A Permutation-based Model for Crowd Labeling: Optimal Estimation and Robustness

Nihar B. Shah*, Sivaraman Balakrishnan♯ and Martin J. Wainwright†

*Department of EECS and †Department of Statistics, University of California, Berkeley
♯Department of Statistics, Carnegie Mellon University

Abstract

The aggregation and denoising of crowd labeled data is a task that has gained increased significance with the advent of crowdsourcing platforms and massive datasets. In this paper, we propose a permutation-based model for crowd labeled data that is a significant generalization of the common Dawid-Skene model, and introduce a new error metric by which to compare different estimators. Working in a high-dimensional non-asymptotic framework that allows both the number of workers and tasks to scale, we derive optimal rates of convergence for the permutation-based model. We show that the permutation-based model offers significant robustness in estimation due to its richness, while surprisingly incurring only a small additional statistical penalty as compared to the Dawid-Skene model. Finally, we propose a computationally-efficient method, called the OBI-WAN estimator, that is uniformly optimal over a class intermediate between the permutation-based and the Dawid-Skene models, and is uniformly consistent over the entire permutation-based model class. In contrast, the guarantees for estimators available in prior literature are sub-optimal over the original Dawid-Skene model.

1 Introduction

Recent years have witnessed a surge of interest in the use of crowdsourcing for labeling massive datasets. Expert labels are often difficult or expensive to obtain at scale, and crowdsourcing platforms allow for the collection of labels from a large number of low-cost workers. This paradigm, while enabling several new applications of machine learning, also introduces some key challenges: first, low-cost workers are often non-experts and the labels they produce can be quite noisy, and second, data collected in this fashion has a high amount of heterogeneity with significant differences in the quality of labels across workers and tasks. Thus, it is important to develop realistic models and scalable algorithms for aggregating and drawing meaningful inferences from the noisy labels obtained via crowdsourcing.

This paper focuses on objective labeling tasks involving binary choices, meaning that each question or task is associated with a single correct binary answer or label. There is a vast literature

---

Author email addresses: nihar@eecs.berkeley.edu, siva@stat.cmu.edu, wainwrig@berkeley.edu.

1In this paper, we use the terms {question, task}, and {answer, label} in an interchangeable manner.
on the problem of estimation from noisy crowdsourced labels [KOS11b, KOS11a, GKM11, LPI12, GZ13, DDKR13, ZCZJ14, GLZ16]. This past work is based primarily on the classical Dawid-Skene model [DS79], in which each worker $i$ is associated with a single scalar parameter $q_{i}^{DS} \in [0,1]$, and it assumed that the probability that worker $i$ answers any question $j$ correctly is given by the same scalar $q_{i}^{DS}$. Thus, the Dawid-Skene model imposes a homogeneity condition on the questions, one which is often not satisfied in practical applications where some questions may be more difficult than others.

Accordingly, in this paper, we propose and analyze a more general permutation-based model that allows the noise in the answer to depend on the particular question-worker pair. Within the context of such models, we propose and analyze a variety of estimation algorithms. One possible metric for analysis is the Hamming error, and there is a large body of past work [KOS11b, KOS11a, GKM11, GZ13, DDKR13, ZCZJ14, GLZ16] that provide sufficient conditions that guarantee zero Hamming error—meaning that every question is answered correctly—with high probability. Although the Hamming error can be suitable for the analysis of Dawid-Skene style models, we argue in the sequel that it is less appropriate for the heterogeneous settings studied in this paper. Instead, when tasks have heterogeneous difficulties, it is more natural to use a weighted metric that also accounts for the underlying difficulty of the tasks. Concretely, an estimator should be penalized less for making an error on a question that is intrinsically more difficult. In this paper, we introduce and provide analysis under such a difficulty-weighted error metric.

From a high-level perspective, the contributions of this paper can be summarized as follows:

- We introduce a new “permutation-based” model for crowd-labeled data, and a new difficulty-weighted metric that extends the popular Hamming metric.
- We provide upper and lower bounds on the minimax error, sharp up to logarithmic factors, for estimation under the permutation-based model. Our bounds lead to the useful implication that the generality afforded by the proposed permutation-based model as compared to the popular Dawid-Skene model enables more robust estimation, and surprisingly, there is only a small statistical price to be paid for this flexibility.
- We provide a computationally-efficient estimator that achieves the minimax limits over the permutation-based model when an approximate ordering of the workers in terms of their abilities is known.
- We provide a computationally-efficient estimator, termed the OBI-WAN estimator, that is consistent over the permutation-based model class. Moreover, it is optimal over an intermediate setting between the Dawid-Skene and the permutation-based models, which allows for task heterogeneity but in a restricted manner. As a special case, our sharp upper bounds on the estimation error of OBI-WAN also apply uniformly over the Dawid-Skene model, while prior known guarantees fall short of establishing such uniform bounds.

The remainder of this paper is organized as follows. In Section 2, we provide some background, setup the problems we address in this paper and provide an overview of related literature. Section 3 is devoted to our main results. We present numerical simulations in Section 4. We present all proofs in Section 5 and defer more technical aspects to the Appendix. We conclude the paper with a discussion of future research directions in Section 6.
2 Background and model formulation

We begin with some background on existing crowd labeling models, followed by an introduction to our proposed models; we conclude with a discussion of related work.

2.1 Observation model

Consider a crowdsourcing system that consists of $n$ workers and $d$ questions. We assume every question has two possible answers, denoted by $\{-1, +1\}$, of which exactly one is correct. We let $x^* \in \{-1, 1\}^d$ denote the collection of correct answers to all $d$ questions. We model the question answering via an unknown matrix $Q^* \in [0, 1]^{n \times d}$ whose $(i,j)$th entry $Q^*_{ij}$ represents the probability that worker $i$ answers question $j$ correctly. Otherwise, with probability $1 - Q^*_{ij}$, worker $i$ gives the incorrect answer to question $j$. For future reference, note that the Dawid-Skene model involves a special case of such a matrix, namely one of the form $Q^* = q^{DS}1^T$, where the vector $q^{DS} \in [0, 1]^n$ corresponds to the vector of correctness probabilities, with a single scalar associated with each worker.

We denote the response of worker $i$ to question $j$ by a variable $Y_{ij} \in \{-1, 0, 1\}$, where we set $Y_{ij} = 0$ if worker $i$ is not asked question $j$, and set $Y_{ij}$ to the answer provided by the worker otherwise. We also assume that worker $i$ is asked question $j$ with probability $p_{obs} \in [0, 1]$, independently for every pair $(i, j) \in [n] \times [d]$, and that a worker is never asked the same question twice. We also make the standard assumption that given the values of $x^*$ and $Q^*$, the entries of $Y$ are all mutually independent. In summary, we observe a matrix $Y$ which has independent entries distributed as

$$Y_{ij} = \begin{cases} 
  x^*_j & \text{with probability } p_{obs} Q^*_{ij} \\
  -x^*_j & \text{with probability } p_{obs} (1 - Q^*_{ij}) \\
  0 & \text{with probability } (1 - p_{obs}).
\end{cases}$$

Given this random matrix $Y$, our goal is to estimate the binary vector $x^* \in \{-1, 1\}^d$ of true labels.

Obtaining non-trivial guarantees for this problem requires that some structure be imposed on the probability matrix $Q^*$. The Dawid-Skene model is one form of such structure: it requires that the probability matrix $Q^*$ be rank one, with identical columns all equal to $q^{DS} \in \mathbb{R}^n$. As noted previously, this structural assumption on $Q^*$ is very strong. It assumes that each worker has a fixed probability of answering a question correctly, and is likely to be violated in settings where some questions are more difficult than others.

Accordingly, in this paper, we study a more general permutation-based model of the following form. We assume that there are two underlying orderings, both of which are unknown to us: first, a permutation $\pi^*: [n] \to [n]$ that orders the $n$ workers in terms of their (latent) abilities, and second, a permutation $\sigma^*: [d] \to [d]$ that orders the $d$ questions with respect to their (latent) difficulties. In terms of these permutations, we assume that the probability matrix $Q^*$ obeys the following conditions:

- **Worker monotonicity:** For every pair of workers $i$ and $i'$ such that $\pi^*(i) < \pi^*(i')$ and every question $j$, we have $Q^*_{ij} \geq Q^*_{i'j}$. 


• **Question monotonicity:** For every pair of questions \(j \) and \(j'\) such that \(\sigma^*(j) < \sigma^*(j')\) and every worker \(i\), we have \(Q_{ij}^* \geq Q_{ij'}^*\).

In other words, the permutation-based model assumes the existence of a permutation of the rows and columns such that each row and each column of the permuted matrix \(Q^*\) has non-increasing entries. The rank of the resulting matrix is allowed to be as large as \(\min\{n,d\}\). It is straightforward to verify that the Dawid-Skene model corresponds to a particular type of such probability matrices, restricted to have identical columns.

It should be noted that none of these models are identifiable without further constraints. For instance, changing \(x^*\) to \(-x^*\) and \(Q^*\) to \((11^T - Q^*)\) does not change the distribution of the observation matrix \(Y\). In the context of the Dawid-Skene model, several papers [KOS11b, KOS11a, GZ13, ZCZJ14] have resolved this issue by requiring that \(\frac{1}{n} \sum_{i=1}^{n} q_{1}^{DS} \geq \frac{1}{2} + \mu\) for some constant value \(\mu > 0\). Although this condition resolves the lack of identifiability, the underlying assumption—namely that every question is answerable by a subset of the workers—can be violated in practice. In particular, one frequently encounters questions that are too difficult to answer by any of the hired workers, and for which the worker’s answers are near uniformly random (e.g., see the papers [EdV11, SZ15]). On the other hand, empirical observations also show that workers in crowdsourcing platforms, as opposed to being adversarial in nature, at worst provide random answers to labeling tasks [YKL11, EdV11, GKDD15, GFK15]. On this basis, it is reasonable to assume that for every worker \(i\) and question \(j\) we have that \(Q_{ij}^* \geq \frac{1}{2}\). We make this assumption throughout this paper.

In summary, we let \(C_{\text{Perm}}\) denote the set of all possible values of matrix \(Q^*\) under the proposed permutation-based model, that is,

\[
C_{\text{Perm}} := \{Q \in [0.5, 1]^{n \times d} | \text{there exist permutations } (\pi, \sigma) \text{ s.t. question & worker monotonicity hold}\}.
\]

For future reference, we use

\[
C_{DS} := \{Q \in C_{\text{Perm}} | Q = q_{1}^{DS}1^T \text{ for some } q_{1}^{DS} \in [0.5, 1]^n\},
\]

to denote the subset of such matrices that are realizable under the Dawid-Skene assumption.

### 2.2 Evaluating estimators

In this section, we introduce the criteria used to evaluate estimators in this paper. In formal terms, an estimator \(\hat{x}\) is a measurable function that maps any observation matrix \(Y\) to a vector in the Boolean hypercube \([-1,1]^d\). The most popular way of assessing the performance of such an estimator is in terms of its (normalized) Hamming error

\[
d_H(\hat{x}, x^*) := \frac{1}{d} \sum_{j=1}^{d} 1\{\hat{x}_j \neq x^*_j\},
\]

(1)

where \(1\{\hat{x}_j \neq x^*_j\}\) denotes a binary indicator which takes the value 1 if \(\hat{x}_j \neq x^*_j\), and 0 otherwise. A potential deficiency of the Hamming error is that it places a uniform weight on each question. As
mentioned earlier, there are applications of crowdsourcing in which some subset of the questions are very difficult, and no hired worker can answer reliably. In such settings, any estimator will have an inflated Hamming error, not due to any particular deficiencies of the estimator, but rather due to the intrinsic hardness of the assigned collection of questions. This error inflation will obscure possible differences between estimators.

With this issue in mind, we propose an alternative error measure that weights the Hamming error with the difficulty of each task. A more general class of error measures takes the form

$$\mathcal{L}_{Q^*}(\hat{x}, x^*) = \frac{1}{d} \sum_{j=1}^{d} \mathbf{1}\{\hat{x}_j \neq x^*_j\} \Psi(Q^*_{1j}, \ldots, Q^*_{nj}),$$

(2)

for some function $\Psi : [0, 1]^n \rightarrow \mathbb{R}_+$ which captures the difficulty of estimating the answer to a question.

The $Q^*$-loss: In order to choose a suitable function $\Psi$, we note that past work on the Dawid-Skene model [KOS11b, KOS11a, GKM11, GZ13, DDKR13] has shown that the quantity

$$\frac{1}{n} \sum_{i=1}^{n} (2q_{DS}^i - 1)^2,$$

(3)

popularly known as the collective intelligence of the crowd, is central to characterizing the overall difficulty of the crowd-sourcing problem under the Dawid-Skene assumption. A natural generalization, then, is to consider the weights

$$\Psi(Q^*_{1j}, \ldots, Q^*_{nj}) = \frac{1}{n} \sum_{i=1}^{n} (2Q^*_{ij} - 1)^2$$

for each task $j \in [d]$, (4a)

which characterizes the difficulty of task $j$ for a given collection of workers. This choice gives rise to the $Q^*$-loss function

$$\mathcal{L}_{Q^*}(\hat{x}, x^*) := \frac{1}{d} \sum_{j=1}^{d} \left( \mathbf{1}\{\hat{x}_j \neq x^*_j\} \frac{1}{n} \sum_{i=1}^{n} (2Q^*_{ij} - 1)^2 \right) = \frac{1}{dn} \|Q^* - \frac{1}{2} 11^T\| F \|\text{diag}(\hat{x} - x^*)\| F,$$

(4b)

where $\text{diag}(\hat{x} - x^*)$ denotes the matrix in $\mathbb{R}^{d \times d}$ whose diagonal entries are given by the vector $\hat{x} - x^*$. Note that under the Dawid-Skene model (in which $Q^* = q_{DS}^1 1^T$), this loss function reduces to

$$\mathcal{L}_{Q^*}(\hat{x}, x^*) = \left( \frac{1}{n} \sum_{i=1}^{n} (2q_{DS}^i - 1)^2 \right) \left( \frac{1}{d} \sum_{j=1}^{d} \mathbf{1}\{\hat{x}_j \neq x^*_j\} \right),$$

corresponding to the normalized Hamming error rescaled by the collective intelligence.

For future reference, let us summarize some properties of the function $\mathcal{L}_{Q^*}$: (a) it is symmetric in its arguments $(x^*, \hat{x})$, and satisfies the triangle inequality; (b) it takes values in the interval $[0, 1]$;
and (c) if for every question $j \in [d]$, there exists a worker $\ell \in [n]$ such that $Q_{\ell j}^* > \frac{1}{2}$, then $L_{Q^*}$ defines a metric; if not, it defines a pseudo-metric.

Evaluating $L_{Q^*}$ is not feasible in real-world applications, since the underlying matrix $Q^*$ is typically unknown. Rather, this pseudo-metric should be understood as being useful for a more refined theoretical comparison of different algorithms.

**Minimax risk:** Given the loss function $L_{Q^*}$, we evaluate the performance of estimators in terms of their uniform risk properties over a particular class $C$ of probability matrices. More formally, for an estimator $\hat{x}$ and class $C \subseteq [0, 1]^{n \times d}$ of possible values of $Q^*$, the uniform risk of $\hat{x}$ over class $C$ is

$$\sup_{x^* \in \{-1, 1\}^d} \sup_{Q^* \in C} \mathbb{E}[L_{Q^*}(\hat{x}, x^*)],$$

where the expectation is taken over the randomness in the observations $Y$ for the given values of $x^*$ and $Q^*$. The smallest value of the expression (5) across all estimators is the minimax risk.

**Regime of interest:** In this paper, we focus on understanding the minimax risk as well as the risk of various computationally efficient estimators. We work in a non-asymptotic framework where we are interested in evaluating the risk in terms of the triplet $(n, d, p_{\text{obs}})$. We assume that $p_{\text{obs}} \geq \frac{1}{n}$, which ensures that on average, at least one worker answers any question. We also operate in the regime $d \geq n$, which is relevant for many practical applications. Indeed, as also noted in earlier works [ZCZJ14], typical medium or large-scale crowdsourcing tasks employ tens to hundreds of workers, while the number of questions is on the order of hundreds to many thousands. We assume that the value of $p_{\text{obs}}$ is known. This is a mild assumption since it is straightforward to estimate $p_{\text{obs}}$ very accurately using its empirical expectation.

### 2.3 Related work

Having set up our model and notation, let us now relate it to past work in the area. The Dawid-Skene model [DS79] is the dominant model for crowd labeling, and has been widely studied [KOS11b, KOS11a, GKM11, LPI12, GZ13, DDKR13, ZCZJ14]. Some papers have studied models beyond the Dawid-Skene model. In a recent work, Khetan and Oh [KO16] analyze an extension of the Dawid-Skene model where a vector $\tilde{q} \in \mathbb{R}^n$, capturing the abilities of the workers, is supplemented with a second vector $h^* \in [0, 1]^d$, and the likelihood of worker $i$ correctly answering question $j$ is set as $\tilde{q}_i h_{ij}^* + (1 - \tilde{q}_i)$h_{ij}. Although this model now has $(n + d)$ parameters instead of just $n$ as in the Dawid-Skene model, it retains parametric-type assumptions. Each worker and each question is described by a single parameter, and in this model the probability of correctness takes a specific form governed by these parameters. In contrast, in the permutation-based model each worker-question pair is described by a single parameter. Our permutation-based model forms a strict superset of this class. Zhou et al. [ZPBM12, ZLP+15] propose algorithms based on models that are more general than the Dawid-Skene model, governed by a certain minimax entropy principle; however, these algorithms have yet to be rigorously analyzed. While the present paper addresses the setting of binary labels with symmetric error probabilities, several of these prior works also address settings with more than two classes, and where the probability of error of a worker may be asymmetric.
across the classes. We defer a further detailed comparison of our main results with those in earlier works to Section 3.4.

