Thermodynamics in $f(R, \mathcal{L})$ Theories

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ABSTRACT: In this paper we study the recently proposed $f(R, \mathcal{L})$ theories from a thermodynamic point of view. The uniqueness of these theories lies in the fact that the spacetime curvature is coupled to the baryonic matter instead of exotic matter (in the form of scalar field). We investigate the viability of these theories from the point of view of the thermodynamic stability of the models. We consider several models of $f(R, \mathcal{L})$ theories where baryonic matter is coupled with spacetime curvature both minimally and non-minimally. Various thermodynamic quantities like entropy, enthalpy, internal energy, Gibbs free energy, etc. are computed and using their allowed ranges various model parameters are constrained. The study gives us an idea about the viability of these theories from a thermodynamic point of view.

KEYWORDS: Thermodynamics; modified gravity; baryonic matter, non-minimal coupling.
1 Introduction

Late time accelerated expansion [1, 2] of the universe is the biggest riddle of modern cosmology. Logically thinking, gravity being attractive in nature will tend to slow down the expansion of the universe in late times. But the observations are speaking a completely different story. Indeed the universe has entered into a phase of accelerated expansion and quite naturally there has been no satisfactory explanation to this phenomenon till date. Einstein’s theory of General Relativity (GR) is totally inconsistent with this phenomenon and we are left with no choice other than resorting to modifying the field equations of GR so that the modified equations can satisfactorily incorporate the accelerated expansion.

Till date all the attempts of modifying the Einstein’s field equations can be broadly classified into two categories. The first category incorporate exotic nature in the matter content of the universe, and is termed as dark energy. The second category modifies the gravity component of the equation, thus bringing about changes in the space-time geometry. This leads to the concept of modified gravity theory. Here we are interested in this second category which attempts at modifying the curvature of space-time.

The simplest model of modified gravity is the ΛCDM model where cold dark matter is coupled with the cosmological constant Λ. This cosmological constant has an antigravity effect that drives the accelerated expansion. A special class of models attempt to modify the gravitational lagrangian in the Einstein-Hilbert action by replacing \( \mathcal{L}_{GR} = R \) by an analytic function of the scalar curvature given by \( \mathcal{L}_{f(R)} = f(R) \). This model helps us to explore
the non-linear effects of scalar curvature in the evolution of the universe by considering arbitrary functions of $R$ in the gravitational lagrangian. Extensive reviews in $f(R)$ gravity can be found in the Refs. [3, 4]. Another class of models consider non-minimal coupling (NMC) between matter and curvature [5–9]. These models have been quite successful at explaining the post inflationary pre heating [10] and cosmological structure formation [11–13]. Further these models have also been able to successfully mimic dark energy [14–16] and dark matter [17–19].

Most models of NMC have incorporated coupling between curvature and scalar field [20–25]. But extension of this coupling to baryonic matter content has been very rare in literature. Recently, a dynamical system analysis approach was used to analyze a model that incorporated both $f(R)$ theories and a NMC with the baryonic matter content [26]. Ref. [27] extended this coupling to the baryonic matter content and studied a more general class of $f(R, \mathcal{L})$ theories via a dynamical system analysis. Here we are motivated to study the thermodynamical aspects of such $f(R, \mathcal{L})$ group of theories. The motivations to study such theories are quite obvious and lies in the fact that in these models baryonic matter couples with spacetime curvature. The literature is full with models involving coupling between spacetime curvature and exotic matter in the form of scalar field [20–25]. But coupling baryonic matter with curvature seems to be a comparatively alien topic to cosmologists. But logically this should be the more realistic scenario because of the non exotic nature of matter. To be able to describe the universe without resorting to exotic matter will be a very important step in cosmology. Therein lies the motivation in studying $f(R, \mathcal{L})$ theories.

It was back in 1970s when the physicists were first starting to understand that there was a deep connection between gravity and thermodynamics. The early form of these ideas was limited to the study of black hole (BH) thermodynamics. It was found that there was a link between the horizon area and the entropy of BHs. Since horizon area is a geometric quantity and entropy is a thermodynamic quantity, physicists became confident of the deep underlying connection between the Einstein’s field equations and thermodynamics [28]. Moreover the surface gravity of the BHs was found to be associated with its temperature and these quantities followed the first law of thermodynamics (FLT) [29]. Using the fact that entropy is proportional to the horizon area of BH and the first law of thermodynamics $\delta Q = T ds$, Jacobson in 1995 derived the Einstein’s field equations [30]. The literature is filled with studies connecting FLT with Einstein’s field equations in modified gravity theories. Cai and Kim in [31] derived the Friedmann equations from a thermodynamic point of view. Akbar in [32] discussed the relation between FLT and Friedmann equations in scalar-tensor theories and $f(R)$ gravity. Bamba studied the first and the second laws of thermodynamics in $f(R)$ gravity using the Palatini formalism [33]. He also studied the thermodynamics of cosmological horizons in $f(T)$ gravity in [34]. Wu et al studied the laws of thermodynamics for generalized $f(R)$ gravity with curvature matter coupling in an universe described by FRW equations [35]. The laws of thermodynamics at the apparent horizon of FRW spacetime in background of modified gravity theories is discussed in [36, 37]. In these studies non-minimal coupling between matter and spacetime geometry has been considered. Drawing motivations from the above works, here we intend to study
the thermodynamics in $f(R, \mathcal{L})$ gravity theory. We propose to explore the effects of both minimal and non-minimal coupling of matter and geometry on the thermodynamical aspects of the theory of gravity.

The paper is organized as follows. In section 2 we discuss $f(R, \mathcal{L})$ theories and the basic equations involved. Section 3 deals with the basic thermodynamical quantities to be studied for these models. In section 4 we discuss these quantities for general relativity where $R$ is coupled to $\mathcal{L}$ minimally. Section 5 deals with the thermodynamic study of various non-minimally coupled models. Finally the paper ends with a discussion and conclusion in section 6.

