Network Structure and Counterparty Credit Risk

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Abstract

In this paper we offer a novel type of network model, which is capable of capturing the precise structure of a financial market based, for example, on empirical findings. With the attached stochastic framework it is further possible to study how an arbitrary network structure and its expected counterparty credit risk are analytically related to each other. This allows us, for the first time, to model and to analytically analyse the precise structure of a financial market. It further enables us to draw implications for the study of systemic risk. We apply the powerful theory of characteristic functions and Hilbert transforms, which have not been used in this combination before. We then characterise Eulerian digraphs as distinguished exposure structures and we show that considering the precise network structures is crucial for the study of systemic risk. The introduced network model is then applied to study the features of an over-the-counter and a centrally cleared market. We also give a more general answer to the question of whether it is more advantageous for the overall counterparty credit risk to clear via a central counterparty or classically bilateral between the two involved counterparties. We then show that the exact market structure is a crucial factor in answering the raised question.

KEYWORDS — Counterparty credit risk, systemic risk, network structure, network model, analytic function, digraph, graph, Eulerian, characteristic function, Hilbert transform, analytic signal, bilateral & multilateral netting, advantageousness of a central counterparty.
1 Introduction

One risk type that has gained particular attention in recent years, largely due to the credit and financial crisis that started in 2007, is counterparty credit risk. On the one hand, over-the-counter (OTC) markets have been seen to respond heavily to financial distress. On the other hand, centrally cleared markets continued to trade without major disruptions even at the height of the financial crisis. Following the impact of this crisis, the G20 countries therefore decided to thoroughly revamp the OTC derivatives market in 2009 in order to reduce the immanent systemic risk. In Europe the reforms are implemented through the so-called European Market Infrastructure Regulation (EMIR). The US equivalent is called the Dodd-Frank Act. At the core of both new regulations is the obligation of the market participants to clear their standard OTC derivatives through a central counterparty (CCP). Non-centrally cleared contracts should be subject to higher capital requirements. These measures are designed to comprehensively change the market structure. Today, many classes of derivatives are already being cleared through CCPs, for example, LCH.Clearnet clears interest rate swaps and ICE Clear or CME clear credit default swaps.

Several authors such as Nier et al. [Nie+07], Moussa [Mou11], Rosenthal [Ros01] or Gai et al. [GK10] emphasised the importance of the precise market structure in the context of studying counterparty credit risk and therefore systemic risk. Furthermore, empirical studies have shown that network structures in different countries are quite varied. Despite these facts, most previous models in the context of counterparty or systemic risk have assumed a simplistic network structure such as complete or star graphs. However, these simplistic network structures are not able to capture empirical findings such as [UW04] or as in chapter 4 in [Mou11] and tend to over- or underestimate the overall risk.

In section 2 we present a new type of network model, which is capable of capturing the precise structure of any given financial market based, for example, on empirical findings. We further introduce a stochastic framework to study how different network structures and counterparty credit risk are analytically connected to each other. This allows us, for the first time, to model and to analytically analyse the precise network structure of a financial market. We take the perspective of a regulator and are mainly interested in the overall risk of a market for a typical day in the future. In a first step, we incorporate position uncertainty in size and direction in one single distribution. Our model is then capable of dealing with an arbitrary graph as well as accounting for a wide range of distributions that represent a position between two counterparts. In a second step, we use conditional probabilities in order to extend this approach to arbitrary digraphs, where the size and the direction of all positions can be determined independently. That is, the distribution that represents the position value and therefore the exposure can be chosen independently from the exact structure and thus perfectly adapts

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1 See, for instance, [Ros01] or [Ban09].
2 See [http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2012:201:0001:0059:EN:PDF]
3 See [http://www.gpo.gov/fdsys/pkg/PLAW-111publ203/pdf/PLAW-111publ203.pdf]
4 [http://www.lchclearnet.com/]
5 [https://www.theice.com/]
6 [http://www.cmegroup.com/]
7 A comprehensive overview of these studies and the used network models is provided by [FL13] and [Mou11].
8 See section 1.3.1 in [Mou11] for an overview.
to the individual circumstances of a given network. To this end, the model only assumes that each non-zero position is distributed identically by an arbitrary symmetrical distribution with existing mean. We have not incorporated correlations, as suggested by Cont and Kokholm [CK14], because we use the powerful theory of characteristic functions (see Lukacs [Luk70]) for analysing sums of independent random variables. By using this theory, we deduce how we can analytically capture the process of netting in regards to the associated random variables. Afterwards, we show how to determine the expected credit exposure of a netted position by using the so-called Hilbert transform (see King [Kin09a]). To the extent of our knowledge, the combination of both concepts has never been used before in mathematical finance.

In section 5 we provide auxiliary results which can be used for the application of the network model. In the first subsection we prove Proposition 5.1 which contains two very useful formulas about Hilbert transforms, by using the residue theorem. These formulas are particularly useful for calculating the Hilbert transform of intricate functions. We also study the so-called positive and negative absolute values of a distribution in section 5.2. Both types are used to represent the direction of a position. The term ‘analytic signal’, known from the field of signal processing, is then introduced and we show in Proposition 5.3 that the positive absolute value of a distribution is an analytic signal. Some of these insights are used in section 3 to prove both structure theorems 3.2 and 3.3. The theorems basically state that Eulerian digraphs are distinguished exposure structures and that digraphs possess different characteristics compared to graphs in the context of counterparty credit risk. We further reveal that different structures within graphs or digraphs can have a significantly different impact on the overall counterparty risk.

We then apply our network model and its stochastic framework in section 4 to study the features of bilateral and multilateral clearing and to give a more general answer to the question raised by Duffie & Zhu [DZ11], of whether it is more advantageous for the overall counterparty risk to clear via a CCP or classically bilateral between the two involved counterparties. The two authors model the counterparty credit risk of each market participant and for both netting types as an independent and standard normal distributed random variable (r.v.) in order to resolve this question. This web of obligations and claims, described in [DZ11], can be illustrated as a complete graph. With the introduced network model we can answer this question not only for complete graphs, but for arbitrary graphs and digraphs as well. Moreover, the network model introduced in section 2 is not constrained to the normal distribution, it can also employ any (symmetric) distribution with a defined expected value. We finally show in section 4.3 that the question of the advantageousness of one netting type also depends heavily on the precise structure of the market by comparing the implications of our model and the model used by [DZ11].

2 Network Model for a Financial Market

Counterparty credit risk, often known just as counterparty risk or default risk, is usually defined as the risk that the entity with whom one has entered into a financial contract will fail to fulfill their side of the contractual obligations. Credit exposure or simply exposure defines the actual loss in the event of a counterparty defaulting. In the next two subsections we explain the

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9This kind of risk arises in almost every financial market such as the derivatives market, the interbank market, the money market or the repo market.
basic settings for a general financial market which is subject to counterparty risk. We start with a market modelled as a graph where size and the direction of a position is determined by a random variable. We extend this model by using conditional probabilities in order to determine the direction well before any observation is drawn. This allows us to model the exact directed network structure of any financial market, where only the position size is a matter of coincidence. Furthermore, we introduce the stochastic framework and a set of formulas to calculate the expected credit exposure.

2.1 Market Settings

We consider a financial market $\mathcal{M}$ with $N \in \mathbb{N}$ participants and $K \in \mathbb{N}$ different classes of derivatives $C := \{1, \ldots, K\}$. Derivatives classes could be defined by underlying asset classes, but we could also aggregate different underlying asset classes to one derivatives class. Let $k \in C$ and $m_k \in \mathbb{N}$. We model each financial submarket of derivatives class $k$ as a single graph $G_k = (V = \{v_1, \ldots, v_N\}, E^k = \{e^k_1, \ldots, e^k_{m_k}\})$. It consists of a non-empty finite set $V = V(G_k)$ of elements called vertices and a finite set $E^k = E(G_k)$ of unordered pairs of distinct vertices called edges. The vertices of a given graph $G_k$ represent the $N$ market participants and the edges of $E^k$ stand for the trades or simply positions between two different counterparts within derivatives class $k \in C$. A trade position is the net value of a bilateral portfolio within derivatives class $k \in C$.

Furthermore, a financial market $\mathcal{M}$ is usually endowed with a set of market conventions, that apply to each of the $N$ participants within a class of derivatives. For instance, the type of netting or the day-count conventions are typical market conventions. We write $\tilde{E} := \bigcup_{k \in C} E^k$ for the compounded set of edges and $G := (V, \tilde{E})$ shall represent all graphs $G_k$ on the common set of vertices $V$. Here, $\bigcup$ stands for the disjoint union of sets.

We assume that the uncertainty of the value of a future bilateral trade position can be represented by a real-valued distribution $P$, which is symmetric around the origin and with zero mean. That is, we model the uncertainty of size and direction of a position of a counterpart $v \in V$ relative to counterpart $w \in V \setminus \{v\}$ in derivatives class $k \in C$ by a r.v. $X^{(k)}_{v,w} \sim P$. A realisation of the r.v. $X^{(k)}_{v,w}$ is denoted by $x^{(k)}_{v,w} \in \mathbb{R}$. If $x^{(k)}_{v,w}$ is positive then $v$ will claim this amount from $w$, but if $x^{(k)}_{v,w}$ is negative then $v$ will owe the amount of $x^{(k)}_{v,w}$ to $w$. The direction as well as the associated size or weight of an edge $\{v, w\} \in E^k$ is then defined by the observation $x^{(k)}_{v,w}$ of the random experiment.

