On a Relation Between the Integral Image Algorithm and Calculus

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Abstract—This paper is purely theoretical. The main results are: (1) a more rigorous formulation to a parameter at a continuous version of the Integral Image algorithm, and (2) an extension of the algorithm to more general types of domains. The Integral Image algorithm has become an important tool in the image processing community, ever since it was first introduced in the context of objects detection by Viola and Jones in 2001. In this paper, theoretical aspects of this algorithm’s continuous extension from 2007 are being discussed. The paper suggests that a decisive parameter at the formulation of the algorithm should be coherently defined via a novel pointwise operator. This operator examines a function’s instantaneous trend of change, rather than the function’s instantaneous rate of change - as examined by the derivative. The paper also depicts a novel integration method over curves in the plane, whose aim is to naturally extend the algorithm to a more general set of domains.

Index Terms—Calculus, Integral Image, Antiderivative Theorem, Detachment, Tendency, Slanted Line Integral, Semi-discrete

1 PREVIOUS WORK

Ever since the early 1980’s, computer scientists have been using an algorithm named “Summed Area Table”, also known as “Integral Image”. The algorithm suggests a formula for a rapid evaluation of the sum of rectangles in a table (or in an image), given that the ‘image of sums’, or the integral image, is pre-evaluated. The algorithm was first introduced in 1984 by Crow (in [12]), and was reintroduced to the computer vision community in 2001 by Viola and Jones (in [3]).

The algorithm description is as follows. Given a function \( i \) over a discrete domain \( \prod_{j=1}^{2} [m_j, M_j] \subset \mathbb{Z}^2 \), define a new function (\( sat \) stands for summed area table, and \( i \) stands for image):

\[
sat(x, y) \equiv \sum_{x' \leq x \land y' \leq y} i(x', y'),
\]

and now the sum of all the values that the function \( i \) accepts on the grid \( [a, b] \times [c, d] \), where \( m_1 \leq a, b \leq M_1 \) and \( m_2 \leq c, d \leq M_2 \), equals:

\[
\sum_{x'=a}^{b} \sum_{y'=c}^{d} i(x', y') = sat(b, d) + sat(a, c) - sat(a, d) - sat(b, c).
\]

Figure 1. The corners that Wang et. al defined in their paper. The number is the parameter \( \alpha_D \), i.e., the coefficient of the antiderivative in the formulation of the Antiderivative Theorem for the case \( n = 2 \) at the specific corner. The location of the domain with respect to the corner, is highlighted in brown.

The integral image formula, being simple, elegant, efficient, and mathematically interesting (since it forms a natural extension of Newton-Leibniz’s axiom to the plane) has become a very important tool in the past few decades. Many algorithms were driven from the idea of the integral image formula, such as the integral video algorithm (see [10]), and a rotated version of the integral image (see [11]). Popular applications of the integral image formula are efficient face detection (as performed in [3]) and pedestrians detection (see [4]). Another application that relies on a similar idea to that of the integral image (but applied to a different domain) is Integral Histogram (see [8]). Previously known algorithms were also enhanced via the integral image algorithm, such as the SIFT algorithm (see [9]). Finally, there are works that suggest enhancements to the integral image algorithm
itself, such as Hensley et al.’s, see \cite{7}.

In 2007, Wang et al. (in \cite{5}) suggested a rigorous formulation to a natural extension of the integral image algorithm, whose formulation is as follows. Let $D \subset \mathbb{R}^n$ be a ‘generalized rectangular domain’ (a domain that consists of a finite unification of axis aligned boxes - those whose edges are axis aligned), and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an integrable function. Let $F$ be the antiderivative of $f$, i.e., $F(\vec{x}) = \int_{B(\vec{x})} f(\vec{y}) \, d\vec{y}$, where $B(\vec{x}) \subset \mathbb{R}^n$ is an axis aligned box, that is determined according to the point $\vec{x}$ and an arbitrary static point (for example, the origin). Then:

$$\int_D f \, d\vec{x} = \sum_{\vec{x} \in V} \alpha_D(\vec{x}) F(\vec{x}), \quad (1)$$

where $\nabla \cdot D$ is the given domain’s ($D$) set of corners (this notation was suggested by Wang et al. in their work), and $\alpha_D : \mathbb{R}^n \rightarrow \mathbb{Z}$ is a map that depends on $n$. For $n = 2$ it is such that $\alpha_D(\vec{x}) \in \{0, \pm1, \pm2\}$ according to which of the 10 types of corners, depicted in figure 1 in \cite{5} (and for the reader’s convenience, also in this paper’s figure [1]), $\vec{x}$ belongs to. The theorem formulated in equation (1) will be referred to as the “Antiderivative Theorem” in this paper. Further, in their comprehensive work from 2011, Doretto et al. state the Antiderivative Theorem as a key theoretical result (see theorem 1 in \cite{2}).

