A Non-Principal Value Prescription for the Temporal Gauge

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ABSTRACT

A non-principal value prescription is used to define the spurious singularities of Yang-Mills theory in the temporal gauge. Typical one-loop dimensionally-regularized temporal-gauge integrals in the prescription are explicitly calculated, and a regularization for the spurious gauge divergences is introduced. The divergent part of the one-loop self-energy is shown to be local and has the same form as that in the spatial axial gauge with the principal-value prescription. The renormalization of the theory is also briefly mentioned.
1. Introduction

Quantization of constrained systems such as gauge theories is a very delicate problem. In particle physics, one has been interested in quantizing gauge theories in noncovariant gauges [1,2] like the general axial gauges, which can be classified by constant four-vectors. In these gauges, the theories enjoy the possibility of the decoupling of the Faddeev-Popov ghosts. However, these theories possess residual symmetries, which lead to difficulties in defining the propagators for the gauge fields. In order to specify the propagators, one has to define boundary conditions for the gauge fields in the spatial-axial gauge, but to define initial conditions for the gauge fields in the temporal gauge [2].

It is generally known that among the noncovariant gauges, the temporal gauge is the most complicated and cumbersome gauge choice, even from the point of view of the perturbation expansion. It remains a difficult problem to define useful Feynman rules for perturbative calculations in the temporal gauge. Because Feynman rules are Green functions, they are distributions and integrations with respect to loop variables may not commute [3]. For example, integrations in gauge theories in the spatial axial gauge with the principal-value prescription do not necessarily commute. Moreover, for applications in particle physics, it is useful to use a renormalizable gauge formalism for calculating radiative corrections. Thus, one should investigate the renormalization structure of any viable gauge formalism, which may find applications in particle physics. Besides, with the importance of the nontrivial topological vacuum structure of Yang-Mills theory, which has been investigated [4] in the temporal gauge, a perturbative study can be useful to nonperturbative effects in processes involving hadrons.

In this paper, we shall study the renormalization structure of Yang-Mills theory quantized in the temporal gauge. In particular, we shall treat the unphysical singularities in the polarization sum in the gauge propagator by a non-principal value prescription, which enjoys the property of being commutative with respect to integration. Such a prescription has been introduced many years ago [5,6] but has not been analyzed or applied to particle physics. We organize this paper as follows. In section 2, we shall introduce the non-principal value prescription and use it, in section 3, to calculate two typical dimensionally-regularized temporal-gauge integrals for the one-loop gluon self-energy. In section 4, we shall briefly mention the renormalization of the gauge theory in this formalism. Finally, we shall provide a short section for discussion.
2. The Non-Principal Value Prescription

The temporal gauge is defined by the condition $n_\mu A_\mu^a = A_0^a = 0$, where $A_\mu^a$ is the gauge potential, and $n_\mu = (1, 0, 0, 0)$ a constant temporal four-vector. In this gauge, the propagator has a spurious double pole at $q_0 = 0$ and reads ($i, j = 1, 2, 3$)

$$G_{ij}(q) = \frac{i\delta_{ij}}{q^2 + i\epsilon}, \quad C_{i0}(q) = G_{i0}(q) = C_{00}(q) = 0,$$

or, in covariant form,

$$G_{\mu\nu}(q) = \frac{i\delta_{\mu\nu}}{q^2 + i\epsilon} \left\{ \delta_{\mu\nu} + \frac{(q_\mu n_\nu + q_\nu n_\mu)}{q \cdot n} - \frac{q_\mu q_\nu}{(q \cdot n)^2} \right\}, \quad \epsilon > 0,$$

where we use a $(+, -, -, -)$ metric. We note that propagator (2) has a simple pole and a double pole at $q_0 = 0$. In calculations, these unphysical poles should be circumvented with prescriptions in a way that physical quantities should not depend on the prescriptions and on which of the above propagators one uses.

To date, many prescriptions with regularizations for the spurious poles have been introduced. We shall consider a first-order type of prescription, which is defined for the simple pole, with its square for the double pole. For example, we may consider a prescription given by the replacement [7]:

$$\frac{1}{q_0} \to \frac{1}{q_0 - \alpha |\vec{q}|} = \frac{q_0 + \alpha |\vec{q}|}{q_0^2 - \alpha^2 q^2}$$

$$\to \lim_{\epsilon \to 0} \left[ \frac{q_0 + \alpha |\vec{q}|}{q_0^2 - \alpha^2 q^2 + i\epsilon} \right], \quad \epsilon > 0,$$

