Massive 3+1 Aharonov-Bohm fermions in an MIT cylinder

M. De Francia and K. Kirsten

Department of Physics and Astronomy, The University of Manchester,
Theory Group, Schuster Laboratory, Manchester M13 9PL, England

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We study the effect of a background flux string on the vacuum energy of massive Dirac fermions in 3+1 dimensions confined to a finite spatial region through MIT boundary conditions. We treat two admissible self-adjoint extensions of the Hamiltonian. The external sector is also studied and unambiguous results for the Casimir energy of massive fermions in the whole space are obtained.

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I. INTRODUCTION

The influence of background fields or of boundaries on the vacuum structure of a quantum field theory is of continuing importance in various branches of modern physics. The various aspects are often described generically as Casimir effects and many techniques have been developed in order to analyse the different situations. As examples we mention the Green’s function approach \cite{1-3} and the zeta function regularization \cite{4-7}.

In the years since the Casimir effect was first discussed \cite{8} these techniques have been refined considerably. Relatively recently, a contour integral method has been developed which allows the representation of the Casimir or ground state energy in terms of eigenfunctions or of the Jost function of the associated field equation \cite{9-11}. This allows for a relatively direct analysis for configurations where this information can be easily obtained. As a result, spherically or cylindrically symmetric situations have been analysed in various contexts, see e.g. \cite{11-19}. Most of the research done so far concentrates on the influence of background fields or of boundaries separately. Relatively little is known when these two effects are combined. Exceptions are e.g. \cite{20,16}, where a magnetic fluxon and MIT boundary conditions for a massless and massive Dirac field in 2+1 dimensions were considered.

It is our aim to continue the analysis of this combined effect in a 3+1 dimensional cylindrically symmetric situation. We consider a massive Dirac operator in the presence of a magnetic flux string located along the z-axis and with MIT boundary conditions at a cylinder of radius $R$. Because of the presence of the flux string, a self-adjoint extension of the radial Dirac operator is needed, as it is well known from the 2+1 dimensional situation \cite{21,22}. In analogy to \cite{14} we will consider two possible self-adjoint extensions which are both compatible with the presence of a Dirac delta magnetic field at the origin. One extension is constructed by imposing spectral boundary conditions \cite{23-25} at a finite radius which is then shrunk to zero. The second one follows from the zero radius limit of a cylindrical flux shell \cite{26,27}. The Casimir energies differ, reflecting the fact that different self-adjoint extensions describe different physics in the core.

Compared to the calculations in 2+1 dimensions, peculiar differences occur. Most importantly, when considering only the interior of the cylinder, the poles in the Casimir energy depend on the flux for both extensions. This renders an interpretation of the results impossible, in the sense that finite values cannot be extracted in a physically reasonable way. This led us to consider also the exterior space, in an attempt to get a simpler pole structure by adding up both contributions. In doing so, unambiguous results for Casimir energies in different situations can be obtained. Let us stress that in the present situation general heat-kernel arguments \cite{29,30} can not be applied simply because no general answers are known for cases with singular fluxes and boundaries. For this reason we really need to perform explicit calculations.

The paper is organized as follows. First we briefly present the solutions of the Dirac equation in the presence of a flux string. In the following sections self-adjoint extensions are obtained and some details of how to impose spectral boundary conditions are provided. Having chosen the self-adjoint extensions, the MIT boundary conditions are imposed and implicit eigenvalue equations are obtained. These are the basis for the analysis of the Casimir energy. In order to obtain a well-defined Casimir energy (such that a numerical analysis makes sense) we have evaluated differences between the case when the flux is arbitrary and when the flux is integral. Also, we include the exterior space and the pertinent changes in the calculation are briefly explained. A numerical analysis of Casimir energies as a function of the flux, for the Dirac field, is presented afterwards.
II. DESCRIPTION OF THE PROBLEM

We study the Dirac equation for a massive particle in 3+1 dimensional Minkowski space-time in the presence of a flux string located at the origin,
\[ \hat{H} = \nabla \wedge A = \frac{\kappa}{r} \delta(r) \varepsilon_z. \] (1)

We are going to impose boundary conditions on a cylinder and for that reason use cylindrical coordinates. The gamma matrices in the chiral representation are
\[ \gamma^0 = \rho_3 \otimes \sigma_3, \quad \gamma^1 = i \rho_3 \otimes \sigma_2; \quad \gamma^2 = -i \rho_3 \otimes \sigma_1; \quad \gamma^3 = i \rho_2 \otimes 1_2, \]
and in a comoving coordinate frame one has the usual relation
\[ \gamma^r = \cos \theta \gamma^1 + \sin \theta \gamma^2, \quad \gamma^0 = -\frac{1}{r} (\sin \theta \gamma^1 + \cos \theta \gamma^2). \]
Furthermore, for the flux
\[ A^\theta = \frac{\kappa}{r}, \quad \kappa = l + a, \quad 0 \leq a < 1. \]
Here we have introduced the integer part $l$ of the reduced flux $\kappa$ such that $a$ is restricted to the given interval.

