Equilibrium diffusion on the cone of discrete Radon measures

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Abstract

Let $\mathbb{K}(\mathbb{R}^d)$ denote the cone of discrete Radon measures on $\mathbb{R}^d$. There is a natural differentiation on $\mathbb{K}(\mathbb{R}^d)$: for a differentiable function $F : \mathbb{K}(\mathbb{R}^d) \to \mathbb{R}$, one defines its gradient $\nabla^\mathbb{K} F$ as a vector field which assigns to each $\eta \in \mathbb{K}(\mathbb{R}^d)$ an element of a tangent space $T_\eta(\mathbb{K}(\mathbb{R}^d))$ to $\mathbb{K}(\mathbb{R}^d)$ at point $\eta$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a potential of pair interaction, and let $\mu$ be a corresponding Gibbs perturbation of (the distribution of) a completely random measure on $\mathbb{R}^d$. In particular, $\mu$ is a probability measure on $\mathbb{K}(\mathbb{R}^d)$ such that the set of atoms of a discrete measure $\eta \in \mathbb{K}(\mathbb{R}^d)$ is $\mu$-a.s. dense in $\mathbb{R}^d$. We consider the corresponding Dirichlet form

$$E^\mathbb{K}(F,G) = \int_{\mathbb{K}(\mathbb{R}^d)} \langle \nabla^\mathbb{K} F(\eta), \nabla^\mathbb{K} G(\eta) \rangle_{T_\eta(\mathbb{K})} d\mu(\eta).$$

Integrating by parts with respect to the measure $\mu$, we explicitly find the generator of this Dirichlet form. By using the theory of Dirichlet forms, we prove the main result of the paper: If $d \geq 2$, there exists a conservative diffusion process on $\mathbb{K}(\mathbb{R}^d)$ which is properly associated with the Dirichlet form $E^\mathbb{K}$.

Keywords: Completely random measure, diffusion process, discrete Radon measure, Dirichlet form, Gibbs measure

MSC: 60J60, 60G57
1 Introduction

Let $X$ denote the Euclidean space $\mathbb{R}^d$ and let $\mathcal{B}(X)$ denote the Borel $\sigma$-algebra on $X$. Let $\mathcal{M}(X)$ denote the space of all Radon measures on $(X, \mathcal{B}(X))$. The space $\mathcal{M}(X)$ is equipped with the vague topology, and let $\mathcal{B}(\mathcal{M}(X))$ denote the corresponding Borel $\sigma$-algebra on it. A random measure on $X$ is a measurable mapping $\xi : \Omega \rightarrow \mathcal{M}(X)$, where $(\Omega, \mathcal{F}, P)$ is a probability space, see e.g. [8]. A random measure $\xi$ is called completely random if, for any mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{B}(X)$, the random variables $\xi(A_1), \ldots, \xi(A_n)$ are independent [9].

The cone of discrete Radon measures on $X$ is defined by

$$K(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathcal{M}(X) \mid s_i > 0, x_i \in X \right\}.$$ 

Here $\delta_{x_i}$ denotes the Dirac measure with mass at $x_i$. In the above representation, the atoms $x_i$ are assumed to be distinct and their total number is at most countable. By convention, the cone $K(X)$ contains the null mass $\eta = 0$, which is represented by the sum over an empty set of indices $i$. As shown in [6], $K(X) \in \mathcal{B}(\mathcal{M}(X))$. One endows $K(X)$ with the vague topology.

A random measure $\xi$ which takes values in $K(X)$ with probability one is called a random discrete measure. It follows from Kingman’s result [9] that each completely random measure $\xi$ can be represented as $\xi = \xi' + \eta$, where $\xi'$ is a deterministic measure on $X$ and $\eta$ is a random discrete measure. An important example of a random discrete measure is the gamma measure [19], which has many distinguished properties. It should be noted that, for a wide class of random discrete measures (including the gamma measure), the set of atoms of $\eta = \sum_i s_i \delta_{x_i}$, i.e., $\{x_i\}$, is dense in $X$.

In this paper, we will only use the distribution $\mu$ of a random discrete measure. So, below by a random discrete measure we will always mean a probability measure $\mu$ on $(K(X), \mathcal{B}(K(X)))$. (Here $\mathcal{B}(K(X))$ is the Borel $\sigma$-algebra on $K(X)$.)

In [6] Gibbs perturbations of the gamma measure were constructed, and in [16] this result was extended to Gibbs perturbations of a general completely random discrete measure. More precisely, let $\phi : X \times X \rightarrow \mathbb{R}$ be a potential of pair interaction, which satisfies the conditions (C1), (C2) below. In particular, it is assumed that the function $\phi$ is symmetric, bounded, has finite range (i.e., $\phi(x, x') = 0$ if the distance between $x$ and $x'$ is sufficiently large), and the positive part of $\phi$ dominates, in a sense, its negative part. For $\eta \in K(X)$, we heuristically define the energy of $\eta$ (Hamiltonian) by

$$H(\eta) := \frac{1}{2} \int_{X^2 \setminus D} \phi(x, x') \, d\eta(x) \, d\eta(x'),$$ 

where $D = \{(x, x') \in X^2 \mid x = x'\}$. Let $\nu$ be a completely random discrete measure. The Gibbs perturbation of $\nu$ corresponding to the potential $\phi$ is heuristically defined
as a probability measure $\mu$ on $\mathbb{K}(X)$ given by

$$d\mu(\eta) := \frac{1}{Z} e^{-H(\eta)} d\nu(\eta),$$

where $Z$ is a normalizing factor. A rigorous definition of $\mu$ is given through the Dobrushin–Lanford–Ruelle equation. It is proven in [6] that such a Gibbs measure exists. In [16], it was shown that such a Gibbs measure is unique, provided the supremum norm of $\phi$, i.e., $\|\phi\|_{\infty}$, and the first moment of $\nu$ are sufficiently small. In the general case, the uniqueness problem is still open.

Any Gibbs measure $\mu$ satisfies the Nguyen–Zessin identity in which the relative energy of interaction between a single atom measure $\eta = s\delta_x$ and a discrete measure $\eta' \in \mathbb{K}(X)$, with no atom at $x$, is given by

$$H(\eta \mid \eta') = s \int_X \phi(x, x') d\eta'(x').$$

In [10] (see also [7]), some elements of differential geometry on $\mathbb{K}(X)$ were introduced. In particular, for a differentiable function $F : \mathbb{K}(X) \to \mathbb{R}$, one defines its gradient $\nabla^\mathbb{K} F$ as a vector field which assigns to each $\eta \in \mathbb{K}(X)$ an element of a tangent space $T_\eta(\mathbb{K}(X))$ to $\mathbb{K}(X)$ at point $\eta$. It should be stressed that $\mathbb{K}(X)$ is not a flat space, in the sense that the tangent space $T_\eta(\mathbb{K})$ changes with a change of $\eta$.

So, in this paper, we consider the Dirichlet form

$$\mathcal{E}^\mathbb{K}(F, G) := \int_{\mathbb{K}(\mathbb{R}^d)} \langle \nabla^\mathbb{K} F(\eta), \nabla^\mathbb{K} G(\eta) \rangle_{T_\eta(\mathbb{K})} d\mu(\eta).$$

This bilinear form is initially defined on an appropriate set of smooth cylinder functions on $\mathbb{K}(X)$. Using the Nguyen–Zessin identity, we carry out integration by parts with respect to the Gibbs measure $\mu$, and find the $L^2$-generator of the bilinear form $\mathcal{E}^\mathbb{K}$ (containing the potential $\phi$ and its gradient). This, in particular, proves the closability of the bilinear form $\mathcal{E}^\mathbb{K}$ on $L^2(\mathbb{K}(X), \mu)$. This result extends [10] (see also [7]), where the $L^2$-generator of $\mathcal{E}^\mathbb{K}$ (the Laplace operator) was derived in the case of no interaction, $\phi = 0$, and when the completely random measure $\mu = \nu$ is the law of a measure-valued Lévy process.

The main result of the paper is the existence of a conservative diffusion process on $\mathbb{K}(X)$ which is properly associated with the Dirichlet form $\mathcal{E}^\mathbb{K}$. For this, one assumes that the dimension of the underlying space $X$ is $\geq 2$. (It is intuitively clear that in the case where the dimension of $X$ is equal to one, such a result should fail.) We note that this diffusion process has continuous sample paths in $\mathbb{K}(X)$ with respect to the vague topology. The diffusion process has $\mu$ as invariant (and even symmetrizing) measure. To prove the main result, we use the general theory of Dirichlet forms [13] as well as the theory of Dirichlet forms over configuration spaces [14, 18], see also [1, 11].

