PLEIJEL’S THEOREM FOR SCHRÖDINGER OPERATORS

PHILIPPE CHARRON AND CORENTIN LÉNA

ABSTRACT. We are concerned in this paper with the real eigenfunctions of Schrödinger operators. We prove an asymptotic upper bound for the number of their nodal domains, which implies in particular that the inequality stated in Courant’s theorem is strict, except for finitely many eigenvalues. Results of this type originated in 1956 with Pleijel’s Theorem on the Dirichlet Laplacian and were obtained for some classes of Schrödinger operators by the first author, alone and in collaboration with B. Helffer and T. Hoffmann-Ostenhof. Using methods in part inspired by work of the second author on Neumann and Robin Laplacians, we greatly extend the scope of these previous results.

1. Introduction

1.1. Setting. The aim of this paper is to give a generalization of Pleijel’s theorem for the Dirichlet Laplacian to a large class of Schrödinger operators $H_V = -\Delta + V$ in $\mathbb{R}^d$. We want to give upper bounds on the number of nodal domains of an eigenfunction $f_\lambda$ of $H_V$. A nodal domain of a function $f : A \rightarrow \mathbb{R}$ is defined as a connected component of $A \setminus f^{-1}(0)$. Throughout this paper, the number of nodal domains of $f$ will be denoted by $\mu(f)$.

1.2. Past results. Over the years, numerous upper bounds have been found for the number of nodal domains of Laplace and Schrödinger eigenfunctions.

Given $\Omega \subset \mathbb{R}^d$, let $\lambda_n$, $n \geq 1$, be the eigenvalues of the Dirichlet Laplacian on $\Omega$ in increasing order and with multiplicity and let $\{f_n\}$ be a basis of eigenfunctions associated to $\{\lambda_n\}$.

The first upper bound on $\mu(f_n)$ was given by Courant in 1923 [5]: for any $n \geq 1$ and any eigenfunction $f_n$ associated with $\lambda_n$, $\mu(f_n) \leq n$. Using a different method, Pleijel showed in 1956 [12] that the equality in Courant’s theorem can only be attained finitely many times. Furthermore, he gave the following stronger asymptotic result:

$$\limsup_{n \to \infty} \frac{\mu(f_n)}{n} \leq \gamma_d,$$

where $\gamma_d$ is a constant that depends only on the dimension of $\Omega$. Precisely, $\gamma_d$ is given by the following formula:

$$\gamma_d = \frac{2^{2d-2} d^2 \Gamma(\frac{d}{2})}{j^d_1 - 1}.$$  

Here, $j_a$ denotes the first positive zero of the Bessel function of the first kind $J_a$. In particular, $\gamma_d < 1$ for all $d \geq 2$ (see [2, 8]).

Pleijel’s result was generalized to closed manifolds and manifolds with boundary and Dirichlet boundary conditions in 1957 [11] and in 1982 [2]. More recently, Pleijel’s upper bound was proven for domains $\Omega \subset \mathbb{R}^2$ with piecewise-analytic boundary and Neumann boundary conditions in 2009 [13] and then for domains in $\mathbb{R}^d$ with $C^{1,1}$ boundary with either Neumann or some types of Robin boundary conditions in 2019 [10].

In the case of Schrödinger operators, Pleijel’s asymptotic upper bound was found to hold for the quantum harmonic oscillator in 2018 [3] as well as for Schrödinger operators with radial potentials in [4], which included both positive and negative potentials as well as a possible singularity at the origin.

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1.3. **Main results.** In this paper, we consider operators $H_V = -\Delta + V$ in $\mathbb{R}^d$, $V \in C^1(\mathbb{R}^d)$, and we will work with two different cases which will be treated differently.

**Definition 1.1.** *Case A:* There exist $C_1, C_2, C_3 > 0$ and $a, b, c > 0$ such that

\begin{align}
C_1|x|^a & \leq V(x) \leq C_2|x|^b \\
|\nabla V(x)| & \leq C_3|x|^c.
\end{align}

**Definition 1.2.** *Case B:* There exist $C_1, C_2, C_3 > 0$ and $0 < a \leq b < 2$ such that

\begin{align}
-C_1|x|^{-a} & \leq V(x) \leq -C_2|x|^{-b} \\
|\nabla V(x)| & \leq C_3|x|^{-c}.
\end{align}

for $|x| > M$ for a large enough $M$. Also, there exists $x_1, x_2 \ldots x_N$, called singular points, such that $V(x) \sim -C_i|x-x_i|^{-a_i}$ with $0 < a_i < 2$ and $C_i > 0$ for $1 \leq i \leq N$.

**Remark 1.3.** We point out that case A holds for the harmonic oscillator and case B for the hydrogen atom Hamiltonian (that is to say when $V$ is a Coulomb potential).

In both cases, the quadratic form associated to the differential operator $-\Delta + V$, acting on $C^\infty_c(\mathbb{R}^d \setminus X)$ (where $X = \emptyset$ in case A and $Y = \{x_1, \ldots, x_N\}$ in case B), is lower-semi bounded, as we show in section 2.1. We can therefore define the self-adjoint operator $H_V$ as the Friedrichs extension of this differential operator. We restrict our attention to $H_V$ in the rest of the paper, although the differential operator might have other self-adjoint extensions in some cases. The following result is proved in section 2.1.

**Proposition 1.4.** In case A, $\sigma(H_V) = \{\lambda_n\}_{n \geq 1}$, with each $\lambda_n$ an eigenvalue of finite multiplicity and $\lambda_n \rightarrow +\infty$. In case B,

$$\sigma_{ess}(H_V) = [0, +\infty)$$

and

$$\sigma(H_V) \cap (-\infty, 0) = \{\lambda_n\}_{n \geq 1},$$

with each $\lambda_n$ an eigenvalue of finite multiplicity and $\lambda_n \rightarrow 0$. In both cases, we repeat the eigenvalues according to their multiplicity.

The first main result of this paper is the following:

**Theorem 1.5 (Case A).** Let $H_V = -\Delta + V$ with $V$ defined in 1.1 and $a, b, c$ satisfying the following:

$$\frac{d}{a} + \frac{c}{3a} - \frac{1}{2} < \frac{d}{b}.$$ 

Let $\lambda_n$ be the eigenvalues of $H_V$ in increasing order with multiplicity and $f_n$ be a basis of eigenfunctions for $\lambda_n$. Then, we have the following estimate:

$$\limsup_{n \rightarrow \infty} \frac{\mu(f_n)}{n} \leq \gamma_d.$$

Now, these implicit conditions on $a, b$ and $c$ are hard to separate, but it allows us to consider potentials where $a \neq b$. If $a = b$ we get a simpler condition on $c$:

**Corollary 1.6.** Let $V$ be in case A with $a = b$ and $c < \frac{3}{2}a$. Then, theorem 1.5 holds.

The second main result of this paper is the following:
Theorem 1.7 (Case B). Let $H_V = -\Delta + V$ with $V$ defined in 1.2 and $a, b, c$ satisfying the following:

$$\frac{d}{a} + c - \frac{1}{2} > \frac{d}{b}.$$  

Let $\lambda_n$ be the eigenvalues of $H_V$ below the essential spectrum in increasing order with multiplicity and $f_n$ be a basis of eigenfunctions for $\lambda_n$. Then, we have the following estimate:

$$\limsup_{n \to \infty} \frac{\mu(f_n)}{n} \leq \gamma_d$$

Again, if $a = b$, we have the simpler condition on $c$:

Corollary 1.8. Let $V$ be in case B with $a = b$ and $\frac{3}{2}a < c$. Then, theorem 1.7 holds.

In particular, these two corollaries imply the results in [3, 4].

1.4. Constraints and conditions. The conditions on $a, b$ and $c$ defined in equations 7 and 9 appear in our calculations when we try to control the error terms. It is not clear if those conditions are sharp.

Our result can be applied to sums of Coulomb potentials or sums of quantum harmonic oscillators, as well as perturbations of either of them.

Also, our results are stronger than [4] since we allow for potentials which are non-radial, as well as a broader range of exponents on the potential. However, we do not study the case where there is both a singularity at the origin and the potential grows at infinity. It seems very likely that our method could treat such potentials as well.

It seems highly probable that our method could be generalized to manifolds, as long as a suitable version of Faber-Krahn’s inequality and Weyl’s law can be found. Indeed, as seen in [15], the conditions on $V$ given by definitions 1.1 and 1.2 are far from being the most general under which Weyl’s law holds true. Furthermore, there exists classes of potential $V$ such that $H_V$ has a discrete spectrum whose asymptotic distribution does not follow Weyl’s law (see [14, 16] and references therein). It would be worth investigating upper bounds analogous to (1) in those cases.

1.5. Outline of the proof. The current paper expands on the methods from [3], [4] and [10]. In [3] and [4], the two main ideas were to first partition the potentials into layers to optimally control the size of nodal domains inside each layer, and use algebraic geometry to limit the number of nodal domains that cross from one layer to another. Choosing a suitable partition to balance both estimates gave the final result. However, this limited the scope of the proof to potentials which had a basis of eigenfunctions with a polynomial behaviour.

In [10], one of the key ideas was to separate nodal domains depending on where the $L^2$-mass of the eigenfunction was distributed, and then show that the number of nodal domains which have a high $L^2$-mass close to the boundary was negligible.

