LP-Based Covering Games with Low Price of Anarchy

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Abstract. We design a new class of vertex and set cover games, where the price of anarchy bounds match the best known constant factor approximation guarantees for the centralized optimization problems for linear and also for submodular costs. This is in contrast to all previously studied covering games, where the price of anarchy grows linearly with the size of the game. Both the game design and the price of anarchy results are based on structural properties of the linear programming relaxations. For linear costs we also exhibit simple best-response dynamics that converge to Nash equilibria in linear time.

1 Introduction

Distributed algorithms is a developing field that tries to address the novel computational challenges of networked environments, such as the Internet. The purpose of these systems is to coordinate numerous computational resources so as to achieve a common goal, such as solving a large optimization problem. There is a wide spectrum of approaches within distributed algorithm design, depending on the level of coordination between different computational elements. On one extreme, complete coordination is equivalent to centralized algorithms, whereas in the other extreme, sometimes referred to as decentralized computation, there exists no common goal but only local ones that depend on the information available in the immediate neighborhood of each computational element. Such solutions, when they exist, are highly sought after since they can tolerate failures in individual subsystems, as well as evolving network topologies, including the arbitrary addition/deletion of computational elements.

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A fundamental trend in mechanism design, often called *implementation theory*, is to construct games where the Nash equilibria correspond to globally desirable outcomes (see e.g. [16,8,30]). By appropriately setting up the rules and payoffs, we can enforce the selfishly acting agents to arrive at an outcome realizing social goals. An important assumption in these works is the existence of an omnipotent central authority that collects the players’ choices, determines and enforces the outcome. In contrast, we put emphasis on the distributed aspect: we do not allow computations performed by a central authority and use of global information, but require that the incentives of each agent/node need to be efficiently computable based on information available in its immediate neighborhood.

The standard measure of the efficiency in algorithmic game theory is the *price of anarchy* [26], the maximal ratio between the social cost of a Nash equilibrium and that of the global optimal configuration. Intuitively, a low price of anarchy implies that upon converging to a socially stable outcome, the quality of the acquired solution is almost optimal from a central optimization perspective.

Games for vertex cover and set cover problems have already been studied, however, all these approaches exhibit prohibitively high price of anarchy. Specifically, Cardinal and Hoefer in [6] define a vertex cover game where the edges of a network are owned by \( k \) agents. An agent’s goal is to have each of his edges supplied by a service point at least one of its endpoints. There is a cost \( c(v) \geq 0 \) associated to building a service point at vertex \( v \). The strategy of an agent is a vector consisting of offers to the vertices. Service points will be installed at vertices where the total offer exceeds the cost of the vertex. Similar games are defined by Buchbinder et al. [5] and by Escoffier et al. [11] for the more general set cover problem.

A different approach was followed by Balcan et al. [2]. Here the agents are the vertices of the graph, and their strategies are deciding whether they open a service point. If opening a service point, vertex \( v \) incurs a cost \( c(v) \). If he decides not to open, he has to pay a penalty for all edges incident to \( v \) whose other endpoints are uncovered.

The price of anarchy is \( \Theta(k) \) in [6] and \( \Theta(n) \) in [2]. Indeed, if the underlying network is a star, and each edge is owned by a different agent in the first case, we get Nash equilibria with all leaves being service points. These guarantees are significantly worse than the ones available in the centralized setting, where simple factor 2-approximation algorithms exist [3].

We close this gap completely by designing a distributed game with low information burden (i.e. the utilities of the agents depends on the state of their local neighborhood/subnetwork) and high efficiency (i.e. low price of anarchy).

Specifically, we propose games (called “Mafia games”) for covering problems with the price of anarchy being equal to the best constant factor approximation algorithms for the central optimization problems. We prove the following theorem which is the summary of Theorems 3, 4, 5 and 6.

**Theorem 1.** The Mafia games for vertex cover, hitting set and submodular hitting set always have pure Nash equilibria, and the price of anarchy is 2 for vertex cover and \( d \) for (submodular) hitting set, where \( d \) is the maximum set cardinality.