POINCARÉ DUALITY, CAP PRODUCTS AND BOREL-MOORE
INTERSECTION HOMOLOGY

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Abstract. Using a cap product, we construct an explicit Poincaré isomorphism be-
tween the blown-up intersection cohomology and the Borel-Moore intersection homol-
ogy, for any commutative ring of coefficients and second-countable, oriented pseudo-
manifold.

Introduction

Poincaré duality of singular spaces is the “raison d’être” ([9, Section 8.2]) of intersec-
tion homology. It has been proven by Goresky and MacPherson in their first paper on
intersection homology ([12]) for compact PL pseudomanifolds and rational coefficients
and extended to $\mathbb{Z}$ with some hypothesis on the torsion part, by Goresky and Siegel in
[13]. With a similar hypothesis, Friedmann and McClure obtain this isomorphism, from
a cap product with a fundamental class, for any field of coefficients in [11], see also [9]
for a commutative ring of coefficients with restrictions on the torsion.

Using the blown-up intersection cohomology with compact supports, we have estab-
lished in [8] a Poincaré duality for any commutative ring of coefficients, without hypoth-
esis on the torsion part, for any oriented paracompact pseudomanifold. Moreover, we
also set up in [5] a Poincaré duality between the blown-up intersection cohomology and
the Borel-Moore intersection homology of an oriented PL-pseudomanifold $X$.

This paper is the “chainon manquant:” the existence of an explicit Poincaré duality
isomorphism between the blown-up intersection cohomology and the Borel-Moore inter-
section homology, from a cap product with the fundamental class, for any commmutative
ring of coefficients and any second-countable, oriented pseudomanifold.

In Section 1, we recall basic background on pseudomanifolds and intersection homol-
ogy. In particular, we present the complex of blown-up cochains, already introduced and
studied in a series of papers [2, 3, 4, 5, 6, 7, 8] (also called Thom-Whitney cochains in
some works).

Section 2 contains the main properties of Borel-Moore intersection homology: the
existence of a Mayer-Vietoris exact sequence in Theorem A and the recall of some results
established in [5].
Section 3 is devoted to the proof of the main result stated in Theorem B: the existence of an isomorphism between the blown-up intersection cohomology and the Borel-Moore intersection homology, by using the cap-product with the fundamental class of a second-countable, oriented pseudomanifold
\[ \mathcal{D}_X : \mathcal{H}^p_\mathcal{P}(X; R) \to \mathfrak{H}^{n-p}(X; R), \]
for any commutative ring of coefficients. The method is different than the one used in [5], where the sheaf presentation of intersection homology was employed and where PL-pseudomanifolds were involved. Here the duality is realized by a map defined at complexes level by the cap product with a cycle generating the fundamental class.

Mention also that the intersection homology of this work (Definition 1.7) is a general version, called tame homology in [2] and non-GM in [9], which coincides with the original one for the perversities of [12]. Let us also observe that our definition of Borel-Moore intersection homology coincides with the one studied by G. Friedman in [10] for perversities depending only on the codimension of strata.

Homology and cohomology are considered with coefficients in a commutative ring, \( R \). In general, we do not explicit them in the proofs. For any topological space \( X \), we denote by \( cX = X \times [0,1]/X \times \{0\} \) the cone on \( X \) and by \( \mathring{c}X = X \times [0,1[, X \times \{0\} \) the open cone on \( X \). Elements of the cones \( cX \) and \( \mathring{c}X \) are denoted \( [x,t] \) and the apex is \( v = [-,0] \).

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1. Background

1.1. Pseudomanifold. In [12], M. Goresky and R. MacPherson introduce intersection homology for the study of pseudomanifolds. Some basic properties of intersection homology, as the existence of a Mayer-Vietoris sequence, do not require such a structure and exist for filtered spaces.

**Definition 1.1.** A \textit{filtered space of dimension} \( n \), \( X \), is a Hausdorff space endowed with a filtration by closed subsets,
\[ \emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X, \]
such that \( X_n \setminus X_{n-1} \neq \emptyset \). The \textit{strata} of \( X \) of dimension \( i \) are the connected components \( S \) of \( X_i \setminus X_{i-1} \); we denote \( \dim S = i \) and \( \operatorname{codim} S = \dim X - \dim S \). The \textit{regular strata} are the strata of dimension \( n \) and the \textit{singular set} is the subspace \( \Sigma = X_{n-1} \). We denote by \( \delta_X \) (or \( \delta \) if there is no ambiguity) the set of non-empty strata.
In [12], the pseudomanifolds are supposed without strata of codimension 1. Here, we do not require this property.

**Definition 1.2.** An \( n \)-dimensional pseudomanifold is a filtered space of dimension \( n \), \( X \), such that, for any \( i \in \{0, \ldots, n\} \), \( X_i \setminus X_{i-1} \) is an \( i \)-dimensional topological manifold or the empty set. Moreover, for each point \( x \in X_i \setminus X_{i-1}, i \neq n \), there exist

(i) an open neighborhood \( V \) of \( x \) in \( X \), endowed with the induced filtration,
(ii) an open neighborhood \( U \) of \( x \) in \( X_i \setminus X_{i-1} \),
(iii) a compact pseudomanifold \( L \) of dimension \( n-i-1 \), whose cone \( \hat{c}L \) is endowed with the filtration \( (\hat{c}L)_i = \hat{c}L_{i-1} \),
(iv) a homeomorphism, \( \varphi : U \times \hat{c}L \to V \), such that
   (a) \( \varphi(u, v) = u \), for any \( u \in U \), where \( v \) is the apex of the cone \( \hat{c}L \),
   (b) \( \varphi(U \times \hat{c}L_j) = V \cap X_{i+j+1} \), for all \( j \in \{0, \ldots, n-i-1\} \).

The pseudomanifold \( L \) is called a **link** of \( x \).

