TANGENT CONE OF NUMERICAL SEMIGROUP RINGS WITH SMALL EMBEDDING DIMENSION

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Abstract. In this paper, we study the tangent cone of numerical semigroup rings with small embedding dimension \( d \). For \( d = 3 \), we give characterizations of the Buchsbaum and Cohen-Macaulay properties and for \( d = 4 \), we give a characterization of the Gorenstein property. In particular, when \( d = 4 \) and the tangent cone is Gorenstein, the initial form ideal of the defining ideal is 5-generated.

1. Introduction

Throughout this paper we fix \( \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). Recall that a numerical semigroup \( G = \langle n_1, \ldots, n_d \rangle \) generated by \( n_1, \ldots, n_d \in \mathbb{N} \) is the set \( \{ \sum_{i=1}^d a_i n_i : a_i \in \mathbb{N}_0 \} \). For simplicity, we always assume that \( G \) is minimally generated by these generators, with \( n_1 < \cdots < n_d \) and \( \gcd(n_1, \ldots, n_d) = 1 \), unless stated otherwise. Let \( k \) be a field of characteristic 0, and \( t \) an indeterminate over \( k \). Then as a subring of the power series ring \( \mathbb{V} = k[[t]] \), \( R = k[[t^{n_1}, \ldots, t^{n_d}]] \) is the numerical semigroup ring associated to \( G \) with \( \mathfrak{m} = (t^{n_1}, \ldots, t^{n_d}) \) \( R \) being the unique maximal ideal.

\( R \) is the homomorphic image of the power series ring \( S = k[[x_1, \ldots, x_d]] \) by mapping \( x_i \) to \( t^{n_i} \). Let \( \mathfrak{n} \) be the unique maximal ideal of \( S \), and \( I \) the kernel of this surjective map. \( I \) is a binomial ideal. We denote the kernel of the natural map \( \text{gr}_n(S) \to \text{gr}_n(R) \) by \( I' \), and call it the *initial form ideal* of \( I \).

When the embedding dimension \( d = 3 \), Herzog [11] gave a complete characterization of the defining ideal \( I \). In particular the minimal number of generators \( \mu(I) \leq 3 \). It is also proven by Robbiano and Valla [14] and Herzog [12] that the associated graded ring \( \text{gr}_n(R) \) is Cohen-Macaulay if and only if the initial form ideal \( I' \) is generated by at most 3 elements. In this paper, we give another characterization for the Cohen-Macaulay property in terms of the index of nilpotency \( s_Q(\mathfrak{m}) \) and the reduction number \( r_Q(\mathfrak{m}) \), where \( Q = (t^{n_1}) \) \( R \) is a principal reduction of \( \mathfrak{m} \). Recall that \( s_Q(\mathfrak{m}) = \min \{ s : \mathfrak{m}^{s+1} \subseteq Q \} \).

For \( d = 3 \), we also study the 0-th local cohomology module of the tangent cone. And we are able to characterize the \( k \)-Buchsbaum properties of \( \text{gr}_n(R) \) for \( k = 1, 2 \), in terms of the length of this local cohomology module. In particular, this answers the conjectures raised by Sapko [15].

When \( d = 4 \) and the numerical semigroup \( G \) is symmetric, Bresinsky [3] gave a complete description of the defining ideal \( I \). In particular, it is well-known now that \( \mu(I) \leq 5 \). In this paper, we also study the initial form ideal of \( I \). When the tangent cone \( \text{gr}_n(R) \) is Gorenstein, we show that \( I' \) is also 5-generated. Meanwhile, Arslan

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and Shibuta \[16\] showed that $\mu(I^*)$ could be arbitrarily large when $\text{gr}_m(R)$ is only Cohen-Macaulay.

How about $d = 5$? In this situation, Bresinsky \[4\] proved that $\mu(I) \leq 13$, under the condition that $n_1 + n_2 = n_3 + n_4$. Here in $G = \langle n_1, n_2, \ldots, n_5 \rangle$, he didn’t assume the generators to be in increasing order. Furthermore, our computations suggest a positive answer to the following question.

**Question 1.1.** If the symmetric numerical semigroup $G$ has embedding dimension 5, is $\mu(I) \leq 13$ in general? If the associated graded ring $\text{gr}_m(R)$ is Gorenstein, is $\mu(I^*) \leq 13$ as well?

The main technique in this paper is to play with the standard basis of the defining ideal $I$. The standard basis algorithm can generate a set of standard basis from a minimal generating set of $I$. The new generators are $s$-polynomials. Recall that for a polynomial ring $A = k[\overline{X}]$ with monomial ordering $<_A$, the $s$-polynomial for $f_1$ and $f_2$ in $A$ is defined to be

$$\text{spoly}(f_1, f_2) = \frac{\text{lcm}(\text{LM}(f_1), \text{LM}(f_2))}{\text{LM}(f_1)} f_1 - \frac{\text{lcm}(\text{LM}(f_1), \text{LM}(f_2))}{\text{LM}(f_2)} f_2.$$  

$\text{LM}(f)$ is the leading monomial of $f$. When we have a local monomial ordering, we sometimes also call it the initial monomial of $f$. It is part of the initial form of $f$. And in this case, the standard basis algorithm is sometimes called the Mora form algorithm. When the Krull dimension of $A$ is small, it is possible in some cases to carry out the standard basis algorithm by hand.

As an application of the above considerations, we are able to answer a question raised by Heinzer and Swanson which is related to \[10\]. In this paper, they introduced the Goto numbers for parameter ideals. If $(R, \mathfrak{m})$ is a Noetherian local ring and $Q$ is a parameter ideal, then the Goto number $g(Q) = \max \{ i \mid (Q : \mathfrak{m}^i) \subseteq \overline{Q} \}$, where $\overline{Q}$ is the integral closure of $Q$.

For numerical semigroup ring $R$, Heinzer and Swanson \[10, 4.1\] proved that

$$g(t^{f+n_1+1}) = \min \{ g(Q) \mid Q \text{ is a parameter ideal of } R \},$$

where $f = \max \{ x \in \mathbb{N}_0 \mid x \not\in G \}$ is the Frobenius number of $G$.

In particular, one has

$$g(t^{f+n_1+1}) \leq \min \{ g(t^n) \mid i = 1, \ldots, d \}.$$  

Equality holds in (\(\dagger\)) where $d = 2$ (c.f. \[10, 5.10\]). It is natural to ask if equality holds for $d \geq 3$. We note that for $d = 3$, the numerical semigroup ring $R$ corresponding to $(11, 14, 21)$ is Gorenstein. But strict inequality holds for (\(\ast\)). Heinzer pointed out that the tangent cone of this example is Cohen-Macaulay, but not Gorenstein. And we will show that if the tangent cone is Gorenstein, then equality holds for (\(\dagger\)) for $d = 3, 4$. A general proof is given by Lance Bryant.

2. Goto Numbers

The conductor ideal $C$ of $R$ with respect to $V$ is defined by $C = t^{f+1}V$ where $f$ is the Frobenius number.

Let $e$ be an nonzero element in $G$. Recall that the Apéry set of $G$ with respect to $e$ is $\text{Ap}(G, e) = \{ a(0), \ldots, a(e - 1) \}$, where $a(i)$ is the smallest element in $G$ congruent with $i$ modulo $e$. Usually we write the elements of $\text{Ap}(G, e)$ in increasing
order: $\omega_0 = 0 < \omega_1 < \cdots < \omega_{e-1} = e + f$. When $e = n_1$, the multiplicity, the following conditions are equivalent.

(a) The numerical semigroup ring $R$ is Gorenstein.

(b) The numerical semigroup $G$ is symmetric in the sense that for every $z \in \mathbb{Z}$, $z \in G$ if and only if $f - z \notin G$.

(c) $\omega_i + \omega_{e-1-i} = \omega_{e-1}$ where $0 \leq i \leq e-1$.

Lemma 2.1. Let $(R, m)$ be a Gorenstein numerical semigroup ring, and $Q = t^{n_1}R$. Then the index of nilpotency $s_Q(m) = \text{ord}_m(t^{f+n_1})$.

Proof. We always have $s_Q(m) = \max \{\text{ord}(w_i) : 1 \leq i \leq n_1 - 1\}$, and when $G$ is symmetric, $\text{ord}_m(w_{n_1-1})$ is the largest. Hence $s_Q(m) = \text{ord}_m(t^{f+n_1})$.

For every $z \in G$, it is obvious that $\text{ord}_m(t^z) = \max \{\sum a_i : \sum a_in_i = z, a_i \in \mathbb{N}_0\}$. When there is confusion, we also write this number as $\text{ord}_G(z)$ and similarly define min-$\text{ord}_G(z)$ to be $\min \{\sum a_i : \sum a_in_i = z, a_i \in \mathbb{N}_0\}$. The ratio $\frac{\text{ord}_G(z)}{\text{min-ord}_G(z)}$ is called the elasticity of $z$ with respect to $G$.

Lemma 2.2. If $G$ is symmetric, then $\text{min-ord}_G(f+n_1) \leq \text{ord}_m(C)$.

Proof. Set $Q = t^{n_1}R$. Any $z \in G$ can be written uniquely in the form $z = an_1 + w_i$ where $a = \text{ord}_G(z)$ and $w_i \in \text{Ap}(G, n_1)$. If $z > f$, then $an_1 + w_i > w_i + w_{s-1-i} - n_1$. Thus $(a+1)n_1 > w_{s-1-i}$. If we write $w_{s-1-i} = \sum_j a_jn_j$ with $a_j \in \mathbb{N}_0$, then $a_j = 0$, and $w_{s-1-i} > \sum_j a_jn_j$. Consequently $a+1 > \sum_j a_j$, i.e., $a > \sum_j a_j$. Hence $a \geq \text{ord}_G(w_{s-1-i})$. Now $\text{ord}_m(z) \geq a + \text{ord}_G(w_i) \geq \text{ord}_G(w_{s-1-i}) + \text{ord}_G(w_i) \geq \text{min-ord}_G(f+n_1)$. Since the elements of the form $t^z$ with $z > f$ generate the ideal $C$, this completes the proof.

Example 2.3. In general, it is not true that $\text{ord}_G(f+n_1) \leq \text{ord}_m(C)$. For example, if $G = \langle 11, 14, 21 \rangle$, then $G$ is symmetric and the Frobenius number $f = 73$. It is not difficult to see that $\text{ord}_G(f+n_1) = 6$ while $\text{ord}_m(C) = 5$.

Inspired by the proof of [22] we define a partial order $\preceq_G$ on elements in $G$. We say $x \preceq_G x'$ for $x, x' \in G$ if we can write $x = \sum_i a_in_i$ and $x' = \sum_i a'_in_i$ with $a_i, a'_i \in \mathbb{N}_0$, such that $\sum a_i = \text{ord}_G(x)$, $\sum a'_i = \text{ord}_G(x')$, and $a_i \leq a'_i$ for all $i$. Roughly speaking, $x \preceq_G x'$ if and only if "$x$ has a maximal decomposition with respect to $G$ that is dominated by a maximal decomposition of $x'$ with respect to $G$".

Proposition 2.4. Let $(R, m)$ be a Gorenstein numerical semigroup ring, where the associated graded ring $\text{gr}_m(R)$ is Cohen-Macaulay. If for every $x \in \text{Ap}(G, n_1)$, $x \preceq_G (f + n_1)$, then equality holds for (1) in the introduction. In addition, 

\[ g(t^{f+n_1+1}) = g(t^{n_1}) = r_Q(m) = \text{ord}_m(C) = \text{ord}_G(f+n_1). \]

Proof. According to the proof of [22], for any element $z \in G$ with $z > f$, $z - an_1 = w_j \in \text{Ap}(G, n_1)$ with $a = \text{ord}_G(z)$. Since $w_j \preceq_G (f + n_1)$, we can write $w_j = \sum a_in_i$, $f + n_1 = \sum b_in_i$, with $a_i, b_i \in \mathbb{N}_0$, $a_i \leq b_i$ for $2 \leq i \leq d$. Now write $a'_i = b_i - a_i$, and one has $w_{s-1-j} = f + n_1 - w_i = \sum a'_in_i$. Hence by the proof of [22]

\[ \text{ord}_G(z) \geq \text{ord}_G(w_j) + \text{ord}_G(w_{s-1-j}) \geq \sum b_i = \text{ord}_G(f + n_1). \]

This implies that $\text{ord}_m(C) \geq \text{ord}_G(f+n_1)$. By virtue of [22] we have $\text{ord}_G(f+n_1) \geq r_Q(m) = g(t^{n_1})$. On the other hand, Theorem 4.1 and Corollary 5.8 of [10] yield
Lemma 3.1. \( g(t^{n_1}) \geq g(t^{f + n_1 + 1}) \geq \text{ord}_m(C) \). Since

\[
\text{ord}_m(C) = \min \{ \text{ord}_G(z) \mid z \in G, z > f \} ,
\]

\text{ord}_m(C) \geq \text{ord}_G(f + n_1). \) Now the equalities in the assertion follow immediately. In particular, equality holds for \( \square \).

Remark 2.5. If \( \text{ord}_G(f + n_1) = \text{min-ord}_G(f + n_1) \), i.e., the elasticity of \( f + n_1 \) with respect to \( G \) is 1, then every decomposition of \( f + n_1 \) is maximal. Hence for any \( x \in \text{Ap}(G, n_1), x \preceq_G (f + n_1) \). This elasticity condition is satisfied when the embedding dimension equals 4, the tangent cone \( \text{gr}_m(R) \) is Gorenstein but \( R \) is not complete intersection, as we demonstrate in \( \text{5.16} \).

Corollary 2.6. For numerical semigroup ring \( (R, m) \) with embedding dimension \( d \leq 4 \), if the associated graded ring \( \text{gr}_m(R) \) is Gorenstein, then

\[
g(f + n_1) = g(t^{n_1}) = r(m) = \text{ord}_m(C) = \text{ord}_G(f + n_1).
\]

In particular, equality holds for \( \square \).

