Casimir Energies for Spheres in \( n \)-dimensional Minkowski Space and Generalized Bernoulli Polynomials

Patrick Moylan
Department of Physics, The Pennsylvania State University, Abington College, Abington, PA, 19001, USA
E-mail: pjm11@psu.edu

Abstract. We consider a mathematical version of conformal quantum field theory in \( n \) dimensions which give a unified approach to quantum field theory on Minkowski space and the Einstein universe. This unifying treatment enables us to relate massless fields in Minkowski space with massless fields in the Einstein universe into which such fields uniquely and conformally extend. Apart from certain global differences, fields in the Einstein universe approximate fields in Minkowski space as the radius of the Einstein universe tends to infinity and we utilize this fact to describe what seems to be a precise method for determining Casimir energies on spheres of arbitrary radii in \( n \)-dimensional Minkowski space.

1. Conformal geometry in two dimensions and massless representations of \( SO(2,2) \)

For presentational clarity we treat the \( n = 2 \) case in complete detail, since it exhibits all of the essential aspects of our approach for the higher dimensional cases. Let \( \mathbb{RP}^3 \) denote the real projective three space and consider the projective quadric \( \mathcal{M}(2) \) in \( \mathbb{RP}^5 \) defined by the equation \( Q(u, u) = u_{-1}^2 + u_0^2 - u_1^2 - u_2^2 = 0 \) where an equivalence class of points of the form \( u = (u_{-1}, u_0, u_1, u_2) \in \mathbb{RP}^4 \) denotes a point in \( \mathbb{RP}^3 \). Noting that \( u \) and \( -u \) denote the same point of \( Q \), it follows that \( \mathcal{M}(2) \) is homeomorphic to \( (S^1 \times S^1)/\mathbb{Z}_2 \). Clearly the standard action of \( SO(2,2) \) on \( \mathbb{RP}^4 \) induces action on \( \mathcal{M}(2) \). \( \mathcal{M}(2) \) is stable under this action, and it is well-known that this action is transitive on \( \mathcal{M}(2) \).

Next consider the double cover of \( \mathcal{M}(2) \) which is \( S^1 \times S^1 \). We denote this space-time by \( \tilde{\mathcal{M}} \) or, more specifically, by \( \tilde{\mathcal{M}}(2) \). A point \( u \) in \( \tilde{\mathcal{M}}(2) \) is clearly determined by \( u_{-1}^2 + u_0^2 = u_1^2 + u_2^2 = 1 \) and we may introduce spherical coordinates on \( \tilde{\mathcal{M}}(2) \) as follows:

\[
\begin{align*}
    u_{-1} &= \cos \theta, \\
    u_0 &= \sin \theta, \\
    u_1 &= \sin \rho, \\
    u_2 &= \cos \rho
\end{align*}
\] (1.1)
with the ranges of the angular parameters being \( 0 < t < 2\pi \), \( 0 < \rho < 2\pi \). An \( S^1 \times S^1 \) invariant metric on \( \bar{M}(2) \) is

\[
d\sigma^2 = dt^2 - d\rho^2 \tag{1.2}
\]

where \( dt \) and \( d\rho \) denote arc length in radians on \( S^1 \). The standard curved measure on \( \bar{M}(2) \) is given by the form

\[
d^2 u = dt \wedge d\rho . \tag{1.3}
\]

Now define two dimensional Minkowski space, \( M_0(2) \), to be \( \mathbb{R}^2 \) as a vector space, but whose metrical structure is that of a two dimensional Lorentzian manifold with infinitesimal arc length \( ds \) in an inertial coordinate system determined by

\[
ds^2 = dx_0^2 - dx_1^2 . \tag{1.4}
\]