Our main concern is the Casimir energy and for this reason we consider the Dirac Hamiltonian for the system,
\[ H_D \Psi_E = -i \gamma^0 \gamma^r [-\partial_r + \gamma^0 \gamma^\theta (\partial_\theta - i \kappa) + \gamma^r \gamma^3 \partial_z + im \gamma^r] \Psi_E = E \Psi_E. \] (2)
The representation of the Hamiltonian is chosen in such a way as to simplify the implementation of spectral boundary conditions at a later stage. Given the cylindrical symmetry of the configuration the eigenfunctions have the form
\[ \Psi_E(r, \theta, z) = e^{-ik_z z} \psi_e(r, \theta), \quad k_z \in \mathbb{R}. \] (3)
For $\psi_e(r, \theta)$ there are four types of solutions. They are
\[ \psi_e^{(n,1)} = \begin{pmatrix} J_\omega (kr) e^{in\theta} \\ -C_- J_{\omega-1} (kr) e^{(n+1)\theta} \\ \Gamma J_\omega (kr) e^{in\theta} \\ \Gamma C_- J_{\omega-1} (kr) e^{(n+1)\theta} \end{pmatrix}, \quad \psi_e^{(n,2)} = \begin{pmatrix} J_{-\omega} (kr) e^{in\theta} \\ C_- J_{\omega-1} (kr) e^{(n+1)\theta} \\ \Gamma J_{-\omega} (kr) e^{in\theta} \\ -\Gamma C_- J_{\omega-1} (kr) e^{(n+1)\theta} \end{pmatrix}, \]
\[ \psi_e^{(n,3)} = \begin{pmatrix} \Gamma J_\omega (kr) e^{in\theta} \\ \Gamma C_+ J_{\omega+1} (kr) e^{(n+1)\theta} \\ J_\omega (kr) e^{in\theta} \\ -C_+ J_{\omega+1} (kr) e^{(n+1)\theta} \end{pmatrix}, \quad \psi_e^{(n,4)} = \begin{pmatrix} \Gamma J_{-\omega} (kr) e^{in\theta} \\ -\Gamma C_+ J_{\omega+1} (kr) e^{(n+1)\theta} \\ J_{-\omega} (kr) e^{in\theta} \\ C_+ J_{-\omega+1} (kr) e^{(n+1)\theta} \end{pmatrix}, \]
where $n = -\infty, \ldots, \infty$, and we have used the notation
\[ k = \sqrt{e^2 - m^2}, \quad E = \text{sign}(e) \sqrt{k_z^2 + e^2}, \]
\[ C_\pm = \frac{ik}{e \pm m}, \quad \Gamma = -\frac{1}{k_z} (e - E), \quad \omega = n - l - a. \]
Here $e^2 = k^2 + m^2$ are the eigenvalues of the square of the two-dimensional Dirac operator obtained after setting $k_z = 0$.

In order to fix the eigenvalues $E$ we have to choose a self-adjoint extension of the radial Hamiltonian and we have to impose boundary conditions. A family of self-adjoint extensions arises and we will consider two particular cases. The first one is obtained by imposing spectral boundary conditions at some interior cylinder the radius of which is sent to zero. Spectral boundary conditions are imposed as described in \textsuperscript{23} \textsuperscript{25}, see also \textsuperscript{31}. In the present case a suitable choice for the boundary operator $A$ is \textsuperscript{31} \textsuperscript{22}. 

\[ A = \gamma^r \gamma^\theta (\partial_\theta - i \kappa) + \gamma^r \gamma^3 \partial_z + \frac{1}{2r} I_4. \]  

(6)

The term \((1/2r)I_4 = (1/2r)K_1 I_4\), with \(K\) the extrinsic curvature of the boundary, has been added to guarantee that the operator \(A\) leads to a self-adjoint boundary value problem, see [21] for details.