When \( d = 4 \), it follows from \( \text{2.4} \) and \( \text{5.1} \). The case when \( d = 3 \) will be an easy exercise.

It is pointed out by Lance Bryant that the partial order condition on the Apéry set in \( \text{2.4} \) is equivalent to the following condition for multiplicity \( \epsilon = n_1\):

\[
(f) \forall w_i, w_j \in \text{Ap}(G, n_1), w_i + w_j = w_{e - 1} \Rightarrow \text{ord}_m(w_i) + \text{ord}_m(w_j) = \text{ord}_m(w_{e - 1}).
\]

And he gives the following characterization

Theorem 2.7 (\( \text{[6] 3.20} \)). Suppose \( G \) is a symmetric numerical semigroup, and \( \text{gr}_m(R) \) is Cohen-Macaulay. Then \( \text{gr}_m(R) \) is Gorenstein if and only if the condition \( \square \) holds. In particular, Corollary 2.6 holds with no restriction on the embedding dimension.

3. Initial form ideals

Following our previous notation, \( S = K[[x_1, \ldots, x_d]] \) with maximal ideal \( n \) maps onto \( R \). For each nonzero \( x \in S \), let \( o = \text{ord}_n(x) < \infty \) be the \( n \)-adic order of \( x \). We denote by \( x^* \) the residue class of \( x \) in \( n^o/n^{o+1} \) and call it the initial form of \( x \). The initial form ideal \( I^* \subset \text{gr}_n(S) \) is generated by \( x^* \) for all \( x \in I \). For our numerical semigroup ring \( R \), the radical of the initial ideal \( I^* \) is very simple.

Lemma 3.1. \( \sqrt{I^*} = (x_2^*, \ldots, x_d^*) \text{gr}_n(S) \).

Proof. For all \( i \) with \( 2 \leq i \leq d \), \( f_i = x_i^{n_1} - x_1^{n_i} \in I \). Since \( n_1 < n_i \), the initial form of \( f_i^* \) is \( (x_i^*)^{n_1} \in I^* \). Hence \( (x_2^*, \ldots, x_d^*) \subseteq \sqrt{I^*} \). Since \( \text{ht}(x_2^*, \ldots, x_d^*) = \text{ht} I^* = \text{ht} I = d - 1 \) and \( (x_2^*, \ldots, x_d^*) \) is a prime ideal in \( \text{gr}_n(S) \), \( \sqrt{I^*} = (x_2^*, \ldots, x_d^*) \). \( \square \)

Since \( \text{gr}_n(S) \cong k[x_1, \ldots, x_d] \), by abuse of notation, in the sequel of this paper we simply write \( x_i^* \) as \( x_i \) for \( 1 \leq i \leq d \), when there is no confusion.

We need to go over some basic definitions and notations of standard basis. Given a fixed local monomial order \( > \), for a nonzero \( f \in S \), there is a unique way to write

\[
f = a_1 x^{a_1} + a_2 x^{a_2} + \cdots + a_t x^{a_t}, \quad x^{a_1} > x^{a_2} > \cdots > x^{a_t}.
\]

\( \text{LM}(f) := x^{a_1} \) is called the initial monomial of \( f \). The initial ideal \( LI(I) \) of \( I \) is generated by \( \text{LM}(f) \) for \( f \in I \). A subset \( F := \{ f_1, \ldots, f_s \} \subset I \) is a standard basis
of $I$ if $\text{LI}(I) = (\text{LM}(f_1), \ldots, \text{LM}(f_s))$. $F$ is called minimal if $\text{LM}(f_i) \nsubseteq \text{LM}(f_j)$ for $i \neq j$.

A binomial $f = x^\alpha - x^\beta$ with $\alpha = (\alpha(1), \ldots, \alpha(d)), \beta = (\beta(1), \ldots, \beta(d)) \in \mathbb{N}^d$ is called weakly balanced if $\sum_i \alpha(i)n_i = \sum_i \beta(i)n_i$. $f$ is called balanced if it is weakly balanced, $\deg x^\alpha = x^\beta$, and $x^\alpha$ and $x^\beta$ form a regular sequence.

For the numerical semigroup ring $R$, the defining ideal $I$ is generated by weakly balanced binomials. Applying the standard basis algorithm with suitable monomial ordering to this generating set, one is able to get a minimal standard basis $\{f_1, \ldots, f_s\}$. And $I^*$ is minimally generated by the corresponding initial forms: $I^* = (f_1^*, \ldots, f_s^*)$. Since each $f_i$ is also a weakly balanced binomial, $f_i^*$ is either a monomial or a balanced binomial. In the latter case, roughly speaking, $f_i = f_i^*$. In the rest of the paper, when we say $g$ is a minimal generator of $I^*$, it is understood that $g \in \{f_1^*, \ldots, f_s^*\}$ when a minimal generating set of $I$ and the monomial ordering is clear.

From now on, we need to choose the monomial ordering $>$ more carefully. A monomial ordering $>$ is nice in variable $x_1$ if the following holds:

$$\deg x^\alpha < \deg x^\beta, \text{ or } (x^\alpha - x^\beta \text{ is balanced, } \beta(i) > 0) \implies x^\alpha > x^\beta.$$ Being nice is really a mild condition. For instance, the modified negative degree reverse lexicographical ordering in the following is nice in $x_1$:

$$x^\alpha > x^\beta :\iff \deg x^\alpha < \deg x^\beta, \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \alpha(1) = \beta(1), \ldots, \alpha(i-1) = \beta(i-1), \alpha(i) < \beta(i)).$$

When $d = 3$, the normal negative degree reverse lexicographical ordering is also nice in $x_1$.

An ideal $J \subset T := k[x_1, \ldots, x_d]$ is called almost balanced if it satisfies the following conditions:

(a) $\sqrt{J} = (x_2, \ldots, x_d)$;

(b) there is a minimal standard basis $\{f_1, \ldots, f_t\}$ of $J$, such that $f_1$ is either a monomial or a balanced binomial.

The initial form ideal $I^*$ is almost balanced.

Lemma 3.2. Let $>$ be a nice monomial ordering in $x_1$ for $T$ and $J$ an almost $T$-ideal. Then $T/J$ is Cohen-Macaulay if and only if $x_1$ does not divide any of the monomial generators of $J$ constructed above.

Proof. $T/J$ is Cohen-Macaulay if and only if $x_1$ is a regular element.

If in the previous definition that some $f_i = x_1x^\alpha$, then $x^\alpha \notin J$, since $\{f_1, \ldots, f_s\}$ is a minimal standard basis. Hence $J$ is not a perfect ideal.

On the other hand, suppose that $x_1f \in J$ and $0 \neq f \notin J$. We may replace $f$ by the normal form $\text{NF}(f|\{f_1, \ldots, f_s\})$ to assume that none of the monomials in $f$ is divisible by any of $\text{LM}(f_i)$. Notice that $x_1 \text{LM}(f) = \text{LM}(x_1f)$ is divisible by some $\text{LM}(f_i)$. But $\text{LM}(f)$ is not divisible by $\text{LM}(f_i)$, hence $\text{LM}(f_i)$ is divisible by $x_1$. Since the ordering is nice in $x_1$, $f_i$ cannot be a balanced binomial, hence it is a monomial.

Example 3.3. Let $G$ be the numerical semigroup generated by 5, 6 and 13. Then the defining ideal $I = (x_2^5x_3 - x_1^5, x_2^4x_3 - x_1^4x_2, x_1x_3 - x_2^3)$. With the normal negative degree reverse lexicographical ordering, $\{x_2^5x_3 - x_1^5, x_2^4x_3 - x_1^4x_2, x_1x_3 - x_2^3, x_2^3 - x_1^3\}$ form a minimal standard basis. Thus the initial form ideal is $I^* = (x_2^5, x_2^4x_3, x_2^3, x_1x_3)$. 


Since $x_1x_3$ is divisible by $x_1$, $I^*$ is not a Cohen-Macaulay ideal. This property also follows from the fact that $I^*$ is generated by more than 3 elements.

For the rest of this paper, we fix
$$\alpha_i = \min \{\alpha \in \mathbb{N} \mid \alpha n_i \in \langle n_1, \ldots, n_d \rangle\}$$
for $1 \leq i \leq d$. In the previous example, $\alpha_1 = 5$, $\alpha_2 = 3$ and $\alpha_3 = 2$.

4. When the embedding dimension $d = 3$

Throughout this section, we use negative degree reverse lexicographical ordering on $gr_3(S) \cong k[x_1, x_2, x_3]$. Hence if $f = x_2^b - x_1^a x_3^c$ with $a, b, c \in \mathbb{N}$, and $b = a + c$, then the initial monomial of $f$ is $x_2^b$.

We need the following result on the structure of 3-generated numerical semigroups by Herzog.

**Theorem 4.1** ([11 2.1, 3.8]). Let $G$ be a numerical semigroup minimally generated by 3 elements.

(a) If $G$ is symmetric, then by a permutation $(i, j, k)$ of $(1, 2, 3)$, the defining ideal is
$$I = (x_1^{\alpha_i} - x_j^{\alpha_j}, x_k^{\alpha_k} - x_i^{\alpha_i^*} x_j^{\alpha_k})$$
And the Frobenius number of $G$ is
$$f = (\alpha_i - 1)n_i + (\alpha_k - 1)n_k - n_j.$$

(b) If $G$ is not symmetric, then
$$I = (x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, x_3^{\alpha_3} - x_1^{\alpha_1} x_2^{\alpha_2}),$$
where all $\alpha$’s are in $\mathbb{N}$ and $\alpha_i = \alpha_{ji} + \alpha_{ki}$ for all permutation $(i, j, k)$ of $(1, 2, 3)$.

In terms of the defining ideal given by [11 one can quickly give arithmetic conditions on when $gr_3(R)$ will be Cohen-Macaulay.

**Corollary 4.2.**  
(a) If $I = (x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3} - x_1^{\alpha_1^*} x_2^{\alpha_2}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, x_3^{\alpha_3} - x_1^{\alpha_1} x_2^{\alpha_2})$, then $gr_3(R)$ is a complete intersection and $I^*$ is generated by $\{x_2^{\alpha_2}, x_3^{\alpha_3}\}$.

(b) If $I = (x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, x_3^{\alpha_3} - x_1^{\alpha_1} x_2^{\alpha_2})$, then $gr_3(R)$ is Cohen-Macaulay if and only if $\alpha_2 \leq \alpha_2 + \alpha_3$. Whence, $I^*$ is generated by
$$\{x_2^{\alpha_2}, x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}\}.$$

(c) If $I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3})$ with $\alpha_3 < \alpha_3$. Set $f_3 := \text{spoly}(f_1, f_2) = x_2^{\alpha_2 + \alpha_1 - \alpha_3} - x_1^{\alpha_1} x_3^{\alpha_3 - \alpha_3}$. Then $gr_3(R)$ is Cohen-Macaulay if and only if $\alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_3 - \alpha_3$. Whence, $I^*$ is generated by
$$\{x_2^{\alpha_2}, x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, f_3\}.$$

(d) If $I = (x_1^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, x_3^{\alpha_3} - x_1^{\alpha_1} x_2^{\alpha_2})$, then $gr_3(R)$ is
Cohen-Macaulay if and only if $\alpha_2 \leq \alpha_2 + \alpha_3$.

Recall that for a one-dimensional graded ring $A$ with a unique homogeneous maximal ideal $\mathcal{M}$, $A$ is called a $K$-Buchsbaum ring if $\mathcal{M}^k \cdot H^0_{\mathcal{M}}(A) = 0$. 1-Buchsbaum rings are simply called Buchsbaum. And 0-Buchsbaum rings are precisely the Cohen-Macaulay rings. In the following, denote the homogeneous maximal ideal of $gr_3(S)$ by $\mathcal{M}$. Since $gr_3(R) = gr_3(S)/I^*$, we write the image of $f \in gr_3(S)$ in $gr_3(R)$ as $\hat{f}$. 
Sapko [15] investigated the tangent cone of ring $R$ associated with the 3-generated numerical semigroup ring $G$, and made the following conjectures when the tangent cone is Buchsbaum.

**Conjecture 4.3** ([15, Conjecture 24]). If $\text{gr}_m(R)$ is Buchsbaum, then for some $k \geq 1$, $$0 : \text{gr}_m(R) \mathcal{M} = (x_3^k) \text{gr}_m(R).$$

**Conjecture 4.4** ([15, 25]). If $\text{gr}_m(R)$ is Buchsbaum, then the initial form ideal $I^*$ of $I$ is generated by 4 elements.

**Conjecture 4.5** ([15, Conjecture 33]). $\text{gr}_m(R)$ is Buchsbaum if and only if the length $\ell(H_M^0(\text{gr}_m(R))) \leq 1$.

In this note, we want to give positive answers to the above conjectures when the tangent cone is Buchsbaum, and show similar results when the tangent cone is 2-Buchsbaum.

**Lemma 4.6.** Suppose $\text{gr}_m(R)$ is not Cohen-Macaulay. Then the 0-th local cohomology module $H_M^0(\text{gr}_m(R))$ is principal, and generated by $x_3^\gamma$ for suitable $\gamma \in \mathbb{N}$.

**Proof.** The initial form ideal $I^*$ is generated by forms of the following 4 types:

- (a) $x_3^a$,
- (b) $x_2^b$ or balanced $x_2^{72} - x_1^{72_1} x_3^{72_3}$,
- (c) $x_1^a x_3^c$,
- (d) $x_2^b x_3^c$.

There is exactly one generator of the type (a). The same is true for generators of type (b). To see this, notice that if $x_2^{72} - x_1^{72_1} x_3^{72_3}$ is balanced, then $x_2^{72}$ is its initial monomial. There might be more than one generators of type (c) and (d).

