VECTOR-VALUED (SUPER) WEAVING FRAMES

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Abstract. Two frames \( \{ \phi_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) for a separable Hilbert space \( H \) are woven if there are positive constants \( A \leq B \) such that for every subset \( \sigma \subseteq I \), the family \( \{ \phi_i \}_{i \in \sigma} \cup \{ \psi_i \}_{i \in \sigma^c} \) is a frame for \( H \) with frame bounds \( A, B \). Bemrose et al. introduced weaving frames in separable Hilbert spaces and observed that weaving frames has potential applications in signal processing. Motivated by this, and the recent work of Balan in the direction of application of vector-valued frames (or superframes) in signal processing, we study vector-valued weaving frames. In this paper, first we give some fundamental properties of vector-valued weaving frames. It is shown that if a family of vector-valued frames is woven, then the corresponding family of frames for atomic spaces is woven, but the converse is not true. We present a technique for the construction of vector-valued woven frames from given woven frames for atomic spaces. Necessary and sufficient conditions for vector-valued weaving Riesz sequences are given. Several numerical examples are given to illustrate the results.

1. Introduction

Duffin and Schaeffer [12] introduced frames for Hilbert spaces in the context of nonharmonic Fourier series. Today frames have applications in a wide range of areas in applied mathematics. The applications of frames in signal processing are now well-known, for example see [6, 7]. Balan [1] introduced the concept of a vector-valued frame (or "superframe") in the context of multiplexing and further studied in [2]. The vector-valued frame has significant applications in mobile communication, satellite communication, and computer area network. Recently, Bemrose, Casazza, Gröchenig, Lammers and Lynch in [3] proposed weaving frames in a separable Hilbert space. The concept of weaving frames is motivated by a problem regarding distributed signal processing where redundant building blocks (frames) plays an important role. For example, in wireless sensor networks where frames may be subjected to distributed processing under different frames. Motivated by the concept of weaving frames and superframes and their application in Gabor and wavelet analysis, in this paper, we study vector-valued (super) weaving frames. Weaving frames has potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. Notable contribution in the paper is a new technique for the construction of vector-valued weaving frames from frames of atomic spaces. Some necessary and sufficient conditions for vector-valued weaving Riesz sequences are given. Finally, a result for vector-valued weaving Riesz sequences in terms of operators on atomic spaces has been obtained.

1.1. Previous works on weaving frames. For a positive integer \( m \), we write \( [m] = \{1, 2, \ldots, m\} \). We start with the definition of weaving frames in separable Hilbert spaces.

Definition 1.1. [3] A family of frames \( \{ \phi_{ij} \}_{i \in I} \) for \( j \in [m] \) for a Hilbert space \( H \) is said to be woven if there are universal constants \( A \) and \( B \), so that for every partition \( \{ \sigma_j \}_{j \in [m]} \) of \( I \), the family \( \{ \phi_{ij} \}_{i \in \sigma_j, j \in [m]} \) is a frame for \( H \) with lower and upper frame bounds \( A \) and \( B \), respectively.

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2. PRELIMINARIES

2.1. Hilbert space frames: A countable sequence \( \{ f_k \}_{k \in I} \) in a separable Hilbert space \( H \) is called a frame (or Hilbert frame) for \( H \) if there exist positive numbers \( \alpha_0 \leq \beta_0 < \infty \) such that

\[
\alpha_0 \| f \|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq \beta_0 \| f \|^2 \quad \text{for all } f \in H.
\]

(2.1)
The numbers $\alpha_o$ and $\beta_o$ are called lower and upper frame bounds, respectively. If upper inequality in (2.1) is satisfied, then we say that $\{f_k\}_{k \in I}$ is a Bessel sequence (or Hilbert Bessel sequence) with Bessel bound $\beta_o$. The frame $\{f_k\}_{k \in I}$ is tight if it is possible to choose $\alpha_o = \beta_o$.

Let $\{f_k\}_{k \in I}$ be a frame (or a Bessel sequence) for $H$. The pre-frame operator $T : \ell^2(I) \rightarrow H$ given by
\[
T\{c_k\}_{k \in I} = \sum_{k \in I} c_k f_k.
\]
The frame operator $S = TT^* : H \rightarrow H$ is
\[
S f = \sum_{k \in I} \langle f, f_k \rangle f_k.
\]
The frame operator for a frame $\{f_k\}_{k \in I}$ is bounded linear and invertible operator on $H$. This gives the reconstruction formula for all $f \in H$,
\[
f = SS^{-1} f = \sum_{k \in I} \langle S^{-1} f, f_k \rangle f_k = \sum_{k \in I} \langle f, S^{-1} f_k \rangle f_k.
\]
The scalars $\{\langle S^{-1} f, f_k \rangle\}_{k \in I}$ are called frame coefficients of the vector $f \in H$. One may observe that the representation of $f$ in the reconstruction formula need not be unique.

A Riesz basis for $H$ is a family of the form $\{U e_k\}_{k \in I}$, where $\{e_k\}_{k \in I}$ is an orthonormal basis for $H$ and $U : H \rightarrow H$ is a bounded linear bijective operator. The following result for bases in Banach spaces can be found in [20, p. 173]

**Theorem 2.1.** [20] Let $\{x_n\}_{n \in \mathbb{N}}$ be a basis for a Banach space $X$. If $\{y_n\}_{n \in \mathbb{N}} \subset X$ and there exists a constant $\lambda \in [0, 1)$ such that
\[
\left \| \sum_{n=1}^{N} c_n (x_n - y_n) \right \| \leq \lambda \left \| \sum_{n=1}^{N} c_n x_n \right \|, \quad N \in \mathbb{N}, \quad c_1, c_2, \ldots, c_N \in \mathbb{K}.
\]
then $\{y_n\}_{n \in \mathbb{N}}$ is a basis for $X$ and $\{y_n\}_{n \in \mathbb{N}}$ is equivalent to $\{x_n\}_{n \in \mathbb{N}}$. That is, there exists a bounded, linear and bijective operator $T : X \rightarrow X$ such that $T x_n = y_n$ for all $n \in \mathbb{N}$.