Note that the Antiderivative Theorem extends the integral image algorithm in the following three senses: it is stated for continuous domains ($\mathbb{R}$ rather than $\mathbb{Z}$); it is stated for more general types of domains (a finite unification of rectangles, rather than mere rectangles); and it is formulated in higher dimensions. In this paper we will refer only to the two-dimensional version of the Antiderivative Theorem, which is illustrated also in \cite{20}.

2 THIS RESEARCH’S GOALS

Upon observing the Antiderivative Theorem in equation (1), two theoretical questions naturally rise. The first question is, how can the parameter $\alpha_D$ be defined given a parametrization of the domain’s edge, $\partial D$. The second is, how can this theorem be extended to more general types of domains (rather than finite unification of axis aligned boxes). Section 3 introduces a novel, semi-discrete pointwise operator, namely a function’s detachment, which is in turn used for a coherent definition of the parameter $\alpha_D$ from formula (1), given a parametrization of the domain’s edge. The definition is coherent in the sense that it is independent of the curve’s different parametrizations. Section 4 suggests a natural extension of the Antiderivative Theorem for the case $n = 2$, to more general domains than finite unifications of boxes. Based on the tool that enables a coherent definition of $\alpha_D$ (a function’s detachment), the appendix depicts, among others, simple theoretical results for single variable functions, to bring more depth into the theoretical discussion.

3 DEFINING THE PARAMETER $\alpha_D$ COHERENTLY

3.1 The Parameter $\alpha_D$ and the Curve’s Derivatives

Since the parameter $\alpha_D$ at formula (1) is uniquely defined according to the corner’s type, then a tool for classification of corners along a given curve is required. In other words, given a curve $\gamma(t) = (x(t), y(t))$, we would like to point out a parameter that enables to distinguish between different types of corners along this curve and other curves, as depicted in figure 1. Intuitively, given a corner point $\gamma(t_0)$ along a curve, we would expect that the curve’s one-sided derivatives vector at the corner point, that is, $((x_+, x_-, y_+, y_-)^T)_{t=t_0}$, would gain a constant value for corners of that certain type, independently of the curve’s parametrization; thus, we would expect this vector to distinguish between different types of corners. The next example clarifies in what sense this intuition is incorrect, and the consequence will be that the curve’s one-sided derivatives vector is an incoherent tool in the task of corners classification.

![Figure 2. An illustration to the curve C, whose different parametrizations are discussed in example (1)](image)

Example 1. Let us analyze different parametrizations of the same curve $C$ (see figure 2), whose corner occurs at $(0, 0)$. Let us evaluate the curve’s one-sided derivatives at the corner point, for different parametrizations differed by the value of $k$,

$$C : \gamma_k(t), \quad 0 \leq t \leq 2$$

$$\gamma_k(t) = \begin{cases} 
(1 - t)^k, 0 \leq t \leq 1 \\
0, (t - 1)^k, \quad 1 \leq t \leq 2,
\end{cases}$$

where $k \in \mathbb{R}_+^*$. Note that the corner occurs at $t = 1$.

For $k = 1$, $\gamma_1$ forms an arc-length parametrization of the curve. In that case, the curve’s one-sided derivatives at the corner point are $x_+(1) = 0$, $x_-(1) = -1$ and: $y_+(1) = +1$, $y_-(1) = 0$.

For $k \in (0, 1)$, some of the one-sided derivatives of $\gamma_k$ do not exist at the corner point (that is, $x_-(1), y_+(1)$ are undefined).

For $k > 1$, the one-sided derivatives of $\gamma_k$ are all zeroed at the corner point.

The consequence from example 1 is that the vector:

$$((x_+, x_-, y_+, y_-)^T)_{t=t_0}$$
is not a coherent tool in the task of classification of corners (since this vector is dependent of the curve’s parametrization). A possible approach to resolve that lack of consistency is to assume an arc-length parametrization whenever a classification of a corner is required, and evaluate the one-sided derivatives. However, this approach is not optimal, since there are uncountably many other parametrizations of the curve left unhandled. Hence, a different tool is required for the task of corners classification.

Remark 2. Notice that in fact, the derivative inquires superfluous information for this task, since we can settle for less information and inquire the sign of the one-sided derivatives of the functions that form the curve’s parametrization. It is easy to see that the vector of the one-sided derivatives’ signs is not a coherent tool either, due to the previously discussed reason.

In sub-section 3.2 we introduce a simple tool, whose definition results from the following question: why calculate the derivative from the beginning, if all we are interested in - is its sign?

