Volumes of conditioned bipartite state spaces

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Abstract

We analyze the metric properties of conditioned quantum state spaces $\mathcal{M}_{\eta \times \mathbb{H}^m}(n \times m)$. These spaces are the convex sets of $n \times n$ density matrices that, when partially traced over $m$ degrees of freedom, respectively yield the given $n \times n$ density matrix $\eta$. For the case $n = 2$, the volume of $\mathcal{M}_{\eta}(2 \times m)$ equipped with the Hilbert–Schmidt measure can be conjectured to be a simple polynomial of the radius of $\eta$ in the Bloch-ball. Remarkably, for $m = 2, 3$ we find numerically that the probability $p_{\text{sep}}(2 \times m)(\eta)$ to find a separable state in $\mathcal{M}_{\eta}(2 \times m)$ is independent of $\eta$ (except for $\eta$ pure). For $m > 3$, the same holds for $p_{\text{PosPart}}(2 \times m)(\eta)$, the probability to find a state with a positive partial transpose in $\mathcal{M}_{\eta}(2 \times m)$. These results are proven analytically for the case of the family of $4 \times 4$ X-states, and thoroughly numerically investigated for the general case. The important implications of these findings for the clarification of open problems in quantum theory are pointed out and discussed.

Keywords: qubits, qutrits, density matrix, Hilbert–Schmidt metric, entanglement, separability, Monte-Carlo integration

1. Introduction

Open quantum system states $\eta$ are reduced system states $\rho = \rho_S = \text{Tr}_R(\rho_{S+R})$ of some total state $\rho_{S+R}$ of a system $S$ and its environment $R$, where $\text{Tr}_R$ denotes the partial trace over the degrees of freedom of the environment. Open quantum system dynamics refers to the time evolution $\eta \rightarrow \eta(t)$ determined through the unitary evolution of system and environment [2]:

$$\eta(t) = \text{Tr}_R \left( U_{S+R}(t) \rho_{S+R} U_{S+R}^\dagger(t) \right).$$

A crucial issue that is widely discussed (see for example [3, 15, 16, 19]) is how to map the state $\eta$ of the open system $S$ at some initial time to a total
state of $S + R$ (formalized through the so-called assignment map $\pi$). In the literature this assignment is always considered to be linear and most results are obtained on the assumption that $\pi$ maps $\eta$ on a product with a fixed state of the environment, i.e. $\pi(\eta) = \eta \otimes \rho_R$.

While the mathematical properties of $\pi$ have been discussed in detail [8, 12, 19], only little is known [9] about its image, i.e. the set of total states $\rho$ of the closed system $S + R$ that are compatible with a given reduced state $\eta$ of the open system $S$. A thorough investigation of these spaces is therefore necessary to obtain a more complete picture of the properties of assignment maps, and thus, a more complete picture of open quantum system dynamics.

Apart from their relevance for the description of open quantum dynamics, spaces of total states that are conditioned to a given reduced state constitute lower-dimensional sections of the total state space. An analysis of these sections might hence shed light on the properties of the total space. As only little is known about general quantum dynamical state spaces (see e.g. [11] for a discussion of the state space of a qubit and [25, 31] for the Hilbert–Schmidt and the Bures volume of general state spaces) an investigation of a new kind of sections of these spaces might lead the way to a solution of long-standing problems concerning quantum dynamical state spaces.

In their seminal work [30], Życzkowski et al raised the question of the volume of separable states in the total state space of a bipartite system and emphasized that its solution is of both philosophical and experimental interest. Ever since, this problem has been tackled for different measures both numerically and analytically. Analytical results are at hand only for certain lower-dimensional sections of the total state space [20]. For the general problem only conjectures based on extensive numerical research exist [24]. The conjecture that is of most importance for this paper is the belief that $P_{\text{sep}}^{(2 \times 2)}$, the a priori Hilbert–Schmidt-probability for a two-qubit state to be separable, is equal to $\frac{8}{33}$ [22]. In spite of the existence of these analytical and numerical results, a more general geometric picture of state correlations is highly desirable. Our results on conditional state spaces presented here may help to shed light on some long-standing open problems concerning geometrical considerations of state spaces.

Our paper is structured as follows: in sections 2 and 3 the general framework of bipartite systems is introduced. A possible parametrization of these systems that will be used throughout this paper, is presented. The state spaces are equipped with the Hilbert–Schmidt measure as the measure for which all metric results will be derived. Section 4 introduces coupled qubit systems, which are the lowest-dimensional possible bipartite systems and therefore allow for a feasible numerical treatment. The main results of this paper are to be found in sections 5–7. In the first of these, analytical results for the Hilbert–Schmidt volume of spaces of conditioned X-states and the probability to find a separable state in these spaces are derived, while the latter two constitute a numerical investigation of the metric properties of general coupled qubit-systems. A conclusion and a discussion of the implications of the results are given in section 8.

2. State spaces of bipartite quantum systems

An $N \times N$ density matrix $\rho$ is a linear operator acting on the Hilbert space $H_N$ (i.e. $\rho \in B(H_N)$) that satisfies the following three conditions:

1. $\rho^\dagger = \rho$.
2. $\text{Tr} \rho = 1$.
3. $\rho$ is positive semi-definite.
The convex set of all \( \mathbb{N} \times \mathbb{N} \) density matrices is denoted by \( \mathcal{M}^{(N)} \subset B(\mathcal{H}_N) \). Because of the unit trace and the hermiticity of density matrices, the number of free real parameters of \( \rho \) is equal to \( N^2 - 1 \). The demand for positivity restricts the domain of these parameters.

If \( N = n \times m \) is not prime, \( \rho \in B(\mathcal{H}_{nm}) \) can be considered as a bipartite state consisting of an \( n \)-dimensional system \( S \) coupled to an \( m \)-dimensional system \( R \). A natural parametrization of an \( nm \times nm \) density matrix \( \rho \in \mathcal{M}^{(nm)} \) makes use of the traceless generators of the special unitary group \( SU(nm) \) [10] (Einstein summation convention implied):

\[
\rho = \frac{1}{nm} \left( I_{nm} + a_i A_i \otimes I_m + b_j B_j + c_{ij} A_k \otimes B_l \right),
\]

where \( A_i \) and \( B_j \) are the generators of the groups \( SU(n) \) and \( SU(m) \), respectively, \( a_i, b_j, c_{ij} \in \mathbb{R} \) and \( I_{nm} \) is the \( nm \times nm \) identity matrix. \( A_i \) and \( B_j \) are chosen to satisfy the standard orthonormality relations

\[
\delta_{i\ell} = \delta_{i\ell} = 0 \quad \text{and} \quad \text{Tr} (A_i A_j) = 2 \delta_{ij} , \quad \text{Tr} (B_i B_j) = 2 \delta_{ij} .
\]

