GALACTIC METRIC, DARK RADIATION, DARK PRESSURE, AND GRAVITATIONAL LENSING IN BRANE WORLD MODELS

T. HARKO AND K. S. CHENG
Department of Physics, Hong Kong University, 518 Chong Yu Ming Physics Building, Pokfulam Road, Hong Kong, China; harko@hkucec.hku.hk, hrsksc@hkucec.hku.hk
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ABSTRACT

In the brane world scenario, the four-dimensional effective Einstein equation has extra terms that arise from the embedding of the 3-brane in the bulk. These nonlocal effects may provide an explanation for the dynamics of the neutral hydrogen clouds at large distances from the galactic center, which is usually explained by postulating the existence of the dark matter. We obtain the exact galactic metric, the dark radiation, and the dark pressure in the flat rotation curve region in the brane world scenario. Due to the presence of the bulk effects, the flat rotation curves could extend several hundred kiloparsecs. The limiting radius for which bulk effects are important is compared with the numerical values of the truncation parameter of the dark matter halos, obtained from weak-lensing observations. There is a relatively good agreement between the predictions of the model and observations. The deflection of photons in the flat rotation curve region is also considered. The bending angle predicted by the brane world models is much larger than that predicted by dark matter models. The angular radii of the Einstein rings are obtained in the small-angle approximation. The tangential shear is compared with the observational data obtained in the weak lensing of galaxies in the Red-Sequence Cluster Survey. The study of the light deflection by galaxies and the gravitational lensing could discriminate between the different dynamical laws proposed to model the motion of particles at the galactic level and the standard dark matter scenario.

Subject headings: cosmology: theory — galaxies: kinematics and dynamics — gravitational lensing

1. INTRODUCTION

The rotation curves for galaxies or galaxy clusters should, according to Newton’s gravitation theory, show a Keplerian decrease with distance \( r \) of the orbital rotational speed \( v_\text{rot} \) at the rim of the luminous matter, \( v_\text{rot} \propto \frac{M(r)}{r} \), where \( M(r) \) is the dynamical mass. However, one observes instead rather flat rotation curves (Rubin et al. 1980; Binney & Tremaine 1987; Persic et al. 1996; Boriello & Salucci 2001). Observations show that the rotational velocities increase near the center of the galaxy and then remain nearly constant at a value of \( v_\text{rot} \approx 200-300 \) km s\(^{-1}\). This leads to a general mass profile \( M(r) \approx r v_\text{rot}^2 / G \) (Binney & Tremaine 1987). Consequently, the mass within a distance \( r \) from the center of the galaxy increases linearly with \( r \), even at large distances, where very little luminous matter can be detected.

This behavior of the galactic rotation curves is explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weekly interacting massive particles (WIMPs) for a recent review of the particle physics aspects of dark matter see Overduin & Wesson 2004). Their interaction cross sections with normal baryonic matter, while extremely small, are expected to be nonzero, and we may expect to detect them directly. It has been suggested that the dark matter in the universe might be composed of superheavy particles, with mass \( \geq 10^{10} \) GeV. But observational results show that the dark matter can be composed of superheavy particles only if these interact weakly with normal matter or if their mass is above \( 10^{15} \) GeV (Albuquerque & Baudis 2003). Scalar fields or other long-range coherent fields coupled to gravity have also intensively been used to model galactic dark matter (Nucamendi et al. 2000; Matos & Guzman 2001; Mielke & Schunk 2002; Cabral-Rosetti et al. 2002; Lidsey et al. 2002; Mbelek 2004; Matos et al. 2004; Fuchs & Mielke 2004; Lee & Lee 2004; Mielke & Peralta 2004; Hernández et al. 2004).

However, despite more than 20 years of intense experimental and observational effort, up to now no non-gravitational evidence for dark matter has ever been found: no direct evidence of it and no annihilation radiation from it. Moreover, accelerator and reactor experiments do not support the physics (beyond the standard model) on which the dark matter hypothesis is based.

Therefore, it seems that the possibility that Einstein’s (and the Newtonian) gravity breaks down at the scale of galaxies cannot be excluded a priori. Several theoretical models, based on a modification of Newton’s law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves. A modified gravitational potential of the form \( \phi = \frac{GM}{r} [1 + \alpha \exp(-r/r_0)](1 + \alpha)r \), with \( \alpha = -0.9 \) and \( r_0 \approx 30 \) kpc, can explain flat rotational curves for most of the galaxies (Sanders 1984, 1986).

In another model, called MOND and proposed by Milgrom (Milgrom 1983, 2002, 2003; Bekenstein & Milgrom 1984), the Poisson equation for the gravitational potential \( \nabla^2 \phi = 4\pi G\rho \) is replaced by an equation of the form \( \nabla^2 \phi (\nabla \phi / a_0) = 4\pi G\rho \), where \( a_0 \) is a fixed constant and \( \mu(x) \) is a function satisfying the conditions \( \mu(x) = x \) for \( x \ll 1 \) and \( \mu(x) = 1 \) for \( x \gg 1 \). The force law, giving the acceleration \( a \) of a test particle, becomes \( a = a_N \) for \( a_N \gg a_0 \) and \( a = (a_N a_0)^{1/2} \) for \( a_N \ll a_0 \), where \( a_N \) is the usual Newtonian acceleration. The rotation curves of the galaxies are predicted to be flat, and they can be calculated once the distribution of the baryonic matter is known. A relativistic MOND inspired theory was developed by Bekenstein (2004, 2005). In this theory gravitation is mediated by a metric, a scalar field, and a 4-vector field, all three dynamical.

Alternative theoretical models to explain the galactic rotation curves have been proposed by Mannheim (1993, 1997), Moffat & Sokolov (1996), Moffat (2004), and Roberts (2004).
The detection by Brainerd et al. (1996) of weak, tangential distortion of the images of cosmologically distant, faint galaxies due to gravitational lensing by foreground galaxies has opened a new possibility of testing the alternative theories of gravitation and the dark matter hypothesis. Background galaxies are observed to be tangentially aligned around foreground galaxies because of the latter population’s gravitational lensing effect. The angular dependence of the shear signal is consistent with the hypothesis that galaxies are dominated by approximately isothermal halos but can also be explained by assuming some phenomenological modifications of the effective (Newtonian) gravitational force at large distances (Mortlock & Turner 2001).

Galaxy-galaxy lensing could provide very powerful constraints for the alternative gravitational theories (for a general review of week lensing and its applications see Hoekstra et al. 2002). The measurements rely on averaging over many background sources, and the signal is only appreciable at large angular separations from the foreground deflectors. Thus, in the absence of dark matter, the foreground galaxies can be regarded as simple lenses, and the observational data can be used to constrain the deflection law directly. A weak-lensing mass reconstruction of the interacting cluster 1E 0657—558 has been presented in Clowe et al. (2004). Weak lensing has been mainly used to discuss and constrain MOND (Gavazzi 2002; Hoekstra et al. 2002; Clowe et al. 2004). Gavazzi (2002) used the cluster of galaxies MS 2137—23, which presents the most constrained lensing configuration of gravitational images ever detected in a distant cluster of galaxies to constrain MOND-type models. According to this analysis, the MOND model is not compatible with the observations. The need for several more baryons in the MOND model than for the dark matter paradigm implies significant dynamical differences between these models, which can be explored at very large radial distance. Observational evidence against this model was also reported in Clowe et al. (2004). For the interacting cluster 1E 0657—558 the observed offsets of the lensing mass peaks from the X-ray gas directly demonstrate the presence and dominance of dark matter in this cluster. However, this proof of dark matter existence holds true even under the assumption of MOND. Based on the observed gravitational shear-optical light ratios and the mass peak–X-ray gas offsets, the “dark matter” component in a MOND regime, that is, the dynamical MOND mass \( M_n(r) \) related to the dynamical Newtonian mass \( M_B \) by the relation \( M_n(r) = M_B / (1 + (a_0 / a)^2)^{1/2} \), would have a total mass that is at least equal to the baryonic mass of the galactic system. The observed shear and derived mass of the subclump are significantly higher than could be produced by an isothermal sphere with a velocity of 212 km s\(^{-1}\).

