Dispersion and Dispersivity Tensors in
Saturated Porous Media with Uniaxial Symmetry

Leonid G. Fel and Jacob Bear

Department of Civil and Environmental Engineering,
Technion – Israel Institute of Technology, Haifa 3200, Israel

Abstract

The coefficients of dispersion, $D_{ij}$, and the dispersivity, $a_{ijkl}$, appear in the expression for the flux of a solute in saturated flow through porous media. We present a detailed analysis of these tensors in an axially symmetric porous medium and show that in such a medium, the dispersivity is governed by 6 independent moduli. We present also the constraints that have to be satisfied by these moduli. We also show that at least two independent experiments are required in order to obtain the values of these coefficients for any three-dimensional porous medium domain.

1 Introduction

The coefficients of dispersion and dispersivity appear in the expression for the flux, $J$, of a solute in saturated flow through porous media. Here, a porous medium domain is one that is made up of two parts: a solid matrix and a void-space completely occupied by one or more fluid phases. The void-space is interconnected, thus allowing fluid flow through it [3]. Both parts are distributed over the entire porous medium domain. We study the transport of a solute dissolved in a fluid phase. At every point within the fluid, a solute concentration, $\gamma$, and a fluid velocity, $V$, can be identified. A mathematical model describing the solute’s transport within the void space requires information on the fluid-solid interface that bounds the fluid occupied domain. However, in natural materials, such as soil and fractured rock, this information is not available. Therefore, the problem of solute transport is usually described in terms of averaged velocity, $\overline{V}$, and averaged solute concentration, $\overline{\gamma}$. These averaged values, taken over some representative elementary volume, are assigned to every point in the porous medium domain.
With the total (advective) solute flux at a point inside the void-space defined by \( \gamma \mathbf{V} \), and at a point in the porous medium domain, regarded as a continuum, defined by \( \overline{\gamma V} \), we have \( \overline{\gamma V} = \overline{\gamma V} + \overline{\gamma V}' \), where \( \overline{\gamma} = \gamma - \overline{\gamma}, \overline{\mathbf{V}} = \mathbf{V} - \overline{\mathbf{V}} \) are deviations from the average values, with \( \overline{\gamma} = \overline{\mathbf{V}} = 0 \). The above equality states that the total flux of the solute at a point in the porous medium domain, is made up of an advective flux (= product of average concentration and average velocity), and another flux, called dispersive flux, equal to the average of the product of the two deviations, \( \mathbf{J} = \overline{\gamma V}' \).

The above discussion can be extended to any extensive quantity, with \( \gamma \) representing its intensive quantity. Thus, the phenomenon of dispersion occurs whenever an extensive quantity (mass of a solute, in the example considered above) is transported in a fluid phase that flows in the void space of a porous medium. It is a consequence of using an average velocity and average concentration to describe the advective flux of the solute in the void space. The total flux is the sum of the advective flux and the dispersive one. In this article, we use the mass of dissolved chemical species as the considered extensive quantity. For short, we denote the average values: \( \overline{\gamma} = c, \overline{\mathbf{V}} = \mathbf{U} \).

The coefficient of dispersion, \( D_{ij} \), is a symmetric tensor of 2nd rank, which appears in the Fickian-type expression for the dispersive flux of a solute (mass of solute per unit area of void space in a cross-section of the porous medium) \( \overline{\mathbf{J}} = -3 \sum_{j=1}^{3} D_{ij} \nabla j c \), \( D_{ij} = D_{ji}, i, j = 1, 2, 3 \) (1)

where \( J_i \) denotes the \( i \)th component of the solute’s 3-dim flux vector, \( \mathbf{J} \), and \( c \) is the solute’s average concentration. Equation (1) is valid for the general case of any anisotropic porous medium, with isotropic media as a special case.

