Radar Information Theory for Joint Communication and Parameter Estimation with Passive Targets

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Abstract—In this paper, we derive the performance bounds for joint communication data rate and estimation error of radar target parameters from first principles, where the targets are assumed to be passive. Specifically, we let the targets to have control over their passive reflectors in order to transmit their own information back to the radar via reflection-based beamforming or backscattering. Such a setup avoids active radio frequency transmission from battery operated devices such as friendly (or reconnaissance) drones. The concept of target ambiguity function arises naturally from these derivations, which not only poses challenge to waveform designers, but also provides an opportunity for a joint design of waveform and array geometries to achieve an optimal performance. We derive the Cramér-Rao lower bounds for the mean squared error in the estimation of target parameters, and derive lower bounds on the data rates with both radar-only and joint radar and communications scenarios. The challenge of transmit waveform design for joint radar-data communication is illustrated via numerical examples.

Index Terms—Cramér-Rao lower bounds, joint radar and data communications, passive targets, radar information theory, reflecting surfaces, target ambiguity functions.

I. INTRODUCTION

A. Background and Motivation

The tremendous increase in demand for wireless applications and a sharp rise in the number of connected devices has caused severe spectrum shortage and congestion of existing bands. Among several communication systems with under-utilized spectra, radar systems stand out, which have been allotted a large portion of spectral resources. Towards this end, research attention on coexistence of other communication systems in radar bands has significantly grown in the past decade or so. For instance, DARPA funded a research project called shared spectrum access for radar and communications (SSPARC), for eliciting technologies where both radar and communication can co-exist [1]. Joint radar and communications (JRC) finds its utility in several civilian applications such as millimeter wave communications, WiFi-based localization, unmanned aerial vehicles communications, RFID, etc., and also in military applications. Excellent recent surveys on applications and future research directions on JRC are given in [2]-[4]. As discussed in [4], there are two main research directions in JRC, namely (i) radar-communication coexistence (RCC), and (ii) dual-functional radar-communication (DFRC) systems [2], [5]. The goal in RCC is to design efficient interference management techniques for coexistence of a communication system in the radar band ([6]–[12]). On the other hand, the goal in a DFRC system is joint design of sensing and signaling operations through a single hardware for both radar and communications in applications such as indoor radars ([13]-[15]) and radars for vehicular networks ([16], [17]).

Recent radar applications such as automotive radars [18], [19] and drone tracking [20], [21] have introduced new set of challenges and design considerations for spectrum sharing in JRC. Tracking of unmanned aerial vehicles (UAV) and drones finds applications in security and surveillance [22]. Latest developments in joint data transmission and target parameter estimation, by resource sharing, have added new dimensions to this problem [23]. In many such applications, it is necessary to identify a UAV as a friend or a foe, to ensure secure communication [24]. Since UAVs such as drones operate on battery power, passive communication over active is preferred from a UAV to a JRC transceiver [25], [26], for the following two reasons. First, this enables a UAV to hide its presence and identity from adversaries. Second, employing technologies such as opportunistic ambient backscattering communication [27] and intelligent reflecting surfaces [28], [29] that require only a small fraction of power of active transmission, enables the battery power of a UAV to be utilized in an efficient manner.

To the best of our knowledge, a study on the fundamental limits on the performance of JRC with passive targets in terms of joint parameter estimation and communication data rates has not been addressed in literature so far. In this case, crucial design trade-offs between the performance of parameter estimation and data transmission, must be derived and studied from first principles, based on fundamental performance metrics such as mutual information achievable over the JRC communication links and limits on the target parameter estimates, in terms of the Cramér-Rao lower bounds (CRB).

B. Related Work

Studies on radar information theory started with the seminal works by Woodward [30]–[33], and Davies [34]. Woodward and Davies showed that an a posteriori probability receiver is optimal, which reduces to the correlation receiver in the case of additive white Gaussian noise. In particular, as opposed to the signal-to-noise (SNR) maximization, the goal is to maximize the a posteriori probability, which improves the detection probability. The Wigner-Ville transformation was proposed as the unnormalized ambiguity function, which was used to analyse...
the simultaneous range and velocity estimation resolution \cite{35}. Klauder used this ambiguity function to design better radar waveforms which results in improved simultaneous range and velocity resolution, although their practical use case is limited \cite{36}, \cite{37}. The goal in these early works on information theory for radars was information theoretic formulation in waveform design, to improve the performances in terms of target detection and parameter estimation \cite{35}, \cite{38}, \cite{39}. Frost and Shanmugan computed the information content in the synthetic array radar images \cite{40}. The relationship between the mutual information between the target parameters of interest and the transmitted signal, and accuracy of parameter estimation was exploited in \cite{38}, to design optimal radar waveforms. Furthermore, parameterization of targets based on their respective target impulse responses, allows one to design such optimal transmit waveforms. Transmit waveform optimization for frequency diverse array (FDA) radar was considered in \cite{41}, where target localization was done by analyzing the CRBs, and the knowledge of location of the targets was used to design the transmit waveform for FDA and to create range-angle dependent beams. On similar lines, transmit array sub-aperturing was employed in FDA radar \cite{9}, where each sub-array was assigned with different carrier frequencies and their weights were chosen adaptively via cognitive beamforming.

Even in the context of radars, the mean squared error (MSE) for parameter estimation is well-studied in literature. In \cite{42}, CRB for the target range, velocity, and angle-of-arrival (AoA) with a narrow band assumption on the transmit waveform was derived. Further, it was shown that the CRB for AoA is independent of the delay and velocity parameters. Moreover, CRB for AoA was shown to be a function of sensor locations only, through the moment-of-inertia parameters of the sensory array. It was noted in \cite{43} that the CRB and ambiguity functions were related to each other and impact the performance of parameter estimation, as follows – “The ambiguity function establishes global conditions under which the local bounds are accurate predictions of the expected error performance and identifies the regions of the parameter space where large errors may occur.” \cite{43}. Exploiting this key result, we extend the notion of ambiguity functions to wideband signals and general array geometries for our problem at hand.

