Abstract

We present here some results of applying the Cayley-Dickson process to certain alternative algebras (notably built upon Galois fields and congruence rings), in a manner which might yield new building blocks for cryptographic systems. The results presented here are technical but not inherently difficult.

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Introduction

The idea of applying the Cayley-Dickson process to Galois fields is not new ([18, 10, 20]). Finding the number of units and unimodular elements of the resulting structure, and of iterations of the doubling procedure on the base Galois field, is a simple application of a classical exercise ([9]), as we show here, but the relevance of the diophantine equations which are classically presented, to the algebraic structures we are interested in, is not readily apparent in the published literature. Similar results, but using congruence rings as starting points rather than fields (for possible uses in cryptographic systems), are presented here, for what we believe is the first time. One key ingredient of this extension is the Cayley-Dickson process for morphisms, and consequence theorem 1.3 which we believe is new. We focus on enumeration properties rather than the classification and comparison questions which are extensively studied elsewhere, at least for algebras over fields ([19, 22, 5, 20]).

We would like to emphasize the fact that what we present here is at the confluence of several rather distinct mathematical traditions, whose terminologies and notations are sometimes incompatible. We have therefore been led to make choices, and to be sometimes rather verbose so as to disambiguate the notion used. It would be worth one’s while to take a look at the appendices, if only to ascertain the meaning we have chosen here for some terms.

1 Abstract Cayley objects

1.1 Definitions and elementary properties

1.1.1 Order and index

Let \((S, \cdot)\) be a di-associative I.P. loop, whose neutral element we will denote by \(e\). For \(\omega \in S\) we can define the set \(P_{\omega} = \{\omega^n \mid n \in \mathbb{Z}\}\).

\(P_{\omega}\) is an abelian group and \([\mathbb{Z} \to P_{\omega}; n \mapsto \omega^n]\) is a group homomorphism. We can therefore define \(\omega\) to be of infinite order if that homomorphism is injective, and to be of finite order \(\text{card}(P_{\omega})\) otherwise. These definitions are consistent with their counterparts for groups. Note that if \(\omega\) is of finite order \(\alpha\), then for any \(\beta \in \mathbb{Z}\), \(\omega^\beta\) is also of finite order, that order being \(\alpha / \text{PGCD}(\alpha, \beta)\), as \((P_{\omega}, \cdot)\) is an abelian group (recall that PGCD(\(\alpha, \beta\)) designates the least common multiple of \(\alpha\) and \(\beta\)).

\(P_{\omega}\) also is a subloop of \((S, \cdot)\), and we define the two following binary relations on \(S\):

\[
\begin{align*}
\forall (x, y) \in S^2 \quad & x \mathcal{P}_\omega^R y \iff x \cdot y = \omega^n \cdot x \\
\forall (x, y) \in S^2 \quad & x \mathcal{P}_\omega^L y \iff x^{-1} \cdot y = \omega^n \cdot x 
\end{align*}
\]

Proposition 1.1: \(\mathcal{P}_\omega^R\) and \(\mathcal{P}_\omega^L\) are equivalency relations on \(S\), the inversion is a bijection from \(S/\mathcal{P}_\omega^R\) to \(S/\mathcal{P}_\omega^L\) and any element of \(S/\mathcal{P}_\omega^R\) is equipotent to \(P_{\omega}\).

Proof: We only indicate the proof that \(\mathcal{P}_\omega^R\) is an equivalency relation, as the one for \(\mathcal{P}_\omega^L\) is perfectly similar.

\[\forall x \in S \quad x \cdot x^{-1} = x^{-1} \cdot x = e = \omega^0 \in P_{\omega}\] and hence \(\mathcal{P}_\omega^R\) is reflexive.

Let \(x, y \in S\) such that \(x \mathcal{P}_\omega^R y\). Then \((\exists n \in \mathbb{Z}) \quad x = \omega^n \cdot y,\) so \(y = \omega^{-n} \cdot x,\) since we are in an I.P. loop (and \(\omega^{-n} = (\omega^n)^{-1}\) because of the loop di-associativity). Thus \(\mathcal{P}_\omega^R\) is symmetrical.

Let \(x, y, z \in S\) such that \(x \mathcal{P}_\omega^R y\) and \(y \mathcal{P}_\omega^R z\). Then \((\exists n \in \mathbb{Z}) \quad x = \omega^n \cdot y\) and \((\exists m \in \mathbb{Z}) \quad y = \omega^m \cdot z,\) but then \(x = \omega^n \cdot (\omega^m \cdot z) = \omega^{n+m} \cdot z,\) thanks to the di-associativity. Thus \(\mathcal{P}_\omega^R\) is transitive.

This concludes the proof that \(\mathcal{P}_\omega^R\) is an equivalency relation;

Let \(x, y, z \in S\) such that \(x \mathcal{P}_\omega^L y\). Then \((\exists n \in \mathbb{Z}) \quad x \cdot y = \omega^n,\) so \((y^{-1})^{-1} \cdot (x^{-1}) = y^{-1} \cdot x^{-1} = \omega^{-n},\) which means that \(y^{-1} \mathcal{P}_\omega^L x^{-1},\) and the inversion is a bijection from \(S/\mathcal{P}_\omega^R\) to \(S/\mathcal{P}_\omega^L\).

Let now \(Y \in S/\mathcal{P}_\omega^R\) and \(y \in Y,\) and consider \(\phi_y : P_{\omega} \to S; z \mapsto z \cdot y.\) Since we are in a loop, \(\phi_y\) is injective, and since the loop is di-associative, \(\phi_y\) takes its values in \(Y,\) if \(x \in Y,\) then \(x = \omega^m \cdot y\) for some \(m \in \mathbb{Z}\), so \(x \cdot y^{-1} = \omega^{-m} y,\) thanks to the I.P. of the loop. But then \(\phi_y(\omega^m) = \phi_y(x \cdot y^{-1}) = (x \cdot y^{-1}) \cdot y = y,\) again because we are I.P., and thus \(\phi_y\) is surjective. Finally, \(\phi_y\) is bijective.

Remark: There is no reason for the left and right cosets to agree, in particular there is no reason for \(P_{\omega}\) to be a normal subloop of \(S\).
If either \( S/P^B_\omega \) or \( S/P^L_\omega \) is finite, then the other one is too, and they have the same cardinal; in that case we will call that cardinal the index of \( P_\omega \) in \( S \), and denote it by \([S : P_\omega]\). This definition is also consistent with its counterpart for groups.

1.1.2 Conjugation

Recall (2) that, given a commutative and associative ring with unit \((A,+,\cdot)\) and \((E,+,\cdot,\cdot)\) an \( A \)-algebra with unit, with neutral element \( e \), a conjugation over \( E \) is any function \( \sigma : E \rightarrow E \) which is bijective, \( A \)-linear and such that:

1. \( \sigma(e) = e \).
2. \((\forall (x,y) \in E^2)\ \sigma(x \cdot y) = \sigma(y) \cdot \sigma(x) \) (beware the interversion of \( x \) and \( y \)).
3. \((\forall x \in E)\ (x + \sigma(x)) \in A \cdot e \).
4. \((\forall x \in E)\ (x \cdot \sigma(x)) \in A \cdot e \).

These properties imply\(^1\) \((\forall x \in E)\ x \cdot \sigma(x) = \sigma(x) \cdot x \) and\(^2\) \((\forall x \in E)\ (\sigma \circ \sigma)(x) = x \).

Note that if \( E = A \), the identity function is always a conjugation, but that otherwise there is no guaranty a conjugation can exist. Note also that while, for a given \( x \in E \), there exists \( T \in A \) and \( N \in A \) such that \( \mathcal{J}_E(x) = T \cdot e \) and \( \mathcal{N}_E(x) = N \cdot e \), \( T \) and \( N \) may happen not to be unique (see \( \mathbb{A}_2 \)). We will also write \( \bar{x} \) for \( \sigma(x) \) when no confusion is to be feared.

1.1.3 Cayley Algebra

Recall (2) that, given a commutative and associative ring with unit \((A,+,\cdot,\cdot)\) a Cayley algebra over \( A \) is a structure \((E,+,\cdot,\cdot,\cdot)\), where \((E,+,\cdot,\cdot)\) is an \( A \)-algebra with unit, and \( \cdot \) is a conjugation over \( E \).

Given \((E,+,\cdot,\cdot,\cdot)\) an \( A \)-Cayley algebra and \( T \subset E \), we will say \( T \) is a sub-Cayley algebra if and only if it is a sub-algebra of an algebra with unit, of \( E \), and it is stable by conjugation (i.e. \((\forall x \in T)\ \sigma(x) \in T\)).

On a Cayley algebra, it is convenient to consider the Cayley trace and Cayley norm\(^3\) defined respectively by \( \mathcal{J}_E(x) = x + \sigma(x) \) and \( \mathcal{N}_E(x) = x \cdot \sigma(x) \).

We have the following important relations valid for all elements:

\[
\begin{align*}
\mathcal{J}_E(\sigma(x)) &= \mathcal{J}_E(x) \quad (1) \\
\mathcal{N}_E(\sigma(x)) &= \mathcal{N}_E(x) \quad (2) \\
\mathcal{J}_E(y \times x) &= \mathcal{J}_E(x \times y) \quad (3) \\
\mathcal{J}_E(x \times \sigma(y)) &= \mathcal{J}_E(y \times \sigma(x)) = \mathcal{J}_E(x) \times \mathcal{J}_E(y) - \mathcal{J}_E(x \times y) = \mathcal{N}_E(x + y) - \mathcal{N}_E(x) - \mathcal{N}_E(y) \quad (4) \\
x^2 &= \mathcal{J}_E(x) \times x - \mathcal{N}_E(x) \quad (5)
\end{align*}
\]

Relation \( (3) \) is especially noteworthy in light of the fact that no commutativity or associativity has been assumed\(^4\). By contrast, in absence of any form of associativity (or commutativity), little more can be proved about the Cayley norm\(^5\), apart from the following proposition (where \( \text{Mod}_A(x,y) \) denotes the \( A \)-module generated by \( x \in E \) and \( y \in E \), and \( e \) denotes the neutral element of \( E \)):

**Proposition 1.2:**

Given \( x \in E \), \( \text{Mod}_A(e,x) \) is stable for \( \times \); it is a sub-Cayley algebra of \( E \) which is both associative and commutative. Furthermore \((\forall m \in \mathbb{N})\ \mathcal{N}_E(x^n) = (\mathcal{N}_E(x))^n \).

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\(^1\) \((x + \sigma(x)) \in A \cdot e \Rightarrow x \times \sigma(x) = x \times (x + \sigma(x)) - x \times x = (x + \sigma(x)) \times x - x \times x = \sigma(x) \times x\).

\(^2\) Given \( x \in E \), there exists \( \alpha \in A \) such that \( x + \sigma(x) = \alpha \cdot e \); the \( A \)-linearity of \( \sigma \) then implies \( \sigma(x) + (\sigma \circ \sigma)(x) = \sigma(x + \sigma(x)) = \alpha \cdot e \), and finally, \( \sigma(e) = e \).

\(^3\) Yet another unfortunate collision of terms, as this “norm” is actually quadratic...

\(^4\) It is established by noticing that \( \mathcal{J}_E(x \times y) = x \times y + (\mathcal{J}_E(y) \times y) \times (\mathcal{J}_E(x) - x) \) and that therefore \( \mathcal{J}_E(x) \times y + \mathcal{J}_E(y) \times x + (\mathcal{J}_E(x \times y) - \mathcal{J}_E(x) \times \mathcal{J}_E(y)) = x \times y + y \times x = \mathcal{J}_E(y) \times x + \mathcal{J}_E(x) \times y + (\mathcal{J}_E(y \times x) - \mathcal{J}_E(y) \times \mathcal{J}_E(x)) \).

\(^5\) See \( \mathbb{C}_4 \) for a discussion of what one might want out of a “norm”. It should be noted that \((\forall (x,y) \in E^2)\ \mathcal{N}_E(x \times y) = \mathcal{N}_E(y \times x) \) may happen even when \( E \) is not alternative (where this is always true; see \( \mathbb{C}_1 \) Cayley Algebra proposition 1.4), for instance in the case of the hexadecimals.
Remark: Given \( x \in E, (\forall n \in \mathbb{N}) \ x^n \in \text{Mod}_A(e, x) \), and we have:

\[
(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) \ x^{n+m} = x^n \times x^m = x^m \times x^n
\]

which is just property (A') of A.1.5 under different notations.

Invertible elements have a few more properties:

**Proposition 1.3:**
If \( x \in E \) has an inverse for “\( \times \)”, then \( \text{Mod}_A(e, x, x^{-1}) \) is stable for “\( \times \)”; it is a sub-Cayley algebra of \( E \) which is both associative and commutative.

**Proof:** Yet another easy consequence of (1), applied to \( x \) as well as to \( x^{-1} \), and the definition of the inverse.$

**Remark:** With no assumption of associativity for \( E \), not much can be said about the relationship between the Cayley norms of \( x \) and of \( x^{-1} \) in A.4Alternative rings proposition.1.3 Cayley Algebra proposition.1.3

However, if we know that \( \mathcal{N}_E(x) \) is invertible in \( A \cdot e \), then of course \( x \) is invertible in \( E \) and \( x^{-1} = \mathcal{N}_E(x)^{-1} \cdot \sigma(x) \).

With some rather weak hypotheses, however, we can improve the situation, as is well known ([2]):

**Proposition 1.4:**
Let \((A, +, \cdot)\) be a commutative and associative ring with unit, and \((E, +, \cdot, \cdot, \sigma)\) an alternative Cayley algebra over \( A \), with neutral element \( e \). An element \( x \in E \) is invertible if and only if \( \mathcal{N}_E(x) \) is invertible in \( A \cdot e \), in which case the inverse of \( x \) is \( \mathcal{N}_E(x)^{-1} \cdot \sigma(x) \), and \( \text{Mod}_A(e, x, x) \) is an abelian group. Furthermore, \((E, \cdot, \cdot)\) is di-associative and we have \((\forall x \in E)(\forall y \in E) \mathcal{N}_E(x \cdot y) = \mathcal{N}_E(x) \times \mathcal{N}_E(y)\).

**Remark:** The di-associativity of \((E, \cdot, \cdot)\) as a magma (see A.4 Alternative rings proposition A.4) ensures that

\[
(\forall x \in E)(\forall y \in E)(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(\forall p \in \mathbb{N})(\forall q \in \mathbb{N}) \ x^{n+m} y^{p+q} = x^m (x^n y^{p+q}) = (x^{n+m} y^p) y^q
\]

which is just A.1.5 of A.1.5 under different notations.

Alternative Cayley algebras also enjoy a few other nice properties$^6$.

**Proposition 1.5:**
Let \((A, +, \cdot)\) be a commutative and associative ring with unit, and \((E, +, \cdot, \cdot, \sigma)\) an alternative Cayley algebra over \( A \), with neutral element \( e \). Let \( E^* \) the elements of \( E \) which have an inverse for “\( \times \)”, then \((E^*, \cdot, \cdot)\) is a Moufang loop.

**Proof:** An alternative algebra being di-associative, \((E^*, \cdot, \cdot)\) is a di-associative magma with unit. By definition, every \( x \in E^* \) has an inverse \( x^{-1} \), and since \( E \) is an alternative Cayley algebra, \( x^{-1} = \mathcal{N}_E(x)^{-1} \cdot \sigma(x) \)

and \((\exists T \in A) \sigma(x) = T \cdot e - x, \) so \((\exists (\alpha, \beta) \in A^2) x^{-1} = \alpha \cdot e + \beta \cdot x \). Therefore, \( x^{-1} \cdot (x \cdot y) = (\alpha \cdot e + \beta \cdot x) \cdot (x \cdot y) \)

but since the algebra is alternative, \( x \cdot (x \cdot y) = (x \cdot x) \cdot y \), and we finally see that \( x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y = y \). We likewise prove that \( (y \cdot x) \cdot x^{-1} = y \). This proves that \( E^* \) is a quasigroup, hence a loop since it has a neutral element, and has the I.P. property.

Using A.4 Alternative rings proposition A.4 we find that the loop is actually a Moufang loop.$

**Remark:** If \( E \) is actually associative, then of course \( E^* \) is a group.

**Remark:** As a consequence of the di-associativity of loops, we have

\[
(\forall x \in E^*)(\forall y \in E^*)(\forall n \in \mathbb{Z})(\forall m \in \mathbb{Z})(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z}) \ x^{n+m} y^{p+q} = x^m (x^n y^{p+q}) = (x^{n+m} y^p) y^q
\]

which is just A.1.5 of A.1.5 under different notations.

**Remark:** The material developed in A.11 applies to \( E^* \). In particular, if \( \text{card}(E^*) \) is finite we have \((\forall \omega \in E^*) \text{ card}(E^*) = [E^* : P_\omega] \text{ card}(P_\omega)\).

$^6$This is a special case of the theorem which states that the invertible elements of an alternative ring with unit are a Moufang loop ([2]); we only give a proof here because it is slightly simpler than the general case.
1.1.4 Cayley Morphism

Given two Cayley algebras $(E, +, \times, \cdot, \sigma)$ and $(F, \dagger, \ast, \bullet, s)$ on a same commutative and associative ring with unit $(A, +, \cdot)$, a function $\phi : E \rightarrow F$ is a (homo)morphism of Cayley algebras (or simply a Cayley (homo)morphism) if and only if it is a (homo)morphism of algebras with unit and that furthermore $(\forall x \in E) \phi(\sigma(x)) = s(\phi(x))$. Monomorphism, epimorphisms, isomorphisms, endomorphisms and automorphisms of Cayley algebras are built from homomorphism of Cayley algebras in the usual manner.

As a matter of practicality, it is useful to note that a Cayley morphism which is an isomorphism of algebras with unit, actually is a Cayley isomorphism, as is trivially seen.

One verifies immediately that one has a category (as per [14]) (which we will call CAAlg here) by considering Cayley algebras as objects, Cayley morphisms as arrows, and the usual composition of functions as composition of arrows. Of course, if we denote by UAlg the category of algebras with unit and corresponding classical morphisms of algebras with units, then associating every Cayley algebra with its underlying algebra with unit, and every Cayley algebra morphism with its underlying morphism of algebra with unit, yields a covariant functor from CAAlg to UAlg (a stripping functor).

1.2 The Cayley-Dickson process

1.2.1 A fundamental shortcoming

A major problem with CAAlg, as categories go, is that products may fail to exist, if we want the product to be compatible with the stripping functor of [14, 2].

Consider the following Cayley algebra over $\mathbb{R}$: $(E, +, \times, \cdot, \sigma) = (\mathbb{R}, +, \times, \cdot, id_{\mathbb{R}})$, with $id_{\mathbb{R}}$ the identity function on $\mathbb{R}$. Assume we have built the product of E with itself, yielding $(G, \boxplus, \boxtimes, \boxdot)$. Since we want the product creation to be compatible with the stripping functor, we see at once that $(G, \boxplus, \boxtimes, \boxdot)$ must be the product algebra of $(E, +, \times, \cdot, \sigma)$ with itself, and that the Cayley algebra morphisms from from E to $(E, +, \times, \cdot, \sigma)$ as first components) and E (as second component) must be the projections of the algebra $G$ onto $E$. Hence $(x, y) \boxplus (x', y') = (x + x', y + y')$, $(x, y) \boxtimes (x', y') = (xx', yy')$ (with $(1, 1)$ being the neutral element for "\boxtimes"), $\lambda \boxdot (x, y) = (\lambda x, \lambda y)$ and $\boxdot((x, y)) = (x, y)$. But then $(1, 2) \boxdot ((1, 2)) = (1, 4) \notin \mathbb{R} \boxdot (1, 1)$.

