Second-Order Slepian-Wolf Coding Theorems
for Non-Mixed and Mixed Sources

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Abstract

The second-order achievable rate region in Slepian-Wolf source coding systems is investigated. The concept of second-order achievable rates, which enables us to make a finer evaluation of achievable rates, has already been introduced and analyzed for general sources in the single-user source coding problem. Analogously, in this paper, we first define the second-order achievable rate region for the Slepian-Wolf coding system to establish the source coding theorem in the second-order sense. The Slepian-Wolf coding problem for correlated sources is one of typical problems in the multi-terminal information theory. In particular, Miyake and Kanaya, and Han have established the first-order source coding theorems for general correlated sources. On the other hand, in general, the second-order achievable rate problem for the Slepian-Wolf coding system with general sources remains still open up to present. In this paper we present the analysis concerning the second-order achievable rates for general sources which are based on the information spectrum methods developed by Han and Verdú. Moreover, we establish the explicit second-order achievable rate region for i.i.d. correlated sources with countably infinite alphabets and mixed correlated sources, respectively, using the relevant asymptotic normality.

Index Terms

Asymptotic Normality, Correlated Sources, Second-Order Achievability, Slepian-Wolf Data Compression System

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I. Introduction

We establish the second-order source coding theorems for the Slepian-Wolf system. In the single-user source coding system, Han and Verdú [1] and Steinberg and Verdú [2] have shown the source coding theorem for general sources using the information spectrum methods. Since the class of general sources is quite large, their results are very fundamental and useful. On the other hand, there are several researches concerning a finer evaluation of achievable rates called rates of the second-order. In variable-length source coding, Kontoyiannis [3] has established the second-order source coding theorem. In channel coding problems, Strassen [4] (see, also Csiszár and Körner [5]), Hayashi [6], Polyanskiy, Poor and Verdú [7], have determined the second-order capacity. In addition, Hayashi [8] has given the optimal second-order achievability theorem for the fixed-length source coding problem for general sources, and actually computed it for i.i.d. sources by invoking the asymptotic normality. From his analyses we know that the information spectrum methods [9] are still effective also for the evaluation on second-order achievable rates. Nomura and Han [10] have computed the optimal second-order rate for mixed sources, which is a typical case of nonergodic sources, again by invoking the relevant asymptotic normality.

In the area of multi-source coding problems, there are various types of source coding problems [11]. The Slepian-Wolf coding problem for two correlated sources is one of such typical problems [11], [12]. In particular, there are two typical settings in the Slepian-Wolf coding problem. One is the setting that the decoder decodes only the sequence emitted from one source, and the sequence emitted from the other source is used as side-information. The other is the setting that the decoder have to decode both of two sequences emitted from two correlated sources. We call the former one the full side-information problem.

In the full side-information problem, the first-order achievable rates for general correlated sources have been determined by Steinberg and Verdú [13]. On the other hand, Watanabe, Matsumoto and Uyematsu [14], and Nomura and Matsushima [15] have considered the second-order achievable rate problem with full side-information; in this special case, [14] has given only a “necessary” condition on the optimal second-order achievable rates for i.i.d. correlated sources, while Nomura and Matsushima [15] with full-side information has exactly determined the second-order achievable rates for i.i.d. correlated sources. These results hold also on the basis of information spectrum methods and the asymptotic normality.

In this paper, we consider the ordinary Slepian-Wolf coding problem, not necessarily with full
side-information. In this setting, Miyake and Kanaya [16] with finite source alphabets, as well as Han [9] with countably infinite source alphabets, has established the first-order source coding theorem for general correlated sources. The second-order achievable rate region not necessarily with full side-information for the Slepian-Wolf coding problem was first considered by Nomura and Matsushima [17]; they have focused only on i.i.d. correlated sources, where, unfortunately, the sufficient condition and the necessary condition do not coincide. On the other hand, in this paper, we shall give the necessary and sufficient condition for general correlated sources with countably infinite alphabets. Furthermore, for i.i.d. correlated sources with countably infinite alphabets and mixed i.i.d. correlated sources with finite alphabets, we apply this fundamental result to derive the explicit second-order achievable rate region. In the second-order analysis for i.i.d. correlated sources, the multivariate normal distribution function due to the central limit theorem plays the key role, while in the previous literature on the second-order achievable rates (cf. [10], [14], [15]), only one-dimensional normal distribution functions were enough to consider. Recently, Tan and Kosut [18] has independently established a counterpart of the second-order source coding theorem for i.i.d. correlated sources, which is derived partly via the method of information spectra in addition to the standard multi-dimensional central limit theorem. It should be noted that the result in [18] heavily relies on the method of types, which necessitates the assumption of finiteness of source alphabets.

The analyses here are based wholly on the information spectrum methods to invoke the multi-dimensional normal distribution functions. In the second-order analysis, we extend our results for i.i.d. correlated sources to the mixed sources consisting of i.i.d. correlated sources. The class of mixed sources is very important, because all of stationary sources can be regarded as forming mixed sources obtained by mixing stationary ergodic sources with respect to an appropriate probability measure. The first-order achievable rate region for mixed correlated sources has already been shown by Han [9] on the basis of the information spectrum methods. We demonstrate in this paper the second-order achievable rate region also by using the information spectrum methods.

The present paper is organized as follows. In Section II, we define the general correlated sources and the achievable rate region. Then, we review the previous results on the first-order achievable rate region. In Section III we derive the second-order achievable rate region for general sources by invoking the information spectrum methods. In Section IV based on the general formula established in Section III we compute the second-order achievable rate region for i.i.d. correlated sources to establish Theorem 4.1 the canonical representation of which is also given for the first time in
this paper. In Section V, a comparison of the second-order approach with the error exponent type bound is shown to elucidate the significance of Theorem 4.1. In Section VI, we establish the formula for the second-order achievable rate region for mixed correlated sources with general mixture, again by using the relevant asymptotic normality. Finally, we conclude our results in Section VII.

II. Preliminaries

A. Correlated Sources

Let $X_1$ and $X_2$ be alphabets of two correlated sources, where $X_1$ and $X_2$ may be countably infinite. Let $(X_1, X_2) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}$ denote a general correlated source pair, i.e., $(X_1^n, X_2^n)$ taking values in $X_1^n \times X_2^n$ is a pair of correlated source variables of block length $n$, and we write as

$$(X_1^n, X_2^n) = ((X_{11}, X_{21}), (X_{12}, X_{22}), \cdots, (X_{1n}, X_{2n})),$$

and let $x_j = x_{j1}, x_{j2}, \cdots, x_{jn}$ be a realization of random variable $X_j^n$ ($j = 1, 2$). The probability distribution of $(x_1, x_2)$ is denoted by $P_{X_1^n, X_2^n}(x_1, x_2)$. In particular, in the case that the pair of correlated sources has an i.i.d. property, it holds that

$$P_{X_1^n, X_2^n}(x_1, x_2) = \prod_{i=1}^{n} P_{X_1, X_2}(x_{1i}, x_{2i}),$$

with generic correlated random variable $(X_1, X_2)$, where we use the convention that $P_Z(\cdot)$ denotes the probability distribution of $Z$, and $P_{Z|W}(\cdot|\cdot)$ denotes the conditional probability distribution of $Z$ given $W$.

B. $\varepsilon$-Achievable Rate Region

The fixed-length codes for correlated sources are characterized by a pair of encoders $(\phi^{(1)}_n, \phi^{(2)}_n)$ and a decoder $\psi_n$. The encoders are mappings such as $\phi^{(1)}_n : \mathcal{X}_1^n \rightarrow \mathcal{M}_n^{(1)}$, $\phi^{(2)}_n : \mathcal{X}_2^n \rightarrow \mathcal{M}_n^{(2)}$, where

$$\mathcal{M}_n^{(1)} = \{1, 2, \cdots, M_n^{(1)}\}, \quad \mathcal{M}_n^{(2)} = \{1, 2, \cdots, M_n^{(2)}\}$$

denote the code sets. The decoder is defined as a mapping $\psi_n : \mathcal{M}_n^{(1)} \times \mathcal{M}_n^{(2)} \rightarrow \mathcal{X}_1^n \times \mathcal{X}_2^n$. In the sequel, we focus on this Slepian-Wolf type source coding problem.

The performance of fixed-length codes is evaluated in terms of the error probability and the codeword length. The error probability is given by

$$\varepsilon_n = \Pr\{(X_1^n, X_2^n) \neq \psi_n(\phi^{(1)}_n(X_1^n), \phi^{(2)}_n(X_2^n))\}.$$
We call such a pair of encoders \((\phi_n^{(1)}, \phi_n^{(2)})\) and decoder \(\psi_n\) along with error probability \(\varepsilon_n\) an 
\((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) code.

**Definition 2.1:** A rate pair \((R_1, R_2)\) is called an \(\varepsilon\)-achievable rate pair if there exists an 
\((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) code satisfying

\[
\limsup_{n \to \infty} \frac{1}{n} \log M_n^{(1)} \leq R_1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \log M_n^{(2)} \leq R_2 \quad \text{and} \quad \limsup_{n \to \infty} \varepsilon_n \leq \varepsilon.
\]

Then, the \(\varepsilon\)-achievable rate region is defined as the set of all \(\varepsilon\)-achievable rate pairs:

**Definition 2.2:** (\(\varepsilon\)-Achievable Rate Region)

\[
R(\varepsilon|X_1, X_2) = \{(R_1, R_2)| (R_1, R_2) \text{ is } \varepsilon\text{-achievable rate}\}.
\]

For general correlated sources, Miyake and Kanaya \cite{miyake1995information} with finite source alphabets have determined the \(0\)-achievable rate region, while Han \cite{han1979source} with countably infinite source alphabets have established the \(\varepsilon\)-achievable rate region for \(0 \leq \varepsilon < 1\). Before describing their results, we need to define the spectral sup-entropy rate as follows.

\[
H(X_1|X_2) \equiv \text{p- lim sup}_{n \to \infty} \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)},
\]

\[
H(X_2|X_1) \equiv \text{p- lim sup}_{n \to \infty} \frac{1}{n} \log \frac{1}{P_{X_2^n|X_1^n}(X_2^n|X_1^n)},
\]

\[
H(X_1X_2) \equiv \text{p- lim sup}_{n \to \infty} \frac{1}{n} \log \frac{1}{P_{X_1^nX_2^n}(X_1^nX_2^n)}.
\]

The following *first-order* achievable rate theorem reveals the rate region for *general* correlated sources.

**Theorem 2.1 (Miyake and Kanaya \cite{miyake1995information}):** For any general sources with finite alphabets, the \(0\)-achievable rate region \(R(0|X_1, X_2)\) is given as the set of \((R_1, R_2)\) satisfying

\[
R(0|X_1, X_2) = \{(R_1, R_2)| R_1 \geq H(X_1|X_2), R_2 \geq H(X_2|X_1), \quad R_1 + R_2 \geq H(X_1X_2)\}.
\]

Han \cite{han1979source} has generalized their theorem to the case \(0 \leq \varepsilon < 1\) with *countably infinite* alphabets; set

\^For any sequence \(\{Z_n\}_{n=1}^{\infty}\) of real-valued random variables, we define the limit superior in probability of \(\{Z_n\}_{n=1}^{\infty}\) by 
\(\text{p-lim sup}_{n \to \infty} Z_n = \inf \{\beta| \lim_{n \to \infty} \text{Pr}\{Z_n > \beta\} = 0\}\). (cf. \cite{han1979source}).
the function $F_n(R_1, R_2)$ as

$$F_n(R_1, R_2) = \Pr \left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \geq R_1 ight. $$

$$\left. \quad \text{or} \quad \frac{1}{n} \log \frac{1}{P_{X_2^n|X_1^n}(X_2^n|X_1^n)} \geq R_2 \right\},$$

then we have:

**Theorem 2.2 (Han [9]):** For any general sources with countably infinite alphabets, the first-order $\varepsilon$-achievable rate region is given by

$$R(\varepsilon|X_1, X_2) = \text{Cl} \left( \left\{ (R_1, R_2) \mid \limsup_{n \to \infty} F_n(R_1, R_2) \leq \varepsilon \right\} \right),$$

(2.1)

where $\text{Cl}(\cdot)$ denotes the closure operation.

This theorem has been established using the following key lemmas:

**Lemma 2.1 (Han [9]):** Let $M_n^{(1)}$ and $M_n^{(2)}$ be arbitrarily given positive integers. Then, for all $n = 1, 2, \cdots$, there exists an $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code satisfying

$$\varepsilon_n \leq \Pr \left\{ z_n P_{X_1^n|X_2^n}(X_1^n|X_2^n) \leq \frac{1}{M_n^{(1)}} \right.$$

$$\left. \quad \text{or} \quad z_n P_{X_2^n|X_1^n}(X_2^n|X_1^n) \leq \frac{1}{M_n^{(2)}} \right\} + 3 z_n,$$

where $\{z_n\}_{n=1}^{\infty}$ is a sequence of arbitrary real numbers such that $z_i > 0 \ (\forall i = 1, 2, \cdots)$. ■

**Lemma 2.2 (Han [9]):** Any $(n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)$ code satisfies

$$\varepsilon_n \geq \Pr \left\{ P_{X_1^n|X_2^n}(X_1^n|X_2^n) \leq \frac{z_n}{M_n^{(1)}} \right.$$

$$\left. \quad \text{or} \quad P_{X_2^n|X_1^n}(X_2^n|X_1^n) \leq \frac{z_n}{M_n^{(2)}} \right\} - 3 z_n,$$

for all $n = 1, 2, \cdots$, where $\{z_n\}_{n=1}^{\infty}$ is a sequence of arbitrary real numbers such that $z_i > 0 \ (\forall i = 1, 2, \cdots)$. ■

Notice here that these lemmas are valid for general correlated sources with countably infinite alphabets. In this paper too we shall invoke these lemmas in order to derive the second-order achievable rate region as in the subsequent sections.
C. \((a_1, a_2, \varepsilon)\)-Achievable Rate Region

In the single-user source coding problem, a finer evaluation called the second-order achievable rate has been studied. Accordingly, we define the second-order achievable rate pair as follows.

