Hamilton–Jacobi mechanics from pseudo-supersymmetry

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Abstract
For a general mechanical system, it is shown that each solution of the Hamilton–Jacobi equation defines an $N=2$ pseudo-supersymmetric extension of the system, such that the usual relation of the momenta to Hamilton’s principal function is the ‘BPS’ condition for preservation of 1/2 pseudo-supersymmetry. The examples of the relativistic and non-relativistic particle, in a general potential, are worked through in detail and used to discuss the relation to cosmology and to supersymmetric quantum mechanics.

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1. Introduction

The reparametrization invariant dynamics of a Hamiltonian system with $s$ degrees of freedom is determined by a Hamiltonian constraint on a $(2s+2)$-dimensional phase space. Given local Darboux coordinates $(X^m; P_m) \ (m = 0, 1, \ldots, s)$, the Lagrangian takes the form

$$ L = \dot{X}^m P_m - \ell \mathcal{H}(X; P), $$

where the overdot indicates differentiation with respect to an arbitrary time parameter, and $\ell$ is a Lagrange multiplier for the Hamiltonian constraint. The Hamilton equations of motion are then

$$ \ell^{-1} \dot{X}^m = \frac{\partial \mathcal{H}}{\partial P_m}, \quad \ell^{-1} P_m = -\frac{\partial \mathcal{H}}{\partial X^m}. $$

The constraint function $\mathcal{H}$ will be assumed to be a polynomial in the momenta $P$ that is at most quadratic. This is the case of most interest and most others can be put in this form by increasing the dimension of the phase space. Thus,

$$ \mathcal{H} = \frac{1}{2} K^{mn}(X) P_m P_n + J^m(X) P_m + U(X), $$

for symmetric tensor field $K(X)$, vector field $J(X)$ and scalar potential function $U(X)$. If $K$ is invertible it may be interpreted as the metric on a target space with local coordinates $X$. The
case in which $K$ is not invertible\textsuperscript{1} is also of interest because this allows for some components of $P$ to appear linearly, but then we must insist that $v \cdot J \neq 0$ for any co-vector field $v(X)$ in the kernel of $K$; in other words
\[ v_m J^m \neq 0 \quad \text{if} \quad K^{mn} v_n = 0. \] (1.4)
Otherwise, there are components of $P$ that do not appear in $\mathcal{H}$, so their conjugate variables are constants and we may rewrite the model in terms of a lower-dimensional phase space.

The action on the constraint surface, viewed as a function of $X$ at a ‘final’ time, is Hamilton’s ‘principal’ function:
\[ S(X) = \int_X dX^m P_m, \] (1.5)
from which we deduce that
\[ P_m = \partial_m S \quad (m = 0, 1, \ldots, s), \] (1.6)
where $\partial_m = \partial/\partial X^m$. These equations typically yield first-order differential equations for $X$ when combined with the equation of motion for $P$, so we will sometimes refer to them as the ‘first-order’ equations of HJ theory. Using (1.6) in the Hamiltonian constraint, we find the Hamilton–Jacobi (HJ) equation
\[ \frac{1}{2} K^{mn} \partial_n S \partial_m S + J^m \partial_m S + U(X) = 0. \] (1.7)
The solutions of this equation are in one-to-one correspondence with solutions of (1.2). This is not obvious, and Hamilton (who found the equation) did not appreciate this point which is due to Jacobi. The main aim of this paper is to present a new derivation of HJ theory with various novel features, one of which is that it makes this feature of the HJ formalism manifest.

The new derivation relies on a correspondence between solutions of the HJ equation for a given mechanical system and $N = 2$ pseudo-supersymmetric extensions of it.

The concept of pseudo-supersymmetry has its origins in supergravity \cite{1, 2}, and can be viewed as a complex analytic continuation of ‘standard’ supersymmetry. Its relevance to the Hamilton–Jacobi approach to inflationary cosmology was noticed in \cite{3, 4}, where the ‘first-order’ equations of the HJ approach to cosmology \cite{5} were recovered from an effective relativistic particle mechanics model, and interpreted as integrability conditions for the existence of ‘pseudo-Killing’ spinors\textsuperscript{2}. This prompted several explicit realizations of pseudo-supersymmetric cosmologies in a (‘variant’) supergravity context, where the pseudo-Killing spinors become ‘genuine’ Killing spinors associated with the partial preservation of a symmetry \cite{7–9}. In this sense, the equations (1.6) for these models may be interpreted as ‘BPS’ conditions, but this interpretation is not intrinsic to the effective particle mechanics model because it involves consideration of spacetime spinors. However, the supergravity examples do suggest the possibility of an intrinsic BPS interpretation, via an explicit pseudo-supersymmetric extension of the effective particle mechanics model. This was the starting point for the work reported here\textsuperscript{3}.

It turns out that the required pseudo-supersymmetric extension of mechanics is not difficult to construct. As will be explained, there is a close relationship to standard supersymmetric

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\textsuperscript{1} The possibility that $K(X)$ is invertible for some $X$ and non-invertible for other $X$ or, more generally, that the dimension of its kernel is position dependent, will not be considered here because neither is it considered in expositions of Hamilton–Jacobi theory.

\textsuperscript{2} Because of the ‘Domain-Wall/Cosmology Correspondence’ \cite{6, 3}, there is an analogous story for domain walls and many of the observations summarized here were first made in that context. However, in this introduction we focus on cosmology for simplicity of presentation.

\textsuperscript{3} A possible alternative starting point might be the effective supersymmetric mechanics models discussed in the context of supersymmetric quantum cosmology, e.g. \cite{10–12}.
mechanics [13], but this suggests a difficulty: supersymmetry implies restrictions on the possible potentials, and one therefore expects the same to be true of pseudo-supersymmetry. In contrast, the HJ formalism does not involve any such restriction. Here, the key observation is that the superpotentials required for pseudo-supersymmetry may be multi-valued functions with branch points [3, 14]. In the context of the non-relativistic particle, the relation of this observation to HJ theory is obvious, and it leads directly to the general pseudo-supersymmetry formalism described here. We present this formalism from a classical perspective, initially, because the intention is to gain insight into standard results of classical mechanics, which we do by focusing on solutions of the equations of motion for which all anti-commuting variables are zero.

In the quantum theory, one cannot set the anti-commuting variables to zero because this is not consistent with their anti-commutation relations. They could be ‘integrated out’ (in a path-integral formulation) but this still implies some contribution to the wavefunction. The standard semi-classical wavefunction has the form $\rho \exp iS$ for some variable modulus $\rho$ determined by quantum fluctuations, so one would expect the anti-commuting variables to contribute to $\rho$. This is indeed the case and the net result is that $\rho$ is constant: the quantum fluctuations of the new anti-commuting variables cancel the quantum fluctuations of the original model. In effect, the quantum theory of the pseudo-supersymmetric mechanics model is equivalent to classical mechanics of the original model! This is reminiscent of the reformulation of classical mechanics of Gozzi et al [15, 16] but the details appear to be rather different; in part because of the central role of the Hamilton–Jacobi equation in the formalism presented here.

