HYPERGEOMETRIC THETA FUNCTIONS AND ELLIPTIC MACDONALD POLYNOMIALS

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Abstract. Elliptic Macdonald polynomials of $sl_2$-type and level 2 are introduced. Suitable limits of elliptic Macdonald polynomials are the standard Macdonald polynomials and conformal blocks. Identities for elliptic Macdonald polynomials, in particular their modular properties, are studied.

1. Introduction

Conformal field theory and its $q$-deformation provide new classes of interesting special functions. Here we study special functions arising in the $q$-deformation of conformal blocks on elliptic curves with one marked point. These functions are given by elliptic versions of $q$-hypergeometric integrals and we call them hypergeometric theta functions. We study here the simplest non-trivial case, corresponding to the 3-dimensional irreducible representation of the Lie algebra $sl_2$, but one should expect the same picture to hold in the more general situation of the $m$-th symmetric tensor power of the vector representation of $sl_{n+1}$. Then hypergeometric theta functions are functions on the Cartan subalgebra of $sl_{n+1}$ depending on an integer parameter $\kappa$, the level, and two complex parameters: $\tau$, parametrizing the elliptic curve and $\eta$, the deformation parameter. In the classical limit $\eta \to 0$, hypergeometric theta functions are supposed to converge to conformal blocks, realized as theta functions obeying vanishing conditions and obeying the KZB equation. In the trigonometric limit $\tau \to i\infty$ one expects to recover Macdonald polynomials. More precisely, the quotient of hypergeometric theta functions by a suitable product of ordinary theta functions is expected to converge to $A_n$-Macdonald polynomials in the trigonometric limit. We call these quotients elliptic Macdonald polynomials. From the point of view of representation theory, hypergeometric theta functions are expected to be related to traces of intertwining operators for the quantum affine Lie algebra.

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algebra $U_q(\hat{sl}_{n+1})$, $q = e^{-2\pi i \eta}$, over integrable simple modules of level $\kappa - n - 1$. Thus we conjecture that our definition of elliptic Macdonald polynomials coincides with the definition of Etingof and Kirillov \cite{EK2} in terms of traces of intertwining operators.

The classical limit is the limit in which classical Lie algebras are recovered from their quantum version, and in the trigonometric limit one goes from affine to simple finite dimensional Lie algebras. Thus, we expect the following picture to hold.

\[
\text{Conformal blocks} \quad \xrightarrow{\tau \to i\infty} \quad \text{Jack polynomials} \quad \xrightarrow{\eta \to 0} \quad \text{Hypergeometric theta functions} \quad \xrightarrow{\eta \to 0} \quad A_n\text{-Macdonald polynomials}
\]

In this paper, we establish this picture in the simplest non-trivial $sl_2$-case. For $sl_2$, the hypergeometric theta functions may be considered as a degeneration of the elliptic hypergeometric integrals studied for generic parameters in \cite{PV2, PV3}. The elliptic hypergeometric integrals obey several identities involving their values at points related to each other by an action of $\text{SL}(3, \mathbb{Z})$. Some of these identities survive in the degenerate case and result in the known identities in the trigonometric and classical limits. In the case we consider, these identities are the following: first of all, we see that the KZB equation obeyed by conformal blocks is a limiting case of two equations, the qKZB equations, both of which are solved by hypergeometric theta functions. One is an integral equation, which in the trigonometric limit becomes a Macdonald-Mehta type identity for Macdonald polynomials. The other is an infinite difference equation, which in the trigonometric limit expresses the fact that Macdonald polynomials are eigenvectors of the Macdonald–Ruijsenaars difference operators. Other identities in \cite{PV3} give the transformation properties of hypergeometric theta functions under the modular group $\text{SL}(2, \mathbb{Z})$. In the classical limit, we recover the known modular properties of conformal blocks. Finally, the orthogonality relations of \cite{PV3} also have a counterpart for elliptic Macdonald polynomials and reduce to the characterizing orthogonality properties of Macdonald polynomials.

2. QKZB Heat Equation

2.1. Theta functions of level $\kappa$. Let $\text{Im} \, \tau > 0$. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called a theta function of level $\kappa \in \mathbb{Z}_{\geq 0}$ if

$$f(\lambda + 2r + 2s\tau) = e^{-2\pi i s(2r + s\lambda)} f(\lambda), \quad r, s \in \mathbb{Z}.$$ 

Let $\Theta_\kappa(\tau)$ denote the space of theta functions of level $\kappa$. If $\kappa = 0$, $\Theta_\kappa(\tau) = \mathbb{C}$. If $\kappa > 0$, $\Theta_\kappa(\tau)$ has dimension $2\kappa$: a basis is

$$\theta_{j,\kappa}(\lambda, \tau) = \sum_{n \in \mathbb{Z} + \frac{j}{2\kappa}} e^{2\pi i n(\tau + n\lambda)}, \quad j \in \mathbb{Z}/2\kappa\mathbb{Z}.$$
The space $\Theta_\kappa(\tau)$ is the direct sum of the space $\Theta_\kappa(\tau)^{\text{even}}$ of even theta functions and the space $\Theta_\kappa(\tau)^{\text{odd}}$ of odd theta functions, of dimension $\kappa + 1$ and $\kappa - 1$, respectively. In particular, for $\kappa = 2$, Jacobi’s first theta function $\theta(\lambda, \tau) = -\sum_{n \in \mathbb{Z} + 1/2} e^{\pi i(n^2 \tau + n(2\lambda + 1))}$, $(\theta = i(\theta_{-1,1} - \theta_{1,1}))$ spans the space of odd theta functions of level 2.

2.2. The elliptic hypergeometric integral. Let $\lambda, \mu, \tau, \sigma, \eta \in \mathbb{C}$ and $\text{Im} \tau > 0$, $\text{Im} \sigma > 0$.

The basic object is the hypergeometric integral

$$u(\lambda, \mu, \tau, \sigma, \eta) = e^{-\frac{\pi i \mu}{2\eta}} \int_{\gamma} \Omega_{2\eta}(t, \tau, \sigma) \frac{\theta(\lambda + t, \tau)\theta(\mu + t, \sigma)}{\theta(t - 2\eta, \tau)\theta(t - 2\eta, \sigma)} dt,$$

where

$$\Omega_{2\eta}(t, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{(1 - e^{2\pi i(t - 2\eta + j \tau + k \sigma)})(1 - e^{2\pi i(-t - 2\eta + (j + 1) \tau + (k + 1) \sigma)})}{(1 - e^{2\pi i(t + 2\eta + j \tau + k \sigma)})(1 - e^{2\pi i(-t + 2\eta + (j + 1) \tau + (k + 1) \sigma)})}.$$

The integration path is the interval $[0, 1]$ as long as $\eta$ has positive imaginary part. For general $\eta$ the function $u$ is defined by analytic continuation. The analytic properties of $u$ are given by the following refinement of a result in [FV3].

**Proposition 2.1.** Let $H_+ = \{z \in \mathbb{C} \mid \text{Im} \, z > 0\}$ be the upper half plane. Then $u$ is a meromorphic function on $\mathbb{C} \times \mathbb{C} \times H_+ \times H_+ \times \mathbb{C}$. It is regular on the complement of the hyperplanes $4\eta + l + m\tau + n\sigma = 0$, $l \in \mathbb{Z}$, $m, n \in \mathbb{Z}_{\geq 0}$, where it has at most simple poles.

