Augmented Probability Simulation Methods for Non-cooperative Games

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We present a comprehensive robust decision support framework with novel computational algorithms for decision makers in a non-cooperative sequential setup. Existing simulation based approaches such as Monte Carlo methods can be inefficient under certain conditions like in presence of a high number of decision alternatives and uncertain outcomes. Hence, we provide a novel augmented probability simulation alternative to solve non-cooperative sequential games. We cover approaches to approximate subgame perfect equilibria under common knowledge conditions, assess the robustness of such solutions and, finally, approximate adversarial risk analysis solutions when lacking common knowledge. The proposed approach can be especially beneficial in application domains such as cybersecurity and counter-terrorism.

Key words: Sequential decision analysis, Noncooperative games, Augmented probability simulation.

1. Introduction

Non-cooperative game theory refers to conflict situations in which two or more agents make decisions whose payoffs, which they aim at maximising, depend on the actions implemented by all of them, and, possibly, on some random outcomes. Under common knowledge assumptions about the agents preferences and beliefs, the analysis is pervaded by Nash equilibria, and related refinements, which provide a prediction of the decisions to be made by the agents. Ozdaglar and Menache (2011) provide a review, whereas Heap and Varoufakis (2004) include an in-depth critical assessment. Adversarial risk analysis (ARA), Banks et al. (2015), provides an alternative decision analytic approach aimed at one-sided prescriptive support to one of the intervening agents based on a subjective expected utility model treating the adversaries decisions as uncertainties. Their (random)
optimal actions are predicted taking into account the uncertainty about the adversaries probabilities and utilities in an expected utility model of their behaviour. In contrast with game theoretic approaches, the standard common knowledge hypothesis is not assumed.

Our realm is within algorithmic decision (Rossi and Tsoukias 2009) and game (Nisan et al. 2007) theories, in that we propose efficient algorithms to approximate solutions for game theoretic problems. For cases in which an analytical solution is not available or is computationally expensive, simulation based approaches can be utilized. Among those, Monte Carlo (MC) methods are straightforward to use and widely implemented. However, they can be inefficient under certain conditions such as in presence of a high number of decision alternatives for the agents. For instance, counter-terrorism and (cyber) security problems may involve thousands of possible decisions, and there could be large uncertainties associated with the goals and resources of the attackers (Zhuang and Bier (2007)). This can result in computational challenges especially in cases where model uncertainty dominates, Rios Insua et al. (2009). Sampling procedures that focus on high-probability events, would have the potential to handle such computational challenges. Augmented probability simulation (APS), Bielza et al. (1999), is a powerful simulation based methodology used to approximate optimal solutions in decision analytic problems. In particular, we analyse how APS may be used to efficiently compute game theoretic solutions in both the standard and ARA settings. On the whole, we present a comprehensive robust decision support framework with novel computational algorithms for decision makers in a non-cooperative sequential setup. We cover approaches to approximate subgame perfect equilibria under common knowledge conditions, assess the robustness of such solutions and, finally, approximate ARA solutions when lacking common knowledge. The proposed approach can be especially beneficial in application domains such as cybersecurity and counter-terrorism.

We first focus, Section 2, on how APS may be used to approximate equilibria and how to criticise the underlying common knowledge assumptions through sensitivity analysis. When such solutions are not robust, we propose APS schemes to find the decision analytic ARA solution, as discussed in
Section 3. We provide examples to illustrate the whole framework (Section 4), compare the efficiency of standard MC and APS approaches (Section 5) and present a realistic case study (Section 6). Appendices review key concepts in APS and provide proofs of propositions. Code to reproduce the paper is available in a GitHub repository (Torres-Barrán and Naveiro 2019), including parameter choices.

2. Equilibria in sequential games

2.1. The basic approach

We consider sequential games with two agents. As an example, consider a case in which a company deploys cyber security controls and then, having observed them, a hacker decides whether to launch a DDoS attack against such company. These games have received various names in the literature including sequential Defend-Attack (Brown et al. 2006) or Stackelberg (Gibbons 1992).

To fix ideas, consider a case in which a Defender (she) chooses her defense \( d \in D \) and, then, an Attacker (he) chooses his attack \( a \in A \), after having observed \( d \). We assume that both \( D \) and \( A \) are finite, except when noted. The corresponding bi-agent influence diagram (BAID) is shown in Figure 1. The arc between decision nodes \( D \) and \( A \) reflects that the Defender choice is observed by the Attacker. The consequences for both participants depend on the outcome \( \theta \in \Theta \) of the attack. Each agent has a different assessment on the probability of the outcome, which depends on \( d \) and \( a \), respectively designated \( p_D(\theta | d, a) \) and \( p_A(\theta | d, a) \). The Defender’s utility function \( u_D(d, \theta) \) depends on her chosen defense and the result of the attack. Similarly, the Attacker’s utility function has the form \( u_A(a, \theta) \).

The standard game theoretic solution does not require the Attacker to know the Defender’s probabilities and utilities, since he observes the Defender’s actions. However, the Defender must know \( (u_A, p_A) \), the common knowledge condition in this case. We first compute both agents’ expected utilities at node \( \Theta \) in Figure 1,

\[
\psi_A(a, d) = \int u_A(a, \theta) p_A(\theta | d, a) \, d\theta \quad \text{and} \quad \psi_D(d, a) = \int u_D(d, \theta) p_D(\theta | d, a) \, d\theta.
\]
We then compute the Attacker’s best response to the Defender’s action $d$,

$$a^*(d) = \arg \max_{a \in A} \psi_A(d, a)$$

Knowing this, the Defender’s optimal action is

$$d^*_D = \arg \max_{d \in D} \psi_D(d, a^*(d)).$$

The solution $(d^*_D, a^*(d^*_D))$ is a Nash equilibrium and, indeed, a sub-game perfect equilibrium, Heap and Varoufakis (2004). Algorithm 1 is a generic MC approach to solve the problem.

```
input: P, Q
for d ∈ D do
    for a ∈ A do
        Generate Q samples $\theta_1, \ldots, \theta_Q \sim p_A(\theta | a, d)$
        Approximate $\hat{\psi}_A(a, d) = \frac{1}{Q} \sum u_A(a, \theta_i)$
        Find $a^*(d) = \arg \max_a \hat{\psi}_A(a, d)$
    Generate P samples $\theta_1, \ldots, \theta_P \sim p_D(\theta | a^*(d), d)$
    Approximate $\hat{\psi}_D(d) = \frac{1}{P} \sum u_D(d, \theta_i)$
    Compute $d^*_G = \arg \max_d \hat{\psi}_D(d)$
```

