Inverse source problem and null controllability for multidimensional parabolic operators of Grushin type

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Abstract
The approach to Lipschitz stability for uniformly parabolic equations introduced by Imanuvilov and Yamamoto in 1998 based on Carleman estimates, seems hard to apply to the case of Grushin-type operators of interest to this paper. Indeed, such estimates are still missing for parabolic operators degenerating in the interior of the space domain. Nevertheless, we are able to prove Lipschitz stability results for inverse source problems for such operators, with locally distributed measurements in an arbitrary space dimension. For this purpose, we follow a mixed strategy which combines the approach due to Lebeau and Robbiano, relying on Fourier decomposition and Carleman inequalities for heat equations with non-smooth coefficients (solved by the Fourier modes). As a corollary, we obtain a direct proof of the observability of multidimensional Grushin-type parabolic equations, with locally distributed observations—which is equivalent to null controllability with locally distributed controls.

Keywords: inverse source problem, observability, null controllability, degenerate parabolic equations, Carleman estimates

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1. Introduction

1.1. Main results

The relevance of the Heisenberg group to quantum mechanics has long been acknowledged. Indeed, it was recognized by Weyl [22] that the Heisenberg algebra generated by the momentum and position operators comes from a Lie algebra representation associated with a corresponding group—namely the Heisenberg group (Weyl group in the traditional language of physicists). In such a group, the role played by the so-called Heisenberg Laplacian is absolutely central, being analogous to the standard Laplacian in Euclidean spaces, see [13]. On an even larger scale, deep connections have been pointed out between the properties of sub-Riemannian operators, like the Heisenberg Laplacian, and other topics of interest to current mathematical research such as isoperimetric problems and systems theory, see, for instance, [10].

Another important example of a sub-Laplacian is the Grushin operator which takes the form

$$ Gu = -\left(\partial_x^2 u + x^2 \partial_y^2 u\right) $$

on the plane. As a matter of fact, the Heisenberg Laplacian and the Grushin operator are deeply connected by an integral map, see [15].

This paper is a part of a general project which we are pursuing, and consists of investigating the possibility of extending the known controllability, observability and Lipschitz stability properties of the heat equation to degenerate parabolic problems. On all such topics, several results are available for parabolic operators which degenerate at the boundary of the space domain in low dimension, see, for instance, [1, 4–8].

In two space dimensions, a fairly complete analysis of the Grushin operator is presented in [2] as far as controllability and observability are concerned. To the best of our knowledge, there are no results on inverse source problems for Grushin-type equations. Specifically, we shall investigate the following questions.

- **The inverse source problem.** For \( g(t, x, y) \) given by \( R(t, x)f(x, y) \), is it possible to recover the source term \( f \) from \( \partial_t u|_{(T, T)} \times \omega \), where \( \omega \) is a nonempty open subset of \( \Omega \) and \( R \) is suitably given?
- **The null controllability problem.** Is it possible to steer the solution to zero by applying an appropriate control \( g(t, x, y) = v(t, x, v) \) localized on an open subset \( \omega \) of \( \Omega \)?

First, we recall the well-posedness and regularity results for such equations. To this aim, we introduce the space \( H^1_j(\Omega) \), which is the closure of \( C_0^\infty(\Omega) \) for the topology defined by the norm

$$ \| f \|_{H^1_j(\Omega)} := \left( \int_\Omega \left( |\nabla_x f|^2 + |x|^\gamma |\nabla_y f|^2 \right) \, dx \, dy \right)^{1/2} $$
and the Grushin operator \( G_\gamma \) defined by
\[
D(G_\gamma) := \{ f \in H^1_0(\Omega); \exists c > 0 \text{ such that } 
\int_{\Omega} (\nabla_x f \cdot \nabla_x g + |x|^{2\gamma} \nabla_x f \cdot \nabla_x g) \, dx \, dy \leq c\|g\|_{L^2(\Omega)} \text{ for all } g \in H^1_0(\Omega) \}.
\]
\[
G_\gamma u := -\Delta_x u - |x|^{2\gamma} b(x) \Delta_y u.
\]

**Proposition 1.** Let \( \gamma > 0 \). For every \( u_0 \in L^2(\Omega) \) and \( g \in L^2((0, T) \times \Omega) \), there exists a unique weak solution \( u \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \) of (2) such that
\[
u(0, x, y) = u_0(x, y) \quad (x, y) \in \Omega.
\]
Moreover, \( u \in C^0((0, T]; D(G_\gamma)) \).

We refer to [2] for the proof with \( N_1 = N_2 = 1 \); the general case can be treated similarly.

1.1.1. **Inverse source problem.** Taking a source term of the form
\[
g(t, x, y) = R(t, x) f(x, y) \quad \text{where } R \in C^0([0, T] \times \Omega) \quad \text{and } f \in L^2(\Omega)
\]
we will obtain the Lipschitz stability estimates for (2) in the following sense.

**Definition 1** (Lipschitz stability). Let \( T > 0 \), \( 0 \leq T_0 < T_1 \leq T \) and \( \omega \) be an open subset of \( \Omega \). We say that system (2) satisfies a Lipschitz stability estimate on \((T_0, T_1) \times \omega\) if there exists \( C > 0 \) such that, for every \( f \in L^2(\Omega) \) and \( u_0 \in L^2(\Omega) \), the solution of (2), (3) satisfies
\[
\int_{\Omega} |f(x, y)|^2 \, dx \, dy \leq C \left( \int_{T_0}^{T_1} \int_{\Omega} |\partial_t u(t, x, y)|^2 \, dx \, dy + \int_{\Omega} |G_\gamma u(T_1, x, y)|^2 \, dx \, dy \right).
\]

A few comments of the formulation of the above stability notion are now in order. In the inverse source problem, we are requested to determine a source \( g(t, x, y) \) by our data. We note that it is extremely difficult to prove even the uniqueness of the source when such a term depends on all the variables \( t, x, y \). Thus, we assume \( g \) to satisfy (4), which amounts to a sort of separation of variables. Such a form is justified by applications since, in some situations, one could be more interested in the determination of the spatial dependent factor \( f(x, y) \) of the source term \( g(t, x, y) \) than in the time-dependent factor \( R(t) \). This is the case, for instance, of a heat source that is generated by the decay of a radioactive isotope. Then, we know that \( R(t) \) is given by \( e^{-\lambda t} \) with \( \lambda > 0 \), and our main goal is to find the spatial distribution of the radioactive isotope.

Moreover, observe that, in definition 1, we require the known factor \( R(t, x) \) to be independent of \( y \). From the proof, however, it will be clear that our method can be easily adapted to recover a source of the form \( f = f(x) \) when \( R = R(t, x) \).

As is easily seen from the right-hand side of the above inequality, for the inverse problem discussed in this paper, measurements in \( \omega \) are taken before time \( T_1 \), and spatial data are measured in \( \Omega \) at the same time \( T_1 > 0 \). Since we assume \( T_1 > 0 \) and the initial value \( u_0 \) in (3) is unknown, our determination problem is not an inverse problem for the initial-boundary value problem (2)–(3). In general, one cannot expect the Lipschitz stability to hold for the inverse source problem for an initial-boundary problem with interior data in \( \omega \times (0, T) \) even in the case of uniformly parabolic equations. A fortiori, one can make the same conjecture for the inverse problem in the degenerate case of interest to this paper. For the inverse source problem for a uniformly parabolic equation, if one replaces the initial value by the value of \( u \) over \( \Omega \) at a positive time, then one can prove a Lipschitz stability estimate (see [16]).
below corresponds to such a result. In particular, even in the case of the heat equation, the full measurement of the solution on $\Omega$ is required by the technique of [16]. Furthermore, when one seeks for the on-line determination of the source, it is natural to assume that the identification process will start after some time has passed. In that case, since the initial value would be unknown, taking the data $G_p u(T_1, \cdot)\cdot$ at $T_1 > 0$ over the whole spatial domain seems reasonable from a practical point of view. Consequently, removing this term from the above right-hand or relaxing such a requirement by reducing the domain of measurement, remains a challenging open problem—just as it is for the heat operator—and will not be further discussed in this paper.

Finally, we would also like to point out that although for an ill-posed inverse problem it is quite frequent that one can prove the Hölder stability, the Lipschitz stability is a much stronger result. It is, therefore, of greater theoretical interest. Moreover, such a stability property can be very useful for numerical purposes. For example, when one applies Tikhonov’s regularization to a numerical scheme, Lipschitz stability guarantees the linear convergence of approximate solutions under suitable conditions (see, e.g., [11]).

When $\omega$ is a strip, parallel to the $y$-axis, we obtain the Lipschitz stability under general assumptions on $R$.

**Theorem 1.** Assume $\omega = \omega_1 \times \Omega_2$ where $\omega_1$ is an open subset of $\Omega_1$. Suppose further that

$$R, \partial_r R \in C^0([0, T] \times \overline{\Omega_1})$$

and there exist $T_1 \in (0, T]$ and $R_0 > 0$ such that $R(T_1, x) \geq R_0, \forall x \in \Omega_1$. (5)

1. If $\gamma \in (0, 1)$, then system (2) satisfies the Lipschitz stability estimate on $(T_0, T_1) \times \omega$ for every $T_0 \in [0, T_1)$.

2. If $\gamma = 1$, then there exists $T^* > 0$ such that system (2) satisfies the Lipschitz stability estimate on $(T_0, T_1) \times \omega$ for every $T_0 \in [0, T_1 - T^*)$.

**Remark 1.** In theorem 1 above, $T_1 - T_0$ is assumed to be sufficiently large when $\gamma = 1$. Indeed, in this case, the validity of a Lipschitz stability estimate on $(T_0, T_1) \times \omega$ is an open problem for a general $T_0 \in [0, T_1)$ even when $\omega$ is a strip, that is, $\omega = \omega_1 \times \Omega_2$.

On the other hand, it is known that Grushin’s operator with $\gamma = 1$ fails to be observable in arbitrary time, as an example from [2] shows. However, such a counterexample does not apply to the present context because of the source term in (4).

The validity of the Lipschitz stability estimate when $\gamma > 1$, is an open problem for the same reason.

When $\omega$ is an arbitrary open subset of $\Omega$ and $\gamma \in (0, 1)$, we can still prove Lipschitz stability under an additional smallness assumption of the source term, which is probably just due to technical reasons.

**Theorem 2.** Let $\gamma \in (0, 1)$ and $\omega$ be an open subset of $\Omega$. Then, for every $T_0 \in [0, T_1)$, there exists $\eta = \eta(T_0) > 0$ such that for every $R$ satisfying (5) and

$$\frac{1}{R_0} \left( \int_{T_0}^{T_1} \| \partial_r R(t) \|_{L^\infty(\omega)}^2 \, dt \right)^{1/2} < \eta$$

system (2) satisfies the Lipschitz stability estimate on $(T_0, T_1) \times \omega$. (6)

**1.1.2. Observability and null controllability.** In this paper, we are also interested in the observability problem for (2).
Theorem 4. We recall that the null controllability of (2), in the 2D case (i.e., 1.2. Motivation and bibliographical comments

Definition 2 (Observability). Let $T > 0$. System (2) is observable in $\omega$ in time $T$ if there exists $C > 0$ such that, for every $u_0 \in L^2(\Omega)$, the solution of

\[
\begin{align*}
\partial_t u - \Delta u - |x|^{2\gamma} b(x) \Delta_y u &= 0 \quad (t, x, y) \in (0, T) \times \Omega, \\
u(t, x, y) &= 0 \quad (t, x, y) \in (0, T) \times \partial \Omega, \\
u(0, x, y) &= u_0(x, y) \quad (x, y) \in \Omega,
\end{align*}
\]

satisfies

\[
\int_{\Omega} |u(T, x, y)|^2 \, dx \, dy \leq C \int_0^T \int_{\omega} |\nabla u(t, x, y)|^2 \, dx \, dy \, dt.
\]

As a corollary of the analysis developed for the proof of theorems 1 and 2 (see remarks 2 and 4), we obtain a direct proof of observability for Grushin-type parabolic equations. The following statement is a generalization to the multidimensional case of [2, theorem 2] (where $N_1 = N_2 = 1$ is assumed).