In contrast to similar linear flux laws, e.g., Ohm’s and Fourier’s laws, with corresponding tensorial coefficients of electroconductivity, \( \sigma_{ij} \), and of thermoconductivity, \( \lambda_{ij} \), respectively, which depend only on properties of the considered medium, here, in the case of the dispersive flux in saturated flow through a porous medium, the dispersion tensor \( D_{ij} \) depends also on the fluid’s average velocity \( \mathbf{U} \) in the void space \( \overline{\mathbf{U}} \),

\[
D_{ij} = \frac{1}{\overline{\mathbf{U}}} \sum_{k,l=1}^{3} a_{ijkl} U_k U_l, \quad \mathbf{U}^2 = \sum_{k=1}^{3} U_k^2, \quad \mathbf{U}'^2 = \mathbf{U}^2,
\] (2)

where \( a_{ijkl} \) is a property, called dispersivity, of the porous medium only, and \( U_i \) is the \( i \)th component of the fluid’s average velocity vector \( \mathbf{U} \). The dispersivity, is a 4th rank tensor, with intrinsic symmetry \( [V^2]^2 \) in Jahn notations \( [V^2]^2 \)

\[
[V^2]^2 : a_{ijkl} = a_{ijlk}, \quad a_{ijkl} = a_{jikl}.
\] (3)
Additional simplifications in dealing with the dispersivity tensor can be obtained by making use of certain auxiliary algebraic relations among its components. These relations appear under the action of the crystallographic point symmetry group $\mathcal{G} \subset O(3)$ of the porous medium that makes some (not all) of the 36 elements of $a_{ijkl}$ survive, or at least independent of the remaining ones. According to [9], the decrease in the number $N(\mathcal{G})$ of $a_{ijkl}$-components is significant when we increase the symmetry, i.e.,

$$N(C_1) = 36 \rightarrow \ldots \rightarrow N(D_{2h}) = 12 \rightarrow N(D_{4h}) = 7 \rightarrow N(D_{\infty h}) = 6 \rightarrow N(O(3)) = 2. \quad (4)$$

In thermodynamics, the rate of entropy production, $\sigma$, is related to the thermodynamic driving force, $X$, and to the thermodynamic flux, $Y$. According to [4], two vectors $X$ and $Y$ are considered conjugated if they obey the relationship

$$\sigma = \sum_{i=1}^{3} Y_i X_i. \quad (5)$$

In the case considered here, the dispersive flux of the solute, $J$, is driven by $\nabla c$, which acts as a ‘driving force.’ In other words, $X = g \nabla c$ and $Y = g J$, where $g$ denotes a (dimensional) scalar parameter, which depends on the considered transport phenomenon. In this case, the rate of entropy production is expressed by

$$\sigma = g^2 \sum_{i,j=1}^{3} D_{ij} \nabla_i c \nabla_j c. \quad (6)$$

By requiring that the quadratic form (6) be positive definite, we derive the following constraints on the $D_{ij}$-matrix,

$$D_{ii} > 0, \quad \det \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} > 0, \quad \det \begin{pmatrix} D_{11} & D_{13} \\ D_{31} & D_{33} \end{pmatrix} > 0, \quad \det \begin{pmatrix} D_{22} & D_{23} \\ D_{23} & D_{33} \end{pmatrix} > 0, \quad \det D_{ij} > 0.$$

The above constraints hold irrespective of the fluid’s averaged velocity $U$. Furthermore, by (2), they provide another set of constraints imposed on elements of the dispersivity tensor, $a_{ijkl}$. Such straightforward way to establish these constraints is not effective, since it requires a lot of routine algebra and does not prevent repetitive inequalities. In section [4] we shall present a much more effective method for deriving the sought constraints.

In the meantime, let us estimate the total number of such inequalities. Making use of the 2-dim matrix representation of the $a_{ijkl}$-tensor [9], one can show that the number $M(\mathcal{G})$ of such constraints decreases also with growth of the symmetry $\mathcal{G}$ of the porous medium,

$$M(C_1) = 63 \rightarrow \ldots \rightarrow M(D_{2h}) = 10 \rightarrow M(D_{4h}) = 7 \rightarrow M(D_{\infty h}) = 6 \rightarrow M(O(3)) = 2. \quad (7)$$
The Tensor $a_{ijkl}$ in a Porous Medium with Uniaxial Symmetry

In this section, we study the 3-dim tensor $a_{ijkl}$ whose components depend only on porous medium properties, actually, for saturated flow, only on the geometry of the medium’s void space. We shall focus on porous media with uniaxial symmetry. Our interest in this kind of porous medium is motivated by two reasons. First, in saturated flow in ground water geological formations, called aquifers, the porous medium comprising the latter is often axially symmetric \[8\], with a vertical axis, e.g., in a layered aquifer. On the other hand, four decades of continuing discussions (see \[7\] and references therein) on explicit expressions of the tensors $a_{ijkl}$ and $D_{ij}$ in porous media with uniaxial symmetry have not provided any rigorous analysis of these entities.

