Relations between the single-pass and multi-pass qubit probabilities

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In quantum computation the target fidelity of the qubit gates is very high, with the admissible error being in the range from $10^{-4}$ to $10^{-3}$, and even less, depending on the protocol. The direct experimental determination of such an extremely small error is very challenging. Instead, it is often determined by sequentially repeating the same gate multiple times, which leads to the accumulation of the error, until it reaches large enough values to be measured reliably. If the transition probability is $p = 1 - \epsilon$ with $\epsilon \ll 1$ in the single process, then classical intuition dictates that the probability after $N$ passes should be $P_N \approx 1 - N\epsilon$. However, this classical expectation is misleading because it neglects interference effects. This paper presents a rigorous theoretical analysis based on the SU(2) symmetry of the qubit propagator, resulting in explicit analytic relations that link the $N$-pass propagator to the single-pass one. In particular, the relations suggest that in most cases of interest the $N$-pass transition probability degrades as $P_N = 1 - N^2 \epsilon$, i.e. dramatically faster than the classical probability estimate. Therefore, the actual single-pass fidelities in various experiments, calculated from $N$-pass fidelities, might have been far greater than the reported values.

I. INTRODUCTION

In quantum computation the admissible error of gate operations is very small — usually in the range of $10^{-4}$ to $10^{-3}$, depending on the specific protocol. The highest qubit fidelities reported hitherto have been achieved with trapped ions. Recently, single-qubit gate errors as small as $10^{-5}$ [1, 2] and even $10^{-6}$ [3], and two-qubit gate errors below $10^{-3}$ [2, 4] have been reported in trapped ion experiments. The unwanted cross-talk to neighboring qubits has been reduced to $10^{-5}$ [2] and $10^{-6}$ [3] in other trapped ion experiments. Very recently, ion transport with an error of less than $10^{-5}$ has been reported too [5]. With superconducting qubits, single-qubit gate fidelity of 99.9% and two-qubit gate fidelity of 99.4% have been achieved [6].

The direct experimental determination of such tiny probability errors can be very challenging. Instead, it is often deduced by sequentially repeating the same gate many times, and hence amplifying the error and making its accurate measurement feasible. To this end, it is of crucial importance to have a relation that links the single-pass probability $p$ to the $N$-pass probability $P_N$. If the transition probability for the single process is $p = 1 - \epsilon$ with $0 < \epsilon \ll 1$, then classical intuition dictates that the $N$-pass probability should be $P_N = p^N \approx 1 - N\epsilon$. A more accurate calculation, which takes into account the exchange of probabilities between the qubit states adds correction terms to $p^N$ (see below) but in the limit of a tiny error the above estimate for $P_N$ remains in place. However, this classical expectation is misleading because it neglects interference effects caused by the dynamical phases in the propagator.

Recently [6], I analyzed the double-pass transition probability in two-state and three-state quantum systems by using, respectively, the SU(2) and SU(3) symmetry of the corresponding propagator. The correct double-pass probability is obtained by multiplying the sequential propagators, rather than probabilities. The conclusion was that the quantum estimate for the error generally exceeds the classical estimate $2\epsilon$, i.e. the quantum probability degrades faster than the classical one. I have derived the exact relationship (for any value of $p$) between the single-pass and double-pass probabilities, which in the general case depend on a dynamic phase. A recipe for the determination of the single-pass probability from the double-pass probability from a pair of two different measurements was proposed. In two special cases of interest, when the Hamiltonian possesses certain symmetries, the single-pass probability can be determined unambiguously from a single double-pass signal.

These results are of interest in physical situations when it is much easier to measure the initial-state population rather than the target one, e.g. in the formation of ultracold ground-state molecules [7] from ultracold atoms, and in atomic excitation to Rydberg levels [11]. However, these results are less relevant for the objective mentioned in the beginning — the determination of a tiny transition probability error $\epsilon$ ($0 < \epsilon \ll 1$) — because the double-pass probability remains still small.

In the present paper, I extend this earlier approach to $N$ repeated processes in a qubit. I use a rigorous theoretical analysis based on the SU(2) symmetry of the qubit propagator, resulting in an explicit analytic relation that links the $N$-pass propagator to the single-pass one. This relation suggests that in most cases of interest the $N$-pass probability degrades as $P_N = 1 - N^2 \epsilon$, i.e. dramatically faster than the classical probability estimate $1 - N\epsilon$. Therefore, the actual single-pass probabilities in various experiments calculated from $N$-pass probabilities might have been far greater than the reported values.

