Orthogonal polynomials associated with Coulomb wave functions

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Abstract

A class of orthogonal polynomials associated with Coulomb wave functions is introduced. These polynomials play a role analogous to that the Lommel polynomials do in the theory of Bessel functions. The measure of orthogonality for this new class is described explicitly. In addition, the orthogonality measure problem is also discussed on a more general level. Apart from this, various identities derived for the new orthogonal polynomials may be viewed as generalizations of some formulas known from the theory of Bessel functions. A key role in these derivations is played by a Jacobi (tridiagonal) matrix $J_L$ whose eigenvalues coincide with reciprocal values of the zeros of the regular Coulomb wave function $F_L(\eta, \rho)$. The spectral zeta function corresponding to the regular Coulomb wave function or, more precisely, to the respective tridiagonal matrix is studied as well.

Keywords: orthogonal polynomials, measure of orthogonality, Lommel polynomials, spectral zeta function

2010 Mathematical Subject Classification: 33C47, 33E15, 11B37

1 Introduction

In [10], Ikebe showed the zeros of the regular Coulomb wave function $F_L(\eta, \rho)$ and its derivative $\partial_\rho F_L(\eta, \rho)$ (regarded as functions of $\rho$) to be related to eigenvalues of certain compact Jacobi matrices (see [1] Chp. 14] and references therein for basic information about the Coulomb wave functions). He applied an approach originally suggested by
Grad and Zakražek for Bessel functions [9]. In more detail, reciprocal values of the nonzero roots of $F_L(\eta, \rho)$ coincide with the nonzero eigenvalues of the Jacobi matrix

$$J_L = \begin{pmatrix}
\lambda_{L+1} & w_{L+1} & w_{L+2} & w_{L+3} \\
\lambda_{L+2} & w_{L+2} & \lambda_{L+3} & w_{L+4} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$  \hspace{1cm} (1)

where

$$w_n = \frac{\sqrt{(n+1)^2 + \eta^2}}{(n+1)\sqrt{(2n+1)(2n+3)}} \quad \text{and} \quad \lambda_n = -\frac{\eta}{n(n+1)}$$  \hspace{1cm} (2)

for $n = L, L+1, L+2, \ldots$. Similarly, reciprocal values of the nonzero roots of $\partial_\rho F_L(\eta, \rho)$ coincide with the nonzero eigenvalues of the Jacobi matrix

$$\tilde{J}_L = \begin{pmatrix}
\tilde{\lambda}_L & \tilde{w}_L & \tilde{w}_{L+1} & \tilde{w}_{L+2} \\
\tilde{w}_L & \lambda_{L+1} & \lambda_{L+2} & \lambda_{L+3} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$  \hspace{1cm} (3)

where

$$\tilde{w}_L = \sqrt{\frac{2L+1}{L+1}} w_L \quad \text{and} \quad \tilde{\lambda}_L = -\frac{\eta}{(L+1)^2}.$$  \hspace{1cm} (4)

The parameters have been chosen so that $L \in \mathbb{Z}_+$ (non-negative integers) and $\eta \in \mathbb{R}$. This is, however, unnecessarily restrictive and one may extend the set of admissible values of $L$. Note also that $J_L$ and $\tilde{J}_L$ are both compact, even Hilbert-Schmidt operators on $\ell^2(\mathbb{N})$.

Ikebe uses this observation for evaluating the zeros of $F_L(\eta, \rho)$ and $\partial_\rho F_L(\eta, \rho)$ approximately by computing eigenvalues of the respective finite truncated Jacobi matrices. In this paper, we are going to work with the Jacobi matrices $J_L$ and $\tilde{J}_L$ as well but pursuing a fully different goal. We aim to establish a new class of orthogonal polynomials (shortly OPs) associated with Coulomb wave functions and to analyze their properties. Doing so, we intensively use a formalism which has been introduced in [21] and further developed in [22]. The studied polynomials represent a two-parameter family generalizing the well known Lommel polynomials associated with Bessel functions. Let us also note that another generalization of Lommel polynomials, in a completely different direction, has been pointed out by Ismail in [11], see also [17, 15].

Our primary intention in the study of the new class of OPs was to get the corresponding orthogonality relation. Before approaching this task we discuss the problem of finding the measure of orthogonality for a sequence of OPs on a more general level. In particular, we consider the situation when a sequence of OPs is determined by a three-term recurrence whose coefficients satisfy certain convergence condition. Apart of solving the orthogonality measure problem, various identities are derived for the new class of OPs which may be viewed as generalizations of some formulas well known from the theory of Bessel functions. Finally, the last section is devoted to a study
of the spectral zeta functions corresponding to the regular Coulomb wave functions or, more precisely, to the respective tridiagonal matrices. In particular, we derive recursive formulas for values of the zeta functions. Let us remark that this result can be used to localize the smallest in modulus zero of \( F_L(\eta, \rho) \), and hence the spectral radius of the Jacobi matrix \( J_L \).

2 Preliminaries and some useful identities

2.1 The function \( \mathfrak{F} \)

To have the paper self-contained we first briefly summarize some information concerning the formalism originally introduced in \([21]\) and \([22]\) which will be needed further. Our approach is based on employing a function \( \mathfrak{F} \) defined on the space of complex sequences. By definition, \( \mathfrak{F} : D \rightarrow \mathbb{C} \),

\[
\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1},
\]

where

\[
D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C}; \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}, \tag{5}
\]

For \( x \in D \) one has the estimate

\[
|\mathfrak{F}(x)| \leq \exp \left( \sum_{k=1}^{\infty} |x_k x_{k+1}| \right). \tag{6}
\]

We identify \( \mathfrak{F}(x_1, x_2, \ldots, x_n) \) with \( \mathfrak{F}(x) \) where \( x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) \), and put \( \mathfrak{F}(\emptyset) = 1 \) where \( \emptyset \) stands for the empty sequence.

Further we list from \([21]\) \([22]\) several useful properties of \( \mathfrak{F} \). First,

\[
\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k) \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x), \quad k = 1, 2, \ldots, \tag{7}
\]

where \( x \in D \) and \( T \) denotes the shift operator from the left, i.e. \( (Tx)_k = x_{k+1} \). In particular, for \( k = 1 \) one gets the rule

\[
\mathfrak{F}(x) = \mathfrak{F}(T x) - x_1 x_2 \mathfrak{F}(T^2 x). \tag{8}
\]

Second, for \( x \in D \) one has

\[
\lim_{n \to \infty} \mathfrak{F}(T^n x) = 1, \quad \lim_{n \to \infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x). \tag{9}
\]

Third, one has (see \([22]\) Subsection 2.3)

\[
\mathfrak{F}(x_1, x_2, \ldots, x_d) \mathfrak{F}(x_2, x_3, \ldots, x_{d+s}) - \mathfrak{F}(x_1, x_2, \ldots, x_{d+s}) \mathfrak{F}(x_2, x_3, \ldots, x_d) = \left( \prod_{j=1}^{d} x_j x_{j+1} \right) \mathfrak{F}(x_{d+2}, x_{d+3}, \ldots, x_{d+s}) \tag{10}
\]

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where \(d, s \in \mathbb{Z}_+\). By sending \(s \to \infty\) in (10) one arrives at the equality

\[
\mathfrak{F}(x_1, \ldots, x_d) \mathfrak{F}(Tx) - \mathfrak{F}(x_2, \ldots, x_d) \mathfrak{F}(x) = \left( \prod_{k=1}^{d} x_k x_{k+1} \right) \mathfrak{F}(T^{d+1}x) \quad (11)
\]

which is true for any \(d \in \mathbb{Z}_+\) and \(x \in D\).

2.2 The characteristic function and the Weyl m-function

Let us consider a semi-infinite symmetric Jacobi matrix \(J\) of the form

\[
J = \begin{pmatrix}
\lambda_0 & w_0 & & \\
& \lambda_1 & w_1 & \\
& & \lambda_2 & w_2 \\
& & & \ddots & \ddots & \ddots
\end{pmatrix}
\quad (12)
\]

where \(w = \{w_n\}_{n=0}^{\infty} \subset (0, +\infty)\) and \(\lambda = \{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}\). In the present paper, such a matrix \(J\) is always supposed to represent a unique self-adjoint operator on \(\ell^2(\mathbb{Z}_+)\), i.e. there exists exactly one self-adjoint operator such that the canonical basis is contained in its domain and its matrix in the canonical basis coincides with \(J\). For example, this hypothesis is evidently fulfilled if the sequence \(\{w_n\}\) is bounded. With some abuse of notation we use the same symbol, \(J\), to denote this unique self-adjoint operator.

In [22] we have introduced the characteristic function \(\mathcal{F}_J\) for a Jacobi matrix \(J\) provided its elements satisfy the condition

\[
\sum_{n=0}^{\infty} \frac{w_n^2}{|\left(\lambda_n - z\right)(\lambda_{n+1} - z)|} < \infty \quad (13)
\]

for some (and hence any) \(z \in \mathbb{C} \setminus \text{der}(\lambda)\) where \(\text{der}(\lambda)\) denotes the set of all finite cluster points of the diagonal sequence \(\lambda\), i.e. the set of limit values of all possible convergent subsequences of \(\lambda\). By Corollary 17 in [22], the condition (13) also guarantees that the matrix \(J\) represents a unique self-adjoint operator on \(\ell^2(\mathbb{Z}_+)\). The definition of the characteristic function reads

\[
\mathcal{F}_J(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=0}^{\infty} \right) \quad (14)
\]

where \(\{\gamma_n\}_{n=0}^{\infty}\) is determined by the off-diagonal sequence \(w\) recursively as follows: \(\gamma_0 = 1\) and \(\gamma_{k+1} = w_k / \gamma_k\), for \(k \in \mathbb{Z}_+\). The zeros of the characteristic function have actually been shown in [22] to coincide with the eigenvalues of \(J\). More precisely, under assumption (13) it holds true that

\[
\text{spec}(J) \setminus \text{der}(\lambda) = \text{spec}_p(J) \setminus \text{der}(\lambda) = \mathfrak{Z}(J) \quad (15)
\]

where

\[
\mathfrak{Z}(J) := \left\{ z \in \mathbb{C} \setminus \text{der}(\lambda); \lim_{u \to z} (u - z)^p(z) \mathcal{F}_J(u) = 0 \right\} \quad (16)
\]
and \( r(z) := \sum_{k=0}^{\infty} \delta_{z, \lambda_k} \in \mathbb{Z}_+ \) is the number of occurrences of an element \( z \) in the sequence \( \lambda \). Moreover, the eigenvalues of \( J \) have no accumulation points in \( \mathbb{C} \setminus \operatorname{der}(\lambda) \) and all of them are simple.

