Accounting for uncertainty of non-linear regression models by divisive data resorting

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Abstract

This paper focuses on building models of stochastic systems with aleatoric uncertainty. The nature of the considered systems is such that the identical inputs can result in different outputs, i.e. the output is a random variable. This paper suggests a novel algorithm of boosted ensemble training of multiple models for obtaining a probability distribution of an individual output as a function of a system input. The deterministic component in the ensemble can be an arbitrarily-chosen regression model. The algorithm does not require any special properties of the model, other than having descriptive capabilities with some expected accuracy for the chosen dataset type. The efficiency of the suggested algorithm is demonstrated by comparing it with the Monte-Carlo simulations and by modelling of a complex social system — sports betting choices adjusted for obtaining a monetary gain.

Keywords: uncertainty quantification, Kolmogorov-Arnold representation, ensemble methods, divisive data resorting.

1 Introduction

An ordinary regression model is a predictor that converts a vector of inputs into an output scalar. In many real-life modelling scenarios, having such predictor (i.e. a single scalar) is not sufficient, even when the model can predict accurately the expected value of the output. For example, in models of sport outcomes, the predicted expected outcome can be good enough for a sports commentator, but bookmakers need probabilities of possible outcomes to offer attractive bets, while making a profit.

The problem can be defined more precisely by considering a dataset that contains individual records\textsuperscript{1} with vector inputs and corresponding scalar outputs. Different records are considered to be independent and can be used in modelling in any order. The output may take different values if an experiment is repeated with the identical inputs, which means that modelling of such data falls into the category of aleatoric uncertainty quantification. The goal is to obtain the probability distribution of the output as a function of the vector input. Such goal is generally hard to achieve, and, in this paper, it is loosened to obtaining multiple estimations of an individual output, which can qualify as a sample from the probability distribution. Briefly, this is achieved by partitioning the dataset into disjoints in a specific way and building an ensemble of models, which converts an input into a vector of the estimations.

Stochastic systems are typically modelled using Bayesian neural networks (BNNs)\textsuperscript{11} and probabilistic neural networks (PNNs)\textsuperscript{2}. They require conditional probabilities for graphs of connected events, which narrows the field of their application to cases when such information is available or can be assumed. Unfortunately, many recorded datasets are only matrices of numbers, where the last element in each row is the output and the others are the inputs, without any information on conditional probabilities.

Another option for modelling is the ensemble training\textsuperscript{3}. It includes so-called bagging\textsuperscript{15}, boosting\textsuperscript{6,7} and stacking\textsuperscript{8,9}. Ensembles provide multiple outputs for the same input and, in some cases, it is possible to customise ensemble models such that they return an input-dependent distribution; however,

\textsuperscript{1}Data records are also called ‘instances’ or ‘entries’ in literature.
such customisation is data-specific. For example, the random forest \cite{5} is statistically defined by a classifier, which may be designed to provide input-dependent outputs; however, it may not fit the data collected for a different natural phenomena. The goal of this research is to develop a universal approach, applicable to a wide range of data without conceptual changes of the algorithm.

As will be evident later, the approach proposed in this paper falls into the boosting category, since a series of models is built sequentially were the next group of models depends on the previous. It has similarities to some known methods, but, in the form proposed here, it is novel to the best knowledge of the authors. A possibly close concept is called anticlustering \cite{10,12}, where clusters are built to have maximum similarity, while having maximum diversity of the elements within them. In the anticlustering approach, the records with the same inputs and different outputs are partitioned to appear in different disjoints and different models are trained on these different disjoints.

Another important part of the problem is the choice of the regression model as a deterministic element in the ensemble. The authors of this paper have previously published on the Urysohn \cite{13} and the Kolmogorov-Arnold models \cite{14}, the latter can be viewed as a hierarchical tree of the Urysohn models. As shown in \cite{14}, the descriptive capabilities of the Kolmogorov-Arnold model are similar to that of the neural networks; furthermore, it has been used in a variety of applications \cite{13,15,16}. Therefore, it has been chosen to be a deterministic component of the proposed approach for numerical experiments. The source codes are shared by the authors online\footnote{https://github.com/andrewpolar}.

