ON UNIFORMIZATION OF N=2 SUPERCONFORMAL AND N=1 SUPERANALYTIC DEWITT SUPER-RIEMANN SURFACES

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Abstract. We prove a general uniformization theorem for N=2 superconformal and N=1 superanalytic DeWitt super-Riemann surfaces, showing that in general an N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surface is N=2 superconformally (resp., N=1 superanalytically) equivalent to a manifold with transition functions containing no odd functions of the even variable if and only if the first Čech cohomology group of the body Riemann surface with coefficients in the sheaf of holomorphic vector fields over the body is trivial. As a consequence, we give a constructive proof that there is a countably infinite family of N=2 superconformal equivalence classes of N=2 superconformal DeWitt super-Riemann surfaces with genus-zero compact body, and that N=2 superconformal DeWitt super-Riemann surfaces with simply connected body are classified up to N=2 superconformal equivalence by holomorphic line bundles over the underlying body Riemann surface, up to conformal equivalence. In addition, we prove that N=2 superconformal DeWitt super-Riemann surfaces with compact genus-one body and transition functions which correspond to the trivial cocycle in the first Čech cohomology group of the body Riemann surface with coefficients in the sheaf of holomorphic vector fields over the body are classified up to N=2 superconformal equivalence by theta functions associated to the underlying torus up to type modulo the trivial theta functions, or in other words, by holomorphic line bundles over the torus modulo conformal equivalence. We also give the corresponding results for the uniformization of N=1 superanalytic DeWitt super-Riemann surfaces of genus zero or one.

1. Introduction

In this paper, we prove a general uniformization theorem for N=2 superconformal and N=1 superanalytic DeWitt super-Riemann surfaces. In N=2 superconformal field theory, the surfaces swept out by propagating strings with N=2 superconformal symmetry are N=2 superconformal DeWitt super-Riemann surfaces, [F], [DPZ], [FMS], [W], [Ge]. In order to construct an N=2 superconformal field theory, one needs, in particular, a precise description of the moduli space of N=2 superconformal DeWitt super-Riemann surfaces, under N=2 superconformal equivalence. There are two main approaches to supermanifolds, the “concrete” or “DeWitt” approach [F], [DeW], [Ro] and the “ringed-space” approach [L], [M1], [M2]. The DeWitt approach and the ringed-space approach to supermanifolds are equivalent.
if one restricts the supermanifolds in the DeWitt approach to only allow for transition functions which do not include components that are odd functions of an even variable [Bat], [Ro]. However, for applications to superconformal field theory, one needs to include these more general transition functions, which are naturally incorporated using the DeWitt approach. Using the ringed-space approach, in order to incorporate the more general transition functions allowed in the DeWitt approach and in superconformal field theories, one must consider families of ringed-space supermanifolds over a given supermanifold.

The problem of classifying N=2 superconformal super-Riemann surfaces which are restricted to those that have transition functions which do not involve odd functions of an even variable, i.e. those that arise in the the ringed-space approach, has been studied before, in for instance [FaR], [M2], and [BR].

The main result we prove in this paper states that, in general, an N=2 superconformal DeWitt super-Riemann surface with body Riemann surface $M_B$ is N=2 superconformally equivalent to an N=2 superconformal DeWitt super-Riemann surface with transition functions which do not include components that are odd functions of an even variable if and only if the first Čech cohomology of $M_B$ with coefficients in the space of holomorphic vector bundles over $M_B$ is trivial.

As shown in [DRS], N=2 superconformal super-Riemann surfaces are equivalent to N=1 superanalytic super-Riemann surfaces, and thus our results apply to these surfaces as well, under this equivalence. In fact it is interesting to note, that our main uniformization theorem is proved in the N=2 superconformal “homogeneous coordinate setting” where the dependency on the Čech cohomology associated to the underlying body Riemann surface is transparent when one considers consistency conditions of local coordinate transformations on triple overlaps. This dependency is not transparent if one looks at these consistency conditions for N=1 superanalytic super-Riemann surfaces or N=2 superconformal super-Riemann surfaces in the “nonhomogeneous coordinate setting”.

N=1 superanalytic ringed-space manifolds have been classified and studied in for instance [M1] and [M2]. Using our main uniformization theorem which provides criteria for when an N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surface is N=2 superconformally (resp. superanalytically) equivalent to a ringed-space supermanifold, we further study uniformization for genus-zero and certain genus-one supermanifolds and give concrete realizations of the uniformization obtained in the ringed-space approach in the N=1 superanalytic setting. That is, we classify, up to N=2 superconformal (resp. N=1 superanalytic) equivalence, N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces with simply connected body, giving a constructive proof. In addition, we classify N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces with genus-one body and with transition functions restricted to those that do not contain components involving odd functions of an even variable.

In particular, we show that there are unique, up to N=2 superconformal (resp. N=1 superanalytic) equivalence, N=2 superconformal (resp. N=1 superanalytic) DeWitt structures over the complex plane and complex upper-half plane, and a countably infinite number of inequivalent N=2 superconformal (resp. N=1 superanalytic) DeWitt structures over the Riemann sphere. For N=2 superconformal (resp. N=1 superanalytic) DeWitt structures over a complex torus, we show that if we restrict to supermanifolds where the transition functions, in particular, do
not contain components involving odd functions of an even variable, then there is a doubly infinite family of N=2 superconformal (resp. N=1 superanalytic) equivalence classes of genus-one N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces over a given torus, and that the N=2 superconformal (resp. N=1 superanalytic) equivalence classes are determined by theta functions over the underlying lattice defining the complex torus, up to type and up to equivalence with respect to the trivial theta functions.

We then show that our classification of genus-zero N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces and this subclass of genus-one N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces can be restated as follows: N=2 superconformal (resp. N=1 superanalytic) DeWitt super-Riemann surfaces with genus-zero body or with genus-one body and transition functions restricted, in particular, to contain no odd functions of an even variable are classified up to N=2 superconformal (resp. N=1 superanalytic) equivalence by holomorphic line bundles over the underlying Riemann surface up to conformal equivalence.

This work has important implications for N=2 superconformal field theory [BPZ], [F], [DPZ], [EMS], [W], [Ge], and its connections to mathematics through, for instance, the phenomenon of mirror symmetry [GP], [CGP], [GI], [CR], [HKK], and the theory of vertex operator superalgebras [FLM], [KT], [DPZ], [YZ], [LVW], [FPR], [DL], [BS], in addition to having applications, through the theory of vertex algebra, to aspects of number theory [Mi], [TT] and integrable systems [MR], [BR].

This uniformization result is a crucial step in constructing an N=2 superconformal field theory following the work of Huang [H1]–[H9] and Huang and Lepowsky [HL1]–[HL7] in the nonsuper case, and the author [B1]–[B6] in the N=1 superconformal case. The extension of this program to the N=2 superconformal case was initiated in [B7]. The present paper allows the results of [B7] to be put into the context of the description of the entire moduli space of genus-zero N=2 superconformal super-Riemann surfaces. In [B10], we have determined the Lie supergroups of automorphisms of each N=2 superconformal equivalence class of genus-zero N=2 superconformal super-Riemann surface, and this work along with that of the present paper provides the necessary results to proceed in defining the moduli space of N=2 superspheres with tubes and a sewing operation modeling the worldsheet approach to genus-zero two-dimensional N=2 superconformal field theory.

In particular, we note here that in both the nonsuper case and N=1 super case there is a unique genus-zero surface up to conformal or N=1 superconformal equivalence, respectively. However, as proved in this paper, there is a countably infinite number of equivalence classes of N=2 superconformal genus-zero super-Riemann surfaces. Thus, in the N=2 superconformal case, the operad structure on the moduli space of genus-zero worldsheets will have considerably more structure in comparison to the nonsuper and N=1 super cases. Consequently the algebraic structure imposed on the space of particle states by the worldsheet operad structure has considerably more structure in the N=2 superconformal case than in the analogous conformal and N=1 superconformal cases, where the structure of a vertex operator algebra [H1] or N=1 Neveu-Schwarz vertex operator superalgebra [B3], [B4], [B5], respectively, is imposed. In fact, this work indicates that an algebra over the operadic structure of genus-zero N=2 superconformal worldsheets will necessarily
carry much more structure than that of an N=2 Neveu-Schwarz vertex operator superalgebra [BS].

This paper is organized as follows: In Section 2, the basic notions of superfunctions, superanalyticity and N=1 and N=2 superconformality are presented as well as the notion of DeWitt supermanifold and N=2 superconformal super-Riemann surface. As subclasses of these notions, we also present notation for superfunctions and supermanifolds found in the more restricted ringed-space approach. In Section 3, we recall the construction of an equivalence between N=2 superconformal DeWitt super-Riemann surfaces and N=1 superanalytic DeWitt super-Riemann surfaces following [DRS].

In Section 4, we prove the main theorem of this paper, namely that, in general, an N=2 superconformal DeWitt super-Riemann surface with body Riemann surface $M_B$ is N=2 superconformally equivalent to an N=2 superconformal DeWitt super-Riemann surface with transition functions which do not include components that are odd functions of an even variable if and only if the first Čech cohomology of $M_B$ with coefficients in the space of holomorphic vector bundles over $M_B$ is trivial. We then use the equivalence between N=2 superconformal and N=1 superanalytic DeWitt super-Riemann surfaces to state an analogous uniformization theorem for N=1 superanalytic DeWitt super-Riemann surfaces. In Section 4.1, we interpret our uniformization results in terms of certain vector bundles over the the underlying body Riemann surface.

In Section 5, we study N=2 superconformal DeWitt super-Riemann surfaces over the Riemann sphere with certain restricted N=2 superconformal coordinate transformations. We then use the uniformization theorem proved in Section 4, to classify genus-zero N=2 superconformal DeWitt super-Riemann surfaces and give a constructive uniformizing procedure. In particular, we show that compact genus-zero N=2 superconformal DeWitt super-Riemann surfaces are classified up to N=2 superconformal equivalence by a countably infinite family of superspheres, and there is a bijection between these equivalence classes and conformal equivalence classes of holomorphic line bundles over the Riemann sphere. We also state the analogous results for genus-zero N=1 superanalytic DeWitt super-Riemann surfaces.

In Section 6, we first recall some preliminary results about complex tori and theta functions. Then in Section 6.2, we study N=2 superconformal DeWitt super-Riemann surfaces over a fixed complex torus with certain restricted N=2 superconformal coordinate transformations. In particular, we show that there is a doubly infinite family of distinct N=2 superconformal equivalence classes of compact genus-one N=2 superconformal DeWitt super-Riemann surfaces and that these distinct equivalence classes are characterized by theta functions associated to the torus up to type modulo the trivial theta functions, or equivalently by holomorphic line bundles over the underlying complex torus up to conformal equivalence.

In Section 7, we reformulate our results in the “nonhomogeneous coordinate system” as opposed to the “homogeneous coordinate system” we had been using. We use this reformulation to give some intuition as to why our uniformization theorems hold. In particular, we discuss the infinitesimal N=1 and N=2 superconformal coordinate transformations, the presence of a representation of the affine unitary Lie algebra and an action of the $GL(1)$ loop group on the moduli spaces of genus-zero and genus-one N=2 superconformal super-Riemann spheres.
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2. Preliminaries: Superanalytic functions, N=2 superconformal functions and supermanifolds

2.1. Superanalytic functions. In this section, we recall the notion of superalgebra, Grassmann algebra and superanalytic function following, for instance [B7], [DeW], [Rq].

Let \( \mathbb{C} \) denote the complex numbers, let \( \mathbb{Z} \) denote the integers, and let \( \mathbb{Z}_2 \) denote the integers modulo 2. For a \( \mathbb{Z}_2 \)-graded vector space \( V = V^0 \oplus V^1 \), over \( \mathbb{C} \), define the sign function \( | \cdot | \) on the homogeneous subspaces of \( V \) by \( |v| = j \), for \( v \in V^j \) and \( j \in \mathbb{Z}_2 \). If \( |v| = 0 \), we say that \( v \) is even, and if \( |v| = 1 \), we say that \( v \) is odd.

A superalgebra is an (associative) algebra \( A \) (with identity \( 1 \in A \)), such that: (i) \( A \) is a \( \mathbb{Z}_2 \)-graded algebra; (ii) \( ab = (-1)^{\eta(a) \eta(b)} ba \) for \( a, b \) homogeneous in \( A \). Note that when working over a field of characteristic zero or of characteristic greater than two, property (ii), supercommutativity, implies that the square of any odd element is zero.

Let \( V \) be a vector space. The exterior algebra generated by \( V \), denoted \( \wedge(V) \), has the structure of a superalgebra. Let \( \mathbb{N} \) denote the nonnegative integers. For \( L \in \mathbb{N} \), fix \( V_L \) to be an \( L \)-dimensional vector space over \( \mathbb{C} \) with basis \( \{ \zeta_1, \zeta_2, \ldots, \zeta_L \} \) such that \( V_L \subset V_{L+1} \). We denote \( \wedge(V_L) \) by \( \wedge_L \) and call this the Grassmann algebra on \( L \) generators. In other words, from now on we will consider the Grassmann algebras to have a fixed sequence of generators. Note that \( \wedge_L \subset \wedge_{L+1} \), and taking the limit as \( L \to \infty \), we have the infinite Grassmann algebra denoted by \( \wedge_\infty \). Then \( \wedge_L \) and \( \wedge_\infty \) are the associative algebras over \( \mathbb{C} \) with generators \( \zeta_j, \) for \( j = 1, 2, \ldots, L \) and \( j = 1, 2, \ldots, \) respectively, and with relations

\[
\zeta_j \zeta_k = -\zeta_k \zeta_j, \quad \zeta_j^2 = 0.
\]

Note that \( \dim_{\mathbb{C}} \wedge_L = 2^L \), and if \( L = 0 \), then \( \wedge_0 = \mathbb{C} \). We use the notation \( \wedge_\ast \) to denote a Grassmann algebra, finite or infinite. The reason we take \( \wedge_\ast \) to be over \( \mathbb{C} \) is that we will be interested in complex supergeometry. However, formally, we could just as well have taken \( \mathbb{C} \) to be any field of characteristic zero.

Let

\[
J^0_L = \{(j) = (j_1, j_2, \ldots, j_{2n}) \mid j_1 < j_2 < \cdots < j_{2n}, \ j_i \in \{1, 2, \ldots, L\}, \ n \in \mathbb{N}\},
\]

\[
J^1_L = \{(j) = (j_1, j_2, \ldots, j_{2n+1}) \mid j_1 < j_2 < \cdots < j_{2n+1}, \ j_i \in \{1, 2, \ldots, L\}, \ n \in \mathbb{N}\},
\]

and \( J_L = J^0_L \cup J^1_L \). Let \( \mathbb{Z}_+ \) denote the positive integers, and let

\[
J^0_\infty = \{(j) = (j_1, j_2, \ldots, j_{2n}) \mid j_1 < j_2 < \cdots < j_{2n}, \ j_i \in \mathbb{Z}_+, \ n \in \mathbb{N}\},
\]

\[
J^1_\infty = \{(j) = (j_1, j_2, \ldots, j_{2n+1}) \mid j_1 < j_2 < \cdots < j_{2n+1}, \ j_i \in \mathbb{Z}_+, \ n \in \mathbb{N}\},
\]

and \( J_\infty = J^0_\infty \cup J^1_\infty \). We use \( J^0_\ast \), \( J^1_\ast \), and \( J_\ast \) to denote \( J^0_L \) or \( J^0_\infty \), \( J^1_L \) or \( J^1_\infty \), and \( J_L \) or \( J_\infty \), respectively. Note that \( (j) = (j_1, \ldots, j_{2n}) \) for \( n = 0 \) is in \( J^0_\ast \), and we denote
this element by \((\emptyset)\). The \(Z_2\)-grading of \(\Lambda_*\) is given explicitly by
\[
\begin{align*}
\Lambda_*^0 &= \left\{ \sum_{(j) \in J_*^0} a_{(j)} \zeta_{j_1} \cdots \zeta_{j_{2n}} \mid a_{(j)} \in \mathbb{C}, n \in \mathbb{N} \right\}, \\
\Lambda_*^1 &= \left\{ \sum_{(j) \in J_*^1} a_{(j)} \zeta_{j_1} \cdots \zeta_{j_{2n+1}} \mid a_{(j)} \in \mathbb{C}, n \in \mathbb{N} \right\}.
\end{align*}
\]

We can also decompose \(\Lambda_*\) into body, \((\Lambda_*)_B = \{a_{(\emptyset)} \in \mathbb{C}\}\), and soul
\[
(\Lambda_*)_S = \left\{ \sum_{(j) \in J_*\setminus\{\emptyset\}} a_{(j)} \zeta_{j_1} \cdots \zeta_{j_n} \mid a_{(j)} \in \mathbb{C} \right\}
\]
subspaces such that \(\Lambda_* = (\Lambda_*)_B \oplus (\Lambda_*)_S\). For \(a \in \Lambda_*\), we write \(a = a_B + a_S\) for its body and soul decomposition. We will use both notations \(a_B\) and \(a_{(\emptyset)}\) for the body of a supernumber \(a \in \Lambda_*\) interchangeably.

For \(n \in \mathbb{N}\), we introduce the notation \(\Lambda_*^{>n}\) to denote a finite Grassmann algebra \(\Lambda_L\) with \(L > n\) or an infinite Grassmann algebra. We will use the corresponding index notations for the corresponding indexing sets \(J_*^{>n}\), \(J_*^{>n}\), and \(J_*^{>n}\).

Let \(m, n \in \mathbb{N}\), and let \(U\) be a subset of \((\Lambda_*^0)^m \oplus (\Lambda_*^1)^n\). A \(\Lambda_*\)-superfunction \(H\) on \(U\) in \((m, n)\)-variables is given by
\[
H : U \longrightarrow \Lambda_*
\]
\[
(z_1, z_2, \ldots, z_m, \theta_1, \theta_2, \ldots, \theta_n) \mapsto H(z_1, z_2, \ldots, z_m, \theta_1, \theta_2, \ldots, \theta_n)
\]
where \(z_k\), for \(k = 1, \ldots, m\), are even variables in \(\Lambda_*^0\) and \(\theta_k\), for \(k = 1, \ldots, n\), are odd variables in \(\Lambda_*^1\). If \(H\) takes values only in \(\Lambda_*^0\), respectively in \(\Lambda_*^1\), we say that \(H\) is an even, respectively odd, superfunction. Let \(f((z_1)_B, (z_2)_B, \ldots, (z_m)_B)\) be a complex analytic function in \((z_k)_B\), for \(k = 1, \ldots, m\). For \(z_k \in \Lambda_*^0\), and \(k = 1, \ldots, m\), define
\[
f(z_1, z_2, \ldots, z_m) = \sum_{l_1, \ldots, l_m \in \mathbb{N}} \frac{1}{l_1! \cdots l_m!} \frac{\partial^{l_1} \partial^{l_2} \cdots \partial^{l_m}}{\partial (z_1)_B \cdots \partial (z_2)_B \cdots \partial (z_m)_B} f((z_1)_B, (z_2)_B, \ldots, (z_m)_B).
\]

Consider the projection
\[
\pi_B^{(m,n)} : (\Lambda_*^{>n})^m \oplus (\Lambda_*^{>n})^n \longrightarrow \mathbb{C}^m
\]
\[
(z_1, \ldots, z_m, \theta_1, \ldots, \theta_n) \mapsto ((z_1)_B, (z_2)_B, \ldots, (z_m)_B).
\]

**Definition 2.1.** Let \(m, n \in \mathbb{N}\). Let \(U \subseteq (\Lambda_*^{>n})^m \oplus (\Lambda_*^{>n})^n\), and let \(H\) be a \(\Lambda_*^{>n}\)-superfunction in \((m, n)\)-variables defined on \(U\). Then \(H\) is said to be superanalytic if \(H\) is of the form
\[
H(z_1, z_2, \ldots, z_m, \theta_1, \theta_2, \ldots, \theta_n) = \sum_{(j) \in J_\emptyset} \theta_{j_1} \cdots \theta_{j_t} f_{(j)}(z_1, z_2, \ldots, z_m),
\]
where each \(f_{(j)}\) is of the form
\[
f_{(j)}(z_1, z_2, \ldots, z_m) = \sum_{(k) \in J_{-n}} a_{(j), (k)} (z_1, z_2, \ldots, z_m) \zeta_{k_1} \zeta_{k_2} \cdots \zeta_{k_s},
\]
and each $f_{(j),(k)}((z_1)_B, (z_2)_B, \ldots, (z_m)_B)$ is analytic in $(z_1)_B$, for $l = 1, \ldots, m$ and $((z_1)_B, (z_2)_B, \ldots, (z_m)_B) \in U_B = U_B^{(m,n)}(U) \subseteq \mathbb{C}^m$.

We require the even and odd variables to be in $\bigwedge_{*>n-1}$, and we restrict the coefficients of the $f_{(j),(k)}$’s to be in $\bigwedge_{*} \subseteq \bigwedge_{*>n-1}$ in order for the partial derivatives with respect to each of the $n$ odd variables to be well defined and for multiple partials to be well defined (cf. [DeW], [B4], [Ro], [B7]).

**Remark 2.2.** In the language of [Ro], these superanalytic $\bigwedge_{*>n-1}$-superfunctions are called $GC^\omega$ functions on $\mathbb{C}^{m,n}_S$ if $\bigwedge_{*} = \bigwedge_{\infty}$, and are called $GHC^\omega$ functions on $\mathcal{C}^{m,n}_{S[\mathcal{L}]}$ if $\bigwedge_{*} = \bigwedge_{L}$. In the language of [Ro], the class of $HC^\omega$ functions on $\mathbb{C}^{m,n}_S$ are those superanalytic $\bigwedge_{*}$-superfunctions in $(m, n)$-variables for which the coefficient functions $f_1$ are complex analytic, i.e., they take values in $\mathbb{C}$ rather than more generally in $\bigwedge_{\infty}$. In other words, the $HC^\omega$ functions on $\mathbb{C}^{m,n}_S$ are the subclass of $GC^\omega$ functions on $\mathbb{C}^{m,n}_S$ for which $f_{(j),(k)} = 0$ if $(k) \neq (\emptyset)$, and thus are a subclass of the functions we call superanalytic $\bigwedge_{*>n-1}$-superfunction in $(m, n)$-variables. Similarly, the class of $HC^\omega$ functions on $\mathbb{C}^{m,n}_S[\mathcal{L}]$ in [Ro] are those superanalytic $\bigwedge_{L>n-1}$-superfunctions in $(m, n)$-variables for which the coefficient functions $f_1$ are complex analytic, i.e., they take values in $\mathbb{C}$ rather than more generally in $\bigwedge_{L-n}$, and thus are also a subclass of the functions we call superanalytic $\bigwedge_{*>n-1}$-superfunction in $(m, n)$-variables. Often the ringed-space approach to supermanifolds restricts to functions in this subclass $HC^\omega$ of the more general $GC^\omega$.

We define the DeWitt topology on $(\bigwedge^0_{*>n-1})^m \oplus (\bigwedge^1_{*>n-1})^n$ by letting a subset $U$ of $(\bigwedge^0_{*>n-1})^m \oplus (\bigwedge^1_{*>n-1})^n$ be an open set in the DeWitt topology if and only if $U = (f_B^{(m,n)})^{-1}(V)$ for some open set $V \subseteq \mathbb{C}^m$. Note that the natural domain of a superanalytic $\bigwedge_{*>n-1}$-superfunction in $(m, n)$-variables is an open set in the DeWitt topology.

