SHARP ISOPERIMETRIC COMPARISON ON NON-COLLAPSED SPACES WITH LOWER RICCI BOUNDS

COMPARAISON ISOPÉRIMÉTRIQUE OPTIMALE POUR LES ESPACES NON EFFONDÉS À COURBURE DE RICCI MINORÉE

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Abstract. This paper studies sharp isoperimetric comparison theorems and sharp dimensional concavity properties of the isoperimetric profile for non-smooth spaces with lower Ricci curvature bounds, the so-called \( N \)-dimensional RCD(\( K, N \)) spaces.

The absence of most of the classical tools of geometric measure theory and the possible non-existence of isoperimetric regions on non-compact spaces are handled via an original argument to estimate first and second variation of the area for isoperimetric sets, avoiding any regularity theory, in combination with an asymptotic mass decomposition result of perimeter-minimizing sequences.

Most of our statements are new even for smooth, non-compact manifolds with lower Ricci curvature bounds and for Alexandrov spaces with lower sectional curvature bounds. They generalize several results known for compact manifolds, non-compact manifolds with uniformly bounded geometry at infinity, and Euclidean convex bodies.

Résumé. Cet article étudie les théorèmes de comparaison isopérimétrique et les propriétés de concavité du profil isopérimétrique pour les espaces non lisses avec à courbure de Ricci minorée: les espaces RCD(\( K, N \)) de dimension \( N \).

L’absence de la plupart des outils classiques de théorie géométrique de la mesure et la non-existence possible de régions isopérimétriques dans les espaces non compacts sont traitées au moyen d’un argument original pour estimer la première et la deuxième variation de l’aire pour les ensembles isopérimétriques, en évitant la théorie de régularité. Cet argument est combiné avec un résultat de décomposition asymptotique de masse pour les suites minimisant le périmètre.

La plupart de nos énoncés sont nouveaux même pour les variétés lisses non compactes avec à courbure de Ricci minorée, et pour les espaces d’Alexandrov à courbure sectionnelle minorée. Ils généralisent plusieurs résultats connus pour les variétés compactes, les variétés non compactes avec à géométrie uniformément bornée à l’infini, et les corps convexes euclidiens.

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1. Introduction

Isoperimetry and lower Ricci curvature bounds. There is a celebrated connection between Ricci curvature and the isoperimetric problem in geometric analysis, going back at least
to the Lévy–Gromov inequality [60, Appendix C]. The primary goal of this paper is to extend several results about the isoperimetric problem on compact Riemannian manifolds with lower Ricci curvature bounds to non-compact Riemannian manifolds and non-smooth spaces with lower Ricci curvature bounds. In order to deal with the possible non-existence of isoperimetric regions and with the lack of regularity we develop a series of new tools with respect to the classical literature. Non-smooth spaces enter into play naturally when dealing with smooth non-compact Riemannian manifolds, via the analysis of their pointed limits at infinity.

We consider the setting of $N$-dimensional RCD($K$, $N$) metric measure spaces $(X, d, \mathcal{H}^N)$, for finite $N \in [1, \infty)$ and $K \in \mathbb{R}$, see [48, 67] after [89, 90, 72, 9, 53, 6, 50, 13, 39]. Here $K \in \mathbb{R}$ plays the role of (synthetic) lower bound on the Ricci curvature, $N \in [1, \infty)$ plays the role of (synthetic) upper bound on the dimension and $\mathcal{H}^N$ indicates the $N$-dimensional Hausdorff measure. This class includes (convex subsets of) smooth Riemannian manifolds with lower Ricci curvature bounds endowed with their volume measure, their noncollapsing measure Gromov–Hausdorff limits [45], and finite dimensional Alexandrov spaces with sectional curvature lower bounds [32, 86].

We shall rely on the theory of sets of finite perimeter in RCD($K$, $N$) spaces, as developed in [3, 5, 30, 29]. For the sake of this introduction we just remark that it is fully consistent with the Euclidean and Riemannian ones. In particular, (reduced) boundaries of sets of finite perimeter are rectifiable, the perimeter coincides with the restriction of the $(N - 1)$ – dimensional Hausdorff measure to the (reduced) boundary and it does not charge the boundary of the ambient space.

Given an RCD($K$, $N$) metric measure space $(X, d, \mathcal{H}^N)$ such that $\mathcal{H}^N(B_1(x)) \geq v_0$ for any $x \in X$ for some $v_0 > 0$, we introduce the isoperimetric profile $I_X : [0, \mathcal{H}^N(X)) \to [0, \infty)$ by

$$I_X(v) := \inf \left\{ \operatorname{Per}(E) : E \subset X, \mathcal{H}^N(E) = v \right\},$$

where we drop the subscript $X$ when there is no risk of confusion. When $E \subset X$ attains the infimum in (1.1) for $v = \mathcal{H}^N(E)$, we call it an isoperimetric region. In this setting we obtain:

- sharp second order differential inequalities for the isoperimetric profile, corresponding to equalities on the model spaces with constant sectional curvature. These inequalities are new even in the case of non-compact Riemannian manifolds and use in a crucial way the non-smooth approach. The proof bypasses the possible non-existence of isoperimetric regions on the space, that is classically used for such arguments, employing a concentration-compactness argument;
- a sharp Laplacian comparison theorem for the distance function from $\partial E$, which is a fundamental tool to prove the above items since it corresponds to the bounds usually obtained via first and second variation of the area in this low regularity setting;
- Gromov–Hausdorff stability and perimeters’ convergence of isoperimetric regions along non-collapsing sequences of $N$-dimensional RCD($K$, $N$) spaces. In order to prove these statements we deduce uniform regularity estimates for isoperimetric sets from uniform concavity estimates of the isoperimetric profiles.

Many of the above results are new even for smooth, non-compact manifolds with lower Ricci curvature bounds and for Alexandrov spaces with lower sectional curvature bounds. They answer several open questions in [23, 73, 70, 83, 21]. We expect the techniques developed in this paper to have a broad range of applications in geometric analysis under lower curvature bounds. For instance, in the study of more general geometric variational problems, the isoperimetric problem on weighted Riemannian manifolds with lower bounds on the Bakry-Émery curvature tensor, other geometric and functional inequalities.

**Main results.** On model spaces with constant sectional curvature $K/(N - 1) \in \mathbb{R}$ and dimension $N \geq 2$ the isoperimetric profile $I_{K,N}$ solves the following second order differential
equation on its domain:

\[-I''_{K,N} I_{K,N} = K + \frac{\left(I'_{K,N}\right)^2}{N-1}.\]  

(1.2)

Equivalently, setting \(\psi_{K,N} := I_{K,N}^{N-1}\), we have

\[-\psi''_{K,N} = \frac{K N}{N-1} \psi_{K,N}^{2-N}.\]  

(1.3)

Combining the existence of isoperimetric regions for any volume, the regularity theory in geometric measure theory, and the second variation of the area (1.4), in [22, 23, 24, 79, 84] it was proved that the isoperimetric profile of a smooth, compact, \(N\)-dimensional Riemannian manifold with \(\text{Ric} \geq K\) verifies the inequality \(\geq\) in (1.2) and (1.3) in a weak sense.

Here we obtain the following extension to the setting of RCD\((K,N)\) metric measure spaces \((X,d,\mathcal{H}^N)\) with a uniform lower bound on the volume of unit balls, without any assumption on the existence of isoperimetric regions. We stress again that the classical argument to show Theorem 1.1 in the compact setting uses in a crucial way the existence of isoperimetric regions for every volume, that we do not have at disposal in the present setting.

**Theorem 1.1** (cf. with Theorem 4.4). Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) space. Assume that there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\).

Let \(I : (0,\mathcal{H}^N(X)) \rightarrow (0,\infty)\) be the isoperimetric profile of \(X\). Then:

1. the inequality

\[-I'' I \geq K + \frac{(I')^2}{N-1}\]

holds in the viscosity sense on \((0,\mathcal{H}^N(X))\),

2. if \(\psi := I^{N-1}\) then

\[-\psi'' \geq \frac{K N}{N-1} \psi^{2-N}\]

holds in the viscosity sense on \((0,\mathcal{H}^N(X))\).

In particular, the above holds for non-compact smooth Riemannian manifolds with Ricci curvature bounded from below and volume of unit balls uniformly bounded away from zero. In the smooth noncompact setting, Theorem 1.1 was previously known only under the additional assumption of existence of isoperimetric sets, or under strong conditions on the asymptotic geometry, see [76]. To the best of our knowledge, this is the first application of the theory of RCD spaces to prove a new sharp geometric inequality on smooth Riemannian manifolds.

The proof of Theorem 1.1 combines the generalized existence of isoperimetric regions (cf. with Theorem 4.1), the interpretation of the differential inequalities in the viscosity sense and the forthcoming Laplacian comparison Theorem 1.2 to estimate first and second variation of the area via equidistants in the non-smooth setting.

The main new tool that we develop to prove Theorem 1.1 is a sharp bound on the Laplacian of the signed distance function from isoperimetric regions inside RCD\((K,N)\) metric measure spaces \((X,d,\mathcal{H}^N)\).

If \(E\) is a (smooth) isoperimetric region inside a smooth Riemannian manifold \((M^N,g)\) the first variation formula implies that the mean curvature \(H\) of its boundary \(\partial E\) is constant. Moreover, if \(t \mapsto E_t\) denotes the parallel deformation of \(E\) via equidistant sets, \(\nu\) and II denote a choice of the unit normal to \(\partial E\) and its second fundamental form, respectively, and \(\text{Ric}(\nu,\nu)\) indicates the Ricci curvature of \(M\) in the direction of \(\nu\), then the second variation formula yields that

\[
\frac{d^2}{dt^2}|_{t=0}\text{Per}(E_t) = \int_{\partial E} \left(H^2 - ||\text{II}||^2 - \text{Ric}(\nu,\nu)\right) d\text{Per},
\]  

(1.4)

where we denoted by \(\text{Per}\) the perimeter, which coincides with the Riemannian surface measure for sufficiently regular sets.
If we further assume that $\text{Ric} \geq Kg$, for some $K \in \mathbb{R}$, then the Cauchy–Schwarz inequality applied to the eigenvalues of $\Pi$ yields

$$\frac{d^2}{dt^2} \bigg|_{t=0} \text{Per}(E_t) \leq \left( \frac{N-2}{N-1} H^2 - K \right) \text{Per}(E). \quad (1.5)$$

One of the main technical achievements of this work is to develop a counterpart of (1.5) for isoperimetric regions in non-smooth spaces with lower Ricci curvature bounds. Our argument departs from the classical literature, following the general scheme outlined above, and it requires a different, global rather than infinitesimal, perspective.

Let us introduce the comparison functions

$$s_{k,\lambda}(r) := \cos_k(r) - \lambda \sin_k(r), \quad (1.6)$$

where

$$\cos''_k + k \cos_k = 0, \quad \cos_k(0) = 1, \quad \cos'_k(0) = 0, \quad (1.7)$$

and

$$\sin''_k + k \sin_k = 0, \quad \sin_k(0) = 0, \quad \sin'_k(0) = 1. \quad (1.8)$$

**Theorem 1.2** (cf. with Theorem 3.3). Let $(X,d,\mathcal{H}^N)$ be an $\text{RCD}(K,N)$ metric measure space for some $K \in \mathbb{R}$ and $N \geq 2$ and let $E \subset X$ be an isoperimetric region. Then, denoting by $f$ the signed distance function from $E$, and $N' := N - 1$, there exists $c \in \mathbb{R}$ such that

$$\Delta f \geq -(N-1)\frac{\lambda''_k/c}{\lambda'_{k'/c}} \circ (-f) \quad \text{on } E, \quad \text{and} \quad \Delta f \leq (N-1)\frac{\lambda''_k/c}{\lambda'_{k'/c}} \circ (-f) \quad \text{on } X \setminus \overline{E}. \quad (1.9)$$

The bounds in (1.9) are understood in the sense of distributions, and we always consider open representatives for isoperimetric regions (see Theorem 2.16). They are sharp, since equalities are attained in the model spaces with constant sectional curvature.

Notice that the distance function might not be globally smooth even when $(X,d)$ is isometric to a smooth Riemannian manifold and $E \subset X$ has smooth boundary, in which case (1.9) is equivalent to the requirement that $\partial E$ has constant mean curvature equal to $c$. We will indicate any $c \in \mathbb{R}$ such that (1.9) holds as a mean curvature barrier for $E$.

**Consequences and strategies of the proofs.** Several consequences of Theorem 1.1 are investigated in the rest of the paper:

- uniform semi-concavity and Lipschitz properties of the isoperimetric profile in a fixed range of volumes, only depending on the lower Ricci curvature bound, the dimension and a lower bound on the volume of unit balls, see Proposition 4.9, Corollary 4.13, and Corollary 4.14;
- the existence of the limit $\lim_{v \to 0} I(v)/v^{N-1} \in (0, N\omega^N_k)$ on any $\text{RCD}(K,N)$ space $(X,d,\mathcal{H}^N)$ with volume of unit balls uniformly bounded from below, see Proposition 4.9 and Remark 4.12;
- the strict subadditivity of the isoperimetric profile for small volumes (only depending on $K$, $N$ and the uniform lower bound on the volume of unit balls), see Proposition 4.11. This implies in turn that isoperimetric regions with small volume are connected, see Corollary 4.16. Moreover, in the asymptotic mass decomposition, minimizing sequences for small volumes do not split: either they converge to an isoperimetric region, or they drift off to exactly one isoperimetric region in a pointed limit at infinity, see Lemma 4.18. All the previous conclusions hold for every volume when $K = 0$;
- uniform, scale invariant diameter estimates for isoperimetric regions of small volume, without further assumptions, and for any volume when $K = 0$ and $(X,d,\mathcal{H}^N)$ has Euclidean volume growth, see Proposition 4.21;
• uniform density estimates and uniform almost minimality properties for isoperimetric sets, see Corollary 4.15. They allow to bootstrap $L^1$-convergence to Gromov–Hausdorff convergence and convergence of the perimeters for sequences of isoperimetric sets and to prove the stability of mean curvature barriers obtained with Theorem 1.2, see Theorem 4.23. We remark that uniform almost minimality properties are fundamental in several circumstances in geometric measure theory, see for instance [91], and that the classical strategies to achieve them break in the present setting, as they heavily rely on smoothness.

Let us further comment on Theorem 1.2. When $\partial E$ is a smooth constant mean curvature hypersurface in a smooth Riemannian manifold, the Laplacian of the signed distance function from $E$ equals the mean curvature of $\partial E$ along $\partial E$. Then (1.9) can be proved with a classical computation using Jacobi fields and one dimensional comparison for Riccati equations away from the cut locus, finding its roots in [92, 37]. The singular contribution coming from the cut locus has the right sign, in great generality.

The original proof of the Lévy–Gromov isoperimetric inequality [60, Appendix C] builds on a variant of (1.9). The key additional difficulty with respect to smooth constant mean curvature hypersurfaces is that boundaries of isoperimetric regions might be non-smooth when $N \geq 8$ and it is handled relying on a deep regularity theorem in geometric measure theory [2]. This strategy seems out of reach in the setting of RCD spaces. Our proof of Theorem 1.2 partially avoids the regularity theory even on smooth Riemannian manifolds. It is inspired by [36, 35, 85] and the recent study of perimeter minimizing sets in [77].

We start proving adimensional versions of (1.9), corresponding to the limit of (1.9) as $N \to \infty$. Exploiting the equivalence between distributional and viscosity bounds on the Laplacian from [77], we prove that if the bounds fail there exists a volume fixing perturbation of $E$ with strictly smaller perimeter, a contradiction with the isoperimetric condition. The perturbations are built by sliding simultaneously level sets of distance-like functions with well controlled Laplacian. The argument can be thought as a highly non-linear version of the moving planes method, with no symmetries on the background and in a very low regularity setting.

The stability of Laplacian bounds under the Hopf–Lax duality (equivalently, along solutions of the Hamilton–Jacobi equation), proved in [77] building on [68, 10], plays the role of the classical computation with Jacobi fields in Riemannian geometry. It crucially enters into play in the construction of the perturbations. The adimensional versions of (1.9) can be improved to sharp dimensional bounds thanks to the well established localization technique [40, 41]. The additional difficulty with respect to the case of local perimeter minimizers considered in [77] is that only volume fixing perturbations are admissible for the isoperimetric problem.

We do not claim that Theorem 1.2 carries all the information provided by the first and second variation formulas for the area in the smooth setting, however one of the main novelties of the present work will be to show that it is a valid replacement in several circumstances. Moreover, its global nature makes it more suitable for stability arguments. Among its direct consequences we mention sharp Heintze–Karcher type bounds for perimeters Proposition 3.11 and volumes Corollary 3.13 of variations of isoperimetric sets via equidistant sets, that can be obtained by integration.

The ability to deduce information about the isoperimetric behaviour of an RCD$(K,N)$ metric measure space $(X,d,\mathcal{H}^N)$ from Theorem 1.2 is related to the existence of isoperimetric regions, which is not guaranteed, when $\mathcal{H}^N(X) = \infty$, see, e.g., [16, Example 3.6], or the introduction of [82]. We overcome this issue thanks to the asymptotic mass decomposition result recently proved in [17], extending the previous [88, 82, 76, 16] and building on a concentration-compactness argument. If $(X,d)$ is compact, a minimizing sequence for the isoperimetric problem (1.1) for volume $V > 0$ converges, up to subsequences, to an isoperimetric set, by lower semicontinuity
of the perimeter. This is not true, in general, when $(X,d)$ is not compact as elementary examples illustrate. However, [17] shows that, if $(X,d,H^N)$ is a non-compact RCD$(K,N)$ space with $H^N(B_1(p)) > v_0$ for any $p \in X$ for some $v_0 > 0$, then, up to subsequences, every minimizing sequence for the isoperimetric problem at a given volume $V > 0$ splits into finitely many pieces. One of them converges to an isoperimetric region in $X$, possibly of volume less than $V$ (possibly zero). The others converge to isoperimetric regions inside pointed measured Gromov–Hausdorff limits at infinity $(Y,d_Y,H^N,q)$ of sequences $(X,d,H^N,p_i)$, where $d(p,p_i) \to \infty$ as $i \to \infty$ for some reference point $p \in X$. Moreover, there is no loss of total mass in this process, see Theorem 4.1, hence the result can be seen as a generalized existence of isoperimetric regions.

An observation going back to [82] (see also the subsequent [76]) is that if $(X,d)$ and all its pointed limits at infinity are isometric to smooth Riemannian manifolds, then generalized isoperimetric regions can be used as isoperimetric regions in the compact case in combination with (1.4) to deduce useful information. However, the assumption that $(X,d)$ is isometric to a smooth Riemannian manifold with $\text{Ric} \geq K$ and $\mathcal{H}^N(B_1(p)) > v_0$, for every $p \in X$ and some $v_0 > 0$, does not guarantee any regularity of its pointed limits at infinity besides them being non-collapsed Ricci limit spaces.

The ability to estimate the second variation of the perimeter via equidistant sets for isoperimetric sets in general RCD$(K,N)$ spaces $(X,d,H^N)$ as in Theorem 1.2 makes this heuristic work without further assumptions in the smooth Riemannian case and in greater generality, obtaining Theorem 1.1. This is a fundamental new contribution of the present work.

Let us briefly comment on the possible extensions of the present work to more general settings. The generalization of (some of) the results obtained in this paper to general RCD$(K,N)$ metric measure spaces $(X,d,m)$ with $N < \infty$ is left to the future investigation, due to some additional difficulties with respect to the case $m = H^N$ considered here that we outline below.

The two main ingredients for the proof of Theorem 1.1 are the asymptotic mass decomposition Theorem 4.1 and the Laplacian comparison Theorem 1.2. The asymptotic mass decomposition Theorem 4.1 has been obtained in [17] by the first and the third authors together with Nardulli, leveraging on the techniques developed in [82, 16]. We expect that the results of [17] generalize to arbitrary RCD$(K,N)$ metric measure spaces $(X,d,m)$. However, when $m = H^N$, in the proof of [17] we exploit the fact that the density of the measure is 1 almost everywhere, and the perimeters of the balls centered at almost every point are infinitesimally equivalent to the perimeters of the balls in $\mathbb{R}^N$. Both these statement fail in general when $m \neq H^N$.

We also expect that Theorem 1.2 holds for arbitrary reference measures $m$. In particular, we notice that the Laplacian comparison holds for isoperimetric sets in smooth weighted Riemannian manifolds verifying the CD$(K,N)$ condition. However, the extension would require again some new insights, in particular in reference to the mild topological regularity for isoperimetric sets obtained in [18].

Besides the case of general RCD$(K,N)$ metric measure spaces $(X,d,m)$ with finite $N$, two natural directions for the future investigation are the infinite-dimensional and the non-linear settings.