There are interesting cases, however, where products do exist, which are compatible with the stripping functor:

Theorem 1.1 (Dominated Product):

Let I be a set, $\{(E_i, +, \times, \cdot, \sigma_i)\}_{i \in I}$ a family of Cayley algebras on a same commutative and associative ring with unit $(A, +, \cdot)$, and $(E, +, \times, \cdot, \sigma)$ the product algebra of that family, seen as mere algebras with unit. If there exists a Cayley algebra $(F, \dagger, \ast, \bullet, s)$ and a family of Cayley morphisms $\{(\phi_i)_{i \in I}\}$, with $\phi_i : F \rightarrow E_i$ for all $i \in I$, such that the morphism of algebras with unit $\Phi : F \rightarrow E; y \mapsto (\phi_i(y))_{i \in I}$ is surjective, then the function $\sigma : E \rightarrow E; (x_i)_{i \in I} \mapsto (\sigma(x_i))_{i \in I}$ is a conjugation on $(E, +, \times, \cdot, \sigma)$ is a Cayley algebra, for all $i \in I$ the projections of $E$ onto $E_i$ is a Cayley epimorphism and $\Phi$ is a Cayley epimorphism.

Proof: It is obvious that $\sigma$ is an A-linear involution (and thus bijective). Given $(x_i)_{i \in I} \in E$ and $(x'_i)_{i \in I} \in E$, $\sigma((x_i)_{i \in I} \times (x'_i)_{i \in I}) = \sigma((x_i \times x'_i)_{i \in I}) = (\sigma_i(x_i \times x'_i))_{i \in I} = (\sigma(x_i) \times \sigma(x'_i))_{i \in I} = \sigma((x'_i)_{i \in I} \times \sigma((x_i)_{i \in I})$.

Let $y \in F$. We see that $\sigma(\Phi(y)) = \sigma((\phi_i(y))_{i \in I}) = (\sigma_i(\phi_i(y)))_{i \in I}$, but since all the $\phi_i$ are Cayley morphisms, $\sigma_i(\phi_i(y)))_{i \in I} = (\phi_i(s(y)))_{i \in I} = \Phi(s(y))$, and thus $\sigma(\Phi(y)) = \Phi(s(y))$.

Let $(x_i)_{i \in I} \in E$; since $\Phi$ is surjective, there exists $y \in F$ such that $\Phi(y) = (x_i)_{i \in I}$. We see then that $(x_i)_{i \in I} + \sigma((x_i)_{i \in I}) = \Phi(y) + \sigma(\Phi(y))$, but we have just proved that $\sigma(\Phi(y)) = \Phi(s(y))$, so $\Phi(y) + \sigma(\Phi(y)) = \Phi(y) + \Phi(s(y))$, and since $\Phi$ is A-linear, $\Phi(y) + \Phi(s(y)) = \Phi(y \dagger s(y))$. As $(F, \dagger, \ast, \bullet, s)$ is a Cayley algebra, there exists $e \in A$ such that, writing $\varepsilon$ for the neutral element of $F$, $y \dagger s(y) = e \cdot s$ and $\varepsilon = \phi_i(\varepsilon)$ for each i, so $\varepsilon \in A$. Since $\Phi$ is A-linear, we find at once that $\Phi(y \dagger s(y)) = \alpha \cdot (\phi_i(\varepsilon)))_{i \in I}$ and, denoting by $e_i$ the neutral element of $E_i$ (for each $i$), $\alpha \cdot (\phi_i(e_i))_{i \in I} = \alpha \cdot (e_i)_{i \in I}$ since the $\phi_i$ are all Cayley morphisms. Since $e = (e_i)_{i \in I}$ is the neutral element for $E$, we have proved that $(x_i)_{i \in I} + \sigma((x_i)_{i \in I})$ is the (external) product the neutral element of $E$.

This proves that $\sigma$ is a conjugation, and thus that $(E, +, \times, \cdot, \sigma)$ is a Cayley algebra. We have also shown that $\Phi$, which is a morphism of algebra with unit, verifies $\sigma(\Phi(y)) = \Phi(s(y))$, so is a Cayley morphism. Since we have assumed $\Phi$ to be surjective, it is a Cayley epimorphism.

The fact that for all $i \in I$ the projections of $E$ onto $E_i$ is a Cayley epimorphism is trivial.
1.2.2 Cayley-Dickson for algebras

This well-known procedure is a way around the shortcoming described in [12.3]. Notably, there is no compatibility with the stripping functor of [1.1.4] (though there is compatibility if we strip all the way to the module), and only the product of a Cayley algebra with itself is defined. We present here a version which differs only in notation from that of [2]. Note that for our intended uses, we must consider algebras over rings which may happen not to be fields (in contrast to, e.g. [5]).

Let \((A, +, \cdot)\) be a commutative and associative ring with unit (with neutral element 1), and \((E, +, \times, \cdot, \sigma)\) a Cayley algebra over \(A\) (with neutral elements \(e\) and 0 for “\(\times\)” and “\(+\)” respectively). Let \(F = E \times E\), \(\varepsilon = (e, 0) \in F\), and choose an element \(\zeta \in A\). Define now
\[
\begin{align*}
\tilde{\cdot} & : F \times F \to F \\
((x, y), (x', y')) & \mapsto (x + x', y + y') \\
\ast & : F \times F \to F \\
((x, y), (x', y')) & \mapsto (x \times x' - \zeta \cdot (\sigma(y') \times y), y \times \sigma(x') + y' \times x) \\
\bullet & : A \times F \to F \\
(\lambda, (x, y)) & \mapsto (\lambda \cdot x, \lambda \cdot y) \\
\ast & : F \to F \\
(x, y) & \mapsto (\sigma(x), -y)
\end{align*}
\]

The following proposition is entirely classical ([2] and [12]).

**Proposition 1.6 (Structure):**
\(\Phi((x, y)) = (\mathcal{T}_E(x, 0))\) and \(\mathcal{M}_E((x, y)) = (\mathcal{M}_E(x) + \zeta \cdot \mathcal{M}_E(y), 0)\). \(\mathcal{R}_E((x, y)) = (\mathcal{R}_E(x, 0))\)

An especially important use case of the Cayley-Dickson process is when we start with some associative and commutative ring with unit \((A, +, \cdot)\), and repeatedly iterate the process, starting with \((E, +, \times, \cdot, \sigma) = (A, +, \cdot, 1)\), yielding Cayley algebra structures on \(A, A \times A, \ldots\). When we do thus with \(A = \mathbb{R}\), and choose \(\zeta = 1\) at each step, we build \(\mathbb{C}, \mathbb{H}, \mathcal{O}, \mathbb{X}\) (see [4]). We will see other applications of this in the remainder of this text. In such cases, it is convenient to identify \(E\) with \(E \times \{0\}\), or \(F\) with a superset of \(E\), at each step.

This is one case where one sees at once that the Cayley norm and trace actually takes their values in \(E\), at each step, i.e. \(A\), so we can drop the reference to precisely which algebra we consider when computing them (which makes for far more readable \(\mathcal{R}(x, y) = \mathcal{R}(x)\) and \(\mathcal{M}(x, y) = \mathcal{M}(x) + \zeta \cdot \mathcal{M}(y)\)).

We will refer to the Cayley-Dickson process as “doubling” when no confusion is to be feared.

1.2.3 Cayley-Dickson for morphisms, and consequences

**Theorem 1.2:**

Let \((E, +, \times, \cdot, \sigma)\) and \((F, \tilde{\cdot}, \ast, \bullet, s)\) be two Cayley algebras over the commutative and associative ring with unit \((A, +, \cdot)\), and \(\phi : E \to F\) an \(A\)-Cayley morphism. Let \(\Phi : E \times E \to F \times F; (x, y) \mapsto (\phi(x), \phi(y))\). For any \(\alpha \in A\), denote by \((E_\alpha, +_\alpha, \times_\alpha, \cdot_\alpha, \sigma_\alpha)\) (respectively \((F_\alpha, \tilde{\cdot}_\alpha, \ast_\alpha, \bullet_\alpha, s_\alpha)\)) the result of applying the Cayley-Dickson process to \((E, +, \times, \cdot, \sigma)\) (respectively \((F, \tilde{\cdot}, \ast, \bullet, s)\)) using \(\alpha\); then \(\Phi\) is also an \(A\)-Cayley morphism from \(E_\alpha\) to \(F_\alpha\).

**Proof:** The \(A\)-linearity of \(\Phi\) is trivial.

\[
\Phi(\sigma_\alpha(x, y)) = \Phi((\sigma(x), -y)) = (\phi(\sigma(x)), -\phi(y)),
\]
and as \(\phi\) is an \(A\)-Cayley morphism, \(\phi(\sigma(x)) = s_\alpha(\phi(x))\) and \(\phi(-y) = -\phi(y)\). Thus \(\Phi(\alpha(x, y)) = (s_\alpha(\phi(x)), -\phi(y)) = s_\alpha(\phi(x), \phi(y)) = s_\alpha(\Phi((x, y)))\).

\[
\Phi((x, y) \times_\alpha (z, t)) = \Phi((x \times z - \alpha \cdot (\sigma(t) \times y), y \times \sigma(z) + t \times x))
\]
\[
= (\phi(x) \ast_\alpha (\phi(z) - \alpha \cdot (s_\alpha(\phi(t)) \ast_\alpha (\phi(y), \phi(y)) \ast s_\alpha(\phi(z))) + \phi(t) \ast_\alpha \phi(x))
\]
\[
= (\phi(x), \phi(y)) \ast_\alpha (\phi(z), \phi(t))
\]
\[
= \Phi((x, y)) \ast_\alpha \Phi((z, t))
\]

This proves that \(\Phi\) is a Cayley morphism.
Remark: If \( \phi \) is injective, so is \( \Phi \); if \( \phi \) is surjective, so is \( \Phi \).

**Theorem 1.3:**

Let \( I \) be a set, \( ((E_i, +, \times, i, \sigma))_{i \in I} \) a family of Cayley algebras on a same commutative and associative ring with unit \((A, +, \cdot, \sigma)\), such that \((E_i, +, \cdot, \sigma)\), the product algebra of that family together with \( \sigma : E \to E_i; (x_i)_{i \in I} \mapsto (\sigma(x_i))_{i \in I} \) form a Cayley algebra and that every projection of \( \pi_i : E \to E_i \) is a Cayley (epi)morphism.

Let \( \alpha \in A \), and consider \((E_\alpha, +, \cdot, \cdot, \cdot, \sigma_\alpha)\) (respectively \((E_{i, \alpha}, +, \cdot, \cdot, \cdot, \sigma_{i, \alpha})\)) the result of applying the Cayley-Dickson process to \((E_i, +, \cdot, \cdot, \sigma_i)\) (respectively \((E_{i, \alpha}, +, \cdot, \cdot, \cdot, \sigma_{i, \alpha})\)) using \( \alpha \), \((F, \cdot, \cdot, \cdot, \cdot, s)\) the product algebra of the family \((E_{i, \alpha}, +, \cdot, \cdot, \cdot, \sigma_{i, \alpha})_{i \in I}\) and \( s : F \to F; ((x_i, y_i))_{i \in I} \mapsto (\sigma_{i, \alpha}(x_i, y_i))_{i \in I}. \) Then \( s \) is a conjugation on \( F \), \((F, \cdot, \cdot, \cdot, \cdot, s)\) is a Cayley algebra, and \( \Phi : E_\alpha \to F; ((x_i, y_i))_{i \in I} \mapsto (x_i, y_i)_{i \in I}. \) is a Cayley isomorphism from \((E_\alpha, +, \cdot, \cdot, \cdot, \sigma_\alpha)\) to \((F, \cdot, \cdot, \cdot, \cdot, s)\).

**Remark:**

The Cayley-Dickson for morphisms, and consequence theorem, simply states that, provided the product is dominated, the doubling of a product of Cayley algebras is Cayley isomorphic to the product of the doubling of the algebras.

**Proof:**

Cayley-Dickson for algebras proposition proves that \((E_\alpha, +, \cdot, \cdot, \cdot, \sigma_\alpha)\) and \((E_{i, \alpha}, +, \cdot, \cdot, \cdot, \sigma_{i, \alpha})\) are Cayley algebras. Cayley-Dickson for morphisms, and consequence theorem proves that all the \( \Pi_i : E_\alpha \to E_{i, \alpha}; (x, y) \mapsto (\pi_i(x), \pi_i(y)) \) are Cayley epimorphisms.

\( \Phi \) as defined above is trivially an isomorphism of algebra with unit. Since it is, in particular, surjective, 1A fundamental shortcoming theorem proves \( s \) is a conjugation, \((F, \cdot, \cdot, \cdot, \cdot, s)\) is a Cayley algebra and \( \Phi \) is a Cayley epimorphism. And since \( \Phi \) is actually bijective, it is a Cayley isomorphism, as noted earlier.

1.3 Unimodulars and Quadratic Residues

Let \((A, +, \cdot, \cdot, \cdot, \sigma)\) be a commutative and associative ring with unit, \((E, +, \cdot, \cdot, \cdot, \sigma)\) an alternative Cayley algebra over \( A \) (with neutral elements \( e \) and \( 0 \) for \( "+" \) and \( "\cdot" \) respectively), and \( \mathcal{A} = A \cdot e. \) We know that \((\mathcal{A}, +, \cdot, \cdot)\) is an associative and commutative ring with unit (with neutral element \( e \)), and denoting by \( \mathcal{A}^* \) the elements of \( \mathcal{A} \) which are invertible, we see at once that \((\mathcal{A}^*, \cdot)\) is an abelian group. Furthermore, the Cayley norm takes its values in \( \mathcal{A}^* \), and the Cayley norm, restricted to \( E^* \) takes its values in \( \mathcal{A}^* \).

**Proposition 1.7:**

\( \mathcal{N}_E|_{E^*} \) is a loop homomorphism.

**Proof:**

This is merely a restatement of part of 1.4 Cayley Algebroposition.

The unimodulars of \( E \), which we will denote by \( \Omega_E \), are defined to be the kernel of \( \mathcal{N}_E|_{E^*} \), i.e. \( \Omega_E = \{ x \in E^* | \mathcal{N}_E(x) = e \}. \) This definition is of course consistent with what is defined under the same name in \( \mathbb{C} \) (and \( \mathbb{H} \) and \( \mathbb{O} \)). It is known that \((10)\) that the kernel of a loop homomorphism is a normal subgroup. Among other consequences, this entails that left and right coset decompositions modulo \( \Omega_E \) exist, that they coincide, and that all cosets are equipotent (even when the loop is not finite, as is readily seen). Furthermore, if \( E^* \) happens to be finite, then \( \text{card}(E^*) = \text{card}(\Omega_E) \text{card}(E^*/\Omega_E) \) (where \( \text{card}(X) \) denotes the cardinal of the set \( X \)). Also, as \( E^* \) is a di-associative I.P. loop, then so is \( \Omega_E \), and thus the material of \((11)\) apply (note that if \( \omega \in \Omega_E \) then \( P_\omega \subset \Omega_E \)). In particular, if \( \text{card}(\Omega_E) \) is finite we have \( \forall \omega \in \Omega_E \) \( \text{card}(\Omega_E) = [\Omega_E : P_\omega] \text{card}(P_\omega) \).

The quadratic residues of \( E \), which we will denote by \( \mathcal{R}_E \), are defined to be the range of \( \mathcal{N}_E|_{E^*} \), i.e. \( \mathcal{R}_E = \{ y \in \mathcal{A}^* | \exists x \in E^* \mathcal{N}_E(x) = y \} = \{ y \in E^* | \exists x \in E^* \mathcal{N}_E(x) = y \}. \) This definition is of course consistent with the classical definition of quadratic residues \((9, 11, \ldots)\), where one has \( E = A = \mathbb{Z}/n\mathbb{Z} \) for some non-zero integer \( n \), and the conjugation is the identity. It is easy to verify\(^7\) that \((\mathcal{R}_E, \cdot, \cdot, \cdot, \cdot)\) is an abelian subgroup of \((\mathcal{A}^*, \cdot, \cdot)\). Since \( \Omega_E \) is the kernel of \( \mathcal{N}_E|_{E^*} \) and \( \mathcal{R}_E \) is its range, we also know \((10)\) that \( \mathcal{R}_E \) is loop-isomorphic (and hence group-isomorphic) to \( E^*/\Omega_E \). We have therefore proved the following:

**Proposition 1.8:**

If \( E \) is an alternative Cayley algebra and \( E^* \) is finite, then \( \mathcal{R}_E \) is group-isomorphic to \( E^*/\Omega_E \) and

\[
\text{card}(E^*) = \text{card}(\Omega_E) \text{card}(\mathcal{R}_E) \quad \text{card}(\Omega_E) | \mathcal{A}^* \quad \text{card}(\mathcal{R}_E) \quad \mathcal{A}^* \\
\]

\(^7\) The homomorphic image of a quasigroup into an associative quasigroup is a quasigroup \((10 \text{ page } 29)\); the rest is immediate.
The following is also useful.

**Proposition 1.9:**
If $E$ is an alternative Cayley algebra, $[\mathbb{Z} \to \mathcal{R}_E; n \mapsto \mathcal{N}_E(x^n)]$ is a group homomorphism; if in addition $E$ is finite, then $(\forall x \in E^*) \text{card}(\mathcal{N}_E(P_x)) = \text{card}(\mathcal{R}_E)$. §

**Proof:** Obviously, the range of the group homomorphism is $\mathcal{N}_E(P_x)$. §

Note that, in particular, if $E$ is a Galois Field, seen as an algebra over itself and with the identity for conjugation, the quadratic residues are simply the non-zero squares, and we have additional results.

**Proposition 1.10:**
Let $E$ be a Galois Field such that card($E$) is even, then card($\mathcal{U}_E$) = $1$, card($\mathcal{R}_E$) = card($E^*$) = card($E$) - 1; in particular, every non-zero element is a square. §

**Proof:** If card($E$) is even then $[E \to E; x \mapsto x^2]$ is a bijection as follows immediately from B.1.2, so $\mathcal{U}_E = \{1_F\}$. §

To deal with Galois fields of odd cardinal, we will first state a simple lemma.

**Lemma 1.1:**
Let $E$ be a Galois Field such that card($E$) is odd, let $\gamma$ be a generator of $E^*$, and let $\alpha \in E^*$. The set $\{n \in \mathbb{Z} | \alpha = \gamma^n\}$ is always non-empty. It contains either only even numbers or only odd numbers. $\alpha$ is a square if and only if $\{n \in \mathbb{Z} | \alpha = \gamma^n\}$ contains only even numbers. §

**Proof:** The first statement is a direct consequence of the existence of generators for $E^*$, and does not depend upon the parity of card($E$).

Let $n_0 \in \mathbb{Z}$ such that $\alpha = \gamma^{n_0}$; we then have $\{n \in \mathbb{N} | \alpha = \gamma^n\} = n_0 + \text{card}(F^*) \mathbb{Z}$, also irrespective of the parity of card($E$), as if $n$ verifies $\alpha = \gamma^n$ then $\gamma^n - n_0 = 1_F$, and $\gamma$ is a generator of $E^*$.

As card($E^*$) = card($E$) - 1, the statement above parity of the elements of $\{n \in \mathbb{Z} | \alpha = \gamma^n\}$ follows.

If $\alpha = \gamma^{n_0}$ with $n_0 = 2m_0$, then $\alpha = (\gamma^{m_0})^2$ so is a square. Conversely, if $\alpha$ is a square, there exists $\beta \in E^*$ such that $\alpha = \beta^2$. But there exists $m_0 \in \mathbb{Z}$ such that $\beta = \gamma^{m_0}$, so $\alpha = \gamma^{2m_0}$, and the set $\{n \in \mathbb{Z} | \alpha = \gamma^n\}$ contains the even number $2m_0$. §

**Proposition 1.11:**
Let $E$ be a Galois Field such that card($E$) is odd, then the product of two non-zero squares or two non-zero non-squares is a square and the product of a non-zero square by a non-zero non-square is a non-square.

There are exactly $(\text{card}(E) - 1)/2$ non-zero squares and exactly as many non-zero non-squares. §

**Proof:** The first statement is a trivial consequence of 1.1 Unimodulars and Quadratic Residues. Lemma 1.1

The second statement is simple the constatation that (card($E$) - 1) is even, therefore that there are as many even numbers that there are odd numbers in $\{1, \ldots, (\text{card}(E) - 1)\}$, and that each number in that set gives rise to a different element of $E^*$. §

## 2 Applications of finite alternative Cayley algebras

The main goal of this section is to build concrete examples of finite alternative Cayley algebras, and explicitly compute some important numbers related to them (such as the number of invertible elements and the number of unimodular elements).