3.2 Definition of a Function’s Detachment

In order to illustrate the incoherency that rises from example 1 in a clearer manner, let us apply a relaxation to this problem. Let us transform it from a problem on curves, to a problem on single variable monomials. The simpler problem is formulated in the following example.

Example 3. Let us consider the following family of monomials:

\[ f_k : \mathbb{R}^+ \to \mathbb{R} \]
\[ f_k(x) = x^k, \]

where \( k \in \mathbb{R}^+ \) is a positive real number. The right derivative at zero equals:

\[ (f_k)'_+ (0) = \begin{cases} 
\text{undefined}, & k \in (0, 1) \\
+1, & k = 1 \\
0, & k > 1.
\end{cases} \]

Thus, when applied at the point \( x = 0 \), the function’s right derivative depends on the value of \( k \), and in uncountably many cases it vanishes - either undefined or zeroed.

Let us introduce a robust pointwise operator, in the sense that it is independent of the parameter \( k \), while supplying a superficial (yet in some cases, sufficient) information regarding the function’s local monotony behavior.

Definition 4. Detachment of a function. Let us define the one-sided detachments of a function at a point as:

\[ f_{\pm} : \mathbb{R} \to \{0, \pm 1\} \]
\[ f_{\pm}^r (x) = \lim_{h \to 0^\pm} \text{sgn}[f(x + h) - f(x)], \]

if the one-sided limits exist. The definition is illustrated in [21] and in figure 4.

Figure 3. Let us observe the term \( f(x + h) - f(x) \). It is clear that if the limit process is applied immediately, then for any continuous function, it holds that: \( \lim_{h \to 0^\pm} [f(x + h) - f(x)] = 0 \). The derivative, however, manages to supply abounding information regarding the function’s local behavior by inquiring more information: it compares \( dy \) and \( dx \), via a fraction. The detachment uses less information, and \( dy \) is quantized, via the \( \text{sgn} (\cdot) \) function. A function’s detachment reveals a superficial information regarding the function’s instantaneous trend of change, rather than its instantaneous rate of change. Among other properties, this approach demonstrates a trade-off between the set of functions for whom the operator is defined, and the information level it supplies. A simple theory that relies on the definition of the detachment, similarly to the the Calculus that relies on the derivative, is then introduced (but exceeds the paper’s main goal) - see the appendix, sub-sections 6.1 and 6.2.

Remark 5. In terms of example 3, it is easy to validate that the one-sided right detachment to the function \( f_k \) at zero equals:

\[ (f_k)'_+ (0) = +1, \]

independently of the parameter \( k \). Thus, the detachment is capable of supplying information regarding the local monotony behavior of the function near a point, even in cases where the derivative vanishes (in terms of example 3 this could be the case at cusps, where \( k \in (0, 1) \), or at extrema points, where \( k > 1 \)). An extended discussion is held at figure 3 and at the appendix.

3.3 The Parameter \( \alpha_D \) and the Curve’s Detachments

Going back to example 1, the one-sided detachments vector:

\[ (x_+^i, x_-^i, y_+^i, y_-^i)^T_{i=1} \]

equals \( (0, +1, +1, 0)^T \) regardless of the curve’s parametrization, and it distinguishes between different corners, as depicted in figure 4.

The consequence is that the detachment is a coherent tool in the task of classification of corners, hence the parameter \( \alpha_D \) can be defined via the one-sided detachments of the curve at its corner point.

4 Extending the Antiderivative Theorem

In this section we consider the two-dimensional case of the Antiderivative Theorem stated in equation (1). This
Figure 4. Classification of corners along a curve according to the curve’s one-sided detachments.

Theorem states a connection between the double integral of a function over a finite unification of axis aligned boxes, and a linear combination of the given function’s antiderivative at the domain’s corners.

A natural question that arises from the Antiderivative Theorem is, how can it be naturally extended to more general types of domains, not only those that are formed by a finite unification of boxes. Note that in a recent work by Pham et al. (see [6]), the theorem is suggested to be extended to polygonal domains via dynamic programming. We will seek to extend the theorem to even more general domains, not necessarily polygonal, via a different approach.

This research suggests that to extend the theorem, it is required to establish a novel integration method over curves in the plane. This integration method results with an ‘automated’ division of the domain bounded by the curve - into a finite unification of axis aligned boxes (for whom the Antiderivative Theorem is applied) and ‘all the rest’, for whom the double integral of the original function is evaluated separately. Let us build this integration method, step by step.

