Generalized coherent-squeezed-state expansion for the quantum Rabi model

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We develop a systematic variational coherent-squeezed-state expansion for the ground state of the quantum Rabi model, which includes an additional squeezing effect with comparisons to previous coherent-state approach. For finite large ratio between the atomic and field frequency, the essential feature of the ground-state wave function in the super-radiant phase appears, which has a structure of two delocalized wake packets. The single-peaked wave function with one coherent-squeezed state works well even around the critical regime, exhibiting the advantage over the coherent-state method. As the coupling increases to form strong correlations physics in the vicinity of phase transition, we develop an improved wave function with a structure of two Gaussian wave packets, which is a linear superposition of two coherent-squeezed state. The ground-state energy and the average photon number agree well with numerical ones even in the strong-correlated regimes, exhibiting a substantial improvement over the coherent-state expansion. The advantage of the coherent-squeezed-state expansion lies in the inclusion of the second coherent-squeezed state and the additional squeezed deformation of the wave function, providing a useful tool for multi-modes spin-boson coupling systems of greater complexity.

I. INTRODUCTION

The quantum Rabi model describes the interaction of a two-level atom with a single mode of the quantized electromagnetic field, which is the simplest mode describe the interaction between spin systems and boson systems. It has a wide application in cavity and circuit quantum electrodynamics (QED). Recently, an emergence of quantum phase transition is found in the quantum Rabi model when a ratio of the atomic transition frequency \( \Delta \) to the field frequency \( \omega \) approaches infinity, \( \Delta/\omega \to \infty \). This kind of complex behavior is challenged to be captured by an accurate wave function when the coupling to the field becomes sufficiently strong correlations physics.

Despite the fact that the quantum Rabi model has been solved exactly by a Bargmann space technique [11, 13] and an coherent state method [14], respectively, where a numerical search for the zeros of complicated transcendental functions is needed, an accurate expression of the wave function remains elusive. Much works have developed an expansion of the wave function of the field into a set of coherent states, such as a generalized rotating-wave approximation (GRWA) [15], a generalized variational method [16, 17], and adiabatic approximation [18, 19]. As previously studied, the oscillator state was considered as a displaced state, and the squeezing effect of the oscillator part is underestimated. With the consideration of the squeezing deformation of the wave function, we have developed the generalized squeezing rotating-wave approximation (GSRWA) with the coherent-squeezed state to solve the quantum Rabi analytically, exhibiting an improvement over coherent state [20, 21]. The physical role of the squeezing effect in the coherent-squeezed state can not be overlooked in the intermediate coupling regime. The purpose of the paper is to develop an alternative to a reasonable wave function in the challenging regime of strong coupling to the field. We shall focus on the problem of the strong-correlated ground state of the quantum Rabi model, for which a super-radiant phase transition occurs in the limit \( \Delta/\omega \to \infty \). A superposition of deforming polaron and antipolaron has been used to explain the ground-state physics of the single- and multi-modes spin-boson model [22, 24].

We develop a coherent-squeezed expansion to capture the strong-correlated physics of the quantum Rabi model for finite large value of \( \Delta/\omega \). We first analyze the structure of the displacements and squeezing deformations at strong coupling by exact diagonalization, showing occurrence of two Gaussian wave packets for large value of \( \Delta/\omega \). The proliferation of the second Gaussian wave packet at increasing coupling is associated with spin tunneling in the vicinity of the super-radiant phase transition. A second aspect of our study will be to emphasize the improvement of the coherent-squeezed state by including the squeezing deformations over previous coherent state especially in the strong-correlated regime. The optimal displacements and squeezing parameters are determined variationally. The validity of our improved coherent-squeezed expansion is discussed by comparing with the coherent-state expansion as well as numerical exact diagonalization, especially around the phase transition regimes.

The paper is organized in the following. We explore the structure of the wave function in Sec. II by exact diagonalization. In Sec. III, we begin to study the single-peak wave function with one coherent-squeezed state, and compare with results obtained by one coherent state. Furthermore, we develop the two coherent-squeezed expansion to study the ground state for large value of \( \Delta/\omega \).
In Sec. V, we study the first-order phase transition of the anisotropic Rabi model by the two coherent-squeezed state with parity symmetry. We conclude the paper with a brief discussion.