Spectral boundary conditions amount to setting to zero the projection of \(\Psi_E\) onto all eigenfunctions of \(A\) with negative eigenvalues at the boundary. Again, due to the symmetry, the eigenfunctions have the form

\[ A = e^{-ik_z z} \alpha, \]

(7)

where again four different types of solutions exist,

\[ \alpha_1 = \begin{pmatrix} e^{in\theta} \\ 0 \\ 0 \\ d_+ e^{i(n+1)\theta} \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 \\ d_+ e^{i(n+1)\theta} \\ 0 \\ e^{in\theta} \end{pmatrix}, \]

\[ \alpha_2 = \begin{pmatrix} 0 \\ e^{in\theta} \\ d_- e^{i(n-1)\theta} \\ 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} d_- e^{i(n-1)\theta} \\ 0 \\ 0 \\ e^{in\theta} \end{pmatrix}. \]

Here we have introduced

\[ d_\pm = \frac{i}{k_z r_0} \left[ a_\pm - \left( \omega \pm \frac{1}{2} \right) \right], \quad a_\pm = \text{sgn} \left( \omega \pm \frac{1}{2} \right) \sqrt{(k_z r_0)^2 + \left( \omega \pm \frac{1}{2} \right)^2} \]

and \(\alpha_1, \alpha_3\) can be seen to have eigenvalues \(a_{1,3} = (1/r_0)a_+\) and \(\alpha_2, \alpha_4\) have instead \(a_{1,4} = -(1/r_0)a_-\). The eigenvectors and eigenvalues of \(A\) in \(2 + 1\)-dimensions are obtained by taking \(k_z \to 0\).

Having these eigenfunctions \(\alpha\) at our disposal, projections with \(\Psi_E\) can easily be performed. Due care has to be taken in order to implement the boundary conditions correctly for all values of \(n\). In the limit \(r_0 \to 0\), the vanishing of the projection rules out some of the \(\psi^{(n,1)}\) in Eqs. (4) and (5) because they diverge at the origin. In detail, one finds that for \(n \geq l + 1\) the function \(\psi^{(n,1)}\) and \(\psi^{(n,3)}\) form a suitable basis, whereas for \(n < l\) the relevant ones are \(\psi^{(n,2)}\) and \(\psi^{(n,4)}\). The remaining one, \(n = l\), allows the identification as a member of the one parameter family of self-adjoint extensions [22]. Writing the spinors in the form

\[ \psi^{n}(r, \theta) = \begin{pmatrix} f^+_{n}(r)e^{in\theta} \\ g^+_{n}(r)e^{i(n+1)\theta} \\ f^-_{n}(r)e^{in\theta} \\ g^-_{n}(r)e^{i(n+1)\theta} \end{pmatrix}, \]

(8)

self-adjoint extensions are classified according to the conditions

\[ i \lim_{r_0 \to 0} (mr_0)^{1-a} g^{\pm}_{1}(r_0) \sin \left( \frac{\pi}{4} + \frac{\Theta^\pm}{2} \right) = \lim_{r_0 \to 0} (mr_0)^{a} f^{\pm}_{1}(r_0) \cos \left( \frac{\pi}{4} + \frac{\Theta^\pm}{2} \right). \]

(9)

In our case, for \(0 < a < 1/2\),

\[ f^+_1(r_0) = \left[ A^+_1 + \Gamma A^+_1 \right] J_{-a}(kr_0), \]
\[ g^+_1(r_0) = \left[ -A^+_1 C_- + \Gamma A^+_1 C_+ \right] J_{-a+1}(kr_0), \]
\[ f^-_1(r_0) = \left[ A^+_1 \Gamma + A^+_1 \right] J_{-a}(kr_0), \]
\[ g^-_1(r_0) = \left[ A^+_1 \Gamma C_- - A^+_1 C_+ \right] J_{-a+1}(kr_0), \]

and, for \(1/2 < a < 1\),

\[ f^+_1(r_0) = \left[ B^+_1 + \Gamma B^+_1 \right] J_{a}(kr_0), \]
\[ g^+_1(r_0) = \left[ B^+_1 C_- - \Gamma B^+_1 C_+ \right] J_{a-1}(kr_0), \]
\[ f^-_1(r_0) = \left[ B^+_1 \Gamma + B^+_1 \right] J_{a}(kr_0), \]
\[ g^-_1(r_0) = \left[ -B^+_1 \Gamma C_- + B^+_1 C_+ \right] J_{a-1}(kr_0). \]
Considering the above condition \( \Theta \) this imposes
\[
\Theta^\pm = \begin{cases} 
+\frac{\pi}{2} & 0 < a < \frac{1}{2} \\
-\frac{\pi}{2} & \frac{1}{2} < a < 1
\end{cases}
\]  
(10)

We refer to this choice as I. Our second choice, called II, corresponds to
\[
\Theta^\pm = \begin{cases} 
-\frac{\pi}{2} & \kappa > 0 \\
\frac{\pi}{2} & \kappa < 0
\end{cases}
\]  
(11)

This extension has been considered in [27] and it arises when a finite radial flux, which is ultimately shrunk to zero, is considered.

