THE TOPOLOGY OF CALABI–YAU THREEFOLDS

P.M.H. WILSON

ABSTRACT. We ask about the simply connected compact smooth 6-manifolds which can support structures of Calabi–Yau threefolds. In particular, we study the interesting case of Calabi–Yau threefolds $X$ with the second Betti number $b_2(X) = 3$. We have a cubic form $D \mapsto D^3$ on $H^2(X, \mathbb{Z})$ given by cup-product, a linear form $D \mapsto D \cdot c_2(X)$ given by the second Chern class, and the integral middle cohomology $H^3(X, \mathbb{Z})$, and if $X$ is simply connected with torsion free homology this information determines precisely the diffeomorphism class of the underlying 6-manifold by a result of Wall. For simplicity, we assume that the cubic form defines a smooth real elliptic curve whose Hessian is also smooth. Under a further relatively mild assumption that there are no non-movable surfaces $E$ on $X$ with $1 \leq E^3 \leq 8$, we prove that the real elliptic curve must have two connected components rather than one, and that the Kähler cone is contained in the open positive cone on the bounded component; we show moreover that the linear form $c_2$ is also non-negative on this cone. Using Wall’s result, for any given third Betti number we therefore have an abundance of examples of smooth compact oriented 6-manifolds which support no Calabi–Yau structures, both in the cases when the cubic defines a real elliptic curve with one or two connected components.

INTRODUCTION

About 35 years ago, the author was asked by two colleagues in physics what restrictions there were on the cup-product cubic form on $H^2(X, \mathbb{Z})$ given by $D \mapsto D^3$ for a Calabi–Yau threefold $X$. Until now, we were only able to make rather weak statements in response to this question, but recent work now enables us to give an answer with more content. The question becomes interesting when $b_2(X) \geq 3$, and in this paper we concentrate on the case $b_2(X) = 3$: the intervening three decades suggested the following Motivating Question to the author:

Motivating Question. Is the following true? If $X$ is a Calabi–Yau threefold with $b_2(X) = 3$ and the cubic form defines a smooth real elliptic curve $C \subset \mathbb{P}^2(\mathbb{R})$, then either $C$ has two real connected components and the Kähler cone $\mathcal{K}(X)$ is contained in the cone on the bounded component of $C$ on which the cubic is positive, or $C$ has one real component and the Hessian curve is singular, consisting of three non-concurrent lines.

Remark 0.1. The cases with singular Hessian do commonly occur: for one possibility we consider the desingularization of a quintic in $\mathbb{P}^4(\mathbb{C})$ with two singularities, each of which is analytically a cone on a del Pezzo surface, and for another we take the desingularization of a hypersurface of bidegree $(3, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with one such singularity. The Hessian curve in the first case consists of three real lines and in

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the second case one real and two complex lines. In the real coordinates introduced in Section 1, the corresponding invariant \( k \) is 0 and \(-2\) respectively.

The answer however to the question as posed is in fact no, as there are some known examples where the elliptic curve has one component and the Hessian is smooth. Three such examples arise from hypersurfaces in weighted projective space and are included in the list given in Appendix C of [3]. If we consider for instance a quasi-smooth hypersurface \( X_{10} \) of degree 10 in weighted projective space \( \mathbb{P}(1,1,2,3,3) \), there is a curve (isomorphic to \( \mathbb{P}^1 \)) of singularities. Taking a crepant resolution, we obtain a Calabi–Yau threefold \( X \) with \( b_2(\mathcal{X}) = 3 \). With respect to a suitable basis of \( H^2(X, \mathbb{Z}) \) the cubic form is given explicitly in Appendix C of [3] as

\[
15x^3 + 60x^2y + 30x^2z + 78xy^2 + 78xyz + 18xz^2 + 32y^3 + 48y^2z + 18yz^2.
\]

This may be checked to be an example where the real elliptic curve has one connected component, and the Hessian is also smooth.

We recall that the movable cone of a smooth threefold \( X \) is the closure of the cone in \( H^2(X, \mathbb{R}) \) generated by the classes of mobile divisors, and it follows from Lemma 3.2 of [6] that when \( b_2(X) = 3 \), the Hessian is non-negative on the movable cone. As we saw in the earlier paper [3], the rigid non-movable surfaces \( E \) on a Calabi–Yau threefold \( X \), which were defined as the irreducible surfaces on \( X \) that deform with any small deformation of the complex structure on \( X \) but for which no multiple moves in the threefold (see Section 2 of [6] for a discussion of these) play a crucial role in understanding possible Calabi–Yau structures on a compact 6-manifold. If we impose the extra condition that there are no rigid non-movable surfaces on \( X \) with \( E^3 > 0 \), then the answer to the above question is yes.

Recall the result of Wall ([5], Theorem 5) that under an assumption (\( H \)) that the compact simply connected oriented 6-manifolds studied have torsion free homology and class \( w_2(M) = 0 \) (the latter assumption holding if \( M \) supports a Calabi–Yau structure), then the diffeomorphism classes of compact oriented manifolds \( M \) satisfying (\( H \)) correspond bijectively to isomorphism classes of invariants:

- two free abelian groups \( H \) and \( G \) (corresponding to \( H^i(M, \mathbb{Z}) \) for \( i = 1, 2 \)) with the rank of \( G \) being even,
- a symmetric trilinear map \( \mu : H \times H \times H \to \mathbb{Z} \),
- a homomorphism \( p_1 : H \to \mathbb{Z} \),
- subject to: for all \( x, y \in H \),

\[
\mu(x, x, y) \equiv \mu(x, y, x) \pmod{2} \quad \text{and} \quad p_1(x) \equiv 4\mu(x, x, x) \pmod{24}.
\]

We shall be interested in the case when there is a Calabi–Yau structure \( X \) on the manifold, in which case the linear form on the given free abelian group \( H \) may be identified as \( p_1(X) = -2c_2(X) \). Wall first constructs the relevant 6-manifold \( M_0 \) when \( G = 0 \), and he then forms a connected sum of \( M_0 \) with \( b_3/2 \) copies of \( S^3 \times S^3 \), and so at the smooth level the information on \( H^3(M, \mathbb{Z}) \) and the invariants on \( H^3(M, \mathbb{Z}) \) are independent. This is not true for Calabi–Yau threefolds, since at least for \( b_2(X) \leq 2 \) we know that the invariants on \( H^2(X, \mathbb{Z}) \) (non-effectively) bound \( b_3(X) \) (for \( b_2 = 1 \) this follows using Hilbert schemes, and see [6] for the \( b_2 = 2 \) case). When \( M \) supports an almost complex structure with \( c_1 = 0 \), it is unique up to homotopy by Theorem 9 of [3].

If \( E \) is a rigid non-movable surface with \( E^3 > 0 \) on a Calabi–Yau threefold \( X \), then by the results of Section 2 from [6], any such \( E \) would have \( E^3 \leq 9 \), and
if $E^3 = 9$ there would be a contraction of $E \cong \mathbf{P}^2$ to a point and the Hessian would be reducible. Moreover by applying (maybe a number of times) formula (1') in Section 2 of [6], we deduce under the further assumption that the Hessian curve is non-singular that either $E^3 = 1$ and $c_2(X) \cdot E = -2$, or $1 \leq E^3 \leq 8$ and $c_2(X) \cdot E = 12 - 2E^3$. Only when the integral cubic and linear forms $\mu$ and $c_2 = -p_1/2$ represent one of the above nine pairs of values at some point of $\mathbf{Z}^3$ with the correct index could there exist a rigid non-movable surface $E$ with $E^3 > 0$. If $\mu$ and $p_1$ satisfy the above congruences (for instance arising from some Calabi–Yau threefold), then so too do integral multiples of them, and taking an appropriate integral multiple will rule out the possibility of classes with the above pairs of invariants — for instance scaling both $\mu$ and $p_1$ by a factor of 4 will suffice, since then $E^3$ could only be 4 or 8, but $c_2(X) \cdot E$ would be a multiple of 8.

**Main Theorem.** Let $X$ be a Calabi–Yau threefold with $b_2(X) = 3$, where the cubic form $F$ defines a smooth cubic curve in $\mathbf{P}^2(\mathbf{R})$ with smooth Hessian and there are no rigid non-movable surfaces $E$ on $X$ with $E^3 > 0$. Then the real elliptic curve determined by $F$ has two connected components and the Kähler cone of $X$ is contained in the cone on the interior of the bounded component on which the cubic is positive. Moreover the linear form $c_2$ is non-negative on the open (positive) cone in $H^2(X, \mathbf{R})$ on the bounded component.

In particular, we note using Wall’s result that, for any given even $b_3 > 0$, we have an abundance of examples of smooth compact oriented 6-manifolds which support no Calabi–Yau structures, both in the case when the cubic defines a real elliptic curve with one component and in the case of two components — for the latter we shall also need to choose the linear form appropriately, so that it is negative somewhere on the open (positive) cone on the bounded component. The last sentence of the theorem will also be relevant for boundedness questions; either $c_2$ is numerically trivial and so by a well-known result due to S.-T. Yau [7] $X$ is an étale quotient of an abelian threefold, or there is a unique real ray $\mathbf{R}_+ D$ in the closed cone with $c_2$ a positive multiple of $D^2$.

1. **Components of the positive index cone for real ternary cubics**

In [6], a central role was played by the positive index cone corresponding to the cubic on $H^2(X, \mathbf{Z}) = \mathbf{Z}^r$, namely the real classes $L$ for which $L^3 > 0$ and the quadratic form given by $D \mapsto L \cdot D^2$ has index $(1, \rho - 1)$. For $\rho = 3$ the cubic defines a curve in the real projective plane, and our assumptions say that this is a real elliptic curve. To study real elliptic curves, the Hesse normal form for the curve will be useful, the theory of which may be found in Theorem 6.3 [1] or Section 3 of [2]. Normally we might take real coordinates so that the real elliptic curve takes the classical Hesse normal form $x^3 + y^3 + z^3 = 3kxyz$ with parameter $k$, but for our purposes it will be more convenient to make a change of coordinates, closely related to the canonical form described in Remark 6.9 of [1], so that the cubic takes the form

$$F(x, y, z) = -x^3 - y^3 - (z - x - y)^3 + 3kxy(z - x - y),$$  \hfill (1)

and hence the ‘triangle of reference’ of $F$ is now in the affine plane $z = 1$ with vertices $(0, 0), (1, 0)$ and $(0, 1)$. We shall write $F_k$ if we wish to indicate the dependence on $k$. An easy check verifies that the cubic $F$ is strictly negative at all points on the three affine lines $x = 0$, $y = 0$ and $x + y = 1$. Recall that if $k > 1$, then the real
curve $F = 0$ has two components, the bounded component (lying in the triangle of reference) and the unbounded component. The cone in $\mathbb{R}^3$ corresponding to the bounded component has two connected components when one removes the origin, a positive part inside which $F > 0$ and a negative part inside which $F < 0$, whilst the cone on the unbounded component only has one connected component in $\mathbb{R}^3$, even after removing the origin. In the case $k > 1$, the unbounded component has three affine branches, one of which lies in the negative quadrant $x < 0$, $y < 0$, one in the sector $y < 0$, $x + y > 1$ and the third in the sector $x < 0$, $x + y > 1$. The (real) inflexion points of the cubic are at $B_1 = (0 : 1 : 0)$, $B_2 = (1 : 0 : 0)$ and $B_3 = (1 : -1 : 0)$, i.e. the intersection of the line at infinity $z = 0$ with the curve (a further reason why the chosen change of coordinates is helpful). The asymptotes for the affine branches of the unbounded component may be found by calculating the tangents to the curve at the inflexion points, and are
\[
x = -\frac{1}{k-1}, \quad y = -\frac{1}{k-1} \quad \text{and} \quad x + y = \frac{k}{k-1}.
\]
Noting that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$, where $\omega$ is a primitive cube root of unity, when $k = 1$ the cubic (1) splits into the real line $z = 0$ and two complex lines (meeting at the centroid $(\frac{1}{3} : \frac{1}{3} : 1)$ of the triangle of reference).

When $k < 1$, the cubic $F = 0$ is smooth but with only one real component, with three affine branches, one in the region $x > 0$, $y > 0$, $x + y > 1$, one in the region $x > 0$, $y < 0$, $x + y < 1$ and one in the region $x < 0$, $y > 0$, $x + y < 1$. The asymptotes are calculated as before and are given by the equations (2).

By Remark 2.11 of [1], the Hessian of the cubic $-x^3 - y^3 - z^3 + 3kxyz$ is given by
\[
27(2k^3(x^3 + y^3 + z^3) - (8 - 2k^3)xyz).
\]
Thus if $H_k$ denotes the Hessian of the cubic $F_k$, the fact that our change of coordinates was unimodular shows for $k \neq 0$ that $H_k = -54k^2F_{k'}$, with parameter $k' = \frac{4 - k^3}{3k^2}$. In particular we see that if $k > 1$, then $k' < 1$, and so the Hessian curve of a real elliptic curve with two components has only one component. For $k < 1$, we have two notable values: $k = 0$ for which the Hessian curve is the three real lines given by $-xy(z - x - y) = 0$, and $k = -2$ for which the Hessian curve is the three lines (two of them complex) corresponding to $k' = 1$ described before. Apart from these two values, for any real elliptic curve with one component, the Hessian curve is a real elliptic curve with two components, the bounded component lying in the triangle of reference. Given the simplifying assumptions made in the Introduction, we have however assumed that the Hessian cubic is also smooth, namely that $k \neq -2$ or 0; note that the above calculations then imply that $H$ is strictly positive on the three affine lines $x = 0$, $y = 0$ and $x + y = 1$.

The case of the curve having two components is illustrated in Figure 1 (which shows $F_5$, $H_5$ and the three asymptotes for $F_5$). The picture for one component when $k \neq -2$ or 0 is not dissimilar, where the roles of the cubic and its Hessian are interchanged — see for instance Figure 3 below. For $k \neq -2$, 0 or 1, we note that the asymptotes to the cubic $F_k = 0$ are tangent to the affine Hessian curve $H_k = 0$; this is just a special case of a classical result that the double polar with respect to the cubic at a point on its Hessian is tangent to the Hessian at the image of the point under the Steinian involution (see [2], Section 3.2 and Exercise 3.8) — the Steinian involution on the Hessian will be explained at the start of Section
2. When the point is an inflexion point of the cubic, the double polar is just the tangent line to the cubic, in our case the asymptote — the corresponding points on the Hessian are labelled $Q_i$, $i = 1, 2, 3$. This gives more precise information about the affine regions where the Hessian curve can lie.

