Pauli component erasing operations

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Decoherence of quantum systems is described by quantum channels. However, a complete understanding of such channels, specially in the multi-particle setting, is still an ongoing difficult task. We propose the family of quantum maps that preserve or completely erase the components of a multi-qubit system in the basis of Pauli strings, which we call Pauli component erasing (PCE) maps. For the corresponding channels, it is shown that the preserved components can be interpreted as a finite vector subspace, from which we derive several properties and complete the characterization. Moreover, we show that the obtained family of channels form a semigroup and derive its generators. We use this simple structure to determine physical implementations and connect the obtained family of channels with Markovian processes.

I. INTRODUCTION

Quantum correlations [1–3], including entanglement [4], are an important resource for a wide variety of tasks that include teleportation [5], quantum computation [6], and others [7]. However, this resource is also extremely delicate [6, 8], specially for multi-particle systems [4], an important part of the efforts of the community implementing quantum technologies is devoted to tackle this issue from an experimental [9–11] and theoretical [11–13] point of view. The process by which quantum correlations are unintentionally dissipated is called decoherence [8, 14]. One of the main tools to study the effects of decoherence are quantum channels. Quantum channels can describe quantum noise [15, 16], open quantum systems dynamics [16, 17], and recently even coarse graining [18, 19]. One of the main difficulties in characterizing quantum channels is that, like for quantum states, the number of parameters required for their description increases quite rapidly with Hilbert space dimension. Moreover, such parameters are constrained in a complicated way by physical conditions, such as complete positivity [20]. Describing in detail families of channels having a given property provides insight into the jungle of quantum operations. For the qubit case there are several studies concerning the unital case, for which non-trivial properties can be described using only 3 parameters, which in turn form the well known tetrahedron of Pauli channels [15, 16, 21]. More generally, in Ref. [22] the authors study families of convex combinations of quantum-classical channels that relate to unital qubit channels with positive eigenvalues, and give a generalization of the Bloch sphere. Similarly, a generalization of Pauli channels based on mutually unbiased measurements is introduced and studied in Ref. [23]. Other studies of channels beyond the qubit can be found [24–27].

In this work we present a generalization of idempotent Pauli channels, i.e. the qubit flip operations (bit, phase and bit-phase when the flip probability is 1/2), total depolarizing qubit channel and the identity channel, to the case of N qubits. The generalization is done by extending Pauli observables to Pauli strings (tensor products of Pauli matrices) [28, 29], the resulting maps are unital and diagonal in the Pauli strings’ basis. In addition, these channels describe the decoherence process in a similar way as the aforementioned ones for the single-qubit case. From now on we refer to these maps as Pauli component erasing (PCE) maps. We derive necessary and sufficient conditions for such mappings to be completely positive trace preserving, thus, showing which idempotent operations are valid quantum channels (PCE channels). Additionally, we characterize the algebraic properties of the set of PCE channels, showing that it is equipped with a semigroup structure, which is later used to derive physical implementations.

The paper is organized as follows, in section II we recall the properties of quantum channels needed to proceed with the definition of PCE maps. In section III we diagonalize analytically the Choi matrix for arbitrary PCE maps and characterize completely positive maps by interpreting PCE maps as finite vector spaces. We study the generators of the semigroup structure associated to the set of PCE channels in section IV and we use them to derive meaningful physical interpretations of PCE channels in section V as well Kraus operators of the generators. To finish, we conclude in section VI.

II. PAULI COMPONENT ERASING MAPS

In this section we introduce the family of Pauli component erasing (PCE) maps. Let us start our discussion with a brief review of several basic concepts of quantum
channels that will allow us to introduce some notation, and finish with the definition of PCE maps and some generalities. We further introduce a useful graphical representation for them.

A. Quantum channels

Quantum channels are the most general linear operations that a quantum system undergo independently of its past \[20, 30\]. The physical system under study will be associated with a Hilbert space denoted by \(H\), and the set of linear operators over such space will be denoted by \(\mathcal{B}(H)\). That way, a density matrix \(\rho\) of such system is an element of \(\mathcal{B}(H)\).

The construction of quantum channels includes basically three ingredients: linearity, trace preservation and complete positivity. Linearity is needed to map every convex combination of density matrices into a convex combination of the evolution of such density matrices. The trace preserving property is required for the process \(\mathcal{E}\) to happen with probability 1, and reads \(\text{tr} \mathcal{E}[\rho] = \text{tr} \rho = 1\). The complete positivity condition is needed to preserve positive semidefiniteness and handle the non-local nature of quantum theory. A linear map \(\mathcal{E}\) is positive if it maps density operators to density operators, i.e. if \(\mathcal{E}[\rho] \geq 0\) for all density matrices \(\rho\). On the other hand, if one extends a positive map to include an ancilla, the resulting map is not always positive. If, for an ancillary system of arbitrary dimension, such extension results in a positive map, we say that the original map is completely positive \[31\]. Quantum channels are required to be completely positive so as to allow the proper evolution of potentially entangled states with an ancilla; to test this condition we require some additional steps.

A simple algorithm to test the complete positivity of a quantum channel was developed by Jamiołkowski \[32\] and Choi \[33\]. One first exploits the isomorphism that maps a channel \(\mathcal{E}\) to the state \(D = (\text{id} \otimes \mathcal{E})|\Omega\rangle\langle \Omega|\), where \(|\Omega\rangle = 1/\text{dim}(H) \sum_{i=1}^{\text{dim}(H)} |i\rangle \langle i|\) is a maximally entangled state between the original system and an ancilla and “\(\text{id}\)” is the identity channel. Remarkably, the map \(\mathcal{E}\) is completely positive if and only \(D\) (also called Choi or dynamical matrix of \(\mathcal{E}\)) is positive-semidefinite \[32, 33\].

B. Structure of PCE maps

We have discussed the main features of quantum channels, now we turn our attention to introduce the Pauli component erasing maps. We start by exploring the single-qubit scenario and then we treat the \(N\)-qubit case.

The most general single-qubit density matrix can be written as

\[
\rho = \frac{1}{2} \sum_{\alpha=0}^{3} r_{\alpha} \sigma_{\alpha},
\]

where \(\sigma_{\alpha}\) denotes a multi-index \((\alpha_1, \ldots, \alpha_N)\) and \(\alpha_1 = 0, 1, 2, 3\). These hermitian operators form an orthogonal basis in the space of operators acting on \(N\) qubits. In fact, \(\text{tr} \sigma_{\alpha} \sigma_{\beta} = 2^N \delta_{\alpha\beta}\) and \(\text{tr} \sigma_{\alpha} = 2^N \delta_{\alpha0}\).

