Dislocations in the Field Theory of Elastoplasticity

Markus Lazar

Abstract

By means of linear theory of elastoplasticity, solutions are given for screw and edge dislocations situated in an isotropic solid. The force stresses, strain fields, displacements, distortions, dislocation densities and moment stresses are calculated. The force stresses, strain fields, displacements and distortions are devoid of singularities predicted by the classical elasticity. Using the so-called stress function method we found modified stress functions of screw and edge dislocations.

Key words: dislocations; elastoplasticity; moment stress; stress functions.

1 Introduction

The traditional description of elastic fields produced by dislocations is based on the classical theory of linear elasticity. However the classical dislocation theory breaks down in the dislocation core region. The elastic fields due to dislocations which are calculated within the theory of elasticity contain singularities at the centre of the dislocation. This is unfortunate since the dislocation core is an important region in metallic plasticity and dislocations and failure occur because of the high shear stress action in this region. Clearly, such singularities are unphysical and an improved model should eliminate them. On the other hand, in conventional plasticity theory no internal length scale enters the constitutive relations and no size effects are predicted.

A field theory of elastoplasticity is proposed as a means of describing plastic and elastic deformation even in the dislocation core. In this theory the elementary acts of plastic deformation at the microscopical level are investigated. In fact, the dislocations are considered as elementary carriers of plasticity. From the field theoretical point of view, dislocations bring new special degrees of freedom (e.g. anholonomity) and their presence leads to a specific response with the dimension of a moment stress. In straightforward manner, a new internal characteristic length scale enters the constitutive relation between the dislocation density tensor and the moment stress tensor.

2 Theory of elastoplasticity

In this dislocation theory it turns out that the force stresses, elastic and plastic strains, elastic and plastic distortions and displacement fields due to dislocations contain no singularities and they are finite. In fact, they vanish at the centre of dislocations. Moreover, the extremum stress and strain occur at a short distance away from the dislocation line.

The plan of the paper is as follows. In Section 2, I present the basics of the elastoplastic field theory of dislocations. The non-singular solutions of straight screw and edge dislocations calculated in the proposed theory of elastoplasticity are considered in Section 3 and 4, respectively. Section 5 concludes the paper.

1 Present address: Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, Tour 66, 4 Place Jussieu, Case 162, F-75252 Paris Cédex 05, France.
Thus, it is not a formal additive decomposition. Note that the distortion is dimensionless. In elastoplasticity, the linear elastic strain tensor is given by means of the incompatible distortion tensor as

\[ E_{ij} \equiv \beta_{ij} = \frac{1}{2} \left( \partial_i u_j + \partial_j u_i + \phi_{ij} + \phi_{ji} \right), \quad E_{ij} = E_{ji}. \]  

(2)

The (symmetric) force stress is the response quantity (excitation) with respect to the strain and is given by the generalized Hooke’s law for an isotropic medium

\[ \sigma_{ij} = 2\mu \left( E_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} E_{kk} \right), \quad \sigma_{ij} = \sigma_{ji}. \]  

(3)

where \( \mu, \nu \) are shear modulus and Poisson’s ration, respectively. The (symmetric) force stress has to satisfy the force equilibrium condition

\[ \partial_i \sigma_{ij} = 0 = \partial_j \sigma_{ij}. \]  

(4)

The skew-symmetric part of the distortion tensor defines the elastic rotation of a dislocation \([6, 7]\)

\[ \omega_i \equiv -\frac{1}{2} \epsilon_{ijk} \beta_{jk}. \]  

(5)

The rotation vector gives rise to a rotation gradient (de-Wit’s bend-twist tensor)

\[ k_{ij} = \partial_j \omega_i. \]  

(6)

The total strain \( E_{ij}^T \) and distortion \( \beta_{ij}^T \) are defined in terms of the displacement field \( u_i \). They must be compatible. In the presence of dislocations, the total strain is not completely elastic, but a part of it is plastic \([6]\) so that

\[ E_{ij}^T = \partial_i (u_j) = E_{ij} + E_{ij}^P, \]

\[ \beta_{ij}^T = \partial_i u_i = \beta_{ij} + \beta_{ij}^P, \quad \phi_{ij} \equiv -\beta_{ij}^P. \]

(7)

Here \( E_{ij}^P = -\phi_{ij} \) is the plastic strain and \( \beta_{ij}^P \) the plastic distortion which are not derivable from the displacement field \( u_i \). In performing the differentiations of the displacement \( u_i \), we obtain the elastic distortion \( \beta_{ij} \) plus excess terms which we identify with the plastic distortion \( \beta_{ij}^P \). We notice that this decomposition of \( E_{ij}^T \) is close to the Green-Naghdi decomposition (see \([8]\)).

