Scattering of a relativistic scalar particle by a cusp potential

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We solve the Klein-Gordon equation in the presence of a spatially one-dimensional cusp potential. The scattering solutions are obtained in terms of Whittaker functions and the condition for the existence of transmission resonances is derived. We show the dependence of the zero-reflection condition on the shape of the potential. In the low momentum limit, transmission resonances are associated with half-bound states. We express the condition for transmission resonances in terms of the phase shifts.

PACS numbers: 03.65.Pm, 03.65.Nk, 03.65.Ge

The study of low-momentum scattering of nonrelativistic particles by one-dimensional potentials is a well studied and understood problem [1]. Here we have that, as momentum goes to zero, the reflection coefficient goes to unity unless the potential \( V(x) \) supports a zero energy resonance. In this case the transmission coefficient goes to unity, becoming a transmission resonance [2]. Recently, this result has been generalized to the Dirac equation [3], showing that transmission resonances at \( k = 0 \) in the Dirac equation take place for a potential barrier \( V = V(x) \) when the corresponding potential well \( V = -V(x) \) supports a supercritical state. Kennedy [4] has shown that this result is also valid for a Woods-Saxon potential. More recently, transmission resonances and half-bound states have been discussed for a Dirac particle scattered by a cusp potential [5, 6] as well as for a class of short range potentials [7]. The bound states for scalar relativistic particles satisfying the Klein-Gordon equation are qualitatively different from the previous case. Here, for short-range attractive potentials the Schiff-Snyder effect [8–12] takes place, i.e for a given potential strength two bound states appear, one with positive norm and another with negative norm. Such states can be associated with a particle-antiparticle creation process. No antiresonant states appear [11, 12].

The absence of resonant overcritical states for the Klein-Gordon equation in the presence of short-range potential interactions does not prevent the existence of transmission resonances for given values of the potential.

Quantum effects associated with scalar particles in the presence of external potentials have been extensively discussed in the literature [10, 11]. Among quantum effects, we have that transmission resonance is one of the most interesting phenomena. For given values of the energy and the proper choice of the shape of the effective barrier, the probability of transmission reaches a maximum such as that obtained in the study of superradiance [14], where the amplitude of the scattered solutions by a rotating Kerr black hole is even larger than the amplitude of the incident wave. Analogous phenomena can also be obtained due to the presence of strong electromagnetic potentials [15].

Recently, transmission resonances for the Klein-Gordon equation in the presence of a Woods-Saxon potential barrier have been computed [10]. The transmission coefficient as a function of the energy and the potential amplitude shows a behavior that resembles the one obtained for the Dirac equation [3]. This result also holds for the square potential [11].

In this Letter we discuss the scattering of a Klein-Gordon scalar particle by the vector cusp potential [2]

\[ eA^0(x) = V(x) = \begin{cases} V_0 e^{x/a} & \text{for } x < 0, \\ V_0 e^{-x/a} & \text{for } x > 0. \end{cases} \]

(1)

The potential [11] vanishes exponentially for large values of \( x \), the parameter \( V_0 \) determines the strength of the barrier and the constant \( a \) defines the width of the potential. The cusp potential [11] reduces to a repulsive delta interaction of strength \( g \) in the limit \( 2av_0 \to g \) as \( a \to 0 \). It is the purpose of the present Letter to compute the scattering solutions of the one-dimensional Klein-Gordon equation in the presence of the cusp vector potential and show that one-dimensional scalar wave solutions exhibit transmission resonances with a functional dependence on the shape and strength of the potential similar that obtained for the Dirac equation [3]. The cusp vector potential [11] does not possess a square barrier limit and consequently the phase shift \( \delta \), associated with the transmission amplitude, cannot be directly identified with the positions of the transmission resonances. [11]

The one-dimensional Klein-Gordon equation, minimally coupled to a vector potential \( A^\mu \) can be written as

\[ \eta^{\alpha\beta}(\partial_\alpha + ieA_\alpha)(\partial_\beta + ieA_\beta)\phi + \phi = 0, \]

(2)

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where the metric $\eta^{\alpha\beta} = \text{diag}(1, -1)$ and here and thereafter we choose to work in natural units $\hbar = c = m = 1$.

