Control-Barrier-Function-Based Design of Gradient Flows for Constrained Nonlinear Programming

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Abstract—This paper considers the problem of designing a continuous-time dynamical system that solves a constrained nonlinear optimization problem and makes the feasible set forward invariant and asymptotically stable. The invariance of the feasible set makes the dynamics anytime, when viewed as an algorithm, meaning it returns a feasible solution regardless of when it is terminated. Our approach augments the gradient flow of the objective function with inputs defined by the constraint functions, treats the feasible set as a safe set, and synthesizes a safe feedback controller using techniques from the theory of control barrier functions. The resulting closed-loop system, termed safe gradient flow, can be viewed as a primal-dual flow, where the state corresponds to the primal variables and the inputs correspond to the dual ones. We provide a detailed suite of conditions based on constraint qualification under which (both isolated and nonisolated) local minimizers are stable with respect to the feasible set and the whole state space. Comparisons with other continuous-time methods for optimization in a simple example illustrate the advantages of the safe gradient flow.

Index Terms—Control Barrier Functions, Gradient Flows, Optimization Algorithms, Projected Dynamical Systems.

I. INTRODUCTION

OPTIMIZATION problems are ubiquitous in engineering and applied science. The traditional emphasis on the numerical analysis of algorithms is motivated by the implementation on digital platforms. The alternate viewpoint of optimization algorithms as continuous-time dynamical systems taken here also has a long history, often as a precursor of the synthesis of discrete-time algorithms. This viewpoint has been fruitful for gaining insight into qualitative properties such as stability and convergence.

For constrained optimization problems, the picture is complicated by the fact that algorithms may need to ensure convergence to the optimizer as well as enforce feasibility of the iterates. The latter is important in real-time applications, when feasibility guarantees may be required at all times in case the algorithm is terminated before completion, or when the algorithm is implemented on a physical plant where the constraints encode its safe operation. In this paper, we show that, just as unconstrained optimization algorithms can be viewed as dynamical systems, constrained optimization algorithms can be viewed as control systems. Within this framework, the task of designing an optimization algorithm for a constrained problem is equivalent to that of designing a feedback controller for a nonlinear system. We use this connection to derive a novel control-theoretic algorithm for solving constrained nonlinear programs that combines continuous-time gradient flows to optimize the objective function with techniques from control barrier functions to maintain invariance of the feasible set.

A. Related Work

Dynamical systems and optimization are closely intertwined disciplines [2], [3], [4]. The work [5] provides a contemporary review of the dynamical systems approach to optimization for both constrained and unconstrained problems, with an emphasis on applications where the optimization problem is in a feedback loop with a plant, see e.g., [6], [7], [8]. Examples of such scenarios are numerous, including power systems [7], [8], network congestion [9], and transportation [10].

1) Flows for Equality Constrained Problems: For problems involving only equality constraints, [11], [12] employ differential geometric techniques to design a vector field that maintains feasibility along the flow, makes the constraint set asymptotically stable, and whose solutions converge to critical points of the objective function. The work [13] introduces a generalized form of this vector field to deal with inequality constraints in the form of a differential algebraic equation and explores links with sequential quadratic programming.

2) Projected Gradient Methods: Another approach to solving nonlinear programs in continuous time makes use of projected dynamical systems [14] by projecting the gradient of the objective function onto the cone of feasible descent directions, see e.g., [15]. However, projected dynamical systems are, in general, discontinuous, which from an analysis viewpoint requires properly dealing with notions and existence of solutions, cf. [16]. The work [17] proposes a continuous modification of the projected gradient method, whose stability is analyzed in [18]. However, this method projects onto the constraint set itself, rather than the tangent cone, and may fail when it is nonconvex. Another modification is the “constrained gradient flow” proposed in [19], derived using insights from nonsmooth mechanics, and is well-defined outside the feasible set. The resulting method is related to the one presented here and converges to critical points, though the dynamics are once again discontinuous, and stability guarantees are only provided in the case of convexity, which we do not assume.
3) Saddle-Point Dynamics: Convex optimization problems can be solved by searching for saddle points of the associated Lagrangian via a primal-dual dynamics consisting of a gradient descent in the primal variable and a gradient ascent in the dual one. The analysis of stability and convergence of this method has a long history [2], [20], with more recent accounts provided for discrete-time implementations [21] and continuous-time ones [22], [23], [24]. These methods are particularly well suited for distributed implementation on network optimization problems, but they do not leave the feasible set invariant.

B. Contributions

We consider the synthesis of continuous-time dynamical systems that solve constrained optimization problems while making the feasible set forward invariant and asymptotically stable. Our first contribution is the design of the safe gradient flow for constrained optimization using the framework of safety-critical control. The basic intuition is to combine the standard gradient flow to optimize the objective function with the idea of keeping the feasible set safe. To maintain safety, we augment the gradient dynamics with inputs associated with the constraint functions and use a control barrier function approach to design an optimization-based feedback controller that ensures forward invariance and asymptotic stability of the feasible set. The approach is primal-dual, in the sense that the states correspond to the primal variables and the inputs correspond to the dual variables.

Our second contribution unveils the connection of the proposed dynamics with the projected gradient flow. Specifically, we provide an alternate derivation of the safe gradient flow as a continuous modification of the projected gradient flow, based on a design parameter. We show that, as the parameter grows to ∞, the safe gradient flow becomes the projected gradient flow.

In addition to establishing an interesting parallelism, we build on this equivalence in our third contribution for understanding the regularity and stability properties of the safe gradient flow. We show that the flow is locally Lipschitz (ensuring the existence and uniqueness of classical solutions), well defined on an open set containing the feasibility region (which allows for the possibility of infeasible initial conditions), that its equilibria exactly correspond to the critical points of the optimization problem, and that the objective function is monotonically decreasing along the feasible set of the optimization problem. Lastly, we prove that the feasible set is forward invariant and asymptotically stable.

Our fourth contribution consists of a thorough stability analysis of the critical points of the optimization problem under the safe gradient flow. We provide a suite of constraint qualification-based conditions under which isolated local minimizers are either locally asymptotically stable with respect to the feasible set, locally asymptotically stable with respect to the global state space, or locally exponentially stable. We also provide conditions for semistability of nonisolated local minimizers and establish global convergence to critical points of the optimization problem. Our technical analysis for this builds on a combination of the Kurdyaka-Łojasiewicz inequality with a novel angle-condition Lyapunov test to establish the finite arclength of trajectories, which we present in the appendix.

A preliminary version of this work appeared previously as [1]. The present work significantly expands the scope of the stability analysis of isolated local minimizers under weaker assumptions, as well as characterizes the stability of nonisolated local minimizers, global convergence to critical points, and highlights the advantages of the safe gradient flow over other continuous-time methods in optimization.

C. Notation

We let R denote the set of real numbers. For v, w ∈ R^n, v ≤ w denotes v_i ≤ w_i for i ∈ {1, ..., n}. We let ||v|| denote the Euclidean norm and ||v||∞ = max_{1≤i≤n} |v_i| the infinity norm. For y ∈ R, we denote [y]_+ = max{0, y}, and sgn(y) = 1 if y > 0, sgn(y) = −1 if y < 0 and sgn(y) = 0 if y = 0. We let I_m ∈ R^{m×m} denote the vector of all ones. For a matrix A ∈ R^{n×m}, we use ρ(A) and A^† to denote its spectral radius and its Moore-Penrose pseudoinverse, respectively. We write A ≥ 0 (resp., A > 0) to denote A is positive semidefinite (resp., A is positive definite). Given a subset C ⊂ R^n, the distance of x ∈ R^n to C is distC(x) = inf_{y ∈ C} ||x−y||. We let C, int(C), and ∂C denote the closure, interior, and boundary of C, respectively. Given X ⊂ R^n and f : X → R^m, the graph of f is graph(f) = {(x, f(x)) | x ∈ X}. Similarly, given a set-valued map F : X ⇒ R^m, its graph is graph(F) = {(x, y) | x ∈ X, y ∈ F(x)}. Given g : R^n → R, we denote its gradient by ∇g and its Hessian by ∇^2g. For g : R^n → R^m, ∂g(x) denotes its Jacobian. For I ⊂ {1, 2, ..., m}, we denote by ∂g_i(x) the matrix whose rows are {∇_i g_k(x)}_{k∈I}.

II. Preliminaries

We present notions on invariance, stability, variational analysis, control barrier functions, and nonlinear programming. The reader familiar with the material can safely skip the section.

A. Invariance and Stability Notions

We recall basic definitions from the theory of ordinary differential equations [25]. Let F : R^n → R^n be a locally Lipschitz vector field and consider the dynamical system ˙x = F(x). Local Lipschitzness ensures that, for every initial condition x_0 ∈ R^n, there exists T > 0 and a unique trajectory x : [0, T] → R^n such that x(0) = x_0 and ˙x(t) = F(x(t)). If the solution exists for all t ≥ 0, then it is forward complete.

In this case, the flow map is defined by Φ_t : R^n → R^n such that Φ_t(x) = x(t), where x(t) is the unique solution with x(0) = x. The positive limit set of x ∈ R^n is

ω(x) = \bigcap_{T≥0} Φ_t(x) \mid t > T \big}

A set K ⊂ R^n is forward invariant if x ∈ K implies that Φ_t(x) ∈ K for all t ≥ 0. If K is forward invariant and x* ∈ K is an equilibrium, x* is Lyapunov stable relative to K if for every open set U containing x*, there exists an open set U containing x* such that for all x ∈ U ∩ K, Φ_t(x) ∈ U ∩ K for all t > 0. The equilibrium x* is asymptotically stable
relative to $K$ if it is Lyapunov stable relative to $K$ and there is an open set $U$ containing $x^*$ such that $\Phi_t(x) \to x^*$ as $t \to \infty$ for all $x \in U \cap K$. We say $x^*$ is exponentially stable relative to $K$ if it is asymptotically stable relative to $K$ and there exists $\mu > 0$ and an open set $U$ containing $x^*$ such that for all $x \in U \cap K$, $\|\Phi_t(x) - x^*\| \leq e^{-\mu t} \|x - x^*\|$. Analogous definitions of Lyapunov stability and asymptotically stability can be made for sets, instead of individual points.

Consider a forward invariant set $K$ and a set of equilibria $S$ contained in it, $S \subset K$. We say $x^* \in S$ is semistable relative to $K$ if $x^*$ is Lyapunov stable and, for any open set $U$ containing $x^*$, there is an open set $U$ such that for every $x \in U \cap K$, the trajectory starting at $x$ converges to a Lyapunov stable equilibrium in $U \cap S$. Note that if $x^*$ is an isolated equilibrium, then semistability is equivalent to asymptotic stability. For all the concepts introduced here, when the invariant set is unspecified, we mean $K = \mathbb{R}^n$.

B. Variational Analysis

We review basic notions from variational analysis following [26]. The extended real line is $\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$. Given $f : \mathbb{R}^n \to \mathbb{R}$, its domain is $\text{dom}(f) = \{x \in \mathbb{R}^n \mid f(x) \neq -\infty\}$. The indicator function of $C \subset \mathbb{R}^n$ is $\delta_C : \mathbb{R}^n \to \mathbb{R}$,

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

Note that $\text{dom}(\delta_C) = C$. For $x \in \text{dom}(f)$ and $d \in \mathbb{R}^n$, consider the following limits

$$f'(x; d) = \lim_{(h, y) \to (0^+, x)} \frac{f(y + hd) - f(x)}{h}, \quad (1a)$$

$$f''(x; d) = \lim_{(h, y) \to (0^+, x)} \frac{f(y + hd) - f(x) - h f'(y; d)}{h^2}. \quad (1b)$$

If the limit in (1a) (resp. (1b)) exists, $f$ is directionally differentiable in the direction $d$ (resp. twice directionally differentiable in the direction $d$). By definition, $f'(x; d) = \nabla f(x)^\top d$ if $f$ is continuously differentiable at $x$ and $f''(x; d) = d^\top \nabla^2 f(x)d$ if $f$ is twice continuously differentiable at $x$.

Given a dynamical system $\dot{x} = F(x)$ and a function $V : \mathbb{R}^n \to \mathbb{R}$, the upper-right Dini derivative of $V$ along solutions of the system is

$$D^+_V V(x) = \limsup_{h \to 0^+} \frac{1}{h} [V(\Phi_h(x)) - V(x)],$$

where $\Phi_h$ is the flow map of the system. If $V$ is directionally differentiable then $D^+_V V(x) = V'(x; F(x))$, and if $V$ is differentiable then $D^+_V V(x) = \nabla V(x)^\top F(x)$.

