SYMMETRIES OF THE DISSIPATIVE HOFSTADTER MODEL *

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ABSTRACT

The dissipative Hofstadter model, which describes a particle in 2-D subject to a periodic potential, uniform magnetic field, and dissipation, is also related to open string boundary states. This model exhibits an SL(2,\mathbb{Z}) duality symmetry and hidden reparametrization invariance symmetries. These symmetries are useful for finding exact solutions for correlation functions.

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1. The Dissipative Hofstadter Model and Open String Theory

The dissipative Hofstadter model describes the quantum mechanics of a particle confined to two dimensions in a periodic potential and transverse magnetic field, subject to a dissipative force. The particle’s coordinates are taken to be \( \vec{x}(t) = (x(t), y(t)) \), and the magnetic field is given by \( B \hat{z} \). Classically, the dissipation is described by the \(-\eta \ddot{x}\) term in the equations of motion. To treat the dissipation quantum mechanically, we use the Caldeira-Leggett model. In this model, the Euclidean action is given by

\[
S = S_q + S_\eta + S_V,
\]

where \( S_q \) is the usual action of a particle in a constant magnetic field,

\[
S_q = \int_{-T/2}^{T/2} dt \left[ \frac{M}{2} \dot{x}^2 + \frac{ieB}{2c} (\dot{x}y - \dot{y}x) \right].
\]

\( S_\eta \) is a non-local kinetic term that accounts for the friction. It is given by

\[
S_\eta = \frac{\eta}{4\pi} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} dt dt' \left( \frac{\vec{x}(t) - \vec{x}(t')}{t - t'} \right)^2.
\]

\( S_V \) is the action due to the potential, which we are taking to be

\[
S_V = -\int_{-T/2}^{T/2} \left[ V_0 \cos \left( \frac{2\pi x(t)}{a} \right) + V_0 \cos \left( \frac{2\pi y(t)}{a} \right) \right].
\]

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The friction term, $S_{\eta}$, is obtained by first introducing a bath of harmonic oscillators which interact linearly with the particle. The integration over the oscillators in the resulting functional integral yields the $S_{\eta}$ term in the remaining action for the particle.

It is useful to define the dimensionless parameters describing the friction/unit cell, $\alpha$, and the flux/unit cell, $\beta$. They are given by

$$2\pi \alpha = a^2 \eta/\hbar \quad \text{and} \quad 2\pi \beta = eB/\hbar c a^2.$$ (5)

The action in Eq. (1) is also used to obtain the boundary state in open string theory. This boundary state describes a world sheet with a boundary, where all the fields on the interior of the world sheet are free and all the interactions take place on the boundary. Then the potential corresponds to a tachyon field; the magnetic field corresponds to a gauge field; and the mass term acts as a regulator. In order to give a solution to string theory, the theory must be independent of the regulator. Hence it must be scale invariant and lie at a critical point.

2. The Theory at $\alpha = 1$, $\beta = 0$

When $\alpha = 1$ and there is no magnetic field, the theory is expected to be at a critical point. At this point, we can use simple algebraic identities to fermionize the theory as follows:

$$e^{ix(t)} = \psi_L(t)^\dagger \psi_R(t),$$ (6)

$$\dot{x}(t) = i \left[ \psi_L(t)^\dagger \psi_L(t) - \psi_R(t)^\dagger \psi_R(t) \right],$$ (7)

and similarly for $y(t)$. The propagator is then given by

$$\langle \psi_L(t)^\dagger \psi_L(0) \rangle = \langle \psi_R(t)^\dagger \psi_R(0) \rangle = \frac{i}{t} \quad \text{and} \quad \langle \psi_L(t)^\dagger \psi_R(0) \rangle = 0.$$ (8)

The theory is bilinear in $\psi$, so it can be solved exactly. We find that

$$\langle \dot{x}(t_1)\dot{x}(t_2) \rangle = -\mu \frac{2}{(t_1 - t_2)^2},$$ (9)

where $\mu$ is known as the mobility; and all other $m$-point functions of the $\dot{x}(t)$’s are contact terms. That means they are zero unless at least two points are coincident.

The problem with this treatment is that fermionization is only fine for large-time behavior, since the derivation ignored how the short distance behavior was regulated. One would hope that for the contact terms to be well-defined and independent of the regulator, the symmetries and possibly also the large-time behavior of the system are enough to determine them.

3. Duality Symmetry

The first symmetry we expect this system to have is a duality symmetry in $\alpha$ and $\beta$. In Fig. 1, the approximate phase diagram for this system shows
the expected transitions between localized states (for $\alpha > 1$), delocalized states (in the interiors of the circles), and unknown states (in the triangular regions between the circles). The diagram exhibits an $SL(2,\mathbb{Z})$ symmetry. This means that if the theory at one value of $\alpha$ and $\beta$ is critical, then, for any other value of $\alpha$ and $\beta$ that is related to the first by an $SL(2,\mathbb{Z})$ transformation, it will also be critical. In addition, this symmetry relates theories at different values of flux and friction, so if the correlation functions are known at one value of $\alpha$ and $\beta$, then there are simple transformations we can use to obtain them at the other values of $\alpha$ and $\beta$ related by the $SL(2,\mathbb{Z})$ symmetry. For example, if we know the correlation function $\langle \hat{x}^{\mu_1}(k_1) \ldots \hat{x}^{\mu_m}(k_m) \rangle_0$ at the point $\alpha = 1$, $\beta = 0$, then we can obtain all the correlation functions $\langle \hat{x}^{\mu_1}(k_1) \ldots \hat{x}^{\mu_m}(k_m) \rangle_\beta$ at the other multi-critical points on the large circle centered at $\alpha = 1/2$, $\beta = 0$, as follows:\(^6\)

