Multitransgression and regulators

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Abstract

In a parallel way to the work of Wang, we define higher order characteristic classes associated with the Chern character, generalizing the work of Bott-Chern and Gillet-Soulé on secondary characteristic classes. Our formalism is simplicial and the computations are easier. As a consequence, we obtain the comparison of Borel and Beilinson regulators and an explicit formula for the real single-valued function associated with the Grassmannian polylogarithm.

1 Local multitransgression formula

Let $X$ be a complex manifold. We denote by $A(X) = \bigoplus_{p,q} A^{p,q}(X)$ the space of complex differential forms on $X$. Let $T$ be a real manifold. We denote by $d_X = d' + d''$ and $d_T$ the differentiation operator on differential forms on $X \times T$ along $X$ and $T$, respectively.

Let $E$ be a holomorphic vector bundle on $X$. We denote also by $E$ the inverse image of $E$ on $X \times T$ by the projection $X \times T \to X$. Let $V = A(X \times T, E)$ be the global $C^\infty$ sections of $E$ with coefficients differential forms. We denote by $[\cdot]$ the supercommutator on $\text{End}_\mathbb{C}(V)$ associated to the total differential degree of differential forms on $X \times T$. We endow $E$ with a $C^\infty$ metric $\langle \cdot, \cdot \rangle$ on $X \times T$, which is hermitian restricted to each $X \times \{t\}, t \in T$. Let $\nabla = \nabla(t) : A^0(X, E) \to A^1(X, E)$ be the holomorphic unitary connection for this metric, for each point $t \in T$. We can see $\nabla$ as an application $\nabla : A^0(X \times T, E) \to A^1(X \times T, E)$ and we can extend it in the usual manner to an application $\nabla : A^i(X \times T, E) \to A^{i+1}(X \times T, E)$. We denote by $\nabla', \nabla''$ the holomorphic and the anti-holomorphic components of $\nabla$ with respect to $X$.

Let $r$ be a positive integer and $\varphi$ a symmetric $\mathbb{C}$ multilinear map on $M_N(\mathbb{C})^r$, invariant by the adjoint action of $GL_N(\mathbb{C})$. We can associate to $\varphi$ a characteristic class $\alpha := \varphi(\nabla^2, \cdots, \nabla^2)$. If $\varphi = CH_r$, where $CH_r(A_1, \cdots, A_r) =$

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Let $h = \operatorname{m} \alpha \operatorname{d} m$, we obtain $\alpha = ch_r$, the $r$-th Chern character. We define the number operator $N \in A^1(X \times T, \operatorname{End}(E))$ by the formula $d_T < e, f > = \langle e, N f \rangle$, for any two sections $e, f \in A^0(X \times T, E)$. Locally, if $h$ denotes the hermitian metric expressed in a base of local sections of $E$, $N = h^{-1} d_T h$. We introduce, for $1 \leq n \leq 2r - 1$, the local $n$-transgressed forms of the characteristic class $\alpha$

$$\alpha^{(n)} = \sum_{k, p, q \geq 0, k + p + 1 \leq r} (-1)^{(k+p)} (k+p)! (k+q)! (k+p+q+1)!$$

$$\cdot \varphi(\nabla^{r-n}, N_1/2[N, N]^{<k>}, [\nabla^{n'}, N]^{<p>}, [\nabla', N]^{<q>}),$$

where $X^{<m>}$ means $X, \cdots, X$ ($m$ times).

**Theorem 1.1**

$$d_X d_X^{(1)} + d_T \alpha = 0,$$

$$d_X \alpha^{(n)} + n d_T \alpha^{(n-1)} = \varphi(\nabla^{r-n}, [\nabla^{n'}, N]^{<n>} - (-1)^n [\nabla', N]^{<n>}) \quad (n \geq 2). \quad (1)$$

## 2 The definition of the regulator

Let $X$ be a smooth projective complex variety. We denote by $A^*_R(X)$ the set of complex differential forms on $X$ invariant by complex conjugation. We define $A^*_R(X)(r) := (2\pi)^r A^*_R(X) \subset A^*(X)$. We recall from [W], [B1] the definition of the complex $\mathcal{D}(X, r)$ computing the real Deligne cohomology:

$$\mathcal{D}^n(X, r) = A^n_R(X)(r-1) \bigcap \bigoplus_{p+q=n-1, p \leq n, q \leq r} A^{p,q}(X) \quad \text{if } n \leq 2r - 1,$$

$$\mathcal{D}^n(X, r) = A^n_R(X)(r) \bigcap \bigoplus_{p+q=n, p \geq r, q \geq r} A^{p,q}(X) \quad \text{if } n \geq 2r,$$

with the differential of the element $x \in \mathcal{D}^n(X, r)$ given by

$$d_P x = dx \quad \text{if } n \geq 2r,$$

$$d_P x = -2 d' d'' x \quad \text{if } n = 2r - 1,$$

$$d_P x = -\pi(dx) \quad \text{if } n < 2r - 1,$$

where $\pi(d)$ is the restriction of $d$ to the $p \leq n, q \leq n$ pieces of the bigrading.

