Eventually homological isomorphisms in recollements of derived categories

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Abstract

For a recollement \((\mathcal{D}_B, \mathcal{D}_A, \mathcal{D}_C)\) of derived categories of algebras, we investigate when the functor \(j^*: \mathcal{D}_A \to \mathcal{D}_C\) is an eventually homological isomorphism. In this context, we compare the algebras \(A\) and \(C\) with respect to Gorensteinness, singularity categories and the finite generation condition \(Fg\) for the Hochschild cohomology. The results are applied to stratifying ideals, triangular matrix algebras and derived discrete algebras.

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1 Introduction

Recollement of triangulated categories, introduced by Beilinson et al. [2], is an important tool in algebraic geometry and representation theory. In particular, a recollement \((\mathcal{D}_B, \mathcal{D}_A, \mathcal{D}_C)\) of derived categories of algebras provide a useful framework for comparing the algebras \(A\), \(B\) and \(C\) with respect to certain homological properties, such as global dimension [10, 21, 1], finitistic dimension [18, 9], Hochschild homology and cyclic homology [20], Hochschild cohomology [17], Gorensteinness [28, 33], and so on. Meanwhile, like recollement of triangulated categories, recollement of abelian categories attracts a lot of attention in recent years [15, 29, 30, 31]. In particular, recollement
of abelian categories \((A,B,C)\) with the functor \(e : B \to C\) being an eventually homological isomorphism was used as a common context to compare the Gorensteinness, singularity categories and the F\(_g\) condition for the algebras \(A\) and \(eAe\), where \(e\) is an idempotent of \(A\) \cite{30}. The motivation of this paper is to give a derived categories version of this work.

Let \((DB, DA, DC)\) be a recollement of derived categories of algebras. The functor \(j^* : DA \to DC\) is called an eventually homological isomorphism if there is an integer \(t\) such that for every pair of finitely generated right \(A\)-modules \(M\) and \(N\), and every \(j > t\), there is an isomorphism

\[
\text{Hom}_{DA}(M,N[j]) \cong \text{Hom}_{DC}(j^*M,j^*N[j])
\]

of abelian groups. Our first main theorem characterizes when the functor \(j^* : DA \to DC\) in a recollement \((DB, DA, DC)\) is an eventually homological isomorphism.

**Theorem A.** Let \(A, B\) and \(C\) be finite dimensional algebras over an algebraically closed field \(k\), and let \((DB, DA, DC, i^*, i^!, i^\vee, j^*, j^!, j^*\) be a standard recollement defined by \(X \in D^b(C^{op} \otimes A)\) and \(Y \in D^b(A^{op} \otimes B)\). Suppose \(X^* = \text{RHom}_A(X,A)\) and \(Y^* = \text{RHom}_B(Y,B)\). Then the following are equivalent:

1. The functor \(j^*\) is an eventually homological isomorphism;
2. \(\text{gl.dim}_B < \infty\), \(A^eY \in K^b(\text{proj}_A^{op})\) and \(Y_A^* \in K^b(\text{proj}A)\);
3. \(\text{RHom}_B(Y,Y) \in K^b(\text{proj}A^e)\), where \(A^e = A^{op} \otimes_k A\).

Here, we refer \cite{17} or Definition 1 for the concept of standard recollement. In order to describe our second theorem, we recall the following three definitions briefly. A finite dimensional algebra \(A\) is said to be Gorenstein if \(\text{id}_AA < \infty\) and \(\text{id}_{A^{op}}A < \infty\); The singularity category of \(A\) is defined to be the Verdier quotient \(D^b(\text{mod}A)/K^b(\text{proj}A)\), and two algebras are said to be singularly equivalent if there is a triangle equivalent between their singularity categories; \(A\) is said to satisfy the F\(_g\) condition if the Hochschild cohomology ring \(HH^*(A)\) is Noetherian and the Yoneda algebra \(\text{Ext}^*_A(A/\text{rad}A,A/\text{rad}A)\) is a finitely generated \(HH^*(A)\)-module (for more details and backgrounds, see Section 4.3). Our second theorem shows that recollement \((DB, DA, DC)\) with the functor \(j^* : DA \to DC\) being an eventually homological isomorphism is a very good context to compare the algebras \(A\) and \(C\).

**Theorem B.** Let \(A, B\) and \(C\) be finite dimensional algebras over an algebraically closed field \(k\), and let \((DB, DA, DC, i^*, i^!, i^\vee, j^*, j^!, j^*\) be a recollement \((DB, DA, DC)\) with the functor \(j^* : DA \to DC\) being an eventually homological isomorphism.
oollement such that the functor $j^*$ is an eventually homological isomorphism. Then the following assertions hold true:

(a) $A$ is Gorenstein if and only if so is $C$;
(b) The algebras $A$ and $C$ are singularly equivalent;
(c) $A$ satisfies $Fg$ if and only if so does $C$.

Applying Theorem B to recollements induced by idempotents, we recover a result of Nagase, where the algebras $A$ and $e Ae$ are compared, for an idempotent $e$ and a stratifying ideal $AeA$ [26]. Also, we recover some relevant results in triangular matrix algebras. Finally, we show that derived discrete algebras can be reduced to $k$ or 2-truncated cycle algebras, via recollements of derived categories with the functor $j^*$ being an eventually homological isomorphism. As an application, we prove that derived discrete algebras satisfy the $Fg$ condition.

The paper is organized as follows: In section 2, we will recall some notions and results on recollements of derived categories. Section 3 is about eventually homological isomorphisms in recollements of derived categories, in which Theorem A is obtained. In section 4, we will prove Theorem B. In section 5, we apply our main theorem to stratifying ideals, triangular matrix algebras and derived discrete algebras.

2 Definitions and conventions

**Definition 1.** (Beilinson-Bernstein-Deligne [2]) Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$
\begin{array}{c}
\mathcal{T}_1 \\
i^* \\
i_! \Rightarrow \mathcal{T} \\
i^! \Rightarrow \mathcal{T}_2
\end{array}
$$

and denoted by $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_*, i^!, j_!, j^*, j_!, j^!)$ (or just $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2)$) such that

(R1) $(i^*, i_!), (i_*, i^!), (j_!, j^*)$ and $(j^*, j_!)$ are adjoint pairs of triangle functors;
(R2) $i_!, j_!$ and $j_*$ are full embeddings;
(R3) $j^*i_* = 0$ (and thus also $i^!j_* = 0$ and $i^*j_! = 0$);
(R4) for each $X \in \mathcal{T}$, there are triangles

$$
\begin{align*}
& j_! j^* X \rightarrow X \rightarrow i_* i^* X \\
& i_! i^* X \rightarrow X \rightarrow j_* j^* X
\end{align*}
$$
where the arrows to and from $X$ are the counits and the units of the adjoint pairs respectively.

