Elliptic quantum group $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and vertex operators

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Abstract

Introducing an $H$-Hopf algebroid structure into $U_{q,p}(\hat{\mathfrak{sl}}_2)$, we investigate the vertex operators of the elliptic quantum group $U_{q,p}(\hat{\mathfrak{sl}}_2)$ defined as intertwining operators of infinite dimensional $U_{q,p}(\hat{\mathfrak{sl}}_2)$ modules. We show that the vertex operators coincide with the previous results obtained indirectly by using the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\hat{\mathfrak{sl}}_2)$. This shows a consistency of our $H$-Hopf algebroid structure even in the case with a nonzero central element.

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1. The elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$

In this section, we review a definition of the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ and its RLL formulation following [1, 2].

1.1. Definition of $U_{q,p}(\hat{\mathfrak{sl}}_2)$

The elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_2)$ was introduced in [1] as an elliptic analogue of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ in the Drinfeld realization. It was soon realized that $U_{q,p}(\hat{\mathfrak{sl}}_2)$ is isomorphic to the tensor product of $U_q(\hat{\mathfrak{sl}}_2)$ and a Heisenberg algebra $\{P, c^Q\}$ [2]. We here define $U_{q,p}(\hat{\mathfrak{sl}}_2)$ along the latter observation.

Let us fix a complex number $q$ such that $q \neq 0$, $|q| < 1$.

Definition 1.1 [3]. For a field $\mathbb{K}$, the quantum affine algebra $\mathbb{K}[U_q(\hat{\mathfrak{sl}}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the Drinfeld generators $a_n (n \in \mathbb{Z})$, $x_n^\pm (n \in \mathbb{Z})$, $h, c, d$. The defining relations are given as follows:

\[
\begin{align*}
  c & : \text{central}, \\
  [h, d] &= 0, \\
  [d, a_n] &= na_n, \\
  [d, x_n^\pm] &= nx_n^\pm, \\
  [h, a_n] &= 0, \\
  [h, x_n^\pm(z)] &= \pm 2x_n^\pm(z).
\end{align*}
\]
$[a_n, a_m] = \frac{[2n]_q [cn]_q}{n} q^{-cn} \delta_{n+m,0},$

$[a_n, x^+ (z)] = \frac{[2n]_q}{n} q^{-cn} z^n x^+ (z),$

$[a_n, x^- (z)] = -\frac{[2n]_q}{n} z^n x^- (z),$

$(\z - q^{-k} w) x^+ (z) x^+ (w) = (q^{-2k} z - w) x^+ (w) x^+ (z),$

$[x^+ (z), x^- (w)] = \frac{1}{q - q^{-1}} \left( \delta \left( q^{-r} \frac{z}{w} \right) \psi (q^{r/2} w) - \delta \left( q^{-r} \frac{w}{z} \right) \psi (q^{-r/2} w) \right),$

where $[n]_q = \frac{q^n q^n - 1}{q - q^{-1}}, \delta (z) = \sum_{n \in \mathbb{Z}} z^n$ and

$x^\pm (z) = \sum_{n \in \mathbb{Z}} x^\pm z^n,$

$\psi (q^{r/2} z) = q^h \exp \left( (q - q^{-1}) \sum_{n > 0} a_n z^{-n} \right),$

$\psi (q^{-r/2} z) = q^{-h} \exp \left( -(q - q^{-1}) \sum_{n < 0} a_n z^n \right).$

Let $r$ be a complex parameter. We set $r^* = r - c, p = q^{2r}$ and $p^* = q^{2r}$. We define the Jacobi theta functions $[u]$ and $[u]^*$ by

$[u] = \frac{q^{\frac{r^*}{r-u}}}{(p^*; q^2)_{\infty}} \Theta_p (q^{2u}),$  $[u]^* = \frac{q^{\frac{r^*}{r-u}}}{(p^*; q^2)_{\infty}} \Theta_p (q^{2u}),$

where

$\Theta_p (z) = (z; p)_{\infty} (p/z; p)_{\infty} (p^2; p)_{\infty},$

$(z; p_1, p_2, \ldots, p_m)_{\infty} = \prod_{n_1, n_2, \ldots, n_m = 0}^{\infty} \left( 1 - z p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \right).$

Setting $p = e^{-2\pi i/r}, [u]$ satisfies the quasi-periodicity $[u + r] = -[u], [u + r^*] = e^{-\pi i (2r/r^*)} [u].$

We denote by $\{ P, e^0 \}$ a Heisenberg algebra commuting with $\mathbb{C}[U_q (\mathfrak{sl}_2)]$ and satisfying

$[P, e^0] = -e^0. \tag{1.1}$

We take the realization $Q = \frac{\partial}{\partial r^*}$. We set $H = \mathbb{C} \mathbb{P} \oplus \mathbb{C} r^*$ and $H^* = \mathbb{C} Q \oplus \mathbb{C} \frac{n}{mr}$ with the pairing $(\cdot, \cdot)$

$(Q, P) = 1 = \left\{ \frac{\partial}{\partial r^*}, r^* \right\},$

the others are zero.

