Asplund operators and the Szlenk index

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of the Australian National University
For my gorgs.
Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis constitutes an investigation into the Szlenk index of operators acting between Banach spaces, primarily in the context of operator ideal theory. Roughly speaking, the Szlenk index measures the complexity of certain sets in dual Banach spaces, and it has become a standard tool of the trade in Banach space theory since its introduction by W. Szlenk in 1968. The starting point of this thesis is a proof that there is a class of closed operator ideals naturally associated with the Szlenk index. We study their basic operator ideal properties and their relationship with the well-known operator ideals of separable range operators and Asplund operators. We provide a detailed analysis of the behaviour of the Szlenk index under taking direct sums of operators, ultimately applying the techniques we develop to this end in an investigation of factorisation properties of the closed operator ideals associated with the Szlenk index.
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Chapter 1

Introduction

1.1 Historical background

In this thesis we consider some aspects of the theory of ordinal indices of Banach spaces and their operators. It is not feasible to give here in this introduction a comprehensive survey of this now large area of functional analysis, so instead we give a brief historical overview of some aspects of the topic, with a particular emphasis on those developments that are direct ancestors of the present work.

In mathematical analysis in general, ordinal indices are often used to measure certain properties of a given object. For example, given metric spaces $X$ and $Y$, with $Y$ separable, and a Borel function $f : X \rightarrow Y$, one may measure the discontinuity of $f$ by computing the smallest Baire class to which it belongs (see [34, p.190] for the definition). That this provides an ordinal measure of the discontinuity of $f$ follows from the fact that the Baire classes are indexed by the set of countable ordinals. The study of Baire classes of functions has been of interest to analysts for more than a century, following closely the fundamental work of Cantor on point sets and predating the advent of Banach space theory (see [29, p.344-5] for references).

Although the particular use of ordinal indices in Banach space theory has become widespread only in the last few decades or so, the use of such methods actually
goes back to the early days of the subject\textsuperscript{1}. Indeed, in 1930 two papers appeared, each one independently making use of an ordinal index to establish the following result: if \((f_n)\) is a weakly convergent sequence in \(C[0, 1]\), then the weak limit of \((f_n)\) is the norm limit of a sequence whose terms are convex combinations of terms from the sequence \((f_n)\) (here \(C[0, 1]\) is the space of all continuous scalar-valued functions on the compact interval \([0, 1] \subseteq \mathbb{R}\)). One of these papers is by D. Gillespie and W. Hurwitz [25], the other by Z. Zalcwasser [65]. Only a few years later, S. Mazur [41] bypassed the use of ordinal indices in favour of the Hahn-Banach theorem in order to show that this convergence result holds in every Banach space. It would seem that the use of ordinal indices in Banach space theory then lay dormant for some thirty-five years.

It is the work of Szlenk [63] that kick-started the revival of ordinal index techniques in the study of Banach spaces and their operators, and which has ultimately inspired the title and content of the present thesis. In [63], Szlenk uses an ordinal index on the class of separable, reflexive Banach spaces to show that this class contains no universal element; that is, there is no separable reflexive Banach space that contains an isomorphic copy of every separable reflexive Banach space. Roughly speaking, Szlenk's proof involves an analysis of certain procedures that remove points from \(w^*\)-compact sets in separable dual spaces.

In the northern spring of 1979, Michel Talagrand showed that the class of separable Banach spaces with the Radon-Nikodým property contains no universal element (we refer the reader to the lecture notes of J. Diestel [16] for details of the seemingly-otherwise-unpublished proof). In a similar vein to Szlenk's work, the proof involves an analysis of certain procedures that remove points from closed, bounded subsets of separable Banach spaces. Talagrand's achievement arrived on the tail of the flurry of work on the Radon-Nikodým property that occurred in Banach space theory in the 1970s. One of the most famous characterisations of the Radon-Nikodým property from that time is that a Banach space has the Radon-Nikodým property

\textsuperscript{1}The historical observations presented in this paragraph are adapted from the introduction of the article [2] of D. Alspach and E. Odell.
1.1 Historical background

if and only if each of its nonempty, closed, bounded, convex subsets admits open slices\(^2\) of arbitrarily small diameter. The procedures considered by Talagrand in his proof remove points from closed, bounded subsets of a Banach space via removal of small open slices.

It is the work of G. Lancien in his PhD thesis [36] that brought about the notion of Szlenk index that is perhaps most commonly used today. Lancien considered two indices of Banach spaces, with slightly differing definitions. One is defined via a procedure that removes \(w^*\)-slices of small diameter from \(w^*\)-compact subsets of a dual Banach space (note the similarity to Talagrand's procedure above), the other via a procedure that removes \(w^*\)-open subsets of small diameter from \(w^*\)-compact subsets of dual Banach spaces. It is the index associated to the latter of these procedures - the 'small \(w^*\)-neighbourhood index', for want of terminology - that is of interest to us here. As shown by Lancien in [37, Proposition 3.3], this index coincides with the ordinal index introduced by Szlenk in [63] on the class of separable Banach spaces containing no isomorphic copy of \(\ell_1\). In particular, the index introduced by Szlenk coincides with the 'small \(w^*\)-neighbourhood index' on the class of separable reflexive Banach spaces - the class of Banach spaces of particular interest to Szlenk in [63]. Because of this, and because Lancien's index exhibits superior permanence properties on the class of all Banach spaces, the 'small \(w^*\)-neighbourhood index' introduced in [36] has become known as the Szlenk index in the recent literature. We note, however, that Szlenk's original definition of the Szlenk index has its advantages in various contexts, especially in the study of the geometry of \(C(K)\) spaces with \(K\) metrisable (see [55] for a survey). Application of Szlenk's methods has also arisen in the study of seminormalised weakly-null basic sequences [64]. More generally, the Szlenk index and its variants have found numerous further applications in the study of Banach space geometry, many of which are described in the survey article of Lancien [39]. Quite recently, the Szlenk index has been used in the study of the metric geometry [8] and fixed-point theory.

\(^2\)For a real Banach space \(E\) and a subset \(C \subset E\), an open slice of \(C\) is a set of the form \(\{y \in C \mid y^*(y) > \sup\{y^*(u) \mid u \in C\} - \varepsilon\}\), where \(y^* \in E^*\) and \(\varepsilon > 0\). For \(D \subset E^*\), a \(w^*\)-open slice of \(D\) is a set of the form \(\{x^* \in D \mid x^*(x) > \sup\{x^*(z) \mid z^* \in D\} - \varepsilon\}\), where \(x \in E\) and \(\varepsilon > 0\).
The notion of Szlenk index of an operator goes back at least as far as the work of D. Alspach [3] and J. Bourgain [12], who independently studied the Szlenk index of operators acting on \( C(K) \) spaces. In particular, they studied fixing properties of such operators having a 'large' Szlenk index (in the sense of Szlenk [63]). The motivation for such results is the famous unsolved problem of classifying, up to isomorphism, the complemented subspaces of the classical Banach space \( C[0, 1] \).

In addition to the 'positive' fixing results of Alspach and Bourgain noted above, Alspach has established negative results in the same direction, for instance in [4] and [5]. A perhaps optimal positive fixing result in this direction has been obtained by I. Gasparis in [23], where it is shown that an operator of 'large' Szlenk index acting on a separable \( C(K) \) space fixes a copy of a similarly 'large' Schreier-type space. A problem more general than the isomorphic classification of complemented subspaces of \( C[0, 1] \) is that of determining the lattice of closed, two-sided ideals in the Banach algebra \( \mathcal{B}(C[0, 1]) \) of all bounded linear operators on \( C[0, 1] \). A natural tool in the study of such problems is the notion of a closed operator ideal.

In light of the apparent importance that Szlenk's original index seems to play in the study of the geometry of complemented subspaces of \( C[0, 1] \) (through consideration of projections having large index), it is natural to want to know whether there are operator ideals naturally induced by the Szlenk index or one of its variants. In particular, one might hope to be able to define operator ideals by considering classes of operators whose Szlenk index is 'small' in some sense. With regards to this line of inquiry, it is worth noting that the notion of Szlenk index introduced by Lancien in [36] has superior permanence properties to Szlenk's original index. For example, Lancien's version of the Szlenk index is nonincreasing when passing to subspaces and quotients of an arbitrary Banach space, whereas Szlenk's original index lacks this property over the class of all Banach spaces. As this permanence property is almost surely to be essential to any potential operator ideal theory associated with the Szlenk index, throughout this thesis we choose to work primarily with Lancien's notion of Szlenk index from [36]. It seems that the only prior consideration of the
1.2 Overview of the thesis

Szlenk index of an operator in terms of the version of the Szlenk index introduced by Lancien is the work of B. Bossard [11, Theorem 3.9]. A possible important connection between the two aforementioned notions of Szlenk index in the context of operators acting on $C[0, 1]$ will be put forward as Question 3.3.1 of the present thesis. Although the study of operators on spaces $C(K)$, with $K$ metrisable, is a key motivator behind the results presented in this thesis, we shall not address such problems substantially here. Instead, the primary focus is on the development of a general operator ideal theory associated with the Szlenk index.

1.2 Overview of the thesis

Our goal is to establish the existence of closed operator ideals that arise naturally from the Szlenk index and to study their operator ideal properties. Along the way, we study the associated Banach space geometry. We shall establish basic notational conventions in Section 1.3, whilst background material requiring more substantial exposition shall be outlined in Chapter 2.

Chapter 3 contains the first of our main results on operator ideals arising from the Szlenk index. In general, operator ideals are typically defined in terms of some 'smallness' property of an operator that behaves well under ring-theoretic operations. Naturally then, in Chapter 3 we seek to establish as operators ideals classes of operators whose Szlenk index does not exceed some given ordinal. Indeed, in our first main result, Theorem 3.1.2, we show that for $\alpha$ an ordinal, the class $\mathcal{H}_\alpha$ consisting of all operators whose Szlenk index is an ordinal not exceeding $\omega^\alpha$ is a closed operator ideal. Evidently, these operator ideals will be comparable in terms of class containment. In fact, we will see that there is a one-to-one order-preserving correspondence between the class of all such operator ideals $\mathcal{H}_\alpha$ (the order here being given by class containment) and the class of all ordinals with the usual order. We will compare our operator ideals with several well-known closed operator ideals, beginning with the compact operators, and then moving on to the classes of Asplund operators and separable range operators. The Asplund operators
are of particular importance to us since, as we shall see, an operator belongs to $\mathcal{H}_\alpha$ for some ordinal $\alpha$ if and only if it is an Asplund operator. This relationship between the class of Asplund operators and the classes $\mathcal{H}_\alpha$, along with the fact that the class of Asplund operators is well-known and studied, is the reason for the title of this thesis. Moreover, we mention that this relationship provides an ordinal indexing of the space of Asplund operators between a given (ordered) pair of Banach spaces. In Section 3.2 we explicate some examples involving a number of well-known Banach spaces, in particular $L_\infty[0, 1]$, $L_1[0, 1]$, $C[0, 1]$ and Pelczyński's universal unconditional basis space. We conclude Chapter 3 in Section 3.3 with a discussion of a couple of variants of the Szlenk index that have appeared in the literature. In particular, we discuss the possibilities and difficulties associated with using these variants of the Szlenk index to define operator ideals in a similar vein to the theory developed in Section 3.1.

In Chapter 4 we determine a necessary and sufficient condition for a $c_0$-direct sum or $\ell_p$-direct sum of operators (also known as a diagonal operator in the literature) to belong to $\mathcal{H}_\alpha$ for a given ordinal $\alpha$. Many of the techniques and results we establish in this endeavour will subsequently be applied in the following chapter on factorisation of Asplund operators. For $\ell_1$-direct sums and $\ell_\infty$-direct sums, the necessary and sufficient condition is quite strong and requires the norms of the summand operators to exhibit "$c_0$-like" behaviour; this makes our proofs for this case very easy. On the other hand, for $c_0$-direct sums and $\ell_p$-direct sums ($1 < p < \infty$), the situation is far more subtle. The necessary and sufficient condition for a direct sum of operators to belong to $\mathcal{H}_\alpha$ in this case is a natural uniformity condition on the $\varepsilon$-Szlenk indices of the summand operators (the definition of the $\varepsilon$-Szlenk index will be given in Section 2.4). In all cases, our necessary and sufficient conditions give rise to an explicit formulation of the Szlenk index of a direct sum of operators in terms of collective properties of the Szlenk indices and/or $\varepsilon$-Szlenk indices of the summand operators. Our motivation for the consideration of direct sums of operators is the fact that the process of taking direct sums is a well-understood method for creating new spaces and operators from old, and as such provides an avenue...
for creating examples that possess (or lack) a specified property. As a by-product of our investigations on the Szlenk indices of direct sums operators, we obtain in Section 4.1.3 a number of results concerning the Szlenk indices of Banach spaces.

Chapter 5 is the last of the chapters containing our main results. The motivating question for this chapter is the following: Suppose that $T$ belongs to $\mathcal{M}_\alpha$. Does $T$ admit a continuous, linear factorisation through a Banach space whose Szlenk index is 'close' to $\omega^\alpha$ in some sense? In response to this question, we show in Section 5.1 that every element of $\mathcal{M}_\alpha$ factors through a Banach space of Szlenk index not exceeding $\omega^{\alpha+1}$. We then deduce that for a proper class of ordinals $\alpha$, $\mathcal{M}_\alpha$ possesses the factorisation property (that is, every element of $\mathcal{M}_\alpha$ admits a continuous, linear factorisation through a Banach space whose identity operator belongs to $\mathcal{M}_\alpha$). Section 5.2 is then devoted to establishing a similar, but negative, result. In particular, we show that for a proper class of ordinals $\alpha$, $\mathcal{M}_\alpha$ lacks the factorisation property. In Section 5.3 we introduce and study a class of space ideals that are of interest in determining whether the operator ideals $\mathcal{M}_{\alpha+1}$ have the factorisation property. We conclude in Section 5.4 with discussion of possible future directions for work related to the problems addressed in this chapter, in particular the problem of classifying those ordinals $\alpha$ such that $\mathcal{M}_\alpha$ has the factorisation property.

A number of appendices are provided at the back of the thesis for the benefit of the reader. Only one of these appendices, namely Appendix D, contains original research in any substantial sense. The others contain material that either elaborates on (variants of) known proofs, or gives proofs of 'obvious' results that would not necessarily justify a detailed proof being given in the literature.

### 1.3 Basic notation and terminology

- We shall consider Banach spaces over the field $\mathbb{K}$, which is either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers). Banach spaces shall typically be denoted by the letters $D$, $E$, $F$ and $G$. If no ambiguity should arise, we simply write
\| \cdot \| \) for a given norm; in cases where additional clarity is required, \( \| \cdot \|_E \) shall denote the norm of the Banach space \( E \).

- The (continuous) dual of the Banach space \( E \) is denoted \( E^* \). For \( S \subseteq E^* \), the pair \( (S, w^*) \) denotes the set \( S \) equipped with the relative \( w^* \) topology inherited from \( E^* \).

- \text{BAN} is the class of all Banach spaces over \( \mathbb{K} \).

- For a Banach space \( E \), the \textit{closed unit ball of } \( E \) is \( B_E := \{ x \in E \mid \|x\| \leq 1 \} \). For nonempty, bounded \( S \subseteq E \), we define \( |S| := \sup_{x \in S} \|x\| \).

- For a Banach space \( E \) and \( C \subseteq E^* \), the \textit{\( w^* \)-closed convex hull of } \( C \) is the smallest convex, \( w^* \)-closed subset of \( E^* \) containing \( C \), and is denoted \( \text{conv}^*(C) \). If \( C \) is convex, an \textit{extreme point of } \( C \) is any point \( x \in C \) with the property that if \( \delta \in (0, 1) \) and \( y, z \in C \) satisfy \( \delta y + (1 - \delta)z = x \), then \( x = y = z \). If \( C \) is convex, the set of all extreme points of \( C \) is denoted \( \text{ext}(C) \).

- For a compact, Hausdorff space \( K \), \( C(K) \) is the space of all continuous functions \( f : K \rightarrow \mathbb{K} \), equipped with the supremum norm \( \|f\| = \sup_{k \in K} |f(k)| \). For each \( k \in K \), we denote by \( \delta_k \) the element of \( C(K)^* \) satisfying \( \delta_k(f) = f(k) \) for every \( f \in C(K) \).

- An \textit{operator} is a norm-continuous linear map \( T : E \rightarrow F \), where \( E \) and \( F \) are Banach spaces. The adjoint of \( T \) is denoted \( T^* \), and the identity operator of \( E \) is denoted \( I_E \). We write \( \mathcal{B} \) for the class of all operators between Banach spaces, and \( \mathcal{B}(E, F) \) for the set of all operators \( E \rightarrow F \) for given \( E \) and \( F \).

- For Banach spaces \( E, F \) and \( G \) and \( T \in \mathcal{B}(E, F) \), we say that \( T \) \textit{factors through } \( G \) if there exist \( U \in \mathcal{B}(E, G) \) and \( V \in \mathcal{B}(G, F) \) such that \( T = VU \).

- For a Banach space \( F \), the \textit{canonical embedding of } \( F \) is the isometric linear operator \( \mathfrak{J}_F : F \rightarrow \ell_\infty(B_{F^*}) \) satisfying \( \mathfrak{J}_F(y) = (\langle y^*, y \rangle)_{y^* \in B_{F^*}}, y \in F \). For a Banach space \( E \), the \textit{canonical surjection onto } \( E \) is the surjective operator \( \Omega_E : \ell_1(B_E) \rightarrow E : (a_x)_{x \in B_E} \mapsto \sum_{x \in B_E} a_xx \).
1.3 Basic notation and terminology

- Let \( p \in \{0\} \cup (1, \infty) \subseteq \mathbb{R} \) and \( q \in [1, \infty) \subseteq \mathbb{R} \). We say that \( p \) is predual to \( q \), or equivalently, \( q \) is dual to \( p \), if \( (p, q) \in \{(0, 1)\} \cup \{(r, r(r - 1)^{-1}) \mid r \in (1, \infty)\} \).

- The class of all ordinals is denoted \( \text{ORD} \). Ordinals shall typically be denoted by the lower-case Greek letters, especially \( \alpha, \beta \) and \( \gamma \). The least infinite ordinal is denoted by \( \omega \), and the least uncountable ordinal by \( \omega_1 \).

- For \( \alpha \) an ordinal, we write \( cf(\alpha) \) for the cofinality of \( \alpha \). That is, \( cf(\alpha) \) is the least ordinal that is order-isomorphic to a cofinal subset of \( \alpha \). We note the following basic facts regarding \( cf(\alpha) \):
  - \( cf(\alpha) = 0 \) if and only if \( \alpha = 0 \).
  - \( cf(\alpha) = 1 \) if and only if \( \alpha \) is a successor ordinal.
  - \( cf(\omega^\alpha) = \omega \) whenever \( \alpha \) is a successor ordinal.
  - \( cf(\omega^\alpha) = cf(\alpha) \) whenever \( \alpha \) is a nonzero limit ordinal.
  - For any \( \alpha \), either \( cf(\alpha) = 0, cf(\alpha) = 1, cf(\alpha) = \omega \) or \( cf(\alpha) \geq \omega_1 \).

- For \( \Lambda \) a set, \( \Lambda^{<\infty} \) shall denote the set of all nonempty, finite subsets of \( \Lambda \).

- We denote by \( \mathbb{N} \) the set of natural numbers 1, 2, 3, ..., and by the \( \mathbb{Z} \) the set of all integers.

- For \( a \in \mathbb{R} \), \( \lfloor a \rfloor := \inf \{n \in \mathbb{Z} \mid n \geq a\} \).
Chapter 2

Preliminaries

2.1 Direct sums of spaces and operators

Whenever $\Lambda$ and $\Gamma$ are used to denote index sets over which we take direct sums and direct products, we assume for simplicity that $\Lambda$ and $\Gamma$ are nonempty.

For $1 \leq p \leq \infty$, a set $\Lambda$ and Banach spaces $E_\lambda$, $\lambda \in \Lambda$, the $\ell_p$-direct sum of $\{E_\lambda \mid \lambda \in \Lambda\}$, denoted $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$, is the linear space

$$\left\{ (x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} E_\lambda \mid (\|x_\lambda\|)_{\lambda \in \Lambda} \in \ell_p(\Lambda) \right\}$$

equipped with the complete norm $\|(x_\lambda)_{\lambda \in \Lambda}\| = \|(\|x_\lambda\|)_{\lambda \in \Lambda}\|_{\ell_p(\Lambda)}$. The $c_0$-direct sum of $\{E_\lambda \mid \lambda \in \Lambda\}$, denoted $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_{c_0}$, is the linear space

$$\left\{ (x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} E_\lambda \mid (\|x_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda) \right\}$$

equipped with the complete norm $\|(x_\lambda)_{\lambda \in \Lambda}\| = \|(\|x_\lambda\|)_{\lambda \in \Lambda}\|_{c_0(\Lambda)}$. In cases where the index set $\Lambda$ equals the finite set $\{1, \ldots, n\} \subseteq \mathbb{N}$ for some $n \in \mathbb{N}$, if a statement is true for $(\bigoplus_{i=1}^n E_i)_p$ for all $1 \leq p \leq \infty$ (via the identity isomorphism of these spaces), then we may suppress the subscript $p$ and simply write $E_1 \oplus \cdots \oplus E_n$ in place of $(\bigoplus_{i=1}^n E_i)_p$. For the $\ell_p$-direct sum of two Banach spaces, $E$ and $F$ say, we
may write $E \oplus_p F$, or even simply $E \oplus F$ if the choice of $p \in [1, \infty]$ is unimportant.

For a set $\Lambda$, a family of Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$ and nonempty, bounded subsets $S_\lambda \subseteq E_\lambda$, $\lambda \in \Lambda$, we say that $\{S_\lambda \subseteq E_\lambda \mid \lambda \in \Lambda\}$ is uniformly bounded if $\sup_{\lambda \in \Lambda} |S_\lambda| < \infty$. If $\{F_\lambda \mid \lambda \in \Lambda\}$ is also a family of Banach spaces indexed by $\Lambda$, a set of operators $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ is said to be uniformly bounded if $\sup_{\lambda \in \Lambda} \|T_\lambda\| < \infty$. For $1 \leq p \leq \infty$ and a uniformly bounded set of operators $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$, the $\ell^p$-direct sum of $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ is the operator $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ that sends $(x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ to $(T_\lambda x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} F_\lambda)_p$.

The $c_0$-direct sum of $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ is the operator $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_0$ that sends $(x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} E_\lambda)_0$ to $(T_\lambda x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} F_\lambda)_0$.

Throughout, for $1 < p, q < \infty$ satisfying $p + q = pq$, we implicitly identify $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^* \times (\bigoplus_{\lambda \in \Lambda} E_\lambda)_q^*$ with $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_q^*$, so that the dual of a direct sum is the dual direct sum of the duals of the spaces $E_\lambda$. Making this identification allows us to consider direct products of the form $\prod_{\lambda \in \Lambda} K_\lambda$, where $K_\lambda \subseteq E_\lambda^*$ and $|K_\lambda|_{\lambda \in \Lambda} \in \ell_q(\Lambda)$, as subsets of $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_q^*$. Similarly, $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_q^*$ is naturally identified with $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_1^*$ throughout. Thus, for $1 \leq q < \infty$, $p$ predual to $q$, a set $\Lambda$ and a uniformly bounded family of operators $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$, we have

$$\left(\bigoplus_{\lambda \in \Lambda} T_\lambda\right)_q^* B_{\left(\bigoplus_{\lambda \in \Lambda} F_\lambda\right)_q^*} = \bigcup_{(a_\lambda)_{\lambda \in \Lambda} \in B_{\ell_q(\Lambda)}} \prod_{\lambda \in \Lambda} a_\lambda T_\lambda^* B_{F_\lambda^*}. \quad (2.1)$$

For a set $\Lambda$, a family of Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$, a corresponding uniformly bounded family $\{K_\lambda \subseteq E_\lambda^* \mid \lambda \in \Lambda\}$ of nonempty, absolutely convex, $w^*$-compact sets and $1 \leq q < \infty$, we define $B_q(K_\lambda \mid \lambda \in \Lambda) := \bigcup_{(a_\lambda)_{\lambda \in \Lambda} \in B_{\ell_q(\Lambda)}} \prod_{\lambda \in \Lambda} a_\lambda K_\lambda$ and always consider $B_q(K_\lambda \mid \lambda \in \Lambda)$ as a subset of $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$, with $p$ predual to $q$. We claim that $B_q(K_\lambda \mid \lambda \in \Lambda)$ so defined is $w^*$-compact. To verify this claim, we will first show that for each $\lambda \in \Lambda$, we have $T_\lambda^* B_{C((K_\lambda, w^*))} = K_\lambda$ for the operator $T_\lambda : E_\lambda \rightarrow C((K_\lambda, w^*))$ that sends $x \in E_\lambda$ to the $w^*$-to-$K$ continuous
map \( k \mapsto \langle k, x \rangle \) \((k \in K_\lambda)\). We require the following facts:

(F1) If \( K \) is a nonempty, \( w^* \)-compact, convex subset of a dual Banach space, then \( K = \text{conv}^* (\text{ext}(K)) \).

(F2) For Banach spaces \( E \) and \( F \), an operator \( T : E \to F \) and \( w^* \)-compact, convex \( K \subseteq F^* \), we have \( \text{ext}(T^*(K)) \subseteq T^*(\text{ext}(K)) \).

(F3) For \( K \) compact and Hausdorff, \( \text{ext}(B_{C(K)^*}) = \{a\delta_k \mid k \in K, a \in \mathbb{K}, |a| = 1\} \).

(F4) For Banach spaces \( E \) and \( F \) and an operator \( T : E \to F \), the set \( T^*B_{F^*} \) is \( w^* \)-compact.

(F1) is the well-known Kreîn-Mil'man theorem; see [21, Theorem 3.37] for a proof. (F2) is [24, p.94, Exercise 2.4.11]. A proof of (F3) for real scalars is given in [21, Lemma 3.42], and the proof for complex scalars is obtained by making only minor adjustments to the proof of the real case. (F4) is an immediate consequence of the Banach-Alaoglu theorem and the \( w^* \)-continuity of adjoint operators.

Fix \( \lambda \in \Lambda \). By the definition of \( T_\lambda \), for each \( k \in K_\lambda \) we have \( \langle k, x \rangle = \langle T_\lambda^* \delta_k, x \rangle \) for all \( x \in E_\lambda \), hence \( k = T_\lambda^* \delta_k \in T_\lambda^*B_{C((K_\lambda, w^*))^*} \). Thus \( K_\lambda \subseteq T_\lambda^*B_{C((K_\lambda, w^*))^*} \). On the other hand, by (F1)-(F3) and the fact that \( k = T_\lambda^* \delta_k \) for all \( k \in K_\lambda \),

\[
T_\lambda^*B_{C((K_\lambda, w^*))^*} = \text{conv}^* \left( \text{ext}(T_\lambda^*B_{C((K_\lambda, w^*))^*}) \right) \\
\subseteq \text{conv}^* \left( T_\lambda^*\text{ext}(B_{C((K_\lambda, w^*))^*}) \right) \\
= \text{conv}^* \left( \{aT_\lambda^* \delta_k \mid k \in K_\lambda, a \in \mathbb{K}, |a| = 1\} \right) \\
= \text{conv}^* \left( \{ak \mid k \in K_\lambda, a \in \mathbb{K}, |a| = 1\} \right) \\
= K_\lambda.
\]

We have thus established that \( T_\lambda^*B_{C((K_\lambda, w^*))^*} = K_\lambda \) for all \( \lambda \in \Lambda \). Combining this equality with (2.1) yields \( B_q(K_\lambda \mid \lambda \in \Lambda) = (\bigoplus_{\lambda \in \Lambda} T_\lambda^*B(\mathbb{D}_{\lambda \in \Lambda} C((K_\lambda, w^*))^*)^* \), and the \( w^* \)-compactness of \( B_q(K_\lambda \mid \lambda \in \Lambda) \) follows by (F4).

The following definition establishes notation that will be used throughout the thesis.

**Definition 2.1.1.** Let \( \Lambda \) be a set, \( \{E_\lambda \mid \lambda \in \Lambda\} \) a family of Banach spaces and
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$p = 0$ or $1 < p < \infty$. For $\mathcal{R} \subseteq \Lambda$, $U_\mathcal{R}$ denotes the canonical injection of $((\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p$ into $((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$, and $P_\mathcal{R}$ the canonical surjection of $((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ onto $((\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p$. We shall often consider operators $T : (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p \rightarrow (\bigoplus_{v \in \mathcal{V}} F_v)_p$, where $\Lambda$ and $\mathcal{V}$ are sets, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_v \mid v \in \mathcal{V}\}$ families of Banach spaces and $p = 0$ or $1 < p < \infty$. In this setting, for $\mathcal{R} \subseteq \Lambda$ the operators $U_\mathcal{R}$ and $P_\mathcal{R}$ are defined as above (that is, $U_\mathcal{R}$ and $P_\mathcal{R}$ act to and from the domain of $T$, respectively). Similarly, for $\mathcal{S} \subseteq \mathcal{V}$ we denote by $V_\mathcal{S}$ the canonical injection of $((\bigoplus_{v \in \mathcal{S}} F_v)_p$ into $((\bigoplus_{v \in \mathcal{V}} F_v)_p$, and by $Q_\mathcal{S}$ the canonical surjection of $((\bigoplus_{v \in \mathcal{V}} F_v)_p$ onto $((\bigoplus_{v \in \mathcal{S}} F_v)_p$. Thus $V_\mathcal{S}$ and $Q_\mathcal{S}$ act to and from the codomain of $T$, respectively.

2.2 Operator ideals

Throughout this thesis we work within the theory of operator ideals as expounded by A. Pietsch in [49]. The starting point of this theory is the following definition.

Definition 2.2.1. ([49, §1.1.1]) An operator ideal $\mathcal{I}$ is a subclass of $\mathcal{B}$ such that for Banach spaces $E$ and $F$, the components $\mathcal{I}(E, F) := \mathcal{B}(E, F) \cap \mathcal{I}$ satisfy the following three conditions:

(1) $I_\mathcal{R} \in \mathcal{I}$.
(2) $S + T \in \mathcal{I}(E, F)$ whenever $S, T \in \mathcal{I}(E, F)$.
(3) $U \in \mathcal{B}(D, E)$, $T \in \mathcal{I}(E, F)$ and $V \in \mathcal{B}(F, G)$ implies $VTU \in \mathcal{I}$.

The following is a list of some well-known operator ideals:

- $\mathcal{F}$, the finite-rank operators.
- $\mathcal{K}$, the compact operators.
- $\mathcal{W}$, the weakly compact operators.
- $\Gamma_2$, the operators that factor through some Hilbert space.
- $\mathcal{A}_E$, where $E$ is a Banach space isomorphic to its square, is the class of operators that factor through $E$. 

We now describe several well-known hull procedures (see [49, Definition 4.1.2] for the definition of hull procedure; the definition is not important for us here) that give 'new' operator ideals from 'old'. Let \( \mathcal{I} \) be an operator ideal and define

\[
\overline{\mathcal{I}} := \bigcup_{(E,F) \in \text{BAN} \times \text{BAN}} \left\{ T \in \mathcal{B}(E, F) \mid T \in \overline{\mathcal{I}(E, F)} \right\},
\]

\[
\mathcal{I}^{\text{inj}} := \bigcup_{(E,F) \in \text{BAN} \times \text{BAN}} \left\{ T \in \mathcal{B}(E, F) \mid \mathcal{I}T \in \mathcal{I}(E, \ell_\infty(B_F)) \right\}, \text{ and}
\]

\[
\mathcal{I}^{\text{sur}} := \bigcup_{(E,F) \in \text{BAN} \times \text{BAN}} \left\{ T \in \mathcal{B}(E, F) \mid T\Omega_E \in \mathcal{I}(\ell_1(B_E), F) \right\}.
\]

It is easily verified that \( \overline{\mathcal{I}} \), \( \mathcal{I}^{\text{inj}} \) and \( \mathcal{I}^{\text{sur}} \) are operator ideals and that each of them contains \( \mathcal{I} \). We note the following points of terminology:

- \( \overline{\mathcal{I}} \) is the closed hull of \( \mathcal{I} \), and \( \mathcal{I} \) is closed if \( \mathcal{I} = \overline{\mathcal{I}} \).

- \( \mathcal{I}^{\text{inj}} \) is the injective hull of \( \mathcal{I} \), and \( \mathcal{I} \) is injective if \( \mathcal{I} = \mathcal{I}^{\text{inj}} \).

- \( \mathcal{I}^{\text{sur}} \) is the surjective hull of \( \mathcal{I} \), and \( \mathcal{I} \) is surjective if \( \mathcal{I} = \mathcal{I}^{\text{sur}} \).

The concept of a closed operator ideal is quite natural. The notions of injectivity and surjectivity are due to I. Stephani [60], [61]. We mention in particular that both of the operator ideals \( \mathcal{N} \) and \( \mathcal{W} \) are closed, injective and surjective.

The following theorems provide useful characterisations of injective operator ideals and surjective operator ideals; we refer the reader to [49, p.72-73] for the proofs.

**Theorem 2.2.2.** An operator ideal \( \mathcal{I} \) is injective if and only if the following holds: for Banach spaces \( E, F \) and \( G \) and operators \( T \in \mathcal{B}(E, F) \) and \( J \in \mathcal{B}(F, G) \) such that \( J \) is an isomorphic embedding, it follows from \( JT \in \mathcal{I} \) that \( T \in \mathcal{I} \).

**Theorem 2.2.3.** An operator ideal \( \mathcal{I} \) is surjective if and only if the following holds: for Banach spaces \( D, E \) and \( F \) and operators \( Q \in \mathcal{B}(D, E) \) and \( T \in \mathcal{B}(E, F) \) such that \( Q \) is surjective, it follows from \( TQ \in \mathcal{I} \) that \( T \in \mathcal{I} \).
The following definition introduces the notion of a space ideal, also due to I. Stephani.

**Definition 2.2.4.** ([62, Definition 1.3]) A space ideal $I$ is a subclass of $\text{BAN}$ with the following properties:

1. $K \in I$.
2. $E \oplus F \in I$ whenever $E, F \in I$.
3. If $E \in I$ and $F$ is isomorphic to a complemented subspace of $E$, then $F \in I$.

For an operator ideal $\mathcal{J}$, we denote by $\text{Space}(\mathcal{J})$ the class of all Banach spaces whose identity operator belongs to $\mathcal{J}$. For a space ideal $I$, we denote by $\text{Op}(I)$ the class of all operators that factor through an element of $I$. The following result is established in [62, Theorem 1.1].

**Theorem 2.2.5.** Let $\mathcal{J}$ be an operator ideal and $I$ a space ideal. Then:

(i) $\text{Space}(\mathcal{J})$ is a space ideal.
(ii) $\text{Op}(I)$ is an operator ideal.

For operator ideals $\mathcal{I}$ and $\mathcal{J}$, we say that $\mathcal{I}$ has the $\mathcal{J}$-factorisation property if $\mathcal{I} \subseteq \text{Op}(\text{Space}(\mathcal{J}))$; evidently, this implies that $\mathcal{I} \subseteq \mathcal{J}$. An operator ideal $\mathcal{I}$ has the factorisation property if it has the $\mathcal{I}$-factorisation property.

In the following definition, the operators $U_\mathcal{F}$ and $Q_\mathcal{G}$ are as in Definition 2.1.1.

**Definition 2.2.6.** Let $\mathcal{I}$ and $\mathcal{J}$ be operator ideals and $1 < p < \infty$. We say that $(\mathcal{I}, \mathcal{J})$ is a $\Sigma_p$-pair if the following holds for any sequences of Banach spaces $(E_m)_{m \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ and $T \in \mathcal{B}(\bigoplus_{m \in \mathbb{N}} E_m)_p, (\bigoplus_{n \in \mathbb{N}} F_n)_p$: if $Q_\mathcal{G}TU_\mathcal{F} \in \mathcal{I}$ for all $\mathcal{F}, \mathcal{G} \in \mathbb{N}^{<\infty}$, then $T \in \mathcal{I}$.

Heinrich establishes the following impressive result in [30]:

**Theorem 2.2.7.** Let $1 < p < \infty$ and let $\mathcal{I}$ and $\mathcal{J}$ be surjective operator ideals such that $(\mathcal{I}, \mathcal{J})$ is a $\Sigma_p$-pair and $\mathcal{J}$ is injective. Then $\mathcal{I}$ has the $\mathcal{J}$-factorisation property.
We note that Theorem 2.2.7 is presented and proved in [30] under the additional hypothesis that \( \mathcal{I} = \mathcal{J} \). This restriction is, in fact, unnecessary. Indeed, a straightforward notational substitution in Heinrich's proof of the case \( \mathcal{I} = \mathcal{J} \) provides a proof of Theorem 2.2.7 in the generality in which it is presented here. For the benefit of the reader, a complete proof of Theorem 2.2.7, based closely on Heinrich's arguments in [30], is provided in Appendix C. The result actually presented and proved in [30] is thus the following corollary of Theorem 2.2.7.

**Corollary 2.2.8.** Let \( 1 < p < \infty \) and let \( \mathcal{I} \) be an injective, surjective operator ideal such that \( (\mathcal{I}, \mathcal{J}) \) is a \( \Sigma_p \)-pair. Then \( \mathcal{I} \) has the factorisation property.

### 2.3 Asplund spaces and operators

The notion of an Asplund space goes back to the work of E. Asplund in [7], where such spaces are called *strong differentiability spaces*. To give the traditional definition of an Asplund space, we first introduce the notions of Gâteaux differentiability and Fréchet differentiability.

**Definition 2.3.1.** ([14, Definition 1.1]) Let \( f \) be a real-valued function defined on a Banach space \( E \). We say that \( f \) is *Gâteaux differentiable at* \( x \in E \), if for each \( h \in E \),

\[
 f'(x)(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}
\]

exists and is a continuous linear function in \( h \) (that is, \( f'(x) \in E^* \)).

If, in addition, the above limit is uniform in \( h \in \{ x \in E \mid \|x\| = 1 \} \), we say that \( f \) is *Fréchet differentiable at* \( x \).

A real Banach space \( E \) is said to be *Asplund* if every real-valued convex continuous function defined on a convex open subset \( U \) of \( E \) is Fréchet differentiable on a dense \( G_δ \) subset of \( U \). A complex Banach space \( E \) is said to be Asplund if its underlying real Banach space \( E_\mathbb{R} \) is Asplund in the real scalar sense. Of particular importance to the context of our discussion is the following theorem that collects several useful characterisations of Asplund spaces; for a (real or complex) Banach
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space $E, C \subseteq E^*$, $\varepsilon > 0$ and $x \in E$, the $w^*$-open slice of $C$ determined by $x$ and $\varepsilon$ is the set $\{x^* \in C | \mathcal{R}(x^*, x) > \sup \{\mathcal{R}(y^*, x) | y^* \in C\} - \varepsilon\}$.

**Theorem 2.3.2.** Let $E$ be a Banach space. The following are equivalent:

(i) $E$ is an Asplund space;

(ii) Every separable subspace of $E$ is an Asplund space;

(iii) Every separable subspace of $E$ has separable dual;

(iv) Every bounded nonempty subset of $E^*$ admits nonempty $w^*$-open slices of arbitrarily small diameter.

Theorem 2.3.2 is proved for real Banach spaces in Chapter I.5 of [14]. For complex Banach spaces $E$, Theorem 2.3.2 follows from the real scalar case and properties of the canonical linear surjection $\varphi : x^* \mapsto \mathcal{R}x^*$ of $E^*$ onto $(E_\mathbb{R})^*$; for the benefit of the reader, this is demonstrated in detail in Appendix A.

Classical examples of Asplund spaces are reflexive spaces and spaces $C(K)$ with $K$ a scattered, compact Hausdorff space [21, Theorem 12.29].

The following definition of Asplund operators is taken from C. Stegall's paper [59] on the duality between Asplund spaces and Radon-Nikodým spaces. Let $E$ and $F$ be Banach spaces. An operator $T : E \to F$ is Asplund if for any finite positive measure space $(\Omega, \Sigma, \mu)$, any $S \in \mathcal{B}(F, L_\infty(\Omega, \Sigma, \mu))$ and any $\varepsilon > 0$, there exists $B \in \Sigma$ such that $\mu(B) > \mu(\Omega) - \varepsilon$ and $\{f \chi_B | f \in ST(B_E)\}$ is relatively compact in $L_\infty(\Omega, \Sigma, \mu)$ (here $\chi_B$ denotes the characteristic function of $B$ on $\Omega$). The class of all Asplund operators is denoted $\mathcal{D}$. Standard references for Asplund operators are [49] and [59], where it is shown that $\mathcal{D}$ is a closed operator ideal and that a Banach space is an Asplund space if and only if its identity operator is an Asplund operator. We note that some authors, for example in [49] and [30], refer to Asplund operators as decomposing operators. An impressive and noteworthy result in the theory of Asplund operators is that every Asplund operator factors through an Asplund space; this is due independently to O. Reînov [51], S. Heinrich [30] and C. Stegall [59].
2.4 The Szlenk index

In this section we define the Szlenk index and state some of its known properties. Fundamentally, we consider a set $X$ with a topology $\tau$ and a metric $\rho$ (which is usually not compatible with $\tau$), and study the interplay between the topological structures $(X, \tau)$ and $(X, \rho)$. Specifically, we consider the case where $X$ is the dual $E^*$ of a Banach space $E$, $\tau$ is the $w^*$-topology on $E^*$ and $\rho$ is the usual dual norm on $E^*$.

**Definition 2.4.1.** Let $E$ be a Banach space, $K \subseteq E^*$ a $w^*$-compact set and $\varepsilon > 0$. Define

$$s_\varepsilon(K) := \{x \in K \mid \text{diam}(K \cap V) > \varepsilon \text{ for every } w^*-\text{open } V \ni x\}.$$ 

We iterate $s_\varepsilon$ transfinitely as follows: let $s_\varepsilon^0(K) = K$, $s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K))$ for each ordinal $\alpha$ and, if $\alpha$ is a limit ordinal, $s_\varepsilon^\alpha(K) = \bigcap_{\beta < \alpha} s_\varepsilon^\beta(K)$.

