A self-propelled particle in an external potential: is there an effective temperature?

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We study a stationary state of a single self-propelled, athermal particle in linear and quadratic external potentials. The self-propulsion is modeled as a fluctuating force evolving according to the Ornstein-Uhlenbeck process, independently of the state of the particle. Without an external potential, in the long time limit, the self-propelled particle moving in a viscous medium performs diffusive motion, which allows one to identify an effective temperature. We show that in the presence of a linear external potential the stationary state distribution has an exponential form with the sedimentation length determined by the effective temperature of the free self-propelled particle. In the presence of a quadratic external potential the stationary state distribution has a Gaussian form. However, in general, this distribution is not determined by the effective temperature of the free self-propelled particle.

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I. INTRODUCTION

Recently, there has been a lot of interest in the static and dynamic properties of particles that are self-propelled and, thus, can move on their own accord [1–3]. These particles are said to move actively and to form active matter.

There are two motivations for the interest in active matter systems. First, these systems model static and dynamic properties of specific biological and physical systems in which self-propelled motion occurs. For example, a system of particles with the so-called run-and-tumble motion serves as a model for Escherichia coli bacteria [3]. Similarly, a system of so-called active Brownian particles [2] is a model system for Janus colloidal particles [4]. The second motivation for the interest in active matter systems is the fundamental fascination with non-equilibrium physical systems and, in particular, with systems without detailed balance.

The present contribution is inspired by recent studies that showed that, at least in some cases, active matter systems can exhibit phenomena that are commonly found in standard (thermal, non-active) systems. For example, Palacci et al. [5] found that a dilute active colloidal suspension under gravity exhibits qualitatively the same exponential density distribution as a standard dilute thermal colloidal system. Notably, the parameter that replaces the thermal system’s temperature coincides with the effective temperature that was inferred from an independent measurement of the long-time diffusive motion of an active colloidal particle. More interestingly, behavior similar to that common in thermal systems was found in systems consisting of interacting active particles. For example, Bialké et al. [6] used computer simulations to show that a system of active Brownian particles can crystallize at sufficiently high densities. Next, Das et al. [7] used both a computer simulation and an integral equation theory to show that activity promotes phase separation in an active binary mixture. Finally, it was found that active systems can exhibit glassy dynamics. Berthier and Kurchan [8] analyzed a simple model active system inspired by the so-called spherical p-spin model and showed that it can exhibit kinetic arrest. This pioneering study was followed by two computer simulation investigations of systems of active Brownian particles [9, 10] which showed that, generically, active systems exhibit glassy dynamics, but the onset of glassy behavior is pushed towards higher densities compared with systems of non-active particles. In turn, the latter simulations inspired a very recent mode-coupling-like description of glassy dynamics in active systems [11].

Results of some of the investigations mentioned above [5, 6] suggest an emergence of effective thermal behavior and, more importantly, effective temperature [12]. It should be noted, however, that other studies [6, 13] question the usefulness of the notion of effective temperature. In particular, Fily and Marchetti [13] argue that this notion holds only in the dilute limit.

Our goal is to test the validity of effective temperature in a simple model. To keep the model exactly solvable we replace a system of interacting particles by a single particle in an external field. Specifically, we compare the behavior of a single active particle without any external potential (for which an effective temperature can be easily defined) with the behavior of the same particle in two different external potentials.

There is a number of different models of self-propelled motion [2]. Their common feature is that an active particle moves under an influence of an internal self-propulsion force which evolves in some specified way, independently of the state of the particle. Here we will consider the continuous time, one-dimensional version of the model introduced by Berthier [10]. In the original model of Ref. [10] Monte-Carlo dynamics with correlated trial moves was used (in standard Monte-Carlo dynamics subsequent trial moves are un-correlated [15]). In our model, the particle is subjected to a self-propulsion force and, possibly, a conservative force originating from an external potential. The self-propulsion force has a vanishing average, a finite mean-square and a finite persistence (i.e. relax-
II. FREE SELF-PROPELLED PARTICLE

The free self-propelled particle moves in a viscous medium under the influence of an internal self-propulsion force. We assume that viscous dissipation dominates and consequently the motion is over-damped. The medium is characterized by the single-particle friction coefficient \( \xi_0 \). We assume that any random force originating from the solvent’s fluctuations is negligible and, thus, the particle is non-Brownian. The self-propulsion force evolves according to the Ornstein-Uhlenbeck stochastic process. Specifically, the force relaxes to its average zero value on the time scale characterized by the inverse rate \( \gamma^{-1} \) and it also changes by random, uncorrelated increments due to an internal noise. As a consequence, the self-propulsion force acquires a finite, non-zero mean-square.