II. DEFORMED WAVEFUNCTION

The Hamiltonian of anisotropic Rabi model is
\[ H = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + g(a^\dagger \sigma_z + a \sigma_z^+) + g\tau(a^\dagger \sigma_+ + a \sigma_-), \]
where \( \sigma_i \) (i = x, y, z) are the Pauli matrices with the transition frequency \( \Delta \), \( a^\dagger(a) \) is the creation (annihilation) operator of photon with frequency \( \omega \), and \( g \) is the coupling strength of rotating-wave (RW) interactions. In the paper \( \omega \) is set to be 1. Here the parameter \( \tau \) adjusts the relative weight between the RW and counting-rotating-wave (CRW) interactions. The isotropic Rabi model corresponds to \( \tau = 1 \).

The quantum Rabi model has a discrete \( Z_2 \) symmetry associated with parity operator \( P = e^{i\pi N} \), where the excitation number \( N = a^\dagger a + \sigma_z/2 + 1/2 \) counts the total number of excitation quanta. The parity operator satisfies \([H, P] = 0\) and possesses two eigenvalues \( p = \pm 1 \) depending on whether the number of quanta is even or odd. In this paper, we use the scaled coupling strength \( \lambda = (1+\tau)g/\sqrt{\Delta g} \). In the thermodynamic limit \( \Delta \to \infty \), there occurs the super-radiant phase transition at the critical coupling strength \( g_c = \sqrt{\Delta g}/(1 + \tau) \) [9, 10].

It is interesting to explore the wave function in the position representation, especially for the ground state in the vicinity of the quantum phase transition, which can capture the deformation of the harmonic oscillator. The ground state obtained form numerical diagonalization is of the form
\[ |\varphi\rangle = \sum_{n} |n\rangle (c_{n+}|+x\rangle + c_{n-}|-x\rangle), \]
where \( N_{fr} \) is the truncated boson number in the Fock space, \( c_{n\pm} \) are coefficients, and \( |\pm x\rangle \) are the eigenstates of \( \sigma_x \). In the position \( x \) representation, the oscillator state \( |n\rangle \) is the usual harmonic oscillator wave function
\[ \langle x|n\rangle = (\sqrt{\omega}/\sqrt{\pi}2^n n!)^{-1/2} e^{-wx^2/2} H_n(\sqrt{\omega}x), \]
where \( H_n(\sqrt{\omega}x) \) are Hermite polynomials of order \( n \). Consequently, the wavefunction in the \( x \) representation is
\[ |\varphi\rangle = \phi_{+x}|+x\rangle + \phi_{-x}|-x\rangle, \]
with \( \phi_{\pm x} = \sum_{n} c_{n\pm}\langle x|n\rangle \). In the case of \( \Delta = 0 \), the oscillators displace in two directions by projecting onto \( |\pm x\rangle \). It leads to doubly degenerate coherent state for the displaced oscillators. For \( \Delta \neq 0 \), a competition between spin tunneling and oscillator displacement energy affect the oscillator state \( \phi_{\pm x} \). For \( \Delta/\omega = 1 \) in Fig. 1(a), as the scaled coupling \( \lambda \) increases, the wave function \( \phi_{\pm x} \) in the \( |\pm x\rangle \) projection always has a single peak with a displacement. Extending the argument to large ratio \( \Delta/\omega = 100 \), the single-peaked wave function stretches and then splits into two as the coupling strength increases through the critical value \( \lambda_c = 1 \) in Fig. 1(b). On further increasing the coupling \( \lambda = 1.5 \), \( \phi_{\pm x} \) moves away from each other and the second peak disappears.

The key observation is the two-peaked wavefunction above \( \lambda_c \) as the ratio \( \Delta/\omega \) tends infinity, which can play the role of a thermodynamic limit and a super-radiant phase occurs [9, 10]. Around the critical coupling \( \lambda_c \), the wave functions \( \phi_{\pm x} \) become delocalized, which is a superposition of two Gaussian [22] or of two polarons [23, 24]. It ascribes to the strong competition between spin tunneling and oscillator displacement, which can not be fulfilled by a single-peak wave function. The wave packets are consistent with the delocalized wave function in the super-radiant phase of the Dicke model [23, 26]. The essential feature of the wave function in the super-radiant phase in the infinite ratio \( \Delta/\omega \to \infty \) appears for large ratio value, \( \Delta/\omega = 100 \). For finite \( \Delta/\omega \), the system obeys the parity symmetry in the super-radiant phase and thus has two lobes of the wave function, which is different from the discrepancy in the infinite limit \( \Delta/\omega \to \infty \).