**Proof:** The integrand, regarded as a function of $t$ for fixed generic $\tau, \sigma$, has two families of simple poles: the first family consists of the poles at $t = 2\eta + l + m\tau + n\sigma$ and the second at $t = -(2\eta + l + m\tau + n\sigma)$. In both families $l, m, n$ run over integers such that $m, n \geq 0$. If $\eta$ has positive imaginary part, the poles of the first family lie in the upper half plane and the poles of the second family lie in the lower half plane. The integration cycle separates the (projection onto $\mathbb{C}/\mathbb{Z}$) of the two families. As we do an analytic continuation, the integration cycle, originally along the real axis, gets deformed and $u$ is holomorphic as long as no pole of the first family coincides with a pole of the second family. Such a coincidence happens when $4\eta + l + m\tau + n\sigma = 0$ for some integers $l, m, n$ with $m, n \geq 0$. As one approaches a generic point of a hyperplane $4\eta + l + m\tau + n\sigma = 0$, a finite number of poles of the first family meets poles of the second, and the integration cycles gets pinched between simple poles. The integral has then at most a simple pole on the hyperplane.

We shall use the following two properties of the hypergeometric integral.

**Lemma 2.2.** The hypergeometric integral satisfies the relation

$$u(-\lambda, -\mu, \tau, \sigma, \eta) = u(\lambda, \mu, \tau, \sigma, \eta).$$
The lemma follows from identity (A.15) in [FV3], cf. identities in [FV4].

**Lemma 2.3 ([FV3]).** For \( r, s \in \mathbb{Z} \) we have
\[
 u(\lambda, 2\eta + r + sp, \tau, \sigma, \eta) = e^{2\pi i s(\tau - 4\eta)} u(\lambda, -2\eta + r + sp, \tau, \sigma, \eta).
\]

2.3. The qKZB discrete connection. Let
\[
 Q(\mu, \sigma, \eta) = \frac{\theta(4\eta, \sigma)\theta'(0, \sigma)}{\theta(\mu - 2\eta, \sigma)\theta(\mu + 2\eta, \sigma)},
\]
where the prime denotes the derivative with respect to the first argument. In [FV2], [FV3] we studied the integral operator (defined on a suitable space of holomorphic functions)
\[
 (1) \quad U(\tau, \sigma, \eta) : v \mapsto \int_{\eta \in \mathbb{R}} u(\lambda, \mu, \tau, \sigma, \eta)Q(\mu, \sigma, \eta)v(-\mu)d\mu,
\]
for generic values of the parameters. In [FV2] this integral operator was used to define a q-deformation of the KZB heat equation, the qKZB heat equation. In [FV3] the integral operator was used to construct solutions of the qKZB heat equations and describe their monodromy.

In this paper we consider the case when \( \sigma \) is a positive integer multiple of \(-2\eta\). In this case the qKZB heat operator can be defined as a map between finite dimensional vector spaces of theta functions.

For \( \kappa \in \mathbb{Z}, \kappa \geq 4 \), let \( E_\kappa(\tau, \eta) \) be the subspace of \( \Theta_{\kappa+2}(\tau)^{\text{odd}} \) consisting of functions vanishing at \( 2\eta j + \mathbb{Z} + \mathbb{Z}\tau, j = -1, 0, 1 \). Since such functions are divisible by the odd function \( \theta(\lambda - 2\eta, \tau)\theta(\lambda, \tau)\theta(\lambda + 2\eta, \tau) \) in the graded ring \( \bigoplus_{\kappa=0}^{\infty} \Theta_\kappa(\tau) \), we have
\[
 (2) \quad E_\kappa(\tau, \eta) = \{ \theta(\lambda - 2\eta, \tau)\theta(\lambda, \tau)\theta(\lambda + 2\eta, \tau)g(\lambda) \mid g \in \Theta_{\kappa-4}(\tau)^{\text{even}} \}
\]
In particular,
\[
 \dim E_\kappa(\tau, \eta) = \kappa - 3.
\]

**Proposition 2.4.** Assume that \( \kappa \) is an integer \( \geq 4 \), \( \Im \tau > 0, \Im \eta < 0, j\tau + 4\eta \notin \mathbb{Z}, \) for every \( j = 1, 2, \ldots \). Let \( \alpha(\lambda, \eta) = \exp(-\pi i \lambda^2 / 4\eta) \). Denote by \( \alpha(\eta) \) the operator of multiplication by \( \alpha(\lambda, \eta) \). Then the integral operator
\[
 T_\kappa(\tau, \eta) = -\frac{e^{4\pi i \eta}}{2\pi \sqrt{4\eta}} \alpha(\eta) U(\tau, \tau - 2\eta \kappa, \eta) \alpha(\eta)
\]
maps \( E_\kappa(\tau - 2\eta \kappa, \eta) \) to \( E_\kappa(\tau, \eta) \).
Proof. Our goal is to show that for any function \( f \in E_\kappa(\tau - 2\eta\kappa) \) the integral

\[
-\frac{e^{4\pi i\eta}}{2\pi \sqrt{4i\eta}} \int_{\gamma} \int_{\eta \in \mathbb{R}} \frac{e^{-\pi i(\lambda + \mu)^2/4\eta}}{\Omega_{2\eta}(t, \tau, \tau - 2\eta\kappa)} \frac{\theta(\lambda + t, \tau) \theta(\mu + t, \tau - 2\eta\kappa)}{\theta(t - 2\eta, \tau) \theta(t - 2\eta, \tau - 2\eta\kappa)} \times \frac{\theta(4\eta, \tau - 2\eta\kappa)}{\theta(\mu - 2\eta, \tau - 2\eta\kappa)} f(-\mu) \, dt \, d\mu
\]

converges and belongs to \( E_\kappa(\tau) \) as a function of \( \lambda \).

Clearly the integrand is holomorphic in \( \mu \). The integrand has the same poles with respect to \( t \) as the integrand of \( u(\lambda, \mu, \tau, \tau - 2\eta\kappa, \eta) \), so the integration with respect to \( t \) is defined except possibly on the hyperplanes of Proposition 2.1. Under the assumptions of this proposition that condition reduces to the condition \( 4\eta + j\tau \notin \mathbb{Z}, \ j = 1, 2, \ldots \).

Lemma 2.5 (cf. Lemma C.1, [FV3]). For \( f \in \Theta_\kappa(\tau) \), there exists \( C_1, C_2 > 0 \) such that for all \( \lambda \in \mathbb{C} \), we have

\[
|f(\lambda)| \leq C_1 \exp \left( \frac{\pi \kappa}{2} \left( \frac{\text{Im } \lambda}{\text{Im } \tau} \right)^2 + C_2 \text{Im } \lambda \right).
\]

It follows from the lemma that if \( \mu = \eta x, \ x \in \mathbb{R} \), then there exists \( C_3, C_4 > 0 \) such that the absolute value of the integrand is not greater than

\[
C_3 \exp \left( \frac{\pi}{4} x^2 (\text{Im } \eta) \left( \frac{\text{Im } \tau - 4 \text{Im } \eta}{\text{Im } \tau - 2\kappa \text{Im } \eta} \right) + C_4 |x| \right).
\]

Thus the integration with respect to \( \mu \) is defined and the double integral is a holomorphic function of \( \lambda \).

The integral is an odd function of \( \lambda \) by Lemma 2.2.