The time evolution of the system is described by the following equations of motion,
\[
\begin{align*}
\partial_t x(t) & = \xi_0^{-1} f(t), \\
\partial_t f(t) & = -\gamma f(t) + \eta(t).
\end{align*}
\]

Eq. \( 1 \) describes over-damped motion of the particle and Eq. \( 2 \) describes the evolution of the self-propulsion force. In Eq. \( 2 \) \( \eta(t) \) is a white Gaussian noise with the auto-correlation function given by
\[
\langle \eta(t)\eta(t') \rangle_{\text{noise}} = 2D_f \delta(t-t'),
\]
where \( \langle \ldots \rangle_{\text{noise}} \) denotes averaging of a Gaussian white noise \( \eta \).

Equivalently, the motion of the self-propelled particle can be described by a joint probability distribution for the particle’s position and the self-propulsion force, \( P(x,f;t) \). The equation of motion for this distribution reads:
\[
\partial_t P(x,f;t) = -\frac{f}{\xi_0} \frac{\partial P(x,f;t)}{\partial x} + \frac{\partial}{\partial f} \left( \gamma f P(x,f;t) + D_f \frac{\partial}{\partial f} P(x,v;t) \right).
\]

It can be easily showed that in the stationary state
\[
P^{ss}(x,f) \propto \exp \left( -\frac{\gamma f^2}{2D_f} \right)
\]
and \( \langle f^2 \rangle = D_f/\gamma \). Here and in the following \( \langle \ldots \rangle \) denotes averaging over the stationary distribution of the position and self-propulsion force.

We note that Eq. \( 4 \) is equivalent to the so-called Fokker-Planck equation that describes the motion of a Brownian particle on a time scale on which its velocity relaxation can be observed \( \xi_0^2 \). Indeed, replacing \( f/\xi_0 \) by particle’s velocity \( v \) changes Eq. \( 4 \) into the Fokker-Planck equation. Consequently, we can use the well known theoretical analyzes of Brownian motion \( 16,17 \) for the present case of self-propelled motion. We see immediately that the long-time motion of the self-propelled particle is diffusive and the diffusion constant is equal to
\[
D = \langle f^2 \rangle / (\xi_0^2 \gamma) = D_f/(\xi_0 \gamma)^2.
\]
III. SELF-PROPELLED PARTICLE UNDER THE INFLUENCE OF A CONSTANT FORCE: SEDIMENTATION

If there is an external, conservative, time-independent force acting on the particle, the equation of motion for the position of the self-propelled particle has the following form:

$$\partial_t x(t) = \xi_0^{-1} \left( f(t) + F^{ext}(x(t)) \right)$$  \hspace{1cm} (8)

where $F^{ext}(x) = -\partial_x V^{ext}(x)$ is the external, conservative, time-independent force acting on the particle. Eq. (8) needs to be augmented by the equation of motion for the self-propulsion force, Eq. (2). We emphasize that the evolution of the self-propulsion force is un-changed.

As in the case of a free self-propelled particle, we can describe the time dependence of the state of the particle through the joint probability distribution of the position and the self-propulsion force,

$$\partial_t P(x, f; t) = -\frac{1}{\xi_0} \frac{\partial}{\partial x} \left( \left( f + F^{ext}(x) \right) P(x, f; t) \right)$$  \hspace{1cm} (9)

$$+ \frac{\partial}{\partial f} \left( \gamma f P(x, f; t) + D_f \frac{\partial}{\partial f} P(x, f; t) \right).$$

We should emphasize that since the self-propulsion force evolves independently of the external force, equation of motion (9) is qualitatively different from the Fokker-Planck equation for the joint probability distribution of a position and velocity of a Brownian particle moving under the influence of an external force. Thus, we cannot use the theoretical apparatus developed in Refs. [16, 17].

In the remainder of this section we briefly analyze the stationary state of a self-propelled particle under the influence of a constant external force. We note that the consistency between the free particle effective temperature does not determine the sedimentation length. In fact, Tailleur and Cates [18] showed that for the run-and-tumble model of active particles the stationary state distribution in a linear potential has the exponential form, but the free particle effective temperature does not determine the sedimentation length.

