THE DESCRIPTIVE COMPLEXITY OF THE SET OF ALL CLOSED ZERO-DIMENSIONAL SUBSETS OF A POLISH SPACE

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Abstract. Given a space $X$ we investigate the descriptive complexity class $\Gamma_X$ of the set $\mathcal{F}_0(X)$ of all its closed zero-dimensional subsets, viewed as a subset of the hyperspace $\mathcal{F}(X)$ of all closed subsets of $X$. We prove that $\max\{\Gamma_X; X \text{ analytic}\} = \Sigma^1_2$ and $\sup\{\Gamma_X; X \text{ Borel } \Pi^0_\xi\} \supseteq \mathcal{D}\Sigma^0_\xi$ for any countable ordinal $\xi \geq 1$. In particular we prove that there exists a one-dimensional Polish subspace of $2^{\omega} \times \mathbb{R}^2$ for which $\mathcal{F}_0(X)$ is not in the smallest non trivial pointclass closed under complementation and the Souslin operation $A$.

1. Introduction

If $X$ is a compact metrizable space and if we endow the hyperspace space $\mathcal{F}(X)$ of all closed subsets of $X$ with its canonical (compact metrizable) topology then (see Proposition 5.1) the set $\mathcal{F}_0(X)$ of all zero-dimensional closed subspaces of $X$ is a $\Pi^0_2 = G_\delta$ subset of $\mathcal{F}(X)$. When $X$ is an arbitrary separable metrizable space there is no canonical topology on $\mathcal{F}(X)$. Nevertheless in this general context $\mathcal{F}(X)$ is still naturally endowed with a canonical Borel structure (the Effros Borel structure) which is isomorphic to the Borel structure of an analytic (Polish) space if $X$ itself is analytic (Polish). As we shall see for any analytic space $X$ the set $\mathcal{F}_0(X)$ is $\Sigma^1_2$ (Proposition 6.1) and this bound is optimal (Theorem 6.2), which answers Question 5.2 of [11].

Theorem 1.1. There exists an analytic space $X_\infty \subset 2^{\omega} \times [0,1]$ for which the set $\mathcal{F}_0(X_\infty)$ is $\Sigma^1_2$-complete.

Here and in the sequel we denote by $[0,1]$ the interval $[0,1]$, and by $\mathbb{P}$ and $\mathbb{Q}$ the sets of irrational and rational numbers respectively. The definition of the notions of $\Gamma$-complete and $\Gamma$-hard set are recalled in Section 2.1. We do not know whether one can impose on the space $X_\infty$ to be Polish or even Borel. However we shall prove the following lower complexity bound:

Theorem 1.2. For any countable ordinal $\xi \geq 3$ there exists a $\Pi^0_\xi$ set $X_\xi \subset 2^{\omega} \times [0,1]$ for which the set $\mathcal{F}_0(X_\xi)$ is $\mathcal{D}\Sigma^0_\xi$-hard.

For the definition of the classes $\mathcal{D}\Gamma$ we refer the reader to Section 7. We point out that in [11] E. Pol and R. Pol showed that $\mathcal{F}_0([0,1] \times \mathbb{P})$ is actually a $\Pi^1_1 \setminus \Sigma^1_1$

2010 Mathematics Subject Classification. Primary: 03E15, 54H05; Secondary: 54F45.

Key words and phrases. Effros Borel structure, zero-dimensional sets, $\sigma$-ideals, $\mathcal{D}\Gamma$ classes.
set (note that $\Pi^1_1 = \mathcal{C} \Sigma^0_1$ and $\Sigma^1_1 = \mathcal{C} \Pi^0_1$). We do know not whether the previous result holds for $\xi = 2$ but in this case we have the following:

**Theorem 1.3.** There exists a Polish space $Y \subset 2^\omega \times I^2$ for which the set $F_0(Y)$ is $\mathcal{C} \Sigma^0_2$-hard.

We point out that the class $\mathcal{C} \Sigma^0_2$ is strictly larger than the more classical Selivanovski’s class of $C$-sets. In particular the set $F_0(Y)$ in Theorem 1.3 is not in the smallest class containing all open sets and closed under complementation and operation $\mathcal{A}$. This gives a partial answer to Question 9.12 of [1] which asks about the complexity of $F_0(X)$ for an arbitrary Polish space $X$.

It is interesting to observe that determining the exact complexity of $F_0(X)$ for an explicit Polish space $X$ reveals to be a non trivial problem, as long as compactness is faraway. A typical example is the case of the space $I^2 \setminus Q^2$ pointed out in [11] where the authors ask whether the set $F_0(I^2 \setminus Q^2)$ is $\Pi^1_1$, a question still open. However it turns out that the space $Y$ of Theorem 1.3 can in fact be embedded as a closed subset of $I^5 \setminus Q^5$. It follows then:

**Theorem 1.4.** The set $F_0(I^5 \setminus Q^5)$ is $\mathcal{C} \Sigma^0_2$-hard.

The construction of the space $Y$ in Theorem 1.3 relies on a class of subsets of the plane that we call hypergraphs. More precisely the space $Y$ will be a $G_\delta$ subset of $2^\omega \times H$ where $H$ is a $G_\delta$ hypergraph. This contrasts with the fact that the set $F_0(H)$ for $H$ itself is a $G_\delta$ set. The precise definition of an hypergraph is a bit technical. As a first approach one can say that an hypergraph is a subset $H$ of the plane satisfying $G \subset H \subset \overline{G}$ where $G$ is the graph of some maximal continuous function $f : P \subset I \to \mathbb{R}$ whose domain $P$ is a co-countable subset of some interval $I$ and strongly oscillating around any element of $I \setminus P$.

We only state here two simple properties of hypergraphs:

**Theorem 1.5.** An hypergraph is a one-dimensional connected Polish space which contains no non-trivial continuum.

**Theorem 1.6.** A closed subset of an hypergraph is zero-dimensional if and only if it is of empty interior.

2. **General notions and notations**

2.1. **Descriptive classes:** In all the paper we shall work in the frame of separable metrizable, mostly Polish, spaces. We assume the reader to be familiar with basic notions and results from Descriptive Set Theory, for example as presented in [6] or [10].

A set $A$ is always viewed as a subset of some Polish space $X$, and the complexity of $A$ is then relative to the given space $X$. We recall the fundamental notion of reduction: given $A \subset X$ and $B \subset Y$ the pair $(A, X)$ is said to be reducible to the pair $(B, Y)$ if there exists a continuous mapping $\varphi : X \to Y$ such that $A = \varphi^{-1}(B)$. A descriptive class (or more simply a class) is a class of such pairs $(A, X)$ closed under reducibility.
In fact we shall identify a set $A$ with a pair $(A, X)$ for some (any) Polish space $X \supset A$ depending on the notion. In particular we shall view descriptive classes as classes of sets and when we speak about the complexity of $A$ we mean its complexity relatively to any Polish space $X$ in which $A$ embeds, that is the absolute complexity of $A$. But when we say that $A$ is reducible to $B$ we mean that $(A, X)$ is reducible to $(B, Y)$ for some Polish spaces $X \supset A$ and $Y \supset B$.

We shall consider in the sequel the classical classes that we list next, but also the less popular classes $\mathcal{G}$ that we will present later on in Section 7:

1. **$\Sigma^0_\xi$:** the additive Baire class of rank $\xi$
2. **$\Pi^0_\xi$:** the multiplicative Baire class of rank $\xi$
   (for $\xi = 2$ we shall also use the classical notation $F_\sigma$ and $G_\delta$)
3. **$\Sigma^1_1$:** the class of analytic sets.
4. **$\Pi^1_1$:** the class of co-analytic sets.
5. **$\Sigma^1_2$:** the class of projections of co-analytic sets, also denoted by $PCA$.
6. **$\Pi^1_2$:** the class of complements of $PCA$ sets, also denoted by $CPCA$.
7. **$\Delta^1_2 := \Sigma^1_2 \cap \Pi^1_2$**

Given some class $\Gamma$ a set $A$ (not necessarily in $\Gamma$) is said to be $\Gamma$-hard if any set $A' \subset 2^\omega$ with $A' \in \Gamma$ is reducible to $A$; if moreover $A$ is in $\Gamma$ then $A$ is said to be $\Gamma$-complete. If $A$ is $\Gamma$-hard and $A$ is reducible to $B$ (in particular if $A$ is a closed subset of $B$) then $B$ is $\Gamma$-hard too.

To any class $\Gamma$ we associate its dual class $\hat{\Gamma} = \{ A^c : A \in \Gamma \}$. If $\Gamma$ is not self-dual ($\Gamma \neq \hat{\Gamma}$) then any $\Gamma$-complete set is in $\Gamma \setminus \hat{\Gamma}$.

Computing the exact complexity of a given set $A$ is finding a class $\Gamma$ such that $A$ is $\Gamma$-complete. When this is not accessible one then looks for a lower (upper) bound of this complexity that is a class $\Gamma$ such that $A$ is $\Gamma$-hard ($A$ is in $\Gamma$)

For any mapping $f : X \to Y$ we denote by $\text{Gr}(f)$ its graph.

### 2.2. Sections in product spaces:

We shall very often work in product spaces $X \times Y$ and we need to fix a number of practical conventions. Given a set $E \subset X \times Y$ and an element $a \in X$ (the first factor) we denote by

$$E(a) = \{ y \in Y : (a, y) \in E \}$$

the section of $E$ at $a$ parallel to the second factor. We also associate to $E$ the set

$$\tilde{E} = \{ (x, (x, y)) \in X \times (X \times Y) : (x, y) \in E \} \subset X \times E$$

hence

$$\tilde{E}(a) = \{ a \} \times E(a) \subset E.$$  

Observe that as a subset of $Y$ the set $E(a)$ might be as complicated as the set $E$ while $\tilde{E}(a)$ is a closed subset of $E$.

### 2.3. Admissible topologies on $\mathcal{F}(X)$:

For any topological space $(X, \tau)$ we denote by $\mathcal{F}(X)$ the space of all closed subsets of $X$. 

...
We recall that the Vietoris topology on the hyperspace $F(X)$ is the topology $\tilde{\tau}$ generated by the sets of the form:

$$F(V) = \{ F \in F(X) : F \subset V \} \quad \text{and} \quad F^+(V) = \{ F \in F(X) : F \cap V \neq \emptyset \}$$

where $V$ is an arbitrary open subset of $X$. It is well known that if $(X, \tau)$ is compact then $(F(X), \tilde{\tau})$ is compact too; and if moreover $X$ is metrizable then $\tilde{\tau}$ is the topology induced by the Hausdorff metric associated to any compatible metric on $X$.

For an arbitrary separable metrizable space $(X, \tau)$ the topology $\tilde{\tau}$ is no more metrizable. However one can embed $X$ in a compact space $\hat{X}$ (not necessarily densely) and then identify any set $F \in F(X)$ with $\overline{F}$ (its closure in $\hat{X}$). The set $F(X)$ is then identified to the set:

$$F(X, \hat{X}) = \{ K \in F(\hat{X}) : \overline{K \cap X} = K \}$$

Observe that conversely one can recover $F(X)$ from $F(X, \hat{X})$ since:

$$F(X) = \{ K \cap X : K \in F(X, \hat{X}) \}.$$ 

This identification induces on $F(X)$ a topology $\tilde{\tau}_X$ which depends on the specific compactification $\hat{X}$.

A topology on $F(X)$ will be said to be admissible if it is of the form $\tilde{\tau}_X$ for some compactification $\hat{X}$. We point out that any two admissible topologies on $F(X)$ are first Baire class isomorphic. In particular if $\Gamma$ is any descriptive class which is invariant by first Baire class isomorphisms, and $\tilde{\tau}_{X_1}$, $\tilde{\tau}_{X_2}$ are two compactifications of $X$, a set $S \subset F(X)$ is in $\Gamma(\tilde{\tau}_{X_1})$ iff $S$ is in $\Gamma(\tilde{\tau}_{X_2})$. This is in particular the case if $\Gamma$ is a projective class or a Baire class of infinite rank.

In all the sequel $F(X)$ will implicitly be endowed with an admissible topology.

We state next the main properties (see [6] Section 12 C) that we will use in the sequel:

a) If $X$ is analytic (Polish) then $F(X)$ is analytic (Polish).

b) If $A$ is open (closed, compact) in $X$ then the set $\{ F \in F(X) : F \cap A \neq \emptyset \}$ is open (analytic, closed) in $F(X)$.

c) The set $\{ (F, F') \in F(X) \times F(X) : F \subset F' \}$ is Borel in $F(X) \times F(X)$.

We mention that one can find in the literature other natural Polish topologies on the hyperspace of closed subsets of a Polish space $X$, such as the Wijsman topology. In most of these topologies the sets of the form $F^+(V)$ is open for $V$ open subset of $X$; it follows then that the Borel structure generated by such a topology coincides with the Borel structure generated by any admissible topology, that is the Effros Borel structure. Also as far as one is interested in descriptive classes of higher complexity than the Borel class, which is the case for our study, the precise choice of the topology is irrelevant.
2.4. Topological dimension: We recall the following fundamental result from Dimension Theory, that we shall very often use in the sequel:

**Theorem 2.5.** Suppose that \( X = \bigcup_{n \geq 0} X_n \). If for all \( n > 0 \), \( X_n \) is a closed subset of \( X \) and \( \dim(X_n) = 0 \) then \( \dim(X) = 1 + \dim(X_0) \).

In particular if \( X_0 = \emptyset \) one gets:

**Theorem 2.6.** If \( X = \bigcup_{n \geq 0} X_n \) and for all \( n \geq 0 \), \( X_n \) is a closed subset of \( X \) and \( \dim(X_n) = 0 \) then \( \dim(X) = 0 \).