In the case of RCD$(K,\infty)$ metric measure spaces we raise the following:

**Conjecture 1.3.** Let $(X,d,m)$ be an RCD$(K,\infty)$ metric measure space. Then the isoperimetric profile $I$ satisfies the second order differential inequality

$$-I'' I \geq K \quad (1.10)$$

in the viscosity sense on its domain.

We remark that the inequality (1.10) would be saturated by the Euclidean space endowed with the standard metric and a (suitably normalized) weighted Gaussian measure.
In the direction of removing the Hilbertian assumption, to the best of our knowledge, the validity of sharp differential second-order inequalities for the isoperimetric profile has not been investigated before even in the case of smooth Finsler manifolds satisfying curvature-dimension bounds. In this regard we ask the following:

**Question 1.4.** Do the Laplacian comparison for the distance from isoperimetric sets Theorem 1.2 and the sharp second-order differential inequalities for the isoperimetric profile Theorem 1.1 hold for (possibly essentially non-branching) CD($K, N$) metric measure spaces?

Apart from the additional technical challenges with respect to the setting considered in the present paper, we believe that addressing Conjecture 1.3 and Question 1.4 might require the development of new strategies.

**Comparison with the previous literature.** We conclude this introduction with a brief comparison between our results and the previous literature about the isoperimetric problem under lower curvature bounds, without the aim of being comprehensive.

- The difficulty of obtaining second order properties for the isoperimetric profile on non-
  smooth spaces was pointed out in [23, page 99], [70] and in [73, Appendix]. To the best
  of our knowledge, Theorem 1.1 is the first instance of second order properties of the
  isoperimetric profile in a context where no approximation with smooth Riemannian
  manifolds is at disposal.
- The setting of RCD(0, N) spaces $(X, d, \mathcal{H}^N)$ recovers in particular many of the results
  for Euclidean convex cones treated in [88] and for cones with non-negative Ricci
  curvature considered in [80].
- The results of the present paper recover, in a more general setting, many of the results
  proved in [71] for unbounded Euclidean convex bodies of uniform geometry.
- The stability of mean curvature barriers in the sense of Theorem 1.2 for Gromov–
  Hausdorff converging sequences of boundaries inside measured Gromov–Hausdorff conver-
  ging sequences of RCD($K, N$) spaces has been recently observed in [65] (see also
  the previous [28]). In this regards, The main novelty of our work is to provide a large
  and natural class of sets having mean curvature barriers, namely isoperimetric sets.
  Moreover, we prove that $L^1$ convergence (which is guaranteed, up to subsequences,
  for equibounded isoperimetric sets) self-improves to Gromov–Hausdorff convergence
  under very natural assumptions. Therefore the stability of mean curvature barriers
  applies in this setting.

In order to put things into perspective, we stress that this paper heavily relies on the
results of [18, 17] (mild regularity of isoperimetric sets and asymptotic mass decomposition,
respectively), while it is independent of the existence results in [15] by the first and third
authors together with Bruè and Fogagnolo, and it is completely independent of the sharp
isoperimetric inequality obtained in [27, 21]. The key contribution of the present work is to
develop some refined tools of geometric measure theory in a low regularity setting and to
combine them in original way with the asymptotic mass decomposition from [17], obtaining
several consequences for the isoperimetric problem under lower Ricci curvature bounds.

**Addendum.** This is the first of two companion papers, together with [20]. The joint version
of the two papers appeared on arXiv in [19]. In the second one we are going to explore several
consequences of the main results of this paper to the study of asymptotic isoperimetry on
non-collapsed spaces with Ricci lower bounds, especially in the case where $K = 0$: 
• exploiting Theorem 1.2, we give a new proof of the sharp isoperimetric inequality on RCD(0, N) spaces \((X, d, \mathcal{H}^N)\) with Euclidean volume growth that has been considered, with increasing level of generality, in [1, 51, 27, 21, 38]. We prove also the rigidity of the inequality in the setting above without any additional regularity assumption. Namely we prove that equality is achieved if and only if the ambient space is isometric to a cone and the set saturating the inequality is isometric to a ball centered at a tip of the cone;

• we explicitly determine the asymptotic behavior of the isoperimetric profile for volumes tending to zero on RCD\((K, N)\) spaces \((X, d, \mathcal{H}^N)\), see Remark 4.12;

• we analyze the behavior of sequences of isoperimetric sets for volumes tending to zero on RCD\((K, N)\) spaces \((X, d, \mathcal{H}^N)\) and for volumes tending to infinity on Alexandrov spaces with non-negative curvature and Euclidean volume growth.

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2. Preliminaries

In this paper, by a metric measure space (briefly, m.m.s.) we mean a triple \((X, d, \mathfrak{m})\), where \((X, d)\) is a complete and separable metric space, while \(\mathfrak{m} \geq 0\) is a boundedly-finite Borel measure on \(X\). For any \(k \in [0, \infty)\), we denote by \(\mathcal{H}^k\) the \(k\)-dimensional Hausdorff measure on \((X, d)\). We denote with \(C(X)\) the space of all continuous functions \(f : X \to \mathbb{R}\) and \(C_b(X) := \{ f \in C(X) : f \text{ is bounded} \}\). We denote by \(\text{LIP}(X) \subseteq C(X)\) the space of all Lipschitz functions, while \(\text{LIP}_{bs}(X)\) (resp. \(\text{LIP}_c(X)\)) stands for the set of all those \(f \in \text{LIP}(X)\) whose support \(\text{spt}f\) is bounded (resp. compact). More generally, we denote by \(\text{LIP}_{loc}(X)\) the space of locally Lipschitz functions \(f : X \to \mathbb{R}\). Given \(f \in \text{LIP}_{loc}(X)\),

\[
\text{lip} \, f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}
\]

is the slope of \(f\) at \(x\), for any accumulation point \(x \in X\), and \(\text{lip} \, f(x) := 0\) if \(x \in X\) is isolated.

We shall also work with the local versions of the above spaces: given \(\Omega \subseteq X\) open, we will consider the spaces \(\text{LIP}_{\Omega}(\Omega) \subseteq \text{LIP}_{bs}(\Omega) \subseteq \text{LIP}(\Omega) \subseteq \text{LIP}_{loc}(\Omega)\), where by \(\text{LIP}_{bs}(\Omega)\) we mean the space of all \(f \in \text{LIP}(\Omega)\) having bounded support \(\text{spt}f \subseteq \Omega\) that verifies \(d(\text{spt}f, \partial \Omega) > 0\).

Let us define

\[
\text{sn}_{K}(r) := \begin{cases} 
(-K)^{-\frac{1}{2}} \sinh((K)^{\frac{1}{2}}r) \quad & K < 0, \\
K^{-\frac{1}{2}} \sin(K^{\frac{1}{2}}r) \quad & K = 0, \\
K^{-\frac{1}{2}} \sinh((K)^{\frac{1}{2}}r) \quad & K > 0.
\end{cases}
\]

We denote by \(v(N, K, r)\) and \(s(N, K, r)\) the volume and the surface measure, respectively, of the ball of radius \(r\) in the (unique) simply connected Riemannian manifold of sectional curvature \(K\) and dimension \(N\). In particular \(s(N, K, r) = N \omega_N \text{sn}_{K}^{N-1}(r)\) and \(v(N, K, r) = \int_0^r N \omega_N \text{sn}_{K}^{N-1}(r) \, dr\), where \(\omega_N\) is the Euclidean volume of the Euclidean unit ball in \(\mathbb{R}^N\).
2.1 Convergence and stability results. Following the exposition of [5], we introduce a definition of pointed measured Gromov–Hausdorff convergence (via a proper realization) that is fit for our purposes. In our setting, where we always deal with locally uniformly doubling measures, this definition is equivalent to the standard one, see [57, Theorem 3.15 and Section 3.5].

Definition 2.1 (pGH and pmGH convergence). A sequence \( \{(X_i, d_i, x_i)\}_{i \in \mathbb{N}} \) of pointed metric spaces is said to converge in the pointed Gromov–Hausdorff topology, in the pGH sense for short, to a pointed metric space \((Y, d_Y, y)\) if there exist a complete separable metric space \((Z, d_Z)\) and isometric embeddings
\[
\Psi_i : (X_i, d_i) \to (Z, d_Z), \quad \forall i \in \mathbb{N},
\]
\[
\Psi : (Y, d_Y) \to (Z, d_Z),
\]
such that for any \( \varepsilon, R > 0 \) there is \( i_0(\varepsilon, R) \in \mathbb{N} \) such that
\[
\Psi_i(B^Z_R(x_i)) \subset \left[ \Psi(B^Y_R(y)) \right]_{\varepsilon}, \quad \Psi(B^Y_R(y)) \subset \left[ \Psi_i(B^X_R(x_i)) \right]_{\varepsilon},
\]
for any \( i \geq i_0 \), where \( [A]_{\varepsilon} := \{ z \in Z : d_Z(z, A) \leq \varepsilon \} \) for any \( A \subset Z \).

Let \( m_i \) and \( \mu \) be given in such a way \((X_i, d_i, m_i, x_i)\) and \((Y, d_Y, \mu, y)\) are m.m.s. If in addition to the previous requirements we also have \((\Psi_i)_* m_i \to \Psi_* \mu\) with respect to duality with continuous bounded functions on \( Z \) with bounded support, then the convergence is said to hold in the pointed measured Gromov–Hausdorff topology, or in the pmGH sense for short.

We need to recall a generalized \( L^1 \)-notion of convergence for sets defined on a sequence of metric measure spaces converging in the pmGH sense. Such a definition is given in [5, Definition 3.1], and it is investigated in [5] capitalizing on the results in [11].

Definition 2.2 (\( L^1 \)-strong and \( L^1_{\text{loc}} \) convergence). Let \( \{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}} \) be a sequence of pointed metric measure spaces converging in the pmGH sense to a pointed metric measure space \((Y, d_Y, \mu, y)\) and let \((Z, d_Z)\) be a realization as in Definition 2.1.

We say that a sequence of Borel sets \( E_i \subset X_i \) such that \( m_i(E_i) < +\infty \) for any \( i \in \mathbb{N} \) converges in the \( L^1 \)-strong sense to a Borel set \( F \subset Y \) with \( \mu(F) < +\infty \) if \( m_i(E_i) \to \mu(F) \) and \( \chi_{E_i, m_i} \to \chi_{F, \mu} \) with respect to the duality with continuous bounded functions with bounded support on \( Z \).

We say that a sequence of Borel sets \( E_i \subset X_i \) converges in the \( L^1_{\text{loc}} \)-sense to a Borel set \( F \subset Y \) if \( E_i \cap B_R(x_i) \) converges to \( F \cap B_R(y) \) in \( L^1 \)-strong for every \( R > 0 \).

Definition 2.3 (Hausdorff convergence). Let \( \{(X_i, d_i, m_i, x_i)\}_{i \in \mathbb{N}} \) be a sequence of pointed metric measure spaces converging in the pmGH sense to a pointed metric measure space \((Y, d_Y, \mu, y)\). Then we say that a sequence of closed sets \( E_i \subset X_i \) converges in Hausdorff distance (or in Hausdorff sense) to a closed set \( F \subset Y \) if there holds convergence in Hausdorff distance in a realization \((Z, d_Z)\) of the pmGH convergence as in Definition 2.1.

It is also possible to define notions of uniform convergence and \( H^{1,2} \)-strong and weak convergences for sequences of functions of a sequence of spaces \( X_i \) converging in pointed measure Gromov–Hausdorff sense. We refer the reader to [5, 11] for such definitions.

2.2 BV functions and sets of finite perimeter in metric measure spaces. We begin with the definitions of function of bounded variation and set of finite perimeter in a m.m.s.

Definition 2.4 (BV functions and perimeter on m.m.s.). Let \((X, d, m)\) be a metric measure space. Given \( f \in L^1_{\text{loc}}(X, m) \) we define
\[
|Df|(A) := \inf \left\{ \liminf_i \int_A \operatorname{lip} f_i \, dm : f_i \in \operatorname{LIP}_{\text{loc}}(A), f_i \to f \text{ in } L^1_{\text{loc}}(A, m) \right\},
\]
for any open set \( A \subset X \). We declare that a function \( f \in L^1_{\text{loc}}(X, m) \) is of local bounded variation, briefly \( f \in BV_{\text{loc}}(X) \), if \( |Df|(A) < +\infty \) for every \( A \subset X \) open bounded. A function \( f \in L^1(X, m) \) is said to be of bounded variation, briefly \( f \in BV(X) \), if \( |Df|(X) < +\infty \).
If $E \subseteq X$ is a Borel set and $A \subseteq X$ is open, we define the perimeter $\text{Per}(E, A)$ of $E$ in $A$ by

$$\text{Per}(E, A) := \inf \left\{ \liminf_{n \to \infty} \int_A |\lambda| \ dm : \lambda \in \text{LIP}_{loc}(A), \lambda \to \chi_E \text{ in } L^1_{loc}(A, m) \right\},$$

in other words $\text{Per}(E, A) := |D\chi_E|(A)$. We say that $E$ has locally finite perimeter if $\text{Per}(E, A) < +\infty$ for every open bounded set $A$. We say that $E$ has finite perimeter if $\text{Per}(E, X) < +\infty$, and we denote $\text{Per}(E) := \text{Per}(E, X)$.

In the sequel, we shall frequently make use of the following coarea formula, proved in [74].

Theorem 2.5 (Coarea formula). Let $(X, d, m)$ be a metric measure space. Fix $f \in L^1_{loc}(X)$ and an open set $\Omega \subseteq X$. Then $\mathbb{R} \ni t \mapsto \text{Per}(\{ f > t \}, \Omega) \in [0, +\infty]$ is Borel measurable and

$$|Df|(\Omega) = \int_{\mathbb{R}} \text{Per}(\{ f > t \}, \Omega) \ dt.$$  

2.3. Sobolev functions, Laplacians and vector fields in metric measure spaces. The Cheeger energy on a metric measure space $(X, d, m)$ is defined as the $L^2$-relaxation of the functional $f \mapsto \frac{1}{2} \int \text{Lip}^2 f \ dm$ (see [8] after [44]). Namely, for any function $f \in L^2(X)$ we define

$$\text{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int \text{Lip}^2 f \ dm : (f_n)_{n \in \mathbb{N}} \subseteq \text{Lip}_{bs}(X), f_n \to f \text{ in } L^2(X) \right\}.$$  

The Sobolev space $H^{1,2}(X)$ is defined as the finiteness domain $\{ f \in L^2(X) : \text{Ch}(f) < +\infty \}$ of the Cheeger energy. The restriction of the Cheeger energy to the Sobolev space admits the integral representation $\text{Ch}(f) = \frac{1}{2} \int |\nabla f|^2 \ dm$, for a uniquely determined function $|\nabla f| \in L^2(X)$ that is called the minimal weak upper gradient of $f \in H^{1,2}(X)$. The linear space $H^{1,2}(X)$ is a Banach space if endowed with the Sobolev norm

$$\|f\|_{H^{1,2}(X)} := \sqrt{\|f\|^2_{L^2(X)} + 2\text{Ch}(f)} = \sqrt{\|f\|^2_{L^2(X)} + \|\nabla f\|^2_{L^2(X)}},$$

for every $f \in H^{1,2}(X)$.

Following [53], when $H^{1,2}(X)$ is a Hilbert space (or equivalently $\text{Ch}$ is a quadratic form) we say that the metric measure space $(X, d, m)$ is infinitesimally Hilbertian.

Hereafter, the infinitesimal Hilbertianity of $(X, d, m)$ will be our standing assumption.

The results of [7] ensure that $\text{Lip}_{bs}(X)$ is dense in $H^{1,2}(X)$ with respect to the norm topology. We define the bilinear mapping $H^{1,2}(X) \times H^{1,2}(X) \ni (f, g) \mapsto \nabla f \cdot \nabla g \in L^1(X)$ as

$$\nabla f \cdot \nabla g := \frac{(|\nabla(f + g)|^2 - |\nabla f|^2 - |\nabla g|^2)}{2},$$

for every $f, g \in H^{1,2}(X)$.

Given $\Omega \subseteq X$ open, we define the local Sobolev space with Dirichlet boundary conditions $H^{1,2}_0(\Omega)$ as the closure of $\text{Lip}_{bs}(\Omega)$ in $H^{1,2}(X)$. Notice that $H^{1,2}_0(X) = H^{1,2}(X)$. Moreover, we declare that a given function $f \in L^2(\Omega)$ belongs to the local Sobolev space $H^{1,2}_0(\Omega)$ provided $\eta f \in H^{1,2}(X)$ for every $\eta \in \text{Lip}_{bs}(\Omega)$ and the function

$$|\nabla f| := \text{ess sup}\{\chi_{\{\eta = 1\}}|\nabla(\eta f)| : \eta \in \text{Lip}_{bs}(\Omega)\}$$

belongs to $L^2(\Omega)$. Above, we employed the notation $\text{ess sup}_{\lambda \in \Lambda} h_{\lambda}$ to denote the essential supremum of a set $\{h_{\lambda}\}_{\lambda \in \Lambda}$ of measurable functions, and denoted by $\chi$ the indicator function.

Definition 2.6 (Local Laplacian). Let $(X, d, m)$ be an infinitesimally Hilbertian space and $\Omega \subseteq X$ an open set. Then we say that a function $f \in H^{1,2}(\Omega)$ has local Laplacian in $\Omega$, $f \in D(\Delta, \Omega)$ for short, if there exists a (uniquely determined) function $\Delta f \in L^2(\Omega)$ such that

$$\int_{\Omega} g \Delta f \ dm = -\int_{\Omega} \nabla g \cdot \nabla f \ dm,$$

for every $g \in H^{1,2}_0(\Omega)$.

For brevity, we write $D(\Delta)$ instead of $D(\Delta, X)$.

More generally, we work with functions having measure-valued Laplacian in an open set:
Definition 2.7 (Measure-valued Laplacian). Let \((X,d,m)\) be an infinitesimally Hilbertian space and \(\Omega \subseteq X\) an open set. Then we say that a function \(f \in H^{1,2}(\Omega)\) has measure-valued Laplacian in \(\Omega\), if there exists a (uniquely determined) locally finite measure \(\Delta f\) on \(\Omega\) such that
\[
\int_{\Omega} g \Delta f := \int_{\Omega} g \, d\Delta f = -\int_{\Omega} \nabla g \cdot \nabla f \, dm, \quad \text{for every } g \in \text{LIP}_{\text{loc}}(\Omega).
\]
For brevity, we write \(D(\Delta)\) instead of \(D(\Delta, X)\). Moreover, given functions \(f \in \text{LIP}(\Omega) \cap H^{1,2}(\Omega)\) and \(\eta \in C_0(\Omega)\), we say that \(\Delta f \leq \eta\) in the distributional sense if \(f \in D(\Delta, \Omega)\) and \(\Delta f \leq \eta \, m\).

The above two notions of Laplacian are consistent, in the following sense: given any \(f \in H^{1,2}(\Omega)\), it holds that \(f \in D(\Delta, \Omega)\) if and only if \(f \in D(\Delta, \Omega)\), \(\Delta f \ll m\) and \(\frac{\Delta f}{\text{d}m} \in L^2(\Omega)\). If this is the case, then we also have that the m-a.e. equality \(\Delta f = \frac{\Delta f}{\text{d}m}\) holds.

The heat flow \(\{P_t\}_{t \geq 0}\) on \((X,d,m)\) is the gradient flow of the quadratic form \(\text{Ch}\) in \(L^2(X)\). For any \(f \in L^2(X)\), the gradient flow trajectory \([0, +\infty) \ni t \mapsto P_t f \in L^2(X)\) is a continuous curve with \(P_0f = f\) that is locally absolutely continuous in \((0, +\infty)\), and with \(P_t f \in D(\Delta)\) and \(\frac{d}{dt} P_t f = \Delta P_t f\) for a.e. \(t > 0\).

By a bounded Sobolev derivation on \((X,d,m)\) we mean a linear operator \(v : H^{1,2}(X) \to L^2(X)\) for which there exists a function \(g \in L^\infty(X)\) satisfying \(|v(f)| \leq g(\nabla f)|\) m-a.e. for every \(f \in H^{1,2}(X)\). The minimal (in the m-a.e. sense) function \(g\) verifying this condition is denoted by \(|v| \in L^\infty(X)\) and called the pointwise norm of \(v\). We then define the space \(L^\infty(TX)\) of bounded vector fields on \((X,d,m)\) as the family of all bounded Sobolev derivations on \((X,d,m)\). The space \(L^\infty(TX)\) is a module over \(L^\infty(X)\) if endowed with the multiplication operator \(L^\infty(X) \times L^\infty(TX) \ni (h,v) \mapsto h \cdot v \in L^\infty(TX)\) given by \((h \cdot v)(f) := hv(f)\) for any \(f \in H^{1,2}(X)\). Moreover, to any given function \(f \in H^{1,2}(X)\) with \(|\nabla f| \in L^\infty(X)\) we can associate its gradient \(\nabla f \in L^\infty(TX)\), which is characterized as the unique element of \(L^\infty(TX)\) satisfying \(\nabla f(g) = \nabla f \cdot \nabla g\) for every \(g \in H^{1,2}(X)\). In particular, to use the notation
\[
v \cdot \nabla f := v(f), \quad \text{for every } v \in L^\infty(TX) \text{ and } f \in H^{1,2}(X)
\]
will cause no ambiguity. Observe also that the pointwise norm of \(\nabla f\) de facto coincides with the minimal weak upper gradient \(|\nabla f|\) of \(f\). The above notions are essentially borrowed from \([55, 54]\), up to some technical subtleties one can easily figure out and deal with.

