Scalar field theory description of the running vacuum model: the vacuumon

Spyros Basilakos, Nick E. Mavromatos and Joan Solà Peracaula

Abstract. We investigate the running vacuum model (RVM) in the framework of scalar field theory. This dynamical vacuum model provides an elegant global explanation of the cosmic history, namely the universe starts from a non-singular initial de Sitter vacuum stage, it passes smoothly from an early inflationary era to a radiation epoch (“graceful exit”) and finally it enters the dark matter and dark energy (DE) dominated epochs, where it can explain the large entropy problem and predicts a mild dynamical evolution of the DE. Within this phenomenologically appealing context, we formulate an effective classical scalar field description of the RVM through a field $\phi$, called the vacuumon, which turns out to be very helpful for an understanding and practical implementation of the physical mechanisms of the running vacuum during both the early universe and the late time cosmic acceleration. In the early universe the potential for the vacuumon may be mapped to a potential that behaves similarly to that of the scalaron field of Starobinsky-type inflation at the classical level, whilst in the late universe it provides an effective scalar field description of DE. The two representations, however, are not physically equivalent since the mechanisms of inflation are entirely different. Moreover, unlike the scalaron, vacuumon is treated as a classical background field, and not a fully fledged quantum field, hence cosmological perturbations will be different between the two pictures of inflation.

Keywords: dark energy theory, inflation, physics of the early universe

ArXiv ePrint: 1901.06638
1 Introduction

The recent and past analyses of the Planck collaboration [1–3] provide significant support to the general cosmological paradigm, namely that the observed Universe is spatially flat and it is dominated by dark energy (around $\sim 70\%$), while the rest corresponds to matter (dark matter and baryons) [4–8]. From the dynamical viewpoint dark energy (hereafter DE) occupies an eminent role because it provides a theoretical framework towards explaining the cosmic acceleration. The simplest candidate for DE is the cosmological constant. The combination with cold dark matter (CDM) and ordinary baryonic matter builds the so-called $\Lambda$CDM model, which fits extremely well the current cosmological data. Despite its many virtues, the $\Lambda$CDM model suffers from two fundamental problems, namely the fine tuning problem and the cosmic coincidence problem [9–14]. But it also suffers from some persistent phenomenological problems (called “tensions”) of very practical nature, see e.g. [15] for a variety of them. Perhaps the most worrisome ones nowadays are two. One of them is the mismatch between the CMB determination and the local value of the Hubble parameter [16, 17]. Moreover, there is an unexplained $\sigma_8$-tension [18], which is revealed through the fact that the $\Lambda$CDM tends to provide higher values of that parameter than those obtained from large scale structure formation measurements. Dynamical DE models seem to furnish a possible alleviation of such tensions, see e.g. [19–26], and this is a very much motivating ingredient for focusing our attention on these kind of models. Let us note that these models can also appear as effective behavior of theories beyond General Relativity (GR). For instance, it has recently been shown [27] that Brans-Dicke gravity with a cosmological constant, when parameterized as a deviation of GR, can mimic dynamical dark energy, and more specifically running vacuum (see below), and can provide a solution to those tensions, in particular to the acute $H_0$-tension (which is rendered harmless in this framework).

These problems have opened a window for plenty cosmological scenarios which basically generalize the standard Einstein-Hilbert action of GR by introducing new fields in nature ([28–31]), or extensions of GR. Among the large family of DE models, very important in these kind of studies is the framework of the running vacuum model (RVM) [32–41] — see [13, 42–44] and references therein for a detailed review. In fact the idea to have a vacuum that
depends on cosmic time (or redshift) has a long history in cosmology and it is perfectly allowed by the cosmological principle ([45–62]).

In this context, the equation of state is similar to standard $\Lambda$CDM $p_\Lambda = -\rho_\Lambda$, however the vacuum density varies with time $\rho_\Lambda = \rho_\Lambda(t)$. The cosmological implications of various dynamical DE models models have recently been analyzed both for the early universe [63–70] as well as for the late universe [71–73] — see also [74–84] for previous analyses and [19–26] for the most recent ones. In the context of the RVM, the parameter $\rho_\Lambda$ acts as a running parameter for the entire evolution [44] and evolves slowly as a power series of the Hubble parameter. One finds that the spacetime emerges from a non-singular initial de Sitter vacuum stage, while the phase of the universe changes smoothly from early inflation to a radiation era ("graceful exit"). After this period, the universe passes to the dark-matter and $\Lambda$-dominated epochs before finally entering a late-time de Sitter era.

In a previous work [85], we have provided a concrete examples of a class of field theory models of the early Universe that can be connected with RVM. Specifically, we have managed to reformulate the effective action of Supergravity (SUGRA) inflationary models as an RVM, by demonstrating that $\rho_\Lambda$ in such models can be written as an even power series of the Hubble rate, which can be naturally truncated at the $H^4$ term. This is exactly the underlying framework expected in the simplest class of running vacuum models. The RVM can also provide a clue for alleviating the fine tuning problem [13]. Moreover, comparison of the RVM against the latest cosmological data (SNIa+BAO+$H(z)\text{-}\text{LSS}+\text{BBN}+\text{CMB})$ yield a quality fit that is significantly better than the concordance $\Lambda$CDM, see [19–23] for detailed analyses supporting this fact. A summary review is provided in [86, 87]. Interestingly, the RVM can be mimicked by Brans-Dicke theories as well, [88, 89]. Obviously, the aforementioned results have led to growing interest in dynamical vacuum cosmological models of the type $\Lambda = \Lambda(H)$ Therefore, there is every motivation for further studying running vacuum models from different perspectives, with the aim of finding possible connections with fundamental aspects of the cosmic evolution. In point of fact, this is the main goal of the present article. Specifically, we attempt to provide a scalar field description of the RVM which is one of the most popular $\Lambda(H)$CDM models. As we shall show, within this scalar field representation, the RVM can describe inflation and subsequent exit from it, however the underlying physics is different from other approaches, like the Starobinsky approach.

The structure of the paper is as follows. The general framework of the RVM is introduced in II. The basic theoretical elements of the RVM are presented in sections III and IV. The scalar field description of the RVM, based on the vacuumon field is developed in section V. Finally, our conclusions are summarized in section VI.

2 Running vacuum cosmology

The general idea of dynamical vacuum is a useful concept because it may provide and elegant global description of the cosmic expansion. In this section, we briefly present the basic ingredients of a specific realization of this idea which goes under the name of running vacuum model (RVM), based on an effective “renormalization group (RG) approach” of the cosmic evolution, see [32, 33, 39] and [13, 44]. In fact, the RVM grants a unified dynamical picture for the entire cosmic evolution [44, 63–70]. It connects smoothly the first two cosmic eras, inflation and standard Fridman-Lemaître-Robertson-Walker (FLRW) radiation, and subsequently allows the universe to naturally enter the matter and dark energy domination in the
present time, in which it still carries a mild dynamical behavior compatible with the current observations [71, 72].

In the old decaying vacuum models mentioned above (see e.g. [90, 91] for a review) the time-dependent cosmological term was usually some ad hoc function \( \Lambda(t) \) of the cosmic time or of the scale factor. However, the running vacuum idea is inspired in the context of QFT in curved spacetime. It is based on the effective RG approach to the evolution of the universe, and hence on the existence of a running scale, \( \mu_c = \mu_c(t) \), which is connected to the dynamical variables of the cosmic evolution. Such variable is not known a priori. In particle physics, for example, one associates it to the characteristic energy scale of the process in an accelerator, or to the mass of the decaying particle in the proper reference frame. In cosmology, however, it is more difficult, but a natural ansatz within the FLRW metric is to associate the RG running scale to the Hubble parameter, i.e. to the expansion rate of the cosmic evolution, \( \mu_c(t) = H(t) \) — see [13, 44] and references therein. Following the notations in these references, the RG equation takes the general form:

\[
\frac{d\rho_\Lambda(t)}{d \ln H^2} = \frac{1}{(4\pi)^2} \sum_i \left[ a_i M_i^2 H^2 + b_i H^4 + c_i \frac{H^6}{M_i^2} + \ldots \right],
\]  

(2.1)

Let us emphasize that \( \rho_\Lambda = \Lambda / \kappa^2 \) in this equation is treated as the vacuum energy density, due to the presence of \( \Lambda(t) \), with pressure being given by the following equation of state (EoS)

\[
p_\Lambda = -\rho_\Lambda.
\]

(2.2)

We would like to stress that the latter EoS does not depend on whether the vacuum is dynamical or not.

In general \( \mu_c^2 \) can be associated to a linear combination of \( H^2 \) and \( \dot{H} \) and the variety of terms appearing on the r.h.s. of (2.1) can be richer [44], but the canonical possibility is the previous one and hereafter we restrict to it. As shown in specific analyses [19–23], there are no dramatic differences in the phenomenological results obtained in the presence of the additional \( \dot{H} \) term. Furthermore, these terms are not essential either for the description of the early universe, at least within the RVM since \( \dot{H} \approx 0 \) during inflation, see [63, 68–70].

The coefficients \( a_i, b_i, c_i \ldots \) in (2.1) are dimensionless and receive contributions from loop corrections of boson and fermion matter fields with different masses \( M_i \). The expression (2.1) has to be understood as a general ansatz for the vacuum energy density in an expanding universe. The reason for choosing even powers of \( H \) is due to general covariance of the effective action of QFT in a curved background. Although we cannot presently quote its precise form, it has been found [32–41] that it can only include even powers which can emerge from the contractions of the metric tensor with the derivatives of the scale factor. Moreover, the current vacuum models have recently been linked with a potential variation of the so-called fundamental constants of nature [76, 77].

Notice that if the evolution of the Universe is restricted to eras close the Grand Unified Theory (GUT) scale, then for all practical purposes it is at most the \( H^4 \) terms (those with dimensionless coefficients \( b_i \)) that can contribute significantly. There is no possible inflation without the higher order powers of \( H^n (n > 2) \), as explained in detail in [63, 68–70].

The \( H^2 \) term is of course negligible at this point, and the higher powers of \( H^n \) for \( n = 6, 8, \ldots \) are suppressed by the corresponding inverse powers of the heavy masses \( M_i \), which go to the denominator, as required by the decoupling theorem. In the scenarios of dynamical breaking of local supergravity discussed in refs. [92–95], the breaking and the
associated inflationary scenarios could occur around the GUT scale, in agreement with the inflationary phenomenology suggested by the Planck data [96], provided Jordan-frame supergravity models (with broken conformal symmetry) are used, in which the conformal frame function acquired, via appropriate dynamics, some non trivial vacuum expectation value [85]. For these situations, therefore, corrections in (2.1) involving higher powers than \( H^4 \) will be ignored.

