INTEGRAL, DIFFERENTIAL AND MULTIPLICATION OPERATORS ON GENERALIZED FOCK SPACES

TESFA MENGESTIE AND SEI-ICHIRO UEKI

Abstract. Volterra companion integral and multiplication operators with holomorphic symbols are studied for a large class of generalized Fock spaces on the complex plane $\mathbb{C}$. The weights defining these spaces are radial and subject to a mild smoothness condition. In addition, we assumed that the weights decay faster than the classical Gaussian weight. One of our main results show that there exists no nontrivial holomorphic symbols $g$ which induce bounded Volterra companion integral $I_g$ and multiplication operators $M_g$ acting between the weighted spaces. We also describe the bounded and compact Volterra-type integral operators $V_g$ acting between $\mathcal{F}_q^\psi$ and $\mathcal{F}_p^\psi$ when at least one of the exponents $p$ or $q$ is infinite, and extend results of Constantin and Peláez for finite exponent cases. Furthermore, we showed that the differential operator $D$ acts in unbounded fashion on these and the classical Fock spaces.

1. Introduction

We denote by $\mathcal{H}(\mathbb{C})$ the space of all entire functions on the complex plane $\mathbb{C}$. For functions $f$ and $g$ in $\mathcal{H}(\mathbb{C})$, the Volterra-type integral operator $V_g$ and its companion operator $I_g$ with symbols $g$ are defined by

$$V_g f(z) = \int_0^z f(w)g'(w)dw$$

and

$$I_g f(z) = \int_0^z f'(w)g(w)dw.$$

Applying integration by parts in any one of the above integrals gives the relation

$$V_g f + I_g f = M_g f - f(0)g(0),$$

(1.1)

where $M_g f = gf$ is the multiplication operator of symbol $g$. These integral operators have been studied extensively on various spaces of analytic functions over several domains with the aim to explore the connection between their operator-theoretic behaviours with the function-theoretic properties of the symbols $g$, especially after the works of Pommerenke [20], and Aleman and Siskakis [5, 6] on Hardy and Bergman spaces. For more information on the subject, we refer to [3, 4, 16, 23] and the related references therein.

In [19], J. Pau and J. Peláez studied some properties of the Carleson embedding maps and the operator $V_g$ on weighted Bergman spaces $A^p(w)$ over the unit disc $\mathbb{D}$ when $w$ belongs to a large class of rapidly decreasing weights. In [10], Constantin

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and Pelàez modified the approaches in [19] and [9], and studied the generalized Fock spaces $\mathcal{F}_p^\psi$ (see definition below) when the corresponding weight decays faster than the classical Gaussian weight. They obtained several results including complete characterizations of the bounded, compact and trace ideal properties of the operator $V_g$. Interestingly, their results show that there exists a much richer structure of $V_g$ on $\mathcal{F}_p^\psi$ than when it acts on the classical Fock spaces $\mathcal{F}_p$; the spaces which consist of all entire functions $f$ on $\mathbb{C}$ for which

$$\int_\mathbb{C} |f(z)|^p e^{-\frac{p}{2}|z|^2} dm(z) < \infty.$$ 

In this paper, we study some mapping properties of the operators $I_g$, $M_g$, and the differential operator $D$ using the settings in [10]. We will also consider the operator $V_g$ for the cases where it has not been considered in [10]. In contrast to the case of the operator $V_g$, one of our results shows that there exists no richer structure of $I_g$ and $M_g$ when they act on the spaces $\mathcal{F}_p^\psi$ than on the classical Fock spaces $\mathcal{F}_p$. In some cases, it rather shows poorer structure. From the relation in (1.1), we also note in passing that if any two of the operators $V_g$, $I_g$ and $M_g$ are bounded so is the third one. In generalized Fock spaces, more can be said namely that $M_g$ is bounded (compact) if and only if so is $I_g$.

We shall thus first set the setting as in [10]: we consider a twice continuously differentiable function $\psi : [0, \infty) \to [0, \infty)$, and for each point $z$ in $\mathbb{C}$ we extend it to the whole complex plane by setting $\psi(z) = \psi(|z|)$. We also assume that the Laplacian $\Delta \psi$ is positive and set $\tau(z) \simeq 1$ whenever $0 \leq |z| < 1$ and $\tau(z) \simeq (\Delta \psi(z))^{-1/2}$ otherwise, where $\tau(z)$ is a radial differentiable function satisfying the conditions

$$\lim_{r \to \infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} \tau'(r) = 0. \quad (1.2)$$

In addition, we require that either there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large $r$ or

$$\lim_{r \to \infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$ 

Throughout the paper we will assume that $\psi$ and $\tau$ satisfy all the above mentioned admissibility conditions. Observe that there are many examples of weights $\psi$ that satisfy these conditions. The power functions as $\psi(r) = r^m$, $m > 2$ and the exponential type functions $\psi(r) = e^{\alpha r}$, $\alpha > 0$, and the supper exponential functions $\psi(r) = e^{e^{\alpha r}}$, $\alpha > 0$, are all typical examples of such weights.

The generalized Fock spaces $\mathcal{F}_p^\psi$ induced by the weight function $\psi$ consist of all entire functions $f$ for which

$$\|f\|_{\mathcal{F}_p^\psi}^p = \int_\mathbb{C} |f(z)|^p e^{-p\psi(z)} dm(z) < \infty,$$ 

1The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$. 

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where $0 < p < \infty$, and $dm$ denotes the usual Lebesgue area measure on $\mathbb{C}$. For $p = \infty$, the corresponding growth type space $\mathcal{F}_\infty^\psi$ consists of all entire functions $f$ such that
\[
\|f\|_{\mathcal{F}_\infty^\psi} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\psi(z)} < \infty.
\]

1.1. **Integral type and multiplication operators.** We may mention that spaces of the form $\mathcal{F}_p^\psi$ were also studied earlier by other authors with different contexts, for example in [22] when $p = 2$ and $\psi$ belongs to a wider class of radial weights, and in [15] when $\psi$ is nonradial and its Laplacian $\Delta \psi$ is of a doubling measure. In [10], conditions under which $V_g$ becomes bounded and compact when it acts between $\mathcal{F}_p^\psi$ and $\mathcal{F}_q^\psi$ for finite exponents $p$ and $q$ were obtained. Our first main result extends those results when at least one of the exponent is infinite.

**Theorem 1.1.** Let $g$ be an entire function on $\mathbb{C}$ and $0 < p \leq \infty$. Then
\[(i)\] $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_\infty^\psi$ is bounded if and only if
\[
\begin{align*}
\sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi(z)} &< \infty, & p = \infty \\
\sup_{z \in \mathbb{C}} \frac{|g'(z)(\Delta \psi(z))^{\frac{1}{p}}}{1 + \psi'(z)} &< \infty, & 0 < p < \infty.
\end{align*}
\]
\[(ii)\] $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_\infty^\psi$ is compact if and only if
\[
\begin{align*}
\lim_{|z| \to \infty} \frac{|g'(z)|}{1 + \psi(z)} &< \infty, & p = \infty \\
\lim_{|z| \to \infty} \frac{|g'(z)(\Delta \psi(z))^{\frac{1}{p}}}{1 + \psi'(z)} &< \infty, & 0 < p < \infty.
\end{align*}
\]
\[(iii)\] if $0 < p < \infty$, then the following statements are equivalent.
\[
\begin{align*}
&\text{a) } V_g : \mathcal{F}_\infty^\psi \to \mathcal{F}_p^\psi \text{ is bounded;} \\
&\text{b) } V_g : \mathcal{F}_\infty^\psi \to \mathcal{F}_p^\psi \text{ is compact;} \\
&\text{c) } \text{The function } \frac{g'}{1 + \psi'} \text{ belongs to } L^p(\mathbb{C}, dm).
\end{align*}
\]

For the special case when $p = \infty$, parts (i) and (ii) of the theorem are proved in Theorem 3.4 and Theorem 3.5 of [7]. We will provide a different proof in section 3.

