New strings
with
world-sheet supersymmetry

A. Nichols\textsuperscript{1}, R. Manvelyan\textsuperscript{2}, G. K. Savvidy\textsuperscript{3}
National Research Center Demokritos,
Ag. Paraskevi, GR-15310 Athens, Hellenic Republic

Abstract

We suggest a new model of string theory with world-sheet supersymmetry. It possesses an additional global fermionic symmetry which is similar in many ways to BRST symmetry. The spectrum consists of massless states of Rarita-Schwinger fields describing infinite tower of half-integer spins.

\textsuperscript{1}nichols@inp.demokritos.gr
\textsuperscript{2}manvel@moon.yerphi.am
\textsuperscript{3}savvidy@inp.demokritos.gr
1 Introduction

In the recent articles [1, 5] there was suggested a string theory which is described by the following action:

\[ S = m \cdot L = \frac{m}{\pi} \int d^2 \zeta \sqrt{h} \sqrt{K^i_a K^i_b}, \quad (1) \]

here \( m \) has dimension of mass, \( h_{ab} \) is the induced metric and \( K^i_a \) is the second fundamental form (extrinsic curvature) \(^4\). Instead of being proportional to the area of the surfaces, as it is the case in the standard string theory, the action (1) is proportional to the length of the surface \( L \) [1]. Due to the last property the model has two essentially new properties, first of all, when the surface degenerates into a single world line, the action (1) becomes proportional to the length of the world line

\[ S = m A_{xy} \rightarrow m \int_X^Y ds \quad (2) \]

and the functional integral over surfaces naturally transforms into the Feynman path integral for a point-like relativistic particle, thus naturally extending it to relativistic strings and, secondly, the action is equal to the perimeter of the Wilson loop \( S = m(R + T) \), where \( R \) is space distance between quarks, therefore at the classical level string tension is equal to zero. Quantization of this system and its massless spectrum have been derived in [5]. In this string theory all particles, with arbitrary large integer spin, are massless. This pure massless spectrum is consistent with the tensionless character of the model and it was conjectured in [5] that it may describe unbroken phase of standard string theory when \( \alpha' \rightarrow \infty \) and all masses tend to zero \( M^2_n = \frac{1}{\alpha'}(n - 1) \rightarrow 0 \) [2].

Our aim now is to introduce fermions and to suggest supersymmetric extension of this model using world-sheet superfields [6, 7, 8, 9, 10]. The action (1) can be written in the equivalent form [1, 5]

\[ S = \frac{m}{\pi} \int d^2 \zeta \sqrt{h} \sqrt{(\Delta(h)X_\mu)^2}, \quad (3) \]

here \( h_{ab} = \partial_a X_\mu \partial_b X_\mu \) is the induced metric, \( \Delta(h) = 1/\sqrt{h} \partial_a \sqrt{h} h^{ab} \partial_b \) is Laplace operator, \(^5\) \( a, b = 1, 2; \quad \mu = 0, 1, 2, ..., D - 1 \). We shall fix the conformal gauge \( h_{ab} = \rho \eta_{ab} \) using the reparametrization invariance of the action (3) and represent it in two equivalent forms [5]

\[ S = \frac{m}{\pi} \int d^2 \zeta \sqrt{(\partial^2 X)^2} \quad \Leftrightarrow \quad \dot{S} = \frac{1}{\pi} \int d^2 \zeta \{ \Pi^\mu \partial^2 X^\mu - 2\Omega (\Pi^2 - m^2) \}, \quad (4) \]

where the independent field \( \Pi^\mu \) and the Lagrange multiplier \( \Omega \) have been introduced. The system of equations which follows from \( \dot{S} \)

\[ \partial^2 \Pi^\mu = 0, \quad \partial^2 X^\mu - 2\Omega \Pi^\mu = 0, \quad \Pi^\mu \Pi_\mu = m^2 \quad (5) \]

is equivalent to the original equation for \( X^\mu \) and the \( \Pi^\mu \) field takes the form

\[ \Pi^\mu = m \frac{\partial^2 X^\mu}{\sqrt{(\partial^2 X)^2}}. \]

Both forms of the action (4) can be extended to the supersymmetric case as follows.

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\(^4\)This action is essentially different in its geometrical meaning from the action considered in previous studies [4] where it is proportional to the spherical angle and has dimensionless coupling constant.