Let $\bigwedge^\times_{*}$ denote the set of invertible elements in $\bigwedge_{*}$. Then

$$\{(\bigwedge_{*})^\times = \{a \in \bigwedge_{*} \mid a_B \neq 0\}$$

since $\frac{1}{a} = \frac{1}{a_B + a_S} = \sum_{n \in \mathbb{N}} \frac{(-1)^n a_S^n}{a_B^n}$ if $a_B \neq 0$.

**Remark 2.3.** Recall that $\bigwedge_{L} \subseteq \bigwedge_{L+1}$ for $L \in \mathbb{N}$, and note that from [B1], any superanalytic $\bigwedge_{L}$-superfunction, $H_L$, in $(m, n)$-variables for $L \geq n$ can naturally be extended to a superanalytic $\bigwedge_{L}$-superfunction in $(m, n)$-variables for $L' > L$ and hence to a superanalytic $\bigwedge_{\infty}$-superfunction. Conversely, if $H'_L$ is a superanalytic $\bigwedge_{L'}$-superfunction (or $\bigwedge_{\infty}$-superfunction) in $(m, n)$-variables for $L' > n$, then we can restrict $H'_L$ to a superanalytic $\bigwedge_{L}$-superfunction for $L' > n$ by restricting $(z_1, \ldots, z_m, \theta_1, \ldots, \theta_n) \in (\bigwedge^0_{L})^m \oplus (\bigwedge^1_{L})^n$ and setting $f_{(j),(k)} = 0$ if $(k) \notin J_{L-n}$. That is, there is a certain degree of functorality in terms of what underlying Grassmann algebra we are working over.

2.2. **Subclasses of $(1, 1)$- and $(1, 2)$-superfunctions and $N=1$ and $N=2$ superconformal functions.** For the purposes of this paper, our focus will be on superanalytic $\bigwedge_{*>n-1}$-superfunctions in $(1, n)$-variables for $n \leq 2$, i.e., the case of one even variable and one or two odd variables. Here we introduce some notation for some of the subclasses of $(1, n)$-superfunctions for $n = 1, 2$. In particular, distinguishing these subclasses will be useful both for stating our results, and for
relating these results to some of the results from the ringed-space approach which is more restrictive than the general approach we take throughout this paper. In addition, we recall the notions of $N=1$ and $N=2$ superconformal functions following, for instance, [CR, DRS, B2, H4, B7].

A superanalytic $(1,n)$-superfunction $H(z, \theta) = (\tilde{z}, \tilde{\theta})$ from a DeWitt open neighborhood in $\Lambda_{>0}^0 \oplus \Lambda_{>0}^1$ to $\Lambda_{>0}^0 \oplus \Lambda_{>0}^1$ is of the form

\begin{align}
\tilde{z} &= f(z) + \theta \xi(z) \\
\tilde{\theta} &= \psi(z) + \theta \phi(z)
\end{align}

for $f, g$ even and $\xi, \psi$ odd superanalytic $(1,0)$-superfunctions in $z$. We will call this class of functions $N=1$ superanalytic functions, and denote the class of such functions by $G_{>0}(1)$. In the ringed-space approach, one does not consider odd functions of an even variable such as $\xi$ and $\psi$. Let $H_{>0}(1)$, be the subclass of $G_{>0}(1)$ consisting of $(1, n)$-superfunctions $H(z, \theta) = (\tilde{z}, \tilde{\theta})$ of the form \((2.5), \tilde{z} = f(z) + \theta \xi(z)\), where $\psi(z)$ and $\xi(z)$ and the soul portion of $f(z)$ are identically zero. That is, $H(z, \theta)$ is a complex analytic function on an open subset of $\mathbb{C}$. Finally, let $C_{>0}(1)$ be the subclass of $H_{>0}(1)$ consisting of $H_{>0}(1)$ functions such that $g(z_B)$ is a complex analytic functions on an open subset of $\mathbb{C}$. Thus we have $C_{>0}(1) \subseteq H_{>0}(1) \subseteq G_{>0}(1)$, and the only time equality holds is in the case when $* = 1$, i.e., when we are working over the Grassmann algebra $\Lambda_1$. In this case, $C_1(1) = H_1(1) = G_1(1)$. In addition, we have the inclusions $G_L(1) \subseteq G_L(1) \subseteq G_{>0}(1)$, for $0 < L < L'$. Similarly, a superanalytic $(1,2)$-superfunction $H(z, \theta_1, \theta_2) = (\tilde{z}, \tilde{\theta}_1, \tilde{\theta}_2)$ from a DeWitt open neighborhood in $\Lambda_{>1}^0 \oplus (\Lambda_{>1}^1)^2$ to $\Lambda_{>1}^0 \oplus (\Lambda_{>1}^1)^2$ is of the form

\begin{align}
\tilde{z} &= f(z) + \theta_1 \xi_1(z) + \theta_2 \xi_2(z) + \theta_1 \theta_2 g(z) \\
\tilde{\theta}_1 &= \psi_j(z) + \theta_1 g_j(z) + \theta_2 h_j(z) + \theta_1 \theta_2 \varphi_j(z)
\end{align}

for $j = 1, 2$, and $f, g, g_j, h_j$ even and $\xi_j, \psi_j, \varphi_j$ odd superanalytic $(1,0)$-superfunctions in $z$. We will call this class of functions $N=2$ superanalytic functions, and denote the class of such functions by $G_{>1}(2)$. Again, since in the ringed-space approach, one does not consider odd functions of an even variable, we let $H_{>1}(2)$, be the subclass of $G_{>1}(2)$ consisting of $(1,2)$-superfunctions $H(z, \theta_1, \theta_2) = (\tilde{z}, \tilde{\theta}_1, \tilde{\theta}_2)$ of the form \((2.7), \tilde{z} = f(z) + \theta_1 \xi_1(z) + \theta_2 \xi_2(z) + \theta_1 \theta_2 g(z)\), where $\psi_j(z)$, $\xi_j(z)$ and $\varphi_j$, for $j = 1, 2$, are identically zero and $f(z_B)$, is a complex analytic function on an open subset of $\mathbb{C}$. Finally, let $C_{>1}(2)$ be the subclass of $H_{>1}(2)$ consisting of $H_{>1}(2)$ functions such that $g(z_B)$, and $g_j(z_B)$ and $h_j(z_B)$, for $j = 1, 2$, are complex analytic functions on an open subset of $\mathbb{C}$. Thus we have $C_{>1}(2) \subseteq H_{>1}(2) \subseteq G_{>1}(2)$, and the only time equality holds is in the case when $* = 2$, i.e., when we are working over the Grassmann algebra $\Lambda_2$. In this case, $C_2(2) = H_2(2) = G_2(2)$. In addition, we have the inclusions $G_L(2) \subseteq G_L(2) \subseteq G_{>2}(2)$, for $0 < L < L'$.

An $N=n$ superconformal function is a superanalytic $(1,n)$-superfunction on a DeWitt open subset of $\Lambda_{>n-1}^0 \oplus (\Lambda_{>n-1}^1)^n$ to $\Lambda_{>n-1}^0 \oplus (\Lambda_{>n-1}^1)^n$ that transforms the superderivations $D_j = \frac{\partial}{\partial \theta_j} + \theta_j \frac{\partial}{\partial \theta_j}$, for $j = 1, \ldots, n$, in a certain “homogeneous” way.

In particular, an $N=1$ superanalytic function $H(z, \theta) = (\tilde{z}, \tilde{\theta})$ of the form \((2.5), \tilde{z} = f(z) + \theta \xi(z)\) transforms $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \theta}$ to $\tilde{D} = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \theta}$ by $D = (D\theta)\tilde{D} + (D\tilde{z} - \theta D\theta)\tilde{D}^2$. We define $H$ to be $N=1$ superconformal if $H$ transforms $D$ homogeneously of degree one. That is, if $H(z, \theta) = (\tilde{z}, \tilde{\theta})$ satisfies $D\tilde{z} - \theta D\tilde{\theta} = 0$. This is equivalent to $H$
having the form

\begin{align}
(2.9) & \quad \ddot{z} = f(z) + \theta g(z) \psi(z) \\
(2.10) & \quad \ddot{\theta} = \psi(z) + \theta g(z)
\end{align}

for \( f, g \) even and \( \psi \) odd superanalytic \((1,0)\)-superfunctions in \( z \), satisfying the condition

\begin{equation}
(2.11) \quad f'(z) = g(z)g(z) - \psi(z)\psi'(z), \quad \text{i.e.,} \quad g^2(z) = f'(z) + \psi(z)\psi'(z).
\end{equation}

Thus an \( N=1 \) superconformal function \( H \) is uniquely determined by the superanalytic functions \( f(z) \) and \( \psi(z) \) and a choice of square root for \( 2.11 \).

In the \( N=2 \) superconformal setting there are generally two different coordinate systems commonly used. We will first work in what we call the “nonhomogeneous” coordinate setting (see [B7] and [B8]), and then translate to the “homogeneous” coordinate system.

An \( N=2 \) superanalytic function \( H(z, \theta_1, \theta_2) = (\ddot{z}, \ddot{\theta}_1, \ddot{\theta}_2) \) of the form \( 2.7 \)–\( 2.8 \) transforms \( D_1 \) and \( D_2 \) by

\begin{align}
D_1 &= (D_1 \ddot{\theta}_1) \hat{D}_1 + (D_1 \ddot{\theta}_2) \hat{D}_2 + \left( D_1 \dddot{\ddot{z}} \hat{\theta}_1 D_1 \ddot{\theta}_1 - \hat{\theta}_2 D_1 \ddot{\theta}_2 \right) \hat{D}_1^2 \\
D_2 &= (D_2 \ddot{\theta}_1) \hat{D}_1 + (D_2 \ddot{\theta}_2) \hat{D}_2 + \left( D_2 \dddot{\ddot{z}} \hat{\theta}_1 D_2 \ddot{\theta}_1 - \hat{\theta}_2 D_2 \ddot{\theta}_2 \right) \hat{D}_2^2.
\end{align}

We define \( H \) to be \( N=2 \) superconformal if it transforms \( D_1 \), respectively \( D_2 \), as

\begin{align}
(2.12) & \quad D_1 \ddot{\theta}_1 - D_2 \ddot{\theta}_2 = D_1 \ddot{\theta}_2 + D_2 \ddot{\theta}_1 = 0 \\
(2.13) & \quad D_1 \dddot{\ddot{z}} - \hat{\theta}_1 D_1 \ddot{\theta}_1 - \hat{\theta}_2 D_1 \ddot{\theta}_2 = 0 \\
(2.14) & \quad D_2 \dddot{\ddot{z}} - \hat{\theta}_1 D_2 \ddot{\theta}_1 - \hat{\theta}_2 D_2 \ddot{\theta}_2 = 0.
\end{align}

These conditions \( 2.12 \)–\( 2.14 \) imply that an \( N=2 \) superconformal function in the nonhomogeneous coordinate system \( H(z, \theta_1, \theta_2) = (\ddot{z}, \ddot{\theta}_1, \ddot{\theta}_2) \) is of the form

\begin{align}
(2.15) & \quad \ddot{z} = f(z) + \theta_1 (g_1(z) \psi_1(z) + g_2(z) \psi_2(z)) + \theta_2 (g_1(z) \psi_2(z) - g_2(z) \psi_1(z)) \\
& \quad - \theta_1 \theta_2 (\psi_1(z) \psi_2(z))' \\
(2.16) & \quad \ddot{\theta}_1 = \psi_1(z) + \theta_1 g_1(z) - \theta_2 g_2(z) + \theta_1 \theta_2 (\psi_2(z))' \\
(2.17) & \quad \ddot{\theta}_2 = \psi_2(z) + \theta_1 g_2(z) + \theta_2 g_1(z) - \theta_1 \theta_2 (\psi_1(z))',
\end{align}

satisfying

\begin{equation}
(2.18) \quad f'(z) = g_1^2(z) + g_2^2(z) - \psi_1(z) (\psi_1(z))' - \psi_2(z) (\psi_2(z))',
\end{equation}

for even superanalytic \((1,0)\)-superfunctions \( f, \theta_1 \) and \( \psi_1 \) and odd superanalytic \((1,0)\)-superfunctions \( g_1, \theta_2 \). \( \psi_2 \).

It will be convenient for us to work in the “homogeneous” coordinate system denoted by even variable \( z \) and odd variables \( \theta^+ \) and \( \theta^- \), where

\begin{equation}
(2.19) \quad \theta^\pm = \frac{1}{\sqrt{2}} (\theta_1 \pm i \theta_2).
\end{equation}

or equivalently

\begin{equation}
(2.20) \quad \theta_1 = \frac{1}{\sqrt{2}} (\theta^+ + \theta^-) \quad \text{and} \quad \theta_2 = -\frac{i}{\sqrt{2}} (\theta^+ - \theta^-).
\end{equation}
This is a standard transformation in N=2 superconformal field theory (cf. [DRS], [B7], [B8]), however the nomenclature “homogeneous” for the \((z, \theta^+, \theta^-)\) coordinate system and “nonhomogeneous” for the \((z, \theta_1, \theta_2)\) coordinate system was first introduced in by the author in [B7]. In Remark 2.4 we give some reasons for this terminology. Other reasons for this nomenclature involve the algebra of infinitesimal N=2 superconformal transformations and are discussed in Remark 7.1 below as well as in [B7].

We have that
\[
\frac{\partial}{\partial \theta_1} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta^+} + \frac{\partial}{\partial \theta^-} \right) \quad \text{and} \quad \frac{\partial}{\partial \theta_2} = \frac{i}{\sqrt{2}} \left( \frac{\partial}{\partial \theta^+} - \frac{\partial}{\partial \theta^-} \right)
\]
or equivalently
\[
\frac{\partial}{\partial \theta^\pm} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_1} \mp i \frac{\partial}{\partial \theta_2} \right).
\]

Define
\[
(2.21) \quad D^\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\mp \frac{\partial}{\partial z} = \frac{1}{\sqrt{2}} (D_1 \mp i D_2).
\]

Note that
\[
(2.22) \quad [D^\pm, D^\pm] = 2(D^\pm)^2 = 0
\]
\[
(2.23) \quad [D^+, D^-] = D^+ D^- + D^- D^+ = 2 \frac{\partial}{\partial z}.
\]

Let \(H(z, \theta^+, \theta^-) = (\hat{z}, \hat{\theta}^+, \hat{\theta}^-)\) be an N=2 superanalytic function in the homogeneous coordinate system from a DeWitt open neighborhood in \(\Lambda^0_{n>1} \oplus (\Lambda^1_{n>1})^2\) to \(\Lambda^0_{n>1} \oplus (\Lambda^1_{n>1})^2\), i.e., \(\hat{z}\) is an even superanalytic \((1,2)\)-superfunction and \(\hat{\theta}^\pm\) are odd superanalytic \((1,2)\)-superfunctions. Then \(D^+\) and \(D^-\) transform under \(H(z, \theta^+, \theta^-)\) by
\[
(2.24) \quad D^\pm = (D^\pm \hat{\theta}^\pm) \hat{D}^\pm + (D^\pm \hat{\theta}^\mp) \frac{\partial}{\partial \hat{\theta}^\mp} + (D^\pm \hat{z} - \hat{\theta}^\mp D^\pm \hat{\theta}^\pm) \frac{\partial}{\partial \hat{z}}.
\]

An N=2 superanalytic function \(H\) in the homogeneous coordinate system transforms \(D^+\) and \(D^-\) transform homogeneously of degree one. That is, \(H\) transforms \(D^\pm\) by non-zero superanalytic functions times \(\hat{D}^\pm\), respectively. Since such a superanalytic function \(H(z, \theta^+, \theta^-) = (\hat{z}, \hat{\theta}^+, \hat{\theta}^-)\) transforms \(D^+\) and \(D^-\) according to (2.24), \(H\) is superconformal if and only if, in addition to being superanalytic, \(H\) satisfies
\[
(2.25) \quad D^\pm \hat{\theta}^\mp = 0,
\]
\[
(2.26) \quad D^\pm \hat{z} - \hat{\theta}^\mp D^\pm \hat{\theta}^\pm = 0,
\]
for \(\hat{D}^\pm \hat{\theta}^\pm\) not identically zero, thus transforming \(D^\pm\) by \(D^\pm = (D^\pm \hat{\theta}^\pm) \hat{D}^\pm\). These conditions imply that we can write \(H(z, \theta^+, \theta^-) = (\hat{z}, \hat{\theta}^+, \hat{\theta}^-)\) as
\[
(2.27) \quad \hat{z} = f(z) + \theta^+ g^+(z) \psi^-(z) + \theta^- g^-(z) \psi^+(z) + \theta^+ \theta^- (\psi^+(z) \psi^-(z))',
\]
\[
(2.28) \quad \hat{\theta}^\pm = \psi^\pm(z) + \theta^\pm g^\pm(z) \pm \theta^+ \theta^- (\psi^\pm(z))',
\]
for \(f, g^\pm\) even and \(\psi^\pm\) odd superanalytic \((1,0)\)-superfunctions in \(z\), satisfying the condition
\[
(2.29) \quad f'(z) = (\psi^+)'(z) \psi^-(z) - \psi^+(z) (\psi^-)'(z) + g^+(z) g^-(z),
\]
and we also require that $D^+\tilde{\theta}^+$ and $D^-\tilde{\theta}^-$ not be identically zero. Thus an $N=2$ superconformal function $H$ is uniquely determined by the superanalytic functions $f(z)$, $\psi^\pm(z)$, and $g^\pm(z)$ satisfying the condition (2.29).

Note that, transforming between homogeneous and nonhomogeneous coordinate systems for an $N=2$ superconformal functions given by (2.27)–(2.29), or equivalently (2.15)–(2.18), we have that $\psi^\pm(z) = \frac{1}{\sqrt{2}}(\psi_1(z) \pm i\psi_2(z))$ and $g^\pm(z) = g_1(z) \pm ig_2(z)$.

**Remark 2.4.** From the properties derived above for an $N=2$ superconformal function in the nonhomogeneous coordinate system, we see one of the reasons for our terminology. Namely, that in the nonhomogeneous coordinate system an $N=2$ superconformal function does not transform the superderivations $D_1$ and $D_2$, respectively, homogeneously of degree one. Instead it transforms them as $D_1 = (D_1\theta_1)D_1 + (D_1\theta_2)D_2$ and $D_2 = (D_2\theta_1)D_1 + (D_2\theta_2)D_2$, respectively — unlike the homogeneous nature of the transformation of $D^\pm$ under an $N=2$ superconformal function in the homogeneous coordinates. In the latter case the superderivations transform homogeneously as $D^\pm = (D^+\tilde{\theta}^+)D^\pm$.

2.3. Complex DeWitt supermanifolds and $N=2$ superconformal DeWitt super-Riemann surfaces. A DeWitt $(m,n)$-dimensional supermanifold over $\Lambda^*_\ast$ is a topological space $X$ with a countable basis which is locally homeomorphic to an open subset of $(\Lambda^0_\ast)^m \oplus (\Lambda^1_\ast)^n$ in the DeWitt topology. A DeWitt $(m,n)$-chart on $X$ over $\Lambda^*_\ast$ is a pair $(U,\Omega)$ such that $U$ is an open subset of $X$ and $\Omega$ is a homeomorphism of $U$ onto an open subset of $(\Lambda^0_\ast)^m \oplus (\Lambda^1_\ast)^n$ in the DeWitt topology. A superanalytic atlas of DeWitt $(m,n)$-charts on $X$ over $\Lambda^*_{>n-1}$ is a family of charts $\{(U_\alpha,\Omega_\alpha)\}_{\alpha \in A}$ satisfying

(i) Each $U_\alpha$ is open in $X$, and $\bigcup_{\alpha \in A} U_\alpha = X$.

(ii) Each $\Omega_\alpha$ is a homeomorphism from $U_\alpha$ to a (DeWitt) open set in $(\Lambda^0_{>n-1})^m \oplus (\Lambda^1_{>n-1})^n$, such that $\Omega_\alpha \circ \Omega^{-1}_\beta : U_{\alpha \cap U_\beta} \to \Omega_\alpha(U_{\alpha \cap U_\beta})$ is superanalytic for all non-empty $U_{\alpha \cap U_\beta}$, i.e., $\Omega_{\alpha \cap U_{\beta}} = (\tilde{z}_1, \ldots, \tilde{z}_m, \tilde{\theta}_1, \ldots, \tilde{\theta}_n)$ where $\tilde{z}_i$ is the even superanalytic $\Lambda^*_{>n-1}$-superfunction in $(m,n)$-variables for $i = 1, \ldots, m$, and $\tilde{\theta}_j$ is an odd superanalytic $\Lambda^*_{>n-1}$-superfunction in $(m,n)$-variables for $j = 1, \ldots, n$.

Such an atlas is called maximal if, given any chart $(U,\Omega)$ such that

$$\Omega \circ \Omega^{-1}_\beta : \Omega_{\beta}(U \cap U_{\beta}) \to \Omega(U \cap U_{\beta})$$

is a superanalytic homeomorphism for all $\beta$, then $(U,\Omega) \in \{(U_\alpha,\Omega_\alpha)\}_{\alpha \in A}$.

A DeWitt $(m,n)$-superanalytic supermanifold over $\Lambda^*_{>n-1}$ is a DeWitt $(m,n)$-dimensional supermanifold $M$ together with a maximal superanalytic atlas of DeWitt $(m,n)$-charts over $\Lambda^*_{>n-1}$.

Given a DeWitt $(m,n)$-superanalytic supermanifold $M$ over $\Lambda^*_{>n-1}$, define an equivalence relation $\sim$ on $M$ by letting $p \sim q$ if and only if there exists $\alpha \in A$ such that $p,q \in U_\alpha$ and $\pi_B^{(m,n)}(\Omega_\alpha(p)) = \pi_B^{(m,n)}(\Omega_\alpha(q))$ where $\pi_B^{(m,n)}$ is the projection given by (2.2). Let $p_B$ denote the equivalence class of $p$ under this equivalence relation. Define the **body** $M_B$ of $M$ to be the $m$-dimensional complex manifold with analytic structure given by the coordinate charts $\{((U_\alpha)_B, (\Omega_\alpha)_B)\}_{\alpha \in A}$ where $(U_\alpha)_B = \{p_B \mid p \in U_\alpha\}$, and $(\Omega_\alpha)_B : (U_\alpha)_B \to \mathbb{C}^m$ is given by $(\Omega_\alpha)_B(p_B) = \pi_B^{(m,n)} \circ \Omega_\alpha(p)$. We define the genus of $M$ to be the genus of $M_B$. 


Note that $M$ is a complex fiber bundle over the complex manifold $M_B$; the fiber is the complex vector space $(\Lambda^0_{s>n-1})^0_s \oplus (\Lambda^1_{s>n-1})^0_n$. This bundle is not in general a vector bundle since the transition functions are in general nonlinear.

For any DeWitt $(1, n)$-superanalytic supermanifold $M$, its body $M_B$ is a Riemann surface. An $N=n$ superconformal DeWitt super-Riemann surface over $\Lambda^s_{s>n-1}$, for $n = 1, 2$, is a DeWitt $(1, n)$-superanalytic supermanifold over $\Lambda^s_{s>n-1}$ with coordinate atlas $\{(U_\alpha, \Omega_\alpha)\}_{\alpha \in A}$ such that the coordinate transition functions $\Omega_\alpha \circ \Omega_\beta^{-1}$ in addition to being superanalytic are also $N=n$ superconformal for all nonempty $U_\alpha \cap U_\beta$.

