Density of states of a dissipative quantum dot coupled to a quantum wire

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We examine the local density of states of an impurity level or a quantum dot coupled to a fractional quantum Hall edge, or to the end of a single one-dimensional Luttinger-liquid lead. Effects of an Ohmic dissipative bath are also taken into account. Using both analytical and numerical techniques we show that, in general, the density of states exhibits power-law frequency dependence near the Fermi energy. In a substantial region of the parameter space it simply reflects the behavior of the tunneling density of states at the end of a Luttinger-liquid, and is insensitive either to the value of the dot-lead interaction or to the strength of dissipation; otherwise it depends on these couplings too. This behavior should be contrasted with the thermodynamic properties of the level, in particular, its occupancy, which were previously shown to depend on the various interactions in the system only through the corresponding Fermi edge singularity exponent, and thus cannot display any Luttinger-liquid specific power-law. Hence, we can construct different models, some with and some without interactions in the wire (but with equal Fermi edge singularity exponents), which would have very different level densities of states, although they all result in the same level population vs. energy curves.

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I. INTRODUCTION

Understanding the behavior of low-dimensional electronic systems has been one of the main challenges of experimental and theoretical physics in the last years. These systems are important not only as the basic building blocks of nanoelectronic devices, but also for the intricate strongly-correlated phenomena they exhibit. An important subclass is that of metallic (gapless) one-dimensional systems, whose low energy dynamics is governed not by Fermi liquid theory, but instead by the Luttinger liquid (LL) paradigm. This description applies to a wide variety of experimental realizations, including narrow quantum wires in semiconducting heterostructures, metallic nanowires, and carbon nanotubes. Closely related are chiral LLs, formed at the edges of fractional quantum Hall effect (FQHE) systems, and helical LLs, the edges of spin quantum Hall insulators. The effect of impurities on these systems is interesting from both the applicative and fundamental points of view. These impurities could also be intentionally introduced, in the form of, e.g., quantum dots and anti-dots. Hence, there is no wonder that such questions have attracted much effort recently. However, most of these studies were restricted to investigation of transport phenomena, while other effects received much less attention.

In this work we study probably the most basic example of such a system, namely, a single level in the vicinity of a fractional quantum Hall edge, or, equivalently, a level attached to the end of a single LL wire. We will refer to the two components (in both systems) as “dot” and “lead” respectively. We include in our treatment the effects of short range dot-lead interaction, as well as the influence of an Ohmic dissipative bath (e.g., electromagnetic fluctuations in gate electrodes). In a recent work we have studied the thermodynamic properties of the model (e.g., the level population, entropy, and specific heat), and found that they are universal, in the sense that they depend on the various interactions in the model (intra-lead, dot-lead, and dot-bath) only through a single parameter, the Fermi edge singularity exponent. Thus, thermodynamics can neither be used to identify non-Fermi liquid behavior, nor to extract LL parameters. In this work we proceed to study, both analytically and numerically, the level density of states (LDoS), which may be probed by tunneling or absorption spectroscopies. We find that the LDoS is sensitive to LL physics, even though its integral (times the Fermi function) gives the level occupancy, which is universal in the above sense. As we show below, the LDoS features power-law behavior near the Fermi energy. For not too strong interactions the exponent in this power-law is actually determined by LL physics alone, and is independent of the level-lead and level-bath interactions. This and many other results derived below cannot be achieved using perturbative calculations.

The rest of this paper is organized as follows: In Sec. II we present our model, and apply to it the Anderson-Yuval Coulomb-gas (CG) expansion. We then proceed to analytic treatment of the LDoS in Sec. III, and to numerical calculations in Sec. IV. Finally, we summarize our findings in Sec. V.

II. MODEL AND COULOMB-GAS EXPANSION

The system is described by the Hamiltonian $H = H_D + H_L + H_{DL} + H_B + H_{DB}$. The first term is the dot Hamiltonian $H_D = \varepsilon_0 d^\dagger d$, with $d^\dagger$ and $d$ the level creation and annihilation operators, respectively, and $\varepsilon_0$ the level energy. The second term is the lead Hamilto-
nian. It can be written in the form

$$H_L = \frac{v}{4\pi} \int_{-\infty}^{\infty} \left[ \partial_x \phi(x) \right]^2 dx,$$

(1)

using chiral bosonic field $\phi(x)$ obeying the commutation relation $[\phi(x), \phi(y)] = i\pi \text{sgn}(x-y)$, where $v$ is the velocity of excitations. The level and the lead are coupled by:

$$H_{DL} = t_0 d^d \psi(0) + \text{H.c.} + U_0 \left( d^d d^d - \frac{1}{2} \right) \frac{\sqrt{g^2}}{\pi} \partial_x \phi(0).$$

(2)