We are now in a position to impose MIT boundary conditions at the exterior boundary \( r = R \). These boundary conditions ensure that the fermion current across the boundary vanishes. The relevant boundary operator is
\[
B = 1 - i \slashed{n} = \begin{pmatrix} B_+ & 0 \\ 0 & B_-
\end{pmatrix}
\]  
(12)

with
\[
B_\pm = \begin{pmatrix} 1 & \pm i e^{-i\theta} \\ \mp i e^{i\theta} & 1
\end{pmatrix}
\]  
(13)

and the boundary conditions reads
\[
B \Psi_E(R, \theta, z) = 0.
\]

Given the previous discussion for type I, see Eqs. (9) and (10), this boundary condition has to be applied on the following set of spinors,
\[
n \geq l + 1: \quad \psi_e^{(n,1)}, \psi_e^{(n,3)} \\
n \leq l - 1: \quad \psi_e^{(n,2)}, \psi_e^{(n,4)} \\
n = l: \begin{cases} 
0 < a < 1/2 & \psi_e^{(n,1)}, \psi_e^{(n,3)} \\
1/2 < a < 1 & \psi_e^{(n,2)}, \psi_e^{(n,4)}
\end{cases}
\]  
(14)

and similarly for type II. As a result, the eigenvalues take the form
\[
E = \frac{1}{R} \sqrt{(k_z R)^2 + z^2 + x^2}
\]  
(15)

where \( z = mR \) and \( x \) are the solutions of the equations obtained from
\[
J_\mu^2(x) - J_{\mu-1}^2(x) - \frac{2x}{x} J_\mu(x) J_{\mu-1}(x) = 0,
\]  
(16)

with \( \mu = \nu \pm \alpha, \nu = n + 1/2 = 3/2, 5/2, \ldots, \infty \) and \( \alpha = a - 1/2 \). The critical subspace \( \nu = 1/2 \) belongs to the case \( \mu = \nu - \alpha \) for \( -1/2 < \alpha < 0 \) and to the case \( \mu = \nu + \alpha \) when \( 0 < \alpha < 1/2 \). All eigenvalues have degeneracy two coming from particle and antiparticle states.

III. THE CASIMIR ENERGY FOR THE INTERIOR

Now everything is prepared for the calculation of the Casimir energy in the zeta function regularization scheme. Our presentation shows only the Type I case. We comment on the the minor changes needed for Type II where appropriate. Given the translational invariance in the \( z \)-direction, we define the Casimir energy density by
\[
E_C^{(int)} = -\frac{M}{2} M^{2s} \zeta_{\text{int}}(s) \big|_{s=-1/2},
\]  
(17)

where the zeta function has the structure
\( \zeta_{\text{int}}(s) = 2R^{2s-1} \sum_{x} \int_{-\infty}^{\infty} \frac{dy}{2\pi} (y^2 + z^2 + x^2)^{-s} \)
\[= R^{2s-1} \frac{\Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s)} \sum_{x} (z^2 + x^2)^{-(s-1/2)}. \] (18)

Expressed as a sum over the zeros of Eq. (16), we can write
\[ \zeta_{\text{int}}(s) = \sum_{\mu} \zeta_{\mu}^{(\text{int})}(s) \] (19)
with the partial zeta functions
\[ \zeta_{\mu}^{(\text{int})}(s) = R^{2s-1} \frac{\Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s)} \sum_{m=1}^{\infty} (z^2 + x_{\mu,m}^2)^{-(s-1/2)}. \] (20)

Slightly more explicit is the form
\[ \zeta_{\text{int}}(s) = -\zeta_{1/2-|\alpha|}^{(\text{int})}(s) + \sum_{\nu=1/2}^{\infty} \left[ \zeta_{\nu+\alpha}^{(\text{int})}(s) + \zeta_{\nu-\alpha}^{(\text{int})}(s) \right] \] (21)
showing the symmetry under the transformation \( \alpha \to -\alpha \). So it is sufficient to consider \( 0 < \alpha < 1/2 \). Eq. (21) shows furthermore that the Casimir energy is independent of the integer part of the flux.