2 $f(R, \mathcal{L})$ theory of gravity

The action for the $f(R, \mathcal{L})$ theory \cite{27} is given by

$$I = \int d^4x \sqrt{-g} f(R, \mathcal{L})$$

(2.1)

where $f(R, \mathcal{L})$ is a function of both the scalar curvature $R$ and the matter Lagrangian density $\mathcal{L}$. As usual $g$ is the determinant of the metric tensor $g_{\mu\nu}$. This action is a much wider generalization of the Einstein-Hilbert action than the $f(R)$ theories. Here a non-minimal coupling (NMC) between the curvature and baryonic matter has been introduced in the action via the arbitrary function $f(R, \mathcal{L})$. If we put $f(R, \mathcal{L}) = \kappa (R - 2\Lambda) + \mathcal{L}$ (where $\kappa = c^4/(16\pi G)$ is a constant) we recover GR with a cosmological constant $\Lambda$. For any $f(R)$ modification we can consider $f(R, \mathcal{L}) = f(R) + \mathcal{L}$ and for NMC theories between matter and curvature we can consider $f(R, \mathcal{L}) = f_1(R) + f_2(R) \mathcal{L}$.

Varying the action with respect to the metric we get the field equations for the theory as

$$f^R G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} f - f^R R + \Delta_{\mu\nu} f^R + \frac{1}{2} f^{\mathcal{L}} (T_{\mu\nu} - g_{\mu\nu} \mathcal{L})$$

(2.2)

where $f^R = \frac{\partial f}{\partial R}$, $f^{\mathcal{L}} = \frac{\partial f}{\partial \mathcal{L}}$, and $\Delta_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box$. Hence, the energy momentum tensor is given by,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L})}{\delta g^{\mu\nu}}.$$  

(2.3)

The conservation equation for this theory turns out to be

$$\nabla^\mu T_{\mu\nu} = (g_{\mu\nu} \mathcal{L} - T_{\mu\nu}) \left( \frac{f^{R\mathcal{L}}}{f^R} \nabla^\mu R + \frac{f^{\mathcal{L}\mathcal{L}}}{f^R} \nabla^\mu \mathcal{L} \right)$$

(2.4)

This shows that the conservation equation is no longer covariantly conserved.

In order to study the cosmological evolution of the model we can consider the flat Friedmann-Robertson-Lemaitre-Walker (FLRW) line element in flat space, which is given by (in unit of light speed),

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2),$$

(2.5)

where $a(t)$ is the scale factor or the expansion factor of the universe \cite{38, 39}. In that case the Ricci scalar obtained in terms of scale factor as follow,

$$R = 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right).$$

(2.6)
It is possible to consider FLRW metric in non-flat universe as in \[39, 40\]. We can also consider that matter behaves like a perfect fluid whose energy-momentum tensor is given by,

\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu}. \] (2.7)

It has been found from the matter Lagrangian that \[41–43\],

\[ \mathcal{L} = -\rho. \] (2.8)

Here, \( \rho \) and \( p \) are the energy density and pressure of the fluid respectively, and \( u^\mu \) denotes the four-velocity of the fluid. In that case, by using the equation (2.4), one can obtain the following continuity equation,

\[ \dot{\rho} + 3H (1 + \omega) \rho = Q \] (2.9)

where

\[ Q = \frac{3}{8\pi G} H^2 f^R, \] (2.10)

where \( H = \frac{\dot{a}}{a} \) and \( \omega = \frac{p}{\rho} \) (\( p \) is fluid pressure) are the Hubble expansion and the equation of state (EoS) parameters respectively \[44\]. Clearly, in GR, where \( \dot{f}^R = 0 \), we have \( Q = 0 \).

Using the Hubble expansion parameter in the equation (2.6) one can obtain,

\[ R = 6(\dot{H} + 2H^2). \] (2.11)

Using the FLRW metric in the field equations one can find that the first Friedmann equation (the 00– component) as,

\[ H^2 = \frac{1}{3f^R} \left[ \frac{1}{2} f^R R - 3H f^{RR} \dot{R} - \frac{1}{2} f - 9H^2 f^{RL} (1 + \omega) \rho \right], \] (2.12)

Consequently the modified Raychaudhuri equation is given by,

\[ 2\dot{H} + 3H^2 = \frac{1}{2f^R} \left[ f^R R - f - f^C (1 + \omega) \rho - 2\dot{f}^R - 4H \dot{f}^R \right]. \] (2.13)

It is possible to rewrite the above equations as the following forms,

\[ H^2 = \frac{8\pi G_{\text{eff}}}{3} \rho, \] (2.14)

and

\[ \dot{H} = -4\pi G_{\text{eff}} (\rho + p), \] (2.15)

where \( G_{\text{eff}} = \frac{G}{f^R} \), and by comparison of (2.12) with (2.14) we obtain the following fluid density and pressure,

\[ \rho = \frac{\frac{R}{2} - H f^{RR} \dot{R} - \frac{f}{2f^R}}{8\pi G_{\text{eff}} + 9H^2 f^{RL} (1 + \omega)}, \] (2.16)

and

\[ p = -\frac{R f^R - f - 2\dot{f}^R - AH \dot{f}^R - f^C (1 + \omega) \left( \frac{\frac{R}{2} - H f^{RR} \dot{R} - \frac{f}{2f^R}}{8\pi G_{\text{eff}} + 9H^2 f^{RL} (1 + \omega)} \right)}{16\pi G_{\text{eff}} f^R}, \] (2.17)
where
\[ \omega = - \frac{(1 + 8\pi G_{\text{eff}})(Rf^R - f - 2\dot{f}^R - 4H\ddot{f}^R) - f\mathcal{L}}{(16\pi G_{\text{eff}}f^R) \left( \frac{R}{2} - H\dot{f}^R R - \frac{f}{27\pi R^3} \right) + Rf^R - f - 2\dot{f}^R - 4H\ddot{f}^R - f\mathcal{L}}, \] (2.18)
is EoS parameter. Now, combining the equations (2.8) and (2.14) give us the following relation for the matter Lagrangian,
\[ \mathcal{L} = \frac{3H^2 f^R}{8\pi G}, \] (2.19)
which clearly depends on gravity model. Here we are interested in studying the effect of this on thermodynamics of the model specially when matter Lagrangian couple to the curvature scalar.