Since the condition that the coordinate transition functions be $N=n$ superconformal instead of merely superanalytic is such a strong condition (unlike in the nonsuper case), we again stress the distinction between an $N=n$ superanalytic DeWitt super-Riemann surface which has superanalytic transition functions versus an $N=n$ superconformal DeWitt super-Riemann surface which has $N=2$ superconformal transition functions. In the literature one will find the term “super-Riemann surface” or “Riemannian supermanifold” used for both merely superanalytic structures (cf. [DeW]) and for superconformal structures (cf. [E], [CR]). Throughout this paper, we will be mainly dealing with $N=2$ superconformal super-Riemann surfaces and $N=1$ superanalytic super-Riemann surfaces.

In general, the transition functions for an $N=n$ superanalytic DeWitt super-Riemann surface, for $n = 1, 2$, are $G_{s>n-1}(n)$ functions. If however, the functions are in the subclass of $H_{s>n-1}(n)$ functions, or $C_{s>n-1}(n)$ functions, then we will call such a super-Riemann surface a $H_{s>n-1}(n)$-supermanifold or $C_{s>n-1}(n)$-supermanifold, respectively. These subclasses of supermanifolds are those usually studied in the ringed-space approach in, for instance, [M1] and [M2], although at times the transition functions are allowed to have component functions $f(z)$ that are not purely complex analytic when restricted to $z_B$ but can take values in $\Lambda^0_\infty$.

Let $M_1$ and $M_2$ be $N=n$ superanalytic DeWitt super-Riemann surfaces, for $n = 1, 2$, with coordinate atlases $\{(U_\alpha, \Omega_\alpha)\}_{\alpha \in A}$ and $\{(V_\beta, \Xi_\beta)\}_{\beta \in B}$, respectively. A map $F: M_1 \rightarrow M_2$ is said to be $N=n$ superanalytic if $\Xi_\beta \circ F \circ \Omega_\alpha^{-1}: \Omega_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \rightarrow \Xi_\beta(V_\beta)$ is $N=2$ superanalytic for all $\alpha \in A$ and $\beta \in B$ with $U_\alpha \cap F^{-1}(V_\beta) \neq \emptyset$. If in addition, $F$ is bijective, then we say that $M_1$ and $M_2$ are $N=n$ superanalytically equivalent. By $N=2$ superanalytic structure over a Riemann surface $M_B$, we mean an equivalence class of $N=2$ superanalytic equivalent atlases on a DeWitt $(1, n)$-supermanifold $M$ whose body is $M_B$.

Now, let $M_1$ and $M_2$ be $N=n$ superconformal DeWitt super-Riemann surfaces, for $n = 1, 2$, with coordinate atlases $\{(U_\alpha, \Omega_\alpha)\}_{\alpha \in A}$ and $\{(V_\beta, \Xi_\beta)\}_{\beta \in B}$, respectively. A map $F: M_1 \rightarrow M_2$ is said to be $N=n$ superconformal if $\Xi_\beta \circ F \circ \Omega_\alpha^{-1}: \Omega_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \rightarrow \Xi_\beta(V_\beta)$ is $N=n$ superconformal for all $\alpha \in A$ and $\beta \in B$ with $U_\alpha \cap F^{-1}(V_\beta) \neq \emptyset$. If in addition, $F$ is bijective, then we say that $M_1$ and $M_2$ are $N=n$ superconformally equivalent. By $N=n$ superconformal structure over a Riemann surface $M_B$, we mean an equivalence class of $N=n$ superconformally equivalent atlases on a DeWitt $(1, 2)$-supermanifold $M$ whose body is $M_B$.

Here we have two rather trivial examples of $N=2$ DeWitt super-Riemann surfaces: We define the $N=n$ super complex plane over $\Lambda^s_{s>n-1}$, denoted by $S^n \mathbb{C}$, to be $\mathbb{C} \times (\Lambda^0_{s>n-1})^s \times (\Lambda^1_{s>n-1})^n = \Lambda^0_{s>n-1} \times (\Lambda^1_{s>n-1})^n$ with the usual topology on $\mathbb{C}$ dictating the DeWitt topology on $S^n \mathbb{C}$. 
We define the \( N= n \) super upper half-plane over \( \mathbb{A}_{\ast>n-1} \), denoted by \( S^n \mathbb{H} \), to be \( \mathbb{H} \times (\mathbb{A}_{\ast>n-1}^0) \setminus (\mathbb{A}_{\ast>n-1}^1)^n \) with the usual topology on \( \mathbb{H} \) dictating the DeWitt topology on \( S^n \mathbb{H} \).

Note that for \( N= n \) with \( n = 1, 2 \), both the superplane \( S^n \mathbb{C} \) and the super upper half-plane \( S^n \mathbb{H} \) are not only superanalytic as supermanifolds, but also \( N= n \) superconformal. These are also examples of \( \mathcal{C}_{\ast>n-1}(n) \)-supermanifolds.

3. The equivalence of \( N=2 \) superconformal and \( N=1 \) superanalytic DeWitt super-Riemann surfaces

In this section, we recall some results from [DRS] establishing an equivalence between \( N=1 \) superanalytic super-Riemann surfaces and \( N=2 \) superconformal super-Riemann surfaces. Our main result in this paper, Theorem 4.1 in Section 4 is formulated and proved for \( N=2 \) superconformal DeWitt super-Riemann surfaces. In Section 4, we will use the results from [DRS] stated in this section to formulate Corollary 4.2 to Theorem 4.1 which gives a uniformization theorem for \( N=1 \) superanalytic super-Riemann surfaces. The main reason for pursuing this directions is to compare our results to previous results on \( N=1 \) supermanifolds in the ringed-space theory as studied in, for instance, [M1] and [M2].

Although we follow [DRS], there are discrepancies between some of our formulas and those given in [DRS]. For instance, there is a typo in [DRS] in the transformation from the nonhomogeneous coordinate system \((z, \theta_1, \theta_2)\) to the homogeneous coordinate system \((z, \theta^+, \theta^-)\); this typo is a factor of \( 1/2 \) erroneously introduced into the \( D^k \) superderivations after the transformation of coordinates, and this factor is carried throughout their calculations.

Let \( U_B \) be an open set in \( \mathbb{C} \). Let \( \mathcal{S}\mathcal{C}_{\ast>1}(2, U_B) \) be the set of invertible \( N=2 \) superconformal functions defined on the DeWitt open set \( U_B \times ((\mathbb{A}_{\ast>1}^0) \setminus (\mathbb{A}_{\ast>1}^1)^2) \) in \( \mathbb{A}_{\ast>1}^0 \setminus (\mathbb{A}_{\ast>1}^1)^2 \). Let \( \mathcal{S}\mathcal{A}_{\ast>1}(1, U_B) \) be the set of invertible \( N=1 \) superanalytic functions \( H \) defined on the DeWitt open set \( U_B \times (\mathbb{A}_{\ast>1}) \setminus (\mathbb{A}_{\ast>1})^2 \) in \( \mathbb{A}_{\ast>1} \) such that the coefficients of the functions defining \( H \) are restricted to lie in \( \mathbb{A}_{\ast>2} \) rather than just in \( \mathbb{A}_{\ast>1} \); that is in \( (2.4) \), we take \((k) \in J_{s-2} \) rather than \((k) \in J_{s-1} \).

Define the map

\[
F_1 : \mathcal{S}\mathcal{C}_{\ast>1}(2, U_B) \to \mathcal{S}\mathcal{A}_{\ast>1}(1, U_B)
H \mapsto F_1(H)
\]

as follows: For \( H \in \mathcal{S}\mathcal{C}_{\ast>1}(2, U_B) \), then in particular \( H(z, \theta^+, \theta^-) = (\tilde{z}^+, \tilde{\theta}^-) \) is of the form (2.27)-(2.29) for even functions \( f \) and \( g^\pm \) and odd functions \( \psi^\pm \). Define

\[
F_1(H)(z, \theta) = (f(z) + \psi^+(z)\psi^-(z) + 2\theta^+ g^+(z)) \psi^-(z), \quad \psi^+(z) + \theta g^+(z)
\]

This invertible \( N=1 \) superanalytic function \( F_1(H) \) can be thought of as arising from performing the \( N=2 \) superanalytic coordinate transformation

\[
(z, \theta^+, \theta^-) \mapsto (u, \eta, \alpha) = (z + \theta^+ \theta^-, \theta^+, \theta^-).
\]

Under this transformation, we obtain the \( N=2 \) superanalytic function in the even variable \( u \) and the two odd variables \( \eta \) and \( \alpha \) given by

\[
\tilde{u} = f(u) + \psi^+(u)\psi^-(u) + 2\eta g^+(u)\psi^-(u)
\]

\[
\tilde{\eta} = \psi^+(u) + \eta g^+(u)
\]

\[
\tilde{\alpha} = \psi^-(u) + \alpha g^-(u) - 2\eta\alpha(\psi^-)'(u).
\]
Conversely, define the map
\[ F_2 : \mathcal{SA}_{\ast > 1}(1, U_B) \rightarrow \mathcal{SC}_{\ast > 1}(2, U_B) \]
(3.7)
\[ H \mapsto F_2(H) \]
as follows: For \( H \in \mathcal{SA}_{\ast > 1}(1, U_B) \), then \( H(z, \theta) = (f_1(z) + \theta \xi(z), \psi(z) + g(z)) \)
for even functions \( f_1(z) \) and \( g(z) \) and odd functions \( \xi(z) \) and \( \psi(z) \), and with \( g(z) \) nonvanishing. Define \( F_2(H)(z, \theta^+, \theta^-) = (\hat{z}, \hat{\theta}^+, \hat{\theta}^-) \) to be of the form (2.27)-(2.28)
where
\[ f(z) = f_1(z) - \frac{\psi(z)\xi(z)}{2g(z)}, \]
(3.8)
\[ g^+(z) = g(z), \quad \text{and} \quad g^-(z) = \frac{f_1'(z) - \psi'(z)\xi(z)}{g(z)} \]
(3.9)
\[ \psi^+(z) = \psi(z), \quad \text{and} \quad \psi^-(z) = \frac{\xi(z)}{2g(z)}. \]
(3.10)

One can easily check that condition (2.29) is satisfied, and thus \( F_2(H) \) is indeed \( N=2 \) superconformal.

We have that \( F_1 \) and \( F_2 \) are bijections and
\[ F_1 \circ F_2 = id_{\mathcal{SA}_{\ast > 1}(1, U_B)} \quad \text{and} \quad F_2 \circ F_1 = id_{\mathcal{SC}_{\ast > 1}(2, U_B)}. \]

Let \( \mathcal{SCM}_{\ast > 1}(2) \) be the category of \( N=2 \) superconformal DeWitt super-Riemann surfaces over the Grassmann algebra \( \Lambda_{\ast > 1} \), and let \( \mathcal{SAM}_{\ast > 1}(1) \) be the category of \( N=1 \) superanalytic DeWitt super-Riemann surfaces \( M \) over the Grassmann algebra \( \Lambda_{\ast > 1} \) such that the transition functions for \( M \) are in \( \mathcal{SA}_{\ast > 1}(1, U_B) \) for some \( U_B \in \mathbb{C} \).

Define the functor

\[ F : \mathcal{SCM}_{\ast > 1}(2) \rightarrow \mathcal{SAM}_{\ast > 1}(1) \]
(3.12)
\[ M \mapsto F(M) \]
as follows: Let \( M \) be an \( N=2 \) superconformal DeWitt super-Riemann surface over the Grassmann algebra \( \Lambda_{\ast > 1} \) with coordinate atlas \( \{(U_{\alpha}, \Omega_{\alpha})\}_{\alpha \in A} \). Let \( F(M) \) be the \( N=1 \) superanalytic DeWitt super-Riemann surface with body \( M_B \) obtained by patching together DeWitt open domains in \( \Lambda_{\ast > 1} \) with local coordinates \( (z, \theta) \) by means of the transition functions \( F_1(\Omega_{\alpha} \circ \Omega_{\beta}^{-1}) : (\Omega_{\alpha}(U_{\alpha} \cap U_{\beta}))_B \times (\Lambda_{\ast > 1})_S \rightarrow (\Omega_{\alpha}(U_{\alpha} \cap U_{\beta}))_B \times (\Lambda_{\ast > 1})_S \).

From (3.11), it follows that \( F \) is an isomorphism of categories. Thus we have the following proposition:

**Proposition 3.1.** The category \( \mathcal{SCM}_{\ast > 1}(2) \) of \( N=2 \) superconformal DeWitt super-Riemann surfaces over the Grassmann algebra \( \Lambda_{\ast > 1} \) is isomorphic to the category \( \mathcal{SAM}_{\ast > 1}(1) \) of \( N=1 \) superanalytic DeWitt super-Riemann surfaces such that the coefficients of the coordinate transition functions are restricted to lie in \( \Lambda_{\ast > 2} \).

**Remark 3.2.** In \( N=2 \) superconformal field theory, the supermanifolds that arise from superstrings propagating through space time, are \( N=2 \) superconformal DeWitt super-Riemann surfaces with half-infinite tubes attached. These half-infinite tubes are \( N=2 \) superconformally equivalent to punctures on the \( N=2 \) superconformal DeWitt super-Riemann surface with \( N=2 \) superconformal local coordinates vanishing at the punctures. Although, there is a bijection between \( N=2 \) superconformal local coordinates in a neighborhood of a given point on an \( N=2 \) superconformal DeWitt
4. A general uniformization theorem for N=2 superconformal and N=1 superanalytic DeWitt super-Riemann surfaces

We now prove the main theorem of this paper:

**Theorem 4.1.** Every N=2 superconformal DeWitt super-Riemann surface \( M \) with body \( M_B \) is N=2 superconformally equivalent to a \( \mathcal{H}_{>1}(2) \)-supermanifold if and only if the first Čech cohomology group of \( M_B \) with coefficients in the sheaf of holomorphic vector fields over \( M_B \) is trivial, i.e., \( \check{H}^1(M_B, TM_B) = 0 \).

In particular, if \( \check{H}^1(M_B, TM_B) = 0 \), then any N=2 superconformal DeWitt supermanifold \( M \) with body \( M_B \) is N=2 superconformally equivalent to a supermanifold with transition functions of the form

\[
H(z, \theta^+, \theta^-) = (f(z), \theta^+g^+(z), \theta^-g^-(z))
\]

for \( g^\pm(z) \) even superanalytic \((1,0)\)-superfunctions in \( z \), and \( f(z_B) \) complex analytic in \( z_B \), satisfying the condition

\[
f'(z) = g^+(z)g^-(z).
\]

**Proof.** We follow the spirit of the proof of the uniformization theorem in the N=1 superconformal genus-zero case given by Crane and Rabin in [CR]. Let \( M \) be an N=2 superconformal DeWitt super-Riemann surface with coordinate atlas \( \{(U_\alpha, \Omega_\alpha)\}_{\alpha \in A} \). For any \( \alpha, \beta \in A \) with \( U_\alpha \cap U_\beta \neq \emptyset \), we have transition function \( H_{\alpha\beta} = \Omega_\alpha \circ \Omega_\beta^{-1} : \Omega_\beta(U_\alpha \cap U_\beta) \rightarrow \Omega_\alpha(U_\alpha \cap U_\beta) \). We will write \( H_{\alpha\beta}(z, \theta^+, \theta^-) = (z^\alpha, \theta^\alpha, \theta^-\theta^\alpha) \) and denote the three even superfunctions in \( z \) and two odd superfunctions in \( z \) that uniquely determine \( H_{\alpha\beta} \) according to \( \ref{eq:2.27} \)–\( \ref{eq:2.29} \) by \( f_{\alpha\beta}, g_{\alpha\beta}^\pm \) and \( \psi_{\alpha\beta}^\pm \), respectively.

On each triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \), for \( \alpha, \beta, \gamma \in A \), we have the consistency condition \( H_{\alpha\gamma} = H_{\alpha\beta} \circ H_{\beta\gamma} \), which when expanded in terms of component
functions give the conditions

\[(4.3) \quad f_{\alpha\gamma}(z) = f_{\alpha\beta}(f_{\beta\gamma}(z)) + g_{\alpha\beta}^+(f_{\beta\gamma}(z))\psi_{\beta\gamma}^+(z)\psi_{\alpha\beta}^+(f_{\beta\gamma}(z)) - g_{\alpha\beta}^-(f_{\beta\gamma}(z))\]

\[\cdot \psi_{\alpha\beta}^-(f_{\beta\gamma}(z))\psi_{\beta\gamma}^-(z) + (\psi_{\alpha\beta}^+)^2(f_{\beta\gamma}(z))\psi_{\beta\gamma}^+(z)\psi_{\alpha\beta}^+(z)\]

\[(4.4) \quad \psi_{\alpha\beta}^+(z) = g_{\alpha\beta}^+(f_{\beta\gamma}(z))\psi_{\beta\gamma}^+(z) + \psi_{\alpha\beta}^-)^2(f_{\beta\gamma}(z))\psi_{\beta\gamma}^-\psi_{\alpha\beta}^-(z)\]

\[(4.5) \quad g_{\alpha\beta}^+(z) = g_{\alpha\beta}^+(f_{\beta\gamma}(z))g_{\beta\gamma}^+(z) + 2(\psi_{\alpha\beta}^+)^2(f_{\beta\gamma}(z))g_{\beta\gamma}^+(z)\psi_{\beta\gamma}^+(z)\]

\[\cdot -(g_{\alpha\beta}^-)^2(f_{\beta\gamma}(z))g_{\beta\gamma}^+(z)\psi_{\beta\gamma}^+(z)\]

We will use the equations above to first show that, in general, there exist N=2 superconformal changes of coordinates in each coordinate chart that give a new atlas for which the \(\psi_{\alpha\beta}^\pm\) terms in the coordinate transition functions are equal to zero for all \(\alpha, \beta \in A\) if and only if \(\check{H}^1(M_B, TM_B) = 0\). Then we will show that similarly, we can in general set the soul part of the \(f_{\alpha\beta}\) terms equal to zero if and only if \(\check{H}^1(M_B, TM_B) = 0\).

First we expand each \(\psi_{\alpha\beta}^\pm\) into its component functions, writing

\[(4.6) \quad \psi_{\alpha\beta}^\pm(z) = \sum_{(j) \in J^1_{\alpha\beta}} (\psi_{\alpha\beta}^\pm(j)(z)\zeta_j, \zeta_j, \ldots, \zeta_{J^{n+1}}).

We will show by induction on \(n \in \mathbb{N}\), that by N=2 superconformal change of coordinates in each chart, we can set any nonzero \((\psi_{\alpha\beta}^\pm)_{(j)}\) equal to zero for \((j) = (j_1, j_2, \ldots, j_{n+1}) \in J^1_{\alpha\beta}\) if and only if \(\check{H}^1(M_B, TM_B) = 0\). Let \(\zeta^\sigma\) denote the even coordinate on \(\Omega_a(U_\alpha)\) for \(\alpha \in A\).

For \(n = 1\), letting \((j) = (j_1) \in J^1_{\alpha\beta}\) for \(j_1 \in \{1, \ldots, * - 2\}\), the equations (4.4) reduce to

\[(4.7) \quad (\psi_{\alpha\beta}^\pm(j_1)(z^\gamma_{(\emptyset)}) = (\psi_{\alpha\beta}^\pm(j_1)(z^\gamma_{(\emptyset)}) + (\psi_{\beta\gamma}^\pm(j_1)(z^\gamma_{(\emptyset)}))(g_{\alpha\beta}^\pm)(\emptyset)(z^\beta_{(\emptyset)}).

By the N=2 superconformal condition (224), we have

\[(4.8) \quad (f_{\alpha\beta}')(\emptyset)(z^\beta_{(\emptyset)}) = (g_{\alpha\beta}^+)(\emptyset)(z^\beta_{(\emptyset)})(g_{\alpha\beta}^-)(\emptyset)(z^\beta_{(\emptyset)}).

Since \((f_{\alpha\beta})(\emptyset)\) is a local homeomorphism from an open set in the complex plane to an open set in the complex plane, we have that \((f_{\alpha\beta})'(z^\beta_{(\emptyset)})\) is nonzero for all \(z^\beta_{(\emptyset)} \in (U_\beta)_B\), which along with (1.8) implies that the \((g_{\alpha\beta}^\pm)(\emptyset)(z^\beta_{(\emptyset)})\) are nonzero for all \(z^\beta_{(\emptyset)} \in (U_\beta)_B\). Thus we can write

\[(4.9) \quad (g_{\alpha\beta}^\pm)(\emptyset)(z^\beta_{(\emptyset)}) = \frac{1}{(g_{\alpha\beta}^\pm)(\emptyset)(z^\beta_{(\emptyset)})}((f_{\alpha\beta})'(\emptyset)(z^\beta_{(\emptyset)})).

Let \(\sigma^\pm\) each be a section of the holomorphic tangent bundle of \(M_B\) (i.e. holomorphic vector fields on \(M_B\)) such that on coordinate charts we have

\[(4.10) \quad \sigma^\pm_{\beta} = \frac{1}{(g_{\alpha\beta}^\pm)(\emptyset)(z^\beta_{(\emptyset)})}((f_{\alpha\beta})'(\emptyset)(z^\beta_{(\emptyset)})\sigma^\pm_{\alpha},

on \(U_\alpha \cap U_\beta\). Then multiplying equation (4.7) by these holomorphic vector fields, respectively, we have

\[(4.11) \quad \sigma^\pm_{\alpha}(\psi_{\alpha\beta}^+(j_1)(z^\gamma_{(\emptyset)}) = \sigma^\pm_{\alpha}(\psi_{\alpha\beta}^+(j_1)(z^\gamma_{(\emptyset)}) + \sigma^\pm_{\beta}(\psi_{\beta\gamma}^+(j_1)(z^\gamma_{(\emptyset)}).