The basic theory of frames can be found in books of Casazza and Kutyniok [6] and Christensen [7].

### 2.2. Vector-valued Frames (Superframes)

Let $I$ be a countable index set and consider $(\mathcal{F}_1; \pi_1; I), \ldots, (\mathcal{F}_L; \pi_L; I)$, $L$ indexed sets of vectors (not necessarily from the same Hilbert space), where $\pi_k : I \rightarrow F_k$ is the corresponding indexing map. A collection of such countable sets of vectors together with their corresponding indexing maps from a same index set is called a superset. In short, we write $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ for a superset when an indexing by a same index set $I$ for each subset $F_k$ of vectors of some Hilbert space $\mathcal{H}_k$ (or a bigger space $K_k$) is fixed.

We write
\[
\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_L = \{f^1 \oplus \cdots \oplus f^L : i \in I\}, \quad f^i_k = \pi_k(i) \in \mathcal{F}_k.
\]
Recall that the space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$ is a Hilbert space with natural inner product:
\[
\langle f^1 \oplus \cdots \oplus f^L, g^1 \oplus \cdots \oplus g^L \rangle = \sum_{i=1}^{L} \langle f^i, g^i \rangle_{\mathcal{H}_i}, \quad \left( f^i, g^i \in \mathcal{H}_i \ (1 \leq i \leq L) \right).
\]

**Definition 2.2.** [1] The superset $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is called a vector-valued frame or super frame if $\mathcal{F}$ is a frame for the space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$. That is, if there exists finite positive numbers $A_0 \leq B_0$ such that for every $h_k \in \mathcal{H}_k \ (1 \leq k \leq L)$, we have
\[
A_0 (\|h_1\|^2 + \cdots + \|h_L\|^2) \leq \sum_{i \in I} \left | \sum_{k=1}^{L} \langle h_k, f^i_k \rangle \right |^2 \leq B_0 (\|h_1\|^2 + \cdots + \|h_L\|^2).
\]
If $I$ is the common indexing set (possibly countably infinite) and $F_i = \{f_{ik}\}_{k \in I}$, $i = 1, 2, \ldots, L$, then a vector-valued family is represented by $\{F_1, \ldots, F_L, I\}$. The spaces $\mathcal{H}_i$ are called \textit{atomic spaces} (or \textit{components}) of the space $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$.

By using superframes, Balan in [1] discussed a signal transmission problem (namely, multiplexing, or Multiple Access) and analyzed several encoding-decoding schemes (with an example). In recent years, vector-valued wavelet and Gabor frames have attracted mathematician and engineering specialists, for example see [14, 15, 16, 17, 19, 21, 22] and reference therein.

3. \textbf{VECTOR-VALUED WEAVING FRAMES}

We start this section by the definition of a vector-valued weaving frame.

\textbf{Definition 3.1.} A family of vector-valued frames $\{\{F_i^1, \ldots, F_i^m, I\} : i \in [m]\}$ for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$ is said to be \textit{woven} if there exist universal constants $A$ and $B$ such that for any partition $\{\sigma_i\}_{i \in [m]}$ of $I$, the family

$$
\bigcup_{i \in [m]} \{F_i^1, \ldots, F_i^m, \sigma_i\}
$$

is a vector-valued frame for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$ with lower and upper frame bounds $A$ and $B$, respectively. Each family $\bigcup_{i \in [m]} \{F_i^1, \ldots, F_i^m, \sigma_i\}$ is called a \textit{vector-valued weaving} (or simply \textit{weaving}).

\textbf{Remark 3.2.} Throughout the paper we take $F_j \subset \mathcal{H}_j$ for $j \in [L]$. If $\{F_1, \ldots, F_L, I\}$ is vector-valued frame (Bessel sequence) for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$, then $F_j$ ($j = 1, \ldots, L$) are frames (Bessel sequences) for atomic spaces $\mathcal{H}_j$.

Regarding the existence of vector-valued weaving frames, we have the following examples.

\textbf{Example 3.3.} Let $L = m = 2$, $I = \mathbb{N}$ and let $\mathcal{H}_1 = \mathcal{H}_2 = \ell^2(\mathbb{N})$. Define $F_1 = \{f_{1i}\}_{i \in I}$, $G_1 = \{g_{1i}\}_{i \in I} \subset \mathcal{H}_1$ as follows:

$$
f_{1i} = \begin{cases} e_j, & i = 4j - 3 \ (j \in \mathbb{N}) \\ 0, & \text{otherwise}, \end{cases}
$$

and $g_{1i} = \begin{cases} e_j, & i = 4j - 3, 4j - 1 \ (j \in \mathbb{N}) \\ 0, & \text{otherwise}. \end{cases}$

and $F_2 = \{f_{2i}\}_{i \in I}$, $G_2 = \{g_{2i}\}_{i \in I} \subset \mathcal{H}_2$ as follows:

$$
f_{2i} = \begin{cases} e_j, & i = 4j \ (j \in \mathbb{N}) \\ 0 \text{ otherwise}, \end{cases}
$$

and $g_{2i} = \begin{cases} e_j, & i = 4j - 2, 4j \ (j \in \mathbb{N}) \\ 0 \text{ otherwise}. \end{cases}$

where $\{e_i\}_{i \in I}$ is the canonical orthonormal basis of $\ell^2(\mathbb{N})$.

To show the vector-valued families $\{F_1, F_2, I\}$ and $\{G_1, G_2, I\}$ are woven, let $\sigma \subset I$ be any subset.

