AN INEQUALITY FOR THE NORM OF A POLYNOMIAL FACTOR

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Abstract. Let \( p(z) \) be a monic polynomial of degree \( n \), with complex coefficients, and let \( q(z) \) be its monic factor. We prove an asymptotically sharp inequality of the form \( \|q\|_E \leq C_n \|p\|_E \), where \( \| \cdot \|_E \) denotes the sup norm on a compact set \( E \) in the plane. The best constant \( C_E \) in this inequality is found by potential theoretic methods. We also consider applications of the general result to the cases of a disk and a segment.

1. Introduction

Let \( p(z) \) be a monic polynomial of degree \( n \), with complex coefficients. Suppose that \( p(z) \) has a monic factor \( q(z) \), so that
\[
p(z) = q(z) r(z),
\]
where \( r(z) \) is also a monic polynomial. Define the uniform (sup) norm on a compact set \( E \) in the complex plane \( \mathbb{C} \) by
\[
\|f\|_E := \sup_{z \in E} |f(z)|.
\]
We study the inequalities of the following form
\[
\|q\|_E \leq C_n \|p\|_E, \quad \deg p = n,
\]
where the main problem is to find the best (the smallest) constant \( C_E \), such that (1.2) is valid for any monic polynomial \( p(z) \) and any monic factor \( q(z) \).

In the case \( E = D \), where \( D := \{ z : |z| < 1 \} \), the inequality (1.2) was considered in a series of papers by Mignotte [9], Granville [7] and Glesser [6], who obtained a number of improvements on the upper bound for \( C_D \). D. W. Boyd [3] made the final step here, by proving that
\[
\|q\|_D \leq \beta^n \|p\|_D,
\]
with
\[
\beta := \exp \left( \frac{1}{\pi} \int_0^{2\pi/3} \log \left( 2 \cos \frac{t}{2} \right) dt \right).
\]

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The constant $\beta = C_E$ is asymptotically sharp, as $n \to \infty$, and it can also be expressed in a different way, using Mahler’s measure. This problem is of importance in designing algorithms for factoring polynomials with integer coefficients over integers. We refer to [5] and [8] for more information on the connection with symbolic computations.

A further development related to (1.2) for $E = [-a, a]$, $a > 0$, was suggested by P. B. Borwein in [1] (see Theorems 2 and 5 there or see Section 5.3 in [2]). In particular, Borwein proved that if $\deg q = m$ then

$$|q(-a)| \leq \|p\|_{[-a,a]}a^{m-n}2^{n-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right),$$

where the bound is attained for a monic Chebyshev polynomial of degree $n$ on $[-a, a]$ and a factor $q$. He also showed that, for $E = [-2, 2]$, the constant in the above inequality satisfies

$$\limsup_{n \to \infty} \left( \frac{2^{m-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)}{\cap E} \right)^{1/n}$$

which hints that

$$C_{[-2,2]} = \exp \left( \int_{0}^{2/3} \log (2 + 2 \cos \pi x) \, dx \right) = 1.9081 \ldots ,$$

We find the asymptotically best constant $C_E$ in (1.2) for a rather arbitrary compact set $E$. The general result is then applied to the cases of a disk and a line segment, so that we recover (1.3)-(1.4) and confirm (1.6).

### 2. Results

Our solution of the above problem is based on certain ideas from the logarithmic potential theory (cf. [12] or [13]). Let $\text{cap}(E)$ be the logarithmic capacity of a compact set $E \subset \mathbb{C}$. For $E$ with $\text{cap}(E) > 0$, denote the equilibrium measure of $E$ (in the sense of the logarithmic potential theory) by $\mu_E$. We remark that $\mu_E$ is a positive unit Borel measure supported on $E$, supp $\mu_E \subset E$ (see [13, p. 55]).

