Classical probabilistic realization of “Random Numbers Certified by Bell’s Theorem”

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Abstract. We question the commonly accepted statement that random numbers certified by Bell’s theorem carry some special sort of randomness, so to say, quantum randomness or intrinsic randomness. We show that such numbers can be easily generated by classical random generators.

1. Introduction
The idea that quantum randomness (QR) differs crucially from classical randomness (CR) is due to von Neumann who pointed that QR is intrinsic and CR is reducible to variability in an ensemble [1]. At the first stages of the development of quantum theory (QM) von Neumann’s viewpoint to QR and its exceptional nature was merely of the purely foundational value. It was used to support various no-go statements, starting with von Neumann’s no-go theorem [1] (in fact, this no-go statement was called by von Neumann “ansatz” and “theorem” as it is common in modern literature), see [2], [3] for discussions. However, with the development of quantum information theory and its technological implementation the inter-relation between classical and quantum randomness started to play the important role in real applications.

In this note we shall discuss one very novel and important application of quantum randomness, namely, in the form of random number generators certified by Bell’s theory, see, e.g., [4] for details and review. As was claimed in [4]:

“It is thereby possible to design of a new type of cryptographically secure random number generator which does not require any assumption on the internal working of the devices. This strong form of randomness generation is impossible classically and possible in quantum systems only if certified by a Bell inequality violation.”

It seems that this statement is in contradiction with the recent results [5], see also [6], on embedding of the data generated in experimental test demonstrating violations of Bell’s inequality into the classical probability model (based on the Kolmogorov measure-theoretic model [7], 1933). As well as in [4], we consider the case of CHSH inequality which were tested and confirmed in numerous experiments (although it might happen that the final loophole free test will be performed not for Bell’s inequality not in the CHSH form, but in the Eberhard form [8]).
2. Classical probability space for data violating the CHSH inequality

For readers convenience, we present shortly the model from [5].

In the CHSH-test we operate with probabilities \( p_{ij}(\epsilon, \epsilon') \), \( \epsilon, \epsilon' = \pm 1 \), to get the results \( a_{ij} = \epsilon, b_{ij} = \epsilon' \) in the experiment with the fixed pair of orientations \((\theta_i, \theta'_j)\). We borrow probabilities from the mathematical formalism of quantum mechanics:

\[
p_{ij}(\epsilon, \epsilon) = \frac{1}{2} \cos^2 \frac{\theta_i - \theta'_j}{2}, \quad p_{ij}(\epsilon, -\epsilon) = \frac{1}{2} \sin^2 \frac{\theta_i - \theta'_j}{2}.
\]

Let us now consider the set of points \( \Omega \) (the space of “elementary events” in Kolmogorov’s terminology):

\[
\Omega = \{ (\epsilon_1, 0, \epsilon'_1, 0), (\epsilon_1, 0, 0, \epsilon'_2), (0, \epsilon_2, \epsilon'_1, 0), (0, \epsilon_2, 0, \epsilon'_2) \},
\]

where \( \epsilon = \pm 1, \epsilon' = \pm 1 \).

We define the following probability measure\(^1\) on \( \Omega \):

\[
P(\epsilon_1, 0, \epsilon'_1, 0) = \frac{1}{4} p_{11}(\epsilon_1, \epsilon'_1),
\]

\[
P(\epsilon_1, 0, \epsilon'_2, 0) = \frac{1}{4} p_{12}(\epsilon_1, \epsilon'_2),
\]

\[
P(0, \epsilon_2, \epsilon'_1, 0) = \frac{1}{4} p_{21}(\epsilon_2, \epsilon'_1),
\]

\[
P(0, \epsilon_2, \epsilon'_2, 0) = \frac{1}{4} p_{22}(\epsilon_2, \epsilon'_2).
\]

We now define random variables \( A^{(i)}(\omega), B^{(j)}(\omega) \):

\[
A^{(1)}(\epsilon_1, 0, \epsilon'_1, 0) = A^{(1)}(\epsilon_1, 0, 0, \epsilon'_2) = \epsilon_1, A^{(2)}(0, \epsilon_2, \epsilon'_1, 0) = A^{(2)}(0, \epsilon_2, 0, \epsilon'_2) = \epsilon_2;
\]

\[
B^{(1)}(\epsilon_1, 0, \epsilon'_1, 0) = B^{(1)}(0, \epsilon_2, \epsilon'_1, 0) = \epsilon'_1, B^{(2)}(\epsilon_1, 0, 0, \epsilon'_2) = B^{(2)}(0, \epsilon_2, 0, \epsilon'_2) = \epsilon'_2.
\]

and we put these variables equal to zero in other points.