We compute

$$
\sum_{i \in \sigma} \left| \langle f \oplus g, f_{1i} \oplus f_{2i} \rangle \right|^2 + \sum_{i \in \sigma^c} \left| \langle f \oplus g, g_{1i} \oplus g_{2i} \rangle \right|^2
$$

\begin{align*}
&\geq \sum_{i \in \sigma \cap \{(4m - 3)_{m \in \mathbb{N}} \cup (4m)_{m \in \mathbb{N}}\}} \left| \langle f \oplus g, f_{1i} \oplus f_{2i} \rangle \right|^2 + \sum_{i \in \sigma^c \cap \{(4m - 3)_{m \in \mathbb{N}} \cup (4m)_{m \in \mathbb{N}}\}} \left| \langle f \oplus g, g_{1i} \oplus g_{2i} \rangle \right|^2 \\
&= \left| \langle f \oplus g, e_1 \oplus 0 \rangle \right|^2 + \left| \langle f \oplus g, 0 \oplus e_1 \rangle \right|^2 + \left| \langle f \oplus g, e_2 \oplus 0 \rangle \right|^2 + \left| \langle f \oplus g, 0 \oplus e_2 \rangle \right|^2 + \cdots \\
&= \left| \langle f, e_1 \rangle \right|^2 + \left| \langle g, e_1 \rangle \right|^2 + \left| \langle f, e_2 \rangle \right|^2 + \left| \langle g, e_2 \rangle \right|^2 + \cdots \\
&= \|f\|^2 + \|g\|^2
\end{align*}

for all $f \oplus g \in \mathcal{H}_1 \oplus \mathcal{H}_2$. 

This gives a lower universal bound $A = 1$. Similarly, we can show that an upper universal bound is $B = 2$. Hence the vector-valued frames $\{F_1, F_2, I\}$ and $\{G_1, G_2, I\}$ for $H_1 \oplus H_2$ are woven.

**Example 3.4.** Let $L, m, I$, $H_1$ and $H_2$ be same as in Example 3.3.

Define $F_1 = \{f_{1i}\}_{i \in I}, G_1 = \{g_{1i}\}_{i \in I} \subset H_1$ as follows:

$$f_{1i} = \begin{cases} e_j, & i = 6j \ (j \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g_{1i} = \begin{cases} e_j, & i = 6j - 3, 6j - 1, 6j \ (j \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases}$$

and $F_2 = \{f_{2i}\}_{i \in I}, G_2 = \{g_{2i}\}_{i \in I} \subset H_2$ as follows:

$$f_{2i} = \begin{cases} e_j, & i = 6j - 1 \ (j \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g_{2i} = \begin{cases} e_j, & i = 6j - 4, 6j - 1, 6j \ (j \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases}$$

where $\{e_i\}_{i \in I}$ is the canonical orthonormal basis of $H_1$. Then, $\{F_1, F_2, I\}$ and $\{G_1, G_2, I\}$ are vector-valued frames for $H_1 \oplus H_2$ but not woven.

Indeed, suppose $\{F_1, F_2, I\}$ and $\{G_1, G_2, I\}$ are woven with universal bounds $A$ and $B$.

For $\sigma = I \setminus \{5, 6\}$ and $(-1, 0, 0, 0, \ldots) \oplus (1, 0, 0, 0, \ldots) \in H_1 \oplus H_2$, we compute

$$\sum_{i \in \sigma} \left| \left\langle (-1, 0, 0, 0, \ldots), f_{1i} \right\rangle + \left\langle (1, 0, 0, 0, \ldots), f_{2i} \right\rangle \right|^2$$

$$+ \sum_{i \in \sigma'} \left| \left\langle (-1, 0, 0, 0, \ldots), g_{1i} \right\rangle + \left\langle (1, 0, 0, 0, \ldots), g_{2i} \right\rangle \right|^2$$

$$= \left| \left\langle (-1, 0, 0, 0, \ldots), e_1 \right\rangle + \left\langle (1, 0, 0, 0, \ldots), e_1 \right\rangle \right|^2 + \left| \left\langle (-1, 0, 0, 0, \ldots), e_1 \right\rangle + \left\langle (1, 0, 0, 0, \ldots), e_1 \right\rangle \right|^2$$

$$= 0$$

$$< 2A = A\left(\left\| (-1, 0, 0, 0, 0, \ldots) \oplus (1, 0, 0, 0, 0, \ldots) \right\|^2 \right),$$

a contradiction.

Hence the vector-valued frames $\{F_1, F_2, I\}$ and $\{G_1, G_2, I\}$ for $H_1 \oplus H_2$ are not woven.

The following proposition shows that if a family of vector-valued frames is woven, then it is componentwise woven. That is, frames associated with atomic spaces are woven. The converse is not true, see Example 3.6.

**Proposition 3.5.** Suppose the family $\{F_1, \ldots, F_L, I\}$ of vector-valued frames for $H_1 \oplus \cdots \oplus H_L$ is woven with universal bounds $A$ and $B$. Then, for each $j \in [L]$, the frames $\{F_j^i : i \in [m]\}$ for $H_j$ are woven.

**Proof.** Let $j \in [L]$ be arbitrary but fixed and let $\{\sigma_i\}_{i\in[m]}$ be any partition of $I$. Let us write $F_p^i = \{f_{pk}^i\}_{k \in [1], \ p \in [L], \ i \in [m]}$.

Then, for any $f \in H_j$ (note that $0 \oplus \cdots \oplus f \oplus \cdots \oplus 0 \in H_1 \oplus \cdots \oplus H_j \oplus \cdots \oplus H_L$), we compute

$$A\|f\|^2 \leq \sum_{i \in [m]} \sum_{k \in \sigma_i} \left| \left\langle 0, f_{1k}^i \right\rangle + \cdots + \left\langle f, f_{j_k}^i \right\rangle + \cdots + \left\langle 0, f_{Lk}^i \right\rangle \right|^2$$

$$= \sum_{i \in [m]} \sum_{k \in \sigma_i} \left| \left\langle f, f_{j_k}^i \right\rangle \right|^2$$

$$\leq B\|f\|^2.$$

Hence the family of frames $\{F_j^i : i \in [m]\}$ for the atomic space $H_j$ is woven.