This implies that \(\{\sigma^\pm_{\alpha}(\psi_{\alpha\beta}^+(j_1) \mid (\alpha, \beta) \in A \times A\}\) are the local representatives of a cocycle in the first Čech cohomology group of \(M_B\) with coefficients in the sheaf of
holomorphic vector fields. Thus these cocycles are, in general, coboundaries if and only if this cohomology is trivial. If this cohomology is trivial, there exist elements $b^\pm_{(j)}$ in the zeroth cohomology $\tilde{H}^0(M_B, TM_B)$, i.e. global sections, such that
\begin{equation}
\sigma^\pm_\alpha(\psi^\pm_{\alpha\beta})_{(j)} = \sigma^\pm_\beta(b^\pm_{(j)})_\beta - \sigma^\pm_\alpha(b^\pm_{(j)})_\alpha
\end{equation}
or in other words
\begin{equation}
(\psi^\pm_{\alpha\beta})_{(j)} = (g^\pm_{\alpha\beta})(\theta)(b^\pm_{(j)})_\beta - (b^\pm_{(j)})_\alpha.
\end{equation}

For all $\alpha \in A$, define the N=2 superconformal transformation $H^\alpha_{(j)}$ by $f_\alpha(z^\alpha) = z^\alpha$, $g^\pm_{\alpha}(z^\alpha) = 1$ and $\psi^\pm_{\alpha}(z^\alpha) = (b^\pm_{(j)})_\alpha(z^\alpha)\zeta_j$, where $\zeta_j$ is the $j$-th basis element for the underlying Grassmann algebra. That is $H^\alpha_{(j)}(z^\alpha, (\theta^+)^\alpha, (\theta^-)^\alpha) = (\tilde{z}^\alpha, (\tilde{\theta}^+)^\alpha, (\tilde{\theta}^-)^\alpha)$ is given by
\begin{align}
\tilde{z}^\alpha &= z^\alpha + (\theta^+)^\alpha(b^-_{(j)})_\alpha(z^\alpha)\zeta_j + (\theta^-)^\alpha(b^+_{(j)})_\alpha(z^\alpha)\zeta_j, \\
\tilde{\theta}^\pm)^\alpha &= (b^\pm_{(j)})_\alpha(z^\alpha)\zeta_j + (\theta^+)^\alpha\pm(\theta^-)^\alpha((b^\pm_{(j)})_\alpha(z^\alpha)\zeta_j.
\end{align}
Redefining the coordinates for the charts $(U_\alpha, \Omega_\alpha)$ and $(U_\beta, \Omega_\beta)$ by the N=2 superconformal transformations $H^\alpha_{(j)}$ and $H^\beta_{(j)}$, respectively, such that the new coordinate charts are $(U_\alpha, \tilde{\Omega}_\alpha = H^\alpha_{(j)} \circ \Omega_\alpha)$ and $(U_\beta, \tilde{\Omega}_\beta = H^\beta_{(j)} \circ \Omega_\beta)$, we see that the new coordinate transformation $\tilde{H}_{\alpha\beta} = \tilde{\Omega}_\alpha \circ \tilde{\Omega}_\beta^{-1} = H^\alpha_{(j)} \circ \Omega_\alpha \circ \Omega_\beta^{-1} \circ (H^\beta_{(j)})^{-1}$ has
\begin{equation}
(\tilde{\psi})_{(j)}^\pm = (\psi)_{(j)}^\pm - (g^\pm_{\alpha\beta})(\theta)(b^\pm_{(j)})_\beta + (b^\pm_{(j)})_\alpha = 0.
\end{equation}

The new atlas $\{(U_\alpha, \tilde{\Omega}_\alpha) \mid \alpha \in A\}$ now has the $\tilde{\psi}^\pm_j$ terms for $(j) = (j_1) \in J_{-2}^1$ equal to zero. For this new atlas, we again have the compatibility condition on the triple overlaps, and thus the consistency conditions (4.3)–(4.5) again hold for the components of the new coordinate transformation functions, and we can perform the above procedure again for a new $j_1 \in \{1, \ldots, \ast - 2\}$ without changing the fact that the previous $\psi_{\alpha\beta}^\pm_{(j)}$ terms have been set equal to zero. Doing this repeatedly, we can, in general, set the $\psi^\pm_{\alpha\beta}$ terms equal to zero for $(j) = (j_1)$, for all $j_1 = 1, \ldots, \ast - 2$ if and only if $H^1(M_B, TM_B) = 0$.

Now we make the inductive assumption: assume coordinate transformations have been made such that for the new $\psi^\pm_j$ terms, the components $\psi_{\alpha\beta}^\pm_{(j)}$ have been set equal to zero for $(j) = (j_1, j_2, \ldots, j_{2k+1}) \in J_{-2}^{k}$, and for $k = 1, \ldots, n - 1$. For this new atlas, we again have the compatibility condition on the triple overlaps. Thus for the $\zeta_{j_1} \cdots \zeta_{j_{2n+1}}$, level of equation (4.3) applied to $z(\theta)$, we have that equation (4.7) applies to these new $\psi^\pm_j$'s, and thus, so does equation (4.11). Thus again, in general, there exist $b^\pm_{(j)}$, each in the zeroth cohomology of $M_B$ with coefficients in the holomorphic tangent bundle such that equations (4.12) and (4.13) hold for these terms if and only if $H^1(M_B, TM_B)$ is trivial.

For $\alpha \in A$, define the N=2 superconformal transformation $H^\alpha_{(j)}$ by $f_\alpha(z^\alpha) = z^\alpha$, $g^\pm_{\alpha}(z^\alpha) = 1$ and $\psi^\pm_{\alpha}(z^\alpha) = (b^\pm_{(j)})_\alpha(z^\alpha)\zeta_j \cdots \zeta_{j_{2n+1}}$. Redefining the coordinates for the charts $(U_\alpha, \Omega_\alpha)$ by the N=2 superconformal transformations $H^\alpha_{(j)}$, for $\alpha \in A$, it is a straightforward calculation to see that for the new coordinate coordinate charts $(U_\alpha, \tilde{\Omega}_\alpha = H^\alpha_{(j)} \circ \Omega_\alpha)$ for $\alpha \in A$ the new coordinate transformations $\tilde{H}_{\alpha\beta} =
\[ H^{(j)}_\alpha \circ \Omega_{\alpha}^{-1} = H^\beta_\alpha \circ \Omega_{\beta}^{-1} \circ (H^{(j)}_\beta)^{-1}, \]

have \((\tilde{\psi}^\pm)_{(j)} = 0\). We perform this procedure of redefining the coordinate charts for each \((j) = (j_1, \ldots, j_{2k+1}) \in J^*_{n-2}\) with \(k = n\), resulting in an atlas with coordinate transition functions that have all the \(\psi_{(j)}^\pm\) components equal to zero for \((j)\) of length \(2n + 1\), while keeping the transition functions \(\psi_{(j')}^\pm\) for \((j') \in J^*_{n-2}\) of length \(2k + 1\) for \(k < n\) equal to zero.

This proves that, in general, there exist \(N=2\) superconformal coordinate transformations which result in an atlas of charts with coordinate transition functions for which the \(\psi^\pm\) terms are all zero if and only if \(\tilde{H}^1(M_B, TM_B) = 0\); that is, every \(N=2\) superconformal DeWitt super-Riemann surface \(M\) with body \(M_B\) is \(N=2\) superconformally equivalent to a \(H_{\geq 1}(2)\)-supermanifold if and only if \(\tilde{H}^1(M_B, TM_B) = 0\).

The remaining nontrivial consistency conditions (4.13) and (4.15) for the new coordinate atlas reduce to

\[
\begin{align*}
(4.17) & \quad f_{\alpha\gamma}(z) = f_{\alpha\beta}(f_{\beta\gamma}(z)) \\
(4.18) & \quad g^\pm_{\alpha\gamma}(z) = g^\pm_{\alpha\beta}(f_{\beta\gamma}(z))g^\pm_{\beta\gamma}(z).
\end{align*}
\]

We now expand the \(f_{\alpha\beta}\) terms into component functions, writing

\[
(4.19) \quad f_{\alpha\beta}(z) = \sum_{(j) \in J^*_{n-2}} (f_{\alpha\beta})_{(j)}(z)\zeta_{j_1}\zeta_{j_2} \cdots \zeta_{j_{2n}}.
\]

For the \(n = 0\) terms, i.e., the \((\emptyset) \in J^*_{n-2}\) terms, equation (4.17) becomes

\[
(4.20) \quad (f_{\alpha\gamma})_{(\emptyset)}(z_{(\emptyset)}^\beta) = (f_{\alpha\beta})_{(\emptyset)}(f_{\beta\gamma})_{(\emptyset)}(z_{(\emptyset)}^\beta) = (f_{\alpha\beta})_{(\emptyset)}(z_{(\emptyset)}^\beta),
\]

which is just the usual cocycle condition on the Riemann surface \(M_B\).

We will show by induction on \(n \in \mathbb{N}\) that we can, in general, set the soul components of these \(f\) terms equal to zero if and only if \(\tilde{H}^1(M_B, TM_B) = 0\). For \(n = 1\), let \((j) = (j_1, j_2) \in J^*_{n-2}\). The consistency equation (4.17) implies

\[
(4.21) \quad (f_{\alpha\gamma})_{(j)}(z_{(\emptyset)}^\gamma) = (f_{\alpha\beta})_{(j)}(f_{\beta\gamma})_{(\emptyset)}(z_{(\emptyset)}^\beta) + (f_{\alpha\beta})_{(\emptyset)}(z_{(\emptyset)}^\beta)(f_{\beta\gamma})_{(j)}(z_{(\emptyset)}^\gamma).
\]

Let \(\sigma\) be a section of the holomorphic tangent bundle such that on coordinate charts we have

\[
(4.22) \quad \sigma_{\beta} = (f_{\alpha\beta})_{(\emptyset)}(z_{(\emptyset)}^\beta)\sigma_{\alpha}.
\]

Multiplying equation (4.21) by this vector field, we have

\[
(4.23) \quad \sigma_{\alpha}(f_{\alpha\gamma})_{(j)}(z_{(\emptyset)}^\gamma) = \sigma_{\alpha}(f_{\alpha\beta})_{(j)}(z_{(\emptyset)}^\beta) + \sigma_{\beta}(f_{\beta\gamma})_{(j)}(z_{(\emptyset)}^\gamma).
\]

This implies that \(\{\sigma_{\alpha}(f_{\alpha\beta})_{(j)} \mid (\alpha, \beta) \in A \times A\}\) are the local representatives of a cocycle in \(H^1(M_B, TM_B)\). In general, these cocycles are coboundaries if and only if this cohomology group is trivial; in which case, there exists elements \(b_{(j)}\) in the zeroth cohomology group of \(M_B\), such that

\[
(4.24) \quad \sigma_{\alpha}(f_{\alpha\beta})_{(j)} = \sigma_{\beta}(b_{(j)})\beta - \sigma_{\alpha}(b_{(j)})\alpha,
\]

i.e.,

\[
(4.25) \quad (f_{\alpha\beta})_{(j)} = (f_{\alpha\beta})_{(\emptyset)}(b_{(j)})\beta - (b_{(j)})\alpha.
\]

For \(\alpha \in A\), define the \(N=2\) superconformal transformation \(H_{(j)}^\alpha\) by \(f_{\alpha}(z^\alpha) = z^\alpha + (b_{(j)})_\alpha(z^\alpha)\zeta_{j_1}\zeta_{j_2}\), \(g^+_{\alpha}(z^\alpha) = (f_{\alpha})^+(z^\alpha), \ g^-_{\alpha}(z^\alpha) = 1\) and \(\tilde{\psi}^+_{\alpha}(z^\alpha) = 0\). That is \(H_{(j)}^\alpha(z^\alpha, (\theta^+)^\alpha, (\theta^-)^\alpha) = (z^\alpha + (b_{(j)})_\alpha(z^\alpha)\zeta_{j_1}\zeta_{j_2}, \theta^+ (1 + (b_{(j)})_\alpha(z^\alpha)\zeta_{j_1}\zeta_{j_2}), \theta^-).\)
Redefining the coordinates for the charts $(U_\alpha, \Omega_\alpha)$ by the $N=2$ superconformal transformations $H^\alpha_{(j)}$, for $\alpha \in A$, such that the new coordinate charts are $(U_\alpha, \bar{\Omega}_\alpha = H^\alpha_{(j)} \circ \Omega_\alpha)$, we have that the new coordinate transformations $\tilde{H}_{\alpha\beta} = \bar{\Omega}_\alpha \circ \bar{\Omega}_\beta^{-1} = H^\alpha_{(j)} \circ \Omega_\alpha \circ \Omega_\beta^{-1} \circ (H^{\beta}_{(j)})^{-1}$, for $\alpha, \beta \in A$, have

\begin{align}
(4.26) \quad (\tilde{f}_{\alpha\beta})^{(j)} &= (f_{\alpha\beta})^{(j)} - (f_{\alpha\beta})^{(j)}(b_{(j)})_{\beta} + (b_{(j)})_{\alpha} = 0 \\
(4.27) \quad \bar{\psi}^\pm &\equiv 0.
\end{align}

The new atlas $\{(U_\alpha, \bar{\Omega}_\alpha) | \alpha \in A\}$ now has the $\tilde{f}_{(j)}$ terms for $(j) = (j_1, j_2) \in J_{n-2}$ equal to zero and the $\bar{\psi}^\pm$ terms equal to zero. For this new atlas, we again have the compatibility condition on the triple overlaps, and thus the consistency condition (4.17) again holds on the components of the new coordinate transformation functions. Therefore, we can perform the above procedure again for a different $(j) = (j_1, j_2) \in J_{n-2}$ without changing the fact that the previous $(f_{\alpha\beta})^{(j)}$ terms have been set equal to zero. Doing this repeatedly, we can, in general, set the $(f_{\alpha\beta})^{(j)}$ terms equal to zero for all $(j) = (j_1, j_2) \in J_{n-2}$ if and only if $\tilde{H}^1(M_B, TM_B) = 0$.

Now we make the inductive assumption: assume coordinate transformations have been made such that for the new $f$ terms, the components $(f_{\alpha\beta})^{(j)}$ have been set equal to zero for $(j) = (j_1, j_2, \ldots, j_{2k}) \in J_{n-2}$, for $k = 1, \ldots, n-1$. For this new atlas, we again have the compatibility condition on the triple overlaps. Thus for the $\xi_{j_1} \cdots \xi_{j_{2k}}$ level of equation (4.17) applied to $\xi_{(j)}$, we have that equation (4.22) applies to these new $f_{(j)}$’s, and thus, so does equation (4.23). Thus, in general, there exists $b_{(j)}$ in the zeroth cohomology of $M_B$ such that equations (4.23) and (4.24) hold for these terms if and only if $\tilde{H}^1(M_B, TM_B) = 0$.

For $\alpha \in A$, define the $N=2$ superconformal transformation $H^\alpha_{(j)}$ by $f_{\alpha}(z^\alpha) = z^\alpha + (b_{(j)})_{\alpha}(z^\alpha)\xi_{j_1} \cdots \xi_{j_{2n}}$, $g_{\alpha}(z^\alpha) = f'_{\alpha}(z^\alpha)$, $g_{\alpha}^-(z^\alpha) = 1$, and $\bar{\psi}^\pm(\alpha) = 0$. Redefining the coordinates for the charts $(U_\alpha, \Omega_\alpha)$ by the $N=2$ superconformal transformations $H^\alpha_{(j)}$, for $\alpha \in A$, it is a straightforward calculation to see that the new coordinate transformations have $(f_{\alpha\beta})^{(j)} = 0$, for $\alpha, \beta \in A$. In general, we can perform this procedure of redefining the coordinate charts for each $(j) = (j_1, \ldots, j_{2n}) \in J_{n-2}$, resulting in an atlas with coordinate transition functions that have all the $f_{(j)}$ components equal to zero for $(j)$ of length $2n$, in addition to the $f_{(j')}$ terms for $(j') \in J_{n-2}$ of length $2k$ for $0 < k < n$ equal to zero, and the $\bar{\psi}^\pm$ terms equal to zero if and only if $\tilde{H}^1(M_B, TM_B) = 0$.

This proves that, in general, there exist $N=2$ superconformal coordinate transformations which result in an atlas of charts for $M$ with coordinate transition functions for which the soul portion of the $f$ terms and the $\bar{\psi}^\pm$ terms are all zero if and only if $\tilde{H}^1(M_B, TM_B) = 0$.

Using Proposition 3.1, we immediately obtain from Theorem 5.8 a uniformization result for $N=1$ superanalytic DeWitt super-Riemann surfaces over $\Lambda_{n-1}$ with body $M_B$ and coordinate transition functions with coefficients restricted to $\Lambda_{n-1}$. Namely we have that the uniformization of such a $G_{n-1}(1)$-supermanifold to an $H_{n-1}(1)$-supermanifold is dependent on $\tilde{H}^1(M_B, TM_B)$. However, analyzing the compatibility conditions for coordinate transformations on triple overlaps for any $N=1$ superanalytic DeWitt super-Riemann surfaces over $\Lambda_{n-1}$ with body $M_B$, that is where the coordinate transition functions have coefficients restricted to $\Lambda_{n-1}$,
rather than restricted to $\wedge_{L-2}$, we see that for $\wedge_k = \wedge_L$, the conditions on the $k$ levels for $k < L - 1$ are exactly the same as for the $k$ levels with the restricted coefficients, and that the $L - 1$ level gives the same cocycle property as that for the lower levels. Thus we can extend our uniformization result to general $N=1$ superanalytic DeWitt super-Riemann surfaces over $\wedge_{*>0}$. That is we have:

**Corollary 4.2.** Every $N=1$ superanalytic DeWitt super-Riemann surface over $\wedge_{*>0}$ with body $M_B$ is $N=1$ superanalytically equivalent to an $\mathcal{H}_{*>0}(1)$-supermanifold if and only if the first Čech cohomology group of $M_B$ with coefficients in the sheaf of holomorphic vector fields over $M_B$ is trivial, i.e., $\check{H}^1(M_B, TM_B) = 0$.

In particular, if $\check{H}^1(M_B, TM_B) = 0$, then any $N=1$ superanalytic DeWitt supermanifold $M$ with body $M_B$ is $N=1$ superanalytically equivalent to a supermanifold with transition functions of the form

$$H(z, \theta) = (f(z), \theta g(z))$$

(4.28)

where $f(z_B)$ is complex analytic in $z_B$, and $g(z)$ is an even superanalytic $(1,0)$-superfunction in $z$.

**Remark 4.3.** Although from Proposition 3.1, we know that the $N=2$ superconformal and $N=1$ superanalytic settings are essentially equivalent, it is easier to prove our uniformization theorem, Theorem 4.1, first for $N=2$ superconformal DeWitt super-Reimann surfaces and then translate our result to $N=1$ superanalytic DeWitt super-Reimann surfaces, as we did in Corollary 4.2. The reason for this is that the coordinate transformation compatibility conditions on triple overlaps in the homogeneous coordinate setting for $N=2$ superconformal DeWitt supermanifolds clearly gives a cocycle in $\check{H}^1(M_B, TM_B)$, whereas for $N=1$ superanalytic DeWitt supermanifolds this clear dependency between cocycles in $\check{H}^1(M_B, TM_B)$ and the coordinate transformation compatibility conditions on triple overlaps is lost.

### 4.1. Vector bundles and $N=2$ superconformal and $N=1$ superanalytic DeWitt super-Riemann surfaces.

In this section, we assume that $M$ is an $N=2$ superconformal DeWitt super-Riemann surface with transition functions restricted to be of the form (4.1); that is, $M$ is an $\mathcal{H}_{*>1}(2)$-supermanifold. We will now define a certain complex vector bundle over $M_B$ which is a substructure of $M$ that canonically determines $M$. We begin by noting that the transition functions $H_{\alpha\beta}$ for $M$ have components $f_{\alpha\beta}$, $g_{\alpha\beta}^+$ that are completely determined by their value on the body component of the coordinate charts by (2.1). In addition, $f_{\alpha\beta}(z^\beta_{(0)})$ and $g_{\alpha\beta}^+(z^\beta_{(0)})$, completely determine $g_{\alpha\beta}^-(z^\beta_{(0)})$ by the $N=2$ superconformal condition (4.2).

Thus $M$, is canonically determined by the $\wedge_{*>1}$-bundle over $M_B$ with transition functions $f_{\alpha\beta}$ on $M_B$ and transition functions $g_{\alpha\beta}^+$ for the fiber where here we are restricting $g_{\alpha\beta}^+$ to the body of $M$. Viewing $\wedge_{*>1}$ as a complex vector space of dimension $k = 2^{*-1}$, we have that this $\wedge_{*>1}$-bundle over $M_B$ is a rank $k$ holomorphic vector bundle over $M_B$. There is a bijection from such structures to $\check{H}^1(M_B, GL(k, \mathbb{C}))$, and if this cohomology is trivial, there is no obstruction to the uniformization of this vector bundle to a globally trivial vector bundle. If however this cohomology is nontrivial there is a possible obstruction to the trivialization of this vector bundle. We say “possible” obstruction, because it is easy to see that
from (4.1), we do not have all possible holomorphic \(GL(k)\)-bundles over \(M_B\) arising as this vector bundle substructure of \(M\).

We will now analyze the structure group of this holomorphic vector bundle in some detail. We first order \(J^1_{>1}\) as follows: for \((j), (j') \in J^1_{>1}\), let \((j) = (j_1, j_2, \ldots, j_m) > (j') = (j'_1, j'_2, \ldots, j'_m)\) if \(m > m'\) for \(m\) and \(m'\) odd integers. If \(m = m'\), let \((j) = (j_1, j_2, \ldots, j_l, j_{l+1}, \ldots, j_m) > (j') = (j'_1, j'_2, \ldots, j'_l, j'_{l+1}, \ldots, j'_m)\) if \(j_p = j'_p\) for \(p = 1, \ldots, l\), and \(j_{l+1} > j'_{l+1}\). This gives an ordering for the basis \(\{\zeta_i, \ldots, \zeta_m \mid (j) = (j_1, \ldots, j_m) \in J^1_{>1}\}\) of \(\Lambda^1_{>1}\) as a vector space of dimension \(k\) over \(\mathbb{C}\).

For \(z^\alpha_\emptyset \in (U_{\alpha})_{\emptyset} \subseteq \mathbb{C}\), we have \(g^+_{\alpha\beta}(z^\beta_\emptyset) \in (\Lambda^0_{>1})^\times\), and we can use the natural multiplication in \(\Lambda_{>1}\) by \(g^+_{\alpha\beta}(z^\beta_\emptyset)\) on \(\Lambda^1_{>1}\) to define the map \(g^+_{\alpha\beta}(z^\beta_\emptyset) : \Lambda^1_{>1} \to \Lambda^1_{>1}, \theta^+ \mapsto g^+_{\alpha\beta}(z^\beta_\emptyset)\theta^+ = \theta^+ g^+_{\alpha\beta}(z^\beta_\emptyset)\). By viewing \(\Lambda^1_{>1}\) as a \(k\)-dimensional complex vector space, then this multiplication map is a vector space isomorphism; that is, for \(\theta^+ \in \Lambda^1_{>1}\), the multiplication \(g^+_{\alpha\beta}(z^\beta_\emptyset)\theta^+\) gives a linear transformation from \(\Lambda^1_{>1} = \mathbb{C}^k\) to itself which has nonzero determinant. And in this way we can identify \(g^+_{\alpha\beta}(z^\beta_\emptyset) \in GL(k, \mathbb{C})\).

It is easy to see that the matrices \(X \in GL(k, \mathbb{C})\) obtained as structure matrices have the form \(X = cI_k + X_I\) where \(I_k\) is the identity matrix, \(c \in \mathbb{C}\), and \(X_I\) is strictly lower triangular. In addition, the entry in \(X_I\) corresponding to \(a_{i,j}(j'), j' \in J^1_{>1}\) is nonzero in general only if writing \((j) = (j_1, \ldots, j_m)\) and \((j') = (j'_1, \ldots, j'_m)\) we have \(\{j'_1, \ldots, j'_m\} \subseteq \{j_1, \ldots, j_m\}\) and \(*, * - 1 \notin \{j_1, \ldots, j_m\}\). Let the group of all such matrices be denoted by \(G_k\). In fact \(G_k\) is an abelian subgroup of \(GL(k, \mathbb{C})\), since in fact \(g^+_{\alpha\beta}(z^\beta_\emptyset)\) and \(g^+_{\gamma\delta}(z^\delta_\emptyset)\) are also elements of \(\Lambda^0_{>1}\) and thus commute.

Thus, we have the following lemma:

**Lemma 4.4.** Let \(M\) be an N=2 superconformal (resp., N=1 superanalytic) DeWitt super-Riemann surface with coordinate transition functions satisfying (4.1) (resp., (4.2b)). There is a one-to-one correspondence between the set of N=2 superconformal (resp., N=1 superanalytic) equivalence classes of such surfaces and conformal equivalence classes of holomorphic \(G_k\)-bundles over \(M_B\).

5. Uniformization for Simply Connected N=2 Superconformal and N=1 Superanalytic DeWitt Super-Riemann Surfaces

5.1. A family of inequivalent N=2 superconformal structures over the Riemann sphere. Let \(\mathcal{G}\) be the set of functions \(g : (\Lambda^0_{>1})^\times \to (\Lambda^0_{>1})^\times, z \mapsto g(z)\), such that \(g\) is superanalytic for \(z \in (\Lambda^0_{>1})^\times\). That is, \(g\) is an even superanalytic function in \(z\) such that \(g(j)(z(\emptyset))\) for \(j \in J^0_{>2}\) is complex analytic for all \(z(\emptyset) \in \mathbb{C}\), and \(g(0)\) is nonvanishing on \(\mathbb{C}\). Note that \(\mathcal{G}\) is a group under point-wise multiplication.