Weak gravitational lensing is a very promising method for the study of the galactic dark matter halos. Results of the studies of weak lensing by galaxies have been published recently by Hoekstra et al. (2003, 2004). Using weak lensing, the flattening of the galactic dark matter halos was observed. These results suggest that the dark matter halos are rounder than the light distribution, with a halo ellipticity of the order of \( e_{\text{halo}} \approx 0.33 \). The average mass profile around galaxies was also studied in the framework of two halo models, the truncated isothermal sphere model and the Navarro-Frenk-White model. From the point of view of the alternative theories of gravitation, the main implication of these results is that spherical halos are excluded with 99% confidence, since most of the alternative theories predict an almost isotropic weak-lensing signal, which is not observed.

In a series of recent papers (Harko & Mak 2004, 2005; Mak & Harko 2004), several classes of conformally symmetric solutions of the static gravitational field equations in the brane world scenario (Randall & Sundrum 1999a, 1999b), in which our universe is identified with a domain wall in a five-dimensional anti-de Sitter spacetime, have been obtained (for a review of brane world models see Maartens 2004). The static vacuum gravitational field equations on the brane depend on the generally unknown Weyl stresses in the bulk (a higher dimensional spacetime, in which our universe is embedded), which can be expressed in terms of two functions, called the dark radiation \( U \) and the dark pressure \( P \) (the projections of the Weyl curvature of the bulk, generating nonlocal brane stresses; Mak & Harko 2004; Maartens 2004; Harko & Mak 2005). Generally, the vacuum field equations on the brane can be reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms (Harko & Mak 2004). In order to close the system, a functional relation between \( U \) and \( P \) is necessary.

In Harko & Mak (2004, 2005) and Mak & Harko (2004) the solutions of the gravitational field equations on the brane have been obtained by using some methods from Lie group theory. As a group of admissible transformations, the one-parameter group of conformal motions has been considered. The main advantage of imposing geometric self-similarity via a group of conformal motions is that this condition also uniquely fixes the mathematical form of the dark radiation and dark pressure terms, respectively, which describe the nonlocal effects induced by the gravitational field of the bulk. Thus, there is no need to impose an arbitrary relation between the dark radiation and the dark pressure.

As a possible physical application of the solutions of the spherically symmetric gravitational field equations in the vacuum on the brane, the behavior of the angular velocity \( v_{\text{gpc}} \) of the test particles in stable circular orbits has been considered (Mak & Harko 2004; Harko & Mak 2005). The conformal factor \( \psi \) together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances and for a particular set of values of the integration constants the angular velocity tends to a constant value. This behavior is typical for massive particles (hydrogen clouds) outside galaxies (Binney & Tremaine 1987; Persic et al. 1996; Boriello & Salucci 2001) and is usually explained by postulating the existence of the dark matter.

Thus, the rotational galactic curves can be naturally explained in brane world models without introducing any additional hypothesis. The galaxy is embedded in a modified, spherically symmetric geometry, generated by the nonzero contribution of the Weyl tensor from the bulk. The extra terms, which can be described as the dark radiation term \( U \) and the dark pressure term \( P \), act as a “dark matter” distribution outside the galaxy. The existence of the dark radiation term generates an equivalent mass term \( M_{\text{U}} \), which is linearly increasing with the distance and proportional to the baryonic mass of the galaxy, \( M_{\text{U}}(r) \approx M_B(r / r_0) \). The particles moving in this geometry feel the gravitational effects of \( U \), which can be expressed in terms of an equivalent mass. Moreover, the limiting value \( v_{\text{gpc}} \) of the angular velocity can be obtained as a function of the total baryonic mass \( M_B \) and radius \( r_0 \) of the galaxy as \( v_{\text{gpc}} \approx (2/\sqrt{3})(GM_B / r_0)^{1/2} + (1/12\sqrt{3})(GM_B / r_0)^{3/2} \) (Mak & Harko 2004).

For a galaxy with baryonic mass of the order \( 10^9 M_{\odot} \) and radius of the order of \( r_0 \approx 70 \) kpc, we have \( v_{\text{gpc}} \approx 287 \) km s\(^{-1}\), which is of the same order of magnitude as the observed value of the angular velocity of the galactic rotation curves. In the framework of this model all the relevant physical parameters (metric tensor components, dark radiation and dark pressure terms) can be obtained as functions of the tangential velocity, and hence they can be determined observationally.

However, imposing a group of conformal motions on the brane implies a major restriction on the geometrical structure of the spacetime. Therefore, it would be important to analyze the
behavior of the galactic rotation curves in the brane world model without particular assumptions on the geometry of the brane. In the framework of a Newtonian approximation of the brane metric the possibility that the galactic rotation curves can be explained by the presence of the dark radiation and dark pressure was considered in Pal et al. (2005).

It is the purpose of this paper to consider the geometric properties of the spacetime at the galactic level on the brane and to derive the expressions of the physical quantities (dark radiation and dark pressure), which determine the dynamic of the particle in circular orbit. Under the assumption of spherical symmetry the basic equations describing the static gravitational field on the brane depend on two unknown parameters, the dark radiation and the dark pressure. As a starting point in our study we adopt the well-established observation of the constancy of the galactic baryonic matter is of the order of $10^{4.5}$ to $10^{5}$.

Therefore, there is a relatively good agreement between these two quantities, the difference between prediction and observation being in the range of 15%–20%. Secondly, we compare the tangential angle depends on the tangential velocity of the particles in stable circular orbits, the baryonic mass, and the radius of the galaxy. An analytic expression for the deflection angle in the first order of approximation is also obtained. The size of the radius for which the effects of the extra dimensions is important is also derived as a function of the tangential velocity and of the cosmological parameters. The explicit expressions of observationally important parameters, like the tangential shear, are presented. Hence, the theoretical predictions of the deflection of light in the brane world models can be compared with the observations.

The predictions of the brane world model are compared with the observational data in two cases. First, we compare the observational values of the truncation parameter (the size of the dark matter halos), obtained in the framework of the truncated isotothermal sphere model by Hoekstra et al. (2004), with the size of the region for which bulk effects are important on the brane. There is a relatively good agreement between these two quantities, the difference between prediction and observation being in the range of 15%–20%. Secondly, we compare the tangential shear in the current model with the observational values of Hoekstra et al. (2004). Consistency between brane world models and observations is obtained when the mass-radii ratio of the galactic baryonic mass is of the order of $10^{-4.5}$ to $10^{-4}$.

Therefore, the bending of light could provide a powerful method to distinguish between models that assume that dark matter is a form of unknown matter and those that assume that dark matter is a result of a change in the dynamical laws that govern the motion of particles.

The introduction in the past few years of new observational techniques moved cosmology into the era of precision science. From the study of the cosmic microwave background (CMB), large-scale structure of galaxies, and distant Type Ia supernovae, a new paradigm of cosmology has been established. In this new standard model, the geometry of the universe is flat so that $\Omega_{\text{total}} = 1$, and the total density is made up of matter (comprised of baryons [$\Omega_b = 0.005$] and cold dark matter [$\Omega_{\text{CDM}} = 0.25$]) and dark energy with $\Omega_{\Lambda} = 0.7$ (Tegmark et al. 2004). In particular, current cosmological data provide a very precise bound on the physical dark matter density, $\Omega_{\text{DM}}h^2 = 0.115 \pm 0.012$ (Tegmark et al. 2004). This bound provides a very strong input on any particle physics or alternative gravitation model for dark matter. On the other hand, dark matter is assumed to play a very important role in galaxy and large-scale structure formation. Since in the model developed in this paper the basic properties of the galactic rotation curves are explained without resorting to dark matter, it seems that this would imply that all the dark matter in the universe could be related to extradimensional or purely nonstandard gravitational effects, as it has been already proposed (Cembranos et al. 2003; Maroto 2004). In these models the branons, hypothetical particles attached to the scalar fluctuations of the brane, are decoupled from standard model matter, thus playing the role of the dark matter. Therefore, a particle component of the cosmological dark matter cannot be excluded a priori even in the framework of brane world models. Moreover, nonbaryonic particles from the standard model or some of its extensions could also contribute substantially to the cosmological dynamics.

