DECAY AND SCATTERING IN ENERGY SPACE FOR
THE SOLUTION OF WEAKLY COUPLED CHOQUARD
AND HARTREE-FOCK EQUATIONS

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ABSTRACT. We prove decay with respect to some Lebesgue norms for a
class of Schrödinger equations with non-local nonlinearities by showing
new Morawetz inequalities and estimates. As a byproduct, we obtain
large-data scattering in the energy space for the solutions to the sys-
tems of $N$ defocusing Choquard equations with mass-energy intercriti-
cal nonlinearities in any space dimension and of defocusing Hartree-Fock
equations, for any dimension $d \geq 3$.

1. Introduction

The primary target of the paper is the study of the decaying and scat-
tering properties of the solution to the following system of $N \geq 1$ nonlinear
evolution equations in dimensions $d \geq 1$:

\[
\begin{cases}
    i\partial_t \psi_i + \Delta \psi_i - \sum_{k=1}^{N} G(\psi_i, \psi_k) = 0, \\
    (\psi_i(0, \cdot))_{\mu=1}^{N} = (\psi_{i,0})_{\mu=1}^{N} \in H^1(\mathbb{R}^d)^N,
\end{cases}
\]

characterised by defocusing Choquard (NLC) and Hartree-Fock (NLHF)
coupled equations, thus having the following type of nonlinearities

\[
G(\psi_i, \psi_k) = \lambda_{ik} \left| x \right|^{-(d-\gamma_1)} \ast |\psi_k|^p \left| \psi_i \right|^{p-2} \psi_i
+ \beta \left( \left| x \right|^{-(d-\gamma_2)} \ast |\psi_k|^2 \right) \psi_i
- \left| x \right|^{-(d-\gamma_2)} \ast \bar{\psi}_k \psi_i \right| \psi_k
\]

Here, for all $i, k = 1, \ldots, N$, $\psi_i = \psi_i(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $(\psi_i)_{\mu=1}^{N} = (\psi_1, \ldots, \psi_N)$ and $\beta, \lambda_{ik} \geq 0$, $\lambda_{ii} \neq 0$, are coupling parameters. Furthermore we will require that \( \max(0, d-4) < \gamma_1, \gamma_2 < d \) and the nonlinearity parameter $p$ satisfy the following conditions

\[
p > \frac{d + \gamma_1 + 2}{d}, \quad 2 \leq p < p^*(d), \quad p^*(d) = \begin{cases} +\infty & \text{if } \quad d = 1, 2, \\
\frac{d + \gamma_1}{d-2} & \text{if } \quad d \geq 3.
\end{cases}
\]
that is the $L^2$-supercritical and $H^1$-subcritical regime. One recalls also that the system (1.1) enjoys two important conserved quantities: we have the mass

\begin{equation}
M(\psi_i)(t) = \int_{\mathbb{R}^d} |\psi_i(t)|^2 \, dx,
\end{equation}

for any $i = 1, \ldots, N$ and the energy,

\begin{equation}
E(\psi_1, \ldots, \psi_N) = \sum_{i=1}^N \int_{\mathbb{R}^d} |\nabla \psi_i|^2 + \frac{1}{2p} \sum_{i,k=1}^N \lambda_{ik} \int (|x|^{-(n-\gamma_1)} * |\psi_k|^p)|\psi_i|^p \, dx \\
+ \frac{\beta}{2} \sum_{i,k=1}^N \int_{\mathbb{R}^d} \left( |x|^{-(d-\gamma_2)} * |\psi_k|^2 \right) |\psi_i|^2 - \left[ |x|^{-(d-\gamma_2)} * \bar{\psi}_k \psi_i \right] \bar{\psi}_k \psi_i \, dx.
\end{equation}

The equation (1.1) has a strong physical meaning and its role is important in many models of mathematical physics. In fact, the special case of the Hartree-Newton equation, that is when $d = 3$, $p = 2$, $N = 1$, $\gamma_1 = 2$ and $\beta = 0$, was variously introduced in the scenario of quantum mechanics, in order to represent the mean-field limit of large systems of bosons (the so-called Bose-Einstein condensates) by considering the self-interactions of the such charged particles. We remand, in this direction, to [10], [23] and [19] and to the references therein. About the Hartree-Fock equation, that is the case when is $d = 3$, $N \geq 2$ and $\lambda_{ik} = 0$, $i, j = 1, \ldots, N$, it was applied in [12] for certain approximations in the theory of one component, for portraying an exchange term resulting from Pauli’s principle as well as for describing the fermions as an approximation of the equation overlooking the impact of their fermionic nature. Other relevant papers about this topic are [2] and [3] (see also the references inside). Furthermore, in [13] the Hartree-Fock equation was fundamental for developing models of white dwarfs. Turning to the Choquard equation, being the case of $d = 3$, $p$ as in (1.3), $N = 1$ and $\beta = 0$, it was introduced to sketch an electron trapped in its own hole, as showed in [8] and [9] and very recently in [32], to describe self-gravitating matter together with quantum entanglement and quantum information effects. Inspired by this and by [6], where the general case of systems composed of more than one or two interacting particles is studied, we carry on with the analysis of the decay properties of the solution to (1.1) unfolding large-data scattering in $H^1(\mathbb{R}^d)^N$ for systems of NLC and NLHF. By pursuing the ideas initially developed in [5] for systems of Nonlinear Schrödinger equations with local nonlinearities (see also [34], for the fourth-order NLS), we introduce relevant breakthroughs. Namely, the system (1.1) is translation invariant so we can set up either the classical Morawetz viriel and action, or their bilinear analogues. As a main outcome, we are able to present new Morawetz identities, interaction Morawetz identities and their associated inequalities for (1.1), extending to the non-local setting the theory exploited in [5]. The succeeding step is to localize the Morawetz inequalities on space-time slabs
having \(\mathbb{R}^d\)-cubes as space components, utilizing again the translation invariance of the equation and of all the estimates involved. We say, that at level of localized frame, the dichotomy between local and non-local interactions breaks down: the convolution functions appearing in the interaction Morawetz can be handled in the same manner as if we are considering pure power nonlinearities. The corresponding localized estimates differ from the similar obtained in [5] for the pure NLS, only for a term acting like a localized mass, defined by (1.4). This accomplishes a contradiction argument which infers to the decay of \(L^r\)-norms of the solutions \(\psi(t, x)^{\mu=1}\), provided that \(2 < r \leq 2d/(d - 2)\), for \(d \geq 3\) and \(2 < r < \infty\) for \(d = 1, 2\), with \(r = \infty\) included in the case \(d = 1\) (similarly to [5] and [37]). Let us underline that our approach guarantees the possibility to deal with the low spatial dimension \(d = 1, 2\), bypassing the powerful but difficult to apply techniques of [30]. Now, this peculiar behaviour, jointly with a suitable reformulation of the theory developed in [7], bears to the asymptotic completeness and existence of the wave operators in the energy space \(H^1(\mathbb{R}^d)^N\) for solution to (1.1). We point out now the novelties introduced in our paper. Looking at the Hartree and Hartree-Fock systems, that are (1.1) when \(d \geq 3\), \(N \geq 1\), \(p = 2\) and \(d \geq 3\), \(N \geq 1\), \(\lambda_{ik} = 0\), \(i, j = 1, \ldots, N\) respectively, one knows that the aforementioned decay and the consequent scattering are similarly achieved in several papers like [16] [31], [38] (and the references inside), where the pseudo-conformal technique was successfully applied once one assumes the initial data laying in a weighted energy space. We improve all these results by selecting the initial data in \(H^1(\mathbb{R}^d)^N\) only, showing a similar decay for the solution to (1.1) and improving the range of max \((0, d - 4) < \gamma_2 < d\) (which previously had the lower bound \(\gamma_2 > 1\)). We mention [17] and [18], in which the scattering in the energy space for the Hartree equation is acquired without imposing further regularity to the initial data, but with lack of any decay of the solutions. Let us move to the case of the NLC, given by (1.1) with \(d \geq 1\), \(N \geq 1\), \(p = 2\) and \(\beta = 0\). We earn, in this setting and for any spatial dimension, the full decay of solutions of the system and that the wave and scattering operators are well-defined and bijective in the energy-space \(H^1(\mathbb{R}^d)\). Moreover, all such properties are transposed to the special case of \(N = 1\), that is

\[
\begin{cases}
  i\partial_t \psi + \Delta \psi = (|x|^{-(n-\gamma_1)} * |\psi|^p)|\psi|^{p-2}\psi, \\
  \psi(0, x) = \psi_0(x),
\end{cases}
\]

with \(\psi: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}\) and \(p > 2\). Currently, we are unaware of alike results, so we emphasize that they are new in the whole literature. This explains the reason why we can not supply any kind of references.

The first main target of this paper is the following.

**Theorem 1.1.** Let \(d \geq 1\) and \(p > 0\) such that (1.3) holds. If \((\psi_i)_{i=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\) is the unique global solution to (1.1), then, for all \(i =...
1, \ldots, N, one has the decay property
\begin{equation}
\lim_{t \to \pm \infty} \| \psi_i(t) \|_{L_2^r} = 0,
\end{equation}
with \(2 < r \leq 2d/(d-2)\), for \(d \geq 3\) and with \(2 < r < +\infty\), for \(d = 1, 2\).
Furthermore, when \(d = 1\) one gets
\begin{equation}
\lim_{t \to \pm \infty} \| \psi_i(t) \|_{L_\infty^r} = 0.
\end{equation}
The second main result concerns the scattering of the solution in the energy space.

**Theorem 1.2.** Assume \(d \geq 1\), \(p > 0\) such that (1.3) holds and let be \((\psi_i)_{i=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\) the unique global solution to (1.1), then:
- (asymptotic completeness) There exists \((\psi_{i,0}^\pm)^N_{i=1} \in H^1(\mathbb{R}^d)^N\) such that for all \(i = 1, \ldots, N\)
\begin{equation}
\lim_{t \to \pm \infty} \left\| \psi_i(t, \cdot) - e^{it\Delta} \psi_{i,0}^\pm(\cdot) \right\|_{H^1} = 0.
\end{equation}
- (existence of wave operators) For every \((\psi_{i,0}^\pm)^N_{i=1} \in H^1(\mathbb{R}^d)^N\) there exists unique initial data \((\psi_{i,0})^N_{i=1} \in H^1(\mathbb{R}^d)^N\), such that the global solution to (1.1) \((\psi_i)_{i=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\) satisfies (1.9).