Remarkably, it can be shown that the presence of the term \( H^4 \) in the effective expression of the vacuum energy density can be the generic result of the low-energy effective action based on the bosonic gravitational multiplet of string theory, see [97]. The unavoidable presence of the CP-violating gravitational Chern-Simons term associated with that action turns out to lead to an effective \( \sim H^4 \) behavior when averaged over the inflationary spacetime, in the presence of primordial gravitational waves. This higher order term triggers inflation within the context of the RVM, see section 4.1. It follows that the entire history of the universe can be described in an effective RVM language upon starting from the fundamental massless bosonic gravitational multiplet of a generic string theory. The reader is invited to study [97] for details and [98] for a summary of the underlying framework.

In the next sections, we study the running vacuum model by integrating (2.1), following the approach of [63, 64, 68–70]. Before doing so, let us briefly review the main ingredients of RVM.

### 3 Dynamical vacuum and running vacuum model

The aim of this section is to demonstrate that there exists a family of time-dependent effective dynamical vacuum models of \textit{running type}, i.e. the class of the running vacuum models (RVM’s) introduced in section 2, which characterize the evolution of the Universe from the exit of the Starobinsky inflationary phase till the present era.

Following the notations of [32, 33, 63, 64, 68–70] eq. (2.1) is approximated by

\[
\frac{d \rho_{\Lambda}(t)}{d \ln H^2} = \frac{1}{(4\pi)^2} \sum_i \left[ a_i M_i^2 H^2 + b_i H^4 \right].
\]

Performing the integration we find

\[
\rho_{\Lambda}(H) = \frac{\Lambda(H)}{\kappa^2} = \frac{3}{\kappa^2} \left( c_0 + \nu H^2 + \alpha \frac{H^4}{H_I^2} \right),
\]

where \( \kappa^2 = 8\pi G \), \( c_0 \) is an integration constant (with dimension +2 in natural units, i.e. energy squared) which can be constrained from the cosmological data [71, 74]. Also, the dimensionless coefficients are given by

\[
\nu = \frac{1}{48\pi^2} \sum_{i=F,B} a_i \frac{M_i^2}{M_{Pl}^2},
\]

and

\[
\alpha = \frac{1}{96\pi^2} \frac{H_I^2}{M_{Pl}^2} \sum_{i=F,B} b_i.
\]

For the present universe, \( H = H_0 \sim 10^{-42} \text{GeV} \), it is obvious that the \( \sim H^4 \) contribution can be completely neglected as compared to the \( \sim H^2 \) component of the vacuum energy.
density (3.2). However, for the early universe the value of $H$ becomes much larger and the $\sim H^4$ contribution is dominant. We shall confirm this fact quantitatively in the next section in the light of the solution of the cosmological equations.

Practically, the coefficient $\nu$ can be seen as the reduced (dimensionless) beta-function for the RG running of $\rho_\Lambda$ at low energies, while $\alpha$ has a similar behavior at high energies. Notice that the index $i$ depends on whether bosons ($B$) or fermions ($F$) dominate in the loop contributions. Of course, since the dimensionless coefficients ($\nu, \alpha$) play the role of one-loop beta-functions (at the respective low and high energy scales) they are expected to be naturally small because $M_i^2 \ll M_{Pl}^2$ for all the particles, even for the heavy fields of a typical GUT. Indeed, within GUT $\nu$ lies in the range $|\nu| = 10^{-6} - 10^{-3}$ [39], while $\alpha$ is also small ($|\alpha| \ll 1$), because the scale of inflation $H_I$ is certainly below the Planck scale. The latter predictions are in agreement with the observational constraints. In point of fact, from the joint likelihood analysis of various cosmological data (supernovae type Ia data, the CMB shift parameter, and the Baryonic Acoustic Oscillations), it has been found $|\nu| = \mathcal{O}(10^{-3})$ [71–75] (see also [19–23, 99, 100]).

Let us finally clarify the possible effect of the $\dot{H}$ terms, which we mentioned briefly in the previous section. Since $\dot{H} = (1 + q)H^2$ (with $q$ the deceleration parameter), it can been shown that the effect of that term in the measured cosmological observables can be taken into account by “renormalizing” the effect of the $H^2$ terms. Moreover, during inflation we have $H \simeq$ constant and as previously indicated $\dot{H}$ can be neglected. Therefore, terms of the form $\dot{H}^2$ and $\dot{H}H^2$ are negligible and inflation is dominated by the single term proportional to $H^4$ in the context of the RVM.

## 4 Running vacuum versus scalar field: the vacuumon

Let us start here with the Einstein-Hilbert action

$$S_{R,\Lambda} = \int d^4x\sqrt{-g} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) + \mathcal{L}_m \right],$$

where $\kappa = \sqrt{8\pi G}$, with $G$ the four-dimensional Newton constant, $\rho_\Lambda(t) = \Lambda(t)/\kappa^2$ and $\mathcal{L}_m$ is the Lagrangian of matter. Notice that the cosmological equations are expected to be formally equivalent to the standard $\Lambda$CDM case, due to the Cosmological Principle which is embedded in the FLRW metric. The latter is perfectly compatible with the possibility of a time-evolving cosmological term [101]. If we vary the action (4.1) with respect to the metric we obtain the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 \tilde{T}_{\mu\nu},$$

where $\tilde{T}_{\mu\nu} \equiv \rho_{tot} g_{\mu\nu} + (p_{tot} + p_{r}) U_\mu U_\nu$, is the total energy momentum tensor, with with $\rho_{tot} = \rho_m + \rho_\Lambda$ and $p_{tot} = p_m + p_\Lambda = p_m - \rho_\Lambda$. As already mentioned, $\rho_\Lambda = \Lambda(t)/\kappa^2$ is the vacuum energy density related to the presence of $\Lambda(t)$ with EoS (2.2). The quantity $\rho_m$ is the density of matter-radiation and $p_m = \omega_m \rho_m$ is the corresponding pressure, where $\omega_m$ is the EoS parameter: $w_m = 0$ for nonrelativistic matter and $w_m = 1/3$ for relativistic (i.e. for the radiation component).

In the framework, of a spatially flat FLRW spacetime, we obtain the Friedmann equations in the presence of a running $\Lambda$-term:

$$\kappa^2 \rho_{tot} = \kappa^2 \rho_m + \Lambda = 3H^2,$$

$$\kappa^2 p_{tot} = \kappa^2 p_m - \Lambda = -2\dot{H} - 3H^2$$

(4.3)  

(4.4)
and the Ricci scalar
\begin{equation}
R = g^{\mu \nu} R_{\mu \nu} = 6(2H^2 + \dot{H}),
\end{equation}
where the overdot denotes derivative with respect to cosmic time $t$. Following standard lines the Bianchi identities $\nabla^{\mu} T_{\mu \nu} = 0$ for $G =$const. reduce
\begin{equation}
\dot{\rho}_m + 3(1 + \omega_m)H \rho_m = -\dot{\rho}_\Lambda.
\end{equation}
where one may check that there is an exchange between matter and vacuum.

Using equations (4.3), (4.4) and (4.6), we derive the main differential equation that governs the cosmic expansion, namely
\begin{equation}
\dot{H} + 3\frac{1}{2}(1 + \omega_m)H^2 = \frac{1}{2} \kappa^2 (1 + \omega_m)\rho_\Lambda = \frac{(1 + \omega_m)\Lambda}{2}.
\end{equation}
Inserting eq. (3.2) into the above differential equation and solving it, one finds
\begin{equation}
\dot{H} + 3\frac{1}{2}(1 + \omega_m)H^2 \left( 1 - \nu - \frac{c_0}{H^2} - \alpha \frac{H^2}{H^2} \right) = 0.
\end{equation}
Below, we discuss the cosmic history and the scalar field description of RVM.

Before embarking onto that, we feel commenting on an important feature of the RVM approach, namely the fact that terms of order $O(H^4)$ and higher, that appear in the evolution (4.8) of the Hubble parameter, owe their existence to the corresponding higher-order terms of the RG-like vacuum energy density $\rho_\Lambda$ (3.2). Such terms might also be associated with higher curvature terms in the tensor $\tilde{T}_{\mu \nu}$, due, e.g. to quadratic curvature terms of the type appearing in the Starobinsky model [102], induced by conformal anomalies, or, in the context of string-inspired effective actions, by non-constant-dilaton $\Phi(x)$ terms-coupled to Gauss-Bonnet quadratic curvature invariants, $\int d^4x \sqrt{-g} e^{-2\Phi(x)}(R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4R_{\mu \nu} R^{\mu \nu} + R^2)$ or gravitational axion $b(x)$ terms, originating in four space-time dimensions from the antisymmetric tensor field of the massless bosonic string multiplet, coupled to gravitational Chern-Simons anomalous terms in a CP-violating manner, $\int d^4x \sqrt{-g} b(x) R_{\mu \nu \rho \sigma} \tilde{R}^{\mu \nu \rho \sigma}$, where the tilde over the Riemann tensor denotes the dual $\tilde{R}^{\mu \nu \rho \sigma} = \frac{1}{\sqrt{-g}} \epsilon^{\mu \nu \alpha \beta} R_{\alpha \beta \rho \sigma}$, where $\epsilon^{0123} = +1$ etc. Such higher order terms also lead to contributions to the energy density of the Universe of “running vacuum type”, in particular $H^4$ terms.

This is rather remarkable and has been demonstrated explicitly in [97, 98] for the gravitational Chern Simons terms, in the presence of primordial gravitational waves, which induce condensates of the Chern-Simons terms that lead to de Sitter space time, but it can be extended to include all other higher curvature terms. Nonetheless, the presence of extra scalar modes like dilaton and/or the Starobinsky-model scalaron can be distinguished from the scalar mode encoded in the RVM contributions (“vacuumon”), as we shall discuss below, although their co-existence in certain models cannot be excluded. In general, the cosmological perturbation effects of scalaron and “vacuumon” models can be different.
In fact, as we shall see, the vacuumon is only viewed as a classical background field, useful for a unified description of the evolution of the Universe from a de Sitter (inflationary) phase to the present era. The reader should recall that the RVM, which the vacuumon describes in scalar field language, is based on a vacuum energy density function interpolating continuously the various epochs, cf. eq. (3.2). As such, it should be stressed, that this is a very different view as compared to more standard approaches, including Starobinsky inflation. In the latter, for instance, the scalaron appears as a physical particle associated to fluctuations of a quantum field. This is in stark contrast with the classical vacuumon picture since in the RVM there is no reheating caused by the decay of an intermediate state consisting of massive particles, but rather a fast and continuous process of heating up of the early universe whereby the inflationary vacuum decays into radiation and smoothly gives rise to the radiation phase [63–70].