4.1 Definition of a Curve’s Tendency

Definition 6. Tendency Indicator Vector. Let \( \gamma(t) = (x(t), y(t)) \) be a curve in the plane, where \( t \in (\alpha, \beta) \). Let \( t_0 \in (\alpha, \beta) \). If the curve’s one-sided detachments at \( t_0 \) (see definition 4),

\[
\begin{align*}
  x^{+}_+(t_0), & \quad x^{-}_-(t_0), \quad y^{+}_+(t_0), \quad y^{-}_-(t_0), \\
end{align*}
\]

all exist, then the curve is said to be tendable at \( t_0 \), and the curve’s tendency indicator vector there is the following ternary vector:

\[
(x^+, y^+, y^-)^T \mid_{t_0} \in \{0, \pm1\}^4.
\]

Remark 7. For the simplicity of the discussion, the curves will be assumed to be tendable, continuous and simple from now on.

Remark 8. Note that the following four cases:

\[
\begin{align*}
  (1) & \quad x^+(t_0) = y^+(t_0) = 0, \\
  (2) & \quad x^-(t_0) = y^-(t_0) = 0, \\
  (3) & \quad x^+(t_0) = x^-(t_0) = 0 \land y^+(t_0) = y^-(t_0), \\
  (4) & \quad y^+(t_0) = y^-(t_0) = 0 \land x^+(t_0) = x^-(t_0),
\end{align*}
\]

can not take place in case the curve is simple (which is assumed at remark 7). Thus, in this paper’s discussion, a curve’s tendency indicator takes \( 3^4 - (2 \cdot 3^2 - 1 + 2 \cdot 2) = 60 \) values (ternary vectors).

Definition 9. Tendency of a curve. Let \( \gamma(t) = (x(t), y(t)) \) be a (tendable) curve in the plane, where \( t \in (\alpha, \beta) \). We denote the tendency of the curve at a point \( \gamma(t_0) \) on the curve by \( \tau_{\gamma}(\gamma(t_0)) \), and define it as a discrete function \( f \) of the curve’s tendency indicator vector,

\[
\tau_{\gamma} : \quad \gamma \rightarrow \{0, \pm1\}
\]

\[
\tau_{\gamma}(\gamma(t_0)) \equiv f \left( (x^+, x^-, y^+, y^-)^T \mid_{t_0} \right),
\]

where the function \( f : \{0, \pm1\}^4 \rightarrow \{0, \pm1\} \) is determined according to the possible values of the curve’s tendency indicator vector, as detailed at the table in figure 6.
Positive and negative stand for the curve’s tendency indicator vector. The table should be detailed as a function of \( \alpha \) independent of the curve’s parametrization. A curve’s tendency is robust in the sense that it is regarding the term of a function’s detachment above, non-corner points. Further, according to the discussion and it extends it in the sense that it is defined also at \( \alpha \) parameter. This geometric interpretation associated with that corner. This geometric interpretation curve’s tendency is the number \( \alpha \) angle is contained in the left hand-side of the curve; the tendency at a point, we place an axis-aligned right angle curve’s tendency is as follows: to calculate a curve’s tendency at a point such that the quadrant defined by this angle is contained in the left hand-side of the curve.

Remark 10. Note that a curve’s tendency agrees with the parameter \( \alpha_D \) from formula (1) along the curve’s corners, and it extends it in the sense that it is defined also at non-corner points. Further, according to the discussion regarding the term of a function’s detachment above, a curve’s tendency is robust in the sense that it is independent of the curve’s parametrization.

Definition 11. The geometric interpretation of the curve’s tendency is as follows: to calculate a curve’s tendency at a point, we place an axis-aligned right angle at that point such that the quadrant defined by this angle is contained in the left hand-side of the curve; the curve’s tendency is the number \( \alpha_D \) from formula (1), associated with that corner. This geometric interpretation is illustrated in figure 5. A more detailed illustration to this parameter and its geometric interpretation is found in [22].

### 4.2 Definition of Slanted Line Integral

Equipped with the parameter of tendency, let us now establish the following integration method, whose aim is to extend the Antiderivative Theorem (stated in equation (1)) to more general domains than finite unifications of axis aligned boxes. Let us first define the following useful terms.

**Definition 12. Uniformly Tended Curve.** A curve is said to be uniformly tended if its tendency indicator vector (see definition 6) is constant for each of its interior points.

**Definition 13. Positive Domain of a uniformly tended curve.** Given a uniformly tended curve \( \gamma \), let us define its positive domain, \( D^+ (\gamma) \), as the domain bounded by the curve and two axis-aligned lines, such that the domain is contained in a left hand-side of the curve.

Definitions 12 and 13 are illustrated in figure 7.