We also consider the Abelian group $\hat{H}^* = \mathbb{Z} Q$. We denote by $\mathbb{C}[\hat{H}^*]$ the group algebra over $\mathbb{C}$ of $\hat{H}^*$, and by $e^\alpha$ the element of $\mathbb{C}[\hat{H}^*]$ corresponding to $\alpha \in \hat{H}^*$. These $e^\alpha$ satisfy $e^\alpha e^\beta = e^{\alpha + \beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now we take the power series field $\mathbb{F} = \mathbb{C}((P, r^*))$ as $\mathbb{K}$ and consider the semi-direct product $\mathbb{C}$-algebra $U_q (\mathfrak{sl}_2) = \mathbb{F} [U_q (\mathfrak{sl}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\hat{H}^*]$ of $\mathbb{F} [U_q (\mathfrak{sl}_2)]$ and $\mathbb{C}[\hat{H}^*]$, whose multiplication is defined by

$(f (P, r^*) a \otimes e^\alpha) \cdot (g (P, r^*) b \otimes e^\beta) = f (P, r^*) g (P + \langle \alpha, P \rangle, r^*) ab \otimes e^{\alpha + \beta},$

$a, b \in \mathbb{C}[U_q (\mathfrak{sl}_2)], f (P, r^*), g (P, r^*) \in \mathbb{F}, \alpha, \beta \in \hat{H}^*. \tag{2}$
Proposition 1.3. In fact, from definition 1.1 and (1.1), we can derive the following relations.

Let us consider the following generating functions:

\[ u^+(z, p) = \exp \left( \sum_{n=0}^{\infty} \frac{1}{[r+n]_q} a_{-n}(q^r z)^n \right), \quad u^-(z, p) = \exp \left( -\sum_{n>0} \frac{1}{[rn]_q} a_n(q^{-r} z)^{-n} \right). \]

We define a mapping \( \phi_r \) of \( \mathbb{C}[U_q(\widehat{sl}_2)] \) by

\[
\begin{align*}
&c \mapsto c, \quad h \mapsto h, \quad d \mapsto d, \\
x^+(z) \mapsto u^+(z, p)x^+(z), \quad x^-(z) \mapsto x^-(z)u^-(z, p), \\
\psi(z) \mapsto u^+(q^{r/2} z, p)\psi(z)u^-(q^{-r/2} z, p), \\
\varphi(z) \mapsto u^+(q^{-r/2} z, p)\varphi(z)u^-(q^{r/2} z, p).
\end{align*}
\]

Definition 1.2. We define \( E(u), F(u), K(u) \in U_{q, p}(\widehat{sl}_2)[[u]] \) and \( \hat{d} \) by the following formulae:

\[
\begin{align*}
E(u) &= \phi_r(x^+(z)) e^{2qz}z^{-r/2}, \\
F(u) &= \phi_r(x^-(z)) e^{2qz}z^{-r/2}, \\
K(u) &= \exp \left( \sum_{n=0}^{\infty} \frac{1}{[n]_q} a_{-n}(q^r z)^n \right) \exp \left( -\sum_{n>0} \frac{1}{[n]_q} a_n(q^{-r} z)^{-n} \right), \\
\hat{d} &= d - \frac{1}{4r^2}(P - 1)(P + 1) + \frac{1}{4r}(P + h - 1)(P + h + 1),
\end{align*}
\]

where we set \( z = q^{2u} \). We call \( E(u), F(u), K(u) \) the elliptic currents.

In fact, from definition 1.1 and (1.1), we can derive the following relations.

Proposition 1.3.

\[
\begin{align*}
c : \text{central}, \\
[h, a_n] &= 0, \quad [h, E(u)] = 2E(u), \quad [h, F(u)] = -2F(u), \\
[d, h] &= 0, \quad [d, a_n] = na_n, \\
[d, E(u)] &= \left( -z \frac{\partial}{\partial z} - \frac{1}{r^2} \right) E(u), \quad [d, F(u)] = \left( -z \frac{\partial}{\partial z} - \frac{1}{r} \right) F(u), \\
[a_n, a_m] &= \frac{[2n]_q}{n} q^{-cn} \delta_{nm}, \\
[a_n, E(u)] &= \frac{[2n]_q}{n} q^{-cn} z^n E(u), \\
[a_n, F(u)] &= -\frac{[2n]_q}{n} q^{-cn} F(u), \\
E(u)E(v) &= \frac{u - v + 1}{u - v - 1} E(v)E(u), \\
F(u)F(v) &= \frac{u - v + 1}{u - v - 1} F(v)F(u), \\
[E(u), E(v)] &= \frac{1}{q - q^{-1}} \left( \delta \left( q^r \frac{z}{w} \right) H^+(q^{r/2} w) - \delta \left( q^{-r} \frac{z}{w} \right) H^-(q^{-r/2} w) \right),
\end{align*}
\]

where \( z = q^{2u}, w = q^{2v} \).

\[
\begin{align*}
H^\pm(z) &= \kappa K \left( u \pm 1 \left( r - \frac{c}{2} \right) + \frac{1}{2} \right) K \left( u \pm 1 \left( r - \frac{c}{2} \right) - \frac{1}{2} \right), \\
\kappa &= \lim_{z \to q^{-1}} \frac{\xi(z; p^4, q)}{\xi(z; p, q)}, \quad \xi(z; p, q) = \frac{(q^2 z; p, q^4)_{\infty}(pq z; p, q^4)_{\infty}}{(q^2 z; p, q^4)_{\infty}(pz; p, q^4)_{\infty}}.
\end{align*}
\]
In particular, we have the following relations which, together with the last three relations in the above, appeared in [1].