Bounds on data rate between a JRC receiver and a communication transmitter was studied in \cite{44}, where the radar aids to relay the data from the communication transmitter to the receiver which is not co-located with the radar. This system follows a DFRC model, in which the radar performs joint data decoding and parameter estimation of targets using the combined received signal and then modulates the information onto the radar signal. Thus, the communication and radar receivers are able to decode the message and estimate the target parameters, respectively. Further, the data rate and parameter estimation rate were balanced similar to the case of a two user multiple access channel at the radar receiver, and the corresponding rate region was derived. In \cite{34}, a massive MIMO DFRC architecture was proposed, that employs hybrid beamforming to enable joint radar tracking and data communication. A radar transmission signal was designed using orthogonal frequency division multiplexing (OFDM) in \cite{45}. Here, uncertainties in the received data due to data symbols at the radar were mitigated, by dividing the target echo signal by a priori known or decoded data symbols in the frequency domain. In this setup, the transmitter and receiver are not co-located as in the case of bistatic radars. In \cite{11}, a joint waveform design using preambles of OFDM signal as radar detection waveforms were employed to achieve a trade-off in the performance of data transmission and parameter estimation. Even though the usage of communication preambles as radar waveforms allows one to reuse the spectral and temporal resources, it results in non-optimal radar performance due to poor ambiguity function in the transmit waveform. For the same duration of a preamble, multiple smaller periodic waveforms can be designed to achieve a better performance, especially when the targets move rapidly.

### C. Contributions

In this work, we derive lower bounds on the mutual information for the forward and reverse channels between a radar transceiver and targets, and derive the CRB on the minimum MSE of target parameters, namely range, velocity and AoA. We consider generic linear models for forward and reverse channels and target response function, for analytical tractability. The proposed model accommodates both narrow-band and wideband transmit signals, any receiver antenna array geometry, and applicable for both monostatic and bistatic radars. Towards this end, we derive analytical expressions for lower bounds on data rates and parameter estimates for the cases of (a) radar-only, (b) communication-only, and (c) joint radar and communication scenarios, from the first principles. In particular, we compute the Fisher information matrix (FIM) to further calculate the CRB. Motivated by the relationship between CRB and the ambiguity function studied in \cite{43}, we propose novel target ambiguity functions (TAF) for each target parameter, which can be optimized to achieve the corresponding CRB. We illustrate the utility of the proposed framework through applications, which includes (a) optimal waveform design for JRC that minimizes TAFs, (b) resource sharing between radar and communication, in terms of spectral and temporal overlaps in radar transmission. To summarize, the main contributions of this paper are given below.

- From first principles, we derive lower bounds on data rates over the forward and reverse channels between a JRC transceiver and a passive target. The target backscatters the signal received from the radar transmitter, and the lower bound of the data rate on the reverse channel is shown to be a function of the backscattered signal strength, which depends on the radar cross section (RCS) and the target response function.
- We derive the CRB on the minimum MSE of the estimates of relevant target parameters, namely range, velocity and azimuth angle, for radar-only and JRC scenarios. Further, we show that data communication on the forward channel does not affect the CRB on each of these target parameters. However, data communication on the reverse channel has an adverse effect on parameter estimation, in the presence of decoding errors.
We present novel TAFs for range, velocity and azimuth which embed other parameters such as elevation angle, target impulse responses and array geometry.

Through numerical examples, we apply the proposed TAFs for parameter estimation by designing a transmit waveform for radar-only and JRC scenarios. Further, we revisit the analysis on some of the well-known waveforms in the context of TAFs. Moreover, we propose a practical Gaussian code book construction method for JRC.

D. Organization

The remainder of the paper is organized as follows. In Section II, we discuss the data model and formulate the joint radar and communications problem with passive targets. In Section III, we derive lower bounds on communication data rate for forward and reverse channels between the radar transmitter and receiver. In Section IV, we computes the CR lower rate for forward and reverse channels between the radar transceiver and a single passive target. Concluding remarks are provided in Section VI.

II. DATA MODEL AND PROBLEM FORMULATION

A. Radar / Communication Transmit Signal

Consider a radar system as depicted in Fig. 1 consisting of a transceiver equipped with a single antenna joint radar communication transmitter and a joint radar and communication receiver with L antennas. The radar transmits a signal \( x_i(t), i = 1, \ldots, N \) with \( N \) pulses of duration \( T_s \). The average transmit power is \( P_{\text{avg}} \), and the total transmit power \( P_T = N P_{\text{avg}} T_s \). The baseband component of the signal \( x_i(t) \), denoted by \( x_i(t) \), is assumed to be band-limited to \( B \) Hz. In other words, magnitude spectrum of \( x_i(t) \), denoted by \( |X_i(f)|^2 \), is positive for \( |\omega| \leq B \), and zero otherwise. Throughout this paper, we consider our analysis on \( x_i(t) \), even though all the channel effects such as Doppler shift, target impulse response, etc. are applied on \( x_i(t) \). We assume that the transmitted signal comprises of \( N \) Gaussian pulses with \( BT_s > 1 \) \( \iff \). That is,

\[
x_i(t) = \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{\pi^2 (t - iT_s)^2}{\beta^2}\right), \quad -\frac{T_s}{2} \leq t \leq \frac{T_s}{2}.
\]

where \( \beta \) determines the 3 dB bandwidth \( B \) of the pulse. The magnitude of the Gaussian pulse in frequency domain representation can be written as

\[
|X_i(f)| = \exp\left(-\beta^2 f^2\right).
\]

Following [38], we assume that the characteristics of \( k \)-th target is captured in its random target channel response, \( g_k(t), k = 1, \ldots, K \), with the following assumptions.

(i) Finite energy: \( g_k(t) \) satisfies \( \int_{-\infty}^{\infty} |g_k(t)|^2 dt < 1 \), where \( E(X) \) denotes the expectation of a random variable \( X \). We assume that the response \( g_k(t) \) does not include parameters such as the reflection coefficient and the effective radar cross section (RCS) of the \( k \)-th target.

(ii) Causality: \( g_k(t) \) is causal, i.e., \( g_k(t) = 0, \ t < 0 \).

(iii) Fourier transform of \( g_k(t) \), denoted by \( G_k(f) \), exists.

(iv) Independence: \( g_k(t) \) is independent across \( k = 1, \ldots, K \), and is independent of \( x_i(t) \), for \( i - 1, \ldots, N \).

A random code \( c \) of length \( N \) modulates the Gaussian pulses before transmission, which is determined by \( \{a_i, i = 1, \ldots, N\} \), and is assumed to be an element of a set \( \mathcal{C} \). Therefore, a total of \( NB T_s \) resource dimensions are available to be shared between radar and communications. The code \( c \) is agnostic to the modulating source, that is, it can be applied either by the transmitter, by the passive targets, or both. In case of codes used by both the transmitter and the receiver, we assume that a combination (Cartesian product) of transmit and receive codes come from an ensemble \( C = C_r \times C_t \). Accordingly, the length of the radar source and target codes will be reduced to \( N_r \) and \( N_t \) respectively, such that \( N = N_r + N_t \). Furthermore, if the transmitter uses all \( N \) pulses, we assume that \( c \) is an \( N \)-dimensional Gaussian random vector with zero mean vector and a covariance matrix \( \sigma_c^2 I_N \), where \( I_N \) denotes the identity matrix of size \( N \). The transmitter uses the variance \( \sigma_c^2 = \sigma_t^2 = P_{\text{avg}} \). Therefore, the information in each pulse is given by \( \frac{1}{2} \log(2\pi e \sigma_c^2) \) nats. The end-to-end system model is described in Fig. 2 based on which we next present the models for the reflected signal from \( K \) passive targets.