Similarly to the single-qubit case, the density matrix \(\rho\) of a system of \(N\) qubits can be written using Pauli strings in the following way,

\[
\rho = \frac{1}{2^N} \sum_{\vec{\alpha}} r_{\vec{\alpha}} \sigma_{\vec{\alpha}},
\]

with \(\sigma_0 = 1\), and \(\sigma_{1,2,3}\) the usual Pauli matrices. Normalization requires that \(r_0 = 1\) and the remaining \(r_{1,2,3}\) form the Bloch vector. Consider the map that projects each component in the following way,

\[
r_{\alpha} \mapsto r_{\alpha} r_{\alpha},
\]

where \(r_{\alpha}\) is either 0 or 1 (trace preserving requires that \(r_0 = 1\)). From now on we refer to any operation like that described in Eq. \[2\], as single-qubit Pauli component erasing (PCE) maps. Not every such operation is a quantum channel; for example, collapsing the entire Bloch ball to a disk on the \(xy\) plane \((\tau_1 = \tau_2 = 1, \tau_3 = 0)\) leads to a violation of the CP conditions. Indeed a direct evaluation of such conditions yield \[15, 20\],

\[
1 + \tau_1 + \tau_2 + \tau_3 \geq 0,
1 + r_{\alpha} - \tau_{\beta} - \tau_{\gamma} \geq 0 \quad \forall \alpha \neq \beta \neq \gamma,
\]

where trace-preserving is already imposed, and show that five out of the eight 1-qubit PCE maps are quantum channels. These operations are the identity map, the completely depolarizing channel, and complete depolarization, respectively, as the color of each square indicates the value attained by the corresponding \(\tau_{\alpha}\), either 0 (white) or 1 (black). In (e) we show a map that only erases the component \(r_1\), collapsing the Bloch sphere into a disk, and thus does not correspond to a quantum operation.

Figure 1. In (a) we introduce the notation in the diagrams that represent the single-qubit PCE maps, so that each square corresponds to a single \(\tau_{\alpha}\), \(\alpha = 0, 1, 2, 3\). The diagrams in (b), (c) and (d) correspond to the identity map, completely dephasing channel, and complete depolarization, respectively, as the color of each square indicates the value attained by the corresponding \(\tau_{\alpha}\), either 0 (white) or 1 (black). In (e) we show a map that only erases the component \(r_1\), collapsing the Bloch sphere into a disk, and thus does not correspond to a quantum operation.
Figure 2. In (a) we introduce the positions of two qubit diagrams. The diagrams in (b) corresponds to a quantum channel that results from the tensor product of bit flip channels in each qubit [see Fig. 1(c)], and in (c) a diagram of a map that is not a quantum channel is presented.

so \( r_\alpha = \langle \sigma_\alpha \rangle \) is the coefficient corresponding to the expansion of the density matrix in the normalized basis of Pauli strings. Again, normalization of the state requires that \( r_\alpha = 1 \). We shall refer to \( r_\alpha \) as the \textit{Pauli components} of the density matrix of a system of qubits.

In general, a Pauli component erasing (PCE) map is a map that either preserves or completely erases the Pauli components of a density matrix. That is,

\[
\rho_\alpha \mapsto \tau_\alpha \rho_\alpha, \quad \tau_\alpha = 0, 1. \tag{6}
\]

In addition, for the operation to be trace preserving, it is required that \( \tau_0 = 1 \). It is worth noticing that, as for the single-qubit case, not all PCE maps are quantum operations. On the other hand, constructing and evaluating the conditions for complete positivity is non-trivial and is the main problem addressed in this paper. We shall refer to the map \( \rho_\alpha \mapsto \tau_\alpha \rho_\alpha \), with arbitrary values of \( \tau_\alpha \) (only restricted by complete positivity) as \textit{Pauli diagonal operations}.

A graphical representation for PCE maps may be introduced, being the two qubits case what has proved to be the most useful. Consider a \( N \) dimensional Cartesian grid, with \( 4^N \) places. Each place has \( N \) integer coordinates, ranging from 0 to 3, so each place corresponds to a given \( \alpha \) in Eq. (9). For a given PCE, we shall fill the square if the corresponding \( \tau_\alpha = 1 \). Otherwise, we leave it empty. Examples for \( N = 1 \) and \( N = 2 \) are provided in Fig. 1 and Fig. 2, respectively.

It is worth noticing that the set of PCE maps overlaps with the set of “Pauli diagonal channels constant on axes” defined in Ref. 22, consisting of convex combinations of \textit{quantum-classical} channels. In particular, it can be shown that quantum-classical channels defined with the eigenbasis of some set of \( 2^N - 1 \) commuting Pauli observables 29 is a PCE map with exactly \( 2^N \) components equal to 1 in its diagonal. For details, we refer the reader to appendix A.

### III. MATHEMATICAL CONSIDERATIONS

This section is devoted to derive the conditions a Pauli diagonal map needs to satisfy the complete positivity condition i.e. that all the eigenvalues of the Choi matrix associated to the channel are non-negative. To do so, we calculate and diagonalize the Choi matrix of a general Pauli diagonal map, first for a single qubit and then for \( N \) qubits. Finally, we restrict from Pauli diagonal maps to PCE maps, and provide a connection between a vector subspace and the set of coefficients \( \{ \tau_\alpha \} \) in Eq. (6) of a PCE quantum channel. This allows us to derive several important properties of this particular family of channels.

#### A. Diagonalization of Choi matrix

We now construct the Choi matrix of a single-qubit Pauli diagonal map \( \mathcal{E} \). As described above, \( \mathcal{E} \) is a linear map from \( \mathcal{B}(H) \) to itself. We shall denote elements of \( \mathcal{B}(H) \) by the notation \( | \cdot \rangle \rangle \). Thus, for instance, \( | \sigma_\alpha \rangle \rangle \) represents the Pauli matrix \( \sigma_\alpha \) understood as a vector belonging to \( \mathcal{B}(H) \), for the present case, in which \( H = \mathbb{C}^2 \).

Since the scalar product in \( \mathcal{B}(H) \) is given by \( \langle \langle \sigma_\alpha | \sigma_\alpha' \rangle \rangle = \text{tr} (\sigma_\alpha^\dagger \sigma_\alpha') = 2 \delta_{\alpha \alpha'} \). In this language, the state of a single qubit reads \( | \rho \rangle = 2^{-1} \sum_{\alpha=0}^3 r_\alpha | \sigma_\alpha \rangle \rangle \) and the matrix form of the map \( \mathcal{E} \) is

\[
\dot{\mathcal{E}} = \frac{1}{2} \sum_{\alpha=0}^3 \tau_\alpha | \sigma_\alpha \rangle \langle \sigma_\alpha |. \tag{7}
\]

After some steps, detailed from Eq. (B3) to Eq. (B6), it is possible to show that the Choi matrix of \( \mathcal{E} \) reads

\[
\mathcal{D} = \frac{1}{2} \sum_{\alpha=0}^3 \tau_\alpha \sigma_\alpha \otimes \sigma_\alpha^*. \tag{8}
\]

Notice that \( | \sigma_\alpha \rangle \langle \sigma_\alpha | \) and \( \sigma_\alpha \otimes \sigma_\alpha^* \) are different operators. Indeed, the former acts as a linear map upon the vector space \( \mathcal{B}(H) \), whereas the latter acts on the tensor product \( H \otimes H \). Of course, there is a basis dependent identification between these two spaces, which is used in the construction of the Choi matrix. Surprisingly, one can in fact show that \( \mathcal{D} \) is diagonal in the Pauli basis (see Appendix B for details). The eigenvalues are

\[
\lambda_\alpha = \frac{1}{2} \sum_{\beta=0}^3 a_{\alpha \beta} \tau_\beta, \tag{9}
\]

where

\[
a = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}. \tag{10}
\]

We wish to add that one can replace \( a \) with \( H \otimes H \), with \( H \) the Hadamard matrix, and still diagonalize the same Choi matrix \( \mathcal{D} \). This is due to the fact that \( a \) corresponds to a permutation of rows of \( H \otimes H \). However, we chose the aforementioned definition as some later considerations [see Eq. (12)] cannot be easily written in terms of \( H \otimes H \).
The same program can be carried out for $N$ qubits. In this case, one uses the vectorized Pauli strings,

$$|\sigma_{\vec{a}}\rangle = |\sigma_{a_1} \otimes \cdots \otimes \sigma_{a_N}\rangle. \quad (11)$$

