FINITE DISTRIBUTIVE LATTICES, POLYOMINOES AND IDEALS OF KÖNIG TYPE

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Abstract. Finite distributive lattices whose join-meet ideals are of König type will be classified. Furthermore, a class of polyominoes whose polyomino ideals are of König type will be studied.

Introduction

Inspired by König’s theorem in the classical graph theory, an ideal of König type is introduced in [4]. In the present paper, binomial ideals of König type arising from finite distributive lattices ([2],[5]), and those from polyominoes ([7]) will be studied.

1. Ideals of König type

Recall from [4] what is an ideal of König type. Let \( S = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over a field \( K \) with each \( \deg x_i = 1 \) and let \( I \subset S \) be a graded ideal of height \( h \). We say that \( I \) is of König type if there exist (i) a sequence \( f_1, \ldots, f_h \) of homogeneous polynomials which forms part of a minimal system of homogeneous generators of \( I \) and (ii) a monomial order \( \prec \) on \( S \) for which \( \text{in}_{\prec}(f_1), \ldots, \text{in}_{\prec}(f_h) \) is a regular sequence. More precisely, we say that \( I \) is of König type with respect to the sequence \( f_1, \ldots, f_h \) and the monomial order \( \prec \).

2. Finite distributive lattices

Let \( P \) be a finite partially ordered set (poset, for short) with \( |P| = d \) and \( L = J(P) \) the finite distributive lattice ([3] pp. 156–159) consisting of poset ideals of \( P \), ordered by inclusion. In other words, \( P \) is the subposet of \( L \) consisting of join-irreducible elements of \( L \). A subset \( \{a_{i_0}, a_{i_1}, \ldots, a_{i_q}\} \) of \( L \) of the form \( a_{i_0} < a_{i_1} < \cdots < a_{i_q} \) is called a chain of \( L \) of length \( q \). It follows that the length of every maximal chain of \( L \) is equal to \( d \). The rank of \( a \in L \) is the maximal length of chains of \( L \) of the form \( a_{i_0} < a_{i_1} < \cdots < a_{i_r} = a \). Let \( \rho_L(i) \) denote the number of \( a \in L \) with \( \text{rank}_L(a) = i \), where \( 0 \leq i \leq d \). Especially \( \rho_L(0) = \rho_L(d) = 1 \). We say that \( \xi \in L \) is an apex of \( L \) if \( \text{rank}_L(\xi) = i \) and \( \rho_L(i) = 1 \).

In other words, \( \xi \in L \) is an apex of \( L \) if, for each \( a \in L \) with \( a \neq \xi \), one has either \( \xi < a \) or \( \xi > a \). In particular, the unique minimal element \( 0_L \) of \( L \) and the unique

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Let \( I_L \) be the set of binomials \( f_{a,b} \), where \( a \in L \) and \( b \in L \) are incomparable in \( L \) and \( \rho \) is of König type with respect to a sequence \( f_{a_1,b_1}, f_{a_2,b_2}, \ldots, f_{a_h,b_h} \), where \( h = |L| - (d + 1) \), and a monomial order \( \prec \) on \( S \).

**Lemma 2.1.** Let \( L' = \mathcal{J}(P') \) and \( L'' = \mathcal{J}(P'') \) be finite distributive lattice. Let \( L = L' \bigoplus L'' \) be the ordinal sum [3] p. 246 of \( L' \) and \( L'' \). Then the finite distributive lattice \( L \) is of König type if and only if each of \( L' \) and \( L'' \) is of König type.

**Proof.** One has \( L = \mathcal{J}(P) \), where \( P = P' \bigoplus \{a\} \bigoplus P'' \). Let \( |P'| = d' \) and \( |P''| = d'' \).