The tangent cone to $C \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$T_C(x) = \{ d \in \mathbb{R}^n \mid \exists \{t^\nu\}_{\nu=1}^\infty \subset (0, \infty), \{x^\nu\}_{\nu=1}^\infty \subset C, t^\nu \to 0^+, x^\nu \to x, \frac{x^\nu - x}{t^\nu} \to d \text{ as } \nu \to \infty \}.$$ 

If $C$ is an embedded submanifold of $\mathbb{R}^n$, then the tangent cone coincides with the usual differential geometric notion of tangent space. Let $\Pi_C : \mathbb{R}^n \Rightarrow \mathbb{C}$, with $\Pi_C(x) = \{ y \in \mathbb{C} \mid \|x - y\| = \text{dist}_C(x) \}$, be the projection map onto $\mathbb{C}$. The proximal normal cone to $C$ at $x$ is

$$\mathcal{N}^p_C (x) = \{ d \in \mathbb{R}^n \mid \exists \{t^\nu\}_{\nu=1}^\infty \subset (0, \infty), \{x^\nu\}_{\nu=1}^\infty \subset \text{graph}(\Pi_C), \frac{x^\nu - y^\nu}{t^\nu} \to d \text{ as } \nu \to \infty \}.$$ 

C. Safety Critical Control via Control Barrier Functions

We introduce here basic concepts from safety and a method for synthesizing safe controllers using vector control barrier functions. Our exposition here slightly generalizes [27], [28] to set the stage for dealing with constrained optimization problems later. Consider a control-affine system

$$\dot{x} = F_0(x) + \sum_{i=1}^r u_i F_i(x), \quad (2)$$

with locally Lipschitz vector fields $F_i : \mathbb{R}^n \to \mathbb{R}^n$, for $i \in \{0, \ldots, r\}$, and a set $U \subset \mathbb{R}^m$ of valid control inputs. Let $C \subset \mathbb{R}^n$ represent the set of states where the system can operate safely and $u : X \to U$ be a locally Lipschitz feedback controller, with $X \subset \mathbb{R}^n$ a set containing $C$. The closed-loop system (2) under $u$ is safe with respect to $C$ if $C$ is forward invariant under the closed-loop system.

Feedback controllers can be certified to be safe by resorting to the notion of control barrier function, which we here generalize for convenience. Let $C \subset X \subset \mathbb{R}^n$ and $m, k \in \mathbb{Z}_{\geq 0}$. A $(m, k)$-vector control barrier function (VCBF) of $C$ on $X$ relative to $U$ is a continuously differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}^{n+k}$ such that the following properties hold.

(i) The safe set can be expressed using $m$ inequality constraints and $k$ equality constraints:

$$C = \{ x \in \mathbb{R}^n \mid \phi_i(x) \leq 0, 1 \leq i \leq m, \quad \phi_j(x) = 0, m + 1 \leq j \leq m + k \};$$

(ii) there exists $\alpha > 0$ such that the map $K : \mathbb{R}^n \Rightarrow U$, $K_\alpha(x) = \{ u \in U \mid D_{F_0}^+ \phi_1(x) + \sum_{i=1}^r u_i D_{F_i}^+ \phi_1(x) + \alpha \phi_i(x) \leq 0, \quad D_{F_0}^+ \phi_j(x) + \sum_{i=1}^r u_i D_{F_i}^+ \phi_j(x) + \alpha \phi_j(x) = 0, \quad 1 \leq i \leq m, m + 1 \leq j \leq m + k \},$ takes nonempty values for all $x \in X$.

In the special case where $m = 1$ and $k = 0$, this definition coincides with the usual notion of control barrier function [28, Definition 2], where the class $K$ function is linear, and the Lie derivative has been replaced with the upper-right Dini derivative. In general, the problem of finding a suitable VCBF $\phi$ is problem-specific: in many cases, the function naturally emerges from formalizing mathematically the safety specifications one seeks to enforce. The use of vector-valued functions instead of scalar-valued ones allows to consider a broader class of safe sets. If $\phi$ is a VCBF and $u$ is a feedback where $u(x) \in K_\alpha(x)$, it follows that along solutions to (2), $\dot{\phi}_i(x) \leq -\alpha \phi_i(x)$ for
We say that the constant rank condition (CRC) holds at \( x \in \mathbb{R}^n \) if there exists an open set \( U \) containing \( x \) such that for all \( I \subseteq I_0(x) \) the rank of \( \{ \nabla g_i(y) \}_{i \in I} \cup \{ \nabla h_j(y) \}_{j=1}^k \) is constant for all \( y \in U \).

If \( x^* \in C \) is a local minimizer, and any of the constraint qualification conditions hold at \( x^* \), then there exists \( u^* \in \mathbb{R}^m \) and \( v^* \in \mathbb{R}^k \) such that the Karash-Kuhn-Tucker conditions hold,

\[
\nabla f(x^*) + \frac{\partial g(x^*)}{\partial x}^T u^* + \frac{\partial h(x^*)}{\partial x}^T v^* = 0, \\
g(x^*) \leq 0, \\
h(x^*) = 0, \\
u^*_i \geq 0, \\
(u^*)^T g(x^*) = 0.
\]

The pair \((u^*, v^*)\) are called Lagrange multipliers, and the triple \((x^*, u^*, v^*)\) satisfying (6) is referred to as a KKT triple. We denote the set of KKT points of (4) by

\[
X_{\text{KKT}} = \{ x^* \in \mathbb{R}^n \mid \exists (u^*, v^*) \in \mathbb{R}^m \times \mathbb{R}^k \text{ such that } (x^*, u^*, v^*) \text{ solves (6)} \}.
\]

### III. Problem Formulation

Our goal is to solve the optimization problem (4) by designing a dynamical system \( \dot{x} = F(x) \) that converges to its solutions. The dynamics should enjoy the following properties.

(i) trajectories should remain feasible if they start from a feasible point. This can be formalized by asking the feasible set \( C \), defined in (5), to be forward invariant;
(ii) trajectories that start from an infeasible point should converge to the set of feasible points. This can be formalized by requiring that \( F \) is well defined on an open set containing \( C \), and that \( C \) is asymptotically stable with respect to the dynamics.

The requirement (i) ensures that, when viewed as an algorithm, the dynamics is anytime, meaning that it is guaranteed to return a feasible solution regardless of when it is terminated. The requirement (ii) ensures in particular that trajectories beginning from infeasible initial conditions approach the feasible set and, if the solutions of the optimization (4) belong to the interior of the feasible set, such trajectories enter it in finite time, never to leave it again. The problem is summarized next.

**Problem 1:** Find an open set \( X \) containing \( C \) and design a vector field \( F : X \to \mathbb{R}^n \) such that the system \( \dot{x} = F(x) \) satisfies the following properties.

(i) \( F \) is locally Lipschitz on \( X \);
(ii) \( C \) is forward invariant and asymptotically stable;
(iii) \( x^* \) is an equilibrium if and only if \( x^* \in X_{\text{KKT}} \);
(iv) \( x^* \) is asymptotically stable if \( x^* \) is a isolated local minimizer.

### IV. Constrained Nonlinear Programming via Safe Gradient Flow

In this section we introduce our solution to Problem 1 in the form of a dynamical system called the safe gradient flow. We present two interpretations of this system: first from the...
The next result shows that

$$\frac{\partial g}{\partial x} u - \frac{\partial h}{\partial x} v \leq \frac{\partial g}{\partial x} \nabla f(x) - \alpha g(x) - \epsilon \mathbf{1}_m \quad (9a)$$

$$\frac{\partial h}{\partial x} u - \frac{\partial h}{\partial x} v = \frac{\partial h}{\partial x} \nabla f(x) - \alpha h(x). \quad (9b)$$

Let \( \bar{g} = g(x) + \frac{\alpha}{\epsilon} \mathbf{1}_m \). By Farka’s Lemma [31], (9) is infeasible if and only if there exists a solution \((u, v)\) to

$$-\frac{\partial g}{\partial x} u - \frac{\partial h}{\partial x} v \geq 0 \quad (10a)$$

$$\frac{\partial h}{\partial x} u - \frac{\partial h}{\partial x} v = 0 \quad (10b)$$

$$u \geq 0 \quad (10c)$$

$$u^T \left( \frac{\partial g}{\partial x} \nabla f - \alpha \bar{g} \right) + v^T \left( \frac{\partial h}{\partial x} \nabla f - \alpha h(x) \right) < 0. \quad (10d)$$

Then (10a), (10b), and (10c) imply that

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \ker \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \end{bmatrix} = \ker \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \end{bmatrix}. \quad (11)$$

Next, by (11) and that \( x \in C \), (10d) reduces to

$$-u^T (ag(x) + \epsilon \mathbf{1}_m) < 0, \quad (12)$$

and by a second application of Farka’s Lemma, we see that (10c), (11), and (12) are feasible if and only if the following system is infeasible.

$$\frac{\partial g(x)}{\partial x} \xi \leq -\alpha g(x) - \epsilon \mathbf{1}_m \quad (13a)$$

We claim that a solution to (13) can be constructed if MFCQ holds at \( x \). Indeed, by MFCQ, there exists \( \xi \in \mathbb{R}^n \) such that

$$\frac{\partial g(x)}{\partial x} \xi < 0 \quad \text{and} \quad \frac{\partial h(x)}{\partial x} \xi = 0,$$

and for \( \epsilon \) sufficiently small, there exists \( \gamma > 0 \) such that \( \xi = \gamma \xi \) solves (13). Thus (10) is infeasible, and therefore (9) is feasible.

By strict feasibility of (9) and the fact that the matrix \( \frac{\partial h}{\partial x} \frac{\partial h}{\partial x}^T \) has full rank, it can be shown by Lemma C.1 that, for all \( x \in C \), the affine inequalities that parameterize \( K_{\alpha}(x) \) are regular. Finally, since the affine inequalities parameterizing \( K_{\alpha} \) are continuous, \( K_{\alpha}(y) \) is nonempty for any \( y \) sufficiently close to \( x \). Hence there exists an open set \( X \) such that \( K_{\alpha} \) takes nonempty values on \( X \).

Since \( \phi \) is a VCBF, we can design a feedback of the form (3) to maintain safety of \( C \) while modifying the drift term as little as possible. Formally,

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \in \{ (u, v) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^k \} \arg \min_{(u, v) \in K_{\alpha}(x)} \left\{ \| \frac{\partial g(x)}{\partial x} u + \frac{\partial h(x)}{\partial x} v \|^2 \right\}. \quad (14)$$

1Consider a linear system of inequalities of the form \( Cz \leq c; Dz = d \), and a solution \( z_0 \). The system is regular (c.f. [32]) if for \( C', c', D'; d' \) sufficiently close to \( C, c, D, d \), the perturbed system \( C'z \leq c', D'z = d' \) remains feasible, and the distance of \( z_0 \) to the solution set of the perturbed system is proportional to the magnitude of the perturbation.
We refer to the closed-loop system (7) under the controller (14) as the safe gradient flow. In general, the solution to (14) might not be unique. Nevertheless, as we show later, the safe gradient flow is well-defined because, the closed-loop behavior of the system is independent of the chosen solution.

Comparing (7) with the KKT equation (6a) suggests that \((u(x), v(x))\) can be interpreted as approximations of the dual variables of the problem. With this interpretation, the safe gradient flow can be viewed as a primal-dual method. We use this viewpoint later to establish connections between the proposed method and the projected gradient flow.

Remark 4.2 (Connection with the Literature): The work [11] considers the problem of designing a dynamical system to solve (4) when only equality constraints are present using a differential-geometric approach. Here, we show that the safe gradient flow generalizes the solution proposed in [11]. Under the assumption that \(h \in C^r\) and LICQ holds, the feasible set \(C = \{x \in \mathbb{R}^n \mid h(x) = 0\}\) is an embedded \(C^r\) submanifold of \(\mathbb{R}^n\) of codimension \(k\). The approach in [11] proceeds by identifying a vector field \(F: \mathbb{R}^n \to \mathbb{R}^n\) satisfying: (i) \(F \in C^r\) and \(F(x) \in T_C(x)\) for all \(x \in C\); and (ii) \(\hat{h}(x) = -\alpha h(x)\) along the trajectories of \(\dot{x} = F(x)\), where \(\alpha > 0\) is a design parameter. The proposed vector field satisfying both properties is

\[
F(x) = -(I - h^1 \frac{\partial h^1}{\partial x} \nabla f(x) - \alpha h^0 \frac{\partial h^0}{\partial x} h(x),
\]

To see that this corresponds to the safe gradient flow, note that the admissible control set (8) in this case is

\[
K_\alpha(x) = \{v \in \mathbb{R}^k \mid -\frac{\partial h^1}{\partial x} \nabla f(x) - \frac{\partial h^0}{\partial x} h(x) = -\alpha h(x)\}.
\]

By the LICQ assumption, \(K_\alpha(x)\) is a singleton whose unique element is

\[
v(x) = -(\frac{\partial h^1}{\partial x} \frac{\partial h^0}{\partial x})^{-1} (\frac{\partial h^1}{\partial x} \nabla f(x) - \alpha h(x)).
\]

Substituting this into (7), we obtain the expression (15). This provides an alternative interpretation from a control-theoretic perspective of the differential-geometric design in [11], and justifies viewing the safe gradient flow as the natural extension to the case with both inequality and equality constraints. 

Remark 4.3 (Inequality Constraints via Quadratic Slack Variables): The work [13] pursues a different approach that the one taken here to deal with inequality constraints by reducing them to equality constraints. This is accomplished introducing quadratic slack variables. Formally, for each \(i \in \{1, \ldots, m\}\), one replaces the constraint \(g_i(x) \leq 0\) with the equality constraint \(g_i^0(x) = -y_i^2\), and solves the equality-constrained optimization problem in the variables \((x, y) \in \mathbb{R}^{n+m}\) with a flow of the form (15). While this method can be expressed in closed form, there are several drawbacks with it. First, this increases the dimensionality of the problem, which can be problematic when there are a large number of inequality constraints. Second, adding quadratic slack variables introduces equilibrium points to the resulting flow which do not correspond to KKT points of the original problem.

B. Safe Gradient Flow as an Approximation of the Projected Gradient Flow

Here, we introduce an alternative design in terms of a continuous approximation of the projected gradient flow. The latter is a discontinuous dynamical system obtained by projecting the gradient of the objective function onto the tangent cone of the feasible set. Later, we show that this continuous approximation is in fact equivalent to the safe gradient flow.

