FERMIONS IN THE BACKGROUND OF THE SPHALERON BARRIER

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Abstract

We demonstrate the level crossing phenomenon for fermions in the background field of the sphaleron barrier, by numerically determining the fermion eigenvalues along the minimal energy path from one vacuum to another. We assume that the fermions of a doublet are degenerate in mass, allowing for spherically symmetric ansätze for all of the fields, when the mixing angle dependence is neglected.
1 Introduction

In 1976 ’t Hooft [1] observed that the standard model does not absolutely conserve baryon and lepton number due to the Adler-Bell-Jackiw anomaly. The process ’t Hooft considered was spontaneous fermion number violation due to instanton induced transitions. Fermion number violating tunnelling transitions between topologically distinct vacua might indeed be observable at high energies at future accelerators [2,3].

The possibility of fermion number violation in the standard model was considered from another point of view by Manton [4]. Investigating the topological structure of the configuration space of the Weinberg-Salam theory, Manton showed that there are noncontractible loops in configuration space, and predicted the existence of a static, unstable solution of the field equations, a sphaleron [5], representing the top of the energy barrier between topologically distinct vacua.

At finite temperature this energy barrier between topologically distinct vacua can be overcome due to thermal fluctuations of the fields, and fermion number violating vacuum to vacuum transitions involving changes of baryon and lepton number can occur. The rate for such baryon number violating processes is largely determined by a Boltzmann factor, containing the height of the barrier at a given temperature and thus the energy of the sphaleron. Baryon number violation in the standard model due to such transitions over the barrier may be relevant for the generation of the baryon asymmetry of the universe [6-10].

How can baryon and lepton number change when the barrier between topologically distinct vacua is traversed? The answer is seen in the level crossing picture. Let us consider a process which starts in the vacuum sector labelled by the Chern-Simons number $N_{CS}$. During the process the barrier is traversed. The Chern-Simons number changes continuously, ending in the vacuum sector $N_{CS} - 1$. Let us assume a filled Dirac sea in the first vacuum. While the gauge and Higgs field configurations slowly change and with them the Chern-Simons charge, the fermion levels also change. When the bosonic configurations reach the top of the barrier, the sphaleron with Chern-Simons charge $1/2$, one fermion level of the sea has precisely reached zero energy, and when the bosonic fields reach the next vacuum configuration, this occupied energy level has dived out of the Dirac sea.

In this letter we demonstrate the level crossing phenomenon for fermions in the background field of the sphaleron barrier, by numerically determining the fermion eigenvalues along the minimal energy path from one vacuum to another [11,12]. We assume that the fermions of a doublet are degenerate in mass. This assumption, violated in the standard model, allows for spherically symmetric ansätze for all of the fields, when the mixing angle dependence is neglected (which is an excellent approximation [13,14]). At the top of the barrier, in the background field of the sphaleron, the fermions reach a zero mode [15-17].
We briefly review in section 2 the Weinberg-Salam lagrangian with the approximations employed. In section 3 we present the sphaleron energy barrier, providing the background field for the fermions. In section 4 we derive the radial equations for the fermions, and we present our results in section 5.

2 Weinberg-Salam Lagrangian

Let us consider the bosonic sector of the Weinberg-Salam theory in the limit of vanishing mixing angle. In this limit the U(1) field decouples and can consistently be set to zero

\[ \mathcal{L}_b = -\frac{1}{4} F^{a \mu \nu} F_{\mu \nu}^a + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - \frac{v^2}{2})^2 \]  

with the SU(2)$_L$ field strength tensor

\[ F^{a \mu \nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + \frac{i}{2} g \epsilon^{a bc} V^b_\mu V^c_\nu, \]  

and the covariant derivative for the Higgs field

\[ D_\mu \Phi = \left( \partial_\mu - \frac{i}{2} g \tau^a V^a_\mu \right) \Phi. \]  

The SU(2)$_L$ gauge symmetry is spontaneously broken due to the non-vanishing vacuum expectation value \( v \) of the Higgs field

\[ \langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]  

leading to the boson masses

\[ M_W = M_Z = \frac{1}{2} g v, \quad M_H = v \sqrt{2 \lambda}. \]  

We employ the values \( M_W = 80\text{GeV}, \ g = 0.67. \)

For vanishing mixing angle, considering only fermion doublets degenerate in mass, the fermion lagrangian reads

\[ \mathcal{L}_f = \bar{q}_L i \gamma^\mu D_\mu q_L + \bar{q}_R i \gamma^\mu D_\mu q_R - f^{(q)} \bar{q}_L (\tilde{\Phi} u_R + \Phi d_R) - f^{(q)} (\bar{d}_R \Phi^\dagger + \bar{u}_R \tilde{\Phi}) q_L, \]  

where \( q_L \) denotes the lefthanded doublet \((u_L, d_L)\), while \( q_R \) abbreviates the righthanded singlets \((u_R, d_R)\), with covariant derivative

\[ D_\mu q_L = \left( \partial_\mu - \frac{i}{2} g \tau^a V^a_\mu \right) q_L. \]
and with $\tilde{\Phi} = i\tau_2 \Phi^*$. The fermion mass is given by

$$M_F = \frac{1}{\sqrt{2}} f^{(q)} v .$$  \hfill (8)

