Quantum Hydrodynamics of Fermi Fluids

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Abstract

It is shown that gauge theories with fermions are most naturally studied via a polar decomposition of the field variable. This is the fermionic analog of the preprint cond-mat/0210673. The hope is that these two put together will enable the treatment of neutral nonrelativistic matter composed of electrons and nuclei in a nonperturbative manner with nuclei and electrons treated on an equal footing. We recast the electron-phonon (superconductivity) problem in the hydrodynamic language and indicate how it is solved. In particular we focus on the a.c. conductivity.

1 Introduction

Read My Lips!

George Bush Sr., 1993 Campaign.

The program of quantizing hydrodynamics has a long and distinguished history. Landau [1] and his students were among the first to attempt this. Later on Sunakawa et.al. [2] and others - notably Rajagopal and Grest [3] took this program further. Dashen, Sharp, Menikoff and Goldin[4] in the seventies introduced many of these ideas. Recently, Jackiw and collaborators[5] have revived interest in this approach in the context of relativistic quarks. In our earlier work, we introduced the DPVA for fermions[6]. We also note that Rajagopal and Grest[3] had already in the seventies pointed out the need for having a nonzero-phase functional found in the DPVA. In our earlier work[6] we made a first pass at computing the phase functional. This attempt yielded an answer that in retrospect is quite wrong. Upon closer examination the \( U_0(\mathbf{q}) \) of our earlier work[6] is imaginary when it was postulated to be real(for small \( \mathbf{q} \)). So far the author has avoided this issue by taking refuge under the the sea-boson approach that enables us to derive the momentum distribution, anomalous exponents, quasiparticle residue and so on without yielding the full dynamical propagator.
which is of interest only because it contains information about quantities just
mentioned. If one is able to compute them without having to compute the full
propagator so much the better. However, there are physical problems in which
the full propagator is important. The X-ray edge problem\[8\] is one such. In fact
we tried using the DPVA to compute the X-ray edge spectra in a preprint\[7\]
and found that we obtain the right answers in one dimension but the answers
in higher dimensions were inconsistent with Mahan’s exact results\[8\]. Thus we
must now face up to the harsh reality and try and address this (hopefully last)
hurdle. This article is the fermionic analog of the preprint cond-mat/0210673.
In an earlier preprint, after much reflection, we chose to dismiss the approach
that only uses the hydrodynamic variables namely the density and its conjugate
as ‘myopic’ (myopic bosonization). This is because a hamiltonian formulation
in terms of the hydrodynamic variables is unable to distinguish between
fermions and bosons. We have to further decompose these variables in terms of
linear combination of oscillators in order to distinguish between the two statistics.
However, the sea-boson approach is not without its share of problems.
For one it does not generalise to finite temperatures easily. Also the full dy-
namical propagator is not reducible to quadratures due to a technical difficulty.
Both these problems may be resolved in an approach that incorporates only
the hydrodynamical variables. We show in this preprint, that the path integral
approach is an avenue to distinguish between the statistics when using only the
hydrodynamical variables.

The author had this idea in 1993 but conversations with various knowl-
dgeable people suggested that this approach that only uses hydrodynamic variables
is not going to work out for fermions, since one has to take into account the
extended nature of the Fermi surface. However, this idea seems too important
to pass up. In particular, the natural manner in which gauge theory may be
studied in this approach makes this effort for fermions worthwhile and urgent.

2 The Field Operator

In our earlier work\[6\], we showed how the field operator may be expressed in
terms of currents and densities. We reproduce the formula here.

$$\psi(x) = e^{-i\Pi(x)} e^{i\Phi(\rho;x)} \sqrt{\rho(x)} \tag{1}$$

The conjugate $$\Pi$$ obeys canonical commutation rules.

$$[\Pi(x), \Pi(x')] = 0 \tag{2}$$

$$[\Pi(x), \rho(x')] = i \delta^d(x - x') \tag{3}$$

$$[\rho(x), \rho(x')] = 0 \tag{4}$$

There are some technical difficulties associated with the fact that $$\Pi$$ is not strictly
self-adjoint, but we shall operate in the high enough density limit where we may
assume that we expand around a nonzero mean for the density and this means that we may choose \( \Pi \) to be self-adjoint. The conjugate \( \Pi \) may be related to the current as follows.

\[
J(x) = -\rho(x)\nabla \Pi(x) + \rho(x)\nabla \Phi(\rho; x) - \rho(x)[-i\Phi(\rho; x), \nabla \Pi(x)]
\] (5)

The phase functional \( \Phi \) determines the statistics of the field \( \psi \). As shown earlier[6] \( \Phi \) obeys a recursion relation as depicted below. It must be pointed out that the need for having a nonzero \( \Phi \) for fermions was appreciated by Rajagopal and Grest[3] way back in the seventies. However, the constraint below brought about by the imposition of Fermi statistics on \( \psi \) was probably first shown in our earlier work[6].

\[
\Phi([\rho(y) - \delta^d(y - x)]; x) - \Phi([\rho; x]) = \pi \ m(x, x')
\] (6)

for \( x \neq x' \) and \( m(x, x') = -m(x', x) \) is an odd integer. A further constraint on \( \Phi \) emerges when we realise that Eq.( 5) has to be consistent with current algebra. We know that currents and densities obey the current algebra. In other words,

\[
[J(x), \rho(x')] = -i\rho(x)\nabla_x \delta^d(x - x')
\] (7)

\[
[J_i(x), J_j(x')] = -iJ_j(x)\nabla^i_x \delta^d(x - x') + iJ_i(x')\nabla^j_x \delta^d(x' - x)
\] (8)

\[
[\rho(x), \rho(x')] = 0
\] (9)

In what follows we try and impress upon the reader the need for a systematic and general approach by pointing out that reasonable sounding and rather general ansatzs fail to obey the constraints just outlined. Thus it seems that constraint on \( \Phi \) due to current algebra is almost inconsistent with the constraint due to the statistics requirement.

2.1 A Serendipitous Surmise

Many years ago the author had a conversation with then the student now Prof. A.H. Castro-Neto where the latter suggested that maybe the field operator is simply given by,

\[
\psi(x) \approx e^{-i\sum_q e^{i\mathbf{q}\cdot\mathbf{x}} X_q \sqrt{\rho^0}}
\] (10)

where by definition \( X_q = i\mathbf{q}\cdot\mathbf{j}(-\mathbf{q})/(\mathbf{q}^2N^0) \) and \( \rho^0 = N^0/V \). Later he realised that if we choose \( [X_q, X_{q'}] = 0 \) as is in fact mandatory \(^1\), then fermion commutation rules are not obeyed. However one may take solace in the fact that at least one commutation rule does come out right namely \( [\psi(x), \rho_q] = e^{i\mathbf{q}\cdot\mathbf{x}}\psi(x) \).

