Preserving subordination and superordination results of generalized Srivastava-Attiya operator

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ABSTRACT: In this paper, we obtain some subordination and superordination-preserving results of the generalized Srivastava-Attiya operator. Sandwich-type result is also obtained.

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1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...,$ with $H_0 = H[0, 1]$ and $H = H[1, 1].$ Denote $A(p)$ by the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}; \quad z \in U) \quad (1.1)$$

and let $A(1) = A.$ For $f, F \in H(U),$ the function $f(z)$ is said to be subordinate to $F(z),$ or $F(z)$ is superordinate to $f(z),$ if there exists a function $\omega(z)$ analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U),$ such that $f(z) = F(\omega(z)).$ In such a case we write $f(z) \prec F(z).$ If $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [14] and [15]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U.$ If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:

$$\phi \left( p(z), zp'(z) ; z \right) \prec h(z), \quad (1.2)$$

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then \( p(z) \) is a solution of the differential subordination (1.2). The univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination (1.2) if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying (1.2). A univalent dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants of (1.2) is called the best dominant. If \( p(z) \) and \( \phi \left( p(z), zp'(z) ; z \right) \) are univalent in \( U \) and if \( p(z) \) satisfies the first order differential superordination:

\[
h(z) \prec \phi \left( p(z), zp'(z) ; z \right),
\]

then \( p(z) \) is a solution of the differential superordination (1.3). An analytic function \( q(z) \) is called a subordinant of the solutions of the differential superordination (1.3) if \( q(z) \prec p(z) \) for all \( p(z) \) satisfying (1.3). A univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all subordinants of (1.3) is called the best subordinant (see [14] and [15]).

The general Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) is defined by:

\[
\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s},
\]

\((a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |z| < 1; R\{s\} > 1 \text{ when } |z| = 1).\)

For interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) (see [3], [8], [9], [11] and [19]).

Recently, Srivastava and Attiya [18] introduced the linear operator \( L_{s,b} : A \to A \), defined in terms of the Hadamard product by

\[
L_{s,b}(f)(z) = G_{s,b}(z) * f(z) \quad (z \in U; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}),
\]

(1.5)

where for convenience,

\[
G_{s,b} = (1 + b)^s[\Phi(z, s, b) - b^{-s}] \quad (z \in U).
\]

(1.6)

The Srivastava-Attiya operator \( L_{s,b} \) contains among its special cases, the integral operators introduced and investigated by Alexander [1], Libera [7] and Jung et al. [6].

Analogous to \( L_{s,b} \), Liu [10] defined the operator \( J_{p,s,b} : A(p) \to A(p) \) by

\[
J_{p,s,b}(f)(z) = G_{p,s,b}(z) * f(z) \quad (z \in U; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}; p \in \mathbb{N}),
\]

(1.7)

where

\[
G_{p,s,b} = (1 + b)^s[\Phi_p(z, s, b) - b^{-s}]
\]

and

\[
\Phi_p(z, s, b) = \frac{1}{b^p} + \sum_{n=0}^{\infty} \frac{z^{n+p}}{(n + 1 + b)^s}.
\]

(1.8)

It is easy to observe from (1.7) and (1.8) that

\[
J_{p,s,b}(f)(z) = z^p + \sum_{n=1}^{\infty} \left( \frac{1 + b}{n + 1 + b} \right)^s a_{n+p} z^{n+p}.
\]

(1.9)
We note that
(i) $J_{p,0,b}(f)(z) = f(z)$;
(ii) $J_{1,1,0}(f)(z) = Lf(z) = \int_0^z \frac{f(t)}{t} dt$, where the operator $L$ was introduced by Alexander [11];
(iii) $J_{s,b}(f)(z) = L_s,b f(z)$ (s, b $\in \mathbb{C}$, b $\in \mathbb{C}\setminus \mathbb{D}$), where the operator $L_s,b$ was introduced by Srivastava and Attiya [18];
(iv) $J_{p,1,\nu+p-1}(f)(z) = F_{\nu,p}(f(z))$ ($\nu > -p, p \in \mathbb{N}$), where the operator $F_{\nu,p}$ was introduced by Choi et al. [4];
(v) $J_{p,0,p}(f)(z) = I_p^p f(z)$ ($\alpha \geq 0, p \in \mathbb{N}$), where the operator $I_p^p$ was introduced by Shams et al. [17];
(vi) $J_{p,m,p-1}(f)(z) = J_p^m f(z)$ ($m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}$), where the operator $J_p^m$ was introduced by El-Ashwah and Aouf [5];
(vii) $J_{p,m,p+l-1}(f)(z) = J_p^m (l) f(z)$ ($m \in \mathbb{N}_0, p, l \in \mathbb{N}, l \geq 0$), where the operator $J_p^m (l)$ was introduced by El-Ashwah and Aouf [5].

