PT-/non-PT-symmetric and non-Hermitian Hellmann potential: approximate bound and scattering states with any $\ell$-values

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Abstract
We investigated the approximate bound-state solutions of the Schrödinger equation for the PT-/non-PT-symmetric and non-Hermitian Hellmann potential. Exact energy eigenvalues and corresponding normalized wave functions were obtained. Numerical energy eigenvalues for the bound states were compared with ones obtained before. Scattering state solutions were also studied. Phase shifts of the potential are written in terms of the angular momentum quantum number $\ell$.

Keywords: Hellmann potential, Schrödinger equation, PT-symmetry

1. Introduction

PT-symmetric quantum mechanics has been widely studied in recent years. The usual form of quantum mechanics has the Hamiltonian-defining symmetries of a Hermitian system. However, for the PT-symmetric case, it has real spectra, although it is not Hermitian. Bender and Boettcher \cite{1} studied this case, but many later authors have studied PT-symmetric and non-Hermitian cases that have real and/or complex eigenvalues \cite{2–10}.

The Hellmann potential
\begin{equation}
V (r) = \frac{1}{r} \left(-a + b e^{-\lambda r}\right),
\end{equation}
with $b > 0$ was first proposed by Hellmann \cite{11, 12} (it was then called the ‘Hellmann potential’ independently of the sign of $b$), and has many applications in atomic and condensed physics \cite{13}. The Hellmann potential has been used as a potential model to calculate the electronic wave functions of metals and semiconductors \cite{14}. Many authors have studied the electron-core \cite{15–17} and electron-ion \cite{18} interactions by using this potential. In \cite{19}, it was proposed that the Hellmann potential is a suitable ground for study of inner-shell ionization problems. The present potential could be used as a potential model for alkali hydride molecules \cite{20}.

Energy eigenvalues of the Hellmann potential have recently been studied by various authors with the help of different methods, such as the $1/N$ expansion method \cite{21}, the shifted large-$N$ expansion method \cite{22}, the method of potential envelopes \cite{23}, the $J$-matrix approach \cite{24} and the generalized Nikiforov–Uvarov method \cite{25, 26}. In the current work, we solve the Schrödinger–Hellmann problem in terms of the hypergeometric functions by using an approximation instead of the centrifugal term. We also extend the computation, including the solutions of the Hellmann-like potential that have the form
\begin{equation}
V (x) = -\frac{a}{x} + \frac{b}{x} e^{-\lambda x},
\end{equation}
which can be written in a PT-symmetric form, and the energy spectra and eigenfunctions of PT-/non-PT- and non-Hermitian Hellmann potential can be obtained with any angular momentum. Scattering state solutions are also studied. Phase shifts of the potential are written in terms of the angular momentum quantum number $\ell$.

The organization of this work is as follows. In section \textsuperscript{2}, we find the approximate energy eigenvalues and the corresponding normalized wave functions of the Hellmann potential. In section \textsuperscript{3}, we obtain the phase shifts of the
potential under consideration in terms of the quantum number \( \ell \). We give our conclusions in last section.

2. Bound states

2.1. Radial solutions

The radial part of the Schrödinger equation (SE) [27]

\[
\left\{ \frac{d^2}{dr^2} + \frac{\ell (\ell + 1)}{r^2} + \frac{2m}{\hbar^2} [E - V(\mathbf{r})] \right\} R(r) = 0. \tag{3}
\]

where \( R(\mathbf{r}) = R(r)/r, \ell \) is the angular momentum quantum number, \( m \) is the particle mass moving in the potential field \( V(\mathbf{r}) \) and \( E \) is the nonrelativistic energy of the particle.

Using the following approximation instead of the centrifugal term [28],

\[
\frac{1}{r^2} \simeq \frac{\lambda^2}{(1-e^{-\frac{\lambda}{r}})^2}, \tag{4}
\]

inserting equation (1) into equation (3) and defining a new variable \( u = e^{-\frac{\lambda}{r}} (0 \leq u \leq 1) \), equation (3) becomes

\[
u(1-u) \frac{d^2R(u)}{du^2} + (1-u) \frac{dR(u)}{du} \times \left[ -\frac{\ell (\ell + 1)}{1-u} + \left( \frac{2mE}{\lambda^2 \hbar^2} + \frac{2ma\lambda}{\lambda^2 \hbar^2} - \ell (\ell + 1) \right) \frac{1}{u} \right.