1.1. Particle mechanics examples

As an aid to understanding the general formalism to follow, it may help to keep in mind the case of a particle in a space of dimension $s$, which yields various possible mechanical systems with $s$ degrees of freedom. We shall later consider the $s = 1$ case for (i) a relativistic particle and (ii) a non-relativistic particle. When needed, we use the following notation for the components of the phase superspace coordinates:

\[ X^m = (t, x), \quad P_m = (-E, p), \]
\[ \Lambda^m = (\psi, \lambda), \quad \bar{\Lambda}_m = (-\bar{\psi}, \bar{\lambda}). \]  \hspace{1cm} (1.8)

The momentum $P$ is a co-vector in a two-dimensional Minkowski spacetime with signature $(-1, 1)$, and

\[ p^2 \equiv g_{mn}P_mP_n = -E^2 + p^2. \]  \hspace{1cm} (1.9)

The two cases to be considered later are defined as follows:

- **Relativistic particle:**
  \[ \mathcal{H} = \frac{1}{2}P^2 + U(X). \]  \hspace{1cm} (1.10)
  This illustrates the case in which $K$ is non-degenerate and $J = 0$. For $2U = m^2$ we have a free relativistic particle of mass $m$, but we will consider a more complicated potential that is relevant to cosmology.

- **Non-relativistic particle:**
  \[ \mathcal{H} = -E + \frac{1}{2}p^2 + U(t, x). \]  \hspace{1cm} (1.11)
  This illustrates the case in which $K$ is degenerate and $J \neq 0$. For $t$-independent potential we will write $U = V(x)$.
2. The formalism

We begin by supposing that \((X; P)\) phase space is the ‘body’ of a phase superspace with coordinates \((X, \Lambda; P, \bar{\Lambda})\), where the anti-commuting \((s + 1)\)-vector \(\Lambda\) and \((s + 1)\)-covector \(\bar{\Lambda}\) are canonically conjugate. We now allow for an extension of the Hamiltonian constraint function \(H\) on the \((X; P)\) ‘body’ of the phase superspace to a function \(\tilde{H}\) on the full phase superspace, and then consider a Lagrangian of the form:

\[
L = \dot{X}^m P_m + i \dot{\Lambda}_m \bar{\Lambda}^m - \ell \tilde{H} + i \chi Q + i \bar{\chi} \bar{Q},
\]

where \((\chi, \bar{\chi})\) is a pair of real anti-commuting Lagrange multipliers for a pair of constraints with real anti-commuting constraint functions \((Q, \bar{Q})\). The Lagrangian itself is real because we adopt the convention that complex conjugation changes the order of anti-commuting quantities.

It follows from this Lagrangian that the non-zero Poisson brackets of the dynamical variables are,

\[
\{X^m, P_n\}_PB \equiv -\{P_n, X^m\}_PB = \delta^m_n,
\]

\[
\{(\Lambda^m, \bar{\Lambda}_n)\}_PB = \{[\bar{\Lambda}_n, \Lambda^m]_PB = -i\delta^m_n,
\]

and one may use this result to compute the Poisson brackets of the constraint functions, which must be in involution for consistency. We will require that the only non-zero Poisson bracket of constraint functions is

\[
\{Q, \bar{Q}\}_PB = -2iH.
\]

If \(\bar{Q}\) were the complex conjugate of \(Q\) then we would have a standard \(N = 2\) supersymmetric extension of the model defined by (1.1). Instead, both \(Q\) and \(\bar{Q}\) are real, so we have an \(N = 2\) pseudo-supersymmetric extension.

Each of the constraint functions generates a local symmetry. In particular, the infinitesimal pseudo-symmetry variations of any function \(\Phi\) on the phase superspace are given by

\[
\delta_\epsilon \Phi = i[\epsilon Q, \Phi]_PB, \quad \delta_{\bar{\epsilon}} \Phi = i[\bar{\epsilon} \bar{Q}, \Phi]_PB,
\]

where the real anti-commuting parameters \((\epsilon, \bar{\epsilon})\) are arbitrary functions of the arbitrary independent variable.

2.1. Principal function as superpotential

We shall now re-interpret the HJ equation (1.7) as an expression for \(U\) in terms of a superpotential \(S\). We may then rewrite the Hamiltonian constraint function in the factorized form

\[
\mathcal{H} = \frac{1}{2}(K^m(P_m + \partial_m S) + 2J^m(P_m - \partial_m S).
\]

We now introduce a symmetric affine connexion \(\Gamma\) on the coordinate space such that both the tensor \(K\) and the vector \(J\) are covariantly constant:

\[
\partial_p K^mn = -2\Gamma^m_{pq}K^nq, \quad \partial_p J^m = -\Gamma^m_{pq}J^q.
\]

When \(K\) is invertible, and interpreted as a metric, the first of these conditions implies that \(\Gamma\) is the usual Levi-Civita connexion, and the second condition is then a constraint on the allowed choices for \(J\). The curvature tensor constructed from \(\Gamma\) is

\[
R^m_{npq} = 2\partial_q \Gamma^m_{pq} - \Gamma^m_{pq}J^q, \quad \Gamma^\ell_{pn} = 0,
\]

and it satisfies the cyclic identity \(R^m_{[npq]} = 0\).
Now define
\[ P_n = P_n + i \Gamma_{np}^m \Lambda^p \Lambda^m, \]  
(2.8)
and consider the choice
\[ \mathcal{Q} = (\mathcal{P}_m - \partial_m S) \Lambda^m \equiv (P_m - \partial_m S) \Lambda^m \]
\[ \mathcal{Q}' = K^{mn} (P_n + \partial_n S) + 2 J^m \Lambda_m. \]
(2.9)
It is obvious that \( \{ \mathcal{Q}, \mathcal{Q} \}_PB = 0, \) and a computation of \( \{ \mathcal{Q}', \mathcal{Q}' \}_PB \) shows that this is zero too provided that
\[ K^{pq}(P_q - \partial_q S)\Lambda_q \Lambda^p \Lambda^m - \frac{i}{2} K^{pq} (\partial_p S) \Lambda^p \Lambda^m \Lambda_m = 0, \]
(2.10)
which is a consequence of the cyclic identity for invertible \( K. \) Finally, a computation of \( \{ \mathcal{Q}, \mathcal{Q}' \}_PB \) shows that (2.3) holds with
\[ \mathcal{H} = \frac{i}{2} K^{mn} (P_n + \partial_n S) + 2 J^m (P_m - \partial_m S) \quad \text{and} \quad \mathcal{H}' = i K^{mn} (\partial_n S) \Lambda^m \Lambda_m + \frac{1}{2} K^{pq} R^m_{\ell pq} \Lambda^p \Lambda^q \Lambda^m \Lambda_m. \]
(2.11)
As required, \( \mathcal{H} \to \mathcal{H} \) when all anti-commuting variables are set to zero.