3 Main results

We now turn to the statement of our main results. As noted earlier, our results are focused on the practically relevant regime where we have that:

\[ p_{\text{obs}} \geq \frac{1}{n} \quad \text{and} \quad d \geq n. \]  

(R)

We use \( c, c_c, c_L, c_0, c_H \) to denote positive universal constants that are independent of all other problem parameters. Recall that the \( Q^* \)-loss takes values in the interval \([0, 1]\).

3.1 Minimax risk for estimation under the permutation-based model

We begin by proving sharp upper and lower bounds on the minimax risk for the permutation-based model \( \mathcal{C}_{\text{Perm}} \). The upper bound is obtained via an analysis of the following least squares estimator

\[ (\hat{x}_{\text{LS}}, \hat{Q}_{\text{LS}}) \in \arg \min_{x \in \{-1, 1\}^d, Q \in \mathcal{C}_{\text{Perm}}} \| p_{\text{obs}}^{-1} Y - (2Q - 11^T) \text{diag}(x) \|_F^2. \]  

(6)

We do not know of a computationally efficient way to compute this estimate. Nonetheless, our statistical analysis provides a benchmark for comparing other computationally-efficient estimators, to be discussed in subsequent sections. The following result holds in the regime (R):

**Theorem 1.** (a) For any \( x^* \in \{-1, 1\}^d \) and any \( Q^* \in \mathcal{C}_{\text{Perm}} \), the least squares estimator \( \hat{x}_{\text{LS}} \) has error at most

\[ \mathcal{L}_{Q^*}(\hat{x}_{\text{LS}}, x^*) \leq c_U \frac{1}{nP_{\text{obs}}} \log^2 d, \]  

(7a)

with probability at least \( 1 - e^{-c_H d \log(dn)} \).

(b) Conversely, any estimator \( \hat{x} \) has error at least

\[ \sup_{Q^* \in \mathcal{C}_{\text{Perm}}} \sup_{x^* \in \{-1, 1\}^d} \mathbb{E}[\mathcal{L}_{Q^*}(\hat{x}, x^*)] \geq c_L \frac{1}{nP_{\text{obs}}}. \]  

(7b)

The lower bound holds even if the true matrix \( Q^* \) is known to the estimator.

The result of Theorem 1 has a number of important consequences. Since the permutation-based class \( \mathcal{C}_{\text{Perm}} \) is significantly richer than the Dawid-Skene class \( \mathcal{C}_{\text{DS}} \), one might expect that estimation over \( \mathcal{C}_{\text{Perm}} \) might require a significantly larger sample size to achieve the same accuracy. However, Theorem 1 shows that this is not the case: the lower bound (7b) holds even when the supremum over matrices \( Q^* \) is restricted to the Dawid-Skene model \( \mathcal{C}_{\text{DS}} \subset \mathcal{C}_{\text{Perm}} \). Consequently, we see that estimation over the more general permutation-based model leads to (at worst) a logarithmic penalty in the required sample size. Thus, making the restrictive assumption that the data is drawn from
the Dawid-Skene model yields little statistical advantage as compared to making the more relaxed assumption of the permutation-based model.

We note that the least squares estimator analyzed in part (a) also yields an accurate estimate of the probability matrix $Q^*$ in the Frobenius norm, useful in settings where the calibration of workers or questions might be of interest. Again, this result holds in the regime [1]:

**Corollary 1.** (a) For any $x^* \in \{-1, 1\}^d$ and any $Q^* \in \mathbb{C}_{\text{Perm}}$, the least squares estimate $\tilde{Q}_{LS}$ has error at most

$$\frac{1}{dn} \| \tilde{Q}_{LS} - Q^* \|_F^2 \leq c_Y \frac{1}{n \text{obs}} \log^2 d,$$

with probability at least $1 - e^{-c_Yd \log(\frac{1}{e})}$.

(b) Conversely, for any answer vector $x^* \in \{-1, 1\}^d$, any estimator $\hat{Q}$ has error at least

$$\sup_{Q^* \in \mathbb{C}_{\text{Perm}}} \mathbb{E}\left[ \frac{1}{dn} \| \hat{Q} - Q^* \|_F^2 \right] \geq c_Y \frac{1}{n \text{obs}}.$$

This lower bound holds even if the true answer vector $x^*$ is known to the estimator.

We do not know if there exist computationally-efficient estimators that can achieve the upper bound on the sample complexity established in Theorem 1(a) uniformly over the entire permutation-based model class. In the next three sections, we design and analyze polynomial-time estimators that address interesting subclasses of the permutation-based model.

### 3.2 The WAN estimator: When workers’ ordering is (approximately) known

Several organizations employ crowdsourcing workers only after a thorough testing and calibration process. This section is devoted to a setting in which the workers are calibrated, in the sense that it is known how they are ordered in terms of their respective abilities. More formally, recall from Section 2.1 that any matrix $Q^* \in \mathbb{C}_{\text{Perm}}$ is associated with two permutations: a permutation of the workers in terms of their abilities, and a permutation of the questions in terms of their difficulty. In this section, we assume that the permutation of the workers is (approximately) known to the estimation algorithm. Note that the estimator does not know the permutation of the questions, nor does it know the values of the entries of $Q^*$.

Given a permutation $\pi$ of the workers, our estimator consists of two steps, which we refer to as Windowing and Aggregating Naively, respectively, and accordingly term the procedure as the WAN estimator:

- **Step 1 (Windowing):** Compute the integer

$$k_{\text{WAN}} \in \arg \max_{k \in \{p_{\text{obs}}^{1.5}(dn), \ldots, n\}} \sum_{j \in [d]} \mathbb{1}\left\{ \sum_{i \in [k]} Y_{\pi^{-1}(i)j} \geq \sqrt{k \text{obs} \log^{1.5}(dn)} \right\}.$$  

- **Step 2 (Aggregating Naively):** Set $\tilde{x}_{\text{WAN}}(\pi)$ as a majority vote of the best $k_{\text{WAN}}$ workers—that is

$$[\tilde{x}_{\text{WAN}}(\pi)]_j \in \arg \max_{b \in \{-1, 1\}} \sum_{i=1}^{k_{\text{WAN}}} \mathbb{1}\{Y_{\pi^{-1}(i)j} = b\} \quad \text{for every } j \in [d].$$
The windowing step finds a value \( k_{\text{WAN}} \) such that the answers of the best \( k_{\text{WAN}} \) workers to most questions are significantly biased towards one of the options, thereby indicating that these workers are knowledgeable (or at least, are in agreement with each other). The second step then simply takes a majority vote of this set of the best \( k_{\text{WAN}} \) workers. We remark that it is important to choose a reasonably good value of \( k_{\text{WAN}} \) (as done in Step 1) since a much larger value could include many random workers thereby increasing the noise in the input to the second step, whereas too small a value could eliminate too much of the “signal”. Both steps can be carried out in time \( O(nd) \).

For the case when \( \pi \) is an approximate ordering, we establish an oracle bound on the error. For every \( j \in [d] \), let \( Q_j^* \) denote the \( j^{\text{th}} \) column of \( Q^* \); for any ordering \( \pi \) of the workers, \( Q_j^\pi \) denote the vector obtained by permuting the entries of \( Q_j^* \) in the order given by \( \pi \), that is, with the first entry of \( Q_j^\pi \) corresponding to the best worker according to \( \pi \), and so on. Also recall the notation \( \pi_* \) representing the true permutation of the workers in terms of their actual abilities. As with all of our theoretical, results, the following claim holds in the regime (R):

**Theorem 2.** For any \( Q^* \in C_{\text{Perm}} \) and any \( x^* \in \{-1,1\}^d \), suppose the WAN estimator is provided with the permutation \( \pi \) of workers. Then for every question \( j \in [d] \) such that

\[
\|Q_j^* - \frac{1}{2}\|^2 \geq \frac{5 \log^{2.5}(dn)}{p_{\text{obs}}}, \quad \text{and} \quad \|Q_j^\pi - Q_j^{\pi_*}\|_2 \leq \frac{\|Q_j^* - \frac{1}{2}\|_2}{\sqrt{9 \log(dn)}},
\]

we have

\[
P(\hat{x}_{\text{WAN}}(\pi)_j = x_j^*) \geq 1 - e^{-c_H \log^{1.5}(dn)}.
\]

Consequently, if \( \pi \) is the correct permutation of the workers, then

\[
\mathcal{L}(\hat{x}_{\text{WAN}}(\pi), x^*) \leq c_U \frac{1}{np_{\text{obs}}} \log^{2.5} d,
\]

with probability at least \( 1 - e^{-c_H \log^{1.5}(dn)} \).

At this point, we recall the lower bound of Theorem 1(b) on the estimation error in the \( Q^*\)-loss allows for any estimator. Moreover, it applies to estimators that know not only the ordering of the workers, but also the entire matrix \( Q^* \). This lower bound matches the upper bound (10c) of Theorem 2, and the two results in conjunction imply that the bound (10c) is sharp up to logarithmic factors.

The result of Theorem 2 for the WAN algorithm has the following useful implication for the setting when the ordering of workers is unknown (under either of the models \( C_{\text{DS}} \) or \( C_{\text{Perm}} \)). For any \( Q^* \in C_{\text{Perm}} \), there exists a set of workers \( S_{Q^*} \subseteq [n] \) such that an estimator \( \hat{x}_S \) that takes a majority vote of the answers of the workers in \( S_{Q^*} \), has risk at most

\[
\mathcal{L}(\hat{x}_S, x^*) \leq c_U \frac{1}{np_{\text{obs}}} \log^{2.5} d,
\]

with high probability. Consequently, it suffices to design an estimator that only identifies a set of good workers and computes a majority vote of their answers. The estimator need not attempt to infer the values of the entries of \( Q^* \), as is otherwise required, for instance, to compute maximum likelihood estimates. The estimator we propose in the next section is based on this observation.
3.3 The OBI-WAN estimator

In this section, we return to the setting where the ordering of the workers is unknown. We study the estimation problem in an intermediate model that lies between Dawid-Skene and the permutation-based model. In addition to the parameters $\tilde{q} \in \mathbb{R}^n$ associated to the workers as in the Dawid-Skene model, this intermediate model introduces a parameter $h^*_j \in [0, 1]$ that captures the difficulty of each question $j \in [d]$. Under this model, the probability that worker $i \in [n]$ correctly answers question $j \in [d]$ (when the worker is asked the question) is given by

$$P(Y_{ij} = x_j^*) = \tilde{q}_i (1 - h^*_j) + \frac{1}{2} h^*_j, \quad \forall (i, j) \text{ such that } Y_{ij} \neq 0.$$  

Intuitively, the parameter $h^*_j$ corresponds to the difficulty of question $j$. When $h^*_j = 1$, the worker is purely stochastic and provides a random guess, while for smaller values of $h^*_j$ the worker is more likely to provide a correct answer.\footnote{This model is similar to a recent model proposed by Khetan and Oh \cite{KO16}; the difference being that they set the probability of a correct answer as $\tilde{q}_i (1 - h^*_j) + (1 - \tilde{q}_i) h^*_j$.}

This modeling assumption leads to the class $C_{\text{Int}} := \{ Q = \tilde{q}(1 - h)^T + \frac{1}{2} h^T | \text{for some } \tilde{q} \in [\frac{1}{2}, 1]^n, \ h \in [0, 1]^d \}$. Note that we have the nested relations $C_{\text{DS}} \subset C_{\text{Int}} \subset C_{\text{Perm}}$; the Dawid-Skene model is a special case of $C_{\text{Int}}$ corresponding to $h = 0$.

We now describe a computationally efficient estimator for this intermediate model $C_{\text{Int}}$, and establish sharp guarantees on its statistical risk. Our analysis of this estimator also makes contributions in the specific context of the Dawid-Skene model. In particular, the guarantees established for computationally efficient estimators in prior works (e.g., \cite{KOS11a, KOS11b, GKM11, GZ13, DDKR13, ZCZJ14, KO16, GLZ16}) fall short of translating to uniform guarantees over the Dawid-Skene model $C_{\text{DS}}$ in the $Q^*$-loss; see Section 3.4 for further details. Our result in this section fills this gap by establishing sharp uniform bounds on the statistical risk over the entire Dawid-Skene class $C_{\text{DS}}$, and more generally over the entire class $C_{\text{Int}}$.

Our proposed estimator operates in two steps. The first step performs an Ordering Based on Inner-products (OBI), that is, the first step computes an ordering of the workers based on an inner product with the data. The second step calls upon the WAN estimator from Section 3.2 with this ordering. We thus term our proposed estimator as the OBI-WAN estimator, $\hat{x}_{\text{OBI-WAN}}$. In order to make its description precise, we augment the notation of the WAN estimator $\hat{x}_{\text{WAN}}(\pi)$ to let $\hat{x}_{\text{WAN}}(\pi, Y)$ to denote the estimate given by $\hat{x}_{\text{WAN}}(\pi)$ operating on $Y$ when given the permutation $\pi$ of workers.

An important technical issue is that re-using the observed data $Y$ to both determine an appropriate ordering of workers as well as to estimate the desired answers, results in a violation of important independence assumptions. We resolve this difficulty by partitioning the set of questions into two sets, and using the ordering estimated from one set to estimate the desired answers for the other set and vice versa. We provide a careful error analysis for this partitioning-based estimator in the sequel. Formally, the OBI-WAN estimator $\hat{x}_{\text{OBI-WAN}}$ is defined by the following steps:
• Step 0 (preliminary): Split the set of $d$ questions into two sets, $T_0$ and $T_1$, with every question assigned to one of the two sets uniformly at random. Let $Y_0$ and $Y_1$ denote the corresponding submatrices of $Y$, containing the columns of $Y$ associated to questions in $T_0$ and $T_1$ respectively.

• Step 1 (OBI): For $\ell \in \{0, 1\}$, let $u_\ell \in \arg \max_{\|u\|_2 = 1} \|Y_\ell^T u\|_2$ denote the top eigenvector of $Y_\ell Y_\ell^T$; in order to resolve the global sign ambiguity of eigenvectors, we choose the global sign so that $\sum_{i \in [n]} [u_\ell]_i^2 1\{[u_\ell]_i > 0\} \geq \sum_{i \in [n]} [u_\ell]_i^2 1\{[u_\ell]_i < 0\}$. Let $\pi_\ell$ be the permutation of the $n$ workers in order of the respective entries of $u_\ell$ (with ties broken arbitrarily).

• Step 2 (WAN): Compute the quantities

$$\hat{x}_{\text{OBI-WAN}}(T_0) := \hat{x}_{\text{WAN}}(Y_0, \pi_1), \quad \text{and} \quad \hat{x}_{\text{OBI-WAN}}(T_1) := \hat{x}_{\text{WAN}}(Y_1, \pi_0),$$

(11)
corresponding to estimates of the answers for questions in the sets $T_0$ and $T_1$, respectively.

The following theorem provides guarantees on this estimator, again in the regime $[R]$.

**Theorem 3.** (a) Uniformly optimal over $C_{\text{Int}}$: For any $Q^* \in C_{\text{Int}}$ and any $x^* \in \{-1, 1\}^d$, the error incurred by the estimate $\hat{x}_{\text{OBI-WAN}}$ is upper bounded as

$$\mathcal{L}_{Q^*}(\hat{x}_{\text{OBI-WAN}}, x^*) \leq c_U \frac{1}{n \text{obs}} \log^{2.5} d,$$

(12a)
with probability at least $1 - e^{-c_H \log^{1.5}(dn)}$.

(b) Uniformly consistent over $C_{\text{Perm}}$: For any $Q^* \in C_{\text{Perm}}$ and any $x^* \in \{-1, 1\}^d$, the estimate $\hat{x}_{\text{OBI-WAN}}$ has error at most

$$\mathcal{L}_{Q^*}(\hat{x}_{\text{OBI-WAN}}, x^*) \leq c_U \frac{1}{\sqrt{n \text{obs}}} \log d,$$

(12b)
with probability at least $1 - e^{-c_H \log^{1.5}(dn)}$.

Recall that the statistical lower bound established earlier in Theorem 1(b) is also applicable to the classes $C_{\text{DS}}$ and $C_{\text{Int}}$. Consequently, the upper bound of Theorem 3 is sharp over these two classes.

### 3.3.1 Guarantees for OBI-WAN under the Dawid-Skene model for the Hamming error

In this section, we present results relating the performance of the OBI-WAN estimator to the settings considered in most prior works on this topic. Most of our paper focuses on the permutation-based model, the $Q^*$-loss and does not account for adversarial workers. In the following theorem, we present optimality guarantees of the OBI-WAN estimator, in terms of the popular Hamming error, when data is actually faithful to the Dawid-Skene model, and in a setting where the workers may also be adversarial (that is, where $q_i^{\text{DS}} < \frac{1}{2}$ for some workers $i \in [n]$). In particular, we show that the OBI-WAN estimator incurs a zero Hamming error under the Dawid-Skene model when
the collective intelligence (see Equation (3)) is sufficiently high. Our results are optimal up to logarithmic factors.

