ALGEBRAIC CYCLES AND LEHN–LEHN–SORGER–VAN STRATEN EIGHTFOLDS

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Abstract This article is about Lehn–Lehn–Sorger–van Straten eightfolds $Z$ and their anti-symplectic involution $ι$. When $Z$ is birational to the Hilbert scheme of points on a K3 surface, we give an explicit formula for the action of $ι$ on the Chow group of 0-cycles of $Z$. The formula is in agreement with the Bloch–Beilinson conjectures and has some non-trivial consequences for the Chow ring of the quotient.

Keywords: Algebraic cycles; chow group; motive; Bloch–Beilinson filtration; Beauville’s ‘splitting property’ conjecture; multiplicative Chow–Künneth decomposition; hyperkähler varieties; Hilbert schemes of K3 surfaces; Lehn–Lehn–Sorger–van Straten eightfolds; non-symplectic automorphism

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1. Introduction

Given a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups of $X$ (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$-coefficients, modulo rational equivalence).

In this article, we consider the Chow ring of Lehn–Lehn–Sorger–van Straten (=LLSS) eightfolds. These eightfolds (defined in terms of twisted cubic curves contained in a given cubic fourfold $Y$) form a 20-dimensional locally complete family of hyperkähler varieties, deformation equivalent to the Hilbert scheme of points on a K3 surface [2, 36, 38]. One notable feature of the construction is that any LLSS eightfold admits an anti-symplectic involution $ι$ [37].

Our main result gives an explicit formula for the action of $ι$ on the Chow group of 0-cycles for certain LLSS eightfolds:

**Theorem (Theorem 3.11).** Let $Z = Z(Y)$ be an LLSS eightfold, and let $ι \in \text{Aut}(Z)$ be the anti-symplectic involution. Assume $Z$ is birational to a Hilbert scheme...
\( Z' = S^{[4]} \) where \( S \) is a K3 surface. Then the action of \( \iota \) on \( A^8(Z') \) is given by

\[
\iota_\ast [x, y, z, t] = [x, y, z, t] - 2([x, y, z, o] + [x, y, t, o] + [x, z, t, o] + [y, z, t, o]) \\
+ 4([x, y, o, o] + [x, z, o, o] + \cdots + [z, t, o, o]) \\
- 8([x, o, o, o] + [y, o, o, o] + [z, o, o, o] + [t, o, o, o]) \\
+ 16[o, o, o, o].
\]

(Here, \( x, y, z, t \in S \) and \( o \in A^2(S) \) denotes the distinguished 0-cycle of \([7]\).)

This theorem applies to a countably infinite number of divisors in the moduli space of LLSS eightfolds (the condition on \( Z \) is equivalent to asking that the cubic fourfold \( Y \) has discriminant of the form \((6n^2 - 6n + 2)/a^2\) for some \( n, a \in \mathbb{Z} \) \([39, \text{Proposition 5.3}]\). In particular, Theorem 3.11 applies to Pfaffian cubics).

The statement of Theorem 3.11 is similar to a statement for double EPW sextics proven in \([33, \text{Theorem 2}]\) and answers a question raised there \([33, \text{Introduction}]\).

The proof of Theorem 3.11 consists of a two-step argument. First, we prove that for any LLSS eightfold \( Z \) (not necessarily birational to a Hilbert scheme), there is equality

\[
\iota_\ast = -\text{id} : S^8_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \to A^8(Z),
\]

(1)

where \( S^8_{\text{mob}} \) is related to Voisin’s orbit filtration on zero-cycles \([59]\), cf. Definition 2.9. (As explained in Remark 2.10, one expects that the piece \( S^8_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \) is related, via the Bloch–Beilinson conjectures, to \( H^{2,0}(Z) \). Since \( \iota \) acts as \(-\text{id}\) on \( H^{2,0}(Z) \), equality (1) is in keeping with this expectation.) Next, we restrict attention to LLSS eightfolds birational to a Hilbert scheme \( Z' \). In this case, Vial \([54]\) has constructed a multiplicative Chow–Künneth decomposition, giving a bigrading on the Chow ring \( A^*_\iota(Z') \) such that the piece \( S^8_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z') \) is exactly \( A^8_{(2)}(Z') \). Exploiting the good formal properties of this bigrading, one can deduce Theorem 3.11 from equality (1).

Theorem 3.11 admits a reformulation that does not mention LLSS eightfolds:

**Theorem (\(=\)Theorem 3.12).** Let \( X = S^{[4]} \) be a Hilbert scheme, where \( S \) is a K3 surface of Picard number 1 and degree \( d \) such that \( d > 2 \) and \( d = (6n^2 - 6n + 2)/a^2 \) for some \( n, a \in \mathbb{Z} \). Then \( \text{Bir}(X) = \mathbb{Z}/2\mathbb{Z} \) and the non-trivial element \( \iota \in \text{Bir}(X) \) acts on \( A^8(X) \) as in Theorem 3.11.

