The Devil’s Staircase in the Frequency and Amplitude Locking of Nonlinear Oscillators with Continuous Modulated Forcing

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We study the emergence of a devil’s staircase resonance structure in the locking behaviour of driven continuous systems by periodic modulation of the driving force. In particular, we concentrate on systems that, for a constant sinusoidal forcing term, show only Adler-type locking around a single main resonance tongue. We show that, with harmonically modulated forcing, nonlinear oscillations close to a Hopf bifurcation generically reproduce the devil’s staircase known from the discrete circle map. Experimental results on a semiconductor laser driven by a modulated optical signal show excellent agreement with our theoretical predictions. We show that the emergence of higher harmonic resonance tongues in the oscillation frequency is enabled by the locking of the amplitude oscillations. Our results show that by proper implementation of an external drive, additional regions of stable frequency locking can only be introduced in systems which originally show only a single Adler-type resonance tongue.

The synchronization of periodic systems by an external force or coupling to another oscillator has been the topic of intensive research for several centuries [1–10]. The entrainment of the intrinsic oscillation frequency of an oscillator by a driving signal has been observed in a multitude of dynamical systems, from electronic circuits [11, 12], through climate dynamics [13, 14], to physiological [15–17] and neuronal systems [18, 19]. When the detuning \( \nu \) between the driving frequency and an oscillator’s intrinsic frequency is sufficiently small, the oscillation frequency of the driven system becomes entrained by the external signal. Often, outside of this main locking region, no further resonances exist. This is usually the case when the intrinsic oscillation frequency is very large compared to the width of the resonance region, and higher harmonic resonances are unsupported by the system. We refer to this type of locking as Adler-type, as such systems can be approximated by Adler’s equation:

\[
\frac{d}{dt} \phi(t) = -\nu - K \sin \phi(t) .
\]  

(1)

This equation describes phase-locking in a continuously driven oscillator close to resonance [7, 11]. Adler’s model exhibits phase-locking for sufficiently strong driving strength \( K \geq |\nu| \), in which the driven oscillator maintains a constant phase difference \( \phi \) relative to the external force. Beyond this, the only remaining solution is the drifting-phase solution, in which the amplitude of the phase difference monotonically increases in time and the average oscillation frequency approaches its free-running value for \( |\nu| \to \infty \). When instead a periodic, instantaneous forcing at discrete times is considered, of the form \( \sum_n \delta(t-nT) \sin \phi(t) \), Adler’s equation evaluated at the time of the \( n \)-th forcing reduces to the well-known circle map [7],

\[
\phi_{n+1} = \phi_n - \tilde{\nu} - K \sin \phi_n ,
\]  

(2)

where we define \( \phi_n := \phi(nT) \) and \( \tilde{\nu} := T \nu \). The circle map describes the dynamics of periodic systems with pulsed forcing, with a multitude of applications including neuronal [20, 21] and biological systems [22, 23]. Compared to Adler’s equation, the circle map exhibits very different locking scenarios, which we refer to as Arnold-type. In addition to the main resonance tongue, additional Arnold tongues appear at oscillation frequencies corresponding to rational ratios of the driving frequency \( T^{-1} \), leading to a devil’s staircase structure formed by the resonance tongues [24]. This devil’s staircase has been observed in a multitude of different physical and mathematical systems [25], including chemical oscillations [26], interfaces between crystalline solids [27], and particles in periodic potentials [28]. The richer synchronization dynamics of the circle map compared to Adler’s equation stem from the instantaneous driving term and the related discrete phase steps that allow the system trajectory to pass the original fixed points of Eq. (1) without travelling through the points. This leads to the emergence not only of a devil’s staircase locking structure, but also quasiperiodic and chaotic dynamics [29, 30]. In the continuously driven phase oscillator, a possibility to observe Arnold-type locking is the inclusion of a second driving frequency in the forcing scheme, introducing a periodic modulation to the external drive. Even then, an added nonlinearity is required in the coupling term or in the system dynamics. Such nonlinearities can be introduced by a variation of the oscillation amplitude due to the driving signal.

