Relativistic origin of Hertz-form and extended Hertz-form equations for Maxwell theory of electromagnetism

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Abstract: We show explicitly that the Hertz-form Maxwell’s equations and their extensions can be obtained from the non-relativistic expansion of Lorentz transformation of Maxwell’s equations. The explicit expression for the parameter $\alpha$ in the extended Hertz-form equations can be derived from such a non-relativistic expansion. The extended Hertz-form equations, which do not preserve Galilean invariance, origin from Lorentz transformation of Maxwell’s equations and differ from the Galilean-transformed Maxwell equations (the original Hertz equations) by the relative sign differences between the two $\alpha$ terms etc. Especially, the $\alpha$ parameter is of relativistic origin. The superluminal behavior illustrated by the D’Alembert equation from the extended Hertz-form equations should be removed by including all subleading contributions in the $v/c$ expansion, although such a superluminal behavior will not occur in the vacuum because $\alpha = 0$. In the case that the electromagnetic field is a background field, we need not worry about the apparent superluminal behavior of the D’Alembert equation. We should note that in the Hertz form and extended Hertz form equations, the electromagnetic fields should take the forms $\vec{E}(x) = \vec{E}(\Lambda^{-1}x)$ and $\vec{B}(x) = \vec{B}(\Lambda^{-1}x)$ while the derivatives in the equations are taken with respect to $x$. Such a choice of description for the fields is different from the ordinary one with $\vec{E}(x)$ and $\vec{B}(x)$, which are well known to satisfy the ordinary Maxwell’s equations. The descriptions of electromagnetic phenomena using the function set $\{\vec{E}(x), \vec{B}(x)\}$ and the function set $\{(\vec{E}(x), \vec{B}(x))\}$ are equivalent, with the $\{\vec{E}(x), \vec{B}(x)\}$ description satisfying the extended Hertz-form Maxwell’s equations in the low speed approximation. The solution of (extended) Hertz-form Maxwell’s equations describe the traveling wave form electromagnetic field.

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1 Introduction

Maxwell’s equations, which can successfully describe the classical electromagnetic phenomena, play an important rule in fundamental science and practical technologies. There exist various extensions to Maxwell’s equations, mostly by adding the magnetic monopole related terms in the framework of quantum field theory. In the framework of classical limit, the Hertz-form equations [1], which amend the ordinary Maxwell equations by several terms, apparently preserve Galilean invariance. As the Maxwell’s equations should satisfy Lorentz invariance, the Hertz-form Maxwell’s equations seem strange. So, it is interesting to revisit Hertz-form equations to figure out if it is possible to derive them from the non-relativistic approximation of Lorentz transformation for electromagnetic fields.

However, the Hertz-form equations do not agree with the experimental data on the movement of dielectrics in electromagnetic fields [2]. In order to satisfy the experimental data, in [3] an additional factor $\alpha = (\mu_r \varepsilon_r - 1)/\mu_r \varepsilon_r$ is introduced for the terms $\nabla \times (B \times v)$ and $\nabla \times (D \times v)$ appearing in Hertz-form equations of $\nabla \times E$ and $\nabla \times H$, with $\varepsilon_r$ and $\mu_r$ being the relative permittivity and the relative permeability of the medium, and $E$, $H$, $B$, $D$ being the vectors of electric and magnetic field strengths, magnetic induction and electric displacement, respectively. Such extended Hertz-form equations with the parameter $\alpha$, which lose Galilean invariance, can agree with the experimental data [3]. Although the appearance of the factor $\alpha$ is commented in [3], its origin is not specified. So, it is interesting to figure out the origin of the $\alpha$ parameter from the first principle.

Note that the recent work in [4] proposes to extend Maxwell’s equations with new $P_S$ term [5] and velocity-related terms to describe electromagnetic phenomena in the slow-moving media. Formally, after redefining a new electric displacement field $D = D' + P_S$,
the work of [4] can be seen to be equivalent to Hertz-form equations [1]. Maxwell equations in materials have also been studied intensively in [6, 7].

In this note, we derive the non-relativistic approximation for Lorentz transformation of Maxwell’s equations. We find that the (extended) Hertz-form equations can be obtained at the (next) leading order approximation and the explicit form of $\alpha$ can be obtained from the non-relativistic limit. We show that the appearance of the factor $\alpha$, which is commented in [3] and violates Galilean transformation, is the consequence of special relativity. The (extended) Hertz-form Maxwell’s equations give an alternative (traveling-wave form) solution for electromagnetic field.

This paper is organized as follows. In Sec. 2, we derive explicitly the form of $\alpha$ and the extended Hertz-form equations from the non-relativistic expansion of Lorentz transformation for Maxwell’s equations. A typical consequence of the extended Hertz-form equations, i.e., the superluminal behavior, is discussed in Sec. 3. Sec. 4 contains our conclusions. Some important formulas used in the non-relativistic (low speed) approximation are given in the appendix A.

2 Non-relativistic expansion of Lorentz transformation for Maxwell’s equations

The validity of applying special relativity to the description of electromagnetic field and its quanta has been proven to an astonishing accuracy. For example, the theoretical prediction on electron anomalous magnetic moment by QED can fit the experimental data up to $10^{-10}$ accuracy. So, to describe electromagnetic phenomena, it is natural to begin with special relativity. In the following, we try to derive the non-relativistic approximation of Lorentz transformation for Maxwell’s equations to describe the relevant electromagnetic phenomena in slow-moving media.

Lorentz transformation of electromagnetic field is given by

$$F'_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu F_{\rho\sigma},$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which, in components, is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -E_1/c \\ -B_3 & 0 & B_1 & -E_2/c \\ B_2 & -B_1 & 0 & -E_3/c \\ E_1/c & E_2/c & E_3/c & 0 \end{pmatrix}.$$  \hspace{1cm} \text{(2.3)}

We keep the vacuum light speed $c$ in the expressions for later convenience.