Theorem 3. Let $\omega$ be an open subset of $\Omega$.

1. If $\gamma \in (0, 1)$, then system (2) is observable in $\omega$ in any time $T > 0$.
2. If $\gamma = 1$ and $\omega = \omega_1 \times \Omega_1$ where $\omega_1$ is an open subset of $\Omega_1$ then there exists $T^* > 0$ such that for every $T > T^*$ system (2) is observable in $\omega$ in time $T$.

Note that we do not require $0 \in \omega$; the problem would be easily solved by cut-off function arguments, under such an assumption (see [2]). As a consequence, we deduce the following null controllability result.

Definition 3 (Null controllability). Let $T > 0$. System (2) is null controllable in time $T$ if, for every $u_0 \in L^2(\Omega)$, there exists $v \in L^2((0, T) \times \Omega)$ such that the solution of

\[
\begin{align*}
\partial_t u - \Delta u - |x|^{2\gamma} b(x) \Delta_y u &= v(t, x, y) \chi_\omega(x, y) \quad (t, x, y) \in (0, T) \times \Omega, \\
u(t, x, y) &= 0 \quad (t, x, y) \in (0, T) \times \partial \Omega, \\
u(0, x, y) &= u_0(x, y) \quad (x, y) \in \Omega,
\end{align*}
\]

satisfies $u(T, \cdot, \cdot) = 0$.

Here $\chi_\omega$ is the characteristic function of the set $\omega$.

Theorem 4. Let $\omega$ be an open subset of $\Omega$.

1. If $\gamma \in (0, 1)$, then system (2) is null controllable in any time $T > 0$.
2. If $\gamma = 1$ and $\omega = \omega_1 \times \Omega_2$ where $\omega_1$ is an open subset of $\Omega_1$ then there exists $T^* > 0$ such that for every $T > T^*$ system (2) is null controllable in time $T$.

Note that, when $\gamma > 1$, the observability on a horizontal strip $\omega = \omega_1 \times \Omega_2$ (with $\omega_1 \subset \subset \Omega_1$) fails for any $T > 0$ in dimension 2 (i.e. $N_1 = N_2 = 1$). Thus, observability is not expected to hold in higher space dimension.

1.2. Motivation and bibliographical comments

We recall that the null controllability of (2), in the 2D case (i.e., $N_1 = N_2 = 1$), is studied in detail in [2]. In particular, in 2D, null controllability:

- holds in any positive time $T > 0$ with controls supported in an arbitrary open set $\omega$ when $\gamma \in (0, 1)$.
- holds only in large time $T > T^* > 0$ when $\gamma = 1$ and $\omega := \omega_1 \times \Omega_2$ is a strip parallel to the $y$-axis, not containing the line segment $x = 0$, and
- does not hold when $\gamma > 1$. 

The goal of this paper is:
• to generalize the previous positive controllability results to the multidimensional case, and
• to prove a Lipschitz stability estimate for the inverse source problem, by adapting a method by Imanuvilov and Yamamoto [16], for the values of $\gamma$ for which null controllability holds.

Our formulation of the inverse problem corresponds to a single measurement (see also Bukhgeim and Klibanov [3] who first proposed a methodology based on the Carleman estimates). Following [3], many works have been published on this subject. For uniformly parabolic equations we can refer the reader, for example, to [16–18, 23], and the references therein (the present list of references is by no means complete). As for the inverse problems for degenerate parabolic equations, see [8, 9].

1.3. Structure of the paper

This paper is organized as follows.

Section 2 is devoted to the preliminary results concerning the well-posedness of (2), the Fourier decomposition of its solutions and the dissipation speed of the Fourier modes.

In section 3, we state a Carleman estimate for a heat equation with non-smooth coefficients, solved by the Fourier modes of the solution of (2).

Section 4 is devoted to the proof of the Lipschitz stability estimates, for the inverse source problem, i.e. theorems 1 and 2.

In the appendix we prove the Carleman estimate stated in section 3.

1.4. Notation

The Euclidean norm in $\mathbb{R}^N$ is denoted by $|.|$ for every $N \in \mathbb{N^*}$. The notation $\|\cdot\|$ refers to $L^2$-norms in the space variables $x$, $y$ or $(x, y)$, depending on the context. $\omega$ is the characteristic function of the set $\omega$. When there is no ambiguity, the integration elements $d\tau, dx$ or $dy$ are omitted, to simplify formulas.

2. Preliminaries

2.1. Well-posedness

Proposition 2. Let $\gamma \in (0, 1]$, $u_0 \in D(G_{\gamma})$ and $g \in H^1(0, T; L^2(\Omega))$. Then, the weak solution $u$ of (2) and (3) satisfies

$$\partial_t u \in C^0([0, T], L^2(\Omega)) \cap L^2(0, T; H^1_{\gamma}(\Omega)).$$

Consequently, the function $v := \partial_t u$ solves

\[
\begin{aligned}
\partial_t v - \Delta_x v - b(x)\partial_{\gamma}^2 \Delta_y v &= \partial_t g(t, x, y) & (t, x, y) \in (0, \infty) \times \Omega, \\
v(t, x, y) &= 0 & (t, x, y) \in (0, \infty) \times \partial \Omega, \\
v(0, x, y) &= -G_{\gamma} u_0(x, y) + g(0, x, y) & (x, y) \in \Omega.
\end{aligned}
\]  

(9)

2.2. Fourier decomposition

We introduce the operator $A$ defined by

$$D(A) := H^2 \cap H^1_0(\Omega_2), \quad A \psi := -\Delta_y \psi.$$
the non-decreasing sequence \((\mu_n)_{n \in \mathbb{N}^*}\) of its eigenvalues and the associated eigenvectors \((\varphi_n)_{n \in \mathbb{N}^*}\):
\[
\begin{cases}
-\Delta_y \varphi_n(y) = \mu_n \varphi_n(y) & y \in \Omega_2, \\
\varphi_n(y) = 0 & y \in \partial \Omega_2.
\end{cases}
\] (10)

**Proposition 3.** Let \(u_0 \in L^2(\Omega), g \in L^2((0, T) \times \Omega)\) and \(u\) be the solution of (2) and (3). For every \(n \in \mathbb{N}^*,\) the function
\[u_n(t, x) := \int_{\Omega_2} u(t, x, y) \varphi_n(y) \, dy\]
belongs to \(C^0([0, T]; L^2(\Omega_1))\) and is the unique weak solution of
\[
\begin{cases}
\partial_t u_n - \Delta_x u_n + \mu_n |x|^2 b(x) u_n = g_n(t, x) & (t, x) \in (0, T) \times \Omega_1, \\
\partial_t u_n(t, x) = 0 & t \in (0, T) \times \partial \Omega_1, \\
u_n(0, x) = u_{n, 0}(x) & x \in \Omega_1,
\end{cases}
\] (11)
where
\[g_n(t, x) := \int_{\Omega_2} g(t, x, y) \varphi_n(y) \, dy \text{ and } u_{n, 0}(x) = \int_{\Omega_2} u_0(x, y) \varphi_n(y) \, dy.
\]
The proof is done as in [2].

### 2.3. Dissipation speed

We introduce, for every \(n \in \mathbb{N}^*, \gamma > 0,\) the operator \(G_{n, \gamma}\) defined on \(L^2(\Omega_1)\) by
\[
D(G_{n, \gamma}) := H^2 \cap H_0^1(\Omega_1), \quad G_{n, \gamma} u := -\Delta_x u + \mu_n |x|^{2\gamma} b(x) u.
\] (12)
The smallest eigenvalue of \(G_{n, \gamma}\) is given by
\[
\lambda_{n, \gamma} = \min \left\{ \int_{\Omega_1} \left| \nabla \psi(x) \right|^2 + \mu_n |x|^{2\gamma} b(x) \psi(x)^2 \, dx ; \, \psi \in H^1_0(\Omega_1), \, \int_{\Omega_1} \psi(x)^2 \, dx = 1 \right\}.
\]
We are interested in the asymptotic behavior (as \(n \to +\infty\)) of \(\lambda_{n, \gamma},\) which quantifies the dissipation speed of the solutions of (2). The following result turns out to be a key point of the proofs of this paper.

**Proposition 4.** For every \(\gamma > 0,\) there exist constants \(c_* = c_*(\gamma), c^* = c^*(\gamma) > 0\) such that
\[
c_* \mu_n \frac{1}{n} \leq \lambda_{n, \gamma} \leq c^* \mu_n \frac{1}{n}, \quad \forall n \in \mathbb{N}^*.
\]

**Proof of Proposition 4.** First, we prove the lower bound. Let \(\tau_n := \frac{1}{\mu_n \frac{1}{n}}.\) With the change of variable \(\phi(x) = \frac{\tau_n^{1/2}}{\tau_n} \phi(\tau_n x),\) we get
\[
\lambda_{n, \gamma} = \inf \left\{ \int_{\Omega_1} (|\nabla \phi(x)|^2 + \mu_n |x|^{2\gamma} b(x) \phi(x)^2) \, dx ; \, \phi \in C^\infty_c(\Omega_1), \, \||\phi||_{L^2(\Omega_1)} = 1 \right\}
\]
\[
= \tau_n^2 \inf \left\{ \int_{\tau_n \Omega_1} (|\nabla \psi(y)|^2 + |y|^{2\gamma} b(y/\tau_n) \psi(y)^2) \, dy ; \, \psi \in C^\infty_c(\tau_n \Omega_1), \, \||\psi||_{L^2(\tau_n \Omega_1)} = 1 \right\}
\]
\[
\geq c_* \tau_n^2
\]
where
\[c_* := \inf \left\{ \int_{\mathbb{R}^n} (|\nabla \psi(y)|^2 + |y|^{2\gamma} b(y)^2) \, dy ; \, \psi \in C^\infty_c(\mathbb{R}^n), \, \||\psi||_{L^2(\mathbb{R}^n)} = 1 \right\}
\]
is positive (see [21]) and \(b_* := \min \{b(x) ; x \in \overline{\Omega_1}\}.\)
Now, we prove the upper bound in proposition 4. For every \( k > 1 \) let us consider the function \( \psi_k(x) := (1 - k|x|) \) if \( (1 - k|x|) > 0 \) and \( \psi_k(x) := 0 \) otherwise. Then, \( \psi_k \) belongs to \( H^1_0(\Omega) \) for \( k \) large enough (so that \( B_{\mathbb{R}^n}(0, 1/k) \subset \Omega_1 \)). Easy computations show that

\[
\int_{\Omega_1} \psi_k(x)^2 \, dx = C_1(N_1)k^{-N}, \quad \int_{\Omega_1} |\nabla \psi_k(x)|^2 \, dx = C_2(N_1)k^{2-N},
\]

\[
\int_{\Omega_1} \mu_n |x|^{2\gamma} b(x)\psi_k(x)^2 \, dx \leq C_3(N_1, \gamma) \mu_n b^k k^{-N-2\gamma},
\]

where \( b^* := \max\{b(x); x \in \Omega_1\} \). Thus,

\[
\lambda_{n, \gamma} \leq f_{n, \gamma}(k) := [C_2k^2 + C_3\mu_n b^k k^{-2\gamma}] / C_1, \forall k > 1.
\]

Minimizing the right-hand side over \( k \), we get \( \lambda_{n, \gamma} \leq C(N_1, \gamma) \mu_n^\frac{1}{2\gamma} \). \( \square \)

### 3. Carleman inequality for heat equations with non-smooth potentials

For \( \mu > 0 \), let us introduce the operator

\[
P_{\mu, \gamma} u := \frac{\partial u}{\partial t} - \Delta u + \mu |x|^{2\gamma} b(x)u.
\]

The goal of this section is the statement of the following Carleman inequality.