Since height \( I_{L'} = |L'| - (d' + 1) \) and height \( I_{L''} = |L''| - (d'' + 1) \), it follows that

\[
\text{height } I_L = |L| - (d' + d'' + 2) = \text{height } I_{L'} + \text{height } I_{L''}.
\]

Each binomial \( f_{a,b} \in I_L \) belong to either \( I_{L'} \) or \( I_{L''} \). It then follows that \( L \) is of König type if and only if each of \( L' \) and \( L'' \) is of König type, as required. \( \Box \)

A finite distributive lattice \( L \) is decomposable, if \( L \) is of the form \( L = L' \bigoplus L'' \), where each of \( L' \) and \( L'' \) is a distributive lattice. It follows that \( L \) is non-decomposable if and only if, for apexes \( \xi \) and \( \xi' \) of \( L \) with \( \xi < \xi' \), one has \( \text{rank}(\xi') - \text{rank}(\xi) \geq 2 \). Every finite simple distributive lattice is non-decomposable.

In general, we say that a finite simple distributive lattice \( L \) is quasi-thin if \( \rho_L(i) \leq 3 \) for each \( 1 \leq i < d \). When \( L \) is quasi-thin, one introduces \( \theta(L) = |\{i : \rho_L(i) = 3\}| \).

A finite quasi-thin distributive lattice \( L \) is called thin if \( \theta(L) = 0 \).

**Lemma 2.2.** Suppose that a finite simple distributive lattice \( L \) is of König type. Then \( L \) is quasi-thin with \( \theta(L) \leq 2 \).

**Proof.** Let \( L = \mathcal{J}(P) \) with \( |P| = d \). Since \( L \) is simple and since \( |L| \leq 2(d + 1) \), it follows that

\[
2 \geq |L| - 2d = \sum_{i=1}^{d-1} (\rho_L(i) - 2).
\]

Suppose that there is \( 1 \leq i_0 < d \) with \( \rho_L(i_0) \geq 4 \). Then \( \rho_L(i) = 2 \) for \( 1 \leq i < d \) with \( i \neq i_0 \). If \( L \) is planar [1] p. 436, then \( P \) possesses the subposet \( \{a_1, \ldots, a_6\} \).
with the partial order $a_1 < a_2 < a_3$ and $a_4 < a_5 < a_6$. Hence $\rho_L(i_0 - 1) \geq 3$ and $\rho_L(i_0 + 1) \geq 3$, a contradiction. If $L$ is non-planar, then the boolean lattice [3] Example 9.1.6 (a) on $[n] = \{1, \ldots, n\}$ with $n \geq 4$ is an interval of $L$. Thus there is $1 \leq i_0' < d$ with $i_0' \neq i_0$ and $\rho_L(i_0') \geq 4$, a contradiction. It follows that $L$ is quasi-thin. Furthermore, since $\sum_{i=1}^{d-1}(\rho_L(i) - 2) \leq 2$, one has $\theta(L) \leq 2$. \qed

3. Finite simple distributive lattices of König type

Now, in the present section, a classification of finite simple distributive lattices of König type will be done. Lemma 2.2 says that a finite simple distributive lattice $L$ of König type is quasi-thin with $\theta(L) \leq 2$. Hence the first step our job is to classify finite simple quasi-thin distributive lattices $L$ with $\theta(L) \leq 2$.

First, suppose that $L$ is non-planar. Then the boolean lattice on $\{1, 2, 3\}$ is an interval of $L$ and $\theta(L) = 2$. Each of the finite distributive lattices of Figure 1 is simple, non-planar and quasi-thin.

Second, suppose that $L$ is planar. We divide finite simple planar quasi-thin distributive lattices $L$ with $\theta(L) \leq 2$ into 5 classes as follows:

- (type 0) thin;
- (type 1) quasi-thin with $\theta(L) = 1$;
- (type $2_a$) quasi-thin with $\theta(L) = 2$ and $\rho_L(i_0) = \rho_L(i_0 + 1) = 3$;
- (type $2_b$) quasi-thin with $\theta(L) = 2$ and $\rho_L(i_0) = \rho_L(i_0 + 2) = 3$;
- (type $2_c$) quasi-thin with $\theta(L) = 2$ and $\rho_L(i_0) = \rho_L(j_0) = 3$ with $j_0 \geq i_0 + 3$.

Let $L, L'$ and $L''$ denote the distributive lattices in Figure 2 from the right to the left. The interval $[h, a']$ of $L''$ is of type 0. Each of the intervals $[1', a']$ and $[h, 5']$ of $L''$ is of type 1. The distributive lattice $L$ is of type $2_a$ with $i_0 = 2$. The distributive lattice $L'$ is of type $2_b$ with $i_0 = 2$. The distributive lattice $L''$ is of type $2_c$ with $i_0 = 2$ and $j_0 = 13$.