For \(g \in \mathcal{G}\), define the N=2 superconformal map

\[
(5.1) \quad I_g : (\Lambda^0_{>1})^\times \oplus (\Lambda^1_{>1})^2 \to (\Lambda^0_{>1})^\times \oplus (\Lambda^1_{>1})^2
\]

\[
(z, \theta^+, \theta^-) \mapsto I_g(z, \theta^+, \theta^-) = \left(\frac{1}{z}, \frac{i\theta^+ g(z)}{z}, \frac{i\theta^-}{z g(z)}\right).
\]
Define $S^2\hat{\mathcal{C}}(g)$, for $g \in \mathcal{G}$, to be the genus zero $N=2$ superconformal super-Riemann surface over $\Lambda_{s>1}$ with $N=2$ superconformal structure given by the covering of local coordinate neighborhoods $\{U_{\Delta s}, U_{T_s}\}$ and the local coordinate maps

\begin{align}
\Delta_g &: U_{\Delta s} \longrightarrow \Lambda_{s>1}^0 \oplus (\Lambda_{s>1}^1)^2 \\
\Upsilon_g &: U_{T_s} \longrightarrow \Lambda_{s>1}^0 \oplus (\Lambda_{s>1}^1)^2,
\end{align}

which are homeomorphisms of $U_{\Delta s}$ and $U_{T_s}$ onto $\Lambda_{s>1}^0 \oplus (\Lambda_{s>1}^1)^2$, respectively, such that

\begin{align}
\Delta_g \circ \Upsilon_g^{-1} &: (\Lambda_{s>1}^0)^x \oplus (\Lambda_{s>1}^1)^2 \longrightarrow (\Lambda_{s>1}^0)^x \oplus (\Lambda_{s>1}^1)^2 \\
(z, \theta^+, \theta^-) &\mapsto I_g(z, \theta^+, \theta^-).
\end{align}

Thus the body of $S^2\hat{\mathcal{C}}(g)$ is the Riemann sphere, i.e., $(S^2\hat{\mathcal{C}}(g))_{\mathcal{H}} = \hat{\mathcal{C}} = \mathbb{C} \cup \{\infty\}$.

The group of $N=2$ superconformal automorphisms from the $N=2$ superconformal plane $S^2\mathcal{C}$ to itself that preserve the even coordinate is comprised of transformations

\begin{align}
T &: S^2\mathcal{C} \longrightarrow S^2\mathcal{C} \\
(z, \theta^+, \theta^-) &\mapsto \left(z, \theta^+ \varepsilon^+(z), \frac{\theta^-}{\varepsilon^+(z)}\right)
\end{align}

where $\varepsilon^+(z)$ is an even superanalytic function defined for all $z \in \Lambda_{s>1}^0$ such that $\varepsilon^+(z(0))$ is nonzero for all $z(0) \in \mathbb{C}$. The set of all such $N=2$ superconformal automorphisms of the $N=2$ superplane that have even component $z$ is a proper subgroup of the the group of $N=2$ superconformal automorphisms of the $N=2$ superplane, and in fact, is an abelian subgroup.

Let $\mathcal{E}$ be the set of even superanalytic functions $\varepsilon^+ : \Lambda_{s>1}^0 \rightarrow (\Lambda_{s>1}^0)^x$. That is $\varepsilon^+(z(0)) \neq 0$ for all $z(0) \in \mathbb{C}$. Then this set $\mathcal{E}$ is a group under point-wise multiplication of functions and is isomorphic to the group of transformations of the form (5.5). If we restrict the domain of elements of $\mathcal{E}$ to $(\Lambda_{s>1}^0)^x$, we see that $\mathcal{E}$ is a subgroup of $\mathcal{G}$. And since $\mathcal{G}$ is in fact abelian, it is a normal subgroup. Let $\mathcal{E}^{\infty}$ be the subgroup of $\mathcal{G}$ given by the set of functions $\{\varepsilon^+(1/z) : (\Lambda_{s>1}^0)^x \rightarrow (\Lambda_{s>1}^0)^x | \varepsilon^+(z) \in \mathcal{E}\}$. Let $\mathcal{E} = \mathcal{E}^0 \mathcal{E}^{\infty}$. Then $\mathcal{E}$ is a proper normal subgroup of $\mathcal{G}$.

**Lemma 5.1.** Let $g, h \in \mathcal{G}$. If $h(z) = g(z)\varepsilon(z)$ for some $\varepsilon \in \mathcal{E}$, then $S^2\hat{\mathcal{C}}(g)$ is $N=2$ superconformally equivalent to $S^2\mathcal{C}(h)$.

**Proof.** The automorphisms of the $N=2$ superconformal plane $S^2\mathcal{C}$ that preserve the even coordinate are of the form $T_\varepsilon^+(z, \theta^+, \theta^-) = (z, \theta^+ \varepsilon^+(z), \theta^-/\varepsilon^+(z))$ for $\varepsilon^+ \in \mathcal{E}^0 \leq \mathcal{E}$. Any $\varepsilon \in \mathcal{E} = \mathcal{E}^0 \mathcal{E}^{\infty}$ can be expressed as $\varepsilon(z) = \varepsilon^0(z)\varepsilon^{\infty}(z)$ for $\varepsilon^0 \in \mathcal{E}^0$ and $\varepsilon^{\infty} \in \mathcal{E}^{\infty}$.

Let $I^0(z) = 1/z$. The inverse of $\varepsilon^0$ in $\mathcal{E}^0$, given by $1/\varepsilon^0(z) = I^0 \circ \varepsilon^0(z)$, is in $\mathcal{E}^0$ and the function $\varepsilon^{\infty}(1/z) = \varepsilon^{\infty} \circ I^0(z)$ is also in $\mathcal{E}^0$. Thus changing coordinates in the chart $(U_{T_g}, \Upsilon_g)$ of $S^2\hat{\mathcal{C}}(g)$ by $T_{I^0 \circ \varepsilon^0}$ and changing coordinates in the chart $(U_{\Delta g}, \Delta_g)$ of $S^2\hat{\mathcal{C}}(g)$ by $T_{\varepsilon^0 \circ I^0}$, the new change of coordinates from $(U_{T_g}, T_{I^0 \circ \varepsilon^0} \circ \Upsilon_g)$ to $(U_{\Delta g}, T_{\varepsilon^0 \circ I^0} \circ \Delta_g)$ is given by

\begin{align}
T_{\varepsilon^0 \circ I^0} \circ I^0 \circ T_{I^0 \circ \varepsilon^0}^{-1}(z, \theta^+, \theta^-) &= \left(\frac{1}{z}, \frac{i\theta^+ g(z)\varepsilon(z)}{z}, \frac{i\theta^-}{2g(z)\varepsilon(z)}\right) \\
&= I_h(z, \theta^+, \theta^-).
\end{align}

The result follows. \qed
Lemma 5.2. Let \( z \) be an even variable in \( \Lambda_{>1}^0 \). We have \( G/E \cong \{ z^nE \mid n \in \mathbb{Z} \} \cong \mathbb{Z} \). That is, the group of even superanalytic functions from \( (\Lambda_{>1}^0)^\times \) into \( (\Lambda_{>1}^0)^\times \) under point-wise multiplication modulo the subgroup of even superanalytic functions which extend to \( \Lambda_{>1}^0 \) into \( (\Lambda_{>1}^0)^\times \) is isomorphic to the group of integers under addition. In other words, the sequence

\[
\begin{array}{cccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & G & \rightarrow & \exp & E & \rightarrow & 1 \\
& & n & \rightarrow & z^n & & & & \\
\end{array}
\]

is an exact sequence of abelian groups, where \( \exp \) denotes the exponential function.

Proof. If \( \varepsilon^+ \in E^0 \), then \( \varepsilon^+(z) = \exp(f(z)) \) for \( f(z) \) an even superanalytic function defined for all \( z \in \Lambda_{>1}^0 \). Thus if \( \varepsilon \in E = E^0E^\infty \), then \( \varepsilon(z) = \exp(f_1(z) + f_2(1/z)) \) where \( f_j(z) \), for \( j = 1, 2 \), are even superanalytic functions defined for all \( z \in \Lambda_{>1}^0 \).

For \( g \in G \cdot E \), we have that either \( g(z) \) can be continued to \( z = z_B + z_S = 0 + z_S \in \{ 0 \} \times (\Lambda_{>1}^0)S \) and has a zero of order \( n \) for some \( n \in \mathbb{Z}^+ \), in which case \( z^{-n}g(z) \in E \), or \( g(1/z) \) can be continued to \( z \in \{ 0 \} \times (\Lambda_{>1}^0)S \) and has a zero of order \( n \) for some \( n \in \mathbb{Z}^+ \), and thus \( z^{-n}g(1/z) \in E \). In the former case, \( g(z) \in z^nE \) and in the latter case, \( g(1/z) \in z^nE \) implying that \( g(z) \in z^{-n}E \). \( \square \)

Remark 5.3. It is interesting to note that \( E \) does in general contain polynomials, which would not be the case if we were working with coefficients in \( \mathbb{C} \) rather than in \( \Lambda_{>1}^0 \). For instance \( 1 + az \in E \) if and only if \( a = 0 \), i.e., \( a \in \{ 0 \} \times (\Lambda_{>1}^0)S \).

That is, the failure of polynomials to factor over \( \Lambda_{>1}^0 \) is a direct consequence of the degree to which elements in \( \Lambda_{>1}^0 \) fail to be invertible.

Remark 5.4. Lemmas 5.1 and 5.2 imply that each \( S^2\hat{C}(g) \), for \( g \in G \), is \( \text{N}=2 \) superconformally equivalent to \( S^2\hat{C}(z^n) \) for some \( n \in \mathbb{Z} \).

Remark 5.5. In Lemma 5.6 below, we will show that the converse of Lemma 5.1 is also true. We can already see evidence of this by noting that, in particular, we have the following rather interesting inequivalent \( \text{N}=2 \) super-Riemann spheres: \( S^2\hat{C}(z) \) which is a globally trivial fiber bundle in the first fermionic (i.e., odd) component of the soul fiber but not in the second; \( S^2\hat{C}(1/z) \) which is a globally trivial fiber bundle in the second fermionic (i.e., odd) component of the soul fiber but not in the first; and \( S^2\hat{C}(1) \) which is a globally nontrivial fiber bundle in all soul components.

Let

\[
SL(2, \Lambda_{>2}^0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \Lambda_{>2}^0, ad - bc = 1 \right\},
\]

and let \( GL(1, \Lambda_{>2}^0) = (\Lambda_{>2}^0)^\times \).

For each \( n \in \mathbb{Z} \), and

\[
\alpha = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \in SL(2, \Lambda_{>2}^0) \times GL(1, \Lambda_{>2}^0),
\]

define

\[
\alpha \cdot (z, \theta^+, \theta^-) = \left( \frac{az + b}{cz + d}, \theta^+ \epsilon (cz + d)^{n-1}, \theta^- \epsilon^{-1} (cz + d)^{-n-1} \right)
\]

for \( (z, \theta^+, \theta^-) \in (\Lambda_{>1}^0 \setminus \{ -d_B/c_B \}) \times (\Lambda_{>1}^0)S \times (\Lambda_{>1}^0)^2 \).
The group $\text{SL}(2, \Lambda^0_{-2}) \times \text{GL}(1, \Lambda^0_{-2})$ acts on $S^2\hat{\mathbb{C}}(z^n)$ for $n \in \mathbb{Z}$ as global N=2 superconformal transformations as follows:

For each $n \in \mathbb{Z}$, and $\alpha \in SL(2, \Lambda^0_{-2}) \times \text{GL}(1, \Lambda^0_{-2})$, define the map

\begin{equation}
T^\alpha_{\Delta} : (\Lambda^0_{>1} \setminus \{-d_B/c_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2 \rightarrow (\Lambda^0_{>1} \setminus \{a_B/c_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2
\end{equation}

by

\begin{equation}
T^\alpha_{\Delta}(z, \theta^+, \theta^-) = \alpha \cdot_n (z, \theta^+, \theta^-).
\end{equation}

In addition, define

\begin{equation}
T^\alpha_T : (\Lambda^0_{>1} \setminus \{-a_B/b_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2 \rightarrow (\Lambda^0_{>1} \setminus \{d_B/b_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2
\end{equation}

by

\begin{equation}
T^\alpha_T(z, \theta^+, \theta^-) = \begin{pmatrix} c + dz \frac{a + dz}{a + b_z} \theta^+ e(a + b_z)^{n-1}, \theta^- e^{-1}(a + b_z)^{-n-1} \end{pmatrix};
\end{equation}

that is $T^\alpha_T(z, \theta^+, \theta^-) = I_{\Lambda^0_{>1}} \circ T^\alpha_{\Delta} \circ I_{\Lambda^0_{>1}}(z, \theta^+, \theta^-)$ for $(z, \theta^+, \theta^-) \in ((\Lambda^0_{>1})^1 \setminus (\Lambda^0_{>1})^2) \times ((\Lambda^0_{>1})^1 \setminus (\Lambda^0_{>1})^2) \times ((\Lambda^0_{>1})^1 \setminus (\Lambda^0_{>1})^2).

Let $\{(U_\Delta, \Delta), (U_T, T)\}$ be the atlas for $S^2\hat{\mathbb{C}}(z^n)$ given by $[\mathbb{H}^2] - [\mathbb{H}^2]$ with $g(z) = z^n$. We define $T^{\alpha\alpha}_\Delta : S^2\hat{\mathbb{C}}(z^n) \rightarrow S^2\hat{\mathbb{C}}(z^n)$ by

\begin{equation}
T^{\alpha\alpha}_\Delta(p) = \begin{cases} \Delta^{-1} \circ T^\alpha_{\Delta} \circ (\Delta(p)) & \text{if } p \in U_\Delta \setminus X_1, \\ Y^{-1} \circ T^\alpha_T \circ (\alpha(p)) & \text{if } p \in U_T \setminus X_2, \end{cases}
\end{equation}

where $X_1 = \Delta^{-1}((\{-d_B/c_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2)$ and $X_2 = Y^{-1}((\{-a_B/b_B\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2)$. This defines $T^{\alpha\alpha}_\Delta$ for all $p \in S^2\hat{\mathbb{C}}(z^n)$ unless:

(i) $a_B = 0$ and $p \in Y^{-1}((\{0\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2)$; or
(ii) $d_B = 0$ and $p \in \Delta^{-1}((\{0\} \times (\Lambda^0_{>1})s) \oplus (\Lambda^1_{>1})^2).

In case (i), we define

\begin{equation}
T^{\alpha\alpha}_\Delta(p) = \Delta^{-1} \left( \frac{a + dz}{c + dz}, i\theta^+ e(c + dz)^{n-1}, i\theta^- e^{-1}(c + dz)^{-n-1} \right),
\end{equation}

for $Y(p) = (z, \theta^+, \theta^-) = (z^s, \theta^+, \theta^-)$.

In case (ii), we define

\begin{equation}
T^{\alpha\alpha}_\Delta(p) = Y^{-1} \left( \frac{cz + d}{az + b}, -i\theta^+ e(az + b)^{n-1}, -i\theta^- e^{-1}(az + b)^{-n-1} \right),
\end{equation}

for $\Delta(p) = (z, \theta^+, \theta^-) = (z^s, \theta^+, \theta^-)$.

Note that with this definition, $T^{\alpha\alpha}_\Delta$ is uniquely determined by $T^\alpha_{\Delta}$, i.e., by its value on $\Delta(U_\Delta)$. Or equivalently, $T^{\alpha\alpha}_\Delta$ is uniquely determined by $T^\alpha_T$, i.e., by its value on $Y(U_T)$.

The group of transformations determined by this action of $\text{SL}(2, \Lambda^0_{-2}) \times \text{GL}(1, \Lambda^0_{-2})$ is a subgroup of the group of automorphisms of $S^2\hat{\mathbb{C}}(z^n)$ for each $n \in \mathbb{Z}$. In fact it is a proper subgroup, but we do not need this fact here; see [B7] for a proof of this fact in the case $n = 0$ and [B10] for the cases $n \neq 0$. By Lemma 5.1, $\text{SL}(2, \Lambda^0_{-2}) \times \text{GL}(1, \Lambda^0_{-2})$ also acts via N=2 superconformal automorphisms on $S^2\hat{\mathbb{C}}(g)$, for $g \in G$, in the obvious way.
Lemma 5.6. Let \( g, h \in G \). If \( gE \neq hE \), then \( S^2\hat{C}(g) \) and \( S^2\hat{C}(h) \) are \( N=2 \) super-conformally inequivalent.

Proof. Suppose that \( S^2\hat{C}(g) \) and \( S^2\hat{C}(h) \) are \( N=2 \) superconformally equivalent with \( N=2 \) superconformal equivalence \( F : S^2\hat{C}(g) \to S^2\hat{C}(h) \). By Lemma 5.1, we can assume \( g(z) = z^{m} \) and \( h(z) = z^{n} \) for \( m, n \in \mathbb{Z} \). By first acting on \( S^2\hat{C}(z^n) \) by a global \( N=2 \) superconformal transformation \( T^{n,\alpha} \) for \( \alpha \in SL(2,\mathbb{A}_x^{2}) \), we can assume without loss of generality that \( F \) sends the even component of the points \( (0, \theta^{+}, \theta^{-}) \), \( (1, \theta^{+}, \theta^{-}) \) and \( (\infty, \theta^{+}, \theta^{-}) \) to the even points \( 0, 1, \) and infinity, respectively. That is, in terms of the local coordinate charts \( \{(U_{\Delta_{m}}, \Delta_{m}), (U_{Y_{m}}, Y_{m})\} \) and \( \{(U_{\Delta_{n}}, \Delta_{n}), (U_{Y_{n}}, Y_{n})\} \) for \( S^2\hat{C}(z^n) \) and \( S^2\hat{C}(z^m) \), respectively, we have \( F(\Delta_{m}^{-1}(0, \theta^{+}, \theta^{-})) = \Delta_{n}^{-1}(0, \rho^{+}, \rho^{-}), \) \( F(\Delta_{m}^{-1}(1, \theta^{+}, \theta^{-})) = \Delta_{n}^{-1}(1, \rho^{+}, \rho^{-}), \) and \( F(\Delta_{m}^{-1}(0, \theta^{+}, \theta^{-})) = \Delta_{n}^{-1}(0, \rho^{+}, \rho^{-}). \)

Any \( N=2 \) superconformal equivalence that fixes these points is equivalent to a redefinition of the coordinates in the local coordinate charts \( (U_{Y_{m}}, Y_{m}) \) and \( (U_{\Delta_{m}}, \Delta_{m}) \) by automorphisms of the two copies of the \( N=2 \) superconformal plane \( \mathbb{Y}_{m}(U_{Y_{m}}) \) and \( \Delta_{m}(U_{\Delta_{m}}) \) that preserve the even coordinate.

The automorphisms of the \( N=2 \) superconformal plane \( S^2\hat{C}(g) \) that preserve the even coordinate are of the form \( T_{\frac{m}{n}}(z, \theta^{+}, \theta^{-}) = (z, \theta^{+}e^{+}(z), \theta^{-}/e^{+}(z)) \) for \( \epsilon^{+} \in \mathcal{E} \leq \mathcal{E} \).

Thus changing coordinates in the chart \( (U_{Y_{g}}, Y_{g}) \) of \( S^2\hat{C}(g) \) by \( T_{\frac{m}{n}} \) for \( \frac{m}{n} \in \mathcal{E} \) and changing coordinates in the chart \( (U_{\Delta_{g}}, \Delta_{g}) \) of \( S^2\hat{C}(g) \) by \( T_{\frac{m}{n}} \) for \( \frac{m}{n} \in \mathcal{E} \), the new change of coordinates from \( (U_{\Delta_{g}}, \Delta_{g}) \) to \( (U_{\Delta_{g}}, \Delta_{g}) \) is given by

\[
T_{\frac{m}{n}} \circ I_{g} \circ T^{-1}_{\frac{m}{n}}(z, \theta^{+}, \theta^{-}) = \left( \frac{1}{z}, \frac{i\theta^{+}g(z)\epsilon^{-}(z)}{z\epsilon^{+}(z)}, \frac{i\theta^{-}e^{+}(z)}{zg(z)\epsilon^{-}(z)} \right) = I_{h}(z, \theta^{+}, \theta^{-})
\]

for

\[
h(z) = g(z)\frac{\epsilon^{-}(z)}{\epsilon^{+}(z)}.
\]

Since \( \epsilon^{-}(z) \in \mathcal{E} \), we have \( \epsilon^{-}(1/z) \in \mathcal{E}^{\infty} \). And since \( 1/\epsilon^{+}(z) \) is the inverse of \( \epsilon^{+}(z) \) in \( \mathcal{E} \), we have that \( \epsilon^{-}(1/z)/\epsilon^{+}(z) \in \mathcal{E}^{0}\mathcal{E}^{\infty} = \mathcal{E} \). But this implies \( gE = hE. \)

Thus as a corollary to Lemmas 5.1, 5.2 and 5.6, we have the following theorem:

**Lemma 5.7.** There is a bijection between the set of \( N=2 \) superconformal equivalence classes of \( N=2 \) super-Riemann spheres in \( \{S^2\hat{C}(g) \mid g \in G\} \) and the set \( \mathcal{G}/\mathcal{E} \cong \mathbb{Z} \). In other words, the quotient group \( \mathcal{G}/\mathcal{E} \) classifies the \( N=2 \) superconformal structures of the form \( \{S^2\hat{C}(g) \mid g \in G\} \) over the Riemann sphere, and the moduli space of such \( N=2 \) super-Riemann spheres is given by \( \{S^2\hat{C}(z^n) \mid n \in \mathbb{Z}\} \).

5.2. The Uniformization Theorem for simply connected \( N=2 \) superconformal and \( N=1 \) superanalytic super-Riemann surfaces.

**Theorem 5.8.** Any \( N=2 \) superconformal DeWitt super-Riemann surface with simply connected noncompact body is \( N=2 \) superconformally equivalent to the \( N=2 \) super plane \( S^2\mathbb{C} \) or the \( N=2 \) super upper half-plane \( S^2\mathbb{H} \). Any \( N=2 \) superconformal DeWitt super-Riemann surface with genus-zero, simply connected compact body is \( N=2 \) superconformally equivalent to one of the unique \( N=2 \) superconformal structures over the Riemann sphere \( \{S^2\hat{C}(z^n) \mid n \in \mathbb{Z}\} \), where \( S^2\hat{C}(z^n) \) is given explicitly
by the covering of local coordinate neighborhoods \( \{ U_{\Delta_n}, U_{\Upsilon_n} \} \) and the local coordinate maps

\[
\begin{align*}
\Delta_n &: U_{\Delta_n} \longrightarrow \Lambda_{\alpha > 1}^0 \oplus (\Lambda_{\alpha > 1}^1)^2 \\
\Upsilon_n &: U_{\Upsilon_n} \longrightarrow \Lambda_{\alpha > 1}^0 \oplus (\Lambda_{\alpha > 1}^1)^2,
\end{align*}
\]

which are homeomorphisms of \( U_{\Delta_n} \) and \( U_{\Upsilon_n} \) onto \( \Lambda_{\alpha > 1}^0 \oplus (\Lambda_{\alpha > 1}^1)^2 \), respectively, such that

\[
\Delta_n \circ \Upsilon_n^{-1} : (\Lambda_{\alpha > 1}^0)^{\times} \oplus (\Lambda_{\alpha > 1}^1)^2 \longrightarrow (\Lambda_{\alpha > 1}^0)^{\times} \oplus (\Lambda_{\alpha > 1}^1)^2
\]

\[
(z, \theta^+, \theta^-) \longmapsto \left( \frac{1}{z}, \frac{i\theta^+}{z^{n-1}}, \frac{i\theta^-}{z^{n-1}} \right).
\]

In particular, the moduli space of simply connected \( N=2 \) superconformal DeWitt super-Riemann surfaces under \( N=2 \) superconformal equivalence is isomorphic to the moduli space of simply connected \( N=2 \) superconformal \( C_{\alpha > 1}(2) \)-supermanifolds under \( N=2 \) superconformal equivalence.

