Abstract

We presented a new physical model that links the maximum speed of light with the minimal Planck scale into a maximal-acceleration Relativity principle in the spacetime tangent bundle and in phase spaces (cotangent bundle). The maximal proper-acceleration bound is $a = c^2/\Lambda$ in full agreement with the old predictions of Caianiello, the Finslerian geometry point of view of Brandt and more recent results in the literature. Inspired by the maximal-acceleration corrections to the Lamb shifts of one-electron atoms by Lambiase, Papini and Scarpetta, we derive the exact integral equation that governs the Renormalization-Group-like scaling dependence of the fractional change of the fine structure constant as a function of the cosmological redshift factor and a cutoff scale $L_c$, where the maximal acceleration relativistic effects are dominant. A particular physical model exists dominated entirely by the vacuum energy, when the cutoff scale is the Planck scale, with $\Omega_\Lambda \sim 1$. The implications of this extreme case scenario are studied.

I. Introduction

In recent years there has been growing evidence for the Cosmological evolution of the fine structure constant using absorption systems in the spectra of distant quasars. It was found that the fine structure $\alpha$ was smaller in the past. For references see [11]. The purpose of this note is to derive an exact integral equation governing the fractional change $\frac{\Delta}{\alpha}$ of the fine structure due to the Maximal-acceleration Relativistic effects proposed in [1]. The latter were just a consequence of the Extended Scale Relativity theory in C-spaces.

Relativity in C-spaces (Clifford manifolds) [1] is a very natural extension of Einstein’s relativity and Nottale’s scale relativity [2] where the impassible speed of light and the minimum Planck scale are the two universal invariants. An event in C-space is represented by a polyvector, or Clifford-aggregate of
lines, areas, volumes, ...... which bear a one-to-one correspondence to the holographic shadows/projections (onto the embedding spacetime coordinate planes) of a nested family of \( p \)-loops (closed \( p \)-branes of spherical topology) of various dimensionalities: \( p = 0 \) represents a point; \( p = 1 \) a closed string, \( p = 2 \) a closed membrane, etc.... where \( p = 0, 1, 2, ..., D - 1 \).

The invariant “line” element associated with a polyparticle is:

\[
d\Sigma^2 = dX.dX = (d\Omega)^2 + \Lambda^{2D-2}(dx_{\mu}dx^\mu) + \Lambda^{2D-4}(dx_{\mu\nu})(dx^{\mu\nu}) + ... \tag{1.1}
\]

the Planck scale appears as a natural quantity in order to match units and combine \( p \)-branes (\( p \)-loops) of different dimensions. The polyparticle lives in a target background of \( D^2 = D + 1 = p + 2 \) dimensions due to the fact that \( C \)-space has \( two \) times, the coordinate time \( x_o = t \) and the \( \Omega \) temporal variable representing the proper \( p + 1 \)-volume. The fact that the Planck scale is a minimum was based on the real-valued interval \( dX \) when \( dX.dX > 0 \). The analog of photons in \( C \)-space are tensionless branes: \( dX.dX = 0 \). Scales smaller than \( \Lambda \) yield ” tachyonic ” intervals \( dX.dX < 0 \) [1]. Due to the matrix representation of the gamma matrices and the cyclic trace property, it can be easily seen why the line element is invariant under the \( C \)-space Lorentz group transformations:

\[
Trace X^2 = Trace [RX^2R^{-1}] = Trace [RR^{-1}X^2] = Trace X^2 , \tag{1.2}
\]

where a finite polydimensional rotation that reshuffles dimensions is characterized by the \( C \)-space “rotation” matrix:

\[
R = \exp[i(\theta I + \theta^\mu \gamma_\mu + \theta^{\mu\nu} \gamma_{\mu\nu} + ...)] . \tag{1.3}
\]

The parameters \( \theta, \theta^\mu, \theta^{\mu\nu}, ...... \) are the \( C \)-space extension of the Lorentz boost parameters and for this reason the naive Lorentz transformations of spacetime are modified to be:

\[
x'^\mu = L^\mu_\nu [\theta, \theta^\mu, \theta^{\mu\nu}, ...]x^\nu + L^\mu_\nu [\theta, \theta^\mu, \theta^{\mu\nu}, ...]x'^\nu + .... \tag{1.4}
\]

It was argued in [1] that the extended Relativity principle in \( C \)-space may contain the clues to unravel the physical foundations of string and \( M \)-theory since the dynamics in \( C \)-spaces encompass in one stroke the dynamics of
all p- branes of various dimensionalities. In particular, how to formulate a master action that encodes the collective dynamics of all extended objects.

For further details about these issues we refer to [1] and all the references therein. Like the derivation of the minimal length/time string/brane uncertainty relations; the logarithmic corrections to the black- hole area-entropy relation; the existence of a maximal Planck temperature; the origins of a higher derivative gravity with torsion; why quantum-spacetime may be truly infinite dimensional whose average dimension today is close to \(4 + \phi^3 = 4.236\ldots\) where \(\phi = 0.618\ldots\) is the Golden Mean; the construction of the p-brane propagator; the role of supersymmetry; the emergence of two times; the reason behind a running value for \(\hbar\); the way to correctly pose the cosmological constant problem as well as other results.

In [1] we discussed another physical model that links the maximum speed of light, and the minimal Planck scale, into a maximal-acceleration principle in the spacetime tangent bundle, and consequently, in the phase space (cotangent bundle). Crucial in order to establish this link was the use of Clifford algebras in phase spaces. The maximal proper acceleration bound is \(a = c^2/\Lambda\) in full agreement with [4] and the Finslerian geometry point of view in [6]. A series of reasons why C-space Relativity is more physically appealing than all the others proposals based on kappa-deformed Poincare algebras [10] and other quantum algebras was presented.