Definition 2.8 (Essentially bounded divergence measure vector fields). Let \((X,d,m)\) be an infinitesimally Hilbertian space. Then we say that an element \(v \in L^\infty(TX)\) is an essentially bounded divergence measure vector field if there exists a (uniquely determined) finite Radon measure \(\text{div}(v)\) on \(X\) such that
\[
\int g \, \text{div}(v) := \int g \, d(\text{div}(v)) = -\int v \cdot \nabla g \, dm \quad \text{for every } g \in \text{LIP}_c(X).
\]
We denote by \(\mathcal{DM}^\infty(X)\) the family of all essentially bounded divergence measure vector fields.

Similarly, one can also define the space \(\mathcal{DM}^\infty(\Omega)\) of locally essentially bounded divergence measure vector fields in some open set \(\Omega \subseteq X\). Notice that, given any function \(f \in H^{1,2}(X)\) with \(|\nabla f| \in L^\infty(X)\), it holds that \(\nabla f \in L^\infty(TX)\) is an essentially bounded divergence measure vector field if and only if \(f \in D(\Delta)\) and \(\Delta f\) is finite. If this is the case, then we also have that \(\text{div}(\nabla f) = \Delta f\). The analogous property holds for Sobolev functions defined on an open set \(\Omega\).

2.4. Geometric analysis on RCD spaces. The focus of this paper will be on RCD\((K,N)\) metric measure spaces \((X,d,m)\). We avoid giving a detailed introduction to this notion, addressing the reader to the survey \([4]\) and references therein for the relevant background.

For most of the results of this paper we will consider RCD\((K,N)\) spaces of the form \((X,d,\mathcal{H}^N)\), for some \(K \in \mathbb{R}\) and \(N \in \mathbb{N}\). Notice that we are requiring that the dimension
the Hausdorff measure coincides with the upper dimensional bound in the RCD condition. These spaces are typically called non-collapsed RCD spaces (ncRCD\((K,N)\) spaces for short) or \(N\)-dimensional RCD\((K,N)\) spaces (see [67, 48, 63]).

Below we recall some of the less classical properties that will be relevant for our purposes.

In the setting of RCD\((K,N)\) spaces it is possible to employ a viscosity interpretation of Laplacian bounds, in addition to the distributional one, see [77].

**Definition 2.9** (Bounds in the viscosity sense for the Laplacian). Let \((X,d,m)\) be an RCD\((K,N)\) metric measure space and let \(\Omega \subset X\) be an open and bounded domain. Let \(f : \Omega \to \mathbb{R}\) be locally Lipschitz and \(\eta \in C_b(\Omega)\). We say that \(\Delta f \leq \eta\) in the viscosity sense in \(\Omega\) if the following holds. For any \(\Omega' \Subset \Omega\) and for any test function \(\varphi : \Omega' \to \mathbb{R}\) such that

1. \(\varphi \in D(\Delta,\Omega')\) and \(\Delta \varphi\) is continuous on \(\Omega'\);
2. for some \(x \in \Omega'\) it holds \(\varphi(x) = f(x)\) and \(\varphi(y) \leq f(y)\) for any \(y \in \Omega', y \neq x\);

it holds

\[
\Delta \varphi(x) \leq \eta(x).
\]

A function \(\varphi\) as in items (i) and (ii) above will be called lower supporting function of \(f\). When instead of \(\leq\) we consider \(\geq\) in the definition above, a function \(\varphi\) as in items (i) and (ii) will be called upper supporting function of \(f\).

We will rely on the equivalence between distributional and viscosity bounds on the Laplacian. The result is classical in the setting of smooth Riemannian manifolds and it has been extended in [77, Theorem 3.24] to RCD\((K,N)\) metric measure spaces \((X,d,\mathcal{H}^N)\).

**Theorem 2.10.** Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) metric measure space. Let \(\Omega \subset X\) be an open and bounded domain, \(f : \Omega \to \mathbb{R}\) be a Lipschitz function and \(\eta : \Omega \to \mathbb{R}\) be continuous. Then \(\Delta f \leq \eta\) in the sense of distributions if and only if \(\Delta f \leq \eta\) in the viscosity sense.

We refer also to [77, Theorem 3.28] for other equivalent characterizations of bounds on the Laplacian in the setting of RCD spaces.

In the proof of Theorem 1.2 it will be important to relate the synthetic lower Ricci curvature bound to the stability of Laplacian bounds through the Hopf-Lax duality

\[
f^C(x) := \inf_{y \in X} \{ f(y) + d(x,y) \}.
\]  

The following statement corresponds to [77, Theorem 4.9]. On a smooth Riemannian manifold, neglecting the regularity issues, it follows from the two-points Laplacian comparison proved in [14], which is based on a computation with Jacobi fields. The proof in [77], which works in a much more general setting, is based on the interplay between the Hopf-Lax semigroup and the heat flow put forward in [68, 56, 10].

**Theorem 2.11.** Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) metric measure space for some \(K \in \mathbb{R}\) and \(1 \leq N < \infty\). Let \(f : X \to \mathbb{R}\) be a locally Lipschitz function with polynomial growth. Let \(\Omega,\Omega' \subset X\) be open domains and \(\eta \in \mathbb{R}\). Then the following holds. Assume that \(f^C\) is finite and that, for any \(x \in \Omega'\) the infimum defining \(f^C(x)\) is attained at some \(y \in \Omega\). Assume also that

\[
\Delta f \leq \eta \quad \text{on } \Omega.
\]  

Then

\[
\Delta f^C \leq \eta - K \max_{x \in \Omega', y \in \Omega} d(x,y) \quad \text{on } \Omega', \quad \text{if } K \leq 0,
\]

\[
\Delta f^C \leq \eta - K \min_{x \in \Omega', y \in \Omega} d(x,y) \quad \text{on } \Omega', \quad \text{if } K \geq 0,
\]

where the Laplacian bounds are intended either in the distributional or in the viscosity sense.
Given a Borel set $E \subseteq X$ in an RCD($K,N$) space $(X,d,\mathcal{H}^N)$ and any $t \in [0,1]$, we denote by $E^{(t)}$ the set of points of density $t$ of $E$, namely

$$E^{(t)} := \left\{ x \in X \left| \lim_{r \to 0} \frac{\mathcal{H}^N(E \cap B_r(x))}{\mathcal{H}^N(B_r(x))} = t \right\}. \right.$$  

The essential boundary of $E$ is defined as $\partial^e E := X \setminus (E^{(0)} \cup E^{(1)})$. We have that $E^{(t)}$ and $\partial^e E$ are Borel sets. Furthermore, the reduced boundary $\mathcal{F}E \subseteq \partial^e E$ of a given set $E \subseteq X$ of finite perimeter is defined as the set of those points of $E$ where the unique tangent to $E$ (up to isomorphism) is the half-space $\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0 \}$ in $\mathbb{R}^N$; see [5, Definition 4.1] for the precise notion of convergence we are using.

It was proved in [30] after [3, 5] that the perimeter measure $\text{Per}(E, \cdot)$ can be represented as

$$\text{Per}(E, \cdot) = \mathcal{H}^{N-1}|_{\mathcal{F}E}. \tag{2.3}$$

As it is evident from (2.3), the notion of perimeter that we are using does not change the boundary of the space under consideration, if any. We refer to [48, 63, 28] for the relevant background about boundaries of RCD($K,N$) spaces $(X,d,\mathcal{H}^N)$.

Moreover, we recall that, according to [29, Proposition 4.2],

$$\mathcal{F}E = E^{(1/2)} = \left\{ x \in X \left| \lim_{r \to 0} \frac{\mathcal{H}^N(E \cap B_r(x))}{\mathcal{H}^N(B_r(x))} = \frac{1}{2} \right\} \right., \text{ up to } \mathcal{H}^{N-1}\text{-null sets.}$$

We will rely on a Gauss–Green integration by parts formula for essentially bounded divergence measure fields over sets of finite perimeter.

As shown in [29, Section 5] after [31], for any $v \in \mathcal{D}M^\infty(X)$ and any set of finite perimeter $E$, there exist measures $D\chi_E(\chi_E v)$, $D\chi_E(\chi_E v)$ on $X$ such that $((\chi_E v) \cdot \nabla P_L \chi_E) m \to D\chi_E(\chi_E v)$ and $((\chi_E v) \cdot \nabla P_L \chi_E) m \to D\chi_E(\chi_E v)$ as $t \to 0$ with respect to the narrow topology. Moreover, it holds that $D\chi_E(\chi_E v), D\chi_E(\chi_E v) \ll \text{Per}(E, \cdot)$. Then we define

$$\langle v \cdot \nu_E \rangle_{\text{int}} := \frac{1}{2} \frac{d(D\chi_E(\chi_E v))}{d \text{Per}(E, \cdot)}, \quad \langle v \cdot \nu_E \rangle_{\text{ext}} := \frac{1}{2} \frac{d(D\chi_E(\chi_E v))}{d \text{Per}(E, \cdot)}. \tag{2.4}$$

We remark that there is full consistency with the classical setting of Riemannian manifolds.

The following result was proved in [29, Theorem 5.2] by building upon [31, Theorem 6.22].

**Theorem 2.12** (Gauss–Green). Let $(X,d,\mathcal{H}^N)$ be an RCD($K,N$) space and $E \subseteq X$ a set of finite perimeter. Fix $v \in \mathcal{D}M^\infty(X)$. Then $(v \cdot \nu_E)_{\text{int}}, (v \cdot \nu_E)_{\text{ext}} \in L^\infty(\mathcal{F}E, \text{Per}(E, \cdot))$ and

$$\| (v \cdot \nu_E)_{\text{int}} \|_{L^\infty(\mathcal{F}E, \text{Per}(E, \cdot))} \leq \| v \|_{L^\infty(E)}, \quad \| (v \cdot \nu_E)_{\text{ext}} \|_{L^\infty(\mathcal{F}E, \text{Per}(E, \cdot))} \leq \| v \|_{L^\infty(E)}.$$  

Moreover, the Gauss–Green integration-by-parts formulas hold: for any $\varphi \in \text{Lip}_c(X)$ it holds

$$\int_{E^{(1)}} \varphi \text{div}(v) + \int_E v \cdot \nabla \varphi \, dm = - \int_{\mathcal{F}E} \varphi (v \cdot \nu_E)_{\text{int}} \, d\text{Per}(E, \cdot), \quad \int_{E^{(1)} \cap \mathcal{F}E} \varphi \text{div}(v) + \int_E v \cdot \nabla \varphi \, dm = - \int_{\mathcal{F}E} \varphi (v \cdot \nu_E)_{\text{ext}} \, d\text{Per}(E, \cdot).$$

Next we report on the natural behaviour of interior/exterior normal traces over the boundary of the intersection of two finite perimeter sets, which has been investigated in [29].

Given two sets of finite perimeter $E, F \subseteq X$, it is well-known that also $E \cap F$ has finite perimeter. The set $\{ \nu_E = \nu_F \}$ introduced in [29] can be characterized as

$$\{ \nu_E = \nu_F \} = \left\{ x \in \mathcal{F}E \cap \mathcal{F}F \left| \lim_{r \to 0} \frac{\mathcal{H}^N((E \cap F) \cap B_r(x))}{\mathcal{H}^N(B_r(x))} = \frac{1}{2} \right\} \right., \text{ up to } \mathcal{H}^{N-1}\text{-null sets,}$$

as it follows from the results of [29]. The ensuing statement is taken from [29, Proposition 5.4].
Proposition 2.13. Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space. Let \(E, F \subseteq X\) be two sets of finite perimeter. Let \(v \in \mathcal{D}\mathcal{M}^\infty(X)\) be given. Then it holds that
\[
(v \cdot \nu_E)_{\text{int}} = (v \cdot \nu_F)_{\text{int}}, \quad \mathcal{H}^{N-1}\text{-a.e. on } \{\nu_E = \nu_F\},
\]
\[
(v \cdot \nu_{E\cap F})_{\text{int}} = (v \cdot \nu_E)_{\text{int}}, \quad \text{Per}(E, \cdot)\text{-a.e. in } F^{(1)},
\]
\[
(v \cdot \nu_{E\cap F})_{\text{int}} = (v \cdot \nu_F)_{\text{int}}, \quad \mathcal{H}^{N-1}\text{-a.e. on } \{\nu_E = \nu_F\}.
\]

We will apply the previous machinery to the level sets of distance-type functions obtained through Hopf-Lax duality, see Theorem 2.11. More specifically, we will need the following result, which is taken from [29, Proposition 6.1].

Proposition 2.14. Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space. Let \(\Omega \subseteq \Omega' \subseteq X\) be open domains and let \(\varphi: \Omega' \to \mathbb{R}\) be a 1-Lipschitz function. Suppose that \(|\nabla \varphi| = 1\) holds \(\mathcal{H}^N\text{-a.e. on } \Omega'\) and that there exists a constant \(L \leq 0\) such that \(\Delta \varphi \geq L\) in the sense of distributions in \(\Omega'\), thus in particular \(\varphi \in D(\Delta, \Omega')\). Suppose further that \(\Delta \varphi\) is finite. Then \(\{\varphi < t\}\) is a set of locally finite perimeter in \(\Omega\) for a.e. \(t \in \mathbb{R}\) such that \(\{\varphi = t\} \cap \Omega \neq \emptyset\) and it holds that
\[
(\nabla \varphi \cdot \nu_{\{\varphi < t\}})_{\text{int}} = (\nabla \varphi \cdot \nu_{\{\varphi < t\}})_{\text{ext}} = -1, \quad \text{Per}(\{\varphi < t\}, \cdot)\text{-a.e. in } \Omega.
\]

The primary focus of this note will be isoperimetric sets, that, as in the classical Riemannian setting, are much more regular than general sets of finite perimeter.

Definition 2.15. Let \((X, d, m)\) be a metric measure space. We say that a subset \(E \subset X\) is a volume constrained minimizer for compact variations in \(X\) if whenever \(F \subset X\) is such that \(E \Delta F \subset K \subset X\), and \(m(K \cap E) = m(K \cap F)\), then \(\text{Per}(E) \leq \text{Per}(F)\).

We say that a subset \(E \subset X\), with \(m(E) < \infty\), is an isoperimetric set whenever for any \(F \subset X\) with \(m(F) = m(E)\) we have that \(\text{Per}(E) \leq \text{Per}(F)\).

Notice that an isoperimetric set in \(X\) is a fortiori a volume constrained minimizer for compact variations in \(X\).

Let us recall a topological regularity result for volume constrained minimizers borrowed from [18]. A similar regularity result for local perimeter minimizers on PI spaces (without volume constraints) was obtained earlier in [66].

Theorem 2.16 ([18, Theorem 1.3 and Theorem 1.4]). Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space with \(2 \leq N < +\infty\) natural number, \(K \in \mathbb{R}\). Let \(E\) be a volume constrained minimizer for compact variations in \(X\). Then \(E^{(1)}\) is open, \(\partial^e E = \partial E^{(1)}\), and \(\partial E^{(1)}\) is locally uniformly \((N-1)\)-Ahlfors regular in \(X\).

Assume further there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\), and that \(E \subset X\) is an isoperimetric region. Then \(E^{(1)}\) is in addition bounded, and \(\partial E^{(1)}\) is \((N-1)\)-Ahlfors regular in \(X\).

In the following, when \(E\) is an isoperimetric region in a space \(X\) as in Theorem 2.16, we will always assume that \(E\) coincides with its open bounded representative given by \(E^{(1)}\).

2.5. Localization of the curvature-dimension condition. We will rely on the so-called localization of the curvature-dimension condition. We give some basic background about it and address the reader to [39, 40, 41] for a detailed account about this topic, under much more general assumptions.

Let us consider an RCD\((K, N)\) metric measure space \((X, d, m)\) for some \(K \in \mathbb{R}\) and \(1 < N < \infty\). Let \(\Omega \subset X\) be an open subset and let \(f : X \to \mathbb{R}\) be the signed distance function from \(\Omega\), i.e.,
\[
f(x) := d(x, \overline{\Omega}) = \inf\{d(x, y) : y \in \Omega\}, \quad \text{if } x \in X \setminus \Omega,
\]
and
\[
f(x) := -d(x, X \setminus \Omega) = -\inf\{d(x, y) : y \in X \setminus \Omega\}, \quad \text{if } x \in \Omega.
\]
The signed distance function $f$ induces an $m$-almost everywhere partition of $X$ into geodesics $X_\alpha$ indexed over a set $Q$. On a smooth Riemannian manifold, these geodesics would correspond to gradient flow lines of the signed distance function, or, equivalently, to integral curves of $-\nabla f$.

Rays $X_\alpha$ are often identified with intervals of the real line via the ray map $\gamma_\alpha : I_\alpha \to X_\alpha$, where $I_\alpha \subset \mathbb{R}$ is an interval and $\gamma_\alpha$ is an isometry.

The almost-everywhere partition of $X$ into transport rays induced by the signed distance function $f$ determines the following disintegration formula:

$$
\mathfrak{m} = \int_Q h_\alpha \mathcal{H}^1 \llcorner X_\alpha \, q(d\alpha).
$$

(2.7)

The non-negative measure $q$ in (2.7), defined on the set of indices $Q$, is obtained in a natural way from the essential partition $(X_\alpha)_{\alpha \in Q}$ of $X$, roughly by projecting $\mathfrak{m}$ on the set $Q$ of equivalence classes (we refer to [41] for the details).

The key property is that, if $(X, d, m)$ is an $\text{RCD}(K, N)$ metric measure space, then each $h_\alpha$ is a $\text{CD}(K, N)$ density over the ray $X_\alpha$ (see [41, Theorem 3.6]), i.e.,

$$
(\log h_\alpha)'' \leq -K - \frac{1}{N-1}((\log h_\alpha)')^2,
$$

(2.8)

in the sense of distributions and point-wise except countably many points, compare with [39, Lemma A.3, Lemma A.5, Proposition A.10]. Equivalently

$$
\left(h_\alpha^{N-1}\right)'' + \frac{K N}{N-1} h_\alpha^{N-1} \leq 0,
$$

(2.9)

in the sense of distributions. This amounts to say that the curvature-dimension condition of the ambient space $(X, d, m)$ is inherited by the needles of the partition induced by $f$.

With the help of the localization technique, we will be able to turn some estimates into one-dimensional comparison results for solutions of Riccati equations. We introduce here the relevant notation for our purposes.

Let us introduce the comparison functions

$$
s_{k,\lambda}(r) := \cos_k(r) - \lambda \sin_k(r),
$$

(2.10)

where

$$
\cos''_k + k \cos_k = 0, \quad \cos_k(0) = 1, \quad \cos'_k(0) = 0,
$$

(2.11)

and

$$
\sin''_k + k \sin_k = 0, \quad \sin_k(0) = 0, \quad \sin'_k(0) = 1.
$$

(2.12)

Notice that $s_{k,-d}$ is a solution of

$$
v'' + kv = 0, \quad v(0) = 1, \quad v'(0) = d.
$$

(2.13)

Moreover, $s_{0,\lambda}(r) = 1 - \lambda r$.

Let us fix $N > 1$, $H \in \mathbb{R}$ and $K \in \mathbb{R}$. Then we introduce the Jacobian function

$$
\mathbb{R} \ni r \mapsto J_{H, K, N}(r) := \left(\cos_{\frac{K}{N-1}}(r) + \frac{H}{N-1} \sin_{\frac{K}{N-1}}(r)\right)^{N-1} = \left(s_{\frac{K}{N-1}, -\frac{H}{N-1}}(r)\right)^{N-1}.
$$

(2.14)

Notice that, when $K = 0$ the expression for the Jacobian function simplifies into

$$
\mathbb{R} \ni r \mapsto J_{H, N}(r) := \left(1 + \frac{H}{N-1} r\right)^{N-1}.
$$

(2.15)

We stress that the function $J_{H, K, N}$ is precisely the one involved in the one-dimensional comparison of $\text{CD}(K, N)$ densities, see [64, Corollary 4.3].
3. The distance function from isoperimetric sets

Let \((M,g)\) be a smooth \(N\)-dimensional Riemannian manifold with Ricci curvature bounded from below by \(K \in \mathbb{R}\). Let \(E \subset M\) be a set of finite perimeter which is an isoperimetric region. Then, by the classical regularity theory for constrained perimeter minimizers \([2, 59, 78]\), \(E\) has an open representative and \(\partial E\) is smooth away from a set \(\partial_x E\) of Hausdorff dimension \(\dim_H(\partial_x E) \leq N - 8\). Moreover, \(\partial E \setminus \partial_x E\) has constant mean curvature \(c \in \mathbb{R}\), in the classical sense and \(\partial_x E\) can be characterized as the set of those points in \(\partial E\) such that the tangent cone is not included in a half-space, thanks to \([2]\).