We can now identify the components of the positive index cone. Taking $A = (a, b, 1)$ in the affine plane $z = 1$, we wish to know the index of the quadratic form defined by the homogeneous quadratic $G_A(D) = A \cdot D^2$; if $D = (x, y, z)$ in the above coordinates, this is explicitly given by

$$-ax^2 - by^2 - (1 - a - b)(z - x - y)^2 + kay(z - x - y) + kbx(z - x - y) + k(1 - a - b)xy.$$ 

If $F_k = 0$ has two real components, i.e. $k > 1$, then this quadratic form at $A$ has index $(1, 2)$ if $H(A) > 0$, index $(2, 1)$ if $H(A) < 0$ and index $(1, 1)$ if $H(A) = 0$. This may be easily verified by considering sample points such as $A = (0, 0, 1)$, where the index is plainly $(1, 2)$ and points $A = (t, t, 1)$ for $t \gg 0$ where the index is $(2, 1)$. One is therefore looking for the regions for which either $F > 0$ and $H > 0$, or $F < 0$ and $H < 0$, the latter being relevant since for $D$ in such a region, both $F$ and $H$ will be positive at $-D$.

The cubic curve then bounds precisely four (convex) regions of the (real) affine plane $z = 1$ on which both $F > 0$ and $H > 0$ including the bounded component inside which $F > 0$ (contained in the triangle of reference), where the index of the corresponding quadratic form is $(1, 2)$. Moreover the Hessian curve bounds precisely three (convex) regions of the affine plane on which both $F$ and $H$ are negative.
(where the index of the associated quadratic form is \((2,1)\)). Each unbounded affine region on which \(F > 0\) will together with the negative of the appropriate affine region on which \(H < 0\) give rise to a connected component of the positive index cone in \(\mathbb{R}^3\), part of whose boundary is contained in \(F = 0\) and part of whose boundary is contained in \(H = 0\), with the two parts meeting along rays corresponding to two of the inflexion points of the curve \(F = 0\). Moreover such a cone is also convex, since the tangent line to the cubic at \(B_i\) only meets the Hessian again at the point \(Q_i\) (on the affine branch not containing \(B_i\)). For each of these three resulting hybrid cones in \(\mathbb{R}^3\), we have \(F > 0\), \(H > 0\) and a continuity argument verifies that the index is \((1,2)\) on each cone. The other component of the positive index cone corresponds to the bounded component and whose boundary is contained in \(F = 0\).

In the case when \(F = 0\) only has one connected component, i.e. \(k < 1\), we have that the condition \(F > 0\) defines three (convex) regions of the affine plane \(z = 1\), and in these regions the Hessian is positive and the index is \((1,2)\). Under the assumption that the Hessian is also smooth, there are four (convex) regions of the affine plane on which the Hessian is negative (where the cubic is also negative), three unbounded regions on which the index is \((2,1)\), and the region determined by the bounded component of the Hessian, on which \(F < 0\) and \(H < 0\). As before we obtain three hybrid components of the positive index cone, obtained from unbounded affine regions on which \(F > 0\), \(H > 0\) together with the (negative of) unbounded affine regions on which \(F < 0\), \(H < 0\); a continuity argument again ensures that the index is \((1,2)\) on the hybrid components.

For the Fermat cubic, i.e. \(k = 0\), we check easily that the index at any interior point of the triangle of reference is in fact \((0,3)\). Thus by a continuity argument, for any \(-2 < k < 1\), the index of the quadratic form for any \(A\) inside the bounded component of the Hessian remains \((0,3)\), since as \(k \to 0\) the bounded component of the Hessian of \(F_k\) tends to the triangle of reference. For points actually on this bounded component of the Hessian the index is \((0,2)\), apart from the Fermat cubic and the vertices of the triangle of reference, where the index is \((0,1)\). For \(k < -2\), we check that for \(A\) inside the bounded component, the index is \((2,1)\) (check it at a suitable such point for \(k \ll -2\) and use continuity again), and for \(A\) actually on the bounded component the index is \((1,1)\). Thus in this case, points in the negative of the corresponding open cone have index \((1,1)\), and we obtain a further connected component of the positive index cone, whose boundary is contained in \(H = 0\).

In both cases under consideration, namely the real elliptic curve has two components or one component with smooth Hessian, the components of the positive index cone are all convex, and their closures are strictly convex cones.

**Remark 1.1.** Suppose now that in the coordinates chosen, we have a real class \(E = (a,b,c)\) with \(E^3 \leq 0\) and at which the index is \((1,q)\) for \(q \leq 2\), for instance the class of a rigid non-movable surface with \(E^3 \leq 0\). If \(c \leq 0\) and \(E\) does not represent an inflexion point of the cubic curve, then the fact that \(H(E) \geq 0\) implies that \(F(E) = E^3 > 0\). Thus one convenient consequence of assuming in the Main Theorem that any rigid non-movable surface satisfies \(E^3 \leq 0\) is that all such classes \(E = (a,b,c)\) lie in the upper half-space \(c \geq 0\), and \(c = 0\) is only possible if \(E\) represents an inflexion point on the cubic. It is this weaker property, namely that any rigid non-movable surface \(E\) with \(E^3 > 0\) lies in the open upper half-space, which will suffice for the proofs in this paper (except the proof of Corollary 1.5), and so the Main Theorem (apart from its last sentence) remains true under this weaker
assumption. Note that because we have only chosen a real coordinate system, it is unclear when a point \((a, b, c)\) represents a class in \(H^2(X, \mathbb{Z})\); this however will not be a problem in our arguments.

In this paper, we shall therefore make the following simplifying assumptions.

**Simplifying Assumptions.** We assume throughout the paper that \(X\) is a Calabi–Yau threefold with \(b_2(X) = 3\), that the cubic form on \(H^2(X, \mathbb{Z})\) defines a smooth cubic curve in \(\mathbb{P}^2(\mathbb{R})\) with smooth Hessian, and that the class of any rigid non-movable surface \(E\) on \(X\) with \(E^3 > 0\) lies in the open upper half-space (with respect to the real coordinates chosen above). Thus the class \(E\) of a rigid non-movable surface either represents an inflexion point or lies in the open upper half-space.

When \(-2 < k < 0\) or \(0 < k < 1\), we know from Hodge index considerations and the above calculations that the negative closed convex cone on the bounded component of the Hessian cannot contain the class of any surface, and in particular the cone cannot contain the Kähler cone of \(X\). Under our assumptions, we show below that this latter fact continues to be true when \(k < -2\).

Recall that for a convex body \(V\) in \(\mathbb{R}^n\) with non-empty interior, a point \(D\) on the boundary \(\partial V\) is said to be visible (with respect to \(V\)) from a point \(A\) if the line segment \(AD\) does not meet the interior of \(V\). If \(D\) is a smooth point of \(\partial V\), the tangent hyperplane through \(D\) determines a closed half-space whose intersection with the interior of \(V\) is empty; then \(D\) is visible from \(A\) if and only if \(A\) is in this half-space. If \(W\) is a convex cone with vertex at the origin and non-empty interior, then for any \(A \neq 0\), the set of points in \(\partial W\) which are visible from \(A\) and \(-A\) will be called the visible extremity of \(\partial W\) from \(A\); a non-zero smooth point \(D \in \partial W\) is in the visible extremity from \(A \neq 0\) if and only if \(A\) is in the tangent hyperplane. A non-zero \(D \in \partial W\) which is not in the visible extremity from \(A \neq 0\) will be visible from \(-A\) if and only if it is not visible from \(A\), and also if and only if \(-D \in \partial(-W)\) is visible from \(A\) with respect to \(-W\).

**Proposition 1.2.** Under the above simplifying assumptions, if the cubic form corresponds to the case \(k < -2\), and so there is a component of the positive index cone \(P^o\) corresponding to the bounded component of the Hessian, then the Kähler cone of \(X\) is not contained in \(P^o\).

**Proof.** We suppose that the Kähler cone is contained in \(P^o\). Index considerations imply that the interior of the movable cone is then contained in \(P^o\), and without loss of generality we may assume that \(X\) is general in moduli. If there are no rigid non-movable surfaces on \(X\), then any effective divisor must be movable, as each component then corresponds to a nef (and hence mobile) divisor on some minimal model — see the first paragraph of Section 2 from [6]. Choosing any \(D\) on the boundary of the movable cone, by the second proof of Theorem 0.1 in Section 4 of [6], we deduce \(\text{vol}(D) \geq D^3 > 0\), and hence \(D\) is big; this remains true for all nearby rational points \(D'\), including those not in the movable cone, therefore yielding a contradiction.

If there is precisely one rigid non-movable surface \(E\) on \(X\), we choose any \(D\) on the boundary of the movable cone which is not visible from \(E\); then the above argument shows that \(\text{vol}(D') > 0\) for all nearby rational points, including \(D' = D - \epsilon E\) with \(0 < \epsilon \ll 1\) not in the movable cone. Since \(D'\) is big, we have that \(D' - \lambda E\) is movable for some \(\lambda > 0\), and this is a contradiction, since \(D'\) is a convex
Next we suppose that there are precisely two rigid non-movable surfaces $E_1, E_2$ on $X$, where $E_2$ is not allowed to be a negative multiple of $E_1$ in the case of points with $z = 0$ (i.e. representing inflexion points). We observe that the closed line segment joining $-E_1$ and $-E_2$ is disjoint from the movable cone, since a convex combination of $E_1$ and $E_2$ is effective and so its negative cannot also be pseudo-effective, and hence cannot lie in the movable cone. We can then find a point $D$ on the boundary of the movable cone which is visible from both $-E_1$ and $-E_2$, in particular with the interior of the cone generated by $D, -E_1$ and $-E_2$ disjoint from the movable cone. The previous argument implies that $\text{vol}(D') > 0$ for all nearby rational points, and in particular we may take a big such $D'$ in the interior of the cone generated by $D, -E_1$ and $-E_2$. Writing such a $D'$ in terms of its movable part $\Delta$ and rational multiples of the $E_i$ yields a contradiction, since then $\Delta$ could not lie in the movable cone.

Finally we may suppose therefore that there are at least three rigid non-movable surfaces on $X$, say $E_i$ for $i = 1, 2, 3$, and let $L \in P^2$ denote an integral ample class. These four integral classes are linearly dependent in $H^2(X, \mathbb{Z})$, and we cannot have any $E_i$ being a rational convex combination of the other three classes since it is non-movable. Also $L$ is not a rational convex combination of the $E_i$ since they lie in different half-spaces. We deduce without loss of generality that some integral convex combination of say $E_1$ and $E_2$ is an integral convex combination of $E_3$ and $L$, and hence is mobile. As however the cone generated by mobile classes lies inside the closed cone $P$ by index considerations, and hence lies in the open lower half-space $z < 0$, this is a contradiction. \(\square\)

We now consider the case where the real elliptic curve has two components, and the Kähler cone is contained in the (positive) cone on the bounded component. We shall need an elementary lemma in convexity theory, the idea of the proof given being suggested to the author by a colleague Imre Leader.

**Lemma 1.3.** Let $V \subset \mathbb{R}^2$ be a open bounded convex body with a smooth boundary curve, and $x_1, x_2, \ldots$ be a (perhaps infinite) collection of points in $\mathbb{R}^2$ not in $V$. We let $W$ denote the points of the boundary $\partial V$ which cannot be seen from any of the $x_i$, i.e. $W$ consists of points $x \in \partial V$ such that the line segments $xx_i$ all meet $V$. Then $V$ is contained in the closure $Z$ of the convex hull of $W$ and $x_1, x_2, \ldots$.

**Proof.** We suppose that the result is not true, and so in particular by the convexity of $V$ there is a point $x$ in the boundary of $V$ which is not in the closure $Z$ of the convex hull of $W$ and $x_1, x_2, \ldots$. Standard convexity results imply the existence of a line $l$ though $x$ such that $Z$ is strictly on one side of $l$, and in particular all points of $W$ and all the $x_i$ are in the corresponding open half-plane.

If $l$ is tangent to the boundary $\partial V$ at $x$, then $V$ itself must be on one side of $l$. If $W$ and the $x_i$ are contained in the open half-plane disjoint from $V$, we get an immediate contradiction by considering $y$ the other point on $\partial V$ whose tangent is parallel to $l$, which therefore cannot be seen from any of the $x_i$ and hence by definition lies in $W$, contrary to assumption. If however $W$ and the $x_i$ are contained in the same open half-plane that contains $V$, then $x$ cannot be seen from any of the $x_i$ and so $x \in W$, contrary to assumption.
If however $l$ is not tangent to the boundary $\partial V$ at $x$, we consider the point $y \in \partial V$ with tangent parallel to $l$ such that $y$ is on the other side of $l$ to $W$ and the $x_i$. Then $y$ cannot be seen from any of the $x_i$ and hence $y \in W$, a contradiction. □

**Proposition 1.4.** Under the above simplifying assumptions, suppose that $X$ is general in moduli and that the real elliptic curve has two components, with the Kähler cone of $X$ contained in the open positive cone $P^o$ on the bounded component. Then $P^o$ is contained in the interior of the effective cone of $X$.

**Proof.** Consider now the affine plane $z = 1$ and let $V$ be the open convex body given by the bounded component of the above real elliptic curve, so that $P^o$ is the cone on $V$. Suppose that $E = (a, b, c)$ represents the class of a rigid non-movable surface; by assumption, either $E$ represents an inflexion point of the elliptic curve or $c > 0$ and there is a unique point $A$ of the affine plane which is a positive multiple of $E$; moreover $A \notin V$ from Proposition 4.4 of [6].

The points of the boundary of $V$ which cannot be seen from $E$ with respect to $P^o$ are precisely those points $D$ with $E \cdot D^2 > 0$; the points at which $E \cdot D^2 = 0$ corresponding to the tangent lines to the boundary which pass through $A$, with an appropriate interpretation for the limit case of a point $E$ at infinity on the plane $z = 0$ in $\mathbb{R}^3$ (therefore representing an inflexion point). We let $Q$ denote the (convex) subset of $V$ defined by the inequalities $E_i \cdot D^2 > 0$ for all $i$, which contains in its closure the set of all $D_0 \in \partial V$ which cannot be seen from any $E_i$ (note that the cone $Q$ on $\tilde{Q}$ contains the Kähler cone).

Given such a $D_0$, for any ample class $L$, note that any strictly convex combination of $D_0$ and $L$ lies in $\tilde{Q}$. This enables us to find rational points $D_i \in \tilde{Q}$ for $i > 0$ with $D_i \rightarrow D_0$. By the argument in Proposition 4.1 and Lemma 4.3 from [6], each $D_i$ is in the effective cone (the quoted results here use the assumption that $X$ is general in moduli), and so the point $D_0$ is in the pseudo-effective cone. Thus, from Lemma 1.3, it follows by a limiting argument that the closure $P$ of $P^o$ is in the pseudoeffective cone, and hence the open cone $P^o$ is contained in the interior of the effective cone (i.e. the big cone). □

**Corollary 1.5.** Under the hypotheses of the Main Theorem, the linear form $c_2$ is non-negative on the positive cone $P$ on the bounded component of the elliptic curve.

**Proof.** Without loss of generality, we may assume that $X$ is general in moduli. We noted in Section 2 of [6] that if $c_2 \cdot E < 0$ for some rigid non-movable surface, then $E^3 > 0$. Thus the hypotheses imply that $c_2 \cdot E \geq 0$ for all rigid non-movable surfaces $E$ on $X$. Moreover we also noted in Remark 1.2 of [6] that $c_2 \cdot L \geq 0$ for any movable class $L$. Since by the previous result, any element of $D \in P^o$ is effective and $X$ is general in moduli, it follows that $c_2 \cdot D \geq 0$ for any $D \in P$. □

In the light of these results, for the Main Theorem it is sufficient to prove:

**Theorem 1.6.** Under the above simplifying assumptions, the Kähler cone is not contained in a hybrid component of the positive index cone.