It follows from (1.9) that:

\[ z (J_{p,s+1,b}(f)(z))^\prime = (b + 1)J_{p,s,b}(f)(z) - (b + 1 - p)J_{p,s+1,b}(f)(z). \]  (1.10)

To prove our results, we need the following definitions and lemmas.

**Definition 1** [14]. Denote by $F$ the set of all functions $q(z)$ that are analytic and injective on $U \setminus E(q)$ where

\[ E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\} \]

and are such that $q^\prime(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $F$ for which $q(0) = a$ be denoted by $F(a)$, $F(0) \equiv F_0$ and $F(1) \equiv F_1$.

**Definition 2** [15]. A function $L(z,t)$ ($z \in U, t \geq 0$) is said to be a subordination chain if $L(.,t)$ is analytic and univalent in $U$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,1]$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

**Lemma 1** [16]. The function $L(z,t) : U \times [0;1] \rightarrow \mathbb{C}$ of the form

\[ L(z,t) = a_1(t) z + a_2(t) z^2 + ... \quad (a_1(t) \neq 0; t \geq 0) \]

and $\lim_{t \to \infty} |a_1(t)| = \infty$ is a subordination chain if and only if

\[ \text{Re} \left\{ \frac{z \partial L(z,t)}{\partial z} \right\} > 0 \quad (z \in U, t \geq 0). \]

**Lemma 2** [12]. Suppose that the function $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the condition

\[ \text{Re} \left\{ H(is; t) \right\} \leq 0 \]

for all real $s$ and for all $t \leq -n (1 + s^2) / 2, n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ is analytic in $U$ and

\[ \text{Re} \left\{ H \left( p(z); z p'(z) \right) \right\} > 0 \quad (z \in U), \]
Lemma 3 [13]. Let \( \kappa, \gamma \in \mathbb{C} \) with \( \kappa \neq 0 \) and let \( h \in H(U) \) with \( h(0) = c \). If \( \text{Re} \{k \gamma + \gamma\} > 0 \) for \( z \in U \), then the solution of the following differential equation:

\[
q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)
\]

is analytic in \( U \) and satisfies \( \text{Re} \{k \gamma(z) + \gamma\} > 0 \) for \( z \in U \).

Lemma 4 [14]. Let \( p \in F(a) \) and let \( q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \) be analytic in \( U \) with \( q(z) \neq a \) and \( n \geq 1 \). If \( q \) is not subordinate to \( p \), then there exists two points \( z_0 = r_0 e^{i\theta} \in U \) and \( \zeta_0 \in \partial U \setminus E(q) \) such that

\[
q(U_{r_0}) \subset p(U) \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m \zeta_0 p'(\zeta_0) \quad (m \geq n).
\]

Lemma 5 [15]. Let \( q \in H[a; 1] \) and \( \varphi : \mathbb{C}^2 \to \mathbb{C} \). Also set \( \varphi(q(z), zq'(z)) = h(z) \). If \( L(z, t) = \varphi(q(z), t zq'(z)) \) is a subordination chain and \( p \in H[a; 1] \cap F(a) \), then

\[
h(z) \prec \varphi(p(z), z p'(z)).
\]

implies that \( q(z) \prec p(z) \). Furthermore, if \( \varphi(q(z), zq'(z)) = h(z) \) has a univalent solution \( q \in F(a) \), then \( q \) is the best subordinant.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the integral operator \( J_{p,s,b} \). Sandwich-type result involving this operator is also derived.

2. Main results

Unless otherwise mentioned, we assume throughout this section that \( b \in \mathbb{C} \setminus \mathbb{Z}_0^+ \), \( s \in \mathbb{C} \), \( \text{Re}(b) > 0 \), \( p \in \mathbb{N} \) and \( z \in U \).

Theorem 1. Let \( f, g \in A(p) \) and

\[
\text{Re} \left\{ 1 + \frac{z \varphi''(z)}{\varphi'(z)} \right\} > -\delta \quad \left( \varphi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U \right),
\]

where \( \delta \) is given by

\[
\delta = \frac{1 + |b| + 1^2 - |1 - (b + 1)^2|}{4 \left[ 1 + \text{Re}(b) \right]} \quad (z \in U).
\]

Then the subordination condition

\[
\frac{J_{p,s-1,b}(f)(z)}{z^p} \leq \frac{J_{p,s-1,b}(g)(z)}{z^p}
\]
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implies that

\[ \frac{J_{p,s,b}(f)(z)}{z^p} < \frac{J_{p,s,b}(g)(z)}{z^p} \]  

(2.4)

and the function \( \frac{J_{p,s,b}(g)(z)}{z^p} \) is the best dominant.