**Example 3.1.** Each of the distributive lattices of Figure 1 is of König type with respect to the sequence

$$x_1x_1' - x_1 \land 1'x_1 \lor 1', x_2x_2' - x_2 \land 2'x_2 \lor 2', \ldots$$

3
and the lexicographic order induced by the ordering

\[ x_1 > x_2 > \cdots > x_1' > x_2' > \cdots. \]

The distributive lattice \( L'' \) of Figure 2 is of König type with respect to the sequence

\[ x_1x_1' = x_1 \land x_1' \land 1', \ldots, x_ax_a' = x_a \land x_a' \land a', \ldots, x_hx_h' = x_h \land x_h' \land h'. \]

and the lexicographic order induced by the ordering

\[ x_a > \cdots > x_h > x_1 > \cdots > x_8 > x_{a'} > \cdots > x_{h'} > x_{1'} > \cdots > x_{8'}. \]

Each of the distributive lattices \( L \) and \( L' \) of Figure 2 is of König type with respect the sequence (1) and the lexicographic order induced by the ordering (2).

**Lemma 3.2.** Let \( L = \mathcal{J}(P) \) with \(|P| = d \) be a finite distributive lattice of König type with respect to \( f_{a_1,b_1}, f_{a_2,b_2}, \ldots, f_{a_h,b_h} \) and a lexicographic order \(<\) on \( S \). Here \( h = |L| - (d + 1) \). Suppose that \( \text{in}_< (f_{a_1,b_1}) = x_{a_1}x_{b_1} \), \( \text{rank}_L(a_1) = \text{rank}_L(b_1) = d - 1 \) and that \( a_1 \lor b_1 \) is the unique maximal element \( 1_L \) of \( L \). Let \( L' \) denote the finite distributive lattice which is obtained by adding new elements \( c \) and \( e \) to \( L \), where

\[ a_1 \lor b_1 < e, \ b_1 < c < e, \ a_1 \lor b_1 \neq c. \]

Then \( L' \) is of König type with respect to \( f_{a_2,b_2}, f_{a_3,b_3}, \ldots, f_{a_h,b_h}, f', f'' \), where

\[ f' = x_{a_1}x_c - x_{a_1 \land b_1}x_c, \quad f'' = x_{b_1}x_e - x_{a_1 \lor b_1}x_c \]

and a lexicographic order \(<'\) on \( S' = S[x_c, x_e] \) with

\[ \text{in}_{<'}(f') = x_{a_1}x_c, \quad \text{in}_{<'}(f'') = x_{b_1}x_e. \]
Proof. If $x_{a_1}$ is bigger than each of $x_{b_1}, x_{a_1 \land b_1}, x_{a_1 \lor b_1}$ with respect to $<$ and if $<$ is the lexicographic order on $S$ induced by the ordering $\cdots > x_{a'} > x_{a_1} > x_{a''} > \cdots$, then the lexicographic order $<^\prime$ on $S'$ induced by the ordering $\cdots > x_{a'} > x_{a_1} > x_e > x_c > x_{a''} > \cdots$ satisfies $\mathrm{in}_{<^\prime}(f') = x_{a_1} x_e$ and $\mathrm{in}_{<^\prime}(f'') = x_{b_1} x_e$. On the other hand, if $x_{b_1}$ be bigger than each of $x_{a_1}, x_{a_1 \land b_1}, x_{a_1 \lor b_1}$ and if $<$ is the lexicographic order on $S$ induced by the ordering $\cdots > x_{b'} > x_{b_1} > x_{b''} > \cdots$, then the lexicographic order $<^\prime$ on $S'$ induced by the ordering $\cdots > x_{b'} > x_{b_1} > x_c > x_e > x_{b''} > \cdots$ satisfies $\mathrm{in}_{<^\prime}(f') = x_{a_1} x_c$ and $\mathrm{in}_{<^\prime}(f'') = x_{b_1} x_e$. Hence $L'$ is of König type with respect to $f_{a_2,b_2}, \ldots, f_{a_h,b_h}, f', f''$ and $<^\prime$ on $S'$, as required. \hfill \Box