This vectorization must not be confused with the tensor product of all $|\sigma_{a_i}\rangle$, since the tensor product and the vectorization process generally do not commute [35]. The vectors satisfy the orthogonality relation $\langle \sigma_{\vec{a}} | \sigma_{\vec{a}'} \rangle = 2^N \delta_{\vec{a} \vec{a}'}$. The matrix representation of the map corresponding to a Pauli diagonal operation is

$$\mathcal{E}_N = \frac{1}{2^N} \sum_{\vec{a}} \tau_{\vec{a}} |\sigma_{\vec{a}}\rangle \langle \sigma_{\vec{a}}|.$$

(12)

As in the previous case, the Choi matrix $D_N$ may be written in terms of tensor products of Pauli matrices

$$D_N = \frac{1}{2^N} \sum_{\vec{a}} \tau_{\vec{a}} \bigotimes_{j=1}^N \sigma_{a_j} \otimes \sigma^{*}_{a_j}. \quad (13)$$

This superoperator is again diagonal in the (multi-qubit) Pauli basis, with the eigenvalue corresponding to $|\sigma_{\vec{a}}\rangle$ given by

$$\lambda_{\vec{a}} = \frac{1}{2^N} \sum_{\vec{\beta}} A_{\vec{a} \vec{\beta}} \tau_{\vec{\beta}}, \quad (14)$$

where

$$A = a^\otimes N. \quad (15)$$

Again, the proofs are provided in appendix [B]. We wish to add that we could diagonalize $D_N$ with $H^\otimes 2N$ instead of $a^\otimes N$, which might be more convenient for other applications.

**B. PCE quantum channels as vector spaces**

In this subsection we will provide a one-to-one relation between PCE quantum channels and the subspaces of a discrete vector subspace associated with the indices $\vec{a}$ labeling the components of a state; see Eq. (13). Some established facts about vector spaces will allow us to derive the main features of PCE quantum channels.

Let us start by recalling that the problem of determining complete positivity of a PCE map can be recast as determining which coefficients $\tau_{\vec{a}}$ are mapped via $A$ to positive eigenvalues $\lambda_{\vec{a}}$, as in Eq. (14). Using the fact that $a^{-1} = a/4$, so

$$A^{-1} = \frac{1}{4^N} A, \quad (16)$$

we can directly invert Eq. (14) to obtain

$$\sum_{\vec{\beta}} A_{\vec{a} \vec{\beta}} \lambda_{\vec{\beta}} = 2^N \tau_{\vec{a}} \quad (17)$$

which will serve as a starting point for our analysis. This is a remarkable equation, as it provides a method to diagonalize the Choi matrix of any Pauli diagonal map.

Other two simple but crucial observations are the following. For valid quantum channels it holds

$$\sum_{\vec{a} \in \Omega} \lambda_{\vec{a}} = 0 \quad \implies \quad \lambda_{\vec{a}} = 0, \forall \vec{a} \in \Omega \quad (18)$$

for an arbitrary subset of multi-indices $\Omega$, as each member of the sum is greater than or equal to zero, due to complete positivity of the underlying channel. Finally, setting $\vec{a} = 0$ in Eq. (17), and taking into account the normalization condition that $\tau_{\vec{0}} = 1$, we obtain

$$\sum_{\vec{a}} \lambda_{\vec{a}} = 2^N, \quad (19)$$

since, $A_{\vec{0} \vec{\beta}} = 1$ for all $\vec{\beta}$.

Now we need a definition: to each multi-index $\vec{a}$ we associate a set of multi-indices $\Phi(\vec{a})$ as follows

$$\Phi(\vec{a}) = \left\{ \vec{\beta} : A_{\vec{a} \vec{\beta}} = 1 \right\} \quad (20)$$

If we now assume that $\tau_{\vec{a}} = 1$, and calculate the difference between Eq. (19) and $\sum_{\vec{\beta}} A_{\vec{a} \vec{\beta}} \lambda_{\vec{\beta}} = 2^N$ (which follows from Eq. (17) and $\tau_{\vec{a}} = 1$), one obtains

$$\lambda_{\vec{\beta}} = 0, \quad \forall \vec{\beta} \notin \Phi(\vec{a}). \quad (21)$$

Thus, if $\tau_{\vec{a}} = 1$, then $\tau_{\vec{\gamma}}$ and $\tau_{\vec{\gamma}'}$ are equal if

$$A_{\vec{a} \vec{\gamma}} = A_{\vec{a} \vec{\gamma}'}, \quad \forall \vec{\beta} \in \Phi(\vec{a}). \quad (22)$$

This follows from restricting the sum Eq. (17) to the $\lambda_{\vec{\beta}} \neq 0$, given in Eq. (21). Condition (22) therefore connects three multi-indices, $\vec{a}$, $\vec{\gamma}$ and $\vec{\gamma}'$. When such a connection exists, $\tau_{\vec{a}} = 1$ implies $\tau_{\vec{\gamma}} = \tau_{\vec{\gamma}'}$.

Let us now work out the nature of the aforementioned connection. Let the components of $\vec{\beta}$ be such that $\beta_l = \beta \delta_{lk}$ for a particular particle index $k$, with $\beta$ such that $a_{\alpha_k \beta} = 1$. Since $a_{\alpha_0} = 1$ for any $\alpha$, this particular choice of $\beta$ belongs to $\Phi(\vec{a})$, so that Eq. (22) indeed holds. However, for such a choice of $\beta$ the above relation reduces to

$$a_{\beta \gamma_k} = a_{\beta \gamma_k}' \quad (23)$$

for all $\beta$ such that $a_{\alpha_k \beta} = 1$. One can verify, by working out the different cases, that Eq. (23) is equivalently expressed as

$$\gamma_k' = \alpha_k \oplus \gamma_k \quad (24)$$

where the $\oplus$ denotes the operation of the Klein group; see Figure [B] for a detailed description.

It will be useful to think of the multi-index $\vec{a}$ as an element of a vector space. To do so, we notice that any group with the property that $\alpha \oplus \alpha = 0$ is indeed
a vector space under the two-element field \( \{0, 1\} \). We notice that the Klein group described in Fig. 3 is actually isomorphic to the two dimensional vector space over the field of two elements \( \{0, 1\} \). Then, we build the complete vector space, with the same field, and defining \( \vec{\alpha} \oplus \vec{\beta} = (\alpha_1 \oplus \beta_1, \cdots, \alpha_N \oplus \beta_N) \) \[36\]. We can indeed restate Eq. (24), and say that for quantum channels, if \( \tau_{\vec{\alpha}} = 1 \), then \( \tau_{\vec{\gamma}} = \tau_{\vec{\alpha} \oplus \vec{\beta}} \). For example, in Fig. 2(c) the indices that correspond to preserved components are \( \vec{\alpha}^{(0)} = (0, 0), \vec{\alpha}^{(1)} = (0, 2), \vec{\alpha}^{(2)} = (1, 0), \vec{\alpha}^{(3)} = (2, 2) \), and \( \vec{\alpha}^{(4)} = (3, 2) \). However, \( \vec{\alpha}^{(1)} + \vec{\alpha}^{(2)} = (1, 2) \), which is not preserved, and thus this diagram does not correspond to a quantum channel. From this view we can derive several interesting observations that will be presented in the rest of the section.