II. CLASSICAL PROBABILITY

Assuming that initially the system is in state $|1\rangle$ let us denote by $Q_N = P_{1\rightarrow 1}^{(N)}$ and $P_N = P_{1\rightarrow 2}^{(N)}$ the probabilities for, respectively, return to state $|1\rangle$ and transition to state $|2\rangle$.
The exact classical probabilities after \( N \) passes are
\[
Q^c_N = \frac{1 + (1 - 2p)^N}{2}, \quad P^c_N = \frac{1 - (1 - 2p)^N}{2}.
\]
These formulas can easily be proved inductively by using the relations \( Q^c_{N+1} = (1-p)Q^c_N + pP^c_N \) and \( P^c_{N+1} = pQ^c_N + (1-p)P^c_N \). For large probability \( p = 1 - \epsilon \) (0 < \( \epsilon \ll 1 \)), we have
(i) \( Q^c_N \approx 1 - N\epsilon \) and \( P^c_N \approx N\epsilon \) for even \( N \);
(ii) \( Q^c_N \approx N\epsilon \) and \( P^c_N \approx 1 - N\epsilon \) for odd \( N \).

In this limit, this is the same behavior as the simple estimates \( p^N \) and \( 1 - p^N \), which neglect the mutual exchange of probabilities between the two states: the error increases linearly with the number of processes \( N \).

For small probability \( p = \epsilon \) (0 < \( \epsilon \ll 1 \)), we have \( Q^c_N \approx 1 - N\epsilon \) and \( P^c_N \approx N\epsilon \) for any \( N \) (odd or even). Hence the probability of being in state \( |2\rangle \) grows as \( N \).

Below the quantum probabilities after \( N \) sequential passes are derived, discussed and compared to the classical probabilities.

### III. Multi-Pass Probabilities

The Hamiltonian of a coherently driven lossless two-state quantum system, in the rotating-wave approximations \cite{12}, reads
\[
H(t) = \frac{1}{2} \begin{bmatrix} -\Delta(t) & \Omega(t) \\ \Omega(t) & \Delta(t) \end{bmatrix},
\]
where \( \Delta(t) \) is the system-field frequency mismatch (the detuning), and \( \Omega(t) \) is the Rabi frequency, which is a measure of the coupling between the two states. For arbitrary \( \Omega(t) \) and \( \Delta(t) \) the propagator is a SU(2) matrix, which can be expressed in terms of the complex-valued Cayley-Klein parameters \( a \) and \( b \) (\(|a|^2 + |b|^2 = 1\)) as
\[
U = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}.
\]
Assume that the system is initially in state \( |1\rangle \). Then the probabilities for remaining in state \( |1\rangle \) and for transfer to state \( |2\rangle \) are
\[
q = P_{1\rightarrow 1} = |a|^2, \quad p = P_{1\rightarrow 2} = |b|^2.
\]
Obviously, \( p + q = 1 \). If the transition probability is very close to 1, i.e. \( p = 1 - \epsilon \), with 0 < \( \epsilon \ll 1 \), then it is difficult to determine the error \( \epsilon \) precisely. A natural approach is to repeat the process \( N \) times, which amplifies the error. By measuring the population of state \( |1\rangle \) or \( |2\rangle \) after \( N \) passes one can deduce the single-pass transition probability \( p \), and hence the single-pass error \( \epsilon \).

In order to determine the populations after \( N \) passes we need to find the \( N \)-pass propagator \( U_N = U^N \). It has been proved \cite{13} that the \( N \)-th power of any SU(2) propagator, parameterized as in Eq. \( 3 \), reads
\[
U_N = \begin{bmatrix} \cos N\theta + ia \frac{\sin N\theta}{\sin \theta} & -b \frac{\sin N\theta}{\sin \theta} \\ b \frac{\sin N\theta}{\sin \theta} & \cos N\theta - ia \frac{\sin N\theta}{\sin \theta} \end{bmatrix},
\]
where \( a = a_r + ia_i \) and
\[
\theta = \arccos a_r \quad (0 \leq \theta \leq \pi).
\]
Therefore, the two populations after \( N \) passes are
\[
Q_N = P_{1\rightarrow 1}^{(N)} = 1 - p \frac{\sin^2 N\theta}{\sin^2 \theta}, \quad \text{(7a)}
\]
\[
P_N = P_{1\rightarrow 2}^{(N)} = p \frac{\sin^2 N\theta}{\sin^2 \theta}. \quad \text{(7b)}
\]

Measuring \( Q_N \) or \( P_N \) alone is not sufficient to deduce the single-pass transition probability \( p \) because the parameter \( \theta \) is not uniquely linked to \( p \) but it depends also on the phase of the propagator parameter \( a \). However, if \( a \) is real, then this is possible.