3 Thermodynamics in \( f(R, \mathcal{L}) \) gravity

In this section we will compute the basic thermodynamic quantities in a model independent way. Using the allowable region for these quantities we can check the stability and viability of the model. We may also constrain some model parameters using these relations. The apparent horizon radius for FRW universe is given by,
\[ r_A = H^{-1}. \] (3.1)

Hence, the equation (2.11) yields to the following expression,
\[ R = \frac{6}{r_A^2}(2 - \dot{r}_A). \] (3.2)

Therefore, one can obtain,
\[ \dot{R} = 6\frac{(2\dot{r}_A^2 - r_A\ddot{r}_A)}{r_A^3}. \] (3.3)

and
\[ R' = -6\frac{r_A\ddot{r}_A + 2(2 - \dot{r}_A)}{r_A^3}, \] (3.4)
where prime denotes derivative with respect to \( r_A \). The associated temperature of the apparent horizon is given by [31],
\[ T = \frac{1}{4\pi}(2H + \frac{\dot{H}}{H}) = \frac{1 - \frac{r_A}{\tilde{r}_A}}{2\pi r_A}, \] (3.5)
while the entropy is,
\[ S = \frac{Af^R}{4G}, \] (3.6)
where \( A = 4\pi r_A^2 \) is apparent horizon area with volume \( V = \frac{4}{3}\pi r_A^3 \). Hence, all thermodynamic variables could be expressed in terms of apparent horizon area and its derivatives. Temperature variation with respect to horizon radius yields,
\[ \frac{dT}{dr_A} = \frac{1}{4\pi r_A^2} \left( \frac{r_A\ddot{r}_A - \dot{r}_A + 2}{r_A} \right), \] (3.7)
while entropy variation yields,

\[
\frac{dS}{dr_A} = \frac{\pi}{G} r_A \left(2 f_R + r_A \frac{df_R}{dr_A}\right). \tag{3.8}
\]

For example the specific heat, which is an important thermodynamic quantity to study model stability, is given by,

\[
C = T \frac{dS}{dT} = \frac{\pi}{G} \frac{(2 - \dot{r}_A)^2 \dot{r}_A (2 f + r_A \frac{df_R}{dr_A})}{r_A^2 \ddot{r}_A - \dot{r}_A^2 + 2 \dot{r}_A}. \tag{3.9}
\]

If the sign of specific heat is positive then the cosmological model is stable, and it helps us to realize the allowed region of the apparent horizon where the model is stable. By using the equation (3.6), one can obtain,

\[
dS = A d f R + f R d A, \tag{3.10}
\]

where \(dA = 8 \pi r_A dr_A\). In this case the internal energy (total energy) could be expressed as \([44]\),

\[
\hat{E} = \frac{3 f^R R^2}{8 \pi G} V = V \rho \tag{3.11}
\]

Therefore,

\[
d\hat{E} = V d\rho + \rho dV. \tag{3.12}
\]

Hence, the first law of thermodynamics is reduced to the following expression \([44]\),

\[
T d\hat{S} = d\hat{E} - \hat{W} dV, \tag{3.13}
\]

where

\[
d\hat{S} = dS + \frac{(1 - \frac{\pi r_A T}{2G} r_A)}{2G T} d f R, \tag{3.14}
\]

and

\[
\hat{W} = \frac{\rho - P}{2}. \tag{3.15}
\]

So, we can propose the corrected entropy as,

\[
\hat{S} = S - \frac{3}{G} \int \frac{(1 - \frac{\pi r_A T}{2G} r_A)}{T r_A^2} \left(2 \dot{r}_A (2 - \dot{r}_A) + r_A \ddot{r}_A\right) f^{RR} dt. \tag{3.16}
\]

Having time-dependent \(r_A\), we can calculate corrected entropy and other thermodynamic quantities. Therefore, by using the equation (3.9), the specific heat at constant volume (corrected specific heat) is given by the following relation,

\[
C_V = \left(\frac{T \frac{d\hat{S}}{dT}}{V}\right)_V = \left(\frac{d\hat{E}}{dT}\right)_V. \tag{3.17}
\]

Then, the specific enthalpy in unit mass is obtained via,

\[
h = \hat{E} + pV, \tag{3.18}
\]
which is in turn used to obtain specific Gibbs free energy,

\[ g = h - T \dot{S}, \quad (3.19) \]

and the Helmholtz free energy

\[ F = \dot{E} - T \dot{S}. \quad (3.20) \]

We will study the thermodynamics of this theory in detail for some specified models as examples in the sections to follow. We will basically examine three basic thermodynamic requirements in the chosen \( f(R, \mathcal{L}) \) models. First of all we would like to satisfy the first and second laws of thermodynamics, where the entropy is an increasing function of time. The second is cosmological point of view where the temperature of the universe is a decreasing function of time (with positive value). Finally we check thermodynamical stability of model by analyzing specific heat and other thermodynamic potentials. In summary, to have a well defined model, we should find entropy as an increasing function, \( T \geq 0 \) (and decreasing with time), \( C_V \geq 0 \), \( g \) as a decreasing function and \( F \) should have a minimum.