The given graph $G_k$ supplemented by the directions of each of its corresponding realisation $x^{(k)}_{v,w}$ represents a so-called directed graph or just digraph $D_k = (V, A^k = \{a^k_1, \ldots, a^k_{m_k}\})$. $D_k$ also comprises the vertex set $V$ and a set $A^k \subseteq (V \times V)$ of ordered pairs of different vertices called arrows, as well as two maps $h : A \rightarrow V$ and $t : A \rightarrow V$ assigning to every arrow $a \in A$ a head vertex $h(a)$ and a tail vertex $t(a)$. Within a digraph we know that the creditor $h(a)$ claims the position value from the debitor $t(a) \in V$ for all $a \in A^k$. A digraph $D_k = (V, A^k)$ is called an orientation of a graph $G_k = (V, E^k)$, if each edge $\{v, w\} \in E^k$ is replaced by one of the ordered pairs $(v, w)$ or $(w, v)$, i.e., the digraph $D_k$ along with its realisations is an orientation of $G_k$.

We call a graph or digraph weighted, or network, if each link of each derivatives class $k$ is assigned with a real number. A realisation $x^{(k)}_{v,w} \in \mathbb{R}$ of the r.v. $X^{(k)}_{v,w}$ also provides a weight.\footnote{This is, for the sake of simplicity, an abuse of notation.}

\footnote{Also called size within this document.}
for each edge of $G_k$. That is, the outcome of the r.v.s $X_{v,w}^{(k)}$ define a weighted orientation of $G_k$. Analogously, we write $\overrightarrow{A} := \cup_{k \in C} A^k$ for the compounded set of arrows and $D := (V, \overrightarrow{A})$ shall represent all digraphs $D_k$ with $k \in C$ on the common set of vertices $V$.

If we only want to model the weights of a digraph with pre-defined directions by r.v.s, we can use the following technique. We start over with an undirected graph $\overline{G}$ and analogically, we write $G$ for each edge of $\overline{G}$. It can be decomposed into arbitrary counterparty transactions. We do not consider benefits of collateral and default recovery. So-called closeout netting agreements have been a common tool to reduce the credit exposure of an entire market $\mathcal{M}$. A closeout netting agreement is a legally binding contract between two parties. It stipulates that if one counterparty defaults, legal obligations arising from derivative transactions, covered by the netting agreement, must be based on the net value of such transactions. We do not consider benefits of collateral and default recovery.

The applied closeout netting convention of the market $\mathcal{M}$ defines exactly what trades of an arbitrary counterparty $v \in V$ can be aggregated into one net position in the case of a default. Expressed in terms of set theory, this means that the applied netting convention defines a partition $12\mathcal{L}_v$ of all links that are incident $13$ to $v$. That is, $\overrightarrow{A}(v) := \{a \in A^k \mid v \text{ incident with } a; k \in C\}$ can be decomposed into $\overrightarrow{A}(v) = \bigcup_{\Lambda \in \mathcal{L}_v} \Lambda$ with $\Lambda \neq \emptyset$, where all trades of each set $\Lambda \in \mathcal{L}_v$ have $v$ as one of the two counterparts in common. We call $\Lambda \in \mathcal{L}_v$ a netting set of the market participant $v$. An evident partition of the set $\overrightarrow{A}(v)$, as in Fig. graphically suggested by the

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12See [Die03], Chapter 1 for the general definition of a partition of a given set.
13The vertex $v$ of a graph $G_k$ is incident with an edge $e \in E^k$ if $v \in e$. We further call an arrow $a \in A^k$ of the digraph $D_k$ incident with the vertex $v$ if $h(a) = v$ or $t(a) = v$. 

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different types of arrows, is \( \tilde{A}(v) = \{a^1_v\} \cup \{a^2_v, a^3_v\} \). Regarding the similar decomposition of \( \tilde{A}(w) = \{a^1_w, a^2_w\} \cup \{a^3_w\} \) we see that the intersection \( \tilde{A}(v) \cap \tilde{A}(w) \) consists of the trade positions between \( v \) and \( w \).

Each netting set \( \Lambda \in \mathcal{L}_v \) corresponds with a set of r.v.s \( X_\Lambda := \{X_{\lambda}\}_{\lambda \in \Lambda} \), where each element represents a future bilateral trade position with counterpart \( v \). If we want to calculate the expected counterparty risk of a set of r.v. \( \{X_{\lambda}\}_{\lambda \in \Lambda} \) of a market participant \( v \), we need to specify whether an arrow \( \lambda \) is a claim or a debt of \( v \) relative to its counterpart. For this purpose we use the already introduced notation \( X_{v,w}^{(k)} \). Further we write \( \Lambda_v \) to stress that each trade position is meant relative to \( v \), i.e., if the position is positive then \( v \) claims the amount from \( w \) and if it is negative then \( v \) owes the amount to \( w \). For example, the netting set \( \Lambda_v = \{a^1_v, a^2_v, a^3_v\} \) of Fig. 1 corresponds with the set \( X_{\Lambda_v} = \{|X_{v,w}^{(1)}|, |X_{v,w}^{(2)}|, |X_{v,w}^{(3)}|\} \) of random variables. We further designate \( X_{\mathcal{L}_v} := \bigcup_{\Lambda \in \mathcal{L}_v} X_\Lambda \) as the family of sets of r.v.s of \( v \) that implies the counterparty risk of the market participant \( v \) for the entire market. We say that the digraph \( D \) is distributed by \( P \), denoted by \( X \sim \mathsf{d}[P] \), if and only if all trade positions of \( X_{\mathcal{L}_v} \) for all \( v \in V \) are distributed independently and identically (i.i.d.) by \( \mathsf{d}[P] \). We adopt a similar notation \( X \sim P \) for undirected graphs \( G \).

Obviously, a netting set is strongly dependent on the used netting opportunity as part of the market conventions. Within an OTC market, for instance, the ISDA Master Netting Agreement\(^{14}\) is a standard closeout netting agreement, which allows two bilateral counterparts to net across different kinds of derivatives. Although netting across all product categories is often not allowed\(^{15}\), we will in the following net across all classes of derivatives of \( C \). In contrast to that a CCP offers the possibility to net across all its clearing members\(^{16}\). Bilateral as well as multilateral netting will be treated in detail in sections 4.1 and 4.2.

After knowing the relevant netting sets and defining how to interpret the r.v.s, netting is then simply performed by adding the estimated future position values in form of the random variables. By taking the maximum between the netted sum and zero we determine the credit exposure for the counterparty \( v \) and netting set \( \Lambda_v \). Hence, the counterparty risk of \( v \) considering

\[^{14}\text{See \url{http://www2.isda.org/}.}\]
\[^{15}\text{See section 3.4.7 in } \text{Gre10}.\]
\[^{16}\text{See section 3.4.10 and 14.1 in } \text{Gre10}.\]
the netting set $\Lambda_v$ is determined by

$$\max \left\{ \sum_{\lambda \in \Lambda_v} \pm |X_\lambda| ; \; 0 \right\},$$

(2.1)

where $\pm |X_\lambda|$ is the positive or negative absolute value of a real-valued symmetric r.v. $X_\lambda \sim P$ with zero mean. The theory of characteristic functions is a powerful tool for analysing sums of independent random variables. If the r.v.s of the finite sequence $(X_\lambda)_{\lambda \in \Lambda}$ are mutually independent then the c.f. of the sum $Y := \sum_{\lambda \in \Lambda} X_\lambda$ is simply the product

$$\phi_Y = \prod_{\lambda \in \Lambda} \pm |\phi_{X_\lambda}|$$

(2.2)

of the corresponding characteristic functions. In general the function $\phi_Y$ is complex-valued, that is, $\phi_Y(t) = \eta(t) + i\nu(t)$ with real part $\Re(\phi_Y) = \eta$ and imaginary part $\Im(\phi_Y) = \nu$. Apparently, if we want to apply this concept to (2.1), we have to find a way to figure out the c.f. of the positive as well as the negative absolute value of the given distribution $P$. We derive formulas for this purpose in section 5.2.

