ENDOSCOPIC CONGRUENCES MODULO ADJOINT $L$-VALUES FOR 
GSp(4)

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Abstract. We establish the existence of congruences between a fixed endoscopic cuspidal automorphic representation $\Pi$ of GSp(4) of level 1 and stable cuspidal automorphic representations of the same level and weight modulo certain prime factors of the value at 1 of the adjoint $L$-function of $\Pi$ normalized by a suitable period.

1. Introduction

In modern number theory, the study of congruences modulo a prime number $p$ between automorphic forms plays a central role. It gives a basis of the theory of $p$-adic families of automorphic forms and the theory of $p$-adic families of $p$-adic Galois representations and it also gives us a deeper understanding some important arithmetic objects. For example, since the prime number 691 divides the numerator of the rational number $\frac{\zeta(12)}{\pi^{12}}$, it divides the constant term of the unique normalized Eisenstein series $E_{12}$ of weight 12. This yields a congruence modulo 691 between $E_{12}$ and the Ramanujan cuspform $\Delta$ of weight 12 and the existence of such a congruence helps us to understand the action of $\text{Gal}(\mathbb{Q}(\mu_{691})/\mathbb{Q})$.
on the 691-part of the class group of $\mathbb{Q}(\mu_{691})$ (see [Ri76]).

Instead of the Eisenstein series $E_{12}$, we consider an automorphic representation $\Pi$ of $GSp(4, \mathbb{A})$, where $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$, which is cuspidal but whose functorial lift to $GL(4, \mathbb{A})$ is not cuspidal. Such an automorphic representation $\Pi$ is called endoscopic. In this article, we investigate congruences between $\Pi$ and cuspidal automorphic representations $\Pi'$ of $GSp(4, \mathbb{A})$ whose functorial lift to $GL(4, \mathbb{A})$ remains cuspidal. Such automorphic representations $\Pi'$ are called stable.

In [Hi81], Hida considers a holomorphic cuspform $f$ of $GL(2, \mathbb{A})$ and establishes the existence of congruences between $f$ and other cuspforms of the same level and weight as $f$ modulo certain prime numbers dividing a value of the adjoint $L$-function of $f$, normalized by a suitable period. As we need it in the proof of our main result, let us recall Hida’s theorem in the particular case where $f$ has level 1. Let $\kappa \geq 2$ denote the weight of $f$ and let $Z(\kappa, f)$ denote the product

$$Z(\kappa, f) = \prod_{\sigma: E \to \mathbb{C}} L(\kappa, \text{Sym}^2(\sigma f)) = \prod_{\sigma: E \to \mathbb{C}} L(1, \pi(\sigma f), Ad)$$

where $\pi(\sigma f)$ denotes the cuspidal automorphic representation of $GL(2, \mathbb{A})$ attached to $f$. Let $c(f)$ denote the real number defined as

$$c(f) = (u(f)\pi(\kappa+1)\rho)^{-1}((\kappa - 1)! \cdot 3 \cdot 2^{-\kappa})^\rho \cdot Z(\kappa, f)$$

where $\rho$ denotes the degree of the number field $E$ generated by the Fourier coefficients of $f$ and where $u(f)$ denotes the period defined in [Hi81 (6.6)]. It follows from loc. cit. Theorem 6.2 that $c(f)^2 \in \mathbb{Z}\setminus\{0\}$ and the main result of loc. cit. states that if a prime $p$ different from 2 and 3 and satisfying $p > \kappa - 2$ divides $c(f)^2$, then there exists a congruence between $f$ and another cuspform $f'$ modulo a prime ideal above $p$ in $\mathbb{Q}$.

We assume that $\Pi \simeq \Pi_\infty \otimes \Pi_f$ satisfies the conditions (i)-(v) of section 3. In particular $\Pi_\infty$ is a discrete series of $GSp(4, \mathbb{R})$ of Harish-Chandra parameter $(k + 3, k' + 1)$ for two integers $k \geq k' \geq 0$. The conditions on $\Pi_\infty$ imply that $\Pi_f$ is defined over its rationality field, which is a number field $E(\Pi_f)$ of degree $r$ over $\mathbb{Q}$. Considering the Betti cohomology of Siegel threefolds, we define a free $\mathbb{Z}_p$-module $L(\Pi_f, V_{\lambda, \mathbb{Z}_p})$ of finite rank $2r$ endowed with a natural bilinear form

$$\langle , \rangle : L(\Pi_f, V_{\lambda, \mathbb{Z}_p}) \times L(\Pi_f, V_{\lambda, \mathbb{Z}_p}) \to \mathbb{Z}_p$$

and a period $\Omega(\Pi_f) \in \mathbb{R}^\times$ (see equation (32) and Remark [5.6]). Let $\delta_1, \ldots, \delta_{2r}$ be a $\mathbb{Z}_p$-basis of $L(\Pi_f, V_{\lambda, \mathbb{Z}_p})$ and let $d(\Pi_f) = \det((\delta_i, \delta_j))_{1 \leq i, j \leq 2r}$ be the discriminant of $\langle , \rangle$. This is an element of $\mathbb{Z}_p$ whose image in $\mathbb{Z}_p / (\mathbb{Z}_p^\times)^2$ does not depend on the choice of the basis $\delta_1, \ldots, \delta_{2r}$. Let $C_{\infty} \subset \mathbb{C}^\times$ denote the constant appearing in the statement of Theorem [3.1] and let $C_{k, k'}$ be the rational number defined in Proposition [6.4]. For $x, y \in \mathbb{R}$ we write $x \sim y$ if there exists $s \in (\mathbb{Z}_p^\times)^2$ such that $x = sy$. 

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Theorem 1.1. Assume that \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup S_{\text{tors}} \cup S'_{\text{tors}} \) where \( S_{G(\mathbb{Z}/3\mathbb{Z})}, S_{\text{tors}}, S'_{\text{tors}}, S_{\text{weight}} \) are defined by \((15), (16), (21), (22)\) respectively. Then, there exists a prime divisor \( p \) such that \( S \in \mathbb{Z} \), \( P \equiv p \pmod{P} \).

Assume that \( \Pi' \equiv \Pi \pmod{\mathfrak{P}} \) if \( \Pi' \) is congruent to \( \Pi \) modulo \( \mathfrak{P} \) (see Definition \( 7.1 \)).

Theorem 1.2. Let \( p \) be a prime such that \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup S_{\text{tors}} \cup S'_{\text{tors}} \cup S''_{\text{tors}} \) where \( S_{G(\mathbb{Z}/3\mathbb{Z})}, S_{\text{tors}}, S'_{\text{tors}}, S''_{\text{tors}}, S_{\text{weight}} \) and \( S''_{\text{tors}} \) are defined by \((15), (16), (21), (22)\) respectively. Assume the following conditions

(a) the residual Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \))-representations \( \overline{\pi}_f \) and \( \overline{\pi}'_f \) of \( f_1 \) and \( f_2 \) are irreducible,
(b) the prime \( p \) does not divide \( c(f_1) \),
(c) the prime \( p \) divides \( c(f_2) \).

Then, there exists a prime divisor \( \mathfrak{P} \) of \( p \) in \( \overline{\mathbb{Q}} \) and a cuspidal representation \( \Pi' \simeq \bigotimes_v \Pi'_v \) of \( G(\mathbb{A}) \) such that

1. the smooth admissible representation \( \Pi'_v \) is unramified for any prime \( l \),
2. the representation \( \Pi'_v \) is a discrete series with the same parameter as \( \Pi_\infty \),
3. the cuspidal representation \( \Pi' \) is stable,
4. we have \( \Pi' \not\sim \sigma \Pi \) for all \( \sigma \in \text{Aut}(\mathbb{C}) \),
5. we have \( \Pi' \equiv \Pi \pmod{\mathfrak{P}} \).

Before our work, Hida’s theorem has been extended to Hilbert modular cuspforms by Ghate \cite{Gh02} and Dimitrov \cite{Di05}, to cuspforms of \( GL(2) \) over an imaginary quadratic field by Urban \cite{U95}, to cuspforms of \( GL(2) \) over an arbitrary number field by Namikawa \cite{N15} and to cuspforms of \( GL(n) \) over an arbitrary number field by Balasubramanyam and Raghuram \cite{BaR17}.

In the proof of Theorem \( 1.1 \) the work of Ichino \cite{I08} Theorem 1.1] relating the Petersson norm of a suitably normalized cuspform \( \varphi \in \Pi \) to \( L(1, \Pi, Ad) \) plays an important role. The assumptions (i)-(iv) on \( \Pi \) stated above are all already present as hypothesis of loc. cit. Theorem 1.1, which also covers the case where \( \Pi \) is stable. Assumption (v) says that \( \Pi \)
is endoscopic. In our Theorem 1.1 stated above, our contribution is to compute the discriminant \( d(\Pi_f) \) in terms of the constants \( C_{k,k'}, C_{\infty} \), the period \( \Omega(\Pi_f) \) and the Petersson norms of the normalized cuspforms \( \sigma \varphi \in \sigma \Pi \). Note that in [BaR17], the rational number analog of \( C_{k,k'} \) is not explicitly computed as in the present work. It seems difficult to perform a similar explicit computation for higher rank reductive groups. Furthermore, let us emphasize that this work is the first application of Hida’s ideas to a reductive group different from \( \text{GL}(n) \) and in particular to the study of endoscopic congruences.

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2. Notation and Conventions

2.1. The algebraic group \( \text{GSp}(4) \) and its algebraic representations. Let \( I_2 \) be the identity matrix of size two and let \( J \) be the symplectic form whose matrix in the canonical basis of \( \mathbb{Z}^4 \) is

\[
J = \begin{pmatrix} -I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}.
\]

The symplectic group \( \text{GSp}(4) \) is defined as

\[
\text{GSp}(4) = \{ g \in \text{GL}(4) \mid gJg = \nu(g)J, \nu(g) \in \mathbb{G}_m \}.
\]

This is a \( \mathbb{Z} \)-group scheme that we will denote by \( G \) in what follows. Then \( \nu : G \to \mathbb{G}_m \) is a character and the derived group of \( G \) is \( \text{Sp}(4) = \text{Ker} \nu \). We denote by \( T \subset G \) the diagonal maximal torus defined as

\[
T = \{ \text{diag}(\alpha_1, \alpha_2, \alpha_1^{-1} \nu, \alpha_2^{-1} \nu) \mid \alpha_1, \alpha_2, \nu \in \mathbb{G}_m \}
\]

and by \( B = TU \) the standard Borel subgroup of upper triangular matrices in \( G \) where \( U \) is the unipotent radical

\[
U = \left\{ \begin{pmatrix} 1 & x_0 & 1 \\ 1 & 1 & 0 \\ -x_0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 1 & x_2 & x_3 \\ 1 & x_3 & 1 \end{pmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{G}_a \right\}
\]

We identify the group \( X^*(T) \) of algebraic characters of \( T \) to the subgroup of \( \mathbb{Z}^2 \oplus \mathbb{Z} \) of triples \( (k, k', c) \) such that \( k + k' \equiv c \pmod{2} \) via

\[
\lambda(k, k', c) : \text{diag}(\alpha_1, \alpha_2, \alpha_1^{-1} \nu, \alpha_2^{-1} \nu) \mapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{c-k-k'}{2}}.
\]

Let \( \rho_1 = \lambda(1, -1, 0) \) be the short simple root and \( \rho_2 = \lambda(0, 2, 0) \) be the long simple root. Then the set \( R \subset X^*(T) \) of roots of \( T \) in \( G \) is

\[
R = \{ \pm \rho_1, \pm \rho_2, \pm (\rho_1 + \rho_2), \pm (2\rho_1 + \rho_2) \}
\]
and the subset $R^+ \subset R$ of positive roots with respect to $B$ is

$$R^+ = \{p_1, p_2, p_1 + p_2, 2p_1 + p_2\}.$$  

Then, the set of dominant weights is the set of $\lambda(k, k', c)$ such that $k \geq k' \geq 0$. For any dominant weight $\lambda$, there is an irreducible algebraic representation $V_{\lambda, \mathbb{Q}}$ of $G_{\mathbb{Q}}$ in a finite dimensional $\mathbb{Q}$-vector space, of highest weight $\lambda$, unique up to isomorphism, and all isomorphism classes of irreducible algebraic representations of $G_{\mathbb{Q}}$ are obtained in this way. We will be interested in the specific realization of $V_{\lambda, \mathbb{Q}}$ obtained by Weyl’s construction as follows. Let $\text{Std}_Q \simeq V_{\lambda(1,0,1),\mathbb{Q}}$ be the standard 4-dimensional representation of $G_{\mathbb{Q}}$, let $\lambda = \lambda(k, k', c)$ be a dominant weight, let $d = k + k'$. If $d \geq 2$, for two integers $1 \leq p, q \leq d$, let $\Psi_{p,q} : \text{Std}_Q^d \to \text{Std}_Q^{d-2}$ denote the map defined by

$$1) \Psi_{p,q} : v_1 \otimes \ldots \otimes v_d \mapsto v_p \bar{v}_q v_1 \otimes \ldots \otimes \bar{v}_p \otimes \ldots \otimes \bar{v}_q \otimes \ldots \otimes v_d$$

Let $\text{Std}_Q^{d} \subset \text{Std}_Q^{\otimes d}$ be the intersection of the kernels of all the $\Psi_{p,q}$. Let $t = \frac{e^{-k-k'}}{2}$. Then

$$V_{\lambda, \mathbb{Q}} = \left( \text{Std}_Q^d \cap S(\lambda(\text{Std}_Q)) \right) \otimes \nu^\otimes t$$

where $S(\lambda(\text{Std}_Q)) \subset \text{Std}_Q^{\otimes d}$ is the image of an explicit element $c_\lambda \in \mathbb{Z}[\mathcal{S}_d]$ acting on $\text{Std}_Q^{\otimes d}$ (see [FH91, Theorem 17.11]). Here $\mathcal{S}_d$ denotes the symmetric group of order $d$. By replacing $\text{Std}_Q$ by the standard representation $\text{Std}_G$ of $G$ over $\mathbb{Z}$, we obtain a free $\mathbb{Z}$-module $V_{\lambda, \mathbb{Z}}$ endowed with a linear action of $G$ such that $V_{\lambda, \mathbb{Z}} \otimes \mathbb{Q} = V_{\lambda, \mathbb{Q}}$.

2.2. Measures. Consider the unitary group $U(2) = \{g \in \text{GL}(2, \mathbb{C}) | \bar{g}g = I_2\}$ where $\bar{g}$ denotes the complex conjugate of $g$. The map $\kappa : U(2) \to \text{Sp}(4, \mathbb{R})$ defined by

$$g = A + iB \mapsto \begin{pmatrix} A & B \n B & A \end{pmatrix},$$

where $A$ and $B$ denote the real and imaginary parts of $g$, identifies $U(2)$ with a maximal compact subgroup $K_{\infty}$ of $\text{Sp}(4, \mathbb{R})$. Let $K'_{\infty}$ denote the subgroup $\mathbb{R}^+_{\infty} \times K_{\infty}$ of $G(\mathbb{R})$, where $\mathbb{R}^+_{\infty}$ denotes the connected component of the center of $G(\mathbb{R})$. Let $d_{K_{\infty}}$ be the unique Haar measure on $K_{\infty}$ such that $\text{vol}(K_{\infty}, d_{K_{\infty}}) = 1$. Let $d\mu$ be a left translation invariant measure on $G(\mathbb{R})_+/K'_{\infty}$. Given a measurable function $f : G(\mathbb{R})_+/\mathbb{R}^+_\infty \to \mathbb{C}$, the function $\overline{f} : G(\mathbb{R})_+/K'_{\infty} \to \mathbb{C}$ defined as

$$\overline{f}(g_{\infty}) = \int_{K_{\infty}} f(g_{\infty}k)dk_{\infty}$$

where $g_{\infty}$ is a lift of $g_{\infty}$ by the canonical projection $G(\mathbb{R})_+/\mathbb{R}^+_\infty \to G(\mathbb{R})_+/K'_{\infty}$, is well defined. We define the left translation invariant measure $dg_{\infty}$ on $G(\mathbb{R})_+/\mathbb{R}^+_\infty$ by the formula

$$\int_{G(\mathbb{R})_+/\mathbb{R}^+_\infty} f(g_{\infty})dg_{\infty} = \int_{G(\mathbb{R})_+/K'_{\infty}} \overline{f}d\mu.$$  

Let

$$\mathcal{H}_+ = \{Z = X + iY \in \mathfrak{gl}_2(\mathbb{C}) | iZ = Z, Y > 0\} = G(\mathbb{R})_+/K'_{\infty}$$

be Siegel upper half-plane of genus 2.

**Lemma 2.1.** The following statements hold.
The measure \(dXdY/\det(Y)^3\) on \(H_+\) is left translation invariant by \(G(\mathbb{R})_+\).