**Definition 2.3:** A rate pair \((L_1, L_2)\) is called a second-order \((a_1, a_2, \varepsilon)\)-achievable rate pair if there exists an \((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) code satisfying

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n^{(1)}}{e^{na_1}} \leq L_1 \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n^{(2)}}{e^{na_2}} \leq L_2,
\]

\[
\limsup_{n \to \infty} \varepsilon_n \leq \varepsilon.
\]

Notice that, in the definition of \((a_1, a_2, \varepsilon)\)-achievable rate pairs, the condition for the error probability is as the same as in the definition of \(\varepsilon\)-achievable rate pair. Moreover, let us define the set of \((a_1, a_2, \varepsilon)\)-achievable rate pairs given \((a_1, a_2)\) as

**Definition 2.4 (second-order \((a_1, a_2, \varepsilon)\)-achievable rate region):**

\[
L(a_1, a_2, \varepsilon | X_1, X_2) = \{(L_1, L_2) | (L_1, L_2) \text{ is } (a_1, a_2, \varepsilon)-achievable\}.
\]

III. \((a_1, a_2, \varepsilon)\)-Achievable Rate Region for General Correlated Sources

We shall determine the \((a_1, a_2, \varepsilon)\)-achievable rate region for general correlated sources with countably infinite alphabets. Before describing our main result, let us define the function \(F_n(L_1, L_2 | a_1, a_2)\) by

\[
F_n(L_1, L_2 | a_1, a_2) = \Pr \left\{ \frac{-\log P_{X_1^n | X_2^n} (X_1^n | X_2^n) - na_1}{\sqrt{n}} \geq L_1 \right. \quad \text{or} \quad \frac{-\log P_{X_2^n | X_1^n} (X_2^n | X_1^n) - na_2}{\sqrt{n}} \geq L_2
\]

\[
\left. \quad \text{or} \quad \frac{-\log P_{X_1^n X_2^n} (X_1^n X_2^n) - n (a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 \right\},
\]

and set

\[
\overline{F}_n(L_1, L_2 | a_1, a_2) \equiv 1 - F_n(L_1, L_2 | a_1, a_2)
\]

\[
= \Pr \left\{ \frac{-\log P_{X_1^n | X_2^n} (X_1^n | X_2^n) - na_1}{\sqrt{n}} < L_1, \quad \frac{-\log P_{X_2^n | X_1^n} (X_2^n | X_1^n) - na_2}{\sqrt{n}} < L_2, \quad \frac{-\log P_{X_1^n X_2^n} (X_1^n X_2^n) - n (a_1 + a_2)}{\sqrt{n}} < L_1 + L_2 \right\}.
\]
It should be remarked that the function $F_n(L_1, L_2|a_1, a_2)$ thus defined is a multivariate cumulative distribution function. Now, we have

**Theorem 3.1:**

\[ L(a_1, a_2, \varepsilon|X_1, X_2) = \text{Cl}\left( \{ (L_1, L_2) \mid \limsup_{n \to \infty} F_n(L_1, L_2|a_1, a_2) \leq \varepsilon \} \right). \quad (3.1) \]

**Proof:** See Appendix A.

This theorem provides us with the information spectrum basis for establishing Theorem 4.1 (for i.i.d. sources) and Theorems 6.1, 6.2 (for mixed sources) later.

IV. \((a_1, a_2, \varepsilon)\)-Achievable Rate Region for I.I.D. Correlated Sources

Although Theorem 3.1 is valid for general correlated sources, it is hard to actually compute it. In the subsequent sections, we compute the \((a_1, a_2, \varepsilon)\)-achievable rate region for several typical cases of theoretical/practical importance. To do so, first in this section we focus on i.i.d. correlated sources.

In this case, Theorem 2.1 as is well known, reduces to the first-order Slepian-Wolf theorem for i.i.d. correlated sources, i.e., the Slepian-Wolf region:

\[ R(0|X_1, X_2) = \{(R_1, R_2) \mid R_1 \geq H(X_1|X_2), R_2 \geq H(X_2|X_1), R_1 + R_2 \geq H(X_1X_2)\}, \]

with generic correlated source variable \((X_1, X_2)\), the conditional entropies \(H(X_1|X_2), H(X_2|X_1)\) and the joint entropy \(H(X_1X_2)\), which forms the polygon with the following boundary points (see Fig. 4.1):

Case I (Corner Points):

\[ a_1 = H(X_1|X_2) \quad \text{and} \quad a_2 = H(X_2); \]

\[ a_1 = H(X_1) \quad \text{and} \quad a_2 = H(X_2|X_1), \]

Case II (Non-Corner Points): For correlated \(X_1, X_2\) and \(0 < \forall \lambda < 1\),

\[ a_1 = \lambda H(X_1) + (1 - \lambda)H(X_1|X_2) \quad \text{and} \quad a_2 = (1 - \lambda)H(X_2) + \lambda H(X_2|X_1), \]

Case III (Full Side Points):

\[ a_1 = H(X_1|X_2) \quad \text{and} \quad a_2 > H(X_2); \]
In each of these cases, we compute the \((a_1, a_2, \varepsilon)\)-achievable rate region on the basis of Theorem 3.1.

Remark 4.1: It is not difficult to check that \((L_1, L_2)\) can take arbitrary values in \(\mathbb{R}^2\) (\(\mathbb{R} = \) the set of all real numbers) if \((a_1, a_2)\) is an internal point of the polygon, while the set of all achievable rate pairs \((L_1, L_2)\) reduces to the empty set \(\emptyset\) if \((a_1, a_2)\) is outside the polygon. Thus, we may focus only on the above cases.

Now, we define \(\Phi(T_1, T_2, T_3) \equiv \lim_{n \to \infty} \Pr \left\{ - \frac{\log P_{X_1^n|X_2^n}(X_1^n|X_2^n)}{\sqrt{n}} - nH(X_1|X_2) < T_1, \right. \)

\[ \left. - \frac{\log P_{X_2^n|X_1^n}(X_2^n|X_1^n)}{\sqrt{n}} - nH(X_2|X_1) < T_2, \right. \]

\[ \left. - \frac{\log P_{X_1^nX_2^n}(X_1^nX_2^n)}{\sqrt{n}} - nH(X_1X_2) < T_3 \right\}, \]

then, by means of the multi-dimensional central limit theorem based on the i.i.d. property of sources (cf. Feller [19], Sazonov [20], Bentkus [21]), we see that \(\Phi(T_1, T_2, T_3)\) specifies a three-dimensional normal cumulative distribution function; more specifically,

\[ \Phi(T_1, T_2, T_3) \equiv \int_{-\infty}^{T_1} dy_1 \int_{-\infty}^{T_2} dy_2 \int_{-\infty}^{T_3} dy_3 \frac{1}{(\sqrt{2\pi})^3 \sqrt{\det \Sigma}} \exp \left( - \frac{1}{2} y \Sigma^{-1} y^T \right), \]

where \(y = (y_1, y_2, y_3)\) is a three-dimensional row vector, and \(\Sigma = (\sigma_{ij}^2)\) (\(i, j = 1, 2, 3\)) denotes the dispersion matrix (cf. Tan and Kosut [18]) i.e., the covariance matrix given by

\[ \sigma_{ij}^2 = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} P_{X_1, X_2}(x_1, x_2) z_i(x_1, x_2) z_j(x_1, x_2), \]
where
\[
z_1(x_1, x_2) = \log \frac{1}{P_{X_1|X_2}(x_1|x_2)} - H(X_1|X_2),
\]
\[
z_2(x_1, x_2) = \log \frac{1}{P_{X_2|X_1}(x_2|x_1)} - H(X_2|X_1),
\]
\[
z_3(x_1, x_2) = \log \frac{1}{P_{X_1X_2}(x_1, x_2)} - H(X_1X_2).
\]

In order to avoid subtle irregularities, we assume throughout that Σ is positive-definite (As for a treatment of the case with singular Σ, see [18]).

We then consider the marginal cumulative distributions Φ_{23}(T_2, T_3), Φ_3(T_3) of Φ(T_1, T_2, T_3) such that
\[
Φ_{23}(T_2, T_3) \equiv \lim_{T_1 \to \infty} Φ(T_1, T_2, T_3),
\]
\[
Φ_3(T_3) \equiv \lim_{T_1, T_2 \to \infty} Φ(T_1, T_2, T_3).
\]

Other marginal cumulative distributions can be defined in similar manners. Notice that Φ_3(T_3) and (resp. Φ_{23}(T_2, T_3)) are one-dimensional (resp. two-dimensional) normal cumulative distribution functions.

Now, we have the following theorem:

**Theorem 4.1:** For any i.i.d. correlated sources with countably infinite alphabets, we have for all 0 ≤ ε < 1:

**Case I:** \(a_1 = H(X_1|X_2)\) and \(a_2 = H(X_2)\) (without loss of generality):
\[
L(a_1, a_2, ε|X_1, X_2) = \{(L_1, L_2) | Φ_{13}(L_1, L_1 + L_2) ≥ 1 - ε\}.
\]

**Case II:** for correlated \(X_1\), \(X_2\) and 0 < ∀λ < 1:
\[
L(a_1, a_2, ε|X_1, X_2) = \{(L_1, L_2) | Φ_3(L_1 + L_2) ≥ 1 - ε\}.
\]

**Case III:** \(a_1 = H(X_1|X_2)\) and \(a_2 > H(X_2)\) (without loss of generality):
\[
L(a_1, a_2, ε|X_1, X_2) = \{(L_1, L_2) | Φ_1(L_1) ≥ 1 - ε\}.
\]

**Proof:** See Appendix B.

**Remark 4.2:** Here, in view of the definitions of first-order and second-order achievable rates, any achievable rates can be expressed as in the form of (first-order achievable rate) + \(\frac{1}{\sqrt{n}}\) (second-order
Fig. 4.2. First-Order Achievable Rate Region \((\varepsilon = 0)\)

achievable rate). Thus, the two-dimensional set \(A_n(\varepsilon)\) of all achievable rate pairs \((R_1^{(n)}, R_2^{(n)})^T\) in the finite blocklength regime turns out to be expressed, up to the second-order, as

\[
A_n(\varepsilon) = \bigcup_{(a_1, a_2)^T \in \mathbb{R}(0|X_1, X_2)} \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{1}{\sqrt{n}} L(a_1, a_2, \varepsilon|X_1, X_2) \right\},
\]

(4.1)

where the first-order Slepian-Wolf region with \(\varepsilon = 0\) is depicted in Fig. 4.2. Then, with the aid of Theorem 4.1, the boundary \(\partial A_n(\varepsilon)\) of \(A_n(\varepsilon)\) is also depicted in Fig. 4.3 and Fig. 4.4 with broken lines, i.e., the boundaries \(\partial A_n(\varepsilon)\) for the cases of \(\varepsilon < \frac{1}{2}\) and \(\varepsilon > \frac{1}{2}\), respectively, are depicted there.

It should be noted that if \(\varepsilon > \frac{1}{2}\) then the second-order achievable rates \(L_1, L_2\) can be negative, i.e., in all of Cases I, II and III in Theorem 4.1 with \(\varepsilon > \frac{1}{2}\) the optimal achievable rates \((R_1^{(n)}, R_2^{(n)})^T\) in the finite blocklength regime approach (when \(n\) becomes large) the optimal first-order achievable rate region \(R(0|X_1, X_2)\) from outside.

Remark 4.3: Tan and Kosut [18] defined with finite source alphabets the region \(R_{in}(n, \varepsilon) \subset \mathbb{R}^2\) to be the set of rate pairs \((R_1, R_2)\) such that

\[
R \geq \mathbf{H} + \frac{1}{\sqrt{n}} \mathcal{S}(\Sigma, \varepsilon),
\]

(4.2)

where “\(\geq\)” means componentwise inequality, and

\[
R = (R_1, R_2, R_1 + R_2)^T, \quad \mathbf{H} = (H(X_1|X_2), H(X_2|X_1), H(X_1X_2))^T,
\]

\[
\mathcal{S}(\Sigma, \varepsilon) = \{ z \in \mathbb{R}^3 | \Pr \{ Z \leq z \} \geq 1 - \varepsilon \}
\]

with \(Z \sim \mathcal{N}(0, \Sigma)\); then they proved that \(R_{in}(n, \varepsilon)\) is a set of achievable rate pairs up to the second-order. It is not difficult to check that \(R_{in}(n, \varepsilon) = A_n(\varepsilon)\) for finite source alphabets. It should be
Remark 4.4: Notice that both of $\Phi_3 (L_1 + L_2)$ and $\Phi_1 (L_1)$ are one-dimensional normal cumulative distribution functions. Thus, they can be simply written as

$$\Phi_3 (L_1 + L_2) = \int_{-\infty}^{L_1+L_2} \frac{1}{\sqrt{2\pi}\sigma_{33}} \exp \left( -\frac{y_3^2}{2\sigma_{33}^2} \right) dy_3,$$

$$\Phi_1 (L_1) = \int_{-\infty}^{L_1} \frac{1}{\sqrt{2\pi}\sigma_{11}} \exp \left( -\frac{y_1^2}{2\sigma_{11}^2} \right) dy_1,$$

and, similarly

$$\Phi_{13} (L_1, L_1 + L_2) = \int_{-\infty}^{L_1} dy_1 \int_{-\infty}^{L_1+L_2} dy_3 \frac{1}{2\pi \sqrt{\det \Sigma_{13}}} \exp \left( -\frac{1}{2} y_{13} \Sigma_{13}^{-1} y_{13}^T \right),$$

remarked here that inequality “$\geq$” in (4.2) was “$\in$” in the original formula of [18]. However, the use of “$\in$” is misleading, whereas “$\geq$” gives the right notation.
where
\[ \Sigma_{13} = \begin{pmatrix} \sigma_{11}^2 & \sigma_{13}^2 \\ \sigma_{31}^2 & \sigma_{33}^2 \end{pmatrix}, \quad y_{13} = (y_1, y_3). \]

Thus, the second-order achievable rate region in Case II reduces simply to \( L_1 + L_2 \geq T_{II} \) where \( T_{II} \) is specified by
\[
\int_{-\infty}^{T_{II}} \frac{1}{\sqrt{2\pi} \sigma_{33}} \exp \left( -\frac{y_3^2}{2\sigma_{33}^2} \right) dy_3 = 1 - \varepsilon \quad (T_{II} = \Phi_3^{-1}(1 - \varepsilon));
\]
and the second-order achievable rate region in Case III reduces simply to \( L_1 \geq T_{III} \) (\( L_2 \) is arbitrary) where \( T_{III} \) is specified by
\[
\int_{-\infty}^{T_{III}} \frac{1}{\sqrt{2\pi} \sigma_{11}} \exp \left( -\frac{y_1^2}{2\sigma_{11}^2} \right) dy_1 = 1 - \varepsilon \quad (T_{III} = \Phi_1^{-1}(1 - \varepsilon)).
\]
On the other hand, the second-order achievable rate region in Case I is explicitly written as the set of all \((L_1, L_2)\) such that
\[
\Phi_{13}(L_1, L_1 + L_2) = \int_{-\infty}^{L_1} dy_1 \int_{-\infty}^{L_1+L_2} dy_3 \frac{1}{2\sqrt{\det \Sigma_{13}}} \exp \left( -\frac{1}{2} y_{13} \Sigma_{13}^{-1} y_{13}^T \right) \geq 1 - \varepsilon.
\]