3. Computing the descriptive complexity of \( \mathcal{F}_0(X) \)

3.1. Pseudo-graphs and canonical mappings: The graph of a mapping \( \Phi : X \to \mathcal{F}(Y) \) is of course a subset of \( X \times \mathcal{F}(Y) \) but one can also consider the graph of the canonical set-valued mapping (with possibly empty values) associated to \( \Phi \) that is the set:

\[
E = \{(x,y) \in X \times Y : y \in \Phi(x)\}.
\]

that we shall call the *pseudo-graph* of \( \Phi \). Observe that if \( (U_n) \) is any countable basis of topology in \( Y \) and for all \( n \), \( A_n = \{x \in X : \Phi(x) \cap U_n \neq \emptyset\} \) then \( E = \bigcap_n (X \times U_n^c) \cup (A_n \times U_n) \). It follows from 2.3.a) that if the mapping \( \Phi \) is Borel (continuous) then \( E \) is a Borel (\( G_\delta \)) subset of \( X \times Y \) with closed sections.

Conversely any subset \( E \) of \( X \times Y \) with closed sections is the pseudo-graph of a uniquely determined mapping \( \Phi_E : X \to \mathcal{F}(Y) \) that we call the *canonical mapping associated to \( E \).* But if \( E \) is Borel or even \( G_\delta \) the mapping \( \Phi_E \) is not Borel in general. However as we shall see in next result a kind of converse is true if we work in one additional dimension.

We recall that to any set \( E \subset X \times Y \) we associate the closed subset of \( X \times E \) defined by:

\[
\hat{E} = \{(x,(x,y)) \in X \times (X \times Y) : (x,y) \in E\} \subset X \times E.
\]

Since all sections of \( \hat{E} \) are closed subsets of \( E \) one can consider then the canonical mapping \( \Phi_{\hat{E}} : X \to \mathcal{F}(E) \). It turns out that the mapping \( \Phi_{\hat{E}} \) has a better behavior than \( \Phi_E \). Still if \( E \) is Borel or even \( G_\delta \) the mapping \( \Phi_{\hat{E}} \) is not Borel in general. However we have the following:

**Theorem 3.2.** Let \( X \) and \( Y \) be two separable metrizable spaces. Given any set \( E \subset X \times Y \) and any admissible topology on \( \mathcal{F}(E) \) there exists a subset \( \hat{E} \) of \( X \times E \) satisfying:

- a) \( \hat{E} \subset \hat{E} \subset X \times E \).
- b) \( \hat{E} \) is closed in \( X \times E \) and for all \( x \in X \), \( \hat{E}(x) \setminus \hat{E}(x) \) is countable.
- c) The canonical mapping \( \Phi_{\hat{E}} : X \to \mathcal{F}(E) \) is of the first Baire class and even continuous if \( X \) is zero-dimensional.

**Proof.** Let \( Z \) be a compactification of \( E \) defining the admissible topology of \( \mathcal{F}(E) \). We fix two compatible metrics \( d_0 \) on \( X \) and \( d_1 \) on \( Z \) and equip \( X \times Z \) with the sum distance \( d \):

\[
d((x,z), (x',z')) = d_0(x,x') + d_1(z,z').
\]
We fix a countable basis \((W_n)_{n \in \omega}\) of the topology of \(Z\) such that for all \(n\), \(W_n\) is nonempty and for all \(\varepsilon > 0\) the set \(\{n \in \omega : \text{diam}(W_n) > \varepsilon\}\) is finite. We then fix for all \(n\) an element \(e_n \in W_n \cap E\), and set \(A_n = \{x \in X : \hat{E}(x) \cap W_n \neq \emptyset\}\) and \(B_n = \{x \in X : d_0(x, A_n) \leq 2^{-n}\}\). Finally let:

\[
D = \bigcup_{n \in \omega} B_n \times \{e_n\} \quad \text{and} \quad \hat{E} = \hat{E} \cup D
\]

So clause a) is trivially satisfied and for every \(x \in X\), \(D(x) = \{e_n : B_n \ni x\}\) is countable.

We first prove that \(\hat{E}\) is closed in \(X \times E\). For this observe that if \(x \in B_n\) then by definition there exists some \(x' \in A_n\) such that \(d_0(x, x') \leq 2^{-n}\), and since \(x' \in A_n\) then there exists some \(z \in W_n\) such that \((x', z) \in \hat{E}\) hence since \(e_n \in W_n\) we have

\[
d((x, e_n), (x', z)) \leq d((x, e_n), (x', z)) = d_0(x, x') + d_1(e_n, z) \leq 2^{-n} + \text{diam}(W_n)
\]

It follows that if a sequence \((x_j, e_{n_j})_j\) in \(D\) with \(x_j \in B_{n_j}\) converges to some element \((x, z)\) then:

- either for infinitely many \(j\)'s, \(n_j = n\) is constant hence \(e_{n_j} = e_n\) and \(x_j \in B_n\), then \(z = e_n\) and since \(B_n\) is closed then \(x \in B_n\); so \((x, z) \in D \subset \hat{E}\),

- or \(\lim_j n_j = \infty\) and in this case the inequality above shows that \((x, z) \in \hat{E}\).

So any accumulation point of \(D\) lies in \(\hat{E}\), hence \(\overline{D} \times E \subset D \cup \hat{E}\); and since \(\hat{E}\) is a closed subset of \(X \times E\) then \(\hat{E}\) is a closed subset of \(X \times E\). In the same way if \(F = \overline{E} \times Z\) then \(\overline{D} \times F \subset \hat{F} := \hat{F} \cup D\) and \(\hat{F}\) is a closed subset of \(X \times F\).

**Claim 3.3.** : For all \(x \in X\), \(\hat{F}(x) = \overline{E}(x) = \overline{D}(x)\).

**Proof.** Since \(\hat{F}\) is closed then for all \(x \in X\), \(\hat{F}(x) \supset \overline{E}(x)\); and we shall prove that if \(W_n \cap \hat{F}(x) \neq \emptyset\) then \(W_n \cap D(x) \neq \emptyset\), which proves that \(\hat{F}(x) \subset \overline{D}(x)\).

If \(W_n \cap D(x) \neq \emptyset\) then we are done. Otherwise \(W_n \cap \hat{F}(x) \neq \emptyset\) and we can pick some \(z \in W_n\) such that \((x, z) \in \hat{F} = \overline{E}\). Then there exists some \((x', z') \in \hat{E}\) such that \(d_0(x, x') < 2^{-n}\) and \(x' \in W_n\) hence \(\hat{E}(x') \cap W_n \neq \emptyset\) and so \(x' \in A_n\) and \(x \in B_n\) and then \((x, z) \in \hat{E}\) so \(e_n \in W_n \cap D(x)\). \(\square\)

So clause b) is satisfied and we now prove clause c). Since \(\mathcal{F}(E)\) is endowed with the admissible topology associated to the compact space \(Z \supset E\) then by definition, the study of the mapping \(\Phi_E : X \to \mathcal{F}(E)\) reduces to the study of the mapping \(\Psi : X \to \mathcal{F}(Z)\) defined by

\[
\Psi(x) = \Phi_E(x) = \overline{E}(x) = \hat{F}(x)
\]

Since \(\hat{F}\) is closed in \(X \times Z\) and \(Z\) is compact then for all open set \(W\) in \(Z\) the set \(\{x \in X : \hat{E}(x) \subset W\}\) is open. Moreover it follows from the Claim that for all \(n\):

\[
\{x \in X : \hat{F}(x) \cap W_n \neq \emptyset\} = \bigcup \{B_{p} : e_p \in W_n\}
\]

which is an \(F_\sigma\) set, hence \(\Psi\) is of first Baire class.
Finally if $X$ is zero-dimensional we can choose for $d_0$ a metric valued in \{$0\} \cup \{2^{-n} : n \geq 0\}$, for example by embedding $X$ in $2^{\omega}$ equipped with its standard ultrametric defined for $\alpha \neq \beta$ by $d_0(\alpha, \beta) = 2^{-n(\alpha, \beta)}$ with $n(\alpha, \beta) = \min\{n : \alpha(n) \neq \beta(n)\}$. It follows that the sets $B_n$ above are clopen and $\Psi$ is then continuous.

**Corollary 3.4.** Let $X$ be a Polish zero-dimensional space, $Y$ a separable metrizable space and $E \subset X \times Y$. Then the set $E_0 = \{x \in X : \dim(E(x)) = 0\}$ is reducible to $F_0(E)$.

**Proof.** Let $\hat{E} \subset X \times E$ be the set given by Theorem 3.2. Observe that since $\hat{E}$ is closed in $\hat{E}$ then it follows from clause b) and Theorem 2.5 that $x \in E_0 \iff \dim(E(x)) = 0 \iff \dim(\hat{E}(x)) = 0 \iff \dim(\hat{E}(x)) = 0$ then by clause c) the mapping $\Phi_{\hat{E}}$ realizes a reduction of $E_0$ to $F_0(E)$ with all the desired properties.

4. The $\sigma$-ideal structure of $F_0(X)$

**Definition 4.1.** Given a set $X$ and $I \subset A \subset P(X)$ we shall say that $I$ is a $\sigma$-ideal of $A$ if for any set $A \in A$, if $A \subset \bigcup_{n \geq 0} A_n$ with $A_n \in I$ then $A \in I$.

So by Theorem 2.6 for any space $X$ the set $F_0(X)$ is a $\sigma$-ideal of $F(X)$.

For any space $X$ let $\mathcal{K}(X)$ denote the space of all compact subsets of $X$ equipped with the Vietoris topology. Observe that if $X$ is a compact space, a $\sigma$-ideal of $\mathcal{K}(X)$ in the sense of Definition 4.1 is a “$\sigma$-ideal of compact sets” in the sense of [7]. We also recall that in this case (compact) if $C \subset \mathcal{K}(X)$ is compact then the set $C = \bigcup C \subset X$ is compact.

More generally given a set $A \subset F(X)$ (endowed with some admissible topology) we shall say that $A$ is closed under compact unions if for any compact set $C \subset A$ the set $C = \bigcup C$ is a member of $A$ (in particular $C \in F(X)$).

**Lemma 4.2.** For any admissible topology $F(X)$ is closed under compact unions.

**Proof.** Let $\hat{X}$ be a compactification of $X$ defining the given admissible topology on $F(X)$. We recall that $F(X)$ is then homeomorphic to the subset $F(X, \hat{X})$ of $\mathcal{K}(\hat{X})$ (see Section 2.3). We now check that $F(X, \hat{X})$ is closed under compact unions: Suppose that $C \subset F(X, \hat{X})$ is compact and let $C = \bigcup C$ then

$$C \cap X = \bigcup_{K \in C} K \cap X \supset \bigcup_{K \in C} K \cap X = \bigcup_{K \in C} K = C$$

So $C \cap X = C$ and thus $C \in F(X, \hat{X})$.

It turns out that most results of [7] concerning $\sigma$-ideals of compact sets can be extended, with the same proofs, to $\sigma$-ideals of any $\Sigma^1_1$ set $A \subset \mathcal{K}(X)$ closed under compact unions. In particular Theorem 7 and Theorem 11 of [7] can be restated as follows:
Theorem 4.3. (Kechris-Louveau-Woodin) Let $X$ be compact space and $\mathcal{I}$ be a $\sigma$-ideal of $A \subset K(X)$. If $A$ is $\Sigma^1_1$ and is closed under compact unions then:

a) If $\mathcal{I}$ is $\Sigma^1_1$ then $\mathcal{I}$ is a $G_\delta$ subset of $A$

b) If $\mathcal{I}$ is $\Pi^1_1$ then $\mathcal{I}$ is either a $G_\delta$ subset of $A$ or is $\Pi^1_1$-complete.

Corollary 4.4. Let $X$ be an analytic space and $\mathcal{I}$ be a $\sigma$-ideal of $F(X)$.

a) If $\mathcal{I}$ is $\Sigma^1_1$ for some admissible topology then for any admissible topology $\mathcal{I}$ is a $G_\delta$ subset of $F(X)$.

b) If $\mathcal{I}$ is $\Pi^1_1$ for some admissible topology then for any admissible topology $\mathcal{I}$ is either a $G_\delta$ subset of $F(X)$ or $\Pi^1_1$-complete.

Proof. As mentioned above any two admissible topologies on $F(X)$ are first Baire class isomorphic, hence if $\mathcal{I}$ is $\Sigma^1_1$ for some admissible topology then $\mathcal{I}$ is $\Sigma^1_1$ for any admissible topology. Corollary 4.4 follows then from Lemma 4.2 and Theorem 4.3.

We recall that $F(X)$ is implicitly endowed with some admissible topology.

Corollary 4.5. Let $X$ be an analytic space.

a) If $F_0(X)$ is $\Sigma^1_1$ then $F_0(X)$ is a $G_\delta$ subset of $F(X)$.

b) If $F_0(X)$ is $\Pi^1_1$ then $F_0(X)$ is either a $G_\delta$ subset of $F(X)$ or $\Pi^1_1$-complete.

We emphasize that in both statements a) and b) of Corollary 4.5 the $G_\delta$ condition is relative to $F(X)$ and not absolute, unless the space $X$ itself, hence $F(X)$, is Polish. In fact if one wants to ensure that $F_0(X)$ is an (absolute) $G_\delta$ then requiring $X$ to be Polish is not a restriction since $X$ embeds as a closed subset of $F_0(X)$, and in this context we have the following complement to Corollary 4.5.

Proposition 4.6. If $X$ is a Polish space and $F_0(X)$ is an $F_\sigma$ subset of $F(X)$ then $X$ is zero-dimensional, hence $F_0(X) = F(X)$.

Proof. Embed $F(X)$ in $F(\hat{X})$ for some compactification $\hat{X}$ of $X$ and let $\bar{\tau}$ denote the corresponding admissible topology on $F(X)$. Let $U$ be the largest open subset of $X$ such that $\dim(U) = 0$ (such a set exists by Theorem 2.6) and suppose that the set $Y = X \setminus U$ is nonempty.