Let $k$ be a field, $D := \text{Hom}_k(-, k)$ and $\otimes := \otimes_k$. Throughout the paper, all algebras are assumed to be finite dimensional algebras over $k$. Let $A$ be such an algebra. Denote by $\text{Mod}_A$ the category of right $A$-modules, and by $\text{mod}A$ (resp. $\text{proj}A$ and $\text{inj}A$) its full subcategories consisting of all finitely generated modules (resp. finitely generated projective modules and injective modules). For $* \in \{\text{nothing}, b\}$, denote by $D^*(\text{Mod}_A)$ (resp. $D^*(\text{mod}A)$) the derived category of (cochain) complexes of objects in $\text{Mod}_A$ (resp. $\text{mod}A$) satisfying the corresponding boundedness condition. Denote by $K^b(\text{proj}A)$ the homotopy category of bounded complexes of objects in $\text{proj}_A$. Up to isomorphism, the objects in $K^b(\text{proj}A)$ are precisely all the compact objects in $D(\text{Mod}_A)$. For convenience, we do not distinguish $K^b(\text{proj}A)$ from the perfect derived category $D_{\text{per}}(A)$ of $A$, i.e., the full triangulated subcategory of $DA$ consisting of all compact objects, which will not cause any confusion. Moreover, we also do not distinguish $K^b(\text{inj}A)$ and $D^b(\text{mod}A)$ from their essential images under the canonical full embeddings into $D(\text{Mod}_A)$. Usually, we just write $DA$ (resp. $D^b(A)$) instead of $D(\text{Mod}_A)$ (resp. $D^b(\text{mod}A)$). In this paper, all functors between triangulated categories are assumed to be triangulated functors.

**Definition 2.** (Han [17]) Let $A, B$ and $C$ be algebras. An recollement $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j_!, j^*, j_*)$ is said to be standard and defined by $Y \in D^b(A^{\text{op}} \otimes B)$ and $X \in D^b(C^{\text{op}} \otimes A)$ if $i^* \cong - \otimes_C^B Y$ and $j_* \cong - \otimes_A^C X$.

**Proposition 1.** (Han [17]) Let $A, B$ and $C$ be algebras, and $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j_!, j^*, j_*)$ a standard recollement defined by $Y \in D^b(A^{\text{op}} \otimes B)$ and $X \in D^b(C^{\text{op}} \otimes A)$. Then

$$
\begin{align*}
  i^* & \cong - \otimes_C^B Y, \\
  i_* & = \text{RHom}_B(Y, -) = - \otimes_B^A Y^*, \\
  i^! & = \text{RHom}_A(Y^*, -), \\
  j_! & \cong - \otimes_A^C X, \\
  j_* & = \text{RHom}_A(X, -) = - \otimes_A^C X^*, \\
  j^* & = \text{RHom}_{C}(X^*, -),
\end{align*}
$$

where $X^* = \text{RHom}_A(X, A)$ and $Y^* = \text{RHom}_B(Y, B)$.

Two recollements $(\mathcal{T}_1, T_1, \mathcal{T}_2, i^*, i_*, i_!^!, j_!^*, j_*^*)$ and $(\mathcal{T}_1', T_1', \mathcal{T}_2', i^{!*}, i_*^!, i_!'^!, j_!'^*, j_*'^*)$ are said to be equivalent if $(\text{Im}j_!, \text{Im}i_*, \text{Im}j_*) = (\text{Im}j_!'^!, \text{Im}i_*'^!, \text{Im}j_!'^*)$. From [17] Proposition 3 and Remark 1, every recollement of derived categories of algebras is equivalent to a standard one.
3 Proof of Theorem A

Let $A$ and $B$ be two algebras. Given a functor $F : \mathcal{D}A \to \mathcal{D}B$, $F$ is called an \textit{eventually homological isomorphism} if there is an integer $t$ such that for every pair of objects $M$ and $N$ in $\text{mod}A$, and every $j > t$, there is an isomorphism

$$\text{Hom}_{\mathcal{D}A}(M, N[j]) \cong \text{Hom}_{\mathcal{D}B}(FM, FN[j])$$

of abelian groups. This definition is taken from \cite{30} Section 3 with a minor modification.

In this section, we will characterize when the functor $j^* : \mathcal{D}A \to \mathcal{D}C$ in a recollement $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C)$ is an eventually homological isomorphism, that is, we will prove Theorem A. This result is used in Section 4 for comparing Gorensteinness, singularity categories and the Fg condition of the algebras $A$ and $C$. Let's start with the following lemmas.

**Lemma 1.** Assume that $F : \mathcal{D}A \to \mathcal{D}B$ is an eventually homological isomorphism. Let $X, Y \in D^b(\text{mod}A)$ such that $H^i(X) = H^i(Y) = 0$, for any $i < m$ or $i > n$. Then there is an integer $t$ such that $\text{Hom}_{\mathcal{D}A}(X, Y[j]) \cong \text{Hom}_{\mathcal{D}B}(FX, FY[j])$, for every $j > t$.

**Proof.** Since $F$ is an eventually homological isomorphism, there exists some $t_0$ such that

$$\text{Hom}_{\mathcal{D}A}(M, N[j]) \cong \text{Hom}_{\mathcal{D}B}(FM, FN[j]),$$

for any $M, N \in \text{mod}A$ and every $j > t_0$. Up to quasi-isomorphism, we assume that $X$ and $Y$ are of the form

$$X : 0 \to X^m \to X^{m+1} \to \cdots \to X^n \to 0,$$

$$Y : 0 \to Y^m \to Y^{m+1} \to \cdots \to Y^n \to 0.$$  

Using truncation technique just like \cite{19} Lemma 1.6, we can prove that $\text{Hom}_{\mathcal{D}A}(X, Y[j]) \cong \text{Hom}_{\mathcal{D}B}(FX, FY[j])$, for every $j > t_0 + n - m$. \hfill $\square$

**Lemma 2.** Let $A, B$ and $C$ be finite dimensional algebras over a field $k$, and let $(\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j^!, j_*, j^*)$ be a recollement. Then the following hold true.

1. For every $M \in \text{mod}B$, there exist two integers $m$ and $n$ such that $H^i(i_* M) = 0$, for any $i < m$ or $i > n$. 

5
(2) If $i_*$ restricts to $K^b(\text{proj})$, that is, $i_*$ sends $K^b(\text{proj}B)$ to $K^b(\text{proj}A)$, then there exist two integers $m_1$ and $n_1$ such that $H^i(i_* M) = 0$, for any $i < m_1$ or $i > n_1$, and every $M \in \text{mod}A$.

(3) If $i_*$ restricts to $K^b(\text{inj})$, then there exist two integers $m_2$ and $n_2$ such that $H^i(i_* M) = 0$, for any $i < m_2$ or $i > n_2$, and every $M \in \text{mod}A$.

Proof. (1): By [1, Lemma 2.9 (e)], the functor $i_*$ restricts to $K^b(\text{proj})$, and thus, $i_* A \in K^b(\text{proj} B)$. Assume $i_* A$ is quasi-isomorphic to a projective complex $P^\bullet$ of the form:

$$0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^{-m} \to 0.$$  

For any $M \in \text{mod}B$ and $i \in \mathbb{Z}$, we have

$$H^i(i_* M) \cong \text{Hom}_{D^b A}(A, i_* M[i]) \cong \text{Hom}_{D^b B}(i_* A, M[i]) \cong \text{Hom}_{D^b A}(P^\bullet, M[i]).$$  

Therefore, $H^i(i_* M) = 0$, for any $i < m$ or $i > n$.

(2): This can be proved in a similar way as we did in (1).