Remark 4.5: Notice that Case III is substantially the same as the full side-information problem. Actually, in Case III, there is no condition on \( L_2 \) for the achievable rate region. This means that the error probability is independent of the second-order rate \( L_2 \). This is because, for the decoder, the first-order rate \( a_2 > H(X_2) \) is sufficient to reconstruct \( X_2^n = x_2 \). Therefore, the decoder is able to decode \( X_2^n = x_2 \) perfectly without the knowledge of \( X_1^n = x_1 \). Then, using the \( X_2^n = x_2 \) the decoder decodes \( X_1^n = x_1 \). That is, \( X_2^n = x_2 \) can be regarded as providing full side-information. Consequently, the result of Case III coincides with the results in [14], [15], in which full side-information problems are treated. ■

Theorem 4.1 can be equivalently restated as the following corollary: let
\[
K_n(a_1, a_2, \varepsilon|X_1, X_2) \equiv \left\{(L_1, L_2) \mid \Phi \left( \sqrt{n} (a_1 - H(X_1|X_2)) + L_1, \ \sqrt{n} (a_2 - H(X_2|X_1)) + L_2, \ \sqrt{n} (a_1 + a_2 - H(X_1X_2)) + L_1 + L_2 \right) \geq 1 - \varepsilon \right\},
\]
then, we have:
Corollary 4.1: For an i.i.d. correlated sources with countably infinite alphabets, the second-order $(a_1, a_2, \varepsilon)$-achievable rate region is given as the two-dimensional set\(^2\):

\[
L(a_1, a_2, \varepsilon|X_1, X_2) = \lim_{n \to \infty} K_n(a_1, a_2, \varepsilon|X_1, X_2).
\] (4.4)

Remark 4.6: It is easy to see that the right-hand side of (4.4) is rewritten as

\[
\lim_{n \to \infty} K_n(a_1, a_2, \varepsilon|X_1, X_2) = \{(L_1, L_2) \mid \lim_{n \to \infty} \Phi(\sqrt{n}(a_1 - H(X_1|X_2)) + L_1, \\
\sqrt{n}(a_2 - H(X_2|X_1)) + L_2, \\
\sqrt{n}(a_1 + a_2 - H(X_1X_2)) + L_1 + L_2) \geq 1 - \varepsilon\}.
\] (4.5)

Proof of Corollary 4.1. It is not difficult to verify by letting \(n \to \infty\) that \(K_n(a_1, a_2, \varepsilon|X_1, X_2)\) asymptotically coincides with each case in Theorem 4.1 with \(a_1\) and \(a_2\) as in Cases I, II and III, respectively. Moreover, Corollary 4.1 also includes trivial cases such that \((a_1, a_2)\) is inside or outside the polygon (cf. Remark 4.1). Thus, it is concluded that Theorem 4.1 along with Remark 4.1 is equivalent to Corollary 4.1.

Remark 4.7: Compared with Theorem 4.1, it is observed that the description of Corollary 4.1 needs only a single equation (4.4) but no longer any classifications of pairs \((a_1, a_2)\). This is a great advantage of the information spectrum methods. Thus, we call (4.4) the canonical representation for \(L(a_1, a_2, \varepsilon|X_1, X_2)\). This point of view is inherited also to Section VI which enables us to successfully establish the second-order achievable rate region for mixtures of correlated i.i.d. sources. This approach is completely different from that of Tan and Kosut [18].

Remark 4.8: Let us consider with (4.3) the following equation for \(L_1, L_2\), given \(a_1, a_2, \varepsilon, n\):

\[
\Phi(\sqrt{n}(a_1 - H(X_1|X_2)) + L_1, \\
\sqrt{n}(a_2 - H(X_2|X_1)) + L_2, \\
\sqrt{n}(a_1 + a_2 - H(X_1X_2)) + L_1 + L_2) = 1 - \varepsilon,
\] (4.6)

\(^2\)Generally speaking, \(\lim_{n \to \infty} A_n\) denotes the limit of a sequence \(\{A_n\}_{n=1}^\infty\) of sets \(A_n\)’s in the sense that \(\lim_{n \to \infty} A_n := \lim \sup_{n \to \infty} A_n = \lim \inf_{n \to \infty} A_n\) (cf. Billingsley [22]).
and denote the solution (as \( n \) tends to \( \infty \)) by \( L_1^*(a_1,a_2,\varepsilon) \), \( L_2^*(a_1,a_2,\varepsilon) \), respectively. Then, it is not difficult to verify that

\[
\partial A_n(\varepsilon) = \bigcup_{(a_1,a_2)^T \in \partial_{\text{SW}}} \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{1}{\sqrt{n}} \begin{pmatrix} L_1^*(a_1,a_2,\varepsilon) \\ L_2^*(a_1,a_2,\varepsilon) \end{pmatrix} \right\},
\]

where \( \partial_{\text{SW}} \) is the boundary of the Slepian-Wolf region \( R(0|X_1,X_2) \), and \( \partial A_n(\varepsilon) \) is the boundary of \( A_n(\varepsilon) \) defined as in Remark 4.2.

**Remark 4.9:** It is interesting to note that in Cases II and III the condition on \((a_1,a_2,\varepsilon)\)-achievable rate region is described by one-dimensional normal distribution function, while in Case I that is described by two-dimensional normal distribution (cf. Fig. 4.3 and Fig. 4.4).

V. COMPARISON WITH THE KOSHELEV BOUND AND NUMERICAL EXAMPLES

In this section, we give some examples of the second-order achievable rate region for i.i.d. binary correlated sources to elucidate the effectiveness of Theorem 4.1. We compare the region so far derived with the *modified* Koshelev bound ([23]), i.e., the Gallager type bound for the Slepian-Wolf source coding system. In Hayashi [6, Sect. V], the optimal second-order capacity rate in channel coding has been compared with the Gallager bound [24] from the viewpoint of error probabilities vs. achievable rates. Analogously, in this section we compare the error probability guaranteed by Theorem 4.1 with the Koshelev type of error exponent that is derived on the basis of the maximum likelihood rule.

Let

\[
\begin{align*}
E_1(s_1) & \equiv - \log \sum_{x_2 \in X_2} \left( \sum_{x_1 \in X_1} P_{X_1X_2}(x_1,x_2)^{-s_1} \right)^{1+s_1} \\
E_2(s_2) & \equiv - \log \sum_{x_1 \in X_1} \left( \sum_{x_2 \in X_2} P_{X_1X_2}(x_1,x_2)^{-s_2} \right)^{1+s_2} \\
E_3(s_3) & \equiv - \log \left( \sum_{(x_1,x_2) \in X_1 \times X_2} P_{X_1X_2}(x_1,x_2)^{-s_3} \right)^{1+s_3},
\end{align*}
\]

where it is evident that \( E_1(0) = E_2(0) = E_3(0) = 0 \). Then, we have the following lemma, which is a stronger version of the original Koshelev bound [23] (Notice that, on the contrary to here, \( s_1, s_2, s_3 \) are constrained so as to be \( 0 \leq s_1 = s_2 = s_3 \leq 1 \) in [23]):
Lemma 5.1: Let \( R_1 = \frac{1}{n} \log M_n^{(1)} \) and \( R_2 = \frac{1}{n} \log M_n^{(2)} \), then there exists an \((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) code satisfying

\[
\varepsilon_n \leq \min_{0 \leq s_1 \leq 1} \exp \left[ -n \left( R_1 s_1 - E_1(s_1) \right) \right] + \min_{0 \leq s_2 \leq 1} \exp \left[ -n \left( R_2 s_2 - E_2(s_2) \right) \right] + \min_{0 \leq s_3 \leq 1} \exp \left[ -n \left( (R_1 + R_2) s_3 - E_3(s_3) \right) \right].
\]

Proof: See Appendix C.

We call this bound merely the Koshelev bound for simplicity. Achievable rate pairs \((R_1, R_2)\) close to the boundary \(\partial_{SW}\) of the Slepian-Wolf region are classified into the following three cases:

Case I (Corner Points):

\[
R_1 = H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \quad \text{and} \quad R_2 = H(X_2) + \frac{L_2}{\sqrt{n}},
\]

Case II (Non-Corner Points): For correlated \(X_1, X_2\) and \(0 < \forall \lambda < 1\),

\[
R_1 = \lambda H(X_1) + (1 - \lambda) H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \quad \text{and} \quad R_2 = (1 - \lambda) H(X_2) + \lambda H(X_2|X_1) + \frac{L_2}{\sqrt{n}}.
\]

Case III (Full Side Points):

\[
R_1 = H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \quad \text{and} \quad R_2 > H(X_2) + \frac{L_2}{\sqrt{n}}.
\]

In each of these cases, we calculate the Koshelev bound for comparison. These calculations are the Slepian-Wolf source coding counterpart of that of Hayashi [8 Sect. V] for channel coding.

In Case I: the Koshelev bound is given by

\[
\varepsilon_n \leq \min_{0 \leq s_1 \leq 1} \exp \left[ -n \left( H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \right) s_1 - E_1(s_1) \right]
\]

\[
+ \min_{0 \leq s_2 \leq 1} \exp \left[ -n \left( H(X_2) + \frac{L_2}{\sqrt{n}} \right) s_2 - E_2(s_2) \right]
\]

\[
+ \min_{0 \leq s_3 \leq 1} \exp \left[ -n \left( H(X_1X_2) + \frac{L_1 + L_2}{\sqrt{n}} \right) s_3 - E_3(s_3) \right].
\]

(5.1)

In view of

\[
\frac{dE_1(s_1)}{ds_1} \bigg|_{s_1=0} = H(X_1|X_2), \quad \frac{d^2E_1(s_1)}{ds_1^2} \bigg|_{s_1=0} = \sigma_{11}^2
\]

(5.2)

\[
\frac{dE_2(s_2)}{ds_2} \bigg|_{s_2=0} = H(X_2|X_1), \quad \frac{d^2E_2(s_2)}{ds_2^2} \bigg|_{s_2=0} = \sigma_{22}^2
\]

(5.3)

\[
\frac{dE_3(s_3)}{ds_3} \bigg|_{s_3=0} = H(X_1X_2), \quad \frac{d^2E_3(s_3)}{ds_3^2} \bigg|_{s_3=0} = \sigma_{33}^2.
\]

(5.4)
the first and the third term on the right-hand side of (5.1) are given by
\[ \min_{0 \leq s_1 \leq 1} \exp \left[ -n \left( H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \right) s_1 - E_1(s_1) \right] \approx \exp \left[ -\frac{L_1^2}{2\sigma^2_{11}} \right], \] (5.5)
\[ \min_{0 \leq s_2 \leq 1} \exp \left[ -n \left( H(X_1X_2) + \frac{L_1 + L_2}{\sqrt{n}} \right) s_2 - E_3(s_3) \right] \approx \exp \left[ -\frac{(L_1 + L_2)^2}{2\sigma^2_{33}} \right], \] (5.6)
for sufficiently large \( n \), respectively. The derivation of (5.5) and (5.6) is as follows:

In view of (5.2) we have
\[ \left( H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \right) s_1 - E_1(s_1) \]
\[ \approx \left( H(X_1|X_2) + \frac{L_1}{\sqrt{n}} \right) s_1 - H(X_1|X_2)s_1 - \frac{\sigma^2_{11}s_1^2}{2} \]
\[ = \frac{L_1}{\sqrt{n}} s_1 - \frac{\sigma^2_{11}s_1^2}{2} \]
\[ = -\frac{\sigma^2_{11}}{2} \left( s_1^2 - \frac{2L_1s_1}{\sqrt{n}\sigma^2_{11}} \right) \]
\[ = -\frac{\sigma^2_{11}}{2} \left( s_1 - \frac{L_1}{\sqrt{n}\sigma^2_{11}} \right)^2 + \frac{L_1^2}{2n\sigma^2_{11}}, \]
which is maximized at \( s_1 = \frac{L_1}{\sqrt{n}\sigma^2_{11}} \) if \( L_1 \geq 0 \), so that (5.5) follows. Similarly, in view of (5.4) it follows that
\[ \left( H(X_1X_2) + \frac{L_1 + L_2}{\sqrt{n}} \right) s_3 - E_3(s_3) \]
\[ \approx \left( H(X_1X_2) + \frac{L_1 + L_2}{\sqrt{n}} \right) s_3 - H(X_1X_2)s_3 - \frac{\sigma^2_{33}s_3^2}{2} \]
\[ = \frac{L_1 + L_2}{\sqrt{n}} s_3 - \frac{\sigma^2_{33}s_3^2}{2} \]
\[ = -\frac{\sigma^2_{33}}{2} \left( s_3 - \frac{L_1 + L_2}{\sqrt{n}\sigma^2_{33}} \right)^2 + \frac{(L_1 + L_2)^2}{2n\sigma^2_{33}}, \]
which is maximized at \( s_3 = \frac{L_1 + L_2}{\sqrt{n}\sigma^2_{33}} \) if \( L_1 + L_2 \geq 0 \), so that (5.6) holds. It should be noted that two maximum values \( \frac{L_1^2}{2n\sigma^2_{11}} \) and \( \frac{(L_1 + L_2)^2}{2n\sigma^2_{33}} \) here could not simultaneously be attained by \( s_1 \) and \( s_3 \) if they were constrained so as to be \( s_1 = s_3 \), as was in the original Koshelev bound [23] that was used by Tan and Kosut [18]. On the other hand, in view of (5.3) and \( H(X_2) - H(X_2|X_1) > 0 \), we see that the second term on the right-hand side of (5.1) is negligible as \( n \) is large enough with \( s_2 = \frac{L_2}{\sqrt{n}\sigma^2_{22}} \). Thus, in Case I, we have
\[ \varepsilon_n \leq \exp \left[ -\frac{L_1^2}{2\sigma^2_{11}} \right] + \exp \left[ -\frac{(L_1 + L_2)^2}{2\sigma^2_{33}} \right] \quad (L_1 \geq 0, L_1 + L_2 \geq 0). \] (5.7)
In Case II: Similarly to Case I, we have
\[ \varepsilon_n \leq \exp \left[ \frac{-(L_1 + L_2)^2}{2\sigma_{33}^2} \right] (L_1 + L_2 \geq 0). \] (5.8)