Let \(x \in C\) and suppose that MFCQ holds at \(x\). Then the tangent cone of \(C\) at \(x\) is

\[
T_C(x) = \left\{ \xi \in \mathbb{R}^n \mid \frac{\partial h(x)}{\partial x} \xi = 0, \frac{\partial g_{I0}(x)}{\partial x} \xi \leq 0 \right\}.
\]

For \(x \in C\), let \(\Pi_{T_C(x)}(\cdot)\) be the projection onto \(T_C(x)\). In general, the projection is a set-valued map, but the fact that \(T_C(x)\) is closed and convex makes the projection onto \(T_C(x)\) unique in this case. The projected gradient flow is

\[
\dot{x} = \Pi_{T_C(x)}(-\nabla f(x)) = \arg\min_{\xi \in \mathbb{R}^n} \frac{1}{2} \|\xi + \nabla f(x)\|^2 \quad \text{subject to} \quad \frac{\partial g_{I0}(x)}{\partial x} \xi \leq 0, \frac{\partial h(x)}{\partial x} \xi = 0.
\]

In general, this system is discontinuous, so one must resort to appropriate notions of solution trajectories and establish their existence, see e.g., [16]. Here, we consider Carathéodory solutions, which are absolutely continuous functions that satisfy (16) almost everywhere. When Carathéodory solutions exist in \(C\), then the KKT points of (4) are equilibria of (16), and isolated local minimizers are asymptotically stable.

Consider the following continuous approximation of (16) by letting \(\alpha > 0\) and defining \(G_\alpha\) by

\[
G_\alpha(x) = \arg\min_{\xi \in \mathbb{R}^n} \frac{1}{2} \|\xi + \nabla f(x)\|^2 \quad \text{subject to} \quad \frac{\partial g(x)}{\partial x} \xi \leq -\alpha g(x), \quad \frac{\partial h(x)}{\partial x} \xi = -\alpha h(x).
\]

Note that (17) has a similar form to (16), and has a unique solution if one exists. However, as we show later, unlike the projected gradient flow, the vector field \(G_\alpha\) is well defined outside \(C\) and is Lipschitz.

We now show that \(G_\alpha\) approximates the projected gradient flow. Intuitively, this is because for \(j \notin I_0(x)\), one has \(g_j(x) < 0\) and hence the \(j\)th inequality constraint in (17), \(\nabla g_j(x)^T \xi \leq -\alpha g_j(x)\), becomes \(\nabla g_j(x)^T \xi \leq \infty\) as \(\alpha \to \infty\) and the constraint is effectively removed, reducing the problem to (16). This is formalized next.

Proposition 4.4 (\(G_\alpha\) approximates the projected gradient): Let \(x \in C\) and suppose MFCQ holds. Then

\[
(i) \quad G_\alpha(x) \in T_C(x).
(ii) \quad \lim_{\alpha \to \infty} G_\alpha(x) = \Pi_{T_C(x)}(-\nabla f(x)).
\]

Proof: To show (i), note that if \(x \in C\), then \(h(x) = 0\) and \(g_{I0}(x) = 0\), so the constraints in (17) imply \(\frac{\partial h(x)}{\partial x} G_\alpha(x) = 0\) and \(\frac{\partial g_{I0}(x)}{\partial x} G_\alpha(x) \leq 0\), and therefore \(G_\alpha(x) \in T_C(x)\).
Regarding (ii), for fixed \( x \in \mathcal{C} \), let \( J = I_-(x) \) and consider the following quadratic program

\[
P_x(\epsilon) = \arg\min_{\xi \in \mathbb{R}^n} \frac{1}{2} ||\xi + \nabla f(x)||^2
\]

subject to \( \frac{\partial g_i(x)}{\partial x} \xi \leq 0, \frac{\partial h(x)}{\partial x} \xi = 0 \) \( \epsilon \frac{\partial \alpha g(x)}{\partial x} \xi \leq -g_J(x). \)

When \( \epsilon = 0 \), the feasible sets of (18) and (16) are the same. Since the objective functions are also the same, \( P_x(0) = \Pi_{\mathcal{T}_C(x)}(-\nabla f(x)) \). Furthermore, for all \( \alpha > 0 \), \( P_x(\frac{\alpha}{\epsilon}) = G_\alpha(x) \).

Finally, since the QP defining \( P_x \) has a unique solution, and satisfies the regularity conditions in \([33, \text{Def. 2.1]}\), \( P_x \) is continuous at \( \epsilon = 0 \) by \([33, \text{Thm. 2.2}]. \)

\[
\lim_{\alpha \to \infty} G_\alpha(x) = \lim_{\epsilon \to 0^+} P_x(\epsilon) = P_x(0) = \Pi_{\mathcal{T}_C(x)}(-\nabla f(x)).
\]

A consequence of Proposition 4.4 is that solutions of \( \dot{x} = G_\alpha(x) \) approximate the solutions of the projected gradient flow, with decreasing error as \( \alpha \) increases, cf. Figure 2.

C. Equivalence Between the Two Interpretations

Here we establish the equivalence between the two interpretations of the safe gradient flow. Specifically, we show that the control barrier function quadratic program \((14)\) can be interpreted as a dual program corresponding to the continuous approximation of the projected gradient flow in \((17)\). The key to establishing the relationship between the continuous approximation of the projected gradient flow and the safe feedback controller in \((14)\) are the Lagrange multipliers of the problem in \((17)\).

Let \( L : \mathbb{R}^n \times \mathbb{R}_{\geq 0}^m \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R} \) be

\[
L(\xi, u, v; x) = \frac{1}{2} ||\xi + \nabla f(x)||^2 + u^T \left( \frac{\partial g(x)}{\partial x} \xi + \alpha g(x) \right) + v^T \left( \frac{\partial h(x)}{\partial x} \xi + \alpha h(x) \right).
\]

Then for each \( x \in \mathbb{R}^n \), the Lagrangian of the optimization problem \((17)\) is the function \( (\xi, u, v) \mapsto L(\xi, u, v; x) \).

For each \( x \in \mathbb{R}^n \), the KKT conditions corresponding to the optimization \((17)\) are

\[
\xi + \nabla f(x) + \frac{\partial g(x)}{\partial x}^\top u + \frac{\partial h(x)}{\partial x}^\top v = 0 \quad (19a)
\]

\[
\frac{\partial g(x)}{\partial x} \xi + \alpha g(x) \leq 0 \quad (19b)
\]

\[
\frac{\partial h(x)}{\partial x} \xi + \alpha h(x) = 0 \quad (19c)
\]

\[
u \geq 0 \quad (19d)
\]

\[
\xi = G_\alpha(x) = \text{argmin}_{\xi \in \mathbb{R}^n} \{ (u, v) \in \mathbb{R}_{\geq 0}^m \times \mathbb{R}^k \mid \exists \xi \in \mathbb{R}^n \text{ such that } (\xi, u, v) \text{ solves } (19)\}. \quad (19e)
\]

Because the \((17)\) is strongly convex, the existence of a triple \((\xi, u, v)\) satisfying \((19)\) is sufficient for optimality of \(\xi\). Since the optimizer is unique, for any triple \((\xi, u, v)\) satisfying these conditions, \(\xi = G_\alpha(x)\).

Let \( \Lambda_\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}^m \times \mathbb{R}^k \) be defined by

\[
\Lambda_\alpha(x) = \{(u, v) \in \mathbb{R}_{\geq 0}^m \times \mathbb{R}^k \mid \exists \xi \in \mathbb{R}^n \text{ such that } (\xi, u, v) \text{ solves } (19)\}. \quad (20)
\]

By definition, \( \Lambda_\alpha(x) \) is the set of Lagrange multipliers of \((17)\) at \( x \in \mathbb{R}^n \). When \( \Lambda_\alpha(x) \neq \emptyset \), then the conditions \((19)\) are also necessary for optimality of \(\xi\). As we show next, this necessity follows as a consequence of the constraint qualification conditions.

**Lemma 4.5 (Necessity of optimality conditions):** For \( \alpha > 0 \), if \((14)\) satisfies MFCQ at \( x \in \mathcal{C} \) then there is an open set \( U \) containing \( x \) such that \( \Lambda_\alpha(x') \neq \emptyset \) for all \( x' \in U \).

**Proof:** If MFCQ holds at \( x \in \mathcal{C} \), there exists \( \xi \in \mathbb{R}^n \) such that \( \nabla g_i(x)^\top \xi < 0 \) for all \( i \in I_0(x) \) and \( \nabla h_j(x)^\top \xi = 0 \) for all \( j \in \{1, \ldots, k\} \).

Next, for every \( x' \in I_-(x) \), let \( \epsilon_x > 0 \) be defined as

\[
\epsilon_x = \begin{cases} \frac{-\alpha g(x)^\top}{\nabla g_i(x)^\top} & \nabla g_i(x)^\top \xi > 0, \\ 1 & \nabla g_i(x)^\top \xi \leq 0. \end{cases}
\]

Then taking \( 0 < \epsilon \leq \min_{x' \in I_-(x)} \{\epsilon_x\} \) and \( \tilde{\xi} = \epsilon \xi \), satisfies

\[
\frac{\partial g(x)}{\partial x} \tilde{\xi} < -\alpha g(x) \quad \frac{\partial h(x)}{\partial x} \tilde{\xi} = -\alpha h(x). \quad (21)
\]

The previous expression means that the constraints of \((17)\) satisfy Slater’s condition \([34, \text{Ch. 5.2.3}]\) at \( x \), so the affine constraints are regular \([32, \text{Thm. 2}]. \) This implies that there exists an open set \( U \) containing \( x \) on which \((17)\) is feasible and \( \Lambda_\alpha(x') \neq \emptyset \) for all \( x' \in U \).

We use the optimality conditions to show that \((14)\) is actually the dual problem corresponding to \((17)\) in the appropriate sense.

**Proposition 4.6 (Equivalence of two constructions of the safe gradient flow):** If \( \Lambda_\alpha(x) \neq \emptyset \),

(i) If \((u, v) \in \Lambda_\alpha(x)\), then \((u, v)\) solves \((14)\);

(ii) \( G_\alpha \) is the closed-loop dynamics corresponding to the implementation of \((14)\) over \((7)\).

**Proof:** To show (i), let \((u, v) \in \Lambda_\alpha(x)\). Then there is \( \xi \in \mathbb{R}^n \) such that \((\xi, u, v)\) solves \((19)\). By \((19a)\), \( \xi = -\nabla f(x) - \frac{\partial g(x)}{\partial x}^\top u - \frac{\partial h(x)}{\partial x}^\top v \) and substituting \( \xi \) into the constraints of \((17)\), it follows immediately that \((u, v) \in K_\alpha(x)\), defined in \((8)\). We claim that \((u, v)\) is also optimal for \((14)\). To
prove this, let \((u', v')\) be a solution of (14) and, reasoning by contradiction, suppose
\[
\left\| \frac{\partial g(x)}{\partial x}^T u + \frac{\partial h(x)}{\partial x}^T v \right\|^2 > \left\| \frac{\partial g(x)}{\partial x}^T u' + \frac{\partial h(x)}{\partial x}^T v' \right\|^2.
\]
Then, \(\xi' = -\nabla f(x) - \frac{\partial g(x)}{\partial x}^T u' - \frac{\partial h(x)}{\partial x}^T v'\) satisfies the constraints in (17) and \(\|\xi' + \nabla f(x)\| < \|\xi + \nabla f(x)\|\), which contradicts the fact that \(\xi\) is optimal for (17).

To show (ii), suppose that \((u, v)\) solves (14), and \(\xi = -\nabla f(x) - \frac{\partial g(x)}{\partial x}^T u - \frac{\partial h(x)}{\partial x}^T v\). We claim that \(\xi\) is optimal for (17). Indeed, if \(\tilde{\xi}\) is the optimizer of (17), since \(\Lambda_\alpha(x) \neq \emptyset\), there exists \((\tilde{u}, \tilde{v}) \in \Lambda_\alpha(x)\) such that \((\tilde{\xi}, \tilde{u}, \tilde{v})\) solves (19). Note that \((\tilde{u}, \tilde{v})\) is feasible for (14), and
\[
\|\xi + \nabla f(x)\|^2 = \left\| \frac{\partial g(x)}{\partial x}^T u + \frac{\partial h(x)}{\partial x}^T v \right\|^2 
\leq \left\| \frac{\partial g(x)}{\partial x}^T \tilde{u} + \frac{\partial h(x)}{\partial x}^T \tilde{v} \right\|^2 = \|\tilde{\xi} + \nabla f(x)\|^2,
\]
where the inequality follows by optimality of \((u, v)\). It follows that \(\xi\) is optimal, but since the optimizer of (17) is unique, \(\xi = \xi_\alpha(x)\). Hence, \(\xi_\alpha(x) = -\nabla f(x) - \frac{\partial g(x)}{\partial x}^T u - \frac{\partial h(x)}{\partial x}^T v\), which is the closed-loop implementation of (14) over (7).

Remark 4.7 (Lagrange Multipliers of Continuous Approximation to Projected Gradient): The notion of duality in Proposition 4.6 is weaker than the usual notion of Lagrangian duality. While the result ensures that the Lagrange multipliers of (17) are solutions to (14), the converse is not true in general. This is because if \((u, v)\) solves (14), then \((\xi_\alpha(x), u, v)\) might not satisfy the complementarity condition (19e), in which case \((u, v) \notin \Lambda_\alpha(x)\). An example of this is given by the following constrained problem with objective function and inequality constraints \(g(x) \leq 0\), where
\[
f(x) = \|x\|^2 \quad g(x) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x - \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
The constraints satisfy LICQ for all \(x \in \mathbb{R}^n\). The solution is \(x^* = 0\) and \(\Lambda_\alpha(x^*) = \{(0, 0)\}\). However, \((1, 1)\) is an optimizer of (14), even though \((1, 1) \notin \Lambda_\alpha(x^*)\).