If a market participant $w$ claims money from the defaulted counterpart $v$, then $w$ will probably incur a loss. Whilst if $w$ owes money to the defaulted counterpart $v$, then $w$ will still have to honour the contractual payments. That is, in latter situation $w$ cannot gain from the default by being somehow released from their liability. Thus, only a positive trade position implies an exposure greater than zero. The exposure can be figured out by using formula (4) in PINELIS [Pin13] in order to obtain the c.f.

$$\phi_{\max[Y;0]}(t) = \mathbb{E}(e^{it\max[Y;0]}) = \frac{1}{2}[1 + \phi_Y(t)] + \frac{i}{2}[H\{\phi_Y\}(t) - H\{\phi_Y\}(0)]$$

(2.3)

of the r.v. $\max[Y;0]$. Here, $i$ is the imaginary unit and $H\{\phi_Y\}$ is the Hilbert transform\(^{17}\) of the (characteristic) function $\phi_Y$ given by

$$H\{\phi_Y\}(\omega) := \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\phi_Y(t)}{\omega - t} dt := \lim_{\epsilon \to 0} \frac{1}{\pi} \left( \int_{-\infty}^{\omega-\epsilon} \frac{\phi_Y(t)}{\omega - t} dt + \int_{\omega+\epsilon}^{\infty} \phi_Y(t) dt \right)$$

with $t, \omega \in \mathbb{R}$ and provided this integral exists. The $\text{PV}$ in front of the integral denotes the Cauchy principal value\(^{18}\) that expands the class of functions for which the ordinary improper integral exists. When it is clear from the context what is meant we will use the variable $t$ for the argument of the input function as well as for the argument of its Hilbert transform. According to Theorem 2.3.1 and its Corollary 2 in [Luk70] we can derive the expected value of any r.v. $Z$ by

$$\mathbb{E}(Z) = i^{-1} \partial_t[\phi_Z(t)](0),$$

(2.5)

on the condition that the first moment exists. In our case, we set $Z := \max[Y;0]$ in order to compute the desired expectation.

\(^{17}\)See section 3.1 in [Kin09a].

\(^{18}\)See section 2.4 in [Kin09a].
3 Specific Network Structures and Counterparty Risk

To deduce a formula for the expected counterparty credit risk of an arbitrary network structure and for such a wide range of possible distributions, even without considering dependencies between the different positions, is a demanding task. Let us assume that the digraph $D = (V, A)$ represents a network structure and that future positions are distributed by $X \sim \mathcal{U}([P])$. The challenge is then to compute $\mathbb{E}(\max[\sum_{\lambda \in \Lambda_v} \pm |X_{\lambda}|, 0])$ for an arbitrary netting set $\Lambda_v$ of a counterparty $v$ and to deal with various of problems attached to it: The negative as well as the positive absolute value is not distributed by $P$ anymore, because their sample space is restricted either to $]-\infty, 0[$ or to $]0, \infty[$. The probability distribution of the sum $\sum_{\lambda \in \Lambda} \pm |X_\lambda|$ is actually the convolution of their distributions, and in general, little can be said about it. Finally, taking the maximum causes some sort of asymmetry of the problem, and this implies the non-additivity as the following example illustrates.

3.1 Example: Suppose $D = (V, A = \{a_1, a_2\})$ is the digraph as depicted in Fig. 2 and represents a market with $K = 1$ class of derivatives.

$$
\begin{array}{c}
\text{u} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{w} \\
\downarrow \\
\text{a}_1 \\
\downarrow \\
\text{a}_2 \\
\downarrow \\
\text{v}
\end{array}
$$

Figure 2: Path $D$

The associated r.v.s $X_{a_1}$ and $X_{a_2}$ are distributed i.i.d. by the continuous uniform distribution $\mathcal{U}([-1, 1])$. The vertex $u$ has no expected counterparty risk at all, as $\mathbb{E}(\max[-|X_{a_1}|; 0]) = \mathbb{E}(0) = 0$. The end-vertex $w$ of $D$ obviously entails $\mathbb{E}(\max[|X_{a_2}|; 0]) = \mathbb{E}(|X_{a_2}|) = \frac{1}{2}$ expected counterparty credit risk. The c.f. of the sum $Y := |X_{a_1}| - |X_{a_2}|$ equals $\phi_{\mathcal{U}(0,1)}(t)\phi_{\mathcal{U}(-1,0)}(t) = \frac{(1-e^{-it})(1+e^{it})}{t^2}$ and its Hilbert transform $H(\phi_Y)$ is $\frac{2(t \sin(t))}{t^2}$. Taking the limit $\lim_{t \to 0} H(\phi_Y)(t)$ and applying formulas (2.3) to (2.5) we obtain $\mathbb{E}(\max[Y; 0]) = \frac{\partial_{\lambda Y} \phi_{\max[Y;0]}(0)}{i} = \frac{1}{6}$. Please note that $\mathbb{E}(\max[|X_{a_1}| - |X_{a_2}|; 0]) = \mathbb{E}(\max[|X_{a_1}|; 0]) + \mathbb{E}(\max[-|X_{a_2}|; 0])$.

That is, the more liabilities a counterpart $v$ has relative to its claims, the lower the counterparty credit risk will be for $v$. Considering the netting efficiency of a single counterparty the situation changes: the better the balance between claims and liabilities, the greater the offsetting effect of the netting opportunity. In anticipation of section 4.2, a popular example for the netting

$$
\begin{array}{c}
\text{u} \\
\downarrow \\
\text{v} \\
\downarrow \\
\text{w}
\end{array}
$$

Figure 3: Exposure circle

efficiency of a centrally cleared market is shown in Fig. 3. Let us assume that each arrow of the exposure circle represents an exposure of 100 € million. Then, the circle of exposure implies a perfect balance between the claims and liabilities of each participant, because the in- and outgoing arrows offset each other completely. If we generalise this obvious concept of exposure circles and use the language of graph theory we come across Eulerian digraphs. If we further replace the deterministic values by r.v.s that represent the future counterparty credit
risk between two participants, then we come to Theorem 3.2. To be able to do so, however, we need to introduce the degree $\gamma(v)$ of a vertex $v \in V$ within a graph $G_k = (V, E^k)$, which is the number $|E(v)|$ of different edges at $v$. Let us now consider a digraph $D_k = (V, A^k)$, then the in-degree of a single vertex $v$, denoted by $\gamma_+(v)$, is the number of arrows $a \in A^k$ with $h(a) = v$. Similarly, we call the number of arrows $a \in A^k$ with $t(a) = v$ the out-degree of $v$ and denote it by $\gamma_-(v)$. We shall call $\gamma_+: v \mapsto \gamma_+(v)$ the in-degree function and $\gamma_-: v \mapsto \gamma_-(v)$ the out-degree function. Moreover, we define $\gamma(v) := \gamma_+(v) - \gamma_-(v)$ for a digraph $D_k$ and call $\gamma(v)$ the Eulerian degree function and $\gamma(v)$ the Eulerian degree of $v$.

3.2 Theorem: Let $D_k = (V, A^k)$ be a connected digraph with $X \sim \pm|P|$ and a netting set $\Lambda_v$ with $v \in V$. Then the following holds:

(i) $\mathbb{E}(\sum_{\lambda \in \Lambda_v} |X_\lambda|) = 0$ if and only if $\gamma(v) = 0$;

(ii) $\mathbb{E}(\max\{\sum_{\lambda \in \Lambda_v} |X_\lambda|; 0\}) = \frac{1}{2} \phi_t[H\{\phi_Y\}](t)(0)$ if $\gamma(v) = 0$.

Proof. See section 7.2.

A graph is called Eulerian if each vertex of that graph has an even degree. A digraph is called Eulerian if the in-degree equals the out-degree for each vertex $v$ of that digraph, i.e., if $\gamma(v) = 0$ for each $v \in V$.

We have shown in the last theorem that a vertex $v \in V$ of a digraph with $\gamma(v) = 0$ is distinguished in the context of counterparty credit risk. Because of the definition of an Eulerian digraph $D_k = (V, A^k)$ the equation $\gamma(v) = 0$ is valid for every vertex $v \in V$. That is, netting efficiency goes hand in hand with Eulerian digraphs in the context of so-called multilateral netting rules.

For graphs we can state a similar result.

3.3 Theorem: Let $G_k = (V, E^k)$ be a connected graph with $X \sim \pm P$ and a netting set $\Lambda_v$ with $v \in V$. Then the following holds:

$$\mathbb{E}\left(\max\left[\sum_{\lambda \in \Lambda_v} X_\lambda; 0\right]\right) = \frac{1}{2} \phi_t[H\{\phi_Y\}](t)(0).$$

Proof. See section 7.2.

In contrast to digraphs, formula (3.1) of Theorem 3.3 is valid for any vertex of an arbitrary graph. The reason for this mismatch is that the symmetry of the net r.v. $Y = \sum_{\lambda \in \Lambda_v} X_\lambda$ does not depend on the netting set $\Lambda_v$. In the case of a graph, the r.v.s $X_\lambda$ with $\lambda \in \Lambda_v$ are symmetric and so is $Y$.

Thus, using either a graph or a digraph to model a financial market does matter and it should be well-considered.

4 Application of the Network Model

In this section we define and explain measures for counterparty credit risk within an OTC and a centrally cleared market. Afterwards, we are going to derive how to compute the expected counterparty credit risk for both types of markets for a typical day in the future by applying
the model introduced in section 2. For that purpose we clarify, in a first step, the outline of both netting types. In a second step, we apply the notation \(X_{v,w}^{(k)}\) for a r.v. and \(x_{v,w}^{(k)} \in \mathbb{R}\) for its realisation as introduced in section 2.1.