As announced in 1.2.2 we will first consider an associative and commutative ring with unit $A$; in 2.3 $A$ will be a Galois field, and in 2.4 it will be a congruence ring. We then choose three elements in $A$, $\alpha$, $\beta$ and $\gamma$, and repeatedly apply the Cayley-Dickson procedure using these constants. More precisely, we start with $E = A$, seen as a Cayley algebra over $A$, and using the identity as conjugations. Applying the Cayley-Dickson process to $E$ using $\alpha$ yields a Cayley algebra we will denote by $E_\alpha$. Then we apply the Cayley-Dickson process to $E_\alpha$ using $\beta$ and yielding $E_{\alpha,\beta}$. Finally, we apply the Cayley-Dickson process one last time, to $E_{\alpha,\beta}$ using $\gamma$ and yielding $E_{\alpha,\beta,\gamma}$. We could keep doing this indefinitely, of course, but beyond $E_{\alpha,\beta,\gamma}$ the algebraic properties are usually so poor that there is no real incentive to do so, and no outside uses have been identified which would make it worthwhile either; technically, the fact that we could no longer rely on 1.4 Cayley Algebra Proposition 1.4 would also prove bothersome. As indicated in 1.2.2 this is a setting
in which it is sometimes advantageous to consider the successive algebras as supersets of each others, and at any rate most convenient to consider the various Cayley norms as taking their values in \( A \) (the identification of \( A \) with \( A \cdot e \), where \( e \) is the neutral element of one of the successive algebras is possible in this case, as the mapping \([x \mapsto x \cdot e]\) is bijective). In particular, this leads to the following simple expressions:

\[
\mathcal{N}_E(x) = x^2 \\
\mathcal{N}_{E_a}((x,y)) = x^2 + \alpha x^2 \\
\mathcal{N}_{E_{a,b}}((x,y,z,t)) = (x^2 + \alpha y^2) + \beta(z^2 + \alpha t^2) \\
\mathcal{N}_{E_{a,b,c}}((x,y,z,t,u,v,w,s)) = ((x^2 + \alpha y^2) + \beta(z^2 + \alpha t^2)) + \gamma((u^2 + \alpha v^2) + \beta(w^2 + \alpha s^2))
\]

### 2.1 Some common properties

We collect here some properties which are scattered in the previous section, and give some immediate consequences.

Then we know that \((E^*_a, \times)\) and \((\mathcal{U}_E, \times)\) are finite di-associative I.P. loops, that \((\mathcal{A}_a^*, \times)\) and \((\mathcal{R}_E, \times)\) are finite abelian groups, that for all \( x \in E^*, (P_x, \times) \) is an abelian group, and that we have

\[
\begin{align*}
(\forall \omega \in E^*) \quad \text{card}(E^*) = [E^* : P_{\omega}] \text{card}(P_{\omega}) \\ (\forall \omega \in \mathcal{U}_E) \quad \text{card}(\mathcal{U}_E) = [\mathcal{U}_E : P_{\omega}] \text{card}(P_{\omega}) \\ \text{card}(E^*) = \text{card}(\mathcal{U}_E) \text{card}(\mathcal{R}_E) \\ \text{card}(\mathcal{R}_E) \text{ card}(\mathcal{A}_a^*) \\ (\forall x \in E^*) \quad \text{card}(\mathcal{A}_E(P_x)) = \text{card}(\mathcal{R}_E)
\end{align*}
\]

### Proposition 2.1:

\[
\begin{align*}
(\forall x \in E^*)(\exists m \mid \text{card}(\mathcal{R}_E)) \quad \{n \in \mathbb{Z} \mid x^n \in \mathcal{U}_E\} = m\mathbb{Z}; \text{ in particular } (\forall x \in E^*) \quad x^{|\text{card}(\mathcal{R}_E)|} \in \mathcal{U}_E \\
(\forall x \in E^*)(\exists m \mid \text{card}(E^*)) \quad \{n \in \mathbb{Z} \mid x^n = e\} = m\mathbb{Z}; \text{ in particular } (\forall x \in E^*) \quad x^{|\text{card}(E^*)|} = e.
\end{align*}
\]

### Proof:
The first statement is a consequence of Proposition 1.9 Unimodulars and Quadratic Residues and the second statement uses (P3) in addition to the first statement. The third statement uses (P4) in addition. The fourth statement is a consequence of the fact that \((P_x, \times)\) is an abelian group, and the fifth is an application of (P5).

The sixth statement is also a consequence of the fact that \((P_x, \times)\) is an abelian group, and the seventh is an application of (P2). 

### 2.2 Preliminary computations

We will use the notations of Appendix B and the results therein, which are merely rewritten from Appendix B. This subsection essentially builds upon the equivalent to page 106, Exercises 19 and 20 for any finite field. We will write here \( \text{card}(F) \).

Given a finite field \( F, a_1 \in F^*, \ldots, a_r \in F^*, \) and \( b \in F \), we will denote by \( N(a_1 x_1^2 + \cdots + a_r x_r^2 = b) \), or just \( N \) as no confusion is to be feared here, the number of solutions of the equation \( a_1 x_1^2 + \cdots + a_r x_r^2 = b \) in \( F \), and by \( N(a_1 x_1^2 = c) \) the number of solutions of \( a_1 x_1^2 = c \) in \( F \). We impose \( r \geq 1 \) here. Clearly, \( N(a_1 x_1^2 + \cdots + a_r x_r^2 = b) = \sum_{a_1 + \cdots + a_r = b} N(a_1 x_1^2 = a_1) \cdots N(a_r x_r^2 = a_r) \).

If \( \text{card}(F) \) is even, \( |F| \rightarrow |F|: x \mapsto x^2 \) is a bijection and thus \( N = \sum_{a_1 + \cdots + a_r = b} 1 = q^{-1} \).

If \( \text{card}(F) \) is odd, then \( N = \sum_{a_1 + \cdots + a_r = b} \prod_{i=1}^r \chi_i(\frac{b}{a_i}) \), because \( 2 \mid (q - 1) \), and we therefore need to determine the number of multiplicative characters of \( F \) whose order divide 2. As the group of multiplicative characters of \( F \) is cyclic, of cardinal \( q - 1 \), which is even, the number of multiplicative characters of \( F \) whose order divide 2 is exactly 2: the trivial character \( \chi_F \) and one we shall call \( \chi_2 \) and which is merely a generalization of the Legendre symbol as shown below.
Proposition 2.2:
Let $F$ be a finite field such that $q = \text{card}(F)$ is odd; then

\[
\mathcal{P}_2(\alpha) = \begin{cases} 
+1 & \text{if } (\exists \beta \in F^*) \alpha = \beta^2 \\
0 & \text{if } \alpha = 0 \\
-1 & \text{otherwise.}
\end{cases}
\]

is a multiplicative character of order two and $\mathcal{P}_2(-1_F) = (-1)^{\frac{q-1}{4}}$; in particular $\mathcal{P}_2 \neq \epsilon_F$.

**Proof:** It is immediate to check (using \textbf{Unimodulars and Quadratic Residues lemma: 1.1}) that $\mathcal{P}_2$ is a multiplicative character, and since it is trivial that $(\forall \alpha \in F^*) (\mathcal{P}_2(\alpha))^2 = 1$, its order divides 2. By \textbf{Unimodulars and Quadratic Residues proposition: 1.1} there exists non-zero non-squares, $\mathcal{P}_2$ does take the value $-1$, and thus its order can’t be one, so is exactly two. At the same time, $\mathcal{P}_2 \neq \epsilon_F$.

The equation $x^2 = -1_F$ in $F$ has solutions if and only if $(-1_F)^\frac{q-1}{4} = 1_F$ (according to \textbf{B.1.2} in this case $d = \text{PGCD}(2, q - 1) = 2$). Hence $-1_F$ is square if and only if $(-1)^{\frac{q-1}{4}} = 1$ (which is equivalent to $q \equiv 1 \mod 4$), and thus $\mathcal{P}_2(-1_F) = (-1)^{\frac{q-1}{4}}$.

We can now see that

\[
N = \sum_{\alpha_1 + \ldots + \alpha_r = b} \prod_{i=1}^r \left[ \epsilon_F \left( \frac{\alpha_i}{a_i} \right) + \mathcal{P}_2 \left( \frac{\alpha_i}{a_i} \right) \right]
= \sum_{\alpha_1 + \ldots + \alpha_r = b} \sum_{\omega \in \mathcal{P}(\{1, \ldots, r\})} \mathcal{P}_2^{\omega(1)} \left( \frac{\alpha_1}{a_1} \right) \cdots \mathcal{P}_2^{\omega(r)} \left( \frac{\alpha_r}{a_r} \right)
\]

where $\mathcal{P}(X)$ denotes the power set of $X$, and we have $\mathcal{P}_2^0 = \epsilon_F$ as in any (multiplicative) group.

Therefore

\[
N = \sum_{\omega \in \mathcal{P}(\{1, \ldots, r\})} \left[ \mathcal{P}_2^{\omega(1)} \left( a_1 \right) \cdots \mathcal{P}_2^{\omega(r)} \left( a_r \right) \right] \sum_{\alpha_1 + \ldots + \alpha_r = b} \mathcal{P}_2^{\omega(1)} \left( a_1 \right) \cdots \mathcal{P}_2^{\omega(r)} \left( a_r \right)
\]
as both $\epsilon_F$ and $\mathcal{P}_2$ are multiplicative characters (so $\epsilon_F \left( \frac{\omega}{a} \right) = \epsilon_F(\alpha) \epsilon_F(a^{-1})$ and $\mathcal{P}_2 \left( \frac{\omega}{a} \right) = \mathcal{P}_2(\alpha) \mathcal{P}_2(a^{-1})$) and are of an order dividing 2 (so $\epsilon_F(a_i^{-1}) = \epsilon_F(a_i)$ and $\mathcal{P}_2(a_i^{-1}) = \mathcal{P}_2(a_i)$).

We now come to a point where we must distinguish between $b = 0$ and $b \neq 0$.

\[
N = \begin{cases} 
\sum_{\omega \in \mathcal{P}(\{1, \ldots, r\})} \left[ \mathcal{P}_2^{\omega(1)} \left( a_1 \right) \cdots \mathcal{P}_2^{\omega(r)} \left( a_r \right) \right] \sum_{\alpha_1 + \ldots + \alpha_r = b} \mathcal{P}_2^{\omega(1)} \left( a_1 \right) \cdots \mathcal{P}_2^{\omega(r)} \left( a_r \right) & \text{if } b = 0 \\
\sum_{\omega \in \mathcal{P}(\{1, \ldots, r\})} \left[ \mathcal{P}_2^{\omega(1)} \left( b \right) \right] \sum_{\alpha_1 + \ldots + \alpha_r = b} \mathcal{P}_2^{\omega(1)} \left( a_1 \right) \cdots \mathcal{P}_2^{\omega(r)} \left( a_r \right) & \text{if } b \neq 0
\end{cases}
\]

Using the results in \textbf{B.3} we find that

\[
N = \begin{cases} 
q^{r-1} + \mathcal{P}_2(a_1 \cdots a_r) J(\mathcal{P}_2, \ldots, \mathcal{P}_2) & \text{if } b = 0 \\
q^{r-1} + \mathcal{P}_2(b) \mathcal{P}_2(a_1 \cdots a_r) J(\mathcal{P}_2, \ldots, \mathcal{P}_2) & \text{if } b \neq 0
\end{cases}
\]

We now must also distinguish whether $r$ is even or odd.

If $r$ is odd, then $\mathcal{P}_2^{-r-1} = \epsilon_F$ and $\mathcal{P}_2^{-r} = \mathcal{P}_2 \neq \epsilon_F$, so $J(\mathcal{P}_2, \ldots, \mathcal{P}_2) = 0$ and $g(\mathcal{P}_2)^r = J(\mathcal{P}_2, \ldots, \mathcal{P}_2) g(\mathcal{P}_2^r)$, and since $\mathcal{P}_2 = \mathcal{P}_2$, $J(\mathcal{P}_2, \ldots, \mathcal{P}_2) = g(\mathcal{P}_2)^{-1}$. But $\mathcal{P}_2^{-1} = \mathcal{P}_2 \mathcal{P}_2(-1_F)$ and $g(\mathcal{P}_2)^2 = \mathcal{P}_2(-1_F)q$ and $J(\mathcal{P}_2, \ldots, \mathcal{P}_2) = \mathcal{P}_2(-1_F)q^{\frac{r-1}{4}}$. Combining this with the fact that $\mathcal{P}_2(-1_F) = (-1)^{\frac{q-1}{4}}$ we finally find that $N = q^{r-1}$ if $b = 0$, and $N = q^{r-1} + \mathcal{P}_2(b) \mathcal{P}_2(a_1 \cdots a_r) (-1)^{\frac{q-1}{4}} q^{\frac{r-1}{4}}$ otherwise.
If $r$ is even, $\mathcal{K}_2^{-1} = \mathcal{K}_2 \neq \epsilon_F$ and $\mathcal{K}_2^2 = \epsilon_F$, so $J_0(\mathcal{K}_2, \ldots, \mathcal{K}_2) = \mathcal{K}_2(-1_F) (q - 1) J(\mathcal{K}_2, \ldots, \mathcal{K}_2)$ and $J(\mathcal{K}_2, \ldots, \mathcal{K}_2) g(\mathcal{K}_2^{-1}) = g(\mathcal{K}_2)^{-1}$. Together these give $J_0(\mathcal{K}_2, \ldots, \mathcal{K}_2) = \mathcal{K}_2(-1_F) (q - 1) g(\mathcal{K}_2)^{-2} = \mathcal{K}_2(-1_F) (q - 1) [\mathcal{K}_2(1_F) g(\mathcal{K}_2)]^{-1}$. Furthermore $J(\mathcal{K}_2, \ldots, \mathcal{K}_2) = \mathcal{K}_2(-1_F) J(\mathcal{K}_2, \ldots, \mathcal{K}_2)$ and $g(\mathcal{K}_2)^{-1} = g(\mathcal{K}_2)^{-1} [\mathcal{K}_2(-1_F) q]^{-1}$. Also, $g(\mathcal{K}_2)^{-1} = g(\mathcal{K}_2)^{-2}$ and $J(\mathcal{K}_2, \ldots, \mathcal{K}_2) = -\mathcal{K}_2(-1_F) g(\mathcal{K}_2)^{-2} = -\mathcal{K}_2(-1_F) \mathcal{K}_2(-1_F) q)^{-1}$ so finally $N = q^{-1} + \mathcal{K}_2(a_1 \cdots a_r)(-1)^{\frac{q+1}{2}} q^{-1} (q - 1) q^{-1}$ if $b = 0$, and $N = q^{-1} - \mathcal{K}_2(b) \mathcal{K}_2(a_1 \cdots a_r)(-1)^{\frac{q+1}{2}} q^{-1}$ otherwise.

We summarize the value of $N(a_1 x_1^2 + \cdots + a_r x_r^2) = b$ as follows:

| $q$ even | $q$ odd | $q^{-1}$ | $q^{-1}$ |
|----------|----------|----------|----------|
| $b = 0$  | $r$ even  | $q^{-1}$  | $q^{-1}$  |
| $b \neq 0$ | $r$ odd   | $q^{-1} + \mathcal{K}_2(a_1 \cdots a_r) (-1)^{\frac{q+1}{2}} q^{-1}$ |
|         |           | $q^{-1} + \mathcal{K}_2(b) \mathcal{K}_2(a_1 \cdots a_r) (-1)^{\frac{q+1}{2}} q^{-1}$ |
|         |           | $q^{-1} - \mathcal{K}_2(b) \mathcal{K}_2(a_1 \cdots a_r) (-1)^{\frac{q+1}{2}} q^{-1}$ |

### 2.3 Cayley-Galois constructs

As announced at the begining of this section, we consider here $A = F_q$, with $F_q$ be a Galois field of cardinal $q$. We will designate by $F_{q, \alpha}$ etc. what was denoted by $E_{\alpha}$ etc. in the introduction to this section.

We now assume $\alpha \beta \gamma \neq 0$, for brevity’s sake. The preliminary computations above yield the table below.

Note that $F_{q, \alpha, \beta}$ and $F_{q, \alpha, \beta, \gamma}$ never are fields, but that when $q$ is even, $F_{q, \alpha}$ is never a field whereas when $q$ is odd, $F_{q, \alpha}$ sometimes is a field (it is a field if and only if $\mathcal{K}_2(\alpha) (-1)^{\frac{q+1}{2}} = -1$).

It is interesting to note that, when $q$ is even, $F_q$ is of characteristic 2, and thus we always have $-x = +x$, so, as a consequence of Cayley-Dickson for algebraic proposition 1.4 and the form of the conjugation resulting from the Cayley-Dickson process, $F_q$, $F_{q, \alpha}$, $F_{q, \alpha, \beta}$ and $F_{q, \alpha, \beta, \gamma}$ and actually any successor in the doubling scheme, are commutative and associative (irrespective of the value of $\alpha$, $\beta$, $\gamma$ or their successors), and the conjugation at each step is the identity.

| $(\alpha \beta \gamma \neq 0)$ | $q$ odd | $q$ even |
|------------------------------|----------|----------|
| card($\mathcal{K}^*$)        | $q - 1$  | $q - 1$  |
| card($F_q^*$)                | $q - 1$  | $q - 1$  |
| card($F_{q, \alpha}$)        | $q^2 - [q + \mathcal{K}_2(\alpha) (q - 1) (-1)^{\frac{q+1}{2}}] q^2 - q$ |
| card($F_{q, \alpha, \beta}$)  | $q^4 - [q^3 + (q - 1) q] q^4 - q^3$ |
| card($F_{q, \alpha, \beta, \gamma}$) | $q^8 - [q^7 + (q - 1) q^3] q^8 - q^7$ |
| card($\mathcal{U}_{F_q}$)     | $2$      | $1$      |
| card($\mathcal{U}_{F_{q, \alpha}}$) | $q - \mathcal{K}_2(\alpha) (-1)^{\frac{q+1}{2}} q$ |
| card($\mathcal{U}_{F_{q, \alpha, \beta}}$) | $q^3 - q$ |
| card($\mathcal{U}_{F_{q, \alpha, \beta, \gamma}}$) | $q^7 - q^3$ | $q^7$ |
2.4 Cayley integers

As announced at the beginning of this section, we consider here $A = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, with $n \in \mathbb{N}$, not necessarily prime (but still $n \geq 1$, of course). We will designate by $\mathbb{Z}_{n;\alpha}$ etc. what was denoted by $E_\alpha$ etc. in the introduction to this section. Elements of these Cayley algebras will be dubbed Cayley integers.

The situation here is somewhat more complicated than that of 2.3

Let $n = \prod_{i=1}^k n_i$ where the $n_i \in \mathbb{N}^*$ are relatively prime, and let $\pi_{n,n_1} : \mathbb{Z}_n \rightarrow \mathbb{Z}_{n_1}$ such that if $X \in \mathbb{Z}_n$ and $x \in \mathbb{Z}_{n_1}$, then $\pi_{n,n_1}(X) = Y$ such that if $y \in \mathbb{Z}_n, y \in \mathbb{Z}_n$ then $y \equiv x \mod n_i$. The Chinese Remainder Theorem informs us that, as algebras with unit over $A = \mathbb{Z}_{n_1}, \mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ are isomorphic, via $\pi_n : \mathbb{Z}_n \rightarrow \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$; $x \mapsto (\pi_{n,n_1}(x), \ldots, \pi_{n,n_k}(x))$. By using the identity on $\mathbb{Z}$ and the $\mathbb{Z}_{n_i}$ as conjugations, $\mathbb{Z}_n$ and the $\mathbb{Z}_{n_i}$ are Cayley algebras on the same ring $A = \mathbb{Z}_n$, the $\pi_{n,n_i}$ are Cayley epimorphisms. 1.1A fundamental shortcoming theorem.1.1 now informs us that, using the identity on $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ as conjugations, $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is also a Cayley algebra over $A = \mathbb{Z}_n$. In fact we see that $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is Cayley-isomorphic to $\mathbb{Z}$ (it is isomorphic as an algebra with unit by the Chinese remainder theorem, and the conjugation is the identity on both sides). Therefore, 1.3Cayley-Dickson for morphisms, and consequence theorem.1.3 proves that, given $\alpha \in \mathbb{Z}_n, \mathbb{Z}_{n;\alpha} \times \cdots \times \mathbb{Z}_{n;\alpha}$ is a Cayley algebra which is Cayley-isomorphic with the result of applying the Cayley-Dickson procedure to $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ using $\alpha$, i.e., Cayley-isomorphic to $\mathbb{Z}_{n_1;\alpha}$, as detailed in 1.3Cayley-Dickson for morphisms, and consequence theorem.1.3.