The paper is organized as follows. In Section 2, we recall how differentiation on $\mathbb{K}(X)$ is introduced [10], and how the Gibbs measure $\mu$ is constructed [6, 16]. In Section 3, we formulate the results of the paper. Finally, Section 4 contains the proofs.
2 Preliminaries

2.1 Differentiation on \( \mathbb{K}(X) \)

In this subsection, we follow [10]. A starting point to define differentiation on \( \mathbb{K}(X) \) is the choice of a natural group \( \mathfrak{G} \) of transformations of \( \mathbb{K}(X) \). So let \( \text{Diff}_0(X) \) denote the group of \( C^\infty \) diffeomorphisms of \( X \) which are equal to the identity outside a compact set. Let \( C_0(X \to \mathbb{R}_+) \) denote the multiplicative group of continuous functions on \( X \) with values in \( \mathbb{R}_+ := (0, \infty) \) which are equal to one outside a compact set. The group \( \text{Diff}_0(X) \) naturally acts on \( X \), hence on \( C_0(X \to \mathbb{R}_+) \). So we define a group \( \mathfrak{G} \) by

\[
\mathfrak{G} := \text{Diff}_0(X) \ltimes C_0(X \to \mathbb{R}_+)
\]

the semidirect product of \( \text{Diff}_0(X) \) and \( C_0(X \to \mathbb{R}_+) \). As a set, \( \mathfrak{G} \) is equal to the Cartesian product of \( \text{Diff}_0(X) \) and \( C_0(X \to \mathbb{R}_+) \), and the product in \( \mathfrak{G} \) is given by

\[
g_1 g_2 = (\psi_1 \circ \psi_2, \theta_1(\theta_2 \circ \psi_1^{-1})) \quad \text{for} \quad g_1 = (\psi_1, \theta_1), \ g_2 = (\psi_2, \theta_2) \in \mathfrak{G}.
\]

The group \( \mathfrak{G} \) naturally acts on \( \mathbb{K}(X) \): for any \( g = (\psi, \theta) \in \mathfrak{G} \) and any \( \eta \in \mathbb{K}(X) \), we define \( g \eta \in \mathbb{K}(X) \) by

\[
d(g \eta)(x) := \theta(x) d(\psi^* \eta)(x).
\]

Here \( \psi^* \eta \) is the pushforward of \( \eta \) under \( \psi \).

The Lie algebra of the Lie group \( \text{Diff}_0(X) \) is the space \( \text{Vec}_0(X) \) consisting of all smooth vector fields acting from \( X \) into \( X \) which have compact support. For \( v \in \text{Vec}_0(X) \), let \( (\psi^v_t)_{t \in \mathbb{R}} \) be the corresponding one-parameter subgroup of \( \text{Diff}_0(X) \), see e.g. [2]. As the Lie algebra of \( C_0(X \to \mathbb{R}_+) \) we may take the space \( C_0(X) \) of all real-valued continuous functions on \( X \) with compact support. For each \( h \in C_0(X) \), the corresponding one-parameter subgroup of \( C_0(X \to \mathbb{R}_+) \) is given by \( (e^{t h})_{t \in \mathbb{R}} \). Thus, \( \mathfrak{g} := \text{Vec}_0(X) \times C_0(X) \) can be thought of as a Lie algebra that corresponds to the Lie group \( \mathfrak{G} \). For an arbitrary \( (v, h) \in \mathfrak{g} \), we may consider the curve \( \{ (\psi^v_t, e^{th}), \ t \in \mathbb{R} \} \) in \( \mathfrak{G} \). For a function \( F : \mathbb{K}(X) \to \mathbb{R} \) we define its derivative in direction \((v, h)\) by

\[
\nabla_{(v, h)}^\mathbb{K} F(\eta) := \frac{d}{dt} \bigg|_{t=0} F((\psi^v_t, e^{th})\eta), \quad \eta \in \mathbb{K}(X),
\]

provided the derivative on the right hand side of this formula exists.

A tangent space to \( \mathbb{K}(X) \) at \( \eta \in \mathbb{K}(X) \) is defined by

\[
T_\eta(\mathbb{K}(X)) := L^2(X \to X \times \mathbb{R}, \eta),
\]

the \( L^2 \)-space of \( X \times \mathbb{R} \)-valued vector fields on \( X \) which are square integrable with respect to the measure \( \eta \). We then define a gradient of a differentiable function \( F : \mathbb{K}(X) \to \mathbb{R} \) at \( \eta \) as the element \( (\nabla_\mathbb{K} F)(\eta) \) of \( T_\eta(\mathbb{K}) \) which satisfies

\[
\nabla_{(v, h)}^\mathbb{K} F(\eta) = \langle \nabla_\mathbb{K} F(\eta), (v, h) \rangle_{T_\eta(\mathbb{K})} \quad \text{for all} \ (v, h) \in \mathfrak{g}.
\]
Remark 1. Note that, in the above definitions, one could replace $\mathbb{K}(X)$ with the wider space $\mathcal{M}(X)$. This is why, in paper [10], the gradient $\nabla^K$ was actually denoted by $\nabla^M$.

Let us now define a set of test functions on $\mathbb{K}(X)$. Let us denote by $\tau(\eta)$ the set of atoms of $\eta$, and for each $x \in \tau(\eta)$, let $s_x := \eta(\{x\})$. Thus, we have

$$\eta = \sum_{x \in \tau(\eta)} s_x \delta_x.$$ 

We define a metric on $\mathbb{R}_+$ by

$$d\mathbb{R}_+(s_1, s_2) := |\log(s_1) - \log(s_2)|, \quad s_1, s_2 \in \mathbb{R}_+.$$ 

Then $\mathbb{R}_+$ becomes a locally compact Polish space, and any set of the form $[a,b)$, with $0 < a < b < \infty$, is compact. We denote $\hat{X} := \mathbb{R}_+ \times X$, and let $C^\infty_0(\hat{X})$ denote the space of all smooth functions on $\hat{X}$ with compact support. For each $\varphi \in C^\infty_0(\hat{X})$ and $\eta \in \mathbb{K}(X)$, we define

$$\langle\langle \varphi, \eta \rangle\rangle := \sum_{x \in \tau(\eta)} \varphi(s_x, x).$$

Note that the latter sum contains only finitely many nonzero terms.

We denote by $\mathcal{F}\mathcal{C}(\mathbb{K}(X))$ the set of all functions $F : \mathbb{K}(X) \to \mathbb{R}$ of the form

$$F(\eta) = g(\langle\langle \varphi_1, \eta \rangle\rangle, \ldots, \langle\langle \varphi_N, \eta \rangle\rangle), \quad \eta \in \mathbb{K}(X),$$

where $g \in C^\infty_b(\mathbb{R}_N)$, $\varphi_1, \ldots, \varphi_N \in C^\infty_0(\hat{X})$, and $N \in \mathbb{N}$. Here $C^\infty_b(\mathbb{R}_N)$ is the set of all infinitely differentiable functions on $\mathbb{R}_N$ which, together with all their derivatives, are bounded.

Let $F : \mathbb{K}(X) \to \mathbb{R}$, $\eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$. We define

$$\nabla_x F(\eta) := \nabla_y|_{y=x} F(\eta - s_x \delta_x + s_y \delta_y),$$

$$\nabla_{s_x} F(\eta) := \frac{d}{du}|_{u=s_x} F(\eta - s_x \delta_x + u \delta_x),$$

provided the derivatives exist. Here the variable $y$ is from $X$, $\nabla_y$ denotes the gradient on $X$ in the $y$ variable, and the variable $u$ is from $\mathbb{R}_+$.

An easy calculation shows that, for each function $F \in \mathcal{F}\mathcal{C}(\mathbb{K}(X))$, the gradient $\nabla^K F$ exists and is given by

$$(\nabla^K F)(\eta, x) = \left( \frac{1}{s_x} \nabla_x F(\eta), \nabla_{s_x} F(\eta) \right), \quad \eta \in \mathbb{K}(X), \ x \in \tau(\eta).$$
2.2 The Gibbs measures

We start with defining a class of completely random measures. Let \( l : \hat{X} \to \mathbb{R}_+ \) be a measurable function which satisfies the following conditions: for \( dx \)-a.a. \( x \in X \)

\[
\int_{\mathbb{R}_+} \frac{l(s,x)}{s} \, ds = \infty
\]  

(7)

and for each \( \Lambda \in \mathcal{B}_0(X) \),

\[
\int_{\mathbb{R}_+ \times \Lambda} l(s,x) \, ds \, dx < \infty.
\]  

(8)

Here \( \mathcal{B}_0(X) \) denotes the collection of all sets from \( \mathcal{B}(X) \) which have compact closure.

We define a measure \( \sigma \) on \( \hat{X} \) by

\[
d\sigma(s,x) := \frac{l(s,x)}{s} \, ds \, dx.
\]  

(9)

Since (8) holds, we may define a completely random measure \( \nu \) as a probability measure on \( \mathbb{K}(X) \) which has Fourier transform

\[
\int_{\mathbb{K}(X)} e^{i\langle f, \eta \rangle} \, d\nu(\eta) = \exp \left[ \int_{\hat{X}} \left( e^{isf(x)} - 1 \right) \, d\sigma(s,x) \right], \quad f \in C_0(X),
\]

see e.g. [3]. Here we denote \( \langle f, \eta \rangle := \int_X f(x) \, d\eta(x) \). The measure \( \nu \) can also be characterized through the Mecke identity: \( \nu \) is the unique probability measure on \( \mathbb{K}(X) \) which satisfies, for each measurable function \( F : \hat{X} \times \mathbb{K}(X) \to [0, \infty] \),

\[
\int_{\mathbb{K}(X)} \sum_{x \in \tau(\eta)} F(s_x, x, \eta) \, d\nu(\eta) = \int_{\mathbb{K}(X)} d\nu(\eta) \int_{\hat{X}} d\sigma(s,x) \, F(s, x, \eta + s\delta_x).
\]  

(10)

For example, by choosing \( l(s,x) = e^{-s} \), we get the gamma measure \( \nu \) [19]. More generally, we may fix measurable functions \( \alpha, \beta : X \to \mathbb{R}_+ \) and set

\[
l(s,x) = \beta(x)e^{-s/\alpha(x)}.
\]

Then conditions (7), (8) are satisfied when \( \alpha(x)\beta(x) \in L^1_{\text{loc}}(X, dx) \).

