Functional limit theorem for occupation time processes of infinite ergodic transformations

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Abstract

We establish a functional limit theorem for joint laws of occupations near and away from indifferent fixed points of interval maps, in the sense of strong distributional convergence. It is a functional and joint-distributional extension both of Darling–Kac type limit theorem and of Lamperti type generalized arcsine laws at the same time. For the proof, we represent occupation time processes in terms of excursion lengths, and show a functional convergence of sum of excursion lengths to a stable Lévy process.

1 Introduction

In statistical physics (see for example Pomeau–Manneville [28] and Manneville [25]), interval maps with indifferent fixed points have been studied as models of intermittent phenomena, such as intermittent transitions to turbulence in convective fluid. In this context, the occupations near the indifferent fixed points correspond to long regular or laminar phases, while the occupations away from them correspond to short irregular or turbulent bursts.

In the present paper, we study the time evolution of the occupations near and away from the indifferent fixed points, and obtain its scaling limit in the sense of strong distributional convergence. Our limit theorem is a functional and joint-distributional extension both of Darling–Kac type limit theorem [2, 36, 43, 26] and of Lamperti type generalized arcsine laws [35, 36, 43, 31] at the same time.

Let us illustrate earlier studies. Following Thaler [35, Examples], we define an interval map $T : [0, 1] \rightarrow [0, 1]$ by

$$Tx = \begin{cases} \frac{x(1-x)}{1-x-x^2} & (0 \leq x \leq \frac{1}{2}), \\ 1 - T(1-x) & \left(\frac{1}{2} < x \leq 1\right) \end{cases}$$

See Figure [1]. Note that 0 and 1 are indifferent fixed points of $T$, that is, $T0 = 0$, $T1 = 1$ and $T'0 = T'1 = 1$. In addition, we have $Tx = x + x^3 + o(x^3)$, as $x \rightarrow 0$. This map is conjugate to Boole’s transformation, which is a typical example of infinite ergodic transformations. The map $T$ has an ergodic invariant measure

$$\mu(dx) = (x^{-2} + (1-x)^{-2})1_{(0,1)}(x)dx,$$

whose total mass is infinite. Let us fix $\delta \in (0, 1/2)$ from now on. With respect to $\mu$, the interval $[\delta, 1 - \delta]$ has finite mass and its complement has infinite mass. Birkhoff’s
Figure 1: the graph of $T$

Figure 2: the orbit $(x_0, Tx_0, T^2x_0, \ldots)$ starting at $x_0 = 1/10$

Pointwise ergodic theorem implies

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}} \to 0 \quad \text{or equivalently,} \quad \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \notin [\delta, 1-\delta]\}} \to 1 \quad \text{a.e.} \quad (1.3)$$

Roughly speaking, the orbit $(x, Tx, T^2x, \ldots)$ of almost every initial point $x$ is concentrated close to 0 and 1. See Figure 2. We are interested in non-trivial scaling limits of the occupation times for $[0, \delta], [\delta, 1-\delta]$ and $(1-\delta, 1]$. Unfortunately, we cannot obtain them in the sense of a.e. convergence. See [1, 6] for the details. Instead, we can obtain them in the sense of strong distributional convergence, as we shall see in the following.

On the one hand, Aaronson [2, Theorem 1 and Example 4] obtained the following strong distributional convergence of Darling–Kac type:

$$\frac{C}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}} \mathcal{L}(\mu) \mathcal{L}(t) = \lim_{n \to \infty} \frac{1}{\sqrt{2\varepsilon}} \int_0^t \mathbb{1}_{\{|B(s)| \leq \varepsilon\}} ds, \quad \text{in } \mathbb{R}, \quad (1.4)$$

for $t > 0$ and $C = \sqrt{2\pi}/\mu([\delta, 1-\delta])$, where the notation $\mathcal{L}(\mu)$ denotes the strong distributional convergence with respect to $\mu$ (it will be explained in Subsection 4.1), and $B = (B(t) : t \geq 0)$ denotes a one-dimensional Brownian motion started at the origin, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In other words, for any absolutely continuous probability measure $\nu$ on $[0, 1]$, it holds that

$$\nu \left[ x \in [0, 1] : \frac{C}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}} \leq u \right] \to \mathbb{P}[L(t) \leq u] = \int_0^u \frac{e^{-s^2/(4t)}}{\sqrt{\pi t}} ds, \quad u \geq 0.$$

The limit random variable $L(t)$ is called the Brownian local time at the origin up to time $t$ in the Blumenthal–Getoor normalization. Its one-dimensional marginal law is a Mittag-Leffler distribution of order 1/2. Owada–Samorodnitsky [20, Theorem 6.1] showed a functional extension of (1.4):

$$\left( \frac{C}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}} : t \geq 0 \right) \overset{\mathcal{L}(\mu)}{\underset{n \to \infty}{\Rightarrow}} (L(t) : t \geq 0), \quad \text{in } D([0, \infty), \mathbb{R}), \quad (1.5)$$

Figure 1: the graph of $T$

Figure 2: the orbit $(x_0, Tx_0, T^2x_0, \ldots)$ starting at $x_0 = 1/10$
where $D([0, \infty), \mathbb{R})$ denotes the space of càdlàg functions $w : [0, \infty) \to \mathbb{R}$, endowed with the Skorokhod $J_1$-topology. The convergence (1.5) is stronger than (1.4), since distributional convergence in $D([0, \infty), \mathbb{R})$ implies convergence of finite-dimensional marginal distributions.

On the other hand, Thaler [35, Theorem] obtained the following Lamperti type generalized arcsine law for the occupations near 0: for any $t > 0$, 

$$
\frac{1}{n} \sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k x < \delta\}} \xrightarrow{L(\mu)} Z_-(t) = \int_0^t \mathbb{1}_{\{B(s) < 0\}} \, ds, \quad \text{in } \mathbb{R}.
$$

The limit random variable $Z_-(t)$ denotes the amount of time which $B$ spends on the negative side $(-\infty, 0)$ up to time $t$. Its one-dimensional marginal law is an arcsine distribution. The convergence (1.6) is equivalent to the following: for any absolutely continuous probability measure $\nu$ on $[0, 1]$, it holds that

$$
\nu \left[ x \in [0, 1] : \frac{1}{n} \sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k x < \delta\}} \leq u \right] \xrightarrow{n \to \infty} \mathbb{P}[Z_-(t) \leq u] = \frac{2}{\pi} \arcsin \left( \sqrt{\frac{u}{t}} \right), \quad 0 \leq u \leq t.
$$

In the previous study [31], the author and Kouji Yano studied a certain class of interval maps with three or more indifferent fixed points, and obtained the scaling limit of the joint law of the occupations near each of the indifferent fixed points. The scaling limit is a joint law of occupation times of a skew Bessel diffusion process on multiray up to a fixed time, and is interpreted as a multidimensional version of Lamperti’s generalized arcsine distribution.

We now illustrate our main result. In Theorem 4.5 (see also Example 4.7), we obtain a functional convergence of the occupations near and away from the indifferent fixed points:

$$
\left( \frac{1}{n} \sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k x < \delta\}}, \frac{C}{\sqrt{n}} \sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k x \in [\delta, 1-\delta]\}}, \frac{1}{n} \sum_{k=0}^{[nt]} \mathbb{1}_{\{T^k x > 1-\delta\}} : t \geq 0 \right) \xrightarrow{L(\mu)} (Z_-(t), L(t), Z_+(t) : t \geq 0), \quad \text{in } D([0, \infty), \mathbb{R}^3),
$$

where $Z_+(t) = t - Z_-(t)$ denotes the amount of time which $B$ spends on the positive side $(0, \infty)$ up to time $t$. The convergence (1.7) can also be written as the strong distributional convergence in $C([0, \infty), \mathbb{R}^3)$ if we consider the linear interpolation of the occupation time processes. Needless to say, our result (1.7) is a refinement of the earlier results (1.4), (1.5) and (1.6). More generally, we focus on interval maps with two or more indifferent fixed points, and study the time evolution of the occupations near and away from them. We prove that its scaling limit is a joint law of occupation and local time processes of a skew Bessel diffusion process on multiray.

Let us outline the proof of (1.7). First, we note the following dynamically-separating property: if $\delta \in (0, 1/2)$ is sufficiently small, then the orbit $(x, Tx, T^2 x, \ldots)$ cannot pass from $[0, \delta]$ to $(1-\delta, 1)$ (or from $(1-\delta, 1]$ to $[0, \delta]$) without visiting $[\delta, 1-\delta]$. Let us denote by
\( \eta_1(t) \) (respectively, \( \eta_2(t) \)) the amount of time which the orbit spends on \([0, \delta] \) (respectively, \((1 - \delta, 1])\) up to the \( \lfloor t \rfloor \)th return time for \([\delta, 1 - \delta] \). Using the dynamically-separating property, we obtain a discrete version of Williams formulae (Lemma 5.1), which represent the occupation times for \([0, \delta], [\delta, 1 - \delta] \) and \((1 - \delta, 1] \) in terms of \( \eta_1 \) and \( \eta_2 \). Second, we prove that a sequence \((\eta_1(N) - \eta_1(N - 1), \eta_2(N) - \eta_2(N - 1) : N \geq 1)\) is strictly stationary and exponentially continued fraction mixing with respect to \( \mu(\cdot \cap [\delta, 1 - \delta]) / \mu([\delta, 1 - \delta]) \) (Section 6). By virtue of these properties, we can use Tyran-Kamińska’s functional limit theorem \(^{37}\) of Skorokhod \(^{32}\) type, in order to show that \((\eta_1(t), \eta_2(t) : t \geq 0)\) converges to a joint of some independent stable subordinators (Lemma 5.3). Third, we recall the Williams formulae for diffusion processes (Lemma 5.2), which represent \(Z_-, L \) and \(Z_+ \) in terms of some independent stable subordinators. Finally, combining these results, we obtain the convergence \(^{1.7}\) (Theorems 4.1 and 4.5 and Example 4.7).

We give a quick review of the histories of Darling–Kac type limit theorems and of Lamperti type generalized arcsine laws. On the one hand, Darling–Kac \(^{13}\) studied the occupation times of Markov processes on a subset having finite mass, and proved that their scaling limits are Mittag-Leffler distributions. Darling–Kac type limit theorems have been further developed, for example by Karlin–McGregor \(^{20}\) for birth-and-death processes, by Kasahara \(^{21, 22}\) for diffusion processes, and by Aaronson \(^{2}\), Thaler–Zweimüller \(^{36}\) and Zweimüller \(^{43}\) for infinite ergodic transformations. Bingham \(^{9}\) (respectively, Owada–Samorodnitsky \(^{26}\)) studied a functional extension of \(^{13}\) (respectively, \(^{2}\)). On the other hand, Lévy \(^{24}\) studied the amount of time which a simple random walk or a Brownian motion spends on the positive side, and obtained the well-known arcsine law. Lamperti \(^{23}\) generalized Lévy’s arcsine law to some discrete-time stochastic processes. He obtained the class of possible limit distributions which are called Lamperti’s generalized arcsine distributions. Barlow–Pitman–Yor \(^{8}\), Watanabe \(^{39}\) and Yuko Yano \(^{41}\) obtained Lamperti type generalized arcsine laws or joint-distributional extensions of them for diffusion processes. Fujihara–Kawamura–Yuko Yano \(^{15}\) studied a functional generalization of \(^{23}\). Thaler \(^{35}\), Thaler–Zweimüller \(^{36}\) and Zweimüller \(^{43}\) studied Lamperti type generalized arcsine laws for infinite ergodic transformations. In the previous study \(^{31}\), the author and Kouji Yano obtained a joint-distributional extension of \(^{43}\).