and the double pole is given by

$$\frac{1}{q_0^2} \to \lim_{\epsilon \to 0} \left[ \frac{q_0 + \alpha |\vec{q}|}{q_0^2 - \alpha^2 q^2 + i\epsilon} \right]^2, \quad \epsilon > 0,$$

where $\epsilon$ specifies how the $q_0$ integration is to be carried out, and the limit $\epsilon \to 0$ is taken only after the integration. The parameter $\alpha$, whose limit $\alpha \to 0$ should be taken only after a calculation of a given loop order in perturbation theory, depends on the sign of $q_0$ and is used for regularizing the divergences arising from the gauge singularities. Note that the prescription for the double pole is non-negative.
In this paper, we shall consider a non-principal value prescription defined by replacing the simple pole by

$$\frac{1}{q_0} \to \frac{1}{|q_0|} \equiv \lim_{\epsilon \to 0} \left[ \frac{q_0}{q_0 + i \epsilon} \right], \epsilon > 0. \quad (5)$$

Then the double pole is given by

$$\frac{1}{q_0^2} \to \frac{1}{|q_0|^2} \equiv \lim_{\epsilon \to 0} \left[ \frac{q_0}{q_0^2 + i \epsilon} \right]^2, \epsilon > 0. \quad (6)$$

Definitions (5) and (6) can be seen as the reduced forms of (3) and (4) with $\alpha = 0$. The two poles in (6), $q_0^\pm = \mp \sqrt{i \epsilon}$, merge with one another as $i \epsilon \to 0$. These pinching poles are the problem of using the temporal gauge. There is no problem with (5), which has essentially a simple pole. We emphasize that the “$i \epsilon$” in (5) and (6) only provides us with a definition for $q_0$ integration. Dimensionally-regularized integrals with the spurious poles of (6) are not regularized. We should expect that dimensional regularization does not regularize the gauge singularities, since gauge singularities and space-time dimensionality are unrelated, with space-time dimensionality being intimately related to the physical structure of a theory. The need for a regularization for integrals with the double gauge pole may be seen in the following section.

The above definitions, (5) and (6), are also seen to be related to those introduced by Leibbrandt [8], who considers the temporal gauge $n_\mu A_\mu = 0$ with a constant temporal vector $n_\mu \equiv (n_0, n_\perp, n_3 = -i|n_\perp|)$. For his definitions, a dual temporal vector $n_\mu^{\text{dual}} \equiv (n_0, n_\perp, i|n_\perp|)$ is also introduced. The basic one-loop temporal gauge integrals are local, but the one-loop gluon self-energy is nonlocal in the external momentum. For real $n_\perp$ and $|n_\perp| \to 0$, his definitions for the simple and double poles reduce algebraically to (5) and (6) above. In [8], the two constant temporal vectors play the role of a regularization for the gauge divergences.

It is perhaps tempting to define the double pole as

$$\frac{1}{q_0^2} \equiv \lim_{\epsilon \to 0} \left[ \frac{1}{q_0^2 + i \epsilon} \right], \epsilon > 0. \quad (7)$$

This definition was introduced long time ago [5,6] and can be the same as (6) if a particular regularization is used. In general, definitions (6) and (7) do not necessarily lead to finite
temporal-gauge integrals but merely provide calculational procedures. The divergences arising from the gauge singularities, as we shall see, require also a regularization. In short, a prescription amounts to providing definitions for the spurious gauge poles and may or may not include a regularization.

In the following section, we shall employ the above definitions for the poles in the one-loop integrals. We look for a regularization scheme such that the definitions enjoy certain useful properties. They are (under integral signs):

\[ \frac{q_0}{[q_0]} = 1, \quad (8) \]

\[ \frac{q_0}{[q_0^2]} = \frac{1}{[q_0]}, \quad (9) \]

\[ \frac{q_0^2}{[q_0^2]} = 1. \quad (10) \]

In addition, the algebraic identity

\[ \frac{1}{q \cdot n(p - q) \cdot n} = \frac{1}{p \cdot n} \left[ \frac{1}{q \cdot n} + \frac{1}{(p - q) \cdot n} \right], \quad p \cdot n \neq 0, \quad (11) \]

if also satisfied, would be useful for simplifying calculations.

3. Sample Calculations

The purpose of this section is to outline the calculational procedure for the one-loop dimensionally-regularized temporal-gauge integrals and to observe the gauge divergence problem of temporal-gauge theories. At the one-loop level, we shall need a regularization for regularizing the gauge divergences of the integrals. For the usual ultra-violet divergences that we are interested in, we employ dimensional regularization with complex space-time dimensionality \(2\omega\).

