ON SOME MODULES OF COVARIANTS FOR A REFLECTION GROUP

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To Ernest Vinberg on the occasion of his 80th birthday

Abstract. Let \( g \) be a simple Lie algebra with Cartan subalgebra \( h \) and Weyl group \( W \). We build up a graded isomorphism \( (\wedge h \otimes H \otimes h)^W \rightarrow (\wedge g \otimes g)^W \) of \((\wedge g)^W\)-modules, where \( H \) is the space of \( W \)-harmonics. In this way we prove an enhanced form of a conjecture of Reeder for the adjoint representation.

1. Introduction

Let \( W \) be a finite irreducible real reflection group, and let \( S \) be a set of Coxeter generators. Let \( V \) be the euclidean space affording a reflection representation of \( W \). Consider the ring \( A \) of complex valued polynomial functions on \( V \). Let \( 2 \leq d_1 \leq d_2 \leq \cdots \leq d_r = \dim V \) be the degrees of any set of homogeneous generators \( \psi_1, \ldots, \psi_r \) of the polynomial ring \( A^W \). Now consider the ideal \( J \) of \( A \) generated by \( \psi_1, \ldots, \psi_r \) and set \( H = A/J \).

Let \( W = \wedge V \otimes A \) be the Weil algebra, which we regard as graded by \( \deg(q \otimes k) = \deg(q) + 2 \deg(k) \) for \( q \in \wedge V \) and \( k \in A \) homogeneous elements. Consider now the graded ring \( B = \wedge V \otimes H = \bigoplus q B_q \) and its special elements
\[
(1.1) \quad p_i = \pi (d(1 \otimes \psi_i)) \in B^W,
\]
d being the De Rham differential on \( W \) (cf. (3.1)) and \( \pi : W \rightarrow B \) the quotient map. A classical theorem of Solomon states that \( B^W = \wedge (p_1, \ldots, p_r) \) (cf. Proposition 4.2). Let \( D = \text{hom}_W(V, B) \). Fix a \( W \)-invariant non degenerate bilinear \((\cdot, \cdot)\) form on \( V \) (unique up to multiplication by a non zero constant). There is a natural \( W \)-valued bilinear form \( E \) on \( W \otimes V \) defined by
\[
(1.2) \quad E(v_1 \otimes v_1, w_2 \otimes v_2) = (v_1, v_2)w_1w_2,
\]
for \( v_1, v_2 \in V, w_1, w_2 \in W \). Since \( J \) is an ideal, the form pushes down to \( B \otimes V \cong \text{hom}(V, B) \), where we identify \( V \) with \( V^* \) using the bilinear form \((\cdot, \cdot)\). Passing to the invariants, we obtain a \( B^W \)-valued bilinear form, still denoted by \( E \), on the \( B^W \)-module \( D = \text{hom}_W(V, B) \).

Our main result is the following theorem, a more precise version of which is given in Theorem 5.1 (see also Proposition 4.5).

Theorem 1.1. (1) \( D \) is a free module, with explicit generators \( f_i, u_i, i = 1, \ldots, r \), over the exterior algebra \( \wedge (p_1, \ldots, p_{r-1}) \).
(2). There are non zero constants $k_i \in \mathbb{Q}$ such that
\[ E(f_i, u_{r-i+1}) = k_i p_r \]
for each $i = 1, \ldots, r$. The multiplication by $p_r$ is self adjoint for the form $E$. It is given by the formulas
\[
\begin{align*}
(p_r f_i) &= - \sum_{j=1, j \neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) f_j, & i = 1, \ldots, r, \\
(p_r u_i) &= - \sum_{j=1, j \neq i}^{r} k_j^{-1} E(f_i, u_{r-j+1}) u_j, & i = 1, \ldots, r,
\end{align*}
\]
Statement (1) has been proven, for well–generated complex reflection groups, in [14].

As a consequence of Theorem 1.1 we give a positive answer to a special case of a conjecture of Reeder, in an “enhanced” formulation due to Reiner and Shepler: see Section 2 for details.

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2. Preliminaries, Motivations and Outline of Proof of Theorem 1.1

The framework of Reeder’s conjecture is Lie-theoretic, so let us revert to this context and fix notation.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$) of rank $r$. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Let $\Delta$ be the corresponding root system, $W$ the Weyl group, $\Delta^+$ a positive system and $\rho$ the Weyl vector. Observe that as a $W$–module, $\mathfrak{h}$ is the reflection representation. We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form which restricts to a $W$–invariant bilinear form on $\mathfrak{h}$ which we choose as our form $(-,-)$. Let $Q, P$ denote the root and weight lattices, $P^+$ the cone of dominant integral weights.

The exterior algebra $\bigwedge \mathfrak{g}$ has been extensively studied as representation of $\mathfrak{g}$ (see e.g. [8], [9]). We are concerned with Reeder’s paper [13], where the author studies the isotopic components in $\bigwedge \mathfrak{g}$ of representations whose highest weight is “near” $2\rho$ or “near” $0$ w.r.t. the usual partial order on dominant weights. The nearness condition about 0 is made precise in the following

Definition 2.1. A irreducible finite dimensional highest weight module $V_\lambda$ with highest weight $\lambda \in Q \cap P^+$ is said to be small if twice a root of $\mathfrak{g}$ is not a weight of $V_\lambda$.

Given $\lambda \in Q \cap P^+$, the zero-weight space $0 \neq V_\lambda^0 \subset V_\lambda$ is a $W$-module. Introduce the following generating functions:
\[
\begin{align*}
P(V_\lambda, \bigwedge \mathfrak{g}, u) &= \sum_{n \geq 0} \dim \text{hom}_W(V_\lambda, \bigwedge \mathfrak{g}) u^n, & P_W(V_\lambda^0, \mathcal{B}, u) &= \sum_{q \geq 0} \dim \text{hom}_W(V_\lambda^0, \mathcal{B}) u^q.
\end{align*}
\]
In [13, Conjecture 7.1] Reeder proposed the following relation between these generating series when $V_\lambda$ is small:
\[
P(V_\lambda, \bigwedge \mathfrak{g}, u) = P_W(V_\lambda^0, \mathcal{B}, u),
\]
and verified it in rank less or equal than 3. The conjecture has two different motivations. Let $G$ be a compact Lie group with complexified Lie algebra $\mathfrak{g}$ and let $T \subset G$ be a maximal
torus. Consider the \( W \)-action on both factors of the manifold \( T \times G/T \). The Weyl map \( T \times W \rightarrow G/T \) induces an isomorphism in cohomology, which in terms of invariants reads as an isomorphism of graded vector spaces

\[
(\bigwedge \mathfrak{g})^W \cong H^*(G) \cong H^*(T \times G/T)^W \cong (\bigwedge \mathfrak{h} \otimes \mathcal{H})^W = \mathcal{B}^W.
\]

Conjecture (2.1) is the natural extension of this graded isomorphism to covariants of small representations. On the other hand, Broer [1] has shown that, exactly for small representations, Chevalley restriction can be generalized to covariants. Let \( S(\mathfrak{g}) \) (resp. \( S(\mathfrak{h}) \)) denote the symmetric algebra of \( \mathfrak{g} \) (resp. of \( \mathfrak{h} \)). Chevalley restriction theorem gives an isomorphism

\[
S(\mathfrak{g}) \cong S(\mathfrak{h})^W.
\]

Broer proves that restriction also induces an isomorphism of graded \( S(\mathfrak{g}) \) modules between \( \text{hom}_{\mathfrak{g}}(V_\lambda, S(\mathfrak{g})) \) and \( \text{hom}_W(V_\lambda, S(\mathfrak{h})) \).