\(^1\)Since strictly speaking it is the conjugate to \( \rho \) as defined by the line integral of the ratio of the current and density that enters; these commute amongst themselves.
A refinement over this ansatz was attempted in our earlier work\[6\] by introducing an additional phase functional of the density linear in the density and this was also inadequate since by now the author knows that Φ there when computed was imaginary when it was postulated to be real. A compromise was also suggested that involved multiplying and dividing by the free propagator and using the exact version in the numerator and the bosonized free propagator in the denominator. This trick though repugnant to most, gives us an anomalous exponent of the Luttinger liquid as we shall see below. However, this anomalous exponent is off by a factor of two from the exact one obtained by Mattis and Lieb\[9\]. From Eq.(10) we may write,

\[ G(x - x') = \langle \psi^\dagger(x')\psi(x) \rangle \approx e^{i\sum q \left( e^{iqx'} - e^{iqx} \right)} \rho^0 \]

Again using the trick outlined in our earlier work\[6\] we may write,

\[ G(x - x') = G_0(x - x') e^{-\sum q \left( 1 - \cos(q(x - x')) \right)/\rho^0} \]

Here \( G_0(x - x') \) is the propagator obtained from elementary considerations. In one dimension, we may see that \( \langle X_q X_{-q} \rangle \approx k^2_F S_0(q)/q^2 N^0 \). The structure factor \( S_0(q) = |q|/(2k_F) \) for the interacting case. For the interacting case we have, \( S(q) = (v_F/v_{eff}) S_0(q) \).

\[ G(x - x') = G_0(x - x') e^{-\int_0^\infty dq \frac{1 - \cos(q(x - x'))}{|q|} \left( \frac{v_F}{v_{eff}} \right)^\frac{1}{2}} \]

\[ \approx G_0(x - x') \left( \frac{1}{|x - x'|} \right)^\gamma \]

where \( \gamma = \frac{v_F}{v_{eff}} - \frac{1}{2} \). This exponent is exactly one half of the exponent obtained by Mattis and Lieb\[9\]. What is even worse is, we have shown in an earlier preprint\[7\] that when applied to the X-ray edge problem, we obtain the well-known results of Mahan in one dimension but not in higher dimensions. Thus it would appear that there is something amiss in the expression for the field operator. The present attempt is to finally address this difficulty.

### 2.2 No-Go Theorems

We make some observations that render seemingly obvious and promising approaches futile.

**The First No-Go Theorem :**
Here we show that a simple and very reasonable ansatz for \( m(x, x') \) in one dimension, fails. Set \( m(x, x') = \theta(x - x') - \theta(x' - x) \). Clearly \( m = \pm 1 \) is odd and \( m(x, x') = -m(x', x) \). It may be shown using the method of generating functions that the most general solution to Eq. (6) is as shown below.

\[
\Phi([\rho]; x) = \pi \int_{-\infty}^{x} dy \, \rho(y) + C_0([\{\rho(y) - \delta(y - x)\}]) - C_0([\rho]) \quad (14)
\]

Here \( C_0 \) is arbitrary. Unfortunately, the presence of the first term means that current algebra is violated. Using Eq. (14) in Eq. (5) we have

\[
J(x) = -\rho(x) \partial_x \tilde{\Pi}(x) + \rho(x) \pi \rho(x^-) \quad (15)
\]

Then the current-current commutator reads as follows.

\[
[J(x), J(x')] = -i J(x) \partial_x \delta(x - x') + i J(x') \partial_x \delta(x' - x)
\]

\[-i \pi \rho(x) \rho(x^-) \partial_x \delta(x - x') + i \pi \rho(x') \rho(x^-) \partial_x \delta(x' - x) \quad (16)
\]

The last two terms tell us that something is not right. Thus this attempt has come to naught. It is telling us that perhaps \( m \) depends on the density as well.

**The Second No-Go Theorem:**

Here we show that a general looking ansatz for \( \Phi \) that manifestly obeys current algebra fails to reproduce fermion commutation rules.

\[
\Phi([\rho]; x) = B_0([\{\rho(y) - \delta(y - x)\}]) - B_1([\rho]) \quad (17)
\]

Using the fact that \( \nabla B_0([\{\rho(y) - \delta(y - x)\}]) = [-i B_0([\{\rho(y) - \delta(y - x)\}]), \nabla \Pi(x)] \) (this can be shown easily by Fourier decomposing \( \Phi \) with respect to \( \rho \)).

\[
J(x) = -\rho(x) \nabla \tilde{\Pi}(x) \quad (18)
\]

\[
\tilde{\Pi}(x) = \Pi(x) + [i B_1([\rho]), \Pi(x)] \quad (19)
\]

It can be seen that \( \tilde{\Pi} \) is also a canonical conjugate of \( \rho \).

\[
[\tilde{\Pi}(x), \tilde{\Pi}(x')] = 0 \quad (20)
\]

\[
[\tilde{\Pi}(x), \rho(x')] = i \delta^d(x - x') \quad (21)
\]

Thus Eq. (18) obeys current algebra. Unfortunately Eq. (17) fails to obey Eq. (6). To see this we merely plug in Eq. (17) into Eq. (6) and find,

\[
F([\rho]; x) - F([\rho]; x') = \pi (\text{odd integer}) \quad (22)
\]

\[
F([\rho]; x) = B_1([\{\rho(y) - \delta^d(y - x)\}]) - B_0([\{\rho(y) - \delta^d(y - x)\}]) \quad (23)
\]
It can be seen that Eq. (22) is a contradiction. If \( F(\rho; x)/\pi \) is odd then \( F(\rho; x)/\pi \) has to be even for all \( x \neq x' \). In particular, \( F(\rho; x')/\pi \) and \( F(\rho; x)/\pi \) have to be both even. But now their difference can no longer be odd, thus violating the recursion relation. While this is somewhat disappointing, it is not surprising since Eq. (18) suggests that the velocity is irrotational, namely, \( v = -\nabla \Pi \). This is clearly not general enough. We would like to be able to write \( v = -\nabla \Pi + v' \). In what follows we shall do precisely this.

We set \( J = \rho v \) and ask what properties should \( v \) possess? This implies the following commutation rules on the velocity operator.

\[
[v(x), \rho(x')] = -i \nabla_x \delta^d(x - x') \tag{24}
\]

and,

\[
-i \rho(x) \delta^d(x - x') \nabla_x v_j(x) + i \rho(x') \delta^d(x - x') \nabla_x v_i(x') \\
+ \rho(x) \rho(x') [v_i(x), v_j(x')] = 0 \tag{25}
\]

From Eq. (5) it is clear that we may expect to be able to write the velocity as a sum of two parts, one irrotational and the other depending only on the density which has a nonvanishing curl. Thus,

\[
v = v^s + v'^r \tag{26}
\]

Here \( v^* = -\nabla \Pi^* \) for some \( \Pi^* \). Without loss of generality we may choose \([v^*_i(x), v^*_j(x')] = 0 \). In other words we include only those parts of the velocity into the irrotational part that makes it commute amongst itself. Of course, \([v^*_i(x), v^*_j(x')] = 0 \). Crucially however, \([v^r_i(x), v^r_j(x')] \neq 0 \). Indeed,

\[
[v^r_i(x), v^r_j(x')] = -\nabla_x \delta^d(x - x') \tag{27}
\]

\[
[v^s(x), \rho(x')] = -i \nabla_x \delta^d(x - x') \tag{28}
\]

and,

\[
-i \rho(x) \delta^d(x - x') \nabla_x v^r_j(x) + i \rho(x') \delta^d(x - x') \nabla_x v^r_i(x') \\
- \rho(x) \rho(x') [v^r_i(x), v^r_j(x')] - \rho(x) \rho(x') \nabla_x [v^r_i(x), i \frac{\delta}{\delta \rho(x')} v^r_j(x')] = 0 \tag{29}
\]