We now demonstrate by a concrete example that converse of Proposition 3.5 is not true.
Example 3.6. Let $F_1, G_1 \subset \mathcal{H}_1$ and $F_2, G_2 \subset \mathcal{H}_2$ be frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively given in Example 3.4. Then, for any subset $\sigma$ of $I$, the families $\{f_{1i}\}_{i \in \sigma} \cup \{g_{1i}\}_{i \in \sigma}$ and $\{f_{2i}\}_{i \in \sigma} \cup \{g_{2i}\}_{i \in \sigma}$ are frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively with frame bounds 1 and 3. That is, the frames $F_1$ and $G_1$ for $\mathcal{H}_1$ and $F_2$ and $G_2$ for $\mathcal{H}_2$ are vector-valued families for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$, where $F_k = \{f_{ki}\}_{j \in I}$ (where $f_{ki} \in \mathcal{H}_1$, $i \in [m]$) such that $\{f_{ki}\}_{j \in I}$ is a woven Bessel sequence, for all $k \in [L]$. Then, the family $\{F_k, G_k\} : i \in [m]$ is a woven vector-valued Bessel sequence for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$.

Proof. Let $\{f_{ki}\}_{j \in I}$ be a woven Bessel sequence with universal bound $B_k$, for all $k \in [L]$. Then, for any partition $\{\sigma_i\}_{i \in [m]}$ of $I$ and for any $(h_1, \ldots, h_L) \subset \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$, we have

$$
\sum_{i \in [m]} \sum_{j \in \sigma_i} \left| \sum_{k=1}^L \langle h_k, f_{kj} \rangle \right|^2 \leq \sum_{i \in [m]} \sum_{j \in \sigma_i} \sum_{k=1}^L 2^{L-1} \left| \langle h_k, f_{kj} \rangle \right|^2
$$

$$
= \sum_{k=1}^L 2^{L-1} \left( \sum_{i \in [m]} \sum_{j \in \sigma_i} \left| \langle h_k, f_{kj} \rangle \right|^2 \right)
$$

$$
\leq \sum_{k=1}^L 2^{L-1} B_k \|h_k\|^2
$$

$$
\leq 2^{L-1} \max_{k \in [L]} \{B_k\} \left( \|h_1\|^2 + \cdots + \|h_L\|^2 \right).
$$

This concludes the proof. \(\square\)

Corollary 3.8. Suppose that $\{F_k, G_k\} : k \in [L]$ are vector-valued families for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$, where $F_k = \{f_{ki}\}_{j \in I}$ is a Bessel sequence for all $k \in [L], i \in [m]$. Then, $\{F_k, G_k\} : i \in [m]$ is a woven vector-valued Bessel sequence for $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L$.

A natural question arises about the construction of vector-valued woven frames from given woven frames for atomic spaces. In this direction, the following theorem provides a technique for the construction of a family of vector-valued woven frames from given woven frames for atomic spaces. We prove the result for $L = 2$. This can be extended to any finite superset.

Theorem 3.9. Assume that $\Phi_j = \{\phi_j^i\}_{i \in I_j}$ and $\Psi_j = \{\psi_j^i\}_{i \in I_j}$ are woven frames for the Hilbert space $\mathcal{H}_j$ (j = 1, 2). Then, there is a vector-valued family associated with $\Phi_j$ and $\Psi_j$ (j = 1, 2) which constitutes vector-valued woven frames for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Proof. Since $I_1$ and $I_2$ are countably infinite sets, there exist bijective maps $\Theta_1 : \mathbb{Z}^- \to I_1$ and $\Theta_2 : \mathbb{Z}^+ \to I_2$.

Define $F_1 = \{f_{1i}\}_{i \in \mathbb{Z}^-} \subset \mathcal{H}_1$ and $F_2 = \{f_{2i}\}_{i \in \mathbb{Z}^+} \subset \mathcal{H}_2$ as follows:

$$
f_{1i} = \begin{cases} 
\phi_{\Theta_1(i)}^1, & i \in \mathbb{Z}^- \\
0, & i \in \mathbb{Z}^+
\end{cases}
$$

and

$$
f_{2i} = \begin{cases} 
\phi_{\Theta_2(i)}^2, & i \in \mathbb{Z}^- \\
0, & i \in \mathbb{Z}^+
\end{cases}
$$

and define $G_1 = \{g_{1i}\}_{i \in \mathbb{Z}^+} \subset \mathcal{H}_1$ and $G_2 = \{g_{2i}\}_{i \in \mathbb{Z}^-} \subset \mathcal{H}_2$ by

$$
g_{1i} = \begin{cases} 
\phi_{\Theta_1(i)}^1, & i \in \mathbb{Z}^- \\
0, & i \in \mathbb{Z}^+
\end{cases}
$$

and

$$
g_{2i} = \begin{cases} 
\phi_{\Theta_2(i)}^2, & i \in \mathbb{Z}^- \\
0, & i \in \mathbb{Z}^+
\end{cases}
$$
To show that the vector-valued family \( \{F_1, F_2, \mathbb{Z}^*\} \) and \( \{G_1, G_2, \mathbb{Z}^*\} \) are woven, let \( \sigma \) be any subset of \( \mathbb{Z}^* \) and \( f \oplus g \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) be arbitrary. We compute