In view of Theorem 1.1, we conclude that there exists a richer structure of $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_\infty^\psi$ than its action on the classical setting where it was shown that $V_g$ is bounded if and only if $g$ is a complex polynomial of degree not bigger than two [17]. The same conclusion holds for boundedness or compactness of $V_g : \mathcal{F}_\infty^\psi \to \mathcal{F}_p^\psi$, $p < \infty$ than its action on the corresponding classical setting in which case boundedness of $V_g$ has been characterized by the fact that $g$ is of polynomial of degree not bigger than one [1, 17]. Some illustrative examples are the following. The weight function $\psi_\beta(z) = |z|^\beta$ $\beta > 2$ satisfies all the initial admissibility assumptions. Then a consequence of the above result is that the operator $V_g : \mathcal{F}_p^{\psi_\beta} \to \mathcal{F}_\infty^{\psi_\beta}$ is bounded if and only if $g$ is a complex polynomial of
\[
\deg(g) \leq \begin{cases} 
\beta, & p = \infty \\
\frac{\beta(p-1)+2}{p}, & 0 < p < \infty.
\end{cases}
\]
On the other hand, if \( \psi_\alpha(z) = e^{\alpha |z|} \), \( \alpha > 0 \), which is also an admissible weight function, then \( V_g : F^{\psi_\alpha}_p \to F^{\psi_\alpha}_\infty \) is bounded if and only if for all \( z \in \mathbb{C} \):

\[
|g(z)| \lesssim \begin{cases} 
  e^{\alpha |z|}, & p = \infty \\
  e^{\frac{\alpha(p-1)}{p} |z|}, & 0 < p < \infty.
\end{cases}
\]

Furthermore, if we, in particular, take super exponential growth function \( \psi(z) = e^{e^{|z|}} \), then \( \psi'(z) \simeq e^{e^{|z|}} \) and \( \Delta \psi(z) \simeq e^{2|z|+e^{|z|}} \). Simplifying condition (1.3) shows that \( V_g : F^{\psi}_p \to F^{\psi}_\infty \) is bounded if and only if for all \( z \in \mathbb{C} \):

\[
|g'(z)| \lesssim \begin{cases} 
  e^{|z|}, & p = \infty \\
  e^{\frac{2e-2}{p} |z| + \frac{(p-1)}{p} e^{|z|}}, & 0 < p < \infty.
\end{cases}
\]

An important ingredient used in the proofs of the results in [10] when \( V_g \) acts between the spaces \( F^{\psi}_p \) and \( F^{\psi}_q \) with finite exponents \( p \) and \( q \) has been the descriptions of the Carleson and vanishing Carleson measures. One could possibly follow similar approach to prove the above theorem as well. It only requires to describe the corresponding Carleson measures first. In Section 3, we will give a direct proof of the theorem without being resorted to the notion of Carleson measures or embedding mapping techniques.

Our next main result describes the bounded and compact Volterra companion integral operators \( I_g \) and multiplication operators \( M_g \) acting between the generalized Fock spaces.

**Theorem 1.2.** Let \( 0 < p, q \leq \infty \) and \( g \) be an entire function on \( \mathbb{C} \). Then

(i) if \( p \neq q \), then the following statements are equivalent.

(a) \( I_g : F^{\psi}_p \to F^{\psi}_q \) is bounded;
(b) \( M_g : F^{\psi}_p \to F^{\psi}_q \) is bounded;
(c) \( g \) is the zero function.

(ii) if \( 0 < p \leq \infty \), then the following statements are equivalent.

(a) \( I_g : F^{\psi}_p \to F^{\psi}_p \) is bounded;
(b) \( M_g : F^{\psi}_p \to F^{\psi}_p \) is bounded;
(c) \( g \) is a constant function.

(iii) if \( 0 < p \leq \infty \), then the following statements are also equivalent.

(a) \( I_g : F^{\psi}_p \to F^{\psi}_p \) is compact;
(b) \( M_g : F^{\psi}_p \to F^{\psi}_p \) is compact;
(c) \( g \) is the zero function.

These results differ significantly between the cases when \( p = q \) and \( p \neq q \). It has been known that this difference does not exist in the corresponding classical setting [17]. On the other hand, the appearance of such a difference has not been totally unexpected since in the classical Fock spaces, the monotonicity property in the sense of inclusion \( F_p \subset F_q \) whenever \( 0 < p \leq q \leq \infty \), holds [13]. As follows from Corollary 2 of [10], this property fails to hold for the family of generalized Fock spaces \( F^{\psi}_p \). In fact, for finite \( p \) and \( q \) such that \( p \neq q \), it has been proved
that
\[ \mathcal{F}_p^\psi \setminus \mathcal{F}_q^\psi \neq \emptyset \quad \text{and} \quad \mathcal{F}_q^\psi \setminus \mathcal{F}_p^\psi \neq \emptyset. \quad (1.5) \]

As will be seen in the subsequent considerations, this property will be used in the proof of our results while in the classical setting the corresponding results were proved using the reproducing kernel \( K_w \) as a sequence of test functions which rather belong to \( \mathcal{F}_p \) for all possible exponents \( 0 < p \leq \infty \).

We remark that for \( p = q = \infty \), the Volterra-type integral operator \( V_g \) when \( g \) is the identity map and the multiplication operator \( M_g \) have been studied in \([7]\) in a more general setting than ours, as the operators in there act between two growth type spaces where the weight functions defining the two spaces could be different.

1.2. The differential operator \( D \). One striking feature of the differential operator \( Df = f' \) is that it is a typical example of unbounded operators in many Banach spaces. In \([2,12]\), conditions under which the operator becomes bounded on some growth type spaces of analytic functions have been given. It turns out that the operator remains unbounded when it acts on generalized Fock spaces with with weight decaying as at least as fast as the classical Gaussian weight. We formulate this observation as follows.

**Theorem 1.3.** Let \( 0 < p, q \leq \infty \). Then the differential operator \( D : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi \) is unbounded. The same conclusion holds when \( D \) acts between the classical Fock spaces.

For the special case when \( p = q = \infty \), the result follows from Theorem 2.10 of \([2]\) or Theorem 4.1 of \([12]\). Thus our contribution here is when at most one of the exponents is infinity. A different proof for \( p = q = \infty \) will be also provided at the end of Section 3.3.

2. Preliminaries

In this section, we collect some known facts and auxiliary lemmas which will be used in the sequel to prove our main results. Our first lemma gives a complete characterization of the space \( \mathcal{F}_\infty^\psi \) in terms of derivative.

**Lemma 2.1.** Let \( f \) be a holomorphic function on \( \mathbb{C} \). Then \( f \) belongs to \( \mathcal{F}_\infty^\psi \) if and only if
\[ \sup_{z \in \mathbb{C}} \frac{|f'(z)| e^{-\psi(z)}}{1 + \psi'(z)} < \infty. \quad (2.1) \]

In this case, we estimate the norm of \( f \) by
\[ \|f\|_{\mathcal{F}_\infty^\psi} \simeq |f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)| e^{-\psi(z)}}{1 + \psi'(z)}. \quad (2.2) \]

The lemma follows from Corollary 3.3 of \([7]\). We give here a different proof which might be interest of its own.
Proof. For a positive $r$ and entire function $f$, we denote its integral means by

$$M_\infty(r, f) = \max_{|z|=r} |f(z)|.$$ 

Then $f$ belongs to $\mathcal{F}_\infty^\psi$ if and only if

$$M_\infty(r, f) = O(e^{\psi(r)}) \quad \text{as} \quad r \to \infty. \quad (2.3)$$

On the other hand, by Lemma 21 of [10], $M_\infty(r, f) = O(e^{\psi(r)})$ whenever $r \to \infty$ if and only if

$$M_\infty(r, f') = O\left(\psi'(r)e^{\psi(r)}\right) \quad \text{as} \quad r \to \infty. \quad (2.4)$$

Furthermore, by our growth assumption, $\psi(r)$ grows faster than the classical Gaussian weight function $|r|^2/2$ and hence

$$1 + \psi'(r) \simeq \psi'(r), \quad (2.5)$$

for sufficiently large $r$. From this, our first assertion on the lemma follows. Next we prove the estimate in (2.2). We may observe that

$$\|f\|_{\mathcal{F}_\infty^\psi} = \sup_{z \in \mathbb{C}} |f(z)e^{-\psi(z)}| \geq \frac{1}{2} \left( \sup_{z \in \mathbb{C}} |f(z)e^{-\psi(z)} + |f(0)|e^{-\psi(0)}| \right) \geq \frac{e^{-\psi(0)}}{2} \left( \sup_{z \in \mathbb{C}} |f(z)e^{-\psi(z)} + |f(0)| \right) \simeq \sup_{z \in \mathbb{C}} |f(z)e^{-\psi(z)} + |f(0)|. \quad (2.6)$$