The above defined action of \( SO(2, 2) \) on \( M(2) \) is that of (global) conformal transformations on \( \bar{M}(2) \), and there is a conformal embedding \( \pi^{-1} \) from \( M_0(2) \) into \( \bar{M}(2) \), given by \( \pi^{-1}(x) = u \), with [1]

\[
\begin{align*}
  u_{-1} &= p \, F, 
  u_0 &= p \, x_0, 
  u_1 &= p \, x_1, 
  u_2 &= p \, D \tag{1.5}
\end{align*}
\]

where

\[
p = \left( F^2 + x_0^2 \right)^{-\frac{1}{2}}, 
  F = 1 - \frac{x_M^2}{4}, 
  D = 1 + \frac{x_M^2}{4} \tag{1.6}
\]

with \( x_M^2 = x_0^2 - x_1^2 \). The embedding map \( \pi^{-1} \) is conformal, since

\[
  d\sigma^2 = p^2 \, ds^2 . \tag{1.7}
\]

The standard volume form on \( M_0(2) \) is given by

\[
d^2 x = dx_0 \wedge dx_1 . \tag{1.8}
\]

Using \( \pi^{-1} \) we find

\[
d^2 x = (p)^{-2} \, d^2 u . \tag{1.9}
\]

Note that equations (1.5) and (1.1) give

\[
\begin{align*}
x_0 &= 2 \frac{\sin t}{(\cos t + \cos \rho)}, 
  x_1 &= \frac{2 \sin \rho}{(\cos t + \cos \rho)} . \tag{1.10}
\end{align*}
\]

The Einstein energy, \( H \), is just the generator of rotations on the first \( S^1 \) factor, i.e.

\[
H = \frac{\partial}{\partial \tau} \tag{1.11}
\]

on \( \bar{M}(2) \), and thus the action of the one parameter group of conformal transformations generated by \( H \), sends a wave function \( \phi(\tau, \rho) \) into \( \phi(\tau - s, \rho) \), i.e., under Einstein temporal evolution

\[
\tau \rightarrow \tau - s \tag{1.12}
\]

with \( s \in \mathbb{R} \). The action of the one parameter group of time translations in Minkowski space is generated by

\[
P_0 = \frac{\partial}{\partial x_0} . \tag{1.13}
\]
$P_0$ is the Minkowski energy.

Let $C^0_K(\mathcal{M})$ be the space of all $K$ finite elements of $C^\infty(\mathcal{M})$ for which $\phi(u) = \phi(-u)$ ($u \in \mathcal{M}$). ($K \cong SO(2) \times SO(2)$.) Now for $u \in \mathcal{M}$ we have $u = (\tau, \rho)$ with $\tau$ and $\rho \in S^1$. A basis for $C^0_K(\mathcal{M})$ is given by the $K$–finite functions

$$
\phi_{n,m}(\tau, \rho) = e^{i(n\tau + m\rho)}
$$

with $n, m = 0, \pm 1, \pm 2, \pm 3, \ldots$, and with $(-1)^{(n+m)} = 1$ which comes from the above condition that $\phi \in C^0_K(\mathcal{M})$ must be even under the antipodal map, $u \rightarrow -u$, on $\mathcal{M}$.

The subspace of all $\phi$ in $C^0_K(\mathcal{M})$ for which

$$
(\Delta_{S^1} - \Delta_{S^1}) \phi(u) := \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \rho^2}\right) \phi(u) = 0
$$

is an $so(2,2)$ module. Denote this subspace by $H_K$ and the representation of $so(2,2)$ on it by $d\pi_0$. $H_K$ is the space of massless scalar fields on $\mathcal{M}$ and we have established the following result:

**Proposition:**

$H_K$ decomposes into the following $K$–finite subspaces:

