Parametric Resonant Phenomena in Bose-Einstein Condensates:  
Breaking of Macroscopic Quantum Self-Trapping

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Abstract

We analyze the periodic tunneling of a Bose-Einstein condensate in a double-well potential which has an oscillating energy barrier. We show that the dynamics of the Bose condensate critically depends on the frequency $\omega$ of the oscillating energy barrier. In the regime of periodic macroscopic quantum tunneling (PMQT) with frequency $\omega_J$, the population imbalance of the condensate in the two wells can be enhanced under the condition of parametric resonance $\omega = 2\omega_J$. Instead, in the regime of macroscopic quantum self-trapping (MQST), we find that MQST can be reduced or suppressed under the condition of parametric resonance between the frequency $\omega$ of the energy barrier and the frequency $\omega_{ST}$ of oscillation through the barrier of the very small fraction of particles which remain untrapped during MQST.
I. INTRODUCTION

Macroscopic quantum tunneling with dilute Bose-Einstein condensates of alkali-metal atoms has been the subject of many theoretical [1-4] and experimental [5-7] studies. Here we consider the problem of trapped Bosons in a double-well potential and investigate the effect of a periodically varying barrier in the tunneling of the Bose condensate. We study the problem using the two-mode classical-like equations [1,2] and find that remarkable effects are related to parametric resonance [8] both in the periodic macroscopic quantum tunneling regime and in the macroscopic quantum self-trapping regime.

II. TUNNELING IN A DOUBLE-WELL POTENTIAL

In a dilute gas of \( N \) Bosons at zero temperature practically all particles are in the same single-particle state of the density matrix \( \rho(\mathbf{r}, \mathbf{r}'; t) \) [9]. This macroscopically occupied single-particle state is called Bose-Einstein condensate and its wavefunction \( \psi(\mathbf{r}, t) \) is well described by the Gross-Pitaevskii equation

\[
\frac{i\hbar}{\partial t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + g|\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t) ,
\]

where \( U(\mathbf{r}) \) is the external potential, \( g = 4\pi\hbar^2a_s/m \) is the inter-atomic strength with \( a_s \) the s-wave scattering length, and the wavefunction is normalized to \( N \) [10].

Let us consider a double-well external potential \( U(\mathbf{r}) \). In the tunneling regime, the wavefunction of the Bose-condensate can be approximated in the following way

\[
\psi(\mathbf{r}, t) = \sqrt{N_1(t)}e^{i\phi_1(t)}\psi_1(\mathbf{r}) + \sqrt{N_2(t)}e^{i\phi_2(t)}\psi_2(\mathbf{r}) ,
\]

where \( N_1(t) \) is the number of particles in the first well and \( N_2(t) \) is the number of particles in the second well, such as \( N = N_1(t) + N_2(t) \) [1]. If the double-well is symmetric, it is not difficult to show that the time-dependent behavior of the condensate in the tunneling energy range can be described, with a suitable rescaling of time, by the two-mode equations

\[
\dot{\zeta} = -\sqrt{1-\zeta^2} \sin \phi , \quad \dot{\phi} = \Lambda \zeta + \frac{\zeta}{\sqrt{1-\zeta^2}} \cos \phi ,
\]
where $\zeta = (N_1 - N_2)/N$ is the fractional population imbalance of the condensate in the two wells, $\phi = \phi_1 - \phi_2$ is the relative phase (which can be initially zero), and $\Lambda = 2E_I/E_T$ with $E_I$ the inter-atomic energy and $E_T$ the tunneling energy, i.e. the kinetic+potential energy splitting between the ground state and the quasi-degenerate odd first excited state of the GP equation [1,2]. Note that $E_T = \hbar \omega_0$, where $\omega_0$ is the oscillation frequency of the Bose condensate between the two wells when the inter-atomic interaction is zero ($E_I = 0$) [1-4].

The two-mode equations have been studied by many authors and they are the Hamilton equations of the following Hamiltonian

$$H = \frac{\Lambda}{2} \zeta^2 - \sqrt{1 - \zeta^2} \cos \phi,$$  \hspace{1cm} (4)

where $\zeta$ is the conjugate momentum of the generalized coordinate $\phi$. Actually, it is possible to introduce [1] an equivalent Hamiltonian, given by

$$H' = \frac{1}{2} p_\zeta^2 + a \zeta^2 + b \zeta^4,$$  \hspace{1cm} (5)

where

$$a = \frac{1}{2} \sqrt{1 - \Lambda H_0}, \hspace{1cm} b = \frac{\Lambda^2}{8},$$  \hspace{1cm} (6)

with $H_0 = \Lambda \zeta(0)^2/2 - \sqrt{1 - \zeta(0)^2} \cos \phi(0)$. For the Hamiltonian $H'$ the generalized coordinate is $\zeta$ and its conjugate momentum is $p_\zeta$. The equations of motion derived by $H'$ are

$$\dot{\zeta} = p_\zeta, \hspace{1cm} \dot{p}_\zeta = -2a\zeta - 4b\zeta^3.$$  \hspace{1cm} (7)