This means that,

\[
- \frac{\delta^d(x - x')}{\rho(x)} \nabla_x v^r_j(x) - \nabla_x [i \frac{\delta}{\delta \rho(x')}, v^r_j(x')] = f_{ij}(\rho; x; x') \tag{30}
\]

for some \( f_{ij}(\rho; x; x') = f_{ji}(\rho; x'; x) \). Therefore so long as Eq. (30) is obeyed then current algebra is also respected. In light of the two no-go theorems let us set \( \Phi \) to be as shown below.

\[
\Phi(\rho; x) = \Phi_{\text{irr}}(\rho; x) + B_0([\{\rho(y) - \delta^d(y - x)\}] - B_0(\rho) \tag{31}
\]
Here Φ_{irr} stands for some particular solution and Φ would be a general solution. If Φ_{irr} obeys current algebra and the recursion relation so too does Φ. From Eq.(5) we may read off the following formula for the velocity operator \( v = v^s + vr \). Where,

\[
v^r(x) = \nabla \Phi_{irr}([\rho]; x) - \left[ \Phi_{irr}([\rho]; x), \nabla_x \frac{\delta}{\delta \rho(x)} \delta \rho(x) \right]
\]

and \( v^s = -\nabla \tilde{\Pi}^s \) where \( \tilde{\Pi}^s \) is given by Eq.(19). We are looking for a singular solution to Φ that also obeys the statistics requirement. Anticipating some simplifications, let us reparametrise Φ as follows.

\[
\Phi_{irr}([\rho]; x) = \pi L([[\rho(y) - \delta^d(y - x)]; x]) - \pi L([\rho]; x)
\]

In fact it can be shown that only a form such as in Eq.(33) is consistent with the statistics requirement in Eq.(6). The statistics requirement may now be written in exponential form as follows.

\[
e^{-i\pi L([[\rho(y) - \delta^d(y - x)]; x])} e^{i\pi L([\rho]; x)} e^{-i\pi L([[\rho(y) - \delta^d(y - x)]; x])} = -e^{-i\pi L([[\rho(y) - \delta^d(y - x)]; x])} e^{i\pi L([\rho]; x)}
\]

The requirement in Eq.(34) is unfortunately strictly speaking, inconsistent with the constraint from current algebra. To see this, we note that in one dimension, the constraint from current algebra reads as follows.

\[
[-i\partial_x \frac{\delta}{\delta \rho(x)}, v^r(x')] = [-i\partial_x' \frac{\delta}{\delta \rho(x')}, v^r(x)]
\]

This can be satisfied only if,

\[
v^r(x) = -\pi \left[ \partial_x \frac{\delta}{\delta \rho(x)} L([\rho]) \right]
\]

for some L (see appendix A). We see that this means that the L of Eq.(33) has to be now independent of x. This means unfortunately that Eq.(34) is violated. However, one can take the point of view that we would like the current algebra to be ‘almost obeyed’, in other words we allow L to depend extremely weakly on x. This will also open up the possibility that Eq.(34) may now be satisfied.

3 Evaluating the Phase Functional

Here we would like to compute the phase functional that obeys the constraints just described. Computing the phase functional systematically seems particularly difficult. Intuition suggests that the phase functional ought to be related
to the properties of the free theory. Specifically, we should be able to expand it in powers of the density fluctuations and somehow relate the coefficients to the density-density, current-density and current-current correlation functions. From the preceding sections, it appears that we may write,

$$\psi(x) = e^{i\Lambda(\rho; x)} e^{-i\Pi(x)}/\sqrt{\rho(x)}$$  (37)

and $\Lambda$ is some functional that depends weakly on $x$. In fact it has to be strictly independent of $x$ in order for current algebra to be obeyed. However, the weak dependence is needed to recover Fermi statistics. Indeed if we choose $\Lambda$ to be independent of $x$ and then evaluate the equal-time version of the propagator of the Luttinger model, we find an anomalous exponent equal to one-half of the exact one derived by Mattis and Lieb. In other words, an $\Lambda$ independent of $x$ is more or less equivalent to the serendipitous surmise. Thus we have to do better.

In general $\Pi$ may be written as $\Pi(x) = X_0 + \Pi(x)$. Here $X_0$ is conjugate to the total number of particles and $\Pi$ is strictly selfadjoint since it is related to currents and densities. We shall assume that $X_0$ is also self-adjoint but may not be expressed in terms of Fermi bilinears. It may be shown[6] that $X_0$ has an expression shown below.

$$X_0 = \frac{i}{2N^0} \sum_k n_F(k) \ln(c_k) - \frac{i}{2N^0} \sum_k n_F(k) \ln(c_k^\dagger)$$  (38)

It seems that the lagrangian formulation is appropriate here. The hamiltonian formulation has not worked out for unknown reasons. In the lagrangian formulation, all the variables are c-numbers. In the usual path integral formulation, the Fermi fields are complex Grassmann numbers and not ordinary complex numbers. As pointed out earlier, we take the point of view that a complex Grassmann number field may be 'simulated' by a complex ordinary number field provided one introduces an appropriate phase functional $\Lambda$. The value of $\Lambda$ is tuned so as to reproduce the correlation functions of the free theory. The anticommutating nature of the Grassmann number is not present in the polar decomposition in Eq.(37) but is recovered at the level of the propagator. To see this we note that in the path integral formulation the fields obey periodic (KMS) boundary conditions[11] namely,

$$\psi(x, t - i\beta) = -e^{i\beta\mu} \psi(x, t)$$  (39)

Using Eq.(39) in Eq.(38) we find,

$$X_0(t - i\beta) = X_0(t) + i \ln(e^{i\beta\mu}) = X_0(t) - \pi + i\beta\mu$$  (40)

That is, the one-particle propagators obey Fermi commutation rules mainly due to the fermionic boundary conditions obeyed by the global Klein factor. The current operator may be written as follows.

$$J(x) = -\rho(x)\nabla \Pi(x) + \rho(x)\nabla(\{\rho_q - e^{iq.x}\})$$  (41)
where,
\[ \nabla \Lambda([\rho]; x) = V([\rho]; x) \] (42)

First we note that in order for current algebra to be obeyed, \( \Lambda([\rho]; x) \) has to be independent of \( x \). In other words \( V([\rho]; x) \equiv 0 \). Unfortunately this means that Fermi statistics is no longer obeyed. However, we can take the point of view that \( \Lambda \) depends very weakly on \( x \) and thus we may ignore the derivative of \( \Lambda \) since it involves taking the difference between neighboring points. On the other hand, for statistics we are usually interested in the opposite extreme namely the asymptotic regime \( |x - x'| \to \infty \). In other words, we may no longer ignore \( \Lambda([\rho]; x) - \Lambda([\rho]; x') \) for large separations. However ignoring \( V \) also seems to be a bad idea. For example if we do and use the Dashen-Sharp formula for the kinetic energy we get a hamiltonian that describes bosons rather than fermions.