**Theorem 2.1.** Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$. Then the best constant $C_E$ in (1.2) is given by

$$C_E = \max_{u \in \partial E} \frac{\exp \left( \int_{|z-u| \geq 1} \log |z-u| \, d\mu_E(z) \right)}{\text{cap}(E)}.$$  

Furthermore, if $E$ is regular then

$$C_E = \max_{u \in \partial E} \exp \left( \int_{|z-u| \leq 1} \log |z-u| \, d\mu_E(z) \right).$$
The above notion of regularity is to be understood in the sense of the exterior Dirichlet problem (cf. [13, p. 7]). Note that the condition \( \text{cap}(E) > 0 \) is usually satisfied for all applications, as it only fails for very thin sets (see [13, pp. 63-66]), e.g., finite sets in the plane. But if \( E \) consists of finitely many points then the inequality (1.2) cannot be true for a polynomial \( p(z) \) with zeros at every point of \( E \) and for its linear factors \( q(z) \). On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [13, p. 56]).

One can readily see from (1.2) or (2.1) that the best constant \( C_E \) is invariant under the rigid motions of the set \( E \) in the plane. Therefore we consider applications of Theorem 2.1 to the family of disks \( D_r := \{ z : |z| < r \} \), which are centered at the origin, and to the family of segments \([-a, a] \), \( a > 0 \).

Corollary 2.2. Let \( D_r \) be a disk of radius \( r \). Then the best constant \( C_{D_r} \), for \( E = D_r \), is given by

\[
C_{D_r} = \begin{cases} 
\frac{1}{r}, & 0 < r \leq \frac{1}{2}, \\
\frac{1}{r} \exp \left( \frac{\pi}{2} \int_0^{\pi - 2 \arcsin \frac{r}{2}} \log \left( 2r \cos \frac{x}{2} \right) \, dx \right), & r > \frac{1}{2}.
\end{cases}
\]  

(2.3)

Note that (1.3)-(1.4) immediately follow from (2.3) for \( r = 1 \). The graph of \( C_{D_r} \), as a function of \( r \), is in Figure 1.

![Figure 1. \( C_{D_r} \) as a function of \( r \).](image)

Corollary 2.3. If \( E = [-a, a] \), \( a > 0 \), then

\[
C_{[-a,a]} = \begin{cases} 
\frac{2}{a}, & 0 < a \leq \frac{1}{2}, \\
\frac{2}{a} \exp \left( \int_a^a \frac{\log(t + a)}{\pi \sqrt{a^2 - t^2}} \, dt \right), & a > \frac{1}{2}.
\end{cases}
\]  

(2.4)
Observe that (2.4), with $a = 2$, implies (1.6) by the change of variable $t = 2 \cos \pi x$. We include the graph of $C_{[-a,a]}$, as a function of $a$, in Figure 2.

![Figure 2. $C_{[-a,a]}$ as a function of $a$.](image)

We now state two general consequences of Theorem 2.1. They explain some interesting features of $C_E$, which the reader may have noticed in Corollaries 2.2 and 2.3. Let

$$
\text{diam}(E) := \max_{z, \zeta \in E} |z - \zeta|
$$

be the Euclidean diameter of $E$.

**Corollary 2.4.** Suppose that $\text{cap}(E) > 0$. If $\text{diam}(E) \leq 1$ then

$$
C_E = \frac{1}{\text{cap}(E)}.
$$

(2.5)

It is well known that $\text{cap}(D_r) = r$ and $\text{cap}([-a,a]) = a/2$ (see [12, p. 135]), which clarifies the first lines of (2.3) and (2.4) by (2.5).

The next Corollary shows how the constant $C_E$ behaves under dilations of the set $E$. Let $\alpha E$ be the dilation of $E$ with a factor $\alpha > 0$.

**Corollary 2.5.** If $E$ is regular then

$$
\lim_{\alpha \to +\infty} C_{\alpha E} = 1.
$$

(2.6)

Thus Figures 1 and 2 clearly illustrate (2.6).

We remark that one can deduce inequalities of the type (1.2), for various $L_p$ norms, from Theorem 2.1, by using relations between $L_p$ and $L_\infty$ norms of polynomials on $E$ (see, e.g., [11]).

