ON CORRELATIONS OF CERTAIN MULTIPLICATIVE FUNCTIONS

R. BALASUBRAMANIAN, SUMIT GIRI, AND PRIYAMVAD SRIVASTAV

Abstract. In this paper, we study sums of shifted products $\sum_{n \leq x} F(n)G(n-h)$ for any $|h| \leq x/2$ and arithmetic functions $F = f \ast 1$ and $G = g \ast 1$, with $f$ and $g$ small. We obtain asymptotic formula for different orders of magnitude of $f$ and $g$. We also provide asymptotic formula for sums of the type $\sum_{n \leq x} \mu^2(n)G(n-h)$, where $G = g \ast 1$ and $g$ is small. For small order of magnitudes of $f$ and $g$, we improve the error terms and make them independent of $h$.

1. Introduction

Let $F$ and $G$ be two arithmetic functions. In [BG], the first two authors studied the problem of getting an asymptotic formula for the sum $\sum_{n \leq x} F(n)G(n-h)$, where $F = f \ast 1$ and $G = g \ast 1$, under the assumption that for primes $p$, $f(p)$ and $g(p)$ are close to 1. In this paper, we continue the investigation (See Theorem 2.2 and Theorem 2.5). We also show that this method is equally applicable to the asymptotic formula for $\sum_{n \leq x} \mu^2(n)G(n-h)$.

In [Mi], Mirsky considers the general sum $\sum_{n \leq x} F_1(n+k_1) \ldots F_s(n+k_s)$, with $F_j = 1 \ast f_j$ and $f_j(p) = O(p^{-\sigma+\epsilon})$ for each $j$. In [S1], Stepanauskas considers $\sum_{n \leq x} F(n)G(n-h)$ under the weaker assumption $\sum_{p \leq x} f(p)g(p)p^{-2}$ is convergent. In [S2], Stepanauskas and Siaulys also consider the sum $\sum_{p \leq x} F(p+1)G(p+2)$, where the sum runs over the primes.

In [CMS], Coppola, Murty, Saha consider the problem of $\sum_{n \leq x} F(n)G(n-h)$ under a general condition that $F$ and $G$ admit a Ramanujan expansion.

A considerable amount of work has been done for such shifted sums. For instance, one can see papers of Carlitz [Ca], Choi and Schwarz [CS], Katai [Ka] and Rearick [Re].

Since all these results have been proved under different conditions, it is difficult to compare these results. However, functions like $\frac{\varphi(n)}{n}$, $\frac{\sigma(n)}{n}$ are the common threads between these results and the results proved in this paper. We shall later compare these results in section 5.

2. Statement of the theorems

In [BG], the following theorem was proved

Theorem A. Let $E_1(x) = \sum_{n \leq x} |f(n)|$ and $E_2(x) = \sum_{n \leq x} |g(n)|$. For $h \neq 0$, let

$$C(h) = \sum_{(d_1, d_2) | h} \frac{f(d_1)g(d_2)}{[d_1, d_2]}.$$ 

Then

$$\sum_{n \leq x} F(n)G(n-h) = xC(h) + O(hE_1(x)E_2(x)).$$

Our aim is to improve the error term.
Definition 2.1. For $\alpha > 0$, define $A_\alpha$ to be the class of arithmetic functions $g$ satisfying $g(n) = O(n^{-\alpha})$ for each $n$.

For ease of exposition, assume that $f \in A_\alpha$ and $g \in A_\beta$ for some $0 < \alpha \leq \beta < 1$. We also assume that $F(n)$ and $G(n)$ are 0, if $n \leq 0$. Then, Theorem A gives

Corollary I. We have, under the conditions above,

$$\sum_{n \leq x} F(n)G(n - h) = xC(h) + O\left(hx^{2-\alpha-\beta}\right).$$

Let

$$E(x; \alpha, \beta) = \begin{cases} x^{1-\alpha}, & \alpha < \beta \text{ and } \alpha < 1, \\ x^{1-\alpha} \log x, & \alpha = \beta < 1, \\ \log x, & 1 = \alpha < \beta, \\ \log^2 x, & \alpha = \beta = 1, \\ 1, & 1 < \alpha < \beta. \end{cases}$$

Then, we prove

Theorem 2.2. Suppose that $f \in A_\alpha$ and $g \in A_\beta$. Then, uniformly for all $h$, $|h| \leq x/2$, we have

$$\sum_{H \leq n \leq x} F(n)G(n - h) = (x - H)C(h) + O\left(E(x; \alpha, \beta)\right),$$

where

$$H = \begin{cases} 1, & h \leq 0, \\ h, & h > 0, \end{cases}$$

$$C(h) = \sum_{\substack{a,b \geq 1 \atop (a,b) \mid h}} \frac{f(a)g(b)}{|a,b|},$$

and the $O$-constant is absolute.

Remark 2.3. Theorem 2.2 improves Theorem A in all cases (in terms of $h$) and also improves upon Corollary I in terms of $x$ and $h$.

