A CLASS OF GENERALIZED QUASILINEAR SCHROEDINGER EQUATIONS

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Abstract. We establish the existence of nontrivial solutions for the following quasilinear Schrödinger equation with critical Sobolev exponent:

$$-\Delta u + V(x)u - \Delta[l(|u|^2)]l'(u^2)u = \lambda u^{2^* - 1} + h(u), \quad x \in \mathbb{R}^N,$$

where $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $l, h$ are real functions, $\lambda \geq 0$, $\alpha > 1$, $2^* = \frac{2N}{N-2}$, $N \geq 3$. Our results cover two physical models $l(s) = s^\alpha$ and $l(s) = (1 + s)^{\alpha/2}$ with $\alpha \geq 3/2$.

1. Introduction. In this paper, we consider the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - \Delta[l(|u|^2)]l'(u^2)u = \lambda u^{2^* - 1} + h(u), \quad x \in \mathbb{R}^N,$$

where $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $l, h$ are real functions, $\lambda \geq 0$, $\alpha > 1$, $N \geq 3$. Solutions of (1) are related to standing wave solutions of Schrödinger equation

$$iz_t = -\Delta z + W(x)z - \rho(|z|^2)z - \Delta(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N,$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\rho$ is a real function. For $l(s) = s$, (2) was used for a superfluid film equation in plasma physics, which was introduced in Kurihara [14]. In the case $l(s) = (1 + s)^{1/2}$, (2) models the self-channeling of a high-power ultra short laser in matter [15]. Besides, equation (2) also appear in plasma physics and fluid mechanics [16], in dissipative quantum mechanics [11], in the theory of Heisenberg ferromagnetism and magnons [13, 22] and in condensed matter theory [20]. See also [10, 6] for more physical backgrounds.

Let $l(s) = s^\alpha$ or $l(s) = (1 + s)^{\alpha/2}$ in (1), we obtain the following corresponding equation of elliptic type:

$$-\Delta u + V(x)u - \frac{\alpha}{2} \Delta(|u|^\alpha)|u|^\alpha - 2u = \lambda u^{2^* - 1} + h(u), \quad x \in \mathbb{R}^N,$$

or

$$-\Delta u + V(x)u - \frac{\alpha}{2} \left[\Delta(1 + u^2)^{\alpha/2}\right] \frac{u}{(1 + u^2)^{\alpha/2}} = \lambda u^{2^* - 1} + h(u), \quad x \in \mathbb{R}^N.$$

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If \( \alpha = 2 \), equation (3) has been studied extensively recently. See, for example, [21, 7, 17, 18, 9, 1, 2, 23, 25] for \( \lambda = 0 \) (i.e., the subcritical case) and [9, 26, 29, 27, 28] for \( \lambda \neq 0 \) (i.e., the critical case). For general \( \alpha > 0 \), by using a minimization argument, Liu and Wang in [17] studied the following quasilinear Schrödinger equation with subcritical growth

\[ -\Delta u + V(x)u - \Delta(|u|^{2\alpha})|u|^{2\alpha-2}u = \lambda|u|^{p-1}u, \quad x \in \mathbb{R}^N. \]  

(5)

where \( \alpha > 1/2 \). They proved the existence of a solution for a sequence of \( \lambda_n \to \infty \) and a sequence of \( \lambda_n \to 0 \) provided \( 4\alpha \leq p + 1 < 2\alpha 2^* \). In [1] and [2], Adachina and Watanable considered the following changing of unknown variable \( v = f^{-1}(u) \) which was introduced in [18] for \( \alpha = 1 \) in (5) (see also [7]), where \( f \) is defined by ODE:

\[ f'(t) = \frac{1}{\sqrt{1 + 2af^{4\alpha-2}(t)}}, \quad t \in [0, +\infty); \]

\[ f(t) = -f(-t), \quad t \in (-\infty, 0). \]

(6)

Applying the change of variable (6), the quasilinear equation (5) is reduced to a semilinear equation. Then, by using variational techniques, the existence of unique solution and multiple positive solutions were established.

In the mathematical literature, few results are known on (4). For \( \alpha = 1 \), in [8], Bouard, Hayashi and Saut proved the global existence and uniqueness of small solutions in transverse space dimensions 2 and 3. But, they did not studied the existence of standing waves. For standing waves and \( \lambda = 0 \), we refer to [25].

Our goal in this paper is to study the existence of nontrivial solutions for quasilinear Schrödinger equation (1) with critical exponent. Without loss of generality, in what follows, we set \( \lambda = 1 \) in (1).

Let

\[ g(s) = \sqrt{1 + 2[sf'(s^2)]^2}. \]

It is easy to check that (1) is equivalent to equation

\[ -\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = u^{2^* - 1} + h(u), \quad x \in \mathbb{R}^N. \]

(7)

We assume \( g(s) : \mathbb{R} \to \mathbb{R}^+ \) satisfies the following conditions:

\( (g_1) \) \( g(s) \) is a nondecreasing positive function with respect to \( |s| \), \( g(s) = g(-s) \), \( g(0) = 1 \) and \( g(s) \in C^m \) with \( m \in \mathbb{Z}^+ \);

\( (g_2) \) \( \lim_{s \to \infty} \frac{g(s)}{|s|^{\alpha-1}} = c_\alpha > 0 \), where \( \alpha > 1 \);

\( (g_3) \) \( 0 < \frac{sg'(s)}{g(s)} \leq \alpha - 1 \) for all \( s \in \mathbb{R} \);

\( (g_4) \) there exists \( \tau > 0 \) such that for \( s \geq G^{-1}\left(\frac{\tau + \alpha}{\alpha}\right) \), there holds \( G(s) \leq \frac{\tau}{\alpha} (\tau + s)^\alpha \), where \( G(s) = \int_0^s g(t)dt \).

**Example 1.1.** The following functions satisfy \( (g_1) - (g_4) \):

1. \( g(s) = \sqrt{1 + \frac{\alpha^2}{2} s^{2(\alpha-1)}} \) for \( \alpha \geq 3/2 \);
2. \( g(s) = \sqrt{1 + \frac{\alpha^2 s^2}{2(1 + s^2)^2}} \) for \( \alpha \geq 3/2 \).

For the proof of (1) and (2), see section 3.

The potential \( V : \mathbb{R}^N \to \mathbb{R} \) is continuous and satisfies:

\( (V_0) \) \( V(x) \geq V_0 > 0 \), \( \) for all \( x \in \mathbb{R}^N \).
By \( (V_1) \lim_{|x| \to \infty} V(x) = V_\infty \) and \( V(x) \leq V_\infty, V(x) \neq V_\infty \), for all \( x \in \mathbb{R}^N \).

The nonlinearity \( h : \mathbb{R} \to \mathbb{R} \) is continuous, we also suppose the following hypotheses:

\[(h_1)\] 
\[h(s) = o(s) \text{ as } s \to 0;\]

\[(h_2)\] 
\[\text{there exists } q \in (2, 2^*) \text{ such that } |h(s)| \leq C(1 + |s|^{q-1}) \text{ for some constant } C > 0 \text{ and all } s \in \mathbb{R};\]

\[(h_3)\] 
\[\text{there exists } \mu \in (2\alpha, 2^*) \text{ such that for all } s \in \mathbb{R}, 0 < \mu H(s) \leq sh(s), \text{ where } H(s) = \int_0^s h(t)dt.\]

Now, we may state our main result:

**Theorem 1.1.** Assume that \( \alpha > 1, (g_1) - (g_4), (V_0) - (V_1) \) and \( (h_1) - (h_3) \), then \( (1) \) has a nontrivial solution if either \( N \geq 4\alpha + 2 \) and \( \mu > 2\alpha \) or \( 3 \leq N < 4\alpha + 2 \) and \( \mu > 2^* - 1 \).

