Holography in the Flat Space Limit

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Abstract

Matrix theory and the AdS/CFT correspondence provide nonperturbative holographic formulations of string theory. In both cases the finite N theories can be thought of as infrared regulated versions of flat space string theory in which removing the cutoff is equivalent to letting N go to infinity.

In this paper we consider the nature of this limit. In both cases the holographic mapping becomes completely nonlocal. In matrix theory this corresponds to the growth of D0-brane bound states with N. For the AdS/CFT correspondence there is a similar delocalization of the holographic image of a system as N increases. In this case the limiting theory seems to require a number of degrees of freedom comparable to large N matrix quantum mechanics.

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1 Introduction

According to the holographic principle, a physical system of dimensionality \( D \) which includes gravity, should be described by a quantum system which lives in fewer dimensions. We have seen a good deal of evidence for the holographic principle from both matrix theory and the AdS/CFT correspondence but very little real understanding of how it works, in other words, how a general configuration of a \( D \) dimensional system is coded by lower dimensional degrees of freedom. My main purpose in this paper is to provoke discussion about the mechanism of holography [1], [2]. Most of the things that I will discuss I do not understand very well. In trying to formulate them precisely I have mainly encountered frustration. Nevertheless I think they are important and deserve to be discussed.

One of the characteristic features of a real hologram is that it codes information in a highly nonlocal way. For example by casually looking at a hologram of several distinct objects it is impossible to tell how many objects it describes or their size and shape. These details are completely delocalized on the hologram. The point of this paper is to argue that quantum gravity is holographic in exactly this sense.

Two concrete realizations of holographic theories now exist, namely matrix theory [3] and the AdS/CFT correspondence [4],[5],[6],[7]. In both theories the hologram is a large \( N \) super Yang Mills (SYM) theory. Furthermore in both cases \( N \) serves as a kind of infrared regulator. In the limit \( N \to \infty \) keeping the Yang Mills coupling fixed both theories describe physics in infinite flat space. Furthermore, as we shall see, as \( N \) grows, the mapping between the hologram and the system it describes becomes more and more nonlocal. In this respect the mapping is like a real hologram. In this paper I will raise some unanswered questions about the nature of the holographic mapping, especially in the limit of infinite flat spacetime. As we shall see, the large \( N \) limit involved in going to flat space is quite different than the usual ’t Hooft limit in which the coupling shrinks to zero as \( N \) increases. The flat space limits in Matrix and AdS/CFT theories both involve letting \( N \) go to infinity with fixed gauge coupling. Thus the ’t Hooft coupling parameter \( g_{ym}^2 N \) tends to infinity and the fixed point becomes infinitely strongly coupled.

Imagine a system composed of point sources of light (particles). Assume that the light from the different sources is coherent as long as they are within a coherence length \( L_c \). All of this takes place in the 3-dimensional half space \( z > 0 \). At \( z = 0 \) in the \( x, y \) plane there is a photographic film which records the light from the particles. As long as the particles are separated by distance greater than \( L_c \) they form two separate blobs of light on the film. If we made a movie from such photos we could follow the individual particles’ motion from these blobs. However as soon as they approached within \( L_c \) the individual identities would disappear. However the details
would not be lost. At this point the details such as the number and position of point sources would become encoded holographically, that is nonlocally distributed over the coherence length $L_c$. As the coherence length increases the information becomes completely delocalized over the entire hologram. For an ordinary hologram the information is in the interference patterns created by the coherent light sources. For matrix theory and the AdS/CFT correspondence the coding is more obscure but in both cases it involves the $N \times N$ matrix degrees of freedom of Super Yang Mills theories. In both these theories we will see the same kind of delocalization with a coherence length that increases like $N^{1/3}$ in matrix theory and $N^{1/4}$ in the AdS/CFT correspondence.

2 Holography and matrix theory

Let us begin with matrix theory. For the present purposes we are interested in uncompactified matrix theory described by $0+1$ dimensional SYM theory. For a review of matrix theory and notations we refer the reader to [8].

Matrix theory can be thought of as the Discrete Light Cone Quantization (DLCQ) of M-Theory in which the spacetime is compactified on an almost light like circle $X^{-}$. The discrete conjugate momentum is related to the gauge group rank $N$ by $p_{-}R = N$. Thus we see that if we fix the momentum $p_{-}$, removing the IR cutoff (letting $R \to \infty$) is tantamount to letting $N \to \infty$.

What has not been sufficiently realized is that $N$ also plays the role of an infrared cutoff in the transverse dimensions. To see why, let us first consider the 10 dimensional metric and dilaton describing a collection of $N$ coincident D0-branes in the near horizon limit [10]:

$$ds^2 = f^{-1/2} dt^2 + f^{1/2} dx^i dx^i$$
$$\exp(2\phi) = f^{3/2}$$
$$f = \frac{N l_{11}^9}{R^2 r^7}$$

(2.1)

where $l_{11}$ is the 11 dimensional Planck scale.