The classical proof of the Lévy–Gromov inequality for manifolds with positive Ricci curvature \([60, Appendix C]\) combines the regularity results mentioned above with a Heintze–Karcher type estimate \([61]\). In particular, the regularity theorem from \([2]\) is used in a crucial way to overcome the possible lack of smoothness of the isoperimetric boundary. The proof gives in particular the following result, valid for any lower curvature bound.

**Theorem 3.1.** Let \((M,g)\) be a smooth \(N\)-dimensional Riemannian manifold with Ricci curvature bounded from below by \(K \in \mathbb{R}\) and let \(E \subset X\) be an isoperimetric set. Then, denoting by \(f\) the signed distance function from \(\overline{E}\) and by \(c\) the value of the constant mean curvature of \(\partial E \setminus \partial_x E\), it holds

\[
\Delta f \geq -(N - 1) \frac{s^N}{N - 1} \circ (-f) \quad \text{on } E \quad \text{and} \quad \Delta f \leq (N - 1) \frac{s^N}{N - 1} \circ f \quad \text{on } X \setminus \overline{E}.
\]

**Remark 3.2.** The very same conclusion above holds assuming that \(E\) is a domain with smooth boundary and that \(\partial E\) has constant mean curvature \(c\). The proof of this variant does not require the deep regularity theorem from \([2]\), but the assumptions are not natural for the applications to the isoperimetric problem when \(N \geq 8\).

Notice that the bounds in (3.1) make perfectly sense on a metric measure space \((X,d,m)\), even though most of the ingredients of the classical proof that we recalled above do not.

By \([18, Theorem 1.3]\), a volume constrained minimizer \(E\) for compact variations enjoys analogous topological regularity properties of an isoperimetric region. More precisely, \(E^{(1)}\) is open and \(\partial^E E = \partial E^{(1)}\). Hence we will always identify such a set with its representative \(E^{(1)}\).

**Theorem 3.3.** Let \((X,d,\mathcal{H}^N)\) be an \(\text{RCD}(K,N)\) metric measure space for some \(K \in \mathbb{R}\) and \(N \geq 2\) and let \(E \subset X\) be a set of finite perimeter. Assume that \(E\) is a volume constrained minimizer for compact variations in \(X\). Then, denoting by \(f\) the signed distance function from \(\overline{E}\), there exists \(c \in \mathbb{R}\) such that

\[
\Delta f \geq -(N - 1) \frac{s^N}{N - 1} \circ (-f) \quad \text{on } E \quad \text{and} \quad \Delta f \leq (N - 1) \frac{s^N}{N - 1} \circ f \quad \text{on } X \setminus \overline{E}.
\]

**Remark 3.4.** The bounds in (3.2) can be understood in the sense of distributions or in the sense of viscosity, see **Theorem 2.10**. The two perspectives will be both relevant for the sake of the proof while in the applications it will be important to rely mostly on the distributional perspective.

**Remark 3.5.** Notice that the comparison function in (3.2) are not well defined globally on \(\mathbb{R}\), but only on a maximal interval \(I_{K,N,c} \subset \mathbb{R}\). In the course of the proof we will show that the signed distance function from \(\overline{E}\) can attain only values in \(I_{K,N,c}\) where the bounds in (3.2) perfectly make sense.

The proof of **Theorem 3.3** is based on a careful adaptation of the argument in \([77, Theorem 5.2]\) that dealt with local perimeter minimizers without volume constraints. We outline the
strategy for smooth Riemannian manifolds, avoiding the technicalities and focusing on the case $K = 0$.

We consider a weaker statement, corresponding to the limit of (3.2) as $N \to \infty$. The self-improvement of the adimensional bound to the sharp dimensional bound is based on a classical computation with Jacobi fields away from the cut-locus of the distance function.

If this weaker statement fails, then there are points $x \in X \setminus E$ and $y \in E$ such that $\Delta f(x) > \Delta f(y)$ in a weak sense. This is already a subtle point, where we exploit the viscosity theory, since the distance function is not globally smooth. Thanks to a perturbation argument, we construct a smooth function $g$, touching $f$ from below only at $x$, and a smooth function $h$, touching $f$ from above only at $y$, such that $\Delta g(x) > \Delta h(y)$.

Then we introduce the functions

$$
\bar{g}(z) := \sup_w \{g(w) - d(w, z)\}, \quad \bar{h}(z) := \inf_w \{h(w) + d(z, w)\}.
$$

(3.3)

Heuristically, $\bar{g}$ and $\bar{h}$ behave like distance functions from their respective level sets. Moreover, the non-linear transformation in (3.3) maintains the bounds on the Laplacian, if the ambient has non-negative Ricci curvature. Notice that a weak perspective on Laplacian bounds needs to be considered again, since the transformation does not preserve the regularity, in general.

Neglecting the regularity issues, the conclusion would follow from [14], which is based on a computation with Jacobi fields.

The idea is then to slide the level sets of the functions $\bar{g}$ and $\bar{h}$ until they start crossing the isoperimetric set $E$. When this happens, we cut $\partial E$ along the level sets of $\bar{g}$ and $\bar{h}$ making sure to balance the interior and exterior perturbations so that, globally, the perturbation has the same volume of $E$. Some care is needed in order to make sure that the two perturbations have disjoint supports. Eventually, we estimate the perimeter of the perturbation and compare it with the perimeter of $E$, reaching a contradiction.

Notice that we do not need to rely on the full regularity theory for isoperimetric sets on Riemannian manifolds. For the sake of the proof it is sufficient to know that the measure theoretic boundary of the isoperimetric set is closed, which follows from the local density estimates obtained in [18] (see also the previous [66] for the case of local perimeter minimizers without volume constraints).

**Proof.** The proof will be divided into two steps. In the first one we are going to prove the weaker adimensional bounds

$$
\Delta f \geq c - Kf \quad \text{on } E \quad \text{and} \quad \Delta f \leq c - Kf \quad \text{on } X \setminus E,
$$

(3.4)

corresponding to the limit as $N \to \infty$ of the bounds in (3.2).

In the second part of the proof we will show how to obtain the sharp dimensional bounds with a by now standard application of the localization technique from [40, 41].

We remark that if (3.4) (3.2 respectively) holds in a neighbourhood of $\partial E$, then (3.4) (3.2 respectively) holds globally. This statement can be verified with the very same argument that we will discuss in Step 2 of the proof below. We omit the details, as this observation will not be needed for the rest of the proof.

**Step 1.** We will prove (3.4) in the case $K = 0$. The modifications needed to address the case $K \neq 0$ will be discussed at the end of the step, see also Step 6 in the proof of [77, Theorem 5.2].

Observe that, by Theorem 2.10, (3.4) in the case $K = 0$ is equivalent to the following claim.

**Claim:** the supremum of the values of the Laplacians of lower supporting functions of $f$ (as in Definition 2.9) at touching points on $X \setminus E$ is lower than the infimum of the values of Laplacians of upper supporting functions of $f$ at touching points on $E$. Indeed, if this statement holds, then letting $c$ be any value between the supremum and the infimum of the two sets, then $\Delta f \leq c$ holds on $X \setminus E$ in the viscosity sense and $\Delta f \geq c$ holds on $E$, again in the viscosity sense. By Theorem 2.10, (3.4) holds also in the sense of distributions.
Observe that also the converse implication holds, again by the same kind of arguments. Since we will not need this implication we omit its proof.

Let us prove the claim.

We argue by contradiction. If it is not true, then we can find \( x \in X \setminus \overline{E}, \ y \in E, \ \lambda > 0, \ \delta > 0 \) (that we think to be very small, in particular, \( \lambda < f(x) \) and \( \delta < \lambda \) and supporting functions \( \overline{\psi} : X \to \mathbb{R} \) and \( \overline{\chi} : X \to \mathbb{R} \) with the following properties:

\begin{enumerate}
  \item \( \overline{\psi} : X \to \mathbb{R} \) is Lipschitz and it belongs to the domain of the measure-valued Laplacian on \( B_\lambda(x) \);
  \item \( \overline{\psi}(x) = f(x) \);
  \item \( \overline{\psi}(z) < f(z) \) for any \( z \neq x \) and \( \overline{\psi} < f - \delta \) on \( X \setminus B_\lambda(x) \);
\end{enumerate}

and

\begin{enumerate}[continue]
  \item \( \overline{\chi} : X \to \mathbb{R} \) is Lipschitz and it belongs to the domain of the measure-valued Laplacian on \( B_\lambda(y) \);
  \item \( \overline{\chi}(y) = f(y) \);
  \item \( \overline{\chi}(z) > f(z) \) for any \( z \neq y \) and \( \overline{\chi} > f + \delta \) on \( X \setminus B_\lambda(y) \).
\end{enumerate}

Moreover, there exist \( c \in \mathbb{R} \) and \( \varepsilon > 0 \) such that

\[
\Delta \overline{\psi} \geq c + \varepsilon \quad \text{on} \quad B_\lambda(x)
\]

and

\[
\Delta \overline{\chi} \leq c - \varepsilon \quad \text{on} \quad B_\lambda(y).
\]

Notice that Theorem 2.10 yields a priori only the existence of locally defined functions \( \psi \) and \( \chi \) verifying the weak inequalities \( \psi \leq f \) and \( \chi \geq f \) in place of the strict ones iia) and iiib).

In order to obtain the strict inequalities in iia) and iiib) it is sufficient to subtract the function \( \varepsilon \cdot \text{d}(x, \cdot)^2 \) to \( \psi \) and to sum the function \( \varepsilon \cdot \text{d}(y, \cdot)^2 \) to \( \chi \), for \( \varepsilon > 0 \) small enough. In this way we obtain new auxiliary functions \( \hat{\psi} \) and \( \hat{\chi} \). The fact that the inequalities (3.5) and (3.6) are not affected by this additive perturbation, up to slightly decreasing the value of \( \varepsilon \), follows from the Laplacian comparison theorem, see [53].

In order to extend the locally defined functions \( \hat{\psi} \) and \( \hat{\chi} \) to globally defined functions \( \overline{\psi} \) and \( \overline{\chi} \) while keeping their good properties, it is sufficient to employ a truncation argument. Namely, if \( \hat{\psi} : B_{2\lambda}(x) \to \mathbb{R} \) is such that \( \hat{\psi}(x) = f(x), \ \hat{\psi}(z) < f(z) \) for any \( z \in B_{2\lambda}(x) \) with \( z \neq x \), \( \hat{\psi} < f - \delta \) on \( B_{2\lambda}(x) \setminus B_\lambda(x) \) and \( \Delta \hat{\psi} \geq c + \varepsilon \) on \( B_{2\lambda}(x) \), we extend \( \hat{\psi} \) to \( -\infty \) on \( X \setminus B_{2\lambda}(x) \) and set

\[
\overline{\psi} := \max \{ f - 2\delta, \hat{\psi} \}.
\]

An analogous construction gives the sought global extension \( \overline{\chi} \) of \( \hat{\chi} \).

We consider the transform of \( \overline{\psi} \) through the Hopf–Lax duality and introduce \( \varphi : X \to \mathbb{R} \) by letting

\[
\varphi(z) := \sup_{w \in X} \{ \overline{\psi}(w) - \text{d}(w, z) \}.
\]

Analogously, we let \( \eta : X \to \mathbb{R} \) be defined by

\[
\eta(z) := \inf_{w \in X} \{ \overline{\chi}(w) + \text{d}(z, w) \}.
\]

Let \( X_\Sigma \) and \( Y_\Sigma \) be the sets of touching points of minimizing geodesics from \( x \) and \( y \) respectively to \( \Sigma := \partial E \), i.e.

\[
X_\Sigma := \{ w \in \partial E : f(x) - f(w) = \text{d}(x, w) \},
\]

and

\[
Y_\Sigma := \{ w \in \partial E : f(y) - f(w) = \text{d}(y, w) \}.
\]

It is easy to verify that \( X_\Sigma \) and \( Y_\Sigma \) are compact subsets of \( \partial E \).

It is elementary to check that \( \varphi \leq f, \ \eta \geq f \) because \( \overline{\psi} \leq f \) and \( \overline{\chi} \geq f \) respectively. Moreover, both \( \varphi \) and \( \eta \) are 1-Lipschitz functions, because they are defined as suprema and
Theorem 5.2. We repeat the argument below for the sake of readability.

\[ \supremum \text{ defining } \eta \text{ is clearly enough to prove a).} \]

Moreover, the infima of families of 1-Lipschitz functions and they are finite at some point. Moreover, there exist neighbourhoods of \( U_\Sigma \supset X_\Sigma \) and \( V_\Sigma \supset Y_\Sigma \) such that:

1. \( |\nabla \varphi| = 1 \) holds \( \mathcal{H}^N \)-a.e. on \( U_\Sigma \);
2. \( |\nabla \eta| = 1 \) holds \( \mathcal{H}^N \)-a.e. on \( V_\Sigma \);
3. \( \Delta \varphi \geq c + \varepsilon \) on \( U_\Sigma \);
4. \( \Delta \eta \leq c - \varepsilon \) on \( V_\Sigma \);
5. \( \varphi(z) = f(z) \) for any \( z \in X \) such that \( f(x) - f(z) = d(x, z) \);
6. \( \eta(z) = f(z) \) for any \( z \in X \) such that \( f(z) - f(y) = d(z, y) \).

We check the claims relative to \( \varphi \), the verification of the claims relative to \( \eta \) being completely analogous.

The proof is the same as the proof of the analogous claims in Step 3 of the proof of [77, Theorem 5.2]. We repeat the argument below for the sake of readability.

Let \( x \in X_\Sigma \subset \partial E \) be any footpoint of minimizing geodesic from \( x \) to \( \overline{E} \). In particular, \( f(x) = 0 \) and \( f(x) - f(x) = d(x, x) \). Let \( \gamma : [0, d(x, x)] \to X \) be a unit speed minimizing geodesic between \( \gamma(0) = x \) and \( \gamma(d(x, x)) = x \). Observe that

\[ f(\gamma(t)) = t \quad \text{for any } t \in [0, d(x, x)]. \] (3.12)

Moreover,

\[ \varphi(\gamma(t)) = f(\gamma(t)), \quad \text{for any } t \in [0, d(x, x)], \] (3.13)

and, for any such \( t \), the supremum defining \( \varphi(\gamma(t)) \) in (3.8) is attained only at \( x \).

Indeed, by iii\( a \) above, \( \psi < f - \delta \) on \( X \setminus B_\lambda(x) \). Hence, for any \( z \in X \) such that \( \varphi(z) > f(z) - \delta \), we can restrict the supremum defining \( \varphi(z) \) in (3.8) to \( B_\lambda(x) \). Since \( B_\lambda(x) \) is compact, the supremum is attained. In details, if \( \varphi(z) > f(z) - \delta \), then

\[ \varphi(z) = \sup_{y \in B_\lambda(x)} \{ \psi(y) - d(y, z) \} = \psi(y_x) - d(y_x, z) \leq d(y, z) \leq f(z), \] (3.14)

for some \( y_x \in B_\lambda(x) \). In particular, whenever \( \varphi(z) = f(z) \), all the inequalities above become equalities. Hence \( \psi(y_x) = f(y_x) \), that implies \( y_x = x \) by iii\( a \) and iii\( i \), and \( f(z) - f(x) = -d(x, z) \). Vice versa, if \( f(z) - f(x) = -d(x, z) \) then \( \varphi(z) = f(z) \) and the supremum defining \( \varphi(z) \) is attained (only) at \( x \). In particular, these observations prove e).

We claim that

\[ |\nabla \varphi| = 1, \quad \mathcal{H}^N \text{-a.e. on } \{ \varphi > f - \delta \} \setminus B_\lambda(x), \] (3.15)

that is clearly enough to prove a).

In order to verify this claim, we let \( z \in \{ \varphi > f - \delta \} \setminus B_\lambda(x) \). By the argument above, the supremum defining \( \varphi(z) \) is a maximum and it is attained at some \( x_z \in B_\lambda(x) \). By assumption \( x_z \neq z \). Let us consider a minimizing geodesic \( \gamma : [0, d(z, x_z)] \to X \) with unit speed connecting \( z \) with \( x_z \). We claim that

\[ \varphi(\gamma(t)) = \varphi(z) + t, \quad \text{for any } t \in [0, d(z, x_z)]. \] (3.16)

The inequality \( \varphi(\gamma(t)) \leq \varphi(z) + t \) follows from the fact that \( \varphi \) is 1-Lipschitz. We only need to prove that \( \varphi(\gamma(t)) \geq \varphi(z) + t \). To this aim, observe that

\[ \varphi(\gamma(t)) = \sup_{y \in X} (\psi(y) - d(y, \gamma(t))) \]
\[ \geq \psi(x_z) - d(\gamma(t), x_z) \]
\[ = \psi(x_z) - d(z, x_z) + t \]
\[ = \varphi(z) + t. \]

From (3.16) we infer that, for any \( z \in \{ \varphi > f - \delta \} \setminus B_\lambda(x) \), the function \( \varphi \) has slope 1 at \( z \). The conclusion that \( |\nabla \varphi| = 1 \)-a.e. on \( \{ \varphi > f - \delta \} \setminus B_\lambda(x) \) follows from the classical a.e. identification between slope and upper gradient obtained in [44].

We are left to prove the Laplacian bound c). By construction, \( \overline{\psi} \) verifies the Laplacian bound (3.5) on \( B_\lambda(x) \). We already observed that for points \( z \in \{ \varphi > f - \delta \} \setminus B_\lambda(x) \) the
supremum defining $\varphi(z)$ is a maximum attained in $\overline{B_\lambda(x)}$, hence we obtain by Theorem 2.11 (more precisely, by the dual version with infima replaced by suprema) that
\[
\Delta \varphi \geq c + \varepsilon \quad \text{on } \{\varphi > f - \delta\} \setminus B_\lambda(x),
\]
in the sense of distributions. The set $\{\varphi > f - \delta\} \setminus B_\lambda(x)$ is easily seen to be a neighbourhood of $X_\Sigma$ for $\lambda$ small enough, as $\varphi = f$ on $X_\Sigma$, hence we have proved c).

Our next goal is to reduce to the case where $X_\Sigma = \{xE\}$ and $Y_\Sigma = \{YE\}$ are singletons. We discuss the reduction for $X_\Sigma$, the case of $Y_\Sigma$ being completely analogous.

As above, we let $x_E \in X_\Sigma \subset \partial E$ be any footpoint of minimizing geodesic from $x$ to $\overline{E}$. In particular, $f(x_E) = 0$ and $f(x) - f(x_E) = d(x, x_E)$. Let $\gamma : [0, d(x, x_E)] \to X$ be a unit speed minimizing geodesic between $\gamma(0) = x_E$ and $\gamma(d(x, x_E)) = x$.

By the non-branching property for minimizing geodesics in RCD($K,N$) spaces, see [49, Theorem 1.3], for any $t \in [0, d(x, x_E))$ the minimizing geodesic from $\gamma(t)$ to $E$ is unique, and it coincides with the restriction of $\gamma$ to the interval $[0,t]$. In particular, the footpoint of the minimizing geodesic from $\gamma(t)$ to $E$ is unique and coincides with $x_E$.

Moreover, we can substitute any such point $\gamma(t)$ for $0 < t < d(x, x_E) - \lambda$ to $x$ in the contradiction argument. Indeed, the function $\varphi$ satisfies the following properties:

i) it is Lipschitz and it belongs locally to the domain of the measure-valued Laplacian. The second statement has been already verified in $\{\varphi > f - \delta\} \setminus B_\lambda(x)$ and it holds globally by the very definition of $\varphi$ and the Laplacian comparison theorem;

ii) $\varphi \leq f$ and $\varphi(\gamma(t)) = f(\gamma(t))$;

iii) $\Delta \varphi \geq c + \varepsilon$ in a neighbourhood of $\gamma(t)$.

With a perturbation and a truncation argument completely analogous to the one employed before (3.7), we can modify $\varphi$ and assume that the inequality in ii) is strict away from $\gamma(t)$ and uniformly strict away from a small ball centred at $\gamma(t)$.

The effect of this reduction is that we can assume that the original points $x, y$ in the contradiction argument are as close as we wish to $\partial E$. Moreover they have unique minimizing geodesics to $\partial E$, hence in particular unique footpoints on $\partial E$, that we shall denote by $xE$ and $yE$ respectively.

We will not rename the points $x, y$ nor the auxiliary functions $\varphi$ and $\eta$, for the ease of notation.

We need to consider two cases. The case $xE = yE$ and the case $xE \neq yE$.