The parametrization given by equation (1) is not the only one in use. In the literature parametrizations that use the Cholesky decomposition [6, 23] or the Euler angles of the elements of the group \( SU(N) \) [5, 28] can also be found. Which parametrization to employ depends strongly on the problem to be solved. In the context of bipartite systems, parametrization (1) is advantageous, as it directly implements the unity trace and the hermiticity of \( \rho \), and the states \( \rho_S \) and \( \rho_R \) of the systems \( S \) and \( R \) can be inferred directly:

\[
\rho = \text{Tr}_S \rho = \frac{1}{n} \left( I_n + a_i A_i \right) \quad \text{and} \quad \rho_R = \text{Tr}_R \rho = \frac{1}{m} \left( I_m + b_j B_j \right),
\]

where \( \text{Tr}_S \) and \( \text{Tr}_R \) denote the partial traces over the degrees of freedom of the systems \( S \) and \( R \) respectively. In the following, the state \( \rho \in \mathcal{M}^{(nm)} \) will be called the total state, whereas \( \eta = \rho_S = \text{Tr}_S \rho \in \mathcal{M}^{(n)} \) will be called the reduced state of \( \rho \) to emphasize the connection with open quantum system dynamics.

The vector \( \mu = (a_1, \ldots, a_{m^2-1}, b_1, \ldots, b_{n^2-1}, c_{11}, c_{12}, \ldots, c_{n-1,m-1})^T \) completely determines the state \( \rho \in \mathcal{M}^{(nm)} \) and vice versa. The positivity of density matrices restricts the possible vectors \( \mu \) to a proper subset \( \Sigma^{(nm)} \) of \( \mathbb{R}^{(n^2 + m^2 - 1)} \).

In the framework of open system quantum mechanics, so called assignment maps are introduced [15]. These maps assign a compatible total state to each reduced state of the open system. In the language of this article, an assignment map \( \pi: \mathcal{M}^{(n)} \to \mathcal{M}^{(nm)} \) is a map with the property

\[
\eta \in \mathcal{M}^{(n)} \Rightarrow \pi(\eta) = \rho \in \mathcal{M}^{(nm)} \quad \text{and} \quad \text{Tr}_R(\rho) = \eta .
\]

For a given reduced state \( \eta \in \mathcal{M}^{(n)} \), the total state \( \rho \in \mathcal{M}^{(nm)} \) with \( \text{Tr}_R(\rho) = \eta \) is obviously not unique. In order to gain a better understanding of open quantum dynamics, it is therefore necessary to investigate the spaces of total states \( \rho \in \mathcal{M}^{(nm)} \) that are conditioned on a given reduced state \( \eta \). These spaces will be denoted as \( \mathcal{M}^{(n \times m)}_\eta \):

\[
\mathcal{M}^{(n \times m)}_\eta = \left\{ \rho \in \mathcal{M}^{(nm)}; \text{Tr}_R \rho = \eta \in \mathcal{M}^{(n)} \right\} .
\]

Its corresponding subspace of \( \mathbb{R}^{n^2 + m^2 - 1} \) will be denoted as \( \Sigma^{(n \times m)}_\eta \). Given that

\[
\bigcup_{\eta \in \mathcal{M}^{(n)}} \mathcal{M}^{(n \times m)}_\eta = \mathcal{M}^{(nm)} ,
\]
a thorough analysis of conditioned spaces will not only shed light on assignment maps, but also on the properties of the total state space $\mathcal{M}^{(n \times m)}$. Before metric properties of $\mathcal{M}^{(n \times m)}$ can be discussed, it is necessary to introduce the notion of measure in $\mathcal{M}^{(n \times m)}$. Then, for example its volume and the *a priori* probability to find a separable state when choosing a state $\rho \in \mathcal{M}^{(n \times m)}$ at random can be determined.

3. The Hilbert–Schmidt measure

While the space of pure $nm$-dimensional states has a natural measure, the so-called Fubini–Study measure [26]—the measure induced by the unique Haar-measure on $U(nm)$—there is no unique measure to choose in the space $\mathcal{M}^{(n \times m)}$ of mixed states. As for the parametrization, the choice of the employed measure depends on the question that is to be answered. A comprehensive overview over a wide family of measures in $\mathcal{M}^{(n \times m)}$ can be found in [4]. One possibility to introduce the notion of distance that induces a measure in the space $\mathcal{M}^{(n \times m)}$ makes use of the unitarily invariant Hilbert–Schmidt inner product $(\cdot, \cdot)_{\text{HS}}$: $\mathcal{M}^{(n \times m)} \times \mathcal{M}^{(n \times m)} \rightarrow \mathbb{C}$:

$$\rho', \rho \in \mathcal{M}^{(n \times m)}: \quad \langle \rho', \rho \rangle_{\text{HS}} = \text{Tr}(\rho' \rho) = \text{Tr}(\rho \rho').$$

Following this definition, the Hilbert–Schmidt distance $d_{\text{HS}}(\rho', \rho)$ of two arbitrary density matrices $\rho', \rho \in \mathcal{M}^{(n \times m)}$ can be expressed as

$$d_{\text{HS}}(\rho', \rho) = \sqrt{\text{Tr}[(\rho - \rho')^2]}.$$  \hspace{1cm} (8)

The Hilbert–Schmidt distance induces a flat metric in $\Sigma^{(n \times m)}$ because of the tracelessness of the generators of the group $SU(n \times m)$ and the orthonormality relations (2):

$$d_{\text{HS}}(\rho', \rho) = \frac{\sqrt{2}}{nm} \sqrt{\sum_{i=1}^{nm} (a'_i - a_i)^2 + \sum_{j=1}^{nm} (b'_j - b_j)^2 + \sum_{k,l=1}^{n \times m} (c'_{kl} - c_{kl})^2} = \frac{\sqrt{2}}{nm} d_{\text{euclid}}(\bar{\rho}', \bar{\rho}).$$  \hspace{1cm} (9)

Up to an overall constant (which could be set equal to one by a change of the normalization of (1)), the mapping

$$\left( \mathcal{M}^{(n \times m)}, d_{\text{HS}} \right) \rightarrow \left( \Sigma^{(n \times m)}, d_{\text{euclid}} \right)$$  \hspace{1cm} (10)

is bijective and isometric. Therefore, any metric results about $\mathcal{M}^{(n \times m)}$ equipped with the Hilbert–Schmidt distance directly give the corresponding metric result in $\Sigma^{(n \times m)}$ equipped with the flat euclidian metric, and vice versa. This, of course, is also true for any subspaces of $\mathcal{M}^{(n \times m)}$ and $\Sigma^{(n \times m)}$ respectively, in particular for the spaces $\mathcal{M}^{(n \times m)}_\eta$ and $\Sigma^{(n \times m)}_\eta$.

