A Revised Generalized Kolmogorov-Sinai-like Entropy and Markov Shifts

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Abstract
The Kolmogorov-Sinai entropy in the sense of Tsallis under Bernoulli shifts was obtained by Mesón and Vericat [J. Math. Phys. 37, 4480(1996)]. In this paper, we propose a revised generalized Kolmogorov-Sinai-q entropy under Markov shifts. The form of this generalized entropy with factor $q$ is nonextensive. The new generalized entropy contains the classical Kolmogorov-Sinai entropy and Renyí entropy as well as Bernoulli shifts as special cases. Applying the generalized entropy we discuss its approximate behavior qualitatively, the entropy rate and the sensitive value $q^*$ of the nonextensive parameter $q$, which may exit in the interval (-2,2) nearby where the generalized entropy return to the classical K-S entropy.

Key words: generalized Kolmogorov-Sinai entropy, Markov shift, ergodic theory
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1 Introduction

Both entropy theories of Boltzmann-Gibbs (BG) and Shannon are two of the most important foundations of statistical physics and modern information theory respectively. the Kolmogorov-Sinai (KS) entropy[12] in the dynamic
systems make a great progress, which establishes a milestone to modern dynamic systems and affects the development of mathematical physics since then. Many concepts characterized complex multifractal systems such as dimension, dimension spectrum[3], entropy, generalized entropy[15] (such as correlation entropy, particularly, Renyi entropy[6]), Lyapunov exponent, topological pressure[7], etc., have gained very important applications in physics. More recently, a generalized entropy to Boltzmann entropy in physical systems is proposed by Tsallis[8], which has the nonextensive property and hence may be a good description for some complex systems with multifractal properties. The Tsallis entropy has been applied to various physical phenomena which may deviate from equilibrium state for instance[9,10,11,12,13,14] to particularly edge phenomena of chaotic threshold in one-dimensional dissipative systems[15,16,17]. It is worthy to concern the progress of works in two branches to physics at present. On the one hand, the nonextensive forms of the Tsallis entropy in some important branches of mathematics have been developed. A generalized entropy in the Tsallis sense for the dynamic systems has been presented by Mesón and Vericat first time[18]. The generalization does what Kolmogorov did for Shannon entropy under the Bernoulli scheme, and obtains a mathematical parallel which is proved as an isomorphism invariant[18,19].

On the another hand, the vague relation between the BG entropy of physics and the KS entropy of dynamics has been clarified gradually by works of many authors[20] as research of Tsallis entropy. The KS entropy can be interpreted with a rate of physical entropy production per unit time. Thus the position of KS entropy of dynamics in the application of statistical physics becomes more important.

The Mesón-Vericat’s definition of KSq entropy is nice to consider the property of power distribution which is a common character of objects described with Tsallis entropy. Meanwhile that Tsallis entropy can smoothly recover to BG entropy is a well-known advantage. However the Mesón-Vericat’s definition of KSq entropy seems to be lack of the advantage, i.e. it can not recover KS entropy when $q = 1$. To this aim we would revise Mesón-Vericat’s definition and make it more complete. Secondly, the Bernoulli shift is a basic scheme of great importance in measure-preserving dynamic systems. When systems to be studied are extended and enlarged, it is necessary to exploit rather complex shifts beyond Bernoulli ones. So the another aim of this paper attempt to extend the work of Mesón and Vericat from the Bernoulli scheme to the Markov scheme, and present a generalized mathematical form of KS entropy in the sense of Tsallis under Markov shifts. The generalized entropy should be consistent in mathematics with both cases of the classical KS entropy and the Kolmogorov-Sinai-like (KSq) entropy under Bernoulli shifts and our work may be a approach to reach this goal. Finally, interrelation of both Renyi and Tsallis entropy is interested and focused in statistical physics community now. It is one important fact in statistical physics that obtaining same distribution under maximal entropy principle are equivalent because they are monotonic.
each other. We will see also that there is similar situation in the level of abstract dynamics.