Finally, denoting by \( \{e_n; \ n \in \mathbb{Z}_+\} \) the canonical basis in \( \ell^2(\mathbb{Z}_+) \), let us recall that the Weyl \( m \)-function \( m(z) := \langle e_0, (J - z)^{-1} e_0 \rangle \) is expressible in terms of \( \mathfrak{F} \),

\[
m(z) = \frac{1}{\lambda_0 - z} \mathfrak{F}\left( \left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{\infty} \right) \mathfrak{F}\left( \left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=0}^{\infty} \right)^{-1}
\]

for \( z \notin \spec(J) \cup \operatorname{der}(\lambda) \). From its definition it is clear that \( m(z) \) is meromorphic on \( \mathbb{C} \setminus \operatorname{der}(\lambda) \) with only simple real poles, and the set of these poles coincides with \( \mathfrak{F}(J) \).

3 Some general results about orthogonal polynomials

The theory of OPs is now developed to a considerable depth. Let us just mention the basic monographs [2, 4]. If convenient, a sequence of OPs, \( \{P_n\}_{n=0}^{\infty} \), where \( \deg P_n = n \), may be supposed to be already normalized. Then one way how to define such a sequence is by requiring the orthogonality relation

\[
\int_{\mathbb{R}} P_m(x)P_n(x) \, d\mu(x) = \delta_{mn}, \quad m, n \in \mathbb{Z}_+,
\]

with respect to a positive Borel measure \( \mu \) on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} x^{2n} \, d\mu(x) < \infty, \quad \forall n \in \mathbb{Z}_+.
\]

Without loss of generality one may assume that \( \mu \) is a probability measure, i.e. \( \mu(\mathbb{R}) = 1 \), and \( P_0(x) = 1 \). As usual, \( \mu \) is unambiguously determined by the distribution function \( x \mapsto \mu((\infty, x]) \). In particular, the distribution function is supposed to be continuous from the right. With some abuse of notation, the distribution function will again be denoted by the symbol \( \mu \). The set of monomials, \( \{x^n; \ n \in \mathbb{Z}_+\} \), is required to be linearly independent in \( L^2(\mathbb{R}, d\mu) \) and so the function \( \mu \) should have an infinite number of points of increase.

It is well known that a sequence of OPs, if normalized, satisfies a three-term recurrence relation,

\[
xP_n(x) = w_{n-1}P_{n-1}(x) + \lambda_n P_n(x) + w_n P_{n+1}(x), \quad n \in \mathbb{N},
\]

with the initial conditions \( P_0(x) = 1 \) and \( P_1(x) = (x - \lambda_0)/w_0 \), where \( \{\lambda_n\}_{n=0}^{\infty} \) is a real sequence and \( \{w_n\}_{n=0}^{\infty} \) is a positive sequence [2, 4]. However, due to Favard’s theorem, the opposite statement is also true. For any sequence of real polynomials, \( \{P_n\}_{n=0}^{\infty} \) with \( \deg P_n = n \), satisfying the recurrence (19) with the above given initial conditions there exists a unique positive functional on the space of real polynomials.
making this sequence orthonormal. Moreover, if the matrix $J$ given in (12) represents a unique self-adjoint operator on $\ell^2(\mathbb{Z}_+)$ then this functional is induced by a unique positive Borel measure $\mu$ on $\mathbb{R}$. This means that (18) is fulfilled. In other words, in that case Hamburger’s moment problem is determinate; see, for instance, §4.1.1 and Corollary 2.2.4 in [2] or Theorem 3.4.5 in [16].

Using (8) one easily verifies that the solution of (19) with the given initial conditions is related to $F$ through the identity

$$P_n(x) = \prod_{k=0}^{n-1} \left(\frac{x - \lambda_k}{w_k}\right) \mathfrak{F} \left(\left\{\frac{\gamma_k^2}{\lambda_k - x}\right\}_{k=0}^{n-1}\right), \quad n \in \mathbb{Z}_+. \quad (20)$$

A second linearly independent solution of (19) can be written in the form

$$Q_n(x) = \frac{1}{w_0} \prod_{k=1}^{n-1} \left(\frac{x - \lambda_k}{w_k}\right) \mathfrak{F} \left(\left\{\frac{\gamma_k^2}{\lambda_{k+1} - x}\right\}_{k=0}^{n-2}\right), \quad n \in \mathbb{N}. \quad (21)$$

The latter solution satisfies the initial conditions $Q_0(x) = 0$ and $Q_1(x) = 1/w_0$.

Being given a sequence of OPs, $\{P_n\}_{n=0}^\infty$, defined via the recurrence rule (19), i.e. via formula (20), a crucial question is how the measure of orthogonality looks like. Relying on the function $\mathfrak{F}$ we provide a partial description of the measure $\mu$. Doing so we confine ourselves to Jacobi matrices for which the set of cluster points of the diagonal sequence $\lambda$ is discrete. This assumption is not too restrictive, though, since it turns out that $\text{der}(\lambda)$ is a one-point set or even empty in many practical applications of interest.

**Proposition 1.** Let $J$ be a Jacobi matrix introduced in (12) and $\text{der}(\lambda)$ be composed of isolated points only. Suppose there exists $z_0 \in \mathbb{C}$ such that (13) is fulfilled for $z = z_0$. Then the orthogonality relation for the sequence of OPs determined in (19) reads

$$\int_{\mathbb{R}} P_m(x)P_n(x) \, d\nu(x) + \sum_{x \in \mathcal{D}} \frac{P_m(x)P_n(x)}{\|P(x)\|^2} = \delta_{mn}, \quad m, n \in \mathbb{Z}_+, \quad (22)$$

where $\mathcal{D} = \text{spec}_p(J) \cap \text{der}(\lambda)$ and $\|P(x)\|$ stands for the $\ell^2$-norm of the vector $P(x) = (P_0(x), P_1(x), \ldots)$. The measure $d\nu$ is positive, purely discrete and supported on the set $\mathfrak{F}(J)$. The magnitude of jumps of the step function $\nu(x)$ at those points $x \in \mathfrak{F}(J)$ which do not belong to the range of $\lambda$ equals

$$\nu(x) - \nu(x - 0) = \frac{1}{x - \lambda_0} \mathfrak{F} \left(\left\{\frac{\gamma_{k+1}^2}{\lambda_{k+1} - x}\right\}_{k=0}^{\infty}\right) \left[\frac{d}{dx} \mathfrak{F} \left(\left\{\frac{\gamma_k^2}{\lambda_k - x}\right\}_{k=0}^{\infty}\right)\right]^{-1}. \quad (23)$$

**Remark.** In Proposition 1 we avoided considering the points from $\mathfrak{F}(J)$ which belong to the range of $\lambda$. We remark, however, that such points, if any, can be treated as well, similarly to (23), though in somewhat more complicated way. But we omit the details for the sake of simplicity.
Proof. Let $E_J$ stand for the projection-valued spectral measure of the self-adjoint operator $J$. As is well known, the measure of orthogonality $\mu$ is related to $E_J$ by the identity

$$\mu(M) = \langle e_0, E_J(M)e_0 \rangle$$

(24)

holding for any Borel set $M \subset \mathbb{R}$. Here again, $e_0$ denotes the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Moreover, $\text{supp}(\mu) = \text{spec}(J)$. In fact, let us recall that (24) follows from the observation that $e_n = P_n(J)e_0$ for all $n \in \mathbb{Z}_+$ and from the Spectral Theorem since

$$\delta_{mn} = \langle e_m, e_n \rangle = \langle e_0, P_m(J)P_n(J)e_0 \rangle = \int_{\mathbb{R}} P_m(x)P_n(x) d\mu(x).$$

The set $\text{der}(\lambda)$ is closed and, by hypothesis, discrete and therefore at most countable. Further we know, referring to (15), that the part of the spectrum of $J$ lying in $\mathbb{C} \setminus \text{der}(\lambda)$ is discrete, too. Consequently, $\text{spec}(J)$ is countable and therefore the continuous part of the spectral measure $E_J$ necessarily vanishes, i.e. $J$ has a pure point spectrum. In that case, of course, in order to determine the spectral measure $E_J$ it suffices to determine the projections $E_J\{\{x\}\}$ for all $x \in \text{spec}_p(J)$. Since the vector $P(z)$ is a formal solution of $(J - z)P(z) = 0$, unique up to a constant multiplier, one has the well known criterion $x \in \text{spec}_p(J)$ iff $\|P(x)\| < \infty$. Moreover, $P_0(x) = 1$ and so

$$\langle e_0, E_J(\{x\})e_0 \rangle = \frac{|\langle P(x), e_0 \rangle|^2}{\|P(x)\|^2} = \frac{1}{\|P(x)\|^2}.$$

The point spectrum of $J$ may be split into two disjoint sets, $\text{spec}_p(J) = \mathfrak{F}(J) \cup \mathcal{D}$. The Hilbert space and the spectral measure decompose correspondingly. Put

$$J' = JE_J(\mathfrak{F}(J)) \quad \text{and} \quad \nu(x) = \langle e_0, E_{J'}((\infty, x])e_0 \rangle \quad \text{for} \quad x \in \mathbb{R}.$$ 

Then the measure $d\nu$ is supported on $\mathfrak{F}(J)$ and

$$\int_{\mathbb{R}} f(x) d\mu(x) = \int_{\mathbb{R}} f(x) d\nu(x) + \sum_{x \in \mathcal{D}} \frac{f(x)}{\|P(x)\|^2}$$

for all $f \in C(\mathbb{R})$. As pointed out in (15), any $x \in \mathfrak{F}(J)$ is a simple isolated eigenvalue of $J$. As usual, $E_J(\{x\})$ can be written as the Riesz spectral projection. Choosing $\epsilon > 0$ sufficiently small one has

$$\langle e_0, E_J(\{x\})e_0 \rangle = -\frac{1}{2\pi i} \oint_{|x-z|=\epsilon} m(z) dz = -\text{Res}(m, x).$$

If, in addition, $x$ does not belong to the range of $\lambda$ then, in view of (17) and (14), (16) (with $r(x) = 0$), we may evaluate

$$\text{Res}(m, x) = \frac{1}{\lambda_0 - x} \mathfrak{F}\left(\left\{ \frac{\gamma_{k+1}^2}{\lambda_{k+1} - x} \right\}_{k=0}^{\infty} \right) \left| \frac{d}{dz} \right|_{z=x} \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=0}^{\infty} \right)^{-1}.$$ 

This concludes the proof. \qed
Remark 2. Of course, the sum on the LHS of (22) is void if \( \text{der}(\lambda) = \emptyset \). The sum also simplifies in the particular case when \( J \) is a compact operator satisfying (13). One can readily see that this happens iff \( \lambda_n \to 0 \) and \( w \in \ell^2(\mathbb{Z}_+) \). Then Proposition 1 is applicable and the orthogonality relation (22) takes the form

\[
\int_{\mathbb{R}} P_n(x) P_m(x) \, d\nu(x) + \Lambda_0 P_n(0) P_m(0) = \delta_{mn}.
\]

If \( J \) is invertible then \( \Lambda_0 \) vanishes but in general \( \Lambda_0 \) may be strictly positive as demonstrated, for instance, by the example of \( q \)-Lommel polynomials, see [18, Theorem 4.2].