It is thus of foremost importance to realize that our framework does not aim at a microscopic interpretation. In fact, we remain at the level of the classical fields and ultimately rely on a thermodynamical description well along the line of the mentioned papers. In this respect, let us mention that the entropy problem [103] in this kind of models is nonexistent and the cosmic evolution can be shown to be perfectly consistent with the generalized second law of thermodynamics [65, 66]. One of the aims of the present work is to discuss in detail such issues once the above fundamental distinction between the microscopic scalaron (or, in general, inflaton like) picture and the classical vacuumon picture has been clearly established.

4.1 Early de Sitter-radiation: scalar field description

Initially, from eq. (4.8) we easily identify inflation (de Sitter phase), namely there is the constant value solution $H^2 = (1 - \nu)H_I^2/\alpha$ of eq. (4.8), which is valid in the early universe for which $c_0/H^2 \ll 1$. It should be now clear that in the absence of the $\sim H^4$ term in eq. (3.2) such de Sitter solution would not exist. Therefore in order to trigger inflation it is necessary to go beyond the $H^2$ contribution to the vacuum energy-density, although not all higher powers are on equal footing. As explained, covariance requires the participation of an even number of times derivatives of the scale factor, and we have adopted the simplest available possibility, $\sim H^4$. As already remarked, the presence of the $H^4$-term in the effective expression of the vacuum energy density (3.2) can actually be the generic result of gravitational Chern Simons terms (averaged over the inflationary spacetime) that characterise the string-inspired low-energy effective action based on the bosonic gravitational multiplet of string theory [97]. Therefore, the RVM description we are considering here can be associated with the effective treatment of such large class of well-motivated fundamental theories.

The inflationary solution can then be made explicit upon integration of eq. (4.8) when $c_0/H^2 \ll 1$ holds. We find

$$H(a) = \left( \frac{1 - \nu}{\alpha} \right)^{1/2} \frac{H_I}{\sqrt{D_a^q(1-\nu)(1+\omega_m)} + 1},$$

(4.9)

where $D > 0$ is the integration constant. For the early universe we consider that matter is essentially relativistic, hence here we impose $\omega_r \equiv \omega_m = 1/3$, hence $p_r = \rho_r/3$. Interestingly, we observe from (4.9) that in the case of $Da^4(1-\nu) \ll 1$ the universe starts from an unstable early de Sitter phase [inflationary era, $H^2 = (1 - \nu)H_I^2/\alpha$] dominated by the huge value $H_I$ which is potentially related to the GUT scale. Notice that within this regime the ratio

---

1 A study concerning the self-consistent de Sitter stage can be found in [104].
between the \( \sim H^4 \) and the \( \sim H^2 \) terms in eq. (3.2) is \( (\alpha/\nu) H^2/H_I^2 \simeq (\alpha/\nu)(1 - \nu)/\alpha \simeq 1/\nu \gg 1 \) (since \( |\nu| \ll 1 \)) and therefore the \( \sim H^4 \) term is dominant over the \( \sim H^2 \) one. This means that in this regime we can neglect \( \nu \). This confirms quantitatively the statement made in section III. On the other hand, after the early de Sitter epoch, in particular for \( Da^4(1 - \nu) \gg 1 \), the universe enters the standard radiation phase. This behavior is confirmed from the form of \( \rho_\Lambda \) and \( \rho_r \). Upon neglecting the terms \( \nu \) and \( c_0/H^2 \) in this early epoch, which is fully justified, we insert (4.9) into (3.2) and obtain:

\[
\rho_\Lambda(a) = \frac{\rho_I}{\alpha} \frac{1}{(1 + Da^4)^2}. \tag{4.10}
\]

Substituting the vacuum density into eq. (4.6) and solving this differential equation we obtain the following solution:

\[
\rho_r(a) = \frac{\rho_I}{\alpha} \frac{Da^4}{(1 + Da^4)^2}, \tag{4.11}
\]

where \( \rho_I = 3H_I^2/\kappa^2 \) is the critical density during inflation. Obviously, the aforementioned expressions tell us that there is an absence of singularity in the initial state, namely the Universe starts at \( a = 0 \) with a huge vacuum energy density \( \rho_\Lambda/\alpha \) (and \( \rho_r = 0 \)) which is gradually transformed into relativistic matter. Asymptotically, we recover the usual behavior \( \rho_r \sim a^{-4} \), while the vacuum energy density \( \rho_\Lambda \sim a^{-8} \ll \rho_r \) becomes essentially negligible. That is, graceful exit is achieved.

Despite the fact that the origin of the RVM is based on the effective action of QFT in curved space-time, the form the action is not known in general [40, 41]; at present such a task has only been achieved in specific cases [39]. However, using a field theoretical language, via an effective classical scalar field \( \phi \), it is possible to write down the basic field equations for the RVM [85]. In the present work, we term the field \( \phi \) the early vacuumon.\(^2\) Combining Friedmann’s eqs. (4.3)–(4.4) and following the standard approach \( \rho_{\text{tot}} \equiv \rho_\phi = \dot{\phi}^2/2 + U(\phi) \) and \( p_{\text{tot}} \equiv p_\phi = \dot{\phi}^2/2 - U(\phi) \) we find

\[
\dot{\phi}^2 = -\frac{2}{\kappa^2} \dot{H}, \tag{4.12}
\]

\[
U = \frac{3H^2}{\kappa^2} \left( 1 + \frac{\dot{H}}{3H^2} \right) = \frac{3H^2}{\kappa^2} \left( 1 + \frac{aH'}{3H} \right) = \frac{3H^2}{\kappa^2} \left( 1 + \frac{a}{6H^2} \frac{dH^2}{da} \right), \tag{4.13}
\]

where \( U(\phi) \) is the effective potential. Notice that \( \dot{H} = aHH' \) with prime denoting derivative with respect to the scale factor. Integrating eq. (4.12) we obtain

\[
\phi = \pm \int \left( \frac{-2\dot{H}}{\kappa^2} \right)^{1/2} \, dt = \pm \frac{\sqrt{2}}{\kappa} \int \left( -\frac{H'}{aH} \right)^{1/2} \, da. \tag{4.14}
\]

\(^2\)In the past, a scalar field called the cosmon [105] was proposed to solve the old cosmological constant problem [9]. The vacuumon, instead, does a different function, it is not intended for solving the CC fine tuning problem but rather to describe the running vacuum in terms of a scalar effective action. Therefore, it can help alleviate the cosmic coincidence problem, and in general it shares benefits with quintessence models, with the tacit understanding, however, that it just mimics the original RVM, whose fundamental action is not known, except in some specific cases, as mentioned above.
Here we postulate the positive sign, while as we will discuss below in the late era we use the minus.

Now, for $\omega_m = 1/3$ the Hubble parameter (4.9) becomes

$$H(a) = \left(\frac{1}{\alpha}\right)^{1/2} \frac{H_I}{\sqrt{D^4 a^4 + 1}}.$$  (4.15)

We would like to point out that we have imposed $\nu = 0$ in eq. (4.9), which has no effect for the study of the early universe.

Within this framework, integrating eq. (4.14) in the range $[0, a]$ with the aid of eq. (4.15) we arrive at

$$\phi(a) = \frac{1}{\kappa} \sinh^{-1} \left(\sqrt{D} a^2\right),$$

$$= \frac{1}{\kappa} \ln \left(\sqrt{D} a^2 + \sqrt{D} a^4 + 1\right).$$  (4.16)

Also using eqs. (4.13)–(4.15) the effective potential is written as

$$U(a) = \frac{H_I^2}{\alpha \kappa^2} \frac{3 + D a^4}{(1 + D a^4)^2},$$  (4.17)

and thus

$$U(\phi) = \frac{H_I^2}{\alpha \kappa^2} \frac{2 + \cosh^2(\kappa \phi)}{\cosh^4(\kappa \phi)}.$$  (4.18)

It should be stressed once more that the early vacuumon field is classical, hence there is no issue in attempting to considering the effects of quantum fluctuations of $\phi$ on the “hill-top” potential (4.18), depicted in figure 1. In fact, if $\phi$ were a fully fledged quantum field, such as the conventional inflation (which is not the case here), the potential (4.18) provides slow-roll parameters which fit at 2$\sigma$ level the optimal range indicated by the Planck cosmological data on single-field inflation [96].

We would like to finish this section with a brief discussion regarding the recent Planck results. Specifically the results provided by the Planck team [96] have placed tight restrictions on single scalar-field models of slow-roll inflation, supporting basically models with very low tensor-to-scalar fluctuation ratio $r = n_T/n_s \ll 1$, with a scalar spectral index $n_s \simeq 0.96$ and no appreciable running. The upper bound found by Planck team [96] on this ratio, is $r < 0.10$, but their favored regions point towards $r \leq 10^{-3}$. These results are in agreement with the predictions of the Starobinsky-type (or $R^2$-inflation, where $R$ is the Ricci scalar) models of inflation [102, 106, 107]. For this inflationary paradigm the action is

$$S = \int d^4 x \sqrt{-g} \left(R + \beta R^2\right), \quad \beta > 0,$$  (4.19)

while the effective potential $V_{\text{eff}}(\varphi)$ is given by:

$$V_{\text{eff}}(\varphi) = \frac{3 M^2 \left(1 - e^{-\sqrt{\kappa} \varphi} \right)^2}{4 \kappa^2},$$  (4.20)

where $\varphi$ is the scalaron field, characteristic of Starobinsky’s inflation [102]. One may check that the scalaron mass, which can be viewed as the new gravitational degree of freedom that
**Figure 1.** *Upper panel:* the vacuumon classical effective potential (4.18) versus the vacuumon field $\kappa \phi$. The decay of the RVM vacuum is clearly seen by the shape of this potential as the classical vacuumon field increases. The radiation and matter dominated eras occur for some large (but finite) value of $\kappa \phi$. *Lower panel:* the RVM effective potential $\alpha \kappa^2 U/H^2$ (solid line) versus the *scalaron* field $\kappa \varphi$. The dashed line corresponds to $4\kappa^2/(3\mathcal{M}^2)$ times the Starobinsky effective potential, eq. (4.20). The regime of validity of the comparison between the models is the positive axis on both fields, indicated by the perpendicular line. The reader should bear in mind, however, that the apparent mapping in $\kappa \varphi$-space between the two potentials does not imply a similarity of the underlying mechanisms for inflation, which are entirely different. Moreover, when expressed in terms of the scalaron, the kinetic term of the vacuumon would look non canonical.