In this paper, we combine the two methods in a novel way. Let $\Omega$ be a nodal domain of some eigenfunction $f_\lambda$. We create a locally finite partition of unity $\{A_i\}$ of $\mathbb{R}^d$, then we find at least one $i$ such that $||f_\lambda||^2_{L^2(\Omega \cap \text{supp}(A_i))}$ is comparable to $||f_\lambda A_i||^2_{L^2(\Omega \cap \text{supp}(A_i))}$. This will be done in sections 3.1 and 4.1. It will allow us to use Faber-Krahn’s inequality in an optimal way in each element of the partition. This will be made more precise in section 2.2. By applying Faber-Krahn’s inequality inside each region, we bound the total number of nodal domains by a main term which can be computed in terms of the Weyl asymptotics and an error term which can be bounded with suitable assumptions on $V$. This will be done in sections 3.2 and 4.2.
At this point, all that is left to do is to compare the nodal domain estimates to the eigenvalue counting function. For the sake of completeness, we made in both cases an explicit calculation for a lower bound of the counting function, using Dirichlet bracketing. This is done in sections 3.3 and 4.3.

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2. Preliminary results

2.1. Hardy’s inequalities and the Friedrichs extension. We need to establish Hardy-type inequalities, first in order to properly define our self-adjoint operator, which we do at the end of this section, and later to give upper estimates of some Rayleigh quotients. As in Definition 1.2, we assume that we are given a finite set of points \( X = \{x_1, \ldots, x_N\} \) in \( \mathbb{R}^d \) (called singular points or poles). We slightly extend the standard integral Hardy inequality (for \( d \geq 3 \)) and the modified one given in [6] (for \( d = 2 \)) to deal with this multipole situation.

Lemma 2.1. We assume \( d \geq 3 \). There exists a positive weight function \( h : \mathbb{R}^d \to \mathbb{R} \) such that

\[
\int_{\mathbb{R}^d} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^d} h(x)|\varphi|^2
\]

for any \( \varphi \in C^\infty_c(\mathbb{R}^d) \), and

\[
h(x) \sim \frac{c_i}{|x - x_i|^2} \text{ as } x \to x_i,
\]

with \( c_i > 0 \), for all \( 1 \leq i \leq N \).

Proof. According to the standard Hardy inequality, for any \( \varphi \in C^\infty_c(\mathbb{R}^d) \),

\[
\int |\nabla \varphi|^2 \geq \int h_d(x)|\varphi|^2;
\]

with

\[
h_d(x) = \left( \frac{d-2}{2} \right)^2 \frac{1}{|x|^2};
\]

and thus, by translation,

\[
\int |\nabla \varphi|^2 \geq \int h_d(x-x_i)|\varphi|^2.
\]

Summing over \( 1 \leq i \leq N \), we get

\[
\int_{\mathbb{R}^d} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^d} h(x)|\varphi|^2,
\]

with

\[
h(x) = \frac{1}{N} \sum_{i=1}^N h_d(x - x_i) = \frac{1}{N} \left( \frac{d-2}{2} \right)^2 \sum_{i=1}^N \frac{1}{|x - x_i|^2}.
\]
Lemma 2.2. We assume $d = 2$. For any $R > \max_{1 \leq i \leq N} |x_i|$, there exists a non-negative weight function $h_R : \mathbb{R}^2 \to \mathbb{R}^2$ such that
\[
\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^2} h_R(x)|\varphi|^2
\]
for any smooth function $\varphi$ supported in $D(0, R) := \{ x \in \mathbb{R}^2 ; |x| < R \}$, and
\[
h_R(x) \sim \frac{c_i}{|x - x_i|^2 \ln(|x - x_i|^2)} \text{ as } x \to x_i,
\]
with $c_i > 0$, for all $1 \leq i \leq N$.

Proof. From the modified Hardy inequality \[6\], it follows immediately that for any $r > 0$ and any smooth function $\varphi \in C_c^\infty(D(0, r))$,
\[
\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^2} g_r(x)|\varphi|^2,
\]
with
\[
g_r(x) = \frac{1}{4|x|^2 \ln \left( \frac{|x|}{r} \right)^2 \eta \left( \frac{|x|}{r} \right)},
\]
where $\eta$ is a fixed smooth non-increasing function on $[0, \infty]$ such that $\eta(t) = 1$ for $t \leq \frac{1}{2}$ and $\eta(t) = 0$ for $t \geq 3/4$ (otherwise arbitrary).

We now pick $R > \max_{1 \leq i \leq N} |x_i|$. For any $\varphi \in C_c^\infty(D(0, R))$ and any $x_i$, $\varphi$ is supported in $D(x_i, 2R)$ and thus
\[
\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^2} g_{2R}(x - x_i)|\varphi|^2.
\]
As before, we deduce
\[
\int_{\mathbb{R}^2} |\nabla \varphi|^2 \geq \int_{\mathbb{R}^2} h_R(x)|\varphi|^2,
\]
with
\[
h_R(x) = \frac{1}{N} \sum_{i=1}^{N} g_{2R}(x - x_i) = \frac{1}{4N} \sum_{i=1}^{N} \frac{1}{|x - x_i|^2 \ln \left( \frac{|x - x_i|}{2R} \right)^2 \eta \left( \frac{|x - x_i|}{2R} \right)}. \tag*{□}
\]

As stated in the introduction, we define the self-adjoint operator $H_V = -\Delta + V$ as the Friedrichs extension of the sesquilinear form $a(\cdot, \cdot)$ given by
\[
a(u, v) := \int_{\mathbb{R}^d} \nabla u \cdot \nabla v + \int_{\mathbb{R}^d} V(x)u\overline{v}
\]
for $(u, v) \in C_c^\infty(\mathbb{R}^d \setminus X) \times C_c^\infty(\mathbb{R}^d \setminus X)$. This standard method of definition works as soon as $a(\cdot, \cdot)$ is lower semi-bounded. We now check this condition.

Lemma 2.3. There exists a constant $C$, depending only on $d$ and $V$, such that, for all $u \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, $a(u, u) \geq -C\|u\|_{L^2}^2$.

Proof. In case A, $V \geq 0$ and the result is obvious (with $C = 0$). Let us consider case B. We denote by $X = \{ x_1, \ldots, x_N \}$ the set of singular points of $V$. Let us treat separately the cases $d \geq 3$ and $d = 2$.

In the former case, we fix a weight function $h$ associated with the set $X$ by Lemma 2.1. From the behavior of $V$ and $h$ in the neighborhood of the $x_i$’s, given in Definition 1.2 and Lemma 2.1 respectively, $\lim_{r \to x_i} h(x) + V(x) = +\infty$ for each $x_i$. It follows that $h + V$ is bounded from below. We can then take $C = -\inf_{\mathbb{R}^d}(h + V)$. 

\[5\]
In the latter case, we choose a radius $R$ and weight function $h$ associated with $X$ by Lemma 2.2. We fix a smooth function $0 \leq \chi(x) \leq 1$ supported in $D(0, R)$ and equal to 1 in the neighborhood of each $x_i$. Then, for $u \in C_c^\infty(\mathbb{R}^2 \setminus X)$,

$$|\nabla(\chi u)|^2 \leq 2\chi^2|\nabla u|^2 + 2|\nabla \chi|^2|u|^2 \leq 2|\nabla u|^2 + 2C_1|u|^2,$$

with $C_1 = \sup_{\mathbb{R}^2} |\nabla \chi|^2$. It follows that

$$\int |\nabla u|^2 \geq \frac{1}{2} \int |\nabla(\chi u)|^2 - C_1 \int |u|^2 \geq \frac{1}{2} \int h(x)|\chi u|^2 - C_1 \int |u|^2$$

and therefore

$$\int |\nabla u|^2 + V(x)|u|^2 \geq \int \left( \frac{1}{2} h(x) + V(x) \right) |\chi u|^2 + \int V(x)(1 - \chi^2)|u|^2 - C_1 \int |u|^2.$$

Since $\chi$ is 1 near the singular points of $V$, $V(1 - \chi^2)$ is bounded from below. By a similar argument as in the case $d \geq 3$, $\frac{1}{2} h + V$ is also bounded from below. We set $C_2 = -\inf_{\mathbb{R}^2} V(1 - \chi^2)$ and $C_3 = -\min \{0, \inf_{\mathbb{R}^2} (\frac{1}{2} h + V)\}$. We can then take $C = C_1 + C_2 + C_3$.

We can describe the spectrum of the operator $H_V$ according to the nature of the potential $V$ and give some properties of its eigenfunctions. In what follows, we define the form domain $Q_V$ of $H_V$ as the completion of $C_c^\infty(\mathbb{R}^d \setminus X)$ for the Hilbert norm

$$\|u\|_V^2 := \langle u, Hu \rangle + (C + 1)\|u\|^2$$

where $u \in C_c^\infty(\mathbb{R}^d \setminus X)$ and $C$ is the constant given by Lemma 2.3. We summarize our analysis in the following proposition, which will be proved in appendix A.

**Proposition 2.4.** In case A, the spectrum of $H$ consists of eigenvalues of finite multiplicity tending to $+\infty$. In case B,

$$\sigma_{ess}(H_V) = [0, +\infty).$$

In both cases,

$$Q_V = \left\{ u \in H^1(\mathbb{R}^d) ; |V|^{\frac{1}{2}} u \in L^2(\mathbb{R}^d) \right\}.$$

**Remark 2.5.** In case B, the negative part of $\sigma(H_V)$ consists of a sequence of eigenvalues with finite multiplicities. This sequence could a priori be finite or even empty. However, the estimate of Lemma 4.4 shows in particular that we can write

$$\sigma(H_V) \cap (-\infty, 0) = \{ \lambda_n ; n \geq 1 \},$$

where the $\lambda_n$ are repeated according to their multiplicity and $\lambda_n \to 0$ as $n \to \infty$. This concludes the proof of Proposition 4.4 in the introduction.