For the intended applications of Proposition 1 the following particular case is of importance. Let \( \lambda \in \ell^1(\mathbb{Z}_+) \) be real and \( w \in \ell^2(\mathbb{Z}_+) \) positive. Then \( J \) is compact and (13) holds for any \( z \neq 0 \) not belonging to the range of \( \lambda \). Moreover, the characteristic function of \( J \) can be regularized with the aid of the entire function

\[
\phi_\lambda(z) := \prod_{n=0}^{\infty} (1 - z \lambda_n).
\]

Let us define

\[
\mathcal{G}_J(z) := \begin{cases} \phi_\lambda(z) \mathcal{F}_J(z^{-1}) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}
\]

(25)

The function \( \mathcal{G}_J \) is entire and, referring to (15), one has

\[
\text{spec}(J) = \{0\} \cup \{z^{-1}; \mathcal{G}_J(z) = 0\}.
\]

(26)

Since

\[
m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}
\]

where \( d\mu \) is the measure from (24), formula (17) implies that the identity

\[
\int_{\mathbb{R}} \frac{d\mu(x)}{1 - xz} = \frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)}
\]

holds for any \( z \notin \mathcal{G}_J^{-1}(\{0\}) \). Here \( J^{(1)} \) denotes the Jacobi operator determined by the diagonal sequence \( \{\lambda_{n+1}\}_{n=0}^{\infty} \) and the weight sequence \( \{w_{n+1}\}_{n=0}^{\infty} \), see (32) below.

Let us denote by \( \{\mu_n\}_{n=1}^{\infty} \) the set of non-zero eigenvalues of the compact operator \( J \). Remember that all eigenvalues of \( J \) are necessarily simple and particularly the multiplicity of 0 as an eigenvalues of \( J \) does not exceed 1. Since \( d\mu \) is supported by \( \text{spec}(J) \), formula (27) yields the Mittag-Leffler expansion

\[
\Lambda_0 + \sum_{k=1}^{\infty} \frac{\Lambda_k}{1 - \mu_k z} = \frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)}
\]

(28)

where \( \Lambda_k \) denotes the jump of the piece-wise constant function \( \mu(x) \) at \( x = \mu_k \), and similarly for \( \Lambda_0 \) and \( x = 0 \). From (28) one deduces that

\[
\Lambda_k = \lim_{z \to \mu_k^{-1}} (1 - \mu_k z) \frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)} = -\mu_k \frac{\mathcal{G}_{J^{(1)}}(\mu_k^{-1})}{\mathcal{G}_J'(\mu_k^{-1})}
\]

(29)
for $k \in \mathbb{N}$. This can be viewed as a regularized version of the identity (23) in this particular case. We have shown the following proposition.

**Proposition 3.** Let $\lambda$ be a real sequence from $\ell^1(\mathbb{Z}_+)$ and $w$ be a positive sequence from $\ell^2(\mathbb{Z}_+)$. Then the measure of orthogonality $d\mu$ for the corresponding sequence of OPs defined in (17) fulfills

$$\text{supp}(d\mu) \setminus \{0\} = \{z^{-1}; \mathcal{G}_J(z) = 0\}$$

where the RHS is a bounded discrete subset of $\mathbb{R}$ with 0 as the only accumulation point. Moreover, for $x \in \text{supp}(d\mu) \setminus \{0\}$ one has

$$\mu(x) - \mu(x - 0) = -x \frac{\mathcal{G}_{J(i)}(x^{-1})}{\mathcal{G}'_J(x^{-1})}.$$  \hfill (30)

Let us denote $\xi_{-1}(z) := \mathcal{G}_J(z)$ and

$$\xi_k(z) := \left( \prod_{l=0}^{k-1} w_l \right) z^{k+1} \mathcal{G}_{J(k+1)}(z), \quad k \in \mathbb{Z}_+, \quad (31)$$

where

$$J^{(k)} = \begin{pmatrix} \lambda_k & w_k & w_{k+1} \\ w_k & \lambda_{k+1} & w_{k+1} \\ & \lambda_{k+2} & w_{k+2} \\ & & \cdot & \cdot & \cdot \end{pmatrix}. \quad (32)$$

**Lemma 4.** Let $\lambda \in \ell^1(\mathbb{Z}_+)$, $w \in \ell^2(\mathbb{Z}_+)$ and $z \neq 0$. Then the vector

$$\xi(z) = (\xi_0(z), \xi_1(z), \xi_2(z), \ldots)$$

belongs to $\ell^2(\mathbb{Z}_+)$, and one has

$$\frac{1}{z^2} \sum_{k=0}^{\infty} \xi_k(z)^2 = \xi_{-1}(z)\xi'_0(z) - \xi'_{-1}(z)\xi_0(z). \quad (34)$$

Moreover, the vector $\xi(z)$ is nonzero and so

$$\xi_{-1}(z)\xi'_0(z) - \xi'_{-1}(z)\xi_0(z) > 0, \quad \forall z \in \mathbb{R} \setminus \{0\}, \quad (35)$$

provided the sequences $\lambda$ and $w$ are both real.

**Proof.** First, choose $N \in \mathbb{Z}_+$ so that $z^{-1} \neq \lambda_k$ for all $k > N$. This is clearly possible since $\lambda_n \to 0$ as $n \to \infty$. Then we have, referring to (6) and (25),

$$|\mathcal{G}_{J(k)}(z)| \leq \exp \left( \sum_{j=N+1}^{\infty} |z||\lambda_j| + \sum_{j=N+1}^{\infty} \frac{|z|^2 |w_j|^2}{(1-z\lambda_j)(1-z\lambda_{j+1})} \right) \quad \text{for } k > N.$$
Observing that \( w_n \to 0 \) as \( n \to \infty \), one concludes that there exists a constant \( C > 0 \) such that
\[
|z|^{k+1} \prod_{l=0}^{k-1} w_l \leq C \cdot 2^{-k} \quad \text{for} \quad k > N.
\]
These estimates obviously imply the square summability of the vector \( \xi(z) \).

Second, with the aid of (31) one verifies that for all \( z \neq 0 \) and \( k \in \mathbb{Z}_+ \),
\[
w_{k-1} \xi_{k-1}(z) + (\lambda_k - z^{-1}) \xi_k(z) + w_k \xi_{k+1}(z) = 0
\]
where we put \( w_{-1} := 1 \). From here one deduces that the equality
\[
(z^{-1} - x^{-1}) \xi_k(x) = W_k(x, z) - W_{k-1}(x, z),
\]
with
\[
W_k(x, z) = w_k (\xi_{k+1}(z) \xi_k(x) - \xi_{k+1}(x) \xi_k(z)),
\]
holds for all \( k \in \mathbb{Z}_+ \). Now one can derive (31) from (30) in a routine way.

Finally, observe that the first equality in (9) implies the limit
\[
\lim_{k \to \infty} G_{J(w)}(z) = 1.
\]
Referring to (31) this means \( \xi_k(z) \neq 0 \) for all sufficiently large \( k \). \( \square \)

**Proposition 5.** Let \( \lambda \in \ell^1(\mathbb{Z}_+) \) be real, \( w \in \ell^2(\mathbb{Z}_+) \) be positive and \( z \neq 0 \). If \( z^{-1} \) is an eigenvalue of the Jacobi operator \( J \) given in (12), then the vector \( \xi(z) \) is a corresponding eigenvector.

**Proof.** As recalled in (26), \( z^{-1} \) is an eigenvalue of \( J \) iff \( G_J(z) \equiv \xi_{-1}(z) = 0 \). Then one readily verifies, with the aid of (31), that \( \xi(z) \) is a formal solution of the eigenvalue equation \( (J - z^{-1}) \xi(z) = 0 \). By Lemma 4 \( \xi(z) \neq 0 \). It is even true that \( \xi_{0}(z) \neq 0 \). Indeed, if \( \xi_{-1}(z) = \xi_{0}(z) = 0 \) then, by the recurrence, \( \xi_k(z) = 0 \) for all \( k \in \mathbb{Z}_+ \), a contradiction. Moreover, Lemma 4 also tells us that \( \xi(z) \in \ell^2(\mathbb{Z}_+) \).

**Proposition 6.** Let \( \lambda \in \ell^1(\mathbb{Z}_+) \) be real and \( w \in \ell^2(\mathbb{Z}_+) \) be positive. Then the zeros of the function \( G_J \) are all real and simple, and form a countable subset of \( \mathbb{R} \setminus \{0\} \) with no finite accumulation points. Furthermore, the functions \( G_J \) and \( G_{J(1)} \) have no common zeros, and the zeros of the same sign of \( G_J \) and \( G_{J(1)} \) mutually separate each other, i.e. between any two consecutive zeros of \( G_J \) which have the same sign there is a zero of \( G_{J(1)} \) and vice versa.

**Proof.** The first part of the proposition follows from (26). In fact, all zeros of \( G_J \) are surely real since \( J \) is a Hermitian operator in \( \ell^2(\mathbb{Z}_+) \). Moreover, \( J \) is compact and all its eigenvalues are simple. Therefore the set of reciprocal values of nonzero eigenvalues of \( J \) is countable and has no finite accumulation points.

Thus we know that the zeros of \( G_J \) and \( G_{J(1)} \) are all located in \( \mathbb{R} \setminus \{0\} \) and \( \xi_{-1}(z) = G_J(z), \xi_0(z) = z G_{J(1)}(z) \). Hence, as far as the zeros are concerned and we are separated from the origin, we can speak about \( \xi_{-1} \) and \( \xi_0 \) instead of \( G_J \) and \( G_{J(1)} \), respectively.
The remainder of the proposition can be deduced from (35) in a usual way. Suppose a zero of $\xi_{-1}$, called $z$, is not simple. Then $\xi_{-1}(z) = \xi'_{-1}(z) = 0$ which leads to a contradiction with (35). From (35) it is immediately seen, too, that $\xi_{-1}$ and $\xi_0$ have no common zeros in $\mathbb{R} \setminus \{0\}$. Furthermore, suppose $z_1$ and $z_2$ are two consecutive zeros of $\xi_{-1}$ of the same sign. Since these zeros are simple, the numbers $\xi'_{-1}(z_1)$ and $\xi'_{-1}(z_2)$ differ in sign. From (35) one deduces that $\xi_0(z_1)$ and $\xi_0(z_2)$ must differ in sign as well. Consequently, there is at least one zero of $\xi_0$ lying between $z_1$ and $z_2$. An entirely analogous argument applies if the roles of $\xi_{-1}$ and $\xi_0$ are interchanged.

