^{\mathcal{J}} \checkmark \)
\(- d_2 Zd'_2 \rightarrow (d_2 = i^{\mathcal{I}} \iff d'_2 = i^{\mathcal{J}} \checkmark \)
\(- d_3 Zd'_3 \rightarrow (d_3 = i^{\mathcal{I}} \iff d'_3 = i^{\mathcal{J}} \checkmark \)
\(- d_4 Zd'_4 \rightarrow (d_4 = i^{\mathcal{I}} \iff d'_4 = i^{\mathcal{J}} \checkmark \)
\(- d_1 Zd'_1 \rightarrow (d_1 = j^{\mathcal{I}} \iff d'_1 = j^{\mathcal{J}} \checkmark \)
\(- d_2 Zd'_2 \rightarrow (d_2 = j^{\mathcal{I}} \iff d'_2 = j^{\mathcal{J}} \checkmark \)
\(- d_3 Zd'_3 \rightarrow (d_3 = j^{\mathcal{I}} \iff d'_3 = j^{\mathcal{J}} \checkmark \)
\(- d_4 Zd'_4 \rightarrow (d_4 = j^{\mathcal{I}} \iff d'_4 = j^{\mathcal{J}} \checkmark \)

\((\mathcal{U}_1)\) :
\(- d_1 Zd'_1 \checkmark \)
\(- d_2 Zd'_2 \checkmark \)
\(- d_3 Zd'_3 \checkmark \)
\(- d_4 Zd'_4 \checkmark \)

\((\mathcal{U}_2)\) :
\(- d_1 Zd'_1 \checkmark \)
\(- d_2 Zd'_2 \checkmark \)
\(- d_3 Zd'_3 \checkmark \)

\((\mathcal{Q})\) :
\(- d_1 Zd'_1 \rightarrow Z \text{ is a one-to-one between } \{d_3\} \text{ and } \{d'_3\} \checkmark \)
\(- d_2 Zd'_2 \rightarrow Z \text{ is a one-to-one between } \{d_4\} \text{ and } \{d'_4\} \checkmark \)
\(- d_3 Zd'_3 \rightarrow Z \text{ is a one-to-one between } \emptyset \text{ and } \emptyset \checkmark \)
\(- d_4 Zd'_4 \rightarrow Z \text{ is a one-to-one between } \emptyset \text{ and } \emptyset \checkmark \)

\((\text{Self})\) :
\(- d_1 Zd'_1 \rightarrow ((d_1, d_1) \in \mathcal{R}^{\mathcal{I}} \iff (d'_1, d'_1) \in \mathcal{R}^{\mathcal{J}}) \checkmark \)
\(- d_2 Zd'_2 \rightarrow ((d_2, d_2) \in \mathcal{R}^{\mathcal{I}} \iff (d'_2, d'_2) \in \mathcal{R}^{\mathcal{J}}) \checkmark \)
\(- d_3 Zd'_3 \rightarrow ((d_3, d_3) \in \mathcal{R}^{\mathcal{I}} \iff (d'_3, d'_3) \in \mathcal{R}^{\mathcal{J}}) \checkmark \)
\(- d_4 Zd'_4 \rightarrow ((d_4, d_4) \in \mathcal{R}^{\mathcal{I}} \iff (d'_4, d'_4) \in \mathcal{R}^{\mathcal{J}}) \checkmark \)

However, applying the transformation \(mrg(i, j)\) yields the interpretations shown in Figure 13 where \(d_1\) is a model of \((\geq 2 R C)\) but \(d'_1\) is not i.e. \((\geq 2 R C)[mrg(i, j)]\) is not a concept of \(\mathcal{ALCQUO}\) or \(\mathcal{ALCQUO\text{Self}}\).
Example 7. We computed in Example 4 the correctness formula for the client-to-proxy connection example. The formula still contained $\text{App}(\rho)$ and substitutions, however, that we can now replace with their correct expressions. $\text{App}(\rho_0)$ is equivalent to $\exists U. (\text{Client} \land \exists \text{Request}. (\text{Proxy} \land (\leq N \, 2 \, C \, P^- \, T)))$ and $\text{App}(\rho_1)$ is equivalent to $\exists U. (\text{Client} \land \exists \text{Request}. (\text{Proxy} \land (\geq N \, 2 \, C \, P^- \, T)))$. After simplification, $\text{wp}(\alpha_{\rho_0}, \text{Post})$ is equivalent to $\forall U. ((\text{Proxy} \land \text{Active}) \Rightarrow ((j \land \exists U. (i \land \forall C \, 2 \, P. \, \neg j) \Rightarrow (\leq N - 1 \, C \, P^- \, T)) \land (\neg j \lor \exists U. (i \land \exists C \, 2 \, P. j) \Rightarrow (\leq N \, 2 \, C \, P^- \, T)))$ and $\text{wp}(\alpha_{\rho_1}, \text{Post})$ is equivalent to $\forall U. ((\text{Proxy} \lor (k \land \exists U. (j \land \text{Proxy}))) \land (\text{Active} \lor k) \Rightarrow ((k \Rightarrow \top) \land (\neg k \lor \exists C \, 2 \, P^- \, i) \Rightarrow (\leq N \, 2 \, C \, P^- \, T)))$.

