Synchronization of periodic self-oscillators interacting via memristor-based coupling

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(Dated: July 3, 2018)

A model of two self-sustained oscillators interacting through memristive coupling is studied. Memristive coupling is realized by using a cubic memristor model. Numerical simulation is combined with theoretical analysis by means of quasi-harmonic reduction. It is shown that specifics of the memristor nonlinearity results in appearance of infinitely many equilibrium points, which form a line of equilibria in the phase space of the system under study. It is established that possibility to observe the effect of phase locking in the considered system depends both on parameter values and initial conditions. Consequently, boundaries of a synchronization area are determined by the initial conditions. It is demonstrated that addition of a small term into the memristor state equation gives rise to disappearance of the line of equilibria and eliminates the dependence of synchronization on the initial conditions.

PACS numbers: 05.10.-a, 05.45.Xt, 84.30.-r

Keywords: synchronization, adaptive coupling, line of equilibria, memristor

I. INTRODUCTION

A two-terminal element called "memristor" was initially introduced by Leon Chua as realization of a hypothesis of relationship between the electrical charge and the magnetic flux linkage [1]. Then the idea has been transformed into conception of "memristive system" [2], which includes mathematical definition and does not concern physical sense of dynamical variables and their functional dependence. It allows to combine systems with different nature into one group and to study their properties in an unified manner. At the present time, the term "memristor" means a two-terminal resistive element, whose resistance (or conductivity) depends on pre-history of operation. Typically, the current-voltage characteristic of a memristor driven by an external periodic influence represents a pinched hysteresis loop (see for example the characteristic of a cubic memristor model in Fig. 1(a)). There are known many experimental prototypes of the memristors. Development and exploration of such elements are attractive due to potential applications in electronics and neuroscience [3–9].

The memristor attracts attention of specialists in nonlinear dynamics because of its intrinsic properties, which can essentially change the dynamics of electronic oscillatory systems and are responsible for qualitatively new types of the behaviour. There are examples of memristor-based chaotic oscillators [10–13] and Hamiltonian systems including the memristor [14]. A variety of the effects in memristor oscillators is complemented by the existence of hidden attractors [15–17] and manifolds of equilibria (in the simplest case it is a line of equilibria) in the phase space [18–26].

The issue of the collective dynamics in ensembles of coupled oscillators with memristive coupling is of potential interest from perspective of nonlinear theory. It represents a distinguished class of problems concerning the influence of adaptive coupling. This topic is attractive in the context of neurodynamics due to an analogy between the memristor dynamics and the behaviour of a neural cell synapse [27–31]. A key step toward understanding of the dynamics of memristively coupled oscillators is consideration of the phenomenon of synchronization [32]. There are known publications addressing the synchronization of memristively coupled regular [33] and chaotic [34–37] self-oscillators. However, results of the mentioned publications do not allow to reveal distinctive features of the phenomenon as compared to classical synchronization of self-sustained oscillators coupled via dissipative coupling. In addition, the question on how a concrete type of the memristor nonlinearity impacts on the observed effects remains to be actual. Therefore the problem of the mutual synchronization of self-sustained oscillators interacting through the memristor is not studied in full.

In the present work we study synchronization of periodic self-sustained oscillations on the example of two Van der Pol self-sustained oscillators interacting through a memristor. First of all, we aim to answer the question on whether the synchronization through the memristor has intrinsic peculiarities as compared to synchronization in a case of resistive coupling. We combine numerical simulation with theoretical analysis by means of quasi-harmonic reduction.

II. SYSTEM UNDER STUDY

According to the paper [1] the memristor relates the transferred electrical charge, \( q(t) \), and the magnetic flux linkage, \( \varphi(t) \): \( dq = G_M \, d\varphi \). The following dependence is assumed to be satisfied in a model of the cubic memristor:

\[ q(t) = \int_0^t G_M(t') \, d\varphi(t') \]
q(\varphi) = \mu \varphi + \frac{1}{3} \nu \varphi^3. \quad \text{Then it follows:} \quad \frac{dG_M}{d\varphi} = \frac{d}{d\varphi} G_M(\varphi) = \mu + \nu \varphi^2. \quad (1)