The analysis of the Casimir energy proceeds using the methods described in detail in [10, 33, 34, 16] and we follow their procedure. The starting point is the contour integral representation
\[ \zeta_{\mu}^{(\text{int})}(s) = R^{2s-1} \frac{\Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s)} \int_{\Gamma} \frac{dx}{2\pi i} (x^2 + z^2)^{-(s-1/2)} \frac{d}{dx} \log \bar{F}(\mu, x) \] (22)
where \( \bar{F}(\mu, x) \) describes the eigenvalue equation,
\[ \bar{F}(\mu, x) = J_{\mu-1}^2(x) - J_{\mu}^2(x) + \frac{2z}{x} J_{\mu}(x) J_{\mu-1}(x), \] (23)
and the integration path \( \Gamma \) encloses all the zeros of \( \bar{F}(\mu, x) \).

Shifting the contour to the imaginary axis, one obtains the following representation, valid for \( 1/2 < \Re(s-1/2) < 1 \),
\[ \zeta_{\mu}^{(\text{int})}(s) = R^{2s-1} \frac{\Gamma(s - 1/2) \sin \pi(s - 1/2)}{\sqrt{\pi} \Gamma(s)} \int_{z}^{\infty} du (u^2 - z^2)^{-(s-1/2)} T_{\text{int}}(z; \mu, u), \] (24)
where \( T_{\text{int}}(z; \mu, u) \) contains the eigenvalue equation transformed to the imaginary axis,
\[ T_{\text{int}}(z; \mu, u) = \frac{d}{du} \log \left( u^{-2(\mu-1)} F(\mu, u) \right) \] (25)
with
\[ F(\mu, u) = I_{\mu}^2(u) + I_{\mu-1}^2(u) + \frac{2z}{u} I_{\mu}(u) I_{\mu-1}(u). \] (26)

Adding and subtracting the uniform asymptotic Debye expansion of \( F(\mu, u) \), see [10, 16] for details, one arrives at
\[ \zeta_{\text{int}}(s) = -\zeta_{1/2-|\alpha|}^{(\text{int})}(s) + Z_{\text{int}}(s) + \sum_{i=-1}^{N} A_{i}^{(\text{int})}(s), \] (27)
where \( Z_{\text{int}}(s) \) is the zeta function with the asymptotic terms subtracted. With \( t = \nu / \sqrt{\nu^2 + z^2} \), it reads
\[ Z_{\text{int}}(s) = \frac{1}{R^{2s-1} \sqrt{\pi} \Gamma(s) \Gamma(3/2 - s)} \sum_{\nu=1/2, 3/2, \ldots}^{\infty} \int_{z}^{\infty} dx \left( x^2 - z^2 \right)^{1/2-s} \left\{ T_{\text{int}}(z; \nu + \alpha, x) + T_{\text{int}}(z; \nu - \alpha, x) - \Delta_{\text{int}}^{(N)}(\nu, x, t, z) \right\}. \] (29)
The last equation may be seen as the definition of the coefficients $b^{(\text{int})}_{i,j}$ where

$$\Delta^{(\text{int})}_{-1} = \frac{4x}{\nu \nu + t}, \quad \Delta^{(\text{int})}_0 = \frac{2x}{\nu \nu + 1 + t}, \quad \Delta^{(\text{int})}_i = \nu^{-i} \frac{d}{dx} \sum_{j=0}^{2i} b^{(\text{int})}_{i,j} \nu^{i+j}. \quad (31)$$

The last equation may be seen as the definition of the coefficients $b^{(\text{int})}_{i,j}$ and they are easily determined by a simple computer program. Most of the $b^{(\text{int})}_{i,j}$ coefficients needed are listed in Appendix A of [16].

In order to ensure $Z_{\text{int}}(s)$ is finite one has to subtract terms up to $N = 3$ at least. To improve the numerical convergence we have chosen $N = 5$ in the analysis presented later on.

The asymptotic terms have to be analysed further in order to extract the poles of the theory. They are conveniently expressed in terms of the functions $[36, 16]$

$$f(s; a, b, x) = \sum_{\nu=1/2}^{\infty} \nu^a \left[ 1 + \left( \frac{\nu}{\pi} \right)^2 \right]^{-s-b},$$

which naturally arise after performing the $x$-integrals. In detail the full list of useful results is

$$A^{(\text{int})}_{-1}(s) = \frac{R^{2s-1}}{\pi} \frac{1}{s-1} z^{2(1-s)} \int_0^1 \frac{dy}{y \nu} f(s; 0, -1; z \sqrt{y}), \quad (32)$$

$$A^{(\text{int})}_0(s) = \frac{R^{2s-1}}{\pi} z^{-2s} \int_0^1 \frac{dy}{y^{3/2}} f(s; 1, 0; z \sqrt{y}), \quad (33)$$

$$A^{(\text{int})}_i(s) = \sum_{j=0}^{2i} b^{(\text{int})}_{i,j} A_{i+j}(s), \quad (34)$$

where

$$A^{(\text{int})}_{i,j}(s) = -R^{2s-1} \frac{\Gamma(s + \frac{i+j-1}{2})}{\sqrt{\pi} \Gamma(s) \Gamma(\frac{i}{2})} z^{-2s-(i+j-1)} f\left(s; j, \frac{i+j-1}{2}, z\right). \quad (35)$$

These representations are very suitable for a numerical analysis of the Casimir energy as a function of the mass and the flux.