**Remark 1.3.** For the convenience of the reader, we first collect basic topological definitions. A topological space \( X \) is said

1. separable if it contains a countable, dense subset;
2. second-countable if its topology has a numerable basis; that is there exists some numerable collection \( \mathcal{U} = \{U_j| j \in \mathbb{N}\} \) of open subsets such that any open subset of \( X \) can be written as a union of elements of some subfamily of \( \mathcal{U} \);
3. hemicompact if it is locally compact and there exists a numerable sequence of relatively compact open subsets, \( (U_i)_{i \in \mathbb{N}} \), such that \( \bigcup_i U_i \subseteq U_{i+1} \) and \( X = \bigcup_i U_i \).

To relate hypotheses of some following results to ones of previous works, we list some interactions between these notions.

- A second countable space is separable and, in a metric space, the two properties are equivalent ([16, Theorem 16.9]).
- A space is locally compact and second-countable if, and only if, its is metrisable and hemicompact, see [1, Corollaire of Proposition 16].

As second-countability is a hereditary property, any open subset of a second-countable pseudomanifold is one also. Moreover, we also know that a second-countable pseudomanifold is paracompact, separable, metrisable and hemicompact.

### 1.2. Intersection homology

We consider intersection homology relatively to the general perversities defined in [14].

**Definition 1.4.** A **perversity on a filtered space**, \( X \), is an application, \( \mathfrak{p} : S \to \mathbb{Z} \), defined on the set of strata of \( X \) and taking the value 0 on the regular strata. Among them, mention the null perversity \( \mathfrak{1} \), constant with value 0, and the **top perversity** defined by \( \mathfrak{T}(S) = \text{codim } S - 2 \) on singular strata. For any perversity, \( \mathfrak{p} \), the perversity \( D\mathfrak{p} := \mathfrak{T} - \mathfrak{p} \) is called the **complementary perversity** of \( \mathfrak{p} \).

The pair \( (X, \mathfrak{p}) \) is called a **perverse space**. For a pseudomanifold we say **perverse pseudomanifold**.

**Example 1.5.** Let \( (X, \mathfrak{p}) \) be a perverse space of dimension \( n \).
• An open perverse subspace \((U, \varpi)\) is an open subset \(U\) of \(X\), endowed with the induced filtration and a perversity still denoted \(\varpi\) and defined as follows: if \(S \subset U\) is a stratum of \(U\), such that \(S \subset U \cap S'\) with \(S'\) a stratum of \(X\), then \(\varpi(S) = \varpi(S')\). In the case of a perverse pseudomanifold, \((U, \varpi)\) is one also.

• If \(M\) is a connected topological manifold, the product \(M \times X\) is a filtered space for the product filtration, \((M \times X)_i = M \times X_i\). The perversity \(\varpi\) induces a perversity on \(M \times X\), still denoted \(\varpi\) and defined by \(\varpi(M \times S) = \varpi(S)\) for each stratum \(S\) of \(X\).

• If \(X\) is compact, the open cone \(\hat{c}X\) is endowed with the conical filtration, \((\hat{c}X)_i = \hat{c}X_{i-1}, 0 \leq i \leq n+1\), where \(\hat{c}0 = \{v\}\) is the apex of the cone. A perversity \(\varpi\) on \(\hat{c}X\) induces a perversity on \(X\) still denoted \(\varpi\) and defined by \(\varpi(S) = \varpi(S \times [0,1])\) for each stratum \(S\) of \(X\).

For the rest of this section, we consider a perverse space \((X, \varpi)\). We introduce now a chain complex giving the intersection homology with coefficients in \(R\), cf. [2].

Definition 1.6. A regular simplex is a continuous map \(\sigma: \Delta \to X\) of domain an Euclidean simplex decomposed in joins, \(\Delta = \Delta_0 \ast \Delta_1 \ast \cdots \ast \Delta_n\), such that \(\sigma^{-1}X_i = \Delta_0 \ast \Delta_1 \ast \cdots \ast \Delta_i\), for all \(i \in \{0, \ldots, n\}\) and \(\Delta_n \neq \emptyset\).

Given an Euclidean regular simplex \(\Delta = \Delta_0 \ast \cdots \ast \Delta_n\), we denote \(\partial\Delta\) the regular part of the chain \(\partial\Delta\). More precisely, we set \(\partial\Delta = \partial(\Delta_0 \ast \cdots \ast \Delta_{n-1}) \ast \Delta_n\), if \(\dim(\Delta_n) = 0\) and \(\partial\Delta = \partial\Delta\), if \(\dim(\Delta_n) \geq 1\). For any regular simplex \(\sigma: \Delta \to X\), we set \(\partial\sigma = \sigma_s \circ \partial\).

Notice that \(\partial^2 = 0\). We denote by \(C_*(\Delta; R)\) the complex of linear combinations of regular simplices (called finite chains) with the differential \(\partial\).

Definition 1.7. The perverse degree of a regular simplex \(\sigma: \Delta = \Delta_0 \ast \cdots \ast \Delta_n \to X\) is the \((n+1)\)-uple, \(\|\sigma\| = (\|\sigma\|_0, \ldots, \|\sigma\|_n)\), where \(\|\sigma\|_i = \dim \sigma^{-1}X_{n-i} = \dim(\Delta_0 \ast \cdots \ast \Delta_{n-i})\), with the convention \(\dim \emptyset = -\infty\). For each stratum \(S\) of \(X\), the perverse degree of \(\sigma\) along \(S\) is defined by

\[
\|\sigma\|_S = \begin{cases} 
-\infty, & \text{if } S \cap \sigma(\Delta) = \emptyset, \\
\|\sigma\|_{\text{codim} S}, & \text{otherwise.}
\end{cases}
\]

A regular simplex \(\varpi\)-allowable if

\[
\|\sigma\|_S \leq \dim \Delta - \text{codim } S + \varpi(S),
\]

for any stratum \(S\). A finite chain \(\xi\) is \(\varpi\)-allowable if it is a linear combination of \(\varpi\)-allowable simplices and of \(\varpi\)-intersection if \(\xi\) and its boundary \(\partial\xi\) are \(\varpi\)-allowable. We denote by \(C^\varpi_*(X; R)\) the complex of \(\varpi\)-intersection chains and by \(\tilde{H}^\varpi_*(X; R)\) its homology, called \(\varpi\)-intersection homology.