**Case 1:** $xE = yE$.

By construction, it holds $\eta \geq f \geq \varphi$. Moreover, $\eta(x_E) = f(x_E) = \varphi(x_E)$, by e), f).

Let us set $g := \eta - \varphi$. Observe that $g \geq 0$ and $g(x_E) = 0$. Moreover, by c) and d), there exists a neighbourhood of $xE$ where
\[
\Delta g \leq (c - \varepsilon) + (-c - \varepsilon) \leq -2\varepsilon.
\]
We get a contradiction, since $g$ would be a non-constant superharmonic function attaining its minimum at an interior point, see [58, Theorem 2.8].

**Case 2:** $xE \neq yE$.

Let us start by proving that for small values of $s \in (-\delta,0)$, we can cut $E$ along a level set of $\varphi$ to obtain inner perturbations $E_{s,0} \subset E$, compactly supported on suitable balls of arbitrary small radius. The argument is analogous to Step 4 in the proof of [77, Theorem 5.2] and we repeat it here for the sake of readability.

Let us define
\[
E_{s,0} := E \setminus \{\varphi > s\}.
\]
Observe that for $s = 0$ it holds $\{\varphi > 0\} \cap E = \emptyset$, since $\{\varphi > 0\} \subset \{\varphi > 0\} \subset X \setminus E$ by construction. When we decrease the value of $s$, the super-level set $\{\varphi > s\}$ starts cutting $E$.

Recall that $xE \in \partial E$ is the footpoint of the minimizing geodesic from $x$ to $\overline{E}$. We claim that for any $s < 0$ sufficiently close to 0, $E_{s,0}$ is a perturbation of $E$ supported in a small ball
\(B_r(x_E)\), i.e. \(\{\varphi > s\} \cap E \subset B_r(x_E)\). To prove this claim, it is enough to observe that from \(f \leq 0\) on \(E\), \(\varphi(x_E) = 0\), and \(B_\lambda(x) \subset X \setminus E\), we get

\[
\{\varphi > s\} \cap E \subset \{\varphi > f - \delta\} \setminus \overline{B_\lambda(x)} \quad \text{for any } s \in (-\delta, 0).
\]

(3.19)

Moreover, for every \(z \in \{\varphi > s\} \cap E\), the maximum defining \(\varphi(z)\) is attained inside \(\overline{B_\lambda(x)}\), see (3.14) and the nearby discussion.

Now we wish to bound the distance from \(x_E\) to any point in \(\{\varphi > s\} \cap E\). For any \(z \in \{\varphi > s\} \cap E\), there exists \(x_z \in \overline{B_\lambda(x)}\) such that

\[
\varphi(z) = \overline{\psi}(x_z) - d(x_z, z) \leq f(x_z) - d(x_z, z) \leq \lambda + d(x, \overline{E}) - d(x_z, z).
\]

Hence

\[
d(x_z, z) \leq d(x, \overline{E}) + \lambda - \varphi(z) \leq d(x, \overline{E}) + \lambda - s.
\]

(3.20)

In particular, we can bound the distance of any point in \(\{\varphi > s\} \cap E\) from \(x\), and hence from \(x_E\), and obtain

\[
\{\varphi > s\} \cap E \subset B_r(x_E), \quad r := 2d(x, \overline{E}) + 2\lambda + \delta.
\]

(3.21)

The effect of this construction is that for \(x\) close enough to \(E\), and \(\lambda, \delta > 0\) sufficiently small, \(r := 2d(x, \overline{E}) + 2\lambda + \delta\) is arbitrarily small. It follows that, for every \(r > 0\), one can perform the above construction in order to obtain \(x_E \in \partial E\) and a family of inner perturbations \((E_{s,0})_{s \in (-\delta, 0)}\) of \(E\), so that \(E \setminus E_{s,0} \subset B_r(x_E)\).

A completely analogous verification shows that, for \(0 < t < \delta\), the set \(E_{0,t} := E \cup \{\eta \leq t\}\) is a perturbation of \(E\) compactly supported in a small ball \(B_r(y_E)\).

When \(r < d(x_E, y_E)/2\), the interior and the exterior perturbations have disjoint supports. It is also elementary to check that the two perturbations are non-trivial.

Let us set now, for any \(-\delta \leq s \leq 0 < t < \delta\),

\[
E_{s,t} := E \setminus \{\varphi \geq s\} \cup \{\eta \leq t\}.
\]

(3.22)

We claim that there exist values \(s, t\) in the range above such that

\[
\mathcal{H}^N(E_{s,t}) = \mathcal{H}^N(E)
\]

(3.23)

and

\[
\mathcal{H}^N(E_{s,0}) < \mathcal{H}^N(E) < \mathcal{H}^N(E_{0,t}).
\]

(3.24)

In order to establish the claim it is sufficient to prove that

\[
(s, t) \mapsto \mathcal{H}^N(E_{s,t}),
\]

(3.25)

is a continuous function. Indeed, (3.24) follows immediately from the non-triviality of the perturbations. The sought continuity is a direct consequence of the 1-Lipschitz regularity of \(\varphi\) and \(\eta\), together with the compactness of the perturbations \(E_{s,t} \Delta E\) and the properties \(|\nabla \varphi| = |\nabla \eta| = 1\)-a.e., which guarantee that

\[
\mathcal{H}^N \left( \{\varphi = s\} \cap U_\Sigma \right) = \mathcal{H}^N \left( \{\eta = t\} \cap V_\Sigma \right) = 0,
\]

(3.26)

for any \(-\delta \leq s \leq 0 < t < \delta\).

Given the claim, it is easy to find by a continuity argument \(s\) and \(t\) such that (3.23) and (3.24) hold. Moreover letting \(\Omega\) be the open neighbourhood of \(\{x_E, y_E\}\) where the perturbation \(E_{s,t} \Delta E\) is compactly contained, it holds, by [29, Proposition 6.1], see Proposition 2.14,

\[
\left( \nabla \varphi \cdot \nu_{\varphi < s} \right)_{\text{int}} = \left( \nabla \varphi \cdot \nu_{\varphi < s} \right)_{\text{ext}} = -1 \quad \text{Per}_{\{\varphi < s\}}\text{-a.e. on } \Omega,
\]

(3.27)

and

\[
\left( \nabla \eta \cdot \nu_{\eta < t} \right)_{\text{int}} = \left( \nabla \eta \cdot \nu_{\eta < t} \right)_{\text{ext}} = -1 \quad \text{Per}_{\{\eta < t\}}\text{-a.e. on } \Omega.
\]

(3.28)

Notice that in order to apply [29, Proposition 6.1] it is necessary to multiply \(\varphi\) and \(\eta\) by regular cut-off functions compactly supported into \(\Omega\). This can be easily done thanks to the existence of regular cut-off functions on RCD\((K, N)\) spaces, see [12, Lemma 6.7] and [75, 55]. In the rest of the proof we will assume that \(\varphi\) and \(\eta\) have compact supports, without changing the notation.
We are going to reach a contradiction comparing the perimeter of \( E_{s,t} \) with that of \( E \), arguing as in the final part of the proof of [77, Theorem 5.2]. We estimate separately the differences in the perimeter coming by \( \varphi \) and \( \eta \) by disjointedness:

\[
\Per(E_{s,t}) - \Per(E) = \left( \Per(E_{(s,0)}, \Omega) - \Per(E, \Omega) \right) + \left( \Per(E_{(0,t)}, \Omega) - \Per(E, \Omega) \right).
\]  
(3.29)

The two contributions can be estimated as follows. Let us set \( F := E \cap \{ \varphi > s \} \) and \( G := \{ \eta < t \} \setminus E \). Observe that by (3.23) it holds \( \mathcal{H}^N(F) = \mathcal{H}^N(G) \).

On the one hand we can apply [29, Theorem 5.2], see Theorem 2.12, with the sharp trace estimates, with test function identically equal to 1, vector field \( V = \nabla \varphi \) and set of finite perimeter \( F \). We obtain

\[
\int_{F(1)} \Delta \varphi = - \int_F (\nabla \varphi \cdot \nu_F) \text{int} \, d\Per - \int_{F \cap \{ \varphi > s \}} (\nabla \varphi \cdot \nu_E) \text{int} \, d\Per
\leq - \mathcal{H}^{N-1} \left( E(1) \cap F \{ \varphi > s \} \right) - \int_{F \cap \{ \varphi > s \}} (\nabla \varphi \cdot \nu_E) \text{int} \, d\Per
\leq - \mathcal{H}^{N-1} \left( E(1) \cap F \{ \varphi > s \} \right) + \mathcal{H}^{N-1} \left( FE \cap \{ \varphi > s \} \right),
\]  
(3.30)

where the first equality follows from [29, Theorem 5.2], the first inequality follows from [29, Proposition 5.4], see Proposition 2.13, and the fact that

\[
F(\cap \{ \varphi > s \}) \sim \left( E(1) \cap F \{ \varphi > s \} \right) \cup \left( FE \cap \{ \varphi > s \} \right) \cup \left( E(1/2) \cap F(1/2) \right),
\]
as a consequence of [29, Proposition 4.2], while the last inequality follows from the sharp trace bound \(|(\nabla \varphi \cdot \nu_E)| \leq 1 \) in [29, Theorem 5.2], see Theorem 2.12.

The analogous computation with \( \nabla \eta \) in place of \( \nabla \varphi \) and \( G \) in place of \( F \) yields to

\[
\int_{G(1)} \Delta \eta \geq \mathcal{H}^{N-1} \left( E(0) \cap F \{ \eta < t \} \right) - \mathcal{H}^{N-1} \left( FE \cap \{ \eta < t \} \right),
\]  
(3.31)

Now, the bounds on \( \Delta \varphi \) and \( \Delta \eta \) imply

\[
\int_{F(1)} \Delta \varphi \geq (c + \varepsilon) \mathcal{H}^N(F)
\]  
(3.32)

and

\[
\int_{G(1)} \Delta \eta \leq (c - \varepsilon) \mathcal{H}^N(G).
\]  
(3.33)

Hence, by (3.29), (3.30), (3.31), (3.32) and (3.33)

\[
\Per(E_{s,t}) - \Per(E) = \left( \Per(E_{(s,0)}, \Omega) - \Per(E, \Omega) \right) + \left( \Per(E_{(0,t)}, \Omega) - \Per(E, \Omega) \right)
= \left( \mathcal{H}^{N-1} \left( E(1) \cap F \{ \varphi > s \} \right) - \mathcal{H}^{N-1} \left( FE \cap \{ \varphi > s \} \right) \right)
+ \left( \mathcal{H}^{N-1} \left( E(0) \cap F \{ \eta < t \} \right) - \mathcal{H}^{N-1} \left( FE \cap \{ \eta < t \} \right) \right)
\leq \int_{G(1)} \Delta \eta - \int_{F(1)} \Delta \varphi
\leq (c - \varepsilon) \mathcal{H}^N(G) - (c + \varepsilon) \mathcal{H}^N(F) = -2\varepsilon \mathcal{H}^N(F) < 0,
\]
yielding to the sought contradiction.

In order to cover the case of a general lower Ricci curvature bound \( K \in \mathbb{R} \) we can modify the argument above as follows.

In the contradiction argument at the very beginning of the proof we obtain functions \( \tilde{\psi} \) and
The bounds (3.5) and (3.6) are now replaced by the bounds
\[ \Delta \varphi \geq c + \varepsilon - K f \quad \text{on } B_\lambda(x), \]
and
\[ \Delta \chi \leq c - \varepsilon - K f, \quad \text{on } B_\lambda(y), \]
respectively.

Arguing in the very same way as we did above, by employing Theorem 2.11 and taking into account the correction terms due to the general lower Ricci curvature bound $K$, the functions $\varphi$ and $\eta$ satisfy the Laplacian bounds $\Delta \varphi \geq c + \varepsilon/2$ and $\Delta \eta \leq c - \varepsilon/2$ in a neighbourhood of $X_\Sigma$ and $Y_\Sigma$ respectively. The rest of the argument carries over as in the $K = 0$ case.

**Step 2.** Let us see how to pass from the adimensional estimates in (3.4) to the sharp Laplacian comparison in (3.2).

We will rely on the localization technique from [40, 41], following the proof of [77, Theorem 5.2]. Let us prove the sharp Laplacian comparison on $X \setminus E$, the bound in the interior follows from an analogous argument.

From [41, Corollary 4.16], we know that
\[ \Delta f \hspace{1pt} \mathbb{L} X \setminus \mathcal{E} = (\Delta f)^{\text{reg}} \mathbb{L} X \setminus \mathcal{E} + (\Delta f)^{\text{sing}} \mathbb{L} X \setminus \mathcal{E}, \]
where the singular part $(\Delta f)^{\text{sing}} \perp \mathbb{H}^N$ satisfies $(\Delta f)^{\text{sing}} \mathbb{L} X \setminus \mathcal{E} \leq 0$ and the regular part $(\Delta f)^{\text{reg}} \ll \mathbb{H}^N$ admits the representation formula
\[ (\Delta f)^{\text{reg}} \mathbb{L} X \setminus \mathcal{E} = (\log h_\alpha)' \mathbb{H}^N \mathbb{L} X \setminus \mathcal{E}. \] (3.36)

In (3.36), $Q$ is a suitable set of indices, $(h_\alpha)_{\alpha \in Q}$ are suitable densities defined on geodesics $(X_\alpha)_{\alpha \in Q}$, which are essentially partitioning $X \setminus \mathcal{E}$ (in the smooth setting, $(X_\alpha)_{\alpha \in Q}$ correspond to the integral curves of $\nabla d_E$; note that here we are using the reverse parameterization of $X_\alpha$ with respect to [41], hence the reversed sign in the right hand side of (3.36)), such that the following disintegration formula holds:
\[ \mathbb{H}^N \mathbb{L} X \setminus \mathcal{E} = \int_Q h_\alpha \mathbb{H}^1 \mathbb{L} X_\alpha \mathfrak{q}(d\alpha). \] (3.37)

The non-negative measure $\mathfrak{q}$ in (3.37), defined on the set of indices $Q$, is obtained in a natural way from the essential partition $(X_\alpha)_{\alpha \in Q}$ of $X \setminus \mathcal{E}$, roughly by projecting $\mathbb{H}^N \mathbb{L} X \setminus \mathcal{E}$ on the set $Q$ of equivalence classes (we refer to [41] for the details).

The key point for the proof of Step 2 is that each $h_\alpha$ is a CD($K,N$) density over the ray $X_\alpha$ (see [41, Theorem 3.6]), i.e.,
\[ (\log h_\alpha)'' \leq -K - \frac{1}{N-1}((\log h_\alpha)')^2, \] (3.38)
in the sense of distributions and point-wise except countably many points, compare with [39, Lemma A.3, Lemma A.5, Proposition A.10]. Equivalently
\[ \left( \frac{h_\alpha^\frac{1}{N-1}}{N-1} \right)'' \frac{K}{N-1} h_\alpha^\frac{1}{N-1} \leq 0, \] (3.39)
in the sense of distributions. Moreover, from (3.36) and (3.4) we know that
\[ (\log h_\alpha)'(d\mathcal{E}) \leq c - K \leq d\mathcal{E} \text{ a.e. on } X_\alpha, \text{ for } \mathfrak{q}\text{-a.e. } \alpha \in Q. \] (3.40)

The sharp estimates in (3.2) then follow from the standard Riccati comparison (see for instance [33, Lemma 4.10]) applied to the functions $v(r) := (h_\alpha(r)/h_\alpha(0))^{\frac{1}{N-1}}$ which verify
\[ v'' + \frac{K}{N-1} v \leq 0, \] (3.41)
in the sense of distributions by (3.39), $v(0) = 1$ and $v'(0) \leq c/(N-1)$ by (3.40).
Definition 3.6 (Mean curvature barriers for isoperimetric sets). Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) space, let \(E \subset X\) be an isoperimetric set. We call any constant \(c\) such that (3.2) holds a mean curvature barrier for \(\partial E\).

For discussions concerning the uniqueness of \(c\) as in the previous definition, and comparison with the Riemannian setting, we refer the reader to the forthcoming remarks.

Remark 3.7. If \((M^n, g)\) is a smooth Riemannian manifold (with Ricci curvature uniformly bounded from below) and \(E \subset M\) is an isoperimetric set, then the constant \(c\) obtained via Theorem 3.3 is unique and equal to the constant mean curvature of the regular part of \(\partial E\).

The validity of a similar statement for isoperimetric sets in RCD\((K,N)\) metric measure spaces \((X,d,\mathcal{H}^N)\) goes beyond the scope of this note and is left to the future investigation.

Remark 3.8. Let \((D,d,\mathcal{H}^2)\) be a two dimensional flat disk with canonical metric measure structure and boundary \(\partial D\). Let \((K,d_K,\mathcal{H}^2)\) be the doubling of \(D\) along its boundary \(\partial D\) and set \(k > 0\) the curvature of \(\partial D\). It is a classical fact that \((K,d_K,\mathcal{H}^2)\) is an Alexandrov space with non-negative curvature, in particular it is an RCD\((0,2)\) space, but it is not a smooth Riemannian manifold. Observe that each of the two isometric copies of \(D\) inside \(K\) verifies the bound (3.2) for any \(c \in [-k,k]\) (for \(K = 0\) and \(N = 2\)). Even though \(D \subset K\) is not an isoperimetric set, cf. with [46, Theorem 5.4], this example illustrates that the uniqueness of the mean curvature barrier is a delicate issue in the non-smooth setting.

The following mild regularity results are obtained arguing verbatim as in [77, Proposition 5.4, Theorem 5.5, Proposition 6.14]. This can be done since Theorem 3.3 is the counterpart of [77, Theorem 5.2] for isoperimetric sets, while [77, Lemma 6.12] holds for volume constrained minimizers for compact variations as well, and [77, Lemma 2.42] holds for volume constrained minimizers for compact variations since they are quasiminimal sets according to [18, Theorem 3.24].

Proposition 3.9. Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) space for some \(K \in \mathbb{R}\) and \(N \geq 2\). Let \(E \subset X\) be a volume constrained minimizer for compact variations in \(X\). Let \(d_{\mathcal{F}}\) be the distance function from the set \(\mathcal{F}\), and let \(d_E^*\) be the signed distance function from \(E\), with the convention that it is positive outside \(E\) and negative inside \(E\).

Then \(d_{\mathcal{F}}\) and \(d_E^*\) have locally measure valued Laplacian in \(X\) and the following hold

\[
\Delta d_{\mathcal{F}} = \mathcal{H}^{N-1} \mathcal{L} \partial E + \Delta d_{\mathcal{F}} \mathcal{L}(X \setminus \mathcal{F}),
\]

\[
\Delta d_E^* \mathcal{L} \partial E = 0.
\]

Proof. We only provide an indication of the strategy of the proof, that can be obtained with minor modifications with respect to the case of local perimeter minimizers considered in [77].

The proof is divided into two steps: the verification that \(d_{\mathcal{F}}\) and \(d_E^*\) admit locally measure valued Laplacian and the computation of the singular part of their Laplacian along \(\partial E\).

In order to prove that \(d_{\mathcal{F}}\) and \(d_E^*\) admit locally measure valued Laplacian it is sufficient to uniformly bound the volumes of the \(t\)-tubular neighbourhoods of \(\partial E\) as \(Ct\) when \(t \to 0\) for some constant \(C > 0\) and to pass to the limit in the Gauss-Green integration by parts formulae on super-level sets \(\{d_{\mathcal{F}} > t_i\}\) for suitably chosen sequences \(t_i \downarrow 0\).

The uniform volume bound for the tubular neighbourhoods of \(\partial E\) follows from [18, Theorem 3.24], where quasiminimality of isoperimetric sets is proved, and [77, Lemma 2.42]. The conclusion follows arguing as in Step 1 of the proof of [77, Proposition 5.4] (see also the previous [28]).

Thanks to the arguments in the proof of [28, Theorem 7.4], \(\Delta d_{\mathcal{F}} \ll \mathcal{H}^{N-1}\) and \(\Delta d_E^* \ll \mathcal{H}^{N-1}\mathcal{L} \partial E\), which is a locally doubling finite measure. This can be done with a classical blow-up argument, as in Step 2 of the proof of [77, Proposition 5.4]. In order to prove that at regular points of \(\partial E\) the (signed) distance from the boundary converges to the (signed) distance from
the boundary of a Euclidean space after the blow-up we rely on the quasiminimality [18, Theorem 3.24] and on [77, Theorem 2.43].

Finally, in order to prove that \( \Delta d^E \| \partial E = 0 \) one argues precisely as in the last part of the proof of [77, Theorem 5.5]. \( \square \)

**Proposition 3.10.** Let \( (X, d, \mathcal{H}^N) \) be an RCD(\( K, N \)) space for some \( K \in \mathbb{R} \) and \( N \geq 2 \). Let \( E \subset X \) be a volume constrained minimizer for compact variations in \( X \). Let
\[
\mu_\varepsilon^+ := \varepsilon^{-1} \mathcal{H}^N \{ 0 \leq d_E < \varepsilon \}, \\
\mu_\varepsilon^- := \varepsilon^{-1} \mathcal{H}^N \{ 0 \leq d_E < \varepsilon \}.
\]
(3.43)

Then \( \mu_\varepsilon^+ \rightarrow \text{Per}_E \), and \( \mu_\varepsilon^- \rightarrow \text{Per}_E \) weakly as \( \varepsilon \to 0 \). In particular, the Minkowski content of \( E \) coincides with \( \text{Per}(E) \).