In this Letter, we thus study the emergence of Arnold-type locking in the frequency and amplitude in continuous systems which, with a simple coupling term, show only Adler-type locking around a single main resonance tongue. We show that the devil’s staircase structure emerges generically in all nonlinear oscillators close to Hopf bifurcations with periodically modulated forcing.
As an example for a driven nonlinear oscillator, we analyse the dynamics of a semiconductor laser subjected to a periodically modulated external optical input signal. Lasers traditionally have been a testbed for the study of driven nonlinear oscillations [31], showing a variety of different dynamical effects, including synchronization between coupled lasers [32–35], or locking to an external signal [36]. In externally forced lasers, devil’s staircase structures have been observed previously, when the laser is already in a periodic state [37, 38], or with strong pulsed forcing [39]. The simple harmonic coupling term $\sin \phi$ as in Eq. (1) naturally arises in the laser system driven by a constant optical field [40]. In fact, Adler’s equation is the low injection limit of the injected laser system [41, 42]. The laser system itself is nonlinear, which leads to rich dynamic scenarios in optically injected lasers, including high-order periodic, quasiperiodic, and chaotic dynamics [43, 44]. However, in the weak continuous wave injection regime, the locking to the driving signal is of Adler-type, comprising a single resonance tongue.

A possible pathway to Arnold-type locking is revealed by transforming the circle map (2) into an ordinary differential equation by expressing the Dirac comb $\sum_n \delta(t - nT)$ as a Fourier sum:

$$\frac{d}{dt} \phi(t) = -\nu - K \sin \phi(t) \times \sum_{n=-\infty}^{\infty} \cos(n\Delta t). \quad (3)$$

The periodic forcing in time-discrete maps can be understood as a forcing of the continuous system with an infinite number of driving frequencies, each spectrally separated by $\Delta := 2\pi/T$, and each leading to the appearance of a locking region around the respective spectral line. Nevertheless, Eq. (3) will lead to Arnold-type locking only when evaluating the infinite Fourier sum to a Dirac comb, introducing finite discontinuities in the trajectory $\phi(t)$. When restricting the forcing to any finite number of Fourier components, the resulting solution of $\phi(t)$ will remain continuous [45]. Thus, even though the explicit time dependence of the driving term adds an additional dimension to the system, the phase oscillator with a periodic driving will show only Adler-type locking around each of the spectral constituents of the forcing term. In this Letter we show however, that the combination of a finite number of injected Fourier components and intrinsic nonlinearities results in the emergence of a devil’s staircase structure with an Arnold-type locking.

In the case of optically injected lasers, the Fourier sum in Eq. (3) corresponds to the injection of an optical frequency comb, or, equivalently, a periodically modulated continuous wave signal. We thus employ a single-mode master laser and modulate its output using a Mach-Zehnder modulator driven by a 10 GHz sinusoidal signal. The resulting optical signal consists of three optical lines of nearly equal optical power, separated by the modulation frequency, which is then injected into a single-mode semiconductor laser. The driving signal corresponds to taking the three center terms of the Dirac comb with
n ∈ \{-1, 0, 1\} in Eq. (3). We control the frequency detuning between the slave laser and the injection signal via the laser mount temperature, which shifts the frequency of the slave laser cavity mode. We record the power spectra and optical spectra of the laser output under a continuous sweep of the slave laser detuning, shown in Fig. 1. The optical spectra in Fig. 1(a) show the three comb lines with a spacing of 10 GHz and the slave laser frequency, swept from a detuning of −20 GHz to 20 GHz. When the laser is tuned close to one of the three comb lines (vertical dashed lines in Fig. 1), its emission frequency becomes locked to that comb line. At the right edge within these locking regions, more complex features become apparent, with additional spectral components in between the comb lines, but with the dominant frequency component remaining locked to the comb. Between the main locking regions, we observe less pronounced locking regions to frequencies in between two adjacent comb lines. As the resolution of the optical spectrum analyser limits the visibility of these resonances, we additionally record the corresponding power spectra of the laser output, shown in Fig. 1(c). Across the whole detuning range, a strong spectral component at 10 GHz is visible, corresponding to the beating frequency between the injected comb lines. When the laser is locked to the individual comb lines, this beating frequency and its harmonics (not shown) are the only features in the spectrum. Increasing the detuning from these regions, a subharmonic locking to half the comb spacing at 5 GHz can be seen for the two leftmost locking regions. Subsequently, an unlocking and splitting of the signal at 5 GHz can be observed (e.g., between [−10 GHz, −8 GHz]). In these regions, the optical spectrum, however, still suggests a frequency locking of the laser to the respective comb line. In between the locking regions, additional spectral lines appear, increasing from zero frequency to 10 GHz and vice versa, while sweeping the detuning between the edges of adjacent locking regions. Within this unlocked detuning interval, quasiperiodic and chaotic dynamics can be observed, with intermittent harmonic locking to rational fractions of the comb frequency spacing. This structure in the power spectrum of the laser output closely resembles a devil’s staircase structure, with prominent spectral signatures at 5 GHz, 3.3 GHz, and 2.5 GHz, corresponding to \( \frac{1}{2}, \frac{1}{3}, \) and \( \frac{1}{4} \) of the comb spacing. For very large detuning beyond the locking regions of the outermost comb lines, no additional higher order resonances can be observed.