The conformal transformation was able to elucidate from the Starobinsky action, is indeed provided by parameter $\mathcal{M} = \sqrt{8\pi/3\beta}$, where $\left. \frac{d^2 V_{\text{eff}}(\varphi)}{d\varphi^2} \right|_{\varphi=0} = \mathcal{M}^2$.

Since the Starobinsky inflationary model fits extremely well the Planck data on inflation [96] we would like to combine it with that of the RVM model. In other words, we are interested to see if there is a connection between *early vacuumon* and *scalaron* fields for which the RVM potential equals that of Starobinsky. A related formulation to Starobinsky framework is anomaly-induced inflation, which has its own merits for describing inflation and graceful exit [108–110]. One can actually find also a connection with the RVM in this formulation based on a nonlocal effective action generated by the conformal anomaly [39]. A comparison between the two inflationary frameworks can be found in [44].

However, we should stress that such an equality is purely *formal*. The two models correspond to different effective actions [85]. As a concrete example let us consider the RVM in the early de Sitter phase of the Universe. As already mentioned, in such a case the underlying physics is well described by making the approximation that the dominant term in
the RVM energy density $\rho_\Lambda(H)$ is the quartic $\propto H^4$, i.e. we may set $c_0, \nu = 0$ in (3.2). This implies that the RVM effective action can be approximated by

$$S_{R,\Lambda} = \int d^4x\sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \rho_\Lambda(H) \right] \sim \frac{1}{2\kappa^2} \int d^4x\sqrt{-g} \left( R - \frac{6\alpha H^4}{H^2} \right).$$

Upon replacing $H^4$ by the square of the Ricci scalar, which is to be expected during epochs where $H$ is approximately constant, such as the inflationary (de Sitter) era, one may then write

$$S_{R,\Lambda} \simeq \int d^4x\sqrt{-g} \frac{1}{2\kappa^2} \left( R - \alpha \frac{R^2}{24 H^4} \right).$$

Notice that, since $\alpha > 0$ in our case, the RVM model is not equivalent to a Starobinsky-type model, for which the effective Lagrangian has the form (4.19), corresponding to a negative $\alpha$ coefficient in (4.21). The reason lies in the fact that the metric tensors between the two models, (4.19) and (4.21), are different, related by a non-trivial conformal transformation involving the scalar field [85]. Nonetheless, we can always rewrite the potential of the RVM as a "Starobinsky-like potential" via the transformation (4.23), (4.24), without reference to the microscopic Starobinsky higher curvature model. The two models are, therefore, equivalent only at the level of effective potentials, and for this reason the vacuumon can share the same successful description of inflation as the scalaron, but not the same physics since the equivalence is not complete at the level of the scalar field representations, as will be demonstrated below. This is an important novel point of our current work, pointing to the fact that, although within the vacuumon representation the RVM can describe primeval inflation, nonetheless inflation is realised in a different physical context than in the Starobinsky approach.

With the above understanding, we now proceed with equating the right-hand sides of eqs. (4.20) and (4.18). This is understood as valid only in the appropriate range of the respective scalar fields, depicted in figure 1, given the fact that the potential for the vacuumon (4.18) is bounded, while the Starobinsky potential is only bounded for $\varphi > 0$.

In this way, we can express the early vacuumon field as a function of the scalaron. After some algebra, we obtain:

$$\phi(\varphi) = \frac{1}{\kappa} \ln \left[ \chi(\varphi) + \sqrt{\chi(\varphi)^2 - 1} \right]$$

where

$$\chi(\varphi) = \left[ \frac{1}{2F(\varphi)} + \frac{1}{2F(\varphi)} \right]^{1/2},$$

with

$$F(\varphi) = \frac{3\alpha M^2 \left( 1 - e^{-\sqrt{\frac{3}{2}} \kappa \varphi} \right)^2}{4 H^2} = \frac{\alpha \kappa^2}{H^2} V_{\text{eff}}(\varphi) > 0.$$ 

We note that $0 < F(\varphi) \ll 1$ since $\alpha \ll 1$ and the scalaron mass is of order of the inflationary scale. As a result, $\chi(\varphi) > 1$ and hence the former relation between the early vacuumon and the scalaron, eq. (4.23), is well-defined and leads to $\phi > 0$ in the early universe.
In figure 1 we plot the RVM effective potential $\alpha \kappa^2 U/H^2$ (solid curve) versus $\kappa \phi$. On top of that we show the effective Starobinsky (dashed curve) potential, specifically $4(\kappa^2/\mathcal{M}^2)V_{\text{eff}}(\varphi)$. Clearly, although the RVM and Starobinsky’s model have different geometrical origins, namely GR and $R^2$, the potentials for the two models look similar from the viewpoint of those properties of inflation that can be extracted by an effective scalar-field dynamics.

However, as already mentioned, this similarity between the models is confined only to the form of the respective potentials. Apart from the opposite signs with which the two potentials enter the respective effective actions, as discussed previously, below eq. (4.22), when expressed in terms of the scalaron, the kinetic term of the vacuumon would look non canonical, and this already manifests the important difference between the two models. Moreover, as already mentioned, we cannot consider the standard inflaton fluctuations of the vacuumon, since the latter is treated purely as a classical effective description. The slow roll parameters computed naively from the potential (4.18) are compatible with Planck data [96] at $\sim 2\sigma$ level.

In this respect, we mention that within the RVM effective approach, the microscopic origin of the de Sitter space time, which would lead to an understanding of the quantum fluctuations, is not specified. Additional input from specific models is necessary for this purposes. For example, if higher curvature corrections à la Starobinsky are present, inflation might be due to the scalaron (quantum) field mode, which fits excellently the data [96]. Alternatively, in the string-inspired scenario of [97], primordial gravitational waves are responsible for inducing condensates of the anomaly term, which in turn leads to a de Sitter space time with the $H^4$ contributions to the running vacuum energy density. Quantum fluctuations of the condensate are not equivalent to the scalaron quantum fluctuations, and in fact such a computation is pending, although the fluctuations are expected to be strongly suppressed, so that scale invariance should be approximately intact.

We would also like to stress that the RVM renders a simple explanation of both graceful exit and reheating problem and provides a unified view of the cosmic evolution, see [63–69] for details. The process of reheating after the exit of the inflationary epoch in Starobinsky’s model has been studied e.g. in [111–113]. However, in contrast to the conventional scenarios, there is no genuine reheating for the vacuumon. There is, instead, a smooth transition from the early vacuum energy into radiation, without intervening particle decays. Rather than reheating there is a progressive heating up of the universe during the massive conversion of vacuum energy into radiation. This mechanism triggers a very large entropy production and may render an alternative solution to the entropy problem, see [65–67].

### 4.2 Equation of state of the early vacuumon

The EoS of a given cosmological model can be a useful tool to describe important physical properties of such model. From the formulas derived in the previous section it is easy to find the following appropriate expression for the effective EoS of our system:

$$w = \frac{p_\phi}{\rho_\phi} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\phi}^2 + U(a) = -1 - \frac{a}{3H^2} \frac{dH^2}{da},$$

where the kinetic term has been written as $(1/2)\dot{\phi}^2 = (-1/\kappa^2) \dot{H} = (-1/\kappa^2) aH H' = (-1/2\kappa^2) a dH^2/da$ and use has been made of the equation (4.13) for the effective potential. Notice that in the early universe we are dealing with a transition from the primeval
vacuum energy into a heat bath of radiation, and therefore the EoS of the vacuumon should reflect this transition. Because matter is essentially relativistic in the early universe, we have \( \omega_m = 1/3 \) in eq. (4.9). Using this expression we can compute the EoS of the early vacuumon from (4.26). A simple calculation renders

\[
w = -1 + 4 \frac{Da^4}{3 Da^4 + 1},
\]  

(4.27)

where again we neglected the \( H^2 \) terms in the early universe in front of the dominant power \( H^4 \) in eq. (3.2). The previous equation can be rephrased in a more suggestive way as follows. Let us compute the transition point (call it \( a_* \)) where the vacuum energy density and the radiation energy density become equal. It ensues from equating equations (4.10) and (4.11). We find that the coefficient \( D \) becomes determined in terms of the scale factor at the transition point:

\[
D = \frac{1}{a_*^4}.
\]  

(4.28)

As a result the EoS (4.27) can be rewritten as follows:

\[
w = -1 + 4 \frac{(a/a_*)^4}{3 (a/a_*)^4 + 1},
\]  

(4.29)

Notice that the point \( a_* \) represents the nominal end of inflation and the start of the radiation epoch. In fact, from the previous equation we can easily see that for \( a \gg a_* \) (i.e. deep in the radiation epoch) we have \( w \simeq 1/3 \), whereas for \( a \ll a_* \) (i.e. deep in the inflationary epoch) we have \( w \simeq -1 \). This behavior confirms our interpretation of the early cosmic evolution in terms of the vacuumon. As we will see in section 4.4, an EoS analysis (in this case of the “late vacuumon”) can be particularly enlightening for studies of the properties of the DE in the current universe. In the next section we shall study the late running vacuum universe from this perspective.