Since $X$ is Polish then $Y$ as well as $F(Y)$ are Polish too. It follows from the hypothesis that $F_0(Y) = F_0(X) \cap F(Y)$ is an $F_\sigma$ subset of $F(Y)$, hence by Corollary 4.4 $F_0(Y)$ is a also a $G_\delta$ subset of $F(Y)$ which is moreover dense since it contains all finite subsets. Hence by Baire Theorem $F_0(Y)$ has a dense interior in $F(Y)$. In particular since $\emptyset$ is an isolated element of $F(Y)$ the set $F_0^*(Y) = F_0(Y) \setminus \{\emptyset\}$ is of nonempty interior in $F(Y)$. It follows then from the very definition of the topology $\bar{\tau}$ that there exists some finite family $(V_j)_{0 \leq j \leq k}$ of open subsets of $\hat{X}$ such that setting $V = \bigcup_{j=0}^{k} V_j$ then $V \cap Y \neq \emptyset$ and

$$V = \{K \cap Y : K \in K(\hat{Y}), \ K \subset V \text{ and } \forall j \leq k, \ K \cap V_j \neq \emptyset\}$$

is a nonempty subset of $F_0^*(Y)$. We then fix for each $j$ an element $a_j$ in $Y \cap V_j$ (which by the definition of $V$ is nonempty), set $F = \{a_j : 0 \leq j \leq p\}$ and write $V = \bigcup_n W_n$ with $W_0 \supset F$ and for all $n$, $W_n \subset V$. It follows that for each $n,$
\[ \overline{W_n} \cap Y \in F_0^*(Y) \text{ so } \dim(\overline{W_n} \cap Y) = 0 \text{ hence by Theorem } 2.5 \text{ dim}(V \cap Y) = 0. \]

Then \( U' = U \cup (V \cap Y) = U \cup V \) is an open subset of \( X \) and since \( \dim(U) = 0 \) then again by Theorem 2.5 \( \dim(U') = 0. \) It follows then from the definition of \( U \) that \( V \cap Y = \emptyset \) which is a contradiction. Hence \( Y = \emptyset \) and \( X = U \) so \( \dim(X) = 0. \)

5. WHEN \( F_0(X) \) IS OF LOW COMPLEXITY

It follows from the previous results that at least in the frame of Polish spaces, aside in the trivial case of a zero-dimensional space, in which case \( F_0(X) = \mathcal{F}(X) \), the first two possible descriptive complexity classes for the set \( F_0(X) \) are the class \( G_\delta \) and the class \( \Pi^1_1. \) In this section we show that both of these classes can really occur. One of the main goals of this work is to prove the existence of other possible classes.

We first mention the following elementary proposition concerning the class \( G_\delta. \)

We point out that in Section 5 we shall give an example of a Polish space \( X \) which does not satisfy the assumptions of this proposition but for which the set \( F_0(X) \) is \( G_\delta. \)

**Proposition 5.1.** If a space \( X \) is decomposable into a countable union \( X = \bigcup_n F_n \) where each \( F_n \) is either compact or a closed zero-dimensional subset of \( X \) then \( F_0(X) \) is a \( G_\delta \) subset of \( F(X). \)

**Proof.** Fix a countable basis \((U_n)_{n \geq 0}\) of the topology of \( X \) and observe that if a space \( E \) is zero-dimensional then any pair \((A, B)\) of disjoint closed subsets of \( E \) can be separated by a clopen set. Hence for any \( F \in \mathcal{F}(X) \) we can write:

\[
F \in F_0(X) \iff \left\{ \begin{array}{l}
\forall m, n : \overline{U_m} \subset U_n, \exists V \text{ open : } \\
\overline{U_m} \subset V \subset U_n \land F \cap \partial V = \emptyset
\end{array} \right.
\]

\[
\iff \left\{ \begin{array}{l}
\forall m, n : \overline{U_m} \subset U_n, \exists V \text{ open : } \\
\overline{U_m} \subset V \subset U_n \land F \subset V \cup V^c
\end{array} \right.
\]

a) We shall first treat the case where \( X \) itself is compact. In this case the condition \( F \subset V \cup V^c \) is open, hence by the equivalence above \( F(X) \) is \( G_\delta. \)

b) Coming back to the general case, suppose that for all \( k, F_{2k} \) is compact and \( F_{2k+1} \) is zero-dimensional. Then by Theorem 2.5 for any \( F \in \mathcal{F}(X) \) we have:

\[
F \in F_0(X) \iff \forall k, F \cap F_{2k} \in F_0(F_{2k})
\]

Since \( F_{2k} \) is compact then by case a) \( F_0(F_{2k}) \) is \( G_\delta \) and it follows from 2.3b) that the mapping \( F \mapsto F \cap F_{2k} \) is Borel from \( \mathcal{F}(X) \) to \( \mathcal{F}(F_{2k}). \) Hence \( F_0(X) \) is Borel and then by Corollary 4.5 \( F_0(X) \) is a \( G_\delta \) subset of \( F(X). \)

As pointed out in the introduction, E. Pol and R. Pol showed in \([11]\) that the set \( F_0([\mathbb{P} \times I]) \) is \( \Pi^1_1 \) and not \( \Sigma^1_1. \) We shall prove in the rest of this section a generalization of this result, with a different proof.

**Proposition 5.2.** If \( X \) is a closed subset of \( Z \times K \) where \( Z \) is zero-dimensional and \( K \) is compact then the following are equivalent:

(i) \( \dim(X) = 0. \)
(ii) For all \( z \in Z \), \( \dim(X(z)) = 0 \).

In particular if \( Z \) is analytic then \( \mathcal{F}_0(X) \) is \( \Pi^1_1 \).

Proof. (i) \( \implies \) (ii) is obvious.

(ii) \( \implies \) (i): Fix \((z, y) \in X \) and \( W \) a neighbourhood of \((z, y)\). Since \( \dim(X(z)) = 0 \) there exists an open neighbourhood \( V \) of \( y \) in \( K \) such that \( V \cap X(z) \) is clopen in \( X(z) \) and \( V \cap X(z) \subseteq W(z) \) hence we can find \( W_0 \) and \( W_1 \) two disjoint open subsets of \( X \), such that \( W_0 \cap W_1 = V \).

Since \( K \) is compact then \( \pi \) the projection mapping on \( Z \) is closed hence the set \( H = \pi(X \setminus (W_0 \cup W_1)) \) is closed, and since \( z \notin H \) there exists a clopen set \( U \) such that \( z \in U \subseteq Z \setminus H \), hence \( V = W_0 \cap \pi^{-1}(U) \) is clopen in \( X \) and \((z, y) \in V \subseteq W_1 \).

This proves the equivalence, and applying this equivalence to \( F \in \mathcal{F}(X) \) we can write:

\[
F \in \mathcal{F}_0(X) \iff \forall z \in Z, F(z) \in \mathcal{F}_0(K)
\]

equivalently:

\[
F \notin \mathcal{F}_0(X) \iff \exists z \in Z, L \notin \mathcal{F}_0(K) : F(z) = L
\]

Since \( K \) is compact one easily checks that the set \( \{(F, z, L) \in \mathcal{F}(X) \times Z \times \mathcal{F}_0(K) : F(z) = L \} \) is a \( G_d \) subset of \( \mathcal{F}(X) \times Z \times \mathcal{F}_0(K) \). Then since \( Z \) is analytic and by Proposition \[5.1\] \( \mathcal{F}_0(K) \) is Polish, it follows then from the second equivalence above that the set \( \mathcal{F}(X) \setminus \mathcal{F}_0(X) \) is \( \Sigma^1_1 \) hence the set \( \mathcal{F}_0(X) \) is \( \Pi_1^1 \). \( \square \)

**Proposition 5.3.** If \( X = Y \times Z \) is the product of a \( \sigma \)-compact space \( Y \) and an analytic zero-dimensional space \( Z \) then \( \mathcal{F}_0(X) \) is \( \Pi_1^1 \).

**Proof.** Set \( X = \bigcup_n F_n \) where for all \( n, F_n = K_n \times Z \) with \( K_n \) compact. Then by Theorem \[2.3\] we have for \( F \in \mathcal{F}(X) \):

\[
F \in \mathcal{F}_0(X) \iff \forall n, F \cap F_n \in \mathcal{F}_0(F_n)
\]

Since the mapping \( F \mapsto F \cap F_n \) is Borel the conclusion follows then from Proposition \[5.2\]. \( \square \)

Observe that in the following theorem if \( Y \) were zero-dimensional then \( X \) would be zero-dimensional too, and then \( \mathcal{F}_0(X) = \mathcal{F}(X) \) would be \( \Sigma^1_1 \), even \( G_d \) if \( Y \) and \( Z \) are Polish. Also if \( Z \) were \( \sigma \)-compact then by Proposition \[5.1\] \( \mathcal{F}_0(X) \) would be \( G_{\delta} \).

**Theorem 5.4.** If \( X = Y \times Z \) is the product of a non-zero-dimensional analytic space \( Y \) and a zero-dimensional non \( \sigma \)-compact analytic space \( Z \) then \( \mathcal{F}_0(X) \) is \( \Pi_1^1 \)-hard.

**Proof.** Fix a compactification \( \hat{Y} \) for \( Y \), and a zero-dimensional compactification \( \hat{Z} \) of \( Z \). Fix also an enumeration \( (y_n) \) of some countable dense subset of \( Y \). Since \( Z \) is not \( \sigma \)-compact then by the classical Hurewicz Theorem we can find a copy of the Cantor set \( C \subset \hat{Z} \) such that \( D = C \setminus Z \) is a dense countable subset of \( C \).

We then fix an enumeration \( (z_k) \) of \( D \) and for all \( k, \ell \) a clopen neighbourhood \( W_{k, \ell} \) of \( z_k \) in \( \hat{Z} \) of diameter \( < 2^{-k-\ell} \) and an element \( z_{k, \ell} \in W_{k, \ell} \cap Z \). Finally let \( (k, \ell) \mapsto \langle k, \ell \rangle \) denote any bijection from \( \omega^2 \) to \( \omega \).
Let $\mathcal{K}(C)$ the (compact) space of all compact subsets of $C$. For any $K \in \mathcal{K}(C)$ set:

$$\Phi(K) = \{(y(k,\ell), z(k,\ell)) : (k,\ell) \text{ such that } W_{k,\ell} \cap K \neq \emptyset\} \subset Y \times Z$$

and

$$\Psi(K) = \hat{Y} \times K \cup \Phi(K) \subset \hat{Y} \times \hat{Z}.$$ 

**Claim 5.5.** $\overline{\text{Gr}(\Phi)} \subset \text{Gr}(\Psi)$.

**Proof.** Suppose that $(K, y, z) = \lim_j (K_j, y(k_j,\ell_j), z(k_j,\ell_j))$ with $(y(k_j,\ell_j), z(k_j,\ell_j)) \in \Phi(K_j)$. Then for all $j$ there exists some $z_j \in W_{k_j,\ell_j} \cap K_j$, so $z_j \in K_j$ and since $\text{diam}(W_{k_j,\ell_j}) < 2^{-k_j-\ell_j}$ then $d(z_{k_j,\ell_j}, z_j) < 2^{-k_j-\ell_j}$.

- If $\lim(k_j + \ell_j) = \infty$ then $\lim_j d(z_{k_j,\ell_j}, K_j) = 0$ hence $z \in K$ and $(y, z) \in \Psi(K)$.
- If not then for infinitely many $j$’s the sequence $(k_j,\ell_j)$ is constant, with value say $(k,\ell)$, so $(K_j, y(k_j,\ell_j), z(k_j,\ell_j)) = (K_j, y(k_j,\ell_j), z(k_j,\ell_j))$ with $W_{k,\ell} \cap K_j \neq \emptyset$; and since $W_{k,\ell}$ is closed then $W_{k,\ell} \cap K \neq \emptyset$ hence $(y, z) \in \Phi(K) \subset \Psi(K)$.

It follows from the claim that $(\hat{Y} \times K) \cup \overline{\text{Gr}(\Phi)} \subset \text{Gr}(\Psi) \subset (\hat{Y} \times K) \cup \overline{\text{Gr}(\Phi)}$ hence $\text{Gr}(\Psi) = (\hat{Y} \times K) \cup \overline{\text{Gr}(\Phi)}$ is a closed subset of $\mathcal{K}(C) \times \hat{Y} \times C$ and the compact-valued mapping $K \mapsto \Psi(K)$ is u.s.c.. Hence for any open set $U \subset \hat{Y} \times C$ the set $\{K \in \mathcal{K}(C) : \Psi(K) \subset U\}$ is open. Moreover if $U = V \times W$ is a basic open set then

$$\Psi(K) \cap U \neq \emptyset \iff \exists (k,\ell), (y(k,\ell), z(k,\ell)) \in U \text{ and } K \cap W_{k,\ell} \neq \emptyset \text{, or } K \cap W \neq \emptyset$$

so the set $\{K \in \mathcal{K}(C) : \Psi(K) \cap U \neq \emptyset\}$ is open too. Hence the mapping $\Psi : \mathcal{K}(C) \to \mathcal{F}(\hat{Y} \times C)$ is continuous.

Observe that $\hat{X} = \hat{Y} \times \hat{Z}$ is a compactification of $X = Y \times Z$ and for all $K$, $\Psi(K) \cap X$ is a closed subset of $X$ which by the lemma is dense in $\Psi(K)$. Hence if we identify $\mathcal{F}(X)$ to a subspace of $\mathcal{F}(\hat{X})$ then the mapping $\bar{\Psi}$ is identified to the mapping $\bar{\Psi} : \mathcal{K}(C) \to \mathcal{F}(X)$ with $\bar{\Psi}(K) = \Psi(K) \cap X$ which is then continuous for the corresponding topology. Observe finally:

- if $K \in \mathcal{K}(D)$ then $\bar{\Psi}(K) = \Phi(K)$ is countable hence $\bar{\Psi}(K) \in \mathcal{F}_0(X)$
- if not and $z \in K \setminus D$ then $\bar{\Psi}(K) \supset Y \times \{z\}$ hence $\bar{\Psi}(K) \not\in \mathcal{F}_0(X)$

Hence $\bar{\Psi}$ is a continuous reduction of $\mathcal{K}(D)$ to $\mathcal{F}_0(X)$, and it is well known that if $D$ is any dense countable subset of the Cantor space then $\mathcal{K}(D)$ is a $\Pi^1_1$-complete subset of $\mathcal{K}(2^\omega)$, hence $\mathcal{F}_0(X)$ is $\Pi^1_1$-hard. \qed

**Corollary 5.6.** If $X = Y \times Z$ is the product of a $\sigma$-compact non zero-dimensional space $Y$ and an analytic zero-dimensional non $\sigma$-compact space $Z$ then $\mathcal{F}_0(X)$ is $\Pi^1_1$-complete.