\(^5\)The equivalence follows from the relation: \( K^i_a K^i_b = (\Delta(h)X_\mu)^2, \quad i, j = 1, 2, ..., D - 2 \)
\section{N=1 World-sheet Supersymmetry}

For the basic fields \((X, \Pi, \Omega)\) in (4) we shall introduce the corresponding superfields \([8, 9, 10]\)

\begin{align*}
\hat{X}^\mu &= X^\mu + \bar{\vartheta}\Psi^\mu + \frac{1}{2} \bar{\vartheta}\vartheta F^\mu \\
\hat{\Pi}^\mu &= \Pi^\mu + \bar{\vartheta}\eta^\mu + \frac{1}{2} \bar{\vartheta}\vartheta \Phi^\mu \\
\hat{\Omega} &= \omega + \bar{\vartheta}\xi + \frac{1}{2} \bar{\vartheta}\vartheta \Omega,
\end{align*}

(6)

where \(\vartheta\) is an anti-commuting variable and shall define the supersymmetric action simply exchanging basic fields \((X, \Pi, \Omega)\) in (4) by corresponding superfields as follows

\[ S = -\frac{i}{2\pi} \int d^2 \zeta d^2 \theta \{ \hat{\Pi}^\mu D D \hat{X}^\mu - 2\hat{\Omega}(\hat{\Pi}^2 - m^2) \}, \]

(7)

where

\[ D_A = \frac{\partial}{\partial \vartheta^A} - i(\rho^a \partial_a)_{\vartheta} \partial_\vartheta, \quad \Psi^\mu_A(\zeta) \equiv \begin{pmatrix} \Psi^\mu_-(\zeta) \\ \bar{\Psi}^\mu_-(\zeta) \end{pmatrix}, \quad \eta^\mu_A(\zeta) \equiv \begin{pmatrix} \eta^\mu_-(\zeta) \\ \bar{\eta}^\mu_-(\zeta) \end{pmatrix}, \quad \xi_A(\zeta) \equiv \begin{pmatrix} \xi_-(\zeta) \\ \bar{\xi}_-(\zeta) \end{pmatrix}, \]

\[ \mu \text{ is a space-time vector index, } A = 1, 2 \text{ is a two-dimensional spinor index.} \]

\[ \bar{\Psi}^\mu = \Psi^{+\mu} \rho^0 = \Psi^{T\mu} \rho^0 \] and \(\rho^a\) are two-dimensional Dirac matrices

\[ \{ \rho^a, \rho^b \} = -2\eta^{ab}. \]

(8)

In Majorana basis the \(\rho^a\)'s are given by

\[ \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

(10)

and \(i\rho^a \partial_a\) is a real operator. The two-dimensional chiral fields are defined as \(\rho^3 \Psi^\mu_\pm = \mp \Psi^\mu_\pm\), where \(\rho^3 = \rho^0 \rho^1\). We should compute different expressions involved in the action,

\[ \bar{D}^A D_A \hat{X}^\mu = 2F^\mu + 2i\vartheta \rho^a \partial_a \Psi^\mu - \vartheta \vartheta \partial^2 X^\mu, \]

thus

\[ \hat{\Pi}^\mu \bar{D} D \hat{X}^\mu = (\Pi^\mu + \bar{\vartheta}\eta^\mu + \frac{1}{2} \bar{\vartheta}\vartheta \Phi^\mu)(2F^\mu + 2i\vartheta \rho^a \partial_a \Psi^\mu - \vartheta \vartheta \partial^2 X^\mu) \]

and the quadratic part in \(\vartheta\) is equal to

\[ -\vartheta \vartheta \Pi^\mu \partial^2 X^\mu + \vartheta \vartheta F^\mu \Phi^\mu + 2i\vartheta \eta^\mu \vartheta \rho^a \partial_a \Psi^\mu. \]

The integral over Grassmann variables is defined as \(\int d^2 \theta \vartheta \vartheta \vartheta = -2i\), therefore

\[ \frac{-i}{2} \int d^2 \theta \{ -\vartheta \vartheta \Pi^\mu \partial^2 X^\mu + \vartheta \vartheta F^\mu \Phi^\mu + 2i \vartheta \eta^\mu \vartheta \rho^a \partial_a \Psi^\mu \} \]

\[ = \Pi^\mu \partial^2 X^\mu + i\eta^\mu \rho^a \partial_a \Psi^\mu - F^\mu \Phi^\mu, \]
where we have used the relation
\[ 2i\bar{\theta}\eta^\mu \frac{\partial}{\partial a}\Psi^\mu = -i\partial\bar{\theta}\eta^\mu \rho^a\partial_a\Psi^\mu. \]

Now we have to compute also the second term in the Lagrangian
\[ \hat{\Omega}(\hat{\Pi}^2 - m^2) = \omega(\Pi^2 - m^2) + 2\omega(\Pi^\mu \bar{\eta}^\mu + \bar{\theta}\partial \Pi^\mu \Phi^\mu) \]
\[ + \bar{\theta}\xi(\Pi^2 - m^2) + 2\bar{\theta}\xi\Pi^\mu \bar{\eta}^\mu + \frac{1}{2} \bar{\theta}\partial\Omega(\Pi^2 - m^2) \]
and to integrate it over Grassmann variables
\[ -\frac{i}{2} \int d^2\theta \{ -2\hat{\Omega}(\hat{\Pi}^2 - m^2) \} = \Omega(\Pi^2 - m^2) + \omega(2\Pi^\mu \Phi^\mu + \bar{\eta}^\mu \eta^\mu) + 2\Pi^\mu \bar{\eta}^\mu \xi. \]