In Case III: Similarly to Case I, we have
\[ \varepsilon_n \leq \exp \left[ \frac{-L_1^2}{2\sigma_{11}^2} \right] (L_1 \geq 0). \] (5.9)

**Example 5.1:** We are now ready to give numerical examples. Let us consider i.i.d. correlated sources with binary alphabets \( X_1 = X_2 = \{0,1\} \), whose joint probabilities are given by Table I.

| TABLE I | Joint Probability \( P_{X_1,X_2}(x_1,x_2) \) |
|----------|---------------------------------------------|
| \( P_{X_1,X_2}(x_1,x_2) \) | \( x_1 \) | 0 | 1 |
| \( x_2 \) | 0 | 0.5 | 0.15 |
| | 1 | 0.25 | 0.1 |

The entropies in bits are \( H(X_1|X_2) = 0.809 \), \( H(X_2) = 0.934 \) and \( H(X_1X_2) = 1.743 \) and the covariance matrix is as follows:
\[ \Sigma_{13} = \begin{pmatrix} \sigma_{11}^2 & \sigma_{13}^2 \\ \sigma_{31}^2 & \sigma_{33}^2 \end{pmatrix} = \begin{pmatrix} 0.475 & 0.492 \\ 0.492 & 0.690 \end{pmatrix}. \]

In Case I, we compute the error probabilities:
\[ 1 - \Phi_{13}(L_1, L_1 + L_2) = 1 - \int_{-\infty}^{L_1} dy_1 \int_{-\infty}^{L_1+L_2} dy_2 \frac{1}{2\sqrt{\det \Sigma_{13}}} \exp \left( -\frac{1}{2} \mathbf{y}_{13} \Sigma_{13}^{-1} \mathbf{y}_{13} \right) \] (5.10)
and
\[ \exp \left[ \frac{-L_2^2}{2\sigma_{11}^2} \right] + \exp \left[ \frac{-(L_1 + L_2)^2}{2\sigma_{33}^2} \right] (L_1 \geq 0, \ L_1 + L_2 \geq 0). \] (5.11)

Fig. 5.1 illustrates contour lines of these functions of two variables \( L_1 \) and \( L_2 \). From this result we find that there is a difference between the second-order evaluation and the Koshelev bound, which shows that the former approach outperforms the latter approach. On the other hand, Fig. 5.2 gives an enlargement of the broken contours in Fig. 5.1, which is to see the detailed behavior of the curvature for the second-order evaluation. We observe that in this area the slopes of tangential lines on the broken contours are between \( \frac{3\pi}{2} \) and \( \frac{7\pi}{4} \). This is an important observation to reveal that
the values of $L_1$ are more influential on the error probability than those of $L_2$, but the influence of $L_2$ is not negligible.

In particular, the graphs of functions (5.10) and (5.11) with $L_1 = L_2$ are depicted in Fig. 5.3.

Fig. 5.1. Comparison between the Koshelev bound (solid line) and the second-order evaluation (broken line) (Case I): contours of functions (5.10) and (5.11) are drawn. The values of error probabilities are marked.
Fig. 5.2. Contours of the second-order evaluation. The values $1 - \Phi_{13}(L_1, L_1 + L_2)$ of error probabilities are marked.

Fig. 5.3. The graphs of functions (5.10) and (5.11) with $L_1 = L_2$ (Case I).
In Case II we compute

\[ 1 - \Phi_3 (L_1 + L_2) = 1 - \int_{-\infty}^{L_1+L_2} \frac{1}{\sqrt{2\pi\sigma_{33}}} \exp \left( -\frac{y_3^2}{2\sigma_{33}^2} \right) dy_3, \]

and

\[ \exp \left[ -\frac{(L_1 + L_2)^2}{2\sigma_{33}^2} \right] (L_1 + L_2 \geq 0). \]

Similarly, in Case III we compute

\[ 1 - \Phi_1 (L_1) = 1 - \int_{-\infty}^{L_1} \frac{1}{\sqrt{2\pi\sigma_{11}}} \exp \left( -\frac{y_1^2}{2\sigma_{11}^2} \right) dy_1, \]

and

\[ \exp \left[ -\frac{L_1^2}{2\sigma_{11}^2} \right] (L_1 \geq 0). \]

Fig. 5.4 and 5.5 illustrate behaviors of these functions in Case II and Case III, respectively. In each of these figures, the solid line denotes the Koshelev bound and the broken line denotes the second-order evaluation. We again find that the second-order evaluation gives better performance than the Koshelev bound.

![Fig. 5.4](image-url)
Fig. 5.5. Comparison between the Koshelev bound (solid line) and the second-order evaluation (broken line) (Case III)
VI. \((a_1,a_2,\varepsilon)\)-achievable rate region for Mixed correlated sources

In Section \[IV\] we have seen that the second-order achievable rate region of any i.i.d. correlated sources (or more generally, correlated sources whose self-information vector has the multi-dimensional asymptotic normality) is relevantly described on the basis of the information spectrum methods. In this section, we establish the second-order achievable rate region of the correlated sources in which the asymptotic normality of self-information vector does not hold. One of such source classes is the mixed correlated source. Recall that the mixed sources are typical cases of nonergodic sources. We consider two typical cases of mixed correlated sources and determine the second-order achievable rate region in each case explicitly by virtue of “mixed” multi-dimensional asymptotic normality.

A. Mixture of Countably Infinite i.i.d. Correlated Sources

In this subsection, we assume that the correlated sources is a mixture of countably infinite i.i.d. correlated sources. Let \(X_1^{(k)}, X_2^{(k)} (k = 1, 2, \cdots)\) be arbitrary pairs of i.i.d. correlated sources indexed by \(k\). The mixed correlated source that we consider in this subsection is defined by

\[
P_{X_1^n X_2^n}(x_1, x_2) = \sum_{k=1}^{\infty} w(k) P_{X_1^{(k)}_n X_2^{(k)}_n}(x_1, x_2),
\]

(6.1)

where \((X_1, X_2) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}\) and \(w(k) \geq 0 (k = 1, 2, \cdots)\) are constants such that \(\sum_{k=1}^{\infty} w(k) = 1\).

The following lemma, which is implicitly contained in Han [9], plays the key role. We give in this paper the formal proof for the sake of reader’s convenience (see Appendix D).

**Lemma 6.1:** Let \(\left\{z_n^{(1)}\right\}_{n=1}^{\infty}, \left\{z_n^{(2)}\right\}_{n=1}^{\infty}, \left\{z_n^{(3)}\right\}_{n=1}^{\infty}\) be any real-valued sequences. Then, for the mixed correlated sources defined by (6.1), it holds that, for \(k = 1, 2, \cdots\) with \(w(k) > 0\);

\[
Pr\left\{ -\frac{\log P_{X_1^n|X_2^n}(X_1^{(k)n}|X_2^{(k)n})}{\sqrt{n}} < z_n^{(1)}; \quad \frac{\log P_{X_2^n|X_1^n}(X_2^{(k)n}|X_1^{(k)n})}{\sqrt{n}} < z_n^{(2)}; \quad \frac{\log P_{X_1^n X_2^n}(X_1^{(k)n}X_2^{(k)n})}{\sqrt{n}} < z_n^{(3)} \right\}
\]
\[
\leq \Pr \left\{ \frac{-\log P_{X_1^{(k)n} | X_2^{(k)n}} (X_1^{(k)n} | X_2^{(k)n})}{\sqrt{n}} < z_n^{(1)} + \gamma_n, \right. \\
- \frac{-\log P_{X_2^{(k)n} | X_1^{(k)n}} (X_2^{(k)n} | X_1^{(k)n})}{\sqrt{n}} < z_n^{(2)} + \gamma_n, \\
\left. - \frac{-\log P_{X_1^{(k)n} | X_2^{(k)n}} (X_1^{(k)n} | X_2^{(k)n})}{\sqrt{n}} < z_n^{(3)} + \gamma_n \right\} + 3e^{-\sqrt{n} \gamma_n},
\]

2)

\[
\Pr \left\{ \frac{-\log P_{X_1^n | X_2^n} (X_1^n | X_2^n)}{\sqrt{n}} < z_n^{(1)}, \\
- \frac{-\log P_{X_2^n | X_1^n} (X_2^n | X_1^n)}{\sqrt{n}} < z_n^{(2)}, \\
\left. - \log P_{X_1^n | X_2^n} (X_1^n | X_2^n) \right\} \]

\[
\geq \Pr \left\{ \frac{-\log P_{X_1^{(k)n} | X_2^{(k)n}} (X_1^{(k)n} | X_2^{(k)n})}{\sqrt{n}} < z_n^{(1)} - \gamma_n, \\
- \frac{-\log P_{X_2^{(k)n} | X_1^{(k)n}} (X_2^{(k)n} | X_1^{(k)n})}{\sqrt{n}} < z_n^{(2)} - \gamma_n, \\
\left. - \log P_{X_1^{(k)n} | X_2^{(k)n}} (X_1^{(k)n} | X_2^{(k)n}) < z_n^{(3)} - \gamma_n \right\} - 2e^{-\sqrt{n} \gamma_n / 2}
\]

for sufficiently large \( n \), where \( \gamma_n > 0 \) satisfies \( \gamma_1 > \gamma_2 > \cdots > 0, \gamma_n \to 0 \) and \( \sqrt{n} \gamma_n \to \infty \).

Now, we define the normal cumulative distribution functions for \( k = 1, 2, \ldots \), as follows.

\[
\Phi^{(k)}(T_1, T_2, T_3) \equiv \int_{-\infty}^{T_1} dy_1 \int_{-\infty}^{T_2} dy_2 \int_{-\infty}^{T_3} dy_3 \frac{1}{(\sqrt{2\pi})^3 \sqrt{\det \Sigma_k}} \exp \left( -\frac{1}{2} y \Sigma_k^{-1} y^T \right),
\]

where \( y = (y_1, y_2, y_3) \) is a three-dimensional row vector, and \( \Sigma_k = (\sigma_{ij}^{2}(k)) \) (\( i, j = 1, 2, 3; k = 1, 2, \ldots \)) denotes the covariance matrix, which is given by

\[
\sigma_{ij}^{2}(k) = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} P_{X_1^{(k)n} | X_2^{(k)n}} (x_1, x_2) z_{ij}^{(k)}(x_1, x_2) z_{ij}^{(k)}(x_1, x_2),
\]

where

\[
z_{1}^{(k)}(x_1, x_2) = \log \frac{1}{P_{X_1^{(k)n} | X_2^{(k)n}} (x_1 | x_2)} - H \left( X_1^{(k)} | X_2^{(k)} \right),
\]

\[
z_{2}^{(k)}(x_1, x_2) = \log \frac{1}{P_{X_2^{(k)n} | X_1^{(k)n}} (x_2 | x_1)} - H \left( X_2^{(k)} | X_1^{(k)} \right),
\]
second-order achievable rate region is trivial, that is, \( L(m) \) using a hand, it will turn out that in the nontrivial cases the second-order rate region is determined by

\[
\lim_{n \to \infty} \text{achievable rate region is given as the set:}
\]

Then, the following theorem holds.

**Theorem 6.1:** For the mixed correlated sources defined by (6.1), the second-order \((a_1, a_2, \varepsilon)\)-achievable rate region is given as the set defined by using the

\[
K_n^{\text{mix}}(a_1, a_2, \varepsilon | \mathbf{X}_1, \mathbf{X}_2) = \left\{ (L_1, L_2) \left| \sum_{k=1}^{\infty} w(k) \Phi(k) \left( \sqrt{n} \left( a_1 - H \left( X_1^{(k)} \mid X_2^{(k)} \right) \right) + L_1, \right. \\
\sqrt{n} \left( a_2 - H \left( X_2^{(k)} \mid X_1^{(k)} \right) \right) + L_2, \right. \\
\sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(k)} X_2^{(k)} \right) \right) + L_1 + L_2 \right. \geq 1 - \varepsilon \right\}.
\]

Then, the following theorem holds.

**Proof of Theorem 6.1:**

This theorem claims that the second-order achievable rate region is given as the set defined by using the **mixed** three-dimensional normal distribution, while in the previous section the set is defined by using the **single** three-dimensional normal distribution. Although the quantity in (6.2) contains the operation \( \lim_{n \to \infty} \), it is possible also to provide an alternative form not including \( \lim_{n \to \infty} \), which is given as a special case of Theorem 6.3 later.

**Remark 6.1:** It should be noted that, if \( a_1 \neq H \left( X_1^{(k)} \mid X_2^{(k)} \right) \) \((k = 1, 2, \cdots)\), \( a_2 \neq H \left( X_2^{(k)} \mid X_1^{(k)} \right) \) \((k = 1, 2, \cdots)\) and \( a_1 + a_2 \neq H \left( X_1^{(k)} X_2^{(k)} \right) \) \((k = 1, 2, \cdots)\), then it necessarily implies that the second-order achievable rate region is trivial, that is, \( L(a_1, a_2, \varepsilon | \mathbf{X}_1, \mathbf{X}_2) = \mathbb{R}^2 \) or \( \emptyset \). On the other hand, it will turn out that in the nontrivial cases the second-order rate region is determined by using a mixture of two- or one-dimensional normal distributions (cf. Remark 4.4 and Example 6.1).