First consider the following integral with the spurious simple pole:

\[ I = \int \frac{d^{2\omega}q}{(p - q)^2q \cdot n}. \quad (12) \]

Using prescription (5) for the simple pole and Feynman’s parametrization formula, we get

\[ I = \int_0^1 d\alpha \int \frac{d^{2\omega}q}{\alpha ((p - q)^2 + i\epsilon) + (1 - \alpha) (q_0^2 + i\epsilon)^2}. \]
\[
\int_0^1 d\alpha \int d^{2\omega-1}q \int_0^{+\infty} dq_0 \frac{(q_0 + \alpha p_0^0)}{[q_0^2 + \alpha (1 - \alpha) p_0^2 - \alpha (\vec{p} - \vec{q})^2 + i\epsilon]^2}. \tag{13}
\]

The \( q_0 \) integral is carried out first and leads to

\[
I = \frac{ip_0}{2} \int_0^1 \frac{d\alpha}{\sqrt{\alpha}} \int d^{2\omega-1}q \left[ p_0^2(\alpha - 1) + (\vec{p} - \vec{q})^2 \right]^{-3/2}. \tag{14}
\]

To perform the \( \vec{q} \) integral, we use the integral representation

\[
\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty dx x^{n-1} e^{-xA}, A > 0, \tag{15}
\]

and get, taking \( p_0^2 < 0, \)

\[
I = ip_0 \int_0^1 d\alpha \int_0^\infty dx \sqrt{\frac{\pi x}{\alpha}} \int d^{2\omega-1}q e^{-x[p_0^2(\alpha - 1) + (\vec{p} - \vec{q})^2]}. \tag{16}
\]

Finally, we obtain

\[
I = i\pi^\omega \int_0^1 \frac{d\alpha}{\sqrt{\alpha}} \int_0^\infty dx x^{1-\omega} e^{-xp_0^2(\alpha - 1)}
\]

\[
= i\pi^\omega \Gamma(2 - \omega)p_0^{2\omega-3} \int_0^1 d\alpha \alpha^{-1/2}(\alpha - 1)^{\omega-2}
\]

\[
= \frac{2p \cdot n}{n^2 \mathbf{T}}, \quad \mathbf{T} \equiv i\pi^2 \Gamma(2 - \omega), \quad \omega \to 2. \tag{17}
\]

This result has the same form as that for the corresponding spatial-axial gauge integral calculated with the principal-value prescription.

Next we turn to an integral with the double pole:

\[
J = \int \frac{d^{2\omega} q}{(p - q)^2(q \cdot n)^2}. \tag{18}
\]

Using definition (6) for the spurious double pole and Feynman’s parametrization formula, we get

\[
J = 2 \int_0^1 d\alpha (1 - \alpha) \int d^{2\omega-1}q \int_{-\infty}^{+\infty} dq_0 \frac{(q_0 + \alpha p_0^0)^2}{[q_0^2 + \alpha (1 - \alpha) p_0^2 - \alpha (\vec{p} - \vec{q})^2 + i\epsilon]^3}.
\]

6
\[ J = -i\pi \Gamma(2 - \omega) (p_0^2)^{\omega - 2} \frac{1}{2} \int_0^1 d\alpha \alpha^{3/2} (\alpha - 1)^{\omega - 1} \]

\[
\approx -\mathcal{T} \frac{1}{2} \int_0^1 d\alpha \alpha^{3/2}, \quad \omega \to 2. \tag{20}
\]

The integral in the Feynman parameter diverges, independent of \( \omega \). Therefore, we need a regularization, and the simplest regularization is to set

\[
\int_0^1 d\alpha \alpha^{-3/2} = -2. \tag{21}
\]

This result is obtained by considering the following complex-valued function:

\[
f(z) = \int_0^1 d\alpha \alpha^z, \tag{22}
\]

which diverges for real \( z \leq -1 \). We may analytically continue it to real \( z < -1 \):

\[
f(z) \longrightarrow \tilde{f}(z) = \frac{1}{z + 1}. \tag{23}
\]

It has a pole at \( z = -1 \) and \( \tilde{f}(-3/2) = -2 \). Hence we obtain

\[
J = \frac{-2}{n^2} \mathcal{T}, \tag{24}
\]

which has the same form as that for the corresponding spatial axial-type integral calculated with the principal-value prescription. Other temporal-gauge integrals required for the one-loop gluon self-energy are calculated and listed in the appendix. We see that the temporal-gauge integrals requiring a regularization like (21) have the spurious double pole. Integrals
having the spurious simple pole do not exhibit gauge divergences and need no regularization. We observe that the use of (7) leads to the same results as those from using (6), provided that we define the only divergent integral by (21). The identities (8), (9), (10) and the algebraic decomposition formula (11) are satisfied if the same regularization scheme defined by (21) is used.