The dislocation density tensor is defined by means of the distortion tensor

\[ T_{ijk} := \partial_j \beta_{ik} - \partial_k \beta_{ij} = \partial_j \phi_{ik} - \partial_k \phi_{ij}, \quad T_{ijk} = -T_{ikj}. \]  

(8)

and

\[ T_{ijk} = -\partial_j \beta_{ik}^P + \partial_k \beta_{ij}^P. \]  

(9)

Physically, the incompatible distortion corresponds to dislocations. The dislocation density tensor has the dimension of an inverse length. It is a fundamental quantity in plasticity because the dislocation is the elementary carrier of plasticity. The usual dislocation density tensor \( \alpha_{ij} \) is recovered by (see also \([9]\))

\[ \alpha_{ij} := \frac{1}{2} \epsilon_{jkl} T_{kl} = \epsilon_{jkl} \partial_k \beta_{il} = -\epsilon_{jkl} \partial_k \beta_{il}^P. \]  

(10)

Here the index \( i \) indicates the direction of the Burgers vector, \( j \) the dislocation line direction. Thus, the diagonal components of \( \alpha_{ij} \) represent screw dislocations, the off-diagonal components edge dislocations. In this approach the dislocation density is not given a priori as a delta function. It follows from the physics of the problem. Eqs. (9) and (10) justify the identification of \( -\partial_i \phi \) with the plastic distortion. The dislocation density tensor satisfies the translational Bianchi identity

\[ \partial_j \alpha_{ij} = 0, \]  

(11)

which means that dislocations do not end inside the medium.

The moment stress which is the response quantity (excitation) to the dislocation density is given by (see \([4]\))

\[ H_{ijk} = \frac{a_1}{2} \left( T_{ijk} - T_{jki} - T_{kij} \right) + \frac{2\nu}{1-\nu} \left( \delta_{ij} T_{ikk} + \delta_{ik} T_{ijl} \right), \]

(12)

or with \( H_{ij} = \frac{1}{2} \epsilon_{jkl} H_{ikl} \)

\[ H_{ij} = a_1 \left\{ \frac{1}{1-\nu} \alpha_{ij} - \frac{\nu}{1-\nu} \alpha_{ji} - \frac{1}{2} \delta_{ij} \alpha_{kk} \right\}. \]  

(13)

Obviously, (12) and (13) are (linear) constitutive relations between dislocation density and moment stress in an isotropic medium. The coefficient \( a_1 \) has the dimension of a force.

The field equation in elastoplasticity for the force stress in an isotropic medium is proposed as the following inhomogeneous Helmholtz equation \([4]\)

\[ (1 - \kappa^{-2} \Delta) \sigma_{ij} = \tilde{\sigma}_{ij}, \quad \kappa^2 = \frac{2\mu}{a_1}. \]  

(14)

where \( \tilde{\sigma}_{ij} \) is the stress tensor obtained for the same traction boundary-value problem within the “classical” the-
ory of dislocations. In the field theory of elastoplasticity Eq. (14) is obtained from the moment stress equilibrium condition (see [4]). It is important to note that (14) agrees with the field equation for the stress field in Eringen’s nonlocal elasticity [10, 11] and in gradient elasticity [12]. The factor $\kappa^{-1}$ has the physical dimension of a length and therefore it defines an internal characteristic length (dislocation length scale). Using the inverse of the generalized Helmholtz equation for the strain fields (see [4])

$$(1 - \kappa^{-2}\Delta) E_{ij} = \varphi E_{ij}, \quad (15)$$

where $E_{ij}$ is the classical strain tensor. Equation (15) is similar to the equation for strain of the gradient theory used by Gutkin and Aifantis [12–14] if we identify $\kappa^{-2}$ with their corresponding gradient coefficient (see, e.g., equation (4) in [12]). From the fact that the field equation of the force stress (14) has the same form in elastoplastic field theory, strain gradient elasticity and nonlocal elasticity some relations for the corresponding solutions will follow. We assume that the stress and strain fields at infinity should have the same form for both the classical and elastoplastic field theory.

3 Screw dislocation in elastoplasticity

Consider an infinitely long screw dislocation whose dislocation line coincides with the $z$-axis of a Cartesian coordinate system. Due to the symmetry of the problem, we choose the Burgers vector, which is parallel to the dislocation line, in $z$-direction: $b_x = b_y = 0, b_z = b$. We solve the force stress equilibrium condition (4) identical by a so-called stress function ansatz. Using the stress function ansatz [9, 15, 16] the elastic stress of a Volterra screw dislocation, in Cartesian coordinates, reads

$$\sigma_{xx} = -\varphi_y = -\frac{\mu b}{2\pi} \frac{y}{r^2},$$
$$\sigma_{yy} = \sigma_{zy} = \varphi_x = \frac{\mu b}{2\pi} \frac{x}{r^2}, \quad (16)$$

or, in cylindrical coordinates,

$$\sigma_{\varphi\varphi} = \sigma_{\varphi z} = \varphi_r = \frac{\mu b}{2\pi r}, \quad (17)$$

where $r^2 = x^2 + y^2$ and $\varphi = \arctan(y/x)$. Here, $\Phi$ is the well-known stress function of elastic torsion, sometimes called Prandtl’s stress function. It is given by (see, e.g., [9])

$$\Phi = \frac{\mu b}{2\pi} \ln r. \quad (18)$$

This is Green’s function of the two-dimensional potential equation

$$\Delta \Phi = \mu b \delta(r). \quad (19)$$

Here $\delta(r)$ is the two-dimensional Dirac delta function and $\Delta$ denotes the two-dimensional Laplacian $\partial^2_{xx} + \partial^2_{yy}$. Obviously, the “classical” stress fields are singular at the dislocation line. For the modified stress we make the ansatz

$$\sigma_{ij} = \begin{pmatrix} 0 & 0 & -\partial_y F \\ 0 & 0 & \partial_x F \\ -\partial_y F & \partial_x F & 0 \end{pmatrix}, \quad (20)$$