**Proof of Theorem 6.1:**
In view of Theorem 6.1, it suffices to show that

\[
\lim_{n \to \infty} F_n(L_1, L_2|a_1, a_2) = \sum_{k=1} w(k) \lim_{n \to \infty} \Phi^{(k)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(k)}|X_2^{(k)} \right) \right) + L_1, \right.
\]

\[
\sqrt{n} \left( a_2 - H \left( X_2^{(k)}|X_1^{(k)} \right) \right) + L_2,
\]

\[
\sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(k)}X_2^{(k)} \right) \right) + L_1 + L_2. \quad (6.3)
\]

By the definition of mixed correlated sources and the first inequality of Lemma 6.1 with \( n_1 \to \infty \), \( n_2 \to \infty \), we have

\[
\limsup_{n \to \infty} F_n(L_1, L_2|a_1, a_2) = \limsup_{n \to \infty} \sum_{k=1} w(k) \Pr \left\{ \frac{-\log P_{X_1^n|X_2^n} \left( X_1^{(k)n} \right) X_2^{(k)n}}{\sqrt{n}} - na_1 < L_1, \right. \]

\[
-\log P_{X_2^n|X_1^n} \left( X_2^{(k)n} \right) X_1^{(k)n} - na_2 < L_2, \right.
\]

\[
- \log P_{X_1^{(k)n}X_2^{(k)n}} \left( X_1^{(k)n}X_2^{(k)n} \right) - n(a_1 + a_2) < L_1 + L_2 \left\} \right. \]

\[
\leq \sum_{k=1} \cdot \sum_{k=1} \limsup_{n \to \infty} \left( \Pr \left\{ \frac{-\log P_{X_1^{(k)n}|X_2^{(k)n}} \left( X_1^{(k)n} \right) X_2^{(k)n}}{\sqrt{n}} - na_1 < L_1 + \gamma_n, \right. \right.
\]

\[
-\log P_{X_2^{(k)n}|X_1^{(k)n}} \left( X_2^{(k)n} \right) X_1^{(k)n} - na_2 < L_2 + \gamma_n, \right.
\]

\[
- \log P_{X_1^{(k)n}X_2^{(k)n}} \left( X_1^{(k)n}X_2^{(k)n} \right) - n(a_1 + a_2) < L_1 + L_2 + \gamma_n \} \right. \left. + 3e^{-\sqrt{n} \gamma_n} \right) \]

\[
= \sum_{k=1} w(k) \limsup_{n \to \infty} \Pr \left\{ \frac{-\log P_{X_1^{(k)n}|X_2^{(k)n}} \left( X_1^{(k)n} \right) X_2^{(k)n}}{\sqrt{n}} - nH \left( X_1^{(k)n}|X_2^{(k)n} \right) \right.
\]

\[
< \sqrt{n} \left( a_1 - H \left( X_1^{(k)n}|X_2^{(k)n} \right) \right) + L_1 + \gamma_n, \right.
\]

\[
- \log P_{X_2^{(k)n}|X_1^{(k)n}} \left( X_2^{(k)n} \right) X_1^{(k)n} - nH \left( X_2^{(k)n}|X_1^{(k)n} \right) \right.
\]

\[
< \sqrt{n} \left( a_2 - H \left( X_2^{(k)n}|X_1^{(k)n} \right) \right) + L_2 + \gamma_n, \right. \]
\[
- \log P_{X_1^{(k)n}, X_2^{(k)n}}(X_1^{(k)n}, X_2^{(k)n}) + nH(X_1^{(k)n}, X_2^{(k)n}) \\
\sqrt{n} < \sqrt{n} \left( a_1 + a_2 - H(X_1^{(k)n}, X_2^{(k)n}) \right) + L_1 + L_2 + \gamma_n \\
\}
\]

Then, by virtue of the central limit theorem due to the i.i.d. correlated sources \(X_1^{(k)n}, X_2^{(k)n}\) \((k = 1, 2, \cdots)\) as well as the argument similar to the proof of Theorem 6.1 we have

\[
\limsup_{n \to \infty} F_n(L_1, L_2 | a_1, a_2) \\
\leq \sum_{k=1}^{\infty} w(k) \limsup_{n \to \infty} \Phi^{(k)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(k)} \mid X_2^{(k)} \right) \right) + L_1, \right. \\
\sqrt{n} \left( a_2 - H \left( X_2^{(k)} \mid X_1^{(k)} \right) \right) + L_2, \\
\sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(k)} X_2^{(k)} \right) \right) + L_1 + L_2 \left. \right) \\
= \sum_{k=1}^{\infty} w(k) \lim_{n \to \infty} \Phi^{(k)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(k)} \mid X_2^{(k)} \right) \right) + L_1, \right. \\
\sqrt{n} \left( a_2 - H \left( X_2^{(k)} \mid X_1^{(k)} \right) \right) + L_2, \\
\sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(k)} X_2^{(k)} \right) \right) + L_1 + L_2 \right). \tag{6.4}
\]

On the other hand, it can also be verified that, in a manner similar to the above, \(\liminf_{n \to \infty} F_n(L_1, L_2 | a_1, a_2)\) is lower bounded by the right-hand side of (6.3), that is,

\[
\liminf_{n \to \infty} F_n(L_1, L_2 | a_1, a_2) \\
\geq \sum_{k=1}^{\infty} w(k) \lim_{n \to \infty} \Phi^{(k)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(k)} \mid X_2^{(k)} \right) \right) + L_1, \right. \\
\sqrt{n} \left( a_2 - H \left( X_2^{(k)} \mid X_1^{(k)} \right) \right) + L_2, \\
\sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(k)} X_2^{(k)} \right) \right) + L_1 + L_2 \right), \tag{6.5}
\]

where the second inequality of Lemma 6.1 is used instead of the first inequality of Lemma 6.1. Thus, combining (6.4) and (6.5) yields (6.3).

\textit{Remark 6.2}: As shown in the above, the analysis here for mixed correlated sources of i.i.d. sources is based on the asymptotic normality of self-information vector and Lemma 6.1. This means that the similar argument is valid for any mixture of countably infinite sources in which the asymptotic normality of self-information vector holds for each of the component correlated sources.
Example 6.1: Let us consider the mixed correlated sources, for which it holds that \(w(1)+w(2) = 1\) \((w(1) > 0, w(2) > 0)\), and

\[
H\left( X_1^{(1)} \mid X_2^{(1)} \right) > H\left( X_1^{(2)} \mid X_2^{(2)} \right),
\]

\[
H\left( X_2^{(1)} \right) > H\left( X_2^{(2)} \right).
\]

We then can compute the second-order achievable rate region:

\[
L\left( H\left( X_1^{(1)} \mid X_2^{(1)} \right), H\left( X_2^{(1)} \right), \epsilon \mid X_1, X_2 \right)
\]

and

\[
L\left( H\left( X_1^{(2)} \mid X_2^{(2)} \right), H\left( X_2^{(2)} \right), \epsilon \mid X_1, X_2 \right).
\]

Notice that the second-order achievable rate region depends on \(w(1)\) and \(\epsilon\). From Theorem 6.1 and Remark it is easy to verify that if \(w(1) > \epsilon\) then

\[
L\left( H\left( X_1^{(1)} \mid X_2^{(1)} \right), H\left( X_2^{(1)} \right), \epsilon \mid X_1, X_2 \right)
= \left\{ (L_1, L_2) \mid w(1)\Phi_1^{(1)}(L_1, L_1 + L_2) + w(2) \geq 1 - \epsilon \right\},
\]

whereas if \(w(1) < \epsilon\) then

\[
L\left( H\left( X_1^{(1)} \mid X_2^{(1)} \right), H\left( X_2^{(1)} \right), \epsilon \mid X_1, X_2 \right) = \mathbb{R}^2,
\]

\[
L\left( H\left( X_1^{(2)} \mid X_2^{(2)} \right), H\left( X_2^{(2)} \right), \epsilon \mid X_1, X_2 \right) = \left\{ (L_1, L_2) \mid w(2)\Phi_2^{(2)}(L_1, L_1 + L_2) \geq 1 - \epsilon \right\},
\]

and otherwise (i.e., \(w(1) = \epsilon\)) we have

\[
L\left( H\left( X_1^{(1)} \mid X_2^{(1)} \right), H\left( X_2^{(1)} \right), \epsilon \mid X_1, X_2 \right) = \mathbb{R}^2,
\]

\[
L\left( H\left( X_1^{(2)} \mid X_2^{(2)} \right), H\left( X_2^{(2)} \right), \epsilon \mid X_1, X_2 \right) = \emptyset.
\]
B. General Mixture of i.i.d. Correlated Sources

In this subsection, we consider an extension of Theorem 6.1 to the case with general mixture instead of countably infinite mixtures. The mixed correlated source \((X_1, X_2)\) that we consider in this subsection is defined by

$$P_{X_1^n X_2^n}(x_1, x_2) = \int_{\Lambda} P_{X_1^{(\theta)n} X_2^{(\theta)n}}(x_1, x_2) w(d\theta), \tag{6.6}$$

where \((X_1, X_2) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}\) and \(w(d\theta)\) is an arbitrary probability measure on the parameter space \(\Lambda\), and \((X_1^{(\theta)n}, X_2^{(\theta)n}) = \{(X_1^{(\theta)n}, X_2^{(\theta)n})\}_{n=1}^{\infty}\) (\(\theta \in \Lambda\)) are i.i.d. correlated sources with finite alphabets, and also the integrand on the right-hand side is assumed to be a measurable function of \(\theta\).

We use here the following lemma demonstrated by Han \[9\], instead of Lemma 6.1. 

**Lemma 6.2 (Han \[9\]):** Let \(\{z_n^{(1)}\}_{n=1}^{\infty}, \{z_n^{(2)}\}_{n=1}^{\infty}, \{z_n^{(3)}\}_{n=1}^{\infty}\) be any real-valued sequences. Then, for the mixed correlated sources defined by \((6.6)\), it holds that

$$\begin{align*}
\liminf_{n \to \infty} \int_{\Lambda} \frac{-\log P_{X_1^{(\theta)n} X_2^{(\theta)n}}(X_1^{(\theta)n} \mid X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(1)} - \gamma_n, \\
\frac{-\log P_{X_2^{(\theta)n} X_1^{(\theta)n}}(X_2^{(\theta)n} \mid X_1^{(\theta)n})}{\sqrt{n}} &< z_n^{(2)} - \gamma_n, \\
\frac{-\log P_{X_1^{(\theta)n} X_2^{(\theta)n}}(X_1^{(\theta)n} X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(3)} - \gamma_n
\end{align*}$$

$$w(d\theta)$$

$$\begin{align*}
\leq \liminf_{n \to \infty} \int_{\Lambda} \frac{-\log P_{X_1^n X_2^n}(X_1^{(\theta)n} \mid X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(1)}, \\
\frac{-\log P_{X_2^n X_1^n}(X_2^{(\theta)n} \mid X_1^{(\theta)n})}{\sqrt{n}} &< z_n^{(2)}, \\
\frac{-\log P_{X_1^n X_2^n}(X_1^{(\theta)n} X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(3)}
\end{align*}$$

$$w(d\theta)$$

$$\begin{align*}
\leq \limsup_{n \to \infty} \int_{\Lambda} \frac{-\log P_{X_1^n X_2^n}(X_1^{(\theta)n} \mid X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(1)}, \\
\frac{-\log P_{X_2^n X_1^n}(X_2^{(\theta)n} \mid X_1^{(\theta)n})}{\sqrt{n}} &< z_n^{(2)}, \\
\frac{-\log P_{X_1^n X_2^n}(X_1^{(\theta)n} X_2^{(\theta)n})}{\sqrt{n}} &< z_n^{(3)}
\end{align*}$$
where \( \gamma_n > 0 \) satisfies \( \gamma_1 > \gamma_2 > \cdots > 0, \gamma_n \to 0 \) and \( \sqrt{n} \gamma_n \to \infty \).

In this case, we also define the normal cumulative distribution function for each \( \theta \in \Lambda \) as follows.

\[
\Phi^{(\theta)}(T_1, T_2, T_3) \equiv \int_{-\infty}^{T_1} dy_1 \int_{-\infty}^{T_2} dy_2 \int_{-\infty}^{T_3} dy_3 \frac{1}{(2\pi)^{3/2} \sqrt{\det \Sigma_\theta}} \exp \left( -\frac{1}{2} y^T \Sigma_\theta^{-1} y \right),
\]

where \( y = (y_1, y_2, y_3) \) (three-dimensional row vector), and the covariance matrix \( \Sigma_\theta = (\sigma_{ij}^{(\theta)}) \) (i, j = 1, 2, 3, \( \theta \in \Lambda \)) are defined in a similar manner to the previous subsection.

Finally, given \( a_1, a_2 \) and \( \varepsilon \), we set

\[
K_n^{(\Lambda)}(a_1, a_2, \varepsilon | X_1, X_2) = \left\{ (L_1, L_2) \int_\Lambda \Phi^{(\theta)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(\theta)} | X_2^{(\theta)} \right) \right) + L_1, \sqrt{n} \left( a_2 - H \left( X_2^{(\theta)} | X_1^{(\theta)} \right) \right) + L_2, \sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(\theta)} X_2^{(\theta)} \right) \right) + L_1 + L_2 \right\} w(d\theta) \geq 1 - \varepsilon \right\}.
\]

Then, the following theorem holds.

**Theorem 6.2**: For the mixed correlated source with finite alphabet defined by (6.10), the second-order \((a_1, a_2, \varepsilon)\)-achievable rate region is given as the set:

\[
L(a_1, a_2, \varepsilon | X_1, X_2) = \lim_{n \to \infty} K_n^{(\Lambda)}(a_1, a_2, \varepsilon | X_1, X_2) = \left\{ (L_1, L_2) \int_\Lambda \lim_{n \to \infty} \Phi^{(\theta)} \left( \sqrt{n} \left( a_1 - H \left( X_1^{(\theta)} | X_2^{(\theta)} \right) \right) + L_1, \sqrt{n} \left( a_2 - H \left( X_2^{(\theta)} | X_1^{(\theta)} \right) \right) + L_2, \sqrt{n} \left( a_1 + a_2 - H \left( X_1^{(\theta)} X_2^{(\theta)} \right) \right) + L_1 + L_2 \right\} w(d\theta) \geq 1 - \varepsilon \right\}.
\]

(6.7)
**Proof:** It suffices to proceed in parallel with the arguments as made in the proof of Theorem 6.1.

Notice that Lemma 6.2 (again with $z_n^{(1)} = \sqrt{n}a_1 + L_1$, $z_n^{(2)} = \sqrt{n}a_2 + L_2$, $z_n^{(3)} = \sqrt{n}(a_1 + a_2) + L_1 + L_2$) is used here instead of Lemma 6.1.