- \frac{2mb\lambda}{\lambda^2 \hbar^2} \left] \right. \frac{dR(u)}{du} = 0. \tag{5}
\]

Taking a trial wave function as

\[
R(u) = u^{\lambda_1} (1-u)^{\lambda_2} \psi(u), \tag{6}
\]

and inserting into equation (5), we obtain

\[
u(1-u) \frac{d^2\psi(u)}{du^2} + \left[ 1 + 2\lambda_1 - (2\lambda_1 + 2\lambda_2 + 1)u \right] \frac{d\psi(u)}{du} \times \left[ -\lambda_1^2 - \lambda_2^2 - 2\lambda_1\lambda_2 - \frac{2m}{\lambda^2 \hbar^2} [E + b\lambda] \right] \psi(u) = 0, \tag{7}
\]

where

\[
\lambda_1^2 = -\frac{2m}{\lambda^2 \hbar^2} (E + a\lambda) + \ell (\ell + 1), \tag{8}
\]

\[
\lambda_2 = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4\ell (\ell + 1)} \right]. \tag{9}
\]

Comparing equation (7) with the hypergeometric equation of the following form [29]

\[
u(1-u) \frac{d^2y(u)}{du^2} + [c' -(a' + b' + 1)u] \frac{dy(u)}{du} -a'b'y(u) = 0, \tag{10}
\]

we find the solution of equation (7) as the hypergeometric function

\[
\psi(u) = \, _2F_1(a', b'; c'; u), \tag{11}
\]

with

\[
a' = \lambda_1 + \lambda_2 + \Lambda_1, \tag{12}
\]

\[
b' = \lambda_1 + \lambda_2 - \Lambda_1, \tag{13}
\]

\[
c' = 1 + 2\lambda_1, \tag{14}
\]

where \( \Lambda_1 = \frac{1}{2} \sqrt{\frac{8m}{\lambda^2 \hbar^2} (E + b\lambda)} \).

The total wave functions in equation (6) are given as

\[
R(u) = \mathcal{N} u^{\lambda_1} (1-u)^{\lambda_2} \, _2F_1(a', b'; c'; u). \tag{15}
\]

where \( \mathcal{N} \) is normalization constant and will be calculated below. When either \( a' \) or \( b' \) equals a negative integer \( -n \), the hypergeometric function \( \psi(u) \) can give a finite solution form. This gives us a polynomial of degree \( n \) in equation (11) and from the following quantum condition

\[
-n = \lambda_1 + \lambda_2 + 1 \frac{1}{2} \sqrt{\frac{8m}{\lambda^2 \hbar^2} (E + b\lambda)}, \tag{16}
\]

the energy eigenvalue becomes

\[
E = -\frac{1}{8m \hbar^2 (n + \ell + 1)^2} \left\{ 4m^2 \left( a^2 + b^2 \right) + 4m \hbar^2 \ell b \left[ 2\ell^2 + (n + \ell)^2 + \ell (3 + 2n) \right] + \ell^2 \hbar^4 \left[ \ell (1 + 2n) + (n + \ell)^2 \right]^2 + 4am \left[ -2bm + \ell \hbar^2 \left[ \ell (1 + 2n) + (n + \ell)^2 \right] \right] \right\}. \tag{17}
\]

The numerical results obtained from last equation are listed in table 1. We also compare them with the ones given in two different papers. They are in agreement with the previous results, and we should also stress that our results are consistent with the ones given in [25]. Equation (16) gives the wave functions as

\[
R(u) = \mathcal{N} u^{\lambda_1} (1-u)^{\lambda_2} \, _2F_1(-n, n + 2\lambda_1 + 2\lambda_2; 1 + 2\lambda_1; u), \tag{18}
\]

where the normalization constant is obtained from

\[
\int_0^1 \! R(u)^2 du = 1, \tag{19}
\]

which can be written as

\[
\mathcal{N}^2 = \frac{\Gamma(1 + 2\lambda_1)}{m! \Gamma(-n+\ell+1) \Gamma(n + 2\lambda_1 + 2\lambda_2) \sum_{m=0}^{\infty} \frac{\Gamma(-n+m) \Gamma(n + 2\lambda_1 + 2\lambda_2 + m)}{\Gamma(1 + 2\lambda_1 + m)} \int_0^1 u^{m+2\lambda_1} (1-u)^{\lambda_2} \, _2F_1(-n, n + 2\lambda_1 + 2\lambda_2; 1 + 2\lambda_1; u) \, du = 1. \tag{19}
\]