(2) For any measure \(du\) on \(H_+\) which is left translation invariant by \(G(\mathbb{R})_+\), then there exists \(c \in \mathbb{R}^+_+\) such that \(d\mu = cdXdY/\det(Y)^3\).

\[
\text{Proof.} \quad \text{The first statement is a particular case of [An87, Proposition 1.2.9]. Let us prove the second statement. Let } du \text{ be a measure on } H_+ = G(\mathbb{R})_+/K_{\infty} \text{ which is left translation invariant by } G(\mathbb{R})_+. \text{ Let } d\nu \text{ denote the measure } dXdY/\det(Y)^3. \text{ By the construction above, we associate to } du \text{ and } d\nu \text{ Haar measures } dg_{\infty,\mu} = \det(g_{\infty})^{-1/2} \text{ and } dg_{\infty,\nu} = \det(g_{\infty})^{-1/2} \text{ on the Lie group } G(\mathbb{R})_+/K_{\infty}^\times. \text{ There exists } c \in \mathbb{R}^+_+ \text{ such that } dg_{\infty,\mu} = cdg_{\infty,\nu}. \text{ Let } s : G(\mathbb{R})_+/K_{\infty}^\times \rightarrow G(\mathbb{R})_+/K'_{\infty} \text{ denote the canonical projection. By an easy computation we have } s_*(dg_{\infty,\mu}) = \text{vol}(K_{\infty}, dk_{\infty})d\mu = d\mu \text{ and similarly } s_*(dg_{\infty,\nu}) = d\nu. \text{ As a consequence } d\mu = cd\nu \text{ as claimed.} \quad \square
\]

### 2.3. Representations of \(K_\infty\) and discrete series classification.

Let \(\mathfrak{k}\) and \(\mathfrak{k}'\) denote the Lie algebra of \(K_\infty\) and \(K'_\infty\) respectively. Let \(\mathfrak{k}_C\) and \(\mathfrak{k}'_C\) denote their complexifications. The differential of \(\kappa\) induces an isomorphism of Lie algebras \(d\kappa : \mathfrak{gl}_{2,C} \simeq \mathfrak{k}_C\). Let \(\mathfrak{gl}_4\) denote the Lie algebra of \(Sp(4, \mathbb{R})\) and by \(\mathfrak{gl}_4\) its complexification. A compact Cartan subalgebra of \(\mathfrak{gl}_4\) is defined as \(\mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2\) where

\[
\begin{align*}
T_1 & = d\kappa \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
T_2 & = d\kappa \left( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\end{align*}
\]

Define a \(C\)-basis of \(\mathfrak{h}_C\) by \(e_1(T_1) = i, e_1(T_2) = 0, e_2(T_1) = 0, e_2(T_2) = i\). The root system \(\Delta\) of the pair \((\mathfrak{gl}_{4,C}, \mathfrak{h}_C)\) is \(\Delta = \{ \pm e_1, \pm e_2, \pm(e_1 \pm e_2) \}\). We denote by \(\Delta_c\), respectively \(\Delta_{nc}\), the set of compact, respectively non-compact roots in \(\Delta\). We have \(\Delta_c = \{ \pm(e_1 - e_2) \}\) and \(\Delta_{nc} = \Delta - \Delta_c\). We choose the set of positive roots as \(\Delta^+ = \{ e_1 - e_2, e_1 + e_2, 2e_1, 2e_2 \}\). Then, the set of compact, respectively non-compact, positive roots is \(\Delta^+_c = \Delta_c \cap \Delta^+\), respectively \(\Delta^+_{nc} = \Delta_{nc} \cap \Delta^+\). For each symmetric matrix \(Z \in \mathfrak{gl}_{2,C}\), define the element \(p_{\pm}(Z)\) of \(\mathfrak{gl}_4\) by

\[
p_{\pm}(Z) = \begin{pmatrix} Z & \pm iZ \\ \pm iZ & -Z \end{pmatrix}.
\]

Let us consider the generator

\[
p_+ \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \wedge p_+ \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \wedge p_- \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \wedge p_- \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \wedge p_- \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \wedge p_+ \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

of the one-dimensional \(C\)-vector space \(\wedge^6 \mathfrak{gl}_{4,C}/\mathfrak{k}_C\). To this generator is associated a non-zero left translation invariant measure \(d\mu\) on \(Sp(4, \mathbb{R})/K_\infty = G(\mathbb{R})_+/K'_\infty\) in a standard way. Let \(c \in \mathbb{R}^+_+\) be the constant given by the second point of Lemma 2.1 Let us denote...
by \( \sqrt[6]{c} \) the positive sixth root of \( c \) in \( \mathbb{R}_+^\times \). Let \( X_{(\alpha_1, \alpha_2)} \in \mathfrak{sp}_4 \) be defined as

\[
X_{\pm(2,0)} = \sqrt[6]{c} p_{\pm} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X_{\pm(1,1)} = \sqrt[6]{c} p_{\pm} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X_{\pm(0,2)} = \sqrt[6]{c} p_{\pm} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

It follows from an easy computation that \( X_{(\alpha_1, \alpha_2)} \) is a root vector corresponding to the non-compact root \( (\alpha_1, \alpha_2) = \alpha_1 e_1 + \alpha_2 e_2 \). If we set

\[
(3) \quad p^\pm = \bigoplus_{\alpha \in \Delta_+^C} \mathbb{C} X_{\pm \alpha},
\]

we have the Cartan decomposition \( \mathfrak{sp}_4 = \mathfrak{k} \oplus p^+ \oplus p^- \). Furthermore the inclusion \( \mathfrak{sp}_4 \subset \mathfrak{g} \) induces a canonical isomorphism \( \mathfrak{sp}_4 / \mathfrak{k} = \mathfrak{g} / \mathfrak{t}_C^0 \). In particular \( \mathfrak{g} / \mathfrak{t}_C^0 = p^+ \oplus p^- \).

Integral weights are defined as the \( (k, k') = k e_1 + k' e_2 \in \mathfrak{h}_C^* \) with \( k, k' \in \mathbb{Z} \) and an integral weight is dominant for \( \Delta_+^C \) if \( k \geq k' \). Assigning its highest weight to a finite dimensional irreducible complex representation \( \tau \) of \( K_{\infty} \), we define a bijection between isomorphism classes of finite-dimensional irreducible complex representations of \( K_{\infty} \) and dominant integral weights, whose inverse will be denoted by \( (k, k') \mapsto \tau_{(k, k')} \). Let \( (k, k') \) be a dominant integral weight and let \( d = k - k' \). Then \( \dim_C \tau_{(k, k')} = d + 1 \). More precisely, there exists a basis \( (v_s)_{0 \leq s \leq d} \) of \( \tau_{(k, k')} \), such that

\[
\begin{align*}
(4) \quad \tau_{(k, k')} (\begin{pmatrix} 1 \\ 1 \end{pmatrix}) (v_s) &= (s + k') v_s, \\
(5) \quad \tau_{(k, k')} (\begin{pmatrix} 1 \\ 1 \end{pmatrix}) (v_s) &= (-s + k) v_s, \\
(6) \quad \tau_{(k, k')} (\begin{pmatrix} 1 \\ 1 \end{pmatrix}) (v_s) &= (s + 1) v_{s+1}, \\
(7) \quad \tau_{(k, k')} (\begin{pmatrix} 1 \\ 1 \end{pmatrix}) (v_s) &= (d - s + 1) v_{s-1},
\end{align*}
\]

which we call a standard basis of \( \tau_{(k, k')} \). In the identities above, we agree to use the convention \( v_{-1} = v_{d+1} = 0 \). Note that \( (\text{Ad}, \mathfrak{p}^+) \) is equivalent to \( \tau_{(2, 0)} \) and that \( (\text{Ad}, \mathfrak{p}^-) \) is equivalent to \( \tau_{(0, -2)} \). Under the identification \( (\text{Ad}, \mathfrak{p}^+) \simeq \tau_{(2, 0)} \), the basis \( (v_2, v_1, v_0) = (X_{(2, 0)}, X_{(1, 1)}, X_{(0, 2)}) \) is a standard basis and the under the identification \( (\text{Ad}, \mathfrak{p}^-) \simeq \tau_{(0, -2)} \), the basis \( (v_2, v_1, v_0) = (X_{(0, -2)}, -X_{(-1, 1)}, X_{(-2, 0)}) \) is a standard basis. As the weights of \( \bigwedge^2 \mathfrak{p}^+ \otimes \mathfrak{p}^- \) are the sums of two distinct weights of \( \mathfrak{p}^+ \) and of a weight of \( \mathfrak{p}^- \), as \( \mathbb{C}[K_{\infty}] \)-modules we have

\[
\begin{align*}
\bigwedge^2 \mathfrak{p}^+ \otimes \mathfrak{p}^- &= \tau_{(3, -1)} \oplus \tau_{(2, 0)} \oplus \tau_{(1, 1)}, \\
\mathfrak{p}^+ \otimes \bigwedge^2 \mathfrak{p}^- &= \tau_{(1, -3)} \oplus \tau_{(0, -2)} \oplus \tau_{(1, -1)},
\end{align*}
\]
Lemma 2.2. (1) The following is a standard basis of $\tau(3,-1) \subset \bigwedge^2 p^+ \otimes \mathbb{C} p^-$:

- $w_4 = X(2,0) \wedge X(1,1) \otimes X(0,-2)$,
- $w_3 = -X(2,0) \wedge X(1,1) \otimes X(-1,-1) + 2X(2,0) \wedge X(0,2) \otimes X(0,-2)$,
- $w_2 = X(2,0) \wedge X(1,1) \otimes X(-2,0) - 2X(2,0) \wedge X(0,2) \otimes X(-1,-1) + X(1,1) \wedge X(0,2) \otimes X(0,-2)$,
- $w_1 = 2X(2,0) \wedge X(0,2) \otimes X(-2,0) - X(1,1) \wedge X(0,2) \otimes X(-1,-1)$,
- $w_0 = X(1,1) \wedge X(0,2) \otimes X(-2,0)$.

(1) The following is a standard basis of $\tau(2,0) \subset \bigwedge^2 p^+ \otimes \mathbb{C} p^-$:

- $x_2 = X(2,0) \wedge X(1,1) \otimes X(0,-1) + 2X(2,0) \wedge X(0,2) \otimes X(0,-2)$,
- $x_1 = -2X(2,0) \wedge X(1,1) \otimes X(-2,0) + 2X(1,1) \wedge X(0,2) \otimes X(0,-2)$,
- $x_0 = -2X(2,0) \wedge X(0,2) \otimes X(-2,0) - X(1,1) \wedge X(0,2) \otimes X(-1,-1)$.

(1) The following is a standard basis of $\tau(1,1) \subset \bigwedge^2 p^+ \otimes \mathbb{C} p^-$:

- $y_0 = X(2,0) \wedge X(1,1) \otimes X(0,-2) + X(2,0) \wedge X(0,2) \otimes X(-1,-1) + X(1,1) \wedge X(0,2) \otimes X(0,-2)$.

Lemma 2.3. In the basis

$$
\begin{align*}
X(2,0) \wedge X(1,1) \otimes X(0,-2), & \quad X(2,0) \wedge X(1,1) \otimes X(-1,-1), & \quad X(2,0) \wedge X(1,1) \otimes X(-2,0), \\
X(2,0) \wedge X(0,2) \otimes X(0,-2), & \quad X(2,0) \wedge X(0,2) \otimes X(-1,-1), & \quad X(2,0) \wedge X(0,2) \otimes X(-2,0), \\
X(1,1) \wedge X(0,2) \otimes X(0,-2), & \quad X(1,1) \wedge X(0,2) \otimes X(-1,-1), & \quad X(1,1) \wedge X(0,2) \otimes X(-2,0)
\end{align*}
$$

of $\bigwedge^2 p^+ \otimes \mathbb{C} p^-$ the matrix of the projection $p : \bigwedge^2 p^+ \otimes \mathbb{C} p^- \to \tau(3,-1)$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 1 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

We will denote by $W_{K_{\infty}}$ the Weyl group of $(\mathfrak{t}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. According to the classification theorem [Kn86] Thm. 9.20, as $W_{K_{\infty}}$ has 4 elements, we have:

**Proposition 2.4.** Let $G(\mathbb{R})_+$ be the identity component of $G(\mathbb{R})$, let $\xi$ be a character of $\mathbb{R}^+$ and let $(k, k') \in \mathfrak{h}_{\mathbb{Z}}^*$ be an integral weight. Assume $k \geq k' \geq 0$. Then, there exist 4 isomorphism classes $\Pi_{k,0}$, $\Pi_{k,1}$, $\Pi_{k,2}$, $\Pi_{k,3}$ of irreducible discrete series representations of $G(\mathbb{R})_+$ with Harish-Chandra parameter $(k+2, k'+1)$ and central character $\xi$. Furthermore, the restrictions of these representations to $K_{\infty}$ contain as minimal $K_{\infty}$-types the irreducible representations $\tau(k+3, k'+3)$, $\tau(k+3, -k'-1)$, $\tau(k'+1, -k-3)$, $\tau(-k'-3, -k-3)$ respectively and these occur with multiplicity 1.
In the proposition above, the discrete series $\Pi_{\infty}^{3,0}$ is holomorphic, $\Pi_{\infty}^{2,1}$ and $\Pi_{\infty}^{1,2}$ are generic, which means that they have a Whittaker model and $\Pi_{\infty}^{3,3}$ is antiholomorphic. In what follows, we will denote by $\Pi^H_{\infty}$ and $\Pi^W_{\infty}$ the discrete series of $GSp(4, \mathbb{R})$ defined as

$$
\Pi^H_{\infty} = \text{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_{\infty}^{3,0} = \text{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_{\infty}^{0,3},
$$

$$
\Pi^W_{\infty} = \text{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_{\infty}^{2,1} = \text{Ind}_{G(\mathbb{R})_+}^{G(\mathbb{R})} \Pi_{\infty}^{1,2}.
$$

In particular, we have

$$
\Pi^H_{\infty}|_{G(\mathbb{R})_+} = \Pi_{\infty}^{3,0} \oplus \Pi_{\infty}^{0,3},
$$

$$
\Pi^W_{\infty}|_{G(\mathbb{R})_+} = \Pi_{\infty}^{2,1} \oplus \Pi_{\infty}^{1,2}.
$$

2.4. Hecke algebras. Let $l$ be a prime number and let $K_l \subset G(\mathbb{Q}_l)$ be a compact open subgroup. Let $\mathcal{H}_l^{K_l}$ be the Hecke algebra of $\mathbb{Z}$-valued compactly supported functions on $G(\mathbb{Q}_l)$, which are invariant by translation on the left and on the right by $K_l$, with product given by the convolution product with respect to the Haar measure 1 to the maximal compact subgroup $G(\mathbb{Z}_l)$ of $G(\mathbb{Q}_l)$. If $K = \prod_l K_l \subset G(\mathbb{A}_f)$ is a compact open subgroup, we denote by $\mathcal{H}^K$ the restricted tensor product $\mathcal{H}^K = \bigotimes_l \mathcal{H}_{l}^{K_l}$. For any ring $R$, we shall denote by $\mathcal{H}^K_R$, respectively $\mathcal{H}^K_R$, the $R$-algebra $\mathcal{H}^K \otimes R$, respectively $\mathcal{H}^K \otimes R$. A smooth admissible complex representation $\Pi_l$ of $G(\mathbb{Q}_l)$ is unramified if it has a non-zero vector fixed by $G(\mathbb{Z}_l)$. In this case $\Pi^G_{\mathcal{C}}(\mathbb{Z}_l)$ is endowed with the action of $\mathcal{H}^G_{\mathcal{C}}(\mathbb{Z}_l)$ as follows. For $f \in \mathcal{H}^G_{\mathcal{C}}(\mathbb{Z}_l)$ and $\chi \in \Pi_l$ a vector invariant by $G(\mathbb{Z}_l)$, we define $f.\chi$ as

$$
f.\chi = \int_{G(\mathbb{Q}_l)} \Pi_l(x)(\chi)f(x)dx.
$$

Specifying an unramified representation $\Pi_l$ is the same as specifying a character of $\mathcal{H}^G_{\mathcal{C}}(\mathbb{Z}_l)$ (see [ASch01 §3.2] for more details).

Lemma 2.5. For any $g \in G(\mathbb{Q}_l)$ we have

$$
\left[ G(\mathbb{Q}_l) : gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l) \right] = \left[ G(\mathbb{Z}_l) : g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l) \right].
$$

Proof. Let $1_{gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l)}$ and $1_{g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l)}$ denote the characteristic functions of the subgroups $gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l)$ and $g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l)$ of $G(\mathbb{Q}_l)$ respectively. We have

$$
\int_{G(\mathbb{Q}_l)} \frac{1}{gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l)} dx_l = \int_{G(\mathbb{Q}_l)} 1_{gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l)}(x_l)dx_l
$$

$$
= \int_{G(\mathbb{Q}_l)} 1_{gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l)}(gx_lg^{-1})dx_l
$$

$$
= \int_{G(\mathbb{Q}_l)} 1_{g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l)}(x_l)dx_l
$$

$$
= \int_{G(\mathbb{Q}_l)} 1_{g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l)}(x_l)dx_l
$$

where the second equality follows the change of variable $x_l \rightarrow g^{-1}x_lg$ and from the equality $d(gx_lg^{-1}) = dx_l$ which follows from the unimodularity of $G(\mathbb{Q}_l)$ (see [R10 Proposition V.5.4]). \hfill \Box
Let $dx_\infty$ be a Haar measure on $G(\mathbb{R})$ and let $dx = \prod_v dx_v$ be the product measure on $G(\mathbb{A})$. Given two cusp forms $\varphi_1$ and $\varphi_2$ on $G(\mathbb{A})$ with trivial central character, let $\langle \varphi_1, \varphi_2 \rangle$ denote the Petersson inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi_1(x) \overline{\varphi_2(x)} dx.$$

**Proposition 2.6.** Assume that $\varphi_1$ and $\varphi_2$ have trivial central character and are invariant by right translation by $G(\mathbb{Z}_l)$. Let $g \in G(\mathbb{Q}_l)$ and let $T_g \in \mathcal{H}^{G(\mathbb{Z}_l)}$ be the characteristic function of $G(\mathbb{Z}_l)gG(\mathbb{Z}_l)$. Then

$$\langle T_g \varphi_1, \varphi_2 \rangle = \langle \varphi_1, T_g \varphi_2 \rangle.$$

**Proof.** Let $n$ denote the integer

$$(8) \quad n = \left[ G(\mathbb{Z}_l) : gG(\mathbb{Z}_l)g^{-1} \cap G(\mathbb{Z}_l) \right] = \left[ G(\mathbb{Z}_l) : g^{-1}G(\mathbb{Z}_l)g \cap G(\mathbb{Z}_l) \right]$$

where the second equality follows from Lemma 2.5. According to equality (8) and to [DS05, Lemma 5.5.1 (c)] , there exists $\beta_1, \ldots, \beta_n \in G(\mathbb{Q}_l)$ such that

$$(9) \quad G(\mathbb{Q}_l)gG(\mathbb{Q}_l) = \bigcap_{j=1}^n \beta_j G(\mathbb{Q}_l) = \bigcup_{j=1}^n G(\mathbb{Q}_l) \beta_j.$$

Note that [DS05, Lemma 5.5.1 (c)] is proved in a global situation and for SL$_2$ but that the proof works when applied to our situation once $\Gamma$ in loc. cit. is replaced by $G(\mathbb{Q}_l)$ and $\alpha$ in loc. cit. is replaced by $g$. Hence we have

$$\langle T_g \varphi_1, \varphi_2 \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi_1(xh_1) \chi_{G(\mathbb{Z}_l)}gG(\mathbb{Z}_l)(h_1) \overline{\varphi_2(x)} dh_1 dx$$

$$= \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{j=1}^n \int_{G(\mathbb{Z}_l)} \varphi_1(x \beta_j h_1) \overline{\varphi_2(x)} dh_1 dx$$

$$= \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{j=1}^n \varphi_1(x \beta_j) \overline{\varphi_2(x)} dx$$

$$= \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{j=1}^n \varphi_1(x) \overline{\varphi_2(x \beta_j^{-1})} dx$$

$$= \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi_1(x) \sum_{j=1}^n \int_{G(\mathbb{Z}_l)} \varphi_2(x \beta_j^{-1} h_1) dh_1 dx$$

$$= \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi_1(x) \sum_{j=1}^n \int_{G(\mathbb{Z}_l)} \varphi_2(x \nu(g) \beta_j^{-1} h_1) dh_1 dx$$

where the first equality is the definition of the action of $T_g$, the second follows from the first equality in (9), the third follows from the fact that $\varphi_1$ is right translation invariant by $G(\mathbb{Z}_l)$, the fourth follows from an obvious change of variable in the integral, the fifth is similar as the third and the sixth follows from the fact that $\varphi_2$ as trivial central character.
Note that (9) implies that \( G(\mathbb{Z}) \nu(g) g^{-1} G(\mathbb{Z}) = \bigcup_{j=1}^{n} \nu(g) \beta_j^{-1} G(\mathbb{Z}) \). As a consequence \( \langle T g \varphi_1, \varphi_2 \rangle = \langle \varphi_1, T \nu(g) g^{-1} \varphi_2 \rangle \).