All gauge field configurations can be classified by a charge, the Chern-Simons charge. The Chern-Simons current

$$K_\mu = \frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(F_{\nu\rho} V_{\sigma} + \frac{2}{3} i g V_{\nu} V_{\rho} V_{\sigma})$$  \hfill (9)

($F_{\nu\rho} = 1/2\tau^i F^i_{\nu\rho}$, $V_{\sigma} = 1/2\tau^i V^i_{\sigma}$) is not conserved, its divergence $\partial^\mu K_\mu$ represents the U(1) anomaly. The Chern-Simons charge of a configuration is given by

$$N_{\text{CS}} = \int d^3 r K^0 .$$  \hfill (10)

For the vacua the Chern-Simons charge is identical to the integer winding number, while the barriers are characterized by a half integer Chern-Simons charge.

### 3 Sphaleron Energy Barrier

The height of the barrier can be obtained by constructing families of field configurations for the gauge and Higgs fields, which interpolate smoothly from one vacuum to another as a function of the Chern-Simons charge. Each of these families of configurations has a maximal energy along such a path. By finding the minimal value of these maximal energies one has found the height of the barrier, the sphaleron [4,5].

In the limit of vanishing mixing angle the general static, spherically symmetric ansatz for the gauge and Higgs fields is given by [18]

$$\Phi = \frac{v}{\sqrt{2}} \begin{pmatrix} H(r) + i\hat{r} \cdot \hat{r} K(r) \\ 0 \end{pmatrix} ,$$  \hfill (11)

$$V^a_i = \frac{1 - f_A(r)}{gr} \epsilon_{aij} \hat{r}_j + \frac{f_B(r)}{gr} (\delta_{ia} - \hat{r}_i \hat{r}_a) + \frac{f_C(r)}{gr} \hat{r}_i \hat{r}_a ,$$  \hfill (12)

$$V^a_0 = 0 ,$$  \hfill (13)

and involves the five radial functions $H(r)$, $K(r)$, $f_A(r)$, $f_B(r)$ and $f_C(r)$.

This ansatz leads to the energy functional

$$E = \frac{4\pi M_W}{g^2} \int_0^{\infty} dx \left[ \frac{1}{2x^2} (f^2_A + f^2_B - 1)^2 + (f'_A + \frac{f_B f_C}{x})^2 + (f'_B - \frac{f_A f_C}{x})^2 
+ (K^2 + H^2)(1 + f^2_A + f^2_B + \frac{f^2_C}{2}) + 2 f_A (K^2 - H^2) - 4 f_B H K 
+ 2x^2 (H'^2 + K'^2) - 2 x f_C (K'H - K H') + \frac{4\lambda}{g^2 x^2} (H^2 + K^2 - 1)^2 \right] ,$$  \hfill (14)
where \( x = M_Wr \), and to the Chern-Simons number

\[
N_{CS} = \frac{1}{2\pi} \int_0^\infty \ dx \left[ (f_A^2 + f_B^2)(\frac{f_C}{x} - \varphi') - (\frac{f_C}{x} - \Theta') - \left( \sqrt{(f_A^2 + f_B^2)} \sin(\varphi - \Theta) \right)' \right]
\]

with \( \varphi = \arctan(f_B/f_A) \). The function \( \Theta(x) \) is an arbitrary radial function, associated with the residual gauge invariance of the ansatz (11)-(13). This gauge freedom can be used to eliminate one of the functions. Here we choose the radial gauge with the gauge condition \( f_C(x) = 0 \).

Let us now consider families of configurations, which connect one vacuum \( (N_{CS} = 0) \) with another vacuum \( (N_{CS} = 1) \) passing the sphaleron \( (N_{CS} = 1/2) \). Note, that the Chern-Simons number of the sphaleron is independent of the Higgs mass, \( N_{CS} = 1/2 \) [5]. For this purpose we extremize the functional [11,12]

\[
W = E + \frac{8\pi M_W}{g^2} \xi N_{CS},
\]

where \( \xi \) is a lagrange multiplier. The minimal energy path constructed accordingly for \( M_H = M_W \) is shown in Fig. 1. This path is symmetric with respect its top, the sphaleron. For large values of the Higgs mass additional less symmetric sphaleron solutions, bisphalerons, appear [19,20]. The first bisphaleron takes over the role of the sphaleron. It represents the top of an asymmetric barrier [12,21], having a Chern-Simons number different from \( 1/2 \).