The fact that the integral belongs to \( \Theta_{\kappa+2}(\tau) \) easily follows from the theta function properties of the integrand. The fact that the integral is divisible by \( \theta(\lambda - 2\eta, \tau) \theta(\lambda, \tau) \theta(\lambda + 2\eta, \tau) \) follows from Lemma 2.3.

We say that a function \( v(\lambda, \tau) \) is a solution of the qKZB heat equation if

\[
T_\kappa(\tau, \eta)v(\lambda, \tau - 2\eta\kappa) = v(\lambda, \tau),
\]

Solutions \( v(\lambda, \tau) \) belonging to \( E_\kappa(\tau, \eta) \) for each fixed \( \tau \) are constructed below.

3. Hypergeometric theta functions

3.1. Definition. Let \( \kappa \in \mathbb{Z}, \ \kappa \geq 4 \). We say that an integer \( l \) is admissible with respect to \( \kappa \) if \( l \neq \pm 1 \mod \kappa \).

For an admissible \( l \), define the \( l \)-th non-symmetric hypergeometric theta function of level \( \kappa + 2 \) by

\[
\widetilde{\Delta}_{l, \kappa}(\lambda, \tau, \eta) = \sum_{j \in 2\kappa \mathbb{Z} + l} u(\lambda, 2\eta j, \tau, -2\eta\kappa, \eta) Q(2\eta j, -2\eta\kappa, \eta) e^{\pi i \frac{\eta j + \eta}{2\kappa} \tau^2}.
\]
Using the transformation properties of $Q$ we may rewrite this as
\[
\tilde{\Delta}_{l,\kappa}(\lambda, \tau, \eta) = e^{\frac{4\pi i}{\kappa} l^2} Q(2\eta l, -2\eta \kappa, \eta) \sum_{j \in 2\kappa \mathbb{Z} + l} u(\lambda, 2\eta j, \tau, -2\eta \kappa, \eta) e^{\pi i \frac{r - 4\eta j^2}{2\kappa}}.
\]

We have
\[
\tilde{\Delta}_{l,\kappa}(-\lambda, \tau, \eta) = \tilde{\Delta}_{-l,\kappa}(\lambda, \tau, \eta), \quad \tilde{\Delta}_{l,\kappa}(\lambda, \tau, \eta) = \tilde{\Delta}_{l+2\kappa,\kappa}(\lambda, \tau, \eta).
\]

We define the $l$-th hypergeometric theta function of level $\kappa + 2$ by
\[
\Delta_{l,\kappa}(\lambda, \tau, \eta) = \tilde{\Delta}_{l,\kappa}(\lambda, \tau, \eta) - \tilde{\Delta}_{l,\kappa}(-\lambda, \tau, \eta).
\]

**Theorem 3.1.** Let $\text{Im} \eta < 0$. Then

(i) For any fixed $\tau \in H_+$, such that $j\tau + 4\eta \notin \mathbb{Z}$ for every $j = 1, 2, \ldots$, and for any admissible $l$, the series $\tilde{\Delta}_{l,\kappa}$ converges to a holomorphic function $\lambda \mapsto \Delta_{l,\kappa}(\lambda, \tau, \eta).

(ii) Under the same assumptions, we have
\[
\Delta_{l,\kappa}(\lambda, \tau, \eta) = e^{\frac{2\pi i}{\kappa} l^2} I_{l,\kappa}(\lambda, \tau, \eta) Q(2\eta l, -2\eta \kappa, \eta),
\]
where
\[
I_{l,\kappa}(\lambda, \tau, \eta) = \int_{\gamma} \Omega_{2\eta}(t, \tau, -2\eta \kappa) \frac{\theta(\lambda + t, \tau) \theta(2\eta l + t, -2\eta \kappa)}{\theta(t - 2\eta, \tau) \theta(t - 2\eta, -2\eta \kappa)} e^{-2\pi it/\kappa} \theta_{l,\kappa}(\frac{2}{\kappa} t - \lambda, \tau) dt.
\]

(iii) The functions $\Delta_{l,\kappa}$, $l = 2, \ldots, \kappa - 2$, form a basis of $E_{\kappa}(\tau, \eta)$.

**Proof:** One shows that the integral $I_{l,\kappa}(\lambda, \tau, \eta)$ is defined and holomorphic with respect to $\lambda$ as in the proof of Proposition 2.4. The series for $\tilde{\Delta}_{l,\kappa}$ is then obtained by expanding the theta function $\theta_{l,\kappa}$ in a Fourier series in $\lambda$. This proves (i) and (ii).

The fact that $I_{l,\kappa}(\lambda, \tau, \eta)$ (and thus $\tilde{\Delta}_{l,\kappa}$) is a theta function of level $\kappa + 2$ follows easily from the theta function properties of $\theta_{l,\kappa}$. The vanishing condition for $\Delta_{l,\kappa}$ at the translates of $\pm 2\eta, 0$ follows from Lemma 2.3. To complete the proof of (iii) it remains to show that the functions $\Delta_{l,\kappa}$, $l = 2, \ldots, \kappa - 2$, are linearly independent. This follows from a degenerate version of the inversion relation of [FV3]:

\[
\frac{1}{32\pi^2 \eta} \int_0^2 u(-\mu, 2\eta l, \tau, \sigma, -\eta) u(\mu, 2\eta n, \tau, \sigma, \eta) Q(\mu, \tau, \eta) d\mu = Q(2\eta n, \sigma, \eta)^{-1} \delta_{l,n},
\]
for any $l, n \in \mathbb{Z}$. The integration is along a path which does not cross the straight line segment between $2\eta$ and $-2\eta$ or its translates by $\mathbb{Z} + \sigma \mathbb{Z}$. The proof of this identity is the same as the proof of the inversion relation in [FV3]. The only difference is that the integration is over a period instead of an infinite line, which is permitted since the integrand is a 2-periodic function of $\mu$ for integers $l, n$. 
We now set $\sigma = -2\eta\kappa$ in the inversion relation and restrict to $n \not\equiv 0 \mod 2\kappa$ to avoid poles of $Q$. We get, for $l, j = 2, \ldots, \kappa - 2$,

\begin{equation}
\frac{1}{32\pi^2\eta} \int_0^2 u(-\mu, 2\eta l, \tau, -2\eta\kappa, -\eta) \Delta_{j, \kappa}(\mu, \tau, \eta) Q(\mu, \tau, \eta) d\mu = \delta_{l,j} e^{\pi i \frac{4\eta l \eta j}{2\kappa}}.
\end{equation}

In particular, this implies the linear independence of $\Delta_{l,\kappa}$.

3.2. Theta function solutions of the qKZB heat equation.

**Theorem 3.2.** For any admissible $l$ the functions $\tilde{\Delta}_{l,\kappa}$ and $\Delta_{l,\kappa}$ are solutions of the qKZB heat equation:

\begin{align*}
T_\kappa(\tau, \eta)\tilde{\Delta}_{l,\kappa}(\lambda, \tau - 2\eta\kappa, \eta) &= \tilde{\Delta}_{l,\kappa}(\lambda, \tau, \eta), \\
T_\kappa(\tau, \eta)\Delta_{l,\kappa}(\lambda, \tau - 2\eta\kappa, \eta) &= \Delta_{l,\kappa}(\lambda, \tau, \eta).
\end{align*}

The theorem follows from the fact that for a fixed $\mu$ the function $u(\lambda, \mu, \tau - 2\eta\kappa, -2\eta\kappa, \eta)$ gives a solution of the qKZB equation up to a scalar factor depending on $\mu$, see [FV3]. Multiplying by the exponential function in the definition of $\tilde{\Delta}_{l,\kappa}$ we get rid of the scalar factor, so that each term in the series obeys the qKZB equation.