Instead of a 2-dim matrix representation of the $a_{ijkl}$-tensor, presented in \[9\], we suggest another representation, which originates in the theory of uniaxial nematic liquid crystals (LC). The underlying idea is that when we compare the $a_{ijkl}$-tensor with the 4th rank viscosity tensor, $\eta_{ijkl}$, of a uniaxial LC, its intrinsic symmetry is $[[V^2]]^2$, i.e., slightly ‘stronger’ than $[[V^2]]^2$, with,

$$
[[V^2]]^2 : \eta_{ijkl} = \eta_{jikl}, \quad \eta_{ijkl} = \eta_{jikl}, \quad \eta_{ijkl} = \eta_{klij}.
$$

(8)

According to \[6\], the known representation of $\eta_{ijkl}$ is

$$
\eta_{ijkl} = \eta_1 \delta_{ij} \delta_{kl} + \eta_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \eta_3 (e_i e_j \delta_{kl} + e_k e_l \delta_{ij}) + \eta_4 (e_i e_k \delta_{jl} + e_j e_l \delta_{ik}) + \eta_5 (e_i e_j \delta_{kl} + e_k e_l \delta_{ij}) + \eta_6 (e_i e_j e_k e_l),
$$

(9)

in which the $e_i$’s denote the three components of the unit vector $e$, along the symmetry axis of the medium.

From (9), it follows that there exists only one term, $(e_i e_j \delta_{kl} + e_k e_l \delta_{ij})$, which reduces its intrinsic symmetry: $[[V^2]]^2 \rightarrow [V^2]^2$, after decomposition of this term into two separate terms: $e_i e_j \delta_{kl}$ and $e_k e_l \delta_{ij}$. This explains why, in accordance with \[7\], the tensor $a_{ijkl}$ has six independent moduli and how to construct its most generic form:

$$
a_{ijkl} = a_1 \delta_{ij} \delta_{kl} + \frac{a_2}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + a_3 e_i e_j \delta_{kl} + a_4 e_k e_l \delta_{ij} + \frac{a_5}{2} (e_i e_k \delta_{jl} + e_j e_k \delta_{il} + e_i e_l \delta_{jk} + e_j e_l \delta_{ik}) + a_6 e_i e_j e_k e_l.
$$

(10)

Note that by making $e_i \rightarrow 0$ in (10), we obtain the dispersivity tensor for an isotropic porous medium,

$$
a_{ijkl} = a_1 \delta_{ij} \delta_{kl} + \frac{a_2}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
$$

(11)

The axisymmetric tensor (10) has to be supplemented by a set of constraints that results from the thermodynamic requirement imposed on the rate of entropy production, $\sigma$, defined in \[6\].
3 The Tensor $D_{ij}$ in a Porous Medium with Uniaxial Symmetry

In this section, we consider the 3-dim tensor $D_{ij}$ and certain interesting related phenomena. Based on (2) and (10), we rewrite this tensor as

$$UD_{ij} = a_1 U^2 \delta_{ij} + a_2 U_i U_j + a_3 U^2 e_i e_j + a_4 \langle e, U \rangle (U_i e_j + e_i U_j) + a_5 \langle e, U \rangle ^2 e_i e_j,$$

(12)

where $\langle e, U \rangle$ denotes a scalar product of the two vectors: $e$ and $U$.

