Relating Symmetrizations of Convex Bodies: Once More the Golden Ratio

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Abstract. Similar to the arithmetic-harmonic mean inequality for numbers, the harmonic mean of two convex sets \( K \) and \( C \) is always contained in their arithmetic mean. The harmonic and arithmetic means of \( C \) and \( -C \) define two different symmetrizations of \( C \), each keeping some useful properties of the original set. We investigate the relations of such symmetrizations, involving a suitable measure of asymmetry—the Minkowski asymmetry, which, besides other advantages, is polynomial time computable for (reasonably given) polytopes. The Minkowski asymmetry measures the minimal dilatation factor needed to cover a set \( C \) by a translate of its negative. Its values range between 1 and the dimension \( \dim(C) \) of \( C \), attaining 1 if and only if \( C \) is symmetric and \( \dim(C) \) if and only if \( C \) is a simplex. Restricting to planar compact, convex sets, positioned so that the translation in the definition of the Minkowski asymmetry is 0, we show that if the asymmetry of \( C \) is greater than the golden ratio \( (1 + \sqrt{5})/2 \approx 1.618 \), then the harmonic mean of \( C \) and \( -C \) is a subset of a dilatate of their arithmetic mean with a dilatation factor strictly less than 1; and for any asymmetry less than the golden ratio, there exists a set \( C \) with the given asymmetry value, such that the considered dilatation factor cannot be less than 1.

The golden ratio \( \varphi = (1 + \sqrt{5})/2 \approx 1.618 \) has a history of 2400 years and wide roots in mathematics, music, architecture, biology, and philosophy (see, e.g., [16]). It was first studied by the ancient Greeks because of its frequent appearance in geometry. For example, if one considers a regular pentagon of edge-length 1, its diagonals have length \( \varphi \). No wonder that the regular pentagram was the Pythagorean symbol [16]. The first known definition is given in Euclid’s Elements, II.11: “If a straight line is cut in extreme and mean ratio, then as the whole line is to the greater segment, the greater is to the lesser segment.” Expressed algebraically, this transfers to the (probably) best-known definition of the golden ratio:

\[
\text{if } a > b > 0 \text{ such that } \frac{a + b}{a} = \frac{a}{b}, \text{ then } \frac{a}{b} = \varphi. \tag{1}
\]

Among the fundamental inequalities in mathematics, a special place is reserved for the arithmetic-geometric-harmonic mean inequality, which in the two-argument case, together with the minimum and maximum, states that

\[
\min\{a, b\} \leq \left(\frac{a^{-1} + b^{-1}}{2}\right)^{-1} \leq \sqrt{ab} \leq \frac{a + b}{2} \leq \max\{a, b\} \tag{2}
\]

for any real numbers \( a, b > 0 \) (see [13, 21]). We may identify means of numbers with means of segments by associating \( a, b > 0 \) with \([-a, a]\) and \([-b, b]\). By doing so we identify, e.g., the arithmetic mean of \( a \) and \( b \) with the segment \([-\frac{1}{2} (a + b), \frac{1}{2} (a + b)] \). In this way means of convex bodies can be introduced.

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Let $\mathcal{K}^n$ denote the set of convex bodies, i.e., full-dimensional compact convex sets in $\mathbb{R}^n$. For $X \subseteq \mathbb{R}^n$ let $\text{conv}(X)$ (respectively, $\text{pos}(X)$ or $\text{aff}(X)$) be the convex hull (respectively, positive hull or affine hull) of $X$, i.e., the smallest convex set in $\mathbb{R}^n$ (respectively, convex cone or affine subspace) containing $X$. A line segment is the convex hull of a two-point set $\{x, y\} \subseteq \mathbb{R}^n$, which we denote by $[x, y]$. For any $K, C \subseteq \mathbb{R}^n$, $\rho \in \mathbb{R}$, let $K + C = \{a + b : a \in K, b \in C\}$ be the Minkowski sum of $K, C$ and $\rho C = \{\rho x : x \in C\}$ the $\rho$-dilatation of $C$. We abbreviate $(-1)C$ by $-C$.