4.3 Scalar field description of the total cosmic fluid in the late universe

In the previous sections we have seen that the scalar field language helps to show that the RVM can describe inflation in a successful way, comparable to the scalaron. In the following two sections we show that it also helps to describe the current universe, comparable to scalar or phantom DE fields. In this section let us again focus on eq. (4.8). As long as the radiation component starts to become sub-dominant the matter dominated epoch appears. At this point since the early de Sitter era is left well behind (\( H \ll H_I \)), the quantity \( c_0/H^2 \) in eq. (4.8) starts to dominate over \( \alpha H^2/H_I^2 \). Therefore, eq. (3.2) reduce to

\[
\Lambda(H) = \tilde{\Lambda}_0 + 3\nu(H^2 - H_0^2),
\]  

(4.30)

where \( \tilde{\Lambda}_0 = 3c_0 + 3\nu H_0^2 \) is the current value of vacuum (cosmological constant) energy density. Using the operator \( d/dt = aH \, d/da \) and taking into account that after recombination the cosmic fluid consists dust (\( \omega_m = 0 \)) and running vacuum with \( H \ll H_I \), we can rewrite eq. (4.8) as follows

\[
a \frac{dH^2}{da} + 3(1 - \nu)H^2 - 3c_0 = 0.
\]
Therefore, the corresponding solution obeying the boundary condition $H = H_0$ at the present time ($a = 1$) is:

$$H^2(a) = H_0^2 \left[ \tilde{\Omega}_{m0} a^{-3(1-\nu)} + \tilde{\Omega}_{\Lambda0} \right], \tag{4.31}$$

with

$$\tilde{\Omega}_{\Lambda0} = 1 - \tilde{\Omega}_{m0} = \frac{\Omega_{\Lambda0} - \nu}{1 - \nu},$$

where $\Omega_{\Lambda0} = 1 - \Omega_{m0}$. The above boundary condition fixes the value of the parameter $c_0$ as follows:

$$c_0 = H_0^2 (\Omega_{\Lambda0} - \nu).$$

Here the matter and vacuum densities are given by

$$\rho_m(a) = \rho_{m0} a^{-3(1-\nu)} \tag{4.32}$$

$$\rho_\Lambda(a) = \rho_{\Lambda0} + \frac{\nu \rho_{m0}}{1 - \nu} [a^{-3(1-\nu)} - 1]. \tag{4.33}$$

Of course in the case of $\nu = 0$ we fully recover the concordance $\Lambda$CDM. It is interesting to mention that even for small values of $\nu$ the Universe contains a mildly evolving vacuum energy that could appear as dynamical dark energy. It has been found that the current cosmological model is in excellent agreement with the latest expansion data and it provides a growth rate of clustering which is compatible with the observations (for more details [71, 72, 74, 75]).

Furthermore, substituting eq. (4.31) as well as its derivative ($H'$) into eq. (4.14) we obtain

$$\phi(a) = -\frac{\sqrt{3(1-\nu)\tilde{\Omega}_m}}{\kappa} \int \frac{da}{a [\tilde{\Omega}_{m0} + \tilde{\Omega}_{\Lambda0} a^{3(1-\nu)}]^{1/2}}, \tag{4.34}$$

where we have used the minus sign here in order to ensure continuity of the effective potential. At this point it is important to notice that in order to derive eq. (4.34) we have set $\rho_{\text{tot}} \equiv \rho_m + \rho_\Lambda = \phi^2/2 + U(\phi)$ and $p_{\text{tot}} \equiv p_m + p_\Lambda = \phi^2/2 - U(\phi)$. This implies that the current scalar field description refers to the total cosmic fluid, hence the corresponding effective equation of state parameter is given by

$$w_{\text{tot}} = \frac{p_m + p_\Lambda}{\rho_m + \rho_\Lambda} = \frac{1}{2} \frac{\phi^2 - U(a)}{\phi^2 + U(a)},$$

where for non-relativistic matter we have $p_m = 0$. Notice that here we are well inside the matter (non-relativistic) era, and thus the radiation contribution to the cosmic expansion is negligible. In the next section we will introduce another dynamical field (we shall call it $\chi$) as an effective description of $\Lambda(H)$ in the late Universe.

Changing the integration variable as follows

$$a = \left( \frac{\Omega_{m0}}{\Omega_{\Lambda0}} \right)^{\frac{1}{3(1-\nu)}} \sinh^{\frac{2}{3(1-\nu)}}(u), \tag{4.35}$$

the integral can be performed analytically. Taking the range $(0, a)$ for the scale factor, which corresponds to $(0, u)$ in the transformed variable, we find

$$\phi = A \ln \left| \frac{\sinh u}{1 + \cosh u} \right| = A \ln \left| \frac{e^u - 1}{e^u + 1} \right|, \tag{4.36}$$
Figure 2. The evolution of the late $\phi$ field. The area between the two vertical lines corresponds to that era for which the cosmic fluid consists of running vacuum and matter. Here we use $(\Omega_{m0}, \nu) = (0.30, 10^{-3})$.

where $A = -2/\kappa \sqrt{3(1-\nu)}$. With the aid of eq. (4.35) the evolution of the late vacuumon field can be found explicitly in terms of the scale factor:

$$
\phi(a) = A \ln \left[ \frac{\sqrt{r a^{3(1-\nu)}} + 1 + \sqrt{r a^{3(1-\nu)}} - 1}{\sqrt{r a^{3(1-\nu)}} + 1 + \sqrt{r a^{3(1-\nu)}} + 1} \right].
$$

(4.37)

where $r$ is the ratio

$$
r = \frac{\tilde{\Omega}_{\Lambda 0}}{\Omega_{m0}} = \frac{\Omega_{\Lambda 0} - \nu}{\Omega_{m0}}.
$$

(4.38)

Such ratio coincides very approximately with the current ratio of vacuum energy density to matter energy density since $|\nu| \ll \Omega_{\Lambda 0}$. In figure 2 we provide the evolution of the vacuumon field. As far as $\Omega_{m0}$ and $\nu$ are concerned we utilize the values $(\Omega_{m0}, \nu) = (0.30, 10^{-3})$ which are in agreement with the recent analyses [19–23, 114].

Let us note from eq. (4.37) that $\phi \to +\infty$ for $a \to 0$ (i.e. deep in the past), whereas $\phi \to 0$ for $a \to \infty$ (in the remote future). In actual fact, this result applies strictly only to the late scalar field, i.e. whenever $H^4$ can be neglected in front of $H^2$ in eq. (3.2). In practice this means well after the inflationary epoch, so it comprises most of the radiation epoch and the entire matter-dominated epoch and the DE epoch. We have not studied the interpolation between the early and late regimes here, so the two types of fields behave as we have described only in the mentioned periods.

Concerning the evolution of the potential, utilizing eqs. (4.31) and (4.13) we easily find that

$$
U(a) = \frac{3H_0^2}{\kappa^2} \left[ \tilde{\Omega}_{\Lambda 0} + \frac{1 + \nu}{2} \tilde{\Omega}_{m0} a^{-3(1-\nu)} \right].
$$

(4.39)

Notice that in the far future $a \gg 1$ the potential tends to a constant value $U \to \frac{3H_0^2 \tilde{\Omega}_{\Lambda 0}}{\kappa^2}$, which means that the universe enters the final de Sitter era [63, 70]. This is perfectly consistent
Figure 3. The effective potential $U/U_0$ versus the late vacuumon field $\kappa \phi$. In order to produce the line we use $\nu = 10^{-3}$. The potential has a minimum $U_{\text{min}} = U_0 = \frac{3H_0^2\Omega_\Lambda}{\kappa^2}$ at $\phi = 0$ which corresponds to the far future ($a \gg 1$), corresponding to the late de Sitter era. Despite we show the symmetric form of the potential around $\phi = 0$, by definition the late scalar field of the total cosmic fluid can not take negative values. See section IVC for details.

with the constant asymptotic behavior of the running vacuum energy density (4.33). At this point we attempt to write the potential in terms of the total field $\phi$. Inserting eq. (4.35) into eq. (4.39) the potential is written as

$$U(u) = \frac{3H_0^2\Omega_\Lambda}{\kappa^2} \left[ 1 + \frac{1 + \nu}{2\sinh^2(u)} \right]. \quad (4.40)$$

Inverting eq. (4.36), namely $e^u = \frac{1 + e^{\omega \phi}}{1 - e^{\omega \phi}}$ and using the definition $\sinh(u) = \frac{e^u - e^{-u}}{2}$, it is easy to prove that

$$\sinh(u) = -\frac{1}{\sinh(\omega \phi)}. \quad (4.41)$$

This equation shows once more that $\phi$ cannot take positive values as $\sinh u$ must always be positive in order to have a well defined scale factor, see eq. (4.35). Equation (4.41) implies

$$U(\phi) = U_0 \left[ 1 + \frac{1 + \nu}{2\sinh^2(\omega \phi)} \right], \quad (4.42)$$

where $\omega = 1/A$ and $U_0 = U(0) = \frac{3H_0^2\Omega_\Lambda}{\kappa^2}$. Obviously, the potential has a minimum ($U_{\text{min}} = U_0$) at $\phi = 0$ which corresponds to $a \gg 1$.

In order to visualize the RVM effective potential $U(\phi)$ as a function of $\phi$ in figure 3 we plot the ratio $U/U_0$ at late times as a function of $\kappa \phi$. Note that although the potential (4.40) is an even function of $\phi$, only the branch $\phi \leq 0$ is physically meaningful, as we explained before. The evolution of the universe terminates at the infinite future, corresponding to $\phi = 0$. As expected, the limit $\nu \to 0$ of all the formulas in this section corresponds to the $\Lambda$CDM. While the deviation of the RVM and of its effective scalar description from the $\Lambda$CDM is, of course, small at the present time (as shown by the value $\nu \sim 10^{-3}$ preferred by the current fits to the data), the departure of the vacuum energy density from a strict
constant is exactly the reason why a mild dynamical dark energy behavior is possible at present and is also responsible for the improved fits as compared to the ΛCDM. [19–23, 114]. The theoretical reason for such effect has been explained in detail in [99, 100].