**Proposition 5.** Let \( \gamma \in (0, 1] \). There exist a weight function \( \beta \in C^1(\overline{\Omega}_1; (0, \infty)) \) and positive constants \( C_1, C_2 \) such that for every \( \mu \in (0, \infty) \), \( 0 \leq T_0 < T_1 \leq T \), and \( u \in C^0([0, T]; L^2(\Omega_1)) \cap L^2(0, T; H^1_0(\Omega_1)) \) the following inequality holds

\[
C_1 \int_{T_0}^{T_1} \int_{\Omega_1} \left( \frac{M}{(t - T_0)(T_1 - t)} |\nabla_x u(t, x)|^2 + \frac{M^3}{(t - T_0)(T_1 - t)^3} |u(t, x)|^2 \right) e^{-M\alpha(t, x)} \, dx \, dt
\]

\[
\leq \int_{T_0}^{T_1} \int_{\Omega_1} |P_{\mu, \gamma} u(t, x)|^2 e^{-M\alpha(t, x)} \, dx \, dt
\]

\[
+ \int_{T_0}^{T_1} \int_{\Omega_1} \frac{M^3}{(t - T_0)(T_1 - t)^3} |u(t, x)|^2 e^{-M\alpha(t, x)} \, dx \, dt \tag{13}
\]

where

\[
\alpha(t, x) := \frac{\beta(x)}{(t - T_0)(T_1 - t)},
\]

\[
M := \begin{cases} C_2 \max\{T + T^2; \sqrt{\mu}T^2\} & \text{if } \gamma \in [1/2, 1], \\ C_2 \max\{T + T^2; \mu^{2/3}T^2\} & \text{if } \gamma \in (0, 1/2), \end{cases}
\]

and \( T := T_1 - T_0 \).

Note that we can have sharp dependence of \( M = O(\mu^{1/2}) \) and \( T \) in the case of \( 1/2 \leq \gamma \leq 1 \). In particular, if we treat the term \( \mu |x|^{2\gamma} b(x)u \) as lower-order term to apply the Carleman estimate for the operator \( \frac{\partial}{\partial t} - \Delta u \), then we can obtain less sharp dependence \( M = O(\mu^{2/3}) \) and we need a sharper estimate for \( 1/2 \leq \gamma \leq 1 \), to prove the Lipschitz stability estimate.

The proof of this Carleman inequality is given in [2] in the case \( N_1 = 1 \). In the 1D case, the sharp dependence \( M = O(\mu^{1/2}) \) is proved for any \( \gamma \in (0, 1) \) and the case \( \gamma \in (0, 1/2) \) requires a weight adapted to the degeneracy.

A proof in the multidimensional case is presented in the appendix. It relies on the usual weight of heat equations.
4. Inverse source problem

4.1. Uniform observability of frequencies

**Proposition 6.** Let $\gamma \in (0, 1)$ and $\omega_1$ be an open subset of $\Omega_1$. There exists $C > 0$ and functions $\epsilon_n : (0, +\infty) \to (0, +\infty)$, $n \in \mathbb{N}^*$ with

- $\epsilon_n(T) \to 0$ when $n \to \infty$, for every $T > 0$,
- $\epsilon_n(T) \leq \epsilon^* < +\infty$, for every $n \in \mathbb{N}^*$ and $T > 0$,

such that, for every $n \in \mathbb{N}^*$, $g_n \in L^2((0, T) \times \Omega_1)$, $u_{0,n} \in L^2(\Omega_1)$ the solution of (11) satisfies

$$
\int_{\Omega_1} |u_n(T, x)|^2 \, dx \leq C (1 + T^\gamma) \left( \int_0^T \int_{\Omega_1} |u_n(t, x)|^2 \, dx \, dt + \epsilon_n(T) \int_0^T \int_{\Omega_1} |g_n(t, x)|^2 \, dx \, dt \right)
$$

where

$$
p = p(\gamma) := \begin{cases} 
\frac{1+\gamma}{2(1+\gamma)}, & \text{if } \gamma \in [1/2, 1], \\
\frac{1+\gamma}{1-\gamma}, & \text{if } \gamma \in (0, 1/2).
\end{cases}
$$

**Proof of proposition 6.** The proof is in five steps.

**Step 1.** We prove

$$
\|u_n(T)\|_{L^2(\Omega_1)}^2 \leq \frac{6}{T} e^{-2\lambda_{n,T}} T^{2/3} \int_{T/3}^{2T/3} \|u_n(t)\|_{L^2(\Omega_1)}^2 \, dt + \frac{1}{\lambda_{n,T}} \|g_n\|_{L^2((0,T) \times \Omega_1)}^2.
$$

From Duhamel’s formula, i.e.,

$$
u_n(T) = e^{-\lambda_{n,T} (T-t)} u_n(t) + \int_t^T e^{-\lambda_{n,T} (T-\tau)} g_n(\tau) \, d\tau, \quad \forall t \in (0, T)
$$

and the Cauchy–Schwarz inequality, we get

$$
\|u_n(T)\|_{L^2(\Omega_1)} \leq e^{-\lambda_{n,T} (T-t)} \|u_n(t)\|_{L^2(\Omega_1)} + \int_t^T e^{-\lambda_{n,T} (T-\tau)} \|g_n(\tau)\|_{L^2(\Omega_1)} \, d\tau
$$

$$
\leq e^{-\lambda_{n,T} (T-t)} \|u_n(t)\|_{L^2(\Omega_1)} + \frac{1}{\sqrt{2\lambda_{n,T}}} \|g_n\|_{L^2((0,T) \times \Omega_1)}.
$$

Thus

$$
\|u_n(T)\|_{L^2(\Omega_1)}^2 \leq 2 e^{-2\lambda_{n,T} (T-t)} \|u_n(t)\|_{L^2(\Omega_1)}^2 + \frac{1}{\lambda_{n,T}} \|g_n\|_{L^2((0,T) \times \Omega_1)}^2.
$$

Integrating this relation over $t \in (T/3, 2T/3)$ gives (16).

**Step 2.** We prove the existence of a constant $C_3 > 0$ such that for every $T > 0$, $n \in \mathbb{N}^*$, $g_n \in L^2((0, T) \times \Omega_1)$ and $u_{0,n} \in L^2(\Omega_1)$, the solution of (11) satisfies

$$
\int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \leq C_3 T e^{\frac{4M^2}{T^2}} \left( \int_0^T \int_{\Omega_1} |g_n|^2 \, dx \, dt + \int_0^T \int_{\Omega_1} |u_n|^2 \, dx \, dt \right)
$$

where $\beta$, $C_2$ and $M$ are as in proposition 5 (with $\mu$ replaced by $\mu_n$), and $\beta^* := \max\{\beta(x); x \in \Omega_1\}$. From proposition 5, we get

$$
C_1 \left( \frac{4M^2}{T^2} \right)^\gamma e^{\frac{4\mu_0^*}{T^2}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt
$$

$$
\leq C_1 \int_{T/3}^{2T/3} \int_{\Omega_1} \frac{M^2}{(t(T-t)^3)} |u_n|^2 e^{\frac{4\mu_0^*}{T^2}} \, dx \, dt
$$

$$
\leq C_1 \int_0^T \int_{\Omega_1} \frac{M^3}{(T(t-T))^\gamma} |u_n|^2 e^{\frac{4\mu_0^*}{T^2}} \, dx \, dt
$$


\[ \leq \int_0^T \int_{\Omega_1} |u_n|^2 e^{-\frac{M^2}{T^2} \|T\|^2} \, dx \, dt + \int_0^T \int_{\Omega_1} \frac{M^3}{(T^2 + t)^{\frac{3}{2}}} |u_n|^2 e^{-\frac{M^2}{T^2} \|T\|^2} \, dx \, dt \]
\[ \leq \int_0^T \int_{\Omega_1} |u_n|^2 \, dx \, dt + C \int_0^T \int_{\Omega_1} |u_n|^2 \, dx \, dt \]  
(18)

where \( C := \sup[x^3 e^{-\beta(x)}: x \geq 0] \) and \( \beta_n := \min[\beta(x); x \in \omega_1] \). We deduce from (18) that

\[ \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n(t)|^2 \, dx \, dt \leq \frac{\max[1, C]}{4C_1} \frac{T^6}{M^3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \]

We remark that \( M \geq C_1 T \) and \( C_2 T^2 \) thus \( T^6/M^3 \leq T/C_3^2 \). Then, the previous inequality gives (17) with \( C_3 := \max[1, C_1/(4^2C_1^2)] \).

**Step 3. We put together (16) and (17).** Using (16) in the first inequality, (17) in the second one, and proposition 4 in the third one, we get

\[ \int_{\Omega_1} |u_n(T)|^2 \leq \frac{6}{T} e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n(t)|^2 \, dr + \frac{1}{\lambda_n, \text{for } \gamma \in [1/2, 1]} \int_{\Omega_1} |g_n|^2 \, dx \, dt \]
\[ \leq 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \]
\[ + \frac{1}{\lambda_n, \text{for } \gamma \in [1/2, 1]} \int_{\Omega_1} |g_n|^2 \, dx \, dt \]
\[ \leq 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \]
\[ + \frac{1}{\lambda_n, \text{for } \gamma \in [1/2, 1]} \int_{\Omega_1} |g_n|^2 \, dx \, dt \]
(19)

where \( C_3 := 2c_2/3 \).

**Step 4. End of the proof when \( \gamma \in [1/2, 1] \).**

First case: \( \mu_n \geq 1 + \frac{1}{T} \). Then we have that \( M = C_2 \sqrt{T} \). For any constant \( c_2 > 0 \), the maximum value of the function \( z \mapsto c_2z - c_1 T^{\frac{3}{2}} \) on \((0, +\infty)\) is of the form \( c_2 T^{\frac{1}{15}} \) for some constant \( c_1 > 0 \) (independent of \( T \)). Thus,

\[ \|u_n(T)|^2 \leq 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \]
\[ + \frac{1}{c_2^{2/3}} + 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt. \]

This proves (14) with any constant \( C \) large enough so that

\[ 6C_3 \leq e^C \]

and

\[ \epsilon(T) := \frac{1}{c_2^{2/3}} + 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}}. \]

Second case: \( \mu_n < 1 + \frac{1}{T} \). Then, \( M = c_2 (T + T^2) \) and

\[ \|u_n(T)|^2 \leq 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt \]
\[ + \frac{1}{c_2^{2/3}} + 6C_3 e^{-\frac{2}{T} + \frac{1}{T^3}} \int_{T/3}^{2T/3} \int_{\Omega_1} |u_n|^2 \, dx \, dt. \]
Note that $\frac{1+\gamma}{T^{\frac{1}{1+\gamma}}} > 1$ thus $\frac{1}{T} \leq 1$ when $T > 1$ and $\frac{1}{T} \leq T^{-\frac{1}{1+\gamma}}$ when $T < 1$; in any case $\frac{1}{T} \leq 1 + T^{-\frac{1}{1+\gamma}}$. Thus, we have

$$\int_{\Omega} |u_n(T)|^2 \leq \left( \frac{1}{c_n\mu_0} + 6C_3 e^{\frac{4\gamma}{2+\gamma}c_2(2+T^{-\frac{1}{1+\gamma}})} \right) \left( \int_0^T \int_{\Omega} |u_n|^2 + \int_0^T \int_{\Omega} |g_n|^2 \right).$$

This proves (14) with $\epsilon_n = 1$ and any constant $C$ large enough so that

$$\frac{1}{c_n\mu_0} \leq \frac{1}{2} e^{C}, \quad 6C_3 e^{\frac{4\gamma}{2+\gamma}c_2} \leq \frac{1}{2} e^{C} \quad \text{and} \quad \frac{9\beta^2 c_2^2}{2} \leq C.$$

Step 5. End of the proof when $\gamma \in (0, 1/2)$. One can proceed as in step 4 observing that the maximum value of the function $z \mapsto c_2 z - c_1 z \frac{1}{\sqrt{1+z}} T/2$ on $(0, +\infty)$ is of the form $c_3 T^{-\frac{1}{1+\gamma}}$ for the first case, and $\frac{1}{2(1+\gamma)} > 1$ for the second one. □

Proposition 7. Assume $\gamma = 1$ and let $\omega_1$ be an open subset of $\Omega_1$. Then there exists $T_*>0$ such that, for every $T>T_*$, there exist $C>0$ and $(\epsilon_n(T))_{n\in\mathbb{N}^*} \in (\mathbb{R}^*_+)^{\mathbb{N}^*}$ with $\epsilon_n(T)\to0$ such that, for every $n\in\mathbb{N}^*$, $g_n \in L^2((0, T) \times \Omega_1)$, $u_{0,n} \in L^2(\Omega_1)$, the solution of (11) satisfies

$$\int_{\Omega} |u_n(T, x)|^2 \leq C \int_0^T \int_{\Omega} |u_n(t, x)|^2 \, dt + \epsilon_n(T) \int_0^T \int_{\Omega} |g_n(t, x)|^2 \, dx \, dt.$$

Remark 2. The above proposition, together with the Bessel–Parseval equality, proves statement 3 of theorem 3.