Now, the arithmetic mean of compact convex bodies $K$ and $C$ is defined by $\frac{1}{2}(K + C)$, the minimum by $K \cap C$, and the maximum by $\text{conv}(K \cup C)$. For any $K \in \mathcal{K}^n$ let $K^o = \{a \in \mathbb{R}^n : a^T x \leq 1, x \in K\}$ be the polar of $K$. Since the polarity can be regarded as the higher-dimensional replacement of the inversion operation $x \to 1/x$ (see [17]), the harmonic mean of $K$ and $C$ is defined by $\left(\frac{1}{2}(K^o + C^o)\right)^\circ$. The geometric mean has been extended in several ways (see [4] or [17]); thus it would need a separate, more involved treatment, which is the reason why we focus on the four other means here. The study of means of convex bodies started in the 1960s [8–10], but there also exist several recent papers [17, 18, 19].

Probably the most essential result of Firey is the extension of the harmonic-arithmetic mean inequality from positive numbers to convex bodies containing 0 in their interior in [8]. Moreover, one can easily show that Firey’s inequality again may be extended involving the minimum and maximum:

**Proposition 1.** For all $K, C \in \mathcal{K}^n$ with 0 in their interior we have

$$K \cap C \subseteq \left(\frac{K^o + C^o}{2}\right)^\circ \subseteq \frac{K + C}{2} \subseteq \text{conv}(K \cup C).$$  \hspace{1cm} (3)

Let us mention an application given in [11]. For two positive definite symmetric matrices $A, B \in \mathbb{R}^{n \times n}$ we denote by $A \succ B$ if $A - B$ is also positive definite. Moreover, $A \succ B$ is strict if $A - B$ is not a zero matrix. Since means of ellipsoids correspond to combinations of the corresponding matrices, (3) also results in a (generalized) harmonic-arithmetic mean inequality:

$$(1 - \lambda)A + \lambda B \succ ((1 - \lambda)A^{-1} + \lambda B^{-1})^{-1}$$

for any $\lambda \in [0, 1]$. This inequality is strict, except of the trivial cases $A = B$ or $\lambda \in \{0, 1\}$.

Moreover, the well-known Brunn–Minkowski determinantal inequality [14]

$$((1 - \lambda)\det(A) + \lambda \det(B))^{\frac{1}{n}} \geq \det(((1 - \lambda)A)^{\frac{1}{n}} + \det(\lambda B)^{\frac{1}{n}},$$

can be further developed using the means of convex bodies as follows [11]: Let $k \in \{1, \ldots, n\}$ and $|A|_k$ denote the product of the $k$ greatest eigenvalues of $A$; then

$$|(1 - \lambda)A^{-1} + \lambda B^{-1}|_k^{\frac{1}{k}} \leq ((1 - \lambda)|A|_k^{-\frac{1}{k}} + \lambda |B|_k^{-\frac{1}{k}})^{-1}.$$

For any $K, C \in \mathcal{K}^n$ we say that $K$ is optimally contained in $C$, and denote it by $K \subset^{opt} C$, if $K \subseteq C$ and $K \not\subset t + \rho C$ for any $0 \leq \rho < 1$ and $t \in \mathbb{R}^n$. If $C = t - C$ for some $t \in \mathbb{R}^n$, we say $C$ is symmetric, and if $C = -C$, we say $C$ is 0-symmetric. The family of 0-symmetric convex bodies is denoted $\mathcal{K}_0^n$. By $T \in \mathcal{K}^n$ we denote a regular simplex with (bary-)center 0.

The goal of this article is to consider optimal containments of means of $C$ and $-C$ of a convex body $C$, i.e., symmetrizations of $C$. These kinds of symmetrizations are used...
frequently in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [3], which bounds from above the ratio of the circumradius (\( \min_{x \in \mathbb{R}^n} \max_{y \in K} |x - y| \)) and the diameter (\( \max_{x, y \in K} |x - y| \)) of convex bodies for general norms \(| \cdot |\) by \( n/(n + 1) \), and for which equality is reached in spaces with \( T \cap (-T) \) or \( \frac{1}{2} (T - T) \) as the unit ball [7].

Or consider the characterization of normed spaces in which \( C \) is complete or reduced, if the unit ball is sandwiched between suitable rescalings of two different means of \( C \) and \(-C \) [6, Propositions 3.5–3.10].

Also, well-known geometric inequalities have been re-investigated, replacing one mean by another. Consider, e.g., the Rogers–Shephard-type inequalities, which bound the ratio of the products of the volumes of the maximum and harmonic (respectively, arithmetic) means of \( K \) and \( C \) with the product of their volumes [1, 2, 20].