\[
\sum_{i \in \sigma} \left| \left\langle f \oplus g, f_{1i} \oplus f_{2i} \right\rangle \right|^2 + \sum_{i \in \sigma^c} \left| \left\langle f \oplus g, g_{1i} \oplus g_{2i} \right\rangle \right|^2 \\
= \sum_{i \in \sigma} \left| \left\langle f, f_{1i} \right\rangle + \left\langle g, f_{2i} \right\rangle \right|^2 + \sum_{i \in \sigma^c} \left| \left\langle f, g_{1i} \right\rangle + \left\langle g, g_{2i} \right\rangle \right|^2 \\
= \sum_{i \in \sigma^c \cap \mathbb{Z}^-} \left| \left\langle f, f_{1i} \right\rangle + \left\langle g, f_{2i} \right\rangle \right|^2 + \sum_{i \in \sigma^c \cap \mathbb{Z}^+} \left| \left\langle f, g_{1i} \right\rangle + \left\langle g, g_{2i} \right\rangle \right|^2 \\
+ \sum_{i \in \sigma \cap \mathbb{Z}^-} \left| \left\langle f, f_{1i} \right\rangle + \left\langle g, f_{2i} \right\rangle \right|^2 + \sum_{i \in \sigma \cap \mathbb{Z}^+} \left| \left\langle f, g_{1i} \right\rangle + \left\langle g, g_{2i} \right\rangle \right|^2 \\
= \sum_{i \in \sigma^c \cap \mathbb{Z}^-} \left| \left\langle f, \phi_{\Theta^1(i)}^1 \right\rangle \right|^2 + \sum_{i \in \sigma^c \cap \mathbb{Z}^+} \left| \left\langle f, \phi_{\Theta^1(i)}^2 \right\rangle \right|^2 + \sum_{i \in \sigma \cap \mathbb{Z}^-} \left| \left\langle f, \psi_{\Theta^1(i)}^1 \right\rangle \right|^2 + \sum_{i \in \sigma \cap \mathbb{Z}^+} \left| \left\langle f, \psi_{\Theta^1(i)}^2 \right\rangle \right|^2 \\
= \left( \sum_{i \in \sigma^c \cap \mathbb{Z}^-} \left| \left\langle f, \phi_{\Theta^1}^1 \right\rangle \right|^2 + \sum_{i \in \sigma \cap \mathbb{Z}^+} \left| \left\langle f, \psi_{\Theta^1}^1 \right\rangle \right|^2 \right) + \left( \sum_{i \in \sigma \cap \mathbb{Z}^-} \left| \left\langle f, \phi_{\Theta^1}^2 \right\rangle \right|^2 + \sum_{i \in \sigma^c \cap \mathbb{Z}^+} \left| \left\langle f, \psi_{\Theta^1}^2 \right\rangle \right|^2 \right) \\
= \left( \sum_{i \in \Theta^1(\sigma^c \cap \mathbb{Z}^-)} \left| \left\langle f, \phi_{\Theta^1}^1 \right\rangle \right|^2 + \sum_{i \in \Theta^1(\sigma \cap \mathbb{Z}^+)} \left| \left\langle f, \psi_{\Theta^1}^1 \right\rangle \right|^2 \right) + \left( \sum_{i \in \Theta^1(\sigma^c \cap \mathbb{Z}^-)} \left| \left\langle g, \phi_{\Theta^1}^2 \right\rangle \right|^2 + \sum_{i \in \Theta^1(\sigma \cap \mathbb{Z}^+)} \left| \left\langle g, \psi_{\Theta^1}^2 \right\rangle \right|^2 \right) \\
= \left( \sum_{i \in \Theta^1(\sigma^c \cap \mathbb{Z}^-)} \left| \left\langle f, \phi_{\Theta^1}^1 \right\rangle \right|^2 + \sum_{i \in \Theta^1(\sigma \cap \mathbb{Z}^+)} \left| \left\langle f, \psi_{\Theta^1}^1 \right\rangle \right|^2 \right) \\
+ \left( \sum_{i \in \Theta^1(\sigma^c \cap \mathbb{Z}^-)} \left| \left\langle g, \phi_{\Theta^1}^2 \right\rangle \right|^2 + \sum_{i \in \Theta^1(\sigma \cap \mathbb{Z}^+)} \left| \left\langle g, \psi_{\Theta^1}^2 \right\rangle \right|^2 \right). \tag{3.1}
\]

Let \( \sigma_1 = \Theta^1(\sigma \cap \mathbb{Z}^-) \), \( \sigma_2 = \Theta^2(\sigma \cap \mathbb{Z}^+) \). Then, \( \sigma_1 \) is a subset of \( I_1 \) such that the compliment of \( \sigma_1 \) in \( I_1 \) is \( \Theta^1(\sigma^c \cap \mathbb{Z}^-) \) and \( \sigma_2 \) is a subset of \( I_2 \) such that the compliment of \( \sigma_2 \) in \( I_2 \) is \( \Theta^2(\sigma^c \cap \mathbb{Z}^+) \). By using the fact that \( \Phi_i \) and \( \Psi_i \) are woven frames for the Hilbert space \( \mathcal{H}_i \) \((i = 1, 2)\) with universal bounds \( A_i, B_i \) (say) and using (3.1), we compute

\[
\min\{A_1, A_2\} \|f\|^2 + \|g\|^2 \\
\leq A_1 \|f\|^2 + A_2 \|g\|^2 \\
\leq \left( \sum_{i \in \sigma_1} \left| \left\langle f, \phi_{\Theta^1}^1 \right\rangle \right|^2 + \sum_{i \in I_1 \setminus \sigma_1} \left| \left\langle f, \psi_{\Theta^1}^1 \right\rangle \right|^2 \right) + \left( \sum_{i \in \sigma_2} \left| \left\langle g, \phi_{\Theta^1}^2 \right\rangle \right|^2 + \sum_{i \in I_2 \setminus \sigma_2} \left| \left\langle g, \psi_{\Theta^1}^2 \right\rangle \right|^2 \right) \\
= \left( \sum_{i \in \sigma} \left| \left\langle f \oplus g, f_{1i} \oplus f_{2i} \right\rangle \right|^2 + \sum_{i \in I_1 \setminus \sigma} \left| \left\langle f \oplus g, g_{1i} \oplus g_{2i} \right\rangle \right|^2 \right) \\
\leq B_1 \|f\|^2 + B_2 \|g\|^2 \\
\leq \max\{B_1, B_2\} \|f\|^2 + \|g\|^2.
\]

Hence the vector-valued frames \( \{F_1, F_2, \mathbb{Z}^*\} \) and \( \{G_1, G_2, \mathbb{Z}^*\} \) are woven with universal frame bounds \( \min\{A_1, A_2\} \) and \( \max\{B_1, B_2\} \). \qed
4. Vector-valued Weaving Riesz Bases

**Definition 4.1.** The family \( \{F_1, \ldots, F_L, I\} \) \( (F_i = \{f_{ik}\}_{k \in I} \subseteq \mathcal{H}_i \) \( (1 \leq i \leq L) \)\) is called a vector-valued Riesz basis (or super-Riesz basis) for the space \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) if there exists a bounded bijective operator \( U : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \rightarrow \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) such that \( U(e_{1k} \oplus \cdots \oplus e_{Lk}) = f_{1k} \oplus \cdots \oplus f_{Lk} \) \( (k \in I) \), where \( \{e_{1k} \oplus \cdots \oplus e_{Lk}\}_{k \in I} \) is an orthonormal basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \).