In components, we have Lorentz transformation of electromagnetic fields as

$$\vec{E}_\perp \rightarrow \gamma(\vec{E} + \vec{v} \times \vec{B})_\perp, \hspace{1cm} \vec{B}_\perp \rightarrow \gamma \left( \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right)_\perp,$$  \hspace{1cm} \text{(2.4)}
\[ E_n \rightarrow \tilde{E}_n, \quad B_n \rightarrow \tilde{B}_n, \quad (2.5) \]

with
\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (2.6) \]

From the transformation law, it is obvious that we only need to discuss the simplest case with \( \vec{E} \) and \( \vec{B} \) perpendicular to the constant velocity \( \vec{v} \) of the reference frame. The extensions to general cases are straightforward.

In our analysis we consider two reference frames, one is \( \Sigma \) (laboratory) frame with coordinates \( \{x\} \), and the other is \( \Sigma' \) (loop-rest) frame with coordinates \( \{x'\} \) which is at rest with respect to the medium and is moving at a constant velocity \( \vec{v} \) with respect to \( \Sigma \).

Then we have the relations between the fields
\[ \vec{E}_\perp(x) \rightarrow \vec{E}'_{\perp}(x') = \gamma(\vec{E}(x) + \vec{v} \times \vec{B}(x))_\perp, \]
\[ \vec{B}_\perp(x) \rightarrow \vec{B}'_{\perp}(x') = \gamma\left(\vec{B}(x) - \frac{\vec{v} \times \vec{E}(x)}{c^2}\right)_\perp, \quad (2.7) \]

and the relation between the coordinates
\[ x' = \Lambda x, \quad (2.8) \]

with \( \Lambda \) the Lorentz transformation for the coordinates. Alternatively, we can write in the forms of \( \vec{E}(x') \) and \( \vec{B}(x') \)
\[ \vec{E}'_\perp(x') = \gamma(\vec{E}(x') + \vec{v} \times \vec{B}(x'))_\perp, \]
\[ \vec{B}'_\perp(x') = \gamma\left(\vec{B}(x') - \frac{\vec{v} \times \vec{E}(x')}{c^2}\right)_\perp, \quad (2.9) \]

where
\[ \vec{E}(x') = \vec{E}[x(x')] \, , \quad \vec{B}(x') = \vec{B}[x(x')] \, , \quad (2.10) \]

with
\[ x(x') = \Lambda^{-1} x', \quad (2.11) \]

being the inverse coordinate transformation. The explicit function forms of \( \vec{E}(x) \) and \( \vec{E}(x) \) are different.

In the non-relativistic approximation up to \( o(v/c) \), we have the approximate Galilean transformation for coordinates
\[ \vec{x}' = \vec{x} - \vec{v}t, \quad t' = t - \frac{\vec{x} \cdot \vec{v}}{c^2}, \quad (2.12) \]
and thus we have
\[ \vec{E}'(x') \equiv \vec{E}[\vec{x}' + \vec{v}t', t' + \frac{\vec{x}' \cdot \vec{v}}{c^2}], \quad \vec{B}'(x') \equiv \vec{B}[\vec{x}' + \vec{v}t', t' + \frac{\vec{x}' \cdot \vec{v}}{c^2}], \tag{2.13} \]
which, after using a new notation of variable \( 'z' \) to avoid confusion, can be written as
\[ \vec{E}(z) = \vec{E}[\vec{x} + \vec{v}z_0, z_0 + \frac{\vec{v} \cdot \vec{z}}{c^2}], \quad \vec{B}(z) = \vec{B}[\vec{x} + \vec{v}z_0, z_0 + \frac{\vec{v} \cdot \vec{z}}{c^2}]. \tag{2.14} \]

This is naively the (inverse) Galilean transformed coordinate variables for electromagnetic fields if we neglect the \((\vec{v} \cdot \vec{z})/c^2\) term in the inverse transformation of \(z_0\) variable.

One of Maxwell’s equation at the loop-rest \(\Sigma\) frame is
\[ \nabla' \times \vec{E}'(x') = -\frac{\partial}{\partial t} \vec{B}'(x'). \tag{2.15} \]
Considering eq.(2.9), the above equation can be rewritten as \(^{1}\)
\[ \nabla' \times (\vec{E}(x') + \vec{v} \times \vec{B}(x')) = -\frac{\partial}{\partial t} \left( \vec{B}(x') - \frac{\vec{v} \times \vec{E}(x')}{c^2} \right), \]
\[ \Rightarrow \nabla \times (\vec{E}(x) + \vec{v} \times \vec{B}(x)) = -\frac{\partial}{\partial t} \left( \vec{B}(x) - \frac{\vec{v} \times \vec{E}(x)}{c^2} \right). \tag{2.16} \]

The forms of the functions \(\vec{E}\) and \(\vec{B}\) are given by eq.(2.10). Here \(\nabla'\) and \(\nabla\) denote the derivative with respect to the \(x' \equiv (\vec{x}', t')\) coordinates and \(x \equiv (\vec{x}, t)\) coordinates, respectively. Note that the replacement in eq.(2.16) is fairly non-trivial if it is to be understood as the changing of reference frame instead of simply changing of variables. The derivative of the original \(x'\) coordinates should be recasted into the \(x\) variables. The \(\vec{E}'\) and \(\vec{B}'\) fields, which are given in terms of \(\vec{E}\) and \(\vec{B}\) fields with inversely Lorentz transformation coordinates, are also in the original \(x'\) coordinates. New terms from the change of the derivative cancel the terms from the change of variables for the \(\vec{E}'\) and \(\vec{B}'\) fields up to the order of \(v/c\). Details of the cancelation are given in Appendix A.