Moreover, it is observed that the wave packets not only shift the displacement, but also have additional squeezing deformations. However, in previous studies associated with coherent states [13, 17], the squeezing effects on the wave function of the oscillators is overlooked. Recently, analytical solutions with an coherent-squeezed state to the quantum Rabi model has been developed [20, 21], which exhibits an improvement over coherent-state approaches. For finite large ratio of \( \Delta/\omega \), whether the coherent-squeezed state or the coherent state can capture the wave functions around the critical region \( \lambda = 1 \) remains unclear, where the system becomes strong correlated.


III. SINGLE COHERENT-SQUEEZED STATE ANSATZ

For the scaled coupling strength $\lambda < \lambda_c$, the single-peaked wave function works well in the ground state for the normal phase in Fig. 1(b). In previous studies, coherent-state ansatz for the quantum Rabi model has been employed to capture the oscillator state for small ratio of $\Delta/\omega$ and limiting coupling regimes [13–17]. As an improvement, we consider the additional squeezing effects of the oscillator by comparing with the coherent state. The single-peaked wave function can be expressed by one coherent-squeezed state (1CSS) as

$$\psi_{1CSS}^+ = C_1 |+x\rangle \otimes |f^{(1)}\rangle + C_2 |-x\rangle \otimes |f^{(2)}\rangle,$$  \hspace{1cm} (5)

where the oscillator state is the coherent-squeezed state

$$|f^{(k)}\rangle = U^\dagger(\beta_k)S^\dagger(\xi_k)\ket{0}.$$  \hspace{1cm} (6)

Here, the coherent operator is $U(\beta_k) = e^{\beta_k(a^\dagger - a)}$ with the variational displacements $\beta_k$, and the squeezing operator is $S(\xi_k) = e^{\xi_k(a^2 - a^2)}$ with the squeezing parameters $\xi_k$. Since the ground state is associated with the even parity, satisfying $P \psi_{1CSS} = \psi_{1CSS}$, the variational parameters are constrained as $C_1 = -C_2 = 1/\sqrt{2}$, $\beta_1 = -\beta_2 = \beta$, and $\xi_1 = \xi_2 = \xi$. This more flexible ansatz facilitates us to obtain the variables $\beta$ and $\xi$. Especially, by setting the squeezing parameter $\xi = 0$, the one coherent-squeezed state $\psi_{1CSS}$ reduces to one coherent state (1CS) $|\psi_{1CS}\rangle$ with the coherent state $|f^{(k)}\rangle = U^\dagger(\beta_k)|0\rangle$.

The Hamiltonian of the isotropic and anisotropic quantum Rabi model is rewritten as

$$H = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + \alpha (a^\dagger + a) \sigma_x + \gamma (a^\dagger - a) i \sigma_y.$$  \hspace{1cm} (7)

where parameters $\alpha = g(1 + \tau)/2$ and $\gamma = g(\tau - 1)/2$. With the one coherent-squeezed state $\psi_{1CSS}$ in Eq. (5), the corresponding energy is given by

$$E_g^{CSS} = \omega (\sinh^2 2\xi + \beta^2) - 2\beta \alpha - \frac{\Delta}{2} + 2\gamma \eta^2 - 2\beta^2 \eta^2,$$  \hspace{1cm} (8)

where $\eta = \cosh(2\xi) - \sinh(2\xi)$. The variational energy is minimized according to $\partial E_{1g}/\partial \beta = 0$ and $\partial E_{1g}/\partial \xi = 0$, respectively. Hence, the displacement $\beta$ and squeezing parameter $\xi$ are variationally determined. One obtain the equations for the isotropic case $\tau = 1$

$$\omega (e^{4\xi} - e^{-4\xi}) - 4\Delta \beta^2 e^{-4\xi} - 2\beta^2 \eta^2 = 0,$$  \hspace{1cm} (9)

$$\omega \beta - g + \Delta \beta e^{-4\xi} - 2\beta^2 \eta^2 = 0.$$  \hspace{1cm} (10)

In the limit $\Delta/\omega \rightarrow \infty$, the values of $\beta$ and $\xi$ are given approximately

$$\beta \simeq \frac{g}{\Delta}.$$  \hspace{1cm} (11)

and

$$\xi = \frac{1}{8} \ln(1 + \frac{4g^2}{\omega \Delta}).$$  \hspace{1cm} (12)

We observe that the squeezing influence $\xi$ plays a more crucial role as the coupling strength approaches the critical value $\lambda_c = 1$, for which the atom becomes strongly correlated with the oscillator.

