Universal Scaling Relations in Strongly Anisotropic Materials

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We consider the critical temperature in strongly anisotropic antiferromagnetic materials, with weak coupling between stacked planes, in order to determine the interplane coupling constant from experimentally measured susceptibilities. We present theoretical arguments for a universal relation between interplane coupling and susceptibility shown numerically by Yasuda et. al., Phys. Rev. Lett. 94, 217201 (2005). We predict a more general scaling function if the system is close to a quantum critical point, a similar relation for other susceptibilities than considered in Yasuda et. al., and the validity of these relations for more general phase transitions.

Many materials display at low temperatures strongly spatially anisotropic responses to magnetic or electronic probes. This fact has motivated the theoretical study of low dimensional quantum systems on their own right. Solving one- or two-dimensional quantum systems can be useful to understand intermediary regimes of temperature in which fluctuations are dominated by the subsystem of lower dimensionality. Three-dimensionality is effectively restored once the temperature is lowered below the lowest energy scale characterizing the anisotropy.

For magnetic systems this scale can be the temperature $1/\beta_{AF} (k_B = \hbar = 1)$ below which antiferromagnetic (AF) long-range order manifests itself. In this context, one of the most studied model is perhaps a stacking net, one of the most studied model is perhaps a stacking net of which a nearest-neighbor quantum spin-S Heisenberg model $H_J$ with AF exchange coupling $J > 0$ is defined.

To model a strong spatial anisotropy, one assumes that there exists a nearest-neighbor AF exchange coupling $J'$ in the directions transverse to the chains or planes that is much weaker than $J$, $J \gg J' > 0$. The three-dimensional quantum Hamiltonian is $H_{3d}$. Many efforts have been invested for the last 30 years in calculating the $J'$-dependence of $1/\beta_{AF}$ in $H_{3d}$. In this letter we shall address the question: are there some universal relations that relate $J'$ and some observables of $H_J$ or $H_{3d}$?

The motivation for this question comes from the work by Yasuda et al. in Ref.\textsuperscript{5}, in which the Néel temperature $1/\beta_{AF}$ of $H_{3d}$, the n-dimensional static staggered susceptibility $\chi^{(n)} = \chi^{zz}(Q, \omega = 0; \beta_{AF})$ of $H_J$ where $Q = \pi n$ if $n = 1$ or $Q = (\pi, \pi, \pi)$ if $n = 2$, and $J' \chi^{(n)} = 1/\zeta_n(J')$ were computed numerically as a function of $0 < J'/J \leq 1$. In quasi-two-dimension, it was observed that

$$J' \chi^{(n=2)} = 1/\zeta_{n=2} \quad (1)$$

becomes independent of $J'/J$ when $J'/J < 0.1$ and that the constant value that it takes, although not $1/2$ as predicted by mean-field theory, is independent of the magnitude of the spin and even takes the same value for a classical ($S = \infty$) model. Although the evidence is less pronounced the same conclusion was reached in quasi-one-dimension.

We want to construct a tractable model that reproduces qualitatively these findings and we want to understand how these results can be useful to establish experimentally the implied universality. We present a theoretical argument that as $J'/J \downarrow 0$, the function $\zeta_{n=2}$ converges to a constant. We make the following additional predictions.

First, we consider more general AF models in the plane, and we consider the case in which, by tuning parameters, it is possible to tune the planar model close to a quantum phase transition, so that the zero-temperature AF order of the two-dimensional model (without interplane couplings) becomes small. Then, we predict the scaling function

$$J' \chi^{(2)}_s = F_1(c_{AF}/\xi^{(2)}), \quad (2)$$

for some scaling function $F_1$, in the limit $J'/J \downarrow 0$, where $\xi^{(2)}$ is the correlation length of the two-dimensional model at temperature $1/\beta_{AF}$ and $c$ is a spin-wave velocity defined below. Note that in the system considered by Yasuda et. al. the planar model is in the renormalized classical regime so that $c_{AF}/\xi^{(2)}$ is exponentially small in $c_{AF}$ and converges to zero as $J'/J \downarrow 0$. Therefore, $F_1(c_{AF}/\xi^{(2)} = 0) = 1/\zeta_2$.