A characterization of vector-valued Riesz basis for a Hilbert space \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) can be obtained by using [7, Theorem 3.6.6]

**Theorem 4.2.** For a sequence \( \{F_1, \ldots, F_L, I\} \) \( (F_i = \{f_{ik}\}_{k \in I} \subseteq \mathcal{H}_i) \), the following conditions are equivalent:

(i) \( \{F_1, \ldots, F_L, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \).

(ii) \( \{F_1, \ldots, F_L, I\} \) is complete in \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) and there exist finite constants \( A, B > 0 \) such that for every finite scalar sequence \( \{c_i\}_{i \in I} \), one has

\[
A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i(f_{1i} \oplus \cdots \oplus f_{L_i}) \right\|^2 \leq B \sum_{i \in I} |c_i|^2. \tag{4.1}
\]

The scalars \( A \) and \( B \) are called lower and upper Riesz bounds, respectively.

**Definition 4.3.** The family \( \{F_1, \ldots, F_L, I\} \) \( (F_i = \{f_{ik}\}_{k \in I} \subseteq \mathcal{H}_i) \) is called a vector-valued Riesz sequence (or super-Riesz sequence) in the space \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) if (4.1) holds for all finite scalar sequences \( \{c_i\}_{i \in I} \).

The following proposition provides a sufficient condition for vector-valued Riesz basis.

**Proposition 4.4.** Suppose \( \{F_1, \ldots, F_L, I\} \) \( (\text{where } F_i = \{f_{ik}\}_{k \in I}) \) is a vector-valued frame for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) such that for some \( n \in [L] \), the sequence \( F_n \) is a Riesz basis for \( \mathcal{H}_n \). Then, the family \( \{F_1, \ldots, F_L, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \).

**Proof.** Suppose that \( F_n \) is a Riesz basis for \( \mathcal{H}_n \) with lower and upper Riesz bounds \( A_n \) and \( B_n \), respectively. Then, for any finite scalar sequence \( \{c_i\}_{i \in I} \), we compute

\[
\left\| \sum_{i \in I} c_i (f_{1i} \oplus \cdots \oplus f_{Li}) \right\|^2 = \left\| \sum_{i \in I} c_i f_{1i} \right\|^2 + \cdots + \left\| \sum_{i \in I} c_i f_{Li} \right\|^2 \\
\geq \left\| \sum_{i \in I} c_i f_{Ni} \right\|^2 \\
\geq A_n \sum_{i \in I} |c_i|^2.
\]

Therefore, (4.1) is satisfied. By hypothesis, the family \( \{F_1, \ldots, F_L, I\} \) is a vector-valued frame for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \). Thus, the family \( \{F_1, \ldots, F_L, I\} \) is complete in \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \). Hence \( \{F_1, \ldots, F_L, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \). \( \square \)

**Remark 4.5.** If a family \( \{F_1, \ldots, F_L, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \), then its component \( F_k \) need not be a Riesz basis for the atomic space \( \mathcal{H}_k \). Indeed, let \( L = 2, I = \mathbb{N} \) and \( \mathcal{H}_1 = \mathcal{H}_2 = L^2(I, \mu) \), where \( \mu \) is the counting measure. Define \( F_1 = \{f_{1i}\}_{i \in I}, F_2 = \{f_{2i}\}_{i \in I} \) as follows:

\[
f_{1i} = \begin{cases} e_j, & i = 2j - 1 \ (j \in \mathbb{N}) \\ 0, & i = 2j \ (j \in \mathbb{N}) \end{cases} ; \\
f_{2i} = \begin{cases} 0, & i = 2j - 1 \ (j \in \mathbb{N}) \\ e_j, & i = 2j \ (j \in \mathbb{N}) \end{cases}
\]

where \( \{e_i\}_{i \in I} \) is the canonical orthonormal basis of \( \mathcal{H}_1 \).

Then, \( \{F_1, F_2, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \mathcal{H}_2 \). But \( F_1 \) and \( F_2 \) are not Riesz bases for \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively.
Definition 4.6. A vector-valued family \( \{F_i^i, \ldots, F_L^i, I\} : i \in [m] \) \((F_j^i) = (f_k^j)_{k \in I}\) for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L\) is said to be a woven Riesz sequence if there exist universal constants \( A \) and \( B \) such that for any partition \( \{\sigma_i\}_{i \in [m]} \) of \( I \), one has
\[
A \sum |c_k|^2 \leq \left\| \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k (f_{i_k}^1 + \cdots + f_{i_k}^L) \right\|^2 \leq B \sum |c_k|^2
\]
for all finite scalar sequences \( \{c_k\}_{k \in I} \).

Let us have a look at Example 3.6 which shows that if the family of frames for atomic spaces are woven, then the family of vector-valued frames need not be woven. But this is not the case for woven Riesz sequences. More precisely, if a family of Bessel sequences for atomic spaces is a woven Riesz sequence, then its associated family of vector-valued Bessel sequences is a woven Riesz sequence.

Theorem 4.7. Suppose \( \{F_1^i, \ldots, F_L^i, I\} \) is a vector-valued family for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) such that \( F_j^i = \{f_k^j\}_{k \in I} \) is a Bessel sequence for all \( i \in [m] \) and \( j \in [L] \). If for some \( n \in [L] \), the family \( \{F_n^i : i \in [m]\} \) is a woven Riesz sequence, then \( \{F_1^i, \ldots, F_L^i, I\} : i \in [m]\) is a woven Riesz sequence.