The Hilbert–Schmidt volume $V_{\text{HS}}^{(n \times m)}$ of the space $\mathcal{M}^{(n \times m)}$ has been calculated by Życzkowski and Sommers in [31]:

$$V_{\text{HS}}^{(n \times m)} = \sqrt{nm} (2\pi)^{nm(nm-1)/2} \prod_{k=3}^{n \times m} \frac{\Gamma(k)}{\Gamma(\frac{nm^2}{k})} ,$$  \hspace{1cm} (11)

where $\Gamma(k)$ is the Gamma function of $k$. The derivation of (11) makes use of the fact that any density matrix $\rho \in \mathcal{M}^{(n \times m)}$ can be represented as $\rho = U\Lambda U^\dagger$, where
$A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{nm})$ is a positive diagonal matrix with $\text{Tr} A = 1$ and $U \in U(nm)$ is a unitary $nm \times nm$ matrix. As the Hilbert–Schmidt distance $d_{\text{HS}}$ is unitarily invariant, its corresponding volume element $dV_{\text{HS}}$ can be written as a product measure

$$
dV_{\text{HS}} = d\mu(\lambda_1, \ldots, \lambda_{nm}) \times d\varphi_{\text{Haar}} ,
$$

where $d\mu(\lambda_1, \ldots, \lambda_{nm})$ is a measure on the space of positive diagonal $nm \times nm$-matrices $A$ with $\text{Tr} A = 1$, i.e. the $(nm-1)$-simplex, and $d\varphi_{\text{Haar}}$ is a measure on the space of unitary $nm \times nm$ matrices that is induced by the Haar-measure on $U(nm)$. Both these measures can be expressed analytically and the Hilbert–Schmidt volume of $\mathcal{M}^{(n \times m)}$ can be calculated without resorting to the particular parametrization (1). For conditioned spaces, the property that the Hilbert–Schmidt measure is of product form fails to apply. It is still true that any density matrix $\rho \in \mathcal{M}^{(n \times m)}$ can be represented as $\rho = U\rho U^\dagger$, but for a given diagonal matrix $A$, only certain matrices $U \in U(nm)$ lead to a density matrix $\rho \in \mathcal{M}^{(n \times m)}$, while most $U$ result in a state $\rho' \notin \mathcal{M}^{(n \times m)}$. The volume of the total space $\mathcal{M}^{(n \times m)}$ can be obtained by integrating over the whole space of unitary matrices$^1$ independently of the entries of the matrix $A$. In the case of conditioned spaces, however, this independence no longer exists. Therefore, the considerations which led to the result (11) cannot be used in order to find the Hilbert–Schmidt volume $V^{(n \times m)}_{\text{HS}}(\eta)$ of the conditioned spaces $\mathcal{M}^{(n \times m)}_\eta$.

While the Hilbert–Schmidt volume of $\mathcal{M}^{(n \times m)}$ has been calculated, there only exist conjectures for the a priori probabilities $P^{(2 \times 2)}_{\text{sep}}(\eta)$ (see [24] and references therein) and $P^{(2 \times 3)}_{\text{sep}}$ [21] to find a separable state in $\mathcal{M}^{(2 \times 2)}$ (the state space of two coupled qubits) and $\mathcal{M}^{(2 \times 3)}$ (the state space of a qubit coupled to a qutrit) equipped with the Hilbert–Schmidt measure. As for the calculation of the volume of conditioned state spaces, the problem of finding general analytical results for $P^{(n \times m)}_{\text{sep}}$ is related to the fact that the Hilbert–Schmidt measure in the space of separable states is not of product form. Moreover, there are no easy-to-use criteria for the distinction between separable and entangled states beyond the $2 \times 3$ case [7, 17]. The same holds of course for the corresponding a priori probabilities $P^{(n \times m)}_{\text{sep}}(\eta)$ to find a separable state in the respective conditioned spaces $\mathcal{M}^{(n \times m)}_\eta$.

The calculation of $V^{(n \times m)}_{\text{HS}}(\eta)$ as well as the investigation of the a priori probabilities $P^{(n \times m)}_{\text{sep}}(\eta)$ are important in the context of open quantum dynamics, but they also shed further light on the structure and the properties of the total state space $\mathcal{M}^{(n \times m)}$. As the measures involved are complicated and not explicitly known, it is unlikely that the techniques used in [25] and [31] can be employed in order to solve these problems. It proves fruitful, however, to exploit the isometry of the metric spaces $(\mathcal{M}^{(n \times m)}_\eta, d_{\text{HS}})$ and $(\Sigma^{(n \times m)}_\eta, d_{\text{euclid}})$, i.e. make use of the particular parametrization (1), in order to find both numerical results and analytical conjectures for the Hilbert–Schmidt volume of $\mathcal{M}^{(n \times m)}_\eta$ and the a priori probabilities $P^{(n \times m)}_{\text{sep}}(\eta)$. It is important to mention, that both $V^{(n \times m)}(\eta)$ and $P^{(n \times m)}_{\text{sep}}(\eta)$ crucially depend on the measure that is used. In the following, assertions will solely be made for the Hilbert–Schmidt measure.

4. Coupled qubit-systems

The only quantum dynamical state space whose structure is completely known is the three-dimensional one-qubit state space $\mathcal{M}^{(1)}$ (see for example [11] for a thorough discussion of its

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$^1$ In fact, the integral in [31] is not evaluated over the total group $U(nm)$, but merely over the flag manifold $U(nm)/U(1)^m$. This does however not change the argument.
properties). Expressed in the parametrization (1), adapted to a monopartite system, any qubit-state can be written as

$$\rho_{\text{Qubit}} = \frac{1}{2} (I_2 + a_i \sigma_i),$$

(13)

where the operators $\sigma_i$ are the well-known Pauli-matrices. The constraint of positive semidefiniteness restricts the so-called Bloch-vector $\vec{r} = (a_1, a_2, a_3)^T$ to a solid ball (the Bloch-ball) of radius $|\vec{r}| \leq 1$. The pure qubit-states make up the surface of this ball, the mixed states lie in the interior and the completely mixed state $\rho = \frac{1}{2} I$ is at the center of the ball.