Notice that for any \( C \in \mathcal{K}^n \) we have

\[
C \cap (-C) \subset^{\text{opt}} \left( \frac{C^\circ - C^\circ}{2} \right)^\circ \quad \text{and} \quad \frac{C - C}{2} \subset^{\text{opt}} \text{conv}(C \cup (-C)).
\]

Moreover,

\[
\left( \frac{C^\circ - C^\circ}{2} \right)^\circ \subset^{\text{opt}} \frac{C - C}{2}
\]

is also possible, i.e., all containments in (3) may be optimal at the same time even for nonsymmetric \( C \). In particular, if \( T \in \mathcal{K}^3 \) is a regular simplex with center 0, then we have the nice situation that the four means are a cross polytope (minimum), a rhombic dodecahedron (harmonic mean), a cuboctahedron (arithmetic mean), and a cube (maximum), such that even the cross polytope is optimally contained in the cube.

However, in the planar case, optimal containment of the harmonic mean of \( T \) and \(-T \) in their arithmetic mean for an equilateral triangle \( T \) implies that the center of the triangle cannot be 0. In contrast, for the equilateral triangle \( T \subset \mathbb{R}^2 \) with center 0, we have

\[
\left( \frac{T^\circ - T^\circ}{2} \right)^\circ \subset^{\text{opt}} \frac{8}{9} \cdot \frac{T - T}{2} \quad \text{and} \quad T \cap (-T) \subset^{\text{opt}} \frac{2}{3} \cdot \text{conv}(T \cup (-T)).
\]

Clearly, symmetrizations of a symmetric \( C \) should coincide with \( C \), which is always true for the arithmetic mean of \( C \) and \(-C \), but for the other means, which we consider, this holds only if 0 is the center of symmetry of \( C \). This indicates the need to fix a meaningful center for every convex body first and then concentrate on translates with that center at 0.

Since we want to investigate the optimality of the inequality chain (3) in dependence of asymmetry, we will introduce one of the most common asymmetry measures, which is best suited to our purposes, and choose the center definition matching it. The \textit{Minkowski asymmetry} of \( C \) is defined by \( s(C) := \inf \{ \rho > 0 : C - c \subset \rho(c - C), \ c \in \mathbb{R}^n \} \) [12] and a \textit{Minkowski center} of \( C \) is any \( c \in \mathbb{R}^n \) such that \( C - c \subset s(C)(c - C) \) [5]. Moreover, if \( c = 0 \) is a Minkowski center, we say \( C \) is \textit{Minkowski centered}. Note that \( s(C) \in [1, n] \) for \( C \in \mathcal{K}^n \), where \( s(C) = 1 \) if and only if \( C \) is centrally symmetric, while \( s(C) = n \) if and only if \( C \) is an \( n \)-dimensional simplex [12]. Moreover, the Minkowski asymmetry \( s : \mathcal{K}^n \to [1, n] \) is continuous with respect to the Hausdorff metric (see [12, 21] for some basic properties) and invariant under nonsingular affine transformations.

The main contribution of this article is that the golden ratio is the largest asymmetry such that (3) can be optimal in the planar case.
Theorem 2. Let $C \in \mathcal{K}^2$ be Minkowski centered such that
\[
\left( \frac{C^\circ - C^\circ}{2} \right)^\circ \subset \text{opt } C - C^2;
\]
then $s(C) \leq \varphi$. Moreover, if $s(C) = \varphi$, then there exists a nonsingular linear transformation $L$ such that $L(C) = \mathcal{GH} := \text{conv} \{p^1, \ldots, p^5\}$, where $p^1 = (-1, -1)^T$, $p^2 = (-1, 0)^T$, $p^3 = (0, \varphi)^T$, $p^4 = (1, 0)^T$, $p^5 = (1, -1)^T$ form the golden house.

Figure 1. Left: $\mathcal{GH}$ (red), $-s(\mathcal{GH})\mathcal{GH}$ (blue), and parallel supporting halfspaces in $p^2$ and $p^4 = -p^2$ (dashed). Right: $\text{conv}(\mathcal{GH} \cup (-\mathcal{GH}))$ (orange), $\mathcal{GH}$ (red), $\left( \frac{1}{2} (\mathcal{GH}^\circ + (-\mathcal{GH})^\circ) \right)^\circ$ (violet), and $\mathcal{GH} \cap (-\mathcal{GH})$ (blue). The golden house and its symmetrizations.