Finally, to compute the cardinals of interest to us in this segment, we simply have to do so for integers $s$ of the form $n = p^s$, with $p$ prime and $s \geq 1$.

If $s = 1$, then we are face to a special case of the Cayley-Galois constructs we have just studied.

The elements of $\mathbb{Z}_{p^s}$ which are not invertible, are, as is well-known, exactly those of the form $\bar{x} = kp$, for some integer $k$, where $\bar{a}$ represents the congruence class of the integer $a$ modulo $p^s$, a convention we will use throughout this segment (and not to be confused with the notation for the conjugation). Hence there are $p^{s-1}$ elements which are not invertible, and thus $\text{card}(\mathbb{Z}_{p^s}) = \text{card}(\mathbb{Z}_p^*) = p^{s-1}(p-1)$.

Assume $p \neq 2$. We will further assume, for brevity’s sake, that $\alpha \in \mathbb{Z}_{p^s}^*, \beta \in \mathbb{Z}_{p^s}^*, \gamma \in \mathbb{Z}_{p^s}^*$. Let us first determine the number of invertible elements. As we will see, finding the number of unimodulars will use essentially the same techniques.

More to the point, we will determine the number of elements which are not invertible, and assume $s > 1$. 1.4Cayley Algebra proposition.1.4 informs us that an elements of $\mathbb{Z}_{p^s}$ (respectively $\mathbb{Z}_{p^s;\alpha}, \mathbb{Z}_{p^s;\alpha,\beta}, \mathbb{Z}_{p^s;\alpha,\beta,\gamma}$) is not invertible if and only if its Cayley norm is not invertible in $\mathbb{N}$, which here is Cayley-isomorphic to $\mathbb{Z}_{p^s}$. Taking into account the expression of the Cayley norm on $\mathbb{Z}_{p^s;\alpha}$ etc. given at the beginning of this section,
we see that finding the non-invertible element of \( \mathbb{Z}_p^{*} \) amounts to finding couples etc. of elements of \( \mathbb{Z}_p^{*} \) such that \( x_i^2 + \alpha x_j^2 \) etc. is not invertible in \( \mathbb{Z}_p^{*} \). As an element \( x_i \in \mathbb{Z}_p^{*} \) is completely characterized by the integer \( x_i \in \{0, \ldots, p^s - 1\} \) such that \( x_i \equiv x_i \mod p \), we are (unsurprisingly) led to solving a congruence equation in \( \mathbb{Z} \). We will find it convenient to write \( x_i \) in the basis \( p \).

Finding the non-invertible elements of \( \mathbb{Z}_p^{*} \), for instance, is therefore equivalent to finding all the \( (x_1;0, \ldots, x_{1;1}, x_{2;0}, \ldots, x_{2;1}) \in \mathbb{Z}^{2s} \) such that there exists \( k \in \mathbb{Z} \) such that

\[
(x_1;0 p^0 + x_{1;1} p^1 + \cdots + x_{1;1} p^{s−1})^2 + (α x_0 p^0 + α_1 x_1 p^1 + \cdots + α_{s} p^{s−1})(x_{2;0} p^0 + x_{2;1} p^1 + \cdots + x_{2;1} p^{s−1})^2 \equiv k p \mod p^s
\]

Solving this kind of equation is classical and elementary (11).

In the case of \( \mathbb{Z}_p^{*} \), it is even especially simple, as we see that in that particular case our equation is equivalent to \( x_1^2, x_0^2, \cdots, x_2^2, \cdots, x_s^2 \equiv 0 \mod p \), which we have already solved in (29).

In brief, there are \( p + \left( \frac{α}{p} \right) (p - 1) (−1)^s−1 \) possible choices for \( (x_1;0, x_{2;0}) \), with \( \left( \frac{α}{p} \right) \) denoting the Legendre symbol of \( x \) with respect to \( p \), and \( p \) choices for every other variable. Hence \( \text{card}(\mathbb{Z}_p^{*}) = p^{2s} - p^{(s−1)} \left( p + \left( \frac{α}{p} \right) (p - 1) (−1)^s−1 \right) \).

In the case of \( \mathbb{Z}_p^{*,0,β} \) (and \( \mathbb{Z}_p^{*,0,β,γ} \)), the method is identical, and the results are summarized in the table below.

Let us now turn to the case of the unimodulars. By definition, this means we look for the elements for which the Cayley norm is equal to \( 1 \in \mathbb{Z}_p^{*} \).

Let us first determine the number of unimodulars in \( \mathbb{Z}_p^{*} \). By the same token as above, we are led to solve the equation

\[
(x_1;0 p^0 + x_{1;1} p^1 + \cdots + x_{1;1} p^{s−1})^2 \equiv 1 \mod p^s
\]

This is of course equivalent to the following expanded equation

\[
\sum_{j=0}^{s−1} \left( \sum_{k=0}^{j} x_{1;k} x_{1;j−k} \right) p^j \equiv 1 \mod p^s
\]

The first thing we note is that (U) implies

\[
x_{1;0}^2 \equiv 1 \mod p^1
\]

(U1)

If \( s = 1 \) these equations are actually equivalent, of course, but otherwise (U) is equivalent to the conjunction of (U1) and (U).

We know (by B.1.2) that the solutions to (U) are \( x_{1;0} = 1 \) and \( x_{1;0} = p − 1 \), both of which happen to be invertible in \( \mathbb{Z}_p^{*} \). If \( s = 1 \), we are done, as noted above, otherwise, (U1) is a constraint which our solutions must verify.

The second thing we note is that (U) also implies

\[
[x_{1;0}^2] p^0 + [2 x_{1;0} x_{1;1}] p^1 \equiv 1 \mod p^2
\]

(U2)

As above, if \( s = 2 \), (U) is equivalent to (U2), but even if such is not the case (U) is equivalent to (U1) and (U1) and (U2). We note also that (U1) implies that \( x_{1;0}^2 \equiv 1 + k_0 p^1 \mod p^2 \), for some integer \( k_0 \) dependent only on \( x_{1;0} \). But this means that (U1) and (U2) is equivalent to the conjunction of (U1) and \( 2 x_{1;0} x_{1;1} \equiv k_0 \mod p^1 \). Since \( x_{1;0} \) always is invertible in \( \mathbb{Z}_p \) for the solutions of (U), this implies a single possible value for \( x_{1;1} \) given a value for \( x_{1;0} \).

We may iterate this reasoning until we have reached the value of \( s \). At each step save the first, the value of \( x_{1;1} \) can be seen to be uniquely determined by the values of \( x_{1;0}, \ldots, x_{1;1} \). Finally, \( \text{card}(\mathbb{Z}_p^{*}) = 2 \).

We briefly sketch the computation of \( \text{card}(\mathbb{Z}_p^{*,0,β}) \), which is essentially similar to the above, albeit yet more computationally unpleasant. \( \text{card}(\mathbb{Z}_p^{*,0,β}) \) and \( \text{card}(\mathbb{Z}_p^{*,0,β,γ}) \) appear in the table below.

To compute that number, we have to solve

\[
(x_{1;0} p^0 + x_{1;1} p^1 + \cdots + x_{1;1} p^{s−1})^2 + (α x_{1;0} p^0 + α_1 x_{1;1} p^1 + \cdots + α_{s−1} p^{s−1})(x_{2;0} p^0 + x_{2;1} p^1 + \cdots + x_{2;1} p^{s−1})^2 \equiv 1 \mod p^s
\]

(UU)
The first step, as above, it to note that \((UU)\) is equivalent to the conjunction of itself and 
\[x_1^2 + \alpha_0 x_2^2 \equiv 1 \mod p\]  
(UU1)

We have solved this equation too, in \(2.3\) and we know it has \(p - \left(\frac{\alpha}{p}\right)(-1)^{\frac{k-1}{2}}\) solutions, all of which evidently differ from \((0, 0)\).

The second step, which is only necessary if \(s \geq 2\), is to see that \((UU)\) is equivalent to the conjunction of itself, \((UU)\), and 
\[\left[x_1^2 + \alpha_0 x_2^2, (x_1, x_2)\right] \equiv 1 \mod p^2\]  
(UU2)

Since we know that \(x_1^2 + \alpha_0 x_2^2 \equiv 1 + k_0 p^2 \mod p^2\) for some integer \(k_0\) dependant only upon \(x_1, 0\) and \(x_2, 0\), we see that we actually have to solve is \(2x_1x_1, 0 + 2\alpha_0 x_2, 0 x_2, 1 + \alpha_1 x_2^2, 0 \equiv k_0 - \alpha_1 x_2^2, 0\). Hence for each \((x_1, 0, x_2, 1)\) solution of \((UU)\) there are \(p^2\) choices for \((x_1, 1, x_2, 1)\).

We may iterate this reasoning until we have reached the value of \(s\). At each step save the first, there are \(p\) values of \((x_{1,i}, x_{2,i})\) which are uniquely determined by the values of \(x_1, 0, \ldots, x_{1,i-1}, x_2, 0, \ldots, x_{2,i-1}\). Finally, \(\text{card}(U_{Z\alpha}) = p^{s-1}\left[p - \left(\frac{\alpha}{p}\right)(-1)^{\frac{k-1}{2}}\right]\).

We now assume \(p = 2\), where things keep getting ugly. We will also assume, for brevity’s sake, that \(\alpha \in \mathbb{Z}_{2^s}, \beta \in \mathbb{Z}_{2^s}, \gamma \in \mathbb{Z}_{2^s}\). Note that \(\alpha \in \mathbb{Z}_{2^s}\) is equivalent to \(\alpha_0 \equiv 1 \mod 2\), and likewise for \(\beta\) and \(\gamma\). Recall also we have already shown that \(\text{card}(U_{Z\alpha}) = 2^{s-1}\).

We again look for elements of \(\mathbb{Z}_{2^s}\) etc. which are not invertible. As above, we look for them in by identifying members of their class in \(\{0, \ldots, 2^s - 1\}\), and writing them in base 2. However in this case the expansion of the equation is different, precisely because the number 2 which is the base of expansion also is a coefficient of elements in the square of a sum!

For \(\mathbb{Z}_{2^s}\), we have to solve
\[(x_1, 0) 2^1 + x_{1,1} 2^1 + \cdots + x_{1,s-1} 2^{s-1}\]  
\[\alpha_0 2^0 + \alpha_1 2^1 + \cdots + \alpha_{s-1} 2^{s-1}\]  
\[\equiv k 2 \mod 2^s\]

for all integers \(k\).

Expanding, we find that this is equivalent to \(x_1^2 + \alpha_0 x_2^2 \equiv 0 \mod 2\), and we have determined that \(\text{card}(U_{Z\alpha}) = 2^{s-1}\).

We proceed likewise for \(\mathbb{Z}_{2^s}\) and give the relevant values in the table below.

We now turn to the unimodulars.

In the case of \(\mathbb{Z}_2\), the relevant numbers have been computed already in \(2.3\).

Since \(\mathbb{Z}_{2^s}\) has just four elements, we list them explicitly and discover that there are exactly 2 unimodulars.

We note that the Cayley norm of an element of \(\mathbb{Z}_{2^s}\) is always either 0 or 1 (and in particular all invertibles are unimodular), and never either 2 or 3 (the only remaining possibilities).

If \((x, y) \in \mathbb{Z}_{2^s}\), we know that (with our above conventions) \(N_{\mathbb{Z}_{2^s}}((x, y)) = N_{\mathbb{Z}_2}(x) + (\alpha_0 + \alpha_1 2)N_{\mathbb{Z}_2}(y)\), and we have assumed here that \(\alpha_0 = 1\).

Therefore, if \(\alpha_1 = 0\) the Cayley norm of an element of \(\mathbb{Z}_{2^s}\) can only take the value \(\bar{0}\), \(\bar{1}\) or \(\bar{2}\), hence here too unimodulars and invertible are the same elements (and so \(\text{card}(U_{\mathbb{Z}_{2^s}}) = 8\).

On the other hand, if \(\alpha_1 = 1\) (the only other possible value), then we have \(N_{\mathbb{Z}_{2^s}}((\bar{1}, 0)) = \bar{1}\) and \(N_{\mathbb{Z}_{2^s}}((\bar{0}, 1)) = 3\), and we see that \(\mathbb{Z}_{2^s} \subset \{1, 3\} \subset \mathbb{R}_{\mathbb{Z}_{2^s}} \subset \mathbb{Z}_{2^s}\), where \(\mathbb{R}_{\mathbb{Z}_{2^s}}\) is the ring of integers of \(\mathbb{Z}_{2^s}\).

On \(\mathbb{Z}_{2^s}\), for example, \(\text{card}(\mathbb{Z}_{2^s}) = 4\) and that \(\mathbb{R}_{\mathbb{Z}_{2^s}} = \{a \mid a \in \mathbb{Z}, a \equiv 1 \mod 8\} = \{\bar{1} + k \bar{8} \mid k \in \mathbb{Z}\}\). On the other hand, an element of \(\mathbb{Z}_{2^s}\) is not invertible if and only if it is of the form \(k \bar{2}\), with \(0 \leq k < 2^{s-1}\), and these have squares of the form \(k^2 \bar{4}\), and these
the methods we have developed above, we see that actually the values \( \bar{y} \in \mathbb{Z}_2^2 \) are all of the form \( \bar{y} = \bar{y}^2 \). By considering elements of the form \( (x, 2y) \), with \( x \in \mathbb{Z}_2^2 \), we see that \( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) \) is a multiple of 2 (and \( x, y \notin \mathbb{Z}_2^2 \)) if \( x \in \mathbb{Z}_2^2 \), \( y \notin \mathbb{Z}_2^2 \) or \( x \notin \mathbb{Z}_2^2 \) and \( y \in \mathbb{Z}_2^2 \), then \( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) \) is of the form \( 1 + k^2 4 + l 8 \) for some integers \( k \) and \( l \), which means that they are of the form \( 1 + m 8 \) or \( 5 + m 8 \) for some integer \( m \), according to whether \( k \) is even or odd.

Therefore card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = 2^{s+1} \).

Assume now \( \alpha_1 = 0, \alpha_2 = 1 \) (and still \( \alpha_0 = 1 \)). We thus have \( \alpha = 3 + k_0 \bar{8} \) for some integer \( k_0 \).

If \( s = 3 \), by listing all the elements explicitly, we see that the Cayley norm can reach the value 5 (and 1, respectively), but not the values \( 3 \) and 7. Hence here card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = 2^{s+1} \).

Assume therefore that \( s \geq 4 \).

By using the projection \( \pi_{2^s} : \mathbb{Z}_2^s \to \mathbb{Z}_{2^s} \) we see that \( \pi_{2^s}((x, y)) \) is invertible in \( \mathbb{Z}_{2^s} \) and we can, for any value of \( t \) in \{0, \ldots, 2^{s-3} - 1\}, solve the equation \( l(5 + k_0 \bar{8}) \equiv t - k_0 \mod 2^{s-3} \) yielding \( l_t \). For each \( l_t \), we know that \( 1 + l_t 8 \in \mathcal{N}_{\mathbb{Z}_2^2}, \bar{8} \) and therefore that there exists \( y_t \in \mathbb{Z}_2^2 \), such that \( y_t^2 = bar1 + l_t 8 \). But then \( \alpha \equiv y_t^2 = 5 + l 8 \), so all elements of the form \( 5 + l 8 \) are reached by the Cayley norm.

A discussion similar to the above shows that no value of the form \( 3 + k 8 \) or \( 7 + k 8 \) can be reached by the Cayley norm. Therefore, as above, card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = 2^{s+1} \).

Assume now \( \alpha_1 = 1, \alpha_2 = 0 \) (and still \( \alpha_0 = 1 \)). We thus have \( \alpha = 3 + k_0 \bar{8} \) for some integer \( k_0 \).

We know all values of the form \( 1 + k 8 \) can be reached by the Cayley norm, and by considering elements of the form \( (x, 2) \) with \( x \in \mathbb{Z}_2^2 \) we see all values of the form \( 5 + k 8 \) can also be reached. If \( s = 3 \) we see that the values \( 3 \) and \( 7 \) are reached (by \((0, 1) \) and \((2, 1) \) respectively, for instance). If \( s \geq 4 \), solving the congruence equations \( l(3 + k_0 \bar{8}) \equiv t \mod 2^{s-3} \) proves as above that all values of the form \( 5 + k 8 \) are reached too. Since \( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) \) is a subgroup of \( \mathbb{Z}_2^2 \), we see that we actually have \( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = \mathbb{Z}_2^2 \). Therefore card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = 2^s \).

The final case, \( \alpha_1 = 1, \alpha_2 = 1 \) (and still \( \alpha_0 = 1 \)), for which we have have \( \alpha = 7 + k_0 \bar{8} \) for some integer \( k_0 \) is entirely similar to the above, and we thus find that card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y)) = 2^s \).

We will gloss over the unimodulars of \( \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4, \mathbb{Z}_2^5, \mathbb{Z}_2^6 \) and \( \mathbb{Z}_2^7, \mathbb{Z}_2^8, \mathbb{Z}_2^9, \mathbb{Z}_2^{10} \) to find the values we want we either investigate the explicit list of the elements or we adapt what follows. We therefore assume \( s \geq 4 \).

If either \( \alpha = 3 + k 4 \) or \( \beta = 3 + k 4 \), by considering elements of the form \((x, y, 0, 0)\) or \((x, 0, y, 0)\) and using the methods we have developed above, we see that actually \( \mathcal{N}_{\mathbb{Z}_2^2}((x, y, u, uv)) = x^2 + y^2 + \frac{2}{6} \bar{v} \) for some integer \( m \), and that we can immediately see that card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y, u, uv)) = 2^{7s} \).

Since by considering elements of the form \((x, y, z, t, 0, 0, 0, 0)\) the Cayley norm reaches all of \( \mathbb{Z}_2^2 \), we can immediately see that card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y, z, t, 0, 0, 0, 0)) = 2^{8s} \).

Therefore, as above, card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y, z, t, 0, 0, 0, 0)) = 2^{8s} \).

We can thus see that \( 3, 7 \) or 13 is reached, and reasoning as above, card(\( \mathcal{N}_{\mathbb{Z}_2^2}((x, y, z, t, 0, 0, 0, 0)) = 2^{8s} \).
2.5 Further results

We collect here some remarks and results about the objects we have studied in this section and relatives thereof, which are too few to warrant a section of their own.

**Proposition 2.3:**
Let \( p \) be an odd prime, \( k \geq 1 \) an integer, \( q = p^k \) and \( \alpha \in F_q^* \) such that \( \mathcal{X}_2(\alpha)(-1)^{\frac{q-1}{2}} = -1 \). Then \( F_q^* \) is cyclic and in particular there is an element of order \( q^2 - 1 \). Furthermore, if \( z_0 \) is a generator of \( F_q^* \), then \( N_{F_q^*}(z_0) \) is a generator of \( \mathcal{R}_{F_q^*} \).

**Proof:** Under these hypotheses, \( F_q^* \) is the finite field of cardinal \( q^2 \), hence its invertible elements are cyclic. Since \( \text{card}(F_q^*) = q^2 - 1 \) this means that there is an element of order \( q^2 - 1 \). The last assertion is trivial by contraposition.

**Proposition 2.4:**
Let \( p \) be an odd prime, \( k \geq 1 \) an integer, \( q = p^k \) and \( \alpha \in F_q^* \) such that \( \mathcal{X}_2(\alpha)(-1)^{\frac{q-1}{2}} = +1 \). Then the order of any element of \( \in F_q^* \) divides \( q - 1 \), and \( F_q^* \) is not cyclic; however \( U_{F_q^*} \) is cyclic and isomorphic to \( F_q^* \).