An immediate consequence of Theorem 6.2 is the following compact formula *not* including $\lim_{n \to \infty}$, where $\Phi_i^{(\theta)}$, $\Phi_{ij}^{(\theta)}$ are the marginal commutative distribution functions of $\Phi^{(\theta)}$ indicated by $i$, $ij$, respectively (cf. $\Phi_i$, $\Phi_{ij}$ in Section 17):

**Theorem 6.3:**

$$L(a_1, a_2, \varepsilon | X_1, X_2) = \{ (L_1, L_2) | \Phi^\Lambda(a_1, a_2; L_1, L_2) \geq 1 - \varepsilon \},$$

(6.8)

where

$$\Phi^\Lambda(a_1, a_2; L_1, L_2) = \int_{\Lambda_0(a_1, a_2)} w(d\theta) + \int_{\Lambda_1(a_1, a_2)} \Phi_1^{(\theta)}(L_1) w(d\theta)$$

$$+ \int_{\Lambda_2(a_1, a_2)} \Phi_2^{(\theta)}(L_2) w(d\theta) + \int_{\Lambda_3(a_1, a_2)} \Phi_3^{(\theta)}(L_1 + L_2) w(d\theta)$$

$$+ \int_{\Lambda_4(a_1, a_2) \setminus \Lambda_5(a_1, a_2)} \Phi_{13}^{(\theta)}(L_1, L_1 + L_2) w(d\theta) + \int_{\Lambda_5(a_1, a_2) \setminus \Lambda_4(a_1, a_2)} \Phi_{23}^{(\theta)}(L_2, L_1 + L_2) w(d\theta)$$

$$+ \int_{\Lambda_4(a_1, a_2) \cap \Lambda_5(a_1, a_2)} \Phi_{12}^{(\theta)}(L_1, L_2) w(d\theta);$$

and

\[
\begin{align*}
\Lambda_0(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 > H(X_1^{(\theta)} | X_2^{(\theta)}), \ a_2 > H(X_2^{(\theta)} | X_1^{(\theta)}), \ a_1 + a_2 > H(X_1^{(\theta)} X_2^{(\theta)}) \}, \\
\Lambda_1(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 = H(X_1^{(\theta)} | X_2^{(\theta)}), \ a_2 > H(X_2^{(\theta)}) \}, \\
\Lambda_2(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 > H(X_1^{(\theta)}), \ a_2 = H(X_2^{(\theta)} | X_1^{(\theta)}) \}, \\
\Lambda_3(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 > H(X_1^{(\theta)} | X_2^{(\theta)}), \ a_2 > H(X_2^{(\theta)} | X_1^{(\theta)}), \ a_1 + a_2 > H(X_1^{(\theta)} X_2^{(\theta)}) \}, \\
\Lambda_4(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 = H(X_1^{(\theta)} | X_2^{(\theta)}), \ a_2 = H(X_2^{(\theta)}) \}, \\
\Lambda_5(a_1, a_2) &= \{ \theta \in \Lambda \mid a_1 = H(X_1^{(\theta)}), \ a_2 = H(X_2^{(\theta)} | X_1^{(\theta)}) \}.
\end{align*}
\]

**Proof:** It suffices to scrutinize in Theorem 6.2 the situation with $n \to \infty$ and take account of Theorem 6.1.

**Remark 6.3:** It is easily seen that if $X_1^{(\theta)}$ and $X_2^{(\theta)}$ are correlated for all $\theta \in \Lambda$ then all $\Lambda_i(a_1, a_2)$ (i =
0, 1, \ldots, 5) are mutually disjoint, which, in particular, implies that

\[ \Lambda_4(a_1, a_2) \setminus \Lambda_5(a_1, a_2) = \Lambda_4(a_1, a_2) \]
\[ \Lambda_5(a_1, a_2) \setminus \Lambda_4(a_1, a_2) = \Lambda_5(a_1, a_2) \]
\[ \Lambda_4(a_1, a_2) \cap \Lambda_5(a_1, a_2) = \emptyset. \]

On the other hand, if \( X_1^{(\theta)} \) and \( X_2^{(\theta)} \) are independent for some \( \theta \in \Lambda \) then \( \Lambda_4(a_1, a_2) \cap \Lambda_5(a_1, a_2) \neq \emptyset \); in this case it should be noted that if \( \theta \in \Lambda_4(a_1, a_2) \cap \Lambda_5(a_1, a_2) \) then

\[ \Phi_{13}^{(\theta)}(L_1, L_1 + L_2) = \Phi_{23}^{(\theta)}(L_2, L_1 + L_2) = \Phi_{12}^{(\theta)}(L_1, L_2). \]

\[ \Box \]

VII. Concluding Remarks

So far we have established the second-order source coding theorem for the Slepian-Wolf system with general correlated sources. On the other hand, in the single-user source coding problem, Hayashi [8] has shown the optimal second-order achievable rate for general sources, and actually computed it for an i.i.d. source using the asymptotic normality. In the case that the source is assumed to be in a class of mixed sources, Nomura and Han [10] have demonstrated the second-order achievable rates explicitly.

Analogously, with the Slepian-Wolf coding system, we have established the second-order achievable rate region for general correlated sources, and actually computed it for i.i.d. correlated sources and mixed correlated sources, respectively. As we have mentioned in the above, in order to compute the achievable rate region for i.i.d. correlated sources, the property of multivariate normal distribution played the key role. In particular, the condition for the second-order achievable rate region turned out to be expressed in terms of the marginal distributions of a three-dimensional normal distribution.

In order to elucidate the effectiveness of Theorem 4.1, we have numerically compared it with a slight strengthening of the original Koshelev bound, and it turned out that the former outperforms the latter.

Notice here that in [10] the optimal second-order rates for mixed sources in the single-user source coding problem has also been expressed in terms of the single equation such as (6.2) and (6.7), which we called there the canonical representation, introduced for the first time in this paper. This is indeed a great advantage of the information spectrum methods. While the equation in [10] is
based on the single-normal distribution functions, the similar formulas in the present paper are based on the three-dimensional normal distribution functions. From this viewpoint, the results in the present paper can be considered as a reasonable generalization of the results in [10]. In the channel coding context, Polyanskiy, Poor and Verdú [25] have derived the optimal second-order capacity for the Gilbert-Elliott channel by using a form similar to canonical equations.

Finally, it should be emphasized that our analysis is based on the information spectrum methods and hence the results in this paper are valid with countably infinite alphabet excepting Theorem 6.2 with general mixtures.

**APPENDIX A**

**Proof of Theorem 3.1**

It is obvious by virtue of De Morgan’s law that (3.1) in Theorem 3.1 is equivalent to

\[ L(a_1, a_2, \varepsilon |X_1, X_2) = \text{Cl} \left( \left\{(L_1, L_2) \left| \lim_{n \to \infty} F_n(L_1, L_2 | a_1, a_2) \geq 1 - \varepsilon \right\} \right) \],

(A.1)

so that in the sequel we will give the proof of (A.1) instead of (3.1).

1) **Direct Part:**

For any fixed \((L_1, L_2)\) satisfying

\[ \lim_{n \to \infty} F_n(L_1, L_2 | a_1, a_2) \geq 1 - \varepsilon, \]  

(A.2)

set

\[ M_n^{(1)} = e^{n a_1 + L_1 \sqrt{n} + 2 \frac{1}{n} \gamma}, \quad M_n^{(2)} = e^{n a_2 + L_2 \sqrt{n} + 2 \frac{1}{n} \gamma}, \]

where \(\gamma > 0\) is an arbitrary small constant. It is obvious that

\[ \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n^{(1)}}{e^{n a_1}} \leq L_1, \]

and

\[ \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \frac{M_n^{(2)}}{e^{n a_2}} \leq L_2 \]

hold. Thus, in this direct part it suffices to show the existence of an \((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) code such that \(\limsup_{n \to \infty} \varepsilon_n \leq \varepsilon\).
Hence, by means of De Morgan’s law, we have

\[
\varepsilon_n \leq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{z_n P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{na_1 + L_1 \sqrt{n} + 2 \sqrt{n} \gamma}{\sqrt{n}}, \right.
\]

or

\[
\frac{1}{\sqrt{n}} \log \frac{1}{z_n P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{na_2 + L_2 \sqrt{n} + 2 \sqrt{n} \gamma}{\sqrt{n}},
\]

or

\[
\frac{1}{\sqrt{n}} \log \frac{1}{z_n P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{n(a_1 + a_2) + (L_1 + L_2) \sqrt{n} + 4 \sqrt{n} \gamma}{\sqrt{n}} \right\} + 3z_n.
\]

Since \( \{z_n\}_{n=1}^\infty \) is a sequence of arbitrary numbers satisfying \( z_i > 0 \) (\( \forall i = 1, 2, \cdots \)), we set \( z_n = e^{-\sqrt{n} \gamma} \). Then, we have

\[
\varepsilon_n \leq \Pr \left\{ \frac{1}{\sqrt{n}} \log \frac{1}{P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{na_1 + L_1 \sqrt{n} + 2 \sqrt{n} \gamma}{\sqrt{n}}, \right.
\]

or

\[
\frac{1}{\sqrt{n}} \log \frac{1}{P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{na_2 + L_2 \sqrt{n} + 2 \sqrt{n} \gamma}{\sqrt{n}},
\]

or

\[
\frac{1}{\sqrt{n}} \log \frac{1}{P_{X_n^1 | X_n^2} (X_n^1 | X_n^2)} \geq \frac{n(a_1 + a_2) + (L_1 + L_2) \sqrt{n} + 4 \sqrt{n} \gamma}{\sqrt{n}} \right\} + 3e^{-\sqrt{n} \gamma}.
\]

Hence,

\[
\varepsilon_n \leq \Pr \left\{ \frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_1}{\sqrt{n}} \geq L_1 + \frac{\gamma}{\sqrt{n}}, \right.
\]

or

\[
\frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_2}{\sqrt{n}} \geq L_2 + \frac{\gamma}{\sqrt{n}},
\]

or

\[
\frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - n(a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 + \frac{3 \gamma}{\sqrt{n}} \right\} + 3e^{-\sqrt{n} \gamma}
\]

\[
\leq \Pr \left\{ \frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_1}{\sqrt{n}} \geq L_1 + \frac{\gamma}{\sqrt{n}}, \right.
\]

or

\[
\frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_2}{\sqrt{n}} \geq L_2 + \frac{\gamma}{\sqrt{n}},
\]

or

\[
\frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - n(a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 + \frac{2 \gamma}{\sqrt{n}} \right\} + 3e^{-\sqrt{n} \gamma}. \tag{A.3}
\]

Hence, by means of De Morgan’s law, we have

\[
\Pr \left\{ \frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_1}{\sqrt{n}} \geq L_1 + \frac{\gamma}{\sqrt{n}}, \right.
\]

or

\[
\frac{- \log P_{X_n^1 | X_n^2} (X_n^1 | X_n^2) - na_2}{\sqrt{n}} \geq L_2 + \frac{\gamma}{\sqrt{n}};
\]

\[
\geq L_1 + \frac{\gamma}{\sqrt{n}},
\]

\[
\geq L_2 + \frac{\gamma}{\sqrt{n}};
\]

\[
\geq L_1 + L_2 + \frac{2 \gamma}{\sqrt{n}} \right\} + 3e^{-\sqrt{n} \gamma}. \tag{A.3}
\]
or 
\[ - \log P_{X^n_1 X^n_2} (X^n_1 X^n_2) - n(a_1 + a_2) \geq L_1 + L_2 + \frac{2\gamma}{\sqrt{n}} \]

\[ = 1 - \Pr \left\{ \frac{- \log P_{X^n_1 X^n_2} (X^n_1 X^n_2) - n(a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 + \frac{2\gamma}{\sqrt{n}} \right\} \]

\[ = 1 - \mathcal{F}_n \left( L_1 + \frac{\gamma}{\sqrt{n}}, L_2 + \frac{\gamma}{\sqrt{n}} \mid a_1, a_2 \right). \quad (A.4) \]

Substituting (A.4) into (A.3), we have

\[ \varepsilon_n \leq 1 - \mathcal{F}_n \left( L_1 + \frac{\gamma}{\sqrt{n}}, L_2 + \frac{\gamma}{\sqrt{n}} \mid a_1, a_2 \right) + 3e^{-\frac{3\sqrt{n}}{2}}. \]

By taking \( \limsup_{n \to \infty} \) of both sides, we have

\[ \limsup_{n \to \infty} \varepsilon_n \leq 1 - \liminf_{n \to \infty} \mathcal{F}_n \left( L_1, L_2 \mid a_1, a_2 \right) \leq \varepsilon, \]

where the last inequality follows from (A.2). Thus, the direct part has been proved.

2) **Converse Part:**

Suppose that a pair \((L_1, L_2)\) is \((a_1, a_2, \varepsilon)\)-achievable. Then, from the assumption, for any small \( \gamma > 0 \) we have a code \((n, M_n^{(1)}, M_n^{(2)}, \varepsilon_n)\) such that

\[ \frac{1}{\sqrt{n}} \log \frac{M_n^{(1)}}{e^{na_1}} \leq L_1 + \gamma, \quad (A.5) \]

\[ \frac{1}{\sqrt{n}} \log \frac{M_n^{(2)}}{e^{na_2}} \leq L_2 + \gamma, \quad (A.6) \]

for all sufficiently large \( n \), and that

\[ \limsup_{n \to \infty} \varepsilon_n \leq \varepsilon. \quad (A.7) \]

Thus, substituting (A.5) and (A.6) into Lemma 2.2, the error probability is lower bounded by

\[ \varepsilon_n \geq \Pr \left\{ \frac{- \log P_{X^n_1 X^n_2} (X^n_1 X^n_2) - n(a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 + 2\gamma - \frac{\log z_n}{\sqrt{n}} \right\} - 3z_n, \]

\[ = 1 - \mathcal{F}_n \left( L_1 + L_2 + 2\gamma - \frac{\log z_n}{\sqrt{n}} \right) - 3z_n. \]
for all $n = 1, 2, \cdots$, where $\{z_n\}_{n=1}^\infty$ is a sequence of an arbitrary numbers satisfying $z_i > 0$ ($\forall i = 1, 2, \cdots$). Set $z_n = e^{-\sqrt{n} \gamma}$ and substituting it into the above, we have

$$
\varepsilon_n \geq \Pr \left\{ \frac{- \log P_{X_1^n | X_2^n} (X_1^n | X_2^n) - na_1}{\sqrt{n}} \geq L_1 + 2\gamma \right. \\
\left. \quad \text{or} \quad \frac{- \log P_{X_2^n | X_1^n} (X_2^n | X_1^n) - na_2}{\sqrt{n}} \geq L_2 + 2\gamma \\
\left. \quad \text{or} \quad \frac{- \log P_{X_1^n X_2^n} (X_1^n X_2^n) - n(a_1 + a_2)}{\sqrt{n}} \geq L_1 + L_2 + 3\gamma \right\} - 3e^{-\sqrt{n} \gamma}
$$