On the other hand, we argued why the truly bicovariant quantum algebras based on inhomogeneous quantum groups developed by Castellani [15] had a very interesting feature related to the T-duality in string theory; the deformation \(q\) parameter could be written as: \(q = \exp[\Lambda/L]\) and, consequently, the classical limit \(q = 1\) is attained when the Planck scale \(\Lambda\) is set to zero, but also when the upper impassible scale \(L\) goes to infinity! This entails that there could be two dual quantum gravitational theories with the same classical limit! Nottale has also postulated that if there is a minimum Planck scale, by duality, there should be another upper impassible upper scale \(L\) in Nature [2]. For a recent discussion on maximal-acceleration and kappa-deformed Poincare algebras see [10]. It was also argued in [1] why the theories based on kappa-deformed Poincare algebras may in fact be related to a Moyal star-product deformation of a classical Lorentz algebra whose deformation parameter is precisely the Planck scale \(\Lambda = 1/\kappa\).

In section II, we discuss the work in [1] and show how to derive the Nesterenko action [5] associated with a sub-maximally accelerated particle
in spacetime directly from phase-space Clifford algebras and present a full-fledged C-phase-space generalization of the Nesterenko action.

In section 3 we review the maximal-acceleration relativistic corrections to the Lamb shift of one electron atoms \[12\] (of fractional order of \(10^{-5}\)) as a preamble to section 4 where we derive the exact integral equation that governs the Renormalization-Group-like scaling dependence of the fractional change of the fine structure constant as a function of the cosmological redshift factor and a cutoff scale \(L_c\) where the maximal acceleration relativistic effects are dominant. To conclude, we derive the corrections to the electric charge due to a running Planck constant induced by Extended Scale Relativistic effects in C-spaces and discuss the Cosmological model with \(\Omega_\Lambda \sim 1\).

II. Maximal-Acceleration from Clifford algebras

We will follow closely the procedure described in the book [3] to construct the phase space Clifford algebra. For simplicity we shall begin with a two-dimensional phase space, with one coordinate and one momentum variable and afterwards we will generalize the construction to higher dimensions.

Let \(e_p, e_q\) be the Clifford basis elements in a two-dimensional phase space obeying the following relations:

\[
e_p.e_q \equiv \frac{1}{2}(e_q e_p + e_p e_q) = 0. \quad e_p.e_p = e_q.e_q = 1. \tag{2.1}
\]

The Clifford product of \(e_p, e_q\) is by definition the sum of the scalar product and wedge product furnishing the unit bivector:

\[
e_p e_q \equiv e_p.e_q + e_p \wedge e_q = e_p \wedge e_q = j. \quad j^2 = e_p e_q e_p e_q = -1. \tag{2.2}
\]
due to the fact that \(e_p, e_q\) anticommute, eq. (2.1).

In this fashion, using Clifford algebras one can justify the origins of complex numbers without introducing them ad-hoc. The imaginary unit \(j\) is \(e_p e_q\). For example, a Clifford vector in phase space can be expanded, setting aside for the moment the issue of units, as:

\[
Q = q e_q + p e_p. \quad Q e_q = q + p e_p e_q = q + j p = z. \quad e_q Q = q + p e_q e_p = q - j p = z^\ast, \tag{2.3}
\]
which reminds us of the creation/annihilation operators used in the harmonic oscillator case and in coherent states.

The analog of the action for a massive particle is obtained by taking the scalar product:

\[ dQ.dQ = (dq)^2 + (dp)^2 \Rightarrow S = m \int \sqrt{dQ.dQ} = m \int \sqrt{(dq)^2 + (dp)^2}. \quad (2.4) \]

One may insert now the appropriate length and mass parameters in order to have consistent units:

\[ S = m \int \sqrt{(dq)^2 + \left(\frac{\Lambda}{m}\right)^2(dp)^2}. \quad (2.5) \]

where we have introduced the Planck scale \( \Lambda \) and the mass \( m \) of the particle to have consistent units, \( \hbar = c = 1 \). The reason will become clear below.

Extending this two-dimensional action to a higher 2\( n \)-dimensional phase space requires to have \( e_{p\mu}, e_{q\mu} \) for the Clifford basis where \( \mu = 1, 2, 3...n \). The action in this 2\( n \)-dimensional phase space is:

\[ S = m \int \sqrt{(dq^\mu dq_\mu) + \left(\frac{\Lambda}{m}\right)^2(dp^\mu dp_\mu)} = m \int d\tau \sqrt{1 + \left(\frac{\Lambda}{m}\right)^2(dp^\mu/d\tau)(dp_\mu/d\tau)} \quad (2.6) \]

in units of \( c = 1 \), one has the usual infinitesimal proper time displacement \( d\tau^2 = dq^\mu dq_\mu \).

One can easily recognize that this action (2.6), up to a numerical factor of \( m/a \), is nothing but the action for a sub-maximally accelerated particle given by Nesterenko [5]. It is sufficient to rewrite: \( dp^\mu/d\tau = md^2x^\mu/d\tau^2 \) to get from eq. (2.6):

\[ S = m \int d\tau \sqrt{1 + \Lambda^2(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}. \quad (2.7) \]

Using the postulate that the maximal-proper acceleration is given in a consistent manner with the minimal length principle (in units of \( c = 1 \)):

\[ a = c^2/\Lambda = 1/\Lambda \Rightarrow S = m \int d\tau \sqrt{1 + \left(\frac{1}{a}\right)^2(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}, \quad (2.8) \]
which is exactly the action of \([5]\), up to a numerical factor of \(m/a\), when the metric signature is \((+,-,-,-)\).