where $F$ is called the modified Prandtl stress function. Substituting the stress functions into (14), we get the following inhomogeneous Helmholtz equation

$$(1 - \kappa^{-2}\Delta) F = \frac{\mu b}{2\pi} \ln r. \quad (21)$$

The solution of the modified stress function of a screw dislocation is given by

$$F = \frac{\mu b}{2\pi} \left\{ \ln r + K_0(\kappa r) \right\}, \quad (22)$$

where $K_n$ is the modified Bessel function of the second kind and of order $n$. Consequently, we find the force stresses

$$\sigma_{xx} = -\frac{\mu b}{2\pi} \frac{y}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\},$$
$$\sigma_{yy} = \frac{\mu b}{2\pi} \frac{x}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\}, \quad (23)$$

and in cylindrical coordinates

$$\sigma_{\varphi\varphi} = \frac{\mu b}{2\pi} \frac{1}{r} \left\{ 1 - \kappa r K_1(\kappa r) \right\}. \quad (24)$$

The appearance of the modified Bessel function in (23) and (24) leads to the elimination of classical singularity $\sim r^{-1}$ at the dislocation line. The modified stress field (23) agrees with the stress field calculated by Eringen [10, 11] within his version of nonlocal elasticity. Additionally, it is interesting to note that the stress field (23) is the same as the one obtained by Gutkin and Aifantis [12] in their version of gradient elasticity. The stress $\sigma_{yy}$ has its extreme value $|\sigma_{yy}(x, 0)| \approx 0.399\kappa b^2 \mu$ at $|x| \approx 1.114\kappa^{-1}$, whereas the stress $\sigma_{xx}$ has its extreme value $|\sigma_{xx}(0, y)| \approx 0.399\kappa \frac{b^2 \mu}{r}$ at $|y| \approx 1.114\kappa^{-1}$. We notice that the extremum stress may be identified with the theoretical shear strength. The factor $\kappa$ should be fitted by comparing predictions of the theory with experimental results. In general, the parameter $\kappa$ can
be used to determine the width of a dislocation and the amplitude of the force stress.

Let us now calculate the distortion of a screw dislocation. The distortion \( \beta_{ij} \) is given in terms of the stress function

\[
\beta_{ij} = \frac{1}{2\mu} \begin{pmatrix}
0 & 0 & -\partial_y F + 2\mu \omega_1 \\
0 & 0 & \partial_x F - 2\mu \omega_2 \\
-\partial_y F - 2\mu \omega_1 & \partial_x F + 2\mu \omega_2 & 0
\end{pmatrix},
\]

where the two functions \( \omega_1 \) and \( \omega_2 \) are used to express the antisymmetric part of the distortion \( \beta_{[r\ell]} \equiv \omega_1 \) and \( \beta_{[y\ell]} \equiv -\omega_2 \). Eventually, \( \omega_1 \) and \( \omega_2 \) are determined from the conditions

\[
\alpha_{xy} = T_{xx} = -\frac{1}{2\mu} \partial_x (2\mu \omega_1 - \partial_y F) \equiv 0,
\]

\[
\alpha_{yx} = T_{yy} = -\frac{1}{2\mu} \partial_y (2\mu \omega_2 - \partial_x F) \equiv 0.
\]

One finds for the distortion tensor of the screw dislocation

\[
\beta_{xx} = -\frac{b}{2\pi} \frac{x}{r^2} \left(1 - \kappa r K_1(\kappa r)\right),
\]

\[
\beta_{xy} = \frac{b}{2\pi} \frac{y}{r^2} \left(1 - \kappa r K_1(\kappa r)\right),
\]

and for elastic rotation vector

\[
\omega_x \equiv \omega_2 = \frac{b}{2\pi} \frac{x}{r^2} \left(1 - \kappa r K_1(\kappa r)\right),
\]

\[
\omega_y \equiv \omega_1 = \frac{b}{2\pi} \frac{y}{r^2} \left(1 - \kappa r K_1(\kappa r)\right).
\]

The rotation vector is in agreement with the result calculated in the Cosserat theory [17]. The incompatible elastic strain reads

\[
E_{xx} = -\frac{b}{4\pi} \frac{x}{r^2} \left(1 - \kappa r K_1(\kappa r)\right),
\]

\[
E_{xy} = \frac{b}{4\pi} \frac{y}{r^2} \left(1 - \kappa r K_1(\kappa r)\right).
\]

By means of the distortion tensor (27) the effective Burgers vector, \( b_1(r) = \oint \beta_{ij} dx_j \), for a circular circuit of radius \( r \) is given by

\[
b_z(r) = \oint \gamma (\beta_{zx} dx + \beta_{zy} dy) = b \left(1 - \kappa r K_1(\kappa r)\right),
\]

where \( \gamma \) is the Burgers circuit. It depends on the radius \( r \). In fact, we find \( b_z(0) = 0 \) and \( b_z(\infty) = b \). This effective Burgers vector differs appreciably from the constant value \( b \) in the core region from \( r = 0 \) up to \( r \approx 6\kappa^{-1} \). Thus, it is suggestive to take \( r_c \approx 6\kappa^{-1} \) as the characteristic length (dislocation core radius). Therefore, in the field theory of elastoplasticity the dislocation has a core in quite natural manner. Outside this core region the Burgers vector reaches its constant value. Accordingly, the classical and the elastoplastic solution coincide outside the core region.