### IV. Special Case: Real \( a \)

For real \( a \), we have \( q = a^2 = \cos^2 \theta \) and \( p = \sin^2 \theta \). The \( N \)-pass transition probabilities \( 7 \) become
\[
P_N = \sin^2[(N/2) \arccos(1 - 2p)], \quad \text{(8a)}
\]
\[
Q_N = \cos^2[(N/2) \arccos(1 - 2p)]. \quad \text{(8b)}
\]

This return probability \( Q_N \) is plotted in Fig. \( 2 \) and compared to the classical probability \( Q^c_N \) of Eq. \( 1 \). For \( p = 1 - \epsilon \) and even \( N \) (meaning \( N/2 \) double passes) we find the initial-state population to be
\[
Q_N \approx 1 - N^2\epsilon + O(\epsilon^2) \quad (\epsilon \ll 1), \quad \text{(9)}
\]
while the classical probability is \( Q^c_N \approx 1 - N\epsilon \). Obviously, the quantum probability error increases much faster (as

![FIG. 1. Set of measurements required to measure the single-pass transition probability (a), by multiple application of several double-pass processes: (b) with the same Rabi frequencies, (c) with different signs of the Rabi frequencies, (d) with different signs of the detunings.](image)
of the number of passes than the classical probability (as $N$). This is clearly visible near $p = 1$ in all frames of Fig. 2.

In the opposite limit, when $p = \epsilon \ll 1$ is small, we find

$$P_N \approx N^2 \epsilon + O(\epsilon^2) \quad (\epsilon \ll 1),$$

i.e. the $N$-pass transition probability grows with $N^2$. The Cayley-Klein parameter $a$ is real in two important special cases.

(i) On exact resonance ($\Delta = 0$) we have $a = \cos(A/2)$, with $A$ being the pulse area. Hence $\theta = A/2$ and therefore, after $N$ passes we find

$$P_N = \sin^2(NA/2).$$

Of course, this result can be found directly from the resonant solution because the total pulse area after $N$ passes is $NA$.

(ii) When the Rabi frequency is symmetric and the detuning is anti-symmetric function of time, $\Omega(-t) = -\Omega(t)$ and $\Delta(-t) = -\Delta(t)$ the parameter $a$ is real [14]. A number of analytically soluble models belong to this special class: the original Landau-Zener-Stückelberg-Majorana (LZSM) model [15–18], the symmetric finite LZSM model [19], the Allen-Eberly-Hioe model [20, 21], and the linearly-chirped Gaussian model [22] — all related to the popular technique of rapid adiabatic passage (RAP) via a level crossing [23].

In either of these cases (i) and (ii), the multi-pass probabilities $P_N$ and $Q_N$ depend on the single-pass probability $p$ only and do not depend on the propagator phases. Hence the mappings $p \rightarrow P_N$ or $p \rightarrow Q_N$ are single-valued: knowing $p$ means knowing the multi-pass probabilities.

However, the opposite correspondences $P_N \rightarrow p$ and $Q_N \rightarrow p$ are not single-valued (because the arccos function is not single-valued), as can be deduced also from Fig. 2. Therefore, if one wants to find the single-pass probability $p$ from the multi-pass ones $P_N$ or $Q_N$, some additional knowledge is required. For instance, if the probability $p$ is known (or measured) to be close to 1 then the largest value for $p$ stemming from the value of $P_N$ should be retained. Alternatively, one can determine $p$ from two measurements, e.g. after $N$ and $2N$ passes, which should produce the same value for $p$.

When $a$ is not real, $\theta$ is not uniquely linked to $p$ but it depends also on a dynamical phase. This undesired dependence can be eliminated by using double-pass processes, in which the second process is different from the first one, as done in Ref. [9].