Remark 2.4. Theorem 2.2 also covers the case $h = 0$. Also, since $f(a) \ll a^{-\alpha}$, $g(b) \ll b^{-\beta}$, it follows that $C(h)$ is well defined. If $f$ and $g$ are multiplicative, then $C(h)$ admits a product expansion

$$C(h) = \prod_p \left( \sum_{\min\{e_1,e_2\} \leq \nu_p(h)} \frac{f(p^{e_1})g(p^{e_2})}{p^\min\{e_1,e_2\}} \right).$$

The method of proof of Theorem 2.2 also applies to study sums of the form $\sum_{n \leq x} \mu^2(n)G(n-h)$.

Let

$$E_1(x; \alpha) = \begin{cases} x^{1-\alpha}, & 0 < \alpha \leq 1/2, \\ x^{1/2}, & \alpha > 1/2. \end{cases}$$

We prove

Theorem 2.5. Let $G(n) = \sum_{d \mid n} g(d)$, where $g \in A_\alpha$ for some $\alpha > 0$. Let $\epsilon > 0$. Then, uniformly for all $|h| \leq x/2$, we have

$$\sum_{n \leq x} \mu^2(n)G(n - h) = (x - H)K(h) + O\left(x^\epsilon E_1(x; \alpha)\right),$$
where
\[ K(h) = \sum_{a,b \geq 1 \atop (a^2, b)|h} \mu(a)g(b) \frac{\varphi(n)}{[a^2, b]}, \]
and \( H \) is as defined in Theorem 2.2.

**Remark 2.6.** In the appendix, we shall remark how to remove the \( x^\epsilon \) from the error term when \( \alpha \) is not in a neighborhood of \( 1/2 \).

**Remark 2.7.** Theorem 2.5 also covers the case \( h = 0 \). Also, \( K(h) \) is well-defined because of the condition \( g \in A_\alpha \). Again, if \( g \) is multiplicative, then \( K(h) \) admits a product expansion
\[ K(h) = \prod_p \left( \sum_{\max\{2e_1, e_2\} \leq v_p(h)} \mu(p^{e_1})g(p^{e_2}) \right). \]

By taking \( G(n) = \frac{\varphi(n)}{n} \), we have

**Corollary 2.8.** Uniformly for \( |h| \leq x/2 \), we have
\[ \sum_{n \leq x} \mu^2(n)\frac{\varphi(n-h)}{n-h} = (x-H)\prod_p \left( 1 - \frac{2}{p^2} \right) \prod_p \left( 1 + \frac{1}{p^3 - 2p} \right) + O\left(x^{1/2}\right). \]

In particular, for \( h = 0 \),
\[ \sum_{n \leq x} \mu^2(n)\frac{\varphi(n)}{n} = x \prod_p \left( 1 - \frac{2}{p^2} \right) \left( 1 + \frac{1}{p^3 - 2p} \right) + O\left(x^{1/2}\right). \]

**Remark 2.9.** We observe that the Dirichlet series of \( \mu^2(n)\frac{\varphi(n)}{n} \) is
\[ \sum_{n=1}^{\infty} \mu^2(n)\frac{\varphi(n)}{n^{1+s}} = \frac{\zeta(s)H(s)}{\zeta(2s)\zeta(4s)}, \]
where \( H(s) \) is absolutely convergent in \( \Re(s) \geq 1/8 \). Consequently, by Landau’s theorem, the error term for the case \( h = 0 \) in Corollary 2.8 is \( \Omega(x^{1/2-\epsilon}) \) if the zeta function were to have a zero close to \( \text{Re}(s) = 1 \). This shows that Corollary 2.8 cannot be improved except for terms like \( \exp(-c(\log x)^{2/5}(\log \log x)^{3/5}) \) unless a good zero-free region for the Riemann zeta function is assumed.

We also note that, by partial summation and using Theorem 2.2, we can also write an asymptotic formula for \( \sum_{n \leq x} Q(n)F(n)G(n-h) \), for any function \( Q(n) \) such that \( Q(t) \) is differentiable for \( 1 \leq t \leq x \) and \( Q'(t) \) is bounded in \( 1 \leq t \leq x \). It is as follows
\[ \sum_{n \leq x} Q(n)F(n)G(n-h) = C(h) \int_1^x Q(t) \, dt + Q(x)E(x; \alpha, \beta) + O\left(\int_1^x |Q'(t)||E(t; \alpha, \beta)| \, dt\right). \]
3. Preliminary Lemmas

In this section, we start with some preliminary lemmas for the proof of Theorem 2.2. We assume throughout $\beta \geq \alpha > 0$. Recall that

$$E(x) = E(x; \alpha, \beta) = \begin{cases} 
  x^{1-\alpha}, & \alpha < \beta \text{ and } \alpha < 1, \\
  x^{1-\alpha} \log x, & \alpha = \beta < 1, \\
  \log x, & 1 = \alpha < \beta, \\
  \log^2 x, & \alpha = \beta = 1, \\
  1, & 1 < \alpha < \beta.
\end{cases}$$

The statements of the lemmas in this section hold true for all $0 < \alpha \leq \beta$. However, we restrict the proofs only to the case $\beta > \alpha$ and $\alpha < 1$. The proof works mutandis-mutandis for $\alpha \geq 1$ and the case $\beta = \alpha$ will have an extra log factor.