If the forms \(\vec{E}(x), \vec{B}(x)\) (instead of the form of the functions \(\vec{E}(x)\) and \(\vec{B}(x)\)) are adopted, we can recover the Maxwell’s equations in the laboratory \(\Sigma\) frame. For example, we can check one of the equation in Maxwell’s equations
\[ \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}. \tag{2.17} \]
In the \(\Sigma'\) reference (loop-rest) frame, the left-handed side of eq.(2.15) is given by
\[ \gamma \nabla_{\Sigma'} \times \left( \vec{E}(x) + \vec{v} \times \vec{B}(x) \right) \]
\[ = \gamma^2 \left( \nabla + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \times \left( \vec{E}(x) + \vec{v} \times \vec{B}(x) \right) \]
\[ = \gamma^2 \left( \nabla \times \vec{E}(x) + \frac{\vec{v}}{c^2} \times \frac{\partial \vec{E}(x)}{\partial t} - (\vec{v} \cdot \nabla) \vec{B}(x) - \frac{\vec{v}}{c^2} \times \left( \nabla \times \vec{E}(x) \right) \right), \tag{2.18} \]
\(^{1}\)The second equation can be obtained from the first equation by simply changing variables from \(x'\) to \(x\). As different variables correspond to different reference frame, the change of variables also corresponds to the transformations between different reference frames, as discussed in Appendix A.
while the right-handed side of eq.(2.15) is

\[-\gamma \frac{\partial}{\partial t^{'}} \left( \mathbf{B}(x) - \frac{\mathbf{v} \times \mathbf{E}(x)}{c^2} \right)\]

\[= -\gamma^2 \left( \mathbf{v} \cdot \nabla + \frac{\partial}{\partial t} \right) \left( \mathbf{B}(x) - \frac{\mathbf{v} \times \mathbf{E}(x)}{c^2} \right)\]

\[= \gamma^2 \left( -\frac{\partial}{\partial t} \mathbf{B}(x) - (\mathbf{v} \cdot \nabla) \mathbf{B}(x) + \frac{\mathbf{v}}{c^2} \times \frac{\partial \mathbf{E}(x)}{\partial t} - \frac{\mathbf{v}}{c^2} \times \left( \nabla \times \mathbf{E}(x) \right) \right) \quad (2.19)\]

where

\[(\mathbf{v} \cdot \nabla) (\mathbf{v} \times \mathbf{E}) = \nabla \left[ \mathbf{v} \cdot \left( \mathbf{v} \times \mathbf{E} \right) \right] - \mathbf{v} \times \left( \nabla \times (\mathbf{v} \times \mathbf{E}) \right) ,\]

\[= -\mathbf{v} \times \left( \nabla \times (\mathbf{v} \times \mathbf{E}) \right) ,\]

\[= \mathbf{v} \times \left[ (\mathbf{v} \cdot \nabla) \mathbf{E} \right] ,\]

\[= -\mathbf{v} \times \left[ \mathbf{v} \times \left( \nabla \times \mathbf{E} \right) \right] . \quad (2.20)\]

Comparing the above equations we just reproduce the ordinary form of Maxwell’s equations in eq.(2.17).

So it is important to notify which forms of functions for electromagnetic fields, \{\mathbf{E}(x), \mathbf{B}(x)\} or \{\mathbf{\bar{E}}(x), \mathbf{\bar{B}}(x)\}, are used in the equations. In the Hertz and extended Hertz equations, we should use the transformed function forms \mathbf{\bar{E}}(x) and \mathbf{\bar{B}}(x) in eq.(2.10). The ordinary \mathbf{E}(x) and \mathbf{B}(x) should be used in the ordinary forms of Maxwell’s equations.

Unless otherwise specified, the coordinates for the fields and the derivatives take the value \((\mathbf{x}, t)\); the functions for electromagnetic fields take the form in eq.(2.10).

The (extended) Hertz-form Maxwell’s equations can be deduced from ordinary Maxwell’s equations order by order in \(v\). From eq.(2.16), after neglecting the \(v/c\) term, we can reproduce the Hertz equation

\[\nabla \times (\mathbf{\bar{E}} + \mathbf{v} \times \mathbf{\bar{B}}) = -\frac{\partial}{\partial t} \mathbf{\bar{B}}. \quad (2.21)\]

Taking into account the \(v/c\) correction, we will have an additional term

\[\frac{\mathbf{v}}{c^2} \times \frac{\partial}{\partial t} \mathbf{\bar{E}}. \quad (2.22)\]

The expression \(\partial \mathbf{\bar{E}}/\partial t\) can be deduced as

\[\nabla \times \left[ \mathbf{\bar{B}} - \frac{\mathbf{v} \times \mathbf{\bar{E}}}{c^2} \right] = \mu \mathbf{\bar{J}} + \mu \epsilon \frac{\partial}{\partial t} \left( \mathbf{\bar{E}} + \mathbf{v} \times \mathbf{\bar{B}} \right) - \gamma \mu \mathbf{\bar{v}}, \quad (2.23)\]

from one of the Maxwell’s equations in the loop-rest \(\Sigma'\) frame after the substitution with coordinates-transformed electromagnetic fields and sources

\[\nabla' \times \mathbf{\bar{B}}'(x') = \mu \mathbf{\bar{J}}(x') + \mu \epsilon \frac{\partial}{\partial t} \mathbf{\bar{E}}'(x'). \quad (2.24)\]
We have
\[
\frac{\partial}{\partial t} \vec{E} = \frac{1}{\mu \epsilon} \nabla \times \left( \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) - \frac{1}{\epsilon} \vec{J} - \vec{v} \times \left( \frac{\partial}{\partial t} \vec{B} \right) + \mu \vec{v} \cdot \vec{v}.
\] (2.25)

Substituting back into the expressions of (2.16), we have
\[
\nabla \times (\vec{E} + \vec{v} \times \vec{B}) = -\frac{\partial}{\partial t} \vec{B} + \frac{\vec{v}}{c^2} \times \frac{\partial}{\partial t} \vec{E}
\]
\[
\approx -\frac{\partial}{\partial t} \vec{B} + \frac{1}{\mu \epsilon c^2} \times \left( \nabla \times \vec{B} \right) - \frac{1}{\epsilon c^2} \vec{v} \times \vec{J},
\] (2.26)

after neglecting the \( \vec{v}^2 \) terms. We also have
\[
\vec{v} \times \left( \nabla \times \vec{B} \right) = \nabla (\vec{v} \cdot \vec{B}) - (\vec{v} \cdot \nabla) \vec{B}
\]
\[
= - (\vec{v} \cdot \nabla) \vec{B}
\]
\[
= \nabla \times \left( \vec{v} \times \vec{B} \right),
\] (2.27)

for constant velocity \( \vec{v} \) and \( \vec{v} \cdot \vec{B} = 0 \).