The proper acceleration is orthogonal to the proper velocity as a result of differentiating the timelike proper velocity squared:

\[
V^2 = \frac{dx\mu}{d\tau} \frac{dx\mu}{d\tau} = 1 = V^\mu V_\mu > 0 \Rightarrow \frac{dV^\mu}{d\tau} V_\mu = \frac{d^2x^\mu}{d\tau^2} V_\mu = 0 , \tag{2.9}
\]

which means that if the proper velocity is timelike the proper acceleration is spacelike so that:

\[
g^2(\tau) \equiv -(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2) > 0 \Rightarrow S = m \int d\tau \sqrt{1 - \frac{g^2}{a^2}} \equiv m \int d\omega , \tag{2.10}
\]

where we have defined:

\[
d\omega \equiv \sqrt{1 - \frac{g^2}{a^2}} d\tau . \tag{2.11}
\]

The dynamics of a submaximally accelerated particle in Minkowski spacetime can be reinterpreted as that of a particle moving in the spacetime tangent bundle background whose Finslerian-like metric is:

\[
d\omega^2 = g_{\mu\nu}(x^\mu, dx^\nu)dx^\mu dx^\nu = (d\tau)^2(1 - \frac{g^2}{a^2}) . \tag{2.12}
\]

For uniformly accelerated motion, \(g(\tau) = g = \text{constant}\) the factor:

\[
\frac{1}{\sqrt{1 - \frac{g^2}{a^2}}} \tag{2.13}
\]

plays a similar role as the standard Lorentz time dilation factor in Minkowski spacetime.

The action is real valued if, and only if, \(g^2 < a^2\) in the same way that the action in Minkowski spacetime is real valued if, and only if, \(v^2 < c^2\). This explains why the particle dynamics has a bound on proper accelerations. Hence, for the particular case of a uniformly accelerated particle whose trajectory in Minkowski spacetime is a hyperbola, one has an Extended Relativity of uniformly accelerated observers whose proper acceleration have
an upper bound given by $c^2/\Lambda$. Rigorously speaking, the spacetime trajectory is obtained by a canonical projection of the spacetime tangent bundle onto spacetime. The invariant time, under the pseudo-complex extension of the Lorentz group [8], measured in the spacetime tangent bundle is no longer the same as $\tau$, but instead, it is given by $\omega(\tau)$.

This is similar to what happens in C-spaces, the truly invariant evolution parameter is not $\tau$ nor $\Omega$, the Stuckelberg parameter [3], but it is $\Sigma$ which is the world interval in C-space and that has units of $\text{length}^D$. The group of C-space Lorentz transformations preserve the norms of the Polyvectors and these have units of hypervolumes; hence C-space Lorentz transformations are volume-preserving.

Another approach to obtain the action for a sub-maximally accelerated particle was given by [8] based on a pseudo-complexification of Minkowski spacetime and the Lorentz group that describes the physics of the spacetime tangent bundle. This picture is not very different from the Finslerian spacetime tangent bundle point of view of Brandt [6]. The invariant group is given by a pseudo-complex extension of the Lorentz group acting on the extended coordinates $X = ax^\mu + Ix^\mu$ with $I^2 = 1$ (pseudo-imaginary unit) where both position and velocities are unified on equal footing. The invariant line interval is $a^2d\omega^2 = (dX)^2$.

A C-phase-space generalization of these actions (for sub-maximally accelerated particles, maximum tidal forces) follows very naturally by using polyvectors:

$$Y = q^\mu e_{q^\mu} + q^{\mu\nu} e_{q^\mu} \wedge e_{q^\nu} + q^{\mu\rho\nu} e_{q^\mu} \wedge e_{q^\rho} \wedge e_{q^\nu} + ....$$

$$+ p^\mu e_{p^\mu} + p^{\mu\nu} e_{p^\mu} \wedge e_{p^\nu} + ... ,$$

where one has to insert suitable powers of $\Lambda$ and $m$ in the expansion to match units.

The C-phase-space action reads then:

$$S \sim \int \sqrt{dY.d\overline{Y}} = \int \sqrt{dq^\mu dq^\mu + dq^{\mu\nu} dq^{\mu\nu} + ... + dp^\mu dp^\mu + dp^{\mu\nu} dp^{\mu\nu} + ...} .$$

(2.15)

This action is the C-phase-space extension of the action of Nesterenko and involves quadratic derivatives in C-spaces which from the spacetime perspective are effective higher derivatives theories [1] where it was shown why the
scalar curvature in C-spaces is equivalent to a higher derivative gravity. One should expect a similar behaviour for the extrinsic curvature of a polyparticle motion in C-spaces. This would be the C-space version of the action for rigid particles [7]. Higher derivatives are the hallmark of W-geometry (higher conformal spins).

Born-Infeld models have been connected to maximal-acceleration [8]. Such models admits an straightforward formulation using the geometric calculus of Clifford algebras. In particular one can rewrite all the nonlinear equations of motion in precise Clifford form [9]. This lead that author to propose the nonlinear extension of Dirac’s equation for massless particles due to the fact that spinors are nothing but right/left ideals of the Clifford algebra: i.e., columns, for example, of the Maxwell-Field strength bivector $F = F_\mu\nu \gamma^\mu \wedge \gamma^\nu$.

III. Maximal-Acceleration corrections to the Lamb-shift

The maximal-acceleration corrections to the Lamb-shift of one electron atoms were calculated by Lambiase et al in [12]. They started from the Dirac equation and splitted the spinor into a large and small component. The crucial point behind this calculation was based on the fact that the spacetime metric is only flat up to a conformal factor which depends on the acceleration:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \sqrt{1 - \frac{g^2(\tau)}{A_{\text{max}}^2}}. \quad (3.1)$$

This is a Finslerian-type metric.

