ILL-POSEDNESS FOR SUBCRITICAL HYPERDISSIPATIVE NAVIER-STOKES EQUATIONS IN THE LARGEST CRITICAL SPACES

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ABSTRACT. We study the incompressible Navier-Stokes equations with a fractional Laplacian and prove the existence of discontinuous Leray-Hopf solutions in the largest critical space with arbitrarily small initial data.

1. INTRODUCTION

In this paper we study the supercritical 3D Navier-Stokes equations with a fractional power of the Laplacian

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nabla p &= -\nu(-\Delta)^\alpha u, \\
\nabla \cdot u &= 0, \\
u(0) &= u_0,
\end{aligned}
\]

where the velocity \(u(x, t)\) and the pressure \(p(x, t)\) are unknowns, \(u_0 \in L^2(\mathbb{T}^3)\) is the initial condition, \(\nu > 0\) is the kinematic viscosity coefficient of the fluid, and \(\alpha > 0\). The case \(\alpha = 1\) corresponds to the classical Navier-Stokes equations, which has been studied extensively for decades. We refer to [7, 17] for the classical theory for these equations. In the case \(\alpha \geq 5/4\) the equations are well-posed, as the dissipative term simply dominates the nonlinear term. Moreover, the global regularity is known even in a slightly supercritical case, i.e., when logarithmic corrections to the Fourier multiplier of the dissipative term are present (see [16, 4]). However, a finite time blow up of solutions to (1) remains a possibility for \(\alpha < 5/4\) due to a supercritical nature of the equations. Nevertheless, a partial regularity result [3] has been established in the supercritical case \(\alpha = 1\), later extended to \(\alpha \in (1, 5/4)\) in [11]. There are also various regularity criteria in the case \(\alpha = 1\), most of which are of Ladyzhenskaya-Prodi-Serrin type [8, 13, 14, 15, 10, 6, 4], which can also be extended to \(\alpha \in (1, 5/4)\).

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One of the open questions studied extensively is whether solutions bounded in the largest critical case ($\dot{B}_{-1}^{-1,\infty}$ for $\alpha = 1$) are regular. A positive answer to this question would extend the famous $L_\infty^t L_3^x$ result due to Isett, Seregin, and Šverák [10]. In addition, the best small initial result for the 3D NSE, due to Koch and Tataru [12], is in the space $BMO^{-1}$, and it is not known either if its extension to the $B_{-1}^{-1,\infty}$ is possible.

In view of these problems two “negative” results have been obtained in the space $\dot{B}_{-1}^{-1,\infty}$. First, Bourgain and Pavlović [2] proved that there are solutions to the 3D NSE equations, with arbitrary small initial data in $\dot{B}_{-1}^{-1,\infty}$ that become arbitrarily large in $\dot{B}_{-1}^{-1,\infty}$ in arbitrarily small time. Second, Leray-Hopf solutions with arbitrary small initial data, but discontinuous in $\dot{B}_{-1}^{-1,\infty}$ were obtained in [5].

The largest critical space for the fractional NSE (1) is $\dot{B}_{-1}^{1-2\alpha}$. Recently Yu and Zhai [18] obtained a small initial data result in this space in the hypodissipative case $\alpha \in (1/2, 1)$. Heuristically, the hypodissipative NSE behaves better because it is closer to the fractional heat semigroup in critical spaces. In the hyperdissipative case it is therefore natural to expect ill-posedness results of the type mentioned above. Indeed, in this paper we demonstrate this in the case $\alpha \in [1, 5/4)$ by constructing a Leray-Hopf solution with arbitrarily small initial data, which is discontinuous in the critical Besov space $\dot{B}_{-1}^{1-2\alpha}$. It is thus a direct extension of our previous result stated in [5]. The method breaks down either when $\alpha$ passes beyond the value of 1, which is consistent with the result of Yu and Zhai, and at $5/4$ and beyond, which is consistent with the global regularity in that range.