The lowest-dimensional state-space of a bipartite system is the space $\mathcal{M}^{(2 \times 2)}$, i.e. the state space of two coupled qubits. Systems of coupled qubits play an important role in the context of quantum computation (see [14]) and the dimensions of $\mathcal{M}^{(2 \times 2)}$ and $\mathcal{M}^{(2 \times 2)}_{\text{sep}}$ are low enough to allow for direct calculations in the coordinates of parametrization (1). Therefore the following investigations will be mainly restricted to the $2 \times 2$ case. Any two-qubit-state $\rho_{\text{2Qubits}} \in \mathcal{M}^{(2 \times 2)}$ can be written as

$$\rho_{\text{2Qubits}} = \frac{1}{4} (1_{\text{2Qubits}} + a_i \sigma_i \otimes 1_2 + b_j 1_2 \otimes \tau_j + c_{ij} \sigma_i \otimes \sigma_j),$$

(14)

where $\sigma_i$ and $\tau_j$ are the Pauli-matrices of the qubits, respectively. The reduced state $\eta = \text{Tr}_R(\rho_{\text{2Qubits}})$ is completely defined by the vector $\vec{a} = (a_1, a_2, a_3)^T$. All further considerations will be simplified by the observation that both the Hilbert–Schmidt measure in the space $\mathcal{M}^{(2 \times 2)}_{\eta}$ and the separability of a state $\rho_{\text{2Qubits}} \in \mathcal{M}^{(2 \times 2)}_{\eta}$ are invariant under a transformation $W \otimes 1_m$, where $W$ is an arbitrary special unitary matrix $\in SU(2)$. As the group $SU(2)$ is the double cover of the group of three-dimensional rotations, $SO(3)$ [27], $V_{\text{HS}}^{(2 \times m)} (\eta)$ and $p_{\text{sep}}^{(2 \times m)} (\eta)$ only depend on the radius of $\eta$ in the Bloch-ball. Accordingly, it is sufficient to calculate the volume $V_{\text{HS}}^{(2 \times m)} (r)$ and the probability $p_{\text{sep}}^{(2 \times m)} (r)$ for a ray from the center of the Bloch-ball to its surface, i.e. $r \in [0, 1]$. In the following, this ray will be chosen to be the ray from the center of the Bloch-ball to its north-pole, i.e. $a_1 = a_2 = 0, a_3 = r$. As the calculations are too involved to be carried out analytically even for the two-qubit case, they will be conducted in a first step for the seven-dimensional family of two-qubit X-states.

5. Conditioned volume $V_{\text{HS}}^{(X)} (r)$ and a priori Hilbert–Schmidt separability probability $p_{\text{sep}}^{(X)} (r)$ for X-states

A possible family of states in $\mathcal{M}^{(2 \times 2)}$ that allows for analytical conclusions, is the family of X-states. These are $4 \times 4$ density matrices of the form

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}.$$ 

(15)

They have been introduced in [13, 29] as a seven-dimensional family of states that contains maximally entangled pure states, as well as separable states. Because of their simple form, it is possible to carry out various analytical computations, like for example the calculation of the quantum discord of an X-state [1]. Note that the definition of X-states requires the choice of a fixed basis. Here, the basis is chosen such that the Bloch-vector of the reduced state has a $z$-component only (see below). X-states do not constitute a ‘random’ subset of $\mathcal{M}^{(2 \times 2)}$, but
possess an underlying symmetry [18], which might help to generalize the results found for X-states to arbitrary systems. A comparison of (14) and (15) shows that any X-state can be represented as

$$\rho_X = \frac{1}{4} \left(1 + a_3 \sigma_3 \otimes 1 + b_3 1 \otimes \tau_3 + c_{11} \sigma_1 \otimes \tau_1 + c_{12} \sigma_1 \otimes \tau_2 + c_{21} \sigma_2 \otimes \tau_1 + c_{22} \sigma_2 \otimes \tau_2 + c_{33} \sigma_3 \otimes \tau_3 \right).$$

(16)

As both $a_1$ and $a_2$ are equal to zero, the reduced states of X-states lie by construction on the ray from the center of the Bloch-ball to its north pole. The eigenvalues of $\rho_X$ can be expressed analytically [1], and the two eigenvalues that are of importance for the definiteness of $\rho_X$ lead to the conditions:

$$\sqrt{(a_3 + b_3)^2 + (c_{11} - c_{22})^2 + (c_{12} + c_{21})^2} \leq (1 + c_{33}),$$

(17)

$$\sqrt{(a_3 - b_3)^2 + (c_{11} + c_{22})^2 + (c_{12} - c_{21})^2} \leq (1 - c_{33}).$$

(18)

The inequalities (17) and (18) define two hypersurfaces in $\mathbb{R}^6$ that confine the space $\Sigma_{a_3}^{(X)}$ of X-states. The conditioned volume $V_{\text{HS}}^{(X)}(a_3) \equiv V_{\text{HS}}^{(X)}(r)$ can be calculated by evaluating the integral

$$V_{\text{euclid}}^{(X)}(r) = \int_{c_{33}} \int_{c_{11}} \int_{c_{22}} \int_{c_{12}} \int_{c_{21}} \int_{b_3} \text{Volume defined by the inequalities (17) and (18)}$$

(19)

and multiplying it by the appropriate factor, i.e. $V_{\text{HS}}^{(X)}(r) = \left(\frac{1}{2\sqrt{2}}\right)^6 V_{\text{euclid}}^{(X)}(r)$. The explicit calculation is carried out in appendix A. It yields the surprisingly simple result

$$V_{\text{HS}}^{(X)}(r) = \frac{\pi^2}{2304} \left(1 - r^2\right)^3.$$

(20)

As a by-product of this formula, the Hilbert–Schmidt volume $V_{\text{HS}}^{(X)}$ of the space of X-states can be derived:
\[ V_{\text{HS}}^{(X)} = \int_0^1 dr \ V_{\text{HS}}^{(X)}(r) = \frac{\pi^2}{5040}. \]  

(21)

In figure 1 we show the analytical curve \( V_{\text{HS}}^{(X)}(r) \) from (20) in comparison with numerical results obtained from a Monte-Carlo integration.

For the special case of \( X \)-states, the partial transpose with respect to the second qubit only changes the signs of \( c_{12} \) and \( c_{22} \). The PPT-criterion (see [7, 17]) for the separability of a state together with inequalities (17) and (18) then allows for a direct calculation of the volume of separable \( X \)-states, \( V_{\text{HS, sep}}^{(X)}(r) \). The ratio determines \( p_{\text{sep}}^{(X)}(r) = V_{\text{HS, sep}}^{(X)}(r)/V_{\text{HS}}^{(X)}(r) \), the probability to find a separable state in \( \mathcal{M}_2^{(X)} \). This calculation is carried out in appendix B. It yields:

\[ p_{\text{sep}}^{(X)}(r) = \frac{2}{5}, \quad r \in [0, 1) \quad \text{and} \quad p_{\text{sep}}^{(X)}(1) = 1. \]  

(22)

We also performed numerical Monte-Carlo calculations which confirmed these values. Most remarkably, the probability to find a separable state in a conditioned state space \( \mathcal{M}_2^{(X)} \) is independent of the reduced state, i.e. independent of the radius \( r \) for \( r < 1 \) and jumps to one in a discontinuous way at \( r = 1 \). The latter fact that \( p_{\text{sep}}^{(X)}(1) = 1 \) is clear: a pure reduced state \( r = 1 \) can only be realized by a product and thus, a separable total state.