Let us now recall the definition of a Gibbs measure from [6,16]. Additionally to (7) and (8), we assume that, for each \( \Lambda \in \mathcal{B}_0(X) \),

\[
\int_{\mathbb{R}_+ \times \Lambda} l(s,x)s \, ds \, dx < \infty.
\]  

(11)

Let \( \phi : X \times X \to \mathbb{R} \) be a pair potential which satisfies the following two conditions:
(C1) $\phi$ is a symmetric, bounded, measurable function which satisfies, for some $R > 0$,

$$\phi(x, y) = 0 \text{ if } |x - y| > R.$$ 

(C2) There exists $\delta > 0$ such that

$$\inf_{x, y \in X: |x - y| \leq \delta} \phi(x, y) > \varepsilon \|\phi^-\|_\infty.$$ 

Here

$$\|\phi^-\|_\infty := \sup_{x, y \in X} (-\phi(x, y) \vee 0)$$

and $\varepsilon := 2v_d d^{d/2} (R/\delta + 1)$, where $v_d := \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of a unit ball in $X$.

Remark 2. Note that condition (C2) excludes the potential $\phi = 0$. Note also that conditions (C1) and (C2) are trivially satisfied if $\phi(x, y) = \psi(x - y)$, where $\psi \in C_0(X)$, $\psi(x) = \psi(-x)$, and $\psi(0) > v_d d^{d/2} \|\psi^-\|_\infty$.

For any $\eta, \xi \in \mathbb{K}(X)$ and $\Lambda \in \mathcal{B}_0(X)$, we define the relative energy (Hamiltonian)

$$H_\Lambda(\eta \mid \xi) := \frac{1}{2} \int_{\Lambda^c \setminus D} \phi(x, y) \, d\eta(x) \, d\eta(y) + \int_{\Lambda^c} \int_{\Lambda^c} \phi(x, y) \, d\eta(x) \, d\xi(y),$$

where $\Lambda^c := X \setminus \Lambda$. Note that $H_\Lambda(\eta \mid \xi)$ is well defined and finite.

For each $\Lambda \in \mathcal{B}(X)$, we denote $\mathbb{K}(\Lambda) := \{\eta \in \mathbb{K}(X) \mid \tau(\eta) \subset \Lambda\}$. Note that $\mathbb{K}(\Lambda) \in \mathcal{B}(\mathbb{K}(X))$. Let $\nu_\Lambda$ denote the pushforward of the completely random measure $\nu$ under the canonical projection

$$\mathbb{K}(X) \ni \eta \mapsto \eta_\Lambda := \sum_{x \in \tau(\eta) \cap \Lambda} s_x \delta_x \in \mathbb{K}(\Lambda).$$

The measure $\nu_\Lambda$ has Fourier transform

$$\int_{\mathbb{K}(\Lambda)} e^{i(f, \eta)} \, d\nu_\Lambda(\eta) = \exp \left[ \int_{\mathbb{R}^+ \times \Lambda} (e^{isf(x)} - 1) \, d\sigma(s, x) \right], \quad f \in C_0(X).$$

Proposition 3 ([6, 16]). Let (7)–(9), (11) hold and let conditions (C1) and (C2) be satisfied. Then, for any $\Lambda \in \mathcal{B}_0(X)$ and $\xi \in \mathbb{K}(X)$,

$$0 < Z_\Lambda(\xi) := \int_{\mathbb{K}(\Lambda)} e^{-H(\eta \mid \xi)} \, d\nu_\Lambda(\eta) < \infty.$$
For each $\Lambda \in \mathcal{B}_0(X)$ with $\int_\Lambda dx > 0$, the local Gibbs state with boundary condition $\xi \in \mathcal{K}(X)$ is defined as a probability measure on $\mathcal{K}(\Lambda)$ given by
\[
d\mu_\Lambda(\eta \mid \xi) := \frac{1}{Z_\Lambda(\xi)} e^{-H(\eta \mid \xi)} d\nu(\eta).
\]
For each $B \in \mathcal{B}(\mathcal{K}(X))$, $\Lambda \in \mathcal{B}_0(X)$, and $\xi \in \mathcal{K}(X)$, we define
\[
B_{\Lambda, \xi} := \{\eta \in \mathcal{K}(\Lambda) \mid \eta + \xi_{\Lambda^c} \in B\} \in \mathcal{B}(\mathcal{K}(\Lambda))
\]
and hence we can define the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \in \mathcal{B}_0(X)}$ on $\mathcal{K}(X)$ as the family of stochastic kernels
\[
\mathcal{B}(\mathcal{K}(X)) \times \mathcal{K}(X) \ni (B, \xi) \mapsto \pi_\Lambda(B \mid \xi) \in [0, 1]
\]
given by $\pi_\Lambda(B \mid \xi) := \mu_\Lambda(B_{\Lambda, \xi})$.

**Definition 4.** A Gibbs perturbation of a completely random measure $\nu$ corresponding to a pair potential $\phi$ is defined as a probability measure $\mu$ on $(\mathcal{K}(X), \mathcal{B}(\mathcal{K}(X)))$ which satisfies the following Dobrushin–Lanford–Ruelle (DLR) equation:
\[
\int_{\mathcal{K}(X)} \pi_\Lambda(B \mid \xi) d\mu(\xi) = \mu(B),
\]
for any $B \in \mathcal{B}(\mathcal{K}(X))$ and $\Lambda \in \mathcal{B}_0(X)$. We denote by $G(\nu, \phi)$ the set of all such probability measures $\mu$.

**Theorem 5** ([6, 16]). *Let the conditions of Proposition 3 be satisfied. Then the set $G(\nu, \phi)$ is non-empty. Furthermore, each measure $\mu \in G(\nu, \phi)$ has finite moments: for each $\Lambda \in \mathcal{B}_0(X)$ and $n \in \mathbb{N}$,
\[
\int_{\mathcal{K}(X)} \eta(\Lambda)^n d\mu(\eta) < \infty.
\]
*Since (7) holds, for each $\Lambda \in \mathcal{B}_0(X)$ with $\int_\Lambda dx > 0$, for $\nu$-a.a. $\eta \in \mathcal{K}(X)$, the set $\tau(\eta) \cap \Lambda$ is infinite. Using the DLR equation, we therefore obtain the following result.

**Proposition 6.** *Let the conditions of Proposition 3 be satisfied, and let $\mu \in G(\nu, \phi)$. Let $\Lambda \in \mathcal{B}_0(X)$ with $\int_\Lambda dx > 0$. Then, for $\mu$-a.a. $\eta \in \mathcal{K}(X)$, the set $\tau(\eta) \cap \Lambda$ is infinite. In particular, the set $\tau(\eta)$ is $\mu$-a.s. dense in $X$.*

By analogy with [15], the Gibbs measures have the following property.

**Theorem 7.** *Let the conditions of Proposition 3 be satisfied, and let $\mu \in G(\nu, \phi)$. Then $\mu$ satisfies the following Nguyen–Zessin identity: for each measurable function $F : \tilde{X} \times \mathcal{K}(X) \to [0, \infty)$,
by (6) and (13), we indeed have

\[
\int_{\mathcal{K}(X)} \sum_{x \in \tau(\eta)} F(s_x, x, \eta) \, d\mu(\eta)
\]

\[
= \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \exp \left[ -s \int_X \phi(x, x') \, d\eta(x') \right] F(s, x, \eta + s\delta_x) \, d\sigma(s, x) \, d\mu(\eta).
\]

(14)

Proof. By the same arguments as in the proof of [6, Theorem 6.3], it is enough to show that, for each \( \Lambda \in \mathcal{B}_0(X) \), equality (14) holds for all functions \( F \) of the form \( F(s, x, \eta) = f(s, x)g(\eta_\Lambda) \), where \( f \in C_0(\mathcal{K}) \), \( f \geq 0 \), the support of \( f \) is a subset of \( \mathbb{R}_+ \times \Lambda \) and \( g : \mathcal{K}(\Lambda) \to [0, \infty) \) is bounded and measurable. By the DLR equation (12) and the Mecke identity (10), we have

\[
\int_{\mathcal{K}(X)} \sum_{x \in \tau(\eta)} F(s_x, x, \eta) \, d\mu(\eta) = \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \sum_{x \in \tau(\eta) \cap \Lambda} f(s_x, x)g(\eta) \, \pi_\Lambda(\eta | \xi) \, d\mu(\xi)
\]

\[
= \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \sum_{x \in \tau(\eta)} f(s_x, x)g(\eta) \frac{1}{Z_\Lambda(\xi)} e^{-H_\Lambda(\eta | \xi \Lambda)} \, d\nu_\Lambda(\eta) \, d\mu(\xi)
\]

\[
= \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \int_{\mathbb{R}_+ \times \Lambda} f(s, x)g(\eta + s\delta_x) \frac{1}{Z_\Lambda(\xi)} e^{-H_\Lambda(\eta + s\delta_x | \xi \Lambda)} \, d\sigma(s, x) \, d\nu_\Lambda(\eta) \, d\mu(\xi)
\]

\[
= \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \int_{\mathbb{R}_+ \times \Lambda} F(s, x, \eta + s\delta_x) \exp \left[ -s \int_{X\backslash\{x\}} \phi(x, x') \, d\eta(x') \right] \pi_\Lambda(\eta | \xi) \, d\mu(\xi) \, d\sigma(s, x)
\]

\[
= \int_{\mathcal{K}(X)} \int_{\mathcal{K}(\Lambda)} \int_{X \backslash \{x\}} \exp \left[ -s \int_{X \backslash \{x\}} \phi(x, x') \, d\eta(x') \right] F(s, x, \eta + s\delta_x) \, d\sigma(s, x) \, d\mu(\eta) \, d\sigma(s, x)
\]

where the last line is obtained by applying the DLR equation (12) again. Note that, for a fixed \( \eta \in \mathcal{K}(X) \), since the set \( \tau(\eta) \) is countable, we have \( \sigma(\tau(\eta) \times \mathbb{R}_+) = 0 \). Hence, in formula (15), instead of the integral \( \int_{X \backslash \{x\}} \phi(x, x') \, d\eta(x') \), we may write \( \int_X \phi(x, x') \, d\eta(x') \).