By taking $\limsup_{n \to \infty}$ in (A.7) we have

$$
\varepsilon_n \geq 1 - \Pr \left\{ \frac{- \log P_{X_1^n | X_2^n} (X_1^n | X_2^n) - na_1}{\sqrt{n}} < L_1 + 2\gamma, \\
\frac{- \log P_{X_2^n | X_1^n} (X_2^n | X_1^n) - na_2}{\sqrt{n}} < L_2 + 2\gamma, \\
\frac{- \log P_{X_1^n X_2^n} (X_1^n X_2^n) - n(a_1 + a_2)}{\sqrt{n}} < L_1 + L_2 + 4\gamma \right\} - 3e^{-\sqrt{n} \gamma}
$$

$$
= 1 - \bar{F}_n \left( L_1 + 2\gamma, L_2 + 2\gamma | a_1, a_2 \right) - 3e^{-\sqrt{n} \gamma}.
$$

By taking $\limsup_{n \to \infty}$ in (A.7) we have

$$
\varepsilon \geq \limsup_{n \to \infty} \varepsilon_n \geq 1 - \liminf_{n \to \infty} \bar{F}_n \left( L_1 + 2\gamma, L_2 + 2\gamma | a_1, a_2 \right).
$$

Since $\gamma > 0$ is arbitrary, this means that

$$
L(a_1, a_2, \varepsilon | X_1, X_2) \subset \text{Cl} \left\{ \left( L_1, L_2 \right) \left| \liminf_{n \to \infty} \bar{F}_n (L_1, L_2 | a_1, a_2) \geq 1 - \varepsilon \right. \right\}.
$$

Thus, we have proved the converse part. ■
Appendix B

Proof of Theorem 4.1

1) Case I:

In view of Theorem 3.1 in the form of (A.1), it suffices to calculate

\[
\lim_{n \to \infty} \Pr_n(L_1, L_2|a_1, a_2) = \lim_{n \to \infty} \Pr \left\{ \frac{-\log P_{X^n|X_2^n}(X_i^n|X_2^n) - na_1}{\sqrt{n}} < L_1, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_1^n}(X_i^n|X_1^n) - na_2}{\sqrt{n}} < L_2, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_2^n}(X_i^n X_2^n) - n(a_1 + a_2)}{\sqrt{n}} < L_1 + L_2 \right\}.
\]

Substituting \(a_1 = H(X_1|X_2)\) and \(a_2 = H(X_2)\) into \(\Pr_n(L_1, L_2|a_1, a_2)\), we have

\[
\Pr_n(L_1, L_2|a_1, a_2) = \Pr \left\{ \frac{-\log P_{X^n|X_2^n}(X_i^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_1^n}(X_i^n|X_1^n) - nH(X_2)}{\sqrt{n}} < L_2, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_2^n}(X_i^n X_2^n) - n(H(X_1|X_2) + H(X_2))}{\sqrt{n}} < L_1 + L_2 \right\}
\]

\[
= \Pr \left\{ \frac{-\log P_{X^n|X_2^n}(X_i^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_1^n}(X_i^n|X_1^n) - nH(X_2)}{\sqrt{n}} < L_2, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_2^n}(X_i^n X_2^n) - nH(X_1 X_2)}{\sqrt{n}} < L_1 + L_2 \right\}
\]

\[
= \Pr \left\{ \frac{-\log P_{X^n|X_2^n}(X_i^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_1^n}(X_i^n|X_1^n) - nH(X_2)}{\sqrt{n}} < L_2, \right. \\
    \left. \quad \frac{-\log P_{X^n|X_2^n}(X_i^n X_2^n) - n(H(X_1|X_2) + H(X_2))}{\sqrt{n}} < L_1 + L_2 \right\}
\]

where the second equality is derived by the chain rule \(H(X_1|X_2) + H(X_2) = H(X_1 X_2)\). Moreover, \(H(X_2) - H(X_2|X_1) = I(X_1; X_2) > 0\) holds, since we are considering correlated sources (recall that we have assumed that \(\Sigma\) is positive-definite; also cf. the observation below). Thus, for any constant \(W > 0\), we have \(\sqrt{n}(H(X_2) - H(X_2|X_1)) > W\) for sufficiently large \(n\).
As a consequence, noting that the correlated sources has an i.i.d. property, and taking account of the asymptotic normality (due to the central limit theorem), we have

\[
\mathcal{T}_n(L_1, L_2|a_1, a_2) \geq \Pr \left\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right. \\
\left. \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - nH(X_2|X_1)}{\sqrt{n}} < W + L_2, \right. \\
\left. \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - nH(X_1X_2)}{\sqrt{n}} < L_1 + L_2 \right\} \to \Phi (L_1, W + L_2, L_1 + L_2) \quad (as \ n \to \infty).
\]

Furthermore, as \( W > 0 \) can be arbitrarily large, this implies that

\[
\lim_{n \to \infty} \mathcal{T}_n(L_1, L_2|a_1, a_2) \geq \lim_{W \to \infty} \Phi (L_1, W + L_2, L_1 + L_2) = \Phi_{13} (L_1, L_1 + L_2). \quad (B.1)
\]

On the other hand, it is obvious that

\[
\mathcal{T}_n(L_1, L_2|a_1, a_2) \leq \Pr \left\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right. \\
\left. \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - nH(X_2|X_1)}{\sqrt{n}} < W + L_2, \right. \\
\left. \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - nH(X_1X_2)}{\sqrt{n}} < L_1 + L_2 \right\},
\]

so that we have

\[
\lim_{n \to \infty} \sup \mathcal{T}_n(L_1, L_2|a_1, a_2) \leq \Phi_{13} (L_1, L_1 + L_2). \quad (B.2)
\]

Thus, summarizing (B.1) and (B.2) leads to

\[
\lim_{n \to \infty} \mathcal{T}_n(L_1, L_2|a_1, a_2) = \Phi_{13} (L_1, L_1 + L_2). \quad (B.3)
\]

Therefore, the proof of Case I has been completed with the observation that in the case of \( I(X_1; X_2) = 0 \) the right-hand side of (B.3) turns out to coincide with \( \Phi_{12}(L_1, L_2) \).

2) Case II:

First, notice that

\[
a_1 + a_2 = \lambda H(X_1) + (1 - \lambda)H(X_1|X_2) + (1 - \lambda)H(X_2) + \lambda H(X_2|X_1)
\]

\[
= \lambda (H(X_1) + H(X_2|X_1)) + (1 - \lambda) (H(X_1|X_2) + H(X_2))
\]

\[
= H(X_1X_2).
\]

Moreover, \( \Delta_1 \equiv a_1 - H(X_1|X_2) > 0 \) because \( X_1 \) and \( X_2 \) are correlated and

\[
a_1 - H(X_1|X_2) = \lambda H(X_1) + (1 - \lambda)H(X_1|X_2) - H(X_1|X_2) = \lambda (H(X_1) - H(X_1|X_2)) > 0.
\]
The same argument yields $\Delta_2 \equiv a_2 - H(X_2|X_1) > 0$, so that $\mathcal{T}_n(L_1, L_2|a_1, a_2)$ is given by

$$
\mathcal{T}_n(L_1, L_2|a_1, a_2) = \Pr \left\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < \Delta_1 \sqrt{n} + L_1, \right.
\left. \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - nH(X_2|X_1)}{\sqrt{n}} < \Delta_2 \sqrt{n} + L_2, \right.
\left. \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - nH(X_1X_2)}{\sqrt{n}} < L_1 + L_2 \right\}.
$$

Here, $\Delta_1 \sqrt{n}$ and $\Delta_2 \sqrt{n}$ go to $\infty$ as $n \to \infty$. Therefore, again, by virtue of the asymptotic normality it holds that

$$
\lim_{n \to \infty} \mathcal{T}_n(L_1, L_2|a_1, a_2) = \Phi_3(L_1 + L_2).
$$

Therefore, the proof of Case II has been completed.

3) Case III:

Notice that setting $\delta = a_2 - H(X_2) > 0$ leads to

$$
a_1 + a_2 - H(X_1X_2) = H(X_1|X_2) + H(X_2) + \delta - H(X_1X_2) = \delta > 0.
$$

Moreover, $\Delta_2 = a_2 - H(X_2|X_1) > 0$ holds, because

$$
a_2 - H(X_2|X_1) = H(X_2) + \delta - H(X_2|X_1) > 0.
$$

holds. Thus, $\mathcal{T}_n(L_1, L_2|a_1, a_2)$ is given by

$$
\mathcal{T}_n(L_1, L_2|a_1, a_2) = \Pr \left\{ \frac{-\log P_{X_1^n|X_2^n}(X_1^n|X_2^n) - nH(X_1|X_2)}{\sqrt{n}} < L_1, \right.
\left. \frac{-\log P_{X_2^n|X_1^n}(X_2^n|X_1^n) - nH(X_2|X_1)}{\sqrt{n}} < \Delta_2 \sqrt{n} + L_2, \right.
\left. \frac{-\log P_{X_1^nX_2^n}(X_1^nX_2^n) - nH(X_1X_2)}{\sqrt{n}} < \delta \sqrt{n} + L_1 + L_2 \right\},
$$

Here, $\Delta_2 \sqrt{n}$ and $\delta \sqrt{n}$ goes to $\infty$ as $n \to \infty$. Then, again, by virtue of the asymptotic normality as well as the same discussion in Cases I and II, it is concluded that

$$
\lim_{n \to \infty} \mathcal{T}_n(L_1, L_2|a_1, a_2) = \Phi_1(L_1),
$$

thus completing the proof. □
Appendix C
Proof of Lemma 5.1

We slightly modify the Koshelev’s argument [23]. We first define the encoders \( \phi_n^{(1)}, \phi_n^{(2)} \) and the decoder \( \psi_n \) as follows.

Encoder:

For each source output \( x_1 \in X_1^n \) we randomly generate an index \( i_1 \in M_n^{(1)} \equiv \{1, 2, \cdots, M_n^{(1)}\} \) according to the uniform distribution and set \( \phi_n^{(1)}(x_1) = i_1 \). The encoder \( \phi_n^{(2)}(x_2) = i_2 \in M_n^{(2)} \equiv \{1, 2, \cdots, M_n^{(2)}\} \) is defined similarly.

Decoder:

Suppose that a decoder \( \psi_n \) receives a pair of the encoder outputs \( (i_1, i_2) \). We define the maximum likelihood decoder

\[
\psi_n(i_1, i_2) = \arg \max_{(x_1, x_2) \in X_1^n \times X_2^n} Q_{i_1, i_2}(x_1, x_2)
\]

where

\[
Q_{i_1, i_2}(x_1, x_2) = P_{X_1^n \times X_2^n}(x_1, x_2) \mathbb{I} \left\{ \phi_n^{(1)}(x_1) = i_1, \phi_n^{(2)}(x_2) = i_2 \right\},
\]

and \( \mathbb{I}\{\cdot\} \) denotes the indicator function.

Analysis of error probability:

The error probability \( \varepsilon_n \) is evaluated as follows where \( \Pr\{\cdot\} \) denotes the probability due to the random code:

\[
\varepsilon_n \leq \sum_{(x_1, x_2) \in X_1^n \times X_2^n} P_{X_1^n \times X_2^n}(x_1, x_2) \Pr \left\{ \bigcup_{(x_1', x_2') \neq (x_1, x_2)} \left\{ \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1', x_2')}{P_{X_1^n \times X_2^n}(x_1, x_2)} \geq 1 \right\} \right\}
\]

\[
\leq \sum_{(x_1, x_2) \in X_1^n \times X_2^n} P_{X_1^n \times X_2^n}(x_1, x_2) \Pr \left\{ \bigcup_{x_1' \neq x_1} \left\{ \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1', x_2)}{P_{X_1^n \times X_2^n}(x_1, x_2)} \geq 1 \right\} \right\}
\]

\[
+ \sum_{(x_1, x_2) \in X_1^n \times X_2^n} P_{X_1^n \times X_2^n}(x_1, x_2) \Pr \left\{ \bigcup_{x_2' \neq x_2} \left\{ \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1, x_2')}{P_{X_1^n \times X_2^n}(x_1, x_2)} \geq 1 \right\} \right\}
\]

\[
+ \sum_{(x_1, x_2) \in X_1^n \times X_2^n} P_{X_1^n \times X_2^n}(x_1, x_2) \Pr \left\{ \bigcup_{x_1' \neq x_1, x_2' \neq x_2} \left\{ \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1', x_2')}{P_{X_1^n \times X_2^n}(x_1, x_2)} \geq 1 \right\} \right\}.
\]
Then, with any $t_1, t_2, t_3 \geq 0$ and any $0 \leq s_1, s_2, s_3 \leq 1$, we obtain

$$
\varepsilon_n \leq \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2) \left[ \sum_{x'_1 \neq x_1} \mathbb{E} \left( \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x'_1, x_2)}{P_{X^n_1 X^n_2}(x_1, x_2)} \right)^{t_1} \right]^{s_1} \\
+ \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2) \left[ \sum_{x'_2 \neq x_2} \mathbb{E} \left( \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1, x'_2)}{P_{X^n_1 X^n_2}(x_1, x_2)} \right)^{t_2} \right]^{s_2} \\
+ \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2) \left[ \sum_{x'_1 \neq x_1, x'_2 \neq x_2} \mathbb{E} \left( \frac{Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x'_1, x'_2)}{P_{X^n_1 X^n_2}(x_1, x_2)} \right)^{t_3} \right]^{s_3},
$$

where $\mathbb{E}$ denotes the expectation due to the random code, from which it follows that

$$
\varepsilon_n \leq \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-t_1 s_1} \left( \sum_{x'_1 \neq x_1} \mathbb{E} Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x'_1, x_2)^{t_1} \right)^{s_1} \\
+ \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-t_2 s_2} \left( \sum_{x'_2 \neq x_2} \mathbb{E} Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x_1, x'_2)^{t_2} \right)^{s_2} \\
+ \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-t_3 s_3} \left( \sum_{x'_1 \neq x_1, x'_2 \neq x_2} \mathbb{E} Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x'_1, x'_2)^{t_3} \right)^{s_3}.
$$

(C.1)

Here, the first term of the right-hand side of the above inequality is evaluated as follows with $t_1 = \frac{1}{1+s_1}$:

$$
\sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-t_1 s_1} \left( \sum_{x'_1 \neq x_1} \mathbb{E} Q_{\phi_n^{(1)}(x_1), \phi_n^{(2)}(x_2)}(x'_1, x_2)^{t_1} \right)^{s_1} \\
= \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-t_1 s_1} \left( \sum_{x'_1 \neq x_1} P_{X^n_1 X^n_2}(x'_1, x_2)^{1} \mathbb{1} \left\{ \phi_n^{(1)}(x_1) = \phi_n^{(1)}(x'_1) \right\} \right)^{s_1} \\
= \left( \frac{1}{M_n^{(1)}} \right)^{s_1} \sum_{(x_1, x_2) \in X^n_1 \times X^n_2} P_{X^n_1 X^n_2} (x_1, x_2)^{1-s_1} \left( \sum_{x'_1 \neq x_1} P_{X^n_1 X^n_2}(x'_1, x_2)^{1} \right)^{s_1} \\
\leq \left( \frac{1}{M_n^{(1)}} \right)^{s_1} \sum_{x_2 \in X^n_2} \left( \sum_{x_1 \in X^n_1} P_{X^n_1 X^n_2}(x_1, x_2)^{1} \right)^{1+s_1}.
$$

The second and the third terms on the right-hand side of (C.1) are similarly bounded from above by setting $t_2 = \frac{1}{1+s_2}$, $t_3 = \frac{1}{1+s_3}$, respectively.