Then, the Theorem 1.2 leads directly to other consequences. First we have the immediate one for the single NLC equation (1.6):

**Corollary 1.3.** Let be \(d \geq 1\), \(N = 1\) and as in (1.3) then, if \(\psi_0 = \psi_{1,0} \in H^1(\mathbb{R}^d)\), the unique global solution to (1.1) \(\psi = (\psi_1) \in C(\mathbb{R}, H^1(\mathbb{R}^d))\), for all \(i = 1, \ldots, N\), is such that:
- the decay property,
\begin{equation}
\lim_{t \to \pm \infty} \| \psi(t, \cdot) \|_{L_r^r} = 0,
\end{equation}
is verified for \(2 \leq p < p^*, \ 2 < r \leq 2d/(d-2), \ d \geq 3, \ for \ 2 \leq p < +\infty, \ 2 < r < +\infty, \ d = 2\) and for \(2 \leq p < +\infty, \ 2 < r \leq +\infty, \ d = 1\);
- the scattering occurs, i.e. there exists \(\psi_0^\pm = \psi_{1,0}^\pm \in H^1(\mathbb{R}^d)\) such that
\begin{equation}
\lim_{t \to \pm \infty} \left\| \psi(t, \cdot) - e^{-it\Delta} \psi_0^\pm(\cdot) \right\|_{H^1} = 0.
\end{equation}

In the NLHF systems framework we have:

**Corollary 1.4.** Let be \(d \geq 3\), \(N \geq 2\) and as in (1.3) then, if \((\psi_{i,0})^N_{i=1} \in H^1(\mathbb{R}^d)^N\), then the unique global solution to (1.1) \((\psi_i)_{i=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\), is such that:
- the decay property,
\begin{equation}
\lim_{t \to \pm \infty} \| \psi_i(t, \cdot) \|_{L_r^r} = 0,
\end{equation}
is fulfilled for \(2 \leq p < p^*, \ 2 < r \leq 2d/(d-2), \ d \geq 3\);
the scattering occurs, i.e. there exists $(\psi_{i,0})_{i=1}^{N} \in H^{1}(\mathbb{R}^{d})^{N}$ such that

$$\lim_{t \to \pm \infty} \|\psi_{i}(t, \cdot) - e^{it\Delta} \psi_{i,0}(\cdot)\|_{H^{1}} = 0.$$  

The literature related to these subjects is not so wide and according to our knowledge, Morawetz and interaction Morawetz estimates were available for systems of NLS for the first time in [5] and successively in [34]. We come to an end by itemizing briefly some other achievements, different from the already cited [16] [31], [38], [17] and [18], which regard particular versions of (1.1). The well posedness for the NLHF, both local and global, was considered in [7], [21] and [27] while the existence of the standing waves was discussed in [24]. The scattering in the focusing critical case was examined in [26], while the blow up of the solutions in the focusing framework, in [28] (we suggest the references contained therein also). As we said, little is known for the NLC. We cite here, for the well-posedness [11] and for the well posedness and blow-up in the case of a Choquard equation perturbed by an inverse square potential, we relegate to [24]. We recall also [14], in which local and global well-posedness, existence of standing waves and blow up solutions were investigated for (1.6) with $\gamma_{1} = 2$ in the critical case $p = (d + 4)/d$. We mention also [4], [15] and [29], for more general informations about the solitary waves solution of (1.6).

**Outline of paper.** After some preliminaries in Section 2, through the Section 3 we build, in Lemma 3.1 and Lemma 3.2, the Morawetz inequalities and their bilinear counterpart, respectively. The principal target of the Section 4 is to display the decay of some Lebesgue-norms of the solutions to the systems (1.1), which is a fundamental property for catching the scattering states and is included in Proposition 1.1. Finally, all the remaining scattering theory associated to (1.1) takes place in Section 5. The last section is the Appendix A, in which a localized Gagliardo-Nirenberg inequality, an ancillary tool used extensively beside the paper, is obtained.

2. Preliminaries

We indicate by $L^{r}_{x}$ the Lebesgue space $L^{r}(\mathbb{R}^{d})$, and respectively by $W^{1,r}_{x}$ and $H^{1}_{x}$ the inhomogeneous Sobolev spaces $W^{1,r}(\mathbb{R}^{n})$ and $H^{1}(\mathbb{R}^{n})$ (for more details see [1]). For any $N \in \mathbb{N}$, we also define $L^{r}_{x} = L^{r}(\mathbb{R}^{d})^{N}$ and introduce the Sobolev spaces $W^{1,r}_{x} = W^{1,r}(\mathbb{R}^{d})^{N}$ and $H^{1}_{x} = H^{1}(\mathbb{R}^{d})^{N}$. From now on and in the sequel we adopt the following notations: for any two positive real numbers $a, b$, we write $a \lesssim b$ (resp. $a \gtrsim b$) to indicate $a \leq Cb$ (resp. $Ca \geq b$), with $C > 0$, we unfold the constant only when it is essential. We recall also some of the results concerning the well-posedness of the system (1.1) already available in the literature, such as [11], [24] for the NLC and as [18], [31], [38] for the NLHF framework. Then we can state

**Proposition 2.1.** Let $d \geq 1$ and $p > 0$ be such that (1.3) holds. Then for all $(\psi_{i,0})_{i=1}^{N} \in \mathcal{H}^{1}_{x}$ there exists a unique $(\psi_{i})_{i=1}^{N} \in C(\mathbb{R}, \mathcal{H}^{1}_{x})$ solution to (1.1),
moreover

\begin{equation}
M(\psi_i)(t) = \|\psi_i(0)\|_{L^2_x}
\end{equation}

for all \(i = 1, \ldots, N\) and

\begin{equation}
E(\psi_1(t), \ldots, \psi_N(t)) = E(\psi_1(0), \ldots, \psi_N(0)),
\end{equation}

with \(E(\psi_1(t), \ldots, \psi_N(t))\) as in (1.5).

The proposition above can be obtained by standard theory (see Theorem 3.3.9 and Remark 3.3.12 in [7]) combined with the classical Gagliardo-Nirenberg

\begin{equation}
\int_{\mathbb{R}^d} (|x|^{-(d-\gamma)} * |\psi|^p) |\psi|^p \, dx \leq C_{d,p} \|\nabla \psi\|_{L^2}^{dp-(d+\gamma)} \|\psi\|_{L^2}^{(d+\gamma)-p(n-2)},
\end{equation}

as well as the defocusing nature of the system.

\section{Morawetz identities and interaction Morawetz inequalities}

We provide, thorough this section, the fundamental tools for the proof of our first main theorem. We start by obtaining Morawetz-type identities, which are similar to the ones holding for the single NLS. From now on we hide the variable \(t\) for simplicity, spreading it out only when necessary. Moreover, we find suitable to set up the following notations: given a function \(h \in H^1(\mathbb{R}^d, \mathbb{C})\), we denote by

\begin{equation}
\begin{align*}
m_h(x) &:= |h(x)|^2, \\
j_h(x) &:= \text{Im} \left[ h \nabla h(x) \right],
\end{align*}
\end{equation}

the mass and momentum densities, respectively. We have the Morawetz identities for non-local nonlinearities.

\begin{lemma}
Let \(d \geq 1\), and \((\psi_i)_{i=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\) be a global solution to the system (1.1), let \(a = a(x) : \mathbb{R}^d \to \mathbb{R}\) be a sufficiently regular and decaying function, and indicate by

\[\mathcal{N}(t) := \sum_{i=1}^N \int_{\mathbb{R}^d} a(x) \, m_{\psi_i}(x) \, dx.\]

\end{lemma}
The following identities hold:

\begin{equation}
\hat{V}(t) = \sum_{i=1}^{N} \int_{\mathbb{R}^d} a(x) \hat{m}_{\psi_i}(x) \, dx = 2 \sum_{i=1}^{N} \int_{\mathbb{R}^d} \hat{j}_{\psi_i}(x) \cdot \nabla a(x) \, dx
\end{equation}

\begin{equation}
\hat{V}(t) = \sum_{i=1}^{N} \int_{\mathbb{R}^d} a(x) \hat{m}_{\psi_i}(x) \, dx
\end{equation}

\begin{align*}
= \sum_{i=1}^{N} & \left[ - \int_{\mathbb{R}^d} m_{\psi_i}(x) \Delta^2 a(x) \, dx + 4 \int_{\mathbb{R}^d} \nabla \psi_i(x) D^2 a(x) \cdot \nabla \psi_i(x) \, dx \right] \\
& + \frac{2(p-2)}{p} \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \Delta a(x) \left[ \left| x \right|^{-(d-\gamma_1)} * |\psi_k|^p \right] |\psi_i(x)|^p \, dx,
\end{align*}

\begin{align*}
& - 2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla \left[ \left| x \right|^{-(d-\gamma_1)} * |\psi_k|^p \right] |\psi_i(x)|^p \, dx, \\
& - 2 \beta \sum_{i,k=1}^{N} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla F(x, \psi_i, \bar{\psi}_k, \psi_j, \bar{\psi}_j) \, dx,
\end{align*}

with, for any \( i, k = 1, \ldots, N \),

\begin{equation}
F(x, \psi_i, \bar{\psi}_k, \psi_j, \bar{\psi}_j) = \left[ \left| x \right|^{-(d-\gamma_2)} * |\psi_k|^2 \right] |\psi_i(x)|^2 - \left[ \left| x \right|^{-(d-\gamma_2)} * \bar{\psi}_k \psi_i \right] \psi_k(x) \bar{\psi}_i(x),
\end{equation}

where \( D^2 a \in \mathcal{M}_{n \times n}(\mathbb{R}^d) \) is the Hessian matrix of \( a \) and \( \Delta^2 a = \Delta (\Delta a) \) the bi-laplacian operator.

