A NOTE ON REGULAR SUBGROUPS OF THE AUTOMORPHISM GROUP OF THE LINEAR HADAMARD CODE

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ABSTRACT. We consider the regular subgroups of the automorphism group of the linear Hadamard code. These subgroups correspond to the regular subgroups of $GA(r, 2)$, w.r.t action on the vectors of $F_2^n$, where $n = 2^r - 1$ is the length of the Hadamard code. We show that the dihedral group $D_{2^{r-1}}$ is a regular subgroup of $GA(r, 2)$ only when $r = 3$. Following the approach of [13] we study the regular subgroups of the Hamming code obtained from the regular subgroups of the automorphism group of the Hadamard code of length 15.

Keywords: error-correcting code, automorphism group, regular action, affine group.

1. INTRODUCTION

Let $x$ be a binary vector of the $n$-dimensional vector space $F_2^n$, $\pi$ be a permutation of the coordinate positions of $x$. Consider the transformation $(x, \pi)$ acting on a binary vector $y$ by the following rule:

$$(x, \pi)(y) = x + \pi(y),$$

where $\pi(y) = (y_{\pi(1)}, \ldots, y_{\pi(n)})$. The composition of two automorphisms $(x, \pi)$, $(y, \pi')$ is defined as follows

$$(x, \pi) \cdot (y, \pi') = (y + \pi(x), \pi' \circ \pi),$$

where $\circ$ is the composition of permutations $\pi$ and $\pi'$.

The automorphism group of the Hamming space $F_2^n$ is defined to be $\text{Aut}(F_2^n) = \{(x, \pi) : x \in C, \pi \in S_n, x + \pi(F_2^n) = F_2^n \}$ with the operation composition, here $S_n$ denotes the group of symmetries of order $n$.

The automorphism group $\text{Aut}(C)$ of a code $C$ is the setwise stabilizer of $C$ in $\text{Aut}(F_2^n)$. In sequel for the sake of simplicity we require the all-zero vector, which we denote by $0^n$ to be always in the code. Then we have the following representation

$$\text{Aut}(C) = \{(x, \pi) : x \in C, \pi \in S_n, x + \pi(C) = C\}.$$

The symmetry group (also known as the permutation automorphism group) of $C$ is defined as

$$\text{Sym}(C) = \{\pi \in S_n : \pi(C) = C\}.$$
A code $C$ is called \textit{transitive} if there is a subgroup $H$ of $\text{Aut}(C)$ acting transitively on the codewords of $C$. If the order of $H$ coincides with the size of $C$, then $H$ acting on $C$ is called a \textit{regular group} \cite{14} (sometimes called sharply-transitive) and the code $C$ is called \textit{propelinear}.

Propelinear codes provide a general view on linear and additive codes, many of which are optimal. The concept is specially important in cases where there are many nonisomorphic codes with the same parameters, separating the codes that are "close" to linear. In particular, among propelinear codes there are $Z_2Z_4$-linear codes that could be obtained from a Hadamard matrix of order $n$. Some researchers consider Hadamard codes of length $n$, augmented by all-ones vector, others study their shortenings of length $n - 1$. $Z_2Z_4$-linear perfect codes were classified in \cite{5}, \cite{10}, along with the description of their automorphism groups in \cite{9}. In work \cite{15} $Z_2Z_4Q_8$-Hadamard codes are discussed.

In below by the \textit{Hadamard code} $A_n$ we mean the linear Hadamard code, i.e. of length $n = 2^r - 1$, dimension $r$ and minimum distance $(n + 1)/2$. The code is dual to the Hamming code, which we denote by $H_n$, so their symmetry groups coincide and $A_n$ is unique up to a permutation of coordinate positions.

In Section 2 of the current paper we give auxiliary statements. In particular, we show that the regular subgroups of $\text{Aut}(A_n)$ correspond to those of $GA(\log(n+1), 2)$ and give a bound on the order of an element of a regular subgroup of $GA(r, 2)$.

There are few references on regular subgroups of the affine group from strictly algebraic point of view. A regular subgroup of $GA(r, q)$ without nontrivial translations was constructed in \cite{8}. In works \cite{6}, \cite{7} it was shown that the abelian regular subgroups of $GA(r, q)$ correspond to certain algebraic structures on the vector space $F_q^r$. In work \cite{7} the following example of abelian regular subgroup of $GA(r, q)$ was mentioned: the group is the centralizer of the Jordan block of size $r + 1$ in the group of upper triangular matrices \cite{7}.