Using the single-peaked wave function $|\psi_{1CSS}\rangle$, the mean photon number can be expressed as

$$\langle a^\dagger a \rangle = \sinh^2 2\xi + \beta^2.$$  \hspace{1cm} (13)

For large ratio of $\Delta/\omega = 100$, the one coherent-squeezed state $|\psi_{1CSS}\rangle$ works relatively well for the ground state by compared with one coherent state $|\psi_{1CS}\rangle$ in Fig. 2. The ground-state energy $E_g$ and the mean photon number for the isotropic $\tau = 1$ and anisotropic $\tau = 1.5$ cases agree well with numerics for the scaled coupling $\lambda < 1$, but the results of the single coherent state $|\psi_{1CS}\rangle$ get worse as $\lambda$ increases to the critical value 1. The success of the coherent-squeezed state comes from the squeezing effects especially as the coupling increases to the strong-correlated regime. However, the single-peaked state $|\psi_{1CSS}\rangle$ presents deficiencies for $\lambda > 1$, where the competition between the spin tunneling and the oscillator displacement becomes strong.

IV. TWO COHERENT-SQUEEZED STATE ANSATZ

As the coupling strength exceeds the critical value $\lambda > \lambda_c$, it becomes a strongly-correlated system and two Gaussian of the wave function emerges in Fig. 1(b). As an improvement, a two-peaked state $|\psi_{2CSS}\rangle$ is proposed by a linear combination of two coherent-squeezed state (2CSS)

$$|\psi_{2CSS}^+\rangle = |+x\rangle \otimes (C_1 |+f^{(1)}\rangle + C_2 |+f^{(2)}\rangle) - |-x\rangle \otimes (C_1 |-f^{(1)}\rangle + C_2 |-f^{(2)}\rangle),$$  \hspace{1cm} (14)

where the coherent-squeezed state $|f^{(k)}\rangle = e^{\pm \beta_k(a^\dagger - a)} e^{\xi_k(a^2 - a^2)}/0\rangle$. The symmetry of the even parity enforces the chosen relative sign between the +x components of the state in Eq. (14). Since the parity breaks at the critical point in the limit $\Delta/\omega \rightarrow \infty$, there the ground state remains in the even parity due to finite large value of $\Delta/\omega = 100$. Variables $\beta_1(2)$ and $\xi$ in the coherent-squeezed states are taken as free parameters and can be determined by minimizing the energy $E_g^{2CSS} = \langle \psi_{2CSS}^+ \rangle H \psi_{2CSS}^+ \rangle$ in the Appendix. Meanwhile, the two-coherent state (2CS) $|\psi_{2CS}\rangle$ can be given by the coherent state $|\pm f^{(k)}\rangle = e^{\mp \beta_k(a^\dagger - a)} |0\rangle$ instead of the coherent-squeezed state in Eq. (14).

Fig. 2 shows the 2CSS ansatz including the addition of a second coherent-squeezed state dramatically improves
FIG. 2: Ground-state energy $E_g/(\Delta \omega)$ and mean photon number $\langle a^\dagger a \rangle$ for the single coherent-squeezed state (1CSS) (blue solid line) and two coherent-squeezed state (2CSS) (red solid line) as a function of the rescaled coupling strength $\lambda$ for the isotropic $\tau = 1$ (a) (c) and anisotropic $\tau = 1.5$ Rabi model (b) (d) with the large detuning $\Delta/\omega = 100$. The results obtained by the single coherent state (1CS) (blue dashed line) and two coherent state (2CS) (red dashed line) are listed for comparison. The inset shows the results from $\lambda = 0.9$ to 1.1.