We use the following representation of the hypergeometric
functions [29]
\[ 3F_1(p, q; r; z) = \frac{\Gamma(r)}{\Gamma(p)\Gamma(q)} \times \sum_{m=0}^{\infty} \frac{\Gamma(p + m)\Gamma(q + m)}{\Gamma(r + m)} \frac{z^m}{m!}. \]

By using the following identity [30]
\[ \int_0^1 s^{\nu-1}(1-s)^{\mu-1} \frac{\Gamma(\nu)(\mu)}{\Gamma(\mu + \nu)} \times 3F_2(v, \alpha, \beta; \mu + \nu; \gamma; a), \]
we obtain the normalization constant as
\[ |\mathcal{V}|^2 = \frac{\Gamma(1 + 2\lambda_1)\Gamma(1 + 2\lambda_2)}{m!\Gamma(-n)\Gamma(n + 2\lambda_1 + 2\lambda_2)} \times \sum_{m=0}^{\infty} \frac{\Gamma(-n + m)\Gamma(n + 2\lambda_1 + 2\lambda_2 + m)}{\Gamma(1 + 2\lambda_1 + m)} \times \frac{3F_2(1 + 2\lambda_1 + m, -n, n + 2\lambda_1 + 2\lambda_2; 2 + 2\lambda_1 + 2\lambda_2 + m, 1 + 2\lambda_1; 1)}. \]

For completeness, we give the energy eigenvalues of the Coulomb potential, which corresponds to the case where \( b = 0 \). We write the energy levels of this potential from equation (17) as \( \lambda = 0, \hbar = 1 \)
\[ E = -\frac{ma^2}{2(n + \ell + 1)^2}. \]

### 2.2. PT-symmetric solutions

Inserting equation (2) into the following one-dimensional SE [27]
\[ \frac{\hbar^2}{2m} \frac{d^2\phi(x)}{dx^2} + [V(x) - E]\phi(x) = 0. \]
using the following approximation instead of \( 1/x \) in the potential (see figure 1)
\[ \frac{1}{x} \sim \frac{\lambda}{1 - e^{-i\lambda}}, \]
and taking a new variable as \( u = 1/(1 - e^{-i\lambda}) \) we obtain
\[ u(1 - u) \frac{d^2\phi(u)}{du^2} + (1 - 2u) \frac{d\phi(u)}{du} \times \left[ \frac{2ma}{\lambda \hbar^2} + \frac{2mE}{\lambda \hbar^2} \right] \frac{1}{1 - u} \phi(u) = 0. \]
Defining the wave function as

\[ \phi(u) = u^{\lambda}(1 - u)^{\lambda} \psi(u), \]  

and following the same procedure in the above we get

\[ u(1 - u) \frac{d^2 \psi(u)}{du^2} + \left[ 1 + 2\lambda_1 - (2\lambda_1 + 2\lambda_2 + 1)u \right] \frac{d\psi(u)}{du} - \left[ \lambda_1(\lambda_1 + 1) + \lambda_2(\lambda_2 + 1) + 2\lambda_1\lambda_2 \right] \psi(u) = 0, \]  

where

\[ \lambda_2 = -\frac{2m}{\lambda\hbar^2} \left( b + \frac{E}{\lambda} \right), \]  

\[ \lambda_2 = -\frac{2m}{\lambda\hbar^2} \left( a + \frac{E}{\lambda} \right). \]  

The solution of equation (28) and the total function is given as, respectively,

\[ \psi(u) \sim 2F_1(a', b'; c'; u), \]  

\[ \phi(u) \sim u^{\lambda_2}(1 - u)^{\lambda_2} 2F_1(a', b'; c'; u), \]  

where

\[ a' = 1 + \lambda_1 + \lambda_2, \]  

\[ b' = \lambda_1 + \lambda_2. \]  

The energy eigenvalues are written as

\[ E = \frac{1}{4A(1 + n)^2} \left\{ a^2 A^2 + 2aA [ -Ab + (1 + n)^2 ] \right\} + \left\{ Ab + (1 + n)^2 \right\}^2. \]  

where \( A = 2m/\lambda\hbar^2 \).