**Remark 1.** When \( g(s) = \sqrt{1 + \frac{\alpha^2}{2} s^{2(\alpha - 1)}} \), we have

\[g'(s) = \frac{(\alpha - 1)\alpha^2 s^{2\alpha - 3}}{\sqrt{4 + 2\alpha^2 s^{2(\alpha - 1)}}},\]

To guarantee \( (g_1) \) holding, we require \( \alpha \geq 3/2 \) since \( g'(s) \) is not continuous at \( s = 0 \) for \( \alpha < 3/2 \). On the other hand, it should point out that \( \alpha \geq 3/2 \) is also necessary to guarantee \( (g_4) \) holding. For \( \alpha < 3/2 \), to our knowledge, there are few results on unbounded domain.

**Theorem 1.2.** Assume that \( \alpha \geq 3/2, (V_0) - (V_1) \) and \( (h_1) - (h_3) \), then \( (3) \) has a nontrivial solution if either \( N \geq 4\alpha + 2 \) and \( \mu > 2\alpha \) or \( 3 \leq N < 4\alpha + 2 \) and \( \mu > 2^* - 1 \).

**Theorem 1.3.** Assume that \( \alpha \geq 3/2, (V_0) - (V_1) \) and \( (h_1) - (h_3) \), then \( (4) \) has a nontrivial solution if either \( N \geq 4\alpha + 2 \) and \( \mu > 2\alpha \) or \( 3 \leq N < 4\alpha + 2 \) and \( \mu > 2^* - 1 \).

In order to prove our Theorems, motivated by the arguments used in [26, 25], we first use a change of variable to reformulate the problem by a semilinear problem which the associated functional is well-defined in the Sobolev space \( H^1(\mathbb{R}^N) \) and satisfies the geometric hypotheses of the mountain pass theorem. Using this fact, we obtain a Palais-Smale sequence converging weakly to a solution \( v \). In order to prove that \( v \) is nontrivial, arguing by contradiction that \( v = 0 \), we combine Lions’s compactness lemma together with some classical arguments used by H. Brezis and L. Nirenberg [5] to establish that the Palais-Smale sequence has a nonvanishing behaviour. Finally, a translated Palais-Smale sequence converges to a nontrivial critical point of an associated functional at infinity. Then, this critical point is used to construct a path related to mountain pass theorem to find a contradiction.

In this paper, we shall work on the space \( H^1(\mathbb{R}^N) \) with the norm

\[\|u\| = \left[ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \right]^{1/2}.\]

By \( (V_0) \) and \( (V_1) \), the norm \( \| \cdot \| \) is equivalent to the usual one in \( H^1(\mathbb{R}^N) \). \( C, C_i, i = 1, 2, \cdots \) denote positive (possibly different) constant.
2. Proof of Theorem 1.1. We observe that (1) is the Euler-Lagrange equation of the following functionals:

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ g^2(u) |\nabla u|^2 + V(x)u^2 \right] dx - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} |u|^\alpha 2^* dx - \int_{\mathbb{R}^N} H(u) dx, \]

where \( g(s) = \sqrt{1 + 2|s'|^2} \). From the variational point view, the first difficulty associated with (8) is to find an appropriate function space where it is well defined. To overcome this difficulty, as in [24, 25], we make change of variables as the following:

\[ v = G(u) = \int_0^u g(s) ds, \quad u = G^{-1}(v). \]

Then, after the changing of variables, \( I \) can be written by the following functional

\[ J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} |G^{-1}(v)|^\alpha 2^* dx \]

\[ - \int_{\mathbb{R}^N} H(G^{-1}(v)) dx. \]

Under the hypotheses (\( g_1 \)) - (\( g_4 \)), (\( V_0 \)) - (\( V_1 \)) and (\( h_1 \)) - (\( h_3 \)), \( J \) is well defined in \( H^1(\mathbb{R}^N) \) and \( J \in C^1 \).

If \( u \) is a nontrivial solution of (7), then it should satisfy

\[ \int_{\mathbb{R}^N} \left[ g^2(u) \nabla u \nabla \varphi + g(u)g'(u)|\nabla u|^2 \varphi + V(x)u \varphi - u^{\alpha 2^*-1} \varphi - h(u) \varphi \right] dx = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \]

We show that (10) is equivalent to

\[ J'(v) \psi = \int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{G^{-1}(v)^{\alpha 2^*-1}}{g(G^{-1}(v))} \psi - \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi \right] dx = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \]

Indeed, if we choose \( \varphi = \frac{1}{g(u)} \psi \) in (10), then we get (11). On the other hand, since \( u = G^{-1}(v) \), if we let \( \psi = g(u) \varphi \) in (11), we get (10). Therefore, in order to find the nontrivial solutions of (7), it suffices to study the existence of the nontrivial solutions of the following equation

\[ -\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{G^{-1}(v)^{\alpha 2^*-1}}{g(G^{-1}(v))} + \frac{h(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \]

We collect some properties of the change of variable \( G^{-1}(t) \).

Lemma 2.1. (1) \( G(t) \in C^{m+1} \) and \( G^{-1}(t) \in C^{m+1} \), \( m \in \mathbb{Z}^+ \).

(2) \( \lim_{t \to 0} \frac{G^{-1}(t)}{t} = 1 \);

(3) \( \lim_{t \to \infty} \frac{G^{-1}(t)^{\alpha}}{t} = \frac{\alpha}{c_\alpha} \);

(4) \( |G^{-1}(t)|^\beta \leq 1, \text{ for all } t \in \mathbb{R}; \)

(5) \( \frac{1}{g(G^{-1}(t))} \leq \frac{G^{-1}(t)}{t} \leq 1, \text{ for all } t \in \mathbb{R}; \)

(6) \( \frac{|G^{-1}(t)|^2}{g(G^{-1}(t))} \leq G^{-1}(t) t \leq |G^{-1}(t)|^2 g(G^{-1}(t)), \text{ for all } t \in \mathbb{R}. \)

(7) \( |G^{-1}(t)|^\alpha \leq \frac{\alpha}{c_\alpha} |t|, \text{ for all } t \in \mathbb{R}; \)

(8) \( \text{there exists } \delta > 0 \text{ such that } |G^{-1}(t)|^\alpha \geq \delta |t|, \text{ for } |t| \geq 1. \)
Proof. By \((g_1)-(g_3), (1)-(6)\) are obviously. By \((g_3),\) we get \([\alpha t-g(G^{-1}(t))G^{-1}(t)]' \geq 0.\) Thus
\[
\left(\frac{|G^{-1}(t)|^\alpha}{|t|}\right)' = \frac{|G^{-1}(t)|^{\alpha-1}[\alpha t-g(G^{-1}(t))G^{-1}(t)]}{t^2g(G^{-1}(t))} \geq 0, \quad \text{if } t > 0, \quad \text{leq 0, } \quad \text{if } t < 0.
\] (13)

Inequality (7) is a consequence of (3) and (13). Moreover, by (13), for \(|t| \geq 1,\) we have
\[
\frac{|G^{-1}(t)|^\alpha}{|t|} \geq |G^{-1}(1)|^\alpha.
\]
Thus (8) is proved. \(\square\)