2 The Revised K-S-q entropy

Now we consider the generalized Markov shift in the measure-preserving dynamic system. Let \( k \geq 2 \) be a fixed integer which is cardinal of the symbolic set \( Y = \{0, 1, \ldots, k-1\} \). Let a direct product space \( (X, \mathcal{B}) = \Pi^\infty_{n=1} \{0, 1, \ldots, k-1\} \) be a space of doubly infinite, where \( \mathcal{B} \) is the subset family equipped the product \( \sigma \)-algebra on the space \( X \). The shift transformation \( T : X \to X \) is \( T\{x_n\} = \{x_{n+1}\} \) For each natural number \( n \) the cylinder set

\[
\{(x_i)_{n=1}^\infty | x_j = i_j, \text{ for } |j| \leq n \} \quad (i_j \in Y)
\]

forms the basic set family with the product \( \sigma \)-algebra. We can introduce a unique probability measure \( m \) on \( (X, \mathcal{B}) \) as

\[
m(\{x_i \in X | x_i = i_0, x_{i+1} = i_1, \ldots, x_{i+n} = i_{n-1}\}) = p_n(i_0, i_1, \ldots, i_{n-1}),
\]

where probability \( p_n(i_0, i_1, \ldots, i_n) \) is a positive real number. According to Kolmogorov consistency theorem \[21\] the probability measure \( m \) in \[2\] can be extended to the entire space \( (X, \mathcal{B}) \). The Markov shift is a quadruple \( (X, \mathcal{B}, m, T) \) and one of the important cases of measure-preserving dynamic systems. If we may endow some explanations to it, \( X \) is regarded as a sample space made up of the whole results of individual random experiments. Then the product \( \sigma \)-algebra \( \mathcal{B} \) on \( X \) and the probability measure \( m \) associated with the cylinder set are the standard frame for the random experiments. For one individual experiment the sample leads to the probability of the cylinder \( p_n(i_0, i_1, \ldots, i_n) \). Different from the Bernoulli scheme, if we suppose that the outcomes of every experiment are not independent, the transition probability \( P_{ij} \geq 0 \) between both states \( (i, j) \) depends only on the states \( (i, j) \).

Then we have an irreducible non-negative \( k \times k \) stochastic matrix \( P = \{P_{ij}\} \) \((P_{ij} \geq 0, i, j \in Y)\) with its row normalizing condition \( \Sigma_{j=0}^{k-1} P_{ij} = 1 \) for each \( i, j \) and there is a unique positive left vector \( p = (p_0, \ldots, p_{k-1}) \) which satisfy the invariance probability

\[
p_j = \Sigma_{i=0}^{k-1} p_i P_{ij}.
\]

We will assign the transition with Markov property to the probability of the cylinder as

\[
p_n(i_0, i_1, \ldots, i_{n-1}) = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}},
\]

with the initial probability vector i.e. \( p_0(i_0) = (p_0, p_1, \ldots, p_{k-1}) \) we obtain a two-sided \((p, P)\) Markov shift. The measure preserving is guaranteed by two normalizing conditions: \( \Sigma_{j=0}^{k-1} P_{ij} = 1 \) and \( \Sigma_{i_0 \in Y} p_0(i_0) = 1 \). If the transition
probability is independent of the initial state \(i\), i.e., taking \(P_{i,j} = p_j\), then the probability of cylinder (4) becomes

\[
p_n(i_0, i_1, \ldots, i_{n-1}) = p_{i_0}p_{i_1}p_{i_2} \ldots p_{i_{n-1}}.
\]

The p-Bernoulli shift is a special case of the \((p, P)\) Markov shift.

According to Shannon’s information theory[22], if we already know a point \(x \in X\) belongs to some fixed set of partition \(\mathcal{A} = \{A_0, A_1, \ldots, A_{m-1}\}\), the quantity of information is

\[
H_1(\mathcal{A}) = -\sum_{i=0}^{m-1} \mu(A_i) \log \mu(A_i),
\]

where partition \(\mathcal{A}\) should satisfies the normalizing condition \(\sum_{i=0}^{m-1} \mu(A_i) = 1\). This entropy possess additivity. Tsallis presented a generalization entropy[8] of the physics entropy as

\[
H_q(\mathcal{A}) = (q - 1)^{-1}\left(1 - \sum_{i=0}^{m-1} [\mu(A_i)]^q\right),
\]

where \(q\) is an arbitrary real number, and the entropy has pseudo-additivity

\[
H_q(A + B) = H_q(A) + H_q(B) + (1 - q)H_q(A)H_q(B)
\]