Remark 4.8 (Lagrangian Dual of Continuous Approximation to Projected Gradient): The safe gradient flow can also be implemented using the Lagrangian dual of (17). This is obtained by replacing the feedback controller (14) with
\[
\left\{ \begin{array}{l}
\arg\min_{(u(x), v(x)) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ \frac{1}{2} \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial x} \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix} \right. \\
+ u^T \begin{bmatrix} \frac{\partial g}{\partial x} \nabla f - \alpha g(x) \\ \frac{\partial h}{\partial x} \nabla f - \alpha h(x) \end{bmatrix} + v^T \begin{bmatrix} \frac{\partial h}{\partial x} \nabla f - \alpha h(x) \\ \frac{\partial h}{\partial x} \nabla f - \alpha h(x) \end{bmatrix} \end{array} \right\}
\]
and considering its closed-loop implementation over (7). Though this controller no longer has the same intuitive interpretation as the CBF-QP (14), it has the advantage that its values correspond exactly with \(\Lambda_\alpha(x)\).

Proposition 4.6 shows that there are two equivalent interpretations of the safe gradient flow. The first is as the closed-loop system corresponding to (7) with the controller (14), which maintains forward invariance of the feasible set \(C\) while ensuring the dynamics is as close as possible to the gradient flow of the objective function. The second interpretation is as an approximation of the projection of the gradient flow of the objective function onto the tangent cone of the feasible set. Both interpretations are related by the fact that the Lagrange multipliers corresponding to the approximate projection are the optimal control inputs solving (7). Beyond the interesting theoretical parallelism, this interpretation is instrumental in our ensuing discussion when characterizing the equilibria, regularity, and stability properties of the safe gradient flow.

V. Stability Analysis of the Safe Gradient Flow

Here we conduct a thorough analysis of the stability properties of the safe gradient flow and show that it solves Problem 1. We start by characterizing its equilibria and regularity properties, then focus on establishing the stability properties of local minimizers, and finally characterize the global convergence properties of the flow.

A. Equilibria, Regularity, and Safety

We rely on the necessary optimality conditions introduced in Section IV-C to characterize the equilibria of \(\xi_\alpha\).

Proposition 5.1 (Equilibria of safe gradient flow correspond to KKT points): If MFCQ holds at \(x^* \in C\), then
(i) \(\xi_\alpha(x^*) = 0\) if and only if \(x^* \in X_{\text{KKT}}\);
(ii) If \(x^* \in X_{\text{KKT}}\), then \(\Lambda_\alpha(x^*)\) is the set of Lagrange multipliers of (4) at \(x^*\).

Proof: Suppose that \(\xi_\alpha(x^*) = 0\). By Lemma 4.5, there exists \((u^*, v^*) \in \Lambda_\alpha(x^*)\) such that \((0, u^*, v^*)\) satisfies the necessary optimality conditions in (19), which reduce to
\[
\nabla f(x^*) + \frac{\partial g(x^*)}{\partial x}^T u^* + \frac{\partial h(x^*)}{\partial x}^T v^* = 0 \quad (22a)
\]
\[
\alpha g(x^*) \leq 0 \quad (22b)
\]
\[
\alpha h(x^*) = 0 \quad (22c)
\]
\[
u^* \geq 0 \quad (22d)
\]
\[
(u^*)^T (\alpha g(x^*)) = 0 \quad (22e)
\]
Because \(\alpha > 0\), it follows immediately that (22) implies that \((x^*, u^*, v^*)\) satisfy (6) and \(x^* \in X_{\text{KKT}}\).

Conversely, if \(x^* \in X_{\text{KKT}}\), then for any \((u^*, v^*)\) such that \((u^*, v^*)\) solves (6), we have that \((0, u^*, v^*)\) solves (19), which implies that \(\xi_\alpha(x^*) = 0\) and \((u^*, v^*) \in \Lambda_\alpha(x^*)\).

Proposition 5.1(i) shows that the safe gradient flow meets Problem 1(iii). The correspondence in Proposition 5.1(ii) between the Lagrange multipliers of (17) and the Lagrange multipliers of (4) means that the proposed method can be interpreted as a primal-dual method when implemented via (17). This is because the state of the system (7) corresponds to the primal variable of (4), and the inputs to the system (7) correspond to the dual variables.

We next establish that \(\xi_\alpha\) is locally Lipschitz on an open set containing \(C\) when the MFCQ and CRC conditions hold. This ensures the existence and uniqueness of classical solutions to the safe gradient flow.
Proposition 5.2 (Lipschitzness of safe gradient flow): Let \( \alpha > 0 \) and suppose that (4) satisfies MFCQ and CRC for all \( x \in \mathcal{C}, f, g \) and \( h \) are continuously differentiable, and their derivatives are locally Lipschitz. Then \( \mathcal{G}_\alpha \) is well defined and locally Lipschitz on an open set \( X \) containing \( \mathcal{C} \).

Proof: By the proof of Lemma 4.5, if MFCQ holds at \( x \in \mathcal{C} \), there is an open neighborhood \( U_x \) containing \( x \) on which the constraints of (17) satisfy Slater’s condition. Next, for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, k \),

\[
\frac{\partial}{\partial \xi} (\nabla g_i(x) \xi + \alpha g_i(x)) = \nabla g_i(x) \xi,
\]

\[
\frac{\partial}{\partial \xi} (\nabla h_j(x) \xi + \alpha h_j(x)) = \nabla h_j(x) \xi,
\]

so the gradients of the constraints in (17) are the same as those in (4) and therefore (17) satisfies CRC. Then, \( \mathcal{G}_\alpha \) is the unique solution to (17) on \( U_x \), and by [35, Thm. 3.6], \( \mathcal{G}_\alpha \) is locally Lipschitz on \( U_x \). The desired result follows by letting \( X = \bigcup_{x \in \mathcal{C}} U_x \).

Proposition 5.2 verifies that the safe gradient flow meets Problem 1(i). Next, we show that under slightly stronger constraint qualification conditions at KKT points, the triple satisfying (19) is unique and Lipschitz near them.

Proposition 5.3 (Lipschitzness of the solution to (19)): Let \( x^* \in \mathcal{X}_{\mathcal{KKT}} \) and suppose (4) satisfies LICQ at \( x^* \). Then, there exists an open set \( U \) containing \( x^* \) and Lipschitz functions \( u : U \to \mathbb{R}^n \), \( v : U \to \mathbb{R}^m \) such that \( (\mathcal{G}_\alpha(x), u(x), v(x)) \) is the unique solution to (19) for all \( x \in U \).

Proof: We claim that the variational equation (19) is strongly regular [36] for all \( x^* \in \mathcal{X}_{\mathcal{KKT}} \). Strong regularity implies, cf. [36, Cor. 2.1], that there exists an open set \( U \) containing \( x^* \) and Lipschitz functions \( \xi : U \to \mathbb{R}^n \), \( u : U \to \mathbb{R}^n \), \( v : U \to \mathbb{R}^k \) such that \( (\xi(x), u(x), v(x)) \) is the unique triple solving (17). Since the solution (17) is unique, if such a triple exists, then \( \xi(x) = \mathcal{G}_\alpha(x) \). To prove the claim, we begin by noting that (17) satisfies the strong second-order sufficient condition since \( \nabla^2 \mathcal{L}(\xi, u, v; x) = I \prec 0 \). Let \( (x^*, u^*, v^*) \) be a KKT triple of (4). By Proposition 5.1, \( (0, u^*, v^*) \) satisfies (19). Since the ith inequality constraint of (17) is \( \nabla g_i(x^*)^{\top} \xi + \alpha g_i(x^*) \leq 0 \), when \( \xi = 0 \) the constraint is active if and only if \( g_i(x^*) = 0 \). It follows that when \( x^* \in \mathcal{X}_{\mathcal{KKT}} \), the indices of the active constraints of (17) are the same as those of (4). Moreover, by the reasoning in the proof of Proposition 5.2, gradients of the binding (i.e., the active inequality and equality) constraints of (17) and (4) are also the same. By LICQ, the gradients of the binding constraints are linearly independent, which along with the strong second-order condition implies that (19) is strongly regular by [36, Thm. 4.1].

The significance of Proposition 5.3 is twofold. First, it establishes that, under certain conditions, the Lagrange multipliers of (17) are Lipschitz as a function of \( x \), which ensures the existence of a locally Lipschitz continuous feedback solving (14). Secondly, the result establishes conditions for uniqueness of the Lagrange multipliers in a neighborhood of an equilibrium \( x^* \). These facts will play an important role in the stability analysis of local minimizers in the sequel.

We now show in the next result that the safe gradient flow also meets Problem 1(ii). The result follows by applying Lemma 2.1 with \( \phi = (g, h) \) as a VCBF and local Lipschitz continuity of the closed-loop dynamics, cf. Proposition 5.2.

Theorem 5.4 (Safety of feasible set under safe gradient flow): Consider the optimization problem (4). If MFCQ and CRC are satisfied on \( \mathcal{C} \), then \( \mathcal{C} \) is forward invariant and asymptotically stable under the safe gradient flow.

Remark 5.5 (Advantages of safe gradient flow over projected gradient flow): Unlike the projected gradient flow, the vector field \( \mathcal{G}_\alpha \) is locally Lipschitz, so classical solutions to \( \dot{x} = \mathcal{G}_\alpha(x) \) exist, and the continuous-time flow can be numerically solved using standard ODE discretization schemes. Secondly, under mild conditions, \( \mathcal{G}_\alpha \) is well defined for initial conditions outside \( \mathcal{C} \), allowing us to guarantee convergence to a local minimizer starting from infeasible initial conditions. Finally, because both (16) and (17) are least-squares problems of the same dimension subject to affine constraints, the computational complexity of solving either one is equivalent.

Remark 5.6 (Discretization of safe gradient flow and role of parameter \( \alpha \)): When considering discretizations of the safe gradient flow, the parameter \( \alpha \) plays an important role. By construction, trajectories of the safe gradient flow beginning at infeasible initial conditions converge to \( \mathcal{C} \) at an exponential rate \( \alpha > 0 \), so larger values of \( \alpha \) ensure faster convergence. On the other hand, smaller values of \( \alpha \) result in a design that enforces safety more conservatively and hence, intuitively, this should allow for larger stepsizes. Our preliminary numerical experiments with the forward-Euler discretization \( x^{n+1} = x^n + h \mathcal{G}_\alpha(x) \) confirm these intuitions, showing that larger choices of \( \alpha \) reduce the range of allowable stepsizes \( h \) that preserve the invariance of the feasible set \( \mathcal{C} \) and stability of local minimizers. In particular, we have noticed that the maximal allowable stepsizes \( h^* \) such that \( 0 < h < h^* \) ensures stability and approximate safety, satisfies \( h^* \to 0 \) as \( \alpha \to \infty \). For space reasons, we leave to future work the formal characterization of suitable stepsizes.

B. Stability of Isolated Local Minimizers

Here we characterize the stability properties of isolated local minimizers under the safe gradient flow. The following result shows that the safe gradient flow meets Problem 1(iv). To ensure existence and uniqueness of solutions, we assume throughout the rest of the paper that CRC holds on \( \mathcal{C} \).

Theorem 5.7 (Stability of isolated local minimizers): Consider the optimization problem (4). Let \( x^* \) be a local minimizer and an isolated KKT point, and let \( \bar{U} \) be an open set such that \( x^* \) is the only KKT point contained in \( \bar{U} \).

(i) If MFCQ holds for all \( x \in \bar{U} \cap \mathcal{C} \), then \( x^* \) is asymptotically stable relative to \( \mathcal{C} \);
(ii) If EMFCQ holds for all \( x \in \bar{U} \), then \( x^* \) is asymptotically stable relative to \( \mathbb{R}^n \);
(iii) If LICQ, strict complementarity, and the second-order sufficient condition hold at \( x^* \), then \( x^* \) is exponentially stable relative to \( \mathbb{R}^n \).

We divide the technical discussion leading up to the proof of the result in three parts, corresponding to each statement.
1. Stability of Isolated Local Minimizers Relative to $\mathcal{C}$: Here we analyze the stability of local minimizers relative to the feasible set. We start by characterizing the growth of the objective function along solutions of the safe gradient flow.

Lemma 5.8 (Growth of objective function along safe gradient flow): Let $x \in \mathbb{R}^n$ such that $\Lambda_\alpha(x) \neq \emptyset$. Then, for all $(u, v) \in \Lambda_\alpha(x)$,

$$D^+_{\partial \alpha} f(x) = -\|G_\alpha(x)\|^2 + \alpha u^\top g(x) + \alpha v^\top h(x).$$

and if $x \in \mathcal{C}$ then,

$$D^+_{\partial \alpha} f(x) \leq 0,$$

with equality if and only if $x \in X_{\text{KKT}}$.

Proof: For $x \in X$ (with $X$ given by Proposition 5.2) such that $(u, v) \in \Lambda_\alpha(x) \neq \emptyset$, $(G_\alpha(x), u, v)$ solves (19). Next,

$$D^+_{\partial \alpha} f(x) = G_\alpha(x)^\top \nabla f(x)$$

(a) $-G_\alpha(x)^\top (G_\alpha(x) + \frac{\partial g(x)}{\partial x} u + \frac{\partial h(x)}{\partial x} v)$

(b) $-\|G_\alpha(x)\|^2 + \alpha u^\top g(x) + \alpha v^\top h(x),$

where (a) follows by rearranging (19a), and (b) follows from (19c) and (19e).

