Localization in non-chiral network models for two-dimensional disordered wave mechanical systems

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(August 6 1998)

Scattering theoretical network models for general coherent wave mechanical systems with quenched disorder are investigated. We focus on universality classes for two dimensional systems with no preferred orientation: Systems of spinless waves undergoing scattering events with broken or unbroken time reversal symmetry and systems of spin 1/2 waves with time reversal symmetric scattering. The phase diagram in the parameter space of scattering strengths is determined. The model breaking time reversal symmetry contains the critical point of quantum Hall systems but, like the model with unbroken time reversal symmetry, only one attractive fixed point, namely that of strong localization. Multifractal exponents and quasi-one-dimensional localization lengths are calculated numerically and found to be related by conformal invariance. Furthermore, they agree quantitatively with theoretical predictions. For non-vanishing spin scattering strength the spin 1/2 systems show localization-delocalization transitions.

PACS numbers: 42.25.Kb coherence; 73.23.-b mesoscopic systems; 72.15.Rn quantum localization; 61.43.Hv fractals

Large intensity fluctuations and spatial localization are fascinating universal features in any coherent wave mechanical system subject to quenched disorder. A modeling that covers essential symmetries and characteristic length scales, but does not rely on particular dispersion relations and specific details is highly desirable in various fields of theoretical physics, e.g. in optics, mesoscopic electronics, and quantum chaos (see e.g. [1]). Already in 1982 Shapiro [2] pointed out that a convenient modeling of disordered coherent wave mechanical systems can be given by networks of unitary random scattering matrices. Only recently it was fully appreciated that such network models (NWMs) have a number of advantages over more traditional Hamiltonian models (e.g. the Anderson model used in mesoscopic electronics or the Helmholtz equation with random refraction index used in optics). NWMs yield directly transport quantities [3], wave packet dynamics [4], and quasi-energy eigenvalues and eigenstates [5]. Furthermore, real space renormalization group (RG) treatments of NWMs [6] open new perspectives for investigating localization-delocalization (LD) transitions. A well known NWM introduced by Chalker and Coddington (CC-model) [7] (see also [8]) describes the situation of disordered two-dimensional (2D) electrons undergoing the quantum Hall LD transition in a strong perpendicular magnetic field.

In contrast to the CC-model here we deal with systems in the absence of fields that would introduce a handedness (chirality). However we do allow for the breaking of time reversal symmetry. In particular, we address three classes of non-chiral (NC) 2D NWMs with respect to their quantum localization properties. The first model describes time reversal symmetric scattering and is denoted as O2NC-model, where 'O' stands for 'orthogonal' (a corresponding Hamiltonian can be diagonalized by orthogonal matrices). Second, a similar model with broken time reversal symmetry ('U' for 'unitary') is introduced denoted as U2NC-model. It describes e.g. the motion of mesoscopic (spinless) electrons in the presence of random magnetic fields with zero mean and disorder potentials. The third model deals with time reversal symmetric scattering also in spin degrees of freedom. It is denoted as S2NC-model ('S' for 'symplectic') and a detailed analysis for this model is given in [10]. Here we focus on the U/O2NC-model, discuss their construction and the phase diagram in detail, and present a quantitative check of analytical results for quasi-1D localization lengths, multifractal exponents and conformal invariance in the weak localization regime [8].

Quite generally, a NWM can be constructed as follows. Take a regular network of \( N \) sites and \( N \) bonds. Each bond \( \alpha \) carries propagating wave modes \( (n^+_{\alpha}, n^-_{\alpha}) \) outgoing modes) represented by complex amplitudes, \( \psi_{n^\pm_{\alpha}} \). On the sites unitary \( S \)-matrices map incoming to outgoing amplitudes. The elements of each \( S \)-matrix are (in general) random quantities respecting the symmetries of \( S \) and are characterized by typical scattering strengths (see Fig. 1). Random phases are attached to the amplitudes on the bonds. They simulate the random distances between scatterers in realistic systems.

