The Signals and Systems Approach to Animation

Caleb Reach and Chris North

Abstract—Animation is ubiquitous in visualization systems, and a common technique for creating these animations is the transition. In the transition approach, animations are created by smoothly interpolating a visual attribute between a start and end value, reaching the end value after a specified duration. This approach works well when each transition for an attribute is allowed to finish before the next is triggered, but performs poorly when a new transition is triggered before the current transition has finished. In particular, interruptions introduce velocity discontinuities, and frequent interruptions can slow down the resulting animation. To solve these problems, we model the problem of animation as a signal processing problem. In our technique, animations are produced by transformations of signals, or functions over time. In particular, an animation is produced by transforming an input signal, a function from time to target attribute value, into an output signal, a function from time to displayed attribute value. We show that well-known signal-processing techniques can be applied to produce animations that are free from velocity discontinuities even when interrupted.

1 Introduction

Transitions [39] are widely used in information visualization to smoothly change visual attributes, such as position, color, opacity, size, or line width. Transitions are useful in situations where these attributes would otherwise be changed abruptly, in response to, for example, new data becoming available, browsing, dynamic queries, or navigation, and are recommended for fluid interaction [19].

As a simple example, imagine a visualization of a quarterly sales dataset in which a slider allows the user to select a quarter and sales by department for the selected quarter are shown in a bar chart. Suppose that the goal is to animate the heights of the bars as the selected quarter is changed. This animation can be produced by triggering a transition each time a new quarter is selected. Each transition, when triggered, smoothly interpolates between the current height and the target height, reaching the target height after a fixed duration has elapsed. The specific motion of the transition is controlled by an easing function, which can be used to produce animations that begin from rest by smoothly increasing velocity and end at rest after smoothly decreasing velocity. However, if the user swiftly drags the slider across many quarters, new transitions will be triggered before the current transition has been reached. In these cases, the new transition will begin from rest with a zero velocity, even if the current velocity is nonzero, causing the bar heights to abruptly change velocity and to grow or shrink at a much slower rate than in cases where transitions are not interrupted. This limits the use of animation to cases where animations are unlikely to be interrupted. We seek to improve this animation algorithm so that it can be used during rapid user interaction.

To produce animations that are smooth in both position and velocity, even when interrupted, this paper introduces the signal framework for animations. In signal framework, animations are not produced by triggering transitions, but instead by transforming signals, which are functions from time to attribute values. An animation system in the signal framework takes as input a signal that maps from time to target attribute value and produces as output a signal that maps from time to displayed attribute value.

The contributions of this paper are as follows:

1. It is shown in section 2 that transitions behave poorly when interrupted. In particular, it is shown that in the limit as the duration between interruptions goes to zero, transitions produce a constant output.

2. The signals and systems framework for animation is introduced in section 4. In this framework, smooth animations are produced by transforming signals rather than by triggering transitions. This abstract technique is made concrete by techniques introduced in the following three sections.

3. The finite impulse response (FIR) animation technique is introduced in section 5. It is shown that in cases where transitions all have the same duration and easing function and are never interrupted, the the FIR technique produces the same output as the transition technique. However, the FIR technique handles interruptions more gracefully than the transition technique for cases where transitions are interrupted.

4. The infinite impulse response (IIR) animation technique is introduced in section 6. An application of a spring-mass-damper system for the purposes of animation is presented.

5. A set of examples in which the above techniques are applied are presented in section 8.

2 Transitions

In this section, the transition algorithm is analyzed, both in theory and in practice. It is shown that in cases where transitions are interrupted, the transition algorithm fails to produce output that is smooth in both position and velocity.

Before the transition algorithm can be analyzed, it must be defined, and to define the transition algorithm we must first define an easing function.

Definition 1 (Easing function). A easing function is a function $s : \mathbb{R} \rightarrow [0,1]$ from time to a real number such that $s(t) = 0$ for all $t \leq 0$ and $s(t) = 1$ for all $t \geq d$, where $d$ is the duration of the easing function.

This definition differs from slightly from the traditional definition of an easing function. Traditionally, an easing function traditionally ends at $t = 1$. In our definition, the easing function is allowed to have an arbitrary duration. The transition algorithm can now be defined.

Definition 2 (Transition algorithm). The transition algorithm algorithm 1 takes an initial state $x_0 \in \mathbb{R}$, change requests $u_1, u_2, \ldots, u_n$ at respective times $t_1, t_2, \ldots, t_n$, and an easing function $s$, and produces an output function $y : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$y_0(t) = x_0$$
$$y_i(t) = y_{i-1}(t) + s(t - t_i)(x_i - y_{i-1}(t_i))$$
$$y(t) = \sum_{i=0}^{n} y_i(t)$$

where $\|p\|$ is 1 if $p$ is true and 0 if $p$ is false.

We will now analyze the behavior of the transition algorithm when interrupted and show that as the frequency of interruptions increases, the speed of the resulting animation decreases. This prevents the

---

1The CSS animation specification [22] instead uses the term “animation timing function”.

• Andrew Reach is with Virginia Tech. E-mail: caleb.reach@cs.vt.edu.
• Chris North is with Virginia Tech. E-mail: north@cs.vt.edu.
transition algorithm from producing desirable results in cases where interruptions are common. Such interruptions can occur when transition triggers are introduced in response to interactive dynamic query changes or in response to new data arriving.

**Claim 1.** If the derivative of the easing function \( s(t) \) vanishes at \( t = 0 \) and the input is bounded, then in the limit as the duration between interruptions goes to zero, the output of the simple transition algorithm is constant.

**Proof.** Let \( x \) be the target function, let \( n \) be the number of samples per unit of time, and let \( T = 1/n \). Our samples occur at \( t_i = iT \) for \( i \) in \( 1 \) to \( n \). Let \( y[i] \) be defined as \( y(IT) \), and let \( x[i] \) be defined similarly. The simple transition algorithm is defined as

\[
y(t) = \sum_{i=0}^{n} y[i] 1_{t < t_i + 1}
\]

Since for \( t = iT \), the condition \( t_k \leq t < t_{k+1} \) is true only for \( k = i \), it follows that \( y[i] = y[i] \). The definition of \( y[i] \) can be further simplified to

\[
y[i] = y[i] = y[i] \cdot (kT)
\]

Notice that \( y[i] = y[i-1] \) and that \( y[i+1] = y[i] \). We rewrite the formula as

\[
y(0) = x[0]
y[i+1] = \alpha y[i] + \beta x[i]
\]

where \( \alpha = 1 - s(T) \) and \( \beta = s(T) \). This can be recognized as a linear discrete-time system, which has the solution [36, Chapter 21]

\[
y[i] = x[0] \sum_{k=0}^{i-1} \alpha^{i-k-1} \beta x[k]
\]

We now need to find \( y[n] \) in the limit as \( n \). Since we have assumed the easing function \( s(t) \) is continuous with zero value and derivative at \( t = 0 \), the first two terms of its Taylor series are zero, and therefore \( s(T) = o(T^2) \) as \( T \to 0 \) where \( o \) is the Little-O Landau symbol. We therefore have

\[
\lim_{n \to \infty} \left( a^n x[0] + \sum_{k=0}^{n-1} a^{n-k} \beta x[k] \right)
\]