Now consider the limits of validity of (2.1). At small $r$ the ten dimensional supergravity description breaks down because the effective string coupling gets large. In 11 dimensional terms, the local value of the radius of the 11th direction becomes bigger than $l_{11}$. This happens at $r \sim N^{1/7} l_{11}$.

While we have to give up the duality between 10D supergravity and D0-brane physics at this point we can replace it with a duality between D0-branes and 11D supergravity. This is the basis for matrix theory. Thus there is no limit on the matrix theory/Supergravity duality at $r \sim N^{1/7} l_{11}$.

At the large distance end another limitation is reached. The scalar curvature $\mathcal{R}$ of the 10D metric satisfies
\[ R \sim \frac{r^{3/2} R}{N^{1/2} l_{11}^{9/2}} \]

It is monotonically increasing with \( r \) and exceeds the string scale at \( r \sim N^{1/3} l_{11} \). At this point the supergravity description completely breaks down. The region \( r > N^{1/3} l_{11} \) is the region where the D0-brane quantum mechanics can be treated perturbatively. From the supergravity point of view \( r = N^{1/3} l_{11} \) represents an infrared cutoff beyond which classical supergravity is no longer applicable. This means that when two colliding objects in matrix theory approach each other from infinity, semiclassical gravity will not generally describe their interactions correctly until \( r < N^{1/3} l_{11} \). Although it is true that matrix theory with 16 supersymmetries sometimes agrees with DLCQ tree graph supergravity to asymptotic distances this probably has more to do with the tight constraints of maximal supersymmetry than with any general reason for agreement. A more typical example is matrix theory on a blown up orbifold where for finite \( N \) the supergravity and one loop matrix theory disagree [11]. If this case is typical, we would expect agreement only when \( N > (\frac{r}{l_{11}})^3 \).

These considerations suggest although the compactification radius \( R \) is allowed to be vanishingly small, the D0-branes create a bubble of space whose transverse size grows as \( N^{1/3} \) so that the limit \( N \to \infty \) is effectively decompactified. It is therefore interesting to ask if we can see the scale \( N^{1/3} l_{11} \) occurring in matrix theory. In the original matrix theory conjecture [3] it was speculated that the threshold bound state describing a supergraviton would grow with \( N \). One estimate was based on the well known \( v^4/r^7 \) velocity dependent effective interaction between D0-brane clusters and suggested that the bound state radius grows like \( N^{1/9} l_{11} \). A second estimate based on a perturbative large \( N \) argument gave the even more rapid growth \( N^{1/3} l_{11} \). Recently Polchinski has given a rigorous proof [12] that the growth is at least as fast as \( N^{1/3} l_{11} \). Polchinski’s argument is based on the virial theorem. The argument I will give here is a less rigorous paraphrase of Polchinski’s but gives some intuition about the nature of the bound state.

Let us use the gauge freedom of matrix theory to work in a basis in which one of the 9 \( X \)-coordinates, say \( X_1 \), is diagonal. The eigenvalues can be thought of as the locations of the constituent D0-branes along the \( X_1 \) axis. Let us suppose that they are smoothly spread over a region of size \( L \). Now consider the quantity \( \langle Tr(X_1)^2 \rangle \). This obviously satisfies

\[ \langle Tr(X_1)^2 \rangle \sim NL^2 \] (2.2)

Consider the quantity \( \langle TrY^2 \rangle \) where \( Y \) is any of the other 8 \( X \)'s. The off diagonal elements of the matrix\( Y \) are described by harmonic oscillators
in the background of $X$ with frequency of order

$$\omega \sim \frac{LR}{l_{11}^3} \quad (2.3)$$

and fluctuation $(\Delta Y)^2 \sim \frac{l_{11}^2}{L}$. Since there are of order $N^2$ such elements we find

$$\langle Tr Y^2 \rangle \sim \frac{N^2 l_{11}^3}{L} \quad (2.4)$$

But now we can use rotational symmetry to equate $\langle Tr (X_1)^2 \rangle$ and $\langle Tr Y^2 \rangle$ giving

$$L \sim N^{1/3} l_{11} \quad (2.5)$$

The typical conjugate momentum of a matrix element is also easily estimated and is given by

$$\Delta P_{ij} = \frac{N^{1/6}}{l_{11}} \quad (2.6)$$

Thus we see that the bound state grows large with $N$, extending to the boundaries of the region of validity of 10D supergravity. As seen from eq(2.3) the matrices $X$ have very high frequency oscillations reminiscent of the high frequency zero point oscillations of free strings which also lead to a growth of the wave function but in this case only a logarithmic growth [13],[14]. Finally, the kinetic energy of the D0-branes is estimated as follows. The total kinetic energy is $\frac{R}{2} Tr P_{\perp}^2$. Using (2.6) and the fact that there are $N^2$ matrix elements we find the total kinetic energy to be of order $\frac{R N^{7/3}}{l_{11}^3}$. This is to be compared with the typical energy scale in DLCQ M-theory $\frac{R}{N l_{s}^2}$. Evidently, on the scale of the energies of physical processes the kinetic energies are huge. The kinetic energy per D0-brane is

$$E/N = \frac{R N^{4/3}}{l_{11}^2} \quad (2.7)$$

This enormous energy is cancelled by the quartic and fermionic terms in the hamiltonian but this estimate gives an idea of the energy scales involved.