The two terms in this equation describe, respectively, dot-lead hopping (with $t_0$ the tunneling matrix element), and local dot-lead interaction whose strength is $U_0$. The electronic annihilation operator at the end of the lead can be written as $\psi(0) = \chi e^{i\phi(0)/\sqrt{2}} / \sqrt{\sqrt{2}\pi a}$, where $\chi$ is a Majorana operator, $a$ is a short distance cutoff (e.g., the lattice spacing), and $g$ is the LL interaction parameter ($g < 1$ for repulsion, $g > 1$ for attraction). For a FQHE system with filling $\nu$, $g = \nu$ for electron tunneling (i.e., a dot outside the FQHE bar). Finally, the level is coupled to a bath of harmonic oscillators (describing, e.g., electromagnetic fluctuations in control gates), governed by $H_B = \sum_k \omega_k a_k^{\dagger} a_k$. The dot-bath coupling can be written as $H_{DB} = (d^d d^d - \frac{1}{2}) \sum_k \lambda_k (a_k^{\dagger} + a_k)$. We assume Ohmic dissipation, i.e., linear low-frequency behavior of the bath spectral function: $J_B(\omega) \equiv \sum_k \lambda_k^2 \delta(\omega - \omega_k) = K \omega$.

We examine this model employing the Anderson-Yuval CG expansion. In this approach, any quantum of interest is expanded to all orders in $t_0$. This results in a series of correlation functions, which need to be evaluated for vanishing $t_0$ (the level-lead interaction gives rise to a potential at the end of the lead, which alternates between $U_0/2$ and $-U_0/2$ whenever an electron tunnels in or out of the level). Similarly, the $k$th bath oscillators experience a shift in its equilibrium position, proportional to $\lambda_k$. We thus have a sequence of Fermi edge singularity events. The solution of this latter problem enables the calculation of all the terms in the series of correlation functions.

Recently, we have studied in this way the partition function $Z$ of the model, whose derivitives with respect to the parameters of the system (for example, the level energy $\varepsilon_0$ and the temperature $T$) give us the thermodynamic properties (e.g., the level population, entropy, and specific heat). We were able to rewrite the series expansion for $Z$ in the form of a grand canonical partition function of a classical system of particles. These represent hopping events generated by $t_0$, and thus reside on the imaginary time axis of the original quantum model, which is a circle with circumference $1/T$. Each particle is assigned a positive (negative) charge if it represents tunneling of an electron from the lead to the dot (from the dot to the lead). Hence, there must be an even number of charges, which have to appear in alternating order of signs. The position of the $i$th particle is $\tau_i$, and the sign of the charge of the first particle is denoted by $s$.

The partition function then reads:

$$Z = \sum_{N=0}^{\infty} \sum_{s=\pm 1}^{2N} y^{2N} \frac{1}{\xi} \int \frac{d\tau_2 N - \xi}{\xi} \int \frac{d\tau_{2 N - 1}}{\xi} \cdots \int \frac{d\tau_2}{\xi} \int \frac{d\tau_1}{\xi} e^{-S_{CG}(s, \{\tau_i\})},$$

(3)

The charges have a fugacity $y = \sqrt{\Gamma_0 \xi / \pi}$, where $\Gamma_0 = \pi |t_0|^2 \rho_L$ is the noninteracting level width ($\rho_L = 1/(\pi v)$ is the corresponding lead local density of states), and $\xi \sim a/v$ is a short-time cutoff. The CG action is given by:

$$S_{CG}(s, \{\tau_i\}) = \sum_{i<j=1}^{2N} \hat{c}_i \cdot \hat{c}_j V_C(\tau_j - \tau_i) + \varepsilon_0 \left[ 1 - s \frac{1}{2T} + s \sum_{i=1}^{2N} (-1)^i \tau_i \right].$$

(4)
The first term of this classical Hamiltonian describes an interaction between the particles, with \( V_C(\tau) = \ln[\pi T\xi / \sin[\pi T |\tau|]] \). This interaction is similar in form to 2D Coulomb interaction, and is the origin of the name “CG expansion”. The charges are two component vectors, where the two components correspond to the effects of the coupling with the lead and the bath, respectively. They are given by \( \varepsilon_i = s(-1)^{i+1}e_0 \), where the squared-magnitude of the charges, to be denoted by \( \alpha_{\text{FES}} \equiv |\varepsilon_0|^2 \), is the Fermi edge singularity exponent of the model. It is defined by behavior of the zero-temperature correlator of \( \hat{e}_\tau \), where the two components correspond to the energetic cost of \( t_0 \) charges of sizes \( \pm \varepsilon_d, \varepsilon_d = \varepsilon_0 - (1/\sqrt{g},0) \), inserted at \( \tau' \) and \( \tau \), respectively. These charges correspond to the level creation and annihilation operators appearing in the definition of the Green function. In the following we will refer to these as “\( \pm \) charges”, to distinguish them form the other “\( t_0 \) charges”, which originate from the \( t_0 \) term. The contribution of each such configuration is to be multiplied by \( \text{sgn}(\tau' - \tau) \) to account for the Fermi statistics. Thus, for \( \tau > \tau' \) the full CG expression for the dot Green function is:

\[
G_D(\tau > \tau') = -\frac{1}{Z} \sum_{s=\pm 1} \sum_{N=0}^{\infty} y^{2N} \sum_{M=0}^{N-s} \sum_{M'=0}^{N-s'-M} \int \frac{d\tau_0}{\xi} \cdots \int \frac{d\tau_{2N}}{\xi} \times \\
\left. \frac{\tau_{2(M+M')+\xi}^{-\varepsilon}}{\xi} \right|_{\tau=0} \left. \frac{\tau_{2M+s+1}^{-\varepsilon}}{\xi} \right|_{\tau=0} \cdots \left. \frac{\tau_{2M+s}^{-\varepsilon}}{\xi} \right|_{\tau=0} \right|_{\tau=0} e^{-S_{\text{CG},D}(s,\tau,\tau',\{\tau_i\})},
\]

where \( s' \equiv (1-s)/2 \). The first \( 2M + s' \) \( t_0 \) charges occupy the interval \( [0,\tau] \), the following \( 2M' \) charges reside in the interval \( [\tau',\tau] \), and the last \( 2(N-M-M') - s' \) \( t_0 \) charges are in the interval \( [\tau,1/T] \). The classical action is given by:

\[
S_{\text{CG},D}(s,\tau,\tau',\{\tau_i\}) = |\varepsilon_0|^2 \sum_{i<j=1}^{2N} s_is_j V_C(\tau_j - \tau_i) + \varepsilon_0 \cdot \varepsilon_d \varepsilon_0 \sum_{i=1}^{2N} s_i [V_C(\tau_i - \tau') - V_C(\tau_i - \tau)] - |\varepsilon_0|^2 V_C(\tau' - \tau) + \\
\varepsilon_0 \left[ \frac{1-s}{2T} - \sum_{i=1}^{2N} s_i \tau_i + \tau - \tau' \right],
\]

where the sign of the \( i \)th \( t_0 \)-charge is \( s_i = (-1)^{i+1} \text{sgn}(\tau_i - \tau') \text{sgn}(\tau_i - \tau) \). A typical configuration is shown in Fig. 1(b). Similar expressions hold for \( \tau < \tau' \).

Comparing the two CG expansions, the following observation emerges: the CG expansion for the partition function contains only three parameters: \( \Gamma_0, \varepsilon_0, \alpha_{\text{FES}} \), while expansion for the Green function depends on \( g \) too (through \( \varepsilon_d \)). Hence, the different interaction types (i.e., interactions in the wire, the dot-wire interaction, and the dot-bath coupling) affect the partition function through a single parameter, the Fermi edge singularity exponent \( \alpha_{\text{FES}} \). Thus, thermodynamic measurements cannot be used to distinguish between the different interaction types. In other words, one can construct very different models, whose interactions differ in strength and even in sign, which will have the same thermodynamic
properties, provided $\Gamma_0$, $\varepsilon_0$, and $\alpha_{\text{FES}}$ are indeed the same. On the other hand, the LDoS, which depends explicitly on $g$, will exhibit different behavior for these different systems. Hence, it can be used to extract the strength of intra-wire interactions, as we show below.

III. ANALYSIS OF THE LEVEL DENSITY OF STATES

As noted in our earlier work, the CG obtained here is identical to the original Anderson-Yuzav expansion for the anisotropic single-channel Kondo model, demonstrating that the models are equivalent. Under this mapping the level population becomes the magnetization of the Kondo spin (plus one half). Hence, $\varepsilon_0$ is analogous to a local magnetic field. Similarly, $J_{xy}$ is related to $\Gamma_0$, and $J_z$ to $\omega_{\text{FES}}$. The CG parameters obey the famous Kondo renormalization group (RG) equations, which read, in our notations:

$$\frac{dy}{d\ln \xi} = 2 - 2\alpha_{\text{FES}} y,$$

$$\frac{d\omega_{\text{FES}}}{d\ln \xi} = -4y^2\alpha_{\text{FES}}.$$

Thus, the system considered can be in one of two phases, a strong coupling (antiferromagnetic-Kondo-like) phase and a weak coupling (ferromagnetic-Kondo-like) phase. The transition occurs, for small $\Gamma_0$ and a weak coupling (ferromagnetic-Kondo-like) phase. The population being discontinuous as a function of $\varepsilon_0$ at zero temperature. Under this mapping the level population becomes the magnetization of the Kondo spin (plus one half). Hence, $\varepsilon_0$ at zero temperature. In the strong-coupling phase the Coulomb charges form tightly-bound pairs. The level is thus effectively decoupled at low energies, resulting in its population being discontinuous as a function of $\varepsilon_0$ at zero temperature. In the strong-coupling phase the CG charges free Coulomb charges proliferate. The impurity is well-coupled with the lead, so that the level population is analytic in $\varepsilon_0$, and could be extracted from the Bethe ansatz solution of the Kondo problem. In particular, for small values of $\varepsilon_0$ one has $n(\varepsilon_0) = (v/a) - 1/2 \sim -\varepsilon_0/T_K$, where $T_K = (v/a)(\Gamma_0a/v)^{1/(2+\alpha_{\text{FES}})}$ is the Kondo temperature (effective level width, reducing to $T_0$ in the noninteracting case). Hence, in this phase the population does not exhibit any nontrivial power-law dependence on $\varepsilon_0$ or $T$. The same applies to other thermodynamic quantities.

What are the implications of this on the LDoS? As we now show, we typically find that at zero temperature we have a power-law behavior $\rho_D(\omega) \sim |\omega|^\delta$ in the vicinity of the Fermi energy, i.e., when $|\omega|$ is much smaller than $T_K$ in the strong-coupling phase, and than the bandwidth $\sim v/a$ in the weak-coupling phase. The values of $\delta$ in the different regimes are summarized in Table I. It should be noted that when $T > 0$, or when the lead length $L$ is finite, such power-law singularity will be smeared and become $[\max(\omega, T, v/L)]^\delta$. We will now consider each phase separately.

| Phase          | $|\omega| \ll |\varepsilon_0|$ | $|\omega| \gg |\varepsilon_0|$ |
|----------------|---------------------------------|----------------------------------|
| Strong-coupling| $1/g - 1$                        | $1/g - 1$                        |
| Weak-coupling  | $1/g - 1$                        | $\alpha_{\text{FES}} - 1$ or $1/g - 1$ |

A. The strong-coupling phase

Let us start from the strong-coupling phase. When $|\varepsilon_0|$ is large enough (with respect to $T_K$), CG charges must appear in tightly-bound pairs, since large intra-pair separation is suppressed by the level energy, as dictated by the last term of Eq. (9). The two d-charges added in the calculation of the level Green function will also be accompanied by two screening $\ell_0$-charges for the same reason. The resulting configuration should thus resemble Fig. 2(a). The leading contribution to the Green function will then come from the residual interaction of these partially screened d-charges, whose charges are $(\pm 1)$ times $\varepsilon_0 - \varepsilon_d$. The usual tunneling density of states singularity at the end of a LL wire.

When $\varepsilon_0$ is small (with respect to $T_K$), renormalization effects are significant. In addition to the usual CG RG Eq. (9) representing the screening of interaction between $\ell_0$-charges by pairs of nearby $\ell_0$-charges (separated by $\xi^{27}$, one can write down similar equations for the flow of the Green function parameters. It is easy to see that the coefficients of the logarithmic interaction between any two charges (either both $\ell_0$-charges, both d-charges, or a mixed pair) are renormalized in the same way by the pairs of nearby $\ell_0$-charges, i.e.,

$$\frac{d(\vec{\varepsilon}_d, \vec{\varepsilon}_d)}{d\ln \xi} = -2y^2(\vec{\varepsilon}_d \cdot \vec{\varepsilon}_d + \vec{\varepsilon}_d \cdot \vec{\varepsilon}_d),$$

where $\mu, \nu = 0, d$. Thus, the combination $|\vec{\varepsilon}_0 - \vec{\varepsilon}_d|^2 = |\vec{\varepsilon}_0|^2 + |\vec{\varepsilon}_d|^2 - 2\vec{\varepsilon}_d \cdot \vec{\varepsilon}_d$ is invariant, and retains its initial value of $1/g$. At the strong coupling fixed point $g$ is large, so by Eq. (9), $|\vec{\varepsilon}_d|^2 = \sqrt{\alpha_{\text{FES}}} = 0$, where asterisks denote fixed-point values. Thus, $|\varepsilon_0| = 1/\sqrt{g}$, so that $G_D(\tau - \tau') \sim (\tau - \tau')^{-1/2} = (\tau - \tau')^{-1/2}$, resulting again in $\rho_D(\omega) \sim |\omega|^{1/2}$. Since this behavior holds at both large and small $\varepsilon_0$ values, it should also apply at all intermediate values.