**Proposition 1.4.**

\[
K(u)K(v) = \rho(u - v)K(v)K(u),
\]

\[
K(u)E(v) = \left[ u - v - 1 + \frac{c}{2} \right] \left[ u - v + 1 + \frac{c}{2} \right] F(v)K(u),
\]

\[
K(u)F(v) = \left[ u - v - 1 + \frac{c}{2} \right] \left[ u - v + 1 + \frac{c}{2} \right] F(v)K(u),
\]

\[
H^+(u)H^-(v) = \left[ u - v - 1 - \frac{c}{2} \right] \left[ u - v + 1 + \frac{c}{2} \right] F(v)K(u),
\]

where

\[
\rho(u) = \frac{\rho^+(u)}{\rho^-(u)}, \quad \rho^+(u) = z^{1/2} \frac{(pq^2z)^2}{(pz)(pq^2z)} \frac{(z^{-1})q^{4z^{-1}}}{(q^2z^{-1})^2}, \quad [z] = \{z, p, q^4\}_\infty.
\]

**Definition 1.5.** We call a set \((F[U_q(\widehat{sl}_2)] \otimes \mathbb{C}[\bar{H}^\tau], \phi_r)\) the elliptic algebra \(U_{q,p}(\widehat{sl}_2)\).

The following relations are also useful.

**Proposition 1.6.**

\[
[K(u), P] = K(u), \quad [E(u), P] = 2E(u), \quad [F(u), P] = 0,
\]

\[
[K(u), P + h] = K(u), \quad [E(u), P + h] = 0, \quad [F(u), P + h] = 2F(u).
\]

1.2. The RLL relation for \(U_{q,p}(\widehat{sl}_2)\)

We next summarize the RLL relation for \(U_{q,p}(\widehat{sl}_2)\) [2]. In the following section, the \(L\) operator is used to discuss the \(H\)-Hopf algebroid structure of \(U_{q,p}(\widehat{sl}_2)\).

Let us define the half currents in the following way.

**Definition 1.7.**

\[
K^+(u) = K \left( u + \frac{r + 1}{2} \right),
\]

\[
E^+(u) = a^* \oint_{C^*} E(u') \left[ u - u' + c/2 - P + 1 \right] [1]^* \frac{dz'}{2\pi i z'},
\]

\[
F^+(u) = a \oint_{C} F(u') \left[ u - u' + P + h - 1 \right] [1] \frac{dz'}{2\pi i z'}.
\]

Here the contours are chosen such that

\[
C^*: |p^* q^* z| < |z'| < |q^* z|, \quad C: |pz| < |z'| < |z|,
\]

and the constants \(a, a^*\) are chosen to satisfy \(a^* a [1]^* \frac{\bar{z}}{q - q^{-1}} = 1\).
Definition 1.8. We define the operator $\hat{L}^+(u) \in \text{End}_\mathbb{C} V \otimes U_{q, p}(\hat{\mathfrak{sl}}_2)$ with $V \cong \mathbb{C}^2$, by
\[
\hat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u - 1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.
\]

Proposition 1.9. The operator $\hat{L}^+(u)$ satisfies the following RLL relation:
\[
R^{\ast(12)}(u_1 - u_2, P + h) \hat{L}^+(u_1) \hat{L}^+(u_2) = \hat{L}^+(u_2) \hat{L}^+(u_1) R^{\ast(12)}(u_1 - u_2, P),
\]
where $R^+(u, P + h)$ and $R^{\ast(u, P)} = R^+(u, P)|_{r \rightarrow r}$, denote the elliptic dynamical $R$ matrices given by
\[
R^+(u, s) = \rho^+(u) \begin{pmatrix} 1 & b(u, s) c(u, s) \\ \bar{c}(u, s) & \bar{b}(u, s) \end{pmatrix},
\]
with
\[
b(u, s) = \frac{[s + 1][s - 1]}{[s]^2} \frac{[u]}{[1 + u]}, \quad c(u, s) = \frac{[1]}{[s]} \frac{[s + u]}{[1 + u]},
\]
\[
\bar{c}(u, s) = \frac{[1]}{[s]} \frac{[s - u]}{[1 + u]}, \quad \bar{b}(u, s) = \frac{[u]}{[1 + u]}.
\]

Note that if we set $L^+(u, P) = \hat{L}^+(u) e^{-h_0 \otimes 0}$, $L^+(u, P)$ is independent of $Q$ and satisfies the dynamical RLL relation [2] characterizing the quasi-Hopf algebra $B_{q, \lambda}(\mathfrak{sl}_2)$ [4]. Moreover, with the parametrization $\lambda = (r + 2) \Delta_0 + (P + 1) \Lambda_1$, where $\Delta_0, \Lambda_0, \Lambda_1$ are the fundamental weights of $\mathfrak{sl}_2$, $B_{q, \lambda}(\mathfrak{sl}_2)$ is isomorphic to $\mathbb{F}[U_q(\mathfrak{sl}_2)]$, as an associative algebra. These two facts lead to the isomorphism $U_{q, p}(\mathfrak{sl}_2) \cong B_{q, \lambda}(\mathfrak{sl}_2) \otimes \mathbb{C}[\hat{H}^*]$ as a semi-direct product $\mathbb{C}$-algebra. However, this semi-direct product breaks down the quasi-Hopf algebra structure, so that $U_{q, p}(\mathfrak{sl}_2)$ is not a quasi-Hopf algebra. In the following section, we show that a relevant co-algebra structure of $U_{q, p}(\mathfrak{sl}_2)$ is the $H$-Hopf algebroid.

Note also that the $c = 0$ case of the dynamical RLL relation for $B_{q, \lambda}(\mathfrak{sl}_2)$ coincides with the one studied by Felder [5, 6], whereas the $c = 0$ case of (1.2) coincides with the RLL relation studied in [7–9] for the trigonometric $R$ and in [10] for the elliptic $R$.

2. $H$-Hopf algebroid structure of $U_{q, p}(\mathfrak{sl}_2)$

In this section, we introduce an $H$-Hopf algebroid structure into $U_{q, p}(\mathfrak{sl}_2)$. The detailed discussion will be published elsewhere [11]. We follow the definition of $H$-Hopf algebroid given in [7–10] with a modification which makes it applicable in the case with nonzero central element.