One of main problems, arising in the theory of propelinear codes is a construction of codes with regular subgroups in their automorphism group that are abelian or "close" to them in a sense, such as for example $Z_4^r$, cyclic or dihedral groups. The same question could be asked for the regular subgroups of the affine group. In Section 3, we see that the dihedral group is a regular subgroup of the affine group if and only if $r = 3$, with the nontrivial case of the proof being when $r$ is 4, when a there is a dihedral subgroup of the affine group, which is not regular.

The Hamming code $H_n$ is known to have the largest order of the automorphism group in the class of perfect binary codes of any fixed length \cite{17} and it would be natural to suggest that it has the maximum number of regular subgroups of its automorphism group among propelinear perfect codes. However, the fact that

$$|\text{Aut}(H_n)| = |GL(\log(n + 1), 2)|2^{n-\log(n+1)}$$

makes attempts of even partial classification of regular subgroups impossible for ordinary calculational machinery starting with the smallest nontrivial length $n = 15$.

Regular subgroups of the Hamming code could be constructed from the regular subgroups of the automorphism group of its subcodes whose automorphism groups
are embedded into that of the Hamming code in a certain way. In work [13], this idea was implemented for the Nordstrom-Robinson code in case of extended length \( n = 16 \). In Section 4 we embed regular subgroups of the Hadamard code into those of the Hamming code of length \( 15 \).

2. Preliminaries

We begin with the following two well-known facts, e.g. see [12].

**Proposition 1.** Let \( C \) be a linear code of length \( n \). Then
\[
\text{Aut}(C) = F_2^n \rtimes \text{Sym}(C) = \{(x, \pi) : x \in C, \pi \in \text{Sym}(C)\}.
\]

The Hadamard code is known to be the dual code of the Hamming code of length \( n = 2^r - 1 \), which implies that their symmetry groups coincide and is isomorphic to the general linear group of \( F_2^r \).

**Proposition 2.** Let \( A_n \) and \( H_n \) be the Hadamard and the Hamming codes of length \( n = 2^r - 1 \). Then
\[
\text{Sym}(H_n) = \text{Sym}(A_n) \cong GL(r, 2).
\]

As far as the automorphism groups are concerned, the following fact holds.

**Proposition 3.** Let \( A_n \) be the Hadamard code of length \( n = 2^r - 1 \). Then \( \text{Aut}(A_n) \cong GA(r, 2) \) and the action of \( \text{Aut}(A_n) \) on the codewords of \( A_n \) is equivalent to the natural action of \( GA(r, 2) \) on the vectors of \( F_2^r \). In particular, the regular subgroups of \( \text{Aut}(A_n) \) correspond to the regular subgroups of \( GA(r, 2) \).

**Proof.** We use a well-known representation of the Hadamard code, see e.g. [12]. For a vector \( a \in F_2^r \) consider the vector \( c_a \) of values of the function \( \sum_{i=1}^{n} x_ia_i \) of variable \( x \) from \( F_2 \setminus \{0\} \) to \( F_2 \). It is easy to see that the code \( A_n = \{c_a : a \in F_2^r\} \) is linear of length \( n = 2^r - 1 \), dimension \( r \) and minimum distance \( (n + 1)/2 \), i.e. \( A_n \) is the Hadamard code. By Propositions [1] and [2] any automorphism of \( \text{Aut}(A_n) \) is \((c_a, \pi_A)\) for a vector \( a \in F_2^r \) and \( A \in GL(r, 2) \), therefore the mapping \((c_a, \pi_A) \rightarrow (a, A)\) is an isomorphism from \( \text{Aut}(A_n) \) to \( GA(r, 2) \).

\[ \square \]

In [2] the maximal orders of elements of \( GL(r, q) \) were described. In particular, the following was shown:

**Proposition 4.** The maximum of orders of elements of \( GL(r, 2) \) of type \( 2^l \) is
\[
2^{1 + \lceil \log_2 (r-1) \rceil}.
\]

This implies that the order of the element of regular subgroup of \( GA(r, 2) \) does not exceed
\[
2^{2 + \lceil \log_2 (r-1) \rceil}.
\]

This fact solely implies the nonexistence of regular dihedral subgroups of \( GA(r, 2) \) for \( r \geq 6 \). In fact we can tighten the bound to \( r \geq 5 \).