According to Eq. (11), the self-propulsion distribution is different from that of a free self-propelled particle,

$$P^{ss}(f) = \int dx P^{ss}(x, f)$$  \hspace{1cm} (13)

$$= \left( \frac{b}{\pi} \right)^{1/2} \exp \left( -\frac{b f^2}{\pi} - cf - c^2/(4b) \right).$$

We note that to show that the above distribution is consistent with Eq. (10) one should pay attention to the $x = 0$ boundary term.

In particular, in the present case there is a non-zero local stationary state self-propulsion:

$$\langle f \rangle_{ss} = \int df f P^{ss}(f|x) = -\frac{c}{2b} = mg,$$  \hspace{1cm} (14)

where $\langle \rangle_{ss}$ denotes the local stationary state average or, more precisely, the stationary state average over self-propulsion under the condition that the particle is at position $x$. In other words, $P^{ss}(f|x)$ in Eq. (14) is the conditional stationary state distribution of the self-propulsion force,

$$P^{ss}(f|x) = P^{ss}(x, f)/P^{ss}(x),$$  \hspace{1cm} (15)

where $P^{ss}(x)$ is the stationary state distribution of particle’s positions, $P^{ss}(x) = \int df P^{ss}(x, f)$. 

be showed that the following distribution is a stationary solution of Eq. (10):

$$P^{ss}(x, f) \propto \exp \left( -ax - bf^2 - cf \right),$$  \hspace{1cm} (11)

where $a = mg\xi_0^2/D_f$, $b = \gamma/(2D_f)$ and $c = -a/(\xi_0\gamma)$. 

It follows from Eq. (11) that the stationary state distribution of positions is exponential,

$$P^{ss}(x) = \int df P^{ss}(x, f) \propto \exp(-x/\delta_{eff})$$  \hspace{1cm} (12)

where the so-called sedimentation length $\delta_{eff} = 1/a = D_f / (mg\xi_0\gamma^2)$. We note that the sedimentation length of a dilute system of non-active Brownian particles at temperature $T$ is given by $\delta = T/(\langle mg \rangle)$. We can thus conclude that the sedimentation length of a dilute system of self-propelled particles has the same form as that of non-active Brownian particles if instead of the equilibrium temperature one uses effective temperature $T_{eff}$ of a free self-propelled particle, $T_{eff} = D_f / (\xi_0\gamma^2)$. This agrees with the experimental result of Palacci et al. [2] obtained for a slightly different system of the active Brownian particles. We note that the consistency between the free particle effective temperature and the sedimentation length is not obvious. In fact, Tailleur and Cates [18] showed that for the run-and-tumble model of active particles the stationary state distribution in a linear potential has the exponential form, but the free particle effective temperature does not determine the sedimentation length.

For a single self-propelled particle under the influence of a constant gravitational force, the equation of motion has the following form:

$$\partial_t P(x, f; t) = -\frac{1}{\xi_0} \frac{\partial}{\partial x} \left( (f - mg) P(x, f; t) \right)$$  \hspace{1cm} (10)

$$+ \frac{\partial}{\partial f} \left( \gamma f P(x, f; t) + D_f \frac{\partial}{\partial f} P(x, f; t) \right),$$

where $g$ is the gravitational acceleration and $m$ is the mass of the particle. Note that this equation is only valid above a lower wall, which we assume to be located at $x = 0$. In principle, Eq. (10) has to be accompanied by a term that ensures that the current through the lower wall vanishes. This term is not needed for finding the stationary state distribution since in the stationary state the current vanishes everywhere.

We note that the so-called drift coefficients [10] in Eq. (10) are linear in $x$ and $f$. This fact suggests looking for a stationary distribution having a Gaussian form. It can be showed that the following distribution is a stationary solution of Eq. (10):

$$P^{ss}(x, f) \propto \exp \left( -ax - bf^2 - cf \right),$$  \hspace{1cm} (11)
Non-zero average self-propulsion follows from the condition that there should be no net current in the stationary state. Let’s define the current density,

$$\partial_t P(x; t) = -\partial_x j(x; t)$$

(16)

where $P(x; t) = \int df P(x, f; t)$. Thus the current density is given by

$$\xi_0^{-1} \left( \int df f P(x, f; t) - mg P(x; t) \right)$$

(17)

and therefore in the stationary state we need to have $\langle f \rangle_{ss} = mg$.

