EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR CHOQUARD EQUATION WITH VARIABLE EXPONENT GROWTH

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Abstract. In this paper, we investigate the following Choquard equation

\[-\Delta u = (I_\alpha * |u|^{f(x)})|u|^{f(x)-2}u \quad \text{in } \mathbb{R}^N,\]

where $N \geq 3$, $\alpha \in (0, N)$ and $I_\alpha$ is the Riesz potential. If

\[ f(x) = \begin{cases} p, & x \in \Omega, \\ \frac{(N + \alpha)}{(N - 2)}, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \]

where $1 < p < \frac{N + \alpha}{N - 2}$ and $\Omega \subset \mathbb{R}^N$ is a bounded set with nonempty, we obtain the existence of positive ground state solutions by using the Nehari manifold.

1. Introduction. The stationary Choquard equation

\[ \begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1} \]

where $N \geq 3$, $I_\alpha$ is the Riesz potential of order $\alpha \in (0, N)$ defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

\[ I_\alpha = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{in which } A_\alpha = \frac{\Gamma\left(\frac{N-2}{2}\right)}{2^{\alpha} \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)}, \]

and $\Gamma$ is the Gamma function (see [37, p.19]), arises in many interesting physical situations in quantum theory and plays an important role in the theory of Bose-Einstein condensation where it accounts for the finite-range many-body interactions. Especially, $N = 3$ and $\alpha = p = 2$ in equation (1), that is,

\[-\Delta u + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{2} \]

It was investigated by Pekar in [34] to study the quantum theory of a polaron at rest. In [19], Choquard applied it as approximation to Hartree-Fock theory of one-component plasma. This equation was also proposed by Penrose in [28] as a model of self gravitating matter and was known in that context as the Schrödinger-Newton equation.

Mathematically, the existence of solutions for problem (2) has been studied with variational techniques by Lieb [19], Lions [21] and Menzala [27]. Ma and Zhao [25] supported by National Natural Science Foundation of China (No.11471267).

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investigated problem (2) and obtained the uniqueness of the positive solutions by the radial symmetry and monotone decreasing property of all positive solutions. Later, Wang and Yi [41] based on the classification results of Ma and Zhao, showed the uniqueness of the positive solutions of problem (2) for $N = 3, 4, 5$.

For problem (1), there exist lots of papers in which the authors considered the existence and qualitative properties of solutions. To illustrate, Moroz and Van Schaftingen [31] obtained the existence of a positive ground state solution for problem (1) with $p \in (\frac{N + \alpha}{N}, \frac{N + \alpha}{N} - \frac{2}{N})$, then Ruiz and Van Schaftingen [38] proved that least energy nodal solution of problem (1) is odd with respect to a hyperplane of $\mathbb{R}^N$ where $\alpha$ is either close to 0 or close to $N$ and $p \in \{2, \frac{N + \alpha}{N} - \frac{2}{N}\}$.

The variants of problem (1) that include the introduction of an external potential and the treatment of more general nonhomogeneous nonlinearities instead of $|u|^p$ were also considered. Such as Alves, Nobrega and Yang [5] obtained Multi-bump solutions for problem (1) with deepening potential well (also see [24]), Van Schaftingen and Xia [40] showed the existence of a groundstate, of an infinite sequence of solutions of unbounded energy and the existence of least energy nodal solution with the coercive potential, Zhang et al. [43] assumed that a more weak condition than coercivity on potential, precisely, there exists a constant $r > 0$ such that, for any $M > 0$, $\{x \in \mathbb{R}^N : |x - y| \leq r, V(x) \leq M\} \to 0$ as $|y| \to +\infty$. Moroz and Van Schaftingen [32] considered problem (3) with $p = \frac{N + \alpha}{N}$ and an external potential, [30] studied problem (1) when the homogeneous nonlinearities $|u|^p$ is replaced by $F(u)$, [13, 44] studied problem (1) with an external potential and a more general nonlinearity $G(x, u)$, [14, 15] considered problem (1) in bounded domains, and so on. For a complete and updated discussion upon the current literature of such problems, we refer the interested reader to the guide [29]. We also mention [3, 4, 33], where the semiclassical case is considered.

In recent years, there has been an increasing interest in the study about problem of elliptic equation with variable exponent growth conditions, who are motivated by their applications to the mathematical models. For example, non-Newtonian fluids [8] (in particular, electro-rheological fluids [1]), nonlinear Darcy’s law in porous medium [7], image processing [12]. For the precise applied background we refer to [9, 34, 36, 45]. Moreover, the qualitative properties of solutions for these problems were considered by many authors. Further results for related problems can be founded in [2, 6, 11, 17, 18, 23, 26, 35] and references therein. However, it is a new topic that the study on the existence of solutions for equations (3) up until now. Thus in this paper, we are interested in the existence of solutions for the following Choquard equation with variable exponent growth

$$
\begin{align*}
-\Delta u &= (I_\alpha * |u|^{f(x)})|u|^{f(x)-2}u, \\
&u \in D^{1,2}(\mathbb{R}^N),
\end{align*}
$$

(3)

where $f$ is a given function. Inspired by above papers, we assume that $f$ satisfies the following conditions

$(f_1)$ there exists a bounded set $\Omega \subset \mathbb{R}^N$ with nonempty and $1 < p < \frac{N + \alpha}{N - 2}$ such that $f(x) = \begin{cases} 
 p, & x \in \Omega, \\
 (N + \alpha)/(N - 2), & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}$

The exponent $\frac{N + \alpha}{N - 2}$ (or $\frac{N + \alpha}{N}$) is called the upper (or lower) critical exponent with respect to the Hardy-Littlewood-Sobolev inequality (see [20, 21]).