We note that in the explicit example given in Remark 0.1, it is straightforward to check that the Hessian matrix is negative definite at say $(-4, 2, 1)$, from which we may deduce that the invariant $-2 < k < 1$ and the Kähler cone must be contained in a hybrid component of the positive index cone.
For the examples of 6-manifolds not supporting any Calabi–Yau structure, we use Wall’s Theorem; we may take the cubic form on $H^2(M, \mathbb{Z}) = \mathbb{Z}^3$ to be an appropriate integral multiple of $F$ as in equation (1), with $k < 1$ an integer other than $-2$ or $0$, and any suitable integral linear form satisfying the congruence conditions — we note that the trilinear form corresponding to $F$ does satisfy the first of the congruence conditions. We may also take the cubic form on $H^2(M, \mathbb{Z}) = \mathbb{Z}^3$ to be an appropriate integral multiple of $F$ as in equation (1) with any integer $k > 1$ and any suitable integral linear form satisfying the congruence conditions but which is negative somewhere on the open positive cone on the bounded component of the real elliptic curve. In these cases, if the integral multiple of $F$ has been chosen so that there are no integral classes $E$ with $1 \leq E^3 \leq 8$, or indeed as we saw in the Introduction if we scale both $F$ and a linear form satisfying the required congruence conditions by say a factor of 4, our Main Theorem rules out the possibility of any Calabi–Yau structures, independent of any choice for the even rank free abelian group $H^3(M, \mathbb{Z})$. Being even more concrete, the integral cubic form $nF_k$ for any integer $k < 1$ ($k \neq -2$ or $0$) and any integer $n > 8$ can for instance never be the cubic form arising from a Calabi–Yau threefold.

**Notation.** We shall now fix on the notation that will be used in the rest of this paper to describe a hybrid component $P^o$ of the positive index cone, with $P$ denoting its closure. Taking the affine slice $z = 1$, where the cubic is of the form described above, with slight abuse of notation concerning the points at infinity, we denote the projectivised boundary of $P$ by $C = C_1 \cup C_2$, where $C_1$ is an affine branch of $F = 0$ and $C_2$ is an associated affine branch of $H = 0$, with $C_1$ and $C_2$ meeting at two inflexion points (at infinity). When $k > 1$, without loss of generality we may by symmetry take $C_1$ in the negative quadrant and then $C_2$ in the region $x > 0, y > 0, x + y > 1$. The branches $C_1$ and $C_2$ meet (at infinity) at the inflexion points $B_1 = (0 : 1 : 0)$ and $B_2 = (1 : 0 : 0)$. More specifically, the cone $P$ then has boundary the positive cone on $C_1$ together with the negative cone on $C_2$, the two parts meeting in two rays corresponding to positive multiples of $(0, -1, 0)$ and $(-1, 0, 0)$. When $k < 1$ and $k \neq 0, -2$, we take $C_1$ in the region $x > 0, y > 0, x + y > 1$ and $C_2$ in the negative quadrant. The branches $C_1$ and $C_2$ again meet (at infinity) at the inflexion points $B_1$ and $B_2$. The cone $P$ then has boundary the positive cone on $C_1$ together with the negative cone on $C_2$, the two parts meeting in the two rays corresponding to positive multiples of $(0, 1, 0)$ and $(1, 0, 0)$. In both cases, we shall denote by $V_1$ the open convex subset of the affine plane bounded by $C_1$ and $V_2$ the open convex subset bounded by $C_2$.

**Proposition 1.7.** Under the above simplifying assumptions, suppose that $P^o$ is a hybrid component of the positive index cone as described above, $E_1, E_2, \ldots$ are the (perhaps infinitely many) rigid non-movable classes in $H^2(X, \mathbb{Z})$ and $Q$ a connected component of the subcone of $P^o$ defined by the inequalities $E_i \cdot D^2 > 0$ for all $i$. If $Q$ contains the Kähler cone then there cannot exist a non-trivial open arc of points $-D \in C_2$ which are visible with respect to the cone $-Q$ from every $E_i$, but with each $D$ representing a point of the boundary $\partial Q$.

**Proof.** Again, we may without loss of generality assume that $X$ is general in moduli. We have that $Q = \bigcap_{i \geq 1} Q(i)$, where $Q(i)$ is the component of the subcone of $P^o$ defined by $E_i \cdot D^2 > 0$ which contains the Kähler cone. Were such an arc of points in $C_2$ to exist, we choose a point $-D_0$ in this arc; note that no point $E_i$ lies on
the tangent plane to the cone on $C_2$ along the ray $R_+(-D_0)$, since otherwise some points of the arc would not be visible from $E_i$. Thus $D_0$ is not visible from any of the $E_i$ with respect to $Q$. For any real ample divisor $L$, we note that any strictly convex combination of $D_0$ and $L$ lies in each $Q(i)$, and hence in $Q$. In this way we show, using the argument in Proposition 4.1 and Lemma 4.3 from [3], as in the proof of Proposition 1.4, where the quoted results use that $X$ is general in moduli, that $D_0$ is a limit of effective rational divisors $D_j \in Q$; therefore $D_0$ is pseudoeffective.

For any prime divisor $\Gamma$, we have a function $\sigma_\Gamma$ on the pseudoeffective cone as in Definition 1.6 of Chapter III from [3]; all the following references to [3] will be from Chapter III. Moreover by Proposition 1.10 and Corollary 1.11 of the given Chapter, there are at most three $\Gamma$ with $\sigma_\Gamma(D_0) > 0$, whose classes are moreover linearly independent in $H^2(X, \mathbb{R})$. Under our assumption that $X$ is general in moduli these will be rigid non-movable surfaces $E_i$, without loss of generality $E_1, \ldots, E_r$ with $r \leq 3$. Thus in the $\sigma$-decomposition (see Definition 1.12 from the cited Chapter) $D_0 = P + N$ of $D_0$, we have $N = \sum_{i=1}^r \sigma_{E_i}(D_0)E_i$. In particular, by Lemma 1.8 of the Chapter, we have $\sigma_{E_i}(P) = 0$ for $i = 1, \ldots, r$, and indeed also that $\sigma_{E_i}(P) = 0$ for any other surface $\Gamma$. If we knew that $D_0$ is big, then by Lemma 1.4 (4) of the cited Chapter, applied $r$ times, we have that $D_0 = \Delta + \sum_{i=1}^r \sigma_{E_i}(D_0)E_i$, with $\Delta$ also big and in particular pseudoeffective, and $\sigma_{\Gamma}(\Delta) = 0$ for all surfaces $\Gamma$. Thus by Lemma 1.14 (1) of the Chapter, we would deduce that $\Delta$ is movable.

Suppose first that $r = 0$; then by Lemma 1.14 (1) of [3], Chapter III, we deduce that $D_0$ is movable. We saw then in the second proof of Theorem 0.1 in Section 4 of [2] that $\text{vol}(D_0) \geq D_0^3 > 0$ and hence $D_0$ is big. If there are no rigid non-movable surfaces $E$ on $X$, we noted in [6] that this gives an immediate contradiction, since then there would exist (rational) points $D$ near $D_0$ at which the Hessian is negative but which are big and hence (recalling that $X$ is assumed general in moduli) in this case also movable; this is a contradiction. However, if there exists a rigid non-movable surface $E$, we deduce that $D_0 - \epsilon E$ is also big for $0 < \epsilon \ll 1$, and hence (using a previous argument) of the form $\Delta + \mathcal{E}'$, with $\Delta$ big and movable and $\mathcal{E}'$ supported on at most three of the $E_i$. Therefore $D_0 = \Delta - \mathcal{E}' + \epsilon E$.

We suppose now that $r > 0$ and verify that $D_0$ is big. We know that there is an $E = E_i$ with $\sigma_{E_i}(D_0) > 0$; choose $\delta = \sigma_{E_i}(D_0)/2$. Thus for all $0 < \epsilon \ll 1$, the big divisor $D_0 + \epsilon L$ has $\sigma_{E_i}(D_0 + \epsilon L) > \delta$, straight from the definition of $\sigma_{E_i}$ as a limit in Definition 1.6 of Chapter III from [3]. Then by Lemma 1.4 (4) of the Chapter, we deduce that $D_0 + \epsilon L - \delta E$ is big for all $0 < \epsilon \ll 1$. Thus $D_0 - \delta E$ is pseudoeffective. From our choice of $D_0$, we know that $D_0$ is not visible from $E$ with respect to $Q$ and hence $D_0 + tE \in Q$ for $0 < t \ll 1$, and hence effective (as we saw above). Since all points $-D$ of the original arc in $C_2$ have $D$ pseudoeffective, we must have that $D_0$ lies in the interior of the pseudoeffective cone, and hence is big as claimed. We saw above that then $D_0 = \Delta - \mathcal{E}$, for some big class $\Delta$ in the movable cone and some real non-zero convex combination $\mathcal{E}$ of at most three of the $E_i$.

Summing up therefore, we can in all cases write $D_0 = \Delta + \mathcal{E}$, for some big class $\Delta$ in the movable cone and some real non-zero convex combination $\mathcal{E}$ of finitely many of the $E_i$. Thus some positive multiple of $-\Delta = -D_0 + \mathcal{E}$ is in one of the regions in the affine plane where $H \leq 0$ and hence $F < 0$, and in particular we deduce that $\Delta^3 > 0$. If $L$ denotes an ample divisor in $P^3$, then $\Delta + tL$ is movable for all $t \geq 0$, and $(\Delta + tL)^3 > 0$ for all $t \geq 0$, from which it follows by connectedness that $\Delta = D_0 - \mathcal{E} \in P$. Recall that $C_2$ has been assumed smooth; since $-D_0 \in C_2$
is visible from every $E_i$ with respect to $V_2$, it follows that the Hessian at $-D_0 + \varepsilon E$ is strictly positive for $0 < \varepsilon \ll 1$, and hence the Hessian at $D_0 - \varepsilon E$ is strictly negative; this contradicts the convexity of $P$. \hfill \Box

Our technique for ruling out such a hybrid component $P^o$ containing the Kähler cone is as follows; in the remaining sections of the paper, we prove the following result in the various cases for the elliptic curve. In the light of this combined with Proposition 1.7, the only conclusion then is that under the above simplifying assumptions $P^o$ cannot contain the Kähler cone, and so Theorem 1.6 will be proven. The proof of Proposition 1.8 is slightly technical when written out in full, but the basic mathematical input consists just of classical facts about the Steinian involution on the Hessian of a real elliptic curve; the basic properties of the $E_i$ we use are that the index at $E_i$ is $(1, q_i)$ with $q_i \leq 2$, and that either $E_i$ represents an inflexion point of the cubic or it lies in the open upper half-space.

**Proposition 1.8.** Under the above simplifying assumptions, let $P^o$ be a hybrid component of the positive index cone. Suppose $E_1, E_2, \ldots$ denote the (perhaps infinitely many) rigid non-movable surface classes in $H^2(X, \mathbb{R})$ and let $Q$ be a connected component of the subcone of $P^o$ defined by the inequalities $E_i \cdot D^2 > 0$ for all $i$, where $Q$ is assumed to have non-empty interior. Then there exists a non-trivial open arc in $C_2$ of points $-D$ which are visible from all the $E_i$ with respect to the cone $-Q$, with each $D$ representing a point of the boundary $\partial Q$.

We have seen in Theorem 1.6, that if the Kähler cone is contained in a hybrid component, then there must exist rigid non-movable surfaces $E$ with $E^3 > 0$ lying in the closed lower half-space. There will however exist at most two of these.

**Proposition 1.9.** Suppose $X$ is a Calabi–Yau threefold whose corresponding cubic and Hessian are smooth, with the Kähler cone is contained in a hybrid component of the positive index cone. Then there are either one or two rigid non-movable surfaces $E$ with $E^3 > 0$ lying in the closed lower half-space.

**Proof.** Assuming that the Kähler cone is contained in the hybrid component $P^o$ (with the usual convention that the boundary is determined by the two affine curves $C_1$ and $C_2$), we know from Proposition 4.4 of [1] that for any such class $E$, we have $E \notin P$. Hence if $k > 1$ any such class lies in the open half-space $x + y > \frac{k'}{k - 1} z$ (where as usual $k' = \frac{4 - k^3}{3k^2}$) corresponding to the relevant asymptote to the Hessian. If $k < 1$ we have a tangent line $x + y = e_2$ to the bounded component of the Hessian with the bounded component on the opposite side to $C_2$ (see Section 3) and then any such class lies in the half-space $x + y \leq e_2 z$, in the case of $k = -2$ possibly lying in the negative of the closed cone on the bounded component of the Hessian. Suppose now that there were three classes $E_1, E_2, E_3$ as above; choose an integral ample class $L \in P^o$; since there is an integral dependence relation between these four classes, we deduce that some class $E$ which is a convex combination of two or three of the $E_i$ (and hence in the appropriate half-space $x + y > \frac{k}{k - 1} z$, respectively $x + y \leq e_2 z$) is in fact mobile (cf. the last part of the proof of Proposition 1.2), and hence has non-negative value for the Hessian. We deduce that $-E$ is also in the closed upper half-space and has non-positive Hessian. If $k < -2$, we note that $E$ is not in the negative cone on the bounded component of the Hessian, since then by index considerations this cone would also contain the mobile cone. Thus $-E$ lies in the closed cone on one of the two affine regions bounded by the unbounded branches
of the Hessian other than $C_2$ (cf. Sections 3 and 4) and so lies in a component of the positive index cone distinct from $P$. This however is impossible: since $\mathcal{E}^3 > 0$, any convex combination of $\mathcal{E}$ and $L$ will have both a positive cube and non-negative value for the Hessian (as it is mobile), which by connectedness of $P$ would imply that $\mathcal{E} \in P$ and hence a contradiction. \hfill $\Box$

2. **Hybrid components when elliptic curve has two real components**

In this section, we study the case when $k > 1$, i.e. the real elliptic curve $F = 0$ has two components, and so in particular the Hessian is smooth. For simplicity, from now on we often use $H$ to denote both the Hessian and the Hessian curve.

With notation as in the previous section, we let $P^\circ$ denote a hybrid component of the positive index cone, with closure $P$. Without loss of generality, we may by symmetry adopt the notation explained in the previous section, where the component is determined by the branches $C_1$ and $C_2$ in the affine plane given by $z = 1$. Recall also that we denote by $V_i$ (for $i = 1, 2$) the open convex affine region bounded by $C_i$.

Let $E$ denote a class of a rigid non-moveable surface on $X$; thus the index at $E$ is $(1, q)$ with $q \leq 2$ and by assumption $E = (a, b, c)$ with $c \geq 0$, and $c = 0$ only if $E$ represents one of the inflexion points of the cubic. We now specify where the homogeneous quadratic given by $G_E(D) = E \cdot D^2$ vanishes on the boundary of $P$. If $E \in \mathbb{R}^3$ represents one of the inflexion points, the function $G_E$ will be easy to understand explicitly. The main case to consider is when $c > 0$ and so some positive real multiple $A$ of $E$ lies in the affine plane $z = 1$. We shall therefore need to study real classes $A$ in the affine plane which have index $(1, q)$ for $q \leq 2$.