As we have a canonical bijection \( G(\mathbb{Z}) \backslash G(\mathbb{Z}[l^{-1}]) / G(\mathbb{Z}) \cong G(\mathbb{Z}) \backslash G(\mathbb{Q}_l) / G(\mathbb{Z}) \) we can assume that \( g \in G(\mathbb{Z}[l^{-1}]) \). It follows from [An87, Lemma 3.3.6] that \( G(\mathbb{Z}) g G(\mathbb{Z}) = G(\mathbb{Z}) g^{-1} G(\mathbb{Z}) \). According to [ASch01, Lemma 8] which identifies the classical Hecke algebra with \( H_{G(\mathbb{Z})} \), this implies that \( T \nu(g) g^{-1} = T g \) and the conclusion follows. \( \square \)

3. Adjoint \( L \)-value and Petersson norm after Ichino

For the convenience of the reader, let us recall a particular case of the main result of [I08] expressing the value at 1 of the adjoint \( L \)-function of some cuspidal automorphic representations of \( G(\mathbb{A}) \) in terms of a Petersson norm.

Let \( \Pi \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}) \). Let us fix an isomorphism

\[
(10) \quad \Pi \simeq \bigotimes_v' \Pi_v.
\]

We make the following assumptions on \( \Pi \):

(i) the central character of \( \Pi \) is trivial,
(ii) \( \Pi \) is globally generic,
(iii) \( \Pi_l \) is unramified for every prime \( l \),
(iv) \( \Pi_\infty |_{\text{Sp}(4, \mathbb{R})} = D(\lambda_1, \lambda_2) \oplus D(-\lambda_2, -\lambda_1) \),

where \( D(\lambda_1, \lambda_2) \) is the (limit of) discrete series representation of \( \text{Sp}(4, \mathbb{R}) \) with Blattner parameter \((\lambda_1, \lambda_2)\) such that \( 1 - \lambda_1 \leq \lambda_2 \leq 0 \). According to [AS06], the automorphic representation \( \Pi \) has a functorial lift \( \Sigma \) to \( \text{GL}(4, \mathbb{A}) \) and we say that \( \Pi \) is endoscopic if \( \Sigma \) is not cuspidal. From now on, we assume that

(v) \( \Pi \) is endoscopic.

Let \( \eta = \otimes_v e_v \) be the standard additive character of \( \mathbb{Q} \backslash \mathbb{A} \), so that \( \eta_\infty(x) = \exp(2\pi ix) \) for \( x \in \mathbb{R} \). By abuse of notation, we denote by \( \eta \) the character of \( U(\mathbb{Q}) \backslash U(\mathbb{A}) \to \mathbb{C}^\times \) defined by

\[
\begin{pmatrix} 1 & x_0 \\ 1 & -x_0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \mapsto \eta(-x_0 - x_3)
\]

Let \( \Pi \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}) \). The Whittaker function of a cusp form \( \psi \in \Pi \) is

\[
W_\psi(g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \psi(ug) \overline{\eta(u)} du
\]

for \( g \in G(\mathbb{A}) \), where \( du \) is the standard \( U(\mathbb{A}) \)-invariant measure on \( U(\mathbb{Q}) \backslash U(\mathbb{A}) \). Let us consider the unique element \( \varphi = \bigotimes_v' \varphi_v \in \Pi \) normalized as in [I08, §1] in the following way:

(a) for every prime \( l \) the vector \( \varphi_l \) is invariant by \( G(\mathbb{Z}_l) \),
(b) the archimedean component \( \varphi_{\infty} \) is a lowest weight vector of the minimal \( U(2) \)-type of \( D(-\lambda_2, -\lambda_1) \),
(c) \( W_{\varphi}(1) = 1 \).

Let \( \langle \varphi, \varphi \rangle \) denote the Petersson norm
\[
\langle \varphi, \varphi \rangle = \int_{Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} |\varphi(g)|^2 dg_{\text{Tam}}
\]
where \( dg_{\text{Tam}} \) is the Tamagawa measure on \( G(\mathbb{A}) \). The following statement is a particular case of Theorem 1.1 in [I08].

**Theorem 3.1.** There exists a constant \( C_\infty \in \mathbb{C}^\times \) which depends only on \( \Pi_{\infty} \) such that
\[
C_\infty L(1, \Pi, Ad) = \langle \varphi, \varphi \rangle.
\]

**Remark 3.2.** The constant \( C_\infty \), in the statement above is denoted by \( 4C_\infty \) in Theorem 1.1 of [I08].

4. Integral Betti cohomology of Siegel threefolds

4.1. Integral local systems on Siegel threefolds. Siegel threefolds are the Shimura varieties associated to the group \( G \). Let us briefly recall their definition. Let \( S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C} \) be the Deligne torus and let \( H \) be the \( G(\mathbb{R}) \)-conjugacy class of the morphism \( h : S \to G(\mathbb{R}) \) given on \( \mathbb{R} \)-points by
\[
(x + iy) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.
\]
The pair \( (G, H) \) verifies the axioms of Deligne-Shimura (see [L05, Lemme 2.1]). For any integer \( N \geq 1 \), let us denote by \( K(N) \) the principal congruence subgroup of level \( N \) defined as \( K(N) = \text{Ker}(G(\mathbb{Z}) \to G(\mathbb{Z}/N \mathbb{Z})) \). Let \( S_N \) the Siegel threefold of level \( K(N) \). For \( N \geq 3 \), this is a smooth quasiprojective \( \mathbb{Z}[1/N] \)-scheme \( S_N \) such that, as complex analytic varieties, we have \( S_N \cong G(\mathbb{Q}) \backslash (H \times G(\mathbb{A}_f)/K(N)) \), and carrying a principally polarized universal abelian surface \( a : A \to S_N \). These statements follow from [L05, Théorème 3.1]. Furthermore, as topological spaces we have
\[
S_N \cong \bigsqcup_{(\mathbb{Z}/N \mathbb{Z})^\times} \Gamma(N) \backslash \mathcal{H}_+ \tag{11}
\]
where \( \Gamma(N) = \ker(\text{PSp}(4, \mathbb{Z}) \to \text{PSp}(4, \mathbb{Z}/N \mathbb{Z})) \).

Let \( \lambda(k, k', c) \) be a dominant weight. We associate to the representation \( V_{\lambda, \mathbb{Z}} \) of \( G \) a local system of \( \mathbb{Z} \)-modules for the classical topology on \( S_N \) as follows. To the standard representation \( \text{Std}_{\mathbb{Z}} \) we associate the first relative homology group \( T_{\mathbb{Z}} = \text{Hom}(R^1a_* \mathbb{Z}, \mathbb{Z}) \) of \( a \). The principal polarization on \( A \) induces a non-degenerate alternate pairing \( \Psi : T_{\mathbb{Z}} \otimes T_{\mathbb{Z}} \to \mathbb{Z} \). If \( d \geq 2 \), let \( T_{\mathbb{Z}}^{(d)} \subset T_{\mathbb{Z}}^{\otimes d} \) be the intersection of the kernels of all the contractions \( T_{\mathbb{Z}}^{\otimes d} \to T_{\mathbb{Z}}^{\otimes d-2} \) defined analogously as the \( \Psi_{p,q} \) (\( \Pi \)) using \( \Psi \) instead of \( J \) and let \( V_{\lambda, \mathbb{Z}} = T_{\mathbb{Z}}^{(d)} \cap S_{\lambda}(T_{\mathbb{Z}}) \otimes (2\pi i)^d \mathbb{Z} \), where \( S_{\lambda}(T_{\mathbb{Z}}) \) denotes the image of \( c_{\lambda} \in \mathbb{Z}[S_d] \) acting on
The Betti cohomology with compact support $H^3_c(S_N, V_{\lambda,Z})$ and in the inner cohomology $H^3(S_N, V_{\lambda,Z})$ defined as

$$H^3(S_N, V_{\lambda,Z}) = \text{Im}(H^3_c(S_N, V_{\lambda,Z}) \rightarrow H^3(S_N, V_{\lambda,Z})).$$

These are finitely generated $\mathbb{Z}$-modules. To prove that they are endowed with a natural action of $\mathcal{H}^K(N)$, we shall make use of the formalism of Grothendieck's functors $(f^*, f_!, f^!)$ on the derived categories of sheaves of $\mathbb{Z}$-modules on locally compact topological spaces (see [Ay08], [Ay10] and [KaS94]).

### 4.2. Hecke action on integral cohomology

Let $g \in G(\mathbb{A}_f)$. Then we have a Hecke correspondence

$$
\begin{array}{ccc}
S_N(g) & \xleftarrow{c_1(g)} & S_N \\
c_2(g) & \downarrow & \downarrow \\
S_N & \xrightarrow{c_2(g)} & S_N
\end{array}
$$

where the notation is as in [L05] p. 8. This diagram depends only on the class of $g$ in $K(N)\backslash G(\mathbb{A}_f)/K(N)$. As $c_i(g)$ is finite and étale for $i = 1, 2$, we have canonical isomorphisms of functors $c_i(g)_! \simeq c_i(g)_*$ and $c_i(g)_!^1 \simeq c_i(g)^*$.

**Lemma 4.1.** There exists a canonical morphism

$$\varphi_* : c_1(g)^* V_{\lambda,Z} \rightarrow c_2(g)^* V_{\lambda,Z}.$$

**Proof.** It follows from the modular description of $c_1(g)$ and $c_2(g)$ given at p. 9 of [L05] that we have an isogeny $\varphi : c_1(g)^* A \rightarrow c_2(g)^* A$ over $S_N(g)$. Taking homology, we obtain a morphism $\varphi_* : c_1(g)^* H_Z \rightarrow c_2(g)^* H_Z$ which induces the morphism of the statement. \(\square\)

Let us denote by $\bullet$ the topological space reduced to a point and let $p_N : S_N \rightarrow \bullet$ the canonical continuous map. Then $H^*(S_N, V_{\lambda,Z})$ is the cohomology of the complex $p_N V_{\lambda,Z}$ and $H^*_c(S_N, V_{\lambda,Z})$ is the cohomology of the complex $p_N V_{\lambda,Z}$.

**Definition 4.2.** The endomorphism

$$T_g : H^*(S_N, V_{\lambda,Z}) \rightarrow H^*(S_N, V_{\lambda,Z})$$

is defined as the endomorphism induced by the map

$$p_N V_{\lambda,Z} \rightarrow p_N (c_1(g)_* c_1(g)^* V_{\lambda,Z}) \rightarrow p_N (c_2(g)_* c_2(g)^* V_{\lambda,Z})$$

$$\varphi_* \rightarrow p_N (c_2(g)_* c_2(g)^* V_{\lambda,Z}) \rightarrow p_N (c_2(g)_! c_2(g)^! V_{\lambda,Z}) \rightarrow p_N V_{\lambda,Z}$$

in the derived category of $\mathbb{Z}$-modules, where the first arrow is induced by the adjunction morphism, the second arrow is induced by the isomorphism of functors

$$p_N (c_1(g)_* \simeq (p_N \circ c_1(g))_*, p_N (c_2(g)_* \simeq (p_N \circ c_2(g))_*.$$
the third arrow is induced by the canonical morphism $\varphi_*$ of Lemma \cite{Ay08}, the fourth arrow is induced by the isomorphism of functors $c_2(g)_* c_2(g)^* \simeq c_2(g) c_2(g)^1$ and the last arrow is induced by trace map. The endomorphism
\[ T_{g,c} : H^*_c(S_N, V_{\lambda,Z}) \to H^*_c(S_N, V_{\lambda,Z}) \]
is defined similarly as the endomorphism induced by
\[ p_N! V_{\lambda,Z} \to p_N! c_1(g)_* c_1(g)^* V_{\lambda,Z} \to p_N! c_2(g)_* c_1(g)^* V_{\lambda,Z} \]
\[ \varphi \to p_N! c_2(g)_* c_2(g)^* V_{\lambda,Z} \to p_N! c_2(g)_* c_2(g)^1 V_{\lambda,Z} \to p_N! V_{\lambda,Z}. \]

**Proposition 4.3.** Let $N \geq 3$ be an integer. For any $g \in G(\lambda_f)$ the diagram
\[ H^*_c(S_N, V_{\lambda,Z}) \longrightarrow H^*(S_N, V_{\lambda,Z}) \]
\[ \downarrow T_{g,c} \quad \quad \downarrow T_g \]
\[ H^*_c(S_N, V_{\lambda,Z}) \longrightarrow H^*(S_N, V_{\lambda,Z}) \]
where the maps are defined above is commutative and depends only on the double class $K(N)gK(N)$.

**Proof.** The statement follows from the commutativity of the following diagram in the derived category of $\mathbb{Z}$-modules
\[ p_N! V_{\lambda,Z} \longrightarrow p_N! c_1(g)_* c_1(g)^* V_{\lambda,Z} \]
\[ \downarrow \quad \quad \downarrow \sim \]
\[ p_N! c_2(g)_* c_1(g)^* V_{\lambda,Z} \longrightarrow p_N! c_2(g)_* c_2(g)^* V_{\lambda,Z} \]
\[ \downarrow \varphi_+ \quad \downarrow \varphi_+ \]
\[ p_N! c_2(g)_* c_2(g)^1 V_{\lambda,Z} \longrightarrow p_N! c_2(g)_* c_2(g)^1 V_{\lambda,Z} \]
\[ \downarrow \quad \quad \downarrow \]
\[ p_N! V_{\lambda,Z} \longrightarrow p_N! V_{\lambda,Z}, \]
where the vertical lines are defined in Definition \cite{Ay08} and where the first, the second, the fifth and the sixth horizontal arrows are induced by the morphism of functors $p_N! \to p_N*$ and the third and the fourth horizontal arrows are induced by the morphism of functors $p_N! c_2(g)_! \to p_N! c_2(g)_*$ and $p_N! c_2(g)_* \to p_N! c_2(g)_*$.

The commutativity of all but the second square follows from functoriality. Let us prove the commutativity of the second square. According to \cite[Proposition 1.7.3]{Ay08}, the following
diagram of functors

\[
\begin{array}{ccc}
p_{N!c_1(g)}! & \longrightarrow & p_{N* c_1(g)}_* \\
\sim & & \sim \\
(p_{N \circ c_1(g)})! & \longrightarrow & (p_{N \circ c_1(g)})_*
\end{array}
\]

where the upper horizontal map is the composite

\[
p_{N!c_1(g)}! \longrightarrow p_{N!c_1(g)}_* \longrightarrow p_{N* c_1(g)}_*
\]

is commutative. As the first map above is an equality, the diagram

\[
\begin{array}{ccc}
p_{N!c_1(g)}_* & \longrightarrow & p_{N* c_1(g)}_* \\
\sim & & \sim \\
(p_{N \circ c_1(g)})! & \longrightarrow & (p_{N \circ c_1(g)})_*
\end{array}
\]

is commutative. Furthermore, the diagram

\[
\begin{array}{ccc}
(p_{N \circ c_1(g)})! & \longrightarrow & (p_{N \circ c_1(g)})_* \\
\| & & \| \\
(p_{N \circ c_2(g)})! & \longrightarrow & (p_{N \circ c_2(g)})_* \\
\sim & & \sim \\
p_{N!c_2(g)}! & \longrightarrow & p_{N* c_2(g)}_*
\end{array}
\]

where the last horizontal map is defined as above, is commutative. By combining the commutativity of the two previous diagrams, we complete the proof.

As each function \( f \in \mathcal{H}^{K(N)} \) is a finite linear combination with \( \mathbb{Z} \)-coefficients of characteristic functions of double cosets \( K(N)gK(N) \) with \( g \in G(\mathbb{A}_f) \), by the previous result we define a ring homomorphism

\[
(12) \quad \mathcal{H}^{K(N)} \to \text{End}_{\mathbb{Z}}(H^*_i(S_N, V_{\lambda, \mathbb{Z}}))
\]

by sending the characteristic function \( \mathbf{1}_{K(N)gK(N)} \) to the endomorphism of \( H^*_i(S_N, V_{\lambda, \mathbb{Z}}) \) induced by \( T_g \).

4.3. Isotypical component of cohomology. For any \( \mathbb{Z} \)-algebra \( R \) let us denote by \( V_{\lambda, R} \) the representation of \( G_R \) deduced from \( V_{\lambda} \) by extending the scalars to \( R \). By abuse of notation, we will denote by the same symbol the local system of \( R \)-modules on \( S_N \) naturally associated to \( V_{\lambda, R} \). By the universal coefficients theorem, we have \( H^*_i(S_N, V_{\lambda, \mathbb{C}}) = H^*_i(S_N, V_{\lambda, \mathbb{Z}}) \otimes \mathbb{C} \). Let

\[
P(V_{\lambda, \mathbb{C}}) = \{ \Pi_{\infty}^{3,0}, \Pi_{\infty}^{2,1}, \Pi_{\infty}^{1,2}, \Pi_{\infty}^{0,3} \}
\]

be the discrete series \( L \)-packet attached to the complex representation \( V_{\lambda, \mathbb{C}} \). If \( \lambda = \lambda(k, k', c) \) with \( k \geq k' \geq 0 \) and \( c \equiv k + k' \mod 2 \) then the set \( P(V_{\lambda, \mathbb{C}}) \) is the set of discrete series of \( G(\mathbb{R})_+ \) with Harish-Chandra parameter \( (k + 2, k' + 1) \) and central character \( x \mapsto x^{-c} \) given by Proposition \([2,3]\). Let us denote by \( m(\Pi) \) the cuspidal multiplicity of an automorphic representation \( \Pi \) of \( G(\mathbb{A}) \).
Proposition 4.4. There is a canonical isomorphism of $\mathcal{H}_{\mathbb{C}}^{K(N)}$-modules

$$H^3_{\cdot}(S_N, V_{\lambda, C}) = \bigoplus_{\Pi = \Pi_{\infty} \otimes \Pi_f} H^3(\mathfrak{g}_{\mathbb{C}}, K_{\infty}, V_{\lambda, C} \otimes \Pi_{\infty}) \otimes C \Pi_f^{K(N)}$$

where the sum is indexed by isomorphism classes of cuspidal automorphic representations $\Pi \simeq \Pi_{\infty} \otimes \Pi_f$ of $G(\mathbb{A})$ such that $\Pi|_{G(\mathbb{R})^+} \in P(V_{\lambda, C})$.