4.4 Equation of state of the late vacuumon and the phantom effective EoS

Here we repeat the analysis of the EoS made in section 4.2, but now for the late vacuumon, which is specially pertinent since it is sensitive to the features of the DE at present, and hence potentially measurable. From the equations of the previous section and the general equation (4.26), along with eq. (4.31), we find

$$w = -1 + (1 - \nu)\frac{H_0^2}{H^2} a^{-3(1-\nu)}.$$  (4.43)

The latter can be worked out as follows:

$$w(a) = -1 + \frac{1 - \nu}{1 + r a^3(1-\nu)}.$$  (4.44)

where \(r\) is the ratio defined before in eq. (4.38). The above formula (4.44) shows in a transparent way the physical behavior: deep in the matter-dominated epoch (\(a \ll 1\)) the EoS tends to a very small value \(w \to -\nu\) (such value would be exactly zero in the ΛCDM), whereas deep in the DE epoch (\(a \gg 1\)) the EoS \(w \to -1\). This kind of evolution was expected since the late vacuumon field describes the combined system of matter and vacuum energy. Therefore the EoS (4.44) of the compound system transits from a situation of matter dominance (\(w \simeq 0\)) into a future one of vacuum dominance (\(w = -1\)). This is similar to the role played by the early vacuumon, which described the transition from the epoch of inflation into the radiation dominated epoch. At present we find ourselves in a mixed EoS state of nonrelativistic matter and DE. Let us expand formula (4.44) linearly in \(\nu\) in order to identify what is the leading contribution from the vacuum dynamics to the EoS around the present time. We find:

$$w(a) \simeq -1 + \frac{1 - \nu}{1 + r a^3} \left(1 + 3\nu r \frac{a^3 \ln a}{1 + r a^3}\right).$$  (4.45)

The terms of \(O(\nu)\) give the leading correction introduced by the vacuum dynamics. However, in this formulation matter and vacuum are in interaction and one cannot disentangle one from another, in particular we cannot read off the effective EoS of the vacuum. As indicated above, only in the asymptotic regime the pure matter or vacuum EoS’s are recovered in opposite ends of the cosmological evolution. On the other hand, while the universe is in transit between the remote past and the remote future, the vacuumon field is evolving in a nontrivial way (see eq. (4.37) and figure 2). Hence, the vacuumon field can describe the vacuum state when it is near these two ends of the Universe’s history, but in the intervening period its EoS is a mixed one, as we have seen above.

The vacuumon representation, therefore, proves to be particularly useful for a description of the physics of the early universe during inflation and at the first stages of the transition into the radiation dominated epoch. During those eras, it provides a sufficiently faithful description of the vacuum evolution, with the rapid inflation period being triggered by the \(\sim H^4\) term in the RVM energy density (3.2). In this period, the vacuumon might naively be thought of as effectively playing the rôle of the inflaton, but, as we have mentioned previously, it is quite different from a traditional inflaton field in that it is a classical field, not a fully fledged quantum field degree of freedom and hence it does not decay into massive particles.
(that is to say, there is no conventional reheating mechanism). For this reason, the vacuumon
is distinct from the scalaron of \(^R^2\)-driven inflation\) and furnishes a different mechanism of
inflation \([67]\). The reader should recall from section 4.1 that this was confirmed on formal
grounds through a mapping between the scalar field potentials of the two models, which,
however, is \textit{not} extendable to the complete Lagrangians underlying the two formulations.
Nonetheless, the two inflationary mechanisms are equally efficient and can both implement
graceful exit \([63–70]\). Only through more detailed analysis and subsequent confrontation
with the CMB data it will be possible to distinguish between these two frameworks. As
noted previously, in both cases a fundamental theory underlies those frameworks: in the
RVM case, the form of the vacuum energy density is naturally triggered by the CP-violating
gravitational anomalous (Chern Simons) terms that characterise the effective action of the
bosonic gravitational multiplet of string theory in a de Sitter background \([97, 98]\), whereas
the scalaron is long known to be associated with the traditional Starobinsky type of inflation
linked to the conformal anomaly \([102, 106]\).

Let us now focus on the late universe. We remind the reader that in the previous
section we have introduced the field \(\phi\) in order to describe the total cosmic fluid, namely
matter and \(\Lambda(H)\) in the late universe. Here the vacuumon EoS becomes entangled with
that of nonrelativistic matter. Hence, some strategy must be devised to track the vacuum
evolution. Notice that, since the EoS of the vacuum is always given by eq. (2.2) and the EoS
of the vacuumon becomes now a time-evolving mixture, we need a different strategy to isolate
the genuine effects of the vacuum dynamics in the current era. To this end, an alternative
scalar field formulation of the combined system of matter and dynamical vacuum proves
convenient. Despite the fact that \(\Lambda\)CDM has exactly the same EoS (2.2) as the running
vacuum, the rigid nature of the (constant in time) vacuum energy density would make it
impossible to mimic any form of DE other than the pure vacuum one. On the contrary, in
the RVM framework, the time dependence on both sides of eq. (2.2), which remains valid at
all times, enables one to introduce a dynamical scalar field \(\chi\) as an effective description of the
RVM in the late Universe. This is more suitable for tracking the DE effects near our time.
In such an alternative representation, it is natural to assume the absence of any interaction
of \(\chi\) with matter, in accordance to the usual minimal assumption made in phenomenological
studies of possible dynamical DE effects on the observational data. The \(\chi\) field, being self-
conserved, satisfies

\[
\dot{\rho}_\chi + 3 (1 + w_\chi) H \rho_\chi = 0. \tag{4.46}
\]

The nontrivial character of \(\chi\) now resides in the dynamical form of the EoS as a function of
the scale factor or redshift, \(w_\chi = w_\chi(a)\), which will be computed below.

Before doing that, we feel stressing once mote that, while the vacuumon \(\phi\) describes the
entire system of matter and dynamical vacuum in mutual interaction, \(\chi\) specifically describes
the dynamics of vacuum independent of matter and tracks possible dynamical DE effects
from the departure of \(w_\chi\) from \(-1\). As already noted, despite the fact that the underlying
RVM EoS is still eq. (2.2), this strategy allows us to obtain the same cosmological system
as in the case of a noninteracting quintessence or phantom DE field together with locally
conserved matter \([10, 37, 38, 79, 80]\).

The solution of eq. (4.46) in terms of the scale factor is

\[
\rho_\chi(a) = \rho_\chi(0) \exp\left\{3 \int_a^1 da' \frac{1 + w_\chi(a')}{a'} \right\}. \tag{4.47}
\]
It follows from this expression that

$$w_\chi(a) = -1 - \frac{a}{3} \frac{1}{\rho_\chi(a)} \frac{d\rho_\chi(a)}{da}.$$  \hspace{1cm} (4.48)

To compute the EoS of $\chi$ such that it mimics the running vacuum, let us first note that the Hubble function $H(a)$ of the model in which the DE is represented by $\rho_\chi$ obviously satisfies

$$\rho_\chi(a) = \frac{3H^2(a)}{8\pi G} - \rho_{m0} a^{-3}.$$  \hspace{1cm} (4.49)

Since the expansion history of the RVM is to be matched by that of the scalar field cosmology based on $\chi$, we can insert $H^2(a)$ from the RVM model, eq. (4.31), in the previous expression and compute explicitly the derivative involved in (4.48). This yields the effective EoS of $\chi$ that matches the running vacuum. After some calculations one finds that

$$w_\chi(a) = -1 + (1 - \nu) \frac{f(a)}{g(a)},$$  \hspace{1cm} (4.50)

where

$$f(a) = \Omega_{m0} \left( a^{-3(1 - \nu)} - a^{-3} \right)$$  \hspace{1cm} (4.51)

and

$$g(a) = \Omega_{m0}^0 \left[ a^{-3(1 - \nu)} - 1 \right] - (1 - \nu) [\Omega_{m0} a^{-3} - 1].$$  \hspace{1cm} (4.52)

If we expand straightforwardly these expressions linearly in $\nu$ and reexpress the result in terms of the redshift variable $z = a^{-1} - 1$, which is more convenient for observations, we arrive at the leading form of the desired EoS:

$$w_\chi(z) \simeq -1 - 3\nu \frac{\Omega_{m0}}{\Omega_\Lambda} (1 + z)^3 \ln(1 + z).$$  \hspace{1cm} (4.53)

The departure of the above EoS from $-1$ precisely captures the dynamical vacuum effects of the RVM in the language of quintessence and phantom DE models. As we can see very obviously from (4.53), the effective behavior is quintessence-like if $\nu < 0$, or phantom-like if $\nu > 0$. As an example, let us take the recent fitting results of the RVM from refs. [22, 23] (cf. table 1 of this reference). If we take three redshift points near the transition between deceleration and acceleration, e.g. $z = 0.7, 1.1, 1.5$, we find $w_\chi \simeq -1.005, -1.016, -1.030$, respectively, hence a mild phantom-like behavior. As it is well-known, phantom behavior of the DE is perfectly compatible with the observational data [96] and it has been a bit controversial if interpreted in terms of fundamental scalar fields, even if playing around with more than one field.

In contrast, the effective description of vacuum in interaction with matter as performed here shows that one can mimic quintessence or phantom DE in models without fundamental fields of this sort. In particular, a phantom-like behavior in our context is completely innocuous since the underlying model is not a fundamental scalar field model. In the current study the underlying model is the running vacuum model, which, as we noted, grants improved fits to the overall observations as compared to the $\Lambda$CDM [19–23, 114]. Therefore, the mere observation of quintessence or phantom DE need not be associated to fundamental fields of this kind, it could be the effective behavior of a nontrivial (and phenomenologically successful) theory of vacuum.
5 Conclusions

In this article we have provided for the first time a classical scalar field description of the running vacuum model (RVM) throughout the entire cosmic evolution. That model is based on the renormalization-group approach in curved spacetime. Specifically, we have found that the RVM can be described with the aid of an effective classical scalar field that we called the vacuumon. At early enough times the $H^4$-term of the RVM dominates the form of running vacuum and it is responsible for inflation. Such $H^4$-driven mechanism for inflation is typical of string theories when their effective action (which contains the gravitational Chern-Simons term) is averaged over the inflationary spacetime, in the presence of primordial gravitational waves. Within this framework, the effective early vacuumon field can be used towards describing the cosmic expansion, namely the universe starts from a non-singular initial de Sitter vacuum stage and it smoothly passes from inflation to radiation epoch, hence graceful exit is achieved. Also, we have shown that under certain conditions the early vacuumon potential can be made formally equivalent to the Starobinsky potential, despite the fact that the origins of the two models (RVM and Starobinsky) are quite different. However, the two frameworks, namely $H^4$-driven inflation and $R^2$-driven inflation, are, however, not physically equivalent, but both provide a successful description of inflation with graceful exit and can be parametrized in terms of scalar fields, the vacuumon and the scalaron, respectively. Nonetheless, the classical nature of the vacuumon does not allow cosmological perturbations to be discussed, unlike the inflaton case. One needs to consider the underlying microscopic models, e.g. string theory as in [97], for this.