\[ K = \int \frac{d^d x}{2m} \left( \frac{\mathbf{J}^2}{\rho} + \frac{(\nabla \rho)^2}{4 \rho} \right) \]

\[ \int \frac{d^d x}{2m} \left( \rho (\nabla \Pi)^2 + \frac{(\nabla \rho)^2}{4 \rho} \right) \approx \sum_q N \epsilon_q X_q X_{-q} + \sum_{q \neq 0} \frac{\epsilon_q}{4 N} \rho_q \rho_{-q} \] (43)

Thus it appears that we have to retain \( V \) as well. We would like to write down the action in the lagrangian formulation. For this we first write,

\[ H = \int \frac{d^d x}{2m} \left( \rho (\nabla \Pi - V')^2 + \frac{(\nabla \rho)^2}{4 \rho} \right) \] (44)

where,

\[ V'([\rho]; x) = V([\rho_q - e^{i q \cdot x}]; x) \] (45)

Till now the discussion has been at the quantum level. Now we would like to recast this theory in the path integral language. For this we have to compute the classical action in terms of the collective coordinates \( \Pi, \rho \). This means we have to ‘de-quantize’ the quantum hamiltonian in Eq. (44). Happily, while there are many inequivalent ways of quantizing a given classical hamiltonian there is likely a unique way of de-quantizing a quantum hamiltonian. Thus we may make the following observations. The Hamilton equations for the canonical variables are,

\[ \partial_t \Pi(x, t) = -\frac{\delta H(\rho, \Pi)}{\delta \rho(x, t)} \] (46)

\[ \partial_t \rho(x, t) = \frac{\delta H(\rho, \Pi)}{\delta \Pi(x, t)} \] (47)

\[ \partial_t \rho(x, t) = -\frac{1}{2m} \nabla_x \left( 2 \rho(x)(\nabla_x \Pi(x) - V'([\rho]; x)) \right) \] (48)

\[ L(\rho, \partial_t \rho) = \int d^d x \Pi(x, t) \partial_t \rho(x, t) - H(\rho, \Pi) \] (49)
\[ L(\rho, \partial_t \rho) = - \int d^d x \Pi(x,t) \frac{1}{2m} \nabla_x \left( 2\rho(x)(\nabla_x \Pi(x) - \mathbf{V}'([\rho]; x)) \right) - \int \frac{d^d x}{2m} \left( \rho(\nabla \Pi - \mathbf{V}')^2 + \frac{(\nabla \rho)^2}{4\rho} \right) \]

\[ = \frac{1}{m} \int d^d x \nabla_x \Pi(x,t) \left( \rho(x)(\nabla_x \Pi(x) - \mathbf{V}'([\rho]; x)) \right) - \int \frac{d^d x}{2m} \left( \rho(\nabla \Pi - \mathbf{V}')^2 + \frac{(\nabla \rho)^2}{4\rho} \right) \]

\[ = \frac{1}{m} \int d^d x \rho(x) \mathbf{V}'([\rho]; x) \left( \nabla_x \Pi(x) - \mathbf{V}'([\rho]; x) \right) + \int \frac{d^d x}{2m} \left( \rho(\nabla \Pi - \mathbf{V}')^2 - \frac{(\nabla \rho)^2}{4\rho} \right) \]

\[ (50) \]

4 One Dimension

In one dimension we have,

\[ \partial_t \rho(x, t) = -\frac{1}{m} \partial_x \left( \rho(x)(\partial_x \Pi(x) - \mathbf{V}'([\rho]; x)) \right) \]  

\[ (51) \]

\[ (\partial_x \Pi(x, t) - \mathbf{V}'([\rho]; x)) = -\frac{m}{\rho(x,t)} \int_{-\infty}^{x} dx' \partial_t \rho(x', t) \]

\[ (52) \]

The Lagrangian is given by,

\[ L(\rho, \partial_t \rho) = -\int_{-\infty}^{\infty} dx \mathbf{V}'([\rho_t]; x) \int_{-\infty}^{x} dx' \partial_t \rho(x', t) \]

\[ + \int_{-\infty}^{\infty} \frac{dx}{2m} \left( \frac{m^2}{\rho(x, t)} \left( \int_{-\infty}^{x} dx' \partial_t \rho(x', t) \right)^2 - \frac{(\partial_x \rho(x, t))^2}{4\rho(x, t)} \right) \]

\[ (53) \]

The current operator is then given simply by,

\[ J(x, t) = m \int_{-\infty}^{x} dx' \partial_t \rho(x', t) \]

\[ (54) \]

So far the discussion has been exact. Now we would like to make some approximations. In particular we assume that we are in the degenerate regime so that \( \rho(x, t) = \rho_0 + \tilde{\rho}(x, t) \) and \( \tilde{\rho}(x, t) \ll \rho_0 \) and \( \rho_0 = \frac{N^0}{L} \) the number of electrons per unit length.

\[ L(\rho, \partial_t \rho) = -\int_{-\infty}^{\infty} dx \mathbf{V}'([\rho_0 + \tilde{\rho}_t]; x) \int_{-\infty}^{x} dx' \partial_t \tilde{\rho}(x', t) \]

\[ + \int_{-\infty}^{\infty} \frac{dx}{2m} \left( \frac{m^2}{\rho_0} \left( \int_{-\infty}^{x} dx' \partial_t \tilde{\rho}(x', t) \right)^2 - \frac{(\partial_x \tilde{\rho}(x, t))^2}{4\rho_0} \right) \]

\[ (55) \]
\[ \rho(x,t) = \frac{1}{L} \sum_{q \neq 0} \rho_q(t) e^{-iqx} \] (56)

\[ L(\rho, \partial_t \rho) = -\int_{-\infty}^{\infty} dx V'([\rho_0 + \tilde{\rho}_t]; x) \int_{-\infty}^{x} dx' \partial_t \rho(x', t) + \int_{-\infty}^{\infty} \frac{dx \, m^2}{2 \rho_0} \left( \int_{-\infty}^{x} dx' \partial_t \rho(x', t) \right)^2 - \sum_{q \neq 0} \frac{\epsilon_q}{4N^0} \rho_q(t) \rho_{-q}(t) \] (57)

\[ \int_{-\infty}^{x} dx' \partial_t \rho(x', t) = \frac{1}{L} \sum_{q \neq 0} \partial_t \rho_q(t) \frac{e^{-iqx}}{-iq} \]

\[ L(\rho, \partial_t \rho) = \sum_{q \neq 0} v'([\rho_0 + \tilde{\rho}_t]; q) \frac{\partial_t \rho_q(t)}{iq} + \sum_{q \neq 0} \frac{\partial_t \rho_q(t) \partial_t \rho_{-q}(t)}{4N^0 \epsilon_q} - \sum_{q \neq 0} \frac{\epsilon_q}{4N^0} \rho_q(t) \rho_{-q}(t) \] (58)

\[ j_q(t) = \frac{im}{q} \partial_t \rho_q(t) \]

\[ v'([\rho_0 + \tilde{\rho}_t]; q) = \int_{-\infty}^{\infty} \frac{dx}{L} V'([\rho_0 + \tilde{\rho}_t]; x) e^{-iqx} \] (60)

We would now like to determine \( v' \) and hence \( \Lambda \) by forcing this action to reproduce the correct current-current, current-density and density-density correlation functions of the noninteracting Fermi theory. Unfortunately since the time dependence of \( v' \) is through \( \rho \) alone this is not possible as we have found after repeated attempts by choosing various \( v' \), linear, quadratic and so on in \( \rho \). Thus we now have to form another reinterpretation. We have the luxury of reinterpreting these formulas anyway we wish so long as the properties of the free theory are properly recovered. Thus we shall transfer the time dependence on to the function \( \Lambda \) itself thereby making the action depend on the history of the density of the system.