The full action is now equal to the following expression
\[ S = \frac{1}{\pi} \int d^2\zeta \{ \Pi^\mu \partial^2 X^\mu + i\bar{\eta}^\mu \rho^a\partial_a\Psi^\mu - F^\mu \Phi^\mu \]
\[ - \Omega(\Pi^2 - m^2) - \omega(2\Pi^\mu \Phi^\mu + \bar{\eta}^\mu \eta^\mu) - 2\Pi^\mu \bar{\eta}^\mu \xi \}, \quad (11) \]

The equations of motion are:
\[(I)\]
\[ \Phi^\mu = 0 \]
\[ \partial^2 \Pi^\mu = 0 \]
\[ i\rho^a \partial_a \eta^\mu = 0 \]
\[ 2\omega \Pi^\mu + F^\mu = 0 \]
\[ \partial^2 X^\mu - 2\Omega \Pi^\mu - 2\omega \Phi^\mu - 2\bar{\eta}^\mu \xi = 0 \]
\[ i\rho^a \partial_a \Psi^\mu - 2\Pi^\mu \xi - 2\omega \eta^\mu = 0, \quad (12) \]

and the variation over Lagrange multipliers gives constraints
\[(II)\]
\[ \Pi^2 - m^2 = 0 \]
\[ 2\Pi^\mu \Phi^\mu + \bar{\eta}^\mu \eta^\mu = 0 \]
\[ 2\Pi^\mu \eta^\mu = 0. \quad (13) \]

The SUSY transformation is:
\[ \delta X^\mu = \bar{\epsilon} \Psi^\mu, \quad \delta \Pi^\mu = \bar{\epsilon} \eta^\mu, \quad \delta \omega = \bar{\epsilon} \xi, \]
\[ \delta \Psi^\mu = -i\rho^a \partial_a X^\mu \epsilon + F^\mu \epsilon, \quad \delta \eta^\mu = -i\rho^a \partial_a \Pi^\mu \epsilon + \Phi^\mu \epsilon, \quad \delta \xi^\mu = -i\rho^a \partial_a \omega \epsilon + \Omega \epsilon, \quad (14) \]
\[ \delta F^\mu = -i\bar{\epsilon} \rho^a \partial_a \Psi^\mu, \quad \delta \Phi^\mu = -i\bar{\epsilon} \rho^a \partial_a \eta^\mu, \quad \delta \Omega = -i\bar{\epsilon} \rho^a \partial_a \xi, \]

where the anti-commuting parameter $\epsilon$ is a two-dimensional spinor
\[ \epsilon \equiv \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}. \]

The action (11), equations (12) and the constraints (13) completely define the system which exhibits the supersymmetry (14).
3 Fermionic BRST-like symmetry

As we shall see below the action (11) possesses surprisingly new global fermionic symmetry which is similar in many ways to the BRST symmetry. This can be seen in the light-cone coordinates. In the light-cone coordinates the action takes the form

\[
S = \frac{2}{\pi} \int d^2 \xi \left\{ -2\Pi^{\mu} \partial_+ \partial_- X^\mu + i\eta_+^{\mu} \partial_- \psi_+^\mu + i\eta_-^{\mu} \partial_+ \psi_-^\mu - \frac{1}{2} F^{\mu} \Phi^\mu \right. \\
- \left. \frac{1}{2} \Omega (\Pi^2 - m^2) - \omega (\Pi^\mu \Phi^\mu + i\eta_+^\mu \eta_-^\mu) - i\Pi^\mu \eta_-^\mu \xi_- + i\Pi^\mu \eta_+^\mu \xi_+ \right\}. 
\] 

(15)

As one can check in addition to the SUSY transformation (14) this system is invariant under fermion transformation laws \( \delta \) and \( \bar{\delta} \):

\[
\begin{align*}
\delta X^\mu &= 0, & \bar{\delta} X^\mu &= 0, & \delta \omega &= i\epsilon_+ \xi_-, \\
\delta \Psi^\mu &= -2\epsilon_+ \partial_- X^\mu, & \bar{\delta} \Psi^\mu &= 0, & \delta \xi_- &= 0, \\
\delta \Psi_+^\mu &= 0, & \bar{\delta} \Psi_+^\mu &= -2\epsilon_- \partial_+ X^\mu, & \delta \xi_+ &= - \epsilon_+ \Omega, \\
\delta F^\mu &= -2i\epsilon_+ \partial_+ \Psi_+^\mu, & \bar{\delta} F^\mu &= 2i\epsilon_- \partial_- \Psi_-^\mu, & \delta \Omega &= 0, \\
\delta \Pi^\mu &= i\epsilon_+ \eta_-^\mu, & \bar{\delta} \Pi^\mu &= i\epsilon_- \eta_+^\mu, & \delta \omega &= i\epsilon_- \xi_+, \\
\delta \eta_-^\mu &= 0, & \bar{\delta} \eta_-^\mu &= \epsilon_- \Phi^\mu, & \delta \xi_- &= \epsilon_- \Omega, \\
\delta \eta_+^\mu &= -\epsilon_+ \Phi^\mu, & \bar{\delta} \eta_+^\mu &= 0, & \delta \xi_+ &= 0, \\
\delta \Phi^\mu &= 0, & \bar{\delta} \Phi^\mu &= 0, & \delta \Omega &= 0, 
\end{align*}
\] 

