Entanglement analysis of a two-atom nonlinear Jaynes–Cummings model with nondegenerate two-photon transition, Kerr nonlinearity, and two-mode Stark shift

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Abstract
An entangled state, as an essential tool in quantum information processing, may be generated through the interaction between light and matter in cavity quantum electrodynamics. In this paper, we study the interaction between two two-level atoms and a two-mode field in an optical cavity enclosed by a medium with Kerr nonlinearity in the presence of a detuning parameter and Stark effect. It is assumed that the atom–field coupling and third-order susceptibility of the Kerr medium depend on the intensity of the light. In order to investigate the dynamics of the introduced system, we obtain the exact analytical form of the state vector of the considered atom–field system under initial conditions which may be prepared for the atoms (in a coherent superposition of their ground and upper states) and the fields (in a standard coherent state). Then, in order to evaluate the degree of entanglement between the subsystems, we investigate the dynamics of the entanglement by employing the entanglement of formation. Finally, we analyze in detail the influences of the Stark shift, the deformed Kerr medium, the intensity-dependent coupling, and also the detuning parameter on the behavior of this measure for different subsystems. The numerical results show that the amount of entanglement between the different subsystems can be controlled by choosing the evolved parameters appropriately.

Keywords: entanglement of formation, two-atom Jaynes–Cummings model, intensity-dependent coupling, two-mode Stark shift

(Some figures may appear in colour only in the online journal)
reported that atom–field entangled states have been experimentally generated via a single atom interacting with a mesoscopic field in a high-Q microwave cavity [11].

The full quantum mechanical approach to the study of a two-level atom interacting with a single-mode quantized field in the electric-dipole and rotating wave approximations was first introduced via the well-known Jaynes–Cummings model (JCM) [12, 13]. Many generalizations of the JCM have been proposed in various ways, for instance, considering different initial conditions [14], entering the effects of dissipation and damping into the model [15], adopting a multi-level atom [16–18], as well as a multi–photon transition [19] and a multi-atom [20, 21]. The dependence of atom–field coupling on the intensity of the light is also considered as another generalization of the JCM which was suggested by Buck and Sukumar [22, 23] and then was used by others [24–26]. In detail, the quantum properties of a $\Lambda$-type three-level atom interacting with a single-mode field in a Kerr medium with intensity-dependent coupling and in the presence of the detuning parameters have been studied by us [27]. Also, the authors discussed the nonlinear interaction between a three-level atom (in a $\Lambda$ configuration) and a two-mode cavity field in the presence of a cross-Kerr medium and its deformed counterpart [28], an intensity-dependent atom–field coupling, and the detuning parameters [29, 30].

From another perspective of this field of research, multi-photon transitions in the JCM may be taken into account. The multi-photon process in atomic systems has attracted a great deal of attention, since this phenomenon results in a high degree of correlation between the emitted photons, which may lead to the nonclassical behavior of the emitted light [31]. The concept of a multi-photon transition can be further clarified after considering the Stark shift. The importance of this effect will be apparent when two atomic levels are coupled with comparable strength to the intermediate level [32]. In this case, by applying the method of the adiabatic elimination of the intermediate level(s) of a multi-level atom, the Stark shift phenomenon is revealed [33]. Indeed, by using this method, one arrives at a simple effective Hamiltonian, in which a two-level atomic system interacts with a single-mode quantized field with a multi-photon (at least two-photon) transition in the presence of the Stark shift [34]. It is instructive to state that the Stark shift in two-photon transitions can be regarded as an intensity-dependent detuning [35].

In particular, and in direct relation to the present work, the state vector of the system containing the interaction between two identical two-level atoms and a two-mode quantized radiation field via a nondegenerate two-photon transition has been explicitly found in [36]. In this attempt, a few nonclassicality features such as photon statistics and squeezing have been numerically discussed. An exact solution for two two-level atoms interacting with a two-mode radiation field containing nondegenerate two-photon and Raman transitions has been proposed [37] in which the amount of atom–field entanglement via the reduced atomic entropy has been examined. The interaction between two two-level atoms and a single-mode field with a degenerate two-photon transition in the presence of the Stark shift has been studied in [38], in which the authors showed that the degree of entanglement (DEM) may be improved after increasing the value of the Stark shift parameter. Recently, the dynamic behavior of two two-level atoms interacting with a single-mode binomial field was studied by one of the authors [39], in which the role of the parameter related to the dimension of the binomial state on various dynamical properties such as the atomic population inversion, sub-Poissonian statistics, and also entropy squeezing for Pauli operators were evaluated.