4 Lommel polynomials

4.1 Basic properties and the orthogonality relation

In this section we deal with the Lommel polynomials as one of the simplest and most interesting examples to demonstrate the general results derived in Section 3. This is done with the perspective of approaching our main goal in this paper, namely a generalization of the Lommel polynomials established in the next section. Let us note that although the Lommel polynomials are expressible in terms of hypergeometric series, they do not fit into Askey’s scheme of hypergeometric orthogonal polynomials [14].

Recall that the Lommel polynomials arise in the theory of Bessel function (see, for instance, [25, ?? 9.6–9.73] or [8, Chp. VII]). They can be written explicitly in the form

$$ R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k} $$

(37)

where $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $-\nu \notin \mathbb{Z}_+$ and $x \in \mathbb{C} \setminus \{0\}$. Here we stick to the traditional terminology though, obviously, $R_{n,\nu}(x)$ is a polynomial in the variable $x^{-1}$ rather than in $x$. Proceeding by induction in $n \in \mathbb{Z}_+$ one easily verifies the identity

$$ R_{n,\nu}(x) = \left(\frac{2}{x}\right)^n \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \mathfrak{S} \left(\left\{\frac{x}{2(\nu+k)}\right\}_{k=0}^{n-1}\right). $$

(38)

As is well known, the Lommel polynomials are directly related to Bessel functions,

$$ R_{n,\nu}(x) = \frac{\pi x}{2} (Y_{-1+\nu}(x) J_{n+\nu}(x) - J_{-1+\nu}(x) Y_{n+\nu}(x)) $$

$$ + \frac{\pi x}{2 \sin(\pi \nu)} (J_{1-\nu}(x) J_{n+\nu}(x) + (-1)^n J_{-1+\nu}(x) J_{-n-\nu}(x)). $$

From here and relation (11) below it is seen that the Lommel polynomials obey the recurrence

$$ R_{n+1,\nu}(x) = \frac{2(n+\nu)}{x} R_{n,\nu}(x) - R_{n-1,\nu}(x), \quad n \in \mathbb{Z}_+, $$

(39)

with the initial conditions $R_{-1,\nu}(x) = 0$, $R_{0,\nu}(x) = 1$. 

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The original meaning of the Lommel polynomials reveals the formula
\[ J_{\nu+n}(x) = R_{n,\nu}(x)J_{\nu}(x) - R_{n-1,\nu+1}(x)J_{\nu-1}(x) \quad \text{for } n \in \mathbb{Z}_+. \] (40)

As firstly observed by Lommel in 1871, (40) can be obtained by iterating the basic recurrence relation for the Bessel functions, namely
\[ J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x) - J_{\nu-1}(x). \] (41)

Let us remark that (40) immediately follows from (11), (38) and formula (56) below.

The orthogonality relation for Lommel polynomials is known explicitly and is expressed in terms of the zeros of the Bessel function of order \( \nu - 1 \) as explained, for instance, in [5, 7], see also [4, Chp. VI, S 6] and [11]. This relation can also be rederived as a corollary of Proposition 3. For \( \nu > -1 \) and \( n \in \mathbb{Z}_+ \), set temporarily
\[ \lambda_n = 0 \quad \text{and} \quad w_n = 1/\sqrt{(\nu + n + 1)(\nu + n + 2)}. \]

Then the corresponding Jacobi operator \( J \) is compact, self-adjoint and 0 is not an eigenvalue. In fact, invertibility of \( J \) can be verified straightforwardly by solving the formal eigenvalue equation for 0. Referring to (56), the regularized characteristic function of \( J \) equals
\[ G_J(z) = F_J(z^{-1}) = \Gamma(\nu + 1) z^{-\nu} J_{\nu}(2z). \]

Consequently, the support of the measure of orthogonality turns out to coincide with the zero set of \( J_{\nu}(z) \). Remember that \( x^{-\nu} J_{\nu}(x) \) is an even function. Let \( j_{k,\nu} \) stand for the \( k \)-th positive zero of \( J_{\nu}(x) \) and put \( j_{-k,\nu} = -j_{k,\nu} \) for \( k \in \mathbb{N} \). Proposition 3 then tells us that the orthogonality relation takes the form
\[-2(\nu + 1) \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{J_{\nu+1}(j_{k,\nu})}{J_{k,\nu} J_{\nu}'(j_{k,\nu})} P_m \left( \frac{2}{j_{k,\nu}} \right) P_n \left( \frac{2}{j_{k,\nu}} \right) = \delta_{mn} \]
where \( J_{\nu}'(x) \) denotes the partial derivative of \( J_{\nu}(x) \) with respect to \( x \).

Furthermore, (20) and (38) imply
\[ R_{n,\nu+1}(x) = \sqrt{\frac{\nu + 1}{\nu + n + 1}} P_n \left( \frac{2}{x} \right). \] (42)

Using the identity
\[ \partial_x J_{\nu}(x) = \frac{\nu}{x} J_{\nu}(x) - J_{\nu+1}(x), \]
the orthogonality relation simplifies to the well known formula
\[ \sum_{k \in \mathbb{Z} \setminus \{0\}} j_{k,\nu}^{-2} R_{n,\nu+1}(j_{k,\nu}) R_{m,\nu+1}(j_{k,\nu}) = \frac{1}{2(n + \nu + 1)} \delta_{mn}, \] (43)
valid for \( \nu > -1 \) and \( m, n \in \mathbb{Z}_+ \).
4.2 Lommel Polynomials in the variable $\nu$

The Lommel polynomials can also be dealt with as polynomials in the parameter $\nu$. Such polynomials are also orthogonal with the measure of orthogonality supported by the zeros of a Bessel function of the first kind regarded as a function of the order.

Let us consider a sequence of polynomials in the variable $\nu$ and depending on a parameter $u \neq 0$, \( \{Q_n(u; \nu)\}_{n=0}^{\infty} \), determined by the recurrence

\[
u Q_{n-1}(u; \nu) - nQ_n(u; \nu) + uQ_{n+1}(u; \nu) = \nu Q_n(u; \nu), \quad n \in \mathbb{Z}_+,
\]

with the initial conditions $Q_{-1}(u; \nu) = 0$, $Q_0(u; \nu) = 1$. According to (20),

\[
Q_n(u, \nu) = R_n,\nu(2u), \quad \forall n \in \mathbb{Z}_+.
\]

Comparing the last formula with (38) one observes that

\[
Q_n(u, \nu) = R_n,\nu(2u), \quad \forall n \in \mathbb{Z}_+.
\]

The Bessel function $J_\nu(x)$ regarded as a function of $\nu$ has infinitely many simple real zeros which are all isolated provided that $x > 0$, see [22, Subsection 4.3]. Below we denote the zeros of $J_{\nu-1}(2u)$ by $\theta_n = \theta_n(u)$, $n \in \mathbb{N}$, and restrict ourselves to the case $u > 0$ since $\theta_n(-u) = \theta_n(u)$.

The Jacobi matrix $J$ corresponding to this case, i.e. $J$ with the diagonal $\lambda_n = -n$ and the weights $w_n = u$, $n \in \mathbb{Z}_+$, is an unbounded self-adjoint operator with a discrete spectrum (see [22]). Hence the orthogonality measure for $\{Q_n(u; \nu)\}$ has the form stated in Remark 2. Thus, using (23) and equation (56) below, one arrives at the orthogonality relation

\[
\sum_{k=1}^{\infty} \frac{J_{\theta_k}(2u)}{u} \left( \frac{\partial}{\partial z} \bigg|_{z=\theta_k} J_{\nu-1}(2u) \right) R_{n,\theta_k}(2u)R_{m,\theta_k}(2u) = \delta_{mn}, \quad m,n \in \mathbb{Z}_+.
\]

Concerning the history, let us remark that initially this was Dickinson in 1958 who proposed the problem of seeking a construction of the measure of orthogonality for the Lommel polynomials in the variable $\nu$ [6]. Ten years later, Maki described such a construction in [19].

5 A new class of orthogonal polynomials

5.1 Characteristic functions of the Jacobi matrices $J_L$ and $\tilde{J}_L$

In this section we work with matrices $J_L$ and $\tilde{J}_L$ defined in (11), (12) and (13), (14), respectively. To have the weight sequence $w$ positive and the matrix Hermitian, we assume, in the case of $J_L$, that $-1 \neq L > -3/2$ if $\eta \in \mathbb{R} \setminus \{0\}$, and $L > -3/2$ if $\eta = 0$. Similarly, in the case of $\tilde{J}_L$ we assume $L > -1/2$ and $\eta \in \mathbb{R}$.