B. Reflected Signal from Passive Targets

Fig. 2 shows the linear model of the forward and reverse channel between the radar transmitter and \( k \)-th target as well the target response channel. First, let us consider a single antenna at the radar receiver. The reflected signal from \( k \)-th target for the \( i \)-th pulse can be written as

\[
w_{k,i}(t) = a_i \alpha_k^{(i)} \gamma_k \int g_k(t') x_i \left(t - t' - \tau_k^{(i)}\right) dt',
\]

where \( \gamma_k \) is the propagation delay in the forward channel from the transmitter to the target, \( \alpha_k^{(i)} \) \( \iff \) \( e \) \( \iff \) \( \tau_k^{(i)} \) denotes the pathloss in the forward channel with \( e \) as the pathloss.
\[ z_i(t) = a_i \sum_{k=1}^{K} \alpha_k^2 \gamma_k w_{k,i} (t - \tau_k) + a_i \alpha_k \sum_{j \neq k} \alpha_j \gamma_j w_{j,i} (t - \tau_j) + j_i(t) + n_i(t), \]  

(9)

Interference for \( k \)th target

exponent, and \( \gamma_k \) denotes the reflection coefficient for \( k \)th target. For convenience, this continuous time system can be approximated by a discrete model as

\[ \int g_k(t') x_i (t - t' - \tau_k^{(i)}) dt' \approx \sum_{p=1}^{D} \zeta_{k,p} \delta (t - t_{k,p}), \]  

(4)

with \( D \) denoting the number of delay taps, and \( \zeta_{k,p} \) being the amplitude in the \( p \)th tap due to \( k \)th target. The signal received at the radar due to non-moving targets can be written as

\[ z_i(t) = a_i \alpha_k^2 \sum_{k=1}^{K} \gamma_k w_{k,i} (t - \tau_k^{(i)}) + j_i(t) + n_i(t), \]  

(5)

where \( \tau_k^{(i)} \) and \( \alpha_k^{(i)} \) denote the propagation delay and the pathloss factor in the reverse channel, \( n_i(t) \) denotes the additive white Gaussian noise (AWGN) with zero mean and unit variance, and \( j_i(t) \) denotes the jamming signal. For the ease of analysis, we assume that the transmitter and receiver are collocated, in which case \( \tau_k = \tau_k^{(i)} = \tau_k \) and \( \alpha_k^{(i)} = \alpha_k = \alpha_k \).

Therefore, (5) simplifies to

\[ z_i(t) = a_i \left( \sum_{k=1}^{K} \alpha_k^2 \gamma_k w_{k,i} (t - \tau_k) + j_i(t - \tau_k) \right) + n_i(t). \]  

(6)

On the other hand, if the \( k \)th target is moving at a constant speed \( v_k \) radially from the radar transmitter, we can write

\[ z_i(t) = a_i \left( \sum_{k=1}^{K} \alpha_k^2 \gamma_k w_{k,i} ([t - \tau_k] T_k) \right) + j_i(t - \tau_k) + n_i(t), \]  

(7)

where

\[ T_k = \sqrt{\frac{c + 2v_k}{c - 2v_k}} \approx \frac{c + v_k}{c - v_k}, \]  

(8)

denotes the time-axis compression factor due to the Doppler shift in frequency from a moving target.\(^3\) Now, separating the reflected signal from \( k \)th target, we get (9), which is given in the top of this page. Note that the reflection from each target may undergo different time compression, depending on the velocity of the target. Moreover, as discussed earlier, it is possible that some of the transmitted data pulses from these passive targets can be used for sending data information, while the remaining can be used to detect the presence of a target and estimate its parameters. The trade-off between the performance of radar target parameter estimation and the data transmission is viable by proportioning the resources such as energy, bandwidth and codes. In this sense, it is not necessary that all the \( N \) transmitted pulses need to be identical. However, all possible realizations of \( x_i(t) \) comes from the finite combinations of waveforms selected from two ensembles; one corresponding to radar parameter estimation and another for data transmission.

Next, we consider the general case where the receiver has \( L \) antennas, and the received signal vector can be written as

\[ z_i(t) = a_i \mathbf{X}_i(t) \mathbf{\Gamma}_i(t) + n_i(t), \]  

(10)

where \( a_i \) is a random variable representing amplitude of the \( i \)th Gaussian pulse, \( \mathbf{X}_i(t) \) is a \( L \times K \) matrix, whose \((l,k)\)th entry is given by

\[ X_{i(l,k)}(t) = x([t - i T_s - 2 \tau_k - \varphi_l] T_k), \]  

(11)

for \( k = 1, \ldots, K, \ l = 1, \ldots, L, \) where \( \varphi_l \triangleq \frac{p_{l}^T p_l}{p_{l}^T p_i} \), with \( p_l \) denoting the relative Cartesian coordinates – with respect to a reference element in the receive antenna array as origin – of the \( l \)th antenna element and \( p_k \) denoting the direction cosines of the \( k \)th target with respect to the antenna. Using (4), the \( k \)th entry of the \( L \)-length vector \( \mathbf{\Gamma}_i(t) \) can be written as

\[ \mathbf{\Gamma}_i^{(k)}(t) = \gamma_k \alpha_k^2 \sum_{p=1}^{D} \zeta_{k,p} \delta (t - t_{k,p}). \]  

(12)

The \( L \)-length vector \( n_i(t) \) is a spatio-temporally white Gaussian random vector with mean being the zero vector and covariance matrix \( \sigma_n^2 \mathbf{I}_L \). The mean vector of \( z_i(t) \) for all \( i = 1, \ldots, N \) is also assumed to be the zero vector. In the next section, we derive the lower bounds on achievable data rates in the forward and reverse channels based on the above model for the reflected signal.