Algorithm 1: MC approach to solve a sequential Defend-Attack problem

From a computational perspective, it requires generating $|D| \times (|A| \times Q + P)$ samples, where $|\cdot|$ designates the cardinality of the corresponding set, in addition to the cost of the final optimization.
and the \(|D|\) inner loop optimizations. When the sets \(A\) and/or \(D\) of alternatives are continuous, we
would introduce further discretization and/or sampling steps to appropriately explore the available
alternatives, or we could use a regression metamodel (Chen et al. 2013). When dealing with decision
dependent uncertainties, as in our games, MC approaches may turn out to be computationally
expensive: they require sampling from \(p_D(\theta \mid d, a)\) and \(p_A(\theta \mid d, a)\), respectively entailing loops over
the decision spaces \(D\) and \(A\). When \(D\) or \(A\) are high dimensional, considering the whole decision
space as in MC can be unfeasible.

2.2. An APS approach

We provide APS approaches to approximate game theoretic solutions for sequential defend-attack
games mitigating the above issue. In relation with the Attacker’s problem, for a given action \(d\) we
introduce an artificial distribution over \((a, \theta)\), assuming that \(u_A(a, \theta)\) is positive and integrable,
declared through

\[
\pi_A(a, \theta \mid d) \propto u_A(a, \theta) p_A(\theta \mid d, a). \tag{1}
\]

Its marginal in \(a\) is proportional to the Attacker’s expected utility, \(\pi_A(a \mid d) \propto \psi_A(a, d)\). Conse-
sequently, the optimal attack given the defense \(d\) may be computed through \(a^*(d) = \text{mode}[\pi_A(a \mid d)]\).

Moving backwards along the tree, and assuming that \(u_D(d, \theta)\) is positive and integrable, we intro-
duce the artificial distribution

\[
\pi_D(d, \theta \mid a^*(d)) \propto u_D(d, \theta) p_D(\theta \mid d, a^*(d)). \tag{2}
\]

Its marginal in \(d\) is proportional to the defender’s expected utility, i.e. \(\pi_D(d \mid a^*(d)) \propto \psi_D(a^*(d), d)\),
so that

\[
d^*_{GT} = \text{mode } [\pi_D(d \mid a^*(d))]. \tag{3}
\]

Thus, we introduce an APS scheme to sample from \(\pi_D(d, \theta \mid a^*(d))\) such that, in a prepro-
cessing phase, we compute \(a^*(d)\) for each \(d\) using another APS based on \(\pi_A(a, \theta \mid d)\). The resulting
procedure is summarized in Algorithm 2. In principle, we sample from both artificial distributions
input: $N$, $M$, $K$
initialize: $a^{(0)}$, $\theta^{(0)}$

for $d \in \mathcal{D}$ do

for $j = 1$ to $M$ do

Draw $\theta^{(j)}$ from $\pi_A(\theta | d, a^{(j-1)})$

Draw $a^{(j)}$ from $\pi_A(a | \theta^{(j)}, d)$

Compute mode of $M$ draws $\{a^{(j)}\}$ and record it as $a^*(d)$

initialize: $d^{(0)}$, $\theta^{(0)}$

for $i = 1$ to $N$ do

Draw $\theta^{(i)}$ from $\pi_D(\theta | d^{(i-1)}, a^*(d^{(i-1)}))$

Draw $d^{(i)}$ from $\pi_D(d | \theta^{(i)}, a^*(d))$

Discard first $K$ samples, compute mode of rest of draws $\{d^{(i)}\}$ and propose it as $\hat{d}_{GT}^*$

Algorithm 2: Nested APS to solve a sequential Defend-Attack problem.

through a Gibbs sampler (Casella and George 1992): in the attacker’s APS, for each $d$, we sample iteratively from $\pi_A(a | \theta, d)$ and $\pi_A(\theta | d, a)$, whereas, in the defender’s APS, we sample iteratively from $\pi_D(d | \theta, a^*(d))$ and $\pi_D(\theta | d, a^*(d))$.

Proposition 1 ensures the convergence of the algorithm to the optimal solution. A proof can be found in Appendix B.

**Proposition 1** If the attacker’s and defender’s utility functions are positive and integrable, $p_A(\theta | d, a), p_d(\theta | d, a) > 0 \ orall a, \theta$ and $\mathcal{A}$, $\mathcal{D}$, $\Theta$ are either discrete or intervals in $\mathbb{R}^n$, the output of Algorithm 2 converges to $d_{GT}^*$.

Observe that the computational complexity of Algorithm 2 does not depend on the dimension of the attacker’s decision space. This could be crucial when $|\mathcal{A}|$ is very large or $A$ is continuous. Recall that MC requires discretization of such space, whereas APS does not. In particular, Algorithm 2 requires $2 \times (|\mathcal{D}| \times M + N)$ samples plus the cost of convergence checks and $|\mathcal{D}| + 1$ mode approximations. Thus Algorithm 2, would be more efficient when the number of attacker alternatives is large and that of defender is small.

If draws from the conditional distributions in Algorithm 2 are not readily available, we could use Metropolis-Hastings variants, Chib and Greenberg (1995). Let us call $d$ and $\theta$ the current samples.
in the Metropolis scheme of the defender’s APS. Within each iteration, we sample a candidate \( \tilde{d} \) for the defender’s alternative from a proposal distribution. Once sampled, we use an inner APS to estimate \( a^*(\tilde{d}) \). Finally, we accept the sample with probability \( \frac{\pi_D(\tilde{d}, \tilde{\theta} | a^*(\tilde{d}))}{\pi_D(d, \theta | a^*(d))} \), being \( \tilde{\theta} \) the uncertainty parameter candidate. The approach is specified in Algorithm 3.