4 General relativity

Here we consider the simplest model of the \( f(R, \mathcal{L}) \) theories. We minimally couple \( R \) with \( \mathcal{L} \) in such a way that we eventually construct general relativity with a cosmological constant. A study of this model will simply help us to check the sanity of the model from the thermodynamical point of view. The model is given by,

\[ f(R, \mathcal{L}) = \kappa_1 (R - 2\Lambda) + \mathcal{L}, \quad (4.1) \]

where \( \kappa_1 \) is a constant. Hence, \( f^R = \kappa_1, \ f^\mathcal{L} = 1 \) are the only non-zero derivatives. In that case, by using the relations (2.19) and (3.1) one can find,

\[ \mathcal{L} = \frac{3\kappa_1}{8\pi G r_A^2}. \quad (4.2) \]

Hence, the equation (3.6) yields the following entropy,

\[ S = \frac{\pi \kappa_1}{G} r_A^2 = \dot{S}, \quad (4.3) \]

where in the last equality we used the equation (3.16). Therefore, energy density (2.16) is reduced to the following relation,

\[ \rho = \frac{\kappa_1}{8\pi G} \left( \Lambda - \frac{3}{16\pi G r_A^2} \right). \quad (4.4) \]

Also the pressure given by eqn.(2.17) gets reduced to the following relation,

\[ p = -\frac{\kappa_1}{16\pi G} \left( \Lambda - \frac{3}{16\pi G r_A^2} \right) \left( 2 - \frac{1 + \omega}{8\pi G} \right). \quad (4.5) \]
where $\omega = -1$ is obtained from the equation (2.18), because

$$1 + \omega = \frac{16\pi G_{\text{eff}} (\dddot{f} + 2H \dot{f} - H f^{RR} \dot{R})}{(16\pi G_{\text{eff}} f^{R}) \left( \frac{R}{2} - H \frac{f^{RR}}{f} \dot{R} - \frac{1}{2f} \right) + R f^{R} - f - 2f^{R} - 4H \dddot{f} - f^{E}} = 0. \quad (4.6)$$

This corresponds to the EoS for the $\Lambda$CDM model quite expectedly. Therefore, we have $p = -\rho$ and hence from the equation (3.15) we obtain $W = \rho$ which means,

$$\dot{W} = \frac{k_1}{8\pi G} \left( \Lambda - \frac{3}{16\pi G r_A^2} \right). \quad (4.7)$$

Hence, the internal energy is obtained as,

$$\dot{E} = \frac{k_1}{96\pi G^2} (16\pi G \Lambda r_A^2 - 3) r_A. \quad (4.8)$$

Satisfying the first law of thermodynamics gives us the following relation,

$$\dot{r}_A = 2 - \frac{1}{8\pi G} - r_A^2 (1 - r_A^2) (\Lambda - \frac{3}{16\pi G r_A^2}). \quad (4.9)$$

In the rationalized Planck units ($4\pi G = 1$) we can solve above equation numerically and the result is represented in Fig.1 (solid red line). We can see that the apparent horizon grows suddenly and settles to a constant value at the late time. The initial value of the apparent horizon radius can be fixed by using the positivity of specific heat which will be discussed below.

**Figure 1.** The figure shows the plot of apparent horizon radius in terms of $t$ with the general relativity condition for $\Lambda = 1$, $4\pi G = 1$ and $r_A(0) = 0.5$. 

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[Raw text not provided in the image]
By using the equation (3.17) one can obtain,

\[ C_V = \frac{2\pi \kappa_1}{G} \left( \frac{r_A^2 \dot{r}_A}{\dot{r}_A^2 - r_A \ddot{r}_A - 2 \dot{r}_A} \right) \left( \Lambda r_A^2 - \frac{1}{16\pi G} \right). \] (4.10)

In order to have positive \( C_V \), both numerator and denominator should be positive which yields to a lower and upper bounds for the apparent horizon radius (if we assume both negative, then we find discrete apparent horizon radius). From the numerator we can find,

\[ r_A^2 \geq \frac{1}{16\pi GA}. \] (4.11)

In the rationalized Planck units and under assumption \( \Lambda = 1 \) one can obtain \( r_A \geq 0.5 \). Hence, we used it as lower bound in the Fig.1. On the other hand, from denominator we have,

\[ r_A \ddot{r}_A - \dot{r}_A^2 + 2 \dot{r}_A < 0. \] (4.12)

We can solve above equation analytically to obtain,

\[ r_A < c_1 e^{2c_2 t} \frac{2 - 1.7e^{-0.6t}}{c_2}. \] (4.13)

Using the boundary condition \( (r_A \geq 0.5) \) we can write the following approximate apparent horizon radius,

\[ r_A \approx \frac{2 - 1.7e^{-0.6t}}{0.6}. \] (4.14)

This result has been indicated by the blue dashed line in Fig.1. It is indeed the apparent horizon radius where the model is stable and the first law of thermodynamics satisfied. Also, we find the temperature to be a decreasing function of time, making the model cosmologically viable. Finally it is quite straightforward to investigate the nature of thermodynamic quantities like internal energy (Fig.2(a)), Helmholtz free energy (Fig.2(b)), enthalpy (Fig.2(c)) and Gibbs free energy (Fig.2(d)). We see from the figures that Gibbs free energy is a decreasing function of time (see Fig.2(d)) and the Helmholtz free energy has a minimum (see Fig.2(b)). These indicate the stability of the model which is obtained by using the relation (4.14). Hence, by using the equation (3.1) we can obtain Hubble expansion parameter which is decreasing function of time.