Proof of proposition 7. One can follow the lines of the previous proof until (19). Then, when $\mu_n \geq 1 + \frac{1}{\gamma}$, we have $M = C_2\sqrt{\mu_n} T^2$. Thus

$$\nu_n(T)_{\L^2(\Omega)} \leq 6C_3 e^{c_2-c_1 \sqrt{T\mu_n}} \int_0^T \int_{\Omega} |u_n|^2 \, dt + \left( \frac{1}{c_n\mu_0} + C_3 e^{c_2-c_1 \sqrt{T\mu_n}} \right) \int_0^T \int_{\Omega} |g_n|^2 \, dx \, dt$$

for some constant $c_2 > 0$. This gives the conclusion with $T_* = c_2/c_1$. □

4.2. Lipschitz stability estimate when $\omega$ is a strip

The goal of this section is the proof of theorem 1. We focus on the uniform Lipschitz stability for systems (11). We assume the source term $g_n$ in (11) takes the form $g_n(t, x) = f_n(x) R(t, x)$, where $f_n \in L^2(\Omega_2)$ and $R \in C^0([0, T] \times \Omega_2)$.

Definition 4 (Uniform Lipschitz stability). Let $\omega_1$ be an open subset of $\Omega_1$, $T > 0$ and $0 \leq T_0 < T_1 \leq T$. We say the system (11) satisfies a uniform Lipschitz stability estimate on $(T_0, T_1) \times \omega_1$ if there exists $C > 0$ such that, for every $n \in \mathbb{N}^*$, $f_n \in L^2(\Omega_1)$, $u_{0,n} \in L^2(\Omega_1)$, the solution of (11) satisfies

$$\int_{\Omega} |f_n(x)|^2 \, dx \leq C \left( \int_{T_0}^{T_1} \int_{\Omega} |\partial_t u_n(t, x)|^2 \, dx \, dt + \int_{\Omega} |G_{n,y} u_n(T_1, x)|^2 \, dx \right).$$

Theorem 1 is a consequence of the following proposition and the Bessel–Parseval equality.

Proposition 8. Assume (5) and let $\omega_1$ be an open subset of $\Omega_1$.

1. If $\gamma \in (0, 1)$ then, for every $T_0 \in [0, T_1)$, system (11) satisfies a uniform Lipschitz stability estimate on $(T_0, T_1) \times \omega_1$.
2. If $\gamma = 1$, then there exists $T^*>0$ such that, for every $T_0 \in [0, T_1 - T^*)$, system (11) satisfies a uniform Lipschitz stability estimate on $(T_0, T_1) \times \omega_1$. 

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Remark 3. The inequality (20) with a constant $C$ that may depend on $n$ is already known (see [16]). The goal of this section is to prove that (20) holds with a constant $C$ which is independent of $n$.

Proof of proposition 8. In this proof, $T^*$ is as in proposition 7 if $\gamma = 1$ and $T^* := 0$ if $\gamma \in (0, 1)$. Assume $(T_1 - T_0) > T_*$. It results from (5) that

$$R_0|f_n(x)| \leq |R(T_1, x)f_n(x)| \leq |\partial_u u_0(T_1, x)| + |G_{n, \gamma} u(T_1, x)|$$

and

$$\int_{\Omega_1} |f_n(x)|^2 \, dx \leq \frac{2}{R_0^2} \left( \int_{\Omega_1} |\partial_u u_0(T_1, x)|^2 \, dx + \int_{\Omega_1} |G_{n, \gamma} u(T_1, x)|^2 \, dx \right).$$

(21)

By propositions 1 and 6 or 7 (applied to $\partial \mu$), we get, for every $T_0 \in (0, T_1 - T^*)$,

$$\int_{\Omega_1} |\partial_u u_n(T_1)|^2 \, dx \leq C \int_{T_0}^{T_1} \int_{\Omega_1} |\partial_u u_0|^2 \, dx \, dt + \epsilon_n |T_1 - T_0| \|\partial_u R\|_\infty^2 \int_{\Omega_1} |f_n|^2 \, dx$$

(22)

with $\epsilon_n = \epsilon_0(T_1 - T_0)$, where $\|\partial_u R\|_\infty := \|\partial_u R\|_{L^\infty((0, 1) \times \Omega)}$. There exists $n^* > 0$ such that, for every $n \geq n^*$, $\gamma (2/R_0^2) \epsilon_n |T_1 - T_0| \|\partial_u R\|_\infty^2 < 1/2$. Using (21) and (22) we get, for $n \geq n^*$

$$\int_{\Omega_1} |f_n|^2 \, dx \leq \frac{4C}{R_0^2} \int_{T_0}^{T_1} \int_{\Omega_1} |\partial_u u_0|^2 \, dx \, dt + \frac{4}{R_0^2} \int_{\Omega_1} |G_{n, \gamma} u_n(T_1)|^2 \, dx.$$

This ends the proof of proposition 8.

4.3. Lipschitz stability estimate when $\omega$ is arbitrary

The goal of this section is the proof of theorem 2. In the whole section, $T > 0$ and $\gamma \in (0, 1)$ are fixed. For simplicity, we take $T_0 = 0$ and $T_1 = T$. In this section, $\| \cdot \|$ denotes the norm in $L^2(\Omega)$; for other types of $L^p$-norms, an additional subscript is written, for instance $\| \cdot \|_{L^2((0, T) \times \Omega)}$.

For $n \in \mathbb{N}^*$, $\varphi_n(\gamma)$ is defined by (10) and $H_n := L^2(\Omega_1) \otimes \varphi_n$ is a closed subspace of $L^2(\Omega)$. For $j \in \mathbb{N}^*$, we define

$$E_j := \bigoplus_{\mu_1 \leq n \leq 2n} H_n$$

and denote by $\Pi_j$ the orthogonal projection onto $E_j$. Moreover, $I_d$ stands for the identity operator on $L^2(\Omega)$.

Proposition 9. Let $\omega$ be an open subset of $\Omega$. Then there exists $C > 0$ such that, for every $T > 0$, $j \in \mathbb{N}^*$, $h_0 \in E_j$, and $\gamma \in L^2(0, T; E_j)$, the solution of (2) satisfies

$$\int_{\Omega} \|u(T, x, y)\|^2 \, dx \, dy \leq e^{C(T^2 + T^p)} \int_0^T \int_{\Omega} \|u(t, x, y)\|^2 \, dx \, dy + e^{C^{(1 + T^p}}} \int_0^T \int_{\Omega} \|g(t, x, y)\|^2 \, dx \, dy,$$

where $p = p(\gamma)$ is defined by (15).

For the proof of proposition 9 we shall need the following inequality obtained in [19] (see also [20]).

Proposition 10. Let $\omega_2$ be an open subset of $\Omega_2$. There exists $C > 0$ such that, for every $(b_k)_{\mu \in \mathbb{N}} \subset \mathbb{R}$ and $\mu > 0$,

$$\sum_{\mu \leq \mu_0} |b_k|^2 \leq Ce^{\sqrt{\pi}} \int_{\omega_2} \left( \sum_{\mu \leq \mu_0} b_k \Phi_k(y) \right)^2 \, dy.$$
Proof of proposition 9. Let \( \omega_j \) be an open subset of \( \Omega_j \) for \( j = 1, 2 \) such that \( \omega_1 \times \omega_2 \subset \omega \). Using proposition 6 and the orthonormality of the functions \( (\varphi_n) \) in \( L^2(\Omega_2) \), we get
\[
\int_{\Omega} |u(T, x, y)|^2 \, dx \, dy = \sum_{\mu_n \leq 2^j} \int_{\Omega_1} |u_n(T, x)|^2 \, dx
\]
where \( p \in \mathbb{N} \) and \( \alpha \) and \( \gamma \) is as in proposition 4.

Proof of proposition 11. Let \( \omega_j \) be an open subset of \( \Omega_j \) for \( j = 1, 2 \) such that \( \omega_1 \times \omega_2 \subset \omega \). Using proposition 6 and the orthonormality of the functions \( (\varphi_n) \) in \( L^2(\Omega_2) \), we get
\[
\int_{\Omega} |u(T, x, y)|^2 \, dx \, dy = \sum_{\mu_n \leq 2^j} \int_{\Omega_1} |u_n(T, x)|^2 \, dx
\]
where \( p \in \mathbb{N} \) and \( \alpha \) and \( \gamma \) is as in proposition 4.

**Proof of proposition 11.** Let \( n \in \mathbb{N}^* \), \( g \in L^2((0, T) \times \Omega) \), \( u_0 \in L^2(\Omega) \), the solution of (2), (3) satisfies
\[
e^{-C_1 \lambda(2^n)} \| \Pi_n u(T - \alpha_{n-1}) \|^2 \leq \int_{I_{\alpha_0}} |u|^2 + C_2 \int_{I_{\alpha_0}} |g|^2 + C_3 e^{-\lambda(2^n)} \| (I - \Pi_n) u(T - \alpha_0) \|^2.
\]

Proof of proposition 11. Let \( n \in \mathbb{N}^* \), \( g \in L^2((0, T) \times \Omega) \) and \( u_0 \in L^2(\Omega) \). By proposition 9, the solution of (2) and (3) satisfies
\[
\| \Pi_n u(T - \alpha_{n-1}) \|^2 \leq e^{C_1 (2^n + \tau_1)} \int_{I_{\alpha_0}} |u|^2 + e^{C_1 (1 + \tau_1)} \int_{I_{\alpha_0}} |g|^2.
\]
Moreover, we have

\[
\int_{L_t \times \omega} |\Pi_n u|^2 \leq \int_{L_t \times \omega} 2|u|^2 + \int_{L_t \times \Omega} 2|(Id - \Pi_n)u|^2. \tag{29}
\]

For every \( t \in I_n \), the Duhamel formula, i.e.

\[
(Id - \Pi_n)u(t) = e^{-G_r(t-T+a_n)}(Id - \Pi_n)u(T - \alpha_n) + \int_{T-a_n}^t e^{-G_r(t-\tau)}(Id - \Pi_n)g(\tau) \, d\tau,
\]

proposition 4 and the Cauchy–Schwarz inequality give

\[
\| (Id - \Pi_n)u(t) \| \leq \| (Id - \Pi_n)u(T - \alpha_n) \| e^{-\lambda(2^r)(t-T+a_n)} + \int_{T-a_n}^t e^{-\lambda(2^r)(t-\tau)} \| (Id - \Pi_n)g(\tau) \| \, d\tau \]

\[
\leq \| (Id - \Pi_n)u(T - \alpha_n) \| e^{-\lambda(2^r)t_n} + \frac{1}{\sqrt{2\lambda(2^r)}} \| (Id - \Pi_n)g \|_{L^2(J_t \times \Omega)}. \tag{30}
\]