The important facts about the construction of the golden house are the following:
1. $p^2 = -p^4$;
2. $\|p^2 - p^3\| = \|p^4 - p^3\|$;
3. $\text{conv}(\{p^1, -s(\mathcal{GH})p^3, p^5\})$ and $\text{conv}(\{p^2, p^3, p^4\})$ are similar up to reflection.

Let $g := [p^1, p^3] \cap [p^3, -s(\mathcal{GH})p^3]$, $\alpha := \|p^3 - g\|$, and $\beta := \|p^3\|$. Then we have on the one hand
\[
s(\mathcal{GH}) = \frac{\| - s(\mathcal{GH})p^3\|}{\|p^3\|} = \frac{\|p^3 - g\|}{\|p^3\|} = \frac{\alpha}{\beta}, \quad (4)
\]
and on the other hand
\[
s(\mathcal{GH}) = \frac{\| - s(\mathcal{GH})p^3 - p^3\|}{\|p^3 - g\|} = \frac{\alpha + \beta}{\alpha}. \quad (5)
\]
Combining (4) and (5) we see that $s(\mathcal{GH}) = \varphi$ (see Left in Figure 1).

To the best of our knowledge, this is the first explicit mention of a set with the properties of the golden house. Theorem 3 demonstrates that items 1 and 2 above suffice to show that, in the case of the golden house (and its negative), optimal containment is reached in (3) throughout the full chain (see Right in Figure 1). Even more: from Theorem 3 it directly follows that the minimum is optimally contained in the maximum.

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For any $C \in \mathcal{K}^n$ let $\text{bd}(C)$ be the boundary of $C$ and for any $a \in \mathbb{R}^n \setminus \{0\}$ and $\rho \in \mathbb{R}$, let $H^\leq_{a,\rho} = \{x \in \mathbb{R}^n : a^T x \leq \rho\}$ denote a halfspace. We say that the halfspace $H^\leq_{a,\rho}$ supports $C \in \mathcal{K}^n$ at $q \in C$ if $C \subset H^\leq_{a,\rho}$ and $q \in \text{bd}(H^\leq_{a,\rho})$.

**Theorem 3.** Let $C \in \mathcal{K}^n$ be Minkowski centered. Then the following are equivalent:
1. $C \cap (-C) \subset \text{opt conv}(C \cup (-C))$;
2. $\left(\frac{1}{2}(C^\circ - C^\circ)\right)^\circ \subset \text{opt} \frac{1}{2}(C - C)$;
3. there exist $p, -p \in \text{bd}(C)$, parallel halfspaces $H^\leq_{a,\rho}$ and $H^\leq_{-a,\rho}$ supporting $C$ at $p$ and $-p$, respectively.

Let us mention that for any regular Minkowski centered $(2n + 1)$-gon $P$, the vertices of $-\frac{1}{s(P)} P$ are the midpoints of the edges of $P$. Hence, they obviously do not satisfy part (iii) of Theorem 3. By letting $n$ grow, we see that there exist Minkowski centered $C \in \mathcal{K}^2$ with $s(C)$ arbitrary close to 1 such that not all containments in the inequality chain (3) are optimal for $C$. Furthermore, one may observe that a Minkowski centered regular pentagon has asymmetry $2/\varphi \approx 1.236 < \varphi$.

**1. CHARACTERIZATIONS OF OPTIMAL CONTAINMENT.** Let us first collect some simple set identities under affine transformations.

**Lemma 4.** Let $K, C \in \mathcal{K}^n$ and $A$ be a nonsingular affine transformation. Then
\[
A(K) \cap A(C) = A(K \cap C),
\]
\[
\left(\left(\frac{(A(K))}{2} - \left(\frac{(A(C))}{2}\right)\right)\right)^\circ = A\left(\left(\frac{(K^\circ - C^\circ)}{2}\right)\right)^\circ,
\]
\[
\left(\frac{(A(K) + A(C))}{2}\right) = A\left(\left(\frac{(K + C)}{2}\right)\right),
\]
\[
\text{conv}\left(\frac{(A(K) \cup (A(C))}{2}\right) = A\left(\text{conv}(K \cup C)\right).
\]

The following proposition characterizes the optimal containment $K \subset^\text{opt} C$ between two convex sets $K, C \in \mathcal{K}^n$ in terms of common boundary points and corresponding supporting halfspaces (see [7, Theorem 2.3]).

