ON THE $\beta$-DISTORTION OF COUNTABLY BRANCHING HYPERBOLIC TREES

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Abstract. In this note we show that the distortion incurred by a bi-Lipschitz embedding of the countably branching hyperbolic tree of height $N$ into a Banach space admitting a norm satisfying Rolewicz property ($\beta$) with power type $p > 1$ is at least of the order of $\log(N)^{1/p}$. An application of our result gives a quantitative version of the non-embeddability of countably branching hyperbolic trees into reflexive Banach spaces admitting an equivalent asymptotically uniformly smooth norm and an equivalent asymptotically uniformly convex norm from $\ell_1$.

1. Introduction

Let us first introduce a few notation and definitions. For a positive integer $N$, we denote $T_N = \bigcup_{n=0}^N \mathbb{N}$, where $\mathbb{N}^0 := \{0\}$. Then $T_\infty = \bigcup_{N=1}^\infty T_N$ is the set of all finite sequences of positive integers. In other words $T_N$ (resp. $T_\infty$) is the countably branching rooted tree of height $N$ (resp. infinite height). For $s \in T_\infty$, we denote by $|s|$ the length of $s$. There is a natural ordering on $T_\infty$ defined by $s \leq t$ if $t$ extends $s$. If $s \leq t$, we will say that $s$ is an ancestor of $t$. If $s \leq t$ and $|t| = |s| + 1$, we will say that $s$ is the predecessor of $t$ and $t$ is a successor of $s$ and we will denote $s = t^-$.

Then we equip $T_\infty$, and by restriction every $T_N$, with the hyperbolic distance $\rho$, which is defined as follows. Let $s$ and $s'$ be two elements of $T_\infty$ and let $u \in T_\infty$ be their greatest common ancestor. We set

$$\rho(s, s') = |s| + |s'| - 2|u| = \rho(s, u) + \rho(s', u).$$

For any sequence $(x_n)_{n \geq 1}$ in a Banach space $X$ let $\text{sep}[(x_n)_{n \geq 1}] := \inf\{\|x_m - x_n\| : m \neq n; n, m \geq 1\}$ and say that $(x_n)_{n \geq 1}$ is $\epsilon$-separated if $\text{sep}[(x_n)_{n \geq 1}] \geq \epsilon$.

According to Kutzarova [6] the norm of a Banach space satisfies Rolewicz property ($\beta$) if and only if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every $x \in B_X$ and every $\epsilon$-separated sequence $(y_n)_{n \geq 1} \in B_X$ there exists some $n_0 \in \mathbb{N}$ such that $\|x + y_{n_0}\| \leq 1 - \delta(\epsilon)$. It is convenient to introduce the ($\beta$)-modulus as follows

$$\overline{\beta}_X(t) := \inf \left\{ \frac{\|x - y_n\|}{2} : n \geq 1 \right\} \quad (y_n)_{n \geq 1} \in B_X : \text{sep}[(x_n)_{n \geq 1}] \geq t; x \in B_X \right\}. $$

The norm of a Banach space satisfies Rolewicz property ($\beta$) if and only if $\overline{\beta}_X(t) > 0$ for every $t > 0$. When $\overline{\beta}_X(t) \geq ct^p$ for some universal constant $c > 0$ and some exponent $p \in (1, \infty)$ one says that the norm satisfies Rolewicz property ($\beta$) with power type $p$, or that the norm has a ($\beta$)-modulus of power type $p$.

The distortion of a bi-Lipschitz embedding $f : (\mathcal{M}, d) \to (Y, \|\cdot\|)$ is defined as

$$\text{dist}(f) := \text{Lip}(f)\text{Lip}(f^{-1}) = \sup_{x \neq y \in \mathcal{M}} \frac{\|f(x) - f(y)\|}{d(x, y)} \sup_{x \neq y \in \mathcal{M}} \frac{d(x, y)}{\|f(x) - f(y)\|}.$$
As usual we denote the $Y$-distortion of $\mathcal{M}$ by

$$c_Y(\mathcal{M}) := \inf \{ \text{dist}(f) \mid f : \mathcal{M} \to Y \}.$$ 

In this article we are concerned with embedding the countably branching trees into Banach spaces satisfying Rolewicz property ($\beta$). Let $p \in (1, \infty)$ and define

$$\mathcal{C}(\beta_p) := \{ Y \text{ has an equivalent norm with } (\beta)-\text{modulus of power type } p \}$$

and $\mathcal{C}(\beta) := \cup_{1 < p < \infty} \mathcal{C}(\beta_p)$. We want to study the asymptotic behavior of the ($\beta$)-distortion of the countably branching trees, namely $c_\beta(T_N) := \inf \{ c_Y(T_N) \mid Y \in \mathcal{C}(\beta) \}$. We will prove that if $Y \in \mathcal{C}(\beta_p)$ then $c_Y(T_N) \geq (\log(N))^{1/p}$ where as usual the symbol $\gtrsim$ is meant to hide a constant depending eventually on the geometry of the receiving space $Y$ but not on $N$. The proof is a combination of an asymptotic version of the tip contraction lemma from [4] (see also [8] for a similar argument), and of a self-improvement argument of Johnson and Schechtman [3] which was elegantly implemented in the case of binary trees by Kloeckner [4].

