Solving Cantilever Beam Models by Reliable Iterative Methods

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Abstract. In this article, we implemented two reliable Cantilever beam models of solution methods. The iterative approaches are the Adomain Decomposition Method (ADM) and the Variation Iteration Method (VIM). The ADM and VIM solve several problems at various places, including the precision and reliability of the results. The solution given with an approximate solution is accuracy, since we demonstrate that various figures and tables are available to investigate maximum error reminders. The technology used for the study calculations was MATHEMATICA 11.

1. Introduction
Non-linear differential equations in various sciences are the most important topic of different problems in engineering fields, and in physics nonlinear ordinary differential equations like Duffing equations can be represented. In engineering and physics, duffing equations were extremely significant, since they were used in many problems such as solid mechanics and structural and fluid mechanics [1, 2].

Duffing equations used in cantilever beam models, where Hamdan and Shabahen [3] were introduced to investigate the large free amplified vibrations of a uniform cantilever beam with an intermediate lumped mass and votive inertia [4].

Hu et al [5] also developed work on the nonlinear vibration of a cantilever with a connecting terminal Derjaguin-Muller-Toporov, considering the key rebound. Many methods have solved this model, such as the differential method of transformation [6] and the method of energy balance [7].

Fig. (1): The physical model of the Duffing equation with constant coefficients [8].
The system in fig. (1) can be represented by the following equation [8]:

\[ u'' + \frac{k_1}{m} u + \frac{k_2}{2mh^2} u^3 = \frac{F_0}{m} \sin w_0 x, \]

With initial condition:

\[ u(0) = A, \quad \text{and} \quad u'(0) = 0. \]

Where \( k_1 = 300, k_2 = 200, F_0 = 1, w_0 = 2, m = 10, h = 1 \) and \( A = 0.5 \), then get

\[ u'' + 30 u + 10 u^3 = 0.1 \sin 2x, \]

With initial conditions:

\[ u(0) = 0.5, \quad \text{and} \quad u'(0) = 0. \]

In this article, two iterative methods will be used to find some new approximate solutions for the Cantilever beam models. George Adomian introduced the first in 1980 [9]. In 1999 Ji-Huan He is now used for a broad range of linear and nonlinear, homogeneous and uniform equations [10]. In many various fields of science like physics, chemistry, Mechanics and other sciences, this mechanism solved many different formulas.

2. The fundamental procedure of the ADM

Let us take into account the following ADM ODE[11]:

\[ Lu + Ru + Nu = G \]

Where \( N \) is a nonlinear operator, \( L \) is an invertible high-order derivative, \( R \) is a linear differential operator in the order of less than \( L \), and \( G \) is a non-homogeneous term. The method begins by formally applying the operator \( L^{-1} \) to the expression

\[ L^{-1}u = G - R^{-1} - N^{-1}u, \]

With initial condition:

\[ u(0) = a, \quad \text{and} u'(0) = b. \]

So by using the given conditions, we have:

\[ u = b + L^{-1}G - R^{-1} - N^{-1}u, \]

Where \( L^{-1}(.) = \int_0^x \cdot dx \), and \( u'(0) = b \), is the initial condition.

The ADM technique consists of approximating the solution of Eq. (2) as an infinite series:

\[ u = \sum_{n=0}^{\infty} u_n, \]

Along with nonlinear concepts, the nonlinear differential equation cannot be resolved directly by ADM. We have to calculate the so-called Adomian polynomial in order to deal with the nonlinear concept. Adomian’s polynomials are arranged to have the form:

\[ F(u) = Nu = \sum_{n=0}^{\infty} A_n, \]

Where \( A_n \) are the Adomain polynomial of \( u_0, u_1, ..., u_n \) that are the terms of the analytical expansion of \( Nu \), where

\[ u = \sum_{i=0}^{\infty} \lambda i u_i, \]
around $\lambda=0$ [11]. That is:

$$A_n = \frac{1}{n!} \left( \frac{\partial^n}{\partial \lambda^n} F \left( \sum_{i=0}^{n} \lambda^i u_i \right) \right)_{\lambda=0}, \quad n = 0, 1, 2, \ldots$$

Adomian polynomials are arranged to have the form [11]:

$$A_0 = F(u_0),$$
$$A_1 = u_1 F'(u_0),$$
$$A_2 = u_2 F''(u_0) + \frac{u_1^2}{2!} F''(u_0),$$
$$A_3 = u F'(u_0) + u_1 u_2 F''(u_0) + \frac{u_1^3}{3!} F'''(u_0),$$

Now, we parameterize Eq. (2) in the form:

$$u = b + L^{-1} g - \lambda L^{-1} R u - \lambda L^{-1} N u,$$

where $\lambda$ is just an identifier of collecting terms in a suitable way such that $u_n$ depends on $u_0, u_1, \ldots, u_n$ and we will later set $\lambda=1$.

$$\sum \lambda^n u^n = b + L^{-1} g - \lambda L^{-1} R \sum \lambda^n u^n - \lambda L^{-1} \lambda^n A^n,$$