Curiously enough, conjecture (2.1) in type \( A \) was implicitly proven in literature before [13] appeared: the left hand side has been computed by Stembridge [16], whereas the right hand side appears in [7], [11] (in a more general context). Further related work appears in [17], where Stembridge provides methods which can be reasonably applied for a case by case proof of the conjecture (see the discussion at the end of Section 3 in [14]).

Set \( \Gamma = (\bigwedge \mathfrak{g})^W \cong \mathcal{B}^W \). In Corollary 5.2 we prove that Theorem 1.1 implies the following

**Theorem 2.2.** There is a degree preserving isomorphism of \( \Gamma \)-modules

\[
(\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes \mathfrak{h})^W \rightarrow (\bigwedge \mathfrak{g} \otimes \mathfrak{g})^W.
\]

We are also able to build up a module isomorphism like (2.2) for the little adjoint representation, i.e., the highest weight module \( \mathfrak{g}_s \) with highest weight the highest short root of \( \Delta \) (provided two different root lengths exist): see Corollary 6.6. Indeed, in Section 6, we prove an analogue of Theorem 1.1 for the “Weyl group side” of the little adjoint representation: see Theorem 6.5.

The statement of Theorem 2.2 cannot be extended from the adjoint representation to a general small representation: the small module \( S^3(\mathbb{C}^3) \) for \( \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) \) admits as zero-weight space the sign representation \( \text{sign} \) of the symmetric group \( S_3 \), but an easy analysis shows that a graded isomorphism of \( \Gamma \)-modules

\[
\text{hom}_{S_3}(\text{sign}, \bigwedge \mathfrak{h} \otimes \mathcal{H}) \cong \text{hom}_{\mathfrak{sl}(3, \mathbb{C})}(S^3(\mathbb{C}^3), \bigwedge \mathfrak{sl}(3, \mathbb{C}))
\]

cannot exist. Nevertheless in Section 7 we provide a speculative approach to a possible extension of Reeder’s conjecture. We build up, for any \( \mathfrak{g} \)-module \( V \), a map \( \Phi^V \) from covariants of type \( V \) in \( \bigwedge \mathfrak{g} \) to covariants of type \( V^0 \) in \( \bigwedge \mathfrak{h} \otimes \mathcal{H} \) (see (7.2)). We conjecture that \( \Phi^V \) is injective for any \( V \). A result of Reeder would then imply that \( \Phi^V \) is an isomorphism of graded vector spaces when \( V \) is small, hence implying Reeder’s conjecture.

Our approach to Theorem 1.1 is motivated by our previous work with Procesi on covariants of the adjoint representation in \( \bigwedge \mathfrak{g} \) [3]. It is a classical fact that the invariant algebra \( \Gamma \) is an exterior algebra \( \bigwedge (P_1, \ldots, P_r) \) over primitive generators \( P_i \) of degree \( 2d_i - 1 \). The main subject of [3] is the study of the module of covariants \( \mathcal{A} = \text{hom}_{\mathfrak{g}}(\mathfrak{g}, \bigwedge \mathfrak{g}) \); we prove the following three facts (assume for simplicity of exposition that all exponents \( 1 = m_1 \leq \ldots \leq m_r \) of \( \mathfrak{g} \) are distinct).
(1). \( A \) is a free module over \( \bigwedge(P_1, \ldots, P_{r-1}) \) of rank \( 2r \). A set of free generators is given by the \( g \)-equivariant maps

\[
 f_i^\wedge(x) = \frac{1}{\text{deg}(P_i)} \iota(x) P_i, \quad u_i^\wedge(x) = \frac{2}{\text{deg}(P_i)} \iota(d(x)) P_i, \quad i = 1, \ldots, r
\]

where \( x \in g \), \( \iota \) denotes interior multiplication in the exterior algebra and \( d \) is the usual Chevalley-Eilenberg coboundary operator for Lie algebra cohomology.

(2). The Killing form on \( g \) induces an invariant graded symmetric bilinear \( \bigwedge g \)-valued form on \( \bigwedge g \otimes g \) given, for \( a, b \in \bigwedge g, x, y \in g \), by

\[
 e(a \otimes x, b \otimes y) = (x, y)a \wedge b,
\]

which restricts to a \( \Gamma \)-valued form on \( A \). Then, for each pair \( i, j \) there exists a non-zero rational constant \( c_{i,j} \) such that

\[
e(f_i^\wedge, f_j^\wedge) = e(u_i^\wedge, u_j^\wedge) = 0.
\]

(2.4)

\[
e(f_i^\wedge, u_j^\wedge) = e(f_j^\wedge, u_i^\wedge) = \begin{cases} 
 c_{i,j} P_k & \text{if } m_i + m_j - 1 = m_k \text{ is an exponent}, \\
 0 & \text{otherwise}.
\end{cases}
\]
Consider the ring $A$ of complex valued polynomial functions on $V$ (which is also $S(V)$, under the identification $V \cong V^*$). One knows that $A^W$ is a polynomial ring on $\dim V = r$ homogeneous generators $\psi_1, \ldots, \psi_r$ of degrees $2 \leq d_1 \leq d_2 \leq \cdots \leq d_r$. Now consider the ideal $J$ of $A$ generated by $\psi_1, \ldots, \psi_r$ and set $\mathcal{H} = A/J$. This is a graded representation of $W$ whose ungraded character is the regular character. It is a well known fact that $A \cong A^W \otimes \mathcal{H}$ as a $A^W$-module.

Let $\mathcal{W} = \bigwedge V \otimes A$ be the Weil algebra, which we regard as graded by $\deg(q \otimes k) = \deg(q) + 2 \deg(k)$ for $q \in \bigwedge V$ and $k \in A$ homogeneous elements. Using the duality between $V$ and $V^*$ we think of $\mathcal{W}$ as the algebra of differential forms on $V$ with polynomial coefficients.

So, $\mathcal{W}$ is equipped with the usual de Rham differential $d$ given by

\begin{equation}
(3.1) \quad d(q \otimes k) = \sum_{i=1}^r (x_i \wedge q) \otimes \frac{\partial k}{\partial x_i},
\end{equation}

where $\{x_1, \ldots, x_r\}$ is an orthonormal basis of $V$. Under our grading, $d$ has clearly degree $-1$.

Consider now the graded ring

$$
\mathcal{B} = \bigwedge V \otimes \mathcal{H} = \mathcal{W}/\mathcal{W} J.
$$

We denote by $\pi : \mathcal{W} \to \mathcal{B}$ the quotient homomorphism; sometimes, abusing notation, we also denote by $\pi$ the quotient map $A \to \mathcal{H}$. It is clear that $\mathcal{B}$ inherits a grading from $\mathcal{W}$.

Together with $d$ we also have the Koszul differential $\delta$ given by the derivation

\begin{equation}
(3.2) \quad \delta(x_i \otimes 1) = 1 \otimes x_i, \quad \delta(1 \otimes f) = 0, f \in A,
\end{equation}

which has degree $1$ under our grading.

Since $\delta$ is $W$-equivariant and the ideal $\mathcal{W} J$ is preserved by $\delta$, $\delta$ induces a differential on $\mathcal{B}$. On the other hand, $\mathcal{W} J$ is clearly not preserved by $d$ and we need to introduce a further differential on $\mathcal{W}$.

