ON SEMITOPOLOGICAL INTERASSOCIATES OF THE BICYCLIC MONOID

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ABSTRACT. Semitopological interassociates \( \mathcal{C}_{m,n} \) of the bicyclic semigroup \( \mathcal{C}(p,q) \) are studied. In particular we show that for arbitrary non-negative integers \( m, n \) and every Hausdorff topology \( \tau \) on \( \mathcal{C}_{m,n} \) such that \((\mathcal{C}_{m,n}, \tau)\) is a semitopological semigroup, is discrete and hence \( \mathcal{C}_{m,n} \) is a discrete subspace of any topological semigroup containing it. Also, we prove that if \( \mathcal{C}_{m,n} \) is any interassociate of the bicyclic monoid such that \( \mathcal{C}_{m,n} \) is a dense subsemigroup of a Hausdorff semitopological semigroup \((S, \cdot)\) and \( I = S \setminus \mathcal{C}_{m,n} \neq \emptyset \) then \( I \) is a two-sided ideal of the semigroup \( S \) and show that for arbitrary non-negative integers \( m, n \) and any Hausdorff locally compact topology \( \tau \) on the interassociate \( \mathcal{C}_{m,n} \) with an adjoined zero \( 0 \) of the bicyclic monoid \( \mathcal{C}_{0,0} \) such that \((\mathcal{C}_{0,0}, \tau)\) is a semitopological semigroup, is either discrete or compact.

We shall follow the terminology of \[9\] \[10\] \[14\] \[27\]. By \( N \) and \( \mathbb{N} \) we denote the sets of non-negative integers and positive integers, respectively. If \( A \) is a subset of a topological space \( X \) then by \( \text{cl}_X(A) \) and \( \text{int}_X(A) \) we denote the topological closure and interior of \( A \) in \( X \), respectively.

A **semigroup** is a non-empty set with a binary associative operation.

The **bicyclic semigroup** (or the **bicyclic monoid**) \( \mathcal{C}(p,q) \) is the semigroup with the identity \( 1 \) generated by two elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The bicyclic monoid \( \mathcal{C}(p,q) \) is a combinatorial bisimple \( F \)-inverse semigroup (see \[23\]) and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known O. Andersen’s result \[11\] states that a \((0-)\)simple semigroup is completely \((0-)\)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup does not embed into stable semigroups \[22\].

An interassociate of a semigroup \((S, \cdot)\) is a semigroup \((S, \ast)\) such that for all \( a, b, c \in S \), \( a \ast (b \ast c) = (a \ast b) \ast c \) and \( a \ast (b \cdot c) = (a \cdot b) \cdot c \). This definition of interassociativity was studied extensively in 1996 by Boyd et al \[8\]. Certain classes of semigroups are known to give rise to interassociates with various properties. For example, it is very easy to show that if \( S \) is a monoid, every interassociate must satisfy the condition \( a \ast b = acb \) for some fixed element \( c \in S \) (see \[8\]). This type of interassociate was called a variant by Hickey \[20\]. In addition, every interassociate of a completely simple semigroup is completely simple \[8\]. Finally, it is relatively easy to show that every interassociate of a group is isomorphic to that group.

In the paper \[10\] the bicyclic semigroup \( \mathcal{C}(p,q) \) and its interassociates are investigated. In particular, if \( p \) and \( q \) are the generators of the bicyclic semigroup \( \mathcal{C}(p,q) \) and \( m \) and \( n \) are fixed nonnegative integers, the operation \( a \ast_{m,n} b = aq^mp^n b \) is known to be an interassociate. There was shown that for distinct pairs \((m,n)\) and \((s,t)\), the interassociates \((\mathcal{C}(p,q), \ast_{m,n})\) and \((\mathcal{C}(p,q), \ast_{s,t})\) are not isomorphic. Also in \[10\] the authors generalized a result regarding homomorphisms on \( \mathcal{C}(p,q) \) to homomorphisms on its interassociates.