**Corollary 5.7.** If $X$ is a non zero-dimensional and non $\sigma$-compact analytic space homeomorphic to its square $X^2$ then $\mathcal{F}_0(X)$ is $\Pi^1_1$-hard.

**5.8. The Erdős space:** As an application we mention the case of Erdős space $S ([S])$. We recall that this is the closed subset $S$ of the Hilbert space $\ell^2 = \ell^2(\mathbb{N})$. 

defined by:

\[ S = \{(x_n)_{n \geq 0} \in \ell^2 : \forall n, x_n = 0 \text{ or } x_n^{-1} \in \mathbb{N}\}. \]

We claim that the set \( \mathcal{F}_0(S) \) is \( \Pi_1^1 \)-hard. Indeed since any compact subset of \( S \) is zero-dimensional then by Theorem 2.5 the Polish space \( S \) is non \( \sigma \)-compact. Moreover the isometry \( \Psi : \ell^2 \times \ell^2 \to \ell^2 \) defined by \( \psi(e_k,0) = e_{2k} \) and \( \psi(0,e_k) = e_{2k+1} \) induces an homeomorphism from \( S^2 \) onto \( S \), hence by Corollary 5.7 \( \mathcal{F}_0(S) \) is \( \Pi_1^1 \)-hard. However we do not know the exact complexity of \( \mathcal{F}_0(S) \).

6. THE MAXIMUM COMPLEXITY OF \( \mathcal{F}_0(X) \) WHEN \( X \) IS \( \Sigma_1^1 \)

**Proposition 6.1.** If \( X \) is an analytic space then \( \mathcal{F}_0(X) \) is \( \Sigma_1^2 \).

**Proof.** Fix a countable basis \((U_n)_{n \geq 0}\) of the topology of \( X \). Resuming the analysis in the proof of Proposition 5.1 we have for any \( F \in \mathcal{F}(X) \):

\[
F \in \mathcal{F}_0(X) \iff \begin{cases} 
\forall m, n : \overline{U_m} \subset U_n, \exists V \text{ open} : \\
\quad \overline{U_m} \subset V \subset U_n \land F \subset V \cup \overline{V}^c 
\end{cases}
\]

\[
\iff \begin{cases} 
\forall m, n : \overline{U_m} \subset U_n, \exists V \text{ open} : \\
\quad \overline{U_m} \subset V \subset U_n \land F \cap V^c \cap \overline{V} = \emptyset 
\end{cases}
\]

\[
\iff \begin{cases} 
\forall m, n : \overline{U_m} \subset U_n, \exists F' \in \mathcal{F}(X) : \\
\quad U_n^c \subset F' \subset \overline{U_m}^c \land F \cap F' \cap (\text{Int}(F'))^c = \emptyset 
\end{cases}
\]

Then observe that by 2.3(b) condition “\( U_n^c \subset F' \subset U_m^c \)” on \( F' \) is \( \Sigma_1^1 \) while condition “\( F \cap F' \cap (\text{Int}(F'))^c = \emptyset \)” on \( (F,F') \) is co-analytic since the dual condition “\( F \cap F' \cap (\text{Int}(F'))^c \neq \emptyset \)” is equivalent to:

\[ \exists x \in F \cap F' : \forall p (x \notin U_p \lor (\exists q : U_q \subset U_p \land F' \cap U_q = \emptyset)) \]

which is analytic by 2.3(b). It follows from the previous analysis that \( \mathcal{F}_0(X) \) is the projection of the intersection of an analytic set with a co-analytic set, hence \( \mathcal{F}_0(X) \) is \( \Sigma_1^2 \). \( \square \)

**Theorem 6.2.** There exists an analytic subspace \( X \) of \( 2^\omega \times I \) for which \( \mathcal{F}_0(X) \) is \( \Sigma_1^1 \)-complete.

**Proof.** Fix an enumeration \((I_n)_{n \geq 0}\) of all nonempty subintervals of \( I \) with endpoints in \( \mathbb{Q} \), and fix for all \( n \), \( G_n \subset I_n \) a copy of \( \omega^\omega \) and a homeomorphism \( h_n : G_n \to \omega^\omega \).

Let \( A \) be any \( \Sigma_1^1 \) subset of \( 2^\omega \), that we represent as the projection on the first factor of some \( \Pi_1^1 \) set \( B \subset 2^\omega \times \omega^\omega \), and consider the set:

\[ X = \{(\alpha, z) \in 2^\omega \times I : \forall n, \ z \notin G_n \lor (z \in G_n \land (\alpha, h_n(z)) \notin B)\} \]

which is clearly \( \Sigma_1^1 \).

**Claim 6.3.** \( \alpha \in A \) if and only if \( \dim(X(\alpha)) = 0 \).

**Proof.** If \( \alpha \in A \) fix \( \beta \in \omega^\omega \) such that \( (\alpha, \beta) \in B \) and set for all \( n \), \( z_n = h_n^{-1}(\beta) \in G_n \) then \( (\alpha, h_n(z_n)) = (\alpha, \beta) \in B \) hence \( (\alpha, z_n) \notin X \). Since the set \( D = \{z_n; n \in \omega\} \) is dense in \( I \) then the set \( E = I \setminus D \) is of empty interior in \( I \) hence \( \dim(E) = 0 \) and since \( X(\alpha) \subset E \) then \( \dim(X(\alpha)) = 0 \).
– If \( \alpha \not\in A \) then for all \( \beta \in \omega^\omega \) \((\alpha, \beta) \not\in B \) hence for all \( z \in I \), if \( z \in G_n \) then \((\alpha, h_n(z)) \not\in B \), hence \( X(\alpha) \supset I \) hence \( \dim(X(\alpha)) \geq 1 \). \( \square \)

It follows then from Claim 6.3 and Corollary 3.4 that the set \( A \) is reducible to \( F_0(X) \). So starting with a \( \Sigma^1_2 \)-complete set \( A \) and considering the corresponding space \( X \), the set \( F_0(X) \) is then \( \Sigma^1_2 \)-hard, hence by Proposition 6.1 \( \Sigma^1_2 \)-complete. \( \square \)

7. The maximum descriptive class of \( F_0(X) \) when \( X \) is Borel

We do not know whether Theorem 6.2 holds for some Polish space \( X \). We are only able to prove that in this more restrictive context the maximum complexity of \( F_0(X) \) is much beyond the class \( \Pi^1_1 \). More generally we shall prove a similar result for the case where \( X \) is a \( \Pi^0_3 \) set, with a lower bound increasing with \( \xi \), still below the class \( \Delta^1_2 \).

7.1. The classes \( \mathcal{D} \Gamma \):

7.1.a Games and strategies: Given any set \( A \subset \omega^\omega \) we denote by \( G_A \) the standard game on \( \omega \) with \( A \) as the win condition for Player I. More explicitly \( G_A \) is the game in which the two Players choose alternatively an element in \( \omega \):

\[
I: \quad n_0 \quad n_2 \quad \cdots \\
II: \quad n_1 \quad n_3 \quad \cdots
\]

Thus an infinite run in the game is an infinite sequence \( \gamma = (n_0, n_1, n_2 \cdots) \in \omega^\omega \) and Player I wins the run if \( \gamma \in A \).

We view a strategy in such a game as a mapping \( \sigma: \omega^< \omega \to \omega \) (for Player I) or \( \tau: \omega^< \omega \setminus \{\emptyset\} \to \omega \) (for Player II). We denote by \( \Sigma_I (\Sigma_{II}) \) the set of all strategies for Player I (Player II) that we endow with their natural product topologies for which \( \Sigma_I \cong \Sigma_{II} \cong \omega^\omega \).

Given any pair \((\sigma, \tau) \in \Sigma_I \times \Sigma_{II} \) we define inductively the infinite run

\[
\sigma \star \tau = \gamma = (n_0, n_1, n_2 \cdots) \in \omega^\omega
\]

by \( n_0 = \sigma(\emptyset), n_{2k} = \sigma(n_1, n_3, \cdots, n_{2k-1}) \) and \( n_{2k+1} = \tau(n_0, n_2, \cdots, n_{2k}) \). The mapping \( \star: \Sigma_I \times \Sigma_{II} \to \omega^\omega \) thus defined is then clearly continuous and:

– the strategy \( \sigma \in \Sigma_I \) is winning in the game \( G_A \) iff for all \( \tau \in \Sigma_{II}, \sigma \star \tau \in A \)

– the strategy \( \tau \in \Sigma_{II} \) is winning in the game \( G_A \) iff for all \( \sigma \in \Sigma_I, \sigma \star \tau \not\in A \)

7.1.b The game operator \( \mathcal{D} \): Let \( B \subset X \times \omega^\omega \) where \( X \) is an arbitrary Polish space then for all \( x \in X, B(x) \subset \omega^\omega \) and one can consider the game \( G_{B(x)} \), and then define

\[
\mathcal{D}B = \{ x \in X : \text{ Player I wins the game } G_{B(x)} \}
\]

In all the sequel \( \Gamma \) denotes a non trivial class \((\Gamma \text{ and } \not\Gamma \neq \emptyset)\) which admits a \( \Gamma \)-complete set (that is \( \Gamma \) is a Wadge class). In particular \( \Gamma \) could be any Baire class or projective class.

Given such a class \( \Gamma \), a set \( A \subset X \) is said to be in \( \mathcal{D} \Gamma \) if it is of the form \( A = \mathcal{D}B \) for some \( B \subset X \times \omega^\omega \) in \( \Gamma \). The reader can find in [10] a detailed study of the operator \( \mathcal{D} \). We state next only some elementary properties:
(i) $\mathcal{D}\Gamma$ is closed under countable unions and intersections.
(ii) If $\Gamma$ is closed under unions with closed sets then $\mathcal{D}\Gamma$ is closed under operation $\mathcal{A}$.
(iii) If all games in $\Gamma$ are determined then $\mathcal{D}\Gamma$ is the dual class of $\mathcal{D}\Gamma$.
(iv) If $\Gamma \subseteq \Pi_1^0$ then $\mathcal{D}\Gamma \subseteq \Sigma_1^1$ and if $\Gamma = \Pi_1^0$ then $\mathcal{D}\Gamma = \Sigma_1^1$

Observe that since all Borel games are determined then $\mathcal{D}\Delta_1^1 \subseteq \Delta_1^1$ with strict inclusion in general. In fact it turns out that sets in $\mathcal{D}\Delta_1^1$ possess all classical descriptive regularity properties such as Baire property or universal measurability, while for an arbitrary $\Delta_1^1$ set such a property might depend on Set Theory axioms.

It is easy to see that $\mathcal{D}\Sigma_1^0 = \Pi_1^0$ and $\mathcal{D}\Pi_1^0 = \Sigma_1^1$ (see 6D.2 in [10] edition 2009), and for Baire classes $\Gamma$ of low rank the classes $\mathcal{D}\Gamma$ coincide with some more classical classes considered much before the intrusion of games in Descriptive Set Theory. In particular the classes $\mathcal{D}\Sigma_2^0$ and $\mathcal{D}\Sigma_3^0$ are related to Kolmogorov’s $R$-sets (we refer the reader to [3] and [4] for definitions and more details). Also relying on the classical transfinite analysis of the class $\Delta_2^0$ one easily derive from [7,4, b. (ii)] that $\mathcal{D}\Delta_2^0$ is the smallest non trivial class closed under complementation and operation $\mathcal{A}$, also known as Selivanovski’s class of C-sets.

We also point out that by a result due to Solovay $\mathcal{D}\Sigma_2^0$ is exactly the class of sets admitting a $\Sigma_2^1$-inductive definition (see [10]). This probably explains why $\mathcal{D}\Sigma_2^0$ sets arise naturally in analysis, though very few natural examples of $\mathcal{D}\Sigma_2^0$-complete sets are known, while no such examples are known for higher $\mathcal{D}\Gamma$ classes.

We mention the following two simple examples:

- (Louveau [2]): There exists a $G_\delta$ set $G \subset 2^\omega \times \omega^\omega$ such that if $\pi$ denotes the projection mapping on the first coordinate and $B = \{\pi(K); K \in \mathcal{K}(G)\}$ then the $\sigma$-ideal of $\mathcal{K}(2^\omega)$ generated by $B$ is $\mathcal{D}\Sigma_2^0$-complete.

- (Saint Raymond [12]): Let $T \subset 2^\omega \times \omega^\omega$ denote the set of all trees on $\omega$. Then the set $\mathcal{Q}B = \{T \in T : \exists \alpha \in \omega^\omega, \forall \beta \in \pi(T) \alpha \leq \beta\}$ (where $\leq$ denotes the product order on $\omega^\omega$) is $\mathcal{D}\Sigma_2^0$-complete.

The following well known result is the only result we will use in this section.

**Lemma 7.2.** For any Baire class $\Gamma$ there exists a set $U \subset 2^\omega \times \omega^\omega$ in $\Gamma$ such that the set $\mathcal{D}U$ is $\mathcal{D}\Gamma$-complete.

**Proof.** Fix a homeomorphism $\varphi : 2^\omega \times 2^\omega \to 2^\omega$ and for all $\alpha$ let $\varphi_\alpha$ denote the partial mapping $\varphi(\alpha, \cdot) : 2^\omega \to 2^\omega$. Starting from any $\Gamma$-set $V \subset 2^\omega \times (2^\omega \times \omega^\omega)$ which is universal for $\Gamma$-subsets of $(2^\omega \times \omega^\omega)$ define

$$ U = \{((\varphi(\alpha, \beta), \gamma) \in 2^\omega \times \omega^\omega : (\alpha, \beta, \gamma) \in V\} $$

Then for any $\Gamma$-set $B \subset 2^\omega \times \omega^\omega$ there exists some $\alpha \in 2^\omega$ such that $B = V(\alpha)$ so for any $(\beta, \gamma) \in 2^\omega \times \omega^\omega$ we have:

$$ (\beta, \gamma) \in B \iff (\alpha, \beta, \gamma) \in V \iff (\varphi(\alpha, \beta), \gamma) \in U \iff (\varphi_\alpha(\beta), \gamma) \in U $$
hence $U$ is $\Gamma$-complete. Moreover for any $\beta \in 2^\omega$, since $B(\beta) = U(\varphi_\alpha(\beta))$, then:

$$\beta \in \partial B \iff \text{Player I wins } G_{B(\beta)}$$

$$\iff \text{Player I wins } G_{U(\varphi_\alpha(\beta))} \iff \varphi_\alpha(\beta) \in \partial U$$

hence $\varphi_\alpha$ reduces $\partial B$ to $\partial U$ and $\partial U$ is $\partial \Gamma$-complete.