The $\varepsilon$-Szlenk index of $K$, denoted $S_{\varepsilon}(K)$, is the class of all ordinals $\alpha$ such that $s_\varepsilon^\alpha(K) \neq \emptyset$. The Szlenk index of $K$ is the class $\bigcup_{\varepsilon > 0} S_{\varepsilon}(K)$. Note that $S_{\varepsilon}(K)$ (resp., $Sz(K)$) is either an ordinal or the class $\text{ORD}$ of all ordinals. If $S_{\varepsilon}(K)$ (resp., $Sz(K)$) is an ordinal, then we write $Sz_{\varepsilon}(K) < \infty$ (resp., $Sz(K) < \infty$), and otherwise we write $Sz_{\varepsilon}(K) = \infty$ (resp., $Sz(K) = \infty$). For a Banach space $E$, the $\varepsilon$-Szlenk index of $E$ is $Sz_{\varepsilon}(E) = Sz_{\varepsilon}(B_{E^*})$, and the Szlenk index of $E$ is $Sz(E) = Sz(B_{E^*})$. If $T : E \to F$ is an operator, the $\varepsilon$-Szlenk index of $T$ is $Sz_{\varepsilon}(T) = Sz_{\varepsilon}(T^*B_{F^*})$, whilst the Szlenk index of $T$ is $Sz(T) = Sz(T^*B_{F^*})$.

In a sense, the $\varepsilon$-Szlenk derivation $s_\varepsilon$ can be thought of as a ‘coarsening’ of the Cantor-Bendixon derivation. Indeed, the Cantor-Bendixon derivation (on a $w^*$-compact set) is obtained by taking $\varepsilon = 0$ in the definition of the derivation $s_\varepsilon$. However, for many $w^*$-compact sets of interest, the Cantor-Bendixon derivation removes no points, so it is not necessarily useful as a notion of the limiting behaviour of the derivations $s_\varepsilon$ as $\varepsilon$ goes to 0. It turns out that it can be very useful to instead consider the limiting behaviour of the associated indices $S_{\varepsilon}$ as $\varepsilon$ goes to 0 - this
limit being the Szlenk index $\text{Sz}$.

We now note some elementary properties of the Szlenk index. It is clear that the Szlenk index of a nonempty $w^*$-compact set cannot be $0$. By $w^*$-compactness, the $\varepsilon$-Szlenk index of a nonempty $w^*$-compact set $K$ is never a limit ordinal. A straightforward transfinite induction yields the following homogeneity property: for a $w^*$-compact set $K$, $c > 0$ a real scalar and $\varepsilon > 0$, we have $c s_\varepsilon(\alpha) = s_\varepsilon(c \alpha) \text{ for all ordinals } \alpha$, hence $S\varepsilon(K) = S\varepsilon(c \alpha) \text{ and } S\varepsilon(K) = S\varepsilon(c K)$. A more general result is proved below in Proposition 2.4.7; to obtain the above-mentioned homogeneity result, take $Q = c I_E$ in the statement of Proposition 2.4.7. We also have the following basic permanence property for the Szlenk index: for $w^*$-compact sets $K$ and $L$ with $K \subseteq L$, ordinals $\alpha$ and $\beta$ with $\alpha \leq \beta$ and $\varepsilon \geq \delta > 0$, the inclusion $s_\varepsilon(\delta) \subseteq s_\varepsilon(\delta) \text{ holds, hence } S\varepsilon(K) \subseteq S\varepsilon(L) \text{ and } S\varepsilon(L)$. Finally, we mention that the derivation $s_\varepsilon$ is closely related to the derivation $t_\varepsilon$ defined by setting

$$t_\varepsilon(K) := \{x \in K \mid \exists \text{ net } (x_i)_{i \in I} \text{ in } K \text{ such that } x_i \overset{w^*}{\to} x \text{ and } \|x - x_i\| > \varepsilon (i \in I)\}.$$ 

for $K$ a $w^*$-compact set. Indeed, the inclusions $s_{\varepsilon/2}(K) \subseteq t_\varepsilon(K) \subseteq s_\varepsilon(K)$ are easily verified and it follows that for $\varepsilon$ varying over the set of all positive real numbers, the derivations $s_\varepsilon$ and $t_\varepsilon$ are, in a sense, equivalent. We shall frequently use throughout the thesis the aforementioned inclusion $s_{\varepsilon/2}(K) \subseteq t_\varepsilon(K)$.

The following proposition collects further known facts about the Szlenk index.

**Proposition 2.4.2.** Let $E$ and $F$ be Banach spaces.

(i) If $E$ is isomorphic to a quotient or subspace of $F$, then $S\varepsilon(E) \leq S\varepsilon(F)$. In particular, the Szlenk index is an isomorphic invariant of a Banach space.

(ii) $S\varepsilon(E) < \infty$ if and only if $E$ is Asplund.

(iii) If $K \subseteq E^*$ is a nonempty, absolutely convex, $w^*$-compact set, then either $S\varepsilon(K) = \infty$ or there exists an ordinal $\alpha$ such that $S\varepsilon(K) = \omega^\alpha$. In particular, for $T \in \mathcal{B}$ either $S\varepsilon(T) = \infty$ or $S\varepsilon(T) = \omega^\alpha$ for some ordinal $\alpha$.

(iv) If $E$ is separable, then $E^*$ is norm separable if and only if $S\varepsilon(E) < \omega_1$, if and
only if $\text{Sz}(E) < \infty$.

(v) $\text{Sz}(E \oplus F) = \max\{\text{Sz}(E), \text{Sz}(F)\}$.

(vi) $\text{Sz}(E) = 1$ if and only if $E$ is finite dimensional.

We briefly indicate the origins of the various assertions of Proposition 2.4.2. Part (i) is well-known; see, for example, [28, p.63]. We note that (i) also follows from Theorem 3.1.2 of the present thesis. Part (ii) follows from the equivalence (i) $\iff$ (iv) of Theorem 2.3.2 above. Part (iii) is due to G. Lancien [38]; we note that although Lancien’s proof is given for the case where $K$ is the closed unit ball of a dual Banach space, his argument works equally well in the more general setting presented above. For the benefit of the truly skeptical reader, a fully detailed proof of (iii) (based on Lancien’s arguments) is provided in Appendix B. For (iv), see [43, Theorem 3.1] and its proof. Part (v) follows from Lemma 3.1.6 stated later in this thesis (Lemma 3.1.6 is due to P. Hájek and G. Lancien [26]). Finally, (vi) is well-known (for example, it is noted in [38, p.65]); it can be seen to follow from the Heine-Borel theorem and Proposition 3.1.4.

We note here the following lemma for future reference.

**Lemma 2.4.3.** Let $E$ be an infinite dimensional Asplund space and $\varepsilon > 0$. Then $\text{Sz}(E) > \text{Sz}_\varepsilon(E)$.

**Proof.** We trivially have $\text{Sz}(E) \geq \text{Sz}_\varepsilon(E)$. By Proposition 2.4.2(iii) and (vi), $\text{Sz}(E) = \omega^\alpha$ for some $\alpha > 0$, hence $\text{Sz}(E)$ is a limit ordinal. On the other hand, $\omega^*-\text{compactness}$ ensures that $\text{Sz}_\varepsilon(E)$ is a successor ordinal, hence $\text{Sz}(E) > \text{Sz}_\varepsilon(E)$. □

For each ordinal $\alpha$, define $\text{SZL}_\alpha := \{E \in \text{BAN} \mid \text{Sz}(E) \leq \omega^\alpha\}$. It follows readily from the properties of the Szlenk index listed in the statement of Proposition 2.4.2 that $\text{SZL}_\alpha$ is a space ideal for each ordinal $\alpha$.

We now give some examples of known results concerning the Szlenk indices of specific classes of Banach spaces.

**Example 2.4.4.** $\text{Sz}(E) = \omega$ for every infinite dimensional, superreflexive $E$ [37].
Example 2.4.5. Let $K$ be a countable compact Hausdorff space. Then, by the Mazurkiewicz-Sierpiński theorem [58, Theorem 8.6.10] and the Bessaga-Pelczyński isomorphic classification of spaces of continuous functions on countable compacta [9, Theorem 1], there is a unique $\alpha < \omega_1$ such that $C(K)$ is isomorphic to $C(\omega^\alpha +1)$, the space of continuous scalar-valued functions on the set of ordinals not exceeding $\omega^\alpha$, equipped with its order topology. Using deep results of D. Alspach and Y. Benyamini [6], C. Samuel has shown [56] that $\text{Sz}(C(\omega^\beta +1)) = \omega^{\beta+1}$ for every $\beta < \omega_1$, hence $\text{Sz}(C(K)) = \omega^{\alpha+1}$. A more direct computation of the Szlenk index of $C(\omega^\alpha +1)$, $\beta < \omega_1$, is given in [26]. The authors there extend Samuel’s result to include the case $\beta = \omega_1$.

Example 2.4.6. For $\alpha < \omega_1$, let $T_\alpha$ denote the $\alpha$th Tsirel’son space. It is shown in [44] that $\text{Sz}(T_\alpha) = \omega^\alpha$.

To conclude the chapter, and the section, we prove the following proposition and corollary regarding the behaviour of the Szlenk index under adjoints of surjective operators. Although a proof of this specific result does not seem to have appeared in the literature (due to its quite elementary nature, we presume), it is essentially the argument used to show that the Szlenk index does not increase when passing to quotients of Banach spaces (Proposition 2.4.2(i)). As we would like to refer to it in subsequent chapters, we state it here and provide a detailed proof.

Proposition 2.4.7. Let $D$ and $E$ be Banach spaces, $Q : D \rightarrow E$ a surjective operator and $K \subseteq E^*$ a w*-compact set. Choose a real scalar $c > 0$ such that $\|Q^*x^*\| \geq c\|x^*\|$ for all $x^* \in E^*$. Then $\text{Sz}_\varepsilon(K) \leq \text{Sz}_{\varepsilon\alpha}(Q^*(K))$ for every $\varepsilon > 0$, hence $\text{Sz}(K) \leq \text{Sz}(Q^*(K))$.

Proof. Fix $\varepsilon > 0$. It suffices to show that for all ordinals $\alpha$,

$$Q^*(s^\alpha_\varepsilon(K)) \subseteq s^\alpha_\varepsilon(Q^*(K)). \quad (2.2)$$

We proceed by induction on $\alpha$. It is trivial that (2.2) holds for $\alpha = 0$. Suppose that $\beta$ is an ordinal such that (2.2) holds for $\alpha = \beta$; we will show that (2.2)
2.4 The Szlenk index

holds for $\alpha = \beta + 1$. To this end, let $x \in E^*$ be such that $Q^*x \notin \mathcal{s}^{\beta+1}(Q^*(K))$. Then there exists $w^*$-open $W \ni Q^*x$ such that $\text{diam}(W \cap \mathcal{s}_e^\beta(Q^*(K))) \leq c\varepsilon$. Let $W_0 = (Q^*)^{-1}(W)$. Then $x \in W_0$, $W_0$ is $w^*$-open in $E^*$ and, since $Q^*$ is bounded below by $c$,

$$c \text{diam}(W_0 \cap \mathcal{s}_e^\beta(K)) \leq \text{diam}(W \cap Q^*(\mathcal{s}_e^\beta(K))) \leq \text{diam}(W \cap \mathcal{s}_e^\beta(Q^*(K))) \leq c\varepsilon.$$ 

That is, $\text{diam}(W_0 \cap \mathcal{s}_e^\beta(K)) \leq \varepsilon$. Thus $x \notin \mathcal{s}_e^\beta+1(K)$, and the injectivity of $Q^*$ ensures that $Q^*x \notin Q^*(\mathcal{s}_e^\beta+1(K))$. We have now shown that (2.2) passes to successor ordinals.

If $\beta$ is a limit ordinal and (2.2) holds for all $\alpha < \beta$, then

$$Q^*(\mathcal{s}_e^\beta(K)) = Q^* \left( \bigcap_{\alpha < \beta} \mathcal{s}_e^\alpha(K) \right) \subseteq \bigcap_{\alpha < \beta} Q^*(\mathcal{s}_e^\alpha(K)) \subseteq \bigcap_{\alpha < \beta} \mathcal{s}_e^\alpha(Q^*(K)) = \mathcal{s}_e^\beta(Q^*(K)),$$

which completes the induction. □

We consider the following corollary to be basic and important to our methods; we shall use it throughout the thesis without further explicit reference.

**Corollary 2.4.8.** Let $\Lambda$ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda \mid \lambda \in \Lambda\}$ a uniformly bounded family of operators, $p = 0$ or $1 < p < \infty$ and $\varepsilon > 0$. Let $T = (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$. Then $\mathcal{Sz}_e(T_\lambda) \leq \mathcal{Sz}_e(T)$ for every $\lambda \in \Lambda$.

**Proof.** For each $\lambda \in \Lambda$ we have $P^*_{\{\lambda\}}(T_\lambda^*B_{F_\lambda}) = (T_\lambda P_{\{\lambda\}})^*B_{F_\lambda} \subseteq T^*B(\bigoplus_{\lambda \in \Lambda} F_\lambda)_p$. For each $\varepsilon > 0$, it thus follows by Proposition 2.4.7 (with $c = 1$) that

$$\mathcal{Sz}_e(T_\lambda) = \mathcal{Sz}_e(T_\lambda^*B_{F_\lambda}) \leq \mathcal{Sz}_e(P^*_{\{\lambda\}}(T_\lambda^*B_{F_\lambda})) \leq \mathcal{Sz}_e(T^*B(\bigoplus_{\lambda \in \Lambda} F_\lambda)_p) = \mathcal{Sz}_e(T).$$ □
Chapter 3

Operator ideals associated with the Szlenk index

3.1 \(\alpha\)-Szlenk operators

In this section we show that the Szlenk index can be used in a natural way to define a class of closed operator ideals indexed by the class of all ordinals. We begin with the following definition.

Definition 3.1.1. For each ordinal \(\alpha\), define \(\mathcal{I}_a := \{T \in \mathcal{B} | \text{Sz}(T) \leq \omega^\alpha\}\). An element of \(\mathcal{I}_a\) shall be known as an \(\alpha\)-Szlenk operator. For each ordinal \(\alpha\) and pair of Banach spaces \((E, F)\), define \(\mathcal{I}_a(E, F) := \mathcal{B}(E, F) \cap \mathcal{I}_a\).

It is trivial that \(\mathcal{I}_a \subseteq \mathcal{I}_\beta\) whenever \(\alpha\) and \(\beta\) are ordinals satisfying \(\alpha \leq \beta\). Later (Proposition 3.1.9) we shall see that this inclusion is strict whenever \(\alpha < \beta\).

The following theorem is the main result of the current chapter.

Theorem 3.1.2. For \(\alpha\) an ordinal, \(\mathcal{I}_a\) is a closed, injective and surjective operator ideal.

For \(\alpha = 0\), the assertion of Theorem 3.1.2 follows from the following proposition and the well-known fact that \(\mathcal{K}\) is closed, injective and surjective.

Proposition 3.1.3. \(\mathcal{I}_0 = \mathcal{K}\).
Recall the classical theorem of Schauder asserting that an operator is compact if and only if its adjoint is compact. Proposition 3.1.3 is a consequence of Schauder's theorem and the following general result:

**Proposition 3.1.4.** Let $E$ be a Banach space, $K$ a nonempty, $w^*$-compact subset of $E^*$. Then $K$ is norm-compact if and only if $\text{Sz}(K) = 1$.

**Proof.** We use the fact that $K$ is norm-compact if and only if the relative norm and $w^*$ topologies of $K$ are the same (see, for example, [20, Corollary 3.1.14]).

First suppose that $\text{Sz}(K) = 1$. Let $(x_i)_{i \in I}$ be a $w^*$-convergent net in $K$; the norm-compactness of $K$ will follow if $(x_i)_{i \in I}$ is necessarily norm convergent. Let $x = w^* - \lim_i x_i \in K$ and note that, as $x \notin \bigcup_{\varepsilon > 0} s_{\varepsilon}(K)$, for every $\varepsilon > 0$ there exists $w^*$-open $U_\varepsilon \ni x$ such that $\text{diam}(U_\varepsilon \cap K) \leq \varepsilon$. For each $\varepsilon > 0$ let $j_\varepsilon \in I$ be such that $j_\varepsilon < j'$ implies $x_{j'} \in U_\varepsilon \cap K$. Then $j_\varepsilon < j'$ implies $\|x - x_{j'}\| \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary, $\|x - x_i\| \downarrow 0$.

Now suppose $\text{Sz}(K) > 1$. Then there is $x \in K$ and $\varepsilon > 0$ such that $x \in s_\varepsilon(K)$, and so for each $w^*$-open $U \ni x$ there is $x_U \in U \cap K$ such that $\|x - x_U\| > \varepsilon/2$. Since $x_U \overset{w^*}{\rightarrow} x$ and $x_U \overset{\|\cdot\|}{\rightarrow} x$ (here, the set of $w^*$-open sets containing $x$ carries the usual order induced by reverse set inclusion), the relative norm and $w^*$ topologies of $K$ are not the same. Hence $K$ is not norm-compact. □

We now prove the general case.

**Proof of Theorem 3.1.2** Let $\alpha$ be an ordinal. We must first show that $\mathcal{IL}_\alpha$ satisfies OI$_1$-OI$_3$ of Definition 2.2.1. To see that $\mathcal{IL}_\alpha$ satisfies OI$_1$, note that by Proposition 3.1.3 we have $I_K \in \mathcal{K} = \mathcal{IL}_0 \subseteq \mathcal{IL}_\alpha$.

Next we show that $\mathcal{IL}_\alpha$ satisfies OI$_3$. Let $D$, $E$, $F$ and $G$ be Banach spaces and $U \in \mathcal{B}(D, E)$, $T \in \mathcal{IL}_\alpha(E, F)$ and $V \in \mathcal{B}(F, G)$ operators. We want to show that $VTU \in \mathcal{IL}_\alpha$; this is clearly true if either $U$ or $V$ is zero, so we henceforth assume that $U$ and $V$ are nonzero. It suffices to show separately that $TU \in \mathcal{IL}_\alpha$ and $VT \in \mathcal{IL}_\alpha$. The fact that $TU \in \mathcal{IL}_\alpha$ will be deduced from the following generalisation of [27, Lemma 2].
Lemma 3.1.5. Let $D$ and $G$ be Banach spaces, $S \in \mathcal{B}(D, G)$ a nonzero operator, $K \subseteq G^*$ $w^*$-compact, $\alpha$ an ordinal and $\varepsilon > 0$. Then

$$s^\alpha_\varepsilon(S^*K) \subseteq S^*(s^\alpha_\varepsilon(2\|S\|)(K)).$$

Proof. We proceed by induction on $\alpha$. The assertion of the lemma is trivially true for $\alpha = 0$. Suppose that $\beta > 0$ is an ordinal such that the assertion of the lemma is true for all $\alpha < \beta$; we show that it is then true for $\alpha = \beta$. First suppose that $\beta$ is a successor, say $\beta = \gamma + 1$. Let $x \in s^\beta_\varepsilon(S^*K)$. Then there is a net $(x_i)_{i \in I}$ in $s^\gamma_\varepsilon(S^*K)$ with $x_i \xrightarrow{w^*} x$ and $\|x_i - x\| > \varepsilon/2$ for all $i$ (for example, let $I$ be the set of all $w^*$-neighbourhoods of $x$, ordered by reverse set inclusion). By the induction hypothesis, for each $i$ there is $y_i \in s^\gamma_\varepsilon(2\|S\|)(K)$ such that $S^*y_i = x_i$. Passing to a subnet, we may assume that the net $(y_i)_{i \in I}$ has a $w^*$-limit $y \in s^\gamma_\varepsilon(2\|S\|)(K)$. Then $S^*y = x$ and for all $i$ we have $\|y_i - y\| > \|x_i - x\|/\|S\| > \varepsilon/(2\|S\|)$, hence $y \in s^\beta_\varepsilon(2\|S\|)(K)$. It follows that the assertion of the lemma passes to successor ordinals.

Now suppose that $\beta$ is a limit ordinal. Let $x \in s^\beta_\varepsilon(S^*K) = \bigcap_{\alpha < \beta} s^\alpha_\varepsilon(S^*K)$. For each $\alpha < \beta$ there is $y_\alpha \in s^\alpha_\varepsilon(2\|S\|)(K)$ with $S^*y_\alpha = x$. The net $(y_\alpha)_{\alpha < \beta}$ admits a subnet $(y_j)_{j \in J}$ with $w^*$-limit $y \in \bigcap_{\alpha < \beta} s^\alpha_\varepsilon(2\|S\|)(K) = s^\beta_\varepsilon(2\|S\|)(K)$. Since $S^*y = x$, we are done. □

By Lemma 3.1.5,

$$Sz(TU) = \sup_{\varepsilon > 0} Sz_\varepsilon((TU)^*B_{F^*}) \leq \sup_{\varepsilon > 0} Sz_\varepsilon(2\|U\|)(T^*B_{F^*}) = Sz(T) \leq \omega^\alpha,$$

hence $TU \in \mathcal{I}^{-}\mathcal{L}_\alpha$.

As $VT = (\|V\|^{-1}V)T(\|V\|I_E)$ and $T(\|V\|I_E) \in \mathcal{I}^{-}\mathcal{L}_\alpha$ (take $U = \|V\|I_E$ above), to show that $VT \in \mathcal{I}^{-}\mathcal{L}_\alpha$ we may assume that $\|V\| \leq 1$. Then

$$(VT)^*B_{G^*} = T^*(V^*B_{G^*}) \subseteq T^*B_{F^*},$$

hence $Sz(VT) = Sz((VT)^*B_{G^*}) \leq Sz(T^*B_{F^*}) = Sz(T) \leq \omega^\alpha$, as desired. We have
now shown that $\mathcal{I}_\alpha$ satisfies $\text{OI}_3$.

To show that $\mathcal{I}_\alpha$ satisfies $\text{OI}_2$, we make use of the following lemma of P. Hájek and G. Lancien\(^1\) [26, Equation (2.3)] (note that Hájek and Lancien work in the context that $E = F$, although this assumption is not necessary). See also [44, Lemma 13(iii)] for the separable case. As stated in Section 2.1, for Banach spaces $E$ and $F$ we identify $(E \oplus_1 F)^*$ with $E^* \oplus_\infty F^*$. Thus, for sets $A_E \subseteq E^*$ and $A_F \subseteq F^*$ we have $A_E \times A_F \subseteq (E \oplus_1 F)^*$, and an element of $(E \oplus_1 F)^*$ can be considered as an ordered pair $(f, g) \in E^* \times F^*$.

Lemma 3.1.6. Let $E$ and $F$ be Banach spaces and $K \subseteq E^*$ and $L \subseteq F^*$ w*-compact sets. Consider $K \times L$ as a subset of $(E \oplus_1 F)^*$. Then, for all $\varepsilon > 0$ and ordinals $\alpha$,

$$s_\varepsilon^{\alpha}(K \times L) \subseteq (K \times s_\varepsilon^{\alpha}(L)) \cup (s_\varepsilon^{\alpha}(K) \times L).$$

Let $E$ and $F$ be Banach spaces and let $S, T \in \mathcal{B}(E, F)$ be operators such that $S + T \notin \mathcal{I}_\alpha$. Define operators $Q : E \rightarrow E \oplus_1 E$ and $R : E \oplus_1 E \rightarrow F$ by setting $Qx = (x, x)$ for $x \in E$, and $R(y, z) = Sy + Tz$ for $(y, z) \in E \oplus_1 E$. Then for every $x \in E$ we have $(RQ)x = R(x, x) = Sx + Tx = (S + T)x$, hence $RQ = S + T \notin \mathcal{I}_\alpha$. This means that $S_\varepsilon(Q^*(R^*B_{\varepsilon^*})) > \omega^\alpha$, which implies that $S_\varepsilon(R^*B_{\varepsilon^*}) > \omega^\alpha$ since $\mathcal{I}_\alpha$ satisfies $\text{OI}_3$. We have

$$R^*B_{\varepsilon^*} = \{(S^*x, T^*x) \mid x \in B_{\varepsilon^*}\} \subseteq S^*B_{\varepsilon^*} \times T^*B_{\varepsilon^*},$$

hence $S_\varepsilon(S^*B_{\varepsilon^*} \times T^*B_{\varepsilon^*}) > \omega^\alpha$. In particular, $s_\varepsilon^{\alpha}(S^*B_{\varepsilon^*} \times T^*B_{\varepsilon^*}) \neq \emptyset$ for some $\varepsilon > 0$. For such $\varepsilon$, Lemma 3.1.6 ensures that either $s_\varepsilon^{\alpha}(S^*B_{\varepsilon^*})$ or $s_\varepsilon^{\alpha}(T^*B_{\varepsilon^*})$ is nonempty, hence either $S_\varepsilon(S) > \omega^\alpha$ or $S_\varepsilon(T) > \omega^\alpha$. In other words, either $S \notin \mathcal{I}_\alpha$ or $T \notin \mathcal{I}_\alpha$. Thus $\mathcal{I}_\alpha$ satisfies $\text{OI}_2$, and is an operator ideal.

The injectivity of $\mathcal{I}_\alpha$ follows from the fact that for Banach spaces $E$ and $F$ and an operator $T \in \mathcal{B}(E, F)$, the Szlenk indices of $T$ and $J_F T$ are determined by the same set, namely $T^*B_{\varepsilon^*} = (J_F T)^*B_{\varepsilon^*}$.

\(^1\)The author thanks Professor G. Lancien for communicating to him a corrected proof of Lemma 3.1.6 (indeed, the proof of Lemma 3.1.6 given in [26] seems to be slightly incorrect). The proof of Lemma 5.2.9 uses some similar arguments.
For the surjectivity of $\mathcal{S}_{\alpha}$, let $E$ and $F$ be Banach spaces and let $T \in \mathcal{B}(E, F)$. By Proposition 2.4.7, $\text{Sz}(T) = \text{Sz}(T^*B_{F^*}) \leq \text{Sz}(\Omega_E^*T^*B_{F^*}) = \text{Sz}(T\Omega_E)$. Thus $T\Omega_E \in \mathcal{S}_{\alpha}$ implies $T \in \mathcal{S}_{\alpha}$, hence $\mathcal{S}_{\alpha}$ is surjective.

Finally, we turn our attention to showing that $\mathcal{S}_{\alpha}$ is a closed operator ideal. Note that for a Banach space $E$, a nonempty, $w^*$-compact set $K \subseteq E^*$ and $x \in E^*$, there exists $y \in K$ such that $\|x - y\| = d(x, K)$ (here $d(x, K)$ denotes the norm distance of $x$ to $K$, which is defined as $d(x, K) = \inf \{\|x - z\| \mid z \in K\}$). Indeed, the set $\{y \in K \mid \|x - y\| = d(x, K)\} = \bigcap_{r > 0} \{z \in K \mid \|x - z\| \leq d(x, K) + r\}$ is the intersection of a downward-filtering family of nonempty, $w^*$-compact sets, hence is itself nonempty. Our proof that $\mathcal{S}_{\alpha}$ is closed will be a straightforward application of the following lemma.

**Lemma 3.1.7.** Let $D$ be a Banach space, $\epsilon > 0$ and $K, L \subseteq D^*$ nonempty, $w^*$-compact sets with $\sup \{d(x, L) \mid x \in K\} \leq \epsilon/8$. Then $\text{Sz}_{\epsilon}(K) \leq \text{Sz}_{\epsilon/4}(L)$.

**Proof.** It clearly suffices to show that for all $\gamma \in \text{Sz}_{\epsilon}(K)$,

$$s_{\epsilon/4}^\gamma(L) \neq \emptyset \quad \text{and} \quad \sup \{d(x, s_{\epsilon/4}^\gamma(L)) \mid x \in s_{\epsilon}^\gamma(K)\} \leq \epsilon/8. \quad (3.1)$$

The assertions of (3.1) hold trivially for $\gamma = 0$. Suppose that $\beta \in \text{Sz}_{\epsilon}(K)$ is such that (3.1) holds for all $\gamma < \beta$; we will show that (3.1) holds for $\gamma = \beta$.

First suppose that $\beta$ is a successor, say $\beta = \zeta + 1$, and let $x \in s_{\epsilon}^\beta(K)$. Then there exists a net $(x_i)_{i \in I}$ in $s_{\epsilon}^\zeta(K)$ with $x_i \rightharpoonup^* x$ and $\|x_i - x\| > \epsilon/2$ for all $i$ (for example, take $I$ as the set of all $w^*$-neighbourhoods of $x$, ordered by reverse set inclusion). By the induction hypothesis, for each $i \in I$ there is $y_i \in s_{\epsilon/4}^\zeta(L)$ with $\|x_i - y_i\| \leq \epsilon/8$. Passing to a subnet, we may assume that $(y_i)_{i \in I}$ has a $w^*$-limit, $y$ say, in $s_{\epsilon/4}^\zeta(L)$. The $w^*$-lower semicontinuity of $\|\cdot\|_{D^*}$ yields $\|x - y\| \leq \liminf_{i \in I} \|x_i - y_i\| \leq \epsilon/8$.

Since for all $i \in I$ we have

$$\|y - y_i\| \geq \|x - x_i\| - \|x_i - y_i\| - \|x - y\| > \frac{\epsilon}{2} - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{\epsilon}{4},$$

$y \in s_{\epsilon/4}^\zeta(s_{\epsilon/4}^\zeta(L)) = s_{\epsilon/4}^\beta(L)$. Thus $s_{\epsilon/4}^\beta(L) \neq \emptyset$ and $d(x, s_{\epsilon/4}^\beta(L)) \leq \|x - y\| \leq \epsilon/8$. 

As $x \in s_{e}^{\beta}(K)$ was arbitrary, $\sup \{d(x, s_{\varepsilon/4}^{\beta}(L)) \mid x \in s_{e}^{\beta}(K)\} \leq \varepsilon/8$. We have now shown that (3.1) passes to successor ordinals in $S_{e}(K)$.

Now suppose that $\beta$ is a limit ordinal. Then $s_{\varepsilon/4}^{\beta}(L)$ is nonempty by the induction hypothesis and $w^{*}$-compactness. For the second assertion of (3.1), we again let $x \in s_{e}^{\beta}(K)$. By the induction hypothesis, for each $\zeta < \beta$ there is $y_{\zeta} \in s_{\varepsilon/4}^{\zeta}(L)$ such that $\|x - y_{\zeta}\| \leq \varepsilon/8$. Let $(z_{j})_{j \in J}$ be a $w^{*}$-convergent subnet of $(y_{\zeta})_{\zeta < \beta}$, with $w^{*}$-limit $y$, say. Then $y \in \bigcap_{\zeta < \beta} s_{\varepsilon/4}^{\zeta}(L) = s_{\varepsilon/4}^{\beta}(L)$ and

$$\|x - y\| \leq \liminf_{j \in J} \|x - z_{j}\| \leq \sup_{j \in J} \|x - z_{j}\| \leq \sup_{\zeta < \beta} \|x - y_{\zeta}\| \leq \varepsilon/8,$$

hence $d(x, s_{\varepsilon/4}^{\beta}(L)) \leq \|x - y\| \leq \varepsilon/8$. As $x \in s_{e}^{\beta}(K)$ is arbitrary, the second assertion of (3.1) holds for $\gamma = \beta$. This completes the proof of the lemma. □

Let $E$ and $F$ be Banach spaces and $T \in \mathcal{B}(E, F)$ an operator such that $T \notin \mathcal{L}_{\alpha}$. Then there is $\varepsilon > 0$ such that $S_{e}(T) > \omega^\alpha$. Let $S \in \mathcal{B}(E, F)$ be such that $\|T - S\| < \varepsilon/8$. Taking $K = T^{*}B_{F}^{*}$ and $L = S^{*}B_{F}^{*}$ in the statement of Lemma 3.1.7 yields $\omega^\alpha < S_{e}(T) \leq S_{e/4}(S) \leq S_{\varepsilon}(S)$, hence $S \notin \mathcal{L}_{\alpha}$. In particular, the open ball in $\mathcal{B}(E, F)$ centred at $T$ and of radius $\varepsilon/8$ has trivial intersection with $\mathcal{L}_{\alpha}(E, F)$. It follows that $\mathcal{L}_{\alpha}(E, F)$ is closed in $\mathcal{B}(E, F)$, and the proof of Theorem 3.1.2 is complete. □

Proposition 3.1.8. Let $\alpha$ be an ordinal. Then there exists a Banach space of Szlenk index $\omega^\alpha + 1$.

The proof of Proposition 3.1.8 will be given in Section 4.1.3. We have the following.

Proposition 3.1.9. Let $\beta$ be an ordinal and $\alpha < \beta$. Then $\mathcal{L}_{\alpha} \subseteq \mathcal{L}_{\beta}$.

Proof. The inclusion $\mathcal{L}_{\alpha} \subseteq \mathcal{L}_{\beta}$ is trivial. To show that the inclusion is strict, it suffices to show that the inclusion $\mathcal{L}_{\alpha} \subseteq \mathcal{L}_{\alpha+1}$ is strict. By Proposition 3.1.8, there is a Banach space $E$ such that $S_{\varepsilon}(E) = \omega^\alpha + 1$; the identity operator of $E$ belongs to $\mathcal{L}_{\alpha+1} \setminus \mathcal{L}_{\alpha}$. □
It is conceivable that one might obtain further operator ideals by considering the classes $\bigcup_{\beta \in C} \mathcal{I}_\beta$, where $C$ is a nonempty set of ordinals. Indeed, for such $C$, the linear ordering of $C$ ensures that the class $\bigcup_{\beta \in C} \mathcal{I}_\beta$ is an injective, surjective operator ideal. If $C$ contains a maximal element, $\gamma$ say, then $\bigcup_{\beta \in C} \mathcal{I}_\beta = \mathcal{I}_{\gamma}$, and we do not obtain any new operator ideal in this way. On the other hand, if $C$ contains no maximal element then there exists a nonzero limit ordinal $\alpha$ such that $\sup C = \alpha$. It is then easy to see that $\bigcup_{\beta \in C} \mathcal{I}_\beta = \bigcup_{\beta < \alpha} \mathcal{I}_\beta$. In this case, the following proposition determines precisely when $\bigcup_{\beta \in C} \mathcal{I}_\beta$ is not equal to $\mathcal{I}_\gamma$ for any ordinal $\gamma$, and precisely when $\bigcup_{\beta \in C} \mathcal{I}_\beta$ is closed. In particular, $\bigcup_{\beta \in C} \mathcal{I}_\beta$ is not equal to $\mathcal{I}_\gamma$ for any ordinal $\gamma$ if and only if $cf(\alpha) < \omega_1$, if and only if $\bigcup_{\beta < \alpha} \mathcal{I}_\beta$ is not closed.

**Proposition 3.1.10.** Let $\alpha > 0$ be an ordinal. The following are equivalent:

1. $cf(\alpha) \geq \omega_1$.
2. $\omega^\alpha$ is not the Szlenk index of any operator between Banach spaces.
3. $\mathcal{I}_\alpha = \bigcup_{\beta < \alpha} \mathcal{I}_\beta$.
4. $\alpha$ is a limit ordinal and $\bigcup_{\beta < \alpha} \mathcal{I}_\beta$ is closed.

**Proof.** We will show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i).

For (i)$\Rightarrow$(ii), note that if $T \in \mathcal{B}$ is such that $\omega^\alpha = \text{Sz}(T) = \sup_{n \in \mathbb{N}} \text{Sz}_{1/n}(T)$, then $cf(\alpha) \leq cf(\omega^\alpha) = \omega < \omega_1$.

The implication (ii)$\Rightarrow$(iii) is immediate from Proposition 2.4.2(iii).

Now suppose that (iii) holds. Then $\bigcup_{\beta < \alpha} \mathcal{I}_\beta$ is closed by Theorem 3.1.2. Moreover, $\alpha$ is a limit ordinal. Indeed, otherwise we may write $\alpha = \zeta + 1$, where $\zeta$ is an ordinal, and by Proposition 3.1.8 there exists a Banach space $E$ such that $I_E \in \mathcal{I}_{\zeta+1} \setminus \mathcal{I}_\zeta = \mathcal{I}_\alpha \setminus \bigcup_{\beta < \alpha} \mathcal{I}_\beta = \emptyset$, which is absurd.

To show (iv)$\Rightarrow$(i), suppose by way of contraposition that $cf(\alpha) = \omega$ and let $\{\alpha_n \mid n < \omega\} \subseteq \alpha$ be cofinal in $\alpha$. Then $\{\alpha_n + 1 \mid n < \omega\}$ is also cofinal in $\alpha$, and $\bigcup_{n < \omega} \mathcal{I}_{\alpha_n+1} = \bigcup_{\beta < \alpha} \mathcal{I}_\beta$. So to complete the proof, it suffices to construct some $T \in \bigcup_{n < \omega} \mathcal{I}_{\alpha_n+1} \setminus \bigcup_{n < \omega} \mathcal{I}_{\alpha_n+1}$. To this end, for each $n < \omega$ let $E_n$ be a Banach space with $\text{Sz}(E_n) = \omega^{\alpha_n+1}$ (Proposition 3.1.8). Set $E = (\bigoplus_{n < \omega} E_n)_{\ell_1}$ and define
$T \in \mathcal{B}(E)$ by setting $T(x_n)_{n<\omega} = ((n+1)^{-1}x_n)_{n<\omega}$, $(x_n)_{n<\omega} \in E$. As $T$ factors $I_{E_n}$ for each $n < \omega$, $Sz(T) \supseteq \sup \{Sz(E_n) \mid n < \omega\} = \sup \{\omega^{\alpha_{n+1}} \mid n < \omega\} = \omega^\alpha$. Thus $T \notin \bigcup_{n<\omega} \mathcal{L}_{\alpha_{n+1}}$. On the other hand, with $U_m$ ($m < \omega$) denoting the operator on $E$ that sends $(x_n)_{n<\omega} \in E$ to the element $(y_n)_{n<\omega}$ of $E$ that satisfies $y_n = x_n$ if $n \leq m$, and $y_n = 0$ otherwise, we have that $I_{E_1 \oplus \ldots \oplus E_m}$ factors $U_m T$ for all $m < \omega$, hence

$$Sz(U_m T) \leq Sz(E_1 \oplus \ldots \oplus E_m) = \max \{\omega^{\alpha_i+1} \mid 1 \leq i \leq m\}.$$ 

In particular, $U_m T \in \bigcup_{n<\omega} \mathcal{L}_{\alpha_{n+1}}$ for $m < \omega$. As $\lim_{m \to \omega} \|U_m T - T\| = 0$, it follows that $T \in \bigcup_{n<\omega} \mathcal{L}_{\alpha_{n+1}}(E)$. \hfill $\square$

We now describe the relationship between the classes $\mathcal{L}_\alpha$ and the operator ideal of Asplund operators. For this we shall call on the following characterisation of Asplund operators that follows readily from work of C. Stegall, in particular [59, Proposition 2.10 and Theorem 1.12].

**Proposition 3.1.11.** Let $E$ and $F$ be Banach spaces and $T : E \to F$ an operator. Then $T$ is Asplund if and only if for every separable Banach space $D$ and every operator $S : D \to E$, the set $S^*T^*B_{F^*}$ is norm separable.

We also require the following result concerning metrisable $w^*$-compact sets; the proof is essentially contained in the proof of Proposition 2.4.2(iv).

**Lemma 3.1.12.** Let $K$ be a $w^*$-compact set that is metrisable in the $w^*$ topology and nonseparable in the norm topology. Then $Sz(K) = \infty$.

The following proposition asserts that the class of Asplund operators coincides with $\bigcup_{\alpha \in \text{Ord}} \mathcal{L}_\alpha$.

**Proposition 3.1.13.** Let $E$ and $F$ be Banach spaces and $T : E \to F$ an operator. The following are equivalent:

(i) $T$ is an $\alpha$-Szlenk operator for some ordinal $\alpha$.

(ii) $T$ is an Asplund operator.
Proof. First suppose that $T$ is Asplund. Then, by the Reînov–Heinrich–Stegall factorisation theorem for Asplund operators (c.f. Section 2.3), there exists an Asplund space $G$ such that $I_G$ factors $T$. By Proposition 2.4.2(iii), there is an ordinal $\alpha$ such that $Sz(G) = \omega^\alpha$, hence $Sz(T) \leq Sz(I_G) = Sz(G) = \omega^\alpha$. That is, $T$ is $\alpha$-Szlenk.

Now suppose that $T$ is not Asplund. By Proposition 3.1.11, there exists a separable Banach space $D$ and an operator $S : D \to E$ such that $S^*T^*B_{F^*}$ is nonseparable in the norm topology. As $D$ is norm separable, we have that $S^*T^*B_{F^*}$ is $w^*$-metrisable, hence $Sz(TS) = Sz(S^*T^*B_{F^*}) = \infty$ by Lemma 3.1.12. That is, $TS$ fails to be $\alpha$-Szlenk for any ordinal $\alpha$. As the classes $\mathcal{L}_\alpha$ are operator ideals, $T$ fails to be $\alpha$-Szlenk for any $\alpha$. \hfill $\square$

For every pair of Banach spaces $(E, F)$, there is an ordinal $\alpha$ such that if $T \in \mathcal{B}(E, F)$ is $\beta$-Szlenk for some ordinal $\beta$, then $T$ is $\alpha$-Szlenk. Indeed, we may take $\alpha$ to satisfy $\omega^\alpha = \sup \{Sz(T) \mid T \in \mathcal{L}_\beta(E, F) \text{ for some ordinal } \beta\}$. By Proposition 3.1.13, with $\alpha$ so defined we have $\mathcal{D}(E, F) = \mathcal{L}_\alpha(E, F)$.

Following G. Birkhoff [10, p.63], for a $T_1$ topological space $X$ and a partially-ordered set $(A, \leq)$, a function $f : X \to A$ is said to be lower semicontinuous if for each $a \in A$ the set $\{x \in X \mid f(x) \leq a\}$ is closed in $X$.

**Proposition 3.1.14.** Let $E$ and $F$ be Banach spaces.

(i) Let $\alpha$ be an ordinal such that $\mathcal{D}(E, F) = \mathcal{L}_\alpha(E, F)$. Then

$$Sz : \mathcal{D}(E, F) \to \omega^\alpha + 1$$

is lower semicontinuous.

(ii) Let $(T_n)_{n=1}^\infty$ be a norm-convergent sequence in $\mathcal{D}(E, F)$. Then

$$Sz(\lim_n T_n) \leq \liminf_n Sz(T_n).$$

*Proof.* (i) follows from Theorem 3.1.2.
For (ii), let $\xi$ be such that $\omega^\xi = \liminf_n \text{Sz}(T_n)$ and let $T = \lim_n T_n$. Let

$$n_1 = \inf \left\{ m \in \mathbb{N} \mid \text{Sz}(T_m) = \inf_{n \in \mathbb{N}} \text{Sz}(T_n) \right\}.$$ 

If $n_k \in \mathbb{N}$ is given, define

$$n_{k+1} = \inf \left\{ m \in \mathbb{N} \mid m > n_k, \text{Sz}(T_m) = \inf_{n > n_k} \text{Sz}(T_n) \right\}.$$ 

We thus obtain a strictly increasing sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}$ with $\text{Sz}(T_{n_k}) \leq \omega^\xi$ for all $k$. In particular, $\{T_{n_k} \mid k \in \mathbb{N}\} \subseteq \mathcal{I} \mathcal{X}_\xi$ and $T_{n_k} \xrightarrow{\text{lip}} T$, hence $T \in \mathcal{I} \mathcal{X}_\xi$ since $\mathcal{I} \mathcal{X}_\xi$ is closed. \hfill \square

We now determine the relationship between the operator ideals $\mathcal{I} \mathcal{X}_\alpha$ and the operator ideal $\mathcal{X}$ of operators having separable range. In what follows, $\mathcal{X}^*$ denotes the operator ideal of operators $T$ with $T^* \in \mathcal{X}$.