If we use the decomposition (1) of the distortion into the compatible and purely incompatible distortion, the displacement field of a screw dislocation turns out to be (see also [18])

\[
u_z = \frac{b}{2\pi} (1 - \kappa r K_1(\kappa r)) \varphi,
\]

where \( \varphi \) is multi-valued. Thus, the proper incompatible part of the distortion is the (negative) plastic distortion which is confined in the dislocation core region

\[
\phi_{xx} = -\frac{b\kappa^2}{2\pi} \kappa \varphi K_0(\kappa r), \quad \phi_{zy} = -\frac{b\kappa^2}{2\pi} y \varphi K_0(\kappa r).
\]

It fulfils Eq. (10). The plastic strain reads

\[
E_{xx}' = -\frac{1}{2} \phi_{xx}, \quad E_{zy}' = -\frac{1}{2} \phi_{zy}.
\]

Now we are able to calculate the dislocation density by means of the distortion tensor. We obtain

\[
\alpha_{zz} = T_{xxy} = \frac{1}{\mu} \Delta F = \frac{b\kappa^2}{2\pi} K_0(\kappa r).
\]

Of course, this dislocation density satisfies the condition (11). In the limit as \( \kappa^{-1} \to 0 \), the elastoplastic result (34) converts to the classical dislocation density \( \alpha_{zz} = b \delta(r) \). By the help of (13) the localised moment stresses caused by the screw dislocation can be expressed in terms of the dislocation density as

\[
H_{xx} = -\frac{\mu b}{2\pi} K_0(\kappa r), \quad H_{yy} = -\frac{\mu b}{2\pi} K_0(\kappa r),
\]

\[
H_{zz} = \frac{\mu b}{2\pi} K_0(\kappa r),
\]

and

\[
H_{kk} = -\frac{\mu b}{2\pi} K_0(\kappa r).
\]

Accordingly, moment stresses of twisting-type occur at all positions where the dislocation density \( \alpha_{zz} \) is non-vanishing. When \( \kappa^{-1} \to 0 \), the moment stresses vanish.
The far-reaching rotation gradients of a screw dislocation read
\[ k_{xx} = \frac{b}{4\pi r^2} \left\{ (y^2 - x^2)(1 - \kappa r K_1(\kappa r)) + \kappa^2 x^2 r^2 K_0(\kappa r) \right\}, \]
\[ k_{yy} = \frac{b}{4\pi r^2} \left\{ (x^2 - y^2)(1 - \kappa r K_1(\kappa r)) + \kappa^2 y^2 r^2 K_0(\kappa r) \right\}, \]
\[ k_{xy} = \frac{b}{4\pi r^2} xy \left\{ 2(1 - \kappa r K_1(\kappa r)) - \kappa^2 r^2 K_0(\kappa r) \right\}, \]
\[ k_{yy} = k_{xy}, \]
\[ k_{jj} = \frac{1}{2} \alpha_{zz}. \]  

(37)

They are in agreement with the expressions calculated within the theory of Cosserat media (see [19–21]).

4 Edge dislocation in elastoplasticity

Consider an infinitely long edge dislocation whose dislocation line coincides with the \( z \)-axis while the Burgers vector is parallel to the \( x \)-axis: \( b_x = b, \ b_y = b_z = 0 \). The extra half plane lies in the plane \( x = 0 \). In order to satisfy the equilibrium condition (4) we use the second order stress function \( f \) and specialize to the plane problem of an edge dislocation by setting \( \partial_z = 0 \). The classical stress field of a straight edge dislocation can be given in terms of Airy’s stress function according to

\[ \sigma_{ij} = \begin{pmatrix} \partial_{yy}\chi - \partial_{xy}^2 \chi & 0 \\ -\partial_{xy}^2 \chi & \partial_{xx}^2 \chi & 0 \\ 0 & 0 & \nu \Delta \chi \end{pmatrix}. \]  

(38)

Airy’s stress function [9, 15]

\[ \chi = -A y \ln r, \quad A = \frac{\mu b}{2\pi(1-\nu)}. \]  

(39)

fulfills the following inhomogeneous bipotential (or biharmonic) equation

\[ \Delta \Delta \chi = -4\pi A \partial_y \delta(r). \]  

(40)

For the modified stress we make the following stress function ansatz

\[ \sigma_{ij} = \begin{pmatrix} \partial_{yy}^2 f - \partial_{xy}^2 f & 0 \\ -\partial_{xy}^2 f & \partial_{xx}^2 f & 0 \\ 0 & 0 & \nu \Delta f \end{pmatrix}. \]  

(41)

In addition, the strain is given in terms of the stress function as

\[ E_{ij} = \frac{1}{2\mu} \begin{pmatrix} \partial_{yy}^2 f - \nu \Delta f & -\partial_{xy}^2 f & 0 \\ -\partial_{xy}^2 f & \partial_{xx}^2 f - \nu \Delta f & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]  

(42)

Substituting (41) and (38) into (14) we get the inhomogeneous Helmholtz equation

\[ \left( 1 - \kappa^{-2} \Delta \right) f = -A y \ln r. \]  

(43)

The solution of the modified stress function of a straight edge dislocation is given by [4]

\[ f = -\frac{\mu b}{2\pi(1-\nu)} y \ln r + \frac{2}{\kappa^2 r^2} \left( 1 - \kappa r K_1(\kappa r) \right), \]  

(44)

where the first piece is Airy’s stress function.