(3): Assume $i_*(DA)$ is quasi-isomorphic to a injective complex $I^\bullet$ of the form:

$$0 \to I^{-n_2} \to I^{-n_2+1} \to \cdots \to I^{-m_2} \to 0.$$  

For any $M \in \text{mod}A$ and $i \in \mathbb{Z}$, we have

$$DH^i(i_* M) \cong H^{-i}(D(i_* M)) \cong \text{Hom}_{D^b k}(i_* M, k[-i]) \cong \text{Hom}_{D^b A}(i_* M, DA[-i]).$$  

Here, the last isomorphism follows by adjunction. Using adjointness again, we get

$$DH^i(i_* M) \cong \text{Hom}_{D^b A}(i_* M, DA[-i]) \cong \text{Hom}_{D^b B}(M, i_* DA[-i]).$$  

Therefore, $H^i(i_* M) = 0$, for any $i < m_2$ or $i > n_2$.  

Now we are ready to prove Theorem A, which is divided into Theorem 1 and Theorem 2.

**Theorem 1.** Let $A$, $B$ and $C$ be finite dimensional algebras over a field $k$, and let $(D^b, DA, DC, i^*, i_*, j^*, j_*)$ be a recollement. Then the following are equivalent:

(a) The functor $j^*$ is an eventually homological isomorphism;

(b) $\text{gl.dim}B < \infty$, and $i_*$ restricts to both $K^b(\text{proj})$ and $K^b(\text{inj})$;

(b') $\text{gl.dim}B < \infty$, and $j^*$ restricts to both $K^b(\text{proj})$ and $K^b(\text{inj})$.  


Proof. (a)⇒ (b): For any $M, M' \in \text{mod}B$, and any $i \in \mathbb{N}$, we have

$$\text{Ext}^i_B(M, M') \cong \text{Hom}_DB(M, M'[i]) \cong \text{Hom}_{DA}(i_*M, i_*M'[i]).$$

By (Lemma 2.9 (e)), we have $i_*M, i_*M' \in D^b(\text{mod}A)$. Therefore, Lemma[1] and Lemma[2] (1) yield that there exists some integer $t$ such that

$$\text{Hom}_{DA}(i_*M, i_*M'[i]) \cong \text{Hom}_{DC}(j^*i_*M, j^*i_*M'[i]), \quad \forall i > t.$$ 

Since $j^*i_* = 0$, we obtain $\text{Ext}^i_B(M, M') \cong 0$, for any $i > t$. Therefore, $\text{gl.dim}B < \infty$.

Now we claim $i_*$ restricts to $K^b(\text{proj})$, and the statement $i_*$ restricts to $K^b(\text{inj})$ can be proved dually. For any $P \in K^b(\text{proj}B)$, we want to show $i_*P \in K^b(\text{proj}A)$. For this, since $i_*P \in D^b(\text{mod}A)$, it is equivalent to show that for any simple $A$-module $S$, there are only finite many integers $n$ such that $\text{Hom}_{DA}(i_*P, S[n]) \neq 0$ (see the proof of (Lemma 2.4 (c))). Clearly, $\text{Hom}_{DA}(i_*P, S[n]) = 0$ for sufficiently small $n$. On the other hand, by Lemma[1] there exists some integer $t$ such that $\text{Hom}_{DA}(i_*P, S[n]) \cong \text{Hom}_{DC}(j^*i_*P, j^*S[n])$, for any $n > t$. Since $j^*i_* = 0$, we obtain that $\text{Hom}_{DA}(i_*P, S[n]) \cong 0$, for any $n > t$. Therefore, $i_*P \in K^b(\text{proj}A)$.

(b)⇒ (a): For any $M, N \in \text{mod}B$ and $i \in \mathbb{N}$, applying the functor $\text{Hom}_{DA}(-, N[i])$ to the triangle $jj^*M \to M \to i_*i^*M \to$, we get exact sequence

$$\text{Hom}_{DA}(i_*i^*M, N[i]) \to \text{Hom}_{DA}(M, N[i]) \to \text{Hom}_{DA}(jj^*M, N[i]).$$

By Lemma[2] there exist $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ such that $H^i(i^*M) = 0$, for any $i < m_1$ or $i > n_1$, and $H^i(i^*M) = 0$, for any $i < m_2$ or $i > n_2$. Since $\text{gl.dim}B < \infty$, it follows from (Lemma 1.6) that there is an integer $t$ such that $\text{Hom}_{DB}(i_*M, i^*N[i]) = 0$, for any $i > t$. Using adjointness, we have $\text{Hom}_{DA}(i_*i^*M, N[i]) \cong 0$, for any $i > t$. Therefore, $\text{Hom}_{DA}(M, N[i]) \cong \text{Hom}_{DA}(jj^*M, N[i]) \cong \text{Hom}_{DA}(j_*j^*M, N[i])$, for any $i > t$. That is, $j^*$ is an eventually homological isomorphism.

(b)⇔ (b'): It follows from (Lemma 2.5 and Lemma 4.3) that $i_*$ restricts to $K^b(\text{proj})$ if and only if $j^*$ restricts to $K^b(\text{proj})$. Dually, $i_*$ restricts to $K^b(\text{inj})$ if and only if $j^*$ restricts to $K^b(\text{inj})$.

Corollary 1. Let $A, B$ and $C$ be finite dimensional algebras over a field $k$, and let $(DB, DA, DC, i^*, i_*, i^!, i^!, j^*, j_*)$ be a recollement such that the functor $j^*$ is an eventually homological isomorphism. Then there is a standard recollement $(DB, DA, DC, i'^*, i'_*, i'^!, i'^!, j'^*, j'_*)$ such that the functor $j'^*$ is an eventually homological isomorphism.
Proof. Since $j^*$ is an eventually homological isomorphism, it follows from Theorem \[\ref{thm:2}\] that $\text{gl.dim} B < \infty$, and $i_*$ restricts to both $K^b(\text{proj})$ and $K^b(\text{inj})$. Therefore, the recollement $(DB, DA, DC, i^*, i_*, i^1, j_1, j^*, j_*)$ can be extended one-step upwards and one-step downwards, see \[\cite{33}, Lemma 3 and Lemma 4\]. On the other hand, it follows from \[\cite{17}, Proposition 3 and Remark 1\] that $(DB, DA, DC, i^*, i_*, i^1, j_1, j^*, j_*)$ is equivalent to a standard recollement $(DB, DA, DC, i'^*, i'_*, i''_1, j'_1, j'^*, j''_*)$, which can also be extended one-step upwards and one-step downwards. Therefore, $i'_*$ restricts to both $K^b(\text{proj})$ and $K^b(\text{inj})$. Using Theorem \[\ref{thm:1}\] again, we obtain that the functor $j^{*'}$ is an eventually homological isomorphism. \qed

Owing to Corollary \[\ref{cor:1}\] we will restrict our discussions on standard recollement in the following text.

Theorem 2. Let $A$, $B$ and $C$ be finite dimensional algebras over a field $k$, and let $(DB, DA, DC, i^*, i_*, i^1, j_1, j^*, j_*)$ be a standard recollement defined by $X \in D^b(C^{op} \otimes A)$ and $Y \in D^b(A^{op} \otimes B)$. Suppose $X^* = \text{RHom}_A(X, A)$ and $Y^* = \text{RHom}_B(Y, B)$. Then the following are equivalent:

(a) The functor $j^*$ is an eventually homological isomorphism;
(b) $\text{gl.dim} B < \infty$, $A^*_Y \in K^b(\text{proj}A^{op})$ and $Y^*_A \in K^b(\text{proj}A)$;
(b') $\text{gl.dim} B < \infty$, $C^*_X \in K^b(\text{proj}C^{op})$ and $X^*_C \in K^b(\text{proj}C)$.