Let $\mathfrak{h} = \mathbb{C} h$ be the Cartan subalgebra, $\alpha_1$ the simple root and $\Lambda_1$ be the fundamental weight of $\mathfrak{sl}_2$. We set $Q = 2\alpha_1$, and $\hat{h}^* = \mathbb{C}\Lambda_1$. Let us use the same symbol $\langle \cdot, \cdot \rangle$ to denote the standard paring of $\mathfrak{h}$ and $\hat{h}^*$. Using the isomorphism $\phi : Q \rightarrow \hat{H}^*$ by $n\alpha_1 \mapsto nQ$, we define the $\hat{H}^*$-bigrading structure of $U_{q, p} \equiv U_{q, p}(\mathfrak{sl}_2)$ by
\[
U_{q, p} = \bigoplus_{\alpha, \beta \in \hat{H}^*} (U_{q, p})_{\alpha \beta},
\]
\[
(U_{q, p})_{\alpha \beta} = \left\{ x \in U_{q, p} \left| \begin{array}{l}
q^h x q^{-h} = q^{\langle \alpha, h \rangle} x, \alpha = \phi(\tilde{\alpha}) + \beta \\
q^p x q^{-p} = q^{\langle \beta, p \rangle} x
\end{array} \right. \right\}.
\]

Noting $\langle \tilde{\alpha}, h \rangle = \langle \phi(\tilde{\alpha}), P \rangle$, we have $q^{p+h} x q^{-(p+h)} = q^{\langle \alpha, p \rangle} x$ for $x \in (U_{q, p})_{\alpha \beta}$.
Then the tensor product
\[ \tilde{f} = f(P, r^*) \in \mathbb{F} \]
as a meromorphic function on \( H^* \) by
\[ \tilde{f}(\mu) = f((\mu, P), (\mu, r^*)), \quad \mu \in H^* \]
and consider the field of meromorphic functions \( M_{H^*} \) on \( H^* \) given by
\[ M_{H^*} = \{ \tilde{f} : H^* \to \mathbb{C} | \tilde{f} = f(P, r^*) \in \mathbb{F} \} \]
We define two embeddings (the left and right moment maps)
\[ \mu_\ell, \mu_r : M_{H^*} \to (U_{q,p})_{00} \]
by
\[ \mu_\ell(\tilde{f}) = f(P + h, r^* + c), \quad \mu_r(\tilde{f}) = f(P, r^*). \quad (2.2) \]
From (2.1), one finds
\[ \mu_\ell(\tilde{f}) = f(P + h + (\alpha, P), r^* + c) = x\mu_\ell(T_\alpha \tilde{f}), \]
\[ \mu_r(\tilde{f}) = f(P, r^*)x = x\mu_r(T_\beta \tilde{f}), \]
where we regard \( T_\alpha = e^{\alpha} \in \mathbb{C}[\hat{H}^*] \) as a shift operator \( M_{H^*} \to M_{H^*} \)
\[ (T_\alpha \tilde{f}) = e^{\alpha} f(P, r^*) e^{-\alpha} = f(P + (\alpha, P), r^*). \]
Hereafter, we abbreviate \( f(P + h, r^* + c) \) and \( f(P, r^*) \) as \( f(P + h) \) and \( f^*(P) \), respectively.

Then equipped with the bigrading structure (2.1) and two moment maps (2.2), the elliptic algebra \( U_{q,p}(\hat{sl}_2) \) is an \( H \)-algebra [7, 8].

In addition, we need the \( H \)-algebra \( \mathcal{D} \) of the shift operators given by
\[ \mathcal{D} = \left\{ \sum_i \tilde{f}_i T_\alpha | \tilde{f}_i \in M_{H^*}, \alpha_i \in \hat{H}^* \right\}, \]
\[ (\mathcal{D})_{\alpha \alpha} = [\tilde{f} T_{-\alpha}], \quad (\mathcal{D})_{\alpha \beta} = 0 \quad \alpha \neq \beta, \]
\[ \mu_\ell^\mathcal{D}(\tilde{f}) = \mu_\ell^\mathcal{D}(\tilde{f}) = \tilde{f} T_0, \quad \tilde{f} \in M_{H^*}. \]

Let \( A \) and \( B \) be two \( H \)-algebras, \( U_{q,p} \) or \( \mathcal{D} \). The tensor product \( A \otimes B \) is the bigraded vector space with
\[ (A \otimes B)_{\alpha \beta} = \bigoplus_{\gamma \in \hat{H}^*} (A_{\alpha \gamma} \otimes B_{\beta \gamma}). \]
where \( \otimes_{M_{H^*}} \) denotes the usual tensor product modulo the following relations:
\[ \mu_\ell^A(\tilde{f}) a \otimes b = a \otimes \mu_\ell^B(\tilde{f}) b, \quad a \in A, \quad b \in B. \quad (2.3) \]

Then the tensor product \( A \otimes B \) is again an \( H \)-algebra with the multiplication \( (a \otimes b)(c \otimes d) = ac \otimes bd \) and the moment maps
\[ \mu_\ell^A \otimes B = \mu_\ell^A \otimes 1, \quad \mu_r^A \otimes B = 1 \otimes \mu_r^B. \]
Note that we have the \( H \)-algebra isomorphism \( U_{q,p} \otimes \mathcal{D} \cong U_{q,p} \cong \mathcal{D} \otimes U_{q,p} \) by \( x \otimes T_{-\beta} = x = T_{-\alpha} \otimes x \) for \( x \in (U_{q,p})_{\alpha \beta} \).