3. **Proofs**

**Proof of Theorem 2.1.** The proof of this result is based on the ideas of [3] and [10]. For $u \in \mathbb{C}$, consider a function

$$
\rho_u(z) := \max(|z - u|, 1), \quad z \in \mathbb{C}.
$$
One can immediately see that \( \log \rho_u(z) \) is a subharmonic function in \( z \in \mathbb{C} \), which has the following integral representation (see [12, p. 29]):

\[
(3.1) \quad \log \rho_u(z) = \int \log |z - t| \, d\lambda_u(t), \quad z \in \mathbb{C},
\]

where \( d\lambda_u(u + e^{i\theta}) = d\theta/(2\pi) \) is the normalized angular measure on \(|t - u| = 1|.

Let \( u \in \partial E \) be such that

\[
\|q\|_E = |q(u)|.
\]

If \( z_k, \ k = 1, \ldots, m, \) are the zeros of \( q(z) \), counted according to multiplicities, then

\[
(3.2) \quad \log \|q\|_E = \sum_{k=1}^{m} \log |u - z_k| \leq \sum_{k=1}^{m} \log \rho_u(z_k)
\]

by (3.1).

We use the well known Bernstein-Walsh lemma about the growth of a polynomial outside of the set \( E \) (see [12, p. 156], for example):

Let \( E \subset \mathbb{C} \) be a compact set, \( \text{cap}(E) > 0 \), with the unbounded component of \( \mathbb{C} \setminus E \) denoted by \( \Omega \). Then, for any polynomial \( p(z) \) of degree \( n \), we have

\[
(3.3) \quad |p(z)| \leq \|p\|_E e^{ng_\Omega(z, \infty)}, \quad z \in \mathbb{C},
\]

where \( g_\Omega(z, \infty) \) is the Green function of \( \Omega \), with pole at \( \infty \). The following representation for \( g_\Omega(z, \infty) \) is found in Theorem III.37 of [13, p. 82]).

\[
(3.4) \quad g_\Omega(z, \infty) = \log \left\{ \frac{1}{\text{cap}(E)} + \int |z - t| \, d\mu_E(t) \right\}, \quad z \in \mathbb{C}.
\]

It follows from (3.1)-(3.4) and Fubini’s theorem that

\[
\frac{1}{n} \log \left( \frac{\|q\|_E}{\|p\|_E} \right) \leq \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} \, d\lambda_u(t) \leq \int g_\Omega(t, \infty) \, d\lambda_u(t)
\]

\[
= \log \left\{ \frac{1}{\text{cap}(E)} + \int |z - t| \, d\mu_E(t) \right\}
\]

\[
= \log \left\{ \frac{1}{\text{cap}(E)} + \int \log \rho_u(z) \, d\mu_E(z) \right\}.
\]

Using the definition of \( \rho_u(z) \), we obtain from the above estimate that

\[
\|q\|_E \leq \left( \max_{u \in \partial E} \exp \left( \frac{\int \log \rho_u(z) \, d\mu_E(z)}{\text{cap}(E)} \right) \right)^n \|p\|_E
\]

\[
= \left( \max_{u \in \partial E} \exp \left( \frac{\int_{|z-u| \geq 1} \log |z - u| \, d\mu_E(z)}{\text{cap}(E)} \right) \right)^n \|p\|_E.
\]
Hence

\[ C_E \leq \max_{u \in \partial E} \exp \left( \frac{\int_{|z-u| \geq 1} \log |z-u| \, d\mu_E(z)}{\text{cap}(E)} \right). \]

(3.5)

In order to prove the inequality opposite to (3.5), we consider the \( n \)-th Fekete points \( \{a_{k,n}\}_{k=1}^{n} \) for the set \( E \) (cf. \[12\] p. 152). Let

\[ p_n(z) := \prod_{k=1}^{n} (z - a_{k,n}) \]

be the Fekete polynomial of degree \( n \). Define the normalized counting measures on the Fekete points by

\[ \tau_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{a_{k,n}}, \quad n \in \mathbb{N}. \]

It is known that (see Theorems 5.5.4 and 5.5.2 in \[12\] pp. 153-155)

\[ \lim_{n \to \infty} \|p_n\|_{E}^{1/n} = \text{cap}(E). \]

(3.6)

Furthermore, we have the following weak* convergence of counting measures (cf. \[12\] p. 159):

\[ \tau_n \rightharpoonup \mu_E, \quad \text{as } n \to \infty. \]

(3.7)

