A Note on Transfinite M Theory and the Fine Structure Constant

Carlos Castro

Revised, July, 2001

Abstract

In this short note, using concepts of $p$-Adic QFT and $p$-branes, we derive the transfinite $M$ theory generalization of the inverse fine structure constant given by $(\alpha_M)^{-1} = 100 + 61\phi$. The original El Naschie and Selvam-Fadnavis inverse fine structure constant value $(\alpha_{HS})^{-1} = 100 + 60\phi$ was based on a transfinite heterotic string theory and a quasiperiodic Penrose tiling formalism, respectively. Here $\phi$ is the Golden Mean $0.6180339...$. Our results are consistent with the recent astrophysical observations of the Boomerang and Maxima experiments, the previous results based on the four dimensional gravitational conformal anomaly calculations, and with an enhanced spacetime hierarchy of a suitable number of lines living on Del Pezzo surfaces.

Motivated by the fact that the bosonic membrane is devoid of anomalies in $d = 27$, and the supermembrane is anomaly free in $d = 11$ [2], and that the anomaly free (super) string actions ($d = 26, 10$) are directly obtained by a double-dimensional reduction process of both the world-volume of the (super) membrane and the target spacetime dimension, where the (super) membrane is embedded, we shall derive the transfinite $M$ theoretical generalization $(\alpha_M)^{-1}$ to El Naschie’s inverse fine structure constant $(\alpha_{HS})^{-1}$ which was based on a transfinite Heterotic string theory formalism [1].

Selvam and Fadnavis [15], independently, obtained this value using a quasiperiodic Penrose tiling model (a quasicrystal) associated with the logarithmic spiral, with a golden-mean winding number, which represents a bidirectional

*Center for Theoretical Studies Clark Atlanta University Atlanta, GA. 30314, USA
vortex (eddy) circulation process with a five-fold symmetry. Taking into account that there are 5 steps in this process, both in a clockwise and counterclockwise motion, giving effectively a factor of $5 \times 2 = 10$ times the fundamental variance which in turn is $2(1 + \phi)^4$, they obtained the value of the total variance of the fractal structure associated with the 5 successive growth-steps modelled by the Penrose quasi-crystal:

$$\text{Total Variance} = 5 \times 2 \times 2(1 + \phi)^4 = 100 + 60\phi$$

which agrees exactly with El Naschie’s value based on a transfinite heterotic string theory formalism. This is no coincidence. Cantorian fractal spacetime has a $p$-adic topology and this topology is precisely the one present in those quasicrystals obtained as limiting quasiperiodic point sets with $p$-adic internal symmetries. The Penrose tiling is a special case of a variety of well known quasiperiodic tilings like the chair tiling and the Robinson square tiling [17]. Roughly speaking, the electron is a quasiperiodic process or a quasicrystal, as we shall see below.

We will see as well that the inverse fine structure constant obtained by the author [3] based on a four dimensional gravitational conformal anomaly calculation [4] $(\alpha_{\text{conformal}})^{-1}$ has for lower/upper bounds the values $(\alpha_{\text{HS}})^{-1}$ and $(\alpha_{M})^{-1}$ respectively. All one needs is to use the following fundamental identities:

$$(1 + \phi)^k = F_{k+1} + F_k \phi, \quad \phi^k = (-1)^k F_{k-1} + (-1)^{k+1} F_k \phi \quad (1)$$

$$\phi(1 + \phi) = 1 \Rightarrow \phi = \frac{\sqrt{5} - 1}{2} = 0.6180339.... \quad (2)$$

where $F_k$ are the Fibonacci numbers obeying:

$$F_k + F_{k+1} = F_{k+2}, \quad F_k = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.... \quad (3)$$

$$\phi = \lim_{k \to \infty} \frac{F_k}{F_{k+1}}. \quad (4)$$

The identities (1) can be derived by setting:

$$(1 + \phi)^k = A_k + B_k \phi \Rightarrow (1 + \phi)^k + (1 + \phi)^{k+1} = (1 + \phi)^{k+2} = A_{k+2} + B_{k+2} \phi = [A_k + A_{k+1}] + [B_k + B_{k+1}] \phi. \quad (5)$$
after an induction process and after using eqs-(3) which define the Fibonacci numbers. A $p$-adic QFT argument \[5,6,7\] allows us to show that the $M$ theoretical generalization to the inverse fine structure constant is uniquely determined by:

$$ (\alpha_M)^{-1} = 1 + (1 + \phi)^2 + (1 + \phi)^5 + (1 + \phi)^{10} = $$

$$ (F_1 + F_3 + F_6 + F_{11}) + \phi(F_2 + F_5 + F_{10}) = 100 + 61\phi. \tag{7a} $$