5 Nonminimally coupled (NMC) Theories

Although minimally coupled theories are widely found in literature because of their computational convenience, it is believed that at high space-time curvatures non-minimal coupling will have a big role to play, specially in quantum field theory. In fact non-minimal coupling is introduced by quantum fluctuations and so they are almost non-existent in classical action \([45]\). The coupling is actually required if a scalar field theory is to be renormalized in a classical gravitational background \([46]\). Here we proceed to study some models where non-minimal coupling is in action. The results obtained from such models will be more generic and realistic in nature and hence very important for the present study.
5.1 Exponential model

An interesting model involving NMC coupling between matter and curvature is the exponential model [27, 47], given by

\[ f(R, \mathcal{L}) = M^4 e^{\frac{R}{6H_0^2} + \frac{\mathcal{L}}{6\kappa_2 H_0^2}}, \tag{5.1} \]

where \( \kappa_2 \) is a constant, \( M \) denotes mass scale, while \( H_0 \) is related to the expansion rate (current value of the Hubble parameter). Here we can see that on expanding the exponential function in series we will get terms of the form \( \xi f(R)g(\mathcal{L}) \) where \( \xi \) is a constant. So here the coupling between \( R \) and \( \mathcal{L} \) is non-minimal in nature. It is to be noted that this model does not simplify to general relativity with a cosmological constant for small \( R \) and \( \mathcal{L} \). In this case, by using the relations (2.19) and (3.1) one can find,

\[ \mathcal{L} e^{-\frac{\mathcal{L}}{6\kappa_2 H_0^2}} = \frac{3M^4}{48\pi Gr^2 H_0^2} e^{\frac{R}{6H_0^2}}, \tag{5.2} \]
which yields the following relation,
\[
\mathcal{L} = -6\kappa_2 H_0^2 W(X_e),
\tag{5.3}
\]
where \( W(X_e) \) is Lambert W function with the argument,
\[
X_e = -\frac{M^4}{96\pi\kappa_2 G r_A^2 H_0^2} e^{\frac{r}{6H_0^2}}.
\tag{5.4}
\]

If we assume \( H_0 \gg 1 \) (according to the latest observational data \( H_0 \approx 67 [48], \) hence our approximation is fairly reasonable) we can obtain the following approximate matter Lagrangian,
\[
\mathcal{L} = \frac{M^4}{16\pi G H_0^2 r_A^4} + \mathcal{O}(\frac{1}{H_0^4}).
\tag{5.5}
\]

In that case one can obtain,
\[
\dot{f}^R = \frac{f}{36H_0^2} (\dot{R} + \frac{\dot{\kappa}}{\kappa}),
\]
\[
\ddot{f}^R = \frac{f}{36H_0^2} (\ddot{R} + \frac{\ddot{\kappa}}{\kappa}) + \frac{f}{216H_0^4} (\dot{R} + \frac{\dot{\kappa}}{\kappa})^2,
\tag{5.6}
\]

Here \( \dot{R} \) is given by the equation (3.4) and
\[
\dot{\kappa} = 12H_0^2 \dot{r}_A - \dot{R} r_A W(X_e) \approx -\frac{M^4}{8\pi G H_0^2 r_A^4},
\tag{5.7}
\]

where in the last term we neglect \( \mathcal{O}(\frac{1}{H_0^4}) \). The leading order terms yield similar results as in the previous case (General relativity) and so we consider the first order approximation. Here the equation of state parameter is obtained as,
\[
\omega = -1
+ \frac{1}{36H_0^2 r_A^2} \left[ 12r_A \ddot{r}_A + (72r_A^2 - 576)r_A^3 - 192r_A^3 \dot{r}_A + 96\dot{r}_A^3 \right]
+ \frac{1}{H_0^2 r_A^2} \left[ 2r_A \ddot{r}_A - 2r_A (\dot{r}_A - 1)(r_A^2 + 4\dot{r}_A)\dot{r}_A + 8\dot{r}_A^4 \right]
+ \mathcal{O}(\frac{1}{H_0^4}).
\tag{5.8}
\]

Hence, one can obtain,
\[
\rho = \frac{3M^4}{4\pi G H_0^2 r_A^2} \left( \frac{r_A^3}{2} (2 - H_0^2 \dot{r}_A^2) - r_A^2 \dot{r}_A^2 + 4r_A \dot{r}_A (\dot{r}_A - 1)\ddot{r}_A - 4\dot{r}_A^4 + 8\dot{r}_A^3 + (\frac{\dot{r}_A}{\dot{H}_A} - 1)\dot{r}_A^3 \dot{r}_A \right)
- \frac{M^8}{256\kappa_2^2 G^2 H_0^2 r_A^2} + \mathcal{O}(\frac{1}{H_0^4}).
\tag{5.9}
\]

Therefore using \( p = \omega \rho \) and the equation (3.6) we get the following expression for entropy,
\[
S = \frac{\pi r_A^2 M^4}{6H_0^2} e^{\frac{r}{6H_0^2} + \frac{\dot{r}}{6\dot{r}_A}} \approx \frac{\pi M^4}{6G} \left( \frac{r_A^2}{H_0^2} - \frac{\dot{r}_A - 2}{H_0^2} \right),
\tag{5.10}
\]
where $R$ is given by the equation (3.2) and in the last expression we neglected $\mathcal{O}(\frac{1}{H_0^4})$. If we consider leading order terms and neglect $\mathcal{O}(\frac{1}{H_0^4})$, then we can reproduce entropy (4.3) where $\kappa_2$ is replaced by $\frac{M^4}{6H_0^6}$. Then, we can use the equation (3.15) to obtain,

$$\hat{W} = \frac{M^4}{16\pi G} + \mathcal{O}(\frac{1}{H_0^4}).$$

(5.11)

Putting all the above results in the equation (3.13) and using the power series method it is evident that the apparent horizon may be a polynomial of the form,

$$r_A = \sum_{n=0}^{n} a_n t^n = \frac{a_n}{t} (t^{n+1} - 1).$$

(5.12)

Unfortunately, this model yields negative temperature and specific heat at the late time which means that the model is unstable except the case $n = 1$, which corresponds to general relativity and leading order approximation. In the case of $n = 1$ the model begin at unstable phase and specific heat tends to zero at the late time. It is similar to the previous section where the lower bound for the apparent horizon was obtained. It is illustrated in Fig.3 where the variation of specific heat is shown w.r.t. temperature for different values of $n$. From the Fig.3 we can see that $n > 1$ leads to negative $C_V$ at the late time.