After neglecting the source \( \vec{J} \) term or the source vector is parallel to \( \vec{v} \), we can arrive at the coefficient of \( \nabla \times (\vec{v} \times \vec{B}) \) within eq. (2.26)
\[
\nabla \times (\vec{E} + \alpha \vec{v} \times \vec{B}) = -\frac{\partial}{\partial t} \vec{B},
\] (2.28)

with
\[
\alpha = 1 - \frac{1}{\epsilon \mu c^2} = 1 - \frac{\epsilon_0 \mu_0}{\epsilon \mu} = \frac{\epsilon \mu - \epsilon_0 \mu_0}{\epsilon \mu}
\] (2.29)

and the expression for light speed in the vacuum
\[
c^2 = \frac{1}{\epsilon_0 \mu_0}.
\] (2.30)

We see that the expression for \( \alpha \) is just the form given in [3].

Similarly, we can substitute the Hertz equation for \( \nabla \times \vec{E} \) into eq. (2.23)
\[
\nabla \times \left[ \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right] = \mu \left( \vec{J} - \vec{v} \hat{\rho} \right) + \mu \epsilon \frac{\partial}{\partial t} \left( \vec{E} + \vec{v} \times \vec{B} \right),
\] (2.31)

to obtain
\[
\nabla \times \left[ \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right] = \mu \left( \vec{J} - \vec{v} \hat{\rho} \right) + \mu \epsilon \frac{\partial}{\partial t} \vec{E} - \mu \epsilon \vec{v} \times \left[ \nabla \times (\vec{E} + \alpha \vec{v} \times \vec{B}) \right],
\] (2.32)

with
\[
\frac{\partial}{\partial t} \left( \vec{v} \times \vec{B} \right) = \vec{v} \times \frac{\partial}{\partial t} \vec{B} = - \vec{v} \times \left[ \nabla \times (\vec{E} + \alpha \vec{v} \times \vec{B}) \right].
\] (2.33)
Again, after neglecting the $v^2$ term, we arrive at

$$\nabla \times \left[ \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right] = \mu \left( \vec{J} - \vec{v} \rho \right) + \mu \frac{\partial}{\partial t} \vec{E} - \mu \varepsilon \vec{v} \times \left( \nabla \times \vec{E} \right). \quad (2.34)$$

Using

$$\vec{v} \times \left( \nabla \times \vec{E} \right) = \nabla (\vec{v} \cdot \vec{E}) - (\vec{v} \cdot \nabla) \vec{E}$$

$$= - (\vec{v} \cdot \nabla) \vec{E}$$

$$= \nabla \times \left( \vec{v} \times \vec{E} \right) - (\nabla \cdot \vec{E}) \vec{v}$$

$$= \nabla \times \left( \vec{v} \times \vec{E} \right) - \frac{1}{\varepsilon} \vec{\rho} \vec{v}, \quad (2.35)$$

we can arrive at

$$\nabla \times \left[ \vec{B} - \mu_0 \varepsilon_0 \varepsilon \vec{v} \times \vec{E} \right] = \mu \vec{J} + \mu \frac{\partial}{\partial t} \vec{E} - \mu \varepsilon \nabla \times \left( \vec{v} \times \vec{E} \right), \quad (2.36)$$

which is just the form

$$\nabla \times \left[ \vec{B} + \mu \varepsilon \alpha \vec{v} \times \vec{E} \right] = \mu \vec{J} + \mu \frac{\partial}{\partial t} \vec{E},$$

$$\nabla \times \left[ \vec{H} + \alpha \vec{v} \times \vec{D} \right] = \vec{J} + \frac{\partial}{\partial t} \vec{D}, \quad (2.37)$$

with

$$\alpha \equiv \frac{\epsilon \mu - \epsilon_0 \mu_0}{\epsilon \mu}. \quad (2.38)$$

So the generalization of Hertz-form equations to include $\alpha$ can be seen to come from the non-relativisitic expansion of Lorentz-transformed Maxwell’s equations, using an alternative form of functions for electromagnetic fields.

Other extended Herz-form Maxwell equations can be obtained in a similar way

$$\nabla \cdot \left( \vec{E} + \vec{v} \times \vec{B} \right) = \frac{1}{\varepsilon} \left( \rho - \frac{\vec{v} \cdot \vec{J}}{c^2} \right)$$

$$\Rightarrow \nabla \cdot \vec{E} - \vec{v} \cdot \left( \nabla \times \vec{B} \right) = \frac{1}{\varepsilon} \left( \rho - \frac{\vec{v} \cdot \vec{J}}{c^2} \right)$$

$$\Rightarrow \nabla \cdot \vec{E} - \vec{v} \cdot \left( \mu \vec{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} + \alpha (\vec{v} \cdot \nabla) \vec{E} \right) = \frac{1}{\varepsilon} \rho - \mu \vec{v} \cdot \vec{J}$$

$$\Rightarrow \nabla \cdot \vec{E} - \frac{\vec{v}}{c^2} \cdot \left( \frac{\partial}{\partial t} \vec{E} + \alpha (\vec{v} \cdot \nabla) \vec{E} \right) = \frac{1}{\varepsilon} \rho$$

$$\Rightarrow \nabla \cdot \vec{E} = \frac{1}{\varepsilon} \rho, \quad (2.39)$$

with

$$\frac{d}{dt} (\vec{v} \cdot \vec{E}) \equiv 0 \Rightarrow \vec{v} \cdot \frac{\partial}{\partial t} \vec{E} = 0. \quad (2.40)$$
We neglect the order $v^2$ term in the fourth line of eq. (2.39) in the last step.

We also have

$$\nabla \cdot \left( \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) = 0 ,$$

$$\Rightarrow \nabla \cdot \vec{B} - \frac{\vec{v}}{c^2} \cdot \left( \nabla \times \vec{E} \right) = 0 ,$$

$$\Rightarrow \nabla \cdot \vec{B} - \frac{\vec{v}}{c^2} \cdot \left( -\frac{\partial}{\partial t} \vec{B} + \alpha (\vec{v} \cdot \nabla) \vec{B} \right) = 0 ,$$

$$\Rightarrow \nabla \cdot \vec{B} = 0 ,$$

(2.41)

with

$$\frac{d}{dt}(\vec{v} \cdot \vec{B}) \equiv 0 \Rightarrow \vec{v} \cdot \frac{\partial}{\partial t} \vec{B} = 0 ,$$

after neglecting the order $v^2$ term.