It follows from Corollary 3.2 that $I^*$ is Cohen-Macaulay if and only if generators of type (c) do not exist. If $\text{gr}_m(R)$ is not Cohen-Macaulay, then $H_M^0(\text{gr}_m(R)) \neq 0$. We claim that this local cohomology module is generated by $x_3^\gamma$ where

$$\gamma = \min \{ c | c x_3^a x_3^b \text{ is a generator of } I^* \text{ of type (c) for some } a, b \in \mathbb{N} \}.$$ 

Since $\sqrt{I^*} = (x_2, x_3)$, this $x_3^\gamma \in H_M^0(\text{gr}_m(R))$. On the other hand, $I^* + (x_3^\gamma)$ is (not necessarily minimally) generated by $x_3^\gamma$ together with the remaining generators of $I^*$ of type (b) and (d). This last ideal is Cohen-Macaulay by 3.2.

**Corollary 4.7.** $\text{gr}_m(R)$ is Buchsbaum if and only if $\ell(H_M^0(\text{gr}_m(R))) \leq 1$.

**Proposition 4.8.** If $\text{gr}_m(R)$ is Buchsbaum, then the initial form ideal $I^*$ is 4-generated.

**Proof.** When $\text{gr}_m(R)$ is Buchsbaum and not Cohen-Macaulay, $I^*$ has at least one minimal generator of the form $x_1^{71} x_3^{73}$. $x_3^{73} \in H_M^0(\text{gr}_m(R))$, hence in $\text{gr}_m(S)$, $\mathcal{M} x_3^{73} \subseteq I^*$. Since $x_1^{71} x_3^{73}$ is a minimal generator, one must have $\gamma_1 = 1$. One also observe that $x_2 x_3^{73}, x_3^{73+1} \in I^*$. By the minimality of $x_3^{73}$, one has $\alpha_3 = \gamma_3 + 1$.

It is also clear now that the only minimal monomial generator that involves positive exponent in $x_1$ is exactly $x_1 x_3^{73-1}$. Since $x_2 x_3^{73-1} \in I^*$, there exists a generator $f = x_2 x_3^{73} - x_1^{\gamma} x_3^{73-1}$ of the minimal standard basis with $\gamma \leq \alpha_3 - 1$. However, since $\gamma < \alpha_3$, this $f$ should not be new basis generated from the standard basis algorithm. It should belong to one of the minimal binomial generators of $I$. 


Claim. If $R$ is Gorenstein, then $\text{gr}_m(R)$ is Cohen-Macaulay if and only if it is Buchsbaum.

By Proposition 4.9 when $R$ is Gorenstein, the defining ideal, by a permutation $(i, j, k)$ of $(1, 2, 3)$, is

$$I = (x_i^{\alpha_i} - x_j^{\alpha_j}, x_k^{\alpha_k} - x_i^{\alpha_i}x_j^{\alpha_j}).$$

By symmetry, we can always assume that $i < j$. Now one can characterize when the associated graded ring is Buchsbaum in terms of these $\alpha$'s. By our discussion for $x_2x_3^{\alpha_3-1}$, the claim is set by checking the minimal binomial generators for the cases when $(i, j, k) = (1, 2, 3)$ or $(1, 3, 2)$.

If $(i, j, k) = (2, 3, 1)$, then

$$I = (f_1 := x_3^{\alpha_3} - x_2^{\alpha_2}, f_2 := x_1^{\alpha_1} - x_2^{\alpha_2}x_3^{\alpha_3}).$$

$f_1^*$ and we can assume that $0 \leq \alpha_1 < \alpha_3$, hence $\alpha_2 > 0$. Now $f_2^* = -x_2^{\alpha_2}x_3^{\alpha_3}$ and it is non-comparable with $f_1^*$. Apply the standard basis algorithm, we get $f_3 := \text{spoly}(f_1, f_2) = x_2^{\alpha_2 + \alpha_3} - x_1^{\alpha_1}x_3^{\alpha_3-\alpha_3}$. Now a necessary and sufficient condition for $I^*$ to be Cohen-Macaulay is $\alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_3 - \alpha_3$. Hence, if $I^*$ is Buchsbaum and not Cohen-Macaulay, then $f_3^* = -x_1^{\alpha_1}x_3^{\alpha_3-\alpha_3}$. Notice that $f_3^*$ must belong to the minimal standard basis. Now by our discussion above, $\alpha_1 = \alpha_1 = 1$ and $\alpha_3 - \alpha_3 = \alpha_3 - 1$. But if $G$ is minimally generated by 3 elements, then $\alpha_1 > 2$, and this is a contradiction. Thus, the claim if proved.

If follows immediately that if $\text{gr}_m(R)$ is Buchsbaum, not Cohen-Macaulay, then $R$ is not Gorenstein. Now we can write the defining ideal

$$I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_1}x_3^{\alpha_3}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_2}x_3^{\alpha_3}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_3}x_2^{\alpha_3}).$$

Obviously, $f_1^* = -x_2^{\alpha_1}x_3^{\alpha_3}$ and $f_3^* = x_3^{\alpha_3}$. We can assume that $\alpha_1, \alpha_3 < \alpha_3$. Now by Proposition 4.9 and our discussion in the beginning, if $\text{gr}_m(R)$ is Buchsbaum and not Cohen-Macaulay, then $\alpha_2 > \alpha_1 + \alpha_2$, $\alpha_2 = \alpha_2 = 1$ and $\alpha_2 = \alpha_3$. Now by Theorem 4.1 $\alpha_3 = 1$, $\alpha_3 = \alpha_1 - 1$ and $\alpha_2 = \alpha_2 - 2$. Obviously for $f_4 := \text{spoly}(f_1, f_2) = x_1^{\alpha_1+1}x_3^{\alpha_3-1} - x_2^{\alpha_2+1}$. The Buchsbaumness would require that $\alpha_2 + 1 \leq (\alpha_1 + 1) + (\alpha_3 - 2)$. An easy calculation would show that the standard basis algorithm will stop at this step, and $\{f_1, \cdots, f_4\}$ form a minimal standard basis for $I$. Now

$$I^* = (x_2x_3, x_1x_3^{\alpha_3-1}, x_3^{\alpha_3}, x_2^{\alpha_2+1} \text{ or } x_2^{\alpha_2+1} - x_1^{\alpha_1+1}x_3^{\alpha_3-2}).$$

□

Next, we study the 2-Buchsbaumness of the tangent cone, and still assume that the numerical semigroup $G$ is minimally 3-generated. We want to show the following:

Proposition 4.9. $\text{gr}_m(R)$ is 2-Buchsbaum if and only if the length

$$\ell(H_M^0(\text{gr}_m(R))) \leq 2.$$

Proof. Notice that we assume that the monomial ordering is nice in $x_1$. Hence if a monomial involving $x_1$ is the initial monomial of a minimal generator constructed as before, this generator as to be a monomial.

Now if the length is at most 2, the tangent cone is trivially 2-Buchsbaum. On the other hand, we assume that $\text{gr}_m(R)$ is 2-Buchsbaum. Without loss of generality, we may assume that $\text{gr}_m(R)$ is not Cohen-Macaulay. Hence in $I^*$, there is a monomial
minimal generator $x_1^2 x_3^2$. Since $x_1^2 x_3^2$, $x_1^2 x_3^5$ ∈ $I^*$, one necessarily have $1 ≤ a ≤ 2$ and $\alpha_3 - 2 ≤ c ≤ \alpha_3 - 1$.

We claim that there is exactly one such minimal generator with the initial monomial $x_1^2 x_3^5$. It is easy to see that this could fail only when both $x_1 x_3^{\alpha_3-1}$ and $x_2^2 x_3^{\alpha_3-2}$ are minimal generators of $I^*$. Since they are minimal, there exist $\beta_1, \beta_2 ∈ \mathbb{N}$ such that both $x_2^{\beta_1} - x_1 x_3^{\alpha_3-1}$ and $x_2^{\beta_2} - x_1^2 x_3^{\alpha_3-2}$ are both in $I$. Since $n_3 > n_2 > n_1$, we must have $\beta_1 > \beta_2$. Hence $x_2^{\beta_1} - \beta_2 x_1 = x_3$, and $G$ is at most 2-generated, which contradicts our assumption.

Similarly, $x_2^2 x_3^2 ∈ I^*$. Hence either it is divisible by the initial monomial $x_1^2 x_3^5$ of a minimal generator of $I^*$ with $1 ≤ b ≤ 2$ and $1 ≤ c ≤ c$, or $\alpha_2 = 2$. If $\alpha_2 = 2$, then [4.2] implies that $I^*$ is Cohen-Macaulay, and there is no need for further discussion.

If $\alpha_2 > 2$, then by an argument similar to that in the previous paragraph, there is exactly one such minimal generator in $I^*$ with initial monomial of the form $x_2^2 x_3^2$ with $1 ≤ b ≤ 2$ and $1 ≤ c ≤ c$. And this generator must be a monomial.

Now we prove that $\ell(H_M^0(\text{gr}_m(R))) ≤ 2$.

(a) Suppose that $x_1^2 x_3^2 = x_1 x_3^{\alpha_3-2}$. Notice that $x_2^2 x_3^{\alpha_3-2}$, $x_2 x_3^{\alpha_3-1} ∈ I^*$. Each of them has to be divisible by some monomial minimal generator of $I^*$ in the form $x_1^2 x_3^2$ with $1 ≤ b ≤ 2$, $1 ≤ c ≤ c$. But there are at most one such generator. Hence this generator must divide the $\gcd(x_2^2 x_3^{\alpha_3-2}, x_2 x_3^{\alpha_3-1}) = x_2 x_3^{\alpha_3-2}$. In particular, $x_2 x_3^{\alpha_3-2} ∈ I^*$. Hence the vector space $H_M^0(\text{gr}_m(R)) = (x_3^{\alpha_3-1}) \text{gr}_m(R)$ is generated by $\{x_3^{\alpha_3-2}, x_3^{\alpha_3-1}\}$.

(b) The case $x_1 x_3^5 = x_2^2 x_3^{\alpha_3-2}$ can never happen. Notice that the image of $x_3^{\alpha_3-2}$ generates the local cohomology module. Hence $x_1 x_3^5 : x_3^{\alpha_3-2} \not∈ I^*$. We know that there are no two such minimal generators of the form $x_1 x_3^5$ in $I^*$. Since $x_1 x_3^{\alpha_3-2}$ is assumed to be a minimal generator, hence $x_1 x_3^{\alpha_3-2}$ has to divide $x_1 x_3^{\alpha_3-1}$, which is impossible.

(c) Assume that $x_1 x_3^5 = x_1 x_3^{\alpha_3-1}$. Notice that $x_3^{\alpha_3} ∈ I^*$. Hence the local cohomology module is generated as vector space by $\{x_3^{\alpha_3-1}\}$ or $\{x_3^{\alpha_3-1}, x_2 x_3^{\alpha_3-1}\}$.

(d) Assume that $x_1 x_3^5 = x_2 x_3^{\alpha_3-1}$. Notice $x_1 x_2 x_3^{\alpha_3-1} ∈ I^*$ by the 2-Buchsbaumness. But it can never be a minimal generator. Hence either $x_1 x_3^{\alpha_3-1} \not∈ I^*$ or $x_2 x_3^{\alpha_3-2} \not∈ I^*$. Because $x_1 x_3^{\alpha_3-1}$ is a minimal generator, the first option cannot happen. Hence $x_2 x_3^{\alpha_3-1} ∈ I^*$ and the local cohomology module as a vector space is generated by $\{x_3^{\alpha_3-1}, x_1 x_3^{\alpha_3-1}\}$.

Proposition 4.10. If $\text{gr}_m(R)$ is 2-Buchsbaum, the initial form ideal $I^*$ is 4-generated.

Proof. We may assume from the beginning that $\text{gr}_m(R)$ is not Cohen-Macaulay. In particular, $\alpha_2 ≥ 3$. If $\text{gr}_m(R)$ is 2-Buchsbaum, then $R$ is not Gorenstein, by an argument similar to that used in the proof for [4.8] Hence we can assume that $R$ is not Gorenstein as well.

(a) Suppose that $x_1 x_3^{\alpha_3-2}$ is one minimal generator for $I^*$, then $I = (f_1 := x_1^{\alpha_1} - x_2 x_3^2, f_2 := x_2^{\alpha_2} - x_1 x_3^{\alpha_3-2}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_1-1} x_2^{\alpha_2-1})$ by case (a) of the proof for [4.9] together with 3.2 of [11], spoly$(f_1, f_3)$ and $(f_1, f_2)$ will not contribute to the standard basis. Since $\text{gr}_m(R)$ is not Cohen-Macaulay, $\alpha_2 > 1 + (\alpha_3 - 2)$.
α₃ - 2 is an exponent in f₂, thus α₃ - 2 ≥ 1. If α₃ = 3, then x₃ generates the local cohomology module. Hence x₃² x₃ ∈ I*. And there is a standard basis generator f = x₂α₂x₃ - x₁α₁ ∈ I with β = 1 or 2. But then f₄ := spoly(f₁, f₂) = x₂α₂+1 x₃ - x₁α₁+1. Since n₂ > n₁ and α₂ ≥ 3 > β, this will imply that γ < α₁, which is a contradiction.

Hence α₃ ≥ 4 and f₄ := spoly(f₂, f₁) = x₂α₂+1 - x₁α₁+1 x₃α₃-4. By the 2-Buchsbaumness, x₂α₂+1 has to be the initial monomial. And the standard basis algorithm will stop at this step.

(b) Suppose that x₁x₃α₃-1 is one minimal generator for I*. Then

I = (f₁ := x₁α₁ - x₂α₁+1 x₃, f₂ := x₂α₂ - x₁x₃α₃-1, f₃ := x₃α₃ - x₁α₁-1 x₂α₂-α₁₂),

with α₁₂ = 1 or 2. The standard basis algorithm generates f₄ := spoly(f₂, f₁) = x₂α₂+α₁₂ - x₁α₁+1 x₃α₃-2. If the tangent cone is 2-Buchsbaum, then α₂ + α₁₂ ≤ α₁ + α₃ - 1. And then the algorithm stop at this step.

