BEST APPROXIMATION WITH WAVELETS IN WEIGHTED ORLICZ SPACES

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Abstract. Democracy functions of wavelet admissible bases are computed for weighted Orlicz Spaces $L^\Phi(w)$ in terms of the fundamental function of $L^\Phi(w)$. In particular, we prove that these bases are greedy in $L^\Phi(w)$ if and only if $L^\Phi(w) = L^p(w)$, $1 < p < \infty$. Also, sharp embeddings for the approximation spaces are given in terms of weighted discrete Lorentz spaces. For $L^p(w)$ the approximation spaces are identified with weighted Besov spaces.

1. Introduction

Let $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ be a quasi-Banach space with a countable unconditional basis $\mathcal{B} = \{ e_j : j \in \mathbb{N} \}$; that is, every $x \in \mathcal{B}$ can be uniquely represented as an unconditionally convergent series $x = \sum_{j \in \mathbb{N}} s_j e_j$, for some sequences of scalars $\{ s_j : j \in \mathbb{N} \}$. Let $\Sigma_N$ denote the set of all elements $y \in \mathcal{B}$ with at most $N$ non-null coefficients in the basis representation $y = \sum_{j \in \mathbb{N}} s_j e_j$. For $x \in \mathcal{B}$, the $N$-term error of approximation (with respect to $\mathcal{B}$) is defined by

$$\sigma_N(x)_\mathcal{B} \equiv \inf_{y \in \Sigma_N} \| x - y \|_\mathcal{B}.$$  

(1.1)

Two main questions in approximation theory concern the construction of efficient algorithms for $N$-term approximation and the characterization of the approximation spaces $A^q_\alpha(\mathcal{B}, \mathcal{B})$, which consists of all $x \in \mathcal{B}$ such that the quantity

$$\| x \|_{A^q_\alpha(\mathcal{B}, \mathcal{B})} = \begin{cases} \left( \sum_{N \geq 1} (N^\alpha \sigma_N(x)_\mathcal{B})^q \frac{1}{N} \right)^{\frac{1}{q}}, & \text{if } 0 < q < \infty; \\ \sup_{N \geq 1} [N^\alpha \sigma_N(x)_\mathcal{B}], & \text{if } q = \infty, \end{cases}$$

(1.2)

is finite. A computational efficient method to produce $N$-term approximations, which has been widely investigate in recent years, is the so called greedy algorithm (see e.g [21]). If $x = \sum_{j \in \mathbb{N}} s_j e_j$ and we order the basis elements in such a way that

$$\| s_{j_1} e_{j_1} \|_\mathcal{B} \geq \| s_{j_2} e_{j_2} \|_\mathcal{B} \geq \ldots$$

(1.3)
(handling ties arbitrarily), the **greedy algorithm of step** \(N\) is defined by the correspondence

\[
x = \sum_{j \in \mathbb{N}} s_j e_j \in \mathbb{B} \rightarrow G_N(x) = \sum_{k=1}^{N} s_{jk} e_{jk} \in \Sigma_N.
\]

(1.4)

S.V. Konyagin and V. N. Temlyakov (21) defined the basis \(B\) to be **greedy** in \((\mathbb{B}, \| \cdot \|_\mathbb{B})\) if the greedy algorithm is optimal in the sense that \(G_N(x)\) is essentially the best \(N\)-term approximation to \(x\) using the basis vectors, i.e., there exists a constant \(C\) such that for all \(x \in \mathbb{B}\) we have

\[
\|x - G_N(x)\|_\mathbb{B} \leq C \sigma_N(x)_\mathbb{B}, \quad N = 1, 2, \ldots.
\]

Thus, for such bases the greedy algorithm produces an almost optimal \(N\)-term approximation, which leads often to a precise identification of the approximation spaces \(A^c_n(B, \mathbb{B})\). In [21] greedy basis in a quasi-Banach space \((\mathbb{B}, \| \cdot \|_\mathbb{B})\) are characterized as those which are unconditional and **democratic**, the latter meaning that there exists some constant \(\Delta > 0\) such that

\[
\left\| \sum_{j \in \Gamma'} \frac{e_j}{\| e_j \|_\mathbb{B}} \right\|_\mathbb{B} \leq \Delta \left\| \sum_{j \in \Gamma} \frac{e_j}{\| e_j \|_\mathbb{B}} \right\|_\mathbb{B},
\]

holds for all finite sets of indices \(\Gamma, \Gamma' \subset \mathbb{N}\) with the same cardinality. Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, V.N. Temlyakov showed in [31] that the Haar basis (and any wavelet system \(L^p\)-equivalent to it) is greedy in the Lebesgue space \(L^p([0, 1])\) for \(1 < p < \infty\). When wavelet have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Triebel-Lizorkin classes (see e.g [17, 13]).

The purpose of this paper is to study the efficiency of wavelet greedy algorithms in the weighted Orlicz spaces \(L^\Phi(w)\) defined for functions on \(\mathbb{R}^d\). In Theorem 2.2 (see section 2) we show that wavelet bases are unconditional in weighted Orlicz spaces \(L^\Phi(w)\) with nontrivial Boyd indices for all \(w \in A_{\Phi^\infty}(\mathbb{R}^d)\). We give in section 3 a simple proof of the fact that admissible wavelet bases (see definition below) are not democratic in weighted Orlicz spaces \(L^\Phi(w)\) if \(L^\Phi(w) \neq L^p(w)\).

In view of this result it have interest to ask how far wavelet bases are from being democratic in \(L^\Phi(w) \neq L^p(w)\). To quantify democracy of a basis \(B = \{ e_j \}_{j \in \mathbb{N}}\) we shall study the following functions:

\[
h_r(N; B, \mathbb{B}) = \sup_{\text{Card}(\Gamma) = N} \left\| \sum_{\gamma \in \Gamma} \frac{e_{\gamma}}{\| e_{\gamma} \|_\mathbb{B}} \right\|_\mathbb{B} \quad \text{and} \quad h_l(N; B, \mathbb{B}) = \inf_{\text{Card}(\Gamma) = N} \left\| \sum_{\gamma \in \Gamma} \frac{e_{\gamma}}{\| e_{\gamma} \|_\mathbb{B}} \right\|_\mathbb{B}
\]

which we call right and left democracy functions of \(B\) (see also [8, 13, 14]). Observe that a basis is democratic if and only if these two quantities are comparable for all \(N \geq 1\). Our main result gives a precise value (except for multiplicative constants) of these functions in terms of intrinsic properties of the space \(L^\Phi(w)\). Namely, let

\[
h^+_\varphi(t) = \sup_{s > t} \frac{\varphi(s)}{\varphi(t)}
\]

denote the dilation function associated with the **fundamental function** \(\varphi\) of \(L^\Phi (w)\), and let \(h^-_\varphi(t)\) be the same quantity with "sup" replaced by "inf" (see Section 2 for precise definitions).
Theorem 1.1. Let $L^p(w)$ be a weighted Orlicz space with non-trivial Boyd indices, $w \in A_p$, a weight on $\mathbb{R}^d$, and $\mathcal{B} = \{ \psi_Q : Q \in \mathcal{D} \}$ be an admissible wavelet basis. Then for all $\Gamma \subset \mathcal{D}$

\[ h_r(N; \mathcal{B}, L^p(w)) \approx h_r^-(N), \quad h_l(N; \mathcal{B}, L^p(w)) \approx h_r^+(N). \tag{1.5} \]

(Here $p^* = \frac{1}{p}$, where $I_p$ is the upper Boyd index of $L^p(w)$. See definition of Boyd indices in subsection 2.1.)

This result will have applications in the study of approximation spaces (defined using admissible wavelet basis) in weighted Orlicz spaces. We take up this task in the section 4, where we investigate Jackson and Bernstein type estimates and corresponding inclusions for $N$-term approximation spaces. In the $L^p$ case, these estimates are naturally given in terms of the class of discrete Lorentz spaces $\ell^{r,q}$ (see [6, 13, 15, 17, 19]). In the case of weighted Orlicz spaces we shall need weighted Lorentz sequence spaces $\Lambda^q_\eta$, defined by

\[ \Lambda^q_\eta = \left\{ s : \|s\|_{\Lambda^q_\eta} = \left[ \sum_{k \geq 1} (\eta_k s_k^*)^q \frac{1}{k^q} \right]^{\frac{1}{q}} < \infty \right\}. \tag{1.6} \]

where $\{s_k^*\}$ is the non-increasing rearrangement of $s$ and the weight $\eta = \{\eta_k\}$ is a fixed increasing and doubling sequence (see [14]). In particular, $\Lambda^q_\eta = \ell^{r,q}$ when $\eta_k = k^{1/r}$.

For $f \in L^p(w)$, and $\mathcal{B} = \{ \psi_Q : Q \in \mathcal{D} \}$ a wavelet basis in $L^p(w)$, write $f = \sum_{Q \in \mathcal{D}} \langle f, \psi_Q \rangle \psi_Q$. Then we define $\Lambda^q_{\eta}(\mathcal{B}, L^p(w))$ as the set of all $f \in L^p(w)$ such that the sequence $\{ \|\langle f, \psi_{Q_k} \rangle \psi_{Q_k} \|_{L^p(w)} : k \geq 1 \} \subset \Lambda^q_{\eta}$ and

\[ \| f \|_{\Lambda^q_{\eta}(\mathcal{B}, L^p(w))} = \left\| \|\langle f, \psi_{Q_k} \rangle \psi_{Q_k} \|_{L^p(w)} \| \right\|_{\Lambda^q_{\eta}} \]

where $\|\langle f, \psi_{Q_k} \rangle \psi_{Q_1} \|_{L^p(w)} \geq \|\langle f, \psi_{Q_2} \rangle \psi_{Q_2} \|_{L^p(w)} \geq \ldots$ (handling ties arbitrarily).  

Theorem 1.2. Let $L^p(w)$ be a weighted Orlicz space with Boyd indices $0 < \iota_\varphi \leq \iota_p < 1$, and $w \in A_p$, a weight on $\mathbb{R}^d$, where $p^* = \frac{1}{\alpha}$. Then

\[ \Lambda^{\alpha}_{\iota_\varphi h_r(k)}(\mathcal{B}, L^p(w)) \hookrightarrow \mathcal{A}^{\alpha}_{\iota_p h_l(k)}(\mathcal{B}, L^p(w)) \hookrightarrow \Lambda^{\alpha}_{\iota_\varphi h_l(k)}(\mathcal{B}, L^p(w)), \tag{1.7} \]

These embeddings are optimal, in the sense that the largest and smallest weighted Lorentz spaces $\Lambda^{\alpha}_{\iota_\varphi x(k)}(\mathcal{B}, L^p(w))$ that one can place on the left- and right-hand side of (1.7) are respectively $\Lambda^{\alpha}_{\iota_\varphi h_r(k)}(\mathcal{B}, L^p(w))$ and $\Lambda^{\alpha}_{\iota_\varphi h_l(k)}(\mathcal{B}, L^p(w))$ (see section 4). We point out that a sufficient condition for these two spaces to be equal is that $h_r(N) \approx h_l(N)$, in which case the basis is necessarily democratic and $L^p(w) = L^p(w)$ (see Lemma 5.2 in [14]). Then Theorem 1.2 leads the following identification of Approximation spaces for $L^p(w)$ in terms of classical Lorentz spaces.