For small masses, including vanishing mass, as well as for reading off the poles, the expansions in powers of $z$ are more suitable. Again, the techniques to obtain these have been described in [10]. In the present context we find

$$A^{(\text{int})}_{-1}(s) = \frac{R^{2s-1}}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + s - 1)}{\Gamma(n + 1) \Gamma(s)} \frac{1}{n + s - 1/2} z^{2n} \zeta_H \left(2n + 2s - 2, \frac{1}{2}\right), \quad (36)$$

$$A^{(\text{int})}_0(s) = \frac{R^{2s-1}}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n + s)}{\Gamma(n + 1) \Gamma(s)} \frac{1}{n + s - 1/2} z^{2n} \zeta_H \left(2n + 2s - 1, \frac{1}{2}\right), \quad (37)$$

$$A^{(\text{int})}_{i,j}(s) = \frac{R^{2s-1}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma(n + s + \frac{1}{2}(i+j-1))}{\Gamma(n + 1) \Gamma(s) \Gamma(\frac{1}{2}(i+j))} z^{2n} \zeta_H \left(2n + 2s + i - 1, \frac{1}{2}\right), \quad (38)$$

which makes the pole structure very explicit.

Before studying this, let us indicate the treatment of the ‘extra’ contribution in Eq. [21], $-\epsilon^{(\text{int})}_{1/2 - |\alpha|}(s)$. Given no $\nu$-sum is involved, it is actually sufficient to just subtract the large argument expansion of $T_{\text{int}}(z; 1/2 - |\alpha|, x)$. The leading terms in this expansion are

$$T_{\text{int}}(z; 1/2 - |\alpha|, x) \sim 2 + \frac{2|\alpha|}{x} + \frac{\alpha^2 - z}{x^2} + \frac{z^2}{x^3} + \mathcal{O}(x^{-4}), \quad (39)$$
and a suitable representation to find the analytical continuation in the critical subspace is

$$\zeta_{1/2-|\alpha|}(s) = R^{2s-1} \frac{1}{\sqrt{\pi} \Gamma(s) \Gamma(3/2-s)} \times$$

$$\left\{ \int_0^1 dx (x^2 - z^2)^{1/2-s} T_{int}(z; 1/2 - |\alpha|, x) + \int_0^\infty dx (x^2 - z^2)^{1/2-s} \left[ T_{int}(z; 1/2 - |\alpha|, x) - 2 - \frac{2|\alpha|}{x} - \frac{\alpha^2 - z}{x^2} - \frac{z^2}{x^3} \right] + \int_0^\infty dx (x^2 - z^2)^{1/2-s} \left[ 2 + \frac{2|\alpha|}{x} + \frac{\alpha^2 - z}{x^2} + \frac{z^2}{x^3} \right] \right\}. \tag{40}$$

While the first two terms are suitable for a numerical evaluation at $s = -1/2$, the asymptotic terms are standard representations of hypergeometric functions \cite{28}.

We are now fully prepared to analyse the pole structure and, if a physically senseful interpretation of finite parts is possible, also for a numerical analysis of the Casimir energy. As mentioned, the poles are best read off in the representations (36)--(38) and (40). The residues of the single asymptotic terms are

$$\text{Res}_{s \to \frac{1}{2}} A_{-1}^{\text{int}}(s) = \frac{1}{R^2} \left[ -\frac{z^4}{8\pi} - \frac{z^2}{24\pi} \right], \tag{41}$$

$$\text{Res}_{s \to \frac{1}{2}} A_0^{\text{int}}(s) = 0, \tag{42}$$

$$\text{Res}_{s \to \frac{1}{2}} A_1^{\text{int}}(s) = \frac{1}{R^2} \left[ -\frac{z^3}{2\pi} + \frac{z^2}{12\pi} + \frac{z^2\alpha^2}{2\pi} \right], \tag{43}$$

$$\text{Res}_{s \to \frac{1}{2}} A_2^{\text{int}}(s) = 0, \tag{44}$$

$$\text{Res}_{s \to \frac{1}{2}} A_3^{\text{int}}(s) = \frac{1}{R^2} \left[ \frac{z^3}{3\pi} - \frac{z^2}{12\pi} - \frac{z}{30\pi} + \frac{1}{126\pi} \right]. \tag{45}$$

the extra contribution adds

$$\text{Res}_{s \to \frac{1}{2}} - \zeta^{\text{int}}_{1/2-|\alpha|}(s) = -\frac{z^2}{2\pi R^2} \left( |\alpha| - \frac{1}{2} \right).$$