Here we give our main result.
**Theorem 1.1.** Assume that $N \geq 3$, $\alpha \in ((N-4)^+, N)$ and $(f_1)$ holds. Then problem (3) has a positive ground state solution.

**Remark 1.** In the present paper, we consider problem (3) in the whole space, generally speaking, whose difficulties lies in two main aspects. Firstly, the compactness fails in the whole space, and secondly $D^{1,2}(\mathbb{R}^N)$ is embedded in $L^q(\mathbb{R}^N)$ if and only if $q = \frac{N}{N-2}$, moreover, the embedding is not compact. Due to the lack of compactness, bounded $(PS)_m$ sequence could not converge, and the methods in [14, 15] do not apply to Choquard equation (3) in the whole space. Under the assumption $(f_1)$, the equation (3) exponent varies with the change of $x$, so it is not autonomous, and then it is meaningless that working in spaces of radial functions to recover the compactness by translations.

To overcome those difficulties, we will use some analysis methods and the Nehari manifold. Finally, we obtain the existence of positive ground state solutions of equation (3) through the critical point theory.

**Remark 2.** It is known that the functional associated with problem (3) is well defined if and only if $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$. The restriction $\frac{N+\alpha}{N} \leq p$ is required in problem (3) in order to control the nonlocal term in terms of the $L^2(\mathbb{R}^N)$ norm (see [31]). However for our problem, the restriction $\frac{N+\alpha}{N} < p$ isn’t necessary and we only need $1 < p \leq \frac{N+\alpha}{N-2}$ from clear proof in Section 2 to sure that the functional is well defined.

The present paper is organized as follows. In the next section we present some preliminary results. In Section 3, we give the proof of Theorem 1.1.

2. Preliminaries. From now on, we will use the following notations.

- $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
  $$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm
  $$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$$
  and $|u|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|$ for all $p \in [1, +\infty)$.

- $\text{meas } \Omega$ denotes the Lebesgue measure of the set $\Omega$.

- $\langle \cdot, \cdot \rangle$ denotes action of dual.

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| \leq r\}$ and $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.

- $C, C_i (i = 0, 1, \ldots)$ denote various positive constants.

Formally, solutions of problem (3) should arise as the critical points of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \ dx - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^{\frac{2^{\alpha}_*}{2}} |u(x)|^{\frac{2^*_\alpha}{2}} \ dy \ dx$$
$$- \frac{2}{2^*_\alpha} \int_{\Omega} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^{\frac{2^{\alpha}_*}{2}} |u(x)|^{\frac{2^*_\alpha}{2}} \ dy \ dx$$
$$- \frac{1}{2p} \int_{\Omega} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^p |u(x)|^p \ dy \ dx,$$

where $2^{\alpha}_\alpha = \frac{2(N+\alpha)}{N-2}$. Recall the Hardy-Littlewood-Sobolev inequality. Let $q, r > 1$ and $0 < \lambda < N$ with $1/q + 1/r + \lambda/N = 2$. Let $g \in L^q(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then
there exists a sharp constant $C(N, \lambda, q)$, independent of $g$ and $h$, such that
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g(x) |x-y|^{-\lambda} h(y) \, dy \, dx \right| \leq C(N, \lambda, q) |g|_{q} |h|_{r}.
\tag{4}
\]
Let $q = r = (N + \alpha)/(N - 2) > 1$ and $\lambda = N - \alpha$. Let
\[
g(x) = \begin{cases} |u(x)|^{(N+\alpha)/(N-2)}, & x \in \mathbb{R}^N \setminus \Omega, \\
0, & x \in \Omega,
\end{cases}
\tag{5}
\]
and
\[
h(x) = \begin{cases} 0, & x \in \mathbb{R}^N \setminus \Omega, \\
|u(x)|^p, & x \in \Omega,
\end{cases}
\tag{6}
\]
where $1 < p < (N + \alpha)/(N - 2)$. Then from (4), (5) and (6) we have
\[
\left| \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_{\alpha}(x-y)|u(y)|^{\frac{N+\alpha}{N}}|u(x)|^p \, dy \, dx \right|
\leq C(N, \alpha) |g|_{2N/(N+\alpha)} |h|_{2N/(N+\alpha)},
\tag{7}
\]
where
\[
|h|_{2N/(N+\alpha)} = \left( \int_{\Omega} |u|^{\frac{2Np}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{2N}} \leq \left[ \text{meas } \Omega \left( \int_{\Omega} |u|^{2^{*}} \, dx \right)^{(N-2)p/(N+\alpha)} \right]^{\frac{N+\alpha}{N}}.
\]
and
\[
|g|_{2N/(N+\alpha)} = \left[ \int_{\mathbb{R}^N \setminus \Omega} |u|^{2^{*}} \, dx \right]^{\frac{N+\alpha}{2N}}.
\]
At the same time, combining (4) with the Hölder inequality, we also get
\[
\left| \int_{\Omega} \int_{\Omega} I_{\alpha}(x-y)|u(y)|^p|u(x)|^p \, dy \, dx \right| \leq \left( \int_{\Omega} |u|^{\frac{2Np}{N+\alpha}} \, dx \right)^{\frac{N+\alpha}{N}} \leq C |\text{meas } \Omega|^{\frac{N+\alpha}{N}} \left( \int_{\Omega} |u|^{2^{*}} \, dx \right)^{\frac{2p}{N}}.
\tag{8}
\]
Similarly, we have
\[
\left| \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} I_{\alpha}(x-y)|u(y)|^{\frac{2p}{p}}|u(x)|^{\frac{2p}{p}} \, dy \, dx \right| \leq C_{N, \alpha} \left( \int_{\mathbb{R}^N} |u|^{2^{*}} \, dx \right)^{\frac{N+\alpha}{N}}.
\tag{9}
\]
Then the energy functional $I : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (3) is well defined. The following Lemma 2.1 implies that $I$ is $C^1$ functional whose derivative is given by
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^N} \left( I_{\alpha} * |u|^{f(x)} \right) |u|^{f(x)-2}uv \, dx
\tag{10}
\]
for all \( v \in D^{1,2}(\mathbb{R}^N) \), where
\[
\int_{\mathbb{R}^N} \left( I_\alpha \ast |u|^{f(x)} \right) |u|^{f(x) - 2} v dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^{f(y)}|u(x)|^{f(x) - 2} u(x)v(x) dy dx.
\]