By means of Eqs. (41) and (44), the modified stress of a straight edge dislocation is given as

\[ \sigma_{xx} = -\frac{\mu b}{2\pi(1-\nu)} \frac{y}{r^4} \left\{ (y^2 + 3x^2) + \frac{4}{\kappa^2 r^2} (y^2 - 3x^2) \right\} - 2y^2 \kappa r K_1(\kappa r) - 2(y^2 - 3x^2) K_2(\kappa r), \]

\[ \sigma_{yy} = -\frac{\mu b}{2\pi(1-\nu)} \frac{y}{r^4} \left\{ (y^2 - x^2) - \frac{4}{\kappa^2 r^2} (y^2 - 3x^2) \right\} - 2x^2 \kappa r K_1(\kappa r) + 2(y^2 - 3x^2) K_2(\kappa r), \]

\[ \sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)} \frac{x}{r^4} \left\{ (x^2 - y^2) - \frac{4}{\kappa^2 r^2} (x^2 - 3y^2) \right\} - 2y^2 \kappa r K_1(\kappa r) + 2(x^2 - 3y^2) K_2(\kappa r), \]

\[ \sigma_{zz} = -\frac{\mu b}{\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\}. \]  

(45)

The trace of the stress tensor \( \sigma_{kk} \) produced by the edge dislocation in an isotropic medium is

\[ \sigma_{kk} = \frac{\mu b(1+\nu)}{\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \kappa r K_1(\kappa r) \right\}. \]  

(46)

Let us now discuss some details of the stress fields in the core region. The stress fields have no artificial singularities at the core and the extremum stress occurs at a short distance away from the dislocation line. In fact, when \( r \to 0 \), we have

\[ K_1(\kappa r) \to \frac{1}{\kappa r}, \quad K_2(\kappa r) \to -\frac{1}{2} + \frac{2}{(\kappa r)^2}, \]  

(47)
and thus $\sigma_{ij} \rightarrow 0$. It can be seen that the stresses have the following extreme values: $|\sigma_{xx}(0, y)| \approx 0.546\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 0.996\kappa^{-1}$, $|\sigma_{yy}(0, y)| \approx 0.260\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 1.494\kappa^{-1}$, and $|\sigma_{zz}(0, y)| \approx 0.399\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 1.114\kappa^{-1}$. The stresses $\sigma_{xx}, \sigma_{yy}$ and $\sigma_{xy}$ are modified near the dislocation core ($0 \leq r \leq 12\kappa^{-1}$). Note that $|\sigma_{xy}(x, 0)| \approx 0.260\kappa\frac{b}{2\pi(1-\nu)}$ can be identified with the theoretical shear strength. The stresses $\sigma_{zz}$ and $\sigma_{kk}$ are modified in the region: $0 \leq r \leq 6\kappa^{-1}$. Far from the dislocation line ($r \gg 12\kappa^{-1}$) the elastoplastic and the classical solutions of the stress of an edge dislocation coincide. Thus, the characteristic internal length $\kappa^{-1}$ determines the position and the magnitude of the stress extremum. It is interesting and important to note that the solution (45) agrees precisely with the gradient solution given by Gutkin and Aifantis [12] (with $\kappa^{-2} = c, c$ is the gradient coefficient).

For the elastic strain of an edge dislocation we find

$$E_{xx} = -\frac{b}{4\pi(1-\nu)} \frac{y}{r^2} \left\{ (1 - 2\nu) + \frac{2x^2}{r^2} + \frac{4\kappa^2r^2}{\nu} (y^2 - 3x^2) - 2 \left( \frac{y^2}{r^2} - \nu \right) \kappa r K(\kappa r) \right\},$$

$$E_{yy} = -\frac{b}{4\pi(1-\nu)} \frac{y}{r^2} \left\{ (1 - 2\nu) - \frac{2x^2}{r^2} - \frac{4\kappa^2r^2}{\nu} (y^2 - 3x^2) - 2 \left( \frac{x^2}{r^2} - \nu \right) \kappa r K(\kappa r) \right\},$$

$$E_{xy} = \frac{b}{4\pi(1-\nu)} \frac{x}{r^2} \left\{ 1 - \frac{2y^2}{r^2} - \frac{4\kappa^2r^2}{\nu} (x^2 - 3y^2) - 2 \frac{y^2}{r^2} \kappa r K(\kappa r) + \frac{2}{r^2} (x^2 - 3y^2) K(\kappa r) \right\},$$

which is in agreement with the solution given by Gutkin and Aifantis [12, 14] in the framework of strain gradient elasticity. The plane-strain condition $E_{zz} = 0$ of the classical dislocation theory is valid. The components of the strain tensor have the following extreme values ($\nu = 0.3$): $|E_{xx}(0, y)| \approx 0.308\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 0.922\kappa^{-1}$, $|E_{yy}(0, y)| \approx 0.010\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 0.218\kappa^{-1}$, $|E_{xy}(0, y)| \approx 0.054\kappa\frac{b}{2\pi(1-\nu)}$ at $|y| \approx 4.130\kappa^{-1}$, and $|E_{xy}(x, 0)| \approx 0.260\kappa\frac{b}{2\pi(1-\nu)}$ at $|x| \approx 1.494\kappa^{-1}$. It is interesting to note that $E_{yy}(0, y)$ is much smaller than $E_{xx}(0, y)$ within the core region (see also [12]). The dilatation $E_{kk}$ reads

$$E_{kk} = -\frac{b(1 - 2\nu)}{2\pi(1-\nu)} \frac{y}{r^2} \left\{ 1 - \kappa r K(\kappa r) \right\},$$