The new extended Hertz-form equations can be written together as

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon} \rho ,$$

(2.42)

$$\nabla \cdot \vec{B} = 0 ,$$

(2.43)

$$\nabla \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial}{\partial t} \vec{E} + \mu \epsilon \alpha (\vec{v} \cdot \nabla) \vec{E} ,$$

(2.44)

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} + \alpha (\vec{v} \cdot \nabla) \vec{B} .$$

(2.45)

In the case of conductors with $\mu, \epsilon \to \infty$, the value of $\alpha$ tends to 1. We should note that, even when $\alpha \to 1$, our new extended Hertz-form Maxwell’s equations can not recover the original Hertz-form Maxwell’s equations. In the original Hertz-form Maxwell’s equations, the $\partial_t$ derivative always accompany with the $\vec{v} \cdot \nabla$ term so as that they can be combined into a $D_t$ derivative

$$D_t \equiv \frac{\partial}{\partial t} - (\vec{v} \cdot \nabla) .$$

(2.46)

However, in our new (extended) Hertz-form Maxwell’s equations, there is a sign difference for the $\alpha$ term in eq.(2.44), which can not be written as $D_t$ in the $\alpha \to 1$ limit. Using the $-\vec{v}$ instead of $\vec{v}$ can not change both eq.(2.44) and eq.(2.45) into the $D_t$ form in the $\alpha \to 1$ limit due to the relative sign difference in $\partial_t$ for the two equations. Such new extended Hertz-form Maxwell’s equations can not recover the Galilean transformed Maxwell’s equations (the original Hertz-form Maxwell’s equations) in the $\alpha \to 1$ limit.

So, it is obvious that the extended Hertz-form equations can be deduced from the non-relativistic expansion of Lorentz transformation of Maxwell’s equations, which adopt an alternative form of functions for electromagnetic fields. Although the original Hertz-form equations preserve Galilean invariance, the extend Hertz-form equations lose such an invariance. As the propagating electromagnetic waves are intrinsically relativistic, the extended
Hertz form (from non-relativistic expansion) should be used when the electromagnetic fields can be seen as the background.

It is worth noting that the \(\{\vec{E}, \vec{B}\}\) set description for electromagnetic field is an equivalent traveling wave form description for electromagnetic field. Such a traveling wave form description satisfies the extended Hertz-form Maxwell’s equations. That is, the solution of extended Hertz-form Maxwell’s equations describe the traveling wave form electromagnetic field.

3 Discussions about the superluminal behavior

From the previous extended Hertz equations involving \(\alpha\), we can deduce the D’Alembert equation for \(\vec{E}\) (and \(\vec{B}\)). For simply, we consider only the plane wave solutions with

\[
\vec{E}(x) = \vec{E}_0 \exp \left[ i \left( \vec{k} \cdot \vec{x} - \omega t \right) \right], \quad \vec{B}(x) = \vec{B}_0 \exp \left[ i \left( \vec{k} \cdot \vec{x} - \omega t \right) \right]. \tag{3.1}
\]

The extended Hertz equations without sources can be rewritten as

\[
\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{B} = 0,
\]

\[
\nabla \times \vec{B} = \mu \varepsilon \frac{\partial}{\partial t} \vec{E} + \mu \varepsilon \alpha (\vec{v} \cdot \nabla) \vec{E},
\]

\[
\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} + \alpha (\vec{v} \cdot \nabla) \vec{B}. \tag{3.2}
\]

Take the \textit{curl} operation for the fourth equation, we can obtain

\[
-\nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left[ \mu \varepsilon \frac{\partial}{\partial t} \vec{E} + \mu \varepsilon \alpha (\vec{v} \cdot \nabla) \vec{E} \right] + \alpha \nabla \times \left[ (\vec{v} \cdot \nabla) \vec{B} \right],
\]

\[
= -\frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} \vec{E} + i \alpha \left( \vec{v} \cdot \vec{k} \right) \frac{\partial}{\partial t} \vec{E} \right] + i \alpha \left( \vec{v} \cdot \vec{k} \right) \left[ \nabla \times \vec{B} \right],
\]

\[
= -\frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} \vec{E} + i \alpha \left( \vec{v} \cdot \vec{k} \right) \frac{\partial}{\partial t} \vec{E} \right] + i \alpha \left( \vec{v} \cdot \vec{k} \right) \frac{1}{c^2} \left[ \frac{\partial}{\partial t} + i \alpha \left( \vec{v} \cdot \vec{k} \right) \right] \vec{E},
\]

\[
= -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} - \frac{\alpha^2}{c^2} \left( \vec{v} \cdot \vec{k} \right)^2 \vec{E}. \tag{3.3}
\]

with the light speed in the media \(1/c^2 = \mu \varepsilon\).

So the resulting D’Alembert equation can be written as

\[
\left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\alpha^2}{c^2} \left( \vec{v} \cdot \vec{k} \right)^2 \right] \vec{E} = 0. \tag{3.4}
\]

We have the following discussions:

- \(\vec{v} \cdot \vec{k} = 0\), that is, the direction \(\vec{v}\) is perpendicular to propagating direction \(\vec{k}\). The D’Alembert equation takes the ordinary form with the light speed \(c^2 = 1/(\mu \varepsilon)\).

- \(\vec{v} \parallel \vec{k}\). The D’Alembert equation takes the form

\[
\left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\alpha^2}{c^2} \vec{v}^2 \vec{k}^2 \right] \vec{E} = 0. \tag{3.5}
\]
In momentum space, we have
\[-k^2 + \frac{\omega^2}{c^2} - \frac{\alpha^2}{c^2} v^2 k^2 = 0\].
(3.6)

So
\[\left(1 + \frac{\alpha^2}{c^2} v^2\right)k^2 = \frac{\omega^2}{c^2} \].
(3.7)

We can define a new quantity, apparent light speed \(c'\), to satisfy
\[c'^2 = c^2 + \alpha^2 v^2\],
(3.8)
so as that the ordinary relation \(k^2c'^2 = \omega^2\) is preserved for massless photon. It is obvious that the quantity \(c'\) is always larger than the light speed in the media.