$$
H_K = (H_K^+ \oplus H_K^-) \oplus H_0 \oplus (H_K^{+'} \oplus H_K^{-'})
$$

where $H_K^+$ and $H_K^-$ are the spaces spanned by the basis elements (1.14) for which $n = m$ ($m > 0$) and $n = -m$ ($m < 0$), respectively, and $H_K^{+'}$ and $H_K^{-'}$ are the spaces spanned by the basis elements (1.14) for which $-n = m$ ($m > 0$) and $-n = -m$ ($m < 0$), respectively, and $H_0$ is the one dimensional space spanned by the constant function, which is given by (1.14) with $n = m = 0$. The irreducible quotients of the $so(2,2)$ invariant subspaces $H_K^+$ and $H_K^-$ ($H_K^{+'}$ and $H_K^{-'}$) by subspace $H_0$ are infinitesmally unitarizable, irreducible lowest (highest) weight representations of $so(2,2)$, respectively. The restrictions of these four quotients to $so(1,2)$ are holomorphic or anti-holomorphic discrete series representations, and their completion with respect to the $so(2,1)$ discrete series norms are representations spaces for irreducible, positive and negative energy representations of $SO_0(2,2)$. The $H_K^+ / H_0$ and $H_K^- / H_0$ are spaces for positive and negative energy representations of $SO_0(2,2)$, respectively. The positive energy representations describes massless particles and the negative energy representations describe the corresponding antiparticles on $\mathcal{M}$.

In order to transfer fields on $\mathcal{M}$ (i.e. functions in $C^0_K(\mathcal{M})$) to fields "living" on $\mathcal{M}_0(2)$ we need to introduce the "flat parallelization." For this we make use of the Bruhat decomposition of $SO(2,2)$ along lines similar to the treatment given in [2], [3] for the four dimensional case. For scalar fields of weight $d$, i.e. representations of $SO(2,2)$ induced from one dimensional representations of the scale extended Poincaré group of $\mathcal{M}_0$ which act trivially on the translations, it leads to the following [1]:

$$
\phi_0(x_0(\tau, \rho), x_1(t, \rho)) = p^d \phi(\tau, \rho)
$$

where $\phi_0(x_0, x_1)$ is the function on $\mathcal{M}_0(2)$ defined by this equation, $\phi(\tau, \rho) \in C^0_K(\mathcal{M})$ and $p = \frac{1}{2}(u_1 + u_2)$ (cf. Eq. (1.6)). For the massless fields considered in the Proposition the conformal weight, $d$, is zero.
2. Casimir energy of a massless scalar field on $M_0(2)$ vanishing on the endpoints of an interval and its relationship to the corresponding problem on $\overline{M}(2)$

The parallelization map (Eq. (1.17) with $d = 0$) establishes a precise relationship between fields on $\overline{M}(2)$ and fields on $M_0(2)$ and it enables us to relate the quantum field theories on the respective manifolds. The Minkowski energy operator for a massless scalar field on $M_0(2)$ is

$$H_0 = \frac{1}{2} \int_{R} \left[ (\partial \phi_0 / \partial x_0)^2 + (\partial \phi_0 / \partial x_1)^2 \right] dx_1 . \quad (2.1)$$

The energy operator (Einstein energy operator) for a massless scalar field on $M(2)$ is given by

$$H = \frac{1}{2} \int_{S^1} \left[ (\partial \phi / \partial \tau)^2 + (\partial \phi / \partial \rho)^2 \right] ds \quad (2.2)$$

For the Casimir energy on $M(2)$ we consider a real massless scalar field $\phi(t, s)$ on $M(2)$ and consider the interval which is an arc of the circle, $S^1$, of radius $R$ satisfying $0 \leq s \leq L$, $S^1$ denoting the spatial part of the Einstein universe. (In what was done above we worked with $R=1$ for computational facility. We continue this practice except for final results or when it is important to note the dependence on the radius $R$ of $M(2)$, in which case we reinstate the explicit $R$ dependence.) We shall impose Dirichlet boundary conditions on the region $0 \leq s \leq L$: $\phi(t, 0) = \phi(t, L) = 0 . \quad (2.3)$