We now fix our notation. We assume periodic boundary conditions in all 3 dimensions, so $\mathbb{T}^3$ will denote the 3D torus, while $| \cdot |_p$, $p \geq 1$, denotes the $L^p$-norm in $\mathbb{T}^3$. We let $\hat{f}$ and $\dot{f}$ stand for the forward and, respectively, inverse Fourier transforms on the torus. The Fourier multiplier with symbol $|\xi|^\alpha$, where $\xi$ stands for the frequency vector and $\alpha > 0$, is denoted by $|\nabla|^\alpha$. The fractional Laplacian operator $(-\Delta)^\alpha$ has symbol $|\xi|^{2\alpha}$. We write $p(\xi) = \text{id} - |\xi|^{-2} \xi \otimes \xi$, $\xi \neq 0$, $p(0) = \text{id}$, for the symbol of the Leray-Hopf projection on the divergence-free fields. We fix notation for the dyadic a-dimensional wavenumbers $\lambda_q = 2^q$. We use extensively the classical dyadic decomposition throughout: $u = \sum_{q \geq 0} u_q$, where $u_q$ is the Littlewood-Paley projection with the Fourier support contained in $\{ \lambda_{q-1} < |\xi| < \lambda_{q+1} \}$. The definitions are standard and can be found in the references above. We often will be using the extended projection defined by $\tilde{u}_q = u_{q-1} + u_q + u_{q+1}$, $q \geq 1$, and projection onto the dyadic ball, $u_{\leq q} = \sum_{p \leq q} u_p$. Thus, $u_q$ is
supported on $\{\lambda_{q-2} < |\xi| < \lambda_{q+2}\}$ and we have the identity

$$\int_{T^3} u \cdot u_q \, dx = \int_{T^3} \tilde{u}_q \cdot u_q \, dx. \tag{2}$$

With the Littlewood-Paley decomposition we define Besov spaces $\dot{B}^s_{r,\infty}$, $s \in \mathbb{R}$, $r \geq 1$ by requiring

$$\|u\|_{\dot{B}^s_{r,\infty}} = \sup_{q \geq 0} \lambda_q^s \|u_q\|_r < \infty.$$  

We will frequently refer to Bernstein’s inequalities, which state that for all $1 \leq r < r' \leq \infty$, and in three dimensions, one has

$$|u_q|_{r'} \lesssim \lambda_q^{3(1/r - 1/r')}|u_q|_r,$$

where here and throughout $\lesssim$ denote inequality up to an absolute constant. Finally, let $\vec{e}_1$, $\vec{e}_2$, etc., stand for the vectors of the standard unit basis.

2. ILL-POSEDNESS OF NSE

The Navier-Stokes equation with a fractional power of the Laplacian is given by

$$u_t + (u \cdot \nabla)u = -\nu(-\Delta)\alpha u - \nabla p. \tag{3}$$

Here $u$ is a three dimensional divergence free field on $T^3$, and $\alpha \in [1, 5/4)$. Let us recall that for every field $U \in L^2(T^3)$ there exists a weak solution $u \in C_w((0,T); L^2) \cap L^2((0,T); H^1)$ to (3) such that the energy inequality

$$|u(t)|_2^2 + 2\nu \int_0^t |\nabla|^\alpha u(s)|_2^2 \, ds \leq |U|^2_2, \tag{4}$$

holds for all $t > 0$ and $u(t) \rightarrow U$ strongly in $L^2$ as $t \rightarrow 0$. In what follows we do not actually use inequality (4) which allows us to formulate a more general statement below in Proposition 2.2.

Let us choose a strictly increasing sequence $\{q_j\} \in \mathbb{N}$ with elements sufficiently far apart so that at least $\lambda_{q_{j-1}}^2 \lambda_{q_{j+1}}^5 < 1$. We consider the following lattice blocks:

$$A_j = \left[\frac{9}{10} \lambda_{q_j}, \frac{11}{10} \lambda_{q_j}\right] \times \left[\frac{1}{10} \lambda_{q_j}, \frac{1}{10} \lambda_{q_j}\right] \times \mathbb{Z}^3$$

$$B_j = \left[-\frac{1}{10} \lambda_{q_j-1}, \frac{1}{10} \lambda_{q_j-1}\right] \times \left[\frac{9}{10} \lambda_{q_j-1}, \frac{11}{10} \lambda_{q_j-1}\right] \cap \mathbb{Z}^3$$