Moreover, if $k$ is algebraically closed, these occur precisely when

(c) $\text{RHom}_B(Y, Y) \in K^b(\text{proj}A^e)$, where $A^e = A^{op} \otimes_k A$.

Proof. (a)$\iff$ (b): It follows from \[\cite{11}, Lemma 2.8\] that $A^*_Y \in K^b(\text{proj}A^{op})$ if and only if $i^* \cong - \otimes_A^L Y$ restricts to $D^b(\text{mod})$, and this occurs precisely when $i_*$ restricts to $K^b(\text{inj})$, see \[\cite{33}, Lemma 4\]. By \[\cite{11}, Lemma 2.5\] and Proposition \[\ref{prop:1}\] $Y^*_A \in K^b(\text{proj}A)$ if and only if $i_* \cong - \otimes_B^L Y^*$ restricts to $K^b(\text{proj})$. Now the statement follows from Theorem \[\ref{thm:1}\]

(a)$\iff$ (b'): As above, we obtain that $C^*_X \in K^b(\text{proj}C^{op})$ if and only if $j^*$ restricts to $K^b(\text{inj})$, and $X^*_C \in K^b(\text{proj}C)$ if and only if $j^*$ restricts to $K^b(\text{proj})$. Thus, the statement follows from Theorem \[\ref{thm:1}\]

(b)$\iff$ (c): According to \[\cite{17}, Theorem 1\] and \[\cite{17}, Theorem 2\], a recollement of derived categories of algebras induces those of tensor product algebras and opposite algebras respectively. Therefore, we have the following two recollements:

\[
\begin{align*}
\mathcal{D}(B^e) & \xrightarrow{L_1} \mathcal{D}(A^{op} \otimes_k B) & \mathcal{D}(C^{op} \otimes_k B) \xrightarrow{F_1} \mathcal{D}(A^{op} \otimes_k B)
\end{align*}
\]
\[
\begin{array}{c}
\mathcal{D}(A^{\text{op}} \otimes_k B) \xrightarrow{L_2} \mathcal{D}(A) \xrightarrow{F_2} \mathcal{D}(A^{\text{op}} \otimes_k C),
\end{array}
\]

where \(L_1 \cong Y^* \otimes_{A^*} - \), \(F_1 \cong Y \otimes_B - \), \(L_2 \cong - \otimes_A Y \), \(F_2 \cong - \otimes_B Y^* \).

Now we claim (b) \(\Rightarrow\) (c). Since \(B\) is a finite dimensional algebra over an algebraically closed field, the condition \(\text{gl.dim}\, B < \infty\) is equivalent to \(B \in K^b(\text{proj}\, B^e)\) (Ref. [34, Lemma 7.2]). On the other hand, \(A^Y \in K^b(\text{proj}\, A^{\text{op}})\) and \(Y^*_A \in K^b(\text{proj}\, A)\) implies that both \(F_1\) and \(F_2\) preserve compactness. Therefore, \(F_2 F_1(B) \cong Y \otimes_B^L Y^* \cong \text{RHom}_B(Y, Y) \in K^b(\text{proj}\, A^e)\).

Next, we prove (c) \(\Rightarrow\) (b). Because \(F_2 F_1(B) \cong \text{RHom}_B(Y, Y) \in K^b(\text{proj}\, A^e)\) and both \(L_1\) and \(L_2\) preserve compactness (Ref. [1, Lemma 2.9 (e)]), we have that \(B \cong L_1 F_1 B \cong L_1 L_2 F_2 F_1 B \in K^b(\text{proj}\, B^e)\). By [34, Lemma 7.2], we get \(\text{gl.dim}\, B < \infty\). Due to [1, Lemma 4.2], \(F_2 F_1(B) \in K^b(\text{proj}\, A^e)\) yields that \(F_1(B) \in K^b(\text{proj}(A^{\text{op}} \otimes_k B))\), that is, \(A^Y_B \in K^b(\text{proj}(A^{\text{op}} \otimes_k B))\). Therefore, we get \(A^Y_B \in K^b(\text{proj}\, A^{\text{op}})\). Since \(i^* A = Y_B\) is a compact generator of \(\mathcal{D}B\), it follows that \(\text{thick} Y_B = \text{thick} B\), where \(\text{thick} Y_B\) is the smallest triangulated subcategory of \(\mathcal{D}B\) containing \(Y_B\) and closed under direct summands. By d\'evissage, \(\text{RHom}_B(A^Y_B, Y_B) \in K^b(\text{proj}\, A)\) yields that \(\text{RHom}_B(A^Y_B, B) \in K^b(\text{proj}\, A)\), that is, \(Y^*_A \in K^b(\text{proj}\, A)\).

\[\Box\]

4 Proof of Theorem B

In this section, we will compare the Gorensteinness, singularity categories and the Fg condition of the algebras \(A\) and \(C\), where there is a recollement \((\mathcal{D}B, \mathcal{D}A, \mathcal{D}C)\) such that the functor \(j^*: \mathcal{D}A \to \mathcal{D}C\) is an eventually homological isomorphism.

4.1 Comparison on Gorensteinness

Recall that a finite dimensional algebra \(A\) is said to be \textit{Gorenstein} if \(\text{id}_A A < \infty\) and \(\text{id}_{A^{\text{op}}} A < \infty\).

**Definition 3.** (\textcite{33}) Let \(\mathcal{T}_1\), \(\mathcal{T}\) and \(\mathcal{T}_2\) be triangulated categories, and \(n\) a positive integer. An \(n\)-recollement of \(\mathcal{T}\) relative to \(\mathcal{T}_1\) and \(\mathcal{T}_2\) is given by \(n+2\)
layers of triangle functors

\[ \begin{array}{c}
T_1 \longrightarrow T \longrightarrow T_2 \\
\vdots & & \vdots
\end{array} \]

such that every consecutive three layers form a recollement.

We mention that the ideal of this definition comes from \[3, 1\], where the concept “ladder” was introduced to study mixed categories. In terms of \(n\)-recollement, the relationship between recollement of derived categories and the Gorensteinness of algebras are expressed as follows.

**Proposition 2.** (See [33, Theorem III]) Let \(A, B\) and \(C\) be finite dimensional algebras, and \(DA\) admit an \(n\)-recollement relative to \(DB\) and \(DC\).

1. \(n = 3\): if \(A\) is Gorenstein then so are \(B\) and \(C\);
2. \(n \geq 4\): \(A\) is Gorenstein if and only if so are \(B\) and \(C\).

**Lemma 3.** Let \(A, B\) and \(C\) be finite dimensional algebras over a field \(k\), and let \((DB, DA, DC, i^*, i_*, i^!, j^*, j_* )\) be a recollement such that the functor \(j^*\) is an eventually homological isomorphism. Then this recollement can be extended to a 5-recollement of \(DA\) relative to \(DB\) and \(DC\).