When $\beta > \alpha$ and $\alpha < 1$, we find that $E(x) = O(x^{1-\alpha})$.

Lemma 3.1.

(a) If $y \geq 1$, then

$$\sum_{mn \leq y} \frac{1}{m^{1+\alpha} n^{1+\beta}} = O\left( \frac{E(y)}{y} \right).$$

(b) If $x \geq 1$, then

$$S = \sum_{[a,b] \geq x} \frac{1}{a^\alpha b^\beta [a,b]} = O\left( \frac{E(x)}{x} \right).$$

Proof. We first prove (a). As $\beta > \alpha$, we get the sum to be equal to

$$\sum_{n \geq 1} \sum_{m \geq y/n} \frac{1}{m^{1+\alpha} n^{1+\beta}} = \sum_{n \leq y} \frac{1}{n^{1+\beta}} \sum_{m \geq y/n} \frac{1}{m^{1+\alpha}} + \sum_{n > y} \frac{1}{n^{1+\beta}} \sum_{m \geq 1} \frac{1}{m^{1+\alpha}}$$

$$= O \left( \frac{1}{y^\alpha} \sum_{n \leq y} \frac{1}{n^{1+\beta-\alpha}} \right) + O \left( \sum_{n > y} \frac{1}{n^{1+\beta}} \right) = O(y^{-\alpha})$$

and hence (a).

To prove (b), we split the sum depending upon the value of $l = \gcd(a, b)$. Write $a = ml$ and $b = nl$. Then

$$S \ll \sum_{l \geq 1} \frac{1}{l^{1+\alpha+\beta}} \sum_{mn \geq x/l} \frac{1}{m^{1+\alpha} n^{1+\beta}}.$$

Thus, using part (a),

$$S \ll \sum_{l \leq x} \frac{1}{l^{1+\alpha+\beta}} \frac{E(x/l)}{x} + \sum_{l > x} \frac{1}{l^{1+\alpha+\beta}} \sum_{m,n \geq 1} \frac{1}{m^{1+\alpha} n^{1+\beta}}$$

$$\ll x^{-\alpha} \sum_{l \leq x} \frac{1}{l^{1+\beta}} + \sum_{l > x} \frac{1}{l^{1+\alpha+\beta}} \ll x^{-\alpha}.$$

This completes the proof.

Lemma 3.2.

(a) If $y \geq 1$, then

$$\sum_{mn \leq y} \frac{1}{m^\alpha n^\beta} = O(E(y)).$$
(b) If \( x \geq 1 \), then
\[
\sum \frac{1}{a^\alpha b^\beta} = O \left( \frac{E(x)}{l^{1+\beta}} \right).
\]

(c) If \( x \geq 1 \), then
\[
\sum \frac{1}{a^\alpha b^\beta} = O(E(x)).
\]

**Proof.** We have
\[
\sum_{mn \leq y} \frac{1}{m^\alpha n^\beta} = \sum_{n \leq y} \frac{1}{n^\beta} \sum_{m \leq y/n} \frac{1}{m^\alpha} = O \left( \sum_{n \leq y} \frac{1}{n^{1-\alpha}} \right) = O(y^{1-\alpha})
\]
and hence (a).

To prove (b), again split the sum depending upon the value of \( l = \gcd(a, b) \) and use (a). Write \( a = ml \) and \( b = nl \) as before. Then the given sum equals
\[
\frac{1}{l^{\alpha+\beta}} \sum_{mn \leq x/l} \frac{1}{m^{\alpha}n^{\beta}} \ll \frac{1}{l^{\alpha+\beta}} \left( \frac{x}{l} \right)^{1-\alpha}
\]
and this proves (b).

Now, (c) is obtained easily from (b).

\[\square\]

**Lemma 3.3.**

(a) Let \( y \geq 1 \) and \( |k| \leq y/2 \). Then
\[
S_1 = \sum_{m \leq y} \sum_{a|m \atop b|m-k \atop ab \geq y} a^{-\alpha}b^{-\beta} = O(E(y))
\]
and the \( O \)-constant is absolute.

(b) Let \( x \geq 1 \) and \( |h| \leq x/2 \). Then
\[
S_2 = \sum_{n \leq x} \sum_{c|n \atop d|n-h \atop |c,d| \geq x} c^{-\alpha}d^{-\beta} = O(E(x))
\]
and the \( O \)-constant is absolute.