To summarize: for every solution \( S \) of the HJ equation of some given mechanical model, we have an \( N = 2 \) pseudo-supersymmetric extension of that model in which Hamilton’s principal function \( S \) is re-interpreted as a superpotential. There is something odd about this result: there was no restriction on the initial choice of potential \( U \) but we are now saying that it should be expressible in terms of a superpotential, so should this not restrict the potential in some way? The sharpest illustration of this ‘paradox’ is provided by the non-relativistic particle constrained to move on the \( x \)-axis in a time-independent potential \( V(x) \). We shall be studying this example in detail; to anticipate, \( N = 2 \) pseudo-supersymmetry implies that \( V \) is given in terms of a superpotential \( W(x) \) by the formula
\[ V = E_0 - \frac{1}{2} (W')^2, \]
(2.12)
where \( E_0 \) is the particle’s energy. Clearly, there are many potentials \( V \) that cannot be written in this form; for example, the harmonic oscillator potential \( V = x^2 \). However, the potential \( V \) is only constrained if we assume that the superpotential \( W \) is defined for all values of \( x \). There is no difficulty if we allow multi-valued superpotentials. In fact, the formula (2.12) is nothing other than the ‘reduced’ Hamilton–Jacobi equation for Hamilton’s characteristic function \( W \) and, as is well known, the characteristic function has branch points at turning points of the motion. Thus, allowing for multi-valued superpotentials, a mechanical model has an \( N = 2 \) pseudo-supersymmetric extension for every solution of its HJ equation.

2.2. The BPS condition

Given a solution of the HJ equation, and hence an \( N = 2 \) pseudo-supersymmetric mechanics, we are faced with the problem of solving the constraints. One obvious way to do this is to set
\[ P = \partial S, \quad \Lambda = 0. \]
(2.13)
Note that these equations are invariant under both pseudo-supersymmetries. This is obvious for \( \mathcal{Q} \) and true for \( \mathcal{Q}' \) because of cancellations in the Poisson bracket with \( (P - \partial S) \).

Another way to solve the constraints is to set
\[ K (P + \partial S) + 2 J = 0, \quad \Lambda = 0. \]
(2.14)

\( ^4 \) This shift of \( P \) ‘covariantizes’ the \( \Lambda \Lambda \) term in the Lagrangian, so all constraint functions become manifestly covariant when expressed in terms of \( P. \)
If $K$ is non-invertible then this implies $v_m J^n = 0$ for $v$ in the kernel of $K$, which contradicts (1.4), so this alternative is viable only if $K$ is invertible, and in this case it is equivalent to

$$\tilde{\mathcal{P}} + K^{-1} J + \partial S = 0, \quad \Lambda = 0,$$

(2.15)

where

$$\tilde{\mathcal{P}} = P + K^{-1} J.$$  

(2.16)

In terms of $\tilde{\mathcal{P}}$ the formula (2.5) becomes

$$\mathcal{H} = \frac{1}{2} (\tilde{\mathcal{P}} + K^{-1} J + \partial S) K (\tilde{\mathcal{P}} - K^{-1} J - \partial S)$$

(2.17)

from which we see that $\mathcal{H}$ is invariant under $\tilde{\mathcal{P}} \rightarrow -\tilde{\mathcal{P}}$. This means that one gets equivalent physics by taking $\tilde{\mathcal{P}} \rightarrow -\tilde{\mathcal{P}}$, but making this transformation in (2.15) we recover the condition $P = \partial S$.

Generically, there are no other ways to solve the constraints. For example, if one tries to consider a combination of the two alternatives just discussed, setting to zero some of the components of $\Lambda$ and the complementary components of $\tilde{\Lambda}$, then one finds that the Hamiltonian constraint is not solved because there remain $\Lambda \tilde{\Lambda}$ terms. This may not happen for special choices of $S$ but then one expects additional symmetries, which plausibly render any additional possibilities equivalent to (2.13). Given this, we conclude that the 'first-order' equations (1.6) are consequences of the pseudo-supersymmetry constraints.

Although, the conditions (2.13) preserve both pseudo-supersymmetries, a generic solution of the equations of motion for $(X, \Lambda)$ will break both of them. To see this, first note that,

$$\delta_{\varepsilon} \Lambda = 2 (K \partial S + J) \varepsilon,$$

(2.18)

for configurations satisfying (2.13), which shows that the $\tilde{Q}$ pseudo-supersymmetry will be broken unless $K \partial S + J = 0$. This condition cannot be met when $K$ is non-invertible, because of (1.4), so in this case the $\tilde{Q}$ pseudo-supersymmetry is broken for all solutions of the equations of motion. Secondly, note that

$$\delta_{\varepsilon} X = -i \varepsilon \Lambda,$$

(2.19)

which shows that the $Q$ pseudo-supersymmetry will be broken unless $\Lambda = 0$.

Our primary interest is in the original model, with $(X; P)$ phase space, and solutions of this model are found from solutions of the pseudo-supersymmetric model by setting

$$\Lambda = 0, \quad \tilde{\Lambda} = 0.$$  

(2.20)

Solutions of the equations of motion that have $\Lambda = \tilde{\Lambda} = 0$ initially will have $\Lambda = \tilde{\Lambda} = 0$ at all times, so we may consistently restrict attention to configurations of this type. In this case, the condition for partial preservation of pseudo-supersymmetry is that the variations of $\Lambda$ and $\tilde{\Lambda}$ vanish for non-zero $\varepsilon$ or non-zero $\tilde{\varepsilon}$. The latter option is either not possible or equivalent to the former, so we assume that $\tilde{\varepsilon} = 0$. The only non-zero pseudo-supersymmetry variation is then

$$\delta_{\varepsilon} \tilde{\Lambda} = (P - \partial S) \varepsilon.$$  

(2.21)

The condition for partial preservation of pseudo-supersymmetry is therefore $P = \partial S$. The ‘first-order’ equation (1.6) of the HJ formalism may thus be interpreted as a ‘BPS’ condition for preservation of 1/2 pseudo-supersymmetry. Note, however, that this interpretation is simply a consequence of the constraints that define the model and the restriction to configurations satisfying (2.20): all solutions of the original mechanical model are BPS solutions of its pseudo-supersymmetric extension.
2.3. Pseudo-superspace

Superfield methods may be used to make the pseudo-supersymmetries manifest. This is easily done for the Q pseudo-supersymmetry because the corresponding ‘supercovariant derivative’ D satisfies $D^2 \equiv 0$, which can be realized as $D = \partial / \partial \theta$ for independent real anti-commuting variable $\theta$. We now interpret $X$ and $\Lambda_1$ as superfields with $\theta$-components:

$$DX^m = \Lambda^m, \quad D\Lambda_m = i(p_m - \partial_m S).$$

(2.22)

