ALGEBRAIC $G$-THEORY IN MOTIVIC HOMOTOPY CATEGORIES

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ABSTRACT. We prove that algebraic $G$-theory is representable in unstable and stable motivic homotopy categories; in the stable category we identify it with the Borel-Moore theory associated to algebraic $K$-theory, and show that such an identification is compatible with the functorialities defined by Quillen and Thomason.

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1. INTRODUCTION

1.0.1. The study of Algebraic $G$-theory (or $K'$-theory) originated from Grothendieck’s definition of the group $G_0$ in [SGA6, IV 2.2], which associates to a noetherian scheme $X$ the Grothendieck group of the abelian category of coherent sheaves on $X$. It was Quillen who first defined higher $G$-groups for noetherian schemes in [Qui73, §7], in the same way as he defined higher $K$-groups. He proved some properties of $G$-theory which fail to hold for $K$-theory in general, including homotopy invariance and localization property. This definition was generalized to all schemes by Thomason ([TT90]) by combining Waldhausen’s construction ([Wal83]) and Grothendieck’s work in [SGA6], which for noetherian schemes agrees with Quillen’s definition. Thomason improved the functoriality of $G$-theory, and his definition is widely used together with his reformulation of algebraic $K$-theory.

1.0.2. Ever since its birth, $G$-theory has been conceived as a cohomological theory that accompanies $K$-theory, and the two theories share many properties such as Brown-Gersten property and projective bundle formula. The definition of the two are quite similar as well: for example in Quillen’s definition, $G$-theory is constructed in the same way as $K$-theory by replacing vector bundles by coherent sheaves; since any coherent sheaf over a regular noetherian scheme has a finite resolution by vector bundles ([SGA2, VIII 2.4]), it follows that Quillen $K$-theory and $G$-theory agree over regular schemes (which also turn out to be the same as Thomason $K$-theory). We recall some important properties of (Thomason) $G$-theory, due to Quillen, Nisnevich and Thomason:
Recall 1.0.3 (Properties of $G$-theory). For any scheme $X$, denote by $G(X)$ Quillen-Thomason’s $G$-theory spectrum ([TT90, 3.3]), whose homotopy groups $G_n(X)$ are $G$-theory groups. Then the following properties hold:

1. (contravariant functoriality) For any morphism of schemes $f : X \to X'$ of finite Tor-dimension, there is a map $f^* : G(X') \to G(X)$. ([TT90, 3.14.1])

2. (proper functoriality) For any proper morphism between noetherian schemes $f : X \to X'$ there is a map $f_* : G(X) \to G(X')$. ([TT90, 3.16.1])

3. (homotopy invariance) For any noetherian scheme $X$, the pull-back by the canonical projection induces a homotopy equivalence $G(X) \xrightarrow{p^*} G(X \times \mathbb{A}^1)$. ([Qui73, §7 4.1])

4. (Brown-Gersten property) For any Cartesian diagram of noetherian schemes

\[
\begin{array}{ccc}
V & \to & Y \\
\downarrow & & \downarrow p \\
U & \xrightarrow{j} & X
\end{array}
\]

where $p$ is étale, $j$ an open immersion such that $p$ induces an isomorphism over $X - U$ (i.e. the square is a distinguished Nisnevich square), the square

\[
\begin{array}{ccc}
G(X) & \xrightarrow{j^*} & G(U) \\
p^* \downarrow & & \downarrow \\
G(Y) & \to & G(V)
\end{array}
\]

is a homotopy pullback square. ([Nis87, 4.4])

5. (localization) Let $X$ be a noetherian scheme and $i : Z \to X$ a closed immersion with complementary open immersion $j : U \to X$. Then there is a homotopy fiber sequence

\[
(1.0.3.3)
\]

\[
G(Z) \xrightarrow{i^*} G(X) \xrightarrow{j_*} G(U).
\]

In other words, there is a long exact sequence of $G$-theory groups

\[
(1.0.3.4)
\]

\[
\cdots \to G_{n+1}(U) \to G_n(Z) \xrightarrow{i^*} G_n(X) \xrightarrow{j_*} G_n(U) \to \cdots
\]

([Qui73, §7 3.2])

6. (action by $K$-theory) For any noetherian scheme $X$ there is a map $K(X) \wedge G(X) \to G(X)$ which makes $G(X)$ a module over $K(X)$, which is compatible with the contravariant functoriality. ([TT90, 3.15.3])

7. (coincidence with $K$-theory for regular schemes) For any regular noetherian scheme $X$, there is a canonical homotopy equivalence $K(X) \simeq G(X)$. ([TT90, 3.21])

8. (projective bundle formula) Let $X$ be a noetherian scheme and $E$ be a vector bundle of rank $r$ over $X$. Denote by $p : \mathbb{P}(E) \to X$ the projection of the projective bundle associated to $E$ and $\alpha \in K_0(\mathbb{P}(E))$ the class of the line bundle $\mathcal{O}(1)$. Then the map

\[
\bigoplus_{i=0}^{r-1} G_n(X) \to G_n(\mathbb{P}(E))
\]

\[
(1.0.3.5)
\]

\[
(x_0, \ldots, x_{r-1}) \mapsto \sum_{i=0}^{r-1} \alpha^i \cdot p^*(x_i)
\]

is an isomorphism. ([Qui73, §7 4.3])

\[\footnote{The condition is to be understood in the sense of [Qui73, Section 7 2.5], where a morphism $f : X \to Y$ has finite Tor-dimension if $\mathcal{O}_X$ has finite Tor-dimension as a module over $f^{-1}(\mathcal{O}_Y)$, which is slightly stronger than the condition of being perfect, see [SGA6, III 4.1].}

\[\footnote{Throughout the article, we use the convention $\mathbb{P}(E) = \text{Proj}(\text{Sym}(E^\vee))$, the projectivization of the symmetric algebra of the dual of its sheaf of sections.} \]
1.0.4. On the other hand, in the framework of motivic homotopy theory developed by Morel and Voevodsky ([MV99]), it is known that Borel-Moore theories have similar functorialities: they are covariant with respect to proper morphisms and contravariant with respect to local complete intersection morphisms, and possess a localization sequence in the form of (1.0.3.4) (see [Dég18], [Jin16]). Since $G$-theory is a module over $K$-theory ring spectrum and agrees with the latter for regular schemes, the properties enlisted in Recall 1.0.3 suggest that $G$-theory should be considered as a Borel-Moore theory associated to $K$-theory seen as a cohomology theory. The main purpose of this paper is to prove such a result, using the six functors formalism established in [Ayo07] and [CD19].

1.0.5. The first problem concerning $G$-theory in motivic homotopy categories is representability problem, which is already well-studied for Thomason’s $K$-theory: Morel and Voevodsky showed in [MV99, Theorem 4.3.13] that over a regular noetherian base scheme $S$, $K$-theory is represented in the pointed $\mathbb{A}^1$-homotopy category $\mathbb{H}_{\ast}(S)$ by the group completion of the infinite Grassmannian space $\mathbb{Z} \times Gr$; the representability can be promoted to the stable homotopy category $\mathbb{SH}(S)$, where $K$-theory is represented by a $\mathbb{P}^1$-spectrum $KGL_S$ constructed from $\mathbb{Z} \times Gr$ in [Voe98] (see [Rio06], [PPR09]).

1.0.6. In Section 2 we follow the same steps to deal with $G$-theory over an arbitrary noetherian base scheme $S$: we first show the representability of $G$-theory in $\mathbb{H}_{\ast}(S)$ (Corollary 2.2.7). The model we construct uses axioms of a module over the $K$-theory spectrum of free vector bundles, which is used to construct a strict model for the proper functoriality. We then imitate Voevodsky’s construction to obtain an object $GGL_S$ in $\mathbb{SH}(S)$ (Definition 2.2.12), show that $GGL_S$ is indeed an $\Omega$-spectrum (Corollary 2.2.15), and therefore deduce the representability of $G$-theory in $\mathbb{SH}(S)$ by $GGL_S$:

**Theorem 1.0.7** (see Corollary 2.2.16). The spectrum $GGL_S$ represents the $G$ theory in the stable homotopy category $\mathbb{SH}(S)$. In other words, for any smooth $S$-scheme $X$, there is a natural isomorphism

$$Hom_{\mathbb{SH}(S)}(\Sigma^\infty X_+[n], GGL_S) \simeq G_n(X).$$

1.0.8. Then it comes to study the functorialities of $G$-theory as a Borel-Moore theory. Apart from the proper functoriality, there is a refined Gysin functoriality for abstract Borel-Moore theories, which has been constructed in [Dég18] in for oriented theories (see Recall 3.3.1) and recently generalized in [DJK18] to a more general setting. We would like to identify these functorialities with the ones of $G$-theory in Recall 1.0.3.