Corollary 1.3. Let $\alpha > 0$, $1 < p < \infty$, $0 < q \leq \infty$, and $w \in A_p$ a weight on $\mathbb{R}^d$. Then, for a wavelet basis $\mathcal{B}$, we have

\[ \mathcal{A}^{\alpha}_{\iota_p}(\mathcal{B}, L^p(w)) = \ell^{r,q}(\mathcal{B}, L^p(w)), \quad \frac{1}{r} = \alpha + \frac{1}{p}. \tag{1.8} \]

Finally we point out that the inclusions in Corollary 1.3 can be described in terms of weighted Besov spaces ($[29, 30]$), namely

\[ \hat{B}_{p,q}^{\alpha}(w) = \{ f \in \mathcal{D} : \| f \|_{L^p(w)} \in \ell_q(\mathbb{Z}) \}. \tag{1.9} \]
Theorem 1.4. Let $\gamma > 0$, $1 < p < \infty$, and $w \in A_p$ a weight on $\mathbb{R}^d$. Suppose that $B = \Psi$ is a family of $d$-dimensional Lemarié-Meyer wavelets or a family of $d$-dimensional compactly supported Daubechies $D_N$ wavelets with $N$ sufficiently large. Then

$$A^\gamma(\Psi, L^p(w)) = B^\gamma(w^\tau) \quad \text{whenever} \quad \frac{1}{\tau} = \frac{\gamma}{d} + \frac{1}{p}. \quad (1.10)$$

The organization of this article is as follows. Basic facts concerning weights, wavelet bases and greedy bases are given in section 2. Section 3 is devoted to prove Theorem 1.1. Jackson and Bernstein type estimates, as well as the inclusions described in Theorem 1.2 are proved in section 4. Corollary 1.3 and Theorem 1.4 are proved in section 5.

2. Preliminaries

2.1. Basics in weighted Orlicz spaces. In this subsection we recall some basic facts about weights, weighted Orlicz spaces and wavelet bases on weighted Orlicz spaces, referring to [2, 11] for a complete account on these topics. By a weight on a given measure space, we shall always mean a non-negative locally integrable function $w$ with values in $[0, \infty)$ a.e. Let $w(x)$ be a weight on $\mathbb{R}^d$, and for a measurable $Q \subset \mathbb{R}^d$ write $w(Q) = \int_Q w(x)dx$. We say that $w \in A_p = A_p(\mathbb{R}^d)$, $(1 < p < \infty)$ if there exists a constant $C_w$ such that

$$\left(\frac{1}{|Q|} \int_Q w(x)dx\right)^p \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p}}dx\right)^{p-1} \leq C_w, \quad (2.1)$$

for all $Q \subset \mathbb{R}^d$, where $|Q|$ denote the usual Lebesgue measure of $Q$. The condition $A_1$ can be viewed as limiting case of the condition $A_p$ for $p \downarrow 1$, i.e., (2.1) is viewed as

$$\left(\frac{1}{|Q|} \int_Q w(x)dx\right) \text{ess} \sup\{w^{-1}\} \leq C_w. \quad (2.2)$$

If $w \in A_p$ for some $p \in [1, \infty)$, then there exist $C_w^1, C_w^2 > 0$ and $\delta > 0$ such that

$$C_w^g \left(\frac{|A|}{Q}\right)^p \leq w(A) \leq C_w^s \left(\frac{|A|}{Q}\right)^\delta \quad (2.3)$$

for all subsets $A \subset Q$. (For the left hand inequality take $f = \chi_A$ in part b) of Theorem 2.1, Chapter IV, of [11]; for the right hand inequality see Theorem 2.9, Chapter IV, of [11]).

A Young function is a convex non-decreasing function $\Phi : [0, \infty) \rightarrow [0, \infty]$ so that $\lim_{t \rightarrow +\infty} \Phi(t) = \infty$. Throughout this paper we shall assume that $\Phi(0) = 0$, $\Phi$ is strictly increasing and everywhere finite, so that it is a continuos bijection of $[0, \infty)$. Given such $\Phi$ and $w \in A_\infty = \cup_{p \geq 1} A_p$, the weighted Orlicz space $L^\Phi(w)$ is the class of all measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ so that $\Phi\left(\frac{|f(x)|}{\lambda}\right) \in L^1(w)$ for some $\lambda > 0$. The space $L^\Phi(w)$ becomes a weighted rearrangement invariant Banach function space when endowed with the corresponding Luxemburg norm

$$\|f\|_{L^\Phi(w)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)dx \leq 1 \right\}. \quad (2.4)$$
It is not difficult to prove that if $E \subset \mathbb{R}^d$ is any measurable set
\[
\|\chi_E\|_{L^p(w)} = \frac{1}{\Phi^{-1}(w(E))}.
\]
(2.5)
The function $\varphi(t) = \frac{1}{\Phi^{-1}(\frac{t}{w(E)})}$, $0 < t < \infty$, satisfies $\varphi(t) = \|\chi_E\|_{L^p(w)}$ for any measurable set $E \subset \mathbb{R}^d$ such that $w(E) = t$, and it is called the fundamental function of $L^\Phi(w)$.

The Boyd indices of the weighted Orlicz space $L^\Phi(w)$ can be computed directly from the Young function $\Phi$ or from the fundamental function $\varphi$. Set
\[
h_\varphi^+(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)}, \quad 0 < t < \infty.
\]
(2.6)
Then, the lower and upper Boyd indices $i_\varphi$ and $I_\varphi$ of $L^\Phi(w)$ are given by
\[
i_\varphi = \lim_{t \to 0^+} \frac{\log h_\varphi^+(t)}{\log t} = \sup_{0 < t \leq 1} \frac{\log h_\varphi^+(t)}{\log t}
\]
and
\[
I_\varphi = \lim_{t \to \infty} \frac{\log h_\varphi^+(t)}{\log t} = \inf_{1 < t < \infty} \frac{\log h_\varphi^+(t)}{\log t}
\]
(2.8)
respectively (see [2], p. 277 or [22], p. 54). It is known that $0 \leq i_\varphi \leq I_\varphi \leq 1$ (see Proposition 5.15 of [2], p. 149). Assuming further that $i_\varphi > 0$ it follows that
\[
\varphi(st) \leq C_\epsilon \max\{s^{i_\varphi - \epsilon}, s^{I_\varphi + \epsilon}\}\varphi(t), \quad s, t > 0
\]
(2.9)
and
\[
\varphi(st) \geq C_\epsilon \min\{s^{i_\varphi - \epsilon}, s^{I_\varphi + \epsilon}\}\varphi(t), \quad s, t > 0
\]
(2.10)
for every $\epsilon > 0$ and some constant $C_\epsilon > 0$ (see [20], p. 3). In this paper we shall only consider weighted Orlicz spaces with non trivial Boyd indices, that is $0 < i_\varphi \leq I_\varphi < 1$.

**Example 2.1.** When $\Phi(t) = t^p$, $1 \leq p < \infty$, then $L^\Phi(w) = L^p(w)$ and $\varphi(t) = t^\frac{1}{p}$. Hence, $h_\varphi^+(t) = t^\frac{1}{p}$, which implies $i_\varphi = I_\varphi = \frac{1}{p}$.

### 2.2. Wavelet bases and weighted Orlicz spaces.

Let $D = \{Q_{j,k} = 2^{-j}(0,1)^d + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ denote the set of all dyadic cubes in $\mathbb{R}^d$. We say that a finite collection of functions $\{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^d)$ is an orthonormal wavelet family if the system
\[
\left\{\psi^l_{Q_{j,k}}(x) = 2^\frac{jd}{2} \psi^l(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{R}^d, l = 1, \ldots, L\right\},
\]
(2.11)
forms an orthonormal basis of $L^2(\mathbb{R}^d)$. We will say that the wavelet family is admissible if for all $1 < p < \infty$,
\[
\|S_\psi f(.)\|_{L^p(\mathbb{R}^d)} \approx \|f(.)\|_{L^p(\mathbb{R}^d)},
\]
(2.12)
where
\[
S_\psi f(.) = \left(\sum_{l=1}^{L} \sum_{l \in D} |\langle f(.) , \psi_l(.) \rangle|^2 \chi_l(.) |I|^{-1}\right)^{\frac{1}{2}}.
\]
(2.13)
This implies that wavelet admissible bases are unconditional in $L^p(\mathbb{R}^d)$, $1 < p < \infty$. The reader can consult [5, 16, 23, 25], for constructions, examples, and properties of
orthonormal wavelets. Admissible wavelets include the $d$-dimensional Haar system, wavelets arising from multiresolution analysis (see [25], p. 22), wavelets belonging to the regularity class $R^d$ (as defined in [16], p. 64 for $d = 1$), and actually any orthonormal wavelet in $L^2(\mathbb{R}^d)$ with mild decay conditions (see [32, 28]).

In the following result we prove that wavelet admissible basis are also unconditional basis of weighted Orlicz spaces $L^p(w)$, for appropriate $w$, since the norm can be characterize in terms of a square function. Without loss of generality we assume $L = 1$ in the rest of this work.

**Theorem 2.2.** Let $L^p(w)$ be a weighted Orlicz space, with the Boyd indices satisfying $0 < i_p \leq I_p < 1$, and $\mathcal{B} = \{\psi_Q : Q \in \mathcal{D}\}$ an admissible wavelet basis. Then, if $w \in A_{p*}(\mathbb{R}^d)$, where $p^* = \frac{1}{I_p}$, we have

$$\|f(.)\|_{L^p(w)} \simeq \|S_\psi f(.)\|_{L^p(w)}, \quad \text{for all } f \in L^p(w). \quad (2.14)$$

For the proof we shall use the following extrapolation theorem adapted to our situation.