To show the second statement, note that if $x \in \mathcal{C}$, then $g(x) \leq 0$ and $h(x) = 0$. Since $u \geq 0$, it follows $\alpha u^\top g(x) + \alpha v^\top h(x) \leq 0$ and therefore $D^+_{\partial \alpha} f(x) \leq -\|G_\alpha(x)\|^2$. Finally, $D^+_{\partial \alpha} f(x) = 0$ if and only if $G_{\partial \alpha}(x) = 0$, which by Proposition 5.1, is equivalent to $x \in X_{\text{KKT}}$.

As a consequence of Lemma 5.8, the objective function decreases monotonically along the solutions starting in $\mathcal{C}$. Thus, the objective function can be used as a Lyapunov function certifying asymptotic stability of an isolated equilibria relative to $\mathcal{C}$.

Proof of Theorem 5.7(ii): By hypothesis and using Lemma 4.5, $\Lambda_\alpha(x) \neq \emptyset$ for all $x \in U \cap \mathcal{C}$. Because $x^*$ is the unique strict minimizer of $f$ on $U \cap \mathcal{C}$, and by Lemma 5.8, $D^+_{\partial \alpha} f(x^*) < 0$ for all $x \in U \cap \mathcal{C} \setminus \{x^*\}$, it follows from Lemma B.1 that $x^*$ is asymptotically stable relative to $\mathcal{C}$.

2) Stability of Isolated Local Minimizers Relative to $\mathbb{R}^n$: Here we establish the asymptotic stability of isolated local minima relative to $\mathbb{R}^n$. To do so, we cannot rely any more on the objective function $f$ as a Lyapunov function. This is because outside of $\mathcal{C}$, there may exist points $x \in \mathbb{R}^n \setminus \mathcal{C}$ where $f(x) < f(x^*)$. Therefore, to show stability relative to $\mathbb{R}^n$, we need to identify an alternative function whose unconstrained minimizer is $x^*$. In fact, the problem of finding a function whose unconstrained minimizers correspond to the local minimizers of a nonlinear program is well studied in the optimization literature [37]: such functions are called exact penalty functions. Our discussion proceeds by constructing an exact penalty function that is also a Lyapunov function for the safe gradient flow.

Let $\Omega \subset \mathbb{R}^n$ be a compact set. A function $V : \Omega \times (0, \infty) \rightarrow \mathbb{R}$ is a strong exact penalty function relative to $\Omega$ if there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$, $x^* \in \text{int}(\Omega)$ is a local minimizer of $V_\epsilon(x) := V(x, \epsilon)$ if and only if $x^*$ is a local minimizer of (4). The following result gives a strong exact penalty function for (4).

Lemma 5.9 (Existence of strong exact penalty function): Let $\Omega \subset \mathbb{R}^n$ be compact such that $\text{int}(\Omega) \cap \mathcal{C} \neq \emptyset$. Suppose (4) satisfies EMFCQ at every $x \in \Omega$ and let $V : \Omega \times (0, \infty) \rightarrow \mathbb{R}$,

$$V(x, \epsilon) = f(x) + \frac{1}{\epsilon} \sum_{i=1}^m |g_i(x)|_+ + \frac{1}{\epsilon} \sum_{j=1}^k |h_j(x)|. \quad (23)$$

Then, $V$ is a strong exact penalty function relative to $\Omega$, $V$ is directionally differentiable on $\Omega$, and

$$D^+_{\partial \alpha} V_\epsilon(x) = D^+_{\partial \alpha} f(x)$$

$$+ \frac{1}{\epsilon} \sum_{i \in I(x)} D^+_{\partial \alpha} g_i(x) + \frac{1}{\epsilon} \sum_{j \in I(x)} \text{sgn}(h_j(x)) D^+_{\partial \alpha} h_j(x),$$

for all $x \in \Omega$.

Proof: The fact that $V$ is a strong exact penalty function relative to $\Omega$ readily follows from [37, Thm. 4]. From [37, Prop. 3], $V_\epsilon$ is directionally differentiable on $\Omega$ and its directional derivative in the direction $x \in \mathbb{R}^n$ is

$$V_\epsilon'(x; \xi) = \nabla f(x)^\top \xi$$

$$+ \frac{1}{\epsilon} \sum_{i \in I(x)} \nabla g_i(x)^\top \xi + \frac{1}{\epsilon} \sum_{j \in I(x)} |\nabla g_i(x)^\top \xi|_+$$

$$+ \frac{1}{\epsilon} \sum_{j \text{ such that } h_j(x) \neq 0} \text{sgn}(h_j(x)) \nabla h_j(x)^\top \xi + \frac{1}{\epsilon} \sum_{j \text{ such that } h_j(x) = 0} |\nabla h_j(x)^\top \xi|_+.$$
Next, choose $0 < \varepsilon_2 < \frac{1}{B}$ where $B > 0$ satisfies the bound given by Lemma D.1. Then, for $\varepsilon < \varepsilon_2$, 
\[ \sum_{i \in I \setminus \{x\}} (u_i - \frac{1}{\varepsilon}) g_i(x) + \sum_{j=1}^{k} \left( \frac{|v_j| - 1}{\varepsilon} \right) |h_j(x)| < 0. \]
Finally, since $u \geq 0$, we have $\alpha \sum_{i \in I \setminus \{x\}} u_i g_i(x) \leq 0$. Thus, 
\[ D^+_{\alpha} V_\varepsilon(x) \leq -\|G_\alpha(x)\|^2 < 0, \]
for all $x \in U \setminus \{x^*\}$, whenever $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore $V_\varepsilon$ is a Lyapunov function on $U$ and asymptotic stability of $x^*$ relative to $\mathbb{R}^n$ follows by Lemma B.1. 

Remark 5.10 (Relationship to merit functions in numerical optimization): In numerical optimization, the $\ell^1$ penalty function in (23) is often used as a merit function, i.e., a function that quantifies how well a single iteration of an optimization algorithm balances the two goals of reducing the value of the objective function and reducing the constraint violation (cf. [38, Sec. 15.4]). Typically, the stepsize on each iteration is chosen so that the merit function is nonincreasing. Thus, if the algorithm is viewed as a discrete-time dynamical system, the merit function is a Lyapunov function. The $\ell^1$ penalty plays a similar role for the continuous-time system described here. 

3) Exponential Stability of Isolated Local Minimizers: We now discuss the exponentially stability of isolated local minimizers. Our first step is to identify conditions under which the safe gradient flow is differentiable. To do so, we introduce the notions of strict complementarity and second-order condition on the optimization problem.

Definition 5.11 (Strict complementarity and second-order sufficient conditions): Let $(x^*, u^*, v^*)$ be a KKT triple of (4).

- The strict complementarity condition holds if $u_i^* > 0$ for all $i \in I_0(x^*)$;
- The second-order sufficient condition holds if $z^T Q z > 0$ for all $z \in \ker \left( \frac{\partial g_i(x^*)}{\partial x} \right) \cap \ker \left( \frac{\partial h_i(x^*)}{\partial x} \right)$, where
\[ Q = \nabla^2 f(x^*) + \sum_{i=1}^{m} u_i^* \nabla^2 g_i(x^*) + \sum_{j=1}^{k} v_j^* \nabla^2 h_j(x^*). \]

When LICQ holds, strict complementarity together with the second-order sufficient condition can be used to establish the differentiability of a KKT triple of a nonlinear parametric program with respect to the parameters [39]. When these conditions are satisfied by (4), we show next that $G_\alpha$ is differentiable and provide an expression for its Jacobian. A step-by-step computation of the Jacobian is in Appendix E.

Lemma 5.12 (Jacobian of safe gradient flow): Let $x^* \in X_{KKT}$ and $(u^*, v^*)$ be the associated Lagrange multipliers. Suppose

- LICQ holds at $x^*$;
- $(x^*, u^*, v^*)$ satisfies the strict complementarity condition;
- $(x^*, u^*, v^*)$ satisfies the second-order sufficient condition.

Then $G_\alpha$ is differentiable at $x^*$ and
\[ \frac{\partial G_\alpha(x^*)}{\partial x} = -PQ - \alpha(I - P), \]
where $I$ is the $n \times n$ identity matrix, $P$ is the orthogonal projection matrix onto $\ker \left( \frac{\partial g_i(x^*)}{\partial x} \right) \cap \ker \left( \frac{\partial h_i(x^*)}{\partial x} \right)$, and $Q$ is defined in (24).

Proof: By Proposition 5.3, there is a neighborhood $U$ of $x^*$ where the unique KKT triple of (17) corresponding to $x \in U$ is $(G_\alpha(x), u(x), v(x))$. From the proof of that result, LICQ and the second-order sufficient condition hold for (17) at $x^*$. Further, the indices of the active constraints of (4) are the same as those of (17). Because $(x^*, u^*, v^*)$ satisfies the strict complementarity condition for (4), $(0, u^*, v^*)$ satisfies the strict complementarity condition for (17). Thus, by [40, Cor. 1], $G_\alpha$ is continuously differentiable at $x^*$, and by following the steps in Appendix E, we obtain (25).

Using the result in Lemma 5.12, stability of an isolated local minimizer can be inferred by showing that the eigenvalues of the Jacobian of the safe gradient flow are all strictly negative.

Proof of Theorem 5.7(iii): By the second-order sufficient condition, $z^T P Q P z > 0$ for all $z \in \ker P \setminus \{0\}$. It follows that $P Q P z = 0$ if and only if $z \in \ker P$. Therefore 0 is an eigenvalue of $P Q P$ with multiplicity $r$ and $P Q P$ has $n - r$ strictly positive eigenvalues, where $r = \dim \ker P$.

Let $z_1, \ldots, z_r$ be the eigenvectors corresponding to the zero eigenvalues, and $z_{r+1}, \ldots, z_n$ be eigenvectors corresponding to the positive eigenvalues, denoted $\lambda_{r+1}, \ldots, \lambda_n$. Then
\[ P z_i = \begin{cases} 0 & i = 1, \ldots, r, \\ z_i & i = r + 1, \ldots, n. \end{cases} \]

Let
\[ \mu_i = \begin{cases} 0 & i = 1, \ldots, r, \\ \lambda_i - \alpha & i = r + 1, \ldots, n. \end{cases} \]

Then, it follows that $(P Q P - \alpha P) z_i = \mu_i z_i$ for all $1 \leq i \leq n$.

Observe that $P Q P - \alpha P = (P Q - \alpha I) P$ has precisely the same eigenvalues as $P (P Q - \alpha I) = P Q - \alpha P$. Therefore, since $\mu_i$ is an eigenvalue of $P Q - \alpha P$, it follows that $\mu_i + \alpha$ is an eigenvalue of
\[ P Q - \alpha P + \alpha I = P Q + \alpha (I - P) = -\frac{\partial G_\alpha(x^*)}{\partial x}. \]

Hence the eigenvalues of $\frac{\partial G_\alpha(x^*)}{\partial x}$ are
\[ \{-\alpha, -\alpha, \ldots, -\alpha, -\lambda_{r+1}, \ldots, -\lambda_n\}, \]
where $-\alpha$ appears with multiplicity $r$. Since all the eigenvalues are strictly negative, $x^*$ is exponentially stable.

C. Stability of Nonisolated Local Minimizers

We have characterized in Section V-B the stability under the safe gradient flow of local minimizers that are isolated KKT points. In general, if $x^*$ is strict local minimizer that is not an isolated KKT point (for example, if there are an infinite number of local maximizers arbitrarily close to $x^*$, cf. [41, page 5]), or if $x^*$ is only a local minimizer, then there are no guarantees on Lyapunov stability. However, as we show here, nonisolated minimizers are stable under the safe gradient flow under additional assumptions on the problem data.

When there are no constraints, the safe gradient flow reduces to the classical gradient flow, where conditions for semistability of local minimizers are well known: if the objective function is real-analytic, then all trajectories of the gradient flow have finite arclength, cf. [42], in which case the objective
function can be used to construct an arclength-based Lyapunov function satisfying the hypotheses of Lemma B.2 to establish semistability. In this section, we conduct a similar analysis for the constrained case. Our main result is as follows.

**Theorem 5.13 (Stability of nonisolated local minima):** Consider the optimization problem (4), and assume $f$, $g$, and $h$ are real-analytic. Let $S$ be a bounded set of local minimizers on which $f$ is constant and equal to $f^*$ such that

(i) There is an open set $U$ and $\beta > 0$ such that $U \cap X_{KKT} = S$ and $f(x) - f^* \geq \beta \operatorname{dist}_S(x)^2$ for all $x \in U \cap S$;

(ii) LICQ is satisfied at all $x^* \in S$;

(iii) $T_S(x^*) \cap N_{\mathcal{S}^{\text{prox}}}(x^*) = \{0\}$ for all $x^* \in S$.

Then there is $\alpha^* > 0$ such that every $x^* \in S$ is semistable relative to $\mathbb{R}^n$ under the safe gradient flow $G_\alpha$, for $\alpha > \alpha^*$.

To prove this result, we first discuss various intermediate results. In particular, the growth condition in Theorem 5.13(i) plays a crucial role in the construction of a Lyapunov function to prove the result. Any $x^* \in S$ satisfying this property is called a weak sharp minimizer of $f$ relative to $S$. Weak sharp minimizers play an important role in sensitivity analysis for nonlinear programs as well as convergence analysis for numerical methods in optimization [43], [44].