**Proof.** To prove (a), put \( m = ac, m-k = bd \). Then writing the sum in terms of \( c \) and \( d \), we have
\[
S_1 = \sum_{m \leq y} \sum_{c|m \atop d|m-k \atop cd \leq m(m-k) \atop m \equiv 0 \pmod{c} \atop m \equiv k \pmod{d} \atop cd \leq m \leq y} \left( \frac{m}{c} \right)^{-\alpha} \left( \frac{m-k}{d} \right)^{-\beta}.
\]
We note that \( cd \leq \frac{m(m-k)}{y} \leq m \). This implies \( m \geq cd \). Thus,
\[
S_1 \ll \sum_{c \leq y} \sum_{m \equiv 0 \pmod{c} \atop m \equiv k \pmod{d} \atop cd \leq m \leq y} c^\alpha d^\beta m^{-\alpha}(m-k)^{-\beta}.
\]
The congruence on \( m \) gives \( m \equiv r \pmod{[c,d]} \). Thus the \( m \)-sum is at most 
\[
\ll \sum_{m \equiv r \pmod{[c,d]}} m^{-\alpha - \beta} \ll \sum_{m \equiv 0 \pmod{[c,d]}} m^{-\alpha - \beta}.
\]
Let \( \gcd(c,d) = l \) and write \( m = j[c,d] \), with \( l \leq j \leq \frac{2y}{[c,d]} \). The \( m \)-sum is then
\[
\ll [c,d]^{-\alpha - \beta} \sum_{l \leq j \leq \frac{2y}{[c,d]}} j^{-\alpha - \beta}.
\]
Hence
\[
S_1 \ll \sum_{l \leq j} j^{-\alpha - \beta} \sum_{l \leq j \leq \frac{2y}{[c,d]}} \frac{c^\alpha d^\beta}{(cd)^{\alpha + \beta}}.
\]
The second sum above is
\[
\sum_{l \leq j \leq \frac{2y}{[c,d]}} \frac{c^\alpha d^\beta}{(cd)^{\alpha + \beta}} \ll l^{\alpha + \beta} \sum_{l \leq j \leq \frac{2y}{[c,d]}} c^{-\beta} d^{-\alpha}.
\]
From Lemma 3.2(b), the above sum is
\[
= O \left( \frac{E(y/j)}{l^{1-\alpha}} \right).
\]
For \( \alpha < 1 \), this error is
\[
O \left( \frac{y^{1-\alpha}}{(jl)^{1-\alpha}} \right).
\]
Thus,
\[
S_1 \ll y^{1-\alpha} \sum_{j} \frac{1}{j^{1+\beta}} \sum_{l \leq j} \frac{1}{l^{1-\alpha}} \ll y^{1-\alpha} \sum_{j \leq y} \frac{1}{j^{1+\beta-\alpha}}
\]
and this proves (a).

Now, we prove (b) by splitting the sum into \( (c, d) = l \). Write the given sum as
\[
S_2 = \sum_{n \leq x} \sum_{l|n} \sum_{l|n-h} \sum_{d|n-h} \sum_{l|d} \sum_{l|d-h} \sum_{(c,d)=l} (lc)^{-\alpha} (ld)^{-\beta}
\]
\[
\ll \sum_{l|h} l^{-\alpha - \beta} \sum_{n \leq x} \sum_{l|n/l} \sum_{l|n-h} \sum_{l|d-h} \sum_{(c,d)=l} (cd)^{\alpha - \beta}.
\]
Write \( n/l = n' \) and \( h/l = h' \) so that the given sum reduces to
\[
S_2 \ll \sum_{l|h} l^{-\alpha - \beta} \sum_{n' \leq x/l} \sum_{l|n'/l} \sum_{d|n'-h'} \sum_{d \geq x/l} (cd)^{\alpha - \beta}.
\]
Therefore, by part (a), it follows that
\[
S_2 \ll \sum_{l|h} l^{-\alpha - \beta} E(x/l) \ll x^{1-\alpha} \sum_{l|h} \frac{1}{l^{1+\beta}} \ll x^{1-\alpha}.
\]
This proves (b). \( \square \)
Lemma 3.4.

(a) \[ S = \sum_{H \leq n \leq y} \sum_{a \sim A} b^{-\alpha} = O \left( \frac{y^\varepsilon}{c} \left( \frac{y}{z} \right)^\alpha E_1(x) \right). \]

(b) \[ \sum_{H \leq n \leq x} \sum_{a \sim A} b^{-\alpha} = O(x^\varepsilon E_1(x)). \]

Proof. We first prove (a). Observe that since \( ca^2 \mid n \), we have \( ca^2 \leq y \). Break the sum over \( a \) and \( b \) dyadically i.e. let \( a \sim A \) and \( b \sim B \). Then

\[ S_{A,B} = \sum_{H \leq n \leq y} \left( \sum_{a \sim A} 1 \right) \left( \sum_{b \sim B} b^{-\alpha} \right) \ll B^{-\alpha} y^\varepsilon \sum_{H \leq n \leq y} \sum_{a \sim A} 1 \ll y^\varepsilon B^{-\alpha} \sum_{a \sim A} \sum_{H \leq n \leq y} \left( \frac{y}{ca^2} + O(1) \right) \ll \frac{y^{1+\varepsilon}}{cAB^\alpha}. \]

Now, summing over \( A \) and \( B \) in geometric progressions with \( A \leq y^{1/2}, B \leq y \) and \( A^2B > z \), we obtain the desired result.