The present paper is organized as follows. In Section 2 we set up notations and state our assumptions in a general setting. In Section 3 we recall the definition and basic properties of skew Bessel diffusion processes on multiray. In Section 4 we formulate our main result in the general setting and then apply it to interval maps with indifferent fixed points. The proofs of our general limit theorem and of its application are given in Sections 5 and 6 respectively. In Appendices, we recall several facts about uniformly expanding interval maps, the Skorokhod \(J_1\)-topology and measure-valued processes.

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2 Notations and assumptions in a general setting

Let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space with \(\mu(X) = \infty\). Let \(T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)\) be a conservative, ergodic, measure preserving transformation, which will be abbreviated as \(\text{CEMPT}\). Equivalently, we assume that \(\mu \circ T^{-1} = \mu\) and, for any \(A \in \mathcal{B}\) with \(\mu(A) > 0\),

\[
\sum_{n \geq 0} 1_{\{T^n x \in A\}} = \infty, \quad \mu\text{-a.e.} x.
\]

For the details, see [4, Chapter 1]. We always use the summation signs to denote unions of disjoint sets; for example, \(A_1 + A_2, \sum_j A_j\) and so on. Let \(d \geq 1\) be a positive integer and \(A_1, \ldots, A_d, Y \in \mathcal{B}\) be disjoint sets. We will say that \(Y\) dynamically separates \(A_1, \ldots, A_d\) if the condition \([x \in A_i \text{ and } T^n x \in A_j \text{ for some } i \neq j \text{ and some } n \geq 1\) implies the existence of some \(k \in \{1, \ldots, n-1\}\) for which \(T^k x \in Y\).

**Assumption 2.1** (dynamical separation). Let \(d \geq 1\) be a positive integer. The state space \(X\) can be decomposed into \(X = \sum_{j=1}^d A_j + Y\) for \(A_1, \ldots, A_d, Y \in \mathcal{B}\) with \(\mu(A_j) = \infty\) \((j = 1, \ldots, d)\) and \(\mu(Y) \in (0, \infty)\). Furthermore, \(Y\) dynamically separates \(A_1, \ldots, A_d\).

We will assume Assumption 2.1 from now on. Roughly speaking, \(A_1, \ldots, A_d\) are rays and \(Y\) is a junction; the orbit cannot pass from \(A_i\) to \(A_j\) \((i \neq j)\) without visiting \(Y\). For \(A \in \mathcal{B}\) and \(t \geq 0\), we define a measurable function \(S_A(t) : X \to [0, \infty)\) by

\[
S_A(t)(x) := \sum_{k=0}^{\lfloor t \rfloor - 1} 1_{\{T^k x \in A\}} + (t - \lfloor t \rfloor)1_{\{T^0 x \in A\}}, \quad x \in X.
\]

In other words, \(S_A(n) \quad (n \in \mathbb{N})\) denotes the amount of time which the orbit spends on \(A\) up to time \(n - 1\), and \(S_A(t) \quad (t \geq 0)\) denotes the linear interpolation of \(S_A(n)\) at time \(t\). Birkhoff’s pointwise ergodic theorem implies

\[
\frac{1}{n} S_Y(n) \to 0 \quad \left(\text{or equivalently, } \frac{1}{n} S_{A_1+\ldots+A_d}(n) \to 1\right), \quad \mu\text{-a.e.} \tag{2.2}
\]

Let us define measurable functions \(\varphi : Y \to \mathbb{N} \cup \{\infty\}\) and \(\ell_j : Y \to \mathbb{N} \cup \{\infty\}\) by

\[
\varphi(x) := \min\{k \geq 1 : T^k x \in Y\}, \quad x \in Y, \tag{2.3}
\]

\[
\ell_j(x) := S_{A_j}(\varphi(x))(x) = \sum_{k=0}^{\varphi(x)-1} 1_{\{T^k x \in A_j\}}, \quad x \in Y. \tag{2.4}
\]

In other words, \(\varphi(x)\) denotes the first return time for \(Y\), and \(\ell_j(x)\) denotes the length of time spent in \(A_j\) by the first excursion \((T^k x : 0 \leq k < \varphi(x))\) away from \(Y\). Since \(T\) is a \(\text{CEMPT}\), we have \(\varphi, \ell_j < \infty, \mu(\cdot \cap Y)\text{-a.e.}\) By virtue of Assumption 2.1, we have

\[
\{\ell_j = n\} = Y \cap T^{-1} A_j \cap \{\varphi = n + 1\} = \{x \in Y : Tx, \ldots, T^n x \in A_j \text{ and } T^{n+1} x \in Y\}, \quad n \geq 1.
\]

In addition, set

\[
\ell(x) := (\ell_1(x), \ldots, \ell_d(x)), \quad x \in Y. \tag{2.5}
\]
Example 2.2. Assume that
\[(T^k x)^3_{k=0} \in Y \times A_1 \times A_1 \times Y, \quad \text{and} \quad (T^k y)_{k=0}^{1} \in Y \times Y.\]
Then we have \(\varphi(x) = 3, \ell(x) = (2, 0, \ldots, 0), \varphi(y) = 1\) and \(\ell(y) = (0, 0, \ldots, 0)\).

Let \(f, g : (0, \infty) \to (0, \infty)\) be measurable functions. For a constant \(c \in [0, \infty]\), we will write \(f(x) \sim cg(x)\) if it holds that \(\lim_{x \to x_0} f(x)/g(x) = c\). We note that \(cg(x)\) has only a symbolic meaning if \(c = 0\) or \(c = \infty\). See [10, p. xix]. Let \(\alpha \in \mathbb{R}\). We will write \(f \in \mathcal{R}_\alpha(\infty)\) if \(f\) is regularly varying of index \(\alpha \in \mathbb{R}\) at \(\infty\), that is, \(f(\lambda r) \sim \lambda^\alpha f(r)\) for each \(\lambda > 0\).

Similarly, we will write \(f \in \mathcal{R}_\alpha(0+)\) if \(f\) is regularly varying of index \(\alpha\) at 0. Let \((a_n)_{n \geq 0}\) be a \((0, \infty)\)-valued sequence. We will write \((a_n)_{n \geq 0} \in \mathcal{R}_\alpha(\infty)\) if the function \([r \mapsto a_(\lfloor r \rfloor)]\) is regularly varying of index \(\alpha\) at \(\infty\). For basic discussions of regular variation, we refer the reader to [10, Chapter 1].

Let us define a probability measure \(\mu_Y\) on \((X, \mathcal{B})\) by
\[\mu_Y(A) := \mu(A \cap Y)/\mu(Y), \quad A \in \mathcal{B}.\] (2.6)

**Assumption 2.3** (regular variations of tail probabilities). For constants \(\alpha \in (0, 1), \beta = (\beta_1, \ldots, \beta_d) \in [0, 1]^d\) with \(\sum_{j=1}^d \beta_j = 1\) and a sequence \((a_n)_{n \geq 0} \in \mathcal{R}_\alpha(\infty)\), it holds that
\[\mu_Y[\ell_j \geq n] \sim \frac{1}{\Gamma(1 - \alpha)a_n} \beta_j, \quad j = 1, \ldots, d,\] (2.7)
where \(\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt\) \((z > 0)\) denotes the gamma function.

**Remark 2.4.** As shown in [36 (6.6)], we have
\[\mu\left(\bigcup_{k=0}^{n} T^{-k} Y \cap A_j\right) - \mu\left(\bigcup_{k=0}^{n-1} T^{-k} Y \cap A_j\right) = \mu(Y)\mu_Y[\ell_j \geq n], \quad n \geq 1, \quad j = 1, \ldots, d.\]
Hence, by Karamata’s Tauberian theorem (see for example [36 Proposition 4.2]), the asymptotic relation (2.7) is equivalent to the following:
\[\mu\left(\bigcup_{k=0}^{n} T^{-k} Y \cap A_j\right) \sim \beta_j \frac{\mu(Y)n}{\Gamma(2 - \alpha) a_n}, \quad j = 1, \ldots, d.\] (2.8)

We will also assume Assumption 2.3 from now on. We will denote by \(T_Y : Y \to Y\) an induced map for \(Y\):
\[T_Y x := T^{\varphi(x)} x, \quad x \in Y.\] (2.9)

The map \(T_Y\) is a CEMPT on the probability space \((X, \mathcal{B}, \mu_Y)\). Therefore, a sequence \((\ell \circ T^n_Y : n \geq 0)\) is strictly stationary with respect to \(\mu_Y\). Note that \((\ell_j \circ T^n_Y)\) is the length of time spent in \(A_j\) by the \((n+1)\)th excursion \((T^k(T^n_Y) x) : 0 \leq k < \varphi(T^n_Y x)\) away from \(Y\).
Example 2.5. Assume that

\[(T^k x)_k^{0} \in Y \times A_1 \times Y \times A_2 \times Y \times Y.\]

Then we have \(\ell(x) = (1, 0, \ldots, 0), \ell \circ T_Y(x) = (0, 2, \ldots, 0)\) and \((\ell \circ T^2_Y)(x) = (0, 0, \ldots, 0)\).

For \(0 \leq n \leq m \leq \infty\), we define a sub-\(\sigma\)-field \(\mathcal{F}_n \subset \mathcal{B}\) by \(\mathcal{F}_n := \sigma\{\ell \circ T^k_Y : n \leq k \leq m\}\). For \(n \geq 0\), set

\[\phi_0(n) := \sup\{|\mu_Y(A \cap B) - \mu_Y(A)\mu_Y(B)| : k \geq 0, A \in \mathcal{F}_0^k, B \in \mathcal{F}_k^{\infty + n}\}.\]

Following [37, Theorem 1.1], we will impose the following assumption:

**Assumption 2.6 (local dependence).** For any \(\varepsilon > 0\), there exist \(N\)-valued sequences \((r_n)_{n \geq 0}\) and \((s_n)_{n \geq 0}\) such that

\[s_n, \frac{r_n}{s_n}, \frac{a_n}{r_n} \to \infty, \quad \text{and} \quad \frac{a_n}{r_n} \phi_0(s_n) \to 0,\]

and

\[\mu_Y\left[\max_{1 \leq k \leq r_n} |\ell \circ T^k_Y| > \varepsilon n \mid |\ell| > \varepsilon n\right] \to 0,\]

where \(\mu_Y[\cdot]\) denotes the conditional probability, and \(|\cdot|\) denotes the Euclid norm of \(\mathbb{R}^d\).

**Remark 2.7.** In earlier studies [2, 36, 43, 26, 31], Assumption 2.6 was not imposed. Instead, the assumption of pointwise dual ergodicity was imposed in [2, Theorem 1] and [26, Theorem 6.1]; the assumptions associated with asymptotic entrance densities were imposed in [36, Theorems 3.1 and 3.2], [43, Theorems 2.1 and 2.2] and [31, Theorems 2.7 and 2.8]. We don’t know whether these assumptions are weaker than Assumption 2.6 or not. Anyway, in our setting of interval maps (see Subsection 4.2), all of these assumptions are satisfied under a certain choice of \(A_1, \ldots, A_d\) and \(Y\). See [3, Theorem 3], [36, Theorem 8.1] and Section 6.