Lemma 2.2. There exist \(C_1, C_2 > 0\) such that for \(t > C_1,\) there holds
\[
G^{-1}(t)^{\alpha^2} - \left(\frac{\alpha}{C_1}\right)^{2^*} \geq -C_2t^{2^*-1}\alpha^2.
\]

Proof. By the mean value theorem, for \(0 \leq \tau \leq x,\) where \(\tau\) is from \((g_4),\) we have
\[
x^{\alpha^2} - (x - \tau)^{\alpha^2} = \alpha^2(x - \theta \tau)^{\alpha^2-1}\tau, \quad 0 \leq \theta \leq 1.
\]
Now, we choose \(x = \left(\frac{\alpha}{C_1}\right)^{\frac{1}{\alpha^2}},\) it follows that
\[
\alpha^2\left(\frac{\alpha}{C_1}\right)^{\frac{2^*}{\alpha^2}} \tau \geq \alpha^2\left[\left(\frac{\alpha}{C_1}\right)^{\frac{1}{2}} - \theta \tau\right]^{\alpha^2-1}\tau
\]
\[
= \left(\frac{\alpha}{C_1}\right)^{2^*} - \left(\left(\frac{\alpha}{C_1}\right)^{\frac{1}{2}} - \tau\right)^{\alpha^2}
\]
\[
\geq \left(\frac{\alpha}{C_1}\right)^{2^*} - G^{-1}(t)^{\alpha^2},
\]
which implies the result. \(\square\)

Now, we establish the geometric hypotheses of the mountain pass theorem.

Lemma 2.3. Assume \((V_0) - (V_1)\) and \((h_1) - (h_3).\) Then,

(i) there exist \(\rho_0, a_0 > 0,\) such that \(J(v) \geq a_0\) for \(||v|| = \rho_0.\)

(ii) There exists \(c \in H^1(\mathbb{R}^N)\) such that \(J(c) < 0.\)

Proof. (i) Let
\[
K(x, s) := -\frac{1}{2}V(x)|G^{-1}(s)|^2 + \frac{1}{\alpha^2}|G^{-1}(s)|^{\alpha^2} + H(G^{-1}(s)).
\]
Then, by Lemma 2.1-(2) and \((h_1),\) we have
\[
\lim_{s \to 0} \frac{K(x, s)}{s^2} = \lim_{s \to 0} \left[ -\frac{1}{2}V(x)\left(\frac{G^{-1}(s)}{s}\right)^2 + \frac{1}{\alpha^2}\left(\frac{G^{-1}(s)}{s}\right)^2|G^{-1}(s)|^{\alpha^2-2} + \frac{H(G^{-1}(s))}{s^2} \right]
\]
\[
= -\frac{1}{2}V(x).
\]
By Lemma 2.1—(3) and (b2), we get
\[
\lim_{s \to \infty} \frac{K(x, s)}{s^{2^*}} = \lim_{s \to \infty} \left[ -\frac{1}{2} V(x) \left( \frac{(G^{-1}(s))^{2\alpha}}{s^{2}} \right)^{1/\alpha} + \frac{1}{\alpha 2^*} \left( \frac{|G^{-1}(s)|^\alpha}{|s|} \right)^{2^*} H(G^{-1}(s)) \right] + \frac{1}{\alpha 2^*} \left( \frac{\alpha}{c_\alpha} \right)^{2^*}.
\]
Thus, for \( \varepsilon > 0 \) sufficiently small, there exists a constant \( C_\varepsilon > 0 \) such that
\[
K(x, s) \leq \left( -\frac{1}{4} V(x) + \varepsilon \right) s^2 + C_\varepsilon s^{2^*}.
\]
Then, we have
\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha 2^*} dx - \int_{\mathbb{R}^N} H(G^{-1}(v)) dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V(x)v^2 dx - \varepsilon \int_{\mathbb{R}^N} v^2 dx - C_\varepsilon \int_{\mathbb{R}^N} v^2 dx
\]
\[
\geq C\|v\|^2 - C\|v\|^{2^*}.
\]
Therefore, by choosing \( \rho_0 \) small, we get (1) when \( \|v\| = \rho_0 \).
(ii) Given \( \varphi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) with \( \text{supp}\varphi = \bar{B}_1 \). By Lemma 2.1—(8), we have
\[
J(t\varphi) \leq \frac{1}{2} t^2 \int_{\mathbb{R}^N} (|\nabla \varphi|^2 + V(x)\varphi^2) dx - t^2 C \int_{\{|t\varphi| \geq 1\}} \varphi^2 dx
\]
\[
\to -\infty, \quad \text{as } t \to \infty.
\]
Let \( e = t\varphi \) with \( t \) large enough, we get the result. \( \square \)

In consequence of Lemma 2.3 and of Ambrosetti–Rabinowitz mountain mass theorem [3], for the constant
\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J(\gamma(t)) > 0,
\]
where \( \Gamma = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0 \} \), there exists a Palais-Smale sequence at level \( c \), that is, \( J(v_n) \to c \) and \( J'(v_n) \to 0 \) as \( n \to \infty \).

Lemma 2.4. The Palais-Smale sequence \( \{v_n\} \) for \( J \) is bounded.

Proof. Note that \( \{v_n\} \subset H^1(\mathbb{R}^N) \) satisfies
\[
J(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) dx - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{\alpha 2^*} dx
\]
\[
- \int_{\mathbb{R}^N} H(G^{-1}(v_n)) dx
\]
\[
=c + o_n(1)
\]
and for any $\psi \in C_0^\infty(\mathbb{R}^N)$,
\[
J'(v_n)\psi = \int_{\mathbb{R}^N} \left[ \nabla v_n \nabla \psi + V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \psi - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} \psi \right] dx
\]
\[
= o_n(1) \|\psi\|.
\] (15)
where $o_n(1) \to 0$ as $n \to \infty$. Now, we consider the function $G^{-1}(v_n)g(G^{-1}(v_n))$.

From $(g_3)$ and Lemma 2.1–(6), we have
\[
|\nabla (G^{-1}(v_n))g(G^{-1}(v_n))| \leq \left[ 1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right] |\nabla v_n| \leq \alpha|\nabla v_n|.
\]

So, we have $G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, by choosing $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ in (15), we deduce that
\[
o_n(1) \|v_n\| = J'(v_n)G^{-1}(v_n)g(G^{-1}(v_n))
\]
\[
= \int_{\mathbb{R}^N} \left[ 1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right] |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2
\]
\[
- |G^{-1}(v_n)|^{\alpha^2} - h(G^{-1}(v_n))G^{-1}(v_n) \right] dx
\]
\[
\leq \int_{\mathbb{R}^N} \left[ \alpha|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 - |G^{-1}(v_n)|^{\alpha^2}
\]
\[
- h(G^{-1}(v_n))G^{-1}(v_n) \right] dx.
\] (16)

Therefore, by $(h_3)$, (14) and (16), we have
\[
\mu c + o_n(1) \|v_n\| = \mu J(v_n) - J'(v_n)G^{-1}(v_n)g(G^{-1}(v_n))
\]
\[
\geq \left( \frac{\mu}{2} - \alpha \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \left( \frac{\mu}{2} - 1 \right) \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx
\]
\[
+ \left( 1 - \frac{\mu}{\alpha^2} \right) \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{\alpha^2} dx
\]
\[
+ \int_{\mathbb{R}^N} \left[ h(G^{-1}(v_n))G^{-1}(v_n) - \mu H(G^{-1}(v_n)) \right] dx
\]
\[
\geq \left( \frac{\mu}{2} - \alpha \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \left( \frac{\mu}{2} - 1 \right) \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx.
\] (17)