It suffices to take $L$ to be a half circle with arc length $\pi R = C/2$, where $C$ denotes the circumference of $S^1$. The condition $\phi(t, 0) = 0$ forces $\phi(t, s)$ to be an odd function of $s$. Thus by taking suitable linear combinations of the K-finite functions (1.14) we obtain the following positive and negative energy solutions of the wave equation (1.15):

$$\phi_\pm^{\pm}(t, s) = \left( \frac{c}{2\pi R \omega_n} \right)^{1/2} e^{\pm i \omega_n t} \sin k_n s , \quad (2.4)$$

$$\omega_n = ck_n , \quad k_n = \frac{2\pi n}{C} . \quad (2.5)$$

Notice that the boundary conditions (2.3) impose no restrictions on the allowed values of $n$ for the functions $\phi_\pm^{\pm}(t, s)$ of Eq. (2.4) i.e. the boundary conditions are automatically satisfied due to the periodicity of the sine function on $S^1$. The $\phi_\pm^{\pm}(t, s)$ are orthonormal with respect to the following $SO(2,2)$ invariant sesquilinear form:

$$\langle \phi_1, \phi_2 \rangle = \frac{i}{c} \int_{S^1} ds \left( \phi_1^* \partial_t \phi_2 - \phi_2^* \partial_t \phi_1 \right) . \quad (2.6)$$

The standard quantization of the field $\phi$ is performed by means of the expansion

$$\phi(t, s) = \sum_{n=1}^{\infty} \left\{ a_n \phi_n^{-}(t, s) + a_n^{\dagger} \phi_n^{+}(t, s) \right\} , \quad (2.7)$$

where now $\phi(t, s)$ is an operator valued distribution which for fixed $(t, s) \in \overline{M}(2)$ acts on the Fock space as a densely defined self-adjoint operator. The quantities $a_n$ and $a_n^{\dagger}$ satisfy the commutation relations

$$[a_n, a_m^{\dagger}] = \delta_{n,m} \quad (2.8)$$
\[ [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad . \quad (2.9) \]

Substitution of \( \phi(t, s) \) into Eq. (2.2) gives after some standard calculations

\[ H = \sum_{n=1}^{\infty} \hbar \omega_n \left[ a_n^\dagger a_n + \frac{1}{2} \right] \cdot I \quad . \quad (2.10) \]

where \( I \) is the identity operator on Fock space. The vacuum state is defined by the equation

\[ a_n|0> = 0 \quad . \quad (2.11) \]

Thus the total vacuum or zero point energy is thus

\[ E = \langle 0|H|0> = \frac{1}{2} \hbar \sum_{n=1}^{\infty} \omega_n \quad . \quad (2.12) \]

In order to determine the Casimir energy for the massless, scalar field we regularize Eq. (2.12) by inserting an exponential damping function \( e^{-n\alpha/R} \) inside the summation, and obtain for the finite part in the limit \( \alpha \rightarrow 0 \) the result:

\[ E(R) = -\frac{1}{12} \frac{\hbar c}{R} \quad . \quad (2.13) \]

Now we analyze the problem in the Minkowski space parallelization. According to Eqs. (1.5) an arc of the circle of length \( L = \pi R \) corresponds to a line segment of length \( 4R \) in the spatial part of \( M_0(2) \), so we want to relate our results to the standard result for the Casimir energy of a massless scalar field with Dirichlet boundary conditions on the endpoints of an interval of length \( L = 4R \) in \( IR^1 \) (the spatial part of \( M_0(2) \)), which is:

\[ E_0(L) = -\pi \frac{\hbar c}{24 L} \quad . \quad (2.14) \]

(Note the scaling property of Eqn. (2.14): \( E_0(L) \) is an homogeneous function of \( L \) of degree \(-1\). This scaling property seems to hold true in all even dimensions i.e. for \( n \) even \([4, 5, 6] \).)

Using Eqs. (1.9) and (1.10) and (1.17) with \( d = 0 \) we rewrite the Einstein energy operator, \( H \), in Eq. (2.2) as an operator on \( M_0(2) \):

\[ H = \frac{1}{2} \int_{ IR^1 } p^{-1} [ (\partial \phi_0/\partial x_0)^2 + (\partial \phi_0/\partial x_1)^2 ] \ dx_1 \quad . \quad (2.2') \]