2 Divisive data resorting (DDR)

Consider a dataset that contains independent records \((X^i, y^i), i \in \{1, \ldots, N\}\), where \(X^i \in \mathbb{R}^m\) is the input of the \(i\)-th record, \(y^i \in \mathbb{R}\) is the output of the \(i\)-th record and \(N\) is the number of records. The records are the observations of a system with an aleatoric uncertainty. The independence implies that each individual record is not related to other records\footnote{An example of a system with order-dependent records is a dynamical system.}.

If the probability distribution of the output arbitrarily depends on the input and all inputs in the dataset are different, then it is not possible to build a model, as only one outcome (output) per one input is available. Therefore, only such stochastic systems are considered, for which the probability distribution of the output is a continuous function of the input. To write this formally, the cumulative distribution function of system output \(y\) is denoted as \(F_y\). It depends on system input \(X\), i.e. \(F_y = F_y(X)\). It is required that for every \(X_0 \in \mathbb{R}^m\) and for every \(\epsilon > 0\), there exists \(\delta > 0\) such that \(|X - X_0| < \delta\), where \(X \in \mathbb{R}^m\), implies that \(|F_y(X) - F_y(X_0)| < \epsilon\). Here, \(||\) denotes the \(L^2\)-norm of a vector and \(||\) denotes the \(L^2\)-norm of a function.

Suppose some deterministic regression model \(M_0 : \mathbb{R}^m \rightarrow \mathbb{R}\) can be built using some error minimisation process,

\[
M_0 = \arg\min_{M \in \mathcal{A}} \sum_{i} (y^i - M(X^i))^2,
\]

where \(\mathcal{A}\) is the space of possible models.

The simplest version of the DDR method starts by building one model for the entire dataset \(M_{1,1}\), where the first subscript is the step number and the second is the index of the model within the step. Then, the data records are resorted according to residual

\[
r^i = y^i - M_{1,1}(X^i)
\]

and divided into two even clusters over the median of the residual in the sorted list. At the second step, a new model is built for each cluster, resulting in \(M_{2,1}\) and \(M_{2,2}\). The records are resorted according to the residual error within each cluster. Afterwards, each cluster is subdivided into two clusters over the median error, resulting in four clusters in total, for which new models \(M_{3,1}, M_{3,2}, M_{3,3}\) and \(M_{3,4}\) are built. The process is continued in the similar way. The average error for each cluster should decline in the division process. When models become sufficiently accurate, the process stops. The ensemble of models obtained at the last divisive step can be now used to approximate the distribution of the output for each individual record — the outputs of the ensemble can be handled as a sample from the probability distribution of the real output.

The intuitive justification of such divisive data resorting can be explained using the following elementary example. Assume that all inputs in the entire dataset are identical and all outputs take any of two values \(C_1\) or \(C_2\). A single model minimises the error and will give a prediction between these values, but
after division into clusters, the records with different outputs will fall into different disjoints. Thus, the
ensemble of models, built on these disjoints, will return a collection of $C_1$ and $C_2$ values in a proportion
that should be close to the proportion in the original dataset.

In the simplest version of the DDR method introduced above, the number of clusters doubles at each
step. This can be relaxed, leading to a general version of the DDR algorithm. The entire process of
constructing the ensemble of models can be defined by a sequence of increasing integers that denote the
number of clusters at each step, for example $w_1, w_2, \ldots, w_p$, where $p$ is the total number of steps of the
algorithm and $w_1 = 1$, which indicates that there is only one cluster at the first step (the entire dataset).
It is also useful to denote $w_p = W$, indicating that $W$ models must constitute the final ensemble at the
last step. The general algorithm is as follows:

1. Start with $j = 1$;
2. Split the dataset into $w_j$ disjoint clusters of equal size, such that all records belonging to one cluster
   are encountered consecutively within the entire dataset; if $N$ is not divisible by $w_j$, there will be
   some remaining records;
3. Build $w_j$ deterministic models $M_{j,k}$, $k \in \{1, \ldots, w_j\}$, one for each cluster;
4. For each record, calculate the residual using the model corresponding to the parent cluster of the
   record;
5. Within each cluster, resort the records according to the residuals;
6. If $j = p$, then finish with models obtained at step 3 being the final ensemble of models; if $j < p$
   then increase $j$ by 1 and go to step 2.

