Gravitational Force by Point Particle in Static Einstein Universe

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Abstract

The gravitational force produced by a point particle, like the sun, in the background of the static Einstein universe is studied. Both the approximate solution in the weak field limit and exact solution are obtained. The main properties of the solution are i) near the point particle, the metric approaches the Schwarzschild one and the radius of its singularity becomes larger than that of the Schwarzschild singularity, ii) far from the point particle, the metric approaches the static Einstein closed universe. The maximum length of the equator of the universe becomes smaller than that of the static Einstein universe due to the existence of the point particle. These properties show the strong correlation between the particle and the universe.

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1. Introduction

Since originally Einstein introduced cosmological constant term in his gravitational field equations to obtain the static universe, the existence of the cosmological constant has been paid noticeable attention. The observed cosmological constant is many order of magnitude smaller than the theoretically expected value. This is known as the cosmological constant problem [1], [2]. Recently, it was reported that models of cosmology with nonzero cosmological constant can successfully account for the deep galaxy counts [3] and the observed flat rotation curve of galaxies [4]. Independently of these, recent observation of the Hubble parameter strongly suggests the existence of the positive value of the cosmological constant [5]. It has been widely investigated to apply the Einstein equations with a cosmological term to obtain a gravitational field produced by a mass density and to study a whole universe [6], [7].

In this paper, we apply the Einstein equations with a cosmological term to the general isotropic metric and derive the gravitational potential produced by the spherically symmetric point matter in the closed Einstein universe. From another point of view of this problem, we study the effect of the existence of the background universe to the gravitational potential produced by a point matter. Nature of the closed universe may affect strongly to the local gravitational potential. The correlation between the local quantity (the gravitational potential by the point matter) and the global quantity (the static Einstein universe) are examined.

The paper is organized as follows. In Sec.2 the Einstein equations with the cosmological term and the constraint on energy density and the cosmological constant of our model are presented. In Sec.3 we study the approximate solutions of weak static field in order to understand the physical meaning well. In Sec.4 we consider the exact solution. Concluding discussions are given in Sec.5.

2. Einstein Equations and Constraint on Energy Density

We start with the Einstein equations with cosmological term $\Lambda$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

(1)

where $G$ is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor of the matter. Throughout this paper we follow the conventions of Ref. [8]. We now apply the theory to
a static isotropic metric in order to represent the gravitational fields produced by a point particle in the background of the static universe. The general form of the static isotropic metric in the standard form is

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2\{d\theta^2 + \sin^2 \theta d\phi^2\} , \]  

(2)

where \( \nu \) and \( \lambda \) are functions of radial coordinate \( r \) only.

The energy-momentum tensor is assumed to be composed of two parts, i.e., that of the perfect fluid \( T^\mu_\nu^F \) and that of a point particle \( T^\mu_\nu^P \):

\[ T^\mu_\nu = T^\mu_\nu^F + T^\mu_\nu^P , \]  

(3)

\[ T^\mu_\nu^F = p(r) g^\mu_\nu + (\rho(r) + p(r)) u^\mu u^\nu , \]  

(4)

\[ T^\mu_\nu^P = M \delta^{(3)}(\vec{r}) , \]  

(5)

where \( \rho \), \( p \) and \( u^\mu \) denote the energy density, the pressure and the four-velocity of the perfect fluid respectively and \( M \) denotes the mass of the particle. The energy-momentum tensor obeys the local conservation equation,

\[ \nabla_\mu T^\mu_\nu = 0 . \]  

(6)

Using the metric Eq. (2) and the energy-momentum tensor Eqs. (3)-(5), the independent component of the Einstein equations are

\[ \frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\nu'}{r} \right) - \Lambda = 8\pi G \{ \rho + M \delta^{(3)}(\vec{r}) \} , \]  

(7)

\[ \frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \Lambda = -8\pi G p , \]  

(8)

\[ \frac{1}{2} e^{-\lambda} \left( \frac{1}{2} \nu' \lambda' - \nu'' \right) - \frac{1}{2} \nu'^2 - \frac{\nu'}{r} + \frac{\lambda'}{r} - \Lambda = -8\pi G p , \]  

(9)

where prime stands for derivative with respect to \( r \). The local conservation of the energy-momentum tensor becomes

\[ p' + \frac{\nu'}{2} \{ p + \rho + M \delta^{(3)}(\vec{r}) \} = 0 . \]  

(10)

Here we impose the constraint between the energy density and the cosmological constant, in order to obtain the solution of the gravitational potential in the static Einstein universe, as

\[ \rho = \rho_0 \quad \text{(constant)} , \]  

(11)
\[
\Lambda = 4\pi G\rho_0 = \frac{1}{a^2},
\]
where \(a\) is a kind of the constant radius of the universe.