On the other hand, by Sobolev imbedding inequality, we have
\[
\int_{|x|:v_n(x)|>1} V(x)v_n^2 dx \leq C \int_{|x|:v_n(x)|>1} v_n^2 dx \leq C \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{2/2}.
\] (18)

In the case $\{ x : |v_n(x)| \leq 1 \}$, since $g(v_n)$ is nondecreasing, we get
\[
\int_{|x|:v_n(x)|\leq1} V(x)v_n^2 dx \leq g^2(G^{-1}(1)) \int_{|x|:v_n(x)|\leq1} V(x)|G^{-1}(v_n)|^2 dx
\]
\[
\leq C \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx.
\] (19)
Combining (17)–(19), recalling $\mu > 2\alpha$, we deduce that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. \hfill \Box

By Lemma 2.4, since $\{v_n\}$ is a bounded Palais-Smale sequence, up to a subsequence if necessary, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2^*)$. We shall prove that $J'(v) = 0$, that is, $v$ is a weak solution of (12). To prove this, it suffices to show for $\forall \psi \in C_0^\infty(\mathbb{R}^N)$ that

$$
\int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{G^{-1}(v)^{\alpha - 1}}{g(G^{-1}(v))} \psi - \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi \right] dx = 0. \quad (20)
$$

Analogous to the arguments in [26], by the Lebesgue Dominated Theorem, we have

$$
J'(v_n)\psi - J'(v)\psi = \int_{\mathbb{R}^N} \left( \nabla v_n - \nabla v \right) \nabla \psi dx + \int_{\mathbb{R}^N} V(x) \left[ \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right] \psi dx
$$

$$
- \int_{\mathbb{R}^N} \left[ \frac{G^{-1}(v_n)^{\alpha - 1}}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)^{\alpha - 1}}{g(G^{-1}(v))} \right] \psi dx
$$

$$
- \int_{\mathbb{R}^N} \left[ \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \right] \psi dx
$$

$$
= o_n(1).
$$

Hence, $v$ is a weak solution of (1). If $v \equiv 0$, then Theorem 1.1 is proved. Otherwise, we further establish the Palais-Smale sequence has a nonvanishing behaviour. To this end, we state the result which provides an appropriate estimate on the minimax level.

**Lemma 2.5.** The minimax level $c$ satisfies

$$
c < \frac{1}{\alpha N} \left( \frac{c_0^2}{\alpha S} \right)^{\frac{2}{N}},
$$

where $S$ is the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

**Proof.** It suffices to show that there exists $v_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$
\max_{t \geq 0} J(tv_0) < \frac{1}{\alpha N} \left( \frac{c_0^2}{\alpha S} \right)^{\frac{2}{N}}.
$$

We follow the strategy used in [5]. First, we choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $0 \leq \varphi(x) \leq 1$ on $B_2(0)$.

Let $\psi_\varepsilon(x) = \varphi(x) w_\varepsilon(x)$, where

$$
w_\varepsilon(x) = \frac{[N(N - 2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}, \quad \forall \varepsilon > 0.
$$

It is known that $w_\varepsilon(x)$ satisfies the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$ and

$$
\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 dx = \int_{\mathbb{R}^N} w_\varepsilon^{2^*} = S^{N/2}, \quad (21)
$$

$$
\int_{\mathbb{R}^N \setminus B_1(0)} |\nabla w_\varepsilon|^2 = O(\varepsilon^{N-2}), \quad \text{as} \quad \varepsilon \to 0. \quad (22)
$$
Thus, if we define the function $v_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon|_2^2}$, then, by (21) and (22), as $\varepsilon \to 0$, we have

$$
\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = S + O(\varepsilon^{N-2}), \quad |v_\varepsilon|^2 - \frac{1}{\varepsilon} = O(\varepsilon^{\frac{N-2}{2N}}), \quad \text{if } N \geq 3
$$

and

$$
|v_\varepsilon|^2 = \begin{cases} 
O(\varepsilon), & \text{if } N = 3, \\
O(\varepsilon^2 \log \varepsilon), & \text{if } N = 4, \\
O(\varepsilon^2), & \text{if } N \geq 5.
\end{cases}
$$

Since $\lim_{\varepsilon \to 0} J(t_\varepsilon v_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$ such that $J(t_\varepsilon v_\varepsilon) = \max_{t \geq 0} J(tv_\varepsilon)$. We claim that there exist $\varepsilon_0 > 0$ and positive constants $t_0, t_1 > 0$ such that $t_0 \leq t_\varepsilon \leq t_1$ for every $0 < \varepsilon < \varepsilon_0$. First, we prove that $t_\varepsilon$ is bounded from below by a positive constant. Otherwise, we could find a sequence $\varepsilon_n \to 0$ such that $t_{\varepsilon_n} \to 0$. Up to a subsequence (still denote by $\varepsilon_n$), we have $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$. Therefore, $0 < c \leq \sup_{t \geq 0} J(t_{\varepsilon_n} v_{\varepsilon_n}) = J(0) = 0$, which is a contradiction. On the other hand, similar to the arguments used in [26] and Lemma 2.1-(8), we have

$$
c \leq J(t_\varepsilon v_\varepsilon)
\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(t_\varepsilon v_\varepsilon)|^2 dx - \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |G^{-1}(t_\varepsilon v_\varepsilon)|^{\alpha^2} dx
\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx - \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |G^{-1}(t_\varepsilon v_\varepsilon)|^{\alpha^2} dx
\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx - C t_{\varepsilon}^{\alpha^2}
\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx - C t_{\varepsilon}^{\alpha^2},
$$

where $C > 0$ is independent of $\varepsilon$. Since $|v_\varepsilon|$ is uniformly bounded for $0 < \varepsilon < \varepsilon_0$, the claim is proved.

Now, by Lemma 2.2, we have

$$
\int_{\mathbb{R}^N} |G^{-1}(t_\varepsilon v_\varepsilon)|^{\alpha^2} dx
= \int_{\{x \in \mathbb{R}^N : t_\varepsilon v_\varepsilon(x) \leq C_1\}} \left[ |G^{-1}(t_\varepsilon v_\varepsilon)|^{\alpha^2} - \left( \frac{\alpha}{c_\alpha} t_\varepsilon v_\varepsilon \right)^{\alpha^2} \right] dx + \int_{\{x \in \mathbb{R}^N : t_\varepsilon v_\varepsilon(x) > C_1\}} |G^{-1}(t_\varepsilon v_\varepsilon)|^{\alpha^2} dx
\geq - \frac{\alpha^2}{c_\alpha^2} \int_{\{x \in \mathbb{R}^N : t_\varepsilon v_\varepsilon(x) \leq C_1\}} (t_\varepsilon v_\varepsilon)^{2^*} dx
- C_1 \int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^* - \frac{1}{\alpha}} dx + \left( \frac{\alpha}{c_\alpha} t_\varepsilon \right)^{\alpha^2}
\geq - C \int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^* - \frac{1}{\alpha}} dx + \left( \frac{\alpha}{c_\alpha} t_\varepsilon \right)^{\alpha^2}.
$$