Now, given some input $X$, the models of the ensemble can be used to calculate $W$ different outputs.
These outputs can be used to calculate some statistics, for example, the mean or the standard deviation.
Furthermore, these outputs can be used to build the so-called empirical cumulative distribution function
(ECDF) of the output. This ECDF will obviously be input-dependent and is the ultimate goal of the
considered uncertainty quantification problem for the considered stochastic system.

It should be noted that if $W$ is relatively small, the ECDF will be rather coarse. A technique
resembling a ‘sliding’ window concept can be used to build a finer ECDF. Having the final order of data
records (i.e. after the final resorting), additional models $M_{\text{df}}^k$, $k \in \{1, \ldots, W_{\text{df}}\}$ can be built, such
that each model is identified on a set of consecutive records, which contains $r$ records and which starts
from record with index $(dk - d + 1)$. It can be seen that these additional models are no longer identified
on disjoint subsets, but rather on overlapping subsets, each shifted by $d$ records with respect to the
neighbouring subsets. Given some input $X$, a finer ECDF can be then constructed using the outputs of
these additional models (or the additional ensemble).

3 Kolmogorov-Arnold model as deterministic component

In 1950s, Andrei Kolmogorov and Vladimir Arnold showed that any continuous multivariate function
can be represented as a composition of functions of a single variable [17][18]. More precisely, function
$F : \mathbb{R}^m \to \mathbb{R} \in C([0,1]^m)$ can be represented as

$$F(X_1, X_2, \ldots, X_m) = \sum_{k=1}^{n} \Phi^k \left( \sum_{j=1}^{m} f^{kj}(X_j) \right),$$  \hspace{1cm} (1)

where $f^{kj} : \mathbb{R} \to \mathbb{R} \in C[0,1]$ and $\Phi^k : \mathbb{R} \to \mathbb{R} \in C(\mathbb{R})$. In the original work [17][18], it has been shown
that $n = 2m + 1$ in order to represent any continuous multivariate function. This decomposition can be
used for a general problem of data modelling, where the output is a continuous function of the inputs. \footnote{More recently the restrictions on continuity have been somewhat relaxed, e.g. see discussion in [19].}

Given a set of data records $(X^i, y^i)$, the Kolmogorov-Arnold model can be simply written as \footnote{More recently the restrictions on continuity have been somewhat relaxed, e.g. see discussion in [19].}:\footnote{More recently the restrictions on continuity have been somewhat relaxed, e.g. see discussion in [19].}

$$\hat{y}^i = F \left( X^i_1, X^i_2, \ldots, X^i_m \right),$$  \hspace{1cm} (2)

where $\hat{y}^i$ is the calculated model output of the $i$-th record, $X^i_j$ denote the $j$-th component of vector $X^i$,
and function $F$ is given by equation (1). Without loss of generality, it is assumed that $X^i_j \in [X_{\min}, X_{\max}]$.\footnote{More recently the restrictions on continuity have been somewhat relaxed, e.g. see discussion in [19].}
\(X_{\min}, X_{\max} \in \mathbb{R}\), extending domains of functions \(f^{k j}\). For convenience of the notation, it is also useful to introduce an intermediate variable,

\[
\theta_k^i = \sum_{j=1}^{m} f^{k j} \left( X_i^j \right),
\]

and to assume that \(\theta_k^i \in [\theta_{\min}, \theta_{\max}], \theta_{\min}, \theta_{\max} \in \mathbb{R}\).

To handle the model, the underlying functions, \(f^{k j}\) and \(\Phi^k\), must be specified in a form allowing computations. In the previous publication by the authors [14], it has been suggested to use the piecewise-linear representation of the functions. Having such functions, the identification of the model is reduced to estimation of a finite number of the nodal values (i.e. points where the functions change the slope). The identification problem is described in detail in [14], and, since this paper focuses on the DDR algorithm, within which the deterministic model can be arbitrary, only the general idea of the identification procedure is outlined below.