Lemma 3.3. Let $L = \mathcal{J}(P)$ with $|P| = d$ be a finite distributive lattice of König type with respect to $f_{a_1,b_1}, f_{a_2,b_2}, \ldots, f_{a_h,b_h}$ and a lexicographic order $<$ on $S$. Here $h = |L| - (d + 1)$. Suppose that $\mathrm{in}_<(f_{a_1,b_1}) = x_{a_1 \land b_1} x_{a_1 \lor b_1}$, $\mathrm{rank}_L(a_1) = \mathrm{rank}_L(b_1) = d - 1$ and that $a_1 \lor b_1$ is the unique maximal element $1_L$ of $L$. Let $L'$ denote the finite distributive lattice which is obtained by adding new elements $c$ and $e$ to $L$, where

$$a_1 \lor b_1 < e, \quad b_1 < c < e, \quad a_1 \lor b_1 \neq c.$$ 

Then $L'$ is of König type with respect to $f_{a_2,b_2}, f_{a_3,b_3}, \ldots, f_{a_h,b_h}, f', f''$, where

$$f' = x_{a_1 \land b_1} x_e - x_{a_1} x_c, \quad f'' = x_{a_1 \lor b_1} x_c - x_{b_1} x_e$$

and a lexicographic order $<^\prime$ on $S' = S[x_c, x_e]$ with

$$\mathrm{in}_{<^\prime}(f') = x_{a_1 \land b_1} x_e, \quad \mathrm{in}_{<^\prime}(f'') = x_{a_1 \lor b_1} x_c.$$ 

Proof. In the proof of Lemma 3.2, replacing the orderings with $\cdots > x_{a'} > x_{a_1 \lor b_1} > x_e > x_c > x_{a''} > \cdots$ and $\cdots > x_{b'} > x_{a_1 \land b_1} > x_c > x_e > x_{b''} > \cdots$ yields the desired result. \hfill \Box

Figure 3. $L' = L \cup \{c, e\}$
Remark 3.4. A dual version of Lemmata 3.2 and 3.3, which can be easily obtained by replacing 1\_L with 0\_L, is also valid with its dual proof.

Lemma 3.5. A finite simple distributive lattice \( L \) is of König type if and only if \( L \) is quasi-thin with \( \theta(L) \leq 2 \).

Proof. The “only if” part is known (Lemma 2.2). On the other hand, the observation made in Example 3.1 together with repeated application of the technique introduced in Lemma 3.2 and Lemma 3.3 guarantees that every finite quasi-thin distributive lattice with \( \theta(L) \leq 2 \) is of König type, as desired. \( \square \)

4. Classification of distributive lattices of König type

Let \( L = \mathcal{J}(P) \) with \(|P| \geq 2 \) be a finite non-decomposable distributive lattice and

\[ 0_L = \xi_0 < \xi_1 < \cdots < \xi_s = 1_L \]

the apexes of \( L \). Let \( L_i = [\xi_{i-1}, \xi_i] = \mathcal{J}(P_i) \) with \(|P_i| = d_i \geq 2 \), where \( 1 \leq i \leq s \). It then follows that \( P = P_1 \bigoplus \cdots \bigoplus P_s \) and \( d = d_1 + \cdots + d_s \).

Theorem 4.1. Let \( L \) be a finite non-decomposable distributive lattice and

\[ 0_L = \xi_0 < \xi_1 < \cdots < \xi_s = 1_L \]

the apexes of \( L \). Let \( L_i = [\xi_{i-1}, \xi_i] \), where \( 1 \leq i \leq s \). Then \( L \) is of König type if and only if the following conditions are satisfied:

- Each \( L_i \) is quasi-thin with \( \theta(L_i) \leq 2 \).
- If \( \theta(L_i) = 2 \) and \( \theta(L_{i'}) = 2 \) with \( 1 \leq i < i' \leq s \), then there is \( i < j < i' \) with \( \theta(L_j) = 0 \).