6. Conditioned volume \( V_{\text{HS}}^{(2 \times m)}(r) \)

While the eigenvalues of \( X \)-states can be easily expressed analytically, the eigenvalues of a general two-qubit state could, in principle, be calculated. So far, however, a direct derivation of the volume \( V_{\text{HS}}^{(2 \times 2)}(r) \) from these expressions is beyond reach. For higher dimensional cases, i.e. for a qubit coupled to a \( m \)-dimensional environment, not even the eigenvalues can be found analytically. Accordingly, for \( V_{\text{HS}}^{(2 \times m)}(r) \) and \( p_{\text{sep}}^{(2 \times m)}(r) \) only numerical results are provided here. The knowledge of \( V_{\text{HS}}^{(X)}(r) \) and \( V_{\text{HS}}^{(2 \times m)}(r) \), however, allows for conjectures of the analytical expressions for \( V_{\text{HS}}^{(2 \times m)}(r) \) and \( p_{\text{sep}}^{(2 \times m)}(r) \).

The Hilbert–Schmidt volume \( V_{\text{HS}}^{(2 \times 2)}(r) \) is numerically estimated by a Monte-Carlo integration. It can be readily derived that the range of each of the parameters \( a_i, b_j, \) and \( c_{ij} \) in the normalization of (14) is \([-1, 1]\). Therefore, a cube of edge length \( d = 2 \) centered around the origin completely encompasses each conditioned two-qubit space \( \Sigma_2^{(2 \times 2)} \) and its euclidian volume can be estimated by a simple rejection sampling. Figure 1 shows the accuracy of this procedure for the six-dimensional case of conditioned \( X \)-states, where we compare to analytical results. The full two-qubit problem is 12-dimensional. The higher dimension requires a larger number of samples in order to achieve similar accuracy. For the general case of a qubit coupled to an \( m \)-dimensional environment, the enormous number of samples necessary to obtain representative numerical results renders rejection sampling methods useless.

A sampling method that goes without the rejection of sampling points makes use of the fact that \( N \times N \) density matrices can be sampled uniformly distributed according to the Hilbert–Schmidt measure, by sampling pure \( N^2 \)-dimensional states uniformly distributed according to the Fubini–Study measure and partially tracing them over \( N \) degrees of freedom (see [4]). Applied to the case of \( 2m \times 2m \) density matrices, this means that they can be sampled according to Hilbert–Schmidt measure, by sampling \( 4m^2 \)-dimensional pure states according to the Fubini–Study measure (see e.g. [4], chapter 7, for a description of how to sample pure states uniformly distributed according to the Fubini–Study measure) and partially tracing these states over \( 2m \) degrees of freedom. In order to sample states \( \rho \in \mathcal{M}_2^{(2 \times m)} \)
conditioned on a given qubit state $\eta$ according to Hilbert–Schmidt measure, it is sufficient to restrict the sampling of the pure states to the subset of pure states that yield the given qubit state $\eta$ when partially traced over $m^2$ degrees of freedom. By construction, every sample then gives a valid $2m \times 2m$ density matrix, which increases the accuracy of the results and makes it independent of $r$. This sampling method is hence well suited to estimate $P_{\text{sep}}^{(2 \times m)}(r)$ and the $r$-dependence of $V_{\text{HS}}^{(2 \times m)}(r)$. However, it does not yield any estimates of the absolute values of $V_{\text{HS}}^{(2 \times m)}(r)$ and, therefore, has to be combined with the results of the rejection sampling.

The result for $V_{\text{HS}}^{(2 \times 2)}(r)$ obtained from a Monte-Carlo integration with $10^8$ samples is displayed in figure 2. The course of the numerical result resembles the analogous analytical result for $V_{\text{HS}}^{(2 \times 2)}(r)$. For scaling reasons, the highest power of $r$ in an analytical expression for $V_{\text{HS}}^{(2 \times 2)}(r)$ has to be $r^{12}$ which leads to the conjecture that $V_{\text{HS}}^{(2 \times 2)}(r)$ is given by

$$V_{\text{HS}}^{(2 \times 2)}(r) = V_{\text{HS}}^{(2 \times 2)}(0)\left(1 - r^2\right)^6. \quad (23)$$

The fit of the conjectured curve to the numerical data is shown in figure 2. Assuming that equation (23) is correct, the value of $V_{\text{HS}}^{(2 \times 2)}(0)$ can be calculated by connecting $V_{\text{HS}}^{(2 \times 2)}(r)$ to the volume $V_{\text{HS}}^{(2 \times 2)}$ of the total two-qubit state space:

$$V_{\text{HS}}^{(2 \times 2)} = 2^{-3} \cdot 4\pi \int_0^1 dr \ r^2 V_{\text{HS}}^{(2 \times 2)}(r) = \frac{2^9\pi}{45045} V_{\text{HS}}^{(2 \times 2)}(0), \quad (24)$$

where the factor $2^{-3}$ is necessary to convert from the euclidean to the Hilbert–Schmidt volume\textsuperscript{2}. From (11) it can then be deduced that

$$V_{\text{HS}}^{(2 \times 2)}(0) = \frac{45045}{2^9\pi} V_{\text{HS}}^{(2 \times 2)} \approx 3.16241 \times 10^{-5} . \quad (25)$$

\textsuperscript{2} Note that equation (24) is not a relation of volumes. $V_{\text{HS}}^{(2 \times 2)}(0)$ is a 12-dimensional volume, while $V_{\text{HS}}^{(2 \times 2)}$ is a 15-dimensional one. They cannot be compared in a meaningful way. Therefore (24) is merely an equation to calculate the numerical value of $V_{\text{HS}}^{(2 \times 2)}(0)$. 

---

**Figure 2.** Hilbert–Schmidt volume $V_{\text{HS}}^{(2 \times 2)}(r)$ of the space of two coupled qubits. Shown are conjecture (23) (green) and the numerical result of a Monte-Carlo integration with $10^8$ samples (blue).
This value coincides perfectly with the analogous value found via the Monte-Carlo sampling:
\[
V_{\text{HS, num}}^{(2 \times 2)}(0) = (3.16333 \pm 0.05713) \times 10^{-5}.
\] (26)

The agreement of the conjectured formula and the numerical results, and the fact that \( V_{\text{HS}}^{(2 \times 2)}(r) \) seems to be described by a simple polynomial, suggest the following generalization of (23) for the \( 2 \times m \) case:
\[
V_{\text{HS}}^{(2 \times m)}(r) = V_{\text{HS}}^{(2 \times m)}(0) \left( 1 - r^2 \right)^{2m^2 - 1}.
\] (27)

The corresponding conjecture for \( V_{\text{HS}}^{(2 \times m)}(0) \) follows as in (24):
\[
V_{\text{HS}}^{(2 \times m)}(0) = \sqrt{m} \cdot 2^{2m^2 - m - \frac{2}{3}} \cdot r^{2m^2 - m - \frac{1}{3}} \cdot \prod_{k = 1}^{2m} \frac{\Gamma(k) \cdot \Gamma\left(\frac{1}{2} + 2m^2\right)}{\Gamma\left(4m^2\right) \cdot \Gamma\left(-1 + 2m^2\right)}.
\] (28)

It is rather difficult to investigate the validity of (27) and (28) with a numerical procedure that involves the rejection of samples, as the dimension of the corresponding state spaces grows rapidly and the huge number of required samples to obtain meaningful results cannot be reached within an acceptable amount of time even for the case \( m = 3 \). However, by employing the method described above (not relying on the rejection of samples) at least (27) can be verified numerically. This is done by sampling states uniformly distributed according to the Hilbert–Schmidt measure, and recording their radius in the Bloch-ball, respectively. The resulting histogram then has an envelope that is described by a function proportional to \( 4\pi r^2 V_{\text{HS}}^{(2 \times m)}(r) \), where \( V_{\text{HS}}^{(2 \times m)}(r) \) is the conjectured formula (27). It is necessary to multiply by the factor \( 4\pi r^2 \), which is the area of the respective spheres, in order to correctly describe the envelope of the histograms as they display the number of states sampled for a given radius of the reduced states. The histograms for the \( 2 \times 3 \) and the \( 2 \times 4 \) case are shown in figure 3. They coincide perfectly with (27).