From this readily follows an amusing property: the set of all multi-indices \( \vec{\gamma} \) for which \( \tau_{\vec{\gamma}} = 1 \), is closed under binary vector addition, in other words, it forms a vector subspace of the set of all multi-indices. A moment’s consideration will further show that the above reasoning can be inverted, that is, that if we set all \( \tau_{\vec{\gamma}} \) equal to one whenever the \( \vec{\gamma} \) belongs to a given vector subspace of the set of all indices, then \( \tau \) indeed has an image which has only positive components. In other words, there is a one-to-one correspondence between a quantum channel, and a vector subspace of the aforementioned space.

With this information, we present a procedure to generate all solutions: we start out from the solution having \( \tau_{\vec{\alpha}} = 1 \), with everything else 0. We may then successively switch \( \tau_{\vec{\alpha}} \)'s to 1 for various values of \( \vec{\alpha} \), taking care immediately to set equal to one the components of \( \tau \) that correspond to values of \( \vec{\beta} \) generated by the previously switched values of \( \vec{\alpha} \) via the operation \( \oplus \). Doing so, in an ordered way, allows one to generate all PCE quantum channels with a given set of preserved components, without the need of exploring the exponentially large space of all PCE maps.

We can show that all PCE quantum channels preserve \( 2^K \) components. First recall that a vector space of dimension \( d \) over a field of \( q \) elements has \( q^d \) elements \[37\]. Now \( V \) is a vector space on a field of two elements having dimension \( 2N \). We have seen earlier that

\[
W = \{ \vec{\alpha} : \vec{\alpha} \in V, \tau_{\vec{\alpha}} = 1 \}
\]  

is a subspace of \( V \). As such, \( W \) has a given dimension \( K \), which means that \( W \) has \( 2^K \) elements. In other words, a set of indices \( \tau_{\vec{\alpha}} \) with the property discussed above can only have \( 2^K \) elements equal to 1, for a given integer \( K \).

It is natural to ask how many PCE quantum channels exist that preserve \( 2^K \) components. One can calculate such number, \( S_{N,K} \), by examining the number of different independent subsets of vectors that spawn a given vector subspace. In appendix C we show that

\[
S_{N,K} = \prod_{m=0}^{K-1} 2^{2N-m} - \frac{1}{2K-1}. 
\]

From the above expression, it is easy to see the symmetry relation

\[
S_{N,K} = S_{N,2N-K} 
\]

which suggest a relation between individual channels that preserve \( K \) and \( 2N - K \) Pauli components that for the time being has escaped our efforts to identify.

### IV. GENERATORS

We now discuss the existence of a generator set for all PCE quantum channels and how to label each of them uniquely as \( G_{\vec{\alpha}} \) (according to its local action on every qubit in the system). Finally we will discuss a symmetry of PCE quantum channel generators and a connection between them and \( A \), see Eq. (15).

There exists a subset of PCE quantum channels that generates the entire set; the nature of these generators may be studied, as we shall see, with the properties of the aforementioned vector space. By standard theorems
of linear algebra, any proper subspace \( W \), see Eq. (25), can be extended to a maximal non-trivial subspace of dimension \( 2N - 1 \) by adjoining appropriate additional basis elements. This can be done in different ways. We therefore arrive to the set of maximal extensions of \( W \), where every maximal subspace corresponds to a PCE quantum channel that preserves half of the Pauli components. The intersection of all the elements of this set reduces to \( W \) itself, and since intersection of subspaces translates to composition of PCE channels, this implies that all PCE quantum channels can be obtained as compositions of PCE channels corresponding to maximally non-trivial subspaces, plus the identity map. In other words, the set of PCE quantum channels that preserve half of the components plus the identity map, are a generator set for all PCE channels. Consider Fig. 4 subfigures (c), (d) and (e) represent nontrivial PCE generators and the composition of any two of them yield the PCE channel corresponding to (b).

PCE generators (hereafter PCEGs) may be characterized by its local action on every qubit in the system. This action can be encoded using a multi-index \( \vec{\alpha} \), as in [4], hence each of the different \( 4^N \) multi-indices may be uniquely related to each of the PCE generators and thus denoted as \( G_{\vec{\alpha}} \), see figures 5 and 6. The proof is simplified if one uses the Kraus representation developed in Sec. IV so we postpone the demonstration to appendix D.

For single qubits, the identity corresponds to \( G_0 \), shown in Fig. 1(b), whereas \( G_3 \) is shown in Fig. 1(c). The 2-qubits PCE channel represented in subfigure (c) of Fig. 4 acts on the first qubit (first column) as mapping its Bloch sphere to be \( x \)-axis, and on the second qubit (first row) acts an identity, hence it is labeled \( G_{(1,0)} \). See Fig. 6 for the notation of all 2-qubit PCE generators.

A reflection symmetry is identified for PCE generators. Consider the map \( \Sigma(k) \) that reflects a multi-index \( \vec{\alpha} \) with respect to the \( k \)-th axis. This map leaves all components of \( \vec{\alpha} \) invariant, except the \( k \)-th component, which is transformed according to

\[
0 \mapsto 3, \quad 3 \mapsto 0, \quad 1 \mapsto 2, \quad 2 \mapsto 1. \quad (28)
\]

The maps have the following properties:

1. \( \Sigma^{(k)}(\vec{\alpha}) \oplus \Sigma^{(k)}(\vec{\beta}) = \vec{\alpha} \oplus \vec{\beta}, \) and
2. \( \Sigma^{(k)}(\vec{\alpha}) \neq \vec{\alpha}. \)

From the first property, we now obtain

\[
\vec{\alpha} = \Sigma^{(k)}(\vec{0}) \oplus \Sigma^{(k)}(\vec{0}). \quad (29)
\]

This implies that if \( \Sigma^{(k)}(\vec{0}) \) belongs to a channel then \( \tau_{\vec{\alpha}} = \tau_{\Sigma^{(k)}(\vec{\alpha})} \), where we used the fact that for channels, the non-zero elements are closed under \( \oplus \). In other words, the components of a PCE channel are symmetric under reflection over the \( k \)-th axis. Now consider the case in which \( \Sigma^{(k)}(\vec{0}) \) does not belong to generator. Then \( \tau_{\vec{\alpha}} \neq \tau_{\Sigma^{(k)}(\vec{\alpha})} \), since the case \( \tau_{\vec{\alpha}} = \tau_{\Sigma^{(k)}(\vec{\alpha})} = 1 \) is forbidden due to Eq. (28) and the case \( \tau_{\vec{\alpha}} = \tau_{\Sigma^{(k)}(\vec{\alpha})} = 0 \) is also forbidden because for generators, the codimension of the associated vector space is 1. This means that the components of a PCE channel are antisymmetric with respect to reflection over the \( k \)-th axis. Indeed, the 2-qubit PCE generators \( G_{(1,0)}, \ G_{(3,2)} \) and \( G_{(2,2)} \) represented in subfigures (c), (d) and (e) of Fig. 2, respectively, are either symmetric or anti-symmetric under reflection with respect to lines that divide the diagram in half vertically and horizontally.