We recall the Nehari manifold
\[
\mathcal{N} = \left\{ u \in D^1(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{I}'(u), u \rangle = 0 \right\}
\]
\[
= \left\{ u \in D^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} \left( I_\alpha \ast |u|^{f(x)} \right) |u|^{f(x)} dx \right\}.
\]

In order to complete the proof, inspired by [10, 22] we introduce the following equation
\[
-\Delta u = \left( I_\alpha \ast \left| u \right|^{\frac{2^*}{2}} \right) \left| u \right|^{\frac{2^*}{2} - 2} u.
\] (11)

Then the associated energy functional is
\[
\mathcal{I}_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} \left( I_\alpha \ast |u|^{\frac{2^*}{2}} \right) |u|^{\frac{2^*}{2}} dx.
\]

Now, giving the following lemmas to support the proof of Theorem 1.1.

**Lemma 2.1.** Assume that \( N \geq 3 \), \( \alpha \in (0, N) \) and \((f_1)\) holds. Then the functional \( \mathcal{I} \) is of class \( C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R}) \) and \( \mathcal{I}'(\cdot) \) satisfies (10).

**Proof.** Indeed, define
\[
\Phi(u) = \frac{1}{2^*} \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y)|u(y)|^{\frac{2^*}{2}}|u(x)|^{\frac{2^*}{2}} dy dx
\]
\[
+ \frac{2}{2^* + p} \int \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y)|u(y)|^{\frac{2^*}{2}}|u(x)|^p dy dx
\]
\[
+ \frac{1}{2p} \int \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y)|u(y)|^p|u(x)| dy dx.
\]

In fact, we only need to prove \( \Phi \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R}) \). Let \( u, h \in D^{1,2}(\mathbb{R}^N) \). Given \( x \in \mathbb{R}^N \) and \( 0 < |t| < 1 \), from the mean value theorem, there exists \( \theta_1 \in (0, 1) \) such that
\[
\frac{|u + th||f(x)| - |u||f(x)|}{t} = f(x)|u + \theta_1 th||f(x) - 1|h
\]
\[
\leq \frac{2^*}{2} (|u| + |h|)|f(x)|^{-1}|h|.
\] (12)

Hence, we have
\[
\frac{1}{t} \int_{\mathbb{R}^N} I_\alpha(x - y) \left( |u(y)| + |h(y)||f(y) - 1|h(y) \right) |u(x) + th(x)||f(x) - 1| dy
\]
\[
\leq C_0 \int_{\mathbb{R}^N} I_\alpha(x - y) \left( |u(y)| + |h(y)||f(y) - 1|h(y) \right) \left( |u(x)||f(x) - 1|h(x) \right) dy.
\] (13)
It follows from (4) that (13) is integrable in \(L^1(\mathbb{R}^N)\). From the Lebesgue dominated convergence theorem and the Fubini lemma that
\[
\lim_{t \to 0} \int_\Omega \int_{\Omega^c} I_\alpha(x - y) \left( |u(y) + th(y)|^{\frac{\alpha}{p^*}} |u(y) + th(y)|^{\frac{\alpha}{p^*}} - |u(y)|^{\frac{\alpha}{p^*}} |u(y)|^{\frac{\alpha}{p^*}} \right) dy dx
\]
\[
= \lim_{t \to 0} \frac{1}{t} \int_\Omega \int_{\Omega^c} I_\alpha(x - y) |u(y) + th(y)|^{\frac{\alpha}{p^*}} |u(y) + th(y)|^{\frac{\alpha}{p^*}} dy dx
- \int_\Omega \int_{\Omega^c} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(y)|^{\frac{\alpha}{p^*}} dy dx
+ \int_\Omega \int_{\Omega^c} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(y)|^{\frac{\alpha}{p^*}} dy dx
- \int_\Omega \int_{\Omega^c} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} dy dx
= 2^\alpha \int_\Omega \int_{\Omega^c} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} - u(x) h(x) dy dx,
\]
where \(\Omega = \mathbb{R}^N \setminus \Omega\). By almost the same process above, we have
\[
\langle \Phi'(u), h \rangle = \lim_{t \to 0} \frac{\Phi(u + th) - \Phi(u)}{t}
= \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} - u(x) h(x) dy dx
+ \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} - u(x) h(x) dx dy
+ \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} - u(x) h(x) dx dy
+ \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x - y) |u(y)|^{\frac{\alpha}{p^*}} |u(x)|^{\frac{\alpha}{p^*}} - u(x) h(x) dx dy
= \int_{\mathbb{R}^N} (I_\alpha * |u|^{|f(x)|}) |u| f(x) - u h dx.
\]
Assume that \(u_n \to u\) in \(D^{1,2}(\mathbb{R}^N)\), then \(u_n \to u\) in \(L^2(\mathbb{R}^N)\) and \(L^{2Np/(N+\alpha)}(\Omega)\). From (7), (8) and (9), one notices that
\[
\|\Phi'(u_n) - \Phi'(u)\|_{D^{-1}} = \sup_{\|h\|=1, h \in D^{1,2}(\mathbb{R}^N)} \langle \Phi'(u_n) - \Phi'(u), h \rangle
= \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{|f(x)|}) |u_n| f(x) - u h dx
+ \int_{\mathbb{R}^N} (I_\alpha * |u|^{|f(x)|}) (|u_n| f(x) - u_n - |u| f(x) - u) h dx
= o(1).
\]
This completes the proof.