If \((U, \varpi)\) is an open perverse subspace of \((X, \varpi)\), we define the complex of relative \(\varpi\)-intersection chains as the quotient \(C^\varpi_*(X, U; R) = C^\varpi_*(X; R)/C^\varpi_*(U; R)\). Its homology is denoted \(\tilde{H}^\varpi_*(X, U; R)\). Finally, if \(K \subset U\) is compact, we have \(\tilde{H}^\varpi_*(X, X \setminus K; R) = \tilde{H}^\varpi_*(U, U \setminus K; R)\) by excision, cf. [2, Corollary 4.5].

Remark 1.8. This homology is called tame intersection homology in [2]. As we are only using it in this work, for sake of simplicity, we call it intersection homology. It coincides with the non-GM intersection homology of [9] (see [2, Theorem B]) and with intersection homology for the original perversities of [12], see [2, Remark 3.9].
1.3. Blown-up intersection cohomology. Let \( N_*(\Delta) \) and \( N^*(\Delta) \) be the simplicial chains and cochains, with coefficient in \( R \), of an Euclidean simplex \( \Delta \). Given a face \( F \) of \( \Delta \), we write \( 1_F \) the element of \( N^*(\Delta) \) taking the value 1 on \( F \) and 0 otherwise. We denote also by \((F,0)\) the same face viewed as face of the cone \( c\Delta = [v]*\Delta \) and by \((F,1)\) the face \( cF \) of \( c\Delta \). The apex is denoted \((\emptyset,1) = c\emptyset = [v] \).

If \( \Delta = \Delta_0 \times \cdots \times \Delta_n \) is a regular Euclidean simplex, we set
\[
\tilde{N}^*(\Delta) = N^*(c\Delta_0) \otimes \cdots \otimes N^*(c\Delta_{n-1}) \otimes N^*(\Delta_n).
\]
A basis of \( \tilde{N}^*(\Delta) \) is formed of the elements \( 1_{(F,\varepsilon)} = 1_{(F_0,\varepsilon_0)} \otimes \cdots \otimes 1_{(F_{n-1},\varepsilon_{n-1})} \otimes 1_{F_n} \), where \( \varepsilon_i \in \{0,1\} \) and \( F_i \) is a face of \( \Delta_i \) for \( i \in \{0,\ldots,n\} \) or the empty set with \( \varepsilon_i = 1 \) if \( i < n \). We set \( |1_{(F,\varepsilon)}| > s = \sum_{i>s}(\dim F_i + \varepsilon_i) \).

**Definition 1.9.** Let \( \ell \in \{1,\ldots,n\} \). The \( \ell \)-perverse degree of \( 1_{(F,\varepsilon)} \in \tilde{N}^* (\Delta) \) is
\[
||1_{(F,\varepsilon)}||_\ell = \begin{cases} -\infty & \text{if } \varepsilon_{n-\ell} = 1, \\ |1_{(F,\varepsilon)}|_{>n-\ell} & \text{if } \varepsilon_{n-\ell} = 0. \\
\end{cases}
\]

For a cochain \( \omega = \sum_b \lambda_b 1_{(F_b,\varepsilon_b)} \in \tilde{N}^*(\Delta) \) with \( \lambda_b \neq 0 \) for all \( b \), the \( \ell \)-perverse degree is
\[
||\omega||_\ell = \max_b ||1_{(F_b,\varepsilon_b)}||_\ell.
\]

By convention, we set \( ||0||_\ell = -\infty \).

Let \((X,\p)\) be a perverse space and \( \sigma : \Delta = \Delta_0 \times \cdots \Delta_n \to X \) a regular simplex. We set \( \tilde{N}_\sigma^* = \tilde{N}^*(\Delta) \). Let \( \delta_i : \Delta' \to \Delta \) be the inclusion of a face, we set \( \delta_i \sigma = \sigma \circ \delta_i : \Delta' \to X \) with the induced filtration \( \Delta' = \Delta_0 \times \cdots \times \Delta_n \).

The **blown-up complex** of \( X \) is the cochain complex \( \tilde{N}^*(X;R) \) composed of the elements \( \omega \) associating to each regular simplex \( \sigma : \Delta_0 \times \cdots \Delta_n \to X \) an element \( \omega_\sigma \in \tilde{N}_\sigma^* \) such that \( \delta_i^* (\omega_\sigma) = \omega_{\delta_i \sigma} \), for any regular face operator \( \delta_i : \Delta' \to \Delta \). The differential \( d\omega \) is defined by \( (d\omega)_s = d(\omega_s) \). The **perverse degree of \( \omega \) along a singular stratum \( S \)** equals
\[
||\omega||_S = \sup \{||\omega_s||_{\codim S} \mid \sigma : \Delta \to X \text{ regular such that } \sigma(\Delta) \cap S \neq \emptyset \}.
\]

By setting \( ||\omega||_S = 0 \) for any regular stratum \( S \), we get a map \( ||\omega|| : S \to \N \).

**Definition 1.10.** A cochain \( \omega \in \tilde{N}^*(X;R) \) is \( \p \)-allowable if \( ||\omega|| \leq \p \) and of \( \p \)-intersection if \( \omega \) and \( d\omega \) are \( \p \)-allowable. We denote \( \tilde{N}_\p^*(X;R) \) the complex of \( \p \)-intersection cochains and \( \p \) its homology, called **blown-up \( \p \)-intersection cohomology** of \( X \).