(16)

The algebra obeyed by the fermionic symmetries is nilpotent and is very similar to BRST transformations

\[
\delta_\epsilon \delta_\epsilon (H) = \bar{\delta}_\epsilon \bar{\delta}_\epsilon (H) = 0, \quad (\delta_\epsilon \delta_\epsilon - \bar{\delta}_\epsilon \bar{\delta}_\epsilon) (H) = 0. 
\] 

(17)

where \( H \) is any of the fields \( (X, \Psi, F, \Pi, \eta, \Phi, \omega, \xi, \Omega) \). The important fact now is that the Lagrangian is a variation of the super-potentials \( W \) and \( \bar{W} \)

\[
W = \Pi^\mu \partial_+ \Psi_-^\mu + \frac{1}{2} \eta_-^\mu F^\mu, \quad \bar{W} = \Pi^\mu \partial_- \Psi_+^\mu - \frac{1}{2} \eta_+^\mu F^\mu, 
\] 

(18)

so that

\[
\delta W = \epsilon_+ \mathcal{L}, \quad \bar{\delta} \bar{W} = \epsilon_- \mathcal{L}. 
\] 

(19)

It is also true that there exists a potential \( V \) such that

\[
\delta V = -i\epsilon_- \bar{W}, \quad \bar{\delta} V = i\epsilon_+ W, \quad V = \frac{1}{2} \Pi^\mu F^\mu, 
\] 

(20)

thus

\[
i\delta \bar{\delta} V = \epsilon_+ \epsilon_- \mathcal{L}, \quad i\bar{\delta} \delta V = \epsilon_+ \epsilon_- \mathcal{L}. 
\]

The constrains (II) can also be represented by the \( \bar{\delta} \delta \) transformation and therefore the full Lagrangian in (15) can be represented as

\[
2\epsilon_+ \epsilon_- \mathcal{L}_{tot} = i\bar{\delta} \delta \left( \Pi^\mu F^\mu + \omega (\Pi^2 - m^2) \right). 
\] 

(21)

From (17) it follows that the action is invariant under fermionic symmetries (16) and can be represented as BRST commutator \( \mathcal{L} = \{ Q, W \} = \{ \bar{Q}, \bar{W} \} \) where \( \delta = \epsilon_+ Q, \bar{\delta} = \epsilon_- \bar{Q} \), as it takes place in topological field theories [3].
4 SUSY Solution

As one can see the SUSY solution of equations (12) is:

\[ i) \Omega = \omega = \xi = 0 \]

and the rest of the equations (1) reduce to the following form

\[
(I) \quad \partial^2 \Pi^\mu = 0, \quad i\rho^a \partial_a \eta^\mu = 0, \quad \partial^2 X^\mu = 0, \quad i\rho^a \partial_a \Psi^\mu = 0, \quad F^\mu = \Phi^\mu = 0 \tag{22}
\]

and should be accompanied by the constraints

\[
(II) \quad \Pi^2 - m^2 = 0, \quad \bar{\eta}^\mu \eta^\mu = 0, \quad \Pi^\mu \eta^\mu = 0. \tag{23}
\]

In the light-cone coordinates these equations are easy to solve since they take the form

\[
\partial_+ \partial_- \Pi^\mu = 0, \quad \partial_+ \eta_+^\mu = 0, \quad \partial_- \Pi^\mu = 0, \quad \partial_- \psi_+^\mu = 0,
\]

\[
\Pi^2 - m^2 = 0, \quad \eta_+^\mu \eta_-^\mu - \eta_-^\mu \eta_+^\mu = 0 \quad \Pi^\mu \eta_\pm^\mu = 0. \tag{24}
\]

The mass-shell supersymmetry transformation of the action

\[
\dot{S} = \frac{2}{\pi} \int d^2 \zeta \left\{ -2\Pi^\mu \partial_+ \partial_- X^\mu + i\eta_+^\mu \partial_+ \psi_+^\mu + i\eta_-^\mu \partial_- \psi_-^\mu \right\} \tag{25}
\]

is:

\[
(III) \quad \delta X^\mu = i\epsilon^+ \psi_-^\mu - i\epsilon_- \psi_+^\mu, \quad \delta \Pi^\mu = i\epsilon^+ \eta_-^\mu - i\epsilon_- \eta_+^\mu,
\]

\[
\delta \psi_-^\mu = -2\epsilon^+ \partial_- X^\mu, \quad \delta \eta_-^\mu = -2\epsilon^+ \partial_- \Pi^\mu,
\]

\[
\delta \psi_+^\mu = 2\epsilon^+ \partial_+ X^\mu. \tag{26}
\]

