Time dependent quantum generators for the Galilei group

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In 1995 Doebner and Mann introduced an approach to the ray representations of the Galilei group in (1 + 1)-dimensions, giving rise to quantum generators with an explicit dependence on time. Recently (2004) Wawrzycki proposed a generalization of Bargmann’s theory: in his paper he introduce phase exponents that are explicitly dependent by 4-space point. In order to find applications of such generalization, we extend the approach of Doebner and Mann to higher dimensions: as a result, we determine the generators of the ray representation in (2 + 1) and (3 + 1) dimensions. The differences of the outcome formal apparatus with respect to the smaller dimension case are established.

I. INTRODUCTION

In 2004 Wawrzycki proposes a generalization of Bargmann’s theory, introducing the phase exponents that explicitly depends on time, for non-relativistic groups, and phase exponents that explicitly depends on 4-space point, for relativistic groups.

On the other hand, in 1995, Doebner and Mann, studying Galilei group in (1 + 1) dimensions, calculated the generators group’s representations that explicitly depends on time, while for Galilei group in (2 + 1) dimensions they founded the generators group’s representations just calculated by Bose and Grigore.

At this point, in order to generalize Doebner and Mann work about higher dimensions, in Section III we calculate the generators of the Galilei group’s representations and we show that they explicitly depends on time. These representations are valid for (2+1) and (3+1) dimensions. At the end, from these representations, we calculate the ray representations of Galilei group, and we show that they explicitly depends on time.

First of all, we introduce the ray representations of a given Lie group (Section I A), and we briefly remind the application of the Bargmann’s theory to Galilei group (Section II). In the Section III we find the time dependence of the ray representations of galilean group in (3+1) and (2+1) dimensions, with phase exponents that depend on time, and we propose a physical interpretation either for it, either for Bargmann’s phase exponent.

A. Ray representations

The vector ray notion, introduced by Weyl, can be extended also to the operators, in particular to unitary operators. According to Wigner’s theorem, every symmetry’s transformation $T_r$ can be represented by an unitary (or anti-unitary) operator, which is unique up at a phase factor, id est $T_r$ can be represented by an unitary operator ray defined by:

$$U_r = \{e^{i\theta} U_r, \theta \in \mathbb{R}\}. \quad (1)$$

So, $U_r$ is the ray containing the identity operator and $U_r^{-1}$ is the inverse of $U_r$, i.e. the ray containing the operators $U_r^{-1}$ inverse of $U_r \in U_r$,

$$U_r^{-1} = \{V_r, \text{ such that } V_r = U_r^{-1}, U_r \in U_r\}$$

then $U_r \cdot U_r^{-1} = U_r^{-1} \cdot U_r = U_e$.

Now, let $G$ be a symmetry group with elements $r, s, t, \ldots$; then, for a given choice of the unitary operator $U_r \in U_r$ representing the elements of $G$, in general the following composition law holds:

$$U_r U_s = \omega(r, s) U_{rs}$$

that we can rewrite for the operator rays as:

$$U_r \cdot U_s = U_{rs}$$

where $\omega(r, s)$ is a factor of modulus 1 ($|\omega(r, s)| = 1$), $r, s \in \mathcal{N}_0 \subset G$, and $\mathcal{N}_0$ is a neighbourhood of $G$ identity.

The correspondence $r \mapsto U_r$ realizes a ray representation (or projective representation) of $G$. It is equivalent to an usual unitary representation of $G$ if a correspondence $r \mapsto U_r \in U_r$ exists such that $\omega(r, s) \equiv 1$, $\forall r, s \in G$.

In order to classify the ray representations of a given Lie group $G$, it is sufficient classify the phase factor equivalence classes; indeed, for a different admissible set of representatives $U'_{rs}$, we obtain

$$U'_{rs} \cdot U'_{rs} = \omega(r, s) U'_{rs}$$

where

$$\omega'(r, s) = \omega(r, s) \frac{\phi(r) \phi(s)}{\phi(rs)}. \quad (2)$$

But it is more advantageous to replace (local) factors with (local) exponents by setting $\omega(r, s) = e^{i\xi(r, s)}$. So, a phase exponent of a group $G$ is a real valued continuous
function $\xi(r,s)$ which is defined for all $r, s$ in $G$ and which satisfies the relations:

$$\xi(e,e) = 0 \quad (3a)$$

$$\xi(r,s) + \xi(rs,t) = \xi(s,t) + \xi(r, st) \quad (3b)$$

For every phase exponent defined on the group (or on a neighbourhood), a function called infinitesimal exponent $\Xi$ defined on the Lie algebra of the group exists, that is in one-to-one linear correspondence with phase exponent:

$$\Xi(a,b) = \lim_{\tau \to 0} \tau^{-2} \left( \xi((\tau a)(\tau b), (\tau a)^{-1}(\tau b)^{-1}) + \xi(\tau a, \tau b) + \xi((\tau a)^{-1}, (\tau b)^{-1}) \right). \quad (4)$$

II. THE GALILEI’S GROUP

Now, we remind the study about the Galilei’s group, its Lie algebra and ray representations in $(3+1)$-dimensions (Section II A), $(2+1)$-dimensions (Section II B) and $(1+1)$-dimensions (Section II C). In the last case we show how Doebner and Mami calculate the generators of Galilean group, which turn out to depend on time, and we propose the Galilei’s group’s ray representations that depend on time.

A. The Galilei’s group in $(3+1)$-dimensions

The Galilean group is constituted by all space-time transformations from an inertial reference frame to another one. The most general Galilean transformation of the Galilean group $G$ is:

$$x' = Wx + vt + u \quad (5a)$$

$$t' = t + \eta \quad (5b)$$

where $x'$, $x$ are spatial vectors, $v$ is the relative velocity, $u$ is a space translation, $t$ is time and $\eta$ a time translation, with $W$ an orthogonal transformation (e.g. rotation).

In order to classify the ray representations of Galilei’s group, we can represent, following Bargmann, the generic element $r$ of the Galilean group $G$ as:

$$r = (W_r, \eta_r, v_r, u_r). \quad (6)$$

So, the group multiplication is given by

$$rs = (W_r, \eta_r, v_r, u_r) \cdot (W_s, \eta_s, v_s, u_s) = (W_r W_s, \eta_r + \eta_s, W_r v_s + v_r, W_r u_s + u_r + \eta_s v_r). \quad (6)$$

Now, to classify the ray representations of $G$, we first must describe the algebra of Galilei’s group: algebra standard basis is constituted by $a_{ij}$, anti-symmetric $3 \times 3$ matrix where only elements $ij$ are non-null; $b_i$, the translations generators; $d_i$, the pure galilean transformations generator ($1 \leq i \leq n, 1 \leq j \leq n$, where $n$ is the space dimension); $f$, the time translations generators. The generators algebra is given by:

$$[a_{ij}, a_{kl}] = \delta_{jk}a_{il} - \delta_{ik}a_{jl} + \delta_{il}a_{jk} - \delta_{jl}a_{ik} \quad (7a)$$

$$[a_{ij}, b_i] = \delta_{jk}b_i - \delta_{ik}b_j \quad b_i, b_j = 0 \quad (7b)$$

$$[a_{ij}, d_k] = \delta_{jk}d_i - \delta_{ik}d_j \quad [d_i, d_j] = 0; \quad [d_i, b_j] = 0 \quad (7c)$$

$$[a_{ij}, f] = 0; \quad [b_i, f] = 0; \quad [d_k, f] = b_k \quad (7d)$$

At this point we can calculate all infinitesimal exponent of the Galilei’s group. The only non-null exponent is:

$$\Xi(b_i, d_k) = -\Xi(d_k, b_i) = \gamma \delta_{ik}. \quad (8)$$

The corresponding phase exponent $\xi$ is a multiple of the function $\xi_0$:

$$\xi(r,s) = \gamma \xi_0(r,s) = \frac{1}{2} \gamma (\langle u_r | W_r v_s \rangle + \langle v_r | W_r u_s \rangle + \eta_r \langle v_r | W_r v_s \rangle + \eta_s \langle u_r | W_r u_s \rangle), \quad (9)$$

by a multiplicative factor $\gamma$, which is interpreted as the mass of a free particle.