We also introduce the ‘fermion number’ superfield

$$N = \Lambda^m \Lambda_m,$$

(2.23)

which has the $\theta$-component

$$DN = -iQ.$$

(2.24)

The other charge $\bar{Q}$, now viewed as a superfield, has $\theta$-component

$$D\bar{Q} = 2i\tilde{H}.$$

(2.25)

Now consider the Lagrangian

$$\tilde{L} = -iD [\dot{X}^m \Lambda_m - \frac{1}{2} \ell \bar{Q} + i\chi N].$$

(2.26)

The Lagrange multipliers $\ell$ and $\chi$ are now superfields with $\theta$-components

$$D\ell = 2\bar{\chi}, \quad D\chi = iq,$$

(2.27)

where $q$ is a new commuting variable. One finds, on omitting a total derivative, that

$$\tilde{L} = L + iqN,$$

(2.28)

where $L$ is the Lagrangian of (2.1). The new variable $q$ therefore imposes a constraint of vanishing ‘fermion’ number. This constraint is satisfied automatically by the solution (2.13) of the pseudo-supersymmetry constraints, which explains why we did not have to consider it previously. Also, it is consistent to set $q = 0$, in which case we recover (2.1) directly.

We have now shown how the Q pseudo-supersymmetry may be made manifest. To do the same for the $\bar{Q}$ supersymmetry, the superfields $X^m$ and $\Lambda_m$ of the above discussion would have to be combined into a single $N = 2$ superfield depending on real anti-commuting variables $\theta$ and $\bar{\theta}$, but this will not be attempted here, in part because it is not obvious how to proceed when $K$ is non-invertible.

2.4. Quantum theory

Let us now consider the quantum theory of the $N = 2$ pseudo-supersymmetric extension of some model of mechanics. In contrast to the viewpoint adopted so far, the additional anti-commuting variables cannot be ‘set to zero’ at the end because they are now operators acting on Hilbert space. In particular, this means that $Q$, $\bar{Q}$ and $\tilde{H}$ become operators (denoted by a ‘hat’) that span a superalgebra for which the only non-zero (anti)commutator is

$$\{\hat{Q}, \hat{\bar{Q}}\} = 2\hat{H}.$$ 

(2.29)

As a consequence, we have the operator identities

$$\hat{Q}^2 \equiv 0, \quad \hat{\bar{Q}}^2 \equiv 0.$$ 

(2.30)

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As is customary, we use the same symbol to denote a superfield and its lowest component. No confusion arises as long as one employs a formalism (as we do) that allows any component equation to be viewed as a superfield equation.
For standard supersymmetry, $\hat{Q}$ is the Hermitian adjoint of $\hat{Q}$. For pseudo-supersymmetry, $\hat{Q}$ and $\hat{Q}$ are independent Hermitian operators, given that $\hat{H}$ is Hermitian. However there are no nilpotent Hermitian operators that act on a Hilbert space with positive definite norm. This might have been anticipated from the non-unitarity of pseudo-supersymmetric field theories since one gets field equations from the first-quantization of particles. However, we set aside this difficulty for the moment.

Upon quantization, the Poisson bracket relations of (2.3) become the (anti) commutation relations

$$[\hat{X}^m, \hat{P}_n] = i\delta^m_n, \quad \{\hat{A}^m, \hat{A}_n\} = \delta^m_n,$$

which we may realize by

$$\hat{X}^m = X^m, \quad \hat{A}^m = A^m, \quad \hat{P}_m = -i\partial_m, \quad \hat{A}_m = \partial/\partial A^m.$$

It may be verified that the following operators are nilpotent:

$$\hat{Q} = -i\Lambda^m (\partial_m - i\partial_m S),$$

$$\hat{Q} = -i \left[ K^{mpn}(\partial_n + i\partial_n S) + 2iJ^m - \Gamma^m_{pq} K^{qn} \partial/\partial A^p \Lambda^p \right] \partial/\partial A^m.$$

Nilpotency of $\hat{Q}$ is manifest and $\hat{Q}$ is nilpotent provided that

$$K^{pq \mid prq} = 0,$$

which, for invertible $K$, is the statement that the Ricci tensor is symmetric. The operator $\hat{H}$ may now be defined by (2.29), and one finds that

$$\hat{H} = \hat{H} + \ldots$$

where the dots indicate terms that are annihilated by $\partial/\partial A$, and

$$\hat{H} = \frac{1}{2} [K^{mn}(\hat{P}_n + \partial_n S) + 2iJ^m](\hat{P}_m - \partial_m S).$$

The constraints are realized in the quantum theory by the following physical-state conditions on the wavefunction $\Psi(X, A)$:

$$\hat{Q}\Psi = 0, \quad \hat{Q}\Psi = 0.$$

These are the only independent physical-state conditions since they imply that $\hat{H}\Psi = 0$. We solve them along the same lines as for the classical theory. Specifically, the classical conditions (2.13) become the following constraints on the wavefunction:

$$\partial_m \Psi = i(\partial_m S)\Psi, \quad \partial\Psi/\partial A^m = 0,$$

for which the solution is

$$\Psi = e^{iS(X)}/0.$$

for constant $\Psi_0$. This is a rather surprising result because it implies that $\Psi$ has constant modulus; this is possible because $\hat{H}$ is not Hermitian, despite the (formal) hermiticity of $\hat{H}$.

Another way of stating the above result is to observe that the effective action, after inclusion of quantum effects, equals the classical action, which means that the anti-commuting variables cancel out the quantum fluctuations of the original variables. In effect, they allow a quantum description of a classical theory. This is what is also achieved by the formalism of Gozzi et al [15, 16] which involves additional anti-commuting variables that are interpreted as ghosts and anti-ghosts associated with a BRST and anti-BRST invariance. The phase superspace in that formalism has dimension $(4s|4s)$ rather than $(2s|2s)$, and there is no special role for the HJ equation, so it is unclear whether there is any connection to the formalism presented here. Nevertheless, the possibility of such a connection suggests that the pseudo-supersymmetry charges introduced here should be interpreted as BRST charges. In this case the superspace expression (2.26) shows that the Lagrangian is BRST exact, and hence that we are dealing with a ‘topological’ theory, as argued in [17] for the formalism of [15, 16].
3. Relativistic particle

Let
\[ Q = (P_m - \delta_m S) \Lambda^m, \quad \bar{Q} = (P_m + \delta_m S) \bar{\Lambda}^m, \] (3.1)
where \( \bar{\Lambda}^m = \eta^{mn} \bar{\Lambda}_n \). The only non-zero Poisson bracket of these functions is that of (2.3) provided that we choose
\[ \mathfrak{H} = \frac{1}{2} \eta^{mn} (P_m + \delta_m S)(P_n - \delta_n S) - i(\delta_m \delta_n S) \Lambda^m \bar{\Lambda}^n. \] (3.2)
This corresponds to a potential given by
\[ U = -\frac{1}{2} (\partial S)^2, \] (3.3)
which is just the HJ equation for our model.