Finally, it is worth pointing out that \( A \) (and thus \( a \) [see Eq. (15)] encodes all the information of PCE generators \( G_{\vec{\alpha}} \), and, therefore, of all PCE quantum channels. From \( A \), the tensor power of matrix \( a \), one can infer the components \( \{ \tau_{\vec{\alpha}} \} \) of a PCE generator \( G_{\vec{\alpha}} \) by taking row (or column) \( \vec{\alpha} \) of \( A \) and replacing \( -1 \) with 0. The proof of the connection between PCEGs and \( A \) is given in appendix D and in Fig. 5 we illustrate this connection for the single-qubit case.

**V. PCE CHANNELS AND DECOHERENCE**

Lindblad processes arise naturally in many theoretical [15, 30, 38–41] and experimental [42] settings and are archetypical in decoherence dynamics. Moreover, these processes lead to a monotonic (continuous) lose of information [43] and describe non-invertible channels in the asymptotic limit \( t \to \infty \) (this can be seen from the monotonically (continuous) decreasing of the determinant, see Ref. [30]). It is known that not every quantum channel can be seen as a snapshot of a process arising from a traditional Lindblad equation or even a time-dependent Lindblad equation [15, 30]. Therefore, an interesting question is whether PCE channels can be seen as limit points of some Markovian processes. In this section we prove that in fact they are, and give two examples of Markovian implementations. The first of them consists in identifying each PCE channel as a fixed point of some pure dissipative process, and in the second implementation we relate each PCE channel to fixed points of some memory-less collision model.
A. Kraus representation

To derive the aforementioned implementations, we exploit the existence of the PCEGs and their Kraus representation (or operator-sum representation) which, for an arbitrary channel $\mathcal{E}$, reads

$$\mathcal{E}[\rho] = \sum_i K_i \rho K_i^\dagger,$$

(30)

with $\sum_i K_i^\dagger K_i = \mathbb{I}$ (the trace-preserving condition) [44]. Inspection of the Kraus operators for two-qubit PCEGs leads to the ansatz that the Kraus operators for the generator of $\mathcal{G}_\alpha$ are

$$K_0 = \frac{\mathbb{I}}{\sqrt{2}}, \quad K_1 = \frac{\sigma_\alpha}{\sqrt{2}}$$

(31)

since the Kraus operators corresponding to a single-qubit PCE are $\{\mathbb{I}/\sqrt{2}, \sigma_\alpha/\sqrt{2}\}$, corresponding to the operation that leaves the component $\sigma_\alpha$ invariant [3]. Notice that according to Kraus operators, the generators $\mathcal{G}_\alpha$ are $N$-qubit flip channels with flip probability of $1/2$, where the joint flip is $\rho \mapsto \sigma_\alpha \rho \sigma_\alpha$. In fact, tracing out all particles except the $k$-th one gives the well known qubit flip channels, i.e. $\text{tr}_k \mathcal{G}_\alpha = \mathcal{G}_{\alpha_0}$, see Eq. (11). And more generally, tracing out $m$ particles leaves a $N-m$ particles flip channel (completely dephasing).

We shall first show that the Kraus operators in Eq. (31) produce a PCE. Notice that $\sigma_\alpha \sigma_\beta \sigma_\alpha = a_{\alpha \beta} \sigma_\beta$, see Eq. (10), which in turn implies that

$$\sigma_\alpha \sigma_\beta \sigma_\alpha = A_{\alpha \beta} \sigma_\beta.$$  

(32)

Next, consider the action of a channel with Kraus representation (31) on a $N$-qubit system:

$$\rho \mapsto \frac{1}{2^N} \sum_{\beta}^{|\beta|} \left( \left( \frac{1}{2} \sigma_\beta + \frac{1}{2} \sigma_\alpha \sigma_\beta \sigma_\alpha \right) \right)$$

$$= \frac{1}{2^N} \sum_{\beta}^{|\beta|} \left( 1 + A_{\alpha \beta} \right) \sigma_\beta.$$  

(33)

However, since $A_{\alpha \beta} = \pm 1$, the channel characterized by the Kraus operators in Eq. (31) is a PCE channel. Moreover, one can notice that, except for the first row, half of the matrix elements of each row is $+1$ and half $-1$, which implies that the channel is a PCEG.

Observe also that a different choice of $\alpha$ in Eq. (31) leads to different channels. This follows from the fact that if two channels were the same, this would imply that the matrix representation of the corresponding superoperator of $\rho \mapsto \sigma_\alpha \rho \sigma_\alpha$ would have to be the same, which is clearly false. Since there are $4^n$ different $\alpha$, this implies that all PCEG (plus the identity map) are in one-to-one correspondence.

B. Pure dissipative implementation

In this section we show that any PCE channel can be seen as the fixed point of some decoherence process, starting with PCEGs and then extending to more general channels. Consider the following dynamical process that implements $\mathcal{G}_\alpha$ when $t \to \infty$,

$$\mathcal{G}_{\alpha,t}[\rho] = e^{-\gamma t} \rho + \frac{(1 - e^{-\gamma t})}{2} \sigma_\alpha \rho \sigma_\alpha \sigma_\alpha.$$  

(34)

where $\gamma > 0$. It is easy to show that the family of channels $\mathcal{G}_{\alpha,t}$ parametrized with $t \geq 0$ forms a one-parametric semigroup, i.e. $\mathcal{G}_{\alpha,t_1} \mathcal{G}_{\alpha,t_2} = \mathcal{G}_{\alpha,t_1 + t_2}$. Therefore $\mathcal{G}_{\alpha,t}$ describes a dissipative time-homogeneous Markovian process, which are always characterized by some Lindblad generator [38]. The Lindblad generator of $\mathcal{G}_{\alpha,t}$, denoted by $\mathcal{L}_0$, can be obtained using the standard procedure,

$$\frac{d \mathcal{G}_{\alpha,t}[\rho]}{dt} \bigg|_{t=0} = \frac{\gamma (\sigma_\alpha \rho \sigma_\alpha - \rho)}{2}$$  

(35)

where the unique Lindblad operator associated with the relaxation ratio $\gamma/2$ is simply $\sigma_\alpha$. Notice that $\sigma_\alpha$ is traceless, therefore the process is pure dissipative [39].

Since PCEGs commute, we can describe easily any other PCE channel as a fixed point of a decoherence process. For them, the Lindblad generators are the sum of the Lindbladians of the corresponding generators. As an example, consider the channel depicted in Fig. 2 b), it is equal to $\mathcal{G}_{\alpha,0} \mathcal{G}_{\beta,3}$, therefore it is the fixed point of the dissipation process described with the following Lindbladian,

$$\mathcal{L} [\rho] = \frac{\gamma_{0,3} (\sigma_{0,3} \rho \sigma_{0,3} - \rho) + \gamma_{3,3} (\sigma_{3,3} \rho \sigma_{3,3} - \rho)}{2}$$  

(36)

where $\gamma_{0,3}$ and $\gamma_{3,3}$ are positive and correspond to the Lindblad operators $\sigma_{0,3}$ and $\sigma_{3,3}$. Notice that such election of Lindblad operators is not unique, as the PCE channel described here is also equal to $\mathcal{G}_{\alpha,3} \mathcal{G}_{\beta,0}$.