III. BOUNDS ON ACHIEVABLE DATA RATES

A. Mutual Information of Forward Channel

Consider the forward channel between the transmitter and \( k \)th target. Let \( \mathbb{E}[a_i^2] = \sigma_x^2 = P_{\text{avg}} \). The reflected signal \( w_{k,i}(t), \ i = 1, \ldots, N, \) from the target can be written as

\[ w_{k,i}(t) = a_i \alpha_k \gamma_k \int_0^{T_k} g_k(t') x([t - i T_s - \tau_k - t'] T_k) dt' + j_{k,i}(t) + n_i(t), \]  

(13)

where \( j_{k,i}(t) \) is the interference for \( k \)th target. For the ease of analysis, we assume that \( j_{k,i}(t) \) is a Gaussian random process with zero mean and variance \( \sigma_j^2 \). However, the following

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\(^3\)Note that this is a generic model which includes narrow-band and wide-band transmitted signals. Moreover, in the case of multiple antennas at the receiver, this model is independent of the array geometry. It is easy to show that for \( v_k \ll c \), the difference between the observed and actual frequencies simplifies to \( f_0 - f = (1 - \frac{1}{v_k}) f \approx \frac{v_k}{c} f. \)
analysis can be extended to different statistical models for clutter and jammer, such as Gamma or Rayleigh, by suitable modifications. Now, the signal-to-noise ratio (SNR) due to the reflected signal from the $i^{th}$ target can be written as

$$\rho_k = \frac{(\sigma_x \alpha_k)k^2 \mathcal{G}_k}{\sigma_j^2 + (\sigma_n(k))^2 + \sigma_x^2 \sum_{j \neq k} \mathcal{G}_j \left[ \frac{\alpha_j}{\alpha_k} \right]^2},$$  \hspace{1cm} (14)

where

$$\mathcal{G}_k \equiv \int_{-B}^{B} \left| X_i \left( \frac{\omega}{\omega_k} \right) G_k(\omega) \right|^2 d\omega$$  \hspace{1cm} (15)

denotes a factor that determines the variance of the reflected signal due to target channel response [46] and $(\sigma_n(k))^2$ denotes the noise variance at the $k^{th}$ target’s receiver. Assuming that the $N$-length code $c$ is applied at the transmitter, recall that the differential entropy in the transmit waveform is given by $\frac{N}{2} \log(2\pi e \sigma^2)$ nats. Additional scaling factors due to path loss and reflection coefficients only affect the variance of the reflected signal from the targets. For this reason, the strength of the reflected signal is taken as the received signal power in the reverse channel. In the next section, we consider the radar performance and derive lower bounds on the MSE of the parameter estimates, namely range, velocity and azimuth.

IV. BOUNDS ON PARAMETER ESTIMATION

A. Radar-Only Transmission

The Cramér-Rao lower bounds (CRB) on the variance of the radar target parameters in our setup can be obtained by finding the inverse of the Fisher information matrix (FIM). Following [10], recall that the $L$-length vector $z_i$ follows a Gaussian distribution with mean vector $a_i X_i \Gamma_i$, and covariance matrix $R_n \equiv \mathbb{E}(n_i n_i^T)$, independent across $i = 1, \ldots, N$. Let $\Theta \equiv [r \ v \ \phi]^T$ denote the vector of desired parameters, and $r_k, v_k$ and $\phi_k$ denote the $k^{th}$ entries of vectors $r, v$ and $\phi$, respectively, which correspond to the range, Doppler and azimuth angle of the $k^{th}$ target.

Towards this end, the second order partial derivatives of the logarithm of joint PDFs of $z_i, i = 1, \ldots, N$ computed with respect to the parameters $r_k, v_k$ and $\phi_k, k = 1, \ldots, K$ are derived in Appendix A and the final expressions are given in Eqs. (18)-(20) at the top of next page. Here, $\circ$ denotes the Hadamard (or element-wise) product of two matrices of same size, $\rho_t = \left( \frac{\beta \alpha_i}{\sigma_n} \right)$ is the transmit SNR, $\beta$ is the parameter defined in [1], $c$ is the speed of light, $x_{i,k}$ is the $k^{th}$ column of $X_i$, and $\Phi_k$ is defined as

$$\Phi_k \equiv \left\{ \sin \theta_k (y_i \cos \phi_k - x_i \sin \phi_k) + z_i \cos \theta_k \right\},$$  \hspace{1cm} (21)

for $i = 1, \ldots, L$ is a vector of dimension $L \times 1$ The Cartesian coordinates of the $i^{th}$ receive antenna is denoted by $(x_i, y_i, z_i)$ and $t_i$ is the vector of relative time instants at which the wavefront from $k^{th}$ target hits the receiver antenna elements.

\footnote{Apart from the desired parameters, there are also nuisance and unknown parameters such as elevation angle $\theta$, random code vector $c$, i.e., the vector of random amplitudes $a_i$s and tap coefficients and amplitude of target impulse response function. Tighter CR bounds can be computed when we condition them out in the FIM computation, which is beyond the scope of this paper.}
It can be observed that the lower bound on the MSE across all the parameters depend on the key factors such as $N$, $\rho_s$, RCS value $\mathbb{E}[\gamma_k^2]$, extension nature of the target $\mathbb{E}[||\zeta_k||^2]$, and the target ambiguity functions (TAF). Therefore, by designing a radar transit waveform which minimizes the TAFs, lower bounds on the minimum MSE can be written as

$$\mathbb{E}(r_k - \hat{r}_k)^2 \geq \frac{-1}{\mathbb{E}\left[\frac{\partial^2 \log f_k(z_k)}{\partial \zeta_k^2}\right]}$$ (18)

$$\mathbb{E}(\tau_k - \hat{\tau}_k)^2 \geq \frac{-1}{\mathbb{E}\left[\frac{\partial^2 \log f_k(z_k)}{\partial \gamma_k^2}\right]}$$ (19)

$$\mathbb{E}(\phi_k - \hat{\phi}_k)^2 \geq \frac{-1}{\mathbb{E}\left[\frac{\partial^2 \log f_k(z_k)}{\partial \phi_k^2}\right]}$$ (20)

Note that the cross terms in the FIM are ignored to derive the lower bounds in (18)-(20). The limits seen in practice may be higher (or tighter), due to the influence of non-diagonal terms on the inverse of FIM. In general, the achievable CRBs (local bounds) are influenced by the ambiguity functions (global conditions), as observed in [48]. That is, if one designs waveforms that achieve the least possible ambiguity functions, the corresponding CRBs listed in (18)-(20) becomes achievable, which otherwise may result in a loose bound. Therefore, the approach considered here – optimizing the ambiguity functions – is flexible, in the sense that the performance of the radar can be designed by individual optimization of the TAF for range, velocity and angle, for selected set of targets within the given range, velocity or angle-of-arrival. This is advantageous in contrast to the conventional ambiguity functions defined to optimize all parameters using one transmit waveform function.