The following result is essentially an operator-theoretic generalisation of Proposition 2.4.2(iv).

**Proposition 3.1.15.** The following chain of equalities holds.

$$\mathcal{X}^* = \mathcal{X} \cap \mathcal{D} = \mathcal{X} \cap \bigcup_{\alpha \in \text{ORD}} \mathcal{I} \mathcal{X}_\alpha = \mathcal{X} \cap \mathcal{I} \mathcal{X} = \mathcal{X} \cap \mathcal{I} \mathcal{X}_{\omega_1}.$$ 

**Proof.** By Proposition 3.1.10 and Proposition 3.1.13,

$$\mathcal{X} \cap \mathcal{I} \mathcal{X}_{\omega_1} = \mathcal{X} \cap \bigcup_{\alpha < \omega_1} \mathcal{I} \mathcal{X}_\alpha = \mathcal{X} \cap \mathcal{I} \mathcal{X}_\alpha = \mathcal{X} \cap \mathcal{D}.$$ 

So it suffices to show that $\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{X}^*$ and $\mathcal{X}^* \subseteq \mathcal{X} \cap \mathcal{I} \mathcal{X}_{\omega_1}$.

To prove $\mathcal{X} \cap \mathcal{D} \subseteq \mathcal{X}^*$, we first note that S. Heinrich [30] has shown that $\mathcal{D}$ satisfies the $\Sigma_p$-condition for all $1 < p < \infty$. We claim that $\mathcal{X}$ also satisfies the $\Sigma_p$-condition for all $1 < p < \infty$. To verify our claim, we note that if $(E_m)_{m \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ are sequences of Banach spaces, $1 < p < \infty$ and $T$ is an element of
3.2 Examples

\[ \mathcal{B}(\bigoplus_{m \in \mathbb{N}} E_m)_p, (\bigoplus_{n \in \mathbb{N}} F_n)_p \] such that \( T \notin \mathcal{X} \), then the set

\[ \bigcup_{\mathcal{F}, \mathcal{G} \in \mathbb{N}^{<\infty}} Q_g T U_{\mathcal{F}} (\bigoplus_{m \in \mathbb{N}} E_m)_p \]

is nonseparable since its uniform closure contains \( T(\bigoplus_{m \in \mathbb{N}} E_m)_p \). As \( \mathbb{N}^{<\infty} \) is countable, it follows that are \( \mathcal{F}, \mathcal{G} \in \mathbb{N}^{<\infty} \) such that \( Q_g T U_{\mathcal{F}} (\bigoplus_{m \in \mathbb{N}} E_m)_p \) is nonseparable. That is, \( Q_g T U_{\mathcal{F}} \notin \mathcal{X} \). This completes the proof of the claim, and it follows that \( \mathcal{X} \cap \mathcal{D} \) satisfies the \( \Sigma_p \)-condition for all \( 1 < p < \infty \). Moreover, \( \mathcal{X} \cap \mathcal{D} \) is injective and surjective since the same is true for \( \mathcal{X} \) and \( \mathcal{D} \). Thus, by Corollary 2.2.8, every element of \( \mathcal{X} \cap \mathcal{D} \) factors through a separable Asplund space. By Theorem 2.3.2, this implies that every element of \( \mathcal{X} \cap \mathcal{D} \) factors through a Banach space with separable dual, and the inclusion \( \mathcal{X} \cap \mathcal{D} \subseteq \mathcal{X}^* \) follows.

We now show that \( \mathcal{X}^* \subseteq \mathcal{X} \cap \mathcal{I}_{\omega_1} \). The inclusion \( \mathcal{X}^* \subseteq \mathcal{X} \) is well-known (see, for example, [49, Proposition 4.4.8]), so we need only show that \( \mathcal{X}^* \subseteq \mathcal{I}_{\omega_1} \). To this end, note that similar arguments to those used above show that \( \mathcal{X}^* \) satisfies the \( \Sigma_p \)-condition for any \( 1 < p < \infty \). Moreover, \( \mathcal{X}^* \) is injective and surjective, hence Corollary 2.2.8 implies that every element of \( \mathcal{X}^* \) factors through a Banach space with separable dual. By Proposition 2.4.2(iv), this means that every element of \( \mathcal{X}^* \) factors through a Banach space of countable Szlenk index; the inclusion \( \mathcal{X}^* \subseteq \mathcal{I}_{\omega_1} \) follows. □

3.2 Examples

In this section we discuss the algebras \( \mathcal{I}_{\omega_1}(E) \) for a number of well-known Banach spaces \( E \). In particular, we study the place of the ideals \( \mathcal{I}_{\omega_1}(E) \) in the lattice of closed, two-sided ideals of \( \mathcal{B}(E) \) by relating them to other well-known closed ideals (for example, the ideal of weakly compact operators).

Example 3.2.1. The space \( L_\infty = L_\infty[0, 1] \). We will show that the operator ideals \( \mathcal{W}, \mathcal{F}_2, \mathcal{D}, \mathcal{I}_{\omega_1} \) and \( \mathcal{X}^* \) coincide on \( L_\infty \). For this purpose, we state the following impressive result of H. Jarchow [31]:
Theorem 3.2.2. Let $A$ be a $C^*$-algebra and $F$ a Banach space. Then

$$\mathcal{W}(A, F) = \overline{\mathcal{W}^{inj}}(A, F).$$

We also require the following lemma.

Lemma 3.2.3. Let $\mathcal{H}$ be a Hilbert space and $E$ a Banach space such that every weakly compact subset of $E$ is norm separable. Then $\mathcal{B}(\mathcal{H}, E) = \mathcal{A}_2(\mathcal{H}, E)$.

Proof. Let $T \in \mathcal{B}(\mathcal{H}, E)$. The reflexivity of $\mathcal{H}$ implies $T \in \mathcal{W}(\mathcal{H}, E)$, and the norm separability of $T(B_{\mathcal{H}})$ implies $T \in \mathcal{X}$. Thus, by the (separable) DFJP factorisation theorem [13, Lemma 1(xi)], there is a separable, reflexive Banach space $F$ and operators $A : \mathcal{H} \to F$ and $B : F \to E$ such that $T = BA$. Since $F^*$ is separable, we have that $A^*(F^*)$ is isometric to a separable closed subspace of $\mathcal{H}^*$, hence isometric to a closed subspace of $\ell_2$. Thus $A^* \in \mathcal{A}_2$. Making the identifications $\mathcal{H} = \mathcal{H}^{**}$ and $F = F^{**}$ via the canonical injections of $\mathcal{H}$ and $F$ into their second duals, we have $A = A^{**} \in \mathcal{A}_2 = \mathcal{A}_2$, hence $T = BA \in \mathcal{A}_2$. □

We have $\mathcal{W}(L_\infty) \subseteq \mathcal{X}(L_\infty)$ since weakly compact subsets of $L_\infty$ are norm separable [52, Proposition 4.7], and $\mathcal{X}(L_\infty) \subseteq \mathcal{W}(L_\infty)$ since $L_\infty$ is a Grothendieck space. Thus $\mathcal{X}(L_\infty) = \mathcal{W}(L_\infty)$. By Example 2.4.4 we have $\text{Sz}(\mathcal{H}) = \omega$ for $\mathcal{H}$ a Hilbert space, and since $L_\infty$ is both a $C^*$-algebra and an injective Banach space, Theorem 3.2.2 yields $\mathcal{X}(L_\infty) = \mathcal{W}(L_\infty) = \overline{\mathcal{W}^{inj}}(L_\infty) \subseteq \mathcal{A}_2(L_\infty) \subseteq \mathcal{D}(L_\infty)$. We have $\mathcal{X}^* = \mathcal{X} \cap \mathcal{D}$ by Proposition 3.1.15, hence $\mathcal{X}^*(L_\infty) = \mathcal{X}(L_\infty)$. Thus, to show that $\mathcal{W}$, $\mathcal{A}_2$, $\mathcal{D}$, $\mathcal{D}_1^*$ and $\mathcal{X}^*$ coincide on $L_\infty$, it now suffices to show that $\mathcal{D}(L_\infty) \subseteq \mathcal{W}(L_\infty)$ and $\Gamma_2(L_\infty) \subseteq \mathcal{A}_2(L_\infty)$. The first of these inclusions is justified by the fact that nonweakly compact operators on $L_\infty$ fix a copy of the non-Asplund space $L_\infty$ (see [53, Proposition 1.2] and the main result of [45]). The second inclusion follows from Lemma 3.2.3 and the fact that every weakly compact subset of $L_\infty$ is norm separable [52, Proposition 4.7].

Example 3.2.4. The space $L_1 = L_1[0, 1]$. Similarly to the previous example, we will show that the operator ideals $\mathcal{W}$, $\overline{\mathcal{A}_2}$, $\mathcal{D}$, $\mathcal{D}_1^*$ and $\mathcal{X}^*$ coincide on $L_1$. 
Let $Q : L_1 \hookrightarrow L_1^*$ denote the canonical embedding and let $P : L_1^{**} \to L_1$ be a projection (that $L_1$ is complemented in its bidual is well-known; see, for example, [1, Proposition 6.3.10]). Since $\text{Sz}(\ell_2) = \omega$, we have $\mathcal{K}_2(L_1) \subseteq \mathcal{L}_1(L_1) \subseteq \mathcal{B}(L_1)$. Moreover, as $\mathcal{K}^*$ and $\mathcal{B}$ coincide on separable Banach spaces by Proposition 3.1.15, it suffices to show that $\mathcal{B}(L_1) \subseteq \mathcal{W}(L_1)$ and $\mathcal{W}(L_1) \subseteq \mathcal{K}_2(L_1)$. The first of these inclusions is justified by the fact that nonweakly compact operators into $L_1$ fix a copy of $\ell_1$ [46, Theorem 1], and therefore fail to be Asplund. For the second inclusion, let $T \in \mathcal{W}(L_1)$. Then, by Gantmacher's theorem, $T^*$ is a weakly compact operator on the (up to isomorphism) $C^*$-algebra $L_1^*$. Moreover, $L_1^*$ is an injective Banach space, hence Theorem 3.2.2 ensures the existence of a sequence $(S_n)$ in $\Gamma_2(L_1^*)$ satisfying $\|T^* - S_n\| \to 0$. It follows that, since $T = PT^{**}Q$, we have $\|T - PS^n_nQ\| = \|P(T^{**} - S^n_n)Q\| \to 0$. In particular, $T \in \Gamma_2(L_1)$ since $S^n_n \in \Gamma_2$ for all $n$. As $L_1$ is separable, it follows that $T \in \mathcal{K}_2(L_1)$.

**Example 3.2.5.** The space $C[0, 1]$. The lattice of closed, two-sided ideals in $\mathcal{B}(C[0, 1])$ contains the following linearly ordered chain, where $0 < \beta < \omega_1$:

$$
\{0\} \subsetneq \mathcal{K}(C[0, 1]) = \mathcal{L}_0(C[0, 1]) \subsetneq \mathcal{W}(C[0, 1]) \subsetneq \mathcal{L}_1(C[0, 1]) \subsetneq \ldots
$$

$$
\ldots \subseteq \bigcup_{\gamma < \beta} \mathcal{L}_\gamma(C[0, 1]) \subseteq \mathcal{L}\alpha(C[0, 1]) \subsetneq \mathcal{L}_\beta(C[0, 1]) \subsetneq \mathcal{L}_{\beta+1}(C[0, 1]) \subseteq \ldots
$$

$$
\ldots \bigcup_{\alpha < \omega_1} \mathcal{L}_\alpha(C[0, 1]) = \mathcal{B}(C[0, 1]) = \mathcal{K}^*(C[0, 1]) \subsetneq \mathcal{B}(C[0, 1]).
$$

Note that the ideal $\mathcal{K}^*(C[0, 1])$ is the unique maximal ideal in $\mathcal{B}(C[0, 1])$ since each element of $\mathcal{B}(C[0, 1]) \setminus \mathcal{K}^*(C[0, 1])$ factors the identity operator of $C[0, 1]$. Indeed, combining theorems of H. Rosenthal [54, Theorem 1] and A. Pelczyński [47, Theorem 1], we have the following: for any $T \in \mathcal{B}(C[0, 1]) \setminus \mathcal{K}^*(C[0, 1])$ there exists a closed subspace $E \subseteq C[0, 1]$ such that $T|_E$ is an isomorphism, $E$ is isomorphic to $C[0, 1]$ and $T(E)$ is complemented in $C[0, 1]$. With this fact in hand, let $R$ be an isomorphism of $C[0, 1]$ onto $E$, let $P : C[0, 1] \to E$ be a continuous projection and set $V = (TR)^{-1}P$. Then $I_{C[0, 1]} = VTR$.

To justify the other claims above regarding the structure of the lattice of closed,
two-sided ideals in $\mathcal{B}(C[0, 1])$, we note the following. Let $A : C[0, 1] \longrightarrow \ell_2$ be a surjective operator and $B : \ell_2 \longrightarrow C[0, 1]$ an isomorphic embedding. Then $BA \in \mathcal{W}(C[0, 1]) \setminus \mathcal{K}(C[0, 1])$. The inclusion $\mathcal{W}(C[0, 1]) \subseteq \mathcal{S}_1^0(C[0, 1])$ follows from Theorem 3.2.2 and the fact that, since Hilbert spaces have Szlenk index $\omega$ and $\mathcal{S}_1^0$ is closed and injective, $\Gamma_2^{\text{inj}} \subseteq \mathcal{S}_1^0$. The difference $\mathcal{S}_1^0(C[0, 1]) \setminus \mathcal{W}(C[0, 1])$ contains any projection of $C[0, 1]$ onto a subspace isomorphic to $c_0$ (of which there are many) since $c_0$ is nonreflexive and of Szlenk index $\omega$. Similarly, the difference $\mathcal{S}_1^0(C[0, 1]) \setminus \mathcal{S}_1^0(C[0, 1])$ contains any projection of $C[0, 1]$ onto a subspace isomorphic to $C(\omega^{\omega^\omega} + 1)$. That the operator ideals $\cup_{\alpha < \omega_1} \mathcal{S}_1^0, \mathcal{D}$ and $\mathcal{K}^*$ coincide on $C[0, 1]$ follows from Proposition 3.1.15.

**Example 3.2.6.** Let $V$ denote the complementably universal unconditional basis space of Pelczyński [48] (see also [57] and [1, Section 13.3]). Then, as in the case of $C[0, 1]$ above, we have $\mathcal{S}_1^0(V) \subseteq \mathcal{S}_1^0(V)$ for every $\beta < \omega_1$. To show this, it suffices to find, for each $\beta < \omega_1$, a Banach space $E_\beta$ having an unconditional basis and Szlenk index $\omega^{\beta+1}$. Indeed, the existence of such a space ensures the existence of a projection of $V$ onto a complemented subspace isomorphic to $E_\beta$, and such a projection clearly belongs to $\mathcal{S}_1^0(V) \setminus \mathcal{S}_1^0(V)$. For the existence of the desired spaces $E_\beta$, we turn to the following construction of W. Szlenk [63] of a family of separable, reflexive Banach spaces whose Szlenk indices are (collectively) unbounded above in $\omega_1$.

**Construction 3.2.7.** Define $G_0 = \{0\}$ (the trivial Banach space), $G_{\beta+1} = G_\beta \oplus_1 \ell_2$ for each ordinal $\beta$ and, if $\beta$ is a limit ordinal, $G_\beta = (\bigoplus_{\xi < \beta} G_\xi)_2$.

A straightforward transfinite induction on $\alpha < \omega_1$ shows that $G_\alpha$ possesses a $1$-unconditional basis for $0 < \alpha < \omega_1$. As will be shown in the proof of Proposition 3.1.8, for each $\beta < \omega_1$ there exists $\alpha(\beta) < \omega_1$ such that $\text{Sz}(G_{\alpha(\beta)}) = \omega^{\beta+1}$. Thus, taking $E_\beta = G_{\alpha(\beta)}$ gives the desired spaces $E_\beta$, $\beta < \omega_1$.

Finally, we note that $\bigcup_{\alpha < \omega_1} \mathcal{S}_1^0(V) = \mathcal{D}(V) = \mathcal{K}^*(V) \subseteq \mathcal{B}(V)$ by Proposition 3.1.15 and the existence of $P \in \mathcal{B}(V) \setminus \mathcal{K}^*(V)$ with $P^2 = P$ and $P(V)$ isomorphic to $\ell_1$. 
3.3 Variants of the Szlenk index

In the final section of this chapter we discuss a couple of variants of the Szlenk index that have appeared in the literature. Our reasons for doing this are as follows. On the one hand, we wish to demonstrate that the definition of the Szlenk index that we have chosen to use throughout this thesis is the best notion to use for the development of a theory of operator ideals associated with the Szlenk index. On the other hand, the variants of the Szlenk index we shall introduce in the current section are often better for practical purposes since they are defined in terms of sequential convergence properties. It is thus important to know when the various notions of Szlenk index coincide, as well as the shortcomings of the sequential notions of Szlenk index that make them unsuitable for developing an operator ideal theory that encompasses operators acting between arbitrary Banach spaces.

We shall consider the following two derivations on subsets of dual Banach spaces. For $E$ a Banach space, $K \subseteq E^*$ and $\varepsilon > 0$, define

$$m_\varepsilon(K) := \{x^* \in K \mid \exists (x_n^*)_{n \in \mathbb{N}} \subseteq K, x_n^* \xrightarrow{w^*} x^*, \|x_n^* - x^*\| > \varepsilon \text{ for all } n\}$$

and

$$n_\varepsilon(K) := \{x^* \in K \mid \exists (x_n^*)_{n \in \mathbb{N}} \subseteq K, \exists (x_n)_{n \in \mathbb{N}} \subseteq B_E, x_n^* \xrightarrow{w^*} x^*, x_n \xrightarrow{w} 0, \limsup_n |\langle x_n^*, x_n \rangle| \geq \varepsilon\}.$$ 

As with the derivations $s_\varepsilon$, we transfinitely iterate the derivations $m_\varepsilon$ and $n_\varepsilon$ to obtain derivations $m_\varepsilon^\alpha$ and $n_\varepsilon^\alpha$ for each $\varepsilon > 0$ and ordinal $\alpha$. That is, $m_\varepsilon^0(K) = K$, $m_\varepsilon^{\alpha+1}(K) = m_\varepsilon(m_\varepsilon^\alpha(K))$ for all ordinals $\alpha$ and, if $\alpha$ is a limit ordinal, $m_\varepsilon^\alpha = \bigcap_{\beta < \alpha} m_\varepsilon^\beta(K)$. The iterates $n_\varepsilon^\alpha$ are defined similarly. For $\varepsilon > 0$, $M_{\varepsilon}(K)$ (resp., $N_{\varepsilon}(K)$) is the class of all ordinals such that $m_\varepsilon^\alpha(K)$ (resp., $n_\varepsilon^\alpha(K)$) is nonempty, and $M_\varepsilon(K) = \bigcup_{\varepsilon > 0} M_{\varepsilon}(K)$ (resp., $N_\varepsilon(K) = \bigcup_{\varepsilon > 0} N_{\varepsilon}(K)$). For Banach spaces $E$ and $F$ and $T \in \mathscr{B}(E, F)$, set $M_\varepsilon(E) = M_\varepsilon(B_{E^*})$ and $M_\varepsilon(T) = M_\varepsilon(T^*B_{F^*})$. Similarly, $N_\varepsilon(E) = N_\varepsilon(B_{E^*})$ and $N_\varepsilon(T) = N_\varepsilon(T^*B_{F^*})$. For $\alpha$ an ordinal, let
\( \mathcal{M}_\alpha = \{ T \in \mathcal{B} \mid \text{Mz}(T) \leq \omega^\alpha \} \) and \( \mathcal{N}_\alpha = \{ T \in \mathcal{B} \mid \text{Nz}(T) \leq \omega^\alpha \} \).

The main obstacle to proving that the classes \( \mathcal{M}_\alpha \) form operator ideals is that we do not seem to have an analogue of Lemma 3.1.5 for the derivations \( m_\varepsilon^\alpha \). Indeed, in the proof of Lemma 3.1.5 we use the fact that every net in a \( w^* \)-compact set \( K \) has a \( w^* \)-convergent subnet, whereas it need not be the case that every sequence in \( K \) has a \( w^* \)-convergent subsequence. However, we may form operator ideals from the classes \( \mathcal{M}_\alpha \) by taking their intersection with the class \( \mathcal{M} \) consisting of all operators having \( w^* \)-sequentially compact adjoint. That is, an operator \( T : E \to F \) belongs to \( \mathcal{M} \) if and only if \( T^*B_{F^*} \) is \( w^* \)-sequentially compact. In Appendix D a proof is given of the fact that \( \mathcal{M} \) is a closed, injective, surjective operator ideal with the factorisation property. The arguments of the proof of Theorem 3.1.2, but with nets replaced by sequences, show that \( \mathcal{M}_\alpha \cap \mathcal{M} \) is a closed, injective, surjective operator ideal for every ordinal \( \alpha \).

We note that the indices Sz and Mz coincide for operators \( T : E \to F \) with the property that \( T^*B_{F^*} \) is \( w^* \)-metrisable; indeed, for \( w^* \)-metrisable, \( w^* \)-compact \( K \), \( \alpha \) an ordinal and \( \varepsilon > 0 \), the chain of inclusions \( m_\varepsilon(K) \subseteq s_\varepsilon(K) \subseteq m_{\varepsilon/2}(K) \) holds (recall the discussion of the derivations \( t_\varepsilon \) preceding Proposition 2.4.2). It turns out that the class of all such operators is precisely \( \mathcal{X} \), the separable range operators. This characterisation of \( \mathcal{X} \) follows easily from the following facts:

- \((F1)\) A Banach space \( E \) is norm separable if and only if \( B_{E^*} \) is metrisable in the relative \( w^* \) topology [21, Proposition 3.24].
- \((F2)\) The image of a compact metric space under a continuous mapping into a Hausdorff space is metrisable [18, p.254, Exercise XI.5.7].
- \((F3)\) For a compact, Hausdorff space \( K \), \( C(K) \) is norm separable if and only if the topology of \( K \) is metrisable [21, Lemma 3.23].
- \((F4)\) For Banach spaces \( E \) and \( F \) and \( T \in \mathcal{B}(E, F) \), the operator \( T \) admits a (canonical) continuous, linear factorisation through a subspace of \( C((T^*B_{F^*}, w^*)) \). Indeed, let \( A : E \to C((T^*B_{F^*}, w^*)) \) be the operator that sends \( x \in E \) to the \( (T^*B_{F^*}, w^*) \)-to-\( K \) continuous map \( y^* \mapsto \langle y^*, x \rangle \) \( (y^* \in (T^*B_{F^*}, w^*)) \). The linear map \( B : A(E) \to F : Ax \mapsto Tx \) is well-defined and norm preserving,
and thus admits a continuous linear extension $\hat{B}$ to the closed subspace $\overline{A(E)}$ of $C((T^*B_{F^*}, w^*))$. Clearly $T = \hat{B}A$.

Now, on the one hand, if $T \in \mathcal{X}$ then $T^*B_{F^*}$ is a $w^*$-continuous image of the $w^*$-compact, $w^*$-metrisable (by $(F1)$) set $B_{\overline{T(E)}}$, hence is itself $w^*$-metrisable by $(F2)$. On the other hand, if $T^*B_{F^*}$ is $w^*$-metrisable, then $C((T^*B_{F^*}, w^*))$ is norm separable by $(F3)$, and it follows then that $T \in \mathcal{X}$ since $T$ admits a factorisation through a subspace of $C((T^*B_{F^*}, w^*))$ by $(F4)$. Thus, when dealing with operators having separable range, one may usually work with $M_z$ in place of $S_z$ if it is more convenient.

We now turn our discussion to the index $N_z$ and the associated classes $\mathcal{N}_\alpha$, $\alpha$ an ordinal. The index $N_z$ is in fact that introduced by Szlenk in [63], and it coincides with $S_z$ and $M_z$ for operators whose domain is a separable Banach space containing no subspace isomorphic to $\ell_1$ (as noted in the introduction, we refer the reader to the proof of [37, Proposition 3.3] for the technical details). The classes $\mathcal{N}_\alpha$ are thus operator ideals over the class of separable Banach spaces not containing an isomorphic copy of $\ell_1$. However, the index $N_z$ lacks sufficiently good permanence properties for the classes $\mathcal{N}_\alpha$ to be operator ideals over the class of all Banach spaces. We illustrate this claim by way of the following example, which will show that $\mathcal{N}_\alpha$ fails to be an operator ideal for all $\alpha < \omega_1$. For each $\alpha < \omega_1$ we have that $C(\omega^\alpha + 1)$ is separable and contains no subspace isomorphic to $\ell_1$, hence $N_z(C(\omega^\alpha + 1)) = S_z(C(\omega^\alpha + 1)) = \omega^{\alpha+1}$ for each $\alpha < \omega_1$. Moreover, there exists an isometric linear embedding $U : C(\omega^\alpha + 1) \longrightarrow \ell_{\infty}$ since $\ell_{\infty}$ contains an isometric copy of every separable Banach space [21, Proposition 5.11]. As observed by J. Bourgain [12, p.88], if $E$ is a Grothendieck space with the Dunford-Pettis property, then $N_z(E) = 1$. In particular, $N_z(\ell_{\infty}) = 1$, hence $I_{\ell_{\infty}} \in \mathcal{N}_\alpha$. On the other hand,

$$N_z(I_{\ell_{\infty}}U) = N_z(U^*B_{\ell_{\infty}}) = N_z(B_{C(\omega^\alpha + 1)^*}) = N_z(C(\omega^\alpha + 1)) = \omega^{\alpha+1},$$

so that $I_{\ell_{\infty}}U \notin \mathcal{N}_\alpha$. In particular, $\mathcal{N}_\alpha$ fails to satisfy $OI_3$ of Definition 2.2.1.
Despite the apparent deficiency of the index $\text{Nz}$ from the point of view of developing a theory of operator ideals associated with the Szlenk index, it is worth emphasising the importance of the index $\text{Nz}$ in the study of the structure of operators acting on spaces $C(K)$, where $K$ is a metrisable compact space. Indeed, a number of authors have studied the connections between the $\text{Nz}$ index of operators acting on $C(K)$ spaces and 'fixing' properties of such spaces; we refer to [55] for a survey, and to the work of I. Gasparis [23] for more recent results. In fact, we believe that both of the indices $\text{Sz}$ and $\text{Nz}$ are of interest in the study of operators in $\mathcal{B}(C[0, 1])$. For example, the following question is of interest in studying the closed ideal structure of the Banach algebra $\mathcal{B}(C[0, 1])$:

**Question 3.3.1.** Let $R \in \mathcal{X}^*(C[0, 1])$. Does there exist $S \in \mathcal{W}(C[0, 1])$ such that $\text{Sz}(R + S) = \text{Nz}(R + S)$?

The motivation for Question 3.3.1 is the fact that $\mathcal{W}$ is a closed operator ideal and that, for $T \in \mathcal{B}(C[0, 1])$, $\text{Nz}(T)$ is minimal (that is, is equal to 1) if and only if $T$ is weakly compact; this latter fact regarding minimality of $\text{Nz}(T)$ for $T \in \mathcal{B}(C[0, 1])$ is due to D. Alspach [3, Remark 2].
Chapter 4

Direct sums and the Szlenk index

This chapter consists of two sections. In Section 4.1 we present our main results, first establishing necessary and sufficient conditions for direct sums of operators to belong to $\mathcal{Sz}_\alpha$ for a given ordinal $\alpha$, and then giving applications of this characterisation to Banach space theory. Section 4.2 is devoted to proving the main technical lemma used in Section 4.1, namely Lemma 4.1.5.

4.1 Main results

It is obvious that a direct sum of operators factors any of its summands. Since Proposition 3.1.13 implies that $\{T \in \mathcal{B} \mid Sz(T) < \infty\}$ is the class of Asplund operators, it is only interesting to consider the Szlenk index of a direct sum of operators in the case that all of the summands are Asplund. With this in mind, we henceforth consider direct sums of Asplund operators only.

4.1.1 $\ell_1$-direct sums and $\ell_\infty$-direct sums

The task of determining the Szlenk index of $\ell_1$-direct sums and $\ell_\infty$-direct sums of operators is made considerably easier by the fact that the Banach spaces $\ell_1$ and $\ell_\infty$ fail to be Asplund, for this ensures that the norms of the summand operators must exhibit $c_0$-like behaviour in order for the direct sum operator to be Asplund. More
Proposition 4.1.1. Let $\Lambda$ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda \in B(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ a uniformly bounded family of Asplund operators and $p = 1$ or $p = \infty$. The following are equivalent:

(i) $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) < \infty$ (that is, $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is Asplund).

(ii) $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{\text{Sz}(T_\lambda) \mid \lambda \in \Lambda\}$.

(iii) $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.

Proof. We prove (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

Suppose (iii) holds; we will show that $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{\text{Sz}(T_\lambda) \mid \lambda \in \Lambda\}$. By Proposition 2.4.2(iii), there exist ordinals $\alpha_\lambda, \lambda \in \Lambda$, with $\text{Sz}(T_\lambda) = \omega^{\alpha_\lambda}$ for each $\lambda$. Let $\alpha = \sup\{\alpha_\lambda \mid \lambda \in \Lambda\}$, so that $\sup\{\text{Sz}(T_\lambda) \mid \lambda \in \Lambda\} = \omega^\alpha$. To see that $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p \in \mathcal{L}_{\alpha}$, for $n \in \mathbb{N}$ and $\lambda \in \Lambda$ let

$$T_{\lambda, n} = \begin{cases} T_\lambda & \text{if } \|T_\lambda\| > 1/n, \\ 0 & \text{otherwise} \end{cases}$$

and $V_n = (\bigoplus_{\lambda \in \Lambda} T_{\lambda, n})_p$. As $\{T_{\lambda, n} \mid \lambda \in \Lambda, n \in \mathbb{N}\} \subseteq \mathcal{L}_{\alpha}$, we have $V_n \in \mathcal{L}_{\alpha}$ since each $V_n$ can be written as a (finite) sum of operators that factor some element of $\{T_{\lambda, n} \mid \lambda \in \Lambda, n \in \mathbb{N}\}$. From the definitions, $\|V_n - (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p\| \leq 1/n$ for each $n \in \mathbb{N}$, hence $V_n \to (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$. As $V_n \in \mathcal{L}_{\alpha}$ for all $n$ and $\mathcal{L}_{\alpha}$ is closed, we have $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p \in \mathcal{L}_{\alpha}$. Thus $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) \leq \omega^\alpha = \sup\{\text{Sz}(T_\lambda) \mid \lambda \in \Lambda\}$. The reverse inequality follows by the fact that $\mathcal{L}_{\alpha}$ satisfies $\text{OI}_3$ of Definition 2.2.1, and that $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ factors each of the operators $T_\lambda, \lambda \in \Lambda$.

It is trivial that (ii) $\Rightarrow$ (i), so it remains only to show that (i) $\Rightarrow$ (iii). To this end, suppose that (iii) does not hold. Then there exists $\delta > 0$ and infinite $\Lambda' \subseteq \Lambda$ such that $\|T_\lambda\| > \delta$ for all $\lambda \in \Lambda'$, hence $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ factors an isomorphic embedding of the non-Asplund space $\ell_p$. By Proposition 2.4.2(ii), $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \infty$. □
4.1.2 \(c_0\)-direct sums and \(\ell_p\)-direct sums (1 < \(p < \infty\))

We now consider the Szlenk index of an operator \((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p\), where \(p = 0\) or \(1 < p < \infty\). As in the cases \(p = 1\) and \(p = \infty\), if \((\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)\), then 
\[ Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\}. \]
However, the situation is not so clear if \((\|T_\lambda\|)_{\lambda \in \Lambda} \notin c_0(\Lambda)\). To illustrate by way of an example, for \(0 < n < \omega\) let \(T_n\) denote the identity operator on \(C(\omega^n + 1)\). By the Bessaga-Pelczyński isomorphic classification of spaces of continuous functions on countable compacta (see [9, Theorem 1]), \(C(\omega^n + 1)\) is isomorphic to \(C(\omega + 1)\) for all \(0 < n < \omega\), hence 
\[ Sz(T_n) = Sz(C(\omega + 1)) = \omega \text{ for all } 0 < n < \omega \] (see Example 2.4.5). On the other hand, as \((\bigoplus_{0 < n < \omega} C(\omega^n + 1))_0\) is linearly isomorphic to \(C(\omega^\omega + 1)\), by Example 2.4.5 we have

\[ Sz((\bigoplus_{0 < n < \omega} T_n)_0) = Sz(C(\omega^\omega + 1)) = \omega^2 > \omega = \sup\{Sz(T_n) \mid 0 < n < \omega\}. \]

Thus the situation under consideration in this section is more subtle than the cases of \(\ell_1\)-direct sums and \(\ell_\infty\)-direct sums. Our goal is to determine precisely the Szlenk index of a \(c_0\)-direct sum or \(\ell_p\)-direct sum (1 < \(p < \infty\)) of operators in terms of the overall behaviour of the \(\varepsilon\)-Szlenk indices of the summand operators. We first deal explicitly with the case where the Szlenk index of a direct sum of operators has Szlenk index \(\omega^0 = 1\). The following result describes the situation for this case.

**Proposition 4.1.2.** Let \(\Lambda\) be a set, \(\{E_\lambda \mid \lambda \in \Lambda\}\) and \(\{F_\lambda \mid \lambda \in \Lambda\}\) families of Banach spaces, \(\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}\) a uniformly bounded family of operators and \(p \in \{0\} \cup [1, \infty]\). The following are equivalent:

(i) \(Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = 1\).

(ii) \(Sz(T_\lambda) = 1\) for every \(\lambda \in \Lambda\) and \((\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)\).

Proposition 4.1.2 follows immediately from Proposition 3.1.3 and the following proposition.

**Proposition 4.1.3.** Let \(\Lambda\) be a set, \(\{E_\lambda \mid \lambda \in \Lambda\}\) and \(\{F_\lambda \mid \lambda \in \Lambda\}\) families of Banach spaces, \(\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}\) a uniformly bounded family of operators
and $p \in \{0\} \cup [1, \infty]$. The following are equivalent:

(i) $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is compact.

(ii) $T_\lambda$ is compact for every $\lambda \in \Lambda$ and $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.

We omit the straightforward proof of Proposition 4.1.3, as it is very similar to the proof of Proposition 4.1.1 presented earlier.

The general case for $c_0$-direct sums and $\ell_p$-direct sums of operators ($1 < p < \infty$) will be deduced from the following key result (in the statement of Proposition 4.1.4 below, the sets $B_q(K_\lambda | \lambda \in \Lambda)$ and $B_q(K_\lambda | \lambda \in \mathcal{F})$ are as defined in Section 2.1).

**Proposition 4.1.4.** Let $\Lambda$ be a set, $\{E_\lambda | \lambda \in \Lambda\}$ a family of Banach spaces and $\{K_\lambda \subseteq E_\lambda^* | \lambda \in \Lambda\}$ a uniformly bounded family of nonempty, absolutely convex, $w^*$-compact sets. For $1 \leq q < \infty$ and $\alpha > 0$ an ordinal, the following are equivalent:

(i) $\text{Sz}(B_q(K_\lambda | \lambda \in \Lambda)) < \omega^\alpha$.

(ii) $\sup \{\text{Sz}_\varepsilon(K_\lambda) | \lambda \in \Lambda\} < \omega^\alpha$ for every $\varepsilon > 0$.

(iii) $\sup \{\text{Sz}_\varepsilon(B_q(K_\lambda | \lambda \in \mathcal{F})) | \mathcal{F} \in \Lambda^{< \infty}\} < \omega^\alpha$ for every $\varepsilon > 0$.

To establish Proposition 4.1.4, we prove (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). When proving the implication (ii)$\Rightarrow$(iii), we shall call upon the following technical result:

**Lemma 4.1.5.** Let $E_1, \ldots, E_n$ be Banach spaces, $K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^*$ nonempty, absolutely convex, $w^*$-compact sets, $1 \leq q < \infty$ and $\varepsilon > 0$. Let $d = \max_{1 \leq i \leq n} \text{diam}(K_i)$ and let $m, M \in \mathbb{N}$ be such that $M \geq m \geq 2$ and $(2^q - 1)\varepsilon^q M \geq 8^q d^q (m - 1)$. Suppose that $\alpha$ is an ordinal such that $s_\varepsilon^{\alpha, M}(B_q(K_i | 1 \leq i \leq n)) \neq \emptyset$. Then for every $\delta \in (0, \varepsilon/16)$ there is $i \leq n$ such that $s_\delta^{\alpha - \delta}(K_i) \neq \emptyset$.

The proof of Lemma 4.1.5 is delayed until Section 4.2. To prove (iii)$\Rightarrow$(i), we require the following discrete variant of [26, Lemma 3.3]:

**Lemma 4.1.6.** Let $\Lambda$ be a set, $(E_\lambda)_{\lambda \in \Lambda}$ a family of Banach spaces, $1 \leq q < \infty$, $p$ predual to $q$ and $K \subseteq (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$ nonempty and $w^*$-compact. Let $\alpha$ be an ordinal, $\mathcal{R} \subseteq \Lambda$ and $\varepsilon > \delta > 0$. If $x \in s_\varepsilon^*(K)$ and $\|U^*_R x\|^q > |K|^q - (\varepsilon/2)^q$, then $U^*_R x \in s_\delta^*(U^*_R K)$.
Proof. We fix $\epsilon$, $\delta$ and $R$ and proceed by transfinite induction. The conclusion of the lemma holds trivially for $\alpha = 0$. So suppose that $\beta$ is an ordinal such that the conclusion of the lemma holds with $\alpha = \beta$; we show that it holds then also for $\alpha = \beta + 1$. Let $x \in K$ be such that $\|U^*_x x\|_q > |K|_q - (\frac{\epsilon - \delta}{2})^q$ and $U^*_x x \notin s^\beta_{\delta}(U^*_x K)$. Our goal is to show that $x \notin s^\beta_{\epsilon}(K)$, so we may assume that $x \in s^\beta_{\epsilon}(K)$, hence $U^*_x x \in s^\beta_{\delta}(U^*_x K)$ by the induction hypothesis. It follows that there is $w^*$-open $V \subseteq (\bigoplus_{\lambda \in R} E_\lambda)^*_p$ such that $U^*_x x \in V$ and $d := \text{diam}(V \cap s^\beta_{\delta}(U^*_x K)) \leq \delta$. As $U^*_x x$ does not belong to the $w^*$-closed set $(|K|_q - (\frac{\epsilon - \delta}{2})^q)^{1/q} B(\bigoplus_{\lambda \in R} E_\lambda)^*_p$, we may assume that $V \cap (|K|_q - (\frac{\epsilon - \delta}{2})^q)^{1/q} B(\bigoplus_{\lambda \in R} E_\lambda)^*_p = \emptyset$. Let $W = (U^*_x)^{-1}(V)$ and let $u \in W \cap s^\beta_{\epsilon}(K)$. Then $\|U^*_x u\|_q > |K|_q - (\frac{\epsilon - \delta}{2})^q$ and $u \in s^\beta_{\epsilon}(K)$, hence by the induction hypothesis $U^*_x u \in V \cap s^\beta_{\delta}(U^*_x K)$. So for $u_1, u_2 \in W \cap s^\beta_{\epsilon}(K)$, we have $\|U^*_x u_1 - U^*_x u_2\|_q \leq d^q \leq \delta^q$. Moreover, as $\|U^*_x u_1\|_q > |K|_q - (\frac{\epsilon - \delta}{2})^q$, it follows that

$$\|u_1 - P^*_R U^*_x u_1\| \leq (|K|_q - \|P^*_R U^*_x u_1\|_q)^{1/q} = (|K|_q - \|U^*_x u_1\|_q)^{1/q} < \frac{\epsilon - \delta}{2}.$$ 

Similarly, $\|u_2 - P^*_R U^*_x u_2\| < \frac{\epsilon - \delta}{2}$. Since $\|z\|_q = \|P^*_R U^*_x z\|_q + \|z - P^*_R U^*_x z\|_q$ for $z \in (\bigoplus_{\lambda \in R} E_\lambda)^*_q$, we may now deduce that

$$\|u_1 - u_2\|_q = \|P^*_R U^*_x u_1 - P^*_R U^*_x u_2\|_q + \|(u_1 - P^*_R U^*_x u_1) - (u_2 - P^*_R U^*_x u_2)\|_q \leq \|U^*_x u_1 - U^*_x u_2\|_q + \left(2 \cdot \frac{\epsilon - \delta}{2}\right)^q \leq \delta^q + (\epsilon - \delta)q \leq \epsilon^q.$$ 

In particular, $\text{diam}(W \cap s^\beta_{\epsilon}(K)) \leq \epsilon$. It follows that $x \notin s^\beta_{\epsilon+1}(K)$, as desired.

Now suppose that $\beta$ is a limit ordinal such that the conclusion of the lemma holds for all $\alpha < \beta$. Suppose that $x \in s^\alpha_{\epsilon}(K)$ is such that $\|U^*_x x\|_q > |K|_q - (\frac{\epsilon - \delta}{2})^q$. Then, since $x \in s^\alpha_{\epsilon}(K)$ for all $\alpha < \beta$, the induction hypothesis implies $U^*_x x \in s^\beta_{\delta}(U^*_x K)$ for $\alpha < \beta$, hence $U^*_x x \in \bigcap_{\alpha < \beta} s^\beta_{\delta}(U^*_x K) = s^\beta_{\delta}(U^*_x K)$. This completes the induction. $\square$
In order to state the third (and final) lemma required in the proof of Proposition 4.1.4, we give the following definition.

**Definition 4.1.7.** For real numbers $a \geq 0$, $b > c > 0$ and $1 \leq d < \infty$, define

$$\sigma(a, b, c, d) := \inf \left\{ n \in \mathbb{N} \mid n \geq \left( \frac{2a}{b-c} \right)^d - \left( \frac{b}{b-c} \right)^d + 1 \right\}.$$  

With regards to Definition 4.1.7, it is worth noting explicitly that $\sigma(a, b, c, d) = 1$ whenever $2a < b$.