Theorem 7.3. Let $H$ be a perfect non zero-dimensional Polish space and $G$ a dense $G_\delta$ subset of $H$. Assume that for every dense subset $D$ of $G$ the space $H \setminus D$ is zero-dimensional. Then for any countable ordinal $\xi \geq 3$ there exists a $\Pi^0_\xi$ set $X \subset \omega^\omega \times H$ such that $F_0(X)$ is $\partial \Sigma^0_\xi$-hard.

If moreover $G$ is nowhere relatively $\sigma$-compact in $H$ then the same conclusion holds for $\xi = 2$.

Proof. Fix a countable basis $(W_n)$ of the topology in $H$. Since $G$ is dense in $H$ then for all $n$ the set $G \cap W_n$ is perfect so we can fix a copy of $\omega^\omega$ say $G_n$ in $G \cap W_n$ and a homeomorphism $h_n : G_n \to \Sigma_1^1 \approx \omega^\omega$.

Given any Borel set $U \subset 2^\omega \times \omega^\omega$ let

$$X_U = \{(\alpha, \tau, z) \in 2^\omega \times \Sigma_\Pi^1 \times H : \forall n, \ z \in G_n \to (\alpha, h_n(z) \star \tau) \notin U\}$$

Claim 7.4. $\alpha \in \partial U$ if and only if $\dim(X_U(\alpha)) = 0$.

Proof. - If $\alpha \in \partial U$ then Player I has a winning strategy in the game $G_{U(\alpha)}$. Fix such a strategy $\sigma^* \in \Sigma_\Pi^1$ and let for all $n$, $z_n = h_n^{-1}(\sigma^*) \in G_n$ then for all $\tau \in \Sigma_\Pi$, $h_n(z_n) \star \tau = \sigma^* \star \tau \in U(\alpha)$ hence $(\alpha, \tau, z_n) \notin X_U$. Then by construction the set $D = \{z_n : n \in \omega\}$ is dense in $G$ hence the set $E = H \setminus D$ is zero-dimensional. But by the observation above $X_U(\alpha) \subset \Sigma_\Pi \times E$ hence $\dim(X_U(\alpha)) = 0$.

- If $\alpha \notin \partial U$ then, since the game $G_{U(\alpha)}$ is determined, Player II has a winning strategy. Fix such a strategy $\tau^* \in \Sigma_\Pi$; then for all $z \in H$, if $z \in G_n$ then $\sigma_n = h_n(z) \in \Sigma_1$, and since $\tau^*$ is winning for Player II then $h_n(z_n) \star \tau^* = \sigma_n \star \tau^* \notin U(\alpha)$ hence $X_U(\alpha, \tau^*) \subset H$ and so $\dim(X_U(\alpha)) \geq \dim(X_U(\alpha, \tau^*)) \geq \dim(H) > 0$. 

From now on we let $U \subset 2^\omega \times \omega^\omega$ be a $\Sigma^0_\xi$ set given by Lemma 7.2 and set $X = X_U$. Then as in the proof of Theorem 6.2 applying Corollary 3.3 one concludes that the set $F_0(X)$ is $\partial \Sigma^0_\xi$-hard.

To finish the proof we only need to check that $X$ has the expected complexity. Observe first that since $2^\omega \times \Sigma_\Pi \approx 2^\omega \times \omega^\omega \approx \omega^\omega$ then $X$ can indeed be viewed as a subset of $\omega^\omega \times H$. Moreover for a given $n$:

$$\{(z \in G_n \to (\alpha, h_n(z) \star \tau) \notin U) \iff (z \notin G_n \lor (z \in G_n \land (\alpha, h_n(z) \star \tau) \notin U)\}$$

Since $U$ is $\Sigma^0_\xi$ and since $h_n : G_n \to 2^\omega \times \omega^\omega$ is continuous and its domain $G_n \approx \omega^\omega$ is a $\Pi^0_\xi$ set then for all $\xi \geq 2$ the condition “$z \in G_n \land (\alpha, h_n(z) \star \tau) \notin U$” on $(\alpha, \tau, z) \in 2^\omega \times \Sigma_\Pi \times H$ is $\Pi^0_\xi$. Moreover if $\xi \geq 3$ the condition “$z \notin G_n$” on $z \in H$ defines a $\Sigma^0_\xi$ subset of $H$, hence an (absolute) $\Pi^0_\xi$ set. This proves the first part of the theorem.

If $G$ is nowhere relatively $\sigma$-compact in $H$ then no $W_n \cap G$ is contained in a $\sigma$-compact subset of $H$. Hence by Hurewicz’ theorem for all $n$ there exists a subset
exists a continuous function \( f \). Fix an enumeration \(( X, \leq )\) the complexity analysis of Lemma 8.1. Given any countable subset \( G \subset \omega^\omega \) of \( X \), hence an (absolute) \( \Pi_2^0 \) set, and it follows that if \( \xi = 2 \) then \( X \) is a \( \Pi_2^0 \) set.

**Corollary 7.5.** For any positive countable ordinal \( \xi \neq 2 \) there exists a \( \Pi_2^0 \) set \( X \subset \omega^\omega \times \mathbb{I} \) such that \( \mathcal{F}_0(X) \) is \( \mathcal{O}_{\Sigma_\xi^0} \)-hard.

**Proof.** For \( \xi = 1 \), \( \mathcal{O}_{\Sigma_1^0} = \Pi_2^0 \) and the result follows from Corollary [5.8] with \( X = \mathbb{P} \times \mathbb{I} \); and for \( \xi \geq 3 \) the result follows from Theorem [7.3] with \( H = G = \mathbb{I} \). □

We do not know whether Corollary [7.5] holds for \( \xi = 2 \), but in Section [9] we shall construct a \( \Pi_2^0 \) subset of \( \omega^\omega \times \mathbb{I}^2 \) for which the set \( \mathcal{F}_0(X) \) is \( \mathcal{O}_{\Sigma_2^0} \)-hard.

### 8. Hypergraphs

Given any function \( f : P \subset \mathbb{R} \to \mathbb{R} \) and any element \( a \in \overline{P} \) we denote by:

\[
\Omega f(a) := \{ \lim_j f(x_j) : x_j \in P \text{ and } \lim_j x_j = a \}
\]

\[
\Omega^- f(a) := \{ \lim_j f(x_j) : x_j \in P, x_j \leq a \text{ and } \lim_j x_j = a \}
\]

\[
\Omega^+ f(a) := \{ \lim_j f(x_j) : x_j \in P, x_j \geq a \text{ and } \lim_j x_j = a \}
\]

the sets of all (left, right) cluster values of \( f \) at \( a \), so \( \Omega f(a) = \Omega^- f(a) \cup \Omega^+ f(a) \).

Observe that if \( G \) is the graph of \( f \) and \( \overline{G} \) its closure in \( \mathbb{R}^2 \) then \( \Omega f(a) = \overline{G}(a) \).

**Lemma 8.1.** Given any countable subset \( Q \) of a non trivial interval \( I \subset \mathbb{R} \) there exists a continuous function \( f : P = I \setminus Q \to \mathbb{R} \) satisfying for all \( a \in Q \):

a) \( \Omega f(a) = \Omega^- f(a) \) is a non trivial interval.

b) There exists a continuous function \( g : I \setminus \{ a \} \to \mathbb{R} \) such that the function \( f - g : P \to \mathbb{R} \) admits a limit at \( a \).

**Proof.** Fix an enumeration \((a_n)_{n \geq 0}\) of \( Q \) and set for all \( n \), \( g_n(x) = 2^{-n} \sin \left( \frac{1}{a_n - x} \right) \).

Then \( g_n : I \setminus \{ a_n \} \to [-2^{-n}, 2^{-n}] \) is a continuous function and \( \Omega g_n(a_n) = \Omega^- g_n(a_n) = \Omega^+ g_n(a_n) = [-2^{-n}, 2^{-n}] \).

Finally set for \( x \in P = I \setminus Q \):

\[
f(x) = \sum_{n=0}^{\infty} g_n(x).
\]

Since \( |g_n(x)| \leq 2^{-n} \) for all \( x \in P \) the series is uniformly convergent on \( P \), hence the function \( f \) is well defined and continuous on \( P \). Similarly for all \( n \), the function \( h : P \cup \{ a_n \} \to \mathbb{R} \) defined by \( h(x) = \sum_{k \neq n} g_k(x) \) is also continuous, so \( f(x) - g_n(x) = h(x) \) tends to \( h(a_n) \) when \( x \) tends to \( a_n \) and \( \Omega f(a_n) = \Omega^- f(a_n) = \Omega^+ f(a_n) = [h(a_n) - 2^{-n}, h(a_n) + 2^{-n}] \). □

It was pointed to us by E. Pol and R. Pol that the function \( f \) used in the previous proof was considered by K. Kuratowski and W. Sierpinski in [8].
Definition 8.2. We shall say that a set $H \subset \mathbb{R}^2$ is an hypergraph if $G \subset H \subset \overline{G}$ where $G$ is the graph of a continuous function $f : I \setminus Q \to \mathbb{R}$ satisfying clauses a) and b) of Lemma 8.1 and for all $a \in Q$, the set $H(a)$ is dense and of empty interior in $\overline{G(a)}$. We shall then refer to the set $G$ as the graph part of $H$.

8.3. Remarks: a) If $H$ is an hypergraph and $I, Q, P, f, G$ are as in Definition 8.2 we shall write $H \equiv (H, I, Q, P, f, G)$. Observe that this tuple is uniquely determined by $H$ since $I$ is the projection of $H$ on the first factor, $Q = I \setminus P$, and $P$ is the set of all elements $x \in I$ such that $H(x)$ is a singleton $\{f(x)\}$, which defines $f$ and $G = \text{Gr}(f)$.

b) If $I'$ and $J'$ are intervals such that $I' \cap P \neq \emptyset$ and $f(I' \cap P) \subset J'$ then clearly $H' = H \cap (I' \times J')$ is an hypergraph too. It follows that any nonempty open subset $W$ of an hypergraph $H$ contains an hypergraph. Indeed since the graph part $G$ is dense in $H$ one can pick some element $(a, f(a))$ in $W \cap G$ and then by continuity of $f$ at $a$ find intervals $I' \ni a$ and $J' \ni f(a)$ such that $f(I' \cap P) \subset J'$ and $H \cap (I' \times J') \subset W$.

c) By Baire Theorem an hypergraph cannot be an $F_\sigma$ set of $\mathbb{R}^2$: indeed if $H$ were the union of countably many compact sets $F_n$ then there would be some $n$ such that $G \cap F_n$ has nonempty interior in $G$ hence some subinterval $J$ of $I$ such that $\forall x \in P \cap J \ (x, f(x)) \in F_n$. And we would have $\overline{G} \cap (J \times \mathbb{R}) \subset F_n \subset H$. In particular for $x \in Q \cap J$, $H(x)$ would contain an interval and would not be zero-dimensional.

But if $Q'$ is any countable dense subset of $\mathbb{R}$ then $\overline{G \setminus Q} \times Q'$ is a $G_\delta$ hypergraph. In particular there are $G_\delta$ hypergraphs which are relatively closed subsets of $\mathbb{R}^2 \setminus \mathbb{Q}^2$.

In all the sequel $(H, I, Q, P, f, G)$ denotes an arbitrary given hypergraph and $\pi : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection mapping on the first factor.

Theorem 8.4. If $H$ is an hypergraph and $E \subset H$ is such that $\pi(E)$ is a non trivial interval then the set $E$ is connected. In particular any hypergraph is connected.

Proof. Replacing $I$ by the interval if necessary we may suppose that $\pi(E) = I$ (see Remark 8.3 b)). So let $U_0$ be an open subset of $I \times \mathbb{R}$ such that $U_0 \cap E \neq \emptyset$ with $\partial U_0 \cap E = \emptyset$ and set $U_1 = I \times \mathbb{R} \setminus U_0$. So $U_0 \cap E$ and $U_1 \cap E$ are two clopen subsets of $E$ and we shall prove that $E \subset U_0$.

Observe that since $\pi(E) = I$ then necessarily $E \supset G$ and set $K = \overline{G}$ (the closure of $G$ in $\mathbb{R}^2$). Since $K$ is a closed subset of $I \times \mathbb{R}$ with compact sections the set-valued mapping $x \mapsto \tilde{K}(x)$ is u.s.c. (we recall that $\tilde{K}(x) = \{x\} \times K(x) \subset K$) hence for $i \in \{0, 1\}$ the set $V_i = \{x \in I : \tilde{K}(x) \subset U_i\}$ is open in $I$; and since $\tilde{K}(x) \neq \emptyset$ for all $x \in I$ and $U_0 \cap U_1 = \emptyset$ then $V_0 \cap V_1 = \emptyset$. Moreover since $G = \text{Gr}(f) \subset U_0 \cup U_1$ then $P \subset V_0 \cup V_1$. Hence $F = I \setminus (V_0 \cup V_1) \subset Q$ is a countable closed subset of $I$ and we shall denote, for any countable ordinal $\xi$, by $F^{(\xi)}$ its Cantor derivative of order $\xi$.

Claim 8.5. For any interval $J \subset I \setminus F^{(\xi)}$ there exists $i \in \{0, 1\}$ such that for all $x \in J$, $\tilde{E}(x) \subset U_i$. 

Proof. We prove the Claim by induction on $\xi$:

- For $\xi = 0$, $F^{(0)} = F$ so $J \subset V_0 \cup V_1$ hence $J \subset V_i$ for some $i$; so for all $x \in J$, $\tilde{E}(x) \subset \tilde{J}(x) \subset U_i$.