In the following, the variable \varphi is considered as a state variable defined mathematically as being 

\varphi(t) = \int_{-\infty}^{t} U(t)dt \quad \text{and is not associated with a magnetic field. By using the formulas} \quad \frac{d\varphi}{dt} = U(t) \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{d}{dt} \int_{-\infty}^{t} U(t)dt \quad \text{we get the equation for \varphi:}

\frac{d\varphi}{dt} = \mu + \nu \varphi^2. \quad (2)

The system under study is depicted in Fig. 1(b). It consists of two coupled self-sustained oscillators. Each partial self-oscillator represents a parallel oscillatory circuit including the capacitor \( C \), the inductor \( L \), a resistor with the conductance \( g \), and the nonlinear element \( N \) with the N-type current-voltage characteristic described by the formula: \( i(U) = -\alpha U + \beta U^3 \). The dynamics of partial self-sustained oscillator is described by a model of the Van der Pol self-sustained oscillator. The memristive coupling is realized by the cubic memristor \( G_M(\varphi) \) and additional resistors with the conductances \( g_p \) and \( g_u \). Adjusting the conductances \( g_p \) and \( g_u \), one can change the summary conductance, which is responsible for the coupling strength and can be presented in the form \( kG_M(\varphi) \).

By using the Kirchhoffs current law the following differential equations for the voltages \( U_{1,2} \) across the capacitors \( C_{1,2} \) and the currents \( i_{1,2} \) through the inductances \( L_{1,2} \) can be derived:

\[
\begin{align*}
\frac{dU_1}{dt} &+ \frac{1}{C_1} i_1 + \frac{g_1}{C_1} U_1 + \frac{kG_M(\varphi)}{C_1} (U_1 - U_2) = 0, \\
\frac{dU_2}{dt} &+ \frac{1}{C_2} i_2 + \frac{g_2}{C_2} U_2 + \frac{kG_M(\varphi)}{C_2} (U_2 - U_1) = 0, \\
\frac{1}{C_1} \frac{di_1}{dt} &+ \frac{1}{C_1} i_1 = \frac{1}{C_1} U_1, \\
\frac{1}{C_2} \frac{di_2}{dt} &+ \frac{1}{C_2} i_2 = \frac{1}{C_2} U_2, \\
\frac{d\varphi}{dt} &+ \frac{1}{\omega_0} \varphi = \frac{1}{\omega_0} U_1 - U_2,
\end{align*}
\]

where \( t \) is a physical time. The following parameters are assumed to be equal: \( \alpha_1 = \alpha_2 = \alpha, \, \beta_1 = \beta_2 = \beta, \, g_1 = g_2 = g, \, C_1 = C_2 = C \). Let us denote \( \omega_1^2 = \frac{1}{L_1 C_1} \) and \( \omega_2^2 = \frac{1}{L_2 C_2} \), \( p = \omega_1^2/\omega_2^2 \) and introduce dimensionless time and variables as being:

\[
\begin{align*}
t &= \omega_1 t, \quad x_1 = \sqrt{\frac{\beta}{C_1 \omega_1}} U_1, \quad x_2 = \sqrt{\frac{\beta}{C_2 \omega_1}} U_2, \\
y_1 &= \frac{1}{\omega_1 C_1} i_1, \quad y_2 = \frac{1}{p \omega_1 C_2} i_2, \\
z &= \omega_1 \sqrt{\frac{\beta}{C_1 \omega_1}} \varphi.
\end{align*}
\]

Then the system (3) becomes:

\[
\begin{align*}
\dot{x}_1 + y_1 - (\gamma - x_1^2) x_1 + kG(z)(x_1 - x_2) &= 0, \\
\dot{x}_2 + py_2 - (\gamma - x_2^2) x_2 + kG(z)(x_2 - x_1) &= 0, \\
\dot{y}_1 &= x_1, \\
\dot{y}_2 &= x_2, \\
\dot{z} &= x_1 - x_2,
\end{align*}
\]
where \( \dot{x}_{1,2} = \frac{dx_{1,2}}{dt}, \dot{y}_{1,2} = \frac{dy_{1,2}}{dt}, \gamma = \frac{\alpha-g}{\sigma}, \)
\( G(z) = \frac{\mu+\nu z^2}{\kappa z}, \)
\( z \in (-\infty, \infty). \) It means the system \([4]\) has a line of equilibria in its phase space, i.e., each point on the axis \( OZ \) is an equilibrium point.