This expansion of $(\alpha_M)^{-1}$ in integer powers of $1 + \phi$ is the analog of the expansion of 137 in powers of $p = 2$:

$$ 137 = 1 + 8 + 128 = 2^0(1 + 2^3 + 2^7) \tag{7b} $$

and whose $p$-adic norm is $||137||_2 = (1/2^0) = 1$. Notice that one could have performed the lacunar series expansion of the form

$$ \sum a_n(1 + \phi)(1 + \phi)^n \tag{7c} $$

a lacunar series is given in powers of the form $p^n$ but this has not the desired $p$-Adic form because $(1 + \phi)^n$ is not an integer as verified by the fundamental identities eq-(1). Also we must restrict the coefficients $a_n$ to be integer valued $0, 1 < 1 + \phi$.

El Naschie’s value based on string theory is given by ten copies of the complexified dimensions of the transfinite space $E(5)$ whose real-valued Hausdorff dimension is $(1 + \phi)^4$:

$$ (\alpha_{HS})^{-1} = 10 \times 2(1 + \phi)^4 = 137 + \phi^5(1 - \phi)^5 = 100 + 60\phi = 137.082 \ldots \tag{8} $$

The hierarchy of dimensions obtained by El Naschie were given by suitable powers of $10\phi^n$ for $n = 0, \pm 1, \pm 2, \ldots$. The $n = 0$ power corresponds naturally to the core dimension of the Heterotic string $d = 10$ and the hierarchy of dimensions for $n = \pm 1, \pm 2$ were $4 - k; 6 + k; 10; 16 + k; 26 + k$ respectively, with the value for $k = \phi^3(1 - \phi^3)$.

However, a close inspection reveals that the quantity $(\alpha_{HS})^{-1} = 100 + 60\phi$ does not admit a $p$-Adic expansion in the form given by eq-(5a); i.e like the value $(\alpha_M)^{-1} = 100 + 61\phi$ does. This is a direct consequence of the fundamental identities (1). The value of $4 - k$ is not the Hausdorff dimension consistent with the four dimensional gravitational anomaly calculations [3, 4] because the latter value is greater than four. The average dimension of
the world $d \sim 4 + \phi^3 = 4.236$. This is another reason why one should have for core dimension the value of $d = 11$, the dimension of the anomaly free supermembrane. We will go back to this crucial point shortly. The value obtained from the gravitational conformal anomaly calculation was $\left[3\right]$:  

$$(\alpha_{\text{conformal}})^{-1} = 2 \frac{(\delta^2 + 1)^2}{\delta (\delta^2 - 1)} = 137.6414382326. \quad D = 4 + \epsilon = 4\delta = 4 + \frac{\phi^3}{2}. $$

$$\delta = 1 + \frac{\phi^3}{8}. \quad (9)$$

This value of the inverse fine structure constant is associated with a fractal spacetime dimension of $4 + \epsilon = 4 + (\phi^3/2)$. It corresponds to an infrared fixed point of the renormalization group where scale invariance is restored $\left[4\right]$. In ordinary (super) string theory, a regularization procedure breaks conformal invariance except in the critical dimensions $d = 26; d = 10$. From the conformal field theory point of view this implies that the beta function associated with the worldsheet couplings of the non-linear $\sigma$ model corresponding to the (super) string have a fixed point; i.e the beta function vanishes in the critical dimensions and conformal invariance is restored in those special dimensions. Hence we have finally:

$$(\alpha_{\text{HS}})^{-1} = 100 + 60\phi = 137.082 < (\alpha_{\text{con}})^{-1} = 137.6414 < (\alpha_M)^{-1} = 100 + 61\phi = 137.700 \quad (10)$$