**Theorem 2.3.** (H) Let $\mathcal{F}$ be a family of couples of measurable non-negative functions $(f,g)$. Suppose that for some $1 \leq p_0 < \infty$, and every weight $w \in A_{p_0}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x)^{p_0}w(x)dx \leq C \int_{\mathbb{R}^d} g(x)^{p_0}w(x)dx, \quad \text{for all } (f,g) \in \mathcal{F}. \quad (2.15)$$

Then, if $L^p(w)$ is a weighted Orlicz space such that the Boyd indices satisfies, $0 < i_p \leq I_p < 1$ and $w \in A_{p*}(\mathbb{R}^d)$, $p^* = \frac{1}{I_p}$, we have

$$\|f\|_{L^p(w)} \leq C\|g\|_{L^p(w)}, \quad \text{for all } (f,g) \in \mathcal{F}. \quad (2.16)$$

**Proof.** (of Theorem 2.2) It is proved in [12] (see also [H]) that

$$\|f\|_{L^p(w)} \simeq \|S_\psi f(.)\|_{L^p(w)}, \quad (2.17)$$

for all $1 < p < \infty$ and $w \in A_p$. We consider the family $\mathcal{F} = \{(f,S_\psi(f)) : S_\psi(f) \in L^p(w)\}$. From the equivalence (2.17), we obtain

$$\int_{\mathbb{R}^d} |f(x)|^p w(x)dx \leq C_1 \int_{\mathbb{R}^d} |S_\psi(f)|^p w(x)dx$$

for all $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d)$. Then, by Theorem 2.3 we obtain

$$\|f\|_{L^p(w)} \leq C_1\|S_\psi(f)\|_{L^p(w)}$$

when $w \in A_{p*}$. The other inequality is proved similarly taking $\mathcal{F} = \{(S_\psi(f),|f|), f \in L^p(w)\}$. \hfill \Box

2.3. **Greedy basis and democracy.** We defined in the introduction the notion of greedy basis in a quasi-normed Banach space $\langle \mathbb{B}, \| \cdot \|_\mathbb{B} \rangle$. We also mentioned the result of Konyagin and Temlyakov [21] characterizing greedy bases as those which are unconditional and democratic. For simplicity, given a basis $\mathcal{B} = \{e_j : j \in \mathbb{N}\}$ in $\mathbb{B}$ we shall denote the (normalized) characteristic function of a finite set of indices $\Gamma \in \mathbb{N}$ by

$$\tilde{1}_\Gamma = \tilde{1}_\Gamma^{\mathbb{B},\mathbb{B}} = \sum_{j \in \Gamma} \frac{e_j}{\|e_j\|_\mathbb{B}}.$$
The basis \( \mathcal{B} \) is democratic in \( \mathbb{B} \) if there exists \( C \geq 1 \) such that

\[
\| \tilde{1}_\Gamma \|_{\mathbb{B}} \leq C \| 1_\Gamma \|_{\mathbb{B}}
\]  

(2.18)

for all finite sets of indices \( \Gamma, \Gamma' \subset \mathbb{N} \) with \( \# \Gamma = \# \Gamma' \) (the symbol \( \# \Gamma \) denotes the cardinality of the set \( \Gamma \)). Quite often one can show democracy by finding a function \( h: \mathbb{N} \rightarrow \mathbb{R}^+ \) for which

\[
\frac{1}{C} h(\#\Gamma) \leq \| \tilde{1}_\Gamma \|_{\mathbb{B}} \leq Ch(\#\Gamma), \quad \forall \ \Gamma \subset \mathbb{N}, \text{ finite}.
\]  

(2.19)

In the case of wavelet bases, many classical function and distribution spaces satisfy (2.19) with \( h(\#\Gamma) = (\#\Gamma)^{\delta} \). Indeed, this is the situation for the Lebesgue spaces \( L^p(\mathbb{R}^d) \) when \( 1 < p < \infty \), for the Hardy spaces \( H^p(\mathbb{R}^d) \), \( 0 < p \leq 1 \) and for the Sobolev spaces \( \dot{W}^{s,p}(\mathbb{R}^d) \), \( 1 < p < \infty \) (see [17]), and more generally for the family of Triebel-Lizorkin spaces \( \dot{F}^{s,p}_q(\mathbb{R}^d) \) with \( 0 < p < \infty \), \( s \in \mathbb{R} \), \( 0 < r \leq \infty \) (under the usual smoothness assumptions, and with the standard modification of the basis in the case of inhomogeneous spaces; see [13]). Thus, wavelet bases are democratic and hence greedy in all these spaces.

The Haar system is not greedy in rearrangement invariant spaces defined in \([0,1]\) other than \( L^p[0,1] \) (see [33]). Moreover, wavelet bases are not democratic in other classical spaces, such as \( \text{BMO} \), the Besov spaces \( \dot{B}^{s,p}_q(\mathbb{R}^d) \) with \( p \neq q \), Orlicz spaces \( L^\Phi(\mathbb{R}^d) \) distinct from \( L^p(\mathbb{R}^d) \), and as we shall see below, weighted Orlicz spaces \( L^\Phi(w) \) distinct from \( L^p(w) \).

**Definition 2.4.** Let \( \mathcal{B} \) be a collection of elements in a quasi-Banach space \( \mathbb{B} \). The right-democracy function associated with \( \mathcal{B} \) is defined by

\[
h_r(N; \mathbb{B}, \mathcal{B}) = \sup_{\text{Card}(\Gamma) = N} \| \tilde{1}_\Gamma \|_{\mathbb{B}};
\]  

(2.20)

analogously, the left-democracy function associated with \( \mathcal{B} \) is defined by

\[
h_l(N; \mathbb{B}, \mathcal{B}) = \inf_{\text{Card}(\Gamma) = N} \| \tilde{1}_\Gamma \|_{\mathbb{B}}
\]  

(2.21)

Observe that a basis \( \mathcal{B} \) is democratic in \( \mathbb{B} \) if and only if, \( h_r(N; \mathbb{B}, \mathcal{B}) \leq Ch_l(N; \mathbb{B}, \mathcal{B}) \) for all \( N \geq 1 \) and some \( C > 0 \).

We want to show that, in general, admissible wavelet bases are not democratic in weighted Orlicz spaces. In order to do so one needs to estimate \( \| \tilde{1}_\Gamma \|_{L^\Phi(w)} \) in terms of \( \#\Gamma \). This can be done when \( \Gamma \) is a collection of pairwise disjoint dyadic cubes \( \{Q_j\}_{j=1}^N \), such that \( w(Q_j) \approx \tau \), for any \( \tau > 0 \).

We state and prove the following results.

**Lemma 2.5.** Let \( w \in A_\infty(\mathbb{R}^d) \) be a weight. If \( \{Q_k\}_{k=-\infty}^\infty \) is a family of dyadic cubes such that \( Q_k \subset Q_{k+1} \) and \( |Q_{k+1}| = 2^d |Q_k| \) for all \( k \in \mathbb{Z}^+ \), then

\[
\lim_{k \to \infty} w(Q_k) = \infty \quad \text{and} \quad \lim_{k \to -\infty} w(Q_k) = 0.
\]  

(2.22)

Proof. Because \( w \in A_\infty \), if \( k \geq 0 \), by (2.23) we obtain

\[
\frac{w(Q_0)}{w(Q_k)} \leq C_w \left( \frac{|Q_0|}{|Q_k|} \right)^\delta = C_w \left( \frac{1}{2^{kd}} \right)^\delta.
\]
Then, \( w(Q_k) \geq (C_{w})^{-1}2^{kd\delta}w(Q_0) \) and \( \lim_{k \to \infty} w(Q_k) = \infty \). On the other hand, if \( k \leq 0 \), by (2.3) we obtain
\[
\frac{w(Q_k)}{w(Q_0)} \leq C_w \left( \frac{|Q_k|}{|Q_0|} \right)^\delta = C_w 2^{kd\delta}.
\]
Then, \( w(Q_k) \leq C_w 2^{kd\delta} w(Q_0) \) and \( \lim_{k \to -\infty} w(Q_k) = 0 \).

**Lemma 2.6.** Let \( w \in A_\infty(\mathbb{R}^d) \) be a weight. Given \( \tau > 0 \) there exists a pairwise disjoint sequence of cubes \( \{R_j\}_{j=1}^{\infty} \subset \mathcal{D} \) such that
\[
C \tau < w(R_j) \leq \tau
\]
where \( C > 0 \) is a constant depending only on \( w \).

**Proof.** Let \( Q_k = [0, 2^k)^d \), \( k \in \mathbb{Z} \). By lemma 2.5 there exists \( k_1 \in \mathbb{Z} \) such that
\[
w(Q_{k_1}) \leq \tau < w(Q_{k_1+1}).
\]
Choose \( R_1 = Q_{k_1} \). We have \( w(R_1) = w(Q_{k_1}) \leq \tau \). On the other hand, by (2.3), we obtain
\[
\frac{w(Q_{k_1})}{w(Q_{k_1+1})} \geq C_w \left( \frac{|Q_{k_1}|}{|Q_{k_1+1}|} \right)^p = C_w 2^{-dp},
\]
so that
\[
w(R_1) = w(Q_{k_1}) \geq C_w 2^{-dp} w(Q_{k_1+1}) > C_w 2^{-dp} \tau.
\]
Thus, we can take \( C = C_w 2^{-dp} \).

Suppose we have chosen disjoint cubes \( R_1, R_2, \ldots, R_{m-1} \) such that \( C \tau < w(R_j) \leq \tau \) for all \( j = 1, 2, \ldots, m-1 \). Without loss of generality we can assume that all the \( R_j \) are contained in the positive cone of \( \mathbb{R}^d \), that is, the set of points of \( \mathbb{R}^d \) with non-negative coordinates.