**Definition 14. Uniformly Tended Division.** Let \( \gamma (t) \) be a tendable curve, where \( t \in (\alpha, \beta) \). Let us observe the following sub-curves of \( \gamma \):

\[
\gamma_i \equiv \{ \gamma (t) \mid t_{i-1} \leq t \leq t_i \},
\]

where \( 1 \leq i \leq n, t_0 = \alpha, t_n = \beta \) and \( t_0 < t_1 < \ldots < t_n \). If each \( \gamma_i \) is uniformly tended (see definition 12), then \( \{ \gamma_i \}_{i=1}^n \) is called a uniformly tended division of \( \gamma \).

Remark 15. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be an integrable function. Suppose that we try to define the following integration method of \( f \) along a uniformly tended curve \( \gamma \):

\[
\int_{\gamma} f \equiv \int_{D^+ (\gamma)} f \, d\vec{x},
\]

and suggest to extend it to general tendable curves in an intuitive manner: Let \( \gamma \) be a tendable curve. Then we try to define the suggested integration method of \( f \) along the curve simply as:

\[
\int_{\gamma} f \equiv \sum_{i=1}^{n} \int_{\gamma_i} f,
\]

where \( \{ \gamma_i \}_{i=1}^n \) is a uniformly tended division of \( \gamma \). The "definitions" suggested at equations (2) and (3) seem like intuitive definitions at first glance. However, in the appendix (see subsection 6.3) it is explained in what sense those are, in fact, not successful definitions.
The problems associated with the suggested “definitions” at equations \(\text{(2)}\) and \(\text{(3)}\) could be resolved via a more careful definition of the above integration method, that involves the antiderivative’s values at key points along the positive domain’s edge (that will, in turn, enable a calculation of the original function’s double integral over a rectangular domain – according to the Antiderivative Theorem).

**Definition 16. Slanted Line Integral over a uniformly tended curve.** Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be an integrable function, and let \(F\) be its antiderivative. Let \(\gamma\) be a uniformly tended curve, contained in another curve, \(\Gamma \supset \gamma\). Then we define the slanted line integral of \(F\) along the curve \(\gamma\) in the context of the curve \(\Gamma\) as follows:

\[
\oint_{\gamma \subset \Gamma} F = \int_{D^+(\gamma)} f \, dx - \tau(\gamma) F(B) + \frac{1}{2} \left[ \tau^+(A) F(A) + \tau^-(C) F(C) \right],
\]

where \(\tau(\gamma)\) is the uniformly tended curve’s tendency, \(D^+(\gamma)\) is the given curve’s positive domain, and \(\tau^+(A), \tau^-(C)\) are the tendencies of the curve \(\Gamma\) at the points \(A\) and \(C\) respectively along the edge of the domain, as illustrated in figure \(\text{7}\).

**Definition 17. Slanted Line Integral over a tendable curve.** Let \(f : \mathbb{R}^2 \to \mathbb{R}\) be an integrable function, and let \(F\) be its antiderivative. Let \(\gamma\) be a tendable curve. Then we define the slanted line integral of \(F\) along the curve \(\gamma\) as follows:

\[
\oint_{\gamma} F = \sum_{i=1}^{n} \oint_{\gamma_i} F,
\]

where \(\{\gamma_i\}_{i=1}^{n}\) is a uniformly tended division of \(\gamma\). This integration method is illustrated in \(\text{23}\).

### 4.3 Algebraic Properties of Slanted Line Integral

The definition of the slanted line integral not only resolves the problems that were raised at the "definitions" depicted at equations \(\text{(2)}\) and \(\text{(3)}\) (see in the appendix, subsection \(\text{6.3}\), but also depicts familiar algebraic properties. For example, let us quote two properties of slanted line integral, that can be shown to hold via applying the Antiderivative Theorem (applied to a single box). See \(\text{19}\), pp. 114-119, for a detailed discussion.

**Claim 18.** Let \(\gamma\) be a uniformly tended curve contained in another curve, \(\Gamma \supset \gamma\), and let \(\gamma_1\) and \(\gamma_2\) be its sub-curves, such that \(\gamma = \gamma_1 \cup \gamma_2\). Then:

\[
\oint_{\gamma \subset \Gamma} F = \oint_{\gamma_1 \subset \Gamma} F + \oint_{\gamma_2 \subset \Gamma} F.
\]

This claim also asserts that the slanted line integral over a tendable curve (see definition \(\text{17}\)) is indeed well defined, because it is independent of the uniformly tended division associated with the curve.

### 4.4 Applying Slanted Line Integral to Extend the Antiderivative Theorem

Using the slanted line integral we can now formulate a theorem that forms a natural extension to the Antiderivative Theorem. Let us begin by stating a lemma.