Since the potential $V(x)$ in Eq. (11) does not depend on time, we have that $\phi = \phi(x) \exp(-iEt)$, and the problem of solving the one-dimensional Klein-Gordon equation (2) reduces to that of finding solutions to the second-order differential equation (10)

$$\frac{d^2 \phi(x)}{dx^2} + \left[(E - V(x))^2 - 1\right] \phi(x) = 0. \quad (3)$$

Let us consider the scattering solutions for $x < 0$ with $E^2 > 1$ of the Klein-Gordon equation. We proceed to solve the differential equation

$$\frac{d^2 \phi_L(x)}{dx^2} + \left[(E - V_0e^{x/a})^2 - 1\right] \phi_L(x) = 0. \quad (4)$$

On making the variable change $y = 2iaV_0e^{x/a}$, Eq. (4) becomes

$$y \frac{d}{dy} \left(y \frac{d \phi_L}{dy}\right) - \left[(iaE - y/2)^2 + a^2\right] \phi_L = 0. \quad (5)$$

Setting $\phi_L = y^{-1/2}f(y)$, Eq. (5) reduces to the Whittaker equation (17), p. 505

$$\frac{d^2 f(y)}{dy^2} + \left[\frac{1}{4} + \frac{iaE}{y} + \frac{1/4 - \mu^2}{y^2}\right] f(y) = 0. \quad (6)$$

The general solution of Eq. (6) can be written as

$$\phi_L(y) = c_1y^{-1/2}M_{\kappa,\mu}(y) + c_2y^{-1/2}M_{\kappa,-\mu}(y), \quad (7)$$

where $M_{\kappa,\mu}(y)$ is the Whittaker functions (17), p. 505 and

$$\kappa = iaE, \quad \mu = ia\sqrt{E^2 - 1}. \quad (8)$$

Now we consider the solution for $x > 0$. In this case, the differential equation to solve is

$$\frac{d^2 \phi_R(x)}{dx^2} + \left[(E - V_0e^{-x/a})^2 - 1\right] \phi_R(x) = 0. \quad (9)$$

On making the variable change $z = 2iaV_0e^{-x/a}$, Eq. (9) can be written as

$$\frac{d}{dz} \left(2iaV_0e^{-x/a} \frac{d \phi_R}{dz}\right) - \left[(iaE - z/2)^2 + a^2\right] \phi_R = 0. \quad (10)$$

Putting $\phi_R = z^{-1/2}g(z)$ we obtain the Whittaker differential equation

$$\frac{d^2 g(z)}{dz^2} + \left[-\frac{1}{4} + \frac{iaE}{z} + \frac{1/4 - \mu^2}{z^2}\right] g(z) = 0. \quad (11)$$

whose solution is

$$\phi_R(z) = c_3z^{-1/2}M_{\kappa,-\mu}(z) + c_4z^{-1/2}M_{\kappa,\mu}(z), \quad (12)$$

Using the asymptotic behavior of the Whittaker function $M_{\kappa,\mu}(y) \to e^{-y/2}y^{1/2+\mu}$, as $y \to 0$ (17), p. 504, we can write the the incoming wave solution $\phi_{\text{inc}}(x)$ in the form

$$\phi_{\text{inc}}(x) = c_1(2iaV_0)^{-1/2}e^{-x/2a}M_{\kappa,\mu}(2iaV_0e^{x/a}). \quad (13)$$

As $x \to -\infty$, $\phi_{\text{inc}}$ behaves like a plane wave traveling to the right

$$\phi_{\text{inc}} \to c_1(2iaV_0)^\mu e^{i\sqrt{E^2 - 1}x}. \quad (14)$$
Analogously, we have that the reflected \( \phi_{\text{ref}}(x) \) solution can be written as

\[
\phi_{\text{ref}}(x) = c_2 (2iaV_0)^{-1/2} e^{-x/2a} M_{\kappa,-\mu}(2iaV_0 e^{x/a}).
\]  

(15)

As \( x \to -\infty \), \( \phi_{\text{ref}}(x) \) has the asymptotic behavior

\[
\phi_{\text{ref}} \to c_2 (2iaV_0)^{-1/2} e^{-i\sqrt{E^2 - 1}x}.
\]  

(16)

Finally, using the right solution \( \phi_R \) [12], we have that the transmitted solution \( \phi_{\text{trans}}(x) \) can be expressed as

\[
\phi_{\text{trans}}(x) = c_3 (2iaV_0)^{-1/2} e^{x/2a} M_{\kappa,-\mu}(2iaV_0 e^{-x/a}),
\]  