Now we shall go back to the crucial point of having $d = 11$ as the core dimension of the transfinite nonperturbative $M$ theory. We mentioned earlier that the bosonic membrane is free of anomalies in $d = 27$ and the supermembrane is anomaly free in $d = 11 \left[2\right]$ and that strings are obtained automatically by dimensional reduction. The nonperturbative $M$ theoretic generalization of the inverse fine structure constant are due to the following $p$-brane configurations for values of $p = 0, 2, 5, 10$ living in $d = 11$ topological dimensions. The 5-brane and the membrane, $p = 2$, are Electro-Magnetic dual to each other in $d = 11$ since EM brane duality requires the numerical relation between the values of the spatial dimensions to be: $11 = 2 + 5 + 4$. The $p = 0$-brane corresponds to the center of mass of the system and the $p = 10$ brane is the spacetime filling since $10 + 1 = 11$. Since among our $p$-brane configurations we have the membrane ($a \ p = 2$ brane) living in $d = 11$ dimensions (and its EM dual 5-brane
this transfinite $M$ theory does naturally contain heterotic strings in $d = 10$ dimensions by a straightforward dimensional reduction of the membrane. Following Eddington’s view that the inverse fine structure constant is an internal dimension we can see that the net value $100 + 61\phi = (\alpha_M)^{-1}$ is the sum of the (fractal) Hausdorff dimensions of the four transfinite spaces associated with certain $p$-branes embedded in $d = 11$ topological dimensions:

$$\mathcal{E}^{(11)}, \mathcal{E}^{(6)}, \mathcal{E}^{(3)}, \mathcal{E}^{(1)}$$

(11)

The (spacetime filling) 10-brane spans an 11-dimensional worldvolume; the 5-brane spans a six dimensional one; its Electromagnetic-Dual Membrane spans a three-dimensional worldvolume and the center of mass (0-brane), spans a one-dimensional line, the normal set $\mathcal{E}^{(1)}$. These (topological) dimensions corresponding to the worldvolumes, worldlines, spanned by these $p$-branes as they evolve in time, match precisely the topological dimensions associated with the four transfinite sets given in eq-(11).

It is suggestive to think that the four transfinite sets present in the decomposition (11) could be related to the electromagnetic, electroweak, strong and quantum gravity phases in Nature. The number $100 + 61\phi$ can be recast explicitly as:

$$(\alpha_M)^{-1} = 137 + \phi^5(1 - \phi^5) + \phi = 100 + 61\phi.$$  

(12)

The first term 137 is the usual $U(1)$ contribution; the second term $\phi^5(1 - \phi^5) = 60\phi - 37$ is the electroweak and strong interaction contributions found by El Naschie [1] and the $\phi$ additional term is the Nonperturbative $M$ theoretic corrections; i.e a quantum gravity effect.

At first hand one may be inclined to object to such an "erroneous" claim because the contribution of $\phi$ is not small. In fact it is larger than $\phi^5(1 - \phi^5)$. However, one must realize that in this Nonperturbative transfinite $M$ theory there is a nontrivial ultraviolet/infrared entanglement (mixing) like that occurring in the construction of Quantum Field Theories in Noncommutative Geometry. It is reminiscent of the large-scale/small-scale $T$ duality in string theory. Therefore, due to this highly nontrivial ultraviolet/infrared entanglement/mixing the large $\phi$ correction is truly an infrared quantum gravity contribution, as it should be, since adding this term is precisely what is needed to match the value obtained from the gravitational conformal anomaly calculation [3,4] of the inverse fine structure constant. Such contribution was due precisely to an infrared long distance effect as a result of the quantum
conformal mode fluctuations of the spacetime metric over large scales whose overall effect is to \emph{screen} the electric charge at large distances and cause the \emph{inverse} value of the fine structure constant to increase. Within the framework of the fine structure value obtained independently by Selvam and Fadnavis \cite{15} we can interpret the additional value of the Golden Mean $\phi$, to their value of $100 + 60\phi$, due to the intrinsic initial perturbation which originated the logarithmic spiraling growth process in successive 5 steps (clockwise/counterclockwise). When one speaks of growth steps we must specify with respect to what the initial perturbation of this growth process refers to.

