On a multiple harmonic power series.

Michel Émery *

August 14, 2018

Abstract

The results obtained in the first version are not new.

MSC: 11M35, 33E20, 40A25, 40B05.

Keywords: Multiple harmonic series, Lerch function, Polylogarithm.

New version

The claimed results are not new. I am thankful to Professor Prodinger who kindly pointed out to me references on earlier publication. The lemma of the first version was already published in 1995 as Corollary 3 of [1], and the formulas stated in the abstract of the first version are trivial consequences of it.

References

[1] K. Dilcher, Some q-identities related to divisor functions, Discrete Math. 145 (1995) 83–93.

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Abstract
If \( \text{Li}_s \) denotes the polylogarithm of order \( s \), where \( s \) is a natural number, and if \( z \) belongs to the unit disk,
\[
\text{Li}_s \left( \frac{-z}{1-z} \right) = - \sum_{1 \leq i_1 \leq \cdots \leq i_s} \frac{z^{i_s}}{i_1 i_2 \cdots i_s}.
\]
In particular,
\[
\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s} = \sum_{1 \leq i_1 \leq \cdots \leq i_s} \frac{1}{i_1 \cdots i_s 2^s}.
\]

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Introduction
Equalities and identities between multiple harmonic series and polylogarithms have been investigated by many authors; see for instance [1] and the references therein. These series usually involve summations over all \( s \)-tuples \((i_1, \ldots, i_s)\) of natural numbers such that \( i_1 < \cdots < i_s \), where \( s \) is fixed. We shall be concerned with an instance where the summation indices may be equal to each other, that is, a sum over all integers \( i_1, \ldots, i_s \) verifying \( 1 \leq i_1 \leq \cdots \leq i_s \).

Definition. For \( \alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\} \) and \( s \in \{1, 2, \ldots\} \), the Lerch function of order \( s \) with shift \( \alpha \) is defined by
\[
\text{Li}_s^\alpha(w) = \sum_{n \geq 1} \frac{w^n}{(\alpha + n)^s}.
\]

The power series converges in the unit disk only, but the analytic function \( \text{Li}_s^\alpha \) extends to the whole complex plane minus a cut along for instance the

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half-line \([1, \infty)\); see §1.11 of [2]. When \(\alpha = 0\), \(\Li_s^\alpha\) is just \(\Li_s\), the usual polylogarithm of order \(s\). We shall be interested in the values of \(\Li_s^\alpha(w)\) in the half-plane \(\Re(w) < \frac{1}{2}\) only; they are given by the next proposition (which establishes anew the existence of the analytic extension of \(\Li_s^\alpha\) to that half-plane).

**Proposition.** For \(\alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\}\) and \(s \in \{1, 2, \ldots\}\), one has the following power series expansion, which converges for \(|z| < 1\):

\[
\Li_s^\alpha \left( \frac{-z}{1-z} \right) = - \sum_{1 \leq i_1 \leq \ldots \leq i_s} \frac{(i_s - 1)!}{(\alpha+1)(\alpha+2)\ldots(\alpha+i_s)} \frac{z^{i_s}}{(\alpha+i_1)(\alpha+i_2)\ldots(\alpha+i_{s-1})}.
\]

When \(\alpha = 0\), the right-hand side becomes much simpler:

**Corollary.** Fix \(s \in \{1, 2, \ldots\}\). For \(z\) in the unit disk,

\[
\Li_s \left( \frac{-z}{1-z} \right) = - \sum_{1 \leq i_1 \leq \ldots \leq i_s} \frac{z^{i_s}}{i_1 i_2 \ldots i_s}.
\]

Equivalently, for \(\Re w < \frac{1}{2}\),

\[
\Li_s(w) = - \sum_{1 \leq i_1 \leq \ldots \leq i_s} \frac{1}{i_1 i_2 \ldots i_s} \left( \frac{-w}{1-w} \right)^{i_s}.
\]

For instance, choosing \(z = \frac{1}{2}\) in the corollary gives

\[
\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s} = \sum_{1 \leq i_1 \leq \ldots \leq i_s} \frac{1}{i_1 \ldots i_s} 2^{i_s};
\]

as the left-hand side is \((1 - 2^{1-s})\zeta(s)\), and the right-hand one is \(\sum a_p/(p 2^p)\) with \(a_p \leq (1 + \log p)^{s-1}\), this is a reasonably fast series expansion of \(\zeta(s)\). More generally, taking \(z = \frac{1}{2}\) in the proposition gives a more rapidly convergent series for \(\sum_n (-1)^n (\alpha + n)^{-s}\).

