Self-organization of dissipationless solitons in negative refractive index materials

V. Skarka, N. B. Aleksic, and V. I. Berzhiani

Laboratory POMA, UMR 6136 CNRS, University Angers, 2, boulevard Lavoisier, 49015 Angers, France
Institute of Physics, Pregrevica 118, 11000 Belgrade, Serbia and
Andronikashvili Institute of Physics, 6 Tamarashvili, Tbilisi 0177, Georgia

General nonlinear and nonparaxial dissipative complex Helmholtz equations for magnetic and electric fields propagating in negative refractive index materials (NIMs) are derived from Maxwell equations. In order to describe nonconservative soliton dynamics in NIMs, such coupled equations are reduced into generalized Ginzburg-Landau equation. Cross-compensation between the excess of saturating nonlinearity, losses, and gain renders these self-organized solitons dissipationless and exceptionally robust. The presence of such solitons makes NIMs effectively dissipationless.

Metamaterials are novel artificial composite structures manufactured in order to display various peculiar very promising properties [1]. The most studied nowadays are negative-refractive-index materials (NIMs) assembled in such a way to exhibit simultaneously negative effective permittivity $\varepsilon$ and permeability $\mu$ [2,3]. A number of challenging optical NIM devices is supposed to work in interaction with lasers [4]. As a consequence, nonlinear and nonparaxial dissipative complex Helmholtz equations for magnetic and electric fields propagating in negative refractive index materials (NIMs) are derived [5,6,7]. A very promising class of self-organized dissipative complex Helmholtz equations for magnetic and electric fields are considered [5], [6], [7], [8]. A wide class of dissipative systems, ranging from nonlinear optics, plasma physics, and Bose-Einstein condensates, can be modeled by complex Schrödinger type equations [3], [4]. A number of Bose-Einstein condensates, can be modeled by complex fluid dynamics to superfluidity, superconductivity, and the cloaking [3], are either altered or annihilated by the dissipation [4]. A wide class of dissipative systems, ranging from nonlinear optics, plasma physics, and Bose-Einstein condensates, can be modeled by complex Ginzburg-Landau equation [10].

In order to study propagation of EM field in optical NIMs, we derive, in this letter, ab initio from Maxwell equations (ME) general nonlinear and nonparaxial dissipative complex Helmholtz equations for magnetic and electric fields. A very promising class of self-organized localized EM structures are spatiotemporal solitons rendered dissipationless due to the cross-compensation of the excess of saturating nonlinearity, losses, and gain [11]. Obtained coupled equations are reduced into generalized Ginzburg-Landau equation in order to describe for the first time dissipationless soliton dynamics in NIMs. Such a NIM in presence of dissipationless solitons may be considered as an effectively dissipationless novel active composite metamaterial.

The propagation of EM radiation in a nonlinear media is described by ME $\text{rot} E = -\partial B / \partial t$, $\text{rot} H = \partial D / \partial t$ and constitutive relations between the magnetic induction $B$ and the magnetic field $H$ as well as between the electric field $E$ and the electric induction $D$. The response of the medium to a quasi-monochromatic EM wave is considered. Real vectorial fields $F = (E, B, H, D)$ read $F(r, t) = F \exp (i k r - \omega t) + c.c.$, where $F$ is slowly varying function in space and time and $\omega$ is the carrying pulse frequency. The complex wave vector $k = k_0 + i k_1$ is determined by the linear dispersion relation $k^2 c^2 / \omega^2 = \varepsilon (\omega) \mu (\omega)$. Respectively $\varepsilon$ and $\mu$ are complex permittivity and permeability of a dissipative medium.

In NIMs the refractive index is negative $n = -(\Re [\varepsilon])^{1/2} = k_0 c / \omega$, hence, the real part of wave vector $k_0$ is negative too. In the case of temporal dispersive media $\varepsilon (\omega)$ and $\mu (\omega)$ are expanded in the series around the carrying frequency up to the second order

$$
\partial D / \partial t \approx \varepsilon_0 (\alpha_x \partial E / \partial t + i \alpha'_{E} / 2 \partial^2 E / \partial t^2 - i \omega \varepsilon_0 E - i \omega \varepsilon_1 E) e^{-i \omega t} \quad \text{and} \quad \partial B / \partial t \approx \mu_0 (\alpha_y \partial H / \partial t + i \alpha'_{H} / 2 \partial^2 H / \partial t^2 - i \omega \mu_0 H - i \omega \mu_1 H) e^{-i \omega t},
$$