### 2.2. Partition and Rayleigh quotient estimates.

We gather in this section several lemmas which will be used in the rest of the paper. Our proof relies, as in Pleijel’s original paper, on the use of Faber-Krahn’s inequality to have lower bounds on the volume of each nodal domain. However, in the case of Schrödinger operators, the Rayleigh quotient over each nodal domains depends on $V$ as well as $\lambda$ and using Faber-Krahn directly will not give a good enough upper bound in the final estimate.

In order to get better bounds, we will show that for every nodal domain $\Omega$ of $f_\lambda$, it is possible to localize $f_\lambda$ with a partition of unity $\{ A_i^2 \}$ and use Faber-Krahn’s inequality in its refined form inside the support of at least one $A_i$. This will show that the volume of $\Omega \cap \text{supp} A_i$ is bounded from below.

However, by using this method, the Rayleigh quotient of $A_i f_\lambda$ will be affected by the norms of $|\nabla A_i|^2$ and by the size of the overlap in the partition. We will need to choose both carefully in order to get the best estimates.

As the partitions that we will be using in our proof may depend on both $\lambda$ and $V$, we shall prove general results for partitions under suitable conditions, and make sure that these conditions are met in our constructions.
Let us denote by \( \{ A^2_i \}_{i \in I} \) a smooth partition of unity (meaning that \( A_i \in C^\infty_c(\mathbb{R}^d) \) and \( \sum A^2_i = 1 \)) and by \( \{ U_i \}_{i \in I} \) an open covering of \( \mathbb{R}^d \) such that \( \text{supp}(A_i) \subset U_i \). We assume that the index set \( I \) is countable and the covering \( \{ U_i \} \) is locally finite.

Let us now consider an eigenvalue \( \lambda \) of \( H_f \), an associated real-valued eigenfunction and \( \Omega \) one of its nodal domain. For all \( i \in I \), we write \( \Omega_i := \Omega \cap U_i \).

**Definition 2.6.** For any \( M > 0 \), we say the nodal domain \( \Omega \) of \( f_\lambda \) is \( M \)-localized in \( U_i \) when
\[
\frac{\int_{\Omega_i} f^2_\lambda}{\int_{\Omega_i} A^2_i f^2_\lambda} \leq M.
\]

The next lemma tells us how to choose \( M \), for a given partition, so that that any nodal domain is \( M \)-localized in at least one \( U_i \).

**Lemma 2.7.** Suppose every point in \( \mathbb{R}^d \) is contained in at most \( M \) distinct \( U_i \)'s. Then, for each nodal domain \( \Omega \) of \( f_\lambda \), there exists \( i_0 \) such that
\[
\frac{\int_{\Omega_i} f^2_\lambda}{\int_{\Omega_i} A^2_i f^2_\lambda} \leq M.
\]

**Proof.** If it is not the case, then
\[
\int_{\Omega_i} f^2_\lambda > M \int_{\Omega_i} A^2_i f^2_\lambda
\]
for all \( i \) such that \( \int_{\Omega_i} A^2_i f^2_\lambda > 0 \). Summing over all these \( i \)'s we find
\[
\int_{\Omega} \left( \sum 1_{U_i} \right) f^2_\lambda > M \int_{\Omega} \left( \sum A^2_i \right) f^2_\lambda = M \int_{\Omega} f^2_\lambda
\]
(\( 1_{U_i} \) is the characteristic function of the set \( U_i \)). According to the hypothesis, \( \sum 1_{U_i} \leq M \) pointwise and we have a contradiction. \( \square \)

**Remark 2.8.** Note that in the rest of the paper, the partition \( \{ A^2_i \} \) may depend on \( \lambda \) and \( V \). However, the partitions will be chosen such that \( M \) is fixed.

As in Pleijel’s paper [12], we use the Faber-Krahn inequality in the following form. If \( \Omega \subset \mathbb{R}^d \) and \( g \) is a function which is smooth in \( \Omega \) and vanishes on \( \partial \Omega \), we define the Rayleigh quotient
\[
R_{\Omega}(f) := \frac{\int_{\Omega} |\nabla f|^2}{\int_{\Omega} f^2} = \frac{\langle -\Delta f, f \rangle_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}^2}. \tag{12}
\]
Then, we have
\[
R_{\Omega}(f) \geq \lambda_1^d(\Omega) \geq K_d |\Omega|^{-\frac{d}{2}},
\]
where $K_d$ is a constant depending only on the dimension $d$. Explicitly,

$$K_d = \lambda_d^2 (\mathbb{B}^d) w_d^2,$$

where $\mathbb{B}^d$ denotes the unit ball in $\mathbb{R}^d$ and $w_d$ its volume. In the following, we will apply this inequality when $f$ is an eigenfunction of $H$ multiplied by an element of the partition of unity. This will yield lower bounds for the volume of a nodal domain and ultimately an upper bound of the nodal count.

The next lemma will help us derive estimates for the Rayleigh quotient of $A_i f_\lambda$.

**Lemma 2.9.** We have, for all $i$,

$$\int_{\Omega_i} |\nabla (A_i f_\lambda)|^2 = \int_{\Omega_i} (\lambda - V(x)) (A_i f_\lambda)^2 + \int_{\Omega_i} |\nabla A_i|^2 f_\lambda^2. \tag{14}$$

**Proof.** Expanding the gradient of a product on the left,

$$\int_{\Omega_i} |\nabla (A_i f_\lambda)|^2 = 2 \int_{\Omega_i} A_i f_\lambda \nabla A_i \cdot \nabla f_\lambda + \int_{\Omega_i} A_i^2 |\nabla f_\lambda|^2 + \int_{\Omega_i} |\nabla A_i|^2 f_\lambda^2.$$

In the first term on the right, we recognize the gradient of the square of $A_i$. Using this and Green’s identity, recalling that $A_i = 0$ on $\partial \Omega_i$,

$$2 \int_{\Omega_i} A_i f_\lambda \nabla A_i \cdot \nabla f_\lambda = \int_{\Omega_i} \nabla (A_i^2) \cdot f_\lambda \nabla f_\lambda = -\int_{\Omega_i} A_i^2 |\nabla f_\lambda|^2 - \int_{\Omega_i} A_i^2 f_\lambda \Delta f_\lambda.$$

Since $-\Delta f_\lambda = (\lambda - V(x)) f_\lambda$, the result follows. \qed

Identity (14) is actually the so-called Localization Formula or IMS Formula. We refer the reader to [7 Th. 3.2] (and reference therein) as well as [9 Th. 1.1] in a semi-classical context.

**Lemma 2.10.** Let $M > 0$ and let us assume that $\Omega$ is $M$-localized in $U_i$. Then,

$$R_{\Omega_i}(A_i f_\lambda) \leq \sup_{U_i} (\lambda - V + M|\nabla A_i|^2) \tag{15}$$

**Proof.** This follows from dividing Identity (14) by $\int A_i^2 f_\lambda^2$ and using Inequality (11). \qed

**Lemma 2.11.** For given $A_i$, $U_i$ and $M > 0$, we denote by $\mu_i$ the number of nodal domains which are $M$-localized in $U_i$. Then,

$$\mu_i \leq K_d^d |U_i| \sup_{U_i} (\lambda - V + M|\nabla A_i|^2)^{\frac{d}{4}}$$

**Proof.** For such a domain $\Omega$, it follows from Inequalities (12) and (15) that

$$1 \leq K_d^d |\Omega_i| R_{\Omega_i}(A_i f_\lambda)^{\frac{d}{2}} \leq K_d^d |\Omega_i| \sup_{U_i} (\lambda - V + C|\nabla A_i|^2)^{\frac{d}{4}}.$$

We obtain the result by summing over all such $\Omega$, recalling that the corresponding $\Omega_i = \Omega \cap U_i$ are contained in $U_i$ and mutually disjoint. \qed

**Remark 2.12.** It is possible for a nodal domain to be $M$-localized in more than one $U_i$. Our method of counting is not optimal in that it counts these multiple times. However, we do not know at this moment if any improvement could be obtained by removing the nodal domains that have been counted more than once.

3. POTENTIALS GROWING AT INFINITY

3.1. Construction of the partition. Take the following partition: let $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ and $\delta > 0$ small. Set $\mu = \lambda^{-m}$, with $m$ to be determined later.

Let $J_{z, \delta, \mu}$ be the hypercube $\mu(z_1 - \frac{1}{2} - \delta, z_1 + \frac{1}{2} + \delta) \times \mu(z_2 - \frac{1}{2} - \delta, z_2 + \frac{1}{2} + \delta) \ldots \mu(z_d - \frac{1}{2} - \delta, z_d + \frac{1}{2} + \delta).$

We know that for any $0 < \delta \leq \frac{1}{2}$ and $\mu > 0$, the collection $\{ J_{z, \delta, \mu}, z \in \mathbb{Z}^d \}$ covers $\mathbb{R}^d$ and each $x \in \mathbb{R}^d$ is covered by at most $2^d$ hypercubes. We set $M := 2^d$ throughout.
Let \( \{\chi_{z,\delta,\mu}^2\} \) be a partition of unity subordinated to the cover \( J_{z,\delta,\mu} \). For each \( \delta \), we can construct it so that there exists a constant \( C(d, \delta) \) such that \( |\nabla \chi_{z,\delta,\mu}| \leq C(d, \delta) \mu^{-1} \).

### 3.2 Nodal count estimates.

We know that for any \( \delta > 0, \mu > 0 \) and a given nodal domain \( \Omega \) of \( f_{\lambda} \), there exists at least one \( z \in \mathbb{Z}^d \) such that \( \Omega \) is \( M \)-localized in \( J_{z,\delta,\mu} \).