## 5 Lagrangian Formulation

The discussion in the above sections shows that for fermions, starting from a Hamiltonian formulation and then moving to the Lagrangian formulation does not work out. Thus we would like to start with a Lagrangian formulation at the outset. For this we postulate that the field operator is given by,

\[ \psi(x,t) = e^{i\Lambda([\rho]; x,t)} e^{-i\Pi(x,t)} \sqrt{\rho(x,t)} \] (61)
The notation $\Lambda([\rho]; xt)$ is meant to imply that the phase functional potentially depends on the history of the density configurations of the system. It does not imply that the phase functional is explicitly time dependent. Also $\Lambda$ depends extremely weakly on $x$ thus we may ignore $\nabla \Lambda \approx 0$. However as pointed out before, we may not ignore $\Lambda([\rho]; xt) - \Lambda([\rho]; x't)$ for $|x - x'| \to \infty$. All the variables are c-numbers in the Lagrangian formulation. Even the field variable in Eq.(61) is a c-number. The anticommuting nature of the field variable is captured in the global Klein factor as already pointed out earlier. The current is now given in terms of the canonical variables as follows. Since we may ignore the gradient of $\Lambda$ we have

$$J(xt) = -\rho(xt) \nabla \Pi(xt)$$  \hspace{1cm} (62)$$

The action may now be recast as follows.

$$S_{free} = \int_{0}^{\beta} dt \int d^4x \left[ \rho \partial_t \Pi - \rho \partial_t \Lambda - \rho (\nabla \Pi)^2 - \frac{(\nabla \rho)^2}{4\rho} \right]$$  \hspace{1cm} (63)$$

After this reinterpretation we would now like the action to be purely quadratic in the canonical variables and yet be able to reproduce the correct fermion correlation functions. Thus we make the following assumption. In the high density limit we may ignore the density fluctuations whenever appropriate and this is equivalent to working in the asymptotic regime.

$$\rho(xt) = \frac{1}{V} \sum_{qn} e^{-iq \cdot x} \rho_{qn} e^{-z_n t}$$  \hspace{1cm} (64)$$

$$\Pi(xt) = \sum_{qn} e^{iq \cdot x} X_{qn} e^{z_n t}$$  \hspace{1cm} (65)$$

$$\Lambda([\rho]; xt) = \sum_{q \neq 0} e^{iq \cdot x} \lambda([\rho]; qn) e^{z_n t}$$  \hspace{1cm} (66)$$

The Klein factor which is the position in dependent part of $\Pi$ ensures that the fermion KMS boundary conditions are obeyed by the field variable since we still have $z_n = 2\pi n/\beta$. Here $\rho$ is a real function hence $[\rho(x, t)]^* = \rho(x, t^*) = \rho(x, -t)$ since $t \in [0, -i\beta]$. This means $\rho^*_{qn} = \rho_{-q, n}$. For a similar reason $X_{q, n} = X_{-q, n}$. Note however that $\Pi$ is not self adjoint due to the first ($q = 0$) term, namely the non-self adjoint Klein factor. The position dependent part of $\Pi$ is self adjoint. Also we have $[\lambda([\rho]; q, n)]^* = \lambda([\rho]; -q, n)$.

The action in Eq.(63) would be identical to the bosonic action were it not for the term $-\int \rho \partial_t \Lambda$. Thus this term is crucial and leads to the fermionic correlation functions for the current-current, current-density and density-density correlations since the current is still given simply as in the bosonic case. The action may now be written similar to the bosonic case,

$$S_{free} = \sum_{qn} (-i\beta z_n) \rho_{qn} X_{qn} + i\beta N^0 \sum_{qn} q^2 X_{qn} X_{-q, -n}$$

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\[ \lambda(\rho; q_n) = C(q_n) \rho - q_{-n} + \delta_{n,0} D(\rho; q) \]  

where \( \rho^0 = N^0/V \) is the mean density of fermions. We have included the leading anharmonic corrections since this is going to be important in future works where we may choose to go beyond RPA. We choose,

\[ S_{\text{free}} = \sum_{q_n} (-i\beta z_n) \rho_{q_n} X_{q_n} + i\beta N^0 \sum_{q_n} q^2 X_{q_n} X_{-q_{-n}} \]

\[ + i\beta N^0 \sum_{q_n} q^2 \rho_{q_n} \rho_{-q_{-n}} + i\beta \sum_{q_n \neq 0} z_n C(q_n) \rho_{q_n} \rho_{-q_{-n}} \]  

We may now integrate out \( X \) to arrive at an effective action,

\[ S_{\text{eff}} = i\beta \sum_{q_n} \left( \frac{z_n^2}{4N^0 q^2} \right) \rho_{q_n} \rho_{-q_{-n}} \]

\[ + i\beta N^0 \sum_{q_n} q^2 \rho_{q_n} \rho_{-q_{-n}} + i\beta \sum_{q_n \neq 0} z_n C(q_n) \rho_{q_n} \rho_{-q_{-n}} \]  

Thus we have,

\[ 2 \langle \rho_{q_n} \rho_{-q_{-n}} \rangle_0 = \frac{1}{\beta z_n^2 + \frac{\beta q^2}{4N^0} + \beta z_n C(q_n)} \]  

From this, we may read off a formula for \( C \) in terms of the density-density correlation function we already know how to compute.

\[ \beta z_n C(q_n) = \frac{1}{2 \langle \rho_{q_n} \rho_{-q_{-n}} \rangle_0} - \frac{\beta z_n^2}{4N^0 q^2} - \frac{\beta q^2}{4N^0} \]  

In one dimension we may write,

\[ \rho_{q,n} = \frac{v_F^2}{4N^0 z_n} \]

where \( v_F \) is the Fermi velocity. Since we taken care to ensure that current algebra is obeyed we may expect to find that the current-current, current-density and current-current correlation functions are also properly recovered. The above action does not give everything right. For example, the three body correlations,
\( \langle \rho_{q,n}\rho_{-q+q',-n+n'}\rho_{-q',-n'} \rangle \) are zero from the above action but clearly they are nonzero in the full free Fermi theory. This means we have to go beyond the quadratic action to capture three and higher body correlations. This is not important in the RPA sense and shall ignore this.