2. Embeddability into spaces with ($\beta$)-modulus of power type

We denote by $K_{1,\omega}$ the star graph with countably many branches, i.e., the bipartite graph that has a partition into exactly 2 classes, one consisting of a singleton called the center, the other one consisting of countably many vertices called the tips. In the sequel one will denote the center by $e$, fix a tip that we will denote by $o$, and fix a labeling $(k_i)_{i \geq 1}$ of the (countably many) remaining Tips. We this labeling in mind $K_{1,\omega}$ can be seen as a fork with countably many tips. As usual $K_{1,\omega}$ is equipped with the shortest path metric.

**Lemma 1** (Asymptotic tip contraction lemma). Let $Y$ be a Banach space whose norm satisfies Rolewicz property ($\beta$) with power type $p$ ($p > 1$), then there exists $\gamma := \gamma(Y) > 0$ such that for every non-contractive map $f : K_{1,\omega} \to Y$ there exists some $i_0 \in \mathbb{N}$ such that

$$\| f(o) - f(k_{i_0}) \| \leq 2 \left( \text{Lip}(f) - \frac{\gamma}{\text{Lip}(f)^{p-1}} \right).$$

**Proof.** One may assume after an appropriate translation that $f(o) = 0$. Fix some $\eta \in (0, \infty)$ to be chosen later and assume also that for every $i \in \mathbb{N}$ one has $\| f(k_i) \| \geq 2(\text{Lip}(f) - \eta)$. Then

$$\| f(e) \| = \| f(e) - f(k_i) + f(k_i) \| \geq 2(\text{Lip}(f) - \eta) - \text{Lip}(f)$$

$$\geq \text{Lip}(f) - 2\eta,$$

and

$$\| f(e) - f(k_i) \| \geq \| f(k_i) \| - \| f(e) \| \geq \| f(k_i) \| - \| f(e) - f(o) \|$$

$$\geq 2(\text{Lip}(f) - \eta) - \text{Lip}(f) = \text{Lip}(f) - 2\eta.$$

Let $v_i = \frac{\| f(e) \|}{\| f(k_i) - f(e) \|} (f(k_i) - f(e))$, then

$$\| f(e) + v_i - f(k_i) \| = \left\| \left( f(k_i) - f(e) \right) \frac{\| f(e) \| - \| f(k_i) - f(e) \|}{\| f(k_i) - f(e) \|} \right\|$$

$$\leq \| f(e) \| \leq \| f(k_i) - f(e) \|$$

$$\leq \text{Lip}(f) - (\text{Lip}(f) - 2\eta) = 2\eta$$

and
can be seen as countably many copies of $K$ boarding. Let $k \modulus \text{power type}$

**Proof.**

Let $x = \frac{f(e)}{\|f(e)\|}, y_i = \frac{v_i}{\|f(e)\|} = \frac{f(k_i) - f(e)}{\|f(k_i) - f(e)\|}$. Clearly $\|x\| = \|y_i\| = 1, \|\frac{x+y_i}{2}\| \leq 1$ and

$$\left\| \frac{x + y_i}{2} \right\| = \frac{f(e) + v_i}{2\|f(e)\|} \geq \frac{2\text{Lip}(f) - 4\eta}{2\text{Lip}(f)}$$

$$\geq 1 - \frac{2\eta}{\text{Lip}(f)}$$

Since the norm of $Y$ satisfies Rolewicz property ($\beta$) with power type $p$ it implies that $\text{sep}[(y_i)_{i \geq 1}] < \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p}$ for some universal constant $c \in (0, \infty)$, and hence there exist $n \neq m$ such that $\|y_n - y_m\| < \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p}$. In particular

$$\|v_n - v_m\| < \text{Lip}(f) \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p}.$$  

But

$$\|f(k_n) - f(k_m)\| = \|f(k_n) - (f(e) + v_n) + v_n - v_m + (f(e) + v_m) - f(k_m)\|$$

$$\leq \|f(k_n) - (f(e) + v_n)\| + \|v_n - v_m\| + \|f(e) + v_m) - f(k_m)\|$$

$$\leq 2\eta + \text{Lip}(f) \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p} + 2\eta$$

$$\leq 4\eta + \text{Lip}(f) \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p}$$

Since $f$ is non-contracting one has $2 < 4\eta + \text{Lip}(f) \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p}$ which is a contradiction for $\eta = \frac{\gamma}{\text{Lip}(f)^{p-1}}$ for $\gamma$ small enough, indeed

$$4\eta + \text{Lip}(f) \left(\frac{2\eta}{c\text{Lip}(f)}\right)^\frac{1}{p} = 4\frac{\gamma}{\text{Lip}(f)^{p-1}} + \text{Lip}(f) \left(\frac{2\gamma}{c\text{Lip}(f)^{p-1}}\right)^\frac{1}{p}$$

$$\leq 4\gamma + \left(\frac{2\gamma}{c}\right)^\frac{1}{p}$$

which can be made arbitrarily small. 