Later for fixed non-negative integers \( m \) and \( n \) the interassociate \((\mathcal{C}(p,q), \ast_{m,n})\) of the bicyclic monoid \( \mathcal{C}(p,q) \) we shall denote by \( \mathcal{C}_{m,n} \).

A **(semi)topological semigroup** is a topological space with a (separately) continuous semigroup operation.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup \( S \) contains it as a dense subsemigroup then \( \mathcal{C}(p,q) \) is an open subset of \( S \) \[13\]. Bertman and West in \[7\] extend this result for the case of Hausdorff semitopological semigroups. Stable and \( \Gamma \)-compact

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topological semigroups do not contain the bicyclic semigroup \([2, 21]\). The problem of an embedding of the bicyclic monoid into compact-like topological semigroups studied in \([5, 6, 19]\). Also in the paper \([15]\) proved that the discrete topology is the unique topology on the extended bicyclic semigroup \(C_0\) such that the semigroup operation on \(C_0\) is separately continuous. Amazing dichotomy for the bicyclic monoid with adjoined zero \(C^0 = C(p, q) \cup \{0\}\) was proved in \([18]\): every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero \(C^0\) is either compact or discrete.

In this paper we study semitopological interassociates \((C(p, q), \ast_{m,n})\) of the bicyclic monoid \(C(p, q)\) for arbitrary non-negative integers \(m\) and \(n\). Some results from \([7, 13, 18]\) obtained for the bicyclic semigroup are extended to its interassociate \((C(p, q), \ast_{m,n})\). In particular we show that for arbitrary non-negative integers \(m, n\) every Hausdorff topology \(\tau\) on \(C_{m,n}\) such that \((C_{m,n}, \tau)\) is a semitopological semigroup, is discrete and hence \(C_{m,n}\) is a discrete subspace of any semitopological semigroup containing it. Also, we prove that if \(C_{m,n}\) is any interassociate of the bicyclic monoid such that \(C_{m,n}\) is a dense subsemigroup of a Hausdorff semitopological semigroup \((S, \cdot)\) and \(I = S \setminus C_{m,n} \neq \emptyset\) then \(I\) is a two-sided ideal of the semigroup \(S\) and show that for arbitrary non-negative integers \(m\) and \(n\) any Hausdorff locally compact topology \(\tau\) on the interassociate \(C_{m,n}\) with an adjoined zero \(0\) of the bicyclic monoid \(C^0_{m,n}\) such that \((C_{m,n}, \tau)\) is a semitopological semigroup, is either discrete or compact.

For arbitrary \(i, j \in N\) we denote 
\[ C_{m,n}^* = \left\{ q^{n+k}p^{m+l} : k, l \in N \right\}. \]

The semigroup operation \(\ast_{m,n}\) of \(C_{m,n}\) implies that \(C_{m,n}^*\) is a subsemigroup of \(C_{m,n}\).

We need the following trivial lemma.

**Lemma 1.** For arbitrary non-negative integers \(m\) and \(n\) the subsemigroup \(C_{m,n}^*\) of \(C_{m,n}\) is isomorphic to the bicyclic semigroup \(C(p, q)\) under the map \(\iota : C(p, q) \rightarrow C_{m,n}^* : q^ip^j \mapsto q^{i+j+k}p^{m+l}, \ i, j \in N\).

**Proof.** It is sufficient to show that so defined above map \(\iota : C(p, q) \rightarrow C_{m,n}^*\) is a homomorphism, because \(\iota\) is bijective. Then for arbitrary \(i, j, k, l \in N\) we have that
\[
\iota(q^i p^j \ast q^k p^l) = \begin{cases} 
\iota(q^{i+j+k} p^{m+l}), & \text{if } j < k; \\
\iota(q^{i+j+k} p^{m+l}), & \text{if } j \geq k 
\end{cases}
\]
and
\[
\iota(q^i p^j) \ast_{m,n} \iota(q^k p^l) = q^{n+i+p^{m+l}} \ast_{m,n} q^{n+k+p^{m+l}} = q^{n+i+p^{m+l}} \ast q^{n+k+p^{m+l}} = q^{n+i+p^{m+l}} \ast q^{n+k+p^{m+l}} = q^{n+i+p^{m+l}} \ast q^{n+k+p^{m+l}} = \begin{cases} 
q^{n+i+j+k+p^{m+l}}, & \text{if } j < k; \\
q^{n+i+j+k+p^{m+l}}, & \text{if } j \geq k, 
\end{cases}
\]
which completes the proof of the lemma. \(\square\)