![Figure 3](image.png)

**Figure 3.** Specific heat at constant volume in terms of time $t$ for the exponential model with $M = 1$, $H_0 \approx 67$ and $4\pi G = 1$ in unit of $a_n$.

### 5.2 Power law Model

Here, we consider the model [27],

$$f(R, \mathcal{L}) = (\kappa M^2)^{-\epsilon} (\kappa_3 R + \mathcal{L})^{1+\epsilon},$$

(5.13)
where $\kappa_3$ is a constant, $M$ denotes mass scale, while $\varepsilon$ is infinitesimal correction parameter. At the $\varepsilon \to 0$ limit we recover GR with $\Lambda = 0$. Here for $\varepsilon \neq 0$ we get non-minimal terms on power expansion. Using the condition $\varepsilon \ll 1$ one can obtain,

$$\mathcal{L} \approx \frac{3}{8\pi G}(1 + \varepsilon)\kappa_3 M^{-2}\frac{(6(2 - \dot{r}_A))^{\varepsilon}}{r_A^{2(1+\varepsilon)}},$$

(5.14)

Power law model of cosmic expansion suggests the following expression for Ricci scalar,

$$R = R_0 t^n.$$  

(5.15)

For the $f(R)$ gravity the most favored value of $n$ is found to be $n = -2$ \cite{49}, but in $f(R, \mathcal{L})$ we can fix $n$ by using thermodynamic requirements. Combination of (3.2) and (5.15) give us apparent horizon as follow,

$$r_A = -2\sqrt{-3R_0} \frac{\left[ J_m(X_p) + c_1 Y_m(X_p) \right]}{R_0^{1/n} \left[ J_l(X_p) + c_1 Y_l(X_p) \right]},$$

(5.16)

where $c_1$ is the integration constant, $m = \frac{1}{n+2}$, $l = -\frac{n+1}{n+2}$ and

$$X_p = \frac{2\sqrt{-3R_0}}{3(n+2)} t^{1+\frac{n}{2}}.$$  

(5.17)

In the above expressions $J_m(X_p)$ is Bessel function of the first kind while $Y_m(X_p)$ is Bessel function of the second kind. In order to have real valued apparent horizon and other thermodynamic quantities, we should choose $c_1 = 0$ and we can analyze for both positive and negative Ricci scalar. Hence, we consider,

$$r_A = -2\sqrt{-3R_0} \frac{J_m(X_p)}{R_0^{1/n} J_l(X_p)},$$

(5.18)

and study thermodynamics for separate case of $R_0 > 0$ and $R_0 < 0$. By analyzing the temperature we can fix $n$ in the relation (5.15) and hence we can study all cosmological parameters. Actually we use two requirements: the temperature must be real positive and must be a decreasing function of time. We will discuss about these two conditions graphically in the next subsections.

### 5.2.1 $R_0 > 0$

In this case the curvature is positive ($R_0 > 0$) and we find a range for parameter $n$ where temperature is a decreasing function of time with a positive real value. It is illustrated by the Fig.4(a). We can see that for the $n < -2$ temperature is negative, which is unphysical. Also for the $n \geq 4$ temperature is not a decreasing function of time throughout the domain. Hence, we can choose $-2 < n < 4$. The special case of $n = -2$ corresponds to a singular point. Then, we can obtain entropy as follows,

$$S = -\frac{12\pi \kappa_3 (1 + \varepsilon)}{R_0 GM^{2\varepsilon} t^n} \left( \frac{J_m(X_p)}{J_l(X_p)} \right)^2 \left[ R_0^{2/n} - \frac{(1 + \varepsilon) R_0^{1+\varepsilon} t^{n(1+\varepsilon)}}{\pi GM^{2\varepsilon}} \frac{J_l(X_p)}{J_m(X_p)} \right]^\varepsilon.$$  

(5.19)
We can obtain further constraint on $n$ by analyzing entropy. Hence we draw entropy in terms of time in the Fig. 5(a) and expect that it is an increasing function of time to satisfy the second law of thermodynamics. According to the Fig. 5(a) we can see that $n > 0$ gives us a maximum for the entropy, while $-2 < n \leq 0$ yields an increasing entropy with time. In the case of $n = 0$ the entropy settles to a constant value which indicates the expansion of the universe coming to a halt. Hence, in order to have expanding universe we should choose $n = -1 \pm \epsilon$, where $\epsilon < 1$ is an infinitesimally small constant. Finally in Fig. 5(b) we can see the effect of $\epsilon$ on the entropy in power law model. From the above analysis it is quite natural to consider,

$$ R = \frac{R_0}{t} > 0. $$

which gives a well-defined thermodynamic relation.

![Figure 4](image)

**Figure 4.** Temperature in terms of $t$ for the power law model for $c_1 = 0$; (a) $R_0 = 1$, (b) $R_0 = -1$.

### 5.2.2 $R_0 < 0$

In the case of negative curvature ($R_0 < 0$) we find a range for parameter $n$ where temperature is totally a decreasing function of time and is real positive. From Fig. 4(b) it is seen that the allowable range is $-3 < n < -2$. Specially we can write $n = -2 - \delta$, where $\delta > 0$ and $\delta \to 0$ is an infinitesimally small constant. The best choice may be $\delta = 0.1$ which makes $n = -2.1$ as the best fit value. However, it results in imaginary entropy and specific heat, for $\epsilon \neq 0$, which is a sign of thermodynamic instability. Therefore, we may conclude that power law model of $f(R, \mathcal{L})$ gravity is thermodynamically unstable model for this case.

In the previous sections we considered models with non-minimal coupling between matter and curvature which was generated as a result of series expansion of the functions. Now, we will consider models involving non minimal coupling of the form

$$ f(R, \mathcal{L}) = \kappa_4 f_1(R) + f_2(R)\mathcal{L}, $$

(5.21)
where $\kappa_4$ is a constant. We will study two separate models of the above form namely the logarithmic model and Starobinsky’s model in the following subsections. In both these models there will be a different $f(R)$ term with a NMC term added to it. We name the models depending on the particular form of the $f(R)$ term that we have considered.