**Lemma 20.** Let us consider a positively oriented and tendable curve \(\gamma = \bigcup_{i=1}^{n} \gamma_i \subset \mathbb{R}^2\), where \(\{\gamma_i\}_{i=1}^{n}\) is a uniformly tended division of the curve \(\gamma\). Let us consider a function \(f : \mathbb{R}^2 \to \mathbb{R}\) which is integrable there. Let \(F : \mathbb{R}^2 \to \mathbb{R}\) be its antiderivative. Let \(M,N,O\) be the endpoints of the subcurves \(\gamma_1,\gamma_2\) respectively (where the point \(N\) is joint to both subcurves). Let \(\vec{MO}\) be the straight line connecting the points \(M,O\). Let:

\[
\Delta \equiv \Delta_1 \cup \gamma_2 \cup \vec{MO},
\]

and:

\[
\gamma^- \equiv \gamma \setminus \Delta \cup \vec{OM}.
\]
Then:
\[
\int_{\gamma} F = \int_{\gamma^+} F + \int_{\gamma^-} F.
\]

A full proof is omitted here due to the paper’s length restrictions (but is available in [19], pp. 119-121). In short, the correctness follows by separating to cases and applying the definition of slanted line integral. Please refer to figure 8 for an illustration of the lemma.

**Theorem 21.** Let \( D \subset \mathbb{R}^2 \) be a given simply connected domain whose edge, \( \partial D \), is continuous, simple and tendable. Let \( f \) be an integrable function in \( \mathbb{R}^2 \). Let \( F \) be its antiderivative. Then:

\[
\int_{D} \int_{\partial D} f \overrightarrow{dx} = \int_{\partial D} F.
\]  

Formula (4) can be shown to hold via induction on the number of uniformly tended sub-curves that form the domain’s edge, \( \partial D \). The induction’s basis can be shown to hold by separating to cases of domains whose edge consists of merely 3 uniformly tended sub-curves. The induction’s step can be proved via applying both lemma 20 and the induction’s hypothesis - now the choice of \( \Delta \) (at the formulation of lemma 20) as a curve that consists of 3 uniformly tended sub-curves, is clearer. A detailed proof is found in [19], pp. 121-128. This theorem is illustrated in [24]. The following example is designated to clarify formula (4).

**Example 22.** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be an integrable function, and let \( F : \mathbb{R}^2 \to \mathbb{R} \) be its antiderivative. Let \( \gamma \) be a curve as depicted in figure 7.

Let us put in details the slanted line integral of the antiderivative along each of the sub-curves \( \gamma_i \) in the context of the curve \( \gamma \), according to definition 16:

\[
\int_{\gamma_1 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = \int_{\partial D} F - \frac{1}{2} F(L) + F(L'),
\]

\[
\int_{\gamma_2 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = F(M') - F(M),
\]

\[
\int_{\gamma_3 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = F(N') - F(M'),
\]

\[
\int_{\gamma_4 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = \frac{1}{2} F(P) - F(O'),
\]

\[
\int_{\gamma_5 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = \frac{1}{2} F(P) - F(P'),
\]

\[
\int_{\gamma_6 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = F(Q'),
\]

\[
\int_{\gamma_7 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = F(R') - F(Q'),
\]

\[
\int_{\gamma_8 \subset \gamma} \int_{D^+} f \overrightarrow{dx} = \frac{1}{2} F(L) + F(S').
\]

Summing up those equations, while considering the equality:

\[
\int_{L^{S'}_{S''}} f \overrightarrow{dx} = F(S) - F(S') + F(S'') - F(L'),
\]
(thus deducting the twice-calculated double integral over the box $L S' S'' L'$), and applying the Antiderivative Theorem, results with $\int F \equiv \sum_{i=1}^{n} \int f \text{d}x$, as stated at formula (4).

Remark 23. The thought behind the formulation of theorem 21 and the suggested proof are similar to those of Pick’s theorem (see [18]). Also note that formula (1) is a withered case of formula (4), in the following sense: If $D$ is a finite unification of axis aligned boxes, then the slanted line integral over its uniformly tended sub-curves consists of a linear combination of the antiderivative’s values alone, where the double integral over the positive domain is omitted, because the positive domain of each such sub-curve is degenerated.

5 FUTURE WORK

5.1 Theory

The detachment is defined for discontinuous functions (see the appendix, subsection 6.1 and figure 10), and in advanced theories of classical analysis where the derivative is undefined – such as metric spaces, because given a function $f : X \rightarrow Y$, where $X$ and $Y$ are different metric spaces, then it is not always possible to evaluate the term $d_Y(f(x_1), f(x_2))$ (as required at the definition of the derivative), however the term $Q \{ f(x_1), f(x_2) \}$ where $Q$ is a quantization function, similar to the definition of the detachment – is well defined (because the calculation involves only the term $d_Y(\cdot, \cdot)$), and enables the establishment of novel definitions and theorems.