\begin{algorithm}
\textbf{function} solve\_attacker\((M, d, g_A)\):
\begin{algorithmic}
\STATE \textbf{initialize}: \( a^{(0)} \)
\STATE Draw \( \theta^{(0)} \sim p_A(\theta|d,a^{(0)}) \)
\FOR {\( i = 1 \) to \( M \)}
\STATE Propose new attack \( \tilde{a} \sim g_A(\tilde{a}|a^{(i-1)}) \).
\STATE Draw \( \tilde{\theta} \sim p_A(\theta|d,\tilde{a}) \).
\STATE Evaluate acceptance probability \( \alpha = \min\left\{1, \frac{u_A(\tilde{a},\tilde{\theta})}{u_A(a^{(i-1)},\theta^{(i-1)})}\right\} \).
\STATE With probability \( \alpha \) set \( a^{(i)} = \tilde{a}, \theta^{(i)} = \tilde{\theta} \).
\STATE Discard the first \( K \) samples and compute mode of the rest of draws \( \{a^{(i)}\}, a^*(d) \)
\ENDFOR
\STATE \textbf{return} \( a^*(d) \)
\end{algorithmic}
\end{algorithm}

\textbf{input}: \( d, M, K, N, R, g_D \) and \( g_A \) symmetric distributions
\textbf{initialize}: \( d^{(0)}, a^*(d^{(0)}) = \text{solve\_attacker}(M, d^{(0)}, g_A) \)
\STATE Draw \( \theta^{(0)} \sim p_D(\theta|d^{(0)}, a^*(d^{(0))) \)
\FOR {\( i = 1 \) to \( N \)}
\STATE Propose new defense \( \tilde{d} \sim g_D(\tilde{d}|d^{(i-1)}) \).
\STATE \( a^*(\tilde{d}) = \text{solve\_attacker}(M, \tilde{d}, g_A) \) if not previously computed
\STATE Draw \( \tilde{\theta} \sim p_D(\theta|\tilde{d}, a^*(\tilde{d})) \).
\STATE Evaluate acceptance probability \( \alpha = \min\left\{1, \frac{u_D(\tilde{d},\tilde{\theta})}{u_D(d^{(i-1)},\theta^{(i-1)})}\right\} \).
\STATE With probability \( \alpha \) set \( d^{(i)} = \tilde{d}, a^*(d^{(i)}) = a^*(\tilde{d}) \) and \( \theta^{(i)} = \tilde{\theta} \).
\STATE Discard first \( R \) samples and compute mode of rest of draws \( \{d^{(i)}\} \). Record it as \( \hat{d}_{GT} \).
\end{algorithmic}

\textbf{Algorithm 3}: APS alternative to solve the sequential Defend-Attack problem.

Convergence of this algorithm follows similarly to that in Proposition 1, provided that the proposal distributions, \( g_A \) and \( g_D \), are symmetric. Interestingly, this also allows us to get rid of the loop over the defender’s decision space. Thus, the computational complexity of this procedure does not depend on the dimensions of the attacker and defender decision spaces. This would be
an excellent choice when facing a problem where the cardinality of these spaces is large or are continuous.

2.3. Sensitivity of the game theoretic solution

In the above setting, we could contend that since we are supporting the Defender, we would know \((u_D, p_D)\) reasonably well. However, information about \((u_A, p_A)\) may be not that precise, since it would essentially require the Attacker to reveal his judgments. This is questionable in application areas in which information is concealed and hidden to adversaries, including cybersecurity and counterterrorism.

To do so, we may perform a sensitivity analysis considering that the Attacker’s preferences and beliefs are modeled through classes of utilities \(u \in \mathcal{U}_A\) and probabilities \(p \in \mathcal{P}_A\), summarizing the information available about the Attacker’s judgments obtained from leakage, earlier interactions or informants. For each pair \((u, p)\), we could compute the Nash defense \(d^*_{u,p}\), using the techniques in Sections 2.1 or 2.2. After that, we need to assess whether the game theoretic solution remains reasonably stable for the allowed perturbations of \(u\) and \(p\). One possibility could be to focus on the regret \(r_{u,p}(d^*_\text{GT})\) given by the difference in expected utility between \(d^*_\text{GT}\) and \(d^*_{u,p}\) for \((u, p) \in \mathcal{U}_A \times \mathcal{P}_A\). A small value of \(r_{u,p}(d^*_\text{GT})\) corresponds to robustness with respect to the choice of utility and probability of the Attacker: any pair \((u, p)\) could be chosen with no significant changes in the attainable expected utilities. Otherwise, the relevance of the proposed Nash defense \(d^*_\text{GT}\) should be criticized and further investigated. At a deeper level, this also questions the appropriateness of \((u_A, p_A)\), serving to criticize the common knowledge assumption. Operationally, a threshold on the maximum regret might be fixed such that if exceeded, that assumption must be questioned.

The whole procedure is illustrated in Algorithm 4, in which threshold is based on the available computational budget. As this approach is implemented in an exploratory sense, we do not need very big sample sizes (in Algorithm 2 or 3) thus enabling us to allocate more resources in exploring a larger sample of \((u, p)\’s.\)
input: \(d^*_GT, U_A, P_A, R, \text{threshold}\)

for \(i = 1 \) to \(R\) do
- Randomly sample \(u\) and \(p\) from \(U_A\) and \(P_A\), respectively
- Compute \(d^*_u,p\) using Algorithm 2 (or its variants)
- Compute \(r_{u,p}(d^*_GT)\)
  - if \(r_{u,p}(d^*_GT) > \text{threshold}\) then
    - Robustness requirements not satisfied
    - Stop
  - Robustness requirements satisfied.

Algorithm 4: Robustness assessment of game theoretic solution

3. Adversarial risk analysis

We explore now the case when the game theoretic solution is not robust. In particular, we introduce a decision analytic solution and investigate its sensitivity to inputs.

3.1. ARA and the basic solution

One way of addressing the lack of robustness of the game theoretic solution is to perform a decision analytic approach based on ARA concepts, Banks et al. (2015). For this, we weaken the common knowledge assumption: the Defender does not know \((p_A, u_A)\). The problem she faces is depicted in Figure 2a as an influence diagram, Shachter (1986). To solve it, besides \(p_D(\theta \mid d, a)\) and \(u_D(d, \theta)\), available from our discussion in Section 2, the Defender requires \(p_D(a \mid d)\), her assessment of the probability that the Attacker will implement attack \(a\) after having observed that she has chosen the defense \(d\). Then, we proceed as follows. The expected utility of defense \(d\) would be

\[
\psi_D(d) = \int \psi_D(a, d) p_D(a \mid d) da = \int \left[ \int u_D(d, \theta) p_D(\theta \mid d, a) d\theta \right] p_D(a \mid d) da,
\]

and her optimal decision would be \(d^*_{ARA} = \arg\max_{d \in D} \psi_D(d)\). This solution does not necessarily correspond to a Nash equilibrium as both solutions are based on different information and assumptions, see an example in Section 4.