At this point Bargmann determined the Galilean ray representations:

$$\phi^i(p) = U_r \phi(r^{-1} p) =$$

$$= e^{-i(\langle p | u_r \rangle - \frac{1}{2} \gamma \langle v_r | v \rangle - \gamma \langle u_r | v \rangle - \gamma \langle u_r | u \rangle)} \phi(r^{-1} p) \quad (10)$$

where the functions $\phi(p)$ are the wave functions in the Heisenberg representation (\phi(p) \in \mathcal{L}_2(\mathbb{R}^2, p)), and $p$ being the linear momentum.

B. The Galilei’s group in $(2+1)$-dimensions

Now we can study the Galilean group in lower dimensions. By the results obtained by Bargmann about the pseudo-orthogonal groups, we must expect the emergence of new phase exponents in $(2+1)$ dimensions. Indeed, in this case, there are two non-equivalent non-trivial phase exponents besides $\xi_0$ in (9):

$$\xi_1(r,s) = \frac{1}{2} (\langle u_r | W_r v_s \rangle \quad (11a)$$

$$\xi_2(r,s) = \theta_r \eta_s - \theta_s \eta_r \quad (11b)$$

where $\eta_r v_s - v_r u_s$, with $u, v$ two-dimensional vectors.

All inequivalence classes are multiple of (11) with multiplicative factors $\lambda$ and $S$ respectively. In correspondence with different values of the three multiplicative factors $\gamma, \lambda, S$, we can have non-equivalent ray representations of the Galilean group. In particular, we are interested in the ray representations with $S = 0$; so we have the two following cases:

$$(U(W, \eta, v, u, f)(p) = e^{i(\langle u | p \rangle + \frac{1}{2} \langle u | v \rangle + \frac{1}{2} \langle v | p \rangle - \frac{1}{2} \gamma \langle v | u \rangle + s \theta)} f(W^{-1}(p + \gamma v)) \quad (12)$$

$$(U(W, \eta, v, u, f)(p) = e^{i(\langle u | p \rangle + \frac{1}{2} \langle u | v \rangle + \frac{1}{2} \langle v | p \rangle + \rangle - \frac{1}{2} \gamma \langle v | u \rangle + s \theta)} f(W^{-1}(p + \gamma v)) \quad (13)$$
where (12) corresponds to a localizable Schrödinger system for which the boost representation is not abelian, and (13) describes a Schrödinger non-relativistic particle, where \(s(h)\) denotes an irreducible representation of the sub-group \(\{ (W, 0, v, 0) \}\), and the functions \( f(p) \in L_2(X_r, p) \), where \(X_r = \{(p_0, p)\}, \) with \( \langle p | p \rangle = t^2 \), is an orbit.

### C. The Galilei’s group in (1 + 1)-dimensions

According to Doebner and Mann\(^2\), we write the generic element of the Galilean group in (1 + 1) dimensions as

\[
\mathbf{r} = (u_r, v_r, \eta_r)
\]

where \(u_r\) is a space translation, \(v_r\) a velocity translation, \(\eta_r\) a time translation. The corresponding phase exponent is:

\[
\xi_\theta(r, s) = \frac{\gamma_1}{2}(a_r v_s - a_s v_r + \eta_r v_r v_s) + \\
+ \frac{\gamma_2}{2}(u_r \eta_s - u_s \eta_r - \eta_r \eta_s v_r)
\]

(14)

with \(a_{1,2}\) are real numbers.

To introduce a time dependence in Hilbert space we use a kind of Heisenberg picture. For any self-adjoint operator \(R(X)\) we set:

\[
\frac{d}{dt} R_t(X) = K R_t([H, X])
\]

(15)

with initial condition

\[
R_{t=0}(X) = R(X)
\]

(where \(K\) is a complex constant, \(H\) time translations generator, \(R_t(X)\) is the time depending representation of the generator \(X\) and \(\frac{dX}{dt} = 0\)) we can calculate generators of Galilean group:

\[
R_t(H) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + f x + V_0,
\]

\[
R_t(P) = i\hbar \frac{\partial}{\partial x} - ft,
\]

\[
R_t(N) = mx - i\hbar \frac{\partial}{\partial x} - \frac{1}{2} ft^2
\]

(16)

where \(H\) is the time translations generator, \(P\) the space translations generator, \(N\) the boost translation generator, \(\hbar\) the Planck constant, \(m = \hbar_1\) the mass particle, \(f = a_2\) an external force, \(V_0 = \frac{\gamma_1}{2}\).

