PRODUCTIVITY OF $[\mu, \lambda]$-COMPACTNESS

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Abstract. We show that if $\mu \leq \text{cf} \lambda$ and $\lambda$ is a strong limit singular cardinal, then $[\mu, \lambda]$-compactness is productive if and only if either $\mu = \omega$, or $\mu$ is $\lambda$-compact.

If $\mu$ and $\lambda$ are infinite cardinals, a topological space is $[\mu, \lambda]$-compact if every open cover by at most $\lambda$ sets has a subcover by less than $\mu$ sets. We say that $[\mu, \lambda]$-compactness is productive if every product of $[\mu, \lambda]$-compact topological spaces is $[\mu, \lambda]$-compact. By a well known theorem of Stephenson and Vaughan [SV, Theorem 1.1], if $\lambda$ is a strong limit singular cardinal, then $[\omega, \lambda]$-compactness (that is, initial $\lambda$-compactness) is productive. We generalize this result to the case when $\mu$ is a strongly compact cardinal, and show that some amount of strong compactness is indeed necessary to obtain productivity.

Among the many possible definitions of strong compactness, we shall use the following one. A cardinal $\mu > \omega$ is said to be $\lambda$-compact if there exists a $\mu$-complete $(\mu, \lambda)$-regular ultrafilter. Recall that an ultrafilter $D$ is $(\mu, \lambda)$-regular provided there exists a family of $\lambda$-many members of $D$ every intersection of $\mu$-many of which is empty. See [L3] for a survey on regularity of ultrafilters, and for applications to topology, as well as to other fields. The cardinal $\mu$ is strongly compact if and only if it is $\lambda$-compact for all cardinals $\lambda$. It is well known that the above definitions are equivalent to the more usual ones (see, e. g., Kanamori and Magidor [KM, Section 15]). Notice that, in the above terminology, a cardinal $\mu$ is measurable if and only if it is $\mu$-compact.

We shall state some results in a general form relative to a class of topological spaces. We say that $[\mu, \lambda]$-compactness is productive for members of a class $\mathcal{K}$ if every product $\prod_{j \in J} X_j$ of $[\mu, \lambda]$-compact topological spaces belonging to $\mathcal{K}$ is $[\mu, \lambda]$-compact. Notice that we are not assuming that $\mathcal{K}$ is closed under products; hence we are only asking that $\prod_{j \in J} X_j$ is $[\mu, \lambda]$-compact, but it does not necessarily belong to $\mathcal{K}$. Notice also that, trivially, if $\mathcal{K}' \subseteq \mathcal{K}$, then productivity
of \([\mu, \lambda]\)-compactness for members of \(K\) implies productivity of \([\mu, \lambda]\)-compactness for members of \(K'\). In particular, for every class \(K\) of topological spaces, productivity of \([\mu, \lambda]\)-compactness implies productivity of \([\mu, \lambda]\)-compactness for members of \(K\).

For every infinite cardinal \(\mu\), let \(C_\mu\) be the class of all infinite regular cardinals \(\mu' < \mu\), each one endowed with the order topology.

Recall that if \(D\) is an ultrafilter over some set \(I\), then a topological space \(X\) is said to be \(D\)-compact if every \(I\)-indexed sequence \((x_i)_{i \in I}\) of elements of \(X\) has some \(D\)-limit point in \(X\), that is, there is a point \(x \in X\) such that \(\{ i \in I \mid x_i \in U \} \in D\), for every open neighborhood \(U\) of \(x\).

\textbf{Theorem 1.} Suppose that \(\omega < \mu \leq \lambda\) are cardinals.

1. If \([\mu, \lambda]\)-compactness is productive (even, only for members of class \(C_\mu\)), then \(\mu\) is a \(\lambda\)-compact cardinal.
2. If \(\mu\) is \(\lambda\)-compact, and \(\lambda\) is a strong limit singular cardinal of cofinality \(\geq \mu\), then \([\mu, \lambda]\)-compactness is productive.

\textbf{Proof.} (1) First, notice that every member of \(C_\mu\) is trivially \([\mu, \lambda]\)-compact. By assumption, every product of members of \(C_\mu\) is \([\mu, \lambda]\)-compact. Applying Caicedo [Ca, Theorem 3.4] to \(T = C_\mu\), we get a \((\mu, \lambda)\)-regular ultrafilter \(D\) such that every member of \(C_\mu\) is \(D\)-compact; that is, every regular cardinal \(\mu' < \mu\), with the order topology, is \(D\)-compact (notice that [Ca] uses a notation in which the order of the cardinals is reversed).