**Proof.** We will proceed similarly to [5] (see also [34], [36] and references therein). We shall assume that \((\psi_i)_i\) is a Schwartz solution to (1.1), taking into account that the case \((\psi_i)_i \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)\) can be established by a density argument (we remind, for example, to [7] or [18]). The equation (3.2) is simple to obtain. We carry out some details for providing (3.3) only. By means of an integration by parts and thanks to (1.1), we have

\begin{equation}
2 \sum_{i=1}^{N} \partial_t \int_{\mathbb{R}^d} \hat{j}_{\psi_i}(x) \cdot \nabla a(x) \, dx
\end{equation}

\begin{align*}
& = - 2 \sum_{i=1}^{N} \text{Im} \int_{\mathbb{R}^d} \partial_t \psi_i(x) \left[ \Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x) \right] \, dx \\
& = 2 \sum_{i=1}^{N} \text{Re} \int_{\mathbb{R}^d} i \partial_t \psi_i(x) \left[ \Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x) \right] \, dx \\
& = 2 \sum_{i=1}^{N} \text{Re} \int_{\mathbb{R}^d} \left[ - \Delta \psi_i(x) + G(\psi_i, \psi_k) \right] \left[ \Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x) \right] \, dx.
\end{align*}
First, one can get (we refer again to [5] and [34], for example)

\[
\frac{2}{N} \sum_{i=1}^{N} \text{Re} \int_{\mathbb{R}^d} -\Delta \psi_i(x) [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] \, dx \\
= -\sum_{i=1}^{N} \int_{\mathbb{R}^d} \Delta^2 a(x) |\psi_i(x)|^2 \, dx + 4 \sum_{i=1}^{N} \int_{\mathbb{R}^d} \nabla \psi_i(x) D^2 \psi_i(x) \nabla \bar{\psi}_i(x) \, dx.
\]

Furthermore we obtain

\[
\frac{2}{N} \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} G(\psi_i, \psi_k) \cdot [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] \, dx \\
= 2 \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} G_1(\psi_i, \psi_k) [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] \, dx \\
+ 2 \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} G_2(\psi_i, \psi_k) [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] \, dx,
\]

with

\[
\frac{2}{N} \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} G_1(\psi_i, \psi_k) \cdot [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] \, dx \\
= 2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \Delta a(x) \left[ |x|^{-(d-\gamma_1)} * |\psi_k|^p \right] |\psi_i|^p \, dx \\
+ 4 \sum_{i,k=1}^{N} \lambda_{ik} \text{Re} \int_{\mathbb{R}^d} \nabla a(x) \left[ |x|^{-(d-\gamma_1)} * |\psi_k|^p \right] \cdot |\psi_i|^p \nabla \bar{\psi}_i(x) \, dx \\
= 2 \left( 1 - \frac{2}{p} \right) \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \nabla a(x) \left[ |x|^{-(d-\gamma_1)} * |\psi_k|^p \right] |\psi_i|^p \, dx
\]
and

\begin{align}
(3.9) \quad 2 \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} G_2(\psi_i, \psi_k) \cdot [\Delta a(x) \bar{\psi}_i(x) + 2 \nabla a(x) \cdot \nabla \bar{\psi}_i(x)] dx \\
= 2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \Delta a(x) \left| |x|^{-(d-\gamma_2)} \ast |\psi_k|^2 \right| |\psi_i(x)|^2 dx \\
+ 2 \sum_{i,k=1}^{N} \lambda_{ik} \text{Re} \int_{\mathbb{R}^d} 2 \nabla a(x) \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^2 \right] \cdot \psi_i(x) \nabla \bar{\psi}_i(x) dx \\
- 2 \beta \sum_{i,k=1}^{N} \int_{\mathbb{R}^d} \Delta a(x) \left[ |x|^{-(d-\gamma_1)} \ast \bar{\psi}_k \psi_i \right] \psi_k(x) \bar{\psi}_i(x) dx \\
- 2 \beta \sum_{i,k=1}^{N} \text{Re} \int_{\mathbb{R}^d} 2 \nabla a(x) \left[ |x|^{-(d-\gamma_1)} \ast \bar{\psi}_k \psi_i \right] \psi_k(x) \cdot \nabla \bar{\psi}_i(x) dx.
\end{align}

An integration by parts of the the second term on the r.h.s. of the above identity (3.9) enhances to

\begin{align}
(3.10) \quad 2 \sum_{i,k=1}^{N} \lambda_{ik} \text{Re} \int_{\mathbb{R}^d} \nabla a(x) \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^2 \right] \cdot \nabla |\psi_i(x)|^2 dx \\
= -2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \Delta a(x) \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^2 \right] |\psi_i(x)|^2 dx \\
- 2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^2 \right] |\psi_i(x)|^2 dx.
\end{align}

By integrating by parts the last term of (3.9), one has, instead,

\begin{align}
(3.11) \quad -2 \beta \sum_{i=1}^{N} \text{Re} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla a(x) \frac{1}{|x-y|^{(d-\gamma_2)}} \cdot \nabla |\psi_i(y)\bar{\psi}_i(x)|^2 dx \\
= -2 \beta \sum_{i,k=1}^{N} \int_{\mathbb{R}^d} \Delta a(x) \left[ |x|^{-(d-\gamma_2)} \ast \bar{\psi}_k \psi_i \right] \psi_k(x) \bar{\psi}_i(x) dx \\
- 2 \beta \sum_{i,k=1}^{N} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla \left[ |x|^{-(d-\gamma_2)} \ast \bar{\psi}_k \psi_i \right] \psi_k(x) \bar{\psi}_i(x) dx.
\end{align}
We can utilize now (3.8) in combination with (3.9), (3.10) and (3.11) to rewrite (3.7) as

\[ 2 \sum_{i=1}^{N} \text{Re} \int_{\mathbb{R}^d} \mathcal{G}(\psi_{i}, \psi_{k}) \cdot [\Delta a(x)\bar{\psi}_{i}(x) + 2\nabla a(x) \cdot \nabla \bar{\psi}_{i}(x)] \, dx \]

\[ = \frac{2(p-2)}{p} \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \Delta a(x) \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^p \right] |\psi_i|^p \, dx, \]

\[ -2 \sum_{i,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^p \right] |\psi_i|^p \, dx, \]

\[ -2\beta \sum_{i,k=1}^{N} \int_{\mathbb{R}^d} \nabla a(x) \cdot \nabla F(x, \psi_{i}, \bar{\psi}_{i}, \psi_{j}, \bar{\psi}_{j}) \, dx, \]

with $F(x, \psi_{i}, \bar{\psi}_{i}, \psi_{j}, \bar{\psi}_{j})$ as in (3.4). Then the above identities (3.6) and (3.12) brings us to the proof of (3.2). \hfill \square

One can now apply the previous lemma for proving the following interaction Morawetz identities and inequalities for non-local nonlinearities.

**Lemma 3.2.** Let $(\psi_{\mu})_{\mu=1}^{N} \in C(\mathbb{R}, L^1(\mathbb{R}^d)^N)$ be a global solution to system (1.1), let $a = a(|x|) : \mathbb{R}^d \to \mathbb{R}$ be a convex radial, sufficiently regular and decaying function. Denote by $a^* = a^*(x, y) := a(|x - y|) : \mathbb{R}^{2d} \to \mathbb{R}$ and by

\[ \mathcal{I}(t) := \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^*(x, y)m_{\psi_{i}}(x)m_{\psi_{j}}(y) \, dx \, dy. \]

The following holds:

\[ \dot{\mathcal{I}}(t) = 2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_{\psi_{i}}(x) \cdot \nabla_x a^*(x, y) m_{\psi_{j}}(y) \, dx \, dy, \]

\[ N^{C}(p, a)(t) + N^{HF}(p, a)(t) + R^{C}(p, a)(t) \leq \mathcal{I}(t), \]

where

\[ N^{C}(p, a^*)(t) = \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^p \right] |\psi_i|^p m_{\psi_{j}}(y) \, dx \, dy, \]

with $\lambda_{ik} = 4\lambda_{ik}(p-2)/p$, \hfill (3.15)

\[ R^{C}(p, a^*)(t) = -4 \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x a^*(x, y) \cdot \nabla_x \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^p \right] |\psi_i|^p m_{\psi_{k}}(y) \, dx \, dy, \]

\[ \sum_{i,j,k=1}^{N} \lambda_{ik} (p-2) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x a^*(x, y) \cdot \nabla_x \left[ |x|^{-(d-\gamma_1)} \ast |\psi_k|^p \right] |\psi_i|^p m_{\psi_{k}}(y) \, dx \, dy, \]

\[ \text{as in (3.4). Then the above identities (3.6) and (3.12) brings us to the proof of (3.2).} \hfill \square \]
and

\begin{equation}
N_{(p,a^*)}^{HF}(t) = -4\beta \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x a^*(x,y) \nabla_x F(x,\psi_i,\bar{\psi}_i,\psi_j,\bar{\psi}_j) m_{\psi_k}(y) \, dxdy,
\end{equation}

with \(F(x,\psi_i,\bar{\psi}_i,\psi_j,\bar{\psi}_j)\) as in \((3.4)\).

**Proof.** We argue, here, in a similar way as in \([5]\) and \([34]\), introducing however some novelties in the proof. As formerly done, we prove the identities for a Schwartz solution \((\psi_i)_{i=1}^{N}\), moving to the general case \((\psi_i)_{i=1}^{N} \in \mathcal{C}(\mathbb{R},H^1(\mathbb{R}^d)^N)\) by an usual density argument. First, we point out that \((3.13)\), because of the symmetry of the function \(a^*(x,y) = a(|x - y|)\), is equivalent to

\begin{equation}
\dot{I}(t) = \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^*(x,y)m_{\psi_i}(x)m_{\psi_j}(y) \, dxdy.
\end{equation}