**Proposition 5.** Let $K, C \in \mathcal{K}^n$ and $K \subset C$. Then the following are equivalent:
1. $K \subset^\text{opt} C$;
2. There exist $k \in \{2, \ldots, n + 1\}$, $p^j \in K \cap \text{bd}(C)$, $a^j$ outer normals of supporting halfspaces of $K$ and $C$ at $p^j$, $j = 1, \ldots, k$, such that $0 \in \text{conv}(\{a^1, \ldots, a^k\})$.

Moreover, in case that $K, C \in \mathcal{K}_0^n$, items 1, 2 are equivalent to $K \cap \text{bd}(C) \neq \emptyset$.

**Lemma 4** together with **Proposition 5** obviously yield the following corollary.

**Corollary 6.** Let $C \in \mathcal{K}^n$ and let $L$ be a nonsingular linear transformation. Then
1. $C$ is Minkowski centered if and only if $L(C)$ is Minkowski centered.
2. $C \cap (-C) \subset^\text{opt} \text{conv}(C \cup (-C))$ if and only if $L(C) \cap L(-C) \subset^\text{opt} \text{conv}(L(C) \cup L(-C))$.

Let us now add a proposition that is a result of Klee [15] reduced to the two-dimensional case.

**Proposition 7.** Let $P, C \in \mathcal{K}^2$, where $P$ is a polygon and $C$ is 0-symmetric, such that $P \subset^\text{opt} C$. Then $0 \in P$.

Taking the two preceding propositions together we obtain the corollary below.
Corollary 8. Let $C \in K^2$ be Minkowski centered, but not 0-symmetric. Then there exist $p^1, p^2, p^3 \in \text{bd}(C) \cap (-s(C)\text{bd}(C))$ such that $0 \in \text{conv}([p^1, p^2, p^3])$.

Proof. Let us first mention that the existence of two or three such touching points of $\text{bd}(C) \cap (-s(C)\text{bd}(C))$ is a direct consequence of Proposition 5, and if it were only two it would follow that $s(C) = 1$.

Now let $S$ be the intersection of the three common supporting halfspaces of $C$ and $-s(C)C$ at the points $p^i, i = 1, 2, 3$. In addition, $C$ (together with $-1/s(C)C$) is also supported in $1/s(C)p^i$ by halfspaces with outer normals being the negatives of the outer normals of the starting three. Hence, we obtain that $\text{conv}([p^1, p^2, p^3])$ is optimally contained in the minimum $S \cap (-S)$ of $S$ and $-S$ and therefore, by Proposition 7, that $0 \in \text{conv}([p^1, p^2, p^3])$.

Proof of Theorem 3. (1) $\Rightarrow$ (2): This part of the proof follows directly from Proposition 1.

(2) $\Rightarrow$ (3): Assuming that $(\frac{c - c^*}{2})^o \subseteq \text{opt} \frac{c - c^*}{2}$, we obtain from Proposition 5 that there exists a common boundary point $p$ of the two sets. Let $\rho_1, \rho_2 > 0$ be the smallest factors such that $\frac{1}{\rho_1} p \in \text{bd}(C)$ and $\frac{1}{\rho_2} p \in \text{bd}(-C)$, respectively. On the one hand, this implies

$$\frac{1}{2} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) p \in \frac{C - C}{2}$$

and since $p \in \text{bd} \left( \frac{c - c^*}{2} \right)$, we have that

$$1 \leq \left( \frac{1}{2} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \right)^{-1}. \quad (6)$$

On the other hand, from $\frac{1}{\rho_1} p \in \text{bd}(C)$ it follows that $C^o \subseteq \{ a \in \mathbb{R}^n : a^T p \leq \rho_1 \}$ and that there exists some $a^1 \in \text{bd}(C^o)$ such that $(a^1)^T p = \rho_1$. Similarly, we obtain $-C^o \subseteq \{ a \in \mathbb{R}^n : a^T p \leq \rho_2 \}$ and the existence of $a^2 \in \text{bd}(-C^o)$ such that $(a^2)^T p = \rho_2$. Hence, \(\frac{1}{2} (C^o - C^o) \subseteq \{ a \in \mathbb{R}^n : a^T p \leq \frac{1}{2} (\rho_1 + \rho_2) \} \) and $\frac{1}{2} (a^1 + a^2) \in \text{bd} \left( \frac{1}{2} (C^o - C^o) \right)$ with $\frac{1}{2} (a^1 + a^2)^T p = \frac{1}{2} (\rho_1 + \rho_2)$. This means