**Proof.** Since \(j^*\) is an eventually homological isomorphism, it follows from Theorem [1] that \(\text{gl.dim} B < \infty\), and \(i_*\) restricts to both \(K^b(\text{proj})\) and \(K^b(\text{inj})\). Therefore, the recollement \((DB, DA, DC, i^*, i_*, i^!, j^*, j_* )\) can be extended one-step upwards and one-step downwards, see [33, Lemma 3 and Lemma 4]. Thus, we obtain a 3-recollement

\[ \begin{array}{c}
DB \longrightarrow DA \longrightarrow DC \\
\downarrow j^* & & \downarrow j_*
\end{array} \quad \text{(R)} \]

Since \(\text{gl.dim} B < \infty\), it follows from [1, Lemma 2.9 (e)] that \(i^*(DA) \in D^b(\text{mod} B) = K^b(\text{inj} B)\), and by [33, Lemma 4], (R) can be extend one step upwards. Similarly, we have \(i^! A \in D^b(\text{mod} B) = K^b(\text{proj} B)\), and thus, (R) can be extend one step downwards.

Now we get the main result of this section.

**Theorem 3.** Let \(A, B\) and \(C\) be finite dimensional algebras over a field \(k\), and let \((DB, DA, DC, i^*, i_*, i^!, j^*, j_* )\) be a recollement such that the functor \(j^*\) is an eventually homological isomorphism. Then the algebra \(A\) is Gorenstein if and only if so is \(C\).
Proof. According to Theorem 1, we have \( \text{gl.dim}B < \infty \), and thus, \( B \) is a Gorenstein algebra. Then it follows from Lemma 3 and Proposition 2 that \( A \) is Gorenstein if and only if so is \( C \).

\[ \text{dim}B < \infty \]

4.2 Comparison on singular categories

Let \( A \) be a finite dimensional algebra over \( k \). The singularity category \( D_{sg}(A) \) of \( A \) is defined to be the following Verdier quotient category:

\[ D_{sg}(A) := D^b(\text{mod}A)/K^b(\text{proj}A). \]

Clearly, the singularity category \( D_{sg}(A) \) carries a triangulated structure, and \( D_{sg}(A) = 0 \) if and only if \( \text{gl.dim}A < \infty \).

From [10, 11], two algebras are said to be singularly equivalent if there is a triangle equivalent between \( D_{sg}(A) \) and \( D_{sg}(B) \).

Proposition 3. (See [25, Proposition 2.5]) Let \( (D_1, D, D_2, i^*, i_*, j^*, j_*) \) be a recollement of triangulated categories and \( T \) be a thick subcategory of \( D \). Set \( T_1 = i^*T \) and \( T_2 = j^*T \). If \( i_*T_1 \subseteq T \) and \( j_*T_2 \subseteq T \), then there exists an induced recollement of triangulated quotient categories \( (D_1/T_1, D/T, D_2/T_1) \).

Proposition 4. Let \( A, B \) and \( C \) be finite dimensional algebras, and \( \mathcal{D}A \) admit a 4-recollement relative to \( \mathcal{D}B \) and \( \mathcal{D}C \). Then there exists an induced recollement of singularity categories \( (D_{sg}(C), D_{sg}(A), D_{sg}(B)) \).

Proof. Let

\[
\begin{array}{c}
D_B \\
\downarrow j_1 \\
\downarrow i_2 \\
D_A \\
\downarrow j_3 \\
\downarrow j_2 \\
D_C
\end{array}
\]

be a 4-recollement of \( DA \) relative to \( DB \) and \( DC \). By [11, Lemma 2.9 (e)], this 4-recollement restricts to the following recollement of \( D^b(\text{mod}) \)-level

\[
D^b(\text{mod}C) \\
\downarrow i_1 \\
D^b(\text{mod}A) \\
\downarrow j_2 \\
D^b(\text{mod}B),
\]

where all the sixes functors restrict to \( K^b(\text{proj}) \). Therefore, we have that \( i_1(K^b(\text{proj}A)) = K^b(\text{proj}C) \), \( j_2(K^b(\text{proj}A)) = K^b(\text{proj}B) \), \( i_2(K^b(\text{proj}C)) \subseteq K^b(\text{proj}A) \) and \( j_3(K^b(\text{proj}B)) \subseteq K^b(\text{proj}A) \). Now the statement follows from Proposition 3.

Now we get the main result of this section.
Theorem 4. Let $A$, $B$ and $C$ be finite dimensional algebras over a field $k$, and let $(DB, DA, DC, i^*, i_*, j^*, j_!, j^*, i_!, i^*)$ be a recollement such that the functor $j^*$ is an eventually homological isomorphism. Then $j^*$ induces a singularly equivalent between $A$ and $C$.

Proof. According to the proof of Lemma 3, there is a 4-recollement

$\xymatrix{ & DC \ar[r]^{j^*} & DA \ar[l]_i \ar[r]^j & DB \ar[l]_{i^*} }$

From Proposition 4, there exists an induced recollement of singularity categories $(D_{sg}(B), D_{sg}(A), D_{sg}(C))$. On the other hand, it follows from Theorem 1 that $\text{gl.dim}B < \infty$, that is, $D_{sg}(B) = 0$. Therefore, the functor $j^*$ induces a singularly equivalent between $A$ and $C$. □

### 4.3 Comparison on Fg condition

Let $A$ be a $k$-algebra and $X$ a complex of $A$-module, then we define

$\mathcal{E}_A^*(X) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_A}(X, X[n]).$

Clearly, $\mathcal{E}_A^*(X)$ is a graded $k$-algebra with multiplication given by Yoneda product. For some $d \in \mathbb{Z}$, we consider the graded ideals of the form

$\mathcal{E}_A^{\geq d}(X) = \bigoplus_{n \geq d} \text{Hom}_{D_A}(X, X[n]).$

From [7], the Hochschild cohomology ring of $A$ is the extension ring $\text{HH}^*(A) := \mathcal{E}_{A^e}(A)$, where $A^e := A^{op} \otimes_k A$ is the enveloping algebra. For convenience, we denote $\text{HH}^{\geq d}(A) := \mathcal{E}_{A^e}^{\geq d}(A) = \bigoplus_{n \geq d} \text{Hom}_{D(A^e)}(A, A[n])$.

To describe the finite generation condition Fg, we first need to define a $\text{HH}^*(A)$-module structure on $\mathcal{E}_A^*(X)$, for any complex $X$ in $D^b(A)$. Indeed, this module structure is given by the graded ring homomorphism $\varphi_X : \text{HH}^*(A) \to \mathcal{E}_A^*(X)$, where $\varphi_X = X \otimes_A^L$.

Support varieties for modules over artin algebras were defined by Snashall and Solberg in [36], using the Hochschild cohomology ring. In [13], Erdmann et al. introduced some finiteness conditions (Fg1) and (Fg2) for an algebra $A$, which ensure many results for support varieties over a group algebra also hold for support varieties over a selfinjective algebra. Later, these conditions were called Fg and were studied by many authors [23, 26, 30, 35].
Definition 4. Let $A$ be an algebra over a field $k$. We say that $A$ satisfies the Fg condition if the following is true:

(Fg1) The ring $\text{HH}^*(A)$ is noetherian.

(Fg2) The $\text{HH}^*(A)$-module $E_A^*(A/\text{rad}A)$ is finitely generated.