Thus

\[
\int_{L_t \times \Omega} |(Id - \Pi_n)u|^2 \leq 2\tau_n \| (Id - \Pi_n)u(T - \alpha_n) \|^2 e^{-2\lambda(2^r)t_n} + \frac{\tau_n}{\lambda(2^r)} \| (Id - \Pi_n)g \|^2_{L^2(J_t \times \Omega)}.
\]

Using (28), (29) and (30), we get

\[
\| \Pi_n u(T - \alpha_n-1) \|^2 \leq 2 e^{C(2^r+\tau_n)} \int_{L_t \times \omega} |u|^2 + \left( \frac{2\tau_n}{\lambda(2^r)} + e^{C(1-2^r)} \right) e^{C(2^r+\tau_n)} \int_{L_t \times \Omega} |g|^2 + 4\tau_n e^{C(2^r+\tau_n)} \| (Id - \Pi_n)u(T - \alpha_n) \|^2.
\]

In view of (23), we have that

\[
\tau_n = K^{-p} 2^{np} \leq C 2^n, \quad \forall n \in \mathbb{N}^+,
\]

for some constant \( C > 0 \). Thus, there exists \( C_1 > 0 \) such that

\[
2 e^{C(2^r+\tau_n)} \leq e^{C_1 2^n}, \quad \forall n \in \mathbb{N}^+.
\]

Using (24), (26) and (23), we obtain, for some constants \( C_2, C_3 > 0, \)

\[
\frac{1}{2} \left( \frac{2\tau_n}{\lambda(2^r)} + e^{C(1-2^r)} \right) \leq C_2 2^{\eta + (\frac{r}{2} + \frac{1}{2})}, \quad \forall n \in \mathbb{N}^+
\]

and

\[
\tau_n e^{-2\lambda(2^r)t_n} \leq C_3 e^{-\lambda(2^r)t_n}, \quad \forall n \in \mathbb{N}^+.
\]

Therefore, we have that

\[
\| \Pi_n u(T - \alpha_n-1) \|^2 \leq e^{C_1 2^n} \int_{L_t \times \omega} |u|^2 + C_2 2^{\eta + (\frac{r}{2} + \frac{1}{2})} e^{C_1 2^n} \int_{L_t \times \Omega} |g|^2 + C_3 e^{C_2 - \lambda(2^r)t_n} \| (Id - \Pi_n)u(T - \alpha_n) \|^2.
\]

for every \( n \geq 1 \). This gives the conclusion. \( \square \)

**Proposition 12.** Let \( T > 0 \). Then, there exists \( C = C(T) > 0 \) such that, for every \( u_0 \in L^2(\Omega) \), \( g \in L^2((0, T) \times \Omega) \) of the form \( g(t, x, y) = R(t, x)f(x, y) \) with \( R \in L^\infty((0, T) \times \Omega_1) \) and \( f \in L^2(\Omega) \), the solution of (2) and (3) satisfies

\[
\int \int |u(T, x, y)|^2 \, dx \, dy \leq C \left( \int \int |u_0|^2 + \int_0^T \| R(t) \|^2_{L^\infty(\Omega)} \, dt \int \int |f|^2 \right).
\]
Remark 4. The above proposition can be used to prove statement 1 of theorem 3.

Proof of proposition 12. Let $C_1$, $C_2$, $C_3$ be as in proposition 11.

Step 1. We prove by induction on $n \in \mathbb{N}^*$ that, for every $n \in \mathbb{N}^*$,

\[
(P_n) : \quad \sum_{k=1}^{n} e^{-C_2 \omega} \| \Pi_k u(T - \alpha_{k-1}) \|^2 \leq \sum_{k=1}^{n} \delta_k \int_{|x| \omega} |u|^2 + A_n \left( \int_{T - \alpha_n} \| R(t) \|^2_{L^\infty(\Omega_1)} \, dt \right) \| f \|^2 + B_n \| u(T - \alpha_n) \|^2
\]

where

\[
\delta_1 := 1, \quad A_1 := C_2 \left( 1 + \frac{\rho}{\pi T} \right), \quad B_1 := C_3 e^{-\lambda(2^1)\tau_1}
\]

and

\[
\delta_{n+1} := \max\{2, 1 + B_n e^{C_2 \omega} \},
\]

\[
A_{n+1} := \max \left\{ A_n, \frac{B_n}{\lambda(2^{n+1})} + \delta_{n+1} C_2 2^{-(n+1)} \left( 1 + \frac{\rho}{\pi T} \right) \right\},
\]

\[
B_{n+1} := 2B_n e^{-\lambda(2^{n+1})\tau_{n+1}} + \delta_{n+1} C_3 e^{-\lambda(2^{n+1})\tau_{n+1}}.
\]

The inequality $(P_1)$ is given by proposition 11 with $n = 1$. Indeed,

\[
\int_{|x| \omega} |g|^2 = \int_{|x| \omega} |R(t, x)f(x, y)|^2 \, dx \, dy \, dt
\]

\[
\leq \int_{|x| \omega} \| R(t) \|^2_{L^\infty(\Omega_1)} |f(x, y)|^2 \, dx \, dy \, dt
\]

\[
\leq \left( \int_{T - \alpha_1} \| R(t) \|^2_{L^\infty(\Omega_1)} \, dt \right) \| f \|^2.
\]

Let us now assume that $(P_n)$ holds for some $n \in \mathbb{N}^*$ and prove $(P_{n+1})$. We have

\[
B_n \| u(T - \alpha_n) \|^2 = B_n \Pi_{n+1} u(T - \alpha_n) \|^2 + B_n \| (Id - \Pi_{n+1}) u(T - \alpha_n) \|^2.
\]

Moreover, using Duhamel’s formula as in the previous proof, we get

\[
\| (Id - \Pi_{n+1}) u(T - \alpha_n) \| \leq \| (Id - \Pi_{n+1}) u(T - \alpha_{n+1}) \| e^{-\lambda(2^{n+1})\tau_{n+1}}
\]

\[
+ \int_{T - \alpha_{n+1}}^{T - \alpha_n} e^{-\lambda(2^{n+1})(T - \alpha_n - s)} \| (Id - \Pi_{n+1}) R(s)f \| \, ds
\]

\[
\leq \| (Id - \Pi_{n+1}) u(T - \alpha_{n+1}) \| e^{-\lambda(2^{n+1})\tau_{n+1}}
\]

\[
+ \left( \int_{T - \alpha_{n+1}}^{T - \alpha_n} e^{-\lambda(2^{n+1})(T - \alpha_n - s)} \, ds \right)^{1/2}
\]

\[
\times \left( \int_{T - \alpha_{n+1}}^{T - \alpha_n} \| R(s) \|^2_{L^\infty(\Omega_1)} \, ds \right)^{1/2} \| f \|
\]

\[
\leq \| (Id - \Pi_{n+1}) u(T - \alpha_{n+1}) \| e^{-\lambda(2^{n+1})\tau_{n+1}}
\]

\[
+ \frac{1}{\sqrt{2\lambda(2^{n+1})}} \left( \int_{\omega_1} \| R(s) \|^2_{L^\infty(\Omega_1)} \, ds \right)^{1/2} \| f \|.
\]
Therefore,
\[ B_n \| u(T - \alpha_n) \|^2 \leq B_n \| \Pi_{n+1} u(T - \alpha_n) \|^2 + 2B_n \| (I - \Pi_{n+1}) u(T - \alpha_{n+1}) \|^2 e^{-\delta (2n+1) \tau_{n+1}} \]
\[ + \frac{B_n}{\lambda(2n+1)} \left( \int_{J_{n+1}} \| R(s) \|_{L^\infty(\Omega_1)}^2 \, dx \right) \| f \|^2. \]

Thus, \((P_n)\) yields
\[ \sum_{k=1}^n e^{-C_1 \delta_k^2} \| \Pi_k u(T - \alpha_{k-1}) \|^2 - B_n \| \Pi_{n+1} u(T - \alpha_n) \|^2 \]
\[ \leq \sum_{k=1}^n \delta_k \int_{J_k \times \Theta} |u|^2 \]
\[ + \left( A_n \int_{T-\alpha_n}^T \| R(t) \|_{L^\infty(\Omega_1)}^2 \, dt + \frac{B_n}{\lambda(2n+1)} \int_{J_{n+1}} \| R(t) \|_{L^\infty(\Omega_1)}^2 \, dt \right) \| f \|^2 \]
\[ + 2B_n e^{-\delta (2n+1) \tau_{n+1}} \| u(T - \alpha_{n+1}) \|^2. \]

Moreover, by proposition 11, we also have
\[ e^{-C_1 \delta_k^2} \| \Pi_{n+1} u(T - \alpha_n) \|^2 \leq \int_{J_{n+1} \times \Theta} |u|^2 \]
\[ + C_2 \delta_k^{-(n+1)\left(\rho + \eta_2\right)} \left( \int_{J_{n+1}} \| R(t) \|_{L^\infty(\Omega_1)}^2 \, dt \right) \| f \|^2 \]
\[ + C_3 e^{-\lambda (2n+1) \tau_{n+1}} \| u(T - \alpha_{n+1}) \|^2. \]

Note that \(\delta_{n+1}\) is chosen so that
\[ \delta_{n+1} e^{-C_1 \delta_{n+1}^2} = B_n \geq e^{-C_1 \delta_{n+1}^2}. \]

Thus, summing \((35)\) and \(\delta_{n+1} \times (36)\), we get \((P_{n+1})\). This ends the first step.

**Step 2.** We prove that \(\delta_n = 2\) for \(n\) large enough. Let \(\tilde{B}_n := B_n e^{C_1 \delta_n^2}\). For every \(n \in \mathbb{N}^*\) we have either
\[ \tilde{B}_{n+1} = 2 \tilde{B}_n e^{-\delta (2n+1) \tau_{n+1} + C_1 \delta_n^2} + 2C_3 e^{-\lambda (2n+1) \tau_{n+1} + C_1 \delta_n^2} \]
if \(\delta_{n+1} = 2\), or
\[ \tilde{B}_{n+1} = 2 \tilde{B}_n e^{-\delta (2n+1) \tau_{n+1} + C_1 \delta_n^2} + (1 + \tilde{B}_n) C_3 e^{-\lambda (2n+1) \tau_{n+1} + C_1 \delta_n^2} \]
if \(\delta_{n+1} = 1 + B_n e^{C_1 \delta_n^2}\). Using proposition 4, \((24)\) and the inequality \(\frac{\tau_2}{\tau_1} - \rho > 1\) (see \((23)\)), we get a constant \(C > 0\) such that \(\tilde{B}_{n+1} \leq C (\tilde{B}_n + C_3)\), \(\forall n \in \mathbb{N}^*\). We deduce the existence of another constant \(C > 0\) such that \(\tilde{B}_n \leq C^n\) for every \(n \in \mathbb{N}^*\). Then, we have either
\[ \tilde{B}_{n+1} \leq 2C^n e^{-\delta (2n+1) \tau_{n+1} + C_1 \delta_n^2} + 2C_3 e^{-\lambda (2n+1) \tau_{n+1} + C_1 \delta_n^2} \]
if \(\delta_{n+1} = 2\), or
\[ \tilde{B}_{n+1} \leq 2C^n e^{-\delta (2n+1) \tau_{n+1} + C_1 \delta_n^2} + (1 + C^n) C_3 e^{-\lambda (2n+1) \tau_{n+1} + C_1 \delta_n^2} \]
if \(\delta_{n+1} = 1 + B_n e^{C_1 \delta_n^2}\). In any case, \(\tilde{B}_n \to 0\) when \(n \to \infty\) because \(\frac{\tau_2}{\tau_1} - \rho > 1\). Thus \(\delta_n = 2\) for \(n\) large enough.