In this paper, we intend to outline the nonlinear interaction between two two-level atoms and a two-mode quantized cavity field within an optical cavity surrounded by a centrosymmetric medium with Kerr nonlinearity (containing its $f$-deformed counterpart) in the presence of a detuning parameter and the Stark effect. Our main goal is to investigate the effects of these parameters, namely intensity-dependent coupling, the Kerr (and also deformed Kerr) medium (containing self- and cross-action), the Stark shift and a detuning parameter on the temporal behavior of the entanglement of formation (EOF) measure. As will be further demonstrated in this paper, this is an appropriate measure which helps us to evaluate the DEM between different bipartites of the subsystems evolved in our considered model.

To make our motivations more clear, a few words on the notability of the introduced model should be given. In this respect, it has been shown that, from the standpoint of the quantum information theory, atomic systems can be regarded as inherent computational hardware necessary for the future implementation of quantum information protocols [40, 41]. In addition, the photons of the field can be considered as a fundamental building blocks of quantum communication [42] and cryptography [43]. Accordingly, it may be expressed that generating and manipulating the nonclassical correlations arising from the interaction between atomic systems and quantized radiation fields puts this kind of interaction at the cutting edge of the field of quantum information. In the particular case related to atomic systems where two atoms participate in the interaction, it is worth mentioning that, since a two-level atom can represent a qubit, an atomic system consisting of two (identical) two-level atoms (two qubits) can be applied as quantum gates in protocols in quantum information processing. In this relation, it is remarkable to state that two-atom entangled states have experimentally been realized via ultracold trapped ions [44] and cavity QED schemes [45].

The remainder of paper is organized as follows: in the next section, the analytical form of the state vector of the whole system is obtained. Section 3 deals with two different regimes of entanglement (atom–field and atom–atom entanglement) by evaluating the EOF measure. Finally, section 4 contains a summary and concluding remarks.

2. Introducing the model Hamiltonian and Its solution

This section is devoted to finding the explicit form of the state vector of the system, since based on the fundamentals of quantum mechanics, possible information when studying
any physical (quantum) system is hidden in its wave function. For this purpose, it is necessary to take all the interactions between subsystems into account by using the fully quantum mechanical approach. Then, with the help of the Schrödinger equation or other appropriate (equivalent) methods, the state vector of the whole system may be found. So, let us consider a model in which a two-mode quantized radiation field oscillating with frequencies \( \Omega_1 \) and \( \Omega_2 \) interacts with two two-level atoms (atom \( A \) and atom \( B \)) with ground states \( |g_A \rangle \) and \( |g_B \rangle \) and excited states \( |e_A \rangle \), \( |e_B \rangle \), in an optical cavity surrounded by a centrosymmetric medium with Kerr nonlinearity in the presence of the Stark shift and a detuning parameter. Accordingly, the Kerr susceptibility of the medium is accompanied by the absorption/emission of two laser photons, its ground (excited) state to an excited (ground) state by the simultaneous absorption (emission) of two laser photons (the physical motivation of this typical change was established in [28]). In order to arrive at a universal formalism for the considered atom–field system, the intensity-dependent functions are applied in the general form \( f_j (\hat{n}_j) \) and \( g_j (\hat{n}_j) \). Accordingly, it is worth mentioning that selecting various nonlinearity functions leads to different Hamiltonians, and as a result different state vectors may be obtained.