The construction of a NWM is fixed by the choice of a certain type of random \( S \)-matrix and a \( N \times N \) connectivity matrix \( C \) which has elements \( C_{ij} = 1 \) if a wave mode can propagate from site \( i \) to site \( j \) and \( C_{ij} = 0 \) otherwise. The NWM defines a dynamical system since \( S \) and \( C \) give rise to a unitary matrix \( U \) [8,11] that maps all incoming to outgoing wave modes. Its dimension \( B \) equals the to-
tial number of amplitudes defined on bonds, \( B \propto N \). We choose a convenient time unit and denote the vector of the \( B \) amplitudes as \( \psi \) such that \( \psi(t+1) = U\psi(t) \). The eigenphases \( \phi_n \) (\( n = 1 \ldots B \)) and corresponding eigenvectors \( \psi_n \) of \( U \), \( U\psi_n = \exp(i\phi_n)\psi_n \), can then be interpreted as quasi-energies and corresponding eigenstates.

Let us now report on the U2NC- and the O2NC-model. The corresponding \( S \)-matrix is graphically represented in Fig. 1a. It maps four ingoing modes to four outgoing modes consistent with the connectivity matrix that corresponds to a regular 2D square lattice where each site is connected to four nearest neighbors. Apart from random phases the parameters of each \( S \) matrix are the strengths of transmission \( t \), reflection \( r \), and deflection \( d \) (\( r, t \) and \( d \) denote the modulus of the corresponding scattering amplitudes) \([2]\). Deflection to the left and right are set equal. This choice classifies the model to be non-chiral.

The unitarity of the \( M \)-matrix \([23]\) corresponds to a length \( M \). The second contribution, \( l_\beta(2 - \beta)/M \), describes the sign of the RG-flow equation \( d\Lambda/d(\ln M) = -(\Lambda - \Lambda_0) \), hence the weak localization (anti-localization) correction for \( \beta = 1 \) (\( \beta = 4 \)). The localization correction for \( \beta = 2 \) is not contained in this approximation.

Fig. 2 displays a contour plot of the function \( \Lambda(r, t) \) for the U2NC-model with \( M = 32 \). A increases dramatically for \( r \to 1 \), decreases for \( r \to 1 \), and vanishes at \( r = 1 \). For \( r, t \to 0 \), it drops down to the value \( \Lambda \approx 1.23 \pm 0.02 \) which is known as the critical value in the CC-model (see e.g. \([4]\)). Varying \( M \) leads to qualitatively similar contour plots, leaving the three fixed point values constant (up to small finite-size effects). We found no indication for the existence of further fixed points. Instead we found that LLs converge to finite values \( \xi_\infty \) as long as scattering parameters are in a regime close to the LFP. As expected there, the scaling flow towards the LFP can be described by a one-parameter scaling function. The parameter is just \( \xi_\infty(r, t)/M \). The same qualitative findings, indicating strong localization in the thermodynamic limit, were obtained for the restricted phase space of the O2NC-model (see Fig. 1b). Close to the CC-FP in the U2NC-model, where \( r, t \ll 1 \), a different one-parameter scaling function of the parameter \( M^2(r^2 + t^2) \) gives a reasonable fit to the RG flow away from the CC-FP. This parameter can be interpreted as the total tunneling probability between two critical CC-models that are uncoupled at \( r = t = 0 \).