1.0.9. In Section 3, we study functorial behaviors of the spectrum $GGL_S$. In Section 3.1 we establish its contravariance with respect to the exceptional inverse image functor $f^!$. There are two simple cases: on the one hand, it follows from the proper functoriality of $G$-theory and our definition that for any proper morphism $p : W \to X$, there is a canonical map $\phi_p : p_*GGL_W \to GGL_X$; on the other hand, for any open immersion $j : U \to X$ there is a canonical identification $j^*GGL_X \simeq GGL_U$ by restriction to an open subset. We show that these two cases can be glued, and with some extra work studying the case of projective morphisms, we obtain:

**Theorem 1.0.10** (see Proposition 3.1.11). For any separated morphism of finite type $f : Y \to X$, there is a canonical isomorphism $GGL_Y \simeq f^!GGL_X$.

1.0.11. In Section 3.2, we show that for smooth morphisms, the map obtained above agrees with another isomorphism obtained from Bott periodicity and purity isomorphisms (see Proposition 3.2.13). The proof uses compatibility between Gysin morphisms to show the quasi-projective case, and deduce the general case by noetherian induction. In Section 3.3 we apply the results above to show the compatibility between functorialities: over a regular base scheme, we identify $G$-theory with the Borel-Moore theory associated to $K$-theory, and deduce the following result:
Theorem 1.0.12 (see Corollary 3.3.7). For $S$ a regular scheme, $f : X \to S$ a separated morphism of finite type and integers $m, n$, we have an isomorphism $KGL_{n,m}^B(X/S) \simeq G_{n-2m}(X)$, compatible with proper covariance and lci \footnote{We say that a morphism of schemes $f : X \to Y$ is local complete intersection (abbreviated as “lci”) if it factors as the composition of a regular closed immersion followed by a smooth morphism.} contravariance on both sides.

1.0.13. Note that the result above has been used in [Dégl18] to establish a Riemann-Roch theorem for singular schemes, which generalizes [Ful98, Theorem 18.3], see Remark 3.3.8 below.

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Notations and Conventions.

(1) Throughout the article we assume that all schemes are noetherian.

(2) A smooth morphism stands for a separated smooth morphism of finite type. We denote by $Sch$ the category of schemes, and for any scheme $X$, we denote by $Sch_X$ (resp. $Sm_X$) the category of separated $X$-schemes of finite type (resp. the category of smooth $X$-schemes).

(3) For any pair of adjoint functors $(F, G)$ between two categories, we denote by $\text{ad}((F, G)) : 1 \to GF$ and $\text{ad}'((F, G)) : GF \to 1$ the unit and counit maps of the adjunction.

2. REPRESENTABILITY OF ALGEBRAIC $G$-THEORY

In this section we prove the representability of algebraic $G$-theory in motivic homotopy categories. We construct a special model which will be used later to study the functorial properties.

2.1. Construction of the model.

2.1.1. In what follows we deal with the action by algebraic $K$-theory. Following [TT90], the action of the $K$-theory spectrum over $G$-theory stems from the tensor product of a cohomologically bounded pseudo-coherent complex by a perfect complex. We would like to rectify this action into a morphism of strict presheaves. To do this, we start with the rectification of the tensor product by a free $O_X$-module. This is done by strictifying the direct sum operation.

2.1.2. For any scheme $X$, we denote by $Coh^{fl}(X)$ the category of cohomologically bounded, pseudo-coherent, bounded below complexes of flasque quasi-coherent $O_X$-modules ([TT90, 3.11.5]). If $S$ is a scheme, any $S$-morphism between smooth $S$-schemes is lci ([Ful98, B.7.6]), therefore of finite Tor-dimension ([SGA6, VIII 1.7]).

Definition 2.1.3. Let $S$ be a scheme. For any scheme $X \in Sm_S$, we denote by $Sm_S/X$ the full subcategory of $Sm_S$ whose objects are also $X$-schemes, i.e. the overcategory of $X$-objects in $Sm_S$. We define $C_*(X)$ as the category where

- Objects are data of
  - for every $Y \in Sm_S/X$, a complex $E_Y$ in $Coh^{fl}(Y)$,
  - for every morphism $p : Y \to Z$ in $Sm_S/X$, a map of complexes $\alpha_p : E_Z \to p_*E_Y$,

such that

1. for every morphism $p : Y \to Z$ in $Sm_S/X$, the map $\tilde{\alpha}_p : Lp^*E_Z \to E_Y$ obtained from $\alpha_p$ by adjunction in the derived category is a quasi-isomorphism,
2. for every $Y \in Sm_S/X$, the map $\alpha_{Id_Y} : E_Y \to E_Y$ is the identity map.
(3) for composable morphisms

\[(2.1.3.1)\]
\[\begin{array}{c}
Y \xrightarrow{p} Z \xrightarrow{q} W
\end{array}\]
in \(Sm_S/X\), the composition

\[(2.1.3.2)\]
\[E_W \xrightarrow{\alpha_p} q_*E_Z \xrightarrow{q_*\alpha_p} q_*p_*E_Y \simeq (q \circ p)_*E_Y\]
agrees with the map \(\alpha_{qop}\).

- A morphism \(\eta\) from \(((E_Y)_{Y \in Sm_S/X}, (\alpha_p)_{p:Y \to Z \in Sm_S/X})\) to \(((F_Y)_{Y \in Sm_S/X}, (\beta_p)_{p:Y \to Z \in Sm_S/X})\)
  is the data of a map of complexes \(\eta_Y : E_Y \to F_Y\) for every \(Y \in Sm_S/X\), such that for every
  morphism \(p : Y \to Z\) in \(Sm_S/X\), the following diagram commutes:

\[(2.1.3.3)\]
\[
\begin{array}{ccc}
E_Z & \xrightarrow{\eta_Z} & F_Z \\
\downarrow{\alpha_p} & & \downarrow{\beta_p} \\
p_*E_Y & \xrightarrow{p_*\eta_Y} & p_*F_Y
\end{array}
\]

2.1.4. According to the definition in [TT90, 1.2.11], the category \(C_S(X)\) is a complicial biWaldhausen
category with respect to the abelian category of all diagrams of the form \(A_Z \to p_*A_Y\) where \(p : Y \to Z\)
is a morphism in \(Sm_S/X\) and \(A_Y\) (resp. \(A_Z\)) is a \(\mathcal{O}_Y\)-module (resp. \(\mathcal{O}_Z\)-module). Recall the following
result in [Cis10, Théorème 2.15]:

**Proposition 2.1.5.** Let \(F : C \to D\) be a right exact functor between Waldhausen categories
([TT90, 1.2.5]) such that

1. Every morphism \(u\) in \(C\) and \(D\) factors as \(u = pi\) where \(i\) is a cofibration and \(p\) is a weak
equivalence.
2. Every morphism in \(C\) whose image in \(Ho(C)\) is an isomorphism is indeed a weak equivalence,
and the same property holds for \(D\).
3. \(F\) induces an equivalence between the homotopy categories \(Ho(C) \simeq Ho(D)\).

Then the induced map on \(K\)-theory spaces \(K(F) : K(C) \to K(D)\) is a homotopy equivalence.

**Lemma 2.1.6.** For any scheme \(X\), the canonical map between Waldhausen categories

\[(2.1.6.1)\]
\[a_S(X) : C_S(X) \to Coh^{fl}(X)\]
induces a homotopy equivalence between the associated Waldhausen \(K\)-theory spaces ([TT90, 1.5.2]).
Consequently, by [TT90, 3.11], for any scheme \(X\) the \(K\)-theory space associated to \(C(X)\) is homotopy
equivalent to the algebraic \(G\)-theory space ([TT90, 3.3]).

**Proof.** It suffices to check that the conditions in Proposition 2.1.5 are satisfied. The first two axioms are
standard (see [Cis10, Exemple 2.5]). For the third axiom, we construct a functor \(b_S(X) : Coh^{fl}(X) \to C_S(X)\)
which is an inverse of \(a_S(X)\) up to quasi-isomorphism: for \(E \in Coh^{fl}(X)\) let \(P(E)\) be a
factorial resolution of \(E\) by a cohomologically bounded, pseudo-coherent, bounded above complex of
flat quasi-coherent \(\mathcal{O}_X\)-modules, and let \(b_S(X)\) send \(E\) to the following data:

- for every \(Y \in Sm_S/X\), the complex \(Go(g^*P(E)) \in Coh^{fl}(Y)\), where \(Go\) is the Godement
  resolution functor,
- for every morphism \(p : Y \to Z\) in \(Sm_S/X\), the canonical map

\[(2.1.6.2)\]
\[Go(g^*P(E)) \to p_*Go(p^*g^*P(E)) \simeq p_*Go((g \circ p)^*P(E)).\]

The functor \(b_S(X)\) is well-defined since any morphism \(f\) in \(Sm_S\) has finite \(Tor\)-dimension, and therefore
\(f^*\) preserves cohomological boundedness and pseudo-coherence. It is easy to see that \(b_S(X)\) is an inverse
to \(a_S(X)\) up to quasi-isomorphism, and the result follows.