In light of the above, let us consider a collision between two gravitons. Most of the literature on scattering in matrix theory makes the implicit assumption that the ”wave function effects” are not important. What this means is that the scattering objects are described by little clusters of D0-branes which are much smaller than the distance separating them. As we shall see this is completely incorrect.

For simplicity take the gravitons to have equal light cone momenta and therefore equal values of $N$. In the transverse center of mass frame they have equal and opposite transverse momenta $P_{\perp}$ and $-P_{\perp}$. The light cone energy is

$$E_{lc} = \frac{P_{\perp}^2}{P_{-}} = R \frac{P_{\perp}^2}{N} \quad (2.8)$$
and the Mandelstam invariant center of mass energy is

\[ S = 2P_\perp^2 \]  

(2.9)

Suppose \( P_\perp \) is fixed and of order \( 1/l_{11} \). If Matrix theory is consistent then the scattering amplitude must tend to a finite limit in 11D Planck units as \( N \) increases. But as we have seen the size of the bound state wave functions grow as \( N^{1/3} \). Each particle is huge blob of eigenvalues and the blobs begin to overlap long before the particles come close in the usual sense. During the period of overlap the constituents of each blob lose their identity. This is obvious because of the very large energy scales involved in the D0-brane dynamics eq(2.7). The puny available energy in eq (2.8) is not enough to significantly modify the correlations in the ground state. Therefore the state of the system should more closely resemble the ground state of the \( 2N \times 2N \) matrix theory than two overlapping but distinguishable subsystems.

Thus the history of the scattering process has two very different but equivalent descriptions. In the usual space time supergravity description two small particles come in from infinity and remain essentially noninteracting until they come within a distance of order \( l_{11} \). They interact for a short time and then separate into final particles which cease to interact as soon as they are separated by \( l_{11} \). In light cone units the interaction lasts for a time \( \frac{l_{11}N}{P_\perp R} \).

The holographic matrix description also begins with asymptotically distant noninteracting objects. In this description the constituents begin to merge and interact when their separation is of order \( N^{1/3}l_{11} \). As they approach, the many body wave function begins to more and more resemble the ground state. The system remains in this entangled state for a light cone time of order \( \frac{l_{11}N^{4/3}}{P_\perp R} \) and then separate into noninteracting final clusters. The situation is particularly perplexing if the energy is not very large and the impact parameter is much larger than \( l_{11} \). In this case the gravity description the particles miss each other and just continue without significant deflection. Exactly how this miracle happens from the SYM description is still a mystery. We will see exactly the same puzzles in the AdS/CFT correspondence.

### 3 Holography and the AdS/CFT Duality

String theory in \( AdS_5 \times S_5 \) is dual to SYM theory on the boundary of the space \([4], [5]\). As pointed out by Witten, this is another example of a holographic connection \([6]\). For our purposes AdS is best thought of as a finite cavity with reflecting walls. The metric is given by

\[ ds^2 = R^2 dS^2 \]  

(3.1)
where $R$ is the radius of curvature of the AdS and $dS^2$ is the metric of a "unit" AdS. The unit AdS metric is

$$dS^2 = \frac{(1 + r^2)^2}{(1 - r^2)^2}dt^2 - \frac{4}{(1 - r^2)^2}(dr^2 + r^2d\Omega)$$

(3.2)

with $d\Omega$ being the unit 3-sphere.

The full geometry is $AdS_5 \times S_5$. The $S_5$ factor is a 5-sphere of radius $R$. Although the boundary, $r = 1$ is an infinite proper distance from any point in the interior of the ball $r < 1$ the time for a light signal to reflect off the boundary is finite. A light signal originating at $r = 0$ (with vanishing $S_5$ momentum) will return after a coordinate time $\pi$. Thus as far as light signals are concerned the space behaves like a finite radiation cavity with reflecting walls. The effect of the inhomogeneous metric is to slow the light velocity at the center to half its value at the boundary. In other words the bulk sphere has a varying dielectric constant. A very simple example of the equivalence of bulk physics with the boundary theory is given by the restriction of causality.

Consider a signal originating at a point on the boundary. At a later time it will reappear at the antipodal point on the boundary. In the SYM description it travels with the speed of light on the boundary taking a time $\pi$ to get to the antipode. In the dual bulk theory the signal travels through the center of the ball, $r = 0$, along a light like geodesic. A simple calculation shows that it again arrives after coordinate time $\pi$.