IV. SELF-PROPELLED PARTICLE IN A HARMONIC POTENTIAL

We show in this section that the effective temperature defined through the long-time diffusive motion of a free self-propelled particle does not always determine the stationary state probability distribution of the particle’s position in an external harmonic potential. To analyze this finding a little further, we investigate the particle’s position auto-correlation function and the linear response to an external perturbation, and use these analyses to examine a fluctuation-dissipation relation-based effective temperature.

A. Stationary state probability distribution

For a single self-propelled particle in a harmonic potential, the equation of motion for the joint probability distribution of the position and self-propulsion force has the following form:

$$\partial_t P(x, f; t) = -\frac{1}{\xi_0} \frac{\partial}{\partial x} \left( (f - kx) P(x, f; t) \right)$$

$$+ \frac{\partial}{\partial f} \left( \gamma f P(x, f; t) + D_f \frac{\partial}{\partial f} P(x, v; t) \right),$$

(18)

where $k$ is the force constant that determines the strength of the potential.

Again, we note that the so-called drift coefficients [16] in Eq. (18) are linear in $x$ and $f$ and therefore a stationary distribution has a Gaussian form,

$$P^{ss}(x, f) \propto \exp \left( -ax^2 - bf^2 - cf x \right),$$

(19)

where $a = k\xi_0 (\gamma + k/\xi_0)^2 / (2D_f)$, $b = (\gamma + k/\xi_0) / (2D_f)$ and $c = -k (\gamma + k/\xi_0) / D_f$.

It follows from Eq. (19) that the stationary distribution of particle’s positions is also Gaussian,

$$P^{ss}(x) = \int df P^{ss}(x, f)$$

$$= \left( \frac{a - c^2/(4b)}{\pi} \right)^{1/2} \exp \left( - (a - c^2/(4b)) x^2 \right),$$

(20)

where $a - c^2/(4b) = (k/2) (\gamma + k/\xi_0) \gamma \xi_0 / D_f$. If we were to define effective temperature through the relation $P^{ss}(x) \propto \exp(-V^{ext}(x)/T_{eff})$, we would get

$$T_{eff} = D_f / (\gamma \xi_0 (\gamma + k/\xi_0)).$$

(21)

We note that this effective temperature is different from that defined through the long-time diffusive motion of the free self-propelled particle, Eq. (4). We note, furthermore, that in the present problem there are two different time scales. First, there is the time scale on which the self-propulsion force forgets its initial value. As for the free particle, this time scale is proportional to $\gamma^{-1}$. Second, there is the characteristic time scale for the relaxation of a particle moving in a viscous medium under the influence of a harmonic potential. This time scale is proportional to $\xi_0/k$. If the former time scale is much shorter than the latter, $\gamma^{-1} \ll \xi_0/k$, the effective temperature coincides with the effective temperature of the free self-propelled particle. In the opposite case, $\gamma^{-1} \gg \xi_0/k$, which is the interesting strong self-propulsion limit, the effective temperature approaches $D_f / (k\gamma)$ and can be significantly lower than that of the free self-propelled particle.

In contrast to the case of a constant external force, the self-propulsion distribution of a particle moving under the influence of a harmonic force agrees with that of the free self-propelled particle,

$$P^{ss}(f) = \int dx P^{ss}(x, f)$$

$$= \left( \frac{b - c^2/(4a)}{\pi} \right)^{1/2} \exp \left( - (b - c^2/(4a)) f^2 \right),$$

(22)

where $b - c^2/(4a) = \gamma/(2D_f)$.

However, there is still non-zero local stationary state self-propulsion,

$$\langle f \rangle_{ss} = \int df \int P^{ss}(f, x) = -\frac{cx}{2b} = kx.$$  

(23)

This result is not unexpected since the joint stationary state distribution does not factorize into distributions of positions and self-propulsions. Physically, this happens because particles with larger (albeit temporary) self-propulsions are able to venture farther into the high potential energy regions.