3.3. Theta functions as eigenfunctions of a difference operator. The integral operator $U(\tau, \sigma, \eta)$ has a discrete version,

\[ U(\tau, \sigma, \eta) : v \mapsto \sum_{m \in \mathbb{Z}} u(\lambda, -\lambda + 2\eta m, \tau, \sigma, \eta)Q(-\lambda + 2\eta m, \sigma, \eta)v(\lambda - 2\eta m). \]

**Theorem 3.3.** Assume that $\kappa$ is an integer $\geq 4$, $\text{Im} \tau > 0$, $\text{Im} \eta < 0$, $j\tau + 4\eta \not\in \mathbb{Z}$ for every $j = 1, 2, \ldots$. Then the difference operator

\begin{equation}
\tilde{T}_\kappa(\tau, \eta) = C\alpha(\eta) \tilde{\Delta}_{l,\kappa}(\lambda, \tau - 2\eta\kappa, \eta) \alpha(\eta),
\end{equation}

\[ C = \frac{i e^{4\pi i \eta}}{2\pi} \left( \sum_{m \in \mathbb{Z}} e^{-\pi i \eta m^2} \right)^{-1}, \]

is well defined on $E_{\kappa}(\tau - 2\eta\kappa, \eta)$ and maps $E_{\kappa}(\tau - 2\eta\kappa, \eta)$ to $E_{\kappa}(\tau, \eta)$. Moreover, for any $l = 2, \ldots, \kappa - 2$ we have

\begin{align*}
\tilde{T}_\kappa(\tau, \eta)\tilde{\Delta}_{l,\kappa}(\lambda, \tau - 2\eta\kappa, \eta) &= \tilde{\Delta}_{l,\kappa}(\lambda, \tau, \eta), \\
\tilde{T}_\kappa(\tau, \eta)\Delta_{l,\kappa}(\lambda, \tau - 2\eta\kappa, \eta) &= \Delta_{l,\kappa}(\lambda, \tau, \eta).
\end{align*}

The proof is the same as the proof of Theorem 2.1 in [FV3], cf. Remark in Section 3.3 in [FV2].
4. Modular transformation properties

4.1. Formulas.

Theorem 4.1. (i) If \( \text{Im} \eta < 0, \ \text{Im} \tau > 0 \), then for \( l = 2, \ldots, \kappa - 2 \) we have

\[
\Delta_{l,\kappa}(\lambda, \tau + 1, \eta) = e^{\frac{\pi i l^2}{2 \kappa}} \Delta_{l,\kappa}(\lambda, \tau, \eta).
\]

(ii) If \( \text{Im} \eta < 0, \ \text{Im} \tau > 0, \ \text{Im} \eta/\tau < 0 \), then for \( l = 2, \ldots, \kappa - 2 \) we have

\[
C^{-}(\tau, \eta) e^{-\frac{\pi i (\kappa+2)\lambda^2}{2 \kappa}} \Delta_{l,\kappa}\left(\frac{\lambda}{\tau}, -\frac{1}{\tau}, \frac{\eta}{\tau}\right) = \sum_{j=2}^{\kappa-2} \Delta_{j,\kappa}(\lambda, \tau, \eta) S_{j,l}^{-}(\tau, \eta),
\]

where

\[
S_{j,l}^{-}(\tau, \eta) = Q \left( \frac{l}{\kappa}, -\frac{\tau}{2 \eta \kappa}, \frac{\eta}{2 \kappa} \right) \times
\left( u \left( J_{j} - \frac{l}{\kappa}, \frac{\tau}{2 \eta \kappa}, -\frac{1}{2 \kappa} \right) - u \left( J_{j} - \frac{l}{\kappa}, \frac{\tau}{2 \eta \kappa}, \frac{\eta}{2 \kappa} \right) \right),
\]

\[
C^{-}(\tau, \eta) = -2\pi i \sqrt{\frac{2\kappa i}{\tau}} e^{\pi i \frac{2\eta}{\kappa} + \frac{\pi i}{6 \kappa} \psi(\tau, -2 \eta \kappa)},
\]

\[
\psi(\tau, p) = 2 \left( 8\eta^2 + \tau^2 + p^2 - 3p + 3\tau + 3\tau p + 1 \right).
\]

Here the square root is with positive real part.

(iii) If \( \text{Im} \eta < 0, \ \text{Im} \tau > 0, \ \text{Im} \eta/\tau > 0 \), then for \( l = 2, \ldots, \kappa - 2 \),

\[
C^{+}(\tau, \eta) e^{-\frac{\pi i (\kappa+2)\lambda^2}{2 \kappa}} \Delta_{l,\kappa}\left(\frac{\lambda}{\tau}, -\frac{1}{\tau}, -\frac{\eta}{\tau}\right) = \sum_{j=2}^{\kappa-2} \Delta_{j,\kappa}(\lambda, \tau, \eta) S_{j,l}^{+}(\tau, \eta),
\]

where

\[
S_{j,l}^{+}(\tau, \eta) = Q \left( \frac{l}{\kappa}, -\frac{\tau}{2 \eta \kappa}, \frac{1}{2 \kappa} \right) \times
\left( u \left( J_{j} - \frac{l}{\kappa}, \frac{\tau}{2 \eta \kappa}, -\frac{1}{2 \kappa} \right) - u \left( J_{j} - \frac{l}{\kappa}, \frac{\tau}{2 \eta \kappa}, \frac{1}{2 \kappa} \right) \right),
\]

\[
C^{+}(\tau, \eta) = -2\pi i \sqrt{\frac{2\kappa i}{\tau}} e^{\pi i \frac{2\eta}{\kappa} + \frac{\pi i}{6 \kappa} \psi(\tau, +2 \eta \kappa)},
\]

Here the square root is with positive real part.

Notice that in (9) and (10) the matrix \( S \) has the property \( S(\tau - 2 \eta \kappa, \eta) = S(\tau, \eta) \). Hence the right hand sides of formulas (8) - (10) are solutions of the qKZB equation with parameters \( \tau, \eta \). Therefore formulas (8) - (10) give transformations of theta function solutions of the qKZB equation.
Proof. Property (i) is a direct corollary of definitions. We prove (ii). The proof of (iii) is analogous.

The proof of (ii) is based on a discrete version of the following identity for the function $u_{\lambda, \mu, \tau, p, \eta}$. If $\text{Im} \left( \frac{\eta \tau}{p} \right) < 0$ and $\text{Im} \left( \frac{p}{\tau} \right) > 0$, then

$$\int u(\lambda, \mu, \tau, p, \eta) u \left( -\frac{\mu}{p}, \frac{\nu}{p}, -\frac{1}{p}, -\frac{\tau}{p}, \frac{\eta}{p} \right) Q(\mu, p, \eta) \rho(\mu, p, \eta) e^{-\frac{\pi i \rho}{4p} \mu^2} d\mu =$$

$$2\pi i \sqrt{\frac{4\eta \tau}{ip}} \rho(\lambda, \tau, \eta) \rho \left( \frac{\nu}{p}, -\frac{\tau}{p}, \frac{\eta}{p} \right) e^{-\frac{\pi i p}{4p} \left( \lambda^2 + \nu^2 / p^2 \right)} e^{-\frac{\pi i \eta \rho(\tau, p)}{4p} \left( \frac{\lambda}{\tau}, -\frac{\nu}{\tau}, \frac{1}{p}, \frac{\eta}{\tau} \right)}.$$  

Here $\rho(\lambda, \tau, \eta) = e^{-\frac{\pi i \rho}{4p} \left( \lambda^2 - 4\eta^2 \right)}$. The integration over $\mu$ is over the path $x \mapsto x \eta + \epsilon$, $x \in \mathbb{R}$, for any generic real $\epsilon$.