As a result, the Dirac equation is modified accordingly due to the conformally-scaled vierbiens $e^a_\mu(x) = \sigma(x) \delta^a_\mu$ used to write down the Dirac equation in terms of the (spacetime) gamma matrices $\gamma^\mu(x) = e^a_\mu(x) \sigma^a$:

$$[i\hbar \gamma^a(\partial_a + i\frac{e}{\hbar c} A_a) + i\frac{3\hbar}{2\sigma} \gamma^a(\partial_a \sigma) - m]\psi(x) = 0. \quad (3.2)$$

As usual, the tangent spacetime indices are represented by $a, b = 1, 2, 3, 4$. and one introduces a minimal EM coupling. The Dirac Hamiltonian is modified and includes the small acceleration-dependent perturbation:

$$\Delta H = -i\frac{3\hbar c}{2} \gamma^0 \gamma^a \partial_a (-\sigma^{-1}). \quad (3.3)$$
The conformal factor can be expressed in terms of the maximal-acceleration and a cutoff scale \( r_o \) as follows:

\[
\sigma(x) = \sqrt{1 - \frac{g^2}{A^2}} = \sqrt{1 - \left(\frac{r_o}{r}\right)^4}.
\]

(3.4)

where:

\[
\frac{r_o^2}{r_o} = \frac{e^2}{mA} \sim 3.3 \times 10^{16}\text{cm}.
\]

(3.5)

The corrections to the Lamb-shift were obtained via perturbation theory by splitting the spinors into a large/small parts. The corrections to the energy spectrum were:

\[
\Delta E = \langle \text{nljm} | \Delta H | \text{nljm} \rangle.
\]

(3.6)

In the special case of non-relativistic electrons in an electrostatic field the maximal acceleration corrections for a one-electron atom were:

\[
\Delta E = 6K \int d^3r \phi^+ \frac{1}{r_o^6} \phi - 4K \int d^3r \phi^+ \frac{\partial}{r^5 \partial r} \phi.
\]

(3.7)

with

\[
K \equiv 3\hbar^2/4m(e^2/mA)^2.
\]

(3.8)

that has units of energy \( \times \) length\(^6\).

The quantity defined in terms of the Bohr radius \( a_o \) has energy units:

\[
\frac{K}{a_o^6} = 1.03 \text{ kHz}.
\]

(3.9)

All the corrections to the Lamb shift were of the form:

\[
\frac{K}{a_o^6} F\left(\frac{a_o}{\Lambda}\right) e^{-\Lambda/a_o}.
\]

(3.10)

where \( \Lambda \leq a_o \) was a suitable cutoff and the function \( F \) was a polynomial one. Lamb-shifts of the states \( 2p \) and \( 1s \) were calculated for a series of values of the \( a_o/\Lambda \) ratios.

(i) the \( \Delta E^{(2,0)} - \Delta E^{(2,1)} \) and
The most salient feature is that all the maximal-acceleration corrections to the Lamb-shifts were found to be positive. For example, taking the cutoff \( \Lambda \) to be \( a_0/2.5 \), the Lamb-shifts were found to be 6.9 kHz and 50.95 kHz respectively. Comparing these shifts with the experimental values of 1057851(4) kHz and 8172874(60) kHz one obtains the positive-valued ratios

\[
\frac{6.9}{1057851} = 0.65 \times 10^{-5}, \quad \frac{50.95}{8172874} = 0.62 \times 10^{-5}.
\]

Therefore, the fractional corrections to the fine structure constants were of the order of \( 10^{-5} \). Notice that these corrections are positive-valued and coincidentally agree, in orders of magnitude, with the observed changes in Astrophysical observations of the fine structure constant in quasar sources with a positive variation in [11].

The main purpose of this review of [12] is to indicate how, coincidentally, the maximal-acceleration corrections to Lamb-shift of one electron atoms yield fractional changes to the fine structure constant of the same order of magnitude as those provided by Astrophysical observations [11], with the main difference that there is a positive variation in [12]. A Planck scale cutoff yields lower corrections to the Lamb shift as expected. In this case the cutoff \( \Lambda \) cannot be of the order of the Bohr radius [12]. In addition, at Planck scales EM has to be replaced by a more fundamental theory.

### IV Variable-Fine Structure in Astrophysics from Maximal-Acceleration

To explain and derive the observed values of the variable fine structure constant in Astrophysics from this new physical model based on a Maximal-acceleration Phase Space Relativity principle, we must rely on the recent observations that the Universe expansion is in fact accelerating, contrary to past expectations.

To find the maximal-acceleration corrections to the observed variable fine structure in Astrophysics we must recall the maximal-proper-acceleration Lorentz-like dilation factors of sections 2, 3:

\[
d\omega = d\tau \sqrt{1 - \frac{(d^2x_\mu/d\tau^2)(d^2x_\mu/d\tau^2)}{A^2}}.
\]
we interpreted in 2,3 the dilation/contraction factors stemming from the conformal factor associated with the effective Finslerian-like metric:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \sqrt{1 - \frac{g^2(\tau)}{A^2}}. \quad (4.2)$$

Next, we must find the cosmological dilation/contraction factor which is responsible for the variation of the electric charge over cosmological time (redshifts). The crux is to find the explicit relation between the scaling factor in eq-(4-1) and the cosmological redshift factor $1 + z = a(t_0)/a(t_1)$, given by the ratios of the Universe sizes in the present and past epoch. The past and present hypersphere (hyper-pseudopshere) radius $a(t_1)R, a(t_0)R$ associated with the Robertson-Walker line element are given by:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - k(r^2/R^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \quad (4-3)$$

where the parameter $k$ takes the values $0, -1$ for an open and noncompact universe and $k = +1$ for a closed compact one. The size of the universe at any moment of time is represented by the scaling factor $a(t)$ times the characteristic length parameter $R$. The cosmological equations for the expansion rates are obtained after using Einstein gravitational equations [13]:

$$\frac{(d^2a/dt^2)}{a} = \frac{\Lambda}{3} - \frac{4}{3} \pi G (\rho + 3p) \quad (4.4)$$

$$\left(\frac{da/dt}{a}\right)^2 = \frac{8}{3} \pi G \rho \pm \frac{1}{a^2R^2} + \frac{\Lambda}{3}. \quad (4.4)$$

The ± signs in front of the $R$ terms in eq-(4-4) correspond to an open or closed Universe, respectively. If the mean mass density is dominated by nonrelativistic matter [13], then $p << \rho \sim \rho_o(a(t_0)/a(t))^3$. The redshift factor is defined as:

$$1 + z = a(t_0)/a(t). \quad (4.5)$$

where $a(t_0)$ is the present scale size and $a(t)$ is the past scale size of the Universe.