To this end, we need to define the terms adjacent and neighbourhood. Two different vertices \(v\) and \(w\) of a graph \(G_k = (V,E^k)\) are adjacent if \(\{v,w\}\) is an edge of \(G_k\). In the case of a digraph \(D_k = (V,A^k)\), the two vertices are adjacent if either \((v,w) \in A^k\) or \((w,v) \in A^k\) is valid. The neighbourhood \(U \subseteq V\) of a vertex \(v\) in a graph \(G_k\) or a digraph \(D_k\) is the set of all vertices adjacent to \(v\).

4.1 Counterparty Risk within an OTC Market

Within an OTC market \(\mathcal{M}\) we are allowed to offset positions across all kinds of derivatives classes, but only between one single pair of counterparties. Thus, a bilateral netting set of a given market participant \(v \in V\) will correspond to a bilateral portfolio of \(v\) and one of its counterparts \(w \in U_v^C\). Here, \(U_v^C\) is the neighbourhood of \(v\) across all classes of derivatives \(k \in C\).

The real number \(z_{v,w}^{(k)}\) shall now represent the current observable position value of \(v\) relative to \(w \notin V \setminus \{v\}\) within derivatives class \(k\). The deterministic function value \(y_{v,w}(C) := \sum_{k \in C} x_{v,w}^{(k)}\) is called the current bilateral position of \(v\) to \(w\). Obviously, the function \(y_{v,w}\) can be positive or negative, and an immediate consequence of the definition is the validity of \(y_{v,w}(C) = -y_{w,v}(C)\), meaning that the claims of the one are the liabilities of the other counterparty. Thus, the actual current bilateral counterparty risk of \(v\) to \(w\) is \(\max\{y_{v,w}(C);0\}\). For a reasonable measure of a bilateral counterparty risk of an entire market \(\mathcal{M}\), that can be used by supervisory authorities, we have to consider that either \(\max\{y_{v,w}(C);0\} = 0\) or \(\max\{y_{w,v}(C);0\} = 0\) is valid. Hence, we call \(z_k(D) := \sum_{v \in V} \sum_{w \in U_v^C} \max\{y_{v,w}(C);0\}\) the current bilateral counterparty risk of \(D = (V,A)\).

We now assume that the trade position that corresponds with an arrow of the digraph \(D = (V,A)\) is not yet realised but represented abstractly by a r.v., i.e., \(x \sim \mathcal{P}\). For all other relations between two different counterparties, we set the (future) position value deterministically to zero. The direction of each arrow in each class \(k \in C\) determines whether the positive or the negative absolute value of the associated r.v. is relevant for the calculation.

We shall call \(\mathbb{E}(\max\{Y_{v,w}(C);0\})\) the expected bilateral counterparty risk of \(v\) to \(w\), where \(Y_{v,w}(C) = \sum_{k \in C} z_{v,w}^{(k)}\) is now a conditional r.v. with mean and variance depending on the information about the directions of the positions. In section 2.2 we have demonstrated how to compute the expectation of such a random variable. Let \(\phi_{Y_{v,w}}\) and \(\phi_{\max\{Y_{v,w};0\}}\) be the c.f.s of the r.v. \(Y_{v,w}(C)\) and \(\max\{Y_{v,w}(C);0\}\), respectively. If we apply formulas (2.2) to (2.5) we obtain

\[
\mathbb{E}(\max\{Y_{v,w}(C);0\}) = \frac{\partial_i[\phi_{\max\{Y_{v,w};0\}}(t)](0)}{i}.
\]

Putting the parts together and considering the possible asymmetry of the r.v.s of a digraph, as described in section 3 we finally obtain the formula

\[
\mathbb{E}(Z_k(D)) = \sum_{v \in V} \sum_{w \in U_v^C} \mathbb{E}(\max\{Y_{v,w}(C);0\}) = \sum_{v \in V} \sum_{w \in U_v^C} \frac{\partial_i[\phi_{\max\{Y_{v,w};0\}}(t)](0)}{i}
\]

for the expected bilateral counterparty risk of an arbitrary digraph \(D\).

We denote the Laplace distribution by \(\mathcal{L}(\mu,b)\), where \(\mu\) is the mean and \(b\) the scaling parameter.
4.1 Example: Financial markets can be organised in different layers. In Germany, for instance, UPPER and WORMS [UW04] describe a two-tier structure of the German interbank market. The directed two-tier structure shown in Fig. 4 with $X \sim \mathcal{N}(0,1)$ is a digraph with $N = 6$ market participants and two classes of derivatives $C = \{1, 2\}$, where the different classes are depicted by different looking arrows. Calculating the expected bilateral counterparty risk for the digraph $D = (V, \mathcal{A})$ with $\mathcal{A} = A^1 \cup A^2$ as depicted in Fig. 4 is straightforward, because all bilateral portfolios have the same simple structure. There is only one claim and one debt within each of the ten bilateral portfolios. For instance, $Y_{v,w}(C) = |X^{(1)}_{v,w}| - |X^{(2)}_{v,w}|$, whereby the r.v.s on the right hand side of the last equation are distributed by the positive and negative absolute value of the Laplace distribution. The associated c.f. $\phi$ on the right hand side of the last equation are distributed by 

$$E(Y_{v,w}(C)) = \max\{Y_{v,w}(C); 0\} = \frac{\frac{1}{1+it} + \frac{1}{1-2it}}{2}$$

Obviously, if we leave out the information about the direction of the r.v.s and if we further consider the symmetry of the r.v.s of a graph we will receive a similar formula for the underlying graph $G_D$ of $D$. The next example will demonstrate this obvious result.

4.2 Example: Consider the graph $G := (\{v, w\}, \mathcal{E})$ with $N = 2$, $X \sim \mathcal{N}(0, \sigma)$ and $\mathcal{E} = E^1 \cup \ldots \cup E^K$ as sketched in Fig 5, where the different looking edges represent the netted positions between $v$ and $w$ within one of $K$ derivatives classes. We have assumed $X_{v,w}^{(k)} \sim \mathcal{N}(0, \sigma)$ for all $k \in C$ and we do not have any additional information about the direction of the edges. The c.f. $\phi_{Y_{v,w}(C)}(t) = e^{-\frac{2K\sigma^2}{2}}$ of the sum $Y := Y_{v,w}(C) = \sum_{k \in C} X_{v,w}^{(k)}$ equals the even and real-valued c.f. of $\mathcal{N}(0, \sqrt{K}\sigma)$. Because of

$$\text{Please note that the positive absolute value of the Laplace distribution is the exponential distribution. The imaginary part of the c.f. of the exponential distribution equals the Hilbert transform of } \frac{1}{1+it} \text{ because of equation (5.8). An alternative way to derive the Hilbert transform of } \frac{1}{1+it} \text{ is to use formula (5.1) or (5.2). In our case we get } H\left(\frac{1}{1+it}\right)(\omega) = 2i\text{Res}\left[\frac{1}{1+it}\right] + i\text{Res}\left[\frac{1}{1+it}\right] = \frac{1}{\omega+it} + \frac{1}{\omega-it} = \frac{1}{\omega^2+t^2} \text{ and by turning } \omega \text{ to } t \text{ we get the same result. Please refer to Example 4.6.}$$

Figure 4: Digraph with a directed two-tier structure and two classes of derivatives.
Figure 5: Two-vertex graph $G$ with $K$ edges

equation (2.3) and $H\{φ_Y\}(0) = 0$ we obtain

$$\phi_{\max}[Y;0](t) = \frac{1}{2}[1 + φ_Y(t)] + \frac{1}{2}[H\{φ_Y\}(t) - H\{φ_Y\}(0)]$$

$$= \frac{1}{2} + \frac{e^{-t^2Kσ^2}}{2} + i\int t \sqrt{\frac{Kσ}{2}} \frac{F(t/\sqrt{2})}{\sqrt{π}}$$

where $H\{φ_Y\}(t) = \frac{2}{\sqrt{π}}F\left(\frac{t\sqrt{Kσ}}{\sqrt{2}}\right)$ and $F(t) := e^{-t^2} \int_0^t e^{s^2} ds$ is the so-called Dawson function. The expected bilateral counterparty risk of $v$ to $w$ is then given by

$$E(\max[Y;0]) = \frac{∂}{∂t} [\phi_{\max}[Y;0]](0) = \frac{∂}{∂t} \left[\frac{F(t/\sqrt{2})}{\sqrt{π}}\right](0) = \frac{σ}{\sqrt{2π}}.$$
4.2 Counterparty Risk within a Centrally Cleared Market

By introducing a CCP we do not need to extend the preceding setting, because the network contains enough information to calculate the counterparty credit risk of each single entity and therefore of the entire network. That is, we imagine a CCP as an abstract entity which is not part of the market participant’s network. We assume that a CCP clears exactly one class of derivatives. Within a multilateral derivatives market $\mathcal{M}$ we are not allowed to offset positions across different derivatives classes. But we can aggregate positions of all market participants of one single class $k \in C$ that is cleared by a CCP instead. Thus, a multilateral netting set of a market participant $v \in V$ equals the neighbourhood $U_v^{(k)}$ of $v$ within derivatives class $k \in C$.

We shall call $y_{v,k}(U_v^{(k)}) := \sum_{w \in U_v^{(k)}} x_{v,w}^{(k)} \in \mathbb{R}$ the current multilateral position of $v$ within derivatives class $k$. The function $y_{v,k}$ therefore represents the netted sum of the observable trade positions of $v$ relative to all adjacent vertices within class $k$, but the actual current multilateral counterparty risk of $v$ to the CCP is $\max[y_{v,k}(U_v^{(k)}); 0]$.