We introduce some notation in order to describe the result involving adversarial workers. For the vector \( q_{\text{DS}} \in [0, 1]^n \), we define two associated vectors \( q_{\text{DS}+}, q_{\text{DS}-} \in [0, 1]^n \) as \( q_{\text{DS}+} = \max\{q_{\text{DS}}^i, \frac{1}{2}\} \) and \( q_{\text{DS}-} = \min\{q_{\text{DS}}^i, \frac{1}{2}\} \) for every \( i \in [n] \). Then we have \( q_{\text{DS}} - \frac{1}{2} = (q_{\text{DS}+} - \frac{1}{2}) + (q_{\text{DS}-} - \frac{1}{2}) \), with \( q_{\text{DS}+} \) representing normal workers and \( q_{\text{DS}-} \) representing adversarial workers who are inclined to provide incorrect answers. As with all our theorems, the following result holds in the regime \( \mathcal{R} \):

**Theorem 4.** Consider any Dawid-Skene matrix of the form \( Q^* = q_{\text{DS}}^T \) for some \( q_{\text{DS}} \in [0, 1]^n \). Then:

(a) If \( \|q_{\text{DS}+} - \frac{1}{2}\|_2 \geq \|q_{\text{DS}-} - \frac{1}{2}\|_2 + \sqrt{\frac{4 \log^2(n)}{\rho_{\text{obs}}}} \) and \( (q_{\text{DS}} - \frac{1}{2})^T 1 \geq 0 \), then for any \( x^* \in \{-1, 1\}^d \), the OBI-WAN estimator satisfies

\[
P(\hat{x}_{\text{OBI-WAN}} = x^*) \geq 1 - e^{-c \rho_{\text{obs}} \log^2 (dn)}.
\]  

(13a)

(b) Conversely, there exists a positive universal constant \( c \) such that for any \( q_{\text{DS}} \in \left[\frac{1}{2}, \frac{9}{10}\right]^n \) with \( \|q_{\text{DS}} - \frac{1}{2}\|_2 \leq \sqrt{\frac{c}{\rho_{\text{obs}}}} \), any estimator \( \hat{x} \) has (normalized) Hamming error at least

\[
\sup_{x^* \in \{-1, 1\}^d} \mathbb{E} \left[ \sum_{i=1}^{d} \frac{1}{d} 1\{\hat{x}_i \neq x_i^*\} \right] \geq \frac{1}{10}.
\]  

(13b)

One application of the above corollary is to the setting that has been the focus of our paper, where we have no adversarial workers. In this case, \( q_{\text{DS}-} = 0 \), and \( q_{\text{DS}+} = q_{\text{DS}} \), and the upper and lower bounds match up to a logarithmic factor. The upper bound shows that when \( \|q_{\text{DS}} - \frac{1}{2}\|_2 \geq \sqrt{\frac{4 \log^2(n)}{\rho_{\text{obs}}}} \) the Hamming error is vanishingly small while the lower bound shows that there is a universal constant \( c \) such that the Hamming error is essentially as large as possible when \( \|q_{\text{DS}} - \frac{1}{2}\|_2 \leq \sqrt{\frac{c}{\rho_{\text{obs}}}} \).

The results of Theorem 3 and Theorem 4 in conjunction show that the OBI-WAN estimator not only has optimal guarantees (up to logarithmic factors) in terms of the models and metrics popular in past literature, but is also efficient in terms of the more general models and metric introduced here.

### 3.4 Past work and the \( Q^* \)-loss

Several pieces of past work have introduced computationally-efficient estimation algorithms, and provided theoretical guarantees for these algorithms under the Dawid-Skene model. These guarantees apply to the Hamming metric, and usually quantify the sample complexity required for exact recovery of all the questions with high probability. In this section, we consider the implications for such guarantees for the goal of this paper—namely, that of establishing uniform guarantees under the \( Q^* \)-loss. We find that guarantees from earlier works—for the purposes of establishing uniform guarantees over the Dawid-Skene model in the \( Q^* \)-loss—are either inapplicable, or lead to sub-optimal guarantees.
To be fair, some of this past work applies to settings more general than our paper, including problems with more than two classes, and problems where the probability of error of a worker may be asymmetric across the classes. The present paper, on the other hand, considers the setting of binary labels with symmetric error probabilities, and accordingly, all comparison made in this section pertain to this setting. We note that the various prior works make different assumptions regarding the choice of questions assigned to each worker, and in order to bring these works under the same umbrella, we assume that each of the $n$ workers answers each of the $d$ questions (that is, $p_{\text{obs}} = 1$). As indicated earlier, in this section we restrict attention to the Dawid-Skene model $C_{\text{DS}}$.

Note that when the guarantee claimed in a past work requires certain additional conditions that are not satisfied, one can always appeal to the naïve bound

$$\mathcal{L}_{Q^*}(\hat{x}, x^*) \leq \frac{1}{n} \left\| q^{\text{DS}} - \frac{1}{2} \right\|_2^2. \tag{14}$$

Thus, in all of comparisons with past work, we take the minimum of this bound, and the bound provided by their work. We show below that in each of the prior works, this augmented guarantee has weaker scaling than the bound strictly weaker scaling than the scaling of

$$\mathcal{L}_{Q^*}(\hat{x}, x^*) \leq \frac{1}{n} \log^{2.5} d, \tag{15}$$

achieved by the OBI-WAN estimator for the Dawid-Skene model (see Theorem 3(a)) when $p_{\text{obs}} = 1$.

**Ghosh et al. [GKM11]:** The guarantees for recovery provided in the paper [GKM11] require the lower bound

$$\left\| q^{\text{DS}} - \frac{1}{2} \right\|_2^2 \geq c_0 \sqrt{n \log n} \tag{16}$$

to be satisfied, where $c_0$ is a positive universal constant. This requirement means that it is not possible to translate the bounds of [GKM11] to a uniform bound over the entire Dawid-Skene class in the $Q^*$-loss. For instance, for a DS matrix given by the vector

$$q_i^{\text{DS}} = \begin{cases} 1 & \text{if } i \leq \sqrt{n} \\ \frac{1}{2} & \text{otherwise,} \end{cases} \tag{17}$$

the guarantees of [GKM11] are inapplicable, and the naïve bound of $\frac{1}{n} \left\| q^{\text{DS}} - \frac{1}{2} \right\|_2^2 = \frac{1}{\sqrt{n}}$ is sub-optimal.

**Karger et al. [KOS11b, KOS11a], Khetan and Oh [KO16]:** The guarantees from this set of works assume that $p_{\text{obs}} = O\left(\frac{\log d}{d}\right)^{\frac{3}{2}}$. The assumption stems from the use of message passing algorithms, where the analysis requires a certain “locally tree-like” worker-question assignment

4The setting analyzed in these papers is slightly different from ours when $p_{\text{obs}} < 1$. Specifically, the paper [KO16] assumes that the sets of questions assigned to the workers are chosen based on a certain regular random bipartite graph, with each worker answering $d_{\text{obs}}$ questions and each question being answered by $n_{\text{obs}}$ workers. We think that the assumptions on the worker-question assignment in [KO16] and those made in the present paper may have similar guarantees. In the spirit of allowing for a comparison between the two works, we consider their guarantees as applicable for our setting as well.
graph which is guaranteed to hold in this regime. Moreover, the results of \cite{KOS11a} apply to a particular subset of the Dawid-Skene model, for which it is assumed that $q_{DS} \in \{\frac{1}{2}, 1\}^n$.

Let us evaluate these guarantees from the perspective of our requirements, namely to obtain uniform guarantees on the $Q^*$-loss under the Dawid-Skene model across different values of the problem parameters. When $p_{\text{obs}} = O(\log d)$, then the trivial upper bound of 1 on the $Q^*$-loss is only a logarithmic factor away from the lower bound of $\frac{1}{n p_{\text{obs}}}$ given by Theorem 1(b) in the present paper. Consequently, any result will then be sandwiched between these two bounds, and can yield at most a logarithmic improvement over the trivial upper bound in this regime. On the other hand, the guarantees derived in \cite{KOS11b, KOS11a, KO16} are loose when $p_{\text{obs}}$ takes larger values. For instance, when $p_{\text{obs}} \geq \frac{1}{\sqrt{n}}$, these bounds reduce to the trivial property that the number of answers decoded incorrectly is upper bounded by $d$. Consequently, in this regime, these analyses yield an upper bound of $\frac{1}{n} \|q_{DS} - \frac{1}{2}\|^2_2$; note that this bound could be as large as $\frac{1}{4}$.

**Dalvi et al. \cite{DDKR13}:** For the setting described in equation (17), the bound of Dalvi et al. only guarantees that the number of answers estimated incorrectly is upper bounded by $cd$, for some constant $c > 0$. This guarantee translates to a suboptimal bound of order $\frac{1}{\sqrt{n}}$ on the $Q^*$-loss.

**Zhang et al. \cite{ZCZJ14}:** Zhang et al. \cite{ZCZJ14} assume the existence of three groups of workers such that the second largest singular value of a certain set of matrices capturing the correlations between the probabilities of correctness of workers in the groups are all lower bounded by a parameter, denoted as $\sigma_L$. Their results require, among other conditions, that $d \geq (\sigma_L)^{-13}$. It turns out that for a large number of settings of interest, this condition is quite prohibitive. Here is a simple example to illustrate this issue. Suppose that

$$q_{DS}^i = \begin{cases} 1 & \text{if } i \leq \sqrt{n} \log d \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad (18)$$

In order to apply the bounds of \cite{ZCZJ14} to this setting, we must have $d \geq n^{14}$. One can see that this condition is prohibitive, even when the number of workers $n$ is as small as 10. The naive bound of $\frac{1}{n} \|q_{DS} - \frac{1}{2}\|^2_2 = \frac{\log d}{4n}$ is also suboptimal. We note that on the other hand, the problem (18) is not actually hard: a simple analysis of the majority voting algorithm leads to a guarantee that all the questions will be decoded correctly with a high probability.

**Gao et al. \cite{GLZ16}:** Gao et al. \cite{GLZ16} present an algorithm and associated guarantees to estimate the true labels under the Dawid-Skene model when the worker abilities $q_{DS}$ are (approximately) known. In order to estimate the value of $q_{DS}$, they employ one of the two following methods: (a) The algorithm of Zhang et al. \cite{ZCZJ14}, which results in the same limitations as those for the guarantees of \cite{ZCZJ14} discussed earlier; and (b) An estimator based on the work of Gao and Zhou \cite{GZ13} that prohibits settings where most labels in may have the same true value, thereby yielding only the naive bound of 1 on the minimax risk of estimation under the $Q^*$-loss.
**Majority voting:** Finally, let us comment on a relatively simple estimator—namely, the majority voting estimator. It computes the sign vector $\tilde{x}_{MV} \in \{-1, +1\}^d$ with entries

$$[\tilde{x}_{MV}]_j \in \arg \max_{b \in \{-1, +1\}} \sum_{i=1}^n 1\{Y_{ij} = b\} \quad \text{for all } j \in [d].$$

Here we use $1\{\cdot\}$ to denote the indicator function. One can show that for the Dawid-Skene parameters defined in equation (17), the majority voting estimator incurs a normalized Hamming error of $\Theta(1)$, and a $Q^*$-loss of order $\Theta(\frac{1}{\sqrt{n}})$, in expectation. We refer the reader to Appendix D for more details on these claims as well as some other properties of the majority voting estimator.

### 4 Simulations

In this section, we present numerical simulations comparing the following three estimators:

(i) Majority voting.

(ii) The Spectral-EM estimator due to Zhang et al. [ZCZJ14], which to the best of our knowledge, has the strongest established guarantees in the literature. We used an implementation provided by the authors of the paper [ZCZJ14].

(iii) Our proposed OBI-WAN estimator (introduced in Section 3.3). The code for the OBI-WAN estimator as well as the constituent WAN estimator is available on the first author’s website.

The results from our simulations are plotted in Figure 1. The plots in the six panels (a) through (f) of the figure are discussed below.

(a) **Easy:** $Q^* = q_{DS}^1 T \in \mathbb{C}_{DS}$ where $q_{i}^{DS} = \frac{9}{10}$ if $i < \frac{n}{2}$, and $q_{i}^{DS} = \frac{1}{2}$ otherwise. The parameter $n$ is varied, and the regime of operation is $(d = n, p_{obs} = 1)$. In this setting, each of the estimators correctly recover $x^*$.

(b) **Few Smart:** $Q^* = q_{DS}^1 T \in \mathbb{C}_{DS}$ where $q_{i}^{DS} = \frac{9}{10}$ if $i < \frac{n}{4}$ or $j < \frac{d}{2}$, and $q_{i}^{DS} = \frac{1}{2}$ otherwise. The parameter $n$ is varied, and the regime of operation $(d = n, p_{obs} = 1)$. Even though the data is drawn from the Dawid-Skene model, the error of Spectral-EM is much higher than that of the OBI-WAN estimator. Recall that the OBI-WAN estimator has uniform guarantees of recovery over the entire Dawid-Skene class, unlike the estimators in prior literature.

(c) **Adversarial:** $Q^* = q_{DS}^1 T \in \mathbb{C}_{DS}$ where $q_{i}^{DS} = \frac{9}{10}$ if $i < \frac{n}{4} + \sqrt{n}$, $q_{i}^{DS} = \frac{1}{10}$ if $i > \frac{3n}{4}$, and $q_{i}^{DS} = \frac{1}{2}$ otherwise. The parameter $n$ is varied, and the regime of operation is $(d = n, p_{obs} = 1)$. This set of simulations moves beyond the assumption that the entries of $Q^*$ are lower bounded by $\frac{1}{2}$, and allows for adversarial workers. The OBI-WAN estimator is successful in such a setting as well.

(d) In $\mathbb{C}_{Perm}$ but outside $\mathbb{C}_{Int}$: $Q^*_{ij} = \frac{9}{10}$ if $(i < \sqrt{n}$ or $j < \frac{d}{2})$, and $Q^*_{ij} = \frac{1}{2}$ otherwise. The parameter $n$ is varied, and the regime of operation is $(d = n, p_{obs} = 1)$. Here we have $Q^* \in \mathbb{C}_{Perm} \setminus \mathbb{C}_{Int}$. The $Q^*$-loss incurred by the majority voting and the OBI-WAN estimators decays as $\frac{1}{\sqrt{n}}$, whereas the $Q^*$-loss of Spectral-EM grows remains a constant.
Figure 1. Results from numerical simulations comparing OBI-WAN, Spectral-EM and majority-voting estimators. The plots in panels (a)-(d) measure the $Q^*$-loss as a function of $n$, and the plots in panels (e)-(f) measure the $Q^*$-loss as a function of $p_{\text{obs}}$. Each point is an average of over 20 trials. Recall that when $Q^*$ follows the Dawid-Skene model, as in panels (a)-(c), (e)-(f), the Hamming error is proportional to the $Q^*$-loss. Also note that the Y-axis of panel (d) is plotted on a logarithmic scale.

(e) Minimax lower bound: $Q^* = q_{\text{DS}}^T \in C_{\text{DS}}$ where $q_{\text{DS}}^i = \frac{9}{10}$ if $i \leq \frac{5}{p_{\text{obs}}}$ and $q_{\text{DS}}^i = \frac{1}{2}$ otherwise. The parameter $p_{\text{obs}}$ is varied, and the regime of operation is $(d = 1000, n = 1000)$. This setting is the cause of the minimax lower bound of Theorem 1(b). The error of each of the three estimators, in this case, behaves in an identical manner with a scaling of $\frac{1}{p_{\text{obs}}}$.

(f) Super sparse: $Q^* = q_{\text{DS}}^T \in C_{\text{DS}}$ where $q_{\text{DS}}^i = \frac{9}{10}$ if $i \leq \frac{n}{10}$ and $q_{\text{DS}}^i = \frac{1}{2}$ otherwise. The parameter $p_{\text{obs}}$ is varied, and the regime of operation is $(d = 1000, n = 1000)$. We see that the OBI-WAN estimator performs poorly when data is very sparse — more generally, we have observed a higher error when $p_{\text{obs}} = o(\frac{\log^2(n)}{n})$, and this gap is also reflected in our upper bounds for the OBI-WAN estimator in Theorem 3(a) and Theorem 4(a) that are loose by precisely a polylogarithmic factor as compared to the associated lower bounds.
The relative benefits and disadvantages of the proposed OBI-WAN estimator, as observed from the simulations, may be summarized as follows. In terms of limitations, the error of OBI-WAN is higher than prior works when $p_{\text{obs}}$ is small (as observed in the super-sparse case) or when $n$ and $d$ are small (for instance, less than 200). On the positive side, the simulations reveal that the OBI-WAN estimator leads to accurate estimates in a variety of settings, providing uniform guarantees over the $C_{\text{DS}}$ and $C_{\text{Int}}$ classes, and demonstrating significant robustness in more general settings in comparison to the best known estimator in the literature.

5 Proofs

In this section, we present the proofs of all our theoretical results. In the proofs, we use $c$, $c_1$, $c'$ etc. to denote positive universal constants, and ignore floors and ceilings unless critical to the proof. We assume that $n$ and $d$ are bigger than some universal constants; the case of smaller values of these parameters are then directly implied by only changing the constant prefactors.