**Lemma 4.1.8.** Let $\Lambda$ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ a family of Banach spaces, $1 \leq q < \infty$, $p$ predual to $q$, $K \subseteq (\bigoplus_{\lambda \in \Lambda} E_\lambda)^*_p$ a nonempty, $w^*$-compact set and $\varepsilon > \delta > 0$. Suppose $\eta_\delta$ is a nonzero ordinal such that $s_\delta^{\eta_\delta}(U_F^*K) = \emptyset$ for every $F \in \Lambda^{<\infty}$. Then $s_\delta^{\eta_\delta \cdot \sigma(|K|, \varepsilon, \delta, q)}(U_F^*K) = \emptyset$, hence $Sz_\varepsilon(K) \leq \eta_\delta \cdot \sigma(|K|, \varepsilon, \delta, q)$.

**Proof.** We claim that for each $n < \omega$, either $s_\varepsilon^{\eta_\delta \cdot n}(K)$ is empty or

$$|s_\varepsilon^{\eta_\delta \cdot n}(K)|^q \leq |K|^q - n \left( \frac{\varepsilon - \delta}{2} \right)^q. \tag{4.1}$$

To prove the claim, we proceed by induction on $n$. (4.1) holds trivially for $n = 0$. Suppose the claim holds for $n = m$; we will show that it holds for $n = m + 1$. For every $F \in \Lambda^{<\infty}$ we have

$$s_\varepsilon^{\eta_\delta \cdot (m+1)}(U_F^*s_\varepsilon^{\eta_\delta \cdot m}(K)) \subseteq s_\varepsilon^{\eta_\delta}(U_F^*K) = \emptyset. \tag{4.2}$$

If $s_\varepsilon^{\eta_\delta \cdot m}(K) = \emptyset$, we are done. Otherwise, by the induction hypothesis,

$$|s_\varepsilon^{\eta_\delta \cdot (m+1)}(K)|^q \leq |K|^q - m \left( \frac{\varepsilon - \delta}{2} \right)^q. \tag{4.3}$$

If $s_\varepsilon^{\eta_\delta \cdot (m+1)}(K) \neq \emptyset$, then applying (4.2), (4.3) and Lemma 4.1.6 implies that for every $x \in s_\varepsilon^{\eta_\delta \cdot (m+1)}(K)$ and $F \in \Lambda^{<\infty}$, we have

$$\|U_F x\|^q \leq |K|^q - m \left( \frac{\varepsilon - \delta}{2} \right)^q - \left( \frac{\varepsilon - \delta}{2} \right)^q = |K|^q - (m + 1) \left( \frac{\varepsilon - \delta}{2} \right)^q.$$
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Thus \( x \in s_{\varepsilon}^{\eta_{(m+1)}}(K) \) implies

\[
\|x\|^q = \sup_{\mathcal{F} \in \mathcal{A}_{< \infty}} \|U_{\mathcal{F}}^* x\|^q \leq |K|^q - (m + 1) \left( \frac{\varepsilon - \delta}{2} \right)^q
\]

and so (4.1) holds for \( n = m + 1 \). The inductive proof of the claim is complete.

By definition (precisely, Definition 4.1.7), we have

\[
|K|^q - (\sigma(|K|, \varepsilon, \delta, q) - 1) \left( \frac{\varepsilon - \delta}{2} \right)^q \leq \left( \frac{\varepsilon}{2} \right)^q.
\]  

Thus, by (4.4) and the claim proved above we have

\[
\text{diam}(s_{\varepsilon}^{\eta_{(\sigma(|K|, \varepsilon, \delta, q) - 1)}}(K)) \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon,
\]

and we now conclude that

\[
s_{\varepsilon}^{\eta_{\sigma(|K|, \varepsilon, \delta, q) - 1}}(K) \subseteq s_{\varepsilon}^{\eta_{\sigma(|K|, \varepsilon, \delta, q) - 1} + 1}(K) = s_{\varepsilon}^{\eta_{\sigma(|K|, \varepsilon, \delta, q) - 1}}(K) = \emptyset. \]

We now give the proof of Proposition 4.1.4, assuming Lemma 4.1.5.

**Proof of Proposition 4.1.4.** We prove (i)\(\Rightarrow\)(ii)\(\Rightarrow\)(iii)\(\Rightarrow\)(i). Throughout, \( p \) shall denote the real number predual to \( q \).

To show (i)\(\Rightarrow\)(ii), suppose by way of a contraposition that there is \( \varepsilon > 0 \) such that \( \sup \{ S_{\varepsilon}(K_\lambda) \mid \lambda \in \Lambda \} \geq \omega^\alpha \). Since \( B_q(K_\lambda \mid \lambda \in \Lambda) \supseteq P_{\{\lambda\}'}(K_{\lambda'}) \) for every \( \lambda' \in \Lambda \) (here, \( P_{\{\lambda\}} : (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p \rightarrow E_{\lambda'} \) is as defined in Definition 2.1.1), it follows by Proposition 2.4.7 (with \( c = 1 \)) that

\[
S_{\varepsilon}(B_q(K_\lambda \mid \lambda \in \Lambda)) \supseteq \sup \{ S_{\varepsilon}(P_{\{\lambda\}'}(K_{\lambda'})) \mid \lambda \in \Lambda \}
\]

\[
\sup \{ S_{\varepsilon}(K_\lambda) \mid \lambda \in \Lambda \}
\]

\[
\geq \omega^\alpha.
\]
As $\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda))$ cannot be a limit ordinal, we deduce that

$$\text{Sz}(B_q(K_\lambda \mid \lambda \in \Lambda)) \geq \text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) > \omega^\alpha.$$  

This proves (i)$\Rightarrow$(ii).

Suppose that (ii) holds. Then for every $\varepsilon > 0$ there is $1 < m_\varepsilon < \omega$ and $\beta_\varepsilon < \alpha$ such that $\sup\{\text{Sz}_\varepsilon/32(K_\lambda) \mid \lambda \in \Lambda\} < \omega^{\beta_\varepsilon} \cdot m_\varepsilon$. Let $d = \sup\{\text{diam}(K_\lambda) \mid \lambda \in \Lambda\}$ and for each $\varepsilon \in (0, 1)$ let $M_\varepsilon \in \mathbb{N}$ be such that $(2^q - 1)\varepsilon^q M_\varepsilon \geq 8^q d^q (m_\varepsilon - 1)$. By Lemma 4.1.5, for each $\mathcal{F} \in \Lambda^{<\infty}$ we have $\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \mathcal{F})) < \omega^{\beta_\varepsilon} \cdot M_\varepsilon$, hence

$$\sup\{\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \mathcal{F})) \mid \mathcal{F} \in \Lambda^{<\infty}\} \leq \omega^{\beta_\varepsilon} \cdot M_\varepsilon < \omega^\alpha.$$  

Thus, (ii)$\Rightarrow$(iii).

Suppose that (iii) holds. Since $U^\varepsilon_\mathcal{F} B_q(K_\lambda \mid \lambda \in \Lambda) = B_q(K_\lambda \mid \lambda \in \mathcal{F})$ for each $\mathcal{F} \in \Lambda^{<\infty}$, applying Lemma 4.1.8 with $K = B_q(K_\lambda \mid \lambda \in \Lambda)$, $\delta = \delta(\varepsilon) = \varepsilon/2$ and $\eta_6(\varepsilon) = \sup\{\text{Sz}_\varepsilon/2(U^\varepsilon_\mathcal{F} B_q(K_\lambda \mid \lambda \in \Lambda)) \mid \mathcal{F} \in \Lambda^{<\infty}\} (< \omega^\alpha)$ yields

$$\text{Sz}(B_q(K_\lambda \mid \lambda \in \Lambda)) = \sup\{\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) \mid \varepsilon > 0\}$$

$$\leq \sup \left\{ \eta_6(\varepsilon) \cdot \sigma\left( \sup\{|K_\lambda| \mid \lambda \in \Lambda, \varepsilon, \varepsilon/2, q\} \mid \varepsilon > 0\right) \right\}$$

$$\leq \omega^\alpha,$$

hence (iii)$\Rightarrow$(i). \hfill \Box

**Remark 4.1.9.** The idea that a result whose proof uses an iterated implementation of Lemma 4.1.6 (in particular, Lemma 4.1.8, whose proof features an iterated application of Lemma 4.1.6 via induction on $n < \omega$) might be used to prove the implication (iii)$\Rightarrow$(i) in the proof of Proposition 4.1.4 was essentially suggested to me by Professor Gilles Lancien; previous versions of the main results of this chapter used a slightly different argument (also using Lemma 4.1.6, but just a single direct application) and required the additional hypothesis that $K_\lambda = B_{E_\lambda}$ for all $\lambda$ (see Theorem 4.1.11).
The following result, along with Proposition 4.1.2, determines precisely the Szlenk index of a $c_0$-direct sum or $\ell_p$-direct sum of operators ($1 < p < \infty$) in terms of properties of the $\varepsilon$-Szlenk indices of the summands.

**Theorem 4.1.10.** Let $\Lambda$ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda : E_\lambda \to F_\lambda \mid \lambda \in \Lambda\}$ a uniformly bounded family of Asplund operators, $\alpha > 0$ an ordinal and $p = 0$ or $1 < p < \infty$. The following are equivalent:

(i) $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) \leq \omega^\alpha$.

(ii) $\sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha$ for every $\varepsilon > 0$.

It follows that if $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is noncompact, then

$$Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \inf \{\omega^\alpha \mid \sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for every } \varepsilon > 0\}.$$ 

**Proof.** Let $q \in [1, \infty)$ be dual to $p$ and set $T = (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$. The equivalence of (i) and (ii) follows from an application of Proposition 4.1.4 with $K_\lambda = T_\lambda^*B_{F_\lambda}$ for all $\lambda \in \Lambda$, for in this case $T^*B_{(\bigoplus_{\lambda \in \Lambda} E_\lambda)_q}^* = B_q(T_\lambda^*B_{F_\lambda}^* \mid \lambda \in \Lambda)$.

For each $\lambda \in \Lambda$, let $\alpha_\lambda$ be such that $Sz(T_\lambda) = \omega^{\alpha_\lambda}$. Let $\alpha_\Lambda = \sup\{\alpha_\lambda \mid \lambda \in \Lambda\}$ and note that the set $\{\omega^\alpha \mid \sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for every } \varepsilon > 0\} \ni \omega^{\alpha_\Lambda + 1}$ is nonempty. We have $Sz(T) \leq \inf \{\omega^\alpha \mid \sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for every } \varepsilon > 0\}$ by the implication (ii)$\Rightarrow$(i) above.

To complete the proof, we now suppose that $T$ is noncompact. As $Sz(T)$ is a power of $\omega$, it is enough to show that $Sz(T) > \omega^\beta$ holds for ordinals $\beta$ satisfying $\omega^\beta < \inf \{\omega^\alpha \mid \sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for every } \varepsilon > 0\}$. Take such $\beta$. If $\beta = 0$, then we have $Sz(T) > \omega^\beta$ by noncompactness of $T$. If $\beta > 0$, then there is $\varepsilon > 0$ so small that $Sz_\varepsilon(T) \geq \sup\{Sz_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} \geq \omega^\beta$; since $Sz_\varepsilon(T)$ cannot be a limit ordinal, we conclude that $Sz(T) \geq Sz_\varepsilon(T) > \omega^\beta$. \hfill $\square$

### 4.1.3 Applications to Banach spaces

Our first result here is the following Banach space analogue of Theorem 4.1.10 which determines precisely the Szlenk index of a $c_0$-direct sum or $\ell_p$-direct sum of Banach spaces in terms of the behaviour of the $\varepsilon$-Szlenk indices of the summand spaces.
Theorem 4.1.11. Let $\Lambda$ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ a family of Asplund spaces, $\alpha > 0$ an ordinal and $p = 0$ or $1 < p < \infty$. The following are equivalent:

(i) $\text{Sz}((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p) \leq \omega^\alpha$.

(ii) $\sup \{\text{Sz}_\varepsilon(E_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha$ for every $\varepsilon > 0$.

It follows that if $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ is infinite dimensional, then

$$\text{Sz}((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p) = \inf \{\omega^\alpha \mid \sup \{\text{Sz}_\varepsilon(E_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for every } \varepsilon > 0\}.$$ 

Proof. The conclusions of the theorem follow by taking $T_\lambda$ to be the identity operator of $E_\lambda$ for each $\lambda \in \Lambda$ in the statement of Theorem 4.1.10. □

Theorem 4.1.12. Let $\Lambda$ be a set, $E$ an infinite dimensional Banach space and $1 < p, r < \infty$. Then

$$\text{Sz}(E) = \text{Sz}(c_0(\Lambda, E)) = \text{Sz}(\ell_p(\Lambda, E)) = \text{Sz}(\ell_r(\Lambda, E)).$$

Proof. Apply Theorem 4.1.11 with $E_\lambda = E$ for all $\lambda \in \Lambda$. □

We now give the proof of Proposition 3.1.8.

Proof of Proposition 3.1.8 Our proof is based on Construction 3.2.7 and the following proposition.

Proposition 4.1.13. Let $\beta$ be an ordinal and let $G_\beta$ be as in Construction 3.2.7. Then $\text{Sz}_1(G_\beta) > \beta$.

Proposition 4.1.13 is proved in [39, p.216]. As Proposition 3.1.8 is known to be true for $\alpha = 0$ (for example, $\text{Sz}(\ell_2) = \omega$), we assume that $\alpha > 0$ and let $\beta'$ denote the least ordinal such that $\text{Sz}(G_{\beta'}) > \omega^\alpha$ (Proposition 4.1.13 guarantees the existence of $\beta'$). Then, by Proposition 2.4.2(iii), $\text{Sz}(G_{\beta'}) \geq \omega^{\alpha+1}$. By Proposition 2.4.2(v) and the definition of $\beta'$, it must be that $\beta'$ is a limit ordinal, hence $G_{\beta'} = (\bigoplus_{\beta'' < \beta'} G_{\beta''})_2$. It follows that $\text{Sz}(G_{\beta'}) = \text{Sz}((\bigoplus_{\beta'' < \beta'} G_{\beta''})_2) \leq \omega^{\alpha+1}$, where the final inequality here follows from Theorem 4.1.11 and the fact that, for all $\varepsilon > 0$,

$$\sup \{\text{Sz}_\varepsilon(G_{\beta''}) \mid \beta'' < \beta'\} \leq \sup \{\text{Sz}(G_{\beta''}) \mid \beta'' < \beta'\} \leq \omega^\alpha < \omega^{\alpha+1}.$$
It is now clear that $\text{Sz}(G^{\alpha'}) = \omega^{\alpha+1}$, so we are done. \hfill \Box

Remark 4.1.14. Implicit in the proof of Proposition 3.1.8 is the following fact: for a set $\Lambda$, Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$, $p = 0$ or $1 < p < \infty$ and $\alpha$ an ordinal satisfying $\sup\{\text{Sz}(E_\lambda) \mid \lambda \in \Lambda\} \leq \omega^\alpha$, we have $\text{Sz}(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p \leq \omega^{\alpha+1}$. This follows easily from Theorem 4.1.11, but seems to have been known for some time. For example, the separable case was established in [44, Proposition 15], and the result is also implicit in the proof of [39, Proposition 5]. Theorem 4.1.10 yields an operator theoretic analogue for uniformly bounded families of Asplund operators.

The following proposition asserts that the set of all countable values of the Szlenk index of Banach spaces is attained by the class of Banach spaces with a shrinking basis.

**Proposition 4.1.15.** Let $0 < \alpha < \omega_1$. The following are equivalent:

(i) There exists a Banach space $E$ with $\text{Sz}(E) = \omega^\alpha$.  
(ii) There exists a Banach space $E$ with a shrinking basis and $\text{Sz}(E) = \omega^\alpha$.

To prove Proposition 4.1.15, we shall call on the following result regarding subspaces and quotients, due to G. Lancien [38] and [33, Theorem III.1]:

**Proposition 4.1.16.** Let $\beta < \omega_1$ and let $E$ be a Banach space such that $\text{Sz}(E) > \beta$.  

(i) There is a separable closed subspace $F$ of $E$ such that $\text{Sz}(F) > \beta$.  
(ii) If $E^*$ is norm separable, then for every $\delta > 0$ there is a closed subspace $F$ of $E$ such that $\text{Sz}(E/F) > \beta$ and $E/F$ has a shrinking basis with basis constant not exceeding $1 + \delta$.

With the exception of the basis constant assertion of part (ii), Proposition 4.1.16 is proved in [38]. Lancien’s proof follows closely the proof of [33, Theorem III.1], and the extra assertion above regarding the basis constant is easily added to Lancien’s result using the observations regarding basis constants in the proof of [33, Theorem III.1].

Proposition 4.1.15 is an immediate consequence of the following
Proposition 4.1.17. Let $\alpha > 0$ be a countable ordinal and $E$ a Banach space with $\text{Sz}(E) = \omega^\alpha$. Then there exist closed subspaces $F \subseteq E$ and $G \subseteq \ell_2(F)$ such that $\ell_2(F)/G$ has a shrinking basis and $\text{Sz}(\ell_2(F)/G) = \omega^\alpha$.

Proof. For each $n \in \mathbb{N}$, Lemma 2.4.3 and Proposition 4.1.16(i) yield a separable closed subspace $D_n$ of $E$ such that $\text{Sz}(D_n) > \text{Sz}_1/n(E)$. Let $F = \text{span}(\bigcup_{n \in \mathbb{N}} D_n)$. Then

$$\omega^\alpha = \text{Sz}(E) = \sup_n \text{Sz}_1/n(E) \leq \sup_n \text{Sz}(D_n) \leq \text{Sz}(F) \leq \text{Sz}(E) = \omega^\alpha,$$

hence equality holds throughout. In particular, $\text{Sz}(F) = \omega^\alpha$ and, as $F$ is a separable Asplund space (indeed, $\text{Sz}(F) < \infty$), $F^*$ is norm separable by Theorem 2.3.2. For each $n \in \mathbb{N}$ let $F_n = F$. Then, by Lemma 2.4.3 and Proposition 4.1.16(ii), for each $n \in \mathbb{N}$ there is a closed subspace $G_n$ of $F_n$ such that $\text{Sz}(F_n/G_n) > \text{Sz}_1/n(E)$ and $F_n/G_n$ has a shrinking basis with basis constant not exceeding 2. Let $G$ denote the image of $\left( \bigoplus_{n \in \mathbb{N}} G_n \right)_2$ under its natural embedding into $\left( \bigoplus_{n \in \mathbb{N}} F_n \right)_2$. Then $\left( \bigoplus_{n \in \mathbb{N}} F_n \right)_2/G$ is naturally isometrically isomorphic to $\left( \bigoplus_{n \in \mathbb{N}} F_n/G_n \right)_2$. Note that $\left( \bigoplus_{n \in \mathbb{N}} F_n/G_n \right)_2$ has a shrinking basis since it is the $\ell_2$-direct sum of a countable family of Banach spaces with shrinking bases that have uniformly bounded basis constants. On the one hand, by Theorem 4.1.12 we have

$$\text{Sz}(\left( \bigoplus_{n \in \mathbb{N}} F_n \right)_2/G) \leq \text{Sz}(\left( \bigoplus_{n \in \mathbb{N}} F_n \right)_2) = \text{Sz}(F) = \omega^\alpha.$$

On the other hand,

$$\text{Sz}(\left( \bigoplus_{n \in \mathbb{N}} F_n/G_n \right)_2) = \text{Sz}(\left( \bigoplus_{n \in \mathbb{N}} F_n/G_n \right)_2) \geq \sup_n \text{Sz}_1/n(E_n) = \text{Sz}(E) = \omega^\alpha.$$

Thus $\left( \bigoplus_{n \in \mathbb{N}} F_n \right)_2/G$ has a shrinking basis and Szlenk index $\omega^\alpha$. \hfill \Box

Proposition 3.1.8 and Proposition 4.1.15 concern themselves with the existence of Banach spaces having a particular Szlenk index. Whilst the values taken by the Szlenk index on the class $\mathcal{B}$ is known ($\text{Sz}(\mathcal{B}) = \{ \omega^\alpha \mid \alpha \in \text{ORD}, \text{cf}(\alpha) < \omega_1 \}$ by Proposition 2.4.2(iii) and Proposition 3.1.10), the author is not aware of a complete
classification of the values taken by the Szlenk index on the class BAN of all Banach spaces. Proposition 2.4.2(iii) asserts that the Szlenk index of a Banach space is a power of \( \omega \). On the other hand, as the Szlenk index of a Banach space \( E \) is the supremum of the countable set \( \{ Sz_{1/n}(E) \mid n \in \mathbb{N} \} \), it follows that the Szlenk index of a Banach space is of countable cofinality. In light of these observations and Proposition 3.1.8, the classification of the values taken by the Szlenk index on BAN will be achieved if one establishes an affirmative answer to the following question, which we believe to be open:

**Question 4.1.18.** Let \( \alpha \) be an ordinal with \( cf(\alpha) = \omega \). Does there exist a Banach space with Szlenk index equal to \( \omega^\alpha \)?

### 4.2 Proof of Lemma 4.1.5

Our goal in this section is to prove Lemma 4.1.5. We proceed via a sequence of lemmas, whose general theme is to establish upper bounds (in terms of set containment) on various derived sets \( s^K, \alpha \), where \( K \) is \( w^* \)-compact, \( \alpha \) is an ordinal and \( \varepsilon > 0 \). The sets \( K \) that we shall consider are typically direct products, for it will be seen later that the set \( B_q(K_i \mid 1 \leq i \leq n) \) in the statement of Lemma 4.1.5 can be 'approximated' from above (with respect to set containment) in a convenient way by a finite union of direct products of \( w^* \)-compact sets. Indeed, this so-called approximation of \( B_q(K_i \mid 1 \leq i \leq n) \) plays a key role in our proof.

We mention another important aspect of our results in this section. As noted earlier, Lemma 4.1.5 is used to establish the implication (ii)\( \Rightarrow \)(iii) in the proof of Proposition 4.1.4. Note that in the statement of Proposition 4.1.4(iii), there is no (finite) upper bound on the cardinality of the finite sets \( F \in \Lambda^{<\infty} \). It is thus important for us in this section, when aiming for estimates of \( \varepsilon \)-Szlenk indices of direct products, to obtain estimates that are independent of the (finite) number of factors in a given direct product. Our efforts in this regard are reflected in the fact that the numbers \( M \) and \( n \) in the statement of Lemma 4.1.5 are independent of one another.
We first establish the following general result regarding the behaviour of $s^\alpha_\varepsilon$ derivatives of finite unions of $w^*$-compact sets.

**Lemma 4.2.1.** Let $E$ be a Banach space, $K_1, \ldots, K_n \subseteq E^*$ $w^*$-compact sets and $\varepsilon > 0$. Let $\alpha$ be an ordinal and $m < \omega$. Then

(i) $s^\alpha_\varepsilon(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s^\alpha_{\varepsilon/2}(K_i)$.

(ii) $s^{\alpha n}_\varepsilon(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s^{\alpha m}_\varepsilon(K_i)$.

(iii) If $\alpha$ is a limit ordinal, then $s^\alpha_\varepsilon(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s^\alpha_\varepsilon(K_i)$.

**Proof.** (i) holds trivially for $\alpha = 0$. Suppose that $\beta$ is an ordinal such that (i) holds for all $\alpha \leq \beta$ and let $x \in E^* \setminus \bigcup_{i=1}^n s^{\beta+1}_{\varepsilon/2}(K_i)$. Then for $1 \leq i \leq n$ there is a $w^*$-open set $U_i \subseteq E_i^*$ such that $x \in U_i$ and $\text{diam}(U_i \cap s^{\beta}_{\varepsilon/2}(K_i)) \leq \varepsilon/2$. It follows that for $x_1, x_2 \in (\bigcap_{i=1}^n U_i) \cap (s^{\beta}_{\varepsilon}(\bigcup_{i=1}^n K_i))$, we have

$$\|x_1 - x_2\| \leq \|x_1 - x\| + \|x - x_2\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\text{diam}((\bigcap_{i=1}^n U_i) \cap (s^{\beta}_{\varepsilon}(\bigcup_{i=1}^n K_i))) \leq \varepsilon$. In particular, $x \notin s^{\beta+1}_{\varepsilon}(\bigcup_{i=1}^n K_i)$, and so (i) passes to successor ordinals.

Suppose that $\beta$ is a limit ordinal such that (i) holds for all $\alpha < \beta$. Then

$$s^\beta_{\varepsilon}\left(\bigcup_{i=1}^n K_i\right) = \bigcap_{\alpha < \beta} s^\alpha_{\varepsilon}\left(\bigcup_{i=1}^n K_i\right) \subseteq \bigcup_{\alpha < \beta} \bigcap_{i=1}^n s^\alpha_{\varepsilon/2}(K_i). \quad (4.5)$$

Let $x \in s^\beta_{\varepsilon}(\bigcup_{i=1}^n K_i)$. Then for each $\alpha < \beta$ we may choose $i_\alpha \in \{1, \ldots, n\}$ such that $x \in s^\alpha_{\varepsilon/2}(K_{i_\alpha})$, and for some $i' \in \{1, \ldots, n\}$ the set $\{\alpha < \beta \mid i_\alpha = i'\}$ is cofinal in $\beta$. Hence

$$x \in \bigcap_{i_\alpha = i'} s^{\alpha}_{\varepsilon/2}(K_{i'}) = \bigcap_{\alpha < \beta} s^{\alpha}_{\varepsilon/2}(K_{i'}) = s^{\beta}_{\varepsilon/2}(K_{i'}) \subseteq \bigcup_{i=1}^n s^{\beta}_{\varepsilon/2}(K_i). \quad (4.6)$$

Since $x \in s^\beta_{\varepsilon}(\bigcup_{i=1}^n K_i)$ was arbitrary, (i) passes to limit ordinals, and thus holds for all ordinals $\alpha$.

Statement (ii) is trivial for $m = 0$. To see that it is true for $m = 1$, we first let
\( \mathbb{P}_k = \{ \mathcal{F} \subseteq \{1, \ldots, n\} \mid |\mathcal{F}| = k \}, \ k \in \mathbb{N}. \) It suffices to show that for all \( l < \omega, \)
\[
\mathcal{S}_l^i \left( \bigcup_{i=1}^{n} K_i \right) \subseteq \left( \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) \right) \cup \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i \right),
\]
(4.7)
Indeed, taking \( l = n \) in (4.7) gives (ii) with \( m = 1 \) (since \( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i = \emptyset \) when \( l = n \)). It is clear that (4.7) holds for \( l = 0 \). Suppose now \( l' < \omega \) is such that (4.7) holds for \( l = l' \); we show that it holds also for \( l = l' + 1 \). Let
\[
x \in \mathcal{B}^* \setminus \left( \left( \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) \right) \cup \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+2}} \bigcap_{j \in \mathcal{G}} K_j \right) \right).
\]
We want to show that \( x \notin \mathcal{S}_{l+1}^l(\bigcup_{i=1}^{n} K_i) \), so by the induction hypothesis it suffices to assume that
\[
x \in \mathcal{S}_l^l \left( \bigcup_{i=1}^{n} K_i \right) \subseteq \left( \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) \right) \cup \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i \right),
\]
hence
\[
x \in \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i \right) \setminus \left( \left( \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) \right) \cup \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+2}} \bigcap_{j \in \mathcal{G}} K_j \right) \right).
\]
By (4.8) there is (a unique) \( \mathcal{F}_x \in \mathbb{P}_{l+1} \) such that \( x \in (\bigcap_{i \in \mathcal{F}_x} K_i) \setminus (\bigcup_{l' \notin \mathcal{F}_x} K_{l'}) \). For each \( i \in \mathcal{F}_x \) let \( U_i \ni x \) be \( w^* \)-open and such that \( \text{diam}(U_i \cap K_i) \leq \epsilon \) and \( U_i \cap \bigcup_{l' \notin \mathcal{F}_x} K_{l'} = \emptyset \). Set \( U = \bigcap_{i \in \mathcal{F}_x} U_i \), so that \( U \cap \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) = \emptyset \). Then \( U \) is a \( w^* \)-neighbourhood of \( x \) and
\[
U \cap \left( \left( \bigcup_{i=1}^{n} \mathcal{S}_i(K_i) \right) \cup \left( \bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i \right) \right) = U \cap \bigcup_{i \in \mathcal{F}_x} K_i = \bigcap_{i \in \mathcal{F}_x} U_i \cap K_i
\]
has norm diameter not exceeding \( \epsilon \) (because \( \text{diam}(U_i \cap K_i) \leq \epsilon \) for \( i \in \mathcal{F}_x \)). It
follows then by (4.7) and the induction hypothesis on $l = l'$ that
\[ x \notin s_{\varepsilon} \left( \bigcup_{i=1}^{n} s_{\varepsilon}(K_i) \cup \left( \bigcup_{F \in \mathcal{P}_{l+1}} \bigcap_{i \in F} K_i \right) \right) \supseteq s_{\varepsilon}^{l+1} \left( \bigcup_{i=1}^{n} K_i \right), \]
as required. In particular, (4.7) holds for all $l < \omega$ and (ii) holds for $m = 1$.

Suppose $h < \omega$ is such that (ii) holds for all $m \leq h$. Then
\[ s_{\varepsilon}^{(h+1)n} \left( \bigcup_{i=1}^{n} K_i \right) \subseteq s_{\varepsilon}^{n} \left( \bigcup_{i=1}^{n} s_{\varepsilon}^{h}(K_i) \right) \subseteq \bigcup_{i=1}^{n} s_{\varepsilon}^{h+1}(K_i), \]
so that (ii) holds for $m = h + 1$, and thus for all $m$ by induction.

For (iii), we prove the case $n = 2$, with the general case then following from this case and a straightforward induction on $n$. So we want to show that if $\alpha$ is a limit ordinal, then
\[ s_{\varepsilon}^{\alpha}(K_1 \cup K_2) \subseteq s_{\varepsilon}(K_1) \cup s_{\varepsilon}(K_2). \tag{4.9} \]
To this end, it suffices to consider the case $\alpha = \omega^\beta$, $\beta > 0$, since the general case follows from finitely many iterations of this case. Indeed, every limit ordinal $\alpha$ is the sum of finitely many ordinals of the form $\omega^\beta$, $\beta > 0$. We proceed by induction on $\beta$. For $\beta = 1$ we note that, by (ii),
\[ s_{\varepsilon}^{\omega}(K_1 \cup K_2) = \bigcap_{m<\omega} s_{\varepsilon}^{2m}(K_1 \cup K_2) \subseteq \bigcap_{m<\omega} (s_{\varepsilon}^{m}(K_1) \cup s_{\varepsilon}^{m}(K_2)), \tag{4.10} \]
and then a similar argument to that used to obtain (4.6) from (4.5) yields (iii) for $\alpha = \omega$. Suppose now that (4.9) holds for $\alpha = \omega^\beta$, some $\beta > 0$. Then a straightforward induction on $l < \omega$ shows that for all such $l$ we have
\[ s_{\varepsilon}^{\omega^\beta l}(K_1 \cup K_2) \subseteq s_{\varepsilon}^{\omega^\beta l}(K_1) \cup s_{\varepsilon}^{\omega^\beta l}(K_2). \tag{4.11} \]
(4.11) and an argument similar to that used to obtain (4.6) from (4.5) yields
\[ s_{\varepsilon}^{\omega^\beta+1}(K_1 \cup K_2) \subseteq s_{\varepsilon}^{\omega^\beta+1}(K_1) \cup s_{\varepsilon}^{\omega^\beta+1}(K_2); \]
in particular, (iii) passes to successor ordinals. The straightforward proof that (iii) passes to limit ordinals uses, once again, a similar cofinality argument to that used to obtain (4.6) from (4.5) above.

The next three lemmas are specifically concerned with $s^*_\alpha$ derivatives of direct products of $w^*$-compact sets, considered as $w^*$-compact subsets of dual spaces of direct sums of Banach spaces. We require the following notation. Given Banach spaces $E_1, \ldots, E_n$, nonempty $w^*$-compact sets $K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^*$, $1 \leq q < \infty$ and $a_1, \ldots, a_n \geq 0$ real numbers such that $\sum_{i=1}^n a_i^q < 1$, for each $\varepsilon > 0$ define

$$A_\varepsilon := \left\{ (\varepsilon_i)_{i=1}^n \in \mathbb{R}^n \left| \sum_{i=1}^n a_i^q \varepsilon_i^q \geq \varepsilon^q \text{ and } 0 \leq \varepsilon_i \leq \text{diam}(K_i), 1 \leq i \leq n \right. \right\}.$$  

In all places where we use the notation $A_\varepsilon$, the $w^*$-compact sets $K_1, \ldots, K_n$, real numbers $a_1, \ldots, a_n$ and $1 \leq q < \infty$ will be fixed, so no ambiguity should arise from this notation. It is easy to see that $A_\varepsilon = \emptyset$ if and only if $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$. Indeed, if $(\varepsilon_i)_{i=1}^n \in A_\varepsilon$, then $\varepsilon^q \leq \sum_{i=1}^n a_i^q \varepsilon_i^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$. On the other hand, if $\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, then $(\text{diam}(K_i))_{i=1}^n \subseteq A_\varepsilon$.

We adopt the notational convention that $s^*_\alpha(K) = K$ for every ordinal $\alpha$ and $w^*$-compact $K$.

**Lemma 4.2.2.** Let $E_1, \ldots, E_n$ be Banach spaces and $K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^*$ $w^*$-compact sets. Let $1 \leq q < \infty$, $\varepsilon > 0$ and let $a_1, \ldots, a_n \geq 0$ be real numbers such that $\sum_{i=1}^n a_i^q \leq 1$. Let $p$ be predual to $q$ and consider $\prod_{i=1}^n a_i K_i$ as a subset of $(\bigoplus_{i=1}^n E_i)_p^*$. Then, for every $\delta \in (0, \varepsilon)$,

$$s_\varepsilon \left( \prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{(\varepsilon_i) \in A_\varepsilon} \prod_{i=1}^n a_i s_{\varepsilon_i}(K_i).$$

**Proof.** We first suppose that $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$. Then $s_\varepsilon(\prod_{i=1}^n a_i K_i)$ is empty since $\text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon$. The assertion of the lemma follows.

Suppose now that $\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, so that $A_\varepsilon' \neq \emptyset$ for $0 < \varepsilon' \leq \varepsilon$. Let
\[ \delta \in (0, \varepsilon), \quad (a_i x_i)^n_{i=1} \in s_{\varepsilon}(\prod_{i=1}^n a_i K_i) \quad \text{and, for } 1 \leq i \leq n, \quad \delta_i := \inf \{ \text{diam}(K_i \cup U_i) \mid U_i \text{ a } w^*\text{-neighbourhood of } x_i \} . \]

Since \((a_i x_i)^n_{i=1} \in s_{\varepsilon}(\prod_{i=1}^n a_i K_i)\), it follows from the definition of \(\delta_i\) (1 \( \leq i \leq n\)) that \(\sum_{i=1}^n a_i^q \delta_i^q \geq \varepsilon^q > \delta^q\). Let \(f : \{1, \ldots, n\} \rightarrow \mathbb{R}\) be such that \(\sum_{i=1}^n a_i^q f(i)^q \geq \delta^q\) and \(f(i) \in \{0\} \cup (0, \delta_i)\) for each \(i\) (note that \([0, \delta_i]\) is empty whenever \(\delta_i = 0\)). We claim that with \(f\) so defined, \(x_i \in s_{f(i)}(K_i)\) for 1 \( \leq i \leq n\). Indeed, if \(\delta_i = 0\) then \(f(i) = 0\), hence \(x_i \in K_i = s_{f(i)}(K_i)\) by convention. On the other hand, if \(\delta_i > 0\) then for all \(w^*\)-open \(U_i \ni x_i\) we have \(\text{diam}(K_i \cap U_i) \geq \delta_i > f(i)\), hence \(x_i \in s_{f(i)}(K_i)\) in this case too. Note that \((f(i))_{i=1}^n \in A_{\delta}^p\) since \(f(i) \leq \delta_i \leq \text{diam}(K_i)\) for all \(i\) and \(\sum_{i=1}^n a_i^q f(i)^q \geq \delta^q\), hence

\[
(a_i x_i)^n_{i=1} \subseteq \bigcup_{i=1}^n a_i s_{f(i)}(K_i) .
\]

\hfill \(\Box\)

**Lemma 4.2.3.** Let \(E_1, \ldots, E_n\) be Banach spaces and \(K_1 \subseteq E_1^*\), \(K_n \subseteq E_n^*\) \(w^*\)-compact sets. Let \(1 \leq q < \infty\), \(\varepsilon > 0\) and let \(a_1, \ldots, a_n \geq 0\) be real numbers such that \(\sum_{i=1}^n a_i^q \leq 1\). Let \(p\) be predual to \(q\) and consider \(\prod_{i=1}^n a_i K_i\) as a subset of \((\bigoplus_{i=1}^n E_i)^p\). Then, for every \(\delta \in (0, \varepsilon)\), \(0 < m < \omega\) and ordinal \(\alpha\),

\[
s^{\omega^\alpha \cdot m}_{\varepsilon}
left(\prod_{i=1}^n a_i K_i\right)
\subseteq
\bigcup_{(\varepsilon_i, \ldots, \varepsilon_{i,m}) \in A_{\delta/2}} \prod_{i=1}^n a_i s^{\omega^\alpha}_{\varepsilon_i,m}(s^{\omega^\alpha}_{\varepsilon_i,m-1}(\ldots s^{\omega^\alpha}_{\varepsilon_i,1}(K_i) \ldots)) .
\]

**Proof.** If \(\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q\), then \(s^{\omega^\alpha \cdot m}_{\varepsilon}(\prod_{i=1}^n a_i K_i)\) is empty since \(\omega^\alpha \cdot m \geq 1\) and \(\text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon\). The assertion of the lemma follows.

Now suppose that \(\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q\), so that \(A_{\epsilon'} \neq \emptyset\) for \(0 < \epsilon' \leq \varepsilon\). For \(\alpha = 0\) and \(m = 1\), (4.12) is a consequence of Lemma 4.2.2. Suppose that \(\alpha\) is an ordinal such that (4.12) holds for \(m = 1, 2, \ldots, k\), for some \(0 < k < \omega\). We will show that (4.12) holds for \(\alpha\) and \(m = k + 1\). Fix \(\delta \in (0, \varepsilon)\) and note that \(A_{(\epsilon + \delta)/4} \subseteq A_{\delta/2}\) since \(\delta/2 < (\epsilon + \delta)/4\). We now detail a method that assigns to each \((\varepsilon_i)^n_{i=1} \in A_{(\epsilon + \delta)/4}\) an element \((\bar{\varepsilon}_i)^n_{i=1}\) of a certain finite subset of \(A_{\delta/2}\). For
4.2 Proof of Lemma 4.1.5

\[(\varepsilon_i)_{i=1}^n \in A(\varepsilon+\delta)/4 \text{ and } 1 \leq i \leq n, \text{ define } j_i := \max \{j \in \mathbb{N} \cup \{0\} \mid j(\varepsilon - \delta) \leq 4\varepsilon_i\} \text{ and set } \bar{\varepsilon}_i = j_i(\varepsilon - \delta)/4. \text{ Note that } \bar{\varepsilon}_i \leq \varepsilon_i \leq \text{diam}(K_i) \text{ and}
\]

\[
\left(\sum_{i=1}^{n} a_i^q \bar{\varepsilon}_i^{q} \right)^{1/q} \geq \left(\sum_{i=1}^{n} a_i^q \varepsilon_i^{q} \right)^{1/q} - \left(\sum_{i=1}^{n} a_i^q (\varepsilon_i - \bar{\varepsilon}_i)^{q} \right)^{1/q} \geq \frac{\varepsilon + \delta}{4} - \frac{\varepsilon - \delta}{4} = \frac{\delta}{2},
\]

hence \((\bar{\varepsilon}_i)_{i=1}^n \in A_{\delta/2}. \) Moreover, for \((\varepsilon_{i,1})_{i=1}^n, \ldots, (\varepsilon_{i,m})_{i=1}^n \in A(\varepsilon+\delta)/4 \) we have

\[
s^\omega_{\varepsilon_{i,m}} \left( s^\omega_{\varepsilon_{i,m-1}} (\ldots s^\omega_{\varepsilon_{i,1}} (K_i) \ldots) \right) \subseteq s^\omega_{\varepsilon_{i,m}} \left( s^\omega_{\varepsilon_{i,m-1}} (\ldots s^\omega_{\varepsilon_{i,1}} (K_i) \ldots) \right),
\]

all \(1 \leq i \leq n. \) Let \(A = \{(\bar{\varepsilon}_i)_{i=1}^n \mid (\varepsilon_i)_{i=1}^n \in A(\varepsilon+\delta)/4\} \subseteq A_{\delta/2}. \) Then \(A\) is finite, with

\[
|A| \leq \left[ \frac{4 \cdot \max_{1 \leq i \leq n} \text{diam}(K_i)}{\varepsilon - \delta} + 1 \right]^n.
\]

The finiteness of \(A\) will allow us to invoke Lemma 4.2.1 in the next step of our proof.

To complete our demonstration that (4.12) holds for \(m = k+1, \) we henceforth treat the cases \(\alpha = 0\) and \(\alpha > 0\) separately.

