REGULARITY AND SPECTRAL METHODS FOR TWO-SIDED FRACTIONAL DIFFUSION EQUATIONS WITH A LOW-ORDER TERM

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Abstract. We study regularity and numerical methods for two-sided fractional diffusion equations with a lower-order term. We show that the regularity of the solution in weighted Sobolev spaces can be greatly improved compared to that in standard Sobolev spaces. With this regularity, we improve higher-order convergence of a spectral Galerkin method. We present a spectral Petrov-Galerkin method and provide an optimal error estimate for the Petrov-Galerkin method. Numerical results are presented to verify our theoretical convergence orders. Regularity and Pseudo eigenfunctions and Non-uniformly weighted Sobolev spaces and Spectral methods and Optimal error estimates and Riemann-Liouville fractional operators

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1. Introduction

Anomalous diffusion has been widely used to investigate transport dynamics in complex systems, such as underground environmental problem [10], fluid flow in porous materials [4], anomalous transport in biology [11], etc. Many mathematical models are developed to study anomalous diffusion. Some of these models are based on a linear equation for diffusion on fractals [24], a linear differential Fisher’s information theory [29] and Levy description of anomalous diffusion in dynamical systems[14]. In particular, fractional differential equations (FDEs) can serve as an accurate model of the anomalous diffusion, e.g. super-diffusion process in [23].

Explicit solutions for FDEs are mostly not available and thus it is essential to develop efficient numerical methods. Extensive numerical methods have been investigated in recent decades e.g. finite difference method [9, 17, 22, 28, 31], finite element method [7, 6, 13, 32], spectral method [5, 7, 16, 20, 12, 34, 35], discontinuous Galerkin method [30], finite volume method [27] etc.

Despite of rich numerical methods for FDEs, regularity of solutions to FDEs is not thoroughly investigated, especially the regularity well suited for error analysis. In literature, it is assumed that solutions are sufficiently smooth. However, it has been pointed out in [5, 13, 15, 32] that the regularity of solutions to FDEs can be very low.

In this paper, we consider the following two-sided fractional diffusion equation with a reaction term

$$\mathcal{L}_\theta^\alpha u + \mu u = f(x), \quad x \in I = (a, b),$$  \hspace{1cm} (1.1)

with homogeneous boundary conditions

$$u(a) = u(b) = 0,$$ \hspace{1cm} (1.2)
and a given function $f(x)$, $L_\theta := -[\theta \, a D_x^\alpha + (1 - \theta) \, x D_x^\alpha]$ with $\theta \in [0, 1]$ and $\alpha \in (1, 2)$. Here $a D_x^\alpha$ and $x D_x^\alpha$ are left- and right-sided Riemann-Liouville operators, defined as follows (see e.g. [25, 26])

$$a D_x^\alpha u(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_a^x \frac{u(\xi)}{(x - \xi)^{\alpha-1}} d\xi, \quad x > a, \quad (1.3)$$

and

$$x D_x^\alpha u(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_x^b \frac{u(\xi)}{(\xi - x)^{\alpha-1}} d\xi, \quad x < b. \quad (1.4)$$

Ervin and Roop [6] established uniqueness and existence of the solution to (1.1). Ervin et al. [7] discussed the regularity of the solution in standard Sobolev space for $\mu = 0$ in (1.1).

This work devotes to the study of regularity and spectral methods for (1.1). From the weakly singular kernel of fractional derivatives, solutions of fractional differential equations naturally inherit weak singularity. According to [7], the solution to (1.1) when $\mu = 0$ can be written as the product $(1 - x)^\gamma (1 + x)^\beta \tilde{u}$, $\gamma$ and $\beta$ are some constants depending on $\alpha$ and $\theta$ in (1.1). Based on the relation in Lemma 2.1, we expand the function $\tilde{u}$ using Jacobi polynomials and apply Fourier-type analysis of $\tilde{u}$. For $\tilde{u}$, we show that the solution has a limited regularity, more precisely $2\alpha + 1$ in non-uniformly weighted Sobolev space even for smooth $f$. This work is motivated by [36] which gave the regularity for the fractional diffusion equation with fractional Laplacian by Fourier analysis and bootstrapping technique. We also notice that in [21], a detailed error estimate for a Petrov-Galerkin method is given when $\mu = 0$. All these works can be considered as special cases in the current work.

We present a spectral Petrov-Galerkin method which is discussed in [7, 21] where $\mu = 0$ and also a spectral Galerkin method. Based on the obtained regularity, we prove error estimates of these methods. For the spectral Galerkin method, an optimal error estimate is obtained in the case $\theta = 0.5$, similar to that in [36]. We get optimal error estimate for the spectral Petrov-Galerkin method when $\mu$ is small; see Theorem 5.3.

The rest of this paper is arranged as follows. In Section 2, we introduce some basic notations and recall some properties of Jacobi polynomials and non-uniformly weighted Sobolev spaces. Some lengthy but important auxiliary materials are presented in Appendix. In Section 3, we present the regularity of the two-sided fractional diffusion equations using Fourier type analysis and a bootstrapping technique. We consider a spectral Galerkin method and carry out its error analysis in Section 4. In Section 5, we present a Petrov-Galerkin method and provide error estimates as well. Several numerical results are showed to verify the theoretical convergence order in Section 6. Finally, we make some concluding remarks.

## 2. Preliminary

We consider the interval $I = (-1, 1)$ for simplicity. Denote by $L^2_{\omega^{\gamma, \beta}, I}(I)$ the space with the inner product and norm defined by

$$(u, v)_{\omega^{\gamma, \beta}, I} = \int_I uv \omega^{\gamma, \beta} dx, \quad \|u\|_{\omega^{\gamma, \beta}, I} = (u, u)_{\omega^{\gamma, \beta}, I}^{\frac{1}{2}},$$

where $\omega^{\gamma, \beta} = (1-x)^\gamma (1+x)^\beta$, $\gamma, \beta > -1$. When $\gamma = \beta = 0$, we will drop $\omega$ from the above notations. We also drop the domain $I$ from the notation for simplicity without incurring confusion.
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The Jacobi polynomials \( P_n^{\gamma,\beta}(x) \) are mutually orthogonal: for \( \gamma, \beta > -1 \),
\[
\int_{-1}^{1} (1 - x)^\gamma (1 + x)^\beta P_m^{\gamma,\beta}(x) P_n^{\gamma,\beta}(x) \, dx = h_n^{\gamma,\beta} \delta_{nm}. \tag{2.1}
\]
Here \( \delta_{nm} \) is equal to 1 if \( n = m \) and zero otherwise, and
\[
h_n^{\gamma,\beta} = \| P_n^{\gamma,\beta} \|_{\omega^{\gamma,\beta}}^2 = \frac{2^{\gamma+\beta+1}}{2n+\gamma+\beta+1} \times \frac{\Gamma(n+\gamma+1)\Gamma(n+\beta+1)}{\Gamma(n+\gamma+\beta+1)n!}. \tag{2.2}
\]

To incorporate singularities at the endpoints, we introduce the following non-uniformly weighted Sobolev space, see e.g. [8],
\[
B_m^{\omega^{\gamma,\beta}} := \{ u \mid \partial^k_x u \in L_2^{\omega^{\gamma+k,\beta+k}}, k = 0, 1, \ldots, m \}, \quad m \text{ is a nonnegative integer} \tag{2.3}
\]
which is equipped with the following norm
\[
\| u \|_{B_m^{\omega^{\gamma,\beta}}} = \left( \sum_{k=0}^{m} |u|_{L_2^{\omega^{\gamma+k,\beta+k}}}^2 \right)^{1/2}, \quad |u|_{L_2^{\omega^{\gamma+k,\beta+k}}} = \| \partial^k_x u \|_{\omega^{\gamma+k,\beta+k}}. \tag{2.4}
\]
When \( m \) is not an integer, the space is defined by interpolation, see e.g. [8].

The following pseudo-eigenfunctions for the fractional diffusion operator in [7] are essential to analyze the regularity.

Lemma 2.1 ([7, 21]). For the \( n \)-th order Jacobi polynomial \( P_n^{\sigma,\sigma^*}(x) \), it holds that
\[
\mathcal{L}_0^{\sigma}[\omega^{\sigma,\sigma^*}(x)P_n^{\sigma,\sigma^*}(x)] = \lambda_{0,n}^{\sigma} P_n^{\sigma,\sigma^*}(x), \tag{2.5}
\]
\[
\mathcal{L}_0^{\sigma^{-1}}[\omega^{-1,\sigma^*}(x)] = 0, \tag{2.6}
\]
where
\[
\lambda_{0,n}^{\sigma} = \frac{\sin(\pi\alpha)}{\sin(\pi\sigma) + \sin(\pi\sigma^*)} \times \frac{\Gamma(\alpha + n + 1)}{n!},
\]
and \( \sigma^* = \alpha - \sigma \) and \( \sigma \) is determined by the following equation:
\[
\theta = \frac{\sin(\pi\sigma^*)}{\sin(\pi\sigma^*) + \sin(\pi\sigma)}. \tag{2.7}
\]

Remark 2.2. To ensure that (2.7) is uniquely solvable, we restrict \( \sigma \) and \( \sigma^* \) into the interval \((0,1]\). In particular, \( \sigma = 1 \) and \( \sigma^* = \alpha - 1 \) for \( \theta = 1 \); \( \sigma = \alpha - 1 \) and \( \sigma^* = 1 \) for \( \theta = 0 \), and \( \lambda_{0,n}^{\alpha} = \lambda_{1,n}^{\alpha} = \frac{\Gamma(\alpha+n+1)}{n!} \).

Throughout the paper, \( C \) or \( c \) denote generic constants and are independent of the truncation parameter \( N \).

3. Regularity

We present the regularity of the two-sided FDE (1.1).