**Step 3.** We prove that \((A_n)\) is bounded. By definition \((A_n)_{n \in \mathbb{N}^*}\) is a non-decreasing sequence. Moreover,
\[ \frac{B_n}{\lambda(2n+1)} + \delta_{n+1} C_2^{-\lambda (n+1) \left(\rho + \eta_2\right)} \xrightarrow{n \to \infty} 0. \]
Thus, for \(n\) large enough, we have that
\[
\frac{B_n}{\lambda(2n+1)} + \delta_{n+1} C 2^{-n(\rho+\frac{1}{2})} \leq A_1 \leq A_n.
\]
This implies \(A_{n+1} = A_n\) for all \(n \in \mathbb{N}^*\).

Step 4. We pass to the limit as \(n \to \infty\) in \((P_n)\). The last term on the right-hand side of \((P_n)\) converges to zero because \(B_n \leq \delta^* e^{-C_2^n}\). Thus, we get
\[
\sum_{n=1}^{\infty} e^{-C_2^n} \|\Pi_n u(T - \alpha_{n-1})\|^2 \leq C \left( \int_0^T \int_\omega |u|^2 + \int_0^T \|R(t)\|_{L^2(\Omega, \lambda)}^2 \int_\Omega |f|^2 \right)
\]
with
\[
C := \max\{\delta^*; \max\{A_n; n \in \mathbb{N}^*\}\}.
\]

Step 5. Conclusion. Using Duhamel’s formula and the convention \(\Pi_0 = 0\), we get
\[
\|u(T)\|^2 = \sum_{n=1}^{\infty} \|\Pi_n - \Pi_{n-1}) u(T)\|^2 \\
\leq \sum_{n=1}^{\infty} 2 \|\Pi_n - \Pi_{n-1}) u(T - \alpha_{n-1})\|^2 e^{-2\lambda(2n+1)\rho_{n-1}} \\
+ \sum_{n=1}^{\infty} \frac{1}{\lambda(2n+1)} \left( \int_{T-\alpha_{n-1}}^T \|R(t)\|_{L^2(\Omega, \lambda)}^2 \right) \|\Pi_n - \Pi_{n-1}) f\|^2.
\]
Then, by proposition 4, (24) and the inequality \(\frac{2}{1+\rho} - \rho > 1\), we obtain, for some constant \(C > 0\)
\[
\|u(T)\|^2 \leq C \sum_{n=1}^{\infty} e^{-C_2^n} \|\Pi_n u(T - \alpha_{n-1})\|^2 + C \left( \int_0^T \|R(t)\|_{L^2(\Omega, \lambda)}^2 \right) \|f\|^2.
\]
Thus, (37) implies the conclusion. \(\square\)

**Proof of theorem 2.** Using the equation in (2), and (5), we obtain
\[
\|f\|^2 \leq \frac{1}{R_0^2} \int_\Omega |R(T_1, x) f(x, y)|^2 \, dx \, dy \\
\leq \frac{2}{R_0^2} \int_\Omega \left( |\partial_x u(T_1, x, y)|^2 + |G_y u(T_1, x, y)|^2 \right) \, dx \, dy.
\]
Applying propositions 2 and 12, we obtain, for some constant \(C > 0\),
\[
\|\partial_x u(T_1)\|^2 \leq C \left( \int_0^T \int_\omega |u|^2 \, dx \, dy \, dt + \|\partial_x R\|_{L^2(0, T; L^\infty(\Omega, \lambda))} \|f\|^2 \right)
\]
From (39) and (38), we get another constant \(C > 0\) such that
\[
\|f\|^2 \leq \frac{C}{R_0^2} \left( \int_0^T \int_\omega |u|^2 + \|\partial_x R\|_{L^2(0, T; L^\infty(\Omega, \lambda))} \|f\|^2 + \|G_y u(T_1)\|^2 \right).
\]
We get the Lipschitz stability estimate if \(\sqrt{C} \|\partial_x R\|_{L^2(0, T; L^\infty(\Omega, \lambda))}/R_0 < 1\). \(\square\)

**Remark 5.** The smallness assumption on \(\partial_x R\) is used only to absorb the source term of the right-hand side by the left-hand side, in the previous estimates. In other references (for example [8]), the parameter \(M\) in the Carleman estimate is chosen large enough for this absorption to be possible without additional smallness assumptions on \(R\). In our situation, to use the same trick, one would need dissipation estimates in weighted \(L^2\)-spaces (weight given by the Carleman estimate), which may be quite difficult to prove.
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Appendix. Proof of the Carleman estimate

The goal of this section is the proof of proposition 5. To simplify notations, we take $T_0 = 0$ and $T_1 = T$. Subsection A.1 is devoted to the properties of an appropriate weight function; the Carleman estimate is proved in section A.2.

A.1. Properties of the weight function

**Proposition 13.** Let $\tilde{\omega}_1$ be an open subset of $\Omega_1$. There exist $a, C_1, C_3 > 0$ and $\beta \in C^4(\overline{\Omega}_1; (0, \infty))$ such that

\[
\frac{\partial \beta}{\partial v} \geq 0 \quad \text{on} \quad \partial \Omega_1, \quad (A.1)
\]

\[
(1 - a)(\Delta \beta)(x)|Z|^2 - 2D^2 \beta(x)(Z, Z) \geq C_1|Z|^2, \quad \forall Z \in \mathbb{R}^N, \quad x \in \Omega_1 \setminus \tilde{\omega}_1,
\]

\[
(a - 1)(\Delta \beta)(x)|\nabla \beta(x)|^2 - 2D^2 \beta(x)(\nabla \beta(x), \nabla \beta(x)) \geq C_3, \quad \forall x \in \Omega_1 \setminus \tilde{\omega}_1, \quad (A.2)
\]

where $D^2 \beta(x)$ is the Hessian matrix of the scalar function $\beta$ at point $x$ and $D^2 \beta(x)(Z, Z) = Z^T D^2 \beta(x)Z$ is the associated quadratic form.

Hereafter, for any $N \times N$ matrix $A$ and $Z \in \mathbb{R}^N$, we denote by $A(Z, Z)$ the scalar $Z^T AZ$.

**Proof of proposition 13.** Let $a \in (1, 3)$ and let $\psi \in C^4(\overline{\Omega}_1)$ be such that

\[
\psi > 0 \quad \text{on} \quad \overline{\Omega}_1, \quad \psi = 0 \quad \text{on} \quad \partial \Omega_1 \quad \text{and} \quad |\nabla \psi(x)| > 0, \quad \forall x \in \overline{\Omega}_1 \setminus \tilde{\omega}_1 \quad (A.3)
\]

(see [14] or [12, lemma 2.68 on p 80] for the existence of such a function). Note that the $C^4$-regularity of the boundary of $\Omega_1$ ensures the $C^4$-regularity of the distance to the boundary of $\Omega_1$, which in turn allows the construction of a $C^4$-function $\psi$ with the same construction as in [12, lemma 2.68 on p 80].

There exist numbers $m_*, m^* > 0$ such that

\[
|\nabla \psi(x)| > m_*, \quad |\nabla \psi(x)|, |\Delta \psi(x)|, |D^2 \psi(x)| \leq m^*, \quad \forall x \in \overline{\Omega}_1 \setminus \tilde{\omega}_1.
\]

The function $\beta$ in proposition 13 will be of the form

\[
\beta(x) := e^{2\lambda |\psi|_{L^\infty(\Omega_1)}} - e^{\lambda \psi(x)} \quad (A.4)
\]

for an appropriate parameter $\lambda > 0$. From (A.4), we get

\[
\nabla \beta(x) = -\lambda \nabla \psi(x) e^{\lambda \psi(x)},
\]

\[
D^2 \beta(x) = -(\lambda^2 \nabla \psi(x) \otimes \nabla \psi(x) + \lambda D^2 \psi(x)) e^{\lambda \psi(x)},
\]

\[
\Delta \beta(x) = -((\lambda^2 |\nabla \psi(x)|^2 + \lambda \Delta \psi(x)) e^{\lambda \psi(x)}.
\]

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Using the above relations we get, for any \( x \in \Omega_1 \setminus \tilde{\omega}_1 \),

\[
(1 - a)(\Delta \beta) |Z|^2 - 2D^2 \beta (Z, Z) \\
= (a - 1)[\lambda(\Delta \beta) |\nabla \psi|^2 + \lambda \Delta \psi]|Z|^2 + 2(\lambda^2 |\nabla \psi|^2 + \lambda D^2 \psi (Z, Z)) e^{\lambda \psi} \\
= \lambda^2 ((a - 1)|\nabla \psi|^2 |Z|^2 + 2(\nabla \psi, Z)^2) e^{\lambda \psi} + \lambda ((a - 1) \Delta \psi |Z|^2 + 2D^2 \psi (Z, Z)) e^{\lambda \psi} \\
\geq (\lambda^2 (a - 1)m_2^3 - \lambda (a + 1)m^3)|Z|^2 \\
\]  \hspace{1cm} \text{(A.5)}
and

\[
(a - 1)(\Delta \beta) |\nabla \psi|^2 - 2D^2 \beta (\nabla \psi, \nabla \psi) \\
= [(a - 1)(\lambda^2 |\nabla \psi|^2 + \lambda \Delta \psi) \lambda^2 |\nabla \psi|^2 + 2(\lambda^4 |\nabla \psi|^4 + \lambda^3 D^2 \psi (\nabla \psi, \nabla \psi)) e^{3\lambda \psi} \\
= [\lambda^4 (3 - a)|\nabla \psi|^4 + \lambda^3 ((a - 1) \Delta \psi |\nabla \psi|^2 + 2D^2 \psi (\nabla \psi, \nabla \psi)) e^{3\lambda \psi} \\
\geq (\lambda^4 (3 - a)m_2^3 - \lambda^3 (a + 1)(m^3)^3). \hspace{1cm} \text{(A.6)}
\]

The conclusion follows taking, for example,

\[
\lambda := \max \left\{ \frac{2(a + 1)m^3}{(a - 1)m_2^3}, \frac{2(a + 1)(m^3)^3}{(3 - a)m_2^3} \right\}, \hspace{1cm} C_1 := \frac{(a - 1)m_2^3 \lambda^2}{2}, \hspace{1cm} C_3 := \frac{(3 - a)m_2 \lambda^4}{2}. \hspace{1cm} \square
\]

\section*{A.2. Proof of the Carleman inequality}

Let \( \tilde{\omega}_1 \) be an open subset such that \( \tilde{\omega}_1 \subset \subset \omega_1 \). All the computations of the proof will be made by assuming first that \( u \in H^1(0, T; L^2(\Omega_1)) \cap L^2(0, T; H^1(\Omega_1)) \). Then, the conclusion will follow by a density argument.