If \(\alpha = 0\) then the induction hypothesis, (4.13), Lemma 4.2.1(i) and Lemma 4.2.2 yield

\[
s_{\varepsilon}^{k+1} \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq s_\varepsilon \left( \bigcup_{(\varepsilon_{i,1}, \ldots, (\varepsilon_{i,k}) \in A(\varepsilon+\delta)/4} \prod_{i=1}^{n} a_i s_{\varepsilon_{i,k}} (s_{\varepsilon_{i,k-1}} (\ldots s_{\varepsilon_{i,1}} (K_i) \ldots)) \right)
\]

\[
\subseteq s_\varepsilon \left( \bigcup_{(\varepsilon_{i,1}, \ldots, (\varepsilon_{i,k}) \in A(\varepsilon+\delta)/4} \prod_{i=1}^{n} a_i s_{\varepsilon_{i,k}} (s_{\varepsilon_{i,k-1}} (\ldots s_{\varepsilon_{i,1}} (K_i) \ldots)) \right)
\]

\[
\subseteq \bigcup_{(\varepsilon_{i,1}, \ldots, (\varepsilon_{i,k}) \in A(\varepsilon+\delta)/4} s_{\varepsilon/2} \left( \prod_{i=1}^{n} a_i s_{\varepsilon_{i,k}} (s_{\varepsilon_{i,k-1}} (\ldots s_{\varepsilon_{i,1}} (K_i) \ldots)) \right)
\]

\[
\subseteq \bigcup_{(\varepsilon_{i,1}, \ldots, (\varepsilon_{i,k}), (\varepsilon_{i,k+1}) \in A_{\delta/2}} \prod_{i=1}^{n} a_i s_{\varepsilon_{i,k+1}} (s_{\varepsilon_{i,k}} (\ldots s_{\varepsilon_{i,1}} (K_i) \ldots)),
\]

On the other hand, if \(\alpha > 0\) then it follows from the induction hypothesis, (4.13)
and Lemma 4.2.1(iii) that

\[
\mathcal{S}_\varepsilon^{\omega^{\alpha}(k+1)} \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq \mathcal{S}_\varepsilon^{\omega^{\alpha}} \left( \bigcup_{(\varepsilon_i,1),\ldots,(\varepsilon_i,k) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}} \left( s_{\varepsilon_i^{\omega^{\alpha}-1}}^{\omega^{\alpha}}(K_i) \ldots \right) \right)
\]

\[
\subseteq \mathcal{S}_\varepsilon^{\omega^{\alpha}} \left( \bigcup_{(\varepsilon_i,1),\ldots,(\varepsilon_i,k) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}} \left( s_{\varepsilon_i^{\omega^{\alpha}-1}}^{\omega^{\alpha}}(K_i) \ldots \right) \right)
\]

\[
\subseteq \bigcup_{(\varepsilon_i,1),\ldots,(\varepsilon_i,k) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}} \left( s_{\varepsilon_i^{\omega^{\alpha}-1}}^{\omega^{\alpha}}(K_i) \ldots \right)
\]

\[
\subseteq \bigcup_{(\varepsilon_i,1),\ldots,(\varepsilon_i,k),\ldots,(\varepsilon_i,k+1) \in A_{\delta/2}} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}} \left( s_{\varepsilon_i^{\omega^{\alpha}}}(K_i) \ldots \right),
\]

as we would like.

Finally, suppose that \( \beta \) is a nonzero ordinal (either limit or successor) such that

\[(4.12) \]

holds for all \( m < \omega \) and \( \alpha < \beta \); we show that \((4.12)\) then holds for \( m = 1 \) and \( \alpha = \beta \). Fix \( \delta \in (0, \varepsilon) \) and let \( A \) be defined as above. Then, since \( A \subseteq A_{\delta/2} \), to complete the induction it suffices to show that

\[
\mathcal{S}_\varepsilon^{\omega^{\alpha}} \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq \bigcup_{(\varepsilon_i) \in A} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}}(K_i). \tag{4.14}
\]

To prove (4.14), we shall establish the following two inclusions:

\[
\mathcal{S}_\varepsilon^{\omega^{\alpha}} \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq \bigcap_{(l, \alpha) \in (0, \omega) \times \beta} \bigcup_{(\varepsilon_i) \in A} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}-1}(K_i) \tag{4.15}
\]

and

\[
\bigcap_{(l, \alpha) \in (0, \omega) \times \beta} \bigcup_{(\varepsilon_i) \in A} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}-1}(K_i) \subseteq \bigcup_{(\varepsilon_i) \in A} \prod_{i=1}^{n} a_i s_{\varepsilon_i}^{\omega^{\alpha}}(K_i). \tag{4.16}
\]

We first deal with (4.15). To this end, let

\[
x \in \mathcal{S}_\varepsilon^{\omega^{\alpha}} \left( \prod_{i=1}^{n} a_i K_i \right) = \bigcap_{(m, \alpha) \in (0, \omega) \times \beta} \mathcal{S}_\varepsilon^{\omega^{\alpha}-m} \left( \prod_{i=1}^{n} a_i K_i \right).
\]
Then, since \( \varepsilon + k < \varepsilon \), it follows from the induction hypothesis and (4.13) that

\[
x \in \bigcap_{(m,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\varepsilon_{1},1),\ldots,\varepsilon_{m} \in A_{(\varepsilon+1)/4}} \prod_{i=1}^{n} a_{i} s_{\varepsilon_{i},m \alpha}^{\omega \alpha} (s_{\varepsilon_{i},m-1 \alpha}^{\omega \alpha} (\ldots s_{\varepsilon_{i},1 \alpha}^{\omega \alpha} (K_{i}) \ldots))
\]

\[
\subseteq \bigcap_{(m,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\varepsilon_{1},1),\ldots,\varepsilon_{m} \in A} \prod_{i=1}^{n} a_{i} s_{\varepsilon_{i},m \alpha}^{\omega \alpha} (s_{\varepsilon_{i},m-1 \alpha}^{\omega \alpha} (\ldots s_{\varepsilon_{i},1 \alpha}^{\omega \alpha} (K_{i}) \ldots)).
\]

So, for each \((m, \alpha) \in (0, \omega) \times \beta\) there are \((\varepsilon_{1},1,m,\alpha)_{i=1}^{n}, \ldots, (\varepsilon_{m},m,\alpha)_{i=1}^{n} \in A\) such that

\[
x \in \prod_{i=1}^{n} a_{i} s_{\varepsilon_{i},m \alpha}^{\omega \alpha} (s_{\varepsilon_{i},m-1 \alpha}^{\omega \alpha} (\ldots s_{\varepsilon_{i},1 \alpha}^{\omega \alpha} (K_{i}) \ldots)). \tag{4.17}
\]

Suppose that \( l \in (0, \omega) \) and \( \alpha < \beta \) and set \( m_{l} = |A| \cdot l \). Then there is a subset \( J_{l,\alpha} \subseteq \{1, 2, \ldots, m_{l}\} \) with \( |J_{l,\alpha}| = l \) and \( \{|(\varepsilon_{i},j,m,\alpha)_{i=1}^{n} | j \in J_{l,\alpha}\} = 1 \); let \((\tilde{\varepsilon}_{i},l,\alpha)_{i=1}^{n}\) denote the unique element of \( \{(\varepsilon_{i},j,m,\alpha)_{i=1}^{n} | j \in J_{l,\alpha}\} \). We may write \( J_{l,\alpha} = \{j_{1} < j_{2} < \ldots < j_{l}\} \), and then by (4.17) we have, in particular,

\[
x \in \prod_{i=1}^{n} a_{i} s_{\tilde{\varepsilon}_{i},l \alpha}^{\omega \alpha} (s_{\tilde{\varepsilon}_{i},l \alpha-1 \alpha}^{\omega \alpha} (\ldots s_{\tilde{\varepsilon}_{i},1 \alpha}^{\omega \alpha} (K_{i}) \ldots))
\]

\[
\subseteq \prod_{i=1}^{n} a_{i} s_{\tilde{\varepsilon}_{i},l \alpha}^{\omega \alpha-1} (\ldots s_{\tilde{\varepsilon}_{i},1 \alpha}^{\omega \alpha} (K_{i}) \ldots))
\]

\[
= \prod_{i=1}^{n} a_{i} s_{\tilde{\varepsilon}_{i},l \alpha}^{\omega \alpha-1} (K_{i})
\]

\[
\subseteq \bigcup_{(\tilde{\varepsilon}_{i}) \in A} \prod_{i=1}^{n} a_{i} s_{\tilde{\varepsilon}_{i}}^{\omega \alpha-1} (K_{i}).
\]

As \( l \in (0, \omega) \) and \( \alpha < \beta \) were arbitrary, (4.15) follows.
and for each \( l \in (0, \omega) \) and \( \alpha < \beta \) let \((\varepsilon_i, (l, \alpha))_{i=1}^n \in A\) be such that

\[ y \in \prod_{i=1}^n a_i s_{\varepsilon_i, (l, \alpha)}^{\omega \cdot l} (K_i). \]

For \((\varepsilon_i)_{i=1}^n \in A\), let \( A[\{(\varepsilon_i)_{i=1}^n\}] = \{ \omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta, (\varepsilon_i, (l, \alpha))_{i=1}^n = (\varepsilon_i)_{i=1}^n \}\). Since \( \{\omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta\}\) is cofinal in \( \omega^\beta \) and \( \{A[(\varepsilon_i)_{i=1}^n] \mid (\varepsilon_i)_{i=1}^n \in A\}\) is a finite partition of \( \{\omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta\}\), there exists \((\bar{\varepsilon}_i)_{i=1}^n \in A\) such that \( A[(\varepsilon_i)_{i=1}^n]\) is cofinal in \( \omega^\beta \). It follows that

\[ y \in \bigcap_{\xi \in A[(\bar{\varepsilon}_i)_{i=1}^n]} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega \cdot \alpha} (K_i) \subseteq \prod_{i=1}^n a_i \left( \bigcup_{\xi \in \omega^\beta} s_{\bar{\varepsilon}_i}^{\omega \cdot \alpha} (K_i) \right) \]

\[ = \prod_{i=1}^n a_i \left( \bigcup_{\xi \in \omega^\beta} s_{\bar{\varepsilon}_i}^{\omega \cdot \alpha} (K_i) \right) \]

\[ = \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega \cdot \alpha} (K_i) \]

\[ \subseteq \bigcup_{(\bar{\varepsilon}_i)_{i=1}^n \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega \cdot \alpha} (K_i). \]

At last, the proof of Lemma 4.2.3 is complete. \(\square\)

**Lemma 4.2.4.** Let \( E_1, \ldots, E_n \) be Banach spaces and let \( K_1 \subseteq E_1, \ldots, K_n \subseteq E_n^* \) be nonempty, \( w^* \)-compact sets. Let \( 1 \leq q < \infty, \varepsilon > 0 \) and let \( a_1, \ldots, a_n \geq 0 \) be real numbers such that \( \sum_{i=1}^n a_i^q \leq 1 \). Let \( d = \max_{1 \leq i \leq n} \text{diam}(K_i) \) and let \( m \in \mathbb{N} \) be such that \( M \geq m \geq 2 \) and \( (2^q - 1)\varepsilon M \geq 8^q d^q (m - 1) \). Let \( p \) be predual to \( q \) and consider \( \prod_{i=1}^n a_i K_i \) as a subset of \( (\bigoplus_{i=1}^n E_i)^* \). If \( \alpha \) is an ordinal such that \( s_{(2^q - 1)\varepsilon}^{\omega \cdot \alpha}(K_i) = 0 \) for all \( 1 \leq i \leq n \), then \( s_{\varepsilon/8}^{\omega \cdot \alpha}(\prod_{i=1}^n a_i K_i) = 0 \).

**Proof.** If \( \varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q \), then \( \text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon \). As \( \omega^\alpha \cdot M \geq 1 \), the set \( s_{\varepsilon/8}^{\omega \cdot \alpha}(\prod_{i=1}^n a_i K_i) \) is empty and the assertion of the lemma follows.

So suppose now that \( \varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q \). Then \( A_{\varepsilon'} \neq \emptyset \) for \( 0 < \varepsilon' \leq \varepsilon \). Applying Lemma 4.2.3 with \( \delta = \varepsilon/2 \), we see that \( s_{\varepsilon/8}^{\omega \cdot \alpha}(\prod_{i=1}^n a_i K_i) \) is contained in
a union of sets of the form

\[ \prod_{i=1}^{n} a_i s_{\varepsilon, M}^{\alpha}(s_{\varepsilon, M-1}^{\alpha}(\ldots s_{\varepsilon, 1}^{\alpha}(K_i)\ldots)), \tag{4.18} \]

where \((\varepsilon_{i,1})_{i=1}^{n}, (\varepsilon_{i,2})_{i=1}^{n}, \ldots, (\varepsilon_{i,M})_{i=1}^{n} \in A_{\varepsilon/4}\). For each such product (4.18),

\[ a_1^q \left( \sum_{j=1}^{M} \varepsilon_{1,j}^q \right) + a_2^q \left( \sum_{j=1}^{M} \varepsilon_{2,j}^q \right) + \ldots + a_n^q \left( \sum_{j=1}^{M} \varepsilon_{n,j}^q \right) \geq \frac{M \varepsilon^q}{4^q}. \]

Since \(\sum_{i=1}^{n} a_i^q \leq 1\), there is \(h \in \{1, \ldots, n\}\) such that \(\sum_{j=1}^{M} \varepsilon_{h,j}^q \geq M \varepsilon^q / 4^q\). We claim that at least one of the following two conditions holds for such \(h\):

(a) There exist \(1 \leq j_1 < j_2 < \ldots < j_m \leq M\) with \(\min \{\varepsilon_{h,j_1}, \ldots, \varepsilon_{h,j_m}\} \geq \varepsilon/8\).

(b) There exists \(j \leq M\) such that \(\varepsilon_{h,j} > d\).

Indeed, suppose that (a) does not hold. Then there are distinct \(j_h, \ldots, j_{m-1}\) in \(\{1, \ldots, M\}\) such that \(\varepsilon_{h,j} < \varepsilon/8\) whenever \(j \in \{1, \ldots, M\} \setminus \{j_1, \ldots, j_{m-1}\}\). It follows then that

\[ \sum_{k=1}^{m-1} \varepsilon_{h,j_k}^q > \frac{M \varepsilon^q}{4^q} - (M - m + 1) \left( \frac{\varepsilon}{8} \right)^q \]

\[ = M \left( \left( \frac{\varepsilon}{4} \right)^q - \left( \frac{\varepsilon}{8} \right)^q \right) + (m - 1) \left( \frac{\varepsilon}{8} \right)^q \]

\[ > M \left( \left( \frac{\varepsilon}{4} \right)^q - \left( \frac{\varepsilon}{8} \right)^q \right) \]

\[ \geq d^q (m - 1). \]

Thus \(\varepsilon_{h,j_k}^q > d^q\) for some \(k \leq m - 1\), hence \(\varepsilon_{h,j_k} > d\) for some \(k \leq m - 1\). In particular, (b) holds whenever (a) does not.

If (b) holds, then the factor \(a_h s_{\varepsilon, M}^{\alpha}(s_{\varepsilon, M-1}^{\alpha}(\ldots s_{\varepsilon, 1}^{\alpha}(K_h)\ldots))\) is empty since \(\text{diam}(K_h) \leq d < \varepsilon_{h,j}\) for \(j\) satisfying (b). It follows then that the product \(\prod_{i=1}^{n} a_i s_{\varepsilon, M}^{\alpha}(s_{\varepsilon, M-1}^{\alpha}(\ldots s_{\varepsilon, 1}^{\alpha}(K_i)\ldots))\) is empty also. On the other hand, if (a) holds then

\[ s_{\varepsilon, M}^{\alpha}(s_{\varepsilon, M-1}^{\alpha}(\ldots s_{\varepsilon, 1}^{\alpha}(K_h)\ldots)) \subseteq s_{\varepsilon, M}^{\alpha}(s_{\varepsilon, M-1}^{\alpha}(\ldots s_{\varepsilon, j_m}^{\alpha}(K_h)\ldots)) \subseteq s_{\varepsilon/8}^{\alpha,m}(K_h). \]
We conclude that $s_{\omega^n-m}(\prod_{i=1}^n a_i K_i)$ is contained in a union of direct products of the form (4.18), with each such direct product having a factor contained in a scalar multiple of one of the sets $s_{\omega^n-m}(K_i)$, $1 \leq i \leq n$. From this it is clear that if $s_{\omega^n-m}(K_i) = \emptyset$ for all $1 \leq i \leq n$, then $s_{\omega^n-m}(\prod_{i=1}^n a_i K_i) \subseteq \emptyset$. \hfill \Box

The next and final lemma required for our proof of Lemma 4.1.5 shows how we can put a set $B_q(K_i \mid 1 \leq i \leq n)$ inside a finite union of direct products of $w^*$-compact sets in a way that will be useful for us.

**Lemma 4.2.5.** Let $E_1, \ldots, E_n$ be Banach spaces, $K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^*$ nonempty, absolutely convex, $w^*$-compact sets, $1 \leq q < \infty$, $l \in \mathbb{N}$ and $L = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q^n}$. Then

$$B_q(K_i \mid 1 \leq i \leq n) \subseteq \bigcup_{(k_i)_{i=1}^n \in L} \prod_{i=1}^n \frac{k_i}{l} K_i.$$ 

**Proof.** Let $(a_i)_{i=1}^n \in B_{\ell_q^n}$ and set $j_i = \inf\{j \in \mathbb{N} \mid l |a_i| < j\}$, $1 \leq i \leq n$. Then $j_i - 1 \leq l |a_i|$ for all $i$, hence $\|(j_i)_{i=1}^n\|_{\ell_q^n} \leq \|(a_i)_{i=1}^n\|_{\ell_q^n} + n^{1/q} \leq l + n^{1/q}$. In particular, $(j_i)_{i=1}^n \in L$. As the sets $K_i$, $1 \leq i \leq n$, are absolutely convex, we have $a_i K_i \subseteq \frac{k_i}{l} K_i$ for all $i$, hence $\prod_{i=1}^n a_i K_i \subseteq \prod_{i=1}^n \frac{k_i}{l} K_i$. It follows that

$$B_q(K_i \mid 1 \leq i \leq n) = \bigcup_{(a_i)_{i=1}^n \in B_{\ell_q^n}} \prod_{i=1}^n a_i K_i \subseteq \bigcup_{(k_i)_{i=1}^n \in L} \prod_{i=1}^n \frac{k_i}{l} K_i. \hfill \Box

We note a few points of interest regarding the sets $\bigcup_{(k_i)_{i=1}^n \in L} \prod_{i=1}^n \frac{k_i}{l} K_i$ from Lemma 4.2.5. For each $l \in \mathbb{N}$, let $L_l = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q^n}$. Then the intersection of the collection $\{\bigcup_{(k_i)_{i=1}^n \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i\}_{l \in \mathbb{N}}$ is precisely $B_q(K_i \mid 1 \leq i \leq n)$; this follows from the observation that for $l \in \mathbb{N}$, each point of $\bigcup_{(k_i)_{i=1}^n \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i$ is no greater than $n^{1/q} \cdot l^{-1} \cdot \max\{\text{diam}(K_i) \mid 1 \leq i \leq n\}$ in norm distance from $B_q(K_i \mid 1 \leq i \leq n)$. We may thus think of $\{\bigcup_{(k_i)_{i=1}^n \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i\}_{l \in \mathbb{N}}$ as a sequence of increasingly closer approximations to the set $B_q(K_i \mid 1 \leq i \leq n)$, and our need to closely approximate $B_q(K_i \mid 1 \leq i \leq n)$ is reflected by our choice of $l$ in the following proof of Lemma 4.1.5.

**Proof of Lemma 4.1.5.** Fix $\delta \in (0, \varepsilon/16)$. Let $l = \lceil 16\delta n^{1/q}(\varepsilon - 16\delta)^{-1} \rceil$ and let
4.2 Proof of Lemma 4.1.5

\[ L = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q}. \]

By Lemma 4.2.5 and the hypothesis of Lemma 4.1.5,

\[
s_{\varepsilon}^{\omega,M}(\bigcup_{(k_i)\in L} \prod_{i=1}^{n} \frac{k_i}{l}K_i) \supseteq s_{\varepsilon}^{\omega,M}(B_q(K_i \mid 1 \leq i \leq n)) \neq \emptyset.
\]

Thus, since \( L \) is finite, by Lemma 4.2.1(i) there exists \((h_i)_{i=1}^{n} \in L\) such that

\[
s_{\varepsilon/2}^{\omega,M}\left(\prod_{i=1}^{n} \frac{h_i}{l}K_i\right) \neq \emptyset. \tag{4.19}
\]

Let \( \rho = (1 + \frac{n^{1/q}}{l})^{-1} \). By (4.19) and the homogeneity of the derivations \( s_{\varepsilon}^{\gamma} \) (where \( \gamma \) is an ordinal and \( \varepsilon' > 0 \)), we have

\[
s_{\rho\varepsilon/2}^{\omega,M}\left(\prod_{i=1}^{n} \frac{\rho h_i}{l}K_i\right) = \rho s_{\varepsilon/2}^{\omega,M}\left(\prod_{i=1}^{n} \frac{h_i}{l}K_i\right) \neq \emptyset. \tag{4.20}
\]

Thus, since \( \|\prod_{i=1}^{n} \frac{h_i}{l}\|_{\ell_q} \leq 1 \), it follows from (4.20) and Lemma 4.2.4 that there is \( i \leq n \) such that \( s_{\rho\varepsilon/16}^{\omega,m}(K_i) \neq \emptyset \). As \( \rho\varepsilon/16 \geq \delta \), we have \( s_{\delta}^{\omega,m}(K_i) \supseteq s_{\rho\varepsilon/16}^{\omega,m}(K_i) \neq \emptyset \).

This completes the proof. \( \square \)
Chapter 5

Quantitative factorisation of Asplund operators

An important, basic question in operator ideal theory is whether a given operator ideal \( \mathcal{I} \) has the factorisation property; that is, whether every element of \( \mathcal{I} \) factors through a Banach space whose identity operator belongs to \( \mathcal{I} \). The most well-known and widely applied result in this direction is the celebrated Davis-Figiel-Johnson-Pelczyński factorisation theorem [13] asserting that every weakly compact operator factors through a reflexive Banach space. In the absence of the factorisation property, one may then ask whether \( \mathcal{I} \) satisfies some nontrivial 'weak' factorisation property. For example, W. Johnson has shown in [32] that there exists a separable, reflexive Banach space \( G \) with the following property: for arbitrary Banach spaces \( E \) and \( F \) and arbitrary \( T \in \mathcal{F}(E, F) \), there exist operators \( T_1 \in \mathcal{F}(E, G) \) and \( T_2 \in \mathcal{F}(G, F) \) with \( T = T_2T_1 \). In the current chapter we study factorisation properties of the operator ideals \( \mathcal{I}\mathcal{L}_\alpha, \alpha \in \text{ORD} \). In some sense, the results presented here can be considered a quantitative refinement of the independent efforts of O. Reînov, S. Heinrich and C. Stegall showing that the operator ideal of Asplund operators possesses the factorisation property (cf. Section 2.3).

Throughout, we rely heavily on techniques developed in Chapter 4. Indeed, forming direct sums of Banach spaces and their operators is central to the proofs of
both the positive and the negative results that we present in the current chapter. We also note before we begin that the results of Section 5.1 make significant use of the factorisation methods developed by S. Heinrich in [30] and stated in Section 2.2.

5.1 $\mathcal{L}_\alpha$ has the $\mathcal{L}_{\alpha+1}$-factorisation property

Our first task in this section is to establish the following weak factorisation result for the operator ideals $\mathcal{L}_\alpha$. As an operator is Asplund if and only if it is $\alpha$-Szlenk for some ordinal $\alpha$, the theorem below can be thought of as a quantitative factorisation result for Asplund operators.

Theorem 5.1.1. Let $\alpha$ be an ordinal and $T \in \mathcal{L}_\alpha$. Then $T$ factors through a Banach space whose Szlenk index is at most $\omega^{\alpha+1}$.

Before embarking on a proof of Theorem 5.1.1, we mention a similar result due to B. Bossard. It is shown in [11, Theorem 3.9] that there is a universal function $\varphi : \omega_1 \rightarrow \omega_1$ such that for any separable Banach space $E$ and Asplund operator $T : E \rightarrow C[0, 1]$, there exists a Banach space $G$ and operators $A : E \rightarrow G$ and $B : G \rightarrow C[0, 1]$ such that $G$ has a shrinking basis, $\text{Sz}(G) \leq \varphi(\text{Sz}(T))$ and $T = BA$ (recall from Proposition 3.1.15 that an Asplund operator with separable range has countable Szlenk index). It will be shown at the end of Section 5.2 that $\varphi$ necessarily exceeds the identity function of $\omega_1$ at uncountably many points of $\omega_1$.

We shall deduce Theorem 5.1.1 from the following proposition.

Proposition 5.1.2. Let $\Lambda$ and $\Upsilon$ be sets, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\upsilon \mid \upsilon \in \Upsilon\}$ families of Banach spaces, $p = 0$ or $1 < p < \infty$, $T : (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p \rightarrow (\bigoplus_{\upsilon \in \Upsilon} F_\upsilon)_p$ an operator and $\alpha > 0$ an ordinal. The following are equivalent:

(i) $\text{Sz}(T) \leq \omega^\alpha$.

(ii) $\sup\{\text{Sz}_\varepsilon(\mathcal{F}U_{\mathcal{F}}) \mid \mathcal{F} \in \Lambda^{\prec \infty}\} < \omega^\alpha$ for every $\varepsilon > 0$.

(iii) $\sup\{\text{Sz}_\varepsilon(QG\mathcal{F}) \mid G \in \Upsilon^{\prec \infty}\} < \omega^\alpha$ for every $\varepsilon > 0$.

(iv) $\sup\{\text{Sz}_\varepsilon(QG\mathcal{F}U_{\mathcal{F}}) \mid \mathcal{F} \in \Lambda^{\prec \infty}, G \in \Upsilon^{\prec \infty}\} < \omega^\alpha$ for every $\varepsilon > 0$. 
5.1 $\mathcal{S}_\alpha$ has the $\mathcal{S}_{\alpha+1}$-factorisation property

Let us now see how Theorem 5.1.1 follows from Proposition 5.1.2. We begin by fixing $1 < p < \infty$. By Theorem 3.1.2 and Theorem 2.2.7, it suffices to show that $(\mathcal{S}_\alpha, \mathcal{S}_{\alpha+1})$ is a $\Sigma_p$-pair. To this end, let $(E_m)_{m \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ be sequences of Banach spaces and suppose $T \in \mathcal{B}((\bigoplus_{m \in \mathbb{N}} E_m)_p, (\bigoplus_{n \in \mathbb{N}} F_n)_p)$ is such that $QgTU_\mathcal{F} \in \mathcal{S}_\alpha$ for all $\mathcal{F}, \mathcal{G} \in \mathbb{N}^{<\infty}$. Then

$$\forall \varepsilon > 0 \sup \{Sz_\varepsilon(QgTU_\mathcal{F}) \mid \mathcal{F}, \mathcal{G} \in \mathbb{N}^{<\infty}\} \leq \omega^\alpha < \omega^{\alpha+1},$$

hence $T \in \mathcal{S}_{\alpha+1}$ by Proposition 5.1.2, and we are done.

To prove Proposition 5.1.2, we draw on several preliminary results. The first of these is the following variant of [26, Proposition 2.2], which can be useful for obtaining an upper estimate on the Szlenk index of an operator.

**Proposition 5.1.3.** Let $E$ and $F$ be Banach spaces, $T : E \rightarrow F$ an operator and $\beta$ an ordinal. Suppose that for every $\varepsilon > 0$ there exists $\beta_\varepsilon < \omega^\beta$ and $\delta_\varepsilon \in (0, 1)$ such that $s_{\varepsilon}^{\beta_\varepsilon}(T^*B_{F^*}) \subseteq \delta_\varepsilon T^*B_{F^*}$. Then $Sz(T) \leq \omega^\beta$.

**Proof.** Fix $\varepsilon > 0$. We claim that $s_{\varepsilon}^{\beta_\varepsilon n}(T^*B_{F^*}) \subseteq \delta_\varepsilon^n T^*B_{F^*}$ for all $n \in \mathbb{N}$. Indeed, it is true for $n = 1$ by assumption, and if true for $n \leq k$ then

$$s_{\varepsilon}^{\beta_\varepsilon(k+1)}(T^*B_{F^*}) \subseteq s_{\varepsilon}^{\beta_\varepsilon}(s_{\varepsilon}^{\beta_\varepsilon k}(T^*B_{F^*})) \subseteq s_{\varepsilon}^{\beta_\varepsilon} s_{\varepsilon}^{\beta_\varepsilon k}(T^*B_{F^*}) \subseteq \delta_\varepsilon^{k+1} T^*B_{F^*},$$

so that the above claim holds by induction on $n$.

For each $\varepsilon > 0$ let $N_\varepsilon \in \mathbb{N}$ be large enough that $s_{\varepsilon}^{\beta_\varepsilon N_\varepsilon}(T^*B_{F^*}) \subseteq \frac{\varepsilon}{2} B_{F^*}$. Then $s_{\varepsilon}^{\beta_\varepsilon N_\varepsilon+1}(T^*B_{F^*}) = \emptyset$ for each $\varepsilon > 0$, hence $Sz(T) \leq \sup_{\varepsilon > 0} (\beta_\varepsilon \cdot N_\varepsilon + 1) \leq \omega^\beta$. □

The next lemma is concerned with the behaviour under an adjoint operator of the $\varepsilon$-Szlenk derivations of a $w^*$-compact set in the dual of a direct sum of Banach spaces.

**Lemma 5.1.4.** Let $\mathcal{Y}$ be a set, $\{F_v \mid v \in \mathcal{Y}\}$ a family of Banach spaces, $E$ a Banach space, $1 \leq q < \infty$, $p$ predual to $q$, $K \subseteq (\bigoplus_{v \in \mathcal{Y}} F_v)^*$ a nonempty $w^*$-compact set, $T : E \rightarrow (\bigoplus_{v \in \mathcal{Y}} F_v)_p$ a nonzero operator and $\varepsilon > 0$. Let $\alpha$ be an ordinal and let
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\( x \in s^\alpha_\varepsilon(T^*K) \). Then there is \( y \in s^\alpha_{\varepsilon/(2\|T\|)}(K) \) such that \( T^*y = x \). Further, if \( S \subseteq Y \) is such that \( \|Q^*_S V^*_S y\|^q > |K|^q - (\varepsilon/(8 \|T\|))^q \), then \( T^*Q^*_S V^*_S y \in s^\alpha_{\varepsilon/(4)}(T^*Q^*_S V^*_S K) \).

**Proof.** The lemma is clearly true for \( \alpha = 0 \). We now assume it true for some \( \alpha = \gamma + 1 \) and show that it is also true for \( \alpha = \gamma + 1 \). Let \( x \in s^\gamma_\varepsilon(T^*K) = s^\gamma_\varepsilon(T^*K) \). Then there exists a net \((x_i)_{i \in I} \) in \( s^\gamma_\varepsilon(T^*K) \) such that \( x_i \xrightarrow{w^*} x \) and \( \|x_i - x\| > \varepsilon/2 \) for all \( i \in I \) (for example, take \( I \) to be the set of all \( w^* \)-neighbourhoods of \( x \), ordered by reverse set inclusion). For each \( i \in I \) let \( y_i \in s^{\gamma+1}_{\varepsilon/(2\|T\|)}(K) \) be such that \( T^*y_i = x_i \).

Passing to a subnet, we may assume that \((y_i)_{i \in I} \) has a \( w^* \)-limit \( y \in s^{\gamma+1}_{\varepsilon/(2\|T\|)}(K) \), and then \( T^*y = x \). Moreover, since \( \|y_i - y\| \geq \|x_i - x\| / \|T\| > \varepsilon/(2 \|T\|) \) for all \( i \), we have \( y \in s^{\gamma+1}_{\varepsilon/(2\|T\|)}(K) \). Now suppose that \( \|Q^*_S V^*_S y\|^q > |K|^q - (\varepsilon/(8 \|T\|))^q \), where \( S \subseteq Y \). Passing to a subnet, we may assume\(^1\) that \( \|Q^*_S y\|^q > |K|^q - (\varepsilon/(8 \|T\|))^q \) for all \( i \). Then, since \( \|z - Q^*_S V^*_S z\|^q + \|Q^*_S V^*_S z\|^q = \|z\|^q \) for \( z \in \bigoplus_{t \in T} E_t \), it follows that for all \( i \) we have

\[
\|y_i - Q^*_S V^*_S y_i\| \leq (\|y_i\|^q - \|Q^*_S V^*_S y_i\|^q)^{1/q} \leq (|K|^q - \|Q^*_S V^*_S y_i\|^q)^{1/q} \leq \frac{\varepsilon}{8 \|T\|},
\]

hence \( \|y - Q^*_S V^*_S y\| \leq \varepsilon/(8 \|T\|) \) by \( w^* \)-lower semicontinuity. Thus, for all \( i \),

\[
\|T^*Q^*_S V^*_S y_i - T^*Q^*_S V^*_S y\| \\
\geq \|T^*y_i - T^*y\| - \|T^*y - T^*Q^*_S V^*_S y\| - \|T^*y_i - T^*Q^*_S V^*_S y_i\| \\
\geq \|x_i - x\| - \|T\| \|y - Q^*_S V^*_S y\| - \|T\| \|y_i - Q^*_S V^*_S y_i\| \\
= \frac{\varepsilon}{2} - 2 \|T\| \cdot \frac{\varepsilon}{8 \|T\|} \\
= \frac{\varepsilon}{4}.
\]

Since \( T^*Q^*_S V^*_S y_i \in s^\gamma_{\varepsilon/(4)}(T^*Q^*_S V^*_S K) \) for each \( i \) by the induction hypothesis, and since \( T^*Q^*_S V^*_S y_i \xrightarrow{w^*} T^*Q^*_S V^*_S y \), we have \( T^*Q^*_S V^*_S y \in s^{\gamma+1}_{\varepsilon/(4)}(T^*Q^*_S V^*_S K) \). Thus, in

\(^1\)Indeed, the following is true: Let \( E \) be a Banach space, \((I, \leq)\) a directed set and \( g : I \rightarrow E^* \) a \( w^* \)-convergent net, with \( w^* \)-limit \( x^* \), say. Then, for every \( \varepsilon \in (0, \|x^*\|) \), \( g \) admits a subnet whose image is disjoint from the ball \( B_{E^*} \varepsilon \). To see why this is so, let \( x_\varepsilon \in B_{E^*} \) be such that \( \langle x^*, x_\varepsilon \rangle \geq \varepsilon \) and let \( j_\varepsilon \in I \) be such that \( |\langle g(j), x_\varepsilon \rangle| \geq \varepsilon \) whenever \( j \leq j_\varepsilon \). Then the restriction \( g|_{\{i \in I^* \mid j \leq j_\varepsilon \}} \) provides the desired subnet of \( g \).
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particular, the assertion of the lemma passes to successor ordinals.

Finally, let $\gamma$ be a limit ordinal and suppose that the assertion of the lemma holds whenever $\alpha < \gamma$. Let $x \in s_\varepsilon^\alpha(T^*K) = \bigcap_{\alpha < \gamma} s_\varepsilon^\alpha(T^*K)$ and for each $\alpha < \gamma$ let $y_\alpha \in s_\varepsilon^{\alpha\downarrow(2\|T\|)}(K)$ be such that $T^*y_\alpha = x$. By $w^*$-compactness of $K$, there is a directed set $J$ and a mapping $f : J \to \gamma$ such that $(y_{f(j)})_{j \in J}$ is a $w^*$-convergent subnet of $(y_\alpha)_{\alpha < \gamma}$. Let $y$ denote the $w^*$-limit of $(y_{f(j)})_{j \in J}$. Then $T^*y = x$, and since $f(J)$ is cofinal in $\gamma$ (by definition of a subnet), $y \in \bigcap_{j \in J} s_\varepsilon^{f(j)}(T^*QsV^K) = s_\varepsilon^{f(j)}(T^*QsV^K)$. Now suppose that $\|QsV^K\| > \|K\| - (\varepsilon/(8\|T\|))^q$, where $S \subseteq \mathcal{T}$. Passing to a subnet, we may assume that $\|QsV^Ky_j\| > \|K\| - (\varepsilon/(8\|T\|))^q$ for all $j$, hence by the induction hypothesis $T^*QsV^Ky_{f(j)} \in s_\varepsilon^{f(j)}(T^*QsV^K)$ for all $j$. Again, by the cofinality of $f(J)$ in $\gamma$,

$$T^*QsV^Ky = w^* - \lim_{j \to f} T^*QsV^Ky_{f(j)} \in \bigcap_{j \in J} s_\varepsilon^{f(j)}(T^*QsV^K) = s_\varepsilon^{f(j)}(T^*QsV^K).$$

The assertion of the lemma thus passes to limit ordinals, and we are done. \qed

At last, we are ready to prove Proposition 5.1.2.

**Proof of Proposition 5.1.2.** We prove (i)$\Rightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(i), assuming, as we may, that $T \neq 0$. Throughout, $q \in [1, \infty)$ is dual to $p$.

To prove (i)$\Rightarrow$(ii), suppose $\varepsilon > 0$ is such that $\sup \{Sz_e(TU_\mathcal{F}) \mid \mathcal{F} \in \Lambda^{<\infty} \} \geq \omega^\alpha$. We want to show that $Sz(T) > \omega^\alpha$, so to this end note that by Lemma 3.1.5 we have

$$Sz_\varepsilon/2(T) = Sz_\varepsilon/2(T^*B(\bigoplus_{v \in \mathcal{T}} F_v)_p) \geq \sup \{Sz_e(U^*T^*B(\bigoplus_{v \in \mathcal{T}} F_v)_p) \mid \mathcal{F} \in \Lambda^{<\infty} \}
= \sup \{Sz_e(TU_\mathcal{F}) \mid \mathcal{F} \in \Lambda^{<\infty} \}
\geq \omega^\alpha.$$

As $Sz_\varepsilon/2(T)$ cannot be a limit ordinal, it follows that $Sz(T) \geq Sz_\varepsilon/2(T) > \omega^\alpha$.

We now show that (ii)$\Rightarrow$(iv). Let $\mathcal{F} \in \Lambda^{<\infty}$. Then for $G \in \mathcal{T}^{<\infty}$ we have $U^*T^*QgB(\bigoplus_{v \in \mathcal{T}} F_v)_p \subseteq U^*T^*B(\bigoplus_{v \in \mathcal{T}} F_v)_p$, hence $Sz_e(QgTU_\mathcal{F}) \leq Sz_e(TU_\mathcal{F})$ for all
\( \mathcal{G} \in \mathcal{T}^{<\infty} \) and \( \varepsilon > 0 \). Thus, for each \( \varepsilon > 0 \),

\[
\sup \left\{ S_{\varepsilon}(Q \mathcal{G}TU_\mathcal{F}) \mid \mathcal{F} \in \Lambda^{<\infty}, \mathcal{G} \in \mathcal{T}^{<\infty} \right\} \leq \sup \left\{ S_{\varepsilon}(TU_\mathcal{F}) \mid \mathcal{F} \in \Lambda^{<\infty} \right\},
\]

and the implication (ii)\( \Rightarrow \) (iv) follows.

Suppose that (iv) holds and fix \( \mathcal{G} \in \mathcal{T}^{<\infty} \). An application of Lemma 4.1.8 with 
\( K = T^*Q^*B(\Theta_{v \in T} F_v)_p \), \( \delta = \delta(\varepsilon) = \varepsilon/2 \) and

\[
\eta(\varepsilon) = \sup \left\{ S_{\varepsilon/2}(Q \mathcal{G}TU_\mathcal{F}) \mid \mathcal{F} \in \Lambda^{<\infty}, \mathcal{G} \in \mathcal{T}^{<\infty} \right\} (< \omega^\alpha)
\]
yields

\[
S_{\varepsilon}(T^*Q^*B(\Theta_{v \in T} F_v)_p) \leq \eta(\varepsilon) \cdot \sigma(\|T\|, \varepsilon, \varepsilon/2, q).
\]

As \( \mathcal{G} \in \mathcal{T}^{<\infty} \) was arbitrary and \( \eta(\varepsilon) \) and \( \sigma(\|T\|, \varepsilon, \varepsilon/2, q) \) do not depend on our choice of \( \mathcal{G} \), we deduce that

\[
\sup \{ S_{\varepsilon}(Q \mathcal{G}T) \mid \mathcal{G} \in \mathcal{T}^{<\infty} \} \leq \eta(\varepsilon) \cdot \sigma(\|T\|, \varepsilon, \varepsilon/2, q) < \omega^\alpha,
\]

hence (iv)\( \Rightarrow \) (iii).

Suppose that (iii) holds. The implication (iii)\( \Rightarrow \) (i) will follow from Proposition 5.1.3 if we can show that for every \( \varepsilon > 0 \) there is \( \beta_\varepsilon < \omega^\alpha \) and \( \delta_\varepsilon \in (0, 1) \) with

\[
s_{\varepsilon}(T^*B(\Theta_{v \in T} F_v)_p) \leq \delta_\varepsilon T^*B(\Theta_{v \in T} F_v)_p.
\]

If \( \varepsilon \geq 2 \|T\| \), then \( s_{\varepsilon}(T^*B(\Theta_{v \in T} F_v)_p) = 0 \), hence \( \beta_\varepsilon = 1 \) and any \( \delta_\varepsilon \in (0, 1) \) suffice. So it remains to find suitable \( \beta_\varepsilon \) and \( \delta_\varepsilon \) for \( \varepsilon \in (0, 2 \|T\|) \). For each \( \varepsilon \in (0, 2 \|T\|) \), let \( \xi_\varepsilon = \sup \{ S_{\varepsilon}(Q \mathcal{G}T) \mid \mathcal{G} \in \mathcal{T}^{<\infty} \} \).

As \( T^*Q^*B(\Theta_{v \in T} F_v)_p = T^*Q^*V^*B(\Theta_{v \in T} F_v)_p \) for each \( \mathcal{G} \in \mathcal{T}^{<\infty} \), it follows that

\[
\sup \{ S_{\varepsilon}(V \mathcal{G}Q \mathcal{G}T) \mid \mathcal{G} \in \mathcal{T}^{<\infty} \} = \xi_\varepsilon \text{ for each } \varepsilon \in (0, 2 \|T\|).
\]

Now fix \( \varepsilon \in (0, 2 \|T\|) \) and suppose \( x \in s_{\varepsilon}^{\xi_\varepsilon/4}(T^*B(\Theta_{v \in T} F_v)_p) \) (if \( s_{\varepsilon}^{\xi_\varepsilon/4}(T^*B(\Theta_{v \in T} F_v)_p) = 0 \), then taking \( \beta_\varepsilon = \xi_\varepsilon/4 \) and any \( \delta_\varepsilon \in (0, 1) \) will do). Since \( s_{\varepsilon}^{\xi_\varepsilon/4}(T^*Q^*V^*B(\Theta_{v \in T} F_v)_p) = 0 \) for all \( \mathcal{G} \in \mathcal{T}^{<\infty} \), an appeal to Lemma 5.1.4 gives us \( y \in s_{\varepsilon}^{\xi_\varepsilon/4}(B(\Theta_{v \in T} F_v)_p) \) such that

\[
T^*y = x \text{ and } \|y\|^q = \sup_{\mathcal{G} \in \mathcal{T}^{<\infty}} \|Q^*V^*_\mathcal{G}y\|^q \leq 1 - \left( \frac{\varepsilon}{8 \|T\|} \right)^q.
\]
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In particular, since $x \in s_{\xi/4}^\xi \left( T^* B(\bigoplus_{v \in T} F_v)^p \right)$ was arbitrary,

$$s_{\xi/4}^\xi \left( T^* B(\bigoplus_{v \in T} F_v)^p \right) \subseteq \left( 1 - \left( \frac{\varepsilon}{8 \|T\|} \right)^q \right)^{1/q} T^* B(\bigoplus_{v \in T} F_v)^p.$$

Taking $\beta_\varepsilon = \xi/4$ and $\delta_\varepsilon = (1 - (\varepsilon/(8 \|T\|))^q)^{1/q}$ for each $\varepsilon \in (0, 2 \|T\|)$ completes the proof. □

Corollary 5.1.5. Let $\alpha$ be a nonzero limit ordinal. Then $\bigcup_{\beta < \alpha} \mathcal{L}_\beta$ is an operator ideal having the factorisation property.