**Proposition 5.** 1. Let \( A \) be an element of \( GL(r, 2) \) of order \( 2^l \). Then \((I + A)^r = 0\).

2. The order of an element of a regular subgroup of \( GA(r, 2) \) is not greater then \( 2^\lceil \log_2 r \rceil + 1 \).
Proof. 1. The order of $A$ is $2^d$, then $(\lambda + 1)2^d = 0$ for any eigenvalue $\lambda$ of $A$, which implies that all eigenvalues of $A$ are 1's and w.r.g. $A$ is in the Jordan form with the Jordan blocks $J_1, \ldots, J_s$ corresponding to 1. It is easy to see that the polynomial $(I + J_i)^r$ of the Jordan cell $J_i$ is zero for $J_i$ of size not greater then $r$.

2. Suppose that $(a, A)$ is an element of a regular subgroup of $GA(r, 2)$ of order greater then $2^{[\log_2 r]+1}$.

We see that $(a, A)^i = (\sum_{j=0}^{i-1} A^j a, A^i)$, which combined with the fact that binomials $(2^i - 1) = 1$ in $F_2$ for any $i : 0 \leq i \leq 2^s - 1$, implies that:

$$(a, A)^{2^i} = ((I + A)^{2^i - 1}a, A^{2^i}).$$

In particular, using that $(I + A)^r = 0$, we have that

$$(a, A)^{2^{[\log_2 r]+1}} = ((I + A)^{2^{[\log_2 r]+1} - 1}a, A^{2^{[\log_2 r]+1}}) = (0^r, A^{2^{[\log_2 r]+1}}).$$

Therefore distinct elements $(0^r, I)$ and $(0^r, A^{2^{[\log_2 r]+1}})$ of a regular subgroup both preserve $0^r$, a contradiction.

We finish the section by noting that a version of the direct product construction works for regular subgroups.

**Proposition 6.** Let $G$ and $G'$ be regular subgroups of $GA(r, 2)$ and $GA(r', 2)$. Then there is a regular subgroup of $GA(r + r', 2)$ isomorphic to $G \times G'$.

**Proof.** Given elements $\alpha = (a, A)$ of $GA(r, 2)$ and $\beta = (b, B)$ of $GA(r', 2)$, define $\alpha \cdot \beta$ to be $((a|b), \begin{pmatrix} A & 0^{rr'} \\ 0^{r'r} & B \end{pmatrix})$, where $(a|b)$ is the concatenation of vectors $a$ and $b$. Obviously, the elements $\{\alpha \cdot \beta : \alpha \in G, \beta \in G'\}$ form a regular subgroup of $GA(r + r', 2)$, isomorphic to $G \times G'$.

3. **Dihedral regular subgroups of $GA(r, 2)$**

The cyclic group $\mathbb{Z}_2^r$ is not a regular subgroup of $GA(r, 2)$ for any $r$, as we see from the bound in Proposition 5. Therefore, we address the question of being a regular subgroup of the affine group to other groups, that are "close" to cyclic. The dihedral group, which we denote by $D_n$, is the group composed by all $2n$ symmetries of the $n$-sided polygon. It is well-known that any group, generated by an element $\alpha$ of order $n$ and an involution $\beta$ satisfying $\beta\alpha\beta = \alpha^{-1}$ is isomorphic to $D_n$.

**Theorem 1.** $D_{2^{r-1}}$ is a regular subgroup of $GA(r, 2)$ if and only if $r = 3$.

**Proof.** Consider the subgroup $G$ of $GA(3, 2)$ generated by $(a, A), (b, I)$, where $a = (101)^T, b = (011)^T, A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The orbit of $(000)^T$ under the action of the subgroup generated by $(a, A)$ consists of vectors $a = (101)^T, a + aA = (100)^T, Aa = (001)^T, (000)^T$. Since the orbit is a subspace that does not contain the vector $b$, $G$ acts transitively on the elements of $F_2^3$. Moreover, $(b, I)(a, A)(b, I) = (Ab + a + b, A) = ((001)^T, A) = (a, A)^{-1}$, so $G$ is $D_4$ and it is regular.
Suppose there is a regular subgroup of $GA(4, 2)$, generated by an element $(a, A)$ of order 8 and an element $(b, B)$ of order 2, satisfying relation 

(1) \[(b, B)(a, A)(b, B) = (a, A)^{-1}\].

Note that the order $A$ is 4 by Proposition 4.

Since $(b, B)^2 = (0^4, I)$, we have that

(2) \[b = Bb\].

Taking into account relation $b = Bb$ we have the following:

\[(a, A)^{-1} = (a, A)^7 = (\sum_{j=0}^{6} A^j a, A^3) = (A^3 a, A^3) = (b, B)(a, A)(b, B) = (Bb + BA^3b + b, BAB),\]

therefore using (2) and $BAB = A^3$, we obtain:

(3) \[Ba = A^3 b + A^3 a + b\].