Second, we predict a similar scaling relation that will be valid for quantities which are easier to access experimentally. The susceptibility $\chi^{(3)}_{s, \pi, \pi}$ defined above is that of the two-dimensional model without the interlayer couplings, and cannot be measured in most real materials. We define $\chi^{(3)}_{s, \pi, 0} = \chi^{zz}(\pi, \pi, 0, \omega = 0; \beta_{AF})$ to be the static susceptibility in the layered system at wave vector $(\pi, \pi)$ in the plane and wave vector 0 perpendicular to the plane at temperature $1/\beta_{AF}$. Then, we predict that

$$J' \chi^{(3)}_{s, \pi, 0} = F_2(c^{(3)}_{AF}/\xi^{(3)}_{s, \pi, 0}), \quad (3)$$

for some scaling function $F_2$, in the limit $J'/J \downarrow 0$, where $c^{(3)}$ and $\xi^{(3)}_{s, \pi, 0}$ are the in-plane spin-wave velocity and cor-
relation length of $H_{3d}$ at temperature $1/\beta_{AF}$ near wave vector $(\pi, \pi, 0)$, respectively. As it is the instantaneous structure factor $S_{\pi, \pi, 0}$ that is most readily measured \cite{3}, we note that the product $J'S_{\pi, \pi, 0}\beta_{AF}$ also should obey a scaling law of the form \cite{4} for some scaling function $F_3$. Third, we predict that similar scaling results hold for other layered models.

In quasi one-dimension, we expect that similar scaling results will also hold. This does not, however, help us understand the results of Yasuda et. al. in quasi one-dimension. The scaling functions $F_1, F_2$ imply that the classical and quantum models will show the same $\zeta^{(1)}$ only if $c\beta_{AF} \ll \xi^{(1)}$. However, as the one-dimensional Heisenberg model on a chain is not in the renormalized classical regime but rather quantum critical, it should have some non-zero $c\beta_{AF}/\xi^{(1)}$ and should show a different $\zeta^{(1)}$ than the classical model. Thus, the one-dimensional results remain a puzzle.

**Physical Motivation** — Here, we present a physical motivation for the results above and a brief microscopic derivation of the relevant non-linear sigma model. In the next section, we show these scaling results using a renormalization group (RG) for this non-linear sigma model. The reason for which we use this model is that we want to illustrate the effects of field renormalization and the non-linear sigma model RG already has a nontrivial field renormalization at leading order in the coupling constant, while such a renormalization is not seen until order $\epsilon^2(1/N)$ in a $4 - \epsilon$ (large $N$) expansion.

Since the interplane interaction is weak, we can treat it perturbatively at the microscopic level. Following standard steps, in the absence of the interplane interaction, we can first derive the partition function for the two-dimensional $O(N)$ quantum non-linear sigma model (2dQNSLM) with field $n_k(r, \tau)$, where $k$ is a discrete index labelling individual planes, $r$ is a two-dimensional vector describing coordinates in the plane, and $\tau$ is imaginary time. The relevant action for plane $k$ is $S_k = S_k^{(1)} + S_k^{(2)}$ where

$$S_k^{(1)} := \int \mathcal{L}_k^{(1)} \equiv \int_0^\beta d\tau \int_0^L d^2r \frac{c}{2ag} (\partial_\mu n_k)^2 \tag{4a}$$

and

$$S_k^{(2)} := \int \mathcal{L}_k^{(2)} \equiv -\int_0^\beta d\tau \int_0^L d^2r \frac{c}{a^3} Z_k h \cdot n_k. \tag{4b}$$

Here, the lattice spacing $a$ plays the role of the microscopic ultra-violet (UV) cutoff, i.e., $\Lambda \sim 1/a$ that of an upper cutoff on momenta. The linear size $L$ of the plane is the largest length scale of the problem. The derivative $\partial_\mu = (\partial_\tau, \nabla)$ depends on the spin-wave velocity $c$ in the plane and is of order $Ja$. The dimensionless coupling constant $g$ depends on the microscopic details of the intraplane interactions. The dimensionless background field $h$, where $h = |h|$, is the external source for a static staggered magnetic field conjugate to the planar AF order parameter of the underlying lattice model. It breaks the $O(N)$ symmetry of Lagrangian \cite{4} down to $O(N - 1)$ and as such acts as an infra-red (IR) regulator. The dimensionless coupling $Z_k$ is the field renormalization constant associated to $n_k$. The use of the continuum limit within each of the planes labelled by $k$ is justified if we are after the physics on length scales much longer than $a$.