7. Conditioned a priori Hilbert–Schmidt separability probability \( p_{\text{sep}}^{(2 \times m)}(r) \)

Numerical results for the probability \( p_{\text{sep}}^{(2 \times m)}(r) \) can easily be obtained with high accuracy by employing the sampling method that does not require the rejection of samples. To this end, for each radius \( r, k \) total states with the given radius are sampled, and, at least for \( m = 2 \) and

Figure 3. Histograms of the Hilbert–Schmidt volume distribution as a function of the radius \( r \) in the Bloch-ball for the \( 2 \times 3 \) and the \( 2 \times 4 \) cases for \( 10^7 \) samples. In blue are the numerical data, in green the conjectured envelopes.
m = 3, the separability of each of these states is checked via the PPT-criterion. The ratio of the number of separable states to the total number of sampled states then gives an estimate for \( p_{\text{sep}}(2 \times 2) \). As it is not the absolute volume of the space of separable states that is to be estimated, but rather the ratio of the two volumes \( V_{\text{HS, sep}}(2 \times m) \) and \( V_{\text{HS}}(2 \times m) \), this sampling procedure in this case does not only give qualitative but also quantitative results. For the 2 × 2 case they are displayed in figure 4.

The resemblance to the corresponding results for the X-states is striking: our numerical evidence strongly suggests that \( p_{\text{sep}}(2 \times 2) \) is constant for \( r \in [0, 1) \) and jumps to 1 in a discontinuous way. As for X-states it is clear why \( p_{\text{sep}}(2 \times 2) (1) = 1 \). The fact that this jump is discontinuous cannot yet be formally proven, but the thorough numerical investigation of the region \( r \leq 1 \), also shown in figure 4, strongly suggests this conjecture. Obviously, from a knowledge of \( p_{\text{sep}}(2 \times 2) \), the value \( p_{\text{sep}}(2 \times 2) \) of the total state space could be computed to be

\[
p_{\text{sep}}(2 \times 2) = \int_0^1 dr \ p_{\text{sep}}(2 \times 2) (r).
\]

The numerical results for \( p_{\text{sep}}(2 \times 2) \) obtained above yield

\[
p_{\text{sep, num}}(2 \times 2) = 0.24262 \pm 0.01340 .
\]

Apart from a postulated but not yet formally proven formula [24], there do not exist any analytical results for \( p_{\text{sep}}(2 \times 2) \). However, based on extensive numerical research, a value of

\[
p_{\text{sep}}(2 \times 2) \approx 0.24242
\]

has been conjectured (see [24] and references therein). The agreement between (30) and (31) further supports the conjecture of this value.

The accuracy of the sample method without rejection even allows for an expansion of the numerical investigation of \( p_{\text{sep}}(2 \times 3) \) (r) and the probability to find a state with positive partial trace, \( p_{\text{PosPart}}(2 \times 4) \). The respective results are shown in figure 5.

They correspond qualitatively to the parallel results for \( p_{\text{sep}}(2 \times 2) \). Most remarkably, within numerical confidence, \( p_{\text{sep}}(2 \times 3) \) (r) and \( p_{\text{PosPart}}(2 \times 4) \) (r) are independent of \( r \) — except for the

Figure 4. Numerical results for the \textit{a priori} Hilbert–Schmidt separability probability \( p_{\text{sep}}(2 \times 2) \) for 10^5 samples. The small box shows the numerical results for \( r \leq 1 \) in detail.
case $r = 1$. For the $2 \times 3$ case a conjecture for $p^{(2 \times 3)}_{\text{sep}}(r)$ was made by Slater in [22] based on an extensive numerical investigation:

$$p^{(2 \times 3)}_{\text{sep}, [24]} = \frac{32}{1199} \approx 0.02669 .$$

(32)

From the numerical results above we find that

$$p^{(2 \times 3)}_{\text{sep, num}} = 0.02700 \pm 0.00016 ,$$

(33)

which is in good correspondence with (32).

The independence of the functions $p^{(2 \times 2)}_{\text{sep}}(r)$, $p^{(2 \times 3)}_{\text{sep}}(r)$ and $p^{(2 \times 4)}_{\text{PosPart}}(r)$ of $r$ leads to the conjecture that

$$p^{(2 \times m)}_{\text{sep}}(r) = p^{(2 \times m)}_{\text{sep}} \text{ for } r \in [0, 1) \text{ and } p^{(2 \times m)}_{\text{sep}}(1) = 1 .$$

(34)

As beyond the $2 \times 3$ case there is no simple criterion to decide whether or not a bipartite state is separable, it is easier to test the more conservative conjecture

$$p^{(2 \times m)}_{\text{PosPart}}(r) = p^{(2 \times m)}_{\text{PosPart}} \text{ for } r \in [0, 1) \text{ and } p^{(2 \times m)}_{\text{PosPart}}(1) = 1 ,$$

(35)

where $p^{(2 \times m)}_{\text{PosPart}}$ denotes the a priori Hilbert–Schmidt probability for a state in $\mathcal{M}^{(2 \times m)}$ with a positive partial transpose.

8. Conclusion and discussion

Although certain lower dimensional sections of the space $\mathcal{M}^{(2 \times m)}$ have already been studied analytically (see e.g. [20]), not much attention has been given to the conditioned spaces $\mathcal{M}^{(2 \times m)}_{\eta}$ yet. A thorough knowledge of their properties is crucial for the understanding of assignment maps in the framework of open quantum systems and might help to shed light on fundamental open questions in quantum state geometry.

In this work, the metric properties of the conditioned spaces $\mathcal{M}^{(2 \times m)}_{\eta}$ equipped with the Hilbert–Schmidt measure have been investigated numerically. The results of this investigation led to the conjecture that the Hilbert–Schmidt volume $V_{\text{HS}}^{(2 \times m)}(\eta)$ follows a simple polynomial of the radius $r$ in the Bloch sphere of the reduced state $\eta$, while the probability to find a
separable state in a conditioned space $\mathcal{M}^{(2 \times m)}_r$ is independent of $r$—except for the case $r = 1$. Both these results can be proven analytically for the case of the seven-dimensional family of $X$-states.