\hspace{1cm} \Box

**Lemma 2.2.** Assume that \(N \geq 3\), \(\alpha \in (0, N)\) and \((f_1)\) holds. Then
(a) for any \(u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}\), there exists a unique \(t \in (0, +\infty)\) such that \(tu \in N\);
(b) for any \(u \in N\), there exists a constant \(\delta > 0\) such that \(\|u\| \geq \delta\);
(c) \(I\) is bounded from below on \(N\) by a positive constant;
(d) \(N\) is a \(C^1\) manifold.
Proof. (a) Define for $t > 0$,
\[ \Psi(t) = \mathcal{I}(tu) \]
\[ = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^{\frac{2^*}{2}} |u(x)|^{\frac{2^*}{2}} dydx \]
\[ - \frac{2t^{2^* + p}}{2^* + p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^{\frac{2^*}{2}} |u(x)|^p dydx \]
\[ - \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x - y)|u(y)|^p |u(x)|^p dydx. \]

Then we have
\[ \Psi'(t)t = \langle \mathcal{I}'(tu), tu \rangle \]
\[ = t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \left( I_\alpha * |tu|^p(x) \right) |tu|^p(x) dx. \]

Combining with $f(x) > 1$, one can easily see that $\Psi'(t) > 0$ as $t$ small enough, and $\Psi'(t) < 0$ as $t$ large enough, which implies that there exists $t > 0$ such that $\langle \mathcal{I}'(tu), tu \rangle = 0$, namely $tu \in \mathcal{N}$. Moreover, the uniqueness of $t$ is obvious.

(b) For $u \in \mathcal{N}$, according to (4) and the Sobolev embedding, one obtains
\[ \|u\|^2 = \int_{\mathbb{R}^N} \left( I_\alpha * |u|^p(x) \right) |u|^p(x) dx \]
\[ \leq C_{p,\alpha} \left( \int_{\mathbb{R}^N} |u|^{\frac{2Np}{N + \alpha}} dx \right)^{\frac{N + \alpha}{N}} \]
\[ \leq C_{p,\alpha} \left( \left( \int \operatorname{meas} \Omega \right)^{1 - \frac{p(N - 2)}{N + \alpha}} \left( \int_{\Omega} |u|^{2^*} dx \right)^{\frac{p(N - 2)}{2N - 2p}} + \int_{\mathbb{R}^N \setminus \Omega} |u|^{2^*} dx \right)^{\frac{N + \alpha}{N}} \]
\[ \leq C \left( \|u\|^{2p} + \|u\|^{2^*(N + 1) - \frac{2p}{N}} \right). \]

In fact, we only discuss the case that $\|u\| < 1$. Recall that $u \neq 0$ by the definition of the Nehari manifold. Since $2 < 2p < \frac{2^*(N + 1)}{N}$, there exists $\delta > 0$ such that $\|u\| \geq \delta$ and $\delta$ is independent of $u$.

(c) For any $u \in \mathcal{N}$, one has
\[ \mathcal{I}(u) = \mathcal{I}(u) - \frac{1}{2p} \langle \mathcal{I}'(u), u \rangle \]
\[ \geq \left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \]
\[ \geq \frac{p - 1}{2p} \delta^2. \]

It is quite clear that for any $u \in \mathcal{N}$ such that $\mathcal{I}(u) > 0$ from $(f_1)$.

(d) Considering the derivative of $J(u) = \langle \mathcal{I}'(u), u \rangle$ at $u$ and applied in $u$, we obtain
\[ \langle J'(u), u \rangle = \langle J'(u), u \rangle - 2p \langle \mathcal{I}'(u), u \rangle \]
\[
\leq (2-2p) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]
\[< 0.\]
This completes the proof. \(\square\)

From Lemma 2.2, there exists a positive constant \(c\) such that
\[
c = \inf_{u \in \mathcal{N}} \mathcal{I}(u).
\]

Use \(S_\alpha\) to denote the best constant defined by
\[
S_\alpha := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{2\alpha}{N}) |u|^\frac{2\alpha}{N} \, dx = 1 \right\}.
\]
From [14, 15, 31, 39], we know that \(S_\alpha\) is achieved by
\[
\phi(x) = \frac{C_{N,\alpha}}{(1 + |x|^2)^{\frac{N-2}{2}}},
\]
where \(C_{N,\alpha} > 0\) is a fixed constant.