**Proof:** Under these hypotheses, \( F_q^* \), \( U_{F_q^*} \) and \( \mathcal{R}_{F_q^*} \) are abelian groups, and \( \text{card}(U_{F_q^*}) = \text{card}(\mathcal{R}_{F_q^*}) = q - 1 \). Therefore, given any \( (n, u) \in \text{card}(U_{F_q^*}) \times \text{card}(\mathcal{R}_{F_q^*}) \), \( (n, u)^{q-1} = (1, 1) \), with \( 1 \) the neutral element for the multiplication in \( F_q^* \). The first statement then results from Unimodulars and Quadratic Residues Proposition 1.8. The second statement then results from the fact that \( q - 1 < (q - 1)^2 = \text{card}(F_q^*) \).

Note that \( \mathcal{X}_2(\alpha)(-1)^{\frac{q-1}{2}} = +1 \) is equivalent to the existence of \( \omega \in F_q^* \) such that \( \omega^2 = -\alpha \). Therefore \( (x, y) \in \mathcal{R}_{F_q^*} \Leftrightarrow (x + \omega y)(x - \omega y) = e \). Let now \( z = x + \omega y \in F_q^* \). We compute immediately that \( (x, y) = 2^{-1}(z + z^{-1}, \omega^{-1}(z - z^{-1})) \), with \( 2 = 1 + 1 \) (which is invertible because \( q \) is odd). Better yet, we immediately verify that \( [F_q^* \rightarrow U_{F_q^*}; z \mapsto 2^{-1}(z + z^{-1}, \omega^{-1}(z - z^{-1}))] \) is a group isomorphism.
Appendices

A Some common and not-so-common algebraic structures

Algebra is rife with named structures. Unfortunately, authors tend to disagree about what a given name refers to precisely (always with good reason, but easily resulting in confusion nonetheless; \[1\]), even for the many structures found in most introductory-level material (\[15 \ldots\]), and even for reference material (\[2 13 16\]). This appendix is therefore here to describe, precisely, the meaning we give to various terms used in the main text of this document. We have included in this discussion a few other familiar structures to serve as “landmarks”, but this is not a comprehensive catalogue of all algebraic structures (not even of the most common ones). We will endeavor to follow what we feel is current usage (\[12\]), though we take this opportunity to perform a few generalizations. One particular emphasis of this appendix is the careful treatment of associativity, in some (again, not all) of its various shades.

A.1 Structures involving only one law

A.1.1 A protean foundation: the magma

Given a set \( S \), an internal law on \( S \) (or simply a law on \( S \)) is a function \( f \) defined on all of \( S \times S \) with values in \( S \). A magma is a structure \((S,f)\) where \( S \) is a (perhaps empty) set and \( f \) is a law on \( S \). A groupoid is a non-empty magma. Given a set \( T \subset S \), if \( f \) source-restricted to \( T \times T \) takes its values in \( T \), we call the structure \((T,f|_{T\times T})\) a submagma of \((S,f)\). Given any (non-empty) collection \((T_i)_{i \in I}\) of subsets of \( S \) such that \( f \) source-restricted to \( T_i \times T_i \) takes its values in \( T_i \) for all \( i \in I \), if we name \( U = \bigcap_{i \in I} T_i \), we find that \((U,f|_{U\times U})\) is a submagma of \((S,f)\) and of every \((T_i,f|_{T_i\times T_i})\). Therefore, given any (perhaps empty) set \( X \subset S \), there exist a (perhaps empty) set \( Y \subset S \) such that \( X \subset Y \), \((Y,f|_{Y\times Y})\) is a submagma of \((S,f)\), and such that if \((T,f|_{T\times T})\) is a submagma of \((S,f)\) such that \( X \subset T \), then \( Y \subset T \) (in other words, \( Y \) is the smallest submagma of \( S \) containing \( X \)); we call this the submagma generated by \( X \). Given a collection \((a_j)_{j \in J}\) of elements of \( S \), we call the submagma generated by the \( a_j \), the submagma generated by \( \{a_j \mid j \in J\} \).

A magma \((S,f)\) is monogenic (or cyclic) if and only if there exists some \( a \in S \) such that the submagma generated by \( a \) is the whole of \( S \).

A law \( f \) on \( S \) is\footnote{This definition generalizes that \[13\]. A definition equivalent to that of \[13\] involves separating left and right alternativity, and flexibility (though left and right alternativity together imply flexibility, \[19\] \[3\]).} alternative if and only if \((\forall x \in S)(\forall y \in S)(\forall z \in S) [(x = y \text{ or } x = z \text{ or } y = z) \Rightarrow f(f(x,y),z) = f(f(x,y),z)] \). If the law of a magma is alternative we say that the magma itself is alternative.\footnote{Consider:}

\[
\begin{array}{cccc}
  f & a & b & c \\
  a & a & a & a \\
  b & a & d & a \\
  c & a & a & a \\
  d & a & c & a \\
\end{array}
\]

In this case, the law is alternative, but the submagma generated by \( b \) is the whole of \( \{a,b,c,d\} \) and the law is not associative on it because \( f(b,f(d,c)) = f(b,a) = a \neq c = f(c,b) = f(f(b,d),c) \), hence the law not power-associative.\footnote{Consider:}

\[
\begin{array}{ccc}
  f & a & b & c \\
  a & a & c & b \\
  b & c & b & a \\
  c & b & a & c \\
\end{array}
\]

In this case the law is power-associative, as the submagma generated by any element is reduced to that element, but the law is not alternative, as for instance \( f(a,f(a,b)) = f(a,c) = b \neq c = f(a,b) = f(f(a,a),b) \).
It is perhaps surprising that a monogenic magma need not be power-associative\textsuperscript{11}! A power-associative magma need not be monogenic, of course\textsuperscript{12}.

A law \( f \) on \( S \) is \textit{di-associative} if and only if for each couple of (not necessarily distinct) elements of \( S \), \( a_1 \) and \( a_2 \), the submagma generated by \( \{a_1, a_2\} \) is associative. If the law of a magma is di-associative we say that the magma itself is \textit{di-associative}. A di-associative magma is always both alternative and power-associative, but an alternative magma need not be di-associative\textsuperscript{13}, and a power-associative magma need not be di-associative\textsuperscript{14}. Likewise, an associative magma is always di-associative, but a di-associative magma need not be associative\textsuperscript{15}.

Given a law \( f \) on \( S \), an \( e_1 \in S \) is a \textit{left identity element} for the law \( f \) if and only if \( (\forall x \in S) \ f(e_1, x) = x \); and an \( e_2 \in S \) is a \textit{right identity element} for the law \( f \) if and only if \( (\forall x \in S) \ f(x, e_2) = x \). An \( e \in S \) is an \textit{identity element (or neutral element)} for the law \( f \) if and only if it is both a left identity element for \( f \) and a right identity element for \( f \). A neutral element is necessarily unique. A neutral element is frequently called a “unit” for the law, but we will not use this terminology here due as this would conflict with other possible meanings of this word. A \textit{magma with unit} is a magma whose law has a neutral element.

A law \( f \) on \( S \) is \textit{commutative} if and only if \( (\forall x \in S) (\forall y \in S) \ f(x, y) = f(y, x) \). If the law of a magma is commutative we say that the magma itself is \textit{commutative}.

Given two magmas \((S, f)\) and \((S', f')\), a \textit{magma (homo)morphism} is a function \( \theta : S \rightarrow S' \) such that \( (\forall (x,y) \in S^2) \ \theta(f(x,y)) = f'(\theta(x),\theta(y)) \). If \((S, f)\) and \((S', f')\) are magmas with units, whose neutral elements are respectively \( e \) and \( e' \), then a function \( \theta : S \rightarrow S' \) is a \textit{(homo)morphism of magmas with unit} if and only if it is a magma morphism and \( \theta(e) = e' \). A surjective morphism is called an \textit{epimorphism}, and an injective one is called a \textit{monomorphism}. An \textit{endomorphism} is a morphism from one set to itself. A function \( \theta : S \rightarrow S' \) is an \textit{isomorphism} if and only if it is a homomorphism, it is bijective and the reciprocal function \( \theta^{-1} : S' \rightarrow S \) is a homomorphism. An \textit{automorphism} is an isomorphism from one set to itself.

If \( I \) is a set and \((S_i, f_i)_{i \in I}\) is a (non-empty) family of magmas, we define a magma structure on the product set \( S \) by \( f : S \times S; ((x_i)_{i \in I}, (x'_i)_{i \in I}) \mapsto (f_i(x_i, x'_i))_{i \in I} \). If all the magmas in the family are commutative

\textsuperscript{11} [\textsuperscript{11}] \textsuperscript{11} gives the following example:

| \( f \) | a | b | c | d | e |
|-----|---|---|---|---|---|
| a   | b | a | c | d | e |
| b   | b | a | e | c | d |
| c   | c | e | d | b | a |
| d   | d | c | a | e | b |
| e   | e | d | b | a | c |

In this case \( e \) generates the magma (a loop, actually, see further on for the definition), but \( f(c, f(c, c)) = f(c, d) = b \neq a = f(d, c) = f(f(c, c), c) \).

\textsuperscript{12}As the example in footnote\textsuperscript{11} attests.

\textsuperscript{13}Consider:

| \( f \) | a | b | c | d |
|-----|---|---|---|---|
| a   | a | a | a | a |
| b   | a | a | a | c |
| c   | a | a | c | a |
| d   | a | a | a | a |

In this case, the law is alternative, but if \( a_1 = b \) and \( a_2 = d \), then the generated submagma is \( \{a, b, c, d\} \), which is not associative, since \( f(f(b, d), c) = a \neq c = f(b, f(d, c)) \).

\textsuperscript{14}Consider:

| \( f \) | a | b | c |
|-----|---|---|---|
| a   | a | b | a |
| b   | b | a | a |
| c   | a | a | a |

In this case the law is power-associative, but not di-associative because if \( a_1 = a \) and \( a_2 = c \) then the submagma generated by these two element is the whole of \( \{a, b, c\} \), and the law is not associative on it (because \( f(a, f(c, b)) = f(a, a) = a \neq b = f(a, b) = f(f(a, c), b) \)).

\textsuperscript{15}Consider:

| \( f \) | a | b | c |
|-----|---|---|---|
| a   | a | a | c |
| b   | a | b | b |
| c   | a | b | c |

In this case the law is di-associative, but is not associative because \( f(a, f(b, c)) = a \neq c = f(f(a, b), c) \).
(respectively power-associative, alternative, di-associative, associative, with unit), then the product magma is likewise commutative (respectively power-associative, alternative, di-associative, associative, with unit). Even if all the magmas in the family are monogenic, however, the product clearly does not have to be monogenic.

A.1.2 Strongly associative subsets

We present this interesting variation on the idea of associativity which we will use later on in this appendix. This is essentially a simplified presentation of notions found in \cite{2}.

Let \((S, f)\) be a magma, and \(T \subset S\). We will say \(T\) is a \textit{strongly associative subset} of \(S\) if and only if

\[
(\forall (u, v, w) \in S^3) \left[\left( (u, v) \in T^2 \right) \lor \left( (u, w) \in T^2 \right) \lor \left( (v, w) \in T^2 \right) \right] \Rightarrow f(u, f(v, w)) = f(f(u, v), w)
\]

Notice that \(\emptyset\) is a strongly associative subset of \(S\)!

Before coming to the crux of this segment, let us recall that an ordered set \((E, \preceq)\), whose order may perhaps only be partial, is said to be \textit{inductive} if and only if any totally ordered subset of \(E\) has a majorant in \(E\) for \(\preceq\). The very important Zorn theorem\footnote{Given the axiomatic framework of Zermelo-Frankel, the axiom of choice and the theorem of Zorn are actually equivalent.} then states that every inductive set has maximal elements. We can now state the following simple proposition:

\textbf{Proposition A.1 (Inductivity of the Strongly Associative):}

\emph{Let \((S, f)\) be a magma, and }\(\mathcal{E}\)\emph{ the set of all strongly associative subsets of }\(S\); \emph{then \((\mathcal{E}, \subset)\) is inductive.}

\textbf{Proof:} Let \(\mathcal{T} \subset \mathcal{E}\), totally ordered by \(\subset\), and let \(U = \bigcup_{T \in \mathcal{T}} T\), then \(U\) is a majorant of \(\mathcal{T}\) in \(\mathcal{P}(S)\) (the set of subsets of \(S\)).

If \(U = \emptyset\), then \(U\) is strongly associative.

If \(U \neq \emptyset\), let \((u, v, w) \in S^3\). Assume that \(u \in U, v \in U\). Then \((\exists F_u \in \mathcal{T}) u \in F_u, (\exists F_v \in \mathcal{T}) v \in F_v\). Since \((\mathcal{T}, \subset)\) is totally ordered by hypothesis, then either \(F_u \subset F_v\) or \(F_v \subset F_u\), so \(u \in F_v\) and \(v \in F_u\) with \(F_u = F_v\) or \(F_v = F_u\) as the case may be. But \(\mathcal{T} \subset \mathcal{E}\) and \(F_u \in \mathcal{T}\), \(F_v \in \mathcal{T}\), and therefore \(f(u, f(v, w)) = f(f(u, v), w)\). The other possible branches of assumption are dealt with in the same manner. Finally \(U\) is strongly associative in this case too.

A.1.3 Beyond the magma

A \textit{quasigroup} is a groupoid \((S, f)\) such that for all \(a \in S\) the functions \(l_a = [S \to S; x \mapsto f(a, x)]\) (left translation, or multiplication, by \(a\)) and \(r_a = [S \to S; x \mapsto f(x, a)]\) (right translation, or multiplication, by \(a\)) are both bijections\footnote{Which \textit{does not} imply the existence of either left or right inverses, or even that of a neutral element! Consider:}

\[
\begin{array}{c|ccc}
  f & a & b & c \\
  \hline
  a & b & c & a \\
  b & a & b & c \\
  c & c & a & b \\
\end{array}
\]

\textit{Given the axiomatic framework of Zermelo-Frankel, the axiom of choice and the theorem of Zorn are actually equivalent.}

\[
\begin{array}{c|ccc}
  f & a & b & c \\
  \hline
  a & a & c & b \\
  b & c & b & a \\
  c & b & a & c \\
\end{array}
\]

\textit{gives the following example:}

\[\text{16}\]
A quasigroup which has both the L.I.P. property and the R.I.P. property is said to have the inverse property (or I.P. for short), or to be an I.P. quasigroup.

A loop is a quasigroup whose law has a neutral element. A quasigroup need not be a loop\(^{19}\). In a loop \((L, \times)\) therefore, every element \(a \in L\) has both a right inverse \(a^\lambda\) and a left inverse \(a^\rho\) (which are necessarily unique), but it may happen\(^{20}\) that there exists \(b \in L\) such that \(a^\lambda \times (a \times b) \neq b\) or \((b \times a) \times a^\rho \neq b\). We define subloops in a like manner as subquasigroups. Note that one can prove\(^{21}\) that the neutral element of a loop is included in every subloop and is necessarily identical to that subloop’s neutral element. The intersection of subloops being a loop, we define likewise the subloop generated by a subset or a collection of elements. A loop is commutative (respectively monogenic, alternative, L.I.P., R.I.P., I.P.) if, as a quasigroup, it is such. It is power-associative, respectively di-associative, if the subloop generated respectively by any one or any two (not necessarily distinct) elements is associative.

Note that in an I.P. loop, \(J_a = J_{p_a}\), and therefore every element has an inverse.

Given a loop \((L, \times)\), the following properties are equivalent (\(\text{I}3\)):

\[
\begin{align*}
\forall x \in L & \forall y \in L \forall z \in L \quad (x \times y) \times (z \times x) = [x \times (y \times z)] \times x \quad (\text{M1}) \\
\forall x \in L & \forall y \in L \forall z \in L \quad (x \times y) \times (z \times x) = x \times [(y \times z) \times x] \quad (\text{M2}) \\
\forall x \in L & \forall y \in L \forall z \in L \quad [x \times (z \times x)] \times y = x \times [z \times (x \times y)] \quad (\text{M3}) \\
\forall x \in L & \forall y \in L \forall z \in L \quad [(x \times z) \times x] \times y = x \times [z \times (x \times y)] \quad (\text{M4}) \\
\forall x \in L & \forall y \in L \forall z \in L \quad [(y \times x) \times z] \times x = y \times [z \times (x \times x)] \quad (\text{M5})
\end{align*}
\]

Any loop which verifies one (and therefore all) of the above properties is called a Moufang loop. A Moufang loop is commutative (respectively monogenic) if, as a loop, it is such. We define sub-Moufang loops in a like manner as subloops. Any subloop of a Moufang loop is also a Moufang loop.

A Moufang loop always is \((\text{I}5\text{ Theorem IV.1.4 and Corollary IV.2.9})\) a di-associative I.P. loop (which is equivalent to saying that all subloops generated by any two, not necessarily distinct, elements always is a group; this is a corollary of Moufang’s Theorem).

A monoid is a magma with unit for which the law \(f\) is associative. A monoid is commutative (respectively monogenic) if it is commutative (respectively monogenic) as a magma. A monoid need not be a quasigroup\(^{22}\), hence need not be a loop, and a quasigroup need not be a monoid\(^{23}\); a loop, even a Moufang loop, need not be associative\(^{24}\), hence need not be a monoid either.

\(^{19}\)Consider:

\[
\begin{array}{c|ccc}
 f & a & b & c \\
\hline
 a & b & a & c \\
b & a & c & b \\
c & c & b & a \\
\end{array}
\]

\(^{20}\)\(\text{I}6\) gives the following example:

\[
\begin{array}{c|cccc}
 f & a & b & c & d \\
\hline
 a & a & b & c & d \\
b & b & a & e & c \\
c & c & d & a & e \\
d & d & e & b & a \\
e & e & c & d & b \\
\end{array}
\]

In this case \((\forall x \in L) \ x^\lambda = x^\rho = x\), but \(f(f(c, d), d) \neq c\) and \(f(c, f(c, d)) \neq d\).

\(^{21}\)Let \((L, \times)\) be a loop with neutral element \(e\), and \((H, \times)\) be a subloop. Then there exists \(a \in H\) such that \(a \times a = a\), but as \(H \subset L\), \(a \in H\) and \(a \times a = a \times e\).

\(^{22}\)Consider:

\[
\begin{array}{c|ccc}
 f & a & b & c \\
\hline
 a & c & a & a \\
b & a & b & c \\
c & a & c & c \\
\end{array}
\]

\(^{23}\)See the example in footnote \(\text{I}7\) which is not even alternative.

\(^{24}\)The multiplication of the classical octonions is an example of this; see also \((\text{I}6\text{ page 89})\) for a reference to the smallest possible example of a non-associative Moufang loop.
A group is a monoid in which every element has an inverse. One can prove ([16]) that being a group is equivalent to being a quasigroup whose law is associative, or a loop whose law is associative, or a Moufang loop whose law is associative. We define subgroups in a like manner as submagmas. As with loops, groups and subgroups share their neutral element. A group is commutative (respectively monogenic) if, as a monoid, it is such. A commutative group is also called an abelian group.

Morphisms (including isomorphisms, etc.) of quasigroups, loops, monoids and groups are simply morphisms for the underlying magma (or magma with unit). While these notions are quite interesting for groups, they are not so for poorer structures (a quasigroup can perfectly be homomorphic to something which is not a quasigroup, and a loop can be homomorphic to something which is not a loop, see [16, page 28]), and are too strict a tool to be very useful in classifying these later (a better tool for this is isotopism; see [16]).

A.1.4 Derivation graph for one law

We present here a graph representing the “is-a” relationship for some of the most important structures in this first part of the appendix. We have abbreviated “power-associative magma” into “P.-A. Magma”, “di-associative magma” into “Di-A. Magma”, “alternative magma” into “Alt. Magma”, “associative magma” into “Ass. Magma”, “magma with unit” into “Magma w. U.” etc.

---

25 However, since a loop, even a Moufang loop, needs not be associative, it needs not be a group; see footnote 24
A.1.5 A festival of pitfalls: powers (or multiples) of an element

We will now take an interlude with an eye towards applications (however remote). Several successful methods in cryptography involve taking powers (or multiples) of an element belonging to rather rich structures, groups at the least, usually commutative ([21]). One can’t help but wonder what can be salvaged of these methods in far poorer settings. Of course, the crux of these techniques usually is a variation on Fermat’s little theorem, and we will not investigate this here, but the simple idea of taking powers bears some investigating.

Given any magma \((S, f)\), for any \(a \in S\) and any non-zero integer \(n\), we define \(a^n\) by induction, i.e., \(a^1 = 1\), and having defined \(a^p\) for some non-zero integer \(p\), we define \(a^{p+1} = f(a, a^p) = f(a^p, a)\).