The dynamical variable \( z \) can be excluded from the system \([4]\). Indeed, it results from the last equation that \( \dot{z} = y_1 - y_2. \) Then one can derive \( z(t) = z(0) + y_1(t) - y_2(t) - y_1(0) + y_2(0). \) It means a value of the memristor conductance at any time depends on both instantaneous values \( y_1 \) and \( y_2 \) and initial values \( y_1(0), y_2(0) \) and \( z(0). \)

Hence, it follows that the system \([4]\) describes two interacting self-oscillators with dissipative coupling, whose strength depends on both instantaneous and initial values of dynamical variables:

\[
\begin{align*}
\dot{x}_1 + y_1 - (\gamma - x_1^2) x_1 &= kG(z(0) + y_1 - y_2 - y_1(0) + y_2(0)) (x_2 - x_1), \\
\dot{x}_2 + y_2 - (\gamma - x_2^2) x_2 &= kG(z(0) + y_1 - y_2 - y_1(0) + y_2(0)) (x_1 - x_2), \\
\dot{y}_1 &= x_1, \\
\dot{y}_2 &= x_2.
\end{align*}
\]

It gives rise to possibility to control the coupling strength by changing of the initial conditions. By this way one can induce (or destroy) the effect of phase locking at fixed values of the parameters.

### III. RESULTS

The system of two coupled Van der Pol self-oscillators \([4]\) is considered at fixed parameters of the memristor characteristic \( a = 0.02, b = 0.8 \) and the self-oscillation excitation parameter \( \gamma = 0.1. \) The coupling strength, \( k, \) and the frequency mismatch parameter, \( p, \) are variable.

#### A. Numerical modelling

Numerical simulations were carried out by the integration of Eqs. \([4]\) using the Runge-Kutta fourth-order method with the time step \( \Delta t = 0.001. \) Numerically obtained time realizations were used for plotting of phase portraits and calculation of an instantaneous phase of self-oscillations in partial systems. The instantaneous phase of self-oscillators \( \Psi_1(t) \) and \( \Psi_2(t) \) are calculated as being:

\[
\Psi_i(t) = \arctg \frac{y_i(t)}{x_i(t)} \pm \pi n(t), \quad i = 1, 2,
\]

where \( n(t) \) is an integer variable defined by the condition of phase continuity. Using the instantaneous phases, one can calculate the phase difference \( \Delta \Psi(t) = \Psi_2(t) - \Psi_1(t) \) and the mean difference frequency:

\[
\Omega = \lim_{T \to \infty} \frac{\Delta \Psi(t + T) - \Delta \Psi(t)}{T}.
\]

It is evident that the calculated quantity \( \Omega \) tends to zero in a region of synchronization. In a case of large coupling strength the synchronization can be realized through suppression of self-oscillations of either self-oscillator. Then calculation of \( \Omega \) using the formula \([7]\) gives rise to incorrect results.

The following results have been obtained by means of numerical modelling of the system \([4]\) in the absence of the frequency mismatch \((p = 1). \) The in-phase regime of synchronization corresponding to \( x_1(t) \equiv x_2(t), \)
\( y_1(t) \equiv y_2(t) \) is achieved at any positive values of the coupling strength \( k > 0 \) and any initial conditions. In this case interaction through a memristor leads to the same phenomenon as compared to usual dissipative coupling. The difference takes place only in the context of transient time. Duration of a transient process in the system \([4]\) strongly depends on initial conditions.