(17)

with \( c_4 = 0 \). As \( x \to \infty \), \( \phi_{\text{trans}}(x) \) takes the asymptotic plane wave behavior

\[
\phi_{\text{trans}} \to c_3 (2iaV_0)^{-1/2} e^{i\sqrt{E^2 - 1}x}.
\]  

(18)

The electrical current density for the one-dimensional Klein-Gordon equation [2] is given by the expression:

\[
j^\mu = \frac{i}{2} \left( \phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right)
\]  

(19)

The current as \( x \to -\infty \) can be decomposed as \( j_L = j_{\text{inc}} - j_{\text{refl}} \) where \( j_{\text{inc}} \) is the incident current and \( j_{\text{refl}} \) is the reflected one. Analogously we have that, on the right side, as \( x \to \infty \) the current is \( j_R = j_{\text{trans}} \), where \( j_{\text{trans}} \) is the transmitted current.

Using the reflected \( j_{\text{refl}} \) and transmitted \( j_{\text{trans}} \) currents, we have that the reflection and transmission coefficients \( R \) and \( T \) can be expressed as

\[
R = \frac{j_{\text{refl}}}{j_{\text{inc}}}, \quad T = \frac{j_{\text{trans}}}{j_{\text{inc}}}
\]  

(20)

The quantities \( R \) and \( T \) are not independent, they are related via the unitarity condition \( R + T = 1 \).

In order to obtain \( R \) and \( T \) we proceed to equate at \( x = 0 \) the right \( \phi_R \) and left \( \phi_L \) wave functions and their first derivatives. From the matching condition we derive the following system of equations governing the dependence of coefficients \( c_1 \) and \( c_2 \) on \( c_3 \). Such set of equations we solved numerically

\[
c_1 M_{\kappa,\mu}(2iaV_0) + c_2 M_{\kappa,-\mu}(2iaV_0) = c_3 M_{\kappa,-\mu}(2iaV_0),
\]  

(21)

\[
c_1 \left[ \left( -\frac{1}{2a} + iV_0 - \frac{\kappa}{a} \right) M_{\kappa,\mu}(2iaV_0) \right] + \frac{1}{2a} \left( \frac{1}{2} + \mu + \kappa \right) M_{\kappa+1,\mu}(2iaV_0)
\]

\[
+ c_2 \left[ \left( -\frac{1}{2a} + iV_0 - \frac{\kappa}{a} \right) M_{\kappa,-\mu}(2iaV_0) \right] + \frac{1}{2a} \left( \frac{1}{2} - \mu + \kappa \right) M_{\kappa+1,-\mu}(2iaV_0) = 0,
\]  

\[
= -c_3 \left[ \left( -\frac{1}{2a} + iV_0 - \frac{\kappa}{a} \right) M_{\kappa,-\mu}(2iaV_0) \right] + \frac{1}{2a} \left( \frac{1}{2} - \mu + \kappa \right) M_{\kappa+1,-\mu}(2iaV_0).
\]  

(22)

The reflection coefficient \( R \), and the transmission coefficient \( T \), are calculated by

\[
R = \frac{j_{\text{refl}}}{j_{\text{inc}}} = \left| \frac{c_2 (2iaV_0)^{-\mu}}{c_1 (2iaV_0)^\mu} \right|^2,
\]  

(23)

and

\[
T = \frac{j_{\text{trans}}}{j_{\text{inc}}} = \left| \frac{c_3 (2iaV_0)^{-\mu}}{c_1 (2iaV_0)^\mu} \right|^2.
\]  

(24)

From the system of equations [21] and [22] we obtain that the condition for transmission resonances \( T = 1 \) can be written as
\[
\left( \frac{1}{2a} - iV_0 + \frac{\kappa}{a} \right) M_{\kappa,\mu}(2iaV_0)M_{\kappa,-\mu}(2iaV_0) - \frac{1}{2a} \left( \frac{1}{2} + \kappa + \mu \right) M_{\kappa+1,\mu}(2iaV_0)M_{\kappa,-\mu}(2iaV_0) - \frac{1}{2a} \left( \frac{1}{2} + \kappa - \mu \right) M_{\kappa+1,-\mu}(2iaV_0)M_{\kappa,\mu}(2iaV_0) = 0. \tag{25}
\]