We set \( \Omega_z := \Omega \cap J_{z,\delta,\mu} \).

Let \( B_z \) be the set of nodal domains \( \Omega \) which are \( M \)-localized in \( J_{z,\delta,\mu} \).

Since any nodal domain is contained in at least one set \( B_z \), using lemma 2.11 and summing over all hypercubes gives us the following:

\[
\mu(f_{\lambda}) \leq \sum_{z \in \mathbb{Z}^d} (1 + 2\delta)^d \lambda^{-md} K_d^{-\frac{d}{2}} \sup_{J_{z,\delta,\mu}} [\lambda - V + C\lambda^{2m}]^{\frac{d}{2}}_+
\]

with \( K_d \) defined in equation 13.

Now, let

\[
M_{\lambda} := \sum_{z \in \mathbb{Z}^d} \lambda^{-md} \sup_{J_{z,\delta,\mu}} [\lambda - V + C\lambda^{2m}]^{\frac{d}{2}}_+
\]

\[
W_{\lambda} := \int_{\mathbb{R}^d} (\lambda - V)^{\frac{d}{2}}_+
\]

and

\[
m_{\lambda} := \sum_{z \in \mathbb{Z}^d} \lambda^{-md} \inf_{J_{z,\delta,\mu}} [\lambda - V]^{\frac{d}{2}}_+
\]

We want to show that

\[
A_{\lambda} := M_{\lambda} - W_{\lambda} = o(W_{\lambda})
\]

when \( \lambda \to \infty \) for a fixed \( V \), which would imply \( \mu(f_{\lambda}) \leq (1 + 2\delta)^d K_d^{-\frac{d}{2}} W_{\lambda}(1 + o(1)) \). Since \( m_{\lambda} \leq W_{\lambda} \leq M_{\lambda} \), it is sufficient to show \( M_{\lambda} - m_{\lambda} = o(W_{\lambda}) \).

We recall our assumptions on \( V \) and \( \nabla V \):
\[ V(x) \geq C_1|x|^a \]
\[ V(x) \leq C_2|x|^b \]
\[ |\nabla V(x)| \leq C_3|x|^c \]

Using these assumptions, we can find an upper bound on \( A_\lambda \):

**Lemma 3.1.** We assume \( c < \frac{a}{2} \). By choosing \( \mu = \lambda - \frac{c}{3} \) in the partition and for fixed \( \delta > 0 \),

\[ A_\lambda \leq C(d, V, \delta) \lambda^{\frac{d}{2} + \frac{a}{d} + \frac{c}{d} - 1}. \]

The proof of this lemma will be done in appendix B.

Now, we estimate the size of \( W_\lambda \) from below:

\[ W_\lambda \geq \int_{B(0, (\lambda/C_2)^{1/b})} \left( \lambda - C_2|x|^b \right)^{\frac{d}{d}} dx = C(d, V) \lambda^{\frac{d}{2} + \frac{a}{d}} \]

### 3.3. Weyl’s law for positive exponents.

We will do a Dirichlet bracketing argument since we only need lower bounds for \( N(\lambda) := \sharp \{ \lambda_n < \lambda \} \). Let \( R_\Omega \) be the Rayleigh quotient for the Dirichlet Laplacian on \( \Omega \) and \( Q_\Omega(f) \) the modified Rayleigh quotient for \( H = -\Delta + V \):

\[ Q_\Omega(f) = \frac{< -\Delta f, f >_{L^2(\Omega)} + < Vf, f >_{L^2(\Omega)}}{< f, f >_{L^2(\Omega)}}. \]

Let \( J_{z, \mu} \) be the hypercube \( \mu(z_1 - \frac{1}{2}, z_1 + \frac{1}{2}) \times \mu(z_2 - \frac{1}{2}, z_2 + \frac{1}{2}) \ldots \mu(z_d - \frac{1}{2}, z_d + \frac{1}{2}) \). Since they do not overlap, we can use them to do a Dirichlet bracketing for the eigenvalue count. We have the following lower bound for the eigenvalue count:

\[ N(\lambda) = \max_{V \subset H^1(\mathbb{R}^d)} \dim V \]
\[ \forall f \in V, Q_\Omega(f) < \lambda \]

\[ \geq \sum_{z \in \mathbb{Z}^d} \max_{V_z \subset H^1_0(J_{z, \mu})} \dim V_z \]
\[ \forall f \in V_z, Q_\Omega(f) < \lambda \]

\[ \geq \sum_{z \in \mathbb{Z}^d} \max_{V_z \subset H^1_0(J_{z, \mu})} \dim V_z \]
\[ \forall f \in V_z, R_{J_{z, \mu}}(f) < \lambda - \sup_{J_{z, \mu}} V \]

We can use the explicit formula for the number of Dirichlet eigenvalues of a hypercube with remainder:

\[ \max_{V_z \subset H^1_0(J_{z, \mu})} \dim V_z = |J_{z, \mu}| (2\pi)^{-d} w_d \inf_{J_{z, \mu}} (\lambda - V)^{\frac{d}{2}} + O \left( \frac{|\partial J_{z, \mu}|}{J_{z, \mu}} \inf_{J_{z, \mu}} (\lambda - V)^{\frac{d}{2} - \frac{1}{2}} \right) \]

Here, \( w_d \) is the volume of the unit ball in \( \mathbb{R}^d \). By summing over all cubes, we obtain the following estimate:
Lemma 3.2. By choosing \( \mu = \lambda^{-\frac{c}{3q}} \) in the partition, we obtain the following estimate for \( N(\lambda) \):

\[
N(\lambda) \geq (2\pi)^{-d}w_dW_\lambda + O(\lambda^{d + \frac{d}{2} + \frac{c}{3q} - \frac{1}{2}})
\]

The proof of this lemma will be given in appendix D.

3.4. Completing the proof. Now, since \( c < \frac{3}{2}a, \frac{d}{2} + \frac{d}{2} + \frac{c}{3q} - \frac{1}{2} < \frac{d}{2} + \frac{d}{2} + \frac{c}{3q} - 1 \).

(****Philippe: afin de garder les deux preuves des lemmes séparées à l’annexe.)

If \( \frac{d}{2} + \frac{d}{2} + \frac{c}{3q} - \frac{1}{2} < \frac{d}{2} + \frac{d}{b} \) we can combine lemmas 3.1 and 3.2 as well as equation (22) to get the following:

**Lemma 3.3.** If \( a, b, c \) are chosen such that

\[
\frac{d}{2} + \frac{d}{a} + \frac{c}{3q} - \frac{1}{2} < \frac{d}{2} + \frac{d}{b},
\]

then we have the following estimates:

\[
\mu(f_\lambda) \leq K_d^{-\frac{d}{2}}(1 + 2\delta)^dW_\lambda(1 + o_\lambda(1))
\]

\[
N(\lambda) \geq (2\pi)^{-d}w_dW_\lambda(1 + o_\lambda(1))
\]

Now, since \( N(\lambda_n) \leq n \), we get the following estimate:

\[
\limsup_{n \to \infty} \frac{\mu(f_\lambda)}{n} \leq \limsup_{n \to \infty} \frac{\mu(f_\lambda)}{N(\lambda_n)} \leq \limsup_{n \to \infty} \frac{K_d^{-\frac{d}{2}}(1 + 2\delta)^dW_\lambda(1 + o_\lambda(1))}{(2\pi)^{-d}w_dW_\lambda(1 + o_\lambda(1))} \leq (1 + 2\delta)^d\gamma_d,
\]

where \( \gamma_d \) is Pleijel’s constant.

Now, since this estimate is true for any \( \delta > 0 \), we can let \( \delta \to 0 \) to obtain Theorem 1.5.

4. Potential vanishing at infinity

4.1. Construction of the partition. We want to construct a partition of unity of \( \mathbb{R}^d \) with the following properties:

(1) The overlap between the supports of any two functions is small.
(2) Each \( x \in \mathbb{R}^d \) is contained in the support of finitely many elements of the partition.
(3) If \( x \) is in the support of some function, the the diameter of the support will be \( \asymp |x|^q \) with \( q < 1 \).
(4) The gradient of each function in the partition can be bounded by \( C|x|^{-q} \), where \( x \) is some point in the support of the function.

We will start with a simple lemma that enables us to construct such a partition:

**Lemma 4.1.** For any \( q < 1 \) and \( r > 0 \), there exist a sequence of pairs of positive numbers \( (r_i, d_i) \), \( i \geq 1 \) and positive constants \( M, M' \) such that

(1) \( r_1 = r > 0 \),
(2) \( r_{i+1} = r_i + d_i \),
(3) \( \frac{d_i}{d_i} \in \mathbb{N} \),
(4) \( M r_i^q \leq d_i \leq M' r_i^q \),
(5) \( r_i \to \infty \).

The proof of this lemma will be done in appendix D.

We will now use this sequence (with \( q \) still undetermined) to construct a covering of \( \mathbb{R}^d \).

Let \( r = r(V) > 0 \) be large enough such that all the singularities of \( V \) are contained in the ball of radius \( r \) centered at 0.

Let \( D(x_1, x_2 \ldots x_d) := \sup |x_d| \). This norm is equivalent to the standard Euclidean norm.

Let \( C_i := \{ x | r_i \leq D(x) \leq r_i + d_i \} \) for \( i \geq 1 \) and \( B_0 := \{ x | D(x) \leq r \} \).
We now cover $C_i$ with hypercubes $B_{i,j}$ of side-length $d_i$. Since $r_i/d_i$ is an integer, the cubes fit perfectly.

We will use an abuse of notation and label the $B_{i,j}$ as simply $B_i$.

Here is an example of such a partition around the origin:

Let $x_i$ be the center of each hypercube $B_i$.