2.3. Non-Hermitian PT-symmetric form

Changing the potential parameters as \( a \to ia, b \to ib, \beta \to i\beta \) in equation (2), the potential satisfies

\[ V^*(-x) = \left( \frac{ia}{x} - \frac{ib}{x} e^{i\beta} \right)^* = V(x), \]  

which shows that we have obtained the non-Hermitian PT-symmetric form of the Hellmann-like potential. Inserting equation (37) into equation (24) and using the variable \( u = [1 - e^{-i\lambda x}]^{-1} \), we obtain

\[ u(1 - u) \frac{d^2 \phi(u)}{du^2} + (1 - 2u) \frac{d\phi(u)}{du} \times \left[ \left( \frac{2ma}{\lambda\hbar^2} - \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{1 - u} \right. \left. + \frac{2mb}{\lambda\hbar^2} - \frac{2mE}{\lambda^2\hbar^2} \frac{1}{u} \right] \phi(u) = 0. \]  

where

\[ \lambda_1 = -\frac{2m}{\lambda\hbar^2} \left( b - \frac{E}{\lambda} \right), \]  

\[ \lambda_2 = -\frac{2m}{\lambda\hbar^2} \left( a - \frac{E}{\lambda} \right). \]  

Following the same steps, we obtain the wave function for the non-Hermitian PT-symmetric Hellmann-like potential

\[ \phi(u) \sim u^{\lambda_1}(1 - u)^{\lambda_2} 2F_1(a', b'; c'; u), \]  

and the energy spectrum as

\[ E = \frac{1}{8m\hbar^2(1 + n)^2} \left\{ 4m^2a^2 + 4ma [-2mb \right. \left. + \lambda\hbar^2(1 + n)^2 \right] + \left. \left[ 2mb + \lambda\hbar^2(1 + n)^2 \right]^2 \right\}. \]  

2.4. Non-Hermitian non-PT-symmetric form

Case 1: \( a \) and \( b \) real, \( \lambda \to i\lambda \).
In this case the potential satisfies \( [V(x)]^\dagger \neq V(x) \) so it has a non-Hermitian non-PT-symmetric form given as
\[
V(x) = -ia\lambda \frac{1}{1 - e^{-i\lambda x}} + ib\lambda \frac{e^{-i\lambda x}}{1 - e^{-i\lambda x}},
\]
(43)

Using the variable \( u = [1 - e^{-i\lambda x}]^{-1} \), we obtain
\[
(1 - u) \frac{d^2\phi(u)}{du^2} + (1 - 2u) \frac{d\phi(u)}{du} \times \left\lbrack - \left( \frac{2m\lambda a}{\lambda h^2} + \frac{2mE}{\lambda^2 h^2} \right) \frac{1}{1 - u} + \left( \frac{2mb}{\lambda h^2} + \frac{2mE}{\lambda^2 h^2} \right) \frac{1}{u} \right\rbrack \phi(u) = 0.
\]
(44)

with
\[
\lambda_1^2 = \frac{2m}{\lambda h^2} \left( ib + \frac{E}{\lambda} \right),
\]
(45)
\[
\lambda_2^2 = \frac{2m}{\lambda h^2} \left( ia + \frac{E}{\lambda} \right).
\]
(46)

Following the same steps, we obtain the wave function for the non-Hermitian non-PT-symmetric Hellmann-like potential
\[
\phi(u) \sim u^{\lambda_1^2}(1 - u)^{\lambda_2^2} \; {}_2F_1(a', b'; c'; u),
\]
(47)
and the energy spectrum as
\[
E = \frac{1}{8m\hbar^2(1 + n)^2} \left\{ 4m\alpha^2 - 4ma \left[ 2mb - i\lambda h^2(1 + n)^2 \right] \right\}.
\]
(48)

Case 2: \( \beta \) real, \( a \to ia \) and \( b \to ib \)

By using the variable \( t = 1/(1 - e^{-\beta x}) \) we obtain the following in the current case
\[
V(x) = -ia\lambda \frac{1}{1 - e^{-i\lambda x}} + ib\lambda \frac{e^{-i\lambda x}}{1 - e^{-i\lambda x}},
\]
(49)