Let \( u \in \partial E \) be a point, where the maximum on the right hand side of (3.5) is attained. Define the factor \( q_n(z) \) for \( p_n(z) \), with zeros being the \( n \)-th Fekete points satisfying \( |a_{k,n} - u| \geq 1 \). Then we have by (3.7) that

\[ \lim_{n \to \infty} \|q_n\|_{E}^{1/n} \geq \lim_{n \to \infty} |q_n(u)|^{1/n} = \lim_{n \to \infty} \exp \left( \frac{1}{n} \sum_{|a_{k,n} - u| \geq 1} \log |u - a_{k,n}| \right) \]

\[ = \exp \left( \lim_{n \to \infty} \int_{|z-u| \geq 1} \log |u - z| \, d\tau_n(z) \right) \]

\[ = \exp \left( \int_{|z-u| \geq 1} \log |u - z| \, d\mu_E(z) \right). \]

Combining the above inequality with (3.6) and the definition of \( C_E \), we obtain that

\[ C_E \geq \lim_{n \to \infty} \|q_n\|_{E}^{1/n} \geq \exp \left( \int_{|z-u| \geq 1} \log |z-u| \, d\mu_E(z) \right) \]

\[ \frac{\text{cap}(E)}{\text{cap}(E) \cdot \int_{|z-u| \geq 1} \log |z-u| \, d\mu_E(z)}. \]

This shows that (2.1) holds true. Moreover, if \( u \in \partial E \) is a regular point for \( \Omega \), then we obtain by Theorem III.36 of \[13\] p. 82) and (3.4) that

\[ \log \frac{1}{\text{cap}(E)} + \int \log |u - t| \, d\mu_E(t) = g_{\Omega}(u, \infty) = 0. \]

Hence

\[ \log \frac{1}{\text{cap}(E)} + \int_{|z-u| \geq 1} \log |u - t| \, d\mu_E(t) = - \int_{|z-u| \leq 1} \log |u - t| \, d\mu_E(t), \]

which implies (2.1) by (2.4). \( \square \)
Proof of Corollary 2.2. It is well known [13, p. 84] that \( \text{cap}(D_r) = r \) and \( \frac{d}{d\theta}(r e^{i\theta}) = d\theta/(2\pi) \), where \( d\theta \) is the angular measure on \( \partial D_r \). If \( r \in (0, 1/2] \) then the numerator of \( (2.1) \) is equal to 1, so that

\[
C_{D_r} = \frac{1}{r}, \quad 0 < r \leq 1/2.
\]

Assume that \( r > 1/2 \). We set \( z = re^{i\theta} \) and let \( u_0 = re^{i\theta_0} \) be a point where the maximum in \( (2.1) \) is attained. On writing

\[
|z - u_0| = 2r \left| \sin \theta - \frac{\theta_0}{2} \right|
\]

we obtain that

\[
C_{D_r} = \frac{1}{r} \exp \left( \frac{1}{2\pi} \int_{\theta_0 + \frac{2\pi}{2}}^{2\pi + \theta_0 - \frac{2\pi}{2}} \log \left| 2r \sin \frac{\theta - \theta_0}{2} \right| d\theta \right)
\]

\[
= \frac{1}{r} \exp \left( \frac{1}{2\pi} \int_{\frac{\pi}{2} - \frac{2\pi}{2}}^{\frac{2\pi}{2} - \pi} \log \left( 2r \cos \frac{x}{2} \right) dx \right)
\]

\[
= \frac{1}{r} \exp \left( \frac{1}{\pi} \int_{0}^{\pi} \log \left( 2r \cos \frac{x}{2} \right) dx \right),
\]

by the change of variable \( \theta - \theta_0 = \pi - x \). \( \square \)

Proof of Corollary 2.3. Recall that \( \text{cap}([-a, a]) = a/2 \) (see [13, p. 84]) and

\[
d\mu_{[-a,a]}(t) = \frac{dt}{\pi \sqrt{a^2 - t^2}}, \quad t \in [-a, a].
\]

It follows from (2.1) that

(3.8) \[
C_{[-a,a]} = \frac{2}{a} \exp \left( \max_{u \in [-a,a]} \int_{[-a,a] \setminus (u-1,u+1)} \frac{\log |t-u|}{\pi \sqrt{a^2 - t^2}} dt \right).
\]