### III. EXTENSION OF DOEBNER AND MANN CALCULATION

Now we extend the Doebner and Mann approach to determine the time depending Galilean generators in (2 + 1) and (3 + 1) dimensions; by using these generators, we derive the unitary time depending representations for the Galilean group. As a result, besides (9), we find ray representations with a phase exponent which explicitly depends on time.

Now, applying (15) to generators of Galilei’s group in (2 + 1)- and (3 + 1)-dimensions\(^2,3\), we find that \(R_t(H) = R(H), R_t(P_i) = R(P_i), R_t(M) = R(M)\), while for \(N_i\) generators we find:

\[
R_t(N_1) = -ip_x t + \gamma \frac{\partial}{\partial p_1} - i\frac{\lambda}{2\gamma^2} p_2,
\]

\[
R_t(N_2) = -ip_y t + \gamma \frac{\partial}{\partial p_2} - i\frac{\lambda}{2\gamma^2} p_1
\]

for representation (12), and we find:

\[
R_t(N_3) = -ip_z t + \gamma \frac{\partial}{\partial p_3}
\]

for representation (13).

Now, for \(v = (1, 0, 0) \in G\), the corresponding unitary operator time depending will be

\[
(1 - i \langle p | v \rangle t - i\frac{\lambda}{2\gamma^2} (v \wedge p) + \gamma v_1 \frac{\partial}{\partial p_1} + \gamma v_2 \frac{\partial}{\partial p_2}) f(p) \approx
\]

\[
(1 - i \langle p | v \rangle t - i\frac{\lambda}{2\gamma^2} (v \wedge p)) \cdot
\]

\[
(1 + \gamma v_1 \frac{\partial}{\partial p_1} + \gamma \frac{\partial}{\partial p_2}) f(p) \approx
\]

\[
e^{-i\langle p | v \rangle t - i\frac{\lambda}{2\gamma^2} (v \wedge p)} f(p) + \gamma v = (U_t(v)f)(p)
\]

for representation (12), and

\[
(U_t(v)f)(p) = e^{-i\langle p | v \rangle t} f(p) + \gamma v
\]

for representation (13). So the representation of Galilean group, time depending, in (2 + 1) (for \(\gamma \neq 0, \lambda \neq 0, S = 0\), and for \(\gamma \neq 0, \lambda = S = 0\)) and (3 + 1) dimensions is

\[
(U_t(r)f)(p) = e^{-i\langle p | v_r \rangle t} (U(r)f)(p).
\]

(17)

So, \(\forall r \in G\), the corresponding \(U_t(r)\) time depending unitary operator is given by the unitary operator \(U(r)\) of the ray representation up to a time depending phase factor, \(e^{-i\langle p | v_r \rangle t}\).

Now we can see whether the introduction of the phase \(e^{-i\langle p | v \rangle t}\) in the ray representation of Galilean group generates a further projective phase factor. To this purpose we calculate:

\[
((U_t(r)U_t(s))f)(p) = e^{-i\langle p | v_r \rangle t} U_t(s)((U(r)f)(p))
\]

\[
= e^{-i\langle p | v_r \rangle t} e^{-i\langle W_r^{-1}(p + \gamma v_r) | v_r \rangle t} \omega(r, s) (U(r s)f)(p).
\]

On the other hand

\[
(U_t(r s)f)(p) = e^{-i\langle p | v_r \rangle t} (U(r s)f)(p) =
\]

\[
e^{-i\langle p | v_r \rangle t} e^{-i\langle p | W_r v_r \rangle t} (U(r s)f)(p).
\]

Comparing these two equations we can find that

\[
((U_t(r)U_t(s))f)(p) = \phi(r, s, t) \omega(r, s) (U_t(r s)f)(p)
\]

(18a)
where \( \omega(r, s) \) is the usual projective phase factor, while \( \phi(r, s, t) \) is defined by:

\[
\phi(r, s, t) = e^{-i\gamma(v_r | W_r v_s) t} = e^{i\xi(r, s)}
\]

(18b)

with

\[
\xi(r, s) = -\gamma \langle v_r | W_r v_s \rangle t = -\gamma \xi_0, t(r, s)
\]

(18c)

is a bilinear continuous function in \( r, s \) coordinates which satisfies equations (3), like every projective phase exponent.