We review second-order optimality conditions for weak sharp minimizers. Let $x^* \in X_{KKT}$, suppose that LICQ holds at $x^*$, and let $(u^*, v^*)$ be the unique Lagrange multipliers of (4) associated to $x^*$. Define the index set of strongly active constraints as

$$I_0^+(x^*) = \{1 \leq i \leq m \mid u^*_i > 0\}.$$  

The critical cone is

$$\Gamma(x^*) = \{d \in \mathbb{R}^n \mid \nabla h_j(x^*)^\top d = 0, j = 1, \ldots, k, \quad \nabla g_i(x^*)^\top d = 0, i \in I_0^+(x^*), \quad \nabla g_j(x^*)^\top d \leq 0, j \in I_0(x^*) \setminus I_0^+(x^*)\}.$$  

**Lemma 5.14 (Second-order necessary condition for constrained weak sharp minima [44, Prop. 3.5]):** Consider (4) and let $S \subset C$ be a set on which $f$ is constant. Suppose that $x^* \in \partial S$ is a weak sharp local minimizer of $f$ relative to $S$ and LICQ is satisfied at $x^*$. Let $u^*, v^*$ be the Lagrange multipliers and define $\ell(x) = f(x) + (u^*)^\top g(x) + (v^*)^\top h(x)$. Then, there exists $\gamma > 0$ such that, for all $d \in \Gamma(x^*)$,

$$\ell''(x^*; d) \geq \gamma \operatorname{dist}_S(x^*) (d^2).$$

**Lemma 5.15 (Second-order sufficient condition for unconstrained weak sharp minima [44, Thm. 2.5]):** Consider $W : \mathbb{R}^n \to \mathbb{R}$ and suppose that $W$ is constant on $S$. Suppose $x^* \in \partial S$ and $W''(x^*; d) > 0$ for all $d \in N_{\mathcal{S}^{\text{prox}}}(x^*) \setminus \{0\}$, then $x^*$ is a weak sharp local minimizer of $W$ relative to $S$.

We now proceed with the construction of the Lyapunov function. Let $T_\mathcal{C}(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ be the set-valued map where, for each $x \in \mathbb{R}^n$, $T_\mathcal{C}(\alpha)(x)$ is the constraint set of (17). Let $J_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be

$$J_\alpha(x, \xi) = \alpha f(x) + \nabla f(x)^\top \xi + \frac{1}{2} \|\xi\|^2.$$  

Consider the optimization problem

$$\minimize_{\xi \in T_\mathcal{C}(\alpha)(x)} J_\alpha(x, \xi)$$  

(27)

As we show next, the solution to (27) is (17).

**Lemma 5.16 (Correspondence between (27) and (17)):** Let $x \in \mathbb{R}^n$. Then the program (17) has a solution at $x$ if and only if (27) has a solution, in which case $G_\alpha(x) = \arg \min_{\xi \in T_\mathcal{C}(\alpha)(x)} \{J_\alpha(x, \xi)\}$.

**Proof:** Note that the feasible sets of (27) and (17) coincide. Next, for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$

$$J_\alpha(x, \xi) - \frac{1}{2} \|\xi + \nabla f(x)\|^2 = \alpha f(x) - \frac{1}{2} \|\nabla f(x)\|^2.$$  

Since the difference of the objectives in (27) and (17) does not depend on $\xi$, both problems have the same optimizer.

**Lemma 5.16** shows that (27) is another characterization of the safe gradient flow in terms of a parametric quadratic program. Let $W_\alpha : X \to \mathbb{R}$ be the value function

$$W_\alpha(x) = \inf_{\xi \in T_\mathcal{C}(\alpha)(x)} \{J_\alpha(x, \xi)\} = \alpha f(x) + \nabla f(x)^\top G_\alpha(x) + \frac{1}{2} \|G_\alpha(x)\|^2.$$  

(28)

Our strategy to prove Theorem 5.13 consists of showing that $W_\alpha$ is a Lyapunov function satisfying the hypotheses in Lemma B.3 whenever $\alpha$ is sufficiently large. Towards this end, we begin by computing the directional derivative of $W_\alpha$. Let $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the matrix-valued function,

$$Q(x, u, v) = \nabla^2 f(x) + \sum_{i=1}^m u_i \nabla^2 g_i(x) + \sum_{j=1}^k v_j \nabla^2 h_j(x).$$  

Since the Lagrange multipliers, $(u(x), v(x))$ are unique in a neighborhood of $S$, we slightly abuse notation by defining $Q(x) := Q(x, u(x), v(x))$. By Lipschitzness of $u$ and $v$, $Q$ is continuous on $X$. The proof of the next result follows from [40, Thm. 2] and [45, Cor. 4.1] and is omitted for brevity.

**Lemma 5.17 (Differentiability of $W_\alpha$):** Suppose that $S$ satisfies the hypotheses in Theorem 5.13, and $X$ is an open set containing $S$ on which $(G_\alpha(x), u(x), v(x))$ is the unique solution to (19). Then

(i) For all $x \in X$, $W_\alpha$ is differentiable with

$$\nabla W_\alpha(x) = -(\alpha I - Q(x)) G_\alpha(x);$$  

(29)

(ii) For all $x^* \in S$, $W_\alpha$ is twice directionally differentiable in any direction $d \in \mathbb{R}^n$, where

$$W_\alpha''(x^*; d) = \min_{\zeta \in \mathbb{R}^n} \left[\begin{array}{c} d^\top (\alpha Q(x^*) - Q(x^*)) \\ (Q(x^*) - I) \zeta \end{array}\right]^\top \left[\begin{array}{c} d \\ \zeta \end{array}\right].$$  

s.t. $\alpha \nabla h_j(x^*)^\top d + \nabla h_j(x^*)^\top \zeta = 0$, $\forall j = 1, \ldots, k$,

$$\alpha \nabla g_i(x^*)^\top d + \nabla g_i(x^*)^\top \zeta = 0,$$  

$$\forall i \in I_0^+(x^*),$$  

$$\alpha \nabla g_i(x^*)^\top d + \nabla g_i(x^*)^\top \zeta \leq 0,$$  

$$\forall i \in I_0(x^*) \setminus I_0^+(x^*).$$

**Remark 5.18 (Dependence of $Q(x)$ on $\alpha$):** In general, given $x \in X$, the value of $Q(x)$ depends on the choice of $\alpha$, since $(u(x), v(x))$ depends on $\alpha$. However, if $x^* \in X_{KKT}$, then $(u(x^*), v(x^*))$ corresponds to the Lagrange multipliers of (4) and $Q(x^*)$ is the Hessian of the Lagrangian of (4). In
particular, this means that for $x^* \in X_{KKT}$, the value of $Q(x^*)$ depends only on the problem data and is independent of $\alpha$. 

We now proceed with the proof of Theorem 5.13.

**Proof of Theorem 5.13:** Let $\alpha^* = \sup_{x^* \in S}\{\rho(Q(x^*))\}$. For $\alpha > \alpha^*$, we have $\alpha I - Q(x^*) \succ 0$ for all $x^* \in S$. Assume without loss of generality that $\alpha I - Q(x^*) \succ 0$ for all $x \in U$ (if not, since $Q$ is continuous, we can always find an open subset of $U$ containing $S$ for which this property holds). We claim that $W_\alpha$ satisfies each of the conditions (i)-(iii) in Lemma B.3 with $K = \mathbb{R}^n$.

We begin by showing condition (iii). If $x^* \in U$ is a local minimizer of $W_\alpha$, then $\nabla W_\alpha(x^*) = (\alpha I - Q(x^*))\mathcal{G}_\alpha(x^*) = 0$. Since $\alpha I - Q(x^*) \succ 0$, from (29) we deduce $\mathcal{G}_\alpha(x^*) = 0$, so $x^* \in X_{KKT}$ and therefore $x^* \in U \cap X_{KKT} = S$.

Conversely, suppose that $x^* \in S$. Note that, by Proposition 5.1, $W_\alpha(x) = \alpha f(x)$ for all $x \in S$. Therefore, if $x^* \in \text{int}(S)$, it follows that $x^*$ is a local minimizer of $W_\alpha$. On the other hand, suppose that $x^* \in \partial S$. For $d \in \mathbb{R}^n$, let $\zeta_d$ be the unique optimizer of (30). Then

$$W_\alpha''(x^*; d) = \alpha d^T Q(x^*)d + 2\zeta_d^T Q(x^*)d + \|\zeta_d\|^2. \quad (31)$$

From the constraints in (30), $\zeta_d + \alpha d \in \mathcal{G}(x^*)$. Because $x^* \in \partial S$ is a weak sharp minimizer of $f$ relative to $S$, by Lemma 5.14, there exists $\gamma > 0$ such that

$$\ell''(x^*; \zeta_d + \alpha d) = (\zeta_d + \alpha d)^T \nabla^2 \ell(x^*)(\zeta_d + \alpha d), \quad (32)$$

Since $\nabla^2 \ell(x^*) = Q(x^*)$, we combine (31) and (32) to get

$$\alpha W_\alpha''(x^*; d) \geq \zeta_d^T (\alpha I - Q(x^*)) \zeta_d + \gamma \text{dist}_{T_S(x^*)}(\zeta_d + \alpha d)^2.$$ 

Because $\alpha I - Q(x^*) \succ 0$, if $W_\alpha''(x^*; d) > 0$, then $\zeta_d = 0$ and $d \in T_S(x^*)$. But $T_S(x^*) \cap N_S^{\text{prox}}(x^*) = \{0\}$, which means $W_\alpha''(x^*; d) > 0$ for all $d \in N_S^{\text{prox}}(x^*) \setminus \{0\}$, so by Lemma 5.15, $x^*$ is a weak sharp local minimizer of $W_\alpha$.

Next we verify condition (ii) in Lemma B.3. For all $x \in U$,

$$D^+_{\mathcal{G}_\alpha} W_\alpha(x) = -\mathcal{G}_\alpha(x)^T (\alpha I - Q(x)) \mathcal{G}_\alpha(x).$$

Without loss of generality, we can assume that $U$ is bounded. Then, we can choose $c_1, c_2 > 0$ so that

$$c_1 < \inf_{x \in U} \{\lambda_{\min}(\alpha I - Q(x))\},$$

$$c_2 > \sup_{x \in U} \{\lambda_{\max}(\alpha I - Q(x))\}.$$ 

It follows that $D^+_{\mathcal{G}_\alpha} W_\alpha(x) \leq -c_1 \|\mathcal{G}_\alpha(x)\|^2$ for all $x \in U$, but since $\|\nabla W_\alpha(x)\| \leq c_2 \|\mathcal{G}_\alpha(x)\|$, we have for all $x \in U$,

$$D^+_{\mathcal{G}_\alpha} W_\alpha(x) \leq -\frac{c_1}{c_2} \|\nabla W_\alpha(x)\| \|\mathcal{G}_\alpha(x)\|.$$ 

Finally, we claim that $W_\alpha|_U$ is a globally subanalytic function, and therefore condition (i) holds by [46, Thm. 1] and the fact that the class of globally subanalytic sets is $\omega$-minimal structure (cf. [46, Definition 1]). To prove the claim, first note that, since $f$ is real-analytic, $J_\alpha$ is real-analytic, and therefore subanalytic [47, Definition 3.1]. Since $U$ is bounded, and the restriction of any subanalytic function to a bounded open set is globally subanalytic [48], it follows that $J_\alpha|_U$ is globally subanalytic. Finally, since $T^{(\alpha)}_C|_U : U \Rightarrow \mathbb{R}^n$ is a globally subanalytic set valued map, and

$$W_\alpha|_U(x) = \inf_{\xi \in \mathcal{T}^{(\alpha)}_C|_U(x)} \{\mathcal{J}_\alpha|_U(x, \xi)\},$$

it follows by application of Lemma A.2 that $W_\alpha|_U$ is globally subanalytic. The statement follows by applying Lemma B.3 with $K = \mathbb{R}^n$.

**D. Global Convergence**

Finally, we turn to the characterization of the global convergence properties of the safe gradient flow. We show that when the problem data are real-analytic and the feasible set is bounded, every trajectory converges to a KKT point.

**Theorem 5.19 (Global convergence properties):** Consider the optimization problem (4), and assume $C$ is bounded, $f$, $g$, and $h$ are real-analytic functions, and LICQ holds everywhere on $C$. Let $X$ be an open set containing $C$ on which the safe gradient flow is well defined. Then there is $\alpha^* > 0$ such that for $\alpha > \alpha^*$, every trajectory of the safe gradient flow starting in $X$ converges to some KKT point.

To prove Theorem 5.19, we use the next result characterizing the positive limit set of solutions of the safe gradient flow.

**Lemma 5.20 (Convergence to connected component):** Consider the optimization problem (4), and assume $C$ is bounded, $f$, $g$, and $h$ are real-analytic functions, and MFQC holds everywhere on $C$. Let $X$ be an open set containing $C$ on which the safe gradient flow is well defined. Then for all $x \in X$, $\omega(x)$ is contained in a unique connected component of $X_{KKT}$.

**Proof:** By Theorem 5.4, $C$ is asymptotically stable and forward invariant on $X$, and by Lemma 5.8, $D^+_{\mathcal{G}_\alpha} f(x) \leq 0$ for all $x \in C$. Using the terminology from [49], $\tilde{f}$ is a height function of the pair $(C, \mathcal{G}_\alpha)$.