where $\alpha_x = \partial (\varepsilon_0 / \partial \omega)$, $\alpha'_{E} = \partial \varepsilon_0 / \partial \omega$, $\alpha_{y} = \partial (\mu_0 / \partial \omega)$, $\alpha_{H} = \partial (\mu_0 / \partial \omega)$, and $\alpha'_{H} = \partial \mu_0 / \partial \omega$. In order to take into account the nonlinear loss and gain the nonlinear permittivity $\varepsilon_{nl} (|E|^2) = \Re [\varepsilon_{nl}] + i 3 [\varepsilon_{nl}]$ and permeability $\mu_{nl} (|H|^2) = \Re [\mu_{nl}] + i 3 [\mu_{nl}]$ are considered as complex. Taking curl of ME and neglecting vectorial terms $(\nabla \cdot E = 0)$ yield following equations rewritten in new variables $\zeta = z$ and $\tau = t - z / v_g$ with real group velocity $v_g = (\partial (\omega / n) / \partial \omega)^{-1} = (\Re [\omega (\alpha_x + \varepsilon_0) (2 k c^{-1})])^{-1}$,

$$
2 i k \frac{\partial E}{\partial \zeta} - \frac{W}{v_g} \frac{\partial^2 E}{\partial \tau^2} + \Delta E + \frac{\omega^2}{c^2} \varepsilon_{nl} E = 0,
$$

(1)

$$
-\omega \mu_0 \mu_{nl} k \times H + \frac{\omega^2}{c^2} \Im [\varepsilon] \mu E = 0,
$$

and

$$
2 i k \frac{\partial H}{\partial \zeta} - \frac{W}{v_g} \frac{\partial^2 H}{\partial \tau^2} + \Delta H + \frac{\omega^2}{c^2} \mu_{nl} H
$$

$$
+ \omega \varepsilon_{nl} k \times E + i \frac{\omega^2}{c^2} \Im [\mu] H = 0,
$$

(2)

with the complex function $W = c^2 V^{-2} - 0.5 \omega \mu_0 \alpha'_{H} - 0.5 \omega \varepsilon_0 \alpha'_{E} - \alpha_x \alpha_e$. The second order $\omega$ derivatives in $\Delta E$ and $\Delta H$ should be kept in Eqs. (1-2) not only to account for nonparaxial pulse dynamics but also to have an adequate description of the strong dissipation. In best of our knowledge it is for the first time that such gen-
eralized complex Helmholtz equations describing simultaneously nonparaxial and dissipative effects in NIMs are ab initio derived from ME. However, investigation of the EM filed dynamics based on the generalized Helmholtz equations is beyond of intended scope of the current letter and will be presented elsewhere. Therefore, in what follows only weak dissipation is considered. As a consequence, the imaginary part of the dissipative parameter ε, permeability μ, and wave vector k are small. Using Drude model of free electron collisions (νe and νμ) and permeability and permittivity read respectively ε = 1 − ωp2[ω(ω + iνe)]−1 and μ = 1 − ωm2[ω(ω + iνμ)]−1 where ωp and ωm are electric and magnetic plasma frequencies [13]. For linearly polarized EM pulses (E = xE, H = yH) propagating along the z axis (k||z) Eqs. (1-2) become scalar ones. Applying paraxial approximation (k ≫ |∇|) and neglecting higher order z derivatives Eqs. (1-2) can be reduced to generalized Ginzburg-Landau equation in NIMs (GLENIM) for electric field

\[ i \frac{\partial E}{\partial z} - k^p \frac{\partial^2 E}{\partial t^2} + \frac{1}{\omega} \Delta E + i \varepsilon \frac{\partial |E|^4 E}{\partial t} = 0 \]  

(3)