Eqs-(4-4) become:
\[
\frac{\left(\frac{d^2a}{dt^2}\right)}{a} = H_o^2[\Omega_{\Lambda} - \frac{\Omega_m}{2}(1 + z)^3]
\]

\[
\frac{\left(\frac{da}{dt}\right)}{a} = H_o[\Omega_m(1 + z)^3 \pm \Omega_R(1 + z)^2 + \Omega_{\Lambda}]^{1/2}.
\] (4.6)

The fractional contributions to the present value of the Hubble constant \(H_o\) due to the mass density, the radius of curvature and the cosmological constant are respectively:

\[
\Omega_m = \frac{8\pi G \rho_o}{3H_o^2}, \quad \Omega_R = \frac{1}{(H_o a_o R)^2}, \quad a_o R = R_o = R(t_o), \quad \Omega_{\Lambda} = \frac{\Lambda}{3H_o^2}.
\] (4.7)

With

\[\Omega_m \pm \Omega_R + \Omega_{\Lambda} = 1\]

( for an open/closed Universe respectively ).

Rigorously speaking we should take derivatives w.r.t the proper-time rather than the cosmological time \(t\) and instead of solving the Einstein equations one should be solving the full-fledged C-space Gravitational equations which are equivalent to a Higher Derivative Gravity with Torsion. This is a very difficult task. For this reason to get an estimate of the maximal-acceleration corrections to the electric charge as a result of the cosmological expansion we shall study the simplest model above.

Inserting the values \(R(t) = a(t)\overline{R}\) into eqs- ( 4-1, 4-2 ) yields after some straightforward algebra the relativistic factor analog of \(\beta^2 = (v/c)^2\):

\[
\beta^2 = F(z)^2 = \left(\frac{d^2R(t)/dt^2}{A_{\max}}\right)^2 = \left(H_o R(t)\right)^2(H_o L)^2[\Omega_{\Lambda} - \frac{\Omega_m}{2}(1 + z)^3]^2.
\] (4.8)

Where we define the maximal-acceleration in terms of a length cutoff \(L_c\) in units of \(c = 1\):

\[
A_{\max} = \frac{c^2}{L_c} = \frac{1}{L_c}.
\] (4.9)

Such as:
\[ L_{\text{Planck}} < L_e < R(t_{\text{past}}) < R(t_{\text{today}}). \]  

(4.10)

Let us suppose that we wrote the ordinary electrostatic Coulomb energy:

\[ e^2/L. \]  

(4.11)

with the fundamental difference now that the length interval \( L \) is no longer equal to the flat space one, but it changes as a result of an effective metric that is now conformally flat. The scaling factor is dependent on the acceleration:

\[ dL = cd\tau \sqrt{1 - g^2(\tau)A^2}. \]  

(4.12)

For a small and uniform acceleration \( g(\tau) = g = \text{constant} \), we may expand the square root in a powers series and integrate to get:

\[ L \sim c\tau(1 - \frac{g^2}{2A^2}). \]  

(4.13)

\[ \frac{L}{c\tau} \sim 1 - \frac{g^2}{2A^2} \Rightarrow \frac{L - c\tau}{c\tau} \sim -\frac{g^2}{2A^2}. \]  

(4.14)

Hence, in the small and uniform acceleration limit we have fractional length changes:

\[ \frac{\Delta L}{L} = -\frac{g^2}{2A^2}. \]  

(4.15)

When the maximal proper acceleration is taken to \( \infty \) the fractional length change is 0. Since the proper lengths are now scaled by a conformal factor dependent on the acceleration, one could reabsorb such scalings by a suitable scaling of the electric charges, which in turn, will modify the fine structure constant:

\[ \frac{\Delta \alpha}{\alpha} = -\frac{g^2}{2A^2}. \]

And in magnitude we may write:
\[
|\frac{\Delta\alpha}{\alpha}| = \frac{g^2}{2A^2}.
\] (4.16)

Therefore, we propose that the physical mechanism responsible for a variation of the fine structure constant is due to the Maximal-acceleration Relativity principle !. This is the most important conclusion of this work. We have presented a very specific example for a small and uniform acceleration that permits us to expand in a power series and to integrate trivially. In general this is not the case.

The more general expression is:

\[
\frac{\Delta\alpha}{\alpha} = -1 + \frac{c}{(L_1 - L_o)} \frac{1}{\sqrt{1 - g^2(t_o)/A^2}} \int_{t_1}^{t_o} d\tau \sqrt{1 - \frac{g^2(\tau)}{A^2}}. 
\] (4.17)

When the maximal acceleration \( A = \infty \) one gets:

\[
\frac{\Delta\alpha}{\alpha} = \frac{L_1 - L_o}{L_1 - L_o} - 1 = 0.
\] (4.18)

as it should. We shall now use these results to obtain the precise expression for the variations of the fine structure constant within the Cosmological scenario of a Robertson-Walker-Friedmann model with a nonvanishing cosmological constant.