For an attractive cusp potential \( V_0 < 0 \), the bound state solution of the Klein-Gordon equation can be written in terms of Whittaker functions as \[18\]
\[
\phi_L(y) = c_1(-y)^{-1/2}M_{-\kappa,\mu}(-y), \quad \phi_R(z) = c_2(-z)^{-1/2}M_{-\kappa,\mu}(-z), \tag{26}
\]
where \( c_1 \) and \( c_2 \) are normalization constants, and \( \phi_L \) and \( \phi_R \) correspond to \( x < 0 \) and \( x > 0 \) respectively. Imposing the continuity of the solution \[26\] and of its derivative at \( x = 0 \), we obtain the bound energy condition
\[
[(1 + 2iaV_0 - 2\kappa)M_{-\kappa,\mu}(-2iaV_0) - (1 - 2\kappa + 2\mu)M_{-\kappa+1,\mu}(-2iaV_0)]M_{-\kappa,\mu}(-2iaV_0) = 0. \tag{27}
\]
In the low momentum limit, the transmission resonance condition \[25\] reduces to the expression
\[
[(1 - 2iaV_0 + 2\kappa)M_{\kappa,0}(2iaV_0) - (1 + 2\kappa)M_{\kappa+1,0}(2iaV_0)]M_{\kappa,0}(2iaV_0) = 0, \tag{28}
\]
It is not difficult to verify that, in the low-momentum limit \( \mu = 0 \), after the transformation \( V_0 \rightarrow -V_0 \) and \( E \rightarrow -E \) equation \[28\] reduces to the bound energy condition \[27\] showing that, when the Klein-Gordon equation exhibits a low-momentum transmission resonance \( E = m \) in the presence of the repulsive cusp potential \[11\] the attractive cusp potential supporting a half-bound state for \( E = -m \). This state corresponds to an antiparticle emerging from the negative-energy continuum \[11, 18\].

\[\text{FIG. 1: Transmission coefficient versus the energy for a square barrier of height } V_0 = 4 \text{ and width } 2L = 2\]

It is instructive to compare the scattering of a scalar Klein-Gordon particle by a cusp potential with the scattering by a square barrier. The transmission amplitude for a square barrier of height \( V_0 \) and width \( 2L \) is
\[
t = \frac{2k_0k_1 e^{-2ik_0L}}{-i(k_0^2 + k_1^2)\sin(2k_1L) + 2k_0k_1 \cos(2k_1L)}, \tag{29}
\]
where
\[
k_0 = \sqrt{E^2 - 1}, \quad k_1 = \sqrt{(E - V_0)^2 - 1}. \tag{30}
\]
Low momentum transmission resonances for a square barrier of height $V_0$ and width $2L$ can be obtained analytically from equation (29), they satisfy the relation
\[ \sin(2k_1L) = 0, \quad \text{with} \quad k_0 = 0. \]

From equations (29)–(30) we have that, in the low momentum limit, a potential well of depth $V_0$ supports half-bound states at $E = -m$ if $V_0$ satisfies equation (31), as it was observed for the cusp potential.

Fig. 1 shows the transmission coefficient as a function of the energy for a square barrier of height $V_0 = 4$ and width $2L = 2$. It should be noticed that $T(E)$ exhibits a behavior analogous to the one observed for the Dirac particle by a Woods-Saxon potential and its square barrier limit.

From Figures 2 and 3, we can see that, analogous to the scalar and Dirac cases in a Woods-Saxon potential,
the Klein-Gordon equation exhibits transmission resonances in the presence of a cusp potential. Figure\textsuperscript{2} shows that, for \( V_0 = 4 \) and \( a = 1 \), a transmission resonance is reached for \( E = 1.694 \). Figure\textsuperscript{3} shows the existence of a transmission resonance for \( V_0 = 4.378 \). It should be noticed that in both cases the condition \( T = 1 \) is reached for states with an energy lower the potential strength \( V_0 \). It is worth mentioning that the cusp potential does not possess a square well limit. The form of the transmission resonances in Fig.\textsuperscript{4} shows that, in contrast to the Woods-Saxon potential, there is no potential strength \( V_0 \) making the cusp barrier completely impenetrable.