If we focus on the low energy physics, the RVM has also been shown to fit the current observational data rather successfully as compared to the ΛCDM, and here we have also presented an effective scalar field description. The late time cosmic expansion is still dominated by the constant additive term of the running vacuum density $\rho_\Lambda(H)$, what makes the RVM to depart only slightly from the ΛCDM. However, the power dependence $H^2$ in $\rho_\Lambda(H) = \rho_0^\Lambda + (3\nu/\kappa^2)(H^2 - H_0^2)$ makes the current vacuum slightly dynamical, and this is the clue for some improvement of the RVM fit to the data as compared to the ΛCDM. Since the coefficient in front $H^2$ is small the evolution of the vacuum energy density is mild prior to the present time. The remnant of the RVM at present epoch is precisely that mild quadratic dynamical behavior of $\rho_\Lambda(H)$ around the present value $\rho_0^\Lambda$ which is affected by the coefficient $|\nu| \ll 1$. The current dark energy dominated era can also be described in the framework of an alternative scalar field whose equation of state can appear in the form of quintessence or phantom dark energy. Using the fitting data existing in the literature on the RVM we find that the phantom option is favored in this representation, which is perfectly compatible with the current observations. The overall picture of the universe is that the vacuum is dominant in the early stages, it decays into radiation, proceeds into matter era and enters the current epoch of mild dynamical DE until the universe whimpers into a final de Sitter stage. The difference with the ΛCDM is that here the vacuum is dynamical at all stages of the evolution, and this also helps in a better description of the data.

To conclude, in this work we have argued that the effective vacuumon scalar-field representation of the RVM provides an efficient way for understanding the underlying mechanism of early inflation, as is the case of $R^2$-inflation, in which the scalaron plays such a rôle. However, we have demonstrated that the vacuumon and the scalaron describe two physically different mechanisms of inflation. We have also shown how to describe the intervening period between the two asymptotic de Sitter epochs of the Universe, the inflationary and the
current one, by means of an alternative scalar field representation, which is able to track the
dynamical character of the vacuum through an ostensible effective phantom behavior, very
near \( w = -1 \) (approaching it from below) around our time.

Acknowledgments

SB acknowledges support from the Research Center for Astronomy of the Academy of Athens
in the context of the program “Tracing the Cosmic Acceleration”. The work of NEM is
supported in part by the U.K. Science and Technology Facilities research Council (STFC)
under the research grant ST/P000258/1. The work of JS has been partially supported
by projects FPA2016-76005-C2-1-P (MINECO), 2017-SGR-929 (Generalitat de Catalunya)
and MDM-2014-0369 (ICCUB). NEM also acknowledges a scientific associateship (“Doctor
Vinculado”) at IFIC-CSIC-Valencia University, Valencia, Spain.

References

[1] Planck collaboration, Planck 2018 results. VI. Cosmological parameters, arXiv:1807.06209
[2] Planck collaboration, Planck 2015 results. XIII. Cosmological parameters, Astron.
Astrophys. 594 (2016) A13 [arXiv:1502.01589] [inspire].
[3] Planck collaboration, Planck 2013 results. XVI. Cosmological parameters, Astron.
Astrophys. 571 (2014) A16 [arXiv:1303.5076] [inspire].
[4] Supernova Cosmology Project collaboration, Measurements of \( \Omega \) and \( \Lambda \) from 42 high
redshift supernovae, Astrophys. J. 517 (1999) 565 [astro-ph/9812133] [inspire].
[5] Supernova Search Team collaboration, Observational evidence from supernovae for an
accelerating universe and a cosmological constant, Astron. J. 116 (1998) 1009
[astro-ph/9805201] [inspire].
[6] A.G. Riess et al., New Hubble space telescope discoveries of type Ia supernovae at \( z \geq 1 \):
narrowing constraints on the early behavior of dark energy, Astrophys. J. 659 (2007) 98
[astro-ph/0611572] [inspire].
[7] Supernova Cosmology Project collaboration, The Hubble space telescope cluster
supernova survey: V. Improving the dark energy constraints above \( z > 1 \) and building an
early-type-hosted supernova sample, Astrophys. J. 746 (2012) 85 [arXiv:1105.3470]
[inspire].
[8] WMAP collaboration, Seven-year Wilkinson Microwave Anisotropy Probe (WMAP)
observations: cosmological interpretation, Astrophys. J. Suppl. 192 (2011) 18
[arXiv:1001.4538] [inspire].
[9] S. Weinberg, The cosmological constant problem, Rev. Mod. Phys. 61 (1989) 1.
[10] V. Sahni and A.A. Starobinsky, The case for a positive cosmological \( \Lambda \)-term, Int. J. Mod.
Phys. D 9 (2000) 373 [astro-ph/9904398].
[11] P.J.E. Peebles and B. Ratra, The cosmological constant and dark energy, Rev. Mod. Phys. 75
(2003) 559 [astro-ph/0207347] [inspire].
[12] T. Padmanabhan, Cosmological constant: the weight of the vacuum, Phys. Rept. 380 (2003)
235 [hep-th/0212290] [inspire].
[13] J. Solà, Cosmological constant and vacuum energy: old and new ideas, J. Phys. Conf. Ser.
453 (2013) 012015 [arXiv:1306.1527] [inspire].
[14] A. Padilla, *Lectures on the cosmological constant problem*, arXiv:1502.05296 [inSPIRE].
[15] L. Perivolaropoulos, *Six puzzles for LCDM cosmology*, arXiv:0811.4684 [inSPIRE].
[16] A.G. Riess et al., *A 2.4% determination of the local value of the Hubble constant*, *Astrophys. J.* 826 (2016) 56 [arXiv:1604.01424] [inSPIRE].
[17] A.G. Riess et al., *New parallaxes of galactic cepheids from spatially scanning the Hubble space telescope: implications for the Hubble constant*, *Astrophys. J.* 855 (2018) 136 [arXiv:1801.01120] [inSPIRE].
[18] E. Macaulay, I.K. Wehus and H.C. Eriksen, *Lower growth rate from recent redshift space distortion measurements than expected from Planck*, *Phys. Rev. Lett.* 111 (2013) 161301 [arXiv:1303.6583].
[19] J. Solà, A. Gómez-Valent and J. de Cruz Pérez, *Hints of dynamical vacuum energy in the expanding Universe*, *Astrophys. J.* 811 (2015) L14 [arXiv:1506.05793] [inSPIRE].
[20] J. Solà, A. Gómez-Valent and J. de Cruz Pérez, *First evidence of running cosmic vacuum: challenging the concordance model*, *Astrophys. J.* 836 (2017) 43 [arXiv:1602.02103] [inSPIRE].
[21] J. Solà, A. Gómez-Valent and J. de Cruz Pérez, *The H0 tension in light of vacuum dynamics in the Universe*, *Phys. Lett.* B 774 (2017) 317 [arXiv:1705.06723] [inSPIRE].
[22] J. Solà Peracaula, J.d.C. Perez and A. Gomez-Valent, *Possible signals of vacuum dynamics in the Universe*, *Mon. Not. Roy. Astron. Soc.* 478 (2018) 4357 [arXiv:1703.08218] [inSPIRE].
[23] J. Solà Peracaula, J. de Cruz Pérez and A. Gómez-Valent, *Dynamical dark energy vs. Λ = const in light of observations*, *EPL* 121 (2018) 39001 [arXiv:1606.00450] [inSPIRE].
[24] J. Solà Peracaula, Tensions in the ΛCDM and vacuum dynamics, *Int. J. Mod. Phys.* A 33 (2018) 1844009 [inSPIRE].
[25] J. Solà Peracaula, A. Gomez-Valent and J.d.C. Perez, *Signs of dynamical dark energy in current observations*, *Phys. Rev.* D 100 (2019) 023539 [arXiv:1905.00100] [inSPIRE].
[26] M. Rezaei, M. Malekjani and J. Solà, *Can dark energy be expressed as a power series of the Hubble parameter?*, *Phys. Rev.* D 100 (2019) 023539 [arXiv:1905.00100] [inSPIRE].
[27] J. Solà, A. Gómez-Valent, J.d.C. Perez and C. Moreno-Pulido, *Brans-Dicke gravity with a cosmological constant smoothes out ΛCDM tensions*, *Astrophys. J.* 886 (2019) L6 [arXiv:1909.02554] [inSPIRE].
[28] G.W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, *Int. J. Theor. Phys.* 10 (1974) 363 [inSPIRE].
[29] C. Brans and R.H. Dicke, *Mach’s principle and a relativistic theory of gravitation*, *Phys. Rev.* 124 (1961) 925.
[30] A. Nicolis, R. Rattazzi and E. Trincherini, *The Galileon as a local modification of gravity*, *Phys. Rev.* D 79 (2009) 064036 [arXiv:0811.2197] [inSPIRE].
[31] L. Arturo Ureña-López, *Scalar fields in cosmology: dark matter and inflation*, *J. Phys. Conf. Ser.* 761 (2016) 012076.
[32] I.L. Shapiro and J. Solà, *Scaling behavior of the cosmological constant: Interface between quantum field theory and cosmology*, *JHEP* 02 (2002) 006 [hep-th/012227] [inSPIRE].
[33] I.L. Shapiro and J. Solà, *On the scaling behavior of the cosmological constant and the possible existence of new forces and new light degrees of freedom*, *Phys. Lett.* B 475 (2000) 236 [hep-ph/9910462] [inSPIRE].
[34] I.L. Shapiro and J. Solà, *Cosmological constant, renormalization group and Planck scale physics*, *Nucl. Phys. Proc. Suppl.* 127 (2004) 71 [hep-ph/0305279].
[35] I.L. Shapiro and J. Solà, A Friedmann-Lemaître-Robertson-Walker cosmological model with running \( \Lambda \), astro-ph/0401015 [inSPIRE].

[36] I.L. Shapiro, J. Solà and H. Štefančič, Running \( G \) and \( \Lambda \) at low energies from physics at \( M(X) \): Possible cosmological and astrophysical implications, JCAP 01 (2005) 012 [hep-ph/0410095] [inSPIRE].

[37] J. Solà and H. Štefančič, Effective equation of state for dark energy: mimicking quintessence and phantom energy through a variable \( \Lambda \), Phys. Lett. B 624 (2005) 147 [astro-ph/0505133] [inSPIRE].

[38] J. Solà and H. Štefančič, Dynamic equation of state for dark energy: mimicking quintessence and phantom energy through a variable \( \Lambda \), Phys. Lett. B 624 (2005) 147 [astro-ph/0505133] [inSPIRE].
[57] J.A.S. Lima and J.M.F. Maia, Some cosmological consequences of a $\Lambda$-term varying as $\beta H^2 + \alpha R^n$, Mod. Phys. Lett. A 8 (1993) 591.