Proof. The fact that $\mathcal{L}_\gamma \subseteq \mathcal{L}_\zeta$ whenever $\gamma \leq \zeta$ ensures that $\bigcup_{\beta < \alpha} \mathcal{L}_\beta$ is an operator ideal. To see that $\bigcup_{\beta < \alpha} \mathcal{L}_\beta$ has the factorisation property, suppose $T \in \bigcup_{\beta < \alpha} \mathcal{L}_\beta$. Then there is $\beta < \alpha$ such that $T \in \mathcal{L}_\beta$. By Theorem 5.1.1 there exists a Banach space $E$ such that $Sz(E) \leq \omega^{\beta+1}$ and the identity operator of $E$ factors $T$. As $\beta + 1 < \alpha$, the identity operator of $E$ belongs to $\bigcup_{\beta < \alpha} \mathcal{L}_\beta$. □

In light of Theorem 3.1.2 and Proposition 2.2.7, it is of interest to know for which ordinals $\alpha$ the pair $(\mathcal{L}_\alpha, \mathcal{L}_\alpha)$ is a $\Sigma_p$-pair for some $1 < p < \infty$. We now work towards the complete classification of such ordinals, for which we shall make use of Szlenk's construction of Banach spaces [63] (c.f. Construction 3.2.7).

Proposition 5.1.6. Let $p = 0$ or $1 < p < \infty$ and let $\alpha$ be an ordinal of countable cofinality. Then there exist Banach spaces $D_n \in Sz\iota_\alpha$, $n \in \mathbb{N}$, such that $Sz((\bigoplus_{n \in \mathbb{N}} D_n)_p) > \omega^\alpha$.

Proof. First we assume that $\alpha = 0$. Let $D_n = K \in Sz\iota$ for all $n$. Then $(\bigoplus_{n \in \mathbb{N}} D_n)_p$ is infinite dimensional, hence $Sz((\bigoplus_{n \in \mathbb{N}} D_n)_p) > \omega > \omega^0$ by assertions (iii) and (vi) of Proposition 2.4.2. The assertion of the proposition thus holds for $\alpha = 0$.

We now assume $\alpha > 0$. For each ordinal $\beta$, let $G_\beta$ be as in Construction 3.2.7. By Proposition 4.1.13, the ordinal $\beta' = \inf \{ \beta \mid Sz(G_\beta) > \omega^\alpha \}$ exists. Moreover, Proposition 2.4.2(v) implies that $\beta'$ is a limit ordinal, hence $G_{\beta'} = (\bigoplus_{\beta < \beta'} G_\beta)_2$. An appeal to Theorem 4.1.11 yields $\varepsilon > 0$ such that $\sup_{\beta < \beta'} Sz_\varepsilon(G_\beta) \geq \omega^\alpha$. That $cf(\alpha)$ is countable ensures that $cf(\omega^\alpha)$ is countable, and so there exists a sequence
(βₙ) in β' such that supₙ∈ℕ Szₑ(□ βₙ) ≥ ω¹. For each n ∈ ℕ set Dₙ = G βₙ. Then Dₙ ∈ SzL α for all n and Szₑ((□ₙ∈ℕ Dₙ)p) ≥ ωα. As Szₑ((□ₙ∈ℕ Dₙ)p) cannot be a limit ordinal, Sz((□ₙ∈ℕ Dₙ)p) ≥ Szₑ((□ₙ∈ℕ Dₙ)p) > ωα.

We now give the classification of those ordinals α for which (Szₑ, Szₑ) is a Σₚ-pair.

**Proposition 5.1.7.** Let α be an ordinal and 1 < p < ∞. The following are equivalent:

(i) α is of uncountable cofinality.

(ii) (Szₑ, Szₑ) is a Σₚ-pair.

**Proof.** Assume first that (i) holds. Let (Eₘ)m and (Fₙ)n be sequences of Banach spaces and let T ∈的操作符ₜ((□ₙ∈ℕ Eₘ)p), (□ₙ∈ℕ Fₙ)p) be such that Qₑ TUₑ ∈ Szₑ for all F, G ∈ N<∞. By Proposition 3.1.10 and Proposition 2.4.2(iii), for each pair (F, G) ∈ N<∞ × N<∞ there is α(F, G) < α such that Sz(Qₑ TUₑ) = ωα(F, G). Let α' = sup {α(F, G) | F, G ∈ N<∞}, so that α' ≤ α. Then, since N<∞ × N<∞ is countable, cf(α') is countable also, hence α' < α. As α is of uncountable cofinality, it is also a limit ordinal, hence α' + 1 < α. Moreover,

∀ε > 0 sup {Szₑ(Qₑ TUₑ) | F, G ∈ N<∞} ≤ ωα' < ωα'+1,

and so Proposition 5.1.2 yields T ∈ Szₑα'+1 ⊆ Szₑα. Thus (Szₑ, Szₑ) is a Σₚ-pair.

Now suppose that (i) fails; that is, the cofinality of α is countable. Then by Proposition 5.1.6 there exist Dₙ ∈ SzL α, n ∈ ℕ, with Sz((□ₙ∈ℕ Dₙ)p) > ωα. Let S denote the identity operator of (□ₙ∈ℕ Dₙ)p, and for each E ∈ N<∞ let Sₑ denote the identity operator of (□ₙ∈ℕ Dₙ)p. Then Sₑ ∈ Szₑ for all E ∈ N<∞ by Proposition 2.4.2(v). The fact that Sₑ factors Qₑ SUₑ for all F, G ∈ N<∞ thus implies that Qₑ SUₑ ∈ Szₑ for all F, G ∈ N<∞. However S ∉ Szₑ by definition, hence (Szₑ, Szₑ) is not a Σₚ-pair. □
5.2 Counterexamples to the factorisation property

The following is an immediate consequence of Theorem 3.1.2, Corollary 2.2.8 and Proposition 5.1.7.

**Corollary 5.1.8.** Let $\alpha$ be an ordinal of uncountable cofinality. Then $\mathcal{I}^{\alpha}$ has the factorisation property.

The following is open:

**Problem 5.1.9.** Let $\alpha$ be an ordinal. Are the following equivalent?

(i) $\alpha$ is of uncountable cofinality.

(ii) $\mathcal{I}^{\alpha}$ has the factorisation property.

With Corollary 5.1.8 we have already just established the implication (i)⇒(ii) of Problem 5.1.9. The remainder of this chapter is motivated by the search for a proof of the reverse implication (ii)⇒(i). Although we do not obtain the full answer, we give some further partial results and anticipate that further development of the ideas presented here may eventually lead to a complete solution.

### 5.2 Counterexamples to the factorisation property

Our goal in this section is to prove the following theorem.

**Theorem 5.2.1.** Let $\beta$ be an ordinal of countable cofinality. Then $\mathcal{I}^{\omega_\beta}$ does not have the factorisation property.

One of the main ingredients in our construction of counterexamples to the factorisation property is the following result concerning the submultiplicity of the $\varepsilon$-Szlenk index of a Banach space, due to G. Lancien.

**Proposition 5.2.2 ([39, p.212]).** Let $E$ be a Banach space and $\varepsilon, \varepsilon' > 0$. Then

$$S_{\varepsilon \varepsilon'}(E) \leq S_{\varepsilon}(E) \cdot S_{\varepsilon'}(E).$$
Some remarks concerning the use of Proposition 5.2.2 are in order. Suppose that \( E \) is an infinite-dimensional Asplund space and let \( \alpha \) denote the unique ordinal satisfying \( Sz(E) = \omega^\alpha \). The submultiplicity of the \( \varepsilon \)-Szlenk index seems to be of use in analysis of \( E \) only in the case that the ordinal \( \omega^\alpha \) is closed under ordinal multiplication, which is true if and only if \( \alpha \) is closed under ordinal addition, which is true if and only if \( \alpha = \omega^{\beta} \) for some ordinal \( \beta \). Indeed, suppose that \( \alpha \) is not a power of \( \omega \); then there is \( \gamma < \alpha \) such that \( \gamma \cdot 2 \geq \alpha \). Let \( \varepsilon \) be so small that \( Sz_\varepsilon(E) \geq \omega^\gamma \). Then for \( 0 < \varepsilon', \varepsilon'' \leq \varepsilon \) we have

\[
Sz_{\varepsilon',\varepsilon''}(E) \leq \omega^\alpha \leq \omega^{\gamma \cdot 2} \leq Sz_{\varepsilon'}(E) \cdot Sz_{\varepsilon''}(E),
\]

so that submultiplicity of the \( \varepsilon \)-Szlenk index of \( E \) is essentially trivial in this case. In particular, in this case the submultiplicity of the \( \varepsilon \)-Szlenk index of \( E \) does not yield any information regarding the growth of \( Sz_\varepsilon(E) \) as \( \varepsilon \) goes to zero. By contrast, if \( \alpha = \omega^{\beta} \) for some \( \beta \), then it is possible to use the submultiplicity of the \( \varepsilon \)-Szlenk index to obtain a certain growth condition on \( Sz_\varepsilon(E) \), and similar growth conditions on the \( \varepsilon \)-Szlenk indices of operators in \( Op(Sz_{\omega^\alpha}) \) (see Proposition 5.2.4 below). By constructing an element of \( \mathcal{J} \mathcal{L}_\alpha \) that cannot satisfy any such growth condition, we will show that the containment \( Op(Sz_{\omega^\alpha}) \subseteq \mathcal{J} \mathcal{L}_\alpha \) is proper.

**Definition 5.2.3.** Let \( \beta \) be an ordinal of countable cofinality. A sequence \( (\beta_n)_{n \in \mathbb{N}} \) in \( \omega^\beta \) is called a *superadditive cofinal sequence for \( \omega^\beta \)* if \( \{\beta_n \mid n \in \mathbb{N}\} \) is cofinal in \( \omega^\beta \) and \( \beta_{n_1} + \beta_{n_2} \leq \beta_{n_1+n_2} \) for all \( n_1, n_2 \in \mathbb{N} \) (including when \( n_1 = n_2 \)).

It is easy to see that each ordinal \( \beta \) of countable cofinality admits a superadditive cofinal sequence for \( \omega^\beta \). Indeed, for such an ordinal \( \beta \) we have that \( \omega^\beta \) is also of countable cofinality, and so we may choose a sequence \( (\gamma_m)_{m \in \mathbb{N}} \) in \( \omega^\beta \) such that \( \{\gamma_m \mid m \in \mathbb{N}\} \) is cofinal in \( \omega^\beta \). We shall inductively define a strictly increasing sequence \( (m_n)_{n \in \mathbb{N}} \) in \( \mathbb{N} \) such that \( (\beta_n = \gamma_{m_n})_{n \in \mathbb{N}} \) is a superadditive cofinal sequence for \( \omega^\beta \). Let \( m_1 = 1 \). Suppose \( j \in \mathbb{N} \) and that \( m_n \) has been defined for \( 1 \leq n \leq j \).
5.2 Counterexamples to the factorisation property

Define

\[ m_{j+1} := \inf \{ m \in \mathbb{N} \mid m > m_j \text{ and } \gamma_m \geq \max\{\gamma_{m_1} + \gamma_{m_2} \mid n_1 + n_2 = j + 1\} \}. \]

With the sequence \((m_n)_{n \in \mathbb{N}}\) so defined, \((\beta_n = \gamma_m)_{n \in \mathbb{N}}\) is a superadditive cofinal sequence for \(\omega^\beta\).

For a nonzero ordinal \(\beta\) of countable cofinality, the following proposition establishes a necessary condition for membership of the operator ideal \(\text{Op}(SZL_{\omega^\beta})\), and will be used in the proof of Theorem 5.2.1. The proposition asserts that elements of \(\text{Op}(SZL_{\omega^\beta})\) must possess a certain restricted-growth property defined in terms of arbitrary superadditive cofinal sequences for \(\omega^\beta\).

**Proposition 5.2.4.** Let \(\beta\) be a nonzero ordinal of countable cofinality. For each \(T \in \text{Op}(SZL_{\omega^\beta})\) and superadditive cofinal sequence for \(\omega^\beta\), \((\beta_n)_{n \in \mathbb{N}}\) say, there exists \(n_0 \in \mathbb{N}\) such that

\[
S_{Z_{1/2^n}}(T) \leq \omega^{\beta n_0 - n}
\]

for all \(n \in \mathbb{N}\).

**Proof.** The result if trivial if \(T = 0\), so we assume henceforth that \(T \neq 0\). Let \(D\), \(E\) and \(F\) be Banach spaces and \(A \in \mathcal{B}(E, D)\) and \(B \in \mathcal{B}(D, F)\) operators such that \(D \in SZL_{\omega^\beta}, T = BA\) and, without loss of generality, \(\|B\| \leq 1\). The bound \(\|B\| \leq 1\) and Lemma 3.1.5 ensure that

\[
\forall \varepsilon > 0 \quad S_{Z_\varepsilon}(T) \leq S_{Z_\varepsilon}(A) \leq S_{Z_{\varepsilon/(2\|A\|)}}(D). \quad (5.1)
\]

Let \(s = \inf \{n \in \mathbb{N} \mid S_{Z_{1/(2\|A\|)}}(D) \leq \omega^{\beta n}\}\), \(t = \inf \{n \in \mathbb{N} \mid S_{Z_{1/2}}(D) \leq \omega^{\beta n}\}\) and set \(n_0 = s + t\) (the existence of such \(s\) and \(t\) is guaranteed by the cofinality of \(\{\beta_n \mid n \in \mathbb{N}\}\) in \(\omega^\beta\), Lemma 2.4.3 and our assumption that \(\beta > 0\)). Then by
Proposition 5.2.2 and (5.1), for each \( n \in \mathbb{N} \) we have

\[
\text{Sz}_{1/2^n}(T) \leq \text{Sz}_{1/(2^{n+1}\|A\|)}(D) \leq \left( \text{Sz}_{1/2}(D) \right)^n \leq \omega^{\beta_n} \cdot \omega^{\beta_0 \cdot n} \leq \omega^{\beta_{n_0 \cdot n}}.
\]

For the remainder of this section, let \( r = 0 \) or \( 1 < r < \infty \) be fixed. We now detail a construction, inspired by Szlenk’s construction (c.f. Construction 3.2.7), that takes a given operator \( T \) and yields an operator \( T_\alpha \) for each ordinal \( \alpha \) in such a way that \( T_0 = T \) and \( T_\alpha \) is obtained via direct sums of predecessors in the construction. For Banach spaces \( D \) and \( G \) and an operator \( S \in \mathcal{B}(D, G) \), we write \( S[n] = (\bigoplus_{i=1}^n S)_1 \) for each \( n \in \mathbb{N} \) and \( S^+ = (\bigoplus_{n \in \mathbb{N}} S[n])^r \).

Construction 5.2.5. For Banach spaces \( E \) and \( F \) and an operator \( T : E \rightarrow F \), define \( T_0 = T \), \( T_{\alpha+1} = (T_\alpha)^+ \) for each ordinal \( \alpha \) and \( T_\alpha = (\bigoplus_{\xi < \alpha} T_\xi)^r \) whenever \( \alpha \) is a limit ordinal.

With respect to Construction 5.2.5, note that \( \|T_\alpha\| = \|T\| \) for all ordinals \( \alpha \). Our counterexamples to the factorisation property shall be obtained as direct sums of operators obtained via this construction. For this we shall require some estimates on the Szlenk and \( \varepsilon \)-Szlenk indices of the operators \( T_\alpha \) in terms of Sz(T).

For a noncompact Asplund operator \( T \), let \( \alpha_T \) denote the unique ordinal satisfying \( \text{Sz}(T) = \omega^{\alpha_T} \). Then we may write \( \alpha_T = \eta_T + \omega^{\zeta_T} \), where \( \zeta_T \) is uniquely determined by the Cantor normal form of \( \alpha_T \) and \( \eta_T = \inf \{ \eta \mid \alpha_T = \eta + \omega^{\zeta_T} \} \). The following result gives the required estimates of \( \text{Sz}(T_\alpha) \) and \( \text{Sz}_\varepsilon(T_\alpha) \), \( \alpha \in \text{ORD} \).

Proposition 5.2.6. Let \( T \) be a noncompact Asplund operator.

(i) Suppose \( \varepsilon > 0 \) is so small that \( \text{Sz}_\varepsilon(T) > \omega^{\eta_T} \). Then \( \text{Sz}_\varepsilon(T_\alpha) > \omega^{\eta_T + \alpha} \) for every ordinal \( \alpha \).

(ii) \( \text{Sz}(T_\alpha) = \text{Sz}(T) \) for all \( \alpha < \omega^{\zeta_T} \).

To prove part (i) of Proposition 5.2.6, we require the following lemma.
Lemma 5.2.7. Let $E_1, \ldots, E_n$ be Banach spaces and let $K_1 \subseteq E_1^*$, \ldots, $K_n \subseteq E_n^*$ be w*-compact sets. Consider $\prod_{i=1}^n K_i$ as a subset of $(\bigoplus_{i=1}^n E_i)_1^* = (\bigoplus_{i=1}^n E_i^*)_\infty$. Then for all $\varepsilon > 0$, ordinals $\alpha$ and $1 \leq j \leq n$,

$$K_1 \times \ldots \times K_{j-1} \times s_\varepsilon^\alpha(K_j) \times K_{j+1} \times \ldots \times K_n \subseteq s_\varepsilon^\alpha \left( \prod_{i=1}^n K_i \right). \quad (5.2)$$

It follows that for all $\varepsilon > 0$ and ordinals $\alpha$,

$$\prod_{i=1}^n s_\varepsilon^\alpha(K_i) \subseteq s_\varepsilon^{\alpha+n} \left( \prod_{i=1}^n K_i \right). \quad (5.3)$$

Proof. We prove (5.2), with (5.3) then following from $n$ applications of (5.2). Trivially, (5.2) holds for $\alpha = 0$. We now suppose that $\beta$ is an ordinal such that (5.2) holds for $\alpha = \beta$, and show that (5.2) then holds for $\alpha = \beta + 1$. Fix $j \in \{1, \ldots, n\}$. Let $(k_1, \ldots, k_n) \in \prod_{i=1}^n K_i$ be such that $k_j \in s_\varepsilon^{\beta+1}(K_j)$ (if $s_\varepsilon^{\beta+1}(K_j)$ is empty then we are done) and let $V \ni (k_1, \ldots, k_n)$ be w*-open. Then there are w*-open sets $V_i \subseteq E_i^*$, $1 \leq i \leq n$, such that $(k_1, \ldots, k_n) \in V_1 \times \ldots \times V_n \subseteq V$. For $1 \leq l \leq m \leq n$ we shall write $K^{l,m} = \prod_{i=l}^m K_i$ and $W^{l,m} = \prod_{i=l}^m (V_i \cap K_i)$. Assuming $1 < j < n$ (the argument for the other two cases being similar), we have

$$\operatorname{diam} \left( V \cap s_\varepsilon^\beta \left( \prod_{i=1}^n K_i \right) \right) \geq \operatorname{diam} \left( \prod_{i=1}^n V_i \cap (K^{1,j-1} \times s_\varepsilon^\beta(K_j) \times K^{j+1,n}) \right)$$

$$= \operatorname{diam} \left( W^{1,j-1} \times (V_j \cap s_\varepsilon^\beta(K_j)) \times W^{j+1,n} \right)$$

$$\geq \operatorname{diam} (V_j \cap s_\varepsilon^\beta(K_j))$$

$$> \varepsilon.$$  

It follows that $(k_1, \ldots, k_n) \in s_\varepsilon^{\beta+1}(\prod_{i=1}^n K_i)$, thus (5.2) holds for $\alpha = \beta + 1$.

Now suppose that $\beta$ is a limit ordinal and that (5.2) holds for every $\alpha < \beta$. 
Assuming once again, for notational convenience, that \(1 < j < n\), we have

\[
K^{1,j-1} \times s^{\alpha}_{\varepsilon}(K_j) \times K^{j+1,n} = K^{1,j-1} \times \left( \bigcap_{\alpha < \beta} s^{\alpha}_{\varepsilon}(K_j) \right) \times K^{j+1,n}
\]

\[
= \bigcap_{\alpha < \beta} (K^{1,j-1} \times s^{\alpha}_{\varepsilon}(K_j) \times K^{j+1,n})
\]

\[
\subseteq \bigcap_{\alpha < \beta} s^{\alpha}_{\varepsilon}\left( \prod_{i=1}^{n} K_i \right)
\]

\[
= s^{\beta}_{\varepsilon}\left( \prod_{i=1}^{n} K_i \right).
\]

The inductive proof is now complete. \(\square\)

To prove Proposition 5.2.6(i), we fix \(\varepsilon\) and proceed via transfinite induction on \(\alpha\). Part (i) is trivially true for \(\alpha = 0\). So suppose that (i) holds for some \(\alpha = \gamma\); we show that it then holds for \(\alpha = \gamma + 1\). We have \(S_{\varepsilon}(T_{\gamma}) > \omega^{\eta \tau + \gamma} \cdot n\) for all \(n \in \mathbb{N}\) by Lemma 5.2.7, hence

\[
S_{\varepsilon}(T_{\gamma+1}) \supseteq \sup_{n \in \mathbb{N}} S_{\varepsilon}(T_{\gamma}[n]) \supseteq \sup_{n \in \mathbb{N}} \omega^{\eta \tau + \gamma} \cdot n = \omega^{\eta \tau + \gamma + 1}.
\]

As \(S_{\varepsilon}(T_{\gamma+1})\) cannot be a limit ordinal, we conclude that \(S_{\varepsilon}(T_{\gamma+1}) > \omega^{\eta \tau + \gamma + 1}\). In particular, assertion (i) of Proposition 5.2.6 passes to successor ordinals.

Now suppose that \(\gamma\) is a limit ordinal and that assertion (i) of Proposition 5.2.6 holds for all \(\alpha < \gamma\). Then

\[
S_{\varepsilon}(T_{\gamma}) \supseteq \sup_{\alpha < \gamma} S_{\varepsilon}(T_{\alpha}) \supseteq \sup_{\alpha < \gamma} \omega^{\eta \tau + \alpha} = \omega^{\eta \tau + \gamma},
\]

hence \(S_{\varepsilon}(T_{\gamma}) > \omega^{\eta \tau + \gamma}\). This concludes the inductive proof of Proposition 5.2.6(i).

The proof of assertion (ii) of Proposition 5.2.6 will take substantially more effort. In particular, we require the following three lemmas. We mention for the sake of clarity that in what follows, we shall often consider unions of sets over \(n\)-tuples of finite ordinals \((g_1, \ldots, g_n)\) such that \(g_j = 0\) for all but one \(j\), and for that \(j\), \(g_j = 1\);
5.2 Counterexamples to the factorisation property

this shall be represented in subscripts by writing \( g_1, \ldots, g_n < \omega, g_1 + \ldots + g_n = 1 \).

**Lemma 5.2.8.** Let \( E_1, \ldots, E_n \) be Banach spaces and \( K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^* \) \( w^* \)-compact sets. Consider \( \prod_{i=1}^n K_i \) as a subset of \((\bigoplus_{i=1}^n E_i)^*\). Then for all \( \varepsilon > 0 \) and ordinals \( \alpha \),

\[
\mathcal{s}_\varepsilon^{\omega^\alpha} \left( \prod_{i=1}^n K_i \right) \subseteq \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = 1} \prod_{i=1}^n \mathcal{s}_\varepsilon^{\omega^\alpha} \mathcal{g}_i(K_i). \tag{5.4}
\]

Lemma 5.2.8 is simply the \('n\)-factor' generalisation of Lemma 3.1.6, and its proof is essentially the same. Note that the reverse inclusion to (5.4) also holds; this is achieved by substituting \( \omega^\alpha \) in place of \( \alpha \) in (5.2).

**Lemma 5.2.9.** Let \( E_1, \ldots, E_n \) be Banach spaces and \( K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^* \) nonempty \( w^* \)-compact sets. Let \( 1 \leq q < \infty \) and \( a_1, \ldots, a_n \geq 0 \) be real numbers such that \( \sum_{i=1}^n a_i^q \leq 1 \). Let \( p \) be predual to \( q \) and consider \( \prod_{i=1}^n a_i K_i \) as a subset of \((\bigoplus_{i=1}^n E_i)^*\). Then for all \( \varepsilon > 0 \) and ordinals \( \alpha \),

\[
\mathcal{s}_\varepsilon^{\omega^\alpha} \left( \prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = 1} \prod_{i=1}^n a_i \mathcal{s}_\varepsilon^{\omega^\alpha} \mathcal{g}_i(K_i). \tag{5.5}
\]

**Proof.** We fix \( \varepsilon > 0 \) and proceed via induction on \( \alpha \). To see that (5.5) holds for \( \alpha = 0 \), let

\[
(a_i k_i)_{i=1}^n \in \prod_{i=1}^n a_i K_i \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = 1} \prod_{i=1}^n a_i \mathcal{s}_\varepsilon^{\omega^\alpha} \mathcal{g}_i(K_i).
\]

Then for each \( i = 1, 2, \ldots, n \) there is \( w^* \)-open \( V_i \ni k_i \) such that \( \text{diam}(V_i \cap K_i) \leq \varepsilon \). It follows that for the \( w^* \)-neighbourhood \( V = \prod_{i=1}^n a_i V_i \) of \((a_i k_i)_{i=1}^n\), we have

\[
\text{diam} \left( V \cap \prod_{i=1}^n a_i K_i \right) = \text{diam} \left( \prod_{i=1}^n a_i (V_i \cap K_i) \right) \leq \left( \sum_{i=1}^n a_i^q (\text{diam}(V_i \cap K_i))^q \right)^{1/q} \leq \varepsilon.
\]

Thus \((a_i k_i)_{i=1}^n \notin \mathcal{s}_\varepsilon(\prod_{i=1}^n a_i K_i)\), as required.
Now suppose that $\beta$ is an ordinal such that (5.5) holds for $\alpha = \beta$; we show that (5.5) then holds for $\alpha = \beta + 1$. For each $j \in \mathbb{N}$ let $r_j$ be the (finite) cardinality of the set $\{(g_1, \ldots, g_n) \mid g_1, \ldots, g_n < \omega, g_1 + \ldots + g_n = j\}$. Let $m_1 = 1$ and define $m_j$, $j \in \mathbb{N}$, by setting $m_{j+1} = m_j + r_j$ for all $j$. We will now show that for all $j \in \mathbb{N}$,

$$
S_\varepsilon^{\omega^\beta}m_j \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = j} \prod_{i=1}^{n} a_is_\varepsilon^{\omega^\beta}g_i(K_i).
$$

(5.6)

For $j = 1$, (5.6) is simply the induction hypothesis that (5.5) holds for $\alpha = \beta$. So suppose $h \in \mathbb{N}$ and that (5.6) is true for $j \leq h$. Then, by Lemma 4.2.1 (part (ii) if $\beta = 0$, part (iii) if $\beta > 0$),

$$
S_\varepsilon^{\omega^\beta}m_{h+1} \left( \prod_{i=1}^{n} a_i K_i \right) \subseteq S_\varepsilon^{\omega^\beta}h \left( \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = h} \prod_{i=1}^{n} a_is_\varepsilon^{\omega^\beta}g_i(K_i) \right)
$$

$$
\subseteq \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = h} S_\varepsilon^{\omega^\beta} \left( \prod_{i=1}^{n} a_is_\varepsilon^{\omega^\beta}g_i(K_i) \right)
$$

$$
\subseteq \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = h} \bigcup_{g'_1, \ldots, g'_n < \omega \atop g'_1 + \ldots + g'_n = 1} \prod_{i=1}^{n} a_is_\varepsilon^{\omega^\beta}(g_i + g'_i)(K_i)
$$

$$
= \bigcup_{g_1, \ldots, g_n < \omega \atop g_1 + \ldots + g_n = h+1} \prod_{i=1}^{n} a_is_\varepsilon^{\omega^\beta}g_i(K_i).
$$

Thus, by induction, (5.6) holds for all $j \in \mathbb{N}$.

Suppose that $(a_i k_i)_{i=1}^{n} \in \prod_{i=1}^{n} a_i K_i$ is such that

$$(a_i k_i)_{i=1}^{n} \in S_\varepsilon^{\omega^\beta+1} \left( \prod_{i=1}^{n} a_i K_i \right) = \bigcap_{j \in \mathbb{N}} S_\varepsilon^{\omega^\beta,m_j} \left( \prod_{i=1}^{n} a_i K_i \right).$$

(5.7)

Then, by (5.6), for each $j \in \mathbb{N}$ there is a sequence $g_{j,1}, \ldots, g_{j,n}$ of finite ordinals such that $g_{j,1} + \ldots + g_{j,n} = j$ and $k_i \in S_\varepsilon^{\omega^\beta,g_{j,i}}(K_i)$, $1 \leq i \leq n$. Moreover, there is
therefore some natural number $i' \leq n$ such that $(g_{j, i'})^\infty_{j=1}$ is unbounded, hence

$$k_{i'} \in \bigcap_{j \in \mathbb{N}} s^\omega_{e} g_{j, i'}(K_{i'}) = s^\omega_{e+1}(K_{i'}).$$

With $g_{i'} = 1$ and $g_i = 0$ otherwise ($1 \leq i \leq n$), $(a_i k_i)_{i=1}^n \in \prod_{i=1}^n a_i s^\omega_{e+1} g_i(K_i)$. As $(a_i k_i)_{i=1}^n$ was chosen arbitrarily to satisfy (5.7), it follows that

$$s^\omega_{e+1} \left( \prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{g_i, \ldots, g_n \prec \omega} \prod_{i=1}^n a_i s^\omega_{e+1} g_i(K_i). \quad (5.8)$$

That is, (5.4) passes to successor ordinals.

Now suppose that $\beta$ is a limit ordinal such that (5.4) holds for all $\alpha < \beta$. Let

$$(a_i k_i)_{i=1}^n \in s^\omega_{e} \left( \prod_{i=1}^n a_i K_i \right) = \bigcap_{\alpha < \beta} s^\omega_{e} \left( \prod_{i=1}^n a_i K_i \right).$$

Then for each $\alpha < \beta$ there exists $i_\alpha \in \{1, \ldots, n\}$ such that $k_{i_\alpha} \in s^\omega_{e} g_{i_\alpha}(K_{i_\alpha})$. For some $i' \in \{1, \ldots, n\}$, the set \{\(\alpha < \beta \mid i_\alpha = i'\}\} is cofinal in $\beta$. Moreover, for such $i'$ we have

$$k_{i'} \in \bigcap_{i_\alpha = i'} s^\omega_{e} g_{i'}(K_{i'}) = \bigcap_{\alpha < \beta} s^\omega_{e} g_{i'}(K_{i'}) = s^\omega_{e} g_{i'}(K_{i'}).$$

Thus, with $g_{i'} = 1$, and $g_i = 0$ otherwise ($1 \leq i \leq n$), $(a_i k_i)_{i=1}^n \in \prod_{i=1}^n a_i s^\omega_{e} g_i(K_i)$.

In particular, (5.4) passes to limit ordinals, and the proof is complete. \qed

**Lemma 5.2.10.** Let $E_1, \ldots, E_n$ be Banach spaces, $K_1 \subseteq E_1^*, \ldots, K_n \subseteq E_n^*$ nonempty, absolutely convex, $w^*$-compact sets, $\varepsilon > 0$, $\alpha$ a nonzero ordinal and $1 < q < \infty$. If $s^\omega_{\varepsilon} (B_q(K_i \mid 1 \leq i \leq n)) \neq \emptyset$, then for every $\delta \in (0, \varepsilon)$ there is $i \leq n$ such that $s^\omega_{\delta} (K_i) \neq \emptyset$.

**Proof.** Fix $\delta \in (0, \varepsilon)$. Let $l = [\delta n^{1/q}(\varepsilon - \delta)^{-1}]$ and $L = \mathbb{N}^n \cap (l + n^{1/q})B_{e_1}$. By
Lemma 4.2.5 and the hypothesis of Lemma 5.2.10,
\[ s^\omega_\varepsilon \left( \bigcup_{(k_i) \in L} \prod_{i=1}^n \frac{k_i}{l} K_i \right) \supseteq s^\omega_\varepsilon (B_q(K_i | 1 \leq i \leq n)) \supseteq \emptyset. \]

Thus, since $L$ is finite and $\omega^\alpha$ is a limit ordinal, Lemma 4.2.1(iii) ensures the existence of $(h_i)_{i=1}^n \in L$ such that
\[ s^\omega_\varepsilon \left( \prod_{i=1}^n \frac{h_i}{l} K_i \right) \neq \emptyset. \]  

The remainder of the proof uses essentially the same arguments as those in the proof of Lemma 4.1.5. Let $\rho = (1 + \frac{n^{1/\alpha}}{l})^{-1}$. By (5.9) and the homogeneity of the derivations $s^\gamma_\varepsilon$ (where $\gamma$ is an ordinal and $\varepsilon' > 0$), we have
\[ s^\omega_\rho \left( \prod_{i=1}^n \frac{p h_i}{l} K_i \right) = \rho s^\omega_\varepsilon \left( \prod_{i=1}^n \frac{h_i}{l} K_i \right) \neq \emptyset. \]  

Thus, since $|| (\frac{p h_i}{l})_{i=1}^n ||_{\varepsilon_q} \leq 1$, it follows from (5.10) and Lemma 5.2.9 that there is $i \leq n$ such that $s^\omega_\rho (K_i) \neq \emptyset$. As $\rho \varepsilon \geq \delta$, we have $s^\omega_\delta (K_i) \supseteq s^\omega_\rho (K_i) \supseteq \emptyset$. □

We will now prove Proposition 5.2.6(ii). Let $T$ be a noncompact Asplund operator, with $Sz(T) = \omega^\alpha_T$ (c.f. the paragraph preceding Proposition 5.2.6). If $\alpha_T$ is a successor ordinal, then $\omega^\alpha_T = 1$, hence (ii) holds in this case since $T_0 = T$.

Suppose $\alpha_T$ is a limit ordinal. For each $\varepsilon > 0$, let $\beta_\varepsilon = \inf \{ \beta : Sz_\varepsilon (T) < \omega^\beta \}$ and $\nu_\varepsilon = \inf \{ \beta_\delta : 0 < \delta < \varepsilon \}$ (note that $\beta_\varepsilon$ and $\nu_\varepsilon$ exist for all $\varepsilon > 0$ since the set $\{ \omega^\beta : \beta < \alpha_T \}$ is cofinal in $\omega^\alpha_T$). Our immediate goal is to show the following:
\[ \forall \varepsilon > 0 \ \forall \alpha \in \text{ORD} \ \ Sz_\varepsilon (T_\alpha) < \omega^{\nu_\varepsilon + \alpha + 1}. \]  

We proceed by induction on $\alpha$. For $\varepsilon > 0$ we have
\[ Sz_\varepsilon (T_0) < \omega^\beta_\varepsilon \leq \omega^{\nu_\varepsilon} < \omega^{\nu_\varepsilon + 0 + 1}, \]
hence the estimate of (5.11) holds for \( \alpha = 0 \) and all \( \varepsilon > 0 \).

Now suppose that \( \gamma \) is an ordinal such that the estimate of (5.11) holds for \( \alpha = \gamma \) and all \( \varepsilon > 0 \); we will show that it then holds for \( \alpha = \gamma + 1 \) and all \( \varepsilon > 0 \). By the induction hypothesis, for every \( \varepsilon > 0 \) we have \( S_{\varepsilon}(T_{\gamma}) < \omega^{\nu + \gamma + 1} \). It follows then by Lemma 5.2.8 that

\[
\forall \varepsilon > 0 \quad \forall n \in \mathbb{N} \quad S_{\varepsilon}((T_{\gamma})[n]) < \omega^{\nu + \gamma + 1}.
\]

Thus, Lemma 5.2.10 yields

\[
\forall \varepsilon > \rho > 0 \quad \forall F \in \mathbb{N}^{< \infty} \quad S_{\varepsilon}\left((\bigoplus_{n \in F}(T_{\gamma})[n])_{\tau}\right) < \omega^{\nu + \gamma + 1}.
\] (5.12)

Moreover, (5.12) implies that

\[
\forall \varepsilon > \rho > 0 \quad \forall F \in \mathbb{N}^{< \infty} \quad S_{\varepsilon}\left((\bigoplus_{n \in F}(T_{\gamma})[n])_{\tau}\right) \leq S_{\varepsilon}(\varepsilon + \rho)/2 \left((\bigoplus_{n \in F}(T_{\gamma})[n])_{\tau}\right) < \omega^{\nu + \gamma + 1}.
\] (5.13)

Let \( D \) denote the domain of \( T_{\gamma + 1} \) and let \( K = T_{\gamma + 1}^{*}B_{D^{*}} \), so that \( s_{\rho}^{\nu + \gamma + 1}(U_{F}K) = \emptyset \) for every \( F \in \mathbb{N}^{< \infty} \) by (5.13) (here \( U_{F} \) denotes the canonical embedding of the \( \ell_{r}\)-direct sum of the domains of the operators \( (T_{\gamma})[n], \ n \in F, \) into the \( \ell_{r}\)-direct sum of the domains of the operators \( (T_{\gamma})[n], \ n \in \mathbb{N} \)). It follows then by an application of Lemma 4.1.8 with \( \delta = (\varepsilon + \rho)/2 \) and \( \eta_{\delta} = \omega^{\nu + \gamma + 1} \) that

\[
\forall \varepsilon > \rho > 0 \quad S_{\varepsilon}(T_{\gamma + 1}) \leq \omega^{\nu + \gamma + 1} \cdot \sigma(\|T\|, \varepsilon, (\varepsilon + \rho)/2, r(r - 1)^{-1}).
\] (5.14)

For each \( \varepsilon > 0 \) there exists \( \pi(\varepsilon) \in (0, \varepsilon) \) such that \( \nu_{\pi(\varepsilon)} = \inf\{\nu_{\rho} \mid 0 < \rho < \varepsilon\} \). We have

\[
\nu_{\pi(\varepsilon)} = \inf_{\rho \in (0, \varepsilon)} \nu_{\rho} = \inf_{\rho \in (0, \varepsilon)} \inf_{\tau \in (0, \rho)} \beta_{\tau} = \inf_{\rho \in (0, \varepsilon)} \beta_{\rho} = \nu_{\varepsilon},
\] (5.15)

and so from (5.15) and (5.14) (with \( \rho = \pi(\varepsilon) \)) we have

\[
\forall \varepsilon > 0 \quad S_{\varepsilon}(T_{\gamma + 1}) < \omega^{\nu_{\varepsilon} + \gamma + 1} \cdot \sigma(\|T\|, \varepsilon, (\varepsilon + \pi(\varepsilon))/2, r(r - 1)^{-1}) < \omega^{\nu_{\varepsilon} + (\gamma + 1) + 1}.
\]
In particular, the estimate of (5.11) passes to successor ordinals for every \( \varepsilon > 0 \).

Let \( \gamma \) be a limit ordinal and suppose that the estimate of (5.11) holds for every \( \alpha < \gamma \) and \( \varepsilon > 0 \). By Lemma 5.2.10 we have

\[
\forall \varepsilon > \rho > 0 \quad \forall F \in \gamma^< \infty \quad S\varepsilon \left( \bigoplus_{\alpha \in F} T_\alpha \right) \leq \omega^{\nu_\rho + (\max F) + 1} < \omega^{\nu_\rho + \gamma}.
\]  

(5.16)

Moreover, (5.16) implies that

\[
\forall \varepsilon > \rho > 0 \quad \forall F \in \gamma^< \infty \quad S\varepsilon \left( \bigoplus_{\alpha \in F} T_\alpha \right) \leq S\left( \left( \bigoplus_{\alpha \in F} T_\alpha \right)_{\rho/2} \right) < \omega^{\nu_\rho + \gamma}.
\]  

(5.17)

Let \( D \) denote the domain of \( T_\gamma \) and let \( K = T_\gamma^* B_{D^*} \), so that \( \omega^{\nu_\rho + \gamma} (U_F K) = \emptyset \) for every \( F \in N^< \infty \) by (5.17) (here \( U_F \) denotes the canonical embedding of the \( \ell_r \)-direct sum of the domains of the operators \( T_\alpha \), \( \alpha \in F \), into the \( \ell_r \)-direct sum of the domains of the operators \( T_\alpha \), \( \alpha < \gamma \)). It follows then by an application of Lemma 4.1.8 with \( \delta = (\varepsilon + \rho)/2 \) and \( \eta_\delta = \omega^{\nu_\rho + \gamma} \) that

\[
\forall \varepsilon > \rho > 0 \quad S\varepsilon (T_\gamma) \leq \nu^{\nu_\rho + \gamma} \cdot \sigma(||T||, \varepsilon, (\varepsilon + \rho)/2, r(r - 1)^{-1}).
\]  

(5.18)

With \( \pi(\varepsilon) \in (0, \varepsilon) \) as above, taking \( \rho = \pi(\varepsilon) \) in (5.18) yields

\[
\forall \varepsilon > 0 \quad S\varepsilon (T_\gamma) < \nu^{\nu_\rho + \gamma} \cdot \sigma(||T||, \varepsilon, (\varepsilon + \pi(\varepsilon))/2, r(r - 1)^{-1}) < \omega^{\nu_\rho + \gamma + 1}.
\]

This concludes the inductive proof of (5.11).