Now, let us construct a partition of unity of $\mathbb{R}^d$. Let $B = [-\frac{1}{2}, \frac{1}{2}]^d$ and $B_\delta = (-\frac{1}{2} - \delta, \frac{1}{2} + \delta)^d$. There exists a smooth function $\phi_\delta$ with the following properties:

1. $\phi_\delta = 1$ in $B$.
2. $\text{supp}(\phi_\delta) \subset B_\delta$.
3. $\sup |\nabla \phi_\delta| \leq C(d)\delta^{-1}$.

Let $B_i$ be one hypercube, and $\tilde{\phi}_i$ be defined as the rescaled and translated $\phi_\delta$ such that $\tilde{\phi}_i = 1$ on $B_i$. With the rescaling, for $i \neq 0$, $\sup |\nabla \tilde{\phi}_i| \leq C_{d,M,\delta}|x|^{-q}$, with $x$ being any $x$ in the support of $\tilde{\phi}_i$.

Since the closure of the union of all $B_i$ covers $\mathbb{R}^d$, we have $\sum_i \tilde{\phi}_i^2 \geq 1$. Hence, the functions $A_i^2$, with

$$A_i := \frac{\tilde{\phi}_i}{\left(\sum_i \tilde{\phi}_i^2\right)^\frac{1}{2}},$$

form a partition of unity, and they have the following properties:

1. For each $x \in \mathbb{R}^d$ and $\delta$ small enough, there are at most $2^d$ indices $i$ such that $x \in \text{supp}A_i$. Indeed, since the ratio $d_{i+1}/d_i$ is bounded from above and away from 0 (see remark [D.1]), then choosing $\delta$ small enough insures that any $x$ will only be covered by functions from at most two successive rows.
2. $\sup |\nabla A_i| \leq C_{d,r,\delta} \sup_{x \in \text{supp}(\phi_i)} |x|^{-q}$.

We define $B_{i,\delta}$ as the rescaled and translated $B_\delta$ such that $B_i \subset B_{i,\delta}$ and $\text{supp}(A_i) \subset B_{i,\delta}$. 


4.2. **Nodal count.** Let \( M := 2^d \). We know that for any \( \delta > 0 \) and a given nodal domain \( \Omega \), there exists at least one \( i \) such that \( \Omega \) is \( M \)-localized in \( B_{i, \delta} \).

Now, since \( \sup_{B_{0, \delta}} [\lambda - V] = +\infty \), we need to treat the central cube differently than the others. However, by fixing \( r \), we can control the number of nodal domains that are \( M \)-localized in \( B_{0, \delta} \):

**Lemma 4.2.** Let \( r \) be large enough such that all the singularities of \( V \) are contained in the ball of radius \( r \) centered at zero. Then, the number \( \mu_0 \) of nodal domains which are \( M \)-localized in \( B_{0, \delta} \) is less or equal than a constant \( C \) which depends on \( V, d, r \) and \( \delta \), but not on \( \lambda \).

**Proof.** We first remark that there exists a function \( h : \mathbb{R}^2 \to \mathbb{R} \) such that \( x \mapsto \alpha V(x) + h(x) \) is bounded from below for all \( \alpha \in \mathbb{R} \), and

\[
\int |\nabla \varphi|^2 \geq \int h(x)|\varphi|^2
\]

for all \( \varphi \in C_c^\infty(B_{0, \delta}) \). Indeed, when \( d \geq 3 \), we can take directly \( h \) as given by Lemma 2.1. When \( d = 2 \), we choose \( R > 0 \) such that \( B_{0, \delta} \subset D(0, R) \) and take \( h := h_R \), as given by Lemma 2.2.

We now assume that \( \Omega \) is a nodal domain of \( f_\lambda \) which is \( K \)-localized in \( B_{0, \delta} \). From Identity (14), we have

\[
\int_{\Omega_0} |\nabla (A_0 f_\lambda)|^2 = \int_{\Omega_0} (\lambda - V(x))(A_0 f_\lambda)^2 + |\nabla A_0|^2 f_\lambda^2
\]

and thus

\[
\frac{1}{2} \int_{\Omega_0} |\nabla (A_0 f_\lambda)|^2 = \int_{\Omega_0} \left( (\lambda - V(x))(A_0 f_\lambda)^2 - \frac{1}{2}|\nabla (A_0 f_\lambda)|^2 + |\nabla A_0|^2 f_\lambda^2 \right).
\]

Using Inequality (31) and recalling that \( \lambda < 0 \), it follows

\[
\int_{\Omega_0} |\nabla (A_0 f_\lambda)|^2 \leq \int_{\Omega_0} \left(-2V(x) + h(x))(A_0 f_\lambda)^2 + 2|\nabla A_0|^2 f_\lambda^2 \right).
\]

Since \( \Omega \) is \( M \)-localized in \( B_{0, \delta} \), we get

\[
R_{\Omega_0}(A_0 f_\lambda) \leq C,
\]

with

\[
C := -\inf_{\mathbb{R}^2} (2V + h) + 2M \sup_{\mathbb{R}^d} |\nabla A_0|^2,
\]

which depends on \( d, V, r \) and \( \delta \), but not on \( \lambda \).

Reasoning as in the proof of Lemma 2.1, we find

\[
\mu_0 \leq (CK^{-\frac{1}{d}})^\frac{d}{2} |B_{0, \delta}|.
\]

Since \( |B_{0, \delta}| \) depends only on \( d, r \) and \( \delta \), this is the desired result. \( \square \)

As in the previous section, using Faber-Krahn’s inequality and summing over all \( i \) we obtain an upper bound for the number of nodal domains of \( f_\lambda \):

\[
\mu(f_\lambda) \leq \mu_0 + K_d^{-\frac{d}{2}} \sum_{i \geq 1} |B_{i, \delta}| \sup_{B_{i, \delta}} \left[ \lambda - V + 2C|x|^{-2q} \right]_{+}^\frac{d}{2}
\]

\[
\leq \mu_0 + K_d^{-\frac{d}{2}} \sum_{i \geq 1} (1 + 2\delta)^d |B_i| \sup_{B_{i, \delta}} \left[ \lambda - V + 2C|x|^{-2q} \right]_{+}^\frac{d}{2}
\]

Again, let

\[
M_\lambda := \sum_{i \geq 1} \text{Vol}(B_i) \sup_{B_{i, \delta}} \left[ \lambda - V + C_{d, \delta}|x|^{-2q} \right]_{+}^\frac{d}{2},
\]

\[
W_\lambda := \int_{\mathbb{R}^d \setminus B_{0, \delta}} (\lambda - V)^{\frac{d}{2}},
\]
\[ m_\lambda := \sum_{i \geq 1} \text{Vol}(B_i) \inf_{B_i, \delta} \left[ \lambda - V \right]^{\frac{d}{2}} \]

and

\[ A_\lambda := M_\lambda - W_\lambda. \]

We recall the conditions imposed on \( V \):

\[
\begin{align*}
V(x) & \geq -C_1|x|^{-a} \\
V(x) & \leq -C_2|x|^{-b} \\
|\nabla V(x)| & \leq C_3|x|^{-c}
\end{align*}
\]

Using these assumptions, we have the following estimate for \( A_\lambda \):

**Lemma 4.3.** By choosing \( q = c/3 \) in the partition, assuming \( c > 3/2a \) and for fixed \( \delta > 0 \),

\[ A_\lambda \leq C(d, V, \delta)\lambda^{\frac{d}{2} - \frac{d}{2} + \frac{c}{6} - 1} \]

The proof of this lemma will be done in section C.1.

Now, we estimate the size of \( W_\lambda \) from below. For \( R > 0 \) large enough so that \( B_{0, \delta} \subset B(0, R) \),

\[ W_\lambda \geq \int_{B\left(0, \left(\frac{C_2}{\delta}\right)^{1/b}\right) \setminus B(0, R)} \left[ \lambda + C_2|x|^{-b} \right]^{\frac{d}{2}} dx \]

\[ \geq C(V, d)\lambda^{\frac{d}{2} - \frac{d}{2}} \]

We also note that since \( b < 2 \), \( \mu_0 = o(W_\lambda) \).

4.3. **Weyl's law for negative exponents.** As in the case with positive potentials, we will use a Dirichlet bracketing argument for the number of eigenvalues. Let \( R_\Omega \) be the Rayleigh quotient for the Dirichlet laplacian on \( \Omega \). Since the hypercubes \( B_i \) do not overlap, we have the following lower bound:

\[ N(\lambda) \geq \sum_{i \geq 1} \max_{V_i \subset H^1_0(B_i)} \dim V_i \]

Again, the sets \( V_i \) are taken as linear spaces.

**Lemma 4.4.** By choosing \( q = c/3 \) in the partition and assuming \( c > 3/2a \), we obtain the following estimate for \( N(\lambda) \):

\[ N(\lambda) \geq (2\pi)^{-d}w_dW_\lambda + O(\lambda^{\frac{d}{2} - \frac{d}{2} + \frac{c}{6} - \frac{1}{2}}), \]

The proof of this lemma will be given in section C.2.

4.4. **Completing the proof.** Now, since \( \frac{c}{3a} > \frac{1}{2} \), then \( \frac{d}{2} - \frac{d}{a} + \frac{c}{3a} - \frac{1}{2} < \frac{d}{2} - \frac{d}{3a} + \frac{2c}{3a} - 1 \).