The discrete version has the following form. For any admissible $l$ and generic $\epsilon$ we have

$$\sum_{\mu \in 2\eta(\mathbb{Z} + \epsilon)} u(\lambda, \mu, \tau, -2\eta \kappa, \eta)Q(\mu, -2\eta \kappa, \eta) e^{\frac{\pi i}{2\kappa} \left( \frac{\mu}{2\kappa} \right)^2} Q \left( \frac{l}{\kappa}, \frac{\tau}{2\eta \kappa}, -\frac{1}{2\kappa} \right) \times$$

$$u \left( \frac{\mu}{2\eta \kappa}, -\frac{l}{\kappa}, \frac{1}{2\eta \kappa}, \frac{\tau}{2\eta \kappa}, -\frac{1}{2\kappa} \right) = C^- (\tau, \eta) e^{-\frac{\pi i}{2\kappa} (\kappa + 2) \lambda^2} e^{-\frac{\pi i}{2\kappa} \left( \frac{l}{\kappa} - \frac{1}{2\kappa} \right) \Delta_{l, \kappa} \left( \frac{\lambda}{\tau}, -\frac{\epsilon}{\kappa}, -\frac{1}{p}, \frac{\eta}{\tau} \right)},$$

where

$$\Delta_{l, \kappa} (\lambda, \epsilon, \tau, \eta) = e^{-\frac{2\pi i}{\kappa} l^2} Q(2\eta l, -2\eta \kappa, \eta) \times$$

$$\int_\gamma \Theta_{2\eta}(t, \tau, -2\eta \kappa) \frac{\theta(\lambda + t, \tau) \theta(2\eta l + t, -2\eta \kappa)}{\theta(t - 2\eta, \tau) \theta(t - 2\eta, -2\eta \kappa)} e^{-2\pi i t l / \kappa} \Theta_{l, \kappa} \left( \frac{2}{\tau} t - \lambda + \epsilon, \tau \right) dt.$$

The proof of this identity is the same as the proof in [FV3] of identity (11) but instead of evaluating the Gaussian integral one applies the Poisson summation formula.

Identity (12) implies

$$\sum_{\mu \in 2\eta(\mathbb{Z} + \epsilon)} u(\lambda, \mu, \tau, -2\eta \kappa, \eta)Q(\mu, -2\eta \kappa, \eta) e^{\frac{\pi i}{2\kappa} \left( \frac{\mu}{2\kappa} \right)^2} Q \left( \frac{l}{\kappa}, \frac{\tau}{2\eta \kappa}, -\frac{1}{2\kappa} \right) \times$$

$$\left[ u \left( \frac{\mu}{2\eta \kappa}, -\frac{l}{\kappa}, \frac{1}{2\eta \kappa}, \frac{\tau}{2\eta \kappa}, -\frac{1}{2\kappa} \right) - u \left( -\frac{\mu}{2\eta \kappa}, -\frac{l}{\kappa}, \frac{1}{2\eta \kappa}, \frac{\tau}{2\eta \kappa}, -\frac{1}{2\kappa} \right) \right] =$$

$$C^- (\tau, \eta) e^{-\frac{\pi i}{2\kappa} (\kappa + 2) \lambda^2} e^{-\frac{\pi i}{2\kappa} \Delta_{l, \kappa} \left( \frac{\lambda}{\tau}, -\frac{\epsilon}{\kappa}, -\frac{1}{p}, \frac{\eta}{\tau} \right)} - e^{-\frac{\pi i}{2\kappa} \Delta_{-l, \kappa} \left( \frac{\lambda}{\tau}, -\frac{\epsilon}{\kappa}, -\frac{1}{p}, \frac{\eta}{\tau} \right)}.$$
Let $\epsilon$ tend to zero and $\mu = 2\eta(m+\epsilon)$, $m \in \mathbb{Z}$. Then the limit of each factor of the corresponding term in the left hand side of (13) is well defined unless $m \equiv \pm 1 \mod \kappa$. Moreover, if $m \equiv 0 \mod \kappa$, then the limit is zero, since the function $u\left(\frac{m}{\kappa}, -\frac{l}{2\eta\kappa}, \frac{\tau}{2}, -\frac{1}{2\kappa}\right)$ is multiplied by $-1$ under the shift $m \mapsto m + \kappa$.

If $m \equiv \pm 1 \mod \kappa$ and $\epsilon$ tends to zero, then $Q(2\eta(m+\epsilon), -2\eta\kappa, \eta)$ tends to infinity, but the factor $\left[u\left(\frac{m+\epsilon}{\kappa}, -\frac{l}{2\eta\kappa}, \frac{\tau}{2}, -\frac{1}{2\kappa}\right) - u\left(-\frac{m+\epsilon}{\kappa}, -\frac{l}{2\eta\kappa}, \frac{\tau}{2}, -\frac{1}{2\kappa}\right)\right]$ tends to zero by Lemma 2.3. Hence the limit of the corresponding term in the left hand side of (13) is well defined in this case as well. Taking the limit $\epsilon \to 0$ we observe that for any $r \in \mathbb{Z}$ the terms corresponding to $m = r\kappa + 1$ and $m = r\kappa - 1$ are canceled by Lemma 2.3 applied to the factor $u(\lambda, 2\eta(m+\epsilon), \tau, -2\eta\kappa, \eta)$. Now taking the limit $\epsilon \to 0$ in (13) we get the formula of part (ii) of the theorem.

**Theorem 4.2.** For every admissible $l$ we have

$$
\Delta_{l,k}(\lambda + 1, \tau, \eta) = (-1)^{l+1}\Delta_{l,k}(\lambda, \tau, \eta),
$$

$$
e^{\pi i(k+2)(\lambda+\tau/2)} \Delta_{l,k}(\lambda + \tau, \tau, \eta) = \Delta_{l+k,k}(\lambda, \tau, \eta).
$$

This theorem gives additional transformations of the theta function solutions of the qKZB equation with parameters $\tau, \eta$.

### 4.2. Modular action.

The group $\text{SL}(2, \mathbb{Z})$ is generated by matrices

$$
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

satisfying the relations $S^2 = -Id$, $(ST)^3 = Id$. Let $\mathcal{H}_+$ be the upper half plane of complex numbers with positive imaginary part, and let $\mathcal{H}_-$ be the lower half plane. The group $\text{SL}(2, \mathbb{Z})$ acts on $\mathcal{H}_+ \times \mathbb{C}$ with coordinates $\tau, \eta$ by the formulas

$$
S: (\tau, \eta) \mapsto (-1/\tau, \eta/\tau), \quad T: (\tau, \eta) \mapsto (\tau + 1, \eta).
$$

An orbit of this action will be called admissible if it does not contain a point $(\tau, \eta)$ with $\eta \in \mathbb{R}$. Let $X$ be the intersection of $\mathcal{H}_+ \times \mathcal{H}_-$ with the union of all admissible orbits.