Now let us define an \( H \)-Hopf algebroid structure on \( U_{q,p} \) as its co-algebra structure. For this purpose, it is convenient to use the \( L \) operator \( \hat{L}^*(u) \). We shall write the entries of \( \hat{L}^*(u) \) as
\[ \hat{L}^*(u) = \left( \begin{array}{cc} \hat{L}_{+$+}(u) & \hat{L}_{+$-}(u) \\ \hat{L}_{-$+}(u) & \hat{L}_{-$-}(u) \end{array} \right). \]
From proposition 1.6 and definition 1.8, one finds
\[ \hat{L}^*_{+\mp}(u) \in (U_{q,p})_{-\gamma, -\gamma, -\gamma}. \]
It is also easy to check the relations
\[
f(P + h)\hat{T}_{e_{1}e_{2}}(u) = \hat{T}_{e_{1}e_{2}}(u) f(P + h - \varepsilon_{1}),
\]
\[
f^*(P)\hat{T}_{e_{1}e_{2}}(u) = \hat{T}_{e_{1}e_{2}}(u) f^*(P - \varepsilon_{2}).
\]

**Definition 2.1.** We define $H$-algebra homomorphisms, $\varepsilon : U_{q,p} \to \mathcal{D}$ and $\Delta : U_{q,p} \to U_{q,p} \otimes U_{q,p}$ by
\[
\varepsilon(\hat{T}_{e_{1}e_{2}}(u)) = \delta_{e_{1},e_{2}} T_{-e_{1}Q}, \quad \varepsilon(e^{0}) = e^{0},
\]
\[
\varepsilon(\mu_{1}(\hat{f})) = \varepsilon(\mu_{2}(\hat{f})) = \hat{T}_{0},
\]
\[
\Delta(\hat{T}_{e_{1}e_{2}}(u)) = \sum_{e'} \hat{T}_{e_{1}e'}(u) \hat{T}_{e'e_{2}}(u),
\]
\[
\Delta(e^{0}) = e^{0} \hat{\otimes} e^{0}, \quad \Delta(\mu_{1}(\hat{f})) = 1 \hat{\otimes} \mu_{1}(\hat{f}), \quad \Delta(\mu_{2}(\hat{f})) = 1 \hat{\otimes} \mu_{2}(\hat{f}).
\]

We also define an $H$-algebra anti-homomorphism $S : U_{q,p} \to U_{q,p}$ by
\[
S(\hat{T}_{e_{1}e_{2}}(u)) = \hat{T}_{e_{1}e_{2}}(u - 1), \quad S(\hat{T}_{+}(u)) = -\frac{[P + h + 1]}{[P + h]} \hat{T}_{+}(u - 1),
\]
\[
S(\hat{T}_{-}(u)) = -\frac{[P]}{[P + 1]} \hat{T}_{-}(u - 1), \quad S(\hat{T}_{+}(u)) = \frac{[P + h + 1][P]}{[P + h][P + 1]} \hat{T}_{+}(u - 1),
\]
\[
S(e^{0}) = e^{-0}, \quad S(\mu_{1}(\hat{f})) = \mu_{1}(\hat{f}), \quad S(\mu_{2}(\hat{f})) = \mu_{2}(\hat{f}).
\]

In fact, one can show that $\Delta$ and $S$ preserve the $RLL$ relation (1.2). Moreover, we have the following lemma indicating that $\varepsilon$, $\Delta$ and $S$ satisfy the axioms for the counit, the comultiplication and the antipode. Hence the $H$-algebra $U_{q,p}(\mathfrak{sl}_{2})$ with $(\Delta, \varepsilon, S)$ is an $H$-Hopf algebroid [7–9].

**Lemma 2.2.** The maps $\varepsilon$, $\Delta$ and $S$ satisfy
\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,
\]
\[
(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta,
\]
\[
m \circ (\text{id} \otimes S) \circ \Delta(x) = \mu_{1}(\varepsilon(x)), \quad \forall x \in U_{q,p},
\]
\[
m \circ (S \otimes \text{id}) \circ \Delta(x) = \mu_{2}(T_{0}(\varepsilon(x))), \quad \forall x \in (U_{q,p})_{2}.\]

**Definition 2.3.** We call the $H$-Hopf algebroid $(U_{q,p}(\mathfrak{sl}_{2}), H, M_{H}, \mu_{1}, \mu_{2}, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(\mathfrak{sl}_{2})$.

3. Representations

We consider the dynamical representations, i.e. the representations as $H$-algebras [7, 8, 12], of the elliptic algebra $U_{q,p}(\mathfrak{sl}_{2})$.

3.1. Evaluation representation

We construct the evaluation representation of $U_{q,p}(\mathfrak{sl}_{2})$ by using the one of $\mathbb{F}[U_{q}(\mathfrak{sl}_{2})]$. We define the $(l+1)$-dimensional vector space over $\mathbb{F}$ by $V^{(l)} = \bigoplus_{m=0}^{l} \mathbb{F}v_{m}$. Here, $v_{m}$ $(0 \leq m \leq l)$ denote the weight vectors satisfying $hv_{m} = (l - 2m)v_{m}$. Consider the operator $S^{\pm}$ acting on
V(l) by $S^l v_m^l = v_{m+l}^l$, $v_m^l = 0$ for $m < 0$, $m > l$. In terms of the Drinfeld generators, the evaluation representation $(\pi_{l,w}, V_w = V(l) \otimes \mathbb{C}[w, w^{-1}])$ of $\mathbb{F}[U_q(\widehat{sl}_2)]$ is given by [2]

\[
\pi_{l,w}(c) = 0, \quad \pi_{l,w}(d) = 0,
\]

\[
\pi_{l,w}(a_n) = \frac{w^n}{n} \frac{1}{q - q^{-1}} ((q^n + q^{-n})q^{nh} - (q^{(l+1)n} + q^{-(l+1)n})),
\]

\[
\pi_{l,w}(x^+(z)) = S^l \left[ \frac{\pm h + l + 2}{2} \right] \delta \left( q^{h+1} \frac{w}{z} \right).
\]