It is interesting to note that Batchelor [1], in his work on axisymmetric turbulence, derived a general expression for $D_{ij}$ that is based on five tensorial terms: $e_i e_j$, $\delta_{ij}$, $U_i U_j$, $U_i e_j$, $U_j e_i$, and four scalar functions, $C_i (U, e)$,

$$UD_{ij} = C_1(U, e) e_i e_j + C_2(U, e) \delta_{ij} + C_3(U, e) U_i U_j + C_4(U, e) (U_i e_j + U_j e_i).$$

(13)

Keeping in mind that there exist only two quadratic-in-$U$ scalar invariants, $U^2$ and $\langle U, e \rangle ^2$, and only one linear-in-$U$ scalar invariant, $\langle U, e \rangle$, we can verify the equivalence of the two expressions, (12) and (13), by noting that

$$C_1(U, e) = a_3 U^2 + a_6 \langle U, e \rangle ^2, C_2(U, e) = a_1 U^2 + a_4 \langle U, e \rangle ^2, C_3(U, e) = a_2, C_4(U, e) = a_5 \langle U, e \rangle.$$

We introduce the two 6-dim vectors, $D_6$ and $a_6$,

$$D_6 = \{D_{11}, D_{22}, D_{33}, D_{12}, D_{23}, D_{31}\}, a_6 = \{a_1, a_2, a_3, a_4, a_5, a_6\},$$

(14)

and denote their transposed counterparters by $D_6^T$ and $a_6^T$. Next, we rewrite (12) in the form

$$D_6^T = \frac{1}{U} \Delta_1 (U, e) \cdot a_6^T,$$

(15)

where the $6 \times 6$ matrix $\Delta_1 (U, e)$ is defined as

$$\Delta_1 (U, e) = \begin{bmatrix} U^2 & U^2_1 & e_1^2 U^2 & \langle e, U \rangle ^2 & 2 \langle e, U \rangle e_1 U_1 & \langle e, U \rangle ^2 e_1^2 \\ U^2 & U^2_2 & e_2^2 U^2 & \langle e, U \rangle ^2 & 2 \langle e, U \rangle e_2 U_2 & \langle e, U \rangle ^2 e_2^2 \\ U^2 & U^2_3 & e_3^2 U^2 & \langle e, U \rangle ^2 & 2 \langle e, U \rangle e_3 U_3 & \langle e, U \rangle ^2 e_3^2 \\ 0 & U_1 U_2 & e_1 e_2 U^2 & 0 & \langle e, U \rangle (U_1 e_2 + e_1 U_2) & \langle e, U \rangle ^2 e_1 e_2 \\ 0 & U_2 U_3 & e_2 e_3 U^2 & 0 & \langle e, U \rangle (U_2 e_3 + e_2 U_3) & \langle e, U \rangle ^2 e_2 e_3 \\ 0 & U_3 U_1 & e_3 e_1 U^2 & 0 & \langle e, U \rangle (U_3 e_1 + e_3 U_1) & \langle e, U \rangle ^2 e_3 e_1 \end{bmatrix}.$$

(16)

Equation (15) opens the way to the derivation of the 6 dispersivity $a_i$-moduli from values of $D_{ij}$. The latter can be obtained from experimental measurements, assuming that the symmetry
axis \(\mathbf{e}\) is known, and so is the velocity \(\mathbf{U}\), in an appropriate reference frame, \(x_1, x_2, x_3\). However, a straightforward calculation shows that the matrix \(\Delta_1(\mathbf{U}, \mathbf{e})\) is degenerated,

\[
\det \Delta_1(\mathbf{U}, \mathbf{e}) = 0, \quad \text{rank} \Delta_1(\mathbf{U}, \mathbf{e}) = 4.
\] (17)

Equality (17) leads to two most important consequences and raises a question:

1. At most, four independent components \(D_{ij}\) can be obtained from a single experiment. The remaining two \(D_{ij}\)’s are linearly representable by the first four.

2. One cannot get all six dispersivity \(a_i\)-moduli from a single experiment.

3. How many experiments do we have to perform in order to obtain all six \(a_i\)-values?

In what follows, we shall demonstrate that two different experiments suffice for obtaining all six \(a_i\)-values. We consider two cases (= experiments):

1. \(U_1 = 0, U_2 = U_3 = u\) and \(e_1 = 1, e_2 = e_3 = 0\). Then,

\[
D_{11} = 2(a_1 + a_3)u, \quad D_{22} = D_{33} = (2a_1 + a_2)u, \quad D_{23} = a_2u, \quad D_{12} = D_{31} = 0.
\] (18)

The above relationships provide three moduli: \(a_1, a_2\) and \(a_3\).