Let us recall its main properties. First, the canonical projection \( \pr : X \times \R \to X \) induces an isomorphism (\cite[Theorem D]{BOREL-MOORE INTERSECTION HOMOLOGY})
\[
\pr^* : \p(\tilde{N}^*(X;R)) \to \p(X \times \R;R).
\]

Also, if \( L \) is a compact pseudomanifold and \( \p \) a perversity on the cone \( \tilde{c}L \), inducing \( \p \) on \( L \), we have \cite[Theorem E]{BOREL-MOORE INTERSECTION HOMOLOGY}:
\[
\p(\tilde{c}L;R) = \begin{cases} \p(L;R), & \text{if } k \leq \p(v), \\ 0, & \text{if } k > \p(v). \\
\end{cases}
\]

where \( v \) is the apex of the cone. If \( k \leq \p(v) \), the isomorphism \( \p(\tilde{c}L;R) \cong \p(L;R) \) is given by the inclusion \( L \times [0,1] = \tilde{c}L \setminus \{v\} \to \tilde{c}L \).
Definition 1.11. Let \( U \) be an open cover of \( X \). A \( U \)-small simplex is a regular simplex, \( \sigma: \Delta = \Delta_0 \ast \cdots \ast \Delta_n \to X \), such that there exists \( U \in U \) with \( \text{Im} \sigma \subset U \). The blow-up complex of \( U \)-small cochains of \( X \) with coefficients in \( R \), written \( \tilde{N}^* U(X; R) \), is the cochain complex made up of elements \( \omega \), associating to any \( U \)-small simplex, \( \sigma: \Delta = \Delta_0 \ast \cdots \ast \Delta_n \to X \), an element \( \omega_\sigma \in N^*(\Delta) \), so that \( \delta_\ell(\omega_\sigma) = \omega_{0,\sigma} \), for any face operator, \( \delta_\ell: \Delta_0 \ast \cdots \ast \Delta_n \to \Delta_0 \ast \cdots \ast \Delta_n \), with \( \Delta'_n \neq \emptyset \). If \( \overline{p} \) is a perversity on \( X \), we denote by \( \tilde{N}^* U(X; R) \) the complex of \( U \)-small cochains verifying \( \|\omega\| \leq \overline{p} \) and \( \|\delta_\ell \omega\| \leq \overline{p} \).

Proposition 1.12. [6, Corollary 9.7] The restriction map, \( \rho_U: \tilde{N}^* U(X; R) \to \tilde{N}^* U(X; R) \), is a quasi-isomorphism.

Finally, the blow-up intersection cohomology verifies the Mayer-Vietoris property.

Proposition 1.13. [6, Theorem C] Let \((X, \overline{p})\) be a paracompact perverse space, endowed with an open cover \( U = \{W_1, W_2\} \) and a subordinated partition of the unity, \((f_1, f_2)\).

For \( i = 1, 2 \), we denote by \( U_i \) the cover of \( W_i \) consisting of the open subsets \((W_1 \cap W_2, f_i^{-1}(1/2, 1))\) and by \( U \) the cover of \( X \), union of the covers \( U_i \). Then, the canonical inclusions, \( W_i \subset X \) and \( W_1 \cap W_2 \subset W_i \), induce a short exact sequence, where \( \varphi(\omega_1, \omega_2) = \omega_1 - \omega_2 \),

\[
0 \to \tilde{N}^* U(X; R) \to \tilde{N}^* U(W_1; R) \oplus \tilde{N}^* U(W_2; R) \xrightarrow{\varphi} \tilde{N}^* U(W_1 \cap W_2; R) \to 0.
\]

2. Borel-Moore intersection homology

In a topological space \( X \), locally finite chains are sums, perhaps infinite, \( \xi = \sum_{j \in J} \lambda_j \sigma_j \), such that every point in \( X \) has a neighborhood \( U_x \) for which all but a finite number of the regular simplices \( \sigma_j \), with support intersecting \( U_x \), have a coefficient \( \lambda_j \) equal to 0.

Definition 2.1. Let \((X, \overline{p})\) be a perverse space. We denote by \( \mathcal{C}^\infty \overline{p}(X; R) \) the complex of locally finite chains of \( \overline{p} \)-intersection with the differential \( \partial \). Its homology, \( H_*^\infty \overline{p}(X; R) \), is called the locally finite (or Borel-Moore) \( \overline{p} \)-intersection homology.

Recall a characterization of locally finite \( \overline{p} \)-intersection chains.

Proposition 2.2. [5, Proposition 3.4] Let \((X, \overline{p})\) be a perverse space. Suppose that \( X \) is locally compact, metrizable and separable. Then, the complex of locally finite \( \overline{p} \)-intersection chains is isomorphic to the inverse limit of complexes,

\[
\mathcal{C}^\infty \overline{p}(X; R) \cong \lim_{\overset{\longrightarrow}{K \subset X}} \mathcal{C}^\overline{p}(X, X \setminus K; R),
\]

where the limit is taken over all compact subsets of \( X \).

Since locally finite chains in an open subset of \( X \) can be not locally finite in \( X \), the complex \( \mathcal{C}^\infty \overline{p}(-; R) \) is not functorial for the inclusions of open subsets. To get round this defect, as in [10, Remark 2.3.2], we introduce the complex

\[
\mathcal{C}^\infty_X \overline{p}(U; R) := \lim_{\overset{\longrightarrow}{K \subset U}} \mathcal{C}^\overline{p}(X, X \setminus K; R),
\]

where \( K \) runs over the family of compact subsets of \( U \). An element \( \alpha \in \mathcal{C}^\infty_X \overline{p}(U; R) \) is a family \( \alpha = \{\alpha_K\}_K \), indexed by the family of compacts of \( U \), with \( \alpha_K \in \mathcal{C}^\overline{p}(X; R) \).
and \( \alpha_{K'} - \alpha_K \in \mathcal{C}_p(X \setminus K; R) \), if \( K \subseteq K' \). In particular, \( \alpha = 0 \) if, and only if, \( \alpha_K \in \mathcal{C}_p(X \setminus K; R) \) for every \( K \).

For the construction of the projective limit, an exhaustive family of compacts suffices. Therefore, if \( X \) is hemicompact, we may use a numerable increasing sequence of compacts, \( (K_i)_{i \in \mathbb{N}} \), and get \( \alpha = \langle \alpha_i \rangle_i \), with \( \alpha_i \in \mathcal{C}_p(X; R) \) and \( \alpha_{i+1} - \alpha_i \in \mathcal{C}_p(X \setminus K_i; R) \).