Further, the detachment enables a natural way of classifying corners (independently of the curve’s parametrization), hence for a coherent definition of the parameter $\alpha_D$ from the Antiderivative Theorem. The detachment operator can be applied to other monotony classification tasks. Moreover, the suggested integration method (slanted line integral) over curves in $\mathbb{R}^2$ can be thought of as considering the macro aspect of the curve (since at this integration method it is required to divide the curve into a finite number of sub-curves), rather than its micro aspect as at the line integral method. This integration method can be formulated in higher dimensions, to form an extended version of discrete Stokes’s theorem. Finally, the theory suggests a combination between discrete and continuous mathematics, as suggested by Lovasz in [14].

5.2 Practice

In [19], pp. 97-101, it is suggested that the numerical approximation of the detachment operator is, to some extents, cheaper than that of the derivative. This claim should be verified experimentally, as well as following claim: the slanted line integral’s nature of definition enables the calculation of a double integral over a domain via parallel computing.

6 APPENDIX

6.1 General Comments Regarding the Detachment Operator

Remark 24. While the detachment seems as a withered derivative at first glance, this is not always the case. As we could see in examples 1 and 2, in cases where the derivative vanishes (either zeroed or non-existent), the detachment is still capable of supplying information regarding the function’s local monotony behavior. This anomaly occurs due to the discontinuity of the sign function at zero, from which the inequality at equation (5) follows:

$$sgn \left[ f_{\pm} \right] = sgn \left[ \lim_{\Delta x \to 0^\pm} \frac{\Delta f}{\Delta x} \right] \neq \lim_{\Delta x \to 0^\pm} sgn \left[ \frac{\Delta f}{\Delta x} \right] \quad (5)$$

A simple example to this anomaly is $f(x) = x^2$, where the derivative at $x = 0$ is zeroed, while the one-sided detachments are both +1. Other examples are given in figure 10. See figure 3 for more details.

Remark 25. Due to their simplicity, the one-sided detachments are defined for a broad set of functions, that may be not differentiable and even discontinuous. This fact is illustrated in figure 10 and in [25]. Note that while there are many known extensions to the term of a function’s derivative, such as Fréchet derivative (see [16]), Gâteaux derivative (see [1]) and more modern approaches (see [15]), those approaches usually deal with abstract (for example, Banach) spaces, or impose non-trivial conditions on the function (such as BV - bounded variation, see [17]). On the other hand, the term of a function’s detachment is intuitive and requires merely that the function is defined over an Euclidean space.

6.2 Detachment Based Analogues of Calculus Theorems

First let us introduce an operator that is defined whenever a function is one-sided detachable (see definition 4 and figure 10) from both sides.

Definition 26. Tendency of a function. Let us define the tendency of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ at a point $x \in \mathbb{R}$ as:

$$\tau_f : \mathbb{R} \rightarrow \{0, \pm1\}$$

$$\tau_f \left( x \right) \equiv sgn \left[ f_+ \left( x \right) - f_- \left( x \right) \right] ,$$

if the one-sided detachments exist. If indeed both the one-sided detachments exist then the function is said to be tendable.

The tendency operator is zeroed at a function’s extreme points (where $f_+ = f_-,$), and everywhere else it forms a weighted combination of its one-sided detachments, if those exist: an averaged instantaneous trend of change of the function at a point. In many cases, a function’s tendency agrees with the sign of its derivative – and since this operator is defined also for non-differentiable and even discontinuous functions – it can be
thought of as an extension to the sign of the derivative. Semi-discrete analogues to familiar theorems in Calculus can be formulated via applying this operator. For example, let us recall Lagrange's mean value theorem, followed by its semi-discrete analogue: (note that due to length restrictions, full proofs to theorems are omitted. Proofs are suggested in [19]).

**Theorem 27. Mean Value Theorem.** Let \( f : [a,b] \to \mathbb{R} \) be a continuous function in \([a,b]\) and differentiable in \((a,b)\). Then there exists a point \( c \in (a,b) \) such that:

\[
f'(c) = \frac{f(b) - f(a)}{b-a}.
\] (7)

**Theorem 28.** Let \( f : [a,b] \to \mathbb{R} \) be a continuous function in \([a,b]\) and tendable in \((a,b)\). Suppose that \( f(a) \neq f(b) \). Then for each threshold \( v \in (f(a), f(b)) \) there exists a point \( c_v \in f^{-1}(v) \) such that:

\[
\tau_f(c_v) = \text{sgn} [f(b) - f(a)].
\] (8)

Theorem 28 can be shown to hold by inspecting the finite number of cases, and applying the intermediate value theorem for each case, see [19], pp. 90-91.