Looking deeply at the relation (1) implies the fact that the introduced Hamiltonian signifies the intensity-dependent two-atom two-mode two-photon JCM in the presence of \( f \)-deformed Kerr nonlinearity and the Stark shift. It is valuable to express that the third part in the Hamiltonian (1) indicates the effect of the two-mode Stark shift [46], which may be interpreted as the intensity-dependent energy shifts of the atomic levels [35]. It is worth mentioning that the process of the two-mode, two-photon transition can be physically demonstrated via a non-degenerate two-photon process, in which the atom makes a transition from its ground (excited) state to an excited (ground) state by the simultaneous absorption (emission) of two laser photons [47]. Indeed, in the case of two-photon absorption, for instance, the atom first absorbs a photon of frequency \( \Omega_1 \) and jumps from a real level to the higher virtual one, and then by absorbing a photon of frequency \( \Omega_2 \), it jumps to the nearest real level.

For the next purpose it is suitable to rewrite the Hamiltonian (1) in the interaction picture, from which one arrives at:

\[
\hat{H}_I = \sum_{i=A,B} \omega_i \hat{a}_i^\dagger \hat{a}_i + \sum_{j=1}^2 \Omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_{i=A,B} \lambda_i \left( \hat{A}_i \hat{A}_2^\dagger \sigma_{11}^{(\uparrow)} + \hat{A}_1^\dagger \hat{A}_2 \sigma_{22}^{(\uparrow)} \right) \\
+ \sum_{j=1}^2 \chi_j \hat{R}_j^\dagger \hat{R}_j^\dagger + \chi \hat{R}_1^\dagger \hat{R}_2^\dagger \hat{R}_1 \hat{R}_2 \\
+ \sum_{i=A,B} \left( \beta_{i1}^{(\uparrow)} \hat{a}_i^\dagger \hat{a}_2 \sigma_{12}^{(\uparrow)} + \beta_{i2}^{(\uparrow)} \hat{a}_1 \hat{a}_2^\dagger \sigma_{21}^{(\uparrow)} \right),
\]

(1)

In the above relation, \( \hat{a}_i^{(\uparrow)} \) and \( \hat{a}_2^{(\uparrow)} \) are the atomic pseudospin operators for the \( i \)-th atom, \( \hat{a}_i^\dagger \) is the bosonic annihilation (creation) operator of the field mode \( j \), \( \omega_i \) shows the frequency of the atomic transition, \( \lambda_i \) is related to the atom–field coupling constant, and \( \beta_{i1}^{(\uparrow)} \) and \( \beta_{i2}^{(\uparrow)} \) are the effective Stark shift coefficients. Also, \( \chi_j \) and \( \chi_{12} \) denote the cubic susceptibility of the medium; \( \chi_j \) represents the Kerr self-action for mode \( j \), while \( \chi_{12} \) is related to the Kerr cross-action process. In addition, \( \hat{A}_i = \hat{a}_i^\dagger f(\hat{n}_i) \) and \( \hat{R}_i = \hat{a}_i^\dagger g(\hat{n}_i) \) are the nonlinear (\( f \)-deformed) annihilation operators with \( \hat{A}_i^\dagger \) and \( \hat{R}_i^\dagger \) as their respective Hermitian conjugates, where \( \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \) and \( f(\hat{n}_i) \) and \( g(\hat{n}_i) \) correspond to the Hermitian operator–valued functions responsible for the intensity-dependent atom–field coupling and \( f \)-deformed Kerr nonlinearity. These operators satisfy the communication relations \( \{ \hat{A}_i, \hat{n}_j \} = \hat{A}_i \), \( \{ \hat{A}_i^\dagger, \hat{n}_j \} = -\hat{A}_i \), \( \{ \hat{R}_i, \hat{n}_j \} = \hat{R}_i \) and \( \{ \hat{R}_i^\dagger, \hat{n}_j \} = -\hat{R}_i^\dagger \). By comparing the Hamiltonian in (1) with the standard JCM it can be seen that we have in fact made the transformations \( \lambda_i \rightarrow \lambda_i \hat{f}_i (\hat{n}_i) (\hat{f}_2 (\hat{n}_2) - \hat{f}_1 (\hat{n}_1)) \), \( \hat{f}_l \rightarrow \hat{f}_l \chi_l / \rho_l (\hat{n}_l) \chi_l (\hat{n}_l - 1) \) and \( \chi \rightarrow \chi \chi / \rho^2 (\hat{n}) \chi (\hat{n} - 1) \) (the physical motivation of this typical change was established in [28]).
$|\psi(t)\rangle$ associated with the entire system at any time $t$ is in the following form:

$$|\psi(t)\rangle = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} q_n q_m |A(n, m, t)| e^{i\xi n m} |e_A, e_B, n, m\rangle
+ B(n + 1, m + 1, t) |g_A, g_B, n + 1, m + 1\rangle
+ C(n + 1, m + 1, t) |e_A, e_B, n + 1, m + 1\rangle
+ D(n + 2, m + 2, t) |g_A, g_B, n + 2, m + 2\rangle,$$  \hspace{1cm} (3)

where $q_n$ and $q_m$ are the probability amplitudes of the initial state of the radiation field of the cavity. Clearly, $A, B, C,$ and $D$ are the time-dependent atomic probability amplitudes which have to be evaluated. Considering the probability amplitude technique, and after some lengthy but straightforward manipulations, the atomic probability amplitudes $A, B, C,$ and $D$ (leading to the explicit form of the wave function of the whole system) are given by:

$$A(n, m, t) = \sum_{j=1}^{3} \eta_j e^{i\xi_j},$$

$$B(n + 1, m + 1, t) = C(n + 1, m + 1, t) = -\frac{\alpha}{2k_1} \sum_{j=1}^{3} (V_j + \xi_j) \eta_j e^{i\xi_j},$$

$$D(n + 2, m + 2, t) = \frac{\alpha}{2k_2} \sum_{j=1}^{3} \left[ (\xi_j + V_A) (\xi_j - 4) + (\xi_j + V_B) V_B - 2k_2^2 \right] \eta_j e^{i\xi_j},$$  \hspace{1cm} (4)

where:

$$\xi_j = -\frac{1}{x_1} + \frac{2}{3} \sqrt{x_1^2 - 3x_2} \cos \left[ \theta + \frac{2}{3} (j - 1) \pi \right],$$

$$\theta = \frac{1}{3} \cos^{-1} \left[ \frac{9x_2 - 2x_1^2 - 27x_3}{2(x_1^2 - 3x_2)^{3/2}} \right],$$  \hspace{1cm} (5)

with:

$$x_1 = V_B + V_B + V_B - 3\Delta,$$

$$x_2 = (V_B - \Delta) (V_B - 2\Delta) + V_A (V_B + V_B + 3\Delta) - 2 (k_1^2 + k_2^2),$$

$$x_3 = (2\Delta - V_B) (V_B (\Delta - V_B) + 2k_2^2) - 2V_B k_2^2.$$  \hspace{1cm} (6)

In the above relations we have defined:

$$V_A = V_1 + 2\beta_1 m,$$

$$V_B = V_2 + \beta_1 (n + 1) + \beta_2 (m + 1),$$

$$V_D = V_3 + 2\beta_2 (n + 2)$$  \hspace{1cm} (7)

with:

$$V(n, m) = \chi_1 (n - 1) g_1^3 (n - 1) + \chi_2 (m - 1) g_2^3 (m - 1) + \chi_3 (n) g_2^3 (m).$$

$$V_1 = V(n, m),$$

$$V_2 = V(n + 1, m + 1),$$

$$V_3 = V(n + 2, m + 2),$$

$$k_j = j k_j (n + j) (m + j),$$  \hspace{1cm} (8)

Notice that the coefficients $\eta_j$ are still unknown. They can be determined via determining the initial conditions of the atoms. Suppose that the atoms are initially prepared in the coherent superposition of the excited and ground states $|e_A, e_B\rangle$ and $|g_A, g_B\rangle$, respectively, that is:

$$|\psi(t = 0)\rangle_{\text{atoms}} = \cos (\varphi / 2) |e_A, e_B\rangle + \sin (\varphi / 2) |g_A, g_B\rangle,$$  \hspace{1cm} (9)

where $0 \leq \varphi \leq \pi$, i.e. $A(0) = \cos (\varphi / 2)$, $B(0) = 0 = C(0)$, and $D(0) = \sin (\varphi / 2)$. This superposition is specially known as the Bell state (maximally entangled state) if one sets $\varphi = \pi / 2$. However, by considering the general value of $\varphi$, the following relation may be obtained:

$$\eta_j = \frac{2 \sin (\varphi / 2) k_j k_2 \cos (\varphi / 2) (2k_2^2 + (\xi_j + V_A) (\xi_j + V_A))}{\xi_j \xi_j \xi_j},$$  \hspace{1cm} (10)

where $\xi_j = \xi_j - \xi_j$. Consequently, the probability amplitudes $A, B, C,$ and $D$ are explicitly derived, and as a result the exact form of the wave function of the whole system is analytically obtained.