Due to temporal invariance of Eq. (2.6) (cf. \[2\] (p. 123)) we shall work with \( \tau \) close to zero, and so \( \tau \sim \sin \tau = \frac{x_0}{1 + x_1^2/4} \). Using this we find

\[ \frac{d}{dx_0} e^{\pm i\omega_n \arcsin \left( \frac{x_0}{1 + x_1^2/4} \right)} = \frac{\pm i\omega_n}{(1 + x_1^2/4)} e^{\pm i\omega_n \arcsin \left( \frac{x_0}{1 + x_1^2/4} \right)} + O(x_0^2) = \]

\[ = \pm i\omega_n p(x_0 = 0, x_1) e^{\pm i\omega_n \rho} + O(x_0^2) \quad . \quad (2.15) \]

For the term in Eq. (2.2') involving the spatial derivative, we have \( \sin \rho = \frac{x_1}{1 + x_1^2/4} \) on the surface \( \tau = 0 \) and we compute

\[ \frac{d}{dx_1} \sin (k_n \rho) = -\left( \frac{k_n}{1 + x_1^2/4} \right) \cos \left( k_n \arcsin \left( \frac{x_1}{1 + x_1^2/4} \right) \right) = \]
\[ = -k_n \ p(x_0 = 0, x_1) \cos(k_n \rho) . \]  

(2.16)

Substitution of Eqs. (2.15) and (2.16) into Eq. (2.2') and then computation of the vacuum expectation value of Eq. (2.2') gives:

\[ E(R) = \frac{1}{2} \int_{\mathbb{R}} dx_1 p^{-1} \frac{h c}{2\pi R} \sum_{n=1}^{\infty} n = -\frac{1}{12} \frac{h c}{R} , \]  

(2.17)

which agrees with Eq. (2.13). In obtaining the last term we have regularized the infinite sum over \( n \) by the same method as before (exponential damping function) and extracted the relevant finite part.

Next we compute the vacuum expectation value of the Minkowski energy operator \( H_0 \) in the Minkowski space parallelization. We obtain

\[ < 0 | H_0 | 0 > = \frac{1}{2} \int_{\mathbb{R}} < 0 | (\partial \phi_0 / \partial x_0)^2 + (\partial \phi_0 / \partial x_1)^2 | dx_1 | 0 > = \frac{1}{2} < 0 | H | 0 > , \]  

(2.18)

where the last step follows from use of Eqs. (2.15) and (2.16) and then the fact that \( \int p^2 dx_1 = \frac{1}{2} \int p dx_1 \). Thus the Casimir energy corresponding to \( H_0 \) equals \( \frac{1}{2} E(R) \).

We now compute the regularized vacuum expectation value of the Minkowski energy operator as a power series in inverse powers of \( R \) to lowest order in \( 1/R \), and show that the resulting Casimir energy agrees with Eq. (2.14) for an interval of length \( L = 4R \). For large \( R \), we have:

\[ \sin(k_n s) = k_n x_1 + O(1/R^2) \]  

(2.19)

and

\[ \tau = \frac{x_0}{R} + O(1/R^2) . \]  

(2.20)

By use of these equations in Eq. (2.4) we obtain to order \( O(1/R^2) \):

\[ \phi_n^\pm(t, s) = \left( \frac{c}{2\pi R \omega_n} \right)^{1/2} e^{\pm i\omega_n x_0} \sin(k_n x_1) . \]  

(2.21)

With these results it is easy to see that the problem of computation of the Casimir energy for the operator \( H_0 \) in the flat parallelization is exactly the same as the computation of the Casimir energy on an interval of length \( 4R \) in Minkowski space, and our result for the Casimir energy agrees, for \( L = 4R \), with Eq. (2.14) up to higher order terms in the curvature, provided we use the same Fock space defined by Eqs. (2.8) and (2.9) and the same definition of the vacuum given in Eq. (2.11).

### 3. Further developments and conclusions

Almost everything described above can be generalized to arbitrary \( n \). \( SO(2, 2) \) becomes \( SO(2, n) \) and the generalization of the harmonic component of \( C_K^0(\mathbb{M}) \), i.e. the analog of \( H_K \), becomes the “minimal representation” of \( SO(2, n) \) at least for \( n \) large enough [7]. Sine functions are replaced by Gegenbauer polynomials and we know their asymptotic expansions [8].