In order to perform a deeper analysis of the involved locking dynamics, we formulate a dimensionless rate equation model [41] with an injected optical signal, modeling the dynamics of the complex electric field inside the cavity, \( E(t) \), and the normalized optical gain \( N(t) \):

\[
\frac{d}{dt} E(t) = (1 + i\alpha) N(t) E(t) + \left. \frac{\partial E}{\partial t} \right|_{\text{inj}} + \sqrt{\beta} \xi(t) \tag{4}
\]

\[
T \frac{d}{dt} N(t) = J - N(t) - (N(t) + 1)|E(t)|^2 \tag{5}
\]

Here, \( J = 1.5 \) is the normalized pump current, \( T = 13.2 \) is the relative inversion lifetime, and \( \alpha = 3 \) is the amplitude-phase coupling parameter. The time \( t \) is given in units of the inverse optical cavity lifetime, \( \kappa = 60 \text{ ns}^{-1} \). The optical comb injected into the laser cavity is modeled by an additional driving term:

\[
\left. \frac{\partial E}{\partial t} \right|_{\text{inj}} = KE_0 (1 + 2m \cos(\Delta t)) - i\nu E(t), \tag{6}
\]
where \( m = 1.1 \) is the relative strength of the injected side-modes at frequencies ±\( \Delta \). The injection strength \( K \) is the amplitude ratio of the injected field and the free-running laser intracavity field \( E_0 \). The second term in Eq. (6) transforms the electric field into the rotating frame of the central comb mode, at a frequency detuning of \( \nu \) with respect to the free-running laser frequency [44]. We include the spontaneous emission noise inside the laser cavity by a \( \delta \)-correlated complex Gaussian white noise source term \( \xi(t) \), with a noise strength \( \beta = 4 \cdot 10^{-5} \).

The numerically calculated optical and power spectra of the laser emission under the external comb injection are shown in Fig. 1(b) and (d). The model closely reproduces the measured dynamics and locking behaviour, allowing us to study the observed resonance structure in more detail by numerical exploration of the parameter space. To this end, we calculate two-dimensional bifurcation diagrams in the detuning \( \nu \) and the injection strength \( K \). In Fig. 2, we evaluate the oscillation period \( T_{\text{per}} \) of the electric field amplitude \( |E(t)| \), as well as the average optical frequency, \( \langle \nu_{\text{opt}} \rangle := \langle d\phi/dt \rangle \), where \( \phi \) is the angle of the electric field variable in the complex plane. In the experimental optical spectra, \( \langle \nu_{\text{opt}} \rangle \) denotes the position of the dominating spectral line of the laser emission. In Fig. 2(b) we highlight areas where the lasing frequency changes little with the detuning, i.e., where it is locked. Both the amplitude oscillation period and the lasing frequency show pronounced Arnold tongues around the three comb lines, with higher harmonic locking tongues for detuning frequencies in between. In contrast to continuous wave injection, exact phase locking with a constant phase difference between the slave laser and injection signal is impossible in our setup, due to the time-varying injected signal. The frequency locking that we observe is therefore always a locking of the average optical frequency, while the phase is non-static. In lasers with single-mode injection a similar type of locking exists close to Hopf bifurcations [46–48]. Within the main resonance tongues, at small injection strengths, the light amplitude performs oscillations with a period equal to the inverse comb frequency \( (2\pi\Delta)^{-1} \) and the average lasing frequency is exactly the comb line frequency. In the higher harmonic \( q \) locking tongues, the average optical frequency is locked to the ratio \( q \) of the frequencies of neighboring comb lines, whereas the field amplitude exhibits periodic dynamics with a period \( T_{\text{per}} = q \cdot (2\pi\Delta)^{-1} \).