1) MSE on Range Estimation: Note that, the CR lower bound for the range estimation given in Eqn. (18), decreases with transmit SNR, code length of the code $N$, RCS value $\mathbb{E}[\gamma_k^2]$, mean energy in the target impulse response $\zeta_k$ as well as the effective BW of the signal $\int_{-\infty}^{\infty} \left\{\frac{d^2 x(t)}{dt^2}\right\}^2 dt$ as observed in [48]. Another interesting observation is that, the MSE bound in range increases with higher velocity (i.e., when $T_k < 1$, but not by a great deal for small velocities although it increases as $1/T_k^4$. The $\beta$ parameter is inversely proportional to bandwidth of the signal and hence MSE bound decreases with fourth power bandwidth of the signal, which is an interesting observation from this analysis.

2) MSE on Velocity Estimation: From Eqn. (19), it can be observed that, the velocity estimation also improves with the typical parameters such as transmit SNR, mean RCS value, etc. The key difference is observed in the new term $\int_{-\infty}^{\infty} \left\{\frac{d^2 x(t)}{dt^2}\right\}^2 dt$ in place of the effective bandwidth term in Eqn. (13). With large bandwidth, this quantity will increase exponentially and hence it improves the velocity estimation provided, the wide-band array configuration is used as in the case of STAP filters [49]. That is, narrow-band pulses or shortening the array aperture to meet the narrow-band conditions, greatly impact the velocity and angle estimation.

3) MSE on Angle Estimation: Apart from the effect from the expected parameters, the key difference in the influence of parameters on the MSE comes from the $\Phi_k$ which brings in the influence of array geometry on the performance (See Eqn. (20)). The nature of influence can be understood by studying the angle TAF defined in Appendix A. Also, note that the absence of array geometry parameters in the range and velocity MSE bounds directly, but it appears indirectly in the range TAF and velocity TAF functions. This is also an interesting result from the analysis in contrast with existing results (e.g. [42]), where the array geometry does not play any role in the CR bounds on the range and velocity estimation and bound on the angle estimation error does not depend on the range and velocity estimation error.

B. Joint Radar and Communication

It can be observed from rate equations (16) and (17) that the bounds on data rates and MSE are linear functions of the length of the transmission, $N$. Therefore, by dividing the total number of pulses per frame into $N = N_d + N_t$, where $N_d$ pulses are used for data transmission and $N_t$ pulses are used for target parameter estimation, a trade-off on the achievable data rate and parameter estimation error can be studied. The corresponding bounds on data rate and MSE can be easily computed.

1) Joint Radar and Communication on the Forward Channel: Consider the scenario where radar pulses are modulated by the $N$-dimensional Gaussian code designed in Appendix B. At the radar receiver, the amplitudes $\{a_i, i = 1, \ldots, N\}$ are known and hence can be used in the parameter estimation procedure. The targets sample the peak of Gaussian pulses at the periodic intervals of the received signal and compare them with the scaled codebook for the final decision. The bit error rate (BER) depends on the noise at the target receiver, jamming signal level and the interference due to other nearby targets. The previously computed data rate bounds and the radar parameter estimation bounds are applicable even in this case. This result is in contrast with the work in [44], where the radar and the communication transmitter and receiver are not

\[ E\left[\frac{\partial^2 \log f_k(z_k)}{\partial \zeta_k^2}\right] = \frac{-16 \rho_s N \pi^2 T_k^4 \mathbb{E}[\gamma_k^2]}{\beta^4 \epsilon^2} \mathbb{E}[\|\zeta_k\|^2] \epsilon^4 r_k^{-4c} E\left(\|x_{i,k} \circ t_k\|^2\right). \] (18)

\[ E\left[\frac{\partial^2 \log f_k(z_k)}{\partial \gamma_k^2}\right] = \frac{-16 \rho_s N \pi^2 \epsilon^2 T_k^2 \mathbb{E}[\gamma_k^2]}{\beta^4 (c - \nu_k)^2} \mathbb{E}[\|\zeta_k\|^2] \epsilon^4 r_k^{-4c} E\left(\|x_{i,k} \circ t_k\circ t_k\|^2\right). \] (19)

\[ E\left[\frac{\partial^2 \log f_k(z_k)}{\partial \phi_k^2}\right] = \frac{-4 \rho_s N \pi^2 T_k^2 \mathbb{E}[\gamma_k^2]}{\beta^4} \mathbb{E}[\|\zeta_k\|^2] \epsilon^4 r_k^{-4c} E\left(\|x_{i,k} \circ t_k \circ \Phi_k\|^2\right). \] (20)
co-located. The parameter estimation at the joint data-radar receiver and information transmitted to the communication receiver using data pulses which are overlaid with the radar pulses are treated as part of a two user MAC channel and bounds can be computed \cite{47}.

2) Joint Radar and Communication on the Reverse Channel: In this case, radar transmits constant modulus pulses and the targets modulate them using their back-scattering control elements. Let \( \{ b_{ki}, i = 1, \ldots, N \} \) denote the reflection amplitudes due to data modulation from the \( k \)th target to the radar receiver. The radar-data receiver compares the amplitude of the pulses at the peaks of the received signal and uses them to compare with the scaled codebook made known to it, apriori. The Euclidean distance is computed for minimum distance decoding. The radar receiver then compensates for the amplitude variation in the received pulses as per the decoded data. Thus, amplitude corrected data is used to estimate the target parameters. To obtain the parameter estimation bounds for this joint data-radar case, we need to also compute the partial derivatives for the for the unknown data symbols \( b_{ki} \) and use them along with the derivatives for the radar parameters to invert the larger FIM. The second derivative with respect to the symbols \( b_{ki} \) can be written as

\[
E \left[ \frac{\partial^2 \log f_x(z_i)}{\partial b_{ki}^2} \right] = -\rho T E[b_{ki}^2] E[\gamma_k^2] E[\| \zeta_k \|^2] 
+ \epsilon^4 r_k^4 e^{4\epsilon} E[\| x_{i,k} \|^2], \tag{25}
\]

and the cross-term second order derivative can be written (only one parameter is shown here for illustration) as

\[
E \left[ \frac{\partial^2 \log f_x(z_i)}{\partial b_{ki} \partial r_k} \right] = -\rho T 4\pi^2 E[b_{ki}^2] T_s^2 E[\gamma_k^2] E[\| \zeta_k \|^2] 
+ \epsilon^4 r_k^4 e^{4\epsilon} E[(x_{i,k} \circ t_{ik})^T x_{i,k}]. \tag{26}
\]

It can be noticed that the cross-terms are not zero and hence it impacts the inverse of FIM and the MSE bounds. That is, the decision on each and every bit transmitted by the individual targets impact the performance all radar parameters for all targets. Even for a single target, the FIM for all parameters and data does not simplify and all data decision errors impact the performance of the radar. Hence, some level of orthogonality in resource allocation is a must for getting good performance in terms of bit error rate (BER) and parameter estimation error, where previously computed data rate bounds and parameter estimation bounds remain valid.