We will now show that the summation goes to zero and that \( \beta^n x[0] \) goes to \( x[0] \) as \( n \) goes to infinity. Because the magnitude of \( \alpha \) is near zero for sufficiently large \( n \), \( \beta \) is positive for sufficiently large \( n \), and therefore \( \beta^n \) is monotonically increasing with respect to \( \beta \). Since \( \alpha = o(T) \), we have that \( cT^2 < \alpha < cT^2 \) for some \( c \). Therefore \( (1 - cT^2)^n - k < \beta < (1 + cT^2)^{n-k} \). It can be easily verified that the limit of both bounds as \( n \) goes to infinity is \( 1 \), and therefore \( \beta \) must also approach 1. Since we have assumed that \( u \) is bounded, \( \Sigma_{k=0}^{n-1} u[k-1] = O(n) \), and therefore

\[
\sum_{k=0}^{n} a\beta^{n-k} u[k-1] = o(T^2)O(n) + o(1/n^2)O(n) = O(1/n)
\]

and therefore, in the limit, goes to zero. The term \( \beta^n x[0] \) goes to \( x[0] \) because \( \beta^n \) goes to one, as previously discussed.

---

**Algorithm 1 Simple transitions for easing function \( f \)**

- \( t_1 \leftarrow t_0 \) \quad \rightarrow \text{Time of last target change}
- \( u_1 \leftarrow u_0 \) \quad \rightarrow \text{Target value at } t_1
- \( x_1 \leftarrow x_0 \) \quad \rightarrow \text{Output value at } t_1
- \( x \leftarrow x_0 \) \quad \rightarrow \text{Current output value}

**for each frame at time } t \text{ do**

- **if** target changed to \( u \text{ then**}
  - \( t_1 \leftarrow t \) \quad \rightarrow \text{Set } t_1 \text{ to current time}
  - \( u_1 \leftarrow u \) \quad \rightarrow \text{Set } u_1 \text{ to current input value}
  - \( x_1 \leftarrow x \) \quad \rightarrow \text{Set } u_1 \text{ to current output value}

**end if**

\( x \leftarrow x_1 + ((u_1 - x_1) f(t - t_0)) \) \quad \rightarrow \text{Set output}

**end for**

---

### 3 Splines

In this section, we analyze an alternative approach based on splines. Splines are often used in computer graphics to smoothly animate between user-specified key-frames. In such systems, all key-frames are given in advance. By contrast, the problem considered in this paper requires an algorithm that can process changes as they become available.

Splines can be easily adapted to this task, and so before presenting our technique, we first consider a solution to the animation problem based on splines.

One approach to allowing animations to be smoothly interrupted is to specify the easing function as a spline and modify the velocity of the starting point on the spline to account for the current velocity of the object. We now analyze this approach.

A cubic Hermite spline beginning at \( p_0 \) with slope \( p_0 \) at \( t = 0 \) and ending at \( p_1 \) with slope \( p_1 \) at \( t = 1 \) is given by

\[
p(t) = (2t^3 - 3t^2 + 1)p_0 + (t^3 - 2t^2 + t)p_1 \quad + (-2t^3 + 3t^2)p_0 + (t^3 - t^2)p_1
\]

**Definition 3** (Spline transition algorithm). The spline transition algorithm introduces an initial state \( x_0 \), initial velocity \( v_0 \), and change requests \( x_1, x_2, \ldots, x_n \) at respective times \( t_1, t_2, \ldots, t_n \), and produces an output function \( y : \mathbb{R} \to \mathbb{R} \) given by

\[
y_0(t) = x_0
\]

\[
y_i(t) = r(y_{i-1}(t), y_{i-1}(t), x_i, t - t_i)
\]

\[
y(t) = \sum_{i=0}^{n} y_i(t) 1_{t < t_{i+1}}
\]

where \( r \) is given by

\[
r(p_0, p_0, p_1, t) = (2t^3 - 3t^2 + 1)p_0 + (t^3 - 2t^2 + t)p_1 + (-2t^3 + 3t^2)p_0 + (-2t^3 + 3t^2)p_1
\]

In other words, the spline transition algorithm responds to each input change by constructing a spline segment that begins at the current point and with the current velocity and ends \( d \) seconds in the future at the new point with zero velocity.

We now show that the spline transition algorithm successfully approaches the target value. Let \( x \) be the target function, \( n \) be the number of samples per unit of time, and let \( T = 1/n \). Let \( t_i = iT \) and let \( y[i] = y(IT) \) and \( x[i] = x(IT) \).

The spline transition algorithm is defined as

\[
y(t) = \sum_{i=0}^{n} y_i(t) 1_{t < t_{i+1}}
\]

which simplifies as before to

\[
y(0) = x_0
\]

\[
y[i+1] = \Delta(y[i], y[i], x[i], T)
\]

**end if**

\( x \leftarrow x_1 + ((u_1 - x_1) f(t - t_0)) \) \quad \rightarrow \text{Set output}

**end for**
This can be seen as a discrete-time state-space system with two state variables: position ($y$) and velocity ($\dot{y}$). Let $A$ be the matrix and $B$ be the vector such that

$$
\begin{bmatrix}
\dot{y}[k+1] \\
y[k+1]
\end{bmatrix} =
\begin{bmatrix}
A & B
\end{bmatrix}
\begin{bmatrix}
y[k] \\
\dot{y}[k]
\end{bmatrix} + Bu[k]
$$

Let

$$y[k] = \begin{bmatrix} \dot{y}[k+1] \\ y[k+1] \end{bmatrix}$$

Then $y[k+1] = Ay[k] + Bu[k]$. By making the substitution $t = kT$ and rearranging, this can be rewritten as

$$
\frac{y(t + T) - y(t)}{T} = Ay(t) + Bu(t) - y(t)
$$

Notice that in the limit as $k$ approaches infinity, and therefore $T$ approaches zero, the left side becomes the derivative of $y$, and so taking the limit of both sides gives

$$
\dot{y}(t) = \lim_{T \to 0} \frac{Ay(t) + Bu(t) - y(t)}{T} = \lim_{T \to 0} \left( A - \frac{I}{T} \right) y(t) + \frac{B}{T} u(t)
$$

The matrix $A$ is given by

$$A = \begin{bmatrix} 2T^3 - 3T^2 + 1 & T^3 - 2T^2 + T \\ 6T^2 - 6T & 3T^2 - 4T + 1 \end{bmatrix},$$

and $B$ is given by

$$B = \begin{bmatrix} -2T^3 + 3T^2 \\ -6T^2 + 6T \end{bmatrix}.$$

Let $A'$ and $B'$ be the matrices that result from $(A - I)/T$ and $B/T$ respectively in the limit as $T$ goes to zero. The matrix $A'$ evaluates to

$$A' = \lim_{T \to 0} \frac{(A - I)}{T} = \lim_{T \to 0} \begin{bmatrix} 2T^2 - 3T & T^2 - 2T + 1 \\ 6T - 6 & 3T - 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -4 \end{bmatrix},$$

and $B'$ evaluates to

$$B' = \lim_{T \to 0} \frac{B}{T} = \lim_{T \to 0} \begin{bmatrix} -2T^2 + 3T \\ -6T + 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$ 

This forms the continuous-time state-space system

$$
\begin{align*}
\dot{x}(t) &= A'x(t) + B'u(t) \\
y(t) &= x_1(t)
\end{align*}
$$

which has the solution

$$y(t) = \int_0^\infty e^{A(t-\tau)}B'u(\tau) d\tau$$

in which the matrix exponential is used. Evaluating this with a unit step input $u(t)$ produces the step response. This step response does approach one as time increases, and so the system does eventually reach the target value. Interestingly, the system overshoots the target very slightly before settling.