**Proof.** Since \( M_B \) is simply connected, we have that \( \check{H}^1(M_B, TM_B) = 0 \). Thus by Theorem 4.1, \( M \) is given by a coordinate atlas with coordinate transition functions of the form \( (\ref{eq:pt}) \). By the uniformization theorem for Riemann surfaces, we know that \( M_B \) is conformally equivalent to \( \mathbb{C}, \mathbb{H} \) or \( \hat{\mathbb{C}} \). This implies that for the body there exist coordinate redefinitions \( f_0^\alpha : \Omega_\alpha(\theta)(\Omega_\alpha) \longrightarrow \mathbb{C} \) for \( \alpha \in A \) such that the new atlas under these coordinate transformations is equivalent to the standard coordinate atlas for \( \mathbb{C}, \mathbb{H} \), or \( \hat{\mathbb{C}} \).

It remains to show that there exist \( N=2 \) superconformal coordinate redefinitions that uniformize the body of \( M \). Furthermore, we must show that we can further reduce, under \( N=2 \) superconformal transformations, the coordinate atlas on \( M \) to be of the form \( (\ref{eq:pt}), \ (\ref{eq:pt}) \) for the compact case, and to be the usual coordinate atlases on \( \mathbb{C} \) and \( \mathbb{H} \) with trivial transition functions in the soul directions in the noncompact case.

Letting \( f^\alpha_0 : (\Omega_\alpha)(\theta)(\Omega_\alpha) \longrightarrow \mathbb{C} \) for \( \alpha \in A \) be the coordinate redefinitions of \( M_B \) taking \( M_B \) to \( \mathbb{C}, \mathbb{H} \), or \( \hat{\mathbb{C}} \), let \( H^\alpha : (\Omega_\alpha)(\theta)(\Omega_\alpha) \longrightarrow \Lambda_{\alpha > 1}^0 \times (\Lambda_{\alpha > 1}^1)^2 \) be given by \( H^\alpha(z, \theta^+, \theta^-) = (f_0^{\alpha})(z), \theta^+ \gamma f_0^\alpha(z), \theta^- \) for \( \alpha \in A \). Under these \( N=2 \) superconformal coordinate redefinitions, we see that \( M \) is \( N=2 \) superconformally equivalent to an \( N=2 \) super-Riemann surface whose body is \( \mathbb{C}, \mathbb{H} \), or \( \hat{\mathbb{C}} \), respectively, and with atlas \( \{ (U_\alpha, \Omega_\alpha) \mid \alpha \in A \} \) with transition functions given by

\[
H_{\alpha\beta}(z, \theta^+, \theta^-) = \Omega_{\alpha \beta}^{-1}(z, \theta^+, \theta^-) = (f_{\alpha\beta}(z), \theta^+ g_{\alpha\beta}^+(z), \theta^- g_{\alpha\beta}^-(z))
\]

for \( \alpha, \beta \in A \), and where each \( (f_{\alpha\beta})_\theta(z) \) is given by \( \gamma \) in the case of \( M_B = \mathbb{C} \) and \( M_B = \mathbb{H} \), or by \( z \) or \( z^{-1} \) in the case of \( M_B = \hat{\mathbb{C}} \), and where the \( g_{\alpha\beta}^\pm \) satisfy the cocycle condition

\[
g_{\alpha\beta}^\gamma(z) = g_{\alpha\beta}^+(z^\gamma) g_{\beta\gamma}^+(z^\gamma),
\]

on the coordinate atlas \( \{ (U_\alpha, \Omega_\alpha) \}_{\alpha \in A} \), for all \( \alpha, \beta, \gamma \in A \) with \( U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \). Note that by the superconformal condition \( (\ref{eq:pt}) \) the \( g_{\alpha\beta}^- \) are completely determined by \( g_{\alpha\beta}^+ \).

Define retractible submanifold(s) of \( M_B \), each denoted by \( M_{\text{ref}} \), to be \( M_B \) itself if \( M_B = \mathbb{C} \) or \( \mathbb{H} \), to be \( \hat{\mathbb{C}} \setminus \{ \infty \} \) or \( \mathbb{C} \times \{ \infty \} \) if \( M_B = \hat{\mathbb{C}} \). Writing the \( g_{\alpha\beta}^\pm \) for
\( \alpha, \beta \in A \) in component form

\[
g_{\alpha\beta}^+(z) = \sum_{(j) \in J^0_{-2}} (g_{\alpha\beta}^+(z_{j1}, z_{j2}, \ldots, z_{j2n}),
\]

we note that the cocycle condition \((5.26)\) at the zero level reduces to

\[
g_{\alpha\beta}^+(z_{\emptyset}) = (g_{\alpha\beta}^+(z_{\emptyset}))(z_{\emptyset})^\beta_g(z_{\emptyset}^\gamma),
\]
on the body of \( M_B \). On \( M_{ret} \subseteq M_B \), this cohomology is trivial; that is, there exists a global section which gives a trivialization of the line bundle associated to these transition maps. Let \( h^\alpha : (U_\alpha) \times \mathbb{C} \rightarrow \mathbb{C}^2 \) be the coordinate redefinitions which trivialize this line bundle. Let \( H^\overline{\alpha} : \Omega_\alpha(U_\alpha) \rightarrow \bigwedge^0_{n \geq 1} \oplus (\bigwedge^1_{n \geq 1})^2 \) be given by \( H^\alpha(z, \theta^+, \theta^-) = (z, \theta^+ h^\alpha(z), \theta^- (h^\alpha(z))^{-1}) \). With these coordinate redefinitions, the new coordinate transition functions over any retractible submanifold \( M_{ret} \) of \( M_B \) are of the form \((4.11)\) with \( g_{\alpha\beta}^\pm \) of the form

\[
g_{\alpha\beta}^+(z) = 1 + \sum_{(j) \in J^0_{-2} \setminus \{\emptyset\}} (g_{\alpha\beta}^+(z_{j1}, z_{j2}, \ldots, z_{j2n}).
\]

We will show by induction on \( n \in \mathbb{Z}_+ \) that we can set the soul components of these \( g^\pm \) terms equal to zero by \( N=2 \) superconformal coordinate redefinitions. For \( n = 1 \), let \( (j) = (j_1, j_2) \in J^0_{-2} \). The consistency equation \((5.28)\) implies

\[
(g_{\alpha\beta}^+(z_{\emptyset}))(z_{\emptyset}^\gamma) = (g_{\alpha\beta}^+(z_{j1}, z_{j2}))(z_{\emptyset}^\gamma) + (g_{\alpha\beta}^+(z_{\emptyset}))(z_{\emptyset}^\gamma).
\]

This implies that \( g_{\alpha\beta}^+(z_{\emptyset}) \) is a cocycle in the first Čech cohomology of \( M_{ret} \). Since \( M_{ret} \) is retractible this cohomology is trivial, and there exists elements \( h_{(j)} \) in the zeroth cohomology of \( M_{ret} \), such that

\[
(g_{\alpha\beta}^+(z_{\emptyset}))(h_{(j)}) = (g_{\alpha\beta}^+(z_{\emptyset})) = (h_{(j)}) \gamma - (h_{(j)}) \beta.
\]

For \( \alpha \in A \), define the \( N=2 \) superconformal coordinate transformation \( H^\alpha_{\overline{\beta}} \) by \( f_{\alpha}(z^\alpha) = z^\alpha, \ g_{\alpha}^+(z^\alpha) = 1 + (h_{(j)})_{\alpha}(z^\alpha) \right( z_{j1}, z_{j2}, \ldots, z_{j2n} \right) and \( \psi_\alpha^{\pm}(z^\alpha) = 0 \). That is \( H^\alpha_{\overline{\beta}}(z^\alpha, (\theta^+)^\alpha, (\theta^-)^\alpha) = (z^\alpha, \theta^+ (1+(h_{(j)})_{\alpha}(z^\alpha) \right( z_{j1}, z_{j2}, \ldots, z_{j2n} \right), \theta^- (1-(h_{(j)})_{\alpha}(z^\alpha) \right( z_{j1}, z_{j2}, \ldots, z_{j2n} \right)) \). Redefining the coordinates for the charts \((U_\alpha, \Omega_\alpha)\) by the \( N=2 \) superconformal transformations \( H^\alpha_{(j)} \), for \( \alpha \in A \), such that the new coordinate charts are \((U_\alpha, \Omega_\alpha = H^\alpha_{(j)} \circ \Omega_\alpha)\), we have that the new coordinate transformations \( \tilde{H}_{\alpha\beta} = \Omega_\alpha \circ \tilde{\Omega}_{\overline{\beta}}^{-1} = H^\alpha_{(j)} \circ \Omega_{\alpha} \circ \tilde{\Omega}_{\overline{\beta}}^{-1} \circ (H^\beta_{(j)})^{-1} \), for \( \alpha, \beta \in A \), are now of the form

\[
(h_{(j)}) \gamma - (h_{(j)}) \beta \right( z, \theta^+ \tilde{g}_{\alpha\beta}^+(z), \theta^- \tilde{g}_{\alpha\beta}^-(z) \right)
\]

with

\[
\tilde{g}_{\alpha\beta}^+(z_{\emptyset}) = 1
\]

\[
\tilde{g}_{\alpha\beta}^+(z_{(j)}) = (g_{\alpha\beta}^+(z_{(j)})) - (h_{(j)})_{\beta} + (h_{(j)})_{\alpha} = 0
\]

\[
\tilde{g}_{\alpha\beta}^-(z_{(j)}) = 0.
\]

For this new atlas of \( M_{ret} \), we again have the compatibility condition on the triple overlaps, and thus the consistency condition \((5.29)\) again holds on the components of the new coordinate transformation functions, and we can perform the above procedure again for a different \( (j) = (j_1, j_2) \in J^0_{-2} \) without changing the fact that the previous \( g_{\alpha\beta}^+(z_{(j)}) \) terms have been set equal to zero. Doing this repeatedly, we can set the \( g_{\alpha\beta}^+(z_{(j)}) \) terms equal to zero for all \( (j) = (j_1, j_2) \in J^0_{-2} \).
Now we make the inductive assumption: assume coordinate transformations have been made such that the for the new \( g^\pm \) terms, the components \((g^\pm_{\alpha\beta})_{(j)}\) have been set equal to zero for \((j) = (j_1, j_2, \ldots, j_{2k}) \in J^0_{n=2}\), for \(k = 1, \ldots, n-1\). For this new atlas, we again have the compatibility condition on the triple overlaps. Thus for the \(\zeta_1, \ldots, \zeta_{2n}\) level of equation (5.29) applied to \(z(\theta)\), we have that equation (5.29) applies to these new \(g^\pm_{(j)}\)'s. Thus there exists \(h_{(j)}\) in the zeroth cohomology of \(M_{ret}\) such that equation (5.30) holds for these terms.

For \(\alpha \in A\), define the \(N=2\) superconformal transformation \(H^\alpha_{(j)}\) by \(f_{\alpha}(z^\alpha) = z^\alpha, g^\alpha_{(j)}(z^\alpha) = 1 + (h_{(j)}(z^\alpha) \zeta_{j_1} \cdots \zeta_{j_{2n}}), g^-_{\alpha}(z^\alpha) = 1 - (h_{(j)}(z^\alpha) \zeta_{j_1} \cdots \zeta_{j_{2n}})\), and \(\psi^\alpha_{(j)}(z^\alpha) = 0\). Redefining the coordinates for the charts \((U_\alpha, \Omega_\alpha)\) by the \(N=2\) superconformal transformations \(H^\alpha_{(j)}\), for \(\alpha \in A\), it is a straightforward calculation to see that the new coordinate transformations are of the form \((z, \theta^+g^+(z), \theta^-g^-(z))\) with \(g^\pm_{(j)} = 1\) and \(g^\pm_{(j)} = 0\). We perform this procedure of redefining the coordinate charts for each \((j) = (j_1, \ldots, j_{2n}) \in J^0_{n=2}\), resulting in a new atlas with coordinate transition functions that have all the \(g^\pm_{(j)}\) components of length \(2k\) for \(k = 1, \ldots, n\) equal to zero. By induction, we have that the \(g^\pm_{(j)}\) terms of the transition functions on \(M_{ret}\) are equal to the constant function 1.

In the case where \(M_B = M_{ret}\), i.e., \(M_B = \mathbb{C} \text{ or } \mathbb{H}\), this proves that \(M\) is \(N=2\) superconformally equivalent to \(S^2\mathbb{C}\) or \(S^2\mathbb{H}\), respectively. In the case where \(M_{ret} = \mathbb{C} \setminus \{0\} \text{ or } \mathbb{C} \setminus \{\infty\}\), we obtain an \(N=2\) super-Riemann surface whose body is \(\mathbb{C}\), and such that \(M\) is covered by two coordinate charts \((U_\Delta, \Delta)\) and \((U_\Upsilon, \Upsilon)\) with coordinate transition function \(\Delta \circ \Upsilon^{-1}(z, \theta^+, \theta^-) = (z^{-1}, i\theta^+g^+(z), i\theta^-/(zg^-(z)))\), for \(g^+\) an even superanalytic \((1,0)\)-superfunction defined and nonzero for all \(z \in (\Lambda^0_{n=2})^\times\); in other words \(g^+ \in G\). The condition that the transition functions be \(N=2\) superconformal, and thus in particular, satisfy (2.29), implies that

\[
(5.34) \quad \frac{-1}{z^2} = g^+(z)g^-(z).
\]

Letting \(g(z) = -izg^+(z)\), we have

\[
(5.35) \quad \Delta \circ \Upsilon^{-1}(z, \theta^+, \theta^-) = (z^{-1}, i\theta^+g(z)/z, i\theta^-/(zg^-(z))) = I_g(z, \theta^+, \theta^-).
\]

Thus \(M\) is \(N=2\) superconformally equivalent to \(S^2\mathbb{C}(g)\) for some \(g \in G\). By Lemma 5.27, the result follows. 

It is also easy to see that the classification of genus-zero \(N=2\) superconformal DeWitt super-Riemann surfaces up to \(N=2\) superconformal equivalence in fact coincides with the classification of holomorphic line bundles over the underlying Riemann surface up to holomorphic equivalence. We can see this by observing that holomorphic line bundles are classified by \(H^1(M_B, \mathbb{C}^\times)\). For the genus-zero case, this cohomology is trivial for \(M_B\) noncompact and \(\mathbb{Z}\) for \(M_B \cong \mathbb{C}\).

One can see this bijective correspondence between genus-zero \(N=2\) super-Riemann surfaces and holomorphic line bundles over \(M_B \cong \mathbb{C}\) explicitly, by noting that the \(N=2\) super-Riemann sphere \(S^2\mathbb{C}(z^n)\) for \(n \in \mathbb{Z}\) has, as a substructure, the \(GL(1, \mathbb{C})\)-bundle over \(\mathbb{C}\) given by the transition function \(iz^{n-1} : \mathbb{C}^\times \rightarrow \mathbb{C}^\times\), corresponding to the transition function for the first fermionic component of \(S^2\mathbb{C}(z^n)\) restricted to the fiber in the first component of \(\theta^+ = \theta^+_1(1)\zeta_1 + \theta^+_2(2)\zeta_2 + \cdots\). (Or equivalently, one can restrict to the \((j)\)-th component for \((j) \in J^1_{n>1}\).) Moreover, the \(GL(1, \mathbb{C})\)-bundle
over \( \hat{\mathbb{C}} \) with transition function \( iz^{n-1} : \mathbb{C}^\times \to \mathbb{C}^\times \), for \( n \in \mathbb{Z} \), picks out a unique \( S^2\hat{\mathbb{C}}(z^n) \). Under this bijection between equivalence classes of \( N=2 \) super-Riemann surfaces and equivalence classes of holomorphic line bundles over the body, the \( N=2 \) super-Riemann surface \( S^2\hat{\mathbb{C}}(z^n) \) corresponds to the holomorphic line bundle over \( \hat{\mathbb{C}} \) of degree \( -n + 1 \). In particular, the Uniformization Theorem 5.8 in conjunction with Lemma 4.4 can be thought of as proving that there is a canonical isomorphism \( H^1(\hat{\mathbb{C}},G_k) \cong H^1(\hat{\mathbb{C}},\mathbb{C}^\times) \). That is in the genus-zero case, these \( G_k \)-bundles reduce to line bundles, and thus we have the following corollary:

**Corollary 5.9.** Let \( M_B \) be a Riemann surface of genus zero. \( N=2 \) superconformal DeWitt super-Riemann surfaces with body \( M_B \) are classified up to \( N=2 \) superconformal equivalence by holomorphic line bundles over \( M_B \) up to conformal equivalence.

Using Proposition 3.1 and Corollary 4.2 we have the following corollary to Theorem 5.8 and Corollary 5.9, which gives the uniformization for simply connected \( N=1 \) superanalytic DeWitt super-Riemann surfaces:

**Corollary 5.10.** Any \( N=1 \) superanalytic DeWitt super-Riemann surface with simply connected noncompact body is \( N=1 \) superanalytically equivalent to the \( N=1 \) superanalytically equivalent to the \( N=1 \) superanalytically equivalent sphere given explicitly by the covering of local coordinate neighborhoods \( \{U_{\Delta_n}, U_{\Upsilon_n}\} \) and the local coordinate maps \( \Delta_n : U_{\Delta_n} \to \Lambda_{s>0} \) and \( \Upsilon_n : U_{\Upsilon_n} \to \Lambda_{s>0} \), which are homeomorphisms of \( U_{\Delta_n} \) and \( U_{\Upsilon_n} \) onto \( \Lambda_{s>0} \), respectively, such that

\[
\Delta_n \circ \Upsilon_n^{-1} : \Lambda_{s>0} \to \Lambda_{s>0} \quad (z, \theta) \mapsto \left( \frac{1}{z}, \frac{i\theta}{z^n} \right).
\]

In particular, the moduli space of simply connected \( N=1 \) superanalytic DeWitt super-Riemann surfaces under \( N=1 \) superanalytic equivalence is isomorphic to the moduli space of simply connected \( C_{s>0}(1) \)-supermanifolds under \( N=1 \) superanalytic equivalence, and thus, is isomorphic to the moduli space of holomorphic line bundles over a simply connected Riemann surface under holomorphic equivalence.

### 6. Uniformization for Certain Supermanifolds in the Genus-One Case

We note that \( \hat{H}^1(M_B, TM_B) = \mathbb{C} \) if \( M_B \) is a complex torus. Thus by Theorem 4.1 there is, in general, an obstruction to uniformizing an \( N=2 \) superconformal or \( N=1 \) superanalytic DeWitt supertorus to one with transition functions of the form (4.1), i.e., to a \( \mathcal{H}_{s>1}(2) \)-supermanifold. However, we can analyze the moduli space of genus-one \( N=2 \) superconformal DeWitt super-Riemann surfaces which have coordinate transition functions that correspond to the trivial cocycle in \( H^1(M_B, TM_B) \).

In this section, we show that these supertori are classified up to \( N=2 \) superconformal equivalence, by holomorphic line bundles over the underlying complex torus, up to holomorphic equivalence, cf. [M2].

#### 6.1. The moduli space of complex tori, automorphisms, and theta functions

In this section we present some standard facts about complex tori and theta
functions, following for example [Br], [Mir], [Deb]. Throughout this section and this section only, $z$ denotes a complex variable rather than an even supervariable.

Let $\tau \in \mathbb{H}$, and let $\Gamma \tau = \mathbb{Z} \oplus \tau \mathbb{Z}$. The group $\Gamma \tau$ acts on $\mathbb{C}$ by translation, and the quotient $\mathbb{C}/\Gamma \tau$ defines a complex torus, also known as an elliptic curve. The moduli space (up to conformal equivalence) of complex tori is given by $PSL(2,\mathbb{Z}) \backslash \mathbb{H}$, where the action of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$ is given by

$$ (a \ b \ c \ d) \cdot \tau = \frac{a\tau + b}{c\tau + d} $$

for $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = 1$. From now on, when we refer to a lattice $\Gamma \tau$ for some $\tau \in \mathbb{H}$, it is implied that we mean the equivalence class of $\tau$ in the moduli space $PSL(2,\mathbb{Z}) \backslash \mathbb{H}$.

Let $\omega_n$ be a primitive $n$-th root of unity for $n \in \mathbb{Z}^+$. The group of automorphisms for a complex torus $\mathbb{C}/\Gamma \tau$ are given by translations by elements of $\Gamma \tau$ along with the following groups written multiplicatively and acting on $\mathbb{C}/\Gamma \tau$ via multiplication:

(i) $\langle \omega_4 \rangle = \{ \omega_4^k \mid 1 \leq k \leq 4 \}$ if $\tau = i$;
(ii) $\langle \omega_6 \rangle = \{ \omega_6^k \mid 1 \leq k \leq 6 \}$ if $\tau = e^{2\pi i/3}$;
(iii) $\langle \omega_2 \rangle = \{ 1, -1 \}$ otherwise.

Now fix $\tau \in \mathbb{H}$. Let $\pi_\tau : \mathbb{C} \rightarrow \mathbb{C}/\Gamma \tau$ be the canonical projection map. Let $\{((U_\alpha),B_\alpha)\}_{\alpha \in A}$ be a coordinate atlas on $\mathbb{C}/\Gamma \tau$ given by taking $\pi_\tau^{-1}(U_\alpha)B$ to be an open set in $\mathbb{C}$ such that $\gamma_1(\pi_\tau^{-1}(U_\alpha)B) \cap \gamma_2(\pi_\tau^{-1}(U_\alpha)B) = \emptyset$ for distinct $\gamma_1, \gamma_2 \in \Gamma \tau$. Then the coordinate transition functions for the chart on $\mathbb{C}/\Gamma \tau$ are given by translation by elements of the lattice $\Gamma \tau$.

A theta function associated to $\Gamma \tau$, denoted $\vartheta_\tau$, is an entire function on $\mathbb{C}$ that is not identically zero such that for each $\gamma \in \Gamma \tau$ there exist constants $a_\gamma, b_\gamma \in \mathbb{C}$ satisfying

$$ \vartheta_\tau(z + \gamma) = e^{2\pi i (a_\gamma z + b_\gamma)} \vartheta_\tau(z) \quad \text{for all } \gamma \in \Gamma \tau \text{ and } z \in \mathbb{C}. $$

The constants $\{ a_\gamma, b_\gamma \}_{\gamma \in \Gamma \tau}$ are called the type of the theta function $\vartheta_\tau$, and are equivalently defined as maps

$$ a : \Gamma \tau \rightarrow \mathbb{C}, \quad b : \Gamma \tau \rightarrow \mathbb{C} $$

satisfying

$$ a_{\gamma_1 + \gamma_2} = a_{\gamma_1} + a_{\gamma_2} \quad \text{(6.4)}
$$

$$ b_{\gamma_1 + \gamma_2} = (b_{\gamma_1} + b_{\gamma_2} + a_{\gamma_1} a_{\gamma_2}) \mod \mathbb{Z}, \quad \text{(6.5)}
$$

for all $\gamma_1, \gamma_2 \in \Gamma \tau$. Thus we can also refer to the type of the theta function as the pair of maps $(a, b)$.