Let us consider the system \([4]\) in the presence of the weak frequency mismatch \((p \neq 1). \) Instantaneous phases of partial self-oscillators and the mean difference frequency \( \Omega \) (see the formula \([7]\)) were calculated in order to detect mutual phase and frequency locking. The dependence \( \Omega(p) \) allows to reveal an effect of phase and frequency locking and to estimate an interval of synchronization. It has been shown in numerical experiments that memristive coupling provides an opportunity for mutual locking of phases and frequencies of self-oscillators similarly to usual dissipative coupling. There is a certain interval of the frequency mismatch where the
FIG. 3: Phase trajectories of system (4) in the \((x_1,z)\) (the panel (a)) and \((x_1,x_2)\) (the panel (b)) planes. The red curves correspond to the regime of synchronization, the black trajectory traces quasi-periodical oscillations. Initial conditions are: 
\[x_1(0) = 0.5, \ y_1(0) = 0.5, \ x_2(0) = -0.3, \ y_2(0) = -0.1, \ z(0) = 1.5 \text{ (the curve 1)}, \]
\[x_1(0) = 0.5, \ y_1(0) = 0.5, \ x_2(0) = -0.3, \ y_2(0) = 0.1, \ z(0) = 0.4 \text{ (the curve 2)}, \]
\[x_1(0) = 0.5, \ y_1(0) = 0.5, \ x_2(0) = -0.5, \ y_2(0) = -0.4, \ z(0) = 0.0 \text{ (the curve 3)} \]
Parameters are: \(p = 1.05, \ k = 0.1, \ \gamma = 0.1, \ a = 0.02, \ b = 0.8\).

Mean difference frequency \(\Omega\) equals to zero. This effect was observed in electronic experiments described in the paper [33]. However, there is the significant feature of the synchronization via the memristor: width of the phase-frequency locking area continuously depends on the initial conditions. Numerically obtained dependences \(\Omega(p)\) corresponding to fixed coupling strength \(k = 0.02\) and different initial values of the variable \(z(0) = z_0\) are depicted in Fig. 2 (a). It is seen that boundaries of the synchronization area are essentially different for different values \(z_0\). The width of the synchronization area increases with growth of the absolute value \(|z_0|\). The dependence of the mean difference frequency on the initial value \(z_0 = z(0)\) indicates the influence of initial conditions [Fig. 2 (b)]. It was calculated in numerical experiments at fixed parameters \(p = 1.05, \ k = 0.1\) and different initial values of other variables. On each curve \(\Omega(z_0)\) depicted in Fig. 2 (b) one can distinguish an interval of values \(z_0\), where the effect of synchronization is absent. Boundaries of this interval vary depending on the initial values of other dynamical variables, but all the curves presented in Fig. 2 (b) have identical shape.

Projections of phase trajectories corresponding to occurrence or nonoccurrence of synchronization are depicted in Fig. 3. The trajectories were obtained from different initial conditions at the same parameter values. Two identical red closed curves in Fig. 3 illustrate synchronous oscillations. Projections of the synchronous oscillations are identical in the space of variables \(x_1, x_2, y_1, y_2\) (see for example Fig. 3 (b)). However, there is shift along the \(OZ\) axis in the full phase space (compare the curves 1 and 3 in Fig. 3 (a)). Projections of non-synchronous oscillations (the black trajectory in Fig. 3) trace a figure being topologically equivalent to two-dimensional torus. The figures obtained from different initial conditions have different shape.

The results presented above have shown that possibility to observe the regime of synchronization in the system (4) depends on initial conditions. After that the next question on whether characteristics of synchronous and non-synchronous oscillation continuously depend on initial conditions can be raised. It is known that continuous dependence of oscillation characteristics on initial conditions is typical for oscillators with a line of equilibria including memristor-based oscillators [18, 19, 24–26]. Therefore one can assume that boundaries of the synchronization area continuously depend on the initial conditions in some area on the plane \((p,k)\). However, this assumption requires detailed theoretical analysis of the model (4).

B. Theoretical analysis

Self-oscillations in partial self-oscillators of the system (4) are close to harmonic at small positive values of the parameter \(\gamma\). In such a case one can derive reduced equations for instantaneous amplitude and phase by means of the Van der Pol method. In terms of quasi-harmonic reduction a solution of Eqs. (4) is found in the following form:

\[y_{1,2}(t) = \text{Re}\left[\frac{1}{2} (a_{1,2}(t)e^{jt} + a_{1,2}^*(t)e^{-jt})\right],\]
\[x_{1,2}(t) = \frac{j}{2} (a_{1,2}(t)e^{jt} - a_{1,2}^*(t)e^{-jt}),\]
where \(a_1(t)\) and \(a_2(t)\) are instantaneous complex amplitudes of self-oscillations in the partial self-oscillators, \(a_1^*(t)\) and \(a_2^*(t)\) are complex conjugate functions, \(j\) is the imaginary unit. The amplitudes \(a_1(t)\) and \(a_2(t)\) are assumed to be slowly varying during the period of self-oscillations \(T_0 = 2\pi\). In addition, the following condition for the first derivatives is assumed to be satisfied: \(a_{1,2}e^{jt} + a_{1,2}^*e^{-jt} = 0\). The equation for the variable \(z(t)\) can be derived by using the last equation of the system (4) and substitution (8):

\[z(t) = z(0) + \int_0^t (x_1(\tau) - x_2(\tau))d\tau = z(0) + y_1(t) - y_2(t) + y_2(0) - y_1(0) = C_0 + \frac{1}{2}(a_1 - a_2)e^{jt} + \frac{1}{2}(-a_1^* - a_2^*)e^{-jt},\]
where \( C_0 = z(0) + y_2(0) - y_1(0) \) is a constant determined by an initial state of the system. Next, the expressions \( \vec{s} \) and \( \vec{z} \) are inserted into Eqs. \( \vec{A} \). Then a system of equations for complex amplitudes is derived by using the memristor characteristic and the condition for derivatives. After averaging of the complex amplitudes and their derivatives over the period \( T_0 \) the following system of reduced equations is developed:

\[
\dot{a}_1 = \frac{\gamma}{2} a_1 - \frac{3}{8} a_1 a_1^2 + \frac{k}{2}(a + bC_0^2)(a_2 - a_1) + \frac{k b}{\pi} |a_2 - a_1|^2(a_2 - a_1),
\]

\[
\dot{a}_2 = \frac{\gamma}{2} a_2 - \frac{3}{8} a_2 a_2^2 + \frac{k}{2}(a + bC_0^2)(a_1 - a_2) + \frac{k b}{\pi} |a_1 - a_2|^2(a_1 - a_2).
\]

(10)

The system (10) is presented as a system of equations for real amplitudes \( A_1, A_2 \) and phases \( \varphi_1, \varphi_2 \) by using substitution \( a_1, a_2 = A_{1,2} \exp[j\varphi_{1,2}] \):

\[
\dot{A}_1 = \frac{\gamma}{2} A_1 - \frac{3}{8} A_1^2 + \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right](a_2 - a_1),
\]

\[
\dot{\varphi}_1 = \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right] A_1/ A_2 \sin(\varphi_2 - \varphi_1),
\]

\[
\dot{A}_2 = \frac{\gamma}{2} A_2 - \frac{3}{8} A_2^2 + \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right](a_1 - a_2),
\]

\[
\dot{\varphi}_2 = \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right] A_2/ A_1 \sin(\varphi_2 - \varphi_1).
\]

Introducing of the phase difference \( \varphi = \varphi_2 - \varphi_1 \) allows to rewrite Eqs. (11) as being:

\[
\dot{A}_1 = \frac{\gamma}{2} A_1 - \frac{3}{8} A_1^2 + \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right](a_2 - a_1),
\]

\[
\dot{A}_2 = \frac{\gamma}{2} A_2 - \frac{3}{8} A_2^2 + \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right](a_1 - a_2),
\]

(12)

\[
\dot{\varphi} = \frac{k}{2} \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right] \left[ \frac{A_1}{A_2} + \frac{A_2}{A_1} \right] \sin \varphi,
\]

where \( \Delta = \frac{\gamma}{2} - 1 \).

Next, phase reduction is used for description of coupled self-oscillators. It means the real amplitudes of self-oscillators are assumed to be almost constant and equal to a stationary value in the absence of coupling:

\[
A_1 = A_2 = A_0 = \sqrt{\frac{4\gamma}{3}}.
\]

(13)

Using (13) and (12) we obtain the equation for the phase difference:

\[
\dot{\varphi} = \Delta - kF(\varphi) = \Delta - k \left[a + b \left(C_0^2 + \frac{A_1^2 + A_2^2}{4}\right)\right] \sin \varphi.
\]

(14)