\( \square \)

3 The results

In this section, we will introduce the Dirichlet form \( \mathcal{E}^\mathcal{K} \) and formulate the results. We postpone the proofs to Section 4.

Let the conditions of Proposition 3 be satisfied and let us fix any Gibbs measure \( \mu \in G(\nu, \phi) \). For any \( F, G \in \mathcal{F}C(\mathcal{K}(X)) \), we define \( \mathcal{E}^\mathcal{K}(F, G) \) by formula (1). Note that, by (6) and (13), we indeed have

\[
\int_{\mathcal{K}(X)} \left\langle \nabla^\mathcal{K} F(\eta), \nabla^\mathcal{K} G(\eta) \right\rangle_{\mathcal{T}_0(\mathcal{K})} \, d\mu(\eta) < \infty.
\]

Lemma 8. Let \( F, G \in \mathcal{F}C(\mathcal{K}(X)) \) and let \( F = 0 \) \( \mu \)-a.e. Then \( \mathcal{E}^\mathcal{K}(F, G) = 0 \).
Thus, we may consider $\mathcal{E}^\mathbb{K}$ as a symmetric bilinear form on $L^2(\mathbb{K}(X), \mu)$ with domain $\mathcal{F}\mathcal{C}(\mathbb{K}(X))$. Note that $\mathcal{F}\mathcal{C}(\mathbb{K}(X))$ is dense in $L^2(\mathbb{K}(X), \mu)$. Let us now find the $L^2$-generator of this form. Analogously to (4), (5), we define, for each function $F \in \mathcal{F}\mathcal{C}(\mathbb{K}(X)), \eta \in \mathbb{K}(X)$, and $x \in \tau(\eta)$,

$$\Delta_x F(\eta) := \Delta_y \big|_{y=x} F(\eta - s_x \delta_x + s_x \delta_y),$$
$$\Delta_{s_x} F(\eta) := \frac{d^2}{du^2} \bigg|_{u=s_x} F(\eta - s_x \delta_x + u \delta_x),$$

where $\Delta_y$ is the Laplace operator on $X$ acting in the $y$ variable.

The following proposition gives, in particular, the explicit form of the $L^2$-generator of the bilinear form $(\mathcal{E}^\mathbb{K}, \mathcal{F}\mathcal{C}(\mathbb{K}(X)))$.

**Proposition 9.** Assume that $l \in C^1(\hat{X})$ and $\phi \in C^1(X \times X)$. For each $F \in \mathcal{F}\mathcal{C}(\mathbb{K}(X))$, we define a function $L^\mathbb{K} F \in L^2(\mathbb{K}(X), \mu)$ by

$$L^\mathbb{K} F(\eta) = \sum_{x \in \tau(\eta)} \left[ \frac{1}{s_x} \Delta_x F(\eta) + \frac{1}{s_x} \langle \nabla_x \log l(s, x), \nabla_x F(\eta) \rangle_X - \int_X \frac{d(\eta - s_x \delta_x)(x') \langle \nabla_x \phi(x, x'), \nabla_x F(\eta) \rangle_X}{s_x} \right] + s_x \Delta_{s_x} F(\eta) + s_x \left( \langle \nabla_{s_x} \log l(s_x, x) \rangle \nabla_{s_x} F(\eta) \right) - \left( \int_X \frac{d(\eta - s_x \delta_x)(x') \phi(x, x')}{s_x} \right) s_x \nabla_{s_x} F(\eta) \right].$$

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Here $\langle \cdot, \cdot \rangle_X$ denotes the scalar product in $X$. Then, for any $F, G \in \mathcal{F}\mathcal{C}(\mathbb{K}(X))$,

$$\mathcal{E}^\mathbb{K}(F, G) = (-L^\mathbb{K} F, G)_{L^2(\mathbb{K}(X), \mu)}.$$  

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The bilinear form $(\mathcal{E}^\mathbb{K}, \mathcal{F}\mathcal{C}(\mathbb{K}(X)))$ is closable on $L^2(\mathbb{K}(X), \mu)$, and its closure, denoted by $(\mathcal{E}^\mathbb{K}, D(\mathcal{E}^\mathbb{K}))$ is a Dirichlet form. The operator $(-L^\mathbb{K}, \mathcal{F}\mathcal{C}(\mathbb{K}(X)))$ has Friedrichs’ extension, which we denote by $(-L^\mathbb{K}, D(L^\mathbb{K}))$.

**Remark 10.** Note that, in the case where $\mu$ is the Gibbs perturbation of the gamma measure, i.e., when $l(s, x) = e^{-s}$, formula (16) becomes

$$L^\mathbb{K} F(\eta) = \sum_{x \in \tau(\eta)} \left[ \frac{1}{s_x} \Delta_x F(\eta) - \int_X \frac{d(\eta - s_x \delta_x)(x') \langle \nabla_x \phi(x, x'), \nabla_x F(\eta) \rangle_X}{s_x} \right] + s_x \left( \Delta_{s_x} F(\eta) - \nabla_{s_x} F(\eta) \right) - \left( \int_X \frac{d(\eta - s_x \delta_x)(x') \phi(x, x')}{s_x} \right) s_x \nabla_{s_x} F(\eta).$$

We are now ready to formulate the main result of the paper.
Theorem 11. Assume that the conditions of Propositions 3 and 9 be satisfied. Further assume that the dimension $d$ of the space $X$ is $\geq 2$. Then there exists a conservative diffusion process on $\mathbb{K}(X)$ (i.e., a conservative strong Markov process with continuous sample paths in $\mathbb{K}(X)$),

$$M^K = (\Omega^K, \mathcal{F}^K, (\mathcal{F}^K_t)_{t \geq 0}, (\Theta^K_t)_{t \geq 0}, (X^K(t))_{t \geq 0}, (\mathbb{P}^K_{\eta})_{\eta \in \mathbb{K}(X)}),$$

(cf. [4]) which is properly associated with the Dirichlet form $(\mathcal{E}^K, D(\mathcal{E}^K))$, i.e., for all $\mu$-versions of $F \in L^2(\mathbb{K}(X), \mu)$ and all $t > 0$ the function

$$\mathbb{K}(X) \ni \eta \mapsto (p^K_t F)(\eta) := \int_{\Omega} F(X(t)) \, d\mathbb{P}^K_{\eta}$$

is an $\mathcal{E}^K$-quasi-continuous version of $\exp(tL^K)F$ (cf. [13, Chap. 1, Sect. 2]). Here $\Omega^K = C([0, \infty) \to \mathbb{K}(X))$, $X^K(t) = \omega(t) + \theta^K(t)$, $\omega \in \Omega^K$, $t \geq 0$, $\omega \in \Omega^K$, $(\mathcal{F}^K_t)_{t \geq 0}$ together with $\mathcal{F}^K$ is the corresponding minimum completed admissible family (cf. [5, Section 4.1]) and $\Theta^K_t, t \geq 0$, are the corresponding natural time shifts.

In particular, $M^K$ is $\mu$-symmetric (i.e., $\int G p^K_t F \, d\mu = \int F p^K_t G \, d\mu$ for all $F, G : \mathbb{K}(X) \to [0, \infty)$, $\mathcal{B}(\mathbb{K}(X))$-measurable) and has $\mu$ as an invariant measure.

$M^K$ is up to $\mu$-equivalence unique (cf. [13, Chap. IV, Sect. 6]).

Remark 12. In addition to (7)–(11), let us assume that the function $l(s, x)$ satisfies, for each $\Lambda \in \mathcal{B}_0(X)$,

$$\int_{\mathbb{R}_+ \times \Lambda} l(s, x)s^i ds dx < \infty, \quad i = 2, 3.$$  

This implies that the completely random measure $\nu$ satisfies, for each $\Lambda \in \mathcal{B}_0(X)$,

$$\int_{\mathbb{K}(X)} \eta(\Lambda)^n d\nu(\eta) < \infty \quad \text{for } n = 1, 2, 3, 4.$$  

Then it easily follows from the proofs of Proposition 9 and Theorem 11 that these statements remain true when $l \in C^1(\hat{X})$ and the pair potential $\phi$ is equal to zero, i.e., when $\mu = \nu$.

We note that, in paper [10], for a different choice of a tangent space $T_{\eta}(\mathbb{K})$ and in the case where $l(s, x) = l(s)$ is independent of $x$ and $\mu = \nu$, the corresponding diffusion process on $\mathbb{K}(X)$ was constructed explicitly. However, for the choice of the tangent space $T_{\eta}(\mathbb{K})$ as in this paper, even in the case where $\mu = \nu$, an explicit construction of the diffusion process is an open problem, see Subsec. 5.2 in [10].
4 The proofs

4.1 Proofs of Lemma 8 and Proposition 9

We start with the following

Lemma 13. For any $F, G \in \mathcal{FC}(\mathcal{K}(X))$,

$$
\mathcal{E}_K(F, G) = \int_{\mathcal{K}(X)} d\mu(\eta) \int_{\hat{X}} ds \, dx \, l(s, x) \exp \left[ -s \int_X \phi(x, x') \, d\eta(x') \right] \\
\times \left[ \frac{1}{s^2} \langle \nabla_x F(\eta + s\delta_x), \nabla_x G(\eta + s\delta_x) \rangle_X + \left( \frac{d}{ds} F(\eta + s\delta_x) \right) \left( \frac{d}{ds} G(\eta + s\delta_x) \right) \right].
$$

(18)

Proof. Formula (18) follows directly from (1), (2), (4)–(6), and (14).