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Recall that the regular and irregular Coulomb wave functions, $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, are two linearly independent solutions of the second-order differential equation

$$\frac{d^2u}{d\rho^2} + \left[ 1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2} \right] u = 0,$$  \hspace{1cm} (47)

see, for instance, [1, Chp. 14]. One has the Wronskian formula (see [1, 14.2.5])

$$F_{L-1}(\eta, \rho)G_L(\eta, \rho) - F_L(\eta, \rho)G_{L-1}(\eta, \rho) = \frac{L}{\sqrt{L^2 + \eta^2}}.$$  \hspace{1cm} (48)

Furthermore, the function $F_L(\eta, \rho)$ admits the decomposition [1, 14.1.3 and 14.1.7]

$$F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} \phi_L(\eta, \rho)$$  \hspace{1cm} (49)

where

$$C_L(\eta) := \sqrt{\frac{2\pi\eta}{e^{2\pi\eta} - 1}} \frac{\sqrt{(1+\eta^2)(4+\eta^2)\cdots(L^2+\eta^2)}}{(2L+1)!! L!}$$

and

$$\phi_L(\eta, \rho) := e^{-i\rho} F_{1}(L+1-i\eta, 2L+2, 2i\rho).$$  \hspace{1cm} (50)

For $L$ not an integer, $C_L(\eta)$ is to be understood as

$$C_L(\eta) = \frac{2L e^{-\pi\eta/2} |\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}.$$  \hspace{1cm} (51)

In [23], the characteristic function for the matrix $J_L$ has been derived. If expressed in terms of $G_{J_L}$, as defined in (25), the formula simply reads

$$G_{J_L}(\rho) = \left( \prod_{k=L+1}^{\infty} (1 - \lambda_k \rho) \right) \mathfrak{F} \left( \left\{ \frac{\rho^2}{1 - \lambda_k \rho} \right\}_{k=L+1}^{\infty} \right) = \phi_L(\eta, \rho).$$  \hspace{1cm} (52)

For the particular values of parameters, $L = \nu - 1/2$ and $\eta = 0$, one gets

$$\mathfrak{F} \left( \left\{ \frac{\rho}{2(\nu + k)} \right\}_{k=1}^{\infty} \right) = \phi_{\nu-1/2}(0, \rho).$$  \hspace{1cm} (53)

It is also known that, see (50) and Eqs. 14.6.6 and 13.6.1 in [1],

$$F_{\nu-1/2}(0, \rho) = \sqrt{\frac{\pi \rho}{2}} J_\nu(\rho),$$  \hspace{1cm} (54)

$$\phi_{\nu-1/2}(0, \rho) = e^{-i\rho} F_{1}(\nu + 1/2, 2\nu + 1, 2i\rho) = \Gamma(\nu + 1) \left( \frac{2}{\rho} \right)^\nu J_\nu(\rho).$$  \hspace{1cm} (55)

(53) jointly with (55) imply

$$\mathfrak{F} \left( \left\{ \frac{\rho}{\nu + k} \right\}_{k=1}^{\infty} \right) = \Gamma(\nu + 1) \rho^{-\nu} J_\nu(2\rho).$$  \hspace{1cm} (56)
The last formula has already been observed in [21] and it holds for any \( \nu \notin -\mathbb{N} \) and \( \rho \in \mathbb{C} \).

Using the recurrence (58) for \( \tilde{J}_L \), one can also obtain the characteristic function for \( \tilde{J}_L \),

\[
\mathcal{F}_{J_L}(\rho^{-1}) = \mathcal{F}_{J_L}(\rho^{-1}) - \frac{\tilde{w}_L^2}{(\rho^{-1} - \lambda_L)(\rho^{-1} - \lambda_{L+1})} \mathcal{F}_{J_{L+1}}(\rho^{-1}).
\]

(57)

Be reminded that \( \phi_L(\eta, \rho) \) obeys the equations

\[
\frac{\partial}{\partial \rho} \phi_{L+1}(\eta, \rho) = \frac{2L + 3}{\rho} \phi_L(\eta, \rho) - \left( \frac{2L + 3}{\rho} + \frac{\eta}{L + 1} \right) \phi_{L+1}(\eta, \rho),
\]

\[
\frac{\partial}{\partial \rho} \phi_L(\eta, \rho) = \frac{\eta}{L + 1} \phi_L(\eta, \rho) - \frac{\rho}{2L + 3} \left( 1 + \frac{\eta^2}{(L + 1)^2} \right) \phi_{L+1}(\eta, \rho),
\]

(58), (59)
as it follows from [1, 14.2.1 and 14.2.2]. A straightforward computation based on (52), (57) and (59) yields

\[
\left( 1 + \frac{\eta \rho}{(L + 1)^2} \right) \left( \prod_{n=L+1}^{\infty} \left( 1 + \frac{\eta \rho}{n(n + 1)} \right) \right) \mathcal{F}_{J_L}(\rho^{-1}) = \phi_L(\eta, \rho) + \frac{\rho}{L + 1} \frac{\partial}{\partial \rho} \phi_L(\eta, \rho).
\]

(60)

In view of (19), this can be rewritten as

\[
G_{J_L}(\rho) = \phi_L(\eta, \rho) + \frac{\rho}{L + 1} \frac{\partial}{\partial \rho} \phi_L(\eta, \rho) = \frac{1}{(L + 1)C_L(\eta)} \rho^{-L} \frac{\partial}{\partial \rho} F_L(\eta, \rho).
\]

(61)

### 5.2 Orthogonal polynomials associated with \( F_L(\eta, \rho) \)

Following the general scheme outlined in Section 3 (see (19)) we denote by \( \{ P_n^{(L)}(\eta; z) \}_{n=0}^{\infty} \) the sequence of OPs given by the three-term recurrence

\[
z P_n^{(L)}(\eta; z) = w_{L+n} P_{n-1}^{(L)}(\eta; z) + \lambda_{L+n+1} P_n^{(L)}(\eta; z) + w_{L+n+1} P_{n+1}^{(L)}(\eta; z), \quad n \in \mathbb{Z}_+,
\]

(62)

with \( P_{-1}^{(L)}(\eta; z) = 0 \) and \( P_0^{(L)}(\eta; z) = 1 \). We again restrict ourselves to the range of parameters \(-1 \neq L > -3/2 \) if \( \eta \in \mathbb{R} \setminus \{0\} \), and \( L > -3/2 \) if \( \eta = 0 \). Likewise the Lommel polynomials, these polynomials are not included in Askey’s scheme [14]. Further let us denote

\[
R_n^{(L)}(\eta; \rho) := P_n^{(L)}(\eta; \rho^{-1})
\]

(63)

for \( \rho \neq 0, n \in \mathbb{Z}_+ \). According to (20),

\[
P_n^{(L)}(\eta; z) = \left( \prod_{k=1}^{n} \frac{z - \lambda_{L+k}}{w_{L+k}} \right) \tilde{\mathcal{S}} \left( \left\{ \frac{\gamma_{L+k}^2}{z - \lambda_{L+k}} \right\}_{k=1}^{n} \right), \quad n \in \mathbb{Z}_+.
\]

(64)

Alternatively, these polynomials can be expressed in terms of the Coulomb wave functions.
Proposition 7. For \( n \in \mathbb{Z}_+ \) and \( \rho \neq 0 \) one has
\[
R_n^{(L)}(\eta; \rho) = \frac{\sqrt{(L + 1)^2 + \eta^2}}{L + 1} \sqrt{\frac{2L + 2n + 3}{2L + 3}} \times \left( F_L(\eta, \rho)G_{L+n+1}(\eta, \rho) - F_{L+1}(\eta, \rho)G_L(\eta, \rho) \right).
\]

Proof. To verify this identity it suffices to check that the RHS fulfills the same recurrence relation as \( R_n^{(L)}(\eta, \rho) \) does and with the same initial conditions. The RHS actually meets the first requirement as it follows from known recurrence relations for the Coulomb wave functions, see [1, 14.2.3]. The initial condition is a consequence of the Wronskian formula (48).

For the computations to follow it is useful to notice that the weights \( w_n \) and the normalization constants \( C_L(\eta) \), as defined in (2) and (51), respectively, are related by the equation
\[
\prod_{k=0}^{n-1} w_{L+k} = \frac{\sqrt{2L + 2n + 1}}{2L + 1} \frac{C_{L+n}(\eta)}{C_L(\eta)}, \quad n = 0, 1, 2, \ldots.
\]

Proposition 8. For the above indicated range of parameters and \( \rho \neq 0 \),
\[
\lim_{n \to \infty} \sqrt{(2L + 3)(2L + 2n + 1)} C_{L+n}(\eta)\rho^{L+n}R_n^{(L)}(\eta; \rho) = \sqrt{1 + \frac{\eta^2}{(L + 1)^2}} F_L(\eta, \rho).
\]

Proof. Referring to (63) and (64), \( R_n^{(L)}(\eta; \rho) \) can be expressed in terms of the function \( \mathfrak{F} \). The sequence whose truncation stands in the argument of \( \mathfrak{F} \) on the RHS of (64) belongs to the domain \( D \) defined in (5) and so one can apply the second equation in (9). Concerning the remaining terms occurring on the LHS of (66), one readily computes, with the aid of (65), that
\[
\lim_{n \to \infty} \sqrt{(2L + 3)(2L + 2n + 1)} C_{L+n}(\eta)\rho^{L+n}R_n^{(L)}(\eta; \rho) = \sqrt{1 + \frac{\eta^2}{(L + 1)^2}} F_L(\eta, \rho)
\]

Remark 9. Note that the polynomials \( R_n^{(L)}(\eta; \rho) \) can be regarded as a generalization of the Lommel polynomials \( R_{n,\nu}(x) \). Actually, if \( \eta = 0 \) then the Jacobi matrix \( J_{\nu-1/2} \) is determined by the sequences
\[
\lambda_n = 0, \quad w_n = 1/(2\sqrt{(\nu + n + 1)(\nu + n + 2)}).
\]
Thus the recurrence (52) reduces to (39) for \( \eta = 0 \) and \( L = \nu - 1/2 \). More precisely, one finds that \( P_n^{(\nu-1/2)}(0; z) \) coincides with the polynomial \( P_n(2z) \) from Subsection 4.1 for all \( n \). In view of (42) this means

\[
R_n^{(\nu-1/2)}(0; \rho) = \sqrt{\frac{\nu + n + 1}{\nu + 1} R_{n+1}(\rho)}
\]

for \( n \in \mathbb{Z}_+ \), \( \rho \in \mathbb{C} \setminus \{0\} \) and \( \nu > -1 \). In addition we remark that, for the same values of parameters, (66) yields Hurwitz’ limit formula (see §9.65 in [25])

\[
\lim_{n \to \infty} \frac{(\rho/2)^{\nu+n}}{\Gamma(\nu+n+1)} R_{n+1}(\rho) = J_\nu(\rho).
\]

A more explicit description of the polynomials \( P_n^{(L)}(\eta; z) \) can be derived. Let us write

\[
P_n^{(L)}(\eta; z) = \sum_{k=0}^n c_k(n, L, \eta) z^{n-k}.
\]

**Proposition 10.** Let \( \{Q_k(n, L; \eta); k \in \mathbb{Z}_+\} \) be a sequence of monic polynomials in the variable \( \eta \) defined by the recurrence

\[
Q_{k+1}(n, L; \eta) = \eta Q_k(n, L; \eta) - h_k(n, L) Q_{k-1}(n, L; \eta) \quad \text{for} \quad k \in \mathbb{Z}_+,
\]

with the initial conditions \( Q_{-1}(n, L; \eta) = 0 \), \( Q_0(n, L; \eta) = 1 \), where

\[
h_k(n, L) = \frac{k(2L + k + 1)(2n - k + 2)(4L + 2n - k + 3)}{4(2n - 2k + 1)(2n - 2k + 3)}, \quad k \in \mathbb{Z}_+.
\]

Then the coefficients \( c_k(n, L, \eta) \) defined in (65) fulfill

\[
c_k(n, L, \eta) = \sqrt{\frac{2L + 2n + 3}{2L + 3}} \left| \frac{\Gamma(L + 2 + i\eta)}{\Gamma(L + n + 2 + i\eta)} \right| \left| \frac{\Gamma(2n - k + 2)}{\Gamma(2L + k + 2)} \right| \frac{2^{n+k-1}}{k!} Q_k(n, L; \eta)
\]

for \( k = 0, 1, 2, \ldots, n \).