C. Collision model implementation

We show now that PCE channels can also be implemented with simple collision models [45]. To do this, observe that employing the Stinespring dilation theorem [46], PCEGs can be implemented using an unitary over the system and an ancilla. Since PCEG always have Kraus rank 2, one can always choose a qubit as the ancillary system. Concretely,

$$\mathcal{G}_\alpha[\rho] = \text{tr}_{\text{qubit}} \left( U_\alpha (\rho \otimes |0\rangle \langle 0|) U_\alpha^\dagger \right),$$  

(37)
where \( \text{tr}_{\text{qubit}} \) denotes partial trace over the ancillary qubit, with the unitary defined as follows,

\[
U_α |ψ⟩ |0⟩ = \frac{1}{\sqrt{2}} \left( |ψ⟩ |0⟩ + σ_α |ψ⟩ |1⟩ \right), \tag{38}
\]

\[
U_α |ψ⟩ |1⟩ = \frac{1}{\sqrt{2}} \left( |ψ⟩ |0⟩ - σ_α |ψ⟩ |1⟩ \right). \tag{39}
\]

Therefore, any concatenation of PCEGs can be described as a collision model with as many collisions as generators needed. In fact, generators are described with one collision. For the general case consider some PCE channel \( \mathcal{E} \) generated with \( \{G_{α_1}, G_{α_2}, \ldots, G_{α_M}\} \). For this we can define an environment consisting of \( M \) qubits initially in the state \( (|0⟩|0⟩)^{⊗M} \), or equivalently one qubit with the additional assumption that its state is reset to \( |0⟩ \) after each collision (memory-less collisions). The collision with the \( k \)-th particle is described by \( U_{α_k} \), which acts solely over the system and the \( k \)th particle. Therefore \( \mathcal{E} \) can be written as follows,

\[
\mathcal{E}[ρ] = \text{tr}_E \left[ \left( U_{α_1} \ldots U_{α_M} \right) ρ \otimes (|0⟩|0⟩)^{⊗M} \left( U_{α_1} \ldots U_{α_M} \right) \dagger \right], \tag{40}
\]

where \( \text{tr}_E \) is the partial trace over the whole ancillary qubits. Notice that as PCEGs commute, the order of the collisions is irrelevant.

VI. CONCLUSIONS AND OUTLOOK

In this work we introduce and characterize a set of quantum maps which either preserve or completely erase the components of a multi-qubit density matrix, in the basis of Pauli strings; we call those maps Pauli component erasing (PCE) maps. For a single qubit these include the completely depolarizing and dephasing channels. To start the characterization, we note that not all PCE maps are quantum channels, as some are not completely positive. In fact, the most laborious task of this work was to evaluate complete positivity conditions given by the Choi-Jamiolkowski isomorphism, after which we showed that the components of PCE quantum channels form a finite vector space. This in turn allows to unravel several properties, such as the possible number of PCE channels, and the number of components preserved while also providing advantages to study numerically this set, for example by employing an efficient method to construct all quantum channels for a given number of qubits.

Similar to other objects in open quantum systems (for example, Lindblad processes), PCE quantum channels form a semigroup, but finite in this case. For PCE channels, the generators are generalized flip operations, i.e., channels that with probability 1/2 apply a joint flip. This structure allows to link this channel with multi-qubit decoherence processes which can be described, say by simple dissipative processes or memory-less collision models, which in turn may pave a way to either implement these channels or connect them with already existent decoherence families. This, together with the discovered algebraic structure that translates complete positivity into an explicit conditioned preservation of many-body correlations, encompasses an advance in the knowledge of the mathematical structures underlying general quantum channels.

In the future we might consider generalizations (such as going from qubits to qudits) as well as the geometric role of PCE channels within the set of all quantum channels to further advance the understanding of open quantum systems. We have thus described a family of quantum channels with a very special mathematical structure that allows to widen the understanding of quantum channels in the context of many-body systems.

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Appendix A: Quantum-Classical channels

A Quantum-classical (QC) channel is defined by using an orthonormal basis in the Hilbert space, Let \( B = \{|ψ_i⟩\} \) such a basis in \( \mathcal{H} \) with \( \dim(\mathcal{H}) = 2^N \), the QC channel associated to \( B \) is

\[
\mathcal{E}^{QC}_B[ρ] = \sum_{i=1}^{2^N} \langle ψ_i | ρ | ψ_i⟩ |ψ_i⟩⟨ψ_i|, \tag{A1}
\]

this is, QC channels project density matrices onto the corresponding diagonal matrix in the basis \( B \) [22].

Consider now that the basis \( B \) is the simultaneous eigenbasis of a maximal set of commuting Pauli strings denoted by \( \text{set}(B) \), such set contains \( 2^N \) elements (including the identity), and there are \( 2^N + 1 \) of such sets [29]. Now we proceed to demonstrate that the QCs defined in this way are PCE channels with \( 2^N \) 1s on their diagonal.

First we compute the components of \( \mathcal{E}^{QC}_B \) in the basis \( 2^{-N/2} \{σ_α\} \),

\[
\left( \mathcal{E}^{QC}_B \right)_{ij} = \frac{1}{2^N} \sum_{i=1}^{2^N} \langle ψ_i | σ_k | ψ_i⟩ |ψ_i⟩⟨σ_i|ψ_i⟩. \tag{A2}
\]

To evaluate the components, observe that \( \langle ψ_i | σ_k | ψ_i⟩^2 = 1 \) \( \forall σ_k ∈ \text{set}(B) \), and from the formula for the purity of
and its inverse relation

\[ |k⟩|j + l(\text{mod} 2)⟩ = \frac{1}{2} \sum_{j=0}^{1} (-1)^{jk} |σ_{jl}⟩. \]  

On the other hand, the matrix form of an arbitrary Pauli map \( \mathcal{E} \) may be written as

\[ \hat{\mathcal{E}} = \frac{1}{2} \sum_{lm=0}^{1} \tau_{lm} |σ_{lm}⟩⟨σ_{lm}| \]  

\[ = \frac{1}{2} \sum_{jklm} \tau_{lm}(-1)^{l(j+k)} \times |j⟩⟨j + m(\text{mod} 2)| ⟨k|⟨k + m(\text{mod} 2)|. \]  

After applying the reshuffling operation on \( \hat{\mathcal{E}} \), we obtain the Choi matrix associated to the map. It reads

\[ D = \frac{1}{2} \sum_{jklm} \tau_{lm}(−1)^{(l+j+k)} \times |j⟩⟨j + m(\text{mod} 2)|⟨k|⟨k + m(\text{mod} 2)|. \]  

Note furthermore that the expression above may also be written as a combination of tensor products of Pauli matrices

\[ D = \frac{1}{2} \sum_{lm} \tau_{lm}σ_{lm} ⊗ σ^*_{lm}. \]  

Back to the Eq. (B5), let us apply the index relabeling \( k \rightarrow j + k(\text{mod} 2) \), then the Choi matrix reads

\[ D = \frac{1}{2} \sum_{jklm} \tau_{lm}(−1)^{jk} \times |j⟩⟨j + k(\text{mod} 2)|⟨j + m(\text{mod} 2)|⟨j + m + k(\text{mod} 2)|, \]  

since \((-1)^{j+k(\text{mod} 2)} = (-1)^k\). To continue, we use the relation between computational and Pauli elements [Eq. (B2)], and notice that \( \sum_{j} (-1)^{j(m+n)} = 2δ_{mn}. \) We arrive to the simple expression

\[ D = \frac{1}{2} \sum_{jk} \left( \frac{1}{2} \sum_{lm} (-1)^{jm+k} \tau_{lm} \right) |σ_{jk}⟩⟨σ_{jk}|. \]  

Notice that \( D \) is already written in its diagonal form, and one can identify by inspection the eigenvalues. The eigenvalues read

\[ λ_{jk} = \frac{1}{2} \sum_{lm} (-1)^{jm+k} \tau_{lm}, \]  

or more compactly \( \vec{λ} = (1/2)H ⊗ H \vec{σ} \), where \( H \) is the Hadamard matrix.