Proof of Lemma 8. By (C1) and (13), for a fixed $x \in X$, we get

$$
\int_X |\phi(x, x')| \, d\eta(x') \, d\mu(\eta) < \infty.
$$

Hence, for $\mu$-a.a. $\eta \in \mathcal{K}(X)$, we have $\int_X |\phi(x, x')| \, d\eta(x') < \infty$. Therefore, on $\hat{X} \times \mathcal{K}(X)$, the measures

$$
l(s, x) \exp \left[ -s \int_X \phi(x, x') \, d\eta(x') \right] ds \, dx \, d\mu(\eta)
$$

and $ds \, dx \, d\mu(\eta)$ are equivalent.

Let $F \in \mathcal{F}(\mathcal{K}(X))$ be such that $F = 0$ $\mu$-a.e. Then, for any $\Lambda \in \mathcal{B}(X)$, we get by (14)

$$
\int_{\mathcal{K}(X)} d\mu(\eta) \int_{\hat{X}} ds \, dx \, l(s, x) \exp \left[ -s \int_X \phi(x, x') \, d\eta(x') \right] |F(\eta + s\delta_x)| \chi_\Lambda(x)
= \int_{\mathcal{K}(X)} |F(\eta)| \eta(\Lambda) \, d\mu(\eta) = 0.
$$

Here $\chi_\Lambda$ denotes the indicator function of the set $\Lambda$. Hence, $F(\eta + s\delta_x) = 0$ for $ds \, dx \, d\mu(\eta)$-a.a. $(s, x, \eta) \in \hat{X} \times \mathcal{K}(X)$. For each fixed $\eta \in \mathcal{K}(X)$, the function $(s, x) \mapsto F(\eta + s\delta_x)$ is continuous. Therefore, for $\mu$-a.a. $\eta \in \mathcal{K}(X)$, $F(\eta + s\delta_x) = 0$ for all $(s, x) \in \hat{X}$. Hence, by Lemma 13, for each $G \in \mathcal{F}(\mathcal{K}(X))$, $\mathcal{E}_K(F, G) = 0$.

Proof of Proposition 9. We first note that $(\mathcal{E}_K, \mathcal{F}(\mathcal{K}(X)))$ is a pre-Dirichlet form form on $L^2(\mathcal{K}(X), \mu)$, i.e., if it is closable then its closure is a Dirichlet form. This assertion follows, by standard methods, directly from [13, Chap. I, Proposition 4.10] (see also [13, Chap. II, Exercise 2.7]).
4.2 Proof of Theorem 11

We will divide the proof into several steps.

Step 1. To prove the theorem, we will initially construct a diffusion process on a certain subset of the configuration space over \( \hat{X} \). So in this step, we will present the necessary definitions and constructions related to the configuration space.

We denote by \( \tilde{\Gamma}(\hat{X}) \) the space of all \( \mathbb{N}_0 \cup \{ \infty \} \)-valued Radon measures on \( \hat{X} \). Here \( \mathbb{N}_0 := \{ 0, 1, 2, \ldots \} \). The space \( \tilde{\Gamma}(\hat{X}) \) is endowed with the vague topology and let \( \mathcal{B}(\tilde{\Gamma}(\hat{X})) \) denote the corresponding \( \sigma \)-algebra.

The configuration space over \( \hat{X} \), denoted by \( \Gamma(\hat{X}) \), is defined as the collection of all locally finite subsets of \( \hat{X} \):

\[
\Gamma(\hat{X}) := \{ \gamma \subset \hat{X} \mid |\gamma \cap A| < \infty \text{ for each compact } A \subset \hat{X} \}.
\]

Here \( |\gamma \cap A| \) denotes the cardinality of the set \( \gamma \cap A \). One usually identifies a configuration \( \gamma \in \Gamma(\hat{X}) \) with the Radon measure \( \sum_{(s,x) \in \gamma} \delta_{(s,x)} \) on \( \hat{X} \). Thus, one gets the inclusion \( \Gamma(\hat{X}) \subset \tilde{\Gamma}(\hat{X}) \).

Let \( \Gamma_{pf}(\hat{X}) \) denote the subset of \( \Gamma(\hat{X}) \) which consists of all configurations \( \gamma \) which satisfy:
then the mapping \( R : \Gamma \) of \( B \) Poisson measure on \( \hat{\Gamma} \) the Fourier transform of \( \pi \langle \cdot \rangle \). Here we denote \( \rho \) and (19), the measure \( g, \phi \) where the functions \( b \) bilinear form \( (F) \) of the form \( \int \gamma \). Consider a bijective mapping \( R : \Gamma_{pf}(\hat{X}) \to \mathbb{K}(X) \) defined by 

\[
\Gamma_{pf}(\hat{X}) \ni \gamma = \{(s_i, x_i)\} \mapsto R\gamma := \sum_i s_i \delta_{x_i} \in \mathbb{K}(X). \quad (19)
\]

Then the mapping \( R \) and its inverse \( R^{-1} : \mathbb{K}(X) \to \Gamma_{pf}(\hat{X}) \) are measurable.

Note that the pushforward of the completely random measure \( \nu \) under \( R^{-1} \) is the Poisson measure on \( \Gamma(\hat{X}) \) with intensity measure \( \sigma \): if we denote this measure by \( \pi \), the Fourier transform of \( \pi \) is given by

\[
\int_{\Gamma_{pf}(\hat{X})} e^{i(f, \gamma)} d\pi(\gamma) = \exp \left[ \int_{\hat{X}} (e^{if(s, x)} - 1) d\sigma(s, x) \right], \quad f \in C_0(\hat{X}).
\]

Here we denote \( (f, \gamma) := \int_{\hat{X}} f d\gamma = \sum_{s,x} f(s, x) \).

Let \( \rho \) denote the pushforward of the Gibbs measure \( \mu \) under \( R^{-1} \). By Theorem 7 and (19), the measure \( \rho \) satisfies, for each measurable function \( F : \hat{X} \times \Gamma(\hat{X}) \to [0, \infty] \),

\[
\int_{\Gamma_{pf}(\hat{X})} \sum_{(s,x) \in \gamma} F(s, x, \gamma) d\rho(\gamma)
\]

\[
= \int_{\Gamma_{pf}(\hat{X})} d\rho(\gamma) \int_{\hat{X}} d\sigma(s, x) \exp \left[ - \sum (s', x') \in \gamma ss' \phi(x, x') \right] F(s, x, \gamma \cup \{(s, x)\}).
\]

Let \( \mathcal{F}C(\Gamma_{pf}(\hat{X})) \) denote the set of functions on \( \Gamma_{pf}(\hat{X}) \) which are of the form \( F(\gamma) = G(R\gamma) \) for some \( G \in \mathcal{F}C(\mathbb{K}(X)) \). Thus, \( \mathcal{F}C(\Gamma_{pf}(\hat{X})) \) consists of all functions \( F \) of the form

\[
F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle), \quad \gamma \in \Gamma_{pf}(\hat{X}),
\]

where the functions \( g, \varphi_1, \ldots, \varphi_N \) are as in (3). Thus, we may equivalently consider a bilinear form \( \langle \delta^G, \mathcal{F}C(\Gamma_{pf}(\hat{X})) \rangle \) on \( L^2(\Gamma_{pf}(\hat{X}), \rho) \) which is defined by

\[
\delta^G(F, G) := \delta^K(F \circ R^{-1}, G \circ R^{-1}), \quad F, G \in \mathcal{F}C(\Gamma_{pf}(\hat{X})).
\]
As easily seen, for any \( F, G \in \mathcal{F}(\Gamma_{p\bar{f}}(\hat{X})) \), we have
\[
\mathcal{E}(F, G) = \int_{\Gamma(\hat{X})} \sum_{(s,x) \in \gamma} \left[ \frac{1}{s} \langle \nabla_x F(\gamma), \nabla_x G(\gamma) \rangle_X + s \langle \nabla_s F(\gamma), \nabla_s G(\gamma) \rangle_X \right] d\rho(\gamma),
\]
where \( \nabla_x F(\gamma) \) and \( \nabla_s G(\gamma) \) are defined analogously to formulas (4), (5). By Proposition 9, the bilinear form \((\mathcal{E}, \mathcal{F}(\Gamma_{p\bar{f}}(\hat{X})))\) is closable on \( L^2(\Gamma_{p\bar{f}}(\hat{X}), \rho) \), and its closure, denoted by \((\mathcal{E}, D(\mathcal{E}))\), is a Dirichlet form.

Step 2. Our aim now is to construct a diffusion process on \( \Gamma_{p\bar{f}}(\hat{X}) \) which is properly associated with the Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\). We will initially construct such a process on a bigger space \( \hat{\Gamma}_{f}(\hat{X}). \) In this step, we will define the set \( \hat{\Gamma}_{f}(\hat{X}) \) and construct a metric on it such that the set \( \hat{\Gamma}_{f}(\hat{X}) \) equipped with this metric is a Polish space.

For each \( \Lambda \in \mathcal{B}_0(X) \), we define a local mass \( \mathcal{M}_\Lambda \) by
\[
\mathcal{M}_\Lambda(\gamma) := \int_{\hat{X}} \chi(\gamma) s d\gamma(s,x), \quad \gamma \in \hat{\Gamma}(\hat{X}).
\]
We set
\[
\hat{\Gamma}_{f}(\hat{X}) := \{ \gamma \in \hat{\Gamma}(\hat{X}) \mid \mathcal{M}_\Lambda(\gamma) < \infty \text{ for each } \Lambda \in \mathcal{B}_0(X) \}.
\]
We have \( \hat{\Gamma}_{f}(\hat{X}) \in \mathcal{B}(\hat{\Gamma}(\hat{X})) \), and let \( \mathcal{B}(\hat{\Gamma}_{f}(\hat{X})) \) denote the Borel \( \sigma \)-algebra on the space \( \hat{\Gamma}_{f}(\hat{X}) \) equipped with the vague topology.