5.1 Proof of Theorem I(a): Minimax upper bound

In this section, we prove the minimax upper bound stated in part (a) of Theorem I. We begin by rewriting our observation model in a “linearized” fashion that is convenient for subsequent analysis. In particular, let us define a random matrix $W \in \mathbb{R}^{n \times d}$ with entries independently drawn from the distribution

$$W_{ij} = \begin{cases} 1 - p_{\text{obs}}(2Q^*_{ij} - 1)x_j^* & \text{with probability } p_{\text{obs}}\left(Q^*_{ij} \left(\frac{1+x_j^*}{2}\right) + (1 - Q^*_{ij}) \left(\frac{1-x_j^*}{2}\right)\right) \\ -1 - p_{\text{obs}}(2Q^*_{ij} - 1)x_j^* & \text{with probability } p_{\text{obs}}\left(Q^*_{ij} \left(\frac{1-x_j^*}{2}\right) + (1 - Q^*_{ij}) \left(\frac{1+x_j^*}{2}\right)\right) \\ -p_{\text{obs}}(2Q^*_{ij} - 1)x_j^* & \text{with probability } 1 - p_{\text{obs}}. \end{cases} \quad (19)$$

One can verify that $E[W] = 0$, every entry of $W$ is bounded by 2 in absolute value, and moreover that our observed matrix $Y$ can be written in the form

$$\frac{1}{p_{\text{obs}}} Y = (2Q^* - 11^T) \text{ diag}(x^*) + \frac{1}{p_{\text{obs}}} W. \quad (20)$$

Let $\Pi_n$ denote the set of all permutations of the $n$ workers, and let $\Sigma_d$ denote the set of all permutations of the $d$ questions. For any pair of permutations $(\pi, \sigma) \in \Pi_n \times \Sigma_d$, define the set

$$C_{\text{Perm}}(\pi, \sigma) := \left\{ Q \in [0,1]^{n \times d} \mid Q_{ij} \geq Q_{i'j'} \text{ whenever } \pi(i) \leq \pi(i') \text{ and } \sigma(j) \leq \sigma(j') \right\}, \quad (21)$$

corresponding to the subset of $C_{\text{Perm}}$ consisting of matrices that are faithful to the permutations $\pi$ and $\sigma$. For any fixed $x \in \{-1,1\}^d$, $\pi \in \Pi_n$ and $\sigma \in \Sigma_d$, define the matrix

$$\tilde{Q}(\pi, \sigma, x) \in \arg\min_{Q \in C_{\text{Perm}}(\pi, \sigma)} \mathcal{C}(Q, x), \quad \text{where } \mathcal{C}(Q, x) := \frac{1}{p_{\text{obs}}} \| Y - (2Q - 11^T) \text{ diag}(x) \|_F^2.$$
Using this notation, we can rewrite the least squares estimator (6) in the compact form

\[ (\tilde{x}_{LS}, \tilde{\pi}_{LS}, \tilde{\sigma}_{LS}) \in \arg \min_{(\pi, \sigma) \in \Pi_n \times \Sigma_d} \{ C(\tilde{Q}(\pi, \sigma, x), x) : x \in \{-1, 1\}^d \} \]

For the purposes of analysis, let us define the set

\[ P := \left\{ (\pi, \sigma, x) \in \Pi_n \times \Sigma_d \times \{-1, 1\}^d \mid C(\tilde{Q}(\pi, \sigma, x), x) \leq C(Q^*, x^*) \right\}. \] (22)

With this set-up, we claim that it is sufficient to show the following: fix a triplet \((\pi, \sigma, x) \in P\), for this fixed triplet there is a universal constant \(c_1 > 0\) such that

\[ \mathbb{P}\left( \| (2\tilde{Q}(\pi, \sigma, x) - 11^T) \text{diag}(x - x^*) \|^2_{F} \leq c_1 \frac{d \log^2 d}{p_{\text{obs}}} \right) \geq 1 - e^{-4d \log^2 (dn)}. \] (23)

Given this bound, since the cardinality of the set \(P\) is upper bounded by \(e^{3d \log d}\) (since \(d \geq n\)), a union bound over all these permutations applied to (23) yields

\[ \mathbb{P}\left( \max_{(\pi, \sigma, x) \in P} \| (2\tilde{Q}(\pi, \sigma, x) - 11^T) \text{diag}(x - x^*) \|^2_{F} \leq c_1 \frac{d \log^2 d}{p_{\text{obs}}} \right) \geq 1 - e^{-d \log^2 (dn)}. \]

The set \(P\) is guaranteed to be non-empty since the true permutations \(\pi^*\) and \(\sigma^*\) corresponding to \(Q^*\) and the true answer \(x^*\) always lie in \(P\), and consequently, the above tail bound yields the claimed result.

The remainder of our analysis is devoted to proving the bound (23). Given any triplet \((\pi, \sigma, x) \in P\), we define the matrices

\[ V^* := (2Q^* - 11^T) \text{diag}(x^*), \quad \text{and} \quad \tilde{V}(\pi, \sigma, x) := (2\tilde{Q}(\pi, \sigma, x) - 11^T) \text{diag}(x). \]

Henceforth, for brevity, we refer to the matrix \(\tilde{V}(\pi, \sigma, x)\) simply as \(\tilde{V}\) and the matrix \(\tilde{Q}(\pi, \sigma, x)\) simply as \(\tilde{Q}\), since the values of the associated quantities \((\pi, \sigma, x)\) are fixed and clear from context.

Applying the linearized form (20) of our observation model to the inequality that defines the set (22), some simple algebraic manipulations yield

\[ \frac{1}{2} \| V^* - \tilde{V} \|^2_{F} \leq \frac{1}{p_{\text{obs}}} \langle V^* - \tilde{V}, W \rangle. \] (24)

The following lemma uses this inequality to obtain an upper bound on the quantity \(\frac{1}{2} \| V^* - \tilde{V} \|^2_{F} \).

**Lemma 1.** There exists a universal constant \(c_1 > 0\) such that

\[ \mathbb{P}\left( \| V^* - \tilde{V} \|^2_{F} \leq c_1 \frac{d \log^2 d}{p_{\text{obs}}} \right) \geq 1 - e^{-4d \log^2 (dn)}. \] (25)

See Section A.1 for the proof of this lemma.

Our next step is to convert our bound (25) on the Frobenius norm \(\| V^* - \tilde{V} \|^2_{F} \) into one on the error in estimating \(x^*\) under the \(Q^*\)-loss. The following lemma is useful for this conversion:
Lemma 2. For any pair of matrices $A_1, A_2 \in \mathbb{R}^{n \times d}$ and any pair of vectors $v_1, v_2 \in \{-1, 1\}^d$, we have

$$\|A_1 \text{ diag}(v_1 - v_2)\|_F^2 \leq 4 \|A_1 \text{ diag}(v_1) - A_2 \text{ diag}(v_2)\|_F^2.$$  \hspace{1cm} (26)

See Section A.2 for the proof of this claim.

Recall our assumption that every entry of the matrices $Q^*$ and $\tilde{Q}$ is at least $\frac{1}{2}$. Consequently, we can apply Lemma 2 with $A_1 = (Q^* - \frac{1}{2}11^T)$, $A_2 = (\tilde{Q} - \frac{1}{2}11^T)$, $v_1 = x^*$ and $v_2 = x$ to obtain the inequality

$$\| (Q^* - \frac{1}{2}11^T) \text{ diag}(x^* - x)\|_F^2 \leq 4 \| (Q^* - \frac{1}{2}11^T) \text{ diag}(x^*) - (\tilde{Q} - \frac{1}{2}11^T) \text{ diag}(\tilde{x})\|_F^2 = 4 \| V^* - \tilde{V} \|_F^2.$$  

Coupled with Lemma 1, this bound yields the desired result (23).

5.2 Proof of Theorem 1(b): Minimax lower bound

We now turn to the proof of the minimax lower bound. For a numerical constant $\delta \in (0, \frac{1}{4})$ whose precise value is determined later, define the probability matrix $Q^* \in [0, 1]^{n \times d}$ with entries

$$Q^*_{ij} = \begin{cases} \frac{1}{2} + \delta & \text{if } i \leq \frac{1}{p_{\text{obs}}} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$  \hspace{1cm} (27)

The Gilbert-Varshamov bound [Gil52, Var57] guarantees that for a universal constant $c > 0$, there is a collection $\beta = \exp(cd)$ binary vectors—that is, a collection of vectors $\{x^1, \ldots, x^\beta\}$ all belonging to the Boolean hypercube $\{-1, 1\}^d$—such that the normalized Hamming distance (1) between any pair of vectors in this set is lower bounded as

$$d_H(x^\ell, x^{\ell'}) \geq \frac{1}{10}, \quad \text{for every } \ell, \ell' \in [\beta].$$

For each $\ell \in [\beta]$, let $\mathbb{P}^\ell$ denote the probability distribution of $Y$ induced by setting $x^* = x^\ell$. For the choice of $Q^*$ specified in (27), following some algebra, we obtain an upper bound on the Kullback-Leibler divergence between any pair of distributions from this collection as

$$D_{\text{KL}}(P^\ell \| P^{\ell'}) \leq c' d \delta^2 \quad \text{for every } \ell \neq \ell' \in [\beta],$$

for another constant $c' > 0$. Combining the above observations with Fano’s inequality [CT12] yields that any estimator $\tilde{x}$ has expected normalized Hamming error lower bounded as

$$\mathbb{E}[d_H(\tilde{x}, x^*)] \geq \frac{1}{20} \left( 1 - \frac{c'd \delta^2 + \log 2}{\log \beta} \right).$$

Consequently, for the choice of $Q^*$ given by (27), the $Q^*$-loss is lower bounded as

$$\mathbb{E}[\mathcal{L}_{Q^*}(\tilde{x}, x^*)] = \frac{4\delta^2}{p_{\text{obs}}} \mathbb{E}[d_H(\tilde{x}, x^*)] \geq \frac{4\delta^2}{20np_{\text{obs}}} \left( 1 - \frac{c'd \delta^2 + \log 2}{cd} \right) \geq \frac{c''}{np_{\text{obs}}},$$

for some constant $c'' > 0$ as claimed. Here inequality (i) follows by setting $\delta$ to be a sufficiently small positive constant (depending on the values of $c'$ and $c''$).
5.3 Proof of Corollary [I](a)

In the proof of Theorem [I](a), we showed that there is a constant $c_1 > 0$ such that
\[
\| (2Q^* - 11^T)x^* - (2\tilde{Q}_{LS} - 11^T)\tilde{x}_{LS}\|_F^2 \leq c_1 \frac{d}{p_{\text{obs}}} \log^2 d,
\]
with probability at least $1 - e^{-d \log (dn)}$. Since all entries of the matrices $2Q^* - 11^T$ and $2\tilde{Q}_{LS} - 11^T$ are non-negative, and since every entry of the vectors $x^*$ and $\tilde{x}_{LS}$ lies in $\{-1, 1\}$, some algebra yields the bound
\[
((2Q^*_{ij} - 1) - (2[\tilde{Q}_{LS}]_{ij} - 1))^2 \leq ((2Q^*_{ij} - 1)x^*_j - (2[\tilde{Q}_{LS}]_{ij} - 1)[\tilde{x}_{LS}]_j)^2 \text{ for every } i \in [n], j \in [d].
\]
Combining these inequalities yields the claimed bound.

5.4 Proof of Corollary [I](b)

We begin by constructing a set, of cardinality $\beta$, of possible matrices $Q^*$, for some integer $\beta > 1$, and subsequently we show that it is hard to identify the true matrix if drawn from this set. We begin by defining a $\beta$-sized collection of vectors $\{h^1, \ldots, h^\beta\}$, all contained in the set $[\frac{1}{2}, 1]^d$, as follows. The Gilbert-Varshamov bound [Gil52, Var57] guarantees a constant $c \in (0, 1)$ such that there exists set of $\beta = \exp(cd)$ vectors, $v^1, \ldots, v^\beta \in \{-1, 1\}^d$ with the property that the normalized Hamming distance [1] between any pair of these vectors is lower bounded as
\[
d_{\text{Ham}}(v^\ell, v^{\ell'}) \geq \frac{1}{10}, \quad \text{for every } \ell, \ell' \in [\beta].
\]
Fixing some $\delta \in (0, \frac{1}{4})$, let us define, for each $\ell \in [\beta]$, the vector $h^\ell \in \mathbb{R}^d$ with entries
\[
[h^\ell]_j = \begin{cases} \frac{1}{2} + \delta & \text{if } [v^\ell]_j = 1 \\ \frac{1}{2} & \text{otherwise.} \end{cases}
\]
For each $\ell \in [\beta]$, define the matrix $Q^\ell = 1(h^\ell)^T$, and let $P^\ell$ denote the probability distribution of the observed data $Y$ induced by setting $Q^* = Q^\ell$ and $x^* = 1$. Since the entries of $Y$ are all independent, some algebra leads to the following upper bound on the Kullback-Leibler divergence between any pair of distributions from this collection:
\[
D_{\text{KL}}(P^\ell || P^{\ell'}) \leq 4p_{\text{obs}}nd\delta^2 \quad \text{for every } \ell \neq \ell' \in [\beta].
\]
Moreover, some simple calculation shows that the squared Frobenius norm distance between any two matrices in this collection is lower bounded as
\[
\|Q^\ell - Q^{\ell'}\|_F^2 \geq \frac{1}{10} d n \delta^2 \quad \text{for every } \ell \neq \ell' \in [\beta].
\]
Combining the above observations with Fano’s inequality [CT12] yields that any estimator $\hat{Q}$ for $Q^*$ has MSE at least
\[
\mathbb{E}[\|Q^* - \hat{Q}\|_F^2] \geq \frac{1}{20} d n \delta^2 \left(1 - \frac{4p_{\text{obs}}nd\delta^2 + \log 2}{\log \beta}\right) \geq c' \frac{d}{p_{\text{obs}}},
\]
where we have set $\delta^2 = \frac{c''}{p_{\text{obs}}}n$ for a small enough positive constant $c''$, where $c'$ is another positive constant whose value may depend only on $c$ and $c''$. 

20
5.5 Proof of Theorem 2

We begin by stating a key auxiliary lemma, which is somewhat more general than what is required for the current proof. For any matrix $Q^* \in \mathbb{C}_{\text{Perm}}$ and worker permutation $\pi$, we define the set

$$J := \{ j \in [d] \mid \exists k_j \geq 1 \text{ s.t. } \sum_{i=1}^{k_j} (Q^*_{\pi^{-1}(i)j} - \frac{1}{2}) \geq \frac{3}{4} \frac{k_j}{p_{\text{obs}}} \log^{1.5}(dn) \}. $$

(28)

Note that this set corresponds to a subset of questions that are relatively “easy”, in a certain sense specified by $Q^*$.

**Lemma 3.** For the set $J$, the WAN estimator satisfies the bound

$$\mathbb{P} \left( \left[ \hat{x}_{\text{WAN}}(\pi) \right]_{j_0} = x^*_{j_0} \text{ for all } j_0 \in J \right) \geq 1 - e^{-c \log^{1.5}(dn)}. $$

See Appendix B.1 for the proof of this claim.

Lemma 3 guarantees that the WAN estimator correctly answers all questions that are relatively easy. Note that the set (28) is defined in terms of the $\ell_1$-norm of subvectors of columns of $Q^* - \frac{1}{2}$, whereas the conditions

$$\|Q_j^* - \frac{1}{2}\|_2^2 \geq \frac{5 \log^{2.5}(dn)}{p_{\text{obs}}} \quad \text{and} \quad \|Q_j^* - Q_j^{\pi^*}\|_2 \leq \frac{\|Q_j^* - \frac{1}{2}\|_2}{\sqrt{9 \log(dn)}} $$

(29)

in the theorem claim are in terms of the $\ell_2$-norm of the columns of $Q^*$. The following lemma allows us to connect the $\ell_1$ and $\ell_2$-norm constraints for any vector in a general class.

**Lemma 4.** For any vector $v \in [0, 1]^n$ such that $v_1 \geq \ldots \geq v_n$, there must be some $\alpha \geq \lceil \frac{1}{2} \|v\|_2^2 \rceil$ such that

$$\sum_{i=1}^{\alpha} v_i \geq \frac{\sqrt{\alpha \|v\|_2^2}}{2 \log n}. $$

(30)

See Appendix B.2 for the proof of this claim.

Using these two lemmas, we can now complete the proof of the theorem. We may assume without loss of generality that the rows of $Q^*$ are ordered to be non-decreasing downwards along any column, that is, that $\pi^*$ is the identity permutation. Consider any question $j \in [d]$ for which the permutation $\pi$ satisfies the bounds (29). For any $\ell \in [n]$, let $g_\ell \in \mathbb{R}^n$ denote a vector with ones in its first $\ell$ positions and zeros elsewhere. The Cauchy-Schwarz inequality implies that

$$(Q^*_j - \frac{1}{2})^T g_\ell \geq (Q^*_j - \frac{1}{2})^T g_\ell - \sqrt{\ell \|Q_j^* - Q_j^{\pi^*}\|_2}. $$
By applying Lemma 4 to the vector $Q^*_j - \frac{1}{2}$, we are guaranteed the existence of some value $k \geq \frac{5 \log^{2.5}(dn)}{2p_{obs}}$ such that $(Q^*_j - \frac{1}{2})^T g_k \geq \|Q^*_j - \frac{1}{2}\|_2 \sqrt{\frac{k}{2 \log n}}$. Consequently, we have the lower bound 

$$(Q^*_j - \frac{1}{2})^T g_k \geq \|Q^*_j - \frac{1}{2}\|_2 \sqrt{\frac{k}{2 \log n}} - \sqrt{k}\|Q_j - Q^*_j\|_2 \geq \frac{37}{4}\|Q^*_j - \frac{1}{2}\|_2 \sqrt{\log(dn)}$$

where inequalities (i) and (ii) follow from conditions (29). Consequently, we can apply Lemma 3 for every such question $j$, thereby yielding the claimed result.