Hence, \((3.13)\) is straightforward from \((3.2)\) and Fubini’s Theorem. We differentiate w.r.t. time variable working out now the equality

\begin{equation}
\ddot{I}(t) = -4 \sum_{i,j=1}^{N} \Re \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\psi_j}(y)i\partial_t(\bar{\psi}_i(x)\nabla_x \psi_i(x)) \cdot \nabla_x a^*(x,y) \, dxdy
\end{equation}

\begin{equation*}
-2 \sum_{i,j=1}^{N} \Re \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} i\partial_t m_{\psi_i}(x)\bar{\psi}_j(y) \nabla_y \psi_j(y) \cdot \nabla_y a^*(x,y) \, dxdy
\end{equation*}

\begin{equation*}
-2 \sum_{i,k=1}^{N} \Re \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} i\partial_t m_{\psi_j}(y)\bar{\psi}_i(x) \nabla_x \psi_i(x) \cdot \nabla_x a^*(x,y) \, dxdy
\end{equation*}

\begin{equation*}
:= I^I_1(t) + II_2(t).
\end{equation*}
By the identity (3.3), Fubini’s Theorem and the symmetry of \( a^*(x, y) \) we achieve

\[
\mathcal{I}_1(t) = -2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x a^*(x, y)m_{\psi_i}(x)m_{\psi_j}(y) \, dx \, dy \\
+ \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) \left| x \right|^{-(d-\gamma_1)} \left| \psi_k(x) \right|^p \left| \psi_i(x) \right|^p m_{\psi_j}(y) \, dx \, dy,
\]

\[
-4 \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x a^*(x, y) \cdot \nabla_x \left[ \left| x \right|^{-(d-\gamma_1)} \left| \psi_k(x) \right|^p \left| \psi_i(x) \right|^p m_{\psi_j}(y) \right] \, dx \, dy,
\]

\[
-4\beta \sum_{i,j,k=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x a^*(x, y) \nabla_x F(x, \psi_i, \psi_j, \psi_j) m_{\psi_k}(y) \, dx \, dy,
\]

with \( F(x, \psi_i, \psi_j, \psi_j) \) given by (3.4). The first term of (3.19) above arises from the linear part of the equation, while the other terms are connected to the nonlinearity contained in the equation. The linear term can be modified as follows

\[
-2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x a^*(x, y)m_{\psi_i}(t, x)m_{\psi_j}(t, y) \, dx \, dy
\]

\[
=2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_x \partial_{\psi_i} \Delta_x a^*(x, y)m_{\psi_i}(t, x)m_{\psi_j}(t, y) \, dx \, dy
\]

\[
=2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) \nabla_x m_{\psi_i}(t, x) \cdot \nabla_y m_{u_\gamma}(t, y) \, dx \, dy,
\]

by an integration by parts and taking again advantage of the property \( \partial_{x_k} a^* = -\partial_{y_k} a^* \). At the end, we have

\[
\mathcal{I}_1(t) = N_{(p, a^*)}^C(t) + N_{(p, a^*)}^{HF}(t) + R_{(p, a^*)}^C(t)
\]

\[
+2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) \nabla_x m_{\psi_i}(t, x) \cdot \nabla_y m_{u_\gamma}(t, y) \, dx \, dy.
\]
As well, by the Morawetz identities (3.2), (3.3) and the Fubini’s Theorem we get

\[
\mathcal{II}_2(t) = 4 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \psi_i(x) D_x^2 a^*(x,y) \nabla \overline{\psi}_i(x) m_{\psi_j}(y) \, dx \, dy \\
+ 4 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\psi_i}(x) \nabla \overline{\psi}_j(y) D_y^2 a^*(x,y) \nabla \overline{\psi}_j(y) \, dx \, dy \\
+ 8 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j\psi_i(x) D_x^2 a^*(x,y) \cdot j\overline{\psi}_j(y) \, dx \, dy,
\]

(3.22)

here we applied, at this point, the symmetry of \(D^2 a^*\) to drop the real part condition in the first two summands on the r.h.s. of the identity above. Once more, the fact that \(\partial_{x_j} a^* = -\partial_{y_j} a^*\) allows us to reshape (3.22) as follows

\[
\mathcal{II}_2(t) = 4 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_y \psi_j(y) D_x^2 a(|x-y|) \nabla_y \overline{\psi}_j(y) |\psi_i(x)|^2 \, dx \, dy \\
+ 4 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x \psi_i(x) D_x^2 \phi(|x-y|) \nabla_x \overline{\psi}_i(x) |\psi_j(y)|^2 \, dx \, dy \\
- 8 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Im(\overline{\psi}_i(x) \nabla_x \psi_i(x)) D_x^2 a(|x-y|) \Im(\overline{\psi}_j(y) \nabla_y \psi_j(y)) \, dx \, dy,
\]

(3.23)

and lastly to

\[
\mathcal{II}_2(t) = 2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( H_{ij}^1 D_x^2 a(|x-y|) H_{ij}^1 + H_{ij}^2 D_x^2 a(|x-y|) H_{ij}^2 \right) \, dx \, dy,
\]

(3.24)

where we set

\[
H_{ij}^1 := \psi_i(t,x) \nabla_y \overline{\psi}_j(t,y) + \nabla_x \overline{\psi}_i(t,x) \psi_j(t,y),
\]

\[
H_{ij}^2 := \psi_i(t,x) \nabla_y \psi_j(t,y) - \nabla_x \psi_i(t,x) \psi_j(t,y).
\]

Thus by (3.24), and since \(a\) is a convex function one achieves \(\mathcal{II}_2(t) \geq 0\), for any \(t \in \mathbb{R}\). We claim further that

\[
2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x,y) \nabla_x m_{\psi_i}(t,x) \cdot \nabla_y m_{\psi_i}(t,y) \, dx \, dy + \mathcal{II}_2(t) \geq 0.
\]

(3.25)
In fact by (we remand for instance to [33]), we obtain
\[
-2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) m_{\psi_i}(x) m_{\psi_j}(y) \, dx \, dy
\]
\[
= -2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x \Delta_y a(|x - y|) m_{\psi_i}(x) m_{\psi_j}(y) \, dx \, dy
\]
\[
= 2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x m_{\psi_i}(x) D^2_x a(|x - y|) \cdot \nabla_y m_{u_a}(y) \, dx \, dy,
\]
then the l.h.s. of (3.25) becomes equal to
\[
2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x m_{\psi_i}(x) D^2_x a(|x - y|) \cdot \nabla_y m_{u_a}(y) \, dx \, dy
+ 4 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( H^1_{ij} D^2_x a(|x - y|) \overline{H^1_{ij}} + H^2_{ij} D^2_x a(|x - y|) \overline{H^2_{ij}} \right) \, dx \, dy.
\]
A straight computation displays
\[
2 \nabla_x |\psi_i(x)|^2 D^2_x a(|x - y|) \cdot \nabla_y |\psi_j(y)|^2 + 2 H^1_{ij} D^2_x a(|x - y|) \overline{H^1_{ij}} + 2 H^2_{ij} D^2_x a(|x - y|) \overline{H^2_{ij}}
= 2 H^1_{ij} D^2_x a(|x - y|) \overline{H^1_{ij}} - 2 H^2_{ij} D^2_x a(|x - y|) \overline{H^2_{ij}}
+ 2 H^1_{ij} D^2_x a(|x - y|) \overline{H^1_{ij}} + 2 H^2_{ij} D^2_x a(|x - y|) \overline{H^2_{ij}} = 4 H^1_{ij} D^2_x a(|x - y|) \overline{H^1_{ij}} \geq 0,
\]
where we employed along the calculation that the matrix $D^2_x a^* = -D^2_x a^* = D^2_y a^*$ is symmetric. Gathering together (3.26) and (3.27) we have that (3.25) is satisfied. With this last inequality in mind, we sum now $\mathcal{I}_1(t)$ with $\mathcal{I}_2(t)$ realizing that
\[
\tilde{I}(t) = 2 \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x, y) \nabla_x m_{\psi_i}(t, x) \cdot \nabla_y m_{u_a}(t, y) \, dx \, dy
+ \mathcal{I}_2(t) + N^C_{(p,a)}(t) + N^{HF}_{(p,a)}(t) + R^C_{(p,a)}(t)
\geq N^C_{(p,a)}(t) + N^{HF}_{(p,a)}(t) + R^C_{(p,a)}(t),
\]
that is the desired (3.14). \qed

The proof of Lemma 3.2 accomplishes also the following proposition.

**Proposition 3.3.** Let be $(\psi_1)_{\mu=1}^{N} \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)$ and for any $t \in \mathbb{R}$, $N^C_{(p,a)}(t)$, $N^{HF}_{(p,a)}(t)$, $R^C_{(p,a)}(t)$ as in Lemma 3.2, then the following hold.
• (Low regularity Morawetz interaction inequality)

(3.29) \[ \tilde{I}(t) \geq N_{(p,a^*)}^C(t) + N_{(p,a^*)}^{HF}(t) + R_{(p,a^*)}^C(t) + 2 \sum_{j,k=1}^N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x a^*(x,y) \nabla_x m_{\psi_j}(x) \cdot \nabla_y m_{\psi_j}(y) \, dx dy. \]

• (High regularity Morawetz interaction inequality)

(3.30) \[ \tilde{I}(t) \geq N_{(p,a^*)}^C(t) + N_{(p,a^*)}^{HF}(t) + R_{(p,a^*)}^C(t) - 2 \sum_{j,k=1}^N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x^2 a^*(x,y) m_{\psi_j}(x)m_{\psi_j}(y) \, dx dy. \]

**Proof.** We will supply for the proof in few lines. Here we shall make use of the inequality (3.28) along with (3.20), arriving to the inequalities (3.29) and (3.30). \(\square\)

A direct consequence of Lemma 3.2 is that we can prove the following:

**Proposition 3.4.** Let \(d \geq 1\) and let \((\psi_j)_{j=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d))^N\) be a global solution to (1.1). Then, for \(p > 2\) as in (1.3), \(\max(0, d - 4) < \gamma_1, \gamma_2 < d\) and selecting \(a^*(x,y) = |x - y|\), one has the following global estimate

(3.31) \[ \int_{\mathbb{R}} N_{(p,a^*)}^C(t) \, dt \leq C \sum_{i=1}^N \|\psi_{i,0}\|_{H^1}^4, \]

with \(N_{(p,a^*)}^C(t)\) as in (3.15). Moreover, let be \(Q^d_\tilde{x}(r) = x + [-r,r]^d\), with \(r > 0\) and \(\tilde{x} \in \mathbb{R}^d\), one gets the following localized estimates: for \(d \geq 2\),

(3.32) \[ \sum_{i,j,k=1}^N \tilde{\lambda}_{ik} \int_{\mathbb{R}} \sup_{\tilde{x} \in \mathbb{R}^d} \int_{(Q^d_{\tilde{x}}(2r))^3} |\psi_i(t,x)|^p |\psi_j(t,y)|^2 |\psi_k(t,z)|^p \, dx dy dz dt \leq C \sum_{i=1}^N \|\psi_{i,0}\|_{H^1}^4, \]

where \(\tilde{\lambda}_{ik} = 4\lambda_{ik}(p-2)/p\) and \((Q^d_{\tilde{x}}(2r))^3 = Q^d_{\tilde{x}}(2r) \times Q^d_{\tilde{x}}(2r) \times Q^d_{\tilde{x}}(2r)\). For \(d = 1\),