**Lemma 2.** For arbitrary non-negative integers \(m\) and \(n\) and for each elements \(a, b \in C_{m,n}\) the both sets 
\[ \{ x \in C_{m,n} : a \ast_{m,n} x = b \} \quad \text{and} \quad \{ x \in C_{m,n} : x \ast_{m,n} a = b \} \]
are finite; that is, left translation by \(a\) and right translation by \(a\) are finite-to-one maps.

The following theorem generalises the Eberhart–Selden result about semigroup topologization of the bicyclic semigroup (see \([13, \text{Corollary I.1}]\) and corresponding statement for the case semitopological semigroups in \([7]\).

**Theorem 3.** For arbitrary non-negative integers \(m, n\) and every Hausdorff topology \(\tau\) on \(C_{m,n}\) such that \((C_{m,n}, \tau)\) is a semitopological semigroup, is discrete. Thus \(C_{m,n}\) is a discrete subspace of any topological semigroup containing it.
Proof. By Proposition 1 of [2] every Hausdorff topology \( \tau_{\epsilon} \) on the bicyclic semigroup \( \mathcal{C}(p, q) \) such that \( (\mathcal{C}(p, q), \tau_{\epsilon}) \) is a semitopological semigroup, is discrete. Hence Lemma 1 implies that for any element \( x \in \mathcal{C}_{m,n}^* \) there exists an open neighbourhood \( U(x) \) of the point \( x \) in \( (\mathcal{C}_{m,n}, \tau) \) such that \( U(x) \cap \mathcal{C}_{m,n}^* = \{x\} \). Fix an arbitrary open neighbourhood \( U(q^n p^m) \) of the point \( q^n p^m \) in \( (\mathcal{C}_{m,n}, \tau) \) such that \( U(q^n p^m) \cap \mathcal{C}_{m,n}^* = \{q^n p^m\} \). Then the separate continuity of the semigroup operation in \( (\mathcal{C}_{m,n}, \tau) \) implies that there exists an open neighbourhood \( V(q^n p^m) \subseteq U(q^n p^m) \) of the point \( q^n p^m \) in the space \( (\mathcal{C}_{m,n}, \tau) \) such that

\[
V(q^n p^m) *_{m,n} q^n p^m \subseteq U(q^n p^m) \quad \text{and} \quad q^n p^m *_{m,n} V(q^n p^m) \subseteq U(q^n p^m).
\]

Suppose to the contrary that the neighbourhood \( V(q^n p^m) \) is an infinite set. Then at least one of the following conditions holds:

(i) there exists a non-negative integer \( i_0 < n \) such that the set \( A = \{q^{i_0}p^l : l \in \mathbb{N}\} \cap V(q^n p^m) \) is infinite;

(ii) there exists a non-negative integer \( j_0 < m \) such that the set \( B = \{q^l p^{j_0} : l \in \mathbb{N}\} \cap V(q^n p^m) \) is infinite.

In case (i) for arbitrary \( q^{i_0}p^l \in A \) we have that

\[
q^n p^m *_{m,n} q^{i_0}p^l = q^n p^m q^{i_0}p^l = q^n q^{i_0}p^{l+1}  \notin U(q^n p^m);
\]

and similar in case (ii) we obtain that

\[
q^l p^{j_0} *_{m,n} q^n p^m = q^l p^{j_0} q^n p^m = q^l q^j p^{m+1}  \notin U(q^n p^m),
\]

for each \( q^l p^{j_0} \in B \), which contradicts the separate continuity of the semigroup operation in \( (\mathcal{C}_{m,n}, \tau) \). The obtained contradiction implies that \( q^n p^m \) is an isolated point in the space \( (\mathcal{C}_{m,n}, \tau) \).