If \( \frac{d}{2} - \frac{d}{a} + \frac{c}{3a} - \frac{1}{2} > \frac{d}{2} - \frac{d}{b} \) we can combine lemmas 4.3 and 4.4 as well as equation (39) to get the following:

**Lemma 4.5.** If \( a, b, c \) are chosen such that

\[ \frac{d}{2} - \frac{d}{a} + \frac{c}{3a} - \frac{1}{2} > \frac{d}{2} - \frac{d}{b}, \]

then we have the following estimates:
We have \( V \phi \) prove that for any sequence \((\chi_n)\) that is to say \(\sigma_K\) where \(\chi\) then apply Persson’s formula to compute the bottom of the essential spectrum of \( \|
abla \phi \|^p \) such that \( p > d \). (44)

\[
\begin{align*}
\limsup_{n \to \infty} \frac{\mu(f_n)}{n} & \leq \limsup_{n \to \infty} \frac{\mu(f_n)}{N(\lambda_n)} \leq \limsup_{n \to \infty} \frac{K_d^{-\frac{d}{2}}(1 + 2\delta)^d W \phi(1 + o_1(1))}{(2\pi)^{-d} w_d W \phi(1 + o_1(1))} \\
& \leq (1 + 2\delta)^d \gamma_d,
\end{align*}
\]

where \( \gamma_d \) is Pleijel’s constant.

Now, since this estimate is true for any \( \delta > 0 \), we can let \( \delta \to 0 \) to obtain Theorem 1.7

**APPENDIX A. PROOF OF PROPOSITION 2.4**

The following analysis is rather classical (see for instance [1]) and is recalled here for the reader’s convenience. In cases A and B, the potential \( V \) belongs to \( L^p_{loc}(\mathbb{R}^d) \) for some \( p > d/2 \). Indeed, it suffices to choose \( p \) such that \( p > d/2 \) and \( a_i p < d \) for all \( 1 \leq i \leq N \). According to [1, Th. 3.2, pp. 44-45, Eq. (3.16)], we can then apply Persson’s formula to compute the bottom of the essential spectrum of \( H_V \):

\[
\inf \sigma_{ess}(H_V) = \sup_K \inf_{\varphi \in C_c^\infty(\mathbb{R}^d \setminus K)} \frac{\langle \varphi, H_V \varphi \rangle}{\| \varphi \|^2},
\]

where \( K \) ranges over all compact subsets of \( \mathbb{R}^d \). It follows immediately that

\[
\inf \sigma_{ess}(H_V) \geq \sup_{K \subseteq \mathbb{R}^d \setminus K} \inf_{\varphi \neq 0} V.
\]

In case A, \( V(x) \to +\infty \) when \( |x| \to +\infty \), and thus

\[
\inf \sigma_{ess}(H_V) = +\infty,
\]

that is to say \( \sigma_{ess}(H_V) = \emptyset \). Since \( H_V \) is lower semi-bounded, \( \sigma(H_V) \) is a sequence of eigenvalues with finite multiplicities tending to \( +\infty \).

In case B, \( V(x) \to 0 \) when \( |x| \to +\infty \) and we obtain

\[
\sigma_{ess}(H_V) \subset [0, +\infty).
\]

We can easily check the reverse inclusion by constructing appropriate Weyl sequences. Indeed, let us set, for \( \xi \in \mathbb{R}^d \),

\[
\varphi_{\xi,n}(x) := A_n \chi \left( \frac{x - x_n}{R_n} \right) \exp(i \xi \cdot x),
\]

where \( \chi \) is a smooth function in \( \mathbb{R}^d \) with compact support, \( x_n \in \mathbb{R}^d \), \( R_n > 0 \), and \( A_n > 0 \) is defined by \( \| \varphi_{\xi,n} \| = 1 \). By a suitable choice of sequences \( |x_n| \to +\infty \) and \( R_n \to +\infty \), we can ensure that \( \langle \varphi_{\xi,n}, H_V \varphi_{\xi,n} \rangle \to |\xi|^2 \) and \( (\varphi_{\xi,n}) \) converges to 0 weakly in \( L^2(\mathbb{R}^d) \), which implies \( |\xi|^2 \in \sigma_{ess}(H_V) \). We have shown that \( \sigma_{ess}(H_V) = [0, +\infty) \).

Let us sketch briefly how to characterize the form domain \( Q_V \). We use the notation

\[
\mathcal{V} := \left\{ u \in H^1(\mathbb{R}^d) \mid \| V \|^{\frac{1}{2}} u \in L^2(\mathbb{R}^d) \right\}.
\]

We have \( V = V_+ - V_- \) and \( |V|^{\frac{1}{2}} = V_+^{\frac{1}{2}} + V_-^{\frac{1}{2}} \), where \( V_\pm := \max(\pm V, 0) \). To show that \( Q_V \subset \mathcal{V} \), it suffices to prove that for any sequence \( (\varphi_n) \subset C_c^\infty(\mathbb{R} \setminus X) \) which is Cauchy for the norm \( \| \cdot \|_1 \), the sequence \( (V_\pm^{\frac{1}{2}} \varphi_n) \) is Cauchy in \( L^2(\mathbb{R}^d) \).
This follows easily from the following inequality: for any $\epsilon > 0$, there exists a constant $C_\epsilon$ such that

\[
\int V_-(x)|\varphi|^2 \leq \epsilon \int |\nabla \varphi|^2 + C_\epsilon \int |\varphi|^2.
\]  

This last result can be proved using the Hardy inequalities in Section 2.2. Let us treat the case $d \geq 3$.

We first note that in both cases A and B, $V_-(x) \to 0$ when $|x| \to +\infty$ and $V_-(x)|x-x_i|^2 \to 0$ when $x \to x_i$, for all $1 \leq i \leq N$. Thus, there exists $\delta > 0$ such that $V_-(x) \leq \epsilon h(x)$ for all $x \in U := \bigcup_{i=1}^N B(x_i, \delta)$, where $h$ is the weight function given by Lemma 2.1. Inequality (46), with $C_\epsilon = \sup_{\mathbb{R}^d \setminus U} V_-$, then follows from Lemma 2.1. The case $d = 2$ is similar.

The reverse inclusion $V \subset Q_V$ is a direct consequence of the density of $C^\infty_c(\mathbb{R}^d \setminus X)$ in $H^1(\mathbb{R}^d)$. This latter property holds because points have zero capacity in $\mathbb{R}^d$ for $d \geq 2$.

**Appendix B. Proofs of lemmas 3.1 and 3.2**

**B.1. Proof of lemma 3.1** Recall that the side-length of the cubes in the partition is $\mu(1 + 2\delta)$ with $\mu = \lambda - m$.

Also, we had the following conditions on $V$:

\[
\begin{align*}
V(x) &\geq C_1|x|^a, \\
V(x) &\leq C_2|x|^b, \\
|\nabla V(x)| &\leq C_3|x|^c,
\end{align*}
\]

with $c < \frac{3}{2}a$.

Finally, we recall that

\[
M_\lambda := \sum_{z \in \mathbb{Z}^d} \lambda^{-md} \sup_{J_{z,\delta,\mu}} \left[ \lambda - V + C_\lambda 2^m \right]^\frac{d}{2},
\]

\[
W_\lambda := \int_{\mathbb{R}^d} (\lambda - V)^\frac{d}{2},
\]

\[
m_\lambda := \sum_{z \in \mathbb{Z}^d} \lambda^{-md} \inf_{J_{z,\delta,\mu}} [\lambda - V]_+^\frac{d}{2}
\]

and $A_\lambda = M_\lambda - W_\lambda$.

We want to show that $A_\lambda \leq C(d, V, \delta)\lambda^\frac{d}{2} + \frac{d}{2} + \frac{\lambda^d}{d} - 1$.

In this section, $C$ will denote constants which may change from line to line but which never depend on $\lambda$. However we may write down the dependency of $C$ on different parameters to emphasize this point.

We have the following estimates for $A_\lambda$:
We now assume that $m < \frac{1}{2}$ so that $\lambda^{2m} = o(\lambda)$. As we will see later, this condition will be fulfilled for suitable $a$, $b$ and $c$.

The last term is simply the number of cubes of side-length $\lambda^{-m}$ in a ball of radius $C(d, V)\lambda^{\frac{a}{2}}$. We can estimate it directly:

\[
\sum_{z \in \mathbb{Z}^d} \sup_{J_z,d,\mu} |\nabla V| \sup_{J_z,d,\mu} |\lambda - V|^{\frac{d}{2}} - C(d,\delta) \lambda^{2m}\sup_{J_z,d,\mu} |\lambda - V|^{\frac{d}{2} - 1}
\]

\[
\leq \lambda^{-md} \sum_{z \in \mathbb{Z}^d} \sup_{J_z,d,\mu} |\nabla V| \sup_{J_z,d,\mu} |\lambda - V|^{\frac{d}{2}} + C(d,\delta) \lambda^{2m}\sup_{J_z,d,\mu} |\lambda - V|^{\frac{d}{2} - 1}
\]

\[
= \lambda^{-md} \sum_{z \in \mathbb{Z}^d} \sup_{J_z,d,\mu} \left| \sup_{V \leq \lambda + 2C\lambda^{2m}} C(d,\delta) \right|
\]

In order to balance the two left terms, we need that

\[-m + \frac{c}{d} = 2m.\]

Therefore, putting $m = \frac{c}{3d}$ gives us the following estimate for $A_\lambda$:

\[
A_\lambda \leq C(d, V, \delta) \left[ \lambda^{2m} + \lambda^{\frac{d}{2} - 1} + \lambda^{\frac{d}{2} + \frac{d}{2} - 1} \right]
\]
Now, in the case $d = 2$, the exponent in the two terms is equal. In the case $d \geq 3$, since we assumed that $m < \frac{1}{2}$, the term on the left dominates and we obtain that for any dimension,

$$A_\lambda \leq C \lambda^{\frac{2a}{a} + \frac{d}{2} + \frac{d}{2} - 1}.$$ 

We also note that by adding the condition $\frac{2c}{3a} < 1$, then $A_\lambda = o(W_\lambda)$ if $a = b$. Also, this condition ensures that $\lambda^{2m} = o(\lambda)$ in equation \[55\].