Notice the similarity between the formulation of theorem 27 and that of theorem 28 (equations 7 and 8). Let us analyze the relation between the theorems. Both theorems require continuity in the closed interval. While the original version requires differentiability in the open interval, its semi-discrete analogue requires tendability (detachability from both sides) there, which can be thought of as a weaker condition (see figure 10).

Further, the statements of both theorems are similar: the right hand-sides in equations 7 and 8 resemble the definitions of the derivative and the detachment, respectively. Finally, while the original version guarantees the existence of one point \( c \in (a,b) \) that satisfies the theorem’s statement, its semi-discrete analogue is stated for each value in the interval \( v \in (f(a), f(b)) \), hence its statement holds for uncountably many points. To sum up this discussion, the semi-discrete version depicts a trade-off between the amount of information that the theorem supplies, and the set of functions for whom the theorem holds, with respect to the original version. The author would like to suggest to refer to theorem 28 as the ‘trend value theorem’. Some other analogous theorems are formulated below. Detailed proofs and extended discussion are found in [19], pp. 75-96.

**Theorem 29.** (Analogous to Fermat’s theorem). Let \( f : [a,b] \to \mathbb{R} \) and \( x_0 \in (a,b) \) be an extremum of \( f \). Then \( \tau_f(x_0) = 0 \).

**Theorem 30.** (Analogous to Rolle’s theorem). Let \( f : [a,b] \to \mathbb{R} \) be continuous in its definition domain, and suppose that \( f(a) = f(b) \). Then there exists a point \( c \in (a,b) \) such that \( \tau_f(c) = 0 \).

**Theorem 31.** (Analogous to Darboux’s theorem). Let \( f : [a,b] \to \mathbb{R} \) be continuous and tendable in a non-degenerate neighborhood of the point \( x_0 \in (a,b) \), denoted by \( I(x_0) \). If \( x_0 \) is either a local maximum or a local minimum of \( f \), then \( \text{Im}(\tau_f|_{I(x_0)}) = \{0, \pm 1\} \), and there are uncountably many
points in \( I(x_0) \) where the tendency of \( f \) is \( \pm 1 \).

**Theorem 32. (Analogous to Newton-Leibniz’s axiom).** Let \( f: [a, b] \to \mathbb{R} \) be tendable and continuous in \( [a, b] \). Let \( \{x_i\}_{i=1}^n \subset [a, b] \) be the set of all the points in the interval \([a, b]\) where \( f \) is not signpostable detachable (see figure \([10]\)). Let \( \tau_f \) be the tendency of \( f \). Then:

\[
\int_a^b \tau_f(x) \, dx = -\left[ \sum_{i=1}^{n-1} f_+(x_i) x_i + \sum_{i=2}^n f_-(x_i) x_i \right].
\]  

(9)

### 6.3 The Trap in the “Definitions” Suggested at Equations (2) and (3)

Let us note four key problems related to the “definitions” suggested at equations (2) and (3), to show that this integration method is not well defined.

1) The “definition” suggested at equation (2) is sensitive to different uniformly tended divisions of the curve, since \( f \neq \int f + f \). The term to the left includes the function’s integral over a box, that is not included in the terms to the right.

2) This integration method does not depict a curve-domain relation (contrary to the familiar line integral, where Green’s theorem guarantees the relation between integration over a domain’s edge and integration over the domain itself). In simple words, it is easy to see that independently of the uniformly tended division of the domain’s edge, \( \partial D \), it holds that: \( \int f \neq \int \int \frac{\partial f}{\partial x} \). 

(13)

3) In this integration method, overlapping domains are not deducted, in the sense that if the positive domains of two adjacent curves intersect, then the function’s double integral over the domain of intersection is taken twice into account.

4) If \( \gamma \) is a curve and \( -\gamma \) is the same curve with a flipped orientation, then according to the suggested definition it holds that \( \int f \neq -\int f \). This is in contrast with familiar integration methods.

The four problems associated with the “definitions” stated at equations (2) and (3) are all caused due to a mis-calculated, or over-calculated, double integral of the function \( f \) over a rectangular domain.

### 6.4 Etymological Discussion

The term "Detachment" was chosen since applying, for example, a right detachment to a function in an interval, results with a step-function, that is "detached" at the extrema points of the original function. The term "Slanted Line Integral" follows from the fact that when applied to open curves, the integration method is dependent of the axes’ rotation angle.

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