- Assume now that the Claim is proved for all $\eta < \xi$ and fix an interval $J \subset I \setminus F^{(\xi)}$. Then for all $a \in J$ we can fix some $\eta_a < \xi$ such that $a \notin F^{(\eta_a+1)}$ and an open interval $J^a = [a - \epsilon, a + \epsilon]$ such that $J^a \cap F^{(\eta_a)} \subset \{a\}$.

Subclaim: For all $a \in J$ there exists $i \in \{0, 1\}$ and an open neighbourhood $W$ of a such that for all $x \in W_i$, $\tilde{E}(x) \subset U_i$.

We first finish the proof of the Claim assuming the Subclaim: Set for $i \in \{0, 1\}$, $W_i = \{x \in J : \tilde{E}(x) \subset U_i\}$. By the Subclaim $(W_0, W_1)$ is an open covering of $J$ and since $\tilde{E}(x) \neq \emptyset$ for all $x \in J$ and $U_0 \cap U_1 = \emptyset$ then $W_0 \cap W_1 = \emptyset$; hence $J = W_i$ for some $i$, which proves the Claim for $\xi$.

To prove the Subclaim we distinguish two cases:

- If $a \notin F^{(\eta_a)}$ then the conclusion follows from the induction hypothesis for $\eta_a$.

- If $a \in F^{(\eta_a)}$ we distinguish three cases:

  - If $a$ is a right endpoint of $I$ then applying the induction hypothesis for $\eta_a$ with $J = [a - \epsilon, a[$ we can find $i_- \in \{0, 1\}$ such that $\tilde{E}(x) \subset U_{i_-}$ for all $x \in ]a - \epsilon, a[$. Now pick any $b \in E(a) \subset \Omega f(a) = \Omega^- f(a)$; we can then find some sequence $x_j$ in $]a - \epsilon, a[ \cap P$ such that $(a, b) = \lim_j (x_j, f(x_j))$. Then for all $j$, $(x_j, f(x_j)) \in E \cap U_{i_-}$; and since $E \cap U_{i_-}$ is a closed subset of $E$ then $(a, b) \in E \cap U_{i_-}$ too, hence $\tilde{E}(a) \subset U_{i_-}$. Similarly:

  - If $a$ is a left endpoint of $I$ we can find $i_+ \in \{0, 1\}$ such that $\tilde{E}(x) \subset U_{i_+}$ for all $x \in ]a, a + \epsilon[$.

  - If $a$ is an interior point we can find $i_-\ i_+\ i_-\ i_+\ i_-\ i_+$ in $\{0, 1\}$ such that $\tilde{E}(x) \subset U_{i_-}$ for all $x \in ]a - \epsilon, a]$ and $\tilde{E}(x) \subset U_{i_+}$ for all $x \in ]a, a + \epsilon[$, hence $\tilde{E}(a) \subset U_{i_-} \cap U_{i_+}$; and since $\tilde{E}(a)$ is nonempty then necessarily $i_-\ i_+\ i_-\ i_+\ i_-\ i_+$ and $\tilde{E}(x) \subset U_i$ for all $x \in ]a - \epsilon, a + \epsilon[$.

This finishes the proof of Claim \[5.3\]

Then applying Claim \[5.3\] to $J = I$ and $\xi$ such that $F^{(\xi)} = \emptyset$ we get that $E \subset U_i$ with necessarily $i = 0$ since $U_0 \cap E \neq \emptyset$, which proves that the set $E$ is connected. \[5.3\]

Proposition 8.6. If $H$ is an hypergraph with graph part $G$ then $\dim(H) = \dim(G) = 1$.

Proof. Since $H$ is connected then $\dim(H) \geq 1$ and we only have to prove that $\dim(G) \leq 1$. So fix $(a, b) \in G$; given any neighbourhood $V$ of $(a, b)$ we can find two open intervals $I_a = ]a', a''[$ and $I_b = ]b', b''[$ such that $(a, b) \in W = (I_a \times I_b) \cap \overline{G} \subset V$.

And we can impose on the endpoints $a'$ and $a''$ to be in $P = I \setminus Q$ so that $A = (\{a', a''\} \times J_b) \cap \overline{G} = \{(a', f(a')), (a'', f(a'')), (a', f(a'')), (a'', f(a''))\}$ is finite. Also since the function $f$ is nowhere constant on $P$ the set $B = (I \times \{b', b''\}) \cap \overline{G}$ is a closed subset of $\overline{G}$ of empty interior in $\mathbb{R} \times \{b', b''\}$ hence $\dim B = 0$. And since the boundary of $W$ in $\overline{G}$ is contained in $A \cup B$ then $\dim(G) \leq 1$. \[5.3\]
Theorem 8.7. If $H$ is an hypergraph and $E \subset H$ is such that $\dim(\pi(E)) = 0$ then $\dim(E) = 0$.

Proof. We shall first prove the theorem under the additional assumption that $E \cap G$ is dense in $E$.

Fix $(a, b) \in E$ and two intervals $I_a = [a - \varepsilon, a + \varepsilon]$ and $J_b = [b - \varepsilon, b + \varepsilon]$ for some $\varepsilon > 0$. We have to find a clopen neighbourhood $W$ of $(a, b)$ in $E$ such that $W \subset I_a \times J_b$. We distinguish the two cases:

- If $a \in P$ then $(a, b) \in G$ and $b = f(a)$. Then by the continuity of $f$ we can find two open intervals $I', J'$ such that $(a, b) \in I' \times J'$, $I' \times J' \subset I_a \times J_b$ and $f(P \cap I') \subset J'$. Since $\pi(E)$ is of empty interior we can also impose on the endpoints of $I'$ to be in $I' \setminus \pi(E)$ so that $I' \cap \pi(E)$ is clopen in $\pi(E)$. Then $W = E \cap \pi^{-1}(I')$ is a clopen subset of $E$ and by the choice of $I'$ we have $W \cap G \subset I' \times J'$, hence by the density of $G \cap E$ in $E$ we have $W \subset W \cap G \subset I' \times J' \subset I_a \times J_b$.

- If $a \in Q$ then applying clause b) of Lemma 8.1 we can fix two continuous functions $g : I \setminus \{a\} \to \mathbb{R}$ and $h : P \cup \{a\} \to \mathbb{R}$ such that $f(x) = g(x) + h(x)$ for all $x \in P$. Since $H(\mathbb{R})$ is of empty interior we can find $b', b'' \in \mathbb{R} \setminus H(\mathbb{R})$ with $b' < b < b''$ and $b'' - b' < \frac{\varepsilon}{2}$, and an open interval $I'$ containing $a$ on which the oscillation of $h$ is $< \frac{\varepsilon}{2}$; since $\pi(E)$ is zero-dimensional then it is of empty interior and we can impose on the endpoints of $I'$ to be in $I \setminus \pi(E)$. Set $Z = I' \cap \pi(E) \setminus \{a\}$; since $g$ is continuous on $Z$ the three sets

$$U = \{x \in Z : b' - h(a) < g(x) < b'' - h(a)\}$$
$$U_- = \{x \in Z : g(x) < b' - h(a) + |x - a|\}$$
$$U_+ = \{x \in Z : g(x) > b'' - h(a) - |x - a|\}$$

constitute an open covering of $Z$ which is zero-dimensional, hence we can find in $Z$ a clopen covering $(U'_-, U''_-, U'_+)$ finer than $(U, U_-, U_+)$. We claim that each of the following three sets:

$$W = (\{a\} \times [b', b'' \cup \pi^{-1}(U')]) \cap E$$
$$W_- = (\{a\} \times -\infty, b' \cup \pi^{-1}(U'_-)) \cap E$$
$$W_+ = (\{a\} \times b''', +\infty, \cup \pi^{-1}(U'_+)) \cap E$$

is open, hence clopen, in $E$. We detail the proof for $W$, the argument being similar for the other two sets. Since $\pi^{-1}(U') \cap E$ is open in $E$ we only need to show that any element of the form $(a, y) \in E$ with $y \in [b', b'']$ is in the interior of $W$. So suppose by contradiction that $(a, y) = \lim_j (x_j, y_j)$ with $(x_j, y_j) \in W_- \cup W_+$. Then necessarily $x_j \neq a$ for co-finitely many $j$’s; so we may suppose that for all $j$, $x_j \in U_- \cup U_+$. Since $E \cap G$ is dense in $E$ we can also assume that $x_j \in P$ hence $y_j = f(x_j)$. Then

$$\lim_j g(x_j) = \lim_j f(x_j) - \lim_j h(x_j) = y - h(a) \in [b' - h(a), b'' - h(a)]$$
hence for large \( j \) we have:
\[
b' - h(a) + |x_j - a| \leq g(x_j) \leq b'' - h(a) - |x_j - a|
\]
which is a contradiction since \( x_j \in U_\epsilon \cup U_- \).

Hence \( W \) is a clopen subset of \( E \) and clearly \( (a, b) \in W \). To finish the proof we have to show that \( W \subset I_a \times J_b \). Again by the density of \( G \cap E \) in \( E \) it is sufficient to show \( W \cap G \subset I_a \times J_b \). So let \( (x, y) \in W \cap G \) then \( y = f(x) = g(x) + h(x) \) and since the oscillation of \( h \) on \( I' \) is \( < \frac{\epsilon}{2} \) then
\[
b' - \frac{\epsilon}{2} \leq b' - h(a) + h(x) \leq y \leq b'' - h(a) + h(x) \leq b'' + \frac{\epsilon}{2}
\]
and since \( b'' - b' \leq \frac{\epsilon}{2} \) then \( |y - b| < \epsilon \) so \( (x, y) \in I_a \times J_b \).

This finishes the proof under the additional density condition and we go back now to the general case. Starting from an arbitrary set \( E \) such that \( \dim(\pi(E)) = 0 \) consider the closed subset \( E' = E \cap G^E = E \cap G \cap E \) of \( E \). Then \( E' \cap G = E \cap G \) and \( E' \cap G^E = E \cap G^E = E' \) so \( E' \) satisfies the density condition. Hence \( \dim(E') = 0 \) and since \( E = E' \cup \bigcup_{a \in Q} E \cap \{(a) \times \mathbb{R}\} \) and each \( E \cap \{(a) \times \mathbb{R}\} \) is a zero-dimensional closed subset of \( E \), then \( \dim(E) = 0 \).

**Corollary 8.8.** Let \( H \) be an hypergraph. For any set \( E \subset H \) the following are equivalent:

(i) \( \dim(E) = 0 \).

(ii) \( \dim(\pi(E)) = 0 \).

**Proof.** Suppose that \( \dim(E) = 0 \). If \( \pi(E) \) were not zero-dimensional then \( \pi(E) \) would contain a non trivial interval \( J \) hence by Theorem 8.7. the set \( E' = E \cap J \times \mathbb{R} \) would be a non trivial connected subset of \( E \), a contradiction. The converse implication follows from Theorem 8.7.

**Corollary 8.9.** For a closed subset \( F \) of an hypergraph \( H \) the following are equivalent:

(i) \( F \) is of empty interior in \( H \).

(ii) \( \dim(F) = 0 \).

(iii) \( \dim(\pi(F)) = 0 \).

**Proof.** (i) \( \Rightarrow \) (ii): Suppose that \( F \) is of empty interior in \( H \). If \( \pi(F) \) were not zero-dimensional then \( \pi(F) \) would contain a nonempty open interval \( J \). Then for all \( x \in J \cap P \), \( (x, f(x)) \in F \) hence \( F \supset \{(x, f(x)) : x \in J \cap P\} \) and it follows that \( F \) contains the nonempty open subset \( H \cap (J \times \mathbb{R}) \) of \( H \), a contradiction.

(ii) \( \Rightarrow \) (iii): Observe first that by compactness \( \bar{\pi(F)} = \pi(F) \). Also since the graph part \( G \) is a closed subset of \( P \times \mathbb{R} \) then necessarily \( \bar{F} \cap F \subset Q \times \mathbb{R} \) so \( \pi(F) \setminus \pi(F) \subset Q \).

Suppose now that \( \dim(F) = 0 \). If \( \pi(F) \) were not zero-dimensional then it would be of nonempty interior in \( \mathbb{R} \) hence would contain some open interval \( I_0 \) and then \( I_0 \setminus Q \subset \pi(F) \), hence \( F \) would contain the closure in \( H \) of the graph of \( f \mid_{I_0 \cap P} \) which is an hypergraph too, and this is impossible since \( F \) is zero-dimensional.
(iii) $\Rightarrow$ (i): Suppose that $\dim(\pi(F)) = 0$. Then by Corollary 8.8 $\dim(F) = 0$ and if $F$ were not of empty interior then by Remark 8.3 b) $F$ would contain an hypergraph $H'$, and then we would have that $\dim(H') = 0$ which contradicts Theorem 8.4.$\square$

**Corollary 8.10.** If $H$ is a $G_δ$ hypergraph then $F_0(H)$ is a $G_δ$ subset of $F(H)$.

**Proof.** Set $\hat{R} = [-\infty, +\infty]$ and fix a countable basis $(U_n)_n$ of open sets in $\hat{R}$. Observe that for any set $A \subset \mathbb{R}$ the set $A^{\mathbb{R}}$ is of empty interior in $\mathbb{R}$ if and only if the set $\hat{A}$ is of empty interior in $\hat{R}$.

Embed $H$ in the compact space $\hat{R} \times \hat{R}$ and consider on $F(H)$ the corresponding admissible topology $\hat{\tau}_0$. Then by Corollary 8.9 for $F \in F(H)$ we have:

$$ F \in F_0(H) \iff \pi(F)^{\hat{\tau}} \text{ is of empty interior in } \hat{R} $$

$$ \iff \pi(F)^{\hat{\tau}} = (\pi(F))^{\hat{R}} $$

$$ \iff \forall n, \exists m, \overline{U_m}^{\hat{R}} \subset U_n, \pi(F) \cap \overline{U_m}^{\hat{R}} = \emptyset $$

$$ \iff \forall n, \exists m, \overline{U_m}^{\hat{R}} \subset U_n, F \cap \pi^{-1}(\overline{U_m}^{\hat{R}}) = \emptyset $$

where $\pi$ denotes the projection mapping from $\hat{R} \times \hat{R}$ onto the first factor. Since $\pi^{-1}(\overline{U_m}^{\hat{R}})$ is compact then by 2.3 b) condition “$F \cap \pi^{-1}(\overline{U_m}^{\hat{R}}) = \emptyset$” on $F$ is open, hence $F_0(H)$ is a $\Pi_0^2$ subset of $(F(H), \hat{\tau}_0)$.