The spline animation technique has the advantage that interruptions do not produce velocity discontinuities. However, it has the disadvantage that the designer has no creative control over the motion of the animation. In particular, there is no easy way to produce motion that accelerates quickly and decelerates slowly. The techniques presented in the following sections retain creative control offered by the transition algorithm while allowing interruptions without velocity discontinuities.

### 4. Signals and Systems

The techniques we propose in this paper to address the problems with the simple transition algorithm discussed in the introduction follow from the simple idea that the animation problem should be approached as a problem of designing a system that transforms signals. Fortunately, nearly all of the difficult work has been done, and we need only to adapt well-established signal processing techniques to our particular problem of transitions. The field of signal processing is vast, and there are countless ways in which to design a signal processing system to complete a given task. This section introduces some of the basic concepts from signal processing, and discusses their relevance to information visualization.

A signal is any value that changes over time. For example, in visualization, signals could be used to model

1. The mouse position
2. The mouse button state
3. Position of a data point on the screen
4. A streaming data value
5. Any computed intermediate value

Visualizations can be seen as a function from data and program state to an image [1]. Of course, this function is in turn composed of many smaller functions. All of these functions may compute intermediate values, and nearly all of these intermediate values can be regarded as signals. For example, a visualization might assign a position to all nodes in a tree, and then this position might be projected to screen space via some transformation. Each position assigned by the tree layout algorithm can be seen as a signal, and the screen-space position resulting from the projection of the layout position can also be seen as a signal. Therefore, signal processing techniques can be applied at any stage in the visualization pipeline [8, 12].

Systems may be classified as linear or nonlinear. A system is linear if it satisfies the following two properties:

1. If $x(t)$ is transformed to $y(t)$, then $ax(t)$ is transformed to $ay(t)$.
2. If $x_1(t)$ is transformed to $y_1(t)$ and $x_2(t)$ is transformed to $y_2(t)$, then $x_1(t) + x_2(t)$ is transformed to $y_1(t) + y_2(t)$.

It is important to note that the linearity of a system is unrelated to the linearity of its input or output signals, i.e. both linear and nonlinear systems can transform lines to lines, lines to curves, or curves to lines. When using a linear system to animate the movement of an object from one point to another, the animation will appear to take the same amount of time regardless of the distance between the points. The animation produced by a non-linear system, by contrast, may depend on the distance between the points, or even on the location of the points themselves.

Systems can also be classified as time-varying or time-invariant. A time-invariant system has the property that if $x(t)$ is transformed to $y(t)$ then $x(t - a)$ is transformed to $y(t - a)$. A time-varying system is any system for which this property does not hold. In other words, a delayed input produces a equally delayed output for a time-invariant system, whereas a delayed input may produce a completely different output for a time-varying system.

This paper focuses on the class of linear, time-invariant (LTI) systems, which is extremely well-understood. Any LTI system can be perfectly represented by its impulse response $h$, and given $h$, the output $y$ for any input $x$ can be found by

$$y(t) = (h \ast x)(t)$$

where $\ast$ denotes convolution, defined as

$$(h \ast x)(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau) d\tau.$$
Another way to classify systems is by causality. In a causal system, the output at a given instant depends only on past and present input values. Offline systems may be non-causal, but interactive systems such as those discussed in this paper must be causal.

Systems may be classified as finite impulse-response (FIR) or infinite impulse-response (IIR). In a FIR system, the output depends only on recent input values—all sufficiently old input values are forgotten. In IIR system, the output is influenced by all past input values. Most IIR systems encountered in practice can be cast into a state-space representation, in which the output is determined by only the present input value and the state of the system, which evolves over time in a manner depending only on the present input at each instant.

Physical systems, such as mechanical systems composed of springs, masses, and dampers or electrical systems composed of resistors, inductors, and capacitors tend to be IIR. There is a well-established correspondence between these mechanical systems and electrical circuits comprising capacitors, inductors, and resistors [38] [45]. FIR systems, by contrast, are usually associated with digital signal processing techniques. For animation, both FIR filters and IIR filters can be useful.

LTI systems can be classified by the way that they affect sinusoidal inputs. A system that leaves low-frequency sine waves essentially unchanged but greatly reduces the amplitude of high-frequency sine waves is called a low-pass filter. Since the goal of transitions is to smooth an input signal that changes abruptly, we only consider low-pass filters in this paper.

We call a filter whose impulse response integrates to one affine. We call an affine filter whose impulse response is everywhere non-negative convex. A causal affine filter applied to a constant input will eventually approach the input value, and a convex system will never travel outside the range of its input. For the purposes of this paper, we only consider affine filters, both convex and non-convex.

Another way to classify systems is continuous-time or discrete-time. For transitions, it’s often helpful to perform the initial design in continuous space, as it frees the designer from having to think about the frame rate. There are well-known ways to convert continuous-time systems to discrete-time systems [37].

To design an animation, we must make design decisions. FIR and IIR filters provide two low-level building blocks upon which we can construct more elaborate systems.

5 Finite impulse response transitions

The first technique we propose is the FIR transition. Like simple transitions, FIR transitions allow the easing curve to be specified explicitly, giving interaction designers precise control over motion. Unlike simple transitions, FIR transitions maintain velocity continuity even when interrupted, and they will make progress towards a target even if the target is continuously changing.

Definition 4 (Finite impulse response transition). The FIR transition technique gives, for input signal \( x \) and easing function \( s \), the output signal \( x + s \).

While mathematically simple, this definition cannot be implemented as written, since convolution involves an integral over all time, which does not immediately imply a particular implementation. The convolution could, of course, be implemented using numerical integration techniques, but this would be needlessly wasteful for almost all cases.

In the case where the input is a step function, the math can be simplified considerably, and is expressible using sums, which are easy to compute, instead of integrals, which are not. Since the impulse response is finite, we also do not have to retain the entire history of the program, but rather only a small window of history. While mathematically a FIR filter needn’t be discrete-time or have step function inputs, in practice a FIR filter is usually implemented using discrete time.