**Proof.** The proof is analogous to the one of [77, Proposition 6.14].

The fact that the measures \( \mu_\varepsilon^+ \) and \( \mu_\varepsilon^- \) have uniformly bounded mass follows from [77, Lemma 2.42] thanks to [18, Theorem 3.24].

The fact that the perimeter is smaller than any weak limit as \( \varepsilon_i \downarrow 0 \) of the measures \( \mu_\varepsilon_i^+ \) and \( \mu_\varepsilon_i^- \) is general and does not require any regularity of \( \partial E \).

In order to prove the converse inequality we rely on Theorem 3.3, which plays the role of [77, Theorem 5.2] in this setting, on Proposition 3.9, which plays the role of [77, Proposition 5.4] in this setting and argue as in the second part of the proof of [77, Proposition 6.14]. \( \square \)

Our next goal is to turn the Laplacian comparison in Theorem 3.3 into an estimate for the perimeter of the equidistant sets from the boundary of an isoperimetric set.

**Proposition 3.11.** Let us consider an RCD(\( K, N \)) metric measure space \( (X, d, \mathcal{H}^N) \) for some \( K \in \mathbb{R} \) and \( N \geq 2 \). Let \( E \subset X \) be a volume constrained minimizer for compact variations in \( X \), and let \( c \in \mathbb{R} \) be given by Theorem 3.3. Then for any \( t \geq 0 \) it holds
\[
\text{Per}(\{ x \in X : d(x, \overline{E}) \leq t \}) \leq J_{c,K,N}(t) \text{Per}(E),
\]
(3.44)
and, for any \( t \geq 0 \),
\[
\text{Per}(\{ x \in X : d(x, X \setminus E) \leq t \}) \leq J_{-c,K,N}(t) \text{Per}(E),
\]
(3.45)
where we recall that the Jacobian function has been introduced in (2.14).

**Proof.** Let us focus on the estimate for the perimeter of the exterior equidistant set, the estimate for the interior one can be obtained with a completely analogous argument.

The bound can be obtained by applying the Gauss–Green integration by parts formula [29, Theorem 1.6] with vector field the gradient of the distance function in the slab \( E^t \setminus E \) (compare with [29, Theorem 5.2], and Theorem 2.12), where we denoted \( E^t := \{ x \in X : d(x, \overline{E}) \leq t \} \) the \( t \)-enlargement of \( E \). Indeed, taking into account Theorem 3.3 we obtain
\[
\text{Per}(E_t) \leq \text{Per}(E) + \int_{E^t \setminus E} (N - 1) \frac{s' \nu_K - c}{s_K - N - 1} \circ dE \mathcal{H}^N.
\]
(3.46)

Arguing as in the proof of [77, Proposition 6.15] via a classical comparison argument for ODEs (see also [33, Lemma 4.10]) we obtain that
\[
\text{Per}(\{ x \in X : d(x, \overline{E}) \leq t \}) \leq J_{c,K,N}(t) \text{Per}(E), \quad \text{for any } t \geq 0,
\]
(3.47)
as we claimed. \( \square \)

**Remark 3.12.** If \( (M^n, g) \) is a smooth Riemannian manifold and \( E \subset M \) is an isoperimetric set, then
\[
\lim_{t \to 0^+} \frac{\text{Per}(\{ x \in X : d(x, \overline{E}) \leq t \}) - \text{Per}(E)}{t} = c \text{Per}(E)
\]
(3.48)
and an analogous conclusion holds for the perimeters of the interior equidistant sets. This follows directly from the first variation formula for the perimeter when \( \partial E \) is smooth (since
we can consider deformations induced by a smooth compactly supported extension of the unit normal of \( \partial E \). If \( n \geq 8 \) an additional approximation argument (relying on the regularity theory for isoperimetric sets) is required, see for instance \([23, 84]\).

The validity of an analogous statement for isoperimetric sets in \( \text{RCD}(K, N) \) metric measure spaces goes beyond the scope of this note and is left to the future investigation.

Let us point out the expression for the bounds above when \( K = 0 \). Under these assumptions, with the very same notation above we obtain

\[
\text{Per}(E_t) \leq \text{Per}(E) \left( 1 + \frac{ct}{N-1} \right)^{N-1}, \quad \text{for any} \ t \geq 0. \tag{3.49}
\]

Using the coarea formula we can get volume bounds for the tubular neighbourhoods of isoperimetric sets integrating the perimeter bounds in Proposition 3.11.

**Corollary 3.13.** Let us consider an \( \text{RCD}(K, N) \) metric measure space \((X, d, \mathcal{H}^N)\) for some \( K \in \mathbb{R} \) and \( N \geq 2 \). Let \( E \subset X \) be a volume constrained minimizer for compact variations in \( X \), and let \( c \in \mathbb{R} \) be given by Theorem 3.3. Then for any \( t \geq 0 \) it holds

\[
\mathcal{H}^N(\{x \in X \setminus E : d(x, \overline{E}) \leq t\}) \leq \text{Per}(E) \int_0^t J_{c,K,N}(r) \, dr, \tag{3.50}
\]

and, for any \( t \geq 0 \),

\[
\mathcal{H}^N(\{x \in E : d(x, X \setminus E) \leq t\}) \leq \text{Per}(E) \int_0^t J_{-c,K,N}(r) \, dr. \tag{3.51}
\]

4. **Concavity properties of the isoperimetric profile function and consequences**

In Theorem 3.3 we proved sharp bounds on the Laplacian of the distance function from an isoperimetric set. Such bounds encode information about the first and second variation of the area of equidistants from the isoperimetric boundary. As we shall see, this information is sufficient to extend the sharp concavity properties for the isoperimetric profile known for smooth and compact Riemannian manifolds with lower Ricci curvature bounds to the setting of \( N \)-dimensional compact \( \text{RCD}(K, N) \) spaces.

More in general, we are going to join such information together with the generalized asymptotic mass decomposition Theorem 4.1 to get sharp concavity properties of the isoperimetric profile for \( \text{RCD}(K, N) \) spaces \((X, d, \mathcal{H}^N)\) with a lower bound on the volume of unit balls in Theorem 4.4, which is the main result of this section. In particular, this will imply that the sharp concavity properties of the isoperimetric profile hold on complete non-compact manifolds with uniform lower Ricci curvature bounds and uniform lower volume bounds.

Concavity properties of the isoperimetric profile for (weighted) manifolds with lower Ricci curvature bounds have been considered by various authors, see for instance \([22, 23, 24, 25, 73, 84, 79, 52]\). All these works deal with compact manifolds or with weighted manifolds of finite total measure and they heavily rely on the existence of isoperimetric regions for any volume. In all cases smoothness is a relevant assumption, in order to rely on the regularity theory for isoperimetric regions. The case of non-smooth weights in \([73]\) is handled with a careful approximation procedure. The only previous references where the problem is considered in non-compact manifolds with infinite volume are \([87, 82]\) and \([71]\). In \([87]\) the case of surfaces with non-negative Gaussian curvature is treated and existence of isoperimetric regions of any volume is an intermediate step in order to prove concavity of the isoperimetric profile. In \([82]\) the case of complete non-compact manifolds with \( C^{2,\alpha} \) bounded geometry is considered. In \([71]\) the authors consider unbounded convex bodies \( C \subset \mathbb{R}^n \) verifying no further regularity assumptions. Their proof relies on a generalized existence result for isoperimetric regions and on an approximation argument, to deal with convex bodies with non-smooth boundary.

The two statements below are proved in \([17]\) building on top of \([82, 16, 18]\). They will be key ingredients for the proof of Theorem 4.4.
Theorem 4.1 (Asymptotic mass decomposition). Let \((X, d, \mathcal{H}^N)\) be a non-compact RCD\((K, N)\) space. Assume there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\). Let \(V > 0\).

For every minimizing (for the perimeter) sequence \(\Omega_i \subset X\) of volume \(V_i\), with \(\Omega_i\) bounded for any \(i\), up to passing to a subsequence, there exist an increasing and bounded sequence \(\{N_i\}_{i \in \mathbb{N}} \subset \mathbb{N}\), disjoint finite perimeter sets \(\Omega_i^R, \Omega_i^d \subset \Omega_i\), and points \(p_{i,j}\), with \(1 \leq j \leq N_i\) for any \(i\), such that

\[
\begin{align*}
\text{• } & \lim_{\overline{N}} \delta(p_{i,j}, p_{i,\ell}) = \delta(p_{i,j}, o) = \infty, \text{ for any } j \neq \ell < \overline{N} + 1 \text{ and any } o \in X, \text{ where } \overline{N} := \lim_i N_i < \infty; \\
\text{• } & \Omega_i^R \text{ converges to } \Omega \subset X \text{ in the sense of finite perimeter sets, and we have } \mathcal{H}^N(\Omega_i^R) \to \mathcal{H}^N(\Omega), \text{ and } \text{Per}(\Omega_i^R) \to \text{Per}(\Omega). \text{ Moreover } \Omega \text{ is a bounded isoperimetric region for its own volume in } X; \\
\text{• } & \text{for every } j < \overline{N} + 1, (X, d, \mathcal{H}^N, p_{i,j}) \text{ converges in the pmGH sense to a pointed RCD}(K, N) \text{ space } (X_j, d_j, \mathcal{H}^N, p_j). \text{ Moreover there are isoperimetric regions } Z_j \subset X_j \text{ such that } \Omega_i^d_{i,j} \to Z_j \text{ in } L^1\text{-strong and } \text{Per}(\Omega_i^d_{i,j}) \to \text{Per}(Z_j); \\
\text{• } & \text{it holds that} \\
I_{(X, d, \mathcal{H}^N)}(V) = \text{Per}(\Omega) + \sum_{j=1}^{\overline{N}} \text{Per}(Z_j), \quad V = \mathcal{H}^N(\Omega) + \sum_{j=1}^{\overline{N}} \mathcal{H}^N(Z_j). \quad (4.1)
\end{align*}
\]

Proposition 4.2. Let \((X, d, \mathcal{H}^N)\) be a non-compact RCD\((K, N)\) space. Assume there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\). Let \(\{p_{i,j} : i \in \mathbb{N}\}\) be a sequence of points on \(X\), for \(j = 1, \ldots, \overline{N}\) where \(\overline{N} \in \mathbb{N} \cup \{\infty\}\). Suppose that each sequence \(\{p_{i,j}\}\) is diverging along \(X\) and that \((X, d, \mathcal{H}^N, p_{i,j})\) converges in the pmGH sense to a pointed RCD\((K, N)\) space \((X_j, d_j, \mathcal{H}^N, p_j)\). Defining

\[
I_{\overline{X} \cup_{j=1}^{\overline{N}} X_j}(v) := \inf \left\{ \text{Per}(E) + \sum_{j=1}^{\overline{N}} \text{Per}(E_j) : E \subset X, E_j \subset X_j, \mathcal{H}^N(E) + \sum_{j=1}^{\overline{N}} \mathcal{H}^N(E_j) = v \right\}, \quad (4.2)
\]

it holds \(I_{\overline{X} \cup_{j=1}^{\overline{N}} X_j}(v) = I_X(v)\) for any \(v > 0\).

We need to start with a mild regularity property of the isoperimetric profile, namely that it is strictly positive and continuous. Later in the paper these two properties will be sharpened in several directions. We stress that an argument similar to the one discussed in Lemma 4.3 had already appeared in [73, Lemma 6.9], and [52, Lemma 6.2]. By a careful inspection of the proofs, the argument for proving the local Hölder property in Lemma 4.3 is likely to be adapted in the more general case of CD\((K, N)\) spaces \((X, d, m)\) with densities uniformly bounded above and volume of unit balls uniformly bounded below, as kindly pointed out to the authors by E. Milman. Since we do not need such level of generality, we will not give the details of the proof in such a general case.

Lemma 4.3. Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space. Assume that there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\). Let \(I_X : (0, \mathcal{H}^N(X)) \to \mathbb{R}\) be the isoperimetric profile of \(X\). Then \(I_X(v) > 0\) for every \(v > 0\) and \(I_X\) is continuous.

Proof. The first conclusion can be reached arguing as in [16, Remark 4.7], building on the top of Theorem 4.1. The second conclusion can be reached adapting [81, Theorem 2], which shows that \(I\) is locally \((1 - \frac{1}{N})\)-Hölder, cf. [17, Lemma 2.23]. \(\square\)

4.1. Sharp concavity inequalities for the isoperimetric profile. Given a continuous function \(f : (0, \infty) \to (0, \infty)\) and parameters \(K \in \mathbb{R}\) and \(1 < N < \infty\) we are going to consider second order differential inequalities of the form

\[
-f'' f \geq K + \frac{(f')^2}{N - 1} \quad (4.3)
\]
and
\[ -f'' \geq \frac{KN}{N-1} f^{\frac{2}{N}}. \quad (4.4) \]

In general the function \( f \) will be continuous but not twice differentiable everywhere and the inequalities will be understood in the viscosity sense, i.e. we will require that whenever \( \varphi: (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R} \) is a \( C^2 \) function with \( \varphi \leq f \) on \((x_0 - \varepsilon, x_0 + \varepsilon)\) and \( \varphi(x_0) = f(x_0) \) the corresponding inequality (4.3) or (4.4) holds at \( x_0 \) with \( \varphi \) in place of \( f \).

**Theorem 4.4.** Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) space. Assume that there exists \( v_0 > 0 \) such that \( \mathcal{H}^N(B_1(x)) \geq v_0 \) for every \( x \in X \).

Let \( I: (0, \mathcal{H}^N(X)) \to (0, \infty) \) be the isoperimetric profile of \( X \). Then

1. The inequality
   \[ -I'' \geq K + \frac{(I')^2}{N-1} \, \text{ holds in the viscosity sense on } (0, \mathcal{H}^N(X)); \quad (4.5) \]
   2. Let \( N \leq \alpha < +\infty \) and \( \xi := I^{\frac{\alpha}{N-1}} \). Hence the inequality
   \[ -\xi'' \geq \frac{\alpha}{\alpha - 1} \frac{2 - \alpha}{\alpha} \left( \left( \frac{1}{N-1} - \frac{1}{\alpha - 1} \right) (I')^2 + K \right) \]
   \[ = \frac{\alpha}{\alpha - 1} \frac{2 - \alpha}{\alpha} \left( \left( \frac{1}{N-1} - \frac{1}{\alpha - 1} \right) (\alpha - 1)^2 \frac{\xi - 2}{\alpha} (\xi')^2 + K \right), \]
   \[ \text{ holds in the viscosity sense on } (0, \mathcal{H}^N(X)). \]
   In particular, in the case \( \alpha = N \), if \( \psi := I^{\frac{N}{N-1}} \) then
   \[ -\psi'' \geq \frac{KN}{N-1} \psi^{\frac{2}{N}} \, \text{ holds in the viscosity sense on } (0, \mathcal{H}^N(X)). \quad (4.7) \]

**Proof.** Let us assume \( \mathcal{H}^N(X) = +\infty \), the compact case being completely analogous. Let us prove (4.5) first. Let \( v \in (0, \infty) \) be fixed. Take \( \Omega \) a minimizing sequence of bounded sets of volume \( v \) and let \( \Omega, Z_j \) be the isoperimetric regions in \( X, X_j \) respectively, according to the notation of Theorem 4.1.

Let us consider a smooth function \( \varphi \) such that \( \varphi \leq I \) in a neighbourhood of \( v \) and \( \varphi(v) = I(v) \). We wish to prove that
\[ -\varphi''(v) \varphi(v) \geq K + \frac{(\varphi'(v))^2}{N-1}. \quad (4.8) \]

Let \( E \) be one of the isoperimetric sets \( \Omega, Z_j \), and let \( E_t \) be the \( t \)-enlargements, in the associated space \( X, X_j \), of \( E \) for \( t \in (-\varepsilon, \varepsilon) \). To be more precise, for \( t \geq 0 \), the points in \( E_t \) are those points of \( X \) that have distance \( \leq t \) from \( E \), while, for \( t \leq 0 \), the points in \( E_t \) are those points of \( X \) such that the distance from \( X \setminus E \) is \( \geq -t \). Let \( c \) be a mean curvature barrier for \( E \) provided by Theorem 3.3. We now show that \( c = \varphi'(v) \).

As we observed in (3.44),
\[ \operatorname{Per}(E_t) \leq J_{c,K,N}(t) \operatorname{Per}(E), \quad \text{ for any } t \in (-\varepsilon, \varepsilon). \quad (4.9) \]

By simple computations,
\[ J'_{c,K,N}(0) = c \quad \text{ and } \quad J''_{c,K,N}(0) = -K + \frac{N-2}{N-1} c^2. \quad (4.10) \]

Let us show that \( t \mapsto \operatorname{Per}(E_t) \) is differentiable at \( t = 0 \).

Setting \( \beta(t) := \mathcal{H}^N(E_t) + \sum_{T \in \{\Omega, Z_1, \ldots, Z_{j-1}\}, T \neq E} \mathcal{H}^N(T) \), by the coarea formula we get that \( \beta(t) \) is continuous at \( t = 0 \). Hence, by (4.2), and (4.9), we have that, for any \( t \in (-\varepsilon, \varepsilon) \),
\[ \begin{align*}
J_{c,K,N}(t) \operatorname{Per}(E) + \sum_{T \in \{\Omega, Z_1, \ldots, Z_{j-1}\}, T \neq E} \operatorname{Per}(T) & \geq \operatorname{Per}(E_t) + \sum_{T \in \{\Omega, Z_1, \ldots, Z_{j-1}\}, T \neq E} \operatorname{Per}(T) \\
& \geq I_{X, j} \beta(t) = I(\beta(t)) \geq \varphi(\beta(t)).
\end{align*} \quad (4.11) \]
From (4.11), the continuity of $\beta(t)$ at $t = 0$, the fact that

$$\varphi(v) = I(v) = \text{Per}(E) + \sum_{T \in \mathcal{G}} \text{Per}(T),$$

we first deduce that $\text{Per}(E_t)$ is continuous at $t = 0$. Hence, by the continuity of $t \mapsto \text{Per}(E_t)$ at $t = 0$ and the coarea formula, $t \mapsto \beta(t)$ is differentiable at $t = 0$. Exploiting this information again in (4.11) we obtain that $t \mapsto \text{Per}(E_t)$ is differentiable at $t = 0$.

By using the latter differentiability together with (4.9) we get that

$$\frac{d}{dt} \text{Per}(E_t)|_{t=0} = c \text{Per}(E).$$

(4.13)

By coarea, this implies that

$$t \mapsto H^N(E_t),$$

is twice differentiable at $t = 0$ with

$$\frac{d}{dt} H^N(E_t)|_{t=0} = \text{Per}(E)$$

and

$$\frac{d^2}{dt^2} H^N(E_t)|_{t=0} = c \text{Per}(E).$$

(4.14)

(4.15)

Moreover, from (4.1) and the discussion above, $\beta(0) = v$, $\beta'(0) = \text{Per}(E)$, $\beta''(0) = c \text{Per}(E)$ and $\varphi'(v)\beta'(0) = c \text{Per}(E)$, therefore $\varphi'(v) = c$, which is the sought claim.

Hence, we showed that the barrier $c$ given by Theorem 3.3 is unique and $c := \varphi'(v)$ for every isoperimetric set $\Omega, Z_j$ in any of the limit spaces $X, X_j$.

Let us define $\tilde{\beta}(t) := \mathcal{H}^N(\Omega) + \sum_{j=1}^N \mathcal{H}^N((Z_j)_t)$ for every $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small enough.