5.3 Logarithmic model

In relation (5.21) we consider the following expressions

\[ f_1(R) = a_4 \ln bR + c_4 R, \]
\[ f_2(R) = \alpha R, \quad (5.22) \]

where $a_4$, $b$, $c_4$ and $\alpha$ are some constants. Here $f_1(R)$ is taken in the logarithmic form and hence our logarithmic model takes the shape of

\[ f(R, \mathcal{L}) = \kappa_4 (a_4 \ln bR + c_4 R) + \alpha R \mathcal{L}, \quad (5.23) \]

where $\alpha$ plays role of the coupling parameter between matter and curvature. In case of $\alpha \neq 0$, we realize non minimal coupling. For $\kappa_4 = c_4 = 1$ and $a_4 = \alpha = 0$ we recover general relativity. Then, by using the relations (2.19) and (3.1) one can obtain,

\[ \mathcal{L} = \frac{3\kappa_4}{8\pi Gr_A^2 - 3\alpha} \left( \frac{a_4}{R} + c_4 \right). \quad (5.24) \]

Equation (3.6) yields the following expression for entropy,

\[ S = \frac{4\pi^2 \kappa_4}{3} \frac{r_A^2}{2 - r_A} \left( \frac{a_4 r_A^2 - 6c_4 r_A + 12c_4}{8\pi Gr_A^2 - 3\alpha} \right). \quad (5.25) \]
Therefore, using the equation (3.9) we have,

\[
C = \frac{64\pi^2 \kappa_4 r_A^4}{3(8\pi G r_A^2 - 3\alpha)^2 (\dot{r}_A - 2)^2 (r_A \ddot{r}_A - \dot{r}_A^2 + 2 \dot{r}_A)},
\]

where we defined,

\[
B = -\frac{r_A^3}{8}(8\pi G r_A^2 - 3\alpha)\ddot{r}_A
\]

\[- 3(2 - \dot{r})\dot{r} \left(-\frac{c_4}{4}(8\pi G r_A^2 - 9\alpha)\dot{r}_A + \pi G a_4 r_A^4 + (4\pi G c_4 - \frac{5}{8} a_4 \alpha) r_A^2 - \frac{9}{2} c_4 \alpha \right),
\]

(5.27)

Some simple calculations using equation (3.11) leads to

\[
\dot{E} = \frac{\kappa_4 r_A}{2G} \left[ \left( \frac{a_4 r_A^2}{6(\ddot{r}_A - 2)} + c_4 \right) \left( \frac{3\kappa_4 \alpha}{8\pi G r_A^2 - 3\alpha} + 1 \right) \right].
\]

(5.28)

Comparing (5.26) with (4.10) and requiring that both numerator and denominator should be positive, we get the following apparent horizon radius,

\[
r_A = \frac{2 - r_0 e^{-0.6 t}}{0.6},
\]

(5.29)

where \(r_0\) is the integration constant. The case of \(r_0 = 1.7\) coincides with the solution obtained from general relativity condition (see section 4). In that case the entropy (5.25) is an increasing function of time while temperature is a decreasing function with positive value for suitable choice of \(r_0\). In Fig. 6 we generate the trajectories for the equation of state (EoS) parameter which is obtained using the equation (2.18). It reflects typical behavior of the models. It is seen that the EoS parameter \(\omega\) tends to negative unity at the late times which is consistent with the observations. We can also see asymptotic behavior at early times for the case of \(\alpha = 0\) (minimal coupling).

It is obvious that the value of \(r_0\) is important to have a well defined model. Moreover we find small fluctuations in the temperature on variation of \(r_0\). Hence, for some values of \(r_0\) we have increasing temperature while some suitable values like \(r_0 = 1.7\) results in temperature to be a decreasing function of time. In fig. 7 the corrected entropy is plotted against time. We can see that corrected entropy may be negative at the early times which is a sign of instability. We can investigate such possibility by analyzing specific heat.

By using equations (3.5), (3.17) and (5.28) one can easily obtain specific heat at constant volume. In the plots of Fig. 8 we can see the behavior of specific heat for this model. In the case of \(r_0 = 1.7\) and \(\alpha > 0\) with \(a_4 = 1\) we have stable model at the late time. In the case of \(\alpha = 0\) the model is completely stable.

We can extend our study to thermodynamic potentials and find that in order to have a stable model we need a small \(b\), while \(r_0 \approx 1.7\) seems to be the best fit value as in the case of general relativity.
Figure 6. Equation of state parameter \( \omega \) in terms of time \( t \) for the NMC model with logarithmic \( f(R) \). The initial conditions are taken as \( \kappa_4 = a_4 = c_4 = b = 16\pi G = 1 \), and \( r_0 = 1.7 \).

Figure 7. Corrected entropy is plotted against time \( t \) for the logarithmic model. Entropy in case of general relativity is shown by dashed red line, corrected entropy (eqn.3.16) is shown by dotted blue line and solid green lines represent the NMC model with logarithmic form of \( f(R) \). The initial conditions are taken as \( \kappa_4 = c_4 = 16\pi G = 1 \) and \( r_0 = 1.7 \).
5.4 Starobinsky’s model

Here we consider the $f_1(R)$ in the form of the famous Starobinsky’s model [50, 51] as given below

$$f_1(R) = R + aR^2,$$
$$f_2(R) = b_1 R,$$  \hspace{1cm} (5.30)

where $a$ and $b_1$ are constants. This model is consistent with the inflationary scenario of early universe. Studying this model in a coupled form with matter will be very interesting. So, using the above expressions in equation (5.21) we get the ultimate model as,

$$f(R, \mathcal{L}) = \kappa_5(R + aR^2) + b_1 R \mathcal{L},$$  \hspace{1cm} (5.31)