The matrix $A$ is similar to the Jordan block of size 4 with the eigenvalue 1. Since $(a, A)^3 = ((I + A)^3) a$, the vector $(I + A)^3 a$ is nonzero and moreover is the unique eigenvector of $A$. The Jordan chain (the basis for which $A$ is the Jordan block) containing $(I + A)^3 a$ are vectors $a, (I + A)a, (I + A)^2a, (I + A)^3a$, which implies that $a, Aa, A^2a, A^3a$ is a basis of $F_2^4$, so

(4) \[b = c_0 a + c_1 Aa + c_2 A^2a + c_3 A^3a,\]

for some $c_i$ in $F_2$, $i \in \{0, \ldots, 3\}$.

Putting the expression (4) for $b$ into the equality (3), we obtain the following expression for $Ba$:

(5) \[Ba = (c_0 + c_1) a + (c_1 + c_2) Aa + (c_2 + c_3) A^2a + (c_0 + c_3 + 1) A^3a.\]

Putting the expression (4) for $b$ into (2) and using equality $BAB = A^3$, we obtain:

\[c_0 Ba + c_1 (A^3) Ba + c_2 (A^2) Ba + c_3 (A) Ba + c_0 a + c_1 Aa + c_2 A^2a + c_3 A^3a = 0^4.\]

Substituting the expression (5) for $Ba$ in the previous equality, we obtain that:

\[(c_0 c_3 + c_0 c_1 + c_1 c_2 + c_2 c_3 + c_2 + c_3)(a + Aa + A^2a + A^3a) = 0^4.\]

Finally, we see that the only binary vectors $(c_0, c_1, c_2, c_3)$ satisfying

\[c_0 c_3 + c_0 c_1 + c_1 c_2 + c_2 c_3 + c_2 + c_1 = 0\]

are exactly

\[(0000), (0000), (1100), (1110), (1111), (0111), (0011), (0001),\]

that are, in turn, exactly coefficients of linear combinations expressing elements $\sum_{j=0}^{i} A^j a, 0 \leq i \leq 7$ in the basis $a, Aa, A^2a, A^3a$. Therefore, elements $(b, B) = (\sum_{j=0}^{i} A^j a, B)$ and $(a, A)^{i+1} = (\sum_{j=0}^{i+1} A^j a, A^{i+1})$ are distinct elements of the dihedral subgroup, for some $i$, sending $0^4$ to $b$. We conclude that the considered group is not regular.

Suppose there is a regular subgroup $D_{2^{r-1}}$ of $GA(r, 2)$, $r \geq 5$. Then there is an element in $D_{2^{r-1}}$ of order $2^{r-1}$ which is impossible for $r \geq 5$, because the order of an
element in a regular subgroup of $GA(r, 2)$ does not exceed $2^{log r + 1}$ by Proposition 4.

**Remark 1.** The subgroup of $GA(4, 2)$ generated by $((0001)^T, A)$ and $((0000)^T, B)$, where $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a irregular subgroup of $GA(4, 2)$ isomorphic to $D_8$.

**Remark 2.** Consider elements $((0001)^T, A)$ and $((0100)^T, B)$, where $A$ is the same as in Remark 1, $B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. It is easy to see that the element $((0100)^T, B)$ is an involution, $((0001)^T, A)$ is of order 8, they commute and moreover they generate a regular subgroup isomorphic to $Z_2Z_8$. Actually, the group is isomorphic to the abelian regular subgroup, arising from the centralizer of the Jordan block of size 5 in the group of upper triangular $5 \times 5$ matrices, described in [7].

4. **Embedding to regular subgroups of the automorphism group of the Hamming code of length 15**

Regular subgroups of the Hamming code could be constructed from regular subgroups of its subcodes whose automorphism groups are embedded into that of the Hamming code in a certain way. In work [13], narrow-sense embeddings of the regular subgroups of the automorphism group of Nordstrom-Robinson code to those of the Hamming code were considered in case of extended length $n = 16$. Here we apply the idea to embed regular subgroups of the automorphism group of the Hadamard code to regular subgroups of the automorphism group the Hamming code.

Given a subgroup $G$ of $Aut(F_n^u)$, denote by $\Pi_G$ the subgroup of $S_n$ whose elements are $\{ \pi : (x, \pi) \in G \}$. We say that a group $H, H \leq Aut(F_n^u)$ is narrow-sense embedded [13] in a subgroup $G, G \leq Aut(F_n^u)$, if $H \leq G$ and $\Pi_G = \Pi_H$.