(24)
Therefore, we have
\[
J(t_\varepsilon v_\varepsilon) \leq \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx + \frac{1}{2} t_\varepsilon^2 \int_{\mathbb{R}^N} V(x) v_\varepsilon^2 dx - \frac{1}{\alpha^2} \int_{\mathbb{R}^N} \left( \frac{\alpha}{c_\alpha} t_\varepsilon \right)^{2^*} \nabla v_\varepsilon \cdot \nabla \xi(t_\varepsilon) dx + C \int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^* - \frac{4}{N}} dx - \int_{\mathbb{R}^N} H(G^{-1}(t_\varepsilon v_\varepsilon)) dx.
\] (25)

Let \( A = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx \), considering the function \( \xi : [0, +\infty) \to \mathbb{R} \) given by
\[
\xi(t) = \frac{1}{2} t^2 A - \frac{1}{\alpha^2} \left( \frac{\alpha}{c_\alpha} \right)^{2^*} t^{2^*},
\]
we have \( t_0 = (\alpha A)^{\frac{1}{2^* - 2}} \left( \frac{c_\alpha}{\alpha} \right)^{\frac{2^* - 2}{2^* - 1}} \) is the maximum point of \( \xi(t) \) and
\[
\max_{t \in [0, +\infty)} \xi(t) = \xi(t_0) = \frac{1}{\alpha N} \left( \frac{c_\alpha^2}{\alpha} \right)^{N/2} A^{N/2}.
\]
Thus, by (25), we deduce that
\[
J(t_\varepsilon v_\varepsilon) \leq \frac{1}{\alpha N} \left( \frac{c_\alpha^2}{\alpha} \right)^{N/2} \left[ S + O(\varepsilon^{N-2}) \right]^{N/2} + C |v_\varepsilon|^2 + C |v_\varepsilon|^{2^* - \frac{4}{N}} - \int_{\mathbb{R}^N} H(G^{-1}(t_\varepsilon v_\varepsilon)) dx.
\] (26)

Therefore, by using the following inequality:
\[
(a + b)^r \leq a^r + r(a + b)^{r-1}b, \quad \text{for any} \quad a, b > 0, r \geq 1,
\]
we have
\[
J(t_\varepsilon v_\varepsilon) \leq \frac{1}{\alpha N} \left( \frac{c_\alpha^2}{\alpha} \right)^{N/2} \left[ S + O(\varepsilon^{N-2}) \right]^{N/2} + C |v_\varepsilon|^2 + C |v_\varepsilon|^{2^* - \frac{4}{N}} - \int_{\mathbb{R}^N} H(G^{-1}(t_\varepsilon v_\varepsilon)) dx + O(\varepsilon^{N-2}).
\] (27)

By the definition of \( v_\varepsilon \) and since \( G^{-1}(s) \) is increasing, there exists constant \( \delta > 0 \) such that for \( |x| \leq \varepsilon \), it follows that
\[
H(G^{-1}(t_\varepsilon v_\varepsilon)) \geq C \varepsilon^{-1}(t_\varepsilon v_\varepsilon)^{\mu} \geq C \varepsilon^{-1}(t_0 v_\varepsilon)^{\mu} \geq C \varepsilon^{-1}(\varepsilon^{2-N})^{\mu}.
\]

We choose \( 0 < \varepsilon_1 \leq \varepsilon_0 \) such that \( \delta \varepsilon^{2-N}/2 \geq 1 \) for \( |x| \leq \varepsilon \) and \( 0 < \varepsilon \leq \varepsilon_1 \). Then by lemma 2.1–(9), we have
\[
\int_{\mathbb{R}^N} H(G^{-1}(t_\varepsilon v_\varepsilon)) dx \geq C \int_{B_\varepsilon(0)} |G^{-1}(t_\varepsilon v_\varepsilon)|^{\mu} dx \geq C_0 \varepsilon^{\frac{(2-N)\mu}{2\alpha}} + N.
\] (28)

Therefore, by (27),
\[
J(t_\varepsilon v_\varepsilon) \leq \frac{1}{\alpha N} \left( \frac{c_\alpha^2}{\alpha} \right)^{N/2} S^{N/2} + C |v_\varepsilon|^2 + O(\varepsilon^{N-2}) - C_0 \varepsilon^{\frac{(2-N)\mu}{2\alpha}} + N + O(\varepsilon^{N-2}).
\] (29)

Let
\[
B(\varepsilon) = C |v_\varepsilon|^2 + O(\varepsilon^{N-2}) - C_0 \varepsilon^{\frac{(2-N)\mu}{2\alpha}} + N + O(\varepsilon^{N-2}).
\]

We will prove our result if we show that \( B(\varepsilon) < 0 \) if \( \varepsilon > 0 \) is small enough.

For \( N = 3 \), if \( \varepsilon > 0 \) is small enough, we have
\[
B(\varepsilon) = C \varepsilon^3 + C \varepsilon^3 - C \varepsilon^{3-\frac{4}{N}} < 0, \quad \text{if} \quad \mu > 6\alpha - 1.
\] (30)

For \( N = 4 \), if \( \varepsilon > 0 \) is small enough, we have
\[
B(\varepsilon) = C \varepsilon^4 |\log \varepsilon| + C \varepsilon^3 - C \varepsilon^{4-\frac{4}{N}} + C \varepsilon^2 < 0, \quad \text{if} \quad \mu > 4\alpha - 1.
\] (31)
For $N \geq 5$, if $\varepsilon > 0$ is small enough, we have
\[
B(\varepsilon) = C\varepsilon^2 + O(\varepsilon^{N/2}) - C_0\varepsilon^{(2-N)/4} + N + O(\varepsilon^{N-2})
\]
\[
= \varepsilon^{(2-N)/4} + O(\varepsilon^{(2-N)/4} + N) - C_0 + O(\varepsilon^{2N/2N-2}) < 0, \quad \text{if } \mu > \max \left\{2\alpha, \alpha2^* - 1\right\}.
\]
Combining (29)–(32), the result holds for either $N \geq 4\alpha + 2$ and $\mu > 2\alpha$ or $3 \leq N < 4\alpha + 2$ and $\mu > \alpha2^* - 1$, and the Lemma follows.

**Proposition 1.** Assume that $\{v_n\}$ is a Palais-Smale sequence for $J$ at level $c$ with $c < \frac{1}{\alpha N} \left(\frac{\alpha^2}{\alpha} S\right)^{\frac{N}{2}}$. Then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $r, \eta > 0$ such that
\[
\limsup_{n \to \infty} \int_{B_r(y_n)} v_n^2 \, dx \geq \eta > 0.
\]
**Proof.** By contradiction, using Lions Lemma [4], we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^p \, dx = 0,
\]
for $p \in (2, 2^*)$.