**Remark 2.8.** Suppose that Assumptions 2.1 and 2.3 hold. Furthermore, assume that the sequence \((\ell \circ T^n_Y : n \geq 0)\) is **exponentially continued fraction mixing** with respect to \(\mu_Y\), that is, there exist \(C \in (0, \infty)\) and \(\theta \in (0, 1)\) such that, for any \(k, n \geq 0\), \(A \in \mathcal{F}_0^k\) and \(B \in \mathcal{F}_k^{\infty + n}\),

\[|\mu_Y(A \cap B) - \mu_Y(A)\mu_Y(B)| \leq C\theta^n \mu_Y(A)\mu_Y(B).\]

Then, by a similar argument as in the proof of [38, Corollary 4.1], we can show that the conditions of Assumption 2.6 are satisfied.

For a Polish space \(U\), we will write \(C([0, \infty), U)\) for the space of all continuous functions \(w : [0, \infty) \to U\). The space \(C([0, \infty), U)\) is endowed with the Polish topology of uniform convergence on compact subsets of \([0, \infty)\).
3 Skew Bessel diffusion processes on multiray

Following [8], we will define skew Bessel diffusion processes on multiray and give generalized arcsine laws for them. Note that a standard one-dimensional Brownian motion is a typical example of them. For basic properties of Bessel diffusion processes reflected at the origin, see for instance [8, Section 2] and [29, Chapter XII]. For a deeper discussion of diffusion processes on multiray, we refer the reader to [41, Section 2]. For basic discussions of the Itô excursion theory, see for instance [16, 29, Chapter XII] or [30, Section VI.8].

For $\alpha \in (0, 1)$, we will denote by $R^{(\alpha)} = (R^{(\alpha)}(t) : t \geq 0)$ a Bessel diffusion process of dimension $2 - 2\alpha \in (0, 2)$, started and instantaneously reflected at the origin. In other words, $R^{(\alpha)}$ is a $[0, \infty)$-valued regular diffusion process started at the origin with scale function $s^{(\alpha)}(x) = x^{2\alpha}, x \geq 0$, and speed measure $m^{(\alpha)}(dx) = \alpha^{-1}x^{1-2\alpha}\mathbb{1}_{(0,\infty)}(x)dx$, $x \geq 0$. In the special case of $\alpha = 1/2$, this process is nothing else but the absolute value of a standard one-dimensional Brownian motion. Let us denote by $(L^{(\alpha)}(t, x) : t, x \geq 0)$ the local time of $R^{(\alpha)}$. More specifically, a map $[0, \infty)^2 \ni (t, x) \mapsto L^{(\alpha)}(t, x) \in [0, \infty)$ is jointly continuous, and for any bounded and measurable function $f : [0, \infty) \to [0, \infty)$, the following occupation-time formula holds:

$$
\int_0^t f(R^{(\alpha)}(s))ds = C^{(\alpha)}\int_0^\infty f(x)L^{(\alpha)}(t, x)x^{1-2\alpha}dx, \quad t \geq 0,
$$

where $C^{(\alpha)} := \frac{2^{\alpha}\Gamma(\alpha)}{\Gamma(1 - \alpha)}$. We will write $L^{(\alpha)} = (L^{(\alpha)}(t) : t \geq 0)$ for the local time of $R^{(\alpha)}$ at the origin (in the Blumenthal–Getoor normalization in the sense of [30, VI.(45.5)]):

$$
L^{(\alpha)}(t) := L^{(\alpha)}(t, 0) = \lim_{\varepsilon \downarrow 0} \frac{2 - 2\alpha}{C^{(\alpha)}2^{-2\alpha}} \int_0^t \mathbb{1}_{\{R^{(\alpha)}(s) \leq \varepsilon\}}ds, \quad t \geq 0. \quad (3.1)
$$

Then $L^{(\alpha)}$ is a Mittag-Leffler process of order $\alpha$ in the sense of [26, Section 3]. Its one-dimensional distributions are characterized by

$$
\mathbb{E}[\exp(-\lambda L^{(\alpha)}(t))] = \sum_{n \geq 0} \frac{(-\lambda t^{\alpha})^n}{\Gamma(1 + n\alpha)}, \quad \lambda, t \geq 0.
$$

The finite-dimensional distributions of $L^{(\alpha)}$ are characterized by [9, Propositions 1(a) and 1(b)]. We will denote by $\eta^{(\alpha)} = (\eta^{(\alpha)}(s) : s \geq 0)$ the inverse local time of $R^{(\alpha)}$ at the origin:

$$
\eta^{(\alpha)}(s) := (L^{(\alpha)})^{-1}(s) = \inf\{t > 0 : L^{(\alpha)}(t) > s\}, \quad s \geq 0. \quad (3.2)
$$

Then $\eta^{(\alpha)}$ is an $\alpha$-stable subordinator with Laplace transform

$$
\mathbb{E}[\exp(-\lambda \eta^{(\alpha)}(s))] = \exp(-\lambda^\alpha s), \quad \lambda, s \geq 0.
$$

We will denote by $n^{(\alpha)}$ the Itô characteristic measure of excursions of $R^{(\alpha)}$ away from the origin, for the normalization $L^{(\alpha)}$. For $z \in \mathbb{C}$ with $|z| = 1$, we define a map $\sigma_z : C([0, \infty), \mathbb{C}) \to C([0, \infty), \mathbb{C})$ by

$$
\sigma_z((w(t) : t \geq 0)) = (w(t)z : t \geq 0),
$$
that is, $\sigma_z$ is a rotation about the origin. Note that an image measure $n^{(\alpha)} \circ \sigma_z^{-1}$ is the Itô characteristic measure of excursions of $\sigma_z(R^{(\alpha)})$ away from the origin. Let $d \geq 1$ be a positive integer. Set
\[ z_j := \exp\left(2\pi j \sqrt{-1}/d\right), \quad j = 1, \ldots, d, \]
\[ I_j := \{rz_j \in \mathbb{C} : r \geq 0\}, \quad j = 1, \ldots, d. \]

**Definition 3.1** (skew Bessel diffusion process). For $\alpha \in (0, 1)$ and $\beta = (\beta_1, \ldots, \beta_d) \in [0, 1]^d$ with $\sum_j \beta_j = 1$, we will denote by $R^{(\alpha, \beta)} = (R^{(\alpha, \beta)}(t) : t \geq 0)$ the process pieced together from excursions away from the origin associated with Itô characteristic measure
\[ \sum_{j=1}^d \beta_j (n^{(\alpha)} \circ \sigma_z^{-1}). \]

It is called a $(\bigcup_{j=1}^d I_j)$-valued skew Bessel diffusion process of dimension $2 - 2\alpha \in (0, 2)$, with skewness parameter $\beta$, started at the origin.

**Remark 3.2.** Roughly speaking, every time the process $R^{(\alpha, \beta)}$ reaches to the origin, it chooses randomly a ray $I_j$ from rays $I_1, \ldots, I_d$ with probability $\beta_j$, and then it moves like the $I_j$-valued diffusion process $\sigma_z(R^{(\alpha)})$ until it returns again to the origin. It is a multiray diffusion process on $\bigcup_{j=1}^d I_j$ in the sense of [11, Section 2] (see [11, Remark 2.2]).

**Remark 3.3.** In the case of $\alpha = 1/2$, the process $R^{(\alpha, \beta)}$ is also called a Walsh Brownian motion or a Brownian spider. For more direct construction of Walsh Brownian motions, we refer the reader to [7]. In the special case of $d = 2$ and $\alpha = \beta_1 = \beta_2 = 1/2$, the process $R^{(\alpha, \beta)}$ is nothing else but a standard one-dimensional Brownian motion.

Set
\[ Z_j^{(\alpha, \beta)}(t) := \int_0^t 1_{\{R^{(\alpha, \beta)}(s) \in I_j\}} \, ds, \quad t \geq 0, \quad j = 1, \ldots, d, \quad (3.3) \]
\[ L^{(\alpha, \beta)}(t) := \lim_{\varepsilon \downarrow 0} \frac{2 - 2\alpha}{C^{(\alpha)}\varepsilon^{2-2\alpha}} \int_0^t 1_{\{|R^{(\alpha, \beta)}(s)| \leq \varepsilon\}} \, ds, \quad t \geq 0. \quad (3.4) \]

That is, $Z_j^{(\alpha, \beta)}(t)$ denotes the amount of time which $R^{(\alpha, \beta)}$ spends on $I_j$ up to time $t$, and $L^{(\alpha, \beta)}(t)$ denotes the local time of $R^{(\alpha, \beta)}$ at the origin up to time $t$. Note that $L^{(\alpha, \beta)} \overset{d}{=} L^{(\alpha)}$, in $C([0, \infty), \mathbb{R})$. Recall that $R^{(\alpha, \beta)}$ has the Brownian scaling property in the sense that
\[ \left( \frac{R^{(\alpha, \beta)}(\lambda t)}{\sqrt{\lambda}} : t \geq 0 \right) \overset{d}{=} \left( R^{(\alpha, \beta)}(t) : t \geq 0 \right), \quad \text{in } C([0, \infty), \mathbb{C}), \quad \lambda > 0. \]

Hence, its occupation times for each rays and its local time at the origin have the following scaling property:
\[ \left( \frac{Z_1^{(\alpha, \beta)}(\lambda t)}{\lambda}, \ldots, \frac{Z_d^{(\alpha, \beta)}(\lambda t)}{\lambda}, \frac{L^{(\alpha, \beta)}(\lambda t)}{\lambda^{2\alpha}} : t \geq 0 \right) \overset{d}{=} \left( Z_1^{(\alpha, \beta)}(t), \ldots, Z_d^{(\alpha, \beta)}(t), L^{(\alpha, \beta)}(t) : t \geq 0 \right), \quad \text{in } C([0, \infty), \mathbb{R}^{d+1}), \quad \lambda > 0. \]

The following identity in joint-distribution was proved by [8].
Theorem 3.4 ([8] Theorem 1]). Let $\alpha \in (0, 1)$ and $\beta = (\beta_1, \ldots, \beta_d) \in [0, 1]^d$ with $\sum_{j=1}^d \beta_j = 1$. Then, for any $t > 0$, it holds that

$$
\left( \frac{Z_1^{(\alpha,\beta)}(t)}{t}, \ldots, \frac{Z_d^{(\alpha,\beta)}(t)}{t}, \frac{L^{(\alpha,\beta)}(t)}{t^{\alpha}} \right) \overset{d}{=} \left( \frac{\xi_1}{\sum_{j=1}^d \xi_j}, \ldots, \frac{\xi_d}{\sum_{j=1}^d \xi_j}, \frac{1}{(\sum_{j=1}^d \xi_j)^\alpha} \right), \text{ in } \mathbb{R}^{d+1},
$$

where $\xi_1, \ldots, \xi_d$ denote independent $[0, \infty)$-valued random variables with the one-sided $\alpha$-stable distributions characterized by

$$
\mathbb{E}[\exp(-\lambda \xi_j)] = \exp(-\lambda^\alpha \beta_j), \quad \lambda \geq 0, \; j = 1, \ldots, d. \quad (3.6)
$$

Recall that the joint law of $(\xi_1/\sum_{j=1}^d \xi_j, \ldots, \xi_d/\sum_{j=1}^d \xi_j)$ is a multidimensional version of Lamperti’s generalized arcsine distributions (see for example \[31\], Subsection 2.2), and the law of $(\sum_{j=1}^d \xi_j)^{-\alpha}$ is a Mittag-Leffler distribution of order $\alpha$.