By [L1, Proposition 1], \(D\) is not \((\mu', \mu')\)-regular, for every regular cardinal \(\mu' < \mu\). By standard arguments (see, e. g., [L3, p. 344]), if \(\kappa\) is the least infinite cardinal such that a (non principal) ultrafilter \(D\) is \((\kappa, \kappa)\)-regular, then \(\kappa\) is a regular cardinal, and \(D\) is \(\kappa\)-complete. In the present case, \(\mu \leq \kappa\), hence \(D\) is \(\mu\)-complete. Since \(D\) is also \((\mu, \lambda)\)-regular, then \(\mu\) is \(\lambda\)-compact. We have proved (1).

In order to prove (2), we need the following result, whose proof resembles [L4, Theorem 1].

\textbf{Theorem 2.} If \(X\) is a \([\mu, \lambda]\)-compact topological space, \(D\) is a \(\mu\)-complete ultrafilter over some set \(I\), and \(2^{|I|} \leq \lambda\), then \(X\) is \(D\)-compact.

\textbf{Proof.} Suppose by contradiction that \(X\) is \([\mu, \lambda]\)-compact, \(D\) is a \(\mu\)-complete ultrafilter over \(I\), \(2^{|I|} \leq \lambda\), and \(X\) is not \(D\)-compact. Thus, there is a sequence \((x_i)_{i \in I}\) of elements of \(X\) which has no \(D\)-limit point in \(X\). This means that, for every \(x \in X\), there is an open neighborhood \(U_x\) of \(x\) such that \(\{ i \in I \mid x_i \not\in U_x \} \in D\). For each \(x \in X\), choose some \(U_x\) as above, and let \(Z_x = \{ i \in I \mid x_i \not\in U_x \}\). Thus, \(Z_x \in D\).
For each $Z \in D$, let $V_Z = \bigcup \{ U_x \mid x \text{ is such that } Z_x = Z \}$. Notice that $(V_Z)_{Z \in D}$ is an open cover of $X$, and that if $i \in Z \in D$, then $x_i \not\in V_Z$. Since $|D| \leq 2^{|I|} \leq \lambda$, then, by $[\mu, \lambda]$-compactness, there is $\mu' < \mu$, and there is a sequence $(Z_\beta)_{\beta \in \mu'}$ of elements of $D$ such that $(V_{Z_\beta})_{\beta \in \mu'}$ is a cover of $X$. Since $D$ is $\mu$-complete, $\bigcap_{\beta \in \mu'} Z_\beta \in D$, hence $\bigcap_{\beta \in \mu'} Z_\beta \neq \emptyset$. Choose $i \in \bigcap_{\beta \in \mu'} Z_\beta$. Then $x_i \not\in V_{Z_\beta}$, for every $\beta \in \mu'$, but this contradicts the fact that $(V_{Z_\beta})_{\beta \in \mu'}$ is a cover of $X$. $\square$

We are now able to prove Clause (2) in Theorem 1. So, consider a product $\prod_{j \in J} X_j$ of $[\mu, \lambda]$-compact topological spaces. For every $\nu < \lambda$, since $\mu$ is $\lambda$-compact, there is a $\mu$-complete $(\mu, \nu)$-regular ultrafilter $D$, and we can choose $D$ over $\nu^{< \mu}$ (this is a standard fact, see, e. g., [L3, Property 1.1(ii)], in connection with Form II there). Letting $\nu' = \nu^{< \mu}$, we get $\nu' < \lambda$, since $\lambda$ is strong limit.

Again since $\lambda$ is strong limit, $2^{\nu'} < \lambda$, and, by Theorem 2, every $X_j$ is $D$-compact, hence $\prod_{j \in J} X_j$ is $D$-compact, since $D$-compactness is productive. By [Ca, Lemma 3.1], $\prod_{j \in J} X_j$ is $[\mu, \nu]$-compact, and this holds for every $\nu < \lambda$. Since $\mu \leq \text{cf} \lambda < \lambda$, then $\prod_{j \in J} X_j$ is $[\mu, \text{cf} \lambda]$-compact, in particular, $[\text{cf} \lambda, \lambda]$-compact, and this, together with $[\mu, \nu]$-compactness for every $\nu < \lambda$, easily implies $[\mu, \lambda]$-compactness.

**Corollary 3.** Suppose that there exists no measurable cardinal. If $\mu \leq \lambda$ are infinite cardinals, and $[\mu, \lambda]$-compactness is productive (even, only for members of class $C_\mu$), then $\mu = \omega$.