Let $\{ g_\gamma : \mathbb{C} \rightarrow \mathbb{C}^\times \}_{\gamma \in \Gamma \tau}$ be a set of holomorphic functions which satisfy the condition

$$ g_{\gamma_1 + \gamma_2}(z) = g_{\gamma_2}(z) g_{\gamma_1}(z + \gamma_2) \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma \tau \text{ and } z \in \mathbb{C}. \quad \text{(6.6)}
$$

Then

$$ g_\gamma(z) = \frac{\vartheta_\tau(z + \gamma)}{\vartheta_\tau(z)} = e^{2\pi i (a_\gamma z + b_\gamma)} \quad \text{(6.7)}
$$

for some theta function $\vartheta_\tau$ of type $(a, b)$. If $\vartheta_\tau$ and $\tilde{\vartheta}_\tau$ are of the same type, then they define the same function $g_\gamma$ via (6.7).
Let $\Theta_\tau$ denote the space of theta functions associated to $\Gamma_\tau = \mathbb{Z} \oplus \tau \mathbb{Z}$ modulo equivalence up to type. Note that $\Theta_\tau$ is a group under point-wise multiplication.

A theta function $\vartheta_\tau$ associated to $\Gamma_\tau$ is called trivial if it is of the form

$$\vartheta_\tau(z) = e^{az^2 + bz + c} \quad \text{for some } a, b, c \in \mathbb{C}.$$

These are the theta functions that never vanish, and thus for such a $\vartheta_\tau$, we have that $\vartheta_\tau(z + \gamma)/\vartheta_\tau(z) = e^{2az\gamma + a\gamma^2 + b\gamma}$ is a nonvanishing entire function on $\mathbb{C}$. A trivial theta function given by (6.8) is of type $\{a_\gamma, b_\gamma\}_{\gamma \in \Gamma}$ where $a_\gamma = \frac{\pi}{\tau} a\gamma$ and $b_\gamma = \frac{\pi}{\tau}(b\gamma + a\gamma^2)$.

Let $\mathcal{T}_\tau$ denote the set of trivial theta functions associated to $\Gamma_\tau$ modulo equivalence up to type. Then $\mathcal{T}_\tau$ is in fact a subgroup of $\Theta_\tau$. Two theta functions are said to be equivalent if their quotient is a trivial theta function. Thus the quotient group $\Theta_\tau/\mathcal{T}_\tau$ is the moduli space of theta functions up to type modulo equivalence with respect to the trivial theta functions. This space is often referred to as the set of normalized theta functions up to type, and such theta functions are often expressed uniquely via a pair consisting of a Hermitian form associated to the Riemannian form of the theta function and a map $\alpha : \Gamma_\tau \longrightarrow U(1, \mathbb{C})$; this pair is called the Appell-Humbert data of the theta function [Br], [Deb].

6.2. A Family of Inequivalent N=2 Superconformal Structures over $\mathbb{C}/\Gamma_\tau$.

An N=2 superconformal structure over $\mathbb{C}/\Gamma_\tau$ is an N=2 superconformal DeWitt super-Riemann surface $M$ with body $M_B = \mathbb{C}/\Gamma_\tau$, such that if $\{(\Omega_\alpha_B, (\Omega_\beta)_B)\}_{\alpha \in A}$ is the coordinate atlas for $\mathbb{C}/\Gamma_\tau$ as given in the previous section, then $M$ is covered by coordinate charts $\{(\Omega_\alpha_B, \Omega_\alpha)\}_{\alpha \in A}$ where $\Omega_\alpha : U_\alpha \longrightarrow \Delta_{s>0} \oplus (\Delta_{s>1})^2$ maps $U_\alpha$ onto $(\Omega_\alpha_B)((\Omega_\alpha_B) \times (\Delta_{s>0}) \oplus (\Delta_{s>1})^2$ and the coordinate transition functions $\Omega_\alpha \circ \Omega_\beta^{-1}$ are N=2 superconformal for all $\alpha, \beta \in A$.

**Remark 6.1.** The group of automorphisms of $\mathbb{C}/\Gamma_\tau$ extend to $M$ by acting via $H(z, \theta^+, \theta^-) = (z + \gamma, \theta^+, \theta^-)$ on translations, for $\gamma \in \Gamma_\tau$ and by $H(z, \theta^+, \theta^-) = (\omega z, \omega^{\theta^+}, \theta^-)$ for multiplicative automorphisms with $\omega \in \mathbb{C}^\times$.

**Definition 6.2.** A super-theta function on $\Delta_{s>0}^0$ associated to $\Gamma_\tau$, denoted, $S\vartheta_\tau$ is an even superanalytic function $S\vartheta_\tau : \Lambda_{s>0}^0 \longrightarrow \Lambda_{s>0}^0$ satisfying

$$S\vartheta_\tau(z + \gamma) = e^{2\pi i (a \gamma + b_\gamma)} S\vartheta_\tau(z) \quad \text{for all } \gamma \in \Gamma_\tau \text{ and } z \in \Lambda_{s>0}^0,$$

where

$$a : \Gamma_\tau \longrightarrow \Lambda_{s>0}^0 \quad b : \Gamma_\tau \longrightarrow \Lambda_{s>0}^0$$

are even superfunctions on $\Gamma_\tau$ satisfying

$$a_{\gamma_1 + \gamma_2} = a_{\gamma_1} + a_{\gamma_2}$$

$$b_{\gamma_1 + \gamma_2} = (b_{\gamma_1} + b_{\gamma_2} + a_{\gamma_1 \gamma_2}) \mod \mathbb{Z},$$

for all $\gamma_1, \gamma_2 \in \Gamma_\tau$. The pair of maps $(a, b)$ on $\Gamma_\tau$ with values in $\Lambda_{s>0}^0$ is called the type of the super-theta function.

A trivial super-theta function on $\Lambda_{s>0}^0$ associated to $\Gamma_\tau$ is one whose image lies in $(\Lambda_{s>0}^0)^\times$.

In [FR], Freund and Rabin introduce a notion of super-theta function. However, their notion is different than ours given here, in that they are interested in extending
the notion of a classical theta function $\theta_\tau(z)$ to $\theta_{\tau + \delta}(z)$ where $z$ is a complex variable $\theta$ is an odd super variable and $\delta$ is an odd parameter. Rather, we are interested in extending the classical notion of theta function to one whose range is in $\Lambda^0_s$, by extending the maps $(z, b)$ which define the type of the theta function to be even functions on the lattice rather than just complex-valued functions on the lattice. (Note that in addition, we extend the domain to $\Lambda^0_{s>1}$, but this is trivially done via (2.1)).

Remark 6.3. Let $S\theta_\tau$ be a super-theta function on $\Lambda^0_s$ of type $(a, b)$ as in Definition 6.2. Writing $a$ and $b$ as

$$a = \sum_{(j) \in J_2^s} a_{\gamma_1} \zeta_{\gamma_1} \cdots \zeta_{\gamma_n} \quad \text{and} \quad b = \sum_{(j) \in J_2^s} b_{\gamma_1} \zeta_{\gamma_1} \cdots \zeta_{\gamma_n}$$

we have from (6.11) and (6.12) that

$$a_{\gamma_1 + \gamma_2} = a_{\gamma_1} + a_{\gamma_2} \quad \text{and} \quad b_{\gamma_1 + \gamma_2} = (b_{\gamma_1} + b_{\gamma_2} + a_{\gamma_1 + \gamma_2}) \mod Z_s$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $(j) \in J_2^s$. Thus a super-theta function on $\Lambda^0_s$ is equivalent to a $2^{s-1}$-tuple $(\theta_{\tau, (0)}, \theta_{\tau, (12)}, \theta_{\tau, (13)}, \ldots, \theta_{\tau, (1234), \ldots})$ where $\theta_{\tau, (j)}$ for $(j) \in J_2^s$ is an ordinary theta function of type $(a_{(j)}, b_{(j)})$.

Remark 6.4. Let $\pi_B : \Lambda^0_s \longrightarrow \mathbb{C}$ denote the canonical projection onto the body of $\Lambda^0_s = (\Lambda^0_s)_B \oplus (\Lambda^0_s)_S$. The trivial super-theta functions on $\Lambda^0_s$ are exactly those super-theta functions such that the theta function $(S\theta_\tau)_B(z_B) = \pi_B \circ S\theta_\tau(z_B)$ is a trivial theta function.

Let $M$ be an $\mathbb{N}=2$ superconformal DeWitt super-Riemann surface with $M_B = \mathbb{C}/\Gamma$. For the remainder of this section, we restrict to the case when the coordinate transition functions for $M$ have their $\psi^\pm$ components and the soul part of their $f$ components equal to zero; that is, they are of the form

$$H_\gamma(z, \theta^+, \theta^-) = (z + \gamma, \theta^+ g_\gamma(z), \theta^-(g_\gamma(z))^{-1})$$

for $\gamma \in \Gamma$, and thus $M$ is a $\mathcal{H}_{s>1}(2)$-supermanifold. Then the compatibility condition on triple overlaps imposes the following condition on the transition functions

$$H_{\gamma_1 + \gamma_2}(z, \theta^+, \theta^-)$$

for $\gamma \in \Gamma$, and thus $M$ is a $\mathcal{H}_{s>1}(2)$-supermanifold. Then the compatibility condition on triple overlaps imposes the following condition on the transition functions

$$H_{\gamma_1 + \gamma_2}(z, \theta^+, \theta^-) = H_{\gamma_1}(z, \theta^+, \theta^-)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $z \in \Lambda^0_{s>1}$. Conversely, any family of even superanalytic functions $g_\gamma$ for $\gamma \in \Gamma$, satisfying (6.18) defines an $\mathbb{N}=2$ superconformal structure on $\mathbb{C}/\Gamma_\tau$ in this way.

Lemma 6.5. The solutions to equation (6.18) are given by

$$g_\gamma(z) = \frac{S\theta_\tau(z + \gamma)}{S\theta_\tau(z)} = e^{2\pi i (a \gamma + b \gamma)}$$

where $S\theta_\tau$ is a super-theta function on $\Lambda^0_{s-2}$ associated to $\Gamma_\tau$ of type $(a, b)$. 
Proof. Writing \( g_\gamma \) in component form, we have

\begin{equation}
(6.20) \quad g_\gamma(z) = \sum_{(j) \in J_{\gamma-2}} (g_\gamma)_{(j)}(z) \zeta_{j1} \zeta_{j2} \cdots \zeta_{jn}. \end{equation}

Restricting to \( z(0) \in \mathbb{C} \), this imposes the condition

\begin{equation}
(6.21) \quad (g_{\gamma_1+\gamma_2})(z(0)) = (g_{\gamma_2})(z(0)) (g_{\gamma_1})(z(0) + \gamma_2) \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma_\tau \end{equation}

on the complex analytic functions \( (g_\gamma)(z) : \mathbb{C} \rightarrow \mathbb{C}^\times \). This implies that there is a theta function \( \vartheta_\tau : \mathbb{C} \rightarrow \mathbb{C} \) associated to \( \Gamma_\tau \) such that

\begin{equation}
(6.22) \quad (g_\gamma)(z(0)) = \frac{\vartheta_\tau(z(0) + \gamma)}{\vartheta_\tau(z(0))} = e^{2\pi i (a_\gamma z(0) + b_\gamma)} \end{equation}

where \( (a_\gamma, b_\gamma) \gamma \in \Gamma_\tau \) is the type of \( \vartheta_\tau \).

Writing \( g_\gamma(z) = e^{2\pi i h_\gamma(z)} \), we have that the condition \( (6.15) \) is equivalent to the condition

\begin{equation}
(6.23) \quad h_{\gamma_1+\gamma_2}(z) = h_{\gamma_2}(z) + h_{\gamma_1}(z + \gamma_2) \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma_\tau, \end{equation}

and expanding \( h_\gamma \) in component form, we have that for each \( (j) \in J_{\gamma-2}^0 \)

\begin{equation}
(6.24) \quad h_{\gamma_1+\gamma_2,j}(z(0)) = h_{\gamma_2,j}(z(0)) + h_{\gamma_1,j}(z(0) + \gamma_2) \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma_\tau. \end{equation}

Then the fact that any solution to \( (6.21) \) is of the form \( (6.22) \), implies that any solution to \( (6.24) \) is of the form

\begin{equation}
(6.25) \quad h_{\gamma,j}(z(0)) = a_{\gamma,j} z(0) + b_{\gamma,j} \end{equation}

for some

\begin{align}
(a(j) : \Gamma_\tau &\rightarrow \mathbb{C} \quad b(j) : \Gamma_\tau \rightarrow \mathbb{C}) \\
\gamma &\mapsto a_{\gamma,j} \quad \gamma &\mapsto b_{\gamma,j}
\end{align}

maps satisfying

\begin{align}
(6.27) \quad a_{\gamma_1+\gamma_2,j} &= a_{\gamma_1,j} + a_{\gamma_2,j} \\
(6.28) \quad b_{\gamma_1+\gamma_2,j} &= (b_{\gamma_1,j} + b_{\gamma_2,j} + a_{\gamma_1,j} \gamma_2) \mod \mathbb{Z},
\end{align}

for all \( \gamma_1, \gamma_2 \in \Gamma_\tau \). Letting

\[ a = \sum_{(j) \in J_{\gamma-2}^0} a_{\gamma,j} \zeta_{j1} \cdots \zeta_{jn} \]

and

\[ b = \sum_{(j) \in J_{\gamma-2}^0} b_{\gamma,j} \zeta_{j1} \cdots \zeta_{jn} \]

the result follows. \( \square \)

Let \( S_\tau \Theta_\tau \) denote the set of super-theta functions on \( \Lambda_{\gamma-2}^0 \) associated to \( \Gamma_\tau \) modulo equivalence up to type. This set is a group under point-wise multiplication. Let \( S_\tau \tau \) denote the subgroup of \( S_\tau \Theta_\tau \) consisting of the trivial super-theta functions.

If \( M \) is an \( N=2 \) superconformal DeWitt super-Riemann surface over \( \Lambda_{\gamma>1} \), with body \( \mathbb{C}/\Gamma_\tau \) and with transition functions of the form \( (6.16) \), then this structure defines a super-theta function over \( \Lambda_{\gamma-2}^0 \) associated to \( \Gamma_\tau \) which is unique up to type. Conversely, given a super-theta function \( S\vartheta_\tau \) over \( \Lambda_{\gamma>1}^0 \), then \( (6.16) \) and \( (6.19) \) define the transition functions for an \( N=2 \) superconformal DeWitt super-Riemann surface over \( \Lambda_{\gamma>1} \), with body \( \mathbb{C}/\Gamma_\tau \).

Denote by \( S^2 \tau_\tau (S\vartheta_\tau) \) the \( N=2 \) superconformal DeWitt super-Riemann surface over \( \Lambda_{\gamma>1} \), with body \( \mathbb{C}/\Gamma_\tau \) and with transition functions determined by the super-theta function \( S\vartheta_\tau \in S_\tau \Theta_\tau \). That is, the transition functions for \( S^2 \tau_\tau (S\vartheta_\tau) \) are given by

\begin{equation}
(6.29) \quad H_\gamma(z, \theta^+, \theta^-) = \left( z + \gamma, \theta^+ S\vartheta_\tau(z + \gamma) S\vartheta_\tau(z), \theta^- S\vartheta_\tau(z) S\vartheta_\tau(z + \gamma) \right)
\end{equation}
for $\gamma \in \Gamma$. 

**Lemma 6.6.** Let $S\vartheta_{\gamma}^{(1)}, S\vartheta_{\gamma}^{(2)} \in S_{\ast}\Theta_{\tau}$ be two super-theta functions on $\Lambda_{\ast}^{0} - 2$ associated to $\Gamma_{\tau}$. If $S\vartheta_{\gamma}^{(1)}(z) = S\vartheta_{\gamma}^{(2)}(z)S\vartheta_{\tau}^{(1)}(z)$ for some trivial super-theta function $S\vartheta_{\tau}^{(1)} \in S_{\ast}\mathcal{T}$, then the $N=2$ superconformal DeWitt super-Riemann surfaces over $\mathbb{C}/\Gamma_{\tau}$ uniquely determined by $S\vartheta_{\tau}^{(1)}$ and $S\vartheta_{\tau}^{(2)}$, respectively, and denoted by $S_{\tau}^{2T_{\tau}}(S\vartheta_{\tau}^{(1)})$ and $S_{\tau}^{2T_{\tau}}(S\vartheta_{\tau}^{(2)})$, respectively, are $N=2$ superconformally equivalent.

**Proof.** If $S\vartheta_{\tau}^{T}$ is a trivial super-theta function, then $S\vartheta_{\tau}^{T} \in \mathcal{E}^{0}$, and we have an automorphism of the $N=2$ superconformal plane $S\mathbb{C}$ that preserves the even coordinate given by $T(z, \theta^{+}, \theta^{-}) = (z, \theta^{+}S\vartheta_{\tau}^{T}(z), \theta^{-}/S\vartheta_{\tau}^{T}(z))$.

The transition functions for the $N=2$ superconformal DeWitt super-Riemann surface $S_{\tau}^{2T_{\tau}}(S\vartheta_{\tau}^{(1)})$ are given by $H_{\gamma}^{(1)}(z, \theta^{+}, \theta^{-}) = (z + \gamma, \theta^{+}g_{\gamma}^{(1)}(z), \theta^{-}(g_{\gamma}^{(1)}(z))^{-1})$ where $g_{\gamma}^{(j)}(z) = S\vartheta_{\gamma}^{(j)}(z + \gamma)/S\vartheta_{\tau}^{(j)}(z)$, for $j = 1, 2$, respectively. Thus if $S\vartheta_{\tau}^{(1)}(z) = S\vartheta_{\tau}^{(2)}(z)$, we have

\[
H_{\gamma}^{(1)}(z, \theta^{+}, \theta^{-}) = (z + \gamma, \theta^{+}g_{\gamma}^{(1)}(z), \theta^{-}(g_{\gamma}^{(1)}(z))^{-1})
\]

\[
= \left( z + \gamma, \theta^{+}g_{\gamma}^{(2)}(z)S\vartheta_{\gamma}^{T}(z + \gamma), \theta^{-}S\vartheta_{\gamma}^{T}(z) \right)
\]

\[
= \left( z + \gamma, \theta^{+}g_{\gamma}^{(2)}(z), \theta^{-}S\vartheta_{\gamma}^{T}(z) \right)
\]

\[
= T^{-1} \circ H_{\gamma}^{(2)} \circ T(z, \theta^{+}, \theta^{-}).
\]

Therefore the automorphism of the $N=2$ superplane $T$ lifts to an $N=2$ superconformal bijection from $S_{\tau}^{2T_{\tau}}(S\vartheta_{\tau}^{(1)})$ to $S_{\tau}^{2T_{\tau}}(S\vartheta_{\tau}^{(2)})$. □

**Lemma 6.7.** We have that $S_{\ast}\Theta_{\tau}/S_{\tau} \Theta_{\tau} \cong \Theta_{\tau}/\Gamma_{\tau}$. That is, the group of super-theta functions on $\Lambda_{\ast}^{0} - 2$ associated to $\Gamma_{\tau}$ up to type modulo the subgroup of trivial super-theta functions is isomorphic to the group of theta functions associated to $\Gamma_{\tau}$ up to type modulo the subgroup of trivial theta functions.

**Proof.** Let $S_{\ast}\vartheta_{\tau} \in S_{\ast}\Theta_{\tau}$ be of type $(a, b)$ for functions $(a, b)$ of the form (6.10) satisfying (6.11) and (6.12). Let $a = a_{B} + a_{S}$ and $b = b_{B} + b_{S}$ be the body and soul decompositions of the functions $a$ and $b$, respectively. That is $a_{B} = \pi_{B}a$ and $a_{S} = \pi_{S}a$, for $\pi_{B} : \Lambda_{\ast}^{0} \to \mathbb{C}$ and $\pi_{S} : \Lambda_{\ast}^{0} \to (\Lambda_{\ast}^{0})_{S}$ the canonical projections onto the body and soul, respectively, of $\Lambda_{\ast}^{0} = (\Lambda_{\ast}^{0})_{B} \oplus (\Lambda_{\ast}^{0})_{S}$, and similarly for $b$.

From Remark 6.4 we have that the super-theta function of type $(a_{S}, b_{S})$ is trivial. Denoting this trivial super-theta function by $S\vartheta_{\tau}^{T}$, we have

\[
\frac{S\vartheta_{\tau}(z + \gamma)}{S\vartheta_{\tau}^{T}(z + \gamma)} = e^{2\pi i (a_{B}, \gamma z + b_{B} \gamma)} \frac{S\vartheta_{\tau}(z)}{S\vartheta_{\tau}^{T}(z)}
\]

implying that $S\vartheta_{\tau}/S\vartheta_{\tau}^{T}$ is a theta function. □

Lemmas 6.6 and 6.7 imply that any $N=2$ superconformal supertorus with transition functions restricted to be of the form (6.10) (or equivalently transition functions whose $\psi^{\pm}$ and soul $f$ components correspond to the zero cocycle in $H(M_{B}, TM_{B})$) is $N=2$ superconformally equivalent to $S_{\tau}^{2T_{\tau}}(\vartheta_{\tau})$ for some $\vartheta_{\tau} \in \Theta_{\tau}/\Gamma_{\tau}$. The following lemma, will allow us to conclude that these are in fact representatives of distinct $N=2$ superconformal equivalence classes.

**Lemma 6.8.** Let $\vartheta_{\tau}^{(1)}, \vartheta_{\tau}^{(2)} \in \Theta_{\tau}$ be two theta functions associated to $\Gamma_{\tau}$ up to type. If $\vartheta_{\tau}^{(1)} T_{\tau} \neq \vartheta_{\tau}^{(2)} T_{\tau}$ (that is, the ratio of $\vartheta_{\tau}^{(1)}$ to $\vartheta_{\tau}^{(2)}$ is not a trivial theta function)
then the $N=2$ super-Riemann surfaces over $\mathbb{C}/\Gamma_\tau$ uniquely determined by $\vartheta^{(1)}_\tau$ and $\vartheta^{(2)}_\tau$, respectively, and denoted by $S^2\mathbb{T}_\tau(\vartheta^{(1)}_\tau)$ and $S^2\mathbb{T}_\tau(\vartheta^{(2)}_\tau)$, respectively, are $N=2$ superconformally inequivalent.

Proof. Let $F : S^2\mathbb{T}_\tau(\vartheta^{(1)}_\tau) \to S^2\mathbb{T}_\tau(\vartheta^{(2)}_\tau)$ be an $N=2$ superconformal equivalence. By Remark 6.1 acting on $S^2\mathbb{T}_\tau(\vartheta^{(1)}_\tau)$ by global automorphisms, we can assume without loss of generality that $F$ restricted to $\mathbb{C}/\Gamma_\tau$ is the identity. Thus $F$ is a transformation from the fiber bundle over $\mathbb{C}/\Gamma_\tau$ defined by $S^2\mathbb{T}_\tau(\vartheta^{(1)}_\tau)$ to the fiber bundle over $\mathbb{C}/\Gamma_\tau$ defined by $S^2\mathbb{T}_\tau(\vartheta^{(2)}_\tau)$. Then to keep the even component of the transition functions equal to $z + \gamma$ and the $\psi^\pm$ components equal to zero, $F$ must be of the form $F(z, \theta^+, \theta^-) = (z, \theta^+ \epsilon^+(z), \theta^- (\epsilon^+(z))^{-1})$, for some $\epsilon^+ \in \mathbb{E}^n \cap S_T\tau$.