Proof. (“only if”) Suppose that a finite non-decomposable distributive lattice \( L \) is of König type with respect to a sequence \( f_{a_1,b_1}, \ldots, f_{a_h,b_h} \) and a monomial order \( \prec \) on \( S = K[\{x_a\}_{a \in L}] \), where \( L = \mathcal{J}(P) \) with \(|P| = d \geq 2 \) and \( h = |L| - (d + 1) \). Let \( L_i = \mathcal{J}(P_i) \) with \(|P_i| = d_i \geq 2 \). Let \( u_j = \text{in}_<(f_{a_j,b_j}) \). Let \( u_{j_1}, \ldots, u_{j_h} \) belong to \( S_i = K[\{x_a\}_{a \in L_i}] \). It then follows that

\[ g_i = \text{grade}(\text{in}_<(I_{L_i})) = \text{height}(\text{in}_<(I_{L_i})) \leq |L_i| - \dim(S/\text{in}_<(I_{L_i})) = |L_i| - \dim(S/I_{L_i}) = |L_i| - (d_i + 1). \]

Hence

\[ h = \sum_{i=1}^s g_i \leq \sum_{i=1}^s (|L_i| - (d_i + 1)) = |L| + (s - 1) - d - s = |L| - (d + 1). \]

Since \( h = |L| - (d + 1) \), one has \( g_i = |L_i| - (d_i + 1) \) for each \( 1 \leq i \leq s \). Thus each \( L_i \) is of König type. In other words, each \( L_i \) is quasi-thin with \( \theta(L_i) \leq 2 \).
Now, suppose that $\theta(L_{i_0}) = 2$ and $\theta(L_{i'_0}) = 2$ with $1 \leq i_0 < i'_0 \leq s$ and that $\theta(L_j) = 1$ for each $i_0 < j < i'_0$. Since $g_i = |L_i| - (d_i + 1)$, it follows that

$$
\sum_{i=i_0}^{i'_0} 2g_i = \sum_{i=i_0}^{i'_0} |L_i| - (i'_0 - i_0 - 1)
= |L'| + (i'_0 - i_0) - (i'_0 - i_0 - 1)
= |L'| + 1.
$$

On the other hand, one has $\sum_{i=i_0}^{i'_0} 2g_i \leq |L'|$, a contradiction.

("if") Lemma 3.5 says that each of the simple distributive lattices $L_i$ is of König type with respect to a sequence $f_{a_i}^{(i)}, b_i^{(i)}, \ldots, f_{a_{h_i}}^{(i)}, b_{h_i}^{(i)}$, where $h_i = |L_i| - (d_i + 1)$, and a monomial order $<_{(i)}$ on $S_i = K[x_a]_{a \in L_i}$. Let $\text{in}_{<_{(i)}}(f_{a_j}^{(i)} x_{e_j}^{(i)}) = x_{e_j}^{(i)} x_{e_j}^{(i)}$ for each $1 \leq i \leq s$ and $1 \leq j \leq h_i$. Furthermore, set $A_i = \{c_j^{(i)}, e_j^{(i)} : 1 \leq j \leq h_i\}$ for each $1 \leq i \leq s$. The crucial facts observed in Example 3.1 are as follows:

(i) If $\theta(L_i) = 0$, then one can make neither $\xi_{i-1}$ nor $\xi_i$ belongs to $A_i$;

(ii) Let $\theta(L_i) = 1$. Then there exists a lexicographic order $<_s$ on $S_i$ for which $\xi_{i-1} \in A_i$ and $\xi_i \notin A_i$. Furthermore, there exists a lexicographic order $<_s$ on $S_i$ for which $\xi_{i-1} \notin A_i$ and $\xi_i \in A_i$.