**Lemma 2.1.7.**

1. The map \(X \mapsto C_S(X)\) defines a (strict) presheaf of sets over the category \(Sm_S\).
(2) Let \( f : T \to S \) be a proper morphism. For any \( S \)-scheme \( X \), denote by \( f_X : X_T = X \times_S T \to X \) the base change of \( f \). Then there is a covariant map \( f_{X*} : C_T(X_T) \to C_S(X) \), such that for any morphism \( p : Y \to X \) in \( Sm_S \), the following diagram commutes:

\[
\begin{array}{ccc}
C_T(Y_T) & \xrightarrow{p_{Y*}} & C_T(X_T) \\
\downarrow{f_{Y*}} & & \downarrow{f_{X*}} \\
C_S(Y) & \xleftarrow{p_*} & C_S(X)
\end{array}
\]

where \( p_T : X_T \to Y_T \) is the base change of \( p \).

**Proof.**

(1) Let \( q : W \to X \) be a morphism in \( Sm_S \). The functor \( Lq^* \) preserves cohomologically bounded pseudo-coherent complexes, and the composition with \( q \) identifies \( Sm_S/W \) as a full subcategory of \( Sm_S/X \). Then there is a well-defined map \( q^* : C_S(X) \to C_S(W) \) which sends the data \((E_Y)_{Y \in Sm_S/X, (\alpha_p)_{p:Y \to Z \in Sm_S/X}}\) to the subdata \((E_Y)_{Y \in Sm_S/W, (\alpha_p)_{p:Y \to Z \in Sm_S/W}}\).

Such a map defines a strict contravariant functoriality of the map \( X \mapsto C_S(X) \) on \( Sm_S \).

(2) We define the map \( f_{X*} : C_T(X_T) \to C_S(X) \) by sending the data \((E_Y)_{Y \in Sm_T/X_T, (\alpha_p)_{p:Y \to Z \in Sm_T/X_T}}\) to the following data:

- for every \( Y \in Sm_S/X \), the complex of \( \mathcal{O}_Y \)-modules \( f_{Y*}E_Y \),
- for every morphism \( p : Y \to Z \) in \( Sm_S/X \), the map \( f_{p} : f_{Y*}E_Y \to f_{Z*}E_Z \).

(2.1.7.2)

\[
f_{X*} : f_{X*}E_X \xrightarrow{f_{Z*}p_{X*}^{-1}} f_{Z*}p_{T*}^{-1}E_{Z_T} \cong p_{T*}^{-1}f_{Z*}E_{Z_T}.
\]

Since \( f_Y : Y_T \to Y \) is a proper morphism between noetherian schemes, the functor \( Rf_Y \) on the derived category of \( \mathcal{O}_Y \)-modules preserves flasqueness, pseudo-coherence and cohomological boundedness ([TT90, 3.16]). The map \( Lp^* f_{Z*}E_{Z_T} \to f_{Y*}E_{Y_T} \) agrees with the composition

(2.1.7.3)

\[
Lp^* f_{Z*}E_{Z_T} = Lp^* Rf_{Z*}E_{Z_T} \to Rf_{Y*}Lp_{T*}E_{Z_T} \xrightarrow{Rf_{Y*}p_{T*}} Rf_{Y*}E_{Y_T} = f_{Y*}E_{Y_T}.
\]

The map \( Lp^* Rf_{Z*}E_{Z_T} \to Rf_{Y*}Lp_{T*}E_{Z_T} \) is a quasi-isomorphism by the Tor-independent base change theorem ([TT90, 2.5.6]), and by assumptions the map (2.1.7.3) is a quasi-isomorphism. Therefore the map \( f_{X*} : C_T(X_T) \to C_S(X) \) is well-defined. The commutativity of the diagram (2.1.7.1) follows directly from the construction.

\[\square\]

2.18. By [ML78, XI.3. Theorem 1], every monoidal category \( C \) is equivalent to a strict monoidal category \( C^{\oplus} \) in a strongly monoidal way. For any scheme \( S \) and \( X \in Sm_S \) we denote by \( C_S(X)^\oplus \) the strict monoidal category corresponding to the monoidal category \( (C_S(X), \oplus) \). In other words, in the category \( C_S(X)^\oplus \) direct sums are strictly associative, i.e. for any triple \( (A, B, C) \) of objects we have an identification \( (A \oplus B) \oplus C = A \oplus (B \oplus C) \). This also makes \( C_S(X)^\oplus \) a Waldhausen category in a canonical way. By Lemma 2.1.6, we have the following:

**Lemma 2.1.9.** For any scheme \( X \), there is a canonical homotopy equivalence between the Waldhausen \( K \)-theory space of \( C_S(X)^\oplus \) and Thomason’s \( G \)-theory space \( G(X) \).

2.1.10. For any scheme \( X \), denote by \( Vect^f_X \) the category of free vector bundles (of finite rank) over \( X \) as a subcategory of \( Vect_X \). The category \( Vect^f_X \) is a Waldhausen category by taking isomorphisms as weak equivalences and monomorphisms as cofibrations, and has a monoidal structure by tensor products. We define an action of \( Vect^f_X \) on the category \( C_S(X)^\oplus \) by setting

(2.1.10.1)

\[
Vect^f_X \times C_S(X)^\oplus \to C_S(X)^\oplus \\
(V, E) \mapsto E \oplus \cdots \oplus E \oplus E
\]

The map (2.1.10.1) induces a map between \( K \)-theory spectra

(2.1.10.2)

\[
K(Vect^f_X) \wedge K(C_S(X)^\oplus) \to K(C_S(X)^\oplus).
\]
2.2. Representability in motivic homotopy categories.

2.2.1. In what follows, we fix a noetherian scheme $S_0$ and work over schemes in $Sch_{S_0}$. Denote by $F$ the functor in [RS010, Lemma 2.2] over the category $Sch_{S_0}$, which is a lax symmetric monoidal fibrant replacement functor on the category of pointed motivic spaces over $S$ for any $S \in Sch_{S_0}$.

**Definition 2.2.2.** Let $S$ be a scheme of finite type over $S_0$. Denote by $K_S^0$ (respectively $K_S^f$) the pointed presheaf $X \mapsto K(Vect_X)$ (respectively $X \mapsto K(Vect_X^f)$) over the category $Sm_S$, which has a monoidal structure induced by tensor product in $Vect_X$ (respectively $Vect_X^f$). Denote by $K_S = F(K_S^0)$, which has the structure of a monoid.

2.2.3. The canonical map $K_S^f \rightarrow K_S^0$ is a Nisnevich local weak equivalence in the category of pointed motivic spaces over $Sm/S$, since every vector bundle is Zariski locally free. It follows that the canonical map $K_S^f \rightarrow K_S$ is also a weak equivalence. Since this map is compatible with the structure of monoids of both objects, by [SS00, Theorem 4.3] we have the following:

**Lemma 2.2.4.** The derived functors of restriction and base change induce equivalences between homotopy categories

\[(2.2.4.1) \quad Ho(K_S^f - Mod) \simeq Ho(K_S - Mod).\]

2.2.5. Denote by $G_S^\oplus$ the presheaf $X \mapsto K(C_S(X)^\oplus)$ over the category $Sm_S$. The map (2.1.10.2) endows $G_S^\oplus$ with the structure of a module over $K_S^f$. By Lemma 2.2.4, there is a cofibrant replacement $\bar{G}_S^\oplus$ of $G_S^\oplus$ in $K_S^f - Mod$ which is weakly equivalent to the base change $G_S^\oplus \otimes_{K_S^f} K_S$. The object $G_S^\oplus \otimes_{K_S^f} K_S$ is a module over $K_S$, and we choose $G_S$ as a fibrant replacement of the object $\bar{G}_S^\oplus$ in the category $K_S - Mod$.

2.2.6. The object $G_S$ in the unstable motivic homotopy category $H(S)$ represents Thomason-Trobaugh $G$-theory:

**Lemma 2.2.7.** For any $X \in Sm_S$ there is a canonical isomorphism:

\[(2.2.7.1) \quad [S^n \wedge X_+, G_S]_{H(S)} \simeq G_n(X).\]

**Proof.** Since $G$-theory satisfies Nisnevich descent and homotopy invariance, by Lemma 2.1.9 the object $G_S^\oplus$ represents $G$-theory in $H(S)$, that is, for any $X \in Sm/S$ there is a functorial isomorphism:

\[(2.2.7.2) \quad [S^n \wedge X_+, G_S^\oplus]_{H(S)} \simeq G_n(X).\]

Then the lemma follows from the fact that the two objects $G_S^\oplus$ and $G$ are isomorphic in the homotopy category $H_s(S)$.