4. Renormalization

In this section, we shall investigate the renormalization of the temporal gauge theory in the non-principal value formalism. We shall need to calculate the one-loop gluon self-energy. Since we are interested in simple Feynman rules with the decoupling of the Faddeev-Popov ghosts, we shall work with the equivalence theorem [5,9].

This theorem is based on the required property that the gauge propagator be time-translation invariant. Let the time-translation invariant gauge propagator be:

$$G_{\mu\nu}(q) = \frac{i\delta^{ab}}{q^2 + i\epsilon}(-\delta_{\mu\nu} + a_\mu(q)q_\nu - a_\nu(-q)q_\mu),$$

(25)

where $a_\mu(q)$ is an arbitrary function related to the gauge choice. Let the ghost-gluon-ghost vertex be:

$$\Gamma_{\mu}^{abc}(q) = -gf^{abc}((a\cdot q) - 1)q_\mu - q^2 a_\mu(q),$$

(26)

and let the ghost propagator be:

$$G(q) = \frac{-i}{q^2 + i\epsilon}, \quad \epsilon > 0,$$

(27)

with $g$ being the coupling constant, $q_\mu$ the outgoing ghost’s momentum. Then the S-matrix is independent of the gauge-dependent function $a_\mu(q)$. We should mention that definitions (26) and (27) are related, since they always appear in pairs in diagrams with ghost loops.

From this theorem, we get for the temporal gauge

$$a_\mu(q) = \frac{n_\mu}{q\cdot n} - \frac{q_\mu}{2(q\cdot n)^2},$$

(28)

$$\Gamma_{\mu}^{abc}(q) = gf^{abc}\frac{q^2 n_\mu}{q\cdot n}.$$  

(29)

We observe that the ghost-gluon-ghost vertex (29) has a $q^2$ factor, which cancels the $q^2$ dependence of the ghost propagator (27) in diagrams with ghost loops. Thus, the pair (27)
and (29) is seen to be identical to the pair of Feynman rules for the ghost propagator and the ghost-gluon-ghost vertex obtained by the Faddeev-Popov gauge-fixing procedure. Using the procedure, we have a ghost-gluon-ghost vertex that is proportional to $n_{\mu}$ and a ghost propagator that is inversely proportional to $q \cdot n$. It is easy to show that in dimensional regularization the one-loop ghost diagram vanishes in the non-principal value prescription. Therefore, the calculation of the one-loop gluon self-energy requires considering one diagram with an internal gluon loop.

The calculation of the divergent part of the one-loop gluon self-energy is straightforward but tedious and yields

$$i\Pi_{\mu\nu}^{ab}(p) = \frac{11g^2C_A}{3(2\pi)^2} \delta^{ab}(p^2 \delta_{\mu\nu} - p_\mu p_\nu) I,$$

(30)

where $C_A = N$ for $SU(N)$ gauge group. We observe that the self-energy is transverse and independent of the temporal vector $n_{\mu}$. Thus, the renormalization structure is very simple in this formalism and has the same form as that in the spatial axial gauge calculated with the principal-value prescription. The finite part of the self-energy in the temporal gauge in the non-principal value formalism could depend on the temporal vector $n_{\mu}$, since gauge self-energy is a gauge dependent quantity.

Finally, we note that many authors have considered time-translation noninvariant propagators [10]. The renormalizations of the gauge theories with these propagators have yet to be investigated.

5. Discussion

Among the noncovariant gauges, the temporal gauge is generally known to be the most complicated and cumbersome gauge choice. In this work, we have considered a non-principal value prescription as a choice for defining the spurious temporal-gauge poles. This prescription provides us with a calculational procedure for the dimensionally-regularized temporal-gauge integrals. In calculating the integrals, we first transform the divergences arising from the gauge singularities into divergences in the Feynman integration parameters. These divergences are seen not to be regularized by dimensional regularization and hence a regularization is needed. By using the regularization scheme (21), we have showed that the results for the integrals have the same forms as those for the corresponding spatial-axial gauge integrals.
with the principal-value prescription. Within the regularization scheme, the basic integrals are seen to obey the power-counting rule. We see that the identity (11) can be used to simplify integrals. The divergent part of one-loop gluon self-energy is local in the external momentum and has the same simple structure as that in the spatial-axial gauge in the principal-value formalism. The renormalization structure of the temporal-gauge theory in the non-principal value prescription is seen to be very simple.