Therefore, a straightforward $p$-Adic QFT argument \cite{5,6,7} yields the Nonperturbative transfinite $M$ theoretical extensions to El Naschie-Selvam-Fadnavis inverse fine structure constant. The value $100 + 61\phi$ is consistent with the most recent Astrophysical data of the spectral density index obtained from the Boomerang and Maxima experiments measuring the primordial cosmological background curvature perturbations \cite{8,9}. The spectral index of the two-point correlation function of the Cosmic Microwave Background radiation due to the density perturbations, induced by the curvature fluctuations, was given in terms of the quantum trace anomaly \cite{4} and the value is $n \equiv 2\Delta - 3 = 1.0299358$ where $\Delta = 2.0149679 > 2$ in full agreement with the numerical bounds of Covi and Lyth \cite{8}.

In a forthcoming publication \cite{9} we shall present a lengthy and detailed discussion of the results discussed in this short note. Especially, the deep interrelation among $p$-Adic QFT \cite{5,6,7} ; the Bruhat-Tits tree-like $p$-Adic Topology of Cantorian-Fractal Spacetime \cite{1} , the Non-Archimedean Geometry associated with the Algebraic continued-fraction Renormalization group-like procedure of the physical constants and the $p$-Adic hierarchical structures of (symmetry breaking) phases in stochastic processes \cite{10,11}.

What remains is to look at the hierarchy of dimensions generated by this transfinite $M$ theory. The hierarchy generated by the core dimension of $d = 11$ is given by $11\phi^n$ for different values of $n$ is:

\[
\begin{align*}
n = 0 & \Rightarrow d_0 = 11 \\
n = -1 & \Rightarrow d_{-1} = 11(1 + \phi) = 17.7983 \\
n = -2 & \Rightarrow d_{-2} = 11(1 + \phi)^2 = 11(2 + \phi) = 28.7983 \\
n = +1 & \Rightarrow d_{+1} = 11(\phi) = 6.7983
\end{align*}
\]
\[ n = +2 \Rightarrow d_{+2} = 11(\phi)^2 = 11 - 11\phi = 5 - 0.7983 = 4.2017. \] (13)

Hence this hierarchy has the following dimensions:

\[ 28.7983, 17.7983, 11, 6.7983, 5 - 0.7983 = 4.2017. \] (14)

The author has been informed by Metod Saniga [14] that this hierarchy in its integer valued part, is almost exactly reproduced by the sequence of number of lines lying on Del Pezzo surfaces if one adds one to each value: 28; 17; 11; 7; 4 that would mean nothing but that Del Pezzo hierarchy corresponds to spatial dimensions only, while this enhanced hierarchy grasps both space and time dimensions! [14].

As we said previously, the lower dimension 4.2017 is consistent with the four dimensional gravitational anomaly calculation [3, 4] and with the recent Astrophysical data [8, 9]. A \( p \)-Adic interpretation to the fundamental scales in Physics has been given by Pitkanen [7] based on the Mersenne primes: \( M = 2^m - 1 \) for \( m = \text{prime} \). Some values of \( m \) that yield a Mersenne prime are:

\[ m = 2, 3, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521.... \] (14)

Pitkanen’s \( p \)-Adic Length Scale hypothesis is obtained by selecting the electron Compton wavelength as a reference scale which allows to fix the size of an internal \( CP^2 \) space whose isometries are linked to the standard gauge symmetries of the fundamental interactions. The electron Compton wavelength was chosen to be the one associated with the Mersenne prime corresponding to \( M_{127} = 2^{127} - 1 \) which fixes the size of the internal space to be:

\[ l_o \sim 137.6 \times 10^2 \text{ Planck}. \] (15)

The hierarchy of fundamental scales in physics is given by:

\[ L = \sqrt{p} l_o, \quad p = M = 2^m - 1 \]

The following values for the Mersenne primes are:

1. \( M_{127} \Rightarrow L \sim 10^{-10} \text{ cm} = \text{electron} \)
2. \( M_{107} \Rightarrow L \sim 10^{-14} \text{ cm} = \text{proton} \)
3. \( M_{89} \Rightarrow L \sim 10^{-16} \text{ cm} = W \text{ boson} \)
4. \( M_{61} \Rightarrow L \sim 10^{-20} \text{ cm} = \text{Michael Conrad’s Fluctuon particle} [20] \)
The reader may ask, do the remaining scales in between correspond to new particles and forces? This question is perfectly valid. For a discussion of $W_\infty$ Geometry and a master field in infinite dimensions that generates an infinity of higher spin massless gauge fields (interactions) in lower (four) dimensions see [23].

The main conclusion of this work is the following: The value $(\alpha_{HS})^{-1} = 100 + 60\phi$ does admit the decomposition $10 \times 2(1 + \phi)^4$ consistent with the hierarchy of (complexified) dimensions generated by the core $d = 10$ dimensions of superstrings but fails to admit the correct $p$-Adic decomposition exhibited by $(\alpha_M)^{-1} = 100 + 61\phi$ shown in eqs-(5). Since 61 is a prime number one can see automatically that $100 + 61\phi$ has for lower and upper bounds the following (complexified) dimensions belonging to a hierarchy of dimensions generated by the core $d = 10, d = 11$ dimensions of the superstring and the anomaly free supermembrane, respectively:

$$10 \times 2(1 + \phi)^4 < 100 + 61\phi < 11 \times 2(1 + \phi)^4.$$  

(16)

Constraining the inverse fine structure constant to lie in the interval $136 < \alpha^{-1} < 138$; to be compatible with recent Astrophysical data [8, 9] and with the gravitational conformal anomaly results [3, 4], we conclude that one cannot simultaneously satisfy both conditions of $\alpha^{-1}$ having the correct $p$-adic expansion like in eq-(5) and in being a member of the hierarchy of complexified dimensions of the form $\alpha^{-1} = 2d(1 + \phi)^k$ for $d = \text{integer}$ core-dimensions and $k = \text{integer}$.

One would require to have a relation of the type:

$$136 < \sum_n a_n(1 + \phi)^n = 2d(1 + \phi)^k < 138.$$  

(17)

It seems that this equation does not have solutions for $a_n = 0, 1$ and $d, n, k = \text{integers}$ in the range between: $136 < \alpha^{-1} < 138$. However, a value of the inverse fine structure constant like: $11 \times 2(1 + \phi)^4 \sim 150.788...$ is consistent with that value obtained by Nottale using scale relativistic arguments if, and only if, the number of Higgs doublets is 7 [16]. The value obtained by Nottale is:

$$\alpha^{-1} = 137.08 + 2.11(N_H - 1) \pm 0.13.$$  

(18)
in very good agreement with the experimental value of 137.036 provided $N_H = 1$ and with the value $11 \times 2(1 + \phi)^4 \sim 150.788...$ provided $N_H = 7$.

We believe this cannot be just a numerical coincidence but that an underlying $p$-Adic internal symmetry operates in the Fractal structures of Nature. T. Smith has used the Hyperdiamond lattice model associated with the exceptional group $E_8$, and Octonions, to derive the value of the inverse fine structure constant 137.068... . In particular, using the Feynman chessboard construction on Hyper-Diamond structures gives the physics of the Standard Model plus Gravitation [18]. Ord has used a similar Feynman chessboard construction to formulate a spiral gravity model [21]. The $E_8$ lattice can be represented by quaternionic icosians , as described by Conway and Sloane [19]. This $E_8$ lattice can be constructed using the Golden Mean ratio from the $D_4$ lattice which has a 24-cell nearest neighbour polytope. We refer to [19] for details.

Based on the results presented in this work we dare to say that the fundamental particles, themselves, like the electron, can be visualized as a nonlinear dynamical ( multi ) fractal process, represented by a Penrose quasicrystal, for example. It is this lattice structure that is behind the values of the fundamental constants in Nature. Multifractality governs also the prime number distribution [22]. Not suprisingly, we are going to see more and more in the near future how all these disciplines : number theory, algebraic geometry, Cantorian Fractal spacetime, quasicrystals, $p$-Adic QFT, Noncommutative/Nonassociative Geometry, fractal strings.... merge together.