Now, it follows from these facts (i) and (ii) together with $h = \sum_{i=1}^s h_i$ that "if" part turns out to be true. \qed

\textbf{Figure 1}

\textbf{Figure 4.} Repeated application of Lemmata 3.2 and 3.3
5. Polyominoes and polyomino ideals

Recall from [7] fundamental materials on polyominoes and their binomial ideals. One regards \( \mathbb{N}^2 \) as a infinite poset with the natural partial order defined by setting \((i, j) \leq (i', j')\) if \(i \leq i'\) and \(j \leq j'\). Let \(a, b \in \mathbb{N}^2\) with \(a \leq b\). Then the set \([a, b] = \{c \in \mathbb{N}^2 : a \leq c \leq b\}\) is an interval of \(\mathbb{N}^2\). If \(a = (i, j), b = (i', j')\) with \(i < i'\) and \(j < j'\), then the interval \([a, b]\) is called proper. The corners of the proper interval \([a, b]\) are \(a, b\) and \(c = (i', j), d = (i, j')\). We say that \(a\) and \(b\) are the diagonal corners and that \(c\) and \(d\) are anti-diagonal corners of \([a, b]\). The interval \(C = [a, b]\) with \(b = a + (1, 1)\) is called a cell of \(\mathbb{N}^2\). Let \(c, d\) be anti-diagonal corners of the cell \(C = [a, b]\). The set of vertices of \(C\) is \(V(C) = \{a, b, c, d\}\) and the set of edges of \(C\) is \(E(C) = \{(a, c), (a, d), (b, c), (b, d)\}\). Let \([a, b]\) be a proper interval of \(\mathbb{N}^2\). A cell \(C = [a', b']\) of \(\mathbb{N}^2\) is called a cell of \([a, b]\) if \(a \leq a'\) and \(b' \leq b\).

Let \(\mathcal{P}\) be a finite collection of cells of \(\mathbb{N}^2\). The vertex set of \(\mathcal{P}\) is \(V(\mathcal{P}) = \cup_{C \in \mathcal{P}} V(C)\) and the edge set of \(\mathcal{P}\) is \(E(\mathcal{P}) = \cup_{C \in \mathcal{P}} E(C)\). A vertex \(a \in V(\mathcal{P})\) is called an interior vertex of \(\mathcal{P}\) if \(a\) is a vertex of four distinct cells of \(\mathcal{P}\), otherwise it is called a boundary vertex of \(\mathcal{P}\). Let \(A\) and \(B\) be two cells of \(\mathcal{P}\). Then \(A\) and \(B\) are connected in \(\mathcal{P}\) if there is a sequence of cells of \(\mathcal{P}\) of the form \(C_1, \ldots, C_m = D\) for which \(C_i \cap C_{i+1}\) is an edge of \(C_i\) for \(i = 1, \ldots, m - 1\). A polyomino is a finite collection \(\mathcal{P}\) of cells of \(\mathbb{N}^2\) for which any two cells of \(\mathcal{P}\) is connected in \(\mathcal{P}\).

Let \(\mathcal{P}\) be a polyomino and \(S = K[\{x_a\}_{a \in V(\mathcal{P})}]\) the polynomial ring in \(|V(\mathcal{P})|\) variables over a field \(K\). A proper interval \([a, b]\) of \(\mathbb{N}^2\) is called an inner interval of \(\mathcal{P}\) if each cell of \([a, b]\) belongs to \(\mathcal{P}\). Now, for each inner interval \([a, b]\) of \(\mathcal{P}\), one introduces the binomial \(f_{a,b} = x_a x_b - x_c x_d\), where \(c\) and \(d\) are the anti-diagonal corners of \([a, b]\). The binomial \(f_{a,b}\) is called an inner 2-minor of \(\mathcal{P}\). The polyomino ideal of \(\mathcal{P}\) is the binomial ideal \(I_P\) which is generated by the inner 2-minors of \(\mathcal{P}\). Furthermore, we write \(K[\mathcal{P}]\) for the quotient ring \(S/I_P\). We say that \(\mathcal{P}\) is of König type if \(I_P\) is of König type with respect to a sequence \(f_{a_1, b_1}, \ldots, f_{a_h, b_h}\) of inner 2-minors, where \(h = \text{height } I_P\), and a monomial order \(<\) on \(S\).

In order to study polyominoes of König type, to find a combinatorial formula to compute height \(I_P\) is indispensable.

**Theorem 5.1.** Let \(\mathcal{P}\) be a polyomino for which each \(x_{ij}\) with \((i, j) \in V(\mathcal{P})\) is a non-zero divisor modulo \(I_P\). Then height \(I_P \leq |\mathcal{P}|\). In particular, when \(I_P\) is prime, one has height \(I_P \leq |\mathcal{P}|\).