For $f$ in $\mathcal{F}_\infty^\psi$, condition (2.1) along with (2.3) and (2.4) implies that the right-hand side of (2.6) is bounded (up to a constant) from below by

$$|f(0)| + \sup_{z \in \mathbb{C}} \frac{|f'(z)e^{-\psi(z)}|}{1 + \psi'(z)}$$

from which one side of the estimate in (2.2) follows. To prove the remaining estimate, we act as follows. Since $\|f\|_{\mathcal{F}_\infty^\psi} \leq |f(0)| + \|f - f(0)\|_{\mathcal{F}_\infty^\psi}$, it suffices to show that $\|f - f(0)\|_{\mathcal{F}_\infty^\psi}$ is bounded by the quantity $\sup_{z \in \mathbb{C}} \frac{|f'(z)e^{-\psi(z)}|}{1 + \psi'(z)}$. Thus, we write

$$|f(w) - f(0)|e^{-\psi(w)} \leq e^{-\psi(w)} \int_0^1 |w|f'(xw)dx$$

$$\leq e^{-\psi(w)} \sup_{w \in \mathbb{C}} \left( \frac{|f'(w)|e^{-\psi(w)}}{1 + \psi'(w)} \right) \int_0^1 |w|(1 + \psi'(xw))e^{\psi(xw)}dx$$

$$\lesssim e^{-\psi(w)} \left( \sup_{w \in \mathbb{C}} \frac{|f'(w)|e^{-\psi(w)}}{1 + \psi'(w)} \right) e^{\psi(w)} = \sup_{w \in \mathbb{C}} \frac{|f'(w)|e^{-\psi(w)}}{1 + \psi'(w)},$$

and completes the proof of the lemma. \qed
Note that the approximation formula (2.2) is in the spirit of the famous Littlewood–Paley formula for entire functions in the growth type space $F^\psi_{\infty}$. The corresponding formula for $F^\psi_p$ for finite $p$ was obtained in [10] and reads as

$$
\|f\|^p_{F^\psi_p} \simeq |f(0)|^p + \int_\mathbb{C} |f'(z)|^p \frac{e^{-\psi(z)}}{(1 + \psi'(z))^p} dm(z) \tag{2.7}
$$

for any entire function $f$. Both formulas (2.2) and (2.7) will be used repeatedly in our subsequent considerations.

**Lemma 2.2.** Let $0 < q, p \leq \infty$ and $g$ be an entire function on $\mathbb{C}$. Then

(i) $V_g : F^\psi_p \to F^\psi_q$ is compact if and only if $\|V_g f_n\|_{F^\psi_q} \to 0$ as $n \to \infty$ for each uniformly bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $F^\psi_p$ converging to zero uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.

(ii) A similar statement holds when we replace the operator $V_g$ by $I_g$ or $M_g$ in (i).

The lemma can be proved following standard arguments, and will be used repeatedly in what follows without mentioning it over and over again.

We denote by $D(w, r)$ the Euclidean disk centered at $w$ and radius $r > 0$. Then we record the following useful covering lemma.

**Lemma 2.3.** Let $t : \mathbb{C} \to (0, \infty)$ be a continuous function which satisfies $|t(z) - t(w)| \leq \frac{1}{4}|z - w|$ for all $z$ and $w$ in $\mathbb{C}$. We also assume that $t(z) \to 0$ when $|z| \to \infty$. Then there exists a sequence of points $z_j$ in $\mathbb{C}$ satisfying the following conditions.

(i) $z_j \notin D(z_k, t(z_k))$, $j \neq k$;

(ii) $\mathbb{C} = \bigcup_j D(z_j, t(z_j))$;

(iii) $\bigcup_{z \in D(z_j, t(z_j))} D(z, t(z)) \subset D(z_j, 3t(z_j))$;

(iv) The sequence $D(z_j, 3t(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity $N_{\max}$.

This lemma was proved in [10] by adopting an approach used by Oleinik [18]. It will be used in our subsequent proofs being referred as the covering lemma.

As pointed out earlier the reproducing kernel function $K_w(z) = e^{(z, w)}$ has been used as a sequence of test functions to prove the corresponding results mentioned above on the classical Fock spaces. Unfortunately, an explicit expression for the reproducing kernel $K_{(w, \psi)}$ in the generalized space $F^\psi_\infty$ is still unknown and it is not clear if any of the arguments connected to the reproducing kernels in the classical setting could be directly carried over to the generalized case. To prove our mains results, we will rather use another sequence of test functions in the current setting. This sequence has been used by several authors before for example [8, 10, 19]. We introduce the test function as follows. By Proposition A and Corollary 8 of [10], for a sufficiently large positive number $R$, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w, R)}$ such that

(i)

$$
|f_{(w, R)}(z)| e^{-\psi(z)} \leq C \min \left\{ 1, \left( \frac{\min\{\tau(w), \tau(z)\}}{|z - w|} \right)^{\frac{\eta^2}{2}} \right\} \tag{2.8}
$$

Note that the approximation formula (2.2) is in the spirit of the famous Littlewood–Paley formula for entire functions in the growth type space $F^\psi_{\infty}$. The corresponding formula for $F^\psi_p$ for finite $p$ was obtained in [10] and reads as

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(i)

$$
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$$
for all $z \in \mathbb{C}$ and for some constant $C$ that depends on $\psi$ and $R$. In particular when $z \in D(w, R\tau(w))$, the estimate becomes
\[
|f_{(w,R)}(z)| e^{-\psi(z)} \simeq 1. \tag{2.9}
\]

(ii) $f_{(w,R)}$ belongs to $F^\psi_p$ and its norm is estimated by
\[
\|f_{(w,R)}\|_{F^\psi_p}^p \simeq \tau(w)^2, \quad \eta(R) \leq |w| \tag{2.10}
\]
for all $p$ in the range $0 < p < \infty$.

Another important ingredient in our subsequent consideration is the pointwise estimate for subharmonic functions $f$, namely that
\[
|f(z)|^p e^{-\beta \psi(z)} \lesssim \frac{1}{\sigma^2 \tau(z)^2} \int_{D(z, \sigma \tau(z))} |f(w)|^p e^{-\beta \psi(w)} dm(w) \tag{2.11}
\]
for all finite exponent $p$, any real number $\beta$, and a small positive number $\sigma$: see Lemma 7 of [10] for more details.

**Lemma 2.4.** Let $R$ be a sufficiently large number and $\eta(R)$ be as before. If $(z_k)$ is the covering sequence from Lemma 2.3, then the function
\[
F = \sum_{z_k: |z_k| > \eta(R)} a_k f(z_k, R)
\]
belongs to $F^\psi_\infty$ for every $\ell^\infty$ sequence $(a_k)$, and also $\|F\|_{F^\psi_\infty} \lesssim \|(a_k)\|_{\ell^\infty}$.

**Proof.** We estimate the norm of $F$ as
\[
\|F\|_{F^\psi_\infty} = \sup_{z \in \mathbb{C}} |F(z)| e^{-\psi(z)} \lesssim \sup_{z \in \mathbb{C}} \sum_k |a_k| |f(z_k, R)(z)| e^{-\psi(z)}
\[
\leq \|(a_k)\|_{\ell^\infty} \sup_{z \in \mathbb{C}} \sum_k |f(z_k, R)(z)| e^{-\psi(z)}. \tag{2.12}
\]
Invoking (2.8), the right-hand side of (2.12) is bounded by
\[
\|(a_k)\|_{\ell^\infty} \sup_{z \in \mathbb{C}} \left( \sum_{k: z_k \neq z} \frac{\tau(z_k)}{\tau(z)} \frac{R^2}{z^2} + \sum_{k: z_k = z} 1 \right)
\]
\[
\leq \|(a_k)\|_{\ell^\infty} \left( \sup_k \tau(z_k) \frac{R^2}{\tau(z)} \sup_{z \in \mathbb{C}} \sum_{k: z_k \neq z} \frac{1}{|z_k - z| \frac{R^2}{\tau(z)} + N_{\max}} \right)
\]
\[
\lesssim \|(a_k)\|_{\ell^\infty} \left( \sup_k \tau(z_k) \frac{R^2}{\tau(z)} \sup_{z \in \mathbb{C}} \frac{1}{|z_{k_0} - z| \frac{R^2}{\tau(z)} + N_{\max}} \right),
\]
where $k_0$ is the index for which $|z_{k_0}| \leq |z_k|$ for all $z_k \neq z$ and $N_{\max}$ as in Lemma 2.3. Observe that because of (1.2), $\sup_k \tau(z_k)$ is finite and hence
\[
\|(a_k)\|_{\ell^\infty} \left( \sup_k \tau(z_k) \frac{R^2}{\tau(z)} \sup_{z \in \mathbb{C}} \frac{1}{|z_{k_0} - z| \frac{R^2}{\tau(z)} + N_{\max}} \right) \lesssim \|(a_k)\|_{\ell^\infty},
\]
and completes the proof. \qed
We note that the finite exponent version of the above lemma was proved in [10, Proposition 9]. That is, the function

\[ F = \sum_{z_k:|z_k| \geq \eta(R)} a_k \frac{f(z_k,R)}{\tau(z_k)^{\frac{2}{p}}} \]  