Finally, we should mention that our definitions, (5) and (6), are seen to be related to those introduced by Leibbrandt [8]. In Leibbrandt’s definitions for the simple and double poles, the constant temporal vector and its dual temporal vector play the role of a regularization, and no regularization is needed for the gauge divergences. We observe that certain one-loop integrals derived with the definitions in [8] do not reduce to the corresponding integrals derived with (5) and (6) in the limit of vanishing spatial components of the constant temporal vectors. The one-loop gluon self-energy calculated in [8] is transverse but nonlocal in the external momentum.

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Appendix

Here we list the ultra-violet divergent parts of the dimensionally-regularized temporal-gauge integrals in the non-principal value prescription with divergences denoted by $\mathcal{T} \equiv i\pi^2 \Gamma(2 - \omega)$. These integrals, which are the same as the corresponding spatial-axial gauge integrals ($n^2 = -1$) with the principal-value prescription, are used for the evaluation of the one-loop gluon self-energy.

\[
\int \frac{d^2 \omega q}{(p - q)^2 q \cdot n} = \frac{2p \cdot n}{n^2} \mathcal{T} \\
(A.1)
\]

\[
\int \frac{d^2 \omega q q_\mu}{(p - q)^2 q \cdot n} = \frac{2}{n^2} \left[ p \cdot np_\mu - \frac{(p \cdot n)^2 n_\mu}{n^2} \right] \mathcal{T} \\
(A.2)
\]

\[
\int \frac{d^2 \omega q}{q^2 (p - q)^2 q \cdot n} = \text{finite} \\
(A.3)
\]

\[
\int \frac{d^2 \omega q q_\mu}{q^2 (p - q)^2 q \cdot n} = \frac{n_\mu}{n^2} \mathcal{T} \\
(A.4)
\]

\[
\int \frac{d^2 \omega q q_\mu q_\nu}{q^2 (p - q)^2 q \cdot n} = \frac{1}{2n^2} \left[ p \cdot n\delta_{\mu\nu} + p_\mu n_\nu + p_\nu n_\mu - \frac{2p \cdot nn_\mu n_\nu}{n^2} \right] \mathcal{T} \\
(A.5)
\]

\[
\int \frac{d^2 \omega q}{(p - q)^2 (q \cdot n)^2} = -\frac{2}{n^2} \mathcal{T} \\
(A.6)
\]

\[
\int \frac{d^2 \omega q q_\mu}{(p - q)^2 (q \cdot n)^2} = \frac{2}{n^2} \left[ \frac{2p \cdot nn_\mu}{n^2} - p_\mu \right] \mathcal{T} \\
(A.7)
\]

\[
\int \frac{d^2 \omega q q_\mu q_\nu}{(p - q)^2 (q \cdot n)^2} = \frac{2}{n^2} \left[ -p_\mu p_\nu + \frac{2p \cdot n}{n^2} (p_\mu n_\nu + p_\nu n_\mu) \right]
\]
\[
\int \frac{d^2\omega q}{q^2(p-q)^2(q\cdot n)^2} = \text{finite} \tag{A.9}
\]

\[
\int \frac{d^2\omega q q_{\mu}}{q^2(p-q)^2(q\cdot n)^2} = \text{finite} \tag{A.10}
\]

\[
\int \frac{d^2\omega q q_{\mu} q_{\nu}}{q^2(p-q)^2(q\cdot n)^2} = \frac{1}{n^2} \left[ \frac{2n_{\mu}n_{\nu}}{n^2} - \delta_{\mu\nu} \right] T \tag{A.11}
\]

\[
\int \frac{d^2\omega q q_{\mu} q_{\nu} q_{\sigma}}{q^2(p-q)^2(q\cdot n)^2} = -\frac{1}{2n^2} \left[ p_{\mu} \delta_{\nu\rho} + p_{\nu} \delta_{\mu\rho} + p_{\rho} \delta_{\mu\nu} \\
- 2p \cdot n (n_{\mu} \delta_{\nu\rho} + n_{\nu} \delta_{\mu\rho} + n_{\rho} \delta_{\mu\nu}) \\
- 2 (p_{\mu} n_{\nu} n_{\rho} + p_{\nu} n_{\mu} n_{\rho} + p_{\rho} n_{\mu} n_{\nu}) \\
+ \frac{8p \cdot n}{n^2} n_{\mu} n_{\nu} n_{\rho} \right] T \tag{A.12}
\]
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