The fundamental equation that governs the cosmological evolution of the fine structure constant as function of the redshift is then:

\[
\frac{\Delta(\alpha)}{\alpha} = -1 + \frac{1}{(t(z_o) - t(z_1))} \frac{1}{\sqrt{1 - F^2(z_o)}} \int_{z_o}^{z_1} \frac{dz}{dz} \left( \frac{dt(z)}{dz} \right) \sqrt{1 - F^2(z)}.
\] (4.19)

where the time integration is taken from the present \( t_o \) to the past \( t_1 \), which can be converted into an integration over the redshift factor from the present \( z_o \) to the past \( z_1 \) by noticing that \( dt > 0 \leftrightarrow dz < 0 \):

\[
dt(z) = \frac{dt(z)}{dz} dz = \frac{dz}{H_o(1 + z)E(z)} \equiv \frac{dz}{H(z)}.
\] (4.20)
where [13]:

\[ E(z) \equiv [\Omega_m(1+z)^3 \pm \Omega_R(1+z)^2 + \Omega_\Lambda]^{1/2} \]

The redshift dependence of the Hubble parameter is:

\[ H(z) \equiv H_0(1+z)E(z). \tag{4.21} \]

naturally:

\[ E(z = 0) = 1 = [\Omega_m \pm \Omega_R + \Omega_\Lambda]^{1/2} \]

The net fractional contribution to the mass-energy density from the three sources has to add up to unity at the present time. The maximal-acceleration correction terms inside the integral were defined in (4-8):

\[ \beta^2 = F^2(z) \equiv [H_o^2 R(z)L_c]^2[\Omega_\Lambda - \frac{\Omega_m}{2}(1+z)^3]^2 \]

with the requirement that: \(0 \leq F^2(z) \leq 1\), otherwise the square root would have been imaginary in (4-19).

The temporal displacement \( t(z_0) - t(z_1) > 0 \) is:

\[ t(z_0) - t(z_1) = \int_{z_0}^{z_1} \frac{dz}{H(z)} > 0, \quad z_1 > z_0. \tag{4.22} \]

The size \( R(t) \equiv a(t)R \) is recast in terms of the redshift:

\[ R(z) = \frac{R(z_0)}{1+z} = \frac{R_o}{1+z}, \quad z_o = 0 \]

\[ \frac{(dR(t)/dt)}{R} = -\frac{dz}{1+z}, \quad \Rightarrow \frac{dR}{R} = -\frac{dz}{1+z}. \tag{4.23} \]

All the terms involved in the fundamental equation (4-19) that furnish the fractional change of the fine structure constant due to the maximal-acceleration relativistic corrections are now defined. Had the maximal acceleration been \( A = \infty \leftrightarrow L_c = 0 \), the equation (4-19) would have yielded a zero fractional change because in this extreme case \( F^2(z) = 0 \), for all \( z \), the integrand of eq-(4-19) becomes unity and one gets after performing the trivial time integral:
\[
\frac{\Delta \alpha}{\alpha} (A = \infty) = \frac{\left[ (t(z_1) - t(z_o))/ (t(z_1) - t(z_o)) \right]}{1} - 1 = 0. \tag{4.24}
\]

We shall study now the range of parameters in order for the fundamental equation to be well defined and furnish sound physical results compatible with observations. The crux of this work is to set the cutoff scale \( R(t) \geq L_c \geq L_{Planck} \) from the requirement that close to the cutoff \( L_c \) the maximal-acceleration corrections were the most dominant. At the present epoch \( z_o = 0, \ R_o = R(z_o) = a_o R \).

The \( \beta^2 = F^2(z) \) factors must obey the maximal acceleration relativistic constraints \( 0 \leq F^2(z) \leq 1 \). Upon using the definition of the Cosmological redshift:

\[
R(z) = \frac{R_o}{1 + z}. \quad R_o = R(z_o) = a_o R. \tag{4.25}
\]

one gets:

\[
0 \leq [H_o^2 R_o L_c]^2 (1 + z)^{-2} [\Omega_{\Lambda} - \frac{\Omega_m}{2} (1 + z)^3]^2 \leq 1. \tag{4.26}
\]

If one requires that the Maximal-acceleration relativistic effects are dominant at the cutoff scale \( L_c \) such that

\[
R_{min} \equiv R(z_{max}(L_c)) = L_c \geq L_P. \tag{4.27a}
\]

the equation which defines the explicit relation between \( z_{max} \) and \( L_c \) is obtained by demanding that the upper limiting acceleration at \( L_c \) gives:

\[
z = z_{max}(L_c) \Rightarrow F^2(z_{max}) = 1 \Rightarrow [H_o^2 R_o L_c]^2 (1 + z_{max})^{-2} [\Omega_{\Lambda} - \frac{\Omega_m}{2} (1 + z_{max})^3]^2 = 1. \tag{4.27b}
\]

The most interesting case is when we set the minimum scale to coincide precisely with the Planck scale: \( R_{min} = R(z_{max}) = L_c = L_P \) that implies that \( z_{max} \to \infty \). If one takes the characteristic scale of the RWF metric \( R \) to coincide with the Hubble radius-horizon (as observed today) \( R = R_H = (1/H_o) \), in units of \( c = 1 \), to be \( 10^{60} - 10^{61} \) Planck lengths = \( 10^{27} - 10^{28} \) cms. In this case it will be meaningless to have parameter values like \( \Omega_m \sim 0.3 \)

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for the simple reason that $\beta^2 = F^2(z) \gg 1$ contrary to the maximal-acceleration relativistic principle that the upper bound on the acceleration cannot be surpassed, where $A_{max} = c^2/L_P$ is the maximum in this case. The $\Omega_m$ in this extreme case has to be basically zero if we wish to satisfy:

$F^2(z) = [H_o L_P]^2 [H_o R_o] (1 + z_{max})^{-2} [\Omega_{\Lambda} - \frac{\Omega_m}{2} (1 + z_{max})^3]^2 \sim 1$. (4.28)

For the choice $R = R_H = (1/H_o)$, the curvature parameter as the scaling factor $a_o \rightarrow \infty$ gives:

$\Omega_R = (\frac{1}{H_o a_o R})^2 = (\frac{1}{a_o})^2 \rightarrow 0$. (4.29)

Hence one concludes that as $z_{max} \rightarrow \infty$ eq-(4-28) becomes:

$[H_o L_P]^4 (\frac{\Omega_m}{2})^2 (z_{max})_6 \sim 1 \Rightarrow \Omega_m \rightarrow 0$. (4.30)

otherwise it would have been impossible to counter-balance the huge factors stemming from the $z_{max}^6$ terms despite the very small factor $[H_o L_P]^4 \sim 10^{-240} - 10^{-244}$.