2. \(U_1 = U_2 = u, U_3 = 0\) and \(e_1 = 1, e_2 = e_3 = 0\). Then,

\[
D_{11} = (2a_1 + a_2 + 2a_3 + a_4 + 2a_5 + a_6)u, \quad D_{12} = (a_2 + a_5)u, \quad D_{22} = (2a_1 + a_2 + a_4)u, \quad D_{33} = (2a_1 + a_4)u, \quad D_{23} = D_{31} = 0.
\] (19)

The above relations provide three additional moduli: \(a_4, a_5\) and \(a_6\).

Thus, by two experiments, with different setups of \(\mathbf{U}\), and a known \(\mathbf{e}\), we have obtained the entire set of 6 dispersivity moduli in a 3-dim saturated porous medium with uniaxial symmetry.

3.1 The Tensor \(D_{ij}\) in an Isotropic Porous Medium

For an isotropic porous medium, instead of (12), we have

\[
U D_{ij} = a_1 U^2 \delta_{ij} + a_2 U_i U_j.
\] (20)

This means that only two moduli: \(a_1\) and \(a_2\) are required for a full description of the \(D_{ij}\)-components. The expression for \(\Delta_1(\mathbf{U}, \mathbf{e})\)-matrix, which is defined in (16), is even more degenerated, with \(\text{rank} \Delta_1(\mathbf{U}, \mathbf{e}) = 2\). This indicates a strong linear dependence among the \(D_{ij}\)-values: there exist four linear relations among the six \(D_{ij}\)-elements.
Let us consider an isotropic porous medium, with the setup: \( U_1 = u, U_2 = U_3 = 0 \). Then,

\[
\left( (D_{ij}) \right) = u \begin{pmatrix}
  a_1 + a_2 & 0 & 0 \\
  0 & a_1 & 0 \\
  0 & 0 & a_1
\end{pmatrix}.
\]

(21)

In ground water hydrology [3], the modulus \( a_1 \) is called transversal dispersivity (denoted \( a_T \)) and \( a_1 + a_2 \) is called longitudinal dispersivity (denoted \( a_L \)).

4 Thermodynamic constraints

In this section we determine the algebraic constraints that have to be satisfied by the six dispersivity \( a_i \)-moduli for an axisymmetric porous medium. We substitute (12) into (6) and present \( \sigma \) in its invariant form, irrespective of the reference frame,

\[
\frac{\sigma}{g^2} = a_1 U^2 (\nabla c)^2 + a_2 (U, \nabla c)^2 + a_3 U^2 (\langle e, \nabla c \rangle)^2 + a_4 (e, U)^2 (\nabla c)^2 + 2 a_5 \langle e, U \rangle \langle U, \nabla c \rangle \langle \nabla c, e \rangle + a_6 (e, U)^2 (\langle e, \nabla c \rangle)^2.
\]

(22)

We then represent (22) as a quadratic form. This is a standard way to write the necessary and sufficient conditions for \( \sigma \) to be positive definite. For this purpose, we choose the reference frame such that \( e_1 = e_2 = 0 \) and \( e_3 = 1 \). By substituting these into (22), we get

\[
\frac{\sigma}{g^2} = a_1 \sum_{i=1}^{3} U_i^2 \sum_{i=1}^{3} (\nabla_i c)^2 + a_2 \left( \sum_{i=1}^{3} U_i \nabla_i c \right)^2 + a_3 (\nabla_3 c)^2 \sum_{i=1}^{3} U_i^2 + a_4 U_3^2 \sum_{i=1}^{3} (\nabla_i c)^2 + 2 a_5 U_3 \nabla_3 c \sum_{i=1}^{3} U_i \nabla_i c + a_6 U_3^2 (\nabla_3 c)^2.
\]

(23)