Note that $V_w = \bigoplus_{\mu \in (-l, -l + 2, \ldots, l)} V_{\mu}$ with $V_{\mu}$, $\mu = l - 2m$ spanned by $v_m^l \otimes w^n (n \in \mathbb{Z})$.

Let us define the $H$-algebra $\mathcal{D}_{H,V}$ by

\[
\mathcal{D}_{H,V} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H,V})_{\alpha\beta},
\]

\[
(\mathcal{D}_{H,V})_{\alpha\beta} = \left\{ X \in \text{End}_C V_w(l) \left| X(f^s(P)v) = f^s(P - (\beta, P))X(v), \quad v \in V_w(l) \right. \right\},
\]

\[
\mu_{l,w}^D \left( \pi_l \right) v = f(P + \mu)v, \quad \mu_{l,w}^D \left( \pi_l \right) v = f^s(P)v
\]

for $v \in V_{\mu}$, then $\pi_{l,w} = \pi_{l,w} \otimes \text{id} : U_q(\widehat{sl}_2) = \mathbb{F}[U_q(\widehat{sl}_2)] \otimes \mathbb{C}[H^*] \to \mathcal{D}_{H,V}$ with $e^Q v_m^l = v_m^l$ yields the $H$-algebra homomorphism. We call $(\pi_{l,w}, V(l))$ the dynamical evaluation representation. In particular, applying this to definitions 1.2, 1.7 and 1.8, we obtain the following expressions for the images of the $\hat{L}^+(u)$ operator.

**Theorem 3.1.**

\[
\pi_{l,w}(\hat{L}^+_{\alpha}(u)) = -\left[ u - v + \frac{h^l + 1}{2} \right] \frac{q \phi(u - v)[P + h]}{q \phi(u - v)[P + h + 1]} e^Q,
\]

\[
\pi_{l,w}(\hat{L}^+_{\alpha}(u)) = -S^l \left[ u - v + \frac{h^l + 1}{2} \right] \frac{q \phi(u - v)[P + h]}{q \phi(u - v)[P + h + 1]} e^{-Q},
\]

\[
\pi_{l,w}(\hat{L}^+_{\alpha}(u)) = S^l \left[ u - v - \frac{h^l + 1}{2} \right] \frac{q \phi(u - v)[P]}{q \phi(u - v)[P]} e^Q,
\]

\[
\pi_{l,w}(\hat{L}^+_{\alpha}(u)) = -\left[ u - v - \frac{h^l + 1}{2} \right] \frac{q \phi(u - v)[P]}{q \phi(u - v)[P]} e^{-Q},
\]

where we set $z = q^{2n}$, $w = q^{2v}$ and

\[
\phi(u) = z^{-1/2} \rho^v_{l}(u, p)^{-1} \left[ u + l + 1 \right],
\]

\[
\rho^v_{l}(z, p) = q^{l/2} \left\{ pq^{k+2}/(pq^{k+2}) \right\}^{-1} \left\{ pq^{k+2}/(pq^{k+2}) \right\}^{-1} \left\{ pq^{k+2}/(pq^{k+2}) \right\}^{-1} \left\{ pq^{k+2}/(pq^{k+2}) \right\}^{-1}.
\]

The following proposition indicates a consistency of our construction of $\pi_{l,w}$ and the fusion construction of the dynamical $R$ matrices (face-type Boltzmann weights).

**Proposition 3.2.** Let us define the matrix elements of $\pi_{l,w}(\hat{L}^+_{\alpha}(u))$ by

\[
\pi_{l,w}(\hat{L}^+_{\alpha}(u)) v_{m}^l = \sum_{m'} \left( \hat{L}^+_{\alpha}(u) \right)_{\mu_{m'}, \mu_{m}} v_{m'}^l,
\]

where $\mu_m = l - 2m$. Then we have

\[
\left( \hat{L}^+_{\alpha}(u) \right)_{\mu_{\mu_{m'}}} = R^l_{\mu_{m}}(u, v, P c_{\beta_{\mu_{m}}}).
\]
Here, \( R^+_l(u - v, P) \) is the \( R \) matrix from (C.17) in [2]. The case \( l = 1 \), \( R^+_1(u - v, P) \) coincides with the image \( (\pi_{1,z} \otimes \pi_{1,w}) \) of the universal \( R \) matrix \( R^+(\lambda) \) [4] given in (1.3). The case \( l > 1 \), \( R^+_l(u - v, P) \) coincides with the \( R \) matrix obtained by fusing \( R^+_1(u - v, P) \) \( l \)-times. In particular, the matrix element \( R^+_l(u - v, P)_{\mu}^{\mu'} \) is gauge equivalent to the fusion face weight \( W_l(\mu + \epsilon', \mu + \epsilon' + \mu, \mu + \mu, P[u - v]) \) from (4) in [13].