**Proof.** Let \(S = K[x_{ij} : (i, j) \in V(\mathcal{P})]\) and \(x = \prod_{(i,j) \in V(\mathcal{P})} x_{ij} \in S\). One claims that \(I_P S_x\) is generated by the 2-minors corresponding to the cells of \(\mathcal{P}\). The choice of \(x\) guarantees that height \(I_P = \text{height } I_P S_x\). The claim says that \(I_P S_x\) is generated by \(|\mathcal{P}|\) elements, from which height \(I_P S_x \leq |\mathcal{P}|\) follows. Thus height \(I_P \leq |\mathcal{P}|\), as desired.

It remains to prove the claim. Let \(i_1 < i_2 < i_3\) and \(j_1 < j_2\), and assume that \([i_1, j_1), (i_3, j_2)\] is an inner interval of \(\mathcal{P}\). Then each of \([(i_1, j_1), (i_2, j_2)]\) and \([(i_2, j_1), (i_3, j_2)]\) is also an inner interval of \(\mathcal{P}\). Let \(a = (i_1, j_1), c = (i_2, j_1), d = (i_2, j_2)\) and \(b = (i_3, j_2)\). Then

\[
x_{i_1,j_2} f_{c,b} - x_{i_2,j_2} f_{a,b} + x_{i_3,j_2} f_{a,d} = 0.
\]
This equation shows that in $S_x$ the 2-minor $f_{a,b}$ of the big interval $[a,b]$ is a linear combination of the 2-minors $f_{a,d}$ and $f_{c,b}$ of the smaller intervals $[a,d]$ and $[c,b]$.

![Figure 5. Vertical splitting of an interval](image)

Now one proves the claim by showing that, for each inner interval $[a, b]$ of $\mathcal{P}$, the 2-minor $f_{a,b}$ is in $S_x$ a linear combination of the 2-minors corresponding to cells of $[a, b]$. One proceeds by induction on the number of cell columns. If there is only one column, then one proceeds by the length of this column. If this column consists of only one cell, then the job is done. If the column consists of more than one column, then the column can be splitted into two shorter columns as indicated in Figure 6.

![Figure 6. Horizontal splitting of an interval](image)

Then similarly as in the vertical case, in $S_x$ the 2-minor of the whole column is a linear combination of the 2-minors of the two shorter columns. The induction hypothesis guarantees that in $S_x$ each of the two shorter columns is a linear combination of the 2-minors of their inner cells. It then follows that in $S_x$ the 2-minor of the whole column is a linear combination of the 2-minors of its cells.

Now suppose that $[a, b]$ consists of more than one column. Then one splits $[a, b]$ as shown in Figure 5 and applies the similar induction argument as was done for the first column of $[a, b]$, in order to deduce that, in $S_x$, the inner 2-minor $f_{a,b}$ is a linear combination of the 2-minors corresponding to the cells of $[a, b]$. □

Recall from [8] the following definitions and facts: let $\mathcal{P}$ be a polyomino. An interval $[a, b]$ of $\mathcal{P}$ with $a = (i, j)$ and $b = (k, \ell)$ is called a horizontal edge interval of $\mathcal{P}$ if $j = \ell$ and if each $\{(r, j), (r + 1, j)\}$ for $i \leq r < k$ is an edge of a cell of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then it is called maximal. Similarly one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$. Let $h(\mathcal{P})$ denote the number of maximal horizontal edge intervals of $\mathcal{P}$ and $v(\mathcal{P})$ the number of maximal vertical edge intervals of $\mathcal{P}$. In [8, Theorem 2.2] it is shown that if $\mathcal{P}$ is a simple polyomino
Then \( K[\mathcal{P}] \) is isomorphic to the edge ring \( K[G(\mathcal{P})] \), where \( G(\mathcal{P}) \) is the bipartite graph with vertex decomposition \( V(G(\mathcal{P})) = V_1 \cup V_2 \) with \( |V_1| = h(\mathcal{P}) \) and \( |V_2| = v(\mathcal{P}) \), and \( |E(G(\mathcal{P}))| = |V(\mathcal{P})| \). This implies in particular, that \( \mathcal{P} \) is a prime polyomino, i.e., \( K[\mathcal{P}] \) is an integral domain.