Now, since the semigroup \( \mathcal{C}_{m,n} \) is simple (see [16, Section 2]) Lemma 2 implies that the topology \( \tau \) on \( \mathcal{C}_{m,n} \) is discrete. \( \square \)

Theorem 4. If \( m \) and \( n \) are arbitrary non-negative integers, the interassociate \( \mathcal{C}_{m,n} \) of the bicyclic monoid \( \mathcal{C}(p, q) \) is a dense subsemigroup of a Hausdorff semitopological semigroup \( (S, \cdot) \) and \( I = S \setminus \mathcal{C}_{m,n} \neq \emptyset \) then \( I \) is a two-sided ideal of the semigroup \( S \).

Proof. Fix an arbitrary element \( y \in I \). If \( x \cdot y = z \notin I \) for some \( x \in \mathcal{C}_{m,n} \) then there exists an open neighbourhood \( U(y) \) of the point \( y \) in the space \( S \) such that \( \{x\} \cdot U(y) = \{z\} \subseteq \mathcal{C}_{m,n} \). The neighbourhood \( U(y) \) contains infinitely many elements of the semigroup \( \mathcal{C}_{m,n} \) which contradicts Lemma 2. The obtained contradiction implies that \( x \cdot y \in I \) for all \( x \in \mathcal{C}_{m,n} \) and \( y \in I \). The proof of the statement that \( y \cdot x \in I \) for all \( x \in \mathcal{C}_{m,n} \) and \( y \in I \) is similar.

Suppose to the contrary that \( x \cdot y = w \notin I \) for some \( x, y \in I \). Then \( w \in \mathcal{C}_{m,n} \) and the separate continuity of the semigroup operation in \( S \) implies that there exist open neighbourhoods \( U(x) \) and \( U(y) \) of the points \( x \) and \( y \) in \( S \), respectively, such that \( \{x\} \cdot U(y) = \{w\} \) and \( U(x) \cdot \{y\} = \{w\} \). Since both neighbourhoods \( U(x) \) and \( U(y) \) contain infinitely many elements of the semigroup \( \mathcal{C}_{m,n} \), both equalities \( \{x\} \cdot U(y) = \{w\} \) and \( U(x) \cdot \{y\} = \{w\} \) contradict mentioned above Lemma 2. The obtained contradiction implies that \( x \cdot y \in I \). \( \square \)

We recall that a topological space \( X \) is said to be:

- **compact** if every open cover of \( X \) contains a finite subcover;
- **countably compact** if each closed discrete subspace of \( X \) is finite;
- **finitely compact** if each locally finite open cover of \( X \) is finite;
- **pseudocompact** if \( X \) is Tychonoff and each continuous real-valued function on \( X \) is bounded;
- **locally compact** if every point \( x \) of \( X \) has an open neighbourhood \( U(x) \) with the compact closure \( \text{cl}_X(U(x)) \);
- **\( \check{\text{C}} \)ech-complete** if \( X \) is Tychonoff and there exists a compactification \( cX \) of \( X \) such that the remainder of \( X \) is an \( F_\sigma \)-set in \( cX \).
According to Theorem 3.10.22 of [14], a Tychonoff topological space $X$ is feebly compact if and only if $X$ is pseudocompact. Also, a Hausdorff topological space $X$ is feebly compact if and only if every locally finite family of non-empty open subsets of $X$ is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is feebly compact (see [4]).

A topological semigroup $S$ is called $\Gamma$-compact if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is a compactum in $S$ (see [21]). Since by Lemma 1 the semigroup $\mathcal{C}_{m,n}$ contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [5], [6], [19], [21] imply the following corollary

**Corollary 5.** Let $m$ and $n$ be arbitrary non-negative integers. If a Hausdorff topological semigroup $S$ satisfies one of the following conditions:

(i) $S$ is compact;
(ii) $S$ is $\Gamma$-compact;
(iii) the square $S \times S$ is countably compact; or
(iv) the square $S \times S$ is a Tychonoff pseudocompact space,
then $S$ does not contain the semigroup $\mathcal{C}_{m,n}$.