Expression (23) is a quadratic form in a 9-dim space, with orthogonal basis \( U_i \nabla_j c, i, j = 1, 2, 3 \). We decompose it in the form

\[
\frac{\sigma}{g^2} = a_1 \left[ U_1^2 (\nabla_2 c)^2 + U_2^2 (\nabla_1 c)^2 \right] + (a_1 + a_3) (\nabla_3 c)^2 \left[ U_1^2 + U_2^2 \right] + (a_1 + a_4) U_3^2 \left[ (\nabla_1 c)^2 + (\nabla_2 c)^2 \right] + (a_1 + a_2) \left[ U_1^2 (\nabla_1 c)^2 + U_2^2 (\nabla_2 c)^2 \right] + (a_1 + a_2 + a_3 + a_4 + 2 a_5 + a_6) U_3^2 (\nabla_3 c)^2 + 2 a_2 U_1 U_2 \nabla_1 c \nabla_2 c + 2 (a_2 + a_5) U_3 \nabla_3 c (U_1 \nabla_1 c + U_2 \nabla_2 c),
\]

(24)

and require the positive definiteness of the last expression. Then we obtain three inequalities

\[
a_1 > 0, \quad a_1 + a_3 > 0, \quad a_1 + a_4 > 0,
\]

(25)
and requirement of positive definiteness of the $3 \times 3$ matrix $\Gamma$,

$$
\Gamma = \begin{bmatrix}
    a_1 + a_2 & a_2 & a_2 + a_5 \\
    a_2 & a_1 + a_2 & a_2 + a_5 \\
    a_2 + a_5 & a_2 + a_5 & a_1 + a_2 + a_3 + a_4 + 2a_5 + a_6
\end{bmatrix}.
$$

(26)

The last claim leads to three additional algebraically independent restrictions,

$$
a_1 + a_2 > 0, \quad a_1 + a_2 + a_3 + a_4 + 2a_5 + a_6 > 0,
$$

$$
a_1^2 + a_1(3a_2 + a_3 + a_4 + 2a_5 + a_6) + 2a_2(a_3 + a_4 + a_6) > 2a_5^2.
$$

(27)

Thus, we have arrived at 6 constraints imposed on the 6 dispersivity $a_i$-moduli; this is in full agreement with (7) for uniaxial symmetry group $G = D_{\infty h}$. Note that the first two inequalities in (25) and (27) correspond to isotropic porous media.

5 The Tensor $D_{ij}$ in a Planar Porous Medium with Dihedral Symmetry

In this section we consider the dispersion phenomenon in a 2-dim porous medium domain, which is obtained by intersecting a 3-dim uniaxial porous medium with a plane that is parallel to the uniaxial axis. In the resulting 2-dim porous medium domain, there are two mirror lines, which are perpendicular to each other. By these symmetry elements, it is easy to recognize the dihedral point symmetry group $D_2$, which is one of ten 2-dim point symmetry subgroups of $O(2)$.

From the mathematical standpoint, the dispersion tensor, $D_{ij}$, can be represented by a $2 \times 2$ symmetric matrix, with three components: $D_3 = \{D_{11}, D_{22}, D_{12}\}$, instead of the $6 \times 6$ matrix $\Delta_2(U, e)$, as in (15). By doing so, we obtain:

$$
D_3^T = \frac{1}{U} \Delta_2(U, e) \cdot a_6^T,
$$

(28)

where the $3 \times 6$ matrix, $\Delta_2(U, e)$, is defined as

$$
\Delta_2(U, e) = \begin{bmatrix}
    U^2 & U_1^2 & e_2^2U^2 & \langle e, U \rangle^2 & 2\langle e, U \rangle e_1 U_1 & \langle e, U \rangle^2 e_1^2 \\
    U^2 & U_2^2 & e_1^2U^2 & \langle e, U \rangle^2 & 2\langle e, U \rangle e_2 U_2 & \langle e, U \rangle^2 e_2^2 \\
    0 & U_1 U_2 & e_1 e_2 U^2 & 0 & \langle e, U \rangle(U_1 e_2 + e_1 U_2) & \langle e, U \rangle^2 e_1 e_2
\end{bmatrix}.
$$

(29)

Equation (28) does not enable the determination of the six dispersivity $a_i$-moduli, from 3 values of $D_{ij}$, which can be obtained from a single experiment. For comparison, in section 3 we have shown that by two experiments, with different setups of $U$, and a known $e$, we have obtained the entire set of 6 dispersivity moduli in a 3-dim saturated porous medium with uniaxial symmetry.
In this conjunction, the following question arises: how many experiments do we have to perform in a 2-dim saturated porous medium domain with dihedral symmetry, in order to obtain the entire set of 6 dispersivity moduli?