The insight that $V_{HS}^{(2 \times m)} (r)$ seems to follow a simple polynomial of $r$ reveals a generic metric feature of well-defined sub-spaces of $\mathcal{M}^{(2 \times m)}$—with $m$ arbitrary. Hence, it constitutes a further step to a description of the metric and geometric properties of bipartite state spaces. On the other hand, the independence of $p_{sep}^{(2 \times m)} (r)$ of the radius $r$ we found is intriguing, as it, once analytically proven, opens new ways to study properties of the total state space through these conditional cuts. Above all for the $2 \times 2$ case, the lower dimension of conditioned spaces in comparison to the full state space would then allow for a direct calculation of $p_{sep}^{(2 \times 2)} (0)$, and therefore for a direct calculation of $\pi_{sep}^{(2 \times 2)}$.

It is important to point out that all these results and conjectures only hold for the Hilbert–Schmidt measure. This particularity further singles out this measure among all other unitarily invariant measures. The corresponding results for the example of the product measure used in [30] are shown in figure 6. One means to find analytical expressions for all quantities investigated in this paper could be the use of $X$-states. Obviously, there is a deep qualitative connection between the metric properties of this seven-dimensional family of states, and the corresponding total space of states. A thorough investigation of higher-dimensional $X$-states, i.e. in a first step $6 \times 6$ $X$-states, might therefore further support the conjectures made for the general space $\mathcal{M}^{(2 \times m)}$. Furthermore, they might even be helpful from a quantitative point of view. While the conjectured value for $p_{sep}^{(2 \times 2)}$ of $33$ seems plausible, the origins of this simple fraction still remain unclear. From formula (B.5) the volume of conditioned entangled $X$-states can be derived to be equal to $\frac{2}{15} \pi^2 (1 - r^2)^3$. If $d_X$ denotes the number of free parameters for conditioned $X$-states, the denominator 15 is equal to $3 (d_X - 1)$. If $d_{(2 \times 2)}$ denotes the corresponding number for the full problem, it can easily be seen that the denominator $33$ of the conjectured value of $p_{sep}^{(2 \times 2)}$ is equal to $3 (d_{(2 \times 2)} - 1)$. While this is still highly speculative, it nevertheless suggests that the analytical results for $X$-states might be generalizable to the total $2 \times 2$ and even higher-dimensional cases.

Figure 6. Results for the a priori probability $p_{sep, prod}^{(2 \times 2)} (r)$ for the product measure used in [30] obtained from a Monte-Carlo integration with $10^7$ samples. Because of the vanishing volumes that are involved for $r \lesssim 0.9$ the relative error of the data diverges from this value on. The green dotted line shows the numerical value of $p_{sep, prod}^{(2 \times 2)}$ for this measure.
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Appendix A. Volume of the space $\mathcal{M}^X(\mathbf{r})$

Semi-definiteness of $\rho_X$ yields two inequalities that define the seven-dimensional subspace occupied by $X$-states in the full two-qubit parameter space:

$$\sqrt{(a_3 + b_3)^2 + (c_{11} - c_{22})^2 + (c_{12} + c_{21})^2} \leq (1 + c_{33}), \quad (A.1)$$

$$\sqrt{(a_3 - b_3)^2 + (c_{11} + c_{22})^2 + (c_{12} - c_{21})^2} \leq (1 - c_{33}). \quad (A.2)$$

(A.1) and (A.2) set the limits for $c_{33}$: $c_{33} \in [-1, 1]$. Given these limits, (A.1) and (A.2) can be squared, and one obtains:

$$\left(a_3 + b_3\right)^2 + \left(c_{12} + c_{21}\right)^2 + \left(c_{11} - c_{22}\right)^2 - \left(1 + c_{33}\right)^2 \leq 0 \quad (A.3)$$

$$\left(a_3 - b_3\right)^2 + \left(c_{12} - c_{21}\right)^2 + \left(c_{11} + c_{22}\right)^2 - \left(1 - c_{33}\right)^2 \leq 0. \quad (A.4)$$

The task of calculating the volume defined by these two inequalities is remarkably simplified by the following coordinate transformation:

$$x = \frac{1}{\sqrt{1 - a_3^2}} (c_{22} + c_{11}), \quad X = \frac{1}{\sqrt{1 - a_3^2}} (c_{21} + c_{12}),$$

$$y = \frac{1}{\sqrt{1 - a_3^2}} (c_{12} + c_{21}), \quad Y = \frac{1}{\sqrt{1 - a_3^2}} (c_{11} - c_{22}),$$

$$z = \frac{1}{1 + a_3} (b_3 + c_{33}), \quad Z = \frac{1}{1 - a_3} (b_3 - c_{33}).$$

This transformation is singular for $a_3 = 1$. However, for this value of $a_3$ the space $\mathcal{M}^X(a_3)$ is lower-dimensional, and its volume therefore zero. The Jacobian of this transformation is $\frac{1}{8} (1 - a_3^2)^3$ and in the new coordinates (A.3) and (A.4) read as:

$$x^2 + y^2 \leq \frac{1}{1 - a_3^2} \left[(1 - c_{33})^2 - (a_3 - b_3)^2\right] = r^2, \quad (A.5)$$

$$X^2 + Y^2 \leq \frac{1}{1 - a_3^2} \left[(1 + c_{33})^2 - (a_3 + b_3)^2\right] = R^2, \quad (A.6)$$

with

$$r^2 = \frac{1}{1 - a_3^2} \left[1 - c_{33} - a_3 + b_3\right] \left[1 - c_{33} + a_3 - b_3\right] = (1 + Z)(1 - z)$$

and

$$R^2 = \frac{1}{1 - a_3^2} \left[1 + c_{33} - a_3 - b_3\right] \left[1 + c_{33} + a_3 + b_3\right] = (1 - Z)(1 + z).$$

From (A.5) and (A.6) it follows that the integrals in the $x - y$-plane and the $X - Y$-plane will merely give the areas of circles of radius $r$ and $R$, respectively. As both $r^2$ and $R^2$ have to be positive, $z$ and $Z$ range from $-1$ to $1$. The euclidian volume $V_{\text{euclid}}(a_3)$ of the space of
conditioned X-states can then be calculated:

\[
V_{\text{euclid}}^{(X)}(a_3) = \frac{\pi^2}{8} (1 - a_3^2)^3 \int_{-1}^{1} dz \int_{-1}^{1} dZ \left(1 - Z^2\right)^2 \frac{1}{a_1^2 R^2}
\]

\[
= \frac{\pi^2}{8} (1 - a_3^2)^3 \left[ \int_{-1}^{1} dz \left(1 - z^2\right)^2 \right] = \frac{2}{9} \pi^2 (1 - a_3^2)^3 , \quad (A.7)
\]

which is valid for all \(a_3 \in [0, 1]\). From (A.7), the result (20) for \(V_{\text{HS}}^{(X)}(a_3)\) follows directly.