Using the variable \( u = [1 - e^{-i\lambda x}]^{-1} \), we obtain
\[
(1 - u) \frac{d^2\phi(u)}{du^2} + (1 - 2u) \frac{d\phi(u)}{du} \times \left\lbrack - \left( \frac{2m\lambda a}{\lambda h^2} + \frac{2mE}{\lambda^2 h^2} \right) \frac{1}{1 - u} + \left( \frac{2mb}{\lambda h^2} + \frac{2mE}{\lambda^2 h^2} \right) \frac{1}{u} \right\rbrack \phi(u) = 0.
\]
(50)

where
\[
\lambda_1^2 = -\frac{2m}{\lambda h^2} \left( ib + \frac{E}{\lambda} \right),
\]
(51)
\[
\lambda_2^2 = -\frac{2m}{\lambda h^2} \left( ia + \frac{E}{\lambda} \right).
\]
(52)

Following the same steps as in the above section, we obtain the wave function as
\[
\phi(u) \sim u^{\lambda_1^2}(1 - u)^{\lambda_2^2} \; {}_2F_1(a', b'; c'; u),
\]
(53)
and the energy spectrum as
\[
E = \frac{1}{8m\hbar^2(1 + n)^2} \left\{ 4m\alpha^2 - 4ma \left[ 2mb + i\lambda h^2(1 + n)^2 \right] \right\}.
\]
(54)

In the next section, we will study the scattering state solutions of the Hellmann potential.

3. Scattering states and phase shifts

In order to obtain the scattering state solutions, we choose the variable as \( t = 1 - e^{-\beta x} \) in equation (3) and we get the following equation in terms of \( t \)
\[
t(1 - t) \frac{d^2R(t)}{dt^2} + \left[ \frac{2m}{\lambda h^2}(a - e) - \ell(\ell + 1) \right] \frac{1}{1 - t} \]
\[
- \ell(\ell + 1) \frac{1}{t^2} \frac{dR(t)}{dt} + \left[ \mu^2 + \nu^2 + 2\mu \frac{2m}{\lambda h^2}(b - e) \right] R(t) = 0,
\]
(55)

where \( -\epsilon = E/\lambda \). By using a trial wave function as \( R(t) = t^\mu(1 - t)^\nu\psi(t) \), we get
\[
t(1 - t) \frac{d^2\psi(t)}{dt^2} + \left[ 2\mu - (2\mu + 2\nu + 1)t \right] \frac{d\psi(t)}{dt} + \left[ \mu^2 + \nu^2 + 2\mu \frac{2m}{\lambda h^2}(b - e) \right] \psi(t) = 0,
\]
(56)

with
\[
\mu = \begin{cases} -\ell & \nu = -i\kappa; \\ 1 + \ell & \nu = i\kappa; \end{cases}
\]
(57)

By using the following abbreviations
\[
\xi_1 = \mu - i\kappa + A_2,
\]
(58)
\[
\xi_2 = \mu - i\kappa - A_2,
\]
(59)
\[
\xi_3 = 2\mu,
\]
(60)

where \( A_2 = \sqrt{\frac{2m}{\lambda h^2}(e - b)} \), equation (56) can be written as a hypergeometric equation \[30\]
\[
t(1 - t) \frac{d^2\psi(t)}{dt^2} + \left[ \xi_3 - (\xi_1 + \xi_2 + 1)t \right] \frac{d\psi(t)}{dt} - \xi_1\xi_2\psi(t) = 0.
\]
(61)

Its solution is given by
\[
\psi(t) = \sqrt{2} \; {}_2F_1(\xi_1, \xi_2; \xi_3; t),
\]
(62)

where the parameters satisfy the followings
\[ \xi_3 - \xi_1 - \xi_2 = (\xi_1 + \xi_2 - \xi_3)^{\#}; \]
\[ \xi_3 - \xi_1 = \xi_2^{\#}; \quad \xi_3 - \xi_2 = \xi_1^{\#}, \]