The interplane nearest-neighbor AF coupling $J'$ gives the characteristic interplane spin-wave velocity $c' \sim J'a$ and length scale $a' = (J/J')^{1/2}a$. The couplings $J', g$ get renormalized as discussed below, so the velocity $c'$ changes at longer length scales. For a very weak nearest-neighbor interplan AF coupling, $J' \ll J$, the physics on length scales much larger than $a$ but not much larger than $a'$ is captured by the partition function

$$Z = \int_{\mathbb{R}^N} \left[ \prod_k \mathcal{D}[n_k] \delta(n_k^2 - 1) \right] \exp \left( -\sum_k \mathcal{L}_k \right). \tag{5a}$$

$$\mathcal{L}_k = \mathcal{L}_k^{(1)} + \mathcal{L}_k^{(2)} + \mathcal{L}_k^{(3)}.$$

The Lagrangian $\mathcal{L}_k^{(3)}$ encodes the effect of the microscopic nearest-neighbor interplane AF interaction $J'$. To compute this, we use a Hubbard-Stratonovich transformation to replace the microscopic interaction between any two spins in the cubic lattice with coordinates $(i, j, k)$ and $(i, j, k + 1)$ by an interaction of each spin with a fluctuating magnetic field $H_{i,j,k}^{l}$. When $J' \ll J$ and on intermediary length scales $a \ll \lambda \lesssim a'$, the dominant mode for the Hubbard-Stratonovich field is near momentum $(\pi, \pi)$ in the plane. The action for the spins in the presence of this field is the same as Eq. \cite{4} with field $h$ replaced by $H_{l,k}$. In this approximation, integrating the Hubbard-Stratonovich field gives the short-range interplane interaction term

$$\mathcal{L}_k^{(3)} = \frac{J'Z'}{2a^2} (n_k - n_{k+1})^2, \tag{5c}$$

where $Z'$ renormalizes as $Z_k^2$ to lowest order in $J'/J$,

$$Z' = Z_k^2 [1 + \mathcal{O}(J'/J)]. \tag{6}$$

This defines the so-called three-dimensional strongly anisotropic $O(N)$ QNSLM (d3SAQNSLM).

This derivation of the d3SAQNSLM considered interactions between spins positioned directly above and below each other in neighboring planes. It is possible to treat more complicated interplane interactions. For example, going back to the cubic lattice, let there be an AF interaction $J_1$ between the spin at site $(i, j, k)$ with that at $(i, j, k \pm 1)$ and another AF interactions $J_2$ to the spins at sites $(i \pm 1, j, k \pm 1)$ and $(i, j \pm 1, k \pm 1)$. Then,
if $J_1 \ll J$ and $J_2 \ll J$, we can derive the 3dSAQNLSM with the interplane coupling $J' = J_1 - 4J_2$.

Renormalization Group—Here, we present an RG analysis of the non-linear sigma model \[^4\]. The most important result in this section is that the identity \[^5\] is preserved under the RG flow up to the length scale at which the scale dependent effective anisotropy \[^1\] is of order 1.

We perform a RG analysis following Polyakov for convenience \[^2\]. In each plane labelled by $k$, we write

$$\mathbf{n}_k = \mathbf{m}_k \left(1 - \phi_k^{2}\right)^{1/2} + \sum_{a=1}^{N-1} e_k^a \phi_k^a. \tag{7}$$

The field of unit length $\mathbf{m}_k$ encodes the planar AF order expected in the limit $g/c \downarrow 0$, while the $N - 1$ fields $e_k^a$ capture the deviations away from the direction $\mathbf{m}_k$ of symmetry breaking, i.e., the $N - 1$ $e_k^a$ form an orthonormal basis of vectors orthogonal to $\mathbf{m}_k$. The $N - 1$ coefficients $\phi_k^a$ make up the vector $\phi_k$. To leading order in an expansion in powers of $g/c$ of the parametrization \[^4\], the field $\mathbf{m}_k$ is the slow mode while the $N - 1$ fields $\phi_k^a$ represent fast modes with characteristic 2-momenta $\Lambda < |\mathbf{p}| \leq \Lambda$. Substituting Eq. (7) into Eq. (5) gives the Lagrangian

$$\mathcal{L}_k^{(1)} + \mathcal{L}_k^{(2)} = \frac{\nu}{2 N g} \left(\partial_\mu \phi_k^a - A_k^{ab} \partial_\mu \phi_k^b\right)^2 + \left(\partial_\mu \phi_k^a - A_k^{ab} \partial_\mu \phi_k^b\right)^2 \tag{8}$$