$$C_j = A_j + B_j$$

$$A_j^* = -A_j, \quad B_j^* = -B_j, \quad C_j^* = -C_j.$$  

Thus, $A_j$, $C_j$ and their conjugates lie in the $q_j$-th shell, while $B_j$, $B_j^*$ lie in the adjacent $(q_j - 1)$-th shell. The particular choice of scaling exponents...
9/10, 11/10, etc., is unimportant as long as the blocks fit into their respective shells. Let us denote
\[ \vec{e}_1(\xi) = p(\xi)\vec{e}_1, \quad \vec{e}_2(\xi) = p(\xi)\vec{e}_2. \]

We now define the initial condition field to be the following sum
\[ U = \sum_{j \geq 1} (U_{q_j} + U_{q_j-1}), \tag{5} \]
where the components, on the Fourier side, are
\[ \hat{U}_{q_j}(\xi) = \lambda_{q_j}^{2\alpha-4} \left( \vec{e}_2(\xi)\chi_{A_j \cup A_j^*} + i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j} - i(\vec{e}_2(\xi) - \vec{e}_1(\xi))\chi_{C_j^*} \right), \]
and
\[ \hat{U}_{q_j-1}(\xi) = \lambda_{q_j}^{2\alpha-4} \vec{e}_1(\xi)\chi_{B_j \cup B_j^*}. \]

By construction, \( \hat{U}(-\xi) = \hat{U}(\xi) \), which ensures that \( U \) is real. Since \( U \) has no modes in the \((q_j + 1)\)-st shell, then the extended Littlewood-Paley projection of the \( j \)-th component has the form \( \hat{U}_{q_j} = U_{q_j-1} + U_{q_j} \).

**Lemma 2.1.** We have \( U \in B^{1+\frac{4}{r}-2\alpha}_{r,\infty} \), for any \( 1 < r \leq \infty \).

**Proof.** We give the estimate only for one block, the other ones being similar. Using boundedness of the Leray-Hopf projection, we have, for all \( 1 < r < \infty \),
\[ |\lambda_{q_j}^{2\alpha-4} (\vec{e}_2(\cdot)\chi_{A_j})^\vee|_r \lesssim \lambda_{q_j}^{2\alpha-4} |(\chi_{A_j})^\vee|_r. \]
Notice that by construction,
\[ |(\chi_{A_j})^\vee(x_1, x_2, x_3)| = |D_{(c+1)\lambda_{q_j}}(x_1)D_{c\lambda_{q_j}}(x_2)D_{c\lambda_{q_j}}(x_3)|, \]
where \( D_N \) denotes the Dirichlet kernel. Hence,
\[ |(\chi_{A_j})^\vee|^r \leq |D_{(c+1)\lambda_{q_j}}|_r |D_{c\lambda_{q_j}}|_r^3. \]
By a well-known estimate, we have \( |D_N|_r \leq N^{1-\frac{1}{r}} \) (c.f. [9]). Putting the above estimates together implies the desired inclusion in \( B^{1+\frac{4}{r}-2\alpha}_{r,\infty} \). In the case \( r = \infty \) we simply use the triangle inequality to obtain
\[ |U_{q_j}|_\infty \lesssim \lambda_{q_j}^{2\alpha-1}. \]

Let us now examine the trilinear term. We will use the following notation for convenience
\[ u \otimes v : \nabla w = \int_{\mathbb{T}^3} v_i \partial_i w_j u_j dx. \tag{6} \]
Using the antisymmetry we obtain
\[ U \otimes U : \nabla U_{q_j} = \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + \tilde{U}_{q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} + U_{q_j} \otimes U_{q_{j-1}} : \nabla U_{q_j} \]
\[ = \sum_{k \geq j+1} \tilde{U}_{q_k} \otimes \tilde{U}_{q_k} : \nabla U_{q_j} + U_{q_{j-1}} \otimes U_{q_j} : \nabla U_{q_j} - U_{q_j} \otimes U_{q_j} : \nabla U_{q_{j-1}} \]
\[ = A + B + C. \]