For $A$ in the affine plane, there is a simple answer to the question of where on $C_1$ we have vanishing of $G_A$, namely points on of $C_1$ for which the tangent passes through $A$ (including maybe inflexion points at infinity), noting that if $A \in V_1$, then $G_A$ is strictly positive on $C_1$. There will be two such points if $A \notin V_1$ is in the quadrant $x \leq -\frac{1}{c-1}, y \leq -\frac{1}{c-1}$ (including the possibilities of the inflexion points $B_1$ and $B_2$, or a point on $C_1$ taken twice), no such points if $A$ is in the quadrant $x > -\frac{1}{c-1}, y > -\frac{1}{c-1}$, and one point otherwise.

We now recall some very classical theory: when the Hessian curve $H$ of an elliptic curve $F = 0$ is smooth, there is a well-defined base-point free involution $\alpha$ on $H$, known as the Steinian map or Steinian involution ([2], Section 3.2, noting a misprint in Corollary 3.2.5), where the polar conic of $F$ with respect to a point $U$ on $H$ is a line pair with singularity at $U' = \alpha(U)$. We note that this says that $U \cdot U' \equiv 0$ and $\alpha$ induces an involution on the real points of $H$. The Steinian involution has the property that for any point $U' \in H$, the second polar of $F$ with respect to the point $U'$ is the tangent to the Hessian $H$ at $U = \alpha(U')$ ([2], Exercise 3.8, again noting a misprint). So for any point $A$ on the tangent line to the Hessian at $U$, the conic $G_A = 0$ contains the point $U'$; moreover if $A \neq U$, the conic is non-singular at $U'$ with tangent line $L$ at $U'$ independent of $A$; explicitly $L$ is defined by the linear form $W \cdot U'$, where $W$ is any point ($\neq U$) on the tangent line through $U$. Moreover since such a linear form $W \cdot U'$ vanishes at both $U$ and $U'$, the common tangent line $L$ is just the line joining $U$ and $U'$. Moreover $U'$ is clearly the unique point of intersection of the conic with $L$.

In the case currently under consideration, where the Hessian has only one real component, the branch $C_2$ of the Hessian $H = 0$ is in the region $x > 0$, $y >$
0, \ x + y > 1, \text{ passing through } B_1, B_2 \text{ and } R = Q_3, \text{ the affine point with coordinates } \left(\frac{k}{2(k - 1)}, \frac{k}{2(k - 1)}\right). \text{ As } B_3 = (-1 : 1 : 0) \text{ is the third inflexion point, we have already observed that the tangent to } F = 0 \text{ at } B_3 \text{ is tangent to } H = 0 \text{ at } R. \text{ If therefore we take } B_3 \text{ to be the zero of the group law, then } R \text{ is the unique real 2-torsion point of the Hessian and } \alpha \text{ is given by translation in the group law by this point. Let } Q_1 \text{ be the point on the branch of } H \text{ in the region } x < 0, \ y > 0, \ x + y < 1, \text{ which is the 2-torsion point when we take } B_1 \text{ as the zero in the group law, and } Q_2 \text{ the point on the branch of } H \text{ in the region } x > 0, \ y < 0, \ x + y < 1 \text{ corresponding to taking } B_2 \text{ to be the zero is the group law. We note that } \alpha(B_3) = Q_1, \ \alpha(B_2) = Q_2 \text{ and } \alpha(R) = B_3. \text{ Thus the second polar of } F \text{ with respect to the inflexion point } B_1 \text{ is the tangent to the Hessian at } Q_1, \text{ namely given affinely by } x = -\frac{1}{k - 1}, \text{ the second polar of } F \text{ with respect to the inflexion point } B_2 \text{ is the tangent to the Hessian at } Q_2, \text{ namely given affinely by } y = -\frac{1}{k - 1}, \text{ and the second polar of } F \text{ with respect to the point } R \text{ is the tangent to the Hessian at } B_3, \text{ namely the asymptote } x + y = \frac{k'}{k - 1}, \text{ where as before } k' = 4 - \frac{k^3}{6k^2}. \text{ Under the Steiner involution, the arc } Q_1B_3 \text{ of the Hessian corresponds to the arc } B_1R \text{ of } C_2, \text{ whilst the arc } B_3Q_2 \text{ corresponds to the arc } RB_2 \text{ of } C_2.

Setting } A = (a, b, 1), \text{ it is easily checked from this that } G_A(B_1) > 0 \text{ if and only if } a < -\frac{1}{k - 1}, \text{ that } G_A(B_2) > 0 \text{ if and only if } b < -\frac{1}{k - 1} \text{ and that } G_A(R) > 0 \text{ if and only if } a + b < \frac{k}{(k - 1)}. \text{ Moreover if } A = Q_i \text{ (for } i = 1, 2) \text{ then } G_A(B_i) = 0, \text{ and if } A = B_3 \text{ then } G_A(R) = 0. \text{ We are interested in the cases of } B_1 \text{ and } B_2 \text{ since we want to know the sign of } G_A \text{ on points of } C_2 \text{ where either } y \gg 0 \text{ or } x \gg 0. \text{ For future use, we introduce the notation that } B_1 = (0, 1, 0), \ B_2 = (1, 0, 0) \text{ and } \tilde{B}_3 = (-1, 1, 0), \text{ points in } \mathbf{R}^3 \text{ representing the three inflexion points on the real projective cubic.}

When } A \text{ is in the quadrant } x \leq -\frac{1}{k - 1}, \ y \leq -\frac{1}{k - 1}, \text{ then no tangent line at a point on the open arcs } Q_1B_3 \text{ or } B_3Q_2 \text{ contains } A, \text{ and so } G_A \text{ is non-vanishing (and indeed positive) on the affine branch } C_2 — \text{ in the case of equality perhaps vanishing at } B_1 \text{ or } B_2. \text{ In this case we note that all of } C_2 \text{ is visible from } A \text{ with respect to } V_2. \text{ Moreover, when } E \text{ is a positive multiple of } -B_1 \text{ or } -B_2, \text{ then } G_E \text{ is positive on both } C_2 \text{ and } P^0, \text{ and all of } C_2 \text{ remains visible from } E \text{ with respect to } -P^0. \text{ When however } A \text{ is in the open region given by } a > -\frac{1}{k - 1}, \ b > -\frac{1}{k - 1} \text{ and } a + b > \frac{k}{(k - 1)}, \text{ then } G_A \text{ is non-vanishing on all the projectivised boundary of } P, \text{ and hence } G_A \text{ is strictly negative on all of } P \text{ by continuity, since it plainly is for } A \in V_2 \text{ by Lemma 3.3 of } [6]. \text{ Thus for } A \text{ in the corresponding closed region, } G_A \text{ is negative on all of } P^0 \text{ and so } P^0 \cap \{G_A > 0\} \text{ is empty. For } E \text{ a positive multiple of } B_1 \text{ or } B_2, \text{ we also have that } P^0 \cap \{G_E > 0\} \text{ is empty. We shall refer to the above cases as the trivial cases, as for such classes the results in this section will be satisfied trivially.}

The case of } E \text{ representing } B_3 \text{ is more interesting: the conic } B_3 \cdot D^2 = 0 \text{ consists of a line pair with singularity at } R, \text{ one line of which is tangent to } C_2 \text{ there (i.e. the asymptote } x + y = k/(k - 1)) \text{ and one line of which is the line of symmetry } y = x. \text{ Thus the subset } Q \text{ of } P^0 \text{ given by } B_3 \cdot D^2 > 0 \text{ lies on one side of the plane through the origin in } \mathbf{R}^3 \text{ corresponding to the line of symmetry, and the arc in } C_2 \text{ given by the same inequality is just the upper half of } C_2, \text{ which we note is also the half which is visible from } B_3 \text{ with respect to } -Q. \text{ Clearly there is a corresponding statement for the point } -B_3. \text{ The main cases of interest however are where } A \text{ is an affine point, not in the above two closed regions which yield trivial cases, and}
where the index is $(1, q)$ for some $q \leq 2$; here $A$ will be on the tangent line to the Hessian at some point $U$ (not necessarily unique) on one of the open arcs $Q_1B_3$ or $B_3Q_2$. The subcone of $P^\circ$ defined by $A \cdot D^2 > 0$ will consist of one or two (convex) open subcones, and we wish to describe the (convex) components of $V_2 \cap \{G_A > 0\}$ and $P^\circ \cap \{G_A > 0\}$.

Given a point $U_1$ on say the open arc $Q_1B_3$ of the Hessian, we now describe in the Summary below the components of $P^\circ \cap \{G_A > 0\}$ for classes $A$ on the affine tangent line at $U_1$ for which the index is $(1, q)$ for some $q \leq 2$. These may also be described in terms of the components of $V_2 \cap \{G_A > 0\}$ and $V_1 \cap \{G_A > 0\}$.

**Summary:** Suppose that $U_1$ is a point of the arc $Q_1B_3$ of the Hessian, with $U_1' = \alpha(U_1)$ the corresponding point of the arc $B_1R$ of $C_2$. The tangent line $L_1$ to the Hessian at $U$ will intersect the Hessian again at some point $Z$ on the branch $B_3B_1$. Recall that for any point $A$ of the tangent line $L_1$, the conic $A \cdot D^2 = 0$ contains $U_1'$ and that if $A \neq U_1'$, then the conic is smooth at $U_1'$ with tangent line $L$ there being the line joining $U_1$ and $U_1'$. Moreover, for any point $A = (a, b, 1)$ on the tangent line $L_1$ above the third point of intersection $Z$ with the Hessian, the index at $A$ is $(1, q)$ with $q \leq 2$. It follows from Proposition 3.4 of [6] that for such points the components of $V_2 \cap \{A \cdot D^2 > 0\}$ and $P^\circ \cap \{A \cdot D^2 > 0\}$ are convex. We call a component of $V_2 \cap \{A \cdot D^2 > 0\}$ bounded if its boundary intersects $C_2$ in a bounded arc, and unbounded otherwise (in which case its boundary contains $B_1$ or $B_2$).

We consider first the case of $A \in L_1$ having $a < -\frac{1}{k-1}$ and $a + b > \frac{k'}{k-1}$ (and so furthest away from the above third point of intersection). Here $A$ does not lie in any other tangent line at a point on the two arcs $Q_1B_3$ and $B_3Q_2$, and so $U_1' = \alpha(U_1)$ is the unique point of $C_2$ at which $G_A$ vanishes. We know that $G_A$ is positive at $B_1$ and vanishes also at the unique point of $C_1$ for which the tangent passes through $A$. We deduce that $\{G_A > 0\}$ defines a unique component in both $V_2$ and $P^\circ$, the latter containing $-B_1 = (0, -1, 0)$ in its boundary. The unique component of $V_2 \cap \{A \cdot D^2 > 0\}$ is an unbounded component and lies above the common tangent line $L$ to the conic part of the boundary at $U_1'$, with the corresponding region $V_1 \cap \{A \cdot D^2 > 0\}$ lying below $L$. (For the sake of brevity, we shall in future describe components in $V_2$ and leave the reader to formulate the appropriate statements about $V_1 \cap \{A \cdot D^2 > 0\}$.) Moving down the tangent line $L_1$, when we reach the line $a + b = \frac{k'}{k-1}$, we have that $G_A$ also vanishes (twice) at $R$.

For $A$ between here and $U_1$ on the tangent line, the point $A$ lies on two further tangents, one at $U_2$ in the arc $U_1B_3$ and one at $U_3$ on the arc $B_3Q_2$. With $U'_1$ denoting the corresponding points on $C_2$, we have that $R$ lies in the arc $U_2U'_3$, and that $G_A$ vanishes at these three points on $C_2$, in addition to the point on $C_1$ whose tangent line passes through $A$. The open subset $V_2 \cap \{A \cdot D^2 > 0\}$ then has two components, an unbounded one (with boundary passing through $U_1'$ with $L$ tangent there) lying above the line $L$, and a bounded one lying below $L$ whose boundary has points in $C_2$ comprising the arc $U_2U'_3$.

When $A = U_1$, the conic is a real line pair with singularity at $U_1'$, one line passing through the point $\beta(U_1)$ on the arc of $C_1$ between $B_1$ and the halfway point, determined by the tangent to $C_1$ at $\beta(U_1)$ passing through $U_1$, and the other line passing through the point $\gamma(U_1)$ on the arc $RB_2$ of $C_2$ which is the image under the Steinian involution of the point on the arc $B_3Q_2$ for which the tangent to the Hessian passes through $U_1$. In particular, we note that the two components
of \( P^o \cap \{ U_1 \cdot D^2 > 0 \} \) lie on opposite sides of the plane in \( \mathbb{R}^3 \) corresponding to the line \( L \) (for the corresponding points in \( V_2 \), this means ‘above’ and ‘below’ the line \( L \)), since for all \( A \neq U_1 \) near \( U_1 \) on the tangent line \( L_1 \), the quadratic \( A \cdot D^2 \) is negative on \( L \cap V_2 \), and thus by continuity the quadratic \( U_1 \cdot D^2 \) is non-positive on \( L \cap V_2 \).

Between \( U_1 \) and the point where \( a = -1/(k - 1) \), we again get an unbounded and a bounded component, but this time it is the bounded component which passes through \( U'_1 \) with tangent \( L \) there; the unbounded component in \( V_2 \) (which in this case passes through \( U''_2 \)) is still above \( L \) and the bounded component below \( L \).

With \( A \) moving further down \( L_1 \), we have \( a \geq -\frac{1}{k-1} \), \( b \geq -\frac{1}{k-1} \) and \( a+b < \frac{k^2}{k-1} \) and we lose the unbounded component. Here \( A \) lies on two tangents to the Hessian, one at \( U_1 \) on the arc \( Q_1B_3 \) and one at a point \( U_3 \) on the arc \( B_3Q_2 \), with images under the Steinian involution being \( U'_1, U'_3 \), where \( R \) is in the arc \( U'_1U'_3 \) of \( C_2 \). In this case the inequality \( G_A > 0 \) defines a unique component in \( V_2 \), lying below the line \( L \) and whose boundary has intersection \( U'_1U'_3 \) with \( C_2 \), the conic part of the boundary being tangent to \( L \) at \( U'_1 \); there is a corresponding unique component in \( P^o \).