With the constraint Eqs. (11) and (12), and using the dimensionless radial variable
\[\eta = \frac{r}{a},\]
the Einstein equations Eqs. (7)-(9) and the local conservation of the energy-momentum tensor Eq. (10) become as
\[
\frac{1}{\eta^2} - e^{-\lambda}\left(\frac{1}{\eta^2} - \frac{\dot{\lambda}}{\eta}\right) - 3 = 8\pi a^2 GM\delta^{(3)}(\vec{r}) ,
\]
\[
\frac{1}{\eta^2} - e^{-\lambda}\left(\frac{1}{\eta^2} + \frac{\dot{\nu}}{\eta}\right) - 1 = -8\pi a^2 G\rho ,
\]
\[
\frac{1}{2} e^{-\lambda}\left\{\frac{1}{2}\dot{\nu}\dot{\lambda} - \ddot{\nu} - \frac{1}{2}\dot{\nu}^2 - \frac{\dot{\lambda}}{\eta}\right\} - 1 = -8\pi a^2 G\rho ,
\]
and
\[
\dot{p} + \frac{\dot{\nu}}{2}\left\{p + \rho_0 + M\delta^{(3)}(\vec{r})\right\} = 0 ,
\]
where dot stands for derivative with respect to \(\eta\). Among these four equations Eqs. (13)-(15) and (16), three independent equations determine the three unknown functions \(\nu, \lambda\) and \(p\).

The boundary condition is required, in order to represent the Newtonian potential in the Einstein universe, as
\(i)\) near the point particle, the solution represents the Newtonian potential, \(ii)\) far from the point particle, the metric reduces to one of the Einstein universe.

The exact solution of the vacuum Einstein equation with cosmological term is known as the Schwarzschild-de Sitter solution:
\[
e^{\nu} = e^{-\lambda} = 1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2 .
\]

In the following sections, we will study another type of solution of the Einstein equations with cosmological term not in the flat space-time but in the closed universe.

3. Weak Field Approximation
In this section, we study an approximate solution of Eqs. (13)-(15) and (16), which is of first order in the gravitational constant $G$. This weak field solution plays the important role to understand the physical meaning of the gravitational potential produced by the point particle in the background of the closed universe. The metric tensor is expressed as

$$-g_{00} = e^{\nu} = 1 + 2\phi(\eta),$$

$$g_{11} = e^{\lambda} = \frac{1 + 2\psi(\eta)}{1 - \eta^2}. (17)$$

The weak fields $\phi$ and $\psi$ as well as the pressure $p$ are meaningful only within the linear approximation. From the local conservation law Eq. (16), the pressure is expressed by $\phi$ as,

$$p = -\rho_0 \phi. (19)$$

Eliminating the pressure using Eq. (19), the Einstein field equations in Eqs. (13)- (15) become,

$$\frac{(1 - \eta^2)}{\eta} \dot{\psi} + \frac{1}{\eta^2 - 3} \psi = 4\pi a^2 GM \delta^{(3)}(\vec{r}), (20)$$

$$\eta \ddot{\phi} + \frac{\eta^2}{1 - \eta^2} \phi = \psi, (21)$$

and

$$\eta \dddot{\phi} + \frac{1}{1 - \eta^2} \ddot{\phi} + \frac{2\eta}{(1 - \eta^2)^2} \phi = \dot{\psi}. (22)$$