Using Bernstein’s inequalities we estimate
\[ |A| \lesssim \lambda_{q_j} |U_{q_j}|_\infty \sum_{k \geq j+1} |\tilde{U}_{q_k}|^2 \lesssim \lambda_{q_j}^{2\alpha} \lambda_{q_{j+1}}^{4\alpha-5} \leq 1, \]
\[ |C| \lesssim |U_{q_j}|^2 \sum_{k \leq j-1} \lambda_{q_k} |\tilde{U}_{q_k}|_\infty \lesssim \lambda_{q_{j-1}}^{2\alpha} \lambda_{q_j}^{4\alpha-5} \leq 1, \]
where in the latter inequality we used the fact \(|U_{q_j}|_2 \sim \lambda_{q_j}^{2\alpha-5/2} \). On the other hand, a straightforward computation shows that
\[ B \sim \lambda_{q_j}^{6\alpha-5}, \]
which is thus the dominant term of the three, and hence,
\[ U \otimes U : \nabla U_{q_j} \sim \lambda_{q_j}^{6\alpha-5}. \]

**Proposition 2.2.** Let \( u \in C_w([0, T); L^2) \cap L^2([0, T); H^1) \) be a weak solution to the NSE with initial condition \( u(0) = U \). Then there is \( \delta = \delta(u) > 0 \) such that
\[ \limsup_{t \to 0^+} \|u(t) - U\|_{B^2_{w, \infty}} \geq \delta. \]

If, in addition, \( u \) is a Leray-Hopf solution satisfying the energy inequality (4), then \( \delta \) can be chosen independent of \( u \).

**Proof.** Let us test (3) with \( u_{q_j} \). Using (2), we find
\[ \partial_t (\tilde{u}_{q_j} \cdot u_{q_j}) = -\nu |\nabla|^{\alpha} \tilde{u}_{q_j} \cdot |\nabla|^{\alpha} u_{q_j} + u \otimes u : \nabla u_{q_j}, \]
where as defined before, \( \tilde{u}_{q_j} = u_{q_{j-1}} + u_{q_j} + u_{q_{j+1}} \). Denoting \( E(t) = \int_0^t \||\nabla|^{\alpha} u\|_2^2 ds \) we obtain
\[ |\tilde{u}_{q_j}(t)|_2^2 \geq |U_{q_j}|_2^2 - \nu E(t) + c_1 \lambda_{q_j}^{6\alpha-5} t \]
\[ - c_2 \int_0^t |u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j}|_1 ds, \]
for some positive constants $c_1$ and $c_2$. We now show that if the conclusion of the proposition fails then for some small $t > 0$ the integral term the growth of the integral term above becomes less than $c_1 \lambda^{6\alpha-5} t$ for large $j$. This forces $|\tilde{u}_{q_j}(t)|^2 \gtrsim \lambda^{6\alpha-5} t$ for all large $j$. Hence $u$ has infinite energy, which is a contradiction.

So suppose that for every $\delta > 0$ there exists $t_0 = t_0(\delta) > 0$ such that $\|u(t) - \bar{U}\|_{B^{-\alpha}_\infty} < \delta$ for all $0 < t \leq t_0$. Denoting $w = u - \bar{U}$ we write

$$u \otimes u : \nabla u_{q_j} - U \otimes U : \nabla U_{q_j} = w \otimes U : \nabla U_{q_j} + u \otimes w : \nabla U_{q_j} + u \otimes u : \nabla w_{q_j} = A + B + C.$$ 

We will now decompose each triplet into three terms according to the type of interaction (c.f. Bony [1]) and estimate each of them separately.

$$A = \sum_{p', p'' \geq q_j \atop |p' - p''| \leq 2} w_{p'} \otimes U_{p''} : \nabla U_{q_j} + w_{\leq q_j} \otimes \tilde{U}_{q_j} : \nabla U_{q_j} + \tilde{w}_{q_j} \otimes U_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = A_1 + A_2 + A_3.$$ 