Choose \( Q_0 = 2^{km}[0,1)^d \), \( k_m \in \mathbb{Z} \), such that \( R_j \subset Q_0 \) for all \( j = 1, 2, \ldots, m-1 \). Consider the increasing family of dyadic cubes given by \( Q_k = 2^{km+k}[0,1)^d \), \( k = 0, 1, 2, \ldots \). Let \( \tilde{Q}_k \), \( k = 1, 2, \ldots \), be a dyadic cube contained in \( Q_k \) such that \( |Q_k| = |\tilde{Q}_k| \) and \( \tilde{Q}_k \cap Q_{k-1} = \emptyset \). If \( w(Q_k) \leq \tau \) for all \( k = 1, 2, 3, \ldots \) by (2.3) we obtain
\[
\frac{w(\tilde{Q}_k)}{w(Q_k)} \geq C_w \left( \frac{|\tilde{Q}_k|}{|Q_k|} \right)^p = C_w 2^{-dp}.
\]
Thus, \( w(Q_k) \leq (C_w)^{-1}2^{dp} \tau \) for all \( k = 1, 2, \ldots \) contradicting lemma 2.5. Thus, there exists \( k_0 \in \mathbb{Z} \) such that \( w(\tilde{Q}_{k_0}) \). Consider a family of descendants of the dyadic cube \( \tilde{Q}_{k_0} \). By lemma 2.5 there exists \( \tilde{Q}_{k_m} \), \( \tilde{Q}_{k_m} \subset \mathcal{D} \) such that
\[
w(\tilde{Q}_{k_m}) \leq \tau < w(\tilde{Q}_{k_m})
\]
and \( |\tilde{Q}_{k_m}| = \frac{|\tilde{Q}_{k_m}|}{2^d} \). Choose \( R_m = \tilde{Q}_{k_m} \). Since (2.24) is the same relation as (2.23) it follows that
\[
C_w 2^{-dp} \tau < w(R_m) \leq \tau.
\]
Observe that \( R_m \) has been chosen in the positive cone of \( \mathbb{R}^d \) and is disjoint to \( R_1, \ldots, R_{m-1} \). \( \square \)
Proposition 2.7. Let \( L^\Phi(w) \) be a weighted Orlicz space with Boyd indices \( 0 < i_\varphi \leq I_\varphi < 1 \), \( w \in \mathcal{A}_{i_\varphi} \) a weight in \( \mathbb{R}^d \), and let \( \mathcal{B} = \{\psi_Q : Q \in \mathcal{D}\} \) be an admissible wavelet basis.

i) If \( \Gamma = \{Q_1, \ldots, Q_N\} \subset \mathcal{D} \) is a pairwise disjoint family then

\[
\|\tilde{1}_\Gamma\|_{L^\Phi(w)} \approx \left\| \sum_{Q \in \Gamma} \frac{\chi_Q(.)}{\varphi(w(Q))} \right\|_{L^\Phi(w)}.
\] (2.25)

ii) Moreover, for any \( \tau > 0 \), there exist a family of disjoint cubes \( \Gamma = \{R_1, R_2, \ldots, R_N\} \subset \mathcal{D} \), such that

\[
\|\tilde{1}_\Gamma\|_{L^\Phi} \approx \frac{\varphi(\tau)}{\varphi(\tau^*)}.
\] (2.26)

Proof. i) For a single element of the basis \( \mathcal{B} \) we have, by (2.14) that

\[
\|\psi_Q\|_{L^\Phi(w)} \approx \left\| \frac{\chi_Q(.)}{|Q|} \right\|_{L^\Phi(w)} = \frac{\|\chi_Q\|_{L^\Phi(w)}}{|Q|^{\frac{1}{p}}} = \frac{\varphi(w(Q))}{|Q|^{\frac{1}{p}}}.
\] (2.27)

By (2.14) again

\[
\|\tilde{1}_\Gamma\|_{L^\Phi(w)} \approx \left\| \left( \sum_{Q \in \Gamma} \frac{1}{\|\psi_Q\|_{L^\Phi(w)}^2} \chi_Q |Q|^{-1} \right)^{\frac{1}{2}} \right\|_{L^\Phi(w)} \approx \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q}{\varphi(w(Q))^2} \right)^{\frac{1}{2}} \right\|_{L^\Phi(w)} =
\]

\[
= \left\| \sum_{Q \in \Gamma} \frac{\chi_Q}{\varphi(w(Q))} \right\|_{L^\Phi(w)},
\]

where in the last equality we have used that the cubes in \( \Gamma \) are pairwise disjoint.

ii) The existence of the family \( \Gamma = \{R_1, R_2, \ldots, R_N\} \subset \mathcal{D} \) is proved in Lemma 2.6 where it is shown that \( w(R_j) \approx \tau \), \( j = 1, 2, \ldots, N \). In this situation

\[
\|\tilde{1}_\Gamma\|_{L^\Phi(w)} = \left\| \sum_{j=1}^N \frac{\chi_{R_j}(.)}{\varphi(w(R_j))} \right\|_{L^\Phi(w)} \approx \frac{1}{\varphi(\tau)} \left\| \sum_{j=1}^N \chi_{R_j} \right\|_{L^\Phi(w)}
\]

\[
= \frac{1}{\varphi(\tau)} \varphi\left( \left( \bigcup_{j=1}^N R_j \right) \right) \approx \frac{\varphi(\tau) \varphi(\tau)}{\varphi(\tau)} = \frac{\varphi(\tau) \varphi(\tau)}{\varphi(\tau^*)}.
\] (2.28)

\[ \square \]

Remark 2.8. It follows from part ii) of Proposition 2.7 that for admissible wavelet basis \( \mathcal{B} \)

\[
h_+^\varphi(N; L^\Phi(w), \mathcal{B}) \geq \sup_{\tau > 0} \frac{\varphi(\tau)}{\varphi(\tau^*)} = h_+^\varphi(N)
\]

and

\[
h^-_\varphi(N; L^\Phi(w), \mathcal{B}) \leq \inf_{\tau > 0} \frac{\varphi(\tau)}{\varphi(\tau^*)} = h^-_\varphi(N).
\]

Thus, if \( h_+^\varphi(N) \) and \( h^-_\varphi(N) \) are not comparable for \( N \geq 1 \) it follows that admissible wavelet bases are non democratic in weighted Orlicz spaces. On the other hand, if wavelet admissible bases are democratic in \( L^\Phi(w) \), \( h_+^\varphi(N) \leq h^-_\varphi(N) \), and Lemma 5.2 in [14] shows that \( \varphi(t) \approx t^\alpha \) for some \( \alpha \in (0, 1) \); thus, the only democratic weighted Orlicz spaces are the spaces \( L^p(w) \) for some \( p = \frac{1}{\alpha} \in (1, \infty) \).
3. LEFT AND RIGHT DEMOCRACY FUNCTIONS FOR WEIGHTED ORLICZ SPACES

Our main theorem in this section shows that $h_r(N; L^\Phi(w), B) \lesssim h^+_\varphi(N)$ and $h_l(N; L^\Phi, B) \gtrsim h^-_\varphi(N)$ (see theorem 3.1 below) giving us together with remark 2.8 a complete description (up to multiplicative constants) of the left and right democracy functions of wavelet basis on weighted Orlicz spaces.

**Theorem 3.1.** Let $L^\Phi(w)$, be a weighted Orlicz space with Boyd indices satisfying $0 < i_\varphi \leq I_\varphi < 1$, $w \in A_p^{\Phi}$ a weight in $\mathbb{R}^d$, and let $B = \{\psi_Q : Q \in \mathcal{D}\}$ be an admissible wavelet basis. Then for all $\Gamma \subset \mathcal{D}$

$$h^-_\varphi(\# \Gamma) \lesssim \|\tilde{1}_\Gamma\|_{L^\Phi(w)} \lesssim h^+_\varphi(\# \Gamma).$$

(3.1)

This, together with Remark 2.8 gives

$$h_r(N; B, L^\Phi(w)) \approx h^+_\varphi(N) \quad \text{and} \quad h_l(N; B, L^\Phi(w)) \approx h^-_\varphi(N),$$

which is Theorem 1.1.

The rest of this section is devoted to prove Theorem 3.1. We first present a very simple argument for the case of pairwise disjoint cubes.

3.1. **Proof of Theorem 3.1: The case of disjoint cubes.** We assume first that $\Gamma = \{Q_1, \ldots, Q_N\}$ consists of pairwise disjoint cubes. Let $\lambda = h^+_\varphi(N)$, so that $\varphi(N \omega(Q)) \leq \lambda \varphi(\omega(Q)), \forall Q \in \Gamma$. Therefore, since the elements of $\Gamma$ are disjoint, and $\Phi$ is increasing

$$\int_{\mathbb{R}^d} \Phi\left(\frac{\sum_{j=1}^N \chi_{Q_j}(x)}{\lambda \varphi(\omega(Q_j))}\right) \omega(x) dx = \sum_{j=1}^N \Phi\left(\frac{1}{\lambda \varphi(\omega(Q_j))}\right) \omega(Q_j) \leq \sum_{j=1}^N \Phi\left(\varphi(N \omega(Q_j))\right) \omega(Q_j) = 1$$

Then by (2.25) and (2.4) we have

$$\|\tilde{1}_\Gamma\|_{L^\Phi(w)} \lesssim \left\|\sum_{j=1}^N \frac{\chi_{Q_j}(x)}{\varphi(\omega(Q_j))}\right\|_{L^\Phi(w)} \lesssim h^+_\varphi(N).$$

The lower estimate is obtained in a similar way.

3.2. **Proof of Theorem 3.1: The general case.** In the case of disjoint cubes just considered we have two important features. First, Proposition 2.7 allows us to “linearize” the square function in (2.14). Second, for the estimates obtained in the previous argument it is crucial that the sets involved are disjoint. For general families of cubes we are going to follow the same scheme. First we “linearize” the square function and we dominate this by an expression involving only disjoint subsets from the elements of $\Gamma$.