Now we calculate the distortion of an edge dislocation. The distortion $\beta_{ij}$ is given in terms of the stress function (44):

$$\beta_{ij} = \frac{1}{2\mu} \begin{pmatrix} \partial_y^2 f - \nu \Delta f & -\partial_y^2 f + 2\mu \omega \\ -\partial_x^2 f - 2\mu \omega & \partial_x^2 f - \nu \Delta f \end{pmatrix},$$

where $\omega$ is used to express the antisymmetric part of the distortion, $\omega \equiv \beta_{xy}$. Eventually, $\omega$ is determined from the conditions:

$$\alpha_{xz} = T_{xx} = \frac{1}{\mu} \left( 2\mu \partial_x \omega - (1 - \nu) \partial_y \Delta f \right),$$

$$\alpha_{yz} = T_{yy} = \frac{1}{\mu} \left( 2\mu \partial_y \omega + (1 - \nu) \partial_x \Delta f \right).$$

We find for the elastic distortion of the edge dislocation

$$\beta_{xx} = -\frac{b}{2\pi(1-\nu)} \frac{y}{\nu} \left\{ (1 - 2\nu) + \frac{2x^2}{r^2} + \frac{4\kappa^2r^2}{\nu} (y^2 - 3x^2) - 2 \left( \frac{y^2}{r^2} - \nu \right) \kappa r K(\kappa r) - \frac{2\mu}{\kappa^2r^2} (y^2 - 3x^2) K(\kappa r) \right\},$$

$$\beta_{xy} = \frac{b}{4\pi(1-\nu)} \frac{x}{r^2} \left\{ (3 - 2\nu) - \frac{2y^2}{r^2} - \frac{4\kappa^2r^2}{\nu} (x^2 - 3y^2) - 2 \left( (1 - \nu) + \frac{y^2}{r^2} \right) \kappa r K(\kappa r) + \frac{2}{r^2} (x^2 - 3y^2) K(\kappa r) \right\},$$

$$\beta_{yx} = -\frac{b}{4\pi(1-\nu)} \frac{x}{r^2} \left\{ (1 - 2\nu) + \frac{2y^2}{r^2} + \frac{4\kappa^2r^2}{\nu} (x^2 - 3y^2) - 2 \left( (1 - \nu) - \frac{y^2}{r^2} \right) \kappa r K(\kappa r) - \frac{2}{r^2} (x^2 - 3y^2) K(\kappa r) \right\},$$

$$\beta_{yy} = -\frac{b}{4\pi(1-\nu)} \frac{y}{r^2} \left\{ (1 - 2\nu) - \frac{2x^2}{r^2} - \frac{4\kappa^2r^2}{\nu} (y^2 - 3x^2) - 2 \left( \frac{x^2}{r^2} - \nu \right) \kappa r K(\kappa r) + \frac{2}{r^2} (y^2 - 3x^2) K(\kappa r) \right\},$$

and for the rotation

$$\omega \equiv -\omega = -\frac{b}{2\pi} \frac{y}{\nu} \left\{ 1 - \kappa r K(\kappa r) \right\}.$$
fers from the classical one in the region $0 \leq \gamma \leq 1$ and a single-valued displacement field $\phi_{xy} = \frac{b \kappa^2}{2\pi} y \varphi K_0(\kappa r).$ (54)

Of course, it fulfils Eq. (10). This proper incompatible distortion is exactly the same as $\phi_{xx}$ and $\phi_{xy}$ of a screw dislocation (see Eq. (32)). The appearance of such proper incompatible distortion is a typical result in elastoplasticity. The plastic strain reads

$$E_{xx}^p = -\phi_{xx}, \quad E_{xy}^p = -\frac{1}{2}\phi_{xy}. \quad (55)$$

The corresponding compatible part can be given in terms of a multi-valued displacement field

$$u_x = \frac{b}{2\pi} \left\{ (1 - \kappa r K_1(\kappa r)) \varphi + \frac{1}{2(1-\nu)} \frac{xy}{r^2} \left( 1 - \frac{4}{\kappa^2 r^2} + 2 K_2(\kappa r) \right) \right\}. \quad (56)$$

The core radius of the edge dislocation can be estimated as $R_c \approx 6\kappa^{-1}$. Additionally, we find from the distortion the incompatible part

$$\phi_{yx} = \phi_{yy} = 0, \quad (57)$$

and a single-valued displacement field

$$u_y = -\frac{b}{4\pi(1-\nu)} \left\{ (1 - 2\nu)(\ln r + K_0(\kappa r)) + \frac{x^2 - y^2}{2r^2} \left( 1 - \frac{4}{\kappa^2 r^2} + 2 K_2(\kappa r) \right) \right\}. \quad (58)$$

This means that $\beta_{yx}$ and $\beta_{yy}$ are proper compatible distortions. Therefore, the field $u_y$ agrees (up to a constant term) with the corresponding formula in gradient elasticity (see [12, 14]). The displacement field $u_y$ differs from the classical one in the region $0 \leq r \leq 12\kappa^{-1}$. In the framework of elastoplasticity the displacement fields (56) and (58) have no singularity. Therefore, these displacement fields can be used to model the dislocation core. In this way, one can estimate the displacements, elastic strains and stresses near the dislocation cores and compare them with HRTEM micrographs and atomic calculations. The far fields of the displacements (56) and (58) are identical to the classical ones (see, e.g., [16]). It is worth noting that within the Peierls-Nabarro dislocation model the displacement $u_y$ is identically zero.