To illuminate the mechanism responsible for the devil’s staircase locking structure, we compare the dynamics of different periodically driven nonlinear oscillators. In addition to the driven phase oscillator, Eq. (3), we consider a periodically driven Hopf-normal-form oscillator for \( z \in \mathbb{C} \), generically describing the oscillations of all nonlinear systems close to a Hopf bifurcation [49]:

\[
\frac{dz}{dt} = \left[ (1-|z|^2)(1+i\gamma) - i\nu \right] z + K \left( 1+2\cos(\Delta t) \right). \tag{7}
\]

FIG. 3. Comparison of the locking behaviour of different driven systems, showing the average phase velocity for the circle map (2) (orange, \( K = 0.18 \)), the Hopf normal form (7) (green, \( K = 0.46, \gamma = 1, \Delta = 0.5 \)), and the laser equations (blue, \( K = 0.06, \beta = 0 \)). We plot subsequent curves with an artificial horizontal offset of 0.2 for better readability. The driven phase oscillator (3) (dashed line, \( K = 0.18 \)) is shown as comparison for each model.

Here, \( \gamma \) is the shear parameter, equivalent to the amplitude-phase parameter \( \alpha \) in the laser model. We perform one-dimensional parameter sweeps of the detuning \( \nu \) for the circle map (2), the Hopf normal form (7), the phase oscillator (3), and the laser model. We evaluate the average phase velocity, \( \langle d\phi/dt \rangle \), along a sweep of the detuning between two adjacent main resonance tongues, as shown in Fig. 3. The discrete circle map shows consecutive areas of locking to harmonics of the driving frequency, forming the well-known devil’s staircase structure. The time-continuous Hopf normal form and laser models closely reproduce this structure. In the parameter region for \( \nu/\Delta > \frac{1}{2} \), we observe a complete devil’s staircase, with consecutive locking regions at most rational frequency ratios. For \( \nu/\Delta < \frac{1}{2} \), the shear parameter in the time-continuous models leads to a slight asymmetry in the dynamics and a breakup of the staircase. The Hopf normal form thus forms a “harmless” staircase, with discontinuous jumps (cf. [27]). We note that for different choice of parameters, even without shear \( \gamma = 0 \) in Eq. (7)), the devil’s staircase structure persists, albeit less pronounced. While the phase-amplitude coupling in the investigated semiconductor laser leads to pronounced Arnold tongues, this phenomenon can therefore also be observed in oscillators without a comparable shear mechanism. The phase oscillator model (dashed line), on the other hand, shows only Adler-type locking, i.e., a locking to the main resonance tongues around \( \nu = 0 \) and \( \nu = \Delta \), and only a smooth transition of the oscillation frequency in between the main resonances. The devil’s staircase structure observed for the other continuous oscillators is thus a direct consequence of the amplitude dynamics, via both the explicit nonlinearities and the dynamic variation of the relative injection strength when the os-
oscillation amplitude changes. Considering the comparison of continuous systems with only three driving frequencies on the one hand, and the time-discrete circle map with an infinite number of driving frequencies on the other, the similarity of the locking dynamics is remarkable.

In conclusion, we have analysed the transition from Adler-type locking to Arnold-type locking, i.e., the emergence of a devil’s staircase structure in oscillators with continuous periodic forcing. We have shown using both theory and experiment that the periodic modulation of the driving force can introduce higher order locking tongues in systems that show only a single main resonance tongue for constant driving. In particular, we studied the case of a periodically driven semiconductor laser, which shows only a single main resonance tongue when subject to constant optical injection. With a modulated driving signal, we observe a complete devil’s staircase in both the oscillation period of the amplitude and the average lasing frequency. The harmonic resonances in the lasing frequency are induced by the nonlinear amplitude variations, and disappear when neglecting the field amplitude dynamics. We remark, however, that although the locking of the oscillation amplitude is necessary for the devil’s staircase in the frequency, the frequency locking ranges extend beyond the amplitude-locked parameter regions. This can be seen in the unlocked (white) regions in Fig. 2(a), which nevertheless remain frequency-locked to the harmonics of the comb frequency. Our results show that by proper implementation of an external drive, additional regions of stable devil’s staircase frequency locking can be introduced in systems which originally show only a single Adler-type resonance tongue. These higher order tongues may be of technological interest for applications where they could increase the density of injected optical combs [50], among other possibilities. We reveal that the investigated mechanism creating the devil’s staircase in time-continuous systems is generic. By demonstrating in simulations that it arises via the periodic driving of a Hopf-normal-form oscillator, we prove that Arnold-type frequency locking appears in every nonlinear oscillator with periodically modulated forcing close to a Hopf bifurcation.

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