For $s \in T$ consider the operator

$$
\nabla_s = (d \log \alpha)(1 - s) = (\alpha \otimes 1) \frac{1 - s}{\alpha},
$$

where $\alpha = \alpha_s \in \Delta^+$. Remark that $\nabla_s$ does not depend on the choice of $\alpha$ and acts on the Weil algebra $\mathcal{W}$. The following properties of $\nabla_s$ are clear from its definition.

**Lemma 3.1.**

1. If $\omega \in \mathcal{W}^W$, $\nabla_s(\omega) = 0$.
2. $\nabla_s(\omega \nu) = \nabla_s(\omega) \nu + (s \omega) \nabla_s(\nu)$, $\omega, \nu \in \mathcal{W}$.

**Lemma 3.1** implies that the ideal $\mathcal{W} J$ is preserved by $\nabla_s$, so we get an operator on the algebra $\mathcal{B}$.

We now remark that if $\omega = a \otimes b$, $a \in \bigwedge V$, $b \in A$, $a \otimes b - s(a \otimes b) = (a - s(a)) \otimes b + (a) \otimes (b - s(b))$ and we have

**Lemma 3.2.** If $a \in \bigwedge V$, $\alpha_s \wedge (a - s(a)) = 0$.

**Proof.** If $x \in V$, $x - s(x)$ is a multiple of $\alpha_s$ and we are done. Let $a = a' \wedge x'$ with $a'$ of degree $t$ and $x \in V$. Then, by induction, $\alpha_s \wedge (a - s(a)) = \alpha_s \wedge ((a' - s(a')) \wedge x' + s(a') \wedge (x' - s(x'))) = \alpha_s \wedge (a' - s(a')) \wedge x' + (-1)^t s(a') \wedge \alpha_s \wedge (x' - s(x')) = 0$. \(\square\)
It follows that
\[(3.3)\quad \nabla_s(a \otimes b) = (\alpha_s \wedge s(a)) \otimes \frac{(1-s)(b)}{\alpha_s}.\]

We now choose a function \(c : T \to \mathbb{C}\) constant on conjugacy classes and set
\[(3.4)\quad D_c := \sum_{s \in T} c(s)\nabla_s,\]
and consider it as an operator both on \(\mathcal{W}\) and on \(B\). Notice that, since clearly \(D_c(\mathcal{W}^W) = 0\) and any element of \(B^W\) can be lifted to a \(W\)-invariant element of \(\mathcal{W}\), we get

**Proposition 3.3.** If \(u \in \mathcal{B}^W\), \(D_c(u) = 0\).

**Lemma 3.4.** If \(w \in \mathcal{W}\),
\[w^{-1}D_cw = D_c.\]

**Proof.** We have
\[w^{-1}D_cw(\omega) = w^{-1}\left(\sum_{s \in T} c(s)\nabla_s(w\omega)\right) = \sum_{s \in T} c(s)\nabla_{w^{-1}sw}(\omega) = D_c(\omega),\]
since the function \(c\) is constant on conjugacy classes. \(\square\)

**Proposition 3.5.** Let \(U\) be an irreducible \(W\)-module and \(x \in \mathcal{H}\) or \(x \in A\) be such that it generates a copy of \(U\). Fix \(s_\ell \in T_\ell, s_p \in T_p\). Then
\[(3.5)\quad \delta D_c(x) = (c(s_\ell)|T_\ell|1 - \chi_U(s_\ell)\chi_U(1)) + c(s_p)|T_p|(1 - \chi_U(s_p))x.\]

**Proof.** By the definitions
\[\delta D_c(x) = \sum_{s \in T} c(s)(x - s(x)),\]
so that \(\delta D_c(x) \in U\). Since \(U\) is irreducible and \(\delta D_c\) commutes with the \(W\)-action, we get that \(\delta D_c(x) = \gamma x, \gamma\) a constant. Computing traces we get
\[\gamma \chi_U(1) = (c(s_\ell)|T_\ell| + c(s_p)|T_p|)|\chi_U(1) - c(s_\ell)|T_\ell|\chi_U(s_\ell) - c(s_p)|T_p|\chi_U(s_p),\]
from which (3.5) is clear. \(\square\)

Finally we see that \(D_c\) gives a differential both on \(\mathcal{W}\) and on \(B\). Indeed we have

**Proposition 3.6.** \(D_c^2 = 0\).

**Proof.** We have
\[D_c^2 = \sum_{(s,t) \in T \times T} c(s)c(t)\nabla_s \nabla_t.\]

Now
\[\nabla_s \nabla_t(a \otimes b) = (\alpha_s \wedge s(\alpha_t) \wedge st(a)) \otimes \frac{b - t(b)}{\alpha_s \alpha_t} - \frac{s(b) - st(b)}{\alpha_s \alpha_t}\]
If \(s = t\), clearly \(\alpha_s \wedge s(\alpha_s) = -\alpha_s \wedge \alpha_s = 0\), so we can assume \(s \neq t\).

We now consider the space \(V_{s,t}\) spanned by \(\alpha_s\) and \(\alpha_t\) and the dihedral subgroup \(W_{s,t}\) generated by \(s, t\). If we set \(U = \alpha_s^\perp \cap \alpha_t^\perp\), we clearly get that \(V = V_{s,t} \otimes U\) and we can write \(\alpha_s\) as a linear combination of elements of the form \(a = a' \otimes u\) with \(a' \in \bigwedge V_{s,t}\) and \(u \in \bigwedge U\) each...
homogeneous. Then if $a'$ is of positive degree we get $\alpha_s \wedge s(\alpha_s) a' = 0$ so that we get possibly non zero contributions to $\nabla_s \nabla_t (a \otimes b)$ only when $a' = 1$. By linearity we can assume that $a \in \wedge U$ so that $st(a) = a$ and we get

$$\nabla_s \nabla_t (a \otimes b) = (a \otimes 1)(\alpha_s \wedge s(\alpha_t) \otimes \frac{b - t(b)}{\alpha_s \alpha_t} - \frac{s(b) - st(b)}{\alpha_s s(\alpha_t)}).$$

We can even assume that $a = 1$ and look at

$$\nabla_s \nabla_t (1 \otimes b) = (\alpha_s \wedge s(\alpha_t)) \otimes \frac{b - t(b)}{\alpha_s \alpha_t} - \frac{s(b) - st(b)}{\alpha_s s(\alpha_t)}.$$ 

Furthermore notice that all the contributions to the right hand side come from either multiplying or dividing by vectors in $V_{s,t}$ or applying elements in $W_{s,t}$. From these considerations we deduce that we can really assume that $W = W_{s,t}$ and the claim follows from Lemma 3.7 below.

Consider a dihedral group $D$ generated by the reflections $s, t$ subject to the relation $(st)^n = 1$, so that its set of reflections is formed by the $n$ elements $s_1 = s, s_2 = st, s_3 = stst, \ldots, s_n = (st)^{n-1}s = t$.

Let $\mathfrak{h}$ be a reflection representation of $D$ (meaning that is the direct sum $\mathfrak{h}_{s,t} \oplus U$, with $\mathfrak{h}_{s,t}$ the irreducible 2 dimensional reflection representation of $D$ as above). We choose as usual $\alpha_i = \alpha_{s_i}, i = 1, \ldots, n$ and consider the ring $R = \wedge \mathfrak{h}_{s,t} \otimes S(\mathfrak{h})[\prod \alpha_i^{-1}]$ and the twisted group algebra $R[D]$.