V. NUMERICAL EXAMPLES

A. Transmit Waveform Design

From the analysis in Secs. III and IV, we intend to maximize of the average norm \( E[\| x_{i,k} \|^2] \) and minimize the following three inner products

(a) \( E[(x_{i,m} \circ t_m \circ x_{i,n} \circ t_n)]^T R_n^{-1}(x_{i,n} \circ t_n) \),
(b) \( E[(x_{i,m} \circ t_m \circ x_{i,n} \circ t_n)]^T R_m^{-1}(x_{i,n} \circ t_n) \), and
(c) \( E[(x_{i,m} \circ t_m \circ \Phi_m)^T R_n^{-1}(x_{i,n} \circ t_n) \circ \Phi_m] \),

simultaneously, for \( m, n = 1, \ldots, K \) and \( n \neq m \), where the expectation is taken over all possible values of ranges, velocities, and azimuth angles. But, for ideal range estimation, it is necessary to minimize the above mentioned inner product in (a), for the same velocity, and angle but different range values. That is, we wish to minimize the average inner product between two target derivative vectors at different ranges. Since it is not analytically tractable, one can compute and compare the above inner products numerically for some of the well-known radar waveforms shown in Table I along with a set of two newly proposed waveforms motivated by the our analysis, where \( \text{rect}(T_s) \) denotes the rectangular pulse centered around \( t = 0 \) with unit amplitude and duty cycle \( \frac{T_s}{T} \). Here, \( W \)

| No. | Baseband Waveform | Envelope Signal | Constants |
|-----|-------------------|-----------------|-----------|
| 1   | 1                 | \( A \ \text{rect}(T_s) \) | \( A = \sqrt{\frac{T}{T_s}} \) |
| 2   | 1                 | \( B x_i(t) \) in (1) | \( B = 1 \) |
| 3*  | 1                 | \( C t^2 \ \text{rect}(T_s) \) | \( C = \sqrt{\frac{T}{T_s}} \) |
| 4   | \( \cos \pi W \left( \frac{t - T_s}{T_s} \right) t \) | \( A \ \text{rect}(T_s) \) | \( A = \sqrt{\frac{T}{T_s}} \) |
| 5   | \( \cos \pi W \left( \frac{t - T_s}{T_s} \right) t \) | \( B x_i(t) \) in (1) | \( B = 1 \) |
| 6*  | \( \cos \pi W \left( \frac{t - T_s}{T_s} \right) t \) | \( C t^2 \ \text{rect}(T_s) \) | \( C = \sqrt{\frac{T}{T_s}} \) |

Table I. Well-known and newly introduced radar transmit waveforms.

Before computing the TAFs for the above pulse waveforms, consider the following auto-correlation functions – special cases of the ambiguity functions – for the waveforms given in Table II. Let the \( q \)th waveform given in Table II be denoted by \( x^{(q)}(t) \), and the corresponding auto-correlation function be denoted by \( R^{(q)}(\tau) \). Observe that

\[
R^{(1)}(\tau) = \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} x^{(1)}(t)x^{(1)}(t - \tau)dt = (1 - \tau') P, \tag{27}
\]

\[
R^{(3)}(\tau) = \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} x^{(3)}(t)x^{(3)}(t - \tau)dt = c_1 (\tau')^5 P + \mathcal{O}(\tau'^4), \tag{28}
\]

where \( \tau' = \frac{\tau}{T_s} \) denotes the fractional time delay in one pulse duration and \( c_1 \) is a constant. It is observed that \( R^{(3)}(\tau) \) is more sensitive to the fractional time shift parameter \( \tau' \). Consider the modified functions with the time axis weighting given by

\[
R^{(1)}_z(\tau) = \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} tx^{(1)}(t)(t - \tau)x^{(1)}(t - \tau)dt = c_2 (\tau')^3 P + \mathcal{O}(\tau'^2), \tag{29}
\]

\[
R^{(3)}_z(\tau) = \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} tx^{(3)}(t)(t - \tau)x^{(3)}(t - \tau)dt = c_3 (\tau')^7 P + \mathcal{O}(\tau'^6), \tag{30}
\]

where \( c_2 \) and \( c_3 \) are constants. It can be observed that introducing the time weighting in the ambiguity function,

\footnote{Note that, these conditions are arrived at for the Gaussian pulses. For other envelope waveform, the optimization criteria can be different. In this section, we substantiate the selection of Gaussian pulses, by analyzing the three TAFs.}
the sensitivity of the radar for small time delays can be improved significantly. This sensitivity improvement reflects in the nature of various TAFs introduced in Appendix A. For range TAF, we either get the 3rd order sensitivity or the 7th order sensitivity, depending on the choice of the baseband signal. Similarly, the velocity TAF has further improvement in sensitivity due to the presence of $t_m \circ t_m$ term present in the TAF. On the angular TAF, sensitivity is not only improved with $t_m$, but also with $\Phi_m$ which brings differentiation in the time delays owning to the array geometry and the angle of arrival. Thus, one can compare the performance of various transmit waveforms using the three TAFs as performance metric.

1) TAF for a Rectangular Pulse: It is easy to see that for a rectangular pulse, the range TAF will be proportional to the correlator output where the choice of pulse width $T_s$ decides the minimum distance between which two independent targets can be resolved – that is, the off-diagonal elements of FIM corresponding to those two targets will be zero. If we include the Doppler velocity also, it simplifies to the Wigner-Ville transformation. The velocity TAF and angle TAF will not be zero for two different targets, due to slowly reducing amplitude nature of the correlation function with or without the frequency offset (or Doppler shift) which impacts the minimum MSE one can achieve. However, if FMCW pulse is used then ambiguity function goes to zero quickly (as inverse function of the bandwidth of the FMCW waveform).