If the expression in square brackets is changed to a constant \( \eta \), then Eq. (14) is transformed to the Adler equation describing synchronization of quasi-harmonic self-oscillators with dissipative coupling:

\[
\dot{\varphi} = \Delta - \Delta_s \sin \varphi, \quad \Delta_s = k\eta.
\]

(15)

The phase synchronization area corresponds to the existence of a stable solution \( \varphi_0 = \text{const} \). In this case the difference frequency becomes \( \Omega = \dot{\varphi} \equiv 0 \). Then boundaries of the synchronization area can be found in a case of Eq. (15):

\[
|\Delta| \leq \Delta_s.
\]

(17)

Outside the area of synchronization the mean difference frequency is determined by the known formula:

\[
\Omega = \langle \dot{\varphi} \rangle = \sqrt{\Delta_s^2 - \Delta_s^2}, \quad |\Delta| \geq \Delta_s.
\]

(18)

Where the brackets \( \langle ... \rangle \) mean the time-averaging operation. In a case of Eq. (14) the boundaries of the synchronization area cannot be calculated analytically. However, the term of Eq. (14) including \( \cos \varphi \) can be neglected in a case of a large absolute value of the constant \( C_0 \). Then Eqs. (16) and (17) are true and include the parameter \( \Delta_s \) determined by the formula:

\[
\Delta_s \approx k \left[a + b \left(C_0^2 + \frac{2\gamma}{3}\right)\right].
\]

(19)

Figure 4 (a) illustrates the function \( \eta \sin(\varphi) \) and the function \( F(\varphi) \), which determines the right part of Eq. (14). For chosen initial conditions corresponding to \( C_0 = -1.4 \) [Fig. 4 (a), the upper panel] and \( C_0 = -0.4 \) [Fig. 4 (a), the lower panel] the curves are almost identical. It allows to use the condition (16) and the formula (17) for estimation of the dependence \( \Omega(p) \). Results of theoretical analysis and results of numerical modelling of the system (4) are depicted in Fig. 5 (b). For chosen set of the parameters and initial conditions the similarity between the results of numerical modelling and analytical approach is evident.

Figure 5 also demonstrates good correspondence between results of numerical experiments and theoretical data. Figure 5 (a) shows the dependence of the mean difference frequency \( \Omega \) on the initial value \( z(0) = z_0 \) obtained in numerical modelling of the system (4) and the corresponding dependence obtained analytically. Boundaries of the synchronization area obtained by using the condition (16) for two values of the constant \( C_0 \) (the solid lines in Fig. 5 (b)) are close to the estimated in numerical experiments for the same initial conditions (the red
Dependences $\Omega(p)$ of mean difference frequency on frequency mismatch. Results of numerical experiment are shown by the red circles and corresponding theoretical curves (see Eq. (17)) are coloured in black. Initial condition $z(0)$ is $z(0) = -0.5$ (the upper panel) and $z(0) = 0.5$ (the lower panel). Other initial conditions are $x_1(0) = 0.5$, $y_1(0) = 0.5$, $x_2(0) = -0.5$, $y_2(0) = -0.4$. The parameters of the system (4) are $\gamma = 0.1$, $a = 0.02$, $b = 0.8$, $k = 0.02$.

circles in Fig. 5 (b)). If a value of the constant $C_0$ is close to zero, then the term of Eq. (14) including $\cos(\theta)$ cannot be neglected. As a result, visible difference between results of numerical modelling and theoretical approach involving formulas (16) and (17) appears. Nevertheless, the theoretical results presented above allow to conclude that continuous varying of the initial conditions $y_1(0)$, $y_2(0)$ and $z(0)$ in some intervals gives rise to continuous change of the quantity $\Delta_s$ (see Eq. (18)) and to continuous change of the boundaries of the phase locking area. Results of numerical modelling confirm this fact.