### A. Physical interpretation of the phase exponents

The phase exponent (9) of galilean group

\[
\xi(r, s) = -\gamma \xi_0(r, s) = \frac{1}{2} \left( \langle u_r | W_r v_s \rangle - \langle v_r | W_r u_s \rangle + \eta_s \langle v_r | W_r v_s \rangle \right)
\]

has the physical dimensions of a action, and so it can be interpreted like the action of the particle in the frame \( rs \). Indeed, set \( r = (1, 0, 0, u), s = (0, 0, v, 0) \), then

\[
\xi(r, s) = \frac{1}{2} (u | v),
\]

that it has the dimension of an action, and \( rs = (1, 0, v, u) \) is the Galilean transformation that relates an inertial frame system to another with relative velocity \( v \) and origin of axes translated by \( u \).

About the phase exponent (18c), we propose the following interpretation.

Let be \( W_r v_s \) velocity of \( \Sigma_s \), \( \langle v_r | W_r v_s \rangle \) is the velocity of \( \Sigma_s \) along the motion direction of \( \Sigma_r \). So \( \xi(r, s) \) is the contribution of the two coordinate systems \( \Sigma_r, \Sigma_s \) to the total action of the particle.

For example: let be \( r = (1, 0, v, 0), s = (1, 0, v_s, 0) \) two elements of galilean group. Then

\[
(U_r(r)U_i(s)f)(p) = e^{-i\gamma(v_r | v_s) t} (U_i(rs)f)(p)
\]

And the contribution of the two coordinate systems to the total action of the particle is:

\[-\gamma \langle v_r | v_s \rangle t.\]

**In conclusion:** In this work, we briefly remind the theory of phase exponents of ray representation of Lie groups and the application on galilean group in \( (3 + 1) \), \( (2 + 1) \) and \( (1 + 1) \) dimensions. Finally, we find the time dependence of the ray representations of galilean group (17): the action of this representation on physical state \( f(p) \) is given by:

\[
f'(p, t) = e^{-i(p | v_r) t} \left( U(r)f \right)(p)
\]

(19)

For example, if we use (17) on (10) we obtain:

\[
\phi'(p, t) = e^{-i(p | v_r) t} e^{-i(u_r | u_s) - \frac{H}{2} (p | p) + i \gamma (u_r | v_r) - \gamma (u_s | v_s)} \phi(r^{-1} p)
\]

(20)

The phase exponents in (20) represents the total action of the particle in \( \Sigma_{rs} \) frame.

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10. Let \( U_r \) be a continuous ray representations of a group \( G \). For all \( r \) in a suitably chosen neighborhood \( \mathcal{U} \) of the identity \( e \) of \( G \), one may select a strongly continuous set of representatives \( U_r \in \mathcal{U} \). A set of such representations operator will be called an admissible set of representatives.
11. From the study of the full Galilean group in \( (3 + 1) \) dimensions, it is possible to obtain the Bargmann’s super-selection rules.a
12. Let be \( \psi(x, t) \) the solutions of Schrödinger equation, then:

\[
\psi(x, t) \sim (2\pi)^{-\frac{3}{2}} \int \phi(p) e^{-i\frac{\gamma}{2} (p | p) t - i x \cdot p} d^3 p
\]

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14. Where \( a_3 \) is a real number connected to the Casimir element

\[
C_3 = 2H Z_1 - 2K Z_2 - P^2
\]

where \( H, K, P \) are time translations, nonrelativistic boosts and space translations generators, and \( Z_1 \) and \( Z_2 \) are the two central elements.

15. The infinitesimal generators of \( \mathfrak{u}(2) \) are:

\[
H = \frac{1}{2\gamma} (p_1^2 + p_2^2), \quad N_1 = \gamma \frac{\partial}{\partial p_1} + \frac{\lambda}{2\gamma} p_2
\]

(12)

\[
M = is + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2}, \quad N_2 = \gamma \frac{\partial}{\partial p_2} - \frac{\lambda}{2\gamma} p_1
\]

\[
P_i = p_i (i = 1, 2)
\]

The infinitesimal generators of \( \mathfrak{u}(3) \) are:

\[
H = \frac{1}{2\gamma} (p_1^2 + p_2^2), \quad P_i = p_i, \quad M = is + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2}, \quad N_1 = \gamma \frac{\partial}{\partial p_1} (i = 1, 2)
\]

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