Nowadays, the Fg condition is becoming an important property in geometry and representation theory — it is a good criterion for deciding whether a given algebra has a nice theory of support varieties. What’s more, the Fg condition turns out to be related with Gorensteinness — an algebra $A$ is Gorenstein if $A$ satisfies the Fg condition [13]. Therefore, it is of great interest to know whether the Fg condition holds for various algebras, and to find out which relations between algebras preserve the Fg condition. The second question was considered in [26, 30] for algebras $A$ and $eAe$ with $e$ being an idempotent, in [24] for separable equivalence between symmetric algebras, in [35] for singular equivalence between Gorenstein algebras and in [23] for general derived equivalence. In this section, we will consider algebras whose derived categories are related by a recollement of triangulated categories. The following propositions will be used.

Proposition 5. (See [37, Proposition 10.3]) If an artin algebra $A$ satisfies the Fg condition, then $E_A^*(X)$ is a finitely generated $\text{HH}^*(A)$-module, for every $X \in D^b(A)$.

Proposition 6. Let $A$ and $B$ be finite-dimensional $k$-algebras. Set $M = A/\text{rad}A$ and $N = B/\text{rad}B$. Assume that we have the following two commutative diagrams

\[
\begin{array}{ccc}
\text{HH}^{\geq d}(A) & \xrightarrow{\varphi_M} & \mathcal{E}_A^{\geq d}(M) \\
f \downarrow & & \downarrow g \\
\text{HH}^{\geq d}(B) & \xrightarrow{\varphi_Y} & \mathcal{E}_B^{\geq d}(Y)
\end{array}
\quad
\begin{array}{ccc}
\text{HH}^{\geq d}(A) & \xrightarrow{\varphi_X} & \mathcal{E}_A^{\geq d}(X) \\
f' \downarrow & & \downarrow g' \\
\text{HH}^{\geq d}(B) & \xrightarrow{\varphi_N} & \mathcal{E}_B^{\geq d}(N)
\end{array}
\]

of graded nonunital $k$-algebras, for some positive integer $d$, some $X \in D^b(A)$ and $Y \in D^b(B)$, where the vertical maps $f$, $g$, $f'$ and $g'$ are isomorphisms. Then $A$ satisfies Fg if and only if so does $B$.

Proof. This follows from Proposition [5] and [30, Proposition 6.3].

The following lemma is essentially due to [22, Lemma 2.1] and [17, Lemma 5].
Lemma 4. Let $A, B$ be an algebras, and let $X \xrightarrow{u} Y \xrightarrow{v} Z$ be a triangle in $\mathcal{DA}$ such that $Y \in D^b(A)$, $Z \in K^b(\text{proj}A)$ and $Z \in K^b(\text{inj}A)$. Assume that $F: \mathcal{DA} \to \mathcal{DB}$ is a triangulated functor such that $FY \in D^b(B)$, $FZ \in K^b(\text{proj}B)$ and $FZ \in K^b(\text{inj}B)$. Then there is the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}^d_A(Y) & \xrightarrow{\psi} & \mathcal{E}^d_A(X) \\
\downarrow F & & \downarrow F \\
\mathcal{E}^d_B(FY) & \xrightarrow{\psi'} & \mathcal{E}^d_B(FX)
\end{array}
$$

of graded nonunital $k$-algebras, for some positive integer $d$, where the horizontal maps $\psi$ and $\psi'$ are isomorphisms.

Proof. For the sake of simplicity, we just denote the bifunctor $\text{Hom}_{\mathcal{DA}}(-,-)$ and $\text{Hom}_{\mathcal{DB}}(-,-)$ by $(-,-)$, when it may not cause any confusion. Applying the functor $(-,Y[n])$ (resp. $(-,FY[n])$ ) to the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z$ (resp. $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ$ ), we have the following commutative diagram

$$
\begin{array}{ccc}
(Z,Y[n]) & \xrightarrow{v^*} & (Y,Y[n]) \\
\downarrow F & & \downarrow F \\
(FZ,FY[n]) & \xrightarrow{(Fv)^*} & (FY,FY[n]) \\
\downarrow F & & \downarrow F \\
& (FX,FY[n]) & \xrightarrow{(Fu)^*} (FX,FY[n]) \\
& \downarrow F & \downarrow F \\
& (FZ[-1],FY[n]) & \xrightarrow{(Fu)^*} (FX,FY[n])
\end{array}
$$

Since $Z \in K^b(\text{proj}A)$, $Y \in D^b(A)$, $FZ \in K^b(\text{proj}B)$ and $FY \in D^b(B)$, there exists some integer $s$ such that $(Z,Y[n]) \cong 0 \cong (Z[-1],Y[n])$ and $(FZ,FY[n]) \cong 0 \cong (FZ[-1],FY[n])$, for any $n \geq s$. Therefore, we have the following commutative diagram

$$
\begin{array}{ccc}
(Y,Y[n]) & \xrightarrow{u^*} & (X,Y[n]) \\
\downarrow F & & \downarrow F \\
(FY,FY[n]) & \xrightarrow{(Fu)^*} (FX,FY[n])
\end{array}
$$

(I)

for any integer $n \geq s$, where the horizontal maps $u^*$ and $(Fu)^*$ are isomorphisms.

Similarly, applying the functor $X, -[n]$ (resp. $(FX,-[n])$) to the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z$ (resp. $FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ$ ), we have the
following commutative diagram

\[
\begin{array}{ccc}
(X,Z[n-1]) & \xrightarrow{F} & (X,X[n]) \xrightarrow{(u[n])_*} (X,Y[n]) \xrightarrow{F} (X,Z[n]) \\
(FX,FZ[n-1]) & \xrightarrow{F} & (FX,FX[n]) \xrightarrow{(Fu[n])_*} (FX,FY[n]) \xrightarrow{F} (FX,FZ[n]).
\end{array}
\]

On the other hand, \(Z \in K^b(\text{proj} A)\) and \(Y \in D^b(A)\) implies that \(X \in D^b(A)\), and \(FZ \in K^b(\text{proj} B)\) and \(FY \in D^b(B)\) implies that \(FX \in D^b(B)\). Since \(Z \in K^b(\text{inj} A)\) and \(FZ \in K^b(\text{inj} B)\), there exists some integer \(t\) such that \((X,Z[n-1]) \cong 0 \cong (X,Z[n])\) and \((FX,FZ[n-1]) \cong 0 \cong (FX,FZ[n])\), for any \(n \geq t\). Therefore, we have the following commutative diagram

\[
\begin{array}{ccc}
(X,X[n]) & \xrightarrow{(u[n])_*} (X,Y[n]) \\
(FX,FX[n]) & \xrightarrow{(Fu[n])_*} (FX,FY[n]),
\end{array}
\]

for any integer \(n \geq t\), where the horizontal maps \((u[n])_*\) and \((Fu[n])_*\) are isomorphisms. Let \(d = \max\{1,t,s\}\). Combining (I) and (II), we have commutative diagram

\[
\begin{array}{ccc}
(Y,Y[n]) & \xrightarrow{\psi} (X,X[n]) \\
(FY,FY[n]) & \xrightarrow{\psi'} (FX,FX[n]),
\end{array}
\]

for any integer \(n \geq d \geq 1\), where the horizontal maps \(\psi\) and \(\psi'\) are isomorphisms. Now the statement holds obviously.

\[\square\]

**Lemma 5.** Let \(M\) be a complex of \(D^b(A)\). Then the functor \(\varphi_M = M \otimes_A -\) sends \(K^b(\text{proj} A^e)\) to \(K^b(\text{proj} A)\).