The identification algorithm for the Kolmogorov-Arnold model is called record-by-record descent. It makes small sequential modifications of the current nodal values for each individual data record, starting from arbitrary initial approximation. For each record, first, intermediate variables \(\theta_k^i\) are modified to reduce error \(|y^i - \hat{y}^i|\); afterwards, all involved linear segments of the functions are updated to support this improvement. Piecewise-linearity makes these changes obvious, since either the argument or two nodal values of a particular linear block must be updated to either increase or decrease the involved function value. Although changes are applied record-by-record with repetition (in any order), the model converges to one of many local minima. The choice of the number of linear segments of the functions and the number of the inner blocks, \(n\), protects from overfitting. Choosing only one linear segment for each function with condition of crossing the origin makes the model a simple linear regression.

### 4 Numerical example

The evaluation of the performance of the DDR algorithm is made using the synthetic data, since it requires simulating the cumulative distribution functions of the outputs using the Monte-Carlo technique for a given record, which is not possible if data recordings of a real physical system are taken. The generated synthetic data is a set of records with observed input vectors \(X^i\) and corresponding output scalars \(y^i\). The outputs are computed by some formula \(y^i = F \left( X^i + \xi^i \right)\), where inputs \(X^i\) are perturbed by a uniformly-distributed noise \(\xi^i\). When the ensemble of models is obtained and a sample of the outputs is computed for particular \(X^i\), this sample can be compared to the probability distribution or the expected value of \(y^i\).

A synthetic example must be challenging enough to test the proposed procedure to the limit. This means that the stochastic system must be such that the statistical behaviour of the output changes significantly as a function of the inputs — the probability density function (PDF) changes qualitatively not only from symmetric to non-symmetric, but also by changing the number of minima/maxima as the inputs are changing. Thus, the following system is taken:

\[
y = \frac{2 + 2X_1^i}{3\pi} \left( \arctan \left( 20 \left( X_1^i - \frac{1}{2} + \frac{X_2^i}{6} \right) \exp (X_2^i) \right) + \frac{\pi}{2} \right) + \frac{2 + 2X_1^i}{3\pi} \left( \arctan \left( 20 \left( X_1^i - \frac{1}{2} - \frac{X_2^i}{6} \right) \exp (X_2^i) \right) + \frac{\pi}{2} \right),
\]

\[X_2^i = X_j + 0.4 \left( C_j - 0.5 \right), \quad j \in \{1, \ldots, 5\},\]

where \(C_j \sim \text{unif} (0, 1)\) are uniformly distributed random variables. The probability density of the output indeed significantly changes depending on the inputs. In figure 1 in the insets (blue figures), four examples of probability densities of \(y\) are shown for four different inputs:

\[
\begin{align*}
X^1 &= [0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5], \\
X^2 &= [0.65 \ 0.5 \ 0.5 \ 0.5], \\
X^3 &= [0.68 \ 1 \ 0.5 \ 0.5 \ 0.5], \\
X^4 &= [0.74 \ 1 \ 0.5 \ 0.5 \ 1],
\end{align*}
\]

corresponding to subfigures (a)-(d), respectively. The PDFs are built using the Monte-Carlo sampling of \(10^5\) points.
The empirical cumulative distribution functions (ECDFs) for the considered stochastic system obtained using the Monte-Carlo sampling (black) and using the realisations of the DDR algorithm (grey). Averages over the realisations are shown in red. The corresponding probability density functions (PDFs) obtained using the Monte-Carlo sampling are shown in the insets. Subfigures (a)-(d) correspond to inputs $X_1$-$X_4$.

To benchmark the DDR procedure, a total of 40 runs of the programme have been performed. During each run, a dataset of $N = 10^6$ records has been generated, the ensemble of models has been constructed using the DDR algorithm, the ECDFs for the given above four points have been calculated using the ‘sliding’ window technique. For data generation, inputs $X_j \sim \text{unif}(0,1)$ were taken. The structure of the Kolmogorov-Arnold model has been selected based on a separate parametric study — 5 and 7 equidistant nodes for the inner and the outer functions, respectively. The noise-reduction parameter of the identification of the Kolmogorov-Arnold model has been selected to be $\mu = 0.002$. During the identification of each model, 4 runs through the dataset have been performed. To construct the ensemble of models, $w \in \{1, 2, 3, 5, 7, 11, 17, 23, 29\}$ divisions have been made. For the ‘sliding’ window technique, parameters have been selected to be $r = 30000$ and $d = 5000$.