[58] J.A.S. Lima and M. Trodden, Decaying vacuum energy and deflationary cosmology in open and closed universes, Phys. Rev. D 53 (1996) 4280 [astro-ph/9508049] [inSPIRE].

[59] S. Carneiro and J.A.S. Lima, Decaying $\Lambda$ cosmology, varying $G$ and holography, Int. J. Mod. Phys. A 20 (2005) 2465 [gr-qc/0405141] [inSPIRE].

[60] J.M. Salim and I. Waga, Thermodynamic constraints on a time dependent $\Lambda$ model, Class. Quant. Grav. 10 (1993) 1767 [inSPIRE].

[61] R.C. Arcuri and I. Waga, Growth of density inhomogeneities in Newtonian cosmological models with variable Lambda, Phys. Rev. D 50 (1994) 2928 [inSPIRE].

[62] S. Pan, Exact solutions, finite time singularities and non-singular universe models from a variety of $\Lambda(t)$ cosmologies, Mod. Phys. Lett. A 33 (2018) 1850003 [arXiv:1712.01215].

[63] J.A.S. Lima, S. Basilakos and J. Solà, Expansion history with decaying vacuum: a complete cosmological scenario, Mon. Not. Roy. Astron. Soc. 431 (2013) 923 [arXiv:1209.2802] [inSPIRE].

[64] J.A.S. Lima, S. Basilakos and J. Solà, Nonsingular decaying vacuum cosmology and entropy production, Gen. Rel. Grav. 47 (2015) 40 [arXiv:1412.5196] [inSPIRE].

[65] J. Solà, The cosmological constant and entropy problems: mysteries of the present with profound roots in the past, Int. J. Mod. Phys. D 24 (2015) 1544027 [arXiv:1505.05863] [inSPIRE].

[66] J. Solà and H. Yu, Particle and entropy production in the running vacuum universe, arXiv:1910.01638 [inSPIRE].

[67] J.A.S. Lima, S. Basilakos and J. Solà, Thermodynamical aspects of running vacuum models, Eur. Phys. J. C 76 (2016) 228 [arXiv:1509.00163] [inSPIRE].

[68] S. Basilakos, J.A.S. Lima and J. Solà, From inflation to dark energy through a dynamical $\Lambda$: an attempt at alleviating fundamental cosmic puzzles, Int. J. Mod. Phys. D 22 (2013) 1342008 [arXiv:1307.6251] [inSPIRE].

[69] S. Basilakos, J.A. Sales Lima and J. Solà, A viable Starobinsky-like inflationary scenario in the light of Planck and BICEP2 results, Int. J. Mod. Phys. D 23 (2014) 1442011 [arXiv:1406.2201] [inSPIRE].

[70] E.L.D. Perico, J.A.S. Lima, S. Basilakos and J. Solà, Complete cosmic history with a dynamical $\Lambda = \Lambda(H)$ term, Phys. Rev. D 88 (2013) 063531 [arXiv:1306.0591] [inSPIRE].

[71] A. Gómez-Valent, J. Solà and S. Basilakos, Dynamical vacuum energy in the expanding Universe confronted with observations: a dedicated study, JCAP 01 (2015) 004 [arXiv:1409.7048] [inSPIRE].

[72] A. Gomez-Valent and J. Solà, Vacuum models with a linear and a quadratic term in $H$: structure formation and number counts analysis, Mon. Not. Roy. Astron. Soc. 448 (2015) 2810 [arXiv:1412.3785] [inSPIRE].

[73] A. Gomez-Valent, E. Karimkhani and J. Solà, Background history and cosmic perturbations for a general system of self-conserved dynamical dark energy and matter, JCAP 12 (2015) 048 [arXiv:1509.03298] [inSPIRE].

[74] S. Basilakos, M. Plionis and J. Solà, Hubble expansion & structure formation in time varying vacuum models, Phys. Rev. D 80 (2009) 083511 [arXiv:0907.4555] [inSPIRE].

[75] J. Grande, J. Solà, S. Basilakos and M. Plionis, Hubble expansion and structure formation in the ‘running FLRW model’ of the cosmic evolution, JCAP 08 (2011) 007 [arXiv:1103.4632] [inSPIRE].
[76] H. Fritzsch and J. Solà, Matter non-conservation in the Universe and dynamical dark energy, Class. Quant. Grav. 29 (2012) 215002 [arXiv:1202.5097] [inSPIRE].

[77] J. Solà, Dark matter, dark energy and the time evolution of masses in the Universe, Int. J. Mod. Phys. A 29 (2014) 1444016 [arXiv:1408.4427] [inSPIRE].

[78] S. Basilakos, D. Polarski and J. Solà, Generalizing the running vacuum energy model and comparing with the entropic-force models, Phys. Rev. D 86 (2012) 043010 [arXiv:1204.4806] [inSPIRE].

[79] S. Basilakos and J. Solà, Effective equation of state for running vacuum: ‘mirage’ quintessence and phantom dark energy, Mon. Not. Roy. Astron. Soc. 437 (2014) 3331 [arXiv:1307.4748] [inSPIRE].

[80] S. Basilakos and J. Solà, Dark matter, dark energy and the time evolution of masses in the Universe, Class. Quant. Grav. 29 (2012) 215002 [arXiv:1202.5097] [inSPIRE].

[81] J.L. Shapiro, J. Solà, C. Espana-Bonet and P. Ruiz-Lapuente, Variable cosmological constant as a Planck scale effect, Phys. Lett. B 574 (2003) 149 [astro-ph/0303306] [inSPIRE].

[82] J. Solà, A. Gómez-Valent and J.d.C. Pérez, Starobinsky-like inflation and running vacuum in the context of Supergravity, Universe 2 (2016) 14 [arXiv:1505.04434] [inSPIRE].

[83] J. Grande, J. Solà, J.C. Fabris and I.L. Shapiro, Cosmic perturbations with running $G$ and $\Lambda$, Class. Quant. Grav. 27 (2010) 105004 [arXiv:1001.0259] [inSPIRE].

[84] J. Solà, Entropic-force dark energy reconsidered, Phys. Rev. D 90 (2014) 023008 [arXiv:1402.6594] [inSPIRE].

[85] C. Espana-Bonet, P. Ruiz-Lapuente, I.L. Shapiro and J. Solà, Testing the running of the cosmological constant with type-IA supernovae at high $z$, JCAP 02 (2004) 006 [hep-ph/0311171] [inSPIRE].

[86] J. Solà, Brans–Dicke gravity: from Higgs physics to (dynamical) dark energy, Int. J. Mod. Phys. D 27 (2018) 1847029 [arXiv:1805.09810] [inSPIRE].

[87] J. de Cruz Pérez and J. Solà Peracaula, Brans–Dicke cosmology mimicking running vacuum, Mod. Phys. Lett. A 33 (2018) 1850228 [arXiv:1809.03329] [inSPIRE].

[88] J. Ellis and N.E. Mavromatos, Inflation induced by gravitino condensation in supergravity, Phys. Rev. D 88 (2013) 085029 [arXiv:1308.1906] [inSPIRE].
Planck collaboration, Planck 2018 results. X. Constraints on inflation, arXiv:1807.06211 [inSPIRE].

S. Basilakos, N.E. Mavromatos and J. Solà Peracaula, Gravitational and chiral anomalies in the running vacuum universe and matter-antimatter asymmetry, arXiv:1907.04890 [inSPIRE].

S. Basilakos, N.E. Mavromatos and J. Solà, Do we come from a quantum anomaly?, Int. J. Mod. Phys. 28 (2019) 1944002 [arXiv:1905.04685] [inSPIRE].

S. Basilakos, N.E. Mavromatos and J. Solà, Density perturbations for running vacuum: a successful approach to structure formation and to the \( \sigma_8 \)-tension, Mon. Not. Roy. Astron. Soc. 478 (2018) 126 [arXiv:1801.08501] [inSPIRE].

A. Gómez-Valent and J. Solà Peracaula, Density perturbations for running vacuum: a successful approach to structure formation and to the \( \sigma_8 \)-tension, Mon. Not. Roy. Astron. Soc. 478 (2018) 126 [arXiv:1801.08501] [inSPIRE].

A. Gomez-Valent and J. Solà, Relaxing the \( \sigma_8 \)-tension through running vacuum in the Universe, EPL 120 (2017) 39001 [arXiv:1711.00692] [inSPIRE].

J.R. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Time dependent vacuum energy induced by D particle recoil, Gen. Rel. Grav. 32 (2000) 943 [gr-qc/9810086] [inSPIRE].

A.A. Starobinsky, A new type of isotropic cosmological models without singularity, Phys. Lett. 91B (1980) 99 [inSPIRE].

E.W. Kolb and M.S. Turner, The early universe, Addison-Wesley Publishing Company, U.S.A. (1990).

J.S. Dowker and R. Critchley, Effective Lagrangian and energy-momentum tensor in de Sitter space, Phys. Rev. D 13 (1976) 3224.

R.D. Peccei, J. Solà and C. Wetterich, Adjusting the cosmological constant dynamically: cosmons and a new force weaker than gravity, Phys. Lett. B 195 (1987) 183 [inSPIRE].

A.A. Starobinsky, Nonsingular model of the universe with the quantum-gravitational de Sitter stage and its observational consequences, in the proceedings of the Second Seminar Quantum Theory of Gravity, October 13–15, Moscow, Russia (1982), in Quantum Gravity, M.A. Markov and P.C. West, Plenum Press, New York U.S.A. (1984).

A. Vilenkin, Classical and quantum cosmology of the Starobinsky inflationary model, Phys. Rev. D 32 (1985) 2511 [inSPIRE].

I.L. Shapiro and J. Solà, Towards the graceful exit from the anomaly-induced inflation, Russ. Phys. J. 45 (2002) 727 [Izv. Vuz. Fiz. 2002N7 (2002) 75].

D.S. Gorbunov and A.G. Panin, Scalaron the mighty: producing dark matter and baryon asymmetry at reheating, Phys. Lett. B 700 (2011) 157 [arXiv:1009.2448] [inSPIRE].

E.V. Arbuzova, A.D. Dolgov and L. Reverberi, Cosmological evolution in \( R^2 \) gravity, JCAP 02 (2012) 049 [arXiv:1112.4995] [inSPIRE].