**Proof.** Assume \(M\) is of the form \(0 \rightarrow M^k \rightarrow M^{k+1} \rightarrow \cdots \rightarrow M^l \rightarrow 0\) with \(M^i \in \text{mod} A\), and \(P : 0 \rightarrow P^p \rightarrow P^{p+1} \rightarrow \cdots \rightarrow P^q \rightarrow 0\) with \(P^i \in \text{proj} A^e\). Then \(\varphi_M(P) = M \otimes_A \mathbb{L}^2 P \cong (0 \rightarrow X^{i+p} \rightarrow X^{i+p+1} \rightarrow \cdots \rightarrow X^{l+q} \rightarrow 0)\), where \(X^n = \oplus_{i+j=n} M^i \otimes_A P^j\). Clearly, \(M^i \otimes_A (A \otimes k A) \cong M^i \otimes_k A \in \text{proj} A\), and thus, \(M^i \otimes_A P^j \in \text{proj} A\). Therefore, \(\varphi_M(P) \in K^b(\text{proj} A)\). \[\square\]
Theorem 5. Let \((\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j_!, j^!, j^*)\) be a standard recollement defined by \(X \in \mathcal{D}^b(\mathcal{D}^\text{op} \otimes A)\) and \(Y \in \mathcal{D}^b(\mathcal{D}^\text{op} \otimes B)\). Then the functor \(X^* \otimes_C^L - \otimes_C^R X : \mathcal{D}(C^e) \to \mathcal{D}(A^e)\) is fully faithful.

Proof. By [17, Theorem 1 and Theorem 2], there are two recollements

\[
\begin{array}{cccc}
\mathcal{D}(\mathcal{D}^\text{op} \otimes_k B) & \mathcal{D}(\mathcal{D}^\text{op} \otimes_k A) & \mathcal{D}(\mathcal{D}^\text{op} \otimes_k C) & \\
\mathcal{D}(\mathcal{D}^\text{op} \otimes_k A) & \mathcal{D}(A^e) & \mathcal{D}(\mathcal{D}^\text{op} \otimes_k A) & \\
\end{array}
\]

where \(G_1 \cong \mathcal{D}^\text{op} \otimes_k X \) and \(G_2 \cong X^* \otimes_C^L -\). Since \(G_1\) and \(G_2\) are fully faithful, we have that the functor \(G_2G_1 \cong X^* \otimes_C^L - \otimes_C^R X : \mathcal{D}(C^e) \to \mathcal{D}(A^e)\) is fully faithful.

Now we are ready to compare the Fg condition in the framework of recollement.

Theorem 5. Let \(A, B\) and \(C\) be finite dimensional algebras over an algebraically closed field \(k\), and let \((\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j_!, j^!, j^*)\) be a recollement such that the functor \(j^*\) is an eventually homological isomorphism. Then we have

1. \(\text{HH}^{\geq d}(A) \cong \text{HH}^{\geq d}(C)\), for some positive integer \(d\).
2. \(A\) satisfies \(\text{Fg}\) if and only if \(C\) does.

Proof. (1): Due to Corollary [1] we may assume that the recollement \((\mathcal{D}B, \mathcal{D}A, \mathcal{D}C, i^*, i_*, i^!, j_!, j^!, j^*)\) is standard and defined by \(X \in \mathcal{D}^b(\mathcal{D}^\text{op} \otimes A)\) and \(Y \in \mathcal{D}^b(\mathcal{D}^\text{op} \otimes B)\). Let \(X^* = \text{RHom}_A(X, A)\) and \(Y^* = \text{RHom}_B(Y, B)\). From [17, Corollary 3], there is a long exact sequence

\[
\cdots \to \text{Hom}_{\mathcal{D}(A^e)}(\text{RHom}_B(Y, Y), A[n]) \to \text{HH}^n(A) \to \text{HH}^n(C) \to \cdots .
\]

On the other hand, it follows from Theorem [2] that \(\text{RHom}_B(Y, Y) \in \mathcal{K}^b(\text{proj}A^e)\), and thus, there exists some integer \(d\) such that \(\text{Hom}_{\mathcal{D}(A^e)}(\text{RHom}_B(Y, Y), A[n]) \cong 0\), for any \(n \geq d\). As a result, we get \(\text{HH}^n(A) \cong \text{HH}^n(C)\), for any \(n \geq d\).

(2): Assume either \(A\) or \(C\) satisfies \(\text{Fg}\). Then, it follows from [13, Theorem 1.5(a)] that either \(A\) or \(C\) is Gorenstein, and by Theorem [3], both \(A\) and \(C\) are Gorenstein. From [17, Theorem 1], we have the following recollement

\[
\begin{array}{cccc}
\mathcal{D}(\mathcal{D}^\text{op} \otimes_k B) & \mathcal{D}(A^e) & \mathcal{D}(A^e) & \\
\mathcal{D}(A^e) & \mathcal{D}(\mathcal{D}^\text{op} \otimes_k C) & \\
\end{array}
\]

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where \( I^* \cong - \otimes_A^L Y \), \( I \cong - \otimes_B^L Y^* \), \( J_1 \cong - \otimes_C^L X \) and \( J^* \cong - \otimes_A^L X^* \). Therefore, there is a triangle \( X^* \otimes_B^L X \to A \to Y \otimes_B^L Y^* \to \) in \( D(A^e) \). By Theorem 2, \( Y \otimes_B^L Y^* \cong \text{RHom}_B(Y, Y) \in K^b(\text{proj}A^e) \), and by [5, Lemma 2.1], the Gorensteinness of \( A \) implies the Gorensteinness of \( A^e \). Hence, we have \( Y \otimes_B^L Y^* \in K^b(\text{inj}A^e) \). For any \( M \in D^b(A) \), consider the functor \( \varphi_M = M \otimes_A^L - : D(A^e) \to D(A) \). Clearly, \( \varphi_M(A) = M \otimes_A^L A \cong M \in D^b(A) \), and by Lemma 2 \( \varphi_M(Y \otimes_B^L Y^*) \in K^b(\text{proj}A) = K^b(\text{inj}A) \). Applying Lemma 2 to the triangle \( X^* \otimes_C^L X \to A \to Y \otimes_B^L Y^* \to \) and the functor \( \varphi_M \), we obtain the following commutative diagram

\[
\begin{array}{ccc}
\text{HH}^{\geq d}(A) & \xrightarrow{\psi} & \mathcal{E}_A^{\geq d}(X^* \otimes_C^L X) \\
\varphi_M \downarrow & & \varphi_M \\
\mathcal{E}_A^{\geq d}(M) & \xrightarrow{\psi'} & \mathcal{E}_A^{\geq d}(M \otimes_A^L X^* \otimes_C^L X)
\end{array}
\]