Eq. (22) is automatically satisfied by using Eqs. (20) and (21). We can solve Eq. (20) for $\psi$ as

$$\psi = \frac{\psi_0}{\eta(1 - \eta^2)}, (23)$$

where $\psi_0$ is a integration constant. Inserting Eq. (21) into Eq. (20), the equation for $\phi$ is obtained

$$\Delta \phi + \frac{3}{a^2} \phi = 4\pi GM \delta^{(3)}(\vec{r}), (24)$$

where the Laplacian operator in static closed universe is

$$\Delta \equiv \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j) ,$$

$$= \frac{1}{a^2} \left[ (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} + \frac{(2 - 3\eta^2)}{\eta} \frac{\partial}{\partial \eta} \right]. (25)$$
The general solution of this equation for $\phi$ is

$$\phi = b_1 \frac{1 - 2\eta^2}{\eta} + b_2 \sqrt{1 - \eta^2},$$

(26)

where $b_1$ and $b_2$ are integration constants. According to boundary condition, which is in the previous section, the integration constants are determined as

$$\psi_0 = -b_1 = GM/a, \quad b_2 = 0.$$

The invariant distance takes

$$ds^2 = -\{1 - \frac{2GM}{r}(1 - (r/a)^2)\}dt^2 + \frac{1 + \frac{2GM}{r(1 - (r/a)^2)}}{1 - (r/a)^2}dr^2 + r^2\{d\theta^2 + \sin^2 \theta d\phi^2\}.$$

(27)

The comoving radial coordinate, in the linear approximation, is given by

$$\chi(r) = \int_0^r \frac{dr}{\sqrt{1 - (r/a)^2}} = a \sin^{-1}(r/a).$$

(28)

The gravitational potential is expressed by using the comoving radial coordinate

$$\phi(\rho) = -\frac{GM \cos(2\chi/a)}{a \sin(\chi/a)},$$

$$\simeq -\frac{GM}{\chi} \text{ for } \chi/a \ll 1,$$

(29)

and has the symmetry

$$g_{00}(\pi - \chi/a) = g_{00}(\chi/a).$$

(30)

This means that the gravitational potential, as well as the mass of the point particle, is symmetric with respect to the north pole ($\chi \simeq 0$) and the south pole ($\chi \simeq a\pi$) of the closed universe.

So far, we studied some physical implications of our model within the weak field approximation. In the next section, we will explore the exact solution, which will clear the non-linear effects of our model.

4. Exact Solution
In this section, we discuss exact solution of Eqs. (13)-(15) and (16), which are the Einstein equations with cosmological term Eq. (1) in the general static isotropic metric Eq. (2) with the constraint Eqs. (11) and (12). The exact solution is obtained in a similar way as the approximate one. The local energy-momentum conservation Eq. (16) is solved and the pressure is obtained as
\[ p = \rho_0 (e^{-\nu/2} - 1) , \]  
(31)
where integration constant is chosen to coincide with the approximate solution Eq. (19) in the weak field limit. The radial component of the metric is solved from the Eq. (13) as
\[ e^\lambda = \frac{1}{1 - \eta^2 - b/\eta} , \]  
(32)
where \( b \) is the integration constant. Inserting Eqs. (31) and (32) into Eq. (14), the equation for the time component of the metric becomes
\[ \frac{d}{d\eta} e^{\nu/2} + \frac{\eta - b/(2\eta^2)}{1 - \eta^2 - b/\eta} e^{\nu/2} = \frac{\eta}{1 - \eta^2 - b/\eta} . \]  
(33)
The solution is obtained in the integral form:
\[ e^{\nu/2} = 1 + \frac{b}{2} \int \sqrt{1 - \eta^2 - b/\eta} \int^{\eta} \frac{1}{\eta^2(1 - \eta^2 - b/\eta)^{3/2}} d\eta , \]  
(34)
where the lower value of the integration \( \eta_* \) is recognized as a integration constant. The integration in Eq. (34) is a kind of the elliptic integral and is estimated approximately in extreme cases. The resultant asymptotic expression for the time component of the metric is
\[ e^{\nu/2} \simeq \begin{cases} 
1 & \text{for } b = 0 , \\
1 - \frac{b(1 - 2\eta^2)}{2\eta} + b \frac{(1 - 2\eta^2)}{2\eta_*} \sqrt{1 - \eta^2} - \frac{\sqrt{1 - \eta^2}}{\sqrt{1 - b/\eta_*}} & \text{for } b \ll 1 , \\
\frac{\sqrt{1 - b/\eta}}{\sqrt{1 - b/\eta_*}} & \text{for } a \to \infty . 
\end{cases} \]  
(35)
We impose the same boundary condition as in the previous section, and one integration constant is determined as
\[ b = 2GM/a . \]  
(36)
Another integration constant \( \eta_* \) is determined as the solution of the condition
\[ \frac{\partial}{\partial \chi} g_{00} = \sqrt{1 - \eta^2 - b/\eta} \frac{\partial}{\partial \eta} g_{00} = 0 , \]  
(37)
at the equator of the closed universe (which corresponds to the largest value of the radial coordinate \(\eta_+\) defined in Eq.\((39)\) below) and, where, \(\chi\) denotes the comoving radial coordinate. We note that the above condition matches the boundary condition. This integration constant is estimated, in a weak field approximation \((b \ll 1)\), as