Given two open subsets \( V \subset U \subset X \), we denote by

\[
I^X_{V,U}: \lim_{K \subseteq U} \mathcal{C}_p(U, X \setminus K; R) \rightarrow \lim_{K \subseteq V} \mathcal{C}_p(U, X \setminus K; R)
\]

the map induced by the identity. So, the complex \( \mathcal{C}^\infty_{\ast,X}(-; R) \) defines a contravariant functor of domain the category of canonical inclusions between open subsets of \( X \). Moreover, this is an appropriate substitute for the study of locally finite \( \mathcal{I} \)-intersection homology as shows the following result.

**Proposition 2.3.** Let \( (X, \mathcal{I}) \) be a locally compact, second-countable perverse space and \( U \subset X \) an open subset. The natural restriction \( I^U_{V,U}: \mathcal{C}^\infty_{\ast,X}(U; R) \rightarrow \mathcal{C}^\infty_{\ast,X}(U; R) \) is a quasi-isomorphism.

**Proof.** Let \( (K_i)_{i \in \mathbb{N}} \) be a numerable increasing sequence of compacts of \( U \), covering \( U \) and cofinal in the family of compact subsets of \( U \). The maps \( \mathcal{C}_p(U, U \setminus K_i) \rightarrow \mathcal{C}_p(U, U \setminus K_{i+1}) \) and \( \mathcal{C}_p(U, X \setminus K_i) \rightarrow \mathcal{C}_p(U, X \setminus K_{i+1}) \) being surjective, these two sequences verify the Mittag-Leffler condition. Thus the inclusions \( (\mathcal{C}_p(U, U \setminus K_i))_i \rightarrow (\mathcal{C}_p(U, X \setminus K_i))_i \) give a morphism of short exact sequences ([15, Proposition 3.5.8]):

\[
\begin{array}{cccc}
0 \rightarrow & \lim_{K \subseteq U} \mathcal{C}_p(U, U \setminus K_i) & \rightarrow & H_k(\lim_{K \subseteq U} \mathcal{C}_p(U, U \setminus K_i)) \rightarrow \lim_{K \subseteq U} \mathcal{C}_p(U, U \setminus K_i) \rightarrow 0 \\
0 \rightarrow & \lim_{K \subseteq V} \mathcal{C}_p(U, X \setminus K_i) & \rightarrow & H_k(\lim_{K \subseteq V} \mathcal{C}_p(U, X \setminus K_i)) \rightarrow \lim_{K \subseteq V} \mathcal{C}_p(U, X \setminus K_i) \rightarrow 0.
\end{array}
\]

The result is now a consequence of the excision property. \( \square \)

The existence of a Mayer-Vietoris exact sequence in this context can be deduced from a sheaf theoretic argument in the case of perversities depending only on the codimension of strata, as mentioned in [10, Proof of Proposition 2.20]. We provide below a direct proof for general perversities.

**Theorem A.** Let \( (X, \mathcal{I}) \) be a locally compact, second-countable perverse space and \( \{U, V\} \) an open covering of \( X \). Then we have a Mayer-Vietoris exact sequence, with coefficients in \( R \),

\[
\cdots \rightarrow \mathcal{C}_k^\infty(X) \rightarrow \mathcal{C}_k^\infty(V) \oplus \mathcal{C}_k^\infty(U) \rightarrow \mathcal{C}_k^\infty(U \cap V) \rightarrow \mathcal{C}_{k-1}^\infty(X) \rightarrow \cdots
\]

**Proof.** As \( U \) and \( V \) are hemicompact, we choose sequences \( (U_i)_{i \in \mathbb{N}} \) and \( (V_i)_{i \in \mathbb{N}} \) of relatively compact open subsets of \( U \) and \( V \), respectively, such that \( U_i \subset U_{i+1}, \cup_{i \in \mathbb{N}} U_i = U \) and \( V_i \subset V_{i+1}, \cup_{i \in \mathbb{N}} V_i = V \). The maps

\[
\mathcal{C}_p(U_i, X \setminus U_{i+1}) \rightarrow \mathcal{C}_p(U_{i+1}, X \setminus U_{i+1})
\]

induce a short exact sequence for each \( i \). By the excision property, we have

\[
\lim_{i \in \mathbb{N}} \mathcal{C}_p(U_i, X \setminus U_{i+1}) = \mathcal{C}_p(U, X \setminus U) \quad \text{and} \quad \lim_{i \in \mathbb{N}} \mathcal{C}_p(V_i, X \setminus V_{i+1}) = \mathcal{C}_p(V, X \setminus V).
\]

Therefore, the Mayer-Vietoris sequence is a consequence of the excision property. \( \square \)
and \( \overline{V}_i \subset V_{i+1}, \cup_{i \in \mathbb{N}} V_i = V \). Let us notice that \((\overline{U}_i \cup \overline{V}_i)_{i \in \mathbb{N}} \) and \((\overline{U}_i \cap \overline{V}_i)_{i \in \mathbb{N}} \) are sequences of compact subsets such that \( \overline{U}_i \cup \overline{V}_i \subset \overline{U}_{i+1} \cup \overline{V}_{i+1} \) and \( \overline{U}_i \cap \overline{V}_i \subset \overline{U}_{i+1} \cap \overline{V}_{i+1} \) which are exhaustive for \( U \cup V \) and \( U \cap V \) respectively.