(2.13)

belongs to \( \mathcal{F}_\psi^p \) for every \( \ell^p \) summable sequence \((a_k)\) with norm estimated by

\[ \|F\|_{\mathcal{F}_\psi^p} \lesssim \sum_k |a_k|^p. \]  

(2.14)

**Lemma 2.5.** Let \( h \) be a holomorphic function on \( \mathbb{C} \) and \( p \) and \( q \) be to positive numbers. Then \( |h(z)| \lesssim \tau(z)^{\frac{2q}{2q-p}} \) for all \( z \in \mathbb{C} \) if and only if

\[ \sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{\frac{2q}{2q-p}}} \int_{D(w,\sigma \tau(w))} |h(z)|^q dm(z) < \infty. \]  

(2.15)

**Proof.** By Lemma 5 of [10], for each \( w \in \mathbb{C} \) and \( z \in D(w, \sigma \tau(w)) \) it holds that

\[ \tau(w) \simeq \tau(z) \]  

(2.16)

for a small positive number \( \sigma \). The proof is then an immediate consequence of this estimate. We include a little proof for completeness. If \( |h(z)| \lesssim \tau(z)^{\frac{2q}{2q-p}} \), then by (2.16)

\[ \sup_{z \in \mathbb{C}} \frac{1}{\tau(z)^{\frac{2q}{2q-p}}} \int_{D(z,\sigma \tau(z))} |h(w)|^q dm(w) \simeq \sup_{z \in \mathbb{C}} \frac{|h(w)|^q}{\tau(w)^{\frac{2q}{2q-p}}} dm(w) \lesssim \sup_{w \in \mathbb{C}} \frac{|h(w)|^q}{\tau(w)^{\frac{2q}{2q-p}}} < \infty. \]

On the other hand if (2.15) holds, then by subharmonicity of \( |h|^q \) and (2.11) we have

\[ |h(w)|^q \lesssim \frac{1}{\sigma^{2q} \tau(w)^{2q}} \int_{D(w,\sigma \tau(w))} |h(z)|^q dm(z) \]

from which it follows that

\[ |h(w)|^q \tau(w)^{\frac{2q}{2q-p}} \lesssim \tau(w)^{-\frac{2q}{p}} \int_{D(w,\sigma \tau(w))} |h(z)|^q dm(z) \]

and the assertion follows. \( \square \)

3. **Proof of the main results**

We now turn to the proofs of the main results of the paper.
3.1. Proof of Theorem 1.1. We begin with the proof of the sufficiency of the condition in part (i). If \( p = \infty \), then applying (2.2)

\[
\|V_gf\|_{\mathcal{F}_\infty^\psi} \simeq \sup_{z \in \mathbb{C}} \frac{|f(z)g'(z)|}{1 + \psi'(z)} e^{-\psi(z)} \leq \left( \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)} \right) \sup_{z \in \mathbb{C}} |f(z)| e^{-\psi(z)}
\]

\[
= \|f\|_{\mathcal{F}_\infty^\psi} \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)},
\]

from which it also follows that

\[
\|V_g\| \lesssim \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)}. \tag{3.1}
\]

On the other hand, if \( 0 < p < \infty \), then applying (2.11), we have

\[
\|V_gf\|_{\mathcal{F}_\infty^\psi} \simeq \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)} |f(z)| e^{-\psi(z)}
\]

\[
\lesssim \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)} \left( \frac{1}{\sigma^2 \tau(z)^2} \int_{D(z, \sigma \tau(z))} |f(w)|^p e^{-p\psi(w)} dm(w) \right)^{1/p}
\]

\[
\lesssim \sup_{z \in \mathbb{C}} \frac{|g'(z)||f||_{\mathcal{F}_p}}{(1 + \psi'(z)) \tau(z)^{1/p}} \simeq \|f\|_{\mathcal{F}_p} \sup_{z \in \mathbb{C}} \frac{|g'(z)|(\Delta \psi(z))^{1/p}}{1 + \psi'(z)}
\]

as required and also deduce the reverse estimate in (3.1)

\[
\|V_g\| \lesssim \sup_{z \in \mathbb{C}} \frac{|g'(z)|(\Delta \psi(z))^{1/p}}{1 + \psi'(z)}. \tag{3.2}
\]

To prove the necessity, note that from (2.10), the sequence of functions \( f_{(w,R)} \) belong to \( \mathcal{F}_p^\psi \) for all \( p < \infty \) and \( w \in \mathbb{C} \). On the other hand for \( p = \infty \), applying (2.8) and (2.9), we make the corresponding estimate

\[
\|f_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \simeq 1. \tag{3.3}
\]

Applying \( V_g \) to \( f_{(w,R)} \) and making use of Lemma 2.1 and (3.3), we find

\[
\|V_g\| \gtrsim \|V_gf_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \simeq \sup_{z \in \mathbb{C}} \frac{|f_{(w,R)}(z)g'(z)|}{1 + \psi'(z)} e^{-\psi(z)} \geq \frac{|f_{(w,R)}(z)g'(z)|}{1 + \psi'(z)} e^{-\psi(z)}
\]

for all \( w, z \in \mathbb{C} \). In particular, setting \( z = w \) and making use of (2.9) give

\[
\frac{|g'(w)|}{1 + \psi'(w)} \simeq \frac{|g'(w)|}{1 + \psi'(w)} |f_{(w,R)}(w)| e^{-\psi(w)} \lesssim \|V_g\|, \tag{3.4}
\]

and the necessity and the reverse estimate in (3.1) follow for the case \( p = \infty \). Seemingly, for the remaining case where \( 0 < p < \infty \), the estimate in (2.9) and (2.10) imply

\[
\|V_g\| \gtrsim \frac{1}{\tau(w)^{1/p}} \|V_gf_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \simeq \frac{1}{\tau(w)^{1/p}} \sup_{z \in \mathbb{C}} \frac{|f_{(w,R)}(z)g'(z)|}{1 + \psi'(z)} e^{-\psi(z)}
\]

\[
\geq \frac{|f_{(w,R)}(w)g'(w)|}{\tau(w)^{1/p}(1 + \psi'(w))} e^{-\psi(w)} \simeq \frac{|g'(w)|}{\tau(w)^{1/p}(1 + \psi'(w))} \approx \frac{|g'(w)|(\Delta \psi(w))^{1/p}}{1 + \psi'(w)}. 
\]
Observe that this together with (3.1), (3.2) and (3.4) give, in addition, an approximation formulas for the operator norm:

$$\|V_g\| \simeq \left\{ \begin{array}{ll} \sup_{z \in \mathbb{C}} \frac{|g'(z)|}{1 + \psi'(z)}, & p = \infty \\
\sup_{z \in \mathbb{C}} \frac{|g'(z)| |\Delta \psi(z)|^\frac{1}{p}}{1 + |\psi'(z)|}, & 0 < p < \infty. \end{array} \right.$$  

**Part (ii).** We first prove the sufficiency of the condition. Let $f_n$ be a uniformly bounded sequence of functions in $\mathcal{F}_p^\psi$ that converges uniformly to zero on compact subsets of $\mathbb{C}$. We let first $p = \infty$. Then for each positive $\epsilon$, the necessity of the condition implies that there exists $N_1$ such that

$$\frac{|g'(z)|}{1 + \psi'(z)} < \epsilon$$

whenever $|z| > N_1$. From this and Lemma 2.1, we have

$$\frac{|g'(z)f_n(z)| e^{-\psi(z)}}{1 + \psi'(z)} \lesssim \|f_n\|_{\mathcal{F}_\infty^\psi} \frac{|g'(z)|}{1 + \psi'(z)} \lesssim \epsilon$$

for all $|z| > N_1$. We need to conclude the same for all $z$ such that $|z| \leq N_1$. To this end, we may first observe that the function $f^*(z) = 1$ belongs to $\mathcal{F}_p^\psi$ for all $0 < p \leq \infty$. Since (1.4) obviously implies the boundedness condition in part (i), it follows that

$$\|V_g f^*\|_{\mathcal{F}_\infty^\psi} \simeq \sup_{|z| \leq N_1} \frac{|g'(z)| e^{-\psi(z)}}{1 + \psi'(z)} < \infty.$$