**Linearization of the square function.** Given a finite set $\Gamma \subset \mathcal{D}$, we denote

$$S_\Gamma(x) = \left(\sum_{Q \in \Gamma} \frac{\chi_Q(x)}{\varphi(\omega(Q))^2}\right)^{\frac{1}{2}},$$

(3.2)
so that by (2.14) and (2.27), we have \( \| \tilde{1}_\Gamma \|_{L^p(w)} \simeq \| S_{\Gamma}(\cdot) \|_{L^p(w)} \). For every \( x \in \bigcup_{Q \in \Gamma} Q \), we define \( Q_x \) as the smallest (hence unique) cube in \( \Gamma \) containing \( x \). It is clear that

\[
S_{\Gamma}(x) \geq \frac{\chi_{Q_x}(x)}{\varphi(w(Q_x))}, \quad \forall \, x \in \bigcup_{Q \in \Gamma} Q, \tag{3.3}
\]

since the left hand side contains at least the cube \( Q_x \) (and possibly more). We now show that the reverse inequality holds. Indeed, if we enlarge the sum to include all dyadic cubes containing \( Q_x \) we have

\[
S_{\Gamma}(x)^2 \geq \sum_{Q \in \Gamma} \frac{\chi_{Q}(x)}{\varphi(w(Q))} \leq \sum_{Q \in \Gamma} \frac{1}{\varphi(w(Q))^2} \leq \sum_{j=0}^{\infty} \frac{1}{\varphi(w(Q^j_x))},
\]

where \( Q^j_x \) denotes the unique cube of measure \( 2^{jd}|Q_x| \) containing \( Q_x \). Now since \( Q_x = Q^0_x \subset Q^1_x \subset Q^2_x \subset \ldots \) we can use (2.23) to obtain

\[
\frac{w(Q^j_x)}{w(Q_x)} \leq C^2 w \left( \frac{|Q_x|}{|Q^j_x|} \right)^{\delta} = C^2 w^{-jd\delta}.
\]

Hence,

\[
w(Q^j_x) \geq (C^2)^{-1} w(Q_x) 2^{jd\delta},
\]

and

\[
\varphi(w(Q^j_x)) \geq \varphi((C^2)^{-1} w(Q_x) 2^{jd\delta}).
\]

Since \( i_\varphi > 0 \), by (2.10) we can choose \( 0 < \epsilon < i_\varphi \) and find a \( C_\epsilon > 0 \) such that

\[
\varphi((C^2)^{-1} 2^{jd\delta} w(Q_x)) \geq C_\epsilon ((C^2)^{-1} 2^{jd\delta} (i_\varphi - \epsilon)) \varphi(w(Q_x)).
\]

Thus,

\[
S_{\Gamma}(x)^2 \leq \frac{(C^2)^{(i_\varphi - \epsilon)} C_\epsilon}{\varphi(w(Q_x))} \sum_{j=0}^{\infty} 2^{-jd(i_\varphi - \epsilon)} \lesssim \frac{\chi_{Q_x}(x)}{(\varphi(w(Q_x)))^2}.
\]

This and (3.3) show that

\[
S_{\Gamma}(x) \simeq \frac{\chi_{Q_x}(x)}{\varphi(w(Q_x))}. \tag{3.4}
\]

Observe from (3.4) that \( S_{\Gamma}(x) \simeq S_{\Gamma_{\min}}(x) \), where \( \Gamma_{\min}(x) \) denotes the family of minimal cubes in \( \Gamma \), that is,

\[
\Gamma_{\min} = \left\{ Q_x : x \in \bigcup_{Q \in \Gamma} Q \right\}.
\]

3.3. Shaded and Lighted Cubes. Shaded and lighted cubes were introduced in [14]. We recall the definitions. Given a fixed \( \Gamma \subset \mathcal{D} \), for any \( \Gamma \in \mathcal{G} \) we define the Shaded of \( Q \) as the union of all cubes from \( \Gamma \) strictly contained in \( Q \)

\[
\text{Shade}(Q) = \bigcup \left\{ R : R \in \Gamma, R \subsetneq Q \right\}.
\]

We define the Light of \( Q \) as \( \text{Light}(Q) = Q \setminus \text{Shade}(Q) \). It is clear that \( Q \in \Gamma_{\min} \); if and only if, \( \text{Light}(Q) \neq \emptyset \), and moreover

\[
\bigcup_{Q \in \Gamma} Q = \bigcup_{Q \in \Gamma_{\min}} \text{Light}(Q).
\]
Therefore, by (3.4) we can write

\[ S_\Gamma(x) \simeq \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\varphi(w(Q))}, \tag{3.5} \]

where in the last sum there is at most one non-zero term for each \( x \). We shall classify the cubes as shaded if the shade is a big portion of the cube or lighted if this does not happen. Precisely, a cube \( Q \in \Gamma \) is called \text{shaded} if \( |\text{Shade}(Q)| > \frac{2^d-1}{2^d}|Q| \), and we write \( \Gamma_s \) for the collection of cubes from \( \Gamma \) that are shaded. A cube \( Q \) from \( \Gamma \) is called \text{lighted} if it is not shaded, that is, if \( |\text{Light}(Q)| \geq \frac{1}{2^d}|Q| \). We write \( \Gamma_L \) for the collection of all cubes from \( \Gamma \) that are lighted.

\textbf{Remark 3.2.} Observe that \( \Gamma_L \subset \Gamma_{\min} \) and by Lemma 4.3 in [14] we have

\[ \frac{2^d-1}{2^d} (#\Gamma) \leq (\#\Gamma_L) \leq (\#\Gamma_{\min}) \leq (#\Gamma), \quad \forall \Gamma \subset \mathcal{D}. \]

Now we shall conclude the proof of theorem 3.1.

\textbf{Proof. ( of Theorem 3.1)} By (2.14) and (3.5) we know that

\[ \| \widetilde{1}_\Gamma \|_{L^p(w)} \simeq \left\| \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\varphi(w(Q))} \right\|_{L^p(w)}. \tag{3.6} \]

Thus it is enough to estimate the quantity in the right side of (3.6). We let \( \lambda = h^+_\varphi(\#\Gamma_{\min}) \) so that \( \varphi(w(Q)\#\Gamma_{\min}) \leq \lambda \varphi(w(Q)) \) for all \( Q \in \Gamma_{\min} \). Since \{Light(Q) : Q \in \Gamma_{\min}\} is a pairwise disjoint collection and \( \Phi \) is increasing, we have

\[ \int_{\mathbb{R}^d} \Phi\left( \sum_{Q \in \Gamma_{\min}} \frac{\chi_{\text{Light}(Q)}(x)}{\varphi(w(Q))} \right) w(x) dx = \sum_{Q \in \Gamma_{\min}} \Phi\left( \frac{1}{\lambda \varphi(w(Q))} \right) w(\text{Light}(Q)) \]

\[ \leq \sum_{Q \in \Gamma_{\min}} \Phi\left( \frac{1}{\varphi(w(Q)\#\Gamma_{\min})} \right) w(Q) = \sum_{Q \in \Gamma_{\min}} \Phi\left( \Phi^{-1}\left( \frac{1}{w(Q)\#\Gamma_{\min}} \right) \right) w(Q) = 1. \]

Hence by, (3.6), Remark 3.2 and since \( h^+_\varphi \) is non decreasing, we have

\[ \| \widetilde{1}_\Gamma \|_{L^p(w)} \lesssim h^+_\varphi(\#\Gamma_{\min}) \lesssim h^+_\varphi(\#\Gamma). \]

For the left inequality, by (3.6) and using that \( \Gamma_L \subset \Gamma_{\min} \), we can write

\[ \| \widetilde{1}_\Gamma \|_{L^p(w)} \gtrsim \left\| \sum_{Q \in \Gamma_L} \frac{\chi_{\text{Light}(Q)}(x)}{\varphi(w(Q))} \right\|_{L^p(w)}. \]

Now let \( \lambda > h^-_\varphi(2^{-dp}C^1_w(\#\Gamma_L)) \) so that \( \lambda \varphi(w(Q)) < \varphi(2^{-dp}C^1_w(\#\Gamma_L)) \) for all \( Q \in \Gamma_L \). Using (2.3), and since \( |\text{Light}(Q)| > 2^{-d}|Q| \) for \( Q \in \Gamma_L \), we deduce, with \( p = p^\Phi \), that

\[ \int_{\mathbb{R}^d} \Phi\left( \sum_{Q \in \Gamma_L} \frac{\chi_{\text{Light}(Q)}(x)}{\varphi(w(Q))} \right) w(x) dx = \sum_{Q \in \Gamma_L} \Phi\left( \frac{1}{\lambda \varphi(w(Q))} \right) w(\text{Light}(Q)) \]

\[ > \sum_{Q \in \Gamma_L} \Phi\left( \frac{1}{\varphi(2^{-dp}C^1_w(\#\Gamma_L))} \right) C^1_w 2^{-dp} w(Q) \].
\[
= \sum_{Q \in \Gamma_L} \Phi \left( \Phi^{-1} \left( \frac{1}{(2^{-dpC_w(Q)(\#\Gamma_L)} \right) \right) 2^{-dp_w(Q)C_w^1} = 1.
\]

Then by (2.11), and Remark 3.2 we obtain
\[
\| \tilde{1}_\Gamma \|_{L^q(w)} \geq h^-_w(2^{-dpC_w^1(\#\Gamma)}) \geq h^-_w(C_w^1(2^d - 1)2^{-d(p+1)}(\#\Gamma)).
\]

Now using (2.10) it can be shown that
\[
h^-_w(C_w^1(2^d - 1)2^{-d(p+1)}(\#\Gamma)) \geq Ch^-_w(\#\Gamma).
\]

If \( \Phi(t) = t^p \), from Theorem 3.1 and Example 2.1 we deduce that admissible wavelet bases are democratic in weighted Lebesgue spaces \( L^p(w) \).

**Corollary 3.3.** Let \( \Phi(t) = t^p \), \( 1 < p < \infty \), \( w \in A_p \) a weight in \( \mathbb{R}^d \), and \( B = \{\psi_Q : Q \in \mathcal{D}\} \) an admissible wavelet basis. Then
\[
h_r(N; B, L^p(w)) \approx h_l(N; B, L^p(w)) \approx N^\frac{1}{p}
\]

(3.7)

**4. Inclusions for \( N \)-Term Approximation spaces of \( L^q(w) \).**

In this section we investigate Jackson and Bernstein type inequalities and the corresponding inclusions for the \( N \)-term approximation spaces \( A^q_{\eta}(B, L^q(w)) \), \( \alpha > 0 \), \( 0 < q < \infty \), \( w \in A_{p,q}(\mathbb{R}^d) \), where the error of approximation is measured in \( L^q(w) \)(see [12]). These inclusions are given in terms of the discrete Lorentz spaces \( \Lambda^q_{\eta} \) (see definition and properties of this spaces in subsection 4.1 below).

**4.1. Sequence spaces in \( \mathcal{D} \).** We recall the definition of some classical sequence spaces over the index set \( \mathcal{D} \) of all dyadic cubes on \( \mathbb{R}^d \). All of them are subspaces of \( c_0 \) and therefore for each sequence \( \{s_Q\}_{Q \in \mathcal{D}} \) we can find an enumeration of the index set \( \mathcal{D} = \{Q_k\}_{k=1}^\infty \) so that \( |s_{Q_1}| \geq |s_{Q_2}| \geq \ldots \) and in addition \( \lim_{k \to \infty} s_{Q_k} = 0 \). We shall always assume that \( \{s_{Q_k}\}_{k \geq 1} \) corresponds to such ordering, which coincides with the non-increasing rearrangement \( s^* \) of the sequence \( s \).