In figure 1, the given above four inputs are considered, the ECDFs built using the Monte-Carlo sampling are shown in black colour and the ECDFs obtained using the DDR procedure (with the ‘sliding’ window technique) are shown in grey colour. Furthermore, an average ECDF across all realisations (i.e. average of grey curves) is shown in red colour. For the purpose of the discussion, the Monte-Carlo sampling ECDFs are referred to as the ‘exact’. It can be seen that the ensemble of models reproduces the major features of the exact CDFs qualitatively and predicts well the range of the output quantitatively.

The DDR algorithm builds the ensemble of models using the specifically created set of disjoints. Therefore, it is crucial to emphasise its advantage over an ensemble built using disjoints containing random records. To show this, a set of 100 input points have been randomly generated. For each input point, the true mean and the true standard deviation have been calculated using the Monte-Carlo sampling of $10^5$ points. Next, for each input point, the mean and the standard deviation have been calculated using the DDR ensembles (i.e. built using the DDR algorithm) and the random ensembles (i.e. built using random disjoints). All ensembles had the same number of models — 29. In figure 2 the results are compared and it can be seen that the random ensembles can model the means well, which is a known fact, but cannot model the standard deviations at all. Meanwhile, the DDR ensembles predict the standard deviations both qualitatively and quantitatively.
Figure 2: The mean and the standard deviation for 100 input points. The points are sorted based on their $X_1$ value. The values are obtained using the Monte-Carlo sampling (red squares), using the ensemble of models built using the realisations of the DDR algorithm (blue crosses) and using the realisations of the ensemble of models built on randomly selected disjoint sets of records (grey pluses).

To estimate quantitatively whether the DDR ensembles can give the ECDFs that are close to the real CDFs (built using the Monte-Carlo sampling) across a wide range of inputs, the standard Kolmogorov-Smirnov goodness test with level $\alpha = 0.05$ has been used. This test quantifies the distance between the distribution functions and indicates whether they can be regarded to be sampled from the same reference distribution. For the 100 input points generated above, the DDR ensembles have failed the Kolmogorov-Smirnov test at $31.8 \pm 4.3$ points (the spread comes from 40 realisations), when compared to the Monte-Carlo sampling. This might seem to be a large percentage, but the underlying system has specifically been chosen to be extremely challenging for the modelling. Furthermore, the random ensembles have failed the test at $98.8 \pm 1.0$ points.

5 Example of a social system

This paper would have been incomplete without a real-life example, for which the problem of predicting the outcomes of football matches of the English Premier League has been taken. The efficiency of the probabilistic model can be evaluated by matching the predicted probabilities to bookmakers’ bets and showing certain money gain, compared to random betting. It can be explained using a simple example.

Assume that the bookmaker has estimated the probability of a particular outcome to be 0.25. The bet of +280 provides a usual 5% commission. However, if true probability is 0.3 and the model estimates it correctly, then, by following the model, a statistical profit of $0.3 \times $280 - $0.7 \times $100 = $14 per $100 bet can be obtained.

The modelling of betting choices to maximise profit is an example of a rather complex social system with many interacting groups pursuing opposite interests. Bookmakers have competitors and must offer attractive bets while making profit and bettors have freedom to choose the bet from offered list or to refrain. The assumption that bookmakers’ models are inaccurate would be an exaggeration. In order to make profit, their offered bets must be coordinated with the number of actually placed bets on different outcomes, but the public is biased and the bookmakers must adjust the bets by making some of them

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1 The bet of +280 means that the bettor bets $100 against $280, if the probability of 0.25 is true, the bookmaker makes $20 in each 4 matches, which yields 5% commission per bet.
more/less attractive. As can be seen from the example above, a change of probability estimation by 0.05, if correct, can change money loss to gain.

5.1 First model

At the first step, the inputs and the outputs are chosen. The output is the goal difference. The inputs are the ratings \( \rho \) of the teams (i.e. the positions in the standings), which are integers from 0 to 20. The aim is to predict the 2019-2020 season of 380 matches, given the training dataset of 5700 matches, starting from 2004-2005 season and ending with 2018-2019 season.

When the season starts, the ratings of the teams are assigned from the previous season. Within the season, the ratings are updated after each match, since the outcome of the match becomes known. Also, at the start of the season, three new teams are assigned the same ratings as three relegated teams of the previous season.