Therefore in this extreme case scenario because of the condition:

$\Omega_{\Lambda} + \Omega_m \pm \Omega_R = 1$. (4.31)

requires, due to $\Omega_R \rightarrow 0$ and $\Omega_m \rightarrow 0$, that $\Omega_{\Lambda} \rightarrow 1$ ! ! !.

An important remark is in order. One has to be very careful in distinguishing the infinite acceleration case which gives $F^2(z) = 0$, for all values of $z$, and consequently the fractional change is identically $(\Delta \alpha / \alpha) = 0$. And the extreme case scenario $F^2(z_{max}) \sim 1$, for a very particular value of the redshift when the cutoff $L_c = L_P \neq 0$, that yields a very, very large value of $z_{max}$, but not $\infty$. In this case the fractional change after computing the integrals (4-19) is not zero, but in fact, it attains its limiting value in magnitude.

This extreme case $\Omega_{\Lambda} = 1$ is the most interesting because of its implications: Physically it suggests that the universe was an extremely unstable cosmic ? egg ? (or ? atom ?) of Planck size, since there cannot be an
initial point singularity as such due to the minimal Planck scale principle, at the maximum attainable Planck temperature $T_P$, that exploded due to the huge vacuum instability, with an initial speed of $c$ and maximal acceleration $A = c^2/L_P$. It was a true inflationary explosion driven by the enormous vacuum energy $\Omega_\Lambda \sim 1$. Eventually the expansion began to slow down while the acceleration began to decrease, from its maximal value $c^2/L_P$ to the presently observed acceleration, as a result of the ensuing attractive gravitational forces among the emerging material constituents of the Universe. Matter was being created out of the vacuum to halt down the huge initial acceleration $A = c^2/L_P$. In this respect this model is not very different than Hoyle’s steady state cosmology where matter was being created as the Universe expanded in order to maintain the matter density constant.

Closely related to this issue is the cosmological constant problem. Within the framework of the Extended Scale Relativity Theory [1] this is an ill posed problem for the simple reason that in C-spaces the vacuum energy is just one component of a Clifford-valued geometric object, a polyvector, in the same way that the energy is just the component of a four-vector in ordinary Relativity. Therefore, in C-spaces, the vacuum energy is not a constant but it changes under C-space Lorentz transformations. It was shown in [1] why the Conformal group originates from Clifford algebras, henceforth C-space Lorentz transformations contain scalings which imply that the vacuum energy itself is subjected to Renormalization-Group scaling-like flows as the Universe expands. In C-spaces there are two times. The standard coordinate time, and the Stuckelberg-like time represented by the volume of the Universe, the cosmological clock, an arrow of time.

Within the framework of Nottale’s Scale Relativity the cosmological problem is due to the fact that it is meaningless to compare the vacuum energy at two separate scales that differ in $60 - 61$ orders of magnitude, without taking into account scale relativistic corrections, like one does in ordinary Lorentz transformations.

Nottale has shown that the scale relativistic corrections must be such:

$$\frac{\Lambda(\text{Planck})}{\Lambda(R_H)} = \left(\frac{R_H}{L_P}\right)^2 \sim 10^{120} - 10^{122}. \quad (4.32)$$

This explains the origins of such huge orders of magnitude discrepancy between the vacuum energy densities at such different scales.
For a very interesting application of this extreme case scenario, within the context of scaling in cosmology, the arrow of time, the variation of the fundamental constants in Nature and the plausible reason behind the Dirac-Eddington large number coincidences we refer to [14]. The model in [14] is also based on the cosmic egg idea where the Hubble horizon is expanding precisely with the speed of light which suggests that the Universe is the ultimate black hole. If matter truly is being created out of the vacuum, as the Hubble horizon radius grows, its matter content grows accordingly in the same fashion that the Schwarzschild radius grows linearly with the black hole Mass contents inside. There are some important differences. As we said above, the Extended Scale Relativity theory has two times, the ordinary clock time and the scaling cosmological time representing the volume of the Universe. It precludes point singularities as such due to the minimal scale principle, for this reason the Planck temperature is the maximum temperature in Nature; i.e. Hawking evaporation stops at the Planck scale when the black hole attains its limiting Planck temperature [1].

From eq-(4-19), when the maximal-acceleration effects are dominant at the scale $L_c$, we can see why the integral term is less than unity and the fractional change of the fine structure constant is an increasing function of time (decreasing function of $z$). The minus sign in $\Delta \alpha$ indicates that the electric charge $e(t)$ in the past was lower than today $e(t_o)$, and consequently the fine structure constant was lower in the past which is consistent with what has been observed [11].

The present time origin $t_o$ is defined such as whenever one sets $t_1 = t_o \Rightarrow \Delta \alpha = 0$ as required. Since, when $t_1 = t_o$, eq-(4-19) becomes when $z_o = z_1$:

$$\frac{\Delta \alpha}{\alpha} = \left[ \frac{\sqrt{1 - F(z)_o^2}}{\sqrt{1 - F(z)_o^2}} (t_o - t_o)/(t_o - t_o) \right] - 1 = 1 - 1 = 0. \quad (4.33)$$

as it should, by definition.

Collecting all these results we arrive finally at the fundamental equation that yields the cosmological-time variation of the fine structure constant in terms of the cosmological redshift-factors and the cutoff scale $L_c$ such that $R_o > R(z) > R(z_{max}) = R_{\min} = L_c \geq L_{Planck}$.
\[ \frac{(\Delta(\alpha)}{\alpha})(z;L_c) = -1 + \frac{1}{(t_o - t(z))\sqrt{1 - F^2(0)}} \int_0^z \frac{dz}{H(z)\sqrt{1 - F^2(z)}} < 0. \]  

(4.34)

The fractional change is an explicit function of the running variable \( z \) and the cutoff scale \( L_c \) exactly as it occurs in the Standard Renormalization Group methods in QFT. This is not surprising in view of the ultraviolet/infrared entanglement which is the hallmark of Noncommutative Field theories based on Noncommutative Geometry at the Planck scale. The Planck scale \( L_{\text{Planck}} \) pays the role of the Noncommutative parameter of the spacetime coordinates.