### 3.2. Infinite dimensional representation

Let \( V(\lambda_i) \) be the level-\( k(i = k) \) irreducible highest weight \( \mathbb{F}[\mathfrak{sl}(\mathfrak{sl}_2)] \) module of highest weight \( \lambda_i = (k - l)\Lambda_0 + l\Lambda_1 \) \((0 \leq l \leq k)\). Here, \( \Lambda_i \) \((i = 0, 1)\) denote the fundamental weights of \( \mathfrak{sl}_2 \). We regard \( \bar{V}(\lambda) = \bigoplus_{m \in \mathbb{Z}} V(\lambda) \otimes \mathbb{C} e^{-mQ} \) as the \( U_{q,p}(\mathfrak{sl}_2) \)-module [2].

We realize \( \bar{V}(\lambda_i) \) by using the Drinfeld generators \( a_n(n \in \mathbb{Z}) \) and the \( q \)-deformed \( \mathbb{Z}_k \)-parafermion algebra [1, 2, 14]. Let us define \( a_n(n \in \mathbb{Z}_{\neq 0}) \) by

\[
a_n = \begin{cases} a_n & \text{for } n > 0, \\ [rn]_q q^{kn}a_n & \text{for } n < 0, \end{cases}
\]

with \( r^n = r - k \). Then we have

\[
[a_m, a_n] = \frac{[2m]_q [km]_q [rm]_q}{m} \delta_{m+n,0}.
\]

The \( q \)-deformed \( \mathbb{Z}_k \)-parafermion algebra is an associative algebra over \( \mathbb{C} \) generated by \( \Psi_+, \Psi_-, \Psi_{\pm}^{-\mu} (\mu, n \in \mathbb{Z}) \). Consider the generating functions (parafermion fields)

\[
\Psi(z) \equiv \Psi^+(z) = \sum_{n \in \mathbb{Z}} \Psi_n z^{-\mu/k+n-1},
\]

\[
\Psi^\dagger(z) \equiv \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi_n^{-\mu} z^{\mu/k+n-1}
\]

defined on a weight vector \( v \) satisfying \( q^v v = q^\mu v \). The parafermion fields \( \Psi(z) \) and \( \Psi^\dagger(z) \) satisfy

\[
\left( \frac{z}{w} \right)^{2/k (x^2 z/w; x^2)_{\infty}} \Psi^+(z) \Psi^+(w) = \left( \frac{w}{z} \right)^{2/k (x^{-2} z/w; x^{-2})_{\infty}} \Psi^+(w) \Psi^+(z),
\]

\[
\left( \frac{z}{w} \right)^{-2/k (x^2 z/w; x^{-2})_{\infty}} \Psi^+(z) \Psi^-(w) - \left( \frac{w}{z} \right)^{-2/k (x^2 z/w; x^{-2})_{\infty}} \Psi^-(w) \Psi^+(z)
\]

\[
= \frac{1}{x - x^{-1}} \left( \delta \left( x^k \frac{w}{z} \right) - \delta \left( x^{-k} \frac{w}{z} \right) \right).
\]

**Theorem 3.3.** [14] By using the irreducible \( q \)-\( \mathbb{Z}_k \) parafermion module \( \mathcal{H}^P_{\lambda,M} \), the level-\( k \) irreducible highest weight \( U_{q,p}(\mathfrak{sl}_2) \)-module \( \bar{V}(\lambda_i) \) is realized as follows:

\[
\bar{V}(\lambda_i) = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{\text{n \in \mathbb{Z}} \atop \text{M \equiv 0, 2 \mod 2}} \bar{V}(\lambda_i)_{M+2kn+m},
\]

\[
\bar{V}(\lambda_i)_{M+2kn+m} = \mathbb{F}[\alpha_{-m}(m \in \mathbb{Z}_{>0})] \otimes \mathcal{H}^P_{\lambda,M} \otimes \mathbb{C} e^{i(M+2kn)a/2} \otimes \mathbb{C} e^{-mQ}.
\]

The action of the elliptic currents on \( \bar{V}(\lambda_i) \) is given by

\[
K(u) \equiv \exp \left( - \sum_{m \neq 0} \frac{[m]_q}{[2m]_q} \alpha_{-m} \right) : e^{Q z^{-k(2P-1)/4rr'+bh/2}}.
\]
\[ E(u) \mapsto \Psi(z) : \exp \left( - \sum_{m \neq 0} \frac{1}{[km]_q} \alpha_m z^{-m} \right) : e^{2Q^{\gamma}(z^{(h+1)/2-(P-1)/r^*})}, \]

\[ F(u) \mapsto \Psi(z) : \exp \left( \sum_{m \neq 0} \frac{[r^*m]_q}{[km]_q[rm]_q} \alpha_m z^{-m} \right) : e^{-\alpha_1 z^{-(h-1)/2+(P+h-1)/r^*}}, \]

Let \((\tilde{V}, V), (\tilde{W}, W)\) be two dynamical representations of \(U_{q,p}\). We define the tensor product \(V \tilde{\otimes} W\) by

\[ V \tilde{\otimes} W = \bigoplus_{\alpha} (V \otimes W)_{\alpha}, \quad (V \tilde{\otimes} W)_{\alpha} = \bigoplus_{\rho} V_{\rho} \otimes_{M_{\rho}} W_{\alpha-\beta}, \]

where \(\otimes_{M_{\rho}}\) denotes the usual tensor product modulo the relation

\[ f^{*}(P)v \otimes w = v \otimes f(P+h)w, \quad (3.1) \]

then \((\tilde{V} \tilde{\otimes} \tilde{W}) \circ \Delta : U_{q,p} \rightarrow D_{H,V} \tilde{\otimes} D_{H,W}\) is a dynamical representation of \(U_{q,p}\) on \(V \tilde{\otimes} W\).