Further support for this result is obtained by analysis of the particular case $\alpha_{\text{FES}} = 1$ (the Toulouse limit), where the CG, and all thermodynamic properties, reduce to those of a noninteracting resonant level (for which $g = 1$, $U_0 = 0$, and $K = 0$). A nontrivial (i.e., in-
teracting) realization of this condition, which still permits an exact calculation of the LDoS, is the case of no coupling to a bath \((K = 0)\), but with \(g = 1/4\) and a corresponding compensating value of the dot-lead interaction. Then \(\mathcal{E}_d = (-1, 0) = -\mathcal{E}_0\), so the d-charges have the same magnitudes as the corresponding to-charges, but the opposite signs. Comparison of the corresponding CG expansions thus shows that the level Green function of the interacting system is equal to a two-particle Green function of the noninteracting resonant level \(\langle \mathcal{T}_\tau \psi(0, \tau) d(\tau) d(\tau') \psi(0, \tau') \rangle\), up to a factor of \((2\pi a)^{-1} \text{sgn}(\tau' - \tau)\). The operators at \(\tau\) and \(\tau'\) in this noninteracting two-particle Green function are similar to the \(t_0\) term in the Hamiltonian, but with \(\psi(0)\) replaced by \(\psi(0)\) to account for the signs of the d-charges in the interacting system. After a straightforward evaluation of this two-particle Green function by Wick’s theorem, we find for the LDoS the following expression:

\[
\rho_D(\omega) = 2\rho_L a \coth \left( \frac{\omega}{2T} \right) \text{Im} \left\{ \frac{\omega - 2\mathcal{E}_0 + 2i\Gamma_0}{\omega - 2\mathcal{E}_0} \right\} \times \left[ \psi \left( \frac{1}{2} \frac{i \mathcal{E}_0 - i\Gamma_0}{2iT} \right) - \psi \left( \frac{1}{2} \frac{i \mathcal{E}_0 + i\Gamma_0}{2iT} \right) \right] \right\} \}
\]

(10)

where \(\psi(z)\) is the digamma function. For small \(|\omega|\) and \(T\) we indeed recover the \([\max(\tau, |\omega|)]^3\) behavior appropriate for \(g = 1/4\), for all values of \(\mathcal{E}_0\).

Physically, the result is clear: in the strongly-coupled phase the level behaves, at low energies, as the last site of a LL wire, i.e., \(\rho_D(\omega) \sim |\omega|^{1/9-1}\). Interestingly, not only dot-lead interactions, but even coupling to the bath does not modify this behavior. As a result of this, the LDoS exhibits a power-law behavior with exponent which depends only on the LL parameter \(g\) and not on \(\alpha_{\text{FES}}\), i.e., on the interactions in the wire but not on the level-lead and level-bath couplings. This is in contrast with, e.g., the level occupancy, which depends on \(\alpha_{\text{FES}}\) but not on \(g\), as discussed above. Below we also test these predictions numerically.

**B. The weak-coupling phase**

We now turn our attention to the weak-coupling phase. Here all the \(t_0\)-charges are bound in pairs, so the high \(|\mathcal{E}_0|\) results discussed above (which should also hold in this phase) actually carry over to low values of \(|\mathcal{E}_0|\). It should however be remembered that then they compete with the contribution of the \(t_0 = 0\) term in the CG expansion (the weak coupling fixed point), the term representing the interaction of the unscreened d-charges, which gives the LDoS a contribution of the form \(\rho_{\text{pair}}(\omega) \sim |\omega|^{\alpha_{\text{FES}}^d - 1}\), with \(\alpha_{\text{FES}}^d \equiv |\mathcal{E}_d|^2\). This is simply the LDoS of a tunnel-decoupled level, broadened from a delta peak to a power-law by the Anderson orthogonalities in the wire and in the Ohmic bath. Note that since the level is effectively decoupled, the usual \(|\omega|^{1/9-1}\) behavior at the end of a LL wire need not apply anymore.

One could actually proceed to study higher order terms in the weak-coupling regime (and similarly, for large \(|\mathcal{E}_0|\) in the strong-coupling phase). For small \(t_0\) the leading correction is dressing of the above-mentioned charge configurations by a series of pairs of close-by \(t_0\) charges (close-by since \(|\mathcal{E}_0|\) and/or \(\alpha_{\text{FES}}\) are large), as depicted in Fig. 2. One then has to sum over all the terms similar to Fig. 2(a) for large \(|\mathcal{E}_0|\) (i.e., for \(|\omega| \ll |\mathcal{E}_0|\) in the weak-coupling phase) or all the terms similar to Fig. 2(b) for small \(|\mathcal{E}_0|\) (i.e., for \(|\omega| \gg |\mathcal{E}_0|\)). Since each pair has a very small dipole moment (due to the proximity of the charges), inter-pair interactions are negligible in a first approximation. This is actually an imaginary time variant of the noninteracting blip approximation (NIBA). The following argument can spare us the need of explicit calculations. For \(U_0 = 0\) and \(K = 0\), the d-charges are noninteracting, \(\mathcal{E}_d = (0, 0)\). The sum over all the terms with pairs of nearby \(t_0\) charges would be the same (in the current approximation) as the Green function of a noninteracting system consisting of a level tunnel-coupled to a bath of noninteracting fermions with power-law local density of states \(\rho_L(\omega) \sim |\omega|^{\alpha_{\text{FES}}^d - 1}\) (with some appropriate high-energy cutoff and normalization). For the latter system the Green function can be easily
evaluated to give \( G_D^0(\omega_i) = [\omega - \varepsilon_0 - \Sigma^0_D(\omega)]^{-1} \), where the dot self energy is:

\[
\Sigma^0_D(\omega) = (t_0)^2 \int_{-\infty}^{\infty} \frac{\rho(\Omega)}{\omega - \Omega} d\Omega,
\]

so that \( \Sigma^0_D(\omega_i) \sim \omega^{\alpha_{\text{FES}} - 1} \). For small \( |\varepsilon_0| \) we indeed see that \( \Sigma^0_D(\omega) \) is subdominant with respect to the noninteracting contribution only if \( \alpha_{\text{FES}} > 2 \), which is exactly the condition for the weak-coupling phase for small \( t_0 \). Then, exactly at \( \varepsilon_0 = 0 \), a delta-function term appears at \( \omega = 0 \) in the expression for the LDoS (similarly to the situation at \( t_0 = 0 \)), whose coefficient is determined by the requirement that the integral of the entire expression for the LDoS corresponding to \( G^0_D(\omega_i) \) is unity.\(^{22} \)

Before discussing nonzero \( U_0 \) and \( K \), it should be remarked that this NIBA-like approximation exactly reproduces the perturbative (in the tunneling \( t_0 \) approach employed in Ref. 19, and would lead to similar predictions for the behavior of the level population. Both approximations are justified only in the weak-coupling phase (or when \( |\varepsilon_0| \) is large), but not in the strong-coupling phase, where \( t_0 \) grows under RG flow and thus cannot be treated perturbatively, similarly to the exchange \( J_{\text{ex}} \) in the equivalent Kondo problem.\(^{23} \)

In the strong-coupling regime perturbative results predict correctly the dependence of the Kondo temperature on \( t_0 \) (again, just like in the Kondo model) and the qualitative behavior of the LDoS for \( 1/2 < g < 1 \) and \( \varepsilon_0 \neq 0 \) (at \( U_0 = 0 \) and \( K = 0 \)), but deviate from our previous conclusions in many other respects. For example, as we discuss below, at nonzero \( U_0 \) and \( K \) our NIBA calculations indicate that the exponent in the power-law behavior of the LDoS at low energy may depend on these interactions too, in contrast with the situation in the strong-coupling phase, where the corresponding exponent depends only on the LL parameter \( g \), as shown above. Moreover, even for vanishing dot-lead and dot-bath interactions (the case treated in Ref. 19, the NIBA/perturbative expression given above does not agree with our previous analysis of the strong-coupling phase: for (a) \( 1/2 < g < 1 \) and \( \varepsilon_0 = 0 \) or (b) \( g > 1 \) and any \( \varepsilon_0 \), NIBA would suggest that the LDoS varies as \( |\omega|^{1-1/g} \), i.e., with the opposite exponent to the one appearing in the density of states at the end of a LL wire. Perturbative results would thus imply that the LDoS may be enhanced for \( g < 1 \) or suppressed for \( g > 1 \), which is clearly at odds with both our previous results and the behavior of density of states at the lead edge. Moreover, integrating the perturbative LDoS leads to the prediction that the level population may have a power-law dependence on \( \varepsilon_0 \) at the strong-coupling phase (for \( 1/2 < g < 2/3 \)),\(^{24} \) which is in contrast with the analytical behavior expected from the exact mapping of our model onto the Kondo problem, as discussed above. To summarize, NIBA/perturbative (in \( t_0 \) expressions do not hold in general in the strong-coupling phase. It may be noted that the numerical data of Ref. 19 does not cover these regimes of the strong coupling phase for which perturbative calculations disagree with our previous analysis.

Returning to the discussion of the NIBA approximation for the weak-coupling phase, we will now treat the more general case, i.e., nonzero \( U_0 \) and \( K \) (which was not addressed in Ref. 19). Again, screened d-charges terms [Fig. 2(a)] are dominant for \( |\omega| \ll |\varepsilon_0| \), whereas unscreened d-charges terms [Fig. 2(b)] are dominant for \( |\omega| \gg |\varepsilon_0| \). Let us start from the latter case. Now that there is interaction between d-charges, the CG expression for the Green function \( G_D(\tau - \tau') \) contains an additional factor of the form \( |\tau - \tau'|^{-\alpha_{\text{FES}}} \) (interaction of d-charges with the pairs of close-by \( t_0 \)-charges is negligible). Turning to the frequency domain, the LDoS of the unscreened d-charges contribution [Fig. 2(b)] will thus be the convolution of the LDoS \( \rho_D^0(\omega) \) associated with \( G_D^0(\omega_i) \) from the previous paragraph, with a function \( \rho_{\text{pair}}(\omega) \sim |\omega|^{\alpha_{\text{FES}}-1} \), which is simply the LDoS of a decoupled \( (t_0 = 0) \) level as discussed above, i.e.,