**Proposition 6.** Let $m$ and $n$ be arbitrary non-negative integers. Let $S$ be a Hausdorff topological semigroup which contains a dense subsemigroup $\mathcal{C}_{m,n}$. Then for every $c \in \mathcal{C}_{m,n}$ the set

$$D_c(A) = \{(x, y) \in \mathcal{C}_{m,n} \times \mathcal{C}_{m,n} : x \ast_{m,n} y = c\}$$

is an open-and-closed subset of $S \times S$.

**Proof.** By Theorem 3 $\mathcal{C}_{m,n}$ is a discrete subspace of $S$ and hence we have that $D_c(A)$ is an open subset of $S \times S$.

Suppose that there exists $c \in \mathcal{C}_{m,n}$ such that $D_c(A)$ is a non-closed subset of $S \times S$. Then there exists an accumulation point $(a, b) \in S \times S$ of the set $D_c(A)$. The continuity of the semigroup operation in $S$ implies that $a \cdot b = c$. But $\mathcal{C}_{m,n} \times \mathcal{C}_{m,n}$ is a discrete subspace of $S \times S$ and hence by Theorem 3 the points $a$ and $b$ belong to the two-sided ideal $I = S \setminus \mathcal{C}_{m,n}$ and hence $a \cdot b \in S \setminus \mathcal{C}_{m,n}$ cannot be equal to the element $c$. \hfill $\Box$

**Theorem 7.** Let $m$ and $n$ be arbitrary non-negative integers. If a Hausdorff topological semigroup $S$ contains $\mathcal{C}_{m,n}$ as a dense subsemigroup then the square $S \times S$ is not feebly compact.

**Proof.** Since the square $S \times S$ contains an infinite open-and-closed discrete subspace $D_c(A)$, we conclude that $S \times S$ fails to be feebly compact (see [14, Ex. 3.10.F(d)] or [11]). \hfill $\Box$

The following proposition generalizes Theorem I.3 from [13].

For arbitrary non-positive integers $m$ and $n$ by $\mathcal{C}_{m,n}^0$ we denote the interassociate $\mathcal{C}_{m,n}$ with an adjoined zero $0$ of the bicyclic monoid $\mathcal{C}(p, q)$, i.e., $\mathcal{C}_{m,n}^0 = \mathcal{C}_{m,n} \sqcup \{0\}$.

**Example 8.** On the semigroup $\mathcal{C}_{m,n}^0$ we define a topology $\tau_{\text{Ac}}$ in the following way:

(i) every element of the semigroup $\mathcal{C}_{m,n}$ is an isolated point in the space $(\mathcal{C}_{m,n}^0, \tau_{\text{Ac}})$;
(ii) the family $\mathcal{B}(0) = \{U \subseteq \mathcal{C}_{m,n}^0 : U \ni 0 \text{ and } \mathcal{C}_{m,n} \setminus U \text{ is finite}\}$ determines a base of the topology $\tau_{\text{Ac}}$ at zero $0 \in \mathcal{C}_{m,n}^0$.

i.e., $\tau_{\text{Ac}}$ is the topology of the Alexandroff one-point compactification of the discrete space $\mathcal{C}_{m,n}$ with the remainder $\{0\}$. The semigroup operation in $(\mathcal{C}_{m,n}^0, \tau_{\text{Ac}})$ is separately continuous, because all elements of the interassociate $\mathcal{C}_{m,n}$ of the bicyclic semigroup $\mathcal{C}(p, q)$ are isolated points in the space $(\mathcal{C}_{m,n}^0, \tau_{\text{Ac}})$ and left and right translations in the semigroup $\mathcal{C}_{m,n}$ are finite-to-one maps (see Lemma 2).