Finally, the effective Burgers vector of the edge dislocation can be calculated as

$$b_x(\tau) = \oint_{\gamma} (\beta_{xx} dx + \beta_{xy} dy) = b \{ 1 - \kappa r K_1(\kappa r) \}. \quad (59)$$

Again, we find $b_x(0) = 0$ and $b_x(\infty) = b$. The effective Burgers vector $b_x(r)$ differs from the constant Burgers vector $b$ in the region $0 \leq r \leq 6\kappa^{-1}$. Therefore, the core radius can be taken as $R_c \approx 6\kappa^{-1}$. For the value of $\kappa^{-1} = 0.25\alpha$ the core radius is $R_c = 1.5\alpha$ and for the value of $\kappa^{-1} = 0.399\alpha$ the core radius is $R_c = 2.4\alpha$ (a is the lattice constant). Note that the effective Burgers vector $b_x(r)$ of an edge dislocation has the same form as the effective Burgers vector $b_x(r)$ of a screw dislocation (see Eq. (30)).

The proper incompatible part of the elastic distortion gives rise to a localized dislocation density and moment stress tensor. We find for the dislocation density of an edge dislocation

$$\alpha_{xx} = T_{xyy} = \frac{b \kappa^2}{2\pi} K_0(\kappa r) \quad (60)$$

which satisfies the translational Bianchi identity (11). The dislocation density is short-reaching. It is interesting to note that the dislocation density of an edge dislocation has the same form as the dislocation density $\alpha_{zz}$ of a screw dislocation (see Eq. (34)). In the limit as $\kappa^{-1} \to 0$, the elastoplastic result (60) converts to the classical dislocation density $\alpha_{xx} = b \delta(r)$. The localized moment stress of bending type is given by the help of (13) according to

$$H_{xz} = \frac{\mu b}{\pi(1-\nu)} K_0(\kappa r), \quad H_{xx} = -\frac{\mu b}{\pi(1-\nu)} K_0(\kappa r). \quad (61)$$

This expression is very close to the moment stress of an edge dislocation given in [22] (see also the remarks in [23]).

DeWit’s bend-twist tensor of an edge dislocation reads

$$\kappa_{xz} = \frac{b}{2\pi r^3} \left\{ (x^2 - y^2)(1 - \kappa r K_1(\kappa r)) - \kappa^2 x^2 y^2 K_0(\kappa r) \right\},$$

$$\kappa_{xy} = \frac{2}{2\pi r^4} xy \left\{ 2(1 - \kappa r K_1(\kappa r)) - \kappa^2 x^2 y^2 K_0(\kappa r) \right\} \quad (62)$$

It is in agreement with the expression calculated in the theory of Cosserat media (see [19, 20]).

5 Conclusions

The field theory of elastoplasticity has been employed to consider straight screw and edge dislocations. The dislocation densities of straight screw and edge dislocations obtained in elastoplastic field theory agree with Eringen’s two-dimensional nonlocal modulus (nonlocal kernel) used in [10, 11]. Consequently, every component of
the dislocation density tensor is Green’s function of the Helmholtz equation:

\[
(1 - \kappa^{-2} \Delta) \alpha_{ij}(r) = b \delta(r). \tag{63}
\]

The characteristic internal length in elastoplasticity is \(\kappa^{-1}\). This length may be selected to be proportional to the lattice parameter \(a\) for a single crystal, i.e.

\[
\kappa^{-1} = e_0 a, \tag{64}
\]

where \(e_0\) is a non-dimensional constant which can be determined by one experiment [11]. For \(e_0 = 0\) we recover classical elasticity. In [10, 11] the choice \(e_0 = 0.399\) and in [24] the choice \(e_0 = 0.25\) are proposed. That length specifies the plastic region and should be estimated by means of experimental observations and computer simulations of the core region.

Exact analytical solutions for the displacements, elastic and plastic strain fields, and force stresses of dislocations have been reported which demonstrate the elimination of any singularity at the dislocation line. It has been shown that the force stresses achieve their extreme values at a short distance away from the dislocation line. Therefore, we are able to obtain finite strain and stress in the (linear) field theory of elastoplasticity. These maximum values may serve as measures of the critical stress level of fracture. The stress fields are in agreement with the ones obtained by Gutkin and Aifantis [12]. Additionally, we have calculated the moment stresses and deWit’s bend-twist tensor for straight screw and edge dislocations.

Acknowledgements

The author acknowledges the support provided by the Max-Planck-Institute for Mathematics in the Sciences.