**Lemma 2.3.** Assume that \(N \geq 3\) and \(\alpha \in (0,N)\). Then \(\omega = \mu \phi\) is a positive solution of problem (11), where \(\mu = S_{\alpha}^{(N-2)/(2\alpha+4)}\).

**Proof.** Recall some known facts,
\[
\int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx = S_\alpha \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |\phi|^\frac{2\alpha}{N}) |\phi|^\frac{2\alpha}{N} \, dx = 1.
\]
Fix \(v \in D^{1,2}(\mathbb{R}^N)\) and for \(s \in \mathbb{R}\) small enough, say \(s \in (-\varepsilon, \varepsilon)\), the function \(\phi + sv\) is not identically zero. Therefore there exists \(t : (-\varepsilon, \varepsilon) \to (0, +\infty)\) such that
\[
\int_{\mathbb{R}^N} \left( I_\alpha * |t(s)(\phi + sv)|^\frac{2\alpha}{N} \right) |t(s)(\phi + sv)|^\frac{2\alpha}{N} \, dx = 1,
\]
precisely,
\[
t(s) = \left( \frac{1}{\int_{\mathbb{R}^N} \left( I_\alpha * |\phi + sv|^\frac{2\alpha}{N} \right) |\phi + sv|^\frac{2\alpha}{N} \, dx} \right)^{\frac{1}{2\alpha}}.
\]
Notice that the application \(s \mapsto t(s)(\phi + sv)\) defines a curve that passes through \(\phi\) when \(s = 0\). The function \(t\) is differentiable on \((-\varepsilon, \varepsilon)\), and we have
\[
t(0) = 1 \text{ and } \dot{t}(0) = -\int_{\mathbb{R}^N} (I_\alpha * |\phi|^\frac{2\alpha}{N}) |\phi|^\frac{2\alpha}{N} - 2 \phi v \, dx.
\]
We define
\[
\gamma(s) = \|t(s)(\phi + sv)\|^2.
\]
Since \(t(s)(\phi + sv)\) for every \(s \in (-\varepsilon, \varepsilon)\), the point \(s = 0\) is a local minimum for \(\gamma\). The function \(\phi\) is differentiable and
\[
\gamma'(s) = 2 \langle t(s)(\phi + sv), t'(s)(\phi + sv) + t(s)v \rangle.
\]
So that
\[
0 = \gamma'(0) = 2t(0)\dot{t}(0)\|\phi\|^2 + 2t^2(0)\langle \phi, v \rangle
\]
\[
= -2S_\alpha \int_{\mathbb{R}^N} (I_\alpha * |\phi|^\frac{2\alpha}{N}) |\phi|^\frac{2\alpha}{N} - 2 \phi v \, dx + 2\langle \phi, v \rangle.
\]
Thus shown that for every \( v \in D^{1,2}(\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^N} \nabla \phi \cdot \nabla v dx = S_\alpha \int_{\mathbb{R}^N} (I_\alpha * |\phi|^{2^*_\alpha}) |\phi|^{2^*_\alpha} - 2 \phi v dx.
\]
Set \( \phi = \lambda \omega \) with \( \lambda \in \mathbb{R} \) to be determined. Then \( \omega \) satisfies
\[
\int_{\mathbb{R}^N} \nabla \omega \cdot \nabla v dx = S_\alpha \lambda^{2^*_\alpha - 2} \int_{\mathbb{R}^N} (I_\alpha * |\omega|^{2^*_\alpha}) |\omega|^{2^*_\alpha} - 2 \omega v dx.
\]
Choosing \( \lambda = S_\alpha^{-(N+2)/(2\alpha+4)} \), we see that \( \omega \) is positive and satisfies
\[
\int_{\mathbb{R}^N} \nabla \omega \cdot \nabla v dx = \int_{\mathbb{R}^N} (I_\alpha * |\omega|^{2^*_\alpha}) |\omega|^{2^*_\alpha} - 2 \omega v dx, \quad v \in D^{1,2}(\mathbb{R}^N),
\]
namely \( \omega = \mu \phi \) is a positive solution of problem (11) where \( \mu = S_\alpha^{(N-2)/(2\alpha+4)} \).

**Lemma 2.4.** Assume that \( N \geq 3 \) and \( \alpha \in (0, N) \). Then \( \sup_{t \in (0, +\infty)} \mathcal{I}_\infty(t \omega) = c_\infty \), where \( c_\infty = \left( \frac{\alpha+2}{2(N+\alpha)} \right) S_\alpha^{(N+\alpha)/(\alpha+2)} \).

**Proof.** Firstly, we define for \( t \in (0, +\infty) \),
\[
\Psi_\infty(t) := \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \omega|^2 dx - \frac{1}{2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |t\omega|^{2^*_\alpha}) |t\omega|^{2^*_\alpha} dx. \tag{16}
\]
Then from lemmas 2.2 and 2.3, we can see that \( \sup_{t \in (0, +\infty)} \Psi_\infty(t) \) is obtained by \( t = 1 \). This implies
\[
\Psi_\infty(1) = \sup_{t \in (0, +\infty)} \Psi_\infty(t)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \omega|^2 dx - \frac{1}{2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\omega|^{2^*_\alpha}) |\omega|^{2^*_\alpha} dx
\]
\[
= \frac{\mu^2}{2} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx - \frac{\mu^{2^*_\alpha}}{2\alpha} \int_{\mathbb{R}^N} (I_\alpha * |\phi|^{2^*_\alpha}) |\phi|^{2^*_\alpha} dx
\]
\[
= \frac{\alpha+2}{2(N+\alpha)} S_\alpha^{N+\alpha/\alpha+2}.
\]
This completes the proof. \( \square \)