It is easy to show that two experiments are not enough. Indeed, let us consider two different flow setups, with velocities \( \mathbf{W} \) and \( \mathbf{U} \), and the same orientation of \( \mathbf{e} \),

\[
\begin{align*}
\mathbf{D}_1^T &= \frac{1}{U} \Delta_2 (\mathbf{W}, \mathbf{e}) \cdot \mathbf{a}_6^T, \\
\mathbf{D}_2^T &= \frac{1}{U} \Delta_2 (\mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T.
\end{align*}
\]

In (30) the front subscripts 1 and 2 in the notations \( \mathbf{D}_1^T \) and \( \mathbf{D}_2^T \), indicate data obtained from the 1st and 2nd experiments, with velocities \( \mathbf{W} \) and \( \mathbf{U} \), respectively. We now introduce a new 6-dim vector, \( \mathbf{D}_{6,a} \),

\[
\mathbf{D}_{6,a} = \{1D_{11}, 1D_{22}, 1D_{12}, 2D_{11}, 2D_{22}, 2D_{12}\},
\]

composed of two 3-dim vectors \( \mathbf{D}_3 = \{1D_{11}, 1D_{22}, 1D_{12}\} \) and \( \mathbf{D}_3 = \{2D_{11}, 2D_{22}, 2D_{12}\} \), and rewrite the two equations in (30), in the form

\[
\mathbf{D}_{6,a}^T = \frac{1}{U} \Delta_3 (\mathbf{W}, \mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T, \
\text{ where}
\]

\[
\Delta_3 (\mathbf{W}, \mathbf{U}, \mathbf{e}) = \begin{cases} 
\mathbf{W}^2 & \mathbf{W}_1^2 & e_1^2 \mathbf{W}_2^2 & \langle \mathbf{e}, \mathbf{W} \rangle^2 & 2\langle \mathbf{e}, \mathbf{W} \rangle e_1 W_1 & \langle \mathbf{e}, \mathbf{W} \rangle^2 e_1^2 \\
\mathbf{W}^2 & \mathbf{W}_2^2 & e_2^2 \mathbf{W}_2^2 & \langle \mathbf{e}, \mathbf{W} \rangle^2 & 2\langle \mathbf{e}, \mathbf{W} \rangle e_2 W_2 & \langle \mathbf{e}, \mathbf{W} \rangle^2 e_2^2 \\
0 & W_1 W_2 & e_1 e_2 \mathbf{W}_2^2 & 0 & \langle \mathbf{e}, \mathbf{W} \rangle (W_1 e_2 + e_1 W_2) & \langle \mathbf{e}, \mathbf{W} \rangle^2 e_1 e_2 \\
\mathbf{U}^2 & U_1^2 & e_1^2 \mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U} \rangle^2 & 2\langle \mathbf{e}, \mathbf{U} \rangle e_1 U_1 & \langle \mathbf{e}, \mathbf{U} \rangle^2 e_1^2 \\
\mathbf{U}^2 & U_2^2 & e_2^2 \mathbf{U}^2 & \langle \mathbf{e}, \mathbf{U} \rangle^2 & 2\langle \mathbf{e}, \mathbf{U} \rangle e_2 U_2 & \langle \mathbf{e}, \mathbf{U} \rangle^2 e_2^2 \\
0 & U_1 U_2 & e_1 e_2 \mathbf{U}^2 & 0 & \langle \mathbf{e}, \mathbf{U} \rangle (U_1 e_2 + e_1 U_2) & \langle \mathbf{e}, \mathbf{U} \rangle^2 e_1 e_2 
\end{cases}
\]

A straightforward calculation shows that

\[
\det \Delta_3 (\mathbf{W}, \mathbf{U}, \mathbf{e}) = 0, \quad \text{rank } \Delta_3 (\mathbf{W}, \mathbf{U}, \mathbf{e}) = 5.
\]

Thus, among the six elements of the vector \( \mathbf{D}_{6,a} \), given in (31), there are only five independent ones, and equation (32) does not suffice for providing the six dispersivity \( a_i \)-moduli.