4 Main result

4.1 Functional limit theorem for occupation time processes

Let $U$ be a Polish space, $\nu_0$ a probability measure on $(X, \mathcal{B})$, and $(F_n)_{n \geq 0}$ a sequence of $U$-valued measurable functions defined on $(X, \mathcal{B})$. Let $\zeta$ be a $U$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will write $[F_n \overset{\nu_0}{\Rightarrow} \zeta, \text{ in } U]$ if a sequence of pushforward probability measures $(\nu_0 \circ F_n^{-1})_{n \geq 0}$ converge weakly to the law $\mathbb{P}[\zeta \in \cdot]$ of $\zeta$. We say that the $F_n$ converge to $\zeta$ strongly in distribution (with respect to $\mu$) if, for any probability measure $\nu \ll \mu$ on $(X, \mathcal{B})$, the convergence $[F_n \overset{\nu}{\Rightarrow} \zeta, \text{ in } U]$ holds. We will denote this convergence by $[F_n \overset{\mathbb{L}(\mu)}{\Rightarrow} \zeta, \text{ in } U]$.

We can now formulate our main result in the general setting.

Theorem 4.1 (functional convergence of occupation time processes). Let $T$ be a CEMP on $(X, \mathcal{B}, \mu)$ and suppose that Assumptions [2.7], [2.3] and [2.6] hold. Then,

$$
\left( \frac{1}{n} S_{A_1}(nt), \ldots, \frac{1}{n} S_{A_d}(nt), \frac{1}{a_n} S_Y(nt) : t \geq 0 \right)
\overset{\mathbb{L}(\mu)}{\underset{n \to \infty}{\Rightarrow}} \left( Z_1^{(\alpha,\beta)}(t), \ldots, Z_d^{(\alpha,\beta)}(t), L^{(\alpha,\beta)}(t) : t \geq 0 \right), \text{ in } C([0, \infty), \mathbb{R}^{d+1}).
$$

(4.1)

The proof of Theorem 4.1 will be given in Section 5. Combining Theorem 4.5 with Theorem 3.4 we obtain the following corollary, since the distributional convergence in $C([0, \infty), \mathbb{R}^{d+1})$ implies the convergence of finite-dimensional marginal laws.
Corollary 4.2 (marginal convergence). Let $T$ be a CEMPT on $(X, \mathcal{B}, \mu)$ and suppose that Assumptions 2.1, 2.3 and 2.6 hold. Then,

$$
\left( \frac{1}{n} S_{A_1}(n), \ldots, \frac{1}{n} S_{A_d}(n), \frac{1}{a_n} S_Y(n) \right) \xrightarrow{\mathbb{L}(\mu)} \left( \frac{\xi_1}{\sum_{j=1}^d \xi_j}, \ldots, \frac{\xi_d}{\sum_{j=1}^d \xi_j}, \frac{1}{\sum_{j=1}^d \xi_j} \right), \quad \text{in } \mathbb{R}^{d+1},
$$

(4.2)

where $\xi_1, \ldots, \xi_d$ denote independent $[0, \infty)$-valued random variables with the one-sided $\alpha$-stable distributions characterized by (3.6).

Remark 4.3. In addition, suppose that Borel subsets $A_1', \ldots, A_d', Y' \in \mathcal{B}([0, 1])$ satisfy

$$
\mu(A_j \triangle A_j') < \infty, \quad j = 1, \ldots, d, \quad \text{and} \quad \mu(Y') < \infty,
$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of $A$ and $B$. Then the convergence (4.1) implies

$$
\left( \frac{1}{n} S_{A_1'}(nt), \ldots, \frac{1}{n} S_{A_d'}(nt), \frac{1}{a_n} S_{Y'}(nt) : t \geq 0 \right) \xrightarrow{\mathbb{L}(\mu)} \left( Z_1(\alpha, \beta)(t), \ldots, Z_d(\alpha, \beta)(t), \frac{\mu(Y')}{\mu(Y)} L(\alpha, \beta)(t) : t \geq 0 \right), \quad \text{in } C([0, \infty), \mathbb{R}^{d+1}),
$$

which is clear from Birkhoff’s pointwise ergodic theorem and Hopf’s ratio ergodic theorem.

4.2 Application to interval maps with indifferent fixed points

Let $d \geq 2$ be a positive integer and $0 = b_0 = x_1 < b_1 < x_2 < \cdots < x_d = b_d = 1$. Set $J_1 := [b_0, b_1), J_2 := [b_1, b_2), \ldots, J_d := [b_{d-1}, b_d]$. Suppose that an interval map $T : [0, 1] \rightarrow [0, 1]$ satisfies the following two conditions: for each $j = 1, \ldots, d$,

1. the restriction $T|_{J_j}$ over $J_j$ can be extended to a $C^2$-bijective map $T_j : \overline{J}_j \rightarrow [0, 1]$;
2. $T_j x_j = x_j$, $T_j' x_j = 1$ and $(x - x_j) T_j'' x > 0$ for any $x \in \overline{J}_j \setminus \{x_j\}$.

Therefore we have $T_j' x > 1$ on $\overline{J}_j \setminus \{x_j\}$ for $j = 1, \ldots, d$. Each $x_j$ is called an indifferent fixed point and a regular source. In this case, $T$ has a unique (up to scalar multiplication) $\sigma$-finite invariant measure $\mu(dx)$ equivalent to the Lebesgue measure $dx$, and $T$ is a CEMPT on $([0, 1], \mathcal{B}([0, 1]), \mu)$. Any neighborhood of $x_j$ has infinite volume with respect to $\mu$. More specifically, for any $\varepsilon > 0$, it holds that

$$
\mu((x_j - \varepsilon, x_j + \varepsilon)) = \infty, \quad j = 1, \ldots, d, \quad \text{and} \quad \mu\left([0, 1] \setminus \bigcup_{j=1}^d (x_j - \varepsilon, x_j + \varepsilon)\right) < \infty.
$$

(4.3)

The density $\mu(dx)/dx$ has the version $h(x)$ which is continuous and positive on $[0, 1] \setminus \{x_j\}_{j=1}^d$. For the details, we refer the reader to [33, 34].
Assumption 4.4 (regular variation). There exist constants \( \alpha \in (0, 1) \) and \( c = (c_1, \ldots, c_d) \in (0, \infty]^d \setminus \{(\infty, \ldots, \infty)\} \) and an increasing function \( \Psi : (0, \infty) \to (0, \infty) \) such that \( \Psi \in \mathcal{R}_{1+1/\alpha}(0+) \) and

\[
|Tx - x| \sim c_j \Psi(|x - x_j|), \quad j = 1, \ldots, d.
\] (4.4)

We will assume Assumption 4.4 from now on. For \( j = 1, \ldots, d \), let us denote by \( f_j : [0, 1] \to J_j \) the inverse function of \( T_j \). Set

\[
v_j := \frac{\sum_{i \neq j} (h \circ f_j)(x_i)f'_j(x_i)}{\sum_{i \neq j} (h \circ f_j)(x_i)f'_j(x_i)}, \quad j = 1, d,
\]

\[
\beta_j := \frac{c_j^{-\alpha}v_j}{\sum_{i=1}^d c_i^{-\alpha}v_i}, \quad j = 1, \ldots, d, \quad \text{and} \quad \beta := (\beta_1, \ldots, \beta_d),
\]

\[
a_n := \frac{1}{\Gamma(1 - \alpha) \sum_{i=1}^d c_i^{-\alpha}v_i} \inf \left\{ s \geq 1 : \frac{\alpha s^{-1}}{\Psi(s^{-1})} > n \right\}, \quad n \geq 0.
\] (4.7)

Using basic theory of regular variation (see [10, Theorem 1.5.12]), we have \( (a_n)_{n \geq 0} \in \mathcal{R}_{\alpha}(\infty) \). Let \( A_1, \ldots, A_d, Y \in \mathcal{B}([0,1]) \) be disjoint Borel subsets of \([0,1]\) such that, for some \( \varepsilon > 0 \),

\[
(x_j - \varepsilon, x_j + \varepsilon) \cap [0, 1] \subset A_j, \quad j = 1, \ldots, d, \quad \text{and} \quad Y = [0, 1] \setminus \sum_{j=1}^d A_j.
\] (4.8)

For \( A \in \mathcal{B}([0,1]) \) and \( t \geq 0 \), we define a measurable function \( S_A(t) : [0, 1] \to [0, \infty) \) in the same way as (2.1).

Theorem 4.5 (functional convergence of occupation time processes). Suppose that Assumption 4.4 holds. Then, it holds that

\[
\left( \frac{1}{n} S_{A_1}(nt), \ldots, \frac{1}{n} S_{A_d}(nt), \frac{1}{a_n} S_Y(nt) : t \geq 0 \right) \xrightarrow{\mathcal{L}(\mu)} (Z_1^{(\alpha, \beta)}(t), \ldots, Z_d^{(\alpha, \beta)}(t), L^{(\alpha, \beta)}(t)\mu(Y) : t \geq 0), \quad \text{in} \ C([0, \infty), \mathbb{R}^{d+1}).
\] (4.9)

Figure 3: graphs of \( T \) in the cases of \( d = 2 \) and \( d = 3 \)
The proof of Theorem 4.5 will be given in Section 6.

**Corollary 4.6** (marginal convergence). Suppose that Assumption 4.4 holds. Then, it holds that

\[
\left( \frac{1}{n} S_{A_1}(n), \ldots, \frac{1}{n} S_{A_d}(n), \frac{1}{a_n} S_Y(n) \right) \xrightarrow{\mathbb{P}} \left( \frac{\xi_1}{\sum_{j=1}^d \xi_j}, \ldots, \frac{\xi_d}{\sum_{j=1}^d \xi_j}, \mu(Y) / \left( \sum_{j=1}^d \xi_j \right)^{\alpha} \right), \quad \text{in } \mathbb{R}^{d+1},
\]

where $\xi_1, \ldots, \xi_d$ denote independent $[0, \infty)$-valued random variables with the one-sided $\alpha$-stable distributions characterized by (3.6).

**Example 4.7** (Boole’s transformation). Let us define an interval map $T$ by (1.1). This map satisfies our assumptions for $d = 2, \alpha = 1/2, c_1 = c_2 = 1$ and $\Psi(s) = s^3$. It is easy to check that $\beta_1 = \beta_2 = 1/2$ and $a_n = \sqrt{n/(2\pi)}$. Recall Remark 3.3. We use Theorem 4.5 to obtain the convergence (1.7).

**Remark 4.8.** More precisely, we can also prove the following refinement of Theorem 4.5. Let $A_{1}^{+}, A_{2}^{+}, A_{2}^{-}, \ldots, A_{d}^{-}, Y \in \mathcal{B}([0, 1])$ be disjoint Borel subsets of $[0, 1]$ such that, for some $\varepsilon > 0$,

\[
(x_j - \varepsilon, x_j) \subset A_j^{-} \quad \text{and} \quad (x_j, x_j + \varepsilon) \subset A_j^{+}, \quad j = 1, \ldots, d,
\]

and $Y' = [0, 1] \setminus \sum_{j=1}^d A_j^{\pm}$. Then we can formulate a functional limit theorem for

\[
\left( \frac{1}{n} S_{A_1}^+(nt), \frac{1}{n} S_{A_2}^+(nt), \frac{1}{n} S_{A_2}^-(nt), \ldots, \frac{1}{n} S_{A_d}^-(nt), \frac{1}{a_n} S_Y(nt) : t \geq 0 \right),
\]

which is a functional and joint-distributional extension of [31, Corollary 2.12]. The statement and proof are almost the same as those of Theorem 4.5, so we omit them.

Furthermore, Theorem 4.5 implies a certain functional limit theorem for the occupation measure of the orbit. In the following, we will explain it.