Eliciting \(p_D(a \mid d)\) is facilitated if the Defender analyzes the Attacker problem, Figure 2b. For this, the Defender would use all the information and judgment available about the Attacker’s utilities and
probabilities. Instead of using point estimates for $p_A$ and $u_A$ to find the Attacker’s optimal decision $a^*(d)$ for a given $d$ as in Section 2, the Defender’s uncertainty about the Attacker’s decision would derive from her uncertainty about $(p_A, u_A)$ modelled through a distribution $F = (U_A, P_A)$ on the space of utilities and probabilities. This induces a distribution over the Attacker’s expected utility $\psi_A(a, d)$, where the random expected utility for $A$ would be $\Psi_A(a, d) = \int U_A(a, \theta) P_A(\theta | a, d) d\theta$.

Then, the Defender would find

$$p_D(a | d) = \mathbb{P}_F \left[ a = \arg \max_{x \in \mathcal{A}} \Psi_A(x, d) \right],$$

in the discrete case and, similarly, in the continuous one. In general, we use MC simulation to approximate $p_D(a | d)$ by drawing $J$ samples $\{(P_A^i, U_A^i)\}_{i=1}^J$ from $F$ and setting

$$\hat{p}_D(a | d) \approx \frac{\#\{a = \arg \max_{x \in \mathcal{A}} \Psi_A^i(x, d)\}}{J},$$

where $\Psi_A^i(a, d) = \int U_A^i(a, \theta) P_A^i(\theta | a, d) d\theta$.

Operationally, we first use MC simulation to estimate $p_D(A^* = a | d)$ and, then, expected utility maximization, as reflected in Algorithm 5. Note that to solve the optimization problems in $d$ and $a$, we shall also typically use an MC approach. Algorithm 5 requires generating $|D| \times (|A| \times Q \times J + P)$ samples, where $Q$ and $P$ are the number of samples required to approximate $\int u_A^i(a, \theta) p_A^i(\theta | a, d) d\theta$ and $\int \int u_D(d, \theta) p_D(\theta | a, d) p_D(A^* = a | d) d\theta da$, respectively.
input: \( J \)

for \( d \in \mathcal{D} \) do

for \( i = 1 \) to \( J \) do

Sample \( u^i_A(a, \theta) \) \( \sim \) \( U_A(a, \theta) \)

Sample \( p^i_A(\theta \mid a, d) \) \( \sim \) \( P_A(\theta \mid d, a) \)

Compute \( a^*_i(d) = \arg \max_a \int u^i_A(a, \theta) p^i_A(\theta \mid a, a) \, d\theta \)

\( \hat{p}_D(A^* = a \mid d) = \frac{1}{J} \sum_{i=1}^J I[a^*_i(d) = a] \)

Solve \( \max_d \int u_D(d, \theta) p_D(\theta \mid a, d) \hat{p}_D(A^* = a \mid d) \, d\theta \, da \)

Algorithm 5: MC approach to solve the ARA problem

3.2. APS based solutions for ARA

We now consider an APS approach to solve the ARA problem. We first provide several observations concerning the relevant augmented probability models and, then, outline an algorithm. To start with, we replicate the argument in Equation (1), with random (and positive) utilities and probabilities. For a given \( d \), we introduce the artificial random distribution \( \Pi_A(a, \theta \mid d) \propto U_A(a, \theta) P_A(\theta \mid a, d) \). Its marginal \( \Pi_A(a \mid d) \) is proportional to the random expected utility \( \Psi_A(a, d) \). Then, slightly abusing notation, the random optimal attack coincides with the mode of the marginal of the random augmented distribution \( A^*(d) = \text{mode} (\Pi_A(a \mid d)) \). Based on it, we build \( p_D(a \mid d) \) as in Equation (4).

Moving backwards, we introduce the artificial distribution \( \pi_D(d, a, \theta) \propto u_D(d, \theta) p_D(\theta \mid a, d) p_D(a \mid d) \), whose marginal \( \pi_D(d) \) is proportional to the expected utility \( \psi_D(d) \), and, therefore,

\[
d^*_\text{ARA} = \text{mode} (\pi_D(d))
\]

Based on this argument, we propose a nested APS algorithm, Algorithm 6, which estimates \( \pi_D(a \mid d) \) and then optimizes.

Convergence follows from Proposition 2 with proof in Appendix B.

**Proposition 2** If \( u_D \) and all members of \( U_A \) are positive and integrable, and \( p_D(\theta \mid d, a) \) and all members of \( P_A(\theta \mid d, a) \) are also positive \( \forall a, \theta \) and \( A, \mathcal{D} \) and \( \Theta \) are either discrete or intervals in \( \mathbb{R}^n \), the output of Algorithm 6 converges to \( d^*_\text{ARA} \).
input: \( N, M, J \)
for \( d \in \mathcal{D} \) do
  for \( j = 1 \) to \( J \) do
    Sample \( U_A^j, P_A^j \) and define \( \Pi_A^j \)
    Initialize \( \theta^0 \)
    for \( i = 1 \) to \( M \) do
      Sample \( a^{(i)} \) from \( \Pi_A^j(a | \theta^{(i-1)}, d) \)
      Sample \( \theta^{(i)} \) from \( \Pi_A(\theta | a^{(i)}, d) \)
    Estimate \( a^*_j \) as mode of \( \{a^{(i)}\} \)
  Estimate \( p_D(a | d) \) from \( \{a^*_j\} \)
for \( i = 1 \) to \( N \) do
  Draw \( d^{(i)} \) from \( \pi_{d}(d | a^{(i-1)}, \theta^{(i-1)}) \)
  Draw \( \theta_{D}^{(i)} \) from \( \pi_{\theta}(\theta | a^{(i-1)}, d^{(i)}) \)
  Draw \( a^{(i)} \) from \( \pi_{a}(a | d^{(i)}, \theta_{D}^{(i)}) \)
  Estimate \( d^* \) as mode of \( \{d^{(i)}\} \)

Algorithm 6: Nested APS approach to solve the ARA problem

From a computational perspective, this algorithm requires generating \(|\mathcal{D}|(2M \times J) + 3N\) samples from multivariate distributions in addition to the cost of the convergence checks and mode computation.