In the regime close to the MFP we found weak localization corrections. For a qualitative analysis we defined the elastic mean free path in our model in a way consistent with the notions of the quasi-1D Fokker-Planck approach, \( l_e/\delta L = \mathcal{T}/(1 - \mathcal{T}) \), where \( \mathcal{T} \) denotes the transmission strength of a microscopic scattering array of extension \( \delta L \). For the diagonal arrangement of the network in the transfer matrix calculations a suitable microscopic scattering array is the one shown in Fig. 1a. It corresponds to a length \( \delta L = 1/2 \) and yields

\[
l_e = \frac{1}{2} \frac{t^2 + d^2}{r^2 + d^2}.
\]

The scattering array has a Landauer conductance \( \mathcal{T}/(1 - \mathcal{T}) \) per channel that we denote as microscopic conductance, \( g_{mi} = 4l_e \). A classical conductance \( g_{cl} \) can also be defined by using the Einstein relation (in atomic units), \( g_{cl} = 2\pi\rho D \). Here \( \rho \) denotes the density of states per unit volume in (quasi-)energie space, \( D = v\ell_e/2 \) the classical diffusion constant, and \( v \) the wave mode velocity. By construction we have \( v = 1, B = 4V \). The range of quasi-energies is \( 2\pi \). Thus, we find for the U/O2NC-NWM \( g_{cl} = 2l_e = g_{mi}/2 \). The channel number for a given width \( M \) is \( N_c = 2M \). Thus, in the regime of large \( \Lambda \), we expect
\[ \Lambda = g_m \frac{2\beta}{4} + \frac{2 - \beta}{4M} \]  

(2)

to be a good approximation to our data. In Fig. 3a we have plotted about 300 values of \( \Lambda \) versus \( g_m \) for the U2NC-model obtained for \( M = 4, 8, 16, 32 \) and \( r, t \) values distributed over the entire phase space. For \( \Lambda > 3 \) almost all data points are consistent with Eq. (2) (for \( \beta = 2 \)). We checked that those events which are clearly off the line \( \Lambda = g_m \) belong to very low values of \( d \) and thus correspond to almost decoupled 1D chains. For low values of \( \Lambda \) (see inset of Fig. 3a) strong localization can be observed. We made similar observations for the O2NC-model.

As a second quantitative test we calculated the scaling exponent \( \alpha_0 \) describing the scaling of the typical local density of states, \( \rho_L \propto L^{2-\alpha_0} \), in a square geometry for which the 2D LL \( \xi \) is much larger than the (linear) system size \( L \). In 2D these states behave like critical states and are multifractal (for review see [15]). Non-linear \( \sigma \)-model calculations yield analytic results for multifractal exponents in this regime, and the result for \( \alpha_0 = 2 \) reads [16]

\[ \alpha_0 - 2 = (\pi \beta g_{cl})^{-1}. \]  

(3)

This result can also directly be obtained from Eq. (2) and a conformal mapping relation between quasi-1D LLs and multifractal exponents (successfully tested in 2D quantum Hall systems at criticality [13]). For \( \alpha_0 = 2 \) this relation yields \( \alpha_0 - 2 = (\pi \Lambda)^{-1} \). Together with Eq. (2), in the limit where \( \Lambda \) is approximately independent of \( M \), this leads to Eq. (3) [17]. In Fig. 3b the calculated values of \( \alpha_0 \) are plotted versus the values obtained from the conformal mapping relation, denoted as \( \alpha_0^c \). The relation is fulfilled for \( \alpha_0 \simeq 2 \), i.e. for large values of \( \Lambda \). Also the critical point of the CC-model fulfills the relation within the errors. As expected, for intermediate and small values of \( \Lambda \) the localization corrections – in particular for the O2NC-model – lead to deviations.