For a locally compact second countable Hausdorff space $U$, let us denote by $\mathcal{M}_U$ the class of all Radon measures (or equivalently, locally finite measures) on $U$. We endow $\mathcal{M}_U$ with the Polish topology of vague convergence. See Appendix D. For example, $\mu \notin \mathcal{M}_{[0,1]}$ but $\mu \in \mathcal{M}_{[0,1] \setminus \{x_j\}_{j=1}^d}$ because of (4.3). Of course, the space $[0, 1] \setminus \{x_j\}_{j=1}^d$ is not compact but locally compact with respect to the subspace topology induced by $[0, 1]$. For $t \geq 0$, we define a measurable map $S(t) : [0, 1] \to \mathcal{M}_{[0,1]}$ by

\[
S(t)(x) := \sum_{k=0}^{\lfloor t \rfloor - 1} \delta_{T^k x} + (t - \lfloor t \rfloor) \delta_{T^\lfloor t \rfloor x}, \quad x \in [0, 1],
\]

where $\delta_x$ denotes the Dirac measure at $x$. In other words, $S(t)$ denotes the linear interpolation of the occupation measures of the orbit at time $t$. For a positive integer $N \geq 1$ and
a measurable function \( f = (f_1, \ldots, f_N) : [0, 1] \rightarrow \mathbb{R}^N \), let us denote by \( S_f(t) \) the integral of \( f \) with respect to \( S(t) \):

\[
S_f(t)(x) = \sum_{k=0}^{|t|-1} f(T^k x) + (t - |t|) f(T^{|t|} x)
\]

\[
= \left( \sum_{k=0}^{|t|-1} f_i(T^k x) + (t - |t|) f_i(T^{|t|} x) : 1 \leq i \leq N \right), \quad x \in [0, 1]. \tag{4.12}
\]

**Theorem 4.9** (Functional convergence of occupation measure processes). Suppose that Assumption \([4.4]\) holds. Then, it holds that

\[
\left( \frac{1}{n} S(nt), \frac{1}{a_n} S(nt) : t \geq 0 \right) \overset{\mathcal{L}(\mu)}{\underset{n \rightarrow \infty}{\longrightarrow}} \left( \sum_{j=1}^d Z_j^{(a,\beta)}(t) \delta_{x_j}, L^{(a,\beta)}(t) \mu : t \geq 0 \right),
\]

in \( C([0, \infty), \mathcal{M}_{[0,1]} \times \mathcal{M}_{[0,1] \setminus \{x_j\}_{j=1}^d}) \). \tag{4.13}

**Proof.** For a positive integer \( N \geq 1 \), let \( f, g : [0, 1] \rightarrow \mathbb{R}^N \) be continuous functions such that

\[
\{x \in [0, 1] : g(x) \neq (0, \ldots, 0)\} \cap \{x_j\}_{j=1}^d = \emptyset.
\]

Equivalently, \( f \in C_K([0, 1], \mathbb{R}^N) = C([0, 1], \mathbb{R}^N) \) and \( g \in C_K([0, 1] \setminus \{x_j\}_{j=1}^d, \mathbb{R}^N) \). By virtue of Proposition \([D.3]\) it is sufficient to show that

\[
\left( \frac{1}{n} S_f(t), \frac{1}{a_n} S_g(t) : t \geq 0 \right) \overset{\mathcal{L}(\mu)}{\underset{n \rightarrow \infty}{\longrightarrow}} \left( \sum_{j=1}^d Z_j^{(a,\beta)}(t) f(x_j), L^{(a,\beta)}(t) \int_0^1 g(x) \mu(dx) : t \geq 0 \right),
\]

in \( C([0, \infty), \mathbb{R}^{2N}) \). \tag{4.14}

Birkhoff’s pointwise ergodic theorem implies

\[
\frac{1}{n} \left( S_f(nt) - \sum_{j=1}^d f(x_j) S_{A_j}(nt) \right) \rightarrow 0, \quad \text{in } \mathbb{R}^N, \text{ } \mu\text{-a.e., for } t > 0. \tag{4.15}
\]

In addition, we use Hopf’s ratio ergodic theorem to obtain

\[
\frac{S_g(nt)}{S_Y(nt)} \rightarrow \frac{\int_0^1 g(x) \mu(dx)}{\mu(Y)}, \quad \text{in } \mathbb{R}^N, \text{ } \mu\text{-a.e., for } t > 0. \tag{4.16}
\]

Combining Theorem \([4.5]\) with the convergences \((4.15)\) and \((4.16)\), we have the following convergence of finite-dimensional marginal laws: for any \( m \geq 1 \) and \( 0 \leq t_1 \leq \cdots \leq t_m \), it holds that

\[
\left( \frac{1}{n} S_f(nt_i), \frac{1}{a_n} S_g(nt_i) : 1 \leq i \leq m \right) \overset{\mathcal{L}(\mu)}{\underset{n \rightarrow \infty}{\longrightarrow}} \left( \sum_{j=1}^d Z_j^{(a,\beta)}(t_i) f(x_j), L^{(a,\beta)}(t_i) \int_0^1 g(x) \mu(dx) : 1 \leq i \leq m \right), \quad \text{in } \mathbb{R}^{2Nm}. \tag{4.17}
\]

Hence by Lemma \([B.3]\) and Remark \([B.1]\) we obtain the convergence \((4.14)\). \( \square \)
5 Proof of Theorem 4.1

For the proof of Theorem 4.1, we mimic the method of [15].

5.1 Williams formulae

Following [15, Proposition 2.1], we will introduce a discrete version of Williams formulae, which are representation formulae for occupation times of the orbit in terms of excursion lengths. Their names come from Williams [40, Theorem 1].

Let us denote by

\[ D_0 = \{ w : [0, \infty) \to \mathbb{R} \mid w(0) = 0 \text{ and } \lim_{t \to \infty} w(t) = \infty \} \]

with \( w^{-1} = (w^{-1}(s) : s \geq 0) \) the right-continuous inverse of \( w \), that is,

\[ w^{-1}(s) := \inf \{ t > 0 : w(t) > s \}, \quad s \geq 0. \]

Suppose that Assumption 2.1 holds. For \( N \geq 0 \) and \( x \in Y \), let us denote by \( \phi_N = \phi_N(x) \) the \( N \)th return time of the orbit \( (T_k x)_{k \geq 0} \) for \( Y \):

\[ \phi_0(x) := 0 \quad \text{and} \quad \phi_N(x) := \min \{ k > \phi_{N-1}(x) : T_k x \in Y \}, \quad N \geq 1. \]

In addition, set

\[ \eta_j(t)(x) := S_{A_j}(\phi_{\lfloor t \rfloor}(x))(x) = \sum_{k=1}^{\lfloor t \rfloor} (\ell_j \circ T_{Y}^{k-1})(x), \quad t \geq 0, \quad j = 1, \ldots, d, \quad (5.1) \]

that is, \( \eta_j(t) = \eta_j(t)(x) \) denotes the amount of time which the orbit spends on \( A_j \) up to the \( \lfloor t \rfloor \)th return time for \( Y \). Since \( T \) is a CEMPT, we have \( (\eta_j(t) : t \geq 0) \in D_0, \mu_Y\text{-a.e.} \).

Lemma 5.1 (discrete Williams formulae). Suppose that \( T \) is a CEMPT on \( (X, B, \mu) \) and Assumption 2.1 holds. Then it holds that

\[ S_{A_j}^{-1}(t) = t + \sum_{i=1}^{d} \eta_i(\eta_{j}^{-1}(t)), \quad t \geq 0, \quad j = 1, \ldots, d, \quad \mu_Y\text{-a.e.,} \quad (5.2) \]

\[ S_{Y}^{-1}(t) = t + \sum_{i=1}^{d} \eta_i(t), \quad t \geq 0, \quad \mu_Y\text{-a.e.} \quad (5.3) \]

Proof. Set \( N := \eta_j^{-1}(t) \), which is a positive integer, \( \mu_Y\text{-a.e.} \). Then it holds that

\[ S_{A_j}(\varphi_{N-1} + 1) = \eta_j(N - 1) \leq t < \eta_j(N) = S_{A_j}(\varphi_N). \quad (5.4) \]

Hence we have \( \varphi_{N-1} + 1 \leq S_{A_j}^{-1}(t) < \varphi_N \). By virtue of Assumption 2.1, the orbit stays in \( A_j \) from \( \varphi_{N-1} + 1 \) until \( \varphi_N - 1 \). Therefore we have

\[ S_{A_i}(S_{A_j}^{-1}(t)) = S_{A_i}(\varphi_N) = \eta_i(N), \quad i \neq j, \quad (5.5) \]
and

\[ S_Y(S_{A_j}^{-1}(t)) = S_Y(\varphi_N) = N. \]  (5.6)

It also follows immediately that \( S_{A_j}(S_{A_j}^{-1}(t)) = t \). In addition, we have

\[
u = \sum_{i=1}^{d} S_{A_i}(u) + S_Y(u), \quad u \geq 0. \]  (5.7)

Substituting \( u = S_{A_j}^{-1}(t) \) in (5.7), we obtain (5.2). By a similar argument, we can easily obtain (5.3).

Furthermore, following [41], we will introduce Williams formulae for skew Bessel diffusion processes. Recall that \( R^{(\alpha, \beta)}_j, Z^{(\alpha, \beta)}_j \) and \( L^{(\alpha, \beta)}_j \) have been defined in Definition 3.1 (3.3) and (3.4), respectively. We will denote by \( \eta^{(\alpha, \beta)}_j \) the inverse local time of \( R^{(\alpha, \beta)}_j \) at the origin:

\[
\eta^{(\alpha, \beta)}_j(t) := (L^{(\alpha, \beta)}_j)^{-1}(t), \quad t \geq 0. \]  (5.8)

Furthermore, set

\[
\eta^{(\alpha, \beta)}_j(t) := Z^{(\alpha, \beta)}_j(\eta^{(\alpha, \beta)}_j(t)), \quad t \geq 0, \quad j = 1, \ldots, d. \]  (5.9)

It is the amount of time spent on \( A_j \) by the excursions of \( R^{(\alpha, \beta)}_j \) away from 0 up to time \( \eta^{(\alpha, \beta)}_j(t) \). By the Itô excursion theory, we know that \( \eta^{(\alpha, \beta)}_1, \ldots, \eta^{(\alpha, \beta)}_d \) are independent \( \alpha \)-stable subordinators with Laplace transforms

\[
\mathbb{E}[\exp(-\lambda \eta_j^{(\alpha, \beta)}(t))] = \exp(-\lambda^\alpha \beta_j t), \quad t \geq 0, \quad j = 1, \ldots, d. \]  (5.10)

\textbf{Lemma 5.2 (Williams formulae).} Let \( d \geq 1 \) be a positive integer, \( \alpha \in (0, 1) \) and \( \beta = (\beta_1, \ldots, \beta_d) \in [0, 1]^d \) with \( \sum_{j=1}^{d} \beta_j = 1 \). Then, it holds that

\[
(Z^{(\alpha, \beta)}_j)^{-1}(t) = t + \sum_{i=1, i \neq j}^{d} \eta^{(\alpha, \beta)}_i \left( (\eta^{(\alpha, \beta)}_j)^{-1}(t) \right), \quad t \geq 0, \quad j = 1, \ldots, d. \]  (5.11)

\[
(L^{(\alpha, \beta)}_j)^{-1}(t) = \sum_{i=1}^{d} \eta^{(\alpha, \beta)}_i(t) = \eta^{(\alpha, \beta)}_j(t), \quad t \geq 0. \]  (5.12)

For the proof, we refer the reader to [41, Theorem 3.1]. See also [39, Proposition 1].