when two independent subsystems \(A\) and \(B\) form one system. The Tsallis entropy recovers the Shannon entropy when \(q = 1\). We now define a generalized entropy parallel to Ref. [18] for the finite partition partition \(\mathcal{A} = \{A_0, A_1, \ldots, A_{m-1}\}\) of the sample space. Based on Meson & Vericat’s definition[18], a new revised definition of generalized entropy( KSq entropy) of the measure-preserving transformation \(T\) is

\[
h_{KSq}(T) = (q - 1)^{-1}[1 - \exp[(1 - q)\tilde{h}_{KSq}(T)]]],
\]

\[
\tilde{h}_{KSq}(T) = \sup_{\mathcal{A} \subseteq \mathcal{B}, \mathcal{A}\text{ finite}} h_{KSq}(\mathcal{A}, T),
\]

\[
h_{KSq}(\mathcal{A}, T) = \lim_{n \to \infty} \frac{1}{n} \{(1 - q)^{-1}\log[1 + (1 - q)H_q(n^{-1}\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A})]\}.
\]

where the supremum in (10) is taken over the set \(\mathcal{B}\) formed with all finite sub-\(\sigma\)-algebras \(\mathcal{A}\). It is enough to physics, if not, just taking over all countable partitions like current monographs[23]. After getting rid of an infinite point partition entropy we can still work on the finite entropy, because the finite entropy rate is a necessary condition to statistical mechanics. Note that the partition entropy \(H_q\) in (11) takes the form of Tsallis entropy. The revised definition improves the nonextensive property of the parameter \(q\). Since \(q\) is any real number, the case \(q = 1\) will be imposed continuity. By using L’Hospital
rule the definitions of nonextensive entropy recover smoothly to that of classical K-S extensive entropy

\[ h_{KS}(T) = h_{KS1}(T) = \tilde{h}_{KS1}(T), \]  
(12)

\[ \tilde{h}_{KS1}(T) = \sup_{A \subseteq B, A \text{ finite}} h_{KS1}(A, T), \]  
(13)

\[ h_{KS1}(A, T) = \lim_{n \to \infty} \frac{1}{n} H_1(\bigvee_{i=-n}^{n} T^i A). \]  
(14)

Thus the classical definition of KS entropy is contained within the nonextensive parameter \( q \), so the revision enhance a little completeness to the definition of Mesón and Vericat[18]. We will see that some conclusions of classical KS entropy should also be contained in the generalized KSq entropy below.

### 3 the K-S-q entropy of Markov Shifts

We now come to seek the form of a generalized Markov shift. There exist many generalized Markov shifts on the basis of the classical Markov shift, we want to seek such a generalized Markov shift which can generalize KS entropy according to the form of Tsallis entropy. We assume that there is a family of pseudo-stochastic matrices with the real parameter \( q \): \( \{P_{ij}^q\}_{i,j} \), \( P_{ij} \geq 0 \) and still satisfies the condition \( \sum_{j=0}^{k-1} P_{ij} = 1 \). One recognizes in the analysis of multifractal that the Tsallis entropy is analogy to \( q \)-dimension in its mathematical form[24,25]. It is necessary to introduce a new \( L_q \)-probability \( r_j \):

\[
r_j = \left( \sum_{i=0}^{k-1} p_i P_{ij}^q \right)^{1/q}, \]
(15)

which will be used as the partition probability of the generalized Markov shift. Of course, the generalized probability can play the role of \( p_j \) in the classical Markov shift. when \( q \) approaches to 1, it has \( r_j \to p_j \) and recovers to the classical Markov shift. Particularly, if \( P_{ij} = p_j \), then \( r_j^q \) is just the partition probability of the generalized Bernoulli shift defined by Mesón and Vericat[18]. So the partition probability contains the generalized Bernoulli shift as a special case. It will lead to the KSq entropy of the Markov shift we wanted.

We now calculate the KSq entropy of generalized Markov shift. With the definition of product \( \sigma \)-algebra there exits a canonical generator \( G = \bigvee_{i=-\infty}^{\infty} T^i A \). For the generalized entropy the calculation is analogy to Bernoulli scheme [18]. Considering a typical element of the refinement \( \xi(\bigvee_{i=0}^{n-1} T^{-i} A) \)

\[
A_{i_0} \cap T^{-1} A_{i_1} \cap \cdots \cap T^{-n+1} A_{i_{n-1}} = \{ \{x_n\} : x_0 = i_0, x_1 = i_1, \ldots, x_{n-1} = i_{n-1} \}, \quad 0 \leq i_j \leq k - 1
\]
and using the generalized Markov shift and the $L_q$-probability $r_j$ in the previous section, we will assign the new probability measure to the partition $A$, i.e. $\mu(A_j) = C_j$, we can take the $L_q$-probability partition in (11)