$$\frac{2}{\rho_1 + \rho_2} p \in \text{bd} \left( \frac{C^o - C^o}{2} \right)^o,$$

which by the fact that $p \in \text{bd} \left( \frac{c - c^*}{2} \right)^o$ implies

$$\frac{2}{\rho_1 + \rho_2} = 1. \quad (7)$$

Combining (6) and (7), we obtain that the arithmetic mean is not greater than the harmonic mean of $\rho_1$ and $\rho_2$, thus $\rho_1 = \rho_2 = 1$. This proves $p \in C \cap (-C)$.

Finally, let $H_{a, \rho}^\leq$ be a supporting half space of $(C - C)/2$ at $p$ and assume that $H_{a, \rho}^\leq$ does not support $C$. Hence there would exist some $q \in C$ with $a^T q > \rho$. Now, since $p \in -C$, we obtain $(p + q)/2 \in (C - C)/2$, which, because of $a^T \left( \frac{p + q}{2} \right) > \rho$, contradicts the fact that $(C - C)/2 \subset H_{a, \rho}^\leq$. This proves that $C \subset H_{a, \rho}^\leq$ and analogously one obtains $-C \subset H_{a, \rho}^\leq$. However, the convexity of halfspaces now implies $\text{conv}(C \cup (-C)) \subset H_{a, \rho}^\leq$, which shows that condition 3 is satisfied.

(3) $\Rightarrow$ (1): Assuming that $C$ is supported by $H_{a, \rho}^\leq, H_{-a, \rho}^\leq$ at $p, -p$, respectively, the same holds for $-C$. Hence, we have $p, -p \in C \cap (-C)$ and $\text{conv}(C \cup (-C))$...
is supported by $H^\le_{a,\rho}, H^\le_{a,\rho}$ at $p, -p$, respectively. By Proposition 5 this means that $C \cap (-C) \subset_{\text{opt}} \text{conv}(C \cup (-C))$. 

2. THE MAIN RESULT.

Proof of Theorem 2. Let $C \in K^2$ be Minkowski centered, with $s := s(C) > 1$, such that

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset_{\text{opt}} \frac{C - C}{2}.$$  

By Theorem 3 this optimality condition is equivalent to $C \cap (-C) \subset_{\text{opt}} \text{conv}(C \cup (-C))$ and to the existence of $-p, p \in \text{bd}(-sC)$, as well as parallel halfplanes $-H, H$ supporting $-sC$ at $-p$ and $p$, respectively. Since $C$ is Minkowski centered, we have $C \subset_{\text{opt}} -sC$ and therefore by Proposition 5 we obtain the existence of $k \in \{2, 3\}, q^1, \ldots, q^k \in \text{bd}(C) \cap \text{bd}(-sC)$ and outer normals of supporting halfplanes $a^1, \ldots, a^k$ with $0 \in \text{conv}(\{a^1, \ldots, a^k\})$. Moreover, from $s > 1$ it easily follows that $k = 3$.

It cannot be that $\pm p \not\in \{q^1, q^2, q^3\}$. Otherwise, let, e.g., $q^2 = p$. Then we have $q^2 \in C \cap \text{bd}(-sC)$ and thus $-sp = -sq^2 \in \text{bd}(-sC)$, which would imply $s(C) = 1$.

By Corollary 8 we have $0 \in \text{conv}(\{q^1, q^2, q^3\})$. Hence, we can assume without loss of generality that $q^1$ is located on one side, while $q^2, q^3$ are on the other side of the line $\text{aff}((-p, p))$, and moreover even that $-p \in \text{pos}(q^1, q^3)$ and $p \in \text{pos}(q^1, q^2)$. Observe that the lines $\text{aff}((-p, q^3))$ and $\text{aff}((p, q^2))$ intersect in some point $d^1$. Otherwise, we would have $q^3 \in \text{bd}(-H)$ and $q^2 \in \text{bd}(H)$ and therefore $[q^3, -p], [q^2, p] \subset \text{bd}(-sC)$. This would imply that the segment $[-\frac{1}{s}q^2, -\frac{1}{s}p]$, which is parallel to $[q^3, -p]$, belongs to $\text{bd}(C) \cap \text{int}(-sC)$. Together with $q^1 \in \text{bd}(C)$ and $s > 1$, this would contradict the convexity of $C$.