Let \( a \in (1, 3) \) and \( \beta \) be as in proposition 13. Let us consider the weight function

\[
\alpha(t, x) := \frac{\beta(x)}{t(T - t)}, \hspace{1cm} (t, x) \in (0, T) \times \mathbb{R}^n, \tag{A.7}
\]

and set

\[
z(t, x) := u(t, x) e^{-M \alpha(t, x)}, \tag{A.8}
\]

where \( M = M(T, \mu, \beta) > 0 \) will be chosen later on. One has

\[
e^{-M \alpha} P_{\mu, \gamma} u = P_1 z + P_2 z + P_3 z, \tag{A.9}
\]

where

\[
P_1 z := -\Delta z + (M \alpha - M^2 |\nabla \alpha|^2) z + \varepsilon \mu |x|^{2\gamma} b(x) z, \]

\[
P_2 z := \frac{\partial z}{\partial t} - 2M \nabla \alpha \cdot \nabla z - aM (\Delta \alpha) z, \]

\[
P_3 z := (a - 1)M (\Delta \alpha) z + (1 - \varepsilon) \mu |x|^{2\gamma} b(x) z, \tag{A.10}
\]

and \( \varepsilon \in \{0, 1\} \) will be chosen later on. Here \( \alpha := \partial \alpha \). We develop the classical proof (see [14]), taking the \( L^2(Q) \)-norm in the identity (A.9), then developing the double product, which leads to

\[
\int_Q \left( P_1 z P_2 z - \frac{1}{2} |P_3 z|^2 \right) \, dx \, dt \leq \frac{1}{2} \int_Q e^{-M \alpha} |P_{\mu, \gamma} u|^2 \, dx \, dt, \tag{A.11}
\]

where \( Q := (0, T) \times \Omega_1 \) and we compute precisely each term, paying attention to the behavior of the different constants with respect to \( \mu \) and \( T \).
Step 1. Computation of the terms in (A.11).

Terms concerning $\Delta z$ in $P_1 P_2$: Integrating by parts, we get

$$-\int Q \frac{\partial z}{\partial t} \, dx \, dt = \int Q \nabla z \cdot \frac{\partial}{\partial t} \nabla z \, dx \, dt = \frac{1}{2} \int_0^T \left( \int_{\partial Q} \frac{\partial z}{\partial t} \, d\sigma + \frac{1}{2} \nabla (|\nabla z|^2) \cdot \nabla \alpha \right) \, dx \, dt \quad (A.12)$$

because $\frac{\partial z}{\partial t}(t, \cdot) = 0$ on $\partial Q_1$ and $z(0, \cdot) = z(T, \cdot) = 0$, by definition of $z$. Using Green’s formula and the relation $\nabla z = \frac{\partial z}{\partial v}$ on $\partial Q_1$, we get

$$2M \int Q \Delta z \nabla \alpha \cdot \nabla z \, dx \, dt \quad (A.13)$$

Using the Green formula and $z = 0$ on $\partial Q_1$, we get

$$aM \int Q (\Delta z) (\Delta z) \, dx \, dt = -aM \int Q (|\nabla z|^2 \Delta \alpha + \nabla z \cdot \nabla (\Delta \alpha)) \, dx \, dt \quad (A.14)$$

Terms concerning $(M \alpha - M^2 |\nabla \alpha|^2) z$ in $P_1 P_2$: Integrating by parts, we get

$$\int Q (M \alpha - M^2 |\nabla \alpha|^2) z \frac{\partial z}{\partial t} \, dx \, dt = -\frac{1}{2} \int_Q (M \alpha - M^2 |\nabla \alpha|^2) \frac{\partial z}{\partial t} \, dx \, dt \quad (A.15)$$

Using the Green formula and $z(t, \cdot) = 0$ on $\partial Q_1$, we get

$$-2M \int Q (M \alpha - M^2 |\nabla \alpha|^2) \nabla \alpha \cdot \nabla z \, dx \, dt = M \int Q \nabla [\frac{1}{2} (M \alpha - M^2 |\nabla \alpha|^2) \nabla \alpha] |\nabla z|^2 \, dx \, dt \quad (A.16)$$

Finally, the last term concerning $(M \alpha - M^2 |\nabla \alpha|^2) z$ is

$$-aM \int Q (M \alpha - M^2 |\nabla \alpha|^2) (\Delta \alpha) |z|^2 \, dx \, dt \quad (A.17)$$

Terms concerning $\epsilon |x|^{2\gamma} b(x) z$ in $P_1 P_2$: Integrating by parts, we get

$$\int Q \epsilon |x|^{2\gamma} b(x) z \frac{\partial z}{\partial t} \, dx \, dt = \frac{1}{2} \int_0^T \left( \int_{\partial Q} \epsilon |x|^{2\gamma} b(x) \, d\sigma + \epsilon |x|^{2\gamma} b(x) |\nabla \alpha|^2 \right) \, dx \, dt = 0 \quad (A.18)$$

because $z(0, \cdot) = z(T, \cdot) = 0$. Using Green’s formula and $z(t, \cdot) = 0$ on $\partial Q_1$, we get

$$-2\epsilon M \int Q |x|^{2\gamma} b(x) \nabla \alpha \cdot \nabla z \, dx \, dt = \epsilon M \int Q \nabla [\frac{1}{2} |x|^{2\gamma} b(x) \nabla \alpha] |\nabla z|^2 \, dx \, dt \quad (A.19)$$

Finally, the last term concerning $\epsilon |x|^{2\gamma} b(x) z$ is

$$-\epsilon aM \mu \int Q |x|^{2\gamma} b(x) (\Delta \alpha) |z|^2 \, dx \, dt \quad (A.20)$$

Terms concerning $P_3$: We have

$$-\frac{1}{2} \int Q |P_3 z|^2 \, dx \, dt = -\frac{1}{2} \int Q [(a - 1)M (\Delta \alpha) z + (1 - \epsilon) \mu |x|^{2\gamma} b(x) z] |z|^2 \, dx \, dt \quad (A.21)$$

$$\geq -\int Q [(a - 1)^2 M^2 (\Delta \alpha)^2 |z|^2 + (1 - \epsilon)^2 \mu^2 |x|^{4\gamma} b(x)^2 |z|^2] \, dx \, dt \quad (A.22)$$
Using the relations \((A.12)\) to \((A.20)\), the inequality \((A.11)\) may be written
\[
\int_{\Omega} M((1 - a)(\Delta \alpha)|\nabla z|^2 - 2D^2 \alpha(\nabla z, \nabla z))
+ \int_{\Omega} \left(f + \epsilon M \mu(\text{div}([x]^{\nu} b(x) \nabla \alpha] - a|x|^{\nu} b(x) \Delta \alpha) - (1 - \epsilon)^2 \mu^2 |x|^{\nu} b(x)^2 \right)|z|^2
\leq \frac{1}{2} \int_{\Omega} |P_{\mu, \nu} u|^2 e^{-2Ma},
\]
(A.21)
where
\[
f := \frac{Ma}{2}(\Delta^2 \alpha) - \frac{1}{2}(M\alpha_t - M^2 |\nabla \alpha|^2)_t + M \text{ div}([M\alpha_t - M^2 |\nabla \alpha|^2] \nabla \alpha)
- aM(M\alpha_t - M^2 |\nabla \alpha|^2) \Delta \alpha - (a - 1)^2 M^2(\Delta \alpha)^2.
\]

**Step 2. Estimation of the terms in \((A.21)\).**

Using \((A.1)\), we deduce that
\[
\int_{\Omega} M((1 - a)(\Delta \alpha)|\nabla z|^2 - 2D^2 \alpha(\nabla z, \nabla z)) \, dx \, dt
+ \int_{\Omega} \left(f + \epsilon M \mu(\text{div}([x]^{\nu} b(x) \nabla \alpha] - a|x|^{\nu} b(x) \Delta \alpha) - (1 - \epsilon)^2 \mu^2 |x|^{\nu} b(x)^2 \right)|z|^2
\leq \frac{1}{2} \int_{\Omega} |P_{\mu, \nu} u|^2 e^{-2Ma}.
\]
(A.22)

We remark that
\[
f = \frac{1}{t(T - t)} \left\{ M^3[(a - 1)|\nabla \beta|^2 \Delta \beta - 2D^2 \beta(\nabla \beta, \nabla \beta)]
\right.
+ M^2[(2t - T)(|\nabla \beta|^2 + \text{div}(\beta \nabla \beta) - a \beta \Delta \beta) - (a - 1)^2 t(T - t)(\Delta \beta)^2]
\left. + M \left[ \frac{d}{dt} \Delta \beta(t(T - t))^2 - \beta(T^2 - 3Tt + 3t^2) \right] \right\}.
\]
(A.23)

Using \((A.2)\) and the \(C^4\) regularity of \(\beta\) on the compact set \(\overline{\Omega}_1\), we get constants \(C_j = C_j(\beta) > 0\) for \(j = 2, 3, 4, c = c(\beta) > 0\) such that
\[
f \geq \frac{1}{t(T - t)} \left[ C_3 M^3 - c(T + T^2)M^2 - c(T + T^2)^2 M \right], \quad \forall (t, x) \in (0, T) \times (\Omega_1 \setminus \tilde{\omega}_1),
\]
\[
|f| \leq \frac{1}{t(T - t)} \left[ C_4 M^3 + c(T + T^2)M^2 + c(T + T^2)^2 M \right], \quad \forall (t, x) \in (0, T) \times \tilde{\omega}_1,
\]
\[
|(1 - a)(\Delta \alpha)|\nabla z|^2 - 2D^2 \alpha(\nabla z, \nabla z)| \leq \frac{C_5 M}{t(T - t)} |\nabla z|^2, \quad \forall (t, x) \in [0, T] \times \Omega_1.
\]

Thus, there exist \(m_1 = m_1(\beta), C_3 = C_3(\beta), C_4 = C_4(\beta) > 0\) such that, for every \(M \geq M_1(T, \beta)\)
\[
f \geq \frac{C_3 M^3}{t(T - t)} t, \quad \forall (t, x) \in (0, T) \times (\Omega_1 \setminus \tilde{\omega}_1),
\]
\[
|f| \leq \frac{C_4 M^3}{t(T - t)} t, \quad \forall (t, x) \in (0, T) \times \tilde{\omega}_1,
\]
where
\[
M_1(T, \beta) := m_1(\beta)(T + T^2).
\]
(A.24)
By (A.2) and the previous inequalities, we get, for $M \geq M_1(T, \beta)$,

$$\int_0^T \int_{\Omega \setminus \tilde{\omega}} \frac{C_1 M}{t(T - t)} |\nabla z|^2 \, dx \, dt$$

$$+ \int_0^T \int_{\Omega \setminus \tilde{\omega}} \left[ \frac{C_2 M^3}{t(T - t)^3} + \epsilon M \mu (\text{div}[|x|^{2\gamma} b \nabla \alpha] - a|x|^{2\gamma} b \Delta \alpha) - (1 - \epsilon) \mu^2 |x|^{4\gamma} b^2 \right] |z|^2$$

$$\leq \int_0^T \int_{\tilde{\omega}} \frac{C_1 M}{t(T - t)} |\nabla z|^2$$

$$+ \int_0^T \int_{\tilde{\omega}} \left[ \frac{C_2 M^3}{t(T - t)^3} - \epsilon M \mu (\text{div}[|x|^{2\gamma} b \nabla \alpha] - a|x|^{2\gamma} b \Delta \alpha) + (1 - \epsilon) \mu^2 |x|^{4\gamma} b^2 \right] |z|^2$$

$$+ \frac{1}{2} \int_Q |e^{-Ma} P_{\mu, \gamma} u|^2 \, dx \, dt. \tag{A.25}$$

**Step 3. End of the proof when $\gamma \in [1/2, 1]$.** We take $\epsilon = 1$. Then (A.25) is written

$$\int_0^T \int_{\Omega \setminus \tilde{\omega}} \frac{C_1 M}{t(T - t)} |\nabla z|^2 \, dx \, dt$$

$$+ \int_0^T \int_{\Omega \setminus \tilde{\omega}} \left[ \frac{C_2 M^3}{t(T - t)^3} + M \mu (\text{div}[|x|^{2\gamma} b(x) \nabla \alpha] - a|x|^{2\gamma} b(x) \Delta \alpha) \right] |z|^2$$

$$\leq \int_0^T \int_{\tilde{\omega}} \frac{C_1 M}{t(T - t)} |\nabla z|^2 + \int_0^T \int_{\tilde{\omega}} \left[ \frac{C_2 M^3}{t(T - t)^3} - M \mu (\text{div}[|x|^{2\gamma} b(x) \nabla \alpha] - a|x|^{2\gamma} b(x) \Delta \alpha) \right] |z|^2$$

$$+ \frac{1}{2} \int_Q |e^{-Ma} P_{\mu, \gamma} u|^2 \, dx \, dt. \tag{A.26}$$

There exists $C_3 = C_3(\beta) > 0$ such that

$$|M \mu (\text{div}[|x|^{2\gamma} b(x) \nabla \alpha] - a|x|^{2\gamma} b(x) \Delta \alpha)| \leq \frac{C_3 M \mu}{t(T - t)}, \quad \forall (t, x) \in Q. \tag{A.27}$$