Summing up, the total residue reads

$$\text{Res}_{s \to \frac{1}{2}} \zeta_{\text{int}}(s) = \frac{1}{R^2} \left[ -\frac{1}{8\pi} \frac{z^4}{\pi} - \frac{z^3}{6\pi} + \left( \frac{5}{24\pi} + \frac{1}{2\pi} (\alpha^2 - |\alpha|) \right) \frac{z^2}{30\pi} + \frac{1}{126\pi} \right].$$

So the pole in the Casimir energy depends on the mass, as to be expected, as well as on the flux via $\alpha$. This last dependence on $\alpha$ did not occur in $2 + 1$ dimensions, so that the difference of two situations with different fluxes gave a finite answer \cite{16}. Here this happens only in the case $z = 0$ so that we should restrict to this case in order to have unambiguous answers. In the case $z \neq 0$ a flux-dependent counterterm $FR$ has to be introduced and finite ambiguities remain. The numerical results for the Casimir energy difference between the cases with and without flux for $z = 0$ are presented in Fig. \ref{fig:CasimirEnergy} in a black line. Other results, affected by the regularization procedure, are presented as the grey lines, corresponding, from top to bottom, to the values of $z = 1/64, 1/32, 1/16, 1/8, 3/16$. For this numerical analysis a minimal subtraction was considered.
FIG. 1. (Differences of) Interior Casimir Energies - Type I

Our analysis can be straightforwardly extended to the Type II case. It is not difficult to show that just the replacement of $|\alpha|$ by $\alpha$ in the extra term of Eq. (21), $-\zeta_{1/2-\alpha}(s)$, or in the second order of the asymptotic expansion, leads to the wanted results. A similar replacement rule gives the total residue and all the considerations above apply directly. The corresponding numerical results are shown in Fig. 2

FIG. 2. (Differences of) Interior Casimir Energies - Type II

IV. THE CASIMIR ENERGY OF THE EXTERNAL SECTOR

Let us now consider the external sector in order to see if the pole structure considerably simplifies when combining the internal and external spaces. Partly this is to be expected for simple geometric reasons, because the extrinsic curvature has the opposite sign.

The external sector is analysed most effectively using the formulation in terms of a Jost function, see e.g. [9]. As a first step, a large ‘external’ sphere is introduced as an additional boundary to compactify the space. Boundary
conditions are imposed, e.g. MIT ones, and the eigenfunctions are combinations of the ones given in (4) and (5) with the $J_\omega$ and $J_{-\omega}$ replaced by $H^{(1)}_{\omega}$ and $H^{(2)}_{\omega}$. Having the eigenfunctions at hand, it is not difficult to find the Jost function for the associated scattering problem. As observed already in various examples \[36\], the external sector is obtained when $I_\nu$ is replaced by $K_\nu$. This, essentially, is true also in the present situation apart from the fact that the external space does not have a critical subspace because the singular flux is not sensed directly and no self-adjoint extension is involved. The last comment, as it turned out, is just equivalent to the use of $\alpha$ instead of $|\alpha|$ where applicable.

As a result of the above comments, the zeta function of the external part reads

$$\zeta_{\text{ext}} = -\zeta_{1/2-\alpha}(s) + \sum_{\nu=1}^{\infty} \left[ \zeta^{\text{(ext)}}_{\nu+\alpha}(s) + \zeta^{\text{(ext)}}_{\nu-\alpha}(s) \right],$$

where

$$\zeta^{\text{(ext)}}_{\nu}(s) = R^{2s-1} \frac{1}{\sqrt{\pi} \Gamma(s) \Gamma(3/2-s)} \int_{z}^{\infty} dx \left( x^2 - z^2 \right)^{-(s-1/2)} T_{\text{ext}}(z; \mu, u),$$

and

$$T_{\text{ext}}(z; \mu, u) = \frac{d}{dx} \log \left( u^{2\mu} G(\mu, u) \right),$$

and

$$G(\mu, u) = K^{2}_{\mu}(u) + K^{2}_{\mu-1}(u) + \frac{2x}{u} K_{\mu}(u) K_{\mu-1}(u).$$