It should be noted that the prediction of the outcomes of sports events is a large research field, and accurate predictions require more inputs, see e.g. [20]. For football matches, this can include possessions of the ball, shots, corner kicks, wind directions, injuries and even distances travelled by the away team. As discussed in [20], some models use up to 66 inputs and the prediction accuracy approaches 90%. The aim of this example is to demonstrate the efficiency of the DDR algorithm, and the introduced above simplistic approach with two inputs is sufficient for these purposes.

The outcome of a football match is a random variable and proper prediction should not be just an expected value of the output (the goal difference), but a probability distribution of the output. In literature, typically, the outcome is modelled and the number of correct predictions is reported. Here, to account for the probability distribution of the outcome, the money gained by a betting strategy was modelled. It should be noted that the bookmakers odds are designed to provide them a small advantage (they create a so-called Dutch book), which is usually around 5-7% of all bets, which means that placing random bets statistically leads to a loss of money. The chosen accuracy metric is an averaged per match ratio of the gained amount to the bet amount, which will be referred to as the relative gain. The negative and the positive values of the relative gain correspond to the loss and the profit, respectively. Thus, the random betting yields the relative gain of approximately \(-0.07\) to \(-0.05\). Below, betting on the so-called ‘1X2’ odds (home win, draw, away win) is considered.

Before the DDR algorithm is considered, it must be shown that the naive ways of betting will not result in any gains. Since the training data is large, the records with the identical inputs are repeated. Therefore, the first obvious strategy is betting on the most frequent outcome found in the training dataset records with the same inputs as the considered match. It appears that such strategy results in losses close to bookmakers’ commission, i.e. it is similar to the random betting. The second obvious strategy is constructing one deterministic model to predict an expected value of the outcome. The authors took the Kolmogorov-Arnold model (for details see the code) and it achieved the same result — losses similar to the random betting.

The main test is comparing the ensemble of models built using the DDR algorithm and the ensemble of models built using randomly selected disjoint sets of records, which can also be called ‘bagging’. The Kolmogorov-Arnold model was used as the deterministic component. Within the DDR algorithm, \( w \in \{1, 2, 3, 5, 7, 11, 23\} \) divisions were made, resulting in 23 models in the ensemble. In the case of bagging, the training dataset was divided into 23 disjoints, also resulting in 23 models in the ensemble. The ensembles were built using the training dataset described above.

For the matches that must be predicted, each predicted goal difference of each individual model of the ensemble was rounded and used to estimate probabilities \( \hat{P}_h, \hat{P}_d, \hat{P}_a \) for home win, draw, away win, respectively. Then, the estimations of possible money gains \( M_h, M_d, M_a \) were computed as

\[
M_\rho = \hat{P}_\rho W_\rho - \left( 1 - \hat{P}_\rho \right) B_\rho, \quad \rho \in \{h, d, a\},
\]

where \( B_\rho \) is the bet amount (i.e. money that the gambler gives to the bookmaker) and \( W_\rho \) is the gain amount in the case of a win (i.e. the bookmaker gives \( W_\rho + B_\rho \) to the gambler in this case). Values of \( W_\rho \) come from historic bookmakers’ 1X2 odds, given \( B_\rho \). The bet amounts were always fixed \( B_\rho = \$100 \). After this, the bet was made on the home win if \( M_h > M_d \) and \( M_h > M_a \), or on the away win if \( M_a > M_d \) and \( M_a > M_h \), or on the draw if \( M_d > M_h \) and \( M_d > M_a \). Having made the bet, the actual outcome was then used to determine whether the bet has been successful.

\[\text{The data records are formatted as three integer numbers. The first number is the rating (the position in the standings) of the home team, the second number is the rating of the away team, the third number is the goal difference. The records are ordered by the date of the match.}\]

\[\text{The data are obtained from https://www.oddsportal.com}\]\n
7
| Number of bets | 380 | 380 | 380 | 380 | 380 | 380 | 380 | 380 |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|
| Correct predictions | 164 | 161 | 167 | 159 | 165 | 162 | 167 | 159 |
| Relative gain   | -0.07 | -0.08 | -0.05 | -0.10 | -0.07 | -0.08 | -0.06 | -0.12 |

Table 1: Performance of the ensemble of models built on randomly selected disjoint sets of records for the first model. Columns are realisations of the ensemble.

| Number of bets | 380 | 380 | 380 | 380 | 380 | 380 | 380 | 380 |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|
| Correct predictions | 121 | 115 | 115 | 121 | 106 | 120 | 124 | 131 |
| Relative gain | 0.13 | 0.09 | 0.13 | 0.09 | 0.13 | 0.10 | 0.16 | 0.16 |

Table 2: Performance of the ensemble of models built using the DDR algorithm for the first model. Columns are realisations of the ensemble.