A consequence of this is that

$$\sup_{|z| \leq N_1} \frac{|g'(z)f_n(z)| e^{-\psi(z)}}{1 + \psi'(z)} \lesssim \sup_{|z| \leq N_1} |f_n(z)| \rightarrow 0, \quad n \rightarrow \infty. \quad (3.5)$$

Similarity, when $0 < p < \infty$, the condition implies that for some positive number $N_2$ we have

$$\sup_{|z| > N_2} \frac{|g'(z)|}{\tau(z)^\frac{2}{p}} < \epsilon$$

from which and (2.11) we also have

$$\sup_{|z| > N_2} \frac{|g'(z)| |f_n(z)| e^{-\psi(z)}}{1 + \psi'(z)} \lesssim \sup_{|z| > N_2} \frac{|g'(z)|}{1 + \psi'(z)} \left(\frac{1}{\tau(z)^2} \int_{D(z,\sigma \tau(z))} |f_n(w)| e^{-\psi(w)} dm(w)\right)^\frac{1}{p} \lesssim \sup_{|z| > N_2} \frac{|g'(z)|}{1 + \psi'(z)} \frac{1}{\tau(z)^\frac{2}{p}} \|f_n\|_{\mathcal{F}_p} \lesssim \sup_{|z| > N_2} \frac{|g'(z)|}{1 + \psi'(z)} \frac{1}{\tau(z)^\frac{2}{p}} \lesssim \epsilon.$$

On the other hand, when $|z| \leq N_2$, the desired conclusion follows from (3.5).

Conversely, let us now show that if $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_\infty^\psi$ is compact, then the relation in (1.4) holds. To this end, we take the test function

$$f^*_{(w,R)} = \begin{cases} \frac{f_{(w,R)}}{\tau(w)^{2/p}}, & \eta(R) \leq |w|, \quad p < \infty \\
\frac{f_{(w,R)}}{\tau(w)} & \text{for } p = \infty, \end{cases} \quad (3.6)$$
where \( f_{(w,R)} \) is the function described with properties in (2.8), (2.9) and (2.10). As shown in [10], \( f_{(w,R)}^* \to 0 \) as \(|w| \to \infty\) uniformly on compact subsets of \( \mathbb{C} \) and
\[
\sup_{|w| \geq \eta(R)} \| f_{(w,R)}^* \|_{\mathcal{F}_p^\psi} < \infty. \tag{3.7}
\]

It follows from this and compactness of \( V_g \) that
\[
\lim_{|w| \to \infty} \| V_g f_{(w,R)}^* \|_{\mathcal{F}_\infty^\psi} = 0.
\]

Making use of this and (2.9), for \( p = \infty \) we obtain
\[
\lim_{|z| \to \infty} \frac{|g'(z)|}{1 + \psi'(z)} \simeq \lim_{|z| \to \infty} \frac{|g'(z)|}{1 + \psi'(z)} |f_{(z,R)}^*| e^{-\psi(z)} \lesssim \lim_{|z| \to \infty} \| V_g f_{(z,R)}^* \|_{\mathcal{F}_\infty^\psi} = 0.
\]

On the other hand, if \( 0 < p < \infty \), then
\[
\lim_{|z| \to \infty} \frac{|g'(z)| |\Delta \psi(z)|^{\frac{1}{p}}}{1 + \psi'(z)} \simeq \lim_{|z| \to \infty} \frac{|g'(z)|}{1 + \psi'(z)} |f_{(z,R)}^*| e^{-\psi(z)} \lesssim \lim_{|z| \to \infty} \| V_g f_{(z,R)}^* \|_{\mathcal{F}_\infty^\psi} = 0
\]
and completes the proof of part (ii) of Theorem 1.1.

**Part (iii).** Since (b) obviously implies (a), it suffices to show (a) implies (c) and (c) implies (b). For the first, we follow this classical technique where the original idea goes back to Luecking [14]. Let \( 0 < q < \infty \) and \( R \) be a sufficiently large number and \((z_k)\) be the covering sequence as in Lemma 2.3. Then by Lemma 2.4,
\[
F = \sum_{z_k :|z_k| \geq \eta(R)} a_k f_{(z_k,R)}
\]
belongs to \( \mathcal{F}_\infty^\psi \) for every \( \ell^\infty \) sequence \((a_k)\) with norm estimate \( \| F \|_{\mathcal{F}_\infty^\psi} \lesssim \|(a_k)\|_{\ell^\infty}. \)

If \((r_k(t))\) is the Rademacher sequence of function on \([0,1]\) chosen as in [14], then the sequence \((a_k r_k(t))\) also belongs to \( \ell^\infty \) with \( \|(a_k r_k(t))\|_{\ell^\infty} = \|(a_k)\|_{\ell^\infty} \) for all \( t \).

This implies that the function
\[
F_t = \sum_{z_k :|z_k| \geq \eta(R)} a_k r_k(t) f_{(z_k,R)}
\]
belongs to \( \mathcal{F}_\infty^\psi \) with norm estimate \( \| F_t \|_{\mathcal{F}_\infty^\psi} \lesssim \|(a_k)\|_{\ell^\infty}. \)

Then, an application of Khinchine’s inequality [14] yields
\[
\left( \sum_{z_k :|z_k| \geq \eta(R)} |a_k|^2 |f_{(z_k,R)}(z)|^2 \right)^{\frac{q}{2}} \lesssim \int_0^1 \left| \sum_{z_k :|z_k| \geq \eta(R)} a_k r_k(t) f_{(z_k,R)}(z) \right|^q dt. \tag{3.8}
\]
Setting \( \theta_{(\omega,\psi,q)}(z) = |g'(z)|^q e^{-q\psi(z)}(1 + \psi'(z))^{-q} dm(z) \), making use of (3.8), and subsequently Fubini’s theorem, we have

\[
\int_{C} \left( \sum_{z_k:|z_k| \geq \eta(R)} |a_k|^q |f(z_k, R)(z)|^2 \right)^{\frac{q}{2}} \theta_{(\omega,\psi,q)}(z)
\]

\[
\lesssim \int_{C} \int_{0}^{1} \left| \sum_{z_k:|z_k| \geq \eta(R)} a_k r_k(t) f(z_k, R)(z) \right|^q dt \theta_{(\omega,\psi,q)}(z)
\]

\[
= \int_{0}^{1} \int_{C} \left| \sum_{z_k:|z_k| \geq \eta(R)} a_k r_k(t) f(z_k, R)(z) \right|^q \theta_{(\omega,\psi,q)}(z) dt
\]

\[
\approx \int_{0}^{1} \|V_{z} F_{t}\|_{q}^{q} dt \lesssim \|(a_k)\|_{L^{\infty}}^{q}. \quad (3.9)
\]

Then, using (2.9) we get

\[
\sum_{z_k:|z_k| \geq \eta(R)} |a_k|^q \int_{D(z_k, 3\sigma \tau(z_k))} |g'(z)|^q (1 + \psi'(z))^{-q} dm(z)
\]

\[
\lesssim \sum_{z_k:|z_k| \geq \eta(R)} |a_k|^q \int_{D(z_k, 3\sigma \tau(z_k))} \frac{|g'(z)|^q |f(z_k, R)(z)|^q}{(1 + \psi'(z))^q} e^{-q\psi(z)} dm(z)
\]

\[
= \int_{C} \sum_{z_k:|z_k| \geq \eta(R)} |a_k|^q \chi_{D(z_k, 3\sigma \tau(z_k))}(z) |f(z_k, R)(z)|^q \theta_{(\omega,\psi,q)}(z) dt
\]

\[
\lesssim \max\{1, N_{\max}^{1-q/2}\} \int_{C} \left( \sum_{z_k:|z_k| \geq \eta(R)} |a_k|^q |f(z_k, R)(z)|^2 \right)^{\frac{q}{2}} \theta_{(\omega,\psi,q)}(z) dt
\]

\[
\lesssim \|(a_k)\|_{L^{\infty}}^{q}.
\]

Setting, in particular, \( a_k = 1 \) for all \( k \) in the above series of estimates results in

\[
\sum_{z_k:|z_k| \geq \eta(R)} \int_{D(z_k, 3\sigma \tau(z_k))} |g'(z)|^q (1 + \psi'(z))^{-q} dm(z) < \infty. \quad (3.10)
\]

Now we take a positive number \( r \geq \eta(R) \) such that whenever \( z_k \) of the covering sequence belongs to \( \{|z| < \eta(R)\} \), then \( D(z_k, \sigma \tau(z_k)) \) belongs to \( \{|z| < \eta(R)\} \).