(c) If x₁x₃α₃-1 is one minimal generator for I*, then by the proof for 4.9, α₁₂ = 1, and the defining ideal is

I = (f₁ := x₁α₁ - x₂x₃, f₂ := x₂α₂ - x₁x₃α₃-1, f₃ := x₃α₃ - x₁α₁-2 x₂α₂-1).

Similar to the previous case, the standard basis algorithm will only contribute an additional basis element f₄ := spoly(f₁, f₂) = x₂α₂+1 - x₁α₁+2 x₃α₃-2.

□

In the rest of this section, we establish a new characterization on when the tangent cone grₘ(R) will be Cohen-Macaulay in the case d = 3. First of all, we give a remark to 2.7.

Remark 4.11. For symmetric G with arbitrary embedding dimension, if the order values of the Apéry set elements are symmetric in the sense of \([\text{1}]\), and s₉(m) = r₉(m), then grₘ(R) is Gorenstein, even without assuming it to be Cohen-Macaulay in the beginning. For the proof, see [3, 14].

We thank Lance Bryant for the helpful comments regarding 4.12.

Theorem 4.12. If G = (n₁, n₂, n₃) is minimally 3-generated, and for the principal reduction Q = (n₁)R of the maximal ideal m, the index of nilpotency s₉(m) equals the reduction number r₉(m), then the tangent cone grₘ(R) is Cohen-Macaulay.

Proof. In the following, for x ∈ S, write \(\bar{x}\) for its image in R = S/I. First we study the case when G is symmetric, i.e., the numerical semigroup ring R is a complete intersection. Now for \(w_{e-1} = f + n₁\), ordₘ(wₑ-1) = s₉(m) by \([\text{2, 1}]\).

Using \([\text{4, 1}]\) again, we have three cases.

(i) \((i, j, k) = (1, 2, 3)\), then the tangent cone is automatically a complete intersection.

(ii) \((i, j, k) = (1, 3, 2)\). Now the last Apéry set element can be written as

\(wₑ-1 = \alpha₂ - 1)n₂ + (\alpha₃ - 1)n₃\).

This is obviously the unique representation of \(wₑ-1\) with respect to G. Hence the order values of the Apéry set element is symmetric in the sense of \([\text{1}]\). If r₉(m) = s₉(m), then grₘ(R) is Gorenstein by \([\text{4, 11}]\).

(iii) \((i, j, k) = (2, 3, 1)\). Now the associated graded ring grₘ(R) is Cohen-Macaulay if and only if \(\alpha₂ + \alpha₁₂ ≤ \alpha₁ + \alpha₃ - α₁₃\). We want to prove that the following conditions are equivalent:
instance, suppose for contradiction that \( \alpha \) then happen that both \( b = w \)

Notice that the Frobenius number can be written as

\[
f = (a_2 + \alpha_{12} - 1)n_2 + (\alpha_{13} - 1)n_3 - n_1,
\]

hence

\[
w_{e-1} = (a_2 + \alpha_{12} - 1)n_2 + (\alpha_{13} - 1)n_3.
\]

This gives the maximal representation of \( w_{e-1} \) with respect to \( G \), and

\[
\text{ord}_m(w_{e-1}) = (a_2 + \alpha_{12} - 1) + (\alpha_{13} - 1).
\]

That (a) implies (b) is clear.

If (b) holds, then for \( r = r_Q(m) \), \( x_2^{\alpha_2+\alpha_{12}}x_3^{\alpha_{13}} \in \mathfrak{m}^r \). So \( x_2^{\alpha_2+\alpha_{12}}x_3^{\alpha_{13}} = x_2^{\alpha_2}x_3^{\alpha_{13}} \). Hence \( x_2^{\alpha_2}x_3^{\alpha_{13}} \in \mathfrak{m}^r \). We want to show that \( \text{ord}_m(x_2^{\alpha_{12}}x_3^{\alpha_{13}}) = (\alpha_1 - 1) + (\alpha_3 - 1) \), hence (c) holds.

It suffices to show that \( (\alpha_1 - 1)n_1 + (\alpha_3 - 1)n_3 \) is the unique representation of this element with respect to \( G \). Suppose not, then

\[
(\alpha_1 - 1)n_1 + (\alpha_3 - 1)n_3 = an_1 + bn_2 + cn_3,
\]

with \( a, b, c \in \mathbb{N}_0 \) and \( b > 0 \). By the minimality of \( \alpha_1 \) and \( \alpha_3 \), one must have \( a \leq \alpha_1 - 1 \) and \( c \leq \alpha_3 - 1 \). Now

\[
(\alpha_1 - 1 - a)n_1 + (\alpha_3 - 1 - c)n_3 = bn_2
\]

Since \( b > 0 \), \( b \geq \alpha_2 \) by the minimality of \( \alpha_2 \). Hence

\[
(\alpha_1 - 1 - a)n_1 + (\alpha_3 - 1 - c)n_3 = (b - \alpha_2)n_2 + \alpha_3n_3,
\]

thus

\[
(\alpha_1 - 1 - a)n_1 = (b - \alpha_2)n_2 + (c + 1)n_3,
\]

which is against the minimality of \( \alpha_1 \). This shows that (b) implies (c).

If (c) holds, then \( \alpha_2 + \alpha_{12} + \alpha_{13} - 2 = s_Q(m) \leq r_Q(m) \leq \alpha_1 + \alpha_3 - 2 \). It follows immediately that \( \alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_3 - \alpha_{13} \). Hence \( \text{gr}_Q(m) \) is Cohen-Macaulay and (a) holds.

Now we consider the case when the group \( G \) is not symmetric. Recall that the defining ideal shall be

\[
I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}, f_2 := x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, f_3 := x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}).
\]

Our aim is to show that if \( s_Q(m) = r_Q(m) \), then \( \alpha_2 \leq \alpha_{21} + \alpha_{23} \). We first show that the index of nilpotency is

\[
s_Q(m) = \max \{ \alpha_2 + \alpha_{13} - 2, \alpha_3 + \alpha_{12} - 2 \}.
\]

Notice that \( \alpha_2 + \alpha_{13} - 2 = \alpha_2 + \alpha_3 - 2 - \alpha_{23} \), and \( \alpha_3 + \alpha_{12} - 2 = \alpha_2 + \alpha_3 - 2 - \alpha_{32} \).

\[
s_Q(m) = \max \{ \text{ord}_m(w) \mid w \in \text{Ap}(G, e) \}. \quad \text{Every} \ w \in \text{Ap}(G, e) \ \text{can be written as} \ w = bn_2 + cn_3 \ \text{for some} \ b, c \in \mathbb{N}_0. \ \text{Obviously,} \ b < \alpha_2 \ \text{and} \ c < \alpha_3. \ \text{And it cannot happen that both} \ b \geq \alpha_{12} \ \text{and} \ c \geq \alpha_{13}. \ \text{Hence for} \ w \ \text{with this representation, if} \ b = \alpha_2 - 1 \geq \alpha_{12}, \ \text{then} \ c < \alpha_{13} \ \text{because of} \ f_1. \ \text{Similarly, if} \ c = \alpha_3 - 1 \geq \alpha_{13}, \ \text{then} \ b < \alpha_{12} \ \text{because of} \ f_2. \ \text{On the other hand, it is not difficult to see that} \ (\alpha_2 - 1)n_2 + (\alpha_{13} - 1)n_3 \ \text{and} \ (\alpha_{12} - 1)n_2 + (\alpha_3 - 1)n_3 \ \text{are elements of} \ \text{Ap}(G, e). \ \text{For instance, suppose for contradiction that} \ (\alpha_2 - 1)n_2 + (\alpha_{13} - 1)n_3 = an_1 + bn_2 + cn_3 \ \text{is a solution.} \]


with $a, b, c \in \mathbb{N}_0$ and $\alpha > 0$. By the minimality of $\alpha_2$ and $\alpha_3$, it is clear that $\alpha_2 - 1 \geq b$ and $\alpha_3 - 1 \geq c$. Now

$$(\alpha_2 - 1 - b)n_2 + (\alpha_3 - 1 - c)n_3 = an_1.$$ 

$a > 0$, hence $a \geq \alpha_1$. Now

$$(\alpha_2 - 1 - b)n_2 + (\alpha_3 - 1 - c)n_3 = (a - \alpha_1)n_1 + \alpha_1n_2 + \alpha_3n_3.$$ 

This implies

$$(\alpha_2 - 1 - b)n_2 = (a - \alpha_1)n_1 + (c + 1)n_3.$$ 

This contradicts the minimality of $\alpha_2$. One can argue in a similar way for $(\alpha_3 - 1)n_2 + (\alpha_2 - 1)n_3$. Now the formula \(\square\) follows naturally.

The case when $s_Q(m) = \alpha_2 + \alpha_3 - 2$ is easy. Suppose the condition is satisfied, i.e., $\alpha_2 + \alpha_3 - 2 = r = r_Q(m)$. Then $x_2^{\alpha_2}x_3^{\alpha_3-1} \in m^{r+1} = Qm^r$. Notice that $x_2^{\alpha_2}x_3^{\alpha_3-1} = x_1^{\alpha_2}x_3^{\alpha_3-1}$. Hence $x_1^{\alpha_2}x_3^{\alpha_3-1} \in m^r$. Similar to the proof for (3) of Gorenstein case, one can show that this decomposition is unique, hence $\text{ord}_m(x_1^{\alpha_2}x_3^{\alpha_3-1}) = \alpha_2 + \alpha_3 - 2 \geq r = s = \alpha_2 + \alpha_3 - 2$. Hence $\alpha_2 \leq \alpha_2 + \alpha_3 - \alpha_3 = \alpha_2 + \alpha_3$.

If $s_Q(m) > \alpha_2 + \alpha_3 - 2$ and $r_Q(m) = s_Q(m)$, then $\delta := \alpha_2 - \alpha_3 > 0$ and $\alpha_2 + \alpha_3 - 2 = r - \delta$. Now $\text{ord}_m(x_2^{\alpha_2-1}x_3^{\alpha_3-1}) \geq r - \delta$, hence $x_2^{\alpha_2-1}x_3^{\alpha_3-1} \in m^{r+1} = Qm^r$. It follows easily that $g_0 := x_1^{\alpha_2-1}x_2x_3^{\alpha_3-1} \in m^r$. It is not true that this is the unique representation. However, suppose that associated graded ring $gr_m(R)$ is not Cohen-Macaulay, then $\alpha_2 = \alpha_2 + \alpha_3 > \alpha_2 + \alpha_3$. Hence $\delta = \alpha_2 - \alpha_3 < \alpha_2 - \alpha_2 < \alpha_2$. If we can modify the representation of $g_0$, then the smallest possible step to do it for the time being will be to use $f_3$. Now $g$ takes a different representation $g_1 := x_1^{\alpha_2-1}x_2^{\alpha_3-1}x_3^{\alpha_3-1}$. It is by the same reason that the only minimal relation that can be possibly used to modify the representation of $g_1$ is $f_3$. And either you go back to get $g_0$, or you continue to drop the exponents of $x_1$ and $x_2$, but increase the exponent of $x_3$. This argument can continue, i.e., one can only use $f_3$ for a multiple of times to modify the representation of $g_0$ to get some new $g_i$. Notice that $\alpha_3 < \alpha_2 + \alpha_3$, hence $\text{ord}_m(g_i)$ is achieved when it is written as $g_0$. The rest of the proof is similar. $\text{ord}_m(g_0) = \delta = \alpha_2 + \alpha_3 + \alpha_3 - 2 \geq r = s = \alpha_2 + \alpha_3 - 2 + \delta$. Hence $\alpha_2 \leq \alpha_2 + \alpha_3 - \alpha_3 = \alpha_2 + \alpha_3$. \hfill \(\square\)

The previous theorem fails if the embedding dimension is 4.

**Example 4.13.** Let $G = \langle 9, 10, 11, 23 \rangle$. Then $G$ is symmetric, and $s_Q(m) = r_Q(m) = 4$. But $gr_m(R)$ is not Cohen-Macaulay.

If the 1-dimensional local ring $R$ is not associated to numerical semigroup, then theorem might still fail, even when it has embedding dimension 3. The prototype of the following example is due to Lance Bryant.

**Example 4.14.** Let $S = \mathbb{C}[a, b, c]$ and $R = S/I$ where $I = (a^3 + c^3 + b^6, a^2b + ac^3 + b^6)$. Then $R$ is a 1-dimensional reduced ring. The initial form ideal $I^e = (b^2c^3 + aec^3, abc^3, a^2c^3, a^2b, a^3)$, hence $\mathbb{C}[a, b, c]/I^e$ is not Cohen-Macaulay. On the other hand, $Q = (b-c)R$ is a principal reduction of the maximal ideal $m = (a, b, c)R$. And $r_Q(m) = s_Q(m) = 6$. 
In this section, we want to prove the following main result by studying the standard basis of the corresponding defining ideal.

**Theorem 5.1.** Let the numerical semigroup $G = \langle n_1, n_2, n_3, n_4 \rangle$ be minimally generated by 4 elements. If the associated graded ring $\text{gr}_m(R)$ is Gorenstein, then for every $x \in \text{Ap}(G, n_1)$, $x \preceq_G (f + n_1)$.

The proof of the above theorem depends heavily on an important result by Bresinsky [3]:

**Theorem 5.2.** Let $G = \langle n_1, n_2, n_3, n_4 \rangle$ be a symmetric numerical semigroup, minimally generated by 4 elements. Then up to a permutation of generators, the defining ideal $I$ can be classified into the following three cases.

(I) $I = (x_1^{α_1} - x_2^{α_2}, x_3^{α_3} - x_4^{α_4}, x_1^{β_1} - x_2^{β_2}, x_3^{β_3} x_4^{β_4})$.