It is now worth declaring that studying the nonclassicality features of the state vector can be achieved after specifying the amplitudes of the initial states of the field, which may be considered as the number, phase, coherent, or squeezed state. However, since the coherent state (the laser field far above the threshold condition [48]) is more accessible than other typical field states, we shall consider the fields to be initially in the coherent states:

$$|\alpha_1, \alpha_2\rangle = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} q_n q_m |n, m\rangle,$$

$$q_n = \exp \left( -\frac{|\alpha_1|^2}{2} \right) \frac{\alpha_n^n}{\sqrt{n!}},$$

$$q_m = \exp \left( -\frac{|\alpha_2|^2}{2} \right) \frac{\alpha_m^m}{\sqrt{m!}},$$  \hspace{1cm} (11)

in which $|\alpha_1|^2$ and $|\alpha_2|^2$ represent the mean photon number (intensity of light) of mode 1 and 2, respectively. It is worth mentioning that the obtained formalism can be applied for any physical system with an arbitrary nonlinearity function. In this paper, we use the nonlinearity function $f(n) = -\eta$ (associated with the atom–field coupling) where its associated coherent state arises naturally from the Hamiltonian, illustrating the interaction with the intensity-dependent coupling between a two-level atom and a radiation field [49--51]. An experimental verification of this function has been recently reported in [52]. In addition, the physical interest of the nonlinearity function $g(n) = 1 / \sqrt{n}$ (related to the intensity-dependent nonlinear susceptibility), which was derived by Man’ko et al [53] from the coherent states, and the corresponding nonlinear coherent states called harmonic states by Sudarshan [54]. This function is a popular nonlinearity function which has been usually used in the contents of the deformation of bosonic operators in the quantum optics literature [55--57]. We are now in a position to study the nonclassical features of the introduced quantum system. The nonclassicality of the radiation field states which may be generated through the nonlinear coherent states technique [58], has generated a great deal of attention in various fields of research [59]. Among all the nonclassical properties, which are of special interest to the field of quantum optics and quantum information processing, we now pay attention to evaluating the entanglement dynamics of the obtained state.
due to the significant role of quantum entanglement in the implementation of quantum information processing devices [60]. It has been shown that there exist some suitable measures that are well justified and mathematically tractable, for instance, EOF and distillation [61], negativity [62], von Neumann and relative entropies [63], and concurrence [64], from which quantum entanglement can be quantified [65]. In the next section, in order to obtain the DEM between subsystems, the dynamics of the EOF (to understand the atom–field entanglement as well as the atom-atom entanglement) is numerically evaluated. In each case, the effects of the intensity-dependent coupling, the deformed Kerr nonlinearity, the detuning parameter and the Stark shift are examined in detail.

3. Entanglement of formation

The EOF is the most meaningful and physically motivated measure of quantum entanglement which clarifies the minimal cost required to prepare a special quantum state in terms of EPR pairs [66]. The EOF has recently attracted a great deal of attention; for instance, it plays an important role in quantum phase transition for various interacting quantum multi-component systems [67], the capacity of quantum channels [68], and may significantly affect the macroscopic properties of solids [69].