5.6 Proof of Theorem 3 (a)

Define the vector $r^* := \tilde{q} - \frac{1}{2}$. We split the proof into two parts, depending on whether or not the condition

$$\|r^*\|_2 \|1 - h^*\|_2 \geq \sqrt{\frac{Cd \log^{2.5}(dn)}{p_{obs}}}$$

is satisfied. Here $C > 20$ is a constant, whose value is specified later in the proof. (In particular, see equation (51) in Lemma 5.)

5.6.1 Case 1

First, suppose that condition (31) is violated. For each $\hat{x} \in \{-1, 1\}^d$, we then have

$$\mathcal{L}_{Q^*}(\hat{x}; x^*) \leq \frac{1}{dn}\|r^*\|_2^2\|1 - h^*\|_2^2 \leq \frac{6C \log^{2.5} d}{np_{obs}}.$$ 

as claimed, where we have made use of the fact that $d \geq n$.

5.6.2 Case 2

In this second case, we may assume that condition (31) holds, and we do so throughout the remainder of this section. Our proof of this case is divided into three parts, each corresponding to one of the three steps in the OBI-WAN algorithm. The first step is to derive certain properties of the split of the questions. The second step is to derive approximation-guarantees on the outcome of the OBI step. The third and final step is to show that this approximation guarantee ensures that the output of the WAN estimator meets the claimed error guarantee.

Step 1: Analyzing the split. Our first step is to exhibit a useful property of the split of the questions—namely, that with high probability, the questions in the two sets $T_0$ and $T_1$ have a similar total difficulty.
Applying Bernstein’s inequality then guarantees that
\[
E[\sum_{j \in [d]} (1 - h^*_j)^2 \epsilon_j] = \frac{1}{2} \|1 - h^*\|_2^2, \quad \text{and} \quad E[\sum_{j \in [d]} ((1 - h^*_j)^2 \epsilon_j)^2] = \frac{1}{2} \sum_{j \in [d]} (1 - h^*_j)^4 \leq \frac{1}{2} \|1 - h^*\|_2^2.
\]

Applying Bernstein’s inequality then guarantees that
\[
P(\sum_{j \in T_\ell} (1 - h^*_j)^2 > \frac{2}{3} \|1 - h^*\|_2^2) \leq \exp(-c \|1 - h^*\|_2^2) \quad \text{for each} \ \ell \in \{0, 1\},
\]
where \(c\) is a positive universal constant. Consequently we are guaranteed that
\[
\frac{1}{3} \|1 - h^*\|_2^2 \leq \sum_{j \in T_\ell} (1 - h^*_j)^2 \leq \frac{2}{3} \|1 - h^*\|_2^2 \quad \text{for both} \ \ell \in \{1, 2\}, \quad (32)
\]
with probability at least \(1 - e^{-c C \log^{2.5} d / p_{\text{obs}}}\), where we have used the fact that \(\|1 - h^*\|_2^2 \geq C d \log^{2.5} d / p_{\text{obs}}\|p^*\|_2^2 \geq C \log^{2.5} d / p_{\text{obs}}\). Now define the error event
\[
\mathcal{E} : = \left\{ L_{Q^*}(\hat{x}_0, x^*) > \frac{6C \log^{2.5} d}{np_{\text{obs}}} \right\}.
\]
Combining the sandwich relation (32) with the union bound, we find that
\[
P(\mathcal{E}) \leq \sum_{\text{partitions } \tilde{T}_0, \tilde{T}_1} P(\mathcal{E} \mid T_0 = \tilde{T}_0, T_1 = \tilde{T}_1) P(T_0 = \tilde{T}_0, T_1 = \tilde{T}_1)
\]
\[
\leq \sum_{\text{partitions } \tilde{T}_0, \tilde{T}_1 \text{ satisfying } (32)} P(\mathcal{E} \mid T_0 = \tilde{T}_0, T_1 = \tilde{T}_1) P(T_0 = \tilde{T}_0, T_1 = \tilde{T}_1) + e^{-c C \log^{2.5} d / p_{\text{obs}}}.
\]
Consequently, in the rest of the proof we consider any partition \((\tilde{T}_0, \tilde{T}_1)\) that satisfies the sandwich bound (32) and derive an upper bound on the error conditioned on this partition. In other words, it suffices to prove the following bound for any partition \((\tilde{T}_0, \tilde{T}_1)\) satisfying (32):
\[
P(\mathcal{E} \mid T_0 = \tilde{T}_0, T_1 = \tilde{T}_1) \leq e^{-c' \log^{1.5}(dn)}, \quad (33)
\]
for some positive universal constant \(c'\) whose value may depend only on \(C\). We note that conditioned on the partition \((\tilde{T}_0, \tilde{T}_1)\), and for any fixed values of \(Q^*\) and \(x^*\), the responses of the workers to the questions in one set are statistically independent of the responses in the other set. Consequently, we describe the proof for any one of the two partitions, and the overall result is implied by a union bound of the error guarantees for the two partitions. We use the notation \(\ell\) to denote either one of the two partitions in the sequel, that is, \(\ell \in \{0, 1\}\).
Step 2: Guarantees for the OBI step. Assume without loss of generality that the rows of the matrix $Q^*$ are ordered according to the abilities of the corresponding workers, that is, the entries of $\tilde{q}$ are arranged in a non-increasing order. Recall that $\pi_\ell$ denotes the permutation of the workers in order of their respective values in $u_\ell$. Let $\tilde{\tau}_\ell \in \mathbb{R}^n$ denote the vector obtained by permuting the entries of $r^*$ in the order given by $\pi_\ell$. Thus the entries of $\tilde{\tau}_\ell$ are identical to those of $r^*$ up to a permutation; the ordering of the entries of $\tilde{\tau}_\ell$ is identical to the ordering of the entries of $u_\ell$. The following lemma establishes a deterministic relation between these vectors; its proof combines matrix perturbation theory with some careful algebraic arguments.

Lemma 5. Suppose that condition (31) holds for a sufficiently large constant $C > 0$. Then for any split $(T_0, T_1)$ satisfying the relation (32), we have

$$\Pr \left[ \|\tilde{\tau}_\ell - r^*\|_2^2 > \frac{\|r^*\|_2^2}{9 \log(\frac{dn}{\text{obs}})} \right] \leq e^{-c\log^{1.5} d}. \tag{34}$$

See Section C.1 for the proof of this claim.

At this point, we are now ready to apply the bound for the WAN estimator from Theorem 2.

Step 3: Guarantees for the WAN step. Recall that for any choice of index $\ell \in \{0, 1\}$, the OBI step operates on the set $T_\ell$ of questions, and the WAN step operates on the alternate set $T_{1-\ell}$. Consequently, conditioned on the partition $(\tilde{T}_0, \tilde{T}_1)$, the outcomes $Y_{1-\ell}$ of the comparisons in set $(1 - \ell)$ are statistically independent of the permutation $\pi_\ell$ obtained from set $\ell$ in the OBI step.

Consider any question $j \in T_{1-\ell}$ that satisfies the inequality $\|(1 - h_j^*)r^*\|_2^2 \geq \frac{5 \log^{2.5}(dn)}{\text{obs}}$. We now claim that this question $j$ satisfies the pair of conditions (10a) required by the statement of Theorem 2. First observe that $(1 - h_j^*)r^*$ is simply the $j$th column of the matrix $(Q^* - \frac{1}{2})$, we have $\|Q_j^* - \frac{1}{2}\|_2^2 \geq \frac{5 \log^{2.5}(dn)}{\text{obs}}$. The first condition in (10a) is thus satisfied.

In order to establish the second condition, observe that a rescaling of the inequality (34) by the non-negative scalar $(1 - h_j^*)$ yields the bound

$$\|(1 - h_j^*)\tilde{\tau}_\ell - (1 - h_j^*)r^*\|_2^2 \leq \frac{\|(1 - h_j^*)r^*\|_2^2}{9 \log(\frac{dn}{\text{obs}})} \quad \text{for every question } j \in T_{1-\ell}. \tag{35}$$

Recall our notational assumption that the entries of $\tilde{q}$ (and hence the rows of $Q^*$) are arranged in order of the workers' abilities, and that $Q^\pi$ is a matrix obtained by permuting the rows of $Q^*$ according to a given permutation $\pi$. Also observe that the vector $(1 - h_j^*)\tilde{\tau}_\ell$ equals the $j$th column of $(Q^\pi - \frac{1}{2})$, where $\pi_\ell$ is the permutation of the workers obtained from the OBI step. Consequently, the approximation guarantee (35) implies that $\|Q_j^\pi - Q_j^\star\|_2 \leq \frac{\|Q_j^\star\|_2}{\sqrt{9 \log(\frac{dn}{\text{obs}})}}$. Thus the second condition in equation (10a) is also satisfied for the question $j$ under consideration.

Applying the result of Theorem 2 for the WAN step, we obtain that this question $j$ is decoded correctly with a probability at least $1 - e^{-c\log^{1.5}(dn)}$, for some positive constant $c$. Since this argument holds for every question $j$ satisfying $\|(1 - h_j^*)r^*\|_2^2 \geq \frac{5 \log^{2.5}(dn)}{\text{obs}}$, the total contribution from the
remaining questions to the $Q^*$-loss is at most $\frac{5 \log^{2.5}(dn)}{p_{\text{obs}}n}$. A union bound over all questions and both values of $\ell \in \{0, 1\}$ then yields the claim that the aggregate $Q^*$-loss is at most $\frac{5 \log^{2.5}(dn)}{p_{\text{obs}}n}$ with probability at least $1 - e^{-c' \log^{1.5}(dn)}$, for some positive constant $c'$, as claimed in (33).

5.7 Proof of Theorem 3(b)

First, suppose that $p_{\text{obs}} < \frac{\log^{1.5}(dn)}{n}$. In this case, we have

$$L_{Q^*}(\hat{x}_{\text{OBI-WAN}}, x^*) \leq 1 \leq \frac{1}{\sqrt{n}p_{\text{obs}}} \log(dn),$$

and the claim follows immediately.

Otherwise, we may assume that $p_{\text{obs}} \geq \frac{\log^{1.5}(dn)}{n}$. For any index $\ell \in \{0, 1\}$, consider an arbitrary permutation $\pi_\ell$. Observe that conditioned on the split $(T_0, T_1)$, the data $Y_{1-\ell}$ is independent of the choice of the permutation $\pi_\ell$. Now consider any question $j \in T_{1-\ell}$ that satisfies

$$\sum_{i=1}^{n} (Q^*_{ij} - \frac{1}{2})^2 \geq \frac{3}{2} \sqrt{\frac{n}{p_{\text{obs}}}} \log(dn). \quad (36a)$$

Lemma 3 then guarantees that

$$\mathbb{P}([x_{\text{WAN}}(\pi)]_j \neq x^*_j) \leq e^{-c \log^{1.5}(dn)}. \quad (36b)$$

All remaining questions can contribute a total of at most $\frac{3}{2} \sqrt{\frac{n}{p_{\text{obs}}}} \log(dn)$ to the $Q^*$-loss. Consequently, a union bound over the probabilities $(36b)$ for all questions (in $T_0$ and $T_1$) that satisfy the bound $(36a)$ yields the claimed result.

5.8 Proof of Theorem 4(a)

Throughout the proof, we make use the notation previously introduced in the proof of Theorem 3(a). As in this same proof, we condition on some choice of $T_0$ and $T_1$ that satisfies (32). The proof of this theorem follows the same structure as the proof of Theorem 3(a) and the lemmas within it. However, we must make additional arguments in order to account for adversarial workers. In the remainder of the proof, we consider any $\ell \in \{0, 1\}$, and then apply the union bound across both values of $\ell$.

Our proof consists of the three steps:

1. We first show that the vector $u_\ell$ is a good approximation to $(q^{DS} - \frac{1}{2})$ up to a global sign.
2. Second, we show that the global sign of $r^*$ is indeed recovered correctly.
3. Third, we establish guarantees on the performance of the WAN estimator for our setting.

We work through each of these steps in turn.
5.8.1 Step 1

We first show that the vector \( u_\ell \) is a good approximation to \( q^{DS} - \frac{1}{2} \) up to a global sign. When \( Q^* = q^{DS}1^T \), we can set the vector \( h^* = 0 \) in the proof of Theorem 3(a). We also have \( r^* = q^{DS} - \frac{1}{2} \).

With these assignments, the arguments up to equation (51) in Lemma 5 continue to apply even for the present setting where \( q^{DS} \in [0,1]^n \). From these arguments, we obtain the following approximation guarantee (51) on recovering \( q^* \) even for the present setting.

\[
\text{min}\{\|u_\ell - \frac{1}{\rho} r^*\|_2, \|u_\ell + \frac{1}{\rho} r^*\|_2\} \leq \frac{1}{36} \frac{\log^{1.5} d}{p_{\text{obs}}} \tag{37}
\]

with probability at least \( 1 - e^{-c \log^{1.5} d} \).

5.8.2 Step 2

The next step of the proof is to show that the global sign of \( r^* \) is indeed recovered correctly. Define two pairs of vectors \( \{u_\ell^+, u_\ell^-\} \) and \( \{r_\ell^+, r_\ell^-\} \), all lying in the unit cube \([0,1]^n\), with entries

\[
[u_\ell^+]_i := \max\{[u_\ell^+]_i, 0\} \quad \text{and} \quad [u_\ell^-]_i := \min\{[u_\ell^-]_i, 0\} \quad \text{for every } i \in [n],
\]

\[
[r_\ell^+]_i := \max\{[r_\ell^+]_i, 0\}, \quad \text{and} \quad [r_\ell^-]_i := \min\{[r_\ell^-]_i, 0\} \quad \text{for every } i \in [n].
\]

From the conditions assumed in the statement of the theorem, we have \( \|r^+\|_2 \geq \|r^-\|_2 + \sqrt{\frac{4 \log^2(n)}{p_{\text{obs}}}} \), whereas from the choice of \( u \) in the OBI-WAN estimator, we have \( \|u^+\|_2 \geq \|u^-\|_2 \). One can also verify that

\[
\|u_\ell + \frac{1}{\rho} r^*\|_2^2 \geq \|u_\ell^+ + \frac{1}{\rho} r^+\|_2^2 + \|u_\ell^- + \frac{1}{\rho} r^-\|_2^2. \tag{38a}
\]

Now suppose that \( \frac{1}{\rho} r^+ \|_2 \geq \|u_\ell^-\|_2 + \sqrt{\frac{\log^2(n)}{p_{\text{obs}}}} \). Then from the triangle inequality, we obtain the bound

\[
\|u_\ell^- + \frac{1}{\rho} r^+\|_2 \geq \frac{1}{\rho} r^+\|_2 - \|u_\ell^-\|_2 \geq \sqrt{\frac{\log^2(n)}{p_{\text{obs}}}}. \tag{38b}
\]

Otherwise we have that \( \frac{1}{\rho} r^+\|_2 < \|u_\ell^-\|_2 + \sqrt{\frac{\log^2(n)}{p_{\text{obs}}}} \). In this case, we have

\[
\|u_\ell^+ + \frac{1}{\rho} r^-\|_2 \geq \|u_\ell^+\|_2 - \frac{1}{\rho} r^-\|_2 \geq \|u_\ell^-\|_2 - \frac{1}{\rho} r^-\|_2 + 2 \sqrt{\frac{\log^2(n)}{p_{\text{obs}}}} \geq \sqrt{\frac{\log^2(n)}{p_{\text{obs}}}}. \tag{38c}
\]

Putting together the conditions (38a), (38b) and (38c), we obtain the bound

\[
\|u_\ell + \frac{1}{\rho} r^*\|_2 \geq \frac{\log^2(n)}{p_{\text{obs}}}. \tag{39}
\]
In conjunction with the result of equation (37), this bound guarantees the correct detection of the global sign, that is,

\[ \| u^\ell - \frac{1}{\rho} r^* \|_2^2 \leq \frac{1}{36} \frac{1}{\rho^2} \log \left( \frac{1}{\frac{1}{5} d_{\text{obs}}} \right). \]

The deterministic inequality afforded by Lemma 7 then guarantees that

\[ \| \tilde{r}^\ell - r^* \|_2^2 \leq \frac{1}{18} \frac{1}{\rho} \log \left( \frac{1}{\frac{1}{5} d_{\text{obs}}} \right), \]

and this completes the analysis of the OBI part of the estimator.

### 5.8.3 Step 3

In the third step, we establish guarantees on the performance of the WAN estimator for our setting. Recall that since the WAN estimator uses the permutation given by \( \tilde{r}^\ell \) and with this permutation, acts on the observation \( Y_1 - \ell \) of the other set of questions, the noise \( W_{1-\ell} \) is statistically independent of the choice of \( \tilde{r}^\ell \), when conditioned on the split \( (T_0, T_1) \). Assume without loss of generality that \( x^* = 1 \) and that the rows of \( Q^* \) are arranged according to the worker abilities, meaning that \( q_{i,j}^{\text{DS}} \geq q_{i',j}^{\text{DS}} \) for every \( i < i' \), or in other words, \( r^*_i \geq r^*_{i'} \) for every \( i < i' \). Recall our earlier notation of \( g_k \in \{0, 1\}^n \) denoting a vector with ones in its first \( k \) positions and zeros elsewhere.