We now prove (b). Let \((a^2, b) = l_1^2 l_2^2\), with \( l_2 \) square-free. Hence, we have \( a = kl_1 l_2 \) and \( b = ml_1 l_2^2 \) and \([a^2, b] = k^2 m(l_1 l_2)^2\). For a fixed \( l_1, l_2 \), the desired sum is

\[ \sum_{H \leq n \leq x} \sum_{k^2 l_1^2 l_2 | n} \sum_{m(l_1 l_2)^2 \gg x} b^{-\alpha}. \]

Write \( h = h' l_1^2 l_2^2 \) and \( n = n' l_1^2 l_2^2 \). The given sum now becomes,

\[ \ll \sum_{H \leq n \leq x} \sum_{n \equiv 0(l_1^2 l_2)} k^2 l_1^2 l_2 | n \sum_{m(l_1 l_2)^2 \gg x} b^{-\alpha} \ll (l_1^2 l_2)^{-\alpha} \sum_{H/l_1^2 l_2 \leq n' \leq x/l_1^2 l_2} \sum_{k^2 m(l_1 l_2)^2 \gg x} m^{-\alpha}. \]

Applying part (a) to the above sum with \( y = \frac{x}{l_1^2 l_2}, z = \frac{x}{l_1^2 l_2^2} \) and \( c = l_2 \), we obtain that for a fixed \( l_1, l_2 \) the given sum is,

\[ \ll (l_1^2 l_2)^{-\alpha} \left( \frac{x}{l_1^2 l_2^2} \right)^\varepsilon l_2^{\alpha-1} E_1 \left( \frac{x}{l_1^2 l_2} \right). \]

Summing over \( l_1^2 l_2 \leq x \), we obtain the desired result. \( \Box \)

Remark 3.5. In the final step above, we have summed over all \( l_1^2 l_2 \leq x \) instead of \( l_1^2 l_2 \mid h \). This shows that the \( O \)-constant is indeed independent of \( h \).

Lemma 3.6.
Lemma 3.7.

(a) \[ \sum_{a^2 \leq y, b \leq y} b^{-\alpha} = \mathcal{O}(E_1(y)). \]

(b) \[ \sum_{|a^2, b| \leq x} b^{-\alpha} = \mathcal{O}(E_1(x)). \]

Proof. For (a), we follow the proof of Lemma 3.2 (a). For (b), let \((a^2, b) = l_1^2l_2\), with \(l_2\) square-free. Write \(a = kl_1l_2\) and \(b = ml_2^2\) as in the proof of Lemma 3.4 (b). The sum then reduces to the sum in part (a). Summing over \(l_1^2l_2 \leq x\) gives the desired result. \(\square\)

Lemma 3.9. Let \(a, m\) be positive integers, \(h \neq 0\). Let \(g = \gcd(h, m)\). Then the equation \(ax^2 \equiv h \pmod{m}\),

has at most \(L(m)\tau(m)\) solutions modulo \(m\).

Proof. If \((a, m) > 1\), then \(\gcd(a, m) \mid h\). Cancelling the factor, we have

\[ ax_1^2 \equiv h_1 \pmod{m_1}, \]

where \(m_1 = \frac{m}{(a, m)}\) and \((m_1, a_1) = 1\).

Note that any solution of the latter equation lifts to a unique solution of \(ax^2 \equiv h \pmod{m}\). Since \((m_1, a_1) = 1\), the latter equation is the same as \(x^2 \equiv k \pmod{m_1}\). Now, write \(m_1 = q_1q_2\), where \(q_1\) is the product of prime powers \(p^l\) with \(v_p(m_1) \leq v_p(k)\) and \(q_2\) is a product of those prime powers \(p^l\) with \(v_p(m_1) > v_p(k)\).

The equation \(x^2 \equiv k \pmod{q_1}\) is the same as \(x^2 \equiv 0 \pmod{q_1}\) and has at most \(L(q_1)\) solutions. The equation \(x^2 \equiv k \pmod{q_2}\) has at most \(\tau(q_2)\) solutions. Thus, the total number of solutions is at most \(L(q_1)\tau(q_2)\). Since \(q_1 \mid m\), we get \(L(q_1) \leq L(m)\). Since \(\tau(q_2) \leq \tau(m)\), we are through. \(\square\)
4. Proof of Theorem 2.2

Now, we prove Theorem 2.2. We have

\[ S = \sum_{H \leq n \leq x} \sum_{\substack{a|n \\ b|n-h}} f(a)g(b) = \sum_{H \leq n \leq x} \sum_{[a,b] \leq x} f(a)g(b) + \sum_{H \leq n \leq x} \sum_{[a,b] > x} f(a)g(b). \]

The second term on the rightmost side above is \( O(E(x)) \) by Lemma 3.3 (b). The first term is

\[ \sum_{[a,b] \leq x} f(a)g(b) \sum_{H \leq n \leq x} \sum_{\substack{\alpha \equiv n \pmod{a} \\ \beta \equiv n \pmod{b}}} 1 = \sum_{[a,b] \leq x} f(a)g(b) \left( \frac{x-H}{[a,b]} + O(1) \right) \]

and the \( O \)-term is \( O(E(x)) \) by Lemma 3.2 (c). The main term is

\[ (x-H) \sum_{[a,b] \leq x} \frac{f(a)g(b)}{[a,b]} - (x-H) \sum_{[a,b] > x} \frac{f(a)g(b)}{[a,b]}. \]

The first term is \( (x-H)C(h) \) and the second term is \( O(E(x)) \) by Lemma 3.1 (b).