Arguing as in (4.11) for each isoperimetric region separately, we get that $\tilde{\beta}(t)$ is twice differentiable at $t = 0$. Moreover

$$\varphi(\tilde{\beta}(t)) \leq J_{c,K,N}(t) \left( \text{Per}(\Omega) + \sum_{j=1}^N \text{Per}(Z_j) \right)$$

for any $t \in (-\varepsilon, \varepsilon)$,

(4.16)

$$\tilde{\beta}(0) = v, \quad \tilde{\beta}'(0) = I(v), \quad \text{and} \quad \tilde{\beta}''(0) = cI(v).$$

Using (4.16), we can pass to the limit as $t \to 0$ in

$$\frac{\varphi(\tilde{\beta}(t)) + \varphi(\tilde{\beta}(-t)) - 2\varphi(\tilde{\beta}(0))}{t^2} = \frac{\varphi(\tilde{\beta}(t)) + \varphi(\tilde{\beta}(-t)) - 2I(v)}{t^2} \leq \frac{J_{c,K,N}(t) + J_{c,K,N}(-t) - 2}{t^2} I(v),$$

(4.17)

to obtain

$$\varphi''(\tilde{\beta}(0))\tilde{\beta}'(0)^2 + \varphi'(\tilde{\beta}(0))\tilde{\beta}''(0) \leq I(v) \left( c^2 \frac{N - 2}{N - 1} - K \right),$$

(4.18)

Using the previous identities, we obtain the sought conclusion

$$\varphi''(v)\varphi(v) = \varphi''(v)I(v) \leq -\frac{\varphi'(v)^2}{N - 1} - K.$$  

(4.19)

Let us prove (4.6). We need to prove that for every $v \in (0, \mathcal{H}^N(X))$, if $\varphi$ is a smooth function in a neighbourhood $U$ of $v$ such that $\varphi \leq \xi$ on $U$ and $\varphi(v) = \xi(v)$, then (4.6) holds with $\varphi$ in place of $\xi$. The latter inequality is obtained noticing that $\varphi \frac{\alpha}{\alpha - 1}$ $\leq I$ on $U$ and $\varphi \frac{\alpha}{\alpha - 1}(v) = I(v)$, using (4.5) with some easy algebraic computations.

(4.6)

Remark 4.5. Let us point out that (4.6), and thus (4.7), holds also in the sense of second order incremental quotients considered in [23, 24]. Given a continuous function $f : (0, \mathcal{H}^N(X)) \to (0, \infty)$ and $x \in (0, \mathcal{H}^N(X))$ we denote

$$\overline{D^2} f(x) := \limsup_{h \downarrow 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}.$$  

(4.20)
Then, for example, the very same proof of (4.7), where the bound was considered in the viscosity sense, shows that
\[
\overline{D^2\psi} \leq \frac{KN}{N-1} \psi^{\frac{2-N}{N}} \quad \text{on } (0, \mathcal{H}^N(X)).
\] (4.21)

4.2. Fine properties of the isoperimetric profile. In this subsection we derive further regularity properties and asymptotics for small volumes of the isoperimetric profile. They will be particularly useful to study the stability of isoperimetric regions under non-collapsed (pointed) Gromov–Hausdorff convergence, diameter bounds and connectedness properties for isoperimetric regions, and the asymptotic isoperimetric behaviour of non-collapsed spaces with lower Ricci curvature bounds (see also the forthcoming \[\text{[20]}\]).

The arguments essentially rely only on the sharp concavity properties of the isoperimetric profile and on qualitative isoperimetric inequalities.

The following corollary is a standard consequence of Theorem 4.4, and therefore we omit its proof.

**Corollary 4.6.** Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space with \(K \leq 0\). Assume that there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\).

For every \(v \in (0, \mathcal{H}^N(X))\) and for every \(\delta \in (v, \mathcal{H}^N(X) - v)\) there exists \(C > 0\) such that the function \(I_{\mathcal{H}^N(X)}(x) - Cx^2\) is concave on \((v - \delta, v + \delta)\). Hence the isoperimetric profile function \(I\) has right derivative \(I_+^+(v)\) and left derivative \(I^-_-(v)\) defined for every \(v \in (0, \mathcal{H}^N(X))\). Moreover the isoperimetric profile \(I\) is differentiable in all \((0, \mathcal{H}^N(X))\) except at most countably many values, it is locally Lipschitz, and it is twice differentiable almost-everywhere. Moreover (4.5), (4.6), and (4.7) also hold pointwise almost everywhere.

We now aim at giving a slight improvement of Corollary 4.6. We first need an auxiliary result.

The following Lemma 4.7 is rather classical and it holds in the class of locally doubling metric measure spaces satisfying a Poincaré inequality and a uniform noncollapsing assumption on the volumes of unit balls. Such a result has its roots in the papers \[\text{[34, 62, 43, and 47]}\]. Since we only need it in the setting of \(N\)-dimensional RCD\((K, N)\) spaces, we state it in this setting. See also \[\text{[18, Proposition 3.20]}\] (cf. \[\text{[18, Remark 3.21]}\]).

**Lemma 4.7.** Let \(K \in \mathbb{R}\) and \(N \geq 2\). Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space. Let us assume there exists \(v_0 > 0\) such that \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\). Then there exist \(v_1 := v_1(K, N, v_0)\), and \(\bar{v} := \bar{v}(K, N, v_0)\) such that
\[
I(v) \geq \bar{v} v^{\frac{N}{N-1}}, \quad \text{for all } v \leq v_1. \tag{4.22}
\]

**Remark 4.8.** Under the assumptions of Lemma 4.7, by bounding from above \(I\) with the perimeter of balls, we have that there exist constants \(C_0 = C_0(K, N, v_0) > 0\) and \(\bar{v} = \bar{v}(K, N, v_0) > 0\) such that
\[
\frac{I(v)}{v^{\frac{N}{N-1}}} \leq C_0, \quad \text{for every } v \leq \bar{v}. \tag{4.23}
\]

Joining the second order differential inequalities derived in Theorem 4.4 with Lemma 4.7 we derive further analytical properties of the isoperimetric profile.

**Proposition 4.9.** Let \(K \leq 0\) and \(N \geq 2\). Let \((X, d, \mathcal{H}^N)\) be an RCD\((K, N)\) space such that there exists \(v_0 \geq 0\) with \(\mathcal{H}^N(B_1(x)) \geq v_0\) for every \(x \in X\). Then the following hold.

1. There exist \(C := C(K, N, v_0) > 0\) and \(v_1 := v_1(K, N, v_0) > 0\) such that the function \(\eta(v) := I_{\mathcal{H}^N(X)}(v) - Cv^{\frac{2-N}{N}}\) is concave on the interval \([0, v_1]\). Moreover, if \(N = 2\), we can choose \(C = -K\) and the claim holds on \([0, \mathcal{H}^N(X)]\), if \(\mathcal{H}^N(X) < \infty\), or on \([0, \mathcal{H}^N(X)]\), if \(\mathcal{H}^N(X) = \infty\).

As a consequence the function
\[
[0, \mathcal{H}^N(X)) \ni v \mapsto \frac{I(v)}{v^{\frac{N}{N-1}}},
\]
has a finite strictly positive limit as \( v \to 0 \).

(2) There exists \( \tilde{v}_1 := \tilde{v}_1(K,N,v_0) \in (0,v_1) \) such that \( I \) is concave on \([0,\tilde{v}_1]\).

**Proof.** Since the proof follows from Theorem 4.4, Lemma 4.7 and an elementary one-dimensional analysis, we just sketch it.

Let us prove item (1). Let us first deal with the case \( N > 2 \). It suffices to take

\[
C := \frac{-KN^3\bar{\vartheta}^{\frac{2}{N}}}{2(N-1)(N+2)}, \quad \text{where } \bar{\vartheta} \text{ and } v_1 \text{ are the constants in (4.22).}
\]

(4.24)

With this choice of the constant \( C \), it can be readily proved that \( -\eta'' \geq 0 \) holds in the viscosity sense on \((0,v_1)\). This is proved by a straightforward computation using item (2) of Theorem 4.4, (4.22), and the choice of the constant \( C \). The conclusion then follows since \( \eta \) is continuous on \((0,v_1)\) thanks to Lemma 4.3.

Let us deal with the remaining case \( N = 2 \). From Theorem 4.4 we get that the function \( \psi := I^2 \) satisfies

\[
-\psi'' \geq 2K,
\]

in the viscosity sense on \((0,\mathcal{H}^N(X))\). Hence the function \( \eta(x) := \psi(x) + Kx^2 \) satisfies \( -\eta'' \geq 0 \) in the viscosity sense on \((0,\mathcal{H}^N(X))\). Therefore, in this case we can take \( C := -K \) on the whole interval \([0,\mathcal{H}^N(X)]\) or \((0,\mathcal{H}^N(X))\), depending on whether \( \mathcal{H}^N(X) \) is finite or infinite.

The last conclusion in the statement of item (1) readily follows from the fact that \( v \mapsto \eta(v)/v \) is non-increasing on \([0,v_1]\), since \( \eta \) is concave on \([0,v_1]\).

Let us now prove item (2). By concavity of \( \eta \), exploiting (4.22) and (4.23), for \( A > 1 \) we find

\[
\eta'_+(v) \geq \eta(Av) - \eta(v) \geq \frac{1}{A-1}v \left( -K\bar{\vartheta}^{\frac{2}{N}}Av - CA^{\frac{2}{N}}v^{\frac{2}{N}} - C_0^{\frac{N}{N}}v + C_1v^{\frac{2}{N}} \right)
\]

\[
\geq \eta \frac{\bar{\vartheta}^{\frac{2}{N}}}{A-1} - \frac{C_0^{\frac{N}{N}}}{A-1} + \frac{1}{A-1}v \left( -CA^{\frac{2}{N}}v^{\frac{2}{N}} + C_1v^{\frac{2}{N}} \right),
\]

for any \( 0 < v < Av < v_1 \). Hence choosing first \( A \) sufficiently large and then restricting \( v \in (0,\tilde{v}_1) \) for \( \tilde{v}_1 < v_1 \) small enough, we obtain that \( \eta'_+(v) \geq \frac{1}{2}\bar{\vartheta}^{\frac{2}{N}} \) on \( v \in (0,\tilde{v}_1] \). This implies \( \frac{N}{N-1}I_\frac{N}{N-1}(v)I'_+(v) \geq \frac{1}{2}\bar{\vartheta}^{\frac{2}{N}} \) on \( v \in (0,\tilde{v}_1] \), and thus

\[
I'_+(v) \geq \frac{N}{N-1}\frac{1}{2\bar{\vartheta}^{\frac{2}{N}}}(C_0^{\frac{N}{N}})^{-1}v^{\frac{2}{N}}.
\]

(4.26)

Therefore (4.5) implies that \( I'' \leq 0 \) on \((0,\tilde{v}_1]\) in the viscosity sense, up to decrease \( \tilde{v}_1 \), proving item (2).

\( \square \)

**Remark 4.10.** Item (1) of Proposition 4.9 answers in the affirmative to Questions 2 and 3 in [83] in the more general setting of \( N \)-dimensional RCD\((K,N)\) spaces. As a consequence, it is possible to drop the additional hypothesis (H) in [83, Lemma 4.9].

Let us now derive some further consequences from the concavity properties above.

**Proposition 4.11.** Let \( K \leq 0 \) and \( N \geq 2 \). Let \((X,d,\mathcal{H}^N)\) be an RCD\((K,N)\) space such that there exists \( v_0 \geq 0 \) such that \( \mathcal{H}^N(B_1(x)) \geq v_0 \) for every \( x \in X \). Let us denote \( \vartheta := \lim_{v \to \vartheta_0} I(v)/v^{\frac{N}{N-1}} > 0 \), which exists due to item (1) of Proposition 4.9. Hence the following hold.

(1) There holds

\[
\lim_{v \to \vartheta_0} \frac{I'_+(v)}{v^{-\frac{N}{N-1}}} = \frac{N-1}{N}\vartheta.
\]

(2) Let \( \alpha > N \). Hence there exists \( \varepsilon = \varepsilon(K,N,v_0,\alpha) > 0 \) such that \( I^{\frac{N}{N-1}} \) is concave on \((0,\varepsilon)\). As a consequence, \( I \) is strictly subadditive on \((0,\varepsilon)\).
Proof. The first item follows from elementary computations exploiting that $I$ is concave for small volumes by Proposition 4.9. The second claim now easily follows from the first one, employing (4.6), Lemma 4.7 and Remark 4.8. \hfill \Box

Remark 4.12. Under the same assumptions of Proposition 4.11, we prove in [20] that the isoperimetric profile satisfies the asymptotic behavior for small volume given by

$$\lim_{v \to 0} \frac{I_X(v)}{v^{N}} = N (\omega_N \vartheta_{\infty, \text{min}})^{\frac{1}{N}},$$

where, being $\nu(N, K/(N-1), r)$ the volume of the ball of radius $r$ in the simply connected model space with constant sectional curvature $K/(N-1)$ and dimension $N$, we have that

$$\vartheta_{\infty, \text{min}} := \lim_{r \to 0} \inf_{x \in X} \frac{\mathcal{H}^N(B_r(x))}{\nu(N, K/(N-1), r)} > 0$$

is the minimum of all the possible densities at any point in $X$ or in any pmGH limit at infinity of $X$. The limit in (4.27) yields an answer to Questions 4 in [83]. It follows that the limit $\vartheta$ in Proposition 4.11 is now known to be equal to $N(\omega_N \vartheta_{\infty, \text{min}})^{\frac{1}{N}}$. Hence

$$\lim_{v \to 0} \frac{I_+(v)}{v^{N}} = (N-1) (\omega_N \vartheta_{\infty, \text{min}})^{\frac{1}{N}}.$$

The following lower bound appears to be classical, and it could be stated and proved in the class of locally doubling metric measure spaces satisfying a Poincaré inequality, and a uniform noncollapsing assumption on the volumes of unit balls. We refer to [42, Theorem V.2.6], which is stated for smooth Riemannian manifolds with bounded geometry, but whose proof adapts to the latter setting. Since we only need the statement in the setting of $N$-dimensional RCD($K,N$) spaces, we state it in this setting. We stress that an alternative proof of Corollary 4.13 using the results of this paper can be given arguing by contradiction, by exploiting Theorem 4.1 and Proposition 4.11.

Corollary 4.13. Let $0 < V_1 < V_2 < V_3$ and let $K \in \mathbb{R}, N \in \mathbb{N}_{\geq 2}, v_0 > 0$. Then there exists $\mathcal{I} = \mathcal{I}(K, N, v_0, V_1, V_2, V_3) > 0$ such that the following holds. If $(X, d, \mathcal{H}^N)$ is an RCD($K,N$) space with $\inf_{x \in X} \mathcal{H}^N(B_1(x)) \geq v_0 > 0$ and $\mathcal{H}^N(X) \geq V_3$, then

$$I_X(v) \geq \mathcal{I} \quad \forall v \in [V_1, V_2].$$

(4.28)

Corollary 4.14. Let $0 < V_1 < V_2$ and let $K \leq 0, N \in \mathbb{N}_{\geq 2}, v_0 > 0$. Then there exist $\mathcal{C}, \mathcal{L} > 0$ depending on $K, N, v_0, V_1, V_2$ such that the following holds. If $(X, d, \mathcal{H}^N)$ is RCD($K,N$) with $\inf_{x \in X} \mathcal{H}^N(B_1(x)) \geq v_0 > 0$ and $\mathcal{H}^N(X) \geq V_2$, then

$$v \mapsto I_{\frac{N}{N-1}}(v) - \mathcal{C} v^{\frac{2}{N}}$$

is concave on $[0, V_1]$,

$$v \mapsto I_{\frac{N}{N-1}}(v)$$

is $L$-Lipschitz on $[0, V_1]$.

(4.29)

In particular, for any $V \in (0, V_1)$ there is $L > 0$ depending on $K, N, v_0, V_1, V_2, V$ such that

$$v \mapsto I(v)$$

is $L$-Lipschitz on $[V, V_1]$.

(4.30)

Proof. We know from Proposition 4.9 that there exist $v_1, C > 0$ depending on $K, N, v_0$ such that $I_{\frac{N}{N-1}}(v) - C v^{\frac{2}{N}}$ is concave on $[0, v_1]$.

By (4.7) and Corollary 4.13, easy computations give that $I_{\frac{N}{N-1}}(v) - \mathcal{C} v^{\frac{2}{N}}$ is concave on $[0, V_1]$, for a constant $\mathcal{C} = \mathcal{C}(K, N, v_0, V_1, V_2) > 0$.

Let us denote $f(v) := I_{\frac{N}{N-1}}(v) - \mathcal{C} v^{\frac{2}{N}}$. In order to show the Lipschitzianity of $I_{\frac{N}{N-1}}$, it is enough to observe that by concavity, Remark 4.8, Proposition 4.9, Proposition 4.11, and Corollary 4.13, we have that $f_+'$ is bounded above on $[0, V_1]$ and $f$ is bounded below on $[V_1/2, (V_1 + V_2)/2]$. Hence, since $f$ is concave, we get that $f_+'$ is uniformly bounded above and below on $[0, V_1]$ by constants only depending on $K, N, v_0, V_1, V_2$. 


As a direct consequence $I^\mathcal{N}_{\Pi^r}$ is $\mathcal{L}$-Lipschitz on $[0,V_1]$ for some constant $\mathcal{L}$ depending on $K,N,v_0,V_1,V_2$. 

The above uniform bounds on the isoperimetric profile allow to derive the following uniform regularity properties on isoperimetric regions, that is, we prove that isoperimetric sets satisfy almost-minimality properties and density estimates with constants independent of the specific ambient space. This is new even in the case of isoperimetric sets in smooth Riemannian manifolds.

**Corollary 4.15.** Let $0 < V_1 < V_2 < V_3$ and let $K \in \mathbb{R}, N \in \mathbb{N}_{\geq 2}, v_0 > 0$. Then there exist $\Lambda, R > 0$ depending on $K, N, v_0, V_1, V_2, V_3$ such that the following holds.

If $(X,d,\mathcal{H}^N)$ is an $\operatorname{RCD}(K,N)$ space with $\inf_{x \in X} \mathcal{H}^N(B_1(x)) \geq v_0 > 0$, $\mathcal{H}^N(X) \geq V_3$, and $E \subset X$ is an isoperimetric region with $\mathcal{H}^N(E) \in [V_1,V_2]$, then $E$ is a $(\Lambda,R)$-minimizer, i.e., for any $F \subset X$ such that $F \Delta E \subset B_R(x)$ for some $x \in X$, then

$$\operatorname{Per}(E) \leq \operatorname{Per}(F) + \Lambda \mathcal{H}^N(F \Delta E).$$

In particular there exist $R' > 0$, $C_1 \in (0,1)$ and $C_2,C_3 > 0$ depending on $K,N,v_0,V_1,V_2,V_3$ such that

$$\operatorname{Per}(E,B_r(x)) \leq (1+C_3r) \operatorname{Per}(F,B_r(x)), \quad (4.31)$$

for any $x \in X$, $r \in (0,R']$, and any $F$ such that $F \Delta F \subset B_r(x)$. Moreover

$$C_1 \leq \frac{\mathcal{H}^N(B_r(x) \cap E)}{\mathcal{H}^N(B_r(x))} \leq 1 - C_1, \quad C_2^{-1} \leq \frac{\operatorname{Per}(E,B_r(x))}{r^{N-1}} \leq C_2,$$

for any $x \in \partial E$ and any $r \in (0,R']$.

**Proof.** Let $R > 0$ be a radius such that $\mathcal{H}^N(B_R(x)) \leq \min\{V_1/2, (V_3-V_2)/2\}$ for any $x \in X$. Let also $L > 0$ be the Lipschitz constant of $I$ on $[V_1/2,(V_2+V_3)/2]$ given by Corollary 4.14. Then for any $F \subset X$ with $F \Delta E \subset B_R(x)$ it holds

$$\operatorname{Per}(F) \geq I(\mathcal{H}^N(F)) \geq I(\mathcal{H}^N(E)) - L |\mathcal{H}^N(E) - \mathcal{H}^N(F)| \geq \operatorname{Per}(E) - L \mathcal{H}^N(F \Delta E).$$

For the second part of the claim let us exploit [18, Equation (3.51) in Remark 3.25]. According to the latter, one can find $R' > 0$ possibly smaller than $R$ and only depending on $K,N,v_0,V_1,V_2,V_3$ such that, calling $v(N,K/(N-1),r)$ the volume of the geodesic ball of radius $r$ in the model of constant sectional curvature $K/(N-1)$ and dimension $N$, one has, for some constant $\tilde{C}_1$ only depending on $N,K,v_0$, that

$$\operatorname{Per}(E,B_r(x)) \leq \frac{1 + \tilde{C}_1 v(N,K/(N-1),r)^{1/N}}{1 - \tilde{C}_1 v(N,K/(N-1),r)^{1/N}} \operatorname{Per}(F,B_r(x)),$$

for any $x \in X$, $r \in (0,R']$, and any $F$ such that $F \Delta F \subset B_r(x)$. Hence, by taking $R'$ smaller if needed, and by using that there exists $\tilde{C}_2$ only depending on $N,K$ such that $v(N,K/(N-1),r) \leq \tilde{C}_2 r^N$ for every $r \leq 1$, we get the conclusion in (4.31).

The last part of the claim follows arguing as in the proof of [18, Proposition 3.27].

**4.3. Consequences.** From the previous results on the concavity of the isoperimetric profile one can prove that in the $N$-dimensional $\operatorname{RCD}(K,N)$ spaces, isoperimetric regions of sufficiently small volume are connected. If $K = 0$, the conclusion holds for all volumes.