Summarizing the above arguments, it is concluded that the error probability $\varepsilon_n$ is upper bounded
by

\[
\varepsilon_n \leq \left( \frac{1}{M_n^{(1)}} \right)^{s_1} \sum_{x_2 \in X_2^n} \left( \sum_{x_1 \in X_1^n} P_{X_1^n X_2^n}(x_1, x_2) \right)^{1/s_1} + \left( \frac{1}{M_n^{(2)}} \right)^{s_2} \sum_{x_1 \in X_1^n} \left( \sum_{x_2 \in X_2^n} P_{X_1^n X_2^n}(x_1, x_2) \right)^{1/s_2} + \left( \frac{1}{M_n^{(1)} M_n^{(2)}} \right)^{s_3} \left( \sum_{(x_1, x_2) \in X_1^n \times X_2^n} P_{X_1^n X_2^n}(x_1, x_2) \right)^{1/s_3},
\]

thus completing the proof of the lemma, because we are considering stationary memoryless correlated sources (cf. Gallager [24]).

**APPENDIX D**

**PROOF OF LEMMA 6.1**

We first show the first inequality. Set a sequence \( \{ \gamma_n \}_{n=1}^{\infty} \) satisfying \( \gamma_1 \geq \gamma_2 \geq \cdots > 0 \), and \( \gamma_n \to 0, \sqrt{n} \gamma_n \to \infty \) and define three sets

\[
D_n^{(1)}(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| \frac{-\log P_{X_1^n | X_2^n}(x_1 | x_2)}{\sqrt{n}} - \frac{-\log P_{X_1^{(k)} | X_2^{(k)}(x_1 | x_2)}}{\sqrt{n}} \leq -\gamma_n \right. \right\},
\]

\[
D_n^{(2)}(k) = \left\{ (x_1, x_2) \in X_2^n \times X_1^n \left| \frac{-\log P_{X_2^n | X_1^n}(x_2 | x_1)}{\sqrt{n}} - \frac{-\log P_{X_2^{(k)} | X_1^{(k)}(x_2 | x_1)}}{\sqrt{n}} \leq -\gamma_n \right. \right\},
\]

\[
D_n^{(3)}(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| \frac{-\log P_{X_1^n X_2^n}(x_1, x_2)}{\sqrt{n}} - \frac{-\log P_{X_1^{(k)} X_2^{(k)}(x_1, x_2)}}{\sqrt{n}} \leq -\gamma_n \right. \right\}
\]

for \( k = 1, 2, \ldots \). In addition, we set

\[
D_n(k) = D_n^{(1)}(k) \cup D_n^{(2)}(k) \cup D_n^{(3)}(k).
\]

Then, it holds that for \( k = 1, 2, \ldots \)

\[
\Pr \left\{ X_1^{(k)n} X_2^{(k)n} \in D_n^{(3)}(k) \right\} = \sum_{(x_1, x_2) \in D_n^{(3)}(k)} P_{X_1^{(k)n} X_2^{(k)n}}(x_1, x_2) \leq \sum_{(x_1, x_2) \in D_n^{(3)}(k)} P_{X_1^n X_2^n}(x_1, x_2) e^{-\sqrt{n} \gamma_n} \leq e^{-\sqrt{n} \gamma_n}.
\]

(D.1)
Similarly, it holds that

$$\Pr \left \{ \mathbf{X}_1^{(k)n} \mathbf{X}_2^{(k)n} \in D_n^{(1)(k)} \right \} = \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in D_n^{(1)(k)}} P_{\mathbf{X}_1^{(k)n} \mathbf{X}_2^{(k)n}}(\mathbf{x}_1, \mathbf{x}_2)$$

$$= \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in D_n^{(1)(k)}} P_{\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}}(\mathbf{x}_1 | \mathbf{x}_2) P_{\mathbf{X}_2^{(k)n}}(\mathbf{x}_2)$$

$$\leq \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in D_n^{(1)(k)}} P_{\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}}(\mathbf{x}_1 | \mathbf{x}_2) P_{\mathbf{X}_2^{(k)n}}(\mathbf{x}_2) e^{-\sqrt{n}\gamma_n}$$

$$\leq e^{-\sqrt{n}\gamma_n}, \quad (D.2)$$

for \(k = 1, 2, \cdots\) and also that

$$\Pr \left \{ \mathbf{X}_1^{(k)n} \mathbf{X}_2^{(k)n} \in D_n^{(2)(k)} \right \} \leq e^{-\sqrt{n}\gamma_n} \quad (D.3)$$

for \(k = 1, 2, \cdots\). It then follows from (D.1), (D.2) and (D.3) that

$$\Pr \left \{ \mathbf{X}_1^{(k)n}, \mathbf{X}_2^{(k)n} \in \mathcal{D}_n(k) \right \} \leq 3e^{-\sqrt{n}\gamma_n}.$$

This means that

$$\Pr \left \{ \mathbf{X}_1^{(k)n} \mathbf{X}_2^{(k)n} \not\in \mathcal{D}_n(k) \right \}$$

$$= \Pr \left \{ \frac{-\log P_{\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}}(\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n})}{\sqrt{n}} > -\log P_{\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}}(\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}) - \gamma_n, \quad \frac{-\log P_{\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n}}(\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n})}{\sqrt{n}} > -\log P_{\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n}}(\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n}) - \gamma_n, \right \}$$

$$\geq 1 - 3e^{-\sqrt{n}\gamma_n}.$$

Hence, we have

$$\Pr \left \{ \frac{-\log P_{\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n}}(\mathbf{X}_1^{(k)n} | \mathbf{X}_2^{(k)n})}{\sqrt{n}} - \gamma_n < -\sqrt{n}, \right \},$$

$$\frac{-\log P_{\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n}}(\mathbf{X}_2^{(k)n} | \mathbf{X}_1^{(k)n})}{\sqrt{n}} - \gamma_n < -\sqrt{n},$$

$$\frac{-\log P_{\mathbf{X}_1^{(k)n} \mathbf{X}_2^{(k)n}}}{\sqrt{n}} - \gamma_n < -\sqrt{n}.$$
for $k = 1, 2, \cdots$, which is the first inequality of the lemma.

Next, we show the second inequality of the lemma. Set

$$S_n^{(1)} = \left\{ (x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \mid \log P_{X_1^n | X_2^n} (x_1 | x_2) \geq -\sqrt{n} z_n^{(1)} \right\},$$

$$S_n^{(2)} = \left\{ (x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \mid \log P_{X_2^n | X_1^n} (x_2 | x_1) \geq -\sqrt{n} z_n^{(2)} \right\},$$

$$S_n^{(3)} = \left\{ (x_1, x_2) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \mid \log P_{X_1^n X_2^n} (x_1, x_2) \geq -\sqrt{n} z_n^{(3)} \right\}.

In the sequel, setting $S_n = S_n^{(1)} \cap S_n^{(2)} \cap S_n^{(3)}$, we evaluate $P_{X_1^{(k)} n X_2^{(k)} n}(S_n)$. To do so, first rewrite

$$\log P_{X_1^n | X_2^n} (x_1 | x_2) = \log P_{X_1^n X_2^n} (x_1, x_2) - \log P_{X_2^n} (x_2)$$

$$= \log \sum_{k=1}^{\infty} w(k) P_{X_1^{(k)n} X_2^{(k)n}} (x_1, x_2) - \log P_{X_2^n} (x_2)$$

$$\geq \log w(k) + \log P_{X_1^{(k)n} X_2^{(k)n}} (x_1, x_2) - \log P_{X_2^{(k)n}} (x_2)$$

$$\geq -\sqrt{n} \gamma_n + \log P_{X_1^{(k)n} X_2^{(k)n}} (x_1, x_2)$$

$$\geq \delta_1^{(k)} (x_1, x_2),$$

where we have taken account of $\gamma_n \geq -\frac{\log w(k)}{\sqrt{n}}$ for $k$ with $w(k) > 0$ and sufficiently large $n$.

Similarly, we have

$$\log P_{X_2^n | X_1^n} (x_2 | x_1) \geq -\sqrt{n} \gamma_n + \log P_{X_2^{(k)n} | X_1^{(k)n}} (x_2 | x_1)$$

$$\geq \delta_2^{(k)} (x_1, x_2),$$
\[
\log P_{X_1^n X_2^n}(x_1, x_2) \geq -\sqrt{n} \gamma_n + \log P_{X_1^{(k)n} X_2^{(k)n}}(x_1, x_2) \\
\equiv \delta_3^{(k)}(x_1, x_2).
\] (D.6)

On the other hand, again similarly to (D.1), it follows that
\[
\Pr \left\{ X_1^{(k)n} \in E_n^{(1)} \right\} \geq 1 - e^{-\sqrt{n} \gamma_n},
\]
\[
\Pr \left\{ X_2^{(k)n} \in E_n^{(2)} \right\} \geq 1 - e^{-\sqrt{n} \gamma_n},
\]
where
\[
E_n^{(1)} = \left\{ x_1 \in X_1^n \left| - \log P_{X_1^n}(x_1) + \log P_{X_1^{(k)n}}(x_1) \geq -\sqrt{n} \gamma_n \right. \right\},
\]
\[
E_n^{(2)} = \left\{ x_2 \in X_2^n \left| - \log P_{X_2^n}(x_2) + \log P_{X_2^{(k)n}}(x_2) \geq -\sqrt{n} \gamma_n \right. \right\}.
\]

Hence, with \( E_n = E_n^{(1)} \times E_n^{(2)} \) we obtain
\[
\Pr \left\{ X_1^{(k)n} X_2^{(k)n} \in E_n \right\} \geq 1 - 2e^{-\sqrt{n} \gamma_n}. \tag{D.7}
\]

Let us now define the subsets \( G_n^{(1)}(k) \), \( G_n^{(2)}(k) \) and \( G_n^{(3)}(k) \) by
\[
G_n^{(1)}(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| \delta_1^{(k)}(x_1, x_2) \geq -\sqrt{n} \gamma_n \right. \right\}, \tag{D.8}
\]
\[
G_n^{(2)}(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| \delta_2^{(k)}(x_1, x_2) \geq -\sqrt{n} \gamma_n \right. \right\}, \tag{D.9}
\]
\[
G_n^{(3)}(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| \delta_3^{(k)}(x_1, x_2) \geq -\sqrt{n} \gamma_n \right. \right\}. \tag{D.10}
\]

It is then obvious that, for \( k = 1, 2, \cdots \),
\[
S_n^{(1)} \supset G_n^{(1)}(k), \quad S_n^{(2)} \supset G_n^{(2)}(k), \quad S_n^{(3)} \supset G_n^{(3)}(k).
\]

Thus, setting \( G_n(k) = G_n^{(1)}(k) \cap G_n^{(2)}(k) \cap G_n^{(3)}(k) \), we obtain
\[
P_{X_1^{(k)n} X_2^{(k)n}}(S_n) \geq P_{X_1^{(k)n} X_2^{(k)n}}(G_n(k)) \geq P_{X_1^{(k)n} X_2^{(k)n}}(G_n(k) \cap E_n) \\
= P_{X_1^{(k)n} X_2^{(k)n}}(G_n(k)|E_n) P_{X_1^{(k)n} X_2^{(k)n}}(E_n). \tag{D.11}
\]

Now we set
\[
T_n(k) = \left\{ (x_1, x_2) \in X_1^n \times X_2^n \left| - \log P_{X_1^{(k)n} X_2^{(k)n}}(x_1 | x_2) < \sqrt{n} \left( z_n^{(1)} - 2 \gamma_n \right), \right. \right. \\
- \log P_{X_2^{(k)n} X_1^{(k)n}}(x_2 | x_1) < \sqrt{n} \left( z_n^{(2)} - 2 \gamma_n \right), \\
- \log P_{X_1^{(k)n} X_2^{(k)n}}(x_1, x_2) < \sqrt{n} \left( z_n^{(3)} - 2 \gamma_n \right) \right\}.
\]
Summarizing (D.4)–(D.10) yields

\[ P_{X_1^{(k)n}X_2^{(k)n}}(G_n(k)|E_n) \geq P_{X_1^{(k)n}X_2^{(k)n}}(T_n(k)|E_n). \]  
(D.12)

On the other hand, by means of (D.11) and (D.12),

\[
P_{X_1^{(k)n}X_2^{(k)n}}(S_n) \geq P_{X_1^{(k)n}X_2^{(k)n}}(T_n(k)|E_n)P_{X_1^{(k)n}X_2^{(k)n}}(E_n) \\
= P_{X_1^{(k)n}X_2^{(k)n}}(T_n(k)) - P_{X_1^{(k)n}X_2^{(k)n}}(T_n(k)|E_n^c)P_{X_1^{(k)n}X_2^{(k)n}}(E_n^c) \\
\geq P_{X_1^{(k)n}X_2^{(k)n}}(T_n(k)) - 2e^{-\sqrt{n}\gamma_n}, \]  
(D.13)

for sufficiently large \( n \) where “\(^c\)” denotes the complement, and in the last step we have used (D.7). Finally, it suffices only to replace \( 2\gamma_n \) by \( \gamma_n \) and to notice that (D.13) gives the second inequality of the lemma. ■

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