Now if $\hat{\tau}$ is any admissible topology on $F(H)$ then $\hat{\tau}$ is first Baire class isomorphic to $\hat{\tau}_0$, so $F_0(H)$ is a $\Pi_0^2$ subset of $(F(H), \hat{\tau})$, hence by Corollary 4.4 $F_0(H)$ is actually a $\Pi_0^2$ subset of $(F(H), \hat{\tau})$. $\square$

**Corollary 8.11.** Any compact subset of an hypergraph is of empty interior hence zero-dimensional.

**Proof.** Let $K$ be a compact subset of $H$ and suppose that $U \times V$ is an open subset of $\mathbb{R}^2$ such that $\emptyset \neq (U \times V) \cap H \subset K$. Pick any element $a \in U \cap Q$; since $G(a)$ is a non trivial interval we can also pick some element $b \in V \cap \overline{G(a)} \setminus H(a)$, so $(a,b) \in (\overline{G} \setminus H) \cap (U \times V)$. We can then find $(a_j, b_j) \in G \cap (U \times V)$ such that $(a, b) = \lim_j (a_j, b_j)$; and since $G$ is a graph then $H(a_j) = K(a_j) = \{b_j\}$ hence $(a_j, b_j) \in K$ and so $(a,b) \in K \subset H$, which is a contradiction.

Hence $K$ is closed and of empty interior and the last part of the conclusion follows then from Corollary 8.9. $\square$

**Corollary 8.12.** An hypergraph is a connected space which contains no non trivial continuum.

Observe that an hypergraph $H$ cannot be decomposable into a countable union of closed zero-dimensional or compact subsets since by Corollary 8.11 it would be a countable union of closed zero-dimensional subsets hence by Theorem 2.5 $H$ itself would be zero-dimensional which contradicts Proposition 8.5.
9. The maximum descriptive class of $\mathcal{F}_0(X)$ when $X$ is Polish

We can now prove the following complement to Corollary 7.5.

**Theorem 9.1.** For any $G_\delta$ hypergraph $H$ there exists a $\Pi^0_2$ subset $X$ of $2^\omega \times H$ such that $\mathcal{F}_0(X)$ is $\mathcal{D}\Sigma^0_2$-hard.

**Proof.** We check that $H$ and its graph part $G$ satisfy all the assumptions of Theorem 7.3 for $\xi = 2$: By Proposition 8.6 $\dim(H) = 1$ and it is easily seen as in Remark 8.3 (c) that if $W$ is a nonempty open subset of $H$ and if a $\sigma$-compact set $T = \bigcup_n F_n$ (with $T_n$ compact) contains $G \cap W$ then some $F_n$ contains the whole interval $\overline{G}(x)$ for some $x \in Q \cap W$ hence $T$ meets $\mathbb{R}^2 \setminus H$. Thus $G \cap W$ is not relatively $\sigma$-compact in $H$. Finally observe that if $D \subset G$ then $\pi(D) \cap \pi(H \setminus D) = \emptyset$ hence if $D$ is dense in $G$ then $\pi(D)$ is dense in $\pi(H)$ hence $\pi(H \setminus D)$ is of empty interior in $\pi(H)$, so $\dim(\pi(H \setminus D)) = 0$ and by Theorem 8.7 $\dim(H \setminus D) = 0$. \(\square\)

9.2. Back to games: For the next result we need to consider games in which one of the Players, namely Player II, is required to make his moves in $\{0, 1\}$. Thus if $\gamma = (n_0, n_1, n_2, \cdots) \in \omega^\omega$ is any infinite run in such a game then $n_{2k+1} \in \{0, 1\}$. The main advantage of this variation is that in this case the space $\Sigma^*_2$ of all strategies for Player II is now compact: $\Sigma^*_2 = \{0, 1\}(\omega^{<\omega} \setminus \{0\}) \approx 2^\omega$, while for Player I the situation is unchanged: $\Sigma^*_1 = \omega^{(2^\omega)} \approx \omega^\omega$. The basic observation is that any $\Sigma^0_2$ game on $\omega$ can be reduced (in a precise sense) to a $\Sigma^0_2$ game of the previous form. To state this properly let us introduce a notation. Let $\ast$ denote the operation on $\omega^\omega$ defined by:

$$\alpha \ast \beta(n) = \begin{cases} 
\alpha(k) & \text{if } n = 2k \\
\beta(k) & \text{if } n = 2k + 1
\end{cases}$$

Then the mapping $(\alpha, \beta) \mapsto \alpha \ast \beta$ is clearly a homeomorphism from $\omega^\omega \times \omega^\omega$ onto $\omega^\omega$. Notice that if we view $\alpha$ and $\beta$ as two (trivial) strategies for Player I and Player II respectively then $\alpha \ast \beta$ is just the infinite run obtained by the $\ast$ operation introduced in 7.3.a.

Set:

$$\omega^\omega \times 2^\omega = \{\alpha \ast \beta : (\alpha, \beta) \in \omega^\omega \times 2^\omega\}$$

$$= \{\gamma : \omega^\omega : \forall k, \gamma(2k + 1) \in \{0, 1\}\}$$

We need the following variation of Lemma 7.2. It is worth noting that the situation is not symmetrical and one cannot replace in this statement $\Sigma^0_2$ by $\Pi^0_2$, equivalently Player I by Player II, while any Borel game of rank $\xi \geq 3$ on $\omega$ can be reduced to a game where both Players’ moves are in $\{0, 1\}$.

**Lemma 9.3.** There exists a $\Sigma^0_2$ set $V \subset 2^\omega \times (\omega^\omega \ast 2^\omega)$ such that the set $\mathcal{D}V$ is $\mathcal{D}\Sigma^0_2$-complete.

**Proof.** Given any $\varepsilon \in 2^{<\omega} \cup 2^\omega$ let $\nu = \Psi(\varepsilon) \in \omega^{<\omega} \cup \omega^\omega$ be the increasing enumeration of the set $N = \{j < |\varepsilon| : \varepsilon(j) = 1\}$ and let $\beta = \Phi(\varepsilon) \in \omega^{[|\varepsilon|]}$ be defined by: $\beta(0) = \nu(0)$ and $\beta(k + 1) = \nu(k + 1) - \nu(k) - 1$. Observe that $|\Psi(\varepsilon)| = |\Phi(\varepsilon)| \leq |\varepsilon|$. 

Claim 9.4. \( \alpha \star \varepsilon \Phi(\varepsilon) \)

Proof. – If Player I has a winning strategy inverse. Moreover the mapping \( \Phi \mid \varepsilon = G \) some \( \varepsilon \alpha \varepsilon \) \( 2^{\omega} \) and \( \Phi(Q) = \omega^{<\omega} \); and we shall denote by \( \Phi_s : 2^{<\omega} \to 2^{\omega} \) \( \varepsilon ) \) its inverse. Moreover the mapping \( \Phi : P \to \omega^{\omega} \) is a homeomorphism.

Then to any set \( A \subset \omega^{\omega} \) we associate a set \( A^* \subset \omega^{\omega} \star 2^{\omega} \) defined as follows: an element \( \alpha \star \varepsilon \in \omega^{\omega} \star 2^{\omega} \) belongs to \( A^* \) if:

- either \( |\Psi(\varepsilon)| = |\Phi(\varepsilon)| < \infty \)
- or \( |\Psi(\varepsilon)| = |\Phi(\varepsilon)| = \infty \) and if \( \nu = \Psi(\varepsilon) \) and \( \beta = \Phi(\varepsilon) \) then \( \alpha \circ \nu \star \beta \in A \)

Claim 9.5. Player I wins the game \( G_A \) if and only if he wins the game \( G_{A^*} \).

Proof. – If Player I has a winning strategy \( \sigma \) in \( G_A \) he constructs a strategy \( \sigma^* \) in \( G_{A^*} \) by playing as follows: If in an infinite run in \( G_{A^*} \) in which Player II plays some \( \varepsilon \in 2^{\omega} \), then following \( \sigma^* \) Player I plays \( \alpha \in \omega^{\omega} \) defined inductively by:

\[
\alpha(k) = \begin{cases} 
\sigma(\Phi(\varepsilon|k)) & \text{if } \varepsilon(k-1) = 1 \\
0 & \text{if not} 
\end{cases}
\]

If \( \varepsilon \notin P \) then by definition of \( A^* \) Player I wins the run. Otherwise \( \varepsilon \in P \) and if \( \nu = \Psi(\varepsilon) \) and \( \beta = \Phi(\varepsilon) \) then from the definition of \( \sigma^* \) we have \( \nu = \Phi(s) \beta \) and since \( \sigma \) is winning in \( G_A \) then \( (\alpha \circ \nu) \star \beta \in A \) so by definition of \( A^* \), \( \alpha \star \varepsilon \in A^* \), hence \( \sigma^* \) is winning in \( G_{A^*} \).

– Conversely if Player I has a winning strategy \( \sigma^* \) in \( G_{A^*} \) he constructs a strategy \( \sigma \) in \( G_A \) by playing as follows: In an infinite run in \( G_{A^*} \) in which Player II plays some \( \beta \in \omega^{\omega} \) then following \( \sigma \) Player I plays \( \alpha \in \omega^{\omega} \) defined for all \( k \) by:

\[
\alpha(k) = \sigma^*(\Phi_s(\beta|j_k)) \quad \text{where } (j_k) = \Psi(\Phi_s(\beta)) \quad \text{is the enumeration of the set } \{ j : \varepsilon(j) = 1 \} \quad \text{for } \varepsilon = \Phi_s(\beta).
\]

Notice that \( \beta|j_k \in Q \) hence \( \sigma \) is well defined. Moreover if \( \alpha' = \sigma^*(\varepsilon) \) then \( \alpha = \alpha' \circ \nu \) and since \( \sigma^* \) is winning in \( G_{A^*} \), then by definition of \( A^* \) we have \( \alpha \star \beta \in A \), hence \( \sigma \) is winning in \( G_A \). \( \square \)

Let \( U \subset 2^{\omega} \times \omega^{\omega} \) and \( V \subset 2^{\omega} \times (\omega^{\omega} \star 2^{\omega}) \) defined by \( V(\delta) = (U(\delta))^\ast \) for all \( \delta \in 2^{\omega} \).

Claim 9.5. If \( U \) is a \( \Sigma^0_2 \) set then \( V \) is a \( \Sigma^0_2 \) set too.

Proof. Observe that for any set \( A \subset \omega^{\omega} \star 2^{\omega} \) and any \( \gamma = (\alpha \star \varepsilon) \in \omega^{\omega} \star 2^{\omega} \) we have

\[
\gamma \notin A^* \iff \varepsilon \in P \land \alpha \star \Phi(\varepsilon) \notin A
\]

hence for any \( \delta \in 2^{\omega} \):

\[
(\delta, \gamma) \notin V \iff \varepsilon \in P \land (\delta, \alpha \star \Phi(\varepsilon)) \notin U
\]

So letting \( \pi : 2^{\omega} \times \omega^{\omega} \to \omega^{\omega} \) and \( \rho : 2^{\omega} \times P \to 2^{\omega} \times \omega^{\omega} \) denote the mappings defined by:

\[
\pi(\delta, \gamma) = \varepsilon \quad \text{and} \quad \rho(\delta, \gamma) = (\delta, \alpha \star \Phi(\varepsilon))
\]

we have

\[
2^{\omega} \times (\omega^{\omega} \star 2^{\omega}) \setminus V = \pi^{-1}(P) \cap \rho^{-1}(2^{\omega} \times \omega^{\omega}) \setminus U
\]

and since the mappings \( \pi \) and \( \rho \) are continuous and \( P \) is a \( G_\delta \) set it follows that \( 2^{\omega} \times (\omega^{\omega} \star 2^{\omega}) \setminus V \) is a \( G_\delta \) subset, hence \( V \) is a \( \Sigma^0_2 \) subset, of \( 2^{\omega} \times (\omega^{\omega} \star 2^{\omega}) \). \( \square \)
To finish the proof of Lemma 9.3 let $U \subset 2^\omega \times \omega^\omega$ be a $\Sigma_0^3$-complete set given by Lemma 7.2 and consider the set $V \subset 2^\omega \times (\omega^\omega \ast 2^\omega)$ as above. By Claim 9.2 $V$ is a $\Sigma_2^0$ set hence $\partial V$ is $\partial \Sigma_2^0$ set and we now check that the set $\partial V$ is $\partial \Sigma_2^0$-hard.

So let $A \subset 2^\omega$ be any $\partial \Sigma_2^0$ set. By Lemma 9.3 there exists a continuous mapping $\varphi : 2^\omega \to 2^\omega$ which reduces the set $A$ to $\partial U$, hence by Claim 9.3 we have:

\[
\varepsilon \in A \iff \varphi(\varepsilon) \in \partial U \iff \text{Player I wins } G_{(\partial U)(\varphi(\varepsilon))} 
\iff \text{Player I wins } G_{(\partial U)(\varphi(\varepsilon))}^* = G_V(\varphi(\varepsilon)) \iff \varphi(\varepsilon) \in \partial V
\]

which proves that $\varphi$ reduces the set $A$ to $\partial V$. \qed

We recall that it is not known whether the set $F_0(\mathbb{P}^2 \setminus \mathbb{Q}^5)$ is $\Pi_1^1$.