(3.33) \[ \sum_{i,j,k=1}^N \tilde{\lambda}_{ik} \int_{\mathbb{R}} \sup_{\tilde{x} \in \mathbb{R}^d} \int_{(Q^d_{\tilde{x}}(2r))^2} |\psi_i(t,x)|^p |\psi_j(t,x)|^2 |\psi_k(t,z)|^p \, dx dz dt \leq C \sum_{i=1}^N \|\psi_{i,0}\|_{H^1}^4, \]

with \((Q^d_{\tilde{x}}(2r))^2 = Q^d_{\tilde{x}}(2r) \times Q^d_{\tilde{x}}(2r)\).
Proof. We will use, there, the interaction inequality (3.14). Let us start by handling (3.16). Namely, by means of
\[ \nabla_x a^\star(x, y) = \frac{x - y}{|x - y|}, \]
and inspired by [26], we can write, for max \((0, d - 4) < \gamma_1 < d\),
\[ R^C_{(p,|x-y|)}(t) = \sum_{i,j,k=1}^{N} 4\lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(x - y) \cdot (x - z)}{|x - y| |x - z|^{d - \gamma_1 + 2}} |\psi_i(x)|^p |\psi_i(z)|^p m_{\psi_j}(y) \, dx dy dz, \]
where
\[ K(x, z) = (x - z) \cdot \sum_{j=1}^{N} \int_{\mathbb{R}^d} m_u(y) \left( \frac{x - y}{|x - y|} - \frac{z - y}{|z - y|} \right) dy. \]
Then, the elementary inequality
\[ (x - z) \cdot \left( \frac{x - y}{|x - y|} - \frac{z - y}{|z - y|} \right) = (|x - y| |z - y| - (x - y) \cdot (z - y)) \left( \frac{|x - y| + |z - y|}{|x - y| |z - y|} \right) \geq 0, \]
bears to
\[ \inf_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} K(x, z) \geq 0. \]
By combining now the previous (3.37) with (3.35) we obtain that \( R^C_{(p,a^\star)}(t) \geq 0 \), for any \( t \in \mathbb{R} \). We move now on the term (3.17) by proceeding as the step above. Thus we have, for max \((0, d - 4) < \gamma_2 < d\) and \( K(x, z) \) as in (3.36)
\[ N^{HF}_{(p,|x-z|)}(t) \geq 4\beta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x - z|^{d - \gamma_2 + 2}} (\eta(x) \eta(z) - |\eta(x, z)|^2) K(x, z) \, dx dy, \]
where we indicated by
\[ \eta(x, z) = \sum_{i=1}^{N} \psi_i(x) \bar{\psi}_i(z), \quad \eta(x) = \eta(x, x). \]
In addition we infer, by an use of the Cauchy-Schwartz inequality, the bound \(|\eta(x, z)|^2 \leq \eta(x) \eta(z)\), for any \( x, y \in \mathbb{R}^d \). This observation, jointly again with (3.37), implies \( N^{HF}_{(p,a^\star)}(t) \geq 0 \) for any \( t \in \mathbb{R} \). Then we achieved, at this stage, the following pointwise (in time) estimate
\[ N^C_{(p,a^\star)}(t) \leq \tilde{I}(t), \]
which, after an integration w.r.t. time variable over the interval $[t_1, t_2] \subseteq \mathbb{R}$ with $t_1, t_2 \in \mathbb{R}$, becomes

$$\int_{t_1}^{t_2} N^C_{(p, |x-y|)}(t) \, dt \lesssim \sup_{t \in [t_1, t_2]} |\dot{I}(t)|. \quad (3.40)$$

We have also the following

$$\sup_{t \in [t_1, t_2]} |\dot{I}(t)| \leq 2 \sup_{t \in [t_1, t_2]} \sum_{i,j=1}^{N} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_{\psi_i}(t, x) \cdot \nabla_x a^*(x, y) m_{\psi_j}(t, y) \, dx \, dy \right| \lesssim \sup_{t \in [t_1, t_2]} \sum_{j=1}^{N} \|\dot{\psi}_j(t)\|_{H_x^1}^{4} \lesssim \sum_{j=1}^{N} \|\dot{\psi}_{j,0}\|_{H_x^1}^{4} < \infty, \quad (3.41)$$

for the reason that $H_x^1$-norm of the solution is bounded according to the conservation laws (2.1) and (2.2). From the estimates (3.41), (3.40) and allowing $t_1 \to -\infty, t_2 \to +\infty$ we finally get (3.31) which displays, after recalling that

$$\Delta_x |x-y| = \begin{cases} \frac{d-1}{|x-y|} & \text{if } d \geq 2, \\ -2\pi \delta_{x=y} & \text{if } d = 1, \end{cases}$$

as

$$\sum_{i,j,k=1}^{N} \tilde{\lambda}_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d-1}{|x-y||x-z|^{d-\gamma_1}} |\psi_i(x)|^p |\psi_j(y)|^2 |\psi_k(z)|^p \, dx \, dy \, dz, \quad (3.42)$$

with $\mathbb{R}^{3d} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$, for $d \geq 2$ and

$$\sum_{i,j,k=1}^{N} \tilde{\lambda}_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} \frac{1}{|x-z|^{d-\gamma_1}} |\psi_i(x)|^p |\psi_j(x)|^2 |\psi_k(z)|^p \, dx \, dz, \quad (3.43)$$

with $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, for $d = 1$. We are in position to go over the proof of (3.32) and (3.33). We notice that, for any $\tilde{x} \in \mathbb{R}^d$,

$$\inf_{x,y,z \in \mathcal{Q}^d_2(2r)} \left( \frac{1}{|x-y|}, \frac{1}{|z-y|} \right) = \inf_{x,y,z \in \mathcal{Q}^d_0(2r)} \left( \frac{1}{|x-y|}, \frac{1}{|z-y|} \right) > 0, \quad (3.44)$$
as an outcome, we can bound the l.h.s. of (3.42) as
\[
\sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d-1}{|x-y||x-z|^{d-\gamma}} |\psi_i(x)|^p |\psi_j(y)|^2 |\psi_k(z)|^p \, dx dy dz dt
\]
(3.45)
\[
\geq \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}} \sup_{\bar{x} \in \mathbb{R}^d} \int_{(Q_{\bar{x}}^d(2r))^3} |\psi_i(t,x)|^p |\psi_j(t,y)|^2 |\psi_k(t,z)|^p \, dx dy dz dt.
\]
Then the previous (3.42) and (3.45) guarantee that the estimate (3.32) holds.
In a similar way we can manage the l.h.s of (3.43). To be specific we have, by utilizing again (3.44), that
\[
\sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d-1}{|x-y||x-z|^{d-\gamma}} |\psi_i(x)|^p |\psi_j(x)|^2 |\psi_k(z)|^p \, dx dz dt
\]
(3.46)
\[
\geq \sum_{i,j,k=1}^{N} \lambda_{ik} \int_{\mathbb{R}} \sup_{\bar{x} \in \mathbb{R}^d} \int_{(Q_{\bar{x}}^d(2r))^2} |\psi_i(t,x)|^p |\psi_j(t,x)|^2 |\psi_k(t,z)|^p \, dx dz dt,
\]
The above (3.43) and (3.46) give the way to (3.33). The proof of the proposition is finally completed.

In addition we get also the following result for the pure NLHF. More precisely, we earn:

**Proposition 3.5.** Let \( d \geq 3, p = 2 \) and let \((\psi_i)_{i=1}^{N} \in C(\mathbb{R}, H^1(\mathbb{R}^n)^N)\) be a global solution to (1.1). Then we have, if one chooses \( a^*(x,y) = |x-y| \),
\[
\tag{3.47}
- \sum_{i,j=1}^{N} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2 a^*(x,y) |\psi_i(t,x)|^2 |\psi_j(t,y)|^2 \, dx dy dt \leq C \sum_{i=1}^{N} ||\psi_{i,0}||_{H^2_x}^4.
\]
Let be \( Q_{\bar{x}}^d(r) = x + [-r,r]^d \), with \( r > 0 \) and \( \bar{x} \in \mathbb{R}^d \), one gets the following estimates:
- for \( d = 3 \)
\[
\tag{3.48}
\sum_{i=1}^{N} \int_{\mathbb{R}} \sup_{\bar{x} \in \mathbb{R}^3} \int_{Q_{\bar{x}}^3(2r)} |\psi_i(t,x)|^4 \, dx dt \leq C \sum_{i=1}^{N} ||\psi_{i,0}||_{H^2_x}^4;
\]
- for \( d \geq 4 \)
\[
\tag{3.49}
\sum_{i,j=1}^{N} \int_{\mathbb{R}} \sup_{\bar{x} \in \mathbb{R}^d} \int_{(Q_{\bar{x}}^d(2r))^2} |\psi_i(t,x)|^2 |\psi_j(t,y)|^2 \, dx dy dt \leq C \sum_{i=1}^{N} ||\psi_{i,0}||_{H^1}^4,
\]
with \( (Q_{\bar{x}}^d(2r))^2 = Q_{\bar{x}}^d(2r) \times Q_{\bar{x}}^d(2r) \).
Proof. We notice that in the case $p = 2$, the term $N_{2,|x-y|}^C(t)$ will vanish. Hence, by applying the high regularity Morawetz interaction inequality (3.30) with $N_{2,|x-y|}^{HF}(t) \geq 0$, $N_{2,|x-y|}^{HF}(t) \geq 0$ and then arguing as in (3.40) and (3.41), one can easily attain the (3.47), which reads, by recalling that

$$\Delta^2 |x - y| = -\frac{(d - 1)(d - 3)}{|x - y|^3}, \quad \Delta^2 |x - y| = -4\pi \delta_{x=y} \leq 0,$$

as

$$\sum_{i,j=1}^N \int_\mathbb{R} \int_\mathbb{R} d \int_\mathbb{R} d \int_\mathbb{R} d \frac{|\psi_i(t,x)|^2 |\psi_j(t,y)|^2}{|x - y|^3} dt \leq C \sum_{i=1}^N \|\psi_{i,0}\|^4_{H^1},$$

for $d \geq 4$ and

$$\sum_{i,j=1}^N \int_\mathbb{R} \int_\mathbb{R} d \int_\mathbb{R} d \int_\mathbb{R} d \frac{|\psi_i(t,x)|^2 |\psi_j(t,y)|^2}{|x - y|^3} dt \leq C \sum_{i=1}^N \|\psi_{i,0}\|^4_{H^1},$$

for $d = 3$. The proofs of (3.48) and (3.49) are exactly the same as in Proposition 3.4, considering also the bound (3.44). □