This paper is organized as follows. The gravitational field equations for a static, spherically symmetric vacuum brane are written down in § 2. The definition and main properties of the observationally important parameters for the study of the galactic rotation curves are presented in § 3. In § 4 we derive the galactic metric and the basic physical parameters (dark radiation, dark pressure, effective potential, etc.) on the brane. The deflection of light in the flat rotation curve region is considered in § 5. In § 6 we discuss and conclude our results.

2. THE FIELD EQUATIONS FOR A STATIC, SPHERICALLY SYMMETRIC VACUUM BRANE

We start by considering a five-dimensional spacetime (the bulk), with a single four-dimensional brane, on which gravity is confined. The four-dimensional brane world $(\mathcal{M}, g_{\mu\nu})$ is located at a hypersurface $\mathcal{B}(\mathcal{X}^4 = 0)$ in the five-dimensional bulk spacetime $(\mathcal{M}, g_{ab})$, of which coordinates are described by $\mathcal{X}^4, A = 0, 1, \ldots, 4$. The action of the system is given by (Shiromizu et al. 2000)

$$S = S_{\text{bulk}} + S_{\text{brane}},$$

where

$$S_{\text{bulk}} = \int (\mathcal{M}^4) \sqrt{-g} \left( \frac{1}{2k_s^2} R + L_{\text{brane}} + \Lambda_s \right) \, d^4X,$$

and

$$S_{\text{brane}} = \int (\mathcal{M}^4) \sqrt{-g} \left( \frac{1}{k_s^2} K^\pm + L_{\text{brane}}(g_{\alpha\beta}, \psi) + \Lambda_b \right) \, d^4x,$$

where $k_s^2 = 8\pi G_s$ is the five-dimensional gravitational constant and $(\mathcal{M}^4)R$ and $(\mathcal{M}^4)L_{\text{brane}}$ are the five-dimensional scalar curvature and matter Lagrangian in the bulk, respectively. Parameters $x^\mu, \mu = 0, 1, 2, 3$ are the induced four-dimensional coordinates of the brane, $K^\pm$ is the trace of the extrinsic curvature on either side of the brane, and $L_{\text{brane}}(g_{\alpha\beta}, \psi)$ is the four-dimensional Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields $\psi$. In the following capital Latin indices run in the range $0, \ldots, 4$, while Greek indices take the values $0, 1, 2, 3$. The action of the system is given by (Shiromizu et al. 2000)
0, ..., 3. Parameters $\Lambda_4$ and $\Lambda$ are the negative vacuum energy densities in the bulk and in the brane, respectively.

The Einstein field equations in the bulk are given by (Shiromizu et al. 2000; Sasaki et al. 2000; Maeda et al. 2003)

\begin{align}
(5) G_{IJ} &= k_4^2 (5) T_{IJ}, \\
(5) T_{IJ} &= -\Lambda_5 (5) g_{IJ} + \delta(B)[-\dot{\lambda}_b (5) g_{IJ} + T_{IJ}],
\end{align}

where

\begin{equation}
(5) T_{IJ} \equiv -2 \frac{\delta (5) L_m}{\delta g_{IJ}} + (5) g_{IJ} L_m
\end{equation}

is the energy-momentum tensor of bulk matter fields, while $T_{\mu \nu}$ is the energy-momentum tensor localized on the brane, defined by

\begin{equation}
T_{\mu \nu} \equiv -2 \frac{\delta L_{\text{brane}}}{\delta g_{\mu \nu}} + g_{\mu \nu} L_{\text{brane}}.
\end{equation}

The delta function $\delta(B)$ denotes the localization of brane contribution. In the five-dimensional spacetime a brane is a fixed point of the $S_5$ symmetry. The basic equations on the brane are obtained by projection of the variables onto the brane world. The induced four-dimensional metric is $g_{IJ} = (5) g_{IJ} - n_l n_J$, where $n_l$ is the spacelike unit vector field normal to the brane hypersurface $M$. In the following we assume that all the matter fields, except gravitation, are confined to the brane. This implies that $(5) L_m = 0$.

Assuming a metric of the form $ds^2 = \eta_{IJ} dx^I dx^J$, we have $g_{IJ} = g_{IJ} = \eta_{IJ}$, where $\eta_{IJ}$ is the Minkowski metric coefficient given by equation (17) must tend to the standard general relativistic Schwarzschild metric coefficient, which gives $C_1 = 2GM$, where $M = \text{const}$ is the mass of the gravitating body.

\begin{equation}
\chi = 0
\end{equation}

where $\tilde{k} = k_5/k_6$, $h_{\mu \nu} = g_{\mu \nu} + \bar{u}_\mu \bar{u}_\nu$, projects orthogonal to $u^\mu$, the “dark radiation” term $U = -k^2 u^\mu u^\nu$ is a scalar, $Q_\mu = k_4^2 h_{\mu \nu} E_{\alpha \beta}$ is a spatial vector, and $P_{\mu \nu} = -k^4 [h_{(\alpha \beta)} u^\alpha u^\beta - \frac{1}{2} h_{\mu \nu} h^{\alpha \beta}] E_{\alpha \beta}$ is a spatial, symmetric, and trace-free tensor.

In the case of the vacuum state we have $\rho = \rho_0$, $T_{\mu \nu} = 0$, and consequently $S_{\mu \nu} = 0$. Therefore, the field equations describing a static brane take the form

\begin{equation}
R_{\mu \nu} = \Lambda g_{\mu \nu} - E_{\mu \nu},
\end{equation}

with the trace $R$ of the Ricci tensor $R_{\mu \nu}$ satisfying the condition $R = R^\mu_\mu = 4\Lambda$. $E_{\mu \nu}$ is a traceless tensor, $E^\mu_\mu = 0$.

In the vacuum case $E_{\mu \nu}$ satisfies the constraint $D_\nu E_{\mu \nu} = 0$. In an inertial frame at any point on the brane we have $u^\mu = \delta_\theta^\mu$ and $h_{\mu \nu} = (0, 1, 1, 1)$. In a static vacuum $Q_\mu = 0$ and the constraint for $E_{\mu \nu}$ takes the form (Germani & Maartens 2001)

\begin{equation}
\frac{1}{3} D_\mu U + \frac{4}{3} U A_\mu + D^\nu P_{\mu \nu} + A^\nu P_{\mu \nu} = 0,
\end{equation}

where $D_\mu$ is the projection (orthogonal to $u^\mu$) of the covariant derivative and $A_\mu = u^\alpha D_\alpha u_\mu$ is the 4-acceleration.

In the static spherically symmetric case we may choose $A_\mu = A(r) u_\mu$ and $P_{\mu \nu} = P(r) (u_\mu u_\nu - \delta_\mu_\nu h_{\mu \nu})$, where $A(r)$ and $P(r)$ (the “dark pressure”) are some scalar functions of the radial distance $r$ and $r_p$ is a unit radial vector (Dadhich et al. 2000).

We choose the static spherically symmetric metric on the brane in the standard form

\begin{equation}
d^2 s = -e^{\nu(r)} dt^2 + e^{\chi(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\end{equation}

Then the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form (Harko & Mak 2004, 2005; Mak & Harko 2004)

\begin{align}
e^{-\frac{\nu}{2}} \left( \frac{1}{r^2} + \frac{\nu'}{r} + 1 \right) &= \frac{48\pi G}{k_4^2 \lambda_b} U + \Lambda, \\
e^{-\frac{\nu}{2}} \left( \frac{1}{r^2} + \frac{1}{r} \right) &= \frac{16\pi G}{k_4^2 \lambda_b} (U + 2P) - \Lambda, \\
e^{-\frac{\nu}{2}} \left( \frac{\nu'}{2} + \frac{\nu'}{r} - \frac{\nu'}{2} \right) &= \frac{32\pi G}{k_4^2 \lambda_b} (U - P) - 2\Lambda,
\end{align}

where prime denotes $d/dr$. Equation (13) can immediately be integrated to give

\begin{equation}
-\frac{1}{2} = \frac{C_1}{r} - \frac{2GMU(r)}{r} - \frac{\Lambda}{3} r^2,
\end{equation}

where $C_1$ is an arbitrary constant of integration, and we define

\begin{equation}
M_U(r) = \frac{24\pi}{k_4^2 \lambda_b} \int_0^r r^2 U(r) \, dr.
\end{equation}

The function $M_U(r)$ is the gravitational mass corresponding to the dark radiation term (the dark mass). For $U = 0$ and $\Lambda = 0$ the metric coefficient given by equation (17) must tend to the standard general relativistic Schwarzschild metric coefficient, which gives $C_1 = 2GM$, where $M = \text{const}$ is the mass of the gravitating body.
3. STABLE CIRCULAR ORBITS AND FREQUENCY SHIFTS IN STATIC SPACETIMES ON THE BRANE

The galactic rotation curves provide the most direct method of analyzing the gravitational field inside a spiral galaxy. The rotation curves have been determined for a great number of spiral galaxies. They are obtained by measuring the frequency shifts of the light emitted from stars and from the 21 cm radiation emission from the neutral gas clouds. Usually the astronomers report the resulting $z$ in terms of a velocity field $v_{|0|$} (Binney & Tremaine 1987; Persic et al. 1996; Boriello & Salucci 2001).