\section{5.2 Functional convergence of sum of excursion lengths}

Set \( S^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \} \) and \( e^{(i)} := (1_{(i=j)})_{i=1}^{d} \in S^{d-1} \). For \( \alpha \in (0, 1) \) and \( \beta = (\beta_1, \ldots, \beta_d) \in [0, 1]^d \) with \( \sum_{j=1}^{d} \beta_j = 1 \), the stochastic process \( \eta^{(\alpha, \beta)}_1(t), \ldots, \eta^{(\alpha, \beta)}_d(t) :
\( t \geq 0 \) is an \( \mathbb{R}^d \)-valued \( \alpha \)-stable Lévy process with Lévy measure

\[
\Pi_{(\alpha, \sum_{j=1}^d \beta_j \delta_{e(j)})} (A) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty dr \int_{\mathbb{R}^{d-1}} \sum_{j=1}^d \beta_j \left( \delta_{e(j)}(dx) \right) \mathbb{1}_A(rx) \alpha r^{-\alpha-1}
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^d \beta_j \int_0^\infty \mathbb{1}_A(re^{(j)}) \alpha r^{-\alpha-1} dr, \quad A \in \mathcal{B}(\mathbb{R}^d),
\]

where \( \delta_x \) denotes the Dirac measure at \( x \). We will denote by \( D([0, \infty), \mathbb{R}^d) \) the space of \( \mathbb{R}^d \)-valued càdlàg functions defined on \([0, \infty)\). We equip \( D([0, \infty), \mathbb{R}^d) \) with the Skorokhod \( J_1 \)-topology, which is a Polish topology. See Appendix B. By a slight modification of [37, Theorem 1.1], we can show the following lemma.

**Lemma 5.3** (functional convergence of sum of excursion lengths). Suppose that \( T \) is a CEMPT on \((X, \mathcal{B}, \mu)\) and that Assumptions 2.1, 2.3 and 2.6 hold. Then,

\[
\left( \frac{1}{n} \eta_1(a_n t), \ldots, \frac{1}{n} \eta_d(a_n t) : t \geq 0 \right) \xrightarrow{n \to \infty} (\eta_1^{(\alpha, \beta)}(t), \ldots, \eta_d^{(\alpha, \beta)}(t) : t \geq 0), \text{ in } D([0, \infty), \mathbb{R}^d). \tag{5.13}
\]

**Proof.** By the assumptions and Proposition C.3, we obtain the convergence (5.13). \( \square \)

We will recall a sufficient condition for strong distributional convergence:

**Lemma 5.4** ([41, Theorem 1]). Let \( T \) be a CEMPT on \((X, \mathcal{B}, \mu)\), let \( \nu_0 \ll \mu \) be a probability measure on \( X \), and let \((U, \rho)\) be a separable metric space. Assume that measurable functions \( F_n : X \to U \) (\( n \in \mathbb{N} \)) satisfy the following:

(i) \( F_n \xrightarrow{\nu_0} \zeta \) for some \( U \)-valued random variable \( \zeta \).

(ii) For any \( \varepsilon > 0 \) and for any probability measure \( \nu \ll \mu \) on \( X \), it holds that \( \nu[\rho(F_n \circ T, F_n) > \varepsilon] \xrightarrow{n \to \infty} 0. \)

Then, it holds that \( F_n \xrightarrow{\nu} \zeta. \)

**Proof of Theorem 4.1**. For \( a, b > 0 \) and \( w \in D_0 \), we will denote by \((aw(b))^{-1}\) the right-continuous inverse of \((aw(bt)) : t \geq 0\). Then we see at once that \((aw(b))^{-1}(t) = \frac{1}{b} w^{-1}(\frac{1}{a} t)\), \( t \geq 0 \). Therefore Lemma 5.1 implies

\[
\frac{1}{n} S_{A_j}^{-1}(nt) = t + \sum_{i=1}^d \frac{1}{n} \eta_i \left( a_n \left( \frac{1}{n} \eta_j(a_n) \right)^{-1}(t) \right) + \frac{a_n}{n} \left( \frac{1}{n} \eta_j^{-1}(a_n) \right)^{-1}(t), \quad t \geq 0, \ j = 1, \ldots, d, \ \mu_Y \text{-a.e.,}
\]

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and
\[ \frac{1}{n} S_Y^{-1}(a_n t) = a_n t + \sum_{i=1}^{d} \frac{1}{n} \eta_i(a_n t), \quad t \geq 0, \quad \mu_Y \text{-a.e.} \]

Note that \( a_n/n \to 0, \) as \( n \to \infty. \) We recall that \( \eta_1^{(\alpha,\beta)}, \ldots, \eta_d^{(\alpha,\beta)} \) are stochastically continuous, strictly increasing processes and they are independent. Combining Lemmas 5.2, 5.3 and B.5 we have
\[ \left( \frac{1}{n} S_{A_1}(nt), \ldots, \frac{1}{n} S_{A_d}(nt), \frac{1}{n} S_Y^{-1}(a_n t) : t \geq 0 \right) \]
\[ \xrightarrow{\mu_Y \text{-a.e.}} \left( (Z_1^{(\alpha,\beta)})^{-1}(t), \ldots, (Z_d^{(\alpha,\beta)})^{-1}(t), \eta^{(\alpha,\beta)}(t) : t \geq 0 \right), \quad \text{in } D([0, \infty), \mathbb{R}^{d+1}). \quad (5.14) \]

Since \( Z_j^{(\alpha,\beta)} \) and \( L^{(\alpha,\beta)} \) are continuous, their inverse \( (Z_j^{(\alpha,\beta)})^{-1} \) and \( (L^{(\alpha,\beta)})^{-1} = \eta^{(\alpha,\beta)} \) are strictly increasing. Using Lemmas B.4 and 5.4 we have
\[ \left( \frac{1}{n} S_{A_1}(nt), \ldots, \frac{1}{n} S_{A_d}(nt), \frac{1}{a_n} S_Y(nt) : t \geq 0 \right) \]
\[ \xrightarrow{n \to \infty} \left( Z_1^{(\alpha,\beta)}(t), \ldots, Z_d^{(\alpha,\beta)}(t), L^{(\alpha,\beta)}(t) : t \geq 0 \right), \quad \text{in } D([0, \infty), \mathbb{R}^{d+1}). \quad (5.15) \]

By Remark B.1, we obtain the convergence (4.1).

\section{6 Proof of Theorem 4.5}

By virtue of Birkhoff’s pointwise ergodic theorem and Hopf’s ratio ergodic theorem, we only need to consider one particular combination of sets \( A_1, \ldots, A_d, Y \in B([0,1]) \) satisfying (4.8). In the following, we will choose \( A_1, \ldots, A_d, Y \) suitably so that all of the conditions of Assumptions 2.1, 2.3, and 2.6 will be satisfied. Then we will apply Theorem 4.1 to obtain Theorem 4.5. We need to break up the proof into the case of \( d = 2 \) and the case of \( d \geq 3 \) for a certain reason. See Remark 6.5.

\subsection{6.1 Case \( d = 2 \)}

Let us consider the case of \( d = 2. \) Following [35, Section 4] and [42, Section 2], we choose a point \( \gamma \in J_1 \) such that
\[ T\gamma \in J_2 \quad \text{and} \quad T^2\gamma = \gamma. \quad (6.1) \]

Hence \( \gamma \) is a 2-periodic point of \( T. \) Set
\[ A_1 := [0, \gamma), \quad Y := [\gamma, T\gamma] \quad \text{and} \quad A_2 := (T\gamma, 1], \quad (6.2) \]
Then we have
\[ T(A_1) = A_2^c, \quad T(A_2) = A_1^c, \quad T(Y \cap J_1) = A_2, \quad T(Y \cap J_2) = A_1, \quad \text{a.e.} \]
Therefore Assumption 2.1 holds for \( d = 2 \). Let us define \( \varphi, \ell = (\ell_1, \ell_2), \mu_Y \) and \( T_Y \) as in Section 2. For \( n \geq 1 \), we define subsets \( P_{1,n} \) and \( P_{2,n} \subset Y \) by
\[
P_{1,n} := \{ \ell_1 = n \} = \{ \varphi = n + 1 \} \cap J_2, \quad (6.3)
\]
\[
P_{2,n} := \{ \ell_2 = n \} = \{ \varphi = n + 1 \} \cap J_1. \quad (6.4)
\]
Then we have
\[
P_{1,n} = (f_2 \circ f_1^n)(Y) = \left[ (f_2 \circ f_1^{n-1})\gamma, (f_2 \circ f_1^n)\gamma \right], \quad (6.5)
\]
\[
P_{2,n} = (f_1 \circ f_2^n)(Y) = \left[ (f_1 \circ f_2^{n+1})\gamma, (f_1 \circ f_2^n)\gamma \right]. \quad (6.6)
\]
Here we used the fact that \( T\gamma = f_2^{-1}\gamma = f_2\gamma \). It is easy to check that \( Y = \sum_{j,n} P_{j,n} \), a.e.

**Lemma 6.1** ([35, Lemma 5]). Suppose that Assumption 4.4 holds. Then,
\[
\mu_Y[\ell_j \geq n] \sim n \rightarrow \infty \beta_j \frac{1}{\mu(Y)\Gamma(1-\alpha)a_n}, \quad j = 1, 2. \quad (6.7)
\]
where \( \beta_j \) and \((a_n)_{n \geq 0}\) have been defined in (4.6) and (4.7), respectively.

**Lemma 6.2** ([42, Lemma 2]). The map \( T_Y \) satisfies the following conditions:

1. for each \( j = 1, 2 \) and \( n \geq 1 \), the restriction \( T|_{P_{j,n}} \) can be extended to a \( C^2 \)-bijective map from \( P_{j,n} \) to \( Y \).
2. \( \inf \{ T_Y x : x \in \sum_{j,n} P_{j,n} \} > 1 \).
3. \( \sup \{ |T_Y' x| / |T_Y x|^2 : x \in \sum_{j,n} P_{j,n} \} < \infty \).

Combining Lemmas 6.1 and 6.2 with Lemma A.3 and Remark 2.8, we see that Assumption 2.6 is satisfied. Therefore we apply Theorem 4.1 to obtain the desired result.

### 6.2 Case \( d \geq 3 \)

Let us consider the case of \( d \geq 3 \). Set
\[
A_j := f_j(J_j), \quad j = 1, \ldots, d, \quad \text{and} \quad Y := [0, 1] \setminus \sum_{j=1}^d A_j. \quad (6.8)
\]
Then Assumption 2.1 holds. Let us define \( \varphi, \ell = (\ell_1, \ldots, \ell_d) \), \( \mu_Y \) and \( T_Y \) as in Section 2.

The following lemma is a slight modification of Lemma 6.1.
Lemma 6.3. Suppose that Assumption 4.4 holds. Then,
\[ \mu_Y[\ell_j \geq n] \sim \frac{1}{\mu(Y)\Gamma(1-\alpha)a_n} \beta_j \frac{1}{n^{\alpha}}, \quad j = 1, \ldots, d, \] (6.9)
where \( \beta_j \) and \( (a_n)_{n \geq 0} \) have been defined in (4.6) and (4.7), respectively.