If \( a \in (0, 1/2] \) then the integral in (3.8) obviously vanishes, so that \( C_{[-a,a]} = 2/a \). For \( a > 1/2 \), let

(3.9) \[
f(u) := \int_{[-a,a] \setminus (u-1,u+1)} \frac{\log |t-u|}{\pi \sqrt{a^2 - t^2}} dt.
\]

One can easily see from (3.9) that

\[f'(u) = \int_{u+1}^{a} \frac{dt}{u(t-u) \sqrt{a^2 - t^2}} < 0, \quad u \in [-a, 1-a],\]

and

\[f'(u) = \int_{-a}^{u-1} \frac{dt}{u(t-u) \sqrt{a^2 - t^2}} > 0, \quad u \in [a-1,a].\]

However, if \( u \in (1-a, a-1) \) then

\[f'(u) = \int_{u+1}^{a} \frac{dt}{u(t-u) \sqrt{a^2 - t^2}} + \int_{-a}^{u-1} \frac{dt}{u(t-u) \sqrt{a^2 - t^2}}.
\]

It is not difficult to verify directly that

\[
\int \frac{dt}{\pi (u-t) \sqrt{a^2 - t^2}} = \frac{1}{\pi \sqrt{a^2 - u^2}} \log \left| \frac{a^2 - ut + \sqrt{a^2 - t^2} \sqrt{a^2 - u^2}}{t - u} \right| + C,
\]
which implies that
\[ f'(u) = \frac{1}{\pi \sqrt{a^2 - u^2}} \log \left( \frac{a^2 - u^2 + u + \sqrt{a^2 - (u-1)^2 \sqrt{a^2 - u^2}}}{a^2 - u^2 - u + \sqrt{a^2 - (u+1)^2 \sqrt{a^2 - u^2}}} \right), \]
for \( u \in (1-a, a-1) \). Hence
\[ f'(u) < 0, \quad u \in (1-a, 0), \quad \text{and} \quad f'(u) > 0, \quad u \in (0, a-1). \]
Collecting all facts, we obtain that the maximum for \( f(u) \) on \([-a, a]\) is attained at the endpoints \( u = a \) and \( u = -a \), and it is equal to
\[ \max_{u \in [-a, a]} f(u) = \int_{1-a}^{a} \frac{\log(t + a)}{\pi \sqrt{a^2 - t^2}} dt. \]
Thus (2.3) follows from (3.8) and the above equation. □

**Proof of Corollary 2.5.** Note that the numerator of (2.1) is equal to 1, because \(|z-u| \leq 1, \forall z \in E, \forall u \in \partial E\). Thus (2.5) follows immediately. □

**Proof of Corollary 2.4.** Observe that \( C_E \geq 1 \) for any \( E \in \mathbb{C} \), so that \( C_{\alpha E} \geq 1 \). Since \( E \) is regular, we use the representation for \( C_E \) in (2.2). Let \( T : E \to \alpha E \) be the dilation mapping. Then \(|Tz-Tu| = \alpha |z-u|, \ z, u \in E, \) and \( d\mu_{\alpha E}(Tz) = d\mu_E(z) \). This gives that
\[ C_{\alpha E} = \max_{Tu \in \partial(\alpha E)} \exp \left( -\int_{|Tz-Tu| \leq 1} \log |Tz-Tu| d\mu_{\alpha E}(Tz) \right) \]
\[ = \max_{u \in \partial E} \exp \left( -\int_{|z-u| \leq 1/\alpha} \log(\alpha |z-u|) d\mu_E(z) \right) \]
\[ = \max_{u \in \partial E} \exp \left( -\int_{|z-u| \leq 1/\alpha} \log \alpha - \log(\alpha |z-u|) d\mu_E(z) \right) \]
\[ < \max_{u \in \partial E} \exp \left( -\int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) \right), \]
where \( \alpha \geq 1 \). Using the absolute continuity of the integral, we have that
\[ \lim_{\alpha \to +\infty} \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) = 0, \]
which implies (2.6). □

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