Finally, we report briefly on our findings in the S2NC-model (for details see [14]). It is based on the O2NC-model. In addition to the scattering parameters \( t \) and \( r \) a new parameter, the spin scattering parameter \( s \in [0, 1] \), appears. As can be seen in Fig. 4a each scattering matrix of the O2NC-model has twice the number of incoming and outgoing modes to take care of the spin degree of freedom. The \( S \)-matrix denoted as ‘potential’ belongs to two superimposed uncoupled O2NC-models. A time reversal symmetric spin scattering process can occur, however, at each \( S \)-matrix of the ‘spin’ type. This spin \( S \)-matrix is characterized by the spin scattering parameter \( s \). For \( s = 1 \) an entering spin mode becomes randomized in spin space after a single scattering event. The peculiar time reversal behavior of spin 1/2 states leads to the effect of weak anti-localization (see e.g. [18]). This turns the MFP into an attractive fixed point, hence the occurrence of a LD transition in 2D seems to be unavoidable, and was observed in numerical calculations for Hamiltonian models (e.g. [19]). For fixed spin scattering parameter \( s \) we determined via the transfer matrix method the phase diagram in \((r, t)\)-space. For any \( s > 0 \) a metallic phase appears as soon as \( r \) is small and \( t \) is large enough (see Fig. 4b). The area of the metallic phase gradually shrinks to zero for \( s \rightarrow 0 \). On fixing two of the parameters, \( t = 0.6 \) and \( s = 0.4 \), and varying \( r \) we obtained a scaling function (by an iterative fit procedure) for the LD transition, the quality of which was checked by a \( \chi^2 \)-test. The scaling function describes the quantity \( \Lambda(r, M) \) as a two-branch function \( f^\pm(\xi_c(r)/M) \) where \( \xi_c(r) \) is the correlation length of the LD transition. We confirmed the assumption that the LD transition is governed by one-parameter scaling. Approaching the transition point \( r^* \) the correlation length diverges, \( \xi_c \propto |r - r^*|^{-\nu} \), with critical exponent \( \nu = 2.51 \pm 0.18 \). With the help of the conformal mapping relation \( \alpha_0 = 2.174 \pm 0.003 \) follows from the critical value \( \Lambda^* = 1.83 \pm 0.03 \) obtained as the branching point of the scaling function.

In conclusion, we have studied non-chiral network models for coherent wave mechanical systems with quenched disorder in two dimensions. In the parameter space of scattering strengths we found three fixed points of a finite size scaling flow: a metallic fixed point, a localization fixed point and the quantum Hall fixed point. The latter is part of phase space only in systems which break time reversal symmetry (U2NC-model). Only the localization fixed point is attractive and thus almost all states localize in the thermodynamic limit. Furthermore, we showed that a small amount of time reversal symmetric spin scattering (S2NC-model) turns the metallic fixed point into an attractive one, and metallic and localized phases occur.

We thank A. Altland, A. Dohmen, B. Huckestein, R. Klesse, M. Metzler and in particular B. Shapiro for useful discussions. This work was supported by the DFG within the SFB 341.

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FIGURE CAPTIONS:
Figure 1: a) The scattering unit of the U/O2NC-model. The modulus of scattering amplitudes for transmission is $t$, for reflection $r$ and for deflection $d$. b) Phase space of the U/O2NC-models. The area in light grey is only available in the U2NC-model. The dark grey area corresponds to parameters available in both models. Fixed points are marked by bullets and decoupled 1D chains correspond to the line $r^2 + t^2 = 1$.
Figure 2: Contour plot of the scaling variable $\Lambda$ in the parameter space of the U2NC-model for a fixed system width $M = 32$.
Figure 3: Weak Localization effects: a) $\Lambda$ for the U2NC-model as a function of $g_{mi}$ for system widths of $M = 4, 8, 16, 32$ and varying values of $r \in [0, 0.9], t \in [0, 0.9]$. The inset shows an enlargement for $\Lambda < 2$. b) The numerically obtained multifractal exponents $\alpha_0$ corresponding to different scattering parameters, plotted versus $\alpha^c_0$ calculated by Eq. (3).
Figure 4: S2NC-model: a) Section of the S-matrix for the S2NC-NWM. In contrast to the O2NC-NWM the number of channels per bond is doubled. Straight lines correspond to spin-up, dashed lines to spin-down. Potential scatterers (light grey) can change the direction of the motion of electrons while spin scatterers can only scatter in the spin degrees of freedom. b) Phase diagram of the transition for spin scattering strength $s = 0.1$. The grey area shows the delocalized regime and the white area the localized regime.
FIGURE 1b.
FIGURE 3b.
FIGURE 4a.
FIGURE 4b.