**Lemma 3.7.** The element

$$\sum_{r=1}^{n} c(s_r) d \log \alpha_r (1 - s_r)$$

has zero square.

**Proof.** As we have seen, each summand of

$$\left( \sum_{i=1}^{n} c(s_r) \alpha_r \otimes \frac{1}{\alpha_r} (1 - s_r) \right)^2$$

comes from a pair of reflections $(s_i, s_j)$ and is of the form

$$q_i q_j (\alpha_i \wedge s_i(\alpha_j)) \otimes \left( \frac{1 - s_j}{\alpha_i \alpha_j} - \frac{s_i(1 - s_j)}{\alpha_i \alpha_s(\alpha_j)} \right),$$

where we set $c(s_h) = q_h$ for each $h$. So the pairs $(s_i, s_j)$ give to $s_j$ the contribution

$$q_j \sum_{i=1}^{n} q_i (\alpha_i \wedge s_i(\alpha_j)) \otimes \frac{1}{\alpha_i \alpha_j}.$$ (3.6)

On the other hand the pairs $(s_j, s_i)$ give to $s_j$ the contribution

$$q_j \sum_{i=1}^{n} q_i (\alpha_j \wedge s_j(\alpha_i)) \otimes \frac{1}{\alpha_j \alpha_s(\alpha_i)}.$$ (3.7)
We have already seen that we can assume that $j \neq i$. Then setting $\alpha_h = s_j(\alpha_i)$, and observing that $q_h = q_i$, we get that (3.7) becomes

$$q_j \sum_{h=1}^{n} q_h(\alpha_j \wedge \alpha_h) \boxtimes \frac{1}{\alpha_j \alpha_h}.$$ 

On the other hand, $\alpha_i \wedge s_i(\alpha_j) = \alpha_i \wedge \alpha_j$, so that (3.6) equals

$$q_j \sum_{i=1}^{n} q_i(\alpha_i \wedge \alpha_j) \boxtimes \frac{1}{\alpha_i \alpha_j}.$$ 

Thus the coefficient of $s_j$ is clearly 0.

We now pass to the coefficient of $s_i s_j$. This is equal to

$$q_i q_j(\alpha_i \wedge s_i(\alpha_j)) \boxtimes \frac{1}{\alpha_i s_i(\alpha_j)} = q_i q_j(\alpha_i \wedge \alpha_j) \boxtimes \frac{1}{\alpha_i \alpha_j}.$$ 

For each $h = 1, \ldots, n$, $s_i s_j = s_h s_{h+j-i}$ and $q_h q_j = q_h q_{h+j-i}$. So one needs to verify

$$\sum_{h=1}^{n} d \log \alpha_h \wedge d \log s_h(\alpha_{h+j-i}) = \sum_{h=1}^{n} d \log \alpha_h \wedge d \log \alpha_{h+i-j} = 0.$$ 

If we take a cycle $c = (u_1, u_2, \ldots, u_d)$, we claim that, setting $u_{d+1} = u_1$,

$$\sum_{r=1}^{d} d \log \alpha_{u_r} \wedge d \log (\alpha_{u_{r+1}}) = 0.$$ 

If the cycle has length 2 this is obvious. If it has length 3, a simple computation shows that

$$d \log \alpha_{u_1} \wedge d \log \alpha_{u_2} + d \log \alpha_{u_2} \wedge d \log \alpha_{u_3} = d \log \alpha_{u_1} \wedge d \log \alpha_{u_3}$$

which is our relation.

Proceed now by induction and, using above relation, substitute and get the relation using the cycle $(u_1, u_3, \ldots, u_d)$. Let us fix $m = j - i$ and consider the permutation $\sigma(h) = m + h, \text{mod}(n)$ (choosing as remainders $1, \ldots, n$). Decompose it into cycles and apply the previous claim to get the result.

**Remark 3.8.** We are going to call $D_c$ a Dunkl differential. Operators of this kind on differential forms already appear in the paper [5].

### 4. The Bilinear Form

If $W$ is crystallographic, hence it is the Weyl group associated to a simple Lie algebra $\mathfrak{g}$, we recall that, by Chevalley theorem, restriction gives an isomorphism between $S(\mathfrak{g})^W$ and $A^W$, the polynomial ring of $W$ invariant functions on the Cartan subalgebra. We then fix homogenous generators $\psi_1, \ldots, \psi_r$ of the polynomial ring $\mathbb{C}[\mathfrak{g}]^W \simeq A^W$ in such a way that they induce by transgression the generators $P_1, \ldots, P_r$ of $\left(\bigwedge \mathfrak{g}\right)^W$ considered in the Introduction. On the other hand considering $\psi_1, \ldots, \psi_r$ in $A^W$, we can introduce the elements $p_i$ (cf. (1.1)).

In the case $W$ is not crystallographic, we choose the homogenous generators $\psi_1, \ldots, \psi_r$ of the polynomial ring $A^W$ arbitrarily and proceed to define the elements $p_i$, $i = 1, \ldots, r$ as above.
Remark 4.1. A priori the definition of the elements $p_i$ depends on the choice of the generators $\psi_1, \ldots, \psi_r$ of the polynomial ring $A^W$. However, if $J \subset A^W$ denotes as above the ideal of elements of positive degree it is immediate to see that the $p_i$ depend only on the induced basis $\psi_1, \ldots, \psi_r$ of $J/J^2$.

Indeed if $z = \psi_i - \psi'_i \in J^2$ and $z = \sum_j x_j y_j, x_i, y_j \in J$, then
\[
\pi(d(1 \otimes z)) = \pi(\sum_j d(x_j) y_j + \sum_j x_j d(y_j)) = 0,
\]
proving the claim.

We have the following theorem of Solomon, which is reproved here for the reader’s convenience.

**Proposition 4.2.** $B^W$ is the graded exterior algebra generated by the elements $p_i$ defined in (1.1).

**Proof.** First notice that $\dim B^W = \dim \bigwedge V = 2^r$. Secondly, notice that for each $i = 1, \ldots, r$ the element $p_i$ is of degree $(1, 2d_i - 2)$, that is of total degree $2d_i - 1$. It is clear that $p_ip_j = -p_jp_i$, so it suffices to show that $\prod_{j=1}^r p_j \neq 0$. Now let us remark that the element
\[
\Delta = \det(\frac{\partial \psi_j}{\partial x_i})
\]
spans the copy of the sign representation of $W$ of lowest possible degree. Thus $\Delta \notin J$.

Furthermore
\[
\prod_{j=1}^r p_j = x_1 \wedge x_2 \cdots \wedge x_r \otimes \pi(\Delta) \neq 0
\]
and the claim follows. \(\square\)

**Remark 4.3.** In the crystallographic case, we get a natural isomorphism between $(\bigwedge g)^g$ and $B^W$.

We now consider $D = \mathrm{hom}_W(V, B)$, and the following special element of $D$
\[
f_i(v) = \pi(1 \otimes \partial_v \psi_i),
\]
where $\partial_v$ denotes the directional derivative in the direction $v \in V$ and $i = 1, \ldots, r$.