(II) $I = (x_1^{α_1} - x_2^{α_2}, x_3^{α_3} - x_1^{α_1} x_4^{α_2}, x_3^{α_3} - x_1^{α_1} x_2^{α_2} x_4^{α_3})$ with $α_{ij} \in \mathbb{N}_0$.

(III) $I = \{x_1^{α_1} - x_3^{α_3} x_4^{α_4}, x_2^{α_2} - x_1^{α_1} x_4^{α_2}, x_3^{α_3} - x_1^{α_1} x_2^{α_2}, x_3^{α_3} - x_2^{α_2} x_4^{α_3}, x_4^{α_4} - x_2^{α_2} x_3^{α_3}, x_4^{α_4} x_1^{α_1} - x_2^{α_2} x_3^{α_3} x_4^{α_4}\}$ with $0 < α_{ij} < α_j$.

**Remark 5.3.** Let $T = k[x_1, \ldots, x_d]$ be a polynomial ring of dimension $d$ over the field $k$. Let $m = (x_1, \ldots, x_d)$ be the graded maximal ideal of $T$. If an $T$-ideal $J$ is $m$-primary, monomial and Gorenstein, then it is generated by pure powers of the form $\{x_1^{α_1}, \ldots, x_d^{α_d}\}$. The proof is a direct application of linkage theory. When the ideal is Gorenstein, one can use the generators that are pure powers to generate a link. The resulting quotient ideal is an almost complete intersection and monomial. Now use the same pure powers to generate a new link, and we shall get the original monomial ideal. But it will be easy to see that for an almost complete intersection monomial ideal, this link has to be generated by pure powers. See [13] for related discussion.

As a generalization of the above idea, we can prove the following:

**Proposition 5.4.** Let ideal $L \subset S = k[x, y, z]$ be generated by $f = x^a - y^{b'} z^{c'} \in (x, y, z)$ and a finite collection of monomials. If $\text{ht}(L) = 3$ and $L$ is Gorenstein, then $L$ is generated by up to 5 elements.

**Proof.** In the following, fix the binomial $f$. We say an ideal $K$ is *almost monomial* if we can write $K = (f) + K'$ where $K'$ is an monomial ideal in $k[x, y, z]$. We fix a monomial ordering so that the initial monomial of $f$ is $x^a$. Depending on whether $a \geq b' + c'$ or not, this monomial ordering is either global or local. Applying the reduced standard basis algorithm to the existing generators of $I$, we may have a minimal standard basis $\{f_1, \ldots, f_s\}$. At most one of the $f_i$ is a binomial; the remaining are monomials. If $f$ is part of this minimal standard basis, then we say $K$ is *strictly almost monomial*. Following [5, 3] we may always assume that $L$ is strictly almost monomial.

Since $L$ has the maximal possible height, we can write $L$ in a standard form $L = (f, y^b, z^c) + L^\#$, where $L^\#$ is (minimally) generated by the remaining monomial generators of the standard basis. Let $J = (f, y^b, z^c)$, then

$$J : L = J : L^\# = \bigcap_{x^α y^{b'} z^{c'} \in L^\#} J : x^α y^{b'} z^{c'}.$$
Notice that for every $x^αy^βz^γ \in S$
\[
J : x^αy^βz^γ = (f, y^{\max(b-β,0)}, z^{\max(c-γ,0)}),
\]
\[x^{\max(a-α,0)}y^{\max(b'-β,0)}, x^{\max(a-α,0)}z^{\max(c'-c-γ,0)}).
\]
(2)

To prove it, one can argue directly or use standard basis arguments which we give in the Appendix.

Hence the quotient ideal $J : L$ is an intersection of almost monomial ideals. We want to show that it is again almost monomial. This can be done by using induction. Suppose $K_i = (f) + K'_i$, $i = 1, 2$, are two almost monomial ideals, then
\[
K_1 \cap K_2 = (f) + (K'_1 \cap K'_2).
\]
(3)

To see this, observe that for every element $g \in S$, there is $g' = \text{NF}(g|\{f\}) \in S$. For every monomial $m \in \text{mono-Supp}(g')$, $m$ is not divisible by $x^α$. Hence $g \in K$ if and only if $g' \in K'$.

Since $L$ is Gorenstein, $\mu((J : L)/J) = 1$. Hence the almost monomial ideal $J : L = J + x^αy^βz^γ$ for some monomial $x^αy^βz^γ \in S$. And $L = J : x^αy^βz^γ$ is given by the formula (2). In particular, $L$ is generated by up to 5 elements.

**Corollary 5.5 (Structure for Gorenstein strictly almost monomial ideals).** Let $L \subset k[x, y, z]$ be a strictly almost monomial ideal with $f = x^α - y^βz^c$. Then $L$ is Gorenstein and not a complete intersection if and only if $L$ can written as
\[
L = (f, y^b, z^c, x^{a-α}y^{b-b'}, x^{a-α}z^{c-c'}).
\]
(4)

**Proof.** If $L$ is almost monomial and Gorenstein, but not complete intersection, then we can write $L$ in the format (2). Since we can assume that $y^b$ and $z^c$ belong to the minimal generators of $L$, one must have $β = γ = 0$ and $L$ can be written in the given format (4).

On the other hand, if $L$ is written as in (4), then it is generated by the five submaximal Pfaffians of the anti-symmetric matrix
\[
M = \begin{pmatrix}
0 & 0 & -y^b & 0 & x^{a-α} \\
0 & 0 & -z^{c-c'} & y^{b-b'} & 0 \\
y^b & z^{c-c'} & 0 & x^α & 0 \\
0 & -y^{b-b'} & -x^α & 0 & z^c \\
x^{a-α} & 0 & 0 & -z^{c-c'} & 0
\end{pmatrix}.
\]

Since $\text{ht}(L) = 3$ and $L$ is Cohen-Macaulay, by the Buchsbaum-Eisenbud’s theorem [5, 3.4.1], $L$ is Gorenstein.

**Remark 5.6.** With the assumptions as in Corollary 5.5 and equip the polynomial ring $k[x, y, z]$ with a monomial ordering, such that the leading monomial of $f$ is $x^a$. Let $L$ be Gorenstein and strictly almost monomial with standard basis $\{f, y^b, z^c, g_4, \ldots, g_s\}$, where all the $g_i$’s are monomials. One observes the following from (4).

(a) None of the $g_i$’s is divisible by $yz$.

(b) If $L$ is not a complete intersection, then it has the two additional monomial generators that are not pure powers. These two monomials share the same $x$ component.
(c) None of the \( g_i \)'s is a pure power in \( x \). In other words, if a pure power term in \( x \) is a minimal generator for a Gorenstein almost monomial ideal, then this ideal is generated by pure powers in \( x \), \( y \) and \( z \).

These observations will play an essential role in the proof of [5.1]

**Remark 5.7.** Return to our discussion of numerical semigroup ring \( R \). When the embedding dimension \( d = 4 \), we know \( \text{ht}(I) = \text{ht}(I^*) = 3 \). Since \( \sqrt{I} = (x_2, x_3, x_4) \), \( \text{ht}(I^* + (x_1)) = 4 \). Thus if we assume that \( I^* \) is Cohen-Macaulay, then \( x_1 \) is regular modulo \( I^* \). And \( I^* \) is Gorenstein if and only if \( I^* + (x_1) \) is so. Furthermore, if \( f_1, \cdots, f_m \) is a minimal standard basis for \( I \) with respect to suitable local ordering, then the initial forms \( f_1^*, \cdots, f_m^* \) minimally generate \( I^* \). For each \( i \), if \( f_i^* \) is a monomial, since \( I^* \) is Cohen-Macaulay, \( f_i^* \) does not involve any \( x_1 \) term by [3.4]

Let \( \tilde{} \) denote the image by modulo \( x_1 \). If \( f_i^* \) is binomial, then \( f_i = x_3^{\beta_3} - x_2^{\beta_2} x_4^{\beta_4} \) is balanced. On the other hand, it is easy to check that at most one \( f_j \) can be binomial. Hence \( I^* \) is almost monomial where \( x_3 \) plays the same role as \( x \) in [5.5]

It follows from the previous discussion that

**Corollary 5.8.** If \( \text{gr}_m(R) \) is Gorenstein, then \( \mu(I^*) \leq 5 \).

Now we are ready for the proof of [5.1] We proceed according to the three cases in [5.2]. We always assume the monomial order for \( T \) is first nice in \( x \), then nice in \( x_k \). For instance, the usual negative degree reverse lexicographical ordering on \( k[x_1, x_2, x_3, x_4] \) satisfies this requirement. In particular, if \( x_3^{\alpha_3} - x_2^{\alpha_2} x_4^{\alpha_4} \) is balanced with \( \alpha_3 > 0 \), its initial monomial is \( x_3^{\alpha_3} \).

### 5.1. Proof for Case I.

**Remark 5.9.** The original statement of Theorem 4 in [8] contains an unnecessary condition \( 0 < \beta_i < \alpha_i \). This restriction is incorrect, as can be seen in [5.11] But the proof of it still holds without any further change. And the Lemma 5 before it only needs minimal modifications. It follows from this Lemma 5 and its corollary that

\[
(5) \quad f + n_1 = (\alpha_2 - 1)n_2 + (\alpha_3 + \beta_3 - 1)n_3 + (\beta_4 - 1)n_4.
\]

**Proof for case I.** If we insist that \( n_1 < n_2 < n_3 < n_4 \), then by a permutation we need to consider the following three sub-cases:

(i) \( I = (f_1 := x_1^{\alpha_1} - x_2^{\alpha_2}, f_2 := x_3^{\alpha_3} - x_4^{\alpha_4}, f_3 := x_3^{\beta_1} x_2^{\beta_2} - x_3^{\beta_1} x_4^{\beta_4}) \);

(ii) \( I = (f_1 := x_1^{\alpha_1} - x_3^{\alpha_3}, f_2 := x_1^{\alpha_1} x_2^{\beta_2} - x_3^{\beta_3} x_4^{\beta_4}) \);

(iii) \( I = (f_1 := x_1^{\alpha_1} - x_4^{\alpha_4}, f_2 := x_1^{\alpha_1} x_3^{\beta_3} - x_2^{\beta_2} x_3^{\beta_3}) \).

Let’s prove according to these three cases.

(i) Due to the generator \( f_2 \), we may assume that \( 0 \leq \beta_4 \leq \alpha_4 - 1 \) in \( f_3 \). Hence \( \beta_3 \neq 0 \) by the choice of \( \alpha_4 \). Similarly, we assume that \( 0 \leq \beta_3 \leq \alpha_2 - 1 \), and hence \( \beta_1 \neq 0 \). Since \( \text{spoly}(f_1, f_2) \) does not contribute to the standard basis, the initial forms \( f_1^* = -x_2^{\alpha_2}, f_2^* = -x_4^{\alpha_4}, f_3^* = -x_3^{\beta_3} x_4^{\beta_4} \) form part of the minimal basis of \( I^* \).

By [5.7] and [5.9] \( I^* \) is almost monomial. If \( \beta_4 \neq 0 \), then \( I^* \) is strictly almost monomial. Now by applying standard basis algorithm, we get \( f_4 = \text{spoly}(f_2, f_3) = x_3^{\alpha_3 + \beta_3} x_2^{\beta_2} x_4^{\beta_4 - \beta_4} \). By the Cohen-Macaulay condition, we must have \( \alpha_3 + \beta_3 \leq \beta_1 + \beta_2 + \alpha_4 - \beta_4 \). Since \( \beta_1 \neq 0 \), \( I^* \) has one minimal generator \( x_3^{\alpha_3 + \beta_3} \) which is a pure power. This is impossible for
strictly almost monomial ideals. Hence $\beta_3 = 0$ and $I^*$ is generated by pure powers. Notice that $f_1$, $f_2$ and $f_3$ form a standard basis for $I$, and $\beta_3 \geq \alpha_3$. It follows from the formula (5) that

\begin{equation}
 f + n_1 = (\alpha_2 - 1)n_2 + (\beta_3 - 1)n_3 + (\alpha_4 - 1)n_4.
\end{equation}

The decomposition of $f + n_1$ with respect to $G$ might not be unique in general. But it is easy to see that the coefficient $(\alpha_2 - 1)$ of $n_2$ is fixed. Now if $\beta_3 = \alpha_3$, then the above representation is unique. Otherwise, $\beta_3 - 1 \geq \alpha_3$, and we have at least one extra representation

\begin{equation}
 f + n_1 = (\alpha_2 - 1)n_2 + (\beta_3 - 1 - \alpha_3)n_3 + (2\alpha_4 - 1)n_4.
\end{equation}

Since $n_3 < n_4$, we have $\alpha_3 > \alpha_4$ by the relation $\alpha_3n_3 = \alpha_4n_4$. Hence the decomposition in (6) is the unique one such that gives us the maximal length of $f + n_1$ in terms of $n_2$, $n_3$ and $n_4$. Now $\text{ord}_G(f + n_1) = (\alpha_2 - 1) + (\beta_3 - 1) + (\alpha_4 - 1)$.

Similarly, every element $z_2n_2 + z_3n_3 + z_4n_4 \in \text{Ap}(G, n_1)$ can be written in such a way that $0 \leq z_2 \leq \alpha_2 - 1$, $0 \leq z_3 \leq \beta_3 - 1$ and $0 \leq z_4 \leq \alpha_4 - 1$. $\text{ord}_G(z_2n_2 + z_3n_3 + z_4n_4) = z_1 + z_2 + z_3$.

In particular, we have

\begin{equation}
 (z_2n_2 + z_3n_3 + z_4n_4) \preceq_G (f + n_1).
\end{equation}

(ii) We may assume that $0 \leq \beta_3 \leq \alpha_3 - 1$ and $0 \leq \beta_4 \leq \alpha_4 - 1$, so that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. We have $f_1^* = -x_3^{\alpha_3}$ and $f_2^* = -x_4^{\alpha_4}$. By comparing $\beta_1 + \beta_3$ with $\beta_2 + \beta_4$, we have three cases for $f_3^*$.