If $(\tau, \eta) \in \mathcal{H}_+ \times \mathbb{C}$, then the set $\{(\tau - 2\eta kl), \ l = 0, 1, 2, \ldots\}$ will be called the qKZB sequence generated at $(\tau, \eta)$. If $(\tau, \eta) \in X$, then $X$ contains the qKZB sequence generated at $(\tau, \eta)$.

Consider the space $F$ of all functions $u(\lambda, \tau, \eta)$, defined on $\mathbb{C} \times X$ such that for any fixed $(\tau, \eta) \in X$ we have $u \in E_\kappa(\tau, \eta)$.
Introduce transformations $A, B, T, S$ of functions of $F$ by formulas

$$(Au)(\lambda, \tau, \eta) = u(\lambda + 1, \tau, \eta),$$

$$(Bu)(\lambda, \tau, \eta) = e^{\pi i (\kappa + 2)(\lambda + \tau/2)} u(\lambda + \tau, \tau, \eta),$$

$$(Tu)(\lambda, \tau, \eta) = u(\lambda, \tau + 1, \eta),$$

$$(Su)(\lambda, \tau, \eta) = C^\pm (\tau, \eta) e^{-\pi i \kappa/2} \theta\left(\frac{\lambda}{\tau}, -\frac{1}{\tau}, \pm -\frac{\eta}{\tau}\right),$$

where in the definition of $S$ the sign $+$ is chosen if $\text{Im}\, \eta/\tau > 0$ and $-$ if $\text{Im}\, \eta/\tau < 0$.

**Lemma 4.3.** The transformations $A, B, T, S$ preserve the space $F$. Moreover, each of the transformations send solutions of the qKZB equation to solutions.

The lemma is a corollary of Theorems 4.1 and 4.2.

**Lemma 4.4** (cf. [FVS2]). Restricted to the space $F$, the transformations $A, B, T, S$ satisfy the relations

$$A^2 = 1, \quad B^2 = 1, \quad S^2 = 8\pi^2 \kappa, \quad (ST)^3 = 8\pi^3 \kappa^{3/2} e^{2\pi i \kappa - \pi i/4},$$

$$SA = BS, \quad AB = (-1)^k BA, \quad TB = -e^{\pi i \kappa/2} BAT, \quad AT = TA.$$

Lemmas 4.3, 4.4 in particular say that the transformations $S, T$ define a projective representation of $SL(2, \mathbb{Z})$ in the space of theta function solutions of the qKZB equation.

5. Limiting cases

5.1. **Conformal blocks.** It was shown in [FV2] that in the limit $\eta \to 0$, the qKZB heat equation degenerates to the KZB heat equation of [B]. More precisely, if $v(\lambda, \tau, \eta)$ is a family of solutions of [B] such that $v(\lambda, \tau, \eta) = v(\lambda, \tau) + O(\eta)$ then $v(\lambda, \tau)/\theta(\lambda, \tau)$ obeys the KZB heat equation with the three dimensional irreducible representation of $sl_2$ sitting at one marked point:

$$2\pi i \kappa \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \lambda^2} + 2\rho'(\lambda, \tau)v + c(\tau)v, \quad \rho = \theta'/\theta,$$

for some $c(\tau)$ independent of $\lambda$.

Conformal blocks are solutions of the KZB heat equation with $c(\tau) = 0$ taking values in theta functions of level $\kappa - 2$. The integral representation of conformal blocks has the following form [FV1]:

$$v_{l,\kappa}(\lambda, \tau) = \int_0^1 \frac{\theta(t, \tau)}{\theta'(0, \tau)} \left(\frac{\theta(0, \tau)}{\theta(t, \tau)}\right)^{-\frac{l}{\kappa}} \frac{\theta(t - \lambda, \tau)\theta'(0, \tau)}{\theta(t, \tau)\theta(\lambda, \tau)} \theta_{l,\kappa} \left(\frac{2}{\kappa} t + \lambda, \tau\right) dt - (\lambda \to -\lambda).$$

In the limit $\eta \to 0$ the integral representation for elliptic hypergeometric functions turns into the integral representation for conformal blocks. To make the statement precise we first discuss the integral representations.
The integral in (14) is understood in the sense of analytic continuation from the range of parameters where the exponent $-2/\kappa$ is replaced by a positive number. We give an explicit formula for the regularization of the integral

$$F(\lambda, \tau) = \int_0^1 \left( \frac{\theta(t, \tau)}{\theta'(0, \tau)} \right)^{-1-\frac{2}{\kappa}} f(t, \lambda, \tau) dt$$

where $f(t, \lambda, \tau) = \theta(t - \lambda, \tau) \theta_{t, \kappa} \left( \frac{2}{\kappa} t + \lambda, \tau \right)$.

Set $\tilde{f}(t, \tau) = e^{2\pi i l \kappa/\theta(t, \tau)}$. We have

$$f(t + 1, \lambda, \tau) = -e^{2\pi i l/\kappa} f(t, \lambda, \tau), \quad \tilde{f}(t + 1, \tau) = -e^{2\pi i l/\kappa} \tilde{f}(t, \tau),$$

$$(\tilde{f}(t, \tau) \theta(t, \tau)^{-1-2/\kappa})' = -\frac{2}{\kappa} e^{2\pi i l/\kappa} \theta(t, \tau)^{-1-2/\kappa} (\theta'(t, \tau) - \pi i \theta(t, \tau)) .$$

The regularization of the integral can be defined as

$$F(\lambda, \tau) =$$

$$\int_0^1 \left( \frac{\theta(t, \tau)}{\theta'(0, \tau)} \right)^{-1-\frac{2}{\kappa}} (f(t, \lambda, \tau) - f(0, \lambda, \tau) \frac{\theta(t, \tau)}{\theta'(0, \tau)}) e^{2\pi i l/\kappa} (\theta(t, \tau)' - \pi i \theta(t, \tau)) dt .$$

The added terms form a complete differential with respect to $t$ and the second factor is equal to zero at $t = 0, 1$, thus the integral is well defined.

The integral representation for hypergeometric theta functions has the following form,

$$\Delta_{l, \kappa}(\lambda, \tau, \eta) = e^{\frac{2\pi i l}{\kappa}} (I_{l, \kappa}(\lambda, \tau, \eta) - I_{l, \kappa}(-\lambda, \tau, \eta)) Q(2\eta l, -2\eta \kappa, \eta) ,$$

where

$$I_{l, \kappa}(\lambda, \tau, \eta) = \int_\gamma G(t, \tau, \eta) g(t, \lambda, \tau, \eta) dt ,$$

$$G(t, \tau, \eta) = \Omega_{2\eta}(t, \tau, -2\eta \kappa) \frac{\theta(2\eta l + t, -2\eta \kappa)}{\theta(t - 2\eta, \tau) \theta(t - 2\eta, -2\eta \kappa)} e^{-2\pi i l/\kappa},$$

$$g(t, \lambda, \tau, \eta) = \theta(\lambda + t, \tau) \theta_{l, \kappa} \left( \frac{2}{\kappa} t - \lambda, \tau \right) .$$

Set $\tilde{g}(t, \tau, \eta) = e^{2\pi i l/\kappa} \theta(t - 2\eta, \tau)$. We have

$$g(t + 1, \lambda, \tau, \eta) = e^{2\pi i l/\kappa} g(t, \lambda, \tau, \eta), \quad \tilde{g}(t + 1, \tau, \eta) = e^{2\pi i l/\kappa} \tilde{g}(t, \tau, \eta),$$

$$G(t - 2\eta \kappa, \tau, \eta) \tilde{g}(t - 2\eta \kappa, \tau, \eta) - G(t, \tau, \eta) \tilde{g}(t, \tau, \eta) =$$

$$G(t, \tau, \eta) e^{2\pi i l/\kappa} (e^{-4\pi i l \eta} \theta(t + 2\eta, \tau) - \theta(t - 2\eta, \tau)) .$$

We have

$$I_{l, \kappa}(\lambda, \tau, \eta) = \int_\gamma G(t, \tau, \eta) (g(t, \lambda, \tau, \eta) -$$

$$\frac{g(2\eta l, \lambda, \tau, \eta)}{e^{4\pi i l \eta (1-1/\kappa) \theta(4\eta)}} e^{2\pi i l/\kappa} (e^{-4\pi i l \eta} \theta(t + 2\eta, \tau) - \theta(t - 2\eta, \tau)) dt .$$
The integrals (17) and (18) are equal since the added terms form a discrete differential, see [FTV, FV3].