Notice that with respect to a orthonormal basis $\{x_i\}$ of $V$, we have
\[
f_i = \sum_{j=1}^r \pi(1 \otimes \frac{\partial \psi_i}{\partial x_j}) \otimes x_j.
\]
Moreover, by (3.2), for every $v \in V$,
\[
(4.2) \quad \delta(f_i(v)) = 0.
\]

Fix a function $c : T \to \mathbb{C}$ constant on conjugacy classes as in the previous section. Set $|T|_c = c(s_l)|T|_l + c(s_p)|T|_p$ and define
\[
u_i(v) = \frac{r}{2|T|_c} D_c f_i(v).
\]

**Proposition 4.4.** For every $v \in V$, $\delta(u_i(v)) = f_i(v)$. 
Proposition 4.7.

Proposition 1.3]

Consider by Proposition 4.2. 

Hence dihedral groups; for $k$ also the equality $k$ is readily verified.

Remark 4.6



Proof. Let us choose a orthonormal basis \{ \pi \} for $V$. Then, since $E(f_i, f_j) \in B^W$ and

$$E(f_i, f_j) = \pi \sum_{s=1}^{r} \partial \psi_i \partial \psi_j$$

we have that $\sum_{s=1}^{r} (\partial \psi_i / \partial x_s)(\partial \psi_j / \partial x_s) \in J$, hence (1) follows.

To see part (2), notice that $E(u_i, f_j) \in (\wedge^1 V \otimes \mathcal{H})^W$ so that if there is no $s$ for which $d_i + d_j - 2 = d_s$, then by Proposition 4.2 we have $E(u_i, f_j) = 0$.

Assume $d_i + d_j - 2 = d_s$. Then we have that necessarily $E(u_i, f_j) = k_{i,j}p_k$, $k_{i,j} \in C$ again by Proposition 4.2.

We have to prove that $k_{i,j} \neq 0$. Lifting to $W$ and applying $\delta$ we obtain $E(d\psi_i, d\psi_j) = k_{i,j}\psi_k + b$, $b \in J^2$. If $k = r = 2$ this statement is obvious. If $k = r$, so that the indices $i,j$ are complementary, we can then apply the argument of Proposition 2.9 from [3] and deduce $k_{i,j} \neq 0$.

This completes the proof in the non crystallographic case, since the only pairs $d_i, d_j$ with $d_i + d_j - 2 = d_s$ either have $d_i = 2$ or $d_j = 2$ or $i,j$ are complementary: this is clear for dihedral groups; for $H_3$ the degrees are 2, 6, 10 and for $H_4$ they are 2, 12, 20, 30 so everything is readily verified.

It remains to treat the crystallographic case. But this follows from [3, 2.7.2], from which also the equality $k_{i,j} = c_{i,j}$ is easily deduced.

Remark 4.6. Type $D_{2n}$, where a basic degree of multiplicity 2 appears, is handled as in [3, Proposition 1.3].

Proposition 4.7.

(4.6) $E(u_j, u_i) = 0$.

Proof. Consider $u, v \in B$. We have

$$D(uv) = (Du)v + \sum_{s \in S} s(u)\nabla_s(v) = (Du)v - uDv + \sum_{s \in S} (u + s(u))\nabla_s(v).$$
But \((u + s(u)) \nabla_s(v) = (-1)^{\deg(u)}(1 - s)(d \log \alpha_s(u + s(u))v)\) Since \(d \log \alpha_s(u + s(u))\) is fixed by \(s\), we have
\[
s((u + s(u)) \nabla_s(v)) = -(u + s(u)) \nabla_s(v).
\]
Thus, since the usual scalar product is \(W\)-invariant, we deduce that \(s((u + s(u)) \nabla_s(v))\) is orthogonal to the \(W\)-invariants. So, also
\[
(uv) - (Du)v + uDv
\]
is orthogonal to the \(W\)-invariants. From this, reasoning as in [3, Lemma 2.15], we get that
\[
DE(f_j, u_i) - E((1 \otimes D)f_j, u_i) + E(f_j, (1 \otimes D)u_i) = 0
\]
However by Lemma 3.3, \(DE(f_j, u_i) = 0\), by Proposition 3.6 \(E(f_j, (1 \otimes D)u_i) = 0\), so that (4.6) follows.

5. Main Theorem

**Theorem 5.1.** (1) \(D\) is a free module, with basis the elements \(f_i, u_i, i = 1, \ldots, r\), over the exterior algebra \(\bigwedge(p_1, \ldots, p_{r-1})\).

(2) Let \(k_i = k_{i, r-i+1}\) with \(k_{i, j}\) defined as in (4.4). Then for each \(i = 1, \ldots, r\),
\[
E(f_i, u_{r-i+1}) = k_ip_i.
\]
The multiplication by \(p_i\) is self-adjoint for the form \(E\) and it is given by the formulas
\[
p_if_i = - \sum_{j=1, j \neq i}^r k_j^{-1}E(f_i, u_{r-j+1})f_j, \quad i = 1, \ldots, r,
\]
\[
p_iu_i = - \sum_{j=1, j \neq i}^r k_j^{-1}E(f_i, u_{r-j+1})u_j, \quad i = 1, \ldots, r,
\]

**Proof.** (1) Suppose that we have a relation \(\sum_{i=1}^r \lambda_i u_i + \sum_{j=1}^r \mu_j f_j = 0\). Then apply \(1 \otimes \delta\) and by (4.2) and Proposition 4.4 get \(\sum_{i=1}^r \lambda_i f_i = 0\). So if we prove that the \(f_i\) are linearly independent, we get \(\lambda_i = 0\) for all \(i\) and in turn that also all the \(\mu_j\) are 0.

Remark that, if there is a non trivial relation \(\sum_{j=1}^r \mu_j f_j = 0\), we may assume that it is homogeneous. Moreover, given an index \(j\), multiplying by a suitable element of \(\bigwedge(p_1, \ldots, p_{r-1})\) we can reduce ourselves to the case in which \(\mu_j = p_1 \wedge p_2 \wedge \ldots \wedge p_{r-1}\).

Notice now that the coefficient \(\mu_h\) of the terms \(\mu_h f_h\) for which \(d_h < d_j\) has degree higher than the maximum allowed degree, hence it is zero. Thus, if we choose for \(j\) the maximum for which \(\mu_j \neq 0\), we are reduced to prove that
\[
p_1 \wedge p_2 \wedge \ldots \wedge p_{r-1} f_j \neq 0.
\]
By part (2) of Proposition 4.5 we have \(E(f_j, u_{r-j+1}) = k_ip_i\), hence
\[
E(p_1 \wedge p_2 \wedge \ldots \wedge p_{r-1} f_j, u_{r-j+1}) = k_ip_i p_1 \wedge p_2 \wedge \ldots \wedge p_{r-1} \wedge p_r \neq 0.
\]

(2) Using Propositions 4.5, 4.7, one can mimic the proof of [3, Theorem 1.4]. We briefly explain how to proceed, omitting for simplicity the case \(D_{2n}\).
Consider the relation for $u_i$. We have

\begin{equation}
(p_r u_i) = \sum_{j=1}^{r} H_j u_j + \sum_{j=1}^{r} K_j f_j
\end{equation}

where the $H_j, K_j \in \bigwedge (p_1, \ldots, p_{r-1})$. Applying the differential $1 \otimes \delta$ we get

\begin{equation}
p_r f_i = \sum_{j=1}^{r} H_j f_j.
\end{equation}

Thus the relation for $f_i$ involves only the $f_j$'s. Also we have that the relation is homogeneous.