Claim 2. If \( x \) is a step function that starts at \( x_0 \) and changes to \( x_1, x_2, \ldots, x_n \) at respective times \( t_1, t_2, \ldots, t_n \), LTI transitions simplify to

\[
x_0 + \sum_{i=1}^{n} (x_i - x_{i-1}) s(t - t_i)
\]

Proof. The input is given as

\[
\sum_{i=0}^{n} x_i \mathbb{1}_{t_i \leq t < t_{i+1}}
\]

where \( t_0 = -\infty \) and \( t_{n+1} = \infty \). The convolution therefore simplifies to

\[
\sum_{i=0}^{n} x_i (\tilde{s}(t) * \mathbb{1}_{t_i \leq t < t_{i+1}})
\]

where the convolution is taken with respect to \( t \). The convolution terms expand as

\[
\tilde{s}(t) * \mathbb{1}_{t_i \leq t < t_{i+1}} = \int_{-\infty}^{\infty} \mathbb{1}_{t_i \leq t - \tau < t_{i+1}} \tilde{s}(\tau) d\tau
\]

and therefore the transition can be given as

\[
\sum_{i=0}^{n} x_i (s(t - t_i) - s(t - t_{i+1}))
\]

Another way to classify systems is by causality. In a causal system, the output at a given instant depends only on past and present input values. Offline systems may be non-causal, but interactive systems such as those discussed in this paper must be causal.

Systems may be classified as finite impulse-response (FIR) or infinite impulse-response (IIR). In a FIR system, the output depends only on recent input values—all sufficiently old input values are forgotten. In IIR system, the output is influenced by all past input values. Most IIR systems encountered in practice can be cast into a state-space representation, in which the output is determined by only the present input value and the state of the system, which evolves over time in a manner depending only on the present input at each instant.

Physical systems, such as mechanical systems composed of springs, masses, and dampers or electrical systems composed of resistors, inductors, and capacitors tend to be IIR. There is a well-established correspondence between these mechanical systems and electrical circuits comprising capacitors, inductors, and resistors [38] [45]. FIR systems, by contrast, are usually associated with digital signal processing techniques. For animation, both FIR filters and IIR filters can be useful.

LTI systems can be classified by the way that they affect sinusoidal inputs. A system that leaves low-frequency sine waves essentially unchanged but greatly reduces the amplitude of high-frequency sine waves is called a low-pass filter. Since the goal of transitions is to smooth an input signal that changes abruptly, we only consider low-pass filters in this paper.

We call a filter whose impulse response integrates to one affine. We call an affine filter whose impulse response is everywhere non-negative convex. A causal affine filter applied to a constant input will eventually approach the input value, and a convex system will never travel outside the range of its input. For the purposes of this paper, we only consider affine filters, both convex and non-convex.

Another way to classify systems is continuous-time or discrete-time. For transitions, it’s often helpful to perform the initial design in continuous space, as it frees the designer from having to think about the frame rate. There are well-known ways to convert continuous-time systems to discrete-time systems [37].

To design an animation, we must make design decisions. FIR and IIR filters provide two low-level building blocks upon which we can construct more elaborate systems.

5 Finite impulse response transitions

The first technique we propose is the FIR transition. Like simple transitions, FIR transitions allow the easing curve to be specified explicitly, giving interaction designers precise control over motion. Unlike simple transitions, FIR transitions maintain velocity continuity even when interrupted, and they will make progress towards a target even if the target is continuously changing.

Definition 4 (Finite impulse response transition). The FIR transition technique gives, for input signal \( x \) and easing function \( s \), the output signal \( x + s \).

While mathematically simple, this definition cannot be implemented as written, since convolution involves an integral over all time, which does not immediately imply a particular implementation. The convolution could, of course, be implemented using numerical integration techniques, but this would be needlessly wasteful for almost all cases.

In the case where the input is a step function, the math can be simplified considerably, and is expressible using sums, which are easy to compute, instead of integrals, which are not. Since the impulse response is finite, we also do not have to retain the entire history of the program, but rather only a small window of history. While mathematically a FIR filter needn’t be discrete-time or have step function inputs, in practice a FIR filter is usually implemented using discrete time.

Claim 2. If \( x \) is a step function that starts at \( x_0 \) and changes to \( x_1, x_2, \ldots, x_n \) at respective times \( t_1, t_2, \ldots, t_n \), LTI transitions simplify to

\[
x_0 + \sum_{i=1}^{n} (x_i - x_{i-1}) s(t - t_i)
\]

Proof. The input is given as

\[
\sum_{i=0}^{n} x_i \mathbb{1}_{t_i \leq t < t_{i+1}}
\]

where \( t_0 = -\infty \) and \( t_{n+1} = \infty \). The convolution therefore simplifies to

\[
\sum_{i=0}^{n} x_i (\tilde{s}(t) * \mathbb{1}_{t_i \leq t < t_{i+1}})
\]

where the convolution is taken with respect to \( t \). The convolution terms expand as

\[
\tilde{s}(t) * \mathbb{1}_{t_i \leq t < t_{i+1}} = \int_{-\infty}^{\infty} \mathbb{1}_{t_i \leq t - \tau < t_{i+1}} \tilde{s}(\tau) d\tau
\]

and therefore the transition can be given as

\[
\sum_{i=0}^{n} x_i (s(t - t_i) - s(t - t_{i+1}))
\]

In the special case where the input is a step function and each step in the input has a duration at least that of the easing function, i.e. in the case where transitions are not interrupted, the FIR transition algorithm produces results identical to those from the simple transition algorithm.

Claim 3. If each transition is allowed to finish before the next begins, FIR animation is equivalent to simple transitions.

Proof. The simple transition algorithm is defined as

\[
y_0(t) = x_0
\]

\[
y_i(t) = y_{i-1}(t_i) + (x_i - y_{i-1}(t_i)) s(t - t_i)
\]

\[
y(t) = \sum_{i=0}^{n} y_i(t) \mathbb{1}_{t_i \leq t < t_{i+1}}
\]

The condition \( t_i \leq t < t_{i+1} \) will be true for only a single value of \( i \), and so \( y(t) = y_i(t) \) for this \( i \). In the case where \( i = 0 \), \( y(t) = x_0 \). In the case where \( i \neq 0 \),

\[
y_i(t) = y_{i-1}(t_i) + (x_i - y_{i-1}(t_i)) s(t - t_i)
\]
Algorithm 2 Continuous-time FIR transitions for step input and easing function \( f \).

```plaintext
value ← initial value
targets ← Queue()
for each frame at time \( t \) do
  if target changed to \( u \) then
    targets.enqueue(Target(time: \( t \), value: \( u \)))
  end if
  while targets not empty and targets.peek().time < \( t - d \) do
    value ← targets.dequeue().value
  end while
  out ← value
  prev ← value
  for each target in targets (in chronological order) do
    out ← out + \( f(t - \text{target.time})(\text{target.value} - \text{last}) \)
    prev ← target.value
  end for
end for
```

The term \( y_{i-1}(t_i) \) expands as

\[
y_{i-1}(t_i) = y_{i-2}(t_{i-1}) + (x_{i-1} - y_{i-2}(t_{i-1}))s(t_i - t_{i-1})
\]