3.3. Sensitivity analysis of the ARA solution

The above approach leads to a decision analysis problem with the peculiarity that it includes a sampling procedure to forecast the adversary’s actions. A sensitivity analysis would be conducted with respect to its inputs, \((u_D(d, \theta), p_D(\theta | a, d), p_D(a | d))\). We focus on sensitivity with respect to \(p_D(a | d)\), the most critical element as it comes from adversarial calculations based on the random probability distribution \(P_A(\theta | a, d)\) and utility \(U_A(a, \theta)\). For that, we define classes \(\mathcal{U}_A, \mathcal{P}_A\) of random utilities and probabilities. For each pair \((U, P)\) in such classes, we obtain \(p_{D}^{UP}(a | d)\) through the ARA approach which leads to \(d_{ARA}^{UP}\). We then consider the impact of the imprecision about \(U\) and \(P\) over the attained expected utility \(\psi(d_{ARA}^{UP})\). We would say that sensitivity holds if the
maximum expected utility changes considerably when the input parameters change, in which case we also need to check whether the ARA solution changes as well.

Should the results be sensitive, we may opt for gathering additional information to reduce the classes $\mathcal{U}_A$ and $\mathcal{P}_A$. Once all possible sources of information have been exploited to increase robustness about $d^*_\text{ARA}$ without success, an extra criterion would be introduced to make a decision. As an example, we could consider the decision $d^*_R$ minimizing the maximum regret,

$$
\min_d \max_{U \in \mathcal{U}_A, P \in \mathcal{P}_A} \left[ \int \psi_D(a, d^*_\text{UP}) p_D^{UP}(a \mid d^*_\text{UP}) \, da - \int \psi_D(a, d) p_D^{UP}(a \mid d) \, da \right].
$$

In any case, regardless of the chosen criterion, such decision should be reported with a warning of lack of robustness. All of the above may be embedded in a simulation scheme similar to Algorithm 4.

4. An illustrative example

We illustrate the proposed framework through a sequential defend-attack cyber security problem. An organization needs to determine its security protocol, either through a safe but costly route, or through cheaper but more dangerous protocols with weaker protection levels, rendering business performance increasingly at risk.

4.1. Basic elements

Assume the Defender has available ten protocols: $d = 0$, no defensive action is taken; $d = i$, use a level $i$ protection protocol, $i = 1, \ldots, 8$, with increasing levels of protection; $d = 9$, a very safe but cumbersome protocol. The attacker has two alternatives: attack ($a = 1$) or not ($a = 0$). Successful (unsuccessful) attacks are denoted by $\theta = 1$ ($\theta = 0$). Clearly, when there is no attack, we necessarily have $\theta = 0$.

*Defender non strategic judgments.* Table 1a presents the costs $c_D$ associated with each decision and outcome. The Defender’s expected business valuation is 7M euros. Each increase in security level entails 0.05M euros. The Defender’s probability vector of a successful attack for each $a$ and $d$
### Table 1

| \&hline
| \( \theta \) & \( a \) & \( d \) & \( \alpha \) & \( \beta \) |
|---|---|---|---|---|
| \( 0 \) & 0.05 & 7.05 & 0 & 50.0 & 50.0 |
| \( 1 \) & 0.10 & 7.10 & 1 & 40.0 & 60.0 |
| \( 2 \) & 0.15 & 7.15 & 2 & 35.0 & 65.0 |
| \( 3 \) & 0.20 & 7.20 & 3 & 30.0 & 70.0 |
| \( 4 \) & 0.25 & 7.25 & 4 & 25.0 & 75.0 |
| \( 5 \) & 0.30 & 7.30 & 5 & 20.0 & 80.0 |
| \( 6 \) & 0.35 & 7.35 & 6 & 15.0 & 85.0 |
| \( 7 \) & 0.40 & 7.40 & 7 & 10.0 & 90.0 |
| \( 8 \) & 0.45 & 7.45 & 8 & 5.0 & 95.0 |
| \( 9 \) & 0.50 & 7.50 & 9 & 1.0 & 99.0 |

(a) Def. costs; (b) Successful attack probs.; (c) Att. costs; (d) Beta dist. parameters

is in Table 1b, with complementary probabilities for unsuccessful attacks (e.g., \( p_D(\theta = 0 | a = 1, d = 2) = 1 - 0.35 \)). The Defender is constant risk averse in monetary costs; her utility is strategically equivalent to \( u_D(c_D) = -\exp(c \times c_D) \) with \( c > 0 \). Suppose \( c \) is 0.4.

**Attacker judgments.** Consider the Attacker problem. The average cost of an attack is estimated at 0.03M Euros. The eventual average benefit (market share obtained, potential ransom, etc.) is 2M Euros. The cost of attack is 0.5M Euros. Table 1c presents the attacker profit level \( c_A \) associated with each attack \( a \) and outcome \( \theta \). The Defender thinks the Attacker is constant risk prone over money. Therefore, she considers that his utility is strategically equivalent to \( u_A(c_A) = \exp(e \times c_A) \), with \( e > 0 \).

### 4.2. Game theoretic approach

Assuming common knowledge, we set e.g. \( p_A(\theta = 1 | a, d) = p_D(\theta = 1 | a, d) \). Table 1b reflects the Attacker probabilities \( p_A(\theta = 1 | a = 1, d) \). With respect to \( u_A \), the value of the risk proneness coefficient \( e \) is set at 1.
To compute the game theoretic solution, we use the MC and APS approaches described in Algorithms 1 and 2, respectively. Firstly, for each defense $d$, we compute the optimal attack $a^*(d)$. Figure 3a represents the expected MC estimation of the attacker expected utility for each $d$ and $a$. The optimal attack for each $d$ maximizes expected utility. For example, for $d = 5$ the attacker optimal decision is to perform the attack, whereas for $d = 8$ he should not attack. Clearly, when there is no attack ($a = 0$), the cost for the attacker is null, and the utility is constant and equal to 1. Figure 3b represents, for each $d$, the results of sampling from the augmented distribution $\pi_A(a, \theta \mid d)$ marginalized in $a$. The mode of such distribution coincides with the optimal attack. For example, for $d = 9$ the mode is $a = 0$ and the optimal solution is not to attack: for defenses from 0 (no defensive action), until 7 (level 7 protocol), the attack should be implemented; but if stronger defenses are adopted, the attack is not advised.