For each $j$, taking the scalar product with $u_{r-j+1}$, we have

\begin{equation}
p_r E(f_i, u_{r-j+1}) = H_j E(f_j, u_{r-j+1}) + \sum_{h \neq j} H_h E(f_h, u_{r-j+1})
= H_j k_j p_r + \sum_{h \neq j} H_h E(f_h, u_{r-j+1}).
\end{equation}

Since the terms $\sum_{h \neq j} H_h E(f_h, u_{r-j+1})$ do not involve $p_r$, we must have

\begin{equation}
\sum_{h \neq j} H_h E(f_h, u_{r-j+1}) = 0,
\end{equation}

\begin{equation}
-E(f_i, u_{r-j+1}) p_r = H_j k_j p_r.
\end{equation}

If $i \neq j$ we have that $E(f_i, u_{r-j+1})$ is not a multiple of $p_r$ and we deduce that

\begin{equation}
E(f_i, u_{r-j+1}) = -k_j H_j.
\end{equation}

If $i = j$ we deduce $H_j = 0$, so finally (5.5) becomes

\begin{equation}
p_r f_i + \sum_{i \neq j} k_j^{-1} E(f_i, u_{r-j+1}) f_j = 0.
\end{equation}

Since $E(f_i, u_{r-i+1}) = k_i p_i$, formula (5.7) is indeed formula (5.1), as required. We go back to formula (5.4), which we now write:

\begin{equation}
p_r u_i = - \sum_{j=1}^{r} k_j^{-1} E(f_i, u_{r-j+1}) u_j + \sum_{j=1}^{r} K_j f_j.
\end{equation}

Take the scalar product of both sides of (5.8) with $u_{r-j+1}$. We get

\begin{equation}
p_r E(u_i, u_{r-j+1}) = - \sum_{j=1}^{r} k_j^{-1} E(f_i, u_{r-j+1}) E(u_j, u_{r-j+1}) + \sum_{j=1}^{r} K_j E(f_j, u_{r-j+1}).
\end{equation}

Since $E(u_h, u_k) = 0$, we deduce that

\begin{equation}
k_j K_j p_r + \sum_{i, i \neq j} K_j E(f_i, u_{r-j+1}) = 0.
\end{equation}

We claim that all $K_j$ are zero. Indeed the only product containing $p_r$ is $k_j K_j p_r$. Since each element of $\Gamma$ can be written in a unique way in the form $a + b p_r$ with $a, b \in \bigwedge (p_1, \ldots, p_{r-1})$, we deduce $K_j = 0$ as desired.

\hfill \Box
Using (4.5) one gets the following Corollary, which obviously implies Reeder’s conjecture (2.1) for \( \mathfrak{g} \).

**Corollary 5.2.** The map
\[
p_i \mapsto P_i, \quad u_i \mapsto u_i^\wedge, \quad f_i \mapsto f_i^\wedge, \quad 1 \leq i \leq r,
\]
extends to an isomorphism of graded \( B^W \)-modules \((\bigwedge \mathfrak{h} \otimes \mathcal{H} \otimes \mathfrak{h})^W \rightarrow (\bigwedge \mathfrak{g} \otimes \mathfrak{g})^\theta\).

6. The Weyl group side of the little adjoint representation

Suppose that \( W \) contains two distinct conjugacy classes of reflections \( T_\ell, T_p \). Set \( r_\ell = \vert T_\ell \cap S \vert \), \( r_p = \vert T_p \cap S \vert \). Denote by \( H_{T_\ell} \) the subgroup of \( W \) generated by the reflections \( s \in T_\ell \), and by \( W_{T_p} \) the reflection subgroup of \( W \) generated by the reflections \( s \in T_p \cap S \). The following fact is proven in [12, Proposition 2.1].

**Lemma 6.1.** \( W = W_{T_p} \ltimes H_{T_\ell} \) so \( W/H_{T_\ell} \) is canonically isomorphic to \( W_{T_p} \). Symmetrically \( W = W_{T_\ell} \ltimes H_{T_p} \), so \( W/H_{T_p} \) is canonically isomorphic to \( W_{T_\ell} \).

Let us now consider the reflection representation \( U \) of \( W_{T_p} \). Since \( W_{T_p} \) is a quotient of \( W \), we may consider \( U \) as a \( W \)-module.

Consider now \( V \) as a \( H_{T_\ell} \)-module. Since \( H_{T_\ell} \) is generated by reflections, the ring \( A^{H_{T_\ell}} \) is a polynomial ring generated by homogeneous generators \( \psi_1, \ldots, \psi_n \). Let \( J_{H_{T_\ell}} \) be the ideal in \( A^{H_{T_\ell}} \) generated by \( \overline{\psi}_1, \ldots, \overline{\psi}_n \). Clearly \( W \) acts on \( \overline{V} = V_{H_{T_\ell}}/J_{H_{T_\ell}} \), and we have

**Proposition 6.2.** For the \( W \)-module \( \overline{V} \) one has
\[
(1) \quad \overline{V} \simeq U \oplus \overline{V}^W.
\]
\[
(2) \quad \dim \overline{V}^W = \vert T_\ell \cap S \vert.
\]
\[
(3) \quad \text{The submodule } U \subset \overline{V} \text{ is homogeneous of degree } d_n/2 - (r_p - 1)r_\ell.
\]

**Proof.** The proof is a case by case check. Let us start recalling that we have two distinct conjugacy classes of reflections precisely in the following cases: \( B_n = C_n, I_2(2m), F_4 \).

Type \( B_n \). Let us choose an orthonormal basis \( e_1, \ldots, e_n \) of \( V \) in such a way that the conjugacy class \( T_\ell \) is given by the \( n \) reflections with respect to the coordinate hyperplanes, the other class \( T_p \) by the reflections with respect to the hyperplanes of equation \( x_i \pm x_j, i < j \).

The group \( H_{T_\ell} \) is clearly isomorphic to \((\mathbb{Z}/2\mathbb{Z})^n\), and identifying \( A \) with \( K[x_1, \ldots, x_n] \) using the coordinates associated to our basis, it turns out that \( A^{H_{T_\ell}} = K[x_1^2, \ldots, x_n^2] \). Moreover, \( W_{T_p} \) is the symmetric group \( S_n \), acting on \( \overline{V} = \langle x_1^2, \ldots, x_n^2 \rangle \) by the permutation representation. It clearly follows that \( \overline{V} = U \oplus \overline{V}^W \). So \( \overline{V} \) and hence \( U \) is contained in the homogeneous component of degree 2 and the rest is clear since \( d_n = 2n \), \( r_\ell = 1 \), \( r_p = n - 1 \), so that \( q = d_n/2 - (r_p - 1)r_\ell = 2 \).

Let us now exchange the roles of \( T_\ell \) and \( T_p \). In this case \( H_{T_p} \) is a Weyl group of type \( D_n \), so \( H_{T_p} \) has index 2 in \( W \) and \( W_{T_p} \simeq \mathbb{Z}/2\mathbb{Z} \). We have that
\[
A^{H_{T_p}} = K[\psi_0, \psi_1, \ldots, \psi_{n-1}],
\]
where \( \psi_i = \sum_{n=1}^{n} x^{2i} \) is a basic invariant for \( B_n \) of degree 2i for \( i = 1, 2, \ldots, n-1 \), while \( \psi_0 = x_1 \cdots x_n \). It is now clear that \( \nabla = \langle \psi_0, \ldots, \psi_1, \ldots \rangle \) and \( U = K \psi_0 \), while \( \langle \psi_1, \ldots \psi_n \rangle = \nabla^W \).

The remaining statement is clear.