As already observed in the proof of Proposition 2.3, the sequences \((\mathcal{C}_k^\mathcal{F}(X, X \backslash K_i))_{i \in \mathbb{N}} \) satisfy the Mittag-Leffler property, for \( K_i = \overline{U}_i, \overline{V}_i, \overline{U}_i \cap \overline{V}_i \) or \( \overline{U}_i \cup \overline{V}_i \). Therefore, the short exact sequences

\[
0 \rightarrow \mathcal{C}_k^\mathcal{F}(X \backslash (\overline{U}_i \cup \overline{V}_i)) \rightarrow \mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \oplus \mathcal{C}_k^\mathcal{F}(X \backslash \overline{V}_i) \rightarrow \mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \rightarrow 0
\]

induce the short exact sequence

\[
0 \rightarrow \lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X \backslash (\overline{U}_i \cup \overline{V}_i)) \rightarrow \lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \oplus \lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X \backslash \overline{V}_i) \rightarrow \lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \rightarrow 0 \tag{2.3}
\]

with \( \Omega_\mathcal{F}^\mathcal{F}(X, \overline{U}_i, \overline{V}_i) = \mathcal{C}_k^\mathcal{F}(X) / (\mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \oplus \mathcal{C}_k^\mathcal{F}(X \backslash \overline{V}_i)) \). The long exact sequence associated to (2.3) gives (2.2). Let us see that.

- First, by Proposition 2.3, we have \( S_k^\infty \mathcal{F}(U \cup V) \cong H_k(\lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X, X \backslash (\overline{U}_i \cup \overline{V}_i))) \), \( S_k^\infty \mathcal{F}(U) \cong H_k(\lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X, X \backslash \overline{U}_i)) \) and \( S_k^\infty \mathcal{F}(V) \cong H_k(\lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X, X \backslash \overline{V}_i)) \).

- As for the third term of the sequence (2.3), in [2, Proposition 4.1], we proved that the identity map on \( X \) induces a quasi-isomorphism, \( \mathcal{C}_k^\mathcal{F}(X \backslash \overline{U}_i) \oplus \mathcal{C}_k^\mathcal{F}(X \backslash \overline{V}_i) \rightarrow \mathcal{C}_k^\mathcal{F}(X \backslash (\overline{U}_i \cap \overline{V}_i)) \). Therefore, it induces a quasi-isomorphism

\[
\Omega_\mathcal{F}^\mathcal{F}(X, \overline{U}_i, \overline{V}_i) \rightarrow \mathcal{C}_k^\mathcal{F}(X, X \backslash (\overline{U}_i \cap \overline{V}_i))
\]

With the Mittag-Leffler property and Proposition 2.3, the identity map also gives a quasi-isomorphism

\[
\psi : \lim_{\longrightarrow \, i} \Omega_\mathcal{F}^\mathcal{F}(X, \overline{U}_i, \overline{V}_i) \rightarrow \lim_{\longrightarrow \, i} \mathcal{C}_k^\mathcal{F}(X, X \backslash (\overline{U}_i \cap \overline{V}_i)) = \mathcal{C}_k^\infty \mathcal{F}(U \cap V). \tag{2.4}
\]

The following properties have been proven in [5].

**Proposition 2.4.** [5, Proposition 3.5] Let \((L, \mathcal{P})\) be a compact perverse space. Then we have

\[
S_k^\infty \mathcal{F}(\mathbb{R}^m \times L; R) = S_{k-m}(L; R).
\]

**Proposition 2.5.** [5, Proposition 3.7] Let \( L \) be a compact space and \( \mathcal{P} \) be a perversity on the cone \( \hat{c}L \) of apex \( v \). Then we have

\[
S_k^\infty \mathcal{F}(\mathbb{R}^m \times \hat{c}L; R) = \begin{cases} 
0 & \text{if } k \leq m + D\mathcal{P}(v) + 1, \\
S_{k-m-1}(L; R) & \text{if } k \geq m + D\mathcal{P}(v) + 2.
\end{cases}
\]

### 3. Poincaré Duality

#### 3.1. Fundamental class and cap product.

Let \((X, \mathcal{P})\) be a perverse pseudomanifold of dimension \( n \).

Recall from [12] that an \( R \)-orientation of \( X \) is an \( R \)-orientation of the manifold \( X^n := X \backslash X_{n-1} \). For any \( x \in X^n \), we denote by \( o_x \in H_n(X^n, X^n \backslash \{x\}; R) = S_{n}(X, X \backslash \{x\}; R) \) the associated local orientation. We know (see [11] or [9, Theorem 8.1.18]) that, for any compact \( K \subset X \), there exists a unique element \( \Gamma^K_X \in S_{n}(X, X \backslash K; R) \) whose restriction
equals $o_\varepsilon$ for any $x \in K$. These classes give a Borel-Moore homology class, called the fundamental class of $X$,

$$\Gamma_X = \langle \Gamma^K_X \rangle_K \in \mathcal{H}_n^\infty(X; R).$$

The fundamental classes are natural for the injections between open subsets of $X$. Given two open subsets $V \subset U \subset X$, the map induced in Borel-Moore homology by the identity

$$\tau_{V,U}^X: \lim_{K \subset U} c^\tau(X, X \setminus K; R) \to \lim_{K \subset V} c^\tau(X, X \setminus K; R) \quad (3.1)$$

sends $\Gamma_U$ on $\Gamma_V$, see [9, Theorem 8.1.18].

Suppose that $X$ is equipped with two perversities, $\mathfrak{p}$ and $\mathfrak{q}$. In [6, Proposition 4.2], we prove the existence of a map

$$\sim - : \tilde{N}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes \tilde{N}^\mathfrak{q}_\mathfrak{q}(X; R) \to \tilde{N}^{k+\ell}_{\mathfrak{p},\mathfrak{q}}(X; R), \quad (3.2)$$

inducing an associative and commutative graded product, called intersection cup product,

$$\sim - : \mathcal{H}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes \mathcal{H}^\mathfrak{q}_\mathfrak{q}(X; R) \to \mathcal{H}^{k+\ell}_{\mathfrak{p},\mathfrak{q}}(X; R). \quad (3.3)$$

Mention also from [6, Propositions 6.6 and 6.7] the existence of cap products,

$$\sim - : \mathcal{H}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes c^\mathfrak{q}_j(X; R) \to c^{p+\mathfrak{q}}_{j-1}(X; R) \quad (3.4)$$

such that $(\eta \sim \omega) \smile \xi = \eta \smile (\omega \sim \xi)$ and $d(\omega \sim \xi) = d\omega \sim \xi + (-1)^{|\omega|}\omega \sim d\xi$. Thus, this cap product induces a cap product in homology,

$$\sim - : \mathcal{H}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes \mathcal{H}^\mathfrak{q}_\mathfrak{q}(X; R) \to \mathcal{H}^{p+\mathfrak{q}}_\mathfrak{p}(X; R). \quad (3.5)$$