of graded nonunital \( k \)-algebras, for some positive integer \( d \), where the horizontal maps \( \psi \) and \( \psi' \) are isomorphisms. On the other hand, the associativity of the tensor product yields the commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}_A^{\geq d}(X^* \otimes_C^L X) & \xrightarrow{X^* \otimes_C^L X} & \text{HH}^{\geq d}(C) \\
\varphi_M \downarrow & & \varphi_M \\
\mathcal{E}_A^{\geq d}(M \otimes_A^L X^* \otimes_C^L X) & \xleftarrow{- \otimes_C^L X} & \mathcal{E}_C^{\geq d}(M \otimes_A^L X^*)
\end{array}
\]

where the horizontal two maps are isomorphisms, because both the two functors are fully faithful, see Lemma 3. As a result, we obtain the following commutative diagram

\[
\begin{array}{ccc}
\text{HH}^{\geq d}(A) & \xrightarrow{\cong} & \text{HH}^{\geq d}(C) \\
\varphi_M \downarrow & & \varphi_M \\
\mathcal{E}_A^{\geq d}(M) & \xrightarrow{\cong} & \mathcal{E}_C^{\geq d}(M \otimes_A^L X^*)
\end{array}
\]

of graded nonunital \( k \)-algebras, where the horizontal maps are isomorphisms. Now take \( M = A/\text{rad}A \) and \( M = C/\text{rad}C \otimes_C^L X \) respectively, we obtain two desired commutative diagrams in Proposition 6. Therefore, \( A \) satisfies \( Fg \) if and only if so does \( B \).
5 Applications

In this section, we will apply our main theorem to stratifying ideals, triangular matrix algebras and derived discrete algebras, and we prove that derived discrete algebras satisfy the Fg condition.

Let $A$ be an algebra, and let $e \in A$ be an idempotent such that $AeA$ is a stratifying ideal, that is, $Ae \otimes_{eAe} L eA \cong AeA$ canonically. From [12], there is a recollement

\[
\xymatrix{ & D(A/AeA) \ar[dr] & & D(eAe) \ar[dl] & \\
\otimes_{eAe} L A & & & & \otimes_{eAe} L A \\
& DA & & & DA}
\]

By Theorem 2, the functor $- \otimes_{eAe} L A$ is an eventually homological isomorphism if and only if $\text{pd}_{Ae} A/AeA < \infty$. Applying Theorem 3 and Theorem 5, we recover the following result of Nagase.

**Corollary 2.** (Nagase [26]) Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Suppose $AeA$ is a stratifying ideal and $\text{pd}_{Ae} A/AeA < \infty$. Then we have

1. $\text{HH}^d(A) \cong \text{HH}^d(eAe)$, for some positive integer $d$.
2. $A$ satisfies Fg if and only if so does $eAe$.
3. $A$ is Gorenstein if and only if so is $eAe$.

Let $B$ and $C$ be finite dimensional algebras over a field $k$, and let $M$ be a finitely generated $C$-$B$-bimodule. Then we have the triangular matrix algebra $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$, where the addition and the multiplication are given by the ordinary operations on matrices. From [11 Example 3.4], there is a recollement

\[
\xymatrix{ & DB \ar[dr] & & DC \ar[dl] & \\
\otimes_{B e_1 A} L & & & & \otimes_{A e_2 C} L \\
& DA & & & DA}
\]

where $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By Theorem 2, $- \otimes_{A} L A e_2$ is an eventually homological isomorphism if $\text{gl.dim.} B < \infty$ and $\text{pd}_{C}(e_2 A) < \infty$, and the latter holds precisely when $\text{pd}_{C} M < \infty$. Combining Theorem 3, Theorem 4 and Theorem 5, we reobtain the following corollary in [30].

**Corollary 3.** ([30 Corollary 8.17]) Let $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ be a triangular matrix algebra over an algebraically closed field $k$. Suppose $\text{gl.dim.} B < \infty$ and $\text{pd}_{C} M < \infty$. Then the following hold.
(1) The algebras $A$ and $C$ are singularly equivalent.
(2) $A$ satisfies Fg if and only if so does $C$.
(3) $A$ is Gorenstein if and only if so is $C$.

Also, by [1, Example 3.4], we have another recollement

$$\begin{array}{cccc}
DC & \rightarrow & \rightarrow & DA \\
\rightarrow & \rightarrow & \rightarrow & \leftarrow\leftarrow
\end{array}\otimes B_{e_1}A \rightarrow \otimes A_{e_1}B,$$

and similarly, we recover the following result.

**Corollary 4.** ([30, Corollary 8.19]) Let $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ be a triangular matrix algebra over an algebraically closed field $k$. Suppose $\text{gl.dim}. C < \infty$ and $\text{pd} M_B < \infty$. Then the following hold.

(1) (See [9, Theorem 4.1]) The algebras $A$ and $B$ are singularly equivalent.
(2) $A$ satisfies Fg if and only if so does $B$.
(3) $A$ is Gorenstein if and only if so is $B$.

In [26], Gorenstein Nakayama algebras are reduced to (local) selfinjective Nakayama algebras, via idempotents in the situation of Corollary [2]. Thus, Gorenstein Nakayama algebras satisfy the Fg condition because so do selfinjective Nakayama algebras. Now, we will reduce derived discrete algebras by recollements with the functor $j^*$ being an eventually homological isomorphism.

From [39], an algebra $A$ is said to be derived discrete provided for every positive element $d \in K_0(A)^{(2)}$ there are only finitely many isomorphism classes of indecomposable objects $X$ in $\mathcal{D}^b(A)$ of cohomology dimension vector $(\dim H^p(X))_{p \in \mathbb{Z}} = d$. Up to derived equivalent, a basic connected derived discrete algebra is either a piecewise hereditary algebra of Dynkin type or a special gentle algebra $\Lambda(r, n, m)$, see [39, 6] for details.

**Proposition 7.** Let $A$ be a derived discrete algebra. Then we have

(1) $DA$ admits a finite stratification of derived categories along recollements with the functor $j^*$ being an eventually homological isomorphism, where all derived-simple factors are either $k$ or 2-truncated cycle algebras.
(2) $A$ satisfies the Fg condition.

**Proof.** (1): It follows from [32, Theorem 19] that $DA$ admits a finite stratification of derived categories along $n$-recollements, where all derived-simple factors are either $k$ or 2-truncated cycle algebras. In these recollements
\((DB, DA, DC, i^*, i_!, j^!, j_!),\) all sixes functors restrict to both \(K^b(\text{proj})\) and \(K^b(\text{inj}).\) Moreover, either \(\text{gl.dim} A < \infty\) and hence \(\text{gl.dim} B < \infty,\) or \(\text{gl.dim} A = \infty\) and one of \(B\) and \(C\) has finite global dimension (see [32, proposition 8]). By Theorem 1, the functor \(j^!\) in \((DB, DA, DC, i^*, i_!, j^!, j_!, j_\theta, i^\theta, i_\theta)\) or the functor \(i^!\) in \((DC, DA, DB, j^*, j_!, j^\theta, i_!, i^\theta, i_\theta)\) is an eventually homological isomorphism.

(2): By [4, Section 4], selfinjective Nakayama algebras satisfy the \(Fg\) condition. Therefore, it follows from (1) and Theorem 5 that \(A\) satisfies the \(Fg\) condition.

Remark 1. Proposition 7 add derived discrete algebras to the classes of algebras where the \(Fg\) is known to hold, e.g. Gorenstein Nakayama algebras [26], group algebras of finite groups [14, 38] and local finite dimensional algebras which are complete intersections [16].

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