If $\mu$ is an infinite cardinal, and $(X_j)_{j \in J}$ are topological spaces, the box product $\Box_{j \in J} X_j$ is a topological space defined on the Cartesian product $\prod_{j \in J} X_j$, and a base of $\Box_{j \in J} X_j$ is given by the family of all products $\prod_{j \in J} O_j$ such that $O_j$ is open in $X_j$, for every $j \in J$, and $|\{ j \in J \mid O_j \neq X_j \}| < \mu$. Of course, the box product is the usual Tychonoff product.

We say that $[\mu, \lambda]$-compactness is productive for $\Box_{< \mu}$ products (of members of some class $\mathcal{K}$) if every product $\Box_{j \in J} X_j$ of $[\mu, \lambda]$-compact topological spaces (belonging to $\mathcal{K}$) is $[\mu, \lambda]$-compact.

**Corollary 4.** Suppose that $\mu \leq \lambda$ are infinite cardinals, and that $[\mu, \lambda]$-compactness is productive for members of some class $\mathcal{K} \supseteq C_\mu$. Then:

1. $[\omega, \lambda]$-compactness, too, is productive, for members of $\mathcal{K}$.
2. $[\mu, \lambda]$-compactness is productive for $\Box_{< \mu}$ products of members of $\mathcal{K}$.

**Proof.** (1) is a quite easy corollary of Theorem 1 and of Stephenson and Vaughan’s result. Notice that a space is $[\omega, \lambda]$-compact if and only if
it is both (i) \([\mu, \lambda]\)-compact, and (ii) \([\omega, \mu']\)-compact, for every \(\mu' < \mu\).

Since we have proved in Theorem 1(1) that, under the assumptions of the present corollary, if \(\mu > \omega\), then \(\mu\) is (at least) measurable, and since measurable cardinals are inaccessible, (ii) above can be equivalently replaced by (ii)' \([\omega, \mu']\)-compact, for every singular strong limit \(\mu' < \mu\).

By assumption, \([\mu, \lambda]\)-compactness is productive for members of \(\mathcal{K}\). By Stephenson and Vaughan’s theorem, \([\omega, \mu']\)-compactness is productive (in the class of all topological spaces, hence also for members of \(\mathcal{K}\)), for every singular strong limit \(\mu' < \mu\). Hence \([\omega, \lambda]\)-compactness, being the combination of the above properties productive in \(\mathcal{K}\), is productive, too, in \(\mathcal{K}\).

We now prove (2). Let \(\mathcal{K}'\) be the class of those members of \(\mathcal{K}\) which are \([\mu, \lambda]\)-compact. Applying [Ca, Theorem 3.4] to \(T = \mathcal{K}'\), we get a \((\mu, \lambda)\)-regular ultrafilter \(D\) such that every member of \(\mathcal{K}'\) is \(D\)-compact. Since \(\mathcal{K} \supseteq \mathcal{C}_\mu\), hence also \(\mathcal{K}' \supseteq \mathcal{C}_\mu\), then, by the arguments in the proof of Theorem 1(1), we get that \(D\) is \(\mu\)-complete. It is easy to see that if \(D\) is a \(\mu\)-complete ultrafilter, then every \(\Box_{\mu} <\mu\) product of \(D\)-compact spaces is still \(D\)-compact. In the case at hand, we get that every \(\Box_{\mu} <\mu\) product of members of \(\mathcal{K}'\) is \(D\)-compact, hence \([\mu, \lambda]\)-compact, by [Ca, Lemma 3.1], since \(D\) is \((\mu, \lambda)\)-regular.

As far as we know, no complete characterization is known for those cardinals \(\lambda\) such that initial \(\lambda\)-compactness is productive. A fortiori, we are unable to give a characterization for those pairs of cardinals \(\mu\) and \(\lambda\) such that \([\mu, \lambda]\)-compactness is productive. For sure, as a consequence of the next proposition, we know that the assumption \(\text{cf} \lambda \geq \mu\) in Theorem 1(2) is necessary (unless the existence of strongly compact cardinals is inconsistent).

**Proposition 5.** Suppose that \(\omega \leq \text{cf} \lambda < \mu \leq \lambda\).

Then there is a \([\mu, \lambda]\)-compact \(T_5\) topological space \(X\) such that \(X^\kappa\) fails to be \([\lambda, \lambda]\)-compact, for \(\kappa = 2^{\lambda < \lambda}\).

In particular, \([\mu, \lambda]\)-compactness is not productive.