5. Comparison with earlier results

Now, we make comparison of our results with earlier results.

In Theorem 2.2, we take \( F(n) = \frac{n}{\varphi(n)} \) and \( G(n) = \frac{\sigma(n)}{n} \). In this case, \( f(p) = \frac{1}{p-1} \), \( f(p^\alpha) = 0 \) for \( \alpha \geq 2 \) and \( g(n) = 1/n \). Hence, one can take \( \alpha = 1 - \epsilon \) for any \( \epsilon > 0 \) and \( \beta = 1 \). This gives by Theorem 2.2,

**Corollary 5.1.**

(a)

\[ \sum_{n \leq x} \frac{\sigma(n+1)}{n+1} \frac{n}{\varphi(n)} = \prod_p \left( 1 + \frac{2p+1}{p(p^2-1)} \right) + O(x^{1/4}). \]

(b)

\[ \sum_{n \leq x} \frac{\sigma(n+1)}{\varphi(n)} = x \prod_p \left( 1 + \frac{2p+1}{p(p^2-1)} \right) + O(x^{1/4}). \]

For comparison, we note that Stepanauskas [S1] has proved

\[ \sum_{n \leq x} \frac{\sigma(n+1)}{\varphi(n)} = x \prod_p \left( 1 + \frac{2p+1}{p(p^2-1)} \right) + O\left( \frac{x}{(\log x)^2} \right). \]

The method of proof of Theorem 2.2 can also be used to prove an asymptotic formula for

\[ \sum_{p \leq x} F(p+h)G(p+k). \]

We explain this with an example \( F(n) = G(n) = \frac{\varphi(n)}{n} \). We prove

**Theorem 5.2.** Fix \( A > 0 \). Then

\[ \sum_{p \leq x} \frac{\varphi(p+2)}{p+2} \frac{\varphi(p+1)}{p+1} = \frac{\varphi(x)}{2} \prod_{p > 2} \left( 1 - \frac{2}{p(p-1)} \right) + O\left( \frac{x}{(\log x)^{1-\epsilon}} \right). \]

Here the \( O \)-constant depends only upon \( A \).

**Remark 5.3.** The above result can be compared with Corollary 1 of [S2], where the error term \( O\left( \frac{\varphi(x)}{(\log x)^{1-\epsilon}} \right) \) is much larger.
Proof of Theorem 5.2. We have
\[
S = \sum_{p \leq x} \frac{\varphi(p + 2)}{p + 2} \frac{\varphi(p + 1)}{p + 1} = \sum_{p \leq x} \sum_{a \mid p+2, b \mid p+1} \frac{\mu(a)\mu(b)}{ab}
\]
\[
= T_1 + T_2 + T_3,
\]
where \( T_1 \) corresponds for \([a, b] \leq (\log x)^A\), \( T_2 \) for \((\log x)^A < [a, b] \leq x \) and \( T_3 \) for \([a, b] > x \).

Now,
\[
T_3 \leq \sum_{n \leq x} \sum_{\substack{a \mid n + 2 \\ b \mid n + 1 \\ [a, b] \leq x}} \frac{1}{ab} = O \left( \log^2 x \right),
\]
by Lemma 3.3 (b).

Moreover,
\[
T_2 \leq \sum_{n \leq x} \sum_{\substack{a \mid n + 2 \\ b \mid n + 1 \\ (\log x)^A < [a, b] \leq x}} \frac{1}{ab} = \sum_{(a, b) = 1} \frac{1}{(ab + O(1))} = O \left( \frac{x}{(\log x)^A - 1} \right).
\]

Now,
\[
T_1 = \sum_{p \leq x} \sum_{\substack{a \mid n + 2 \\ b \mid n + 1 \\ [a, b] \leq (\log x)^A}} \frac{\mu(a)\mu(b)}{ab} = \sum_{[a, b] \leq (\log x)^A} \frac{\mu(a)\mu(b)}{ab} \sum_{p \leq x} \frac{1}{ab}.
\]

For \( p \neq 2 \), the \( p \)-sum survives only if \((a, b) = 1 \) and \( a \) is odd. Thus,
\[
T_1 = \sum_{a \text{ odd} \geq 1} \sum_{(a, b) = 1} \sum_{ab \leq (\log x)^A} \frac{\mu(a)\mu(b)}{ab} \left( \frac{\varphi(x)}{\varphi(ab)} + O \left( \frac{x}{(\log x)^A} \right) \right),
\]
by Siegel’s theorem on primes in arithmetic progressions. Clearly, the \( O \)-term is \( O \left( \frac{x}{(\log x)^A - 1} \right) \).