**Remark 9.** By Theorem 3 the discrete topology $\tau_d$ is a unique Hausdorff topology on the interassociate $\mathcal{C}_{m,n}$ of the bicyclic monoid $\mathcal{C}(p, q)$, $m, n \in N$, such that $\mathcal{C}_{m,n}$ is a semitopological semigroup. So $\tau_{\text{Ac}}$ is the unique compact topology on $\mathcal{C}_{m,n}^0$ such that $(\mathcal{C}_{m,n}^0, \tau_{\text{Ac}})$ is a Hausdorff compact semitopological semigroup for any non-negative integers $m$ and $n$.

The following theorem generalized Theorem 1 from [18].
Theorem 10. Let m and n be arbitrary non-negative integers. If \( C_{m,n}^0 \) is a Hausdorff locally compact semitopological semigroup, then either \( C_{m,n}^0 \) is discrete or \( C_{m,n}^0 \) is topologically isomorphic to \( (C_{m,n}^0, \tau_{Ac}) \).

Proof. Fix an arbitrary Hausdorff locally compact topology \( \tau \) on \( C_{m,n}^0 \) such that \( (C_{m,n}^0, \tau) \) is a semitopological semigroup and the zero \( 0 \) of \( C_{m,n}^0 \) is not an isolated point of the space \( (C_{m,n}^0, \tau) \). By Lemma 1 the subsemigroup \( C_{m,n}^* \) of \( C_{m,n} \) is isomorphic to the bicyclic semigroup \( C(p,q) \) and hence the sub-semigroup \( (C_{m,n}^*)^0 = C_{m,n}^* \cup \{0\} \) of \( C_{m,n}^0 \) is isomorphic to the bicyclic semigroup with adjoined zero \( C^0 = C(p,q) \cup \{0\} \). Then Theorem 1 implies that \( C_{m,n} \) is a dense discrete subspace of \( C_{m,n}^0 \), and hence by Corollary 3.3.10 of [14] the subspace \( (C_{m,n}^*)^0 \) of \( C_{m,n}^0 \) is locally compact. Now by Theorem 1 from [18] we obtain that \( (C_{m,n}^*)^0 \) is compact. Then for every open neighbourhood \( U(0) \) of the zero \( 0 \) in \( (C_{m,n}^*, \tau) \) we have that the set \( (C_{m,n}^*)^0 \setminus U(0) \) is finite. The semigroup operation of \( C_{m,n}^0 \) implies that the set \( C_{m,n}^0 \setminus \left( p^m \ast_{m,n} (C_{m,n}^*)^0 \cup (C_{m,n}^*)^0 \ast_{m,n} q_{n}^1 \right) \) is finite, and hence the above arguments imply that every open neighbourhood \( U(0) \) of the zero \( 0 \) in \( (C_{m,n}^0, \tau) \) has a finite complement in the space \( (C_{m,n}^0, \tau) \). Thus the space \( (C_{m,n}^0, \tau) \) is compact and by Remark 9 the semitopological semigroup \( C_{m,n}^0 \) is topologically isomorphic to \( (C_{m,n}^0, \tau_{Ac}) \).

Since by Corollary 5 the interassociate \( C_{m,n} \) of the bicyclic monoid \( C(p,q) \) does not embeds into any Hausdorff compact topological semigroup, Theorem 10 implies the following corollary.

Corollary 11. If m and n are arbitrary non-negative integers and \( C_{m,n}^0 \) is a Hausdorff locally compact topological semigroup, then \( C_{m,n}^0 \) is discrete.

The following example shows that a counterpart of the statement of Corollary 11 does not hold when \( C_{m,n}^0 \) is a Čech-complete metrizable topological semigroup for any non-negative integers m and n.

Example 12. Fix an arbitrary non-negative integers m and n. On the semigroup \( C_{m,n}^0 \) we define a topology \( \tau_1 \) in the following way:

(i) every element of the interassociate \( C_{m,n} \) of the bicyclic monoid is an isolated point in the space \( (C_{m,n}^0, \tau_1) \);

(ii) the family \( \mathcal{B}_1(0) = \{U_s : s \in \mathbb{N}\} \), where

\[
U_s = \{0\} \cup \{q^{n+i} p^{m+j} \in C_{m,n}^0 : i, j > s\},
\]

determines a base of the topology \( \tau_1 \) at zero \( 0 \in C_{m,n}^0 \).