**Proof.** Let \(X\) be the disjoint union of \(\text{cf} \lambda\) and \(\lambda^+\), both endowed with the order topology. Observe that \(X\) is trivially \([\mu, \lambda]\)-compact, since \(\text{cf} \lambda < \mu\). Suppose, by contradiction, that \(X^\kappa\) is \([\lambda, \lambda]\)-compact. Let \(I\) be the set of all subsets of \(\lambda\) of cardinality \(< \lambda\). Since \(|I| = \lambda^{< \lambda}\), and \(|X| = \lambda^+\), there are \((\lambda^+)^{\lambda^{< \lambda}} = 2^{\lambda^{< \lambda}}\) \(I\)-indexed sequences of elements of \(X\). Choose a sequence \((x_i)_{i \in I}\) of elements of \(X^\kappa\) in such a way that any \(I\)-indexed sequence of elements of \(X\) can be obtained as the projection of \((x_i)_{i \in I}\) onto some factor. Since we are assuming that \(X^\kappa\) is \([\lambda, \lambda]\)-compact, then, by [Ca, Lemma 3.3], there is a \((\lambda, \lambda)\)-regular
ultrafilter $D$ over $I$ such that $(x_i)_{i \in I} \ D$-converges to some element of $X^\kappa$. Since a sequence in a product $D$-converges if and only if every projection converges, we get that every $I$-indexed sequence of elements of $X$ $D$-converges in $X$, that is, $X$ is $D$-compact. By [L2, Corollary 2] (and [L3, Property 1.1(xi)]), $D$ is either $(\text{cf} \, \lambda, \text{cf} \, \lambda)$-regular, or $(\lambda^+, \lambda^+)$-regular. But then, by [Ca, Lemma 3.1], $X$ is either $[\text{cf} \, \lambda, \text{cf} \, \lambda]$-compact, or $[\lambda^+, \lambda^+]$-compact, but neither possibility occurs, thus we reached a contradiction, hence $X^\kappa$ is not $[\lambda, \lambda]$-compact.

The last statement follows from the trivial fact that, for $\mu \leq \lambda$, $[\mu, \lambda]$-compactness implies $[\lambda, \lambda]$-compactness. □

For $\omega < \mu \leq \lambda$, consider the following conditions:

(A) $[\mu, \lambda]$-compactness is productive.

(B) $\mu$ is $\lambda$-compact, and $[\omega, \lambda]$-compactness is productive.

It follows from Theorem [L1(1)] and Corollary [L1(1)] that Condition (A) above implies Condition (B). However, by Proposition [L5] and Stephenson and Vaughan’s Theorem, if $\mu$ is $\lambda$-compact, and $\lambda$ is singular strong limit of cofinality $< \mu$, then (B) $\Rightarrow$ (A) fails.

The present research is partly motivated by [L5]. Let $\mathcal{H}$ be the set of all $(\mu, \lambda)$-regular ultrafilters over $[\lambda]^{< \mu}$. By a remark in [L5], if $[\mu, \lambda]$-compactness is productive, then $\mathcal{H}$ has a minimum, with respect to the Comfort order (see García-Ferreira [GF]).

References

[Ca] X. Caicedo, The Abstract Compactness Theorem Revisited, in Logic and Foundations of Mathematics (A. Cantini et al. editors), Kluwer Academic Publishers (1999), 131–141.

[GF] S. García-Ferreira, Comfort types of ultrafilters, Proc. Amer. Math. Soc. 120 (1994), 1251-1260.

[KM] A. Kanamori, M. Magidor, The evolution of large cardinal axioms in set theory, in: Higher Set Theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977), edited by G. H. Müller and D. S. Scott, Lecture Notes in Math. Vol. 669 (Springer, Berlin, 1978), 99–275.

[L1] P. Lipparini, Productive $[\lambda, \mu]$-compactness and regular ultrafilters, Topology Proc. 21 (1996), 161–171.

[L2] P. Lipparini, Decomposable ultrafilters and possible cofinalities, Notre Dame J. Form. Log. 49 (2008), 307–312.

[L3] P. Lipparini, More on regular and decomposable ultrafilters in ZFC, MLQ Math. Log. Q. 56 (2010), 340-374.

[L4] P. Lipparini, For Hausdorff spaces, $H$-closed = $D$-pseudocompact for all ultrafilters $D$, arXiv:1107.1435 (2011).

[L5] P. Lipparini, Topological spaces compact with respect to a set of filters, arXiv:1210.2120 (2012).

[SV] R. M. Stephenson, Jr., J. E. Vaughan, Products of initially $m$-compact spaces, Trans. Amer. Math. Soc. 196 (1974), 177–189.
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