The solution of fermionic fields can be represented in the form of mode expansion

\[
\eta_+^\mu = \sum c_n^\mu e^{-in\zeta^+}, \quad \psi_+^\mu = \sum d_n^\mu e^{-in\zeta^+},
\]

\[
\eta_-^\mu = \sum \bar{c}_n^\mu e^{-in\zeta^-}, \quad \psi_-^\mu = \sum \bar{d}_n^\mu e^{-in\zeta^-} \tag{27}
\]

and the basic anti-commutators should be defined as:

\[
\{\eta_\pm^\mu(\zeta), \psi_\mp^\nu(\zeta')\} = 2\pi \eta_\mu^\nu \delta(\zeta - \zeta'), \tag{28}
\]

with all others equal to zero \{\eta_\pm^\mu(\zeta), \eta_\pm^\nu(\zeta')\} = 0, \{\psi_\pm^\mu(\zeta), \psi_\mp^\nu(\zeta')\} = 0. Substituting the mode expansion into the anti-commutators requires following relations between modes

\[
\{c_n^\mu, d_k^\nu\} = \eta_\mu^\nu \delta_{n+k,0}, \quad \{c_n^\mu, c_k^\nu\} = 0, \quad \{d_n^\mu, d_k^\nu\} = 0, \tag{29}
\]

and similar ones for \bar{c}_n^\mu and \bar{d}_n^\mu. The commutation relations for bosonic coordinates X and \Pi remain the same as in [5].

Our aim now is to describe the ground state sector. Let us consider for that the fermion zero mode sector

\[
\{c_0^\mu, d_0^\nu\} = \eta_\mu^\nu, \quad \{c_0^\mu, c_0^\nu\} = 0, \quad \{d_0^\mu, d_0^\nu\} = 0, \tag{30}
\]

\[
\{c_0^\mu, d_0^\nu\} = \eta_\mu^\nu, \quad \{c_0^\nu, c_0^\mu\} = 0, \quad \{d_0^\nu, d_0^\mu\} = 0. \tag{31}
\]
together with the constraint \( \eta_+^\mu \eta_-^\mu - \eta_-^\mu \eta_+^\mu = 0 \), which for the ground state takes the form \( c_0^\mu \bar{c}_0^\mu - \bar{c}_0^\mu c_0^\mu = 0 \). The nontrivial solution of the last equations is:

\[
\begin{align*}
2c_0^\mu &= i\gamma^\mu \gamma^{D+1} + \chi^\mu, \\
2\bar{c}_0^\mu &= \gamma^\mu + i\gamma^{D+1}\chi^\mu,
\end{align*}
\]  
\begin{equation}
(32)
\end{equation}

\[
\begin{align*}
2d_0^\mu &= i\gamma^\mu \gamma^{D+1} - \chi^\mu, \\
2\bar{d}_0^\mu &= -\gamma^\mu + i\gamma^{D+1}\chi^\mu,
\end{align*}
\]  
\begin{equation}
(33)
\end{equation}

where \([\gamma^\mu, \gamma^\nu] = 0\). The matrix \( \gamma^{D+1}\chi^{D+1} \) anticommutes with all these matrices and therefore the nonzero modes should be multiplied by this matrix to fulfill anticommutation relations between zero modes and nonzero modes. Thus we have the solution for boson [5] and fermion fields in the form

\[
\begin{align*}
\Pi^\mu &= m\epsilon^\mu + k^\mu \tau + \Pi^\mu_{\text{oscil}}, \\
X^\mu &= \epsilon^\mu + \frac{1}{m} \pi^\mu \tau + X^\mu_{\text{oscil}},
\end{align*}
\]  
\begin{equation}
(34)
\end{equation}

\[
\begin{align*}
\eta_+^\mu &= c_0^\mu + \eta_+^\mu_{\text{oscil}}, \\
\eta_-^\mu &= \bar{c}_0^\mu + \eta_-^\mu_{\text{oscil}},
\end{align*}
\]  
\begin{equation}
(35)
\end{equation}

where \([\epsilon^\mu, \pi^\nu] = [x^\mu, k^\nu] = i\eta^{\mu\nu} \) is a pair of conjugate variables describing bosonic zero mode sector. The important difference between the standard string theory and the present one is the appearance of conjugate variables \( e^\mu \) and \( \pi^\mu \), where \( e^\mu \) is a polarization vector orthogonal to the momentum vector \( k^\mu \) (\( e^\mu k^\mu = 0 \) ) [5]. It is convenient to denote the ground state wave function as \( |k, e, 0> \), so that the constraint \( L_0 \Psi_{\text{phys}} = \{k \cdot \pi + \text{oscillators} \ldots \} \Psi_{\text{phys}} = 0 \), which take place in this theory (see for details [5]), take the form

\[
\begin{align*}
k \cdot e |k, e, 0 > &= k \cdot \partial_e |k, e, 0 > = 0.
\end{align*}
\]  
\begin{equation}
(36)
\end{equation}