Since transitions are not interrupted, \( t_i - t_{i-1} \) is greater than the duration of \( s \), and therefore \( s(t_i - t_{i-1}) = 1 \), and thus \( y_{i-1}(t_i) = x_{i-1} \). The definition of \( y \equiv y(t) \) therefore simplifies to

\[
y(t) = x_{i-1} + (x_i - x_{i-1})s(t - t_i).
\]

From claim 2, FIR transitions can be defined as

\[
\hat{y}(t) = x_0 + \sum_{k=1}^{n} (x_k - x_{k-1})s(t - t_k).
\]

Clearly, in the case where \( i = 0 \), \( \hat{y}(t) = y(t) = x_0 \). We now consider the case where \( i \neq 0 \). Since \( t_i \leq t \), if \( k \leq i - 1 \), then \( d \leq t_i - t_k \leq t - t_k \) where \( d \) is the duration of the easing curve, and therefore \( s(t - t_k) = 1 \). Similarly, for all \( k \geq i + 1 \), \( s(t - t_k) = 0 \). We can therefore expand the sum to

\[
\hat{y}(t) = x_0 + \sum_{k=1}^{i-1} (x_k - x_{k-1})s(t - t_k).
\]

Noting that \( x_{k-1} \) in the sum cancels \( x_0 \) for \( k = 1 \) and that \( x_k \) is canceled by \( x_{(k+1)-1} \) for all \( k < i - 1 \), the sum further simplifies to

\[
\hat{y}(t) = x_{i-1} + (x_i - x_{i-1})s(t - t_i),
\]

which is clearly equal to \( y(t) \).

An implementation of continuous-time FIR animation is shown in algorithm 2. An implementation of the discrete-time FIR animation algorithm is shown in algorithm 3. These algorithms can be trivially extended to operate on vector input signals to produce 2D or 3D animations by means of a matrix easing function: if the easing function maps from time to a matrix and the input is vector-valued, then the algorithm works as written. The key difference between these algorithms and the simple transition algorithm is that these algorithms retain a queue of in-progress transitions whereas the simple transition algorithm only tracks a single transition.

6 Infinite impulse response transitions

The second technique we propose is the IIR animation. Unlike FIR transitions, IIR animations do not take an explicit easing curve as a parameter. Instead, IIR systems are based on differential equations, and can be used to model mechanical and electrical systems that are linear and time-invariant.

Algorithm 3 Discrete-time FIR algorithm.

```plaintext
buffer ← Array(size: \( n \), initial: 0)
index ← 0
for each input value \( u \) do
  buffer[index] ← \( u \)
  index ← index + 1
  if index \( > n - 1 \) then
    index ← 1
end if
end for
```

Definition 5. An IIR animation is given by

\[
x'(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} C & D \end{bmatrix} x(t)
\]

where \( u(t) \) is the input vector, \( x(t) \) is the state vector, \( A, B, C, \) and \( D \) are matrices and \( y(t) \) is the output vector.

The formula above is simply the well-known state-space representation of a linear system. Such systems can be used to model physical mechanical systems of springs, masses, and dampers, as well as circuits comprising resistors, capacitors, and inductors. Such systems have natural-looking behavior, and are therefore useful for producing natural-looking animations in visualization. An example system is shown in figure 1. For this system, the equations of motion are given by

\[
m\ddot{x} = k(u - x) - \zeta \dot{x}
\]

Putting this equation into state-space form requires introducing an additional state variable to capture the second derivative of \( x \). By setting, \( q = \dot{x} \), the above equation can be written as

\[
\dot{q} = \frac{k}{m}(u - x) - \frac{\zeta}{m} \dot{x}
\]

\[
\dot{x} = q
\]

which is easily put into the form given by (4) using state vector \( (x, q) \). More detailed examples can be found in most signal processing textbooks, e.g. [38][31][36]. These systems can produce fast-in, slow-out animations, as well as spring-type effects where the object wiggles back and forth as it settles. Fast-in/slow-out animations, i.e. animations that accelerate quickly from rest and decelerate slowly, are recommended by Google’s Material Design guidelines [2]. Note that in this paper, fast-in/slow-out refers to an animation that smoothly accelerates from rest and decelerates smoothly to rest, but that attains its maximum speed before the animation is halfway finished. Some papers [16] instead use fast-in to refer to an animation that begins with a nonzero velocity.

The trajectory of such systems can be animated using well-known numerical integration techniques, e.g. Runge-Kutta or Verlet integration. Another option, which we advocate, is converting the continuous-time system to a discrete-time system, which can be accomplished using well-known methods such as the impulse-invariant method, the bilinear transform, or the matched \( z \) transform [38]. When converting systems in this manner, high-order systems (i.e. systems with high-dimension state vectors) are usually broken up into biquads, which are order-two.
Algorithm 4 Discrete-time IIR realized via the Transposed Direct Form II [37] with coefficients $a_i$ and $b_i$.

```plaintext
for each input value $x$ do
    $y \leftarrow s_1 + b_0 x$  \hspace{1em} \text{\textcopyright Compute output}
    $s_1 \leftarrow s_2 + b_1 x - a_1 y$
    $s_2 \leftarrow b_2 x - a_2 y$  \hspace{1em} \text{\textcopyright Update state}
end for
```

IIR filters (i.e. IIR filters with two state variables). One possible implementation of a biquad is shown in algorithm 4, and an implementation of a high-order system in terms of biquads placed in series is shown in figure 2.

7 System diagrams

The preceding two sections discussed two low-level techniques for creating animations. This section discusses system diagrams, which are used in signal processing to represent higher-level systems that are assembled out of simpler components. In a system diagram, blocks represent components (subsystems) and wires represent signals. Each block has an associated state that changes over time, and the state of the composite system is given by the combination of the states of the components. Each block has state and behavior, and is therefore similar to the notion of an object in object-oriented programming.

If all components in the system are LTI, then the composite system is also LTI. If two LTI components are connected in series, then resulting impulse response is the convolution of the impulse responses of the components. If the first component transforms an input sine wave by multiplying the amplitude by $a_1$ and delaying the output by $\tau_1$, and the second component transforms the same input sine wave by multiplying the amplitude by $a_2$ and delaying the output by $\tau_2$, then a connecting these two components in series results in a system that multiplies the amplitude of the input sine wave by $a_1 a_2$ and delays the output by $\tau_1 + \tau_2$. If two LTI components are connected in parallel, then the impulse responses add. Connecting several LTI components in series, each of which smooths its input a moderate amount, can be used to produce a system that smooths its input heavily.

8 Applications

We now discuss several example applications of these ideas as a way to illustrate the signals and systems design process. These examples represent scenarios that involve frequent interruptions during animations.

8.1 Animating histogram bins

Imagine a dynamic query [3] or scented widget [44] system in which there is one histogram per data attribute and each histogram has a selection control that can be used to filter along that dimension. For example, crossfilter [1] provides this type of dynamic query. As the user drags the selection control, points will enter and exit the selection, causing the heights of bins in the other histograms to change abruptly. Our goal is to smoothly animate the heights of bins changing.