From previous discussions, although it is not problematic for the apparent light speed \(c'\) of the electromagnetic field to be larger than the ordinary light speed in the media, the quantity \(c'\) apparently exceed the ordinary vacuum light speed if this D’Alembert equation is applied for the vacuum. However, it is interesting to note that the value \(\alpha = 0\) is hold for vacuum. So, the value of apparent light speed in the vacuum is still equal to the vacuum light speed. Even though \(c'^2 = c^2\) holds in the vacuum, it is still possible that the apparent light speed \(c'\) in the media is larger than the vacuum light speed, depending on the choice of \(\alpha\) and \(v\).

One may worry that the superluminal behavior may cause inconsistency. However, we should note that the previous expressions in eq.(2.28) and eq.(2.36) are just the non-relativistic approximation of Lorentz transformations for Maxwell’s equations. They should not apply to the propagation of electromagnetic wave, which is intrinsically relativistic. By going to the relativistic region, the sub-leading terms in \(v/c\) expansion should be included. The \(\alpha^2 k^2 v^2\) term will be canceled by the inclusion of all the subleading contributions (in \(v/c\) expansion) to keep the vacuum light speed as a constant, which is just one of the two ansatzes of special relativity. In the case that the electromagnetic field is a background field, we need not worry about the apparent superluminal behavior illustrated by the D’Alembert equation.

4 Conclusions

We showed explicitly that the Hertz-form Maxwell’s equations and their extensions can be obtained from the non-relativistic expansion of Lorentz transformation of Maxwell’s equations. The explicit expression for the parameter \(\alpha\) in the extended Hertz-form equations can be derived from such a non-relativistic expansion. The extended Hertz-form equations, which do not preserve Galilean invariance, origin from Lorentz transformation of Maxwell’s equations. Especially, the \(\alpha\) parameter is of relativistic origin. The superluminal behavior illustrated by the D’Alembert equation from the extended Hertz-form equations should be removed by including all subleading contributions in the \(v/c\) expansion, although such
a superluminal behavior will not occur in the vacuum because $\alpha = 0$. In the case that
the electromagnetic field is a background field, we need not worry about the apparent
superluminal behavior of the D’Alembert equation. We should note that in the new Hertz
form and extended Hertz form equations, the electromagnetic fields should take the forms
$\vec{E}(x) = \vec{E}(\Lambda^{-1}x)$ and $\vec{B}(x) = \vec{B}(\Lambda^{-1}x)$ while the derivatives in the equations are taken with
respect to $x$. Such a choice of description for the fields is different from the ordinary one
with $\vec{E}(x)$ and $\vec{B}(x)$, which are well known to satisfy the ordinary Maxwell’s equations.
The descriptions of electromagnetic phenomena using the function set \{\(\vec{E}(x), \vec{B}(x)\)\} and
the function set \(\{E(x), B(x)\}\) are equivalent, with the \{\(\vec{E}(x), \vec{B}(x)\)\} description satisfying
the new extended Hertz-form Maxwell’s equations in the low speed approximation. The
solution of (extended) Hertz-form Maxwell’s equations describe the traveling wave form
electromagnetic field.

We should mention that ordinary magnetic monopole extension of Maxwell’s equations
use the form of functions $\vec{E}(x)$ and $\vec{B}(x)$. If we adopt the $E(x)$ and $B(x)$ descriptions of
the electromagnetic fields, we can also extend the (extended) Hertz-form Maxwell’s equations
with additional topological terms.

Besides, as the (extended) Hertz equations can be readily deduced from special relativity,
the aether assumption is no-longer needed here. It is just the non-relativistic limit
description of Maxwell’s equations with an alternative set of functions for electromagnetic
fields, although they can approximately preserve Galilean invariance.

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A The non-relativistic expansion of typical terms in the Lorentz-transformed Maxwell’s equations

In eq.\((2.16)\), the expressions in both coordinates are unchanged. It can surely be seen as the naive change of variables. It can also be seen physically by directly adopting the transformations of reference frame.

The expansion of electromagnetic fields in \(x'\) variable can be expressed in terms of \(x\) variable as

\[
\vec{E}(\vec{x}', t') \approx \vec{E}(\vec{x} - t\vec{v}, t - \frac{\vec{v} \cdot \vec{x}^{'}}{c^2}) \approx \vec{E}(\vec{x}, t) - t(\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) - \frac{\vec{v} \cdot \vec{x}^{'}}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{x}, t),
\]

(A.1)

for small \(v\). Similarly expansion can be given for other functions in the Hertz and extended Hertz equations. We keep only the leading order of \(v\) expansion without taking into account the \(\gamma\) factor, which will give higher order terms of \(v^2/c^2\) in the expansion. Firstly, we consider the transformation

\[
\nabla \times \vec{B}(\vec{x}, t) \rightarrow \nabla' \times \vec{B}(\vec{x'}, t').
\]

(A.2)

From the Lorentz transformation law of \(\partial \mu\), we can obtain the expressions for \(\nabla\)

\[
\nabla' \rightarrow \gamma \left( \nabla + \frac{\vec{v} \partial}{c^2 \partial t} \right),
\]

\[
\frac{\partial}{\partial t'} \rightarrow \gamma \left( \vec{v} \cdot \nabla + \frac{\partial}{\partial t} \right).
\]

(A.3)

So we have

\[
\nabla' \times \vec{E}(\vec{x}', t') \approx \gamma \left( \nabla + \frac{\vec{v} \partial}{c^2 \partial t} \right) \times \left[ \vec{E}(\vec{x}, t) - t(\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) - \frac{\vec{v} \cdot \vec{x}^{'}}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) \right]
\]

\[
\approx \nabla \times \vec{E}(\vec{x}, t) + \frac{\vec{v} \partial}{c^2 \partial t} \times \vec{E}(\vec{x}, t) - \frac{1}{\mu c^2} \nabla \times \left[ (\vec{v} \cdot \vec{x}) \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) \right] - t\nabla \times \left[ (\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) \right]
\]

\[
\approx \nabla \times \vec{E}(\vec{x}, t) + \frac{1}{\mu c^2} \nabla \times \vec{B}(\vec{x}, t) - \frac{1}{\mu c^2} \nabla \times \left[ (\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) \right] - \nabla \times \left[ (\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) \right]
\]

\[
= \nabla \times \vec{E}(\vec{x}, t) - t\nabla \times \left[ (\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t) \right],
\]