The new constraints \( \Pi \cdot \eta_\pm \Psi_{\text{phys}} = 0 \), which appear now in supersymmetric case, on the ground state wave function \( |k, e, 0> \) will take the form

\[
\begin{align*}
(i\gamma^\mu \gamma^{D+1} + \chi^\mu)k^\mu |k, e, 0> &= 0,
(\gamma^\mu + i\gamma^{D+1}\chi^\mu)k^\mu |k, e, 0> &= 0,
(\gamma^\mu + i\gamma^{D+1}\chi^\mu)e^\mu |k, e, 0> &= 0.
\end{align*}
\]  
\begin{equation}
(37)
\end{equation}

Expanding the ground wave function in \( e^\mu \) series

\[
|k, e, 0> = \psi + e^\mu \phi^\mu + e^{\mu_1} e^{\mu_2} \phi^{\mu_1\mu_2} + ...
\]  
\begin{equation}
(38)
\end{equation}

we shall get Dirac equations on the spin tensor \( \phi^{\mu_1\mu_2 \ldots \mu_J} \)

\[
\begin{align*}
(1 + \gamma^{D+1}\gamma^{D+1})\gamma^\mu k^\mu \phi^{\mu_1\mu_2 \ldots \mu_J} &= 0, \\
(1 + \gamma^{D+1}\gamma^{D+1})\gamma^\mu k^\mu \phi^{\mu_2 \mu_3 \ldots \mu_J} &= 0.
\end{align*}
\]  
\begin{equation}
(39)
\end{equation}

\[ k^{\mu_1} \phi^{\mu_1 \mu_2 \ldots \mu_J} = 0, \]  
\begin{equation}
(40)
\end{equation}

thus describing massless particles of half-integer spin \( J + 1/2 = 1/2, 3/2, \ldots \). One should study in great details full Hilbert space of exited states in order to learn more about complete spin content of the theory and to prove the absence of the negative norm states. The details will be given elsewhere.
5 Second Order Formulation

In this section we shall return back to the original action (4) with its unique variable $X^\mu$. Using superfield formalism one can extend this form of the action to supersymmetric case as well. We need just one superfield $\hat{X}^\mu = X^\mu + \bar{\theta} \Psi^\mu + \frac{1}{2} \bar{\theta} \partial F^\mu$. The worldsheet supersymmetric action can be written in the form

$$S = -\frac{im}{2\pi} \int d^2z d^2\theta \sqrt{(\bar{D}D\hat{X}^\mu)^2}.$$  \hfill (41)

Let us demonstrate now that classically it defines a model which is equivalent to (7). Indeed the superfield equations which follow from (7) are:

$$\bar{D}D\hat{X}^\mu - 4\hat{\Omega} \hat{\Pi}^\mu = 0$$
$$\bar{D}D\hat{\Pi}^\mu = 0$$
$$\hat{\Pi}^2 - m^2 = 0$$ \hfill (42)

and one can see that

$$\hat{\Pi}^\mu = \frac{1}{4\Omega} \bar{D}D\hat{X}^\mu.$$  

From the last equation it follows that

$$\frac{1}{4\Omega} = \frac{m}{\sqrt{(\bar{D}D\hat{X}^\mu)^2}}$$  

and field equations take the form

$$\bar{D}D\{m \frac{\bar{D}D\hat{X}^\mu}{\sqrt{(\bar{D}D\hat{X}^\mu)^2}}\} = 0.$$ \hfill (43)

The last equation simply follows from (41) by direct variation over $\hat{X}$, thus these models are indeed classically equivalent.

In this formulation we have

$$(\bar{D}D\hat{X}^\mu)^2 = 4(F^\mu F^\mu + 2iF^\mu \bar{\rho}^a \partial_a \Psi^\mu - \bar{\theta} \partial F^\mu \partial^2 X^\mu - \bar{\theta} \rho^a \partial_a \Psi^\mu \bar{\rho}^b \partial_b \Psi^\mu)$$

so that

$$\sqrt{(\bar{D}D\hat{X}^\mu)^2} = 2\sqrt{F^\mu F^\mu} \{ 1 - \bar{\theta} \partial F^\mu \partial^2 X^\mu / 2F^2 - (\bar{\theta} \rho^a \partial_a \Psi^\mu)^2 / 2F^2 + i\bar{\theta} \rho^a \partial_a \Psi^\mu F^\mu / F^2 - (1/8)(2i\bar{\theta} \rho^a \partial_a \Psi^\mu F^\mu / F^2)^2 \}$$

and

$$S = \frac{m}{\pi} \int d^2z \frac{1}{\sqrt{F^2}} \{ F^\mu \partial^2 X^\mu + \frac{1}{2} \partial_a \bar{\Psi}^a \rho^a K^{\mu\nu} \rho^b \partial_b \Psi^\nu \}, \quad K^{\mu\nu} = \eta^{\mu\nu} - \frac{F^\mu F^\nu}{F^2}.$$ \hfill (44)