**Lemma 2.2.8.** For any proper morphism $f : W \rightarrow X$ in $Sch_{S_0}$, there is a canonical map $\phi_f : f_*G_W \rightarrow G_X$ in the unstable motivic homotopy category $H(S)$. If $g : V \rightarrow W$ is another proper morphism, then the composition

\[(2.2.8.1) \quad (f \circ g)_*G_V \simeq f_*g_*G_V \xrightarrow{f_*\phi_g} f_*G_W \xrightarrow{\phi_f} G_X\]

agrees with $\phi_{(f \circ g)}$.

**Proof.** Since the construction in 2.1.8 is functorial, for any proper morphism $f : W \rightarrow X$ and $Y \in Sm_X$, the map $f_* : C_W(Y_W) \rightarrow C_X(Y)$ in Lemma 2.1.7 induces a map $f_* : C_W(Y_W)^\oplus \rightarrow C_X(Y)^\oplus$ which is an exact functor between Waldhausen categories. For any morphism $p : Y \rightarrow Z$ in $Sm_X$ with a Cartesian diagram

\[(2.2.8.2) \quad \begin{array}{ccc}
Y_W & \rightarrow & Z_W \\
\downarrow f_Y & & \downarrow f_Z \\
Y & \rightarrow & Z
\end{array}
\]
the following diagram commutes by Lemma 2.1.7:

\[
\begin{align*}
& C_W(Y_W)^\oplus \xrightarrow{p_W} C_W(Z_W)^\oplus \\
& \downarrow f^* \quad \downarrow f^* \\
& C_X(Y)^\oplus \xrightarrow{p^*} C_X(Z)^\oplus.
\end{align*}
\]

Applying the $K$-theory space functor, we get a functorial map of presheaves of $K$-theory spaces $f_*G_W^\oplus \to G_X^\oplus$. By construction the two objects $G^\oplus$ and $G$ are isomorphic in the homotopy category $\mathbf{H}(S)$, and the result follows.

\[\square\]

2.2.9. By definition, $K_S^0$ is the presheaf of Quillen $K$-theory spectra, and since Quillen $K$-theory agrees with Thomason $K$-theory for affine schemes ([TT90, 3.9]), there is Nisnevich local weak equivalence between $K_S^0$ and the presheaf of Thomason $K$-theory. Therefore for any $X \in Sm_S$, there is a canonical map

\[K_n(X) \to [S^n \wedge X^+, K_S]_{H_s(S)}\]

which is an isomorphism if $S$ is regular (see also [Voe98, Theorem 6.5]). Now for any vector bundle $E$ of rank $r$ over $S$, let $x$ be the class $[O(1)]$ in $K_0(P(E \oplus O_X))$, and denote by $\nu(E)$ the class

\[\nu(E) = x^r - [\wedge^1 E]x^{r-1} + \cdots + (-1)^r[\wedge^r E] \in K_0(P(E \oplus O_X)).\]

By abuse of notation, we still denote by $\nu(E)$ its image via the map (2.2.9.1). By [MV99, Proposition 3.2.17] we have an isomorphism

\[Th(E) \simeq \mathbb{P}(E \oplus O_X)/\mathbb{P}(E).\]

Since the restriction of $\nu(E)$ to $\mathbb{P}(E)$ is zero, $\nu(E)$ induces a class in $[Th(E), K_S]_{H_s(S)}$. Via the canonical maps $K_S \wedge K_S \to K_S$ and $K_S \wedge G_S \to G_S$, the class of $\nu(E)$ induces maps in $H_*(S)$:

\[K_S \to \text{Hom}(Th(E), K_S),\]

\[G_S \to \text{Hom}(Th(E), G_S).\]

**Lemma 2.2.10.** The maps (2.2.9.4) and (2.2.9.5) are isomorphisms.

**Proof.** The case of $K_S$ is known by [Rio10, Lemma 6.1.3.3]. The case of $G_S$ is similar: for any $X \in Sm/S$, the map (2.2.9.5) induces a map

\[G_n(X) \xrightarrow{(2.2.7.1)} [S^n \wedge X^+, G_S]_{H_s(S)} \to [Th(E) \wedge S^n \wedge X^+, G_S]_{H_s(S)},\]

where by the isomorphism (2.2.9.3), the group $[Th(E) \wedge S^n \wedge X^+, G_S]_{H_s(S)}$ is identified with the $n$-th homotopy group of the homotopy fiber of the canonical map

\[G(\mathbb{P}(E \oplus O_S) \times_S X) \to G(\mathbb{P}(E) \times_S X).\]

By definition, the composition

\[G_n(X) \xrightarrow{(2.2.10.1)} [Th(E) \wedge S^n \wedge X^+, G_S]_{H_s(S)} \to G_n(\mathbb{P}(E \oplus O_S) \times_S X)\]

is given by $a \mapsto \nu \cdot p^*a$. Therefore the map (2.2.10.1) is an isomorphism by the projective bundle formula for $G$-theory (Recall 1.0.3 (8)) and the result follows. \[\square\]
2.2.11. In particular, if $E$ is a trivial bundle of rank $1$, we have an isomorphism $Th(\mathbb{A}^1) \simeq \mathbb{P}^1$ as pointed motivic spaces, and $\nu(\mathbb{A}^1_S) : \mathbb{P}^1 \to K_S$ is the image of the Bott class $[\mathcal{O}(1)] - 1 \in K_0(\mathbb{P}^1_S)$. Following [Rio06, Définition IV.1] we stabilize the $G$-theory sheaf as follows:

**Definition 2.2.12.** For any $S \in Sch_{S_0}$, we define $KGL_S$ (respectively $GGL_S$) as the $\mathbb{P}^1$-spectrum with $KGL_{S,n} = K_S$ (respectively $GGL_{S,n} = G_S$) for all $n$ and suspension maps

\[(2.2.12.1) \quad \mathbb{P}^1 \wedge K_S \xrightarrow{\nu(\mathbb{A}^1_S)^\wedge 1} K_S \wedge K_S \to K_S\]

\[(2.2.12.2) \quad \mathbb{P}^1 \wedge G_S \xrightarrow{\nu(\mathbb{A}^1_S)^\wedge 1} K_S \wedge G_S \to G_S.\]

2.2.13. Both spectra $KGL_S$ and $GGL_S$ are fibrant for the projective model structure since they are degreewise fibrant. We now determine the cohomology theories they represent in the stable homotopy category. The case of $KGL_S$ has already been well studied in the literature: by [Voe98, Theorem 3.6], the spectrum $KGL_S$ is isomorphic to the Voevodsky spectrum ([Voe98, 6.2]) in $SH(S)$, as explained in [PPR09, Remark 1.2.2]. By [Voe98, Theorem 6.9] we have the following:

**Corollary 2.2.14.** For any smooth $X \in Sm/S$, there is a functorial isomorphism

\[(2.2.14.1) \quad [\Sigma^\infty X_+[n], KGL_S]_{SH(S)} \simeq KH_n(X)\]

where the right hand side is the homotopy $K$-theory group ([Wei13, Definition IV.12.7])

We now go back to the spectrum $GGL_S$. As a consequence of Lemma 2.2.10 we have the following:

**Corollary 2.2.15.** The spectrum $GGL_S$ is an $\Omega$-spectrum in the stable homotopy category $SH(S)$. Consequently, for any pointed motivic space $A$ over $S$ the canonical map

\[(2.2.15.1) \quad [A, G_S]_{H_*(S)} \to [\Sigma^\infty A, GGL_S]_{SH(S)}\]

is an isomorphism.

From Lemma 2.2.7 and Corollary 2.2.15 we deduce that $GGL$ represents algebraic $G$-theory in $SH$:

**Corollary 2.2.16.** The spectrum $GGL_S$ represents the $G$-theory in the stable homotopy category $SH(S)$: for any $X \in Sm/S$, there is a canonical isomorphism

\[(2.2.16.1) \quad [\Sigma^\infty X_+[n], GGL_S]_{SH(S)} \simeq G_n(X).\]

Since Thom spaces are invertible for the $\wedge$-product in $SH$, by Lemma 2.2.10 we have the following:

**Corollary 2.2.17** (Bott periodicity). For any vector bundle $E$ over $S$, there is a canonical isomorphism in $SH(S)$:

\[(2.2.17.1) \quad GGL_S \wedge Th(E) \simeq GGL_S.\]

3. Functorial properties of the $G$-theory spectrum

In this section we study functorial properties of the $G$-theory spectrum, especially its contravariance with respect to the exceptional inverse image functor $f^!$. Using such a result we identify algebraic $G$-theory with the Borel-Moore theory associated to algebraic $K$-theory.