Type \( I_2(2m) \). In this case the roles of \( T_\ell \) and \( T_p \) are completely symmetric, so we shall treat only one case. We have

\[
H_{T_\ell} = I_2(m), \quad W_{T_p} \simeq \mathbb{Z}/2\mathbb{Z}. \quad A^{H_{T_\ell}} = K[\psi_1, \psi_2],
\]

while \( A^W = K[\psi_1, \psi_2] \) with \( \deg \psi_1 = 2, \deg \psi_2 = m \). From this everything follows.

Type \( F_4 \). Also in this case the roles of \( T_\ell \) and \( T_p \) are completely symmetric, so we shall treat only one case. The group \( H_{T_\ell} \) is of type \( D_4 \) and \( W_{T_p} = S_3 \). Let \( \psi_1, \psi_2, \psi_3, \psi_4 \) be basic invariants for \( H_{T_\ell} \) of degrees 2, 4, 4, 6 respectively. The basic invariants for \( W \) occur in degrees 2, 6, 8, 12. We can choose \( \psi_1, \psi_4 \) to be basic invariants for \( W \). We claim that the action of \( W_{T_p} \) on \( \langle \psi_2, \psi_3 \rangle \) is given by its reflection representation. Indeed, since \( \langle \psi_2, \psi_3 \rangle \) cannot contain invariants for \( W_{T_p} \), the only other possibility is that \( W_{T_p} \) acts on \( \langle \psi_2, \psi_3 \rangle \) by two copies of the sign representation. If this were the case we would have that the degree 8 component of \( A^W \) would have dimension at least 5 while we know that it has dimension 3. Finally \( d_n = 12, r_\ell = r_\ell = 2 \), so \( d_n/2 - (r_p - 1) r_\ell = 4 \).

Now take a \( W \)-invariant complement to \( J_{H_{T_\ell}} \) in \( J_{H_{T_\ell}} \) which we can clearly identify with \( \nabla \). Then \( A^{H_{T_\ell}} = K[\nabla] = K[U] \otimes K[\nabla^W] \). Set \( A = K[U] \). Let \( \phi_1, \ldots, \phi_{r_p} \) be homogeneous polynomial generators of \( A^W \). Consider the ideal \( J \) kernel of the quotient \( \pi : A \rightarrow \mathcal{H} \). Then

**Lemma 6.3.** \( \bar{J} := J \cap \bar{A} \) is the ideal generated by \( \phi_1, \ldots, \phi_{r_p} \).

**Proof.** Take a homogeneous basis \( \phi_{r_p+1}, \ldots, \phi_n \) of \( \nabla^W \). Then \( J = (\phi_1, \ldots, \phi_n) \). Take \( a \in J \cap A^{H_{T_\ell}} \), and write \( a = \sum_i b_i \phi_i \), with \( b_i \in A \). Applying to \( a \) the operator \( R = \frac{1}{|H_{T_\ell}|} \sum_{g \in H_{T_\ell}} g \) we get

\[
a = \sum_i R(b_i) \phi_i,
\]

so that, since \( R(b_i) \in A^{H_{T_\ell}} \), \( J \cap A^{H_{T_\ell}} \) is generated by \( \phi_1, \ldots, \phi_n \). But then \( \bar{J} \) is clearly generated by \( \phi_1, \ldots, \phi_{r_p} \). \( \square \)

Let us now double all degrees. The inclusion \( \bar{A} \subset A \) multiplies the degrees by \( q = d_n - 2(r_p - 1) r_\ell \). Furthermore Lemma 6.3 clearly implies that we have an inclusion of \( \mathcal{H} = \bar{A}/\bar{J} \) into \( \mathcal{H} \), which also multiplies the degrees by \( q = d_n - 2(r_p - 1) r_\ell \).

In each case \( W_{T_p} \) is the symmetric group \( S_{r_p+1} \), so that \( \deg \phi_i = 2(i+1)q \), each \( j = 1, \ldots, r_p \). In particular \( \phi_{r_p} \) has degree \( (d_n - 2(r_p - 1) r_\ell)(r_p + 1) \), which one checks easily to equal \( 2d_n \).

We deduce that \( \phi_{r_p} \) is a highest degree generator of both \( \bar{A}^W \) and \( A^W \).

We define for each \( i = 1, \ldots, r_p \), the \( W \)-equivariant map \( g_i : U \rightarrow \bigwedge V \otimes \mathcal{H} \), given, for \( u \in U \), by

\[
g_i(u) = 1 \otimes \pi'(\partial u \phi_i) = \sum_{j=1}^{s} (u, y_j) (1 \otimes \pi'(\partial \phi_i/\partial y_j)).
\]
By the above discussion \( g_i \) is homogeneous of degree \( 2i q \). Let us now take the operator \( D \) introduced in (3.4) (with \( c = 1 \)) and notice that clearly its restriction to \( A^{Fr_\ell} \) equals
\[
D(p) := \sum_{s \in T_p} \nabla_s.
\]
Define
\[
(6.1) \quad v_i(u) = \frac{s}{2|T_p|} D(p) g_i(v).
\]
By repeating the proof of Proposition 4.4, we then get \((1 \otimes \delta)(v_i) = g_i\). Furthermore using a \( W \)-invariant bilinear form on \( U \) and reasoning exactly as in 1.2, we obtain a bilinear form on the module \( E = \text{hom}_W(U, B) \) with values in \( B^W \) which we still denote by \( E \). We have

**Proposition 6.4.** Let \( 1 \leq i, j \leq r_p \). Then

1. \( E(g_i, g_j) = 0 \).
2. \( E(v_i, g_j) = E(v_j, g_i) = \begin{cases} m_{i,j} p_k & \text{if there exists } k \text{ such that } d_i + d_j - 2 = d_k, \\ 0 & \text{otherwise.} \end{cases} \)

with \( m_{i,j} \neq 0 \).

**Proof.** We have seen that any set \( \phi_1, \ldots, \phi_{r_p} \) of homogeneous polynomial generators of \( \tilde{A}^W \) is part of a set of polynomial generators for \( A^W \) and that \( \phi_{r_p} \) is the highest degree generator for both \( \tilde{A}^W \) and \( A^W \). Furthermore, by Proposition 3.5, we have that \((1 \otimes \delta)(v_i) = g_i\) for all \( i = 1, \ldots, r_p \). At this point, everything follows right away from Propositions 6.4 and 4.7 applied to the group \( W_{T_p} \).

Let us now consider the \( B^W \)-module \( D_p := \text{hom}_W(U, B) \). We get, repeating word by word the proof of Theorem 5.1,

**Theorem 6.5.** (1). \( D_p \) is a free module, with basis the elements \( g_i, v_i, i = 1, \ldots, r_p \), over the exterior algebra \( \bigwedge (p_1, \ldots, p_{r_p-1}) \).
(2). The multiplication by \( p_{r_p} \) is self-adjoint for the form \( E \). Setting \( m_i = m_{i,r+1-i} \), it is given by the formulas

\[
(6.3) \quad p_{r_p} g_i = - \sum_{j=1, j \neq i}^{r_p} m_j^{-1} E(g_i, v_{r_p-j+1}) g_j, \quad i = 1, \ldots, r_p,
\]
\[
(6.4) \quad p_{r_p} v_i = - \sum_{j=1, j \neq i}^{r_p} m_j^{-1} E(g_i, v_{r_p-j+1}) v_j, \quad i = 1, \ldots, r_p.
\]

In the case in which \( W \) is the Weyl group of a simple Lie algebra \( \mathfrak{g} \), which is of course non-simply laced, our representation \( U \) is the zero weight space of the irreducible \( \mathfrak{g} \)-module \( \mathfrak{g}_s \) whose highest weight is the dominant short root, which in fact is small. Using Theorem 6.5, and [4], one can then easily deduce the following Corollary which obviously implies Reeder’s conjecture (2.1) for \( \mathfrak{g}_s \).
Corollary 6.6. The map
\[ p_i \mapsto P_i, \quad v_i \mapsto w_i^\wedge, \quad g_i \mapsto f_i^\wedge, \quad 1 \leq i \leq r \]
extends to an isomorphism of graded \( B^W \)-modules \((\Lambda^\bullet \mathfrak{h} \otimes \mathcal{H} \otimes U)^W \to (\Lambda^\bullet \mathfrak{g} \otimes \mathfrak{g})^0\).