FIG. 3: Displacement Variables $\beta_1$ (red solid line) and $\beta_2$ (black dashed line) (a), the coefficients $C_1$ (red solid line) and $C_2$ (black dashed line) (b), and the squeezing $\xi$ (c) for the two coherent-squeezed state (2CSS) for the isotropic Rabi model $\tau = 1$.

the ground-state energy and mean photons, which are in very good agreement with numerical results both for the isotropic and anisotropic case. Since the two-peaked wave function $|\psi_{2CS}\rangle$ correctly predicts an enhancement of the spin tunneling around the critical value $\lambda = 1$ in comparison to the single-peaked state. However, the results of two coherent state $|\psi_{2CS}\rangle$ remains a deviation in the vicinity of the critical coupling strength. The failure of the coherent state attributes to the ignorance of the squeezing variance of the wave functions.

The corresponding variables $\beta_{1(2)}$, $C_{1(2)}$ and squeezing parameter $\xi$ are calculated by minimizing the ground-state energy $E_g^{2CSS}$ in Fig. 3. In the weak coupling regime $\lambda < 1$, coefficient $C_1$ approaches 1 and $C_2$ tends to 0, which corresponds to single-peaked wave function. As coupling exceeds the critical value 1, $C_2$ increases dramatically and the additional second coherent-squeezed state plays an important effect. It corresponds to the occurrence of the second peak of the delocalized wave function for $\lambda > 1$ in Fig. 3(b). Then, the probability of the second peak decreases with further increasement of $\lambda$, which agrees well with vanishing of the second peaks for $\lambda = 1.5$ in Fig. 3(b). Meanwhile, the displacements of two coherent-squeezed state $\beta_1$ and $\beta_2$ become larger as the coupling strength exceeds the critical value $\lambda > 1$. It reveals that two Gaussian wave packets move in the opposite directions in Fig. 3(b). Moreover, the squeezing parameter $\xi$ increases and reaches the maximum value at the critical value $\lambda = 1$. It means that the squeezing effect of the oscillator state can not be overlooked especially in the critical regime. This motivates the inclusion of additional coherent-squeezed state in the ground-state ansatz, which further captures the spin tunneling in the strong-correlated regimes and squeezing influence of the wave functions.

V. FIRST-ORDER QUANTUM PHASE TRANSITION IN THE ANISOTROPIC RABI MODEL

Besides the second-order super-radiant phase transition, a additional first-order phase transition occurs at the critical value $\varrho^{(1)} = \sqrt{\Delta \omega/(1 - \tau^2)}$ in the anisotropic Rabi model [12, 13]. When the atom-photon RW coupling strength becomes stronger than that of the CRW terms, e.g. $\tau < 1$, there exists energy-level crossing between the ground state and the first-excited state. A natural question follows as to the ground-state wave function in the first-order phase transition of the anisotropic quantum Rabi model. As the coupling strength exceeds the critical value $\varrho^{(1)}$, the even parity symmetry of the ground state breaks down and the first-excited state with the odd parity becomes to the lowest-energy state. It is reasonable to describe the first-excited state by the two coherent-squeezed state

$$|\psi_{2CSS}^-(\xi)\rangle = |+x\rangle \otimes [C_1 f^{(1)} + C_2 f^{(2)}] + |-x\rangle \otimes [C_1 f^{(1)} + C_2 f^{(2)}],$$

where the coefficients and the coherent-squeezed state $|\pm f\rangle$ for the spin states $|+x\rangle$ satisfy the odd parity. With its comparison to the ground state $|\psi_{2CSS}^+\rangle$ with the even parity symmetry, $|\psi_{2CSS}^-\rangle$ changes the sign of the wave function for the $|-x\rangle$ part due to the odd parity.