The main term is
\[
\frac{\varphi(x)}{\varphi(ab)} \sum_{a \text{ odd} \geq 1} \sum_{(a, b) = 1} \frac{\mu(a)\mu(b)}{ab} - \frac{\varphi(x)}{\varphi(ab)} \sum_{a \text{ odd} \geq 1} \sum_{(a, b) = 1} \frac{\mu(a)\mu(b)}{ab}.
\]

The second term is \( O \left( \frac{x}{(\log x)^A - 1} \right) \) and the first term is \( \frac{\varphi(x)}{\varphi(ab)} \prod_{p \geq 2} \left( 1 - \frac{2}{p(p-1)} \right) \). This completes the proof.

Remark 5.4. The method of proof of Theorem 5.2 gives the same error term for all sums of the form
\[
\sum_{p \leq x} F(p+h)G(p+k),
\]
whenever \( F = f \ast 1, G = g \ast 1 \) and \( f \) and \( g \) are in \( A_\alpha, A_\beta \) respectively.

In Theorem 2.2, we take \( F(n) = \frac{\sigma_s(n)}{n^s}, G(n) = \frac{\sigma_t(n)}{n^t}, \) with \( s \leq t \), where \( \sigma_s(n) = \sum_{d \mid n} d^s \). Then \( f(n) = \frac{1}{n} \) and \( g(n) = \frac{1}{n^t} \). Taking \( \alpha = s \) and \( \beta = t \) gives
Corollary 5.5. Uniformly for $|h| \leq N/2$, we have
\[
\sum_{n \leq N} \frac{\sigma_s(n) \sigma_t(n+h)}{n^s} \frac{1}{(n+h)^t} = (N - H) \frac{\zeta(s+1) \zeta(t+1)}{\zeta(s+t+2)} \frac{\sigma_{-(s+t+1)}(h)}{\zeta(s+t+2)} + O\left(E(N; s, t)\right),
\]
where the $O$-term depends only on $s$ and $t$ and is independent of $h$. In particular, the error term is
\[
\begin{cases}
O(N^{1-s}), & s < 1 \text{ and } t > s, \\
O(N^{1-s} \log N), & s = t < 1, \\
O(\log N), & 1 \leq s < t, \\
O(\log^2 N), & s = t = 1, \\
O(1), & s > 1.
\end{cases}
\]

We can compare the above result with Corollary 1 of Coppola, Murty, Saha [CMS], where the error term depends on $h$, and as a function of $N$, given by
\[
\begin{cases}
O(N^{1-s}(\log N)^{4-2s}), & s < 1, \\
O(\log^3 N), & s = 1, \\
O(1), & s > 1.
\end{cases}
\]
Similar remarks also apply for Corollary 2 of [CMS].

6. Proof of Theorem 2.5

We have,
\[
S = \sum_{H \leq n \leq x} \mu^2(n)G(n-h) = \sum_{H \leq n \leq x} \sum_{a^2|n, b|n-h} \mu(a)g(b)
\]
\[
= T_1 + T_2,
\]
where $T_1$ corresponds to $[a^2, b] \leq x$ and $T_2$ corresponds to $[a^2, b] > x$.

We note that $T_2 = O(x^\epsilon E_1(x))$ by Lemma 3.4 (b).

Now,
\[
T_1 = \sum_{a, b} \mu(a)g(b) \sum_{n \equiv 0 \pmod{a^2}, b \equiv 0 \pmod{b}} 1 = \sum_{n \equiv 0 \pmod{a^2 \cdot b}} \mu(a)g(b) \left( \frac{x - H}{[a^2, b]} + O(1) \right)
\]
\[
= T_3 + T_4.
\]
Now,
\[
T_3 = (x - H) \sum_{(a^2, b) | h} \mu(a)g(b) \frac{H}{[a^2, b]} + O \left( x \sum_{|a^2, b| \geq x} \frac{|g(b)|}{[a^2, b]} \right).
\]
In $T_3$, the main term is $(x - H)K(h)$ and the $O$-term is $O(E_1(x))$ by Lemma 3.7 (b).

Moreover,
\[
T_4 = O \left( \sum_{|a^2, b| \leq x} |g(b)| \right) = O(E_1(x)),
\]
by Lemma 3.6 (b). This completes the proof.
7. Appendix

We now sketch how $x^{\epsilon}$ could be saved from the error term in Theorem 2.5, if $\alpha$ is not in the neighbourhood of $1/2$. Let us recall the term $x^{\epsilon}$ occurs only in Lemma 3.4, and so we concentrate only on this lemma. Recall that in the proof Lemma 3.4, we have

$$S_{A,B} = \sum_{H \leq n \leq x} \sum_{a^2 | n} \mu(a) b^{-\alpha}$$

and

$$S = \sum_{A = 2^k \leq x^{1/2}} S_{A,B},$$

where $A$ and $B$ are powers of 2 satisfying $A \leq x^{1/2}$, $B \leq x$ and $A^2 B > x$.

Case I. $x^{0.05} \leq A \leq x^{0.45}$. In this case, we rewrite Lemma 3.4 as Claim 1.