and the equivalent magnetic field GLENIM obtained using medium impedance Z = (μ/ε)^1/2 = E/H. The following renormalisation is used: ωp ⊢ t, ω(ω + iνe) ⊢ z, ωp ⊢ ω, and (ε/ωp^2) ΔE ⊢ Δp = \partial^2 E/|\omega|^2 + \partial^2 E/\partial t^2. The renormalized frequency ω runs in Fig.1 till ω/ωp = 0.8 [12]. In this range the group velocity v_g is always positive and the phase velocity is negative. The group velocity dispersion (GVD) k^p = -(nω)\^1\^2R[W] is anomalous, thus, negative till ω = 0.7, changing sign after (see also Fig.2). As many materials, NIMs also can exhibit cubic Kerr nonlinearity which may be saturated [13, 14]. In order to prevent pulse collapse the cubic nonlinearity is usually saturated by a quintic one with the opposite sign εnl (|E|^2 = (ε^3)^2 + ε^3 |E|^2 - ε^3 + ε^3 |E|^4) and μnl (|H|^2 = (μ^3) |H|^2 - (μ^5) |H|^4). The sign of the real part of these indices determines the focusing/defocusing properties of the medium while imaginary parts are related to the nonlinear loss and gain. Nonlinear permeability and permittivity in synergy define the cubic susceptibility χ^3 = ε^3 Z + μ^3 Z^3 as well as the quintic one χ^5 = ε^5 Z + μ^5 Z^5, Corresponding relations for magnetic susceptibility \( \chi^H \) can be obtained using the impedance Z. Therefore, the system of coupled Eqs. (1-2) is reduced to a single GLENIM

\[ i \frac{\partial E}{\partial z} + \alpha \frac{\partial^2 E}{\partial t^2} + \sigma_1 \Delta E + (E^2) E + \nu |E|^2 E + \varepsilon_0 |E|^4 E = 0 \]  

(4)

with the renormalisation (2/k^n)^1/2t → t, \( \Delta \) ⊢ \|ω → \( \Delta \), and \( |E|/(\varepsilon_0^2)^1/2 \) → \( E \). In reality, every material exhibits losses. The dissipative parameter of linear loss 8 = -\varepsilon_0 |\varepsilon_\mu|/2n is always insuring background stability since in NIMs 3 |μ| < 0 [2]. The spectral filtering term is positive taking into account that \( \beta = 3 |W|/(n_\omega k^p) > 0 \) since 3 |W| < 0, as well as the energy diffusion term with \( \theta = 2k_i/k_r^2 > 0 \) where \( k_i > 0 \). Positive \( \eta = \chi_1^3/|\chi_1^3| \) corresponds to the cubic gain compensating linear and nonlinear losses, thus, \( \gamma = \chi_1^3/|\chi_1^3| \) must be negative [11]. The parameter \( \sigma = \text{sgn}(-k^p) \) is either positive for anomalous GVD (AGVD) or negative for normal GVD (NGVD) (see Fig.1). In NIMs the diffraction has a negative sign since the coefficient \( \sigma_1 = \text{sgn}(n) < 0 \). Therefore, bright spatial solitons exist only if the cubic nonlinearity is of the same sign as diffraction, thus, \( \kappa = \text{sgn}(\chi^3) < 0 \). Consequently, the quintic nonlinearity has to be positive \( \nu = \chi_1^5/|\chi_1^3| \) > 0. In order to generate bright spatiotemporal solitons so-called light bullets both diffraction and dispersion must be compensated by saturating nonlinearity [11]. Hence, both linear and nonlinear effects need to have the same sign. As a consequence, light bullets can be generated only in NIMs with NGVD in the range 0.7 < \( \kappa < 0.8 \) for the choice of dissipative parameters \( \beta \) and \( \delta \) as in Fig. 2. The lack of space limits our studies here to temporal solitons described by temporal GLENIM (TGLENIM) corresponding to Eq. (4) without spatial second derivatives. The generation, propagation, and stability of light bullets in NIMs are investigated elsewhere. The obtained nonintegrable TGLENIM can be solved only numerically. However, some analytical approach even thought approximate is needed in order to have a better physical inside. The approximate resolution of TGLENIM is done using variation method extended to dissipative systems [11]. Although it is unable to account structural change of pulse profile, this method can serve as a guideline for simulation. A trial function \( E = AA^0 \exp [j^2 (\omega - 0.5T^2 - T_0^2 + j\delta) z/2A^2 + 2\gamma A^4 - T/2Σ - 2\sigma C]A, \) \( dA \) \( \frac{dT}{dz} = (4\sigma C - \eta A^2 - \gamma A^4)T + \beta - 4\beta T^3 C^2, \) \( dC \) \( \frac{dz}{dz} = -4\sigma C^2 + \sigma T^2 - \delta - \kappa A^2/T^2 - \nu A^4/T^2 - 4\beta T^2 C, \) \( d\omega \) \( \frac{d\omega}{dz} = -\sigma T^2 + j^2 A^2 + 2\nu A^4 + 2\beta C. \) \( \text{Solving Eqs. (5-7) with zero } \) \( \text{derivatives, a double steady-state solution with amplitudes } A_+ \) and \( A_- \) is
obtained $A_{2} = \frac{(\beta \kappa \sigma - 4 \eta) + \sqrt{(\beta \kappa \sigma - 4 \eta)^2 + 8 \eta (3 \gamma - \beta \kappa \sigma)}}{2(\beta \kappa \sigma - 3 \gamma)}$. The family of solitons in conservative systems, reduces into either double or simple solution for each set of dissipative parameters. A double solution ($A_{-} > A_{+}$) exists in the $(\eta, \gamma)$ domain between the parabola $(\beta \kappa \sigma - 4 \eta)^2 + 8 \eta (3 \gamma - \beta \kappa \sigma) = 0$ and straight line $A_{-} = 1$ for NGVD and AGVD in Fig.3. Above $A_{-} = 1$ persists only $A_{+}$. The ratio between dissipative parameters $\beta$ and $\delta$ (given in Fig.2) is kept same for NGVD and AGVD, in order to stress the similarity of dissipative parameters. A double solution $(\text{either double or simple solution for each set of dissipative parameters})$ when coefficients $\alpha$ fulfill Hurwitz conditions $A_{2} = A_{2}^{\pm}$. The ratio between dissipative parameters $\beta$ and $\delta$ (given in Fig.2) is kept same for NGVD and AGVD, in order to stress the similarity of domains. The beam power $P = 3 \sqrt{\pi/2} A^{2} T$ is no more conserved in dissipative systems $\text{(11)}$. However, the temporal width $T = A^{-1} (\sigma \kappa + \alpha \beta A^{2})^{1/2}$ and the power depend, up to $v = \max \{|\beta|, |\delta|, |\eta|, |\gamma|\}$, only on the amplitude. The striking difference from the conservative systems is the nonzero wave front curvature $C = -A_{2}^{2} (\beta \kappa - \eta \sigma + (\beta \nu - \gamma \sigma) A_{2}^{2})^{-1}$. To be soliton a steady state solution has to be stable. Our stability criterion based on the method of Lyapunov’s exponents has to be extended to NIMs in order to check the stability of each steady-state solution. The solution of Eqs. (5-7) is stable if and only if the real parts of the solutions $\lambda$ of the equation $(\lambda^{3} + \alpha_{1} \lambda^{2} + \alpha_{2} \lambda + \alpha_{3}) = 0$ are all nonpositive $\text{(11)}$. The stability criterion for TGLENIM is satisfied when coefficients $\alpha$ fulfill Hurwitz conditions