Proof. This is well known and follows for example from [Le17] (8) and (9).

In this work, we are interested in the level 1 case. The group $G(\mathbb{Z}/N\mathbb{Z})$ acts on $S_N$ by automorphism of schemes and hence on the cohomology $H^3_{\cdot}(S_N, V_{\lambda, C})$. We shall denote by $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C})$ the $\mathbb{C}$-vector space

$$\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C}) = H^3(S_3, V_{\lambda, C})^{G(\mathbb{Z}/3\mathbb{Z})}$$

and by $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$ the full lattice of $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C})$ defined as

$$\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3}) = \text{Im}(H^3_{\lambda}(S_3, V_{\lambda, 3}) \rightarrow H^3_{\lambda}(S_3, V_{\lambda, C})) \cap \tilde{H}^3_{\lambda}(S_1, V_{\lambda, C}).$$

By the definition of $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$, the action of $\mathcal{H}^{K(3)}$ on $H^3_{\lambda}(S_3, V_{\lambda, C})$ induces an action of $\mathcal{H}^{K(1)}$ on the subspace $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$ via the natural map $\mathcal{H}^{K(1)} \rightarrow (\mathcal{H}^{K(3)})^{G(\mathbb{Z}/3\mathbb{Z})}$. This fact and the action of $\mathcal{H}^{K(1)}$ imply that $\mathcal{H}^{K(1)}$ acts on $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$.

It follows from the decomposition of Proposition 4.4 and from the identity $K(1)/K(3) \simeq G(\mathbb{Z}/3\mathbb{Z})$ that we have a $\mathcal{H}^{K(1)}$-equivariant decomposition

$$\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C}) \simeq \bigoplus_{\Pi = \Pi_{\infty} \otimes \Pi_f} H^3(\mathfrak{g}_{\mathbb{C}}, K_{\infty}, V_{\lambda, C} \otimes \Pi_{\infty}) \otimes C \Pi_f^{K(1)}$$

indexed as above. Let us introduce some more notation. We will denote by $\overline{\mathcal{H}}^{K(1)}$ the subring of $\text{End}_{\mathbb{C}}(\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C}))$ defined as the image of the homomorphism defined above $\mathcal{H}^{K(1)} \rightarrow \text{End}_{\mathbb{C}}(\tilde{H}^3_{\lambda}(S_1, V_{\lambda, C}))$. For any $\mathbb{Z}$-algebra $R$, we denote by $\overline{\mathcal{H}}^{K(1)}_R$ the base change $\mathcal{H}^{K(1)} \otimes_{\mathbb{Z}} R$. Note that as $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$ is torsion free, the $\mathbb{Z}$-algebra $\overline{\mathcal{H}}^{K(1)}_\mathbb{Z}$ is torsion free and hence $\overline{\mathcal{H}}^{K(1)}_R$ is canonically isomorphic to the image of $\overline{\mathcal{H}}^{K(1)}_R$ in the endomorphism algebra $\text{End}_R(\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3}) \otimes R).

Lemma 4.5. There exists a finite collection $(E_i)_{i \in I}$ of number fields such that the commutative $\mathbb{Q}$-algebra $\overline{\mathcal{H}}^{K(1)}_\mathbb{Q}$ is isomorphic to $\prod_{i \in I} E_i$.

Proof. It follows from the decomposition of Proposition 4.4 that the $\mathbb{C}$-algebra $\overline{\mathcal{H}}^{K(1)}_\mathbb{C}$ is isomorphic to a finite product of copies of $\mathbb{C}$. In particular it contains no non-zero nilpotent element. As $\overline{\mathcal{H}}^{K(1)}_\mathbb{Q} \subset \overline{\mathcal{H}}^{K(1)}_\mathbb{C}$, the Jacobson radical of the commutative artinian $\mathbb{Q}$-algebra $\overline{\mathcal{H}}^{K(1)}_\mathbb{Q}$ is trivial. Hence, the conclusion follows from [DF04] p. 752, Theorem 3.

Let us fix once and for all $\Pi = \Pi_{\infty} \otimes \Pi_f$ as in the statement of Theorem 4.1. As $\Pi|_{G(\mathbb{R})^+} \in P(V_{\lambda, C})$, the $\mathcal{H}^{K(1)}$-module $\Pi_f^{K(1)}$ appears in the decomposition of $\tilde{H}^3_{\lambda}(S_1, V_{\lambda, 3})$. Hence $\Pi_f^{K(1)}$ is a $\overline{\mathcal{H}}^{K(1)}_f$-module. Let $E(\Pi_f)$ be the rationality field of $\Pi_f$. By definition $E(\Pi_f)$ is
a subfield of \( \mathbb{C} \) and it follows from [BHR94, Theorem 3.2.2] that \( E(\Pi_f) \) is a number field.

Let
\[
S_{G(\mathbb{Z}/3\mathbb{Z})} = \{ p \text{ prime}, p|\#G(\mathbb{Z}/3\mathbb{Z}) \}.
\]
We have \( S_{G(\mathbb{Z}/3\mathbb{Z})} = \{2, 3, 5\} \). Let \( H^3_1(S_3, V_{\lambda}, \mathbb{Z})_{\text{tors}} \) denote the torsion subgroup of \( H^3_1(S_3, V_{\lambda}, \mathbb{Z}) \). Let
\[
S_{\text{tors}} = \{ p \text{ prime}, p|\#H^3_1(S_3, V_{\lambda}, \mathbb{Z})_{\text{tors}} \}
\]
and fix a prime number \( p \) such that \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{tors}} \).

Let \( \theta : \mathbb{H}^{K(1)}_{\mathbb{Z}(p)} \to \mathbb{C} \) the character giving the action of \( \mathbb{H}^{K(1)}_{\mathbb{Z}(p)} \) on \( \Pi_f^{K(1)} \). As \( \Pi_f^{K(1)} \) is one-dimensional, it follows from [Wa85, Lemme I.1] that \( \Pi_f \) is defined over \( E(\Pi_f) \) and so, the character \( \theta \) has values in \( E(\Pi_f) \).

**Lemma 4.6.** We have \( E(\Pi_f) = E_i \) for some \( i \in I \).

**Proof.** The character \( \theta \) factors as
\[
\mathbb{H}^{K(1)}_{\mathbb{Z}(p)} \hookrightarrow \mathbb{H}^{K(1)}_{\mathbb{Q}} \simeq \prod_{i \in I} E_i \hookrightarrow E_i \hookrightarrow E(\Pi_f) \hookrightarrow \mathbb{C}
\]
for some \( i \in I \). In particular \( E(\Pi_f) \) is an extension of \( E_i \). On the other hand the isomorphism class of an unramified representation is determined by the character giving the action of the Hecke algebra on the line of vectors which are invariant by \( K(1) \). For any \( \sigma \in \text{Aut}(\mathbb{C}) \), the representation \( ^{\sigma} \Pi_f \) is unramified and corresponds to the character \( \sigma \circ \theta \). As \( \theta \) factors through \( E_i \), we have \( \sigma \circ \theta = \theta \) and so \( ^{\sigma} \Pi_f \simeq \Pi_f \) for any \( \sigma \in \text{Aut}(\mathbb{C}/E_i) \). As a consequence \( E(\Pi_f) \subset E_i \), which completes the proof. \( \square \)

**Corollary 4.7.** The image of \( \theta \) verifies \( \text{Im} \theta \otimes_{\mathbb{Z}(p)} \mathbb{Q} = E(\Pi_f) \).

**Proof.** The statement is a direct consequence of the fact that \( \mathbb{H}^{K(1)}_{\mathbb{Z}(p)} \otimes_{\mathbb{Z}(p)} \mathbb{Q} = \mathbb{H}^{K(1)}_{\mathbb{Q}} \) and of Lemma 4.5 and 4.6. \( \square \)

Let us denote by \( M(\Pi_f, V_{\lambda}, \mathbb{Q}) \) the finite dimensional \( \mathbb{Q} \)-vector space
\[
M(\Pi_f, V_{\lambda}, \mathbb{Q}) = \left( H^3_1(S_3, V_{\lambda, \mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \otimes_{\mathbb{H}^{K(1)}_{\mathbb{Z}(p)}} \mathbb{H}^{K(1)}_{\mathbb{Z}(p)}/\ker \theta \right) \otimes_{\mathbb{Z}(p)} \mathbb{Q}.
\]
This is a direct factor of the \( \mathbb{Q} \)-vector space \( H^3_1(S_3, V_{\lambda}, \mathbb{Q})^{G(\mathbb{Z}/3\mathbb{Z})} \). Let
\[
p_M : H^3_1(S_3, V_{\lambda}, \mathbb{Q})^{G(\mathbb{Z}/3\mathbb{Z})} \to M(\Pi_f, V_{\lambda}, \mathbb{Q})
\]
denote the projection and let \( M(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) \) be defined as
\[
M(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) = p_M \left( H^3_1(S_3, V_{\lambda, \mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \right).
\]
Let \( L(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) \) denote the \( \mathbb{Z}(p) \)-lattice of \( M(\Pi_f, V_{\lambda, \mathbb{Q}}) \) defined as
\[
L(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) = M(\Pi_f, V_{\lambda, \mathbb{Q}}) \cap H^3_1(S_3, V_{\lambda, \mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})}.
\]
It is clear that \( L(\Pi_f, V_{\lambda, Z_{(p)}}) \) is a sub \( Z_{(p)} \)-module of \( M(\Pi_f, V_{\lambda, Z_{(p)}}) \). For any \( Z_{(p)} \)-algebra \( R \), we will denote by \( L(\Pi_f, V_{\lambda, R}) \) the \( R \)-module \( L(\Pi_f, V_{\lambda, Z_{(p)}}) \otimes_{Z_{(p)}} R \). According to Corollary 4.7, we have
\[
L(\Pi_f, V_{\lambda, \mathbb{C}}) = M(\Pi_f, V_{\lambda, \mathbb{C}}) = \bigoplus_{\sigma : E(\Pi_f) \rightarrow \mathbb{C}} H^3(S_3, V_{\lambda, Z_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})} \otimes_{\theta^\sigma} \mathbb{C}
\]
where for any embedding \( \sigma : E(\Pi_f) \rightarrow \mathbb{C} \) the character \( \theta^\sigma : \mathcal{H}^{K(1)}_{Z_{(p)}} \rightarrow \mathbb{C} \) is the composite \( \theta^\sigma : \mathcal{H}^{K(1)}_{Z_{(p)}} \rightarrow E(\Pi_f) \xrightarrow{\sigma} \mathbb{C} \).

**Theorem 4.8.** Let \( \Pi = \Pi_\infty \otimes \Pi_f \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}) \) with trivial central character and such that \( \Pi_\infty|_{G(\mathbb{R})_+} \in P(V_{\lambda, \mathbb{C}}) \). Then the following statements hold for any \( \sigma \in \text{Aut}(\mathbb{C}) \).

1. There exists an irreducible cuspidal automorphic representation \( \sigma \Pi \simeq \bigotimes_v \Pi_{\sigma,v} \) of \( G(\mathbb{A}) \) with trivial central character such that \( \Pi_{\sigma,\infty}|_{G(\mathbb{R})_+} \in P(V_{\lambda, \mathbb{C}}) \) and whose non-archimedean component is equivalent to \( \sigma \Pi_f \).
2. If one assumes that \( \Pi \) is globally generic and endoscopic then so is \( \sigma \Pi \).

**Proof.** Statement (1) follows from the proof of [GR13, Proposition 2.4] combined with the fact that for Siegel threefolds, \( L^2 \)-cohomology coincides with cuspidal cohomology (see [MT02, Proposition 1]). As PGSp(4) \( \simeq \text{SO}(5) \), statement (2) is a particular case of [GR13, Theorem 10.1 and Theorem 9.5]. \( \square \)

The decomposition (14) and the remark at the end of section 2.4 imply that we have a canonical isomorphism
\[
L(\Pi_f, V_{\lambda, \mathbb{C}}) \simeq \bigoplus_{\sigma : E(\Pi_f) \rightarrow \mathbb{C}} \Pi_\infty|_{G(\mathbb{R})_+} \otimes \Pi_\infty \oplus \bigoplus_{\sigma : E(\Pi_f) \rightarrow \mathbb{C}} \bigoplus_{\Pi \in P(V_{\lambda, \mathbb{C}})} \bigoplus_{m(\Pi_\infty \otimes \sigma \Pi_f) \in \mathbb{C}} \Pi^{K(1)}_f.
\]
Furthermore, for any \( r, s \), the \( \mathbb{C} \)-vector spaces \( H^3(\mathfrak{g}_\mathbb{C}, \mathbb{K}_\infty, V_{\lambda, \mathbb{C}} \otimes \Pi^{r,s}_\infty) \) are one-dimensional ([Le17, Proposition 3.7]). As a consequence, we have
\[
\text{dim}_\mathbb{C} L(\Pi_f, V_{\lambda, \mathbb{C}}) \otimes_{Z_{(p)}} \mathbb{C} = 2 \sum_{\sigma : E(\Pi_f) \rightarrow \mathbb{C}} m(\Pi_\infty^H \otimes \sigma \Pi_f) + m(\Pi_\infty^W \otimes \sigma \Pi_f).
\]
Recall that we say that a globally generic cuspidal automorphic representation \( \Pi \) of \( G(\mathbb{A}) \) is endoscopic if its functorial lift to \( \text{GL}(4, \mathbb{A}) \) is not cuspidal.

**Proposition 4.9.** Let \( \Pi = \Pi_\infty \otimes \Pi_f \) be a globally generic irreducible unitary endoscopic cuspidal automorphic representation of \( G(\mathbb{A}) \). Then

1. we have \( m(\Pi) = 1 \),
2. we have \( m(\Pi_\infty^H \otimes \Pi_f) = 0 \).

**Proof.** The first statement follows from the main result of [JS07]. According to [AS06, Proposition 2.2 (a)], \( \Pi \) is obtained as a Weil lifting from \( \text{GSO}(2,2, \mathbb{A}) \). As \( \Pi \) is assumed to be globally generic, for every prime \( l \) the representation \( \Pi_l \) has a Whittaker model. Hence, applying [We09, Theorem 5.2 (4)] to \( \Pi \) we obtain
\[
m(\Pi_\infty^H \otimes \Pi_f) = m(\Pi_\infty^H \otimes \Pi_f) = \frac{1}{2} (1 + (-1)^{\infty})
\]
where \( \epsilon_\infty = 0 \) or \( 1 \) if \( \Pi^H_\infty \) has or has not a Whittaker model. But it is well known that \( \Pi^H_\infty \) is the archimedean component of an automorphic representation associated to a cuspidal Siegel modular form, which does not have a Whittaker model. As a consequence we obtain\( m(\Pi^H_\infty \otimes \Pi_f) = m(\Pi^H_\infty \otimes \Pi_f) = 0 \).

**Corollary 4.10.** Let the notation and assumptions be as in the previous result. Then
\[
\text{rk}_{\mathbb{C}} L(\Pi_f, V, \lambda, \mathbb{Z}(p)) = 2 [E(\Pi_f) : \mathbb{Q}] .
\]

**Proof.** This follows from the fact that \( \text{rk}_{\mathbb{C}} L(\Pi_f, V, \lambda, \mathbb{Z}(p)) = \dim_{\mathbb{C}} L(\Pi_f, V, \lambda, \mathbb{C}) \) and from \([20]\) combined with Theorem 4.8 and Proposition 4.9 \( \square \)

### 4.4. Poincaré duality

Let \( \tilde{V}_\lambda, \lambda(p) \) denote the dual local system \( \tilde{V}_\lambda, \lambda(p) = \text{Hom}(V, \lambda, \mathbb{Z}(p)) \). Let us define
\[
S^\prime_{\text{tors}} = \{ p \text{ prime }, \, p | \#H^3_c(S, V, \lambda)_{\text{tors}} \}
\]

The following lemma is a standard consequence of the existence of Verdier duality. We give a proof for the sake of completeness.

**Lemma 4.11.** Assume that \( p \notin S^\prime_{\text{tors}} \). Then we have a canonical isomorphism
\[
H^3(S, \tilde{V}_\lambda, \lambda(p)) \simeq \text{Hom}_{\mathbb{Z}(p)}(H^3_c(S, V, \lambda, \mathbb{Z}(p)), \mathbb{Z}(p)).
\]

**Proof.** Recall that \( p_3 : S_3 \to \bullet \) denotes the canonical continuous map from \( S_3 \) to the topological space reduced to a point. According to \([\text{KaS94}], \text{Proposition 3.1.10}\) we have a canonical isomorphism
\[
\text{RHom}(\text{Rp}_3! V, \lambda, \mathbb{Z}(p), \mathbb{Z}(p)) \simeq \text{Rp}_3 \ast \text{RHom}(V, \lambda, \mathbb{Z}(p), \mathbb{Z}(p))
\]
in the derived category of abelian groups. Hence, we have two spectral sequences
\[
E^{p,q}_2 = \text{Ext}^p_{V,\lambda, \mathbb{Z}(p)}(H^{-q}_c(S_3, V, \lambda, \mathbb{Z}(p)), \mathbb{Z}(p)) \implies E^{p+q}_\infty, \\
E^{p,q}_2 = H^p(S_3, \text{Ext}^q(V, \lambda, \mathbb{Z}(p), p_3^! V, \lambda(p))) \implies E^{p+q}_\infty
\]
where \( E^{p+q}_\infty \simeq E^{p+q}_\infty \). According to \([\text{KaS94}], \text{Proposition 3.3.2 (i)}\) , as \( S_3 \) is smooth of real dimension 6, we have \( p_3^! V, \lambda(p) = \mathbb{Z}(p)[6] \). As a consequence, using the fact that \( V, \lambda, \mathbb{Z}(p) \) is a sheaf of free \( \mathbb{Z}(p) \)-modules of finite type, we have \( \text{Ext}^q(V, \lambda, \mathbb{Z}(p), p_3^! V, \lambda(p)) = 0 \) for \( q \neq -6 \) and \( \text{Ext}^{-6}(V, \lambda, \mathbb{Z}(p), p_3^! V, \lambda(p)) = \tilde{V}_\lambda, \lambda(p) \). This implies immediately \( E^{2, -6}_\infty \simeq E^{-6}_\infty \).