For the sake of convenience in writing, we define
\[
\omega_n := \omega(x + x_n),
\]
where \( x \in \mathbb{R}^N \) and \( x_n := (0, 0, 0, \cdots, 0, n) \in \mathbb{R}^N \). Then \( ||\omega_n||^2 = ||\omega||^2 \) and there exists \( \omega' \in D^{1,2}(\mathbb{R}^N) \) such that
- \( \omega_n \to \omega' \in D^{1,2}(\mathbb{R}^N) \);
- \( \omega_n \to \omega' \in L_q^{\infty}(\mathbb{R}^N) \) for all \( q \in [2, 2^*]; \)
- \( \omega_n(x) \to \omega'(x) \) a.e. in \( \mathbb{R}^N \).

Since for any \( x \in \mathbb{R}^N \), \( \omega_n(x) \to 0 \) and \( \omega' = 0 \). Now for estimating the energy \( c \), we give the following lemmas.

**Lemma 2.5.** Suppose that \( N \geq 3 \), \( \alpha \in (0, N) \) and \( (f_1) \) holds. Then there exists \( t_n \geq 1 \) such that \( t_n \omega_n \in \mathcal{N} \) and \( \lim_{n \to \infty} t_n = 1 \).
Proof. It is quite clear that there exists \( t_n \geq 0 \) such that \( t_n \omega_n \in \mathcal{N} \) from Lemma 2.2. Indeed, \( \langle I'(t_n \omega_n), t_n \omega_n \rangle = 0 \) implies

\[
\int_{\mathbb{R}^N} |\nabla \omega_n|^2 dx = t_n^{2r-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^{2s} dydx + 2t_n^{2r+p-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^p dydx + t_n^{2p-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y)|\omega_n(y)|^p |\omega_n(x)|^p dydx.
\]

It follows from (17) and the fact that \( \omega_n \) is also a solution of problem (11) that

\[
\int_{\mathbb{R}^N} |\nabla \omega_n|^2 dx = \int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^{2r}) |\omega_n|^{2s} dx \geq t_n^{2r-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^{2s} dydx,
\]

which implies \( t_n \geq 1 \).

Next, we seek \( \lim_{n \to \infty} t_n = 1 \). Since \( \Omega \) is bounded, there exists \( R \) such that \( \Omega \subset B_R \). One can easily see that

\[
\int_{|x-x_n|<R} \frac{1}{(1+|x|^2)^N} dx \leq \int_{|x-x_n|<R} \frac{2^N}{\eta^{2N}} dx = \frac{2^N}{\eta^{2N}} \text{ meas } B_R.
\]

According to (7) and (18), we have

\[
\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^p dydx \leq C_1 \left( \int_{\mathbb{R}^N \setminus \Omega} |\omega_n|^{2s} dx \right)^{\frac{N+a}{2s}} \left( \int_{\Omega} |\omega_n|^{2r} dx \right)^{\frac{s}{p}}.
\]

At the same time, combining with (8) we also get

\[
\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^p |\omega_n(x)|^p dydx \leq \left( \int_{\Omega} |\omega_n|^{2Np} dx \right)^{\frac{N+a}{Np}}.
\]

According to (19) and (20), for \( n \) large enough one obtains

\[
\int_{\mathbb{R}^N} (I_\alpha * |\omega|^{2r}) |\omega|^{2s} dx = \int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^{2r}) |\omega_n|^{2s} dx \geq \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^{2s} dydx = \int_{\mathbb{R}^N} (I_\alpha * |\omega_n|^{2r}) |\omega_n|^{2s} dx + o(1).
\]

Thus one has

\[
t_n^{2r-2} \int_{\mathbb{R}^N} (I_\alpha * |\omega|^{2r}) |\omega|^{2s} dydx = \int_{\mathbb{R}^N} |\nabla \omega|^2 dx + o(1),
\]

which implies that \( \lim_{n \to \infty} t_n = 1 \) from lemma 2.3. This completes the proof. \( \square \)

Lemma 2.6. Suppose that \( N \geq 3 \), \( \alpha \in (0, N) \) and \( (f_1) \) holds. Then

\[
\lim_{n \to \infty} \left( \int_{\Omega} \int_{\Omega} I_\alpha(x-y)|\omega_n(y)|^p |\omega_n(x)|^p dydx \right) = +\infty.
\]

\[
\int_{\Omega} \int_{\Omega} I_\alpha(x-y)|\omega_n(y)|^{2r} |\omega_n(x)|^{2s} dydx = +\infty.
\]
Proof. Since the interior of $\Omega$ is nonempty, there exist $z_0 \in \mathbb{R}^N$ and $r > 0$ such that

\[ B_r(z_0) := \{ x \in \mathbb{R}^N : |x - z_0| < r \} \subseteq \Omega. \]

Thus for $n$ large enough one has

\[
\int_{B_r(z_0)} \int_{B_r(z_0)} I_\alpha(x - y) \left( \frac{1}{1 + |y + x_n|^2} \right)^{(N-2)p/2} \left( \frac{1}{1 + |x + x_n|^2} \right)^{(N-2)p/2} dydx \\
\geq \int_{B_r(z_0)} \int_{B_r(z_0)} I_\alpha(x - y) \left( \frac{1}{2n^2} \right)^{(N-2)p} dydx \\
= \frac{2(2-N)p}{n^2(N-2)p} \int_{B_r(z_0)} \int_{B_r(z_0)} I_\alpha(x - y) dydx.
\]