For the proof we shall need an auxiliary identity. Note that, if convenient, \( Q_k(n, L; \eta) \) can be treated as a polynomial in \( \eta \) with coefficients belonging to the field of rational functions in the variables \( n, L \).

**Lemma 11.** The polynomials \( Q_k(n, L; \eta) \) defined in Proposition 10 fulfill

\[
Q_k(n, L; \eta) - \alpha_k(n, L) Q_k(n-1, L; \eta) - \beta_k(n, L) \eta Q_{k-1}(n-1, L; \eta) = 0
\]

for \( k = 0, 1, 2, \ldots \), where

\[
\alpha_k(n, L) = \frac{2(2n - 2k + 1)(L + n + 1)}{(2n - k + 1)(2L + 2n - k + 2)} \quad \text{and} \quad \beta_k(n, L) = \frac{k(2L + k + 1)}{(2n - k + 1)(2L + 2n - k + 2)}.
\]
Proof. It is elementary to verify that \( \alpha_k(n, L) \), \( \beta_k(n, L) \) fulfill the identities

\[
\alpha_k(n, L) + \beta_k(n, L) = 1, \tag{72}
\]
\[
\alpha_{k+1}(n, L)h_k(n-1, L) - \alpha_{k-1}(n, L)h_k(n, L) = 0, \tag{73}
\]
\[
\beta_k(n, L)h_{k-1}(n-1, L) - \beta_{k-1}(n, L)h_{k}(n, L) = 0. \tag{74}
\]

Now to show (71) one can proceed by induction in \( k \). The case \( k = 0 \) means \( \alpha_0(n, L) = 1 \) which is obviously true. Furthermore, \( Q_1(n, L; \eta) = \eta \) and so the case \( k = 1 \) is a consequence of (72), with \( k = 1 \). Suppose \( k \geq 2 \) and that the identity is true for \( k - 1 \) and \( k - 2 \). Applying (69) both to \( Q_k(n, L; \eta) \) and \( Q_k(n-1, L; \eta) \) and using (72), (73) one can show the LHS of (71) to be equal to

\[
-h_{k-1}(n, L)(Q_{k-2}(n, L; \eta) - \alpha_{k-2}(n, L)Q_{k-2}(n-1, L; \eta)) \\
+ \eta(Q_{k-1}(n, L; \eta) - Q_{k-1}(n-1, L; \eta)).
\]

Next we apply the induction hypothesis both to \( Q_{k-1}(n, L; \eta) \) and \( Q_{k-2}(n, L; \eta) \) in this expression to find that it equals, up to a common factor \( \eta \),

\[
-\beta_{k-1}(n, L)Q_{k-1}(n-1, L; \eta) + \beta_{k-1}(n, L) \eta Q_{k-2}(n-1, L; \eta) \\
-h_{k-1}(n, L)\beta_{k-2}(n, L)Q_{k-3}(n-1, L; \eta).
\]

Finally, making once more use of (69), this time for the term \( Q_{k-1}(n-1, L; \eta) \), one can prove the last expression to be equal to

\[
(h_{k-2}(n-1, L)\beta_{k-1}(n, L) - h_{k-1}(n, L)\beta_{k-2}(n, L))Q_{k-3}(n-1, L; \eta) = 0
\]

as it follows from (74). \( \square \)

**Proof of Proposition 11.** Write the polynomials \( P^{(L)}_n(\eta; z) \) in the form (68) and substitute the RHS of (70) for the coefficients \( c_k(n, L, \eta) \). Plugging the result into (62) one finds that the recurrence relation for the sequence \( \{P^{(L)}_n(\eta; z)\} \) is satisfied if and only if the terms \( Q_k(n, L; \eta) \) from the substitution fulfill

\[
a_k(n, L)Q_k(n, L; \eta) - b_k(n, L)Q_k(n-1, L; \eta) - c_k(n, L) \eta Q_{k-1}(n-1, L; \eta) \\
+ d_k(n, L, \eta)Q_{k-2}(n-2, L; \eta) = 0 \tag{75}
\]

for \( k, n \in \mathbb{N}, n \geq k \), where we again put \( Q_{-1}(n, L; \eta) = 0, Q_0(n, L; \eta) = 1 \), and

\[
a_k(n, L) = \frac{(2n-k)(2n-k+1)(2L+2n-k+1)(2L+2n-k+2)}{(2L+k)(2L+k+1)(L+n+1)}, \\
b_k(n, L) = \frac{4(2L+2n+1)(n-k)(2n-2k+1)}{(2L+k)(2L+k+1)}, \\
c_k(n, L) = \frac{k(2L+2n+1)(2n-k)(2L+2n-k+1)}{(2L+k)(L+n)(L+n+1)}, \\
d_k(n, L, \eta) = \frac{(k-1)k \left( \eta^2 + (L+n)^2 \right)}{L+n}. \\

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Notice that for \( k = 0 \) one has
\[
a_0(n, L)Q_0(n, L, \eta) - b_0(n, L)Q_0(n - 1, L, \eta) - c_0(n, L)\eta Q_{-1}(n - 1, L, \eta) = a_0(n, L) - b_0(n, L) = 0.
\]

One also observes, as it should be, that the relation (75) determines the terms \( Q_k(n, L; \eta) \) unambiguously. In fact, to see it one can proceed by induction in \( k \).

Suppose \( k > 0 \) and all terms \( Q_j(n, L; \eta) \) are already known for \( j < k \), \( n \geq j \). Putting \( n = k \) in (75), one can express \( Q_k(k - 1, L; \eta) \) in terms of \( Q_{k-1}(k - 1, L; \eta) \) and \( Q_{k-2}(k - 2, L; \eta) \) since \( b_k(k, L) = 0 \). Then, treating \( k \) as being fixed and \( n \) as a variable, one can interpret (75) as a first order difference equation in the index \( n \), with a right hand side, for an unknown sequence \( \{Q_k(n, L; \eta); n \geq k\} \). The initial condition for \( n = k \) is now known as well as the right hand side and so the difference equation is unambiguously solvable.

To prove the proposition it suffices to verify that if \( \{Q_k(n, L; \eta)\} \) is a sequence of monic polynomials in the variable \( \eta \) defined by the recurrence (69) then it obeys, too, the relation (75). To this end, one may apply repeatedly the rule (71) to bring the LHS of (75) to the form
\[
e_0(n, L)Q_k(n - 2, L; \eta) + e_1(n, L)\eta Q_{k-1}(n - 2, L; \eta) + e_2(n, L, \eta)Q_{k-2}(n - 2, L; \eta) \quad (76)
\]
where
\[
e_0(n, L) = a_k(n, L)\alpha_k(n - 1, L)\alpha_k(n, L) - b_k(n, L)\alpha_k(n - 1, L),
\]
\[
e_1(n, L) = a_k(n, L)\alpha_{k-1}(n - 1, L)\beta_k(n, L) + a_k(n, L)\alpha_k(n, L)\beta_k(n - 1, L) - b_k(n, L)\beta_k(n - 1, L) - c_k(n, L)\alpha_{k-1}(n - 1, L),
\]
and
\[
e_2(n, L, \eta) = (a_k(n, L)\beta_{k-1}(n - 1, L)\beta_k(n, L) - c_k(n, L)\beta_{k-1}(n - 1, L))\eta^2 + d_k(n, L, \eta).
\]

Direct evaluation then gives
\[
e_1(n, L)/e_0(n, L) = -1, \quad e_2(n, L, \eta)/e_0(n, L) = h_{k-1}(n - 2, L).
\]

Referring to the defining relation (69), this proves (76) to be equal to zero indeed. \( \square \)

**Remark 12.** Let us shortly discuss what Proposition 11 tells us in the particular case when \( \eta = 0, L = \nu - 1/2 \). The recurrence (69) is easily solvable for \( \eta = 0 \). One has \( Q_{2k+1}(n, L; 0) = 0 \) and
\[
Q_{2k}(n, L; 0) = (-1)^k \frac{(2k)! (n - k)! (2n - 4k + 1)!}{k! (n - 2k)! (2n - 2k + 1)!} \frac{\Gamma(L + k + 1)\Gamma(L + n + 2)}{\Gamma(L + 1) \Gamma(L + n - k + 2)}
\]
for \( k = 0, 1, 2, \ldots \). Whence \( c_{2k+1}(n, \nu - 1/2, 0) = 0 \) and
\[
c_{2k}(n, \nu - 1/2, 0) = \sqrt{\frac{\nu + n + 1}{\nu + 1}} (-1)^k 2^{n - 2k} \binom{n - k}{k} \frac{\Gamma(\nu + n - k + 1)}{\Gamma(\nu + k + 1)}.
\]
Recalling (67) one rederives this way the explicit expression (37) for the usual Lommel polynomials.
Let us mention two more formulas. The first one is quite substantial and shows that the polynomials \( R_{\nu}^{(L)}(\eta, \rho) \) play the same role for the Coulomb wave functions as the Lommel polynomials do for the Bessel functions. It follows from the abstract identity (11) where we specialize \( d = n \),

\[
x_k = \frac{\gamma^2_{L+k-1}}{\rho^{-1} - \lambda_{L+k-1}},
\]

and again make use of (65). Thus we get

\[
R_{\nu}^{(L-1)}(\eta, \rho)F_{\nu}(\eta, \rho) - \frac{L+1}{L} \sqrt{\frac{2L+3}{2L+1}} \frac{\sqrt{\eta^2 + L^2}}{\sqrt{\eta^2 + (L+1)^2}} R_{\nu-1}^{(L)}(\eta, \rho)F_{\nu-1}(\eta, \rho)
= \sqrt{\frac{2L+2n+1}{2L+1}} F_{\nu+n}(\eta, \rho),
\]

where \( n \in \mathbb{Z}_+ \), \( 0 \neq L > -1/2 \), \( \eta \in \mathbb{R} \) and \( \rho \neq 0 \). Moreover, referring to (52) and (67), one observes that relation (10) is a particular case of (78) if one lets \( \eta = 0 \) and \( L = \nu - 1/2 \).