**Proof.** Let us define the functions \( F(z) \) and \( G(z) \) in \( U \) by

\[ F(z) = \frac{J_{p,s,b}(f)(z)}{z^p} \quad \text{and} \quad G(z) = \frac{J_{p,s,b}(g)(z)}{z^p} \quad (z \in U) \]  

(2.5)

and without loss of generality we assume that \( G(z) \) is analytic, univalent on \( \bar{U} \) and

\[ G'(\zeta) \neq 0 \quad (|\zeta| = 1) . \]

If not, then we replace \( F(z) \) and \( G(z) \) by \( F(\rho z) \) and \( G(\rho z) \), respectively, with \( 0 < \rho < 1 \). These new functions have the desired properties on \( \bar{U} \), so we can use them in the proof of our result and the results would follow by letting \( \rho \to 1 \).

We first show that, if

\[ q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U) , \]  

(2.6)

then

\[ \text{Re} \{ q(z) \} > 0 \quad (z \in U) . \]

From (1.10) and the definition of the functions \( G, \phi \), we obtain that

\[ \phi(z) = G(z) + \frac{zG'(z)}{b + 1} . \]  

(2.7)

Differentiating both sides of (2.7) with respect to \( z \) yields

\[ \phi'(z) = \left( 1 + \frac{1}{b + 1} \right) G'(z) + \frac{zG''(z)}{b + 1} . \]  

(2.8)

Combining (2.6) and (2.8), we easily get

\[ 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + b + 1} = h(z) \quad (z \in U) . \]  

(2.9)

It follows from (2.1) and (2.9) that

\[ \text{Re} \{ h(z) + b + 1 \} > 0 \quad (z \in U) . \]  

(2.10)

Moreover, by using Lemma 3, we conclude that the differential equation (2.9) has a solution \( q(z) \in H(U) \) with \( h(0) = q(0) = 1 \). Let

\[ \mathcal{H}(u, v) = u + \frac{v}{u + b + 1} + \delta , \]
where $\delta$ is given by (2.2). From (2.9) and (2.10), we obtain $\text{Re} \left\{ \mathcal{H} \left( q(z); zq'(z) \right) \right\} > 0 \ (z \in U)$.

To verify the condition

$$\text{Re} \left\{ \mathcal{H} (i\vartheta; t) \right\} \leq 0 \quad (\vartheta \in \mathbb{R}; t \leq -\frac{1 + \vartheta^2}{2}) \quad (2.11)$$

we proceed as follows:

$$\text{Re} \left\{ \mathcal{H} (i\vartheta; t) \right\} = \text{Re} \left\{ \vartheta + \frac{t}{b + 1 + i\vartheta} + \delta \right\} = \frac{t (1 + \text{Re} (b))}{|b + 1 + i\vartheta|^2} + \delta \leq -\frac{\Upsilon (b, \vartheta, \delta)}{2 (b + 1 + i\vartheta)^2},$$

where

$$\Upsilon (b, \vartheta, \delta) = [1 + \text{Re} (b) - 2\delta] \vartheta^2 - 4\delta \text{Im} (b) \vartheta - 2\delta |b + 1|^2 + 1 + \text{Re} (b). \quad (2.12)$$

For $\delta$ given by (2.2), the coefficient of $\vartheta^2$ in the quadratic expression $\Upsilon (b, \vartheta, \delta)$ given by (2.12) is positive or equal to zero. To check this, put $b + 1 = c$, so that

$$1 + \text{Re} (b) = c_1 \quad \text{and} \quad \text{Im} (b) = c_2.$$

We thus have to verify that

$$c_1 - 2\delta \geq 0,$$

or

$$c_1 \geq 2\delta = \frac{1 + |c|^2 - |1 - c|^2}{2c_1}.$$

This inequality will hold true if

$$2c_1^2 + |1 - c|^2 \geq 1 + |c|^2 = 1 + c_1^2 + c_2^2,$$

that is, if

$$|1 - c|^2 \geq 1 - \text{Re} (c^2),$$

which is obviously true. Moreover, the quadratic expression $\Upsilon (b, \vartheta, \delta)$ by $\vartheta$ in (2.12) is a perfect square for the assumed value of $\delta$ given by (2.2). Hence we see that (2.11) holds. Thus, by using Lemma 2, we conclude that

$$\text{Re} \{ q(z) \} > 0 \quad (z \in U),$$

that is, that $G$ defined by (2.5) is convex (univalent) in $U$. Next, we prove that the subordination condition (2.3) implies that

$$F (z) \prec G (z),$$
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for the functions $F$ and $G$ defined by (2.5). Consider the function $L(z, t)$ given by

$$L(z, t) = G(z) + \frac{(1 + t) z G'(z)}{b + 1} \quad (0 \leq t < \infty; z \in U).$$  \hspace{1cm} (2.13)