(A.4)

with

\[
\nabla \times \left[ (\vec{v} \cdot \vec{x}) \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) \right] = \nabla(\vec{v} \cdot \vec{x}) \times \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) + (\vec{v} \cdot \vec{x}) \frac{\partial}{\partial t} \left( \nabla \times \vec{E}(\vec{x}, t) \right)
\]

\[
= \frac{1}{\mu c^2} \vec{v} \times (\nabla \times \vec{B} - \frac{1}{\mu} \vec{J}) - (\vec{v} \cdot \vec{x}) \frac{\partial^2}{\partial t^2} \vec{B}.
\]

(A.5)

The term involving \(\partial^2 \vec{E}/\partial t^2\) is \(1/c^2\) suppressed in comparison with the first term as \(c^2 = 1/(\mu \epsilon)\) and \(\vec{J}\) always parallel to \(\vec{v}\). So we can neglect such terms. Note that the cancelation of the \(\partial \vec{E}/\partial t\) terms in the second line of eq.(A.4) is automatic, which do not depend on the explicit form of \(\partial \vec{E}/\partial t\) in Maxwell’s equations.
Up to $v/c$ order, we have
\[
\nabla' \times \left( \vec{v} \times \vec{B}(\vec{x}', t') \right) \approx \nabla \times \left( \vec{v} \times \vec{B}(\vec{x}, t) \right) + o(v^2/c^2).
\quad (A.6)
\]

The right-handed side of eq. (2.16) is
\[
\frac{\partial}{\partial t'} \vec{B}(\vec{x}', t') \approx \left( \vec{v} \cdot \nabla + \frac{\partial}{\partial t} \right) \left[ \vec{B}(\vec{x}, t) - t (\vec{v} \cdot \nabla) \vec{B}(\vec{x}, t) \right]
\approx (\vec{v} \cdot \nabla) \vec{B}(\vec{x}, t) + \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) - t (\vec{v} \cdot \nabla) \vec{B}(\vec{x}, t) - t \frac{\partial}{\partial t} (\vec{v} \cdot \nabla) \vec{B}(\vec{x}, t)
= \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) - t \left( \nabla \times \left( \vec{v} \times \vec{B}(\vec{x}, t) \right) \right)
= \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) + t \left( \nabla \times \left( \vec{v} \times \nabla \vec{E}(\vec{x}) \right) \right)
= \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) + t \left( \nabla \times (\vec{v} \cdot \nabla) \vec{E}(\vec{x}) \right)
= \frac{\partial}{\partial t} \vec{B}(\vec{x}, t) - t \left( \nabla \times (\vec{v} \cdot \nabla) \vec{E}(\vec{x}) \right),
\quad (A.7)
\]

and
\[
\frac{\partial}{\partial t'} \left[ \frac{\vec{v}}{c^2} \times \vec{E}(\vec{x}) \right] \approx \frac{\partial}{\partial t} \left[ \frac{\vec{v}}{c^2} \times \vec{E}(\vec{x}) \right],
\quad (A.8)
\]

up to $o(v)$.

So the Maxwell’s equation
\[
\nabla' \times \vec{E}(\vec{x}') = -\frac{\partial}{\partial t'} \vec{B}(\vec{x}')
\Rightarrow \nabla' \times (\vec{E} + \vec{v} \times \vec{B})(\vec{x}') = -\frac{\partial}{\partial t'} \left( \vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right) \quad (A.9)
\]

can be recast into the form
\[
\nabla \times \vec{E}(\vec{x}) + \nabla \times (\vec{v} \times \vec{B}(\vec{x})) = -\frac{\partial}{\partial t} \vec{B}(\vec{x}) + \frac{\vec{v}}{c^2} \times \frac{\partial}{\partial t} \left( \vec{E}(\vec{x}) \right).
\quad (A.10)
\]

The extension to the formula involving $\nabla \times \vec{B}$ is straightforward. The divergence related expression in the Maxwell’s equations is transformed as
\[
\nabla' \cdot \vec{E}(\vec{x}') \approx \left( \nabla + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \cdot \left[ \vec{E}(\vec{x}, t) - t (\vec{v} \cdot \nabla) \vec{E}(\vec{x}, t) - \frac{\vec{v} \cdot \vec{x}}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) \right]
\approx \nabla \cdot \vec{E}(\vec{x}, t) + \frac{\vec{v}}{c^2} \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) - \frac{1}{c^2} [\vec{v} \cdot \nabla] \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) - \frac{\vec{v} \cdot \vec{x}}{c^2} \nabla \cdot \left[ \frac{1}{\mu} \nabla \times \vec{B} - \frac{1}{\epsilon} \vec{J} \right]
\approx \nabla \cdot \vec{E}(\vec{x}, t) + \frac{\vec{v}}{c^2} \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) - \frac{\vec{v}}{c^2} \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) + \frac{(\vec{v} \cdot \vec{x})}{\epsilon c^2} \nabla \cdot \vec{J}
\approx \nabla \cdot \vec{E}(\vec{x}, t) + \frac{(\vec{v} \cdot \vec{x})}{\epsilon c^2} \nabla \cdot \vec{J},
\quad (A.11)
\]
by using
\[
\nabla \cdot \left[ (\vec{v} \cdot \vec{x}) \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) \right] = \left[ \nabla (\vec{v} \cdot \vec{x}) \right] \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t) + (\vec{v} \cdot \vec{x}) \nabla \cdot \frac{\partial}{\partial t} \vec{E}(\vec{x}, t),
\]
\[
\nabla \cdot ((\vec{v} \cdot \nabla)\vec{E}(\vec{x}, t)) = -\nabla \cdot \left( \nabla \times (\vec{v} \times \vec{E}(\vec{x}, t)) \right) = 0,
\]
(A.12)
and
\[
\nabla \cdot (\nabla \times \vec{B}) = 0,
\]
(A.13)
up to \( o(v) \).