Appendix B. A priori probability \(p_{\text{sep}}^{(X)}(r)\) to find a separable state in \(\mathcal{M}_r^{(X)}\)

In order for a two-qubit-state to be separable, its partial transpose with respect to one of the subsystems has to be positive semi-definite [7, 17]. For the special case of X-states, the partial transpose with respect to the second qubit merely changes the signs of \(c_{12}\) and \(c_{22}\). An X-state is hence separable iff it satisfies the two additional restrictions

\[
x^2 + y^2 \leq (1 - Z)(1 + z) = R^2 \tag{B.1}
\]

and

\[
X^2 + Y^2 \leq (1 + Z)(1 - z) = r^2 . \tag{B.2}
\]

Together with (A.5) and (A.6), these two inequalities allow for a direct calculation of \(V_{\text{euclid, sep}}^{(X)}\), the euclidian volume of separable X-states:

\[
V_{\text{euclid, sep}}^{(X)} = \frac{\pi^2}{8} (1 - a_3^2)^3 \int_{-1}^{1} dz \int_{-1}^{1} dZ \min \left(r^2, R^3\right)^2 , \quad \text{(B.3)}
\]

where \(\min \left(r^2, R^3\right)^2\) denotes the minimum of \(\{r^2, R^3\}\). A short calculation yields that

\[
r^2 < R^2 \iff Z < z
\]

and therefore

\[
V_{\text{euclid, sep}}^{(X)} = \frac{\pi^2}{8} (1 - a_3^2)^3 \int_{-1}^{1} dz \int_{-1}^{1} dZ r^4(z, Z) + \int_{-1}^{1} dZ \int_{-1}^{1} dZ R^4(z, Z) . \quad \text{(B.4)}
\]

It is easy to verify that (B.4) leads to the result

\[
V_{\text{euclid, sep}}^{(X)} = \frac{4\pi^2}{45} (1 - a_3^2)^3 . \quad \text{(B.5)}
\]

Accordingly, for \(r \equiv a_3 \in [0, 1]\), the probability \(p_{\text{sep}}^{(X)}(r)\) is independent of \(r\) and equal to:

\[
p_{\text{sep}}^{(X)}(r) = \frac{V_{\text{euclid, sep}}^{(X)}(r)}{V_{\text{euclid}}^{(X)}(r)} = \frac{2}{5}
\]

For \(r = 1\) all reduced states are pure, and therefore all the corresponding total states are separable. Hence: \(p_{\text{sep}}^{(X)}(1) = 1\).

References

[1] Ali M, Rau A R P and Alber G 2010 Quantum discord for two-qubit X states Phys. Rev. A 81 042105
[2] Alicki R and Lendi K 1987 Quantum Dynamical Semigroups and Applications (Berlin: Springer)
[3] Alicki R 1995 Comment on reduced dynamics need not be completely positive Phys. Rev. Lett. 75 5020
[4] Bengtsson I and Życzkowski K 2008 *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge: Cambridge University Press)
[5] Byrd M 1998 Differential geometry on SU(3) with applications to three state systems *J. Math. Phys.* **39** 6125–36
[6] Daboul J 1967 Conditions on density matrix elements and their application in resonance production *Nucl. Phys.* B **4** 180–8
[7] Horodecki M, Horodecki P and Horodecki R 1996 Separability of mixed states: necessary and sufficient conditions *Phys. Lett.* A **223** 1–8
[8] Jordan T F 2006 Assumptions that imply quantum dynamics is linear *Phys. Rev.* A **73** 022101
[9] Jordan T F, Shaji A and Sudarshan E C G 2004 Dynamics of initially entangled open quantum systems *Phys. Rev.* A **70** 052110
[10] Jordan T F, Shaji A and Sudarshan E C G 2006 Mapping the Schrödinger picture of open quantum dynamics *Phys. Rev.* A **73** 012106
[11] Kimura G 2003 The Bloch vector for $N$-level systems *Phys. Lett.* A **314** 339–49
[12] Masillo F, Scolarici G and Solombrino L 2011 Some remarks on assignment maps *J. Math. Phys.* **52** 012101
[13] Maziero J, Céleri L C, Serra R M and Vedral V 2009 Classical and quantum correlations under decoherence *Phys. Rev.* A **80** 044102
[14] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[15] Pechukas P 1994 Reduced dynamics need not be completely positive *Phys. Rev. Lett.* **73** 1060–2
[16] Pechukas P 1995 Pechukas replies *Phys. Rev. Lett.* **75** 3021–3021
[17] Peres A 1996 Separability criterion for density matrices *Phys. Rev. Lett.* **77** 1413–5
[18] Rau A R P 2009 Algebraic characterization of $X$-states in quantum information *J. Phys. A: Math. Theor.* **42** 412002
[19] Rodríguez-Rosario C A, Modi K and Aspuru-Guzik A 2010 Linear assignment maps for correlated system-environment states *Phys. Rev.* A **81** 012313
[20] Slater P B 2000 Exact Bures probabilities that two quantum bits are classically correlated *Eur. Phys. J.* B **17** 471–80
[21] Slater P B 2003 *A priori* probability that a qubit–qutrit pair is separable *J. Opt. B: Quantum Semiclass. Opt.* **5** 651–6
[22] Slater P B 2007 Dyson indices and Hilbert–Schmidt separability functions and probabilities *J. Phys. A: Math. Theor.* **40** 14279
[23] Slater P B 2010 Radial and azimuthal profiles of two-Qubit/Rebit Hilbert–Schmidt separability probabilities and related 3D visualization analyses arXiv:1010.5180
[24] Slater P B 2013 A concise formula for generalized two-qubit Hilbert–Schmidt separability probabilities *J. Phys. A: Math. Theor.* **46** 445302
[25] Sommers H-J and Życzkowski K 2003 Bures volume of the set of mixed quantum states *J. Phys. A: Math. Gen.* **36** 10083–100
[26] Study E 1905 Kürzeste Wege im komplexen Gebiet *Math. Ann.* **60** 321–78
[27] Taylor R L 1954 Covering groups of nonconnected topological groups *Proc. Am. Math. Soc.* **5** 753–68
[28] Tilma T and Sudarshan E C G 2002 Generalized Euler angle parametrization for SU(N) *J. Phys. A: Math. Gen.* **35** 10467
[29] Vedral V and Plenio M B 1998 Entanglement measures and purification procedures *Phys. Rev.* A **57** 1619–33
[30] Życzkowski K, Horodecki P, Sanpera A and Lewenstein M 1998 Volume of the set of separable states *Phys. Rev.* A **58** 883–92
[31] Życzkowski K and Sommers H-J 2003 Hilbert–Schmidt volume of the set of mixed quantum states *J. Phys. A: Math. Gen.* **36** 10115