As we move further down \( L_1 \) to points \( A \) with \( b < -1/(k - 1) \), one of two things can happen (until we reach the third point of intersection \( Z \) with the Hessian). If the \( x \)-coordinate at \( Z \) is less than the the \( x \)-coordinate at \( Q_2 \), then \( A \) is on a unique tangent line to points on the arc \( Q_1B_3 \) and \( B_3Q_2 \), namely \( L_1 \), and so \( U'_1 \) is the unique point of \( C_2 \) at which \( G_A \) vanishes, with \( G_A \) also vanishing at a unique point of \( C_1 \). Noting that \( G_A(B_2) > 0 \), we have a unique (unbounded) component of \( V_2 \cap \{ G_A > 0 \} \) lying below \( L \), with boundary \( G_A = 0 \) tangent to \( L \) at \( U'_1 \) and with corresponding arc \( C_2 \cap \{ G_A > 0 \} \) containing \( R \), and hence a unique component of \( P^o \cap \{ G_A > 0 \} \), whose boundary contains \( -\hat{B}_2 \). If however the \( x \)-coordinate at \( Z \) is greater than the the \( x \)-coordinate at \( Q_2 \), then \( A \) lies on the tangents to two points \( U_2 \) and \( U_3 \) on the arc \( B_3Q_2 \), with corresponding points \( U'_2 \) and \( U'_3 \) on the arc \( RB_2 \) of \( C_2 \). Here \( V_2 \cap \{ G_A > 0 \} \) has two components, one bounded and one unbounded, with corresponding arcs \( U'_2U''_2 \) (containing \( R \)) and \( U'_3B_2 \) in \( C_2 \). Both components lie below the line \( L \), and the bounded component has boundary tangent to \( L \) at \( U'_1 \).

In the first of the cases above, the conic \( Z \cdot D^2 = 0 \) is a line pair, which intersects the projectivised boundary of \( P \) is precisely two points, one of which corresponds to \( U'_1 \in C_2 \) and one to the relevant point of \( C_1 \) whose tangent line contains \( Z \); the conic is smooth at \( U'_1 \) with tangent line \( L \). Thus one of the line must be \( L \) and joins \( U'_1 \) to the relevant point of \( C_1 \), and the other line is disjoint from \( P \). Therefore \( V_2 \cap \{ Z \cdot D^2 > 0 \} \) corresponds to the points in \( V_2 \) below the line \( L \). With \( L \) we associate a plane through the origin in \( \mathbb{R}^3 \), and then \( P^o \cap \{ Z \cdot D^2 > 0 \} \) is the intersection of \( P^o \) with the associated open half-space containing \( -\hat{B}_2 \). In the case when \( Z = Q_2 \), the conic \( Z \cdot D^2 = 0 \) is a line pair, with singularity at \( B_2 \), one line of which only intersects the projectivised boundary of \( P \) at \( B_2 \) and the other line of which is \( L \) joining \( U'_1 \) to \( B_2 \). Thus again \( V_2 \cap \{ Z \cdot D^2 > 0 \} \) corresponds to the points in \( V_2 \) below the line \( L \). Again we associate with \( L \) the corresponding plane through the origin in \( \mathbb{R}^3 \) and \( P^o \cap \{ Z \cdot D^2 > 0 \} \) is the intersection of \( P^o \) with the open half-space containing \( -\hat{B}_2 \).

In the second of the cases above, the conic \( Z \cdot D^2 = 0 \) is a line pair with singularity at \( Z' = \alpha(Z) \) in the open arc \( RB_2 \) of \( C_2 \). The conic is smooth at \( U'_1 \) with tangent \( L \) there, so the two lines are \( L \) which joins \( U'_1 \) to \( Z' \), and a second line joining \( Z' \) to the
point on \( C_1 \) whose tangent contains \( Z \). Here \( V_2 \cap \{ Z \cdot D^2 > 0 \} \) has two components, one being bounded and the other unbounded containing \( B_2 \) in its boundary.

We now have listed, for the various classes \( E \) in the open upper half-space or representing an inflexion point, with index \((1, q)\) for \( q \leq 2 \), the connected components of \( P^* \cap \{ G_E > 0 \} \). For such a component \( Q \), we first check that we have the required property that the corresponding component \( \Gamma \) of \( C_2 \cap \{ G_E > 0 \} \) is visible from \( E \) with respect to \(-Q\). The crucial general result needed here is that if the Hessian of an elliptic curve is smooth and \( U \) is a point on the Hessian curve with image \( U' = \alpha(U) \) under the Steiner involution, and \( U'' \) is the third point of intersection of the line \( UU' \) with the Hessian, then the tangent lines to the Hessian at \( U \) and \( U' \) intersect at the point \( \alpha(U'') \) of the Hessian (and the line \( UU'' \) is one of the lines of the line pair \( \alpha(U'') \cdot D^2 = 0 \), with \( U'' \) being the singularity) — see [2], Proposition 3.2.7. The intersection point of the two tangents is therefore just the third point of intersection of the tangent line to the Hessian at \( U \) (or \( U' \)) with the Hessian.

**Proposition 2.1.** Given a component \( Q \) of \( P^* \cap \{ E \cdot D^2 > 0 \} \), associated to a real class \( E \) in the open upper half-space or representing an inflexion point, with index \((1, q)\) for \( q \leq 2 \), then all points of \( C_2 \) whose negative multiples are on the boundary of \( Q \) are visible (with respect to the cone \(-Q\)) from \( E \).

**Proof.** This result has already been checked explicitly for the so-called trivial cases, and the cases \( E = \pm B_3 \). We now need to check visibility for the cases when some positive multiple of \( E \) is a point \( A \) of the affine plane, not lying in the two closed regions where the result is trivial. We recall that this then reduced us to considering points \( A \), lying on the tangent line to the Hessian at \( U_1 \), for some \( U_1 \) on the open arc \( Q_1B_3 \) or \( B_3Q_2 \). Without loss of generality, we assume that \( U_1 \) lies on the open arc \( Q_1B_3 \), and we use the explicit description of the possible components \( Q \) from the above Summary. The quoted classical result says that the tangent \( L_1 \) at \( U_1 \) meets the tangent at \( U_3' \) at the point \( Z \) where \( L_1 \) meets the Hessian for the third time. Thus \( U_3' \) is visible from \( A \) with respect to \( V_2 \) for all points on \( L_1 \) above (and to the left of) \( Z \), and is not visible from \( A \) for all points on \( L_1 \) strictly below (and to the right of) \( Z \).

Let us consider the various possibilities for \( A \in L_1 \); the first case is \( A = (a, b, 1) \in L_1 \) having \( a < -\frac{1}{k-1} \) and \( a + b \geq \frac{k}{k-1} \). Not only is \( U_3' \) visible from \( A \) with respect to \( V_2 \), but so also is \( B_1 \), and thus so too is the arc \( B_1U_3' \) as required. The next case will be \( a \leq -\frac{1}{k-1} \) and \( a + b < \frac{k}{k-1} \). Here \( A \) lies on the tangent line \( L_3 \) at a point \( U_3 \) on the arc \( B_3Q_2 \) of the Hessian and \( L_3 \) intersects the Hessian again at a point on the arc \( Q_1B_3 \) (on the other side of \( A \) to \( U_3 \) on \( L_3 \)). From the above quoted classical result it is this point where the tangent lines at \( U_3 \) and \( U_3' \) intersect. This ensures that \( U_3' \) is visible from \( A \) as claimed, and thus the same is true for all points of the arc \( B_1U_3' \) (not just the points of \( C_2 \) where \( G_A \geq 0 \)). A similar argument shows that when \( a > -\frac{1}{k-1} \) and \( b > -\frac{1}{k-1} \), and so \( G_A \geq 0 \) defines just a single arc \( U_3' \) in \( C_2 \), then both endpoints are visible from \( A \), as is the arc inbetween.

When \( A \) has \( b \leq -\frac{1}{k-1} \) but above the third intersection point \( Z \) with the Hessian, we have that \( B_2 \) is also visible from \( A \) with respect to \( V_2 \), and so too are all points of the arc \( U_3'B_2 \). \( \square \)
Crucial for the next proof will be the fact noted before that for any $U$ on the arc $Q_1 B_3$ of the Hessian, and $W$ on the tangent line through $U$, not only does the conic $W \cdot D^2 = 0$ always intersect $C_2$ at $U' = \alpha(U)$, but also when $W \neq U$, the conic is non-singular at $U'$ and the tangent line to the conic at $U'$ does not depend on the choice of $W$, and is in fact just the line joining $U$ and $U'$. The other ingredient that we shall need in the proof below is the explicit description of the line pair when the conic is singular. The Proposition is in fact false if we allow one or both of the $E_i$ to lie in the lower half-space.

**Proposition 2.2.** Let $E_i$ (for $i = 1, 2$) be real classes in the open upper half-space or representing an inflexion point, with indices $(1, q_i)$ for $q_i \leq 2$, and suppose there are components $Q(i)$ of $P^o \cap \{G_{E_i} > 0\}$ for $i = 1, 2$ whose intersection is non-empty; then some non-trivial open arc in $C_2$ is in the boundaries of both $-Q(1)$ and $-Q(2)$.

**Proof.** If the boundary of one of the components has points corresponding to the whole arc $C_2$, then the result is clear. So we assume that neither of the $Q(i)$ corresponds to a trivial case. We now prove that if there is no arc of $C_2$ as described, then $Q(1) \cap Q(2)$ is empty. Suppose $Q(1)$ is a component of $P^o \cap \{D : E_1 \cdot D^2 > 0\}$ and $Q(2)$ is a component of $P^o \cap \{D : E_2 \cdot D^2 > 0\}$. Corresponding to these components, we have components of $C_2 \cap \{E_1 \cdot D^2 > 0\}$ and $C_2 \cap \{E_2 \cdot D^2 > 0\}$, open arcs $\Gamma_1$ and $\Gamma_2$ in $C_2$, where we assume that $\Gamma_1 \cap \Gamma_2$ is empty. We deduce that at least one component has corresponding arc $\Gamma_i$ in $C_2$ not containing $R$. We comment that if one of these arcs has $R$ in its closure but not in its interior, then this is the case when $E_i$ is a positive multiple of $\pm \bar{B}_3$ and the conic $E_i \cdot D^2 = 0$ consists of a line pair with singularity at $R$, one line of which is tangent to $C_2$ there and one line of which is the line of symmetry $y = x$. Thus the corresponding $Q(i)$ lies on one side of the plane $\Lambda$ in $\mathbb{R}^3$ determined by this line of symmetry.

The arc corresponding to the other component cannot by our assumption then contain $R$; of course it may be that the other component corresponds to taking a negative multiple of $\pm \bar{B}_3$, in which case the result is obvious — although of course we cannot have $\pm \bar{B}_3$ both being positive multiples of classes of rigid non-movable surfaces. Otherwise, the argument given in the first basic case below implies that the other component is contained in the complementary half-space, and the result follows.

This enables us to assume that both $E_1$ and $E_2$ lie strictly above the plane $z = 0$, and we let $A_1$ and $A_2$ denote the corresponding points in the affine plane. We can then reduce to considering two cases: when neither $\Gamma_i$ contains $R$, and when one doesn’t and one does. If $R \notin \Gamma_i$, we may assume by the above comments that it is not in the closure. Using symmetry, the above assertion is proved in these cases from the two basic cases below. We shall without further reference repeatedly use the facts detailed in the above Summary.

In the first basic case, $Q(1)$ corresponds to an unbounded component of $V_2 \cap \{G_{A_1} > 0\}$ with the associated arc $\Gamma_1 \subset C_2$ not containing $R$ and having an endpoint $B_1$ (corresponding to a point $A_1$ above the arc $Q_1 B_3$ of the Hessian in the region $x < -1/(k-1)$), and $Q(2)$ corresponds to an unbounded component of $V_2 \cap \{G_{A_2} > 0\}$ with the associated arc $\Gamma_2 \subset C_2$ not containing $R$ and having an endpoint $B_2$ (corresponding to a point $A_2$ to the right of the arc $B_3 Q_2$ in the region $y < -1/(k-1)$). We consider the affine picture; if $\bar{A}_i$ denotes the point of the arc
Suppose that the real elliptic curve $F = 0$ has two connected components, then the statement of Proposition 1.8 holds.

Proof. For each component $Q(i)$ (with associated class $E_i$) we have an open arc $\Gamma_i \subset C_2$. Suppose first that each $\Gamma_i$ is an arc of the form $B_iU'_i$; unless some subsequence of the $U'_i$ tends to $B_1$, we have $\bigcap_{i \geq 1} \Gamma_i$ is an arc of the form $B_1U$ in $C_2$ and the result is proved. Without loss of generality therefore, we may assume $Q_1B_3$ of the Hessian vertically below $A_1$, we have two lines defined by $A_1 \cdot D^2 = 0$, with one line joining $\beta(A_1)$ to the point $\beta(A_1)$ (below the midpoint) on $C_1$ where the tangent contains $A_1$, and this corresponds to a plane through the origin in $\mathbb{R}^3$.

Recalling that we defined $B_1 = (0, 1, 0)$, for all $D$ in the corresponding half-space containing $\bar{B}_1$ we have $A_1 \cdot D^2 > 0$. Since $B_1 \cdot D^2 < 0$ at all points $D$ of $P^\circ$, the component $Q(1)$ is contained in this half-space. A similar statement holds for the component $Q(2)$ — we take $A_2$ to be the point on the arc $B_2Q_2$ of the Hessian horizontally to the left of $A_2$; one the two lines given by $A_2 \cdot D^2 = 0$ (namely the one joining $\alpha(A_2)$ to the appropriate point of $C_1$, this point being above the midpoint) corresponds to a half-space in $\mathbb{R}^3$ containing $(-1, 0, 0)$ and that $Q(2)$ is contained in this half-space. We note that the point of intersection of the two affine lines under consideration is a point of the affine plane not in $V_1 \cup V_2$. Therefore the planes through the origin we have constructed via $A_1$ and $A_2$ meet in a line disjoint from $P^\circ$, from which it follows that $Q(1) \cap Q(2)$ is empty. In fact, since the above two lines do not intersect the line of symmetry $y = x$ inside $V_1 \cup V_2$, the two components lie on opposite sides of the plane of symmetry $\Lambda$ defined above.