\[
\rho_D(\omega) = 2\text{sgn}(\omega) \int_0^\infty \rho_D^0(\omega - \Omega) \rho_{\text{pair}}(\Omega) d\Omega,
\]

at \( t = 0 \). Thus, the \( |\omega|^{\alpha_{\text{FES}}-1} \) behavior of the decoupled level will survive for vanishing \( |\varepsilon_0| \), due to the delta peak in \( G_D^0(\omega) \). This behavior actually applies to the entire \( |\omega| \gg |\varepsilon_0| \) region, which is exactly where the contribution of terms similar to Fig. 2(b) is important. Higher order corrections will give extra powers of \( |\omega|^{\alpha_{\text{FES}}-2} \), which are subleading since \( \alpha_{\text{FES}} > 2 \).

Similar considerations apply to the screened d-charges contribution [Fig. 2(a)] when \( |\omega| \ll |\varepsilon_0| \). For \( U_0 \neq 0 \) and/or \( K \neq 0 \), two corrections are due. The first correction takes into account the factors coming from the interaction between each d-charge and the neighboring \( \varepsilon_0 \)-charge, which are power-laws in the time domain. Since the d-\( t_0 \) charges form tightly-bound pairs, we can take this expression and its value at \( \varepsilon_0 = \varepsilon_0' \) pair \( \varepsilon_0, \varepsilon_0' \) from the previous paragraph, with a function \( \rho_{\text{pair}}(\omega) \sim |\omega|^{\alpha_{\text{FES}}-1} \), which is simply the LDoS of a decoupled \( (t_0 = 0) \) level as discussed above, i.e.,

\[
\int_0^\infty \frac{\varepsilon_0 \cdot e_d}{\varepsilon_0 - \varepsilon_0'} d\varepsilon_0 \cdot e_d = \frac{\Gamma(1 - \varepsilon_0 \cdot e_d)}{\varepsilon_0(1 - |\varepsilon_0|)} \bigg|_{\varepsilon_0 = \varepsilon_0'},
\]

where \( \Gamma(z) \) is the gamma function, and \( \varepsilon_0 \cdot e_d = (\alpha_{\text{FES}} + \alpha_{\text{FES}} - 1/g)/2 \). The correction is then the ratio between this expression and its value at \( \varepsilon_0' = 0 \). Apart from this constant factor, one must compensate for the fact that the inter-pair interaction of these two d-\( t_0 \) pairs gives the Green function a factor which varies as \( |\tau - \tau'|^{-1/g} \) (since \( \varepsilon_0 \cdot e_d = 1/g \)), instead of the \( |\tau - \tau'|^{-\alpha_{\text{FES}}} \) dependence used in the calculation of \( G_D^0(\omega) \). Hence, \( \rho_D^0(\omega) \) should be convoluted here with \( |\omega|^{1/g - \alpha_{\text{FES}}-1} \). For \( |\omega| \ll |\varepsilon_0| \) the LDoS will thus retain the \( |\omega|^{1/g - 1} \) behavior of the strong-coupling phase.

To conclude the discussion of this NIBA-type approximation, a general expression for the LDoS which interpolates between large and small \( |\omega|/|\varepsilon_0| \) limits is of the
form of Eq. (12), but with $\rho_{\text{pair}}(\Omega)$ replaced by

$$\rho'_{\text{pair}}(\Omega) \sim |\Omega|^{\alpha FES - 1} \left| \frac{\epsilon_0}{\Omega} - 1 \right|^{\alpha FES + \alpha d FES - 1/g}.$$  (14)

All the low-energy results of this section are summarized in Table I.

IV. NUMERICAL CALCULATIONS

In this section we present the results of numerical calculations, verifying the conclusions of our previous analysis, i.e., that the LDoS at low energies features a power-law behavior with the power determined by LL physics only (in the strong coupling phase), although, as we have shown before, its integral (the level population) is universal, and cannot be used to extract LL parameters.

To calculate the LDoS we used classical Monte-Carlo (MC) simulations on the CG expansion of dot Green function. The MC update procedure used is similar to the one employed recently for the closely-related continuous time quantum MC algorithm. After obtaining the imaginary-time Green function it was Fourier-transformed to Matsubara frequencies, followed by analytic continuation to real frequencies using the Padé approximant technique. This yields the retarded Green function, whose imaginary part is proportional to the LDoS. Below we present data in the non-perturbative strong-coupling region, which confirms the results of our previous analysis. Actually, in the weak-coupling phase (which is accessible analytically through the NIBA-like
approximation) MC simulations are not efficient, since there CG charges are rare and averaging very slow. In this sense, our analysis and numerical calculations are complementary.