**Proof of the proposition**

Observe first that the right-hand side in the statement of the proposition is a power series \(\sum c_p z^p\), with

\[
|c_p| \leq \frac{(p-1)!}{|\alpha+1|\ldots|\alpha+p|} \sum_{1 \leq i_1 \leq \ldots \leq i_{s-1} \leq p} \frac{1}{|\alpha+i_1|\ldots|\alpha+i_{s-1}|} \leq \frac{(p-1)!}{|\alpha+1|\ldots|\alpha+p|} \frac{p^{s-1}}{C(\alpha)^{s-1}},
\]

where \(C(\alpha)\) is some constant depending on \(\alpha\).
where \( C(\alpha) = \inf \{ |\alpha+1|, |\alpha+2|, \ldots \} > 0 \). By d’Alembert’s test, the series \( \sum |c_p z^p| \) converges in the unit disk; so does also \( \sum c_p z^p \). Consequently, to prove the proposition, it suffices to show that both sides of the claimed identity are equal for \(|z|\) small enough. We shall take \(|z| < \frac{1}{2}\); for such a \( z \), one has \(|-z/(1-z)| < 1\), whence

\[
\text{Li}^\alpha\left(\frac{-z}{1-z}\right) = \sum_{n \geq 1} \frac{(-1)^n}{(\alpha+n)} \left(\frac{z}{1-z}\right)^n = \sum_{n \geq 1} \frac{(-1)^n}{(\alpha+n)} \sum_{p \geq n} \frac{p-1}{n-1} z^p.
\]

Exchanging the summations gives

\[
(*) \quad \text{Li}^\alpha\left(\frac{-z}{1-z}\right) = \sum_{p \geq 1} z^p \sum_{n=1}^{p} \frac{(p-1)}{(n-1)} \frac{(-1)^n}{(\alpha+n)^n}.
\]

This exchange is licit because, setting \( K(\alpha) = 1/\inf_{n \geq 1} |1+\alpha/n| < \infty \), one has the estimate

\[
\sum_{p \geq n \geq 1} \left| \frac{(p-1)}{(n-1)} \frac{(-1)^n}{(\alpha+n)^n} z^p \right| \leq K(\alpha)^s \sum_{p \geq n \geq 1} \frac{|z|^p}{n^s} = K(\alpha)^s \sum_{n \geq 1} \frac{1}{n^s} \left( \frac{|z|}{1-|z|} \right)^n,
\]

which is finite because \( 0 \leq |z|/(1-|z|) < 1 \).

To establish the proposition, it remains to compute the coefficient of \( z^p \) in \((*)\), and more precisely to show that, for all \( \alpha \notin \{ -1, -2, \ldots \} \), \( s \in \{ 1, 2, \ldots \} \) and \( p \geq 1 \),

\[
\sum_{n=1}^{p} \frac{(p-1)}{(n-1)} \frac{(-1)^n}{(\alpha+n)^n} = \left\{ \begin{array}{ll}
\frac{-(p-1)!}{(\alpha+1)_p} \sum_{1 \leq i_1 \leq \ldots \leq i_{s-1} \leq p} \prod_{r=1}^{s-1} \frac{1}{\alpha+i_r} & \text{if } s \geq 2, \\
\frac{-(p-1)!}{(\alpha+1)_p} & \text{if } s = 1,
\end{array} \right.
\]

where \((x)_p\) is the Pochhammer symbol standing for \( \prod_{j=0}^{p-1} (x+j) \).

Putting \( q = p-1 \), \( m = n-1 \) and \( \beta = \alpha+1 \), it suffices to prove the following lemma:

**Lemma.** For all \( \beta \in \mathbb{C} \setminus \{ 0, -1, \ldots \} \) and all integers \( q \geq 0 \) and \( s \geq 1 \), one has

\[
\sum_{m=0}^{q} \binom{q}{m} \frac{(-1)^m}{(\beta+m)^s} = \left\{ \begin{array}{ll}
\frac{q!}{(\beta)_{q+1}} \sum_{0 \leq i_1 \leq \ldots \leq i_{s-1} \leq q} \prod_{r=1}^{s-1} \frac{1}{\beta+i_r} & \text{if } s \geq 2, \\
\frac{q!}{(\beta)_{q+1}} & \text{if } s = 1.
\end{array} \right.
\]
Proof of the lemma

Call \( L(q, \beta) \) and \( R(q, \beta) \) the left- and right-hand sides in this statement. First, for \( q = 0 \) and \( q = 1 \), the lemma is easily verified:

\[
L(0, \beta) = \frac{1}{\beta^s} = R(0, \beta) ;
\]

\[
L(1, \beta) = \frac{1}{\beta} - \frac{1}{(\beta + 1)} = \frac{1}{\beta(\beta + 1)} \sum_{u+v=s-1} \frac{1}{\beta^u (\beta + 1)^v} = R(1, \beta) .
\]