The second basic case to consider is when $Q(1)$ corresponds to a component of $V_2 \cap \{G_{A_1} > 0\}$, with an associated open arc $\Gamma_1 = B_1U'_1$ in $C_2$ with $U'_1$ in the open arc $B_1R$ (corresponding to a point $A_1$ above the arc $Q_1B_3$ of the Hessian in the region $x < -1/(k - 1)$), and $Q(2)$ corresponds to a component of $V_2 \cap \{G_{A_2} > 0\}$, with an associated open arc $\Gamma_2$ in $C_2$ containing $R$; we saw in the above Summary that $Q(2)$ corresponding to either a bounded component or an unbounded component in $V_2$ was possible here. We denote the lefthand end of $\Gamma_2$ by $U'_2$, with $U'_2$ in the open arc $U'_2R$ or equal to $U'_1$. The case when $U'_2 = U'_1 = U'$ is easy, since then the $Q(i)$ lie on opposite sides of the plane in $\mathbb{R}^3$ corresponding to the line joining $U$ to $U'$; so we assume that $U'_2$ is in the open arc $U'_1R$. We have the affine tangent line $L$ to (the conic part of) $-\partial Q(1)$ at $U'_1$, namely the line joining $U_1$ to $U'_1$, and the affine tangent line $M$ to (the conic part of) $-\partial Q(2)$ at $U'_2$, namely the line joining $U_2$ to $U'_2$. We note that the point of intersection of the two affine lines under consideration is a point of the affine plane not in $V_1 \cup V_2$, since if $U$ moves monotonically from $Q_1$ to $B_3$ on the Hessian, its conjugate $U'$ moves monotonically from $B_1$ to $R$ on $C_2$. Moreover, $Q(1)$ corresponds to a subset of the points in $V_2$ above the line $L$ and of the points in $V_1$ below the line $L$, whilst for $Q(2)$ the corresponding points in $V_2$ are below the line $M$ and in $V_1$ are above the line $M$. By the continuity argument from the Summary, this continues to be true even if one or both of the $A_i = U_i$. The plane in $\mathbb{R}^3$ corresponding to $L$ determines an open half-space containing $-\bar{B}_1$, and we denote by $\bar{Q}(1)$ the intersection of this half-space with $P^\circ$, and the plane corresponding to $M$ determines an open half-space containing $-\bar{B}_2$, and we denote by $\bar{Q}(2)$ the intersection of this half-space with $P^\circ$. The above planes meet in a line disjoint from $P^\circ$, and so $\bar{Q}(1) \cap \bar{Q}(2)$ is empty. Since $Q(i) \subset \bar{Q}(i)$ for $i = 1, 2$, the claim follows in this case also.
that the sequence $U'_i$ tends to $B_1$, and also that no $E_i$ is a positive multiple of $\tilde{B}_3$. Thus we have a point $A_i$ of the affine plane corresponding to each $E_i$. To each $U'_i$, we have a corresponding point $U_i = \alpha(U'_i)$ in the arc $B_3Q_1$ of the Hessian, with the $U_i$ tending to $Q_1$, and so the $A_i$ defining $Q(i)$ lies on the tangent line to the Hessian at $U_i$, above (and to the left of) $U_i$. Moreover for each $i$, we have a common tangent line $M_i$ to the conics $A \cdot D^2 = 0$ at $U_i'$ for $A \neq U_i$ on the tangent line at $U_i$, which we saw was just the line joining $U_i$ and $U_i'$. Corresponding to $M_i$, there is a plane in $\mathbb{R}^3$, and the component $Q(i)$ lies in the associated open half-space that also contains $-\tilde{B}_1$; as usual, from the continuity argument in the above Summary, we note that this is also true when $A_i = U_i$. We let $Q(i) \supset Q(i)$ denote the open subcone of $P^\circ$ given by all the points in $P^\circ$ on the same side of the plane corresponding to $M_i$ as $-\tilde{B}_1$; as usual, from the continuity argument in the above Summary, we note that this is also true when $A_i = U_i$. We let $Q(i) \supset Q(i)$ denote the open subcone of $P^\circ$ given by all the points in $P^\circ$ on the same side of the plane corresponding to $M_i$ as $-\tilde{B}_1$; as usual, from the continuity argument in the above Summary, we note that this is also true when $A_i = U_i$. We let $Q(i) \supset Q(i)$ denote the open subcone of $P^\circ$ given by all the points in $P^\circ$ on the same side of the plane corresponding to $M_i$ as $-\tilde{B}_1$; as usual, from the continuity argument in the above Summary, we note that this is also true when $A_i = U_i$. We let $Q(i) \supset Q(i)$ denote the open subcone of $P^\circ$ given by all the points in $P^\circ$ on the same side of the plane corresponding to $M_i$ as $-\tilde{B}_1$; as usual, from the continuity argument in the above Summary, we note that this is also true when $A_i = U_i$.

We now put an ordering on the points of $C_2$ by specifying that $S \leq S'$ if $S$ is in the (closed) arc $B_1S'$. Let $S_1 \in C_2$ denote the infimum of the righthand ends of the arcs $\Gamma_i = \partial(-Q(i)) \cap C_2$ and let $S_2 \in C_2$ denote the supremum of the lefthand ends of arcs $\Gamma_i$.

If $S_2 < S_1$, then the required open arc of $C_2$ is $S_2S_1$. If $S_2 > S_1$, then we can find components $Q(1)$ and $Q(2)$ say, with $\Gamma_1 \cap \Gamma_2$ empty. From the Proposition 2.2, we then have that $Q(1) \cap Q(2)$ is empty, therefore contradicting the assumption that $Q$ has non-empty interior.

Finally, we need to deal the remaining possibility, that for any pair of components $Q(1)$ and $Q(2)$ say, including the case of $Q(1) = Q(2)$ corresponding to a bounded
component in $V_2$, we have that $\Gamma_1 \cap \Gamma_2$ is an arc $U_2U_1'$ of $C_2$ in the boundary of $-Q(1) \cap Q(2)$, but the intersection of all such arcs is a single point $U' = \alpha(U)$ on $C_2$.

We may assume without loss of generality that $U_2' \neq B_1$ is a lefthand end of $\Gamma_2$ and $U_1' \neq B_2$ is a righthand end of $\Gamma_1$, and we have a schematic diagram as in Figure 2, illustrating the regions of $V_2$ corresponding to $-Q(1)$ and $-Q(2)$. For $i = 1, 2$, we saw that for any $A_i$ on the tangent to the Hessian at $U_i$, the conic $A_i \cdot D^2 = 0$ has a zero at $U_i' = \alpha(U_i) \in C_2$ and the tangent $T_i$ to the conic at $U_i'$ is independent of the choice of $A_i \neq U_i$, and was in fact just the line joining $U_i$ and $U_i'$. It follows that all the points of $V_2$ which are negative multiples of elements in $Q(1)$, respectively $Q(2)$, lie above $T_1$, respectively below $T_2$, whilst the points of $V_1$ which are positive multiples of elements in $Q(1)$, respectively $Q(2)$, lie below $T_1$, respectively above $T_2$. This ensures that $Q(1) \cap Q(2)$ is contained in a convex subcone of $P^o$ lying between the planes corresponding to $T_1$ and $T_2$. By the continuity argument from the Summary, this continues to be true even if one or both of the $A_i = U_i$. In the case when $Q(1) = Q(2)$ is a bounded component, the statement also remains true.

If now $L$ is the common tangent line $UU'$ to the conics $A \cdot D^2 = 0$, where $A \neq U$ lies on the tangent to the Hessian at the point $U = \alpha(U')$ defined above, then we can find a subsequence of the $Q(i)$ for which the corresponding tangent lines $T_1$ at the righthand ends of $\Gamma_i$ tend to $L$, and similarly a subsequence of the $Q(i)$ for which the corresponding tangent lines $T_2$ at the lefthand ends also tend to $L$. From this we deduce that the points of $Q = \bigcap_{i \geq 1} Q(i)$ must lie in the plane through the origin in $\mathbf{R}^3$ corresponding to $L$, and so $Q$ would have empty interior, contrary to assumption.

Therefore in the case when the real elliptic curve has two components, we have completed (via Proposition 1.7) the proof of Theorem 1.6, and hence we have proved (via Corollary 1.5) the relevant parts of our Main Theorem.

3. HYBRID COMPONENTS WHEN ELLIPTIC CURVE HAS ONE REAL COMPONENT

We now wish to study the case when the elliptic curve has only one real component, and so the Hessian, assumed smooth, has two components. Our objective will be to prove the analogous results to Proposition 2.1, Proposition 2.2 and Corollary 2.3 in this case, and in this way complete (via Proposition 1.7) the proof of Theorem 1.6, and hence prove the relevant part of our Main Theorem.

The three hybrid components of the positive index cone are as described in Section 1. Recall that there are two special values for $k < 1$ where changes occur, namely $k = 0$ and $-2$. Away from these two values, we wish to describe the Steinian map $\alpha$.

If as before the inflexion points of the cubic (and hence of the Hessian) are denoted $B_1, B_2, B_3$, then the tangents there to the cubic $F$ (which we saw are just the asymptotes to the three affine branches) will be tangent to the Hessian at three distinct points $Q_1, Q_2, Q_3$. These lines may also be characterised as second polars of $F$ with respect to the $B_i$. Having chosen one of the $B_i$ as the zero of the group law, the corresponding point $Q_i$ where the tangent to $F$ at $B_i$ is tangent to the Hessian is just one of the 2-torsion points of the Hessian.

It is easiest to understand what is going on dynamically. For $k > 1$, we found a description of $\alpha$, where the tangent to $F$ at each $B_i$ is tangent to an affine branch of the unique connected component of $H$. If we consider the corresponding points of the upper half-sphere in $S^2$, we note that as $k \to 1$, the affine branches of both
the real curves given by $F$ and $H$ tend to arcs of the equator $z = 0$ between representatives of the relevant inflexion points, whilst the bounded component shrinks to a point, so that for $k = 1$ both $F$ and $H$ vanish on the equator plus an isolated point on the upper half-sphere corresponding to the centroid $(\frac{1}{3} : \frac{1}{3} : 1)$ of the triangle of reference. Deforming away from $k = 1$ towards zero, the singular point then expands to become the bounded component of the Hessian, and the arcs on the equator that were limits as $k \to 1^+$ of the affine branches of $F$ deform to affine branches of $H$ and the arcs that were limits as $k \to 1^-$ of the affine branches of $H$ deform to affine branches of $F$. Recall here from Section 1 that $H_k = -54k^2F_{k'}$ where $k' = \frac{4-k^2}{3k}$, and so one does expect the regions of the affine plane occupied by the unbounded affine branches of $F$ and $H$ to switch over. In particular, for $0 < k < 1$, the tangents to the cubic at each inflexion point are tangents at appropriate points to the unbounded affine branches of $H$, similar therefore to the case $k > 1$. Thus in this case, the Steinian map $\alpha$ sends each inflexion point $B_i$ to a point on the unbounded component of $H$, and hence gives an involution on both connected components of $H$ individually.

The next change occurs at $k = 0$, where the bounded component together with the three unbounded affine branches of $H$ just tend to the three real lines determined by the triangle of reference. To see what happens to the Steinian map, it is probably easiest to look instead at the value $k = -2$; here the Hessian is just the line at infinity together with the isolated point $(\frac{1}{3} : \frac{1}{3} : 1)$ and all three asymptotes of the cubic pass through this point. As one deforms in either direction away from $k = -2$, this point expands to give the bounded component of the Hessian and each asymptote of $F$ will now be tangent to the bounded component of the Hessian, which by continuity will also be the case for all $k < -2$ and $-2 < k < 0$. Thus for $k < -2$ and $-2 < k < 0$, the Steinian map $\alpha$ interchanges the two components of $H$. The bounded component will be contained in (and tangent to) the asymptotic triangle given by the lines $x = \frac{1}{1-k}$, $y = \frac{1}{1-k}$ and $x + y = \frac{k}{1-k}$. For $k > -2$, the asymptotic triangle will be given by the inequalities $x \leq \frac{1}{1-k}$, $y \leq \frac{1}{1-k}$ and $x + y \geq \frac{k}{1-k}$, whilst for $k < -2$, it will be given by $x \geq \frac{1}{1-k}$, $y \geq \frac{1}{1-k}$ and $x + y \leq \frac{k}{1-k}$.

What is occurring here is that for each value of $k' > 1$, there are three possible values of $k$ for which $H_k$ is a multiple of $F_{k'}$, one with $k < -2$, one with $-2 < k < 0$ and one with $0 < k < 1$. If we choose an inflexion point $B_3$ say, the tangent to $F_k$ at $B_3$ will be tangent at one of the three 2-torsion points of $H_k$ and hence $F_{k'}$, the one on the unbounded component if $0 < k < 1$, and the ones on the bounded component in the other two cases.

For any given point $A = (a,b,1)$ in the affine plane $z = 1$ at which the index is $(1,q)$ with $q \leq 2$, we let $G_A$ denote the homogeneous quadratic given by $A \cdot D^2$, explicitly in coordinates

$$-ax^2 - by^2 - (1-a-b)(z-x-y)^2 + kay(z-x-y) + kbx(z-x-y) + k(1-a-b)xy,$$

and we wish to understand how $G_A = 0$ intersects not only the (unbounded) affine branches of $F$ but also the unbounded affine branches of the Hessian. We will therefore need to understand this in all the three cases detailed above, as the Steinian map will be different in the three cases.

In all three cases, we let $C_1$ denote the unbounded branch of $F = 0$ which lies in the region $x > 0$, $y > 0$ and $x + y > 1$, and $C_2$ the unbounded branch of
the Hessian lying in the sector $x < 0, y < 0$. There is then a hybrid component $P^o$ of the positive index cone whose boundary consists of the positive cone on $C_1$ together with the negative cone on $C_2$, the two parts meeting along rays generated by $(0,1,0)$ and $(1,0,0)$, and without loss of generality we may assume that this is the hybrid component which we study.

For the case when the cubic is $F_k$ with $0 < k < 1$, the proofs of the analogous results to those in Section 2 are essentially identical to the arguments for $k > 1$ given in Section 2, modulo the fact that the regions of the affine plane occupied respectively by the unbounded branches of $F$ and $H$ have switched over. This case is illustrated in Figure 3 (which shows $F_k, H_k$ and the three asymptotes for $F_k$ when $k = 0.5$). The tangent to $F_k$ at $B_i$ (i.e an asymptote) is therefore tangent to the unbounded component of the Hessian at $Q_i$, for $i = 1, 2, 3$. As in the case of two components, the asymptote $x + y = k'/(k' - 1)$ to the Hessian will again play a pivotal role. If an affine point $A$ is in the closed region $x \leq 1/(1 - k), y \leq 1/(1 - k)$ and $x + y \leq k'/(k' - 1)$, then $G_A$ is negative on $P^o$. If $A$ is in the closed quadrant $x \geq 1/(1 - k), y \geq 1/(1 - k)$, then $G_A$ is positive on $C_2$ and all of $C_2$ is visible from $A$ with respect to $V_2$. At all other affine points with index $(1, q)$ for some $q \leq 2$, we have that $A$ is on the tangent line to the Hessian at some point $U$ (not necessarily unique) on one of the open arcs $Q_2B_3$ or $B_3Q_1$. Under the Steinian involution, the arc $Q_2B_3$ corresponds to the arc $B_2Q_3$ of $C_2$ and the arc $B_3Q_1$ to the arc $Q_3B_1$ of $C_2$. Analogous arguments to those in Section 2 may then be seen to prove Proposition 1.8 in the case $0 < k < 1$. For $k > 1$, the bounded component of $F$ essentially played no role in the proof of Proposition 1.8, whilst for $0 < k < 1$, it is the bounded component of $H$ that essentially plays no role in the

\[ \text{Figure 3. Cubic with asymptotes and Hessian for } k = 0.5 \]
proof. We shall therefore not give any further details in this case, and from now on concentrate on the other two cases. Even in these other two cases, although the Steinian involution looks very different, the arguments we use are very similar to those in Section 2.

For the rest of this Section, we shall assume that $-2 < k < 0$, and in the final Section we shall study the remaining case when $k < -2$. We shall in this Section prove Proposition 1.8 when $-2 < k < 0$.