We will now construct a bounded metric on \( \hat{\Gamma}_{f}(\hat{X}) \) in which this space will be complete and separable. Let \( d_V(\cdot, \cdot) \) denote the bounded metric on \( \hat{\Gamma}(\hat{X}) \) which was introduced in [14, Section 3]. Recall that this metric generates the vague topology on \( \hat{\Gamma}(\hat{X}) \), and \( \hat{\Gamma}(\hat{X}) \) is complete and separable in this metric.

For each \( k \in \mathbb{N} \), we fix any function \( \phi_k \in C_0^\infty(X) \) such that
\[
\chi_{B(k)} \leq \phi_k \leq \chi_{B(k+1)}, \quad \left| \frac{\partial}{\partial x_i} \phi_k(x) \right| \leq 2 \chi_{B(k+1)}(x), \quad i = 1, \ldots, d, \quad x = (x^1, \ldots, x^d) \in X.
\] \hspace{1cm} (20)

Here
\[
B(k) := \{ x = (x^1, \ldots, x^d) \in X \mid \max_{i=1,\ldots,d} |x_i| \leq k \}.
\]
Next, we fix any \( q \in (0, 1) \). We take any sequence \((\psi_n)_{n \in \mathbb{Z}}\) such that, for each \( n \in \mathbb{Z}, \psi_n \in C_0^\infty(\mathbb{R}) \) and
\[
\chi_{[q^n,q^{n-1}]} \leq \psi_n \leq \chi_{[q^{n+1},q^{n-2}]}, \quad |\psi'_n| \leq \frac{2}{q^n - q^{n+1}} \chi_{[q^{n+1},q^n] \cup [q^{n-2},q^{n-1}]}.
\] \hspace{1cm} (21)
For each $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, we define
\[
\kappa_{kn}(s, x) := \phi_k(x) \psi_n(s) s, \quad (s, x) \in \hat{X}.
\] (22)

Note that $\kappa_{kn} \in C_0^\infty(\hat{X})$. For any $k \in \mathbb{N}$ and $\gamma, \gamma' \in \hat{\Gamma}_f(\hat{X})$, we define
\[
d_k(\gamma, \gamma') := \sum_{n \in \mathbb{Z}} |\langle \kappa_{kn}, \gamma - \gamma' \rangle|.
\] (23)

As follows from (20) and (21), for each $\gamma \in \hat{\Gamma}_f(\hat{X})$,
\[
\sum_{n \in \mathbb{Z}} \langle \kappa_{kn}, \gamma \rangle = \int_{\hat{X}} d\gamma(s, x) \phi_k(x) \left( \sum_{n \in \mathbb{Z}} \psi_n(s) \right) s
\leq 4 \int_{\hat{X}} d\gamma(s, x) \phi_k(x)s \leq 4 M_B(k+1)(\gamma) < \infty.
\] (24)

Therefore, $d_k(\gamma, \gamma') < \infty$ for all $\gamma, \gamma' \in \hat{\Gamma}_f(\hat{X})$. Clearly, $d_k(\cdot, \cdot)$ satisfies the triangle inequality.

Let $\{c_k\}_{k=1}^\infty$ be a sequence of $c_k > 0$ such that $\sum_{k=1}^\infty c_k < \infty$. Below, in formula (35), we will make an explicit choice of the sequence $\{c_k\}_{k=1}^\infty$. We next define
\[
d_f(\gamma, \gamma') := \sum_{k=1}^\infty c_k \frac{d_k(\gamma, \gamma')}{1 + d_k(\gamma, \gamma')}, \quad \gamma, \gamma' \in \hat{\Gamma}_f(\hat{X}).
\]

Clearly, $d_f(\cdot, \cdot)$ also satisfies the triangle inequality. We finally define the metric
\[
d(\gamma, \gamma') := d_V(\gamma, \gamma') + d_f(\gamma, \gamma'), \quad \gamma, \gamma' \in \hat{\Gamma}_f(\hat{X}).
\]

**Proposition 15.** $(\hat{\Gamma}_f(\hat{X}), d(\cdot, \cdot))$ is a complete, separable metric space.

**Proof.** Let $\{\gamma_i\}_{i=1}^\infty$ be a Cauchy sequence in $(\hat{\Gamma}_f(\hat{X}), d(\cdot, \cdot))$. Then $\{\gamma_i\}_{i=1}^\infty$ is a Cauchy sequence in $(\hat{\Gamma}(\hat{X}), d_V(\cdot, \cdot))$. Since the latter space is complete, there exists $\gamma \in \hat{\Gamma}(\hat{X})$ such that $\gamma_i \to \gamma$ vaguely as $i \to \infty$. Denote
\[
a_{kn}^{(i)} := \langle \kappa_{kn}, \gamma_i \rangle, \quad a_{kn} := \langle \kappa_{kn}, \gamma \rangle, \quad k \in \mathbb{N}, \ n \in \mathbb{Z}.
\]

As $\kappa_{kn} \in C_0(\hat{X})$, we therefore get:
\[
\text{for each } k \in \mathbb{N} \text{ and } n \in \mathbb{Z} \quad a_{kn}^{(i)} \to a_{kn} \text{ as } i \to \infty.
\] (25)

Note that, for each $k \in \mathbb{N}$ and $i \in \mathbb{N}$, $a_{kn}^{(i)} \geq 0$ for all $n \in \mathbb{Z}$ and by (24)
\[
\sum_{n \in \mathbb{N}} a_{kn}^{(i)} < \infty.
\]
Hence, \((a_{kn})_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})\). As \(\{\gamma_i\}_{i=1}^{\infty}\) is a Cauchy sequence in \((\hat{\Gamma}_f(\hat{X}), d(\cdot, \cdot))\),

\[
\lim_{i,j \to \infty} \sum_{n \in \mathbb{Z}} |a_{kn}^{(i)} - a_{kn}^{(j)}| = \lim_{i,j \to \infty} d_k(\gamma_i, \gamma_j) = 0, \quad k \in \mathbb{N}.
\]

Hence, \(\{(a_{kn})_{n \in \mathbb{Z}}\}_{i=1}^{\infty}\) is a Cauchy sequence in \(\ell^1(\mathbb{Z})\). Since the latter space is complete, the sequence \(\{(a_{kn})_{n \in \mathbb{Z}}\}_{i=1}^{\infty}\) is convergent in \(\ell^1(\mathbb{Z})\). In view of (25), we therefore conclude that the \(\ell^1(\mathbb{Z})\)-limit of this sequence is \((a_{kn})_{n \in \mathbb{Z}}\). This, in particular, implies that

\[
\sum_{n \in \mathbb{Z}} a_{kn} = \sum_{n \in \mathbb{Z}} \langle \kappa_{kn}, \gamma \rangle < \infty, \quad k \in \mathbb{N}. \tag{26}
\]

By (21), \(\sum_{n=1}^{\infty} \psi_n(s) \geq 1\) for all \(s \in \mathbb{R}_+\). We therefore deduce from (26) that \(\gamma \in \hat{\Gamma}_f(\hat{X})\).

Furthermore,

\[
d_k(\gamma_i, \gamma) = \sum_{n \in \mathbb{Z}} |a_{kn}^{(i)} - a_{kn}| \to 0 \quad \text{as} \quad i \to \infty, \quad k \in \mathbb{N}.
\]

Hence \(d(\gamma_i, \gamma) \to 0\) as \(i \to \infty\). Thus, \((\hat{\Gamma}_f(\hat{X}), d(\cdot, \cdot))\) is complete. The proof of the separability of this space is routine, so we skip it.

**Step 3.** We will now consider \((\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma))\) as a Dirichlet form on \(L^2(\hat{\Gamma}_f(\hat{X})), \rho)\) and prove that is is quasi-regular. For the definition of quasi-regularity of a Dirichlet form, see [13, Chap. IV, Def. 3.1] and [14, subsec. 4.1].

We consider the complete separable metric space \((\hat{\Gamma}_f(\hat{X}), d(\cdot, \cdot))\), and let \(\mathcal{B}(\hat{\Gamma}_f(\hat{X}), d)\) denote the corresponding Borel \(\sigma\)-algebra on \(\hat{\Gamma}_f(\hat{X})\).

**Lemma 16.** We have \(\mathcal{B}(\hat{\Gamma}_f(\hat{X})) = \mathcal{B}(\hat{\Gamma}_f(\hat{X}), d)\).

**Proof.** We have \(d(\gamma, \gamma') \geq d_V(\gamma, \gamma')\) for all \(\gamma, \gamma' \in \hat{\Gamma}_f(\hat{X})\). Therefore, \(\mathcal{B}(\hat{\Gamma}_f(\hat{X})) \subset \mathcal{B}(\hat{\Gamma}_f(\hat{X}), d)\). On the other hand, it follows from the construction of the metric \(d(\cdot, \cdot)\) that, for a fixed \(\gamma' \in \hat{\Gamma}_f(\hat{X})\), the function

\[
\hat{\Gamma}_f(\hat{X}) \ni \gamma \mapsto d(\gamma, \gamma') \in \mathbb{R}
\]

is \(\mathcal{B}(\hat{\Gamma}_f(\hat{X}))\)-measurable. Hence, for any \(\gamma' \in \hat{\Gamma}_f(\hat{X})\) and \(r > 0\),

\[
\{\gamma \in \hat{\Gamma}_f(\hat{X}) \mid d(\gamma, \gamma') < r\} \in \mathcal{B}(\hat{\Gamma}_f(\hat{X})). \tag{27}
\]

But in a separable metric space, every open set can be represented as a countable union of open balls, see e.g. Theorem 2 and its proof in [12, p. 206]. Hence, (27) implies the inclusion \(\mathcal{B}(\hat{\Gamma}_f(\hat{X}), d) \subset \mathcal{B}(\hat{\Gamma}_f(\hat{X}))\).
We will now consider $\rho$ as a probability measure on the measurable space $(\bar{\Gamma}_f(\hat{X}), \mathcal{B}(\bar{\Gamma}_f(\hat{X})))$, and $(\mathcal{E}^\Gamma, D(\mathcal{E}^\Gamma))$ as a Dirichlet form on the space $L^2(\bar{\Gamma}_f(\hat{X}), \rho)$.