Table 1 shows the results of eight consecutive realisations of the ensemble built using bagging. The difference between the realisations is a result of random initial approximations for the models at the beginning of the training process. It can be seen that the relative gain is negative and near the result of random betting. Table 2 shows the results of eight consecutive realisations of the ensemble built using the DDR algorithm. The number of the correct predictions is lower, but the relative gain is positive.

The advantage of the DDR algorithm can only be attributed to a more accurate probability estimations for each individual match, while bagging approach builds an ensemble with each model predicting an expected value of the output. The public prefers betting on the favourite and the bookmakers make the odds on the underdog more attractive, which shifts the odds off the probabilistic balance. The algorithm identified these opportunities and used them. Underestimating of the underdogs is a known phenomenon and some experienced gamblers also use it intuitively; however, straightforward betting exclusively on the underdogs leads to losses. Both the code and the data are available online.

### 5.2 Second model

In the second test, the model has been slightly changed. Since only top 17 teams from one season are participating in the next season, their results from the previous season can be directly reused. In particular, when top 17 teams of a given season are considered, for each pair of teams, the home team has already played 16 matches with the same opponents at home, and similarly for the away team. Some of these matches may have taken place in the current season and some in the previous. However, the model with 32 inputs would have been hard to construct (due to a significant amount of data required in such case), and the usual feature extraction technique has been applied. The number of the inputs per team has been reduced to 3: the first is the sum of all negative numbers, which are the goal differences in the lost matches, the second is the sum of all positive numbers, which are the differences in the won matches, and the third is the number of draws.

The training data were seasons from 2004-2005 to 2019-2020, resulting in 4352 records with 6 inputs and 1 output per record. The predicted season was 2020-2021. For each season, 272 matches from 380 were taken, since only top 17 teams were modelled. When a match had been predicted, the data table was updated and the involved features were recalculated.

One difference in the decision making logic, compared to the previous model, was applied. Other researchers noticed that the prediction of the draws is much less accurate compared to the home win or the away win. While in the previous model it was used anyway, in this model, the predicted draws were replaced by either the home win or the away win, depending whether predicted amount $\hat{M}_h$ or $\hat{M}_a$ is greater.

The results are shown in tables and . The money gain is computed assuming equal bets of $100 each. The advantage of DDR over the ensemble built by random clustering (bagging) is also clearly visible in this second experiment. The money gain in the case of DDR is obtained using significantly smaller number of the correct predictions, which means that the successful bets on the underestimated underdogs exceed other losses. Both the code and the data are available online.

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1 The entire goal of modelling outcomes of the football matches was to back up the provided theory. The authors do not intend to create a tool facilitate profiting from sports betting. The published code is used at users own risk.

2 https://github.com/andrewpolar/premierleague

3 https://github.com/andrewpolar/premierleague7
6 Conclusions

The cases when modelled objects are stochastic, while observed inputs and outputs are exact, are common across a wide variety of scientific and technological fields. The introduced in this paper divisive data resorting (DDR) algorithm attempts to address this uncertainty by building an ensemble of models and by using the models’ outputs to estimate the probability distribution of the real output. The considered approach does not require any additional information (e.g. on the type of the probability distribution), which is usually hard to find, or any prior assumptions, which are usually hard to make. Only observed/recorded inputs and outputs are used, as in common regression models. The approach, however, relies on the probability distribution of the system output being a continuous function of the system input.

The problems remaining for the future research are more rigorous numerical analysis of the DDR algorithm and consideration of stochastic systems with correlated outputs. A typical example of the latter is a physical system, where index $i$ represents moments of time, in which case, the order of the records matters and they cannot be resorted in such straightforward manner.

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