Using two coherent-squeezed state $|\psi_{2CSS}^\pm\rangle$, the ground-state and first-excited-state energy can be ob-
tained as $E^\pm = \langle \psi_{CSS}^\pm | H^{Rabi} | \psi_{CSS}^\pm \rangle$ (see Appendix). The variational displacement, squeezing and coefficients can be determined by minimizing the energy $E^\pm$. Energy levels $E^\pm$ for the ground state $| \psi_{CSS}^+ \rangle$ and the first-excited state $| \psi_{CSS}^- \rangle$ agree well with the numerical ones in Fig. 4(a). Energy-levels crossing is captured at the critical value $g_c^{(1)}$ by two coherent-squeezed state, exhibiting the accuracy of our methods. Moreover, the average photons $\langle a^\dagger a \rangle$ in the ground state shows a discontinuous transition in Fig. 4(b), exhibiting the first-order phase transition. The success of our method lies in the inclusion of the second coherent-squeezed state and the additional squeezing effect of the wave functions.

\[ E_{atom} = \langle \psi_{CSS} | \frac{\Delta}{2} \sigma_z | \psi_{CSS} \rangle = -\Delta [C_1^2 \langle +f^{(1)} | - f^{(1)} \rangle + 2C_1C_2 \langle +f^{(1)} | - f^{(2)} \rangle + C_2^2 \langle +f^{(2)} | - f^{(2)} \rangle], \]

\[ E^{ph} = \langle \psi_{CSS} | \omega a^\dagger a | \psi_{CSS} \rangle = \omega [2C_1^2 \beta_1^2 + 2C_2^2 \beta_2^2 + 2C_1C_2 \beta_1 \beta_2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)] + \omega \sinh^2 2\lambda [2C_1^2 + 2C_2^2 + C_1^2 e^{-2\beta_1^2 \eta^2} + C_2^2 e^{-2\beta_2^2 \eta^2} + 2C_1C_2 [1 - \eta^2 (\beta_1 - \beta_2)^2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)] + \omega \sinh 2\lambda [2C_1C_2 \eta (\beta_1 - \beta_2)^2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)], \]

\[ E^{iso-int} = \langle \psi_{CSS} | \alpha (a^\dagger + a) \sigma_z | \psi_{CSS} \rangle = -2\alpha [C_1^2 \beta_1 + C_2^2 \beta_2 + C_1C_2 (\beta_1 + \beta_2) (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)], \]

\[ E^{uni-int} = \langle \psi_{CSS} | \gamma (a^\dagger - a) \sigma_y | \psi_{CSS} \rangle = -4\gamma^2 [C_1^2 \beta_1 (\langle +f^{(1)} | - f^{(1)} \rangle + C_1C_2 (\langle +f^{(1)} | - f^{(2)} \rangle (\beta_1 + \beta_2) + C_2^2 (\beta_2 + f^{(2)} | - f^{(2)} \rangle)]. \]

where the overlap of the coherent-squeezed state is

\[ \langle +f^{(k)} | \pm f^{(k')} \rangle = \langle -f^{(k)} | \mp f^{(k')} \rangle = e^{-\eta^2 (\beta_1 \mp \beta_2)^2}. \]

VI. CONCLUSIONS

We have developed a coherent-squeezed-state expansion for the isotropic and anisotropic quantum Rabi model, which includes the squeezing effect of the oscillator state and shows an improvement over previous coherent-state methods. The improved coherent-squeezed state allows squeezing distortion of the wave packets as well as displacement. Excellent agreement is found with numerical ones. We have constructed a simple physical picture in terms of displacements and squeezing influence of the wave function in the vicinity of the phase transition, which turns out to be validity of coherent-squeezed-state expansion.

VII. ACKNOWLEDGMENTS

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Appendix A: Energy of the two-coherent-squeezed state

Using the two coherent-squeezed state with even parity $| \psi_{CSS}^\pm \rangle$ in Eq. (14), the energy can be written as $E^+ = E^{atom} + E^{ph} + E^{iso-int} + E^{uni-int}$, consisting of

\[ \langle \psi_{CSS} | \frac{\Delta}{2} \sigma_z | \psi_{CSS} \rangle = \langle \psi_{CSS} | \omega a^\dagger a | \psi_{CSS} \rangle = \omega [2C_1^2 \beta_1^2 + 2C_2^2 \beta_2^2 + 2C_1C_2 \beta_1 \beta_2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)] + \omega \sinh^2 2\lambda [2C_1^2 + 2C_2^2 + C_1^2 e^{-2\beta_1^2 \eta^2} + C_2^2 e^{-2\beta_2^2 \eta^2} + 2C_1C_2 [1 - \eta^2 (\beta_1 - \beta_2)^2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)] + \omega \sinh 2\lambda [2C_1C_2 \eta (\beta_1 - \beta_2)^2 (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)], \]