Theorem 3.1 (Regularity in weighted Sobolev spaces, I). Assume that \( f \in B_{\omega^{\sigma^*},\sigma}^{r} \) and \( u \in L^\infty \).
If \( \mu = 0 \), then \( \omega^{-\sigma,-\sigma^*} u \in B_{\omega^{\sigma^*},\sigma}^{\alpha+r} \). If \( \mu \neq 0 \), then \( \omega^{-\sigma,-\sigma^*} u \in B_{\omega^{\sigma^*},\sigma}^{\alpha+r+\alpha} \). Here \( \alpha \wedge r = \min\{\alpha, r\} \).
Proof. For \( \mu = 0 \), we write \( u = \omega^{\sigma,\sigma'} \sum_{n=0}^{\infty} u_n P_n^{\sigma,\sigma'}(x) \). Then with Lemma 2.1 we have \( u_n = (\lambda_{\theta,n}^\alpha)^{-1} f_n \) from the equation \( \mathcal{L}_\theta^\alpha u = f \) and \( f = \sum_{n=0}^{\infty} f_n P_n^{\sigma,\sigma'}(x) \). Then

\[
\omega^{\sigma,\sigma'} \sum_{n=0}^{\infty} (\lambda_{\theta,n}^\alpha)^{-1} f_n P_n^{\sigma,\sigma'}(x).
\]

By (A.1) and (A.7), we have

\[
\text{Proof.}
\]

Finally, by the fact that \( \tilde{u} \) is \( \omega^{\sigma,\sigma'} \), we have

\[
\sigma = 0, \text{ if } \mu = 0, \text{ we use a bootstrapping technique to obtain higher regularity. First, we can obtain that } u = \omega^{\sigma,\sigma'} \tilde{u} \text{ and } \tilde{u} \in B^\alpha_{\omega^{\sigma,\sigma'}}. \text{ In fact, we have from } u \in L^\infty \text{ that } \tilde{f} = f - \mu u \in B^{r,\alpha}_{\omega^{\sigma,\sigma'}}.
\]

From the equation \( \mathcal{L}_\theta^\alpha u = \tilde{f} \) and using the conclusion when \( \mu = 0 \), we have \( \omega^{-\sigma,-\sigma'} u \in B^{r,\alpha}_{\omega^{\sigma,\sigma'}} \).

Second, by a direct calculation using the definition of weighted space \( B^{r,\alpha}_{\omega^{\sigma,\sigma'}} \), we have \( u \in B^{r,\alpha}_{\omega^{\sigma,\sigma'}} \).

Finally, by the fact that \( \tilde{f} = f - \mu u \in B^{r,\alpha}_{\omega^{\sigma,\sigma'}} \) and the conclusion for \( \mu = 0 \), we have \( \omega^{-\sigma,-\sigma'} u \in B^{2r,\alpha}_{\omega^{\sigma,\sigma'}} \).

\[
\text{Theorem 3.2 (Regularity in weighted Sobolev spaces, II). Assume that } f \in B^{r,\alpha}_{\omega^{\sigma-1,\sigma-1}}, \text{ with } r \geq 0. \text{ If } \mu = 0, \text{ then } \omega^{-\sigma,-\sigma'} u \in B^{r,\alpha}_{\omega^{\sigma-1,\sigma-1}}. \text{ If } \mu \neq 0, \omega^{-\sigma,-\sigma'} u \in B^{r,\alpha}_{\omega^{\sigma-1,\sigma-1}} \text{ and } \mathcal{L}_\theta^\alpha u \in B^{r,\alpha}_{\omega^{\sigma-1,\sigma-1}}.
\]

Proof. Consider first \( \mu = 0 \). From Corollary A.2, we have the following relations: for \( n \geq 0 \)

\[
P_n^{\sigma-1,\sigma'-1} = A_n^{\sigma-1,\sigma'-1} P_n^{\sigma,\sigma'} + B_n^{\sigma-1,\sigma'-1} P_n-1^{\sigma-1,\sigma'-1} + C_n^{\sigma-1,\sigma'-1} P_n^{\sigma,\sigma'}, \quad (3.3)
\]

\[
P_n^{\sigma-1,\sigma'-1} = A_n^{\sigma-1,\sigma'-1} P_n^{\sigma,\sigma'} - B_n^{\sigma-1,\sigma'-1} P_n-1^{\sigma,\sigma'} + C_n^{\sigma-1,\sigma'-1} P_n^{\sigma,\sigma'}, \quad (3.4)
\]

where \( A_n^{\sigma-1,\sigma'-1}, B_n^{\sigma-1,\sigma'-1} \) and \( C_n^{\sigma-1,\sigma'-1} \) are defined in Corollary A.2 and \( P_n^{\gamma,\beta} \equiv P_n^{-1,\beta} \equiv 0 \). Throughout the proof, to simplify the notations but without incurring confusion, we drop the superscript \( \sigma - 1,\sigma' - 1 \) for \( A_n, B_n \) and \( C_n \) and abbreviate \( \lambda_{\theta,n}^\alpha \) as \( \lambda_n \). From (3.3), we have

\[
\sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma'-1} = \sum_{n=0}^{\infty} u_n (A_n P_n^{\sigma,\sigma'} + B_n P_n^{\sigma,\sigma'} + C_n P_n^{\sigma,\sigma'})
\]

\[
= \sum_{n=0}^{\infty} (u_{n+2} A_{n+2} + u_{n+1} B_{n+1} + u_n C_n) P_n^{\sigma,\sigma'}. \quad (3.5)
\]

It follows from Lemma 2.1 that

\[
\mathcal{L}_\theta^\alpha u = \mathcal{L}_\theta^\alpha (\omega^{\sigma,\sigma'} \sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma'-1})
\]
Substituting (3.6) and (3.7) into (1.1) leads to

\[ \sum_{n=0}^{\infty} u_n (A_n \lambda_{n-2} P_n^{\sigma,2} + B_n \lambda_{n-1} P_n^{\sigma,1} + C_n \lambda_n P_n^{\sigma,\cdot}) = \sum_{n=0}^{\infty} \lambda_n (u_{n+2} A_{n+2} + u_{n+1} B_{n+1} + u_n C_n) P_n^{\sigma,\cdot}. \]  

From (3.4), we have

Thus we have

\[ \text{where } h \text{ is defined in (2.2). Notice that } |A_k| \leq C \text{ and } |B_k| \leq C/k \text{ in Corollary A.2. By Lemma A.3, we have} \]

\[ |u_n| \leq \frac{|f_n|}{\lambda_n} + C \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right) |f_{k+2}| + C \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right) \frac{1}{k} |f_{k+1}|. \]  

\[ \frac{1}{\lambda_n} (f_{n+2} A_{n+2} - f_{n+1} B_{n+1} + f_n C_n) P_n^{\sigma,\cdot}. \]  

Substituting (3.6) and (3.7) into (1.1) leads to

\[ \sum_{n=0}^{\infty} \lambda_n (u_{n+2} A_{n+2} + u_{n+1} B_{n+1} + u_n C_n) P_n^{\sigma,\cdot} = \sum_{n=0}^{\infty} (f_{n+2} A_{n+2} - f_{n+1} B_{n+1} + f_n C_n) P_n^{\sigma,\cdot}. \]  

Multiplying by \( P_n^{\sigma,\cdot} \) over both sides of the last equation (3.8) and by the orthogonality of the Jacobi polynomials, we arrive at

\[ u_{n+2} A_{n+2} + u_{n+1} B_{n+1} + u_n C_n \]

\[ = \frac{1}{\lambda_n} (f_{n+2} A_{n+2} - f_{n+1} B_{n+1} + f_n C_n) \]

\[ = F_{n+2} A_{n+2} + F_{n+1} B_{n+1} + F_n C_n + \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+2}} \right) f_{n+2} A_{n+2} \]

\[ - \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}} \right) f_{n+1} B_{n+1}, \]  

\[ \text{where } F_n = f_n/\lambda_n. \]  

Using (3.5) again, we obtain

\[ \sum_{n=0}^{\infty} u_n P_n^{\sigma-1,\sigma-1} = \sum_{n=0}^{\infty} F_n P_n^{\sigma-1,\sigma-1} + \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+2}} \right) f_{n+2} A_{n+2} P_n^{\sigma,\cdot} \]

\[ \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}} \right) f_{n+1} B_{n+1} P_n^{\sigma,\cdot}. \]  

Thus we have

\[ u_n = F_n + \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right) f_{k+2} A_{k+2} (h_n^{\sigma-1,\sigma-1})^{-1} (P_k^{\sigma,\cdot},P_n^{\sigma-1,\sigma-1})_{\omega^{\sigma-1,\sigma-1}} \]

\[ - \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right) f_{k+1} B_{k+1} (h_n^{\sigma-1,\sigma-1})^{-1} (P_k^{\sigma,\cdot},P_n^{\sigma-1,\sigma-1})_{\omega^{\sigma-1,\sigma-1}}, \]  

where \( h_n^{\sigma-1,\sigma-1} \) is defined in (2.2). Notice that \( |A_k| \leq C \) and \( |B_k| \leq C/k \) in Corollary A.2. By Lemma A.3, we have

\[ |u_n| \leq \frac{|f_n|}{\lambda_n} + C \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right) |f_{k+2}| + C \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right) \frac{1}{k} |f_{k+1}|. \]  

\[ (3.11) \]
Similarly, we obtain

$$\sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right) |f_{k+2}| \leq \left[ \sum_{k=n}^{\infty} (|f_{k+2}|^2 \delta_{k+2,r})_1 \right]^{1/2} \left( \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right)^2 (\delta_{k+2,r})^{1-r+1} \right)^{1/2} \leq |f|_{B^r_{\omega^{-1},-1}} \left[ \sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+2}} \right)^2 (\delta_{k+2,r})^{1-r+1} \right]^{1/2} \leq C |f|_{B^r_{\omega^{-1},-1}} n^{-r-\alpha}. \quad (3.12)$$

Similarly, we obtain

$$\sum_{k=n}^{\infty} \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right) \frac{1}{k} |f_{k+1}| \leq C |f|_{B^r_{\omega^{-1},-1}} n^{-r-\alpha}. \quad (3.13)$$

Therefore, we have

$$|u_n| \leq (\lambda_{n+1}^\alpha - 1) |f_n| + C n^{-r-\alpha}. \quad (3.14)$$

Similar to the proof of Theorem 3.1, we have \( \omega^{-\sigma,-\sigma} u \in B^\alpha_{\omega^{-1},-1} \). When \( r \) is not an integer, we can apply standard space interpolation to obtain the conclusion.