It has been demonstrated that, for pure states, the EOF measure is defined as the entropy of either of two subsystems $A$ and $B$ as the following form [61, 66]:

$$E_F (\psi) = - \text{Tr} (\hat{\rho}_A \ln \hat{\rho}_A) = - \text{Tr} (\hat{\rho}_B \ln \hat{\rho}_B),$$  \hspace{1cm} (12)

where $\hat{\rho}_A (\hat{\rho}_B)$ is the reduced density operator of subsystem $A(B)$. The definition of EOF can also be extended to the mixed state $\hat{\rho}$ by using the convex-roof method as [64, 66]:

$$E_F (\hat{\rho}) = \min \sum_i p_i E_F (\psi_i),$$  \hspace{1cm} (13)

where the minimum is taken over all the possible pure-state decompositions with $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. The ensemble that satisfies the minimum value in (13) is known as the decomposition of the optimal mixed state $\hat{\rho}$. Therefore, in order to find the EOF for a mixed state, the fundamental problem is to determine the optimal decomposition. Specifically, the EOF for bipartite mixed states is given by [61]:

$$E_F (C) = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right),$$  \hspace{1cm} (14)

where $h(x)$ is the binary entropy function defined by $h(x) = -x \ln x - (1 - x) \ln (1 - x)$ (Shannon entropy) and $C$ is called the concurrence. Since the function $E_F (C)$ is monotonically increased for $0 \leq C \leq 1$, the concurrence can be considered as a quantity that can measure the quantum entanglement. Unlike the EOF that is a resource-based or information theoretic measure, the concurrence is not [70]. The concurrence for a bipartite system with density matrix $\rho$ reads as:

$$C(\rho) = \max \left\{ \frac{1}{2} \max \left[ \lambda_i \right] - \sum_{j=1}^{4} \lambda_j \right\},$$  \hspace{1cm} (15)

in which $\lambda_i$ denotes the square roots of the eigenvalues of the operator $\hat{\rho} \hat{\rho}$ with $\hat{\rho}$ showing the ‘spin-flipped’ density matrix defined by:

$$\hat{\rho} = (\hat{\sigma}_x \otimes \hat{\sigma}_x) \hat{\rho}^* (\hat{\sigma}_x \otimes \hat{\sigma}_x),$$  \hspace{1cm} (16)

where $\hat{\rho}^*$ and $\hat{\sigma}_i$ are the Hermitian conjugate of $\hat{\rho}$ and the ‘$\gamma$’ Pauli matrix, respectively. It is worth mentioning that the concurrence varies from $C = 0$ (separable state) to $C = 1$ (maximally entangled state) and the case $0 < C < 1$ indicates a partially entangled state. In the next subsections we are going to examine the DEM between different two-partites of the whole system, namely atom–field (pure state) and atom–atom (mixed state), by using the equations (12) and (14), respectively.

3.1. Atom–field entanglement

In order to study the DEM between the subsystems (atoms and fields) of the obtained state, the EOF is a good measure. Because in this case the system is pure state, we use the definition of the EOF for pure states as:

$$E_F (t) = - \text{Tr}_{AF} (\hat{\rho}_{AF}(t) \ln \hat{\rho}_{AF}(t)), \hspace{1cm} (17)$$

with $\hat{\rho}_{AF}(t) = \text{Tr}_F (|\psi(t)\rangle \langle \psi(t)|)$, which denotes the reduced density operator of the atoms (fields). Considering the procedure of [29], the equation (17) may be expressed as follows:

$$E_F (t) = - \sum_{j=1}^{4} \zeta_j \ln \zeta_j,$$  \hspace{1cm} (18)

where $\zeta_j$ represents the eigenvalues of the reduced density operator of the atoms, which is given by Cardano’s method as [71]:

$$\zeta_j = \frac{1}{3} q_0 + \frac{2}{3} \sqrt{q_0^2 - 27 q_0^4 \cos \left( \frac{2}{3} (j - 1) \pi \right)}, \hspace{1cm} j = 1, 2, 3, 4,$$  \hspace{1cm} (19)

with:

$$q_0 = \frac{1}{3} \cos^{-1} \left( \frac{9 q_0 q_2 - 2 q_1^3 - 27 q_0^4}{2 (q_1^3 - q_2^3)^{3/2}} \right),$$  \hspace{1cm} (20)

and:

$$q_1 = - \rho_{12} - 2 \rho_{22} - \rho_{44}, \hspace{1cm} (21a)$$

$$q_2 = 2 \rho_{12} \rho_{21} - \rho_{14} \rho_{41} - 2 \rho_{24} \rho_{42} + 2 \rho_{22} \rho_{44} + \rho_{11} (2 \rho_{22} + \rho_{44}), \hspace{1cm} (21b)$$