3.1. Contravariant functoriality.

**Lemma 3.1.1.** For any smooth morphism $f : Y \to X$, there is a canonical isomorphism

\[(3.1.1.1) \quad \chi_f : GGL_Y \simeq f^*GGL_X.\]
Proof. For any $W \in Sm/Y$, by Corollary 2.2.16 we have a canonical isomorphism
\[(3.1.1.2) \quad [\Sigma^\infty W_+ [n], GGL_Y]_{Sm(Y)} \simeq G_n(W).\]
Since $f$ is smooth, the functor $f^*$ has a left adjoint $f_!$ such that $f_! \Sigma^\infty W_+ = \Sigma^\infty W_+$ ([MV99, Proposition 3.2.9]), and therefore we have
\[(3.1.1.3) \quad [\Sigma^\infty W_+ [n], f^* GGL_X]_{Sm(Y)} = [f_! \Sigma^\infty W_+ [n], GGL_X]_{Sm(X)} \simeq [\Sigma^\infty W_+ [n], GGL_X]_{Sm(X)} = G_n(W),\]
and the result follows. \hfill \Box

3.1.2. The following lemma is straightforward from Lemma 2.2.8 by stabilization:

Lemma 3.1.3. For any proper morphism $f : W \to X$, there is a map $\phi_f : f_* GGL_W \to GGL_X$. If $g : V \to W$ is another proper morphism, then the composition
\[(3.1.3.1) \quad (f \circ g)_* GGL_V \simeq f_* g_* GGL_V \xrightarrow{f_* \phi_g} f_* GGL_W \xrightarrow{\phi_f} GGL_X\]
agrees with $\phi_{f\circ g}$.

3.1.4. The map $\phi_f$ can be understood as follows: for any $V \in Sm/X$, the map on $G$-groups
\[(3.1.4.1) \quad G_n(W \times_X V) \simeq [\Sigma^\infty (W \times_X V)_+ [n], GGL_W]_{Sm(W)} = [f_* \Sigma^\infty V_+ [n], GGL_W]_{Sm(W)} \simeq [\Sigma^\infty V_+ [n], f_* GGL_W]_{Sm(X)} \xrightarrow{\phi_f} [\Sigma^\infty V_+ [n], GGL_X]_{Sm(X)} \simeq G_n(V)\]
induced by $\phi_f$ agrees with the proper functoriality of $G$-theory (Recall 1.0.3 (2)). The following lemma follows from the construction:

Lemma 3.1.5. For any Cartesian square of schemes
\[(3.1.5.1) \quad \begin{array}{ccc} V & \xrightarrow{q} & Y \\ q g \downarrow & & \downarrow f \\ W & \xrightarrow{p} & X \end{array}\]
where $f$ and $g$ are smooth and $p$ and $q$ are proper, the following diagram commutes:
\[(3.1.5.2) \quad \begin{array}{ccc} f^* p_* GGL_W & \xrightarrow{\phi_p} & f^* GGL_X \\ \downarrow i \phi_f & & \downarrow i f \\ q_* GGL_V & \xrightarrow{\chi_q} & q_* GGL_V \\ \end{array}\]

3.1.6. We use the following notation: given two composable morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ and two maps $a : GGL_Y \to f^! GGL_X$, $b : GGL_Z \to g^! GGL_Y$, we denote by $a \cdot b$ the composition
\[(3.1.6.1) \quad GGL_Z \xrightarrow{b} g^! GGL_Y \xrightarrow{g^! a} g^! f^! GGL_X \simeq (f \circ g)^! GGL_X.\]

3.1.7. If $j : U \to X$ is an open immersion, by Lemma 3.1.1 there is a canonical isomorphism
\[(3.1.7.1) \quad \chi_j : GGL_U \simeq j^* GGL_X = j^! GGL_X.\]
On the other hand, by Lemma 3.1.3 and adjunction, for any proper morphism $f : W \to X$, we have a map
\[(3.1.7.2) \quad \psi_f : GGL_W \to f^! GGL_X.\]
It is then clear that the formation of the maps (3.1.7.1) and (3.1.7.2) are compatible with compositions of open immersions and proper morphisms respectively. These two types of maps are compatible by the following lemma:
Lemma 3.1.8. For any commutative square of schemes

\[
\begin{array}{ccc}
V & \xrightarrow{q} & U \\
\downarrow{k} & & \downarrow{j} \\
Y & \xrightarrow{p} & X
\end{array}
\]

where \( j \) and \( k \) are open immersions and \( p \) and \( q \) are proper, we have \( \chi_j \cdot \psi_q = \psi_p \cdot \chi_k \).

Proof. If the diagram is Cartesian, the result follows from Lemma 3.1.5. In the general case the canonical morphism \( V \to Y \times_X U \) is an open and closed immersion, and the result follows from the Cartesian case.

3.1.9. Since every separated morphism of finite type \( f \) has a compactification by our assumptions ([Con07]), namely a factorization \( f = p \circ j \) where \( p \) is proper and \( j \) is an open immersion, using the technique in [SGA4, XVII], we have the following functoriality by gluing maps (3.1.7.1) and (3.1.7.2):

Proposition 3.1.10. There is a unique family of maps \( \psi_f : GGL_Y \to f^!GGL_X \) associated to all separated morphisms of finite type \( f : Y \to X \) such that

1. If \( f \) is an open immersion, \( \psi_f = \chi_f \) is the map (3.1.7.1);
2. If \( f \) is proper, \( \psi_f \) is the map (3.1.7.2);
3. For any two composable morphisms \( f \circ g = h \), we have \( \psi_f \cdot \psi_g = \psi_h \).

Proof. We know that for every morphism the category of its compactifications is non empty and left filtering. For a morphism \( f \) with \( f = p \circ j \) a compactification, we set \( \psi_f = \psi_p \cdot \xi_j \). By Lemma 3.1.8, the map \( \psi_f \) is well-defined and independent on the choice of compactification. The properties (1) and (2) and the uniqueness are satisfied by definition. The property (3) follows by applying Lemma 3.1.8 again.

Proposition 3.1.11. For any separated morphism of finite type \( f \), the map \( \psi_f \) is an isomorphism.

The statement is local, and by localizing \( f \), we only need to show the case where \( f \) is quasi-projective. Therefore by Proposition 3.1.10 we only need to deal with three cases: open immersions, closed immersions, and the projection of a projective space. The open immersion case is Lemma 3.1.1, and the two remaining cases will follow from Proposition 3.1.12 and Proposition 3.1.14 below.

Proposition 3.1.12. Let \( X \) be a scheme and \( i : Z \to X \) a closed immersion with complementary open immersion \( j : U \to X \). Then

1. The map \( \phi_i : i_*GGL_Z \to GGL_X \) identifies \( i_*GGL_Z \) canonically with homotopy fiber of the canonical map \( GGL_X \to j_*GGL_U \);
2. The map \( \psi_i : GGL_Z \to i^!GGL_X \) is an isomorphism.

Proof. (1) For any smooth \( X \)-scheme \( X' \), denote \( Z' = Z \times_X X' \) and \( U' = U \times_X X' \). By Corollary 2.2.16 we have

\[
(3.1.12.1) \quad \Sigma^\infty X_+'[n], i_*GGL_Z|_{SH(X)} = [i^*\Sigma^\infty X_+'[n], GGL_Z]|_{SH(Z)} \simeq G_n(Z'),
\]

\[
(3.1.12.2) \quad \Sigma^\infty X_+'[n], j_*GGL_U|_{SH(X)} = [j^*\Sigma^\infty X_+'[n], GGL_U]|_{SH(U)} \simeq G_n(U').
\]

Then the result follows from the localization sequence (Recall 1.0.3 (5)).

(2) Using the localization distinguished triangle

\[
(3.1.12.3) \quad i_! i^! \to 1 \to j_* j^* \to i_! i^! [1]
\]

([Ayo07, Proposition 1.4.9]), by Lemma 3.1.1 we know that the map \( i_*GGL_Z \to i_! i^! GGL_X \) induced by \( \psi_i \) is an isomorphism. The result then follows from the fact that the functor \( i_* = i_! \) is conservative ([Ayo07, 1.4.11]).
Definition 3.1.13. For any smooth morphism \( f : Y \to X \), by Lemma 3.1.1, Corollary 2.2.17 and purity for \( f \), we denote by \( \xi_f : GGL_Y \simeq f^!GGL_X \) the following composition of isomorphisms

\[
\xi_f : GGL_Y \simeq GGL_Y \wedge Th(T_f) \xrightarrow{\chi_f} f^*GGL_X \wedge Th(T_f) \simeq f^!GGL_X .
\]

Proposition 3.1.14. Let \( p : \mathbb{P}^n_X \to X \) be the projection of a projective space. Then \( \psi_p : GGL_{\mathbb{P}^n_X} \to p^!GGL_X \) is an isomorphism.