Armed with $a^*(d)$, we compute the optimal defense, using again MC and APS. Figure 4a presents the MC estimation of $\psi_D(d, a^*(d))$ for each $d$; and 4b, the frequency of samples from the marginal augmented distribution $\pi_D(d \mid a^*(d))$. Both methods agree that the optimal decision is acquiring level 8 protection.

We next check the robustness of the game theoretic solution by computing the optimal defense for $K = 10,000$ perturbations of $u_A(c_A)$ and $p_A(\theta \mid d, a)$, sampling $e' \sim \mathcal{U}(0, 2)$ and using $u'_A(c_A) = \exp(e' \times c_A)$. Regarding $p_A(\theta \mid d, a)$, to perturb $p_A(\theta \mid d, a = 1)$, for each $d$, we sample from a beta
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Figure 4  Solutions of defender problem

distribution with mean its original value and variance, 0.1% of the corresponding mean. Obviously,
\[ p_A(\theta = 1 \mid d, a = 0) = 0 \] for all \( d \).

Figure 5  Sensitivity analysis of the game theoretic solution

Figure 5 reflects the frequency with which each \( d \) was optimal. In particular, \( d^* = 8 \) emerges
around 25% of the time as optimal. However, the solution is not very stable: inducing small per-
turbations in the utilities and success probabilities leads to other solutions. For example, \( d = 9 \)
appears 42% of the time and \( d = 7, 16\% \). In addition, we observe large variations in the expected
utilities attained. The maximum regret obtained was 42.5% of the total expected utility. Therefore,
we conclude that the game theoretic solution is not robust to small perturbations and consider an ARA next.

4.3. ARA approach

We describe the Defender’s beliefs over the Attacker’s judgments through $P_A$ and $U_A$. Assuming the Attacker knows the Defender’s decision, the probability of success for the Attacker will be modeled as $P_A(\theta = 1 \mid a = 1, d) \sim Beta(\alpha, \beta)$ with parameters $\alpha$ and $\beta$, in Table 1d; their expected values are set to $p_D(\theta = 1 \mid a, d)$ from the game theoretic setting under common knowledge. In addition, we assume uncertainty over $e$ with $e \sim U(0, 2)$, which induces the uncertainty leading to $U_A(e_A)$.

![Figure 6 Estimation of $p_D(a \mid d)$](image)

(a) MC estimation of $p_D(a \mid d)$

(b) APS estimation of $p_D(a \mid d)$

The distribution, $p_D(a \mid d)$, over attacks $a$, given defense $d$, is presented in Figure 6, both estimated using MC (Algorithm 5) and APS (Algorithm 6). They coincide up to numerical errors. With the forecast over attacks, we compute the ARA optimal solution for the defender. Figure 7a shows the MC estimation of the defender’s expected utility. Figure 7b, the frequency of samples from the marginal augmented distribution $\pi_D(d)$. Its mode coincides with the optimal defense, $d_{ARA}^* = 9$, in agreement with the MC solution.
We emphasize that the ARA solution does not correspond to the Nash equilibrium, being based on different informational assumptions. In this case, the ARA solution appears to be more conservative, as it suggests a safer, although more expensive, defense.

5. Comparing APS and MC

We have shown how MC and APS can be used for solving the game theoretic and ARA models and discussed their computational complexity, see Table 2 for a summary. The number of MC samples depends on the cardinality of the Attacker’s decision space, while this dependence is not present in APS. Thus, in problems in which the adversary’s decision space is large, APS would be more efficient than MC.

|                | MC                         | APS                      |
|----------------|---------------------------|--------------------------|
| GT             | $|D|(|A| \times Q + P)$     | $2\left(|D| \times M + N\right)$ |
| ARA            | $|D|(|A| \times Q \times J + P)$ | $|D|\left(2 \times M \times J + 3 \times N\right)$ |

Table 2  Number of samples required by MC and APS algorithms for game theoretic (GT) and ARA solutions.

In the game theoretic framework, Algorithm 3 can be used to remove dependence of the complexity on the cardinality of the Defender’s decision space as well. In addition, with continuous decision
spaces, APS could provide solutions with arbitrary precision, while MC is limited to the precision of the discretization of the corresponding decision space. We provide now empirical evidence supporting the computational properties of the algorithms in Sections 2.1 and 2.2.

We focus on a simple sequential game with continuous decision spaces to compare different discretization steps and control the cardinality. Both agents may choose a parameter measuring the proportion of resources invested in, respectively, protecting or attacking an online server. Consequently, $d$ and $a$ are continuous decisions with $d, a \in [0, 1]$. The parameter $\theta$ measures the proportion of losses for the Defender due to a successful attack. The value of the server is $s$.

The defender’s payoff is $f(d, \theta) = (1-\theta) \times s - c \times d$, where $c$ denotes the unit cost of the defender’s resources. The defender is constant risk averse so her utility function is strategically equivalent to $1 - \exp(-h \times f(d, \theta))$, with $h > 0$. The attacker’s payoff is $f(a, \theta) = \theta \times s - e \times a$, where $e$ denotes the value of the attacker’s resources. Assume the attacker is constant risk prone and his utility is strategically equivalent to $\exp(-k \times g(d, \theta))$, with $k > 0$. Finally, we model $\theta$ with a beta distribution with parameters $\alpha(a, d)$ and $\beta(a, d)$, with $\alpha$ an increasing function in $a$ and decreasing in $d$, and $\beta$ increasing in $d$ and decreasing in $a$.

Figure 8 plots the defender’s expected utility surface approximated with MC, for certain values of the above parameters. This surface is very flat close to the optimal defender’s decision, which could influence mode approximations in APS as commented in Müller et al. (2004). In cases with very flat expected utility surfaces, APS can be improved by replacing the augmented distribution by a power transformation of it. In this way, we sample from a distribution more peaked around the mode (Müller 2005). In order to sample from such distribution, at each iteration of the Defender’s Metropolis Hastings sampler in Algorithm 3, we would draw $H$ different values of $\tilde{\theta}$, where $H$ is the power to which we raise the augmented distribution. Then, we would compute the acceptance probability as

$$\min \left\{ 1, \prod_{i=1}^{H} \frac{u_D(d_{i-1}, \tilde{\theta}_t)}{u_D(d_i, \tilde{\theta}_t)} \right\}.$$ 

We proceed in the same way for the attacker’s APS. We refer to the powers to which we raise the attacker’s and defender’s augmented distributions as inner and outer, respectively.
We compare MC and APS running times on equal foot with the following experiment. As decisions are continuous, we discretize them to approximate the MC solution with discretization step impacting the precision of the solution and the cardinalities of $\mathcal{D}$ and $\mathcal{A}$ which, in turn, affect the computational complexity of both algorithms. However, in the game theoretic setting we are able to get rid of the dependence on $|\mathcal{D}|$ and $|\mathcal{A}|$ on the complexity of APS through Algorithm 3, while this is not possible for MC. Thus, there should be a limit precision such that MC is faster for smaller precisions but we should turn to APS for bigger ones. We investigate this trade-off.