It is well-known that the Hadamard code $A_{15}$ and the punctured Nordstrom-Robinson, which we denote by $N$, are subcodes of the code $H_{15}$ [12], [16]. Obviously, $\Pi_{Aut(C)} = Sym(C)$ if $C$ is linear. So, $Aut(A_{15})$ is narrow-sense embedded into that of the Hamming code $H_{15}$, because their symmetry groups coincide (see Proposition 2). The linear span of the punctured Nordstrom-Robinson code is the Hamming code [16], thus its automorphism group is embedded in that of the Hamming code. Moreover, the inclusion is in narrow sense. We recall a description of symmetry group of $N$ from [4].

**Proposition 7.** $Sym(N) \cong A_7 \leq Sym(H_{15}) \cong GL(4, 2) \cong A_8$.

**Corollary 1.** $Aut(N)$ is narrow-sense embedded in $Aut(H_{15})$.

**Proof.** The punctured Nordstrom-Robinson code is propelinear, then it is not hard to see that $|\Pi_{Aut(N)}| = |Sym(N)||N|/|Ker(N)|$, where $Ker(N) = \{ x \in N : x + N = N \}$ see e.g. [4], Proposition 4.3. Then, since $Ker(N)$ is $A_{15}$ augmented
by all-ones vector, see [16] and the size of $N$ is $2^8$, we see that
\[ |\Pi_{\text{Aut}(N)}| = 8|\text{Sym}(N)| = |\text{Sym}(\mathcal{H}_{15})|. \]

Let $\pi$ be an element of $\Pi_{\text{Aut}(N)}$, in other words, $x + \pi(N) = N$. The linear span of $N$ is $\mathcal{H}_{15}$ [16], therefore $\pi$ is a symmetry of $\mathcal{H}_{15}$. Taking into account the equality (6), we obtain that $\Pi_{\text{Aut}(N)} = \text{Sym}(\mathcal{H}_{15}) = \Pi_{\text{Aut}(\mathcal{H}_{15})}$.

First of all, the regular subgroups of the automorphism group of $\text{Aut}(A_{15})$ (regular subgroups of $GA(4, 2)$) were classified. The results below were obtained using PC.

**Theorem 2.** There are 39 conjugacy classes of regular subgroups of $\text{Aut}(A_{15})$, that fall into 11 isomorphism classes.

**Remark 3.** Four of 11 isomorphism classes of regular subgroups are abelian and are isomorphic to groups $Z_2^2$, $Z_2Z_8$, $Z_2^2Z_4$ and $Z_2^2$. A group isomorphic to $Z_2Z_8$ is given in Remark 2. It is not hard to see that there is a regular subgroup of $GA(2, 2)$, isomorphic to $Z_4$. Then the regular subgroups isomorphic to $Z_2^2Z_4$ and $Z_2^2$ could be constructed using direct product construction (see Proposition 6).

The narrow-sense embeddings into regular subgroups of the automorphism group of the Hamming code were found. The bound on the number of isomorphism classes we obtain by comparing the orders of the centralizers of elements.

**Theorem 3.** The regular subgroups of $\text{Aut}(A_{15})$ are narrow-sense embedded in at least 1207 conjugacy classes of regular subgroups of $\text{Aut}(\mathcal{H}_{15})$, which fall into at least 48 isomorphism classes.

The result is somewhat disappointing, as embeddings of Nordstrom-Robinson code in Hamming code gave significantly better bound for isomorphism classes.

**Theorem 4.** [13] There are 73 conjugacy classes of regular subgroups of $\text{Aut}(\mathcal{N})$ that fall into 45 isomorphism classes. The regular subgroups of $\text{Aut}(N)$ are narrow-sense embedded in exactly 605 conjugacy classes of regular subgroups of $\text{Aut}(\mathcal{H}_{15})$, which fall into at least 219 isomorphism classes.

One might suggest a tighter interconnection of regular subgroups of $\text{Aut}(A_{15})$ and that of $\text{Aut}(\mathcal{N})$. However, despite that $A_{15} \subset \mathcal{N}$, $\text{Aut}(A_{15})$ is not embedded in that of $\text{Aut}(\mathcal{N})$ in narrow-sense, which in turn, follows, for example, from a proper containment of $\text{Sym}(\mathcal{N})$ in $\text{Sym}(A_{15})$, see Proposition 7. Moreover, only 6 of 39 conjugacy classes of regular subgroups of $\text{Aut}(A_{15})$ are subgroups $\text{Aut}(\mathcal{N})$.

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