A topological space \( X \) is called a \( Q \)-space if every subset of \( X \) is of type \( F_{\sigma} \) in \( X \). For \( i \in \{1, 2, 3\} \) let \( q_i \) be the smallest cardinality of a second-countable \( T_i \)-space which is not a \( Q \)-space. It is clear that \( q_1 \leq q_2 \leq q_3 \). For \( i \in \{1, 2\} \) we prove that \( q_i \) is equal to the smallest cardinality of a second-countable \( T_i \)-space which is not perfect. Also we prove that \( q_3 \) is equal to the smallest cardinality of a submetrizable space, which is not a \( Q \)-space. Martin’s Axiom implies that \( q_i = c \) for all \( i \in \{1, 2, 3\} \).

A topological space \( X \) is called

- **perfect** if every open subset is of type \( F_{\sigma} \) in \( X \);
- a **\( Q \)-space** if every subset is of type \( F_{\sigma} \) in \( X \);
- a **non-\( Q \)-space** if \( X \) is not a \( Q \)-space.

A subset of topological space is of type \( F_{\sigma} \) if it can be written as the union of countably many closed sets.

It is clear that every \( Q \)-space is perfect. Under Martin’s Axiom, every metrizable separable space of cardinality \( < c \) is a \( Q \)-space, see [5, 4.2].

For a class \( \mathcal{T} \) of topological spaces, denote by \( q_{\mathcal{T}} \) the smallest cardinality of a non-\( Q \)-space \( X \in \mathcal{T} \). The cardinal \( q_{\mathcal{T}} \) is well-defined only for classes \( \mathcal{T} \) containing non-\( Q \)-spaces. In this paper we study the cardinals \( q_{\mathcal{T}} \) for classes \( \mathcal{T} \) of topological spaces satisfying various separation properties.

A topological space \( X \) is called

- a **\( T_1 \)-space** if every finite subset is closed in \( X \);
- a **\( T_2 \)-space** if \( X \) is Hausdorff, which means that any distinct points in \( X \) have disjoint neighborhoods;
- a **\( T_{2\frac{1}{2}} \)-space** if \( X \) is Urysohn, which means that any distinct points in \( X \) have disjoint closed neighborhoods;
- **functionally Hausdorff** if for any distinct points \( x, y \in X \) there exists a continuous function \( f : X \to \mathbb{R} \) such that \( f(x) \neq f(y) \);
- a **\( T_3 \)-space** if \( X \) is Hausdorff and every neighborhood of any point \( x \in X \) contains a closed neighborhood of \( x \);
- **Tychonoff** if \( X \) is Hausdorff and for any closed set \( F \subseteq X \) and point \( x \in X \setminus F \) there exists a continuous function \( f : X \to \mathbb{R} \) such that \( f(x) = 1 \) and \( f[F] \subseteq \{0\} \);
- **submetrizable** if there exists a continuous bijective map \( f : X \to Y \) onto a metric space \( Y \);
- **second-countable** if \( X \) has a countable base of the topology.
For every topological space we have implications

\[
\begin{align*}
\text{metrizable} & \longrightarrow \text{submetrizable} & \longrightarrow & \text{functionally Hausdorff} \\
\text{Tychoff} & \longrightarrow \text{T}_3\text{-space} & \longrightarrow \text{Urysohn} & \longrightarrow \text{Hausdorff} & \longrightarrow \text{T}_1\text{-space}
\end{align*}
\]

Observe that the one-point compactification of the discrete space of cardinality \(\omega_1\) is not perfect and hence \(q_T = \omega_1\) for the class \(T\) of (compact) Tychoff spaces of weight \(\leq \omega_1\). For classes \(T\) of second-countable spaces the cardinals \(q_T\) are more interesting.

For \(i \in \{1, 2, 2_1, 3\}\) denote by \(q_i\) the smallest cardinality of a second-countable \(T_i\)-space which is not a \(Q\)-space. It is clear that \(q_1 \leq q_2 \leq q_{2_1} \leq q_3\). Since each second-countable \(T_3\)-space (of cardinality < \(c\)) is metrizable (and zero-dimensional), the cardinal \(q_3\) coincides with the well-known cardinal \(q_0\), defined as the smallest cardinality of a subset of \(\mathbb{R}\) which is not a \(Q\)-space. The cardinal \(q_0\) is well-studied in Set-Theoretic Topology, see [5, §4], [3] or [1]. This cardinal has the following helpful property.