Proof. Let \( A \) and \( B \) be universal Riesz bounds for the family \( \{F_i^i : i \in [m]\} \). Let \( \{\sigma_i\}_{i \in [m]} \) be any partition of \( I \). Then, for any \( \{c_k\}_{k \in I} \in \ell^2(I) \), we compute
\[
\left\| \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k (f_{i_k}^1 + \cdots + f_{i_k}^L) \right\|^2 = \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k f_{i_k}^1 \|^2 + \cdots + \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k f_{i_k}^L \|^2
\]
\[
\geq \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k f_{i_k}^1 \|^2
\]
\[
\geq A \sum_{k \in I} |c_k|^2.
\]
By Corollary 3.8, \( \bigcup_{i \in [m]} \{F_1^i, \ldots, F_L^i, \sigma_i\} \) is a Bessel sequence with universal bound \( B_o \) (say). Therefore,
\[
\left\| \sum_{i \in [m]} \sum_{k \in \sigma_i} c_k (f_{i_k}^1 + \cdots + f_{i_k}^L) \right\|^2 \leq B_o \sum_{k \in I} |c_k|^2.
\]
Hence the family \( \{F_1^i, \ldots, F_L^i, I\} : i \in [m] \) is a woven Riesz sequence with universal bounds \( A \) and \( B_o \).

A vector-valued frame with one component as Riesz basis (for the underlying atomic space) cannot be weave with vector-valued frames whose corresponding component is not a Riesz basis.

Proposition 4.8. Suppose the family of vector-valued frames \( \{F_1^i, \ldots, F_L^i, I\} : i \in [m]\) for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) is woven and \( F_j^i \) is a Riesz basis for \( \mathcal{H}_q \) for some \( p \in [L] \) and \( q \in [m] \). Then, \( F_j^i \) is a Riesz basis for \( \mathcal{H}_p \) for all \( i \in [m] \). In particular, \( \{F_1^i, \ldots, F_L^i, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) for all \( i \in [m] \).

Proof. Since family of vector-valued frames \( \{F_1^i, \ldots, F_L^i, I\} : i \in [m]\) is woven, the family \( \{F_j^i : i \in [m]\} \) is woven \((j \in [L]) \). In particular, \( \{F_j^i : i \in [m]\} \) is woven. By hypothesis, \( F_j^i \) is a Riesz basis for \( \mathcal{H}_p \). Therefore, by Theorem 1.4, \( F_j^i \) is a Riesz basis for \( \mathcal{H}_p \) for all \( i \in [m] \). Hence by Proposition 4.4, the family \( \{F_1^i, \ldots, F_L^i, I\} \) is a vector-valued Riesz basis for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \) for all \( i \in [m] \).

□
The following theorem provides sufficient conditions for vector-valued weaving Riesz sequences in terms of operators on atomic spaces.

**Theorem 4.9.** Suppose \( \{F_i^1, \ldots, F_i^L, N\} (i \in [m]) \) is a vector-valued family for \( \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_L \). Let there exists some \( n \in [L] \) such that \( F_i^j = \{f_{jk}^i\}_{k \in K} \) is a Bessel sequence for \( j \in [L] \setminus \{n\} \) (\( i \in [m] \)).

Let \( \{\chi_k\}_{k \in K} \) and \( \{e_k\}_{k \in K} \) be any orthonormal bases for \( \mathcal{H}_n \) such that

\[
f_{nk}^i = \chi_k - \sum_{p \in P} a_k^i T_p(e_k) \quad \text{for all } i \in [m] \text{ and all } k \in \mathbb{N},
\]

and

\[
\lambda_i = \sum_{p \in P} \|T_p\| \sup_{k \in K} |a_k^i| < \frac{1}{3} \quad \text{for all } i \in [m],
\]

where \( T_p \in \mathcal{B}(\mathcal{H}_n) \) for each \( p \in P, \ i \in [m] \) and \( a_k^i \) are scalars for all \( k, p \in P, \ i \in [m] \). Then, \( \{F_i^1, \ldots, F_i^L, I\} : i \in [m] \} \) is a woven Riesz sequence.

**Proof.** Let \( \{\sigma_i\}_{i \in [m]} \) be any partition of \( \mathbb{N} \). Then, for any \( N \in \mathbb{N} \) and scalars \( c_1, c_2, \ldots, c_N \), we compute

\[
\left\| \sum_{i \in [m]} \sum_{k \in [N] \cap \sigma_i} c_k (\chi_k - f_{nk}^i) \right\| = \left\| \sum_{i \in [m]} \sum_{k \in [N] \cap \sigma_i} c_k \left( \sum_{p \in P} a_k^i T_p(e_k) \right) \right\|
\]

\[
= \left\| \sum_{p \in P} \sum_{i \in [m]} \sum_{k \in [N] \cap \sigma_i} c_k a_k^i T_p(e_k) \right\|
\]

\[
\leq \sum_{p \in P} \left\| \sum_{i \in [m]} \sum_{k \in [N] \cap \sigma_i} c_k a_k^i T_p(e_k) \right\|
\]

\[
\leq \sum_{p \in P} \left\| T_p \left( \sum_{k \in [N] \cap \sigma_i} c_k a_k^i e_k \right) \right\|
\]

\[
= \sum_{p \in P} \left\| T_p \left( \sum_{k \in [N] \cap \sigma_i} |c_k a_k^i|^2 \right)^{\frac{1}{2}} \right\|
\]

\[
\leq \sum_{p \in P} \left\| T_p \right\| \sup_{k \in K} |a_k^i| \left( \sum_{k \in [N] \cap \sigma_i} |c_k|^2 \right)^{\frac{1}{2}}
\]

\[
= \sum_{i \in [m]} \lambda_i \left( \sum_{k \in [N] \cap \sigma_i} |c_k|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \max_{i \in [m]} \{\lambda_i\} \left( \sum_{i \in [m]} \left( \sum_{k \in [N] \cap \sigma_i} |c_k|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]

\[
\leq \max_{i \in [m]} \{\lambda_i\} \left( \frac{m}{3} \right)^{\frac{1}{2}} \left( \sum_{i \in [m]} \sum_{k \in [N] \cap \sigma_i} |c_k|^2 \right)^{\frac{1}{2}}
\]

\[
= \lambda \left( \sum_{k \in [N]} |c_k|^2 \right)^{\frac{1}{2}}, \text{ where } \lambda = \frac{m}{3} \max_{i \in [m]} \{\lambda_i\} < 1.
\]