V. GENERAL CASE

A. Double-pass propagators

Consider a second interaction with the same magnitudes but with different signs of $\Omega(t)$ and $\Delta(t)$, cf. Fig. 1. The respective propagators can be obtained from Eq. (4) by simple algebraic operations [14] and, very importantly, can be expressed with the same Cayley-Klein parameters $a$ and $b$,

$$U_{\pm \Omega, \Delta} = \begin{bmatrix} a & b^* \\ -b & a^* \end{bmatrix}, \quad U_{\Omega, -\Delta} = \begin{bmatrix} a^* & b \\ -b^* & a \end{bmatrix}. \quad (12)$$

The respective double-pass propagators read

$$U_{\Omega, \Delta} U_{\Omega, \Delta} = \begin{bmatrix} a^2 - |b|^2 & -2b^* \text{Re}(a) \\ 2b \text{Re}(a) & (a^*)^2 - |b|^2 \end{bmatrix}, \quad (13a)$$

$$U_{-\Omega, \Delta} U_{\Omega, \Delta} = \begin{bmatrix} a^2 + |b|^2 & -2ib^* \text{Im}(a) \\ -2ib \text{Im}(a) & (a^*)^2 + |b|^2 \end{bmatrix}, \quad (13b)$$

$$U_{\Omega, -\Delta} U_{\Omega, \Delta} = \begin{bmatrix} |a|^2 + b^2 & 2ia^* \text{Im}(b) \\ 2ia \text{Im}(b) & |a|^2 + (b^*)^2 \end{bmatrix}, \quad (13c)$$

where $U_{\Omega, \Delta}$ is the same as $U$ of Eq. (3) but the subscripts are added for the sake of consistency with the...
other propagators. We immediately see that the corresponding double-pass probabilities are different in each case; it was this difference that enabled the derivation of the single-pass probability \( p \) from the double-pass probabilities in Ref. [3].

For \( N \) double-pass pairs of this type, i.e. for \( 2N \) passes, we can find the overall propagator in the same manner as Eq. (7) is derived from Eq. (9).

**B. Special case: Imaginary \( b \)**

Imaginary \( b \) occurs when both the Rabi frequency and the detuning are symmetric functions of time, \( \Omega(t) = \Omega(t) \) and \( \Delta(t) = \Delta(t) \). A beautiful analytically soluble model that belongs to this class is the Rosen-Zener model [26], in which the Rabi frequency has a hyperbolic-secant shape, \( \Omega(t) \propto \text{sech}(t/T) \) and the detuning is constant, \( \Delta = \text{const} \). Another (approximately soluble) example is the Gaussian model with constant detuning [27], in which \( \Omega(t) \propto e^{-t^2/T^2} \). In these cases, the Cayley-Klein parameter \( b \) is purely imaginary [14, 24, 25], implying \( p = |b|^2 = |\text{Im}(b)|^2 \). Then the diagonal elements of the product \( U_{\Omega,\Delta} U_{\Omega,\Delta} \) in Eq. (15) are real and we find \( \theta = \arccos(1 - 2p) \). Therefore \( \sin^2 \theta = 4p(1 - p) \) and hence

\[
P_{2N} = \sin^2 N \theta, \tag{14}
\]

which is the same as in the previous case, Eq. (9), with the replacement \( N \to 2N \) (because here we have \( N \) double-pass processes).

For large transition probability, \( p = 1 - \epsilon \), we arrive at Eq. (9) again, with the same conclusions. For small transition probability, \( p = \epsilon \), we arrive at Eq. (11), with the same \( N^2 \) scaling of the probability.

**C. General case: Complex \( a \) and \( b \)**

When the Cayley-Klein parameter \( a \) is complex, \( a = \sqrt{q} e^{i \xi} \) the parameter \( \theta \) cannot be linked to the single-pass transition probability \( p = 1 - q \) alone. Instead we have \( \cos \theta = \sqrt{q} \cos \xi \), i.e. \( \theta \) depends on the phase \( \xi \) (known as St"uckelberg phase). Therefore we cannot use the approaches in the two special cases of real \( a \) or imaginary \( b \) described above. One possibility to proceed is to extend the approach of Ref. [3], which uses the double-pass propagators \( U_{\Omega,\Delta} U_{\Omega,\Delta} \) and \( U_{-\Omega,\Delta} U_{\Omega,\Delta} \), see Eqs. (13a) and (13b). For these double-pass propagators the parameter \( \theta \) is defined as

\[
\theta_{\pm} = \arccos(q \cos 2\xi \mp p), \tag{15}
\]

where \( \theta_{\pm} \) refers to \( U_{\pm\Omega,\Delta} U_{\Omega,\Delta} \). Hence the transition probability after \( N \) double passes depends on both \( p \) (or \( q \)) and \( \xi \);