**Proposition 1.** Every submetrizable space of cardinality < \(q_0\) is a \(Q\)-space.

**Proof.** Let \(X\) be a submetrizable space of cardinality < \(q_0\). Find a continuous bijective map \(f : X \rightarrow Y\) onto a metrizable space \(Y\). The metrizable space \(Y\) has cardinality \(|Y| = |X| < q_0 \leq c\) and weight \(w(Y) \leq \omega \cdot |Y| \leq c\). By [4, 4.4.9], \(Y\) admits a topological embedding \(h : Y \rightarrow J(c)^\omega\) into the countable power of the hedgehog space \(J(c)\) with \(c\) many spikes. Here \(J(c)\) is the set \(\{x \in [0, 1]^c : |\{\alpha \in c : x(\alpha) > 0\}| \leq 1\}\) endowed with the metric

\[
d(x, y) = \max_{\alpha \in c} |x(\alpha) - y(\alpha)|.
\]

It is easy to see that the hedgehog space \(J(c)\) admits a continuous bijective map onto the triangle \(\{(x, y) \in [0, 1]^2 : x + y \leq 1\}\) and hence \(J(c)^\omega\) admits a continuous bijective map \(\beta : J(c)^\omega \rightarrow [0, 1]^\omega\) onto the Hilbert cube \([0, 1]^\omega\). Then \(g \overset{\text{def}}{=} \beta \circ f : X \rightarrow [0, 1]^\omega\) is a continuous injective map. The metrizable separable space \(g[X]\) has cardinality < \(q_0\) and hence is a \(Q\)-space. Then for every set \(A \subseteq X\) it image \(g[A]\) is of type \(F_\sigma\) in \(g[X]\). By the continuity of \(g\), the preimage \(g^{-1}[g[A]] = A\) of \(g[A]\) is an \(F_\sigma\)-set in \(X\), witnessing that \(X\) is a \(Q\)-space. \(\square\)

Now we prove some criteria of submetrizability among “sufficiently small” functionally Hausdorff spaces.

A family \(\mathcal{F}\) of subsets of a topological space \(X\) is called

- **separating** if for any distinct points \(x, y \in X\) there exists a set \(F \in \mathcal{F}\) that contains \(x\) but not \(y\);
- **a network** if for every open set \(U \subseteq X\) and point \(x \in U\) there exists a set \(F \in \mathcal{F}\) such that \(x \in F \subseteq U\).

We say that a topological space \(X\) is

- **Lindelöf** if every open cover of \(X\) has a countable subcover;
- **hereditarily Lindelöf** if every subspace of \(X\) is Lindelöf;
- **nw-countable** if \(X\) has a countable network;
- **sw-countable** if \(X\) has a countable separating family of open sets.

It is clear that every second-countable \(T_1\)-space is both nw-countable and sw-countable.
Lemma 2. Every functionally Hausdorff space $X$ with hereditarily Lindelöf square is submetrizable.

Proof. Since $X$ is functionally Hausdorff, for any distinct points $a, b \in X$, there exists a continuous function $f_{a,b} : X \to \mathbb{R}$ such that $f_{a,b}(a) = 0$ and $f_{a,b}(b) = 1$. By the continuity of $f_{a,b}$, the sets

$$f_{a,b}^{-1}(\downarrow \frac{1}{2}) \overset{\text{def}}{=} \{ x \in X : f_{a,b}(x) < \frac{1}{2} \} \quad \text{and} \quad f_{a,b}^{-1}(\uparrow \frac{1}{2}) \overset{\text{def}}{=} \{ y \in X : f_{a,b}(y) > \frac{1}{2} \}$$

are open neighborhoods of the points $a, b$, respectively. Since the space $X \times X$ is hereditarily Lindelöf, the open cover $\{ f_{a,b}^{-1}(\downarrow \frac{1}{2}) \times f_{a,b}^{-1}(\uparrow \frac{1}{2}) : (a,b) \in \nabla X \}$ of the subspace

$$\nabla X \overset{\text{def}}{=} \{(x,y) \in X \times X : x \neq y \}$$

of $X \times X$ has a countable subcover. Consequently, there exists a countable set $C \subseteq \nabla X$ such that

$$\nabla X = \bigcup_{(a,b) \in C} f_{a,b}^{-1}(\downarrow \frac{1}{2}) \times f_{a,b}^{-1}(\uparrow \frac{1}{2})$$

Consider the metrizable space $\mathbb{R}^C$ and the continuous function

$$f : X \to \mathbb{R}^C, \quad f : x \mapsto (f_{a,b}(x))_{(a,b) \in C}.$$ 

This function is injective because for any distinct $x, y \in X$ there exists a pair $(a,b) \in C$ such that $(x,y) \in f_{a,b}^{-1}(\downarrow \frac{1}{2}) \times f_{a,b}^{-1}(\uparrow \frac{1}{2})$ and hence $f_{a,b}(x) < \frac{1}{2} < f_{a,b}(y)$, witnessing that $f(x) \neq f(y)$. □

Corollary 3. Every nw-countable functionally Hausdorff space of cardinality $< \aleph_0$ is a submetrizable $Q$-space.