Finally, we note that, as in the case of a constant external force, in the stationary state the current density vanishes,

$$\xi_0^{-1} \left( \int df \int P^{ss}(x, f) - kxP^{ss}(x) \right) = 0.$$  

(24)

B. Particle’s position auto-correlation function

We use standard methods [16] to derive coupled equations of motion for the time-dependent auto-correlation
function of the position of the self-propelled particle, $\langle x(t)x(0) \rangle$, and the correlation function between the self-propulsion force at time $t$ and the position at the initial time, $\langle f(t)x(0) \rangle$,

$$
\partial_t \langle x(t)x(0) \rangle = \frac{\xi_0}{\xi_0} \langle f(t)x(0) \rangle - k\xi_0^{-1} \langle x(t)x(0) \rangle \tag{25}
$$

$$
\partial_t \langle f(t)x(0) \rangle = -\gamma \langle f(t)x(0) \rangle \tag{26}
$$

Usually, equations of motion for these two functions would involve other, more complicated time-dependent correlation functions. The equations above are closed due to the simplicity of the external potential. Initial conditions for Eqs. (25) are $\langle x^2 \rangle$ and $\langle f \rangle = k \langle x^2 \rangle$, where, from Eq. (20), $\langle x^2 \rangle = D_f / (k\gamma \xi_0 (\gamma + k/\xi_0))$.

Equations of motion (25, 26) can be easily solved. The solution for the particle’s position auto-correlation function reads

$$
\langle x(t)x(0) \rangle = \left( \frac{\gamma}{\gamma - k/\xi_0} e^{-\frac{k}{\xi_0} t} + \frac{k/\xi_0}{k/\xi_0 - \gamma} e^{-\gamma t} \right) \langle x^2 \rangle \tag{27}
$$

We note two qualitatively different behaviors in two limiting cases identified in the previous subsection. If the self-propulsion force relaxation time is the shortest relevant time scale, $\gamma^{-1} \ll \xi_0/k$, we get

$$
\langle x(t)x(0) \rangle \approx e^{-\gamma t \frac{D_f}{k^2 \gamma}}. \tag{28}
$$

In this case the self-propelled particle’s position auto-correlation function has the same form as the auto-correlation function of a non-active Brownian particle in equilibrium in an external harmonic potential.

In the opposite limit, $\gamma^{-1} \gg \xi_0/k$, we get

$$
\langle x(t)x(0) \rangle \approx e^{-\frac{k}{\xi_0} t \frac{D_f}{k^2 \gamma}}. \tag{29}
$$

We note that in this limit the time dependence of the particle’s position auto-correlation function is slaved to the evolution of the self-propulsion force. Interestingly, the auto-correlation function is independent of the friction coefficient $\xi_0$.

C. Linear response to an external force

To calculate a linear response function we consider the self-propelled particle in the harmonic potential and under an influence of a weak time-dependent force. In this case, the evolution equation for the joint probability distribution of the position and the self-propulsion force has the following form,

$$
\partial_t P(x, f; t) = -\frac{1}{\xi_0} \frac{\partial}{\partial x} (\langle f - kx \rangle P(x, f; t)) \tag{30}
$$

$$
+ \frac{\partial}{\partial f} \left( \gamma f P(x, f; t) + D_f \frac{\partial}{\partial f} P(x, v; t) \right)
$$

$$
- \frac{1}{\xi_0} \frac{\partial}{\partial x} \left( f^{ext}(t) P(x, f; t) \right). \tag{31}
$$

Here $f^{ext}(t)$ is a weak, time-dependent force which, following the analysis of the linear response in equilibrium [19], we take to be position-independent.

To examine the linear response we linearize Eq. (30) with respect to the external force. In this way we get the following equation for the difference between the probability distribution in the presence of the force and its stationary state form, $\delta P(x, f; t) = P(x, f; t) - P^{ss}(x, f)$,

$$
\partial_t \delta P(x, f; t) = -\frac{1}{\xi_0} \frac{\partial}{\partial x} \left( (f - kx) \delta P(x, f; t) \right) \tag{31}
$$

$$
+ \frac{\partial}{\partial f} \left( \gamma f P(x, f; t) + D_f \frac{\partial}{\partial f} \delta P(x, v; t) \right)
$$

$$
- \frac{1}{\xi_0} \frac{\partial}{\partial x} \left( f^{ext}(t) P^{ss}(x, f; t) \right). \tag{32}
$$

We assume that the force is turned on at $t = 0$ and thus the initial condition for $\delta P(x, f; t)$ is $\delta P(x, f; t = 0) = 0$.