While this is a perfectly reasonable definition, which might even be useful in some contexts, without any additional constraints on the magma there not much that can be proved about these objects. For instance, we might want to have the same values if we repeatedly multiply on the right by \(a\) rather than multiply on the left as we have done. Unfortunately, without any assumption of commutativity or associativity, in whatever form, it may happen that this is not the case.

Commutativity will avoid this complication, but we will in this document consider very interesting cases which are not commutative. Furthermore, commutativity on its own will not be sufficient to insure some nice properties such as:

\[
(\forall n \in \mathbb{N} - \{0\})(\forall m \in \mathbb{N} - \{0\}) a^{n+m} = f(a^n, a^m) = f(a^m, a^n)
\]

which some forms of associativity will provide.

Alternativity is a rather poor choice for a shade of associativity, as then \(\boxed{A}\) is true if \(m\) is a power of 2 (as can be proved by induction), but may otherwise fail\(^{27}\). A power-associative magma, on the other hand does verify \(\boxed{A}\) for every element.

If the magma under consideration is not only power-associative, but is also with unit, with neutral element \(e\), we define \(a^0 = e\), in addition to the above, and now have an improvement over \(\boxed{A}\):

\[
(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) a^{n+m} = f(a^n, a^m) = f(a^m, a^n)
\]

A’

If, in addition, the element \(a\) happens to have an inverse \(\alpha\), we define for any integer \(n\), \(a^{-n}\) as \(\alpha^n\). Unfortunately, if the magma is no more than power-associative, it may happen\(^{28}\) that \(f(a^{-1}, a^2) \neq a\). If, however, the magma is actually di-associative, or is a power-associative loop (which is more constraining than being power-associative as a magma), then we can improve upon \(\boxed{A}\):

\[
(\forall n \in \mathbb{Z})(\forall m \in \mathbb{Z}) a^{n+m} = f(a^n, a^m) = f(a^m, a^n)
\]

\(A’’\)

\(^{26}\)Consider:

|   | a | b | c | d |
|---|---|---|---|---|
| a | b | c | a |
| b | a | a | a |
| c | a | a | a |

In this case \(a^2 = f(a, a) = b\) and \(a^3 = f(a, b) = f(a, f(a, a)) = c \neq a = f(b, a) = f(f(a, a), a)\).

\(^{27}\)Consider:

|   | a | b | c | d | e |
|---|---|---|---|---|---|
| a | b | c | d | e | e |
| b | c | d | e | e | e |
| c | d | e | d | e | e |
| d | e | e | e | e | e |
| e | e | e | e | e | e |

In this case, \(a^1 = a, a^2 = b, a^3 = c, a^4 = d, a^5 = e\), but \(a^6 = e \neq d = f(a^3, a^3)\), so \(\boxed{A}\) does not hold here for \(m = 3\).

\(^{28}\)Consider:

|   | a | b | c | d |
|---|---|---|---|---|
| a | a | b | c | d |
| b | b | c | b | a |
| c | c | b | c | a |
| d | d | a | a | a |

In this case, the law is power-associative, has a unit, \(a\), but the element \(b\) has an inverse, \(d\), while \(f(b^{-1}, b^2) = f(d, c) = a \neq b\).
We may want to investigate what can be said for products of powers of two elements which may not be inverses of each other. More precisely, even if the magma is without unit, we would like to have the following property:

\[(\forall n \in \mathbb{N} - \{0\})(\forall m \in \mathbb{N} - \{0\})(\forall p \in \mathbb{N} - \{0\})(\forall q \in \mathbb{N} - \{0\})
\]

\[f(a^{n+m}, b^{p+q}) = f(a^m, f(a^n, b^{p+q})) = f(f(a^{n+m}, b^p), b^q) \quad (B)\]

If the magma is di-associative, then indeed, we do have \[B\]. Furthermore, if the magma also has a unit we can extend \[A\] into:

\[(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(\forall p \in \mathbb{N})(\forall q \in \mathbb{N}) f(a^{n+m}, b^{p+q}) = f(a^m, f(a^n, b^{p+q})) = f(f(a^{n+m}, b^p), b^q) \quad (B')\]

About the best we can hope for, if the magma has a unit and \(a\) and \(b\) have inverses would be:

\[(\forall n \in \mathbb{Z})(\forall m \in \mathbb{Z})(\forall p \in \mathbb{Z})(\forall q \in \mathbb{Z}) f(a^{n+m}, b^{p+q}) = f(a^m, f(a^n, b^{p+q})) = f(f(a^{n+m}, b^p), b^q) \quad (B'')\]

As one might come to expect, if the only shade of associativity which can be mustered is di-associativity, this might fail\(^{29}\). There are, however, very interesting cases where \[B''\] is true without imposing (full) associativity (which is of course enough to guarantee \[B\]).

**Proposition A.2:**

If \((S, f)\) is a di-associative magma with unit, and if \(S\) is finite, then \[B'\] holds for every \(a\) and \(b\) which have inverses.

**Proof:** Let \(e\) be the neutral element of \((S, f)\), and let \(a \in S\) having \(a^{-1}\) as inverse. Since we have assumed \((S, f)\) to be di-associative, we immediately see that the sub-magma generated by \(a\) and \(a^{-1}\) is actually a group, and since it is a subset of \(S\) which is finite, it is a finite group. Hence \((\exists n_0 \in \mathbb{N}) a^{-1} = a^{n_0}\), and \[B''\] reduces to \[B\].

It is also worth mentioning, irrespective of whether \(S\) is finite or not, that \[B''\] holds if \((S, f)\) is a di-associative loop (which is a more stringent condition than being di-associative as a magma), in particular a Moufang loop.

### A.2 Structures involving more than one law

#### A.2.1 Familiar structures

A **ring** is a structure \((A, +, \times)\) such that \((A, +)\) is a commutative group (whose neutral element we will write \(0\)), and that the multiplication (the law “\(\times\)”) is distributive over the addition\(^{30}\). A ring is said to be with unit (respectively commutative, monogenic, power-associative, alternative, associative) if and only if the multiplication is with unit (respectively commutative, monogenic, power-associative, alternative, associative). As we will see later on, if the multiplication is alternative, it will turn out to be di-associative as well, contrary to what happens for magmas (in other words, not all magma laws are fit to play the role of multiplication in rings). Of course, di-associative multiplications are still alternative!

Given two rings \((A, +, \times)\) and \((B, \dagger, \ast)\), a **ring homomorphism** \(\theta : A \to B\) such that it is a group morphism from \((A, +)\) to \((B, \dagger)\), and a magma morphism from \((A, \times)\) to \((B, \ast)\). If \(A\) and \(B\) are with unit, whose neutral elements are respectively \(e\) and \(\varepsilon\), then a **morphism of ring with unit** \(\theta\) is a morphism of rings such that \(\theta(e) = \varepsilon\). We define, epi-, mono-, endo-, iso- and automorphisms for rings as we have for the other structures, from ring homomorphisms.

---

\(^{29}\) Consider the set \(S = \{2^p | p \in \mathbb{Z}\} \cup \{3^q | q \in \mathbb{Z}\} \cup \{5^r | r \in \mathbb{Z}\}\) along with the following law “\(*\)”: \(2^p \ast 2^{p'} = 2^{p+p'}\), \(3^q \ast 3^{q'} = 3^{q+q'}\), \(5^r \ast 5^{r'} = 5^{r+r'}\), \(2^p \ast 3^q = 3^{q+q+2p}\), \(2^p \ast 5^{r'} = 5^{r+2p}\), \(2^p \ast 2^q = 5^{p+q}\), \(2^p \ast 5^r = 5^{p+r}\), \(3^q \ast 5^{r'} = 5^{q+r}\), \(3^q \ast 3^{r'} = 5^{q+r}\).

One verifies that \((S, *)\) is indeed di-associative, has a neutral element \((1)\), and that every element has inverses: an element of the form \(2^p\) has \(2^{-p}\) for only inverse, an element of the form \(3^q\) has \(3^{-q}\) for only inverse, and an element of the form \(5^r\) have exactly three inverses \((5^{-r}, 3^{-r}\) and \(5^{-r})\) if \(r \neq 0\) (if \(r = 0\), of course it only has 1 for inverse).

We now see that \(2^{-1} \ast (2^{-1} \ast 3^{1+}) = 2^{-1} \ast 5^2 = 5^{1+} \neq 3^{-1} = (2^{-1} \ast 2^{1+}) \ast 3^{1+}\).

\(^{30}\) Which means that \((\forall x \in A)(\forall y \in A)(\forall z \in A) [(x + y) \times z = (x \times z) + (y \times z)\) and \(z \times (x + y) = (z \times x) + (z \times y)]\)
Note that if $A$ is a ring with unit, whose neutral element we will write $1_A$, then $1_A \neq 0$ $\iff$ $A \neq \{0\}$. Given a ring with unit $A$ with at least two elements, we will denote by $A^*$ the set of elements of $A$ which have an inverse (for the multiplication in $A$); of course $A^* \subset A - \{0\}$ but the inclusion may be strict. The elements of $A^*$ are also (very unfortunately) called the units of $A$. The magma with unit $(A^*, \cdot)$, has a rather poor structure in general, but when $A$ is associative, it is a group \(\text{[13]}\); when $A$ is only alternative, it is a Moufang loop \(\text{[13]}\); we will show a special case of this result in this document.

Given a ring $(A, +, \cdot)$ and a $T \subset A$, we will say that $T$ is a subring if and only if $(T, +)$ is a subgroup of $(A, +)$, and $T$ is stable for the multiplication of $A$ (i.e. $(\forall (x, y) \in T^2) \ x \cdot y \in T$). If $(A, +, \cdot)$ is a ring with unit, whose neutral element for “$\cdot$” we will denote by $e$, then $T$ will be said to be a subring of a ring with unit if and only if it is a subring and furthermore that $e \in T$\(^{31}\).

A semifield is a structure $(K, +, \cdot)$ which is a ring with unit such that, writing $K^* = K - \{0\}$ (with $0$ being the neutral element for the addition, i.e. the law $+$), $(K^*, \cdot)$ is a loop, whose neutral element which we will call $1_K$ verifies $0 \neq 1_K$. A semifield is commutative (respectively monogenic, power-associative, alternative, associative) if it is such as a ring.

A field is\(^{32}\) an associative semifield. Denoting by $F^*$ the set of elements of $F$ which have an inverse (for the multiplication in $F$) we therefore have $F^* = F - \{0\}$; $(F^*, \cdot)$ is a group. A field is said to be commutative (respectively monogenic) if, as a semifield, it is such.

Morphisms for semifields and fields are those of the underlying rings.

Given an associative and commutative ring with unit $(A, +, \cdot)$, an algebra is a ring, and an algebra with unit is a ring with unit. An algebra is said to be power-associative (respectively alternative, associative).\(^{33}\)

Given a ring $(A, +, \cdot)$ and a $T \subset A$, we will say that $T$ is a subalgebra of $E$ if and only if is is a subring and furthermore that $e \in T$. A subalgebra is commutative (respectively monogenic, power-associative, alternative, associative) if it is such as a ring.

A vector space is a module over a commutative field. A sub-vector space is a submodule over a commutative field.

Given two modules $(M, +, \cdot)$ and $(N, \cdot\cdot)$ over the same commutative ring $A$, a module (homo)morphism is a function $\theta : M \to N$ such that it is a group homomorphism from $(M, +)$ to $(N, \cdot\cdot)$, and such that $(\forall \alpha \in A)(\forall \alpha \cdot m) = \alpha \cdot (\theta(m))$. Morphisms for vector fields are those of the underlying module. The usual morphism declinations apply here as well.

Given a commutative and associative ring with unit $(A, +, \cdot)$, an algebra over $A$ (also called an $A$-algebra) is a structure $(E, +, \cdot, \cdot)$ where $(E, +, \cdot)$ is an $A$-module\(^{34}\), and the law “$\cdot$” (the multiplication) $E$ is $A$-bilinear, i.e. it is distributive over “$+$” and $(\forall \alpha \in A)(\forall \beta \in A)(\forall x \in E)(\forall y \in E) \ (\alpha \cdot x) \times (\beta \cdot y) = (\alpha \cdot \beta) \cdot (x \times y)$. An algebra with unit is an algebra whose multiplication has a neutral element. In particular, an algebra is a ring, and an algebra with unit is a ring with unit. An algebra is said to be commutative (respectively power-associative, alternative, associative) if and only if the multiplication is commutative (respectively power-associative, alternative, associative).

Given an associative ring with unit $(A, +, \cdot)$, an $A$-algebra $(E, +, \cdot, \cdot)$ and a $T \subset E$, we will say $T$ is a sub-algebra if and only if $(T, +, \cdot)$ is a submodule of $(E, +, \cdot)$, and $T$ is stable for “$\cdot$” (i.e. $(\forall (x, y) \in T^2) \ (x \cdot y) \in T$). If $(E, +, \cdot, \cdot)$ is an $A$-algebra with unit, with neutral element $e$, $T$ will be a sub-algebra of an algebra with unit is and only if is is a sub-algebra of $E$ and $e \in T$.

If $E$ is an $A$-algebra with unit, we will also denote by $E^*$ the set of elements of $E$ which have an inverse (for the multiplication in $E$); of course $E^* \subset E - \{0\}$ but the inclusion may be strict. As with rings in general, the magma with unit $(E^*, \cdot)$ usually has a rather poor structure.

\(^{31}\)It is well-known that undesirable things may happen otherwise, and that $(T, +, \cdot)$ may happen to be a ring with unit, without being a “subring of a ring with unit” as per our definition; consider the square matrices of order two on $\mathbb{R}$ for $A$, and the matrices of the form \(\left( \begin{array}{cc} \cdot & \cdot \\ x & \cdot \end{array} \right)\) with $x \in \mathbb{R}$ for $T$.

\(^{32}\)\([13]\) requires the multiplication to be commutative; we do not, however, in accordance with \([2]\).

\(^{33}\)We do not limit ourselves to the case where $A$ is a field, which is the most common situation \([10, \ldots]\), but which would be too restrictive for our needs in this document.
If $E$ is an $A$-algebra with unit, whose neutral element we will denote by $e$, there is a natural epimorphism of rings with unit defined by $[A \to E; a \mapsto a \cdot e]$. It should be noted that this homomorphism need not be injective\footnote{The following example was communicated to me by M. Daniel R. Grayson: $A = \mathbb{Z}$ and $E = \mathbb{Z}/2\mathbb{Z}$; in this case $e$ is not linearly independent on $A$.}

Given two algebras $(E, +, \times, \cdot)$ and $(F, \hat{+}, \hat{\times}, \hat{\cdot})$ over the same ring $A$, an algebra (homo)morphism is a function $\theta: E \to F$ such that it is a morphism of modules from $(E, +, \cdot)$ to $(F, \hat{+}, \hat{\cdot})$ and a morphism of ring from $(E, +, \times)$ to $(F, \hat{+}, \hat{\cdot})$. If $E$ and $F$ are with unit, a morphism of algebra with unit is a morphism of algebra such that it is also a morphism of ring with unit from $(E, +, \times)$ to $(F, \hat{+}, \hat{\cdot})$. The usual offshoots of the morphism family are dealt with as usual.

A.2.2 Alternative rings

We recall here some important properties of alternative rings that we need in this document. In particular, we present here a rather elementary proof of a classical, which is a special case of a theorem of Bruck and Kleinfeld (5), itself an extension of a result due to Artin, classically known for algebras (219). This is mostly for self-sufficiency’s sake, but we believe it fits quite well with this appendix’ emphasis on shades of associativity; it also is an opportunity to remedy what is most likely an unfortunate ellipsis in the presentation found in 2.

Given a ring $(A, +, \times)$ and a function $f: A^n \to A$, we will say that $f$ is additive with respect to the $i$th variable if and only if:

$$(\forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in A^{n-1})(\forall x \in A)(\forall x' \in A) f(x_1, \ldots, x_{i-1}, x + x', x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, x', x_{i+1}, \ldots, x_n)$$

We trivially verify that a function $f$ additive with respect to the $i$th variable takes the value 0 when the $i$th variable takes the value 0, and that furthermore

$$(\forall (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in A^{n-1})(\forall x \in A) f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) = -f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

We will say that a function $f$, additive with respect to each of its variables, is alternated if and only if:

$$(\forall i \in \{1, \ldots, n-1\})(\forall (x_1, \ldots, x_n) \in A^n) [x_i = x_{i+1} \Rightarrow f(x_1, \ldots, x_n) = 0]$$

Let us denote by $\mathfrak{S}_n$ the group of permutations on $\{1, \ldots, n\}$ and, for $\sigma \in \mathfrak{S}_n$, by $\epsilon_\sigma$ the signature of $\sigma$. We trivially prove that if $f$ is alternated then $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \epsilon_\sigma f(x_1, \ldots, x_n)$, and that a function is alternated if and only if it is zero whenever any two variables have the same value, whether these variables are consecutive or not.

Given any ring $(A, +, \times)$, we can always define the alternator by:

$$\mathfrak{a}: A^3 \to A$$

$$(x, y, z) \mapsto x \times (y \times z) - (x \times y) \times z$$

and it is always true that the alternator is additive with respect to each variable. It is also always true that:

$$(\forall (p, q, r, s) \in A^4) \mathfrak{a}(p \times q, r, s) - \mathfrak{a}(p, q \times r, s) + \mathfrak{a}(p, q, r \times s) = p \times \mathfrak{a}(q, r, s) + \mathfrak{a}(p, q, r) \times s \quad \text{(T)}$$

(which is proved by simply evaluating both sides, and is known as the Teichmuller identity).

Another interesting function is the Kleinfeld function (which differ only in sign from the definition in 22 6):

$$\mathfrak{k}(p, q, r, s) = q \times \mathfrak{a}(p, r, s) + \mathfrak{a}(q, r, s) \times p - \mathfrak{a}(p \times q, r, s)$$

which also is an additive function with respect to each variable.

The alternator is interesting as we trivially prove that $A$ is alternative if and only if $\mathfrak{a}$ is alternated, and that $A$ is associative if and only if $\mathfrak{a}$ is the zero function. Likewise, a subset $T$ of $A$ is strongly associative (regardless of whether $A$ is alternative or not) if and only if

$$(\forall (u, v, w) \in A^3) \mathfrak{k}([(u, v) \in T^2] \lor [(u, w) \in T^2] \lor [(v, w) \in T^2]) \Rightarrow \mathfrak{a}(u, v, w) = 0$$
and if $A$ is indeed alternative, then $T$ is strongly associative if and only if

$$\forall (u, v) \in T^2 \forall w \in A \ a(u, v, w) = 0$$

As well we have (5):

**Proposition A.3:**

If $(A, +, \times)$ is alternative then $\mathfrak{t}$ is alternated.

**Proof:** We first compute $\mathfrak{t}(p, q, r) = p \times a(s, q, r) + a(p, q, r) \times s - a(s \times p, q, r)$.

Using (4) we find that $\mathfrak{t}(p, q, r) = a(p \times q, r, s) - a(p, q \times r, s) + a(p, q, r \times s) - a(s \times p, q, r)$.

However, writing (4) for $(q, r, s, p)$ we see that $a(q \times r, s, p) - a(q, r \times s, p) + a(q, r, s \times p)$ and since $A$ is alternated, $a(r, s, p) = a(p, r, s)$, so $\mathfrak{t}(p, q, r) = a(q \times r, s, p) - a(p \times q, r, s) + a(q, r, s \times p) = a(q, r, s, p)$.

Since $A$ is alternative, the associator is alternated, so $a(p, q \times r, s) = a(q \times r, s, p)$, $a(p, q, r \times s) = a(q, r \times s, p)$ and $a(s \times p, q, r) = a(q, r, s \times p)$ and thus $\mathfrak{t}(p, q, r) = a(q \times r, s, p) - a(p \times q, r, s) + a(q, r, s \times p) = a(q, r, s, p)$.

Since the alternator is alternated because the ring is alternative, we have $\mathfrak{t}(p, q, r, s) = 0$ and so $\mathfrak{t}(p, q, r, s) = -\mathfrak{t}(p, q, r, s)$.