## 4 Fermion Equations

Let us now consider the fermions in the background of the sphaleron barrier. To retain spherical symmetry we consider only fermion doublets degenerate in mass. From the fermion lagrangian (7) we obtain the eigenvalue equations for the lefthanded doublet

\[
iD_0q_L + i\sigma^iD_iq_L - f^{(q)}(\Phi u_R + \Phi d_R) = 0,
\]

and for the righthanded singlets

\[
i\partial_0 \left( \frac{u_R}{d_R} \right) - i\sigma^i\partial_i \left( \frac{u_R}{d_R} \right) - f^{(q)} \left( \Phi^\dagger q_L \right) = 0.
\]

Employing the spherically symmetric ansatz for the fermion eigenstates, the hedgehog ansatz,

\[
q_L(\vec{r},t) = e^{-iwt}(G_L(r) + i\vec{\sigma} \cdot \hat{r} F_L(r))\chi_h,
\]

\[
q_R(\vec{r},t) = e^{-iwt}(G_R(r) + i\vec{\sigma} \cdot \hat{r} F_R(r))\chi_h,
\]
with the hedgehog spinor satisfying the spin-isospin relation \( \vec{\sigma} \chi_h + \vec{\tau} \chi_h = 0 \), we obtain the following set of four coupled first order differential equations

\[
\tilde{\omega} G_L - F'_L - \frac{2}{x} F_L + \frac{1-f_A}{x} F_L - \frac{f_B}{x} G_L - \frac{f_C}{2x} G_L - \tilde{M}_F (H G_R + K F_R) = 0 , \tag{21}
\]

\[
\tilde{\omega} F_L + G'_L + \frac{1-f_A}{x} G_L + \frac{f_B}{x} F_L - \frac{f_C}{2x} F_L - \tilde{M}_F (H F_R - K G_R) = 0 , \tag{22}
\]

\[
\tilde{\omega} G_R + F'_R + \frac{2}{x} F_R - \tilde{M}_F (H G_L - K F_L) = 0 , \tag{23}
\]

\[
\tilde{\omega} F_R - G'_R - \tilde{M}_F (H F_L + K G_L) = 0 , \tag{24}
\]

where \( x \) is the dimensionless coordinate, \( \tilde{\omega} \) is the dimensionless eigenvalue \( \tilde{\omega} = \omega/M_W \) and \( \tilde{M}_F \) is the dimensionless fermion mass \( \tilde{M}_F = M_F/M_W \). (Remember the gauge choice \( f_C = 0 \).)

The eigenvalue problem (21)-(24) for the fermions in a sphaleron-like background field requires certain boundary conditions for the fermion functions. At the origin \( G_L(x) \) and \( G_R(x) \) are finite, while \( F_L(x) \) and \( F_R(x) \) vanish, at spatial infinity all functions vanish.

### 5 Results

In the background field of the sphaleron the fermions have a zero mode, i.e. a normalizable eigenstate with zero eigenvalue. In this case, the two functions \( F_L(x) \) and \( F_R(x) \) decouple and are identically equal to zero. When the mass of the fermions vanishes, also \( G_R(x) \) decouples and the zero mode can be given analytically [15-17]. The normalized eigenfunctions are shown in Fig. 2 for fermion masses of \( M_f = 80 \) GeV, 8 GeV and 0.8 GeV. In the background field of the biphazerons the fermions do not have a zero mode, in fact, the fermion eigenvalue depends on the Higgs mass [22].

When the fermion eigenvalue equations are solved in the background field of the sphaleron barrier, given by the minimal energy path discussed above, the level crossing phenomenon is observed. Since the barrier is symmetric about the sphaleron, the fermion eigenvalue \( \omega \) is antisymmetric with respect to the sphaleron configuration. In Fig. 3 we represent the fermion eigenvalue along the barrier for fermion masses of \( M_f = 80 \) GeV, 8 GeV and 0.8 GeV. We observe that light fermions are bound only close to the top of the barrier, the sphaleron configuration, while heavy fermions are bound almost along the full path along the barrier. Denoting by \( M_{f\ell}^c \) the fermion mass, at which for a given Chern-Simons number, the fermion bound state enters the continuum, this observation is illustrated also in Fig. 4, where the critical fermion mass \( M_{f\ell}^c \) is shown as a function of the Chern-Simons number \( N_{CS} \). At zero mass, fermions are bound only by the sphaleron.
6 References

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7 Figure Captions

Figure 1
The Energy $E$ in TeV is shown as a function of the Chern-Simons number $N_{CS}$ along the minimal energy path from one vacuum to another vacuum for $M_H = M_W$.

Figure 2
The fermion zero mode wavefunction components $G_L(x)$ (positive) and $G_R(x)$ (negative) are shown for fermion masses $M_f = 80$ GeV (solid), $M_f = 8$ GeV (dashed), and $M_f = 800$ MeV (dotted) for $M_H = M_W$.

Figure 3
The normalized fermion eigenvalue $\omega/M_f$ is shown as a function of the Chern-Simons number $N_{CS}$ along the minimal energy path from one vacuum to another vacuum for $M_H = M_W$ for the fermions masses $M_f = 80$ GeV, $M_f = 8$ GeV and $M_f = 800$ MeV.

Figure 4
The critical fermion mass $M^c_f$ (in GeV) at which the bound state enters the continuum, for a given Chern-Simons number, is shown as a function of the Chern-Simons number $N_{CS}$ for $M_H = M_W$. 