Massive particle trajectories (timelike geodesics) are all periodic in time with period $2\pi$. These trajectories never reach the boundary. The cavity walls repel massive particles with a force which diverges near $r = 1$. The force is proportional to the mass of the particle as is always the case in gravity. From the point of view of the $AdS_5$ the particles carrying momentum along the 5-sphere are massive. A massive particle which starts at $r = 0$ with velocity $v$ will move outward on a radial trajectory for a time $\pi/2$ at which point it reaches a maximum radial coordinate satisfying

$$v^2 = \frac{4r_{\text{max}}^2}{(1 + r_{\text{max}}^2)^2}$$

(3.3)

In describing the SYM theory we will use the dimensionless metric $dS^2$. This means that all SYM quantities will be treated as dimensionless. The corresponding quantities in the bulk theory carry their usual dimensions. To go from one to the other the conversion factor is $R$. For example an SYM energy of order 1 corresponds to an energy of order $1/R$ in the bulk theory. A coordinate time interval $t$ is an interval $Rt$ in bulk units.

The dimensionless parameters of the bulk theory are the 10 dimensional string coupling constant $g_s$ and the ratio of the radius of curvature to the string length scale $R/l_s$. The parameters of the dual SYM theory are the
SYM coupling $g_{ym}$ and the rank of the gauge group $N$. The connection between these parameters was given by Maldacena,

$$g_s = g_{ym}^2, \quad \frac{R}{l_s} = (Ng_s)^{1/4} \quad (3.4)$$

The fact that by increasing $N$ the radius of curvature in eq(3.11) can be made to increase while keeping the string coupling fixed leads to a conjecture for a new nonperturbative definition of IIB string theory in terms of SYM theory. The $AdS \times S_5$ geometry can be thought of as an infrared regulator for type IIB string theory. As $R \to \infty$ the space becomes locally flat 10 dimensional Minkowski space. To formulate this precisely let us begin with Euclidean SYM theory in the Euclidean version of the metric (3.2).

$$dS^2 = \frac{-(1 + r^2)^2}{(1 - r^2)^2} d\tau^2 - \frac{4}{(1 - r^2)^2} (dr^2 + r^2 d\Omega) \quad (3.5)$$

I will refer to these coordinates and their Minkowski counterparts as "cavity coordinates" where $\tau$ is Euclidean time. It is very convenient to transform to "1/2-plane" coordinates with metric

$$ds^2 = -R^2 \frac{(dx^i dx^i + dy^2)}{y^2} \quad (3.6)$$

The 4 noncompact coordinates $x^i$ are parallel to the boundary and can also be used as coordinates for the SYM theory. The coordinate $y$ runs perpendicular to the boundary and varies from zero to infinity.

The transformation from 1/2-plane to cavity coordinates is given as follows. First transform $(x^i, y)$ to 5 dimensional polar coordinates $\rho, \theta, \alpha, \beta, \gamma$.

$$\begin{align*}
y &= \rho \cos \theta \\
x^1 &= \rho \sin \theta \cos \alpha \\
x^2 &= \rho \sin \theta \sin \alpha \cos \beta \\
x^3 &= \rho \sin \theta \sin \alpha \sin \beta \cos \gamma \\
x^4 &= \rho \sin \theta \sin \alpha \sin \beta \sin \gamma \end{align*} \quad (3.7)$$

Now set $\rho = e^\tau$ and $\cos \theta = \frac{1 - r^2}{1 + r^2}$. The three angles $\alpha, \beta, \gamma$ are the coordinates of the unit sphere $\Omega$.

We will be interested in correlation functions of various fields in the superconformally invariant SYM theory. Thus consider a set of points $x_a$ on the boundary of the 1/2-plane coordinates. For each pair of points $a, b$ define $x_{ab} \equiv |x_a - x_b|^2$. In terms of Euclidean cavity coordinates $x_{ab}$ is given by

$$x_{ab} = e^{(\tau_a + \tau_b)} (\cosh \tau_{ab} - \cos \phi_{ab}) \quad (3.8)$$

where $\tau_{ab} \equiv \tau_a - \tau_b$ and $\phi_{ab}$ is the angular separation between the points in $\Omega$. It is also convenient to define $Z_{ab} = (\cosh \tau_{ab} - \cos \phi_{ab})$. 
Euclidean correlation functions of the SYM theory are typically homogeneous functions of the $x_{ab}$ of degree determined by the dimensions of the operators. To express the corresponding correlators in cavity coordinates just replace each $x_{ab}$ by $Z_{ab}$. The various of $e^\tau$ cancel the Jacobian factors in the transformation of fields with nonvanishing dimensions. Thus the correlators are homogeneous functions of the $Z_{ab}$. As an example, the correlation function of two scalar fields $\Phi$ of dimension 4 is of the form $Z_{ab}^{-4}$

It is now a simple matter to pass to Minkowski signature by replacing $\tau$ by $it$. Thus the correlator becomes

$$\langle \Phi(x_a)\Phi(x_b) \rangle = (\cos t_{ab} - \cos \phi_{ab})^{-4}$$  \hspace{1cm} (3.9) \hspace{1cm}

The singularity when $\cos t_{ab} = \cos \phi_{ab}$ is the usual light cone singularity.