By (h2) and (33), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{h(G^{-1}(v_n))v_n}{g(G^{-1}(v_n))} \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} H(G^{-1}(v_n)) \, dx = 0.
\]
We claim that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) \left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\right] \, dx = 0;
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) \left[|G^{-1}(v_n)|^2 - v_n^2\right] \, dx = 0;
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left[\frac{G^{-1}(v_n)^{\alpha2^*-1}v_n}{g(G^{-1}(v_n))} - \frac{1}{\alpha} s_{\alpha}^{\alpha2^*} v_n^{2^*}ight] \, dx = 0;
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left[\frac{1}{\alpha} s_{\alpha}^{\alpha2^*} v_n^{2^*} - |G^{-1}(v_n)|^{\alpha2^*}\right] \, dx = 0.
\]
To prove (35), by Lemma 2.1–(6), we have
\[
0 \leq \int_{\mathbb{R}^N} V(x) \left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\right] \, dx
\]
\[
= \left[\int_{\{x: v_n(x) \geq \delta\}} + \int_{\{x: v_n(x) < \delta\}}\right] V(x) \left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))}\right] \, dx
\]
\[
\leq V_\infty(1 - \alpha^{-1}) \int_{\{x: v_n(x) \geq \delta\}} |G^{-1}(v_n)|^2 \, dx
\]
\[
+ \int_{\{x: v_n(x) < \delta\}} V(x)|G^{-1}(v_n)|^2 \left[1 - \frac{v_n}{G^{-1}(v_n)g(G^{-1}(v_n))}\right] \, dx
\]
By (33) and Lemma 2.1–(5), we have
\[ \int_{\{x:|v_n(x)| \geq \delta\}} |G^{-1}(v_n)|^2 dx \leq C \int_{\{x:|v_n(x)| \geq \delta\}} v_n^2 dx \leq C \int_{RN} |v_n|^p dx \to 0. \] (40)

By Lemma 2.1–(2) and \( \lim_{t \to 0} g(t) = 1 \), as \( \delta \to 0^+ \), we have
\[ \int_{\{x:|v_n(x)| < \delta\}} V(x)|G^{-1}(v_n)|^2 \left[ 1 - \frac{v_n}{G^{-1}(v_n)g(G^{-1}(v_n))} \right] dx \to 0. \] (41)

Combining (39)–(41), we get (35). Similarly, we can verify (36).

Next, we prove (37). As the proof of (35), we get
\[ \int_{RN} \left[ \frac{G^{-1}(v_n)\alpha^{2-1}v_n}{g(G^{-1}(v_n))} - \left( \frac{\alpha}{c_\alpha} \right)^{2^*} v_n^{2^*} \right] dx = \int_{\{x:|v_n(x)| \geq R\}} \left[ \frac{G^{-1}(v_n)\alpha^{2-1}v_n}{g(G^{-1}(v_n))} - \left( \frac{\alpha}{c_\alpha} \right)^{2^*} v_n^{2^*} \right] dx \]
\[ = \lim_{\{x:|v_n(x)| \geq \delta\}} \frac{v_n}{\|G^{-1}(v_n)\|} = \frac{1}{\alpha}. \] (42)

Combining (43)–(44), we prove (37). Similarly, we can verify (38).

Setting
\[ \lambda = \lim_{n \to +\infty} \int_{RN} \left[ |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 \right] dx, \quad \gamma = \lim_{n \to +\infty} \int_{RN} v_n^{2^*} dx \]
Then, by (14), (15) (choosing \( \psi = v_n \) as a test function), (34)–(38), we deduce we have
\[ c = \frac{\lambda}{2} - \frac{\gamma}{\alpha^{2^*}} \left( \frac{\alpha}{c_\alpha} \right)^{2^*}, \quad \text{and} \quad \lambda = \frac{1}{\alpha} \left( \frac{\alpha}{c_\alpha} \right)^{2^*} \gamma. \] (45)

Now, observing that
\[ S \left( \int_{RN} v_n^{2^*} dx \right)^\frac{2}{2^*} \leq \int_{RN} |\nabla v_n|^2 dx \leq \int_{RN} \left[ |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 \right] dx, \]
we get \( S \gamma \frac{n}{N} \leq \lambda \). Therefore, by (45), we get
\[ c \geq \frac{1}{\alpha N} \left( \frac{\alpha^2 S}{\alpha} \right)^{\frac{n}{N}}, \]
which contradicts the assumption $c < \frac{1}{\alpha N} \left( \frac{\varepsilon^2}{\alpha} S \right)^{\frac{\alpha}{\alpha}}$.

We are now ready to prove Theorem 1.1. We assume that $v \equiv 0$. As in [25], \{v_n\} is also a Palais-Smale sequence for the functional $J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$:

$$J_\infty(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V_\infty |G^{-1}(v_n)|^2) dx - \frac{1}{\alpha 2^*} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{\alpha 2^*} dx$$

$$- \int_{\mathbb{R}^N} H(G^{-1}(v_n)) dx.$$ (46)

Moreover, $J_\infty(v_n) \to c$. By Proposition 1, there exist $r, \eta > 0$, and \{y_n\} $\subset \mathbb{R}^N$ such that

$$\limsup_{n \to \infty} \int_{B_r(y_n)} v_n^2 dx \geq \eta > 0.$$ (47)

Define $\bar{v}_n(x) = v_n(x + y_n)$. Since \{v_n\} is a Palais-Smale sequence for $J_\infty$, \{\bar{v}_n\} is also a Palais-Smale sequence for $J_\infty$. Arguing as in the case of \{v_n\} we get that $\bar{v}_n \to \bar{v}$ in $H^1(\mathbb{R}^N)$ with $J'_\infty(\bar{v}) = 0$. By (47), we have $\bar{v} \neq 0$. Now, by Lemma 2.1–(6), we note that

$$\frac{2}{\alpha 2^*} |G^{-1}(\bar{v}_n)|^{\alpha 2^*} - \frac{G^{-1}(\bar{v}_n)^{\alpha 2^* - 1}}{g(G^{-1}(\bar{v}_n))} \bar{v}_n = G^{-1}(\bar{v}_n)^{\alpha 2^* - 1} \left( \frac{2}{2^*} - 1 \right) \frac{\bar{v}_n}{g(G^{-1}(\bar{v}_n))} \leq 0.$$ (48)

From (h3) and Lemma 2.1–(6), it follows that

$$2H(G^{-1}(\bar{v}_n)) - \frac{h(G^{-1}(\bar{v}_n))}{g(G^{-1}(\bar{v}_n))} \bar{v}_n \leq 2H(G^{-1}(\bar{v}_n))g(G^{-1}(\bar{v}_n)) - h(G^{-1}(\bar{v}_n))\bar{v}_n \leq 0.$$

Therefore, by Fatou’s Lemma, we have

$$2c = \lim_{n \to \infty} [2J_\infty(\bar{v}_n) - J'_\infty(\bar{v}_n) \bar{v}_n]$$

$$= \lim_{n \to \infty} \int_{\mathbb{R}^N} V_\infty G^{-1}(\bar{v}_n) \left[ G^{-1}(\bar{v}_n) - \frac{1}{g(G^{-1}(\bar{v}_n))} \bar{v}_n \right] dx$$

$$- \lim_{n \to \infty} \int_{\mathbb{R}^N} G^{-1}(\bar{v}_n)^{\alpha 2^* - 1} \left[ \frac{2}{\alpha 2^*} G^{-1}(\bar{v}_n) - \frac{1}{g(G^{-1}(\bar{v}_n))} \bar{v}_n \right] dx$$

$$- \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ 2H(G^{-1}(\bar{v}_n)) - \frac{h(G^{-1}(\bar{v}_n))}{g(G^{-1}(\bar{v}_n))} \bar{v}_n \right] dx$$

$$\geq \int_{\mathbb{R}^N} V_\infty G^{-1}(\bar{v}) \left[ G^{-1}(\bar{v}) - \frac{1}{g(G^{-1}(\bar{v}))} \bar{v} \right] dx$$

$$- \int_{\mathbb{R}^N} G^{-1}(\bar{v})^{\alpha 2^* - 1} \left[ \frac{2}{\alpha 2^*} G^{-1}(\bar{v}) - \frac{1}{g(G^{-1}(\bar{v}))} \bar{v} \right] dx$$

$$- \int_{\mathbb{R}^N} \left[ 2H(G^{-1}(\bar{v})) - \frac{h(G^{-1}(\bar{v}))}{g(G^{-1}(\bar{v}))} \bar{v} \right] dx$$

$$= 2J_\infty(\bar{v}) - J'_\infty(\bar{v}) \bar{v}.$$

Thus, we have $J_\infty(\bar{v}) \leq c$. 