\[\alpha_{2} = 4 \kappa A^{4} (\kappa + \nu A^{2}) > 0,\]
\[\alpha_{3} = 2 A^{6} (\kappa + \nu A^{2})^{2} [2A^{2} (\beta \nu \sigma - 3 \gamma) + (\beta \kappa \sigma - 4 \eta)] > 0,\]
\[\alpha_{4} = 4 (\kappa + \nu A^{2}) [A^{6} (12 \kappa \sigma - 8 \eta \nu) + A^{6} (13 \kappa \sigma - 8 \eta \nu)] > 0,\]

where $\alpha_{4} = \alpha_{1} \alpha_{2} - \alpha_{3}$ taking into account that $\alpha_{1} = A^{4} (4 \beta \kappa \sigma - 6 \gamma) + A^{2} (\beta \kappa \sigma - 4 \eta)$. The steady-state solution $A_{-}$ satisfies this stability criterion only between the parabola and curve $s$ corresponding to $\alpha_{4} = 0$ in Fig.3. The solution $A_{+}$ is everywhere unstable. However, the stability of the solution $A_{-}$ is only a prerequisite to obtain a soliton after a self-organizing evolution. Indeed, an input pulse chosen in the established stable domain corresponds to a point on the upper stable branch of analytically obtained bifurcation curve $v$ for power $P$ versus control parameter $\gamma$ in Fig.4. The curve $v$ is only a good approximation of the numerically obtained exact bifurcation curve $n$ composed of solitonic attractors for different parameters. The lower unstable branch corresponds to unstable solutions $A_{+}$. A stable steady state on the curve $v$ taken as the input for numerical simulations evolves toward the soliton on the curve $n$. The domain of stability is checked point by point in order to confirm by numerical simulations that corresponding inputs always lead to a soliton. Indeed, the numerically obtained domain of stability limited by the curve $s_{n}$ in Fig.3 is even slightly larger than the analytical one. Numerical simulations confirm that a soliton is propagating with, analytically predicted, nonzero wave front curvature. As a consequence, the dispersion is overcompensated by saturating nonlinearity leading to the collapse $\text{(11)}$. However, the collapse is prevented by losses equilibrated in turn by gain. Therefore, the self-organization of dissipative solitons is based on crosscompensation. During their propagation such solitons render the medium effectively dissipationless. In order to check soliton robustness its amplitude is increased 40% at each $z$ systematically 3000 times (see Fig.5). Such tremendous perturbations drastically increase the pulse temporal width and amplitude. However, far from being annihilated the pulse, remaining in attraction domain without losses and dispersion, maintains its new form. Indeed, to each increase the self-organized system reacts lowering its amplitude $\text{(11)}$. Perturbations arrested the soliton recovers its initial shape after about 400 steps in $z$ giving evidence of astonishing robustness. Perturbations may correspond to monochromatic EM pulses injected in such novel active medium composed of dissipationless solitons and NIM (SOLINIM). Therefore, it seems that SOLINIM behaves for other pulses as effectively dissipationless.