We choose $d^2 \in \text{bd}(-H), d^3 \in \text{bd}(H)$ such that $q^1 \in [d^2, d^3]$ and $[d^2, d^3]$ is parallel to $[q^2, q^3]$.

Let us first prove that

$$-sq^1 \in \text{conv}(\{q^2, q^3, d^1\}) \tag{8}$$

Since $0 \in \text{conv}(\{q^1, q^2, q^3\})$, we have $-sq^1 \in \text{pos}(\{q^2, q^3\})$. Thus using the fact that $q^2, q^3, -sq^1 \in \text{bd}(-sC)$, the convexity of $-sC$ implies that $-sq^1 \in \text{conv}(\{q^2, q^3, d^1\})$.

The next fact we want to see is

$$-sq^3 \in \text{conv}(\{p, q^1, d^3\}) \tag{9}$$

To see this, remember that $-H$ supports $-sC$ at $-p$. Moreover, directly from $-p \in \text{pos}(\{q^1, q^3\})$ we obtain $-sq^3 \in \text{pos}(\{p, q^1\})$. Now, since $p, q^1, -sq^3 \in \text{bd}(-sC)$, the convexity of $-sC$ implies $-sq^3 \not\in \text{int}(\text{conv}(\{0, p, q^1\}))$. Collecting the facts that $q^1, -sq^2, -sq^3 \in \text{bd}(-sC), q^1 \in \text{pos}((-sq^2, -sq^3))$, and the parallelism of $[-sq^2, -sq^3]$ and $[d^2, d^3]$, we obtain $-sq^3 \in \text{conv}(\{p, q^1, d^3\})$

Similarly to (9), one may prove

$$-sq^2 \in \text{conv}(\{-p, q^1, d^2\}) \tag{10}$$

See Figure 2 for an illustration of the construction and the validness of (8)–(10).

Our goal in the following is to determine the greatest possible $s$ such that $C \cap (-C) \subset_{\text{opt}} \text{conv}(C \cup (-C))$ is still satisfied. We say that the points $q^1, q^2, q^3$ present a valid situation if they satisfy conditions (8), (9), and (10). We make the
following changes to \( q^1, q^2, q^3 \), so that after each step (see Figure 3), we still have a valid situation for the given asymmetry \( s \):

1. Replace \( q^2 \) (respectively, \( q^3 \)) by the point in \([q^2, p]\) (respectively, \([-q^3, -p]\)) such that \(-sq^2 \in -H\) (respectively, \(-sq^3 \in H\)). Since \( s > 1 \), \( q^2 \) belongs in the strip between \( H \) and \(-H\), and \(-sp\) belongs outside the same strip and is closer to \(-H\) than to \( H \), we have that \(-s[q^2, p] = [-sq^2, -sp] \) intersects \(-H\) at a point \(-sq^2\). Let us replace \( q^2 \) by \( \tilde{q}^2 \).

2. Replace \( q^1 \) by \( \mu q^1 \), for some \( \mu < 1 \), such that \( \mu q^1 \in [-sq^2, -sq^3] \).

3. Substitute \( q^1 \) by \(-\gamma d^1 \in [-sq^2, -sq^3] \), for some \( \gamma > 0 \).

Recognize that \( s\gamma d^1 = -sq^1 \in \text{conv}\{d^1, q^2, q^3\} \) implies \( s\gamma \leq 1 \).

Now we can study the maximal possible value for \( s \), which means we want to characterize the situation in which \( s \) becomes maximal such that \( s\gamma \leq 1 \). Thus we need to know the explicit value of \( \gamma \) (depending on \( s \)).