If we change the perspective to the overall counterparty risk of the entire market, we get $z_m(D_k) := \sum_{v \in V} |y_{v,k}(U_v^{(k)})|$ the current multilateral counterparty risk of $D_k$. The equation $\sum_{v \in V} y_{v,k}(U_v^{(k)}) = 0$ is valid since each position is counted twice in $z_m(D_k)$, once as debt and once as claim. Therefore, we receive

$$z_m(D_k) = \sum_{v \in V} |y_{v,k}(U_v^{(k)})| = 2 \cdot \sum_{v \in V} \max[y_{v,k}(U_v^{(k)}); 0].$$

The last equation justifies the novation of the original bilateral contract to a CCP. We further assume that bilateral netting is the prevailing form of netting, unless explicitly stated otherwise. Therefore, if we introduce a CCP in one of the $K$ classes of derivatives, let us say in the class $k \in C$, then we assume that the remaining $(K - 1)$ classes are still cleared bilaterally. Thus the total exposure of the entire market is

$$z_m(D) := z_m(D_k) + z_m(D \setminus D_k),$$

where the last summand of the right-hand side of the equation denotes the current bilateral counterparty risk of the OTC market $D \setminus D_k := (V, A \setminus A^k)$.

Let us now suppose that the position values of $\mathcal{M}$ that corresponds to arrows of the digraph $D = (V, A)$ are not yet realised but represented abstractly by $X \sim \#|P|$. For all other relations between two different counterparts, we set the (future) position value deterministically to zero. The direction of each arrow of $A^k$ determines the condition $\pm$ of the r.v.s $\#|X_{v,w}^{(k)}|$ distributed by $\#|P|$. Further, we denote the c.f. of the r.v. $Y_{v,k}(U_v^{(k)})$ and $\max[Y_{v,k}(U_v^{(k)}); 0]$ by $\phi_{Y_v}$ and $\phi_{\max[Y_v,0]}$, respectively. If we apply again formulas (2.2) to (2.5) the expected multilateral counterparty risk of an arbitrary market participant $v$ of $D_k$ is then determined by

$$\mathbb{E}\left(\max[Y_{v,k}(U_v^{(k)}); 0]\right) = \frac{\partial_t[\phi_{\max[Y_v,0]}(t)](0)}{1},$$

where $Y_{v,k}(U_v^{(k)}) = \sum_{w \in U_v^{(k)}} \#|X_{v,w}^{(k)}|$ is a r.v. with mean and variance depending on the information about the directions of the positions. Putting the parts together, we obtain the formula

$$\mathbb{E}(Z_m(D_k)) = \sum_{v \in V} \mathbb{E}\left(\max[Y_{v,k}(U_v^{(k)}); 0]\right) = \sum_{v \in V} \frac{\partial_t[\phi_{\max[Y_v,0]}(t)](0)}{1}.$$
for the expected multilateral counterparty risk of an arbitrary digraph $D_k$ (of derivatives class $k$). We adopt formula (4.3) to get

$$Z_m(D) := Z_m(D_k) + Z_m(D \setminus D_k)$$

(4.6)

for digraphs, where the weights of the arrows depend on random variables.

Again, if we fade the information about the direction of each r.v. out, we will obtain similar formulas like (4.4) and (4.5) for an undirected underlying graph $G_k$ of $D_k$. As in Example 4.2 we derive a formula for a complete undirected graph with $X \sim \mathcal{N}(0, \sigma)$.

4.4 Example: Suppose $G = (V, E)$ is the undirected complete graph with $N$ vertices, one class of derivatives and $X \sim \mathcal{N}(0, \sigma)$. We do not have any information about the direction of the positions. For the sake of simplicity we write $Y_v$ instead of $Y_{v,1} = \sum_{w \in U_v} X_{v,w}^{(1)}$ and $X_{v,w}$ instead of $X_{v,w}^{(1)}$. The c.f. of the sum $Y_v(U_v^{(k)}) = \sum_{w \in U_v} X_{v,w}^{(k)}$ equals the even function $\phi_{Y_v}(t) = e^{-\frac{1}{2}(N-1)^2 t^2}$ and its Hilbert transform equals the odd function $H\{\phi_{Y_v}\}(t) = \frac{2}{\sqrt{\pi}} F\left(\frac{\sqrt{N-1} \sigma}{\sqrt{2}}\right)$.

Applying formulas (3.1) and (4.4), we obtain

$$\mathbb{E} \left( \max[Y_v(U_v^{(k)}); 0] \right) = \frac{1}{i} \partial_t \left[ \frac{1}{2} \left( 1 + e^{-\frac{1}{2}(N-1)^2 t^2} \right) + \frac{1}{\sqrt{\pi}} F\left(\frac{\sqrt{N-1} \sigma}{\sqrt{2}}\right) \right] (0) = \sqrt{\frac{N-1}{2\pi}}$$

the expected multilateral counterparty risk of $v$ within a complete graph with $N$ vertices. The last equation equals formula (4) in [DZ11]. Be aware of the symmetry of the multilateral portfolios and that $G$ is the underlying graph of $2^{\frac{N(N-1)}{2}}$ different digraphs and the calculated expected counterparty risk is some sort of average of all of them.

Applying Theorem 4.2 to a centrally cleared market we get the following conclusion.

4.5 Conclusion: Let $D = (V, \bar{A})$ be a connected digraph along with $N \geq 2$ market participants, $K$ derivatives classes $C = \{1, \ldots, K\}$ and $X \sim \mathcal{N}(0,1)$ that represents a centrally cleared market. Then the following holds:

(i) $\mathbb{E}[\sum_{w \in U_v^{(k)}} |X_v^{(k)}|] = \mathbb{E}[Y_v(U_v^{(k)})] = 0$ if and only if $\gamma(v) = 0$ within class $k$;

(ii) $\mathbb{E} \left( \max[Y_{v,k}(U_v^{(k)}); 0] \right) = \frac{1}{2} \partial_t [H\{\phi_{Y_v,k}(t)\}(0)]$ if $\gamma(v) = 0$ within class $k$.

4.6 Example: Suppose $G = (V, E = \{e_1, e_2, e_3\})$ is the undirected graph as depicted in Fig. 6 with a single class of derivatives and $X \sim \mathcal{U}(0,1)$. Let $l, m, n \in \{1, 2, 3\}$ with $m \neq n$.

We consider the Eulerian orientation $D = (V, \{a_1, a_2, a_3\})$ as sketched in Fig. 6 which is one out of eight possible orientations of $G$ as depicted in Fig. 7. The orientations define the directions of the edges and therefore the direction of the corresponding trade positions. The c.f. $\frac{1}{1+t^2} = \frac{1}{1+t^2} \cdot 1$ of the netted sum $Y_v := |X_m| - |X_n|$ is real-valued and even. Here, $X_m$ and $X_n$ are r.v.s that represent the corresponding positions. Applying (ii) of Conclusion 4.5 we obtain

$$\mathbb{E} \left( \max[Y_\gamma(U_\gamma^{(1)}); 0] \right) = \frac{1}{2} \partial_t [H\{\phi_{Y_\gamma}(t)\}(0)] = \frac{1}{2} \partial_t \left[ \frac{t}{1+t^2} \right](0) = \frac{1}{2}$$
the expected multilateral counterparty risk of $v_l$ with $l \in \{1, 2, 3\}$ of the Eulerian orientation $D$. The expectation of the centrally cleared market is therefore given by $E(Z_m(D)) = E(Z_m(D_1)) = \sum_{l=1}^{3} E\left(\max[Y_{v_l}(U_{v_l}^{(1)}); 0]\right) = 3 \cdot \frac{1}{2} = \frac{3}{2}$. All other six non-Eulerian orientations entail $\frac{5}{2}$ as expected counterparty risk of the entire market, which is significantly more than the risk of the two Eulerian orientations.

Let us now turn our attention back to the undirected graph $G = (V, E)$. The c.f. of a position equals the real-valued and even function $\frac{1}{1+t^2}$. The positive absolute value of $\frac{1}{1+t^2}$ is an analytic signal, because $\frac{1}{1+t^2}$ as well as its Fourier transform is an absolute integrable function. Thus, the imaginary part of $\frac{1}{1+t^2} + i \frac{t}{1+t^2}$ is the Hilbert transform of $\frac{1}{1+t^2}$.

The c.f. $\phi_{Y_{v_l}}(t) = \frac{1}{(1+t^2)} = \frac{1}{1+t^2} + \frac{t}{1+t^2}$ of the r.v. $Y_{v_l} := X_m + X_n$ of participant $v_l$ is also even and real-valued. Its Hilbert transform $H\{\phi_{Y_{v_l}}\}(t)$ for all $l \in \{1, 2, 3\}$ is given by $\frac{t(3+t^2)}{2(1+t^2)^2}$. Applying formula (3.1) we obtain

$$E\left(\max[Y_{v_l},1(U_{v_l}^{(1)}); 0]\right) = \frac{1}{2} \partial_t \left[ \frac{t(3+t^2)}{2(1+t^2)^2} \right](0) = \frac{3}{4}$$

for one of the three vertices of $G$, so $E(Z_m(G)) = \frac{9}{4}$. This result can also be deduced as an average of all orientations, that is, $\frac{1}{8}(2 \cdot \frac{3}{2} + 6 \cdot \frac{5}{2}) = \frac{9}{4}$.