The analysis of the two-mode equations (3) or (7) has shown that for $\Lambda < 2$ there is a periodic macroscopic quantum tunneling (PMQT) of the Bose-condensate with Josephson-like oscillations of frequency $\omega_J = \omega_0 \sqrt{1 + \Lambda}$ [1,2]. Instead, for $\Lambda > 2$, there exists a critical $\zeta_c = 2\sqrt{\Lambda - 1}/\Lambda$ such that for $0 < \zeta(0) << \zeta_c$ there is PMQT of condensate with frequency $\omega_J$, but for $\zeta_c < \zeta(0) \leq 1$ there is macroscopic quantum self-trapping (MQST) of the condensate: even if the populations of the two wells are initially set in an asymmetric state ($\zeta(0) \neq 0$) they maintain, on the average, the original population imbalance with a very small periodic transfer of particles through the barrier with frequency $\omega_{ST} = \omega_0 \sqrt{2(\Lambda H_0 - 1)}$ [1,2].
III. AN OSCILLATING BARRIER IN THE DOUBLE-WELL POTENTIAL

In this section we analyze the effect of a periodic oscillating energy barrier in the double-well potential. The presence of a oscillating barrier can be modelled by a time-dependent $\Lambda$. We choose the following form

$$\Lambda(t) = \Lambda_0 (1 + \epsilon \sin(\omega t)),$$

where $\Lambda_0$ is the static value, $\epsilon$ is a small perturbation ($\epsilon \ll 1$) and $\omega$ is the oscillation frequency of the barrier.

We numerically solve the two-mode equations (7) with (8) by using a fourth-order Runge-Kutta algorithm. First we consider the case of pulsed macroscopic quantum tunneling (PMQT). We set $\Lambda_0 = 0$, $\zeta(0) = 0.5$ and $\phi(0) = 0$. In Figure 1 we plot the population imbalance $(t, \zeta)$ and its phase-space portrait $(\zeta, p_\zeta)$ for some values of $\epsilon$ and $\omega$. For $\epsilon = 0$ the population imbalance harmonically oscillates between the values $-0.5$ and $0.5$ with frequency $\omega_J = 1.27$, but for $\epsilon \neq 0$ the motion is not fully harmonic. In particular, Figure 1 shows that under the parametric resonance condition $\omega = 2\omega_J$ the dynamics of the population imbalance is modified: the amplitude of the oscillation is modulated and reaches the values $-1$ and $1$.

Now we consider the case of macroscopic quantum self-trapping (MQST). We set $\Lambda_0 = 25$, $\zeta(0) = 0.6$ and $\phi(0) = 0$. Figure 2 confirms that for $\epsilon = 0$ the population imbalance does not change sign and it periodically oscillates between the values $0.47$ and $0.6$ with frequency $\omega_{ST} = 13.53$. As shown in Figure 2, MQST is strongly affected by the parametric resonance between the frequency $\omega_{ST}$ of MQST oscillation and the frequency $\omega$ of oscillation of the energy barrier. Moreover, Figure 3 shows that at the parametric resonance condition $\omega = 2\omega_{ST}$ and with a sufficiently large perturbation ($\epsilon = 0.2$) the system eventually escapes from the self-trapping configuration. Note that this remarkable effect has been confirmed [11] by using the nonpolynomial Schrödinger equation, an effective one-dimensional equation derived from the three-dimensional Gross-Pitaevskii equation [12].
CONCLUSIONS

We have studied the periodic tunneling and the quantum self-trapping of a Bose-Einstein condensate in a double-well potential with an oscillating energy barrier. We have used the two-mode classical-like equations to find that the periodic macroscopic quantum tunneling can be enhanced and the macroscopic quantum self-trapping can be reduced or suppressed by changing the frequency of the oscillating barrier. In the latter case, we have proved that the system escapes from the self-trapping configuration if the the frequency of oscillation of the double-well energy barrier and the frequency of MQST oscillations of the condensate satisfy the parametric resonance condition with a sufficiently large perturbation of the energy barrier.

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FIG. 1. PMQT regime. Fractional population imbalance $\zeta(t)$ and its phase-space portrait $(\zeta, p_\zeta)$. Initial conditions: $\zeta(0) = 0.5, \phi(0) = 0$. $\Lambda = \Lambda_0(1 + \epsilon \sin (\omega t))$ where $\Lambda_0 = 0.8$. From top to bottom: (a) $\epsilon = 0$; (b) $\epsilon = 0.25$ and $\omega = \omega_J$; (c) $\epsilon = 0.25$ and $\omega = 2\omega_J$; where $\omega_J$ is the frequency of unperturbed PMQT oscillations.
FIG. 2. MQST regime. Fractional population imbalance $\zeta(t)$ and its phase-space portrait $(\zeta, p_{\zeta})$. Initial conditions: $\zeta(0) = 0.6$, $\phi(0) = 0$. $\Lambda = \Lambda_0(1 + \epsilon \sin(\omega t))$ where $\Lambda_0 = 25$. From top to bottom: (a) $\epsilon = 0$; (b) $\epsilon = 0.1$ and $\omega = \omega_{ST}/2$; (c) $\epsilon = 0.1$ and $\omega = 2\omega_{ST}$; where $\omega_{ST}$ is the frequency of unperturbed MQST oscillations.
FIG. 3. MQST regime at the parametric resonance. Fractional population imbalance $\zeta(t)$ of the Bose condensate. Initial conditions: $\zeta(0) = 0.6, \phi(0) = 0$. $\Lambda = \Lambda_0(1 + \epsilon \sin(\omega t))$ where $\Lambda_0 = 25$ and $\omega = 2\omega_{ST}$ with $\omega_{ST}$ the frequency of unperturbed MQST oscillations. From top to bottom and from left to right: (a) $\epsilon = 0.01$; (b) $\epsilon = 0.05$; (c) $\epsilon = 0.1$; (d) $\epsilon = 0.2$. 