**Corollary 5.2.** Let \( \mathcal{P} \) be a simple polyomino. Then

\[
\text{height } I_\mathcal{P} = |V(\mathcal{P})| - (h(\mathcal{P}) + v(\mathcal{P}) - 1).
\]

**Proof.** It is known \([6]\) that \( \dim K[G] = n - 1 \) for a bipartite graph with \( n \) vertices. Applied to our case it follows that

\[
\text{height } I_\mathcal{P} = \text{emb dim } K[\mathcal{P}] - \dim K[\mathcal{P}]
= \text{emb dim } K[G(\mathcal{P})] - \dim K[G(\mathcal{P})]
= |V(\mathcal{P})| - (h(\mathcal{P}) + v(\mathcal{P}) - 1),
\]

as desired. \( \square \)

Together with Theorem 5.1 one can now obtain

**Corollary 5.3.** Let \( \mathcal{P} \) be a simple polyomino and suppose that there exist \(|\mathcal{P}| \) inner 2-minors, whose initial monomials form a regular sequence. Then \( \mathcal{P} \) is of König type and

\[
|\mathcal{P}| = |V(\mathcal{P})| - (h(\mathcal{P}) + v(\mathcal{P}) - 1).
\]

Let \( \mathcal{P} \) is a polyomino and \( C = [a, b] \) a cell of \( \mathcal{P} \) for which one of the vertices of \( C \) is a boundary vertex of \( \mathcal{P} \). Let \( a = (i, j) \) and \( b = (i + 1, j + 1) \). Let \( c = (i, j + 1) \) and \( d = (i + 1, j) \) be the anti-diagonal corners of \( [a, b] \). Let, say, \( d \) and \( b \) be boundary vertices of \( \mathcal{P} \). Let \( \mathcal{P}' \) denote the new polyomino which is obtained by adding the new cell \([d, e]\) to \( \mathcal{P} \), where \( e = (i + 2, j + 1) \). It then follows that height \( I_{\mathcal{P}'} = \text{height } I_\mathcal{P} + 1 \). Now, suppose that there is a sequence \( f_{a_1, b_1}, \ldots, f_{a_h, b_h} = f_{a, b} \) together with a lexicographic order \( < \) on \( S \) for which \( \text{in}_< (f_{a_1, b_1}), \ldots, \text{in}_<(f_{a_h, b_h}) \) form a regular sequence.

**Figure 7.** Polyominoes \( \mathcal{P} \) and \( \mathcal{P}' \)

**Lemma 5.4.** Following the above situation and suppose that \( \text{in}_< (f_{a, b}) = x_a x_b \). Then there is a lexicographic order \( <' \) on \( S[x_{d'}, x_e] \), where \( d' = (i + 2, j) \), for which

\[
\text{in}_{<'} (f_{a_1, b_1}), \ldots, \text{in}_{<'} (f_{a_{h-1}, b_{h-1}}), x_a x_e = \text{in}_{<'} (f_{a, e}), x_b x_{d'} = \text{in}_{<'} (f_{d, e})
\]

form a regular sequence.
Imitating the proofs of the Lemmata 3.2 and 3.3, one can prove Lemma 5.4 easily. Furthermore, slight modifications of Lemma 5.4, for example, when a and d are boundary vertices of P and in_<(f_{a,b}) = x_c x_d, can be valid.

A vertex a of a polyomino P is called free if a belongs to exactly one cell of P. A cell C of a polyomino P is called a leaf of P if two of the vertices of C are free. A simple polyomino P is called a tree if P possesses a leaf and if no inner interval of P is of the form [a, a + (2, 2)]. It follows from repeated application of Lemma 5.4 that

**Corollary 5.5.** Every tree P is of König type with height $I_P = |P|$. 

![Figure 8. A tree polyomino](image)

**Example 5.6.** The polyomino P of Figure 9 is of König type with respect to the sequence $f_{1,1'}, \ldots, f_{16,16'}$ and the lexicographic order < on S induced by the ordering $1 > \cdots > 16 > 1' > \cdots > 16'$. 

![Figure 9. A cycle polyomino](image)

It would be of interest to classify the polyominoes which are of König type.

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