MATHEMATICAL DEFINITION OF QUANTUM FIELD THEORY ON A MANIFOLD

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Abstract. We give a mathematical definition of quantum field theory on a manifold, and definition of quantization of classical field theory given by a variational principle.

To the memory of I. M. Gelfand

1. Introduction

In this note we give a definition of quantum field theory (QFT) on a space-time being a manifold $M$. Such definition is necessary for unification of QFT with general relativity. Our definition is almost directly motivated by the definition of dynamical evolution on space-like surfaces in QFT on $M = \mathbb{R}^{3+1}$ given in our previous paper [1]. The only essential difference is that we impose the additional condition that the Hilbert spaces in question be representations of canonical commutation relations, if the theory is quantization of a classical field theory. This condition seems reasonable. Classification of unitary representations of canonical commutation relations can be found, for example, in the book [2] (in the bosonic case).

2. Definition of QFT on a manifold

2.1. Let $M$ be a (pseudo-Riemannian) manifold of dimension $D$, and let $G$ be a Lie group acting on $M$. By definition, a QFT on $M$ assigns

a) to each (space-like) closed connected co-oriented hypersurface $C$ in $M$ (of codimension 1, below we call them simply surfaces) a Hilbert space $\mathcal{H}_C$, and

b) to each closed co-oriented surface $C$ in $M$ with the connected components $C_1, \ldots, C_n$ it assigns the space $\mathcal{H}_C \overset{\text{def}}{=} \mathcal{H}_{C_1} \otimes \ldots \otimes \mathcal{H}_{C_n}$. Here $\otimes$ means bounded tensor product of Banach spaces, so that for two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\overline{\mathcal{H}_1 \otimes \mathcal{H}_2}$ (bar means complex conjugation) is identified with the space $\text{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$;

c) to each manifold $N$ of the same dimension $D$ with the boundary $\partial N$ and a topological type of smooth mappings $N \to M$ which
isomorphically map \( \partial N \) to a surface \( C \) in a compatible way with co-orientation, it assigns a vector \( \Psi_N \in \mathcal{H}_C \), so that the following conditions hold.

(i) Change of co-orientation of \( C \) corresponds to complex conjugation of \( \mathcal{H}_C \).

(ii) If \( N \) is the union of two open submanifolds \( N_1, N_2 \) with the common boundary \( C_1 \), so that \( \partial N_1 = C \cup C_1 \) and \( \partial N_2 = C_1 \cup C' \), then \( \Psi_N \in \mathcal{H}_C \otimes \mathcal{H}_{C'} \) is obtained from \( \Psi_{N_1} \otimes \Psi_{N_2} \in \mathcal{H}_C \otimes \mathcal{H}_{C_1} \otimes \mathcal{H}_{C'} \) by contraction \( \mathcal{H}_C \otimes \mathcal{H}_{C_1} \otimes \mathcal{H}_{C'} \rightarrow \mathcal{H}_C \otimes \mathcal{H}_{C'} \).

Corollary. If we identify \( \mathcal{H}_C \otimes \mathcal{H}_{C_1} \) with \( \text{Hom}(\mathcal{H}_C, \mathcal{H}_{C_1}) \), then \( \Psi_{N_1} \) is a unitary operator from \( \mathcal{H}_C \) to \( \mathcal{H}_{C_1} \), and its composition with \( \Psi_{N_2} : \mathcal{H}_{C_1} \rightarrow \mathcal{H}_{C'} \) equals \( \Psi_N : \mathcal{H}_C \rightarrow \mathcal{H}_{C'} \).

(iii) \( \Psi_N \) smoothly depends on \( C \); hence the bundle with fiber \( \mathcal{H}_C \) over the infinite dimensional manifold of surfaces \( C \) carries a canonical integrable flat connection \( \nabla \).

All these data should be compatible with the action of the group \( G \) in the obvious sense.