Similarly, one can derive the announced second identity from (10) by making the same choice as that in (77) but writing \( z \) instead of \( \rho^{-1} \). Recalling (64) one finds that

\[
P_{\nu}^{(L-1)}(\eta; z)P_{\nu+n}(\eta; z) - P_{\nu+s+1}^{(L-1)}(\eta; z)P_{\nu-1}^{(L)}(\eta; z) = \frac{w_{\nu} \ L}{w_{\nu+n} P_{\nu+n}(\eta; z)}
\]

holds for all \( n, s \in \mathbb{Z}_+ \).

We conclude this subsection by describing the measure of orthogonality for the generalized Lommel polynomials. To this end, we need an auxiliary result concerning the zeros of the function \( \phi_{L}(\eta, \cdot) \) which is in fact a particular case of Proposition 6 if we specialize the sequences \( \rho \) and \( \lambda \) to the choice made in (2). From (52) we know that \( \phi_{L}(\eta, \rho) = G_{J_{L}}(\rho) \) and we note that, obviously, \( J_{L+1} = J_{L}^{(1)} \). Thus we arrive at the following statement.

**Proposition 13.** Let \(-1 \neq L > -3/2 \) if \( \eta \in \mathbb{R} \setminus \{0\} \), and \( L > -3/2 \) if \( \eta = 0 \). Then the zeros of the function \( \phi_{L}(\eta, \cdot) \) form a countable subset of \( \mathbb{R} \setminus \{0\} \) with no finite accumulation points. Moreover, the zeros of \( \phi_{L}(\eta, \cdot) \) are all simple, the functions \( \phi_{L}(\eta, \cdot) \) and \( \phi_{L+1}(\eta, \cdot) \) have no common zeros, and the zeros of the same sign of \( \phi_{L}(\eta, \cdot) \) and \( \phi_{L+1}(\eta, \cdot) \) mutually separate each other.

Let us arrange the zeros of \( \phi_{L}(\eta, \cdot) \) into a sequence \( \rho_{L,n}, \ n \in \mathbb{N} \) (not indicating the dependence on \( \eta \) explicitly). According to Proposition 13 we can do it so that

\[
0 < |\rho_{L,1}| \leq |\rho_{L,2}| \leq |\rho_{L,3}| \leq \ldots .
\]

Thus we have

\[
\{ \rho_{L,n}, n \in \mathbb{N} \} = \{ \rho \in \mathbb{R}; \phi_{L}(\eta, \rho) = 0 \} = \{ \rho \in \mathbb{R} \setminus \{0\}; F_{L}(\eta, \rho) = 0 \}.
\]

**Proposition 14.** Let \(-1 \neq L > -3/2 \) if \( \eta \in \mathbb{R} \setminus \{0\} \), and \( L > -3/2 \) if \( \eta = 0 \). Then the orthogonality relation

\[
\sum_{k=1}^{\infty} \rho_{L,k}^{-2} R_{\nu}^{(L)}(\eta; \rho_{L,k}) R_{\nu-1}^{(L)}(\eta; \rho_{L,k}) = \frac{(L+1)^2 + \eta^2}{(2L+3)(L+1)^2} \delta_{mn}
\]

holds for \( m, n \in \mathbb{Z}_+ \).

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Proof. By Proposition \ref{prop:orthogonality} we have the orthogonality relation
\[
\int_{\mathbb{R}} P_m^{(L)}(\eta; \rho) P_n^{(L)}(\eta; \rho) \, d\mu(\rho) = \delta_{mn}
\]
where $d\mu$ is supported on the set $\{ \rho_{L,n}^{-1}; \, n \in \mathbb{N} \} \cup \{0\}$. Applying formula (\ref{eq:main}) combined with (\ref{eq:mu}) and (\ref{eq:mu-2}) one finds that
\[
\mu(\rho_{L,k}^{-1}) - \mu(\rho_{L,k}^{-1}) = -\rho_{L,k}^{-1} \frac{\partial \phi_{L+1}(\eta, \rho_{L,k})}{\partial \eta} - (2L + 3)(L + 1)^2 \frac{1}{(L + 1)^2 + \eta^2} \rho_{L,k}^{-2}.
\]
We claim that 0 is a point of continuity of $\mu$. Indeed, let us denote by $\Lambda_k$ the magnitude of the jump of $\mu$ at $\rho_{L,k}^{-1}$ if $k \in \mathbb{N}$, and at 0 if $k = 0$. Then, since $d\mu$ is a probability measure, one has
\[
1 = \sum_{k=0}^{\infty} \Lambda_k = \Lambda_0 + \frac{(2L + 3)(L + 1)^2}{(L + 1)^2 + \eta^2} \sum_{k=1}^{\infty} \rho_{L,k}^{-2} = \Lambda_0 + \frac{(2L + 3)(L + 1)^2}{(L + 1)^2 + \eta^2} \| J_L \|^2_2
\]
where $\| J_L \|^2_2$ stands for the Hilbert-Schmidt norm of $J_L$. This norm, however, can be computed directly,
\[
\| J_L \|^2_2 = \sum_{n=1}^{\infty} \lambda_{L+n}^2 + 2 \sum_{n=1}^{\infty} w_{L+n}^2 = \frac{(L + 1)^2 + \eta^2}{(2L + 3)(L + 1)^2}.
\]
Comparing this equality to (\ref{eq:lambda}) one finds that $\Lambda_0 = 0$. To conclude the proof it suffices to recall (\ref{eq:mu-1}).

In the course of the proof of Proposition \ref{prop:continuity} we have shown that 0 is a point of continuity of $\mu$. It follows that 0 is not an eigenvalue of the compact operator $J_L$.

Corollary 15. Let $-1 \neq L > -3/2$ if $\eta \in \mathbb{R} \setminus \{0\}$, and $L > -3/2$ if $\eta = 0$. Then the operator $J_L$ is invertible.

Remark 16. Again, letting $\eta = 0$ and $L = \nu - 1/2$ in (\ref{eq:mu}) and recalling (\ref{eq:mu-1}), one readily verifies that (\ref{eq:mu-1}) is a particular case of (\ref{eq:mu}).

Remark 17. In the same way as above one can define a sequence of OPs associated with the function $\partial_\rho F_L(\eta, \rho)$ or, more generally, with $\alpha F_L(\eta, \rho) + \rho \partial_\rho F_L(\eta, \rho)$ for some $\alpha$ real. But our results in this respect are not complete yet and are notably less elegant than those for the function $F_L(\eta, \rho)$, and so we confine ourselves to this short comment. Let us just consider the former particular case and call the corresponding sequence of OPs $\{ \tilde{\phi}_n^{(L)}(\eta; z) \}_{n=0}^{\infty}$. It is defined by a recurrence analogous to (\ref{eq:phi}) but now the coefficients in the relation are matrix entries of $\tilde{J}_L$ rather than those of $J_L$. The initial conditions are the same, and one has to restrict the range of parameters to the values $L > -1/2$ and $\eta \in \mathbb{R}$. Let us denote $\tilde{R}_n^{(L)}(\eta; \rho) := \tilde{P}_n^{(L)}(\eta; \rho^{-1})$ for $\rho \neq 0$, $n \in \mathbb{Z}_+$. The zeros of the function $\rho \mapsto \partial_\rho F_L(\eta, \rho)$ can be shown to be all real and simple and to form a countable set with no finite accumulation points. One may arrange the zeros into a sequence. Let us call it $\{ \tilde{\rho}_{L,n}; \, n \in \mathbb{N} \}$. Then the corresponding orthogonality
measure $d\mu$ is again supported on the set $\{\tilde{\rho}_{L,k}^{-1}; k \in \mathbb{N}\} \cup \{0\}$. It is possible to directly compute the magnitude $\Lambda_k$ of the jump at $\tilde{\rho}_{L,k}^{-1}$ of the piece-wise constant function $\mu$ with the result

$$\Lambda_k = \frac{L + 1}{\tilde{\rho}_{L,k}^2 - 2\eta\tilde{\rho}_{L,k} - L(L + 1)}.$$  

We propose that $\mu$ has no jump at the point 0 or, equivalently, that the Jacobi matrix $\tilde{J}_L$ is invertible, but we have no proof for this hypothesis yet.

6 The spectral zeta function associated with $F_L(\eta, \rho)$

Let us recall that the spectral zeta function of a positive definite operator $A$ with a discrete spectrum whose inverse $A^{-1}$ belongs to the $p$-th Schatten class is defined as

$$\zeta^{(A)}(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \text{Tr} A^{-s}, \quad \text{Re } s \geq p,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ are the eigenvalues of $A$. The zeta function can be used to approximately compute the ground state energy of $A$, i.e. the lowest eigenvalue $\lambda_1$. This approach is known as Euler’s method (initially applied to the first positive zero of the Bessel function $J_0$) which is based on the inequalities

$$\zeta^{(A)}(s) - 1/s < \lambda_1 < \frac{\zeta^{(A)}(s)}{\zeta^{(A)}(s + 1)}, \quad s \geq p. \quad (83)$$

In fact, the inequalities in (83) become equalities in the limit $s \to \infty$.

In this section we describe recursive rules for the zeta function associated with the regular Coulomb wave function. The procedure can be applied, however, to a wider class of special functions. For example, this approach can also be applied to the Bessel functions resulting in the well known convolution formulas for the Rayleigh function [13]. Not surprisingly, the recurrences derived below can be viewed as a generalization of these known results.

Recall also that the regularized determinant,

$$\det_2(1 + A) := \det (1 + A \exp(-A)),$$

is well defined if $A$ is a Hilbert-Schmidt operator, i.e. belonging to the second Schatten class, on a separable Hilbert space. Moreover, the regularized determinant is continuous in the Hilbert-Schmidt norm [20, Theorem 9.2].