\[
P_N = p_{\pm} \frac{\sin^2(N \theta_{\pm})}{\sin^2(\theta_{\pm})}, \tag{16}
\]

with \( p_{\pm} = 4p(1 - p) \cos^2 \xi \) and \( p_{\mp} = 4p(1 - p) \sin^2 \xi \), cf. Eqs. (15a) and (15b). Next, we measure the transition probability after \( N \) and \( 2N \) double passes and calculate the ratios

\[
R_N^\pm = \frac{P_N^\pm}{P_N} = \frac{\sin^2(2N \theta_{\pm})}{\sin^2(N \theta_{\pm})} = 4 \cos^2(N \theta_{\pm}). \tag{17}
\]

In this manner the probabilities \( p_{\pm} \) (which are very small if \( p = 1 - \epsilon \)) are eliminated and each of the ratios \( R_N^\pm \) depends on the single parameter \( \theta_{\pm} \) only (rather than on \( \theta_{\pm} \) and \( p_{\pm} \), as do \( P_N^\pm \)). From here we find

\[
\theta_{\pm} = \frac{\arccos(R_N^\pm/2 - 1)}{2N}, \tag{18}
\]

and hence we can find \( p \) from [cf. Eq. (15)]

\[
p = \frac{\cos \theta_{\pm} - \cos \theta_{\mp}}{2}. \tag{19}
\]

This procedure allows one to determine any single-pass probability: small, large or in between. As before, of special interest are the limits of large and small \( p \).

**Large \( p \).** When the transition probability is close to unity, \( p = 1 - \epsilon \) with \( 0 < \epsilon \ll 1 \), then we find from Eq. (16)

\[
P_N^+ = 4N^2 \epsilon \cos^2 \xi + O(\epsilon^2), \tag{20a}
\]

\[
P_N^- = 4N^2 \epsilon \sin^2 \xi + O(\epsilon^2), \tag{20b}
\]

and hence, the multi-pass transition probability error scales as \( N^2 \), as before, although now it depends also on the dynamical phase \( \xi \) too. Obviously, \( P_N^+ + P_N^- = 4N^2 \epsilon + O(\epsilon^2) \), and one can determine the single-pass error \( \epsilon \) from here as

\[
\epsilon \approx \frac{P_N^+ + P_N^-}{4N^2}. \tag{21}
\]

**Small \( p \).** When the transition probability is small, \( p = \epsilon \) \( (\epsilon \ll 1) \), we find from Eq. (16) that

\[
P_N \approx \frac{\sin^2(N \xi)}{\sin^2(\xi)} \epsilon + O(\epsilon^2). \tag{22}
\]

Obviously, this probability depends very strongly on the phase \( \xi \). Figure 3 illustrates this dependence. The approximation [22] describes well the transition probability, which is suppressed well below the classical value [14] for most of the range, with the exception of the ranges around \( \xi = 0, \pi, 2\pi \), where it has the approximate value \( N^2 \) \( p \).

**VI. SUMMARY**

In this paper, a relation between the parameters of the single-pass and \( N \)-pass qubit propagators has been presented, Eq. (7). It allows one to determine the single-pass
transition probability $p$ by measuring the population of the initial or final state after $N$ passes. This is particularly important when the single-pass transition probability is very close to unity, $p = 1 - \epsilon$, and hence the determination of the error $\epsilon \ll 1$ is very challenging.

In several important special cases the relation between the single-pass transition probability $p$ and the $N$-pass transition probability $P_N$ allows one to unambiguously determine $p$ from $P_N$. In the most general case, the detrimental multi-pass interference due to the concomitant dynamical phases in the single-pass propagator is eliminated by taking appropriate ratios of probabilities.

The results suggest that for large single-pass transition probability, $p = 1 - \epsilon$, the quantum-mechanical probability degrades due to quantum interference much faster in sequential processes than the classical probability: as $P_N \approx 1 - N^2 \epsilon$ in the quantum case compared to $P_N^c \approx 1 - N \epsilon$ in the classical case. Therefore the actual single-pass probabilities reported in the literature hitherto, which have been calculated as $N$th roots of $N$-pass probabilities, might have been far greater than the reported values.

For small single-pass transition probability, $p \approx \epsilon$, which is important in cross-talk characterization, the relation to the $N$-pass transition probability is more complicated because in the general case it depends on one of the dynamical phases of the single-pass propagator. Yet, in special cases when the Hamiltonian possesses certain symmetry, the $N$-pass probability increases quadratically with $N$: $P_N \approx N^2 \epsilon$.