$$+ B_k^{a b} \left(\partial_\mu \phi_k^a - A_k^{ab} \partial_\mu \phi_k^b\right) + \frac{c}{a^2} Z_h \cdot \mathbf{m}_k \left(1 - \phi_k^{2}\right)^{1/2}$$

to leading order in an expansion in powers of $g/c$. The $N - 1$ coefficients $B_k^{a b}$ are defined by \[\partial_\mu \phi_k^a = \sum_{a=1}^{N-1} B_k^{a b} \mathbf{m}_k \] \[^5\]. The $(N - 1)(N - 2)/2$ independent coefficients $A_k^{ab} = -\langle \partial_\mu \phi_k^a \cdot \mathbf{e}_b \rangle$. The RG flows of the dimensionless couplings $g$, $Z_h$, and $t = 1/(\beta \Lambda)$ that follow after integration over the fast modes $\phi_k$ in the limit of no interplane interactions were computed by Chakravarty, Halperin, and Nelson leading order in $g/c$ (see Fig. 1). To this order, $c$ is unchanged.

To quantify the very weak microscopic interplanar coupling, we define the anisotropy $\tilde{\alpha}$ as the ratio of the importance of $\mathcal{L}_k^{(1)}$ to the $\mathcal{L}_k^{(3)}$ when the upper cutoff on the momenta is $\Lambda$. By assumption, this anisotropy is strong at the microscopic level (upper cutoff $\Lambda$),

$$\alpha = g J' Z' / J \ll 1. \tag{9}$$

Next, we consider the renormalization of $\mathcal{L}_k^{(1)}$ in Eq. (11) when $h = 0$ and of the interplane interaction

$$\mathcal{L}_k^{(3)} = \frac{J' Z'}{2 a^2} \left[\left(1 - \phi_k^{2}\right)^{1/2} \mathbf{m}_k - \left(1 - \phi_{k+1}^{2}\right)^{1/2} \mathbf{m}_{k+1}ight] \tag{10}$$

$$+ \sum_{a=1}^{N-1} \left(\phi_k^a e_k^a - \phi_{k+1}^a e_{k+1}^a\right)^2$$

after averaging over the fast modes $\phi_k$. To this end, we shall introduce the renormalized values

$$\frac{1}{g} = \frac{1}{\tilde{g}} \left(1 + \langle \phi_k^a \phi_k^b - \phi_k^2 \phi_k^a \phi_k^b\rangle\right), \tag{11a}$$

$$\frac{1}{t} = \frac{1}{\tilde{t}} \left(1 + \langle \phi_k^a \phi_k^b - \phi_k^2 \phi_k^a \phi_k^b\rangle\right), \tag{11b}$$

$$\tilde{Z}_h = Z_h \left(1 - \frac{1}{2} \langle \phi_k^2 \rangle + \cdots\right), \tag{11c}$$

at the scale $\tilde{\Lambda}$ as a result of averaging over the fast modes $\phi_k$. For $\alpha \ll 1$, this average over fast modes is $\langle \phi_k^2 \rangle^2 = \delta_{k \mu} \delta_{k \nu} \ln(\Lambda/\Lambda) \frac{\tilde{Z}}{Z} \coth(g/2t)$. Furthermore,

$$\mathcal{L}_k^{(3)} \approx \frac{J' Z'}{2 a^2} \left(\mathbf{m}_k - \mathbf{m}_{k+1}\right)^2 \tag{12}$$

from which follows the renormalizations

$$\tilde{\alpha} = \left(\frac{\Lambda}{\tilde{\Lambda}}\right)^3 \frac{\tilde{g} Z'}{g Z} \alpha, \quad \frac{\tilde{Z}}{Z} = \frac{\tilde{Z}_h}{Z_h} \tag{13}$$

so long as $\tilde{\alpha} \ll 1$.

Let us follow the RG flows encoded by Eqs. (11) and Eqs. (13) starting from the initial values $g > 0$, $t = 1/(\beta \Lambda) < \infty$, and $1 > \alpha > 0$, see Eq. (9), corresponding to a point on the phase boundary between the Néel and paramagnetic phase (see Fig. 1). Aside from the thermal de Broglie wavelength of the spin waves $c_\beta \tilde{\Lambda}$, the initial values $g$ and $t$ define a second characteristic length scale, the correlation length $\xi (\beta \Lambda)$ in the 2dQNLSM, in view of $1 > \alpha > 0$. We shall distinguish two cases. In the renormalized classical regime $c_\beta \tilde{\Lambda} / \xi (\beta \Lambda) \ll 1$. In the quantum critical regime $c_\beta \tilde{\Lambda} / \xi (\beta \Lambda) \sim 1$. Finally, we denote by $\xi_{cross}$ the RG length scale at which $\tilde{\alpha} \sim 1$ and beyond which the RG flows Eqs. (11) and Eqs. (13) should be replaced by the flows of the isotropic 3dQNLSM; naive scaling gives $\xi_{cross} \sim a'$, but the RG flows above