The study of the phase shift \( \delta \) of the transmission amplitude for the cusp potential

\[
t = \frac{c_0 (2iaV_0)^{-i\mu}}{c_1 (2iaV_0)^{i\mu}} = \sqrt{T} e^{i\delta}
\]  

(32)

shows that, in contrast to the square barrier potential case for the Klein-Gordon equation, there is no straightforward relation between the transmission resonances and the values for which the phase shift \( \delta \) goes through the value \( \pi/2 \) or becomes zero. In fact, using equation (29) we have that the phase shift \( \delta \) for the square barrier is

\[
\delta = -2k_0L + \arctan \left( \frac{(k_0^2 + k_1^2) \tan(2k_1L)}{2k_0k_1} \right)
\]

(33)

The condition for transmission resonances

\[
\sin(2k_1L) = 0,
\]

(34)

reduces the equation (33) to

\[
\delta + 2k_0L = 0 \pmod{\pi},
\]

(35)

an analogous result also takes place for Dirac particle in the presence of a square well potential (\textsuperscript{11} p. 67). In order to get a deeper insight of the nature of the phase shift, and since cusp potential is even, we proceed to study the scattering process in terms of solutions of the Klein-Gordon equation (3) with a given parity. Using a parity definite basis, we have that the transmission and reflection amplitudes expressed in terms of the corresponding phase shifts \( \delta_0 \) and \( \delta_1 \) are \textsuperscript{10}

\[
t = \cos(\delta_0 - \delta_1) e^{i(\delta_0 + \delta_1)}, \quad r = \sin(\delta_0 - \delta_1) e^{i(\delta_0 + \delta_1 + \pi/2)}.
\]

(36)

From equation (36) one can see that transmission resonances satisfy the condition

\[
\delta_0 - \delta_1 = 0 \pmod{\pi}
\]

(37)

For the square barrier we have that \( \delta_0 - \delta_1 \) is

\[
\delta_0 - \delta_1 = \arctan \left[ \frac{2(k_0^2 - k_1^2)}{k_0k_1} \sin(2k_1L) \right],
\]

(38)

the transmission resonance condition for the square barrier \textsuperscript{34} reduces the equation (38) to the condition (37).

The analytic computation of the phase shifts \( \delta_0 \) and \( \delta_1 \) for the cusp potential \textsuperscript{11} involves very cumbersome expressions, nevertheless, looking at the expressions in Eq. (36), it becomes clear that for arbitrary values of \( V_0 \) and \( a \), the transmission resonances in the cusp potential \textsuperscript{11} cannot be directly associated with the phase shift \( \delta = \delta_0 + \delta_1 \). Since the phase of the reflected wave is \( \delta + \pi/2 \), the knowledge of the phase shifts does not suffice for computing the transmission resonances of the Klein-Gordon equation in the presence of the cusp potential. Fig.\textsuperscript{4} shows the dependence of the transmission resonances on the phase shifts \( \delta_0 \) and \( \delta_1 \).

Fig.\textsuperscript{5} shows that, for a given value \( V_0 \) of the potential strength, the transmission resonance energy decreases as the cusp becomes sharper. The point interaction delta limit is illustrated in Fig.\textsuperscript{6}. In this case we observe that, as the potential cusp approaches to the Dirac delta, the energy necessary for a transmission resonance increases, diverging in the limit \( a \to 0 \).

From equation (33) we have that, in the low momentum limit, the phase shift \( \delta \) for the square barrier becomes zero if the potential strength satisfies the transmission resonance condition (34), otherwise the phase shift takes the value \( \delta = \pi/2 \). Looking at the system of equations (21) and (22), we obtain that this statement also holds for the cusp potential \textsuperscript{11}. Fig.\textsuperscript{5} shows that, for a given value \( V_0 \), not satisfying the condition (28), the transmission coefficient vanishes in the low momentum limit and the phase \( \delta \) reaches the value \( \delta = \pi/2 \).
We have obtained that, in the low-momentum limit, the transmission coefficient of Klein-Gordon particles incident on the cusp potential is $T = 1$ provided that the potential supports a half-bound state at $E = -m$. We have also shown that, analogous to the Dirac particle scattered by a square barrier, low-momentum transmission resonances of scalar particles are associated with supercritical states. Nevertheless, for the square well and the attractive cusp potential, half-bound states correspond to antiparticle states that join the bound particle state and they both disappear to create a neutral condensate.

In conclusion, in this Letter we have shown that the Klein-Gordon equation exhibits transmission resonances for potentials that do not possess a square-barrier limit. We have seen that, despite the fact that the supercritical behavior of the bound states in the one-dimensional cusp potential is qualitatively different from the one observed for Dirac particles, transmission resonances possess the same structure observed for the Dirac equation.
FIG. 6: Dependence of the transmission resonance energy on the shape parameter $a$ for a cusp potential with $2aV_0 = 1$, notice that the energy increases as the potential becomes sharper.

Acknowledgments

We thank Dr. Ernesto Medina for reading and improving the manuscript. This work was supported by FONACIT under project G-2001000712.

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