To conclude, the results presented in this paper demonstrate that extreme care must be exercised in deducing the single-pass transition probability from multiple sequential passes. While is several important special cases one can use simple estimates, such as the $N$-pass probability formula $P_N \approx 1 - N^2 \epsilon$ in the case of large single-pass probability $p = 1 - \epsilon$, it is advisable to apply the full recipe for the most general case described in Sec. V C.

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[1] K. R. Brown, A. C. Wilson, Y. Colombe, C. Ospelkaus, A. M. Meier, E. Knill, D. Leibfried, and D. J. Wineland, Phys. Rev. A 84, 030303 (2011).

[2] J. P. Gaebler, T. R. Tan, Y. Lin, Y. Wan, R. Bowler, A. C. Keith, S. Glancy, K. Coakley, E. Knill, D. Leibfried, and D. J. Wineland, Phys. Rev. Lett. 117, 060505 (2016).

[3] T. P. Harty, D. T. C. Allcock, C. J. Ballance, L. Guidoni, H. A. Janacek, N. M. Linke, D. N. Stacey, and D. M. Lucas, Phys. Rev. Lett. 113, 220501 (2014).

[4] C. J. Ballance, T. P. Harty, N. M. Linke, M. A. Sepiol, and D. M. Lucas, Phys. Rev. Lett. 117, 060504 (2016).

[5] C. Pilitz, T. Sjarunothai, A.F. Varon, and C. Wunderlich, Nat. Commun. 5, 4679 (2014).

[6] D. P. L. Aude Craik, N. M. Linke, M. A. Sepiol, T. P. Harty, J. F. Goodwin, C. J. Ballance, D. N. Stacey, A. M. Steane, D. M. Lucas, and D. T. C. Allcock, Phys. Rev. A 95, 022337 (2017).

[7] P. Kaufmann, T. F. Gloger, D. Kaufmann, M. Johanning, and C. Wunderlich, Phys. Rev. Lett. 120, 010501 (2018).

[8] R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. O’Malley, P. Roushan, A. Vainsencher, J. Wenner, A. N. Korotkov, A. N. Cieľand, and J. M. Martinis, Nature 508, 500 (2014).

[9] N. V. Vitanov, Phys. Rev. A 97, 053409 (2018).

[10] T. Takekoshi, L. Reichsöllner, A. Schindewolf, J. M. Hutton, C. R. Le Sueur, O. Dulieu, F. Ferlaino, R. Grimm, and H.-C. Nägerl, Phys. Rev. Lett. 113, 205301 (2014).

[11] G. Higgins, F. Pokorny, C. Zhang, Q. Bodart, and M. Hemmerich, Phys. Rev. Lett. 119, 220501 (2017).

[12] B. W. Shore, The Theory of Coherent Atomic Excitation (Wiley, New York, 1990).

[13] N. V. Vitanov and P. L. Knight, Phys. Rev. A 52, 2245 (1995).

[14] N. V. Vitanov and K.-A. Suominen, Phys. Rev. A 59, 4580 (1999).

[15] L. D. Landau, Phys. Z. Sowjetunion 5, 4679 (1932).

[16] C. Zener, Proc. R. Soc. A 137, 696 (1932).

[17] E. C. G. Stückelberg, Helv. Phys. Acta 5, 369 (1932).

[18] E. Majorana, Nuovo Cimento 9, 43 (1932).

[19] N. V. Vitanov and B. M. Garraway, Phys. Rev. A 53, 4288 (1996); Erratum Phys. Rev. A 54, 5458 (1996).

[20] L. Allen and J. H. Eberly, Optical Resonance and Two-Level Atoms (Dover, New York, 1975).

[21] F. T. Hioe, Phys. Rev. A 30, 2100 (1984).

[22] G. S. Vasilyev and N. V. Vitanov, J. Chem. Phys. 123, 174106(2005).
[23] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, Annu. Rev. Phys. Chem. 52, 763 (2001).
[24] J. B. Delos and W. R. Thorson, Phys. Rev. A 6, 729 (1972).
[25] A. Bambini and M. Lindberg, Phys. Rev. 30, 794 (1984).
[26] N. Rosen and C. Zener, Phys. Rev. 40, 502 (1932).
[27] G. S. Vasilev and N. V. Vitanov, Phys. Rev. A 70, 053407 (2004).