The starting point in the analysis of the motion of the stars on the brane is to assume, as usual, that stars behave like test particles, which follow geodesics of a static and spherically symmetric spacetime. Next, we consider two observers $O_{E}$ and $O_{\infty}$, with 4-velocities $u_{E}^{\mu}$ and $u_{\infty}^{\mu}$, respectively. Observer $O_{E}$ corresponds to the light emitter (i.e., the stars placed at a point $P_{E}$ of the spacetime on the brane), and $O_{\infty}$ represents the detector at point $P_{\infty}$ located far from the emitter and that can be idealized to correspond to "spatial infinity."

Without loss of generality, we can assume that the stars move in the galactic plane $\theta = \pi/2$, so that $u_{E}^{\mu} = (i, r, 0, 0)$, where the dot stands for derivation with respect to the affine parameter $\tau$ along the geodesics. In the timelike case $\tau$ corresponds to the proper time. On the other hand, we suppose that the detector is static (i.e., the $O_{\infty}$ 4-velocity is tangent to the static Killing field $\partial / \partial \tau$), and in the chosen coordinate system its 4-velocity is $u_{\infty}^{\mu} = (i, 0, 0, 0)$ (Nucamendi et al. 2001).

The motion of a test particle in the gravitational field on the brane can be described by the Lagrangian (Mak & Harko 2004)

$$2L = \left(\frac{dx}{d\tau}\right)^{2} = -e^{\nu(r)} \left(\frac{dt}{d\tau}\right)^{2} + e^{\nu(r)} \left(\frac{dr}{d\tau}\right)^{2} + r^{2} \left(\frac{d\Omega}{d\tau}\right)^{2},$$

(19)

where $dS^{2} = d\theta^{2} + \sin^{2} \theta \, dr^{2}$. From the Lagrange equations it follows that we have two constants of motion, the energy $E = e^{\nu(r)} I$ and the angular momentum $l = r^{2} \phi$ (Lake 2004). The condition $u^{\mu} u_{\mu} = -1$ gives $-1 = -e^{-\nu(r)} I^{2} + e^{\nu(r)} r^{2} + r^{2} \phi^{2}$, and with the use of the constants of motion we obtain

$$E^{2} = e^{-\nu} I^{2} + e^{\nu} \left(\frac{I^{2}}{r^{2}} + 1\right).$$

(20)

This equation shows that the radial motion of the particles on a geodesic is the same as that of a particle with position-dependent mass and with energy $E^{2}/2$ in ordinary Newtonian mechanics moving in the effective potential $V_{\text{eff}}(r) = e^{\nu(r)}(I^{2}/r^{2} + 1)$. The conditions for circular orbits $\partial V_{\text{eff}} / \partial r = 0$ and $\dot{r} = 0$ lead to (Lake 2004)

$$I^{2} = \frac{1}{2} \frac{r^{3} \nu'}{1 - r \nu'^{2}/2}, \quad E^{2} = \frac{e^{\nu}}{1 - r \nu'^{2}/2}. $$

(21)

The rotation curves of spiral galaxies are inferred from the redshift and blueshift of the emitted radiation by stars moving in circular orbits on both sides of the central region. The light signal travels on null geodesics with tangent $k^{\mu}$. We may restrict $k^{\mu}$ to lie in the equatorial plane $\theta = \pi/2$ and evaluate the frequency shift for a light signal emitted from $O_{E}$ in circular orbit and detected by $O_{\infty}$. The frequency shift associated with the emission and detection of the light signal is given by

$$z = 1 - \frac{\omega_{I}}{\omega_{\infty}}$$

where $\omega_{I} = -k_{\mu} u_{\mu}^{I}$ and the index $I$ refers to emission ($I = E$) or detection ($I = \infty$) at the corresponding spacetime point (Nucamendi et al. 2001; Lake 2004). Two frequency shifts corresponding to maximum and minimum values are associated with light propagation in the same and opposite direction of motion of the emitter, respectively. Such shifts are frequency shifts of a receding or approaching star, respectively. Using the constancy along the geodesic of the product of the Killing field $\partial / \partial t$ with a geodesic tangent gives the expressions of the two shifts as (Nucamendi et al. 2001; Lake 2004)

$$z_{\pm} = 1 - e^{-[\nu_{-}(r)]^{2}/2} \frac{1 \pm \sqrt{\nu_{r}'^{2}/2}}{\sqrt{1 - \nu_{r}'^{2}/2}},$$

(23)

respectively, where $\exp [\nu_{-}(r)]$ represents the value of the metric potential at the radius of emission $r$ and $\exp (\nu_{\infty})$ represents the corresponding value of $\exp [\nu(r)]$ for $r \rightarrow \infty$. It is convenient to define two other quantities $z_{D} = (z_{+} - z_{-})/2$ and $z_{A} = (z_{+} + z{-})/2$, given by

$$z_{D}(r) = e^{-[\nu_{-}(r)]^{2}/2} \frac{\sqrt{\nu_{r}'^{2}/2}}{\sqrt{1 - \nu_{r}'^{2}/2}},$$

(24)

and

$$z_{A}(r) = 1 - e^{-[\nu_{-}(r)]^{2}/2} \frac{1}{\sqrt{1 - \nu_{r}'^{2}/2}}.$$ 

(25)

respectively, which can be easily connected to the observations (Nucamendi et al. 2001). Parameters $z_{A}$ and $z_{D}$ satisfy the relation $(z_{A} - 1)^{2} - z_{D}^{2} = \exp \{2|\nu_{D} - \nu(r)|\}$, and thus in principle $\exp [\nu(r)]$ can be obtained directly from the observations. This could provide a direct observational test of the galactic geometry and, implicitly, of the brane world and other extradimensional models.

The line element on the brane, given by equation (12), can be rewritten in terms of the spatial components of the velocity, normalized with the speed of light, measured by an inertial observer far from the source, as $dS^{2} = -dt^{2} (1 - \nu^{2})$ (Mak & Harko 2004), where

$$\nu^{2} = e^{-\nu} \left[e^{\nu} \left(\frac{dt}{d\tau}\right)^{2} + r^{2} \left(\frac{d\Omega}{d\tau}\right)^{2}\right].$$

(26)

For a stable circular orbit $\dot{r} = 0$, and the tangential velocity of the test particle can be expressed as

$$v_{tg}^{2} = \frac{r^{2}}{e^{2}} \left(\frac{d\Omega}{dt}\right)^{2}.$$

(27)

In terms of the conserved quantities the angular velocity is given by

$$v_{tg}^{2} = \frac{e^{\nu} I^{2}}{r^{2} E^{2}}.$$

(28)

With the use of equation (21) we obtain

$$v_{tg}^{2} = \frac{\nu^{2}}{2}.$$ 

(29)

Thus, the rotational velocity of the test body is determined by the metric coefficient $\exp (\nu)$ only.
4. GALACTIC METRIC, DARK RADIATION, AND DARK PRESSURE ON THE BRANE

The tangential velocity \( v_{tg} \) of stars moving like test particles around the center of a galaxy is not directly measurable but can be inferred from the redshift \( z_\infty \) observed at spatial infinity, for which \( 1 + z_\infty = \exp \left( \left( \nu_\infty - \nu \right)/2 \left( 1 - v_{tg}^2 \right) \right) \). Because of their nonrelativistic velocities in galaxies bounded by \( v_{tg} \leq (4/3) \times 10^{-3} \), we observe \( v_{tg} \approx z_\infty \) (as the first part of a geometric series), with the consequence that the lapse \( \exp (\nu) \) function necessarily tends to unity, i.e., \( e^\nu \approx 1/(1 - v_{tg}^2) \). The observations show that at distances large enough from the galactic center \( v_{tg} \approx \text{const} \) (Binney & Tremaine 1987; Persic et al. 1996; Boriello & Salucci 2001).