Strictly speaking there is no true S matrix in AdS space. As I have emphasized, AdS is for all practical purposes a finite cavity with reflecting walls. Asymptotic states can not be defined in such a geometry. The strategy that we follow is to introduce sources on the walls of the cavity which act as particle sources and detectors. This will allow us to define a finite time version of the S matrix. When the size of the box is allowed to increase, keeping fixed the energies, impact parameters and other physical quantities the finite time S matrix should tend to a true asymptotic scattering amplitude.

Before discussing the boundary sources further we need to determine what quantities should be kept fixed as $R \to \infty$ in order to recover flat space string theory. First of all we must keep the microscopic parameters of string theory fixed. This means letting $R/l_s \to \infty$ with $g_s$ fixed. In terms of Yang Mills quantities

$$g_{ym} = \text{fixed}$$

$$N \to \infty$$

In addition the energy scale of physical processes should be fixed in string units. In terms of the dimensionless energy of the SYM theory $E$

$$E \sim (g_{ym}^2N)^{1/4}$$

Thus we see that the flat space limit involves the high energy limit of large $N$ SYM theory. We will also require restrictions on the angular momenta of particles.

We will define a spacetime region called the ”lab”. The lab is centered at $r = t = 0$. Its linear dimensions $L$ in both space and time are fixed in string units but are much larger than $l_s$. At the end we may take $L/l_s$ as big as we like. As $N \to \infty$ the entire region of the lab becomes accurately described by flat spacetime. The sources will be constructed in such a way as to insure that the entire collision process takes place within the lab.
A particle can carry momentum components in both the AdS$_5$ directions and in the $S_5$ directions. We will call these $p$ and $k$ respectively. For the moment we will ignore $k$. Consider a massless particle that is inside the lab with momentum $p$. Its angular momentum $l$ is necessarily less than $Lp$. Since the cavity is spherically symmetric, the angular momentum of a freely moving incoming particle is conserved. Therefore if the particle is to arrive in the lab it must be emitted from the boundary with $l < Lp$. This restriction guarantees that the "beam" is focused to pass through the lab.

As an example we will consider scattering amplitudes for dilatons carrying vanishing momentum in the $S_5$ directions. The dilatons will be emitted in such a way that they propagate freely toward the region $r = 0$ where they meet and interact within the lab. Since the wavelength of the particles is vanishingly small by comparison with the radius of the AdS space, the propagation of the wave packets toward the lab can be treated by geometrical optics. The time it takes for a wave packet to travel from the boundary to the lab is $\pi/2$, just half the time for a light signal to cross the AdS space. Therefore the initial sources must act at $t = -(\frac{\pi}{2} \pm \frac{L}{R})$. Similarly the final detector-sources must act at $t = + (\frac{\pi}{2} \pm \frac{L}{R})$.

The appropriate SYM operators for emitting all massless 10 dimensional particles are known. In particular the operator that creates a dilaton at the boundary is the dimension 4 operator $Tr F_{\mu \nu} F^{\mu \nu} \equiv FF$. Let us consider the emission operator for a zero angular momentum dilaton of bulk energy $p$. The obvious choice is

$$A^{in}(p) \sim \int dt d\Omega e^{ipRt} FF$$

(3.12)

However in order to build wave packets which arrive at the lab at $t = 0 \pm \frac{L}{R}$ we need modify the definition of $A$. This can be done by replacing the factor $e^{ipRt}$ by a wave packet of finite extent. Let $f_{in} \left[ (t - \frac{\pi}{2}) \frac{R}{L} \right]$ be a smooth function (such as a gaussian) which is peaked at $t = \pi/2$. The definition of $A$ is

$$A^{in}(p) \sim \int dt d\Omega f_{in} \left[ (t - \frac{\pi}{2}) \frac{R}{L} \right] e^{ipRt} FF$$

(3.13)

A similar expression defines the operators representing the final particles.

$$A^{out}(p) \sim \int dt d\Omega f_{out} \left[ (t + \frac{\pi}{2}) \frac{R}{L} \right] e^{-ipRt} FF$$

(3.14)

To create particles of arbitrary angular momentum the integral over $\Omega$ should contain the relevant $O(4)$ spherical harmonic.

The recipe for computing bulk S matrix elements from SYM quantities is straightforward.

$$S = \langle 0 | \prod_{out} Z_{out} A^{out} \prod_{in} Z_{in} A^{in} | 0 \rangle$$

(3.15)
The factors $Z$ are inverse boundary-bulk propagators which are needed to amputate the external AdS propagators.

The above prescription for recovering flat space amplitudes can be generalized to include nonvanishing momenta along the 5-sphere. The operators which create particles with nonvanishing $O(6)$ angular momentum $n$ are schematically of the form $Tr F F XXXX...$ where $F$ represents components of the Yang Mills Field strength and $XXXX..$ is a polynomial of order $n$ in the scalar fields which transform as vectors under $O(6)$. These are operators of mass dimension $4 + n$. We must also integrate these operators with functions of time and $\Omega$ in order to project out definite energy and $O(4)$ angular momentum. Again the frequencies should be of order $g^2_{YM} N^{1/4}$ in order to keep the physical momentum of the bulk particles of order unity in string units.