On the other hand, as \( \mathbb{Z}(p) \) is a discrete valuation ring, we have \( E^{p,q}_2 = 0 \) for any \( p \geq 2 \) and any \( q \in \mathbb{Z} \). Hence \( E^{0,-3}_\infty = E^{-3}_\infty \) and \( E^{2,-5}_\infty = E^{-5}_\infty = 0 \). The assumption that \( p \) is outside \( S^\prime_{\text{tors}} \) implies that \( E^{1,-4}_\infty = E^{-4}_\infty = 0 \). Hence \( E^{0,-3}_\infty = E^{-3}_\infty \). As \( E^{-3}_\infty \simeq E^{-3}_\infty \) we have a canonical isomorphism \( E^{0,-3}_\infty \simeq E^{2,-6}_\infty \) which proves the claim \( \square \)

By the previous Lemma, assuming that \( p \notin S^\prime_{\text{tors}} \), we have a non-degenerate pairing
\[
\langle , \rangle : H^3(S_3, \tilde{V}_\lambda, \lambda(p)) \times H^3_c(S_3, V, \lambda, \mathbb{Z}(p)) \to \mathbb{Z}(p) .
\]

By composing with the natural inclusion \( \text{H}^3_c(S_3, \tilde{V}_\lambda, \lambda(p)) \hookrightarrow H^3(S_3, \tilde{V}_\lambda, \lambda(p)) \) on the first factor, we obtain a pairing
\[
\langle , \rangle : H^3_c(S_3, \tilde{V}_\lambda, \lambda(p)) \times H^3_c(S_3, V, \lambda, \mathbb{Z}(p)) \to \mathbb{Z}(p) .
\]
Lemma 4.12. For any \( x \in H^3_!(S_3, \tilde{V}_{\lambda,z(p)}) \) and \( y \in \ker(H^3_c(S_3, V_{\lambda,z(p)}) \to H^3(S_3, V_{\lambda,z(p)})) \) we have \( \langle x, y \rangle = 0 \).

Proof. Let us denote by \( \langle \cdot, \cdot \rangle : H^3(S_3, \tilde{V}_{\lambda,c}) \times H^3(S_3, V_{\lambda,c}) \to \mathbb{C} \) be the pairing obtained from \( \langle \cdot, \cdot \rangle \) by base change to \( \mathbb{C} \). Let \( x_{\mathbb{C}} \) denote the image of \( x \) in \( H^3(S_3, \tilde{V}_{\lambda,c}) \) and let \( y_{\mathbb{C}} \) denote the image of \( y \) in \( H^3(S_3, V_{\lambda,c}) \). It is enough to prove that \( \langle x_{\mathbb{C}}, y_{\mathbb{C}} \rangle_{\mathbb{C}} = 0 \). By the comparison isomorphism between Betti and de Rham cohomology and by compatibility of the Poincaré duality in Betti and de Rham cohomology, it is enough to prove the following statement: for any \( x'_{\mathbb{C}} \in H^3_{dR,c}(S_3, V_{\lambda}) \) and any \( y'_{\mathbb{C}} \in \ker(H^3_{dR,c}(S_3, V_{\lambda}) \to H^3_{dR}(S_3, V_{\lambda})) \), we have \( \langle x'_{\mathbb{C}}, y'_{\mathbb{C}} \rangle_{dR,\mathbb{C}} = 0 \) where \( V_{\lambda} \) is the complex vector bundle associated with the local system \( V_{\lambda,c} \). The reduction modulo \( H \) by Nakayama’s lemma allows to identify \( H \). Let us denote by \( \langle \cdot, \cdot \rangle_{dR} \) the duality pairing in de Rham cohomology. Let us abusively denote by \( x'_{\mathbb{C}} \) and \( y'_{\mathbb{C}} \) closed differential forms in the cohomology class of \( x'_{\mathbb{C}} \) and \( y'_{\mathbb{C}} \) respectively. Because \( x \in \text{Im}(H^3_c(S_3, \tilde{V}_{\lambda,z(p)}) \to H^3(S_3, V_{\lambda,z(p)})) \), there exists a compactly supported closed differential 3-form \( x'' \) and a differential 2-form \( u \) such that \( x'_{\mathbb{C}} = x'' + du \) and because \( y \in \ker(H^3_c(S_3, V_{\lambda,z(p)}) \to H^3(S_3, V_{\lambda,z(p)})) \) there exists a differential 2-form \( y'' \) such that the compactly supported differential form \( y'_{\mathbb{C}} \) satisfies \( y'_{\mathbb{C}} = dy'' \). We need to prove \( \int_{S_3} x'_{\mathbb{C}} \wedge y'_{\mathbb{C}} = 0 \). But

\[
 x'_{\mathbb{C}} \wedge y'_{\mathbb{C}} = x'' \wedge dy'' + du \wedge y'_{\mathbb{C}} = d(-x'' \wedge y'' + u \wedge y'_{\mathbb{C}}).
\]

As the differential form \(-x'' \wedge y'' + u \wedge y'_{\mathbb{C}}\) is compactly supported, the statement follows from Stokes theorem.

Lemma 4.13. Let \( \lambda = \lambda(k,k',0) \) be a dominant weight with trivial central character. Assume that \( k + k' + 3 \leq p \). Then, there exists a non-degenerate \( G_{Z(p)} \)-equivariant pairing

\[
[\cdot, \cdot] : V_{\lambda,z(p)} \times V_{\lambda,z(p)} \to \mathbb{Z}(p)
\]

where \( V_{\lambda,z(p)} \times V_{\lambda,z(p)} \) is endowed with the diagonal action and \( \mathbb{Z}(p) \) with the trivial action.

Proof. Let us denote by \( \tilde{V}_{\lambda,F_p} \), resp. \( \tilde{V}_{\lambda,F_p} \) the reduction modulo \( p \) of \( V_{\lambda,z(p)} \), resp. of \( V_{\lambda,z(p)} \). It follows from the Lemma in section 1.9 of [PT02] that \( V_{\lambda,F_p} \) is irreducible. Furthermore as isomorphism classes of irreducible representations of \( G_{F_p} \) are determined by their highest weight, we have \( V_{\lambda,F_p} \simeq \tilde{V}_{\lambda,F_p} \). Let \( r \) be the composition of the canonical projection \( V_{\lambda,z(p)} \to V_{\lambda,F_p} \) and of the isomorphism \( V_{\lambda,F_p} \simeq \tilde{V}_{\lambda,F_p} \). Let us fix \( v \in V_{\lambda,z(p)} \) such that \( r(v) \neq 0 \), let \( w \in \tilde{V}_{\lambda,z(p)} \), be a lifting of \( r(v) \) and let \( i : V_{\lambda,z(p)} \to \tilde{V}_{\lambda,z(p)} \) be the unique \( G_{Z(p)} \)-equivariant map sending \( v \) to \( w \). This map \( i \) is well defined because \( V_{\lambda,z(p)} \) is irreducible by Nakayama’s lemma. The reduction modulo \( p \) of \( i \) is the isomorphism an isomorphism \( V_{\lambda,F_p} \simeq \tilde{V}_{\lambda,F_p} \) considered above. Hence \( i \) is an isomorphism by Nakayama’s lemma.

From now on, we fix a dominant weight \( \lambda = \lambda(k,k',0) \) with trivial central character. Let \( S_{\text{weight}} \) denote the finite set

\[
S_{\text{weight}} = \{ p \text{ prime }, p < k + k' + 3 \}
\]

and assume from now on that \( p \notin S_{\text{weight}} \). The non-degenerate bilinear form of the previous lemma allows to identify \( V_{\lambda,z(p)} \) and \( \tilde{V}_{\lambda,z(p)} \). According to Lemma 4.12 there exists a unique pairing

\[
\langle \cdot, \cdot \rangle : H^3_c(S_3, V_{\lambda,z(p)}) \times H^3_c(S_3, V_{\lambda,z(p)}) \to \mathbb{Z}(p).
\]
such that the diagram
\[
\begin{array}{ccc}
H^3(S_3, V_{\lambda, \mathbb{Z}(p)}) \times H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}) & \xrightarrow{\langle , \rangle} & \mathbb{Z}(p) \\
\parallel & & \parallel \\
H^3(S_3, V_{\lambda, \mathbb{Z}(p)}) \times H^3(S_3, V_{\lambda, \mathbb{Z}(p)}) & \xrightarrow{\langle , \rangle} & \mathbb{Z}(p)
\end{array}
\]
(23)
commutes. Let us define
\[
S''_{\text{tors}} = \{ p \text{ prime}, \ p|\#(H^3(S_3, V_{\lambda, \mathbb{Z}})/H^3_c(S_3, V_{\lambda, \mathbb{Z}}))_{\text{tors}} \}
\]
Corollary 4.14. Let \( p \notin S_{\text{weight}} \cup S'_{\text{tors}} \cup S''_{\text{tors}} \). Then the natural map
\[
H^3(S_3, V_{\lambda, \mathbb{Z}(p)}) \rightarrow \text{Hom}_{\mathbb{Z}(p)}(H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}), \mathbb{Z}(p))
\]
induced by the pairing above is an isomorphism.

Proof. The commutative diagram (23) induces the commutative diagram
\[
\begin{array}{ccc}
H^3(S_3, V_{\lambda, \mathbb{Z}(p)}) & \rightarrow & \text{Hom}_{\mathbb{Z}(p)}(H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}), \mathbb{Z}(p)) \\
\parallel & & \parallel \\
H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}) & \rightarrow & \text{Hom}_{\mathbb{Z}(p)}(H^3(S_3, V_{\lambda, \mathbb{Z}(p)}), \mathbb{Z}(p))
\end{array}
\]
(25)
where the upper horizontal map is an isomorphism according to Lemma 4.11. In particular, the lower horizontal map is injective. Let’s prove the surjectivity of this map. The diagram obtained by tensoring over \( \mathbb{Z}(p) \) with \( \mathbb{Q} \) the diagram above is
\[
\begin{array}{ccc}
H^3(S_3, V_{\lambda, \mathbb{Q}}) & \rightarrow & \text{Hom}_{\mathbb{Q}}(H^3(S_3, V_{\lambda, \mathbb{Q}}), \mathbb{Q}) \\
\parallel & & \parallel \\
H^3_c(S_3, V_{\lambda, \mathbb{Q}}) & \rightarrow & \text{Hom}_{\mathbb{Q}}(H^3_c(S_3, V_{\lambda, \mathbb{Q}}), \mathbb{Q})
\end{array}
\]
(26)
Note that the lower horizontal map is an isomorphism because it is an injection between two \( \mathbb{Q} \)-vector spaces of the same finite dimension. Let \( x \in \text{Hom}_{\mathbb{Z}(p)}(H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}), \mathbb{Z}(p)) \) and let \( x' \in H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}) \) denote the image of \( x \) by the composite of the right hand vertical map and of the inverse of the upper horizontal map of (25). The image \( x'_Q \) of \( x' \) in \( H^3(S_3, V_{\lambda, \mathbb{Q}}) \) is in fact an element of \( H^3_c(S_3, V_{\lambda, \mathbb{Q}}) \). As a consequence the image \( x'' \) of \( x' \) in \( H^3(S_3, V_{\lambda, \mathbb{Z}(p)})/H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}) \) is torsion. By our assumption \( p \notin S''_{\text{tors}} \), this implies \( x'' = 0 \). Hence there exists \( y \in H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)}) \) which maps to \( x \) by the lower horizontal map of (25). \( \square \)

Corollary 4.15. Let \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup S'_{\text{tors}} \cup S''_{\text{tors}} \). Then the \( \mathbb{Z}(p) \)-bilinear map
\[
\langle , \rangle : H^3(S_3, V_{\lambda, \mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \times H^3_c(S_3, V_{\lambda, \mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \rightarrow \mathbb{Z}(p)
\]
(27)
obtained by restricting the bilinear map of the lower horizontal line of diagram (23) is non-degenerate.
Proof. For any \( g \in G(\mathbb{Z}/3\mathbb{Z}) \) and any \( x, y \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \), we have \( \langle g^* x, g^* y \rangle = \langle x, y \rangle \) as \( g \) acts as an automorphism of the \( \mathbb{Q} \)-scheme \( S_3 \) and hence is orientation preserving. Let us prove that the map

\[
H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \to \text{Hom}_{\mathbb{Z}(p)}(H^3(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})}, \mathbb{Z}(p))
\]

defined as \( x \mapsto (z \mapsto \langle x, z \rangle) \) is injective. Let \( x \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \). Let \( x \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \). Then, for any \( g \in G(\mathbb{Z}/3\mathbb{Z}) \) and any \( y \in H^3(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \), we have \( \langle x, y \rangle = \langle g^* x, g^* y \rangle = \langle x, g^* y \rangle \). By summing up over all \( g \in G(\mathbb{Z}/3\mathbb{Z}) \) and dividing by \( |G(\mathbb{Z}/3\mathbb{Z})| \), we obtain

\[
\langle x, y \rangle = |G(\mathbb{Z}/3\mathbb{Z})|^{-1} \langle x, \sum_{g \in G(\mathbb{Z}/3\mathbb{Z})} g^* y \rangle.
\]

Assume that \( \langle x, z \rangle = 0 \) for any \( z \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \). Then for any \( y \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \) we have \( \langle x, y \rangle = |G(\mathbb{Z}/3\mathbb{Z})|^{-1} \langle x, \sum_{g \in G(\mathbb{Z}/3\mathbb{Z})} g^* y \rangle = 0 \) as \( \sum_{g \in G(\mathbb{Z}/3\mathbb{Z})} g^* y \) is an element of \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \). According to Corollary 4.14 this implies that \( x = 0 \). Hence (28) is injective.

To prove its surjectivity let \( \chi \in \text{Hom}_{\mathbb{Z}(p)}(H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})}, \mathbb{Z}(p)) \). Let us denote by \( \chi' \in \text{Hom}_{\mathbb{Z}(p)}(H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}), \mathbb{Z}(p)) \) the element \( \chi' = \chi \circ \rho \) of where \( \rho \) is the projection \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \to H^3(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \). By Corollary 4.14 there exists \( x \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \) such that \( \langle x, y \rangle = \chi'(y) \) for any \( y \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \). In particular for any element \( z \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \) we have \( \langle x, z \rangle = \chi'(z) = \langle \chi \circ \rho \rangle(z) = \chi(z) \). But for any such \( z \) we have \( \chi(z) = \langle x, z \rangle = \langle \rho(x), z \rangle \) by (29). Hence (28) is surjective. \( \square \)

For \( g \in G(\mathbb{A}_f) \), we defined the Hecke operator \( T_g : H^3(S_3, V_{\lambda,\mathbb{Z}}) \to H^3(S_3, V_{\lambda,\mathbb{Z}}) \) in definition 1.2. The dual Hecke operator \( T^*_g : H^3_c(S_3, V_{\lambda,\mathbb{Z}}) \to H^3_c(S_3, V_{\lambda,\mathbb{Z}}) \) is deduced from the sequence of maps defining \( T_g \) by applying the Verdier duality functor

\[
\mathbb{D}(X) = R\text{Hom}(X, p^*_X \mathbb{Z})
\]

and using the fact that \( \mathbb{D} f_* = f_! \mathbb{D} \) and \( \mathbb{D} f^* = f^! \mathbb{D} \). Let us denote again by \( T_g \) and \( T^*_g \) the endomorphisms deduced after extending scalars to \( \mathbb{Z}(p) \). Then we have \( \langle T_g x, y \rangle = \langle x, T^*_g y \rangle \) for any \( g \in G(\mathbb{A}_f) \) and any \( x \in H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}), y \in H^3_c(S_3, V_{\lambda,\mathbb{Z}(p)}). \)

Lemma 4.16. Assume that \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup \text{St}_{\text{tors}} \cup \text{St}'_{\text{tors}} \cup \text{St}''_{\text{tors}} \) where \( S_{G(\mathbb{Z}/3\mathbb{Z})}, S_{\text{tors}}, S'_{\text{tors}}, S''_{\text{tors}}, \text{weight} \) and \( \text{St}_{\text{tors}} \) are defined by (15), (16), (21), (22) and (24) respectively. Then we have a \( \mathbb{Z}(p) \)-linear pairing

\[
L(\Pi_f, V_{\lambda,\mathbb{Z}(p)}) \times L(\Pi_f, V_{\lambda,\mathbb{Z}(p)}) \to \mathbb{Z}(p).
\]

Proof. Recall that in particular, \( p \) does not divide the order of \( G(\mathbb{Z}/3\mathbb{Z}) \), and so the \( \mathbb{Z}(p) \)-module \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \) is a direct factor of \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \). As \( L(\Pi_f, V_{\lambda,\mathbb{Z}(p)}) \) is a \( \mathbb{Z}(p) \)-torsion free quotient of \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)})^{G(\mathbb{Z}/3\mathbb{Z})} \), it is also a direct factor of \( H^3_t(S_3, V_{\lambda,\mathbb{Z}(p)}) \). Let \( \hat{\Pi}_f \) be the contragredient of \( \Pi_f \). Note that, as the representation \( V_{\lambda,\mathbb{Z}(p)} \) has trivial central character, the representation \( \hat{\Pi}_f \) contributes to the cohomology with coefficients \( V_{\lambda,\mathbb{Z}(p)} \). Similarly as before, the \( \mathbb{Z}(p) \)-module \( M(\hat{\Pi}_f, V_{\lambda,\mathbb{Z}(p)}) \) is a direct
factor of $H^3(S_3, V_{\lambda,\mathbb{Z}_{(p)}})$. Hence the $\mathbb{Z}_{(p)}$-module $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\tilde{\Pi}_f, V_{\lambda,\mathbb{Z}_{(p)}})$ is a direct factor of $H^3(S_3, V_{\lambda,\mathbb{Z}_{(p)}}) \times H^3(S_3, V_{\lambda,\mathbb{Z}_{(p)}})$ and by restricting the pairing (23) we obtain a pairing $L(\Pi_f, V_{\lambda,\mathbb{Z}_{(p)}}) \times L(\tilde{\Pi}_f, V_{\lambda,\mathbb{Z}_{(p)}}) \to \mathbb{Z}_{(p)}$. One of the assumptions on $\Pi$ in section 3 is that the central character of $\Pi$ is trivial. According to [We05, Lemma 1.1], this implies that $\Pi \simeq \tilde{\Pi}$ where $\tilde{\Pi}$ is the contragredient of $\Pi$. \hfill \Box

The following results will be useful in the next section. Let $T'$ be the maximal compact subtorus of $\text{Sp}(4, \mathbb{R})$ defined by

$$T' = \left\{ \begin{pmatrix} x & y & 0 & 0 \\ -y & x' & 0 & 0 \\ 0 & 0 & x' & x' \\ 0 & 0 & -y' & x' \end{pmatrix} \mid x^2 + y^2 = x'^2 + y'^2 = 1 \right\} .$$

The Lie algebra of $T'$ is the compact Cartan subalgebra of $\mathfrak{sp}_4$ that we denoted by $\mathfrak{h}$ in section 2.3. Let $\mathbb{R}^\times_+$ be the identity component of the center of $G(\mathbb{R})$. For integers $n, n', c$ such that $n + n' \equiv c \pmod{2}$, let $\lambda'(n, n', c) : \mathbb{R}^\times_+ T' \to \mathbb{C}^\times$ denote the character defined by

$$\begin{pmatrix} x & y \\ -y & x' \\ 0 & 0 \\ 0 \end{pmatrix} \mapsto (x + iy)^n (x' + iy')^{n'} (x^2 + y^2)^\frac{c-n-n'}{2} ,$$

and by $\lambda'(n, n')$ the restriction of $\lambda'(n, n', c)$ to $T'$. Note that the simple root $e_1 - e_2$, respectively $2e_2$, defined in section 2.3 coincides with the differential at the identity matrix of the restriction to $T'$ of the character $\lambda'(1, -1, 0)$, respectively $\lambda(0, 2, 0)$. Let $J \in \text{Sp}(4, \mathbb{C})$ be the matrix

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{pmatrix} .$$

Lemma 4.17. Let $w \in V_{\lambda,\mathbb{C}}$ be a vector of weight $\lambda(u, u', c)$ for the action of the algebraic torus $T$. For the action of the torus $\mathbb{R}^\times_+ T'$, the vector $v = Jw$ has weight $\lambda'(u, u', c)$ and the vector $\overline{v} = \overline{J}w$ has weight $\lambda'(-u, -u', c)$.