A CLASS OF GENERALIZED QUASILINEAR SCHröDINGER EQUATIONS 865
It follows the argument used in [12], we get a path \( \gamma(t) : [0, L] \to H^1(\mathbb{R}^N) \) such that
\[
\begin{align*}
\gamma(0) &= 0, J_\infty(\gamma(L)) < 0, \bar{v} \in \gamma([0, L]), \\
\gamma(t)(x) &= 0, \quad \forall x \in \mathbb{R}^N, t \in [0, L], \\
\max_{t \in [0, L]} J_\infty(\gamma(t)) &= J_\infty(\bar{v})
\end{align*}
\] (49)

In fact, as in [12], we define
\[
\bar{v}_t(x) = \begin{cases} 
\bar{v}(x/t), & \text{if } t > 0, \\
0, & \text{if } t = 0.
\end{cases}
\]

Then, we have
\[
\int_{\mathbb{R}^N} |\nabla \bar{v}_t|^2 \, dx = t^{N-2} \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 \, dx, \quad \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_t)|^2 \, dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^2 \, dx
\]
and
\[
\int_{\mathbb{R}^N} |G^{-1}(\bar{v}_t)|^\alpha \, dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^\alpha \, dx.
\]

Since \( \frac{d}{dt} J_\infty(\bar{v}_t) \big|_{t=1} = 0 \), we have
\[
\frac{N - 2}{2N} \int_{\mathbb{R}^N} |\nabla \bar{v}_t|^2 \, dx = - \frac{V_\infty}{2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_t)|^2 \, dx + \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^\alpha \, dx
\]
\[
+ \int_{\mathbb{R}^N} H(G^{-1}(\bar{v})) \, dx.
\]

Let \( \gamma(t)(x) = \bar{v}_t(x) \), we see that
\[
J_\infty(\gamma(t)) = t^{N-2} \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 \, dx - t^N \left[ - \frac{V_\infty}{2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^2 \, dx \right.
\]
\[
+ \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^\alpha \, dx + \int_{\mathbb{R}^N} H(G^{-1}(\bar{v})) \, dx \bigg]
\]

Thus \( \gamma(t) \in C([0, \infty), H^1(\mathbb{R}^N)) \), and
\[
\frac{d}{dt} J_\infty(\gamma(t)) = \frac{N - 2}{2} t^{N-3} \int_{\mathbb{R}^N} |\nabla \bar{v}_t|^2 \, dx - N \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_t)|^2 \, dx
\]
\[
+ \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v})|^\alpha \, dx + \int_{\mathbb{R}^N} H(G^{-1}(\bar{v})) \, dx
\]
\[
= \frac{N - 2}{2} t^{N-3} (1 - t^2) \int_{\mathbb{R}^N} |\nabla \bar{v}_t|^2 \, dx.
\]

So, \( \frac{d}{dt} J_\infty(\gamma(t)) > 0 \) for \( t \in (0, 1) \) and \( \frac{d}{dt} J_\infty(\gamma(t)) < 0 \) for \( t > 1 \). Thus for sufficiently large \( L > 1 \), we get the desired path. Define the set
\[
\Gamma_\infty = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, J_\infty(\gamma(1)) < 0 \}.
\]

After a suitable scale change in \( t \), we can assume \( \gamma(t) \in \Gamma_\infty \). Moreover,
\[
\max_{t \in [0, 1]} J_\infty(\gamma(t)) = J_\infty(\bar{v}) \leq c.
\]

Finally, we take the path \( \gamma \) given by (49). Since \( V(x) < V_\infty, \gamma \in \Gamma_\infty \subset \Gamma \), we have
\[
c \leq \max_{t \in [0, 1]} J(\gamma(t)) := J(\gamma(\bar{t})) < J_\infty(\bar{t}) \leq \max_{t \in [0, 1]} J_\infty(\gamma(t)) = J_\infty(\bar{v}) \leq c,
\]
which is a contradiction, and the result is proved. Therefore, \( v \) is a nontrivial solution.

3. **Proof of Theorems 1.2 and 1.3.** To prove Theorems 1.2 and 1.3, by Theorem 1.1, it suffices to check \( l(s) = s^{\frac{\alpha}{2}} \) and \( l(s) = (1 + s)^{\frac{\alpha}{2}} \) satisfy \( (g_1) - (g_4) \).

For \( l(s) = s^{\frac{\alpha}{2}} \), that is,

\[
g(s) = \sqrt{1 + \frac{\alpha^2}{2}s^{2(\alpha-1)}},
\]

where \( \alpha \geq 3/2 \), we have the following results:

1. \( g(s) \) is a nondecreasing positive function with respect to \( |s| \), \( g(0) = 1 \), \( g(s) \in C^1 \).
2. \( \lim_{s \to \infty} \frac{g(s)}{|s|^{\frac{\alpha}{2}}} = \frac{\alpha}{\sqrt{2}} \).
3. \( 0 \leq \frac{s}{g(s)}g'(s) \leq \alpha - 1 \), for all \( s \in \mathbb{R} \);

**Lemma 3.1.** For \( \alpha \geq 3/2 \), there holds

\[
G(s) \leq \frac{1}{2} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^\alpha, \quad \text{for all } s \geq 0. \tag{50}
\]

Moreover, the condition \( \alpha \geq 3/2 \) is necessary for (50).

**Proof.** Let \( \gamma(s) = G(s) - \frac{1}{\sqrt{2}} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^\alpha \). Then

\[
\gamma'(s) = g(s) - \frac{\alpha}{\sqrt{2}} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{\alpha-1}.
\]

We will show that \( \gamma'(s) \leq 0 \) for all \( s \geq 0 \). It suffices to prove that

\[
g^2(s) \leq \frac{\alpha^2}{2} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{2(\alpha-1)}.
\]

That is,

\[
1 + \frac{\alpha^2 s^{2(\alpha-1)}}{2} - \frac{\alpha^2}{2} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{2(\alpha-1)} \leq 0.
\]

Let \( P(s) = 1 + \alpha^2 s^{2(\alpha-1)} - \frac{\alpha^2}{2} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{2(\alpha-1)} \). Since \( P(0) = 0 \) and \( P'(s) \leq 0 \) for all \( s \geq 0 \), we get \( P(s) \leq 0 \) for all \( s \geq 0 \). Finally, combining \( \gamma(0) < 0 \), we get (50).