We may solve the pseudo-supersymmetry constraints in two equivalent ways:
\[ P = \partial S, \quad \bar{\Lambda} = 0, \quad \text{or} \quad P = -\partial S, \quad \Lambda = 0. \] (3.4)
For either choice we may restrict to solutions of the equations of motion for which both \( \Lambda = 0 \) and \( \bar{\Lambda} = 0 \), and in this case either the \( Q \) or the \( \bar{Q} \) pseudo-supersymmetry will be preserved, according to whether \( P = \partial S \) or \( P = -\partial S \). The two possibilities are physically equivalent. Both pseudo-supersymmetries are preserved if (in addition) \( P = 0 \), but this is the 2-momentum of the vacuum.

3.1. Application to cosmology

The assumptions of homogeneity and isotropy in a model of gravity coupled to scalar fields with potential \( V \), in \( D \) spacetime dimensions, lead to an effective relativistic particle mechanics model with a Hamiltonian constraint of the form (1.10). In the conventions of [3], the scale factor \( a \) is written as \( \exp(\beta t) \), and the scalar potential of the effective particle mechanics model is
\[ U(t, x) = e^{2\alpha t} V(x) - \frac{k}{2\beta^2} e^t, \] (3.5)
where \( k = -1, 0, 1 \) is the normalized curvature of spatial sections and
\[ \alpha = (D - 1) \beta, \quad \beta = 1/\sqrt{2(D - 1)(D - 2)}. \] (3.6)
There are two cases in which the resulting HJ equation may be solved by a separation of variables. The simplest is \( V = 0 \), in which case
\[ S = \pm(D - 1) e^{t/2\alpha} [e^{t/2\alpha} - k e^{-t/2\alpha}] \quad (V = 0). \] (3.7)

The other case for which the HJ equation may be solved by separation of variables is \( k = 0 \), for which
\[ S = \pm 2 e^{\alpha t} W(x) = 2a^{D-1} W(x) \quad (k = 0), \] (3.8)
where the ‘comoving principal function’ \( W \) is \( t \)-independent and satisfies the ‘reduced HJ equation’ [5]
\[ V = -2[(W')^2 - \alpha^2 W^2] \quad \text{(cosmology)}. \] (3.9)
The equations (1.6) are now
\[ E = \mp 2a e^{\alpha t} W(x), \quad p = \pm 2 e^{\alpha t} W'(x). \] (3.10)
Combining these with the equations of motion for $E$ and $p$ we arrive at the following first-order differential equations:

$$\ell^{-1} \dot{t} = -2\alpha e^{\alpha t} W(x), \quad \ell^{-1} \dot{x} = 2 e^{\alpha t} W'(x).$$ \hfill (3.11)

We have focused on cosmology but exactly the same analysis applies to domain walls. The effective action is the same, except for a flip of the sign of the potential, so an application of HJ theory to flat domain walls \cite{18} leads again to the first-order equations (3.11) but with a ‘reduced HJ equation’ of opposite sign for $V$:

$$V = \frac{2}{3} (W')^2 - \alpha^2 W^2.$$ \hfill (domain wall) \hfill (3.12)

Remarkably, this formula coincides with the ‘supergravity-inspired’ formula for $V$ introduced in \cite{19, 20}; in fact, for $D = 3$ it is the supergravity formula for $V$ in terms of a superpotential $W$. In the context of supergravity domain walls, first-order equations consistent with (3.11) arise as ‘BPS-type’ conditions for the preservation of 1/2 supersymmetry \cite{21, 22}, and they can be found precisely in the HJ form (3.11) by ‘supergravity-inspired’ methods \cite{23, 24}. Specifically, they arise as integrability conditions for the existence of Killing spinors defined in the context of ‘fake supergravity’ \cite{25, 26}. It was pointed out in \cite{3} that this ‘coincidence’ between the HJ and fake supergravity approaches to domain walls also applies to cosmology but the (fake) Killing spinors become (fake) pseudo-Killing spinors, defined by an analytic continuation for which $W \to iW$. As mentioned in the introduction, it has been shown recently that fake pseudo-Killing spinors may be ‘genuine’ Killing spinors of a ‘pseudo-supergravity’ theory, defined by analytic continuation of a standard supergravity theory to the one with spinors obeying ‘twisted reality’ conditions.

3.1.1. Non-flat cosmology. It was shown in \cite{3} that non-flat cosmologies are determined by the following first-order equations involving a complex superpotential $Z$:

$$E = \mp 2\alpha e^{\alpha t} \frac{\text{Re}(\bar{Z}'Z)}{|Z'|}, \quad p = \pm e^{\alpha t} |Z'|.$$ \hfill (3.13)

These equations reduce to those of (3.10) when $Z = W$, which applies when $k = 0$. One may ask what the relation of these equations is to the first-order equations of HJ theory,

$$E = -\partial_t S, \quad p = \partial_x S.$$ \hfill (3.14)

If we try to combine (3.14) with (3.13), we arrive at the equation

$$\pm dS = \omega \equiv 2\alpha e^{\alpha t} \frac{\text{Re}(\bar{Z}'Z)}{|Z'|} dt + e^{\alpha t} |Z'| dx.$$ \hfill (3.15)

However, the 1-form $\omega$ is not closed for $k \neq 0$; using equation (4.16) of \cite{4}, one finds that

$$d\omega = \frac{k}{2\beta} |Z'|^{-1} e^{\alpha t-2\beta r} dr \wedge dx.$$ \hfill (3.16)

It follows that (3.15) cannot be integrated to give a function $S(t, x)$. An expression for $S$ for $k \neq 0$ was found in \cite{4}, but the construction assumed that one is given a solution of the equations of motion, in which case the variables $(t, x)$ are not independent.

For $k = 1$ cosmologies, the ‘first-order’ equations that follow from (3.13) on using the equations of motion for $E$ and $p$ are integrability conditions for the existence of pseudo-Killing spinors, and in this ‘field theoretic’ sense they may be interpreted as ‘BPS’ conditions, as for $k = 0$. Again, there is a parallel story for domain walls, most of which came first; we focus here on cosmology for convenience of presentation, but there is one point that is easier

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6 This property distinguishes these ‘first-order equations from others proposed previously \cite{27}.
to understand from the domain-wall perspective: we do not expect to find Killing spinors for ‘de Sitter sliced’ walls, and this corresponds to the statement that we should not expect to find pseudo-Killing spinors for $k = -1$ cosmologies. This is indeed the case, even though there are first-order equations for any $k$. This state of affairs should be contrasted with the particle mechanics BPS interpretation of the equations (3.14), which applies for all $k$. Moreover, in the particle mechanics BPS interpretation, the superpotential is real for all $k$, being identified with the ‘comoving principal function’ $W$.