### IV. ROLE OF MEMRISTOR STATE EQUATION

Appearance of a line of equilibria in the phase space of the system (4) results from peculiarities of the memristor state equation (the last equation of the system (4)). The existence of the line of equilibria is a non-robust effect. It is difficult to imagine its implementation in real physical systems, which inevitable include sources of fluctuations and have their own intrinsic peculiarities [23–25]. The state equation of the system (4) is one of the simplest forms and follows from initial Chua’s introduction of the memristor [1]. In general, the state equation of the memristor can be more complex [2]. Let us consider how changing of a memristive coupling element model affects on the studied phenomenon. Further consideration of the system (4) is carried out for the modified last equation in the following form:

$$\dot{z} = x_1 - x_2 - \delta z,$$

(19)

where $\delta$ is a small parameter. Change of configuration of the memristor state equation results in disappearance of the line of equilibria at any non-zero value of the parameter $\delta$. There is one point of equilibrium in the phase space of the system with the modified last equation. Stability of the equilibrium point is determined by a sign of the parameter $\delta$. In case $\delta > 0$ perturbations along the axis $OZ$ are damped and stationary regimes do not depend on initial conditions. In case $\delta < 0$ the perturbations along the axis $OZ$ increase during time of observation and trajectories tend to $z = 0$ along the axis $OZ$.

In order to reveal the influence of the additional term $-\delta z$ in the memristor state equation, the system (4) with the state equation (19) has been considered in numerical experiments at $\delta = 0.01$. Figure 6 illustrates results of numerical modelling on the example of projections of the phase trajectories obtained from different initial conditions and fixed parameters $\gamma = 0.1$, $p = 1.05$, $k = 0.1$. Two sets of the initial conditions were used. The first one $(x_1(0) = 0.5$, $y_1(0) = 0.5$, $x_2(0) = -0.3$, $y_2(0) = -0.1$, $z(0) = 1.5)$ corresponds to the regime of synchronization in the system (4) with the last equation $\dot{z} = x_1 - x_2$ (see the red trajectory 1 in Fig. 3), while the second one $(x_1(0) = 0.5$, $y_1(0) = 0.5$, $x_2(0) = -0.3$, $y_2(0) = 0.1$, $z(0) = 0.4)$ induces the quasi-periodic dynamics (see the black trajectory 2 in Fig. 3). In a case of the system (4) with the modified last equation (19) both two sets of...
FIG. 6: Phase trajectories of system (1) with modified memristor state equation (19) in the \((x_1, z)\) (the panel (a)) and \((x_1, x_2)\) (the panel (b)) planes. Initial conditions are: \(x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = -0.3, y_2(0) = -0.1, z(0) = 1.5\) (the curve 1 in upper panels), \(x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = -0.3, y_2(0) = 0.1, z(0) = 0.4\) (the curve 2 in lower panels). Parameters are: \(p = 1.05, k = 0.1, \gamma = 0.1, a = 0.02, b = 0.8, \delta = 0.01\).

the initial conditions as well as any other set of the initial conditions reach the same oscillatory regime tracing a quasi-periodic attractor. However, one can observe a long transient process for an extremely small value of the parameter \(\delta\). Character of the transient process and its duration depend on the initial conditions.

V. CONCLUSIONS

Study of a model of two Van der Pol self-oscillators interacting through memristive coupling has shown intrinsic peculiarities of phase-frequency synchronization. The distinctive character of the synchronization is caused by features of the memristive coupling and is associated with the existence of a line of equilibria in the phase space. In a case of absolutely identical interacting self-oscillators (frequency mismatch is absent) a steady regime corresponds to in-phase oscillations in partial systems. Characteristics of oscillatory regimes depend on initial conditions in the presence of frequency mismatch. Starting from different initial conditions one can realize either regime of phase-frequency locking or the quasi-periodical dynamics at the same values of parameters. At the same time boundaries of the synchronization area continuously depend on initial conditions. Analytical results obtained by means of quasi-harmonic reduction have confirmed results of numerical experiments. Consequently, the presence of memristive coupling leads to special kind of the dynamics and allows to control the effect of synchronization by changing of the initial conditions. It has been shown that addition of a small term into the memristor state equation results in disappearance of the line of equilibria and destroys dependence of the synchronization on the initial conditions.

Acknowledgements

This work was supported by DFG in the framework of SFB 910 and by the Russian Ministry of Education and Science (project code 3.8616.2017/8.9).

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[38] More precisely, the variable $\phi_i$ is a slow-varying component of the full phase of self-oscillations $\Phi_i = t + \phi_i$, $i = 1, 2$. 