In clear analogy the equations corresponding to (27)—(31) can be written down, where one has the relations

$$\Delta^{(\text{ext})}_{-1} = -\Delta^{(\text{int})}_{-1}, \quad \Delta^{(\text{ext})}_{0} = -\Delta^{(\text{int})}_{0},$$

and the multipliers $b^{(\text{ext})}_{(i,j)}$ are obtained in a similar way as the internal multipliers $b^{(\text{int})}_{(i,j)}$. Also the ‘extra’ contribution $-\zeta_{1/2-\alpha}$ is dealt with as before, see eq. (39) and (40), but now we have

$$T_{\text{ext}}(z; 1/2 - \alpha, x) \sim -2 - \frac{2\alpha}{x} - \frac{\alpha^2 + z}{x^2} + \frac{z^2}{x^3} + O(x^{-4}).$$

The residues in the exterior space are thus obtained from the interior ones by the simple replacement rules already mentioned, i.e. $z \rightarrow -z$ and $|\alpha| \rightarrow \alpha$. As a result, the Casimir energy difference between the case with and without flux for massless fields is finite and is shown as the black line in Fig. 3. The grey lines show the regularization dependent results for, from bottom to top, $z = 1/64, 1/32, 1/16, 1/8, 3/16$. 

![Fig. 3. (Differences of) Exterior Casimir Energies](image)
V. THE CASIMIR ENERGY IN THE WHOLE SPACE - FINAL COMMENTS

Adding up the interior and the external sector, we find the pole

\[ \text{Res}_{s \rightarrow -\frac{1}{2}} \zeta_{\text{int}}(s) + \text{Res}_{s \rightarrow -\frac{1}{2}} \zeta_{\text{ext}}(s) = \frac{1}{R^2} \left[ -\frac{z^3}{3\pi} - \frac{1}{15} \frac{z}{\pi} - \frac{1}{2} \frac{z^2}{\pi} (-\alpha + |\alpha|) \right] \] (50)

for behaviour I, still showing a flux dependent term. Numerical results for the difference between arbitrary and integer flux (only unambiguous for the black \( z = 0 \) curve) are shown in Fig. 4.

\[ \begin{array}{c}
0.003 \\
-0.003 \\
0.002 \\
-0.002 \\
0.001 \\
-0.001 \\
0 \\
-0.001 \\
0.001 \\
-0.002 \\
0.002 \\
-0.003
\end{array} \]

\[ \begin{array}{c}
0 \\
0.2 \\
0.4 \\
0.6 \\
0.8 \\
1
\end{array} \]

\( a \)

\( E_C \)

**FIG. 4.** (Differences of) Complete Casimir Energies - Type I

On the other hand,

\[ \text{Res}_{s \rightarrow -\frac{1}{2}} \zeta_{\text{int}}(s) + \text{Res}_{s \rightarrow -\frac{1}{2}} \zeta_{\text{ext}}(s) = \frac{1}{R^2} \left[ -\frac{z^3}{3\pi} - \frac{1}{15} \frac{z}{\pi} \right] \] (51)

gives the residue for behaviour II. We see that the Casimir energy difference between two flux-situations is finite and we can fully analyse its dependence on the mass. Numerical results are displayed in Fig. 5 for the values of \( z \) already given, from top to bottom. Whereas for non-vanishing mass the field free situation leads to the lowest Casimir energy, for non-zero mass a non-trivial global minimum of the Casimir energy emerges. Above a certain critical flux, quantum contributions support the stability of the flux, at least for type II.
In summary, we have studied in detail the behaviour of the Casimir energy of Dirac fields in cylindrically symmetric situations.

In contrast to the 2 + 1-case, when studying the interior case, flux-dependent residues do not vanish when just considering the difference between two values of the flux (except for the massless case). In our hope of obtaining finite results when considering the whole space, the external space was analysed too. The geometrically grounded expectation of a simpler pole structure for the whole space eventually proved to be right, at least for the Type II choice of the self-adjoint extension. Completely well-defined results for the massive case were obtained numerically.

The presence of the mass and the choice of the self-adjoint extension for the radial Hamiltonian at the origin are crucial in our results which emphasizes that, for curved boundaries, the contribution of the mass is not exponentially small [13] as it is for parallel plates [37]. Concerning the self-adjoint extensions, the remarkable differences in the resulting Casimir energies reinforce the well-known fact that the one-parameter family of self-adjoint extensions describes nontrivial physics in the core, see also [22]. Furthermore, a better understanding of the physical meaning of the different self-adjoint extensions, by considering how the Casimir energy depends on the parameter $\Theta$ of the one-parameter family of self-adjoint extensions, should be envisaged in the future.

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