Finally, $(1 \ 2 \ 3 \ 4)$ and $(1 \ 2 \ 3 \ 4)$ generate the symmetric group of order 4.

The alternator is fundamental for proving the next theorem, which contrasts strongly with the case of magmas.

**Theorem A.1 (Di-associativity of alternative rings):**

A ring is alternative if and only if it is di-associative, and if and only if all subrings generated by any two (not necessarily distinct) elements are associative. A ring with unit is alternative if and only if it is di-associative, and if and only if all subrings of a ring with unit generated by any two (not necessarily distinct) elements are associative.

We will mostly, but not quite completely, paraphrase the proof in [2], which requires a few lemmas.

We start with a technical result which shows that the presence of an additional law on which the product is distributive imposes some regularity on the product.

**Lemma A.1:**

Let $(A, +, \times)$ be any ring, and let $\mathcal{E}_\times$ the set of all subsets of $A$ which are strongly associative for “$\times$”. Let $G$ be a maximal element (for the inclusion) of $\mathcal{E}_\times$; then if $G \neq \emptyset$, $(G, +)$ is a group.

Note that we know there are maximal elements in $\mathcal{E}_\times$ thanks to the Zorn theorem.

**Proof:** Assume that $G \neq \emptyset$.

Let’s first prove that $0 \in G$. Consider $K = G \cup \{0\}$. Of course $G \subset K$, so as $G$ is maximal, it is enough to prove that $K$ is strongly associative. Let $(u, v, w) \in K^3$; if two of the three are in $G$, then $a(u, v, w) = 0$, because $G$ is strongly associative, and if at least one is zero, then $a(u, v, w) = 0$ as well because the alternator is additive with respect to each of its variables. Hence $K$ is strongly additive.

Let $u \in G$; considering $L = G \cup \{-u\}$, and invoking the additivity of the alternator and the maximality of $G$, we see as well that $-u \in G$.

Let $u \in G$, $v \in G$; considering $M = G \cup \{u + v\}$, and invoking the additivity of the alternator and the maximality of $G$, we also see that $(u + v) \in G$.

**Lemma A.2:**

Let $(A, +, \times)$ be an alternative ring, $H$ a subset of $A$ strongly associative for “$\times$” and $F$ the subring of $A$ generated by $H$; then $F$ is strongly associative.

**Proof:** If $F = \emptyset$, there is nothing to prove. We therefore assume that $F \neq \emptyset$.

As the subsets of $F$ which are strongly associative for “$\times$” are inductive for set inclusion, by A.1 we may consider $G$ the greatest (for set inclusion) strongly associative subset of $F$ which contains $H$. To prove the lemma it therefore suffices to prove that $G = F$. 

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Using **A.1Analyticringslemma.A.1** we know that \((G, +)\) is a group, so all that is left to prove is that it is also stable for \(\times^3\) (as then \(G\) will be a subring of \(A\) containing \(H\), \(F\) is by definition the smallest subring of \(A\) containing \(H\), and \(G \subseteq F\)).

Let \(u \in G\), \(v \in G\), and consider \(L = G \cup \{u \times v\}\). Of course \(H \subseteq L\) (since \(H \subseteq G\)), and \(L \subseteq F\) (as \(F\) is a ring). Consider \(x \in L\), \(y \in L\), \(z \in A\) and \(\alpha = a(x, y, z)\). If \(x \in G\) and \(y \in G\) then \(\alpha = 0\) because \(G\) is strongly associative. Assume therefore that \(x = u \times v\). If we also have \(y = u \times v\) then \(\alpha = 0\) because the alternator is alternated. Assume therefore only that \(y \in G\). Using (1) we find that

\[
-\alpha = a(x, z, y) = a(u \times v, z, y) = a(u, v, z \times y) - a(u, v, y \times z) = a(u, v, z \times y) + a(u, v, z) \times y + u \times a(v, z, y)
\]

However, since \(u \in G\), \(v \in G\) and \(y \in G\) and \(G\) is strongly associative, \(a(v, z, y) = 0\), \(a(u, v, z) = 0\), \(a(u, v, z \times y) = a(u, v, z'') = 0\) and \(a(u, v \times z, y) = a(u, z'', y) = 0\). Hence \(\alpha = 0\). The case where we assume \(x \in G\) and \(y = u \times v\) is deduced from the above by \(a(x, y, z) = -a(y, x, z)\).

**Lemma A.3:**

Let \((A, +, \times)\) be an alternative ring; for any \(x \in A\), the subring generated by \(x\) is strongly associative.

**Proof:** Let \(H = \{x\}\); \(H\) is strongly associative since the alternator is alternated. We conclude by using **A.2Analyticringslemma.A.2**

**Lemma A.4:**

Let \((A, +, \times)\) be an alternative ring, \(T\) a subgroup of \((A, +)\), \(S\) a subset of \(T\), and \(U\) the subring generated by \(S\). If \((\forall x \in S)[x \times T \subseteq T \text{ and } T \times x \subseteq T]\) then \(T \subseteq U\).

**Proof:** Let \(V = \{x \in U \mid x \times T \subseteq T \text{ and } T \times x \subseteq T\}\). By hypothesis \(S \subseteq V\). Let \(x \in V\), \(y \in V\) and \(t \in T\).

Let \(P = (x \times y) \times t\). Obviously, \(P = x \times (y \times t) - [x \times (y \times t) - (x \times y) \times t]\), which by definition of the alternator means that \(P = x \times (y \times t) - a(x, y, t)\), or, as the alternator is alternated, \(P = x \times (y \times t) + a(x, t, y)\), which evaluates into \(P = x \times (y \times t) + x \times (t \times y) = x \times t \times y\).

Since \(y \in V\) and \(t \in T\), we have both \(y \times t \in T\) and \(t \times y \in T\). But \(y \times t \in T\) and \(x \times V\) together imply \(x \times (y \times t) \in T\), and likewise \(x \times (t \times y) \times y \in T\). Since \((T, +)\) is a group, this proves that \((x \times y) \times t = P \in T\).

Considering \(Q = t \times (x \times y)\), we likewise prove that \(t \times (x \times y) \in T\).

Hence \(V\) is a subring of \(A\), and since \(S \subseteq V\) and \(U\) is the subring generated by \(S\), then \(V \subseteq U\). As \(V \subseteq U\) by definition, this means that \(V = U\), and therefore that \((\forall x \in U)[x \times T \subseteq T \text{ and } T \times x \subseteq T]\).

As \(T \subseteq U\), this means that a fortiori, \(T \times T \subseteq T\), and so that \(T\) is a subring of \(A\). As \(S \subseteq T\) by definition, and \(U\) is the subring generated by \(S\), \(T \subseteq U\).

**Remark:** If in **A.4Analyticringslemma.A.4** \(T \subseteq U\) then of course \(T = U\).

**Lemma A.5:**

Let \((A, +, \times)\) be an alternative ring and \(X\) and \(Y\) two strongly associative subrings of \(A\). The subring generated by \(X \cup Y\) is associative.

**Proof:** Let \(W\) be the subring generated by \(X \cup Y\).

Consider \(Z = \{z \in W \mid (\forall x \in X)(\forall y \in Y) a(x, y, z) = 0\}\).

We trivially see that \((Z, +)\) is a group. Since \(X\) is strongly associative, we have \(X \subseteq Z\) and likewise \(Y \subseteq Z\). By definition we have \(Z \subseteq W\). So \(W\) is also the subring generated by \(Z\).

Let \(x \in X\), \(x' \in X\), \(y \in Y\), \(z \in Z\).

Using (1) we find that

\[
a(x' \times x, z, y) - a(x', x \times z, y) + a(x', x, z \times y) = a(x', x, z) \times y + x' \times a(x, z, y)
\]

but \(a(x', x, z) = 0\) and \(a(x', x, z \times y) = 0\) because \(x \in X\) and \(x' \in X\) and \(X\) is strongly associative, \(a(x, z, y) = -a(x, y, z) = 0\) because \(x \in X\) and \(y \in Y\) and by definition of \(Z\). As \(X\) is a subring of \(A\), \(x'' = x' \times x \in X\), so \(a(x'' \times x, y, z) = -a(x'', y, z) = 0\). Hence \(a(x', y, x \times z) = 0\). This proves that \((\forall x \in X)(\forall z \in Z) x \times z \in Z\).

Using (1) with \((p, q, r, s) = (y, z, x, x')\) we likewise prove that \((\forall x \in X)(\forall z \in Z) x \times z \in Z\).

Since the alternator is alternated, \(Z\) can also be defined as \(\{z \in W \mid (\forall x \in X)(\forall y \in Y) a(y, x, z) = 0\}\) and what we just proved also implies that \((\forall y \in Y)(\forall z \in Z) [y \times z \in Z\) and \(z \times y \in Z\).
Using A.4Alternative rings lemma, we conclude that \( Z = W \); also \( (\forall x \in X)(\forall y \in Y)(\forall w \in W) \) \( a(x, y, w) = 0 \).

Consider now \( Z' = \{ z \in W \mid (\forall x \in X)(\forall y \in Y)(\forall w \in W) \) \( a(z, y, w) = a(x, z, w) = 0 \} \).

Again, \( (Z', +) \) is trivially a group. We have just seen that \( (\forall y \in Y)(\forall w \in W)[z \in X \Rightarrow a(z, y, w) = 0] \), and as \( X \) is strongly associative, \( (\forall x \in X)(\forall w \in W)[z \in X \Rightarrow a(x, z, w) = 0] \), so \( X \subset Z' \). Likewise \( Y \subset Z' \).

By definition, \( Z' \subset W \). So \( W \) is also the subring generated by \( Z' \).

Let \( x \in X, y \in Y, y' \in Y, w \in W, z \in Z' \).

Using (T) again, we find that
\[
\forall y,y',w \quad (a(z, y, y') = 0 \text{ and } a(y, y', w) = 0) \text{ because } Y \text{ is strongly associative, } a(z, y, y' \times w) = 0 \text{ because } W \text{ is a subring and by definition of } Z' \text{, and } a(z, y \times y', w) = 0 \text{ because } Y \text{ is a subring and by definition of } Z'. \text{ Hence } a(z \times y, y', w) = 0.
\]

Using (T) once more, we find that
\[
a(z \times y, y', w) - a(z, y \times y', w) + a(z, y, y' \times w) = a(z, y, y') \times w + z \times a(y, y', w)
\]

but \( a(z, y, y') = 0 \) and \( a(y, y', w) = 0 \) because \( Y \) is strongly associative, \( a(z, y, y' \times w) = 0 \) because \( W \) is a subring and by definition of \( Z' \), and \( a(z, y \times y', w) = 0 \) because \( Y \) is a subring and by definition of \( Z' \). Hence \( a(z \times y, y', w) = 0 \).

Using (T) again, we find that
\[
a(y \times z, w, y') - a(y, z \times w, y') + a(y, z, w \times y') = a(y, z, w) \times y' + y \times a(z, w, y')
\]

but \( a(z, w, y') = 0 \) and \( a(y, z, w) = 0 \) by definition of \( Z' \), \( a(y, z, w \times y') = 0 \) because \( W \) is a subring and by definition of \( Z' \), and \( a(y \times z, z, w, y') = 0 \) because \( W \) is a subring and \( Y \) is strongly associative. Hence \( a(y \times z, w, y') = 0 \).

{}\]

Using (T) yet again, we find that
\[
a(y \times z, w, x) - a(y, z \times w, x) + a(y, z, w \times x) = a(y, z, w) \times x + y \times a(z, w, x)
\]

but \( a(z, w, x) = 0 \) and \( a(y, z, w) = 0 \) by definition of \( Z' \), \( a(y, z, w \times x) = 0 \) because \( W \) is a subring and by definition of \( Z' \), and \( a(y \times z, w, x) = 0 \) because \( W \) is a subring and \( Z \). Hence \( a(y \times z, w, x) = 0 \).

These two results prove that \( (\forall y \in Y)(\forall z \in Z') \) \( z \times y \in Z' \).

Using (T) again, we find that
\[
a(y \times z, w, y') - a(y, z \times w, y') + a(y, z, w \times y') = a(y, z, w) \times y' + y \times a(z, w, y')
\]

but \( a(z, w, y') = 0 \) and \( a(y, z, w) = 0 \) by definition of \( Z' \), \( a(y, z, w \times y') = 0 \) because \( W \) is a subring and by definition of \( Z' \), and \( a(y \times z, z, w, y') = 0 \) because \( W \) is a subring and \( Y \) is strongly associative. Hence \( a(y \times z, w, y') = 0 \).

Using (T) as usual, we find that
\[
a(x \times z, v, w) - a(x, z \times v, w) + a(x, z, v \times w) = a(x, z, v) \times w + x \times a(z, v, w)
\]

but \( a(z, v, w) = 0 \) and \( a(x, z, v) = 0 \) by definition of \( Z'' \), and \( a(x, z, v \times w) = 0 \) and \( a(x, z \times v, w) = 0 \) because \( W \) is a subring and by definition of \( Z'' \). Hence \( a(x \times z, v, v) = 0 \). Likewise \( a(y \times z, v, w) = 0 \).

Using (T) one last time, we find that
\[
a(z \times x, v, w) - a(z, x \times v, w) + a(z, x, v \times w) = a(z, x, v) \times w + z \times a(x, v, w)
\]

but \( a(x, v, w) = 0 \) and \( a(z, x, v) = 0 \) by definition of \( Z'' \), and \( a(z, x, v \times w) = 0 \) and \( a(z, x \times v, w) = 0 \) because \( W \) is a subring and by definition of \( Z'' \). Hence \( a(z \times x, v, v) = 0 \). Likewise \( a(z \times y, v, w) = 0 \).

Using A.4Alternative rings lemma, we conclude that \( Z'' = W \); also \( (\forall u \in W)(\forall v \in W)(\forall w \in W) \) \( a(u, v, w) = 0 \), which means that “\( \times \)” is associative on \( W \).
Proof (Di-associativity of alternative rings): Let \((A, +, \times)\) be an alternative ring. Quite obviously, if all subrings generated by any two (not necessarily distinct) elements are associative, then all submagmas generated by any two (not necessarily distinct) elements, being subsets of the subrings generated by the same two (not necessarily distinct) elements, are associative as well, and we already know that di-associative magmas are alternative. It therefore suffices to show that if a ring is alternative, then all subrings generated by any two (not necessarily distinct) elements are associative.

\[ A.3 \text{Alternative rings lemma.} \]

A.3 tells us that given \(x \in A\) and \(y \in A\), if \(X\) is the subring generated by \(x\) and \(Y\) the subring generated by \(y\), then \(X\) and \(Y\) are strongly associative subrings. We use A.5 to conclude that the subring generated by \(\{x, y\}\) is associative.

If now \((A, +, \times)\) is an alternative ring with unit, the same reasoning as above shows that it suffices to prove that all subrings of rings with units generated by two (not necessarily distinct) elements are associative.

Given any two (not necessarily distinct) elements, \(x\) and \(y\), of \(A\), we know that the subring generated by \(\{x, y\}\), which we will call \(S\), is associative. But if \(e\) is the neutral element for \(\times\), then the subring of a ring with unit generated by \(\{x, y\}\) is merely \(S \cup \mathbb{Z} \cdot e\) (where we have denoted by \(\mathbb{Z} \cdot e\) the integer multiples, both positive and negative, of \(e\)), which is clearly also associative.

We immediately deduce from that the classical result that an algebra is alternative if and only if it is di-associative, and if and only if all sub-algebras generated by any two (not necessarily distinct) elements are associative, and that an algebra with unit is alternative if and only if it is di-associative and if and only if all sub-algebras of algebras with unit, generated by any two (not necessarily distinct) elements are associative.

We close this section with another property we will need \([\mathbb{F}]\).

**Proposition A.4:**
Let \((A, +, \times)\) be an alternative ring with unit; then \((A, \times)\) verifies

\[
(\forall (x, y, z) \in A^3) \ a(x^2, y, z) = x \times a(x, y, z) - a(x, y, z) \times x
\]

\((R1)\)

\[
(\forall (x, y, z) \in A^3) \ a(x \times y, x, z) = a(x, y, z) \times x
\]

\((R2)\)

\[
(\forall (x, y, z) \in A^3) \ a(y \times x, x, z) = x \times a(x, y, z)
\]

\((R3)\)

and also verifies \([M1]\), \([M2] \), \([M3]\), \([M4]\) and \([M5]\).

**Proof:** The ring being alternative, A.3 ensures that \(f(x, y, z) = 0\), which is \((R1)\). Likewise \(f(x, y, x, z) = 0\) is \((R2)\) and \(f(y, x, x, z) = 0\) is \((R3)\).

For \([M1]\), we compute

\[x \times [z \times (x \times y)] - [(x \times z) \times x] \times y =
\]

\[x \times [z \times (x \times y)] - (x \times z) \times (x \times y) + (x \times z) \times x \times a(x, y, z)
\]

\[a(x, y, x, z) + a(x \times z, x, y)
\]

Using \([M1]\) we thus find that \(x \times [z \times (x \times y)] - [(x \times z) \times x] \times y = a(x, z \times x, y) + a(x, z \times x) \times y + x \times a(z, x, y),\)

and since the ring is alternative, \(a(x, z, x, y) = -a(z \times x, y, x)\) and we use \([R2]\) (applied to \((x, y, z)\)) to find that \(x \times [z \times (x \times y)] - [(x \times z) \times x] \times y = 0,\)

which is \([M1]\).

For \([M3]\), we compute

\[y \times [x \times (z \times x)] - [(y \times x) \times z] \times x =
\]

\[y \times [x \times (z \times x)] - (y \times x) \times (z \times x) + (y \times x) \times (z \times x) - [(y \times x) \times z \times x] =
\]

\[a(y, x, z \times x) + a(y \times x, z, x)
\]

For \([M2]\), we compute

\[(x \times y) \times (z \times x) - [x \times (y \times z)] \times x =
\]

\[(x \times y) \times (z \times x) - (x \times y) \times (x \times z)] 	imes x + [(x \times y) \times z \times x] - [x \times (y \times z)] \times x =
\]

\[a(x \times y, z, x) - a(x, y, z) \times x
\]

\([M2]\) and \([M3]\) are dealt with in a similar way.
A.2.3 Derivation graph for several laws

We present here a graph representing the “is-a” relationship for some of the most important structures in this second part of the appendix. We have used abbreviations akin to those of A.1.4 (i.e. “P.-A.” stands for “Power-Associative”, “Alt.” stands for “Alternative”, “Ass.” stands for “Associative” and “w. U.” stands for “with Unit”).
B A brief reminder about Gauss and Jacobi sums

We spell out here what is briefly outlined in [9 page 147].

B.1 Basic definitions and results

B.1.1 Multiplicative characters

Let $p$ be a prime number, $k \geq 1$ an integer, $q = p^k$ and $F = \text{GF}(q)$ the finite field of cardinal $q$; as usual $F^* = F - \{0\}$, and we will denote by $1_F$ the neutral element for the multiplication in $F$. Recall that $F$ is of characteristic $p$.

A multiplicative character on $F$ is a function $\chi : F^* \to \mathbb{C}^*$ such that $(\forall (a,b) \in F^{*2}) \chi(a \cdot b) = \chi(a) \chi(b)$. The character $\epsilon_F : [F^* \to \mathbb{C}^*, a \mapsto 1]$ is extended to all of $F$ as $\epsilon_F(0) = 1$, and any character $\chi \neq \epsilon_F$ is extended to all of $F$ as $\chi(0) = 0$. Therefore, $(\forall (a,b) \in F^2) [\chi(a \cdot b) = \chi(a) \chi(b) = \epsilon_F(a) \epsilon_F(b)]$.

One then proves (as an analogue to [9 proposition 8.1.2]) that given a multiplicative character $\chi$, we have:

$$\sum_{t \in F} \chi(t) = \begin{cases} q & \text{if } \chi = \epsilon_F \\ 0 & \text{otherwise.} \end{cases}$$

We also verify that, as in [9 proposition 8.1.3], the group of multiplicative characters is cyclic of cardinal $q - 1$, and that for any $a \in F^*$ there is a multiplicative character $\chi$ such that $\chi(a) = 1$, with the same corollary that if $a \in F^*$ and $a \neq 1_F$ then $\sum_{\chi \in \text{multiplicative characters of } F} \chi(a) = 0$.