The map (3.4) can be extended in a map

$$\sim - : \tilde{N}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes c^\infty_{\mathfrak{q}}(X; R) \to c^{\infty,p+\mathfrak{q}}_{j-1}(X; R) \quad (3.6)$$

as follows:

given $\alpha \in \tilde{N}^\mathfrak{p}_\mathfrak{p}(X; R)$ and $\eta = \langle \eta_K \rangle_K \in c^\infty_{\mathfrak{q}}(X; R)$, we set $\alpha \sim \eta = \langle \alpha \sim \eta_K \rangle_K \in c^{\infty,p+\mathfrak{q}}_{j-1}(X; R)$. This definition makes sense since the cap product (3.4) is natural. Moreover, with the differentials, we get an induced map,

$$\sim - : \mathcal{H}^\mathfrak{p}_\mathfrak{p}(X; R) \otimes \mathcal{H}^\mathfrak{q}_\mathfrak{q}(X; R) \to \mathcal{H}^{\infty,p+\mathfrak{q}}_\mathfrak{p}(X; R). \quad (3.7)$$

As $\Gamma_X \in \mathcal{H}_n^\infty(X; R)$, the cap product with the fundamental class gives a map,

$$\mathcal{D}_X := - \sim \Gamma_X : \mathcal{H}^\mathfrak{p}_\mathfrak{p}(X; R) \to \mathcal{H}^{\infty,p}_n(X; R), \quad (3.8)$$

that is the Poincaré duality map of the next theorem. Let us emphasize that this map exists at the level of chain complexes.

In this paradigm, the Poincaré duality comes from a cap product with the fundamental class, $\Gamma_X$. As this one is a Borel-Moore homology class, we need to adapt (3.4). In fact, there are two ways of working,

- We may keep the blown-up cohomology, for which the cap with $\Gamma_X$ gives a Borel-Moore homology class as in (3.6). This is the subject of this work which brings Theorem B below.
• We may also work with blown-up cohomology with compact supports, for which the cap product with \( \Gamma_X \) gives a (finite) homology class. That is, we extend (3.4) in

\[
- \rightsquigarrow: \tilde{\mathcal{H}}_{\Gamma, c}^i(X; R) \otimes C_j^{\infty,p}(X; R) \to C_{j-i}^{\infty,p}(X; R),
\]

(3.9)

This approach was used in [8] and led us to the Poincaré duality ([8, Theorem B])

\[
- \rightsquigarrow \Gamma_X: \mathcal{H}_{\Gamma, c}^i(X; R) \xrightarrow{\cong} \mathcal{H}_{n-i}^{\infty, p}(X; R).
\]

3.2. Main theorem. We prove the existence of an isomorphism between the Borel-Moore \( \mathcal{P} \)-intersection homology and the blown-up \( \mathcal{P} \)-intersection cohomology.

**Theorem B.** Let \((X, \mathcal{P})\) be an \( n \)-dimensional, second countable and oriented perverse pseudomanifold. The cap product with the fundamental class induces a Poincaré duality isomorphism

\[
\mathcal{D}_X: \mathcal{H}_{\mathcal{P}}^*(X; R) \xrightarrow{\cong} \mathcal{H}_{n-*}^{\infty, \mathcal{P}}(X; R).
\]

The proof uses the following result. (Recall that pseudomanifolds are particular cases of CS sets.)

**Proposition 3.1.** [3, Proposition 13.2] Let \( \mathcal{F}_X \) be the category whose objects are (stratified homeomorphic to) open subsets of a given paracompact and separable CS set \( X \) and whose morphisms are stratified homeomorphisms and inclusions. Let \( \text{Ab} \) be the category of graded abelian groups. Let \( F^*, G^*: \mathcal{F}_X \to \text{Ab} \) be two functors and \( \Phi: F^* \to G^* \) a natural transformation satisfying the conditions listed below.

(i) \( F^* \) and \( G^* \) admit Mayer-Vietoris exact sequences and the natural transformation \( \Phi \) induces a commutative diagram between these sequences,

(ii) If \( \{U_\alpha\} \) is a disjoint collection of open subsets of \( X \) and \( \Phi: F_*(U_\alpha) \to G_*(U_\alpha) \) is an isomorphism for each \( \alpha \), then \( \Phi: F^*(\bigsqcup U_\alpha) \to G^*(\bigsqcup U_\alpha) \) is an isomorphism.

(iii) If \( L \) is a compact filtered space such that \( X \) has an open subset stratified homeomorphic to \( R^i \times \mathcal{L} \) and, if \( \Phi: F^*(R^i \times (\mathcal{L} \setminus \{v\})) \to G^*(R^i \times (\mathcal{L} \setminus \{v\})) \) is an isomorphism, then so is \( \Phi: F^*(R^i \times \mathcal{L}) \to G^*(R^i \times \mathcal{L}) \). Here, \( v \) is the apex of the cone \( \mathcal{L} \).

(iv) If \( U \) is an open subset of \( X \) contained within a single stratum and homeomorphic to an Euclidean space, then \( \Phi: F^*(U) \to G^*(U) \) is an isomorphism.

Then \( \Phi: F^*(X) \to G^*(X) \) is an isomorphism.