We define the simplicial set $T(X)$ of isomorphism classes of hermitian vector bundles on $X$. An element $\tilde{E} = (E_0, \ldots E_n; h_0, \ldots, h_n) \in T_n(X)$ consist of vector bundles $E_0, \ldots E_n$ on $X$ endowed with hermitian metrics $h_0, \ldots, h_n$, and isomorphisms $\varphi_i : E_0 \to E_i$ for each $i = 1, \ldots, n$. 

We denote by $\Delta^n$ the standard $n$--simplex $\Delta^n = \{(t_0, \ldots, t_n), t_i \geq 0, \sum_{i=0}^n t_i = 1\}$. We associate to an element $\tilde{E} \in T_n(X)$ the hermitian vector bundle $\tilde{\mathcal{E}} = (\mathcal{E}, h_t)$ on $X \times \Delta^n$, where the vector bundle $\mathcal{E}$ is the inverse image of the vector bundle $E_0$ by the projection $X \times \Delta^n \to X$, and the metric $h_t$ on $\mathcal{E}$ restricted to $X \times \{(t_0, \ldots, t_n)\}$ is $t_0h_0 + t_1\varphi_1^*h_1 + \ldots + t_n(\varphi_n\varphi_1)^*h_n$. We define:

$$\text{ch}_r^{(n)}(\tilde{E}) = \frac{1}{2n!} \int_{\Delta^n} \text{ch}_r^{(n)}(\tilde{\mathcal{E}}).$$

We observe that $\text{ch}_r^{(n)}(\tilde{E}) \in \mathcal{D}^{2r-n}(X, r) \subset A^n(X)$. We extend by linearity the application $\text{ch}_r^{(n)}$ to an application $\text{ch}_r^{(n)} : ZT_n(X) \to \mathcal{D}^{2r-n}(X, r)$. Theorem 1.1 and Stokes formula imply $d_\mathcal{D}\text{ch}_r^{(n)}(\mathcal{E}) = \text{ch}_r^{(n-1)}(\partial \mathcal{E})$ (the right term of equation (1) disappears because of its complex bigrading), i.e. $\text{ch}_r^{(1)} : ZT(X) \to \mathcal{D}^{2r-1}(X, r)$ is a morphism of complexes.

If we fix a metric $h$ on the trivial rank $N$ vector bundle $1^N$ on $X$ and if we consider only those elements of the form $\tilde{E} = (1^N, \ldots, 1^N; h, \ldots, h)$ in $T_n(X)$, we obtain an inclusion of simplicial sets $BGL_N(X) \to T(X)$. We obtain therefore a morphism:

$$\text{ch}_r^{(1)} : ZBGL_N(X) \to \mathcal{D}^{2r-1}(X, r).$$

(2)

**Proposition 2.1** The morphism (2) is, up to homotopy, independent of the choice of the metric $h$ and compatible with the inclusion $BGL_N(X) \subset BGL_{N+1}(X)$.

So we obtain a morphism $H_n(BGL_\infty(X)) \to H_D^{2r-n}(X, \mathbb{R}(r))$. If $X = \text{Spec} \mathbb{C}$, we compose this morphism with the classical morphism $K_n(\mathbb{C}) = \pi_n(BGL_\infty(\mathbb{C}))^+ \to H_n(BGL_\infty(\mathbb{C})^+) = H_n(BGL_\infty(\mathbb{C}))$ and we obtain a morphism

$$\text{ch}_r^{(n)} : K_n(\mathbb{C}) \to H_D^{2r-n}(\mathbb{C}, \mathbb{R}(r)).$$

(3)

**Theorem 2.2** The morphism (3) coincides with the Beilinson regulator.

**Proof:** Let $S(X)$ be the simplicial set computing the Waldhausen K-theory of $X$. Wang ([W]) constructs an explicit morphism of complexes $\mathbb{Z}S(X) \to \mathcal{D}^{2r+1-n}(X, r)$ and Burgos-Wang ([BW]) proved that the composition $K_n(X) := \pi_{n+1}(S(X)) \to H_{n+1}(\mathbb{Z}S(X)) \to H_D^{2r-n}(X, \mathbb{R}(r))$ is the Beilinson regulator. There is a canonical map $\Sigma BGL_N(X) \to S(X)$, so we deduce a morphism $\text{ch}_r^{(1)} : ZBGL_N(X) \to \mathcal{D}^{2r-1}(X, r)$. We prove that this morphism coincides with the morphism (2) by using an explicit homotopy on the bisimplicial complex $\Sigma BGL_N(X)$. 