The height of each bin can be regarded as a signal, and this signal could be smoothed using the methods discussed in this paper. However, suppose that we wish to also allow the user to zoom in on the y axis using a pinch gesture. Since the height of a bar on the screen depends on both the count and the zoom position, if a smoothing filter were applied to the height of the bar, it would also smooth the zoom gesture. Therefore, the counts should be smoothed before projecting counts to heights (b).

8.2 Animating permutations

Imagine animating a permutation of a collection of objects arranged in a row. If each object is animated along a line to its new destination, then objects will often collide during the course of the animation. If each object is instead animated along an elliptical trajectory, this tendency will be significantly reduced, and the paths of objects will form an arc diagram [24]. In this section, we describe an LTI FIR filter that has this behavior.
Since each input point lies on a line, the input to the system is a one-dimensional vector, i.e. a scalar. The output, however, is a two-dimensional vector. Therefore, the step response for this system is a time-varying two-by-one matrix, i.e. a time-varying vector. To qualify as an easing function, the step response must begin at \((0,0)\) and end at \((1,0)\). An ellipse with a horizontal major axis having these two endpoints is given by
\[
\begin{align*}
x(\theta) &= \frac{1}{2} \begin{bmatrix} \cos(\theta) + 1 \sin(\theta) \end{bmatrix} \\
\end{align*}
\] (5)
where \(\alpha\) is the aspect ratio of the ellipse. At \(\theta = \pi\), this simplifies \((0,0)\), and at \(\theta = 2\pi\), this simplifies to \((1,0)\). To use this trajectory as an easing function, it must be parameterized by time instead of \(\theta\). A tempting choice is to set \(\theta = \pi + t\pi\), i.e. linearly interpolate \(\theta\) between the beginning and ending angles. When \(\alpha = 1\), the object travels a circular trajectory at constant speed. When \(\alpha \neq 1\), however, the object travels an elliptical trajectory at a time-varying speed.

To avoid this coupling between the shape of the ellipse and the speed of the object, a constant-speed parameterization must be found. The arc length of a segment of (5) from \(\theta_1\) to \(\theta_2\) is given by
\[
s(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \| \dot{x}(\theta) \| \, d\theta
\]
where \(\dot{x}\) denotes the derivative of \(x\). Since the trajectory begins at \(\pi\) and ends at \(2\pi\), the arc-length of the entire trajectory is given by \(s(\pi, 2\pi)\). The distance traveled from \(\pi\) to some \(\theta\) between \(\pi\) and \(2\pi\) is given by \(s(\pi, \theta)\). The portion of the total distance traveled at a given \(\theta\) is therefore given by
\[
\sigma(\theta) = \frac{s(\pi, \theta)}{s(\pi, 2\pi)}.
\]
If \(t = \sigma(\theta)\), then the trajectory has constant speed. The complete trajectory can then be given using the inverse of \(\sigma\) as \(x(\sigma^{-1}(t))\).

We now turn our attention to the computation of \(\sigma^{-1}\). Since \(\sigma\) is defined in terms of \(s\), which is in turn defined in terms \(x\), we first find \(\dot{x}\):
\[
\dot{x} = \frac{1}{2} \begin{bmatrix} -\sin \theta \\ \alpha \cos \theta \end{bmatrix}.
\]
Therefore,
\[
\| \dot{x} \| = \sqrt{\sin^2 \theta + \alpha^2 \cos^2 \theta}
\]
and so
\[
s(\theta_1, \theta_2) = \frac{1}{2} \int_{\theta_1}^{\theta_2} \sqrt{\sin^2 \theta + \alpha^2 \cos^2 \theta} \, d\theta
\]
This integral does not, in general, have a closed-form solution. However, it can be rewritten in terms of the incomplete elliptic integral of the second kind, \(E(\varphi, k)\), defined as
\[
E(\varphi | k^2) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta,
\]
by rearranging \(s\) as follows:
\[
\begin{align*}
s(\theta_1, \theta_2) &= \frac{1}{2} \int_{\theta_1}^{\theta_2} \sqrt{\sin^2 \theta + \alpha^2 \cos^2 \theta} \, d\theta \\
&= \frac{1}{2} a \int_{\theta_1}^{\theta_2} \sqrt{\frac{1}{a^2} \sin^2 \theta + \cos^2 \theta} \, d\theta \\
&= \frac{1}{2} a \int_{\theta_1}^{\theta_2} \sqrt{\frac{1}{a^2} \sin^2 \theta + 1 - \sin^2 \theta} \, d\theta \\
&= \frac{1}{2} a \int_{\theta_1}^{\theta_2} \sqrt{1 - \left(1 - \frac{1}{a^2}\right) \sin^2 \theta} \, d\theta \\
&= \frac{1}{2} a \left( E(\theta_2 | k^2) - E(\theta_1 | k^2) \right)
\end{align*}
\]
where \(k^2 = 1 - (1/a^2)\).

We have now specified \(s\), and therefore \(\sigma\), in terms of \(E\). Implementations of \(E\) can be found in many commonly-used mathematical libraries. In order to compute the trajectory, however, we must resort to numerical methods to find the inverse \(\sigma^{-1}\). We now have everything we need to compute \(x(\sigma^{-1}(t))\), which gives movement along the ellipse at a constant rate.

Velocity discontinuities can be avoided by using an extra easing function. Instead of computing the easing function \(f\) in algorithm 2 by \(f = (x \circ \sigma^{-1})(t)\), we can compute \(f = (x \circ \sigma^{-1} \circ g)(t)\), where \(g\) is a scalar easing function. Given that our method for computing this function relies on running a root finding algorithm on a function involving elliptic integrals, this function is almost certainly costly to evaluate. In the discrete-time case, the solution is simple: compute the FIR filter coefficients by evaluating this function at a fixed set of points. This is the so-called impulse invariant method for discretizing a continuous-time system. These coefficients need only be generated once, and once generated, the output at each frame can be computed using one multiplication and addition per coefficient. In the continuous-time case, we can again compute the function at fixed points, and then interpolate between the samples to reconstruct the continuous function.

To complete this example, we discuss a practical example of the inner easing function \(g\). One choice is to model \(g\) on the step response of an IIR system. Consider the system for a one-pole filter
\[
x(t) = x(t) + a(u(t) - x(t)).
\]
The impulse response of this system decays exponentially. Used alone, this filter would not smooth the input sufficiently for animation use. Therefore, we form a four-pole filter by cascading four one-pole filters, each given by the above equation. The step response of this four-pole filter can then be used as the easing function \(g\), and then \(f = (x \circ \sigma^{-1} \circ g)(t)\) can be used in algorithm 2.

8.3 Animating changes to a text document
In this section we discuss the problem of animating the history of a text document in a manner similar to the Diffamation system [10]. An important contrast between our technique and Diffamation is the behavior of the revision selector: in Diffamation, the revision selector is treated as a playback slider for the entire animation, whereas in ours, the animation is dynamically generated using the selected revision as a target. As a consequence, dragging the selector quickly in Diffamation produces a sped-up animation, and so a separate interaction technique
is used to compare distance revisions, whereas with our technique, moving the slider quickly does not cause objects on screen to move excessively fast. A similar tool is Glimpse [17], which animates back and forth between markup code and the rendered document.