At the same time, there exists $R > 0$ such that

\[ B_R(z_0) := \{ x \in \mathbb{R}^N : |x - z_0| < R \} \supseteq \Omega, \]

we also can easily see that

\[
\int_{B_R(z_0)} \int_{B_R(z_0)} I_\alpha(x - y) \left( \frac{1}{1 + |y + x_n|^2} \right)^{N+\alpha/2} \left( \frac{1}{1 + |x + x_n|^2} \right)^{N+\alpha/2} dydx \\
\leq \int_{B_R(z_0)} \int_{B_R(z_0)} I_\alpha(x - y) \left( \frac{1}{n^2} \right)^{N+\alpha} dydx \\
= \frac{1}{n^{2(N+\alpha)}} \int_{B_R(z_0)} \int_{B_R(z_0)} I_\alpha(x - y) dydx.
\]

Combining (21) and (22), one obtains

\[
\lim_{n \to \infty} \frac{\int_{\Omega} \int_{\Omega} I_\alpha(x - y)|\omega_n(y)|^p|\omega_n(x)|^p dydx}{\int_{\Omega} \int_{\Omega} I_\alpha(x - y)|\omega_n(y)|^\frac{2p}{N+\alpha}|\omega_n(x)|^\frac{2p}{N+\alpha} dydx} \\
\geq C \lim_{n \to \infty} \frac{\int_{B_r(z_0)} \int_{B_r(z_0)} I_\alpha(x - y) dydx}{\int_{B_R(z_0)} \int_{B_R(z_0)} I_\alpha(x - y) dydx} \\
= +\infty,
\]

where $C$ is a positive constant and independent of $\omega_n$. This completes the proof. □

Lemma 2.7. Suppose that $N \geq 3$, $\alpha \in (0, N)$ and (f1) holds. Then

\[
\lim_{n \to \infty} \frac{\int_{\Omega} \int_{R^N \setminus \Omega} I_\alpha(x - y)|\omega_n(y)|^\frac{2\alpha}{N+\alpha}|\omega_n(x)|^\frac{2\alpha}{N+\alpha} dydx}{\int_{\Omega} \int_{R^N \setminus \Omega} I_\alpha(x - y)|\omega_n(y)|^\frac{2p}{N+\alpha}|\omega_n(x)|^\frac{2p}{N+\alpha} dydx} = +\infty.
\]

Proof. For $n$ large enough, it is quite clear that

\[
\int_{\Omega} \int_{R^N \setminus \Omega} I_\alpha(x - y) \left( \frac{1}{1 + |y + x_n|^2} \right)^{N+\alpha/2} \left( \frac{1}{1 + |x + x_n|^2} \right)^{(N-2)p/2} dydx \\
\geq \int_{\Omega} \int_{R^N \setminus \Omega} I_\alpha(x - y) \left( \frac{1}{1 + |y + x_n|^2} \right)^{N+\alpha/2} \left( \frac{1}{2n^2} \right)^{(N-2)p/2} dydx \\
= \frac{2(2-N)p}{n^2(N-2)p} \int_{\Omega} \int_{R^N \setminus \Omega} I_\alpha(x - y) \left( \frac{1}{1 + |y + x_n|^2} \right)^{N+\alpha/2} dydx.
\]
At the same time that, we also can easily see that
\[
\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y) \left( \frac{1}{1+|y|+x_n^2} \right)^{\frac{N+\alpha}{2}} \left( \frac{1}{1+|x|} \right)^{\frac{N+\alpha}{2}} \, dydx \\
\leq \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y) \left( \frac{1}{1+|y|+x_n^2} \right)^{\frac{N+\alpha}{2}} \left( \frac{1}{n^2} \right)^{\frac{N+\alpha}{2}} \, dydx \\
= \frac{1}{n^{2(N+\alpha)}} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y) \left( \frac{1}{1+|y|+x_n^2} \right)^{\frac{N+\alpha}{2}} \, dydx. \tag{24}
\]
Combining (23) with (24), we obtain
\[
\lim_{n \to \infty} \frac{\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{\frac{2^*_\alpha}{2}} |\omega_n(x)|^{\frac{2^*_\alpha}{2}} \, dydx}{\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{\frac{2^*_\alpha}{2}} |\omega_n(x)|^{\frac{2^*_\alpha}{2}} \, dydx} \\
\geq \frac{1}{C} \lim_{n \to \infty} \frac{2^{(2-N)p} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y) \left( \frac{1}{1+|y|+x_n^2} \right)^{\frac{N+\alpha}{2}} \, dydx}{n^{2(N+\alpha)} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y) \left( \frac{1}{1+|y|+x_n^2} \right)^{\frac{N+\alpha}{2}} \, dydx} \\
= +\infty,
\]
where $C$ is a positive constant and independent of $\omega_n$. This completes the proof. \hfill \square

Lemma 2.8. Suppose that $N \geq 3$, $\alpha \in (0, N)$ and $(f_1)$ holds. Then $c < c_\infty$.