In the following we use this observational constraint to reconstruct the metric of a galaxy on the brane. The condition

\[
v_{tg}^2 = \frac{1}{2} rv' = \text{const} \tag{30}
\]

immediately allows one to find the metric tensor component \( e^\nu \) in the flat rotation curve region on the brane as

\[
e^\nu = \left( \frac{R_b}{r} \right)^{2v_{tg}^2}, \tag{31}
\]

where \( R_b \) is a constant of integration.

Adding equations (14) and (15) and eliminating the dark radiation term \( U \) from the resulting equation and equation (13) gives the following equation satisfied by the unknown metric tensor components on the brane:

\[
e^{-\beta} \left( \frac{\nu'' + \beta^2}{2} + \frac{2\nu'}{r} - \frac{2\nu^2'}{r^2} + \frac{2}{r^2} \right) - \frac{4}{r^2} + 4\Lambda = 0. \tag{32}
\]

Substituting \( \nu \) given by equation (31) leads to a first-order linear differential equation satisfied by \( e^{-\lambda} \), given by

\[
\frac{d}{dr} e^{-\lambda} = -\frac{2}{v_{tg}^2} - \frac{1}{v_{tg}^2} + \frac{1}{v_{tg}^2 + 2} + \frac{1}{v_{tg}^2 + 2} - \frac{4\Lambda}{v_{tg}^2 + 2} r. \tag{33}
\]

From equation (33) we obtain \( e^{-\lambda} \) as

\[
e^{-\lambda} = \frac{1}{\left( v_{tg}^2 + 2 \right)\alpha} + C_2 r^{-2\alpha} - \frac{2\Lambda}{\left( v_{tg}^2 + 2 \right)\alpha} r^2, \tag{34}
\]

where \( C_2 \) is an arbitrary constant of integration and we denoted

\[
\alpha = \frac{v_{tg}^2 + 2}{v_{tg}^2 + 2}. \tag{35}
\]

Equations (31) and (34) give the complete galactic metric on the brane.

Once the metric tensor components are known, the calculation of the dark radiation and dark pressure terms is straightforward.

Equation (13) immediately gives the dark radiation \( U \) on the brane as

\[\frac{48\pi G}{k_4^2\lambda_b} U(r) = \frac{1}{v_{tg}^2 + 2} \left[ \frac{2(2 - \alpha)}{(\alpha + 1) - v_{tg}^2} \right] \Lambda + \left[ 1 - \frac{1}{\left( v_{tg}^2 + 2 \right)\alpha} \right] \frac{1}{r^2} + (2\alpha - 1)C_2 r^{-2\alpha - 2}. \tag{36}\]

From equations (13) and (14) we obtain the dark pressure on the brane as

\[\frac{96\pi G}{k_4^2\lambda_b} P(r) = e^{-\lambda} \left( \frac{3\nu'}{r} + \frac{4}{r^2} - \frac{\lambda'}{r} \right) - \frac{4}{r^2} + 4\Lambda, \tag{37}\]

or

\[\frac{96\pi G}{k_4^2\lambda_b} P(r) = \frac{4}{v_{tg}^2 + 2} \left[ \frac{2}{v_{tg}^2 + 2} \frac{5v_{tg}^2 + 1}{v_{tg}^2 + 2} - \frac{2v_{tg}^2 - 1}{v_{tg}^2 + 2} \right] \frac{1}{r^2} + \frac{2C_2}{v_{tg}^2 + 2} \frac{5v_{tg}^2 + 1}{v_{tg}^2 + 2} r^{-2\alpha - 2}. \tag{38}\]

The angular momentum and the energy of a star moving in the galactic gravitational field on the brane are given by

\[l = r \frac{v_{tg}}{\sqrt{1 - v_{tg}^2}} \tag{39}\]

and

\[E = \left( \frac{R_b}{r} \right)^{\frac{v_{tg}^2}{2}} \frac{1}{\sqrt{1 - v_{tg}^2}}, \tag{40}\]

respectively. For the effective potential describing the Newtonian motion of a test particle we find

\[V_{eff}(r) = \left( \frac{R_b}{r} \right)^{\frac{v_{tg}^2}{2}} \frac{1}{1 - v_{tg}^2}. \tag{41}\]

From equation (18) it follows that the dark mass associated with the dark radiation term is given by

\[2GM_U(r) = \frac{1}{3} \left( \frac{v_{tg}^2}{2} \right) \left[ \frac{2(2 - \alpha)}{(\alpha + 1) - v_{tg}^2} \right] \Lambda r^3 + \left[ 1 - \frac{1}{\left( v_{tg}^2 + 2 \right)\alpha} \right] r - C_2 r^{-2\alpha - 1}. \tag{42}\]
At distances relatively close to the galactic center, where the effect of the cosmological constant can be neglected, but sufficiently high so that the last term in equation (42) is also negligibly small, the dark mass can be approximated as

\[ 2GM_{D}(r) \approx \frac{\nu_{tg}^{2}(\nu_{tg}^{2} + 1)}{\nu_{tg}^{2} + 2} r \approx \frac{\nu_{tg}^{2}}{2} r. \]  

(43)

The dark mass is linearly increasing with the radial distance from the galactic center.

The behavior of the metric coefficients and of the dark radiation and pressure in the solutions we have obtained depends on two arbitrary constants of integration \( R_{b} \) and \( C_{2} \). Their numerical value can be obtained by assuming the continuity of the metric coefficient \( \exp(-\lambda) \) across the vacuum boundary of the galaxy. For simplicity we assume that inside the "normal" (baryonic) luminous matter, with density \( \rho_{B} \), which forms a galaxy, the non-local effects of the Weyl tensor can be neglected. We define the vacuum boundary \( r_{b} \) of the galaxy (which for simplicity is assumed to have spherical symmetry) by the condition \( \rho_{B}(r_{b}) \approx 0 \).

Therefore, at the vacuum boundary of the galaxy the metric coefficients are \( \nu = 1 - 2GM_{B}/r_{0} \) and \( \exp(-\lambda) = 1 - 2GM_{B}/r_{0} \), where \( M_{B} = 4\pi \int_{0}^{r_{b}} \rho_{B}(r)r^{2} dr \) is the total baryonic mass inside the radius \( r_{b} \). The continuity of \( \exp(\nu) \) through the surface \( r = r_{b} \) gives

\[ R_{b} = r_{b} \left(1 - \frac{2GM_{B}}{r_{0}}\right)^{-1/2} \nu_{tg}^{2}, \]  

(44)

while the continuity of \( \exp(-\lambda) \) fixes the integration constant \( C_{2} \) as

\[ C_{2} = \frac{r_{b}^{2\alpha}}{\nu_{tg}^{2} + 2} \left[ \left(1 - \frac{2GM_{B}}{r_{0}}\right)(\nu_{tg}^{2} + 2) - \frac{1}{\alpha} + \frac{2\Lambda}{\alpha + 1} r_{b}^{2}\right]. \]  

(45)

Thus, at the galactic level the metric coefficients and the dark radiation and pressure on the brane can be obtained in terms of observable quantities.

5. LIGHT DEFLECTION AND LENSING BY GALAXIES IN BRANE WORLD MODELS

One of the ways we could in principle test the galactic metric obtained in the previous section would be by studying the light deflection by galaxies, and in particular by studying the deflection of photons passing through the region where the rotation curves are flat. Let us consider a photon approaching a galaxy from far distances. We compute the deflection by assuming that the metric is given by equations (31) and (34).