**Corollary 4.16.** Let $(X,d,\mathcal{H}^N)$ be an $\operatorname{RCD}(K,N)$ space with $N \geq 2$. Let us assume that $\inf_{x \in X} \mathcal{H}^N(B_1(x)) \geq v_0 > 0$. Let $\varepsilon$ be such that the isoperimetric profile $I$ is strictly subadditive on $(0,\varepsilon)$. Such an $\varepsilon > 0$ exists thanks to item (2) of Proposition 4.11, and, if $K = 0$, one can take $\varepsilon = \mathcal{H}^N(X)$.

Let $E = E^{(1)}$ be an isoperimetric region in $X$ with $\mathcal{H}^N(E) < \varepsilon$. Then $E$ is connected. If in addition $\mathcal{H}^N$ is finite, then $E$ is simple (i.e. $E$ and $X \setminus E$ are indecomposable) and $E^{(0)}$ is connected.
Proof. We recall that when we deal with an isoperimetric region $E$, we are always considering it is open by taking $E = E^{(1)}$. From item (2) of Proposition 4.11 one gets that there exists $\varepsilon > 0$ such that $I$ is strictly subadditive on $(0, \varepsilon)$. Notice that if $K = 0$, we have that $I_{N/(N-1)}$ is concave as a consequence of item (2) of Theorem 4.4, hence in this case $I$ is strictly subadditive on $(0, H^N(X))$.

Assume $\Omega$ is an isoperimetric region of volume $V < \varepsilon$. We prove first that $\Omega$ is indecomposable of volume $V$. Suppose by contradiction it is decomposable. Hence $\Omega = \Omega_1 \cup \Omega_2$ with $H^N(\Omega_1) + H_N(\Omega_2) = V$ and $\text{Per}(\Omega_1) + \text{Per}(\Omega_2) = \text{Per}(\Omega) = I(V)$. Hence

$$I(V) = \text{Per}(\Omega_1) + \text{Per}(\Omega_2) \geq I(H^N(\Omega_1)) + I(H^N(\Omega_2)) > I(V),$$

where the last inequality is due to the fact that $I$ is strictly subadditive on $(0, \varepsilon)$. Hence we reach a contradiction.

To prove that $E$ is connected, we argue by contradiction: suppose there exist non-empty open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $U \cup V = E$. Being $U$, $V$, $E$ open, we have that $\partial E = \partial U \cup \partial V$. Since we know that $\partial E = \partial^* E$ (recall that we are always assuming that $E = E^{(1)}$), we deduce that $H^{N-1}(\partial U) < +\infty$ and accordingly $U$ is a set of finite perimeter, see e.g., [69].

Now consider the BV function $f := \chi_U$. Again thanks to the fact that $U$ and $V$ are open, we get $|D f|(E) = 0$. Given that $\text{RCD}(K, N)$ spaces have the two-sidedness property in the sense of [26, Definition 1.28]) (see [26, Example 1.31]) and $E$ is indecomposable, we deduce from [26, Theorem 2.5] that $f$ is $H^N$-a.e. constant on $E$, which leads to a contradiction.

Therefore, the set $E$ is connected.

Now assume that $H^N(X) < +\infty$. Then $X \setminus E$ is an isoperimetric set of its own volume, thus accordingly the last part of the statement follows from the first one applied to $X \setminus E$. \qed

Remark 4.17. Classical examples show that, even for smooth compact Riemannian manifolds, connectedness of isoperimetric regions might fail for volumes bounded away from zero if the Ricci curvature is negative somewhere, see for instance [23, Remark 2.3.12].

From the strictly subadditivity of the isoperimetric profile (for small volumes, if $K < 0$), we infer that there is at most one component in the asymptotic mass decomposition result in Theorem 4.1 (for small volumes if $K < 0$). This is understood in the following statement.

Lemma 4.18. Let $(X, d, H^N)$ be an $\text{RCD}(K, N)$ space with $N \geq 2$. Let us assume that $\inf_{x \in X} H^N(B_1(x)) \geq v_0 > 0$. Let $\varepsilon$ be such that the isoperimetric profile $I$ is strictly subadditive on $(0, \varepsilon)$. Such an $\varepsilon > 0$ exists thanks to item (2) of Proposition 4.11 and, if $K = 0$, one can take $\varepsilon = H^N(X)$.

1. Let $\{\Omega_i\}_{i \in \mathbb{N}}$ be a minimizing (for the perimeter) sequence of bounded finite perimeter sets of volume $v < \varepsilon$ in $X$. Then, if one applies Theorem 4.1, either $\overline{\Omega} = \emptyset$, or $\overline{\Omega} = \overline{\Omega} = 0$.

2. Let $X_1, \ldots, X_N$ be pmGH limits of $X$ along sequences of points $\{p_i\}_{i \in \mathbb{N}}$, for $j = 1, \ldots, N \in \mathbb{N} \cup \{+\infty\}$. Let $\Omega = E \cup \bigcup_{j=1}^{N} E_j$, with $E \subset X, E_j \subset X_j$ be a set achieving the infimum in (4.2) for some $v < \varepsilon$. Then exactly one component among $E, E_1, \ldots, E_N$ is nonempty.

In particular, for any $v < \varepsilon$ there is an $\text{RCD}(K, N)$ space $(Y, d, H^N)$ which is either $X$ or a pmGH limit of $X$ along a sequence $\{p_i\}_{i \subset X}$, and a set $E \subset Y$ such that $H^N(E) = v$ and $I_X(v) = \text{Per}(E)$.

Proof. Let us prove the first item, and to this aim we adopt the notation of Theorem 4.1. Let us assume that the assertion is not true. Hence there are $j \geq 2$ nonempty sets $E_1, \ldots, E_j$ among $\Omega, Z_1, \ldots, Z_N$ such that, due to (4.1) and the fact that $E_1, \ldots, E_j$ are isoperimetric in their own spaces,

$$I(v) = \sum_{k=1}^{j} I_X(\mathcal{H}^N(E_k)),$$
and $\sum_{k=1}^j \mathcal{H}^N(E_k) = v$. Using that, for every $v > 0$ and every $k \in \{1, \ldots, j\}$, we have $I_{X_k}(v) \geq I(v)$, see [17], we thus conclude that

$$I(v) \geq \sum_{k=1}^j I(\mathcal{H}^N(E_k)),$$

which is in contradiction with the fact that $I$ is strictly subadditive on $(0, v) \subset (0, \varepsilon)$, and $j \geq 2$.

The second item analogously follows from the strict subadditivity of the profile $I$ of $X$ and the identity in Proposition 4.2.

As we already remarked, on a smooth Riemannian manifold $(M^N, g)$, the barrier $c$ obtained applying Theorem 3.3 to an isoperimetric region $E$ is unique and it coincides with the value of the (constant) mean curvature of the regular part of $\partial E$. Moreover, $t \mapsto \text{Per}(E_t)$ is always differentiable at $t = 0$ and

$$\frac{d}{dt}|_{t=0} \text{Per}(E_t) = c \text{Per}(E). \quad (4.32)$$

A well known consequence of this observation is the fact that the isoperimetric profile is differentiable with derivative $I'(v) = c$ at any volume $v \in (0, \infty)$ such that there exists a unique isoperimetric region $E$ of volume $v$ (with constant mean curvature of the boundary equal to $c$). Below we partially generalize this statement to the present context.

**Corollary 4.19.** Let $(X, d, \mathcal{H}^N)$ be an RCD($K, N$) space such that $\mathcal{H}^N(B_1(x)) \geq v_0 > 0$ for any $x \in X$.

Let $v \in (0, \mathcal{H}^N(X))$ and let $\Omega, Z_1, \ldots, Z_N$ be isoperimetric sets given by Theorem 4.1 applied to the minimization problem at volume $v$. Let $c$ be any barrier given by Theorem 3.3 applied on either $\Omega, Z_1, \ldots,$ or $Z_N$. Then

$$I'_+(v) \leq c \leq I'_-(v). \quad (4.33)$$

Hence, if $I$ is differentiable at $v$, if $E \in \{\Omega, Z_1, \ldots, Z_N\}$, then the barrier given by Theorem 3.3 applied to $E$ is unique and equal to $I'(v)$. In particular, if $I$ is differentiable at $v$ and $E$ is an isoperimetric region on $X$ for the volume $v > 0$, we have that the barrier given by Theorem 3.3 applied to $E$ is unique and equal to $I'(v)$.

**Proof.** Let $v \in (0, \mathcal{H}^N(X))$ be fixed. Take $\Omega'$ a minimizing sequence of bounded sets of volume $v$ and let $\Omega, Z_j$ be the isoperimetric regions in $X, X_j$ respectively, according to the notation of Theorem 4.1. Let $c$ be any barrier as in the statement.

Let $E$ be an arbitrary isoperimetric region among $\Omega, Z_j$ in the spaces $X, X_j$. Let us set $\beta(t) := \mathcal{H}^N(E_t) + \sum_{T \in \{\Omega, Z_1, \ldots, Z_N\}, T \neq E} \mathcal{H}^N(T)$, where $E_t$ is the $t$-tubular neighbourhood of $E$ for $t \in (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$ small enough, see the discussion after (4.10). Arguing as in (4.11), we reach the conclusion.

**Remark 4.20.** Letting $X, v \in (0, \mathcal{H}^N(X))$ and $\Omega, Z_1, \ldots, Z_N$ as in Corollary 4.19, if $I$ is differentiable at $v$, then the function $t \mapsto \text{Per}(E_t)$ is differentiable at $t = 0$ and its derivative is $c \text{Per}(E)$, for any $E \in \{\Omega, Z_1, \ldots, Z_N\}$ where $c$ is a barrier for $E$. This follows by the argument in the proof of Corollary 4.19.

Moreover, if $I$ is differentiable at $v$, every isoperimetric set $\Omega, Z_1, \ldots, Z_N$ has only one possible barrier $c = I'(v)$.

Another consequence of the sharp concavity properties of the isoperimetric profile are uniform diameter bounds for isoperimetric regions of small volume, in great generality, and any volume if the underlying space is RCD(0, N) with Euclidean volume growth.

---

1We understand there is just one isoperimetric region $\Omega \subset X$ if $X$ is compact.
Proposition 4.21. For every $N \geq 2$ natural number, $K \leq 0$, and $v_0 > 0$, there exist constants \( \bar{v} = \bar{v}(K, N, v_0) > 0 \) and $C = C(K, N, v_0) > 0$ such that the following holds. Let $(X, d, \mathcal{H}^N)$ be an RCD$(K, N)$ space. Suppose that $\mathcal{H}^N(B_1(x)) \geq v_0$ holds for every $x \in X$. Let $E \subseteq X$ be an isoperimetric region. Then

\[
\text{diam} E \leq C \mathcal{H}^N(E)^{1/N} \quad \text{whenever } \mathcal{H}^N(E) \leq \bar{v}.
\]  

(4.34)

Moreover, for every $N \geq 2$ natural number and $A > 0$, there exists a constant $\bar{C} = \bar{C}(N, A) > 0$ such that the following holds. Let $(X, d, \mathcal{H}^N)$ be an RCD$(0, N)$ space satisfying

\[
\text{AVR}(X, d, \mathcal{H}^N) := \lim_{r \to +\infty} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} \geq A,
\]

where $\bar{x} \in X$ and $\omega_N$ is the Euclidean volume of the unit ball in $\mathbb{R}^N$. Then it holds that

\[
\text{diam} E \leq \bar{C} \mathcal{H}^N(E)^{1/N} \quad \text{for every isoperimetric region } E \subseteq X.
\]  

(4.35)

Proof. By Lemma 4.7, Remark 4.8, and item (2) of Proposition 4.9, there exist constants $\vartheta = \vartheta(K, N, v_0) > 0$, $\bar{v}_1 = \bar{v}_1(K, N, v_0) > 0$, and $C_0 = C_0(K, N, v_0) > 0$ such that:

a) $I(v) \geq \vartheta v^{N-1}$ for every $v \leq \bar{v}_1$.

b) $I(v)/v^{N-1} \leq C_0$ for every $v \leq \bar{v}_1$.

c) $I$ is concave on $[0, \bar{v}_1]$.

Given any $x \in E$ and $r > 0$, we define $m_x(r) := \mathcal{H}^N(E \cap B_r(x))$ and $E^x_r := E \setminus B_r(x)$. Set $v_E := \mathcal{H}^N(E)$ for brevity and suppose $v_E \leq \bar{v}_1$. By c), the function $v \mapsto I(v)/v$ is non-increasing on $[0, \bar{v}_1]$, so that

\[
\frac{I(v_E)}{v_E} \leq \frac{I(v_E - m_x(r))}{v_E - m_x(r)}, \quad \text{for every } r > 0.
\]

Multiplying both sides by $v_E - m_x(r)$, we deduce that

\[
\text{Per}(E) = I(v_E) \leq I(v_E - m_x(r)) + \frac{m_x(r)}{v_E} I(v_E) \leq \text{Per}(E^x_r) + \frac{m_x(r)}{v_E} I(v_E).
\]  

(4.36)

By using [3, Lemma 4.5] we obtain $\text{Per}(E^x_r) \leq \text{Per}(E) - \text{Per}(E \cap B_r(x)) + 2m'_x(r) \leq \text{Per}(E) - \vartheta m_x(r)^{N-1} + 2m'_x(r)$

(4.37)

for a.e. $r > 0$. Combining (4.36) with (4.37), we thus obtain for a.e. $r > 0$ that

\[
\vartheta m_x(r)^{N-1} - \frac{I(v_E)}{v_E} m_x(r) \leq 2m'_x(r).
\]  

(4.38)

Once (4.38) is obtained, one can argue as in [71, Lemma 5.7] to deduce the claim (4.34) with $\bar{v} = \bar{v}_1$. We just sketch the main steps. Let

\[
r_0 := \frac{1}{\omega_N^{1/N}} \frac{\vartheta v_E}{4I(v_E)} \geq \frac{\vartheta}{4\omega_N^{1/N} C_0} v_E^{1/N}.
\]  

(4.39)

Defining $f(r) := 2e^{\frac{I(v_E)}{4v_E} m_x(r)}$, (4.38) implies that $f'(r) \geq 2^{\frac{1}{N-1}} \vartheta e^{\frac{I(v_E)}{4v_E} m_x(r)} r^{N-1}$ for a.e. $r \in (0, r_0)$. Hence (4.39) implies

\[
2^{\frac{1}{N-1}} \vartheta e^{\frac{d}{\omega_N^{1/N}}} m_x(r_0)^{N} = f(r_0)^{\frac{1}{N}} \geq \int_{r_0/2}^{r_0} (f(r))^{\frac{1}{N}} dr \geq c(K, N, v_0)r_0,
\]  

(4.40)

where $c(K, N, v_0) > 0$ may change from line to line. Hence (4.40) implies

\[
m_x(r_0) \geq c(K, N, v_0)r_0^N.
\]  

(4.41)

Considering a maximal family $\mathcal{F}$ of pairwise disjoint open balls $B$ having radius $r_0$ and center $c(B)$ in $E$, since $\{B_{2r_0}(c(B)) : B \in \mathcal{F}\}$ is a covering of $E$, then $\# \mathcal{F} \leq c(K, N, v_0)v_E/\nu_0^N$. The
set $U := \bigcup_{B \in \mathcal{F}} B_{2r_0}(c(B))$ contains $E$, which is connected by Corollary 4.16, hence $U$ must be connected as well. Then

$$\text{diam}(E) \leq \text{diam}(U) \leq \sum_{B \in \mathcal{F}} \text{diam}(B_{2r_0}(c(B))) \leq 4r_0\# \mathcal{F} \leq C \sqrt{v_E'},$$

for a suitable constant $C = C(K,N,v_0)$, proving (4.34).

The second part of the statement is a consequence of the first part, and a scaling argument. Indeed, from the first part one gets that the following holds for every $K \leq 0$, $R > 0$, $N \in \mathbb{N} \cap [2,\infty)$, and $v_0 > 0$. For every $\text{RCD}(KR^{-2},N)$ space $(X,d,\mathcal{H}^N)$ for which $\inf_{x \in X} \mathcal{H}^N(B_R(x)) \geq v_0 R^N$, then every isoperimetric region $E$ with $\mathcal{H}^N(E) \leq \tilde{v}(K,N,v_0) R^N$ has diameter bounded from above by $C(K,N,v_0) \mathcal{H}^N(E)^{\frac{1}{N}}$.

If now $(X,d,\mathcal{H}^N)$ is an $\text{RCD}(0,N)$ space with $\text{AVR}(X,d,\mathcal{H}^N) \geq A > 0$, then one has $\inf_{x \in X} \mathcal{H}^1(B_R(x)) \geq A \omega_N R^N$ for every $R > 0$. Thus, taking $E$ an isoperimetric region and using what we said above with $K = 0$ and $R$ sufficiently large, we get

$$\text{diam}(E) \leq C(0,N,A\omega_N) \mathcal{H}^N(E)^{\frac{1}{N}}.$$  

□

Remark 4.22. Proposition 4.21 is a three-fold generalization of [83, Lemma 4.9]. Indeed, we prove the statement in the non-smooth $\text{RCD}(K,N)$ case with reference measure $\mathcal{H}^N$ and $\mathcal{H}^N(B_1(x)) \geq v_0 > 0$ for every $x \in X$, while the authors in [83] deal with smooth Riemannian manifolds. Moreover, in the smooth case, our setting corresponds to the weak bounded geometry hypothesis in [83], while [83, Lemma 4.9] is proved under the stronger mild bounded geometry hypothesis. Finally, we are able to drop the hypothesis (H) in [83, Lemma 4.9]. Hence, Proposition 4.21 is new even in the smooth setting and sharpens the previous [83, Lemma 4.9].

Notice that Proposition 4.21 generalizes also [71, Lemma 5.5] to the setting of $\text{RCD}(0,N)$ spaces with a uniform bound below on the volume of unit balls.

We conclude by stating a stability result for sequences of isoperimetric sets $E_i$ converging in $L^1$ to a limit set, where the $L^1$ convergence improves to Hausdorff convergence. In Theorem 4.23 below, observe that no uniform hypotheses on the mean curvature barriers for the $E_i$’s are assumed. Instead a uniform bound on such barriers follows from the fine properties we proved on the isoperimetric profile. We mention that analogous stability results for mean curvature barriers have been independently considered in the recent [65].

Theorem 4.23. Let $(X_i,d_i,\mathcal{H}^N,x_i)$ be a sequence of $\text{RCD}(K,N)$ spaces converging to $(Y,d_Y,\mathcal{H}^N,y)$ in $\text{pmGH}$ sense, and let $(Z,d_Z)$ be a space realizing the convergence. Assume that $\mathcal{H}^N(B_1(p)) \geq v_0 > 0$ for any $p \in X_i$ and any $i$. Let $E_i \subset X_i$, $F \subset Y$.

If $E_i$ is isoperimetric, $E_i \subset B_{R}(x_i)$ for some $R > 0$ for any $i$, $c_i$ is a mean curvature barrier for $E_i$ for any $i$, $E_i \to F$ in $L^1$-strong, and $0 < \lim_i \mathcal{H}^N(E_i) < \lim_i \mathcal{H}^N(X_i)$, then $F$ is isoperimetric,

$$|c_i| \leq L \quad \text{for any } i \text{ large enough},$$

$$|D\chi_{E_i}| \to |D\chi_{F}| \quad \text{in duality with } C_{\text{ba}}(Z),$$

$$\partial E_i \to \partial F, \quad E_i \to F \quad \text{in Hausdorff distance in } Z,$$

where $L = L(K,N,v_0,\inf_i \mathcal{H}^N(E_i),\sup_i \mathcal{H}^N(E_i)) > 0$. In particular, the mean curvature barriers $c_i$ converge up to subsequence to a mean curvature barrier for $F$.

The proof of Theorem 4.23 follows by well-established arguments. In fact, by a classical contradiction argument as in [16, 17], it follows that $\text{Per}(E_i) \to \text{Per}(F)$ and $F$ is isoperimetric. In particular, $|D\chi_{E_i}| \to |D\chi_{F}|$ in duality with $C_{\text{ba}}(Z)$.

By Corollary 4.14, Corollary 4.15, and Corollary 4.19, the mean curvature barriers $c_i$ are uniformly bounded and the $E_i$ satisfy uniform density estimates at boundary points, independently of $i$. This readily implies that any converging sequence $q_i \in \partial E_i$ must converge to a
point in $\partial F$ (compare with [77, Theorem 2.43]), and then Kuratowski convergence of $\partial E_i$ to $\partial F$ is achieved, which easily implies (4.43).

Finally, if $c_i \to c \in \mathbb{R}$ up to subsequence, integrating a sequence of strongly $H^{1,2}$-converging Lipschitz functions along $(X_i, d_i, \mathcal{H}^N, x_i) \to (Y, d_Y, \mathcal{H}^N, y)$ with respect to the Laplacian of the signed distance functions from $E_i$, one gets that the signed distance from $F$ satisfies the adimensional bounds (3.4). Arguing as in Step 2 of the proof of Theorem 3.3, the inequalities are readily improved to get that $c$ is a mean curvature barrier for $F$.

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