Proving that the correctness formula is valid amounts to proving that Proxies, including the possible new one $k$, satisfy some conditions. Let us first prove that $\text{Pre} \land \text{App}(\rho_0) \Rightarrow \text{wp}(\alpha_{\rho_0}, \text{Post})$ is valid:

- For all Proxies that are not $j$, nothing has changed
- For $j$, if it had strictly less than $N$ incoming edges labeled with $C \, 2 \, P$, that is if $\rho_0$ was the rule that was applied, it satisfies $(\leq N - 1 \, C \, 2 \, P^- \, T)$. It thus satisfies $\text{Proxy} \land \text{Active} \Rightarrow (\leq N - 1 \, C \, 2 \, P^- \, T)$ if there was an edge from $i$ to $j$ labeled with $C \, 2 \, P$, that is if $\exists U. (i \land \forall C \, 2 \, P. j)$ is satisfied, and $\text{Proxy} \land \text{Active} \Rightarrow (\leq N - 1 \, C \, 2 \, P^- \, T)$ if not.

$\text{Pre} \land \text{App}(\rho_0) \Rightarrow \text{wp}(\alpha_{\rho_0}, \text{Post})$ is thus valid. Let us focus now on $\text{Pre} \land \text{App}(\rho_1) \Rightarrow \text{wp}(\alpha_{\rho_1}, \text{Post})$:

- For all Proxies that are not $j$ or $k$, nothing has changed
- $j$, from $\text{Pre}$, satisfies $(\leq N \, 2 \, C \, P^- \, T)$ and thus $(\text{Proxy} \land \text{Active} \Rightarrow (\neg k \Rightarrow (\leq N \, 2 \, C \, P^- \, T)))$
- As for $k$, $k \Rightarrow \top$ is an obvious tautology.

As both implications are valid, so is their conjunction and thus the correctness formula is valid. We have successfully proved the correctness of the specification.
6 Conclusion

We have presented a class of graph rewriting systems, LDGRSs, where the left-hand sides of the considered rules can express additional application conditions defined as logic formulas and right-hand sides are sequences of actions. The considered actions include node merging and cloning, node and edge addition and deletion among others. We defined computations with these systems by means of rewrite strategies. There is certainly much work to be done around such systems with logically decorated left-hand sides. For instance, the extension to narrowing derivations, which is a matter of future work, would use an involved unification algorithm taking into account the underlying logic. We have also presented a sound Hoare-like calculus for specifications with pre and post conditions and shown that the considered correctness problem is still decidable in most of the logics we used. We also pointed out those logics for which the rules we gave did not provide a proof of closure under substitutions and proved that they were not actually closed under substitutions. Future work include also an implementation of the proposed verification technique as well as the investigation of more expressive logics with connections some SMT solvers.