It is obvious that \( (C_{m,n}^0, \tau_1) \) is first countable space. Then the definition of the semigroup operation of \( C_{m,n}^0 \) and the arguments presented in [17], p. 68 show that \( (C_{m,n}^0, \tau_1) \) is a Hausdorff topological semigroup.

First we observe that each element of the family \( \mathcal{B}_1(0) \) is an open-and-closed subset of \( (C_{m,n}^0, \tau_1) \), and hence the space \( (C_{m,n}^0, \tau_1) \) is regular. Since the set \( C_{m,n}^0 \) is countable, the definition of the topology \( \tau_1 \) implies that \( (C_{m,n}^0, \tau_1) \) is second countable, and hence by Theorem 4.2.9 from [14] the space \( (C_{m,n}^0, \tau_1) \) is metrizable. Also, it is obvious that the space \( (C_{m,n}^0, \tau_1) \) is Čech-complete, as a union two Čech-complete spaces: that are the discrete space \( C_{m,n} \) and the singleton space \{0\}.

Also the following example shows that a counterpart of the statement of Theorem 10 (and hence of Corollary 11) does not hold for any interassociate of the bicyclic semigroup with adjoined zero \( C^0 = C(p,q) \cup \{0\} \).

Example 13. It is obvious that the interassociate of the bicyclic semigroup with adjoined zero \( C^0 \) with the following operation \( a \ast b = a \cdot 0 \cdot b \) is isomorphic to an arbitrary infinite countable semigroup with zero-multiplication, i.e., it is zero semigroup. It is well known that zero semigroup with any topology is a topology is a topological semigroup (see [9], Vol. 1, Chapter 1]).
Later we need the following notions. A continuous map \( f : X \to Y \) from a topological space \( X \) into a topological space \( Y \) is called:

- **quotient** if the set \( f^{-1}(U) \) is open in \( X \) if and only if \( U \) is open in \( Y \) (see [26] and [14] Section 2.4);
- **hereditarily quotient** or **pseudoopen** if for every \( B \subseteq Y \) the restriction \( f|_B : f^{-1}(B) \to B \) of \( f \) is a quotient map (see [24, 25, 3] and [14] Section 2.4);
- **closed** if \( f(F) \) is closed in \( Y \) for every closed subset \( F \) in \( X \);
- **perfect** if \( X \) is Hausdorff, \( f \) is a closed map and all fibers \( f^{-1}(y) \) are compact subsets of \( X \) [28].

Every closed map and every hereditarily quotient map are quotient [14]. Moreover, a continuous map \( f : X \to Y \) from a topological space \( X \) onto a topological space \( Y \) is hereditarily quotient if and only if for every \( y \in Y \) and every open subset \( U \) in \( X \) which contains \( f^{-1}(y) \) we have that \( y \in \text{int}_Y(f(U)) \) (see [14] 2.4.F).

Later we need the following trivial lemma, which follows from separate continuity of the semigroup operation in semitopological semigroups.

**Lemma 14.** Let \( S \) be a Hausdorff semitopological semigroup and \( I \) be a compact ideal in \( S \). Then the Rees-quotient semigroup \( S/I \) with the quotient topology is a Hausdorff semitopological semigroup.

The following theorem generalized Theorem 2 from [18].

**Theorem 15.** Let \( (C^I_{m,n}, \tau) \) be a Hausdorff locally compact semitopological semigroup, \( C^I_{m,n} = C_{m,n} \sqcup I \) and \( I \) is a compact ideal of \( C^I_{m,n} \). Then either \( (C^I_{m,n}, \tau) \) is a compact semitopological semigroup or the ideal \( I \) is open.