Finally, we have to make sure we are able to compute the full propagator. Unfortunately, a straightforward application of the formula for \( \lambda \) does not yield the right propagator. Thus we have to redefine the field as follows. We may decompose the field variable into fast and slow modes. It is the slow modes that contribute to the action. But the fast modes are needed in the one-particle propagator. Thus we write, \( \psi = \Psi_{\text{fast}} \psi_{\text{slow}} \). We postulate that the fast modes are not affected by interactions, since the action only involves hydrodynamic degrees of freedom, in other words the slow modes. In other words,

\[
\psi(x,t) = \Psi_{\text{fast}}(x,t) e^{-i \sum_{q \neq 0,n} e^{i q \cdot x} e^{i n \cdot (x_{qn} - G(qn) \rho_{-q-n})}} \tag{74}
\]

Here \( G(qn) \) has to be recomputed since we are unable to make contact with the free propagator otherwise. Perhaps this suggests that the real theory will involve some very complicated forms of \( \Lambda \). The main aim of this preprint is to convince the reader of the urgency of somehow making this scheme work for then we will be able to compute the observable properties of a theory of neutral matter where the only adjustable parameters are the relative concentrations of the various elements that make up the material. Now we specialise to one dimension where the analysis is somewhat simpler. In further versions of this preprint, we address the 3d case and apply it to the electron-phonon problem briefly discussed in the next section. We would like to write the propagator in a form that may be decomposed into a product of fast and slow propagators. To this end we demand that only the resonant regime of the propagator be properly recovered. In other words when \( (x-x') \approx \pm v_F(t-t') \).

\[
\langle \psi^\dagger(x',t') \psi(x,t) \rangle = \frac{1}{2\pi i} e^{ik_F^2(t-t')} \left[ e^{ik_F [(x-x')-v_F(t-t')]} e^{-ik_F [(x-x') + v_F(t-t')]} \right] \tag{75}
\]

In other words,

\[
\langle \Psi_{\text{fast}}^\dagger(x',t') \Psi_{\text{fast}}(x,t) \rangle = \rho^0 e^{-ik_F^2(t-t')} \sin[k_F|x-x'|] \tag{76}
\]

Thus we may deduce,

\[
\langle \psi^\dagger_{\text{slow}}(x',t') \psi_{\text{slow}}(x,t) \rangle = \frac{|x-x'|}{k_F} \left[ \frac{2}{(x-x')^2 - v_F^2(t-t')^2} \right] \tag{77}
\]

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In what follows we shall only insist on being able to get the resonant part of the slow propagator correctly. In other words, \([ (x - x')^2 - v_F^2 (t - t')^2]^{-1}\). We shall not be too concerned if we fail to get the \(|x - x'|\) in the numerator. From the hydrodynamic approach we have,

\[
\langle \psi^\dagger_{\text{slow}}(x',t') \psi_{\text{slow}}(x,t) \rangle = e^{-\frac{1}{2} \sum_{q \neq 0,n} \left( 2 - e^{i \mathbf{q} \cdot (x-x')} e^{z_n (t-t')} - e^{-i \mathbf{q} \cdot (x-x')} e^{z_n (t-t')} \right) F(q,n)}
\]

(78)

\[
F(q,n) = \langle (X_{q,n} - G(q,n) \rho_{-q,-n}) (X_{-q,-n} - G(-q,-n) \rho_{q,n}) \rangle
\]

\[
= \langle X_{q,n} X_{-q,-n} \rangle - G(q,n) \langle \rho_{-q,-n} X_{-q,-n} \rangle - G(-q,-n) \langle X_{q,n} \rho_{q,n} \rangle + G(q,n) G(-q,-n) \langle \rho_{q,n} \rho_{-q,-n} \rangle
\]

(79)

In one dimension we may deduce,

\[
\langle \rho_{q,n} \rho_{-q,-n} \rangle = \frac{2N^0}{\beta} \left( \frac{z_n^2 + v_F^2}{q^2 + q_F^2} \right)^{-1}
\]

(80)

\[
\langle X_{q,n} X_{-q,-n} \rangle = \frac{1}{2 \beta N^0} \left( \frac{z_n^2}{q^2} + q_F^2 \right)^{-1}
\]

(81)

\[
\langle X_{q,n} \rho_{q,n} \rangle = \left( -\frac{z_n}{\beta q^2} \right) \left( \frac{z_n^2}{q^2} + v_F^2 \right)^{-1}
\]

(82)

If we set,

\[
G(q,n) = \frac{3v_F^2}{2N^0 z_n}
\]

(83)

then,

\[
F(q,n) = \left[ \frac{2v_F^2}{\beta N^0} \right] \left( z_n^2 + v_F^2 q^2 \right)^{-1}
\]

(84)

Here we have taken the liberty to ignore the pole at \(z_n = 0\) since this may be cancelled by an appropriate time-independent additive contribution to \(\Lambda(\rho; x, t)\). This does not affect the action but serves to regularise the above integrals.

\[
\langle \psi^\dagger_{\text{slow}}(x',t') \psi_{\text{slow}}(x,t) \rangle = e^{-\frac{1}{2} \sum_{q \neq 0,n} \left( 2 - e^{i \mathbf{q} \cdot (x-x')} e^{z_n (t-t')} - e^{-i \mathbf{q} \cdot (x-x')} e^{z_n (t-t')} \right) F(q,n)}
\]

(85)
In one dimension, in the hydrodynamic language we may write,

\[
\frac{1}{2} \sum_{q \neq 0,n} \left( 2 - e^{iq(x-x')} e^z_{n}(t-t') - e^{iq(x-x')} e^z_{n}(t-t') \right) F(q,n)
\]

\[
= \int_0^{\infty} \frac{dq}{q} \left( 2 - e^{iq(x-x')} e^{i} v_F |q|(t-t') - e^{iq(x-x')} e^{i} v_F |q|(t-t') \right) \tag{86}
\]

Thus we have,

\[
\langle \psi_{\text{slow}}(x',t')|\psi_{\text{slow}}(xt) \rangle \sim \left[ (x-x')^2 - v_F^2 (t-t')^2 \right]^{-1} \tag{87}
\]

Now we compute the total momentum-total momentum correlation function of the free Fermi theory. This will be important when we compute a.c. conductivity. To this end we write,

\[
P(t) = - \sum_{q,n,n'} (iq) \rho_{q,n} X_{q,n'} e^{z_{n'}-z_n} t \tag{88}
\]

In one dimension, in the hydrodynamic language we may write,

\[
<T \ P(t) \ P(t') > = \sum_{q,n,n'} \sum_{q',m,m'} (iq')(iq) \left\{ \rho_{q,n} X_{q,n'} \rho_{q',m} X_{q',m'} \right\} e^{z_{n'}-z_n} (t-t')
\]

\[
= \sum_{q,n,n'} q^2 \langle \rho_{q,n} \rho_{q,-n} \rangle \langle X_{q,n'} X_{q,-n'} \rangle e^{z_{n'}-z_n} (t-t')
\]

\[
- \sum_{q,n,n'} q^2 \langle \rho_{q,n} X_{q,n} \rangle \langle X_{q,n'} \rho_{q',m} \rangle e^{z_{n'}-z_n} (t-t')
\]

\[
= \sum_{q,n,m} q^4 \frac{v_F^2}{\beta^2} \left( z_n^2 + v_F^2 q^2 \right) \left( z_{n'}^2 + v_F^2 q^2 \right) e^{z_m} (t-t')
\]

\[
- \sum_{q,n,m} q^2 \left( \frac{z_n}{\beta} \right) \left( \frac{z_{n'}^2 + v_F^2 q^2}{\beta^2} \right) e^{z_m} (t-t')
\]

\[
= \sum_{q,n} |q|^3 \frac{v_F}{\beta} \frac{1}{z_n^2 + 4v_F^2 q^2} e^{z_m} (t-t') - \sum_{q,n} |q|^3 \frac{v_F}{\beta} \frac{1}{z_n^2 + 4v_F^2 q^2} e^{z_m} (t-t') = 0 \tag{89}
\]