Thus we see that passing to the flat space limit generally involves operators in the SYM theory which are high frequency components of high dimension operators.

To actually compute scattering amplitudes from conformal field theory data, a useful strategy might be to use the operator product expansion for the operators $A^{in,out}$. Consider for example a two particle scattering process in which the incoming (outgoing) particles are emitted (absorbed) at time $t_{in,out} = \pm \pi/2$. The angular positions of the incoming particles are $\Omega_{1,2}$ and the outgoing particles $\Omega_{3,4}$. The 4 points $(1,2,3,4)$ are far from each other in spacetime and it is not obvious why the operator product expansion is useful. However, consider the case where there is a small momentum transfer $(p_1 - p_3) << p$. Then the locations of 1 and 3 will be almost light-like with respect to each other. In the rules for continuation from Euclidean to Minkowski signature in AdS space the almost light like separation between 1 and 3 maps to an almost vanishing Euclidean separation so that the OPE should provide an expansion for small angle scattering. Obviously, the operators of low dimensionality in the operator product of $A(1)A(3)$ correspond to massless exchange. In addition we also expect contributions corresponding to massive string exchange with masses of order $t_s^{-1}$. From the point of view of the operator product expansion this means operators of dimensionality $\sim g_s N^{1/4}$. We will leave it to a future publication, hopefully by someone else, to work out the detailed rules for computing on shell scattering amplitudes from CFT data in the flat limit.

4 The Infrared Ultraviolet Connection

The connection between the boundary SYM theory and the ideas of holography rely on an important connection between the ultraviolet behavior of the SYM theory and the infrared behaviour of the bulk supergravity [7]. We
will begin by reviewing the argument for counting the number of degrees of freedom of the system. By now it is well known that an ultraviolet cutoff at wavelength $\delta$ in the SYM is equivalent to a cutoff in the radial coordinate $r$ at $r = 1 - \delta$. The cutoff SYM describes the bulk supergravity in the interior of the ball $r < 1 - \delta$. Now the number of cutoff cells of coordinate size $\delta$ on the boundary of this ball is of order $1/\delta^3$. Assuming that each independent SYM field has one degree of freedom per cells, the total number of degrees of freedom is

$$N_{dof} \sim \frac{N^2}{\delta^3} \quad (4.1)$$

If we use

$$R = l_s (Ng_s)^{1/4}$$

$$\text{Area} = \frac{R^8}{\delta^3}$$

$$G = g_s^2 l_s^8 \quad (4.2)$$

where $\text{Area}$ is the area of the cutoff 3-sphere times $S_5$ and $G$ is the 10D gravitational coupling constant we find the typical holographic behaviour

$$N_{dof} \sim \text{Area}/G \quad (4.3)$$

Let us push this reasoning to the extreme and take the cutoff $\delta$ such that the proper volume of the 4 dimensional ball $r < \delta$ is $R^4$ (dimensionless volume $\sim 1$). The area of the cutoff boundary sphere is then $\sim R^3$ and the number of degrees of freedom is just $N^2$. In other words it takes $N^2$ degrees of freedom to describe all the states of the bulk theory which are supported within a sphere of proper size $R$. This means that the physics, within a neighborhood small enough so that curvature can be ignored, is coded by the $N^2$ matrix degrees of freedom and that the inhomogeneous spatial modes of the SYM are unexcited. This suggests the possibility that the states supported within such a neighborhood might be described by an $N \times N$ matrix quantum mechanics.

As an illustration of the IR-UV connection consider a graviton carrying momentum $k$ along the $S_5$ and momentum $p >> k$ in the radial direction along $r$. Its total energy is $E = \sqrt{p^2 + k^2}$. It is created by applying the operator

$$A^{in}(p, k) = \int \, dt \Omega_f \left[ (t - \frac{\pi}{2}) \frac{R}{L} e^{iERt} \text{Tr} F F \ldots \right] \quad (4.4)$$

where there are $n = kR$ factors inside the trace. This means that the total energy is divided among $n$ SYM quanta and the energy $\nu$ of each SYM quantum is

$$\nu = \sqrt{\frac{(p^2 + k^2)}{k^2}} \approx \frac{p}{k} \quad (4.5)$$
This corresponds to a cutoff in the SYM theory $\delta \approx 1/\nu \approx \frac{k}{p}$. The implication is that in the bulk theory the particle created by $A$ appears not at the boundary but at $1 - r \approx \frac{k}{p}$. This makes good sense for the following reason. From the point of view of $\text{AdS}_5$ a particle with $S_5$ momentum $k$ is a massive particle with mass $k$. The classical trajectory of such a massive particle with (bulk) energy $E$ has a turning point (vanishing velocity) given by (3.3). In terms of momentum components the turning point is at $1 - r \approx \frac{k}{p}$. Hence the particle starts out at the outermost point on its trajectory.

The fact that massive particles originate in the interior of the AdS space does not require a modification of the rules for constructing the $A$ operators. Although they start closer to the interaction point at $r = 0$, the time that it takes to arrive at $r = 0$ is independent of the mass.