$$\mu(A_{j_0} \cap T^{-1}A_{j_1} \cap \cdots \cap T^{-(n-1)}A_{j_{n-1}})$$

$$= r(x_0 = j_0)r(x_1 = j_1) \cdots r(x_{n-1} = j_{n-1})$$

$$= r_{j_0}r_{j_1} \cdots r_{j_{n-1}}, \quad (0 \leq j_0, j_1, \ldots, j_{n-1} \leq k - 1).$$

So the generalized refinement entropy of probability space can be derived by the probability $r_j$

$$H_{KSq}^{n-1} (\bigvee_{i=0}^{n-1} T^{-i}A) = \frac{1}{1 - q} \left[ \left( \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} p_i P^q_{ij} \right)^n - 1 \right]. \quad (16)$$

Put (16) into (9), (10) and (11), then we have the generalized KSq entropy of transformation $T$ under Markov shifts:

$$h_{KSq}(T) = \frac{1 - \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} p_i P^q_{ij}}{q - 1} \quad (17)$$

$$= - \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} p_i P^q_{ij} \ln_q P_{ij}. \quad (18)$$

where

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q} \quad (q \neq 1). \quad (19)$$

We see that (18) returns to the original form of KS entropy when $q \to 1$

$$h_{KS1}(T) = - \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} p_i P_{ij} \log P_{ij}, \quad (20)$$

and since the Bernoulli shift is a special case of the Markov shift, when $P_{ij} = P_j$, (18) becomes

$$h_q(T) = \frac{1 - \sum_{i=0}^{k-1} p_i^q}{q - 1}. \quad (21)$$

This expression recovers the generalized entropy of the Bernoulli shift obtained by Ref. [18]. Thus (18) contains classical KS entropy and generalized Bernoulli KSq entropy. Furthermore, it is noted that for the generalized Markov scheme both the entropy of $T$ and entropy of $T$ with respect to $A$ do not equal.

$$h_{KSq}(T) \neq h_{KSq}(A, T). \quad (22)$$
Using the simple calculation with definition (11) for \((q \neq 1)\) we have

\[
h_{KSq}(A, T) = (1 - q)^{-1} \log \left[ \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} p_i P_{ij}^q \right]
\]

This is just the well-known Renyi entropy in the Markov scheme. The entropy of transformation \(T\) with respect to partition \(A\) is one part of the definition of KSq entropy, i.e. The KSq entropy contains Renyi entropy as a component in context of dynamic systems. When \(h_{KSq}(A, T)\) is maximized by supremum through (10) and (9), it leads to KSq entropy. This implies an unification of both Tsallis and Renyi entropy in KSq form, naturally, it includes a mergence of two parameters \(q\) and \(\beta\) which should not be confused in the viewpoint of current issue, because there is a great difference in the additivity of both. Particularly, we have

\[
\lim_{q \to 1} h_{KSq}(T) = \lim_{q \to 1} h_{KSq}(A, T),
\]

it shows that the both are generalizationes of the KS entropy along the line of abstract dynamics although Renyi and Tsallis entropy are different in the sense of Kolmogorov-Nagumo average. For the Bernoulli scheme, as it is a special case of the Markov scheme, \(h_{KSq}(T)\) does also not necessarily agree with the entropy \(h_{KSq}(A, T)\) by (22), even \(h_{KSq}(T) = H_q(p)\) for the Bernoulli scheme of arbitrary \(q\). According to the revised definition (11) we still have \(h_{KSq}(A, T) = (1 - q)^{-1} \log \left[ \sum_{j=0}^{k-1} p_j^q \right]\) to be Renyi entropy in Bernoulli scheme, so the recover (23) must also be available.

Isomorphism equivalent class of the KSq entropy in Ornstein’s sense is a complex problem for Markov shifts. We will not discuss it since it is not a complete invariant for Kolmogorov automorphism. However, isomorphism of the KSq entropy for Bernoulli shifts will be able to discuss as same as Mesón and Vericat.