The bending of light on the brane by the galactic gravitational field results in a deflection angle \( \Delta \phi \) given by

\[ \Delta \phi = 2|\nu(r_{0})| - \pi, \]  

(46)

where \( \nu_{t} \) is the incident direction and \( r_{0} \) is the coordinate radius of the closest approach to the center of the galaxy, and generally (Nucamendi et al. 2001)

\[ \nu(r_{0}) - \nu_{\infty} = \int_{r_{0}}^{\infty} e^{i(r_{0})/2} \left[ e^{i(r_{0}) - i(r)} \left(\frac{r}{r_{0}}\right)^{2} - 1\right]^{-1/2} \frac{dr}{r}. \]  

(47)

By taking into account the explicit expressions of the metric tensor components in the flat rotation curve region, we obtain

\[ \phi(r_{0}) - \phi_{\infty} = \int_{r_{0}}^{\infty} \left[ \frac{1}{\nu_{tg}^{2} + 2} \alpha + C_{2} r^{-2\alpha} - \frac{2\Lambda}{\nu_{tg}^{2} + 2} (\alpha + 1) \right] \times \left[ e^{i(r_{0})} - 2R_{b}^{2}c^{2}r_{b}^{2(1 - r_{b}^{2})} - 1\right]^{-1/2} \right]^{1/2} \]  

(48)

By introducing a new variable \( \eta = r/r_{0} \) and taking into account the matching condition given by equation (65), we can rewrite equation (48) as

\[ \phi(r_{0}) - \phi_{\infty} = \int_{1}^{\infty} \left[ \sqrt{\nu_{tg}^{2} + 2\eta^{-1}(\eta^{2} - 1) - 1} \right]^{-1/2} d\eta \times \left[ \frac{1}{\alpha + 1} + \frac{2\Lambda}{\alpha + 1} r_{0}^{2} \right]^{-1/2}. \]  

(49)

In standard general relativity the bending angle by a galaxy is given by \( \Delta \phi_{GR} = 2|\phi(r_{0}) - \phi_{\infty}| - \pi = 4GM_{B}/r_{0} \). The total deflection angle is given by \( \Delta \phi_{total} = \Delta \phi_{brane} + \Delta \phi_{GR} \).

In order to estimate the magnitude of the bulk contribution to the bending of light on the brane and to compare this contribution to the results from standard dark matter models, we define the parameter

\[ \delta = \frac{(\Delta \phi)_{brane}}{(\Delta \phi)_{DM}}, \]  

(50)

where \( (\Delta \phi)_{DM} \) is the deflection angle as is usually considered in the standard dark matter models. Generally, in the dark halo models, a light ray that goes past the halo without entering it propagates entirely in a Schwarzschild metric and is deflected by an angle \( \Delta \phi_{GR} \). The bending of the light that passes through the dark matter halo is determined by the metric inside the halo, and this depends on the assumed properties of the dark matter.

We compute the parameter \( \delta \) in a semirealistic model for dark matter, in which it is assumed that the galaxy (the baryonic matter) is embedded into an isothermal mass distribution (the dark matter), with the density varying as \( \rho = \sigma^{2}/2\pi Gr^{2} \), where \( \sigma_{v} \) is the line-of-sight velocity dispersion (Binney & Tremaine 1987). In this model it is assumed that the mass distribution of the dark matter is spherically symmetric. In fact, if the rotation curve is flat, then the mass distribution must be that of the isothermal sphere, for which we also have \( v_{tg} = \sqrt{2}\sigma_{v} \). The surface density \( \Sigma \) of the isothermal sphere is \( \Sigma(r) = \sigma_{v}^{2}/2Gr \). For this dark matter distribution the bending angle of light is constant and is given by \( (\Delta \phi)_{DM} = 2\pi v_{tg}^{2} \) (Blanford & Narayan 1992).

Therefore, in this model

\[ \delta = \frac{(\Delta \phi)_{brane}}{2\pi v_{tg}^{2}}. \]  

(51)

The variation of the parameter \( \delta \) as a function of the quantity \( \chi = 4GM_{B}/r_{0} \) is represented in Figure 1.
By assuming that $v_{tg}^2 \ll 1$ and neglecting all terms containing the tangential velocity and the cosmological constant, the light deflection angle on the brane becomes

$$\Delta \phi_\text{brane} \approx 2 \left[ \int_1^\infty \eta^{-1} \left[ (\eta^2 - 1) (2 - \chi \eta^{-1}) \right]^{-1/2} \, d\eta \right] - \pi. \tag{54}$$

The integral can be calculated exactly, and we obtain

$$\Delta \phi \approx 2 \left[ \int_1^\infty \eta^{-1} \left[ (\eta^2 - 1) (2 - \chi \eta^{-1}) \right]^{-1/2} \, d\eta \right] - \pi, \tag{55}$$

where $K(m)$ is the complete elliptic integral of the first kind given by

$$K(m) = \int_0^\pi \frac{1}{\sqrt{1 - m \sin^2 \theta}} \, d\theta$$

and

$$F(a; b; z) = \sum_{k=0}^\infty \frac{(a)_k \cdot \ldots \cdot (a_p)_k}{(b)_k \cdot \ldots \cdot (b_q)_k} z^k / k!.$$

A series expansion of equation (55) gives

$$\Delta \phi \approx 2 \left[ \int_1^\infty \eta^{-1} \left[ (\eta^2 - 1) (2 - \chi \eta^{-1}) \right]^{-1/2} \, d\eta \right] - \pi. \tag{57}$$

Therefore, for the isothermal dark matter model and in the limit of large $\chi$ the parameter $\Delta$ tends to the value $\chi/\sqrt{4 \pi v_{tg}}$. We find that the numerical results presented in Figure 1. are consistent with the two limiting forms of the parameter $\delta$.

If the effect of the cosmological constant can be neglected, the deflection angle of the light in the Weyl stress—dominated region around a galaxy is a function of the tangential velocity of the test particles in stable circular orbit only. To obtain the total value of the bending angle of the light, one must also add the usual general relativistic contribution to the bending from the baryonic mass of the galaxy.

Once the light deflection angle is known, one can study the gravitational lensing on the brane in the flat rotation curve region. The lensing geometry is illustrated in Figure 2.

The light emitted by the source $S$ is deflected by the lens $L$ (a galaxy in our case) and reaches the observer $O$ at an angle $\theta$ to the optic axis $OL$, instead of $\beta$. The lens $L$ is located at a distance $D_L$ to the observer and a distance $D_{LS}$ to the source, respectively, while the observer-source distance is $D_S$. Parameter $r_0$ is the impact factor (distance of closest approach) of the photon beam.

The lens equation is given by (Whisker 2005)

$$\tan \beta = \tan \theta - \frac{D_{LS}}{D_S} \left[ \tan \theta + \tan(\Delta \phi - \theta) \right]. \tag{59}$$
Therefore, the ratio of the angular radii of the Einstein rings in the brane world models and in the isothermal dark matter model is given by the same parameter δ that has been already introduced in equation (50). The variation of the ratio of the Einstein rings in the two models is presented in Figure 1.

6. DISCUSSION AND FINAL REMARKS

The galactic rotation curves continue to pose a challenge to present-day physics as one would like to have a better understanding of some of the intriguing phenomena associated with them, like their universality and the very good correlation between the amount of dark matter and the luminous matter in the galaxy. To explain these observations, models based on particle physics in the framework of Newtonian gravity are the most commonly considered.

In this paper we have considered and further developed an alternative view to the dark matter problem, namely, that the galactic rotation curves can naturally be explained in models in which our universe is a domain wall (a brane) in a multidimensional spacetime (the bulk). The extra terms in the gravitational field equations on the brane induce a supplementary gravitational interaction, which can account for the observed behavior of the galactic rotation curves. By using the simple observational fact of the constancy of the galactic rotation curves, the galactic metric and the corresponding Weyl stresses (dark radiation and dark pressure) can be completely reconstructed.

The form of the galactic metric we have obtained in the framework of the brane world models differs from what would be naively expected, that is, $ds^2 = -(1 + 2r) dt^2 + (1 + 2r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$, with $\Phi$ representing the Newtonian potential. This form is often implicitly assumed (Pal et al. 2005), and the fact that it is not appropriate for the region where the rotation curves are flat can lead to significant errors in the estimation of the magnitude of some important physical effects, like, for example, the bending of light by the galaxies.