Let \( \eta = \{\eta(k)\}_{k \geq 1} \) be a fixed positive increasing sequence so that \( \lim_{k \to \infty} \eta(k) = \infty \) and \( \eta \) is doubling (i.e. \( \eta(2k) \leq C\eta(k), k \geq 1 \)). Then, for each \( 0 < q \leq \infty \) we define a weighted discrete Lorentz space by
\[
\Lambda^q_{\eta} = \left\{ s \in c_0 : \|s\|_{\Lambda^q_{\eta}} = \left[ \sum_{k \geq 1} (\eta(k)|s_{Q_k}|^q)^{\frac{1}{q}} \right]^{\frac{1}{q}} < \infty \right\}.
\]

Note that for \( q = \infty \) one writes \( \|s\|_{\Lambda^\infty_{\eta}} = \sup_k \eta(k)|s_{Q_k}| \). These are quasi-Banach rearrangement invariant spaces, which are Banach when \( q = 1 \) and \( \{\eta(k)^q\}_{k} \) is non-increasing ([3], p. 28). When \( q = 1 \) or \( q = \infty \) we shall write, respectively, \( \Lambda_{\eta} \) and \( M_{\eta} \) (the latter called Marcinkiewicz space). The particular case \( \eta(k) = k^\frac{1}{q} \) gives the classical (discrete) Lorentz space \( \Lambda^q_{\eta} = \ell^{r,q}(\mathcal{D}) \). The spaces \( \Lambda^q_{\eta} \) for general \( \eta \), and in particular, their interpolation properties, have been studied, e.g., in [3, 24, 27]. In our applications we use the sequences \( \{k^\alpha h^+_w(k)\}_{k \geq 1} \), for \( \alpha > 0 \), which always satisfy the required assumptions.
Given a fixed sequence space \( s \) as above, we define a new sequence space \( s(\Phi^w) \) isomorphic to \( s \), by

\[
\mathcal{s}(\Phi^w) = \{ f = \sum_{Q \in \mathcal{D}} \langle f, \psi_Q \rangle \psi_Q \in \Phi^w : \{ \| \langle f, \psi_Q \rangle \psi_Q \|_{L^s(w)} \}_Q \in s \},
\]

with \( \| f \|_{\mathcal{s}(\Phi^w)} = \left\| \sum_{Q \in \mathcal{D}} \langle f, \psi_Q \rangle \psi_Q \right\|_s \). Such definitions appear naturally in relation with approximation when the basis is not normalized (see, e.g., [13]).

4.2. Jackson type inequalities. In order to obtain the left embedding of the inclusions of approximation spaces given in Theorem [13], we start by proving some inequalities of Jackson Type.

**Proposition 4.1.** Let \( \Phi \) be a Young function so that \( 0 < i_\varphi \leq I_\varphi < 1 \), \( w \in A_\Phi \) a weight in \( \mathbb{R}^d \), and let \( \alpha > 0 \). Let \( \mathcal{B} \) be an admissible wavelet basis. Then, there exists \( C > 0 \) such that for every \( f \in \mathcal{M}_{k^\alpha h^\alpha_p (k)} (\mathcal{B}, \Phi^w) \) we have

\[
\| f - G_{N-1}(f) \|_{L^\Phi(w)} \leq CN^{-\alpha} \| f \|_{\mathcal{M}_{k^\alpha h^\alpha_p (k)} (\mathcal{B}, \Phi^w)}, \quad \forall N \geq 1. \tag{4.1}
\]

**Proof.** By the triangle inequality and (1.3) we have

\[
\| f - G_{N-1}(f) \|_{L^\Phi(w)} = \left\| \sum_{k \geq N} \langle f, \psi_k \rangle \psi_k \right\|_{L^\Phi(w)} \leq \sum_{j=0}^\infty \sum_{2^j N \leq k < 2^{j+1} N} \| \langle f, \psi_k \rangle \psi_k \|_{L^\Phi(w)}
\]

\[
\leq \sum_{j=0}^\infty \| \langle f, Q_{2jN} \rangle \psi_{Q_{2jN}} \|_{L^\Phi(w)} \left( \sum_{2^j N \leq k < 2^{j+1} N} \left\| \psi_k \right\|_{L^\Phi(w)} \right)
\]

\[
\leq \sum_{j=0}^\infty \| \langle f, \psi_{2jN} \rangle \psi_{2jN} \|_{L^\Phi(w)} h_p^\alpha (2^j N) \tag{4.2}
\]

where in the last inequality we have used Theorem 3.1. Now using that \( h_p^\alpha (k) \) is non-increasing (this follows from the fact that \( \frac{\alpha(t)}{t} \) is non-increasing for all \( t > 0 \), see [2]) and the definition of the Marcinkiewicz space we have

\[
\sum_{j=0}^\infty \| \langle f, \psi_{2jN} \rangle \psi_{2jN} \|_{L^\Phi(w)} h_p^\alpha (2^j N) = \sum_{j=0}^\infty \sum_{2^j-1 N \leq k < 2^j N} \| \langle f, \psi_k \rangle \|_{L^\Phi(w)} h_p^\alpha (2^j N) \frac{h_p^\alpha (2^j N)}{2^j-1 N}
\]

\[
\leq 2 \sum_{k \geq 2^j N} \| \langle f, \psi_k \rangle \|_{L^\Phi(w)} h_p^\alpha (k) \leq C \| f \|_{\mathcal{M}_{k^\alpha h^\alpha_p (k)} (\mathcal{B}, \Phi^w)} \sum_{k \geq 2^j N} k^{-\alpha} \frac{h_p^\alpha (k)}{k}
\]

\[
\leq CN^{-\alpha} \| f \|_{\mathcal{M}_{k^\alpha h^\alpha_p (k)} (\mathcal{B}, \Phi^w)}. \tag{4.3}
\]

The previous result can be translated as the following inclusion for approximation spaces

\[
\mathcal{M}_{k^\alpha h^\alpha_p (k)} (\mathcal{B}, \Phi^w) \hookrightarrow \mathcal{A}_\infty (\mathcal{B}, \Phi^w). \tag{4.4}
\]
4.3. Bernstein type inequalities. Bernstein type estimates are useful to obtain the right hand inclusions for approximation spaces of Theorem 1.2.

**Proposition 4.2.** Let $\Phi$ be a Young function such that $0 < i_{\Phi} \leq I_{\Phi} < 1$, $w \in A_{\Phi}$ a weight in $\mathbb{R}^d$, and let $\alpha > 0$. Let $B$ be an admissible basis. Then, there exists $C > 0$ so that, for all $N \geq 1$, and all $f \in \Sigma_N$

\[
\|f\|_{\Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w))} \leq C N^\alpha \|f\|_{L^{\Phi}(w)}. \tag{4.5}
\]

*Proof.* Let $f = \sum_{j=1}^{N} (f, \psi_{Q}, \psi_{Q}) \in \Sigma_N$, written in such a way that $\|\langle f, \psi_{Q}, \psi_{Q} \rangle\|_{L^{\Phi}(w)} \geq \|\langle f, \psi_{Q}, \psi_{Q} \rangle\|_{L^{\Phi}(w)} \geq \cdots$. For $1 \leq k \leq N$, using Theorem 3.1

\[
\|\langle f, \psi_{Q} \rangle\|_{L^{\Phi}(w)} \leq C \|\langle f, \psi_{Q} \rangle\|_{L^{\Phi}(w)} \left(\sum_{j=1}^{k} \|\psi_{Q_{j}}\|_{L^{\Phi}(w)}\right)_{L^{\Phi}(w)} \leq C \|G_{N}(f)\|_{L^{\Phi}(w)}. \tag{4.6}
\]

By (4.6) we have

\[
\|f\|_{\Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w))} = \sum_{k=1}^{N} k^{\alpha} h_{\Phi}(k) \|\langle f, \psi_{Q} \rangle\|_{L^{\Phi}(w)} \frac{1}{k} \leq C \|G_{N}(f)\|_{L^{\Phi}(w)} \sum_{k=1}^{N} k^{\alpha} \leq C' N^\alpha \|f\|_{L^{\Phi}(w)}. \]

\[\]

As before, the above result can be stated as an inclusion for approximation spaces. Below, the number $\rho_{\alpha} \in (0, 1]$ is chosen so that the quasi-normed space $\Lambda_{k^n h_{\Phi}(k)}$ satisfies the $\rho_{\alpha}$-triangle inequality, that is,

\[
\|s_{1} + s_{2}\|_{\Lambda_{k^n h_{\Phi}(k)}}^{\rho_{\alpha}} \leq \|s_{1}\|_{\Lambda_{k^n h_{\Phi}(k)}}^{\rho_{\alpha}} + \|s_{2}\|_{\Lambda_{k^n h_{\Phi}(k)}}^{\rho_{\alpha}}. \tag{4.7}
\]

**Corollary 4.3.** Let $\alpha > 0$. Then, with the same hypothesis as in Proposition 4.2, we have

\[
A_{\rho_{\alpha}}(B, L^{\Phi}(w)) \hookrightarrow \Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w)). \tag{4.8}
\]

*Proof.* The argument for (4.8) is standard (see, e.g., [7]). It suffices to prove that

\[
\|f\|_{\Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w))} \leq C \|f\|_{A_{\rho_{\alpha}}(B, L^{\Phi}(w))}, \quad \forall f \in \Sigma_{N}, \ N \geq 1
\]

with a constant $C > 0$ independent of $N$ and one obtains the desired inclusion by letting $N \rightarrow \infty$. We also assume that $N = 2^{j}$. Now, write $f = \sum_{j=0}^{J} [f^{(j)} - f^{(j-1)}]$, where by convection $f^{(j)} = f$, $f^{(j)} = 0$ and $f^{(j)} \in \Sigma_{2^{j}}$ is so that $\|f - f^{(j)}\|_{L^{\Phi}(w)} \leq 2\sigma_{2^{j}}(f)_{L^{\Phi}(w)}$, $0 \leq j < J$. Then, applying, (4.7) and Proposition 4.2 to $f^{(j)} - f^{(j-1)} \in \Sigma_{2^{j}+1}$ we obtain

\[
\|f\|_{\Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w))} \leq \left[\sum_{j=0}^{J} \|f^{(j)} - f^{(j-1)}\|_{\Lambda_{k^n h_{\Phi}(k)}(B, L^{\Phi}(w))}^{\rho_{\alpha}}\right]^\frac{1}{\rho_{\alpha}} \leq C \left[\sum_{j=0}^{J} 2^{j\rho_{\alpha}} \|f^{(j)} - f^{(j-1)}\|_{L^{\Phi}(w)}^{\rho_{\alpha}}\right]^\frac{1}{\rho_{\alpha}}.
\]
Now, by assumption, for \(1 \leq j \leq J\)
\[
\|f^{(j)} - f^{(j-1)}\|_{L^q(w)} \leq \|f^{(j)} - f\|_{L^q(w)} + \|f - f^{(j-1)}\|_{L^q(w)} \leq 4 \sigma_2(j-1) (f)_{L^q(w)}.
\]
On the other hand, for \(j = 0\) we have
\[
\|f(0) - f^{(-1)}\|_{L^q(w)} = \|f(0) - f\|_{L^q(w)} + \|f\|_{L^q(w)} \leq 2 \sigma_1(f)_{L^q(w)} + \|f\|_{L^q(w)}.
\]
Hence,
\[
\|f\|_{\Lambda_{k^{\alpha}(\eta(k))} \ast \Lambda_{k^{\alpha}(\eta(k))}}_{(B, L^q(w))} \leq C \left(\|f\|_{L^q(w)} + \sum_{j=0}^{J-1} (2j^\alpha \sigma_2(j) (f)_{L^q(w)})^q \right)^{\frac{1}{q}} \approx \|f\|_{A^q(B, L^q(w))}.
\]

Finally, using real interpolation we can obtain inclusions for the whole family of approximation spaces \(A^q(B, L^q(w))\), \(0 < q \leq \infty\). For this we consider the interpolation properties of the sequence spaces \(\Lambda_{\rho}^q\), namely,
\[
(\Lambda_{k^{\alpha}(\eta(k))} \ast \Lambda_{k^{\alpha}(\eta(k))})_{\alpha, q} = \Lambda_{k^{\alpha}(\eta(k))}^q, \quad \alpha = (1 - \theta) \alpha_0 + \theta \alpha_1, \quad (4.9)
\]
for all \(0 < q, r \leq \infty\), \(0 < \theta < 1\) (see, e.g., [27] Proposition 6.2, [24], Theorem 3).