Proof. The first statement follows from [Le17, Lemma 4.25]. The second statement follows from the fact that

$$\overline{J}^{-1} \begin{pmatrix} x & y \\ -y & x' \\ -y' & x' \end{pmatrix} J = J \begin{pmatrix} x & y \\ -y & x' \\ -y' & x' \end{pmatrix} \overline{J} = \begin{pmatrix} x - iy \\ x' - iy' \\ x + iy \end{pmatrix} \begin{pmatrix} x + iy \\ x' + iy' \end{pmatrix}$$
and hence
\[
\begin{pmatrix}
  x & y \\
  x' & y'
\end{pmatrix}
\begin{pmatrix}
  x & -iy \\
  x' & iy'
\end{pmatrix}
= \left|J\right|
\begin{pmatrix}
  x - iy & x' + iy' \\
  -iy & iy'
\end{pmatrix}
\begin{pmatrix}
  x & -iy \\
  x' & iy'
\end{pmatrix}
= \left|J\right|
\begin{pmatrix}
  x - iy & x' + iy' \\
  -iy & iy'
\end{pmatrix}
= \chi(k, -k', c)
\begin{pmatrix}
  x & y \\
  -y & x
\end{pmatrix}
\begin{pmatrix}
  x & y' \\
  -y' & x'
\end{pmatrix}
= \left|J\right|w.
\]

Lemma 4.18. Let \( w \in V_{\lambda, C} \) be a vector of weight \( \lambda(-k, k', 0) \), let \([,]_C : V_{\lambda, C} \otimes V_{\lambda, C} \to \mathbb{C} \) denote the pairing obtained from \([,] \) after extending scalars to \( \mathbb{C} \). Then
\[
[Jw, Jw] \neq 0.
\]

Proof. The pairing \([,]_C \) is \( G_C \)-equivariant. In particular, given two vectors \( v_1 \) and \( v_2 \) of weights \( \lambda(u_1, u_1', 0) \) and \( \lambda(u_2, u_2', 0) \), we have
\[
[v_1, v_2]_C \neq 0 \iff u_1 + u_2 = u_1' + u_2' = 0.
\]
The weight \( \lambda(-k, k', 0) \) belongs to the orbit under the action of \( W \) of the dominant weight \( \lambda(k, k', 0) \), hence has multiplicity one in \( V_{\lambda, C} \). Then it follows from Lemma 4.17 that \( \lambda'(-k, k', 0) \), which is the weight of \( Jw \), has multiplicity one in \( V_{\lambda, C} \) and a similar argument applies to \( \lambda(k, -k', 0) \), which is the weight of \( \overline{Jw} \). As a consequence, if \([Jw, \overline{Jw}]_C = 0 \), then \([Jw, w']_C = 0 \) for any \( w' \in V_{\lambda, C} \). This contradicts the fact that \([,]_C \) is non-degenerate. Hence \([Jw, \overline{Jw}] \neq 0 \).

Let
\[
X_{(1, -1)} = dk\begin{pmatrix}
  (1) \\
  (1)
\end{pmatrix} = \frac{1}{2}\begin{pmatrix}
  1 & 1 & -1 \\
  -1 & i & i \\
  i & -1 & 1
\end{pmatrix} \in k_C
\]
and let
\[
X_{(-1, 1)} = dk\begin{pmatrix}
  (1) \\
  (1)
\end{pmatrix} = \frac{1}{2}\begin{pmatrix}
  1 & 1 & i \\
  -1 & i & -i \\
  -i & -1 & 1
\end{pmatrix} \in k_C
\]
These are root vectors corresponding to the positive, resp. negative, compact root. Let us denote \( v = Jw \) and \( \overline{v} = \overline{Jw} \).

Lemma 4.19. For any \( i, j \in \mathbb{Z} \) we have
\[
[ X_{(1, -1)}^i v, X_{(-1, 1)}^j \overline{v} ]_C = \begin{cases}
  0 & \text{if } i \neq j \\
  (-1)^i i! (k+k')! [v, \overline{v}] & \text{if } i = j.
\end{cases}
\]
Lemma 5.1. Let \( \phi \in \Pi_{\infty}^{-1} \) be a lowest weight vector of the minimal \( K_{\infty} \)-type, let \( w \in V_{\lambda, C} \) be a vector of weight \( \lambda(-k, k', c) \) and let \( v = Jw \in V_{\lambda, C} \).

1. There exists a unique non-zero element

\[
[\phi, v] \in \Hom_{K_{\infty}} \left( \bigwedge_3 g_C/\mathfrak{t}'_C, V_{\lambda, C} \otimes \Pi_{\infty}^W \right)
\]

such that

\[
[\phi, v](X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}) = \sum_{i=0}^{k+k'} (-1)^i X^i_{(1,-1)} v \otimes X^{k+k'+4-i}_{(1,-1)} \phi
\]

where \( X_{(2,0)}, X_{(1,1)} \in p^+ \) and \( X_{(0,-2)} \in p^- \) are the root vectors defined in section 2.3.

2. The map \([\phi, v]\) factors through the canonical projection

\[
\bigwedge_3 g_C/\mathfrak{t}'_C \twoheadrightarrow \bigwedge_2 p^+ \otimes C p^- \twoheadrightarrow \tau_{(3,-1)}
\]

where \( \tau_{(3,-1)} \) is the irreducible sub \( \mathbb{C}[K_{\infty}] \)-module of \( \bigwedge_2 p^+ \otimes C p^- \) generated by the highest weight vector \( X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)} \).

Proof. To prove the existence part of statement (1), note that \( X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)} \) is a highest weight vector of weight \( \lambda'(-k, k', c) \) and hence the vector \( X^i_{(1,-1)} v \) has weight \( \lambda'(-k + i, k' - i, c) \). On the other
hand according to Proposition 2.3 the vector $\phi \in \Pi_{\infty}^W$ has weight $\lambda'(-k' - 1, k + 3, -c)$ and hence the vector $X^{k+k'+4-i}\phi$ has weight $\lambda(k+3-i, -k'-1-i, -c)$. As a consequence, the vector $\sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+4-i}\phi$ has the same weight as $X_{(2,0)} \wedge X_{(1,1)} \otimes X_{(0,-2)}$. Furthermore

$$X_{(1,-1)} \left( \sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+4-i}\phi \right)$$

$$= \sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+4-i}\phi + \sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+5-i}\phi$$

$$= \sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+4-i}\phi + \sum_{i=1}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+5-i}\phi$$

$$= \sum_{i=1}^{k+k'} (-1)^{i-1} X_{(1,1)}^i v \otimes X^{k+k'+5-i}\phi + \sum_{i=1}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+5-i}\phi$$

$$= 0.$$

This means that $\sum_{i=0}^{k+k'} (-1)^i X_{(1,1)}^i v \otimes X^{k+k'+4-i}\phi$ is a highest weight vector. As a consequence the element $[\phi, v]$ of statement (1) of the Lemma exists. Its unicity follows from the fact that $\text{Hom}_{\mathbb{K}_\infty} \left( \wedge^3 \mathfrak{g}_C / \mathfrak{p}'_C, V_{\lambda, \mathbb{C}} \otimes \Pi_{\infty}^W \right)$ has dimension 1. Statement (2) is a direct consequence of the 1-dimensionality of $\text{Hom}_{\mathbb{K}_\infty} \left( \wedge^3 \mathfrak{g}_C / \mathfrak{p}'_C, V_{\lambda, \mathbb{C}} \otimes \Pi_{\infty}^W \right)$ and of the construction of $[\phi, v]$.

Let $\Pi = \Pi_{\infty} \otimes \Pi_f$ be as in the statement of Proposition 2.3. Let $\Pi_f^0$ a model of $\Pi_f$ over the rationality field $E(\Pi_f)$. Assume in addition that $\Pi_f$ has level 1. For any embedding $\sigma : E(\Pi_f) \to \mathbb{C}$, let us define the representation $\sigma \Pi_f = \Pi_f^0 \otimes_\sigma \mathbb{C}$. Let us introduce the $\mathbb{C}$-linear map $\sigma_{\omega, \phi, v}$ defined by

$$\sigma_{\omega, \phi, v} : \sigma \Pi_f^{K(1)} \to L(\Pi_f, V_{\lambda, \mathbb{C}}), \psi \mapsto [\phi, v] \otimes \psi.$$  

**Definition 5.2.** Let $M(\Pi_f, V_{\lambda, \mathbb{R}})$ denote the sub $\mathbb{R}$-vector space of $M(\Pi_f, V_{\lambda, \mathbb{C}})$ of vectors which are fixed under the involution defined on each factor

$$H^3(\mathfrak{g}_C, K_\infty, V_{\lambda, \mathbb{C}} \otimes \Pi_\infty) \otimes \sigma \Pi_f^{K(1)} = \text{Hom}_{\mathbb{K}_\infty} \left( \wedge^3 \mathfrak{g}_C / \mathfrak{p}'_C, V_{\lambda, \mathbb{C}} \otimes \Pi_\infty \otimes \sigma \Pi_f^{K(1)} \right)$$

of the decomposition (19) by $\overline{\mathfrak{g}} : X \mapsto \mathfrak{h}(X)$.

**Remark 5.3.** Note that this action is well defined because for any cusp form $\psi \in \Pi_{\infty} \otimes \sigma \Pi_f^{K(1)}$, the cusp form $\overline{\psi}$ still belongs to $\Pi_{\infty} \otimes \sigma \Pi_f^{K(1)}$ via the identification of $\Pi_{\infty} \otimes \sigma \Pi_f$ with its contragredient via the Petersson inner product.

**Lemma 5.4.** Let $\text{Re} : L(\Pi_f, V_{\lambda, \mathbb{C}}) \to L(\Pi_f, V_{\lambda, \mathbb{R}})$ be the $\mathbb{R}$-linear projection defined as $h \mapsto \frac{1}{2}(h + \mathfrak{h})$. 

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(1) For any φ, v as above, the composition
\[ \bigoplus_{\sigma \in E(\Pi_f) \to \mathbb{C}} \sigma \Pi_f^{K(1)} \to L(\Pi_f, V_{\lambda,\mathbb{C}}) \xrightarrow{\text{Re}} L(\Pi_f, V_{\lambda,\mathbb{R}}), \]

is an isomorphism of \( \mathbb{R} \)-vector spaces of dimension \( 2[E(\Pi_f) : \mathbb{Q}] \) where the first map is \( \bigoplus_{\sigma \in E(\Pi_f) \to \mathbb{C}} \sigma \omega_{\phi,v} \).

(2) For any embedding \( \sigma : E(\Pi_f) \to \mathbb{C} \), let \( \sigma \varphi_\infty \in \sigma \Pi_\infty \) and \( \sigma \varphi_f \in \sigma \Pi_f^{K(1)} \) be vectors such that \( \sigma \varphi = \sigma \varphi_\infty \otimes \sigma \varphi_f \) via the isomorphism \( \text{(1)} \). Let \( \sigma_1, \ldots, \sigma_r \) denote the embeddings of \( E(\Pi_f) \) in \( \mathbb{C} \). Then
\[ (\text{Re} \sigma_1 \omega_{\varphi_\infty,n}(\sigma_f), \ldots, \text{Re} \sigma_r \omega_{\varphi_\infty,n}(\sqrt{-1} \sigma_f)) \]
is a basis of the \( \mathbb{R} \)-vector space \( L(\Pi_f, V_{\lambda,\mathbb{R}}) \).

**Proof.** The second statement is an direct consequence of the first. To prove the first, as the dimensions of the \( \mathbb{R} \)-vector spaces \( \bigoplus_{\sigma \in E(\Pi_f) \to \mathbb{C}} \sigma \Pi_f^{K(1)} \) and \( L(\Pi_f, V_{\lambda,\mathbb{R}}) \) are finite and equal to \( 2[E(\Pi_f) : \mathbb{Q}] \), it is enough to prove the surjectivity of the \( \mathbb{R} \)-linear map of the statement. To this end, let us fix a non-zero vector \( \sigma \psi \in \sigma \Pi_f^{K(1)} \) for any \( \sigma \). The vectors \( \sigma \psi \) and \( \sqrt{-1} \sigma \psi \) form a basis of the \( \mathbb{R} \)-vector space underlying \( \sigma \Pi_f^{K(1)} \). Furthermore by the isomorphism \( \text{(1)} \), Theorem \( \text{4.8} \) Proposition \( \text{4.9} \) and the remark at the beginning of section 5 we have
\[ L(\Pi_f, V_{\lambda,\mathbb{C}}) \cong \bigoplus_{\sigma \in E(\Pi_f) \to \mathbb{C}} \left( \text{Hom}_{K_{\infty}}(\bigwedge^3 gC/\psi, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{1.2}) \right) \otimes \sigma \Pi_f^{K(1)} \]
and the complex conjugation exchanges the subspace \( \text{Hom}_{K_{\infty}}(\bigwedge^3 gC/\psi, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{1.2}) \) and the subspace \( \text{Hom}_{K_{\infty}}(\bigwedge^3 gC/\psi, V_{\lambda,\mathbb{C}} \otimes \Pi_\infty^{1.2}) \). As a consequence, the vectors
\[ \left\{ [\phi, v] \otimes \sigma \psi, \sqrt{-1}[\phi, v] \otimes \sigma \psi, [\phi, v] \otimes \sigma \psi, \sqrt{-1} [\phi, v] \otimes \sigma \psi \right\}_{\sigma \in E(\Pi_f) \to \mathbb{C}} \]
form a basis of the \( \mathbb{R} \)-vector space underlying \( L(\Pi_f, V_{\lambda,\mathbb{C}}) \). This implies that the vectors
\[ \left\{ ([\phi, v] + [\phi, v]) \otimes \sigma \psi, ([\phi, v] + [\phi, v]) \otimes \sqrt{-1} \sigma \psi \right\}_{\sigma \in E(\Pi_f) \to \mathbb{C}} \]
form a basis of the \( \mathbb{R} \)-vector space \( L(\Pi_f, V_{\lambda,\mathbb{R}}) \). For any \( \sigma : E(\Pi_f) \to \mathbb{C} \) the vector \( ([\phi, v] + [\phi, v]) \otimes \sigma \psi \), resp. \( ([\phi, v] + [\phi, v]) \otimes \sqrt{-1} \sigma \psi \), is the image of \( \sigma \psi \), resp. of \( \sqrt{-1} \sigma \psi \), by the the composite map
\[ \bigoplus_{\sigma \in E(\Pi_f) \to \mathbb{C}} \sigma \Pi_f^{K(1)} \to L(\Pi_f, V_{\lambda,\mathbb{C}}) \xrightarrow{\text{Re}} L(\Pi_f, V_{\lambda,\mathbb{R}}) \]
of the statement. This proves the surjectivity of this map as claimed. \( \square \)
Remark 5.5. For any $1 \leq i \leq r$, the vectors $\text{Re}^\sigma \omega_{\varphi_{\infty},v}(\varepsilon_i \varphi_f)$ and $\text{Re}^\sigma \omega_{\varphi_{\infty},v}(\sqrt{-1}^\varepsilon_i \varphi_f)$ do not depend on the choice of $\varphi_{\infty}$ and of $\varphi_f$ and only depend on $\sigma, \varphi$ and $v$.

According to Lemma 4.18, the pairing $[v, \mathfrak{f}]$ is a non-zero complex number. As a consequence we can normalize $v$ in such a way that $[v, \mathfrak{f}] = 1$. Following [38], we can introduce the period of interest in this work. Let us choose a basis $(\delta_1, \ldots, \delta_{2r})$ of the free $\mathbb{Z}(p)$-module $L(\Pi_f, V_\lambda, \mathbb{Z}(p))$. Let us denote by $(\omega_1, \ldots, \omega_{2r})$ the basis of the second point in the previous lemma and let $U \in \text{GL}_{2r}(\mathbb{R})$ be such that $(\delta_1, \ldots, \delta_{2r})U = (\omega_1, \ldots, \omega_{2r})$. Then we define

$$
(32) \quad \Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r})) = \det(U).
$$

Remark 5.6. Under the above normalization of $v$, the vector $v$ is unique up to multiplication by $\pm 1$. Furthermore, if we change the basis $(\delta_1, \ldots, \delta_{2r})$ by another basis $(\delta'_1, \ldots, \delta'_{2r})$ the period $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ is changed by an element of $\mathbb{Z}(p)$. Hence, the image of the real number $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ in $\mathbb{R}^\times / \mathbb{Z}(p)^\times$ is independent of the choice of $v$ normalized as above and of the basis $(\delta_1, \ldots, \delta_{2r})$. In what follows, the image of $\Omega(\Pi_f, \varphi, v, (\delta_1, \ldots, \delta_{2r}))$ in $\mathbb{R}^\times / \mathbb{Z}(p)^\times$ will be denoted by $\Omega(\Pi_f)$.