$$q_3 = 2 \rho_{14} (\rho_{22} \rho_{41} - \rho_{21} \rho_{42}) + \rho_{12} (\rho_{21} \rho_{44} - \rho_{24} \rho_{41}) + \rho_{11} (\rho_{22} \rho_{42} - \rho_{24} \rho_{42}), \hspace{1cm} (21c)$$

where the relation (21a) clearly shows that the parameter $q_1$ is precisely equal to $-1$ and the matrix elements of the atomic density operator are as follows:

$$\rho_{ij}(t) = \sum_{n,m=0}^{\infty} \sum_{m,j=0}^{\infty} \langle n, m, i | \psi(t) \rangle \langle \psi(t) | n, m, j \rangle, \hspace{1cm} i, j = 1, 2, 3, 4.$$  \hspace{1cm} (22)
where we have set $|1\rangle = |e_A, e_B\rangle$, $|2\rangle = |e_A, g_B\rangle$, $|3\rangle = |g_A, e_B\rangle$ and $|4\rangle = |g_A, g_B\rangle$. Figure 2 shows the evolution of the EOF against the scaled time $\tau$ for the initial mean photon numbers fixed at $|\alpha_1|^2 = 10 = |\alpha_2|^2$ and the two atoms prepared initially in the ground state ($\varphi = \pi$). The left plots correspond to the absence of the intensity-dependent coupling ($f(n) = 1$) while the right ones correspond to the intensity-dependent coupling with nonlinearity function $f(n) = \sqrt{n}$. Also (a) $\chi = 0$, $\Delta = 0$ and $\beta_1 = \beta_2 = 0$, (b) $\chi = 0$, $\Delta = 0$ and $\beta_1 = \beta_2 = 0$, (c) $\chi = 0$, $\Delta = 10\lambda$, $\beta_1 = \beta_2 = 0$ and (d) $\chi = 0$, $\Delta = 0$, $\beta_1 = \beta_2 = \lambda$.

From figure 2(a), whereas the Kerr effect and detuning are disregarded, a random behavior for the time evolution of the EOF is observed in both the constant and

**Figure 2.** The time evolution of the EOF versus the scaled time $\tau = \lambda t$, when the atoms and fields are assumed to be initially in the ground and the coherent state, $\varphi = \pi$ and $|\alpha_1|^2 = 10 = |\alpha_2|^2$, respectively. The left plots correspond to the absence of the intensity-dependent coupling ($f(n) = 1$), and the right ones show the presence of the intensity-dependent coupling with nonlinearity function $f(n) = \sqrt{n}$. Also (a) $\chi = 0$, $\Delta = 0$ and $\beta_1 = \beta_2 = 0$, (b) $\chi = 0$, $\Delta = 0$ and $\beta_1 = \beta_2 = 0$, (c) $\chi = 0$, $\Delta = 10\lambda$, $\beta_1 = \beta_2 = 0$ and (d) $\chi = 0$, $\Delta = 0$, $\beta_1 = \beta_2 = \lambda$.
intensity-dependent coupling regime; it is seen that intensity-dependent coupling has no considerable effect in the DEM. This situation is repeated in figure 2(b) with the difference that the deformed Kerr medium (which is distinguishable from the Kerr medium through the nonlinearity function $g(n) = 1/\sqrt{n}$) can improve the maximum amount of the DEM in the presence of the intensity-dependent coupling. Figure 2(c) refers to the effect of the detuning parameter. It is shown that in the presence of the detuning parameter, the amount of the EOF and therefore the DEM is increased. According to figure 2(d), one can observe that the amount of entanglement is reduced in the constant coupling regime, although in the presence of the intensity-dependent coupling and in contrast to figure 2(a), the DEM is retained.

In summary, comparing the left plots in figure 2 (in which the atom–field coupling is constant) with the right ones implies that the presence of the intensity-dependent coupling may improve the maxima of the atom–field entanglement. Considering the effect of the $f$-deformed Kerr nonlinearity, it is understood that the existence of the deformed Kerr medium can enhance the amount of DEM as the detuning parameter, while the Stark shift reduces the entanglement between the atoms and the fields.