B.1.2 The $x^n = a$ equation in GF($q$)

As well, one sees that (just like in [9 proposition 7.1.2]), given $a \in F^*$, the equation $x^n = a$ has solutions in $F$ if and only if $a^\frac{q-1}{d} = 1_F$, with $d = \text{PGCD}(n, q - 1)$ (i.e. $d$ is the smallest common multiple of $n$ and $q - 1$), and if there are solutions, then there are exactly $d$ solutions.

From that, one deduces (as [9 proposition 8.1.4]) that given $a \in F^*$ and $n \mid (q - 1)$ (i.e. a $n$ which divides $q - 1$), if $x^n = a$ is not solvable in $F^*$, then there exists a multiplicative character $\chi$ such that $\chi^n = \epsilon_F$ and $\chi(a) \neq 1$.

This, of course, leads to the result (corresponding to [9 proposition 8.1.5]) that the number of solutions of the equation $x^n = a$ in $F$, which we will write $N(x^n = a)$, is given by $N(x^n = a) = \sum_{\chi \in \text{mul. char. of } F} \chi(a)$, if $n \mid (q - 1)$.

B.1.3 Galois trace

Recall that given $p$ a prime number and $k \geq 1$ an integer, then $\text{GF}(p^k)$ is isomorphic, through some isomorphism $\mathcal{J}_{p,k} : \text{GF}(p) \to \text{GF}(p^k)$, to the subfield of $\text{GF}(p^k)$ defined by $\{\alpha \in \text{GF}(p^k) \mid \alpha^p = \alpha\}$. Furthermore, given any $\alpha \in \text{GF}(p^k)$, if we compute $\beta = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{k-1}}$ then we find that $\beta^p = \beta$. Hence, given $p$ and $k$ as above, we define\footnote{The unfortunate verbosity is to ensure there is no confusion with the Cayley trace, as we will be using both notions in this document.} the Galois trace on $\text{GF}(p^k)$ as the function

$$\text{tr}_{p,k} : \text{GF}(p^k) \to \text{GF}(p)$$

$$\alpha \mapsto \mathcal{J}_{p,k}^{-1}(\alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{k-1}})$$

Recall that $\text{GF}(p^k)$ is an algebra over $\text{GF}(p)$, and that $\text{tr}_{p,k}$ is a surjective $\text{GF}(p)$-linear operator.
B.2 Gauss sums

Let us define in this appendix, for convenience’s sake, $\zeta_p = e^{\frac{-2\pi i}{p}}$. Since, given $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and $a \in \alpha$ then $(\forall b \in \alpha) \zeta_p^b = \zeta_p^a$, we can unambiguously define $\zeta_p^a$ as the common value of $\zeta_p^a$ for all $a \in \beta$. We can therefor (identifying GF($\mathbb{Z}/p\mathbb{Z}$)) define the function

$$
\psi: F \rightarrow \mathbb{C}
\alpha \mapsto \zeta_p^{\chi(a)}
$$

The salient properties of $\psi$ are that $(\forall (\alpha, \beta) \in F^2) \psi(\alpha + \beta) = \psi(\alpha)\psi(\beta)$, $(\exists \alpha_0 \in F) \psi(\alpha_0) \neq 1$, and $\sum_{\alpha \in F} \psi(\alpha) = 0$. Another crucial property is:

$$
\sum_{\alpha \in F} \psi(\alpha \cdot \beta) = \begin{cases} g \text{ if } \beta = 0 \\ 0 \text{ otherwise.} \end{cases}
$$

We finally define $g_\alpha(\chi) = \sum_{\beta \in F} \chi(\beta)\psi(\alpha \cdot \beta)$ to be a Gauss sum belonging to the character $\chi$ (and relative to an $\alpha \in F$). We will also define $g(\chi) = g_{1,F}(\chi)$.

One can then prove (as for [3] proposition 8.2.1]) that $g_\alpha(\chi)$ takes the following values:

| $g_\alpha(\chi)$ | $\alpha = 0$ | $\alpha \neq 0$ |
|------------------|-------------|-------------|
| $\chi = \epsilon_F$ | $q$ | $0$ |
| $\chi \neq \epsilon_F$ | $0$ | $\chi(\alpha^{-1})g(\chi)$ |

As well (mirroring [2] proposition 8.2.2]), $g(\chi^{-1}) = g(\bar{\chi}) = \chi(-1_F)g(\chi)$ and if furthermore $\chi \neq \epsilon_F$ then $g(\chi)g(\chi^{-1}) = \chi(-1_F)q$, which together imply that $|g(\chi)| = \sqrt{q}$.

B.3 Jacobi sums

Given $\ell$ (at least two) multiplicative characters on $F = \text{GF}(q)$, $\chi_1, \chi_2, \ldots, \chi_\ell$, the Jacobi sums of these characters are defined to be:

$$
J(\chi_1, \chi_2, \ldots, \chi_\ell) = \sum_{t_1 + t_2 + \cdots + t_\ell = 1_F} \bar{\chi}_1(t_1)\bar{\chi}_2(t_2)\cdots\bar{\chi}_\ell(t_\ell)
$$

$$
J_0(\chi_1, \chi_2, \ldots, \chi_\ell) = \sum_{t_1 + t_2 + \cdots + t_\ell = 0} \bar{\chi}_1(t_1)\bar{\chi}_2(t_2)\cdots\bar{\chi}_\ell(t_\ell)
$$

Additionally we will define $J(\chi) = 1_F$ for any multiplicative character $\chi$.

The Jacobi sums verify the following properties (rewriting of [3] proposition 8.5.1, theorem 3, corollary 1 and corollary 2]):

if $(\chi_1, \ldots, \chi_{\ell-1}) = (\epsilon_F, \ldots, \epsilon_F)$ then

$$
J(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = J_0(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = q^{\ell-1}
$$

else (i.e. $(\chi_1, \ldots, \chi_{\ell-1}) \neq (\epsilon_F, \ldots, \epsilon_F)$)

if $\chi_1 = \epsilon_F$ or $\chi_2 = \epsilon_F$ or $\ldots$ or $\chi_\ell = \epsilon_F$ then

$$
J(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = J_0(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = 0
$$

else (i.e. $\chi_1 \neq \epsilon_F$ and $\chi_2 \neq \epsilon_F$ and $\ldots$ and $\chi_\ell \neq \epsilon_F$)

if $\chi_1\chi_2\cdots\chi_\ell = \epsilon_F$ then

$$
J_0(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = (q-1)\chi_\ell(-1_F)J(\chi_1, \chi_2, \ldots, \chi_{\ell-1})
$$

$$
J(\chi_1, \chi_2, \ldots, \chi_\ell) = -\chi_\ell(-1_F)J(\chi_1, \chi_2, \ldots, \chi_{\ell-1})
$$

$$
g(\chi_1)g(\chi_2)\cdots g(\chi_\ell) = q\chi_\ell(-1_F)J(\chi_1, \chi_2, \ldots, \chi_{\ell-1})
$$

else

$$
J_0(\epsilon_F, \epsilon_F, \ldots, \epsilon_F) = 0
$$

$$
g(\chi_1)g(\chi_2)\cdots g(\chi_\ell) = g(\chi_1\chi_2\cdots\chi_\ell)J(\chi_1, \chi_2, \ldots, \chi_\ell)
$$
There are unfortunately several rather different meanings to the term “quadratic algebra”.

One of these (19, 22, 3) is to say that every $x$ in the algebra with unit $E$ (with neutral element $e$) over the commutative and associative ring with unit $A$, satisfies an equation of the form $x^2 - t(x)x + n(x)e = 0$ for some $t(x) \in A$ and $n(x) \in A$. For this meaning of the term “quadratic algebra”, all Cayley algebras are quadratic algebras.

Another meaning to the term “quadratic algebra” is found in [2]. For this other meaning of “quadratic algebra” however, the main thing to keep in mind is that these two kinds of structure, while related, and sometimes overlapping (some algebras are both quadratic algebras and Cayley algebras), are different things. We will highlight here some of the similarities and differences between them.

Let $(A, +, \cdot)$ be an associative and commutative ring with unit, whose neutral element we will denote by $1_A$. An $A$-algebra $(E, +, \times, \cdot)$ is a quadratic algebra over $A$ if and only if it admits, as an $A$-module, a generating\footnote{We do not impose the family to be linearly independent over $A$, so as to accept perhaps pathological cases such as $A = \mathbb{Z}/2\mathbb{Z}$, which can be seen as a quadratic algebra of type $(1, 0)$ over itself, using the family $(e_1, e_2) = (1, 1)$.} family $(e_1, e_2)$ such that the multiplication (“$\times$”) verifies

\[
e_1^2 = e_1 \quad e_1 \times e_2 = e_2 \times e_1 = e_2 \\
e_2^2 = \alpha \cdot e_1 + \beta \cdot e_2
\]

for some $(\alpha, \beta) \in A^2$, and we then say the quadratic algebra is of type $(\alpha, \beta)$, though this is not an intrinsic property of the algebra, i.e. the algebra can happen\footnote{It should also be born in mind that, where Cayley algebras are concerned, applying the Cayley-Dickson procedure with different constants may happen to yield isomorphic structures.} to also be of type $(\alpha', \beta')$ with $(\alpha', \beta') \neq (\alpha, \beta)$ (using another family).

As such, quadratic algebras are necessarily commutative, associative and with unit (with neutral element $e_1$), and this is of course not true of every Cayley algebra (the classical quaternions or octonions for instance) which means there are structures which are Cayley algebra but are not quadratic algebras (as we give examples of below.). As well, this also means any quadratic $A$-algebra can be made into a Cayley algebra, over itself if perhaps not over $A$, by using the identity as conjugation.

It is worth noting that if one makes a Cayley algebra out of an associative and commutative ring with unit $A$, considering it as an algebra over itself and using the identity as conjugation, then applying the Cayley-Dickson process with a constant $\zeta$, then one gets a structure which is both a Cayley algebra and a quadratic algebra of type $(-\zeta, 0)$, when considering the basis $((1_A, 0), (0, 1_A))$, of $A^2$ over $A$. Of course, then the conjugation over $A^2$ is usually no longer the identity\footnote{Though it may still be, if for instance $A$ is of characteristic 2!} and reiterating the Cayley-Dickson process over this new structure yields another Cayley algebra which is usually not a quadratic algebra over either $A$ or $A^2$ (this is the case with $A = \mathbb{R}$ yielding first $\mathbb{C}$ and then $\mathbb{H}$, for instance).

Like Cayley algebras, quadratic algebras have a “natural” doubling process, which, of course, produces structures which are always quadratic algebras, but are usually not Cayley algebras. If $(A, +, \cdot)$ is an associative and commutative ring with unit (with neutral element $1_A$), $(E, +, \cdot, \cdot)$ is a quadratic $A$-algebra (with neutral element $1_E$), and $\alpha$ and $\beta$ any two elements of $E$, then on $F = E \times E$ we consider the following laws:

\[
\begin{align*}
\dagger: & \quad F \times F \to F \\
((x, y), (x', y')) & \mapsto (x + x', y + y') \\
*: & \quad F \times F \to F \\
((x, y), (x', y')) & \mapsto (x \times x' + \alpha \times (y' \times y), y \times x' + y' \times x + \beta \times (y \times y')) \\
\bullet: & \quad E \times F \to F \\
(\lambda, (x, y)) & \mapsto (\lambda \times x, \lambda \times y)
\end{align*}
\]

it is then trivial to check that $(F, \dagger, *, \bullet)$ is a quadratic $E$-algebra, of type $(\alpha, \beta)$ when considering the basis $((1_E, 0), (0, 1_E))$ of $F$ over $E$. Indeed, when one starts with $E = A$, apply first the above doubling using constants $\alpha$ and $\beta$ in $E = A$ and then apply the Cayley-Dickson process using a constant $\zeta \in E = A$ and the identity on $E = A$ as conjugation, the result is known in literature ([2]) as a quaternion algebra.
of type \((\alpha, \beta, -\zeta)\). This, of course, is a Cayley algebra, but usually not a quadratic algebra (either over \(E = A\), or \(E^2\)). One can therefore usually not indefinitely “knit” the two kinds of doubling, which makes this construction rather ad hoc.

There are also other known ways to build useful algebras in a way reminiscent of the Cayley-Dickson process. One of the better known \(\mathbb{R}\) is to consider a commutative and associative ring with unit \(A\), with neutral element \(1_A\), and build the algebra \(H(A) = A + A I + A J + A K\) with \(I^2 = J^2 = K^2 = IJ K = -1_A\). With \(A = \mathbb{R}\) one obtains \(\mathbb{H}\), the usual quaternions, and with \(A = \mathbb{C}\) one obtains what is known as the complex quaternions (or biquaternions) \([7, 15]\).

There is at least one more construction which deserves consideration, and which does not seem to fit well with those we have considered so far: that of the sedenions.

In \([3, 4]\) Carmody presents constructions originally due to Musès, some of which (the counter-complex, counter-quaternions and counter-octonions) are easily seen as examples of the Cayley-Dickson process in action. Indeed, we can draw the following filliation graph (where \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(\mathbb{O}\) stand for the set of real numbers, the set of complex numbers, the quaternions and the octonions respectively; \(\mathbb{X}\) we have dubbed the hexadecimations, \([8]\), but see below):

There is also a structure on \(\mathbb{R}^{16}\), which has been called by Musès and Carmody, the sedenion. That structure is different from that of \(\mathbb{X}\), as we shall see, but unfortunately the name “sedenion” has also (posteriorly) been used to designate the hexadecimations \([8]\) and, at the time of this writing, \([1]\).

We will keep the name “sedenion” to designate the construct of Musès and Carmody. It has the following Cayley (multiplication) table:

|   | 1 | \(i_1\) | \(i_2\) | \(i_3\) | \(i_4\) | \(i_5\) | \(i_6\) | \(i_7\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) | \(\omega\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | +1 | +1 | +i_1 | +i_3 | +i_4 | +i_5 | +i_6 | +i_7 | +e_1 | +e_2 | +e_3 | +e_4 | +e_5 | +e_6 | +e_7 | +\omega |
| i_1 | +i_1 | -1 | +i_3 | -i_2 | +i_5 | -i_4 | -i_7 | +i_6 | +\omega | +e_3 | -e_2 | +e_5 | -e_4 | -e_7 | +e_6 | -e_1 |
| i_2 | +i_2 | -i_3 | -1 | +i_1 | +i_6 | +i_7 | -i_4 | -i_5 | -e_3 | +\omega | +e_1 | +e_6 | +e_7 | -e_6 | -e_5 | -e_2 |
| i_3 | +i_3 | +i_2 | -i_1 | -1 | +i_7 | -i_6 | +i_5 | -i_4 | +e_2 | -e_1 | +\omega | +e_7 | -e_6 | +e_5 | -e_4 | e_3 |
| i_4 | +i_4 | -i_5 | -i_6 | -i_7 | -1 | +i_1 | +i_2 | +i_3 | -e_5 | -e_6 | -e_7 | +\omega | -e_1 | +e_2 | +e_3 | e_4 |
| i_5 | +i_5 | +i_4 | -i_7 | +i_6 | -i_1 | -1 | -i_3 | +i_2 | +e_4 | -e_7 | +e_6 | -e_1 | +\omega | -e_3 | +e_2 | -e_5 |
| i_6 | +i_6 | +i_7 | +i_4 | -i_5 | -i_2 | +i_3 | -1 | -i_1 | +e_7 | +e_4 | -e_5 | -e_2 | +e_3 | +\omega | e_7 | -e_1 |
| i_7 | +i_7 | -i_6 | +i_5 | +i_4 | -i_3 | -i_2 | i_1 | -1 | -e_6 | +e_5 | +e_4 | -e_3 | -e_2 | +e_1 | +\omega | -e_7 |
| e_1 | +e_1 | +w | +e_3 | -e_2 | +e_5 | -e_4 | -e_7 | +e_6 | +1 | -i_3 | +i_2 | -i_5 | +i_4 | +i_7 | -i_6 | +i_1 |
| e_2 | +e_2 | -e_3 | +\omega | +e_1 | +e_6 | +e_7 | -e_4 | -e_5 | +i_3 | +1 | -i_1 | -i_6 | -i_7 | +i_4 | +i_5 | +i_2 |
| e_3 | +e_3 | +e_2 | -e_1 | -1 | +e_7 | -e_6 | -e_5 | -e_4 | -i_2 | i_1 | +1 | -i_7 | +i_6 | -i_5 | +i_4 | +i_3 |
| e_4 | +e_4 | -e_5 | -e_6 | -e_7 | +\omega | +e_1 | +e_2 | +e_3 | +i_5 | +i_6 | +i_7 | -1 | -i_2 | -i_3 | +i_4 | +i_5 |
| e_5 | +e_5 | +e_4 | -e_7 | +e_6 | -e_1 | +\omega | -e_3 | +e_2 | -i_4 | +i_7 | -i_6 | +i_1 | +i_3 | -i_2 | +i_3 | +i_5 |
| e_6 | +e_6 | +e_7 | +e_4 | +e_5 | -e_2 | +e_3 | +\omega | -e_1 | -i_7 | -i_4 | +i_5 | +i_2 | -i_3 | 1 | +i_4 | +i_6 |
| e_7 | +e_7 | +e_6 | +e_3 | +e_4 | -e_3 | -e_2 | +e_1 | +\omega | +i_6 | -i_5 | -i_4 | +i_3 | +i_2 | -i_1 | 1 | +i_7 |
| \omega | +\omega | -e_1 | -e_2 | -e_3 | -e_4 | -e_5 | -e_6 | -e_7 | +i_1 | +i_2 | +i_3 | +i_4 | +i_5 | +i_6 | +i_7 | -1 |
There is an involutive automorphism of the sedenions defined by
\[
\sigma(x + \alpha_1 i_1 + \cdots + \alpha_7 i_7 + \beta_1 e_1 + \cdots + \beta_7 e_7 + t \omega) = x - \alpha_1 i_1 - \cdots - \alpha_7 i_7 - \beta_1 e_1 - \cdots - \beta_7 e_7 + t \omega
\]
which does not make the sedenions into a Cayley algebra (because the image of \([x \mapsto x + \sigma(x)]\) is not contained in \(\mathbb{R}\)), but such that if we define \(\mathcal{N}\) by \(\mathcal{N}(x) = x \sigma(x)\), then \(\mathcal{N}(xy) = \mathcal{N}(x) \mathcal{N}(y)\). This, actually, is what seems to have driven the creation of the sedenions.

There are varying expectations on how a “norm” defined on an algebra should behave. For norms taking their values in a valuated field, the least one would expect would be that the norm of the product of two elements to be smaller or equal to the product of the norms of the elements (the usual rule for square matrices of reals, seen as linear operators). A more stringent requirement is the “composition” behavior, by which the norm of the product of two elements is equal to the product of the norms of the elements (the usual rule for reals, complex, quaternions and octonions).

Without any assumption on associativity for the multiplication of that algebra, either behavior can fail to be met. For instance, consider in \(\mathbb{X}\), \(x = i + j''\) and \(y = e' + k''\), then \(x \times y = 2j' - 2j''\) and \(\mathcal{N}_\mathbb{X}(x \times y) = 8 > 4 = \mathcal{N}_\mathbb{X}(x) \times \mathcal{N}_\mathbb{X}(y)\), where we have considered the Cayley norm as taking its values in \(\mathbb{R}\), as explained earlier. Also, if we consider the following example given by Carmody, \(z = i + e'e'' = i + e''\) and \(t = j + k'e'' = j + k''\), then \(z \times t = 0\) and \(\mathcal{N}_\mathbb{X}(z \times t) = 0 < 4 = \mathcal{N}_\mathbb{X}(z) \times \mathcal{N}_\mathbb{X}(t)\). There is therefore really no reasonable relation to be expected in \(\mathbb{X}\) between the norm of a product and the product of the norms!

The sedenions, as we have said, display the “composition” behavior; they are not a composition algebra (\([22, 20]\) however, because the norm is not non-degenerate. According to \([3]\), the algebra is alternative, and verifies (M5), (M4), (M3) and (M2). It is also riddled with zero divisors (as it contains the counter-complex, counter-quaternions and counter-octonions; it also contains the octonions).

While the construction of the sedenions could be carried out on other rings, it does appear like a stunning ad’hoc construction.
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