To complete the proof of Proposition 5.2.6, we now only need show how part (ii) follows from (5.11). On the one hand, it is clear from the construction that \( T_\alpha \) factors \( T \) for each ordinal \( \alpha \), hence \( S(T_\alpha) \geq S(T) \). On the other hand, if \( \alpha < \omega^\omega \) then by (5.11) and the fact that \( \nu + \omega^\omega \leq \alpha_T \) whenever \( \nu < \alpha_T \),

\[
S(T_\alpha) = \sup_{\varepsilon > 0} S\varepsilon (T_\alpha) \leq \sup_{\varepsilon > 0} \omega^{\nu_\rho + \alpha + 1} \leq \sup_{\varepsilon > 0} \omega^{\nu_\rho + \omega^{\omega^\omega}} \leq \omega^{\omega_T} = S(T).
\]

Remark 5.2.11. It is now easy to determine precisely the Szlenk index of the
operators $T_\alpha$ in terms of $\alpha$ and $\alpha_T$. Indeed, if $T$ is a noncompact Asplund operator and $\alpha$ an ordinal, then the Szlenk index of $T_\alpha$ is given by the equation

$$
\text{Sz}(T_\alpha) = \begin{cases} 
\omega^{\alpha_T} & \text{if } \alpha < \omega^{\alpha_T}, \\
\omega^{\alpha_T} + (-\omega^{\alpha_T} + \alpha) + 1 & \text{if } \alpha \geq \omega^{\alpha_T},
\end{cases}
$$

(5.19)

where $-\omega^{\alpha_T} + \alpha$ denotes the unique ordinal order isomorphic to $\alpha \setminus \omega^{\alpha_T}$. To prove (5.19), one proceeds via tranfinite induction, making use of the operator theoretic corollary of Theorem 4.1.10 mentioned in the final sentence of Remark 4.1.14. Similar arguments show that if Construction 5.2.5 is applied to a nonzero compact operator $T$, then for all $\alpha > 0$ the Szlenk index of $T_\alpha$ is $\omega^{(-1+\alpha)+1}$, where $-1 + \alpha$ denotes the unique ordinal order isomorphic to $\alpha \setminus 1$. As the calculation of (5.19) is not of any importance to our agenda, we do not explicate it here.

At last, we are ready to prove Theorem 5.2.1. For simplicity we shall assume $\beta > 0$, but note that proof in the case of $\beta = 0$ is achieved by similar arguments to those used here. In fact, a different proof altogether for the case $\beta = 0$ will be presented in Section 5.3, so there is no real loss for us in assuming $\beta$ nonzero. Moreover, there is a saving: we need not establish an analogue of Proposition 5.2.4 for the case $\beta = 0$ (though it is not difficult to do so).

Let $\beta$ be a nonzero ordinal of countable cofinality and fix a superadditive cofinal sequence for $\omega^\beta$, which we denote $(\beta_n)_{n \in \mathbb{N}}$. Since the necessary condition for membership of $\text{Op}(\text{SzL}_{\omega^\beta})$ imposed by Proposition 5.2.4 holds for an arbitrary superadditive cofinal sequence for $\omega^\beta$, it suffices to construct an element of $\text{SzL}_{\omega^\beta}$ that fails this necessary condition for our fixed superadditive cofinal sequence $(\beta_n)_{n \in \mathbb{N}}$. To this end, let $R$ be an operator such that $\text{Sz}(R) = \omega^\omega$; the existence of such an operator is guaranteed by Proposition 3.1.10, and we have $\text{Sz}(m^{-1}R) = \omega^\omega$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ let $s(m) \in \mathbb{N}$ be so large that $\text{Sz}_{1/2s(m)}(m^{-1}R) > \omega^\beta = 1$, and let $W_m = (m^{-1}R)_{\beta_{s(m)}}$ (that is, $W_m$ is the $\beta_{s(m)}$th operator obtained by application of Construction 5.2.5 with initial operator $m^{-1}R$). Finally, set $W = (\bigoplus_{m \in \mathbb{N}} W_m)_0$.

To prove the theorem, we will show that $W \in \mathcal{L} \backslash \text{Op}(\text{SzL}_{\omega^\beta})$. 


For each $m \in \mathbb{N}$, let $E_m$ and $F_m$ denote the domain and codomain of $W_m$ respectively, so that $W \in \mathcal{B}((\bigoplus_{m \in \mathbb{N}} E_m)_0, (\bigoplus_{m \in \mathbb{N}} F_m)_0)$. Since $\beta s(m)^2 < \omega^\beta$ for every $m \in \mathbb{N}$, it follows by Proposition 5.2.6(ii) that $W_m \in \mathcal{L}_{\omega^\beta}$ for all $m$. For each $m \in \mathbb{N}$, let $Z_m := V_{\{1, \ldots, m\}}Q_{\{1, \ldots, m\}}W \in \mathcal{L}_{\omega^\beta}((\bigoplus_{m \in \mathbb{N}} E_m)_0, (\bigoplus_{m \in \mathbb{N}} F_m)_0)$ (here $\{1, \ldots, m\}$ is considered a subset of the underlying index set of $(\bigoplus_{m \in \mathbb{N}} F_m)_0$, and $V_{\{1, \ldots, m\}}$ and $Q_{\{1, \ldots, m\}}$ are as defined in Definition 2.1.1). Since

$$\|W - Z_m\| = \sup_{k > m} \|W_k\| = (m + 1)^{-1} \|R\| \to 0$$

and $\mathcal{L}_{\omega^\beta}$ is closed, it must be that $W \in \mathcal{L}_{\omega^\beta}$.

On the other hand, by Proposition 5.2.6(i) we have that for each $m \in \mathbb{N}$,

$$Sz_{1/2^s(m)}(W) \geq Sz_{1/2^s(m)}(W_m) = Sz_{1/2^s(m)}\left((m^{-1}R)_{\beta s(m)^2}\right) > \omega^{\beta s(m)^2}.$$ 

Moreover, since $\|m^{-1}R\| \to 0$, it follows that $\{s(m) \mid m \in \mathbb{N}\}$ is unbounded in $\mathbb{N}$. Thus, for any $n_0 \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $s(m) \geq n_0$, and for such $m$ we have

$$Sz_{1/2^s(m)}(W) > \omega^{\beta s(m)^2} \geq \omega^{\beta n_0 s(m)}.$$ 

In particular, $W \notin \text{Op}(SzL_{\omega^\beta})$ by Proposition 5.2.4. The proof of Theorem 5.2.1 is complete.

We now return to our earlier discussion regarding a universal function $\varphi$ of B. Bossard (c.f. the paragraph following the statement of Theorem 5.1.1). The proof of Theorem 5.2.1 begins with an appeal to Proposition 3.1.10 for the existence of an operator $R$ having Szlenk index $\omega^{\omega^\beta}$. It is clear from the proof of Proposition 3.1.10 that if $\beta < \omega_1$, then we may assume that the domain and codomain of $R$ are both norm separable. Under this additional assumption, the domain and codomain of the operator $W$ constructed in the proof of Theorem 5.2.1 are also both norm separable. Moreover, if $U$ is an embedding of the codomain of $W$ into $C[0, 1]$, we have $Sz(UW) = Sz(W) = \omega^{\omega^\beta}$ and $UW \notin \text{Op}(SzL_{\omega^\beta})$. As $UW$ has separable domain, and codomain $C[0, 1]$, it follows that $\varphi(\omega^{\omega^\beta}) > \omega^{\omega^\beta}$. Thus $\varphi$ exceeds the
5.3 A class of space ideals associated with the Szlenk index

In this section we consider a family of space ideals indexed by the class of ordinals. In particular, we shall consider the following classes, where \( \alpha \) is an ordinal:

\[
P_{\text{SzL}}^\alpha := \{ E \in \text{BAN} \mid \exists c \in (0, 1) \exists p \geq 1 \forall \varepsilon \in (0, 1), \ s_\varepsilon^\omega(B_{E^*}) \subseteq (1 - c \varepsilon^p)B_{E^*} \}
\]

and

\[
P_{\text{SzL}} := \{ E \in \text{BAN} \mid E \text{ is linearly isomorphic to some } F \in P_{\text{SzL}}^\alpha \}.
\]

The motivation for studying these classes is the following proposition, to be proved at the end of the current section.

**Proposition 5.3.1.** Let \( \alpha \) be an ordinal. Then at most one of the following two statements holds:

(i) \( \text{SzL}_{\alpha+1} = P_{\text{SzL}}^\alpha \).

(ii) \( \mathcal{I}_1 \mathcal{K}_{\alpha+1} \) has the factorisation property.

Thus, with an interest in solving Problem 5.1.9, we are prompted to ask:

**Question 5.3.2.** Let \( \alpha \) be an ordinal. Is \( P_{\text{SzL}}^\alpha = \text{SzL}_{\alpha+1} \) ?

For each ordinal \( \alpha \), the inclusion \( P_{\text{SzL}}^\alpha \subseteq \text{SzL}_{\alpha+1} \) is attained via an application of Proposition 5.1.3 with \( \beta = \alpha + 1, \ \beta_\varepsilon = \omega^\alpha \) and \( \delta_\varepsilon = 1 - c \varepsilon^p \) (see also [26, Proposition 2.2]). The decision to consider the classes \( P_{\text{SzL}}^\alpha \) is not arbitrary, for Question 5.3.2 is known to have an affirmative answer in the case \( \alpha = 0 \), a result due to M. Raja [50]. We thus obtain from Proposition 5.3.1 a proof that \( \mathcal{I}_1 \mathcal{K}_1 \) lacks the factorisation property (Theorem 5.2.1 with \( \beta = 0 \)). We note that prior to
Raja’s work [50], it had been shown by H. Knaust, E. Odell and Th. Schlumprecht [35] that every separable space in $SzL_1$ belongs to $PzL_0$.

The first result to be proved in this section is the following.

**Proposition 5.3.3.** $PzL_\alpha$ is a space ideal for each ordinal $\alpha$.

To prove Proposition 5.3.3, it suffices to establish the following two facts:

(I) Let $E \in PzL_\alpha^0$ and let $F$ be a closed linear subspace of $E$. Then $F \in PzL_\alpha^0$.

(II) Let $E, F \in PzL_\alpha^0$. Then $E \oplus F \in PzL_\alpha^0$.

The proof of (I) is straightforward. Indeed, let $i : F \hookrightarrow E$ denote the isometric linear inclusion operator and let $c' \in (0, 1)$ and $p' \geq 1$ be scalars such that $s^{\omega_0}_\varepsilon(B_E^*) \subseteq (1 - c'e^p')B_E^*$ for all $\varepsilon > 0$. By Lemma 3.1.5, for every $\varepsilon > 0$ we have

$$s^{\omega_0}_\varepsilon (i^*B_E^*) \subseteq i^* (s^{\omega_0}_\varepsilon (B_E^*)) \subseteq i^* \left( (1 - c')^2 \right) B_E^* = \left( 1 - \frac{c'}{2p'} \right) B_E^* . \quad (5.20)$$

As $s^{\omega_0}_\varepsilon (B_F^*) = s^{\omega_0}_\varepsilon (i^*B_E^*)$, it follows from (5.20) that $F$ satisfies the defining property of $PzL_\alpha^0$ with $c = c'/2p'$ and $p = p'$.

The proof of (II) is somewhat more involved. Let $c' \in (0, 1)$ and $p' \geq 1$ be such that $s^{\omega_0}_\varepsilon (B_E^*) \subseteq (1 - c'e^p')B_E^*$ and $s^{\omega_0}_\varepsilon (B_F^*) \subseteq (1 - c'e^p')B_F^*$ for all $\varepsilon > 0$. We introduce the following notation: for $\varepsilon > 0$ and $\alpha \in [0, 1] \subseteq \mathbb{R}$, define

$$A_\varepsilon^\alpha := \{ (b_1, b_2) \in [0, 1] \times [0, 1] \mid ab_1 + (1 - a)b_2 \geq \varepsilon \} .$$

We henceforth adhere to the following notational convention: for a $w^*$-compact set $K$ and ordinal $\alpha$, we write $s^K_\delta (K) = K$. As the next step in our preparation to prove (II), we state the following straightforward corollary of Lemma 4.2.3.

**Corollary 5.3.4.** Let $E$ and $F$ be Banach spaces, $a \in [0, 1] \subseteq \mathbb{R}$, $\varepsilon > 0$ and $\alpha$ an ordinal. Consider $aB_E^* \times (1 - a)B_F^*$ as a subset of $(E \oplus F)^*$ and let $\delta \in (0, \varepsilon)$. Then

$$s^{\omega_0}_\varepsilon (aB_E^* \times (1 - a)B_F^*) \subseteq \bigcup_{(b_1, b_2) \in A_{\delta/2}^\alpha} a s^{\omega_0}_{b_1} (B_E^*) \times (1 - a) s^{\omega_0}_{b_2} (B_F^*) .$$
Continuing towards a proof of (II), we consider the following situation: let \( l \in \mathbb{N} \) and suppose that \( a_1, a_2 \in \mathbb{R} \) are such that \( a_1 + a_2 \leq 1 \). For \( i = 1, 2 \) let \( l_i \) denote the unique integer satisfying \( l_i - 1 < la_i \leq l_i \), so that \( a_i \leq l_i/l \). Then \( l_1 + l_2 - 2 < l(a_1 + a_2) \leq l \), hence \( l_1 + l_2 \leq l + 1 \). By these considerations, the following chain of inclusions and equalities holds for every \( \varepsilon > 0 \) (a number of the steps are explained in the paragraph that follows):

\[
\begin{align*}
& s_{\varepsilon}^\omega (B_{(E \oplus \infty F)^*}) \\
& = s_{\varepsilon}^\omega \left( \bigcup_{a \in [0,1]} aB_{E^*} \times (1-a)B_{F^*} \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} s_{\varepsilon}^\omega \left( \bigcup_{k=0}^{l+1} \left( \frac{k}{l} B_{E^*} \times \frac{l+1-k}{l} B_{F^*} \right) \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} s_{\varepsilon/2}^\omega \left( \frac{k}{l} B_{E^*} \times \frac{l+1-k}{l} B_{F^*} \right) \\
& = \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} s_{\varepsilon/2}^\omega \left( \frac{k}{l+1} B_{E^*} \times \frac{l+1-k}{l+1} B_{F^*} \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \bigcup_{(b_1, b_2) \in A_{\varepsilon/(l+1)}} \left( 1-c' \left( \frac{k}{l+1} s_{\varepsilon/2}^\omega (B_{E^*}) \times \frac{l+1-k}{l+1} s_{\varepsilon/2}^\omega (B_{F^*}) \right) \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \bigcup_{(b_1, b_2) \in A_{\varepsilon/(l+1)}} \left( 1-c' \left( \frac{k}{l+1} B_{E^*} \times \frac{l+1-k}{l+1} B_{F^*} \right) B_{(E \oplus \infty F)^*} \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \bigcup_{(b_1, b_2) \in A_{\varepsilon/(l+1)}} \left( 1-c' \left( \frac{k}{l+1} B_{E^*} \times \frac{l+1-k}{l+1} B_{F^*} \right) B_{(E \oplus \infty F)^*} \right) \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \bigcup_{(b_1, b_2) \in A_{\varepsilon/(l+1)}} \left( 1-c' \left( \frac{k}{l+1} B_{E^*} \times \frac{l+1-k}{l+1} B_{F^*} \right) \right) B_{(E \oplus \infty F)^*} \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \bigcup_{(b_1, b_2) \in A_{\varepsilon/(l+1)}} \left( 1-c' \left( \frac{k}{l+1} B_{E^*} \times \frac{l+1-k}{l+1} B_{F^*} \right) \right) B_{(E \oplus \infty F)^*} \\
& \subseteq \bigcap_{l \in \mathbb{N}} \bigcup_{k=0}^{l+1} \frac{l+1}{l} \left( 1-c' \left( \frac{\varepsilon}{9} \right) \right) B_{(E \oplus \infty F)^*} \\
& = \left( 1-c' \left( \frac{\varepsilon}{9} \right) \right) B_{(E \oplus \infty F)^*}.
\end{align*}
\]
(5.21) is immediate from the identification of \((E \oplus_\infty F)^*\) with \(E^* \oplus_1 F^*\); (5.22) is a consequence of the considerations of the preceding paragraph; (5.23) follows from Lemma 4.2.1(i); (5.24) is a consequence of the homogeneity of the derivations \(s_\epsilon^\alpha\); (5.25) follows from Corollary 5.3.4 and the fact that \(\epsilon/9 < \epsilon l/(4l + 4)\) for every \(l \in \mathbb{N}\); finally, we obtain (5.26) from the previous line by adding the scalar coefficients of \(B_{E^*}\) and \(B_{F^*}\) (since \((E \oplus_\infty F)^*\) is identified with \(E^* \oplus_1 F^*\)).

We have thus shown that \(E \oplus_\infty F\) satisfies the defining property of \(P_{\mathcal{L}_\alpha}\) with \(c = c'/9^p\) and \(p = p'\). This concludes the proof of (II), and \(P_{\mathcal{L}_\alpha}\) is thus a space ideal for each ordinal \(\alpha\).

We now establish two propositions which we shall use to show that \(\text{Op}(P_{\mathcal{L}_\alpha})\) is never closed.

**Proposition 5.3.5.** Let \(\alpha\) be an ordinal, \(E\) and \(F\) Banach spaces and \(T : E \to F\) an operator. If \(T \in \text{Op}(P_{\mathcal{L}_\alpha})\), then there exist real scalars \(c \in (0, 1), d \geq 0\) and \(p \geq 1\) such that

\[
S_z_{1/2^n}(T) \leq \omega^\alpha \cdot \left[1 - \frac{n + d}{\log_2(1 - c2^{-np})}\right]
\]

for every \(n \in \mathbb{N}\).

**Proof.** The result is trivial if \(T = 0\), so we assume henceforth that \(T \neq 0\). As \(T \in \text{Op}(P_{\mathcal{L}_\alpha})\), there is a Banach space \(D \in P_{\mathcal{L}_\alpha}^0\) and operators \(A \in \mathcal{B}(E, D)\) and \(B \in \mathcal{B}(D, F)\) such that \(T = BA\), \(\|A\| \geq 1\) and \(\|B\| \leq 1\). By Lemma 3.1.5, the bound \(\|B\| \leq 1\) ensures that \(S_z^e(T) \leq S_z^e(A) \leq S_z^{e/(2\|A\|)}(D)\) for every \(e > 0\).

Let \(c' \in (0, 1)\) and \(p \geq 1\) be such that \(s_\epsilon^\alpha(B_{D^*}) \subseteq (1 - c'\epsilon^p)B_{D^*}\) for every \(\epsilon \in (0, 1)\), let \(c = c'(2\|A\|)^{-p}\) and let \(d = 2 + \log_2\|A\|\). For each \(\epsilon \in (0, 1)\) define

\[
l_\epsilon := \inf \{l < \omega \mid S_z^{e/(2\|A\|)}(D) \leq \omega^\alpha \cdot l\}
\]

and

\[
m_\epsilon := \inf \{m < \omega \mid (1 - ce^p)^m \leq \frac{\epsilon}{4\|A\|}\}
\]

(Note that \(l_\epsilon\) exists for each \(\epsilon\) by Lemma 2.4.3).
5.3 A class of space ideals associated with the Szlenk index

Fix \( \varepsilon \in (0, 1) \). By the argument used in the proof of Proposition 5.1.3, for each \( m < \omega \) we have

\[
S_{\varepsilon/(2\|A\|)}^{\omega^m}(B_{D^*}) \subseteq \left(1 - c\left(\frac{\varepsilon}{2\|A\|}\right)^p\right)^m B_{D^*} = (1 - c\varepsilon^p)^m B_{D^*}.
\]

In particular,

\[
S_{\varepsilon/(2\|A\|)}^{\omega^m}(B_{D^*}) \subseteq S_{\varepsilon/(2\|A\|)}^{\omega^{m+1}}(B_{D^*}) \subseteq S_{\varepsilon/(2\|A\|)}\left((1 - c\varepsilon^p)^m B_{D^*}\right)
\]

\[
\subseteq S_{\varepsilon/(2\|A\|)}\left(\frac{\varepsilon}{4\|A\|} B_{D^*}\right)
\]

\[
= \emptyset,
\]

hence \( l_\varepsilon \leq m_\varepsilon + 1 \). As \( 1 - c\varepsilon^p \in (0, 1) \), the definition of the logarithm yields

\[
m_\varepsilon = \left\lfloor \log_{1-c\varepsilon^p}\left(\frac{\varepsilon}{4\|A\|}\right) \right\rfloor = \left\lfloor \frac{\log_2 \varepsilon - \log_2 4 - \log_2 \|A\|}{\log_2(1 - c\varepsilon^p)} \right\rfloor = \left\lfloor \frac{\log_2 \varepsilon - d}{\log_2(1 - c\varepsilon^p)} \right\rfloor.
\]

It follows now that for each \( n \in \mathbb{N} \) we have

\[
l_{1/2^n} \leq 1 + \left\lfloor \frac{\log_2 2^{-n} - d}{\log_2(1 - c2^{-np})} \right\rfloor = \left\lfloor 1 - \frac{n + d}{\log_2(1 - c2^{-np})} \right\rfloor,
\]

hence

\[
S_{1/2^n}(T) \leq S_{1/(2^{n+1}\|A\|)}(D) \leq \omega^\alpha \cdot l_{1/2^n} \leq \omega^\alpha \cdot \left\lfloor 1 - \frac{n + d}{\log_2(1 - c2^{-np})} \right\rfloor. \quad \square
\]

**Proposition 5.3.6.** Let \( \alpha \) be an ordinal, \( \Lambda \) a set and for each \( \lambda \in \Lambda \) let \( D_\lambda \in SzL_\alpha \). Then \( \bigoplus_{\lambda \in \Lambda} D_\lambda \) is bounded in \( SzL_\alpha \).

**Proof.** Fix \( \varepsilon \in (0, 1) \) and suppose \( x \in S_{\varepsilon}^{\omega^\alpha}(B(\bigoplus_{\lambda \in \Lambda} D_\lambda)_0) \). As \( D_\lambda \in SzL_\alpha \) for all \( \lambda \), Proposition 2.4.2(v) ensures that \( S_{\varepsilon}^{\omega^\alpha}(U_\varepsilon^* B(\bigoplus_{\lambda \in \Lambda} D_\lambda)_0) = S_{\varepsilon}^{\omega^\alpha}(B(\bigoplus_{\lambda \in \Lambda} D_\lambda)_0) = \emptyset \) for all \( \mathcal{F} \in \Lambda^{<\infty} \). Thus, applying Lemma 4.1.6 with \( K = B(\bigoplus_{\lambda \in \Lambda} D_\lambda)_0 \) and \( \delta = \varepsilon/2 \), we have \( \|U_\varepsilon^* x\| \leq 1 - \varepsilon/2 \) for all \( \mathcal{F} \in \Lambda^{<\infty} \). It follows that

\[
\|x\| = \sup_{\mathcal{F} \in \Lambda^{<\infty}} \|U_\varepsilon^* x\| \leq 1 - \frac{\varepsilon}{2}.
\]
As \( x \in s_t^\alpha \left( B(\bigoplus_{\lambda \in \Lambda} D_\lambda) \right) \) was arbitrary, \( s_t^\alpha \left( B(\bigoplus_{\lambda \in \Lambda} D_\lambda) \right) \subseteq (1 - \epsilon/2)B(\bigoplus_{\lambda \in \Lambda} D_\lambda) \). In particular, \( (\bigoplus_{\lambda \in \Lambda} D_\lambda)_0 \) satisfies the defining property of \( \text{PZL}_0^\alpha \) with \( c = 1/2 \) and \( p = 1 \).

**Lemma 5.3.7.** For \( \alpha \) an ordinal, the class \( \text{PZL}_\alpha \setminus \text{SzL}_\alpha \) is nonempty.

**Proof.** Let \( T = I_{c_0} \), the identity operator on \( c_0 \). For each ordinal \( \alpha \), let \( T_\alpha \) be the \( \alpha \)th operator given by Construction 5.2.5 with \( r = 0 \), and let \( E_\alpha \) denote the Banach space that is the domain and codomain of \( T_\alpha \) (so that \( T_\alpha \) is the identity operator on \( E_\alpha \)). With \( r_T = 0 \) and \( \zeta_T = 0 \) in the notation introduced in the paragraph preceding Proposition 5.2.6 (since \( \text{Sz}(c_0) = \omega \)), it follows from Proposition 5.2.6(i) that there is \( \epsilon > 0 \) such that \( \text{Sz}(E_\alpha) = \text{Sz}(T_\alpha) \geq \text{Sz}_e(T_\alpha) > \omega^\alpha \) for all ordinals \( \alpha \).

We thus have \( E_\alpha \notin \text{SzL}_\alpha \) for all \( \alpha \), and so to complete the proof it suffices to show that \( E_\alpha \in \text{PZL}_\alpha \) for all \( \alpha \). In this endeavour, we proceed by transfinite induction and recall from the paragraph following Question 5.3.2 that \( \text{PZL}_\alpha \subseteq \text{SzL}_{\alpha+1} \) for all ordinals \( \alpha \).

For \( \alpha = 0 \), we have \( E_0 = c_0 \in \text{PZL}_\alpha \) by an application of Proposition 5.3.6 with \( \Lambda = \mathbb{N} \) and \( D_\lambda = \mathbb{K} \) for all \( \lambda \in \Lambda \).

Suppose that \( \alpha \) is an ordinal such that \( E_\beta \in \text{PZL}_\beta \) for all \( \beta < \alpha \). If \( \alpha \) is a successor ordinal, say \( \alpha = \zeta + 1 \), then since \( \text{PZL}_\zeta \subseteq \text{SzL}_{\zeta+1} \), it follows by Proposition 2.4.2(v) that \( (\bigoplus_{i=1}^n E_{\zeta i})_1 \in \text{SzL}_{\zeta+1} \) for all \( n \in \mathbb{N} \). By Proposition 5.3.6, \( E_\alpha = (\bigoplus_{n \in \mathbb{N}} (\bigoplus_{i=1}^n E_{\zeta i}))_0 \in \text{PZL}_{\zeta+1} = \text{PZL}_\alpha \), as required. If \( \alpha \) is a limit ordinal, then for each \( \beta < \alpha \) we have \( E_\beta \in \text{PZL}_\beta \subseteq \text{SzL}_{\beta+1} \subseteq \text{SzL}_\alpha \), hence \( E_\alpha = (\bigoplus_{\beta < \alpha} E_\beta)_0 \in \text{PZL}_\alpha \) by Proposition 5.3.6. This completes the induction.

**Theorem 5.3.8.** For \( \alpha \) an ordinal, the operator ideal \( \text{Op} (\text{PZL}_\alpha) \) is not closed.

**Proof.** Our proof relies on ideas similar to those used to prove Theorem 5.2.1. Let \( D \in \text{PZL}_\alpha \setminus \text{SzL}_\alpha \) (the existence of such \( D \) is guaranteed by Lemma 5.3.7). As \( \text{PZL}_\alpha \) is a space ideal, \( (\bigoplus_{i=1}^m D)_1 \in \text{PZL}_\alpha \) for all \( m \in \mathbb{N} \).

For each \( m \in \mathbb{N} \), let \( s(m) \in \mathbb{N} \) be so large that \( \text{Sz}_{1/2^m}(m^{-1}I_D) > \omega^\alpha \), let

\[
 t(m) = \left\lfloor \frac{-s(m)^2}{\log_2(1 - 2^{-s(m)^2})} \right\rfloor
\]
and let \( J_m = m^{-1} \left( \bigoplus_{i=1}^{t(m)} I_D \right) \) \( \in \text{Op}(PzL_\alpha) \). Finally, we set \( J = (\bigoplus_{m \in \mathbb{N}} J_m)_0 \). To prove the theorem, we will show that \( J \not\in \text{Op}(PzL_\alpha) \). 

For each \( m \in \mathbb{N} \) let \( H_m = \left( \bigoplus_{i=1}^{t(m)} D \right) \) \( \ell_1 \), so that \( J \in \mathcal{B}(\bigoplus_{m \in \mathbb{N}} H_m)_0 \). For each \( m \), let \( L_m = V_{\{1,...,m\}} Q_{\{1,...,m\}} J \in \text{Op}(PzL_\alpha) \) (here \( \{1,...,m\} \) is considered a subset of the underlying index set of \( (\bigoplus_{m \in \mathbb{N}} H_m)_0 \), and \( V_{\{1,...,m\}} \) and \( Q_{\{1,...,m\}} \) are as defined in Definition 2.1.1). Then \( \| L_m - J \| = \sup_{k > m} \| J_k \| = (m + 1)^{-1} m \to 0 \), hence \( J \not\in \text{Op}(PzL_\alpha) \).

On the other hand, by Lemma 5.2.7 we have that for each \( m \in \mathbb{N} \),
\[
S_{1/2^{s(m)}}(J) \geq S_{1/2^{s(m)}} (J_m) > \omega^a \cdot t(m) = \omega^a \cdot \left[ \frac{-s(m)^2}{\log_2(1 - 2^{-s(m)^2})} \right].
\]
Moreover, since \( \| m^{-1} I_D \| \to 0 \), it follows that \( \{ s(m) \mid m \in \mathbb{N} \} \) is unbounded in \( \mathbb{N} \). Thus, for any \( c \in (0, 1) \), \( d \geq 0 \) and \( p \geq 1 \) there is \( m \in \mathbb{N} \) such that
\[
\left[ \frac{-s(m)^2}{\log_2(1 - 2^{-s(m)^2})} \right] \geq \left[ 1 - \frac{s(m) + d}{\log_2(1 - c2^{-s(m)^p})} \right],
\]
and for such \( m \) we have
\[
S_{1/2^{s(m)}} (J) > \omega^a \cdot \left[ 1 - \frac{s(m) + d}{\log_2(1 - c2^{-s(m)^p})} \right].
\]
We have now shown that \( J \) does not satisfy the conclusion of Proposition 5.3.5, hence \( J \not\in \text{Op}(PzL_\alpha) \).

\[\square\]

**Remark 5.3.9.** Earlier in this section it was mentioned that recent work of M. Raja [50], which removed the separability hypothesis from earlier work of H. Knaust, E. Odell and Th. Schlumprecht [35], leads to a different proof of the fact that \( \mathcal{S}^2_1 \) lacks the factorisation property. However, it is not difficult to see that the greater generality of Raja’s result is in fact not needed to establish the alternative proof. To see why this is so, let \( \text{SEP} \) denote the space ideal consisting of all separable Banach spaces. By taking the spaces \( D_n \) in the proof of Theorem 5.3.8 to be the scalar field \( \mathbb{K} \), and thus \( D = c_0 \), one obtains an operator \( J \) such that the domain
of $J$ is separable and $J \in \mathcal{J}_2 \setminus \text{Op}(PzL_0)$. If it were the case that $J \in \text{Op}(SzL_1)$, then it would follow from the separability of the domain of $J$ and the main result of [35] that $J \in \text{Op}(SzL_1 \cap \text{SEP}) = \text{Op}(PzL_0 \cap \text{SEP}) \subseteq \text{Op}(PzL_0)$ - a contradiction. Thus $J \notin \text{Op}(SzL_1)$, hence $SzL_1$ lacks the factorisation property.

We conclude our results with the following proof, promised at the beginning of the section.

**Proof of Proposition 5.3.1.** Trivially, $\text{Op}(PzL_\alpha) \subseteq \text{Op}(SzL_{\alpha+1}) \subseteq \mathcal{J}_{\alpha+1}$. Note that statement (i) of the proposition implies $\text{Op}(PzL_\alpha) = \text{Op}(SzL_{\alpha+1})$, whilst statement (ii) of the proposition implies $\text{Op}(SzL_{\alpha+1}) = \mathcal{J}_{\alpha+1}$. As $\mathcal{J}_{\alpha+1}$ is closed and $\text{Op}(PzL_\alpha)$ is not, the inclusion $\text{Op}(PzL_\alpha) \subseteq \mathcal{J}_{\alpha+1}$ is strict, hence (i) and (ii) cannot both hold.

\[\square\]

### 5.4 Concluding remarks

We have shown that the operator ideals $\mathcal{J}_\alpha$ fail to have the factorisation property for a large (indeed, proper) class of ordinals $\alpha$. However, we have not addressed here the possibility of the operator ideals $\mathcal{J}_\alpha$ possessing some sort of approximate factorisation property. Noting that $\mathcal{J}_\alpha$ is closed, injective and surjective for every $\alpha$, it is worth considering whether there is some composition of the closed, injective and surjective hull procedures that yields $SzL_\alpha$ from $\text{Op}(SzL_\alpha)$ for every ordinal $\alpha$. We give some possible examples of such compositions via the questions below.

**Question 5.4.1.** Let $\alpha$ be an ordinal. Is $\mathcal{J}_\alpha = \overline{\text{Op}(SzL_\alpha)}$?

**Question 5.4.2.** Let $\alpha$ be an ordinal. Is $\mathcal{J}_\alpha = \left(\overline{\text{Op}(SzL_\alpha)}^{\text{inj}}\right)^{\text{sur}}$?

Note that the injective and surjective hull procedures commute; that is, we have $\left(\mathcal{J}^{\text{inj}}\right)^{\text{sur}} = \left(\mathcal{J}^{\text{sur}}\right)^{\text{inj}}$ for every operator ideal $\mathcal{J}$ (c.f. [49, Proposition 4.7.20]). Evidently, Corollary 5.1.8 ensures that the answer to Question 5.4.1 and Question 5.4.2 is yes in both cases when $\alpha$ is of uncountable cofinality. We do not know if the
counterexample constructed in the proof of Theorem 5.2.1 provides a counterexample to either of the two questions above. It is well-known that in the case $\alpha = 0$, the answer to Question 5.4.1 is no and the answer to Question 5.4.2 is yes. Indeed, in this case $\mathcal{H}_\alpha$ is precisely the class of compact operators, whilst $\text{Op}(\text{SzL}_\alpha)$ is the class $\mathcal{F}$ of finite rank operators; it is well-known that $\mathcal{F} \subset \mathcal{F}^{\text{finj}} = \mathcal{K}$. However, nothing appears to be known for Question 5.4.1 and Question 5.4.2 in the case that $0 < cf(\alpha) \leq \omega$.

Besides answering Question 5.3.2 in the affirmative, one could possibly show that the operator ideals $\mathcal{H}_{\alpha+1}$ ($\alpha \in \text{ORD}$) lack the factorisation property by following a line of inquiry such as the following. Let $\alpha$ be an ordinal and $E \in \text{SzL}_{\alpha+1}$. For each $\varepsilon > 0$, let $m_\varepsilon = \inf \{m < \omega \mid \text{Sz}_\varepsilon(E) < \omega^\alpha \cdot m\}$ (note that $m_\varepsilon$ exists for every $\varepsilon$). We ask: What special properties do the numbers $m_\varepsilon$ have? Are they submultiplicative with respect to $\varepsilon$? Do they satisfy some other general property that ensures that the growth of the $\varepsilon$-Szlenk indices of elements of $\text{Op}(\text{SzL}_{\alpha+1})$ is restricted in some useful way? The straightforward homogeneity argument used by Lancien in [39] to establish the submultiplicity of the $\varepsilon$-Szlenk indices of a given Banach space does not seem to be sufficient for a useful analysis of growth properties of the numbers $m_\varepsilon$, so a more subtle argument is likely to be required if this direction of inquiry is to prove fruitful.

More generally, to investigate whether $\mathcal{H}_\alpha$ has the factorisation property for $\alpha$ an ordinal of countable cofinality, and not of the form $\omega^\beta$ for any $\beta$, one possibility would be to consider growth properties of a family of ordinals $\alpha_\varepsilon$, $\varepsilon > 0$, or perhaps of a (finite or infinite) sequence of ordinals $(\alpha_{\varepsilon,n})_n$, defined in terms of the derivations $s^\gamma$ and depending in some way on the Cantor normal form of $\alpha$. In a related direction, variations of Question 5.3.2 may be of interest to Banach space theory in general.
Appendix A

The complex scalar case

This appendix is devoted to establishing Theorem 2.3.2 for complex Banach spaces, relying on the fact that it holds for real Banach spaces (recall from Section 2.3 that the proof of the real scalar case is given in Chapter I.5 of [14]).

In what follows, if $E$ is a complex Banach space, then $E_{\mathbb{R}}$ denotes the underlying real Banach space structure of $E$. For $x^* \in E^*$, $\Re x^*$ denotes the continuous real-linear functional $x \mapsto \Re(x^*, x)$ ($x \in E$).

Let $E$ be a complex Banach space. We shall consider relabelings of the statements (i)-(iv) of Theorem 2.3.2 to distinguish between complex and real scalar cases. Let each of the statements (i)-(iv) of Theorem 2.3.2 (for our given complex Banach space $E$) be written (i$_C$)-(iv$_C$), and let (i$_R$)-(iv$_R$) denote the statements (i)-(iv) of Theorem 2.3.2 with $E$ replaced throughout by $E_{\mathbb{R}}$. Then, since Theorem 2.3.2 is known to be true for real scalars, to prove Theorem 2.3.2 for complex scalars it suffices to show that each of the statements (i$_C$)-(iv$_C$) holds if and only if its real scalar counterpart (i$_R$)-(iv$_R$) holds. That is, (i$_R$) $\iff$ (i$_C$), (ii$_R$) $\iff$ (ii$_C$), and so on. Our main tool in this task is the following proposition

**Proposition A.0.3.** Let $E$ be a complex Banach space and $V_E : E^* \longrightarrow (E_{\mathbb{R}})^*$ the real-linear map satisfying $V_E(x^*) = \Re x^*$ for each $x^* \in E^*$. Then $V_E$ is a surjective isometry.

Proposition A.0.3 is proved in [42, Proposition 1.9.3].
We now recall for convenience some definitions first given in Section 2.3. Let $E$ be a complex Banach space. We say that $E$ is Asplund if $E_{\mathbb{R}}$ is an Asplund space in the real scalar sense. For $C \subseteq E^*$, $\varepsilon > 0$ and $x \in E$, the $w^*$-open slice of $C$ determined by $x$ and $\varepsilon$ is the set \{ $x^* \in C \mid \Re(x^*, x) > \sup \{\Re(y^*, x) \mid y^* \in C\} - \varepsilon$ \}. If $G$ is a closed (real-linear) subspace of $E_{\mathbb{R}}$, we denote by $G_C$ the closed complex-linear hull of $G$ in $E$.

In the remainder of the current appendix, $E$ is a fixed complex Banach space and statements (i$_{\mathbb{R}}$)-(iv$_{\mathbb{R}}$) and (i$_{\mathbb{C}}$)-(iv$_{\mathbb{C}}$) are as defined above. We now set about establishing the desired equivalences.

The equivalence (i$_{\mathbb{R}}$) $\iff$ (i$_{\mathbb{C}}$) holds trivially by the definition of a complex Asplund space.

Suppose that (ii$_{\mathbb{R}}$) holds and let $F$ be a separable, closed (complex-linear) subspace of $E$. Then $F_{\mathbb{R}}$ is Asplund since $F_{\mathbb{R}} \subseteq E_{\mathbb{R}}$, hence $F$ is Asplund by definition of a complex Asplund space. In particular, (ii$_{\mathbb{C}}$) holds whenever (ii$_{\mathbb{R}}$) does. Now suppose on the other hand that (ii$_{\mathbb{R}}$) does not hold. Then there is a separable, closed real-linear subspace $G$ of $E_{\mathbb{R}}$ such that $G$ is not Asplund. Let $G_C$ denote the closed complex-linear span of $G$ in $E$. Then $G_C$ is separable and $(G_C)^*_{\mathbb{R}}$ fails to be Asplund since $G \subseteq (G_C)^*_{\mathbb{R}}$ is not Asplund. Thus $G_C$ is a separable (complex-linear) subspace of $E$ that is not Asplund. That is, (ii$_{\mathbb{C}}$) fails to hold whenever (ii$_{\mathbb{R}}$) fails to hold, and so (ii$_{\mathbb{R}}$) $\iff$ (ii$_{\mathbb{C}}$).

Assume that (iii$_{\mathbb{R}}$) holds and let $F$ be a separable, closed (complex-linear) subspace of $E$. Then $F_{\mathbb{R}}$ is a norm separable real-linear subspace of $E_{\mathbb{R}}$, hence $(F_{\mathbb{R}})^*$ is norm separable. By Proposition A.0.3, this implies that $F^*$ is norm separable, hence (iii$_{\mathbb{R}}$) implies (iii$_{\mathbb{C}}$). Conversely, let us now suppose that (iii$_{\mathbb{R}}$) fails to hold and let $G$ be a separable, closed (real-linear) subspace of $E_{\mathbb{R}}$ such that $G^*$ is nonseparable. Then $(G_C)^*_{\mathbb{R}}$ is a separable real-linear Banach space such that $G \subseteq (G_C)^*_{\mathbb{R}}$. It follows that $G^*$ is a linear quotient of $((G_C)^*_{\mathbb{R}})^*$, hence $((G_C)^*_{\mathbb{R}})^*$ is nonseparable. Since $((G_C)^*_{\mathbb{R}})^*$ is real-linear isometric to $(G_C)^*$ by Proposition A.0.3, it follows then that $G_C$ is a separable complex-linear subspace of $E$ with nonseparable dual, hence (iii$_{\mathbb{C}}$) fails to hold. We have thus established the equivalence (iii$_{\mathbb{R}}$) $\iff$ (iii$_{\mathbb{C}}$).
Finally, we show (iv\(_{\mathbb{R}}\)) \(\iff\) (iv\(_{\mathbb{C}}\)). Let \(V_E\) be as in the statement of Proposition A.0.3 and let \(U_E : E \to E_{\mathbb{R}}\) denote the formal identity map. Then for each bounded \(C \subseteq E^*\), \(x \in E\) and \(\varepsilon > 0\), we have that \(V_E\) maps the \(w^*\)-open slice of \(C\) determined by \(x\) and \(\varepsilon\) isometrically onto the \(w^*\)-open slice of \(V_E(C)\) determined by \(U_E(x)\) and \(\varepsilon\), and conversely (since \(V_E\) and \(U_E\) have well-defined inverses). In particular, bounded subsets of \(E^*\) admit arbitrarily small \(w^*\)-open slices if and only if the same is true for bounded subsets of \((E_{\mathbb{R}})^*\), as we would like.
Appendix B

The Szlenk index of absolutely convex sets

We present here a very detailed proof of Proposition 2.4.2(iii). As stated in the paragraph following the statement of Proposition 2.4.2, Gilles Lancien established this result in the case that $K$ is the unit ball of a dual Banach space. Since it is easy to see that his arguments apply without modification to arbitrary nonempty, absolutely convex, $w^*$-compact sets, I do not claim any originality for the arguments presented below; the goal is to merely to verify the claim that Lancien's arguments hold more generally, and to expand a little on Lancien's arguments from [38, Proposition 5.4] (see also [28, p.64]) for the benefit of the reader. We shall need the following:

**Lemma B.0.4.** Let $E$ be a Banach space, $K \subseteq E^*$ an absolutely convex, $w^*$-compact set, $\alpha$ an ordinal and $\varepsilon > 0$. Then

$$\frac{1}{2}s_\varepsilon^\alpha(K) + \frac{1}{2}K \subseteq s_{\varepsilon/2}^\alpha(K).$$

**Proof.** We proceed by transfinite induction. The lemma is trivially true for $\alpha = 0$. Suppose that $\beta > 0$ is such that the lemma holds for all $\alpha < \beta$. We consider first the case that $\beta = \gamma + 1$, some $\gamma \in \text{ORD}$. To this end, let $z = \frac{1}{2}x + \frac{1}{2}y$, where $x \in s_\varepsilon^\beta(K)$ and $y \in K$. Let $U \ni z$ be $w^*$-open, so that $-\frac{1}{2}y + U$ is a $w^*$
neighbourhood of $\frac{1}{2}x$. We have $\frac{1}{2}y + \frac{1}{2}s^\gamma(K) \subseteq s^\gamma_{\epsilon/2}(K)$ by the induction hypothesis, and since $\frac{1}{2}x \in s^\gamma_{\epsilon/2}(s^\gamma_{\epsilon/2}(\frac{1}{2}K))$ by homogeneity of the Szlenk derivation,

$$\text{diam}(U \cap s^\gamma_{\epsilon/2}(K)) \geq \text{diam}(U \cap (\frac{1}{2}y + \frac{1}{2}s^\gamma(K)))$$

$$= \text{diam}((-\frac{1}{2}y + U) \cap \frac{1}{2}s^\gamma(K))$$

$$= \text{diam}((-\frac{1}{2}y + U) \cap s^\gamma_{\epsilon/2}(\frac{1}{2}K))$$

$$> \frac{\varepsilon}{2}.$$

As $z \in s^\gamma_{\epsilon/2}(K)$ and $U$ was arbitrary, $z \in s^\gamma_{\epsilon/2}(K) = s^\beta_{\epsilon/2}(K)$, as desired. If, on the other hand, $\beta$ is a limit ordinal, then

$$\frac{1}{2}s^\beta(\frac{1}{2}K) + \frac{1}{2}K = \frac{1}{2}\bigcap_{\alpha<\beta} s^\alpha(K) + \frac{1}{2}K \subseteq \bigcap_{\alpha<\beta} (\frac{1}{2}s^\alpha(K) + \frac{1}{2}K) \subseteq \bigcap_{\alpha<\beta} s^\alpha_{\epsilon/2}(K)$$

$$= s^\beta_{\epsilon/2}(K).$$

This completes the induction, hence the proof. \qed

**Proposition B.0.5.** Let $E$ be a Banach space and $K \subseteq E^*$ a nonempty, absolutely convex, $w^*$-compact set such that $Sz(K) < \infty$. Then there is an ordinal $\alpha$ such that $Sz(K) = \omega^\alpha$.