For the sake of convenience in the demonstration of several useful properties of the PCE channels, we shall reorder the eigenvalues, to write

\[ \lambda = \frac{1}{2} a \vec{σ} \]  

with \( a \) the matrix shown in Eq. (10) instead of \( H ⊗ H. \) This can be done due to the fact that both matrices \((a \text{ and } H ⊗ H)\) are equivalent up to a permutation of rows. In other words this operation corresponds to a reordering of the eigenvalues.

1. \textbf{N} qubits

To work out the \( N \) qubit case, we again rely on binary indices. In this case, we replace \( N \) dimensional vector \( \vec{a} \) with a pair of \( N \) dimensional vector binary indices \( j \).
and $\vec{k}$ so that each entry $\alpha_i$ of $\vec{\alpha}$ is identified with the pair $j_i$ and $k_i$ as in the single qubit case of the previous subsection. Then, all the steps leading to Eq. (B8) can be redone.

The tensor product of Pauli matrices, in vector form, will be denoted by $|\sigma_{\vec{k}\vec{l}}\rangle\langle\sigma_{\vec{k}\vec{l}}|$. With this in mind, a $N$-qubit Pauli map can be written as

$$\hat{\mathcal{E}}_N = \frac{1}{2^N} \sum_{\vec{m}\vec{n}} \tau_{\vec{m}\vec{n}} |\sigma_{\vec{m}\vec{n}}\rangle\langle\sigma_{\vec{m}\vec{n}}|.$$  \hfill (B10)

The generalization of Eqs. (B1) and (B2) read

$$|\sigma_{\vec{k}\vec{l}}\rangle = \sum_j (-1)^{\vec{j}\cdot\vec{k}} |j\rangle |j + \vec{l}(\text{mod}2)\rangle,$$

$$|\vec{j}\rangle |\vec{l} + \vec{m}(\text{mod}2)\rangle = \frac{1}{2^N} \sum_{\vec{m}} (-1)^{\vec{m}\cdot\vec{j}} |\sigma_{\vec{m}\vec{n}}\rangle.$$ \hfill (B11)\hfill (B12)

By employing the previous relations, we can write the matrix representation of the map, $\hat{\mathcal{E}}_N$ in the $N$-qubits computational basis as

$$\hat{\mathcal{E}} = \frac{1}{2} \sum_{\vec{j}\vec{k}\vec{l}\vec{m}} \tau_{\vec{j}\vec{k}\vec{l}\vec{m}} (-1)^{\vec{j}+\vec{k}}
\times |\vec{j}\rangle |\vec{j} + \vec{m}(\text{mod}2)\rangle \langle \vec{k} | \langle \vec{k} + \vec{m}(\text{mod}2)|.$$ \hfill (B13)

In this way it is straightforward to apply the reshuffling operation on $\hat{\mathcal{E}}_N$ to obtain the associated Choi matrix, then transform back to the Pauli basis and simplify to obtain

$$\mathcal{D}_N = \frac{1}{2^N} \sum_{\vec{m}\vec{n}} \left( \frac{1}{2^N} \sum_{\vec{l}\vec{n}} \tau_{\vec{l}\vec{n}} (-1)^{\vec{l}+\vec{n}} \right) |\sigma_{\vec{m}\vec{n}}\rangle\langle\sigma_{\vec{m}\vec{n}}|.$$ \hfill (B14)

All intermediate steps, from Eq. (B3) to Eq. (B7) are similar, but with a vectorized version of the indices, and appropriate normalization constants. Again, we are left with an expression that displays explicitly the eigenvalues of the Choi matrix, so we can write

$$\lambda_{\vec{j}\vec{k}} = \frac{1}{2^N} \sum_{\vec{l}\vec{n}} (-1)^{\vec{j}+\vec{k}+\vec{l}} \tau_{\vec{l}\vec{n}},$$ \hfill (B15)

or more compactly $\lambda = (H \otimes H/2)^{\otimes N} \tau$. Again, we prefer to reorganize the indices to be able to write

$$\lambda = \frac{1}{2^N} A \tau,$$ \hfill (B16)

where $A = a^{\otimes N}$.

**Appendix C: Number of PCEs for a fixed number of invariant components**

Finally, we may enumerate straightforwardly the subspaces $W$ of dimension $K$. We do this in 2 steps: first, we evaluate $N_{K,N}$, the number of all linearly independent subsets $V$ with $K$ elements. Each of these is the basis of one subspace of dimension $K$, but each subspace has a number $\mathcal{M}_K$ of different bases. The crucial point is that $\mathcal{M}_K$ is independent of the subspace under consideration: $\mathcal{M}_K$ simply describes the number of linear maps of $W$ onto itself. The total number $S_{N,K}$ of subspaces of dimension $K$ is therefore $N_{K,N}/\mathcal{M}_K$.

To evaluate $N_{K,N}$ we proceed by steps: the first element of the basis can be any non-zero element, of which the number is $2^{2N} - 1$. For the basis element $m + 1$, we must choose from those which do not belong to the $m$ dimensional space generated by the first $m$ basis elements, so that one chooses from $2^{2N} - 2^m$. We thus have

$$N_{K,N} = \prod_{m=0}^{K-1} (2^{2N} - 2^m).$$ \hfill (C1)

On the other hand, any map of a $K$-dimensional vector space $W$ onto itself is uniquely defined by a non-singular binary $K \times K$ matrix over the field $\{0,1\}$. To count these, we proceed as above: the first line is an arbitrary non-zero vector, of which there are $2^K - 1$. For the row $m + 1$ we must choose an arbitrary vector not belonging to those generated by the first $m$ vectors, of which there are $2^K - 2^m$. This eventually yields

$$\mathcal{M}_K = \prod_{m=0}^{K-1} (2^K - 2^m).$$ \hfill (C2)

From this follows that

$$S_{N,K} = \prod_{m=0}^{K-1} \frac{2^{2N-m} - 1}{2^{K-m} - 1}.$$ \hfill (C3)

**Appendix D: Local-action and labeling of PCE generators**

The local-action of a generator $\mathcal{G}_\vec{a}$ on every qubit in the system depends only, as its notation suggests, on the multi-index $\vec{a}$. This index has a simple meaning that can be read from the graphical representation of the channel. Recall the single qubit PCE generators, shown in Fig. 5 denoted by $\mathcal{G}_0$ (corresponding to the identity map) and