\[ \langle \psi_{CSS} | \alpha (a^\dagger + a) \sigma_z | \psi_{CSS} \rangle = -2\alpha [C_1^2 \beta_1 + C_2^2 \beta_2 + C_1C_2 (\beta_1 + \beta_2) (\langle +f^{(1)} | + f^{(2)} \rangle + \langle -f^{(1)} | - f^{(2)} \rangle)], \]

\[ \langle \psi_{CSS} | \gamma (a^\dagger - a) \sigma_y | \psi_{CSS} \rangle = -4\gamma^2 [C_1^2 \beta_1 (\langle +f^{(1)} | - f^{(1)} \rangle + C_1C_2 (\langle +f^{(1)} | - f^{(2)} \rangle (\beta_1 + \beta_2) + C_2^2 (\beta_2 + f^{(2)} | - f^{(2)} \rangle)]. \]
ity $|\psi_{2^{CSS}}\rangle$ in Eq. (15), the energy is similarly obtained as $E^{-} = -E_{atom} + E_{ph} + E_{iso-int} - E_{uni-int}$.

[1] I. I. Rabi, Phys. Rev. 51, 652 (1937).
[2] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) 464, 1301 (2010).
[3] D. Nagy, G. Kónya, G. Szirmai, and P. Domokos, Phys. Rev. Lett. 104, 130401 (2010).
[4] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A 75, 013804 (2007).
[5] A. Wallraff et al., Nature (London) 431, 162 (2004).
[6] T. Niemczyk et al., Nature Physics 6, 772 (2010).
[7] P. Foro-Díaz et al., Phys. Rev. Lett. 105, 237001 (2010).
[8] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito and K. Semba, Nat. Phys. 13, 44 (2017).
[9] M. J. Hwang, R. Puebla, and M. B. Plenio, Phys. Rev. Lett. 115, 180404 (2015).
[10] M. X. Liu, S. Chesi, Z. J. Ying, X. S. Chen, H. G. Luo, and H. Q. Lin, Phys. Rev. Lett. 119, 220601 (2017).
[11] D. Braak, Phys. Rev. Lett. 107, 100401 (2011).
[12] L. T. Shen, Z. B. Yang, H. Z. Wu, and S. B. Zheng, Phys. Rev. A 95, 013819 (2017).
[13] Q. T. Xie, S. Cui, J. P. Cao, L. Amico and H. Fan, Phys. Rev. X 6, 021046 (2014).
[14] Q. H. Chen, C. Wang, S. He, T. Liu, and K. L. Wang, Phys. Rev. A 86, 023822 (2012).
[15] E. K. Irish, Phys. Rev. Lett. 99, 173601 (2007).
[16] C. J. Gan, and H. Zheng, Eur. Phys. J. D 59, 473 (2010).
[17] Y. Zhang, G. Chen, L. Yu, Q. Liang, J.-Q. Liang, and S. Jia, Phys. Rev. A 83, 065802 (2011).
[18] S. Agarwal, S. M. Hashemi Rafsanjani, and J. H. Eberly, Phys. Rev. A 85, 043815 (2012).
[19] S. Ashhab, Phys. Rev. A 87, 013826 (2013).
[20] Y. Y. Zhang Phys. Rev. A 94, 063824 (2016).
[21] Y. Y. Zhang and X. Y. Chen, Phys. Rev. A 96, 063821 (2017).
[22] H. B. Shore, and L. M. Sander, Phys. Rev. B 7, 10 (1973).
[23] Z. J. Ying, M. X. Liu, H. G. Luo, H. Q. Lin, and J. Q. You, Phys. Rev. A 92, 053823 (2015).
[24] S. Bera, S. Florens, H. U. Baranger, N. Roch, A. Nazir, and A. W. Chin, Phys. Rev B 89, 121108 (R) (2014).
[25] T. Liu, Y. Y. Zhang, Q. H. Chen, and K. L. Wang, Phys. Rev. A 80, 023810 (2009).
[26] C. Emary, and T. Brandes, Phys. Rev. E 67, 066203 (2003).
[27] D. Tolkunov and D. Solenov, Phys. Rev. B 75, 024402 (2007).
[28] A. Alvermann and H. Fehske, Phys. Rev. Lett 102, 150601 (2009).