Acknowledgements
We are indebted to M. S. El Naschie and Metod Saniga for numerous discussions that led to this work.

References
1- M. S. El Naschie : " A general theory for the topology of Transfinite Heterotic strings and Quantum Gravity " Jour. Chaos, Solitons and Fractals 12 (5) ( 2001).

2- R. Kaiser, U. Marquard, M. Scholl : Phys. Letts B 227 ] ( 1989 ) 234. U. Marquard, M. Scholl : Phys. Letts B 227 ] ( 1989 ) 227.

3-C. Castro : " On the four-dimensional Conformal Anomaly, Cantorian-
Fractal Spacetime and the Fine Structure Constant ” To appear in Chaos, Solitons and Fractals. Available from physics/0010072

4-I. Antoniadis, P. Mazur, E. Mottola : ” Fractal Geometry of Quantum Spacetime at Large Scales ” Available from hep-th/9808070. I. Antoniadis, P. Mazur, E. Mottola ” Conformal Invariance and Cosmic Background Radiation ” Available from astro-ph/9611208.

5-V. Vladimorov, I. Volovich, E. Zelenov : ” p-Adic Analysis in Mathematical Physics ” World Scientific, Singapore 1999.

6- L. Brekke, P. G. O Freund : ” p-Adic Numbers in Physics ” Physics Reports 233 ( 1) ( 1993)

7- M. Pitkanen : ” Topological Geometrodynamics ” Book available on line http:// www.physics.helsinki/~ matpitka/tgd.htm

8- L. Covi, D. H. Lyth : ” Global Fits for the Spectral Index of the Cosmological Curvature Perturbations ” Available from astro-ph/0008165 v2.

9-C. Castro, A. Granik : ” In preparation ”.

10- W. Karkowski , R. Vilela Mendes : ”Hierarchical structures and asymmetric stochastic proceses on p-Adics and Adeles ” Jour. Math. Phys. 35 (9) ( 1994) 4637.

11- M. Mezard, G. Parisi, M. Virasoro : ” Spin Glass Theory and Beyond ” World Scientific, Singapore 1987.

12-M. Saniga : ” 27 lines on a Cubic Surface and Heterotic String Spacetimes ” Available from physics/0012033. Chaos, Solitons and Fractals 2001; 12 : 1177-1178

13-M. Saniga : “Lines on Del Pezzo surfaces and transfinite heterotic string spacetime “to appear in Chaos, Solitons and Fractals . Available from physics/0101041.

14-M. Saniga : Private Communication . physics/0105049
15-A. M. Selvam, S. Fadnavis: “Superstrings, Cantorian-Fractal spacetime and quantum like chaos in atmospheric flows. “Chaos, Solitons and Fractals 10 (8) (1999) 1321-1334.

16-L. Nottale: Chaos, Solitons and Fractals 1994, 4 (3): 361.

17-M. Baake, R. V. Moody, M. Scottmann: “Limit quasi periodic point sets as quasicrystals with \(p\)-Adic internal spaces “Available from [math-ph/9901008](http://arxiv.org/abs/math-ph/9901008).

18-T. Smith: “From Sets to Quarks “[hep-ph/9708379](http://arxiv.org/abs/hep-ph/9708379).

19-J. Conway, N. Sloane: “Sphere Packings, Lattices and Groups, 2nd edition, Springer-Verlag (1993) “

20-M. Conrad “Chaos, Solitons and Fractals, 1996, 7 (5) 725.

21-G. Ord: The Spiral Gravity Model “Chaos, Solitons and Fractals, 1999, 10 (2-3).

22-M. Wolf: “Multifractality of Prime Numbers “Physica A 160 (1989)24.

23-C. Castro: “On the large N limit, \(W_\infty\) Strings, Star products..... “Available from [hep-th/0106260](http://arxiv.org/abs/hep-th/0106260).