When \( \mu \neq 0 \), we apply the bootstrapping technique. By \( L^\alpha_0 u = f - \mu u \in B^{r+\alpha}_{\omega^{-1},-1} \), the conclusion above, we have \( \omega^{-\sigma,-\sigma} u \in B^{r+\alpha}_{\omega^{-1},-1} \), which leads to \( u \in B^\alpha_{\omega^{-1},-1} \). Thus \( L^\alpha_0 u = f - \lambda u \in B^{r+\alpha}_{\omega^{-1},-1} \), and by the conclusion above, we have \( \omega^{-\sigma,-\sigma} u \in B^{r+\alpha}_{\omega^{-1},-1} \), which further leads to \( u \in B^{\sigma+1}_{\omega^{-1},-1} \). Therefore, we have \( L^\alpha_0 u = f - \mu u \in B^{r+\alpha}_{\omega,0} \). Then by Theorem 3.1, we have \( \omega^{-\sigma,-\sigma} u \in B^{r+\alpha}_{\omega,0} \).

## 4. Spectral Galerkin method and error analysis

The weak formulation is to find \( u \in H^\alpha_0 \), such that

$$\langle L^\alpha_0 u, v \rangle + \mu(u, v) = (f, v), \quad \forall v \in H^\alpha_0, \quad (4.1)$$

where \( H^\alpha_0 \) is the standard fractional Sobolev space in [1] with induced semi-norm \( |\cdot|_{H^\alpha_0} \), and \( (H^\alpha_0)' \) is the dual space of \( H^\alpha_0 \). From [6], we know that there exits a unique solution \( u \in H^\alpha_0 \) such that \( \|u\|_{H^\alpha_0} \leq \|f\|_{(H^\alpha_0)'} \).

We first consider the implementation of the spectral Galerkin method. Define the finite dimensional space

$$U_N := \omega^{-\sigma,-\sigma} \mathbb{P}_N = \text{Span}\{\phi_0, \phi_1, \ldots, \phi_N\},$$

where \( \phi_k(x) := (1 - x)^{\alpha}(1 + x)^{\alpha} P^\alpha_k(x) \), for \( 0 \leq k \leq N \) and \( \mathbb{P}_N \) is the set of all algebraic polynomials of degree at most \( N \). The spectral Galerkin method is to find \( u_N \in U_N \) such that

$$\langle L^\alpha_0 u_N, v_N \rangle + \mu(u_N, v_N) = (f, v_N), \quad \forall v_N \in U_N. \quad (4.2)$$
Plugging \( u_N = \sum_{n=0}^{N} \hat{u}_n \phi_n(x) \) into (4.2) and taking \( v_N = \phi_k(x) \), we obtain from the orthogonality of Jacobi polynomials and Lemma 2.1 that
\[
\sum_{n=0}^{N} S_{k,n} \hat{u}_n + \mu \sum_{n=0}^{N} M_{k,n} \hat{u}_n = (f, \phi_k), \quad k = 0, 1, 2, \ldots, N,
\]
where
\[
S_{k,n} = \lambda^*_n k \int_{-1}^{1} (1-x)^{\sigma} (1+x)^{\sigma} P_n^{\sigma,\sigma}(x) \, dx,
\]
\[
M_{k,n} = \int_{-1}^{1} (1-x)^{2\sigma} (1+x)^{2\sigma} P_n^{\sigma,\sigma}(x) \, dx = \sum_{j=0}^{N} P_n^{\sigma,\sigma}(x_j) P_k^{\sigma,\sigma}(x_j) w_j.
\]
To find \( S_{k,n} \) and \( M_{k,n} \), we apply Gauss-Jacobi quadrature rule, e.g.
\[
M_{k,n} = \int_{-1}^{1} (1-x)^{2\sigma} (1+x)^{2\sigma} P_n^{\sigma,\sigma}(x) \, dx = \sum_{j=0}^{N} P_n^{\sigma,\sigma}(x_j) P_k^{\sigma,\sigma}(x_j) w_j.
\]
Here \( x_j \)'s are the zeros of Jacobi polynomial \( P_{N+1}^{2\sigma,2\sigma}(x) \), \( w_j \)'s are the corresponding quadrature weights. The quadrature rule here is exact since \( n + k \leq 2N \) while the quadrature rule is exact for all \( (2N+1) \)-th order polynomials. The integral in \( S_{n,k} \) can be calculated similarly. To find \( f_k = (f, \phi_k) \), we use a different Gauss-Jacobi quadrature rule: \( f_k \approx \hat{f}_k = \sum_{j=0}^{N} f(x_j) P_k^{\sigma,\sigma}(x_j) w_j \).
Here \( x_j \)'s are the roots of Jacobi polynomial \( P_{N+1}^{\sigma,\sigma}(x) \), \( w_j \)'s are the corresponding quadrature weights.

Next we focus on the analysis of the spectral Galerkin method.

To show the stability of (4.2), we require the following lemma.

**Lemma 4.1.** [6] For any \( v \in H^0_{\alpha/2} \) with \( 1 < \alpha \leq 2 \), we have
\[
(\mathcal{L}_0^\alpha v, v) = c_1^\alpha |v|^2_{H^{\alpha/2}}, \quad c_1^\alpha = -\cos(\alpha/2\pi).
\]

Since \( U_{\alpha} \subseteq H^0_{\alpha/2} \), the well-posedness of the discrete problem (4.2) can be readily shown by the Lax-Milgram theorem.

Before presenting the error estimates, we need the following approximation properties. Define the \( L_{\omega,\alpha}^2 \)-orthogonal projection \( P_N^{\omega,\omega^*} : L_{\omega,\alpha}^2 \rightarrow \mathbb{P}_N \) such that \( (P_N^{\omega,\omega^*} u - u, v)_{\omega,\alpha^*} = 0 \) for any \( v \in \mathbb{P}_N \), or equivalently \( P_N^{\omega,\omega^*}(u)(x) = \sum_{n=0}^{N} \hat{u}_n P_n^{\omega,\omega^*}(x) \). Denote \( \Pi_N^{\omega,\omega^*} u := \omega^{\omega,\omega^*} P_N^{\omega,\omega^*}(\omega^{-\omega,\omega^*} u) \).

**Lemma 4.2.** Let \( \omega^{-\omega,\omega^*} u \in B_{\omega,\alpha}^m \) and \( \mathcal{L}_0^\alpha u \in B_{\omega,\alpha}^m \). Then, for \( 0 \leq m \leq N \), we have the following estimates
\[
\| u - \Pi_N^{\omega,\omega^*} u \|_{\omega^{-\omega,\omega^*}} \leq c N^{-m} |\omega^{-\omega,\omega^*} u|_{B_{\omega,\alpha}^m}.
\]
and
\[
\| \mathcal{L}_0^\alpha (u - \Pi_N^{\omega,\omega^*} u) \|_{\omega,\alpha^*} \leq c N^{-m} |\mathcal{L}_0^\alpha u|_{B_{\omega,\alpha}^m}.
\]

**Proof.** For \( \tilde{u} = \omega^{-\omega,\omega^*} u \in B_{\omega,\alpha}^m(I) \), we have the expansion as
\[
u = \omega^{\omega,\omega^*} \tilde{u} = \omega^{\omega,\omega^*} \sum_{n=0}^{\infty} u_n P_n^{\omega,\omega^*}.
\]
Noticing (4.9), we obtain
\[ u - \Pi_{N}^{\sigma \omega} u = \omega^{\sigma \omega} \sum_{n=N+1}^{\infty} u_{n} P_{n}^{\omega} = \omega^{\sigma \omega} (\tilde{u} - \mathcal{P}_{N}^{\omega} \tilde{u}). \]  
(4.10)

By Lemma 2.1, it holds that
\[ \mathcal{L}_{0}^{\omega} (u - \Pi_{N}^{\sigma \omega} u) = \sum_{n=N+1}^{\infty} \lambda_{n}^{\omega} u_{n} P_{n}^{\omega} = \mathcal{L}_{0}^{\omega} u - \Pi_{N}^{\sigma \omega} (\mathcal{L}_{0}^{\omega} u). \]

Then by the error estimate for the orthogonal projection \( \mathcal{P}_{N}^{\omega} \), e.g. in [19], we reach the conclusion. \( \square \)

In order to show convergence, we also need Hardy-type inequality below.

**Lemma 4.3.** [18] Let \( \Omega \) be a convex set and \( 1 < \alpha < 2 \). For any \( v \in H^{\alpha/2} \) vanishing on the boundary, it holds
\[ |v|_{H^{\alpha/2}}^{2} \geq C \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{2}}{|x - y|^{n+\alpha}} \, dx \, dy \geq k_{n,\alpha} \int_{\Omega} \frac{|v(x)|^{2}}{d_{\Omega}(x)^{\alpha}} \, dx, \]  
(4.11)

where \( C \) and \( k_{n,\alpha} \) are positive constants which only depend on dimension \( n \) and \( \alpha \), and \( d_{\Omega}(x) \) denotes the distance from the point \( x \in \Omega \) to the boundary of the \( \Omega \).

With above Lemma, we can obtain the following result.