Equating the coefficients of equal powers of $\lambda$, we obtain:

$$u_0 = b + L^{-1} G,$$
$$u_1 = -L^{-1} R(u_0) - L^{-1}(A_0),$$
$$u_2 = -L^{-1} R(u_1) - L^{-1}(A_1),$$

and in general

$$u_n = -L^{-1} R(u_{n-1}) - L^{-1}(A_{n-1}).$$

To explain the use of ADM for the Duffing equation. Also, we calculate the remaining error with the remaining maximum error variables in the approximate solution, the remaining error convenience function [7, 10]:

$$E_{R_n}(x) = u_n'(x) + k_1 u_n(x) + k_2 u_n + k_3 u_n^3 - f(x),$$

The maximal error remainder parameters are:

$$M_{E_{R_n}} = \max_x |E_{R_n}(x)|.$$  \hspace{1cm} \text{(3)}$$

2.1 ADM for solving Cantilever beam models

Now, take the taylor series of $\sin 2x$ as following

$$\sin 2x = 2x - \frac{4x^3}{3} + \cdots$$

Now write Eq. (1) in an operator form then have:

$$L_{xx}u = 0.1 \sin 2x - 30 u(x) - 10 u^3(x),$$

Where the differential operator $L$ is given by $L_x = \frac{d}{dx}$, and $L_{xx} = \frac{d^2}{dx^2}$ and therefore the inverse operator is:
\[ L_x^{-1}(.) = \int_0^x (.) dx, \quad \text{and} \quad L_{xx}^{-1}(.) = \int_0^x \int_0^x (.) dx dx. \]

Applying \( L_x^{-1} \) to both sides and using the initial condition then get:
\[ u(x) = 0.5 + 0.033x^3 - 0.0066x^5 - L_x^{-1}(30 u(x)) - L_{xx}^{-1}(10 u^3(x)) \]  \hspace{1cm} (7)

Applying the algorithm of the A domain method
\[ u(x) = \sum_{n=0}^{\infty} u_n(x), \]
and the nonlinear term \( u^3 \), can be express as follows:
\[ u^3 = \sum_{n=0}^{\infty} A_n. \]

Then, Eq. (7) will be obtained:
\[ \sum_{n=0}^{\infty} u_n(x) = 0.5 + 0.033x^3 - 0.0066x^5 - 30 L_x^{-1} \left( \sum_{n=0}^{\infty} u_n(x) \right) - 10 L_{xx}^{-1} \left( \sum_{n=0}^{\infty} A_n \right). \]

The following recursive relation:
\[ u_0 = 0.5 + 0.033x^3 - 0.0066x^5, \]
\[ u_{n+1} = 0.5 + 0.033x^3 - 0.0066x^5 - L_x^{-1}(u_n) - L_{xx}^{-1}(A_n), \quad n \geq 0. \]

Now, find A domain polynomial \( A_n \).
\[ A_0 = f(u_0) = 10 u_0^3, \]
\[ A_1 = u_1 f'(u_0) = 30 u_0^2 u_1, \]
\[ A_2 = u_2 f''(u_0) + \frac{1}{2!} u_1^2 f'''(u_0) = 30 u_0^2 u_1^2 + 30 u_0 u_1^2, \]

Follows immediately. Consequently, we obtain:
\[ u_0 = 0.5 + 0.033x^3 - 0.0066x^5, \]
\[ u_1 = -L_x^{-1}(30 u_0) - L_{xx}^{-1}(A_0), \]
\[ u_2 = -8.125x^2 - 0.0625x^5 + 0.0059x^7 - 0.000297x^8 + \cdots, \]
\[ u_3 = -L_x^{-1}(30 u_1) - L_{xx}^{-1}(A_1), \]
\[ u_4 = 25.390625x^4 + 0.24925x^7 - 0.0256x^9 + 0.0038x^{10} - \cdots, \]

and so on.

Continuing in this way, the approximation solution of Eq.(1):
\[ u(x) = \sum_{n=0}^{\infty} u_n(x), \]
\[ u(x) = 0.5 - 8.125x^2 + 0.033x^3 + 25.3906x^4 - 0.069166x^5 + 0.255208x^7 - 0.000297x^8 + \cdots, \]

The approximate solution of Eq. (1) see that in the table (1) and figure (2).
Table (1): numerical result for equation (1).

| n  | MER            |
|----|----------------|
| 1  | 2.95331        |
| 2  | 0.183281       |
| 3  | 0.0132565      |
| 4  | 0.00130181     |

Fig (2): MER for equation (1).