By Theorem 2.1, \( \bigcup_{i \in [m]} \{f_{nk}^i\}_{k \in \sigma_i} \) is equivalent to orthonormal basis \( \{\chi_k\}_{k \in K} \). Therefore, there exists a bijective linear operator \( U : \mathcal{H}_n \to \mathcal{H}_n \) such that \( U(\chi_k) = f_{nk}^i \) for \( k \in \sigma_i \). Thus, \( \bigcup_{i \in [m]} \{f_{nk}^i\}_{k \in \sigma_i} \)
is a Riesz basis and hence a frame for $\mathcal{H}_n$. Therefore, the family of frames $\left\{ f^i_{nk} \right\}_{k \in \mathbb{N}} : i \in [m]$ is weakly woven and hence woven (by Remark 1.3). Thus, there exist universal frame bounds (note that the Riesz basis bounds coincide with the frame bounds) for the family $\left\{ f^i_{nk} \right\}_{k \in \mathbb{N}} : i \in [m]$. That is, $\left\{ f^i_{nk} \right\}_{k \in \mathbb{N}} : i \in [m]$ is woven Riesz sequence. Hence by Theorem 4.7, the family $\left\{ F^i_1, F^i_2, N \right\} : i \in [m]$ is a woven Riesz sequence. □

To conclude the paper, we illustrate Theorem 4.9 with the following example.

**Example 4.10.** Let $L = m = n = 2$ and let $\mathcal{H}_1 = \mathcal{H}_2 = \ell^2(\mathbb{N})$. Suppose that $\left\{ \chi_k \right\}_{k \in \mathbb{N}}$ and $\left\{ \epsilon_k \right\}_{k \in \mathbb{N}}$ are any two orthonormal basis for $\ell^2(\mathbb{N})$.

Choose $0 < \epsilon < \frac{1}{\sqrt{3} \sum_{k \in \mathbb{N}} 1/k^2}$. Define sequences $\left\{ T^i_p \right\}_{p \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}_2)$ ($i = 1, 2$) by

$$T^1_p : \mathcal{H}_2 \to \mathcal{H}_2 \text{ by } T^1_p(\{\xi_1, \xi_2, \ldots\}) = \{0, 0, \ldots, 0, \xi_p, \xi_{p+1}, \ldots\},$$

$$T^2_p : \mathcal{H}_2 \to \mathcal{H}_2 \text{ by } T^2_p(\{\xi_1, \xi_2, \ldots\}) = \left\{ \frac{\xi_p}{4}, \frac{\xi_{p+1}}{4}, \ldots \right\}.$$ 

Choose $\left\{ \alpha^i_{kp} \right\}_{k \in \mathbb{N}} \subset \mathbb{K}$ ($p \in \mathbb{N}; i = 1, 2$) as follows:

$$\left\{ \alpha^1_{kp} \right\}_{k \in \mathbb{N}} = \left\{ 0, \ldots, 0, \frac{\epsilon}{p^2}, \frac{\epsilon}{(p+1)^2}, \ldots \right\},$$

$$\left\{ \alpha^2_{kp} \right\}_{k \in \mathbb{N}} = \left\{ 0, \ldots, 0, \frac{4\epsilon}{(p+1)^2}, \frac{4\epsilon}{(p+2)^2}, \ldots \right\},$$

Then

$$\lambda_1 = \sum_{p \in \mathbb{N}} ||T^1_p|| \sup_{k \in \mathbb{N}} |\alpha^1_{kp}| \leq \sum_{p \in \mathbb{N}} \frac{\epsilon}{p^2} < \frac{1}{\sqrt{3}},$$

and

$$\lambda_2 = \sum_{p \in \mathbb{N}} ||T^2_p|| \sup_{k \in \mathbb{N}} |\alpha^1_{kp}| \leq \sum_{p \in \mathbb{N}} \frac{1}{4} \left( \frac{\epsilon}{p+1} \right)^2 \leq \sum_{p \in \mathbb{N}} \frac{\epsilon}{p^2} < \frac{1}{\sqrt{3}}.$$}

Define $\left\{ F^1_1, F^1_2, N \right\}$ and $\left\{ F^2_1, F^2_2, N \right\}$ as follows:

$$f^1_{1k} = \left\{ 0, \ldots, 0, \frac{1}{k^4}, 1, 0, 0, \ldots \right\}, \quad k \in \mathbb{N},$$

$$f^1_{2k} = \chi_k - \sum_{p \in \mathbb{N}} \alpha^1_{kp} T^1_p(\epsilon_k), \quad k \in \mathbb{N},$$

$$f^1_{1k} = \left\{ 0, \ldots, 0, \frac{1}{k^4}, 1, 1, 0, 0, \ldots \right\}, \quad k \in \mathbb{N},$$

$$f^2_{2k} = \chi_k - \sum_{p \in \mathbb{N}} \alpha^2_{kp} T^2_p(\epsilon_k), \quad k \in \mathbb{N}.$$ 

It is easy to verify that $F^j_i$ is a Bessel sequence for $\mathcal{H}_i$, for all $i \in [m]$, $j \in [L] \setminus \{n\} = \{1\}$. Hence, by Theorem 4.9, the family $\left\{ \left\{ F^i_1, F^i_2, N \right\} : i \in [2] \right\}$ is a woven Riesz sequence.
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