**Lemma 4.4.** For any \( v \in H_{0}^{\alpha/2} \) with \( 1 < \alpha \leq 2 \), we have
\[ \|v\|_{\omega^{-\sigma_{\omega},-\sigma_{\omega}}} \leq C |v|_{H^{\alpha/2}}, \quad \|v\|_{\omega^{-\sigma_{\omega},-\sigma_{\omega}}} \leq C |v|_{H^{\alpha/2}}. \]  
(4.12)

**Proof.** The fractional Hardy-type inequality in Lemma 4.3 leads to
\[ \|v\|_{\omega^{-\sigma_{\omega},-\sigma_{\omega}}} \leq c \|v\|_{\omega^{-\alpha/2}} \leq C |v|_{H^{\alpha/2}}. \]  
(4.13)

The second inequality follows similarly. \( \square \)

**Theorem 4.5.** Suppose that \( u \) and \( u_{N} \) satisfy the problems (4.1) and (4.2), respectively. If \( f \in B_{r_{-1,\sigma_{\omega}}}^{\alpha/2} \) with \( r \geq 0 \), we have the following error estimates:
\[ \|u - u_{N}\|_{\omega^{-\alpha_{\omega},-\alpha_{\omega}}} \leq C N^{-\gamma} \|\omega^{-\alpha_{\omega},-\alpha_{\omega}} u\|_{B_{r_{\omega}}^{\alpha_{\omega},\gamma}}, \quad \gamma = (\alpha + 1) \wedge r + \alpha \]  
(4.14)

when \( \theta = 0.5 \) and
\[ \|u - u_{N}\|_{\omega^{-\alpha_{\omega},-\alpha_{\omega}}} \leq C N^{\alpha - \gamma} \|\mathcal{L}_{0}^{\omega} u\|_{B_{r_{\omega}}^{\alpha_{\omega},\gamma}} + C N^{-\gamma} \|\omega^{-\alpha_{\omega},-\alpha_{\omega}} u\|_{B_{r_{\omega}}^{\alpha_{\omega},\gamma}} \]  
(4.15)

when \( \theta \neq 0.5 \).

**Proof.** Denote \( \eta_{N} = u - \Pi_{N}^{\sigma \omega} u \) and \( e_{N} = \Pi_{N}^{\sigma \omega} u - u_{N} \), then \( u - u_{N} = \eta_{N} + e_{N} \). Combining (4.1) and (4.2), we can obtain the following error equation:
\[ (\mathcal{L}_{0}^{\omega} e_{N}, v_{N}) + \mu(e_{N}, v_{N}) = -(\mathcal{L}_{0}^{\omega} \eta_{N}, v_{N}) - \mu(\eta_{N}, v_{N}), \quad \forall v_{N} \in U_{N}. \]

Taking \( v_{N} = e_{N} \) leads to
\[ (\mathcal{L}_{0}^{\omega} e_{N}, e_{N}) + \mu\|e_{N}\|^{2} = -(\mathcal{L}_{0}^{\omega} \eta_{N}, e_{N}) - \mu(\eta_{N}, e_{N}). \]  
(4.16)
When $\theta = 0.5$, $\sigma = \sigma^* = \alpha/2$ and the orthogonal property leads to
\[
(L_{0.5,2}^\alpha (u - \Pi_N^{\alpha/2,\alpha/2} u), v_N) = 0, \quad \forall v_N \in U_N = \omega^{\alpha/2,\alpha/2} P_N.
\]
Thus, when $\theta = 0.5$, (4.16) reduces to
\[
(L_{0.5}^\alpha e_N, e_N) + \mu \|e_N\|^2 = -\mu(\eta_N, e_N).
\]
By the norm equivalence (4.6) and fractional Hardy-type inequality (4.12) that
\[
(L_{0.5}^\alpha e_N, e_N) + \mu \|e_N\|^2 \geq C\|e_N\|_{\omega^{-\sigma,-\sigma^*}}^2.
\]
Combining (4.17) and (4.18), we have
\[
\|e_N\|_{\omega^{-\sigma,-\sigma^*}}^2 \leq C \|\eta_N\| \|e_N\| \leq C \|\eta_N\| \|e_N\|_{\omega^{-\sigma,-\sigma^*}}^2.
\]
Thus $\|e_N\|_{\omega^{-\sigma,-\sigma^*}} \leq C \|\eta_N\|_{\omega^{-\sigma,-\sigma^*}}$. By the regularity in Theorem 3.2 and the projection estimate in Lemma 4.2, we arrive at
\[
\|e_N\|_{\omega^{-\sigma,-\sigma^*}} \leq C N^{-\gamma} \|\omega^{-\sigma,-\sigma^*} u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}}, \quad \theta = 0.5.
\]
We now turn to the case $\theta \neq 0.5$. Using the norm equivalence (4.6) and the fractional Hardy-type inequality (4.12), we have
\[
c_1^2 |\eta_N|^2_{H^{\alpha/2}} + \mu \|e_N\|^2 \leq \|L_{\theta}^\alpha \eta_N\|_{\omega^{-\sigma,-\sigma^*}} \|e_N\|_{\omega^{-\sigma,-\sigma^*}} + \mu \|\eta_N\| \|e_N\|
\leq \|L_{\theta}^\alpha \eta_N\|_{\omega^{-\sigma,-\sigma^*}} \|e_N\|_{H^{\alpha/2}} + \mu \|\eta_N\| \|e_N\|
\leq \frac{(c_2^2)^2}{2c_1^2} \|L_{\theta}^\alpha \eta_N\|_{\omega^{-\sigma,-\sigma^*}}^2 + \frac{c_1^2}{2} \|e_N\|_{H^{\alpha/2}}^2 + \frac{\mu}{2} \|\eta_N\|^2 + \frac{\mu}{2} \|e_N\|^2.
\]
Thus it follows that
\[
C \|e_N\|_{\omega^{-\sigma,-\sigma^*}} \leq c_1^2 |e_N|^2_{H^{\alpha/2}} + \mu \|e_N\|^2 \leq \frac{(c_2^2)^2}{2c_1^2} \|L_{\theta}^\alpha \eta_N\|_{\omega^{-\sigma,-\sigma^*}}^2 + c \|\eta_N\|^2_{\omega^{-\sigma,-\sigma^*}}.
\]
From Theorem 3.2 and Lemma 4.2, we obtain
\[
\|e_N\|_{\omega^{-\sigma,-\sigma^*}} \leq C N^{\alpha - \gamma} \|L_{\theta}^\alpha u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}} + C N^{-\gamma} \|\omega^{-\sigma,-\sigma^*} u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}}.
\]
For both cases $\theta = 0.5$ and $\theta \neq 0.5$, we arrive at the desired results by the triangle inequality
\[
\|u - u_N\|_{\omega^{-\sigma,-\sigma^*}} \leq \|e_N\|_{\omega^{-\sigma,-\sigma^*}} + \|\eta_N\|_{\omega^{-\sigma,-\sigma^*}}.
\]
By Theorems 3.1 and 4.5, it is straightforward to obtain the following result.

Corollary 4.6. Suppose that $u$ and $u_N$ satisfy the problems (4.1) and (4.2), respectively. If $f \in B_{\omega^{-\sigma,-\sigma^*}}^{r}$ and $u \in L^{\infty}$, we have the following error estimates:
\[
\|u - u_N\|_{\omega^{-\sigma,-\sigma^*}} \leq C N^{-\gamma} \|\omega^{-\sigma,-\sigma^*} u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}}, \quad \gamma = \alpha \wedge r + \alpha
\]
when $\theta = 0.5$ and
\[
\|u - u_N\|_{\omega^{-\sigma,-\sigma^*}} \leq C N^{\alpha - \gamma} \|L_{\theta}^\alpha u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}} + C N^{-\gamma} \|\omega^{-\sigma,-\sigma^*} u\|_{B_{\omega^{-\sigma,-\sigma^*}}^{\gamma}}
\]
when $\theta \neq 0.5$. 

5. Spectral Petrov-Galerkin method and error analysis

The spectral Petrov-Galerkin method is to find \( u_N \in U_N \) such that
\[
(L^\sigma_\theta u_N, v_N) + \mu(u_N, v_N) = (f, v_N), \quad \forall v_N \in V_N. \tag{5.1}
\]
The method is implicitly discussed in [7] and is fully discussed in [21] when \( mu = 0 \). Here we define the finite dimensional space \( V_N := \omega^{\sigma_\ast,\sigma}P_N = \text{Span}\{\varphi_0, \varphi_1, \ldots, \varphi_N\} \) and \( \varphi_k(x) := (1 - x)^{\sigma_\ast}(1 + x)^{\sigma}P^\sigma_k\sigma_\ast(x) \).

For implementation, plugging \( u_N = \sum_{n=0}^N \hat{u}_n \phi_n(x) \) in (5.1) and taking \( v_N = \varphi_k(x) \), we obtain from Lemma 2.1 and the orthogonality of Jacobi polynomials that
\[
\lambda^\sigma_{\theta, k} h^{\sigma_\ast,\sigma}_k \hat{u}_k + \mu \sum_{n=0}^N M_{k,n} \hat{u}_n = (f, \varphi_k), \quad k = 0, 1, 2, \ldots, N, \tag{5.2}
\]
where \( \lambda^\sigma_{\theta, k} \) is defined in Lemma 2.1 and
\[
M_{k,n} = \int_{-1}^{1} (1 - x^2)^\alpha P_n^{\sigma,\sigma_\ast}(x)P_k^{\sigma_\ast,\sigma}(x) \, dx. \tag{5.3}
\]
Here \( M_{k,n} \) and \( f_k = (f, \varphi_k) \) can be computed similarly as in the last section.

Now we turn to the analysis of the spectral Petrov-Galerkin method.