Let the text document be modeled mathematically as a sequence $c_1, c_2, \ldots, c_t$ of characters that exist in the document’s history. Let $p_i(t)$ be true if $c_i$ is present in the document at time $t$, and let $x_i(t)$ give the on-screen position of $c_i$ at time $t$. The notation $x_i$ denotes the vector at index $i$ in a sequence of vectors, not the $i^{th}$ element of vector $x$. The document can then be drawn at time $t$ by drawing each $c_i$ for which $p_i(t)$ is true at position $x_i(t)$. Consider an interface in which the user can control the revision displayed using a slider. We seek an animation technique that allows the user to easily see where changes have occurred.

There are three types of abrupt changes that occur in the animation-free visualization:

1. Characters appear abruptly at times when $p_i(t)$ changes from false to true.
2. Characters move abruptly at times when $x_i(t)$ changes abruptly.
3. Characters disappear abruptly at times when $p_i(t)$ changes from true to false.

Suppose we transition characters in by gradually enlarging them from an infinitesimal size to their normal size. At they are enlarged, their color slowly changes from blue to black to show that they are being added. When the position of a character changes, the character should move to their new position. And when they are deleted, they should slowly shrink to the infinitesimal size before disappearing completely, with color changing to red to show that they are being deleted.

We first construct the signal $\psi_i(t)$, which is one when $p_i(t)$ is true and zero otherwise. Next, we define the signal $\psi_i$ as a smoothed version of $\psi_i(t)$. Now $\psi_i$ can be used directly to control the size of the character. Similarly, let $x_i$ be defined as a smooth version of $x_i$, and can be used to directly control the position of the character. Let $\chi_i$ be defined as $\psi_i(t)$ filtered by a system having a step response that is smooth and compactly supported.

We have not yet defined what the position $x_i(t)$ of a character should be in the case where it hasn’t yet appeared in the document or if it has been removed from the document. To specify this, we first define a total ordering on all characters in the document. We start by defining a partial ordering between characters in a document. We assume that characters can be added and deleted but not moved or reinserted. We say that $c_i < c_j$ if there exists some time $t$ such that $p_i(t)$ and $p_j(t)$ are both true and $c_j$ appears earlier in the document than $c_i$ at time $t$. We now extend this partial order to a total order. If the ordering of $c_i$ and $c_j$ is left unspecified by the partial order, then the set of times for which $p_i(t)$ is true must not overlap the set of times for which $p_j(t)$ is true, otherwise the partial order would be defined. Therefore, we let $c_i < c_j$ if $c_i$ is inserted at an earlier time than $c_j$, and $c_j < c_i$ if $c_j$ is inserted at an earlier time than $c_i$.

We can now define the position $x_i(t)$ of a character that is not currently visible. Let $c_j$ be the least (in the sense of the total ordering) currently visible character such that $c_j < c_i$. Then $x_i(t)$ is located at position of $c_j$. Assume that the document contains an invisible end-of-file character that is always present, and therefore such a $c_j$ always exists. The total order of characters in an example document is shown in figure 5. The signal processing diagram for processing a character is shown in figure 6.

## 9 Discussion

We have now introduced two low-level building blocks for animations, FIR animations and IIR animations, and a way to combine these low-level building blocks together to produce high-level functionality, the system diagram. Using this approach, animations can be produced that are smooth, even in the presence of interruptions.

Algorithms for implementing discrete-time FIR and IIR filters are well-known, and can be found in most textbooks on digital signal processing, e.g. [37]. In particular, both algorithm 4 and algorithm 3 are well-known algorithms. The implementation given by algorithm 2 is a simple implementation of a FIR filter applied to a step-function input, and is presumably the same algorithm that Apple has used to implement additive animation [29]. One disadvantage of FIR animations when compared with IIR animations are that FIR filters require space and computational time proportional to the duration of the animation. IIR animations, on the other hand, can be made arbitrarily long with constant space and time complexity.

For an existing code base, the easiest way to switch from simple transitions to LTI transitions is to change the algorithm to use algorithm 2. In practice, this means maintaining a list of in-flight transitions rather than a single transition.

Since the frame-rate is usually constant in visualization systems, we recommend using a discrete-time system whenever possible. However, it is often easier to think in terms of continuous time, as this allows a system to be designed without paying attention to the specific frame rate. In other words, the system should have some essential behavior that should exist regardless of the rate at which it is sliced into frames. Therefore, we recommend designing the system in continuous time and then transforming the continuous-time system into a discretized version. For a bandlimited FIR system, this can be done by the impulse invariant method, which simply samples the impulse response. For an IIR system, this can be done using several means. The impulse-invariant method converts the differential equations to difference equations that trace the same path but at discrete points in time. The disadvantage of the impulse-invariant method is aliasing, although this will not be especially problematic for filters used for animations, as they will have low-pass characteristics. The bilinear transform is another technique for transforming a discrete-time system into a continuous-time system. The bilinear transform does not introduce aliasing, but it does warp the frequency response of the continuous-domain filter. The matched z-transform is still another technique for converting from a continuous-time system to a discrete-time system. All of these techniques are covered in many digital signal processing textbooks, e.g. [38].

Another advantage of our technique is what might be called an emergent interaction—an interaction that arises serendipitously from the combination of independent features. By combining dynamic queries with LTI animation, the interface affords a new interaction technique: the user can, by wiggling the query region rapidly over a range of positions, see the average result for queries within this range of positions.

This paper has focused on linear, time-invariant animations. There is also a rich theory for linear, time-varying systems, although such systems tend to be more difficult to analyze, as commonly-used tools for analyzing LTI systems, such as the Fourier, Laplace, and z transforms, no longer apply.

For FIR transitions, the choice of easing function makes a large impact on the aesthetics of the motion. If symmetric motion is desired, B-spline window functions [41] provide a natural solution, and are equivalent to box filters connected in series. If fast-in slow-out motion is desired, as we have recommended, a series combination of one-pole filters, as described in section 8.2, is an attractive alternative.