These comments suggest that there is no simple general connection between the field-theoretic BPS interpretation of the first-order equations for cosmology that arise from the existence of pseudo-Killing spinors and the ‘intrinsic’ BPS interpretation proposed here in the context of an $N = 2$ pseudo-supersymmetric extension of mechanics.

4. Non-relativistic particle

For a non-relativistic particle in a potential $U(t,x)$, the pseudo-supersymmetric extension is found by choosing

$$Q = \left( p - \frac{\partial S}{\partial x} \right) \lambda + \left( E + \frac{\partial S}{\partial t} \right) \psi, \quad \bar{Q} = \left( p + \frac{\partial S}{\partial x} \right) \bar{\lambda} + 2 \bar{\psi}, \quad (4.1)$$

where $S(t,x)$ is a superpotential. The only non-zero Poisson bracket is that of (2.3) provided that we also choose

$$\mathcal{H} = -E + \frac{1}{2} p^2 + U - i \left( \frac{\partial^2 S}{\partial x^2} \right) \lambda \bar{\lambda} + i \left( \frac{\partial^2 S}{\partial x \partial t} \right) \psi \bar{\lambda}, \quad (4.2)$$

where

$$U = -\frac{\partial S}{\partial t} - \frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2, \quad (4.3)$$

which is the HJ equation of our model.

To satisfy the constraints we set

$$E = -\frac{\partial S}{\partial t}, \quad p = \frac{\partial S}{\partial x} \quad (4.4)$$

and

$$\bar{\psi} = 0, \quad \bar{\lambda} = 0. \quad (4.5)$$

From the infinitesimal pseudo-supersymmetry transformations

$$\delta \psi = -2 \varepsilon, \quad \delta \lambda = \left( p + \frac{\partial S}{\partial x} \right) \varepsilon, \quad \delta \bar{\psi} = -\left( E + \frac{\partial S}{\partial t} \right) \varepsilon, \quad \delta \bar{\lambda} = \left( p - \frac{\partial S}{\partial x} \right) \varepsilon, \quad (4.6)$$

we see that the $\bar{Q}$ pseudo-supersymmetry is necessarily broken. Provided that $\psi = \lambda = 0$, the $Q$ pseudo-supersymmetry is preserved.

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7 This is suggested by a ‘worldline supergravity’ Lagrangian proposed in [28].
4.1. Time-independent potential

It will prove instructive to analyse in more detail the special case for which
\[ S(t, x) = W(x) - E_0 t, \]
for constant \( E_0 \), and a function \( W \) that can be interpreted as Hamilton’s ‘characteristic’ function. In this case we have the \( t \)-independent Hamiltonian constraint function
\[ H = -E + \frac{1}{2} p^2 + V + i W'' \bar{\lambda} \bar{\lambda}, \]
where
\[ V = E_0 - \frac{1}{2} \left( W' \right)^2, \]
which is the ‘reduced’ HJ equation appropriate for a principal function of the assumed form.

The pseudo-supersymmetry constraint functions are now
\[ Q = (p - W') \lambda + (E - E_0) \psi, \quad \bar{Q} = (p + W') \bar{\lambda} + 2 \bar{\psi}, \]
and they generate the following infinitesimal pseudo-supersymmetry transformations:
\[ \delta x = -i(\epsilon \lambda + \bar{\epsilon} \bar{\lambda}), \quad \delta p = -i(\epsilon \lambda - \bar{\epsilon} \bar{\lambda}) W'' \]
\[ \delta t = i \epsilon \psi, \quad \delta E = 0 \]
\[ \delta \lambda = (p + W') \bar{\epsilon}, \quad \delta \bar{\lambda} = (p - W') \epsilon \]
\[ \delta \psi = -2 \bar{\epsilon}, \quad \delta \bar{\psi} = -(E - E_0) \epsilon. \]

The Lagrangian (2.1) transforms into a total derivative under these transformations provided the Lagrange multipliers are assigned the transformations
\[ \delta \chi = \dot{\epsilon}, \quad \delta \bar{\chi} = \dot{\bar{\epsilon}}, \quad \delta \ell = -2i(\epsilon \bar{\chi} + \bar{\epsilon} \chi). \]

To simplify things we will now partially fix the local pseudo-supersymmetry transformations by the gauge choice
\[ \chi = 0, \quad \bar{\chi} = 0. \]
This leaves a residual invariance under the transformations of (4.11) for constant \((\epsilon, \bar{\epsilon})\). We must therefore still impose the constraints \( Q = 0 \) and \( \bar{Q} = 0 \) at some initial time, but they will then hold at all times as a consequence of the equations of motion. The advantage of this gauge choice is that we now have the much simpler Lagrangian
\[ L = \dot{x} p + i \bar{\lambda} \dot{\bar{\lambda}} - \ell \left[ E_0 + \frac{1}{2} p^2 - \frac{1}{2} (W')^2 + i W'' \bar{\lambda} \bar{\lambda} \right] - i \bar{\psi} \dot{\psi} + E(\ell - i). \]

We may further simplify by using the equations of motion of the conjugate pair \((\psi, \bar{\psi})\), which imply that
\[ \psi = \psi_0, \quad \bar{\psi} = \bar{\psi}_0, \]
for anti-commuting constants \((\psi_0, \bar{\psi}_0)\), together with the equations of motion of the conjugate pairs \((t, -E)\):
\[ E = 0, \quad \ell = i. \]

Ultimately, one finds that the constant value of \( E \) must be \( E_0 \) and we shall now assume this in order to shorten the presentation. We now have the even simpler Lagrangian
\[ L = \dot{x} p + i \bar{\lambda} \dot{\bar{\lambda}} - i H, \]
where, from (4.8) and (4.9),
\[ H = \frac{1}{2} p^2 - \frac{1}{2} (W')^2 + i W'' \bar{\lambda} \bar{\lambda}. \]
The pseudo-supersymmetry Noether charges are now
\[
\mathbb{Q} = (p - W')\lambda, \quad \bar{\mathbb{Q}} = (p + W')\bar{\lambda} + 2\psi_0,
\]
and they generate the transformations\(^8\)
\[
\begin{align*}
\delta x &= -i(\epsilon \lambda + \bar{\epsilon} \bar{\lambda}), \\
\delta p &= -i(\epsilon \lambda - \bar{\epsilon} \bar{\lambda})W'', \\
\delta \lambda &= (p + W')\bar{\epsilon}, \\
\delta \bar{\lambda} &= (p - W')\epsilon.
\end{align*}
\]
(4.20)

Together with
\[
\delta \psi_0 = -2\bar{\epsilon}.
\]
(4.21)

Finally, we may fix the time-reparametrization invariance by the gauge choice
\[
i = 1
\]
(4.22)
to arrive at the Lagrangian
\[
L = \dot{x}p + i\bar{\lambda}\dot{\lambda} - \bar{\mathbb{H}},
\]
(4.23)
where \(\bar{\mathbb{H}}\), which is still given by (4.18), may now be interpreted as the Hamiltonian. The gauge choice is again only partial because it leaves a residual invariance under rigid time translations, and this means that we must set \(\bar{\mathbb{H}} = 0\) at some initial time, now as a zero-charge condition which holds at all times as a consequence of the equations of motion. The zero charge conditions may be solved in the same way that we previously solved the phase superspace constraints. We must arrange for \(\mathbb{Q} = 0\) and \(\bar{\mathbb{Q}} = 0\) in such a way that one of the two pseudo-supersymmetries is preserved when all anti-commuting variables are zero.