Claim 1:

$$S_{A,B} \ll \frac{x^{1+\epsilon}}{AB^\alpha}.$$  

Summing over $A, B$ in this case, the sum is $O(E_1(x))$ if $\epsilon < 0.01 |1 - 2\alpha|$.

Case II. $A \leq x^{0.05}$. In this case, we make

Claim 2:

$$S_{A,B} \ll \frac{x (\log A)^{10}}{AB^\alpha}.$$  

To see this, since $a^2 | n$, we write $n = a^2 c$. Let

$$T = \{(a, b, c, d) : a^2 c - bd = h, a \sim A, b \sim B\} \quad (7.1)$$

Thus,

$$S_{A,B} \ll B^{-\alpha} |T|.$$  

Now, $bd = a^2 c - h \leq 2x$ and $a^2 b > x$. hence $d \leq 2a^2 \ll x^{0.1}$.

To count the number of elements in $T$, fix $a$ and $d$. Then the equation $a^2 c - h \equiv 0 \pmod{d}$ has at most $(a^2, d)$ solutions for $c \pmod{d}$. Since $c \leq \frac{x}{a^2}$, the total number of choices for $c$ is at most

$$\left(\frac{x}{a^2 d} + O(1)\right) (a^2, d).$$

Since $a \ll x^{0.05}$ and $d \ll x^{0.1}$, the $O$-term can be absorbed into the main term. Therefore,

$$|T| \ll \sum_{a \sim A} \frac{x (a^2, d)}{a^2 d},$$

and hence the claim.

Summing over the relevant $A$ and $B$, we get the desired estimate $O(E_1(x))$.

Case III. $A \geq x^{0.45}, B > x^{0.2}$. In this case, we use Claim 1 and sum over the appropriate range of $A$ and $B$ and we get the upper bound $O(E_1(x))$. 

Case IV. \( A \geq x^{0.45}, B \leq x^{0.2} \). In this case, we make

Claim 3:

\[
S_{A,B} \ll \frac{x(\log B)^{10}}{AB^\alpha}.
\]

As in the previous case, we need to estimate \(|T|\), with \( T \) as given in (7.1).

Since \( a^2c \leq x \) and \( a > x^{0.45} \), it follows that \( c < x^{0.1} \). We fix \( c \) and \( b \). Then from Lemma 3.9, the equation \( a^2c - h \equiv 0 \pmod{b} \) has at most \( L(b)\tau(b) \) solutions in \( a \pmod{b} \). Since \( a \sim A \), the total number of choices for \( a \) is at most

\[
\left( \frac{A}{b} + O(1) \right) L(b)\tau(b).
\]

Again, the \( O \)-term can be ignored. Since \( a^2c \leq x \), we get \( c \ll \frac{x}{A^2} \). Thus, summing the above with \( c \ll \frac{x}{A^2} \) and \( b \sim B \), the claim follows.

Now, summing over \( A, B \) in the desired range, we obtain the required bound.

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References

[BG] R Balasubramanian and S. Giri, Mean-Value Of product Of Shifted Multiplicative functions and average number of points on Elliptic curves, J. Number Theory 157 (2015), 37-53.

[Ca] L. Carlitz, A note on the composition of arithmetic functions. Duke Math. J. 33 (1966), 629-632.

[CS] E.-H. Choi, W. Schwarz, Mean-values of products of shifted arithmetical functions. Analytic and probabilistic methods in number theory (Palanga, 2001), 32-41, TEV, Vilnius, 2002.

[Ka] I. Katai, On the distribution of arithmetical functions., Acta Math. Acad. Sci. Hungar. 20 (1969), 69-87.

[Mi] L. Mirsky, Summation formulae involving arithmetic functions, Duke Math. J. 16, (1949), 261-272.

[CMS] G. Coppola, M.R. Murty, B. Saha, On the error term in a Parseval type formula in the theory of Ramanujan expansions II, J. Number Theory 160 2016, 700-715.

[Re] D. Rearick, Correlation of semi-multiplicative functions. Duke Math. J. 33 (1966), 623-627.

[SS] J. Siaulys and G. Stepanauskas, On the Mean Value of the Product of Multiplicative Functions with Shifted Argument., Monatsh. Math.150, (2007), 343-351.

[S1] G. Stepanauskas, The mean values of multiplicative functions. II, Lithuanian Math. J. 37 (1997), 162-170.

[S2] G. Stepanauskas, The Mean Values of Multiplicative Functions on Shifted Primes, Lithuanian Math. J. 37 (1997), 443-451.

Department of Mathematics, Institute of Mathematical Sciences, Chennai, India-600113

E-mail address, R. Balasubramanian: balu@imsc.res.in

School of Mathematics, Tel Aviv University, P.O.B. 39040, Ramat Aviv, Tel Aviv 69978, Israel.

E-mail address, Sumit Giri: sumit.giri199@gmail.com

Department of Mathematics, Institute of Mathematical Sciences, Chennai, India-600113

E-mail address, Priyamvad Srivastav: priyamvads@imsc.res.in