4 Application

The asymptotic behavior of KSq entropy is interesting, for it ties closely with the physical meaning of the non-extensive parameter \(q\). We discuss it with Beck’s frame of the statistics of dynamic symbol sequences. For a given map \(T\) the Shannon entropy \(H\)

\[
H(\mu, A, n) = - \sum_{i_0, i_1, \ldots, i_{n-1}=0}^{k-1} \mu(i_0, i_1, \ldots, i_{n-1}) \ln \mu(i_0, i_1, \ldots, i_{n-1}),
\]

where \(\mu(i_0, i_1, \ldots, i_{n-1}) = \mu(A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-n+1}A_{i_{n-1}})\). The study of the asymptotic behavior of the dynamic iterate sequences involves the entropy
rate per unit time

\[ h(\mu, A) = \lim_{n \to \infty} \frac{H}{n}. \]

The quantity \( h(\mu, A) \) depends still on the partition \( A \). Maximizing it over all possible partition set \( \{A\} \) in the phase space

\[ h(\mu) = \sup_{\{A\}} h(\mu, \{A\}). \] (24)

The K-S entropy of the physical version is obtained, which is a global quantity to describe chaos solely. When we focus on the numerical calculation, the supremum can be replaced by the maximization if the generating partition exists[28]. In order to clarify the vague connection between the physical entropy and the KS entropy, Latora and Baranger[20] calculate the entropy rate[15]

\[ \kappa_1 = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} \frac{S_1(t)}{t}, \] (25)

where the iterate number \( t = n \), \( W \) is the cell number of partition of the phase space worked. Initially putting \( N(N \gg W) \) points into one cell randomly, then the occupancies \( \{N_i(t)\} \) of all cells can calculate probabilities \( p_i(t) = \frac{N_i(t)}{\sum_{i=0}^{W} N_i(t)} \) numerically. It is found by the numerical experiment that if the \( S(t) \) is given by

\[ S_1(t) = -\sum_{i=0}^{W} p_i(t) \ln p_i(t), \]

then there exists a linear value of the entropy rate which is just the KS entropy[20]. To describe the edge-of-chaos, the Boltzmann-Gibbs-Shannon entropy above is not be appropriate. The non-extensive entropy must be applied[29].

\[ \kappa_q = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} \frac{S_q(t)}{t}. \] (26)

For the Markov scheme here we have the non-extensive entropy to be

\[ S_q(t) = \frac{1 - \sum_{i=0}^{W} r_i^q(t)}{q - 1}. \] (27)

The asymptotic behavior of the nonextensive entropy (27) is monotonic decrease, as the parameter \( q \) approaches to infinity i.e. \( \lim_{q \to -\infty} h_{KSq}(T) \to \infty \) and \( \lim_{q \to \infty} h_{KSq}(T) \to 0 \) (even \( \lim_{q \to \infty} \ln_q x \to \infty \)). It is allowed that there may exist a special sensitive value \( q^* \neq 1 \) between both entropic values (\( \infty, 0 \)). The time evolution of the non-extensive entropy will sharply turn when \( q \) approaches \( q^* \), which is responsible to the power-law with complicated case in nonextensive statistics such as edge-of-chaos and the chaos nearly by eventual periodic orbits. The latter is another power-law example to be discussed. It is noted that the continuous at \( q \to 1 \) guarantees the nonextensive entropy is smooth and applicable. Since it has recognized that the KS entropy is a
production rate of Boltzmann entropy per unit time in physical systems, it
leads to that this production rate will decay as $q$ increases. So the case $q < 1$
will have more high production rate, it enhances the randomness of system,
and the case $q > 1$ decreases the randomness in the Markov shift. Thus the
parameter $q$ may be a factor which can control and influence random, irregu-
lar chaotic behavior. For the Bernoulli case the conclusion is the same as the
Markov one in qualitative. Finally, in order to estimate $q^*$ we discuss another
special value $q = 0$. For the the entropy of $T$ with respect to $\{A\}$, the Renyi
entropy is just the topological entropy $h_{KS0}(A, T) \sim \log k$, which is smaller
than the K-S-q entropy $h_{KSq}(T) \sim (k - 1)$, i.e., $h_{KSq}(A, T) \leq h_{KSq}(T)$ will
always hold. We may reasonably estimate that the sensitive value $q^*$ should
locate in a symmetric interval with the center at $q = 0$ and the two symmetric
terminal points are $(-2, 2)$, i.e., $-2 < q^* < 2$. The prediction is right for the
edge-of-chaos [29]. To confirm the property of the generalized KSq entropy,
we have applied it to eventual periodic orbits and their edge-of-chaos with
symbolic dynamics of the unimodal transformation [30] and numerical results
show us that the KSq entropy indeed has good monotonic property with $q$
and sharply turn at the interval nearby where it recovers to KS entropy at the
limit $q \to 1$.