3. The VIM’s basic procedure

We suggest the following nonlinear equation to explain the basic concepts of VIM.[12]:
\[ Lu(t) + Nu(t) = g(t), \quad t > 0. \]  ... (4)

Where, L is a linear operator, N a nonlinear operator and g(x) is the source the inhomogeneous term. The VIM introduces functional for Eq. (4) in the form

\[ u_{k+1}(x) = u_k(x) + \int_0^t [\lambda(t)(Lu_k(t) - N\bar{u}_k(t) - g(t))] \, dt, \]  ... (5)

Where, \( \lambda \) is a general Lagrangian multiplier calculated in the following form [13, 14]:
\[ \lambda = \frac{(-1)^n}{(n-1)!} (t-x)^{n-1}, \quad n \geq 1. \]  ... (6)

Consequently, the solution is given by:
\[ u(x) = \lim_{n \to \infty} u_n(x). \]

3.1 VIM for solving Cantilever beam models

The correction functional is:
\[ u_{n+1} = u_n(x) + \int_0^x \lambda(t)(u''_n + 30u_n + 10u_n^3 - g(t)) \, dt, \]

By using the formula in Eq. (1. 21) given above leads to:
In this equation, let \( \lambda \) be defined as \( t - x \):

\[
u_{n+1}(x) = u_n(x) + \int_0^x (t - x)(u''_n + 30u_n + 10u_n^3 - g)(t) \, dt,
\]

Can be selected:

\[
u_0 = u(0) + xu'(0) = 0.5,
\]

Now obtain the following successive approximations by using (VIM):

\[
u_1 = 0.5 + \int_0^x (x - t)(u''_0 + 30u_0 + 10u_0^3 - 0.2x - 0.133x^3)(t) \, dt,
\]

\[
u_1 = 0.5 - 8.125x^2 + 0.0333x^3 - 0.0066x^5,
\]

\[
u_2 = \nu_1 + \int_0^x (t - x)(u''_1 + 30u_1 + 10u_1^3 - 0.2x - 0.133x^3)(t) \, dt,
\]

\[
u_2 = 0.5 - 8.125x^2 + 0.0333x^3 + 25.390625x^4 - 0.69166x^5 - 33.00781x^6 + 0.199404x^7 + 95.781x^8 - ... \]

and so on. Then

\[
u(x) = 0.5 - 8.125x^2 + 0.0333x^3 + 25.390x^4 - 0.69166x^5 - 33.00781x^6 + 0.199404x^7 + 95.781x^8 - ... .
\]

**Table (2):** MER for equation (1).

| \( n \) | MER |
|--------|-----|
| 1      | 2.95204 |
| 2      | 0.0884584 |
| 3      | 0.00107684 |
| 4      | 7.04574 \times 10^{-6} |

**Fig (3):** MER for equation (1).
4. Conclusions

Here in this article, we introduced two iterative analytic methods. These methods namely ADM and VIM proposed to solve Cantilever Beam Models. These methods find the approximate solution in the high accuracy of the result in numerical examples.

5. References

[1] M. A. Al-Jawary, S. G. Abd- ALRazaq, Analytic and numerical solution for Duffing equations, International Journal of Basic and Applied Sciences, 5 (2) (2016) 115- 119.

[2] M. A. Al-Jawary, S. G. Abd- ALRazaq, A semi analytical iterative technique for solving Duffing equations, International Journal of Pure and Applied Mathematics, 108 (4) (2016) 871- 885.

[3] M.N. Hamden and N.H. Shabaneh,"On the large amplitude free vibrations of a restrained uniform beam carrying an intermediate lumped mass", Journal of Sound and Vibration 199(5), 711–736, (1997).

[4] M. Khalid, M. Sultana, U. Arshad, M. Shoaib, "A Comparison Between New Iterative Solutions of Non- Linear Oscillator Equation", University Road, Karachi, (2015).

[5] Q.Q. Hu, C.W. Lim and L.Q. Chen, "Nonlinear vibration of a cantilever with a Derjaguin–M’uller–Toporov contact end", Int J Struct Stab Dyn 8(1), 25–40, (2008).

[6] V. Daftardar-Gejji, H. Jafari, "An iterative method for solving nonlinear functional equations", Journal of Mathematical Analysis and Applications, 316, 753–763, (2006).

[7] J. Duan, R. Rach, A.M. Wazwaz," Steady-state concentrations of carbon dioxide absorbed into phenyl glycidyl ether solutions by the Adomian decomposition method", Journal of Mathematical Chemistry, 53, 1054–1067, (2015).

[8] M. Bayat, I. Pakar, "On the approximate analytical solution to non- linear Oscillation systems", Shock and Vibration, 20, 43- 52, (2013).

[9] G. Adomain, "Solving frontier problems of physics: the decomposition method", Springer- science+ Business madia, USA, 238-239, (1994), (book).

[10] M.A. AL-Jawary, G. H. Radhi," The variational Iteration Method for calculating carbondioxide absorbed into phenyl glycidyl ether". IOSR Journal of Mathematics, 11, 99–105, (2015).

[11] A.M. Wazwaz," Linear and Nonlinear Intagral Equations Methods and Applications", Saint Xavier University, Higher Education Press, Beijing and Springer-Varleg Berlin Heidlberg, (2011).

[12] M. Khalid, M. Sultana, U. Arshad, M. Shoaib, "A Comparison Between New Iterative Solutions of Non-Linear Oscillator Equation", University Road, Karachi, (2015).

[13] S. Gani, “Two Reliable Iterative Methods for Solving Chaos synchronization.” Journal of Physics: Conference Series. Vol. 1530. No. 1. IOP Publishing, 2020.

[14] A.M. Wazwaz, "The variational iteration method for solving two forms of Blasius equation on a half-infinite domain", Applied Mathematics and Computations 188, 485-491, (2007).