Proof. Recall that $t_n \omega_n \in \mathcal{N}$ and there exists $T^* > 1$ such that $t_n < T^*$ for any $n$ from Lemma 2.5. Thus combining with Lemmas 2.5-2.7, for $n$ large enough one obtains
\[
c \leq \mathcal{I}(t_n \omega_n) \\
\leq \mathcal{I}_\infty(t_n \omega_n) + \frac{2(T^*)^{2^*_\alpha}}{2^*_\alpha} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{\frac{2^*_\alpha}{2}} |\omega_n(x)|^{\frac{2^*_\alpha}{2}} \, dydx \\
+ \frac{(T^*)^{2^*_\alpha}}{2^*_\alpha} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{\frac{2^*_\alpha}{2}} |\omega_n(x)|^{\frac{2^*_\alpha}{2}} \, dydx \\
- \frac{2T_n^{2^*_\alpha}}{2^*_\alpha + p} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{\frac{2^*_\alpha}{2}} |\omega_n(x)|^{p} \, dydx \\
- \frac{T_n^{2^*_\alpha}}{2^*_\alpha} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} I_\alpha(x-y)|\omega_n(y)|^{p} |\omega_n(x)|^{p} \, dydx < \mathcal{I}_\infty(\omega).
\]
Therefore $c < \mathcal{I}_\infty(\omega) = c_\infty$. This completes the proof. \hfill \square

3. Proof of Theorem 1.1. To prove Theorem 1.1, we have to overcome the lack of compactness. Then the following lemma is central to our proof, which we sketch here for the readers’ convenience.

Lemma 3.1. Suppose that $N \geq 3$, $\alpha \in ((N-4)^{+}, N)$ and $(f_1)$ holds. Then $\mathcal{I}$ satisfies the Palais-Smale condition at level $m$ for any $m \in (0, c_\infty)$.

Proof. A sequence $\{u_n\}$ is called a Palais-Smale sequence for $\mathcal{I}$ at level $m$ such that
\[
\mathcal{I}(u_n) \to m, \quad \mathcal{I}'(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
\( \mathcal{I} \) satisfies the Palais-Smale condition at level \( m \) if \( \{u_n\} \) has a converging subsequence. Note that, \( \{u_n\} \) is bounded obviously. Thus there exists \( u \in D^{1,2}(\mathbb{R}^N) \), up to a subsequence, such that

- \( u_n \to u \in D^{1,2}(\mathbb{R}^N) \);
- \( u_n \to u \in L^{\frac{2Np}{N+p}}(\Omega) \);
- \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \).

For any \( v \in D^{1,2}(\mathbb{R}^N) \), it is easy to see that

\[
0(1) = \langle \mathcal{I}'(u_n), v \rangle = \langle \mathcal{I}'(u), v \rangle.
\]

Hence we obtain

\[
\mathcal{I}(u) \geq \frac{1}{2} \mathcal{I}(u) - \frac{1}{2p} \langle \mathcal{I}'(u), u \rangle \\
\geq \left( \frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
\geq 0.
\]

Define \( v_n = u_n - u \). Thus one gets

\[
\|v_n\|^2 + \|u\|^2 = \|u_n\|^2 + o(1).
\]

It follows from [31, Theorem 2.4] that

\[
\Phi(v_n) + \Phi(u) = \Phi(u_n) + o(1).
\]

Combining with \( \mathcal{I}(u_n) \to m \), we obtain

\[
m = \mathcal{I}(u) + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \Phi(v_n) + o(1)
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \Phi(v_n) + o(1).
\]

At the same time, from \( \langle \mathcal{I}'(u_n), u_n \rangle = 0 \) and \( \langle \mathcal{I}'(u), u \rangle = 0 \), one gets

\[
0 = \|v_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{f(x)}) |v_n|^{f(x)} dx + \langle \mathcal{I}'(u), u \rangle + o(1)
\]

\[
= \|v_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{f(x)}) |v_n|^{f(x)} dx + o(1).
\]

In fact, from \( \frac{2Np}{N+p} \in [1, 2^*) \), one can easily see that for bounded \( \Omega \),

\[
\lim_{n \to \infty} \int_{\Omega} |v_n|^{\frac{2Np}{N+p}} dx = 0.
\]

Hence, according to proof of Lemma 2.5, (25) and (26) it is quite clear that

\[
\|v_n\|^2 = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{f(x)}) |v_n|^{f(x)} dx
\]

\[
= \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{2^*}{2^*}}) |v_n|^{\frac{2^*}{2^*}} dx + o(1).
\]

From the definition of \( S_\alpha \), we have

\[
S_\alpha \left( \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{2^*}{2^*}}) |v_n|^{\frac{2^*}{2^*}} dx \right)^{\frac{2^*}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx.
\]
\[ \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leq \left( \frac{1}{S_\alpha} \right)^\frac{2^*}{2^* - 2} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^\frac{2^*}{2^* - 2} + o(1). \]

This deduces that either

\begin{align*}
\text{case 1.} & \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = 0, \\
\text{case 2.} & \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq S_\alpha^{\frac{N+\alpha}{N}}.
\end{align*}

If \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = 0 \), which implies \( v_n \to 0 \) in \( D^{1,2}(\mathbb{R}^N) \). This completes the proof. Therefore we only seek that the second case is never to happen.

If \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq S_\alpha^{\frac{N+\alpha}{N}} \). Combining with (27), one obtains

\[ m \geq I(v_n) + o(1) \]
\[ = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{2^*}{2}})|v_n|^{\frac{2^*}{2}} dx + o(1) \]
\[ \geq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + o(1) \]
\[ \geq \frac{\alpha + 2}{2(N + \alpha)} S_\alpha^{\frac{N+\alpha}{N}} + o(1) \]
\[ = c_\infty + o(1), \]

which is a contradiction. Thus \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = 0 \), which deduces that \( u_n \to u \in D^{1,2}(\mathbb{R}^N) \). This completes the proof.

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