Set \( J_{j,-} := Y \cap (b_{j-1}, x_j) \) and \( J_{j,+} := Y \cap (x_j, b_j), \quad j = 1, \ldots, d. \) (6.10)
For \( i, j \in \{1, \ldots, d\}, \sigma \in \{+, -\} \) and \( n \geq 0 \), we define a subset \( P_{i,j,\sigma,n} \subset Y \) by
\[ P_{i,j,\sigma,n} := Y \cap J_i \cap T_Y^{-1}(J_{j,\sigma}) \cap \{\ell_j = n\} \]
\[ = Y \cap J_i \cap T_Y^{-1}(J_{j,\sigma}) \cap \{\varphi = n + 1\}. \] (6.11)
We see at once that
\[ J_{i_0,\sigma_0} = \begin{cases} \sum_{j < i_0, \sigma = \pm, n \geq 0} P_{i,j,\sigma,n}, & \text{if } \sigma_0 = -, \quad \text{a.e.}, \\ \sum_{j > i_0, \sigma = \pm, n \geq 0} P_{i,j,\sigma,n}, & \text{if } \sigma_0 = +, \quad \text{a.e.,} \end{cases} \] (6.12)
and hence \( Y = \sum_{i,j,\sigma,n} P_{i,j,\sigma,n}, \) a.e. Set \[ \Theta := \{(i, j, \sigma, n) : P_{i,j,\sigma,n} \neq \emptyset\} = \{(i, j, \sigma, n) : i \neq j \text{ and } (j, \sigma) \neq (1, -), (d, +)\}. \]

Lemma 6.4. The map \( T_Y \) satisfies the following conditions:

1. for each \( (i, j, \sigma, n) \in \Theta \), the restriction \( T_Y|_{P_{i,j,\sigma,n}} \) can be extended to a \( C^2 \)-bijective map from \( P_{i,j,\sigma,n} \) to \( J_{j,\sigma} \).
2. \( \inf\{T_Y'x : x \in \sum_{i,j,\sigma,n} P_{i,j,\sigma,n}\} > 1. \)
3. \( \sup\{|T_Y'x|/|T_Y'x|^2 : x \in \sum_{i,j,\sigma,n} P_{i,j,\sigma,n}\} < \infty. \)
4. For each \( (i, j, \sigma, n) \in \Theta \), it holds that \( T_Y^d(J_{i,j,\sigma,n}) = Y, \) a.e.

Proof. The proofs of (1), (2) and (3) are almost the same as those of Lemma 6.2. So we omit them. Let \( (i, j, -) \in \Theta \). Then, we have \( T_Y(P_{i,j,-,n}) = J_{j,-} \), a.e. Using (6.12), we have \( T_Y(J_{j,-}) \supset J_{1,+} \), a.e. We see at once that
\[ T_Y(J_{i,\sigma}) = Y \setminus J_{i,\sigma} \quad \text{and} \quad T_Y(Y \setminus J_{i,\sigma}) = Y, \text{ a.e.} \]
Therefore we obtain \( T_Y^d(P_{i,j,-,n}) = Y, \) a.e. Similarly, we can obtain \( T_Y^d(P_{i,j',+,n'}) = Y, \) a.e., for \( (i', j', +, n') \in \Theta. \)

Therefore we obtain the desired result in the case of \( d \geq 3 \), as in the case of \( d = 2. \)

Remark 6.5. Let us consider the case \( d = 2 \) and define \( A_j, Y, J_{i,\sigma} \) and \( P_{i,j,\sigma,n} \) by (6.8), (6.10) and (6.11), respectively. Then we have
\[ T_Y(J_{1,+}) = J_{2,-} \quad \text{and} \quad T_Y(J_{2,\sigma}) = J_{1,+}, \quad \text{a.e.} \]
Hence \( T_Y^m(P_{i,j,\sigma,n}) \neq Y, \) a.e., for any \( m \geq 1. \)
A Mixing property of uniformly expanding Markov interval map

We will recall some mixing properties of uniformly expanding Markov interval maps. For basic discussions of Markov interval maps, see for instance Bowen [11], Bowen–Series [12] and Pollicott–Yuri [27, Sections 4 and 12].

Let \((P_i)_{i \geq 1}\) be a countable family of disjoint open subintervals of \((0, 1)\), and let \(Y = \sum_{i \geq 1} P_i\), a.e. Suppose that a map \(F : Y \to Y\) satisfies the following conditions:

1. \((C^2\text{-extension})\) for each \(i \geq 1\), the restriction \(F|_{P_i}\) can be extended to a \(C^2\)-function on \(P_i\).
2. \((\text{Markov map})\) If \(F(P_i) \cap P_j \neq \emptyset\) for some \(i, j \geq 1\), then \(F(P_i) \supset P_j\).
3. \((\text{aperiodicity})\) There exists \(n_0 \geq 1\) such that, for any \(i \geq 1\), it holds that \(F^{n_0}(P_i) = Y\), a.e.
4. \((\text{finite image})\) \(\{F(P_i) : i \geq 1\}\) is a finite collection.
5. \((\text{uniformly expanding})\) \(\inf\{|F'(x)| : x \in \sum_i P_i\} > 1\).
6. \((\text{Rényi’s (or Adler’s) condition})\) \(\sup\{|F^n(x)| / |F'(x)|^2 : x \in \sum_i P_i\} < \infty\).

The following two lemmas are slight modifications of [12, Theorem (I.2)] and [27, Theorem 12.5], respectively. So we omit the proofs of them.

**Lemma A.1.** Assume that the conditions (1)–(6) hold. Then the map \(F\) has a unique invariant probability measure \(\nu_0\) equivalent to the Lebesgue measure on \(Y\).

**Lemma A.2.** Assume that the conditions (1)–(6) hold. Let \(\nu_0\) be the \(F\)-invariant probability measure given by Lemma A.1. Then the map \(F\) is exact (and hence strong mixing) with respect to \(\nu_0\), that is, \(\bigcap_{n=0}^{\infty} F^{-n}A : A \in \mathcal{B}(Y)\} = \{\emptyset, Y\}, \nu_0\)-a.e.

Let us define a sub-\(\sigma\)-field \(\mathcal{F}_n \subset \mathcal{B}(Y)\) by \(\mathcal{F}_n := \sigma\{F^{-k}P_i : n \leq k \leq m\} \cap \{i \geq 1\}\). Combining Lemmas A.1 and A.2 with [3, Theorem 1.(b)] or [4, Corollary 4.7.8], we obtain the following lemma.

**Lemma A.3.** Assume that the conditions (1)–(6) hold. Let \(\nu_0\) be the \(F\)-invariant probability measure given by Lemma A.1. Then there exist \(C \in (0, \infty)\) and \(\theta \in (0, 1)\) such that, for any \(k, n \geq 0\), \(A \in \mathcal{F}_k\) and \(B \in \mathcal{B}(Y)\),

\[
|\nu_0(A \cap F^{-(k+n)}(B)) - \nu_0(A)\nu_0(B)| \leq C\theta^n\nu_0(A)\nu_0(B). \tag{A.1}
\]
B Space of càdlàg functions and Skorokhod $J_1$-topology

We will recall the space of càdlàg functions and the Skorokhod $J_1$-topology. For the details, see Bingham [9 Section 2], Ethier–Kurtz [14 Chapter 3] and Jacod–Shiryaev [17 Chapter VI].

Let $d \geq 1$ be a positive integer. We will denote by $D([0, \infty), \mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued càdlàg functions defined on $[0, \infty)$, that is,

$$D([0, \infty), \mathbb{R}^d) := \{ w : [0, \infty) \to \mathbb{R}^d : w(t+) = w(t) \text{ and } w(t-) \text{ exists in } \mathbb{R}^d \text{ for each } t \geq 0 \}.$$ 

Here, for $t \geq 0$,

$$w(t+) := \lim_{s \uparrow t} w(s) \quad \text{and} \quad w(t-) := \begin{cases} \lim_{s \uparrow t} w(s), & t > 0, \\ w(0), & t = 0. \end{cases}$$

We equip $D([0, \infty), \mathbb{R}^d)$ with the Skorokhod $J_1$-topology, which is a Polish topology. Let $(w_n)_{n \geq 1}$ be a $D([0, \infty), \mathbb{R}^d)$-valued sequence and $w_\infty \in D([0, \infty), \mathbb{R}^d)$. Then it holds that $w_n \to w_\infty$ in $D([0, \infty), \mathbb{R}^d)$ (with respect to the Skorokhod $J_1$-topology) if and only if, there exists a sequence $(\lambda_n)_{n \geq 1}$ of continuous, strictly increasing functions mapping $[0, \infty)$ onto $[0, \infty)$ such that, for any $t_0 > 0$,

$$\sup_{0 \leq t \leq t_0} |\lambda_n(t) - t| \to 0 \quad \text{and} \quad \sup_{0 \leq t \leq t_0} |w_n(\lambda_n(t)) - w_\infty(t)| \to 0.$$

**Remark B.1.** Recall that $C([0, \infty), \mathbb{R}^d)$ denotes the family of continuous functions $w : [0, \infty) \to \mathbb{R}$. It is a closed subset of $D([0, \infty), \mathbb{R}^d)$ (with respect to the Skorokhod $J_1$-topology). The subspace topology on $C([0, \infty), \mathbb{R}^d)$ induced by $D([0, \infty), \mathbb{R}^d)$ coincides with the topology of uniform convergence on compact subsets.

Let $D_0 \subset D([0, \infty), \mathbb{R})$ be the space of non-decreasing càdlàg functions $w : [0, \infty) \to \mathbb{R}$ with $w(0) = 0$ and $\lim_{t \to \infty} w(t) = \infty$. Let $w = (w(t) : t \geq 0) \in D_0$. Recall that $w^{-1} = (w^{-1}(s) : s \geq 0)$ denotes the right-continuous inverse of $w$, i.e., $w^{-1}(s) := \inf\{t > 0 : w(t) > s\}$.

**Lemma B.2** ([15 Lemma 2.3]). Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be $D_0$-valued sequences, and $x_\infty, y_\infty \in D_0$. Assume that the following conditions are satisfied:

(i) $x_\infty$ is a strictly increasing function.

(ii) $(x_n, y_n) \to (x_\infty, y_\infty)$, in $D([0, \infty), \mathbb{R}^2)$.

Then, it holds that

$$y_n(x_n^{-1}(t)) \to y_\infty(x_\infty^{-1}(t)), \quad \text{in } \mathbb{R},$$

at every $t \geq 0$ for which $x_n^{-1}(t)$ is a continuity point of $y_\infty$. 

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Let \( X = (X(t) : t \geq 0) \) be a \( D([0, \infty), \mathbb{R}^d) \)-valued random variable, that is, a stochastic process whose path-function lies in \( D([0, \infty), \mathbb{R}^d) \). We say that \( X \) is stochastically continuous if \( \mathbb{P}[X(t) = X(t^-)] = 1 \), for any \( t \geq 0 \). The following lemma is a slight modification of [9, Theorem 3]. The proof is almost the same and so we omit it.

**Lemma B.3.** Let \( d \geq 1 \) be a positive integer. For each \( n = 1, 2, \ldots, \infty \) and \( i = 1, \ldots, d \), let \( X_{n,i} = (X_{n,i}(t) : t \geq 0) \) be a \( D([0, \infty), \mathbb{R}) \)-valued random variable. Assume that

(i) \( X_{n,i}(t) \) is a non-decreasing process for any \( n = 1, 2, \ldots, \infty \) and \( i = 1, \ldots, d \).

(ii) \( X_{\infty,1}, \ldots, X_{\infty,d} \) are stochastically continuous.