Proof. Let $X$ be an nw-countable functionally Hausdorff space. By [4, 3.8.12], the square $X \times X$ has countable network and is hereditarily Lindelöf. By Lemma 2, $X$ is submetrizable and by Proposition 1, $X$ is a $Q$-space. □

Propositions 1 and Corollary 3 will help us to prove the following characterization of the cardinal $\aleph_0$.

Proposition 4. The cardinal $\aleph_0$ is equal to:

- the smallest cardinality of a submetrizable space which is not a $Q$-space;
- the smallest cardinality of a non-perfect submetrizable space;
- the smallest cardinality of an nw-countable functionally Hausdorff non-$Q$-space;
- the smallest cardinality of an non-perfect nw-countable functionally Hausdorff space;
- the smallest cardinality of second-countable functionally Hausdorff non-$Q$-space.
- the smallest cardinality of non-perfect second-countable functionally Hausdorff space.

Proof. Let

- $\aleph_{sm}$ be the smallest cardinality of a submetrizable non-$Q$-space;
- $\aleph_{pm}$ be the smallest cardinality of a non-perfect submetrizable space;
- $\aleph_{nw}$ be the smallest cardinality of an nw-countable functionally Hausdorff non-$Q$-space;
- $\aleph_{nwp}$ be the smallest cardinality of an non-perfect nw-countable functionally Hausdorff space;
- $\aleph_{w}$ be the smallest cardinality of second-countable functionally Hausdorff non-$Q$-space.
- $\aleph_{w}$ be the smallest cardinality of non-perfect second-countable functionally Hausdorff space.
We should prove that all these cardinals are equal to $q_0$. The inclusions between corresponding classes of topological spaces yield the following diagram in which an arrow $\kappa \to \lambda$ between cardinals $\kappa, \lambda$ indicates that $\kappa \leq \lambda$.

![Diagram]

Proposition 7 implies that $q_0 \leq q_{sm}$. To prove that all these cardinals are equal to $q_0$, it remains to prove that $p_w \leq q_0$.

By the definition of the cardinal $q_0$, there exists a second-countable metrizable non-$Q$-space $X$ and hence $X$ contains a subset $A$ which is not of type $G_\delta$ in $X$. Let $\tau'$ be the topology on $X$, generated by the subbase $\tau \cup \{X \setminus A\}$ where $\tau$ is the topology of the metrizable space $X$. It is clear that $X' = (X, \tau')$ is a second-countable space containing $A$ as a closed subset. Since $\tau \subseteq \tau'$, the space $X'$ is submetrizable and functionally Hausdorff. Assuming that $X'$ is perfect, we conclude that the closed set $\tau = \tau \cup \{X \setminus A\} \subseteq \tau'$.

By the choice of the topology $\tau'$, for every $n \in \omega$ there exists open sets $U_n, V_n \in \tau$ such that $W_n = U_n \cup (V_n \setminus A)$. It follows from $A \subseteq W_n = U_n \cup (V_n \setminus A)$ that $A = A \cap W_n = A \cap U_n \subseteq U_n$. Then

$$ A = \bigcap_{n \in \omega} W_n = A \cap \bigcap_{n \in \omega} W_n = \bigcap_{n \in \omega} (A \cap W_n) = \bigcap_{n \in \omega} (A \cap U_n) \subseteq \bigcap_{n \in \omega} U_n \subseteq \bigcap_{n \in \omega} W_n = A $$

and hence $A = \bigcap_{n \in \omega} U_n$ is a $G_\delta$-set in $X$, which contradicts the choice of $A$. This contradiction shows that the functionally Hausdorff second-countable space $X'$ is not perfect and hence $p_w \leq |X'| = q_0$.

Proposition 5. Is $q_2 = q_0$?

Repeating the argument of the proof of Proposition 7 we can prove the following characterization of the cardinals $q_i$ for $i \in \{1, 2, 2^\frac{1}{2}\}$.

Proposition 6. Let $i \in \{1, 2, 2^\frac{1}{2}\}$. The cardinal $q_i$ is equal to the smallest cardinality of a non-perfect second-countable $T_i$-space.

Proposition 7. Every $sw$-countable space of cardinality $< q_1$ is a $Q$-space.

Proof. Let $X$ be an $sw$-countable space of cardinality $< q_1$. By the $sw$-countability of $X$, there exists a countable separating family $\mathcal{U}$ of open sets in $X$. Consider the topology $\tau$ on $X$ generated by the subbase $\mathcal{U}$ and observe that $X_\tau \overset{\text{def}}{=} (X, \tau)$ is a second-countable $T_1$-space of cardinality $< q_1$. The definition of $q_1$ ensures that $X_\tau$ is a $Q$-space. Then every set $A \subseteq X$ is an $F_\sigma$ set in $X_\tau$. Since the identity map $X \rightarrow X_\tau$ is continuous, the set $A$ remains of type $F_\sigma$ in $X$, witnessing that $X$ is a $Q$-space.