![Figure 6. PCE generators for 2 qubits. Notice that all generators are either symmetric or anti-symmetric under horizontal and vertical reflections.](image)
\( G_{1,2,3} \) (corresponding to the completely bit, phase and bit-phase flip channels respectively). One can easily read the diagrams in the following manner: \( \alpha = 0 \) corresponds to all squares black, whereas for \( \alpha > 0 \) we have only the 0-th and the \( \alpha \)-th squares black. Let us generalize this characterization rule for \( N \)-qubit PCE generators. Consider the reduced density matrix of the \( k \)-th qubit after generator \( G_\alpha \) acts on the entire system,

\[
\text{tr}_k G_\alpha[\rho] = \frac{1}{2} \text{tr}_k(\rho + \sigma(\bar{\gamma})) = \frac{\rho_k}{2} + \frac{\sigma_{\alpha k} \rho_k \sigma_{\alpha k}}{2} = G_{\alpha_k}[\rho_k],
\]

where \( k \) means all qubits except for the \( k \)-th one are traced out. We can read from Eq. (D1) that \( \alpha_k \) not only characterizes \( G_{\alpha_k} \) but actually tells us which single qubit channel is acting locally on the \( k \)-th qubit. The action of \( G_{\alpha_k} \) on the local components of the reduced density matrix \( \rho_k \) reads \( \tau_{0,\ldots,j_k\ldots,0} \mapsto \sum_{\tau_{0,\ldots,j_k\ldots,0}} \tau_{0,\ldots,j_k\ldots,0} \). The general characterization rule for all PCE generators \( G_\alpha \) is clear now: if all \( \tau_{0,\ldots,j_k\ldots,0} = 1 \), then \( \alpha_k = 0 \), otherwise, if \( \tau_{0,\ldots,j_k\ldots,0} = 1 \) (with \( j_k > 0 \)), then \( \alpha_k = j_k \). For two qubit PCE diagrams this means that the multi-index \( \bar{\alpha} \) is encoded in the first column and row of the diagrams. See \( G_{(0,2)} \) in Fig. 6 for all \( \tau_{1,0} = 1 \) and \( \tau_{0,2} = 1 \), then \( \bar{\alpha} = (0,2) \). In Fig. 6 we show all two qubits PCE generators and their corresponding notation \( G_\bar{\alpha} \).

An interesting relation of the generators and the \( A \) matrix can be derived with the tools developed. Consider the generator \( G_{\bar{\alpha}} \) and its Pauli components \( \tau_{\beta} \). We can calculate the former studying the action of the generator on the non-normalized state \( \rho = \sum_\gamma \sigma_\gamma \). Let us proceed with such calculation, using the Kraus decomposition Eq. (31):

\[
\tau_{\beta}^{(\bar{\alpha})} = \text{tr} \sigma_\beta G_{\bar{\alpha}}[\rho] = \frac{1}{2} \sum_\gamma \text{tr} \big[ \sigma_\beta \sigma_\gamma + \sigma_\bar{\alpha} \sigma_\bar{\alpha} \sigma_\gamma \sigma_\bar{\alpha} \big] = \frac{1}{2} \left( 1 + A_{\bar{\alpha}\beta} \right)
\]

where we have used the orthogonality relations of Pauli matrices and Eq. (32). This means that one can read the \( \alpha \)-th generators directly from matrix \( A \), see Fig. 7 for the \( n = 1 \) case. Alternatively one could construct the \( A \) matrix for \( n = 2 \), from Fig. 6 where the first row of this matrix is read from \( G_{(0,0)} \), replacing black (white) squares with 1’s (−1s), the second row from \( G_{(0,1)} \), etc.
[20] T. Heinosaari and M. Ziman. The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement. Cambridge University Press, 2012.

[21] T. Rybár, S. N. Filippov, M. Ziman, and V. Bužek. Simulation of indivisible qubit channels in collision models. J. Phys. B, 45(15):154006, 2012.

[22] Michael Nathanson and Mary Beth Ruskai. Pauli diagonal channels constant on axes. J. Phys. A, 40(28):8171, 2007.

[23] Katarzyna Siudzińska. Generalization of Pauli channels through mutually unbiased measurements. Phys. Rev. A, 102(3):32603, 2020.

[24] Ivan Sergeev. Generalizations of 2-dimensional diagonal quantum channels with constant frobenius norm. Rep. Math. Phys., 83(3):349–372, 2019. http://arxiv.org/pdf/1903.08420.

[25] Dariusz Chruściński and Katarzyna Siudzińska. Generalized pauli channels and a class of non-markovian quantum evolution. Phys. Rev. A, 94(2):022118, 2016. http://arxiv.org/pdf/1606.02616.

[26] H. Ohno and D. Petz. Generalizations of pauli channels. Acta Math. Hungarica, 124(1-2):165–177, 2009. http://arxiv.org/pdf/0812.2668.

[27] Alejandro Fonseca. High-dimensional quantum teleportation under noisy environments. Physical Review A, 100(6):062311, Dec 2019.

[28] Gen Kimura. The bloch vector for n-level systems. Phys. Lett. A, 314(5):339–349, 2003.

[29] Jay Lawrence, Časlav Brukner, and Anton Zeilinger. Mutually unbiased binary observable sets on N qubits. Physical Review A, 65(3):032320, Feb 2002.

[30] M. M. Wolf and J. I. Cirac. Dividing quantum channels. Comm. Math. Phys., 279(1):147–168, 2008.

[31] Ingemar Bengtsson and Karol Zyczkowski. Geometry of Quantum States: An Introduction to Quantum Entanglement. Cambridge University Press, 2nd edition, 2017.

[32] A. Jamiołkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. Reports on Mathematical Physics, 3(4):275 – 278, 1972.

[33] Man-Duen Choi. Completely positive linear maps on complex matrices. Linear Algebra and its Applications, 10(3):285 – 290, 1975.

[34] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, New York, NY, USA, 10th edition, 2011.

[35] Alexei Gilchrist, Daniel R. Terno, and Christopher Wood. Vectorization of quantum operations and its use. arXiv, 3:12, 2009.

[36] We can further unravel the vector space, by identifying each index $\alpha \in \{0, 1, 2, 3\}$ with its binary notation $\text{e.g. identify 2 with (1, 0)}$ such that to each multindex $\vec{\alpha}$ there corresponds a binary string of length $2N$. In fact, our sum $\oplus$ would correspond to addition modulo 2 of each of the $2N$ components.

[37] Steven Roman. Advanced Linear Algebra, volume 135 of Graduate Texts in Mathematics. Springer New York, New York, NY, 3 edition, 2008.

[38] G. Lindblad. On the generators of quantum dynamical semigroups. Comm. Math. Phys., 48(2):119–130, 1976.

[39] A. Kossakowski. On quantum statistical mechanics of non-hamiltonian systems. Rep. Math. Phys., 3(4):247 – 274, 1972.

[40] A. Kossakowski. On necessary and sufficient conditions for a generator of a quantum dynamical semigroup. Bull. Acad. Pol. Sci., 20(12):1021, 1972.

[41] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan. Completely positive dynamical semigroups of N-level systems. J. Math. Phys., 17(5):821, 1976.

[42] N. Boulant, T. F. Havel, M. A. Pravia, and D. G. Cory. Robust method for estimating the lindblad operators of a dissipative quantum process from measurements of the density operator at multiple time points. Phys. Rev. A, 67:042322, Apr 2003.

[43] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer. Markovianity and non-markovianity in quantum and classical systems. New J. Phys., 13(9):093004, 2011.

[44] Karl Kraus, A. Böhm, J. D. Dollard, and W. H. Wooters, editors. States, Effects, and Operations Fundamental Notions of Quantum Theory, volume 190 of Lecture Notes in Physics. Springer Berlin Heidelberg, Berlin, Heidelberg, 1983.

[45] Mario Ziman and Vladimír Bužek. Open system dynamics of simple collision models. In Quantum Dynamics and Information, pages 199–227. World Scientific, Oct 2010.

[46] W. Forrest Stinespring. Positive Functions on C*-Algebras. Proceedings of the American Mathematical Society, 6(2):211, Apr 1955.