2) TAF for a Gaussian Pulse: For the Gaussian pulse, the range TAF can be computed as

$$A_r(\epsilon) = \frac{1}{R_{\text{max}}} \int_{r_1=0}^{R_{\text{max}}} \int_{t_1=0}^{T_s} \sum_{l=1}^{L} \left( \frac{p_1^2}{c^2} \right) \left( \frac{p_2^2}{c^2} \right) \exp \left\{ -S \left[ \frac{T_s^2}{2} \left( t^2 + \frac{p_1^2}{c^2} \right) \right] \right\} dt dr_1 dr_2,$$

which dies down faster even for smaller value of range difference $\epsilon$, even when $T_1 = T_2$ as well as $p_1 = p_2$ depending on the constant $S$ which is a function of the bandwidth of the pulse. The above approximation is obtained after ignoring the first order terms in the expansion of $(t + a + b)^2$, and after ignoring large range differences $|r_1 - r_2| > \epsilon$. Note that, the above expression is valid when $r_2 < r_1$ as the role of $r_1$ and $r_2$ can interchanged, without loss of generality. The analytical expressions for velocity TAF and angle TAF are complex to list here. Instead, numerical computations are done to demonstrate the same. Figure 3 compares the Gaussian pulse, its autocorrelation and range TAF, when velocities and angles are identical for $L = 8$ and uniform linear array with $d = \frac{\lambda}{2}$.  

As it can be noticed that peaks for the TAFs occur at different times, and consequently at different velocity and angles. The parameters are chosen as follows. We choose an FMCW baseband transmit waveform with a Gaussian envelope, $L = 32$, $r_1 = 0$, $r_2 = \epsilon$, $v_1 = 0.15$ ms$^{-1}$, $v_2 = 1.5$ ms$^{-1}$, $\phi_1 = 30$ degrees and $\phi_2 = 35$ degrees. The Gaussian pulse was generated with $T_s = 1 \mu s$. The FMCW signal sweeps 100 MHz bandwidth within $T_s$.

B. Resource Sharing Between Radar and Communication

The joint data transmission and target parameter estimation requires sharing of resources either in time, bandwidth or angle-of-arrival. In many JRC systems, the known transmit waveforms are exploited to carry out successive interference cancellation – by predicting the range parameter – for improving the communication system performance. In some other work, OFDM bandwidth is shared between the radar and communication system (e.g., [52]). However, one can do the joint data and radar parameter estimation, by suitably

7MIMO radar systems can use beam space or beam vectors as resources for sharing between the data transmission and radar as in [50]. [51]
designing the transmit waveforms in which both the data and radar waveforms can co-exist, i.e., overlap in time and spectrum provided TAFs retain their properties even after overlaying both signals.

Consider the transmit waveform shown in Fig. 5 where the central Gaussian pulse conveys the data bits and an exponential pulse is used for target parameter estimation. It can be observed that a significant overlap in time and spectrum exists between these pulses. The system parameters are chosen to be $T_s = 100\mu s$ and envelope $BW = 100$ KHz. Note that both waveforms are normalized to unit energy. Each waveform can be independently modulated by the Gaussian amplitudes $\alpha_i$ as explained previously.

Figure 6 compares the range TAF, velocity TAF and angle TAF with and without the overlaid communication pulse for a uniform linear array. The pulse for radar-only uses an FMCW baseband signal with exponential envelope. It can be observed that the TAFs are unaffected by the presence of the communication Gaussian pulse at the radar transmitter.

A practical code construction for Gaussian channel is described in Appendix B which can be used to modulate the data pulses at the center. However, the total transmitter power needs to be divided between the radar and communication.

VI. Conclusion

A novel analytical approach using linear channel, target response model was employed to derive the lower bounds for data transmission rate and parameter estimation MSE from first principles. This is relevant in the context of broadband JRC with passive targets, where the radar transceiver not only detects the targets, but also conveys information to them. The targets are also allowed to send data by backscattering the signal from the transmitter. It was shown that forward channel data communication does not impact the parameter estimation bounds while the reverse channel data communication adversely affects the radar performance unless orthogonal resource allocation is done. Moreover, the data bounds were shown to be achievable if and only the TAFs are optimized by proper selection of the transmit waveforms. By optimizing three different TAFs, one can tune the performance of the radar for each of the parameters of interest. The proof of achievability of these bounds are a part of the future work. Moreover, design of practical codes to achieve these bounds will considerably enhance the efficiency of JRC systems.

APPENDIX

A. Target Ambiguity Functions

The TAFs are generalized versions of the conventional ambiguity functions, which arise naturally in the computation of CRBs. This requires computation of second order partial derivatives of logarithm of the joint PDF function. For the radar parameter estimation, the partial derivative with respect to the parameters can be computed using

$$\frac{\partial \log f_2(z_i)}{\partial r_m} = \left[ \frac{\partial \log f_2(z_i)}{\partial (z_i - a_i X_i \Gamma_i)} \right] \frac{\partial (z_i - a_i X_i \Gamma_i)}{\partial r_m}.$$  (28)

Upon simplification, we get

$$\frac{\partial \log f_2(z_i)}{\partial r_m} = -\frac{1}{2} \left[ 2 (z_i - a_i X_i \Gamma_i)^T \mathbf{R}_m^{-1} \right] \frac{\partial (z_i - a_i X_i \Gamma_i)}{\partial r_m}.$$  (29)

Further, it can be shown that

$$\left( \frac{\partial \mathbf{X}_i}{\partial r_m} \right) = \frac{4\pi^2 \mathbf{T}_m}{\beta^2} \gamma_m \sum_j \zeta_{mj} \delta(t - t_{mj}) \left[ x_m \circ t_m \right],$$  (30)

$$\left( \frac{\partial \Gamma_i}{\partial r_m} \right) = \left[ 0, \ldots, 0, \gamma_m \sum_j \zeta_{mj} \delta(t - t_{mj}) \right] \left\{ -2e^{-2e^{-1}} r_m^{-2e^{-1}}, 0, \ldots, 0 \right\}^T.$$  (31)
\[ \frac{\partial^2 \log f_z(z_i)}{\partial r m \partial r n} = \frac{a_i^2 16\pi^2 T_m^2 \gamma_m \gamma_n \sum \zeta_m \delta(t - t_m) e^{\frac{r_m - r_n}{c}}}{\beta^2 c^2} [x_m \circ t_n]^T R_n^{-1} [x_m \circ t_m]. \]

\[ A_r(r_m, r_n) = \int_{r_m = r_{\text{min}}}^{r_{\text{max}}} \int_{r_n = r_{\text{min}}}^{r_{\text{max}}} [x_m \circ t_n]^T R_n^{-1} [x_m \circ t_m] f_{R_m, R_n}(r_m, r_n) dr_n dr_m, \]

where the approximation denoted by (a) is obtained by ignoring the derivative of \( \Gamma \) with respect to \( r_m \), since the first term dominates the second. The final expression for \( \frac{\partial^2 \log f_z(z_i)}{\partial r m \partial r n} \) is given in (34) at the top of this page. It can be observed that \( -E \left[ \frac{\partial^2 \log f_z(z_i)}{\partial r m \partial r n} \right] = 0 \), since \( E[\gamma_m \gamma_n] = 0 \).