For $-2 < k < 0$, we have a (schematic) picture as in Figure 4, showing all the affine branches of the Hessian and the branch $C_1$ of the cubic. We note that the tangent line to $F$ at $B_3$ is the line $x + y = e_2 = \frac{k}{1-k} = \frac{-k}{1-k}$, and this is tangent to the bounded component of the Hessian at the point $Q_3 = (e_2/2, e_2/2)$. If we take $B_3$ as the zero of the group law, $Q_3$ is just a 2-torsion point of $H$. Moreover the bounded component of the Hessian in this case lies above (and touches) the asymptote $x + y = \frac{k}{1-k}$. Also playing a role will be the other two lines through $B_3$ that are tangent to the Hessian and yielding 2-torsion points of the Hessian; these have the form $x + y = e_3$, corresponding to the other tangent to the bounded component and $x + y = e_1$ corresponding to the tangent to the unbounded component of $H$. Explicitly, if the Hessian is (up to a multiple) the Hessian of $F_{k_1}$, where $0 < k_1 < 1$, $-2 < k_2 = k < 0$ and $k_3 < -2$, then $e_i = k_i/(k_i - 1)$; moreover $e_1 < 0 < e_2 < e_3 < k'/(k' - 1)$, where $x + y = k'/(k' - 1)$ is the asymptote to the Hessian corresponding to its tangent line at $B_3$.

We have the Steinian involution on the Hessian which in the case being studied interchanges the two components; explicitly we let $Q_1$ be the point on the bounded component of the Hessian whose tangent is also the tangent to $F$ at $B_1$, and $Q_2, Q_3$ defined similarly with respect to the inflexion points $B_2, B_3$. We saw above that $Q_3 = (e_2/2, e_2/2)$ and the tangent line is $x + y = e_2$, where $e_2 = \frac{k}{1-k}$.

Under the Steinian map, the branch $C_2$ (going from $B_1$ to $B_2$) of the Hessian corresponds to the ‘upper’ arc (i.e. not containing $Q_3$) on the bounded component going from $Q_1$ to $Q_2$. We now argue similarly to Section 2. For a given class $A = (a, b, 1)$, it is clear how many times the conic $G_A = A \cdot D^2 = 0$ cuts $C_1$ — if $a \geq \frac{1}{1-k}$ and $b \geq \frac{1}{1-k}$, then either $A \in V_1$ and so $G_A > 0$ on both $C_1$ and $P^e$, or $G_A = 0$ intersects $C_1$ twice (since there will be two tangents to $C_1$ containing $A$, including maybe at points at infinity or the degenerate case $A \in C_1$); it will not meet $C_1$ at all if $a < \frac{1}{1-k}$ and $b < \frac{1}{1-k}$, and will cut it precisely once in the other cases. We now ask how many times and where the conic cuts $C_2$. To answer this question, we are looking for the tangents from $A$ to the upper arc (i.e. not containing $Q_3$) from $Q_1$ to $Q_2$ on the bounded component of the Hessian. Here the answer is twice (with multiplicity) if $A$ is in the region bounded by $a = \frac{1}{1-k}$, $b = \frac{1}{1-k}$ and by the specified arc $Q_1 Q_2$, none for any other points with $a < \frac{1}{1-k}$ and $b < \frac{1}{1-k}$ or with $a > \frac{1}{1-k}$ and $b > \frac{1}{1-k}$, and precisely once otherwise as there is exactly one tangent to the given arc $Q_1 Q_2$. If $a \geq \frac{1}{1-k}$ and $b \geq \frac{1}{1-k}$, then $G_A$ is positive on the affine branch $C_2$, all of which is visible from $A$. If $a \leq \frac{1}{1-k}$ and $b \leq \frac{1}{1-k}$, and $A$ is not in the region specified above, then the same continuity argument we used in Section 2 shows that $G_A$ is strictly negative in $P^e$ since it is when $A \in V_2$. 


The midpoint of the arc $Q_1Q_2$ is the point $R = (e_3/2, e_3/2)$, and under the Steinian map this point corresponds to the midpoint $R' = \alpha(R) = (e_1/2, e_1/2)$ of $C_2$, namely the intersection of $C_2$ with $x = y$. Thus $G_A = 0$ will cut $C_2$ in the part given by $y < x$ if and only if $A$ lies on a tangent to the Hessian at some point on the open arc $RQ_1$.

The reader will easily check that the points $E = \pm \tilde{B}_i$ for $i = 1, 2$ give rise to trivial cases where either $G_E > 0$ on both $C_2$ and $P^o$ and all of $C_2$ is visible from $E$, or $G_E < 0$ on $P^o$. One difference from the first two cases considered for the elliptic curve is that there, the points $E = \pm \tilde{B}_3$ were isolated, in that they did not represent the point at infinity on any tangent to the affine Hessian curve. In the remaining two cases, this is no longer true, in that they represent the point at infinity of the tangent line at $R$, the midpoint of the relevant arc $Q_2Q_1$. It is still the case that $E \cdot D^2 = 0$ is a line pair, with singularity at $R' = \alpha(R) \in C_2$, one line being tangent there and the other being the line of symmetry; in the two cases now being studied, the points $A$ on the affine tangent line give rise to conics $A \cdot D^2 = 0$ containing $R'$, and that for $A \neq R$ the conic is smooth at $R'$ with tangent line $L$ the line of symmetry. The special cases $E = \pm \tilde{B}_3$ are not then as special as previously, in that now they represent a limit of points on the affine tangent line at $R$. The proofs can therefore invoke continuity when dealing with these cases.

Let us consider a point $U$ on the open arc $Q_2R$; the tangent at $U$ will intersect the Hessian again on the branch $B_2B_3$. We assume that $A$ lies on this tangent.
line and the index at $A$ is $(1, q)$ with $q \leq 2$. Thus $A$ lies above (and to the left of) the third intersection point $Z$ with the Hessian, and for any such $A$, the conic $A \cdot D^2 = 0$ passes though the point $U' = \alpha(U)$ on the arc $B_3 R'$ of $C_2$. We have in this case that $Q = P^\circ \cap \{G_A > 0\}$ is always connected. For the point $Z$, the line pair $Z \cdot D^2 = 0$ intersects the projectivised boundary of $P$ in two points, one on $C_2$ (namely $U'$) and one on $C_1$. Moreover it is non-singular at $U'$ with tangent line $L$, and so one of the two lines must be $L$ and the other will be disjoint from the projectivised boundary of $P$. The line $L$ may then be identified not only as the line joining $U$ to $U'$, but also as the line joining $U$ to the point $U''$ on $C_1$ whose tangent line passes through $Z$. As $U$ moves from $Q_2$ to $R$ monotonically, $U'$ moves monotonically from $B_2$ to $R'$ and $U''$ moves from $B_2$ to the midpoint of $C_1$, but the latter progression is not monotonic if $F_k(\frac{k'}{2(k'-1)}, \frac{k'}{2(k'-1)}, 1) < 0$ (which occurs for example when $-\frac{1}{2} \leq k < 0$) — this is the reason why a simple-minded argument just involving tangent lines to the conics does not suffice to prove Proposition 3.2. Note that $P^\circ \cap \{G_Z > 0\}$ is given by intersecting $P^\circ$ with an open half-space corresponding to the plane through the origin determined by the line $L$.

If $A = (a, b, 1)$ above (and to the left of) $Z$ with $a \geq \frac{1}{1-k}$, then the closure of $Q$ contains $\tilde{B}_1 = (0, 1, 0)$, and in the case of strict inequality it contains not only $(0, 1, 0)$ but also points of $C_1$. When $a < \frac{1}{1-k}$, we have that $(0, 1, 0)$ is no longer in the closure of $Q$. When $A$ is between the point with $a = \frac{1}{1-k}$ and $U$, there is a second point $U_1$ on the arc $UQ_1$ of the Hessian for which $A$ also lies on the tangent line at $U_1$; thus there is a point $U'_1$ in the open arc $U'B_1$ at which $G_A$ vanishes. In this case $\partial(-Q) \cap C_2$ consists of the finite arc $U'U'_1$. For $A = U$, the conic is a pair of complex lines with singular point $U'$, and $G_A < 0$ on $P^\circ$. As $A = (a, b, 1)$ passes to the other side of $U$ but with $b < \frac{1}{1-k}$, we again obtain a second zero $U'_2$ of $G_A$ on $C_2$, this time in the arc $B_2 U'$, corresponding to the other point $U_2$ in the arc $Q_2 U$ where the tangent contains $A$, and $\partial(-Q) \cap C_2$ consists of the finite arc $U'_2 U'$. When $b = \frac{1}{1-k}$, then the closure of $Q$ contains $\tilde{B}_2 = (1, 0, 0)$, and then for all further points $A$ we have that the closure of $Q$ contains points of $C_1$ in addition to $(1, 0, 0)$ in its boundary.

We now use similar methods as in the previous section to prove analogous results to Proposition 2.1, Proposition 2.2 and Corollary 2.3.

**Proposition 3.1.** Suppose the elliptic curve has one component, with invariant $-2 < k < 0$, given a component $Q$ of $P^\circ \cap \{E \cdot D^2 > 0\}$, associated to a real class $E$ in the open upper half-space or representing an inflexion point, with index $(1, q)$ for $q \leq 2$, all points of $C_2$ whose negative multiples are on the boundary of $Q$ are visible (with respect to the cone $-Q$) from $E$.

**Proof.** Since visibility is a closed condition, we may assume that $E$ represents a point $A = (a, b, 1)$ in the affine plane. Moreover we may assume that $A$ does not lie in either of the two regions specified above where the result is trivially true. If $A$ lies on the tangent to the Hessian at $R$, strictly above (and to the left of) the point $R$, then it is the upper half $B_3 R'$ of $C_2$ on which $G_A$ is positive, and all these points are visible from $A$ (it is only when $E = B_3$ that this is precisely the set of points visible from $E$). By symmetry, the result holds also when $A$ lies on the tangent to the Hessian at $R$, strictly below (and to the right of) the point $R$.

Otherwise, we may assume without loss of generality that $A$ lies on the tangent line at a point $U$ in the open arc $Q_2 R$, since then the case when $U = Q_2$ follows by
continuity. We noted above, that for such points $A$, the tangent line at $U$ intersects the Hessian at a point on the branch $B_2B_3$. We note that $U' = \alpha(U)$ lies on the arc $B_2R'$ of $C_2$.

By the classical result used repeatedly in Section 2 ([2], Proposition 3.2.7), the point where the tangents to the Hessian at $U$ and $U'$ meet is precisely the third point of intersection of the tangent at $U$ (or $U'$) with the Hessian. Given that the Hessian at $A$ is non-negative, $A$ lies above (and to the left of) this point of intersection $Z$, from which it follows that $U'$ is visible from $A$.

As the point $A = (a, b, 1)$ moves up the tangent line from $Z$ towards $U$, the part of $C_2$ on which $G_A > 0$ is initially just the arc $U'B_1$, but this is contained in the arc of points which are visible from $A$. When we reach points $A$ with $a < \frac{1}{1-k}$, the part of $C_2$ on which $G_A > 0$ is then a bounded arc $U'U_1'$, whilst all of $C_2$ is visible from $A$. The case $A = U$ is not relevant here, and as $A$ passes to the other side of $U$, the part of $C_2$ on which $G_A > 0$ is then initially a bounded arc $U_2U'$, and when $b \geq \frac{1}{1-k}$ this is all of $B_2U'$. With $A$ moving upwards from $U$ on the tangent line, initially all of $C_2$ is visible from $A$, but the set of visible points will always be an arc containing $B_2U'$; hence all points of the arc $B_2U'$ are visible from $A$, which verifies the claimed result.

\[\square\]

Proposition 3.2. \textit{Suppose the elliptic curve has one component, with invariant $-2 < k < 0$. Let $E_i$ (for $i = 1, 2$) be real classes in the open upper half-space or representing an inflexion point, with indices $(1, q_i)$ for $q_i \leq 2$, and suppose there are components $Q(i)$ of $P^2 \cap \{G_{E_i} > 0\}$ for $i = 1, 2$ whose intersection is non-empty; then some non-trivial open arc in $C_2$ is in the boundaries of both $-Q(1)$ and $-Q(2)$.}

\textit{Proof.} As in the proof of Proposition 2.2, we show that if the arcs on $C_2$ corresponding to the $Q(i)$ are disjoint, then $Q(1) \cap Q(2)$ is empty. Suppose $Q(1)$ is $P^2 \cap \{D : E_1 \cdot D^2 > 0\}$ and $Q(2)$ is $P^2 \cap \{D : E_2 \cdot D^2 > 0\}$. Corresponding to these components, we have open arcs $\Gamma_1 = C_2 \cap \{E_i \cdot D^2 > 0\}$ and $\Gamma_2 = C_2 \cap \{E_i \cdot D^2 > 0\}$ in $C_2$. The assumption that $\Gamma_1$ and $\Gamma_2$ are disjoint, means that for the appropriate endpoints $U_1'$ and $U_2'$ of $\Gamma_1$ and $\Gamma_2$ on $C_2$, we may without loss of generality assume that $\Gamma_1$ is a sub-arc of $B_2U_1'$ and that $\Gamma_2$ is a sub-arc of $U_2'B_1$ with $U_1'$ in the arc $B_2U_2'$. We then have corresponding points $U_1, U_2$ on the arc $Q_2Q_1$ of the bounded component ($U_1$ in the arc $Q_2U_2$). We shall also assume that the $E_i$ determine points $A_i$ in the affine plane — as we note below, the limit cases with $E_i$ a positive multiple of $\pm B_3$ (when the corresponding $U_i = R$) will follow by an essentially unchanged argument. Thus $Q(1)$ corresponds to a point $A_1$ on the tangent line $L_1$ at $U_1$, with $A_1$ strictly above (and to the left of) $U_1$, whilst $Q(2)$ corresponds to a point $A_2$ on the tangent line $L_2$ at $U_2$, with $A_2$ strictly below (and to the right of) $U_2$. With these conventions, we show that $Q(1) \cap Q(2)$ is empty. Recall that for all $W \in L_1$ the conic $W \cdot D^2 = 0$ passes though $U_1' \in C_2$, and if $W \neq U_1$ the conic is smooth there with tangent line $L$ independent of choice of $W$, specifically $L$ is the line joining $U_1$ to $U_1'$. Moreover $-Q(1) \cap V_2$ lies above $L$ in $V_2$ (and $Q(1) \cap V_1$ lies below $L$ in $V_1$).