Now from the proof of Lemma 3 the following two properties ensure that the WAN estimator decodes every question correctly with probability at least \( 1 - e^{-c \log(1.5)(dn)} \): (i) There exists some value \( k \geq p_{\text{obs}}^{-1} \log(1.5)(dn) \) such that \( \langle \tilde{r}^\ell, g_k \rangle \geq \sqrt{k} \sqrt{\log(1.5)(dn)} \), and (ii) \( \langle \tilde{r}^\ell, g_k \rangle > \frac{3}{4} \sqrt{k} \sqrt{\log(1.5)(dn)} p_{\text{obs}} \) for every \( k \in [n] \). Let us first address property (i). Lemma 4 guarantees the existence of some value \( k \geq \left\lceil \frac{1}{2} \| r^* \|_2^2 \right\rceil \) such that

\[ \langle r^+, g_k \rangle \geq \frac{\sqrt{k} \| r^+ \|_2}{\sqrt{\log(dn)}}. \]

If there exist multiple such values of \( k \), then choose the smallest such value. Since the vector \( r^* \) has its entries arranged in order, and since \( \| r^+ \|_2 \geq \| r^- \|_2 \), we obtain the following relations for this chosen value of \( k \):

\[ \langle r^+, g_k \rangle = (r^+)^T g_k \geq \sqrt{k} \frac{\| r^+ \|_2}{\sqrt{\log(dn)}} \geq \frac{\| r^+ \|_2}{2} \sqrt{\frac{k}{\log(dn)}} \geq \sqrt{\frac{\log(1.5)(dn)}{p_{\text{obs}}}} \sqrt{\frac{k}{\log(dn)}}. \]

The Cauchy-Schwarz inequality then implies

\[ \langle \tilde{r}^\ell, g_k \rangle \geq \langle r^*, g_k \rangle - \sqrt{k} \| \tilde{r}^\ell - r^* \|_2 \geq \frac{3}{4} \sqrt{\frac{k \log(1.5)(dn)}{p_{\text{obs}}}}, \]

where the inequality (i) also uses our earlier bound (39), thereby proving the first property. Now towards the second property, we use the condition \( \langle r^*, 1 \rangle \geq 0 \). Since the entries of \( r^* \) are arranged
in order, we have $\langle r^*, g_k \rangle \geq 0$ for every $k \in [n]$. Applying the Cauchy-Schwarz inequality yields

$$\langle \tilde{r}_\ell, g_k \rangle \geq \langle r^*, g_k \rangle - \sqrt{k} \| \tilde{r}_\ell - r^* \|_2 \geq -\frac{1}{4} \sqrt{\frac{k \log^{1.5} (dn)}{p_{obs}}},$$

where the inequality (ii) also uses our earlier bound \cite{39}, thereby proving the second property. This argument completes the proof of part (a) of the corollary.

5.9 Proof of Theorem 4(b)

The proof of this part follows on lines similar to that of Theorem 1(b). The Gilbert-Varshamov bound \cite{Gil52, Var57} guarantees existence of a set of $\beta$ vectors, $x^1, \ldots, x^\beta \in \{-1, 1\}^d$ such that the normalized Hamming distance \cite{1} between any pair of vectors in this set is lower bounded as

$$d_H(x^\ell, x^{\ell'}) \geq \frac{1}{4}, \quad \text{for every } \ell, \ell' \in [\beta],$$

where $\beta = \exp(c_1 d)$ for some constant $c_1 > 0$. For each $\ell \in [\beta]$, let $\mathbb{P}_\ell$ denote the probability distribution of $Y$ induced by setting $x^* = x^\ell$. When $Q^* = q_{DS}1^T$ for some $q_{DS} \in [\frac{1}{2}, \frac{9}{10}]^n$, we have the following upper bound on the Kullback-Leibler divergence between any pair of distributions $\ell \neq \ell' \in [\beta]$:

$$D_{KL}(\mathbb{P}_\ell \| \mathbb{P}_{\ell'}) \leq 25 p_{obs} d \left\| q_{DS} - \frac{1}{2} \right\|_2^2 \leq 25 cd,$$

where we have used the assumption $\left\| q_{DS} - \frac{1}{2} \right\|_2^2 \leq \frac{c}{p_{obs}}$. Putting the above observations together into Fano’s inequality \cite{CTT2} yields a lower bound on the expected value of the normalized Hamming error \cite{1} for any estimator $\hat{x}$ as:

$$\mathbb{E}[d_H(\hat{x}, x^*)] \geq \frac{1}{8} \left( 1 - \frac{25 cd + \log 2}{c_1 d} \right) \overset{(i)}{\geq} \frac{1}{10},$$

as claimed, where inequality (i) results from setting the value of $c$ as a small enough positive constant.

6 Discussion

We proposed a flexible permutation-based model for the noise in crowdsourced labels, and by establishing fundamental theoretical guarantees on estimation, we showed that this model allows for robust and statistically efficient estimation of the true labels in comparison to the popular Dawid-Skene model. We hope that this win-win feature of the permutation-based model will encourage researchers and practitioners to further build on the permutation-based core of this model. In addition, we proposed a new metric for theoretical evaluation of algorithms for this problem that eliminates drawbacks of the Hamming metric used in prior works. Using our approach towards estimation under such a general class, we proposed a robust estimator, OBI-WAN, that unlike the estimators in prior literature, has optimal uniform guarantees over the entire Dawid-Skene model.
In more general settings, the OB-I-WAN estimator is uniformly optimal over the class \( C_{\text{Int}} \) that is richer than the Dawid-Skene model, and is uniformly consistent over the entire permutation-based model. The problems of establishing optimal minimax risk under the permutation-based model for computationally-efficient estimators, and of extending this work to settings with multiple (more than two) choice questions, remain open.

Acknowledgements

This work was partially supported by Office of Naval Research MURI grant DOD-002888, Air Force Office of Scientific Research Grant AFOSR-FA9550-14-1-001, Office of Naval Research grant ONR-N00014, as well as National Science Foundation Grant CIF-31712-23800. The work of NBS was also supported in part by a Microsoft Research PhD fellowship. We thank the authors of the paper [ZCZJ14] for sharing their implementation of their Spectral-EM algorithm.

References

[Ber24] S. Bernstein. On a modification of Chebyshev’s inequality and of the error formula of Laplace. *Ann. Sci. Inst. Sav. Ukraine, Sect. Math.*, 1(4):38–49, 1924.

[BvH14] A. S. Bandeira and R. van Handel. Sharp nonasymptotic bounds on the norm of random matrices with independent entries. *arXiv preprint arXiv:1408.6185*, 2014.

[CT12] T. M. Cover and J. A. Thomas. *Elements of information theory*. John Wiley & Sons, 2012.

[DDKR13] N. Dalvi, A. Dasgupta, R. Kumar, and V. Rastogi. Aggregating crowdsourced binary ratings. In *Conference on World Wide Web*, pages 285–294, 2013.

[DS79] A. Dawid and A. Skene. Maximum likelihood estimation of observer error-rates using the EM algorithm. *Applied Statistics*, pages 20–28, 1979.

[EdV11] C. Eickhoff and A. de Vries. How crowdsourcable is your task. In *Crowdsourcing for search and data mining*, 2011.

[Fel43] W. Feller. Generalization of a probability limit theorem of Cramér. *Transactions of the American Mathematical Society*, 54(3):361–372, 1943.

[GFK15] U. Gadiraju, B. Fetahu, and R. Kawase. Training workers for improving performance in crowdsourcing microtasks. In *Design for Teaching and Learning in a Networked World*. 2015.

[Gil52] E. N. Gilbert. A comparison of signalling alphabets. *Bell System Technical Journal*, 31(3):504–522, 1952.

[GKDD15] U. Gadiraju, R. Kawase, S. Dietze, and G. Demartini. Understanding malicious behavior in crowdsourcing platforms: The case of online surveys. In *ACM Conference on Human Factors in Computing Systems*, 2015.
[GKM11] A. Ghosh, S. Kale, and P. McAfee. Who moderates the moderators?: Crowdsourcing abuse detection in user-generated content. In *ACM conference on Electronic commerce*, 2011.

[GLZ16] C. Gao, Y. Lu, and D. Zhou. Exact exponent in optimal rates for crowdsourcing. In *International Conference on Machine Learning (ICML)*, 2016.

[GZ13] C. Gao and D. Zhou. Minimax optimal convergence rates for estimating ground truth from crowdsourced labels. *arXiv preprint arXiv:1310.5764*, 2013.

[KO16] A. Khetan and S. Oh. Reliable crowdsourcing under the generalized Dawid-Skene model. *arXiv preprint arXiv:1602.03481*, 2016.

[KOS11a] D. Karger, S. Oh, and D. Shah. Budget-optimal crowdsourcing using low-rank matrix approximations. In *Annual Allerton Conference on Communication, Control, and Computing*, 2011.

[KOS11b] D. Karger, S. Oh, and D. Shah. Iterative learning for reliable crowdsourcing systems. In *Advances in neural information processing systems*, 2011.

[KR05] T. Klein and E. Rio. Concentration around the mean for maxima of empirical processes. *The Annals of Probability*, 33(3):1060–1077, 2005.

[LPI12] Q. Liu, J. Peng, and A. T. Ihler. Variational inference for crowdsourcing. In *Advances in Neural Information Processing Systems*, pages 692–700, 2012.

[MV01] J. Matoušek and J. Vondrák. The probabilistic method. *Lecture Notes, Department of Applied Mathematics, Charles University, Prague*, 2001.

[SBGW16] N. B. Shah, S. Balakrishnan, A. Guntuboyina, and M. J. Wainwright. Stochastically transitive models for pairwise comparisons: Statistical and computational issues. In *International Conference on Machine Learning*, 2016.

[SS90] G. Stewart and J.-G. Sun. Matrix perturbation theory, 1990.

[SZ15] N. B. Shah and D. Zhou. Double or nothing: Multiplicative incentive mechanisms for crowdsourcing. In *Advances in Neural Information Processing Systems*, 2015.

[Var57] R. Varshamov. Estimate of the number of signals in error correcting codes. In *Dokl. Akad. Nauk SSSR*, 1957.

[YKL11] M.-C. Yuen, I. King, and K.-S. Leung. A survey of crowdsourcing systems. In *IEEE International Conference on Social Computing*, 2011.

[ZCZJ14] Y. Zhang, X. Chen, D. Zhou, and M. I. Jordan. Spectral methods meet EM: A provably optimal algorithm for crowdsourcing. In *Advances in neural information processing systems*, pages 1260–1268, 2014.

[ZLP+15] D. Zhou, Q. Liu, J. C. Platt, C. Meek, and N. B. Shah. Regularized minimax conditional entropy for crowdsourcing. *arXiv preprint arXiv:1503.07240*, 2015.
APPENDICES

We prove the various auxiliary results claimed in the main text.

A Auxiliary results for Theorem 1

In this appendix, we collect the proofs of Lemmas 1 and 2, used in the proof of Theorem 1.

A.1 Proof of Lemma 1

Our proof of this lemma closely follows along the lines of the proof of a related result in the paper [SBGW16]. Denote the error in the estimate as \( \hat{\Delta} := \bar{V} - V^* \). Then from the inequality (24), have

\[
\frac{1}{2} \|\hat{\Delta}\|_F^2 \leq \frac{1}{p_{\text{obs}}} \langle \langle W, \hat{\Delta} \rangle \rangle.
\] (40)

For the quadruplet \((\pi, \sigma, x, V^*)\) under consideration, define the set

\[
V_{\text{DIFF}}(\pi, \sigma, x, V^*) := \left\{ \alpha (V - V^*) \mid V = (2Q - 11^T) \text{diag}(x), \; Q \in \text{Perm}(\pi, \sigma), \; \alpha \in [0, 1] \right\}.
\]

Since the terms \(\pi, \sigma, x\) and \(V^*\) are fixed for the purposes of this proof, we will use the abbreviated notation \(V_{\text{DIFF}}\) for \(V_{\text{DIFF}}(\pi, \sigma, x, V^*)\).

For each choice of radius \(t > 0\), define the random variable

\[
Z(t) := \sup_{D \in V_{\text{DIFF}}, \|D\|_F \leq t} \frac{1}{p_{\text{obs}}} \langle \langle D, W \rangle \rangle.
\] (41a)

Using the basic inequality (40), the Frobenius norm error \(\|\hat{\Delta}\|_F\) then satisfies the bound

\[
\frac{1}{2} \|\hat{\Delta}\|_F^2 \leq \frac{1}{p_{\text{obs}}} \langle \langle W, \hat{\Delta} \rangle \rangle \leq Z(\|\hat{\Delta}\|_F).
\] (41b)

Thus, in order to obtain a high probability bound, we need to understand the behavior of the random quantity \(Z(t)\).

One can verify that the set \(V_{\text{DIFF}}\) is star-shaped, meaning that \(\alpha D \in V_{\text{DIFF}}\) for every \(\alpha \in [0, 1]\) and every \(D \in V_{\text{DIFF}}\). Using this star-shaped property, we are guaranteed that there is a non-empty set of scalars \(\delta_0 > 0\) satisfying the critical inequality

\[
E[Z(\delta_0)] \leq \frac{\delta_0^2}{2}.
\] (41c)
Our interest is in an upper bound to the smallest (strictly) positive solution $\delta_0$ to the critical inequality (41c), and moreover, our goal is to show that for every $t \geq \delta_0$, we have $\|\hat{\Delta}\|_F \leq c\sqrt{t\delta_0}$ with high probability.

Define a “bad” event

$$A_t := \left\{ \exists \Delta \in \mathbb{V}_{\text{diff}} \mid \|\Delta\|_F \geq \sqrt{t\delta_0} \text{ and } \frac{1}{p_{\text{obs}}} \langle \Delta, W \rangle \geq 2\|\Delta\|_F \sqrt{t\delta_0} \right\}. \quad (42)$$

Using the star-shaped property of $\mathbb{V}_{\text{diff}}$, it follows by a rescaling argument that

$$\mathbb{P}[A_t] \leq \mathbb{P}[Z(\delta_0) \geq 2\delta_0 \sqrt{t\delta_0}] \quad \text{for all } t \geq \delta_0.$$ 

The following lemma helps control the behavior of the random variable $Z(\delta_0)$.

**Lemma 6.** For any $\delta > 0$, the mean is upper bounded as

$$\mathbb{E}[Z(\delta)] \leq c_1 \frac{n + d}{p_{\text{obs}}} \log^2(nd), \quad (43a)$$

and for every $u > 0$, its tail probability is bounded as

$$\mathbb{P}\left(Z(\delta) > \mathbb{E}[Z(\delta)] + u\right) \leq \exp\left(\frac{-c_2 u^2 p_{\text{obs}}}{\delta^2 + \mathbb{E}[Z(\delta)] + \delta_0 \sqrt{t\delta_0}}\right), \quad (43b)$$

where $c_1$ and $c_2$ are positive universal constants.

See Appendix A.3 for the proof of this lemma.

Setting $u = \delta_0 \sqrt{t\delta_0}$ in the tail bound (43b), we find that

$$\mathbb{P}\left(Z(\delta_0) > \mathbb{E}[Z(\delta_0)] + \delta_0 \sqrt{t\delta_0}\right) \leq \exp\left(\frac{-c_2 (\delta_0 \sqrt{t\delta_0})^2 p_{\text{obs}}}{\delta_0^2 + \mathbb{E}[Z(\delta_0)] + \delta_0 \sqrt{t\delta_0}}\right), \quad \text{for all } t > 0.$$ 

By the definition of $\delta_0$ in (41c), we have $\mathbb{E}[Z(\delta_0)] \leq \delta_0^2 \leq \delta_0 \sqrt{t\delta_0}$ for any $t \geq \delta_0$, and with these relations we obtain the bound

$$\mathbb{P}[A_t] \leq \mathbb{P}[Z(\delta_0) \geq 2\delta_0 \sqrt{t\delta_0}] \leq \exp\left(-\frac{c_2}{3} \delta_0 \sqrt{t\delta_0 p_{\text{obs}}}\right), \quad \text{for all } t \geq \delta_0.$$ 

Consequently, either $\|\hat{\Delta}\|_F \leq \sqrt{t\delta_0}$, or we have $\|\hat{\Delta}\|_F > \sqrt{t\delta_0}$. In the latter case, conditioning on the complement $A_t^c$, our basic inequality implies that $\frac{1}{2}\|\hat{\Delta}\|_F^2 \leq 2\|\hat{\Delta}\|_F \sqrt{t\delta_0}$ and hence $\|\hat{\Delta}\|_F \leq 4\sqrt{t\delta_0}$. Putting together the pieces yields that

$$\mathbb{P}(\|\hat{\Delta}\|_F \leq 4\sqrt{t\delta_0}) \geq 1 - \exp\left(-\frac{c_2}{3} \delta_0 \sqrt{t\delta_0 p_{\text{obs}}}\right), \quad \text{valid for all } t \geq \delta_0. \quad (44)$$

Finally, from the bound on the expected value of $Z(t)$ in Lemma 6, we see that the critical inequality (41c) is satisfied for $\delta_0 = \sqrt{\frac{2c_1(n+d)}{p_{\text{obs}}} \log(nd)}$. Setting $t = \delta_0 = \sqrt{\frac{2c_1(n+d)}{p_{\text{obs}}} \log(nd)}$ in equation (44) yields the claimed result.
A.2 Proof of Lemma 2

Consider any four scalars \(a_1 \geq 0, a_2 \geq 0, b_1 \in \{-1, 1\}\) and \(b_2 \in \{-1, 1\}\). If \(b_1 = b_2\) then
\[ (a_1 b_1 - a_1 b_2)^2 = 0 \leq (a_1 b_1 - a_2 b_2)^2. \]
Otherwise we have \(b_1 = -b_2\). In this case, since \(a_1\) and \(a_2\) have the same sign,
\[ (a_1 b_1 - a_2 b_2)^2 \geq (a_1 b_1)^2 = \frac{1}{4} (a_1 b_1 - a_1 b_2)^2. \]
The two results above in conjunction yield the inequality \((a_1 (b_1 - b_2))^2 \leq 4 (a_1 b_1 - a_2 b_2)^2\). Applying
the above argument to each entry of the matrices \(A_1 \text{diag}(v_1 - v_2)\) and \((A_1 \text{diag}(v_1) - A_2 \text{diag}(v_2))\)
yields the claim.