**Lemma 5.1.** Let \( \alpha \in (1, 2) \). Suppose \( u \) satisfies \( \omega^{-\sigma, -\sigma_\ast}u \in L^2_{\omega^{-\sigma, -\sigma_\ast}} \) and \( \|L^\sigma_\theta u\|_{\omega^{\sigma_\ast,\sigma}} < \infty \). Then we have
\[
\lambda^\sigma_{\theta, 0} \|u\|_{\omega^{\sigma_\ast,\sigma}} \leq \lambda^\sigma_{\theta, 0} \|u\|_{\omega^{-\sigma, -\sigma_\ast}} \leq \|L^\sigma_\theta u\|_{\omega^{\sigma_\ast,\sigma}}. \tag{5.4}
\]

**Proof.** For \( u \) satisfying \( \omega^{-\sigma, -\sigma_\ast}u \in L^2_{\omega^{-\sigma, -\sigma_\ast}} \), we write
\[
u = \omega^{\sigma,\sigma_\ast} \sum_{n=0}^\infty u_n P_n^{\sigma,\sigma_\ast}. \tag{5.5}
\]
Given the expansion (5.5), we derive from Lemma 2.1 that
\[
\|u\|_{\omega^{-\sigma, -\sigma_\ast}}^2 = \sum_{n=0}^\infty |u_n|^2 h_n^{\sigma_\ast,\sigma}, \quad \|L^\sigma_\theta u\|_{\omega^{\sigma_\ast,\sigma}}^2 = \sum_{n=0}^\infty (\lambda^\sigma_{\theta, n})^2 |u_n|^2 h_n^{\sigma_\ast,\sigma},
\]
where by (2.2), we have \( h_n^{\sigma_\ast,\sigma} = h_n^{\sigma_\ast,\sigma} \). Noticing the sequence \( \{\lambda^\sigma_{\theta, n}\} \) is monotonically increasing, we have
\[
\lambda^\sigma_{\theta, 0} \|u\|_{\omega^{\sigma_\ast,\sigma}} \leq \lambda^\sigma_{\theta, 0} \|u\|_{\omega^{-\sigma, -\sigma_\ast}} \leq \|L^\sigma_\theta u\|_{\omega^{\sigma_\ast,\sigma}}.
\]
This completes the proof. \( \square \)

**Theorem 5.2 (Stability).** Assume that \( |\mu| \leq \lambda^\sigma_{\theta, 0}/2 \). The problem (5.1) admits a unique solution \( u_N \) such that
\[
\|L^\sigma_\theta u_N\|_{\omega^{\sigma_\ast,\sigma}} \leq C \|f\|_{\omega^{\sigma_\ast,\sigma}}.
\]

**Proof.** Take \( v_N = \omega^{\sigma_\ast,\sigma}L^\sigma_\theta u_N \) in (5.1). By Lemma 5.1, we get
\[
\|L^\sigma_\theta u_N\|_{\omega^{\sigma_\ast,\sigma}}^2 = -\mu(u_N, \omega^{\sigma,\sigma_\ast}L^\sigma_\theta u_N) + (f, \omega^{\sigma_\ast,\sigma}L^\sigma_\theta u_N) \leq |\mu| \|u_N\|_{\omega^{\sigma_\ast,\sigma}} \|L^\sigma_\theta u_N\|_{\omega^{\sigma_\ast,\sigma}} + \|f\|_{\omega^{\sigma_\ast,\sigma}} \|L^\sigma_\theta u_N\|_{\omega^{\sigma_\ast,\sigma}}.
\]
respectively. If \( f \in B^r_{\omega-\sigma,\sigma} \) with \( r \geq 0 \) and \( |\mu| \leq \lambda_{\phi,0}^\vartheta / 2 \), then we can obtain the following error equation:

\[
\| u - u_N \|_{\omega-\sigma,\sigma} \leq cN^{-\gamma} |\omega^{-\sigma,-\sigma} u|_{B^r_{\omega-\sigma,\sigma}}, \quad \gamma = (\alpha + 1) \wedge r + \alpha. \tag{5.6}
\]

Proof. Denote \( \eta_N = u - \Pi_N^\omega \sigma^\vartheta u \) and \( e_N = \Pi_N^\omega \sigma^\vartheta u - u_N \), then \( u - u_N = \eta_N + e_N \). Combining (4.1) and (5.1), we can obtain the following error equation:

\[
(\mathcal{L}_0^\omega e_N, v_N) + (e_N, v_N) = -(\mathcal{L}_0^\omega \eta_N, v_N) - \mu(\eta_N, v_N), \quad \forall v_N \in U_N.
\]

Taking \( v_N = \omega^{-\sigma,-\sigma} \mathcal{L}_0^\omega e_N \), and by the orthogonal property, we get

\[
\| \mathcal{L}_0^\omega e_N \|_{\omega-\sigma,\sigma}^2 = -\mu(e_N, \omega^{-\sigma,-\sigma} \mathcal{L}_0^\omega e_N) - (\mathcal{L}_0^\omega \eta_N, \omega^{-\sigma,-\sigma} \mathcal{L}_0^\omega e_N) - \mu(\eta_N, \omega^{-\sigma,-\sigma} \mathcal{L}_0^\omega e_N).
\]

Following a similar derivation in the proof for stability, we obtain

\[
\| \mathcal{L}_0^\omega e_N \|_{\omega-\sigma,\sigma} \leq 2\mu \| \eta_N \|_{\omega-\sigma,\sigma}.
\]

By Lemma 5.1, we have

\[
\| e_N \|_{\omega-\sigma,\sigma} \leq 1/\lambda_{\phi,0}^\vartheta \| \mathcal{L}_0^\omega e_N \|_{\omega-\sigma,\sigma} \leq 2\mu/\lambda_{\phi,0}^\vartheta \| \eta_N \|_{\omega-\sigma,\sigma} \leq \| \eta_N \|_{\omega-\sigma,\sigma}.
\]

Using the triangle inequality leads to

\[
\| u - u_N \|_{\omega-\sigma,\sigma} \leq \| e_N \|_{\omega-\sigma,\sigma} + \| \eta_N \|_{\omega-\sigma,\sigma} \leq 2\| \eta_N \|_{\omega-\sigma,\sigma}.
\]

Since \( f \in B^r_{\omega-\sigma,\sigma} \) with \( r \geq 0 \), we can see from Theorem 3.2 that \( \omega^{-\sigma,-\sigma} u \in B^r_{\omega-\sigma,\sigma} \). Applying Lemma 4.2 leads to the desired result. \(\square\)

**Remark 4.4.** For the spectral Petrov-Galerkin method, we need \( |\mu| \leq \lambda_{\phi,0}^\vartheta / 2 \). However, this assumption seems to be relaxed to the case for all \( \mu > -\lambda_{\phi,0}^\vartheta / 2 \). The key is to show that \( \lambda_{\phi,0}^\vartheta \) is positive for all \( u_N \in U_N \). Unfortunately, we are not able to prove this for technical reasons. However, when \( \theta = 0.5 \), the spectral Petrov-Galerkin method coincides with spectral Galerkin method and we only need \( \mu > -\lambda_{\phi,0}^\vartheta / 2 \).

By Theorems 3.1 and 5.3, it is straightforward to obtain the following result.

**Corollary 5.5 (Convergence order).** Suppose that \( u \) and \( u_N \) satisfy the problems (4.1) and (5.1), respectively. If \( f \in B^r_{\omega-\sigma,\sigma} \), \( u \in L^\infty \), and \( |\mu| \leq \lambda_{\phi,0}^\vartheta / 2 \), then we have the following optimal error estimates:

\[
\| u - u_N \|_{\omega-\sigma,\sigma} \leq cN^{-\gamma} |\omega^{-\sigma,-\sigma} u|_{B^r_{\omega-\sigma,\sigma}}, \quad \gamma = (\alpha + 1) \wedge r + \alpha. \tag{5.7}
\]
6. Numerical results

In this section, we present three examples with different forcing terms $f$: smooth, weakly singular at an interior point and weakly singular at boundary. Since exact solutions are unavailable, we use reference solutions $u_{\text{ref}}$, which are computed with a very fine resolution using the same methods for computing $u_N$. In the computation, we take $\mu = 1$ and measure the error as follows

$$E_1(N) = \| u_{\text{ref}} - u_N \|_{\omega^{-2\alpha,-2\alpha^*}}$$
$$E_2(N) = \| u_{\text{ref}} - u_N \|_{\omega^{-\alpha,-\alpha^*}}$$

(6.1)

where $u_N = \sum_{n=0}^{N} \hat{u}_n \omega_n^\alpha P_n^{\alpha^*}$ and $u_{\text{ref}} =: u_{512}$.

For the first two examples without boundary singularity, we use $E_1(N)$ to measure the error. As $E_2(N) \leq E_1(N)$, the convergence order of $E_1(N)$ is at least the order of $E_2(N)$. For the third example, we use $E_2(N)$ to measure the error.

Here we present numerical results for $\theta \in [0.5, 1]$, in particular, $\theta = 0.5, 0.7, \text{and } \theta = 1$. Since $\sigma$ and $\sigma^*$ depends on the fractional order $\alpha$ and $\theta$, we find the values of $\sigma, \sigma^*$ numerically using Newton’s method with a tolerance $10^{-16}$. We list in Table 6.1 the values of $(\sigma, \sigma^*)$ for different $\theta$’s and $\alpha$’s. For illustration, we present only four digits in the table while in computation we keep fifteen digits for $\sigma$ and $\sigma^*$.

| $\alpha$ | $\theta = 0.5$ | $\theta = 0.7$ | $\theta = 1.0$ |
|----------|----------------|----------------|----------------|
| 1.2      | (0.6000, 0.6000) | (0.7000, 0.7000) | (1.0000, 0.2000) |
| 1.4      | (0.7000, 0.8000) | (0.8602, 0.7100) | (1.0000, 0.4000) |
| 1.6      | (0.8000, 0.9000) | (0.9411, 0.8589) | (1.0000, 0.6000) |
| 1.8      | (0.9000, 0.9000) | (0.8589, 0.8589) | (1.0000, 0.8000) |

Table 6.1. Numerical values for $(\sigma, \sigma^*)$ corresponding to different $\theta$ and $\alpha$

For spectral Petrov-Galerkin method, we present numerical results for all $\theta$’s listed above while we present only numerical results of the Galerkin method for $\theta \neq 0.5$ since the method is the same as the spectral Petrov-Galerkin method when $\theta = 0.5$.

**Example 6.1.** Consider $f = \sin x$. Here $f$ belongs to $B^\infty_{\omega,\sigma,-\alpha,-1}$. By Theorem 3.2, $\omega^{-\alpha,-\alpha^*}u \in B^{2\alpha+1}_{\omega,\sigma^*}$.