$$\langle \hat{x}^{\mu_1}(k_1) \ldots \hat{x}^{\mu_m}(k_m) \rangle_\beta = \left[ \prod_{i=1}^{m} r^{\mu_i,\sigma}(k_i) + \prod_{i=1}^{m} r^{\mu_i,\tau}(k_i) \right] \langle \hat{x}^{\mu_1}(k_1) \ldots \hat{x}^{\mu_m}(k_m) \rangle_0, \quad (10)$$

where $r^{\mu\nu}$ is given by

$$r^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{\beta}{\alpha} \text{sign}(k) \epsilon^{\mu\nu}. \quad (11)$$

It is not difficult to do calculations to $O(V_0^2)$, and, to this order, direct calculations show that Eq. (10) is exact. They also show that for $\beta \neq 0$, not all the
4. Reparametrization Invariance

The second symmetry the system should have comes from the reparametrization invariance of open string theory. This symmetry implies that the generating function for critical dissipative quantum systems should satisfy “hidden” reparametrization Ward identities. The generating function, $W[\vec{J}]$, is given by

$$e^{W[\vec{J}]} = \int D\vec{x}(t) \exp \left[ -\frac{1}{\hbar} (S_q + S_\eta + S_V) \right] \exp \left[ -\frac{1}{\hbar} \int \vec{J} \cdot \dot{\vec{x}} dt \right].$$ (12)

For $n \geq 0$, the Ward identity is

$$\sum_{m=1}^{n-1} \left[ \frac{1}{2} \frac{\partial W}{\partial \vec{J}_{m-n}} \cdot \frac{\partial W}{\partial \vec{J}_{m-n}} - \frac{\partial^2 W}{\partial \vec{J}_{m-n} \cdot \vec{J}_{m-n}} \right] + \sum_{m=-\infty}^{\infty} m \vec{J}_m \cdot \frac{\partial W}{\partial \vec{J}_{m-n}} = 0,$$ (13)

and there is a similar equation for $n < 0$. This identity is satisfied at all orders in $V_0$ when $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha \in Z$. The equations with $n = 0, \pm 1$ imply that, inside the correlation functions, the $\dot{\vec{x}}(t_i)$ should transform as dimension-one operators under $SL(2, R)$ transformations of time. The only other independent equation is with $n = 2$. One problem with the Ward identities is that they do not give enough information for solving for the correlation functions. However, they do say that any correlation function at $\beta = 0, \alpha = 1$ must be $SL(2, R)$ covariant. We can apply the duality transformation to this correlation function to obtain one at another value of $\beta$. This new correlation function must once again exhibit the $SL(2, R)$ symmetry. This should give a lot of information about the form of the correlation functions.

One difficulty with this reasoning is that the regulator used to derive the duality transformation is not the same one used to prove the Ward identities. A rigorous derivation of Eq. (10) using the regulator satisfying the Ward identities gives a slightly weaker version of the transformation.

5. Results for Correlation Functions

If we carefully repeat the derivation of fermionization using the regulator that satisfies the Ward identity, then we can obtain additional symmetries and properties of the correlation functions to all orders in $V_0$. We find that, when $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha \in Z$, the $m$-point functions are given by

$$\langle \dot{x}^{\mu_1}(k_1) \ldots \dot{x}^{\mu_m}(k_m) \rangle_{\beta} = \left[ \prod_{j=1}^{m} r^{\mu_j x}(k_j) + \prod_{j=1}^{m} r^{\mu_j y}(k_j) \right] F(\vec{k}, \beta).$$ (14)
This is the form of the correlation functions predicted by the duality transformation in Eq. (10). The function $F(\vec{k}, \beta) = \vec{a}(\vec{k}) \cdot \vec{k}$ has the following properties: It is finite as the cutoff goes to zero. It is piecewise linear in $\vec{k}$, which suggests a connection with the Duistermaat-Heckman theorem; and it is homogeneous in $\vec{k}$. In real space, this last property gives a new symmetry under non-one-to-one reparametrizations of time. It tells what happens when $z \to z^n$, where $z = e^{2\pi ik}$. In addition, we find that $F(\vec{k}, \beta) \to 0$ if $k_i = 0$; $F$ is symmetric under $k_i \leftrightarrow k_j$, and also $\vec{k} \leftrightarrow -\vec{k}$; and $F$ is continuous. These results restrict the $m$-point functions to lie in a finite-dimensional linear space.

We conjecture that these results, combined with the reparametrization invariance Ward identities, determine all the $m$-point functions when $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha \in Z$. We have found that they do give exact solutions for the two, four and six-point functions, and also for any correlation function with special conditions on the $k_i$'s. For example, the 4-point function is proportional to $\min(|k_1|, |k_2|, |k_3|, |k_4|)$. However, it is still an open question whether these symmetries are enough to determine any arbitrary correlation function. In any case, it does appear that these symmetries, combined with the long-time behavior, are enough to obtain exact solutions for all the correlation functions.

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7. References

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