Although we could omit the transformation (4.21) on the grounds that this is an independent symmetry (trivially since the Lagrangian is independent of \(\psi_0\)) we will retain it for the moment as a relic of the asymmetry between \(\mathbb{Q}\) and \(\bar{\mathbb{Q}}\) that is inherent in the physics of the non-relativistic particle: this asymmetry is due to a choice of positive rather than negative energy in taking the non-relativistic limit. In this case, the symmetry generated by \(\bar{\mathbb{Q}}\) is necessarily broken, so we must preserve the symmetry generated by \(\mathbb{Q}\), and this requires
\[
p = W', \quad \lambda = 0.
\]
(4.24)

We see that the equation relating the momentum to Hamilton’s characteristic function is a BPS condition for 1/2 pseudo-supersymmetry. This should be combined with the equation (4.9) for \(V\) in terms of the superpotential \(W\), which we may rewrite as
\[
W'(x) = \pm\sqrt{2(E_0 - V(x))}
\]
(4.25)
and interpret as the reduced HJ equation for Hamilton’s characteristic function \(W\) in terms of \(V\). There is a solution only for \(E_0 > V\) so if \(V\) is unbounded from above there is no choice of \(E_0\) for which \(W(x)\) is defined for all \(x\). This is of course a well-known result in mechanics. The corresponding result for superpotentials and its implications in the context of supersymmetric domain walls was recently discussed in [14].

4.1.1. Relation to supersymmetric mechanics. If we eliminate \(p\) from the Lagrangian (4.23) by using its equation of motion \(p = \dot{x}\), then we arrive at the equivalent Lagrangian
\[
L = \frac{1}{2}\dot{x}^2 + i\bar{\lambda}\dot{\lambda} + \frac{1}{2}(W')^2 + iW'\bar{\lambda}\bar{\lambda},
\]
(4.26)

\(^8\) The transformation \(\delta t = i\epsilon \psi_0\) is not generated by these simplified Noether charges because of the simplification we made in eliminating the variable \(E\), but this is irrelevant because there is a manifest independent invariance under constant shifts of \(t\) that we may combine with the pseudo-supersymmetry to arrange for \(t\) to be inert.
which is invariant under the $N = 2$ pseudo-supersymmetry transformations
\[
\delta x = -i(\epsilon \lambda + \bar{\epsilon} \bar{\lambda}), \quad \delta \lambda = (\dot{x} + W')\bar{\epsilon}, \quad \delta \bar{\lambda} = (\dot{x} - W')\epsilon. \quad (4.27)
\]
This is similar to a model of $N = 2$ supersymmetric mechanics, but with ‘wrong sign’ potential. In fact, it is related to $N = 2$ supersymmetric mechanics by complex analytic continuation, as we now explain.

Complexify all the dynamical variables $(x, \lambda, \bar{\lambda})$. The result is a complex Lagrangian, but one that depends analytically on them, and it remains invariant under the transformations (4.27) with complexified $(\epsilon, \bar{\epsilon})$, which are now analytic transformations. Clearly, we recover the original Lagrangian on restricting all fields to be real, but we may now ask whether there is some other way to get a real Lagrangian\(^9\). If there is, it must involve an analytic continuation of the real variables of (4.26). Consider the analytic continuation that effects
\[
W \rightarrow iW. \quad (4.28)
\]
This yields the new Lagrangian
\[
L = \frac{1}{2} \dot{x}^2 + i\dot{\lambda} \dot{\bar{\lambda}} - \frac{1}{2} (W')^2 - W''\lambda\bar{\lambda}. \quad (4.29)
\]
This is again real provided we choose $\lambda$ to be complex with complex conjugate $\bar{\lambda}$. The same analytical continuation of the transformations (4.27) yields
\[
\delta x = -i(\epsilon \lambda + \bar{\epsilon} \bar{\lambda}), \quad \delta \lambda = (\dot{x} + iW')\bar{\epsilon}, \quad \delta \bar{\lambda} = (\dot{x} - iW')\epsilon. \quad (4.30)
\]
where $\epsilon$ is now a complex parameter with complex conjugate $\bar{\epsilon}$. This is now a standard $N = 2$ supersymmetric mechanics model with superpotential $W$.

5. Discussion

We have derived the Hamilton–Jacobi formulation of mechanics by considering the possible $N = 2$ locally pseudo-supersymmetric extensions of reparametrization invariant Lagrangians for mechanical systems for which the Hamiltonian is no more than quadratic in momenta, as can usually be arranged. The Hamilton–Jacobi equation arises as a condition for pseudo-supersymmetrizability, with the superpotential taking the role of Hamilton’s principal function; one need not worry about the fact that there may not always be a solution because the superpotential is allowed to be multi-valued, with branch points, and there is no superpotential only when there is no real solution of the Hamilton–Jacobi equation.

The ‘first-order’ equations of the HJ formalism relating the momenta to Hamilton’s principal function arise as solutions to the constraints associated with the local symmetries, and if one sets all anti-commuting variables to zero then these constraints may be interpreted as BPS conditions arising from the preservation of $1/2$ pseudo-supersymmetry. This is very likely related to a ‘field theoretic’ BPS interpretation that arises from consideration of pseudo-Killing spinors in cosmological spacetimes for flat universes, which can be realized in some cases as supersymmetric solutions of pseudo-supergravity theories. However, a comparison with results of this ‘field theoretic’ BPS interpretation for non-flat cosmologies makes it appear unlikely that there is any simple general relation. One possible route to a further investigation of this point would be to extend the considerations of this paper to mechanical models with an infinite number of degrees of freedom. After all, a field theory can be viewed as a model of mechanics on an infinite-dimensional space. From this perspective, the results obtained here should also apply to field theory, although functions such as Hamilton’s principal function then become functionals.