FIG. 1: Phase diagram and RG flows for the 3dQNLSM ($\alpha = 1$) and 2dQNLSM ($\alpha = 0$) after Ref. \[^8\]. Conjectured phase diagram and RG flow for the anisotropic 3dQNLSM ($1 > \alpha > 0$). The shaded regions have long range Néel order. The figure for $1 > \alpha > 0$ shows a two-dimensional slice of the three-dimensional RG flow, as $\tilde{\alpha}$ is changing under this flow.
will change this scaling. Any two of these characteristic length scales, $c_\beta \Lambda$, $\xi(2)$, and $\xi_{cross}$, fix the third one since the RG flows are constrained to the boundary between the Néel and paramagnetic phases by assumption. Without loss of generality, we shall consider the case $\Lambda > 1/(c_\beta \Lambda)$. As we lower the upper momentum cut-off, the RG scale $\tilde{\Lambda}$ will eventually become larger than $c_\beta \Lambda$. We shall consider RG scales $\tilde{\Lambda}^{-1} \gg c_\beta \Lambda$, for which the quantum fluctuations are important.

We begin with the renormalized classical regime of the 3dSAQNLSM. As is illustrated in Fig. 1 the running coupling constants $\tilde{g}$ in Eq. (11a) and $\tilde{t}$ in Eq. (11b) flow towards zero and $\infty$, respectively, as $\tilde{\Lambda}$ decreases but so long $\tilde{\Lambda}^{-1} \lesssim c_\beta \Lambda$. By Eq. (13), the effective anisotropy decreases; using naive scaling which is valid for $\tilde{g} \ll 1$, we have $\alpha \sim (\Lambda/\tilde{\Lambda})^2 \alpha$. Beyond the RG length scale $\tilde{\Lambda}^{-1} \sim c_\beta \Lambda$ we can replace the 2dQNLSM in each plane by a classical 2dNLSM with the effective coupling $\tilde{g}_{d}$, where $\tilde{g}_{d} = \tilde{g}$ at the scale $\tilde{\Lambda}^{-1} \sim c_\beta \Lambda$. The effective anisotropy of the classical 3dSANLSM continues to decrease as $\tilde{\alpha} = (\Lambda/\tilde{\Lambda})^2 (\alpha c_\beta \Lambda)/\tilde{g}_{d}$, $\tilde{Z'} = (\alpha c_\beta \Lambda)\tilde{g}_{d}$, $\tilde{Z} = \tilde{Z'}$, $\tilde{\beta} = \tilde{g}_{d}$, and $\tilde{\alpha}$ continues to grow until it reaches the isotropic RG scale $\tilde{\alpha} \sim 1$. Equation (2) can then no longer hold. However, since there is only a finite range of scales over which $\tilde{\alpha}$ is non-negligible but still less than unity, we deduce that, at the scale $\tilde{\alpha} \sim 1$, $\tilde{Z'} \sim \tilde{Z}^2$, up to some constant of order unity. Furthermore, $\tilde{g}_{d} \sim 1$ at this scale also since it lies at some point on the phase boundary between the Néel and paramagnetic phases. But $\tilde{g}_{d} \sim 1$ tells us that the corresponding RG scale $\Lambda$ is of the order of the correlation length of the 2dQNLSM. In turn, this allows us to infer that the static staggered susceptibility of the 2dQNLSM is given by $\chi^{(2)}_s \sim (\Lambda/\tilde{\Lambda})^2 (\tilde{Z}^2_{\tilde{\beta} c_\beta \Lambda})^2$ up to universal corrections of order unity. We now multiply $\chi^{(2)}$ by $J'$ estimated from the anisotropy $\alpha$ of the classical 3dSANLSM, $J'\chi^{(2)}_s \sim \tilde{\alpha}(\tilde{Z}^{2}_{\tilde{\beta} c_\beta \Lambda})/\tilde{g}_{d}$. Using Eq. (2), and the fact that $\tilde{g}_{d} \sim 1$ and $\tilde{\alpha} \sim 1$, we arrive at Eq. (11) in the renormalized classical regime.