The observations in spiral galaxies usually determine $v_\phi$ from the redshift $z_D$, so that $z_D \approx v_\phi \approx 0$. By assuming that $\exp(v_\phi) \approx 1$, and with the use of equations (24) and (31), it follows that the results we have obtained are self-consistent if the condition

$$\frac{r}{R_b} \approx \frac{v_\phi^2}{4} \left( 1 - v_\phi^2 \right)^{-1/2} \approx 1$$

holds. By using the approximation $(r/R_b)^{v_\phi^2} \approx 1 - v_\phi^2 \ln (r/R_b)$, valid for $|v_\phi^2 \ln (r/R_b)| \ll 1$, it follows that the condition is satisfied if $|\ln (r/R_b)| \ll v_\phi^2$. Since $v_\phi \approx 10^{-3}$ to $10^{-4}$, the approximations we have used are self-consistent as long as $-10^6 \ll \ln (r/R_b) \ll 10^6$, a condition that, for the case of the galaxies, does not impose any practically relevant constraint.

As a second consistency condition we require that the time-like circular geodesics on the brane be stable. Let $r_0$ be a circular orbit and consider a perturbation of it of the form $r = r_0 + \delta$, where $\delta \ll r_0$ (Lake 2004). Taking expansions of $V_{\text{eff}}(r)$ and $\exp (v + \lambda)$ about $r = r_0$, it follows from equation (20) that

$$\delta + \frac{1}{2} e^{v(r_0) + \lambda(r_0)} V_{\text{eff}}(r_0) \delta = 0.$$  

The condition for stability of the simple circular orbits requires $V_{\text{eff}}'(r_0) > 0$ (Lake 2004). This gives

$$2(\frac{r_0}{R_b})^2 v_\phi^2 \approx \left[ -v_\phi^2 \left( 1 - 2v_\phi^2 \right) r_0^2 + I^2 \left( 3 - 5v_\phi^2 + 2v_\phi^4 \right) \right] > 0.$$
By neglecting the small terms containing powers of the tangential velocity $v_{tg}$ with respect to unity, it follows that the circular orbits are stable if their radii satisfy the condition $r_0^2 < 3L/v_{tg}^2$.

In the current model there is a very simple relation between the mass of the intergalactic dark radiation $M_L$, and the luminous (baryonic) mass $M_B$ of the galaxy. By assuming that the tangential velocity of particles in circular orbit is approximately given by $v_{tg}^2 \approx GM_B/r_0$, it follows that $M_L$ is related to $M_B$ via the following simple scaling relation:

$$M_L(r) \approx \frac{r^2}{2r_0} M_B. \quad (68)$$

The mass of the dark radiation is proportional to the mass of the galaxy and is linearly increasing with the distance to the galactic center.

If we assume that the flat rotation curves extend indefinitely, the resulting spacetime is not asymptotically flat, but of de Sitter type. This is due to the presence of the cosmological constant $\Lambda$ on the brane. Observationally, the galactic rotation curves remain flat to the farthest distances that can be observed. On the other hand, there is a simple way to estimate an upper bound for the mean density of the universe and the value of the cosmological parameter only. For a velocity dispersion $\sigma$ of the velocity dispersion, the observed upper bound for the velocity dispersion and of the cosmological parameters only. For a velocity dispersion $\sigma = 146 \text{ km s}^{-1}$ and with $\Omega_m = 0.3$, equation (72) gives $s \approx 245 \text{ h}^{-1} \text{ kpc}$, while the truncation size obtained observationally in Hoekstra et al. (2004) is $s = 213 \text{ h}^{-1} \text{ kpc}$. For $\sigma = 110 \text{ km s}^{-1}$ we obtain $s \approx 184 \text{ h}^{-1} \text{ kpc}$, while $s = 136 \text{ km s}^{-1}$ gives $s \approx 228 \text{ h}^{-1} \text{ kpc}$. All these values are consistent with the observational results reported in Hoekstra et al. (2004), the error between prediction and observation being of the order of 20%. We also have to mention that the observational values of the truncation parameter depend on the scaling relation between the velocity dispersion and the fiducial luminosity of the galaxy. Two cases have been considered in Hoekstra et al. (2004), the case in which the luminosity $L_B$ does not evolve with the redshift $z$ and the case in which $L_B$ scales with $z$ as $L_B \propto (1+z)$. Depending on the scaling relation, slightly different values of the velocity dispersion and truncation parameter are obtained.

The study of the deflection of light (gravitational lensing) in the flat rotation curve region can provide a powerful observational tool for discriminating between standard dark matter and brane world models. Due to the fixed form of the galactic metric on the brane, the light bending angle is a function of the tangential velocity of particles in stable circular orbit and the baryonic mass and radius of the galaxy. The specific form of the bending angle is determined by the brane galactic metric, and this form is very different as compared to the other dark matter models (long-range self-interacting scalar fields, MOND, nonsymmetric gravity, etc.).
When \( \chi = 4GMb/r_0 \ll 1 \), the gravitational light deflection angle is much larger than the value predicted by the standard general relativistic approach. Even when we compare our results with standard dark matter models, like the isothermal dark matter halo model, we still find significant differences in the lensing effect. Therefore, the study of the gravitational lensing may provide evidence for the existence of the bulk effects on the brane. On the other hand, since in this model there is only baryonic matter, all the physical properties at the galactic level are determined by the amount of the luminous matter and its distribution.

Generally, the angular position of the source \( \beta \) is related to the angular position of its image by the lens equation, \( \beta = \theta + \alpha(\theta) \). The standard general relativistic deflection law for a point mass in general relativity is \( \alpha(\theta) = -\theta^2/\theta \) (Mortlock & Turner 2001). In order to test alternative theories of gravity, a more general point mass deflection law, of the form \( \alpha(\theta) = -\theta^2/\theta (\theta_0/\theta + \theta_0)^{-\xi} \), was introduced by Mortlock & Turner (2001; for a more general parameterization see Hoekstra et al. 2002). Here \( \theta_0 \) is a parameter that can be related to the scale \( r_0 \) beyond which the physics becomes non-Newtonian. In the brane world model the function \( \alpha \) can be represented as \( \alpha(\theta) = -[(\theta_0/\theta)^2/\theta (r_0/GB_0)(\Delta \phi)_{brane} \). From the deflection law one can find the tangential shear of the image, \( \gamma_{tan}(\theta) = [(\partial \alpha/\partial \theta) - \alpha(\theta)/\theta]/2 \). For the brane world model we obtain

\[
\gamma_{tan}(\theta) = \frac{\left(\theta^{GR}\right)^2}{\theta^2} \frac{r_0}{GM_b} (\Delta \phi)_{brane}.
\]

By using the definition of the parameter \( \delta \), we obtain \( (\Delta \phi)_{brane} = \delta(\Delta \phi)_{DM} = 2\pi b v_{out}^2/\theta^2 \), where, for simplicity, we have adopted again the isothermal sphere model for “standard” dark matter. Moreover, we assume that the tangential velocity is related to the baryonic mass by the (approximate) Keplerian value, \( v_{out} \approx GM_b/r_0 \). Therefore, we obtain for the tangential shear the simple expression

\[
\gamma_{tan}(\theta) \approx 2\pi \delta \left(\frac{\theta^{GR}}{\theta^2}\right)^2.
\]

This form of the tangential shear is similar to that one resulting from a modification of the deflection law proposed by Hoekstra et al. (2002), which is given by \( \gamma_{tan}(\theta) = (\theta_{out}/\theta_0)(\theta_0/\theta)^{2\delta} \), where \( \theta_{out} \) and \( \theta_0 \) are some ad hoc parameters. The allowable range for the values of \( \theta_0 \), obtained from observations, excludes small values. The brane world model fixes the values of these parameters as \( \theta_{out}/\theta_0 = 2\pi \delta \). Since the ratio \( \theta_{out}/\theta_0 \) is a measure of the “mass discrepancy,” \( \delta \) could also be interpreted as a measure of the extra mass generated by the dark radiation. The best fit with the observational data is obtained for \( \theta_{out} = s \), where \( s \) is the truncation parameter. Therefore, \( \delta \) can also be expressed as \( \delta = R_s/\theta_0 \).