Because $f$, $g$, and $h$ are real-analytic and $C$ is bounded, $C$ is a globally subanalytic set. Let $\tilde{f} = f + \delta C$. Then $\tilde{f}$ is a globally subanalytic function, $\tilde{f}$ is continuous on $\text{dom}(\tilde{f}) = C$, and $X_{KKT}$ is precisely the set of critical points of $\tilde{f}$. By the Morse-Sard Theorem for subanalytic functions [50, Thm. 14], $X_{KKT}$ has at most a countable number of connected components, and $\tilde{f}$ is constant on each connected component. Since $f(x) = f(x)$ for all $x \in C$, $\tilde{f}$ is also constant on each connected component of $X_{KKT}$, meaning that the connected components of $X_{KKT}$ are contained in $f$ (cf. [49, Definition 5]).

Hence, we can apply [49, Thm. 6], and conclude that for all $x \in X$, the positive limit set $\omega(x)$ is nonempty and contained in a unique connected component of $E$, where

$$E = \{x \in C \mid D^+_{\mathcal{G}_\alpha} f(x) = 0\}.$$ 

However, by Lemma 5.8, $E = X_{KKT}$, concluding the result. 

We are ready to prove Theorem 5.19.

**Proof of Theorem 5.19:** By Lemma 5.20, for $x \in X$, there is a connected component $S \subset X_{KKT}$ such that $\omega(x) \subset S$. Since LICQ holds on $S$, by Proposition 5.3 there is an open set $U$ containing $S$ and Lipschitz functions $(u, v) : U \rightarrow \mathbb{R}^m \times \mathbb{R}^k$ such that $U \cap X_{KKT} = S$ and $(\mathcal{G}_\alpha(x), u(x), v(x))$ is the unique solution to (19) on $U$.

Let $W_\alpha$ be given by (28). By Lemma 5.17, $W_\alpha$ is differentiable on $U$, and using the same reasoning as in the proof
of Theorem 5.13, \( W_\alpha \) is a globally subanalytic function, and satisfies the Kurdyka-Lojasiewicz inequality. Furthermore, if \( \alpha > \alpha^* = \sup_{x^* \in \mathcal{S}} \{\rho(Q(x^*))\} \), then there is some \( c > 0 \) such that
\[
D_{\mathcal{G}_\alpha}^- W_\alpha(y) \leq -c \|\nabla W_\alpha(y)\| \|\mathcal{G}_\alpha(y)\| \text{ for all } y \in U.
\]
Thus, we can apply Lemma B.3 with \( \mathcal{K} = \mathbb{R}^n \) to conclude that every trajectory starting in \( U \) that remains in \( U \) for all time converges to a point in \( \mathcal{S} \). However, since \( \omega(x) \subset \mathcal{S} \), there exists a \( T > 0 \) such that \( \Phi_T(x) \in \mathcal{U} \), and for all \( t > 0 \), \( \Phi_t(\Phi_T(x)) = \Phi_{T+t}(x) \in \mathcal{U} \). Thus, there exists \( x^* \in \mathcal{S} \) such that \( \Phi_{T+t}(x) \to x^* \) as \( t \to \infty \), and therefore the trajectory starting at \( x \) converges to \( x^* \).

Remark 5.21 (Lower bounds on the parameter \( \alpha \) to ensure global convergence): Note that the proof of Theorem 5.19 yields the expression \( \alpha^* = \sup_{x^* \in \mathcal{S}} \{\rho(Q(x^*))\} \) for the lower bound on \( \alpha \) that guarantees global convergence. In general, computing this expression requires knowledge of the primal and dual optimizers of the original problem. However, reasonable assumptions on \( f, g, \) and \( h \) allow us to obtain upper bounds of \( \alpha^* \). For instance, if \( C \) is polyhedral and \( \nabla f \) is \( \ell_f \)-Lipschitz on \( C \), it follows that \( \|\nabla^2 f(x)\| \leq \ell_f \), and \( \nabla^2 g_i(x) = 0 \) and \( \nabla^2 h_j(x) = 0 \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, k \). Therefore, \( \alpha^* \leq \ell_f \), and \( \ell_f \) can be used instead as a lower bound on \( \alpha \) to ensure global convergence.

VI. COMPARISON WITH OTHER OPTIMIZATION METHODS

Here we compare the safe gradient flow with other continuous-time flows to solve optimization problems. For simplicity, we restrict our attention to an inequality constrained convex program. Figure 3 shows the outcome of the comparison on the same example problem taken from [5]. The methods compared are the projected gradient flow, the logarithmic barrier method (see e.g., [51, Sec. 3]), the \( \ell^2 \)-penalty gradient flow (see e.g., [52, Ch. 4]), the projected saddle-point dynamics (see e.g., [24]), the globally projected dynamics (see e.g., [18]), and the safe gradient flow.

Under the logarithmic barrier method, the feasible set is forward invariant and the minimizer of the logarithmic barrier penalty \( f_{\text{barrier}}(x; \mu) = f(x) - \mu \sum_{i=1}^m \log(-g_i(x)) \), with \( \mu > 0 \), does not correspond to the minimizer of (4). Under the unconstrained minimizer of the \( \ell^2 \)-penalty, \( f_{\text{penalty}}(x; \epsilon) = f(x) + \frac{\epsilon}{2} \sum_{i=1}^m |g_i(x)|^2 \), with \( \epsilon > 0 \), does not correspond to the minimizer of (4), and the feasible set is not forward invariant under the gradient flow of \( f_{\text{penalty}} \). Under the projected saddle-point dynamics, the feasible set is not forward invariant, but each trajectory converges to \( x^* \). Under the globally projected dynamics, the feasible set is forward invariant, trajectories converge to \( x^* \), and trajectories are smooth. However, unlike the safe gradient flow, the globally projected dynamics may be undefined when the constraints are not convex.

VIII. CONCLUSIONS

We have introduced the safe gradient flow, a continuous-time dynamical system to solve constrained optimization problems that makes the feasible set forward invariant. The system can be derived either as a continuous approximation of the projected gradient flow or by augmenting the gradient flow of the objective function with inputs, then using a control barrier function-based QP to ensure safety of the feasible set. The equilibria are exactly the critical points of the optimization problem, and the steady-state inputs at the equilibria correspond to the dual optimizers of the program. We have conducted a thorough stability analysis of the dynamics, identified conditions under which isolated local minimizers are asymptotically stable and nonisolated local minimizers are semistable. Future work will explore the flow’s robustness properties, and leverage convexity to obtain stronger global convergence guarantees. Further, we hope to explore issues related to the practical implementation of the safe gradient flow, including interconnections of the optimizing dynamics with a physical system, develop discretizations of the dynamics and study their relationship with discrete-time iterative methods for nonlinear programming, and extend the framework to Newton-like flows for nonlinear programs which incorporate higher-order information.
**APPENDIX A**

**The Kurdyka–Łojasiewicz Inequality**

Here we discuss the Kurdyka–Łojasiewicz inequality, which plays a critical role in the stability analysis of the systems considered in this paper. The original formulation of the Łojasiewicz inequality [42] states that for a real-analytic function $V : \mathbb{R}^n \to \mathbb{R}$ and a critical point $x^* \in V^{-1}(0)$, there exists $\rho > 0$, $\theta \in [0, 1)$, and $c > 0$ with $V(x) < c < c\|\nabla V(x)\|$ for all $x$ in a bounded neighborhood of $x^*$ such that $|V(x)| \leq \rho$. This inequality is used to establish that trajectories of gradient flows of real-analytic functions have finite arclength and converge pointwise to the set of equilibria.

In many applications, the assumption of real analyticity is too strong. For example, the value function of a parametric nonlinear program generally does not satisfy this assumption, even when all the problem data is real-analytic. However, generalizations of the Łojasiewicz inequality have since been shown [46], [53] to hold for much broader classes of functions, which can be characterized using the notion of o-minimal structures, which we define next.

**Definition A.1 (o-minimal structures):** For each $n \in \mathbb{N}$, let $\mathcal{O}_n$ be a collection of subsets of $\mathbb{R}^n$. We call $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ an o-minimal structure if the following properties hold.

(i) $\mathcal{O}_n$ is closed under complements, finite unions and finite intersections.

(ii) If $A \in \mathcal{O}_{n_1}$ and $B \in \mathcal{O}_{n_2}$, then $A \times B \in \mathcal{O}_{n_1+n_2}$.

(iii) Let $\pi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ be the projection map onto the first $n$ components. If $A \in \mathcal{O}_{n+1}$, then $\pi(A) \in \mathcal{O}_n$.

(iv) Let $g_1, \ldots, g_m$ and $h_1, \ldots, h_k$ be polynomial functions on $\mathbb{R}^n$ with rational coefficients. Then $\{x \in \mathbb{R}^n | g_i(x) < 0, h_j(x) = 0, 1 \leq i \leq m, 1 \leq j \leq k\} \in \mathcal{O}_n$.

(v) $\mathcal{O}_1$ is precisely the collection of all finite unions of intervals in $\mathbb{R}$.

Examples of o-minimal structures include the class of semi-algebraic sets and the class of globally subanalytic sets. We refer the reader to [48] for a detailed overview of these concepts. The notion of o-minimality plays a crucial role in optimization theory, since the remarkable geometric properties of definable functions allows nonlinear programs involving them to be studied using powerful tools from real algebraic geometry and variational analysis, cf. [54].

Let $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$ be an o-minimal structure. A set $X \subseteq \mathbb{R}^n$ such that $X \in \mathcal{O}_n$ is said to be definable with respect to $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$. When the particular o-minimal structure is obvious from context, then we simply call $X$ definable. Given a definable set $X$ and $f : X \to \mathbb{R}^m$ and $F : X \to \mathbb{R}^m$, we say that $f$ (resp. $F$) is definable if $\text{graph}(f) \in \mathcal{O}_{n+m}$ (resp. $\text{graph}(F) \in \mathcal{O}_{n+m}$). The image and preimage of a definable set with respect to a definable function is also definable, and the class of definable functions is closed with respect to composition and linear combinations. Furthermore, as we show next, the value function of a parametric nonlinear program is definable when the problem data is definable.

**Lemma A.2 (Definability of value functions):** Let $X \subseteq \mathbb{R}^n$, $J : X \times \mathbb{R}^m \to \mathbb{R}$ and $F : X \to \mathbb{R}^m$ be definable. Let $V : X \to \mathbb{R}$ be given by $V(x) = \inf_{x \in F(x)} \{J(x, \xi)\}$, and suppose that $\text{dom}(V) = X$. Then $V$ is also definable.

Finally, functions definable on o-minimal structures satisfy a generalization of the Łojasiewicz inequality [46].

**Lemma A.3 (Kurdyka–Łojasiewicz inequality for definable functions):** Let $X \subseteq \mathbb{R}^n$ be a bounded, open, definable set, and $V : X \to \mathbb{R}$ a definable, differentiable function, and $V^* = \inf_{x \in X} V(y)$. Then there exists $c > 0$, $\rho > 0$, and a strictly increasing, definable, differentiable function $\psi : [0, \infty) \to \mathbb{R}$ such that

$$\psi'(V(x) - V^*)\|\nabla V(x)\| \geq c$$

for all $x \in U$ where $V(x) - V^* \in (0, \rho)$.

**APPENDIX B**

**Lyapunov Tests for Stability**

Here we present Lyapunov-based tests for stability of an equilibrium. The first result is a special case of [55, Cor. 7.1], and establishes the stability of an isolated equilibrium.

**Lemma B.1 (Lyapunov test for relative stability):** Let $K$ be a forward invariant set of $\dot{x} = F(x)$ and $x^*$ an isolated equilibrium. Let $U \subseteq \mathbb{R}^n$ be an open set containing $x^*$ and suppose that $V : U \cap K \to \mathbb{R}$ is a directionally differentiable function such that

- $x^*$ is the unique minimizer of $V$ on $U \cap K$.
- $D_V^2 V(x) < 0$ for all $x \in U \cap K \setminus \{x^*\}$.

Then $x^*$ is asymptotically stable relative to $K$.

The next results provides a test for attractivity and stability of a set of nonisolated equilibria, using an “arclength”-based Lyapunov test [56, Thm. 4.3 and Theorem 5.2].

**Lemma B.2 (Arclength-based Lyapunov test):** Let $K$ be a forward invariant set of $\dot{x} = F(x)$. Let $S \subseteq K$ be a set of equilibria and $U \subseteq \mathbb{R}^n$ an open set containing $S$ where $U \cap F^{-1}(\{0\}) = S$. Let $V : U \cap K \to \mathbb{R}$ be a continuous function. Consider the following conditions.

(i) There exists a $c > 0$ such that for all $x \in U \cap K$,

$$D_V^2 V(x) \leq -c \|F(x)\|.$$  \hspace{1cm} (33)

(ii) $x^*$ is a minimizer of $V$ if and only if $x^* \in S$.

If (i) holds then every bounded trajectory that starts in $U \cap K$ and remains in $U \cap K$ for all time has finite arclength and converges to a point in $S$. If (i) and (ii) hold then, in addition, every $x^* \in S$ is semistable relative to $K$.

In the case where the Lyapunov function $V$ is definable with respect to an o-minimal structure, we show that the condition in (33) for the arclength-based Lyapunov test can be replaced with $D_V^2 V(x) \leq -c \|F(x)\|\|\nabla V(x)\|$. This is referred to as the “angle-condition” and has been exploited [57], [58] to show convergence of descent methods to solve nonlinear programming problems. The name arises from the fact that the inequality implies that the angle between $F(x)$ and $\nabla V(x)$ remains bounded in a neighborhood of the equilibrium. In the next result, we show that the angle condition, together with the Kurdyka–Łojasiewicz inequality, implies that all trajectories of the system have finite arclength.