**Proof.** Let $\beta \in \text{ORD}$. We first show that $Sz(K) > \omega^\beta$ implies $Sz(K) \geq \omega^{\beta+1}$. To this end, suppose $\varepsilon > 0$ is such that $Sz_{\varepsilon\omega}(K) > \omega^\beta$ and let $x \in s^\omega_{\varepsilon\omega}(K)$. We will show by induction that $0 \in s^\omega_{\varepsilon/2n}(K)$ for all $n < \omega$. For $n = 0$, Lemma B.0.4 yields

$$0 = \frac{1}{2}x - \frac{1}{2}x \in \frac{1}{2}s^\omega_{2\varepsilon}(K) + \frac{1}{2}K \subseteq s^\omega_{\varepsilon}(K),$$

as desired. If $n < \omega$ is such that $0 \in s^\omega_{\varepsilon/2n}(K)$, then by homogeneity of the Szlenk derivation we have

$$0 \in \frac{1}{2}s^\omega_{\varepsilon/2n}(K) = s^\omega_{\varepsilon/2n+1}(\frac{1}{2}K) \subseteq s^\omega_{\varepsilon/2n+1}(s^\omega_{\varepsilon/2n}(K) + \frac{1}{2}K) \subseteq s^\omega_{\varepsilon/2n+1}(s^\omega_{\varepsilon/2n}(K))$$

$$\subseteq s^\omega_{\varepsilon/2n+1}(K),$$
completing the induction. It follows now that

\[ Sz(K) = \sup_{n<\omega} Sz_{\omega/2^n}(K) \geq \sup_{n<\omega} \omega^\beta \cdot 2^n = \omega^{\beta+1}. \]

Let \( \alpha = \inf \{ \gamma \in \text{ORD} \mid Sz(K) \leq \omega^\gamma \} \), and note that \( Sz(K) \leq \omega^\alpha \). If \( \alpha \) is a limit ordinal, then \( Sz(K) \geq \sup_{\beta<\alpha} \omega^\beta = \omega^\alpha \). On the other hand, if \( \alpha = \beta + 1 \), some \( \beta \in \text{ORD} \), then \( Sz(K) > \omega^\beta \), hence \( Sz(K) \geq \omega^{\beta+1} = \omega^\alpha \). In either case we have \( Sz(K) \geq \omega^\alpha \), hence \( Sz(K) = \omega^\alpha \). \( \square \)
Appendix C

Interpolation and factorisation à la Heinrich

Our goal here is to prove Theorem 2.2.7. I make no claim of originality over the material presented in this appendix, as the proofs presented here constitute only minor variations on the work of S. Heinrich in [30]. It is presented so that the reader may see how Heinrich’s proof of [30, Theorem 2.1] can be generalised slightly to obtain Theorem 2.2.7 of the present thesis. Aside from notational changes, the main difference between the material presented here and the content of [30] is that in several places we provide a slightly more detailed argument for the benefit of the reader.

We now briefly sketch some background results from interpolation theory that we require. Let $E_b$ and $E_q$ be Banach spaces, both continuously embedded into a common Hausdorff topological vector space $X$. In this case we say that $(E_b, E_q)$ is an interpolation pair. As $E_b$ and $E_q$ are considered as subsets of $X$ via the continuous embeddings, we may consider the vector spaces $E_b \cap E_q$ and $E_b + E_q$ in $X$. In particular, we equip the spaces $E_b \cap E_q$ and $E_b + E_q$ with complete norms defined as follows: for $x \in E_b \cap E_q$, define $\|x\|_{E_b \cap E_q} := \max \left( \|x\|_{E_b}, \|x\|_{E_q} \right)$, and for $x \in E_b + E_q$, define $\|x\|_{E_b + E_q} := \inf \left\{ \|x_0\|_{E_b} + \|x_1\|_{E_q} \mid x_0 \in E_b, x_1 \in E_q, x = x_0 + x_1 \right\}$.

By an intermediate space for the couple $(E_b, E_q)$ we mean a Banach space $E$ such
that \( E_\theta \cap E_\theta \subseteq E \subseteq E_\theta + E_\theta \) and the inclusion maps \( E_\theta \cap E_\theta \hookrightarrow E \) and \( E \hookrightarrow E_\theta + E_\theta \) are norm continuous.

We are interested in the discrete Lions-Peetre interpolation spaces \((E_\theta, E_\theta)_{\theta,p}\), where \(0 < \theta < 1\) and \(1 < p < \infty\) are fixed. These spaces may be defined in a number of equivalent ways, where by ‘equivalent’ we mean that the different definitions give isomorphic spaces. We now give two such definitions. To this end, let \( \xi_\theta, \xi_\theta \in \mathbb{R} \) be such that \( \xi_\theta \xi_\theta = 0 \) and \( \xi_\theta (\xi_\theta - \xi_\theta)^{-1} = \theta \).

The first definition is that \((E_\theta, E_\theta)_{\theta,p}\) is the space of all \( x \in E_\theta + E_\theta \) that admit a representation of the form \( x = \sum_{n \in \mathbb{Z}} x_n (x_n \in E_\theta \cap E_\theta \text{ for all } n, \text{ and convergence with respect to } \| \cdot \|_{E_\theta + E_\theta}) \).

The second definition is that \((E_\theta, E_\theta)_{\theta,p}\) is the space of all \( x \in E_\theta + E_\theta \) such that there exists an infinite sequence of pairs \( ((x_\theta, x_\xi), (x_\theta, x_\xi)) \) in \( E_\theta \times E_\theta \) satisfying \( (\|e^{\xi \delta n} x_\theta, n\|_{E_\theta}, (\|e^{\xi \delta n} x_\xi, n\|_{E_\theta})_{n \in \mathbb{Z}} \in \ell_p(\mathbb{Z}) \), \( (\|e^{\xi \delta n} x_\theta, n\|_{E_\theta}, (\|e^{\xi \delta n} x_\xi, n\|_{E_\theta})_{n \in \mathbb{Z}} \in \ell_p(\mathbb{Z}) \) and \( x = x_\theta, n + x_\xi, n \) for all \( n \in \mathbb{Z} \). The norm on \((E_\theta, E_\theta)_{\theta,p}\) is given by

\[
\|x\| = \inf_{(x_\theta, x_\xi)_{n \in \mathbb{Z}}} \left( \left( \sum_{n \in \mathbb{Z}} \|e^{\xi \delta n} x_\theta, n\|_{E_\theta}^p \right)^{1/p}, \left( \sum_{n \in \mathbb{Z}} \|e^{\xi \delta n} x_\xi, n\|_{E_\theta}^p \right)^{1/p} \right) < \infty. \quad (C.1)
\]

The equivalence of the two definitions of \((E_\theta, E_\theta)_{\theta,p}\) given above is established in [40, p.18].

We have the following generalisation of [30, Proposition 2.2].

**Proposition C.0.6.** Let \(1 < p < \infty\), \(0 < \theta < 1\) and \(\mathcal{I}\) and \(\mathcal{J}\) surjective operator ideals such that \((\mathcal{I}, \mathcal{J})\) is a \(\Sigma_p\)-pair and \(\mathcal{J}\) is injective. Suppose that \((E_\theta, E_\theta)\) is an interpolation pair and let \(I\) be the canonical injection of \(E_\theta \cap E_\theta \) into \(E_\theta + E_\theta\). If \(I \in \mathcal{I}(E_\theta \cap E_\theta, E_\theta + E_\theta)\), then \((E_\theta, E_\theta)_{\theta,p} \in \text{Space}(\mathcal{I}).\)
Proof. Denote $E = E_b \cap E_\ell$, $F = E_\ell + E_b$ and $G = (E_b, E_\ell)_{\theta, p}$, and for each $n \in \mathbb{Z}$ define equivalent norms on $E$ and $F$ as follows:

$$
||x||_{E,n} = \max \left( \|e^{\xi n} x\|_{E_b}, \|e^{\xi n} x\|_{E_\ell} \right), \quad x \in E; \\
||y||_{F,n} = \inf \left\{ \|e^{\xi n} y_b\|_{E_b} + \|e^{\xi n} y_\ell\|_{E_\ell} \mid y = y_b + y_\ell, y_b \in E_b, y_\ell \in E_\ell \right\}, \quad y \in F.
$$

For each $n \in \mathbb{Z}$ set $E_n = (E, \| \cdot \|_{E,n})$ and $F_n = (F, \| \cdot \|_{F,n})$. Then $(E_b, E_\ell)_{\theta, p}$ is isomorphic to the spaces $G_1$ and $G_2$ defined as follows:

$$
G_1 = \left\{ y \in F \mid y = \sum_{n \in \mathbb{Z}} x_n, \; x_n \in E_n, \; \|y\|_{G_1} = \inf_{y = \sum_{n \in \mathbb{Z}} x_n} \left( \sum_{n \in \mathbb{Z}} \|x_n\|_{E,n}^p \right)^{1/p} < \infty \right\},
$$

and

$$
G_2 = \left\{ y \in F \mid \|y\|_{G_2} = \left( \sum_{n \in \mathbb{Z}} \|x_n\|_{F,n}^p \right)^{1/p} < \infty \right\}.
$$

To see that $(E_b, E_\ell)_{\theta, p}$ is indeed isomorphic to the spaces $G_1$ and $G_2$ defined above, one first shows that $\| \cdot \|_{G_1}$ is equivalent to the norm defined at (C.1). To see the equivalence, notice that the norm defined at (C.1) is determined by maxima of two $\ell_p$-sums of norms, whereas $\| \cdot \|_{G_1}$ is determined by $\ell_p$-sums of maxima of pairs of norms; the equivalence of $\| \cdot \|_{G_1}$ and the norm defined at (C.1) is thus seen to follow from essentially the same arguments as those used to show that the map

$$
\ell_p \oplus \ell_p \longrightarrow \ell_p(\ell^2) : ((a_n)_n, (b_n)_n) \mapsto ((a_n, b_n))_n
$$

is an isomorphism. Similarly, one shows that $\| \cdot \|_{G_2}$ is equivalent to the norm defined at (C.1) through consideration of the isomorphism

$$
\ell_p \oplus \ell_p \longrightarrow \ell_p(\ell_1) : ((a_n)_n, (b_n)_n) \mapsto ((a_n, b_n))_n.
$$

By the isomorphism of $G$ with $G_1$, the map

$$
Q : \left( \bigoplus_{n \in \mathbb{Z}} E_n \right)_p \longrightarrow G : (x_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} x_n
$$
is a continuous linear surjection. By the isomorphism of $G$ with $G_2$, the map

$$J : G \longrightarrow \left( \bigoplus_{n \in \mathbb{Z}} F_n \right)_p : z \mapsto (\ldots, z, z, z, \ldots)$$

is an isomorphic embedding. For $m, n \in \mathbb{Z}$, the composition $Q_{\{n\}}JQU_{\{m\}}$ is precisely the canonical injection of $E_m$ into $F_n$ (here $Q_{\{n\}}$ and $U_{\{m\}}$ are as defined in Defintion 2.1.1), hence $Q_{\{n\}}JQU_{\{m\}} \in \mathcal{F}$ by the hypothesis of the proposition. Moreover, it follows by straightforward linear algebra that $Q_gJQU_x \in \mathcal{F}$ for all $\mathcal{F}, G \in \mathbb{Z}^\infty$, hence

$$JI_GQ = JQ \in \mathcal{F} \quad (C.3)$$

since $(\mathcal{F}, \mathcal{F})$ is a $\Sigma_p$-pair. As $\mathcal{F}$ is injective and surjective, we deduce from (C.3), Theorem 2.2.2 and Theorem 2.2.3 that $I_G \in \mathcal{F}$, hence $G \in \text{Space}(\mathcal{F})$. □

**Proof of Theorem 2.2.7.** Let $E$ and $F$ be Banach spaces and $T \in \mathcal{F}(E, F)$. We assume, as we may, that $\|T\| \leq 1$. Let $E_b = T(E)$ be equipped with the norm

$$\|y\|_b = \inf \{\|x\| \mid Tx = y\}, \quad y \in E_b.$$

It is elementary to see that absolutely convergent series in $E_b$ are convergent, hence $E_b$ is a Banach space (in fact, it is not difficult to see that $(E_b, \|\cdot\|_b)$ is, up to isometric isomorphism, $E/\ker(T)$ equipped with the usual quotient norm).

Let $T_0 : E \longrightarrow E_b$ denote the operator induced by setting $T_0x = Tx$, $x \in E$, and $I : E_b \longrightarrow F$ the formal inclusion operator. Then $T_0$ is surjective and $T = IT_0$; as $\mathcal{F}$ is surjective, $I \in \mathcal{F}(E_b, F)$.

$(E_b, F)$ is an interpolation pair, with $E_b \cap F = E_b$ and $E_b + F = F$. It follows from the definition of $\|\cdot\|_b$ and the assumption that $\|T\| \leq 1$, that $\|y\|_F \leq \|y\|_b$ for all $y \in E_b$, hence $\|\cdot\|_{E_b \cap F} = \|\cdot\|_b$ and $\|\cdot\|_{E_b + F} = \|\cdot\|_F$. By Proposition C.0.6, $(E_b, F)_{\theta, p} \in \text{Space}(\mathcal{F})$ for $0 < \theta < 1$. Fix $\theta \in (0, 1)$ and let $I_1 : E_b \longrightarrow (E_b, F)_{\theta, p}$ and $I_2 : (E_b, F)_{\theta, p} \longrightarrow F$ denote the corresponding embedding maps. The equation $T = I_2(I_1T_0)$ provides the desired factorisation. □
A Banach space $E$ is said to have $w^*$-sequentially compact dual ball if $(B_{E^*}, w^*)$ is sequentially compact. There are certainly Banach spaces that lack this property - for example, $\ell_1(2^{\aleph_0})$ and $C(K)$ for non-sequentially compact $K$ (note that $K$ embeds homeomorphically into $(B_{C(K)^*}, w^*)$ via the map $k \mapsto \delta_k$, $k \in K$). On the other hand, the class of Banach spaces that do have $w^*$-sequentially compact dual ball is very large. Indeed, it includes all separable Banach spaces and all Asplund spaces. That separable Banach spaces have $w^*$-sequentially compact dual ball is an easy consequence of the $w^*$-metrisability of the dual ball of a separable space. The fact that Asplund spaces have $w^*$-sequentially compact dual ball is due to C. Stegall [59, Theorem 3.5]. According to Diestel [15, p.226], a useful characterisation of Banach spaces having $w^*$-sequentially compact dual ball seems to be lacking. For further reading on Banach spaces having (or not) $w^*$-sequentially compact dual ball, [15, Chapter XIII] is an excellent starting point.

The following definition introduces a natural extension of the notion of $w^*$-sequentially compact dual ball to the setting of operators.

**Definition D.0.7.** Let $E$ and $F$ be Banach spaces and $T \in \mathcal{B}(E, F)$. We say that $T^*$ is $w^*$-sequentially compact if $T^*B_{F^*}$ is $w^*$-sequentially compact. We denote by
\( \mathcal{M} \) the class of all operators having \( w^* \)-sequentially compact adjoint, and for each pair of Banach spaces \((E, F)\) we define \( \mathcal{M}(E, F) := \mathcal{B}(E, F) \cap \mathcal{M} \).

It seems that there is little or no mention of the class \( \mathcal{M} \) in the literature. At first, this is surprising since the definition of \( \mathcal{M} \) seems quite natural. On the other hand, since a useful characterisation of Banach spaces having \( w^* \)-sequentially compact dual ball seems to be lacking, and since the class of all such Banach spaces includes many Banach spaces of interest to analysts (thus \( \mathcal{M} \) coincides with \( \mathcal{B} \) on many Banach spaces), it is perhaps not too surprising that no systematic study of \( \mathcal{M} \) - from the point of view of operator ideal theory, in particular - seems to have appeared thus far. One paper that has dealt with \( w^* \)-sequential compactness of operators is [22], where a study is made of operators \( T : X \to F \) such that \( X \) is a Banach lattice not containing a complemented copy of \( \ell_1 \), \( F \) is any Banach space and \( T \notin \mathcal{M} \).

Our goal here is to provide a brief initial study into \( \mathcal{M} \) from the point of view of operator ideal theory. We will show that \( \mathcal{M} \) is an operator ideal and study some of its operator ideal properties. In particular, we shall prove the following.

**Theorem D.0.8.** \( \mathcal{M} \) is a closed, injective, surjective operator ideal having the factorisation property.

The following proposition collects together several results for sequentially compact spaces that we shall need to prove Theorem D.0.8.

**Proposition D.0.9.**

(i) Each closed subset of a sequentially compact space is sequentially compact.

(ii) If \( X \) is sequentially compact, \( Y \) is a Hausdorff space and \( f : X \to Y \) is a continuous mapping such that \( f(X) \) is closed in \( Y \), then \( f(X) \) is sequentially compact.

(iii) The Cartesian product (with the product topology) of countably many sequentially compact spaces is sequentially compact.

**Proof.** The permanence properties (i) and (ii) follow immediately from the definition of sequential compactness. A proof of (iii) is given in [20, Theorem 3.10.35].
Let $E$ and $F$ be Banach spaces and $T : E \to F$ an operator. In light of Proposition D.0.9, it is elementary to see that $T \in \mathcal{M}$ if and only if $T^*(K)$ is $w^*$-sequentially compact for all $w^*$-compact $K \subseteq F^*$, if and only if $T^*$ carries bounded sequences to sequences having a $w^*$-convergent subsequence.

We state without proof the following classical result (see [19, p.28])

**Lemma D.0.10.** Let $I_1$ and $I_2$ be directed sets and let $I_1 \times I_2$ be directed by setting $(i_1, i_2) \prec (i'_1, i'_2)$ if and only if $i_1 \prec i'_1$ and $i_2 \prec i'_2$. Let $X$ be a complete metric space and $\varphi : I_1 \times I_2 \to X$ a net such that:

(a) for each $i_2 \in I_2$, the limit $g(i_2) = \lim_{i_1} \varphi((i_1, i_2))$ exists, and

(b) the limit $h(i_1) = \lim_{i_2} \varphi((i_1, i_2))$ exists uniformly on $I_1$.

Then $\lim_{i_1} h(i_1)$ exists.

**Proof of Theorem D.0.8.** We first show that $\mathcal{M}$ is in fact an operator ideal; that is, that $\mathcal{M}$ satisfies $O\text{I}_1$-$O\text{I}_3$ of Definition 2.2.1. It is clear that $O\text{I}_1$ holds. For $O\text{I}_2$, let $S, T \in \mathcal{M}(E, F)$ and let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $F^*$. Then there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $(S^*x_{n_k})_{k \in \mathbb{N}}$ is $w^*$-convergent, and a further subsequence $(x_{n_{k_j}})_{j \in \mathbb{N}}$ such that $(T^*x_{n_{k_j}})_{j \in \mathbb{N}}$ is $w^*$-convergent, hence $((S + T)^*x_{n_{k_j}})_{j \in \mathbb{N}}$ is $w^*$-convergent. It follows that $S + T \in \mathcal{M}$. We now show that $\mathcal{M}$ satisfies $O\text{I}_3$. Let $D, E, F$ and $G$ be Banach spaces and $U \in \mathcal{B}(D, E)$, $T \in \mathcal{M}(E, F)$ and $V \in \mathcal{B}(F, G)$ operators. Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $G^*$. As $V^*$ is bounded, $(V^*x_n)_{n \in \mathbb{N}}$ is a bounded sequence in $G^*$, and so $(T^*V^*x_n)_{n \in \mathbb{N}}$ has a $w^*$-convergent subsequence, say $(T^*V^*x_{n_k})_{k \in \mathbb{N}}$, because $T \in \mathcal{M}$. Hence $(U^*T^*V^*x_{n_k})_{k \in \mathbb{N}}$ is $w^*$-convergent by $w^*$-continuity of $U^*$.

Let $E$ and $F$ be Banach spaces and $T \in \mathcal{B}(E, F)$. The injectivity of $\mathcal{M}$ is then an immediate consequence of the equality $(3F_{T^*})^*B_{\ell_\infty(B_{F^*})^*} = T^*B_{F^*}$.

The injection $\Omega_E^*$ maps $T^*B_{F^*}$ $w^*$-homeomorphically onto $(T\Omega_E)^*B_{F^*}$, hence $T^*B_{F^*}$ is $w^*$-sequentially compact if and only if $(T\Omega_E)^*B_{F^*}$ is. It follows that $\mathcal{M}$ is surjective.

We now show that $\mathcal{M}$ is closed. Let $E$ and $F$ be Banach spaces, $(T_n)_{n}$ a sequence in $\mathcal{M}(E, F)$ and $T$ an element of $\mathcal{B}(E, F)$ such that $\|T_n - T\| \to 0$. 

Suppose that \((x_m)_{m \in \mathbb{N}}\) is a bounded sequence in \(F^*\). Then, since \(T_n \in \mathcal{M}\) for all \(n\), a standard diagonal argument allows us to extract a subsequence \((z_k)_{k \in \mathbb{N}}\) of \((x_m)_{m \in \mathbb{N}}\) such that \((T_n^*z_k)_{k \in \mathbb{N}}\) is \(w^*\)-convergent for every \(n\). Let \(u \in E\) be fixed. Then, since \(\|T_n - T\| \to 0\) and \((z_k)_{k \in \mathbb{N}}\) is bounded, the limit \(\lim_n (T_n^*z_k, u)\) exists uniformly over \(k \in \mathbb{N}\). It follows now by Lemma D.0.10 that the limit

\[
\lim_k (T^*z_k, u) = \lim_k \lim_n (T_n^*z_k, u)
\]

exists. It follows that the map \(z : E \to \mathbb{K}\) given by setting \(z(u) = \lim_k (T^*z_k, u)\) for each \(u \in E\) is well-defined. Moreover, \(z\) is easily seen to be linear and bounded with \(\|z\| \leq \|T\| \cdot \sup_k \|z_k\|\), hence \(z \in E^*\). It is clear now from the definition of \(z\) that \((T^*z_k)_{k \in \mathbb{N}}\) converges \(w^*\) to \(z\). It follows that \(T \in \mathcal{M}\), hence \(\mathcal{M}(E, F)\) is closed in \(\mathcal{B}(E, F)\). In particular, \(\mathcal{M}\) is a closed operator ideal, as claimed.

It now only remains to show that \(\mathcal{M}\) has the factorization property. By Corollary 2.2.8 and the properties of \(\mathcal{M}\) established thus far in the current proof, it suffices to show that for some (or all) \(1 < p < \infty\), \((\mathcal{M}, \mathcal{M})\) is a \(\Sigma_p\)-pair. We shall achieve this via this following two lemmas. In what follows, our notation adheres to the conventions set out in Definition 2.1.1.

**Lemma D.0.11.** Let \(1 < p < \infty\), \(F\) a Banach space, \((E_m)_{m \in \mathbb{N}}\) a sequence of Banach spaces and \(T \in \mathcal{B}((\bigoplus_{m \in \mathbb{N}} E_m)_p, F)\). Let \(\prod_{m \in \mathbb{N}} (U_{(m)}^* T^* B_{F^*}, w^*)\) be equipped with the product topology. Then \((T^* B_{F^*}, w^*)\) is homeomorphic to a closed subset of \(\prod_{m \in \mathbb{N}} (U_{(m)}^* T^* B_{F^*}, w^*)\). It follows that if \(T U_{(m)} \in \mathcal{M}\) for every \(m \in \mathbb{N}\), then \(T \in \mathcal{M}\).

**Proof.** For each \(j \in \mathbb{N}\) denote by \(\tau_j : \prod_{m \in \mathbb{N}} (U_{(m)}^* T^* B_{F^*}, w^*) \to (U_{(j)}^* T^* B_{F^*}, w^*)\) the \(j\)th coordinate projection. Following the convention established in Section 2.1, we consider \(T^* B_{F^*}\) as a subset of \((\bigoplus_{m \in \mathbb{N}} E_m)_q \subseteq \prod_{m \in \mathbb{N}} E_m\), where \(q\) is dual to \(p\). The formal inclusion map \(i : T^* B_{F^*} \hookrightarrow \prod_m (U_{(m)}^* T^* B_{F^*}, w^*)\) is \(w^*\)-to-product continuous since the composition \(\tau_j \circ i = U_{(j)}^* |_{T^* B_{F^*}}\) is \(w^*\)-continuous for each \(j \in \mathbb{N}\). As \(i\) is also one-to-one, it is thus a homeomorphic embedding of the compact space \((T^* B_{F^*}, w^*)\) into \(\prod_{m \in \mathbb{N}} (U_{(m)}^* T^* B_{F^*}, w^*)\).
Suppose that $TU_{\{m\}} \in \mathcal{M}$ for every $m \in \mathbb{N}$. Then $\prod_{n \in \mathbb{N}} (U^*_{\{m\}} T^* B_{F*}, w^*)$ is sequentially compact in the product topology by Proposition D.0.9(iii). By the first assertion of the lemma and Proposition D.0.9(ii), $(T^* B_{F*}, w^*)$ is sequentially compact. That is, $T \in \mathcal{M}$.

Lemma D.0.12. Let $1 < p < \infty$, $E$ a Banach space, $(F_n)_{n \in \mathbb{N}}$ a sequence of Banach spaces and $T \in \mathcal{B}(E, (\bigoplus_{n \in \mathbb{N}} F_n)_p)$. Let $\prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$ be equipped with the product topology. Then $(T^* B(\bigoplus_{n \in \mathbb{N}} F_n)_p, w^*)$ is the continuous image of a closed subset of $\prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$. It follows that if $Q_{\{1, \ldots, n\}} T \in \mathcal{M}$ for every $n \in \mathbb{N}$, then $T \in \mathcal{M}$.

Proof. Let $\varphi : (B(\bigoplus_{n \in \mathbb{N}} F_n)_p, w^*) \rightarrow \prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$ be the map satisfying $\varphi(y^*) = (T^* Q^*_{\{1, \ldots, n\}} V^*_{\{1, \ldots, n\}} y^*)_{n \in \mathbb{N}}$ for $y^* \in B(\bigoplus_{n \in \mathbb{N}} F_n)_p$. For $j \in \mathbb{N}$, denote by $\sigma_j : \prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*) \rightarrow (T^* Q^*_{\{1, \ldots, j\}} B(\bigoplus_{i=1}^j F_i)_p, w^*)$ the $j$th coordinate projection. For each $j \in \mathbb{N}$, the composition $\sigma_j \circ \varphi$ is precisely the restriction of $(V_{\{1, \ldots, j\}} Q_{\{1, \ldots, j\}} T)^* \rightarrow B(\bigoplus_{n \in \mathbb{N}} F_n)_p$, hence each $\sigma_j \circ \varphi$ is continuous. It follows that $\varphi$ is also continuous, and so $\varphi(B(\bigoplus_{n \in \mathbb{N}} F_n)_p)$ is a closed subset of $\prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$ since $(B(\bigoplus_{n \in \mathbb{N}} F_n)_p, w^*)$ is compact.

For $y^* \in B(\bigoplus_{n \in \mathbb{N}} F_n)_p$ we have $\|T^* Q^*_{\{1, \ldots, n\}} V^*_{\{1, \ldots, n\}} y^* - T^* y^*\|_n \rightarrow 0$, hence the map $\psi : \varphi(B(\bigoplus_{n \in \mathbb{N}} F_n)_p) \rightarrow T^* B_{F*}$ satisfying $\psi((T^* Q^*_{\{1, \ldots, n\}} V^*_{\{1, \ldots, n\}} y^*)_{n \in \mathbb{N}}) = T^* y^*$ is well-defined and surjective. Moreover, since $\varphi$ is a quotient mapping (in the general topological sense [20, §2.4]) of $B(\bigoplus_{n \in \mathbb{N}} F_n)_p$ onto its range (as a continuous mapping of a compact Hausdorff space), and since $\psi \circ \varphi = T^* |_{B(\bigoplus_{n \in \mathbb{N}} F_n)_p}$ is $w^*$-continuous, it follows that $\psi$ is a continuous mapping of the closed subset $\varphi(B(\bigoplus_{n \in \mathbb{N}} F_n)_p)$ of $\prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$ onto $(T^* B(\bigoplus_{n \in \mathbb{N}} F_n)_p, w^*)$. This proves the first assertion of the lemma.

Suppose that $Q_{\{1, \ldots, n\}} T \in \mathcal{M}$ for every $n \in \mathbb{N}$. Proposition D.0.9(iii) implies that $\prod_{n \in \mathbb{N}} (T^* Q^*_{\{1, \ldots, n\}} B(\bigoplus_{i=1}^n F_i)_p, w^*)$ is sequentially compact in the product topology. By the first assertion of the lemma and Proposition D.0.9(i) and (ii), $(T^* B_{F*}, w^*)$ is sequentially compact. That is, $T \in \mathcal{M}$.

It is now easily verified that $(\mathcal{M}, \mathcal{M})$ is a $\Sigma_p$-pair for all $p \in (1, \infty)$. Indeed, fix
$1 < p < \infty$ and let $(E_m)_{m \in \mathbb{N}}$ and $(F_n)_{n \in \mathbb{N}}$ be sequences of Banach spaces. Suppose that $T \in \mathcal{B}((\bigoplus_{m \in \mathbb{N}} E_m)_p, (\bigoplus_{n \in \mathbb{N}} F_n)_p)$ is such that $Q \hspace{1mm} T U \in \mathcal{J}$ for all $\mathcal{F}, \mathcal{G} \in \mathbb{N}^{< \infty}$. Then $Q_{\{1,\ldots,n\}} T U_{\{m\}} \in \mathcal{M}$ for all $m, n \in \mathbb{N}$. By Lemma D.0.11, $Q_{\{1,\ldots,n\}} T \in \mathcal{M}$ for all $n \in \mathbb{N}$, hence $T \in \mathcal{M}$ by Lemma D.0.12. The proof of Theorem D.0.8 is complete. \hfill \square

Remark D.0.13. The reasoning behind the proof of Lemma D.0.11 shows that the class of Banach spaces having $w^*$-sequentially compact dual ball is stable under taking $\ell_p$-direct sums ($1 < p < \infty$) over countable index sets. The method extends to the case of $c_0$-direct sums and $\ell_1$-direct sums. For $p \neq 1$, it is easy to extend this result to $\ell_p$-direct sums over index sets of arbitrary cardinality using the fact that the union of the supports of the terms of a sequence in $(\bigoplus_{\lambda \in \Lambda} E_\lambda)^* = (\bigoplus_{\lambda \in \Lambda} E_\lambda^*)_q$ is countable (here, $q$ is dual to $p$). The same arguments yield analogous results for $c_0$-direct sums.
Bibliography

[1] Albiac, F., and Kalton, N. J. *Topics in Banach space theory*, vol. 233 of *Graduate Texts in Mathematics*. Springer, New York, 2006.

[2] Alspach, D., and Odell, E. Averaging weakly null sequences. In *Functional analysis (Austin, TX, 1986–87)*, vol. 1332 of *Lecture Notes in Math*. Springer, Berlin, 1988, pp. 126–144.

[3] Alspach, D. E. Quotients of $C[0, 1]$ with separable dual. *Israel J. Math.* 29, 4 (1978), 361–384.

[4] Alspach, D. E. $C(K)$ norming subsets of $C[0, 1]$. *Studia Math.* 70, 1 (1981), 27–61.

[5] Alspach, D. E. Operators on $C(\omega^\alpha)$ which do not preserve $C(\omega^\alpha)$. *Fund. Math.* 153, 1 (1997), 81–98.

[6] Alspach, D. E., and Benyamini, Y. $C(K)$ quotients of separable $L_\infty$ spaces. *Israel J. Math.* 32, 2-3 (1979), 145–160.

[7] Asplund, E. Fréchet differentiability of convex functions. *Acta Math.* 121 (1968), 31–47.

[8] Baudier, F., Kalton, N., and Lancien, G. A new metric invariant for Banach spaces. arXiv:0912.5113 (2009).
[9] Bessaga, C., and Pełczyński, A. Spaces of continuous functions. IV. On isomorphical classification of spaces of continuous functions. *Studia Math.* 19 (1960), 53–62.

[10] Birkhoff, G. *Lattice Theory*. American Mathematical Society Colloquium Publications, vol. 25, revised edition. American Mathematical Society, New York, N. Y., 1948.

[11] Bossard, B. An ordinal version of some applications of the classical interpolation theorem. *Fund. Math.* 152, 1 (1997), 55–74.

[12] Bourgain, J. The Szlenk index and operators on $C(K)$-spaces. *Bull. Soc. Math. Belg. Sér. B* 31, 1 (1979), 87–117.

[13] Davis, W. J., Figiel, T., Johnson, W. B., and Pełczyński, A. Factoring weakly compact operators. *J. Functional Analysis* 17 (1974), 311–327.

[14] Deville, R., Godefroy, G., and Zizler, V. *Smoothness and renormings in Banach spaces*, vol. 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1993.

[15] Diestel, J. *Sequences and series in Banach spaces*, vol. 92 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984.

[16] Diestel, J. Introduction to classical descriptive set theory (lecture notes). *Notas Mat.* 140 (1994).

[17] Domínguez Benavides, T. The Szlenk Index and the Fixed Point Property under Renorming. *Fixed Point Theory Appl.* (2010). Art. ID 268270.

[18] Dugundji, J. *Topology*. Allyn and Bacon Inc., Boston, Mass., 1966.

[19] Dunford, N., and Schwartz, J. T. *Linear Operators. I. General Theory*. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York, 1958.
[20] Engelking, R. General topology, second ed., vol. 6 of Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, 1989. Translated from Polish by the author.

[21] Fabian, M., Habala, P., Hájek, P., Montesinos Santalucía, V., Pelant, J., and Zizler, V. Functional analysis and infinite-dimensional geometry. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York, 2001.

[22] Figiel, T., Ghoussoub, N., and Johnson, W. B. On the structure of nonweakly compact operators on Banach lattices. Math. Ann. 257, 3 (1981), 317–334.

[23] Gasparis, I. Operators on $C(K)$ spaces preserving copies of Schreier spaces. Trans. Amer. Math. Soc. 357, 1 (2005), 1–30.

[24] Giles, J. R. Convex analysis with application in the differentiation of convex functions, vol. 58 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[25] Gillespie, D. C., and Hurwitz, W. A. On sequences of continuous functions having continuous limits. Trans. Amer. Math. Soc. 32, 3 (1930), 527–543.

[26] Hájek, P., and Lancien, G. Various slicing indices on Banach spaces. Mediterr. J. Math. 4, 2 (2007), 179–190.

[27] Hájek, P., Lancien, G., and Montesinos, V. Universality of Asplund spaces. Proc. Amer. Math. Soc. 135, 7 (2007), 2031–2035.

[28] Hájek, P., Montesinos Santalucía, V., Vanderwerff, J., and Zizler, V. Biorthogonal systems in Banach spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26. Springer, New York, 2008.

[29] Hausdorff, F. Set theory. Chelsea Publishing Company, New York, 1957. Translated by John R. Aumann, et al.
[30] Heinrich, S. Closed operator ideals and interpolation. *J. Funct. Anal.* 35, 3 (1980), 397–411.

[31] Jarchow, H. On weakly compact operators on $C^*$-algebras. *Math. Ann.* 273, 2 (1986), 341–343.

[32] Johnson, W. B. Factoring compact operators. *Israel J. Math.* 9 (1971), 337–345.

[33] Johnson, W. B., and Rosenthal, H. P. On $\omega^*$-basic sequences and their applications to the study of Banach spaces. *Studia Math.* 43 (1972), 77–92.

[34] Kechris, A. S. *Classical descriptive set theory*, vol. 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[35] Knaust, H., Odell, E., and Schlumprecht, T. On asymptotic structure, the Szlenk index and UKK properties in Banach spaces. *Positivity* 3, 2 (1999), 173–199.

[36] Lancien, G. *Applications de la théorie de l’indice en géométrie des espaces de Banach*. PhD thesis, Université de Paris VI - Pierre et Marie Curie, 1992.

[37] Lancien, G. Dentability indices and locally uniformly convex renormings. *Rocky Mountain J. Math.* 23, 2 (1993), 635–647.

[38] Lancien, G. On the Szlenk index and the weak*-dentability index. *Quart. J. Math. Oxford Ser.* (2) 47, 185 (1996), 59–71.

[39] Lancien, G. A survey on the Szlenk index and some of its applications. *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 100, 1-2 (2006), 209–235.

[40] Lions, J.-L., and Peetre, J. Sur une classe d’espaces d’interpolation. *Inst. Hautes Études Sci. Publ. Math.*, 19 (1964), 5–68.
[41] Mazur, S. über konvexe Mengen in linearen normierten Räumen. *Studia Math.* 4 (1933), 70–84.

[42] Megginson, R. E. *An introduction to Banach space theory*, vol. 183 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[43] Odell, E. Ordinal indices in Banach spaces. *Extracta Math.* 19, 1 (2004), 93–125.

[44] Odell, E., Schlumprecht, T., and Zsák, A. Banach spaces of bounded Szlenk index. *Studia Math.* 183, 1 (2007), 63–97.

[45] Pełczyński, A. On the isomorphism of the spaces $m$ and $M$. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 6 (1958), 695–696.

[46] Pełczyński, A. On strictly singular and strictly cosingular operators. II. Strictly singular and strictly cosingular operators in $L(\nu)$-spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 13 (1965), 37–41.

[47] Pełczyński, A. On $C(S)$-subspaces of separable Banach spaces. *Studia Math.* 31 (1968), 513–522.

[48] Pełczyński, A. Universal bases. *Studia Math.* 32 (1969), 247–268.

[49] Pietsch, A. *Operator ideals*, vol. 20 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1980. Translated from German by the author.

[50] Raja, M. On weak* uniformly Kadec-Klee renormings. *Bull. Lond. Math. Soc.* 42, 2 (2010), 221–228.

[51] Reǐnov, O. I. RN-sets in Banach spaces. *Funktsional. Anal. i Prilozhen.* 12, 1 (1978), 80–81, 96.

[52] Rosenthal, H. P. On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measure $\mu$. *Acta Math.* 124 (1970), 205–248.
[53] Rosenthal, H. P. On relatively disjoint families of measures, with some applications to Banach space theory. *Studia Math.* 37 (1970), 13–36.

[54] Rosenthal, H. P. On factors of $C([0, 1])$ with non-separable dual. In *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972)* (1972), vol. 13, pp. 361–378 (1973); correction, ibid. 21 (1975), no. 1, 93–94.

[55] Rosenthal, H. P. The Banach spaces $C(K)$. In *Handbook of the geometry of Banach spaces, Vol. 2*. North-Holland, Amsterdam, 2003, pp. 1547–1602.

[56] Samuel, C. Indice de Szlenk des $C(K)$ ($K$ espace topologique compact dénombrable). In *Seminar on the geometry of Banach spaces, Vol. I, II (Paris, 1983)*, vol. 18 of *Publ. Math. Univ. Paris VII*. Univ. Paris VII, Paris, 1984, pp. 81–91.

[57] Schechtman, G. On Pełczyński’s paper “Universal bases” (Studia Math. 32 (1969), 247–268). *Israel J. Math.* 22, 3-4 (1975), 181–184.

[58] Semadeni, Z. Banach spaces of continuous functions. Vol. I. PWN—Polish Scientific Publishers, Warsaw, 1971. Monografie Matematyczne, Tom 55.

[59] Stegall, C. The Radon-Nikodým property in conjugate Banach spaces. II. *Trans. Amer. Math. Soc.* 264, 2 (1981), 507–519.

[60] Stephani, I. Injektive Operatorenideale über der Gesamttheit aller Banachräume und ihre topologische Erzeugung. *Studia Math.* 38 (1970), 105–124.

[61] Stephani, I. Surjektive Operatorenideale über der Gesamttheit aller Banach-Räume. *Wiss. Z. Friedrich-Schiller-Univ. Jena* 21, 1 (1972), 187–206.

[62] Stephani, I. Operator ideals with the factorization property and ideal classes of Banach spaces. *Math. Nachr.* 74 (1976), 201–210.
[63] Szlenk, W. The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces. *Studia Math.* 30 (1968), 53–61.

[64] Wojtaszczyk, P. Existence of some special bases in Banach spaces. *Studia Math.* 47 (1973), 83–93.

[65] Zalcwasser, Z. Sur une propriété du champ des fonction continues. *Studia Math.* 2 (1930), 63–67.
There are two kinds of mathematical contributions: work that’s important to the history of mathematics, and work that’s simply a triumph of the human spirit.

- Paul Cohen, Stanford, 1996.