(iii) The finite-dimensional marginal laws of \((X_{n,1}, \ldots, X_{n,d})\) converge as \( n \to \infty \) to those of \((X_{\infty,1}, \ldots, X_{\infty,d})\). In other words, for any \( m \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_m \), it holds that

\[
(X_{n,i}(t_j))_{1 \leq j \leq d, 1 \leq i \leq m} \xrightarrow{d, n \to \infty} (X_{\infty,i}(t_j))_{1 \leq j \leq d, 1 \leq i \leq m}, \quad \text{in } \mathbb{R}^{dm}.
\]

Then, it holds that

\[
(X_{n,1}, \ldots, X_{n,d}) \xrightarrow{d, n \to \infty} (X_{\infty,1}, \ldots, X_{\infty,d}), \quad \text{in } D([0, \infty), \mathbb{R}^d).
\]

**Lemma B.4.** Let \( d \geq 1 \) be a positive integer. For each \( n = 1, 2, \ldots, \infty \) and \( i = 1, \ldots, d \), let \( X_{n,i} = (X_{n,i}(t) : t \geq 0) \) be a \( D_0 \)-valued random variable. Assume that

(i) \( X_{\infty,1}, \ldots, X_{\infty,d} \) are strictly increasing processes.

(ii) \( (X_{n,1}, \ldots, X_{n,d}) \xrightarrow{d, n \to \infty} (X_{\infty,1}, \ldots, X_{\infty,d}), \quad \text{in } D([0, \infty), \mathbb{R}^d).\)

Then, it holds that

\[
(X_{n,1}^{-1}, \ldots, X_{n,d}^{-1}) \xrightarrow{d, n \to \infty} (X_{\infty,1}^{-1}, \ldots, X_{\infty,d}^{-1}), \quad \text{in } D([0, \infty), \mathbb{R}^d). \tag{B.1}
\]

**Proof.** By the Skorokhod coupling (see for example [18, Theorem 4.30]), there is no loss of generality in assuming

\[
(X_{n,1}, \ldots, X_{n,d}) \xrightarrow{D, n \to \infty} (X_{\infty,1}, \ldots, X_{\infty,d}), \quad \text{in } D([0, \infty), \mathbb{R}^d), \quad \text{a.s.}
\]

Then Lemma [3.2] shows that, for any \( i = 1, \ldots, d \) and \( t \geq 0 \),

\[
X_{n,i}^{-1}(t) \xrightarrow[n \to \infty]{}, X_{\infty,i}^{-1}(t), \quad \text{in } \mathbb{R}, \quad \text{a.s.}
\]

Hence the finite-dimensional marginal laws of \((X_{n,1}^{-1}, \ldots, X_{n,d}^{-1})\) converge as \( n \to \infty \) to those of \((X_{\infty,1}^{-1}, \ldots, X_{\infty,d}^{-1})\). Since \( X_{\infty,i} \) is strictly increasing, its inverse \( X_{\infty,i}^{-1} \) is a continuous process. Therefore we apply Lemma [B.3] to obtain [B.1].
The proof of the following lemma is almost the same as that of Lemma B.4. So we omit it.

**Lemma B.5.** Let \(d \geq 2\) be a positive integer. For each \(n = 1, 2, \ldots, \infty\) and \(i = 1, \ldots, d\), let \(X_{n,i} = (X_{n,i}(t) : t \geq 0)\) be a \(D_0\)-valued random variable. Assume that

(i) \(X_{\infty,1}, \ldots, X_{\infty,d}\) are stochastically continuous, strictly increasing processes, and they are independent.

(ii) \((X_{n,1}, \ldots, X_{n,d}) \overset{d}{\to}_{n \to \infty} (X_{\infty,1}, \ldots, X_{\infty,d})\), in \(D([0, \infty), \mathbb{R}^d)\).

Then, it holds that

\[
(X_{n,i}(X_{n,j}^{-1}(t)) : i \neq j, t \geq 0) \overset{d}{\to}_{n \to \infty} (X_{\infty,i}(X_{\infty,j}^{-1}(t)) : i \neq j, t \geq 0),
\]

in \(D([0, \infty), \mathbb{R}^{d(d-1)})\).

## C Functional convergence to \(\alpha\)-stable Lévy process

Following [37], we will explain a functional limit theorem for the processes which have stationary increments and local dependence.

Let \(d \geq 1\) be a positive integer, and \((Z_n)_{n \geq 1}\) a strictly stationary sequence of \(\mathbb{R}^d\)-valued random variables. Set \(S_{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}\), where \(|\cdot|\) denotes the Euclid norm on \(\mathbb{R}^d\). We will denote by \(\mathcal{P}_{S_{d-1}}\) the class of probability measures on \(S_{d-1}\). We endow \(\mathcal{P}_{S_{d-1}}\) with the Polish topology of weak convergence.

**Assumption C.1** (regular variation). The random variable \(Z_1\) is regular varying with index \(\alpha \in (0, 1)\) and spectral measure \(\rho \in \mathcal{P}_{S_{d-1}}\), that is, it holds that

\[
\mathbb{P}\left[|Z_1| > r \Big| |Z_1| > r\right] \to_{r \to \infty} \lambda^{-\alpha}, \quad \text{in } \mathbb{R}, \text{ for } \lambda > 0,
\]

and

\[
\mathbb{P}\left(Z_1 \in \cdot \Big| |Z_1| > r\right) \to_{r \to \infty} \rho(\cdot), \quad \text{in } \mathcal{P}_{S_{d-1}},
\]

where \(\mathbb{P}[\cdot|\cdot]\) denotes the conditional probability.

We will assume Assumption C.1 from now on. For \(n \geq 1\), set

\[
a_n := \frac{1}{\Gamma(1-\alpha)\mathbb{P}(|Z_1| > n)} \in (0, \infty). \tag{C.1}
\]

We define a sub-\(\sigma\)-field \(\mathcal{F}_n^m \subset B(\mathbb{R}^d)\) by \(\mathcal{F}_n^m := \sigma\{Z_k : n \leq k \leq m\}\). For \(n \geq 1\), set

\[
\phi_0(n) := \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : k \geq 0, A \in \mathcal{F}_0^k, B \in \mathcal{F}_{k+n}^\infty\}. \tag{C.2}
\]
Assumption C.2 (local dependence). For any \( \varepsilon > 0 \), there exist \( N \)-valued sequences \((r_n)_{n \geq 1}\) and \((s_n)_{n \geq 1}\) such that
\[
s_n, \frac{r_n}{s_n}, \frac{a_n}{r_n} \to \infty \quad \text{and} \quad \frac{a_n}{r_n} \phi_0(s_n) \to 0,
\]
and
\[
P\left[ \max_{2 \leq k \leq r_n} |Z_k| > \varepsilon n \mid |Z_1| > \varepsilon n \right] \to 0.
\]

For \( n \geq 1 \), we define an \( \mathbb{R}^d \)-valued càdlàg process \( X_n = (X_n(t) : t \geq 0) \) by
\[
X_n(t) := \frac{1}{n} \sum_{k=1}^{[nt]} Z_k, \quad t \geq 0.
\]

We will denote by \( X_{(\alpha, \rho)} = (X_{(\alpha, \rho)}(t) : t \geq 0) \) an \( \mathbb{R}^d \)-valued \( \alpha \)-stable Lévy process with Lévy measure \( \Pi_{(\alpha, \rho)} \) given by
\[
\Pi_{(\alpha, \rho)}(A) := \frac{1}{\Gamma(1-\alpha)} \int_0^\infty dr \int_{\mathbb{S}^{d-1}} \rho(dx) \mathbb{1}_A(rx) r^{-\alpha-1}, \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

The following proposition is a slight modification of [37, Theorem 1.1]. So we omit its proof.

Proposition C.3 (functional convergence to stable process). Suppose that Assumptions C.1 and C.2 hold. Then, it holds that
\[
X_n \xrightarrow{d} X_{(\alpha, \rho)}, \quad \text{in } D([0, \infty), \mathbb{R}^d).
\]

D Distributional convergence of measure-valued processes

Let \( U \) be a locally compact second countable Hausdorff space, which is a Polish space. Let \( \mathcal{M}_U \) denote the class of Radon measures (or equivalently, locally finite measures) on \( V \). For a positive integer \( N \geq 1 \), we will denote by \( C_K(U, \mathbb{R}^N) \) the family of continuous functions \( f: U \to \mathbb{R}^N \) with compact support. For \( f \in C_K(U, \mathbb{R}) \), we will write \( \pi_f \) for the mapping \( \mathcal{M}_U \ni \mu \mapsto \int_U f(x) \mu(dx) \in \mathbb{R} \). The topology of vague convergence is defined to be the Polish topology on \( \mathcal{M}_U \) generated by the maps \( \pi_f, f \in C_K(U, \mathbb{R}) \). For more details, see Kallenberg [18, Chapter 16 and Appendix A.2] and [19, Chapter 4].

Let \( X = (X(t, dx) : t \geq 0) \) be a \( C([0, \infty), \mathcal{M}_U) \)-valued random variable. In other words, \( X \) is a measure-valued process whose path-function lies in \( C([0, \infty), \mathcal{M}_U) \). For \( f = (f_1, \ldots, f_N) \in C_K(U, \mathbb{R}^N) \), we define a \( C([0, \infty), \mathbb{R}^N) \)-valued random variable \( Xf = ((Xf)(t) : t \geq 0) \) by
\[
(Xf)(t) := \left( \int_U f_1(x) X(t, dx), \ldots, \int_U f_N(x) X(t, dx) \right), \quad t \geq 0.
\]
Proposition D.1. Let $U$ and $V$ be locally compact second countable Hausdorff spaces, and let $(X_0,Y_0),\ldots,(X_\infty,Y_\infty)$ be $C([0,\infty),\mathcal{M}_U \times \mathcal{M}_V)$-valued random variables. Then the following conditions are equivalent:

(i) $(X_n,Y_n) \xrightarrow{d_{n \to \infty}} (X_\infty,Y_\infty)$, in $C([0,\infty),\mathcal{M}_U \times \mathcal{M}_V)$.

(ii) For any $N \geq 1$, $f \in C_K(U,\mathbb{R}^N)$ and $g \in C_K(V,\mathbb{R}^N)$, it holds that $(X_nf,Y_ng) \xrightarrow{d_{n \to \infty}} (X_\infty f,Y_\infty g)$, in $C([0,\infty),\mathbb{R}^{2N})$.

Proof. The implication [(i) $\Rightarrow$ (ii)] is trivial. Conversely, assume that the condition (ii) is satisfied. Then the finite-dimensional laws of $(X_n,Y_n)$ converge as $n \to \infty$ to those of $(X_\infty,Y_\infty)$ (see [18, Theorem 16.16]): for any $m \geq 1$ and $0 \leq t_1 \leq \cdots \leq t_m < \infty$,

$$(X_n(t_i,dx),Y_n(t_i,dy) : 1 \leq i \leq m) \xrightarrow{d_{n \to \infty}} (X_\infty(t_i,dx),Y_\infty(t_i,dy) : 1 \leq i \leq m), \text{ in } (\mathcal{M}_U \times \mathcal{M}_V)^m.$$ 

In addition, for any $f \in C_K(U,\mathbb{R})$ and $g \in C_K(V,\mathbb{R})$, the sequence $(X_nf,Y_ng : n \geq 0)$ is tight in $C([0,\infty),\mathbb{R}^2)$. Hence the sequence $(X_n,Y_n : n \geq 0)$ is tight in $C([0,\infty),\mathcal{M}_U \times \mathcal{M}_V)$ (see [18, Theorem 16.27]). Therefore we obtain (i) (see [18, Lemma 16.2, Theorem 16.3 and Proposition 16.6]).

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