(a) $f_3^* = x_1^{\beta_1}x_3^{\beta_3}$. Following (5.2), the associated graded ring $\text{gr}_m(R)$ won’t be Cohen-Macaulay unless $\beta_1 = 0$, which is impossible here.

(b) $f_3^* = -x_2^{\beta_2}x_4^{\beta_4}$. Since $I^*$ is almost monomial, if $I^*$ is Gorenstein, then $\beta_4 = 0$. Now $\beta_2 < \beta_3 = 1$. Thus considering the permutation, from (5) we have

\begin{equation}
 f + n_1 = (\beta_2 - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4.
\end{equation}

Similar to the discussion in case (i), this decomposition is the unique maximal one. The rest of the proof is also similar.

(c) $f_3^* = x_1^{\beta_1}x_3^{\beta_3} - x_2^{\beta_2}x_4^{\beta_4}$. After modulo $x_1$, we are reduced to a case similar to (b).

(iii) $f_2^* = -x_3^{\alpha_3}$ is a pure power in $x_3$. Hence $I^*$ is a complete intersection in pure powers. This implies that $\beta_3 = 0$ and $\beta_2 \leq \beta_4 + \beta_1$ if we assume that $0 \leq \beta_4 \leq \alpha_4 - 1$ from the beginning. The remaining discussion is similar to that of case (i). 

\[\square\]

**Corollary 5.10.** In case I of (5.2) if $\text{gr}_m(R)$ is Gorenstein, then it is a complete intersection.

To illustrate the above proposition, we give several examples.

**Example 5.11.**

(i) If $G = (8, 12, 14, 21)$, then the defining ideal $I = (x_4^2 - x_3^2, x_4^2 - x_2^2, x_4^2 - x_1^2)$ and the initial form ideal $I^* = (x_4^2, x_2^2, x_1^2)$.

(ii) If $G = (8, 10, 12, 15)$, then $I = (x_4^2 - x_2^3, x_4^2 - x_1^2, x_2^2 - x_1x_3)$ and $I^* = (x_4^2, x_2^2, x_2^2 - x_1x_3)$. 
(iii) If \( G = \langle 30, 33, 44, 45 \rangle \), then \( I = (x_2^2 - x_1^4 x_4, x_3^4 - x_2^4, x_4^2 - x_1^3) \) and \( I^* = (x_2^2 - x_1^4 x_4, x_3^4, x_4^2) \).

5.2. **Proof for Case II.** In this case, the defining ideal is the complete intersection ideals studied in [11]. In particular, by [11, 2.1] the Frobenius number
\[
5.2.
\]
Proof for Case II.

We prove in accordance with the permutation \((i, j, k, l)\) of \((1, 2, 3, 4)\). By symmetry, we can always conveniently assume that \(i < j\), and consider the following 12 cases.

(i) \((i, j, k, l) = (1, 2, 3, 4)\). Then
\[
I = (f_1 := x_1^{a_1} - x_2^{a_2}, f_2 := x_3^{a_3} - x_1^{a_31} x_2^{a_32}, f_3 := x_4^{a_4} - x_1^{a_41} x_2^{a_42} x_3^{a_43}).
\]

Since \(n_1 < n_2 < n_3 < n_4\), we have \(a_2 < a_1, a_3 < a_31 + a_32\) and \(a_4 < a_41 + a_42 + a_43\). Now for every \(x \in \text{Ap}(G, n_1)\), suppose \(x = a_2 n_2 + a_3 n_3 + a_4 n_4\) with \(a_2 + a_3 + a_4 = \text{ord}_G(x)\). By the maximality of \(\text{ord}_G(x)\), one must have \(a_i < a_i\) for \(i = 2, 3, 4\). On the other hand, by (7),
\[
f + n_1 = (a_2 - 1) n_2 + (a_3 - 1) n_3 + (a_4 - 1) n_4.
\]

Now it follows immediately that \(x \preceq_G (f + n_1)\).

(ii) \((i, j, k, l) = (1, 2, 4, 3)\). Then
\[
I = (f_1 := x_1^{a_1} - x_2^{a_2}, f_2 := x_4^{a_4} - x_1^{a_41} x_2^{a_42} x_3^{a_43} - x_1^{a_41} x_2^{a_42} x_3^{a_43}).
\]

One has \(a_2 < a_1\) and \(a_4 < a_41 + a_42\) automatically. Now we can assume that in \(f_3\), \(0 \leq a_32 < a_2\) and \(0 \leq a_34 < a_3\). If \(a_31 \neq 0\), then by the Cohen-Macaulay condition, \(a_3 \leq a_31 + a_32 + a_34\). If \(a_31 = 0\), by our previous assumption on \(a_32\) and \(a_34\), \(f_3^*\) is part of the minimal generators for \(I^*\). But if \(I^*\) is Gorenstein, terms of the form \(x_1^{\beta_1} x_4^{\beta_4}\) with \(\beta_2, \beta_4 > 0\) will not be part of the minimal generators. Hence either \(a_3 \leq a_32 + a_34\) or one of \(a_32\) and \(a_34\) is zero. The latter cannot happen, since it is against to the choice of \(a_4\) or \(a_2\) respectively. In short, if we assume that \(I^*\) is Gorenstein, then we should have \(a_3 \leq a_31 + a_32 + a_34\). And the rest of the proof is similar to that of (11).

(iii) \((i, j, k, l) = (1, 3, 2, 4)\). Then
\[
I = (f_1 := x_1^{a_1} - x_3^{a_3}, f_2 := x_2^{a_2} - x_1^{a_21} x_3^{a_23}, f_3 := x_4^{a_4} - x_1^{a_41} x_2^{a_42} x_3^{a_43}).
\]

We automatically have \(a_3 < a_1\) and \(a_4 < a_41 + a_42 + a_43\). We can always assume that \(0 \leq a_23 < a_3\). Hence the initial form \(f_3^*\) is part of the minimal generators for \(I^*\). The Cohen-Macaulay condition would force \(a_2 \leq a_21 + a_23\). The rest of the proof is similar to that of (1).

(iv) \((i, j, k, l) = (1, 3, 4, 2)\). Then
\[
I = (f_1 := x_1^{a_1} - x_3^{a_3}, f_2 := x_4^{a_4} - x_1^{a_41} x_2^{a_43}, f_3 := x_2^{a_2} - x_1^{a_21} x_3^{a_23} x_4^{a_24}).
\]

We have \(a_3 < a_1\) and \(a_4 < a_41 + a_43\). In addition, we can assume that \(0 \leq a_23 < a_3\) and \(0 \leq a_24 < a_4\). Now \(f_3^*\) is part of the minimal generators. Notice that \(f_1^* = -x_3^{a_3}\) is also one such minimal generator. Hence the Gorenstein ideal \(I^*\) after modulo \(x_1\) has to be a complete intersection ideal generated by pure powers. This would force that \(a_2 \leq a_21 + a_23 + a_24\). The rest of the proof is similar to that of (1).
(v) \((i, j, h, k) = (1, 4, 2, 3)\). Then
\[
I = (f_1 := x_1^\alpha x_4^\alpha, f_2 := x_2^\alpha x_4^\alpha, f_3 := x_3^\alpha - x_1 x_2 x_3 x_4, f_4 := x_3^\alpha - x_1 x_2 x_3 x_4).
\]
We have \(\alpha_4 < \alpha_1\). Now we can assume that \(0 \leq \alpha_2, \alpha_4 < \alpha_4\). If \(f_2^*\) is part of a system of minimal generators, then the Cohen-Macaulayness would require that \(\alpha_2 < \alpha_2 + \alpha_2\). The rest of the proof is similar to that of (ii).

If \(f_2^*\) is not part of the minimal generating set for \(I^*\), then \(\alpha_3 = 0\), \(\alpha_3 > \alpha_3 + \alpha_3\), \(\alpha_2 > \alpha_2 + \alpha_2\) and \(x_{12} x_{42} x_{14}^2\) is divisible by \(x_1 x_4^2\). The problem will be reduced to the case (vi) with \(\alpha_3 > 0\).

(vi) \((i, j, h, k) = (1, 4, 3, 2)\). Then
\[
I = (f_1 := x_1^\alpha x_4^\alpha, f_2 := x_2^\alpha - x_1 x_2 x_3 x_4, f_3 := x_3^\alpha - x_1 x_2 x_3 x_4, f_4 := x_3^\alpha - x_1 x_2 x_3 x_4).
\]
Clearly \(\alpha_4 < \alpha_1\) and we can assume that \(0 \leq \alpha_2, \alpha_4 < \alpha_4\). Now if initial form \(f_2^*\) belongs to a system of minimal generators for \(I^*\), then the Cohen-Macaulay condition implies \(\alpha_3 \leq \alpha_3 + \alpha_3\). The rest of the proof is similar to that of (iv).

If \(f_2^*\) does not belong to a system of minimal generators for \(I^*\), then \(\alpha_2 = 0\), \(\alpha_2 > \alpha_2 + \alpha_2\), \(\alpha_3 > \alpha_3 + \alpha_3\) and \(x_{12} x_{42} x_{14}^2\) is divisible by \(x_1 x_4^2\). Now the problem is reduced to the case (v) with \(\alpha_3 > 0\).

(vii) \((i, j, h, k) = (2, 3, 1, 4)\). Then
\[
I = (f_1 := x_2^\alpha x_3^\alpha, f_2 := x_1^\alpha - x_2 x_3 x_4, f_3 := x_4^\alpha - x_1 x_2 x_3 x_4, f_4 := x_4^\alpha - x_1 x_2 x_3 x_4).
\]
Since \(n_1 < n_2 < n_3 < n_4\), \(f_1^* = -x_3^\alpha, f_2^* = -x_2 x_3 x_4, f_3^* = x_4^\alpha\). We can always assume that \(0 \leq \alpha_4 < \alpha_3\). Those \(f_i^*\) belong to a system of minimal generators for \(I^*\). Notice that \(f_1^*\) is a pure power in \(x_4\). Hence after modulo \(x_1\), \(I^*\) is generated by pure powers in \(x_2, x_3\) and \(x_4\). \(f_2^*\) does not involve \(x_1\) term, hence need to be a pure power. One cannot have \(\alpha_2 = 0\), since this is against the choice of \(\alpha_1\) and \(\alpha_3\). Hence \(\alpha_1 = 0\) and \(\alpha_3 > \alpha_2\).

Now for every \(x \in \text{Ap}(G, n_1)\), we can write \(x = \sum_i a_i n_i\) with \(\sum_i a_i = \text{ord}_G(x)\). Obviously \(a_1 = 0\), \(a_2 < \alpha_2, a_3 < \alpha_3\) and \(a_4 < \alpha_4\). Meanwhile, by (vii),
\[
f + n_1 = (\alpha_1 - 1)n_2 + (\alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4.
\]
Clearly \(x \leq G(f + n_1)\).

(viii) \((i, j, h, k) = (2, 3, 4, 1)\). Then
\[
I = (f_1 := x_2^\alpha x_4^\alpha, f_2 := x_1^\alpha - x_2 x_3 x_4, f_3 := x_3^\alpha - x_2 x_3 x_4, f_4 := x_3^\alpha - x_2 x_3 x_4).
\]
Obviously \(f_3^* = x_2 x_3 x_4\) and \(f_4^* = x_2 x_3 x_4\). In the \(f_3\), using \(f_1\) and \(f_2\), we can assume that \(0 \leq \alpha_3 < \alpha_3\) and \(0 \leq \alpha_3 < \alpha_4\). The rest of the proof is similar to that of (vii). In particular, we have \(\alpha_3 = \alpha_4 = 0\).

(ix) \((i, j, h, k) = (2, 4, 1, 3)\). Then
\[
I = (f_1 := x_2^\alpha - x_4^\alpha, f_2 := x_1^\alpha - x_2 x_4, f_3 := x_3^\alpha - x_2 x_3 x_4, f_4 := x_3^\alpha - x_2 x_3 x_4).
\]
\(f_1^* = -x_2 x_4\) and \(f_2^* = -x_2 x_4\). Using \(f_1\) we can always assume that \(0 \leq \alpha_4, \alpha_3 < \alpha_4\). Using \(f_2\), we can assume that \(x_2^2 x_4^2\) does not divide \(x_2^2 x_4^2\). Notice that \(\text{gr}_m(R)\) is a Gorenstein, then after modulo \(x_1\), \(I^*\) is an almost monomial ideal where \(x_2^2 x_4^2\) is not part of the minimal
generators, for \( \beta_2, \beta_4 > 0 \). Hence either \( f_2^* \) is not part of the minimal generators or it is a pure power.

The first case happens if and only if \( \alpha_{31} = 0, \alpha_{12} \geq \alpha_{32}, \alpha_{14} \geq \alpha_{34} \) and \( \alpha_3 > \alpha_{32} + \alpha_{34} \). Then this problem can be reduced to the case \([x]\) with \( \alpha_{13} > 0 \).

For the second situation, since \( \alpha_{14} < \alpha_4 \), \( f_2^* \) is a pure power if and only if \( \alpha_{14} = 0 \) and \( \alpha_{12} \geq \alpha_2 \). Now in addition, one may assume that \( 0 \leq \alpha_{32} \leq \alpha_{12} - 1 \). For \( f_3 \), if \( \alpha_3 \leq \alpha_{31} + \alpha_{32} + \alpha_{34} \), the proof can continue as in \([i]\). This condition must be satisfied when \( \alpha_{31} \neq 0 \) and \( \text{gr}_m(R) \) is Cohen-Macaulay. Otherwise, assume \( \alpha_{31} = 0 \) and \( \alpha_3 > \alpha_{32} + \alpha_{34} \). Then \( \alpha_{32}, \alpha_{34} > 0 \) and \( f_3^* = -x_2^{\alpha_{32}}x_4^{\alpha_{34}} \) will be part of the minimal basis of \( I^* \).