To do so, after a suitable linear transformation, suppose that \( p = (1, 0) \), and \( H \) and \(-H\) are vertical lines (perpendicular to \([-p, p]\)). Because of step 1 above we may furthermore assume that \( q^2 = (1/s, -a)^T \) and \( q^3 = (-1/s, -1)^T \) for some \( a \in (0, 1) \). Now we need the coordinates of \( d^1 \), which is the intersection of the lines \( \text{aff}\{p, q^2\} \) and \( \text{aff}\{-p, q^3\} \). We obtain

\[
d^1_2 = -\frac{1}{1 - \frac{1}{s}} (d^1_1 + 1) \quad \text{and} \quad d^1_2 = \frac{a}{1 - \frac{1}{s}} (d^1_1 - 1),
\]

resulting in

\[
d^1 = \left( \frac{a - 1}{a + 1}, \frac{-2a}{(1 - \frac{1}{s})(a + 1)} \right)^T.
\]
Now we compute $\gamma$ such that condition (3) is satisfied, i.e.,

$$-\gamma d^1 \in [-sq^2, -sq^3] = [(-1, sa)^T, (1, s)^T].$$

Hence, for some $\lambda \in [0, 1]$, we have

$$-\gamma \left( \frac{a - 1}{a + 1}, \frac{-2a}{(1 - 1/s)(a + 1)} \right) = (1 - \lambda)(-1, sa)^T + \lambda(1, s)^T$$

$$= (-1 + 2\lambda, s((1 - \lambda)a + \lambda))^T$$

and it is easy to check that this implies

$$\gamma = \frac{(s - 1)(a + 1)^2}{4a - (s - 1)(a - 1)^2}.$$

Thus the problem of finding the maximal $s$ under the condition $s\gamma \leq 1$ may be rewritten as

$$\max s, \text{ such that } \frac{s(s - 1)(a + 1)^2}{4a - (s - 1)(a - 1)^2} \leq 1.$$

The above condition is easily rewritten as

$$(s^2 - 1)(a + 1)^2 - 4as \leq 0.$$  

We are interested in the maximum $s$, i.e., in the larger of the two roots of the equation $(s^2 - 1)(a + 1)^2 - 4as = 0$, which is

$$s = \frac{2a}{(a + 1)^2} \pm \sqrt{1 + \frac{4a^2}{(a + 1)^4}} =: h(a).$$
a ∈ (0, 1]. Hence, the maximum of s coincides with the maximum of h(a) with a ∈ (0, 1]. It is straightforward to verify that h(a) is increasing in (0, 1], and thus we can conclude that

$$\max s = \max_{a \in (0, 1]} h(a) = h(1) = \frac{1 + \sqrt{5}}{2} = \varphi.$$ 

Now, note that equality holds if and only if a = 1, γ = φ – 1, and d1 = (0, −φ / (φ – 1)). Moreover, in the extreme case we have φγ = 1, which is true if and only if −φq1 = d1, q2 = (1/φ, −1), and q3 = (−1/φ, −1). Since q2 ∈ bd(−φC) ∩ [d1, p], we have [d1, p] ⊆ bd(−φC). The same reasoning with q3 replacing q2 shows that [d1, −p] ⊆ bd(−φC). Moreover, q1 = (0, 1/(φ – 1)) := −γd1 ∈ [−φq2, −φq3]. Thus q1 ∈ bd(−φC) implies [−φq2, −φq3] ⊆ bd(−φC). Since it is also clear that [p, −φq3], [−p, −φq2] ⊆ bd(−φC), we obtain a complete description of the boundary of −φC, thus proving

$$−φC = \operatorname{conv}(d1, ±p, −φq2, −φq3)).$$

Finally, since φ = 1/(φ – 1), we obtain

$$C = \operatorname{conv}\left(\left\{\left(0, \frac{1}{\varphi – 1}\right)^T, \left(±\frac{1}{\varphi}, 0\right)^T, \left(±\frac{1}{\varphi}, −1\right)^T\right\}\right) = \left(\begin{array}{cc}
\frac{1}{\varphi} & 0 \\
0 & 1 
\end{array}\right) \mathcal{GH},$$

which concludes the proof of our theorem.

Remark. For every s ∈ [1, φ] there exists C ∈ K2, Minkowski centered with s(C) = s, such that

$$C \cap (−C) \subseteq \operatorname{opt} \operatorname{conv}(C \cup (−C)).$$

To see this, we perform a symmetrization process: making a hexagon from the pentagon GH by adding the point (0, −τ) for τ ∈ [1, φ2] and translating the whole set in the direction of (1, 0)T such that it is Minkowski centered again. In this way we obtain a continuously monotonely shrinking Minkowski asymmetry with growing τ, ending in a 0-symmetric hexagon when τ = φ2, while keeping property 3 of Theorem 3 true.

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