The supersymmetry transformation is:

$$\delta X^\mu = \bar{\epsilon} \Psi^\mu,$$
$$\delta \Psi^\mu = -i\rho^a \partial_a X^\mu \epsilon + F^\mu \epsilon,$$
$$\delta F^\mu = -i\bar{\epsilon} \rho^a \partial_a \Psi^\mu.$$ \hfill (45)
We can write the action in terms of components in the following form

\[ S = \frac{2m}{\pi} \int d^2\zeta \frac{1}{\sqrt{F^2}} \left\{ -2F^\mu \partial_+ \partial_- X^\mu + i \left[ \partial_- \Psi_+^\mu \partial_+ \Psi_-^\mu - \partial_+ \Psi_-^\mu \partial_- \Psi_+^\mu \right] K^{\mu\nu} \right\} \]  

(46)

with the following equations of motion for \( F^\mu, \Psi_\pm^\mu \) and \( X^\mu \):

\[ K^{\mu\nu} \partial_+ \partial_- X^\nu - \frac{i}{2} \left[ \partial_+ \Psi_-^\nu \partial_- \Psi_+^\nu - \partial_- \Psi_-^\nu \partial_+ \Psi_+^\nu \right] \frac{F_{\nu\lambda}(\mu\lambda)}{F^2} = 0, \]  

(47)

\[ \partial_+ \left[ \frac{1}{\sqrt{F^2}} K_{\mu\nu} \partial_+ \Psi_-^\nu \right] = 0, \]  

(48)

\[ \partial^2 \left( \frac{F^\mu}{\sqrt{F^2}} \right) = 0, \]  

(49)

where \( \{\mu\nu\lambda\} = \mu\nu\lambda + \text{cycl.perm.} \). Solving Eq. (48) and using (47) we shall get the following solution and constraints for fermions

\[ \Psi_+^\mu(\zeta^+,\zeta^-) = \psi_+^\mu(\zeta^+) + \int_0^{\zeta^-} d\tilde{\zeta}^- \left( \frac{1}{m} \eta_+^\mu(\tilde{\zeta}^-) \Lambda(\zeta^+\tilde{\zeta}^-) - \frac{1}{2} \Pi^\mu(\zeta^+\tilde{\zeta}^-) \xi_-^\mu(\zeta^+\tilde{\zeta}^-) \right) \]  

(50)

\[ \Psi_-^\mu(\zeta^+,\zeta^-) = \psi_-^\mu(\zeta^-) + \int_0^{\zeta^+} d\tilde{\zeta}^+ \left( \frac{1}{m} \eta_-^\mu(\tilde{\zeta}^+) \Lambda(\zeta^+\tilde{\zeta}^-) - \frac{1}{2} \Pi^\mu(\zeta^+\tilde{\zeta}^-) \xi_+^\mu(\zeta^+\tilde{\zeta}^-) \right) \]  

(51)

\[ \Pi^\mu \eta_\pm^\mu = 0, \quad \eta_+^\mu \eta_-^\mu - \eta_-^\mu \eta_+^\mu = 0 \]  

(52)

where we have introduced instead of vector \( F^\mu \) the normalized variable \( \Pi^\mu = m \frac{F^\mu}{\sqrt{F^2}} \) and scalar variable \( \Lambda = \sqrt{F^2} \) corresponding to the length of \( F^\mu \). The remaining equations in the bosonic sector are the same as in the “first” order formalism described in the previous sections

\[ \partial_+ \partial_- \Pi^\mu = 0, \quad \Pi^2 - m^2 = 0, \]  

(53)

\[ \partial_+ \partial_- X^\mu + \frac{1}{2} \Omega \Pi^\mu + \frac{i}{2} (\eta_+^\mu \xi_- - \eta_-^\mu \xi_+) = 0. \]  

(54)

From the action (46) we can derive the canonical momenta for fermionic fields

\[ P^\mu_\Psi_+ = \frac{\delta L}{\delta \partial_0 \Psi_+^\mu} = - \frac{2m}{\pi \sqrt{F^2}} K_{\mu\nu} \partial_- \Psi_+^\nu \]  

(55)

\[ P^\mu_\Psi_- = \frac{\delta L}{\delta \partial_0 \Psi_-^\mu} = - \frac{2m}{\pi \sqrt{F^2}} K_{\mu\nu} \partial_+ \Psi_-^\nu \]  

(56)

Using solutions (50) and (52) we can see that these momenta are equal to \( \eta_\pm^\mu \), and anticommutators between \( \eta \) and \( \psi \) coincide with anti-commutation relations (29) of the previous section.