\(^9\) Here we follow the logic presented in [7] for supergravity.
It was stated in the introduction that the pseudo-supersymmetric formulation of HJ theory makes it clear why the HJ equation is equivalent, if taken together with the ‘first-order’ equations that define the momenta, to the Hamilton equations of motion. The reason is that in a reparametrization-invariant formalism, as used here, all the dynamics is encoded in the constraints. Thus, the dynamics of the model defined by (1.1) is encoded in the Hamiltonian constraint function \( \mathcal{H} \). However, to get to the HJ equation we had to use the relations (1.6) for the momenta in terms of the principal function, and this needs a separate motivation (e.g. involving considerations of canonical transformations). In contrast, these relations are implied by the constraints in the pseudo-supersymmetrized model, while the model itself is determined by a solution of the HJ equations. Solutions of the original model are then found as solutions of the pseudo-supersymmetry constraints with vanishing anti-commuting variables, all of which preserve (at least) \( \frac{1}{2} \) of the pseudo-supersymmetry. We thus have a correspondence between solutions of the Hamilton equations (1.2) and solutions obtained by integration of the ‘BPS’ equations, alias the ‘first-order’ equations (1.6), for a given a solution of the HJ equation.

Classical mechanics has been the focus of this work but it is natural to ask whether the anti-commuting variables arising from pseudo-supersymmetry have any role to play in the quantum theory. On one hand, this seems to be ruled out by the impossibility of a realization of the quantum pseudo-symmetry algebra via operators acting on a Hilbert space with positive definite norm. On the other hand, a simple computation of the wavefunction shows that the effect of the anti-commuting variables is to cancel the quantum fluctuations of the original variables, so the model is effectively still classical. This suggests a possible BRST interpretation of the pseudo-supersymmetry algebra, as in the formalism of Gozzi et al [15, 16], which has some features in common with the formalism presented here. It would be satisfying if the two formalisms could be related; in any case, a better understanding of the quantum theory is clearly desirable. In view of the many insights into quantum mechanics provided by supersymmetric quantum mechanics (see e.g. [29]) it seems reasonable to hope that quantum pseudo-supersymmetric mechanics will be useful too.

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References

[1] Pilch K, van Nieuwenhuizen P and Sohnius M F 1985 De Sitter superalgebras and supergravity Commun. Math. Phys. 98 105
[2] de Wit B and Zwartkruis A 1987 SU(2,2|1,1) supergravity and \( N = 2 \) supersymmetry with arbitrary cosmological constant Class. Quantum. Grav. 4 L59
[3] Skenderis K and Townsend P K 2006 Hidden supersymmetry of domain walls and cosmologies Phys. Rev. Lett. 96 191301 (Preprint hep-th/0602260)
Skenderis K and Townsend P K 2007 Pseudo-supersymmetry and the domain-wall/cosmology correspondence J. Phys. A: Math. Gen. 40 6733 (Preprint hep-th/0610253)
[4] Skenderis K and Townsend P K 2006 Hamilton–Jacobi for domain walls and cosmologies Phys. Rev. D 74 125008 (Preprint hep-th/0609056)
[5] Salopek D S and Bond J R 1990 Nonlinear evolution of long wavelength metric fluctuations in inflationary models Phys. Rev. D 42 3936
[6] Cvetič M and Soleng H H 1995 Naked singularities in dilatonic domain wall space times Phys. Rev. D 51 5768 (Preprint hep-th/9411170)
[7] Bergshoeff E A, Hartong J, Ploegh A, Rosseel J and Van den Bleeken D 2007 Pseudo-supersymmetry and a tale of alternate realities J. High Energy Phys. JHEP07(2007)067 (Preprint 0704.3559)

[8] Skenderis K, Townsend P K and Van Proeyen A 2007 Domain-wall/cosmology correspondence in adS/dS supergravity J. High Energy Phys. JHEP08(2007)036 (Preprint 0704.3918)

[9] Vaulà S 2007 Domain wall/cosmology correspondence in (AdS/dS)\textit{n+1} geometries Phys. Lett. B 653 95 (Preprint 0706.1361)

[10] Bene J and Graham R 1994 Supersymmetric homogeneous quantum cosmologies coupled to a scalar field Phys. Rev. D 49 799 (Preprint gr-qc/9306017)

[11] Lidsey J E 1995 Scale factor duality and hidden supersymmetry in scalar–tensor cosmology Phys. Rev. D 52 5407 (Preprint gr-qc/9510017)

[12] Lidsey J E and Vargas Moniz P 2000 Supersymmetric quantization of anisotropic scalar–tensor cosmologies Class. Quantum Grav. 17 4823 (Preprint gr-qc/001073)

[13] Witten E 1981 Dynamical breaking of supersymmetry Nucl. Phys. B 188 513

[14] Sonner J and Townsend P K 2007 Axion-dilaton domain walls and fake supergravity Class. Quantum Grav. 24 3479 (Preprint hep-th/0703276)

[15] Gozzi E 1988 Hidden BRS invariance in classical mechanics Phys. Lett. B 201 525

[16] Gozzi E, Reuter M and Thacker W D 1989 Hidden BRS invariance in classical mechanics: II Phys. Rev. D 40 3363

[17] de Boer J, Verlinde E P and Verlinde H L 2000 On the holographic renormalization group J. High Energy Phys. JHEP08(2000)003 (Preprint hep-th/9912012)

[18] Boucher W 1984 Positive energy without supersymmetry Nucl. Phys. B 242 282

[19] Townsend P K 1984 Positive energy and the scalar potential in higher dimensional supergravity theories Phys. Lett. B 148 55

[20] Cveti ć M, Griffies S and Rey S J 1992 Static domain walls in N = 1 supergravity Nucl. Phys. B 381 301 (Preprint hep-th/9201007)

[21] Cveti ć M and Soleng H P 1997 Supergravity domain walls Phys. Rep. 282 159 (Preprint hep-th/9604090)

[22] Skenderis K and Townsend P K 1999 Gravitational stability and renormalization-group flow Phys. Lett. B 468 46 (Preprint hep-th/9909070)

[23] Freedman D Z, Gubser S S, Pilch K and Warner N P 1999 Renormalization group flows from holography supersymmetry and a c-theorem Adv. Theor. Math. Phys. 3 363 (Preprint hep-th/9904017)

[24] Freedman D Z, Núñez C, Schnabl M and Skenderis K 2004 Fake supergravity and domain wall stability Phys. Rev. D 69 104027 (Preprint hep-th/0312055)

[25] Sonner J and Townsend P K 2006 Dilaton domain walls and dynamical systems Class. Quantum Grav. 23 441 (Preprint hep-th/0510115)

[26] de Boer J, Verlinde E P and Verlinde H L 2000 On the holographic renormalization group J. High Energy Phys. JHEP08(2000)003 (Preprint hep-th/9912012)

[27] Boucher W 1984 Positive energy without supersymmetry Nucl. Phys. B 242 282

[28] Townsend P K 1984 Positive energy and the scalar potential in higher dimensional supergravity theories Phys. Lett. B 148 55

[29] Cooper F, Khare A and Sukhatme U 2001 Supersymmetry in Quantum Mechanics (Singapore: World Scientific)