\( T = t_o - t(z) \) is the relative age of the Universe measured with respect to the time given by \( t(z) < t_o \):

\[ t_o - t(z) = \int_0^z \frac{dz}{H(z)}. \]  

(4.35)

with [13]:

\[ H(z) \equiv H_o(1 + z)[\Omega_m(1 + z)^3 \pm \Omega_R(1 + z)^2 + \Omega_\Lambda]^{1/2} \]

Naturally when \( z = z_o = 0 \) one gets \( t_o - t_o = 0 \). For the following sequence of scale-orderings:

\[ 0 < L_{\text{Planck}} \leq L_c \leq R(z) < R(z_o) = R_o \]

one has a lower and upper bound on the fractional change:

\[ -1 < x_{\text{Planck}} \leq (\frac{\Delta(\alpha)}{\alpha})(z, L_c) < 0. \]  

(4.36)

meaning that there is an upper bound in the magnitude of the fractional change when \( R(z_{\text{max}}) = R_{\text{min}} = L_c = L_P \).

The fundamental equation (4-34) is written in terms of all the fundamental cosmological parameters. One can then use this fundamental equation for the variation of the fine structure constant to tune in precisely all the fundamental cosmological parameters

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$H_0, \Omega_m, \Omega_\Lambda, R_o, \ldots$ \hspace{1cm} (4.37)

in terms of the cosmological redshift $z$ and the cutoff scale $L_c$, and to check wether or not, one gets sound numerical results. The fact that the maximal-acceleration corrections to the Lamb-shift [12] gave corrections to the fine structure constant of the order of $10^{-5}$ is very encouraging. From this model we derived the fundamental equation above.

Variations of the fine structure constant have been observed over way much smaller redshifts than $z_{max}$. Consequently the cutoff scale $L_c$ must be taken to be larger than the Planck scale to match the present fractional change observations of the order of $-10^{-5}$. In this case, the cutoff scale $L_c$ becomes another cosmological parameter that must be tuned in accordingly to yield sensible and verifiable predictions.

The main point of this work is that if the maximal-acceleration had been infinity, then $\Delta \alpha = 0$ identically! In the future, to be more rigorous and precise, we shall use the full-fledged C-space Gravitational equations of motion which are tantamount to a Higher derivative Gravity with Torsion plus a C-space generalization of Yang-Mills and Electrodynamics [1]. i.e we must modify entirely the present Cosmological models in order to understand the Planck scale regime of the Universe. Current models based on conventional theories simply do not work. We require then to quantize the C-space Extended Relativistic Field theory using tools based on Braided Hopf Quantum Clifford algebras, q-deformations of Clifford algebras, for example.

In previous work we evaluated the running Planck constant due to the Extended Scale Relativistic Effects in C-spaces and which was responsible for the string/brane minimal length/time uncertainty relations [1]:

$$\hbar_{\text{eff}}(E) = \hbar_o \frac{\sinh(L_P E)}{L_P E}. \quad \hbar_o = c = 1. \hspace{1cm} (4.38)$$

when $L_P \to 0$, or at very low energies, the $\hbar_{\text{eff}} = \hbar_o$ as it should. At Planck's energy we have $EL_P \sim 1$:

$$\hbar_{\text{eff}}(\text{Planck}) = \hbar_o \sinh(1). \hspace{1cm} (4.39)$$

The fine structure constant is then:
\[
\frac{e^2}{\hbar_{\text{eff}}(\text{Planck})c} < \frac{e^2}{\hbar_0c} = \frac{1}{137}.
\]

(4.40)

This means that the fine structure would have been smaller at lower scales, for \textit{fixed} value of the Cosmological time = scaling size of the Universe. To compensate for the running value of \(\hbar\) that would have induced a lower value of the fine structure constant, for \textit{fixed} value of the cosmological clock, we must have a running value of the electric charge (the speed of light is unaltered):

\[
\frac{e_{\text{eff}}^2(\text{Planck})}{\hbar_{\text{eff}}(\text{Planck})c} = \frac{1}{137} \frac{1}{\sinh(1)} \frac{e_{\text{eff}}^2(\text{Planck})}{e^2} = \frac{1}{4\pi^2}.
\]

(4.41)

where we used the Nottale value for the fine structure constant at the Planck scale of \(1/4\pi^2\) From eq- (4-41) we get a fractional increase of the electric charge:

\[
\frac{e_{\text{eff}}^2(\text{Planck})}{e^2(m_e)} = \frac{137}{4\pi^2} \sinh(1) > 1.
\]

(4.42)

which is indeed consistent with the standard Renormalization Group arguments in QED with the extra modification of the \(\sinh(1)\) factor.

Concluding: The fine structure constant can vary either:

- by changing the value of the Cosmological clock = scale size of the Universe and, as we have shown, its value was lower in the past than it is today due to the Maximal-Acceleration Relativistic effects. Had the maximal acceleration been infinity, the \(\Delta \alpha/\alpha = 0\)! The maximal-acceleration corrections to the Lamb-shift were calculated by [12] and yield fractional changes to the fine structure constant of the order of \(10^{-5}\).

or

- by increasing the energy (probing smaller distances) from the electron’s Compton wavelength to the Planck’s scale, keeping fixed the Cosmological clock.

We believe that these two last points are highly nontrivial and may reveal new Physics in the horizon. In particular, we must modify the current Quantum Field Theories to be able to incorporate the maximal-acceleration relativistic effects. Beginning, for example, by studying the appropriate
Noncommutative Quantum Mechanics due to the Extended Scale Relativistic dynamics in C-spaces (Clifford manifolds) [1] and the canonical groups (with their representations) acting on Noncommutative extended phase spaces [16].

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