## 10 Related work

Many systems exist that use animation to help users comprehend changes [7, 25, 39, 5, 33, 46, 4, 32, 9, 40, 35]. The problem we consider in particular, that of interrupted transitions, has been previously discussed. The W3 CSS animation specification [22] mentions interrupted animations and recommends reversing the animation for the specific case where the new value for the transition equals the old value of the current transition. This technique results in an animation with a velocity discontinuity and symmetry about the time $t_1$ at which the animation is reversed. For this case, our technique produces an animation with no velocity discontinuities and symmetry about $t_1 + d/2$. Note that if we want an asymmetric transition for a button such that the transition from the normal state to the hover state and the transition from the hover state to the normal state are reverses of one another, we can simply use a symmetric easing function and then a nonlinear function to the tween signal. For example, squaring
could be used to solve the problem of animating these objects from objects are modeled as propulsion vehicles, then optimal control theory results that may be applied to animation. For example, if on-screen same location as deleted text should be ordered before the deleted text. at the same time. To define the total ordering, shown above as the ordering of the columns, we must further specify that new text inserted in the that exists in the document from some start time to some end time. A natural partial order exists between characters that exist in the document in which complex animations arise from interactions between simple particles, e.g. dust and magnets [47], force directed graph layout [21], and boids [34]. In game design, motion blending [20, 26, 28] techniques are often used to create new animation clips by blending together two or more preexisting animation clips. All of these techniques do not attempt to solve the problem that we consider, which is animating an attribute to a target value that may change over time. The field of signal processing has produced rich body of knowledge about filter design, and there are many excellent references available, e.g. [31, 37, 36]. Note that our FIR implementation differs from the usual implementation of digital filters. Digital filters usually operate on a discrete, bandlimited, uniformly sampled signal. Our filters instead operate directly on the continuous attribute signal. We don’t expect the attribute signal to oscillate, so aliasing isn’t as problematic as it is in classical digital signal processing. We also don’t use an ideal brickwall lowpass filter, as we wish to avoid the ringing associated with the sinc function. The field of control theory [30] has also produced a rich body of results that may be applied to animation. For example, if on-screen objects are modeled as propulsion vehicles, then optimal control theory could be used to solve the problem of animating these objects from one point to another with the least amount of fuel used. While such animations may be useful for visualization, the difficulty of solving optimal control problems makes this approach unappealing in practice. A related field in robotics is motion planning [27], which concerns the problem of controlling navigating a robot from one location to another. Like control theory, motion planning approaches are likely needlessly complex for the problem of animation. Our technique is somewhat related to kernel smoothing [23]. A key difference is that kernel smoothing deals with discrete points, whereas our method deals with a continuous step signal. Also, in kernel smoothing, the output at a given point in time depends on inputs occurring in the future, thus giving a non-causal filter and precluding a realtime implementation. Moreover, some kernels commonly used in kernel smoothing (e.g. the Gaussian kernel) are everywhere nonzero and thus cannot be used for LTI transitions (although they could, of course, be truncated in practice). Functional reactive programming [13, 14, 18, 15, 6] offers a programming interface in which signals are manipulated as first-class values. Functional reactive programming systems are an area of active research, and could provide a promising method of allowing LTI animation to be effortlessly integrated at any step in a visualization pipeline.

### 11 Conclusion

This paper has presented LTI animation, an animation model, based on well-known techniques from signal processing, that responds well to rapid changes in attribute values. We have shown that for cases where attribute values do not change during transitions, our method is equivalent to the transition model, but that our approach handles rapidly-changing signals gracefully and without the bizarre effects of the transition model. The primary insight behind our method is that animations should not be constructed to smoothly connect one state to another: rather, animations should take into account the history of states and the current target state. From this perspective, animations can be seen as a transformation from an input signal that describes the current target over time to a smoothed output signal. In particular, an easing function can be seen as a step response of a linear, time-invariant filter that describes this transformation.
We have also introduced several algorithms for computing the LTI animation convolution. The FIR step method is straightforward and is somewhat similar to FIR filter implementations. It is likely the best choice for most real-world usages. We note that these techniques are largely interchangeable: in practice, the impulse response for an FIR filter can be truncated and then used as the impulse response for a FIR system. These systems will then produce the same results modulo truncation error, which can be made arbitrarily small.

Future work might consider an extension of this technique to staggered transitions; however, there is evidence [11] that staggered transitions provide little perceptual benefit. Future work might also consider extensions to zooming and panning animations [42, 43] for which considerate care is required to produce efficient trajectories.

Acknowledgments

This research was partially supported by NSF grant IIS-1527453.

References

[1] C. Elliott and P. Hudak. Functional reactive animation. In ACM SIGPLAN Notices, 2011.
[2] E. Czaplicki and S. Chong. Asynchronous functional reactive programming. In Proceedings of the ACM SIGPLAN conference on Programming language design and implementation, pages 233–243. ACM, 2012.
[3] T. Hastie, R. Tibshirani, and J. Friedman. The Elements of Statistical Learning: Data Mining, Inference, and Prediction, Second Edition (Springer Series in Statistics). Springer, 2011.
[4] J. Heer, M. Bostock, and V. Ogievetsky. A tour through the visualization zoo. Commun. ACM, 53(6):59–67, 2010.
[5] J. Heer and G. G. Robertson. Animated transitions in statistical data graphics. Visualization and Computer Graphics, IEEE Transactions on, 13(6):1240–1247, 2007.
[6] L. Kovar and M. Gleicher. Flexible automatic motion blending with registration curves. In Proceedings of the 2003 ACM SIGGRAPH/Eurographics symposium on Computer animation, pages 214–224. Eurographics Association, 2003.
[7] J.-C. Latombe. Robot motion planning, volume 124. Springer Science & Business Media, 2012.
[8] T. Mukai and S. Kuriyama. Geostatistical motion interpolation. In ACM Transactions on Graphics (TOG), volume 24, pages 1062–1070. ACM, 2005.
[9] M. Neuburg. Programming iOS 8. O’Reilly Media, Inc., 2014.
[10] N. Nise. Control Systems Engineering. Wiley, 2014.
[11] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab. Signals and systems, volume 2. Prentice-Hall Englewood Cliffs, NJ, 1983.
[12] C. Plaisant, J. Grosjean, and B. B. Bederson. Spacetree: Supporting exploration in large node link tree, design evolution and empirical evaluation. In Information Visualization, 2002. INFOVIS 2002. IEEE Symposium on, pages 57–64. IEEE, 2002.
[13] H. C. Purchase, E. H. Hoggan, and C. Görg. How important is the “mental map”?–an empirical investigation of a dynamic graph layout algorithm. In International Symposium on Graph Drawing, pages 184–195. Springer, 2006.
[14] C. W. Reynolds. Flocks, herds and schools: A distributed behavioral model. ACM SIGGRAPH computer graphics, 21(4):25–34, 1987.
[15] H. Rosling. Gapminder. GapMinder Foundation http://www.gapminder.org, page 91, 2009.
[16] W. J. Rugh. Linear system theory, volume 2. prentice hall Upper Saddle River, NJ, 1996.
[17] J. O. Smith. Introduction to Digital Filters with Audio Applications. W3K Publishing, http://www.w3k.org/books/, 2007.
[18] J. O. Smith. Physical audio signal processing: For virtual musical instruments and audio effects. W3K Publishing, 2010.
[19] J. T. Stasko. The path-transition paradigm: A practical methodology for adding animation to program interfaces. Journal of Visual Languages & Computing, 1(3):213–236, 1990.
[20] B. H. Thomas and P. Calder. Applying cartoon animation techniques to graphical user interfaces. ACM Transactions on Computer-Human Interaction (TOCHI), 8(3):198–222, 2001.
[21] N. Elmqvist, A. V. Moere, H.-C. Jetter, D. Cernea, H. Reiterer, and T. Jankun-Kelly. Fluid interaction for information visualization. Information Visualization, page 1473871611413180, 2011.
variate information visualization using a magnet metaphor. *Information Visualization*, 4(4):239–256, 2005.