On $D(\mathcal{E}^\Gamma)$ we consider the norm
\[
\|F\|_{D(\mathcal{E}^\Gamma)} := \mathcal{E}^\Gamma(F, F)^{1/2} + \|F\|_{L^2(\bar{\Gamma}_f(\hat{X}), \rho)}.
\]

We define a square field operator
\[
S^\Gamma(F)(\gamma) := \sum_{(s, x) \in \gamma} \frac{1}{s} \|\nabla_x F(\gamma)\|_X^2 + s|\nabla_s F(\gamma)|^2,
\]
where $F \in \mathcal{P}\mathcal{E}(\Gamma_{pf}(\hat{X}))$, $\gamma \in \Gamma_{pf}(\hat{X})$, and $\| \cdot \|_X$ denotes the Euclidean norm in $X$. As easily seen, $S^\Gamma$ extends by continuity in the norm $\| \cdot \|_{D(\mathcal{E}^\Gamma)}$ to a mapping $S^\Gamma : D(\mathcal{E}^\Gamma) \to L^1(\bar{\Gamma}_f(\hat{X}), \rho)$, and furthermore $\mathcal{E}^\Gamma(F, F) = \int_{\bar{\Gamma}_f(\hat{X})} S^\Gamma(F) \, d\rho$.

**Lemma 17.** For each $\gamma \in \bar{\Gamma}_f(\hat{X})$, we have $d(\cdot, \gamma) \in D(\mathcal{E}^\Gamma)$. Furthermore, there exists $G \in L^1(\bar{\Gamma}_f(\hat{X}), \rho)$ (independent of $\gamma$) such that $S^\Gamma(d(\cdot, \gamma)) \leq G \rho$-a.e.

**Proof.** Recall that $d(\cdot, \gamma) = d_V(\cdot, \gamma) + d_f(\cdot, \gamma)$. Using the methods of [14, Section 4] (see also [11, Section 6]), one can show that $d_V(\cdot, \gamma) \in D(\mathcal{E}^\Gamma)$ and there exists $G_1 \in L^1(\bar{\Gamma}_f(\hat{X}), \rho)$ (independent of $\gamma$) such that $S^\Gamma(d_V(\cdot, \gamma)) \leq G_1 \rho$-a.e. Hence, we only need to prove that $d_f(\cdot, \gamma) \in D(\mathcal{E}^\Gamma)$ and there exists $G_2 \in L^1(\bar{\Gamma}_f(\hat{X}), \rho)$ (independent of $\gamma$) such that $S^\Gamma(d_f(\cdot, \gamma)) \leq G_2 \rho$-a.e.

Analogously to the proof of [14, Lemma 4.7], we fix any sequence $(\zeta_n)_{n=1}^\infty$ such that $\zeta_n \in C_0^\infty(\mathbb{R})$, $\int_\mathbb{R} \zeta_n(t) \, dt = 1$, $\zeta_n(t) = \zeta_n(-t)$ for all $t \in \mathbb{R}$, supp$(\zeta_n) \subset (-1/n, 1/n)$. We define
\[
u_n(t) := \int_\mathbb{R} |t - t'| \zeta_n(t') \, dt' - \int_\mathbb{R} |t'| \zeta_n(t') \, dt', \quad t \in \mathbb{R}.
\]
It is easy to check that, for each $n \in \mathbb{N}$, $u_n \in C^\infty(\mathbb{R})$, $|u_n(t)| \leq |t|$, $u_n(t) \to |t|$ as $n \to \infty$ for each $t \in \mathbb{R}$, $u'_n(t) \to \text{sign}(t)$ as $n \to \infty$ for each $t \in \mathbb{R} \setminus \{0\}$, and $|u'_n(t)| \leq 2$ for all $t \in \mathbb{R}$.

Recall (22) and (23). For each $N \in \mathbb{N}$, we define
\[
\begin{align*}
d_k^{(N)}(\gamma, \gamma') &:= \sum_{n \in \mathbb{Z} \cap [-N, N]} u_N(\langle \gamma_n, \gamma - \gamma' \rangle), \\
d_f^{(N)}(\gamma, \gamma') &:= \sum_{k=1}^N c_k \frac{d_k^{(N)}(\gamma, \gamma')}{1 + d_k^{(N)}(\gamma, \gamma')}, \quad \gamma, \gamma' \in \bar{\Gamma}_f(\hat{X}).
\end{align*}
\]
Clearly, for a fixed $\gamma' \in \bar{\Gamma}_f(\hat{X})$, the restriction of $d_f^{(N)}(\cdot, \gamma')$ to $\Gamma_{pf}(\hat{X})$ belongs to $\mathcal{P}\mathcal{E}(\Gamma_{pf}(\hat{X}))$. Hence, $d_f^{(N)}(\cdot, \gamma') \in D(\mathcal{E}^\Gamma)$. 18
As easily seen, for each \( \gamma \in \tilde{\Gamma}_f(\hat{X}) \), we have \( d^{(N)}_f(\gamma, \gamma') \rightarrow d_f(\gamma, \gamma') \) as \( N \rightarrow \infty \). Hence, \( d^{(N)}_f(\cdot, \gamma') \rightarrow d_f(\cdot, \gamma') \) in \( L^2(\tilde{\Gamma}_f(\hat{X}), \rho) \) as \( N \rightarrow \infty \). 

Note that, for \( t \geq 0, \left( \frac{1}{1+t} \right)^{t} = \frac{1}{(1+t)^t} \leq 1 \). Hence, by (20)–(22), for each \( \gamma \in \Gamma_{pf}(\hat{X}) \) and each \((s, x) \in \gamma\),

\[
\|\nabla_x d^{(N)}_f(\gamma, \gamma')\|_X \leq \sum_{k=1}^{N} c_k \|\nabla_x d^{(N)}_k(\gamma, \gamma')\|_X \\
\leq 2 \sum_{k=1}^{N} c_k \sum_{n \in \mathbb{Z} \cap [-N, N]} \|\nabla_x \zeta_{kn}(x, s)\|_X \\
= 2 \sum_{k=1}^{N} c_k \|\nabla \phi_k(x)\|_X \sum_{n \in \mathbb{Z} \cap [-N, N]} \psi_n(s) s \\
\leq 4 \sqrt{d} \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z} \cap [-N, N]} \psi_n(s) s \\
\leq 16 \sqrt{d} \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) s.
\]

Hence, using the Cauchy inequality, we conclude that there exists a constant \( C_1 > 0 \) such that

\[
\|\nabla_x d^{(N)}_f(\gamma, \gamma')\|_X^2 \leq C_1 s^2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x). \tag{31}
\]

Analogously, using (20)–(22), we get

\[
|\nabla_s d^{(N)}_f(\gamma, \gamma')| \leq \sum_{k=1}^{N} c_k |\nabla_s d^{(N)}_k(\gamma, \gamma')| \\
\leq 2 \sum_{k=1}^{N} c_k \sum_{n \in \mathbb{Z} \cap [-N, N]} \left| \frac{\partial}{\partial s} \zeta_{kn}(x, s) \right| \\
= 2 \sum_{k=1}^{N} c_k \phi_k(x) \sum_{n \in \mathbb{Z} \cap [-N, N]} \left| \psi'_n(s) s + \psi_n(s) \right| \\
\leq 2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z}} \left( \frac{2}{q^n(1-q)} \chi_{[q^n+1, q^n] \cup [q^n, q^n-1]}(s) s + \chi_{[q^n+1, q^n-2]}(s) \right) \\
\leq 2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \sum_{n \in \mathbb{Z}} \left( \frac{2}{q^n(1-q)} \chi_{[q^n+1, q^n] \cup [q^n, q^n-1]}(s) q^{n-2} + \chi_{[q^n+1, q^n-2]}(s) \right)
\]
\[
\leq 2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x) \left( \frac{8}{q^2(1 - q)} + 4 \right).
\]

Hence, there exists a constant \( C_2 > 0 \) such that
\[
|\nabla s F(\gamma)|^2 \leq C_2 \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x).
\]

We define, for \( \gamma \in \Gamma_{pf}(\hat{X}) \),
\[
G_2(\gamma) := (C_1 + C_2) \sum_{(s,x) \in \gamma} s \sum_{k=1}^{\infty} c_k \chi_{B(k+1)}(x).
\]

By the monotone convergence theorem,
\[
\int_{\Gamma_{pf}(\hat{X})} G_2 \ d\rho = (C_1 + C_2) \sum_{k=1}^{\infty} c_k \int_{\Gamma_{pf}(\hat{X})} \sum_{(s,x) \in \gamma} s \chi_{B(k+1)}(x) \ d\rho(\gamma)
\]
\[
= (C_1 + C_2) \sum_{k=1}^{\infty} c_k \int_{\mathbb{K}(X)} \eta(B(k+1)) \ d\mu(\eta).
\]

By (13), we have, for each \( k \in \mathbb{N} \),
\[
\int_{\mathbb{K}(X)} \eta(B(k+1)) \ d\mu(\eta) < \infty.
\]