Thus,

\[
\int_{\{|w| \geq r\}} \frac{1}{\tau(w)^2} \left( \int_{D(w, \sigma \tau(w))} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) \right) dm(w)
\]

\[
\lesssim \sum_{|z_k| \geq \eta(R)} \int_{D(z_k, \sigma \tau(z_k))} \frac{1}{\tau(w)^2} \left( \int_{D(w, \sigma \tau(w))} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) \right) dm(w)
\]

\[
\lesssim \sum_{|z_k| \geq \eta(R)} \int_{D(z_k, 3\sigma \tau(z_k))} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) < \infty.
\]

(3.11)
On the other hand, applying (2.7), (2.10), and (2.16) we have that
\[
\int_{\{|w|<r\}} \frac{1}{\tau(w)^2} \left( \int_{D(w,\sigma\tau(w))} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) \right) dm(w)
\]
\[
\lesssim \int_{\{|w|<r\}} \frac{1}{\tau(w)^2} \left( \int_{D(w,\sigma\tau(w))} \frac{|g'(z)|^q |f_{R}(z)|^q e^{-q\psi(z)}}{(1 + \psi'(z))^q} dm(z) \right) dm(w)
\]
\[
\lesssim \int_{\{|w|<r\}} \frac{\|V_{g} f_{R}(z)\|_{\mathcal{F}_{\psi}^q}}{\tau(w)^2} dm(w) \lesssim \int_{\{|w|<r\}} \frac{\|f_{R}\|_{\mathcal{F}_{\psi}^q}}{\tau(w)^2} dm(w)
\]
\[
\lesssim \frac{r^2}{\tau(r)^2} < \infty,
\]
by our admissibility assumption on \(\tau\). This together with (3.11), (2.11) as \(|g'|^q\) is subharmonic, and [10, Lemma 20] implies
\[
\int_{\mathbb{C}} \frac{|g'(w)|^q}{(1 + \psi'(w))^q} dm(w) \lesssim \int_{\mathbb{C}} \frac{1}{\tau(w)^2} \left( \int_{D(w,\sigma\tau(w))} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) \right) dm(w) < \infty.
\]
It remains to prove (c) implies (b). Let \(f_n\) be a uniformly bounded sequence of functions in \(\mathcal{F}_{\psi}^q\) that converges uniformly to zero on compact subsets of \(\mathbb{C}\). For each positive \(\epsilon\), the necessity of the condition implies that there exists \(N_2\) for which
\[
\int_{\{|z|>N_2\}} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) < \epsilon.
\]
Therefore, it follows from this, and (2.2), and boundedness of \(V_{g}\) that
\[
\int_{\{|z|>N_2\}} \frac{|g'(z)|^q |f_n(z)|^q e^{-q\psi(z)}}{(1 + \psi'(z))^q} dm(z)
\]
\[
\lesssim \|f_n\|_{\mathcal{F}_{\psi}^q} \int_{\{|z|>N_2\}} \frac{|g'(z)|^q}{(1 + \psi'(z))^q} dm(z) \lesssim \epsilon. \quad (3.12)
\]
We estimate the remaining piece of integral as
\[
\int_{\{|z|\leq N_2\}} \frac{|g'(z)|^q |f_n(z)|^q e^{-q\psi(z)}}{(1 + \psi'(z))^q} dm(z)
\]
\[
\lesssim \sup_{z:|z|\leq N_2} |f_n(z)|^q \int_{\{|z|\leq N_2\}} \frac{|g'(z)|^q e^{-q\psi(z)}}{(1 + \psi'(z))^q} dm(z)
\]
\[
\lesssim \sup_{|z|\leq N_2} |f_n(z)|^q \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.13)
\]
From (3.12) and (3.13), we conclude that \(\|V_{g} f_n\|_{\mathcal{F}_{\psi}^q} \rightarrow 0\).

3.2. Proof of Theorem 1.2. Part (i) and part (ii). Since the statement in (c) obviously implies both the statements in (a) and (b), we plan to show that (a) implies (c) and (b) implies (c). Suppose \(I_g : \mathcal{F}_{\psi}^q \rightarrow \mathcal{F}_{\psi}^q\) is bounded.
Case 1: We consider first the case when $q = \infty$. Then applying (2.2) and the sequence of test functions with properties in (2.8), (2.9), and (2.10) we obtain
\[
\|I_g f_{(w,R)}\|_{F^\infty_q} \simeq \sup_{z \in \mathbb{C}} \frac{|f'_{(w,R)}(z)g(z)|e^{-\psi(z)}}{1 + \psi'(z)} \geq \frac{|f'_{(w,R)}(z)g(z)|e^{-\psi(z)}}{1 + \psi'(z)}
\]
(3.14)
for all $z, w \in \mathbb{C}$. Taking $w = z$, as done before, we have
\[
\frac{|f'_{(w,R)}(w)g(w)|e^{-\psi(w)}}{1 + \psi'(w)} \simeq |g(w)|.
\]
(3.15)
If $p < \infty$, then combining (3.14), (3.15) and (2.10) gives
\[
|g(w)| \lesssim \|I_g f_{(w,R)}\|_{F^\infty_q} \lesssim \|I_g\|\|f_{(w,R)}\|_{F^\infty_q} \lesssim \|I_g\|\tau(w)^{2/p}
\]
from which and (1.2), we see that $|g(w)| \to 0$ as $|w| \to \infty$. Since $g$ is an analytic function, the above holds only if it is the zero function. Similarly, for $p = q = \infty$, the above techniques shows that $I_g$ is bounded on $F^\infty_q$ only if $g$ is a bounded analytic function, and hence a constant by Liouville’s classical theorem.

Case 2: $q < \infty$. Then making use of (2.9) and (2.10), we estimate
\[
\int_{D(w,\tau(w))} |g(z)|^q dm(z) \simeq \int_{D(w,\tau(w))} \frac{|f'_{(w,R)}(z)|^q|g(z)|^q}{(1 + \psi'(z))^q} e^{-q\psi(z)} dm(z)
\]
\[
\leq \|V_g f_{(w,R)}\|_{F^q_q} \lesssim \|I_g\|q\|f_{(w,R)}\|_{F^\infty_q}^q \simeq \|I_g\|q\tau(w)\frac{2q}{p}.
\]
On the other hand, since $|g|^q$ is subharmonic, applying (2.11), and the above yields
\[
|g(w)|^q \lesssim \frac{1}{\tau(w)^2} \int_{D(w,\tau(w))} |g(z)|^q dm(z) \lesssim \|I_g\|q\tau(w)\frac{2q}{p} \frac{2q}{p} < \infty,
\]
(3.16)
from which we obtain a general necessity condition:
\[
\sup_{w \in \mathbb{C}} \tau(w)^{2-\frac{2q}{p}} |g(w)|^q \simeq \sup_{w \in \mathbb{C}} |g(w)|^q(\Delta \psi(w))^{\frac{q}{p} - \frac{2q}{p} < \infty},
\]
(3.17)
for all possible finite exponents $p$ and $q$. Since by our admissibility assumptions, $\Delta \psi$ increases radially and in particular for the case $q \geq p$, the condition in (3.17) holds if and only if $g$ is a bounded analytic function, and hence a constant. We further claim that $g$ is in fact the zero function when $q > p$. If not, then making use of (1.5) and any function $f_p \in F^q_p \setminus F^\infty_q$ leads to
\[
\int_{\mathbb{C}} \frac{|f'_p(z)|^q}{(1 + \psi'(z))^q} e^{-q\psi(z)} dm(z) \simeq \int_{\mathbb{C}} \frac{|g(z)|^q|f'_p(z)|^q}{(1 + \psi'(z))^q} e^{-q\psi(z)} dm(z) \simeq \|I_g f_p\|_{F^\infty_q}^q \lesssim \|I_g\|q\|f_p\|_{F^\infty_q}^q \lesssim \|f_p\|_{F^\infty_q}^q < \infty,
\]
which shows that $f_p \in F^\infty_q$ and gives a contradiction whenever $g$ is a nonzero constant.