This completes the proof of lemma \[3.1\].

**B.2. Proof of lemma 3.2** We will now give bounds for the estimate on the eigenvalue count:

$$N(\lambda) \geq (2\pi)^{-d} w_d \sum_{z \in \mathbb{Z}^d} |J_{z,\mu}| \inf_{J_{z,\mu}} (\lambda - V)^{\frac{d}{2} - \frac{1}{2}} + O \left( \sum_{z \in \mathbb{Z}^d} |\partial J_{z,\mu}| \inf_{J_{z,\mu}} (\lambda - V)^{\frac{d}{2} - \frac{1}{2}} \right)$$

We want to show that $N(\lambda) \geq (2\pi)^{-d} w_d W_\lambda + O(\lambda^{\frac{a}{2} + \frac{d}{2} + \frac{d}{2} - 1})$. By the previous section, by choosing $m = c/3a$, the first term is equal to $(2\pi)^{-d} w_d W_\lambda + O(\lambda^{\frac{a}{2} + \frac{d}{2} + \frac{d}{2} - 1})$.

We now bound the second term:

$$\sum_{z \in \mathbb{Z}^d} |\partial J_{z,\mu}| \inf_{J_{z,\mu}} (\lambda - V)^{\frac{d}{2} - \frac{1}{2}} \leq C(d) \lambda^m \sum_{z \in \mathbb{Z}^d} \lambda^{-m d} \inf_{J_{z,\mu}} (\lambda - V)^{\frac{d}{2} - \frac{1}{2}}$$

$$\leq C(d) \lambda^m \int_{B(0, (\lambda/C_1)^{1/\alpha})} (\lambda - C_1 |z|^\alpha)^{\frac{d}{2} - \frac{1}{2}}$$

$$\leq C(d, V) \lambda^{\frac{2a}{a} + \frac{d}{2} + \frac{d}{2} - \frac{1}{2}}$$

Now, since $2c < 3a$, then $\frac{c}{3a} - \frac{1}{2} > \frac{2c}{3a} - 1$ and we have the final estimate for $N(\lambda)$:

$$N(\lambda) = (2\pi)^{-d} w_d W_\lambda + O \left( \lambda^{\frac{2a}{a} + \frac{d}{2} + \frac{d}{2} - \frac{1}{2}} \right)$$

This completes the proof of lemma \[3.2\].

**Appendix C. Proofs of lemmas 4.3 and 4.4**

**C.1. Proof of lemma 4.3** Recall that the hypercubes $B_i$ in the partition are of diameter $\asymp |x_i|^\eta$ for any $x_i \in B_i$, and that

$$M_\lambda := \sum_{i \geq 1} \text{Vol}(B_i) \sup_{B_{i,\delta}} [\lambda - V + C_{d,\delta} |x|^{-2\eta}]^{\frac{d}{2}}$$

$$W_\lambda := \int_{\mathbb{R}^d \setminus B_0,\delta} (\lambda - V)^{\frac{d}{2}}$$

$$m_\lambda := \sum_{i \geq 1} \text{Vol}(B_i) \inf_{B_{i,\delta}} [\lambda - V]^{\frac{d}{2}}$$

and

$$A_\lambda := M_\lambda - W_\lambda.$$
\[(66) \quad V(x) \geq -C_1|x|^{-a}, \]
\[(67) \quad V(x) \leq -C_2|x|^{-b}, \]
\[(68) \quad |\nabla V(x)| \leq C_3|x|^{-c}, \]
with \(c > \frac{3}{2}a.\)

We want to show that \(A_\lambda \leq C(d, V, \delta)|\lambda|^{\frac{d}{2} - \frac{d}{2} + \frac{2}{3} - 1}.\)

We now fix \(r > 0\) such that all the singularities of \(V\) are contained in the ball of radius \(r\) centered at the origin.

In this section, \(C\) will denote constants which may change from line to line but which never depend on \(\lambda.\) However we may write down the dependency of \(C\) on different parameters to emphasize this point.

We will follow a strategy similar to the one we used for positive potentials in order to bound \(A_\lambda.\)

\[(69) \quad A_\lambda \leq M_\lambda - m_\lambda \]
\[(70) \quad \leq \sum_{i \geq 1} \text{Vol}(B_i) \sup_{B_{i,\delta}} \left[ \left| \lambda - V + C_{d,\delta}|x|^{-2q} \right| + \inf_{B_{i,\delta}} |\lambda - V| \right]^d \]
\[(71) \quad \leq \sum_{i \geq 1} \text{Vol}(B_i) \sup_{B_{i,\delta}} \left[ \left| \lambda - V \right|^d + C_{d,\delta} \sup_{B_{i,\delta}} |x|^{-2q} \sup_{B_{i,\delta}} |\lambda - V + C_{d,\delta}|x|^{-2q} \right] \]
\[(72) \quad \leq \sum_{i \geq 1} \text{Vol}(B_i) \left[ \text{Diam}(B_{i,\delta}) \sup_{B_{i,\delta}} |\nabla V| \sup_{B_{i,\delta}} \left| \lambda - V \right|^d + C_{d,\delta} \sup_{B_{i,\delta}} |x|^{-2q} \sup_{B_{i,\delta}} \left| \lambda - V + C_{d,\delta}|x|^{-2q} \right|^d \right] \]

We break the last term in (72) in two:

\[(73) \quad \sum_{i \geq 1} \text{Vol}(B_i) \sup_{B_{i,\delta}} C_{d,\delta}|x|^{-2q} \sup_{B_{i,\delta}} \left| \lambda - V + C_{d,\delta}|x|^{-2q} \right|^d \]
\[(74) \quad \leq \sum_{i \geq 1} \text{Vol}(B_i) C_{d,\delta} \sup_{B_{i,\delta}} |x|^{-2q} \sup_{B_{i,\delta}} \left| C_{d,\delta}|x|^{-2q} \right|^d \]
\[(75) \quad \leq \sum_{i \geq 1} \text{Vol}(B_i) C_{d,\delta} |x|^{-2q} \sup_{B_{i,\delta}} \left| \lambda - V \right|^d + \sum_{i \geq 1} \sup_{B_{i,\delta}} V(x) - C|x|^{-2q} \leq \lambda \]

Now, the last term in (75) is bounded (up to a constant factor) by the number of hypercubes \(B_i\) which intersect the region \(\{x; V(x) - C|x|^{-2q} \leq \lambda\}\). In order to estimate this, let \(Q(x)\) be the inverse of the volume of the hypercube that contains \(x.\) From the construction of our partition, \(Q(x) \leq C|x|^{-qd}.\) We assume that \(2q > a,\) which will be justified later by a suitable choice of \(q.\) Hence, we can bound the last term by the following:
Now, let $x_i$ be the center of the hypercubes $B_i$. We can group the first term in (72) with the first term on the right in (75):

\[
A_\lambda \leq C|\lambda|^{\frac{ad}{q} - \frac{d}{q}} + \sum_{i \geq 1} \text{Vol}(B_i) C_{d,\delta} \sup_{B_i,\delta} |\lambda - V|^{\frac{d}{q} - 1} \left[ \sup_{B_i,\delta} |x|^{-2q} + \text{Diam}(B_{i,\delta}) \sup_{B_i,\delta} |\nabla V| \right] \\
\leq C|\lambda|^{\frac{ad}{q} - \frac{d}{q}} + \sum_{i \geq 1} \text{Vol}(B_i) C_{d,\delta} \sup_{B_i,\delta} [-V]^{\frac{d}{q} - 1} \left[ \sup_{B_i,\delta} |x|^{-2q} + \text{Diam}(B_{i,\delta}) \sup_{B_i,\delta} |\nabla V| \right]
\]

Now, by the construction of the cubes $B_i$, we have that for every $i, \delta$ and exponent $P < 0$, $\sup_{B_{i,\delta}} |x|^P \leq C(P) \inf_{B_{i,\delta}} |x|^P$. Hence, we can estimate the sum on the right-hand side by an integral:

\[
A_\lambda \leq C|\lambda|^{\frac{ad}{q} - \frac{d}{q}} + \int_{B_0(C|\lambda|^{-1/\alpha}) \setminus B_0(r)} C|x|^{-\frac{ad}{q} + a} \left[ |x|^{-2q} + |x|^{q} |x|^{-c} \right] \, dx \\
\leq C|\lambda|^{\frac{ad}{q} - \frac{d}{q}} + C|\lambda|^{\frac{d}{q} - 1 + \frac{2d}{q} - \frac{d}{q}} + C\|\lambda\|^{\frac{d}{q} - 1 + \frac{2d}{q} - \frac{d}{q} - \frac{d}{q}} + C\|\lambda\|^{\frac{d}{q} - 1 + \frac{2d}{q} - \frac{d}{q} - \frac{d}{q}}
\]

Now, taking $q = \frac{2}{3}$ balances the last two terms and we end up with the following:

\[
A_\lambda \leq C|\lambda|^{\frac{ad}{q} - \frac{d}{q}} + C|\lambda|^{\frac{d}{q} - \frac{d}{q} + \frac{2d}{q} - \frac{d}{q} - \frac{d}{q} - 1}
\]

Let us note that the assumption $a, b < 2$ implies that $c < 3$ and that $q < 1$. In turn, this makes sure that the conditions of lemma 4.3 are fulfilled. Now, if $d = 2$, both terms are equal. If $d \geq 3$, then the right term dominates and $A_\lambda \leq C|\lambda|^{\frac{d}{q} - \frac{d}{q} + \frac{2d}{q} - \frac{d}{q} - \frac{d}{q} - 1}$.