A Single-valued, discontinuous displacement and (plastic) distortion

In the expressions given by (31), (32), (54) and (56) we have considered \(\varphi\) as a multi-valued field. On the other hand, in the defect theory [7, 12–14] \(\varphi\) is usually used as single-valued and discontinuous function. It is made unique by cutting the half-plane \(y = 0\) at \(x < 0\) and assuming \(\varphi\) to jump from \(\pi\) to \(-\pi\) when crossing the cut. If one uses the single-valued discontinuous form for \(\varphi\), we obtain [4] for the screw dislocation

\[
u_z = \frac{b}{2\pi} \left\{ \varphi \left(1 - \kappa r K_1(\kappa r)\right) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right\}, \tag{A.1}
\]

with

\[
\begin{align*}
\phi_{xx} &= -\frac{b}{2\pi} \kappa^2 x K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right), \\
\phi_{xy} &= -\frac{b}{2\pi} \kappa^2 y K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right) \\
&\quad + \pi \text{sign}(y) \left(1 - \text{sign}(x) \left[1 - \kappa r K_1(\kappa r)\right]\right), \tag{A.2}
\end{align*}
\]

and for the edge dislocation

\[
\begin{align*}
\nu_x &= \frac{b}{2\pi} \left\{ \varphi \left(1 - \kappa r K_1(\kappa r)\right) + \frac{\pi}{2} \text{sign}(y) \kappa r K_1(\kappa r) \right\}, \\
\nu_y &= -\frac{b}{4\pi(1 - \nu)} \left\{ (1 - 2\nu) \left(\ln(r) + K_0(\kappa r)\right) \\
&\quad + \frac{x^2 - y^2}{2r^2} \left(1 - \frac{4}{\kappa^2 r^2} + 2K_2(\kappa r)\right) \right\}, \tag{A.3}
\end{align*}
\]

with

\[
\begin{align*}
\phi_{xx} &= -\frac{b}{2\pi} \kappa^2 x K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right), \\
\phi_{xy} &= -\frac{b}{2\pi} \kappa^2 y K_0(\kappa r) \left( \varphi - \frac{\pi}{2} \text{sign}(y) \right) \\
&\quad + \pi \text{sign}(y) \left(1 - \text{sign}(x) \left[1 - \kappa r K_1(\kappa r)\right]\right), \\
\phi_{yx} &= \phi_{yy} = 0. \tag{A.4}
\end{align*}
\]

For a more detailed discussion see [4].

References

[1] M. Lazar, Ann. Phys. (Leipzig) 9 (2000) 461; [cond-math/0006280].
[2] M. Lazar, J. Phys. A: Math. Gen. 35 (2002) 1983.
[3] M. Lazar, Ann. Phys. (Leipzig) 11 (2002) 635; [cond-math/0203058].
[4] M. Lazar, J. Phys. A: Math. Gen. 36 (2003) 1415.
[5] D.G.B. Edelen and D.C. Lagoudas, Gauge theory and defects in solids, in: Mechanics and Physics of Discrete System, Vol. 1, G.C. Sih, ed., North-Holland, Amsterdam (1988).
[6] R. deWit, J. Res. Nat. Bur. Stand. (U.S.) 77A (1973) 49.
[7] R. deWit, J. Res. Nat. Bur. Stand. (U.S.) 77A (1973) 607.
[8] A.E. Green and P.M. Naghdi, Arch. Rat. Mech. Anal. 18 (1965) 251.
[9] E. Kröner, Continuum theory of defects, in: Physics of defects (Les Houches, Session 35), R. Balian et al., eds., North-Holland, Amsterdam (1981) p. 215.
[10] A.C. Eringen, J. Appl. Phys. 54 (1983) 4703.
[11] A.C. Eringen, Nonlocal Continuum Theory for Dislocations and Fracture, in: The Mechanics of Dislocations, Eds. E.C. Aifantis and J.P. Hirth, American Society of Metals, Metals Park, Ohio (1985) p. 101.
[12] M.Yu. Gutkin and E.C. Aifantis, Scripta Mater. 40 (1999) 559.
[13] M.Yu. Gutkin and E.C. Aifantis, Scripta Mater. 35 (1996) 1353.
[14] M.Yu. Gutkin and E.C. Aifantis, Scripta Mater. 36 (1997) 129.
[15] E. Kröner, Kontinuumstheorie der Versetzungen und Eigenspannungen, Erg. Angew. Math. 5 (1958) p. 1.
[16] J.P. Hirth and J. Lothe, Theory of Dislocations, John Wiley, New York (1982).
[17] J.P. Nowacki, Bull. Acad. Polon. Sci., Sér. sci. techn. 21 (1973) 585.
[18] D.G.B. Edelen, Int. J. Engng. Sci. 34 (1996) 81.
[19] S. Kessel, Z. Ang. Math. Mech. 50 (1970) 547.
[20] N. Nowacki, Arch. Mech. 26 (1974) 3.
[21] S. Minagawa, Appl. Eng. Sci. Lett. 5 (1977) 85.
[22] F.W. Hehl and E. Kröner, Z. Naturforsch. 20a (1965) 336.
[23] E. Kröner, The continuized crystal as a model for crystals with dislocations, in: Materials Modelling: From Theory to Technology, Ed. J.R. Matthews, IOP Publishing Ltd (1992) p. 13.
[24] B.S. Altan and E.C. Aifantis, Scripta Metall. Mater. 26 (1992) 319.