Obviously, the electric density changes as
\[
\hat{\rho}(x') \approx \tilde{\rho}(\vec{x}, t) - t (\vec{v} \cdot \nabla) \rho(\vec{x}, t) - \frac{\vec{v} \cdot \vec{x}}{c^2} \frac{\partial}{\partial t} \hat{\rho}(\vec{x}, t).
\]
(A.14)

So, the equation
\[
\nabla' \cdot \vec{D}'(x') = \rho'(x'),
\]
(A.15)
will lead to
\[
\nabla \cdot \vec{D}(x) = \rho'(x),
\]
(A.16)
with the application of the conservation equation
\[
\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{J} = 0.
\]
(A.17)

B The extended Hertz equations with general choice of \( \vec{v} \) direction

In the deduction of extended Hertz equations in the main text, the assumption that \( \vec{v} \) is perpendicular to both \( \vec{E} \) and \( \vec{B} \) is used. In this section, we will generalize the previous results to the general case with arbitrary choice of \( \vec{v} \) direction.

The Lorentz transformation of electromagnetic fields are given as
\[
\vec{E}'(x') = \gamma (\vec{E}_\perp + \vec{v} \times \vec{B}) + \vec{E}_0 \approx \vec{E} + \vec{v} \times \vec{B},
\]
\[
\vec{B}'(x') = \gamma (\vec{B}_\perp - \frac{\vec{v}}{c^2} \times \vec{E}) + \vec{B}_0 \approx \vec{B} - \frac{\vec{v}}{c^2} \times \vec{E},
\]
(B.1)
at order \( O(v) \) in the \( v \) expansion with the Lorentz transformation of the coordinates
\[
x' = \Lambda x.
\]
(B.2)

Following the discussion after eq (2.9), such transformation can be rewritten with the same coordinate variables for the involved fields as
\[
\vec{E}' = \gamma (\vec{E}_\perp + \vec{v} \times \vec{B}) + \vec{E}_0 \approx \vec{E} + \vec{v} \times \vec{B},
\]
\[
\vec{B}' = \gamma (\vec{B}_\perp - \frac{\vec{v}}{c^2} \times \vec{E}) + \vec{B}_0 \approx \vec{B} - \frac{\vec{v}}{c^2} \times \vec{E},
\]
(B.3)
where
\[ \vec{E}(x') \equiv \vec{E}[x(x')] , \quad \vec{B}(x') \equiv \vec{B}[x(x')] , \] (B.4)
with
\[ x(x') = \Lambda^{-1} x' , \] (B.5)
being the inverse coordinate transformation.

Following the approaches in the main text, it is easy to deduce the Hertz form Maxwell’s equations from ordinary Maxwell’s equations order by order in \( v \), which gives
\[ \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} , \]
\[ \Rightarrow \nabla \times (\vec{E} + \vec{v} \times \vec{B}) = -\frac{\partial}{\partial t} \vec{B} + \frac{\nu}{c^2} \times \frac{\partial}{\partial t} \vec{E} , \]
\[ \approx -\frac{\partial}{\partial t} \vec{B} + \frac{1}{\mu \epsilon c^2} \times \left( \nabla \times \vec{B} \right) - \frac{1}{\epsilon c^2} \times \vec{J} . \] (B.6)

After rearranging terms and neglect \( O(v^2) \) terms, we can obtain
\[ \nabla \times (\vec{E} + \alpha v \times \vec{B}) = -\frac{\partial}{\partial t} \vec{B} - \frac{1}{\epsilon} \vec{J} + \frac{1}{\mu \epsilon c^2} \nabla (\vec{v} \cdot \vec{E}) . \] (B.7)

Similarly, replacing the new equivalent form (B.4) into
\[ \nabla \times \vec{B} = \vec{J} + \mu \frac{\partial}{\partial t} \vec{E} , \] (B.8)
and neglecting the \( O(v^2) \) terms, we have
\[ \nabla \times \left[ \vec{B} + \mu \epsilon \alpha \vec{v} \times \vec{E} \right] = \mu \vec{J} + \mu \frac{\partial}{\partial t} \vec{E} - \mu \epsilon \nabla (\vec{v} \cdot \vec{E}) . \] (B.9)

The remaining extended Hertz-form Maxwell’s equations can be readily obtained
\[ \nabla \cdot \vec{E} = \frac{1}{\epsilon} \tilde{\rho} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \vec{v} \cdot \vec{E} \right) , \] (B.10)
and
\[ \nabla \cdot \vec{B} = -\frac{1}{c^2} \frac{\partial}{\partial t} \left( \vec{v} \cdot \vec{B} \right) . \] (B.11)

We should note that the sign in front of \( \frac{\partial}{\partial t} \) within eq.(B.11) is different to that within eq.(B.10), which can not be recast into the form of \( \tilde{\nabla} \) derivative defined by
\[ \tilde{\nabla} \equiv \nabla - \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} , \] (B.12)
at the same time for the two equations. That is, the low speed approximation for the traveling wave form description of electromagnetic field can not reduce to the Galilean
transformed form description of electromagnetic field. That is, the low speed limit of Lorentz transformation for electromagnetic field is not the Galilean transformation.

The extended Hertz-form Maxwell’s equations can be written together as

\[
\nabla \cdot \vec{E} = \frac{1}{\epsilon} \tilde{\rho} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \vec{v} \cdot \vec{E} \right) ,
\]

(B.13)

\[
\nabla \cdot \vec{B} = -\frac{1}{c^2} \frac{\partial}{\partial t} \left( \vec{v} \cdot \vec{B} \right) ,
\]

(B.14)

\[
\nabla \times (\vec{E} + \alpha \vec{v} \times \vec{B}) = -\frac{\partial}{\partial t} \vec{B} - \frac{\vec{v}^2}{\epsilon c^2} \times \vec{J} + \frac{1}{\mu c^2} \nabla (\vec{v} \cdot \vec{B}) ,
\]

(B.15)

\[
\nabla \times \left[ \vec{B} + \mu \epsilon \alpha \vec{v} \times \vec{E} \right] = \mu \epsilon \frac{\partial}{\partial t} \vec{E} + \mu \vec{J} - \mu \epsilon \nabla (\vec{v} \cdot \vec{E}) ,
\]

(B.16)

with

\[
\alpha = \frac{\epsilon \mu - \epsilon_0 \mu_0}{\epsilon \mu} .
\]