Next, we prove that \( \alpha \geq 3/2 \) is necessary for (50). By contradiction, we suppose \( 1 < \alpha < 3/2 \). By (50), we have

\[
\int_0^t g(s) ds \leq \frac{\alpha}{\sqrt{2}} \int_0^t \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{\alpha-1} ds + \left( \frac{\sqrt{2}}{\alpha} \right)^{\frac{1}{\alpha-1}}.
\]

So, we deduce that

\[
L := \frac{\alpha}{\sqrt{2}} \int_0^t \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{\alpha-1} ds - g(s) \geq - \left( \frac{\sqrt{2}}{\alpha} \right)^{\frac{1}{\alpha-1}}, \quad \text{for all } t > 0. \tag{51}
\]

On the other hand, by the inequality \( b^p - a^p \leq p \alpha a^{p-1} (b - a) \) for \( p < 1 \) and \( b > a > 0 \), we have

\[
L = \int_0^t \frac{\alpha^2}{\sqrt{2}} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{2(\alpha-1)} - g^2(s) ds \leq \int_0^t \frac{\alpha^2}{\sqrt{2}} \left[ \left( \frac{2}{\alpha^2} \right)^{\frac{1}{2(\alpha-1)}} + s \right]^{\alpha-1} + g(s) ds.
\]
Let \( t_0 = \left[ 2^{\frac{1}{\alpha - 2}} \alpha^{\frac{2 - \alpha}{2}} (\alpha - 1)^{-1} \right]^{\frac{1}{\alpha - 3}} \) and \( t_1 = 2^{\frac{1}{\alpha - 1}} \alpha^{\frac{1}{\alpha - 2}} \). Then for \( t \geq t_2 := \max\{t_0, t_1\} \) and \( 1 < \alpha < 3/2 \), we have

\[
L \leq \int_0^{t_0} \frac{2^{\frac{1}{\alpha - 2}} \alpha^{\frac{2 - \alpha}{2}} (\alpha - 1) s^{2\alpha - 3} - 1}{s^{\alpha - 1} + g(s)} ds - \frac{1}{2} t_2 \int_{t_0}^{t} \frac{1}{s^{\alpha/2}} \left( s^{\alpha/2} + s \right)^{\alpha - 1} + g(s) ds
\]

\[
\leq L_1 - \frac{1}{\alpha^2} \int_{t_2}^{t} s^{2(1 - \alpha)} ds
\]

\[
= L_1 - \frac{1}{\alpha^2} \left[ \frac{3 - 2\alpha}{2} - t_2^{-2\alpha} \right] \rightarrow -\infty, \text{ as } t \rightarrow +\infty,
\]

which contradicts to (51).

Finally, we consider the case \( l(s) = (1 + s)^{\frac{2}{3}} \), that is,

\[
g(s) = \sqrt{1 + \frac{\alpha^2 s^2}{2(1 + s^2)^{2-\alpha}}},
\]

where \( \alpha \geq 3/2 \). Clearly, \( g(s) \) satisfies

1. \( g(s) \) is a nondecreasing positive function with respect to \( |s| \), \( g(0) = 1 \), \( g(s) \in C^1 \).

2. \( \lim_{s \to \infty} \frac{g(s)}{|s|^{3/2}} = \frac{\alpha}{\sqrt{2}} \).

**Lemma 3.2.** Assume that \( \alpha \geq \alpha_0 \), then,

\[
sg'(s) \leq g(s) - \alpha - 1 \quad \text{for all } s \geq 0,
\]

where \( \alpha_0 (\approx 1.36) \) satisfies the equation

\[
\alpha^3 - 4\alpha^2 + 8\alpha - 6 = 0.
\]

**Proof.** To prove (52), it suffices to prove the following inequality:

\[
\alpha^2 s^2 (1 + s^2)^{\alpha - 3} \left[ 1 + (\alpha - 1) s^2 \right] \leq (\alpha - 1) \left[ 2 + \frac{\alpha^2 s^2}{(1 + s^2)^{2-\alpha}} \right],
\]

that is,

\[
2(\alpha - 1)(1 + s^2)^{3-\alpha} + (\alpha - 2)\alpha^2 s^2 \geq 0.
\]

(54) is obvious if \( \alpha \geq 2 \). Now, we consider \( 1 < \alpha < 2 \). Let

\[
K(s) = 2(\alpha - 1)(1 + s^2)^{3-\alpha} + (\alpha - 2)\alpha^2 s^2,
\]

then we have

\[
K'(s) = 2s \Theta(s) := 2s \left[ 2(\alpha - 1)(3-\alpha)(1 + s^2)^{2-\alpha} + (\alpha - 2)\alpha^2 \right].
\]

Note that \( \Theta'(s) \geq 0 \) for all \( s \geq 0 \) and \( 1 < \alpha < 2 \), we get the result since \( \Theta(0) = \alpha^3 - 4\alpha^2 + 8\alpha - 6 \).

**Lemma 3.3.** Assume that \( \alpha \geq 3/2 \), then,

\[
G(s) \leq \frac{1}{\sqrt{2}} (1 + s)^{\alpha}, \quad \text{for all } s \geq 0.
\]

(55)
Proof. First, we consider the case $3/2 \leq \alpha \leq 2$. Let
\[
\Upsilon(s) = \frac{1}{\sqrt{2}} (1 + s)^\alpha - G(s).
\]
Since $\Upsilon(0) = \frac{1}{\sqrt{2}} > 0$, it suffices to show for all $s \geq 0$ and $3/2 \leq \alpha \leq 2$,
\[
\Upsilon'(s) = \frac{\alpha}{\sqrt{2}} (1 + s)^{\alpha - 1} - g(s) \geq 0.
\]
that is,
\[
\frac{\alpha^2}{2} (1 + s)^{2(\alpha - 1)} \geq g^2(s),
\]
By $\alpha \leq 2$, it follows that
\[
g^2(s) = 1 + \frac{\alpha^2 s^2}{2(1 + s^2)^{2-\alpha}} \leq 1 + \frac{\alpha^2 s^{2(\alpha-1)}}{2},
\]
then, we prove (56) if we show that
\[
\frac{\alpha^2}{2} (1 + s)^{2(\alpha - 1)} \geq 1 + \frac{\alpha^2 s^{2(\alpha-1)}}{2}.
\]
Let
\[
L(s) := \frac{\alpha^2}{2} (1 + s)^{2(\alpha - 1)} - 1 - \frac{\alpha^2 s^{2(\alpha-1)}}{2},
\]
we note that $L(0) > 0$ and
\[
L'(s) = \alpha^2 (\alpha - 1) [ (1 + s)^{2\alpha-3} - s^{2\alpha-3} ] > 0
\]
since $\alpha \geq 3/2$. Thus, we get (57).

Next, we consider $\alpha \geq 2$. By (56), we should prove
\[
A(s) := \frac{\alpha^2}{2} (1 + s)^{2(\alpha - 1)} - g^2(s) \geq 0.
\]
This follows from the fact $A(0) = \frac{\alpha^2}{2} - 1 > 0$ and
\[
A'(s) := &\alpha^2 \left[ (\alpha - 1)(1 + s)^{2\alpha - 3} - s(1 + s^2)^{\alpha - 2} \left( 1 + \frac{(\alpha - 2)s^2}{1 + s^2} \right) \right] \\
\geq &\alpha^2 (1 + s^2)^{\alpha - 2} \left[ (\alpha - 1)(1 + s) - \frac{2s((\alpha - 1)s^2 + 1)}{1 + s^2} \right] \\
= &\alpha^2 (1 + s^2)^{\alpha - 2} \left[ (\alpha - 1)s^2 + (\alpha - 2)s + \alpha - 1 \right] \\
\geq &0.
\]
Lemma 3.3 is proved.

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