**Proof.** Suppose that \( I \) is not open. By Lemma 14 the Rees-quotient semigroup \( C^I_{m,n}/I \) with the quotient topology \( \tau_q \) is a semitopological semigroup. Let \( \pi : C^I_{m,n} \to C^I_{m,n}/I \) be the natural homomorphism which is a quotient map. It is obvious that the Rees-quotient semigroup \( C^I_{m,n}/I \) is isomorphic to the semigroup \( C^0_{m,n} \) and the image \( \pi(I) \) is zero of \( C^0_{m,n} \). Now we shall show that the natural homomorphism \( \pi : C^I_{m,n} \to C^I_{m,n}/I \) is a hereditarily quotient map. Since \( \pi(C_{m,n}) \) is a discrete subspace of \( (C^I_{m,n}/I, \tau_q) \), it is sufficient to show that for every open neighbourhood \( U(I) \) of the ideal \( I \) in the space \( (C^I_{m,n}, \tau) \) we have that the image \( \pi(U(I)) \) is an open neighbourhood of the zero 0 in the space \( (C^I_{m,n}/I, \tau_q) \). Indeed, \( C^I_{m,n} \setminus U(I) \) is an open-and-closed subset of \( C^I_{m,n}, \tau \), because the elements of the semigroup \( C_{m,n} \) are isolated points of the space \( (C^I_{m,n}, \tau) \). Also, since the restriction \( \pi|_{C_{m,n}} : C_{m,n} \to \pi(C_{m,n}) \) of the natural homomorphism \( \pi : C_{m,n} \to C^I_{m,n}/I \) is one-to-one, \( \pi(C^I_{m,n} \setminus U(I)) \) is an open-and-closed subset of \( (C^I_{m,n}/I, \tau_q) \). So \( \pi(U(I)) \) is an open neighbourhood of the zero 0 of the semigroup \( (C^I_{m,n}/I, \tau_q) \), and hence the natural homomorphism \( \pi : C^I_{m,n} \to C^I_{m,n}/I \) is a hereditarily quotient map. Since \( I \) is a compact ideal of the semitopological semigroup \( (C^I_{m,n}, \tau) \), \( \pi^{-1}(y) \) is a compact subset of \( (C^I_{m,n}, \tau) \) for every \( y \in C^I_{m,n}/I \). By Din’ N’e T’ong’s Theorem (see [12] or [14, 3.7.E]), \( (C^I_{m,n}/I, \tau_q) \) is a Hausdorff locally compact space. If \( I \) is not open then by Theorem 10 the semitopological semigroup \( (C^I_{m,n}/I, \tau_q) \) is topologically isomorphic to \( (C^0_{m,n}, \tau_{Ac}) \) and hence it is compact. Next we shall prove that the space \( (C^I_{m,n}, \tau) \) is compact. Let \( \mathcal{U} = \{U_\alpha : \alpha \in \mathcal{F}\} \) be an arbitrary open cover of the topological space \( (C^I_{m,n}, \tau) \). Since \( I \) is compact, there exist \( U_{\alpha_1}, \ldots, U_{\alpha_n} \in \mathcal{U} \) such that \( I \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \). Put \( U = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \). Then \( C^I_{m,n} \setminus U \) is a closed-and-open subset of \( (C^I_{m,n}, \tau) \). Also, since the restriction \( \pi|_{C_{m,n}} : C_{m,n} \to \pi(C_{m,n}) \) of the natural homomorphism \( \pi : C_{m,n} \to C^I_{m,n}/I \) is one-to-one, \( \pi(C^I_{m,n} \setminus U(I)) \) is an open-and-closed subset of \( (C^I_{m,n}/I, \tau_q) \), and hence the image \( \pi(C^I_{m,n} \setminus U(I)) \) is finite, because the semigroup \( (C^I_{m,n}/I, \tau_q) \) is compact. Thus, the set \( C^I_{m,n} \setminus U \) is finite and hence the space \( (C^I_{m,n}, \tau) \) is compact as well. \( \square \)

**Corollary 16.** If \( (C^I_{m,n}, \tau) \) is a Hausdorff locally compact topological semigroup, \( C^I_{m,n} = C_{m,n} \sqcup I \) and \( I \) is a compact ideal of \( C^I_{m,n} \), then the ideal \( I \) is open.
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