Proposition 8. The cardinal $q_1$ is equal to

1. the smallest cardinality of an $sw$-countable non-$Q$-space;
2. the smallest cardinality of a non-perfect $sw$-countable space.

Proof. Let

- $q_{sm}$ be the smallest cardinality of an $sw$-countable non-$Q$-space;
Proposition 9. Every second-countable \( T_1 \)-space \( X \) of cardinality \( |X| < \text{adp} \) is a Q-space and hence
\[
p \leq \text{dp} \leq \text{adp} \leq q_1 \leq q_2 \leq q_{2\aleph_0} \leq q_3 = q_0.
\]

Proof. Given any subset \( A \subseteq X \), we should prove that \( A \) is of type \( G_\delta \) in \( X \). Let \( B = \{U_n\}_{n \in \omega} \) be a countable base of the topology of the space \( X \).

For every \( y \in X \setminus A \), let \( I_y = \{n \in \omega : y \in U_n\} \). Since \( \{U_n\}_{n \in \omega} \) is a base of the topology of \( X \), for every \( x \in A \) there exists an infinite set \( I_x \subseteq \omega \) satisfying two conditions:

- for any numbers \( n < m \) in \( I_x \) we have \( x \in U_m \subseteq U_n \);
- for every neighborhood \( O_x \) of \( x \) in \( X \) there exists \( n \in I_x \) such that \( x \in U_n \subseteq O_x \).
We claim that for any $x \in A$ and $y \in B$ the intersection $I_x \cap I_y$ is finite. Indeed, by the choice of the set $I_y$, there exists $n \in \omega$ such that $U_n \subseteq X \setminus \{y\}$. Then for every $m \geq n$ we have $U_m \subseteq U_n$ and hence $y \notin U_m$ and $m \notin I_y$. Therefore, the families $\{I_x : x \in A\}$ and $\{I_y : y \in X \setminus A\}$ are orthogonal. The same argument shows that the family $\{I_x : x \in A\}$ is almost disjoint.

Since $|A \cup B| = |X| < \mathfrak{d}$, the family $\{I_x : x \in A\}$ can be weakly separated from the family $\{I_y : y \in X \setminus A\}$ and hence there exists a set $D \subseteq \omega$ such that for any $x \in A$ the intersection $I_x \cap D$ is infinite and for any $x \in B$ the intersection $I_y \cap D$ is finite. For every finite set $F \subseteq D$ consider the open subset
\[
W_F \overset{\text{def}}{=} \bigcup_{n \in D \setminus F} U_n
\]
of $X$. For every $x \in A$ the infinite set $I_x \cap D$ contains a number $n \notin F$ and then $x \in U_n \subseteq W_F$. Therefore $G \overset{\text{def}}{=} \bigcap_{F \in |D| < \omega} W_F$ is a $G_{\delta}$-set containing $A$. On the other hand, for every $x \in X \setminus A$, the intersection $F = I_y \cap D = \{n \in D : y \in U_n\}$ is finite and hence $y \notin \bigcup_{n \in D \setminus F} U_n = W_F$.

Therefore, $A = \bigcap_{F \in |D| < \omega} W_F$ is a $G_{\delta}$-set in $X$ witnessing that $X$ is a $Q$-space. \qed

Since $p = \mathfrak{c}$ under Martin’s Axiom, Proposition \ref{prop:iso} implies the following corollary.

**Corollary 10.** Under Martin’s Axiom, $q_i = \mathfrak{c}$ for every $i \in \{0, 1, 2, 2^\omega, 3\}$.

It would be interesting to have any additional information on (im)possible inequalities between the cardinals $q_i$ and other cardinal characteristics of the continuum. In particular, the following questions are natural and seem to be open.

**Problem 11.**
1. Is $\mathfrak{ap} \leq q_2$?
2. Is $q_1 \leq \text{add}(\mathcal{M})$?
3. Is $q_1 = q_2$?
4. Is the strict inequality $q_1 < q_0$ consistent?

Also the position of the new cardinal $\mathfrak{adp}$ in the interval $[\mathfrak{d}, \mathfrak{ap}]$ is not clear.

**Problem 12.**
1. Is $\mathfrak{adp} = \mathfrak{d}$ in ZFC?
2. Is $\mathfrak{adp} = \mathfrak{ap}$ in ZFC?

By [3], the strict inequality $\mathfrak{d} < \mathfrak{ap}$ is consistent, so one of the questions in Problem 12 has negative answer. But which one? Or both?

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