Referring to (1), (52) and (80), we start from the identity

$$\det_2(1 - \rho J_{L,n}) = \exp(\rho \text{Tr} J_{L,n}) \det(1 - \rho J_{L,n})$$

$$= \exp\left(\frac{\eta\rho}{L + n + 1} - \frac{\eta\rho}{L + 1}\right) \left(\prod_{k=1}^{n} (1 - \rho_{L+k})\right) \mathfrak{F}\left(\left\{\frac{\gamma_{L+k}^2}{\lambda_{L+k} - \rho^{-1}}\right\}_{k=1}^{n}\right).$$
where \( J_{L,n} \) stands for the \( n \times n \) truncation of \( J_L \). The formula can be verified straight-forwardly by induction in \( n \) with the aid of the rule (8). Now, sending \( n \) to infinity and using (52) we get

\[
\det_2(1 - \rho J_L) = \exp \left( -\frac{\eta \rho}{L+1} \right) \phi_L(\eta, \rho). \tag{84}
\]

On the other hand, we have the Hadamard product formula

\[
\det_2(1 - \rho J_L) = \prod_{n=1}^{\infty} \left( 1 - \frac{\rho}{\rho_{L,n}} \right) e^{\rho/\rho_{L,n}}, \tag{85}
\]

see [20, Theorem 9.2.]. Combining (84) and (85) one arrives at the Hadamard infinite product expansion of \( \phi_L(\eta, .) \),

\[
\phi_L(\eta, \rho) = e^{\eta \rho/(L+1)} \prod_{n=1}^{\infty} \left( 1 - \frac{\rho}{\rho_{L,n}} \right) e^{\rho/\rho_{L,n}}. \tag{86}
\]

Let us define

\[
\zeta_L(k) := \sum_{n=1}^{\infty} \frac{1}{\rho_{L,n}^k}, \quad k \geq 2.
\]

In view of (86), we can expand the logarithm of \( \phi_L(\eta, \rho) \) into a power series,

\[
\ln \phi_L(\eta, \rho) = \frac{\eta \rho}{L+1} - \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} \left( \frac{\rho}{\rho_{L,n}} \right)^k,
\]

whenever \( \rho \in \mathbb{C}, |\rho| < |\rho_{L,1}|. \) Whence

\[
\frac{\partial \rho \phi_L(\eta, \rho)}{\phi_L(\eta, \rho)} = \frac{\eta}{L+1} - \sum_{k=1}^{\infty} \zeta_L(k + 1) \rho^k.
\]

Comparing this equality to (59) one finds that

\[
\sum_{k=0}^{\infty} \zeta_L(k + 2) \rho^k = \frac{1}{2L + 3} \left( 1 + \frac{\eta^2}{(L+1)^2} \right) \frac{\phi_{L+1}(\eta, \rho)}{\phi_L(\eta, \rho)} \quad \text{for } |\rho| < |\rho_{L,1}|. \tag{87}
\]

So one can obtain the values of \( \zeta_L(k) \) for \( k \in \mathbb{N}, k \geq 2 \), by inspection of the Taylor series of the RHS in (87).

However, an apparently more efficient tool to compute the values of the zeta function would be a recurrence formula. To find it one has to differentiate equation (87) with respect to \( \rho \) and to use both formulas (58) and (59). This way one arrives at the equation

\[
(2L + 3) \left( 1 + \frac{\eta^2}{(L+1)^2} \right)^{-1} \sum_{k=1}^{\infty} k \zeta_L(k + 2) \rho^k = (2L + 3) \left( 1 - \frac{\phi_{L+1}(\eta, \rho)}{\phi_L(\eta, \rho)} \right)
\]

\[
- \frac{2\eta \rho}{L+1} + \frac{\rho^2}{2L+3} \left( 1 + \frac{\eta^2}{(L+1)^2} \right) \left( \frac{\phi_{L+1}(\eta, \rho)}{\phi_L(\eta, \rho)} \right)^2.
\]

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Using (87) to express $\phi_{L+1}(\eta, \rho)/\phi_{L}(\eta, \rho)$ one obtains
\[
\sum_{k=1}^{\infty} k \zeta_{L}(k + 2) \rho^k = 1 + \frac{\eta^2}{(L + 1)^2} - (2L + 3) \sum_{k=0}^{\infty} \zeta_L(k + 2) \rho^k + \frac{2\eta}{L + 1} \sum_{k=1}^{\infty} \zeta_L(k + 1) \rho^k + \sum_{k=0}^{\infty} \sum_{l=0}^{k} \zeta_L(l + 2) \zeta_L(k - l + 2) \rho^{k+2}.
\]
Now it suffices to equate coefficients at the same powers of $\rho$. In particular, for the absolute term we get
\[
\zeta_L(2) = \frac{1}{2L + 3} \left( 1 + \frac{\eta^2}{(L + 1)^2} \right).
\]
Note that $\zeta_L(2)$ is the square of the Hilbert-Schmidt norm of $J_L$. The desired recurrence relation reads
\[
\zeta_L(k + 1) = \frac{1}{2L + k + 2} \left( \frac{2\eta}{L + 1} \zeta_L(k) + \sum_{l=1}^{k-2} \zeta_L(l + 1) \zeta_L(k - l) \right), \quad k = 2, 3, 4, \ldots.
\]
As described above, bounds on the first (in modulus) zero $\rho_{L,1}$ can be determined with the aid of the zeta function. The operator $J_L$ is not positive, however, and so the bounds should be written as follows
\[
\zeta_L(2s)^{-1/s} < \rho_{L,1}^2 < \frac{\zeta_L(2s)}{\zeta_L(2s + 2)}, \quad s \geq 1.
\]
In the simplest case, for $s = 1$, we get the estimates
\[
\frac{(2L + 3)(L + 1)^2}{(L + 1)^2 + \eta^2} < \rho_{L,1}^2 < \frac{(2L + 3)(2L + 5)(L + 2)(L + 1)^2}{(L + 4)\eta^2 + (L + 2)(L + 1)^2}.
\]
Further let us examine the particular case when $\eta = 0$ and $L = \nu - 1/2$. Then the rules (88) and (89) reproduce the well known recurrence relations for the Rayleigh function $\sigma_{2n}(\nu)$, with $n \geq 2$ and $\nu > -1$ [13]. Recall that
\[
\sigma_{2n}(\nu) := \sum_{k=1}^{\infty} j_{\nu,k}^{-2n}
\]
where $j_{\nu,k}$ denotes the $k$-th positive zero of the Bessel function $J_{\nu}$. The recurrence relation reads
\[
\sigma_2(\nu) = \frac{1}{4(\nu + 1)}, \quad \sigma_{2n}(\nu) = \frac{1}{n + \nu} \sum_{k=1}^{n-1} \sigma_{2k}(\nu) \sigma_{2n-2k}(\nu) \text{ for } n = 2, 3, 4, \ldots.
\]

Remark 18. Let us remark that instead of (89) one can derive a recurrence relation in a form which is a linear combination of zeta functions. Rewrite equation (87) as
\[
\left( 1 + \frac{\eta^2}{(L + 1)^2} \right) \phi_{L+1}(\eta, \rho) = (2L + 3) \phi_L(\eta, \rho) \sum_{k=0}^{\infty} \zeta_L(k + 2) \rho^k.
\]
and replace everywhere the function \( \phi_L \) by the power expansion

\[
\phi_L(\eta, \rho) = e^{-i \rho} \sum_{k=0}^{\infty} \frac{(L + 1 - i \eta)_k (2i \rho)^k}{(2L + 2)_k k!}.
\]

After obvious cancellations and equating coefficients at the same powers of \( \rho \) on the both sides one arrives at the identity

\[
\frac{2[(L + 1)^2 + \eta^2]}{(L + 1)(L + 1 - i \eta)} \frac{\Gamma(L + 2 - i \eta + k)}{\Gamma(2L + 4 + k) k!} = \sum_{l=0}^{k} \frac{\Gamma(L + 1 - i \eta + k - l)(2i)^{-l}}{\Gamma(2L + 2 + k - l)(k - l)!} \zeta_L(l + 2),
\]

which holds for any \( k \in \mathbb{Z}_+ \), \( L > -1 \) and \( \eta \in \mathbb{R} \).

**Remark 19.** The orthogonality measure \( d\mu \) for the sequence of OPs \( \{P_n^{(L)}(\eta, \rho)\} \), as described in Proposition 14, fulfills

\[
\hat{R} f(x) d\mu(x) = \frac{(2L + 3)(L + 1)^2}{(L + 1)^2 + \rho^2} \sum_{k=1}^{\infty} \rho_L^{-2} f(\rho_L^{-1})
\]

for every \( f \in C(\mathbb{R}) \). Consequently, the moment sequence associated with the measure \( d\mu \) can be expressed in terms of the zeta function,

\[
m_n := \int_{\mathbb{R}} x^n d\mu(x) = \frac{\zeta_L(n + 2)}{\zeta_L(2)}, \quad n \in \mathbb{Z}_+
\]

(recall also (88)). In view of formulas (88) and (89), this means that the moment sequence can be evaluated recursively.

**Remark 20.** This comment extends Remark 17. We note that it is possible to derive formulas analogous to (89) for the spectral zeta function associated with the function \( \partial_\rho F_L(\eta, \rho) \) though the resulting recurrence rule is notably more complicated in this case. One may begin, similarly to (85), with the identities

\[
\det_2(1 - \rho \tilde{J}_L) = \prod_{n=1}^{\infty} \left(1 - \frac{\rho}{\rho_L,n}\right) e^{\rho/\tilde{\rho}_L,n}
\]

\[
= \exp\left(-\frac{(L + 2)\eta \rho}{(L + 1)^2}\right)\left(\phi_L(\eta, \rho) + \frac{\rho}{L + 1} \partial_\rho \phi_L(\eta, \rho)\right).
\]

Hence for \( \psi_L(\eta, \rho) := \phi_L(\eta, \rho) + (\rho/(L + 1)) \partial_\rho \phi_L(\eta, \rho) \) we have

\[
\ln \psi_L(\eta, \rho) = \frac{(L + 2)\eta \rho}{(L + 1)^2} - \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{\rho}{\rho_L,n}\right)^k
\]

whenever \( \rho \in \mathbb{C}, |\rho| < |\tilde{\rho}_L,1| \). Let us define

\[
\tilde{\zeta}_L(k) := \sum_{n=1}^{\infty} \frac{1}{\rho_L,n}, \quad k \geq 2.
\]
Now one can apply manipulations quite similar to those used in case of the zeta function associated with $F_L(\eta, \rho)$. Differentiating equation (90) twice and always taking into account that $F_L(\eta, \rho)$ solves (47) one arrives, after some tedious but straightforward computation, at the equation

\[
\frac{2(\rho - \eta)}{\rho^2 - 2\eta \rho - L(L + 1)} \left[ -L \rho - \frac{(L + 2)\eta \rho^2}{(L + 1)^2} + \sum_{k=2}^{\infty} \tilde{\zeta}_L(k) \rho^{k+1} \right] \\
+ \left[ -L - \frac{(L + 2)\eta \rho}{(L + 1)^2} + \sum_{k=2}^{\infty} \tilde{\zeta}_L(k) \rho^{k} \right]^2 \\
= L^2 + \frac{2(L^2 + L - 1)\eta \rho}{(L + 1)^2} - \rho^2 + \sum_{k=2}^{\infty} (k + 1) \tilde{\zeta}_L(k) \rho^{k}.
\]

From here the sought recurrence rules can be extracted in a routine way but we avoid writing them down explicitly because of their length and complexity.

**Acknowledgments**

The authors wish to acknowledge gratefully partial support from grant No. GA13-11058S of the Czech Science Foundation.

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