Our goal is to calculate the time-dependent change of the particle’s position, $\delta \langle x(t) \rangle = \int dx df \delta x P(x, f; t)$. To this end we use Eq. (31) to derive coupled equations of motion for $\delta \langle x(t) \rangle$ and $\delta \langle f(t) \rangle = \int dx df \delta f P(x, f; t)$.

$$
\partial_t \delta \langle x(t) \rangle = \frac{1}{\xi_0} \delta \langle f(t) \rangle - \frac{k}{\xi_0} \delta \langle x(t) \rangle + \frac{1}{\xi_0} f^{ext}(t) \tag{33}
$$

The initial conditions for these equations are $\delta \langle x(t = 0) \rangle = 0 = \delta \langle f(t = 0) \rangle$.

Eqs. (32) can be easily solved. We get $\delta \langle f(t) \rangle \equiv 0$ and

$$
\delta \langle x(t) \rangle = \frac{1}{\xi_0} \int_0^t dt' e^{-\frac{k}{\xi_0} (t-t')} f^{ext}(t'), \tag{34}
$$

and thus the response function is given by

$$
R(t) = \frac{1}{\xi_0} e^{-\frac{k}{\xi_0} t}. \tag{35}
$$

D. Fluctuation-dissipation relation

The form of the joint stationary state distribution [19] suggests an effective temperature can be defined for both rapidly evolving self-propulsion force (in which case $T_{eff}$ is the same as the one defined through diffusive motion of the free particle) and for the more interesting slowly evolving self-propulsion force (strong self-propulsion limit). Physically, in the former case the existence of an effective temperature is expected but in the latter case it seems to be related to the special form of the interaction potential. Here, to investigate this a little further, we examine a different way to introduce an effective temperature, one that uses a fluctuation-dissipation relation (FDR).

Following a recent review [20] we define a frequency-dependent fluctuation-dissipation relation-based effective
temperature

\[ T_{\text{eff}}^{\text{FDR}}(\omega) = \frac{\omega \text{Re} C(\omega)}{\chi''(\omega)}, \quad (36) \]

where \( C(\omega) \) is the one-sided Fourier transform of the particle's position auto-correlation function, \( C(\omega) = \int_0^\infty e^{i\omega t} \langle x(t)x(0) \rangle \), and \( \chi''(\omega) \) is the imaginary part of the one-sided Fourier transform of the response function, \( \chi''(\omega) = \text{Im} \int_0^\infty e^{i\omega t} R(t) \).

Using explicit forms of the auto-correlation function and the response function we get

\[ T_{\text{eff}}^{\text{FDR}}(\omega) = \frac{D_f}{\xi_0 (\omega^2 + \gamma^2)}. \quad (37) \]

In principle, the fluctuation-dissipation relation-based effective temperature is frequency-dependent and, thus, the fluctuation-dissipation relation is violated. A more appropriate interpretation of Eq. \[37\] is that, in the limit of small-frequency perturbations, \( \omega \ll \gamma \), the fluctuation-dissipation relation is recovered and \( T_{\text{eff}}^{\text{FDR}}(\omega) \) coincides with the effective temperature obtained from the long-time diffusive motion of the free self-propelled particle. We note that a similar agreement of effective temperatures measured in different ways has been found by Loi et al. \[12\].

We shall emphasize, however, that in the strong self-propulsion limit, \( \gamma^{-1} \gg \xi_0/k \), the free self-propelled particle-based effective temperature does not determine the stationary state distribution.

V. DISCUSSION

We shall emphasize three points which we expect to be applicable beyond our simple toy model. First, the most natural effective temperature defined through the long-time diffusive motion of a free self-propelled particle does not always determine the stationary state distribution in an external field, even in the dilute (single particle) limit. In other words, the result of Palacci et al. \[3\] is highly non-trivial. Second, a well defined low frequency limit of the fluctuation-dissipation relation-based effective temperature may exist and it may be relevant for some properties of the self-propelled particle \[12\].

However, it does not necessarily determine the stationary state of the self-propelled particle in an external potential.

Third, even though the self-propulsion force evolves independently of the state of the self-propelled particle (and independently of the interaction of this particle with an external force or with other particles), non-trivial correlations between self-propulsion force and the particle's position can develop. We can also expect that in the case of many interacting self-propelled particles, correlations between self-propulsions and distances between particles can develop. These correlations imply the appearance of a non-trivial anisotropic pair distribution function which is thought to be responsible for the instability of a single phase uniform state in some systems of self-propelled particles \[21\].

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