Markov shifts are available for use of physics. Applications of the theory of
Markov shifts widely spread in various practical fields, for instance, continuous
Markov processes by the time shift operator and discrete Markov chains. Many
practical applications are discussed in details by many books, for instance in
the book of Bharucha-Reid [31], such as the growth of populations, mutation
progress, spread of epidemics, theory of gene frequencies in biology; theory of
cascading progress in physics; theory of cosmic rays in astronomy; chemical
reaction kinetics in chemistry; theory of queues, and so on. Particularly, it is
interesting in the symbolic dynamics associated with edge of chaotic phenom-
ena in one-dimensional dissipative systems where the topological Markov shift
on discrete chain plays an very important role.

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References

[1] Y. G. Sinai, Dokl. Akad. Nauk. SSSR 124, 768 (1959).
[2] Y. G. Sinai, (Russian)Mat. Sb. (N.S.) 63(105), 23 (1964).
[3] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A 33, 1141 (1986).
[4] P. Grassberger and I. Procaccia, Phys. Rev. A 28, 2591 (1983).
[5] P. Grassberger and I. Procaccia, Physica D 13, 34 (1984).
[6] A. Renyi, Probability Theory, North Holland, Amsterdam, 1970).
[7] T. Bohr and D. Rand, Physica D 25, 387 (1987).
[8] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[9] W. C. Saslaw, Gravitational physics of stellar and galactic systems, Cambridge University Press, Cambridge, (1985).
[10] H. Risken, The Fokker-Planck equation:methods of solution and applications, Springer-Verlag, Berlin, (1984).
[11] M. O. Cyaceres, Braz. J. Phys. 29, 1065 (1999).
[12] E. W. Montroll and M. F. Shlesinger, J. Stat. Phys. 32, 209 (1983).
[13] D. W. Sciama, Phys. Rev. Lett. 18, 1065 (1967).
[14] N. A. Bahcall, Astrophys. J. 462, L49 (1996).
[15] E. P. Borges, C. Tsallis, G. F. J. Ananos, and P. M. C. de Oliveira, Phys. Rev. Lett. 89, 254103 (2002).
[16] F. Baldovin and A. Robledo, Phys. Rev. E 69, 045202 (2004).
[17] G. F. J. Ananos and C. Tsallis, Phys. Rev. Lett. 93, 020601 (2004).
[18] A. M. Mesón and F. Vericat, J. Math. Phys. 37, 4480 (1996).
[19] A. M. Mesón and F. Vericat, J. Math. Phys. 43, 904 (2002).
[20] V. Latora and M. Baranger, Phys. Rev. Lett. 82, 250 (1999).
[21] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, (2001).
[22] C. E. Shannon, Bell System Technol. J. 27, 379 (1948).
[23] Y. G. Sinai, Topics in Ergodic Theory, Lecture 6 Princeton University Press; N.J., (1994).
[24] D. H. E. Gross, Straight way to Thermo-Statistics, Phase Transitions, Second law of thermodynamics, but without thermodynamic limit, (2001).
[25] E. W. Weisstein, *Concise Encyclopedia of mathematics*, CRC Press, (1999).

[26] A. Dukkipati, M. N. Murty and S. Bhatnagar, Nongeneralizability of Tsallis entropy by means of Kolmogorov-Nagumo average under pseudo-additivity, arXiv:math-ph/0505078v1, (2005)

[27] C. Beck and F. Schlogl, *Thermodynamics of Chaotic Systems: An Introduction*, Cambridge University Press, Cambridge, (1995).

[28] P. Cornfeld, S. V. Fomin, and Y. G. Sinai, *Ergodic Theory*, Springer, New York, (1982).

[29] V. Latora, M. Baranger, A. Rapisarda, and C. Tsallis, Phys.Lett. A 273, 97 (2000).

[30] Q. Liu, S. Peng, and S. L. Peng, The generalized Kolmogorov-Sinai entropy under the Markov shift in symbolic dynamics (Preprint).

[31] A. T. Bharucha-Reid, *Elements of the Theory of Markov Processes and Their Applications* Dover Publications; REPRINT edition, (1997).