**Theorem 4.4.** Let \(\Phi\) be a Young function such that \(0 < \varphi \leq I_\varphi < 1, w \in A_\Phi\) a weight in \(\mathbb{R}^d\), \(\alpha > 0\), and \(0 < q \leq \infty\). Let \(B\) be an admissible wavelet basis. Then
\[
\Lambda_{k^{\alpha}(\eta(k))}^q (B, L^\Phi(w)) \hookrightarrow A_{\alpha}^q (B, L^\Phi(w)) \hookrightarrow \Lambda_{k^{\alpha}(\eta(k))}^q (B, L^\Phi(w)). \quad (4.10)
\]

**Proof.** Let \(\alpha_0 < \alpha < \alpha_1\), so that \(\alpha = (\alpha_0 + \alpha_1)/2\). Then, for every \(0 < q, r \leq \infty\) we have (see, e.g., [1])
\[
A_{\alpha}^q (B, L^\Phi(w)) = (A_\alpha^\alpha (B, L^\Phi(w)), A_\alpha^\alpha (B, L^\Phi(w)))_{\frac{1}{2}q}.
\]
Letting \(r = \min(\rho_{\alpha_1}, \rho_{\alpha_1})\) and using \((1.8)\)
\[
A_{\alpha}^q (B, L^\Phi(w)) = (A_\alpha^\alpha (B, L^\Phi(w)), A_\alpha^\alpha (B, L^\Phi(w)))_{\frac{1}{2}q}
\]
\[
\hookrightarrow (\Lambda_{k^{\alpha}(\eta(k))} (B, L^\Phi(w)), \Lambda_{k^{\alpha}(\eta(k))} (B, L^\Phi(w)))_{\frac{1}{2}q}
\]
\[
= \Lambda_{k^{\alpha}(\eta(k))}^q (B, L^\Phi(w)),
\]
where the last equality follows from \((1.9)\). Similarly, by \((1.4)\)
\[
A_{\alpha}^q (B, L^\Phi(w)) = (A_\infty^\alpha (B, L^\Phi(w)), A_\infty^\alpha (B, L^\Phi(w)))_{\frac{1}{2}q}
\]
\[
\hookrightarrow (\mathcal{M}_{k^{\alpha}(\eta(k))} (B, L^\Phi(w)), \mathcal{M}_{k^{\alpha}(\eta(k))} (B, L^\Phi(w)))_{\frac{1}{2}q}
\]
\[
= \Lambda_{k^{\alpha}(\eta(k))}^q (B, L^\Phi(w)).
\]

We now prove that the inclusions \((1.10)\) are optimal. To state the Theorem we write \(D\) for the class of sequences \(\eta = \{\eta(k)\}_{k=1}^\infty\) that are increasing, and doubling.

**Theorem 4.5.** Same hypothesis as in Theorem 4.4. For fixed \(\alpha > 0\) and \(q\), \(0 < q \leq \infty\), the inclusions given in \((4.10)\) are best possible in the scale of weighted Lorentz spaces \(\Lambda_{k^{\alpha}(\eta(k))}^q (B, L^\Phi(w))\) where \(\eta \in D\)
Proof. Suppose \( A^q_\delta(B, L^\Phi(w)) \leftarrow A^\alpha_\delta(B, L^\Phi(w)) \). We want to prove that \( h^+_\varphi(N) \lesssim \eta(N) \) for all \( N = 1, 2, \ldots \). By definition of \( h^+_\varphi(N) \) we can choose \( \tau = \tau(N) > 0 \) such that

\[
\frac{\varphi(N\tau)}{\varphi(\tau)} \leq h^+_\varphi(N) \leq 2 \frac{\varphi(N\tau)}{\varphi(\tau)}.
\] (4.11)

By Lemma 2.7 we can choose a sequence of pairwise disjoint cubes \( \Gamma = \{ R_j \}_{j=1}^{2N} \) such that \( w(R_j) \approx \tau \). Let \( \tilde{\Gamma} = \sum_{j=1}^{2N} \frac{v_{R_j}}{\| \psi_{R_j} \|_{L^\Phi(w)}} \). By Theorem 2.2, \( \| f \|_{L^\Phi(w)} \) is equivalent to the lattice norm \( \| S_\psi(f) \|_{L^\Phi(w)} \); thus there exists \( \Gamma' \subset \Gamma \) with \( \Gamma' = N \) such that \( \sigma_N(\tilde{\Gamma}) \approx \| \tilde{\Gamma}' \|_{L^\Phi(w)} \) (see (2.6) in [13]). Thus, by (2.28) and (4.11)

\[
\sigma_N(\tilde{\Gamma}) \approx \| \tilde{\Gamma}' \|_{L^\Phi(w)} \approx \frac{\varphi(N\tau)}{\varphi(\tau)} \approx h^+_\varphi(N).
\]

Hence,

\[
\| \tilde{\Gamma} \|_{A^q_\delta(B, L^\Phi(w))} \geq \left( \sum_{k=N/2}^N k^{\alpha q} \sigma_k(\tilde{\Gamma})^q \frac{1}{k} \right) \frac{1}{q} \approx \sigma_N(\tilde{\Gamma}) N^\alpha \approx N^\alpha h^+_\varphi(N). \tag{4.12}
\]

On the other hand

\[
\| \tilde{\Gamma} \|_{\Lambda^q_\delta(k; \varphi)(B, L^\Phi(w))} = \left( \sum_{k=1}^{2N} (k^{\alpha q} \eta(k))^q \frac{1}{k} \right) \frac{1}{q} \approx \eta(2N) N^\alpha \lesssim \eta(N) N^\alpha \tag{4.13}
\]

by the doubling property of \( \eta \). The inequalities (4.12) and (4.13) together with our assumption imply the desired result.

Suppose now that \( A^q_\delta(B, L^\Phi(w)) \leftarrow A^\alpha_\delta(B, L^\Phi(w)) \). We want to prove that \( \eta(k) \lesssim h^-_\varphi(N) \) for all \( N = 1, 2, \ldots \). Let \( \Gamma \subset \mathcal{D} \) with \( |\Gamma| = N \). Write \( \tilde{\Gamma} = \sum_{Q \in \Gamma} \frac{v_Q}{\| \psi_Q \|_{L^\Phi(w)}} \).

Since \( \sigma_k(\tilde{\Gamma}) \lesssim \| \tilde{\Gamma} \|_{L^\Phi(w)} \) for all \( k = 1, 2, \ldots, N \), our hypothesis imply

\[
\| \tilde{\Gamma} \|_{\Lambda^\alpha_\delta(k; \varphi)(B, L^\Phi(w))} \lesssim \| \tilde{\Gamma} \|_{A^\alpha_\delta(B, L^\Phi(w))} \lesssim N^\alpha \| \tilde{\Gamma} \|_{L^\Phi(w)}. \tag{4.14}
\]

On the other hand

\[
\| \tilde{\Gamma} \|_{\Lambda^\alpha_\delta(k; \varphi)(B, L^\Phi(w))} \geq \left( \sum_{k=1}^N (\eta(k) k^{\alpha q})^{1/q} \right) \frac{1}{q} \gtrsim N^\alpha \eta(N/2) \gtrsim N^\alpha \eta(N) \tag{4.15}
\]

since \( \eta \) is doubling. By (4.14) and (4.15) we have \( \eta(N) \lesssim \| \tilde{\Gamma} \|_{L^\Phi(w)} \) for all \( \Gamma \subset \mathcal{D} \), with \( |\Gamma| = N \). Taking the infimum over all \( \Gamma \in \mathcal{D} \), with \( |\Gamma| = N \), we obtain \( \eta(N) \lesssim h^-_\varphi(N) \) by Theorem 3.1. \( \square \)
5. Approximation Spaces for $L^p(w)$

Corollary 1.3 is now an easy consequence of Theorem 4.4 and Corollary 3.3. The rest of this section is devoted to prove Theorem 1.4 (see Theorem 5.4 below).

The approximation spaces $A_j^q(B, L^p(w))$ can also be identified with weighted Besov spaces. Our definition of weighted Besov spaces is borrowed from [29, 30], and it is modeled on the corresponding definition of Besov spaces without weights developed in [26] (see also [9] and [10]).

We say that a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ belongs to the class of admissible kernels if $\text{Supp} \hat{\varphi} \subset \{ \xi \in \mathbb{R}^d : \frac{1}{2} < |\xi| < 2 \}$ and $|\hat{\varphi}(\xi)| \geq c > 0$, if $\frac{d}{2} < |\xi| < \frac{5}{2}$. Set $\varphi_k(x) = 2^{kd} \varphi(2^k x)$ for $k \in \mathbb{Z}$.

Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, $\varphi$ admissible kernel, and $w$ an $A_p$ weight on $\mathbb{R}^d$. The homogeneous weighted Besov space $B^{\alpha}_{p,q}(w)$ is the set of all tempered distributions $f \in \mathcal{S}'/\mathcal{D}(\mathbb{R}^d)$ (modulo polynomials) such that

$$
\|f\|_{B^{\alpha}_{p,q}(w)} = \left( \sum_{k \in \mathbb{Z}} (2^{k\alpha}) \|\varphi_k \ast f\|_{L^p(w)} \right)^{\frac{1}{q}} < \infty.
$$

This definition depends initially of the choice of admissible $\varphi$. It can be proved (see Theorem 1.8 in [29] or [30]) that this is independent of the choice of admissible $\varphi$. Also, the spaces $B^{\alpha}_{p,q}(w)$ are (quasi)-Banach spaces (see section 4.4 of [30]).

Let $\Psi = \{ \psi^l : l = 1, 2, \ldots, 2^{d-1} \}$ be an orthonormal wavelet family in $L^2(\mathbb{R}^d)$ constructed from the 1-dimensional Lemarié-Meyer wavelets (see [16, 23, 25]). Write $s_Q^l = \langle f, \psi^l_Q \rangle$, $Q \in \mathcal{D}$, $l = 1, 2, \ldots, 2^{d-1}$ for the wavelet coefficients.

**Proposition 5.1.** (see Theorem 10.2 in [29] or Theorem 6.2 in [30]).

Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and let $w$ be an $A_p$ weight in $\mathbb{R}^d$. Let $\Psi$ be a family of Lemarié-Meyer wavelets as defined above. Then

$$
\|f\|_{B^{\alpha}_{p,q}(w)} \approx \left( \sum_{j \in \mathbb{Z}} \left( \sum_{|Q| = 2^{-jd}} (|Q| \cdot \frac{d}{2})^{-\frac{d+1}{2}} |s_Q^l|^\gamma \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{d}}. \quad (5.2)
$$

**Remark 5.2.** It is also proved in Theorem 10.2 of [29] and Theorem 6.2 of [30] that the condition $w$ doubling, that is, there exists $C > 0$ such that

$$
\int_{B_{2^d}(z)} w(x) dx \leq C \int_{B_{2^d}(z)} w(x) dx, \ \forall z \in \mathbb{R}^d \text{ and } \forall \delta > 0,
$$

is sufficient to guarantee the equivalence (5.2).

**Remark 5.3.** Equivalence (5.2) also holds for the family $N^{\Psi} = \{ N \psi^l : l = 1, \ldots, 2^{d-1} \}$ constructed from the 1-dimensional Daubechies compactly supported wavelets (see [31]), provided $N$ is sufficiently large (see [29]).

**Theorem 5.4.** Let $\gamma > 0$, $1 < p < \infty$. We have

$$
A^{\gamma/d}_{p}(\Psi, L^p(w)) = L^p(\Psi, L^p(w)) = B^{\gamma}_{\gamma,\gamma}(w^p), \ \text{ whenever} \ \frac{1}{\tau} = \frac{\gamma}{d} + \frac{1}{p}, \quad (5.3)
$$
for all \( w \in A_r(\mathbb{R}^d) \) and all orthonormal wavelet families \( \Psi \) for which (5.2) holds for \( B_{r,r}^{\gamma/d}(w) \).

For the proof we shall need the following lemma:

**Lemma 5.5.** Let \( w \in A_r \) be a weight in \( \mathbb{R}^d \), \( r \geq 1 \), \( 0 < \delta < 1 \), and \( u(x) = w(x)^\delta \). Then \( u \in A_r \), and \( w_Q \approx (u_Q)^{\frac{1}{\delta}} \), where

\[
\frac{1}{|Q|} \int_Q w(x) \, dx.
\]

**Proof.** When \( r > 1 \), since \( w \in A_r \) and \( 0 < \delta < 1 \), using Jensen’s inequality we have

\[
\left( \frac{1}{|Q|} \int_Q u(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{r} (x) \, dx \right)^{r-1}
= \left( \frac{1}{|Q|} \int_Q w(x)^\delta \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{\delta(1-r)} \, dx \right)^{r-1}
\leq \left[ \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-r} \, dx \right)^{r-1} \right]^\delta \leq C_w^\delta.
\]

Thus, \( u \in A_r \). Now we shall prove the equivalence \( w_Q \approx (u_Q)^{\frac{1}{\delta}} \). On the one hand, using Jensen’s inequality with \( \delta < 1 \), we have

\[
(u_Q)^{\frac{1}{\delta}} = \left( \frac{1}{|Q|} \int_Q w(x)^\delta \, dx \right)^{\frac{1}{\delta}} \leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) = w_Q.
\]

On the other hand, since \( h(t) = t^{-(r-1)\delta} \), \( t > 0 \), is a convex function, using again Jensen’s inequality we have

\[
\left( \frac{1}{|Q|} \int_Q w^{1-r}(x) \, dx \right)^{-(r-1)\delta} \leq \left( \frac{1}{|Q|} \int_Q w(x)^{\delta} \, dx \right)^{\delta} = u_Q.
\]

From the condition \( w \in A_r \) it follows that

\[
w_Q = \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \leq C_w \left( \frac{1}{|Q|} \int_Q w^{1-r}(x) \, dx \right)^{-(r-1)} \leq C(u_Q)^{\frac{1}{\delta}}.
\]

For \( r = 1 \), on the one hand, since \( w \in A_1 \), for almost all \( x \in Q \), using again Jensen’s inequality we have

\[
\left( \frac{1}{|Q|} \int_Q u(x) \, dx \right) = \left( \frac{1}{|Q|} \int_Q w^\delta(x) \, dx \right) \leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^\delta \leq C w(x)^\delta = C u(x).
\]

Thus \( u \in A_1 \). On the other hand we can use again Jensen’s inequality and obtain, \( (u_Q)^{\frac{1}{\delta}} \leq w_Q \). Moreover, the condition \( w \in A_1 \) implies that

\[
w_Q = \frac{1}{|Q|} \int_Q w(x) \, dx \leq \text{ess} \inf w = C(\text{ess} \inf u)^{\frac{1}{\delta}} \leq C \left( \frac{1}{|Q|} \int_Q u(x) \, dx \right)^{\frac{1}{\delta}} = C(u_Q)^{\frac{1}{\delta}}
\]

\( \square \)

**Proof.** (of Theorem 5.4.) The first equality in (5.3) follows from Corollary 1.3 (with \( \tau = q \) and \( \alpha = \frac{2}{d} \)). For the second equality, observe that for a single element of the basis \( \Psi \), we have

\[
\| \psi_Q \|_{L^p(w)} \approx \left( \left( \frac{\chi_Q(x)}{|Q|} \right)^{\frac{1}{2}} \right)_{L^p(w)} = |Q|^{-\frac{1}{2}} \| \chi_Q(x) \|_{L^p(w)} = |Q|^{-\frac{1}{2}} \left( \int_Q w(x) \, dx \right)^{\frac{1}{2}}
\]
whenever $0 < q < \infty$, and $\delta = \frac{2}{p} < 1$ we deduce that $u \in A_\tau \subset A_p$ and $w_Q^\frac{1}{r} \approx (u_Q^\frac{1}{r})$. Thus, since $w(Q) = |Q|w_Q$ we obtain

$$\|f\|_{\ell^r(\mathcal{V}, L^p(w))} = \left\| \left\| \langle f, \psi_{Q_k}\rangle \psi_{Q_k} \right\|_{L^p(w)} \right\|_{\ell^{r'}} \approx \left( \sum_{Q \in D} (|Q|^{-\frac{1}{r'}} \|f, \psi_Q\|_{(w_Q^\frac{1}{r})}) \right)^{\frac{1}{r'}} \approx \left( \sum_{Q \in D} (|Q|^{-\frac{1}{r'}} \|f, \psi_Q\|_{(w_Q^\frac{1}{r})}) \right)^{\frac{1}{r'}} = \left( \sum_{Q \in D} (|Q|^{-\frac{1}{r'}} \|f, \psi_Q\|_{(w_Q^\frac{1}{r})}) \right)^{\frac{1}{r'}} \approx \|f\|_{\dot{B}_{r'}^{\gamma}(w^\frac{1}{r})}.$$  \hfill \square

As a corollary we prove a non-trivial interpolation result.

**Corollary 5.6.** Let $\gamma > 0$, $1 < p < \infty$, $\frac{1}{r} = \frac{\gamma}{d} + \frac{1}{p}$, and $w \in A_\tau (\mathbb{R}^d)$. Let $\Psi$ be an orthonormal wavelet family for which (5.2) holds for the Besov spaces involved in this Corollary. For $0 < \theta < 1$ we have

$$(L^p(w), \dot{B}_{r'}^\gamma(w^{\tau/p}))_{\theta, \tau_0} = \dot{B}_{r'}^{\theta \gamma}(w^{\tau_0/p})$$

where $\frac{1}{\tau_0} = \frac{\theta}{d} + \frac{1}{p}$.

**Proof.** If $\Phi(t) = t^p$, use Proposition 4.1 and the continuous embedding $l^r \hookrightarrow l^{r \infty}$, and Theorem 5.4 to obtain, for all $N = 1, 2, \ldots$

$$\sigma_N (f)_{L^p(w)} \leq CN^{-\frac{1}{\tau_0}}\|f\|_{l^{r \infty}(\mathcal{V}, L^p(w))} \leq CN^{-\frac{1}{\tau_0}}\|f\|_{l^r(\mathcal{V}, L^p(w))}$$

whenever $\frac{1}{r} = \frac{\gamma}{d} + \frac{1}{p}$.

From Theorem 5.4 and Proposition 4.2 we obtain, for all $g \in \Sigma_N$, $N = 1, 2, \ldots$

$$\|g\|_{\mathcal{B}_{r'}^\gamma (w^{\tau/p})} \leq C\|g\|_{l^r(\mathcal{V}, L^p(w))} \leq CN^{-\frac{1}{\tau_0}} \|g\|_{L^p(w)},$$

whenever $\frac{1}{r} = \frac{\gamma}{d} + \frac{1}{p}$. From 5.5, (5.6), and the general theory developed by R. DeVore and V. A. Popov (see Theorem 3.1 in [7]) we deduce

$${\mathcal{A}}_{\theta q}^p(\Psi, L^p(w)) = (L^p(w), \dot{B}_{r'}^{\gamma}(w^{\tau/p}))_{\theta, q}$$

whenever $0 < q \leq \infty$ and $0 < \theta < 1$. We use again Theorem 5.4 to obtain

$${\mathcal{A}}_{\tau_0 q}^p(\Psi, L^p(w)) = \dot{B}_{r'}^{\theta \gamma}(w^{\tau_0/p}),$$

when $\frac{1}{\tau_0} = \frac{\theta}{d} + \frac{1}{p}$. The result follows from (5.7) with $q = \tau_0$. \hfill \square
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