Let us fix $r \in (1, 3/(4\alpha - 2))$ and use Lemma 2.1 along with Hölder and Bernstein’s inequalities to estimate $A_1$:

$$|A_1| \leq |\nabla U_{q_j}| r \sum |w_{p''}| |U_{p''}| |r \lesssim \lambda_{q_j}^{2\alpha - 3 + \frac{2}{r}} \sum |w_{p''}| \lambda_{p''}^{2\alpha - 1 - \frac{2}{r}} \lesssim \delta \lambda_{q_j}^{2\alpha - 3 + \frac{2}{r}} \lesssim \delta \lambda_{q_j}^{6\alpha - 5}.$$ 

Intergrating by parts we obtain $A_2 = U_{q_j} \otimes \tilde{U}_{q_j} : \nabla w_{\leq q_j}$. Thus, using the same tools,

$$|A_2| \leq |\tilde{U}_{q_j}|^2 |\nabla w_{\leq q_j}| \lesssim \lambda_{q_j}^{4\alpha - 5} \sum_{p \leq q_j} \lambda_p |w_p| < \delta \lambda_{q_j}^{6\alpha - 5}.$$ 

And finally,

$$|A_3| \leq \lambda_{q_j} |U_{\leq q_j}|^2 |U_{q_j}|^2 |\tilde{w}_{q_j}| \lesssim \lambda_{q_j}^{4\alpha - 4} |\tilde{w}_{q_j}| < \delta \lambda_{q_j}^{6\alpha - 5}.$$ 

We have shown the following estimate:

(10) 