6. Discriminant and Adjoint $L$-values

In what follows $\Pi$ denotes an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ satisfying the assumptions of section 3. Let $1$ denote the generator of the one-dimensional $\mathbb{C}$-vector space $\Lambda^6 \mathfrak{sp}_4, \mathbb{C} / \mathbb{T}_\mathbb{C}$ defined as

$$
1 = X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,2)} \wedge X_{(-2,0)} \wedge X_{(-1,-1)} \wedge X_{(0,-2)}.
$$

This determines a left translation invariant measure $d\mu$ on $\text{Sp}(4, \mathbb{R}) / K_\infty = G(\mathbb{R})_+ / K'_\infty$ in a standard way. By our normalization of the vectors $X_{(2,0)}, X_{(1,1)}, X_{(0,2)}, X_{(-2,0)}, X_{(-1,-1)}$ and $X_{(0,-2)}$ (see section 2.3), this measure coincides with the standard invariant measure $dXdY / \det(Y)^3$ via the isomorphism $G(\mathbb{R})_+ / K'_\infty \simeq \mathcal{H}_+$. Let $dg_\infty$ be the left invariant measure on $G(\mathbb{R})_+ / \mathbb{R}_+^\times$ attached to $d\mu$ by the construction (2). For every prime number $p$, let $dg_p$ be the unique translation invariant measure on $G(\mathbb{Q}_p)$ such that $\text{vol}(G(\mathbb{Z}_p), dg_p) = 1$ and let $dg$ be the measure on $Z(\mathbb{A}) \backslash G(\mathbb{A})$ defined by $dg = \prod_v dg_v$.

Proposition 6.1. The pairing obtained from (31) after extending coefficients from $\mathbb{Z}(p)$ to $\mathbb{C}$ is given by

$$
(33) \quad \left( \bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} \bigoplus_{\Pi_\infty \in \mathcal{P}(V_{\lambda, \mathbb{C}})} H^3(\mathfrak{g}_\mathbb{C}, K'_{\infty}(\mathbb{C}) \otimes \Pi_{\infty}) \oplus \mathbb{C}^m(\Pi_{\infty} \otimes \Pi_f) \otimes \mathbb{C}^\sigma \Pi_f^{K(1)} \right) \wedge \left( \bigoplus_{\sigma : E(\Pi_f) \to \mathbb{C}} \bigoplus_{\Pi_\infty \in \mathcal{P}(V_{\lambda, \mathbb{C}})} H^3(\mathfrak{g}_\mathbb{C}, K'_{\infty}(\mathbb{C}) \otimes \Pi_{\infty}) \oplus \mathbb{C}^m(\Pi_{\infty} \otimes \Pi_f) \otimes \mathbb{C}^\sigma \Pi_f^{K(1)} \right) \to \mathbb{C}
$$

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where (33) is induced by the composite

\[
\left( \text{Hom}_{K'_\infty} \left( \bigwedge^3 g_{C'/k'^*}, V_{\lambda,C} \otimes \Pi_\infty \right) \otimes \sigma^{\Pi_f^{(1)}_{\infty}} \right) \otimes \left( \text{Hom}_{K'_\infty} \left( \bigwedge^6 g_{C'/k'^*}, V_{\lambda,C} \otimes \Pi' \otimes \Pi'_\infty \right) \otimes \sigma' \Pi_f^{(1)} \right)
\]

\[
\rightarrow \text{Hom}_{K'_\infty} \left( \bigwedge^6 g_{C'/k'^*}, V_{\lambda,C} \otimes \Pi' \otimes \Pi'_\infty \right) \otimes \sigma' \Pi_f^{(1)} \rightarrow \mathbb{C}
\]

where the first map is the exterior product, the second map is induced by \([,]\) and the third is the composition of the evaluation at 1 followed by \(\int_{R^*_G(A)} dg\).

Proof. This follows from the compatibility of the Poincaré duality with the comparison isomorphism (19) between Betti and de Rham cohomology and the description given in [Bo81 (5)] of the Poincaré duality in de Rham cohomology in terms of \((g_C, K'_\infty)\)-cohomology. □

For any \(Z_{(p)}\)-algebra \(A\), let us denote by

\[
(,)_A : L(\Pi_f, V_{\lambda,A}) \times L(\Pi_f, V_{\lambda,A}) \rightarrow A
\]

the pairing deduced from (30) after extending scalars to \(A\). Then, we have the commutative diagram

\[
\begin{array}{ccc}
L(\Pi_f, V_{\lambda,R}) \times L(\Pi_f, V_{\lambda,R}) & \xrightarrow{(,)_R} & \mathbb{R} \\
\downarrow & & \downarrow \\
L(\Pi_f, V_{\lambda,C}) \times L(\Pi_f, V_{\lambda,C}) & \xrightarrow{(,)_C} & \mathbb{C}
\end{array}
\]

(34)

where the vertical arrows are the natural inclusions.

Lemma 6.2. There exists \(\epsilon_\lambda \in \mathbb{Z}\) such that for any \(v, w \in L(\Pi_f, V_{\lambda,C})\) we have

\[
\langle w, v \rangle_C = (-1)^{\epsilon_\lambda} \langle v, w \rangle_C.
\]

Proof. In the statement of Proposition 6.1, the first map in the sequence defining (33) is alternate as it is defined as the exterior product of differential forms of degree 3. The second map is induced by the \(G_C\)-invariant bilinear form \([,]\) on \(V_{\lambda,C} \otimes V_{\lambda,C}\). As \(V_{\lambda,C}\) is irreducible, it is an easy consequence of Schur Lemma that such a bilinear form is unique up to a scalar. Let \([,]_C\) be the bilinear form on \(V_{\lambda,C} \otimes V_{\lambda,C}\) defined by \([v, w]_C = [w, v]_C\). There exists \(\lambda \in \mathbb{C}\) such that \([v, w]_C = \lambda [v, w]_C\). Let \(v, w \in V_{\lambda,C}\) such that \([v, w]_C \neq 0\). Then

\[
[v, w]_C = \lambda [v, w]'_C = \lambda [v, w]_C = \lambda^2 [v, w]'_C = \lambda^2 [v, w]_C.
\]
Hence $\lambda = \pm 1$ and $[\cdot, \cdot]_C$ is either symmetric or alternate. The conclusion now follows from the fact that the last map in the sequence defining (33) is symmetric. \hfill $\square$

**Lemma 6.3.** As measures on $Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})$, we have

$$dg = \frac{\pi^3}{270}dg^{Tam}.$$ 

**Proof.** By Weil conjecture on Tamagawa numbers proved by Kottwitz (see [Ko88]), we have

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} dg^{Tam} = 1.$$ 

On the other hand, as $\text{vol}(K_\infty, dg) = \text{vol}(G(\mathbb{Z}_p), dg) = 1$ we have

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} dg = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash K_\infty G(\mathbb{Z})} dg.$$ 

It follows from equation (11) and the definition of $dg$ that there is an isomorphism of measured spaces

$$(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty G(\mathbb{Z}), dg) \simeq (\text{PSp}(4,\mathbb{Z})\backslash \mathcal{H}_+, dXdY/\text{det}(Y)^3)$$

and in particular

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty G(\mathbb{Z})} dg = \int_{\text{PSp}(4,\mathbb{Z})\backslash \mathcal{H}_+} dXdY/\text{det}(Y)^3.$$ 

Let $\xi(s)$ denotes the complete Riemann zeta function $\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. According to [Sie43, Theorem 11] the last displayed integral is equal to $2\xi(2)\xi(4)$. The conclusion now follows from the well known equalities $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. \hfill $\square$

**Proposition 6.4.** Let $\sigma : E(\Pi_f) \to \mathbb{C}$ be an embedding, let $\sigma \varphi_f \in \sigma \Pi_f$ be the normalized vector, let $\omega_{\rho_{\infty, \varphi_f}}$ be the normalized vector, let $\varphi_f \in \sigma \Pi_f$ be the normalized vector, let $\omega_{\rho_{\infty, \varphi_f}}(\sigma \varphi_f) \in L(\Pi_f, V_{\lambda, \mathbb{C}})$ be defined by (31). For any $0 \leq i \leq k \pm 4$, any $0 \leq u \leq r \leq 4$, any $0 \leq u \leq r \leq 4$ such that $i - u \geq 0$, let us denote

$$r_{i,u,r}^{k,k'} = \frac{(k + k' + u - i)!(k + k' + 4 - i)!(i + r - u)!}{(i - u)!(k + k' - i)!(k + k' + 4 - i - r + u)!}, \quad (i - u)!$$

$$s_{i,u}^{k,k'} = \frac{(k + k' - u)!}{(i - u)!(i + r - u)!}, \quad (i + r - u)!$$

$$t_{i,u,r}^{k,k'} = \frac{(k + k' + 4 + u - r - i)!}{(i + r - u)!}, \quad (i + r - u)!$$

let us denote $a_0 = -1, a_1 = -\frac{1}{4}, a_2 = \frac{1}{32}, a_3 = -\frac{1}{72}, a_4 = -\frac{1}{576}$, let $C_{k,k'} \in \mathbb{Q}$ be defined as

$$C_{k,k'} = \frac{(-1)^{k+k'}(k + k')!(k + k' + 4)!}{3!^2.135} \sum_{r=0}^{k+k'} \sum_{i=0}^{k+k'} \sum_{0 \leq i' \leq r} (-1)^{r+u+u'} a_r \binom{r}{u} \binom{r}{u'} t_{i,u}^{k,k'} s_{i,u}^{k,k'} r_{i,u,r}^{k,k'}.$$ 

Then

$$\langle \omega_{\rho_{\infty, \varphi_f}}(\sigma \varphi_f), \omega_{\rho_{\infty, \varphi_f}}(\sigma \varphi_f) \rangle_C = \pi^3 C_{k,k'} C_{\infty}(\sigma \varphi, \sigma \varphi)$$

where $C_{\infty} \in \mathbb{C}^\times$ is the constant appearing in the statement of Theorem 3.1.
Proof. We need to calculate the image of the vector $\sigma\omega_{\varphi, v}(\sigma\varphi_f) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)$ by the sequence of maps defined (33). Let us introduce the following notation: $e_1 = X(2,0), e_2 = X(1,1), e_3 = X(0,2), e_4 = X(-2,0), e_5 = X(-1,-1), e_6 = X(0,2)$. Then, by definition of the exterior product, we have

\[
\left(\sigma\omega_{\varphi, v}(\sigma\varphi_f) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)\right)(1)
\]

\[
= \frac{1}{3!^2} \sum_{\sigma \in S_6} \epsilon(\sigma)\sigma\omega_{\varphi, v}(\sigma\varphi_f)(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)} \otimes e_{\sigma(4)} \otimes e_{\sigma(5)} \otimes e_{\sigma(6)})
\]

\[
= \frac{1}{3!^2} \sum_{\sigma \in S_6} \epsilon(\sigma)\sigma\omega_{\varphi, v}(\sigma\varphi_f)(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(e_{\sigma(4)} \otimes e_{\sigma(5)} \otimes e_{\sigma(6)})
\]

According to the second statement of Lemma [5.1] we have $\sigma\omega_{\varphi, v}(\sigma\varphi_f)(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = 0$ whenever the image of $e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ by the projection

\[
\bigotimes_3 g_C / t_C' \to \bigwedge_3 g_C / t_C' \to \bigwedge_2 p^+ \otimes_C p^- \to \tau_{(3,-1)}
\]

is zero. The weight vectors

\[
X(2,0) \land X(1,1) \land X(0,-2), X(2,0) \land X(1,1) \land X(-1,-1), X(2,0) \land X(1,1) \land X(-2,0),
\]

\[
X(2,0) \land X(0,2) \land X(0,-2), X(2,0) \land X(0,2) \land X(-1,-1), X(2,0) \land X(0,2) \land X(-2,0),
\]

\[
X(1,1) \land X(0,2) \land X(0,-2), X(1,1) \land X(0,2) \land X(-1,-1), X(1,1) \land X(0,2) \land X(-2,0)
\]

form a basis of $\bigwedge_2 p^+ \otimes_C p^-$. Computing the signatures, we have

\[
\left(\sigma\omega_{\varphi, v}(\sigma\varphi_f) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)\right)(1)
\]

\[
= \frac{1}{3!^2} \left(\sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(1,1) \land X(0,-2)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(0,2) \land X(-1,-1) \land X(-2,0))
\right.
\]

\[
- \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(1,1) \land X(-1,-1)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(0,2) \land X(0,-2) \land X(-2,0))
\]

\[
+ \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(1,1) \land X(-2,0)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(0,2) \land X(0,-2) \land X(-1,-1))
\]

\[
- \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(0,2) \land X(0,-2)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(1,1) \land X(1,1) \land X(-2,0))
\]

\[
- \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(0,2) \land X(-1,-1)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(1,1) \land X(0,2) \land X(-2,0))
\]

\[
+ \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(1,1) \land X(0,2) \land X(0,-2)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(1,1) \land X(-2,0))
\]

\[
- \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(1,1) \land X(0,2) \land X(-1,-1)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(0,2) \land X(-2,0))
\]

\[
+ \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(1,1) \land X(2,0) \land X(-1,-1)) \otimes \sigma\omega_{\varphi, v}(\sigma\varphi_f)(X(2,0) \land X(0,2) \land X(-2,0))
\]
As a consequence, using that $p^+ \otimes_C \Lambda^2 p^- = \Lambda^2 p^+ \otimes_C p^-$ and $X_{(r,s)} = X_{(-r,-s)}$, we have

$$
\left(\omega_{\psi,0} \wedge \omega_{\psi,0} \right)(1)
= \frac{1}{3!} \left( -\omega_{\psi,0} \wedge \omega_{\psi,0} + \omega_{\psi,0} \wedge \omega_{\psi,0} \right)(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}) + \omega_{\psi,0} \wedge \omega_{\psi,0} + \omega_{\psi,0} \wedge \omega_{\psi,0} \right)(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)})
$$

A standard basis of $\tau_{(3,1)}$ is computed in Lemma 2.2 and the matrix of the projection $p : \Lambda^2 p^+ \otimes_C p^- \rightarrow \tau_{(3,1)}$ is computed in Lemma 2.3. From these two results we deduce the following equalities which give the image by $p$ of the basis vectors in terms of the highest weight vector $X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}$ of $\tau_{(3,1)}$:

$$
p(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}) = X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)},
$$

$$
p(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(1,-1)}) = -\frac{1}{2} \text{Ad}_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(-2,0)}) = \frac{1}{12} \text{Ad}_{X_{(-1,1)}}^2(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(-2,0)}) = -\frac{1}{4} \text{Ad}_{X_{(-1,1)}}(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(2,0)} \wedge X_{(0,2)} \wedge X_{(0,-2)}) = -\frac{1}{6} \text{Ad}_{X_{(-1,1)}}^2(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(2,0)} \wedge X_{(0,2)} \wedge X_{(1,-1)}) = -\frac{1}{24} \text{Ad}_{X_{(-1,1)}}^3(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(1,1)} \wedge X_{(0,2)} \wedge X_{(0,-2)}) = \frac{1}{24} \text{Ad}_{X_{(-1,1)}}^3(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(1,1)} \wedge X_{(0,2)} \wedge X_{(-1,1)}) = -\frac{1}{12} \text{Ad}_{X_{(-1,1)}}^3(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}),
$$

$$
p(X_{(1,1)} \wedge X_{(0,2)} \wedge X_{(-2,0)}) = \frac{1}{24} \text{Ad}_{X_{(-1,1)}}^4(X_{(2,0)} \wedge X_{(1,1)} \wedge X_{(0,-2)}).
$$

Using these equalities, we find

$$
\left(\omega_{\psi,0} \wedge \omega_{\psi,0} \right)(1)
$$
We have

\[ \text{Ad}_{X^{(1-1)}}(X^{(1-1)} v \otimes X^{k+k'-4-i \sigma}) = \sum_{u=0}^{r} \binom{r}{u} \sum_{i \leq j} (r'_{i,u}) \left( X^{j-u} \otimes X^{k+k'-4-i-(r-u) \sigma} \right), \]

where we use the convention that \( X^{i-u} \otimes X^{(1-1)} = 0 \) if \( i < u < 0 \), and similarly

\[ \text{Ad}_{X^{(1-1)}}(X^{(1-1)} \otimes X^{k+k'+4-j \sigma}) = \sum_{u=0}^{r} \binom{r}{u} \sum_{i \leq j} (r'_{j,u}) \left( X^{i-u} \otimes X^{k+k'+4-j-(r-u) \sigma} \right), \]

with the convention that \( X^{j-u} \otimes X^{(1-1)} = 0 \) if \( j < u < 0 \). Then we have

\[ (\omega_{\varphi_{\infty}, v}(\sigma \varphi_f) \wedge \sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f))(1) = \frac{1}{3!} \sum_{r=0}^{4} \sum_{i \leq j} (r'_{i,u}) \left( X^{i-u} \otimes X^{k+k'-4-i-(r-u) \sigma} \right) \otimes \left( X^{j-u} \otimes X^{k+k'+4-j-(r-u) \sigma} \right). \]

Let us denote by \( \left[ (\sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f) \wedge \sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f))(1) \right] \) the element of \( \sigma \Pi \otimes \sigma \Pi \) defined as the image of \( (\sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f) \wedge \sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f))(1) \) by the map induced by the pairing \( [\ , \ ]_C \). Then, thanks to Lemma 4.19 we have

\[ \left[ (\sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f) \wedge \sigma \omega_{\varphi_{\infty}, v}(\sigma \varphi_f))(1) \right] = \frac{(k+k')!}{3!} \sum_{r=0}^{4} \sum_{i \leq j} (r'_{i,u}) \left( X^{i-u} \otimes X^{k+k'-4-i-(r-u) \sigma} \right) \otimes \left( X^{j-u} \otimes X^{k+k'+4-j-(r-u) \sigma} \right). \]

As the pairing \( \sigma \Pi \otimes \sigma \Pi \rightarrow \mathbb{C} \) defined as

\[ \varphi \otimes \psi \mapsto \int_{\mathbb{R}_+^n \backslash G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \psi(g) dg \]

Now note that as $\sigma$ is invariant by $G(\hat{\mathbb{Z}})$ and \text{vol}(G(\hat{\mathbb{Z}}),dg) = 1$ we have
\[
\int_{R_+^* G(\mathbb{Q}) \backslash G(\mathbb{A})} |^\sigma \varphi(g)|^2 dg = \int_{R_+^* G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})} |^\sigma \varphi(g)|^2 dg = 2 \int_{Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}})} |^\sigma \varphi(g)|^2 dg = \frac{\pi^3}{135} \langle ^\sigma \varphi, ^\sigma \varphi \rangle.
\]
where the last equality follows from Lemma [6.3] and the definition of $\langle ^\sigma \varphi, ^\sigma \varphi \rangle$. \hfill \Box

Recall that we have fixed a basis $(\delta_1, \ldots, \delta_{2r})$ of $L(\Pi_f, V_{\lambda, \mathbb{Z}(p)})$.

**Definition 6.5.** The discriminant $d(\Pi_f)$ of the pairing (30) is defined as follows:
\[
d(\Pi_f) = \det((\delta_i, \delta_j)_{\mathbb{Z}(p)})_{1 \leq i, j \leq 2r}.
\]
It is an element of $\mathbb{Z}(p)$ whose image in $\mathbb{Z}(p)/(\mathbb{Z}(p)^\times)^2$ is independent of the choice of the $\mathbb{Z}(p)$-basis $(\delta_i)_{1 \leq i \leq 2r}$ of $L(\Pi_f, V_{\lambda, \mathbb{Z}(p)})$.