Whence \( \text{gr}_m(R) \) will not be Gorenstein.

\((x) \) \((i, j, h, k) = (2, 4, 3, 1)\). Then

\[ I = (f_1 := x_2^{\alpha_2} - x_4^{\alpha_4}, f_2 := x_3^{\alpha_3} - x_2^{\alpha_{32}}x_4^{\alpha_{34}}, f_3 := x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}}). \]

We can assume that \( 0 < \alpha_{14}, \alpha_{34} < \alpha_4 \).

\((A) \) If \( \alpha_3 < \alpha_{32} + \alpha_{34} \), we can also assume that \( 0 < \alpha_{13} < \alpha_3 \). Now \( f_1^*, f_2^* \) and \( f_3^* = -x_2^{\alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}} \) form part of a system of minimal generators for \( I^* \). \( f_2^* \) is a pure power in \( x_3 \), hence after modulo \( x_1, I^* \) is a complete intersection generated by pure powers as well. Thus \( \alpha_{13} = \alpha_{14} = 0 \) and the rest of the proof is similar to that of \([i]\).

\((B) \) If \( \alpha_3 > \alpha_{32} + \alpha_{34} \), like in \([ix]\), either we have \( \alpha_{34} = 0 \) and we have a contradiction, or \( \alpha_{13} = 0 \) and we can reduce the problem to the \([ix]\) with \( \alpha_{31} > 0 \).

\((C) \) If \( \alpha_3 = \alpha_{32} + \alpha_{34} \), then under the monomial ordering we specified early this section, the initial monomial of \( f_2 \) is \( x_3^{\alpha_3} \). Now we further assume that \( 0 < \alpha_{13} < \alpha_3 \). Hence \( f_1^*, f_2^* \) and \( f_3^* \) form part of the system of generators for \( I^* \), which is an almost monomial ideal when modulo \( x_1 \). By the structure of such ideals, \( -f_3^* = x_2^{\alpha_{12}}, \) or \( x_2^{\alpha_{12}}x_3^{\alpha_{13}}, \) or \( x_3^{\alpha_{13}}x_4^{\alpha_{14}}, \) with strictly positive exponents.

\((a) \) If \( -f_3^* = x_2^{\alpha_{12}} \), we can follow the proof in \([i]\).

\((b) \) If \( -f_3^* = x_2^{\alpha_{12}}x_3^{\alpha_{13}} \), we apply standard basis algorithm to find the standard basis for \( I \). During the process, we have \( \text{spoly}(f_2, f_3) = x_2^{\alpha_{32} + \alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}x_4^{\alpha_{14}} \). It is part of a minimal standard basis for \( I \). Hence according to the structure for Gorenstein almost monomial ideal, \( \text{gr}_m(R) \) is not Gorenstein.

\((c) \) If \( -f_3^* = x_3^{\alpha_{13}}x_4^{\alpha_{14}} \), by the structure property again, one has \( \alpha_{14} + \alpha_{34} = \alpha_4 \). Applying the standard basis algorithm, we find the two extra generators \( f_4 := x_2^{\alpha_{22}}x_3^{\alpha_{13}}x_4^{\alpha_{34}} \) and \( f_5 := x_2^{\alpha_{22} + \alpha_{32}} - x_1^{\alpha_{11}}x_3^{\alpha_{13} - \alpha_{13}} \). In particular, we have the necessary conditions \( \alpha_2 + \alpha_{13} \leq \alpha_1 + \alpha_{34} \) and \( \alpha_2 + \alpha_{32} \leq \alpha_1 + \alpha_3 - \alpha_{13} \). Now for every \( x = \sum a_i n_i \in \text{Ap}(G, n_1) \) with \( \sum a_i = \text{ord}_C(x) \), \( a_1 = 0, a_2 < \alpha_2 + \alpha_{32} \) and \( a_4 < \alpha_4 \). Furthermore, \( x_2^{\alpha_{22}}x_3^{\alpha_{13}} \) does not divide \( x_2^{\alpha_{22}}x_3^{\alpha_{13}} \) and \( x_3^{\alpha_{13}}x_4^{\alpha_{14}} \) does not divide \( x_3^{\alpha_{13}}x_4^{\alpha_{14}} \). From \([i]\), one has

\[ f + n_1 = (\alpha_2 + \alpha_{32} - 1)n_2 + (\alpha_{13} - 1)n_3 + (\alpha_4 - 1)n_4 \]
\[ = (\alpha_2 - 1)n_2 + (\alpha_{13} + \alpha_3 - 1)n_3 + (\alpha_4 - 1)n_4. \]
These are two decompositions of $f + n_1$ with respect to $G$ and have the same length. It will be routine to check that $x \preceq_G (f + n_1)$.

(xi) $(i, j, h, k) = (3, 4, 1, 2)$. Then

$$I = \langle f_1 := x_3^{\alpha_1} - x_4^{\alpha_1}, f_2 := x_3^{\alpha_1} - x_4^{\alpha_1}, f_3 := x_3^{\alpha_2} - x_4^{\alpha_2} - x_1^{\alpha_1}, f_4 := x_1^{\alpha_1} - x_2^{\alpha_2} - x_3^{\alpha_3} - x_4^{\alpha_4} \rangle.$$

Obviously $f_1^* = -x_4^{\alpha_1}$ and $f_2^* = -x_3^{\alpha_2} x_4^{\alpha_2}$. We can assume from the beginning that $0 \leq \alpha_{14}, \alpha_{24} < \alpha_4$, and $x_3^{\alpha_2} x_4^{\alpha_2}$ is not divisible by $x_3^{\alpha_1} x_4^{\alpha_2}$.

(A) If $\alpha_{14} = 0$, then $\alpha_{13} \geq 0$. We have two sub-cases.

(a) If $\alpha_{21} \neq 0$ or $\alpha_{24} \neq 0$, then $f_1^*$, $f_2^*$ and $f_3^*$ form part of a minimal system of generators for $I^*$. Since $f_2^*$ is a pure power in $x_3$, $I^*$ has to be a complete intersection after modulo $x_1$. This can happen only when $\alpha_2 \leq \alpha_{21} + \alpha_{23} + \alpha_{24}$. The rest of the proof for this case is similar to that of [i].

(b) If $\alpha_{21} = \alpha_{24} = 0$, then $\alpha_3 \leq \alpha_{23} < \alpha_{13}$. One can use $f_3$ to reduce the problem to the case (xii) with $\alpha_{12} = 0$.

(B) If $\alpha_{14} \neq 0$, since $\alpha_{14} < \alpha_4$, one must have $\alpha_{13} > 0$. Now consider the following two sub-cases.

(a) If $\alpha_2 \leq \alpha_{21} + \alpha_{23} + \alpha_{24}$, then the initial monomial of $f_3$ is $x_2^{\alpha_2}$. Now apply the standard basis algorithm. We get

$$f_4 := \text{spoly}(f_1, f_2) = x_3^{\alpha_1 + \alpha_{13}} - x_1^{\alpha_1} x_4^{\alpha_2 - \alpha_{14}}.$$

None of the two monomial terms is divisible by any of $f_1^*$, $f_2^*$ or $f_3^*$. Hence $f_4$ is one new generator for the standard basis. Since $\alpha_1 > 0$, by the Cohen-Macaulay condition, we need to have $\alpha_3 + \alpha_{13} \leq \alpha_1 + \alpha_4 - \alpha_{14}$. Now spoly($f_2, f_4$) = $x_1^{\alpha_1} f_1$. Hence the standard basis algorithm stops at this step, and the standard basis has only 4 elements. In other words, the initial form ideal $I^*$ is an almost complete intersection, and cannot be Gorenstein. One can also use the structure property for strictly almost monomial ideals, by noticing that it should not contain pure powers in $x_3$.

(b) If $\alpha_2 > \alpha_{21} + \alpha_{23} + \alpha_{24}$, then $f_3^* = -x_3^{\alpha_2} x_3^{\alpha_2} x_4^{\alpha_2}$. If $f_1^*$, $f_2^*$ and $f_3^*$ form part of the minimal system of generators, then by the Cohen-Macaulay condition, we must have $\alpha_{21} = 0$. Since $f_3^*$ is not a pure power, $I^*$ need to be a strictly almost monomial ideal after modulo $x_1$. If $f_3^*$ is a pure power in $x_3$, then $I^*$ need to be a complete intersection. This is impossible, since $\alpha_{14} \neq 0$. If $\alpha_{24} \neq 0$, then $\alpha_{23} > 0$. But in an almost monomial ideal of our kind, monomials in both $x_3$ and $x_4$ only show up once. Hence we have another contradiction.

If $f_1^*$, $f_2^*$ and $f_3^*$ do not form part of the minimal system of generators, since by our assumption $\alpha_{24} < \alpha_4$ and $x_3^{\alpha_2} x_4^{\alpha_2}$ is not divisible by $x_3^{\alpha_1} x_4^{\alpha_4}$, this can only happen when $f_2^*$ is divisible by $f_3^*$. In particular, $\alpha_{21} = 0$. Again, we can use $f_3$ to reduce the problem to the case (xii) with $\alpha_{12} \neq 0$.

(xii) $(i, j, h, k) = (3, 4, 2, 1)$. Then

$$I = \langle f_1 := x_3^{\alpha_1} - x_4^{\alpha_1}, f_2 := x_2^{\alpha_2} - x_3^{\alpha_2} x_4^{\alpha_2}, f_3 := x_3^{\alpha_1} - x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} \rangle.$$
5.3. Proof for case III. Up to a permutation of variables, the defining ideal \( I \) has the following form:

\[
I = (x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_3^{\alpha_{23}} x_4^{\alpha_{24}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, x_3^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{44}}).
\]

In this case, Bresinsky [3, Theorem 5] proved that \( \alpha_1 = \alpha_{21} + \alpha_{31}, \alpha_2 = \alpha_{32} + \alpha_{42}, \alpha_3 = \alpha_{13} + \alpha_{43} \) and \( \alpha_4 = \alpha_{14} + \alpha_{24} \).

\[\text{(A)}\] Suppose \( f_1^*, f_2^* \) and \( f_3^* \) form part of a minimal system of generators for \( I^* \). Since \( I^* \) is almost monomial after modulo \( x_1 \), we have the following cases to consider with all given exponents strictly positive.

\[(a)\] \( f_2^* = -x_2^{\alpha_{12}} \) with \( \alpha_{12} \geq \alpha_2 \). Applying standard basis algorithm, we get \( f_4 = \text{spoly}(f_1, f_2) = x_2^{\alpha_2} x_4^{\alpha_{24}} - x_3^{\alpha_{23}} \) and it should be a new generator for the standard basis. In almost monomial ideal described in 5.4, monomial of the form \( x_2^{\alpha_2} x_4^{\alpha_{24}} \) is not divisible by \( x_3^{\alpha_{23}} \).

\[\text{(B)}\] Suppose \( f_1^*, f_2^* \) and \( f_3^* \) form part of a minimal system of generators for \( I^* \). Since \( I^* \) is almost monomial after modulo \( x_1 \), we have the following cases to consider with all given exponents strictly positive.

\[(a)\] \( f_2^* = -x_2^{\alpha_{12}} \) with \( \alpha_{12} \geq \alpha_2 \). Applying standard basis algorithm, we get \( f_4 = \text{spoly}(f_1, f_2) = x_2^{\alpha_2} x_4^{\alpha_{24}} - x_3^{\alpha_{23}} \) and it should be a new generator for the standard basis. In almost monomial ideal described in 5.4, monomial of the form \( x_2^{\alpha_2} x_4^{\alpha_{24}} \) is not divisible by \( x_3^{\alpha_{23}} \).

\[(b)\] \( f_3^* = -x_3^{\alpha_{13}} x_4^{\alpha_{14}} \). Since \( f_3^* \) is irredundant, \( f_3^* \) cannot have strictly positive exponents for both variables. But \( 0 \leq \alpha_{23} < \alpha_4 \), hence \( \alpha_{23} \neq 0 \). Thus \( \alpha_{23} = 0 \) and \( f_2^* \) is a pure power in \( x_3 \). As a result, \( I^* \) need to be a complete intersection after modulo \( x_1 \). This is impossible, because of \( f_3^* \).

\[(d)\] \( f_3^* = -x_3^{\alpha_{12}} x_3^{\alpha_{13}} x_4^{\alpha_{14}} \). By the structure of almost monomial ideal discussed in 5.4, we must have \( \alpha_{24} \neq 0 \) and \( \alpha_{13} = \alpha_{23} \). Apply the standard basis algorithm, we get two more expected generators

\[
f_4 := \text{spoly}(f_1, f_2) = x_2^{\alpha_2} x_4^{\alpha_{24}} - x_3^{\alpha_{23}} \]
and
\[
f_5 := \text{spoly}(f_2, f_3) = x_2^{\alpha_{23}} x_4^{\alpha_{24}} - x_1^{\alpha_1} x_4^{\alpha_{24}}.
\]

Meanwhile, we get two necessary conditions \( \alpha_3 + \alpha_{23} = \alpha_2 + \alpha_4 - \alpha_{24} \) and \( \alpha_2 + \alpha_{12} \leq \alpha_1 + \alpha_{24} \). By (7), we know

\[
f + n_1 = (\alpha_2 + \alpha_{12} - 1) n_2 + (\alpha_{13} - 1) n_3 + (\alpha_4 - 1) n_4
\]

\[= (\alpha_{12} - 1) n_2 + (2\alpha_{13} + \alpha_3 - 1) n_3 + (\alpha_{24} - 1) n_4.
\]

The rest of the proof is similar to that of [c] in [x].

\[(B)\] If \( f_1^*, f_2^* \) and \( f_3^* \) fail to be part of a system of minimal generators, then \( f_3^* \) must divide \( f_2^* \). This means \( \alpha_{12} = 0 \) as well. Now using \( f_3^* \), the problem can be reduced to the case [xi] with \( \alpha_{21} \neq 0 \).
To simplify the notation, we follow the convention in [2] and write it as

\[ f_1 = (1, (3, 4)), f_2 = (2, (1, 4)), f_3 = (3, (1, 2)), f_4 = (4, (2, 3)), f_5 = ((1, 3), (2, 4)). \]

Hence we have six cases to consider

1. \( f_1 = (1, (3, 4)) \)
   (a) \( f_2 = (2, (1, 4)), f_3 = (3, (1, 2)), f_4 = (4, (2, 3)), f_5 = ((1, 3), (2, 4)) \)
   (b) \( f_2 = (2, (1, 3)), f_3 = (3, (2, 4)), f_4 = (4, (1, 2)), f_5 = ((1, 4), (2, 3)) \)
2. \( f_1 = (1, (2, 3)) \)
   (a) \( f_2 = (2, (3, 4)), f_3 = (3, (1, 4)), f_4 = (4, (1, 2)), f_5 = ((2, 4), (1, 3)) \)
   (b) \( f_2 = (2, (1, 4)), f_3 = (3, (2, 4)), f_4 = (4, (1, 3)), f_5 = ((1, 2), (4, 3)) \)
3. \( f_1 = (1, (2, 4)) \)
   (a) \( f_2 = (2, (1, 3)), f_3 = (3, (1, 4)), f_4 = (4, (2, 3)), f_5 = ((1, 2), (3, 4)) \)
   (b) \( f_2 = (2, (3, 4)), f_3 = (3, (1, 2)), f_4 = (4, (1, 3)), f_5 = ((2, 3), (1, 4)) \)

We also need the following result.