By the low and high regularity Morawetz interaction inequalities (3.29), (3.30), taking into account that $N_{(p,a)}^C(\tau) \geq 0$ and following the Propositions 3.4 and 3.5, one arrives at the following corollary, where some new linear correlation-type estimates associated to (1.1) are achieved. We have thus:

**Corollary 3.6.** Let $d \geq 1$, $N \geq 1$, $p > 0$ be such that (1.3) holds and let $(u_{p,\mu})_{\mu=1}^N \in C(\mathbb{R}, H^1(\mathbb{R}^d)^N)$ be a global solution to (1.1). Then one has,

$$\sum_{i=1}^N \|(-\Delta)^{\frac{d-4}{2}} \nabla |\psi_i(t,x)|^2\|_{L^2((t_1,t_2);L^4)}^2 \lesssim \sup_{t \in [t_1,t_2]} |\hat{\psi}(t)|.$$  

In particular the following estimates are valid with $\beta \geq 0$, $\lambda_{ij} \geq 0$:

- for $d = 3$,

$$\sum_{i=1}^N \|\psi_i(t,x)\|_{L^4((t_1,t_2);L^4)}^4 \lesssim \sup_{t \in [t_1,t_2]} |\hat{\psi}(t)|;$$

- for $d \geq 4$,

$$\sum_{i=1}^N \|(-\Delta)^{\frac{d-4}{2}} |\psi_i(t,x)|^2\|_{L^2((t_1,t_2);L^2)}^2 \lesssim \sup_{t \in [t_1,t_2]} |\hat{\psi}(t)|.$$

4. **The decay of solutions to (1.1)**

Our main purpose in this section is to exhibit some decaying properties of the solution to (1.1) which is a mandatory property in order to study the scattering phenomena. With the aim of doing that, we present thus the proof of the of Theorem 1.1 and of the associated property (1.12) in Corollary 1.4.
Proof of Theorem (1.1). Let us set \( u(t, x) = (v_i(t, x))_{i=1}^N \), utilizing both notations where it is needed. We split the proof in two part: we look first at \( d \geq 2 \), then at \( d = 1 \).

Case \( d \geq 2 \). It is sufficient to prove the property (1.7) for a suitable \( 2 < q < 2d/(d-2) \) (for \( 2 < q < +\infty \), if \( d = 2 \)), since the thesis for the general case can be acquired by the conservation of mass (2.1), the kinetic energy (2.2) and then by interpolation. Let us select \( p = (2d+8)/(d+2) \), we need to prove then

\[
\lim_{t \to \pm \infty} \| u(t) \|_{L^{2d+8}_{x+t^2}} = 0.
\]

We treat only the case \( t \to \infty \), the case \( t \to -\infty \) can be dealt analogously. Proceeding now by absurd as in \([\text{4.4}]\) (we refer also to \([\text{37}]\)), we assume that there exists a sequence \( \{t_n\} \) with \( t_n \to +\infty \) and a \( \delta_0 > 0 \)

\[
\inf_n \| u(t_n, x) \|_{L^{2d+8}_{x+t^2}} = \delta_0.
\]

Next we will make an use of the localized Gagliardo-Nirenberg inequality given in the Appendix A with \( r = 1 \) and \( \nu = 2 \):

\[
\| \phi \|_{L^{2d+8}_{x+t^2}} \leq C \left( \sup_{x \in \mathbb{R}^d} \| \phi \|_{L^2(Q^d_{x_n}(1))} \right)^{d+2} \| \phi \|_{H^1_{x+t^4}}^{d+2},
\]

where \( Q^d_{x_n}(1) \) is the unit cube in \( \mathbb{R}^d \) centered in \( x_n \). By combining (4.2), (4.3), where we selected \( \phi = u(t_n, x) \), with the bound \( \| u(t_n, x) \|_{H^1_{x+t^4}} < +\infty \), we notice that there exists \( x_n \in \mathbb{R}^d \) and a \( \varepsilon_0 > 0 \) such that

\[
\| u(t_n, x) \|_{L^2(Q^d_{x_n}(1))} = \varepsilon_0.
\]

We can assert now that there exists \( t^* > 0 \) such that

\[
\| u(t, x,t) \|_{L^2(Q^d_{x_n}(2))} \geq \varepsilon_0/2,
\]

for all \( t \in (t_n, t_n + t^*) \) and where \( Q^d_{x_n}(2) \) denotes the cube in \( \mathbb{R}^d \) with sidelength 2 centered at \( x_n \). Then (4.5) can be showed as follows. Fix a cut-off function \( \varphi(x) \in C_0^\infty(\mathbb{R}^d) \), so as \( \varphi(x) = 1 \) for \( x \in Q^d_{x_n}(1) \) and \( \varphi(x) = 0 \) for \( x \notin Q^d_{x_n}(2) \). Then by applying (3.2) where we choose \( a(x) = \varphi(x-x_n) \) we get

\[
\left| \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x-x_n) |u(t,x)|^2 \, dx \right| \lesssim \sup_t \| u(t,x) \|_{H^1_{x+t^4}}^2.
\]

Consequently, by (2.2) and the fundamental theorem of calculus we deduce

\[
\int_{\mathbb{R}^d} \varphi(x-x_n) |u(\sigma,x)|^2 \, dx - \int_{\mathbb{R}^d} \varphi(x-x_n) |u(t,x)|^2 \, dx \leq \tilde{C} |t-\sigma|,
\]

for a \( \tilde{C} > 0 \) which does not depend on \( n \). Hence if we choose \( t = t_n \) we get the elementary inequality
\begin{equation}
\int_{\mathbb{R}^d} \varphi(x-x_n)|u(\sigma, x)|^2 dx \geq \int_{\mathbb{R}^d} \varphi(x-x_n)|u(t_n, x)|^2 dx - \bar{C}|t_n - \sigma|,
\end{equation}

which implies, having in mind the support property of the function $\varphi$,

\begin{equation}
\int_{\mathcal{Q}_{2n}^d(2)} |u(\sigma, x)|^2 dx \geq \int_{\mathcal{Q}_{2n}^d(1)} |u(t_n, x)|^2 dx - \bar{C}|t_n - \sigma|.
\end{equation}

Hence (4.5) follows by an application of (4.4), provided that we choose $t^* > 0$ such that $3\varepsilon^2_0 - 4C_0t^* > 0$. The inequality (4.5) is in contradiction with the Morawetz estimates \((3.32)\). In fact, the lower bound (4.5) means that

\begin{equation}
\inf_n \left( \inf_{t \in (t_n, t_n+t^*)} \sum_{j=1}^N \|\psi_j(t)\|_{L^2_{\sigma}(\mathcal{Q}_{2n}^d(2))}^2 \right) \geq \varepsilon^2_0 > 0,
\end{equation}

with $t^*$ as above and the time intervals $(t_n, t_n+t^*)$ chosen to be disjoint. By Hölder inequality we attain also

\begin{equation}
\inf_n \left( \inf_{t \in (t_n, t_n+t^*)} \sum_{j=1}^N \|\psi_j(t)\|_{L^p_{\sigma}(\mathcal{Q}_{2n}^d(2))}^p \right) \geq \varepsilon^2_0 > 0.
\end{equation}

Thus we can formulate the following

\begin{equation}
\sum_{i,j,k=1}^N \tilde{\lambda}_{ik} \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} \int_{(\mathcal{Q}_{2n}^d(2r))^3} |\psi_i(t, x)|^p |\psi_j(t, x)|^2 |\psi_k(t, z)|^p dxdydzdt
\geq \sum_{i,j,k=1}^N \tilde{\lambda}_{ik} \int_{\mathbb{R}^d} \int_{(\mathcal{Q}_{2n}^d(2r))^3} |\psi_i(t, x)|^p |\psi_j(t, x)|^2 |\psi_k(t, z)|^p dxdydzdt
\end{equation}

\begin{equation}
\sum_{i,k=1}^N \tilde{\lambda}_{ik} \sum_n \int_{t_n}^{t_n+t^*} \varepsilon^6_0 dt \geq \sum_n t^* \varepsilon^6_0 dt = \infty,
\end{equation}

where in the last inequality we employed (4.5) in combination with (4.9) and (4.10). This brings us to contradiction with (3.32).