which are used in the determination of phase shifts. By using the following equality of the hypergeometric functions [30]
\[ \sum F_1(a^*, b^*; c^*; z) \equiv \frac{\Gamma(c^*)\Gamma(c^* - a^* - b^*)}{\Gamma(c^*)\Gamma(c^* - a^* + b^*)} \times \sum F_1(a^*, b^*; a^* + b^* - c^* + 1; 1 - z) \]
\[ + (1 - z)^{-a^* - b^*} \frac{\Gamma(a^*)\Gamma(b^*)}{\Gamma(a^*)\Gamma(b^*)} \times \sum F_1(c^* - a^*, c^* - b^*; c^* - a^* - b^* + 1; 1 - z), \]

and \[ \sum F_1(a^*, b^*; c^*; 0) = 1, \]
we write the solution of equation (61) in the limit of \( r \to \infty \) as
\[ \sum F_1(\xi_1, \xi_2, \xi_3; 1 - e^{-jr}) \to \Gamma(2\mu)\Gamma(2\kappa) \]
\[ \frac{\Gamma(\mu + ik - \frac{2m}{\lambda h^2}(e - b)) \Gamma(\mu + ik + \frac{2m}{\lambda h^2}(e - b))}{\Gamma(2\mu)\Gamma(-2i\kappa)} \Gamma(\mu + ik - \frac{2m}{\lambda h^2}(e - b)) \]
\[ e^{2i\mu r}. \]

Defining the following
\[ \frac{\Gamma(c^* - a^* - b^*)}{\Gamma(c^* - a^* - b^*)\Gamma(c^* - b^*)} = \frac{\Gamma(c^* - a^* - b^*)}{\Gamma(c^* - a^*)\Gamma(c^* - b^*)} e^{i\delta}, \]
and also with the help of equation (63)
\[ \left( \frac{\Gamma(c^* - a^* - b^*)}{\Gamma(c^* - a^*)\Gamma(c^* - b^*)} \right)^{\#} = \frac{\Gamma(c^* - a^* - b^*)}{\Gamma(c^* - a^*)\Gamma(c^* - b^*)} e^{-i\delta}, \]
equation (65) becomes
\[ \sum F_1(\xi_1, \xi_2, \xi_3; 1 - e^{-jr}) \to \Gamma(2\mu) \]
\[ \frac{\Gamma(2\kappa)}{\Gamma(\mu + ik - \frac{2m}{\lambda h^2}(e - b)) \Gamma(\mu + ik + \frac{2m}{\lambda h^2}(e - b))} \]
\[ e^{i\delta + 2\mu r} \left[ e^{i\delta + 2\mu r} + e^{-i\delta + 2\mu r} \right]. \]

With the help of this result, we write the total wave function as
\[ R(r \to \infty) = 2\Gamma(2\mu) \]
\[ \times \frac{\Gamma(2\kappa)}{\Gamma(\mu + ik - \frac{2m}{\lambda h^2}(e - b)) \Gamma(\mu + ik + \frac{2m}{\lambda h^2}(e - b))} \]
\[ e^{i\delta + \lambda kr + \frac{\pi}{2}}. \]
Comparing this result with the boundary condition of the scattering state wave function as \( u(r \to \infty) \to 2 \sin \left( kr - \frac{\pi}{2} \ell + \delta_\ell \right) \), we obtain the phase shifts as
\[ \delta_\ell = \frac{\pi}{2} (1 + \ell) + \arg \Gamma(2i\kappa) \]
\[ arg \Gamma \left( \mu + ik - \frac{2m}{\lambda h^2}(e - b) \right) \]
\[ - arg \Gamma \left( \mu + ik + \frac{2m}{\lambda h^2}(e - b) \right). \]

Thus, we see that the phase shifts of the Hellmann potential can be produced using the behavior of the hypergeometric functions at infinity, and that they are dependent on the energy of the particle.

4. Conclusions

We have solved the Schrödinger equation for the PT-/non-PT-symmetric and non-Hermitian Hellmann potential for any angular momentum. The normalized wave functions were obtained in terms of the hypergeometric functions by using an approximation instead of the centrifugal term. We have calculated the energy eigenvalue. Its numerical values for the bound states are listed in table 1, and they are compared with the previous results. We have seen that our results closely agree with previous works, especially for smaller parameter values. The energy eigenvalue for the Coulomb potential was obtained by setting potential parameters. Finally, we have studied the scattering state solutions of the Hellmann potential and have obtained the phase shifts in terms of the angular momentum quantum number \( \ell \).

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