According to Theorem 5.3, the convergence orders are expected to be $2\alpha + 1$ for the spectral Petrov-Galerkin method (5.1). In Table 6.2, we observe that the convergence orders are $2\alpha + 1$ for the spectral Petrov-Galerkin method when the order $\alpha = 1.2, 1.4, 1.6, 1.8$.

In the light of Theorem 4.5, the convergence orders are expected to be $\alpha + 1$ for the spectral Galerkin method (4.2). However, we found numerically the convergence orders lie in between $\alpha + 1$ and $2\alpha + 1$; see Table 6.3. For $\theta = 0.7$ in Table 6.3, the observed convergence orders are $2\alpha + 1$ when $\alpha = 1.4, 1.6, 1.8$, instead of $\alpha + 1$. When $\alpha = 1.2$, the order is not close to $2\alpha + 1$ but still larger than $\alpha + 1$. For $\theta = 1.0$, the convergence orders again lie in between $\alpha + 1$ and $2\alpha + 1$ except for $\alpha = 1.2$. For $\alpha = 1.2$, the convergence order is around 2 and is smaller than the expected $\alpha + 1 = 2.2$. However, the order 2.2 can be observed for $\alpha = 1.2$ if we take $u_{\text{ref}} = u_{256}$ instead of $u_{\text{ref}} = u_{512}$. More precisely, the errors are $3.56e-04$ ($N = 16$), $9.56e-05$ ($N = 32$), $2.44e-05$
We also observe that when $\alpha = 1.8$, the convergence order can match the regularity index $2\alpha + 1$.

In this example, the spectral Petrov-Galerkin method (5.1) has the convergence order $2\alpha + 1$, which suggests that the regularity index $2\alpha + 1$ for the solution and somewhat verifies Theorem 3.2. The spectral Petrov-Galerkin method has higher accuracy than the spectral Galerkin method (4.2) does, especially for small fractional order $\alpha$ when $\theta \neq 0$. An improvement in the convergence order for the spectral Galerkin method will be considered for further research.

### Table 6.2. Convergence orders and errors of the spectral Petrov-Galerkin method (5.1) for Example 6.1 with $f = \sin x$.

| $\theta = 0.5$ | $\alpha = 1.2$ | $\alpha = 1.4$ | $\alpha = 1.6$ | $\alpha = 1.8$ |
|---------------|----------------|----------------|----------------|----------------|
| $N$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ |
| 16 | 1.29e-04 | 1.11e-05 | 1.43e-06 | 1.78e-07 |
| 32 | 1.31e-05 | 3.31 | 8.81e-07 | 4.01 | 8.54e-09 | 4.38 |
| 64 | 1.29e-06 | 3.34 | 6.71e-08 | 3.71 | 5.17e-09 | 4.09 | 3.85e-10 | 4.47 |
| 128 | 1.25e-07 | 3.37 | 4.97e-09 | 3.75 | 2.92e-10 | 4.14 | 1.66e-11 | 4.53 |

Order (Averaged) 3.34 3.71 4.08 4.46

$2\alpha + 1$ (Theorem 5.3) 3.40 3.80 4.20 4.60

$\theta = 0.7$ | $\alpha = 1.2$ | $\alpha = 1.4$ | $\alpha = 1.6$ | $\alpha = 1.8$ |
|---------------|----------------|----------------|----------------|
| $N$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ |
| 16 | 3.84e-05 | 7.39e-06 | 1.24e-06 | 1.75e-07 |
| 32 | 3.90e-06 | 3.30 | 5.82e-07 | 3.67 | 7.68e-08 | 4.02 | 8.46e-09 | 4.37 |
| 64 | 3.80e-07 | 3.36 | 4.40e-08 | 3.73 | 4.50e-09 | 4.09 | 3.83e-10 | 4.47 |
| 128 | 3.60e-08 | 3.40 | 3.23e-09 | 3.77 | 2.54e-10 | 4.15 | 1.66e-11 | 4.53 |

Order (Averaged) 3.35 3.72 4.07 4.46

$2\alpha + 1$ (Theorem 5.3) 3.40 3.80 4.20 4.60

$\theta = 1$ | $\alpha = 1.2$ | $\alpha = 1.4$ | $\alpha = 1.6$ | $\alpha = 1.8$ |
|---------------|----------------|----------------|----------------|
| $N$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ | $E_1(N)$ |
| 16 | 3.87e-06 | 1.99e-06 | 7.04e-07 | 1.63e-07 |
| 32 | 3.94e-07 | 3.29 | 1.56e-07 | 3.67 | 4.42e-08 | 4.00 | 8.14e-09 | 4.32 |
| 64 | 3.80e-08 | 3.38 | 1.17e-08 | 3.74 | 2.61e-09 | 4.08 | 3.75e-10 | 4.44 |
| 128 | 3.55e-09 | 3.42 | 8.56e-10 | 3.78 | 1.49e-10 | 4.13 | 1.65e-11 | 4.51 |

Order (Averaged) 3.36 3.73 4.07 4.42

$2\alpha + 1$ (Theorem 5.3) 3.40 3.80 4.20 4.60

### Example 6.2. Consider $f = |\sin x|$. The function $f$ has a weak singularity at $x = 0$ and $f \in B_{2,\sigma}^{1.5-\epsilon}$ for any $\epsilon > 0$. By Theorem 3.2, $\omega^{-\sigma, -\sigma}u \in B_{2,\sigma}^{\alpha+1.5-\epsilon}$.

According to Theorems 5.3 and 4.5, the convergence orders are expected to be $\alpha + 1.5 - \epsilon$ for the spectral Petrov-Galerkin method (5.1) and $1.5 - \epsilon$ for the spectral Galerkin method (4.2).

From Table 6.4, we can observe that the convergence order is $\alpha + 1.5 - \epsilon$ for the spectral Petrov-Galerkin method (5.1), which is in agreement with the theoretical prediction when the order $\alpha = 1.2, 1.4, 1.6, 1.8$.

In Table 6.5, we observe that the convergence orders for the spectral Galerkin method lie in between $\alpha + 1.5 - \epsilon$ and $1.5 - \epsilon$. For $\theta = 0.7$, the observed convergence order is $\alpha + 1.5 - \epsilon$ when
apply Theorem 3.1 instead of Theorem 3.2 in order to get higher regularity index. From Theorem
the spectral Petrov-Galerkin method are
\[ \alpha = \beta \]
and the Galerkin method are
\[ \alpha = \sigma \]
for small fractional order \( \alpha \). The spectral Petrov-Galerkin method has higher accuracy than the spectral Galerkin method (4.2), especially
end points \( \pm \) which suggests the regularity index \( \alpha = 1 \).

**Example 6.3.** We then consider the singular forcing term \( f = (1-x^2)^\beta \sin x \). Here \( f \in B^{\sigma+\sigma'^+2\beta}_0,1 \) but \( \not\in B^{\sigma+\sigma'^+2\beta+1}_0,1 \). From Theorem
3.2, \( \omega^{-\sigma,-\sigma'} u \in B^{\alpha+(\sigma\land\sigma'^+2\beta+1)\land\alpha}_0,1 \) and by Theorem 3.1, \( \omega^{-\sigma,-\sigma'} u \in B^{\alpha+(\sigma\land\sigma'^+2\beta+1)\land\alpha}_0,1 \).

In this example, we measure the error using \( E_2(N) \) instead of \( E_1(N) \). We test the different \( \beta \)'s in Tables 6.6-6.7 (\( \beta = 0.5 \)) and in Tables 6.8-6.9 (\( \beta = -0.4 \)).

We first test the case \( \beta = 0.5 \) where the forcing term \( f \) has weak singularity and vanishes at both end points \( \pm 1 \). By Theorems 3.3 and 4.5 the theoretical orders for the Petrov-Galerkin method and the Galerkin method are \( \alpha + \sigma + \sigma'^+1 \) and \( \sigma + \sigma'^+1 \). However, we observe in Tables 6.6-6.7 that both methods can have convergence orders as high as \( \alpha + \sigma + \sigma'^+2 \).

We then consider the singular forcing term \( f = (1-x^2)^\beta \sin x \) where \( \beta = -0.4 \). In this case, we apply Theorem 3.1 instead of Theorem 3.2 in order to get higher regularity index. From Theorem
3.2, \( \omega^{-\sigma,-\sigma'} u \in B^{\alpha+(\sigma\land\sigma'^+2\beta+1)\land\alpha}_0,1 \) and by Theorem 3.1, \( \omega^{-\sigma,-\sigma'} u \in B^{\alpha+(\sigma\land\sigma'^+2\beta+1)\land\alpha}_0,1 \). For \( \beta = -0.4 \), \( \omega^{-\sigma,-\sigma'} u \in B^{\alpha+(\sigma\land\sigma'^+0.2)\land\alpha}_0,1 \). According to Corollary 5.5 the convergence orders for the spectral Petrov-Galerkin method are \( \alpha + \sigma + \sigma'^+0.2 \), which is demonstrated in Table 6.8.