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3 Comparison of Borel and Beilinson regulators

The morphism (2) applied to an element of $B_{2r-1}GL_N(X)$ written in the homogeneous form $(g_0, \ldots, g_{2r-1})$ gives

$$h_t = \sum_{i=0}^{2r-1} t_i g_i \bar{g}_i^t,$$

(4)

$$ch_r^{(2r-1)}(g_0, \ldots, g_{2r-1}) = -\frac{(r-1)!}{2(2r-1)!} \int_{\Delta^n} Tr(h_t^{-1} d_T h_t)^{2r-1}. \quad (5)$$

So the morphism $H_{2r-1}(B.GL_N(\mathbb{C})) \to H^D_1(\mathbb{C}, \mathbb{R}(r)) = \mathbb{R}(r-1)$ giving rise to the Beilinson regulator can be explicited by these formulas. On the other side, the Borel regulator comes ([H]) from the explicit morphism $H_{2r-1}(B.GL_N(\mathbb{C})) \to \mathbb{R}$ given by $b'_r(g_0, \ldots, g_{2r-1}) = \text{cst} \int_{\Delta^n} Tr(h_t^{-1} d_T h_t)^{2r-1}$ where $h_t = \sum_{i=0}^{2r-1} t_i g_i \bar{g}_i^t$ ($h = 1$ in (4)). Taking care of the normalizations, we obtain the theorem of [B2]

**Theorem 3.1** The normalized Borel regulator is twice the Beilinson regulator.

4 Explicit presentation of the real period attached to the grasmmanian polylogarithm

The integral (5) still converges if the metric $h$ is degenerate of rank $N-r+1$ and we prove that we obtain the same morphism $H_{2r-1}(B.GL_N(X)) \to H^D_1(X, \mathbb{R}(r))$.

When $N = r$ we can use a metric of rank one $h = v \bar{v}^t$ for a everywhere non-vanishing section $v \in H^0(X, \mathcal{I}^N)$. Let $v_i = g_i v$. We denote by $\mathcal{I}^s$ the set of subsets of $\{0, \ldots, 2r-1\}$ of cardinality $s$ and for $I = \{i_1, \ldots, i_s\} \in \mathcal{I}^s$ we denote $t_I = \prod_{i \in I} t_i$ and $v_I = v_{i_1}, \ldots, v_{i_s}$. The morphism (2) is

**Theorem 4.1**

$$ch_r^{(2r-1)}(g_0, \ldots, g_{2r-1}) = -\frac{(r-1)!}{2(2r-1)!} \int_{\Delta^n} \prod_{j=1}^{2r-1} t_{I_j} dt_{I_j} \det(v_{I_j}, v_{I_j}) \det(v_{I_j}, v_{I_{j+1}}) \sum_{0 \leq i_1, \ldots, i_{2r-1} \leq 2r-1} \left(\prod_{j=1}^{2r-1} t_{I_j} |\det(v_I)|^{2} \right)^{2r-1}. \quad (6)$$

Goncharov ([G]) associates to $2r$ non-zero vectors $v_0, \ldots, v_{2r-1}$ in $\mathbb{C}^r$ a $\mathbb{Z}$ mixed Hodge structure $\mathcal{G}(v_0, \ldots, v_{2r-1})$ of Hodge-Tate type, the maximal period of which is the Grasmannian polylogarithm.

**Conjecture 4.2** The maximal period of the $\mathbb{R}$-MHS attached to $\mathcal{G}(v_0, \ldots, v_{2r-1})$ is the right-hand side of (6).
**Theorem 4.3** The conjecture is true for $r \leq 3$.

For $r = 2$ we have some simplifications in the right-hand side of (6), beginning with the nice classical type

**Proposition 4.4** The following function is antisymmetric with respect to permutations in the variables $v_i$:

$$f(v_0, v_1, v_2, v_3) = \text{Im} (\det(v_0, v_1) \det(v_2, v_3) \overline{\det(v_0, v_3) \det(v_1, v_2)}).$$

We obtain a new presentation of the Bloch-Wigner dilogarithm $R L i_2$:

**Theorem 4.5** Let $v_0, v_1, v_2, v_3$ be non-zero vectors in $\mathbb{C}^2$ and $r(v_0, v_1, v_2, v_3)$ the cross-ratio of the four points they represent in $\mathbb{P}^1$. Then

$$R L i_2(r(v_0, v_1, v_2, v_3)) = 12 i f(v_0, v_1, v_2, v_3) \cdot \int_{\Delta^3} \frac{dt_1 dt_2 dt_3}{(\sum_{i \neq j} t_i t_j |\det(v_i, v_j)|^2)^2}.$$

5 A speculation concerning the Beilinson-Soulé conjecture

It is remarkable that the vanishing of $\text{ch}^{(n)}_r$ for $n \geq 2r$ is a consequence of the local formalism of the section 1 and not of the vanishing of $D^n(X, r)$ for $n \geq 2r$. If instead of the complex $D(X, r)$ we had a complex $B(X, r)$ and a theory $\text{ch}^{(n)}_r : K_n(X) \to H^{2r-n}(B(X, r))$ which: a) could be realized by a formalism as in section 1 and b) is injective on $K_n^{[r]}(X) \otimes \mathbb{Q}$ (the weight $r$ piece of K-theory for the Adams operations), we would have the Beilinson-Soulé conjecture.

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