The second factor of the integrand in (18) is zero at \( t = 2 \eta \). The integrand is holomorphic at \( t = 2 \eta \) since \( t = 2 \eta \) is a simple pole of \( G(t, \tau, \eta) \). By Stokes’ theorem the integration contour \( \gamma \) in (18) can be replaced by the new contour \( \tilde{\gamma} : [0, 1] \to \mathbb{C}, s \mapsto s + 4 \eta \).

**Theorem 5.1.** If \( \Delta_{l, \kappa} \) are given by (16) in which formula (18) is used, then

\[
\lim_{\eta \to 0} \frac{2 \eta \Delta_{l, \kappa}(\lambda, \tau, \eta)}{\theta(\lambda, \tau)} = -\frac{(2\pi)^{-\frac{2}{\kappa}} e^{\pi i 4\kappa^2}}{2 \sin \left( \frac{\pi}{\kappa} (l+1) \right) \sin \left( \frac{\pi}{\kappa} (l-1) \right)} \prod_{j=1}^{\infty} (1 - e^{2\pi i j \tau})^{-3 - 4/\kappa} v_{l, \kappa}(\lambda, \tau),
\]

where \( v_{l, \kappa}(\lambda, \tau) \) is defined in (14) and the integral in (14) is regularized as in (15).

The proof follows from the formula in [FV4] for asymptotics of \( \Omega_{2\eta} \) as \( \eta \) tends to zero.

Under the limit \( \eta \to 0 \) the discussed in Theorem 4.1 the modular properties of hypergeometric theta functions \( \Delta_{l, \kappa} \) degenerate to the calculated in [FVS1] modular properties of conformal blocks \( v_{l, \kappa}(\lambda, \tau) \).

5.2. **Elliptic Macdonald polynomials.** The \( A_1 \)-Macdonald polynomials \( P_j^{(m)}(x) \in \mathbb{C}[x, x^{-1}] \) (or Askey–Wilson polynomials) are Laurent polynomials depending on two non-negative integers \( m \) and \( j \) and a parameter \( q \), see [M]. They are defined by the conditions:

(i) \( P_j^{(m)}(x) = x^j + \text{terms of lower degree} \),

(ii) \( P_j^{(m)}(x^{-1}) = P_j^{(m)}(x) \),

(iii) \( \langle P_j^{(m)}, P_k^{(m)} \rangle = 0 \) if \( j \neq k \), with respect to the inner product \( \langle P, Q \rangle = \text{constant term of} \)

\[
P(x)Q(x^{-1}) \prod_{j=0}^{m-1} (1 - q^{2j}x^2)(1 - q^{2j}x^{-2}) .
\]

The \( A_1 \)- (and more generally \( A_n \)-) Macdonald polynomials are traces of intertwining operators for the quantum group \( U_q(sl(2)) \) (resp. \( U_q(sl(n+1)) \)) [EK1], cf. [EV1, EV2]. The simplest non-trivial case is the case of \( m = 2 \). The Macdonald polynomials have the form

\[
P_j^{(2)}(x, q) = \frac{x^{j+3} - ax^{j+1} + ax^{-j-1} - x^{-j-3}}{\Pi(x, q)},
\]

\[
\Pi(x, q) = (qx - (qx)^{-1})(x - x^{-1})(q^{-1}x - qx^{-1}), \quad a = \frac{q^{j+3} - q^{-j-3}}{q^{j+1} - q^{-j-1}}.
\]
Let us define elliptic Macdonald polynomials for $m = 2$ by

$$P_{j,\kappa}(x, q, p) = e^{-\pi i \frac{2(j+2)^2}{\kappa} + \pi i 3 \tau / 4} \frac{\Delta_{j+2,\kappa}(\lambda, \tau, \eta)}{\theta(\lambda - 2\eta, \tau) \theta(\lambda, \tau) \theta(\lambda + 2\eta, \tau)},$$

$$x = e^{\pi i \lambda}, \quad q = e^{-2\pi i \eta}, \quad p = e^{2\pi i \tau}, \quad j = 0, \ldots, \kappa - 4.$$

The exponential function in the definition of $P_{j,\kappa}$ ensures that the result is 1-periodic in $\tau, \eta$ and can thus be written as a function of $p$ and $q$.

**Theorem 5.2.**

(i) The elliptic Macdonald polynomials form a basis of $\Theta_{\kappa-4}(\tau)$ even.

(ii) As $p \to 0$, we have $P_{j,\kappa}(x, q, p) = A_{j,\kappa}(q) P^{(2)}_{j}(x, q) + O(p)$, for some $A_{j,\kappa}(q) \neq 0$.

**Proof:** Part (i) follows from (20) and Theorem 3.1. Let us prove part (ii). As $\tau \to i\infty$ ($p \to 0$) the denominator in (20) is $i e^{3\pi i \tau / 4} (\Pi(x, q) + O(p))$. The numerator $\Delta_{j+2,\kappa}(\lambda, \tau, \eta)$ behaves as

$$e^{\pi i \frac{2(j+2)^2}{\kappa}} Q(2\eta(j + 2), -2\eta \kappa, \eta)(u(\lambda, 2\eta(j + 2), i\infty, -2\eta \kappa, \eta) - (\lambda \to -\lambda) + O(p)),$$

as $p \to 0$. Here,

$$u(\lambda, 2\eta(j + 2), i\infty, -2\eta \kappa, \eta) = e^{-\pi i \lambda(j+2)} \prod_{j=0}^{\infty} \frac{1 - q^{2j+2} e^{2\pi it}}{1 - q^{2j-2} e^{2\pi it}} \times \frac{\sin \pi (\lambda + t) \theta(2\eta(j + 2) + t, -2\eta \kappa)}{\sin \pi (t - 2\eta) \theta(t - 2\eta, -2\eta \kappa)} dt .$$

Considering the right hand side of this formula as a function of $x$, we see that $\lim_{\tau \to i\infty} P_{j,\kappa}$ has the form (20) for some coefficient $a$ up to a factor independent of $x$. The fact that the numerator is divisible by the denominator determines the value of $a$ uniquely. The fact that $A_{j,\kappa}(q) \neq 0$ follows from the limiting case of the inversion relation (4). □