**Case** $d = 1$. It can be handled in a similar manner, now by seeking for a $2 < q \leq +\infty$. By an application of the Hölder inequality, as expected, one
figures out the bound
\[
\sum_{i,j,k=1}^{N} \tilde{\lambda}_{ik} \int_{\mathbb{R}} \sup_{x \in \mathbb{R}^d} \int_{Q_{x_n}^{d}(2r)} |\psi_i(t, x)|^2 |\psi_j(t, x)|^2 |\psi_k(t, z)|^p \, dx \, dz \, dt
\]
\[
\gtrsim \sum_{i,k=1}^{N} \tilde{\lambda}_{ik} \sum_{n} \int_{t_n}^{t_n+t^*} \int_{Q_{x_n}^{d}(2r)} |\psi_i(t, x)|^2 |\psi_j(t, x)|^2 |\psi_k(t, z)|^p \, dx \, dz \, dt
\]
\[
\gtrsim \sum_{i,k=1}^{N} \tilde{\lambda}_{ik} \sum_{n} \int_{t_n}^{t_n+t^*} \varepsilon_0^4 \, dt \gtrsim \sum_{n} t^* \varepsilon_0^4 \, dt = \infty.
\]
Therefore, we can proceed as above, getting a contradiction with (3.33) instead.

\[\square\]

Besides, we have the following:

**Proof of (1.12).** We follow the same lines of the proof of the above steps. However, one can not use, at this level, the Proposition 3.4 because we are picking up \( p = 2 \). Then we are forced to focus on the Proposition 3.5: for \( d \geq 4 \), we make use of (4.9) attaining that
\[
\sum_{i,j=1}^{N} \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} \int_{Q_{x_n}^{d}(2r)} |\psi_i(t, x)|^2 |\psi_j(t, y)|^2 \, dx \, dy \, dt
\]
\[
\gtrsim \sum_{i,j=1}^{N} \int_{t_n}^{t_n+t^*} \int_{Q_{x_n}^{d}(2r)} |\psi_i(t, x)|^2 |\psi_j(t, y)|^2 \, dx \, dy \, dt
\]
\[
\gtrsim \sum_{n} \int_{t_n}^{t_n+t^*} \varepsilon_0^4 \, dt = \sum_{n} t^* \varepsilon_0^4 \, dt = \infty,
\]
which contradicts (3.49). In the same manner we can treat the case \( d = 3 \). Then by Hölder inequality and (4.9), we arrive at
\[
\sum_{i=1}^{N} \int_{\mathbb{R}^3} \sup_{\tilde{x} \in \mathbb{R}^3} \int_{Q_{\tilde{x}_n}^{3}(2r)} |\psi_i(t, x)|^4 \, dx \, dt \gtrsim \sum_{i=1}^{N} \sum_{n} \int_{t_n}^{t_n+t^*} \int_{Q_{x_n}^{3}(2r)} |\psi_i(t, x)|^2 \, dx \, dt
\]
\[
\gtrsim \sum_{n} \int_{t_n}^{t_n+t^*} \varepsilon_0^2 \, dt = \sum_{n} t^* \varepsilon_0^2 \, dt = \infty,
\]
that is in contradiction with (3.49). Then the proof is now complete. \[\square\]

5. Scattering for NLC and NLHF systems

We carry out, along this section, the proof of Theorem 1.2 and the corresponding scattering property (1.13) in Corollary 1.4. Albeit these results are classic (we suggest [7], [17] and references therein for further reading),
here we disclose them in a more general and self-contained form. We recall from [22], also:

**Definition 5.1.** An exponent pair \((q, r)\) is Schrödinger-admissible (or \((q, r)\)-Str.) if \(2 \leq q, r \leq \infty\), \((q, r, d) \neq (2, \infty, 2)\), and

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.
\]

**Proposition 5.2.** Let be two Schrödinger-admissible pairs \((q, r)\) and \((\bar{q}, \bar{r})\). Then we have for \(\kappa = 0, 1\) and the following estimates:

\[
\|\nabla^\kappa e^{-it\Delta_x} g\|_{L^q_t L^r_x} + \left\| \nabla^\kappa \int_0^t e^{-i(t-\tau)\Delta_x} G(\tau) d\tau \right\|_{L^q_t L^r_x} \\
\leq C \left( \|\nabla^\kappa g\|_{L^2_x} + \|\nabla^\kappa G\|_{L^q_t L^r_x} \right).
\]

We want to prove Theorem 1.2, then we demand to gain the necessary space-time summability for the scattering. This is contained in the following:

**Lemma 5.3.** Assume \(p\) is as in (1.3). Then, for any \(\psi_i^N\) \(i=1\) \(\in C(\mathbb{R}, H^1_x)\) global solution to (1.1), we have

\[
(\psi_i^1)_{i=1}^N \in L^q(\mathbb{R}, W^{1,r}_x),
\]

for every Schrödinger-admissible pair \((q, r)\).

**Proof.** We consider the integral operator associated to (1.1), that is

\[
u(t) = e^{it\Delta_x} u_0 + \int_0^t e^{i(t-\tau)\Delta_x} \Gamma(u(\tau), p) d\tau
\]

where \(t > 0\) and

\[
u(t) = \begin{pmatrix} \psi_1(t) \\ \vdots \\ \psi_N(t) \end{pmatrix}, \quad u_0 = \begin{pmatrix} \psi_{1,0} \\ \vdots \\ \psi_{N,0} \end{pmatrix},
\]

\[
\Gamma(u, p) = \begin{pmatrix} G(\psi_1, \psi_k) \\ \vdots \\ G(\psi_N, \psi_k) \end{pmatrix}.
\]

Let us choose \((q_1', r_1')\) defined like

\[
(q_1, r_1) := \left(\frac{4p}{dp - d - \gamma_1}, \frac{2dp}{d + \gamma_1}\right)
\]

and \((q_2', r_2')\) so that

\[
(q_2, r_2) := \left(\frac{8}{d - \gamma_1}, \frac{4d}{d + \gamma_1}\right).
\]

We introduce also the following auxiliary space

\[
X = L^q_{t} L^r_x + L^q_{t} L^r_x,
\]
having norm
\[ \|G\|_X = \inf_{G=G_1+G_2} \|G_1\|_{L_t^{q_1}L_x^{r_1}} + \|G_2\|_{L_t^{q_2}L_x^{r_2}}. \]

In this way the Strichartz estimates (5.2) enhance, for \( \kappa = 0,1 \), to the following

\[(5.7) \quad \sup_{(q,r)-Str.} \|\nabla^\kappa u\|_{L_t^{q_1}L_x^{r_1}} \lesssim \|\nabla^\kappa g\|_{L_t^{q_2}L_x^{r_2}} + \inf_{G=G_1+G_2} \|\nabla^\kappa G_1\|_{L_t^{q_1}L_x^{r_1}} + \|\nabla^\kappa G_2\|_{L_t^{q_2}L_x^{r_2}}. \]

Then, the above estimate (5.7) with the Hölder and Hardy-Littlewood-Sobolev inequalities gives,

\[(5.8) \quad \sup_{(q,r)-Str.} \|\nabla^\kappa \psi_i\|_{L_t^{q_1}L_x^{r_1}} \lesssim \left\| \sum_{k=1}^{N} \nabla^\kappa \left( \lambda_{ik} \left[ |x|^{-(d-\gamma_1)} * |\psi_k|^p \right] |\psi_i|^{p-2} \psi_i \right) \right\|_{L_t^{q_1}L_x^{r_1}}^q + \beta \left\| \sum_{k=1}^{N} \nabla^\kappa \left( \left[ |x|^{-(d-\gamma_2)} * |\psi_k|^p \right] \psi_i \right) \right\|_{L_t^{q_2}L_x^{r_2}}^q + \beta \left\| \sum_{k=1}^{N} \nabla^\kappa \left( \left[ |x|^{-(d-\gamma_2)} * \psi_i \psi_k \right] \psi_k \right) \right\|_{L_t^{q_2}L_x^{r_2}}^q \lesssim \left\| \sum_{k=1}^{N} \lambda_{ik} \|\nabla^\kappa \psi_i\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_2}L_x^{r_2}} \|\psi_i\|_{L_t^{q_1}L_x^{r_1}}^{p-2} \right\|_{L_t^{q_1}L_x^{r_1}}^q + \beta \left\| \sum_{k=1}^{N} \|\nabla^\kappa \psi_i\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_2}L_x^{r_2}} \right\|_{L_t^{q_1}L_x^{r_1}}^q + \beta \left\| \sum_{k=1}^{N} \|\nabla^\kappa \psi_k\|_{L_t^{q_1}L_x^{r_1}} \|\psi_i\|_{L_t^{q_2}L_x^{r_2}} \right\|_{L_t^{q_1}L_x^{r_1}}^q. \]

Summing up over \( i = 1 \ldots N \), we see that the last term of the previous inequality is not greater than

\[(5.9) \quad \sum_{i,k=1}^{N} \lambda_{ik} \left\| \nabla^\kappa \psi_i\right\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_1}L_x^{r_1}} \|\psi_i\|_{L_t^{q_1}L_x^{r_1}}^{p-2} + \beta \sum_{i,k=1}^{N} \left\| \nabla^\kappa \psi_i\right\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_1}L_x^{r_1}}^{q_2} + \beta \sum_{i,k=1}^{N} \left\| \nabla^\kappa \psi_k\right\|_{L_t^{q_1}L_x^{r_1}} \|\psi_i\|_{L_t^{q_1}L_x^{r_1}} \|\psi_k\|_{L_t^{q_1}L_x^{r_1}}^{r_2} \lesssim \left\| \nabla^\kappa u\right\|_{L_t^{q_1}L_x^{r_1}} \|u\|_{L_t^{q_1}L_x^{r_1}} \left( \frac{2p-2(1-\theta_1)}{2(1-\theta_2)} \|u\|_{L_t^{q_1}L_x^{r_1}} \right) \|u\|_{L_t^{q_1}L_x^{r_1}}^{2(1-\theta_1)} \right\|_{L_t^{q_1}L_x^{r_1}}^q + \beta \left\| \nabla^\kappa u\right\|_{L_t^{q_1}L_x^{r_1}} \|u\|_{L_t^{q_1}L_x^{r_1}} \left( \frac{2p-2(1-\theta_1)}{2(1-\theta_2)} \|u\|_{L_t^{q_1}L_x^{r_1}} \right) \|u\|_{L_t^{q_1}L_x^{r_1}}^{2(1-\theta_2)} \right\|_{L_t^{q_1}L_x^{r_1}}^q. \]
We single out now $\theta_1, \theta_2 \in (0, 1)$ such that $\theta_1 = (q - q')/(2pq - 2q')$ and $\theta_2 = (q - q')/2q'$, yielding for the last term on the r.h.s. of (5.9),

$$
(5.10) \quad \left\| \frac{1}{|\nabla^\kappa u|} \right\|_{L^1_T \cap L^p} \left\| u \right\|_{L^1_T \cap L^p} \left(2^{p-2}(1-\theta_1) \left\| u \right\|^{(2p-2)\theta_1}_{L^p_T} \left\| u \right\|^{(2p-2)\theta_1}_{L^p_T} + \left\| \nabla^\kappa u \right\|_{L^1_T \cap L^p} \left\| u \right\| \right\|_{L^1_T \cap L^p} \left\| u \right\|^{2\theta_2}_{L^p_T} \left\| u \right\|^{2\theta_2}_{L^p_T},
$$

with the constant $C > 0$ independent from $t$ and $T$. An use of (5.8), (5.9), (5.9) leads, by the estimates (5.7), to

$$
\sup_{(q,r)-\text{Str.}} \left\| u \right\|_{L^q_{t>T} W^{1,r}_x} \leq C \left\| u_{0} \right\|_{\mathcal{H}_x} + \chi(T) \sup_{(q,r)-\text{Str.}} \left\| u \right\|^{q-1}_{L^q_{t>T} W^{1,r}_x},
$$

where

$$
\chi(T) = \left\| u \right\|^{2p-1-\frac{2p-1}{q_1}}_{L^q_{t>T} L^p_{x}} + \left\| u \right\|^{3-\frac{2q}{q_2}}_{L^q_{t>T} L^p_{x}},
$$

so that $\chi(T) \to 0$ as $T \to \infty$ by Theorem 1.1. Then, picking up $T$ sufficiently large we infer that

$$
\left\| u \right\|_{L^p((T,T), W^{1,r}_x)} < \infty,
$$

for all the Schrödinger-admissible pairs $(p, r)$ and consequently that $u \in L^q((T, \infty), W^{1,r}_x)$. Likewise, we can earn $u \in L^q((-\infty, -T), W^{1,r}_x)$. In conclusion, by a continuity argument, one has $u \in L^q(\mathbb{R}, W^{1,r}_x)$ for any Schrödinger-admissible pair $(p, r)$.