Note that each of these relations, such as $\tilde{\alpha} \sim 1$ and $\tilde{g}_{d} \sim 1$, is defined up to a multiplicative constant that depends on the details of how we define the RG. However, the dimensionless combination in Eq. (11) is universal. The reason for the universality is that all the microscopic details of the Heisenberg model are encoded into the three independent quantities $\tilde{g}$, $\tilde{Z}'$, and $\tilde{Z}$ on any length scale much larger than $a$. Let us perform the RG flow to some scale such that $\tilde{\alpha}$ is much less than unity. Then, the identity $\tilde{\alpha}$ relates $\tilde{Z}'$ to $\tilde{Z}$, leaving only two quantities independent in the classical regime, say $\tilde{g}_{d}$ and $\tilde{\alpha}$. The requirement of criticality relates $\tilde{g}_{d}$ to $\tilde{\alpha}$, leaving only one independent quantity, say $\tilde{\alpha}$. Choosing the renormalization scale to be some given fraction of the correlation length in the two-dimensional model fixes the last quantity, and thus there are no independent parameters left.

Near a quantum critical point and as is the case for the renormalized classical regime, the length scale at which $\tilde{\alpha} \sim 1$ is of the order $\xi^{(2)}$. Now, however, there is no significant separation of scales between $c_\beta \Lambda$ and $\xi^{(2)}$ anymore, i.e., $\tilde{\alpha} \sim 1$ already at $c_\beta \Lambda$. Correspondingly, there will be universal corrections to $\tilde{\alpha}$ in the form $\tilde{Z}'/\tilde{Z} = \kappa(1, g(1)) \tilde{Z}'_{\beta c_\beta \Lambda}$ where the function $\kappa$ of $\alpha$ and $g$ is universal with $\kappa(0, g) = 1$. The deviations in the quantum critical regime from the limiting value of $J'\chi^{(2)}$ in the classical renormalized regime define the universal scaling function $F_1$ of $c_\beta \Lambda/\xi^{(2)}$.

Similarly, the correlation at the $(\pi, \pi, 0)$ point is of order $\tilde{\alpha}^{-1}$ in the plane while it is of order a single in-plane spacing between the planes. Thus, $\chi^{(3)}_{\pi, \pi, 0} \sim \chi^{(2)}$ and $\xi^{(3)}_{\pi, \pi, 0} \sim \xi^{(2)}$, and so Eq. (3) follows.

We close by noting that all arguments presented here for a non-linear sigma model with $O(N)$ symmetry extend to non-linear sigma models defined on Riemannian manifolds with a positive curvature tensor. For example, we expect similar universal scaling relations for a stacking of AF Heisenberg models on a triangular lattice.

Discussion — We have provided a field-theoretic basis for understanding the result of Yasuda et. al in the quasi-two-dimensional case, generalized it to deal with situations near quantum critical points, and given a version thereof expressed in terms of experimentally accessible quantities. The calculation of the scaling functions $F_{1,2,3}$ within a field theoretic approach will only be approximate, and the best estimate for $F_1(0)$ is given by numerical calculations. For example, it can be shown that the mean-field result $F_2(0) = 1/2$ follows from the large $N$ limit of the $O(N)$ 3dSAQNLSM. It would be very valuable to perform a numerical study of $F_{2,3}$, in both the renormalized classical and quantum critical regimes.

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[1] D. J. Scalapino, Y. Imry, and P. Pincus, Phys. Rev. B 11, 2042 (1975).
[2] H. J. Schulz, Phys. Rev. Lett. 77, 2790 (1996).
[3] V. Yu. Irkin and A. A. Katanin, Phys. Rev. B 55, 12318 (1997); ibid, 57, 379 (1998); V. Yu. Irkin, A. A. Katanin, and M. I. Katsnelson, ibid, 60, 1082 (1999); ibid Phys. Rev. B 61, 6757 (2000).
[4] M. Bocquet, Phys. Rev. B 65, 184415 (2002).
[5] C. Yasuda, S. Todo, K. Hukushima, F. Alet, M. Keller, M. Troyer, and H. Takayama, Phys. Rev. Lett. 94, 217201 (2005).
[6] H. M. Ronnow, D. F. McMorrow, and A. Harrison, Phys. Rev. Lett. 82, 3152 (1999).
[7] A. M. Polyakov, Phys. Lett. 59B, 79 (1975).
[8] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B 39, 2344 (1989).