A.3 Proof of Lemma 6

We need to prove the upper bound (43a) on the mean, as well as the tail bound (43b).

**Upper bounding the mean:** We upper bound the mean by using Dudley’s entropy integral, as well as some auxiliary results on metric entropy. Given a set \(C\) equipped with a metric \(\rho\) and a tolerance parameter \(\epsilon \geq 0\), we let \(\log N(\epsilon, C, \rho)\) denote the \(\epsilon\)-metric entropy of the class \(C\) in the
metric \(\rho\).

With this notation, the truncated form of Dudley’s entropy integral inequality yields
\[ \mathbb{E}[\tilde{Z}] \leq \frac{1}{\rho_{\text{obs}}} \left\{ d^{-8} + \int_{\frac{1}{2} d^{-9}}^{2 \sqrt{nd}} \sqrt{\log N(\epsilon, \mathbb{V}_{\text{diff}}, \|\cdot\|_F)(\Delta \epsilon)} \right\}. \] (45)
The upper limit of \(2 \sqrt{nd}\) in the integration is due to the fact \(\|D\|_F \leq 2 \sqrt{nd}\) for every \(D \in \mathbb{V}_{\text{diff}}\).

It is known [SBGW16] that the metric entropy of the set \(\mathbb{V}_{\text{diff}}\) is upper bounded as
\[ \log N(\epsilon, \mathbb{V}_{\text{diff}}, \|\cdot\|_F) \leq 8 \frac{\max\{n, d\}^2}{\epsilon^2} \left( \log \frac{\max\{n, d\}}{\epsilon} \right)^2 \quad \text{for each } \epsilon > 0. \]
Combining this upper bound with the Dudley entropy integral (45), and observing that the integration has \(\epsilon \geq \frac{1}{2} d^{-9}\), the claimed upper bound (43a) follows.

**Bounding the tail probability of \(Z(\delta)\):** In order to establish the claimed tail bound (43b), we use a Bernstein-type bound on the supremum of empirical processes due to Klein and Rio [KR03, Theorem 1.1c]. In particular, this result applies to a random variable of the form \(X^\dagger = \sup_{v \in \mathbb{V}} \langle X, v \rangle\),
where \(X = (X_1, \ldots, X_m)\) is a vector of independent random variables taking values in \([-1, 1]\), and \(\mathbb{V}\) is some subset of \([-1, 1]^m\). Their theorem guarantees that for any \(t > 0\),
\[ \mathbb{P}(X^\dagger > \mathbb{E}[X^\dagger] + t) \leq \exp \left( -t^2 \frac{1}{2 \sup_{v \in \mathbb{V}} \mathbb{E}[\langle v, X \rangle^2] + 4 \mathbb{E}[X^\dagger] + 3t} \right). \] (46)
In our setting, we apply this tail bound with the choices

\[ X = W, \quad \text{and} \quad X^\dagger = \sup_{D \in \mathcal{V}_{\text{diff}}, \|V_{\text{diff}}\|_F \leq \delta} \langle D, W \rangle = p_{\text{obs}} Z(\delta). \]

The entries of the matrix \( W \) are independently distributed with a mean of zero and a variance of at most \( 4p_{\text{obs}} \), and are bounded in absolute value by 1. As a result, we have \( \mathbb{E}[\|D\|_F^2] \leq 4p_{\text{obs}} \delta^2 \) for every \( D \in \mathcal{V}_{\text{diff}} \). With these assignments, inequality (16) guarantees that

\[ \mathbb{P}(p_{\text{obs}} Z(\delta) > p_{\text{obs}} \mathbb{E}[Z(\delta)] + p_{\text{obs}} u) \leq \exp \left( \frac{-u p_{\text{obs}}^2}{8 p_{\text{obs}} \delta^2 + 4p_{\text{obs}} \mathbb{E}[Z(\delta)] + 3up_{\text{obs}}} \right) \]

for all \( u > 0 \), and some algebraic simplifications yield the claimed result.

**B Auxiliary results for Theorem 2**

In this appendix, we prove Lemmas 3 and 4, both of which were used in the proof of Theorem 2.

**B.1 Proof of Lemma 3**

Observe that the windowing step of the WAN estimator identifies a group of \( k_{\text{wan}} \) workers such that their aggregate responses towards questions are biased (towards either answer \( \{-1, 1\} \)) by at least \( \sqrt{k_{\text{wan}} p_{\text{obs}} \log^{1.5}(dn)} \). We first derive three properties associated with having such a bias. These properties involve the function \( \gamma_{\pi} : [n] \times [d] \times \{\{-1, 1\}\} \to \mathbb{R} \), where \( \gamma_{\pi}(k, j, x) \) represents the amount of bias in the responses of the top \( k \in [n] \) workers for question \( j \in [d] \) towards the answer \( x \in \{-1, 1\} \):

\[ \gamma_{\pi}(k, j, x) := \sum_{i=1}^{k} \left( 1\{Y_{\pi-1(i)j} = x\} - 1\{Y_{\pi-1(i)j} = -x\} \right) = x \sum_{i=1}^{k} Y_{\pi-1(i)j}. \]

A straightforward application of the Bernstein inequality [Ber24], using the fact that the entries of the observed matrix \( Y \) are all independent, with moments bounded as

\[ \mathbb{E}[Y_{ij}] = 2p_{\text{obs}}(Q_{ij}^* - \frac{1}{2})x_j^*, \quad \text{and} \quad \mathbb{E}[Y_{ij}^2] = p_{\text{obs}}, \]

ensures that all three properties stated below are satisfied with probability at least \( 1 - e^{c \log^{1.5}(dn)} \) for every question \( j \in [d] \) and every \( k \in \{p_{\text{obs}}^{-1} \log^{1.5}(dn), \ldots, n\} \). For the remainder of the proof we work conditioned on the event where the following properties hold:

(P1) **Sufficient condition for bias towards correct answer:** If \( \sum_{i=1}^{k}(Q_{\pi-1(i)j}^* - \frac{1}{2}) \geq \frac{3}{4} \sqrt{\frac{k \log^{1.5}(dn)}{p_{\text{obs}}}} \), then \( \gamma_{\pi}(k, j, x_j^*) \geq \sqrt{kp_{\text{obs}} \log^{1.5}(dn)} \).

(P2) **Necessary condition for bias towards any answer \( x \in \{-1, 1\} \):** \( \gamma_{\pi}(k, j, x) \geq \sqrt{kp_{\text{obs}} \log^{1.5}(dn)} \)

only if \( x = x_j^* \) and \( \sum_{i=1}^{k}(Q_{\pi-1(i)j}^* - \frac{1}{2}) \geq \frac{1}{4} \sqrt{\frac{k \log^{1.5}(dn)}{p_{\text{obs}}}} \).
(P3) Sufficient condition for aggregate to be correct: If \( \sum_{i=1}^{k} (Q_{\pi^{-1}(i)}^{\ast} j - \frac{1}{2}) \geq \frac{3}{4} \sqrt{\frac{k p_{\text{obs}}}{\log^{1.5}(d n)}} \), then \( \gamma_{\pi}(k, j, x_{j}^{\ast}) > 0 \).

We now show that when these three properties hold, for any question \( j_{0} \in J \), we must have that \( [\hat{x}_{\text{wan}}(\pi)]_{j_{0}} = x_{j_{0}}^{\ast} \). In particular, we do so by exhibiting a question that is at least as hard as \( j_{0} \) on which the WAN estimator is definitely correct, and use the above properties to conclude that it therefore must also be correct on the question \( j_{0} \).

Recall that by the definition \([28]\) of \( J \), for any question \( j_{0} \in J \), it must be the case that there exists a \( k_{j_{0}} \geq p_{\text{obs}}^{-1} \log^{1.5}(d n) \) such that

\[
\sum_{i=1}^{k_{j_{0}}} (Q_{\pi^{-1}(i)}^{\ast} j_{0} - \frac{1}{2}) \geq \frac{3}{4} \sqrt{\frac{k_{j_{0}} p_{\text{obs}}}{\log^{1.5}(d n)}}. \tag{47}
\]

We define an associated set \( J_{0} \) as the set of questions that are at least as easy as question \( j_{0} \) according to the underlying permutation \( \sigma^{\ast} \), that is,

\[
J_{0} := \{ j \in [d] \mid \sigma^{\ast}(j) \leq \sigma^{\ast}(j_{0}) \}.
\]

By the monotonicity of the columns of \( Q^{\ast} \), every question in \( J_{0} \) also satisfies condition \([47]\). For each positive integer \( k \), define the set

\[
J(k) := \{ j \in [d] \mid \gamma_{\pi}(k, j, x) \geq \sqrt{k p_{\text{obs}} \log^{1.5}(d n)} \text{ for some } x \in \{-1, 1\} \}.
\]

Property \([P1]\) ensures that every question in the set \( J_{0} \) is also in the set \( J(k_{j_{0}}) \). We then have

\[
|J(k_{\text{wan}})|_{(i)} \geq |J(k_{j_{0}})| \geq |J_{0}|,
\]

where step (i) uses the optimality of \( k_{\text{wan}} \) for the optimization problem in equation \([9a]\). Given this, there are two possibilities: either (1) we have the equality \( J(k_{\text{wan}}) = J_{0} \), or (2) the set \( J(k_{\text{wan}}) \) contains some question not in the set \( J_{0} \). We address each of these possibilities in turn.

**Case 1:** It suffices to observe by Properties \([P2]\) and \([P3]\) that the aggregate of the top \( k_{\text{wan}} \) workers is correct on every question in the set \( J(k_{\text{wan}}) \) and this implies that it must be the case that \( [\hat{x}_{\text{wan}}(\pi)]_{j_{0}} = x_{j_{0}}^{\ast} \) as desired.

**Case 2:** In this case, there is some question \( j' \notin J_{0} \) such that \( \gamma_{\pi}(k, j, x) \geq \sqrt{k_{\text{wan}} p_{\text{obs}} \log^{1.5}(d n)} \) for some \( x \in \{-1, 1\} \). Property \([P2]\) guarantees that \( \sum_{i=1}^{k_{\text{wan}}} (Q_{\pi^{-1}(i)}^{\ast} j' - \frac{1}{2}) \geq \frac{1}{4} \sqrt{\frac{k_{\text{wan}} \log^{1.5}(d n)}{p_{\text{obs}}}} \) and that \( x = x_{j'}^{\ast} \). Now, since every question easier than \( j_{0} \) is in the set \( J_{0} \), question \( j' \) must be more difficult than \( j_{0} \), which implies that

\[
\sum_{i=1}^{k_{\text{wan}}} (Q_{\pi^{-1}(i)j_{0}}^{\ast} - \frac{1}{2}) \geq \frac{1}{4} \sqrt{\frac{k_{\text{wan}} \log^{1.5}(d n)}{p_{\text{obs}}}}.
\]

Applying Property \([P3]\) we can then conclude that \( [\hat{x}_{\text{wan}}(\pi)]_{j_{0}} = x_{j_{0}}^{\ast} \) as desired.
B.2 Proof of Lemma 4

We partition the proof into two cases depending on the value of $\|v\|_2^2$.

**Case 1:** First, suppose that $\frac{1}{2}\|v\|_2^2 \geq e$. In this case, we proceed via proof by contradiction. If the claim were false, then we would have

$$\sqrt{\frac{\alpha \|v\|_2^2}{2 \log n}} > \sum_{i=1}^\alpha v_i \geq \alpha v_\alpha \quad \text{for every } \alpha \geq \lceil \frac{1}{2} \|v\|_2^2 \rceil.$$  

It would then follow that

$$\sum_{i=1}^n v_i^2 = \sum_{i=\lceil \frac{1}{2} \|v\|_2^2 \rceil}^{\lceil \frac{1}{2} \|v\|_2^2 \rceil - 1} v_i^2 + \sum_{i=\lceil \frac{1}{2} \|v\|_2^2 \rceil}^n v_i^2 \leq \left\lceil \frac{1}{2} \|v\|_2^2 \right\rceil - 1 + \sum_{i=\lceil \frac{1}{2} \|v\|_2^2 \rceil}^n v_i^2 < \frac{1}{2} \|v\|_2^2 + \sum_{i=\lceil \frac{1}{2} \|v\|_2^2 \rceil}^n \frac{\|v\|_2^2}{2i \log n},$$

where step (i) uses the fact that $v_i \in [0, 1]$. Using the standard bound $\sum_{i=a}^b \frac{1}{i} \leq \log \left( \frac{b}{a} \right)$ and the assumption $\lceil \frac{1}{2} \|v\|_2^2 \rceil \geq e$, we find that

$$\frac{1}{2} \|v\|_2^2 + \sum_{i=\lceil \frac{1}{2} \|v\|_2^2 \rceil}^n \frac{\|v\|_2^2}{2i \log n} \leq \|v\|_2^2.$$ 

The resulting chain of inequalities contradicts the definition of $\|v\|_2^2$.

**Case 2:** Otherwise, we may assume that $\frac{1}{2} \|v\|_2^2 < e$. Observe that the case $v = 0$ trivially satisfies the claim with $\alpha = 1$, and hence we restrict attention to non-zero vectors. Define a vector $v' \in [0, 1]^n$ as

$$v' = \frac{1}{v_1} v.$$ 

We first prove the claim of the lemma for the vector $v'$, that is, we prove that there exists some value $\alpha \geq \lceil \frac{1}{2} \|v'\|_2^2 \rceil$ such that

$$\sum_{i=1}^\alpha v'_i \geq \sqrt{\frac{\alpha \|v'\|_2^2}{2 \log n}}.$$  

(48)

Observe that $1 = v'_1 \geq \cdots \geq v'_n \geq 0$. If $\frac{1}{2} \|v'\|_2^2 \geq e$, then our claim (48) is proved via the analysis of Case 1 above. Otherwise, we have that $\frac{1}{2} \|v'\|_2^2 \leq e$ and $v'_1 = 1$. Setting $\alpha = 1$, we obtain the inequalities

$$\sum_{i=1}^\alpha v'_i = 1 \quad \text{and} \quad \sqrt{\frac{\alpha \|v'\|_2^2}{2 \log n}} \leq 1,$$

where we have used the assumption that $n$ is large enough (concretely, $n \geq 16$). We have thus proved the bound (48), and it remains to translate this bound on $v'$ to an analogous bound on the
vector $v$. Observe that since $v_1 \leq 1$, we have the relation $\|v'\|_2 \geq \|v\|_2$. Using the same value of $\alpha$ as that derived for vector $v'$, we then obtain from (48) that this value
\[
\alpha \geq \left\lceil \frac{1}{2} \|v'\|_2^2 \right\rceil \geq \left\lceil \frac{1}{2} \|v\|_2^2 \right\rceil
\]
satisfies
\[
v_1 \sum_{i=1}^{\alpha} v'_i \geq v_1 \sqrt{\frac{\alpha \|v'\|_2^2}{2 \log n}},
\]
which establishes the claim.

C  Auxiliary results for Theorem 3

In this section, we collect the proofs of various lemmas used in the proof of Theorem 3.

C.1  Proof of Lemma 5

The proof of this lemma consists of three main steps:

(i) First, we show that $u_\ell$ is a good approximation for the vector of worker abilities $r^*$ up to a global sign.

(ii) We then show that the global sign is correctly identified with high probability.

(iii) The final step in the proof is to convert this guarantee to one on the permutation induced by $u_\ell$.

C.1.1  Step 1

We first show that the vector $u_\ell$ approximates $r^*$ up to a global sign. Assume without loss of generality that $x^*_j = 1$ for every question $j \in [d]$. As in the proof of Theorem 1(a), we begin by rewriting the model in a “linearized” fashion which is convenient for our analysis. Let $Q_0^*$ and $Q_1^*$ denote the submatrices of $Q^*$ obtained by splitting its columns according to the sets $T_0$ and $T_1$. Then we have for $\ell \in \{0, 1\}$,
\[
\frac{1}{p_{\text{obs}}} Y_\ell = (2Q_\ell^* - 11^T) \text{diag}(x^*) + \frac{1}{p_{\text{obs}}} W_\ell,
\]
where conditioned on $T_0$ and $T_1$, the noise matrices $W_0, W_1 \in \mathbb{R}^{n \times d}$ have entries independently drawn from the distribution $\mathcal{N}(0, 1)$. One can verify that the entries of $W_0$ and $W_1$ have a mean of zero, second moment upper bounded by $4p_{\text{obs}}$, and their absolute values are upper bounded by 2.