We shall now show that three experiments are sufficient for providing the entire set of \( a_i \)'s. Consider 3 different flow setups, with velocities: \( \mathbf{W}, \mathbf{U} \) and \( \mathbf{Y} \) and with the same orientation of \( \mathbf{e} \),

\[
\begin{align*}
\mathbf{D}_1^T &= \frac{1}{U} \Delta_2 (\mathbf{W}, \mathbf{e}) \cdot \mathbf{a}_6^T, \\
\mathbf{D}_2^T &= \frac{1}{U} \Delta_2 (\mathbf{U}, \mathbf{e}) \cdot \mathbf{a}_6^T, \\
\mathbf{D}_3^T &= \frac{1}{U} \Delta_2 (\mathbf{Y}, \mathbf{e}) \cdot \mathbf{a}_6^T,
\end{align*}
\]

where \( \mathbf{D}_3 \) and \( \mathbf{D}_3 \) were defined earlier and \( \mathbf{D}_3 = \{3D_{11}, 3D_{22}, 3D_{12}\} \). In contrast to (31), we introduce another 6-dim vector, \( \mathbf{D}_{6,b} \),

\[
\mathbf{D}_{6,b} = \{1D_{11}, 1D_{12}, 2D_{11}, 2D_{12}, 3D_{11}, 3D_{12}\}, \quad (35)
\]
composed of two elements, \( iD_{11} \) and \( iD_{12} \), of three 3-dim vectors \( iD_3 \), \( i = 1, 2, 3 \). We rewrite the three equations (34) in the form,

\[
\Delta_4 (W, U, \Upsilon, e) = \begin{pmatrix}
W^2 & W^2 & e_1^2 W^2 & \langle e, W \rangle^2 & 2 \langle e, W \rangle e_1 W_1 & \langle e, W \rangle^2 e_1^2 \\
0 & W_1 W_2 & e_1 e_2 W^2 & 0 & \langle e, W \rangle (W_1 e_2 + e_1 W_2) & \langle e, W \rangle^2 e_1 e_2 \\
U^2 & U^2 & e_1^2 U^2 & \langle e, U \rangle^2 & 2 \langle e, U \rangle e_1 U_1 & \langle e, U \rangle^2 e_1^2 \\
0 & U_1 U_2 & e_1 e_2 U^2 & 0 & \langle e, U \rangle (U_1 e_2 + e_1 U_2) & \langle e, U \rangle^2 e_1 e_2 \\
\Upsilon^2 & \Upsilon^2 & e_1^2 \Upsilon^2 & \langle e, \Upsilon \rangle^2 & 2 \langle e, \Upsilon \rangle e_1 \Upsilon_1 & \langle e, \Upsilon \rangle^2 e_1^2 \\
0 & \Upsilon_1 \Upsilon_2 & e_1 e_2 \Upsilon^2 & 0 & \langle e, \Upsilon \rangle (\Upsilon_1 e_2 + e_1 \Upsilon_2) & \langle e, \Upsilon \rangle^2 e_1 e_2
\end{pmatrix}
\]

Note that the three 2-dim vectors: \( W, U \) and \( \Upsilon \) are always coplanar. However, if at least two of them would be collinear, then (36) cannot be solved uniquely, i.e. \( \det \Delta_4 (kU, U, \Upsilon, e) = 0 \), where \( k \) is a real number. Otherwise, a straightforward calculation shows that if neither two of the three vectors: \( W, U \), and \( \Upsilon \) are collinear, then

\[
\det \Delta_4 (W, U, \Upsilon, e) \neq 0, \quad \text{rank} \Delta_4 (W, U, \Upsilon, e) = 6.
\]

Thus, three special (not any) experiments suffice for providing all six dispersivity \( a_i \)-moduli in a 2-dim saturated porous medium with dihedral symmetry.

6 Concluding Remarks

For the case of an axisymmetric porous medium, we have shown that six independent \( a_i \)-moduli are needed in order to determine all components of the dispersivity tensor, and the constraints that these moduli have to satisfy. This information is required when determining the latter by experiments for specific porous media. We have also found the number of experiments that are required to determine the entire set of dispersivity moduli for 2-dim and 3-dim domains. This is important for experimental determination of the dispersivity.

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e-mail: lfel@tx.technion.ac.il,
e-mail: cvrbear@tx.technion.ac.il