4. Vertex operators

By using the \(H\)-Hopf algebroid structure, we define the types I and II vertex operators of \(U_{q,p}(\hat{sl}_2)\) as intertwiners of \(U_{q,p}(\hat{sl}_2)\) modules. Investigating their intertwining relations, we show that they coincide with those obtained in [2] by using the quasi-Hopf algebra structure of \(B_{q,\lambda}(\hat{sl}_2)\) and the isomorphism \(U_{q,p}(\hat{sl}_2) \cong B_{q,\lambda}(\hat{sl}_2) \otimes \mathbb{C} [\hat{H}]\).

**Definition 4.1.** The types I and II vertex operators of spin \(n/2\) are the intertwiners of \(U_{q,p}\)-modules of the form

\[ \hat{\Phi}(u) : \hat{V}(\lambda) \rightarrow V^{(n)}(\lambda) \hat{\otimes} \hat{V}(v), \]

\[ \hat{\Psi}^{*}(u) : \hat{V}(\lambda) \hat{\otimes} V^{(n)}(\lambda) \rightarrow \hat{V}(v), \]

where \(\lambda = q^{2\alpha}\), and \(\hat{V}(\lambda)\) and \(\hat{V}(v)\) denote the level-\(k\) highest weight \(U_{q,p}\)-modules of highest weights \(\lambda\) and \(v\), respectively. They satisfy the intertwining relations with respect to the comultiplication \(\Delta\) in definition 2.1.

\[ \Delta(x) \hat{\Phi}(u) = \hat{\Phi}(u)x \quad \forall x \in U_{q,p}, \quad (4.1) \]

\[ x \hat{\Psi}^{*}(u) = \hat{\Psi}^{*}(u) \Delta(x) \quad \forall x \in U_{q,p}, \quad (4.2) \]

The physically interesting cases are \(n = k, \lambda = \lambda_l, v = \lambda_{k-l}\) for the type I and \(n = 1, \lambda = \lambda_l, v = \lambda_{l \pm 1}\) for the type II. See, for example, [14].

Let us define the components of the vertex operators as follows:

\[ \hat{\Phi} \left( v - \frac{1}{2} \right) = \sum_{m=0}^{n} v_{m}^{n} \otimes \Phi_{m}(v), \quad (4.3) \]

\[ \hat{\Psi}^{*} \left( v - \frac{c+1}{2} \right) (\cdot \otimes v_{m}^{n}) = \Psi_{m}^{*}(v). \quad (4.4) \]

**Theorem 4.2.** The vertex operators satisfy the following linear equations:

\[ \hat{\Phi}(u) \hat{L}^{*}(v) = \hat{R}_{ln}^{*(12)}(v-u, P+h) \hat{L}^{*}(v) \hat{\Phi}(u), \quad (4.5) \]
Relation (4.5) should be understood on \( V^{(1)}_u \otimes \tilde{V}(\lambda) \), whereas (4.6) on \( V^{(1)}_u \otimes \tilde{V}(\lambda) \otimes V^{(n)}_v \).

**Proof.** Applying \( \Delta \) in definition 2.1 and noting proposition 3.2, we obtain from (4.1)

\[
\hat{\Phi}(u)\hat{\Phi}^+(v) = \Delta(\hat{\Phi}^+(v))\hat{\Phi}(u)
\]

\[
= \sum_{m=0}^{n} \sum_{\varepsilon} \hat{\Phi}^+(v) v_m^\varepsilon \otimes \hat{\Phi}^+(v) \Phi_m(u)
\]

\[
= \sum_{m=0}^{n} \sum_{\varepsilon} \sum_{n=0}^{n} R_{1n}^+(v - u, P) v_n^\varepsilon \otimes \hat{\Phi}^+(v) \Phi_m(u)
\]

\[
= \sum_{m=0}^{n} v_m^0 \otimes \sum_{\varepsilon} \sum_{n=0}^{n} R_{1n}^+(v - u, P + h \varepsilon) \hat{\Phi}^+(v) \Phi_m(u),
\]

where \( \mu_m = n - 2m \), etc. In the last equality we used (3.1). Similarly, for the type II, from (4.2), we obtain

\[
\hat{\Phi}^+(v + \frac{1}{2}) = \hat{\Phi}^+(v + \frac{1}{2}) \Delta(\hat{\Phi}^+(v)) \otimes v_m^0
\]

\[
= \sum_{m=0}^{n} \sum_{\varepsilon} \hat{\Phi}^+(v + \frac{1}{2}) (\hat{\Phi}^+(v + \frac{1}{2}) \otimes R_{1n}^+(u - v, P) v_m^\varepsilon)
\]

\[
= \sum_{m=0}^{n} \sum_{\varepsilon} \hat{\Phi}^+(v + \frac{1}{2}) (R_{1n}^+(u - v, P - \mu_m) v_m^\varepsilon \otimes \hat{\Phi}^+(v + \frac{1}{2}))
\]

\[
= \sum_{m=0}^{n} \sum_{\varepsilon} \hat{\Phi}^+(v + \frac{1}{2}) (R_{1n}^+(u - v, P - \mu_m) v_m^\varepsilon \otimes \hat{\Phi}^+(v + \frac{1}{2}))
\]

Here in the third equality, we used relation (3.1). Note also \( \varepsilon + \mu_m = \varepsilon_2 + \mu_m \). \( \square \)

Equations (4.5) and (4.6) coincide with (5.3) and (5.4) in [2], respectively. Note that the comultiplication used in [2] corresponds to the opposite one of \( \Delta \) here. Under certain analyticity conditions, these equations determine the vertex operators uniquely up to normalization.

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