From the above discussion it seems that the cutoff theory with a given value of $\delta$ can describe the sector of the theory containing particles with $\frac{k}{p} \geq \delta$. Suppose for example, all the particles in a given reaction have $\frac{k}{p} \sim 1$. In this case only the lowest modes of the SYM theory are excited corresponding to configurations which are spatially homogeneous (in the boundary theory). In other words physical processes involving such particles always appear completely smeared and nonlocal in the holographic SYM description. The situation is very similar to the matrix case.

At this point it is interesting to consider just what problems we in principle would know how to set up and solve if we could completely master SYM theory and find all its correlators, and energy levels. First of all we could apply the recipe described in section (3) to compute any scattering amplitude involving 10 dimensional massless particles. Since we do not expect any other stable particles in the theory, this exhausts all IIB flat space scattering amplitudes.

In addition we could compute the thermodynamics of the theory and discover the existence of a phase transition at (dimensionless) temperature $\sim 1$. This corresponds to the formation of a large black hole at bulk temperature $\sim R^{-1}$. This object has no significance in the flat space limit since it has a size of order the radius of curvature.

However most of ordinary flat space physics would remain out of reach even though it implicitly must be described by the SYM theory. As an example consider the problem of describing an ordinary 10 dimensional Schwarzschild black hole of finite mass and entropy as $N \to \infty$. For simplicity the black hole could be located near the center of the AdS at some point on the 5-sphere. If the proper distance of the black hole from the center is kept fixed as $R \to \infty$ then its coordinates will tend to the origin at $r = 0$. The image will become completely symmetric on the 3-sphere. What features of the SYM state contain the information of the exact position or even the fact that it is a black hole or any other object of the same mass and angular momentum is not known. In fact it is not even clear how to distinguish this
configuration from a pair of distant black holes or other objects of the same total mass if their separation is much smaller than the radius of curvature $R$. In principle the AdS/CFT correspondence requires the SYM theory to contain these objects. Recognizing them from their SYM description requires deciphering the holographic code.

5 Instantons in AdS

It has been suggested that a simple place to begin trying to crack the holographic code is the theory of instantons. In the bulk supergravity theory the instantons are D-instantons whose holographic images are expected to be ordinary Yang Mills instantons. In order to discuss instantons we must consider the Euclidean version of AdS. As is well known this is a 5 dimensional ball bounded by a 4-sphere of radius $R$. The Euclidean SYM theory lives on this boundary sphere. As before, when discussing the SYM the 4-sphere will be thought of as having unit radius.

For small $\delta$, a D-instanton in the bulk located at $r = (1 - \delta)$ is represented in the SYM as an localized instanton of size $\rho = \delta$. This is another example of the UV-IR connection. D-instantons near the center of the ball correspond to Y.M. instantons which fill the entire boundary sphere homogeneously. In fact the gauge field for such instantons is completely homogeneous up to a gauge transformation. Note that if we have one or more D-instantons at a fixed separation from one another and from $r = 0$ then as $N$ increases their coordinates get closer and closer to $r = 0$. Thus in the limit all the instantons lying within a finite proper volume are found within an infinitesimal coordinate distance from the origin. Thus in the SYM theory they are described by the largest homogeneous gauge instantons. Again, the features of the SYM description which distinguish the different D-instanton configurations are obscure.

An SYM instanton will appear approximately homogeneous if the corresponding D-instanton is within a ball of volume $R^5$. An interesting paradox occurs if we ask how many D-instantons can we put in such a region. The naive answer if that the number should roughly be the 10 dimensional volume of the product of the ball×$S_5$. In other words the maximum instanton number in this volume is naively $(\frac{R}{L})^{10} = (Ng_s)^{5/2}$. However if we try to build homogeneous instanton configurations on the SYM sphere we find that we can have a maximum instanton number of order $N$. A single large $SU(2)$ instanton is a homogeneous configuration. If we try to put two instantons into the same $SU(2)$ subgroup we find that it can not be done without making the field inhomogeneous. Since their are $N/2$ commuting $SU(2)$ subgroups we can only accommodate this number of homogeneous instantons. Either the D-instantons within volume $R^5$ can not be identified with homogeneous
gauge field configurations or there must be some reason why it is not possible to cram as many D-instantons into a region as the naive argument suggests.

Exactly this conclusion can be reached by an argument similar to that used for D0-branes in section 2. If we are interested in a small region of AdS over which the curvature can be ignored we can use a flat space description. D-instantons are formally D-branes of dimensionality $-1$ and are described by a matrix integral [16] defined by the dimensional reduction of maximally supersymmetric SYM. If the instanton number is $k$ the matrices are $k$ by $k$.