We compute the minimum number of MC and APS samples, both for the attacker problem (referred to as inner samples) and the defender one (outer samples) needed to achieve a solution with at least such precision, and measure times using the corresponding number of samples. To that end, we compute the solution several times in parallel using both MC and APS with an increasing number of samples and the following criteria: as soon as 90% of the solutions coincide with the true solution (computed with MC and a very large number of samples), we say that the algorithm has converged for the corresponding number of iterations. For APS, we also explore the impact of different values of inner and outer powers, and choose the minimum such that convergence is achieved.
We present MC and APS performance results for precisions 0.1 and 0.01 in Table 3. Time computations where performed in a server node with 16 cores Intel(R) Xeon(R) CPU E5-2640 v3 @ 2.60GHz. With precision 0.1, \(|\mathcal{D}| = |\mathcal{A}| = 10\) and we expect MC to perform better than APS, as the cardinality of the decision spaces is low. In fact, just 1000 MC samples for the defender problem and 100 for the attacker one were needed to achieve convergence with MC. The performances change with higher precisions. For instance, with precision 0.01, \(|\mathcal{D}| = |\mathcal{A}| = 100\) and MC becomes much more challenging from a computational point of view, as can be deduced from the required running times. APS outperforms MC, as we are able to get rid of the dependence on \(|\mathcal{D}|\) and \(|\mathcal{A}|\) on the computational complexity, as shown in Section 2.2. Note that, for MC, there is a factor 200 between the time needed to obtain the solution with precision 0.01 and that with precision 0.1. For APS, this factor is just 10, suggesting that it scales much better with precision than MC. For smaller precisions, such as 0.001, we could not even get a stable solution using MC with a large number of samples \((P = 10M, Q = 100k)\). Finally, observe that, as the expected utility is very flat around the optimal decision, MC requires a huge number of iterations to converge to the right solution.

To sum up, in problems with large or continuous decision spaces APS would be preferred over MC because of its scalability. When facing such problems, another possible strategy could be to use MC for an initial broad exploration of the decision space to determine regions of interest and, then, turn to APS to explore for the optimal decisions.
6. An application in cyber security

We apply now the proposed framework to a real cybersecurity problem, a simplified version of the case in Rios Insua et al. (2019), where we just keep the adversarial threats. The problem is depicted through the BAID in Figure 9. We deal with an organisation (Defender) who faces a competitor (Attacker) that may attempt a DDoS to undermine the availability of the Defender site, compromising her customer services.

![Bi-agent influence diagram of the cyber security application.](image)

The Defender has to determine the security controls to be implemented, which have a cost, being constant risk averse over them. Specifically, she has to make a decision about subscribing to a monthly cloud-based DDoS protection system with possible choices including 0 (not subscribing), 5, 10, 15, ..., 195, and 200 gbps. Costs $c_s$ of different protection systems are presented in Figure 10.

The Attacker must decide on the intensity of the DDoS attack, viewed as the number of days per month that he will attempt to launch it. Thus his possible alternatives are \{0, 1, ..., 30\}. The duration of the DDoS may have an impact on the market share of the Defender, due to the eventual
reputational loss. As in Rios Insua et al. (2019), we assume that all market share is lost at a linear rate until all value is gone after a few days of unavailability. The Attacker’s earnings depend on the gained market share. Being the sole competitor, we assume that the Attacker gain is the market share lost by the defender. However, he runs the risk of being detected with significant costs. Both agents aim at maximising their expected utilities. Details on the required models for DDoS duration, impact on market share, attacker costs when detected, attacker earnings, and utility functions are available at Rios Insua et al. (2019).

This is a problem without common knowledge and large decision spaces, therefore we compute the ARA solution using the APS based approach in Algorithm 6. We first assessed the probability $p_D(a \mid d)$ of each attack for each defense. Figure 11 shows $p_D(a \mid d)$ for four possible defenses.

When no defensive action is taken, $d = 0$, the probability of attack is spread out across all possible attacks. However, subscribing to a DDoS protection of just 50 gbps, makes the probability of attack to be concentrated in the region of attacks with lower values. Increasing the protection to 195 gbps makes no big difference.

Finally, Figure 12 shows a histogram of the APS samples of the defender’s decision. As expected, the frequency of samples with value 50 is similar to the ones with value 100 and 195. The mode
Figure 11  Probability of attack for each decision

Figure 12  APS solution of the defender problem

(highlighted in red) is located around 75 and consequently, we would prescribe subscribing to a DDoS protection of 75 gbps.
7. Discussion

We have considered the problem of supporting a decision maker against adversaries in an environment with random consequences that depend on the actions of all participating agents. The prevalent paradigm is game theoretic. We can also view the problem as a decision analytic one through ARA.

We have presented a computational framework for this problem, switching from the game theoretic to the ARA concept when standard common knowledge assumptions are questionable. The procedure is summarized as follows: under common knowledge assumptions, the game theoretic solution is computed and subject to an appropriate sensitivity analysis; if stable, such solution may be used with confidence and no further analysis is required; otherwise, the common knowledge assumption is questioned and we use ARA as an alternative decision analytic approach; if this solution is found to be stable as a result of the sensitivity analysis, it may be used with confidence and the analysis stops; otherwise, more data must be gathered and relevant probability and utility classes must be refined, eventually declaring the robustness of the ARA solution; if not sufficient, a minimum regret (or other robust) analysis can be undertaken.

We have shown how MC and APS can be used for implementing the above framework discussing their computational complexity. We have shown that in problems with large decision spaces, APS would be more efficient as its complexity does not depend on the cardinality of such spaces. In problems with continuous decision sets, an interesting approach consists of using MC to limit the area of the decision space where the optimum is located, and then switch to an APS approach to search within a neighborhood of such solution in more detail. Also, in the continuous case, exploiting information of the gradient of utility functions could be useful as illustrated in Naveiro and Ríos Insua (2019). We have provided empirical evidence supporting these ideas through extensive experiments.