Let us now study the qKZB heat equation in this limit. The integration kernel appearing in the equation involves the limit of $u$ as both $\tau$ and $\sigma$ tend to $i\infty$:

$$u(\lambda, \mu, i\infty, i\infty, \eta) = e^{-i \pi \lambda \mu / 2\eta} \int_{\gamma} \frac{\sin \pi (\lambda + t) \sin \pi (\mu + t)}{\sin \pi (t + 2\eta) \sin \pi (t - 2\eta)} dt = i e^{-i \pi \lambda \mu / 2\eta} \frac{(q^{-2} \cos \pi (\lambda + \mu) - \cos \pi (\lambda - \mu))}{\sin 4\pi \eta} .$$

Combining this with

$$Q(\mu, i\infty, \eta) = \frac{\pi \sin 4\pi \eta}{\sin \pi (\mu - 2\eta) \sin \pi (\mu + 2\eta)} ,$$
we obtain in the limit the Macdonald-Mehta type identity

\[ q^{-\frac{(j+2)^2}{2}} P_j(2)(e^{i\pi \lambda}, q) = \int_{\eta \mathbb{R}} V(\lambda, \mu) P_j(2)(e^{-i\pi \mu}, q) \, d\mu, \]

\[ V(\lambda, \mu) = q^{-2} e^{-\frac{i\pi(\lambda+\mu)^2}{4\eta}} \frac{(q^{-2} \cos \pi(\lambda+\mu) - \cos \pi(\lambda-\mu)) \sin \pi \mu}{\sin \pi(\lambda-2\eta) \sin \pi \lambda \sin \pi(\lambda+2\eta)}, \]

cf. [EV2].

The limit of the difference equation (7) has the following form. Define the infinite order difference operator

\[ T(q) = \]

\[ -q^{-2} \sum_{m \in \mathbb{Z}} q^{-2} \cos \pi(2\eta m) - \cos \pi(2\lambda-2\eta m) \sin \pi(\lambda-2\eta m) q^{-\frac{m^2}{2}} T_{-2\eta m} \]

where \( T_{-2\eta m} : v(\lambda) \mapsto v(\lambda-2\eta m) \) is the operator of the shift of the argument by \(-2\eta m\). Set

\[ \theta_0(x, q) = \sum_{m \in \mathbb{Z}} x^m q^{-\frac{m^2}{2}}. \]

Then

\[ T(q) P_j(2)(e^{i\pi \lambda}, q) = \theta_0(q^{j+2}, q) P_j(2)(e^{i\pi \lambda}, q). \]

5.3. Remarks on the operator \( T(q) \). Let \( x = e^{i\pi \lambda}, q = e^{-2\pi i \eta} \) as before. Following [M] introduce operators acting on Laurent polynomials:

\[ w : f(x) \mapsto f(qx^{-1}), \Gamma : f(x) \mapsto f(qx), \]

\[ Y = \frac{q^{-1} - qx}{1-x} \Gamma + \frac{q - q^{-1}}{1-x} w. \]

Then

\[ Y^{-1} = \Gamma^{-1} \frac{q - q^{-1} x}{1-x} + w x \frac{q - q^{-1}}{1-x}. \]

For any symmetric Laurent polynomial \( f(x), f(x) = f(x^{-1}) \), and any \( j = 0,1, \ldots \), one has

\[ f(Y) P_j(2)(x) = f(q^{j+2}) P_j(2)(x), \]

see [M]. The Macdonald polynomials form a basis in the space of symmetric Laurent polynomials. Comparing (23) and (24) we see that

\[ T(q) = \theta_0(Y, q) \]

as operators on the space of symmetric Laurent polynomials.
5.4. **Orthogonality relation.** In terms of elliptic Macdonald polynomials the inversion relation (5) says that

\[
\frac{1}{32\pi^2\eta} \int_0^2 u(-\mu, 2\eta l, \tau, -2\eta \kappa, -\eta) P_{j,\kappa}(e^{\pi i \mu}, q, p) \theta(\mu, \tau) d\mu = \delta_{l,j} e^{\pi i \frac{4\eta + \tau}{2\eta}} j^2
\]

for \(l, j = 2, \ldots, \kappa - 2\). Note that the integrand is a 2-periodic entire function of \(\mu\). Thus the integral is equal to the constant Fourier coefficient of the integrand multiplied by 2.

We want to present this integral as a pairing of suitable functional spaces. Namely, consider the vector space \(F\) of entire 2-periodic functions of \(\lambda\). Define the subspace \(H\) as the subspace generated by the functions of the form \(h(\lambda) - h(\lambda + 2\tau) e^{-2\pi i (\kappa - 2)(\tau + \mu)}\) and \(h(\lambda) + h(-\lambda)\) where \(h \in F\). Then the map

\[
f \otimes g \mapsto \int_0^2 f(\mu) g(\mu) \theta(\mu, \tau) d\mu
\]

defines a perfect pairing \((F/H) \otimes \Theta_{\kappa-4}(\tau)^{\text{even}} \to \mathbb{C}\). Formula (25) says that the functions \(u(-\mu, 2\eta l, \tau, -2\eta \kappa, -\eta), l = 2, \ldots, \kappa - 2\), considered as elements of \(F/H\), form a basis dual to the basis of elliptic Macdonald polynomials up to multiplication by scalars.

**References**

[B] D. Bernard, *On the Wess-Zumino-Witten model on the torus*, Nucl. Phys. B303 (1988), 77–93; *On the Wess-Zumino-Witten model on Riemann surfaces*, Nucl. Phys. B309 (1988), 145–174.

[EK1] P. Etingof and A. Kirillov, *Macdonald’s polynomials and representations of quantum groups*, Math. Res. Letters 1 (3) (1994), 279–296.

[EK2] P. Etingof and A. Kirillov, *On the affine analogue of Jack and Macdonald polynomials*, Duke Math. J. 78 (1995), 229–256.

[EV1] P. Etingof and A. Varchenko, *Traces of Intertwiners for Quantum Groups and Difference Equations, I*, Duke Math. Journal, 2000 (104), No. 3, 391–432.

[EV2] P. Etingof and A. Varchenko, *The orthogonality and qKZB-heatequation for traces of U_q(g)-intertwiners*, [math.QA/0302071](math.QA/0302071), 1–29.

[FTV] G. Felder, V. Tarasov, and A. Varchenko, *Monodromy of solutions of the elliptic quantum Knizhnik-Zamolodchikov-Bernard difference equations*, Int. J. Math. 10 (1999) 943–975.

[FV1] G. Felder and A. Varchenko, *Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equation*, Int. Math. Res. Notices, No. 5 (1995), 221–233.

[FV2] G. Felder and A. Varchenko, *The q-deformed Knizhnik-Zamolodchikov-Bernard equation*, Commun. Math. Phys. 221 (2001), 549–571.

[FV3] G. Felder and A. Varchenko, *q-deformed KZB heat equation: completeness, modular properties and SL(3, Z)*, Adv. Math. 171 (2002), no. 2, 228–275.

[FV4] G. Felder and A. Varchenko, *The elliptic gamma function and SL(3, Z) \times Z^3*, Adv. Math. 156 (2000), 44–76.

[FVS1] G. Felder, L. Stevens, and A. Varchenko, *Modular transformations of the elliptic hypergeometric functions, Macdonald polynomials, and the shift operator*, [math.QA/0203049](math.QA/0203049), 1–18.

[FVS2] G. Felder, L. Stevens, and A. Varchenko, *Elliptic Selberg Integrals and Conformal Blocks*, [math.QA/02100040](math.QA/02100040), 1–13.

[M] I. G. Macdonald, *Symmetric Functions and Orthogonal Polynomials*, AMS, 1998.