We note that

$$\left. \frac{\partial L(z, t)}{\partial z} \right|_{z=0} = G'(0) \left( 1 + \frac{1 + t}{b + 1} \right) \neq 0 \quad (0 \leq t < \infty; z \in U; \text{Re}\{b + 1\} > 0).$$

This shows that the function

$$L(z, t) = a_1(t) z + ...,$$

satisfies the condition $a_1(t) \neq 0 \quad (0 \leq t < \infty)$. Further, we have

$$\text{Re} \left\{ \frac{z \partial L(z, t)}{\partial z} / \partial t \right\} = \text{Re} \left\{ b + 1 + (1 + t) q(z) \right\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Since $G(z)$ is convex and $\text{Re}\{b + 1\} > 0$. Therefore, by using Lemma 1, we deduce that $L(z, t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{z G'(z)}{b + 1} = L(z, 0)$$

and

$$L(z, 0) \prec L(z, t) \quad (0 \leq t < \infty),$$

which implies that

$$L(\zeta, t) \notin L(U, 0) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U).$$  \hspace{1cm} (2.14)

If $F$ is not subordinate to $G$, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1 + t) \zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$  \hspace{1cm} (2.15)

Hence, by using (2.5), (2.13),(2.14) and (2.3), we have

$$L(\zeta_0, t) = G(\zeta_0) + \frac{(1 + t) \zeta_0 G'(\zeta_0)}{b + 1} = F(z_0) + \frac{z_0 F'(z_0)}{b + 1} = \frac{J_{p,s-1,b}(g)(z_0)}{z_0^p} \in \phi(U).$$

This contradicts (2.14). Thus, we deduce that $F < G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

**Theorem 2.** Let $f, g \in A(p)$ and

$$\text{Re} \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U \right),$$  \hspace{1cm} (2.16)
where $\delta$ is given by (2.2). If the function $J_{p,s-1,b}(f(z))_{zp}$ is univalent in $U$ and $J_{p,s-1,b}(f(z))_{zp} \in F$, then the superordination condition

$$\frac{J_{p,s-1,b}(g(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}} \prec \frac{J_{p,s-1,b}(f(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}}$$

implies that

$$\frac{J_{p,s,b}(g(z))_{zp}}{J_{p,s,b}(f(z))_{zp}} \prec \frac{J_{p,s,b}(f(z))_{zp}}{J_{p,s,b}(f(z))_{zp}}$$

and the function $J_{p,s,b}(g(z))_{zp}$ is the best subordinant.

**Proof.** Suppose that the functions $F$, $G$ and $q$ are defined by (2.5) and (2.6), respectively. By applying similar method as in the proof of Theorem 1, we get

$$\text{Re} \{ q(z) \} > 0 \quad (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be defined by (2.13). Since $G$ is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG'(z)}{b+1} = \varphi(G(z), zG'(z))$$

has a univalent solution $G$, it is the best subordinant. This completes the proof of Theorem 2.

Combining the above-mentioned subordination and superordination results involving the operator $J_{p,s,b}$, the following "sandwich-type result" is derived.

**Theorem 3.** Let $f, g_j \in A(p)$ $(j = 1, 2)$ and

$$\text{Re} \left\{ 1 + \frac{z\phi_j''(z)}{\phi_j'(z)} \right\} > -\delta \quad \left( \phi_j(z) = \frac{J_{p,s-1,b}(g_j(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}} (j = 1, 2); z \in U \right),$$

where $\delta$ is given by (2.2). If the function $J_{p,s-1,b}(f(z))_{zp}$ is univalent in $U$ and $J_{p,s-1,b}(f(z))_{zp} \in F$, then the condition

$$\frac{J_{p,s-1,b}(g_1(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}} \prec \frac{J_{p,s-1,b}(f(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}} \prec \frac{J_{p,s-1,b}(g_2(z))_{zp}}{J_{p,s-1,b}(f(z))_{zp}}$$

implies that

$$\frac{J_{p,s,b}(g_1(z))_{zp}}{J_{p,s,b}(f(z))_{zp}} \prec \frac{J_{p,s,b}(f(z))_{zp}}{J_{p,s,b}(f(z))_{zp}} \prec \frac{J_{p,s,b}(g_2(z))_{zp}}{J_{p,s,b}(f(z))_{zp}}$$

and the functions $J_{p,s,b}(g_1(z))_{zp}$ and $J_{p,s,b}(g_2(z))_{zp}$ are, respectively, the best subordinant and the best dominant.

**Remark.** (i) Putting $b = p$ and $s = \alpha \quad (\alpha > 0, p \in \mathbb{N})$ in our results of this paper, we obtain the results obtained by Aouf and Seoudy [2];

(ii) Specializing the parameters $s$ and $b$ in our results of this paper, we obtain the results for the corresponding operators $F_{\nu,p}, J_{p}^m$ and $J_{p}^m (l)$ which are defined in the introduction.
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