In the Fermi language,

\[
\langle T \ P(t) \ P(t') \rangle = \sum_{k,k'} (k,k') \langle n_k n_{k'} \rangle = \sum_k k^2 \theta(k_F - |k|) = X^0 \frac{k_F^2}{3} \tag{90}
\]

Thus there seems to be a discrepancy. We must then take the point of view that the hydrodynamic approach predicts the right time-dependent part of the total-momentum total momentum correlation function.
6 The Luttinger Model

The action for the Luttinger model reads as follows.

\[ S_{\text{Lutt}} = \sum_{q,n} (-i\beta z_n) \rho_{q,n} X_{q,n} + i\beta N^0 \sum_{q,n} q^2 X_{q,n} X_{-q,-n} \]

\[ + \frac{i\beta}{4N^0} \sum_{q,n} q^2 \rho_{q,n} \rho_{-q,-n} + i\beta \sum_{q \neq 0 n} \frac{v_F^2}{4N^0} \rho_{q,n} \rho_{-q,-n} + i\beta \rho_0^0 V_0 \sum_{q \neq 0 n} \rho_{q,n} \rho_{-q,-n} \]  

Here \( V_0 > 0 \) is a positive constant signifying repulsion. Thus we may write \( v_{\text{eff}} = v_F \left(1 + \frac{V_0}{\pi v_F}\right)^{\frac{1}{2}} \). Unfortunately it seems that we have to redefine \( G \) (we may euphemistically call this ‘renormalization’). Thus we retry \( G(q,n) = \lambda \frac{v_F^2}{N^0} z_n \)

\[ F(q,n) = \left[ \frac{v_{\text{eff}}^2}{2\beta N^0} - \lambda \frac{2v_F^2}{\beta N^0} + 4\lambda^2 \frac{v_F^2/v_{\text{eff}}^2}{2\beta N^0} \right] \left( z_n^2 + v_{\text{eff}}^2 q^2 \right)^{-1} \]  

\[ \frac{1}{2} \sum_{q \neq 0, n} \left( 2 - e^{iq(x-x')} e^{zn(t-t')} - e^{iq(x'-x)} e^{zn(t'-t')} \right) F(q,n) \]

\[ = \int_0^\infty \frac{dq}{q} \left( 2 - e^{iq(x-x')} e^{iv_{\text{eff}}|q|t-t'} - e^{iq(x'-x)} e^{iv_{\text{eff}}|q|t-t'} \right) \left[ \frac{v_{\text{eff}}}{4v_F} - \lambda \frac{v_F}{v_{\text{eff}}} + \lambda^2 \frac{v_F^3}{v_{\text{eff}}^3} \right] \]  

It is clear that this approach will not give the right exponent since to get the right exponent we have to choose some complicated \( \lambda \). Thus we shall not insist on getting the one-particle properties right.

7 Electron-Phonon System

Consider the lagrangian for phonons.

\[ L_{\text{phonons}} = \sum_q \frac{1}{4\Omega_q} \left( \frac{\partial u_q(t)}{\partial t} \frac{\partial u_{-q}(t)}{\partial t} - \Omega_q^2 u_q(t) u_{-q}(t) \right) \]  

Here \( u_q = b_q + b_{-q}^\dagger \) is the phonon displacement and the phonon dispersion is chosen to be acoustic : \( \Omega_q = v_s |q| \). From Mahan [8] we may write down the hamiltonian for the electron-phonon interaction neglecting umklapp processes,

\[ H_{\text{ep}} = \frac{1}{\sqrt{\Omega}} \sum_{q\sigma} M_q \rho_{q\sigma}(t) u_q(t) \]
\[
M_q = -V_{el}(q) |q| \left( \frac{1}{2M_{ion} \eta v_s |q|} \right)^{\frac{1}{2}}
\]  

We choose \(-V_{el}(q) = Z_{ion}^\text{eff} (4\pi e^2)/q^2\). The overall action may be written as,

\[
S = \sum_{q,\sigma,n} (-i\beta z_n) \rho_{q,\sigma,n} X_{q,\sigma,n} + \frac{i\beta N^0}{2} \sum_{q,\sigma,n} q^2 X_{q,\sigma,n} X_{-q,\sigma,-n} + i\beta \sum_{q,n} z_n C(qn) \rho_{q,\sigma,n} \rho_{-q,\sigma,-n}
\]

\[+(i\beta) \sum_{q,n} \left( \frac{z_n^2 + \Omega_n^2}{4\Omega_q} \right) u_{q,n} u_{-q,-n} + \frac{i\beta}{\sqrt{\Omega}} \sum_{q,\sigma,n} M_q \rho_{q,\sigma,n} u_{q,n} \]

Here \(\eta = N_{ion}/N_e = 1/Z_{ion}\) is the ratio of the number of ions to the number of electrons and \(\hbar = 2m_e = 1\). Also \(Z_{ion}^\text{eff} \sim 1\) is the effective charge seen by the outer electrons. In the case of phonons the momentum transfer \(|q| < \Lambda_D\), the Debye cutoff. We may integrate out the phonons first and write down the effective action for the electrons.

\[
S_{\text{eff}} = \sum_{q,\sigma,n} (-i\beta z_n) \rho_{q,\sigma,n} X_{q,\sigma,n} + i\beta \frac{N^0}{2} \sum_{q,\sigma,n} q^2 X_{q,\sigma,n} X_{-q,\sigma,-n} + i\beta \sum_{q,n} z_n C(qn) \rho_{q,\sigma,n} \rho_{-q,\sigma,-n}
\]

\[-\frac{i\beta}{2N^0} \sum_{q,\sigma',n} \frac{2\rho^0 M_q^2 \Omega_n}{z_n^2 + \Omega_n^2} \rho_{q,\sigma,n} \rho_{-q,\sigma',-n} \]

In order to test for superconductivity we have to compute these quantities. The first is the one-particle dynamical density of states which is related to the full dynamical propagator. The second is the momentum distribution which should not have a discontinuity at the Fermi surface. The third is Yang’s off-diagonal long range order correlation function. Unfortunately to get these right we have to get the one-particle propagator right. We shall now postulate that current algebra is more sacred than Fermi statistics and set (following the suggestion of A.H. Castro-Neto),

\[\psi_{\text{slow}}(x,\sigma, t) \approx e^{-i \Pi(x,\sigma, t)} \sqrt{\rho^0} \]

\[J(x,\sigma, t) \approx -\rho^0 \nabla \Pi(x,\sigma, t) \]

In three dimensions we may write,

\[
\left\langle \rho_{q,\sigma,n} \rho_{-q,\sigma,-n} \right\rangle = \frac{1}{-i\beta} \sum_{k} n_\beta(k + q/2)(1 - n_\beta(k - q/2)) \int_{-\infty}^{\infty} dy \int_{-1}^{1} d\cos \theta \left( \frac{1}{e^{\beta[vFy + \frac{2\beta}{2} \cos \theta]} + 1} \right) \frac{e^{\beta[vFy + \frac{2\beta}{2} \cos \theta]} - 1}{ivFy \cos \theta - z_n}
\]

\[= |q| \left( \frac{k_F^2}{-i\beta} \right) \frac{V}{(2\pi)^2} \int_{0}^{1} dx \tanh \left[ \frac{\beta vFq}{4} x \right] \frac{x}{ivFq x + z_n} \]