Moreover, the transition functions for the $N=2$ super-Riemann surface $S^2\mathbb{T}_\tau(\vartheta^{(2)}_\tau)$ are given by $H^{(1)}_\tau(z, \theta^+, \theta^-) = (z + \gamma, \theta^+ g^{(2)}_\gamma(z), \theta^- (g^{(2)}_\gamma(z))^{-1})$ where $g^{(2)}_\gamma(z) = \vartheta^{(2)}_\tau(z + \gamma)/\vartheta^{(2)}_\tau(z)$, for $j = 1, 2$, respectively, and $F$ must satisfy

$$H^{(1)}_\tau(z, \theta^+, \theta^-) = F^{-1} \circ H^{(2)}_\tau \circ F(z, \theta^+, \theta^-),$$

i.e.,

$$\begin{align*}
(z + \gamma, \theta^+ g^{(1)}(z), \theta^- (g^{(1)}(z))^{-1}) & = (z + \gamma, \theta^+ g^{(2)}_\gamma(z) \epsilon^+(z + \gamma), \theta^- \frac{\epsilon^+(z)}{g^{(2)}_\gamma(z) \epsilon^+(z + \gamma)}).
\end{align*}$$

Thus $\epsilon^+ \in \mathbb{T}_\tau$, and $\vartheta^{(1)}_\tau(z) T_\tau = \vartheta^{(2)}_\tau(z) T_\tau$. \hfill \Box

In the case of $M\mathcal{G}$ a complex torus, the Appell-Humbert Theorem states that the holomorphic line bundles over $\mathbb{C}/\Gamma_\tau$ are classified by $\Theta_\tau/\mathcal{T}_\tau$, that is by theta functions associated to $\Gamma_\tau$ up to type and equivalence by a trivial theta function cf. [BE]. The bijection between equivalence classes of $N=2$ superconformal DeWitt super-Riemann surfaces over a given torus with coordinate transition functions of the form (6.16) and equivalence classes of holomorphic line bundles over this torus is given explicitly, for instance, by restricting to one of the $(j)$-th components, for $(j) \in J^1_{>1}$, of the first fermionic component, i.e., the $\theta^+$ term, as we did in Section 6.6. Thus Lemma 6.4 implies that the $G_\mathcal{A}$-bundle giving the $N=2$ superconformal structure for any $S^2\mathbb{T}_\tau(S\vartheta_\tau)$ reduces to a line bundle. Using this fact, Lemmas 6.6, 6.7 and 6.8 we have the following theorem:

**Theorem 6.9.** The $N=2$ superconformal equivalence classes of $N=2$ superconformal DeWitt super-Riemann surfaces with body $\mathbb{C}/\Gamma_\tau$, and which are also $\mathcal{H}_{\gamma > 1}(2)$-supermanifolds, i.e., which have coordinate transition functions of the form (6.16), are in one-to-one correspondence with theta functions associated to $\Gamma_\tau$ of a given type up to equivalence by the trivial theta functions. That is, these $N=2$ superconformal super-Riemann surfaces are classified up to $N=2$ superconformal equivalence by $\Theta_\tau/\mathcal{T}_\tau \cong \{ S^2\mathbb{T}_\tau(\vartheta_\tau) \mid \vartheta_\tau \in \Theta_\tau/\mathcal{T}_\tau \}$, or equivalently by holomorphic line bundles over $\mathbb{C}/\Gamma_\tau$ up to holomorphic equivalence.

Similarly, $N=1$ superanalytic DeWitt super-Riemann surfaces with body $\mathbb{C}/\Gamma_\tau$ and which are also $\mathcal{H}_{\gamma > 0}(1)$-supermanifolds, are in one-to-one correspondence with theta functions associated to $\Gamma_\tau$ of a given type up to equivalence by the trivial theta functions. That is, these $N=1$ superanalytic super-Riemann surfaces are classified
up to $N=1$ superanalytic equivalence by holomorphic line bundles over $\mathbb{C}/\Gamma_r$ up to holomorphic equivalence.

7. The nonhomogeneous $N=2$ superconformal coordinates and an interpretation of uniformization in terms of loop groups

In this section, we transfer some of our results to the nonhomogeneous $N=2$ supercoordinates, and give an interpretation of the Uniformization Theorems [5.8 and 6.9] in terms of $GL(1)$ loop groups over $\Lambda^0_{*,>1}$.

7.1. $N=2$ superconformal structures over $\hat{\mathbb{C}}$ and over $\mathbb{C}/\Gamma_r$ in nonhomogeneous coordinates. In the nonhomogeneous coordinate system, the uniformized $N=2$ superconformal superspheres $S^2\hat{\mathbb{C}}(z^n)$, for $n \in \mathbb{Z}$, are given by the covering of local coordinate neighborhoods $\{U_{\Delta_n}, U_{\Upsilon_n}\}$ and the local coordinate maps $\Delta_n$ and $\Upsilon_n$ which are homeomorphisms of $U_{\Delta_n}$ and $U_{\Upsilon_n}$ onto $\Lambda^0_{*,>1} / (\Lambda^1_{*,>1})^2$, respectively, such that $\Delta_n \circ \Upsilon_n^{-1} : (\Lambda^0_{*,>1})^x \oplus (\Lambda^1_{*,>1})^2 \rightarrow (\Lambda^0_{*,>1})^x \oplus (\Lambda^1_{*,>1})^2$ is given by

\[
\Delta_n \circ \Upsilon_n^{-1}(z, \theta_1, \theta_2) = \left( \frac{1}{z} \frac{i\theta_1}{2z}(z^n + z^{-n}) - \frac{\theta_2}{2z}(z^n - z^{-n}), \right.
\]
\[
\left. \frac{\theta_1}{2z}(z^n - z^{-n}) + \frac{i\theta_2}{2z}(z^n + z^{-n}) \right).
\]

For an $N=2$ superconformal super-Riemann surface with body $\mathbb{C}/\Gamma_r$ and with transition functions in the homogeneous coordinate system given by $H_\gamma(z, \theta^+, \theta^-) = (z + \gamma, \theta^+ g_\gamma(z), \theta^- (g_\gamma(z))^{-1})$ for $g_\gamma(z) = e^{2\pi i(a_\gamma \ast + b_\gamma)} = \vartheta_\gamma(z + \gamma)/\vartheta_\gamma(z)$ with $\vartheta_\gamma \in \Theta_\gamma/T_\gamma$, we have that in the nonhomogeneous coordinate system the transition functions are of the form

\[
H_\gamma(z, \theta_1, \theta_2) = \left( z + \gamma, \frac{\theta_1}{2} \left( g_\gamma(z) + \frac{1}{g_\gamma(z)} \right) + \frac{i\theta_2}{2} \left( g_\gamma(z) - \frac{1}{g_\gamma(z)} \right), \right.
\]
\[
\left. -\frac{i\theta_1}{2} \left( g_\gamma(z) - \frac{1}{g_\gamma(z)} \right) + \frac{\theta_2}{2} \left( g_\gamma(z) + \frac{1}{g_\gamma(z)} \right) \right) = \left( z + \gamma, \theta_1 \cosh(2\pi i(a_\gamma z + b_\gamma)) + i\theta_2 \sinh(2\pi i(a_\gamma z + b_\gamma)), \right.
\]
\[
\left. i\theta_1 \sinh(2\pi i(a_\gamma z + b_\gamma)) + \theta_2 \cosh(2\pi i(a_\gamma z + b_\gamma)) \right).
\]

From (7.1) and (7.2), it seems that the view from the homogeneous coordinate system is far less opaque than that from the nonhomogeneous system. However, as we shall see in Section 7.2, the nonhomogeneous coordinate setting does give us an intuitive explanation for the classification of genus-zero $N=2$ superconformal DeWitt super-Riemann surfaces, and genus-one $N=2$ superconformal DeWitt super-Riemann surfaces corresponding to the trivial cocycle in $\check{H}^1(M_B, TM_B)$, up to $N=2$ superconformal equivalence that we obtained in Sections 5.2 and 6.2.

7.2. $N=1$ superconformal DeWitt super-Riemann surfaces, affine $u(1)$ and the $GL(1)$ loop group over $\Lambda^0_{*,>1}$. Much of the interpretation below was inspired in part by discussions the author had with Yi-Zhi Huang.

Recall that an $N=1$ superconformal DeWitt super-Riemann surface is a DeWitt $(1,1)$-dimensional supermanifold for which the transition functions, in addition to
being superanalytic are N=1 superconformal. As proved in \[B4\], the Lie superalgebra of infinitesimal N=1 superconformal transformations is given by the superderivations

\[ L_n(z, \theta) = - \left( z^{n+1} \frac{\partial}{\partial z} + \frac{n+1}{2} z^n \theta \frac{\partial}{\partial \theta} \right) \]
\[ G_{n-\frac{1}{2}}(z, \theta) = -z^n \left( \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right) \]

for \( n \in \mathbb{Z} \). Define the N=1 Neveu-Schwarz algebra to be the Lie superalgebra with basis consisting of the central element \( G_d \), for even elements \( L_n \), and odd elements \( G_{n+1/2} \) for \( n \in \mathbb{Z} \), satisfying the supercommutation relations

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} (m^3 - m) \delta_{m+n,0} d, \]
\[ [L_m, G_{n+\frac{1}{2}}] = \left( \frac{m}{2} - n - \frac{1}{2} \right) G_{m+n+\frac{1}{2}}, \]
\[ [G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}}] = 2L_{m+n} + \frac{1}{3} (m^2 + m) \delta_{m+n,0} d, \]

As proved in \[B7\], the Lie superalgebra of infinitesimal N=2 superconformal transformations in the nonhomogeneous coordinate system is given by the superderivations

\[ L_n(z, \theta_1, \theta_2) = - \left( z^{n+1} \frac{\partial}{\partial z} + \frac{n+1}{2} z^n (\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2}) \right) \]
\[ J_n(z, \theta_1, \theta_2) = iz^n \left( \theta_1 \frac{\partial}{\partial \theta_2} - \theta_2 \frac{\partial}{\partial \theta_1} \right) \]
\[ G_{n-\frac{1}{2}}^{(1)}(z, \theta_1, \theta_2) = -z^n \left( \frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial z} \right) - nz^{n-1} \theta_1 \theta_2 \frac{\partial}{\partial \theta_2} \]
\[ G_{n-\frac{1}{2}}^{(2)}(z, \theta_1, \theta_2) = -z^n \left( \frac{\partial}{\partial \theta_2} - \theta_2 \frac{\partial}{\partial z} \right) + nz^{n-1} \theta_1 \theta_2 \frac{\partial}{\partial \theta_1} \]

for \( n \in \mathbb{Z} \). Define the N=2 Neveu-Schwarz algebra to be the Lie superalgebra with basis consisting of the central element \( d \), even elements \( L_n, J_n \), and odd elements \( G_{n+1/2}^{(j)} \) for \( n \in \mathbb{Z}, j = 1, 2 \), satisfying the super commutation relations

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} (m^3 - m) \delta_{m+n,0} d, \]
\[ [L_m, G_{n+\frac{1}{2}}^{(j)}] = \left( \frac{m}{2} - n - \frac{1}{2} \right) G_{m+n+\frac{1}{2}}^{(j)}, \]
\[ [G_{m+\frac{1}{2}}^{(j)}, G_{n-\frac{1}{2}}^{(j)}] = 2L_{m+n} + \frac{1}{3} (m^2 + m) \delta_{m+n,0} d, \]
\[ [G_{m+\frac{1}{2}}^{(j)}, J_{n}^{(j)}] = -i(m-n+1)J_{m+n}, \]
\[ [J_{m}, J_{n}] = \frac{1}{3} m \delta_{m+n,0} d, \]
\[ [J_{m}, G_{n+\frac{1}{2}}^{(j)}] = -iG_{m+n+\frac{1}{2}}^{(j)}, \]
\[ [J_{m}, G_{n+\frac{1}{2}}^{(j)}] = -iG_{m+n+\frac{1}{2}}^{(j)}; \]

\[ [L_m, J_n] = -nJ_{m+n}, \]
\[ [J_m, G_{n+\frac{1}{2}}^{(1)}] = -iG_{m+n+\frac{1}{2}}^{(1)}, \]
\[ [J_m, G_{n+\frac{1}{2}}^{(2)}] = iG_{m+n+\frac{1}{2}}^{(1)}. \]
for \( m, n \in \mathbb{Z}, j = 1, 2 \). It is straightforward to check that the superderivations (7.3)–(7.11) give a representation of the \( N=2 \) Neveu-Schwarz Lie superalgebra with central charge zero.

Thus the \( N=2 \) Neveu-Schwarz Lie superalgebra has two copies of the \( N=1 \) Neveu-Schwarz Lie superalgebra (given by the \( L_n \)'s and \( G^{(j)}_{n-1/2} \)'s, for \( j = 1 \) or \( j = 2 \)) along with a copy of a \( u(1) \) affine Lie algebra running between them given by the \( J_n \)'s.

**Remark 7.1.** Defining \( G_{n+\frac{1}{2}}^\pm = \frac{1}{\sqrt{2}}(G_{n+\frac{1}{2}}^{(1)} \mp iG_{n+\frac{1}{2}}^{(2)}) \), and rewriting the supercommutation relations between \( J_n \) and \( G_{n+\frac{1}{2}}^\pm \) in terms of this new basis, we find that \([J_m, G_{n+\frac{1}{2}}^\pm] = \pm G_{m+n+\frac{1}{2}}^\pm\). We call the basis \( \{L_n, J_n, G_{n+\frac{1}{2}}^\pm, d \mid n \in \mathbb{Z}\} \) the homogeneous basis for the \( N=2 \) Neveu-Schwarz algebra. Performing the change of variables (2.19) from nonhomogeneous coordinates to homogeneous coordinates, we have that the corresponding superderivations (7.3)–(7.11), give a representation of the \( N=2 \) Neveu-Schwarz Lie superalgebra in the homogeneous basis. And here we see another motivation for our terminology: in the homogeneous basis, the \( J_0 \) term has homogeneous supercommutation relations with \( G_{n+\frac{1}{2}}^\pm \). Through the exponentiation of infinitesimal \( N=2 \) superconformal transformations, this results in the relative simplicity of the change of coordinate formulas in the homogeneous case versus the nonhomogeneous case for genus-zero (7.1) and restricted genus-one (7.2) uniformized \( N=2 \) super-Riemann surfaces.

Another way of viewing the relationship between the \( N=1 \) and \( N=2 \) Neveu-Schwarz Lie superalgebras realized as superderivations, is by considering that the superderivations (7.3) and (7.4) along with shown in [B9], these superderivations generate the algebra of all \( N=1 \) superanalytic coordinate transformations, and these are generated by (7.3) and (7.4) along with the zero term of the affine \( u(1) \), namely \( J_0(z, \theta) = \theta \frac{dz}{d\theta} \).

From the fact that the \( N=2 \) Neveu-Schwarz Lie superalgebra of infinitesimal \( N=2 \) superconformal transformations contains two subalgebras of infinitesimal \( N=1 \) superconformal transformations, which give rise to \( N=1 \) superconformal submanifold structures for certain \( N=2 \) super-Riemann surfaces, one can in some intuitive sense, think of the \( N=2 \) superconformal moduli space as arising from two copies of the \( N=1 \) superconformal moduli space with an exponentiated copy of affine \( u(1) \) running between them.

More specifically, by the classification of \( N=1 \) superconformal DeWitt super-Riemann surfaces in [CR], up to \( N=1 \) superconformal equivalence, there is only one unique \( N=1 \) superconformal structure over \( \mathbb{C}, \mathbb{H}, \) and \( \tilde{\mathbb{C}} \), respectively. The unique equivalence class of \( N=1 \) superconformal DeWitt super-Riemann surfaces with compact genus-zero body is given by two coordinate charts \( \{(U_\Delta, \Delta), (U_\Upsilon, \Upsilon)\} \) and local coordinate maps \( \Delta : U_\Delta \rightarrow \mathbb{R}_\geq 0 \) and \( \Upsilon : U_\Upsilon \rightarrow \mathbb{R}_\geq 0 \) which are homeomorphisms of \( U_\Delta \) and \( U_\Upsilon \) onto \( \mathbb{R}_\geq 0 \), respectively, such that \( \Delta \circ \Upsilon^{-1} : \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0 \) is given by \( \Delta \circ \Upsilon^{-1}(z, \theta) = (1/z, i\theta/z) \). We denote this genus-zero \( N=1 \) superconformal DeWitt super-Riemann surface by \( S^1\tilde{\mathbb{C}}(1) \). The \( N=2 \) superconformal DeWitt
super-Riemann surface $S^2 \hat{\mathbb{C}}(1)$, has two embeddings of $S^1 \hat{\mathbb{C}}(1)$, given by the identity mapping on the even subspace (the body and the even fiber component) and identifying the one fermionic component to either the fibers corresponding to the first or the second fermionic component.

In the genus-one case, the $N=1$ superconformal DeWitt super-Riemann surfaces over a complex torus $\mathbb{C}/\Gamma_\tau$ are classified by their $N=1$ superconformal transition functions $H_\tau(z, \theta)$ as follows:

\[(7.19)\]
\[H_{m+n\tau}(z, \theta) = (z + m + nb, e^{\pi in} \epsilon_2 \theta) \quad \text{(nontrivial spin structure)}\]
\[(7.20)\]
\[H_{m+n\tau}(z, \theta) = (z + m + nb + n\delta, \theta + n\delta) \quad \text{(trivial spin structure)}\]

for $m, n \in \mathbb{Z}$, where $b \in \bigwedge^1_{s-1}$ with $b_B = \tau$, $\delta \in \bigwedge^1_{s-1}$, and $(\epsilon_1, \epsilon_2) = (\pm 1, \mp 1)$ or $(-1, -1)$ (see [CR], [LTR], [He]). The first case corresponding to the nontrivial spin structure results in one distinct $N=1$ superconformal equivalence class for each $b \in \bigwedge^1_{s-1}$ with $b_B = \tau$. The family of distinct $N=1$ superconformal equivalence classes in the case of trivial spin structure is parameterized by $b_S = b - \tau \in (\bigwedge^1_{s-1})_S$ and $\delta \in \bigwedge^1_{s-1} / <\pm 1>$.

The $N=1$ superconformal DeWitt super-Riemann surface with trivial spin structure and with $b = b_B = \tau$ and $\delta = 0$ is an $N=1$ superconformal submanifold of the $N=2$ superconformal DeWitt super-Riemann surface $S^2 \mathbb{T}_\tau(1)$ with two unique embeddings given by mapping the $N=1$ fermionic component onto either the first or the second $N=2$ fermionic component for this trivial $\bigwedge^1_{s} \times (\bigwedge^1_{s+1})^2$-bundle over $\mathbb{C}/\Gamma_\tau$.

The $N=1$ superconformal DeWitt super-Riemann surfaces with nontrivial spin structure and with $b = b_B = \tau$ are $N=1$ superconformally equivalent to an $N=1$ superconformal submanifold of $S^2 \mathbb{T}_\tau(\partial_\tau)$ with $\partial_\tau(z + m + n\tau)/\partial_\tau(z) = e^{\pi in}$, i.e., where $\partial_\tau$ is the theta function of type $(\alpha, \beta)$ for $\alpha = 0$ and $\beta = b_{m+n\tau} = n/2$. Again the embedding can be done in two different ways: by embedding the $N=1$ fermionic component into either the first fermionic component or the second fermionic component. This $N=2$ superconformal supertorus is the $N=2$ superconformal super-Riemann surface over $\mathbb{C}/\Gamma_\tau$ with transition functions given by $H_{m+n\tau}(z, \theta_1, \theta_2) = (z + m + n\tau, e^{\pi in} \theta_1, e^{\pi in} \theta_2)$ in the homogeneous coordinate system, which are coincidently given by $H_{m+n\tau}(z, \theta_1, \theta_2) = (z + m + n\tau, e^{\pi in} \theta_1, e^{\pi in} \theta_2)$ in the nonhomogeneous coordinate system.

Using the setting and results of [B7], consider the group given by $N=2$ superconformal transformations of the form

\[(7.21)\]
\[\exp\left(- \sum_{n \in \mathbb{Z}^+} A_n J_n(z, \theta^+, \theta^-) \cdot \exp\left(\sum_{n \in \mathbb{Z}^+} A_n z^n\right) \cdot \right)\]
\[= \left(z, \theta^+ a_0^{-1} \exp\left(\sum_{n \in \mathbb{Z}^+} A_n z^n\right) \cdot \right)\]
\[= \left(z, \theta^+ a_0 \exp\left(\sum_{n \in \mathbb{Z}^+} A_n z^n\right) \cdot \right)\]

for $a_0 \in (\bigwedge^1_{s-2})^\times$, and $A_n \in \bigwedge^1_{s}$, for $n \in \mathbb{Z}^+$, where

\[(7.22)\]
\[J_n(z, \theta^+, \theta^-) = -z^n \left(\theta^+ \frac{\partial}{\partial \theta^+} - \theta^- \frac{\partial}{\partial \theta^-}\right)\]

and the series $\sum_{n \in \mathbb{Z}^+} A_n z^n$ has an infinite radius of convergence. It follows from Theorem 6.10 in [B7], there is a bijection between transformations of the form $(5.5)$ and of the form $(7.21)$. Similarly, exponentiating the $J_n(z, \theta^+, \theta^-)$ terms for $n \leq 0$
acting on \((1/z, i\theta^+/z, i\theta^-/z)\), one obtains the transformations in a neighborhood of infinity.

This exponentiation of these infinitesimals \(J_n(z, \theta^+ + \theta^-, \theta^-)\), which represent the \(u(1)\) affine Lie subalgebra of the \(N=2\) Neveu-Schwarz algebra, over \(\mathcal{L}_{*>1}^0\) gives us the full connected component of the identity in the \(GL(1)\) loop group over \(\bigwedge_{*>1}^0\) [PS] in the case of \(n \in \mathbb{Z}\) which is the group \(\mathcal{E}\). And in the case \(n \in \mathbb{N}\), we obtain the subgroup of the connected component of the \(GL(1)\) loop group which, over \(\bigwedge_{*>1}^0\), corresponds to the subgroup \(\mathcal{E}_0^0\) of \(\mathcal{E}\). The case corresponding to \(n \in \mathbb{N}\) and the subgroup \(\mathcal{E}_0\) occurs when \(M_B\) is noncompact, and the case corresponding to \(n \in \mathbb{Z}\) and the subgroup \(\mathcal{E}\) occurs when \(M_B\) is compact. In the compact genus-zero case, we have the group \(\mathcal{G}\) corresponding to the full loop group, and \(\mathcal{G}/\mathcal{E} \cong \mathbb{Z}\) counts the connected components, in addition to classifying the holomorphic \(GL(1)\)-bundles over \(\mathbb{C}\). And similarly in the genus-one case, exponentiating the affine \(u(1)\), over the two genus-one \(N=1\) superconformal DeWitt super-Riemann surfaces with trivial and non-trivial spin structure, respectively, we again arrive at the holomorphic \(GL(1)\)-bundles over the underlying body manifold giving rise to the moduli space of genus-one \(N=2\) superconformal DeWitt super-Riemann surfaces with transition functions restricted to contain no odd functions of an even variable.

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