We now give a brief outline of how the above described method should work for higher dimensions with an aim to determine the Casimir energy for a massless scalar field in \( n \) dimensional Minkowski space (\( n \) even) with Dirichlet boundary conditions on a sphere of arbitrary radius. We start with our exact analytical expression for the Casimir energy of a
massless scalar field on the $n$ dimensional Einstein universe ($n > 3$) with Dirichlet boundary conditions on a sphere of maximal radius, namely [9]:

$$E(R, n) = -\frac{\hbar c}{2n!R}B_n^{(n-1)}\left(\frac{n-2}{2}\right)$$

where the generalized Bernoulli polynomials $B_n^{(\sigma)}(x)$ are explicitly given by [10]:

$$B_n^{(\sigma)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{\sigma + k - 1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \times 2F_1[k-n, k-\sigma; 2k+1; j/(x+j)]$$

with $2F_1[k-n, k-\sigma; 2k+1; j/(x+j)]$ being the Gaussian hypergeometric function (cf. [8]). Using the appropriate asymptotic expansions of Gegenbauer polynomials, which occur in the problem (cf. ref. [11]), we are able to obtain an expansion of the vacuum expectation value of the Minkowski energy operator in inverse powers of $R$, just as in the $n = 2$ case. We should be able to relate the relevant finite part of the regularization of the term of lowest order in $1/R$ to the Casimir energy for a massless scalar field in $n$ dimensional Minkowski space with Dirichlet boundary conditions on a sphere of radius $2R$ just as we did in the $n = 2$ case. Finally, to obtain the value of the Casimir energy for a sphere of arbitrary radius, we recall the scaling property of the Casimir energy, which was mentioned above, and seems to hold true at least for even $n$ [4], [5], [6]. Standard results for Casimir energies on spheres obtained by asymptotic expansions and numerical evaluation are tabulated in ref. [4] (cf. also [12]) up to $n = 6$, and we should be able to compare our results with theirs.

The difficulties with $n$ odd seems to be related to the lack of a rigorous prescription for the removal of divergences in the conventional quantum field theory on Minkowski space. A detailed treatment of the $n = 3$ case is given in [13]. There it is argued that a change in the sign of the Casimir force occurs in the neighborhood of a critical radius $R_c$. When $R > R_c$, the Casimir force is attractive while for $R < R_c$ the force becomes repulsive. Similar pathologies seems to hold generally for odd $n$ [4]. Clearly the scaling property of the Casimir energy is lost in this case, so it seems that all we can get out of our method for the odd $n$ cases are approximations of the Casimir energies for spheres of cosmic proportions in the spatial part of Minkowski space.

References

[1] Orsted B, Segal I E 1989 Jour. Funct. Anal. 83 150-184.
[2] Paneitz S M, Segal I E 1982 Jour. Funct. Anal. 47 78-142.
[3] Moylan P 1995 Jour. Math. Phys. 36 6 2826-2879.
[4] Bordag M, Mohideen U, Mostepanenko V M 2001 Phys. Rep. 353 1.
[5] Miltão M S R, Farias F A 2010 Advances in High Energy Physics 120964.
[6] Nesterenko V V, Pirozhenko I G 1997 arXiv:hep-th/9707253v1.
[7] Kobayashi T, Mano G 2007 Proc. Japan Acad. 83 Ser. A 27-31.
[8] Gradshteyn I C, Rhyshik I M 2000 Tables of Integrals, Sums, Series and Derivatives, (Academic Press, New York).
[9] Moylan P to appear in Proceedings of the 9th International Workshop "Lie Theory and Its Application in Physics" (LT-9, 2011) (Springer-Verlag).
[10] Sirvastava H M, Todorov P G 1988 Jour. Math. Anal. and Appl. 130 509-513.
[11] Moylan P 2008 Jour. Phys. Conf. Ser. 128 012010.
[12] Vahyan M A, Gousheh S S 2009 arXiv:0911.3578v1 [hep-th].
[13] Sen S 1981 J. Math. Phys. 22 2968.