So we may set
\[
c_k := 2^{-k} \left(1 + \int_{\mathbb{K}(X)} \eta(B(k+1)) \ d\mu(\eta)\right)^{-1}, \quad k \in \mathbb{N}.
\]

Then, by (34), we get \( G_2 \in L^1(\tilde{\Gamma}_f(\hat{X}), \rho) \). Furthermore, by (28), (31)–(33), we get
\[
S_{\Gamma}(d_f^{(N)}(\cdot, \gamma')) \leq G_2 \quad \text{point-wise on \( \Gamma_{pf}(\hat{X}) \).}
\]

Using (36) and the dominated convergence theorem, it is not hard to prove that
\[
\mathcal{E}_{\Gamma}(d_f^{(N)}(\cdot, \gamma') - d_f^{(M)}(\cdot, \gamma')) \to 0 \quad \text{as \( N, M \to \infty \).}
\]

Hence, \( (d_f^{(N)}(\cdot, \gamma'))_{N=1}^{\infty} \) is a Cauchy sequence in \( (D(\mathcal{E}_{\Gamma}), \| \cdot \|_{D(\mathcal{E}_{\Gamma})}) \). Hence, by (30) and (37), \( d_f(\cdot, \gamma') \in D(\mathcal{E}_{\Gamma}) \). Furthermore, since \( d_f^{(N)}(\cdot, \gamma') \to d_f(\cdot, \gamma') \) in the \( \| \cdot \|_{D(\mathcal{E}_{\Gamma})} \) norm,
\[
S_{\Gamma}(d_f^{(N)}(\cdot, \gamma')) \to S_{\Gamma}(d_f(\cdot, \gamma')) \quad \text{in} \ L^1(\tilde{\Gamma}_f(\hat{X}), \rho) \quad \text{as \( N \to \infty \).}
\]

Hence, by (36), \( S_{\Gamma}(d_f(\cdot, \gamma)) \leq G_2 \ \rho\text{-a.e.} \)
By [14, Proposition 4.1] (see also [17, Theorem 3.4]), Proposition 15 and Lemma 17 imply the following proposition.

**Proposition 18.** The Dirichlet form \((\mathcal{E}_\Gamma, D(\mathcal{E}_\Gamma))\) on \(L^2(\tilde{\Gamma}_f(\hat{X}), \rho)\) is quasi-regular.

**Step 4.** We will now construct a corresponding diffusion process on \(\tilde{\Gamma}_f(\hat{X})\).

**Lemma 19.** The Dirichlet form \((\mathcal{E}_\Gamma, D(\mathcal{E}_\Gamma))\) has local property, i.e., \(\mathcal{E}_\Gamma(F, G) = 0\) provided \(F, G \in D(\mathcal{E}_\Gamma)\) with \(\text{supp}(|F| \rho) \cap \text{supp}(|G| \rho) = \emptyset\).

**Proof.** Identical to the proof of [14, Proposition 4.12]. \qed

As a consequence of Proposition 18, Lemma 19, and [13, Chap. IV, Theorem 3.5, and Chap. V, Theorem 1.11], we obtain

**Proposition 20.** There exists a conservative diffusion process on the metric space \((\tilde{\Gamma}_f(\hat{X}), d(\cdot, \cdot))\),

\[
M^\Gamma = (\Omega^\Gamma, \mathcal{F}^\Gamma, (\mathcal{F}_t^\Gamma)_{t \geq 0}, (\Theta_t^\Gamma)_{t \geq 0}, (\mathcal{X}_t^\Gamma)_{t \geq 0}, (\mathbb{P}_\gamma^\Gamma)_{\gamma \in \tilde{\Gamma}_f(\hat{X})}),
\]

which is properly associated with the Dirichlet form \((\mathcal{E}_\Gamma, D(\mathcal{E}_\Gamma))\). Here \(\Omega^\Gamma = C([0, \infty) \to \tilde{\Gamma}_f(\hat{X}))\), \(\mathcal{X}^\Gamma(t)(\omega) = \omega(t), t \geq 0, \omega \in \Omega^\Gamma\), \((\mathcal{F}_t^\Gamma)_{t \geq 0}\) together with \(\mathcal{F}^\Gamma\) is the corresponding minimum completed admissible family, and \(\Theta_t^\Gamma, t \geq 0\), are the corresponding natural time shifts. This process is up to \(\rho\)-equivalence unique.

**Step 5.** We will now show that the diffusion process from Proposition 20 lives, in fact, on the smaller space \(\Gamma_{pf}(\hat{X})\). This is where we use that the dimension \(d\) of the underlying space \(X\) is \(\geq 2\).

**Proposition 21.** The set \(\tilde{\Gamma}_f(\hat{X}) \setminus \Gamma_{pf}(\hat{X})\) is \(\mathcal{E}_\Gamma\)-exceptional. Thus, the statement of Proposition 20 remains true if we replace in it \(\tilde{\Gamma}_f(\hat{X})\) with \(\Gamma_{pf}(\hat{X})\).

**Proof.** The proof of this statement is similar to the proof of [18, Proposition 1 and Corollary 1], see also the proof of [11, Theorem 6.3]. \qed

**Step 6.** We will now prove that the mapping \(R\) is continuous with respect to the \(d(\cdot, \cdot)\) metric.

**Proposition 22.** The mapping \(R\) acts continuously from the metric space \((\Gamma_{pf}(\hat{X}), d(\cdot, \cdot))\) into the space \(\mathcal{K}(X)\) endowed with the vague topology.

**Proof.** Let \(\{\gamma_i\}_{i=1}^\infty \subset \Gamma_{pf}(\hat{X})\) and \(\gamma \in \Gamma_{pf}(\hat{X})\). Let \(d(\gamma_i, \gamma) \to 0\) as \(i \to \infty\). We have to prove that \(R\gamma_i \to R\gamma\) vaguely as \(i \to \infty\).

So fix any \(f \in C_0(X)\) and \(\varepsilon > 0\). Choose \(k \in \mathbb{N}\) such that \(\text{supp}(f) \subset B(k)\). Choose \(N \in \mathbb{N}\) such that

\[
\sum_{n \in \mathbb{Z}, |n| \geq N} \langle x_{kn}, \gamma \rangle \leq \varepsilon. \tag{38}
\]
Since $d(\gamma_i, \gamma) \to 0$, we have $d_k(\gamma_i, \gamma) \to 0$. Hence, there exists $I \in \mathbb{N}$ such that
\[ \sum_{n \in \mathbb{Z}, |n| \geq N} \langle \gamma_i, \kappa_{kn} \rangle \leq 2\varepsilon, \quad i \geq I. \quad (39) \]

By (20)–(22), (38), and (39),
\[
\begin{align*}
\int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} s d\gamma_i(x, s) &\leq \varepsilon, \\
\int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} s d\gamma_i(x, s) &\leq 2\varepsilon, \quad i \geq I.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} |f(x)| s d\gamma_i(x, s) &\leq \varepsilon \|f\|_\infty, \\
\int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} |f(x)| s d\gamma_i(x, s) &\leq 2\varepsilon \|f\|_\infty, \quad i \geq I, \quad (40)
\end{align*}
\]

where $\|f\|_\infty$ is the supremum norm of the function $f$. Fix any $\xi \in C_0(\mathbb{R}_+)$ such that
\[ \chi_{[q^N, q^{-N}]} \leq \xi \leq 1. \quad (41) \]

Since the function $f(x)\xi(s)s$ is from $C_0(\hat{X})$, by the vague convergence
\[
\int_{\hat{X}} f(x)\xi(s)s d\gamma_i(x, s) \to \int_{\hat{X}} f(x)\xi(s)s d\gamma(x, s) \quad \text{as } i \to \infty.
\]

Hence, there exists $I_1 \geq I$ such that
\[ \left| \int_{\hat{X}} f(x)\xi(s)s d(\gamma_i - \gamma)(x, s) \right| \leq \varepsilon, \quad i \geq I_1. \quad (42) \]

By (40)–(42), for all $i \geq I_1$,
\[
\begin{align*}
\left| \int_{B(k) \times [q^N, q^{-N}]} f(x) s d(\gamma_i - \gamma)(x, s) \right| &= \left| \int_{B(k) \times [q^N, q^{-N}]} f(x) \xi(s) s d(\gamma_i - \gamma)(x, s) \right| \\
&\leq \left| \int_{\hat{X}} f(x) \xi(s) s d(\gamma_i - \gamma)(x, s) \right| \\
&\quad + \left| \int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} f(x) \xi(s) s d\gamma_i(x, s) \right| \\
&\quad + \left| \int_{B(k) \times ((0, q^N) \cup (q^{-N}, \infty))} f(x) \xi(s) s d\gamma(x, s) \right|
\end{align*}
\]
By (40) and (43), for all $i \geq I_1$,

$$\left| \int_X f(x) \, d(\mathcal{R} \gamma_i - \mathcal{R} \gamma)(x) \right| = \left| \int_{\hat{X}} f(x) s \, d(\gamma_i - \gamma)(x, s) \right| \leq \varepsilon (1 + 6\|f\|_{\infty}).$$

Thus, the proposition is proven. \hfill \Box

**Step 7.** Finally, to construct the process $M^K$ on $\kappa(X)$, we just map the process $M^\Gamma$ from Proposition 20 onto $\kappa(X)$ by using the bijective mapping $\mathcal{R} : \Gamma_{pf}(\hat{X}) \to \kappa(X)$. Proposition 22 ensures that the sample paths of the obtained Markov process are continuous in the vague topology on $\kappa(X)$.

### Acknowledgements

The authors acknowledge the financial support of the SFB 701 “Spectral structures and topological methods in mathematics” (Bielefeld University).

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