**Theorem 5.12** ([2], 2.10). In case III of Theorem 5.12, write \( f_1 = x_1^{a_1} - m_1, f_2 = x_2^{a_2} - m_2, f_3 = x_3^{a_3} - m_3, f_4 = x_4^{a_4} - m_4 \) and \( f_5 \), where \( m_1, m_2, m_3 \) and \( m_4 \) are monomials. If \( a_2 \leq \text{total degree of } m_2 \) and \( a_3 \leq \text{total degree of } m_3 \), then the tangent cone \( \text{gr}_m(R) \) is Cohen-Macaulay.

Their conditions are equivalent to saying:

- case 1(a): \( a_2 \leq a_{21} + a_{24} \);
- case 1(b): \( a_2 \leq a_{21} + a_{23}, a_3 \leq a_{32} + a_{34} \);
- case 2(b): \( a_2 \leq a_{21} + a_{24}, a_3 \leq a_{32} + a_{34} \);
- case 3(a): \( a_2 \leq a_{21} + a_{23}, a_3 \leq a_{31} + a_{34} \).

No similar result is reproduced for 2(a) and 3(b), since \( f_2 = x_2^{a_2} - x_3^{a_{23}}x_4^{a_{24}} \) is an element of the generator set in both cases, and thus \( a_2 > a_{23} + a_{24} \).

It follows from [3, 2] that if \( \text{gr}_m(R) \) is Cohen-Macaulay, then \( I^* \) does not contain any monomial minimal generator that is divisible by \( x_1 \).

**Corollary 5.13.** Assume the setting of [6, 2] if the defining ideal \( I \) is not a complete intersection and the associated graded ring \( \text{gr}_m(R) \) is Cohen-Macaulay, then the following inequalities hold:

- case 1(a): \( a_2 \leq a_{21} + a_{24}, a_3 \leq a_{31} + a_{32}, a_{21} + a_{43} \geq a_{32} + a_{14} \);
- case 1(b): \( a_2 \leq a_{21} + a_{23}, a_4 \leq a_{41} + a_{42}, a_{21} + a_{43} \geq a_{42} + a_{13} \);
- case 2(a): \( a_3 \leq a_{31} + a_{34}, a_4 \leq a_{41} + a_{42}, a_{41} + a_{23} \geq a_{12} + a_{34} \);
- case 2(b): \( a_2 \leq a_{21} + a_{24}, a_3 \leq a_{41} + a_{43}, a_{41} + a_{32} \geq a_{24} + a_{13} \);
- case 3(a): \( a_2 \leq a_{21} + a_{23}, a_3 \leq a_{31} + a_{34}, a_{31} + a_{42} \geq a_{23} + a_{14} \);
- case 3(b): \( a_3 \leq a_{31} + a_{32}, a_4 \leq a_{41} + a_{43}, a_{41} + a_{34} \geq a_{31} + a_{24} \).

Some of the inequalities are redundant, and hold automatically even without assuming the Cohen-Macaulayness of \( \text{gr}_m(R) \). They are kept here for completeness. But as an interesting result, we can see that

**Corollary 5.14.** For cases 1(a) and 3(a), the conditions in [2], 2.10 are necessary and sufficient for \( \text{gr}_m(R) \) to be Cohen-Macaulay.

Due to [2, 5] our proof for [5.1] will be done by showing the following.

**Proposition 5.15.** Assume the notations in [5, 2]. Then \( \text{gr}_m(R) \) is Gorenstein if and only if the following conditions hold:
(a) $\alpha_2 \leq$ total degree of $m_2$,
(b) $f_3 = (3, (2, 4))$ and $\alpha_3 =$ total degree of $m_3$.

In particular, $\text{gr}_m(R)$ could be Gorenstein only for case 1(b) and case 2(b).

**Proof.** Sufficiency. Condition (2) only holds for cases 1(b) or 2(b). According to [2, 2.10], $\text{gr}_m(R)$ is Cohen-Macaulay and the initial form ideal $I^* \subset S = k[x_1, x_2, x_3, x_4]$ of $I$ is generated by the initial form of each generators.

In case 1(b) we have

$$I^* = (x_3^{\alpha_1} x_4^{\alpha_4}, x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3} \text{ or } x_2^{\alpha_2} - x_1^{\alpha_1} x_3^{\alpha_3}, x_4^{\alpha_4}, (x_1^{\alpha_1} x_4^{\alpha_4} - x_2^{\alpha_2} x_3^{\alpha_3})).$$

We want to point out that $\alpha_1 + \alpha_4 \geq \alpha_2 + \alpha_3$. To see this inequality, just note that by the assumptions we have

$$\alpha_2 = \alpha_3 + \alpha_2 \leq \alpha_2 + \alpha_3 \quad \text{and} \quad \alpha_3 = \alpha_3 + \alpha_3 \leq \alpha_3 + \alpha_3.$$ 

Now it follows that

$$(x_1^{\alpha_1} x_4^{\alpha_4} - x_2^{\alpha_2} x_3^{\alpha_3})^* = (x_1^{\alpha_1} x_4^{\alpha_4} - x_2^{\alpha_2} x_3^{\alpha_3}) \text{ or } -x_2^{\alpha_2} x_3^{\alpha_3}.$$ 

Hence after modulo $x_1$, $I^*$ becomes

$$\tilde{I}^* = (x_3^{\alpha_1} x_4^{\alpha_4}, x_2^{\alpha_2} - x_3^{\alpha_3} x_4^{\alpha_4}, x_4^{\alpha_4}, x_2^{\alpha_2} x_3^{\alpha_3}).$$ 

It follows from [5, 3] that $I^*$ is a Gorenstein ideal.

As for case 2(b), we have $f_5 = x_1^{\alpha_1} x_2^{\alpha_2} - x_3^{\alpha_3} x_4^{\alpha_4}$. Hence $\alpha_1 n_1 + \alpha_2 n_2 = \alpha_1 n_3 + \alpha_2 n_4$. But $\alpha_1 n_1 + \alpha_3 n_2 < \alpha_1 n_2 + \alpha_2 n_2$ and $\alpha_1 n_3 + \alpha_2 n_4 > \alpha_1 n_2 + \alpha_2 n_2$. This implies that $\alpha_1 + \alpha_2 > \alpha_3 + \alpha_4$, and $f_5 = -x_3^{\alpha_1} x_4^{\alpha_4}$. The rest of the discussion is similar to that of the previous case.

Necessity. Since $0 < \alpha_{ij} < \alpha_j$, the five generators $f_i$ form part of a minimal standard basis for $I$, and their initial forms $f_i^*$ form part of the minimal basis for the leading ideal $I^*$. By the Cohen-Macaulay assumption, after modulo $x_1$, none of the leading forms will go to zero. Hence $f_5^*$ form part of the minimal basis for $\tilde{I}^* \subset k[x_2, x_3, x_4]$, where $^*$ denotes the image $k[x_1, x_2, x_3, x_4]/(x_1) \cong k[x_2, x_3, x_4]$.

By the Gorenstein assumption, this $\tilde{I}^*$ is again Gorenstein, and not a complete intersection. Hence by [5.7], it is strictly almost monomial and 5-generated. Now the assertion follows from [5.6] and [5.13].

**Corollary 5.16.** Assume that the numerical semigroup $G$ is $4$-generated, and the defining ideal $I$ is not a complete intersection. If the associated graded ring $\text{gr}_m(R)$ is Gorenstein, then the elasticity of the special element $f + n_1 = 1$ and $\text{ord}_G(f + n_1) = \alpha_2 + \alpha_4 + \alpha_3 - 3$.

**Proof.** By [5.15], we only need to consider the two cases 1(b) and 2(b). We can write $f + n_1 = z_1 n_1 + z_2 n_2 + z_3 n_3 + z_4 n_4$. Since $t^{f + n_1} \not\in (t^{n_1})$, we must have $z_1 = 0$, and hence $z_2 \leq \alpha_2 - 1$ and $z_4 \leq \alpha_4 - 1$. Suppose that we can also write $f + n_1 = z_2' n_2 + z_3' n_3 + z_4' n_4$. By symmetry, there are three cases to consider.

(a) $z_2' > z_2$, $z_3' \leq z_3$ and $z_4' \leq z_4$.
(b) $z_3' > z_3$, $z_2' \leq z_2$ and $z_4' \leq z_4$.
(c) $z_4' > z_4$, $z_2' \leq z_2$ and $z_3' \leq z_3$.

By the choice of $\alpha_2$, $\alpha_3$ and $\alpha_4$, the first and third case cannot happen. Because otherwise, for instance in the first case, we would have $(z_2' - z_2) n_2 = (z_3' - z_3) n_3 + (z_4' - z_4) n_4$. Since $z_2' - z_2 < \alpha_2$, this cannot happen.
By the same reason, in the second case, one must have $z_2' = z_2 - k\alpha_3$, $z_3' = z_3 + k\alpha_3$ and $z_4' = z_4 - k\alpha_{34}$ for some $k \in \mathbb{N}_0$. Since $\alpha_3 = \alpha_{32} + \alpha_{34}$, we can conclude that $\sum z_i = \sum z_i'$, i.e., the elasticity of $f + n_1$ is 1.

Now by the proof of [3, Theorem 5], one knows that $f + n_1 = (\alpha_2 - 1)n_2 + (\alpha_{13} - 1)n_3 + (\alpha_{14} - 1)n_4$. It is clear then $(\alpha_2 - 1) + (\alpha_{13} - 1) + (\alpha_{14} - 1)$ is one (and hence the) length of $f + n_1$.

**Corollary 5.17.** In case III of 5.2, if $\text{gr}_m(R)$ is Gorenstein, then the initial form ideal $I^*$ is 5-generated.

**Remark 5.18.** The converse of 5.17 is not true. For instance in case 1(a) of 5.14, when $\text{gr}_m(R)$ is Cohen-Macaulay, $I^*$ is 5-generated. But it can never be Gorenstein by 5.15.

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**Appendix**

Here we use the standard basis method to give a strict proof of formula (2). Without loss of generality, we can assume that $\alpha < a$, $\beta < b - b'$ and $\gamma < c - c'$.

Consider the following monomials and binomials in $k[t, x, y, z]$ with lexicographical monomial ordering:

- $f_1 = tx^a - ty^{b'} z^{c'}$
- $f_2 = ty^b$
- $f_3 = tz^c$
- $f_4 = tx^a y^\beta z^{\gamma} - x^\alpha y^\beta z^{\gamma}$
- $f_5 = ty^{b' + \beta} z^{c' + \gamma} - x^\alpha y^\beta z^{\gamma}$
- $f_6 = x^\alpha y^b z^c$
- $f_7 = x^\alpha y^\beta z^c$
- $f_8 = x^\alpha y^b z^{c'}$
- $f_9 = x^\alpha y^\beta z^{c - c'}$
- $f_{10} = x^{\alpha+a} y^{\beta + a'} z^{\gamma + c'} - x^\alpha y^{b + b'} z^{\gamma + c'}$

The following table shows that every s-polynomial $\text{spoly}(f_i, f_j)$ is either 0 or can be reduced by some $f_k$. By symmetry, we only need to consider for $i < j$.

|     | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| $f_1$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_2$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_3$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_4$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_5$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_6$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_7$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_8$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_9$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
| $f_{10}$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0       |
Since \( f_5 = \text{spoly}(f_1, f_4), \ f_6 = \text{spoly}(f_2, f_4), \ f_7 = \text{spoly}(f_3, f_4), \ f_8 = \text{spoly}(f_2, f_5), \ f_9 = \text{spoly}(f_3, f_5) \) and \( f_{10} = \text{spoly}(f_4, f_5) \), we know immediately that the set \( \{ f_i : 1 \leq i \leq 10 \} \) forms a standard basis for \( I = (f_1, f_2, f_3, f_4) \). Hence by [8, 1.8.10], the intersection ideal is 
\[
(x^a - y^b z' , y^b, z^c) \cap (x^a y^b z^c) = (x^a y^b z^c , x^a y^b z^c) \],
\[
= (x^a y^b z^c , x^a - y^b z' , y^b, z^c)
\]
and by [8, 1.8.12], the quotient ideal is 
\[
(x^a - y^b z', y^b, z^c) : (x^a y^b z^c , x^a - y^b z' , y^b, z^c) = (x^a y^b z^c , x^a y^b z'^c - \gamma , x^a y^b z^c - \gamma , x^a - y^b z').
\]

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