It should also be noted that MC errors associated with approximating the expected utility can overwhelm the calculation of the optimal decision. Samples from $p(\theta | d, a)$ will typically need to be
recomputed for each pair \((a,d)\). In contrast, APS performs the expectation and the optimization simultaneously, sampling \(d\) from regions with high objective function values, whereas draws of \(\theta\) are tilted away from the conditional density \(p(\theta | a,d)\) towards the artificial distribution. Overall, sampling in a utility-tilted way helps to draw the parameter \(\theta\) more frequently from where it leads to higher utility. This reduces the MC error as less optimization effort is wasted in parts of the parameter space with low objective function values, typically resulting also in reducing sample sizes.

Apart from the computational issues, when the surface of the expected utility function is flat, MC simulation may need many draws or result in poor estimates. Something similar can be argued for non-symmetric distributions. APS can deal with those cases since it is based on sampling from the optimizing portions of the decision space. Even in cases with very flat expected utility surfaces, APS can be improved by replacing the expected utility surface by a power transformation (Müller 2005), as has been done in Section 5. This approach is related to simulated annealing (Kirkpatrick et al. 1983). See Müller et al. (2004), Jacquier et al. (2007) and Aktekin and Ekin (2016) for such implementations.

Finally, we have focused on the sequential two stage game problem. The ideas could be extended to other types of games, like the simultaneous Defend-Attack one (Rios and Rios Insua 2012) or the general BAIDs in González-Ortega et al. (2019).

Appendix A: Augmented probability simulation

We provide an outline of APS. It was initially proposed in Bielza et al. (1999) to solve decision analysis problems with the aid of Markov chain Monte Carlo (MCMC) procedures. Ekin et al. (2014) utilized it to solve two stage stochastic programs with recourse, whereas Ekin et al. (2017) and Ekin (2018) extended it to stochastic programming problems. APS treats the decision variables as random and converts the decision analysis problem into a simulation one in the joint space of both decision and random variables, creating an auxiliary distribution proportional to the product of the utility function and the original distribution. The strategy can accommodate general positive utility functions and probability models.

Suppose that we aim at finding the decision \(x^* \in \mathcal{X}\) maximizing the expected utility \(\psi(x) = \int u(x, \theta) p(\theta | x) d\theta\), where \(x\) is the decision to be made; \(\mathcal{X}\) is the feasible set; \(u(x, \theta)\) is the utility
function; $\theta$ is the random state; and $p(\theta \mid x)$ is the incumbent probability distribution, which depends on the decision. Suppose that $u(x, \theta)$ is positive and integrable in $x$ for each $\theta$. We define an artificial auxiliary distribution $\pi(x, \theta)$, called augmented probability, such that

$$\pi(x, \theta) \propto u(x, \theta) p(\theta \mid x).$$

(6)

Its marginal distribution over the decisions is $\pi(x) \propto \int u(x, \theta) p(\theta \mid x) d\theta$, thus being proportional to $\psi(x)$. Then, the optimal decision $x^*$ coincides with the mode of the marginal distribution $\pi(x)$. This suggests a strategy based on simulating from $\pi(x, \theta)$, and finding the mode of the marginal sample on $x$ to approximate $x^*$. The application of APS requires a positive utility function to get a proper probability density function; adding a large enough number to the utility will do that without changing the nature of the aforementioned distributions.

To simulate from the augmented model $\pi(x, \theta)$ we use MCMC methods, Gamerman and Lopes (2006). Particularly, we utilize Gibbs sampling which iteratively samples from the conditional distributions $\pi(x \mid \theta)$ and $\pi(\theta \mid x)$, resulting in samples from the joint distribution in the limit under appropriate conditions (Casella and George 1992). Non-standard conditional distributions may require the use of Metropolis-Hastings steps to draw samples (Chib and Greenberg 1995).

When the Markov chain has converged, the mode of the marginal samples of $x$ approximates the optimal solution. Practical convergence may be assessed, e.g., with the aid of the Brooks-Gelman-Rubin (BGR) statistics, Brooks and Roberts (1998).

Appendix B: Proofs of Propositions

Proposition 1: Under our hypothesis, for each $d$, $\pi_A(a, \theta \mid d)$ is a properly defined probability distribution. Under standard convergence conditions for the Gibbs sampler, Roberts and Smith (1994), the samples generated through the scheme proposed in the first loop of Algorithm 2 define a Markov chain with stationary distribution $\pi_A(a, \theta \mid d)$, for each $d$. Thus, for each $d$, the mode of the marginal samples of $a$ converges to $a^*(d)$.

Similarly, under our hypothesis, $\pi_D(d, \theta \mid a^*(d))$ is a properly defined probability distribution, and the samples generated in the second loop of Algorithm 2 define a Markov chain with stationary distribution $\pi_D(d, \theta \mid a^*(d))$. Thus, the mode of the samples of $d$ converges to the mode of $\pi_D(d \mid a^*(d))$ which, by construction, coincides with $d^*_GT$.

Proposition 2: If all members of $U_A$ are positive and integrable, then, for each $d$, all augmented random distributions $\Pi_A(a, \theta \mid d)$ are well defined. Thus, under standard convergence conditions for the Gibbs sampler (Roberts and Smith 1994), the stationary distribution of the Markov chain defined by the samples $\{a(i), \theta(i) ; i = 1, \ldots, M\}$ in the first loop of Algorithm 6 will be $\Pi_A(a, \theta \mid d)$. Consequently, the mode of the samples of $\{a(i), i = 1, \ldots, M\}$ converges to mode of $\Pi_A(a \mid d)$ which
coincides with \( \arg \max_{x \in A} \Psi_A^j(x, d) \). Finally, the estimate \( \hat{p}(a \mid d) \) constructed as in Equation (5) converges to \( p(A^*(d) = a) \) based on the law of large numbers.

If \( u_D \) is positive and integrable, then \( \pi_D(d, a, \theta) \) is a properly defined distribution and is the stationary distribution of the Markov chain defined by the samples generated in the second part of Algorithm 6. The mode of the samples of \( d \) converges to the mode of \( \pi_D(d) \) which, by construction, coincides with \( d_{\text{ARA}}^* \).

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