7. A Possible Extension of Reeder’s Conjecture

Consider the bracket map \([-,-]: \Lambda^2 \mathfrak{g} \to \mathfrak{g}\). Dualizing and using the isomorphism \( \mathfrak{g} \simeq \mathfrak{g}^* \) given by the Killing form, we get a linear map \( \mathfrak{g} \to \Lambda^2 \mathfrak{g} \). Since \( \Lambda^{\text{even}} \mathfrak{g} \) is a commutative algebra, this linear map extends to homomorphism of algebras \( s: S(\mathfrak{g}) \to \Lambda^{\text{even}} \mathfrak{g} \).

The inclusion \( \mathfrak{h} \subset \mathfrak{g} \) also gives an inclusion of rings \( j: S(\mathfrak{h}) \to S(\mathfrak{g}) \). Composing with \( s \) we get the homomorphism, \( \tau: S(\mathfrak{h}) \to \Lambda^{\text{even}} \mathfrak{g} \).

Let, as in Section 1, \( J \) be the ideal in \( S(\mathfrak{h}) \) generated by the \( W \)-invariants vanishing in 0. Recall that the ideal \( J \) has a canonical complement \( \mathcal{A} \), the so called harmonic polynomials, i.e. those elements in \( S(\mathfrak{h}) \) killed by all constant coefficients \( W \)-invariant differential operators without constant term.

We have

**Proposition 7.1.** The restriction of the homomorphism \( \tau: S(\mathfrak{h}) \to \Lambda^{\text{even}} \mathfrak{g} \) to \( \mathcal{A} \) is injective.

**Proof.** Let \( \Delta^+ \subset \mathfrak{h}^* \simeq \mathfrak{h} \) denote the set of positive roots. Take the Weyl denominator polynomial \( P = \prod_{\alpha \in \Delta^+} \alpha = \prod_{\alpha \in \Delta^+} t_\alpha \) (where \( t_\alpha \in \mathfrak{h} \) is defined by \( \lambda(t_\alpha) = (\alpha, \lambda), \lambda \in \mathfrak{h}^* \)). We know that \( W \) acts on \( P \) by the sign representation and that in degree \( N = |\Delta^+| \) the homogeneous component \( \mathcal{A}_N \) of \( \mathcal{A} \) is spanned by \( P \).

Recall from [9, (89)] that, if \( \{x_i\}, \{x^i\} \) are dual basis of \( \mathfrak{g} \) w.r.t. the chosen invariant form, then \( s(x) = 1/2 \sum_i x_i \wedge [x^i, x], x \in \mathfrak{g} \). Fix now root vectors \( e_\beta, \beta \in \Delta^+ \) and choose \( e_{-\beta} \) such that \( (e_\beta, e_{-\beta}) = 1 \). A simple computation using the above formula for \( s \) shows that for any \( \alpha \in \Delta^+ \) we have
\[ \tau(t_\alpha) = \sum_{\beta \in \Delta^+} (\beta, \alpha)e_\beta \wedge e_{-\beta}. \]

It follows that
\[ \tau(P) = \text{per}(A) \prod_{\beta \in \Delta^+} (e_\beta \wedge e_{-\beta}), \]
where \( \text{per}(A) \) is the permanent of the matrix \( A = ((\beta, \alpha)) \).

Now \( A \) is a positive semidefinite matrix. It follows that its permanent is non zero. Indeed by [10], one has
\[ \text{per}(A) \geq \frac{N!}{(\rho, \rho)^N} P(\rho)^2 = \frac{N!}{(\rho, \rho)^N} \prod_{\alpha \in \Delta^+} (\alpha, \rho)^2 > 0, \]
where \( \rho \) is the half sum of positive roots, which is well-known to be regular. This proves our claim in degree \( N \).

Let us now consider \( \mathcal{A}_m \). We have \( m \leq N \) otherwise \( \mathcal{A}_m = \{0\} \) and there is nothing to prove. So we can assume \( m < N \). Take \( 0 \neq a \in \mathcal{A}_m \). We then know that there is an element \( b \in \mathcal{A}_{N-m} \) such that \( ab = P + r \) with \( r \in J_N \). Assume \( \tau(a) = 0 \). Then \( \tau(r) = -\tau(P) \). Consider the \( W \)-module \( U \) spanned by \( r \). Then \( U \subset J_N \) and \( \tau \) gives a surjective \( W \)-equivariant homomorphism \( U \to \mathbb{C}\tau(P) \). We deduce that \( U \) and hence \( J_N \) contains a copy of the sign
representation of $W$, contrary to the fact that $P$ spans the only copy of the sign representation in degree $N$. It follows that $\tau(a) \neq 0$, proving our claim. 

Recall that there is a $W$-equivariant degree preserving isomorphism between $A^*$ and $\mathcal{H}$. Since $\wedge^{even}g$ is selfdual, dualizing $\tau$ we obtain a surjective degree preserving map

$$\phi : \wedge^{even}g \to \mathcal{H}.$$ 

Let $p$ be the projection to $p : \wedge g \to \wedge h$ and $\pi : \wedge g \to \wedge^{even}g$ the projection on the even part. Using these, we can build up the map

$$(7.1) \Phi : \wedge g \wedge^* \to \wedge g \otimes \wedge g \xrightarrow{Id \otimes \pi} \wedge g \otimes \wedge g \xrightarrow{even, p \otimes \phi} \wedge h \otimes \mathcal{H}.$$ 

Let $V$ be any finite dimensional irreducible $g$-module. Denote by $V^0$ its zero weight space and by $i : V^0 \to V$ the natural inclusion. If $f \in \text{hom}(V, \wedge g)$, we may consider

$$\Phi_V^f := \Phi \circ f \circ i \in \text{hom}(V^0, \wedge h \otimes \mathcal{H}).$$

Clearly, by equivariance, if $f \in \text{hom}(V, \wedge g)$ then $\Phi_V^f \in \text{hom}_W(V^0, \wedge h \otimes \mathcal{H})$.

Hence we have a graded map

$$(7.2) \Phi^V : \text{hom}_g(V, \wedge g) \to \text{hom}_W(V^0, \wedge h \otimes \mathcal{H}), \quad \Phi^V(f) = \Phi^f_V.$$ 

**Conjecture.** For any finite dimensional irreducible $g$-module $V$, the map $\Phi^V$ is injective.

**Remark 7.2.** Since $\dim \text{hom}_g(V, \wedge g) = \dim \text{hom}_W(V^0, \wedge h \otimes \mathcal{H})$ if (and only if) $V$ is small (cf. [13, Corollary 4.2]), the above conjecture implies Reeder’s conjecture.

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