It remains to show when $0 < q < p < \infty$. For this we run a variant of the arguments used in the proof of part (iii) of Theorem 1.1. We include the details here for the convenience of the reader. Let $(r_k(t))_k$ is the Radmecher sequence
of function on \([0, 1]\) as mentioned before. Then because of (2.13) and (2.14) the function

\[ F_t = \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f(z_k, R)}{\tau(z_k) \frac{1}{p}} \]

belongs to \(\mathcal{F}_p^\psi\) for all \(p\) with norm estimate

\[ \|F_t\|_{\mathcal{F}_p^\psi}^p \lesssim \sum_k |a_k|^p. \quad (3.18) \]

An application of Khinchine’s inequality again yields

\[ \left( \sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f(z_k, R)(z)|^2}{\tau(z_k) \frac{1}{p}} \right)^{\frac{p}{2}} \lesssim \int_0^1 \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f(z_k, R)(z)}{\tau(z_k) \frac{1}{p}} \right|^q \, dt. \quad (3.19) \]

Setting \(d\theta_{(g, \psi, q)}(z) = |g(z)|^q e^{-q\psi(z)}(1 + \psi'(z))^{-q} dm(z)\), as before and making use of (3.19), and subsequently Fubini’s theorem, we have

\[
\int_{\mathcal{C}} \left( \sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f(z_k, R)(z)|^2}{\tau(z_k) \frac{1}{p}} \right)^{\frac{p}{2}} \, d\theta_{(g, \psi, q)}(z) \\
\lesssim \int_{\mathcal{C}} \int_0^1 \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f(z_k, R)(z)}{\tau(z_k) \frac{1}{p}} \right|^q \, dt \, d\theta_{(g, \psi, q)}(z) \\
= \int_0^1 \int_{\mathcal{C}} \left| \sum_{z_k: |z_k| \geq \eta(R)} a_k r_k(t) \frac{f(z_k, R)(z)}{\tau(z_k) \frac{1}{p}} \right|^q \, d\theta_{(g, \psi, q)}(z) \, dt \\
\approx \int_0^1 \|I_g F_t\|_{\mathcal{F}_p^\psi}^q \, dt \lesssim \|(a_k)\|_{\ell^p}. \quad (3.20) \]

Now arguing with this, the covering lemma, and (2.9) leads to the series of estimates

\[
\sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k) \frac{1}{p}} \int_{D(z_k, 3\sigma\tau(z_k))} |g(z)|^ q \, dm(z) \\
\lesssim \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k) \frac{1}{p}} \int_{D(z_k, 3\sigma\tau(z_k))} |g(z)|^q \frac{|f(z_k, R)(z)|^q e^{-q\psi(z)}}{(1 + \psi'(z))^q} \, dm(z) \\
= \int_{\mathcal{C}} \sum_{z_k: |z_k| \geq \eta(R)} \frac{|a_k|^q}{\tau(z_k) \frac{1}{p}} \chi_{D(z_k, 3\sigma\tau(z_k))}(z) |f(z_k, R)(z)|^q d\theta_{(g, \psi, q)}(z) \\
\lesssim \max\{1, N_{\text{max}}^{-1-q/2}\} \int_{\mathcal{C}} \left( \sum_{z_k: |z_k| \geq \eta(R)} |a_k|^2 \frac{|f(z_k, R)(z)|^2}{\tau(z_k) \frac{1}{p}} \right)^{\frac{p}{2}} \, d\theta_{(g, \psi, q)}(z) \\
\lesssim \|(a_k)\|_{\ell^p}^q. \]
Applying duality between the spaces \( L^{p/q} \) and \( L^{p/(p-q)} \), we get

\[
\sum_{z_k:|z_k|\geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(z_k,3\sigma\tau(z_k))} |g(z)|^q dm(z) \right)^{p/(p-q)} \tau(z_k)^2 < \infty.
\]

On the other hand, we can find a positive number \( r \geq \eta(R) \) such that whenever a point \( z_k \) of the covering sequence \( (z_j) \) belongs to \( \{|z| < \eta(R)\} \), then \( D(z_k,\sigma\tau(z_k)) \) belongs to \( \{|z| < \eta(R)\} \). Thus,

\[
\int_{|w|\geq r} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} |g(z)|^q dm(z) \right)^{p/(p-q)} dm(w)
\]

\[
\leq \sum_{|z_k|\geq \eta(R)} \int_{D(z_k,\sigma\tau(z_k))} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} |g(z)|^q dm(z) \right)^{p/(p-q)} dm(w)
\]

\[
\leq \sum_{z_k:|z_k|\geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(z_k,3\sigma\tau(z_k))} |g(z)|^q dm(z) \right)^{p/(p-q)} \tau(z_k)^2 < \infty.
\]

It follows from this that

\[
\int_{|w|<r} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} |g(z)|^q dm(z) \right)^{p/(p-q)} dm(w) < \infty \tag{3.21}
\]

By subharmonicity of \( |g|^q \) and hence (2.11), we get the estimate

\[
\int_{\mathbb{C}} |g(w)|^{\frac{mp}{2q}} dm(w) \lesssim \int_{\mathbb{C}} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} |g(z)|^q dm(z) \right)^{p/(p-q)} dm(w) < \infty
\]

Since \( g \) is analytic, the estimate above holds only if \( g \) is the zero function as asserted. Interested readers may consult [21] to see why the zero function is the only \( L^{\frac{mp}{2q}} \) integrable entire function on \( \mathbb{C} \).

Next we prove (b) implies (c). We may consider first the case when \( 0 < q < \infty \). In this case, the problem can be reformulated in terms of embedding maps or Carleson measures for the weighted Fock spaces. To this effect, observe that for any entire function \( f \)

\[
\|M_g f\|_{\mathcal{F}_q^\psi}^q = \int_{\mathbb{C}} |f(z)|^q |g(z)|^q e^{-\psi(z)} dm(z) = \int_{\mathbb{C}} |f(z)|^q d\mu_{(g,\psi)}(z)
\]

\[
\lesssim \|M_g\|^q \|f\|_{\mathcal{F}_p^\psi}^q,
\]

where \( d\mu_{(g,\psi)}(z) = |g(z)|^q e^{-\psi(z)} dm(z) \). This means that the estimate in (3.22) holds if and only if the embedding map \( i_q : \mathcal{F}_p^\psi \to L^q(\mu_{(g,\psi)}) \) is bounded. By Theorem 1 of [10], the latter holds if and only if

\[
\sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\sigma\tau(w))} |g(z)|^q dm(z) < \infty \tag{3.23}
\]
for some small $\sigma > 0$. By Lemma 2.4, (3.23) holds if and only if

$$\sup_{w \in \mathbb{C}} |g(w)| (\Delta \psi(w))^{\frac{2-p}{p}} < \infty,$$

from which we easily see that $g$ is in deed a constant function when $p = q$ and the zero function otherwise.

On the other hand, if $0 < p < q = \infty$, then applying (2.10) we have

$$\tau(w)^{2/p} \|M_g\| \gtrsim \|M_{f(w,R)}\|_{L^\infty} = \sup_{z \in \mathbb{C}} |f(w,R)(z)| g(z) e^{-\psi(z)} \geq |f(w,R)(z)| g(z) e^{-\psi(z)}$$

(3.24)

for all $w$ and $z$ in $\mathbb{C}$. In particular, when $w = z$, (3.24), (3.16) and (2.9) give

$$|g(w)| \lesssim |f(w,R)(w)g(w)| e^{-\psi(w)} \simeq \|M_g\| \tau(w)^{2/p}$$

from which and (1.2), we again conclude that $g$ is the zero function.