The advantage of the last formalism is that it has less fields and therefore some hidden symmetries of the model are much easier to detect. In particular, it is much easier to observe a surprising property of the action (46), that it is BRST exact and might be related to the topological nature of this supersymmetric extension. More precisely we can define the following set of nilpotent Grassmann odd symmetries related with some part of supersymmetry transformations

\[ S \Psi_+^\mu = \partial_+ X^\mu, \quad S X^\mu = 0, \]  

\[ S F^\mu = -i \partial_+ \Psi_+^\mu, \quad S \Psi_-^\mu = 0, \]  

\[ S^2 = 0 \]  

(57)
or the similar in left moving sector

\[ \bar{S}\Psi_\mu = \partial_- X_\mu, \quad \bar{S}X_\mu = 0, \]
\[ \bar{S}F_\mu = i\partial_- \Psi_\mu^+, \quad \bar{S}\Psi_\mu^+ = 0, \]
\[ \bar{S}^2 = 0 \]  

(58)

The action (46) can be expressed in the BRST exact form

\[ S = \frac{-4m}{\pi} \int d^2 \zeta \bar{S}W = \frac{-4m}{\pi} \int d^2 \zeta \bar{S}\bar{W}, \]  

(59)

where we have introduced so called gauge fermions

\[ W = \partial_- \Psi_\mu^+ \frac{F_\mu}{\sqrt{F^2}}, \quad \bar{W} = \partial_+ \Psi_\mu^+ \frac{F_\mu}{\sqrt{F^2}} \]  

(60)

Then we can observe that these two nilpotent symmetries anticommute

\[ \{S, \bar{S}\} = 0 \]  

(61)

and we can express our two gauge fermions in term of one gauge boson and generators of the second symmetry

\[ W = \bar{S}\sqrt{F^2}, \quad \bar{W} = -S\sqrt{F^2}, \]
\[ S = \frac{-4m}{\pi} \int d^2 \zeta \quad \bar{S}S \sqrt{F^2}. \]

(62)  

(63)

So we can deduce that the action (46) is BRST and anti-BRST exact and somehow is equivalent to the sum of gauge fixing term and Faddeev-Popov determinant for some gauge transformation of our fields (may be in this case nonlocal).

One can formally express the partition function in terms of singular determinant, if for a moment ignore its zero modes. Indeed, integrating in \( Z \)

\[ Z = \int \exp \left[ iS \right] DF_\mu^1 \cdots DF_\mu^n D\Psi_\mu D\Psi_\mu^+ \]  

(64)

over \( \Psi_\mu^+ \) and \( X_\mu \) we shall get the following expression

\[ Z = \int \det \left[ \partial_- \frac{\delta \Pi_\mu}{\delta F_\mu} \partial_+ \right] \delta \left( \partial_+ \partial_- \Pi_\mu \right) DF_\mu, \]

(65)

where \( \Pi_\mu = \frac{F_\mu}{\sqrt{F^2}} \) and \( \frac{\delta \Pi_\mu}{\delta F_\mu} = \frac{K_{\mu\nu}}{\sqrt{F^2}} \). It is clear that if one changes the integration variable \( F_\mu \) to a unit length function \( \Pi_\mu \) then two determinants will cancel each other and the partition function \( Z \) will be equal to one. But we have to notice that this transformation is degenerate because the corresponding Jacobian \( \frac{\delta \Pi_\mu}{\delta F_\mu} \) is proportional to the projector \( K_{\mu\nu} \) and therefore one should properly account its zero modes.

As usually in the case of BRST exact actions [3], we can construct observables. For any form \( A = A(F)_\mu^1 \cdots _\mu^n dF^1 \wedge \cdots \wedge dF^n \) we can define the corresponding operator \( O_A \) replacing \( dF^\mu \) by the BRST exact operator \( \partial_- \Psi_\mu^+ \)

\[ O_A = A(F)_\mu^1 \cdots _\mu^n \partial_- \Psi_\mu^+ \cdots \partial_- \Psi_\mu^n. \]  

(66)
From (57) it follows that
\[ i\mathcal{S}\mathcal{O}_A = \frac{\partial A_{\mu_1...\mu_n}}{\partial F_{\mu_0}} \partial_- \Psi^\mu_0 \partial_- \Psi^\mu_1 ... \partial_- \Psi^\mu_n \] (67)
\[ i\mathcal{S}\mathcal{O}_A = \mathcal{O}_{dA} \] (68)
where \( dA \) is the exterior derivative of \( A \). Thus closed forms \( dA = 0 \) induce BRST invariant operators \( \mathcal{O}_A \) and de Rham cohomology classes of forms transform to the BRST cohomology classes, and we have at our disposal a set of nontrivial invariant observables [3].

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