In conclusion, SOLINIM may be considered as a novel very promising active composite medium due to the synergy between dissipative solitons and NIM. In order to describe such SOLINIM systems, newly established coupled nonparaxial Helmholtz equations for electric and magnetic fields are reduced to novel paraxial complex cubic-quintic Ginzburg-Landau equation. In contrast with ordinary media, bright dissipationless light bullets may propagate only in NGVD NIMs. However, bright dissipationless temporal solitons can be generated in NIMs with both NGVD and AGVD. A stability criterion is established. Choosing as input a steady-state with dissipative parameters from obtained stability domains for either NGVD or AGVD NIMs, a self-organized propagation always results in generation of extremely robust dissipationless soliton. Such solitons during their propagation render NIM effectively dissipationless. We hope that in SOLINIMs the practical realization of peculiar effects like the cloaking and the superresolution will be no more prevented.

ACKNOWLEDGMENTS

This research has been in part supported by French–Serbian cooperation, CNRS/MSCI agreement no. 20504. The work of VIB was supported by ISTC grant G1366. Work at the Institute of Physics is supported by the Ministry of Science of the Republic of Serbia, under the project OI 141031.

[1] J. B. Pendry, Contemporary Physics 45, 191 (2004).
[2] V. G. Veselago, Sov. Phys.Usp. 10, 509 (1968); J. B. Pendry, Phys. Rev. Lett. 85, 3966 (2000); R. A. Shelby, D. R. Smith, and S. Schultz, Science 292, 77 (2001).
[3] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, Science 314, 977 (2006).

[4] V. W. Shalaev et al., Opt. Lett. 30, 3356 (2005); G. Dolling et al., Opt. Lett. 32, 53 (2007).

[5] N. Lazarides and G. P. Tsironis, Phys. Rev. E 71, 036614 (2005).

[6] I. Kourakis and P. K. Shukla, Phys. Rev. E 72, 016626 (2005).

[7] G. D’Aguanno, N. Mattiucci, M. J. Bloemer, Journal of the Optical Society of America B 25, 1236 (2008).

[8] S. Wen et al., Phys. Rev. A 77, 033815 (2007).

[9] K. J. Webb, M. Yang, D. W. Ward, and K. A. Nelson, Phys. Rev. E 70, 035602(R) (2004).

[10] N. N. Akhmediev and A. A. Ankiewicz, Dissipative Solitons, (Springer, Berlin, 2005); Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. 61, 763 (1989).

[11] V. Skarka and N. B. Aleksić, Phys. Rev.Lett. 96, 013903 (2006); N.B. Aleksić, V. Skarka, D. V. Timotijević, and D. Gauthier, Phys. Rev. A 75, 061802(R) (2007); V. Skarka, D. V. Timotijević, and N. B. Aleksić, J. Opt. A: Pure Appl. Opt. 10, 075102 (2008).

Figure captions

Fig. 1. Group velocity $v_g$, $GVD$, and phase velocity $v_{ph}$.

Fig. 2. Dissipative parameters $\beta$ and $\delta$ versus frequency $\omega$.

Fig. 3. $AGVD$ and $NGVD$ domains of stable solutions $A^-$.

Fig. 4. Upper stable and lower unstable branches of variational $v$ curve and numerical $n$ curve.

Fig. 5. Dissipationless soliton resisting 40% increase of the amplitude for each $z$ from $z = 1000$ to $z = 4000$. 
\sigma = -1
\kappa = -1
\nu = 1
\delta = 0.0085
\beta = 0.021
\eta = 0.2