Part (iii). This can be verified by arguing as in parts (i) and (ii) of the proofs above and part (iii) of Theorem 1.1. If $I_g$ is compact on $F^p_\psi$, then using the sequence of test functions $f^*_w$ as defined in (3.6), we easily deduce that $g$ is in deed the zero function. On the other hand, if $I_g$ is compact on $F^p_\psi$, then taking into account (3.22) and Theorem 1 of [10] we have that $M_g : F^p_\psi \to F^q_\psi$ is compact if and only if

$$\int \left( \frac{1}{\tau(w)^2} \int_{D(w,\sigma \tau(w))} e^{q\psi(z)} d\mu(z,g)(z) \right)^{\frac{p}{p-q}} dm(w)$$

$$= \int \left( \frac{1}{\tau(w)^2} \int_{D(w,\sigma \tau(w))} |g(z)|^q dm(z) \right)^{\frac{p}{p-q}} dm(w) < \infty.$$

Since $|g|^q$ subharmonic again, by (2.11) we have

$$\int |g(w)|^{\frac{p}{p-q}} dm(w) \lesssim \int \left( \frac{1}{\tau(w)^2} \int_{D(w,\sigma \tau(w))} |g(z)|^q dm(z) \right)^{\frac{p}{p-q}} dm(w) < \infty$$

Since $g$ is analytic, the estimate in above holds only if $g$ is the zero function as asserted. □

3.3. Proof of Theorem 1.3. In this subsection, we will verify that the differential operator $D$ is always unbounded whenever it acts between two generalized Fock spaces. Let us assume that $D : F^p_\psi \to F^q_\psi$ is bounded and argue in the direction of contradiction. Then, for simplicity we may split and analyze the situation in three different cases.
Case 1: if $0 < p \leq q < \infty$, then making use of the estimates in (2.9), (2.10), and (2.11), we have

$$\tau(w)^{\frac{q}{p}} \| D \|^q \geq \left\| Df_{(w,R)} \right\|_{F^q}^q = \int_{C} |f'_{(w,R)}(z)|^q e^{-q\psi(z)} dm(z)$$

$$\geq \int_{D(w,\delta\tau(w))} |f'_{(w,R)}(z)|^q e^{-q\psi(z)} dm(z)$$

$$\geq (\tau(w))^2 |f'_{(w,R)}(w)|^q e^{-q\psi(w)} \simeq (\tau(w))^2 (\psi'(w))^q$$

for all $w \in \mathbb{C}$. It follows from this that

$$\sup_{w \in \mathbb{C}} (\psi'(w))^q (\Delta \psi(w))^q \frac{2}{p} < \infty$$

which gives a contradiction since $\psi$ grows faster than the Gaussian weight function and $\Delta \psi$ radially increases to $\infty$.

Case 2: If $0 < p < q = \infty$ and $D$ were bounded, then following a similar argument as above and making use of (3.3) and (2.10), we would have

$$\tau(w)^{\frac{q}{p}} \geq \left\| Df_{(w,R)} \right\|_{F^q}^q = \sup_{z \in \mathbb{C}} |f'_{(w,R)}(z)| e^{-\psi(z)} \geq |f'_{(w,R)}(z)| e^{-\psi(z)}.$$  

This implies

$$\sup_{w \in \mathbb{C}} \frac{\psi'(w)}{\tau(w)^{\frac{q}{p}}} < \infty,$$

which gives a contradiction as $\tau$ radially decreases to zero and $\psi$ grows faster than the Gaussian weight function again.

Case 3: if $p = q = \infty$, then arguing towards contradiction as in case 2, we have that

$$1 \geq \left\| Df_{(w,R)} \right\|_{F^q}^q = \sup_{z \in \mathbb{C}} |f'_{(w,R)}(z)| e^{-\psi(z)} \geq |f'_{(w,R)}(z)| e^{-\psi(z)}$$

for all $z, w \in \mathbb{C}$. Then as before, for $w = z$, we get the estimates $1 \geq \psi'(w)$ which leads to a contradiction when $w \to \infty$.

Case 4: $0 < q < p < \infty$. For this, we modify the arguments from (3.18) to (3.21) in the proof above.

$$\int_{C} \left( \sum_{z_k, |z_k| \geq \eta(R)} |a_k|^2 \frac{|f'_{(z_k,R)}(z)|^2}{\tau(z_k)^{\frac{q}{p}}} \right)^{\frac{q}{q'}} e^{-q\psi(z)} dm(z) \lesssim \int_{0}^{1} \left\| DF_t \right\|_{F^q}^q dt$$

$$\lesssim \| (a_k) \|_{F^q}^q.$$ (3.25)
Now arguing with this, the covering lemma, and (2.9) leads to the series of estimates
\[
\sum_{z_k:|z_k|\geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q}} \int_{D(z_k,3\sigma\tau(z_k))} (1 + |\psi'(z)|)^q dm(z)
\leq \sum_{z_k:|z_k|\geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q}} \int_{D(z_k,3\sigma\tau(z_k))} (1 + |\psi'(z)|)|f'_{(z_k,R)}(z)|^q \frac{e^{-q\psi(z)}}{(1 + |\psi'(z)|)^q} dm(z)
\leq \int_{\mathbb{C}} \sum_{z_k:|z_k|\geq \eta(R)} \frac{|a_k|^q}{\tau(z_k)^{2q}} \chi_{D(z_k,3\sigma\tau(z_k))}(z)|f'_{(z_k,R)}(z)|^q \frac{e^{-q\psi(z)}}{(1 + |\psi'(z)|)^q} dm(z)
\leq \max\{1, N_{\max}^{1-q/2}\} \int_{\mathbb{C}} \left( \sum_{z_k:|z_k|\geq \eta(R)} |a_k|^2 \frac{|f'_{(z_k,R)}(z)|^2}{\tau(z_k)^{2q}} \right)^{\frac{q}{2}} e^{-q\psi(z)} dm(z)
\leq ||(a_k)||_q^{10}.
\]
Applying duality between the spaces \(\ell^{p/q}\) and \(\ell^{p/(p-q)}\), we again get
\[
\sum_{z_k:|z_k|\geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(z_k,3\sigma\tau(z_k))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty.
\]
On the other hand, we can find a positive number \(r \geq \eta(R)\) such that whenever a point \(z_k\) of the covering sequence \((z_j)\) belongs to \(|z| < \eta(R)\), then \(D(z_k, \sigma\tau(z_k))\) belongs to \(|z| < \eta(R)\). Thus,
\[
\int_{|w|\geq r} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} dm(w)
\leq \sum_{|z_k|\geq \eta(R)} \int_{D(z_k,\sigma\tau(z_k))} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} dm(w)
\leq \sum_{z_k:|z_k|\geq \eta(R)} \left( \frac{1}{\tau(z_k)^2} \int_{D(z_k,3\sigma\tau(z_k))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} \tau(z_k)^2 < \infty.
\]
It follows that
\[
\int_{|w|< r} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} dm(w) < \infty. \tag{3.27}
\]
Taking into account Lemma 20 of [10], (3.26) and (3.27), we obtain
\[
\int_{\mathbb{C}} \left( \frac{1}{\tau(w)^2} \int_{D(w,3\sigma\tau(w))} (1 + |\psi'(z)|)^q dm(z) \right)^{\frac{p}{p-q}} dm(w)
\leq \int_{\mathbb{C}} \left( \frac{1 + |\psi'(z)|}{\tau(w)^{2q}} \right)^{\frac{p}{p-q}} \tau(w)^{\frac{2q}{p-q}} dm(w) = \int_{\mathbb{C}} \left( \frac{1 + |\psi'(z)|}{\tau(w)^{2q}} \right)^{\frac{p}{p-q}} dm(w) < \infty,
\]
which is a contradiction as \( p > q \) and \( |\psi'(z)| \to \infty \) as \( |z| \to \infty \).

Case 5: \( 0 < q < p = \infty \). For this part, we modify the arguments used in the proof of (a) implies (c) in Theorem 1.1 and along with case 4 above. Thus we omit the details and leave it to the interested reader.

It remains to verify that the differential operator \( D \) is always unbounded whenever it acts between two classical Fock spaces. In this case we can use the normalized reproducing kernel \( k_w(z) = e^{(z,w)} - |w|^2/2 \) as a test function. If \( D \) were bounded, then for \( 0 < q < \infty \) and \( 0 < p \leq \infty \), we would get

\[
1 \gtrsim \|Dk_w\|_{F_q} = \int_\mathbb{C} |k'_w(z)|^q e^{-\frac{q}{2}|z|^2} \, dm(z) \geq \int_{D(w,1)} |k'_w(z)|^q e^{-\frac{q}{2}|z|^2} \, dm(z) \gtrsim |wk'_w(z)|^q e^{-\frac{q}{2}|z|^2}
\]

for all \( z, w \in \mathbb{C} \). In particular, setting \( w = z \) here again leads to the estimate

\[
1 \gtrsim |w|^q.
\]

Letting \( |w| \to \infty \) gives a contradiction. Similarly, for \( q = \infty \), taking \( w = z \) again which results a contradiction for large \( w \). \( \square \)

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