We consider various possibilities for $A_1$ and $A_2$: the easy case is when $U_1 = U_2$, for then $-Q(2) \cap V_2$ lies below the common tangent line $L$ in $V_2$ (and $Q(2) \cap V_1$ lies above $L$ in $V_1$), and disjointness of the two components is clear. Let us assume then by symmetry that $U_2$ is in the arc $RQ_1$. We denote by $A$ the point on $L_1$ vertically above $A_2$ as in the main diagram in Figure 5, and hence below (and to
the right of) $U_1$ on $L_1$ (in fact also below the intersection point of $L_1$ and $L_2$). Suppose first that the Hessian is non-negative at $A$. If we consider the subcone of $P^o$ given by $A \cdot D^2 > 0$, this (projectively) lies on the opposite side of $L$ to that given by $A_1 \cdot D^2 > 0$; thus the subcones of $P^o$ given by $A_1 \cdot D^2 > 0$ and $A \cdot D^2 > 0$ lie on opposite sides of the plane in $\mathbb{R}^3$ corresponding to $L$, and hence in particular are disjoint. Since $B_1 \cdot D^2 > 0$ for all $D \in P^o$, we know that the open connected subcone of $P^o$ corresponding to $A_2$ is strictly smaller that the relevant subcone corresponding to $A$, and hence the result follows in this case. This argument is illustrated schematically in Figure 5, where the main diagram shows the configuration of points and tangent lines to the arc $Q_2Q_1$ on the bounded component of the Hessian, where in order not to complicate the diagram we have omitted the arc (which has tangent $L_1$ at $U_1$ and $L_2$ at $U_2$). The smaller subdiagram shows the corresponding regions in $V_2$. Here $L$ is not only tangent to $-Q(1)$ at $U'_1$, with $-Q(1) \cap V_2$ lying above $L$, but it is also tangent to the region defined in $V_2$ by $A \cdot D^2 > 0$, shown in the diagram lying below $L$ and containing $-Q(2) \cap V_2$. In the case where one of the $U_i = R$, without loss of generality $U_1 = R$ and $U_2$ is in the open arc $RQ_1$, the same proof works, even for the limit case when $E_1 = B_3$, where $Q(1)$ consists of the points of $P^o$ lying on the appropriate side of the plane of symmetry.

Figure 5. Schematic diagram for the proof of Proposition 3.2
As a special case, we note that the result is true when both $U_1$ and $U_2$ lie in the (closed) arc $RQ_1$, since the Hessian at $A$ is then clearly positive. By symmetry, the result is also true when both $U_1$ and $U_2$ lie in the (closed) arc $Q_3R$; if we write out the analogous proof to the one above, we shall be considering the point given by the horizontal projection of $A_1$ onto $L_2$.

We are left therefore with the case when $U_1$ is in the open arc $Q_2R$ and $U_2$ is in the open arc $RQ_1$, and the Hessian is negative at the point $A \in L_1$ vertically above $A_2 \in L_2$. Here we define $A_2$ to be the point of the branch $B_2B_3$ of the Hessian immediately above $A_2$, i.e. lying on the line segment $A_2A$. There is then a unique point $\bar{U}_2$ in the open arc $U_1R$ of the Hessian with the tangent $L_2$ at $\bar{U}_2$ containing $A_2$. In passing, we remark that previous arguments show that the region of $V_2$ given by $A_2 \cdot D^2 > 0$ consists of points below the line joining $\bar{U}_2$ and $\bar{U}_2' = \alpha(U_2)$. We denote by $Q$ the subcone of $P^o$ given by $A_2 \cdot D^2 > 0$; since $\bar{B}_1 \cdot D^2 > 0$ for all $D \in P^o$, we note that $Q(2)$ is contained in $Q$. Since now both $U_1$ and $\bar{U}_2$ are contained in the open arc $Q_2R$, we have seen in the second of the above special cases that $Q(1)$ and $Q$ are disjoint, and hence $Q(1) \cap Q(2)$ is empty as required. □

**Corollary 3.3.** Suppose that the real elliptic curve has $-2 < k < 0$, then the statement of Proposition 1.8 holds.

*Proof.* The argument here using limits may be reconstructed from the proof of Corollary 2.3, using the result we have just proved, and is left as an exercise for the reader. □

4. THE CASE OF THE ELLIPTIC CURVE HAVING INARIANT $k < -2$

Let us now consider the remaining possibility with smooth Hessian, namely $k < -2$; here the bounded component of the elliptic curve is tangent to the asymptotic line $x + y = k/(k-1)$ but in this case lies below the line. We illustrate this situation schematically by Figure 6, where again we show the affine branches of the Hessian together with the affine branch $C_1$ of the cubic. As usual, we assume that the hybrid component $P^o$ under consideration has projectivised boundary corresponding to $C_1 \cup C_2$. If the index at a point $E$ in the open upper half-space is $(1,q)$ with $q \leq 2$, we let $A = (a,b,1)$ denote the point in the affine plane $z = 1$ determined by $E$. Let $Q$ denote a connected component of the subcone of $P^o$ given by $E \cdot D^2 > 0$, by Proposition 3.4 of [6], a convex subcone of $P^o$. We now list the points on $C_1$ and $C_2$ where the quadratic $G_A$ vanishes.

As before it is clear for $A = (a,b,1)$ how many times (and where) $G_A = 0$ intersects $C_1$. If $a \geq \frac{1}{1-k}$ and $b \geq \frac{1}{1-k}$, then either $E \in V_1$ and so $G_A > 0$ on both $C_1$ and $P^o$, or $G_A = 0$ intersects $C_1$ twice (possibly at infinity or with multiplicity); it intersects $C_1$ at no points if $a < \frac{1}{1-k}$ and $b < \frac{1}{1-k}$, and at precisely one point otherwise. In the first of these three cases, we note that all of $C_2$ is visible from $A$. For $a < \frac{1}{1-k}$, $b < \frac{1}{1-k}$, there are no zeros of $G_A$ on $C_2$ either, and a previous continuity argument shows that $G_A$ would be negative on all of $P$, since this is true for $A \in V_2$. Thus if $a \leq \frac{1}{1-k}$, $b \leq \frac{1}{1-k}$, then $G_A$ would be negative on $P^o$ and so the subcone defined by $G_A > 0$ is empty. If for instance $a > \frac{1}{1-k}$, $b = \frac{1}{1-k}$, then $G_A$ has a zero on the affine branch $C_1$ and at the point $B_2$ at infinity, and the whole of $C_2$ is visible from $A$. We note that $G_A(B_1) > 0$ if and only if $a > \frac{1}{1-k}$ and $G_A(B_2) > 0$ if and only if $b > \frac{1}{1-k}$.
An additional feature compared with the previous case is that $G_A = 0$ can intersect the projectivised boundary of $P$ at two points on $C_1$ and two on $C_2$, and that will happen when $A$ is in the open region with boundary consisting of a segment of the line $x = \frac{1}{1-k}$, a segment of the line $y = \frac{1}{1-k}$, and the ‘lower’ arc (i.e. not containing $Q_3$) of the bounded component of the Hessian between $Q_1$ and $Q_2$; as before $Q_i$ denotes the point on the bounded component of the Hessian where the asymptote to the cubic at $B_i$ is tangent. In contrast to the previous case, this time the arc of the Hessian between $Q_1$ and $Q_2$ lies in the quadrant $x \geq \frac{1}{1-k}$, $y \geq \frac{1}{1-k}$.

If $A$ lies in the quadrant $x > \frac{1}{1-k}$, $y > \frac{1}{1-k}$ but not on or below the arc $Q_1Q_2$, then $G_A$ is non-zero (and therefore positive) on $C_2$. We note that the possibilities $E = \pm \tilde{B}_i$ for $i = 1, 2$ will give rise to trivial cases where either $G_E > 0$ on both $C_2$ and $P^o$, or $G_E < 0$ on $P^o$. We remark in passing that if $-E$ is in the interior of the bounded component of the Hessian (and hence $E$ has $E^3 > 0$ and index $(1,2)$, and is in the lower half-space), then $G_E$ is negative on $C_2$ but $P^o \cap \{G_E > 0\}$ is non-empty; such an eventuality does not occur under our assumptions.

We now need to understand what happens when $A$ lies on the tangent line at some point $U$ in the lower arc $Q_1Q_2$ of the bounded component of the Hessian. We assume by symmetry that $U$ lies in the open arc $Q_1R$, leaving it to the reader to check what happens when $U$ is $Q_1$ or $R$, including the limit case when $E$ is a
positive multiple of $\pm \hat{B}_3$. Under the given assumption, the tangent line meets the branch $B_1B_3$ of the Hessian. We note that $A = (a, b, 1)$ is below (and to the right of) this intersection point. If $a < \frac{1}{1-\varepsilon}$, we know that $G_A$ vanishes at $U'$, and it is positive at $B_2$, and there is a unique component of $P^0 \cap \{G_A > 0\}$, whose closure contains $\hat{B}_2$; the second point on the projectivised boundary of $P$ corresponds to the point on $C_1$ where the tangent line contains $A$. Thus the set of points in $C_2$ whose negative multiples are in the boundary of $P^0 \cap \{G_A > 0\}$ is just the arc $B_2U'$, where as usual $U' = \alpha(U)$. When $a = \frac{1}{1-\varepsilon}$, we know that $G_A$ vanishes (twice) at $B_1$, and for $A$ between this point and $U$, there is a point $U_1$ on the arc $Q_1U$ whose tangent also contains $A$, and there are two components of $P^0 \cap \{G_A > 0\}$, one of which has its boundary points corresponding in $C_2$ to the arc $B_2U'$ and the other with boundary points corresponding in $C_2$ to the arc $U'B_1$. For $A = U$, we get two real lines meeting at $U'$, each line joining $U'$ to one of the two points of $C_1$ for which the tangent contains $U$. Moving now to $A$ lying below $U$ but with $b > \frac{1}{1-\varepsilon}$, we obtain a point $U_2$ in the arc $UQ_2$ where the tangent contains $A$, and we still have two components, but with the relevant arcs on $C_2$ now being $B_2U''$ and $U''B_1$. By the time we reach the point with $b = \frac{1}{1-\varepsilon}$, the first of these components of $P^0 \cap \{G_A > 0\}$ has shrunk to the empty set (with $G_A$ vanishing along the ray generated by $\hat{B}_2$), and from then on we just have one component of the intersection, and the relevant arc in $C_2$ is now $U'B_1$. With this description in mind, the required result on visibility follows easily.

**Proposition 4.1.** Suppose the elliptic curve has one component, with invariant $k < -2$. Given a component $Q$ of $P^0 \cap \{E \cdot D^2 > 0\}$, associated to a real class $E$ in the open upper half-space or representing an inflexion point, with index $(1, q)$ for $q \leq 2$, all points of $C_2$ whose negative multiples are on the boundary of $Q$ are visible (with respect to the cone $-Q$) from $E$.

**Proof.** Since visibility is a closed property, we may assume that $E$ is represented by a point $A$ in the affine plane. There were two general cases noted above where the result was true for trivial reasons — in all other cases, $A$ lies on the tangent line at $U$ for some $U$ in the lower arc $Q_1Q_2$ on the bounded component of the Hessian.

We now just observe, in the various possibilities for $A$ lying on the tangent line at $U$, which points of $C_2$ are visible from $A$. By symmetry (and noting that visibility is a closed condition) we may assume that $U$ lies in the open arc $Q_1R$. By the classical result used in previous sections, the point of intersection of the tangent at $U$ with the branch $B_1B_3$ of the Hessian is also just the intersection with the tangent line to $C_2$ at $U'$. Thus the whole closed arc $B_2U'$ is visible from $A$. As $A$ moves down the tangent line, the visible points from $A$ constitute a larger arc containing the arc $B_2U'$, until we reach the point with $a = -1/(k'-1)$ where $k' = (4-k^3)/(3k^2) > 1$, when all of $C_2$ is visible from $A$. This then remains true until $A$ has $b < -1/(k'-1)$, at which stage the visible points of $C_2$ nonetheless form an arc containing $U'B_1$. Thus we have verified the Proposition in all cases. □

**Proposition 4.2.** Suppose the elliptic curve has one component, with invariant $k < -2$. Let $E_i$ (for $i = 1, 2$) be real classes in the open upper half-space or representing an inflexion point, with indices $(1, q_i)$ for $q_i \leq 2$, and suppose there are components $Q(i)$ of $P^0 \cap \{G_{E_i} > 0\}$ for $i = 1, 2$ whose intersection is non-empty; then some non-trivial open arc in $C_2$ is in the boundaries of both $-Q(1)$ and $-Q(2)$. 

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Proof. We show that if the arcs in $C_2$ corresponding to the $Q(i)$ are disjoint, then $Q(1) \cap Q(2)$ is empty. We assume that neither $E_i$ correspond to a case where the result is trivially true. Analogous to the proof of Proposition 3.2, we then have points $U_1$, $U_2$ on the lower arc $Q_1Q_2$ of the bounded component (with $U_3$ in the arc $Q_1U_1$), with corresponding images $U'_1$, $U'_2$ on $C_2$ (with $U'_2$ in the arc $U'_1B_1$), where without of generality the open arc $\Gamma_1 \subset C_2$ corresponding to $Q(1)$ is the arc $B_2U'_1$ and the open arc $\Gamma_2 \subset C_2$ corresponding to $Q(2)$ is the arc $U'_2B_1$. As in the proof of Proposition 3.2, we shall assume that the $E_i$ define points $A_i$ in the affine plane, since as we note below the limit cases $E_i = \pm B_3$ follow by only a minimal change to the argument. Thus $Q(1)$ corresponds to a point $A_1$ on the tangent line $L_1$ at $U_1$, with $A_1$ above (and to the left of) $U_1$, whilst $Q(2)$ corresponds to a point $A_2$ on the tangent line $L_2$ at $U_2$, with $A_2$ below (and to the right of) $U_2$. Thus $Q(1)$ contains $\tilde{B}_2$ and $Q(2)$ contains $\tilde{B}_1$ in their boundaries. Under these conventions, we show that $Q(1) \cap Q(2)$ is empty.

In the above notation, $L$ is just the line joining $U_1$ and $U'_1$, whilst $M$ is the line joining $U_2$ and $U'_2$. We observe that as $U$ moves monotonically from $Q_1$ to $Q_2$ on the arc $Q_1Q_2$ of the Hessian, its conjugate $U' = \alpha(U)$ moves monotonically from $B_1$ to $B_2$ on $C_2$. We deduce that $L$ and $M$ intersect at an affine point not in $V_1 \cup V_2$, and so the corresponding planes through the origin in $R^3$ intersect in a line disjoint from $P^\circ$. The plane determined by $L$ gives rise to a cone $\tilde{Q}(1)$, which is the intersection of the relevant open half-space containing $\tilde{B}_2$ with $P^\circ$, and the plane determined by $M$ gives rise to a cone $\tilde{Q}(2)$, which is the intersection of the relevant open half-space containing $\tilde{B}_1$ with $P^\circ$. Thus $Q(1) \subset \tilde{Q}(1)$ and $Q(2) \subset \tilde{Q}(2)$, which by the usual continuity argument from Section 2 remains true even if one or both of the $A_i = U_i$. Since $\tilde{Q}(1) \cap \tilde{Q}(2)$ is empty, we obtain the claimed result. \hfill $\square$

Corollary 4.3. Suppose that the real elliptic curve has $k < -2$, then the statement of Proposition 1.8 holds.

Proof. The argument via limits may again be reconstructed from the proof of Corollary 2.3, using the result we have just proved, and is left as an exercise for the reader. \hfill $\square$

Therefore in the case when the real elliptic curve has one component, we have completed (via Proposition 1.7 and Corollaries 3.3 and 4.3) the proof of Theorem 1.6, and hence we have proved (via Proposition 1.2) the relevant part of our Main Theorem.

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Department of Pure Mathematics, University of Cambridge, 16 Wilberforce Road, Cambridge CB3 0WB, UK

Email address: pmhw@dpmms.cam.ac.uk