Here $b_1$ is the coupling constant which plays role of a controlling parameter. For $b_1 \neq 0$ we get non-minimal coupling. Then, by using the relations (2.19) and (3.1) one can obtain,

$$\mathcal{L} = \frac{3\kappa_5 (1 + 2aR)}{8\pi G r_A^2 - 3b_1}.$$  \hspace{1cm} (5.32)

Further, one can obtain,

$$S = \frac{8\kappa_5 \pi^2 r_A^2 (24a + r_A^2 - 12a \dot{r}_A)}{8\pi G r_A^2 - 3b_1},$$  \hspace{1cm} (5.33)

and then,

$$\dot{S} = \frac{32\kappa_5 \pi^2 r_A}{(8\pi G r_A^2 - 3)^2} \left( 4\pi G r_A^3 \ddot{r}_A + 9ab_1 r_A \ddot{r}_A + 18ab_1 r_A^2 - 24\pi G r_A^2 \dot{r}_A - 3b_1 r_A^2 \dot{r}_A - 36ab_1 \dot{r}_A \right).$$  \hspace{1cm} (5.34)
The second law of thermodynamics require that $\dot{S} \geq 0$. Here we choose,

$$r_A = a_1 t + a_0.$$  \hfill (5.35)

In that case we obtain,

$$T = \frac{2 - a_1}{4\pi(a_1 t + a_0)}. \hfill (5.36)$$

Now, it is clear that $0 < a_1 < 2$ and $a_0 > 0$ yields positive temperature which is decreasing function of time. Therefore, by using the equation (3.16) one can obtain,

$$\dot{S} = \frac{8\kappa_5 \pi^2 (a_1 t + a_0) [a_1^2 t^2 + 2a_0 a_1 t + a_0^2 - 12aa_1 + 24a]}{2a_1^2 t^2 + 4a_0 a_1 t + 2a_0^2 - 3b_1} - 96\kappa_5 \pi^2 a a_1^2 t. \hfill (5.37)$$

![Nonminimal Starobinsky model](image)

**Figure 9.** Gibbs free energy in terms of $t$ for the NMC model with Starobinsky’s form of $f(R)$ by choosing $\kappa_5 = 1$, $a = b_1 = 1$ and $a_0 = 4$.

Satisfying the second law of thermodynamics helps to further constrain the model parameters as,

$$a_0 \geq \sqrt{6 + 6\sqrt{1 + 16 a}}, \hfill (5.38)$$

which suggest $a \geq 0$. By suitable choice of model parameters we can have $F$ with a minimum value and $g$ as a decreasing function of time. Also, we can find positive specific heat which decays with time. In the Fig.9 we represent the behavior of Gibbs free energy in terms of time and see that by suitable choice of model parameters the model can be made stable (Gibbs free energy is totally decreasing function.) In some cases we can see some critical points where,

$$\frac{dg}{dt} = 0, \quad \frac{d^2 g}{dt^2} = 0 \hfill (5.39)$$
It is illustrated by blue solid line of the Fig. 9 at $t = 2$.

6 Discussions and Conclusions

Here we have investigated the thermodynamic properties of $f(R, \mathcal{L})$ gravity, where space-time curvature $R$ is coupled with baryonic matter $\mathcal{L}$. Models involving both minimal and non-minimal coupling have been studied. In our investigation we considered thermodynamic parameters like temperature of the apparent horizon, entropy, specific heat at constant volume, total internal energy, enthalpy, Gibbs free energy and Helmholtz free energy. Depending on the results obtained we can comment on the stability of the model from the thermodynamic point of view. In order to have a well-defined model it should support an increasing entropy, non-negative temperature, non-negative specific heat, a decreasing Gibbs free energy and a minimum value of Helmholtz free energy. These characterize a realistic cosmological model.

We studied various models of $f(R, \mathcal{L})$ theory. The first model studied was general relativity, where the coupling between $R$ and $\mathcal{L}$ is not non-minimal in nature. The study showed that the model is consistent with the stability requirements which is quite obvious. Then we studied four different models involving non-minimal coupling between matter and curvature. The first one of these was the exponential model, which generates non-minimal terms on series expansion. From the study it was seen that the apparent horizon could be given in the form of a power series. It was also seen that for $n > 1$ leads to negative $C_V$ at late times which is not physical. Next we considered the power law model, which again yields non-minimal terms on expansion. For this model we considered power law form of curvature and computed various thermodynamic parameters for both positive and negative curvature. The trend of temperature $T$ and entropy $S$ with respect to time was investigated for this model for various values of the power law parameter $n$. From this the parameter $n$ was constrained to give realistic thermodynamically stable models. Logarithmic model was studied where a logarithmic form of $f(R)$ was considered with a non-minimal term. Various parameters were studied to check the thermodynamical viability of the model. Plots for entropy and specific heat was generated for various values of the coupling parameter $\alpha$. The results were compared with those of the general relativity model and was found to be considerable. Finally we studied an NMC model with Starobinsky’s form of $f(R)$. Here the trend of Gibbs free energy $g$ was checked with respect to time $t$. It was seen that $g$ was a totally decreasing function of time thus indicating the stability of the model. Other thermodynamic parameters were investigated under this model and choosing suitable initial conditions it was seen that it was possible to constrain the model parameters.

This study gives us a detailed thermodynamic prescription of the recently proposed $f(R, \mathcal{L})$ theories. Since we have considered different types of couplings between matter and curvature and kept the models as generic as possible the span of the work covers a large class of $f(R, \mathcal{L})$ theories. It is hoped that this work will considerably develop our understanding of $f(R, \mathcal{L})$ theories and enrich the existing literature on the topic. Cosmological viability of these models will be very important because of its non-exotic nature as discussed earlier.
Therefore a study on the various cosmological aspects of this theory will be very interesting and will be attempted in a future work.

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