**Table 6.3.** Convergence orders and errors of the spectral Galerkin method (4.2)
for Example 6.1 with \( f = \sin x \).

| \( \theta = 0.7 \) | \( N \) | \( \alpha = 1.2 \) | \( E_1(N) \) | \( \alpha = 1.4 \) | \( E_1(N) \) | \( \alpha = 1.6 \) | \( E_1(N) \) | \( \alpha = 1.8 \) | \( E_1(N) \) |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 3.25e-04 | 2.22e-05 | 3.09e-06 | 5.04e-07 | 0.29e-05 | 3.78e-06 | 5.63e-07 | 0.47e-06 |
| 32 | 5.41e-05 | 2.59 | 1.83e-06 | 3.60 | 1.86e-07 | 4.05 | 2.42e-08 | 4.38 |
| 64 | 8.66e-06 | 2.64 | 1.41e-07 | 3.70 | 1.06e-08 | 4.13 | 1.08e-09 | 4.48 |
| 128 | 1.35e-06 | 2.69 | 1.04e-08 | 3.76 | 5.84e-10 | 4.18 | 4.67e-11 | 4.54 |

Order (Averaged) 2.64 3.69 4.12 4.47

| \( \theta = 0.5 \) | \( N \) | \( \alpha = 1.2 \) | \( E_1(N) \) | \( \alpha = 1.4 \) | \( E_1(N) \) | \( \alpha = 1.6 \) | \( E_1(N) \) | \( \alpha = 1.8 \) | \( E_1(N) \) |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 3.45e-04 | 4.77e-05 | 4.77e-06 | 5.21e-07 | 0.43e-05 | 4.82e-06 | 5.96e-07 | 0.73e-06 |
| 32 | 9.45e-05 | 1.87 | 6.76e-06 | 2.82 | 3.51e-07 | 3.76 | 2.47e-08 | 4.40 |
| 64 | 2.48e-05 | 1.93 | 9.07e-07 | 2.90 | 2.40e-08 | 3.87 | 1.09e-09 | 4.50 |
| 128 | 6.23e-06 | 1.99 | 1.17e-07 | 2.95 | 1.57e-09 | 3.93 | 4.63e-11 | 4.56 |

Order (Averaged) 1.93 2.89 3.85 4.47

\( \alpha = 1.8 \). However, we observed that the convergence orders decrease with \( \alpha \) when \( \theta = 0.7 \) and
\( \theta = 1 \).

In this example, the spectral Petrov-Galerkin method (5.1) has the convergence order \( \alpha + 1.5 - \epsilon \), which suggests the regularity index \( \alpha + 1.5 - \epsilon \) for the solution and verifies Theorem 3.2. The spectral Petrov-Galerkin method has higher accuracy than the spectral Galerkin method (4.2), especially
for small fractional order \( \alpha \) when \( \theta \neq 0.5 \). Again better error estimates for the spectral Galerkin method should be considered for further research.
Table 6.4. Convergence orders and errors of the spectral Petrov-Galerkin method (5.1) for Example 6.2 with $f = |\sin x|$.

| $\theta$ | $\alpha$ | $N$ | $E_1(N)$ | rate | $E_1(N)$ | rate | $E_1(N)$ | rate |
|---------|---------|-----|---------|------|---------|------|---------|------|
| 0.5     | 1.2     | 16  | 1.60e-03| 2.80 | 4.88e-04| 2.12 | 1.22e-04| 1.11 |
|         | 1.4     | 32  | 2.70e-04| 2.56 | 7.32e-05| 2.74 | 2.85e-05| 2.89 |
|         |         | 64  | 4.28e-05| 2.66 | 1.01e-05| 2.85 | 3.48e-06| 3.03 |
|         |         | 128 | 6.61e-06| 2.70 | 1.35e-06| 2.91 | 4.05e-07| 3.10 |
|         | (Averaged) |    | 2.64  | 2.83  | 3.01  | 3.16 |
| 0.7     | $\alpha + 1.5 - \epsilon$ (Theorem 5.3) |    | 2.70  | 2.90  | 3.10  | 3.30 |
| 1       | $\alpha + 1.5 - \epsilon$ (Theorem 5.3) |    | 2.70  | 2.90  | 3.10  | 3.30 |

From Corollary 4.6, the convergence orders for the spectral Galerkin method are expected to be $\sigma \land \sigma^* + 0.2$. However, the observed orders are $\alpha + \sigma \land \sigma^* + 0.2$ in Table 6.9.

In this example, the convergence orders for the two methods are almost the same when the forcing term $f(x)$ has both weak boundary singularity ($\beta = 0.5$) or stronger boundary singularity ($\beta = -0.4$), which suggest the error estimate for the Galerkin method can be improved in the non-symmetrical case $\theta \neq 0.5$. For the Petrov-Galerkin method, however, the convergence orders are higher than the theoretical predictions in the case of $\beta = 0.5$.

7. Conclusion

In this paper, we discuss the regularity of the two-sided fractional diffusion equations with Riemann-Liouville operators under the homogeneous Dirichlet boundary conditions. Writing $u = \omega^{\sigma \land \sigma^*} \tilde{u}$, we find that the regularity index of $\tilde{u}$ is shown to be $2\alpha + 1$. We also validate our finding by considering two numerical methods: spectral Galerkin and Petrov-Galerkin methods. With the regularity index, we showed the optimal error estimate for both methods when $\theta = 0.5$. For $\theta \neq 0.5$, we obtained the optimal error estimate when the reaction coefficient $\mu$ is small.

The error estimate for the Galerkin method are not optimal and the estimate for the Petrov-Galerkin method requires some additional condition on $\mu$. Further improvement in the error
Table 6.5. Convergence orders and errors of the spectral Galerkin method (4.2) for Example 6.2 with \( f = |\sin x| \).

| \( \theta = 0.7 \) | \( \alpha = 1.2 \) | \( \alpha = 1.4 \) | \( \alpha = 1.6 \) | \( \alpha = 1.8 \) |
|---|---|---|---|---|
| \( N \) | \( E_1(N) \) rate | \( E_1(N) \) rate | \( E_1(N) \) rate | \( E_1(N) \) rate |
| 16 | 1.73e-03 | 7.77e-04 | 3.53e-04 | 1.96e-04 |
| 32 | 4.47e-04 | 1.96 | 1.40e-04 | 2.48 | 5.18e-05 | 2.77 | 2.47e-05 | 2.99 |
| 64 | 1.06e-04 | 2.08 | 2.29e-05 | 2.61 | 6.90e-06 | 2.91 | 2.80e-06 | 3.14 |
| 128 | 2.36e-05 | 2.17 | 3.56e-06 | 2.86 | 8.73e-07 | 2.98 | 3.00e-07 | 3.22 |

Order (Averaged) 2.07 2.57 2.89 3.12

| \( \theta = 1 \) | \( \alpha + 1.5 - \epsilon \) | \( \alpha + 1.5 - \epsilon \) | \( \alpha + 1.5 - \epsilon \) | \( \alpha + 1.5 - \epsilon \) |
|---|---|---|---|---|
| \( N \) | \( E_1(N) \) rate | \( E_1(N) \) rate | \( E_1(N) \) rate | \( E_1(N) \) rate |
| 16 | 1.73e-03 | 7.77e-04 | 3.53e-04 | 1.96e-04 |
| 32 | 4.47e-04 | 1.96 | 1.40e-04 | 2.48 | 5.18e-05 | 2.77 | 2.47e-05 | 2.99 |
| 64 | 1.06e-04 | 2.08 | 2.29e-05 | 2.61 | 6.90e-06 | 2.91 | 2.80e-06 | 3.14 |
| 128 | 2.36e-05 | 2.17 | 3.56e-06 | 2.86 | 8.73e-07 | 2.98 | 3.00e-07 | 3.22 |

Order (Averaged) 1.87 2.15 2.56 2.96

estimates are needed. The analysis in this paper can be extended to FDEs with different low-order terms, such as FDEs with an advection term.

Appendix A. Some useful relations of Jacobi polynomials

The following relations hold for Jacobi polynomials \( P_{n}^{\alpha,\beta}(x) \), see e.g. [2, Chapter 2],

\[
\partial_x P_{n}^{\alpha,\beta}(x) = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{\alpha+1,\beta+1}(x), \quad \alpha, \beta > -1. \tag{A.1}
\]

By (A.1), we have

\[
\partial_x^l P_{n}^{\alpha,\beta}(x) = d_{n,l}^{\alpha,\beta} P_{n-l}^{\alpha+1,\beta+1}(x), \quad \alpha, \beta > -1, n \geq l, \quad d_{n,l}^{\alpha,\beta} = \frac{\Gamma(n + \alpha + \beta + l + 1)}{2\Gamma(n + \alpha + \beta + 1)}. \tag{A.2}
\]

Theorem A.1. The Jacobi polynomials \( P_{n}^{\alpha,\beta}(x) \) satisfy

\[
P_{n}^{\alpha,\beta} = \tilde{A}_{n}^{\alpha,\beta} \partial_x P_{n-1}^{\alpha,\beta} + \tilde{B}_{n}^{\alpha,\beta} \partial_x P_{n}^{\alpha,\beta} + \tilde{C}_{n}^{\alpha,\beta} \partial_x P_{n+1}^{\alpha,\beta}, \quad n \geq 0, \tag{A.3}
\]

where \( P_{-1}^{\alpha,\beta} = 0 \) and

\[
\tilde{A}_{n}^{\alpha,\beta} = \frac{-2(n + \alpha)(n + \beta)}{(n + \alpha + \beta)(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}, \quad \tilde{B}_{n}^{\alpha,\beta} = \frac{2(\alpha - \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}, \quad \tilde{C}_{n}^{\alpha,\beta} = \frac{2(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}.
\]

The relation (A.1) and Theorem A.1 lead to the following result.