\[
\eta_* \simeq \frac{1}{\sqrt{2}}.
\]  

(38)

From the asymptotic expression of the time component of the metric, three specific features are obtained; 

\(i\) if the mass of the point particle disappears \((b = 2GM/a = 0)\), the solution reduces to one of the Einstein universe as in the case of the approximate solution, 

\(ii\) for small \(b\) \((= 2GM/a \ll 1)\), it reproduces the approximate solution of Eq.\((26)\) in the previous section, 

\(iii\) if the radius of the universe tends to infinity \((a \to \infty)\), it coincides with the Schwarzschild solution. 

Three pole positions of the radial part of the metric, defined as

\[
(g_{11})^{-1} = 1 - \eta^2 - b/\eta,
\]

\[
= -\frac{1}{\eta}(\eta - \eta_+)(\eta - \eta_0)(\eta - \eta_+)\), \quad (\eta_- < \eta_0 < \eta_+),
\]

(39)

are obtained for small \(b = 2GM/a\) as

\[
\eta_- \simeq -1 - b/2, \quad \eta_0 \simeq b + b^3, \quad \eta_+ \simeq 1 - b/2.
\]  

(40)

The following properties are obtained from consideration of the pole positions. (1) The radius of the singularity by the mass of the particle \(\eta_0\) becomes a little larger than that of the Schwarzschild singularity \(b = 2GM/a\). (2) The maximum proper length \(a\eta_+\) becomes smaller than that of the Einstein static universe. As a result, the size of the universe around the equator \((r = a\eta_+, \theta = \pi/2)\) is calculated to be;

\[
\int_0^{2\pi} \sqrt{g_{33}}d\phi = \int_0^{2\pi} r \sin \theta d\phi = 2\pi a\eta_+,
\]

(41)

which is smaller than that of the Einstein universe. The correlation between the part (the point particle) and the whole (the universe) is a specific property of our solution.

5. Discussions
We have obtained the approximate and the exact solution of the spherically symmetric gravitational field produced by the point particle in the background of the static Einstein universe. About this solution, the following points are noticed.

(a) The three dimensional scalar curvature is constant, but the radial part of the metric has singularity at the origin of the radial coordinate, which corresponds to the energy source of the point particle.

(b) The boundary condition of our solution at infinity is not flat space-time. So our solution does not agree with the Schwarzschild solution contrary to the Birkhoff theorem [9]. The Schwarzschild solution is realized only in the asymptotic case ( $a \to \infty$ ) in our solution (see Eq. (35)).

(c) As is well known, the Einstein universe solution is unstable. This property reflects the presence of the pressure $p = -\rho_0 \dot{\phi}$ (Eq. (19)) or $p = \rho_0 (e^{-\nu /2} - 1)$ (Eq. (31)) in our solution. We have treated the energy density of the perfect fluid to be constant $\rho = \rho_0$. If this energy density is treated as a dynamical variable, it will gather toward the position of the point particle by the pressure and the universe will become unstable.

The work to obtain the gravitational field produced by the various types of the local quantities, i.e., mass, charge, etc., in the background of the expanding universe is now under investigation. Also, the topological structure of the singularities of the metric may be important [10] and that of our solution will be discussed elsewhere.

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