We find two dimensional probabilities

\[
P(\omega \in \Omega : A^{(1)}(\omega) = \epsilon_1, B^{(1)}(\omega) = \epsilon'_1) = \frac{1}{4} p_{11}(\epsilon_1, \epsilon'_1),
\]

\[
P(\omega \in \Omega : A^{(2)}(\omega) = \epsilon_2, B^{(2)}(\omega) = \epsilon'_2) = \frac{1}{4} p_{22}(\epsilon_2, \epsilon'_2).
\]

We also consider the random variables which are responsible for selections of pairs settings \((\theta_i, \theta'_j)\). For “the settings at LHS”:

\[
\eta_L(\epsilon_1, 0, 0, \epsilon'_2) = \eta_L(\epsilon_1, 0, \epsilon'_1, 0) = 1, \eta_L(0, \epsilon_2, 0, \epsilon'_2) = \eta_L(0, \epsilon_2, \epsilon'_1, 0) = 2.
\]

For “the settings at RHS”:

\[
\eta_R(\epsilon_1, 0, \epsilon'_1, 0) = \eta_R(0, \epsilon_2, \epsilon'_1, 0) = 1, \eta_R(0, \epsilon_2, 0, \epsilon'_2) = \eta_R(\epsilon_1, 0, 0, \epsilon'_2) = 2.
\]

The points of the space of “elementary events” \( \Omega \) have the following interpretation. Take, for example, \( \omega = (\epsilon_1, 0, \epsilon'_1, 0) \). Here \( \eta_L = 1 \), i.e., the LHS setting is selected as \( \theta_1 \), and \( \eta_R = 1 \), i.e., the RHS setting is selected as \( \theta'_1 \). Then the random variable \( A^{(1)} = \epsilon_1, A^{(1)} = 0 \) and \( B^{(1)} = \epsilon'_1, B^{(1)} = 0 \). Consider now, e.g., \( \omega = (\epsilon_1, 0, 0, \epsilon'_2) \). Here \( \eta_L = 1 \), i.e., the LHS setting is selected as \( \theta_1 \), and \( \eta_R = 2 \), i.e., the RHS setting is selected as \( \theta'_2 \). Then the random variable \( A^{(1)} = \epsilon_1, A^{(1)} = 0 \) and \( B^{(1)} = 0, B^{(1)} = \epsilon'_2 \).

\(^1\) To match completely with Kolmogorov’s terminology, we have to select a \( \sigma \)-algebra \( F \) of subsets of \( \Omega \) representing events and define the probability measure on \( F \). However, in the case of a finite \( \Omega = \{ \omega_1, \ldots, \omega_k \} \) the system of events \( F \) is always chosen as consisting of all subsets of \( \Omega \). To define a probability measure on such \( F \), it is sufficient to define it for one-point sets, \( \{ \omega_m \} \to P(\omega_m) \), \( \sum_m P(\omega_m) = 1 \), and to extend it by additivity: for any subset \( O \) of \( \Omega \), \( P(O) = \sum_{\omega_m \in O} P(\omega_m) \).
3. Classical random generation of “Random Numbers Certified by Bell’s Theorem”

By using the previous classical probabilistic representation of CHSH-probabilities, we can now construct the classical random generator (at least theoretically) producing 6-dimensional vectors \( \xi_m = (A_m^{(1)}, A_m^{(2)}, B_m^{(1)}, B_m^{(2)}, \eta_{Lm}, \eta_{Rm}), m = 1, 2, \ldots \). The last two coordinates take values 1, 2 and the first four coordinates take the values \([-1, 0, +1]\). Now if we filter from these 6-dimensional vectors the zero coordinates, we shall get 4-dimensional vectors \((a_m, b_m, \eta_{Lm}, \eta_{Rm})\) which are indistinguishable from the viewpoint of Bell’s test from those obtained, e.g., in the quantum optics experiments. The corresponding conditional probabilities, conditioning with respect to selection the fixed pairs of settings \((\theta_i, \theta'_j)\), violate the CHSH inequality.

Finally we remark that our random generator is “local”, see [6] for details.

4. Conclusion

In the light of recent author’s results on embedding of quantum probabilistic data into the classical Kolmogorov model of probability von Neumann’s statement about irreducible and intrinsic quantum randomness as opposed to classical reducible randomness which based only on the lack of knowledge has to be seriously reanalyzed (as well as its quantum information technological implementations). In fact, already careful reading of von Neumann’s book [1] makes the impression that his conclusion about novel features of quantum probability and randomness comparing with their classical counter-parts was not completely logically justified. Roughly speaking he said. See, each electron is individually random! However, at the same time he pointed out that this individual randomness can be gained only from an ensemble of electrons. The latter ensemble approach to probability is purely classical. Therefore it would be natural to expect that one might be able to embed quantum probability into the classical probability model. Precisely this was done in [5], [6].

In our approach the CHSH-probabilities appear as classical conditional probabilities. In general I claim that all quantum probabilities can be modeled as classical conditional probabilities. In such a situation it is difficult to believe in some mystical and non-classical features of quantum random generators.

Finally, I remark that in this paper I do not question the quality of “Random Numbers Certified by Bell’s Theorem”. If they pass the standard tests for randomness, e.g., the NIST-test, they are good, if not, then they are bad, irrespective to non/violation of the CHSH-inequality and irrelevant to the problem of closing of experimental loopholes [8]. My main concern is about claims that quantum random generators are in some way better than classical ones.

Acknowledgments

This paper was written with support by the grant Modeling of Complex Hierarchic Systems, the Faculty of Technology, Linnaeus University, and the grant of Quantum BioInformatic Center of Tokyo University of Science. The author would like to thank Luigi Accardi, Irina Basieva, Masanori Ohya, and Anton Zeilinger for critical discussions on comparison of classical and quantum random generators.

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