Let $M_2 = M_2(T, \mu, \beta)$ be defined by

$$M_2 = M_2(T, \mu, \beta) := \sqrt{\frac{2C_3}{C_3}} \sqrt{\mu \left( \frac{T}{2} \right)^2}. \tag{A.28}$$

From now on, we take

$$M = M(T, \mu, \beta) := C_2 \max\{T + T^2; \sqrt{\mu} T^2\} \tag{A.29}$$

where

$$C_2 = C_2(\beta) := \max \left\{ m_1; \sqrt{\frac{C_3}{8C_3}} \right\}$$

so that $M \geq M_1$ and $M_2$ (see (A.24) and (A.28)). It is only at this step that the dependence of $M$ with respect to $\mu$ has to be specified. From $M \geq M_2$, we deduce that

$$|M \mu (\text{div}[|x|^{2\gamma} b(x) \nabla \alpha] - a|x|^{2\gamma} b(x) \Delta \alpha)| \leq \frac{C_3 M^3}{2t(T - t)^3}, \quad \forall (t, x) \in Q.$$
Indeed,
\[
\frac{C_3 M \mu}{i(T - t)} = \frac{C_3 M^3}{2[i(T - t)]^3} \frac{2\mu C_3 |t(T - t)|^2}{C_3 M^2} \\
\leq \frac{C_3 M^3}{2[i(T - t)]^3} \mu C_5 T^4 \frac{2C_3 M^2}{2C_3 M^2} \\
\leq \frac{C_3 M^3}{2[i(T - t)]^3}.
\]

Thus, (A.26) implies
\[
\int_0^T \int_{\Omega_1} \left( \frac{C_3 M}{i(T - t)} |\nabla u|^2 + \frac{C_3 M^3}{2[i(T - t)]^3} |\xi|^2 \right) \, dx \, dt \\
\leq \int_0^T \int_{\Omega_1} \left( \frac{C_3 M}{i(T - t)} |\nabla u|^2 + \frac{C_3 M^3}{(i(T - t))^3} |\xi|^2 \right) \, dx \, dt + \frac{1}{2} \int_Q |\epsilon_\mu^\rho \nabla \beta|^2.
\]  
(A.30)

where \(C_6 = C_6(\beta) := C_4 + C_5/2\). For every \(\epsilon' > 0\), we have
\[
\frac{C_3 M}{i(T - t)} |\nabla u - M(\nabla \alpha)u|^2 + \frac{C_3 M^3}{2[i(T - t)]^3} |u|^2 \geq \left( 1 - \frac{1}{1 + \epsilon'} \right) \frac{C_3 M}{i(T - t)} |\nabla u|^2 \\
+ \frac{M^3}{(i(T - t))^3} \left( \frac{C_3 M}{2} - \epsilon' C_1 |\nabla \beta|^2 \right) |u|^2.
\]  
(A.31)

Hence, choosing
\[
\epsilon' = \epsilon'(\beta) := \frac{C_3}{4C_1 \|\nabla \beta\|_\infty^2},
\]
from (A.30), (A.31) and (A.8) we deduce that
\[
\int_0^T \int_{\Omega_1} \left( \frac{C_3 M}{i(T - t)} |\nabla u|^2 + \frac{C_3 M^3 |u|^2}{4[i(T - t)]^3} \right) e^{-2Ma} \, dx \, dt \\
\leq \int_0^T \int_{\Omega_1} \left( \frac{C_3 M}{i(T - t)} |\nabla u|^2 + \frac{C_3 M^3 |u|^2}{(i(T - t))^3} \right) e^{-2Ma} + \frac{1}{2} \int_Q |\epsilon_\mu^\rho \nabla \beta|^2,
\]  
(A.32)

where \(C_7 = C_7(\beta) := [1 - 1/(1 + \epsilon')]C_1\), \(C_8 = C_8(\beta) := 2C_2\) and \(C_9 = C_9(\beta) := C_6 + 2C_2 \|\nabla \beta\|_\infty^2\). So, adding the same quantity to both sides,
\[
\int_Q \left( \frac{C_3 M}{i(T - t)} |\nabla u|^2 + \frac{C_3 M^3 |u|^2}{4[i(T - t)]^3} \right) e^{-2Ma} \leq \frac{1}{2} \int_Q |\epsilon_\mu^\rho \nabla \beta|^2 \\
+ \int_0^T \int_{\Omega_1} \left( \frac{C_{10} M}{i(T - t)} |\nabla u|^2 + \frac{C_{11} M^3 |u|^2}{(i(T - t))^3} \right) e^{-2Ma},
\]  
(A.33)

where \(C_{10} = C_{10}(\beta) := C_8 + C_7\) and \(C_{11} = C_{11}(\beta) := C_9 + C_7/4\). Let us prove that the second term on the right-hand side can be dominated by terms similarly to the other two ones. We consider \(\rho \in C^\infty(\mathbb{R}_N; \mathbb{R}_+ )\) such that \(0 \leq \rho \leq 1\) and
\[
\rho \equiv 1 \text{ on } \omega_1 \text{ and } \rho \equiv 0 \text{ on } \mathbb{R}_N \setminus \omega_1.
\]
We have
\[
\int_Q (\epsilon_\mu^\rho \nabla \beta) \frac{\mu \rho e^{-2Ma}}{i(T - t)} \, dx \, dt = \int_0^T \int_{\Omega_1} \left[ \frac{\partial u}{\partial t} - \Delta u + \mu |x|^2 b(x) u \right] \mu \rho e^{-2Ma} \frac{i(T - t)}{i(T - t)}.
\]
Integrating by parts with respect to time and space, we obtain
\[
\int_Q \frac{1}{2} \frac{\partial (u^2)}{\partial t} \rho e^{2 Ma} \, dx \, dt = \int_Q \frac{1}{2} |u|^2 \rho \left( \frac{2 Ma}{t(T-t)} + \frac{T-2t}{t(t(T-t))^2} \right) e^{-2 Ma}
\]
and
\[
- \int_Q \Delta u \rho e^{2 Ma} \frac{1}{t(T-t)} \, dx \, dt = \int_Q \rho e^{2 Ma} |\nabla u|^2 - \int_Q \frac{|u|^2 e^{2 Ma}}{2t(T-t)} \left( \Delta \rho - 4M \nabla \rho \cdot \nabla \alpha + \rho \left( 4M^2 |\nabla \alpha|^2 - 2M \Delta \alpha \right) \right).
\]
Thus,
\[
\int_Q P_{\mu,\gamma} u \rho e^{2 Ma} \frac{1}{t(T-t)} \, dx \, dt \gtrless \int_Q \rho e^{2 Ma} |\nabla u|^2 - \int_Q \frac{|u|^2 e^{2 Ma}}{2t(T-t)} \left( \Delta \rho - 4M \nabla \rho \cdot \nabla \alpha + \rho \left( 4M^2 |\nabla \alpha|^2 - 2M \Delta \alpha - \frac{T-2t}{t(T-t)} \right) \right).
\]
Therefore,
\[
\int_0^T \int_{\Omega_1} \frac{C_{10} M}{t(T-t)} |\nabla u|^2 e^{-2 Ma} \, dx \, dt \leq \int_0^T \int_Q \rho e^{2 Ma} |\nabla u|^2 e^{-2 Ma}
\]
\[
\leq \int_Q P_{\mu,\gamma} u \rho e^{2 Ma} \frac{1}{t(T-t)}
\]
\[
+ \int_Q \frac{C_{10} M |u|^2 e^{2 Ma}}{2t(T-t)} \left( \Delta \rho - 4M \nabla \rho \cdot \nabla \alpha + \rho \left( 4M^2 |\nabla \alpha|^2 - 2M \Delta \alpha - \frac{T-2t}{t(T-t)} \right) \right)
\]
\[
\leq \int_Q |P_{\mu,\gamma} u|^2 + \int_0^T \int_{\Omega_1} \frac{C_{12} M^3 |u|^2 e^{2 Ma}}{t(T-t)^3} \, dx \, dt
\]
for some constant \(C_{12} = C_{12}(\beta, \rho) > 0\). Combining (A.33) with the previous inequality, we get
\[
\int_Q \left( \frac{C_{17} M}{t(T-t)} |\nabla u|^2 + \frac{C_{13} M^3 |u|^2}{4(t(T-t))^3} \right) e^{-2 Ma} \, dx \, dt
\]
\[
\leq \int_Q 2 |e^{-Ma} P_{\mu,\gamma} u|^2 + \int_0^T \int_{\Omega_1} \frac{C_{14} M^3 |u|^2}{(t(T-t))^3} e^{-2 Ma},
\]
where \(C_{13} = C_{13}(\beta, \rho) := C_{11} + C_{12}\). Then, the global Carleman estimates (13) holds with \(M\) in (A.36) replaced by \(M/2\) and
\[
C_1 = C_1(\beta) := \min \{C_7, C_3/4 \over \max \{2, C_{13} \}}.
\]

**Step 4. End of the proof when \(\gamma \in (0, 1/2)\).** The left-hand side of (A.27) diverges at \(x = 0\), thus the proof cannot be ended in the same way and we take \(\epsilon = 0\). Then (A.25) is written
\[
\int_0^T \int_{\Omega_1} \frac{C_M}{t(T-t)} |\nabla z|^2 \, dx \, dt + \int_0^T \int_{\Omega_1} \frac{C_M^3}{(t(T-t))^3} \left[ \mu^2 |x|^{4\gamma} b(x) \right] |z|^2 \, dx \, dt
\]
\[
\leq \int_0^T \int_{\Omega_1} \frac{C_{22} M}{t(T-t)} |\nabla z|^2 + \int_0^T \int_{\Omega_1} \frac{C_{23} M^3}{(t(T-t))^3} + \mu^2 |x|^{4\gamma} b(x) \right] |z|^2
\]
\[
+ \frac{1}{2} \int_Q |e^{-Ma} P_{\mu,\gamma} u|^2 \, dx \, dt.
\]
Let
\[ M_2 = M_2(T, \beta, \mu) := \frac{T^2}{4} \mu^{2/3} \sqrt{\frac{2R^4 \|b\|_2^2}{C_3^3}} \] (A.38)
where \( R > 0 \) is such that \( \Omega_1 \subset B(0, R) \). From now on, we take
\[ M = M(T, \mu, \beta) := \mathcal{C}_2 \max \{T + T^2; \mu^{2/3} T^2\} \] (A.39)
where
\[ \mathcal{C}_2 = \mathcal{C}_2(\beta) := \max \left\{ m_1; \frac{1}{4} \sqrt{\frac{2R^4 \|b\|_2^2}{C_3^3}} \right\} \]
so that \( M \geq M_1 \) and \( M_2 \) (see (A.24) and (A.38)). It is only at this step that the dependence of \( M \) with respect to \( \mu \) has to be specified. From \( M \geq M_2 \), we deduce that
\[ \mu^2 |x|^{4/3} b(x)^2 \leq \frac{C_2^2 M^3}{2(t(T-t))^{1/3}}, \quad \forall (t, x) \in Q. \]
From (A.37), we are led to an inequality of the form (A.30) and the proof may be finished as in step 3. \( \square \)

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