The mean signal can be measured in galaxy-galaxy lensing observations. We compare the predictions of the brane world model, given by equation (74), with the observational data obtained in Hoekstra et al. (2004). By fitting an isothermal sphere model to the tangential shear at radii smaller than 2\( \prime \) (corresponding to a maximum radius of \( \approx 350 \ h^{-1} \) kpc), the mean value for the Einstein radius was found to be \( \theta_E = 0'.140 \pm 0'.009 \). For this value of the Einstein radius the variation of \( \gamma_{tan} \) calculated for different values of the parameter \( \delta \) is compared with the observational data presented in Hoekstra et al. (2004) in Figure 3. The best fit with the observational data corresponds to large values of \( \delta \), \( \delta \approx 8\text{–}12 \). Since the velocity dispersion for the considered sample of galaxies is around \( \sigma \approx (128, 150) \text{ km s}^{-1} \), corresponding to \( v_{out} \approx 150\text{–}210 \text{ km s}^{-1} \), it follows that the ratio of the baryonic mass and the radius for these galaxies should be of the order of \( \chi \in (1.5 \times 10^{-5}, 5 \times 10^{-5}) \). If the baryonic mass-radius ratio for the galaxies is significantly smaller than this value, then the predictions of this simple brane world model could not fit satisfactorily the existing observational data. An estimate of the galactic mass-radius ratio can be obtained by approximating the tangential velocity by its Keplerian value, which gives \( \chi \approx 8\sigma^2 \). For \( \sigma = 150 \text{ km s}^{-1} \) we obtain \( \chi \approx 2 \times 10^{-6} \). This simple estimate seems to indicate at first sight that the present observational data do not favor the brane world interpretation for dark matter. However, we have to mention that the estimation of the ratio of the baryonic mass and the radius from the tangential velocity could be affected by significant errors. It would be more realistic to assume that \( \chi \approx 4k^2v_{out}^2 = 8k^2\sigma^2 \), where \( k \) is a factor describing the uncertainty in the determination of the ratio of the “normal” mass and radius of a galaxy from the tangential velocity alone. If this factor is of the order of 2, \( k \approx 2 \), then the brane world model prediction could become consistent with the observations. Hence, to find a definite answer to the possibility of the brane world models to correctly describe the weak-lensing data, a much more careful analysis of data is needed.

Therefore, the study of brane world model galaxy-galaxy lensing and of the dark matter halo properties could provide strong constraints on the brane world model and on related high-energy physics models.

The measurement of the azimuthally averaged tangential shear around galaxies is robust against residual systematics; that is, contributions from a constant or gradient residual shear cancel. However, this is not the case for the quadrupole signal (Hoekstra et al. 2004). If the lens galaxies are oriented randomly with respect to the residual shear, then the average over many lenses will cancel the contribution from systematics. But in real observations the uncorrected shapes of the lenses are aligned with the systematic signal. An imperfect correction can give rise to a small quadrupole signal, which can mimic a cosmic shear signal. Generally, the residual shear has an amplitude \( \gamma \) and is located with respect to the major axis of the lens at an angle \( \phi \). The tangential shear \( \gamma_{tan} \) observed at a point \( (r, \theta) \) is the sum of the lensing signal \( \gamma_{tan} \) and the contribution from the systematics, \( \gamma_{tan} \), so that \( \gamma_{tan} = \gamma_{tan} + \gamma_{tan} \). Parameter \( \gamma_{tan} \) is given by \( \gamma_{tan} = -\gamma \cos [2(\theta - \phi)] \) (Hoekstra et al. 2004). One way to estimate the flattening of the
halo is to measure the shears $\gamma_{\tan}^{(\pm)}$ at $\theta = 0$ and $\pi$ and $\gamma_{\tan}^{(-)}$ at $\theta = \pi/2$ and $3\pi/2$, respectively. The observed ratio is $f_{\text{obs}} = \gamma_{\text{tan}}^{(+)} / \gamma_{\text{tan}}^{(-)}$ and can be written as $f_{\text{obs}} = \left[ \gamma_{\tan}^{(+)}/\gamma_{\tan}^{(-)} + \gamma_{\tan}^{(+)} \cos(2\phi)/\gamma_{\tan}^{(-)} \gamma_{\tan}^{(+)}/\gamma_{\tan}^{(-)} \right]$ (Hoekstra et al. 2004). In the framework of the simple brane world model we have considered that all the shear parameters (tangential as well as residual) are proportional to the values corresponding to the standard dark matter case. On the other hand, the mean value of $\cos(2\phi)$ is proportional to the measure $\alpha$ of the correlation between the position angle of the lens and the direction of the systematic shear of the background galaxies, which is a very small quantity. The brane world model correction to $\alpha$ cannot lead to a significant increase to this quantity, due to the large separation distances between galaxies. Therefore, we expect that $f_{\text{obs}}$ is an invariant and brane world effects do not affect the robustness of the measurement of the average halo shape.

With the use of the approximate relation $(1 + x/n)^{2} \approx \exp (x)$, which is valid for large $n$, we can write equation (44) in the form

$$R_{b} \approx R_{0} \exp \left( \chi/4v_{\chi}^{2} \right).$$ (75)

In the case of a very massive object with $\chi \approx 1$ and for $v_{\chi} \in (10^{-4}, 10^{-3})$ it follows that $R_{b} \gg R_{0}$. But for very small values of $\chi$, of the same order as $v_{\chi}$, $R_{b} \approx R_{0}$. Therefore, flat rotation curves are specific for particles moving in stable circular orbits around galaxies (having $\chi \ll 1$), while the same phenomenon cannot be detected for very compact (stellar or black hole type) objects.

The measurement of the anisotropy in the lensing signal around galaxies and the detection of the flattening of the dark matter halos could pose some serious challenges to the alternative theories of gravitation (Hoekstra et al. 2002, 2004). In many theories proposing modifications of the dynamic law for gravitation the lensing signal caused by the intrinsic shapes of the galaxies decreases as $\propto r^{-2}$, and hence it is negligible at large distances. Therefore, dark halos cannot be modeled as spherical systems. This important observational result can clearly discriminate between the gravitational explanations proposed as substitutes for dark matter. However, in the framework of the model discussed in this paper, the flattening of the dark matter halos and the anisotropic signal can be easily explained, at least at a simple qualitative level. We have obtained all our results under the (unrealistic) assumption of the spherical symmetry, with the galaxy (consisting of baryonic matter only) modeled as a self-gravitating relativistic sphere. A much more realistic model for a galaxy is its representation as a self-gravitating rotating stationary axisymmetric disk, in which generally the metric coefficients depend on both polar coordinates $r$ and $\theta$. The brane world metric outside this baryonic matter distribution is also anisotropic, and we would naturally expect a flattening of the region in which the bulk effects are important, associated with an anisotropic (angular dependent) weak-lensing signal. The “dark matter” is, in this model, a projection of the geometry of the higher dimensional spacetime far away from the baryonic matter distribution, which has to match the geometry of the galaxy. Therefore, disk type or spheroidal distributions will automatically create flattened and anisotropic (non-spherical) geometrical effects. For example, already in standard general relativity in the weak-field limit a cold disk has the associated gravitational potential $-\Phi = -v_{\chi}^{2} \ln |r(1 + |\cos \theta|)/D|$, where $D$ is a fiducial length scale. The bending of light is anisotropic (polar angle dependent), as is the focusing effect of the disk (Cai & Shu 2002). Therefore, more realistic geometric distributions of the baryonic matter inside the galaxy and of the exterior spacetime geometry of the brane can easily explain the flattening effect and the anisotropic lensing observed for dark matter halos.

One of the main advantages of the brane world models, as compared to the other alternative explanations of the galactic rotation curves and of the dark matter, is that it can give a systematic and coherent description of the universe from galactic to cosmological scales. On the other hand, in the current model all the relevant physical quantities, including the dark energy and the dark pressure, which describe the nonlocal effects due to the gravitational field of the bulk, are expressed in terms of observable parameters (the baryonic mass, the radius of the galaxy, and the observed flat rotational velocity curves). Therefore, this opens the possibility of testing the brane world models by using astronomical and astrophysical observations at the galactic scale. In this paper we have provided some basic theoretical tools necessary for the in-depth comparison of the predictions of the brane world model and of the observational results.

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