**Definition 6.6.** For $x, y \in \mathbb{R}$ we write $x \sim y$ if there exists $s \in (\mathbb{Z}(p)^\times)^2$ such that $x = sy$.

**Theorem 6.7.** Assume that $p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup S_{\text{tors}} \cup S'_{\text{tors}}$ where $S_{G(\mathbb{Z}/3\mathbb{Z})}$, $S_{\text{tors}}$, $S'_{\text{tors}}$, $S_{\text{weight}}$ are defined by (15), (16), (21) and (22) respectively. Then
\[
d(\Pi_f) \sim \left( \frac{C_k k! C_\infty \pi^3}{2} \right)^r \Omega(\Pi_f)^{-1} \prod_{E(\Pi_f) \to C} L(1, ^\sigma \Pi, Ad) \right)^2
\]
\[
\text{Proof.} \text{ Recall that } (\omega_1, \ldots, \omega_{2r}) \text{ denotes the } \mathbb{R}-\text{basis of } L(\Pi_f, V_{\lambda, \mathbb{R}}) \text{ normalized in the second point of Lemma [5.4]. Let } T \text{ denote the matrix } T = (\langle \delta_i, \delta_j \rangle_{\mathbb{Z}(p)})_{1 \leq i, j \leq 2r} \text{ and let } S \text{ denote the matrix } S = ((\omega_i, \omega_j)_{\mathbb{R}})_{1 \leq i, j \leq 2r}. \text{ We have }
\[
d(\Pi_f) \sim \det T \sim \Omega(\Pi_f)^{-2} \det S
\]
where the first equality is the definition of the discriminant, and the second from the definition of \( \Omega(\Pi_f) \) (see (22)). Hence we have to compute the pairings \( \langle \omega_i, \omega_j \rangle_R \). By commutativity of the diagram (34) we have \( \langle \omega_i, \omega_j \rangle_R = \langle \omega_i, \omega_j \rangle_C \). Proposition 2.6 implies that \( \langle \omega_i, \omega_j \rangle_C = 0 \) for any \( 1 \leq i, j \leq 2r \) such that \( j \neq i, j \neq i+r \) and \( i \neq j+r \). Furthermore, as the Poincaré duality pairing is a morphism of Hodge structures, for any \( 1 \leq i \leq r \) we have

\[
\langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C = \frac{\langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C}{\langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C} = 0.
\]

As a consequence, for \( 1 \leq i \leq r \) we have

\[
\begin{align*}
\langle \omega_i, \omega_i \rangle_C &= \begin{cases} 
2^{-1} \langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C & \text{if } \epsilon_\lambda \text{ is even}, \\
0 & \text{if } \epsilon_\lambda \text{ is odd}, 
\end{cases} \\
\langle \omega_i, \omega_{i+r} \rangle_C &= \begin{cases} 
-2^{-1} \sqrt{-1} \langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C & \text{if } \epsilon_\lambda \text{ is even}, \\
0 & \text{if } \epsilon_\lambda \text{ is odd}, 
\end{cases} \\
\langle \omega_{i+r}, \omega_{i+r} \rangle_C &= \begin{cases} 
2^{-1} \langle \sigma_i r_\infty(\sigma_i \varphi_f), \sigma_i r_\infty(\sigma_i \varphi_f) \rangle_C & \text{if } \epsilon_\lambda \text{ is even}, \\
0 & \text{if } \epsilon_\lambda \text{ is odd}.
\end{cases}
\end{align*}
\]

The statement now follows from Proposition 6.4.

7. The congruence criterion

Let \( \Pi' \simeq \bigotimes_{\nu} \Pi'_\nu \) a cuspidal representation of \( G(\mathbb{A}) \) which is unramified at every prime \( l \) and such that \( \Pi'_\infty \in P(Vh, C) \). Let \( E(\Pi'_f) \) be the rationality field of \( \Pi'_f \) and let \( \theta_{\Pi'} : \overline{H}^{K(1)} \to E(\Pi'_f) \) be the character such that for any \( g \in \Pi'^{K(1)}_f \) and any \( h \in H^{K(1)}_f \) we have \( hg = \theta_{\Pi'}(h)g \). Let \( O_{E(\Pi'_f)} \) be the ring of integers of \( E(\Pi'_f) \). According to Corollary 4.7 and to the fact that \( \overline{H}^{K(1)}_f \) is a \( \mathbb{Z} \)-module of finite type we have \( \text{Im} \theta_{\Pi'} \subset O_{E(\Pi'_f)} \).

**Definition 7.1.** Let \( \mathfrak{P} \) be a prime ideal of \( \overline{\mathbb{Q}} \). The cuspidal representation \( \Pi' \) is congruent to \( \Pi \) modulo \( \mathfrak{P} \) if there exists a number field \( E \) containing \( E(\Pi'_f) \) and \( E(\Pi_f) \), with ring of integers \( O_E \) such that the following diagram commutes

\[
\begin{array}{ccc}
O_{E(\Pi'_f)} & \xrightarrow{\theta_{\Pi}} & O_{E(\Pi'_f)} \\
\downarrow{\overline{H}^{K(1)}_f} & & \downarrow{\kappa} \\
O_{E(\Pi_f)} & \xrightarrow{\kappa} & O_E
\end{array}
\]

where \( \kappa = O_E/\mathfrak{P} \cap O_E \). In this case we write \( \Pi' \equiv \Pi \pmod{\mathfrak{P}} \).

Recall that \( \Pi \) is a cuspidal representation of \( G(\mathbb{A}) \), which, among other things is assumed to be globally generic and endoscopic. This means that the functorial lift \( \Sigma \) of \( \Pi \) to \( \text{GL}(4, \mathbb{A}) \) is not cuspidal. In particular, according to [AS06 Proposition 2.2], there exist \( \sigma_1 \) and \( \sigma_2 \) two inequivalent unitary cuspidal automorphic representations of \( \text{GL}(2, \mathbb{A}) \) with the same central characters such that \( \Sigma \) is the isobaric sum \( \Sigma = \sigma_1 \boxplus \sigma_2 \) and such that \( \Pi \) is obtained as
respectively. Assume the following conditions

\[(24)\]

Let \( \lambda, c \) denote the real number defined as

\[(35)\]

where \( r_1 \) denotes the degree of the number field \( E_1 \) generated by the Fourier coefficients of \( f_1 \), where \( Z(k_1, f_1) \) denotes the product

\[Z(k_1, f_1) = \prod_{\sigma:E_1 \to \mathbb{C}} L(k_1, \text{Sym}^2(\sigma f_1))\]

and where \( u(f_1) \) denotes the period defined in [Hi81, (6.6.c)]. It follows from loc. cit. Theorem 6.2 that \( c(f_1) \in \mathbb{Z}\setminus\{0\} \). We say that the cuspidal automorphic Siegel representation attached to a holomorphic Siegel modular form \( F \) of level 1 is stable if \( F \) is not a Saito-Kurokawa lift and its functorial lift to \( \text{GL}(4, \mathbb{A}) \) (see [PSS14] for its construction) is cuspidal.

**Theorem 7.2.** Let \( p \) be a prime such that \( p \notin S_{G(\mathbb{Z}/3\mathbb{Z})} \cup S_{\text{weight}} \cup S_{\text{tors}} \cup S'_{\text{tors}} \cup S''_{\text{tors}} \) where \( S_{G(\mathbb{Z}/3\mathbb{Z})}, S_{\text{tors}}, S'_{\text{tors}}, S_{\text{weight}} \) and \( S''_{\text{tors}} \) are defined by \( (15), (16), (21), (22) \) and \( (23) \) respectively. Assume the following conditions

(a) the residual \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-representations \( \overline{\rho}_{f_1} \) and \( \overline{\rho}_{f_2} \) of \( f_1 \) and \( f_2 \) are irreducible,

(b) the prime \( p \) does not divide \( c(f_1) \),

(c) the prime \( p \) divides

\[\left( \frac{C_{k,k'}C_{\infty}k^3}{2} \right)^r \Omega(\Pi_f)^{-1} \prod_{\sigma:E(\Pi_f) \to \mathbb{C}} L(1, \sigma \Pi, \text{Ad}) \right)^2 \]

Then, there exists a prime divisor \( \mathfrak{P} \) of \( p \) in \( \overline{\mathbb{Q}} \) and a cuspidal representation \( \Pi' \simeq \bigotimes_{v} \Pi'_v \) of \( G(\mathbb{A}) \) such that

(1) the smooth admissible representation \( \Pi'_v \) is unramified for any prime \( l \),

(2) we have \( \Pi'_{\infty} \in P(V_{\lambda, C}) \),

(3) the cuspidal representation \( \Pi' \) is stable,

(4) we have \( \Pi' \not\equiv \sigma \Pi \) for all \( \sigma \in \text{Aut}(\mathbb{C}) \),

(5) we have \( \Pi' \equiv \Pi \pmod{\mathfrak{P}} \).

For the proof of this theorem, we need the following result.

**Lemma 7.3.** Let \( (\ , \ )_{\mathbb{Q}} \) denote the bilinear form deduced from \( (30) \) after base change from \( \mathbb{Z}(p) \) to \( \mathbb{Q} \). Let

\[L(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) = \{ (x, y) \mid \forall y \in L(\Pi_f, V_{\lambda, \mathbb{Z}(p)}), (x, y)_{\mathbb{Q}} \in \mathbb{Z}(p) \} \]

be the lattice dual of \( L(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) \). Then as lattices in \( L(\Pi_f, V_{\lambda, \mathbb{Q}}) \), we have

\[M(\Pi_f, V_{\lambda, \mathbb{Z}(p)}) = L(\Pi_f, V_{\lambda, \mathbb{Z}(p)})^* \].
Proof. We would like to apply [H181 (4.6)] so we verify the assumptions of this result. According to Corollary 4.15, the $\mathbb{Z}_{(p)}$-lattice $L = H^3_f(S_3, V_{\lambda, \mathbb{Z}_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})}$ of the $\mathbb{Q}$-vector space $V = H^3_f(S_3, V_{\lambda, \mathbb{Q}})^{G(\mathbb{Z}/3\mathbb{Z})}$ is self-dual for the bilinear form deduced from (27) by base change from $\mathbb{Z}_{(p)}$ to $\mathbb{Q}$, that we also denote by $\langle \cdot, \cdot \rangle_{\mathbb{Q}}$. Let $W_1$ denote the subspace $W_1 = M(\Pi_f, V_{\lambda, \mathbb{Q}})$ of $V$ and let $W_2$ denote the kernel of the projection $V \to W_1$. Then $V = W_1 \oplus W_2$. Note that $\mathcal{H}_Q^{K(1)}$ is a product of fields according to Lemma 4.5. There exists $e \in \mathcal{H}_Q^{K(1)}$ an idempotent such that $W_1 = eV$. Then $W_2 = (1 - e)V$. We claim that $W_1$ is orthogonal to $W_2$ for $\langle \cdot, \cdot \rangle$. In order to prove this, it is enough to prove that $W_1 \otimes_{\mathbb{Q}} \mathbb{C}$ and $W_2 \otimes_{\mathbb{Q}} \mathbb{C}$ are orthogonal for $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. For any $v, w \in V \otimes_{\mathbb{Q}} \mathbb{C}$, we have $\langle ev, (1 - e)w \rangle_{\mathbb{C}} = \langle v, e(1 - e)w \rangle_{\mathbb{C}} = 0$ where the first equality follows from Proposition 2.9 and Proposition 6.1. This proves the claim. By definition, we have $L(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}}) = L \cap W_1$ and $M(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}})$ is the projection of $L$ on $W_1$ along $W_2$. Hence the statement of the lemma follows from [H181 (4.6)].

Proof of Theorem 7.2. Recall that, for any $\mathbb{Z}_{(p)}$-algebra $R$, we denote by $\mathcal{H}_R^{K(1)}$ the algebra $\mathcal{H}_R^{K(1)} \otimes_{\mathbb{Z}_{(p)}} R$ where $\mathcal{H}_R^{K(1)}$ denotes the image of the abstract Hecke algebra $\mathcal{H}_{\mathbb{Z}_{(p)}}^{K(1)}$ with $\mathbb{Z}_{(p)}$-coefficients in $\text{End}_{\mathbb{Z}_{(p)}}(H^3_f(S_3, V_{\lambda, \mathbb{Z}_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})})$. It follows from Lemma 4.5 that the $R$-algebra $\mathcal{H}_R^{K(1)}$ is a product of fields. Hence, any finitely generated $\mathcal{H}_R^{K(1)}$-module decomposes as a direct sum of simple $\mathcal{H}_R^{K(1)}$-modules. So there exists a sub $\mathcal{H}_R^{K(1)}$-module $Y$ of $H^3_f(S_3, V_{\lambda, \mathbb{R}})^{G(\mathbb{Z}/3\mathbb{Z})}$ such that $L(\Pi_f, V_{\lambda, \mathbb{R}}) \oplus Y = H^3_f(S_3, V_{\lambda, \mathbb{R}})^{G(\mathbb{Z}/3\mathbb{Z})}$ as $\mathcal{H}_R^{K(1)}$-modules. Define the finitely generated $\mathbb{Z}_{(p)}$-module $M_Y$ by the equation $M_Y = pY(H^3_f(S_3, V_{\lambda, \mathbb{Z}_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})})$ where $pY : H^3_f(S_3, V_{\lambda, \mathbb{R}})^{G(\mathbb{Z}/3\mathbb{Z})} \to Y$ is the canonical projection. The finitely generated $\mathbb{Z}_{(p)}$-module $L_Y$ is defined as $L_Y = Y \cap H^3_f(S_3, V_{\lambda, \mathbb{Z}_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})}$.

These $\mathbb{Z}_{(p)}$-modules are stable by the action of $\mathcal{H}_{\mathbb{Z}_{(p)}}^{K(1)}$ on $H^3_f(S_3, V_{\lambda, \mathbb{R}})^{G(\mathbb{Z}/3\mathbb{Z})}$ and we have $L_Y \subset M_Y$. According to Theorem 6.7, the prime $p$ divides $d(\Pi_f)$. It follows from [H181 Proposition 4.3] that $|d(\Pi_f)| = |L(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}})^* : L(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}})|$. Hence, it follows from Lemma 7.3 that $p$ divides the index $[M(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}}) : L(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}})]$. By replacing in the proof of [H181 Theorem 7.1] the symbols $L, L_f, M_f$ and $R$ by the symbols $H^3_f(S_3, V_{\lambda, \mathbb{Z}_{(p)}})^{G(\mathbb{Z}/3\mathbb{Z})}, L(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}}), M(\Pi_f, V_{\lambda, \mathbb{Z}_{(p)}})$ and $\mathcal{H}_{\mathbb{Z}_{(p)}}^{K(1)}$ respectively, we obtain a prime divisor $\mathfrak{p}$ of $p$ in $\mathfrak{p}$ and a $\mathcal{H}_{\mathbb{Z}_{(p)}}^{K(1)}$-module contributing to $Y$ which is congruent to $\Pi$ modulo $\mathfrak{p}$. This amounts to the existence of a cuspidal automorphic representation $\Pi' \sim \bigotimes_{\sigma} \Pi_{\sigma}$ such that $\Pi_{\sigma}$ is unramified for every prime $l$, such that $\Pi_{\sigma} \not\cong \sigma \Pi$ for all $\sigma \in \text{Aut}(\mathbb{C})$ and such that $\Pi' \equiv \Pi \pmod{\mathfrak{p}}$.

In the rest of the proof, we show that $\Pi'$ has to be stable under the conditions (a) and (b). In order to show this with contradiction, let us assume that $\Pi'$ is not stable. According to the classification of the cohomological discrete spectrum of PGSp(4) summarized in [Pe54 §4], this means that $\Pi'$ is either of Type 4 (Yoshida lift) or of Type 5 (Saito-Kurokawa lift). As explained above the statement of Theorem 7.2, the modular forms $f_1$ and $f_2$ have level 1. Hence $f_1$ and $f_2$ have even weights, which implies that $k$ and $k'$ are even integers. Now, in the case of Type 5, any Saito-Kurokawa lift with even $k, k'$ does not contribute to
the degree 3 cohomology $H^3_!\left(S_3, V_{\lambda, \mathbb{C}}\right)^{G(\mathbb{Z}/3\mathbb{Z})}$ by the discussion given in [Pe15, p. 56]. This means that $\Pi'$ is not of Type 5. Now, let us assume that $\Pi'$ is of Type 4. By the discussion in [Pe15, p. 55] and by the fact that $\Pi'$ is of level 1, the archimedean component of $\Pi'$ can not be holomorphic and hence it is globally generic. Then $\Pi'$ is obtained as a Weil lifting from $(\sigma'_1, \sigma'_2)$ which correspond to normalized eigen elliptic cuspforms $f'_1$ and $f'_2$ of weight $k'_1 = k + k' + 4$ and $k'_2 = k - k' + 2$ respectively by [We09, Corollary 4.2]. Since we have the congruence $\Pi' \equiv \Pi \pmod{\mathfrak{P}}$, we have

$$\rho_{\Pi} \simeq \rho_{\Pi'} \pmod{\mathfrak{P}}$$

where $\rho_{\Pi}$ and $\rho_{\Pi'}$ are the mod $\mathfrak{P}$ semi-simple Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to $\Pi$ and $\Pi'$. On the other hand, by the assumption that $\rho_{f_1}$ and $\rho_{f_2}$ are irreducible, we have

$$\rho_{\Pi} \simeq \rho_{f_1} \oplus \rho_{f_2}(-1-k'), \quad \rho_{\Pi'} \simeq \rho_{f'_1} \oplus \rho_{f'_2}(-1-k').$$

Since $f_1$ and $f'_2$ are of level 1, the representations restricted to the inertia subgroup $I_p$ are given as

$$\rho_{f_1}|_{I_p} \simeq 1 \oplus 1(1-k_1),$$

By the same reason, we have

$$\rho_{f_2}|_{I_p} \simeq 1 \oplus 1(1-k_2).$$

As $\rho_{f_1}$ and $\rho_{f_2}$ are irreducible, equations (36) and (37) imply either $\rho_{f_1} \simeq \rho_{f'_1}$ or $\rho_{f_1} \simeq \rho_{f'_2}(-1-k')$. The first case is excluded because $p$ does not divide $c(f_1)$ according to [Hi81, Theorem B]. The fact that $p \notin S_{\text{weight}}$ implies that $\rho_{f'_2}(-1-k')$ does not contain the trivial character and hence $\rho_{f_1}|_{I_p} \not\simeq \rho_{f'_2}(-1-k')|_{I_p}$ by (38) and (39). This is a contradiction. Hence $\Pi'$ is stable.

\[\square\]

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