**Theorem 2.** Let $Y$ be a Banach space admitting an equivalent norm with ($\beta$)-
modulus of power type $p > 1$, then $c_Y(T_N) \geq \log(N)^{1/p}$.

**Proof.** Assume as we may that $f : T_N \to Y$ is a non-contractive bi-Lipschitz embedding. Let $k \in \mathbb{N}$ such that $2^k \leq N < 2^{k+1}$. The first two levels of the tree $T_{2^k}$ can be seen as countably many copies of $K_{1,\omega}$ attached to the root. According to Lemma 1 in each of these copies one can select a vertex from the second level of $T_{2^k}$. Countably many copies of $K_{1,\omega}$ are attached to each selected vertices of level 2. In each of these copies select an element at the level 4 of $T_{2^k}$ according to Lemma 1 and repeat this procedure until countably many leaves of $T_{2^k}$ have been selected. The set of selected vertices endowed with the induced metric is clearly isometric to $T_{2^{k-1}}$ (up to a scaling factor of 2) and it is easy to see that an appropriate rescaling
of $f$ induces a non-contracting embedding of $T_{2k-1}$ into $Y$ with Lipschitz constant at most $\text{Lip}(f) - \frac{\gamma}{\text{Lip}(f)^{p-1}}$. Repeating this extraction $k$ times we get a bi-Lipschitz embedding $f_k$ of $T_1$ into $Y$ such that $1 \leq \text{Lip}(f_k) \leq \text{Lip}(f) - k \frac{\gamma}{\text{Lip}(f)^{p-1}}$. Therefore one has that $\text{Lip}(f_k) \gtrsim k^{1/p}$ and it follows easily that $\text{dist}(f) \gtrsim \log(N)^{1/p}$.

Remark. The estimates are tight since for instance $c_{\ell^p}(T_N) \lesssim \log(N)^{1/p}$, for $p \in (1, \infty)$.

3. Application

Recall briefly the asymptotic versions of uniform convexity and uniform smoothness. Let $(X, \| \|)$ be a Banach space and $\tau > 0$. We denote by $B_X$ its closed unit ball and by $S_X$ its unit sphere. For $x \in S_X$ and $Y$ a closed linear subspace of $X$, we define

$$\overline{\rho}(\tau, x, Y) = \sup_{y \in S_Y} \| x + \tau y \| - 1$$

and

$$\underline{\delta}(\tau, x, Y) = \inf_{y \in S_Y} \| x + \tau y \| - 1.$$ 

Then

$$\overline{\rho}(\tau) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \overline{\rho}(\tau, x, Y)$$

and

$$\underline{\delta}(\tau) = \inf_{x \in S_X} \sup_{\dim(X/Y) < \infty} \underline{\delta}(\tau, x, Y).$$ 

The norm $\| \|$ is said to be asymptotically uniformly smooth (a.u.s. in short) if

$$\lim_{\tau \to 0} \frac{\overline{\rho}(\tau)}{\tau} = 0.$$ 

It is said to be asymptotically uniformly convex (a.u.c. in short) if

$$\forall \tau > 0 \quad \overline{\delta}(\tau) > 0.$$ 

These moduli have been first introduced by Milman in [9]. Let us now denote by $\mathcal{R}$ the class of reflexive Banach spaces and define

$$\mathcal{AUC} := \{ Y \mid Y \text{ is separable and has an equivalent a.u.c. norm} \}$$

and

$$\mathcal{AUS} := \{ Y \mid Y \text{ is separable and has an equivalent a.u.s. norm} \}.$$ 

It was shown in [1] that for every Banach space $Y \in \mathcal{R} \cap \mathcal{AUC} \cap \mathcal{AUS}$ one has

$$\lim_{N \to \infty} c_Y(T_N) = \infty.$$ 

The proof in [1] is rather technical and does not give any estimate on the rate of divergence of $(c_Y(T_N))_{N \geq 1}$. With the help of the following equivalences which were shown in [2], and which says that the class $\mathcal{R} \cap \mathcal{AUC} \cap \mathcal{AUS}$ is exactly the class of separable Banach spaces that are $(\beta)$-renormable, it is now possible to give an order of growth.

Theorem 3. Let $X$ be a separable reflexive Banach space. The following assertions are equivalent:

1. $X$ admits an equivalent norm with property $(\beta)$
2. $X$ admits an equivalent norm with property $(\beta)$ with modulus power type $p$ for some $p \in (1, \infty)$
3. $X$ is a.u.s. renormable and a.u.c. renormable

The same equivalences also hold without the separability assumption [5]. Note that the equivalence with assertion (2) is not explicit in [2] but follows from the proof of Theorem 4 in [7]. The next corollary follows easily from Theorem 2 and Theorem 3.
Corollary 4. Let $Y \in \R \cap \mathcal{ALC} \cap \mathcal{ALTS}$, then there exists $p \in (1, \infty)$ such that 

$$c_Y(T_N) \gtrsim \log(N)^{1/p}.$$ 

In particular $\lim_{N \to \infty} c_Y(T_N) = \infty$.

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