The results presented in the different panels Fig. 4. The values of $g$, $U_0$ and $K$ are varied, in a way which keeps $\alpha_{\text{FES}}$ constant at a value of 2/3. Hence, the occupancies as functions of $\varepsilon_0$ are the same, as we also verify below (actually, the CG representation would predict exactly identical occupations). The LDOS curves are, however, markedly different: depending on whether $g > 1$, $g < 1$, or $g = 1$, they have a maximum, a minimum, or no special feature near the Fermi energy, respectively. In the inset we demonstrate that in all cases the LDOS at the Fermi energy exhibits power-law dependence on temperature, $\rho_0(0) \sim T^{1/g-1}$, as found in the previous section (cf. Table II).

For the sake of completeness, we will repeat here some of our previous data on the level occupancy. Since the MC simulation is based on the CG, to have an independent check of the universality of the level population we employed the density matrix renormalization group (DMRG) algorithm, using block-sizes of up to 256. DMRG is also better suited to ground state calculations, and thus complements the necessarily finite-temperature MC in this respect. The model used is a half-filled tight binding chain with nearest-neighbor interactions. It is described by the Hamiltonian:

$$ H_L = \sum_{i=1}^{N-1} \left[ t c_i^\dagger c_{i+1} + \text{H.c.} + U \left( c_i^\dagger c_i - \frac{1}{2} \right) \left( c_{i+1}^\dagger c_{i+1} - \frac{1}{2} \right) \right] $$

(15)

where $c_i^\dagger$ ($c_i$) is the electronic creation (annihilation) operator at the $i$th site of the wire ($i = 1 \ldots N$, with $L = N a$), and $t$ and $U$ are, respectively, nearest-neighbor hopping amplitude and interaction strength.

The low energy physics of this model is known to be governed by LL theory for not too large interactions (i.e., $|U| < 2t$) with $g = \pi/[2 \cos^{-1}(-U/2t)]$ and $v/(2at) = \pi \sqrt{1-(U/2t)^{2}/[2 \cos^{-1}(U/2t)]}$. The dot is still governed by the same $H_D = \varepsilon_0 d^d$, and is coupled to the lead through:

$$ H_{DL} = t d c_1^\dagger d + \text{H.c.} + U_d (d^d - \frac{1}{2}) \left( c_1^\dagger c_1 - \frac{1}{2} \right) $$

(16)

$t_d$, $U_d$ are related to the corresponding parameters of the continuum version Eq. (2) by $t_0 = t_d N a$, and $U_0 = U_d a$, $a$ being the lattice spacing. We have previously shown that boundary confimal field theory arguments and the Bethe ansatz solution yield that here $\delta_0 = \tanh^{-1}(U_d/\sqrt{4 t_d^2 - U_d^2})\Delta t_d$.

The level population is plotted in Fig. 4 as a function of $\varepsilon_0$. Different curves correspond to different values of $\alpha_{\text{FES}}$, as indicated in the legend. On each such curve there are three types of symbols, denoting DMRG data on three different models: (i) $g = 1$ (i.e., $U = 0$) but nonzero $U_d$; (ii) $g \neq 1$ (nonzero $U$) but $U_d = 0$; (iii) both $U$ and $U_d$ are nonzero. All the models are without coupling to the bath ($K = 0$). The values of $U$ and $U_0$ in each model were chosen so as to give the same value of $\alpha_{\text{FES}}$ for each curve. For model (iii) we used $U = \pm 0.5 t$, with sign opposite to that of model (ii). In all cases we chose $t_d$ to get $\Gamma_0 = 10^{-4} t$ and used $N = 100 v/(at)$ sites. The results clearly show that the occupancy is indeed universal, depending only on $\alpha_{\text{FES}}$, and not on the strengths or signs of the interactions ($U$ and $U_d$). It should be noted that the widest curve has similar parameters to those used in the MC simulations (Fig. 4).

V. CONCLUSIONS

To summarize, we have studied, both analytically and numerically, the LDOS of a level coupled to a LL and to an Ohmic bath over the entire parameter space. We have found that in general it exhibits a power-law dependence at low energies. In large parts of the phase space this is just the power-law behavior of the tunneling density of states at the end of a LL wire. Thus, a measurement of the LDOS there can be used to extract the value of the LL interaction parameter $g$. In other regions it is also affected by level-lead and level-bath interactions. In any case the LDOS is explicitly sensitive to the value of the LL parameter $g$, although the LDOS determines the level.
population, which was found before to be universal\textsuperscript{21}, and thus not to feature any LL-specific power-law.

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