Figure 3. The time evolution of the EOF for chosen parameters similar to figure 2.
3.2. Atom–atom entanglement

In this subsection, we study the dynamics of the DEM between the atoms (atom–atom entanglement) via a definition of the EOF measure for the mixed state (equation (14)). Our results in figure 3 represent the time evolution of the EOF against the scaled time \(\tau\) for the same parameters, as shown in figure 2. The left (right) plots again correspond to \(f(n) = 1\) (\(f(n) = \sqrt{n}\)). In figure 3(a) the Kerr and Stark effects are absent (\(\chi = 0\), \(\beta_1 = \beta_2 = 0\)) and the exact resonant case is assumed (\(\Delta = 0\)). In figure 3(b) the effect of a deformed Kerr medium \((\chi = 0.4\lambda, \gamma_{\lambda} = 1/\sqrt{n}\)) in the absence of the detuning parameter and Stark shift is studied. The effect of a detuning parameter (\(\Delta = 10\lambda\)) without Kerr and Stark effects is presented in figure 3(c). Figure 3(d) is plotted to investigate the particular effect of the Stark shift (\(\beta_1 = \beta_2 = \lambda\)) in the absence of a Kerr medium and detuning parameter. From figure 3(a) showing the resonance case, in the absence of the Kerr nonlinearity and Stark shift it is observed that the EOF has a random behavior. A comparison of both the left and right plots of this figure indicates that the intensity-dependent coupling can increase the maximum amount of the EOF. Figure 3(b), which deals with the effect of the deformed Kerr medium, shows that this nonlinearity has no remarkable effect on the behavior of the EOF, as compared with figure 3(a). The effect of the detuning parameter is presented in figure 3(c). According to this figure, it can be clearly seen that the DEM between the atoms is drastically diminished as time goes on. However, this will be faster in the case of the intensity-dependent coupling. In addition, it may be noted that in the presence of the detuning parameter, the maximum value of the EOF for the case \(f(n) = \sqrt{n}\) is smaller than \(f(n) = 1\). The influence of the Stark shift in the resonance case and in the absence of the Kerr nonlinearity is shown in figure 3(d). It can be seen in this figure that, in the constant coupling regime, the amount of the EOF is considerably increased especially when time proceeds, while in the intensity-dependent coupling, the DEM between the atoms is preserved.

4. Summary and conclusion

Due to the fact that the light–matter interaction in the cavity QED is considered as a usual way to generate various classes of entangled states, in this paper we have outlined a particular nonlinear interaction between two two-level atoms and two-mode quantized radiation field in an optical cavity containing a medium with centrosymmetric Kerr nonlinearity in the presence of the Stark shift, detuning parameter and intensity-dependent coupling. After suitably considering all the existing interactions, the explicit form of the state vector of the entire system was obtained. Then, we studied the dynamics of the entanglement of the system consisting of the two atoms and the two-mode field. Consequently, we planned to examine the temporal behavior of the DEM between the available subsystems, which may be observed during the interaction, through the study of the EOF measure.

To sum up, the main outcomes of the considered interaction model and the related numerical calculations are briefly listed as below.

- Tuning the nonclassicality indicators: it is shown that the amount of the EOF measure can be tuned by suitably choosing the nonlinear parameters related to the atom–field system.
- Intensity-dependent coupling: the presented results show that intensity-dependent coupling (which is considered by the function \(f(n) = \sqrt{n}\)) may in general enhance the EOF.
- Deformed Kerr medium: paying attention to the related results implies that the deformed Kerr medium (which is distinguishable from the Kerr medium through the nonlinearity function \(g(n) = 1/\sqrt{n}\)) has an obvious role and a notable effect in improving the entanglement.
- Detuning parameter: looking in depth at the obtained results shows that the detuning parameter can ameliorate the atom–field entanglement, while this parameter reduces the DEM between the atoms drastically.
- Stark shift: from the numerical results, it is revealed that in the constant coupling regime, the amount of the DEM is apparently decreased in the presence of the Stark shift, although in the case of the intensity-dependent coupling, the Stark shift can increase the DEM when it is compared with the constant coupling regime.

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