Corollary A.2. The Jacobi polynomials \( P_{n}^{\alpha,\beta}(x) \) satisfy

\[
P_{n}^{\alpha,\beta} = A_{n}^{\alpha,\beta} P_{n-2}^{\alpha+1,\beta+1} + B_{n}^{\alpha,\beta} P_{n-1}^{\alpha+1,\beta+1} + C_{n}^{\alpha,\beta} P_{n}^{\alpha+1,\beta+1}, \quad n \geq 0, \tag{A.4}
\]
Table 6.6. Convergence orders and errors of the spectral Petrov-Galerkin method (5.1) for Example 6.3 with $f = (1 - x^2)^{0.5} \sin x$.

| $\alpha$ | $\theta = 0.5$ | $\theta = 0.7$ | $\theta = 1$ |
|----------|----------------|----------------|--------------|
|          | $N$ | $E_2(N)$ | rate | $E_2(N)$ | rate | $E_2(N)$ | rate | $E_2(N)$ | rate |
| $\alpha = 1.2$ | 16 | 4.00e-05 | 3.95e-06 | 1.57e-06 | 32 | 3.32e-06 | 3.59e-06 | 4.22e-06 | 64 | 2.61e-07 | 3.99e-08 | 4.29e-07 |
| $\alpha = 1.4$ | 16 | 3.59e-07 | 3.87e-07 | 4.15e-07 | 32 | 3.59e-07 | 3.87e-07 | 4.15e-07 | 64 | 3.59e-07 | 3.87e-07 | 4.15e-07 |
| $\alpha = 1.6$ | 16 | 3.95e-06 | 7.32e-08 | 4.58e-07 | 32 | 3.95e-06 | 7.32e-08 | 4.58e-07 | 64 | 3.95e-06 | 7.32e-08 | 4.58e-07 |
| $\alpha = 1.8$ | 16 | 1.57e-06 | 4.42e-07 | 4.42e-07 | 32 | 1.57e-06 | 4.42e-07 | 4.42e-07 | 64 | 1.57e-06 | 4.42e-07 | 4.42e-07 |

where we let $A_n^{\alpha,\beta} = A_1^{\alpha,\beta} = B_0^{\alpha,\beta} = 0$ and $P_{-2}^{\alpha+1,\beta+1} = P_{-1}^{\alpha+1,\beta+1} = 0$ and

$$A_n^{\alpha,\beta} = \frac{(n + \alpha)(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}, \quad B_n^{\alpha,\beta} = \frac{(\alpha - \beta)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

$$C_n^{\alpha,\beta} = \frac{(n + \alpha + \beta + 1)(n + \alpha + \beta + 2)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}.$$

Lemma A.3. For any $k \geq n \geq 0$, it holds that $|X_n^{\alpha}| \leq C$ where

$$X_n^{\alpha} := \frac{(P_{n+1}^{\alpha+1,\beta+1}, P_n^{\alpha,\beta})_{h_n^{\alpha,\beta}}}{h_n^{\alpha,\beta}}, \quad \alpha > -1, \quad \beta > -1. \tag{A.5}$$

and $h_n^{\alpha,\beta}$ is defined in (2.2).

Proof. By (3.3), we get

$$\delta_{nk} = A_k^{\alpha,\beta} X_{k-2} + B_k^{\alpha,\beta} X_{k-1} + C_k^{\alpha,\beta} X_k. \tag{A.6}$$
Thus we have
\[ X_n^0 = \frac{1}{C_{n}^{\alpha,\beta}}, \quad X_{n+1}^0 = -\frac{B_{n+1}^{\alpha,\beta}}{C_{n+1}^{\alpha,\beta}} X_n^0, \quad X_{k+2}^n = p_k X_{k+1}^n + q_k X_k^n, \quad k \geq n. \]
where \( p_k = -\frac{B_{n+2}^{\alpha,\beta}}{C_{n+2}^{\alpha,\beta}} \) and \( q_k = -\frac{A_{n+2}^{\alpha,\beta}}{C_{n+2}^{\alpha,\beta}} \). Denote \( Y_k^n = (X_{k+1}^n, X_k^n)^\top \) and \( A_k = (p_k, q_k; 1, 0) \). Then we have \( Y_{k+1}^n = A_k Y_k^n \). It follows that
\[ \|Y_{k+1}^n\|_\infty = \|A_k Y_k^n\|_\infty \leq \|A_k\|_\infty \|Y_k^n\|_\infty = \max\{|1, |p_k| + q_k\} \|Y_k^n\|_\infty, \]
where \( \|Y_k^n\|_\infty = \max\{|X_k^n|, |X_{k+1}^n|\} \). Recalling \( A_k^{\alpha,\beta}, B_k^{\alpha,\beta} \) and \( C_k^{\alpha,\beta} \) in Corollary A.2, we have for \( k \geq 2 \)
\[ |p_{k-2}| = \frac{|B_{k-2}^{\alpha,\beta}|}{C_{k-2}^{\alpha,\beta}} = \frac{|\alpha - \beta|(2k + \alpha + \beta + 1)}{(2k + \alpha + \beta)(k + \alpha + \beta + 2)} = \frac{|\alpha - \beta|}{k} + O\left(\frac{1}{k^2}\right), \quad \text{and} \]
\[ q_{k-2} = -\frac{A_{k-2}^{\alpha,\beta}}{C_{k-2}^{\alpha,\beta}} = \frac{(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2)}{(k + \alpha + \beta + 1)(k + \alpha + \beta + 2)(2k + \alpha + \beta)} = 1 - \frac{\alpha + \beta + 2}{k} + O\left(\frac{1}{k^2}\right). \]
Thus we arrive at
\[ |p_k| + q_k = 1 - \frac{\alpha + \beta + 2 - |\alpha - \beta|}{k} + O\left(\frac{1}{k^2}\right) = 1 - \frac{2\min(\alpha, \beta) + 2}{k} + O\left(\frac{1}{k^2}\right). \]
Since \( \alpha > -1 \) and \( \beta > -1 \), \( |p_k| + q_k \leq 1 \). Thus for any \( k \geq n \geq 0 \), we have \( \|Y_{k+1}^n\|_\infty \leq \|Y_k^n\|_\infty \), which leads to the desired result.

The following asymptotic formula for a ratio of two gamma functions holds
\[ \lim_{n \to \infty} \frac{\Gamma(n + \delta)}{n^{\delta - \gamma} \Gamma(n + \gamma)} = \lim_{n \to \infty} \left[ 1 + \frac{(\delta - \gamma)(\delta + \gamma - 1)}{2n} + O(n^{-2}) \right] = 1. \quad (A.7) \]
Table 6.8. Convergence orders and errors of the spectral Petrov-Galerkin method (5.1) for Example 6.3 with $f = (1 - x^2)^{-0.4} \sin x$.

| $\theta$ | $N$ | $\alpha = 1.2$ | $\alpha = 1.4$ | $\alpha = 1.6$ | $\alpha = 1.8$ |
|----------|-----|----------------|----------------|----------------|----------------|
| 0.5      | 16  | 4.83e-03       | 1.03e-03       | 3.11e-04       | 1.12e-04       |
|          | 32  | 1.27e-03       | 1.92           | 2.26e-04       | 2.18           |
|          | 64  | 3.28e-04       | 1.96           | 4.79e-05       | 2.24           |
|          | 128 | 8.31e-05       | 1.98           | 9.92e-06       | 2.27           |
|          |     | Order (Averaged) | 1.95          | 2.23           | 2.51           | 2.79           |
| 0.7      | 16  | 5.75e-03       | 1.13e-03       | 3.20e-04       | 1.12e-04       |
|          | 32  | 1.86e-03       | 1.63           | 2.73e-04       | 2.05           |
|          | 64  | 5.85e-04       | 1.67           | 6.37e-05       | 2.10           |
|          | 128 | 1.80e-04       | 1.70           | 1.47e-05       | 2.12           |
|          |     | Order (Averaged) | 1.67          | 2.09           | 2.46           | 2.78           |
| 1        | 16  | 4.42e-03       | 1.16e-03       | 3.43e-04       | 1.15e-04       |
|          | 32  | 1.56e-03       | 1.50           | 3.15e-04       | 1.88           |
|          | 64  | 5.33e-04       | 1.55           | 8.23e-05       | 1.94           |
|          | 128 | 1.78e-04       | 1.58           | 2.10e-05       | 1.97           |
|          |     | Order (Averaged) | 1.55          | 1.93           | 2.32           | 2.73           |

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Table 6.9. Convergence orders and errors of the spectral Galerkin method (4.2) for Example 6.3 with $f = (1 - x^2)^{-0.4}\sin x$.

| $\theta$ | $N$ | $\alpha = 1.2$ | Rate | $\alpha = 1.4$ | Rate | $\alpha = 1.6$ | Rate | $\alpha = 1.8$ | Rate |
|----------|-----|----------------|-------|----------------|-------|----------------|-------|----------------|-------|
| 0.7      | 16  | 7.90e-03       | 1.58e-03 | 4.90e-04       | 1.94e-04 |
|          | 32  | 2.41e-03       | 1.71   | 9.11e-05       | 2.43   | 2.95e-05       | 2.72   |
|          | 64  | 7.15e-04       | 1.75   | 1.62e-05       | 2.49   | 4.23e-06       | 2.80   |
|          | 128 | 2.08e-04       | 1.79   | 2.83e-06       | 2.52   | 5.89e-07       | 2.85   |
| Order (Averaged) | 1.75 | 2.14 | 2.48 | 2.79 |
| $\alpha + \sigma \wedge \sigma^* + 0.2$ (Theorem 4.5) | 1.72 | 2.14 | 2.51 | 2.86 |
| $\sigma \wedge \sigma^* + 0.2$ (Theorem 4.5) | 0.52 | 0.74 | 0.91 | 1.06 |

| $\theta$ | $N$ | $\alpha = 1.2$ | Rate | $\alpha = 1.4$ | Rate | $\alpha = 1.6$ | Rate | $\alpha = 1.8$ | Rate |
|----------|-----|----------------|-------|----------------|-------|----------------|-------|----------------|-------|
| 1        | 16  | 7.05e-03       | 1.81e-03 | 5.47e-04       | 1.99e-04 |
|          | 32  | 2.37e-03       | 1.57   | 1.08e-04       | 2.34   | 3.09e-05       | 2.69   |
|          | 64  | 7.69e-04       | 1.63   | 2.04e-05       | 2.40   | 4.53e-06       | 2.77   |
|          | 128 | 2.40e-04       | 1.68   | 2.80e-06       | 2.43   | 6.48e-07       | 2.81   |
| Order (Averaged) | 1.63 | 2.02 | 2.39 | 2.76 |
| $\alpha + \sigma \wedge \sigma^* + 0.2$ (Theorem 4.5) | 1.60 | 2.00 | 2.40 | 2.80 |
| $\sigma \wedge \sigma^* + 0.2$ (Theorem 4.5) | 0.40 | 0.60 | 0.80 | 1.00 |

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