References

1. Shqiponja Ahmetaj, Diego Calvanese, Magdalena Ortiz, and Mantas Simkus. Managing change in graph-structured data using description logics. In Proc. of the 28th AAAI Conf. on Artificial Intelligence (AAAI 2014), pages 966–973. AAAI Press, 2014.
2. Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors. The Description Logic Handbook: Theory, Implementation, and Applications. Cambridge University Press, New York, NY, USA, 2003.
3. Philippe Balbiani, Rachid Echahed, and Andreas Herzig. A dynamic logic for termgraph rewriting. In 5th International Conference on Graph Transformations (ICGT), volume 6372 of LNCS, pages 59–74. Springer, 2010.
4. Luciano Baresi and Paola Spoletini. Procs. of ICGT 2006, chapter On the Use of Alloy to Analyze Graph Transformation Systems, pages 306–320. Springer, 2006.
5. Jon Haël Brenas, Rachid Echahed, and Martin Strecker. Ensuring correctness of model transformations while remaining decidable. In Theoretical Aspects of Computing - ICTAC 2016 - 13th International Colloquium, Taipei, Taiwan, ROC, October 24-31, 2016, Proceedings, pages 315–332, 2016.
6. Jon Haël Brenas, Rachid Echahed, and Martin Strecker. On the closure of description logics under substitutions. In Proceedings of the 29th International Workshop on Description Logics, Cape Town, South Africa, April 22-25, 2016., 2016.
7. Jon Haël Brenas, Rachid Echahed, and Martin Strecker. Proving correctness of logically decorated graph rewriting systems. In 1st International Conference on Formal Structures for Computation and Deduction, FSCD 2016, June 22-26, 2016, Porto, Portugal, pages 14:1–14:15, 2016.
8. Egon Börger, Erich Grädel, and Yuri Gurevich. The classical decision problem. Springer, 2000.
9. Ricardo Caferra, Rachid Echahed, and Nicolas Peltier. A term-graph clausal logic:Completeness and incompleteness results. *Journal of Applied Non-classical Logics*,18(4):373–411, 2008.

10. Andrea Corradini, Dominique Duval, Rachid Echahed, Frédéric Prost, and LeilaRibeiro. The pullback-pushout approach to algebraic graph transformation. In Juan de Lara and Detlef Plump, editors, *Graph Transformation - 10th International Conference, ICGT 2017, Held as Part of STAF 2017, Marburg, Germany, July 18-19, 2017, Proceedings*, volume 10373 of *Lecture Notes in Computer Science*, pages 3–19. Springer, 2017.

11. Andrea Corradini, Tobias Heindel, Frank Hermann, and Barbara König. Sesqui-pushout rewriting. In *ICGT 2006*, volume 4178 of *LNCS*, pages 30–45. Springer, 2006.

12. Bruno Courcelle. The monadic second-order logic of graphs. i. recognizable sets offinite graphs. *Inf. Comput.*, 85(1):12–75, 1990.

13. Ali Rezaei Divroodi and Linh Anh Nguyen. On bisimulations for description logics.*Information Sciences*, 295:465 – 493, 2015.

14. Rachid Echahed. Inductively sequential term-graph rewrite systems. In *4th International Conference on Graph Transformations, ICGT*, volume 5214 of *Lecture Notes in Computer Science*, pages 84–98. Springer, 2008.

15. Amir Hossein Ghamarian, Maarten de Mol, Arend Rensink, Eduardo Zambon, andMaria Zimakova. Modelling and analysis using GROOVE. *STTT*, 14(1):15–40, 2012.

16. Erich Gradel, Martin Otto, and Eric Rosen. Two-Variable Logic with Counting isDecidable. In *Proceedings of 12th IEEE Symposium on Logic in Computer ScienceLICS ‘97*, Warschau, 1997.

17. Annegret Habel and Karl-Heinz Pennemann. Correctness of high-level transformation systems relative to nested conditions. *Mathematical Structures in Computer Science*, 19(2):245–296, 2009.

18. C. A. R. Hoare. An axiomatic basis for computer programming. *Commun. ACM*,12(10):576–580, 1969.

19. Daniel Jackson. *Software Abstractions*. MIT Press, 2011.

20. Ruzica Piskac, Leonardo Mendonça de Moura, and Nikolaj Bjørner. Deciding effectivelly propositional logic using DPLL and substitution sets. *J. Autom. Reasoning*,44(4):401–424, 2010.

21. Christopher M. Poskitt and Detlef Plump. A hoare calculus for graph programs. In *Procs. of ICGT 2010*, pages 139–154, 2010.

22. Christopher M. Poskitt and Detlef Plump. Verifying monadic second-order properties of graph programs. In *Procs. of ICGT 2014*, pages 33–48, 2014.

23. Oszkár Semeráth, Ágnes Barta, Zoltán Szatmári, Ákos Horváth, and Dániel Varró.Formal validation of domain-specific languages with derived features and well-formedness constraints. *International Journal on Software and Systems Modeling*, 07/2015 2015.

24. Dániel Varró. Automated formal verification of visual modeling languages by modelchecking. *Software and System Modeling*, 3(2):85–113, 2004.