The last example has shown that information about the direction of positions is essential for the computation of the expectation of an entire market. Disregarding such information could lead to over- or underestimating counterparty credit risk. It is obvious that similar results can be obtained in more complex networks and that the precise structure is essential for the study of counterparty and therefore systemic risk.

\[22\] Please refer to section 5.2.
4.3 Advantageousness of Multilateral Netting

The objective of this section is to generalise a main result of Duffie and Zhu [DZ11]. The authors raised the question of whether it is more advantageous for the overall counterparty credit risk to clear via a CCP or classically bilateral between the two involved counterparties. By applying the introduced network model we can answer this question not only for complete graphs, but also for arbitrary graphs and digraphs. Moreover, the network model introduced in section 2 is not constrained to the normal distribution, it can also employ any (symmetric) distribution with a defined expected value.

Introducing a CCP for a single class of derivatives $k \in C$ within a digraph $D$ with $K \in \mathbb{N}$ classes of derivatives along with $X \sim \mathcal{N}(0, 1)$ improves the netting efficiency if and only if

$$\mathbb{E}(Z_n(D)) = \mathbb{E}(Z_n(D_k)) + \mathbb{E}(Z_k(D \setminus D_k)) < \mathbb{E}(Z_n(D))$$

due to definition (4.6). Because of formulas (4.2) and (4.5) this applies if and only if

$$\sum_{v \in V} \frac{\partial [\phi_{v,K}(t)](0)}{i} + \sum_{v \in V} \frac{\partial [\phi_{v,v}(t)](0)}{i} < \sum_{v \in V} \frac{\partial [\phi_{v,v}(t)](0)}{i},$$

where

$$M \phi_{v,v}(t) := \frac{1}{2} \left[ 1 + \phi_X(t)^M \right] + \frac{i}{2} \left[ H\{\phi_X(t)^M\}(t) - H\{\phi_X(t)^M\}(0) \right]$$

is the c.f. of the r.v. max$[Y, 0]$ with $Y := \sum_{j=1}^{M} X_j$ and $X_j \sim P$. In this section $M \in \mathbb{N}$ is a representative for $N-1, K$ or $K-1$ and we write $X$ instead of $X_j$. We are now able to compute the advantageousness for a specific digraph by applying the introduced stochastic framework to the inequality (4.7).

Let us now focus on a single representative counterpart $v$ of a complete undirected graph with $X \sim P$ as in [DZ11]. Because of the features of a complete graph as well as the equations (4.11) and (4.14) it is then profitable for $v$ to be cleared via a CCP if and only if

$$\frac{\partial [1 - \phi_X(t)](0)}{i} + (N - 1) \frac{\partial [\phi_{v,v}(t)](0)}{i} < (N - 1) \frac{\partial [\phi_{v,v}(t)](0)}{i} \quad \iff \quad (N - 1) \frac{\partial [\phi_{v,v}(t)](0)}{i} < (N - 1) \frac{\partial [\phi_{v,v}(t)](0)}{i} \quad \text{for } v \in V.$$

If we apply formula (4.1) and then put the result into (4.8) we will receive the inequality (4.10) for a complete graph where the positions are distributed by $P$.

4.7 Example: Suppose we have a complete graph $G = (V, E)$ with $N$ market participants, a single class of derivatives and along with $X \sim \mathcal{N}(0, 1)$. Applying formula (4.10) and considering $\partial [H\{\phi_X(t)^M\}(0)] = \sqrt{M} \sqrt{\frac{2}{\pi}}$ we obtain

$$\frac{1}{2} \sqrt{N - 1} \sqrt{\frac{2}{\pi}} < \frac{(N - 1)}{2} \left( \sqrt{K} \sqrt{\frac{2}{\pi}} - \sqrt{K - 1} \sqrt{\frac{2}{\pi}} \right).$$

This inequality can be easily transformed to $K < \frac{N^2}{4(N-1)}$ with $N > 2$, which equals formula (6) in [DZ11].
The Laplace distribution has plainly fatter tails than the normal distribution. However, the results of the next example are very similar compared to these of Example 4.7.

4.8 Example: Again, we consider a complete graph \( G = (V, E) \) along with \( X \sim \mathcal{L}(0, 1) \). The c.f. \( \phi_X(t) \) equals \( \frac{1}{1 + t^2} \). Unfortunately, in this case we can not solve the inequality induced by formula (4.10) exactly. Instead, we apply formula (3.1) in order to obtain

\[
\frac{1}{i} \partial_t \left[ M \phi_{v,k}(t) \right](0) = \frac{1}{2} \partial_t \left[ H \{ \phi_X(t)^M \} \right](0) = \frac{\Gamma(\frac{1}{2} + M)}{\sqrt{\pi} \Gamma(M)} = \frac{M}{2^{2M}} \left( \frac{2M}{M} \right)
\]

the expected multilateral counterparty risk for the representative counterpart \( v \) within derivatives class \( k \). Here, \( \Gamma(t) := \int_0^\infty x^{t-1} e^{-x} \, dx \) is the so-called gamma function, which is an extension of the factorial function. By applying formula (4.8) and (4.11) we can deduce the values of

| \( K \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( N \geq \) | 2 | 6 | 10 | 14 | 18 | 22 | 26 | 30 | 34 | 38 |

Table 1: How many market participants do we need to assure the advantageousness of the central clearing?

We have confirmed formula (6) of [DZ11], and we have also shown how to apply the introduced model to do a similar calculation for a different distribution within a complete graph. However, if we want to study the exact constitution of a financial market we need to study an arbitrary graph or digraph and not a specific type of graphs or digraphs. As a result something as a single representative counterpart for the entire structure can in general not exist. However, we can compare the implications made from the different models in order to validate the associated assumptions.

4.9 Example: Let \( G_D \) be the underlying graph \( G_D \) of Example 4.1. That is, we consider an undirected two-tier market structure with \( X \sim \mathcal{L}(0, 1) \). On the one hand, the expected bilateral counterparty risk \( \mathbb{E}(Z_b(G_D)) = 7.5 \) is significantly smaller than the expected multilateral counterparty risk \( \mathbb{E}(Z_m(G_D)) = 8.875 \). Compared, on the other hand, with the implications drawn in Example 4.8 Tab. 1 for a complete graph with \( K = 2 \) classes of derivatives, we obviously see that the risk profile of the described two-tier market with \( N = 6 \) participants is completely different.

The last example demonstrates that the exact market structure is essential for giving a comprehensive answer to the raised question.

\[23\] The result of Example 4.6 can now be recalculated with formula (4.11). Just set \( M = N - 1 = 2 \) and we get

\[
\frac{1}{i} \partial_t \left[ 2 \phi_{v,k}(t) \right] = \frac{3 \sqrt{\pi}}{4} \sqrt{\pi} = \frac{3}{4}.
\]
5 Auxiliary Results for the Application

In this section we provide auxiliary results that can help to clear hurdles related to the application of the network model presented in section 2. To the extent of our knowledge, Proposition 5.1 about Hilbert transforms is not yet known. The essence of Proposition 5.3 is known from signal processing, but it has not yet been applied in the context of counterparty or systemic risk.

5.1 Taking the Maximum and the Hilbert Transform

In order to tackle the problem of deriving the Hilbert transform we generalise the result in section 3.4 in [Kin09a]. We show how two derive two very useful formulas by using complex analysis and the well-known residue theorem.

In order to take the maximum \( \max(Y; 0) \) between the sum \( Y = \sum_{\lambda \in \Lambda_0} \lambda|X_\lambda| \) and zero, we have to deal with the Hilbert transform as introduced in (2.4). Let \( \phi_Y \) be a real-valued function, and let the function \( C \ni z \mapsto \frac{\phi_Y(z)}{\omega - z} \) be extended into the complex plane and bounded by \( C \). This extended function is required to be analytic within the complex upper half-plane, except for a finite number of poles \( a_1, \ldots, a_m \in \mathbb{C} \) of order \( n_1, \ldots, n_m \in \mathbb{N} \). Further we assume that \( \frac{\phi_Y(z)}{\omega - z} \rightarrow 0 \) as \( z \rightarrow \infty \).

5.1 Proposition: If the previously stated prerequisites are met then the equations

\[
H(\phi_Y(t))(\omega) = 2i \sum_{j=1}^{m} \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, a_j \right] + i \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, \omega \right].
\]

and

\[
H(\phi_Y(t))(\omega) = 2i \sum_{j=1}^{m} \left( \frac{1}{(n_j - 1)!} \lim_{z \to a_j} \frac{d^{n_j-1}}{dz^{n_j-1}} \left[ (z - a_j)^{n_j} \phi_Y(z) \right] \right) - i \lim_{z \to \omega} \phi_Y(z)
\]

are valid.