1) **Target Ambiguity Function in Range**: Now, consider the function,

\[ A_r(r_m, r_n) \triangleq E \left( [x_m \circ t_n]^T R_n^{-1} [x_m \circ t_m] \right), \]

which measures the average weighted inner product between the two derived vectors, \([x_m \circ t_n]\) and \([x_m \circ t_m]\) where the term \( x_m \circ t_n \) is proportional to the derivative of \( x_n \). Here, the expectation is taken over all possible differences in the range of two targets except for \( r_m = r_n \). That is, the expectation can be computed as shown in (35), where \( f(r_m, r_n) \) is the joint PDF between two target ranges. Typically, one can evaluate the integral for uniform distribution for the two target ranges. When \( R_n \) is a diagonal matrix, this function simplifies to

\[ A_r(r_m, r_n) = \frac{1}{\sigma_n^2} E \left( [x_m \circ t_n]^T [x_m \circ t_m] \right), \]

which measures the ambiguity or dissimilarity between the time weighted columns of \( X_i \). We refer this function as **target ambiguity function** (TAF) which is different from the ambiguity function known in radar literature. TAF measures the similarity of the received vectors due to each target rather than the similarity between the ideal and time-frequency shifted transmit waveform. Moreover, TAF operates on the derivative of the vector signal received from each target rather than the received signal vector. Even if the reflection from the targets are correlated \( E[\gamma_m \gamma_n] \neq 0 \), one can design waveform and array geometry which ensures TAF to be very small between two targets, its parameters can be estimated more accurately, as observed in the sequel.

From (34), the expected value of second order partial derivative for range can be written as in (38), which can be the inverse of the MSE if off-diagonal elements of the FIM are zero\(^8\). It can be noticed that MSE is inversely proportional to the SNR, RCS parameter and norm of the derivative of the signal vector from that target. Hence, one should design a transmitter waveform which maximizes the average norm, \( E \left( \| x_m \circ t_n \| \right) \) rather than the waveform with best self-correlation property. At the same time, one also wishes to minimize \( E \left( x_m \circ t_n \right)^T (x_m \circ t_n) \), to reduce the impact due to cross terms in the FIM.

2) **Target Ambiguity Function in Velocity**: On similar lines of derivation, one can find the partial derivatives with respect to the velocity parameter as shown below. Note that,

\[ \frac{\partial x(T_m[t - iT_p - 2\tau_m - \varphi_l])}{\partial v_m} \]

\[ = 2T_m[t - iT_p - 2\tau_m - \varphi_l] \left( \frac{2c}{(c - v_m)^2} \right) \frac{\pi^2}{\beta^2} x(T_m[t - iT_p - 2\tau_m - \varphi_l]) \circ t_m \circ t_m, \]

using the fact that \( T_m = \frac{2c}{(c - v_m)^2} \). We can write the second order derivative with respect to the velocity parameter as given in (39). Thus, for minimizing the MSE for velocity estimation, one should design waveform which maximizes the norm of the second order derivative of the transmitter waveform. Moreover, define the TAF for velocity as

\[ A_v(v_m, v_n) = E \left( [x_m \circ t_n]^T R_n^{-1} [x_m \circ t_n] \right). \]

As in the case of range, the above TAF is computed over all possible velocity values except when \( v_m = v_n \).

3) **Target Ambiguity Function in Azimuth**: Evaluation of the partial derivatives with respect to the azimuth angle gives the TAF in angle domain. From the definition of \( \varphi_l \), we can write

\[ \varphi_l = \frac{\Phi^T_m P_i}{c} = \frac{\cos \phi_m \cos \theta_m x_l + \sin \phi_m \cos \theta_m y_l + \sin \theta_m z_l}{c}, \]

where \((x_l, y_l, z_l)\) are the Cartesian coordinates of the \( l^{th} \) receiver antenna, \( \phi_m \) is azimuth angle of the \( m^{th} \) target and \( \theta_m \).

\(^8\)In general, the FIM is block diagonal in nature, where the 3x3 block matrices corresponding to all parameters for a given target can influence each other on the MSE. However, the parameters are one target does not influence the MSE of another. Also note that inverse of block diagonal matrix is a block diagonal matrix with the individual inverse of the blocks.
is the corresponding elevation angle of the same target. Now, one can write the second order partial derivative as given in (20), and the TAF in angle can be written as

\[
A(\phi, \sigma) = E \left[ x_m(r_1, v_1, \phi_1) \circ t_m(r_1, v_1, \phi_1) \circ \Phi_m(\phi_1) \right]^T R_n^{-1} \left[ x_m(r_2, v_2, \phi_2) \circ t_m(r_2, v_2, \phi_2) \circ \Phi_m(\phi_2) \right].
\]

Thus, this ambiguity function brings in the role of the array geometry to improve the MSE bound by maximizing the TAF.

4) Cross Terms in FIM: Using the above expressions, one can write the expression for the mixed second order derivatives for pairs of terms such as (range, velocity) or (range, angle) and (velocity, angle). Since these terms are not zero, the FIM has the structure of a block diagonal matrix, where each block belongs to one target. That is, under independent scattering assumption, one target does not influence the parameter estimation of the other target provided we design waveforms which minimizes the TAFs. Hence, to achieve the bounds given in Section 11, one needs to minimize all the inner products given in Eqn. (43), so that the non-diagonal terms in FIM are reduced.

5) Generalized Target Ambiguity Function: Although we listed the ambiguity functions that arose from the partial derivatives taken with respect to different parameters, one can combine all of them to define a generalized TAF from which other TAFs can be derived as special cases. That is, consider the following TAF, which includes the range variations, velocity variations as well as array geometry variations as shown in (44).

### B. Code Construction Procedure

It is well-known that Gaussian random source achieves the capacity in Gaussian channel. Hence, the code construction for the data only transmission can adopt any of the well known methods such as Lattice Gaussian coding [53, 54]. We elaborate a practical code construction along the lines of Shannon’s spherical code construction. The \( N \)-dimensional Gaussian vector with components, from i.i.d. Gaussian random variables with zero mean and unit variance, can be scaled by \( \sigma_x \) to obtain the desired codeword with average transmitter power \( P = \sigma_x^2 \). A Gaussian spherical code with desired number of codewords (say \( 2^r \) codewords or \( r \) bits per codeword), can be constructed by partitioning the surface area of an \( N \)-dimensional sphere with unit radius. This can be done by well known methods such as K-means algorithm or any standard vector quantization methods [55]. Now, use the centroids of the \( 2^r \) regions as the desired codeword for data transmission. Such a code is shown to achieve the channel capacity in AWGN channel for large \( N \) [54].

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