We now require a standard result on the perturbation of eigenvectors of symmetric matrices [SS90]. Consider a symmetric and positive semidefinite matrix $M \in \mathbb{R}^{d \times d}$, a second symmetric matrix $\Delta M \in \mathbb{R}^{d \times d}$, and let $\tilde{M} = M + \Delta M$. Let $v \in \mathbb{R}^d$ be an eigenvector associated to the largest eigenvalue of $M$. Likewise define $\tilde{v} \in \mathbb{R}^d$ as an eigenvector associated to the largest eigenvalue of $\tilde{M}$. Then we are guaranteed [SS90] that
\[
\min\{\|\tilde{v} - v\|_2, \|\tilde{v} + v\|_2\} \leq \frac{2\|\Delta M\|_{\text{op}}}{\max\{\lambda_1(M) - \lambda_2(M) - 2\|\Delta M\|_{\text{op}}, 0\}},
\]
where $\|\cdot\|_{\text{op}}$ denotes the operator norm of a matrix.

37
where $\lambda_1(M)$ and $\lambda_2(M)$ denote the largest and second largest eigenvalues of $M$, respectively.

In order to apply the bound (50), we define the matrix $R_\ell^*: = Q_\ell - \frac{1}{2}11^T$, as well as the matrices

$$\tilde{M} := \frac{1}{p_{\text{obs}}} Y_\ell Y_\ell^T, \quad M = 4R_\ell^* (R_\ell^*)^T,$$

and

$$\Delta M := \frac{2}{p_{\text{obs}}} W_\ell (R_\ell^*)^T + \frac{2}{p_{\text{obs}}} R_\ell^* W_\ell + \frac{1}{p_{\text{obs}}} W_\ell W_\ell^T.$$

Using our linearized observation model (49), it is straightforward to verify that these choices satisfy the condition $\tilde{M} = M + \Delta M$, so that the bound (50) can be applied.

Recall that for any matrix $Q^* \in \mathbb{C}_{\text{Int}}$, we have $Q^* = \tilde{q}(1 - h^*)^T + \frac{1}{2}(h^*)^T$ for some vectors $\tilde{q} \in [\frac{1}{2}, 1]^n$ and $h^* \in [0, 1]^d$. Also recall our definition of the associated quantity $r^* \in [0, \frac{1}{2}]^n$ as $r^* = \tilde{q} - \frac{1}{2}$. We denote the magnitude of the vector $r^*$ as $\rho := \|r^*\|_2$.

With the notation introduced above, we are ready to apply the bound (50). First observe that the matrix $R_\ell^*$ has a rank of one, and consequently $\|R_\ell^*\|_{\text{op}} = \rho \sqrt{\sum_{j \in T_\ell} (1 - h_j^*)^2}$. Conditioned on the bound (32), we obtain

$$\sqrt{\frac{T}{3}} \rho \|1 - h^*\|_2 \leq \|R_\ell^*\|_{\text{op}} \leq \sqrt{\frac{2}{3}} \rho \|1 - h^*\|_2.$$

Moreover, the entries of the matrix $W_\ell$ are independent, zero-mean, and have a second moment upper bounded by $4p_{\text{obs}}$. Consequently, known results on random matrices [BvH14, Remark 3.13] guarantee that

$$\|W_\ell\|_{\text{op}} \leq c \sqrt{\max\{d, n\} p_{\text{obs}} \log^{1.5} d} \leq c \sqrt{dp_{\text{obs}} \log^{1.5} d},$$

with probability at least $1 - e^{-c \log^{1.5} d}$, where we have used the fact that $d \geq n$ and $p_{\text{obs}} \geq \frac{1}{n}$. These inequalities, in turn, imply that the top eigenvalue of $M$ is lower bounded as $\lambda_1(M) = \|R^*\|_{\text{op}} \geq \frac{1}{3} \rho^2 \|1 - h^*\|_2^2$, the second eigenvalue vanishes (that is, $\lambda_2(M) = 0$), and moreover that

$$\|\Delta M\|_{\text{op}} \leq \frac{2}{p_{\text{obs}}} \|R^*\|_{\text{op}} \|W\|_{\text{op}} + \frac{1}{p_{\text{obs}}} \|W\|_{\text{op}}^2 \leq \frac{c' \sqrt{d \log^{1.5} d}}{p_{\text{obs}}} (\rho \|1 - h^*\|_2 \sqrt{p_{\text{obs}}} + \sqrt{d \log^{1.5} d}).$$

Recall the lower bound $\rho \|1 - h^*\|_2 \geq \sqrt{\frac{C d \log^{2.5} d}{p_{\text{obs}}}}$, assumed in the statement of the lemma. Using these facts and doing some algebra, we find that with probability at least $1 - e^{-c \log^{1.5} d}$, for any pair of sets $T_0$ and $T_1$ satisfying (32), we have the bound

$$\min\{\|u_\ell - \frac{1}{\rho} r^*\|_2, \|u_\ell + \frac{1}{\rho} r^*\|_2\} \leq \frac{1}{36} \frac{1}{\rho^2 \|1 - h_j^*\|_2^2} \frac{d \log^{1.5} d}{p_{\text{obs}}},$$

where the prefactor $\frac{1}{36}$ is obtained by setting the constant $C > 20$ to a large enough value.

C.1.2 Step 2

We now verify that the global sign is correctly identified. Recall our selection

$$\sum_{j=1}^n [u_{\ell,j}]^2 1\{[u_{\ell,j}] > 0\} \geq \sum_{j=1}^n [u_{\ell,j}]^2 1\{[u_{\ell,j}] < 0\}.$$
Since every entry of the vector \( r^* \) is non-negative, we have the inequality
\[
\|u_\ell + \frac{1}{\rho} r^*\|_2^2 \geq \sum_{j=1}^{n} [u_\ell]_j^2 1\{[u_\ell]_j > 0\} \geq \sum_{j=1}^{n} [u_\ell]_j^2 1\{[u_\ell]_j < 0\},
\]
and consequently,
\[
\|u_\ell + \frac{1}{\rho} r^*\|_2^2 \geq \frac{1}{2} \|u_\ell\|_2^2. \tag{52a}
\]
On the other hand, a version of the triangle inequality yields
\[
2\|u_\ell\|_2^2 + 2\|u_\ell + \frac{1}{\rho} r^*\|_2^2 \geq \frac{1}{\rho} r^*\|_2^2 = 1 \tag{52b}
\]
Now suppose that \( \|u_\ell - \frac{1}{\rho} r^*\|_2^2 \geq \|u_\ell + \frac{1}{\rho} r^*\|_2^2 \). Then from our earlier result (51), we have the bound
\[
\|u_\ell + \frac{1}{\rho} r^*\|_2^2 \leq \frac{d \log 1.5}{36 r^2 \|1 - h^*\|_2^2 p_{\text{obs}}}, \tag{52c}
\]
with probability at least \( 1 - e^{-c \log 1.5 (dn)} \). Putting together the inequalities (52a), (52b) and (52c) and rearranging some terms yields the inequality
\[
\rho^2 \|1 - h^*\|_2^2 \leq \frac{d \log 1.5}{9 p_{\text{obs}}}. \tag{53}
\]
This requirement contradicts our initial assumption \( \rho^2 \|1 - h^*\|_2^2 \geq \frac{Cd \log 2.5}{p_{\text{obs}}} \), with \( C > 20 \), thereby proving that \( \|u_\ell - \frac{1}{\rho} r^*\|_2^2 < \|u_\ell + \frac{1}{\rho} r^*\|_2^2 \). Substituting this inequality into equation (51) yields the bound
\[
\|u_\ell - \frac{1}{\rho} r^*\|_2^2 \leq \frac{1}{36 r^2 \|1 - h^*\|_2^2} \frac{d \log 1.5}{p_{\text{obs}}}. \tag{53}
\]

C.1.3 Step 3

The final step of this proof is to convert the approximation guarantee (53) on \( u_\ell \) to an approximation guarantee on the vector \( \tilde{r}_\ell \) (which, recall, is a permutation of \( r^* \) according to the permutation induced by \( u_\ell \)). An additional lemma is useful for this step:

**Lemma 7.** For any \( \ell \in \{0, 1\} \), we have \( \|\tilde{r}_\ell - r^*\|_2 \leq 2\|pu_\ell - r^*\|_2 \).

See Appendix C.2 for the proof of this claim.

Combining Lemma 7 with the inequality (53) yields that for any choice of the set \( T_0 \) and \( T_1 \) satisfying the condition (32), with probability at least \( 1 - e^{-c \log 1.5 d} \), we have
\[
\|\tilde{r}_\ell - r^*\|_2 \leq \frac{1}{18 \|1 - h^*\|_2^2} \frac{d \log 1.5}{p_{\text{obs}}} \leq \frac{\|r^*\|_2^2}{18 \log (dn)}.
\]
Here, inequality (i) follows from our earlier assumption that \( \|r^*\|_2 \|1 - h^*\|_2 \geq \sqrt{\frac{Cd \log 2.5}{p_{\text{obs}}}} \) with \( C > 20 \).
C.2 Proof of Lemma 7

Recall that the two vectors \( \tilde{r}_\ell \) and \( r^* \) are identical up to a permutation. Now suppose \( \tilde{r}_\ell \neq r^* \). Then there must exist some position \( i \in [n-1] \) such that \( [r^*]_i < [r^*]_{i+1} \) and \( [\tilde{r}_\ell]_i \geq [\tilde{r}_\ell]_{i+1} \). Define the vector \( \tilde{r}' \) obtained by interchanging the entries in positions \( i \) and \( (i + 1) \) in \( r^* \). The difference \( \Delta := \| \tilde{r}' - \rho u_{\ell} \|_2^2 - \| r^* - \rho u_{\ell} \|_2^2 \) then can be bounded as

\[
\Delta = (\tilde{r}'_i - \rho [u_{\ell}]_i)^2 + (\tilde{r}'_{i+1} - \rho [u_{\ell}]_{i+1})^2 - ([r^*]_i - \rho [u_{\ell}]_i)^2 - ([r^*]_{i+1} - \rho [u_{\ell}]_{i+1})^2 \\
= ([r^*]_{i+1} - \rho [u_{\ell}]_i)^2 + ([r^*]_i - \rho [u_{\ell}]_{i+1})^2 - ([r^*]_i - \rho [u_{\ell}]_i)^2 - ([r^*]_{i+1} - \rho [u_{\ell}]_{i+1})^2 \\
= 2\rho ([r^*]_{i+1} - [r^*]_i)([u_{\ell}]_{i+1} - [u_{\ell}]_i) \\
\leq 0,
\]

where the final inequality uses the fact that the ordering of the entries in the two vectors \( \tilde{r}_\ell \) and \( u_{\ell} \) are identical, which in turn implies that \( |u_{\ell}|_i \geq |u_{\ell}|_{i+1} \). We have thus shown an interchange of the entries \( i \) and \( (i + 1) \) in \( r^* \), which brings it closer to the permutation of \( \tilde{r}_\ell \), cannot increase the distance to the vector \( \rho u_{\ell} \). A recursive application of this argument leads to the inequality \( \| \tilde{r}_\ell - \rho u_{\ell} \|_2 \leq \| r^* - \rho u_{\ell} \|_2 \). Applying the triangle inequality then yields

\[
\| \tilde{r}_\ell - r^* \|_2 \leq \| \tilde{r}_\ell - \rho u_{\ell} \|_2 + \| \rho u_{\ell} - r^* \|_2 \leq 2\| \rho u_{\ell} - r^* \|_2,
\]

as claimed.

D Analysis of the majority voting estimator

In this section, we analyze the majority voting estimator, given by

\[
[\tilde{x}_{MV}]_j = \arg \max_{b \in \{-1,1\}} \sum_{i=1}^n 1 \{ Y_{ij} = b \} \quad \text{for every } j \in [d].
\]

Here we use \( 1 \{ \cdot \} \) to denote the indicator function. The following theorem provides bounds on the risk of majority voting under the \( Q^* \)-semimetric loss in the regime of interest \( (R) \).

**Proposition 1.** For the majority vote estimator, the uniform risk over the Dawid-Skene class is lower bounded as

\[
\sup_{x^* \in \{-1,1\}^d} \sup_{Q^* \in C_{DS}} \mathbb{E}[\mathcal{L}_{Q^*}(\tilde{x}_{MV}, x^*)] \geq c_L \frac{1}{\sqrt{n}p_{\text{obs}}}, \quad (54)
\]

for some positive constant \( c_L \).

A comparison of the bound (54) with the results of Theorem 1, Theorem 3(a) and Theorem 4 shows that the majority voting estimator is suboptimal in terms of the sample complexity. Since this suboptimality holds for the (smaller) Dawid-Skene model class, it also holds for the (larger) intermediate model class, as well as the permutation-based model class.

The remainder of this section is devoted to the proof of this claim.
D.1 Proof of Proposition 1

We begin with a lower bound due to Feller [Fel43] (see also [MV01, Theorem 7.3.1]) on the tail probability of a sum of independent random variables.

**Lemma 8** (Feller). *There exist positive universal constants* $c_1$ *and* $c_2$ *such that for any set of independent random variables* $X_1, \ldots, X_n$ *satisfying* $\mathbb{E}[X_i] = 0$ *and* $|X_i| \leq M$ *for every* $i \in [n]$, *if* $\sum_{i=1}^{n} \mathbb{E}[(X_i)^2] \geq c_1$ *then*

$$
\mathbb{P}(\sum_{i=1}^{n} X_i > t) \geq c_2 \exp\left(-\frac{t^2}{12 \sum_{i=1}^{n} \mathbb{E}[(X_i)^2]}\right),
$$

*for every* $t \in [0, \frac{\sum_{i=1}^{n} \mathbb{E}[(X_i)^2]}{M^2 \sqrt{c_1}}]$.

In what follows, we use Lemma 8 to derive the claimed lower bound on the error incurred by the majority voting algorithm. To this end, let $S \subset [n]$ denote the set of some $|S| = \sqrt{\frac{n}{2p_{\text{obs}}}}$ workers. Consider the following value of matrix $Q^*$:

$$
Q^*_{ij} = \begin{cases} 
1 & \text{if } i \in S \\
\frac{1}{2} & \text{otherwise.}
\end{cases}
$$

Then for any question $j \in [d]$, we have $\sum_{i=1}^{n} (2Q^*_{ij} - 1)^2 = \sqrt{\frac{n}{2p_{\text{obs}}}}$.

Now suppose that $x^*_j = -1$ for every question $j \in [d]$. Then for every $i \in S$, the observations are distributed as

$$
Y_{ij} = \begin{cases} 
0 & \text{with probability } 1 - p_{\text{obs}} \\
-1 & \text{with probability } p_{\text{obs}},
\end{cases}
$$

and for every $i \notin S$, as

$$
Y_{ij} = \begin{cases} 
0 & \text{with probability } 1 - p_{\text{obs}} \\
-1 & \text{with probability } 0.5p_{\text{obs}} \\
1 & \text{with probability } 0.5p_{\text{obs}}.
\end{cases}
$$

Consider any question $j \in [d]$. Then in this setting, the majority voting estimator incorrectly estimates the value of $x^*_j$ when $\sum_{i=1}^{n} Y_{ij} > 0$. We now use Lemma 8 to obtain a lower bound on the probability of the occurrence of this event. Some simple algebra yields

$$
\sum_{i=1}^{n} \mathbb{E}[Y_{ij}] = -|S|p_{\text{obs}} \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{E}[(Y_{ij})^2] = np_{\text{obs}}.
$$

In order to satisfy the conditions required by the lemma, we assume that $np_{\text{obs}} > c_1$. Note that this condition makes the problem strictly easier than the condition $np_{\text{obs}} \geq 1$ assumed otherwise,
and affects the lower bounds by at most a constant factor \( c_1 \). An application of Lemma 8 with \( t = -\sum_{i=1}^{n} \mathbb{E}[Y_{ij}] = |S|p_{\text{obs}} \) now yields

\[
\mathbb{P}(\sum_{i=1}^{n} Y_{ij} > 0) \geq c_2 \exp \left( \frac{-|S|^2 p_{\text{obs}}^2}{12np_{\text{obs}}} \right) \geq c',
\]

for some constant \( c' > 0 \) that may depend only on \( c_1 \) and \( c_2 \), where inequality (i) is a consequence of the choice \( |S| = \sqrt{\frac{n}{2p_{\text{obs}}}} \).

Now that we have established a constant-valued lower bound on the probability of error in the estimation of \( x_j^* \) for every \( j \in [d] \), for the value of \( Q^* \) under consideration, we have

\[
\mathbb{P}(\tilde{x}_{\text{MV}} \neq x_j^*) \sum_{i=1}^{n} (Q_{ij}^* - \frac{1}{2})^2 \geq \sqrt{\frac{n}{2p_{\text{obs}}} c'},
\]

and consequently \( \mathbb{E}[L_{Q^*}(\tilde{x}_{\text{MV}}, x^*)] \geq \frac{c'}{\sqrt{2np_{\text{obs}}}} \), as claimed.