$$|A| \lesssim \delta \lambda_{q_j}^{6\alpha - 5}.$$ 

As to $B$ we decompose analogously,

$$B = \sum_{p', p'' \geq q_j \atop |p' - p''| \leq 2} u_{p'} \otimes w_{p''} : \nabla U_{q_j} + u_{\leq q_j} \otimes \tilde{w}_{q_j} : \nabla U_{q_j} + \tilde{u}_{q_j} \otimes w_{\leq q_j} : \nabla U_{q_j} - \text{repeated} = B_1 + B_2 + B_3.$$
The term $B$ is the least problematic. Here we do not even have to use the smallness of $w$ and can just roughly estimate it in terms of the enstrophy $||\nabla^\alpha u||_2^2$. We have
\[
|B_1| \lesssim \sum_{p',p'' \geq q_j} |u_{p'} \otimes u_{p''} : \nabla U_{q_j}| + \sum_{p,p',p'' \geq q_j} |u_{p'} \otimes U_{p''} : \nabla U_{q_j}|
\]
\[
\leq \lambda_{q_j}^{2\alpha} |u_{\leq q_j}|_2^2 + \lambda_{q_j}^{2\alpha} |u_{\geq q_j}|_2 |U_{\geq q_j}|_2
\]
\[
\leq ||\nabla|^{\alpha} u_{\geq q_j}|_2^2 + \lambda_{q_j}^{3\alpha-5/2} ||\nabla|^{\alpha} u_{\geq q_j}|_2
\]
\[
\leq ||\nabla|^{\alpha} u_{\geq q_j}|_2^2 + \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^{\alpha} u_{\geq q_j}|_2.
\]
Again, using Lemma 2.1 Bernstein and Hölder inequalities we obtain
\[
|B_2| = |U_{q_j} \otimes \tilde w_{q_j} : \nabla u_{\leq q_j}| \leq |U_{q_j}|_2 |\tilde w_{q_j}|_\infty |\nabla u_{\leq q_j}|_2
\]
\[
\leq \lambda_{q_j}^{2\alpha-5/2} |\tilde w_{q_j}|_\infty |\nabla|^{\alpha} u_{\leq q_j}|_2 \leq \lambda_{q_j}^{2\alpha-7/2} ||\nabla|^{\alpha} u_{\leq q_j}|_2 \leq \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^{\alpha} u_{\leq q_j}|_2.
\]
\[
|B_3| \leq |u_{q_j}|_2 |w_{\leq q_j}|_\infty |\nabla U_{q_j}|_2 \lesssim \lambda_{q_j}^{2\alpha-3/2} |\tilde u_{q_j}|_2 \sum_{p \leq q_j} |w_p|_\infty
\]
\[
\lesssim \lambda_{q_j}^{3\alpha-5/2} ||\nabla|^{\alpha} u_{\leq q_j}|_2 \leq \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^{\alpha} u_{\leq q_j}|_2.
\]
We thus obtain
\[
|B| \lesssim ||\nabla|^{\alpha} u_{\geq q_j}|_2^2 + \lambda_{q_j}^{6\alpha-5-1/2} ||\nabla|^{\alpha} u_{\leq q_j}|_2.
\]
Continuing in a similar fashion we write
\[
C = \sum_{p',p'' \geq q_j} |u_{p'} \otimes u_{p''} : \nabla w_{q_j} + u_{\leq q_j} \otimes \tilde u_{q_j} : \nabla w_{q_j} + \tilde u_{q_j} \otimes u_{\leq q_j} : \nabla w_{q_j} \quad \text{repeated} = C_1 + C_2 + C_3.
\]
We have
\[
|C_1| \leq |\nabla w_{q_j}|_\infty |u_{\geq q_j}|_2^2 \lesssim \delta ||\nabla|^{\alpha} u_{\leq q_j}|_2^2.
\]
In $C_2$ we move the derivative onto $u_{\leq q_j}$ and estimate as usual,
\[
|C_2| \leq |\nabla u_{\leq q_j}|_2 |\tilde u_{q_j}|_2 |w_{q_j}|_\infty \lesssim ||\nabla|^{\alpha} u_{\leq q_j}|_2 |\tilde u_{q_j}|_2 \lambda_{q_j}^{2\alpha-1} \leq ||\nabla|^{\alpha} u_{\leq q_j}|_2 \lambda_{q_j}^{6\alpha-5-1/2}.
\]
Using a uniform bound on the energy we have for $C_3$,
\[
|C_3| \lesssim \lambda_{q_j} |w_{q_j}|_\infty |\tilde u_{q_j}|_2 \leq \delta \lambda_{q_j} |w_{q_j}|_2 \lesssim \delta \lambda_{q_j}^{6\alpha-5} |\tilde u_{q_j}|_2 \leq \delta \lambda_{q_j}^{6\alpha-5} |\nabla|^{\alpha} \tilde u_{q_j}|_2.
\]
Thus,
\[
|C| \lesssim \delta ||\nabla|^{\alpha} u_{\leq q_j}|_2^2 + ||\nabla|^{\alpha} u_{\leq q_j}|_2 \lambda_{q_j}^{6\alpha-5-1/2} + \delta \lambda_{q_j}^{6\alpha-5} |\nabla|^{\alpha} \tilde u_{q_j}|_2.
\]
Now combining estimates (10), (11), (12) along with the boundedness of $E(t_0)$ we obtain

\begin{equation}
\int_0^{t_0} |A + B + C| \, ds \lesssim \delta \lambda_{\hat{q}_j}^{\alpha - 5} t_0 + \int_0^{t_0} ||\nabla||^\alpha u_{\geq q_j} ||^2 ds + E(t_0)^{1/2} t_0^{1/2} \lambda_{\hat{q}_j}^{\alpha - 5 - 1/2} + \delta E(t_0) + \delta \lambda_{\hat{q}_j}^{\alpha - 5} \int_0^{t_0} ||\nabla||^\alpha \hat{u}_{q_j} ||^2 ds.
\end{equation}

And for large $j$, and fixed $t_0$, this gives

\[ \int_0^{t_0} |A + B + C| \, ds \lesssim \delta \lambda_{\hat{q}_j}^{\alpha - 5} t_0 + \frac{\nu}{2} E(t_0). \]

Pugging this back into (9) gives the estimate

\[ |\hat{u}_{q_j}(t_0)|_2^2 \gtrsim \lambda_{\hat{q}_j}^{\alpha - 5}, \]

for all $j > j_0$, which shows that $u(t_0)$ has infinite energy, a contradiction.

The last statement of the proposition follows from the fact that we have the bounds on $|u(t)|_2 \leq |U|_2$ and $E(t_0) \leq (2\nu)^{-1} |U|_2^2$ which remove dependence of the constants on $u$. \qed

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