Proof. The interval \([-R, R]\), as part of the domain of the Hilbert transform (2.4), is incorporated into the closed path \( C = C_R \cup [-R, \omega - \epsilon] \cup \omega + \epsilon, R \) as sketched in Fig. 8. Obviously, we have \( R, \epsilon \in ]0, \infty[ \). The positively oriented contour \( C \) consists of the semicircle \( \mathbb{C}R \), the semicircle \( \mathbb{C}_e \) and the segments on the real line.

The real-valued integrand is then extended into the complex region bounded by \( C \) and \( \frac{\phi_Y(z)}{\omega - z} \) is required to be analytic within the complex upper half-plane, except for a finite number of poles \( a_1, \ldots, a_m \in \mathbb{C} \) of order \( n_1, \ldots, n_m \in \mathbb{N} \). Furthermore, we assume that \( \frac{\phi_Y(z)}{\omega - z} \rightarrow 0 \) as \( z \rightarrow \infty \) and we obviously have a simple pole on the real line at \( t = \omega \). We can choose \( R \) large and \( \epsilon \) small enough such that the poles of \( \phi_Y \) of the upper half-plane lie within the contour \( C \) and do not intersect with the semicircle \( \mathbb{C}_e \). Applying the residue theorem we get

\[
\oint_C \frac{\phi_Y(z)}{\omega - z} \, dz = 2\pi i \sum_{j=1}^{m} \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, a_j \right].
\]

\[\text{For a situation where poles inside the contour do show up please refer to section 22.10 in [Kin09].}\]

\[\text{See Chapter 1, Holomorphic Functions in [Rem91].}\]
This contour integral can be decomposed into

\[ \int_C \frac{\phi_Y(z)}{\omega - z} \, dz = \int_{-R}^{\omega - \epsilon} \frac{\phi_Y(z)}{\omega - t} \, dt + \int_{C_\epsilon} \frac{\phi_Y(z)}{\omega - z} \, dz + \int_{\omega + \epsilon}^{R} \frac{\phi_Y(t)}{\omega - t} \, dt + \int_{C_R} \frac{\phi_Y(z)}{\omega - z} \, dz. \]

The first and third integral on the right hand side of equation (5.4) equals the principle value integral as \( R \to \infty \) and \( \epsilon \to 0 \), i.e.,

\[ \text{PV} \int_{-\infty}^{\infty} \frac{\phi_Y(t)}{\omega - t} \, dt = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_{-R}^{\omega - \epsilon} \frac{\phi_Y(t)}{\omega - t} \, dt + \int_{\omega + \epsilon}^{R} \frac{\phi_Y(t)}{\omega - t} \, dt \right). \]

Let \( C_\epsilon \) be parameterised by \( \omega + \epsilon e^{i\theta} \) with \( 0 \leq \theta \leq \pi \), then we obtain

\[ \int_{C_\epsilon} \frac{\phi_Y(z)}{\omega - z} \, dz = \lim_{\epsilon \to 0} \left( i \int_{0}^{\pi} \frac{\phi_Y(\omega + \epsilon e^{i\theta})e^{i\theta}}{\omega - (\omega + \epsilon e^{i\theta})} \, d\theta \right) = \pi i \phi_Y(\omega) = -\pi i \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z} \right]. \]

For the last integral we parameterise \( C_R \) by \( R e^{i\theta} \) with \( 0 \leq \theta \leq \pi \) and we bear in mind that \( \frac{\phi_Y(z)}{\omega - z} \to 0 \) as \( z \to \infty \). We then deduce

\[ \int_{C_R} \frac{\phi_Y(z)}{\omega - z} \, dz = \lim_{R \to \infty} \left( i \int_{0}^{\pi} \frac{\phi_Y(R e^{i\theta})R e^{i\theta}}{\omega - R e^{i\theta}} \, d\theta \right) = 0. \]

Equation (5.3) therefore simplifies to

\[ \text{PV} \int_{-\infty}^{\infty} \frac{\phi_Y(t)}{\omega - t} \, dt = 2\pi i \sum_{j=1}^{m} \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, a_j \right] + \pi i \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, \omega \right], \]

which leads to

\[ \text{H}\{\phi_Y(t)\}(\omega) = 2i \sum_{j=1}^{m} \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, a_j \right] + i \text{Res} \left[ \frac{\phi_Y(z)}{\omega - z}, \omega \right]. \]

If we then apply rule 1) and 2) of chapter 13 in [Rem91] we further receive formula (5.2). □

The two equations (5.1) and (5.2) are particularly useful for calculating the Hilbert transform of intricate functions as we can see in the next example.
5.2 Example: Let φ_{|X|}(t) = \frac{1}{2\pi} \frac{1}{1+4\omega^2} + i\frac{2i}{1+4\omega^2} be the c.f. of a r.v. |X| that is distributed by the Gamma distribution \Gamma(\alpha, \beta) with \alpha := 1 and \beta := 2. Here \alpha > 0 is the shape parameter and \beta > 0 the scale parameter. Be aware that the Gamma distribution has only positive samples and that the function φ_{|X|}(z) converges towards zero if z → ∞. Furthermore, the function φ_{|X|}(z) is analytic in the upper half-plane except for the pole \frac{i}{2}. Applying equation (5.1) we get H\{φ_{|X|}(t)\}(ω) = i\text{Res}\left[\frac{φ_{|X|}(z)}{ω−z}, ω\right] = \frac{2ω}{1+4ω^2} − i\frac{1}{1+4ω^2}.

Let us now consider the associated negative absolute value −(|X|) = −|X| with φ_{−|X|}(t) = \frac{1}{1+4t^2} − \frac{2i}{1+4t^2}. We now have \frac{1}{2} as simple pole. Applying again equation (5.1) we receive H\{φ_{−|X|}(t)\}(ω) = 2i\text{Res}\left[\frac{φ_{−|X|}(z)}{ω−z}, \frac{i}{2}\right] + i\text{Res}\left[\frac{φ_{−|X|}(z)}{ω−z}, ω\right] = \frac{2ω}{1+4ω^2} + i\frac{1}{1+4ω^2}.

If we consider the last example carefully, we will find several interesting connections between the positive and the negative absolute value as well as to its Hilbert transforms. We will examine these connections in the next section. The introduced notation of this section shall be valid for the entire article.

5.2 Positive and Negative Absolute Values

Let X ∼ P be a r.v. that is symmetric around the origin and φ_X its characteristic function. In this section we study the connection between the c.f.s φ_X, φ_{|X|} and φ_{−|X|}. The latter two c.f.s are used for the representation of the direction of an arrow of the digraph D.

It is well-known that a c.f. φ_X of a r.v. X is Hermitian, i.e., that φ_X(−t) = \overline{φ_X(t)} for all t ∈ R. Due to Theorem 3.1.2 in [Luk70] the r.v. X is symmetric if and only if its c.f. φ_X is real-valued and even. However, the c.f.s of the positive absolute value |X| as well as of the negative absolute value −|X| are in general complex-valued. We need to take into account the equation −(|X|) = −|X| as well as that φ_X is Hermitian in order to receive

\begin{equation}
φ_{|X|}(t) = \eta(t) + iν(t)
\end{equation}

for the positive absolute and

\begin{equation}
φ_{−|X|}(t) = \eta(t) − iν(t)
\end{equation}

for the negative absolute value for all t ∈ R. Please note that \eta = \text{Re}(φ_{|X|}), \nu = \text{Im}(φ_{|X|}) and that φ_{|X|} = \overline{φ_{−|X|}} is obviously valid. In the following we explain how the functions \eta and \nu are connected to each other. We call any complex c.f. φ(t) whose real and imaginary components satisfy the equation

\begin{equation}
φ(t) = \eta(t) + iν(t) = \eta(t) + iH\{η\}(t) \quad t ∈ R
\end{equation}

an analytic signal.\(^{23}\)

5.3 Proposition: The c.f. φ_{|X|} of a positive absolute value |X| with X ∼ P is an analytic signal in a natural way,\(^{27}\) i.e.,

\begin{equation}
φ_{|X|}(t) = φ_X(t) + iH\{φ_X\}(t).
\end{equation}

\(^{23}\)See section 4.1.4 in [Kin09a].