Proof. It suffices to show that \( \xi_p = \psi_p \) where \( \xi_p : GGL_{\mathbb{P}^n_X} \simeq p^!GGL_X \) is as defined in (3.1.13.1). By considering the diagram

we see that it suffices to show that the composition

\[
p_*GGL_{\mathbb{P}^n_X} \simeq p_*(GGL_{\mathbb{P}^n_X} \wedge Th(T_p)) \simeq p#GGL_{\mathbb{P}^n_X}
\]

agrees with \( \phi_p \). Again by considering the diagram

we see that it suffices to show that the composition

\[
\text{GGL}_{\mathbb{P}^n_X} \xrightarrow{\phi_p} p#GGL_{\mathbb{P}^n_X} \simeq p#p^*GGL_{\mathbb{P}^n_X} \simeq p^*p_#GGL_{\mathbb{P}^n_X} \wedge Th(T_p)
\]

agrees with the map \( \chi_p \). For any scheme \( Y \) smooth over \( \mathbb{P}^n_X \), we have canonical isomorphisms

\[
[\Sigma^\infty_{\mathbb{P}^n_X} Y_+[n], GGL_{\mathbb{P}^n_X}|_{Sh(\mathbb{P}^n_X)}] \simeq G_n(Y) \simeq [\Sigma^\infty_{p^!_X} Y_+[n], p^*GGL_{\mathbb{P}^n_X}]|_{Sh(p^!_X)} ,
\]

(3.1.13.3)

\[
[\Sigma^\infty_{p^!_X} Y_+[n], p^*p^*_X GGL_{\mathbb{P}^n_X}]|_{Sh(p^!_X)} = [p^#_X \Sigma^\infty_{Y_+[n], Y}, p^*_X GGL_{\mathbb{P}^n_X}]|_{Sh(p^!_X)}
\]

(3.1.14.4)

Therefore applying the functor \([\Sigma^\infty_{\mathbb{P}^n_X} Y_+[n], \cdot]|_{Sh(\mathbb{P}^n_X)}\), the map (3.1.14.2) becomes

\[
G_n(Y) \xrightarrow{\psi_p} G_n(p^!_X Y) \xrightarrow{p^*} G_n(Y)
\]

(3.1.14.5)
where \( p_Y : \mathbb{P}^1_Y \to Y \) is the canonical projection. By projection formula for \( G \)-theory ([Qui73, §7 2.10]) and the description of the proper push-forward of a projective space on \( K_0 ([SGA6, VI 5.2]) \), for any \( a \in G_n(Y) \) we have

\[
3.1.14.6 \quad p_Y \cdot p_Y^* a = p_Y \cdot 1 \cdot a = a
\]

where \( 1 \in K_0(\mathbb{P}^1) \) is the unit element. Therefore the map \( 3.1.14.5 \) is indeed the identity map, and by Lemma 3.1.1, the map \( 3.1.14.2 \) agrees with the map \( \chi_p \), which finishes the proof. \( \square \)

This finishes the proof of Proposition 3.1.11, which can also be stated in the form of Theorem 1.0.10.

3.2. On the smooth functoriality. In this section we show that the smooth functoriality in Definition 3.1.13 and the one in Proposition 3.1.10 are compatible.

3.2.1. Recall that by Definition 3.1.13, for any smooth morphism \( f : X \to S \), there is a canonical isomorphism \( \xi_f : \text{GGL}_Y \simeq \psi^f \text{GGL}_X \). In this section we check its compatibility with the map \( \psi_f \). We use the notations in 3.1.6 for compositions of maps between \( \text{GGL} \).

**Lemma 3.2.2.** The map \( p \mapsto \xi_p \) is compatible with composition of smooth morphisms.

**Proof.** The statement follows from the fact that purity isomorphisms, Bott periodicity maps and the map \( \chi_p \) are compatible with compositions. \( \square \)

**Lemma 3.2.3.** For any Cartesian square

\[
3.2.3.1 \quad \begin{array}{ccc}
Y & \xrightarrow{k} & X \\
\downarrow{g} & \Delta & \downarrow{f} \\
T & \xrightarrow{i} & S
\end{array}
\]

where \( i \) and \( k \) are closed immersions and \( f \) and \( g \) are smooth, then \( \xi_f \cdot \psi_k = \psi_i \cdot \xi_g \).

**Proof.** The result follows from Lemma 3.1.8 and the fact that purity isomorphisms and Bott periodicity isomorphisms are compatible with base change. \( \square \)

**Lemma 3.2.4.** If \( f : X \to S \) be a smooth morphism with \( i : S \to X \) a section, then we have \( \xi_f \cdot \psi_i = 1 \). In other words the composition

\[
3.2.4.1 \quad \text{GGL}_S \xrightarrow{i} \text{GGL}_X \xrightarrow{\xi_f} \psi^f \text{GGL}_S \simeq \text{GGL}_S
\]

is the identity map.

**Proof.** By considering the diagram

\[
3.2.4.2 \quad \begin{array}{ccc}
\text{GGL}_S & \xrightarrow{i} & \psi^f \text{GGL}_S \\
\downarrow{\phi_i} & \downarrow{\xi_f} & \downarrow{\xi_f} \\
\text{GGL}_X & \xrightarrow{\phi} & \psi^f \text{GGL}_X
\end{array}
\]

we are reduced to show that the commutativity of the following diagram:

\[
3.2.4.3 \quad \begin{array}{ccc}
i_s^! f^! \text{GGL}_S & \xrightarrow{ad'(i_s^! \cdot i)} & j^* \text{GGL}_S \\
\downarrow{\xi_f} & & \downarrow{\xi_f} \\
i_s \text{GGL}_S & \xrightarrow{\phi_i} & \text{GGL}_X
\end{array}
\]

Denote by \( j : U \to X \) the open complement to \( i : S \to X \). By Proposition 3.1.12, the map \( \phi_i \) identifies \( i_s \text{GGL}_S \) with the homotopy fiber of the canonical map \( \text{GGL}_X \to j_* \text{GGL}_U \). By localization triangle (3.1.12.3), the upper horizontal map in the diagram (3.2.4.3) identifies \( i_s^! f^! \text{GGL}_S \) with the homotopy fiber of the canonical map \( f^! \text{GGL}_S \xrightarrow{ad'(j^! \cdot j^* \phi_i)} j^* j^! \text{GGL}_S \simeq j_* \text{GGL}_U \). Therefore
the left vertical map in the diagram (3.2.4.3) is a canonical isomorphism between the two homotopy fibers $i_*i^! \mathbb{G}_{GL} \simeq i_* \mathbb{G}_{GL}$, and the result follows.

**Corollary 3.2.5.** Let $p : Y \to X$ be a smooth morphism and $i : Z \to Y$ be a closed immersion such that the composition $q : Z \to X$ is a closed immersion. Then $\xi_p \cdot \psi_i = \psi_q$.

**Proof.** We have the following commutative diagram

\[
\begin{array}{ccc}
Z \xrightarrow{i'} Z \times_X Y & \xrightarrow{q'} & Y \\
\downarrow{p'} & & \downarrow{p} \\
Z & \xrightarrow{q} & X
\end{array}
\]

where $q' \circ i' = i$ and the square is Cartesian. By Proposition 3.1.10, Lemma 3.2.3 and Lemma 3.2.4 we have

\[
(3.2.5.1) \quad \psi_q = \psi_q' \cdot \xi_{q'} \cdot \psi_{i'} = \xi_p \cdot \psi_q' \cdot \psi_{i'} = \xi_p \cdot \psi_i
\]

and the result follows.

**Corollary 3.2.6.** Let $p : Y \to X$ be a smooth morphism and $i : Z \to Y$ be a closed immersion such that the composition $q : Z \to X$ is smooth. Then $\xi_p \cdot \psi_i = \xi_q$.

**Proof.** We have the following commutative diagram

\[
\begin{array}{ccc}
Z \xrightarrow{i'} Z \times_X Y & \xrightarrow{q'} & Y \\
\downarrow{p'} & & \downarrow{p} \\
Z & \xrightarrow{q} & X
\end{array}
\]

where $q' \circ i' = i$ and the square is Cartesian. By Lemma 3.2.2, Lemma 3.2.4 and Corollary 3.2.5 we have

\[
(3.2.6.1) \quad \xi_q = \xi_q' \cdot \psi_{i'} = \xi_p \cdot \xi_q' \cdot \psi_{i'} = \xi_p \cdot \psi_i
\]

and the result follows.

**Proposition 3.2.7.** For any quasi-projective smooth morphism $p$, the map $\xi_p$ agrees with $\psi_p$.

**Proof.** Since every quasi-projective morphism factors into a closed immersion in an open subscheme of a projective space, by Proposition 3.1.10, Lemma 3.2.2 and Corollary 3.2.6, it suffices to check the case where $p$ is an open immersion or the projection of a projective space. The first case follows from the definition, and the second case is already proved in Proposition 3.1.14, which proves the result.