Finally, we note that by imposing $c > \frac{3}{2} a$, we fulfill the condition $2q > a$. This completes the proof of lemma 4.3.

C.2. Proof of lemma 4.4. We want to show that $N(\lambda) \geq (2\pi)^{-d} w_d W_\lambda + O(\lambda^{\frac{d}{q} - \frac{d}{q} + \frac{2d}{q} - \frac{d}{q} - \frac{d}{q}})$.

Using equation 40 and the formula for the Dirichlet spectrum of a hypercube with remainder, we obtain the following:

\[
N(\lambda) \geq \sum_{i \geq 1} \max_{V \subset H_0^1(B_i)} \dim V_i \\
\forall f \in V_i, R_{B_i}(f) < \lambda - \sup_{B_i} V
\]

\[
\geq \sum_{i \geq 1} |B_i|(2\pi)^{-d} w_d \inf_{B_i} (\lambda - V)^{\frac{d}{q} +} + O \left( \sum_{i \geq 1} |B_i| \inf_{B_i} (\lambda - V)^{\frac{d}{q} - \frac{1}{2}} \right)
\]
Under the assumption $c > \frac{3}{2}a$, choosing $q = \frac{e}{3}$ in the partition enables us to control the first term (using estimates from the previous section):

\[
\sum_{i \geq 1} |B_i| \inf_{B_i} |\lambda - V|^{\frac{d}{d + 2}} = W_\lambda + O(|\lambda|^{\frac{d}{d + 2} - \frac{2c}{3a} - 1})
\]  

(86)

We now bound the second term by noting that due to the construction of the partition, for any exponents $q > 0$ and $P < 0$, $\sup_{B_i} |x|^P \leq C(q, P) \inf_{B_i} |x|^P$:

\[
\sum_{i \geq 1} |\partial B_i| \inf_{B_i} |\lambda - V|^{\frac{d}{d + 2}} \leq C \sum_{i \geq 1} |B_i| \sup_{B_i} |x|^{-\frac{q}{d} - \frac{1}{2}} \inf_{B_i} |\lambda - V|^{\frac{d}{d + 2}}
\]  

(87)

\[
\leq C \sum_{i \geq 1} \inf_{B_i} |x|^{-\frac{q}{d} - \frac{1}{2}} \inf_{B_i} (C|x|^{-a})^{\frac{d}{d + 2}}
\]  

(88)

\[
\leq C \int_{B_0(C|\lambda|^{-1/a}) \setminus B_0(r)} |x|^{-\frac{q}{d} - \frac{1}{2}} (C|x|^{-a})^{\frac{d}{d + 2}}
\]  

(89)

\[
\leq C(d, V)|\lambda|^{\frac{d}{d - \frac{d - 2}{a} - \frac{2c}{3a} - 1}}
\]  

(90)

Since $\frac{c}{3a} > \frac{1}{2}$, then $\frac{d}{2} - \frac{d}{a} + \frac{c}{3a} - \frac{1}{2} < \frac{d}{2} - \frac{d}{a} + \frac{2c}{3a} - 1$ and

\[
N(\lambda) \geq (2\pi)^{-d} d_w W_\lambda + O(|\lambda|^{\frac{d}{d - \frac{d}{a} + \frac{2c}{3a} - 1}}).
\]  

This completes the proof of Lemma 4.4.

**Appendix D. Proof of Lemma 4.1**

We recall Lemma 4.1:

For any $q < 1$ and $r > 0$, there exist a sequence of pairs of positive numbers $(r_i, d_i)$, $i \geq 1$ and positive constants $M, M'$ such that

1. $r_1 = r > 0$,
2. $r_{i+1} = r_i + d_i$,
3. $\frac{d_i}{r_i} \in \mathbb{N}$,
4. $Mr_i^q \leq d_i \leq M' r_i^q$,
5. $r_i \to \infty$.

**Proof.** Let us assume that we are given a sequence of positive integers $(n_i)$, $i \geq 1$ and that we define $r_i$ and $d_i$ recursively by

\[
r_1 = r, \\
d_i = \frac{r_i}{n_i}, \\
r_{i+1} = r_i + d_i.
\]

Then, Properties (1)–(3) are automatically satisfied. It remains to choose $(n_i)$ in such a way that (4)–(5) hold. Dividing by $r_i^q$, (4) can be rewritten

\[
M \leq \frac{r_i^{1-q}}{n_i} \leq M'.
\]  

(91)

We can define $n_i$ by

\[
n_i = \left\lceil r_i^{1-q} \right\rceil,
\]

where $\lceil x \rceil$ is the smallest integer at least as large as $x$. Note that by combining this definition with the previous one, we construct the sequences $(r_i, d_i)$ and $(n_i)$ recursively, starting from $r_1 = r$. The sequence
\( (r_i) \) is increasing, so it either is bounded or goes to \( \infty \). If it was bounded, then we would have \( n_i \leq N \) for some integer \( N \), and therefore
\[
r_{i+1} \geq \left( 1 + \frac{1}{N} \right) r_i.
\]
This implies \( r_i \to \infty \), in contradiction with the assumption. Therefore \( r_i \to \infty \), so \( (5) \) holds. Finally, we have
\[
r_i^{-q} \leq n_i < r_i^{-q} + 1.
\]
We deduce
\[
\frac{1}{1+ r_i^{q-1}} \leq \frac{1}{1+ \frac{r_i^{-q}}{n_i}} \leq \frac{1}{n_i} \leq 1.
\]
Since \( d_i = r_i/n_i \), this shows that \( (4) \) holds with \( M = 1/(1 + r_i^{q-1}) \) and \( M' = 1 \).

**Remark D.1.** For any sequence \((r_i, d_i)\) satisfying the properties of Lemma 4.1, there exists positive constants \( M'' \), \( M''' \) such that
\[
(92) \quad M'' \leq \frac{d_{i+1}}{d_i} \leq M'''.
\]
Indeed, dividing Property \((4)\) by \( r_i \), we find
\[
Mr_i^{q-1} \leq \frac{d_i}{r_i} \leq M' r_i^{q-1},
\]
and Property \((5)\) then implies \( d_i/r_i \to 0 \). We then get, from Property \((2)\),
\[
\frac{r_{i+1}}{r_i} = 1 + \frac{d_i}{r_i} \to 1.
\]
Since we have
\[
\frac{M}{M'} \left( \frac{r_{i+1}}{r_i} \right)^q \leq \frac{d_{i+1}}{d_i} \leq \frac{M'}{M} \left( \frac{r_{i+1}}{r_i} \right)^q
\]
from Property \((4)\), the desired result follows.

**References**

[1] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*. Princeton University Press, Princeton, NJ (1982).

[2] P. Bérard & D. Meyer, *Inégalités isopérimétriques et applications*. Annales scientifiques de l’École normale supérieure, Sér. 4, 15:3 (1982), 513–541.

[3] P. Charron, *A Pleijel-type theorem for the quantum harmonic oscillator*. Journal of Spectral Theory, 8:2 (2018), 715–732.

[4] P. Charron, B. Helffer, & T. Hoffmann-Ostenhof, *Pleijel’s theorem for Schrödinger operators with radial potentials*. Ann. Math. Québec 42 (2018), 7–29.

[5] R. Courant, *Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke*. Nachr. Ges. Göttingen (1923), 81–84.

[6] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*. Commun. Pure Appl. Anal. 9 (2010), 109–140.

[7] H. L. Cycon, R. G. Froese, W. Kirsch & B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Springer-Verlag, Berlin (1987).

[8] B. Helffer & M. Persson Sundqvist. *On nodal domains in Euclidean balls*. Proc. Amer. Math. Soc. 144 (2016), 4777–4791.

[9] B. Helffer & J. Sjöstrand, *Multiple wells in the semiclassical limit*. I Comm. Partial Differential Equations 9 (1984), 337–408.

[10] C. Léna, *Pleijel’s nodal domain theorem for Neumann and Robin eigenfunctions*. Annales de l’Institut Fourier, Tome 69, no 1 (2019), 283–301.

[11] J. Peetre, *A generalization of Courant’s nodal line theorem*. Math. Scandinavica 5 (1957), 15–20.

[12] A. Pleijel, *Remarks on Courant’s nodal line theorem*. Comm. Pure Appl. Math. 9 (1956), 543–550.

[13] I. Polterovich, *Pleijel’s nodal domain theorem for free membranes*. Proc. Amer. Math. Soc. 137:3 (2009), 1021–1024.

[14] D. Robert, *Comportement asymptotique des valeurs propres d’opérateurs du type Schrödinger à potentiel ”dégénéré”*. (French. English summary) [Asymptotic behavior of the eigenvalues of Schrödinger operators with ”degenerate” potential] J. Math. Pures Appl. (9) 61 (1982), 275–300.

[15] G. Rozenblum, *Asymptotic behavior of the eigenvalues of the Schrödinger operator. (Russian)* Mat. Sb. (N.S.) 93 (135) (1974), 347–367, 487.

[16] B. Simon, *Nonclassical eigenvalue asymptotics*. J. Funct. Anal. 53 (1983), 84–98.
Philippe Charron, Mathematics Department Technion – Israel Institute of Technology, Haifa 32000, Israel
Email address: philippe.ch@campus.technion.ac.il

Corentin Léna, Institut de Mathématiques, Bâtiment UniMail Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland
Email address: corentin.lena@unine.ch