**Theorem 9.6.** $F_0(\mathbb{P}^5 \setminus \mathbb{Q}^5)$ is $\partial \Sigma_2^0$-hard.

**Proof.** We shall actually construct a space $Y$ for which the set $F_0(Y)$ is $\partial \Sigma_2^0$-hard and which can be embedded as a closed subset in $\mathbb{P}^5 \setminus \mathbb{Q}^5$. It will follow that $F_0(Y)$ is a closed subset of $F_0(\mathbb{P}^5 \setminus \mathbb{Q}^5)$ hence $F_0(\mathbb{P}^5 \setminus \mathbb{Q}^5)$ is $\partial \Sigma_2^0$-hard too.

The definition of the space $Y$ follows the same general scheme of definition as for the spaces $X_U$ constructed in the proof of Theorem 7.3 and will be of the form:

\[Y = \{(\alpha, \tau, z) \in 2^\omega \times \Sigma_1^* \times H : \forall n, \ z \in F_n \to (\alpha, h_n(z) \ast \tau) \notin V\}\]

where $V$ is the set given by Lemma 7.3 $H$ is a specific hypergraph that we will construct, the set $\Sigma_1^* \approx \omega^\omega$ of strategies replaces the previous set $\Sigma_1$, and for all $n$, $h_n : F_n \to \Sigma_1^*$ is a homeomorphism defined on a closed subset $F_n$ of $H$ which will replace the $\Pi_3^0$ set $G_n$ of the proof of Theorem 7.3.

Since the $\Sigma_2^0$ set $V$ satisfies also Lemma 7.2 we can apply Claim 7.4 to conclude as in Theorem 7.3 that $F_0(Y)$ is $\partial \Sigma_2^0$-hard. Moreover since the $F_n$’s are closed subsets of $H$ the immediate analysis of the previous definition shows that $X$ is indeed a $\Pi_3^0$ set.

Our plan is to show that for a suitable choice of $H$ and the $F_n$’s the space $Y$ above can be embedded as a closed subset in $\mathbb{P}^5 \setminus \mathbb{Q}^5$. We point out that unlike in the proof of Theorem 7.3 where the $G_n$’s were constructed for a given space $H$ in the present case we will have to construct $H$ and the $F_n$’s simultaneously, which will be done in a series of steps.

We recall that the notation $\overline{A}$ for a subset of $H$ refers to the closure in $\mathbb{R}^2$.

**Lemma 9.7.** In any hypergraph $H$ with graph part $G$ there exists a closed subset $F$ of $G$ satisfying: $F \approx \omega^\omega$, $\overline{F} \approx 2^\omega$, and for all $q \in Q$, $\text{card } (\overline{F}(q)) \leq 1$.

**Proof.** Observe first that given any $q \in Q$ then by Definition 8.2 we can write $f = g + h$ with $h : P \cup \{q\} \to \mathbb{R}$ continuous and $g : I \setminus \{q\} \to \mathbb{R}$ continuous and such that $\Omega g(q) = \overline{G}(q)$ is a non trivial interval. Then for any cluster value $\ell \in \Omega g(q)$ and any neighbourhood $J$ of $q$ one can find an infinite sequence $(J_n)_n$ of pairwise disjoint closed non trivial intervals in $J \setminus \{q\}$ with endpoints in $P$ and of arbitrary small diameter such that $\lim_n d(q, I_n) = 0$ and the function $g|\cup_n I_n$
has limit \( \ell \) at \( q \); and since \( h \) is continuous on \( P \cup \{q\} \) then \( f|_{P \cup \{q\}} \) has a limit (namely \( \ell + h(q) \)) at \( q \).

Fix an enumeration \((q_n)_{n \in \omega} \) of \( Q \); we shall construct a family \((r_s, J_s)_{s \in \omega^<\omega} \) satisfying for all \( s \):

(i) \( J_s \) a closed non trivial interval with endpoints in \( P \),
(ii) \( \text{diam}(J_s) \leq 2^{-|s|} \),
(iii) \( J_{s-n} \subset J_s \) and \( J_{s-m} \cap J_{s-n} = \emptyset \) if \( m \neq n \),
(iv) \( r_s \in J_s \cap Q \),
(v) if \( |s| = k \) and \( q_k \in J_s \) then \( r_s = q_k \),
(vi) \( r_s \notin J_{s-n} \) for all \( n \) and \( \lim_n d(r_s, J_{s-n}) = 0 \),
(vii) \( f|_{P \cup \{q\}} \) has a limit at \( r_s \).

Take for \( J_0 \) any open interval with endpoints in \( P \). Suppose that the family \((J_s)_{s \in \omega^k} \) is constructed we shall define for each \( s \in \omega^k \), the element \( r_s \) and the family \((J_{s-n})_{n \in \omega} \).

First if \( q_k \in J_s \) then we define \( r_s = q_k \); if not we choose \( r_s \) arbitrarily in \( J_s \cap Q \) so conditions (iv) and (v) are satisfied. Once \( q = r_s \in J_s \) is chosen, observe that since \( J_s \) has endpoints in \( P \) then \( q \) is necessarily in the interior of \( J_s \), and following the previous observations, we can find a family \((J_{s-n})_{n \in \omega} \) satisfying the remaining conditions. Let

\[
F^* = \bigcap_k \bigcup_{s \in \omega^k} J_s \quad \text{and} \quad K = \overline{F^*}
\]

**Claim 9.8.** : For any \( z \in K \) there exists a unique maximal sequence \( \sigma = (s_j) \) in \( \omega^<\omega \) such that: \( z \in J_{s_j} \), \( |s_j| = j \) and \( s_j \subset s_{j+1} \). Then:

- if \( \sigma \) is finite with last element \( s \) then \( z \in Q \) and \( z = r_s \)
- if \( \sigma \) is infinite then \( z \in P \) and \( \{z\} = \bigcap_j J_{s_j} \)

**Proof.** The construction is by induction on \( j \). For \( j = 0 \) then necessarily \( s_0 = \emptyset \) which is legal since \( z \in \overline{P} \subset J_0 \). Suppose that \( s_j \) is already defined then again since \( z \in K \subset (r_{s_j}) \cup \bigcup_{n} J_{s_j-n} \) then:

- either \( z = r_{s_j} \) and then \( |s_j| = j \) and \( z = q_j \) and in this case \( \sigma = (s_i)_{i \leq j} \) is maximal since \( z \notin J_s \) for any \( s \supset s_{j} \) with \( |s| = j + 1 \)
- or \( z \in J_{s_j-n} \) for some unique \( n \) and so necessarily \( s_{j+1} = s_j - n \).

This finishes the construction of \( \sigma \) and:

- either \( \sigma \) is finite and \( z = r_s \) where \( s \) is last element of \( \sigma \)
- or \( \sigma \) is infinite and \( \{z\} = \bigcap_j J_{s_j} \) and we have to show that \( z \in P \). If not then we would have \( z = q_k \) for some \( k \) hence \( q_k \in J_{s_k} \) with \( r_{s_k} \neq q_k \) which is in contradiction with condition (v).

This defines a mapping \( h : K \to \omega^<\omega \cup \omega^\omega \approx \mathcal{P}(\omega) \approx 2^\omega \) which is clearly a homeomorphism such that \( F^* = h^{-1}(\omega^\omega) = K \cap P \approx \omega^\omega \) and \( K \setminus F^* = h^{-1}(\omega^<\omega) = K \cap Q = \{r_s : s \in \omega^<\omega\} \); hence \( F^* \) is a closed subset of \( P \). Finally set \( F = \pi^{-1}(F^*) \); since \( \pi \) is a homeomorphism from \( G \) onto \( P \) then \( \omega^\omega \approx F^* \approx F \) and \( F \) is a closed subset of \( G \); and since by condition (vii) \( f|_{F^*} \) admits a continuous
extension \( g : K \to \mathbb{R} \) then \( \overline{F} = \text{Gr}(g) \) hence for all \( q \in Q \), \( \overline{F}(q) \) is either \( \{g(q)\} \) if \( q = r_x \in K \), or else empty. \( \square \)

**Lemma 9.9.** There exists an hypergraph \( H \) with graph part \( G \) and a sequence \( (F_n) \) of subsets of \( H \) satisfying:

a) \( \overline{H} \setminus H \) is countable,

b) any nonempty open subset of \( H \) contains some \( F_n \).

c) for all \( n \), \( F_n \approx \omega^\omega \), \( \overline{F}_n \approx 2^\omega \) and \( F_n = \overline{F}_n \cap H \subset G \)

**Proof.** Start from an hypergraph \( H' \) with graph part \( G \) which is co-countable in \( \mathcal{G} \) (see Remark 8.3c) and let \((W_n)_n \) be a basis of the topology of \( H' \) with for all \( n \), \( W_n = H' \cap (I_n \times J_n) \neq \emptyset \) for some intervals \( I_n, J_n \). Then applying Lemma 9.7 to each hypergraph \( H_n = H' \cap W_n \) with graph part \( G_n = G \cap W_n \), we can find a closed subset \( F_n \) of \( G_n \cap W_n \) such that for all \( q \in Q \), \( \overline{F}_n(q) \) is contained in some singleton \( \{q^{(n)}\} \). Let

\[ H = H' \setminus \{(q, q^{(n)}) : q \in Q \text{ and } n \in \omega\} \]

then \( \mathcal{G} \setminus H = (\mathcal{G} \setminus H') \cup (H' \setminus H) \) is countable hence \( \overline{H} = \mathcal{G} \) and \( \overline{H} \setminus H \) is countable. In particular for all \( q \in Q \), \( \mathcal{G}(q) \setminus H(q) \) is countable, so \( H(q) \) is dense in \( \mathcal{G}(q) \), and since \( H(q) \subset H'(q) \) then \( H(q) \) is of empty interior in \( \mathcal{G}(q) \). Hence \( H \) is an hypergraph with same graph part \( G \). Finally for all \( n \in \omega \) and all \( q \in Q \), \((\{q\} \times \overline{F}_n(q)) \cap H = \emptyset \), hence \( \overline{F}_n \cap H \subset G \) and since \( F_n \) is a closed subset of \( G \) then \( F_n = \overline{F}_n \cap G = \overline{F}_n \cap H \). \( \square \)

**Lemma 9.10.** Let \( H \) and \((F_n)\) be as in Lemma 9.9, let \( V \) be an arbitrary \( \Sigma^0_2 \) subset of \( \omega^\omega \) and for all \( n \), let \( \Phi_n : 2^\omega \times 2^\omega \times F_n \to \omega^\omega \) be a continuous mapping. Then the space:

\[ X = \{(\alpha, \beta, z) \in 2^\omega \times 2^\omega \times H : \forall n, \ z \in F_n \to \Phi_n(\alpha, \beta, z) \notin V\} \]

can be embedded as a closed subset in \( \mathbb{P}^5 \setminus \mathbb{Q}^5 \).

**Proof.** Since \( V \) is a \( \Sigma^0_2 \) set and the \( F_n \)'s are closed subsets of \( H \) then \( X \) is clearly a \( \Pi^0_3 \) subset of the compact space \( K = 2^\omega \times 2^\omega \times \overline{H} \). Observe that if \((\alpha, \beta, z) \in K \setminus X \) then either \( z \in D = \overline{H} \setminus H \) which is countable, or \( z \in F_n \subset \overline{F}_n \approx 2^\omega \) for some \( n \). Hence

\[ K \setminus X \subset \bigcup_{z \in D, n \in \omega} 2^\omega \times 2^\omega \times (\{z\} \cup \overline{F}_n) \]

Since for all \( z \in D \) and all \( n \), the set \( 2^\omega \times 2^\omega \times (\{z\} \cup \overline{F}_n) \) is a zero-dimensional compact space then by Theorem 2.5 \( \dim(K \setminus X) = 0 \), and we can write the \( \sigma \)-compact space \( K \setminus X \) as the union of a countable family \((K_j)_j \) of pairwise disjoint compact sets: \( K \setminus X = \bigcup_{j} K_j \) with \( \lim_j \text{diam}(K_j) = 0 \). Hence if \( \Delta \) denotes the diagonal of \( K \) then \( E = \Delta \cup \bigcup_j K_j \times K_j \) is a closed equivalence relation on \( K \). Let \( L \) be the quotient (compact) space \( K/E \) and \( \rho : K \to L \) be the canonical quotient mapping. Since the restriction of \( \rho \) to \( X \) is a one-to-one perfect mapping on its image \( Y = \rho(X) \) then it is a homeomorphism from \( X \) onto \( Y \); moreover the set \( L \setminus Y \) is countable as the collection of the \( E \)-equivalence classes \( K_j \).
Note that by Proposition 8.6, $\dim(Y) = \dim(X) \leq \dim(K) = \dim(G) = 1$, and since $L \setminus Y$ is countable then by Theorem 2.5 $\dim(L) \leq 2$, hence the compact space $L$ can be embedded in $\mathbb{R}^5$.

Fix now a homeomorphism $h_0 : L \to L_0 \subset \mathbb{R}^5$; then $D_0 = (\mathbb{Q}^5 \setminus L_0) \cup h_0(L \setminus Y)$ is a countable dense subset of $\mathbb{R}^5$, and since $\mathbb{R}^5$ is a CDH space ([2]) then there exists a homeomorphism $h_1 : \mathbb{R}^5 \to \mathbb{R}^5$ such that $h_1(D_0) = \mathbb{Q}^5$ and we can assume that the compact space $L_1 = h_1(L_0)$ is a subset of $I^5$. Hence $h = h_1 \circ h_0 \circ \rho$ induces a homeomorphism from $X$ onto $L_1 \setminus \mathbb{Q}^5 = L_1 \cap (I^5 \setminus \mathbb{Q}^5)$ which is a closed subset of $I^5 \setminus \mathbb{Q}^5$.

End of the proof of Theorem 9.6: Starting from $H$ and $(F_n)$ as in Lemma 9.9 we fix for all $n$, an arbitrary homeomorphism $h_n : F_n \to \Sigma^*_1 \approx \omega^n$. It is clear then that the space $Y$ defined at the beginning of the proof is of the form considered in Lemma 9.10. Hence $Y$ can be embedded as a closed subset in $I^5 \setminus \mathbb{Q}^5$ and since $F_0(Y)$ is $\Sigma^0_2$-hard then $F_0(I^5 \setminus \mathbb{Q}^5)$ is $\Sigma^0_2$-hard too.

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