**Lemma B.3 (Angle-condition-based Lyapunov test):** Let $K$ be a forward invariant set of $\dot{x} = F(x)$. Let $S \subseteq K$ be a bounded set of equilibria and $U \subseteq \mathbb{R}^n$ a bounded open set.
containing \( S \) where \( U \cap F^{-1}(\{0\}) = S \). Let \( V : U \cap K \to \mathbb{R} \) be a differentiable function. Consider the following conditions.

(i) \( V \) is constant and equal to \( V^* \) on \( S \) and definable with respect to some \( \alpha \)-minimal structure;

(ii) There is \( c_2 > 0 \) such that for all \( x \in U \cap K \),
\[
D^2_F V(x) \leq -c_2 \|\nabla V(x)\| \|F(x)\|.
\]

(iii) \( x^* \) is a minimizer of \( V \) if and only if \( x^* \in S \).

If (i) and (ii) hold then every trajectory that starts in \( U \cap K \) and remains in \( U \cap K \) for all time has finite arclength and converges to a point in \( S \). If (i)-(iii) hold then, in addition, every \( x^* \in S \) is semistable relative to \( K \).

Proof: Suppose (i) holds. By Lemma A.3, there exists \( c_1 > 0 \) and a strictly increasing, definable, differentiable function \( \psi : [0, \infty) \to \mathbb{R} \) such that \( \psi([V(x) - V^*])\|\nabla V(x)\| \geq c_1 \) for all \( x \in (U \cap K) \setminus S \). Assume without loss of generality that \( \psi(0) = 0 \), and define \( \tilde{V} : U \cap K \to \mathbb{R} \) by
\[
\tilde{V}(x) = \begin{cases} 
\psi(V(x) - V^*) & V(x) > V^* \\
0 & V(x) = V^* \\
-\psi(V^* - V(x)) & V(x) < V^*.
\end{cases}
\]

Then for all \( x \in U \) with \( V(x) > V^* \), we have
\[
D^2_F \tilde{V}(x) = \psi'(V(x) - V^*) D^2_F V(x) \leq -c_2 \psi'(V(x) - V^*) \|\nabla V(x)\| \|F(x)\| \leq -c_1 c_2 \|F(x)\|.
\]

A similar argument can be used to show that the previous inequality also holds when \( V(x) \leq V^* \). Since \( \psi \) is increasing, \( x^* \in U \cap K \) is a local minimizer of \( \tilde{V} \) if and only if \( x^* \) is a local minimizer of \( V \). Hence, the result follows by applying Lemma B.2 with the Lyapunov function \( \tilde{V} \).

**Appendix C: Regularity of Systems of Linear Inequalities**

The proof of Lemma 4.1, requires the following technical result which gives conditions under which a linear system of inequalities is regular.

**Lemma C.1:** Consider a linear inequality system in the variables \((u, v) \in \mathbb{R}^m \times \mathbb{R}^k\) with the form
\[
\begin{align*}
G_1 u + G_2 v &\leq c \quad (34a) \\
H_1 u + H_2 v &= h \quad (34b) \\
u &\geq 0 \quad (34c)
\end{align*}
\]

where \( c \in \mathbb{R}^m, d \in \mathbb{R}^k, G_1 \in \mathbb{R}^{m \times m}, G_2 \in \mathbb{R}^{m \times k}, H_1 \in \mathbb{R}^{k \times m}, H_2 \in \mathbb{R}^{k \times k} \). The system (34) is regular if

(i) There exists \((u_0, v_0)\) satisfying (34) where \( G_1 u_0 + G_2 v_0 < c \);

(ii) \( H_2 \) is full rank.

Proof: By [32, Theorem 2], the system (34) is regular if and only if there exists \((u_0, v_0)\) satisfying (34) where \( G_1 u_0 + G_2 v_0 < c \) and the following system is regular:
\[
\begin{align*}
H_1 u + H_2 v &= d \quad (35a) \\
u &\geq 0 \quad (35b)
\end{align*}
\]

We claim that (35) is regular whenever \( H_2 \) has full rank. Indeed, by a second application of [32, Theorem 2], (35) is regular if and only if

(a) There exists \((u_1, v_1)\) with \( u_1 > 0 \) and \( H_1 u_1 + H_2 v_1 = d \);

(b) \([H_1, H_2]\) has full rank.

If \( H_2 \) has full rank, (a) holds since for any \( u_1 > 0 \), we can always find some \( v_1 \) such that \( H_2 v_1 = h - H_1 u_1 \) and (b) holds since if \( H_2 \) is full rank, \( \text{range}([H_1, H_2]) = \text{range}(H_2) \).

**Appendix D: Locally Bounded Set of Lagrange Multipliers**

The proof of Theorem 5.7(iii) requires the following result, which establishes conditions under which \( \Lambda_\alpha(x) \) is locally bounded.

**Lemma D.1** (Local boundedness of \( \Lambda_\alpha(x) \)): Let \( x^* \in X_{KKT} \) and suppose MFCQ is satisfied at \( x^* \). Let \( U \) be a bounded, open set containing \( x^* \) on which (17) is well defined and \( \Lambda_\alpha(x) \neq \emptyset \) for all \( x \in U \). Then, there exists \( B < \infty \) with
\[
\sup_{x \in U} \left\{ \sup_{(u,v) \in \Lambda_\alpha(x)} \| (u,v) \|_\infty \right\} < B. \tag{36}
\]

Proof: By [59, Cor. 4.3], the solution map of (17), \( x \to \{G_\alpha(x) \times \Lambda_\alpha(x) \} \), satisfies the Lipschitz stability property that there exists \( \ell > 0 \) where
\[
\| G_\alpha(x) \| + \| \text{dist}_{\Lambda_\alpha(x)}(u,v) \| \leq \ell \| x - x^* \|, \tag{37}
\]
for all \((u,v) \in \Lambda_\alpha(x)\) and all \( x \in U \). By Proposition 5.1, \( \Lambda_\alpha(x^*) \) is precisely the set of Lagrange multipliers of (4) at \( x^* \), so MFCQ implies that \( \Lambda_\alpha(x^*) \) is bounded [30]. Suppose by contradiction that (36) does not hold. Then there exists a sequence \( \{x^\nu\}_{\nu=1}^{\infty} \subset U \) and \((u^\nu, v^\nu) \in \Lambda_\alpha(x^\nu) \) where
\[
\| (u^\nu, v^\nu) \|_\infty \to \infty \quad \text{as} \quad \nu \to \infty.
\]
Since \( \Lambda_\alpha(x^\nu) \) is bounded, \( \| G_\alpha(x^\nu) \| + \| \text{dist}_{\Lambda_\alpha(x^\nu)}(u^\nu,v^\nu) \| \to \infty \), which contradicts (37) and the fact that \( U \) is bounded.

**Appendix E: Jacobian of Safe Gradient Flow**

The proof of Theorem 5.7(iii) relies on analyzing the Jacobian of \( G_\alpha(x) \) at \( x^* \). Here, we flesh out the steps required to obtain the expression for \( \frac{\partial G_\alpha(x^*)}{\partial x} \) in (25). Let \( J = L_x(x^*) \) and assume, without loss of generality, that the rows of \( g(x^*) \) are ordered such that \( I_0 = \{1, 2, \ldots, |I_0|\} \) and \( J = \{I_0 + 1, \ldots, m\} \). Let \( G = \frac{\partial g(x^*)}{\partial x} \) and \( G_\alpha = \frac{\partial g_\alpha(x^*)}{\partial x} \), where \( G_\alpha \) and \( G \) are related by
\[
G_\alpha = \int_G \alpha \frac{\partial h(u^*)}{\partial x} \quad \text{and} \quad H = \int_H. \tag{38}
\]

By the reasoning in the proof of Lemma 5.12, strict complementarity and the strong second order sufficient condition hold for the parametric optimization problem (17). Therefore, by [39, Theorem 2.3] it follows that the KKT triple of (17), written as \((G_\alpha(x), v(x), u(x))\), is differentiable at \( x^* \) and the Jacobian \( J = \int_G \frac{\partial G_\alpha(x)}{\partial x}, v(x), u(x)) \big|_{x=x^*} \) is
\[
J = \begin{bmatrix}
I & H^T & G^T \\
-H & 0 & 0 \\
-D_\alpha G & 0 & -\alpha D_\alpha G
\end{bmatrix}^{-1} \begin{bmatrix}
-Q & \alpha H \\
0 & \alpha D_\alpha G
\end{bmatrix}, \tag{38}
\]

In this section we use the convention that, in a KKT triple, the Lagrange multipliers corresponding to the equality constraints are written before those corresponding to the inequality constraints. This ensures the Schur complement of \( M \) in (39) is block upper triangular, simplifying the computation of \( M^{-1} \).
where $D_u = \text{diag}(u^*)$ and $D_g = \text{diag}(g(x^*))$. By strict complementarity,
\[
D_u = \begin{bmatrix}
\tilde{D}_u & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix},
D_g = \begin{bmatrix}
0 & 0 \\
0 & \tilde{D}_g
\end{bmatrix}
\]
where $\tilde{D}_u = \text{diag}(u_{i_o}^*)$ and $\tilde{D}_g = \text{diag}(g_i(x^*))$ are invertible matrices. Thus the product (38) can be written as
\[
\mathcal{J} = \begin{bmatrix}
I & H^\top & G_1^\top & G_1^\top \\
-H & 0 & 0 & 0 \\
-\tilde{D}_u G_1 & 0 & 0 & 0 \\
0 & 0 & -\alpha \tilde{D}_g & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
-Q \\
\alpha H \\
\alpha \tilde{D}_u G_1 \\
0
\end{bmatrix}.
\]
Let $M$ be defined as
\[
M = \begin{bmatrix}
I & H^\top & G_1^\top & G_1^\top \\
-H & 0 & 0 & 0 \\
-\tilde{D}_u G_1 & 0 & 0 & 0 \\
0 & 0 & -\alpha \tilde{D}_g & 0
\end{bmatrix}.
\]
Partitioning $M$ into a $2 \times 2$ block matrix as in (39) allows us to compute its inverse in closed form. Let $N = M^{-1}$. Then by [60, Theorem 2.1],
\[
N = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix},
\]
where
\[
N_{11} = I - \begin{bmatrix} H^\top & H G_1^\top & G_1^\top & G_1^\top \\
\tilde{D}_u G_1 H^\top & \tilde{D}_u G_1 G_1^\top & 0 & 0
\end{bmatrix}^{-1}
\begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix},
\]
\[
N_{12} = -\left(\begin{bmatrix} H^\top & H G_1^\top & G_1^\top & G_1^\top \\
\tilde{D}_u G_1 H^\top & \tilde{D}_u G_1 G_1^\top & 0 & 0
\end{bmatrix}^{-1} \times \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix},
\]
\[
N_{21} = \begin{bmatrix} H H^\top & HG_1^\top & 0 & 0 \\
\tilde{D}_u G_1 H^\top & \tilde{D}_u G_1 G_1^\top & 0 & 0
\end{bmatrix}^{-1} \times \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix},
\]
\[
N_{22} = \begin{bmatrix} H H^\top & HG_1^\top & 0 & 0 \\
\tilde{D}_u G_1 H^\top & \tilde{D}_u G_1 G_1^\top & 0 & 0
\end{bmatrix}^{-1} \times \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix},
\]
where in the previous expressions we replace values that will eventually be canceled out by $\times$. It follows that
\[
\frac{\partial g_{\alpha i}(x^*)}{\partial x} = -N_{11} Q + \alpha N_{12} \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix},
\]
\[
\frac{\partial \bar{q}(x^*)}{\partial u (x^*)} = -N_{21} Q + \alpha N_{22} \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0 \\
0
\end{bmatrix}.
\]
To simplify (40a), we start by letting $P$ be the orthogonal projection onto $\ker \frac{\partial g_i(x^*)}{\partial x} \cap \ker \frac{\partial \bar{q}(x^*)}{\partial u (x^*)}$. By [61, Proposition 6.1.6],
\[
P = I - \begin{bmatrix} H^\top & H \end{bmatrix} \begin{bmatrix} H \\
G_1
\end{bmatrix}.
\]
Since LICQ holds at $x^*$, the matrix $[H; G_1]$ has full row rank, so by [61, Proposition 6.1.5],
\[
\begin{bmatrix} H^\top & H G_1 H^\top & HG_1^\top & G_1 G_1^\top \end{bmatrix}^{-1}.
\]
Therefore, if we let $D = \text{blkdiag}(I, \tilde{D}_u)$, then $D$ is invertible, and we have
\[
N_{11} = I - \begin{bmatrix} H \\
G_1
\end{bmatrix}^\top \left(\begin{bmatrix} H H^\top & HG_1^\top & G_1 G_1^\top \\
G_1 H^\top & G_1 G_1^\top & G_1 G_1^\top
\end{bmatrix}\right)^{-1} \begin{bmatrix} H \\
G_1
\end{bmatrix},
\]
and,
\[
N_{12} = \begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0
\end{bmatrix} = -\begin{bmatrix} H \\
G_1
\end{bmatrix}^\top \left(\begin{bmatrix} H H^\top & HG_1^\top & G_1 G_1^\top \\
G_1 H^\top & G_1 G_1^\top & G_1 G_1^\top
\end{bmatrix}\right)^{-1} \begin{bmatrix} H \\
G_1
\end{bmatrix} = -\begin{bmatrix} H \\
\tilde{D}_u G_1 \\
0
\end{bmatrix}.
\]
By substituting the previous two expressions into (40a), we obtain
\[
\frac{\partial g_{\alpha i}(x^*)}{\partial x} = -P Q - \alpha (I - P).
\]

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