3.2.8. Recall that a morphism of schemes is **smoothable** if it factors as a closed immersion followed by a smooth morphism. The following proposition says that we can generalize the map $\xi_p$ to smoothable morphisms by gluing the case of closed immersions and smooth morphisms:

**Proposition 3.2.9.** There is a unique family of isomorphisms $\xi_f : \mathbb{G}_{GL} \xrightarrow{\sim} f^! \mathbb{G}_{GL}$ associated to all smoothable morphisms $f : Y \to X$ such that if $f$ factors as a closed immersion followed by a smooth morphism $Y \xrightarrow{i} Z \xrightarrow{g} X$, then $\xi_f = \xi_g \cdot \psi_i$. The map $\xi_f$ is well-defined and does not depend on the choice of factorization.

**Proof.** If $f : Y \to X$ factors as $Y \xrightarrow{i_1} Z \xrightarrow{g_1} X$ as stated, we set $\xi_f = \xi_g \cdot \psi_{i_1}$. If $Y \xrightarrow{i'} Z' \xrightarrow{g'} X$ is another such factorization, we have the following commutative diagram

\[
\begin{array}{ccc}
Y \xrightarrow{i_1} Z \times_X Z' & \xrightarrow{g_1} & Z' \\
\downarrow{g_1} & & \downarrow{g'} \\
Z & \xrightarrow{g} & X
\end{array}
\]

with $i = g_1 \circ i_1$ and $i' = g_1 \circ i_1$. By Lemma 3.2.2 and Corollary 3.2.5 we have

\[
(3.2.9.1) \quad \xi_g \cdot \psi_{i_1} = \xi_g \cdot \xi_{g_1'} \cdot \psi_{i_1} = \xi_{g'} \cdot \xi_{g_1} \cdot \psi_{i_1} = \xi_{g'} \cdot \psi_{i'}
\]
and therefore the map is well-defined and independent of the factorization. The uniqueness follows from definition.

3.2.10. Since every quasi-projective morphism factors as a closed immersion followed by a quasi-projective smooth morphism, the following result follows from Proposition 3.1.10, Proposition 3.2.7 and Proposition 3.2.9:

**Corollary 3.2.11.** For any quasi-projective morphism \( p \), the map \( \xi_p \) agrees with \( \psi_p \).

3.2.12. We are now ready to prove the general case:

**Proposition 3.2.13.** For any smoothable morphism \( p \), the map \( \xi_p \) agrees with \( \psi_p \). In particular, \( \xi_p = \psi_p \) for every smooth morphism \( p \).

**Proof.** If \( p : X \to S \) is a smoothable morphism, there is a non-empty open subscheme \( U \) of \( X \) which is quasi-projective over \( S \), and the reduced complement \( Z = X \setminus U \) is smoothable over \( S \). Denoting by \( j : U \to X \) and \( i : Z \to X \) the two immersions, since the pair of functors \((i^!, j^!)\) is conservative, the result holds for the morphism \( X \to S \) if and only if it holds for both morphisms \( U \to S \) and \( Z \to S \). We conclude by Corollary 3.2.11 and noetherian induction. \( \square \)

**Remark 3.2.14.** The ring spectrum \( KGL \) is indeed orientable ([Dég18, Example 1.1.2 (4)]), and for-tiori so is \( GGL \) as a module over \( KGL \). But this fact has never been used throughout Chapter 3, and in particular we know that the functoriality of \( GGL \) from Proposition 3.1.11 in fact does not depend on the choice of an orientation.

3.3. **\( G \)-theory as a Borel-Moore theory.** In this section we identify algebraic \( G \)-theory as the Borel-Moore theory associated to algebraic \( K \)-theory, in a way compatible with different functorialities. We fix \( S \in Sch_{S_0} \), a regular scheme.

**Recall 3.3.1.** If \( E \) is an oriented absolute ring spectrum ([Dég18, Definition 2.1.1]), we can associate a Borel-Moore theory as follows: for any separated morphism of finite type \( p : X \to S \) and \( n, m \in \mathbb{Z} \), we define

\[
E_{n,m}^{BM}(X/S) := [p^! p^! S_X(m)[n], E_S]_{SH(S)} = [S_Y(m)[n], p^! E_S]_{SH(X)}.
\]

It has the following natural functorialities:

- For every proper morphism \( p : Y \to X \), we have \( p_* : E_{n,m}^{BM}(Y/S) \to E_{n,m}^{BM}(X/S) \) given by

\[
E_{n,m}(Y/S) = [S_Y(m)[n], p_Y^! E_S]_{SH(Y)} \simeq [p_* S_X(m)[n], p_Y^! E_S]_{SH(X)}
\]

(3.3.1.2)

where \( p_X : X \to S, p_Y : Y \to S \) are structure morphisms.

- For every smooth morphism \( f : Y \to X \) of relative dimension \( d \), we have \( f^* : E_{n,m}^{BM}(X/S) \to E_{n+2d,m+d}^{BM}(Y/S) \) given by

\[
f^* E_{n,m}(X/S) \simeq [S_X(m), p_X^! E_S]_{SH(X)} \xrightarrow{ad_{(p_!, p^!)}} [p_# p_* S_X(m), p_X^! E_S]_{SH(X)}
\]

(3.3.1.3)

where \( p_X : X \to S, p_Y : Y \to S \) are structure morphisms.

- For every regular closed immersion morphism \( i : Z \to X \) of codimension \( c \), we have \( i^* : E_{n,m}^{BM}(X/S) \to E_{n-2c,m-c}^{BM}(Y/S) \) ([DJK18, Proposition 4.1.3]).
3.3.2. In what follows, we study these maps for the ring spectrum $E = \text{KGL}$. The first step is to identify its Borel-Moore theory:

**Lemma 3.3.3.** For any separated morphism of finite type $p : X \to S$, we have an isomorphism

$$KGL_{BM}^{n,m}(X/S) \simeq G_{n-2m}(X).$$  

**(3.3.3.1)**

*Proof.* Since $S$ is regular, by Corollary 2.2.16 and Recall 1.0.3 (7) there is a canonical identification $KGL_S \simeq GGL_S$ in $SH(S)$. By Proposition 3.1.11, Corollary 2.2.16 and Corollary 2.2.17, we have the following isomorphism:

$$KGL_{BM}^{n,m}(X/S) = [S_X(m)[n], p^!KGL_S]_{SH(X)} \simeq [S_X(m)[n], p^!GGL_S]_{SH(X)}$$

**(3.3.3.2)**

$$(\psi_p)^{-1} \simeq [S_X(m)[n], GGL_X]_{SH(X)} \xrightarrow{(2.2.17.1)} [S_X[n - 2m], GGL_X]_{SH(X)} \simeq G_{n-2m}(X).$$

□

**Lemma 3.3.4.** For every proper morphism $p : Y \to X$, the isomorphism (3.3.3.1) identifies the map $f_* : KGL_{BM}^{n+2m,m}(Y/S) \to KGL_{BM}^{n+2m,m}(X/S)$ with the proper functoriality of $G$-theory $p_* : G_n(Y) \to G_n(X)$ (Recall 1.0.3 (2)).

*Proof.* By the construction in Lemma 1.3.1, the map $\phi_p : p_*GGL_Y \to GGL_X$ is such that the map

$$G_n(Y) \simeq [S_Y[n], GGL_Y]_{SH(Y)} = [p^*S_X[n], GGL_Y]_{SH(Y)}$$

**(3.3.4.1)**

$$= [S_X[n], p_*GGL_Y]_{SH(X)} \xrightarrow{\phi_p} [S_X[n], GGL_X]_{SH(X)} \simeq G_n(X)$$

agrees with the proper functoriality of $G$-theory. Since the map $\psi_p$ is obtained from $\phi_p$ by adjunction, the map 3.3.4.1 is induced by the proper functoriality of Borel-Moore theory (3.3.1.2), and the result follows. □

**Lemma 3.3.5.** For every smooth morphism $f : Y \to X$ of relative dimension $d$, the isomorphism (3.3.3.1) identifies the map $f^* : KGL_{BM}^{n+2m,m}(X/S) \to KGL_{BM}^{n+2m+2d,m+d}(X/S)$ with the contravariant functoriality of $G$-theory $f^* : G_n(X) \to G_n(Y)$ (Recall 1.0.3 (1)).