**Proof of Theorem B.** As any open subset \( U \subset X \) is an oriented pseudomanifold, we may consider the associated homomorphism defined in (3.8), \( \mathcal{D}_U: \mathcal{H}_{\mathcal{P}}^*(U) \to \mathcal{H}_{n-k}^{\infty, \mathcal{P}}(U) \), where we use the identification \( \mathcal{H}_{\mathcal{P}}^{\infty, \mathcal{P}}(U) \cong \mathcal{H}_{n-k}^{\infty, \mathcal{P}}(U) \) given by Proposition 2.3. Let \( V \subset U \subset X \) be two open subsets of \( X \), endowed with the induced structure of pseudomanifold of \( X \). The canonical inclusion (see (2.1)) and the cap product with the fundamental class give a commutative diagram,
\[\mathcal{H}_P^k(U) \xrightarrow{(\mathcal{L}_U^*)^*} \mathcal{H}_P^k(V)\] (3.10)

We apply Proposition 3.1 to the natural transformation \(\mathcal{D}_U: \mathcal{H}_P^k(U) \to \mathcal{S}_{n-k}^\infty X_P(V)\). The proof is reduced to the verifications of its hypotheses.

- **Property (i).** Let \(\mathcal{U} = \{W_1, W_2\}\) be an open covering of \(X\). Mayer-Vietoris sequences are constructed in Proposition 1.13 and Theorem A. We build a morphism between them with the following diagram.

In the first row, we take over the notations of Proposition 1.13. As in the proof of Theorem A, we choose sequences \((U_i)_{i \in \mathbb{N}}\) and \((V_i)_{i \in \mathbb{N}}\) of relatively compact open subsets of \(W_1\) and \(W_2\), respectively, such that \(\overline{U}_i \subset U_{i+1}\), \(\cup_{i \in \mathbb{N}} U_i = W_1\) and \(\overline{V}_i \subset V_{i+1}\), \(\cup_{i \in \mathbb{N}} V_i = W_2\). The last row corresponds to (2.3).

The maps \(\rho\) denote restriction and the cycles \(\gamma_X \in C_{n-0}^\infty(Y)\) represent the fundamental class of \(Y\), for \(Y = X, W_1, W_2, W_1 \cap W_2\). First, we prove that the above diagram commutes and that the vertical arrows are quasi-isomorphisms.

The square VII is clearly commutative. The vertical maps of squares I, II, V and VI are induced by restrictions. Thus these squares are commutative. By naturality of the fundamental classes, we may choose for \(\gamma_{W_1}\) the restriction of \(\gamma_X\) and similarly for \(\gamma_{W_2}\) and \(\gamma_{W_1 \cap W_2}\). So, diagrams III and IV commute. The diagram VIII is commutative by construction of the map \(\psi\), see (2.4).

Let us observe that the columns of the diagrams I, II, V, VI, VII and VIII are quasi-isomorphisms. This is a consequence of [6, Theorem B], Proposition 2.3 and (2.4) in the
proof of Theorem A, respectively. So, we get the following commutative diagram,

\[
\begin{array}{cccccccc}
\ldots & \mathcal{H}^k_{\mathfrak{F}}(X) & \mathcal{H}^k_{\mathfrak{F}}(W_1) & \mathcal{H}^k_{\mathfrak{F}}(W_2) & \mathcal{H}^k_{\mathfrak{F}}(W_1 \cap W_2) & \mathcal{H}^{k+1}_{\mathfrak{F}}(X) & \ldots \\
\downarrow D_X & \downarrow D_{W_1 \oplus W_2} & \downarrow D_{W_1 \cap W_2} & \downarrow D_X \\
\ldots & \mathcal{I}^{n-k}_{\mathfrak{F}}(X) & \mathcal{I}^{n-k}_{\mathfrak{F}}(W_1) & \mathcal{I}^{n-k}_{\mathfrak{F}}(W_2) & \mathcal{I}^{n-k}_{\mathfrak{F}}(W_1 \cap W_2) & \mathcal{I}^{n-k-1}_{\mathfrak{F}}(X) & \ldots
\end{array}
\]

- Property (iii) We apply (1.1), (1.2), Proposition 2.4 and Proposition 2.5. First, we have the isomorphism \( \mathcal{I}^k_{\mathfrak{F}}(\mathbb{R}^i \times \partial L) = 0 = \mathcal{I}^{n-k}_{\mathfrak{F}}(\mathbb{R}^i \times \partial L) \) for \( k > \mathfrak{F}(v) \), or equivalently \( n - k < i + D\mathfrak{F}(v) + 2 \). (Observe that \( D\mathfrak{F}(v) = n - i - 2 - \mathfrak{F}(v) \).) Next, let \( k \leq \mathfrak{F}(v) \). The following commutative diagram comes from (3.10).

\[
\begin{array}{ccc}
\mathcal{H}^k_{\mathfrak{F}}(\mathbb{R}^i \times \partial L) & \longrightarrow & \mathcal{I}^k_{\mathfrak{F}}(\mathbb{R}^i \times L | 0, 1]) \\
\downarrow D_{\mathbb{R}^i \times \partial L} & & \downarrow D_{\mathbb{R}^i \times L | 0, 1]} \\
\mathcal{I}^{n-k}_{\mathfrak{F}}(\mathbb{R}^i \times \partial L) & \longrightarrow & \mathcal{I}^{n-k}_{\mathfrak{F}}(\mathbb{R}^i \times L | 0, 1]).
\end{array}
\]

The right column is an isomorphism by hypothesis. Since horizontal rows are also isomorphisms, we deduce that \( D_{\mathbb{R}^i \times \partial L} \) is an isomorphism. This proves Property (iii). \( \square \)

Remark 3.2. If \( \mathfrak{F} : \mathbb{N} \to \mathbb{Z} \) is a loose perversity as in [10], we have an inclusion of complexes \( \iota : \mathcal{I}^{\infty}_{\mathfrak{F}}(X; U; R) \hookrightarrow \mathcal{I}^{\infty}_{\mathfrak{F}}(X; R) \), where \( \mathcal{I}^{\infty}_{\mathfrak{F}}(X; R) \) denotes the chain complex studied by Friedman in [10]. With the technique of the proof of Theorem B, using Proposition 3.1, we also deduce that \( \iota \) is a quasi-isomorphism. For the complex \( \mathcal{I}^{\mathfrak{F}}(X; R) \), the existence of a Mayer-Vietoris exact sequence follows from the fact that its homology proceeds from sheaf theory and the computations involving a cone are done in [10, Propositions 2.18 and 2.20].

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