Prooof of Theorem 1.2 and (1.13) in Corollary 1.4. We exploit the proof of Theorem 1.2 and of (1.13) in a unified manner. We start from:

Asymptotic completeness: We write $\tilde{u}(t) = e^{-it\Delta_x} u(t)$ getting then from (5.4)

$$
\tilde{u}(t) = u_0 + i \int_0^t e^{-is\Delta_x} \Gamma(u, p) ds,
$$

as well as one has, for $0 < t_1 < t_2$,

$$
\tilde{u}(t_2) - \tilde{u}(t_1) = i \int_{t_1}^{t_2} e^{-is\Delta_x} \Gamma(u, p) ds.
$$
An use of the Strichartz estimates (5.7) bears to
\[
\| u(t_2) - u(t_1) \|_{H^1} \lesssim \| e^{it\Delta} (u(t_2) - u(t_1)) \|_{L_x^2} 
\]
(5.11)
\[
\sum_{i,k=1}^N \lambda_{ik} \| [x|-(d-\gamma_1)] |\psi_k|^p |\psi_i|^{p-2} \psi_i \|_{L^p((t_1,t_2),W_x^r)} + \beta \sum_{i,k=1}^N \| [x|-(d-\gamma_2)] |\psi_k|^p \psi_i - [x|-(d-\gamma_2)] |\psi_i|^p \psi_k \|_{L^p((t_1,t_2),W_x^r)} ,
\]
with \((q_1,r_1)\) and \((q_2,r_2)\) are Schrödinger-admissible pairs as in (5.5) and (5.6). Then
\[
\lim_{t_1,t_2 \to \infty} \| u(t_2) - u(t_1) \|_{H^1} = 0,
\]
is verified by (5.11) on condition that
\[
\lim_{t_1,t_2 \to \infty} \sum_{i,k=1}^N \lambda_{ik} \| [x|-(d-\gamma_1)] |\psi_k|^p |\psi_i|^{p-2} \psi_i \|_{L^p((t_1,t_2),W_x^r)} + \beta \lim_{t_1,t_2 \to \infty} \sum_{i,k=1}^N \| [x|-(d-\gamma_2)] |\psi_k|^p \psi_i - [x|-(d-\gamma_2)] |\psi_i|^p \psi_k \|_{L^p((t_1,t_2),W_x^r)} = 0,
\]
which can be easily performed following the same lines of the proof of Lemma 5.3. One can see, as a final step, that there are \((\psi_{1,0}^+, \ldots, \psi_{N,0}^+) \in H^1(\mathbb{R}^d)^N\) and a map \((\psi_1(t), \ldots, \psi_N(t)) \to (\psi_1^+, \ldots, \psi_N^+)\) in \(H^1(\mathbb{R}^d)^N\) when \(t \to \pm\infty\). Notice that, by Proposition 2.1, we establish also the following conservation laws
\[
M(\psi_{1,0}^+, \ldots, \psi_{N,0}^+) = \| (\psi_{1,0}, \ldots, \psi_{N,0}) \|_L^2, \\
\sum_{\mu=1}^N \int_{\mathbb{R}^d} (|\Delta \psi_{i,0}^+|^2 + \kappa |\nabla \psi_{i,0}^+|^2) \, dx = E(\psi_{1,0}, \ldots, \psi_{N,0}).
\]

Existence of wave operators: The construction of the wave operators comes from standard arguments, we remand to [5] for more details about the argument. Then we skip the proof. □

Remark 5.4. Once (1.7) is achieved in the range \(2 < r < 2d/(d-2)\), we were able to set up the scattering operator in \(H^1(\mathbb{R}^d)^N\), as we did in the previous section. Now, proceeding in an analogous way to [37], we arrive by Sobolev embedding at
\[
\| \psi_i(t) \|_{L_x^{\frac{2d}{d-2}}} \lesssim \| \psi_i(t) - e^{it\Delta} \psi_{i,0}^+ \|_{H^1} + \| e^{it\Delta} \psi_{i,0}^+ \|_{L_x^{\frac{2d}{d-2}}}.
\]
Now the above estimate (5.12) combined with the classical dispersive estimate for the free propagator
\[
\left\| e^{it\Delta} \psi_{i,0}^\pm \right\|_{L^2_x} \lesssim \frac{1}{t} \left\| \psi_{i,0}^\pm \right\|_{L^2_x},
\]
again the Sobolev-embedding and (1.9), allows also to
\[
\lim_{t \to \infty} \left\| \psi_i(t) \right\|_{L^2_x} = 0.
\]
The proof of Theorem 1.1 is now completed.

**Appendix A. A Gagliardo-Nirenberg inequality**

The principal target of this section is to exhibit (4.3) that is a localized version of the Gagliardo-Nirenberg inequality and which appears in the proof of Proposition 1.1. Although it is available so far in the literature in different forms (let us cite here [5], [37], [25] or [35] in the context of product space \( \mathbb{R}^d \times M \), with \( M \) a compact manifold), we show here a more general new one. We have:

**Proposition A.1.** Let be \( d \geq 1 \), \( \mu, \nu \in \mathbb{N} \) and \( \nu \in \mathbb{N} \cup \{0\} \), then for all vector-valued functions \( \phi = (\phi_\ell)_{\ell=1}^\mu \in H^1(\mathbb{R}^d)^\mu \) one gets the following

(A.1) \[
\| \phi \|_{L^2_x}^{2d+2\nu+4} \lesssim C \left( \sup_{x \in \mathbb{R}^d} \| \phi \|_{L^2(Q^d_x(r))} \right)^{\frac{4}{d+\nu}} \| \phi \|_{H^1(\mathbb{R}^d)^\mu}^2,
\]

with \( Q^d_x(r) = x + [-r, r]^d \), to be a \( r \) dilation of the unit cube centered at \( x \).

**Proof.** Fix \( r > 0 \) and consider \( x_s \in \mathbb{R}^d \) connected to a covering of \( \mathbb{R}^d \) given by a family of cubes \( \{Q^d_s(x_s) \}_{s \in \mathbb{N}} \) such that \( \text{meas}_d \left( Q^d_{x_{s_1}}(r) \cup Q^d_{x_{s_2}}(r) \right) = 0 \) for \( s_1 \neq s_2 \), where \( \text{meas}_d \) is the Lebesgue measure in \( \mathbb{R}^d \). Without loss of generality, we can take \( \phi = (\phi_\ell)_{\ell=1}^N \), such that \( \text{supp} (\phi_\ell) \subseteq Q^d_{x_s}(r) \), \( \ell = 1, \ldots, \mu \), then by the classical Gagliardo-Nirenberg inequality (see [39]) we attain

(A.2) \[
\sum_{\ell=1}^\mu \int_{Q^d_{x_s}(r)} |\phi_\ell|^{\frac{2d+2\nu+4}{d+\nu}} \lesssim \sum_{\ell=1}^\mu \left( \int_{Q^d_{x_s}(r)} |\phi_\ell|^\rho \right)^{\frac{4}{\rho(d+\nu)}} \left( \int_{Q^d_{x_s}(r)} |\nabla \phi_\ell|^\rho \right)^{\frac{2}{\rho}}
\]
and
\[
\rho = \frac{2d(d+\nu+2)}{(d+2)(d+\nu)} > 2.
\]
An application of the Hölder inequality, gives that the r.h.s. of (A.2) is bounded as
\[
\sum_{\ell=1}^{\mu} \left( \int_{\mathbb{Q}_x^d(r)} |\phi_{\ell}|^\mu \right) \frac{4}{d+\nu} \left( \int_{\mathbb{Q}_x^d(r)} |\nabla \phi_{\ell}|^2 \right) \]

\[
\leq C \sum_{\ell=1}^{\mu} \left( \int_{\mathbb{Q}_x^d(r)} |\phi_{\ell}|^2 \right) \frac{4}{d+\nu} \left( \int_{\mathbb{Q}_x^d(r)} |\nabla \phi_{\ell}|^2 \right) \]

\[
\leq C \left( \sum_{\ell=1}^{\mu} \|\phi_{\ell}\|_{L^2(\mathbb{Q}_x^d(r))}^4 \right) \frac{4}{d+\nu} \left( \sum_{\ell=1}^{\mu} \|\phi_{\ell}\|^2_{H^1(\mathbb{Q}_x^d(r))} \right). \]

with \( C > 0 \) a constant depending on \( \text{meas}_d(\mathbb{Q}_x^d(r)) \). From (A.2) and (A.3) one can get

\[
\|\phi\|_{\frac{2d+2\nu+4}{d+\nu} \left( \sum_{\ell=1}^{\mu} \|\phi_{\ell}\|_{L^2(\mathbb{Q}_x^d(r))}^\mu \right)^\frac{4}{d+\nu} \|\phi\|^2_{H^1(\mathbb{Q}_x^d(r))} \mu}. \]

\[
\|\phi\|_{\frac{2d+2\nu+4}{d+\nu} \left( \sup_{s \in \mathbb{N}} \|\phi\|_{L^2(\mathbb{Q}_x^d(r))}^\mu \right)^\frac{4}{d+\nu} \|\phi\|^2_{H^1(\mathbb{Q}_x^d(r))} \mu}, \]

which is the estimate (A.1), with the constants involved independent from \( s \) because the estimate above is translation invariant.

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