Next, observe that

\[
L(q+1, \beta) = \sum_m \binom{q+1}{m} \left( \frac{-1}{\beta + m} \right)^s = \sum_m \left[ \left( \binom{q}{m} + \binom{q}{m-1} \right) \left( \frac{-1}{\beta + m} \right)^s \right]
\]

\[
= \sum_m \binom{q}{m} \left( \frac{-1}{\beta + m} \right)^s + \sum_m \binom{q}{m} \left( \frac{1}{\beta + m + 1} \right)^s
\]

\[
= L(q, \beta) - L(q, \beta+1) .
\]

The rest of the proof will consist in verifying that, for \( q \geq 1 \), the right-hand side satisfies the same relation:

\[
(**) \quad R(q+1, \beta) = R(q, \beta) - R(q, \beta+1) .
\]

When this is done, \( ** \) immediately implies that the property

\[
\forall \beta \not\in \{0, -1, \ldots\} \quad L(q, \beta) = R(q, \beta)
\]

extends by induction from \( q = 1 \) to all \( q \geq 2 \), thus proving the lemma.

The following notation will simplify the proof of \( ** \): for all integers \( n \geq 0 \), \( t \geq 1 \), \( a \geq 0 \) and \( b \geq a \), set

\[
f_n = \frac{1}{\beta + n} ; \quad S^b_a(t) = \sum_{u \leq i_1 \leq \cdots \leq i_t \leq b} \prod_{r=1}^{t} f_{i_r} ; \quad S^b_a(0) = 1 ;
\]

and remark that, for \( t \geq 0 \) and \( 0 \leq a \leq b < c \),

\[
S^c_a(t) = \sum_{u+v=t} S^b_a(u) S^c_{b+1}(v) ;
\]

similarly, for \( 1 \leq a \leq b \),

\[
S^{b+1}_{a-1}(t) = \sum_{u+v+w=t} S^{a-1}_{a-1}(u) S^b_{a}(v) S^{b+1}_{b+1}(w) = \sum_{u+v+w=t} f^u_{a-1} S^b_{a}(v) f^w_{b+1} .
\]

Keeping these remarks in mind, the proof of \( ** \) goes as follows. Put \( t = s - 1 \) and write

\[
R(q, \beta) - R(q, \beta+1) = \frac{q!}{(\beta+1)_{q+1}} S^q_0(t) - \frac{q!}{(\beta+1)_{q+1} + 1} S^{q+1}_1(t)
\]

\[
= \frac{q!}{(\beta+1)_q} \left[ \frac{1}{\beta} S^q_0(t) - \frac{1}{\beta+q+1} S^{q+1}_1(t) \right] .
\]
The quantity in square brackets can be rewritten as

\[
f_0 \sum_{v=0}^{t} S_0^q(t-v)S_1^q(v) - f_{q+1} \sum_{v=0}^{t} S_1^q(v)S_{q+1}^q(t-v) = \sum_{v=0}^{t} S_1^q(v)(f_0^{t-v+1} - f_{q+1}^{t-v+1}) = \sum_{v=0}^{t} S_1^q(v)(f_0 - f_{q+1}) \sum_{u+w=t-v} f_u^w f_{q+1}^w = (f_0 - f_{q+1}) \frac{q+1}{\beta(\beta+q+1)} S_0^{q+1}(t) .
\]

Finally, the difference \( R(q, \beta) - R(q, \beta+1) \) amounts to

\[
\frac{q!}{(\beta+1)_q} \frac{q+1}{\beta(\beta+q+1)} S_0^{q+1}(t) = \frac{(q+1)!}{(\beta)_q(\beta+q+2)} S_0^{q+1}(t) = R(q+1, \beta) .
\]

This proves (**), and at the same time the lemma and the proposition.

**Remark.** In the particular case when \( \alpha = 0 \) and \( z = \frac{1}{2} \), (*) is exactly formula (4) of J. Sondow [3], obtained by accelerating convergence of the alternating zeta series. What the lemma does is computing the numerator in that formula. To that end, (**) is needed for \( \beta \in \{1, 2, \ldots\} \) only; but the general proof is just as easy.

**References**

[1] D. Bailey, P. Borwein, D.J. Broadhurst, and P. Lisoněk, Special values of multiple polylogarithms, *Trans. Amer. Math. Soc.* 353 (2000) 907–941.

[2] *Higher Transcendental Functions*, A. Erdélyi editor (McGraw-Hill, New York, 1953).

[3] J. Sondow, Analytic continuation of Riemann’s zeta function and values at negative integers via Euler’s transformation of series, *Proc. Amer. Math. Soc.* 120 (1994) 421–424.