*Proof.* By construction in Definition 2.2.5 and Lemma 2.2.7, the map

$$G_n(X) \simeq [S^n \land X, G_X]_{H_*(X)} \to [S^n \land Y, G_X]_{H_*(X)} \simeq G_n(Y)$$

**(3.3.5.1)**

induced by $f_# : Y_+ \to X_+$ equals the smooth functoriality of $G$-theory. By Corollary 2.2.15 and stabilization, we know that the map

$$G_n(X) \simeq [S_X[n], GGL_X]_{SH(X)} \xrightarrow{ad_{(f# \cdot p^*)}} [p#p^*S_X[n], GGL_X]_{SH(Y)}$$

**(3.3.5.2)**

$$\simeq G_n(Y)$$

agrees with the smooth functoriality of $G$-theory. It follows from Proposition 3.2.13 that the map 3.3.5.2 is induced by the smooth functoriality of Borel-Moore theory (3.3.1.3), and the result follows. □

**Lemma 3.3.6.** For every regular closed immersion $i : Z \to X$ of codimension $c$, the isomorphism (3.3.3.1) identifies the map $i^* : KGL_{BM}^{n+2m,m}(X/S) \to KGL_{BM}^{n+2m-2c,m-c}(X/S)$ with the contravariant functoriality of $G$-theory $i^* : G_n(X) \to G_n(Z)$ (Recall 1.0.3 (1)).

*Proof.* By the construction in [JDK18, Definition 3.2.3], the Gysin morphism is given by the composition

$$G_n(X) \xrightarrow{\gamma_t} G_{n+1}(\mathbb{G}_m, X) \xrightarrow{\partial} G_n(N_Z X) \simeq G_n(Z),$$

**(3.3.6.1)**

where $\gamma_t : G_n(X) \to G_{n+1}(\mathbb{G}_m, X)$ is given by the multiplication by $t \in K_1(\mathbb{Z}[[t, t^{-1}]]$, and $\partial$ is the boundary map in the long exact sequence of homotopy groups associated to the homotopy fiber sequence $G(N_Z X) \to G(D_Z X) \to G(\mathbb{G}_m, X)$, $D_Z X = Bl_{\mathbb{A}_Z^1}^{\mathbb{A}_Z^1} X = Bl_Z X$ being the deformation space to the
normal cone. \(^4\) We claim that the composition \(G_n(X) \xrightarrow{\gamma_t} G_{n+1}(\mathcal{G}_{m,X}) \xrightarrow{\partial} G_n(N_Z X)\) is given by the contravariant functoriality \(G_n(X) \to G_n(N_Z X)\). Indeed, the very argument of [Jin16, Proposition 2.30] applies: using the (double) deformation to the normal cone, we are reduced to the case where \(i : Z \to \mathbb{A}^n_Z\) is the zero section, and in this case \(\partial\) is a retraction of \(\gamma_t\). \(^5\) The result then follows from the naturality of the contravariant functoriality of \(G\)-theory.

Corollary 3.3.7. For any separated scheme of finite type \(X\) over \(S\), there is a natural isomorphism

\[
\text{KGL}^BM_{n,m}(X/S) \simeq G_{n-2m}(X)
\]

such that

- The proper functoriality on \(\text{KGL}^BM\) for proper morphisms is given by the proper functoriality of \(G\)-theory;
- The Gysin morphisms on \(\text{KGL}^BM\) for lci morphisms ([Dég18, Definition 3.3.2]) is given by the contravariant functoriality of \(G\)-theory.

Remark 3.3.8. Corollary 3.3.7 has been used in [Dég18, Example 3.3.11 (2)] to obtain a Riemann-Roch theorem for singular schemes using \(G\)-theory: by formal properties of Borel-Moore theories, one constructs a map of spectra

\[
\text{ch}: \text{GGL} \to \bigoplus_{i \in \mathbb{Z}} \text{H}^{BM}_{2i}(\gamma)[2i]
\]

where the right hand side is the spectrum representing Borel-Moore motivic homology with rational coefficients, which is a Chern character map that generalizes the Chern character at the level of motivic spectra in [Rio10, Definition 6.2.3.9] to singular schemes. Using the construction of Gysin morphisms ([Dég18, Definition 3.3.2], which has been extended to a more general setting in [DJK18]), it is proved that the map \(\text{ch}\) satisfies compatibilities with proper functoriality and Gysin morphisms, which generalizes the statement in [Ful98, Theorem 18.3]. This result is proved mostly with tools from \(\mathbb{A}^1\)-homotopy theory and the six functors formalism, without using MacPherson’s “graph construction” method in [Ful98, §18].

REFERENCES

[Ayo07] J. Ayoub, Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, Astérisque No. 314-315 (2007). \(\dagger\), \(\dagger\)
[Cis10] D.-C. Cisinski, Invariance de la K-théorie par équivalences dérivées, Journal of K-theory 6 (2010), no. 3, 505-546. \(\dagger\)
[CD19] D.-C. Cisinski, F. Déglise, Triangulated categories of motives, to appear in the series Springer Monographs in Mathematics. \(\dagger\)
[Con07] B. Conrad, Deligne’s notes on Nagata compactifications, J. Ramanujan Math. Soc. 22 (2007), no. 3, 205-257. \(\dagger\)
[Dég18] F. Déglise, Bivariant theories in stable motivic homotopy, Doc. Math. 23, 997-1076 (2018). \(\dagger\), \(\dagger\), \(\dagger\), \(\dagger\), \(\dagger\)
[DJK18] F. Déglise, F. Jin, A. Khan, Fundamental classes in motivic homotopy theory, arXiv:1805.05920. \(\dagger\), \(\dagger\), \(\dagger\), \(\dagger\)
[Ful98] W. Fulton, Intersection theory, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, A Series of Modern Surveys in Mathematics. 2. Springer-Verlag, Berlin, 1998. \(\dagger\), \(\dagger\)
[Jin16] F. Jin, Borel-Moore motivic homology and weight structure on mixed motives, Math. Z. 283 (2016), no. 3, 1149-1183. \(\dagger\), \(\dagger\)
[ML78] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics. 5 (Second ed.), Springer-Verlag New York, 1978. \(\dagger\)
[MV99] F. Morel, V. Voevodsky, \(\mathbb{A}^1\)-homotopy of schemes, Publications Mathématiques de l’IHÉS, 90 (1999), p. 45-143. \(\dagger\), \(\dagger\), \(\dagger\)
[Nis87] Y. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory, in Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 241-342, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, Kluwer Acad. Publ., Dordrecht, 1989. \(\dagger\)

\(^4\)Equivalently, under the canonical isomorphism \(G_{n+1}(\mathcal{G}_{m,X}) \simeq G_n(X) \oplus G_{n+1}(X)\), \(\gamma_t\) is the inclusion of the first summand, see [Wei13, V.6.2].

\(^5\)Alternatively, we can apply the same argument by looking at Borel-Moore motives in the homotopy category of \text{KGL}-modules.
I. Panin, K. Pimenov, O. Röndigs, On Voevodsky’s Algebraic $K$-Theory Spectrum, Algebraic topology, 279-330, Abel Symp., 4, Springer, Berlin, 2009.

D. Quillen, Higher algebraic $K$-theory: I, in Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85-147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

J. Riou, Opérations sur la $K$-théorie algébrique et régulateurs via la théorie homotopique des schémas, Ph. D. thesis at University Paris 7, 2006.

J. Riou, Algebraic $K$-theory, $A^1$-homotopy and Riemann-Roch theorems, J. Topol. 3 (2010), no. 2, 229-264.

O. Röndigs, M. Spitzweck, P. Østvær, Motivic strict ring models for $K$-theory, Proc. Amer. Math. Soc. 138 (2010), no. 10, 3509-3520.

A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, Séminaire de Géométrie Algébrique du Bois Marie, 1962 (SGA 2). With an exposé by Michèle Raynaud. With a preface and edited by Yves Laszlo. Revised reprint of the 1968 French original. Documents Mathématiques (Paris), 4. Société Mathématique de France, Paris, 2005.

M. Artin, A. Grothendieck, et J. L. Verdier, Théorie des topos et cohomologie étale des schémas, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics, Vol. 269, 270, 305. Springer-Verlag, Berlin-New York, 1972.

P. Berthelot, A. Grothendieck, L. Illusie, Théorie des intersections et théorème de Riemann-Roch, Séminaire de Géométrie Algébrique du Bois-Marie 1966-1967 (SGA 6). With the collaboration of D. Ferrand, J. P. Jouanolou, O. Jus-sila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971.

S. Schwede, B. Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. (3) 80 (2000), 491-511.

R. Thomason, T. Trobaugh, Higher algebraic $K$-theory of schemes and of derived categories, in The Grothendieck Festschrift, Vol. III, 247-435. Progr. Math., 88, Birkhäuser Boston, Boston, MA, 1990.

V. Voevodsky, $A^1$ homotopy theory, Doc. Math., Extra Volume ICM 1998(I), 579-604.

F. Waldhausen, Algebraic $K$-theory of spaces, in Algebraic and geometric topology, (New Brunswick, N.J., 1983), 318-419, Lecture Notes in Math., 1126, Springer, Berlin, 1985.

C. Weibel, The $K$-book: an introduction to algebraic $K$-theory, Graduate Studies in Math. vol. 145, AMS, 2013.

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