2.2. Definition of quantization of a classical field theory. Consider a \( G \)-invariant classical field theory on \( M \) given by the action functional

\[
I = \int L(x, \varphi(x), d\varphi(x)) dx,
\]

where \( L \) is the Lagrangian depending on points \( x \in M \), fields \( \varphi(x) \) (we omit the indices of fields), and their first derivatives \( d\varphi(x) \). Then the Euler–Lagrange equations can be written in the covariant Hamiltonian form, as it is described, for example, in [3,4]:

\[
\frac{\delta \Phi}{\delta x^j(s)} = \{ H^j(s), \Phi \},
\]

where \( x(s) = (x^j(s)) \) is a parameterization of the surface \( C \), \( x^j \) are local coordinates on \( M \), \( \Phi = \Phi(x^j(\cdot); \varphi(\cdot), \pi(\cdot)) \) is a functional of fields \( \varphi(s) \) and canonically conjugate variables \( \pi(s) \), which changes together with the surface \( x = x(s) \); \( H^j(s) = H^j(x(s), x^s(s), \varphi(s), \varphi^s(s), \pi(s)) \) are the covariant Hamiltonian densities, and \( \{ , \} \) is the standard Poisson bracket. Then a QFT on \( M \) depending on a parameter \( h \neq 0 \) is said to be a quantization of this classical field theory if the following additional conditions hold:

(iv) each space \( \mathcal{H}_C \) corresponding to a connected surface \( C \) is an irreducible unitary representation (in the sense of [2]) of the canonical
commutation relations between the variables $\hat{\varphi}(s), \hat{\pi}(s)$:

$$[\hat{\varphi}(s), \hat{\varphi}(s')] = [\hat{\pi}(s), \hat{\pi}(s')] = 0, \quad [\hat{\varphi}(s), \hat{\pi}(s')] = i\hbar\delta(s - s'),$$

where $[,]$ is the supercommutator;

(v) Consider the flat integrable connection on the bundle $\text{End}(\mathcal{H}_C) = \text{Hom}(\mathcal{H}_C, \mathcal{H}_C)$ induced from $\nabla$. Denote it by $\nabla_1$. Then in local coordinates $x^j$ on $M$, and for local parameterizations $x = x(s)$ of surfaces $C$, the connection $\nabla_1$ up to $O(h)$ coincides with the differential operator

$$\nabla_1 \frac{\delta}{\delta x^j(s)}(A) = \frac{\delta}{\delta x^j(s)} A - \frac{1}{i\hbar} [H^j(\hat{\varphi}(s), \hat{\pi}(s)), A] \mod O(h)$$

where the operators $\hat{\varphi}(s), \hat{\pi}(s)$ are put in the Hamiltonian density in their natural order (note that the covariant Schrödinger functional differential equation in all standard cases does not contain terms like $\hat{\varphi}(s)\hat{\pi}(s)$ which depend on the order of operators); $A = A(x(\cdot); \hat{\varphi}(\cdot), \hat{\pi}(\cdot))$ is a regular expression, i.e. a polynomial expression of smoothed operators $\int f(s)\hat{\varphi}(s) ds$ and $\int g(s)\hat{\pi}(s) ds$ for some smooth functions $f(s), g(s)$.

(vi) For any smooth density $j(x)$ on $M$ with compact support, called source, and for each co-orientation of the surfaces $C$, the connection

$$\nabla_j = \nabla + \frac{1}{i\hbar} \int_C j(x(s)) \hat{\varphi}(s)$$

on the bundle $\mathcal{H}_C$ is also flat.

The latter condition is necessary for construction of the Green functions $\langle \varphi(x_1) \ldots \varphi(x_n) \rangle$, as in [1].

References

[1] A. V. Stoyanovsky, Quantization on space-like surfaces, [http://arxiv.org/abs/0909.4918](http://arxiv.org/abs/0909.4918) [math-ph].

[2] I. M. Gelfand, N. Ya. Vilenkin, Generalized functions, vol. 4. Some applications of harmonic analysis. Equipped Hilbert spaces. Fizmatlit, Moscow, 1961 (in Russian).

[3] A. V. Stoyanovsky, Introduction to the mathematical principles of quantum field theory, Editorial URSS, Moscow, 2007 (in Russian).

[4] A. V. Stoyanovsky, Generalized Schrodinger equation for free field, [hep-th/0601080](http://arxiv.org/abs/hep-th/0601080).

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