The action for the D-instantons is

$$S = -\frac{1}{g_s l_s^4} \text{Tr}[X^i, X^j]^2$$

An argument exactly paralleling the one in eqs(2.2), (2.3) and (2.4) will give the size of the region occupied by $k$ D-instanton's. As in that case, first diagonalize $X^1$ and assume that the eigenvalues are smeared over a region of size $L$. As before

$$\langle \text{Tr}(X^1)^2 \rangle \sim kL^2$$

Now consider one of the off diagonal elements of $Y$ where $Y$ is any other $X^j$. The average of $Y^2$ in the $X^1$ background is

$$\langle Y^2 \rangle = g_s l_s^4 / L^2$$

and

$$\langle \text{Tr}Y^2 \rangle = k^2 g_s l_s^4 / L^2$$

Using rotational symmetry to equate (5.23) and (5.25) gives

$$L = (k g_s)^{1/4} l_s$$

To find the maximum number of D-instanton’s that we can put into a flat region of size $R$ we set $L = R = (N g_s)^{1/4}$. Thus we find that the maximum number of D-instanton’s is of order $N$ in agreement with the maximum number of homogeneous gauge field instantons. It is also clear that the positions of the D-instanton’s, within the region approximated by flat space, in the limit $N \to \infty$ must be coded somehow in the $N \times N$ matrix degrees of freedom of the SYM and not in the large gauge field inhomogeneities.

Finally, classical gravitational considerations give the same result. The gravitational field of $k$ D-instantons in flat space is given by

$$ds^2 = f^{1/2} dx^i dx^j$$

where

$$f = 1 + k g_s l_s^8 r^{-8}$$

Thus the gravitational field extends out to distance $r = (k g_s)^{1/4} l_s$. If this distance is not to exceed $R$ then $k \leq N$. 

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6 Decoding the Hologram

Exactly how information is holographically stored in either matrix theory or the AdS/CFT correspondence is a mystery. I will try to give some thoughts about it. Let's begin with matrix theory.

The $N^{1/3}$ increase in the size of the low energy wave functions of D0-brane wave functions is caused by the ground state oscillations of an increasing number ($\sim N^2$) of high frequency modes. The situation is parallel to that in free string theory where as the number of modes is increased the ground state expands \[ \text{[13]} \]. In the free string case the expansion is only logarithmic but for any finite coupling it will eventually grow like a power \[ \text{[2]} \]. The increase in the sizes of images eventually blurs the details of a system. For example if the system consists of two distinct objects separated by transverse distance $\Delta$ then when $N^{1/3} > \frac{\Delta}{R}$ the holographic images become entangled. Given a state of the matrix theory at very large $N$ it would be very difficult to decipher its meaning.

The trick in decoding the hologram is to get rid of the high frequency oscillations. This can be done by averaging over time but the right thing has to be averaged. For example we could define a density of D0-branes along the $X^1$ axis in terms of the distribution of its eigenvalues. This however is a very slowly varying quantity which does not have high frequency oscillations. The right thing to average is the Heisenberg operators representing the matrix elements $X_{a,b}$. For example, we may average $X$ over a time $\delta t$. If we work in the eigenbasis of the Hamiltonian, the averaging is equivalent to throwing away all (quantum) matrix elements $\langle E_1|X_{a,b}|E_2 \rangle$ with $|E_1 - E_2| > 1/\delta t$. The resulting quantum operators will have a modified distribution of eigenvalues. Since modes of frequency $> 1/\delta t$ are now absent the distribution should have a smaller spread. Therefore the holographic image of several objects should become clear. It is evident that all of this is a manifestation of the stringy space, time uncertainty relation \[ \text{[17]} \].

In order to be a little more quantitative I will make an assumption that is motivated by a particular view of the large $N$ limit. According to this view, the large $N$ limit is a fixed point of a kind of renormalization group associated with integrating out rows and columns of the $N \times N$ matrix degrees of freedom to produce a theory with smaller matrices. What I will assume is that time averaging over $\delta t$, or equivalently, integrating out high frequency modes is equivalent to replacing the original $N \times N$ matrix system by another with smaller $n \times n$ matrices. The maximum relevant frequency for the original system is given by eq(2.3) with $L = N^{1/3}l_{11}$. We will call this the characteristic frequency $\omega_N$.

\[
\omega_N = \frac{N^{1/3}R}{l_{11}^2}
\]  

(6.8)
If we identify the characteristic frequency of the $n \times n$ model to be $\omega_n = (\delta t)^{-1}$ then
\[
\frac{n}{N} = \frac{\omega^3_n}{\omega^3_N} = \frac{\eta^6_{11}}{(R\delta t)^3N}
\]
(6.9)

Furthermore since the size of the eigenvalue distributions of $X$ scale like $N^{1/3}$ we should find the spread diminished by the factor $\frac{\omega_n}{\omega_N}$. According to this estimate, by averaging over $\delta t = l^2_{11}/R$ resolution of order the Planck length should be restored for a pair of gravitons.

For the AdS/CFT correspondence decoding the hologram seems to be very different. In the flat space limit the SYM dimensionless energy of a given system increases like $(g_s N)^{1/4}$. On the other hand the information is coded in the longest wavelength modes on the unit sphere. These modes have frequency $\sim 1$ in dimensionless units which corresponds to very long bulk time scales of order $R$. In other words, information seems to be coded in extremely slow degrees of freedom. At the moment I have no idea how this works.

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