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8 Conclusions

It should be clear to the reader by now that bosonization is a powerful method that could hold the key to understanding a wide variety of phenomena using an extremely general and economical set of rules. The sea-boson method has reduced what was till recently, difficult research problems into difficult homework problems. Now the hope is that the hydrodynamic approach will reduce everything to homework problems. The preferred method favored by condensed matter theorists involves starting from experiments and working backwards and trying to come up with some minimal description may sound as the most economical way of proceeding. However this approach is fraught with ambiguity and lacks the generality that the present theory has. In fact this approach of starting from experiments may be compared to this humorous situation. Imagine an athlete when asked to run a marathon stands on the finish line. When the umpire shouts ‘ready!’ he takes a few steps back ...’get set’...’go!’ The athlete runs like the wind and declares himself a winner of the marathon after a fraction of a second. Such an athlete would be the laughing stock of the sporting world. Yet it is this approach that most condensed matter theorists favor. The reader may argue that there is no telling where the starting line is, it could be 22 light years away (as in String Theory or M-theory) rather than 22 miles away. While this is a valid objection to my stand in principle, in practice it is not. Rutherford and others have already showed us using the starting-from-experiments approach that (for most purposes) the atom is the building block of matter and the atom is made of positive and negatively charged point particles making the atom neutral and these obey quantum mechanics. Thus there is no need to repeatedly ask the experimentalist for data. All that these experiments in condensed matter will tell us is that material is made of positive and negative charges and quantum mechanics is important. We already know that. We have to now work out the consequences of this knowledge. The main aim of this preprint is to convince the reader of the urgency of somehow making this scheme work for then we will be able to compute the observable properties of a theory of neutral matter where the only adjustable parameters are the relative concentrations of the various elements that make up the material.

9 Appendix A

Here we find the most general solution to the recursion relation for the phase functional that determines the statistics. We find that it has to be in the specific form that has been used in the main text. Consider Eq.( 6) with $m$ being in general, a function of the density and also a function of the pair of points $x, x'$ in such a way that $m([\rho]; x, x') = -m([\rho]; x', x)$. We may formally Fourier
transform the phase functional as follows.

\[ \Phi(\rho; x) = \int D[P] e^{iP \cdot \rho} \phi([P]; x) \]  

Therefore, Eq.( 6) may be rewritten as follows.

\[ \phi([P]; x) (e^{-iP(x')} - 1) - \phi([P]; x') (e^{-iP(x)} - 1) = \pi \tilde{m}([P]; x, x') \]  

This means that \( \tilde{m} \) has to be of the form,

\[ \tilde{m}(\rho; x, x') = (e^{-iP(x')} - 1) (e^{-iP(x)} - 1) \left( F([P]; x) - F([P]; x') \right) \]  

In other words,

\[ \Phi(\rho; x) = \Lambda(\{ \rho(y) - \delta^d(y - x) \}; x) - \Lambda(\rho; x) \]  

Unfortunately for current algebra to be strictly obeyed, we must have \( \Lambda(\rho; x) \equiv \Lambda(\rho) \) independent of \( x \). This violates the statistics requirement. Thus we shall have to make do with recovering the current algebra as a limit.

Next we prove Eq.( 36). To do this we write,

\[ v^r(x) = \int D[P] e^{iP \cdot \rho} \tilde{v}^r(x; P) \]  

We plug this into the equation below

\[ [-i\partial_x \delta \over\delta \rho(x), v^r(x')] = [-i\partial_{x'} \delta \over\delta \rho(x'), v^r(x)] \]  

to obtain,

\[ [-i\partial_x \delta \over\delta \rho(x), \int D[P] e^{iP \cdot \rho} \tilde{v}^r(x'; P)] = [-i\partial_{x'} \delta \over\delta \rho(x'), \int D[P] e^{iP \cdot \rho} \tilde{v}^r(x; P)] \]  

\[ \int D[P] e^{iP \cdot \rho} \left[ \partial_x P(x) \tilde{v}^r(x'; P) - \partial_{x'} P(x') \tilde{v}^r(x; P) \right] \]  

\[ = 0 \]  

Or,

\[ \partial_x P(x) \tilde{v}^r(x'; P) - \partial_{x'} P(x') \tilde{v}^r(x; P) = 0 \]  

In other words,

\[ \tilde{v}^r(x; P) = G([P]) \partial_x P(x) \]  

and Eq.( 36) follows.
10 Appendix B

Here we would like to ascertain whether or not the velocity operator of fermions is irrotational. To this end we note that if,

\[ J = -\rho \nabla \Pi \]  

then in three dimensions,

\[ \rho \nabla \times J = \nabla \rho \times J \]

Thus we would like to verify whether or not this is obeyed. In general we may write,

\[ J(x) = \frac{1}{V} \sum_{kq} c_{k+q/2}^i c_{k-q/2} e^{-i q \cdot x} \]  

\[ \rho(x) = \frac{1}{V} \sum_{kq} c_{k+q/2}^\dagger c_{k-q/2} e^{-i q \cdot x} \]

From this we can see that,

\[ \rho \nabla \times J - \nabla \rho \times J = -\frac{i}{V^2} \sum_{kq,k'q'} [(q-q') \times k] c_{k+q'/2}^i c_{k'-q'/2}^\dagger c_{k+q/2}^\dagger c_{k-q/2} e^{-i(q+q') \cdot x} \]

This operator is not identically zero but is zero in suitably restricted Hilbert space. We have assumed in the text that this is the case even though it is strictly speaking not right. Besides, current algebra is obeyed only if the velocity operator is strictly irrotational as already shown.

Consider the Hamiltonian,

\[ H = \sum_{q \neq 0} N^0 \epsilon_q X_q X_{-q} + \sum_{q \neq 0} \frac{\epsilon_q}{4N^0} \rho_q \rho_{-q} \]

where \([X_q, \rho_q] = i\) and all other commutators involving any two of these is zero. There seems to be only one inequivalent way in which we may diagonalise Eq. (119). It is to decompose Eq. (119) into oscillators with just one momentum label \(b_q\). This means we may write,

\[ X_q = \frac{i}{2\sqrt{N^0}} (b_{-q} - b_{q}^\dagger) \]  

\[ \rho_q = \sqrt{N^0} (b_q + b_{-q}^\dagger) \]

\[ H = \sum_{q \neq 0} N^0 \epsilon_q \left( \frac{i}{2\sqrt{N^0}} \right)^2 (b_{-q} - b_{q}^\dagger) \left( b_q - b_{q}^\dagger \right) + \sum_{q \neq 0} \frac{\epsilon_q}{4N^0 N^0} \left( b_q + b_{-q}^\dagger \right) \left( b_{-q} - b_{q}^\dagger \right) \]

\[ = \sum_{q \neq 0} \epsilon_q b_q^\dagger b_q \]
This decomposition describes bosons. To describe fermions we need a dispersion of the kind $v_F|q|$ rather than $\epsilon_q = q^2/(2m)$. This is impossible to achieve with the Hamiltonian in Eq.( 119). This fact can be verified independently using the equation of motion approach that also suggests that the dispersion of the modes is $\epsilon_q$. Thus we need an additional phase functional $\Lambda$ to give us a linear dispersion.

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