\(^{27}\)Compare with section 18.4 in [Kin09a].
Proof. Let $X$ be a symmetric r.v. with real-valued and even c.f. $\phi_X$ and let us further assume that $\phi_X$ is absolutely integrable. Then, if its Fourier transform $f := \mathcal{F}\{\phi_X\}$ is also absolutely integrable, and we can use the inverse Fourier transform

$$
\mathcal{F}^{-1}\{f\}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\pi t} dx = \phi_X(t)
$$

(5.9)

to recover the input function $\phi_X$ from its Fourier transform. We can imagine $f$ as some sort of probability function that represents the distribution $F$, i.e., $f$ could be a density function, for instance. The definition of $|X|$ basically means that all negative samples are rejected and the probability for the positive samples is doubled. Thus, we set $f^+(x) := f(x) + \text{sgn}(x)f(x)$ for all $x \in \mathbb{R}$, where

$$
\text{sgn}(x) = \begin{cases} 
1, & \text{for } x > 0 \\
0, & \text{for } x = 0 \\
-1, & \text{for } x < 0
\end{cases}
$$

is the signum function. The function $f^+$ is zero for all negative and $2f(t)$ for all positive real numbers. Apparently, the function $f^+$ is a representation for the positive absolute value $|X|$. Applying the inverse Fourier transform to $f^+$ we get

$$
\mathcal{F}^{-1}\{f^+\} = \mathcal{F}^{-1}\{f\} + \mathcal{F}^{-1}\{\text{sgn} \cdot f\} = \phi_X + \mathcal{F}^{-1}\{\text{sgn} \cdot f\} = \phi_{|X|}
$$

because of formula (5.9) and the additivity of the integral. Due to equation (5.5) we further know that $\mathcal{F}^{-1}\{f^+\} = \eta + i\nu$. Comparison of real and imaginary parts yields to $\mathcal{F}^{-1}(\text{sgn} \cdot f) = i\nu$, which means that $-i \cdot \text{sgn} \cdot f$ is related to $\nu$ by the inverse Fourier transform. According to equation (5.2) in [Kin09a] the inverse Fourier transform of $-i \cdot \text{sgn}(x)$ equals $\frac{1}{\pi x}$ and because of (4.154) in [Kin09a] we receive

$$
\nu(t) = \phi_X(x) * \frac{1}{\pi x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_X(t) \frac{dx}{x-t} = H\{\phi_X\}(t),
$$

where $*$ is the convolution. Putting the parts together to

$$
\phi_{|X|}(t) = \phi_X(t) + iH\{\phi_X\}(t),
$$

(5.10)

we easily infer that the c.f. $\phi_{|X|}$ of the positive absolute value $|X|$ is an analytic signal. This basically means that the negative samples of the real-valued distribution $P$ are superfluous in this context. An illustrative explanation for this fact provides the symmetry of the distribution $P$.

Because of $\phi_{-|X|}(t) = \overline{\phi_{|X|}(t)}$ it follows

$$
\phi_{-|X|}(t) = \phi_X(t) - iH\{\phi_X\}(t).
$$

(5.11)

Furthermore, the $n$-th power of the Hilbert transform of the analytic signal $\phi_{|X|}$ can be written as

$$
H\{\phi^n_{|X|}\} = -i\phi^n_{|X|}, \quad n \in \mathbb{N},
$$

(5.12)

---

28See section 2.6.1 in [Kin09a].
29Section 2.6.2 in [Kin09a].
due to equation (4.252) in [Kin09a].

Formulas (5.8) and (5.11) show how the c.f. of a symmetric distribution $P$ and the c.f. of its positive and negative absolute values are connected to each other by the Hilbert transform. Both formulas are also useful when we employ a distribution with only positive or negative outcomes as positive or negative absolute values. We can then use formulas (5.8) and (5.11) to get the Hilbert transform simply by considering the imaginary component of the positive or negative absolute value of the distribution.

### 5.4 Example

We consider a netting set $\Lambda_v$ comprising $m \in \mathbb{N}$ positive trade positions of a counterparty $v$. Each position is represented by a i.i.d. r.v. $|X_i| \sim \Gamma(\alpha, \beta)$ with $i \in \{1, 2, \ldots, m\}$. The absolutely integrable c.f. of the positive absolute value $|X_i|$ is $(1 - \beta it)^{-\alpha}$ and the sum $Y := \sum_{i=1}^{m} |X_i|$ is determined by the c.f. $(1 - \beta it)^{-\alpha m}$. The sum $Y$ is distributed by $\Gamma(m\alpha, \beta)$ and its Hilbert transform equals $-i(1 - \beta it)^{-\alpha m}$, because of equation (5.12). Please also compare these results to Example 5.2. It is quite straightforward to apply formula (2.5) to get the expectation $\alpha \beta m$ of the credit exposure. The situation becomes more interesting, when the netting set contains positive and negative trade positions of the counterparty $v$. Then, we have to take the maximum between the sum of the r.v.s and zero in order to determine the expected counterparty credit risk.

### 6 Conclusion

We endorse the view of several authors that considering the precise market structure for studying counterparty credit risk or systemic risk is essential. We provide a new type of network model which is capable of capturing the precise structure of any given financial market based, for example, on empirical findings. With the attached stochastic framework it is further possible to study how a network structure and counterparty credit risk are connected to each other. This allows us to study different structures and their characteristics relating to, for instance, systemic risk. We show that Eulerian digraphs are distinguished exposure structures in the context of counterparty risk and we reveal that different structures can have a significantly different impact on the overall risk. We therefore suggest that the individual structure of a financial market should be taken into consideration.

We use the powerful theory of characteristic functions as well as the theory of Hilbert transforms. Deriving the specific characteristic function as well as its Hilbert transform can be a great challenge. However, we provide useful insight into both concepts in order to overcome these barriers in many cases. The model presented here is quite flexible and could be easily modified to meet specific requirements. For example, it could be used to study the structure of counterparty credit risk within other types of markets, different netting rules and more complex distributions such as extreme value distributions. One could also use the model to study analytically how shocks affect a specific network by changing the distribution (parameter) in an appropriate way.

### 7 Proofs

This section contains the proofs of the two structure theorems.

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30 Please refer to Example 5.2.
7.1 Proof of Structure Theorem 3.2

Assume $Y_v := \sum_{\lambda \in \Lambda_v} X_\lambda$.

(i) Let $\lambda \in \Lambda_v$ and let further $\Lambda^+_v$ and $\Lambda^-_v$ be the sets with $h(\lambda) = v$ and $t(\lambda) = v$, respectively. The two sets form a partition of $\Lambda_v$, that is, $\Lambda_v = \Lambda^+_v \cup \Lambda^-_v$. Be aware that $|\Lambda^+_v| = \gamma_+(v)$ and $|\Lambda^-_v| = \gamma_-(v)$. Keeping in mind the linearity of the conditional expectation, we deduce from $0 = \mathbb{E}(Y_v) = \mathbb{E}\left(\sum_{\lambda \in \Lambda_v} \mathbb{E}(X_\lambda)\right)$ the equivalent equation $0 = \sum_{\lambda \in \Lambda_v} \mathbb{E}(|X_\lambda|) + \sum_{\lambda \in \Lambda^+_v} \mathbb{E}(|X_\lambda|)$.

Because each r.v. follows the same symmetric distribution around 0, we receive the corresponding expectation $\mathbb{E}(\max_{\lambda \in \Lambda_v} |X_\lambda|) = \mathbb{E}(\max_{\lambda \in \Lambda^+_v} |X_\lambda|)$ and therefore the expectation can be calculated by $E(\max_{\lambda \in \Lambda_v} |X_\lambda|) = \frac{1}{2} \mathbb{E}(Y_v)$.

Because of (i) we have got $E(\max_{\lambda \in \Lambda_v} |X_\lambda|) = 0$. Let us now assume that $\gamma(v) = 0$ then $\gamma_+(v) = |\Lambda^+_v| = |\Lambda^-_v| = \gamma_-(v)$. According to equations (5.3) and (5.4) the product $\phi_{Y_v} = \left(\prod_{\lambda \in \Lambda^+_v} \phi_{|X_\lambda|}\right) \left(\prod_{\lambda \in \Lambda^-_v} \phi_{-|X_\lambda|}\right)$ must be real-valued and even.

The parity property\(^{31}\) of the Hilbert transform implies that $H\{\phi_{Y_v}\}$ is an odd continuous function and therefore $H\{\phi_{Y_v}\}(0) = 0$.

7.2 Proof of Structure Theorem 3.3

The c.f. $\phi_X$ of a r.v. $X \sim P$ is real-valued and even and so is the c.f. $\phi_Y$ of the sum $Y := \sum_{\lambda \in \Lambda_v} X_\lambda$. Because of the parity property the Hilbert transform $H\{\phi_Y(t)\}(t)$ is a continuous odd function. Since $H\{\phi_Y\}(0) = 0$, we obtain $\phi_{\max_{\lambda \in \Lambda_v} |X_\lambda|}(t) = \frac{1}{2} \mathbb{E}\{\phi_Y(t)\} + \frac{1}{2} \mathbb{E}\{\phi_Y(t)\}$ and therefore the expectation can be calculated by $\frac{\partial}{\partial t}\left[H\{\phi_Y(t)\}(0)\right] = \frac{1}{2} \mathbb{E}\{\phi_Y(t)\}(0) + \frac{1}{2} \mathbb{E}\{\phi_Y(t)\}$.

The term $\frac{\partial}{\partial t}\left[H\{\phi_Y(t)\}(0)\right]$ stands for the expectation of the sum of $|\Lambda_v|$ i.i.d. random variables with zero mean. Thus, $\frac{\partial}{\partial t}\left[H\{\phi_Y(t)\}(0)\right] = 0$ and this leads us to the equation

$$\mathbb{E}\left(\max_{\lambda \in \Lambda_v} X_\lambda; 0\right) = \frac{1}{2} \mathbb{E}\{H\{\phi_Y(t)\}\}(0)$$

for an arbitrary vertex $v \in V$ of the graph $G$.

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\(^{31}\)See (4.5) and (4.6) in [Kin09a].
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