As another consequence of Theorem 3.11, we can show that the Chow ring of the quotient of the involution behaves somewhat like the Chow ring of Calabi–Yau varieties:

**Corollary (\(=\)Corollary 3.13).** Let \( Z \) be an LLSS eightfold birational to a Hilbert scheme \( Z' = S^{[4]} \) where \( S \) is a K3 surface. Let \( X := Z/\langle \iota \rangle \) be the quotient under the anti-symplectic involution \( \iota \in \text{Aut}(Z) \). Then

\[
\text{Im}\left(A^2(X)^{\otimes 4} \to A^8(X)\right) = \text{Im}\left(A^3(X) \otimes A^2(X) \otimes A^2(X) \otimes A^1(X)\right) = \mathbb{Q}[h^8],
\]

where \( h \in A^1(X) \) is an ample divisor.
Corollary 3.13 is expected to be true for all LLSS eightfolds, and not merely those birational to a Hilbert scheme. Since for a general LLSS eightfold, we do not know how to construct a multiplicative Chow–Künneth decomposition, this expectation seems difficult to prove.

Conventions. In this article, the word variety will refer to a reduced irreducible scheme of finite type over \( \mathbb{C} \). A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by \( A_j(X) \) the Chow group of \( j \)-dimensional cycles on \( X \) with \( \mathbb{Q} \)-coefficients; for \( X \) smooth of dimension \( n \), the notations \( A_j(X) \) and \( A^{n-j}(X) \) are used interchangeably.

The notations \( A^j_{\text{hom}}(X) \), \( A^j_{\text{AJ}}(X) \) will be used to indicate the subgroups of homologically trivial, respectively, Abel–Jacobi trivial cycles. For a morphism \( f : X \to Y \), we will write \( \Gamma_f \in A_*(X \times Y) \) for the graph of \( f \). The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [42, 48]) will be denoted \( M_{\text{rat}} \).

We will write \( H^j(X) \) to indicate singular cohomology \( H^j(X, \mathbb{Q}) \).

Given an automorphism \( \sigma \in \text{Aut}(X) \), we will write \( A^j(X)^\sigma \) (and \( H^j(X)^\sigma \)) for the subgroup of \( A^j(X) \) (respectively \( H^j(X) \)) invariant under \( \sigma \).

2. Preliminaries

2.1. Quotient varieties

Definition 2.1. A projective quotient variety is a variety

\[
X = Z/G,
\]

where \( Z \) is a smooth projective variety and \( G \subset \text{Aut}(Z) \) is a finite group.

Proposition 2.2 (Fulton [22]). Let \( X \) be a projective quotient variety of dimension \( n \). Let \( A^i(X) \) denote the operational Chow cohomology ring. The natural map

\[
A^i(X) \to A_{n-i}(X)
\]

is an isomorphism for all \( i \).

Proof. This is [22, Example 17.4.10].

Remark 2.3. It follows from Proposition 2.2 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in [22, Example 16.1.13]). We may thus consider motives \( (X, p, 0) \in M_{\text{rat}} \), where \( X \) is a projective quotient variety and \( p \in A^n(X \times X) \) is a projector. For a projective quotient variety \( X = Z/G \), one readily proves (using Manin’s identity principle) that there is an isomorphism

\[
h(X) \cong h(Z)^G := (Z, \Delta_G, 0) \quad \text{in} \ M_{\text{rat}},
\]

where \( \Delta_G \) denotes the idempotent \((1/|G|)\sum_{g \in G} \Gamma_g\).
2.2. LLSS eightfolds

In this subsection, we briefly recall the construction of the Lehn–Lehn–Sorger–van Straten eightfold (see [38] for additional details). Let \( Y \subset \mathbb{P}^5 \) be a smooth cubic fourfold which does not contain a plane. Twisted cubic curves on \( Y \) belong to an irreducible component \( M_3(Y) \) of the Hilbert scheme \( \text{Hill}_{3}^{3m+1}(Y) \); in particular, \( M_3(Y) \) is a ten-dimensional smooth projective variety and it is referred to as the Hilbert scheme of generalized twisted cubics on \( Y \). There exists a hyperkähler eightfold \( Z = Z(Y) \) and a morphism \( u : M_3(Y) \to Z \) which factorizes as \( u = \Phi \circ a \), where \( a : M_3(Y) \to Z' \) is a \( \mathbb{P}^2 \)-bundle and \( \Phi : Z' \to Z \) is the blow-up of the image of a Lagrangian embedding \( j : Y \hookrightarrow Z \). By [2] (or alternatively [36], or [27, Section 5.4]), the manifold \( Z \) is of \( K3^{[4]} \)-type.

For a point \( j(y) \in j(Y) \subset Z \), a curve \( C \in u^{-1}(j(y)) \) is a singular cubic (with an embedded point in \( y \)), cut out on \( Y \) by a plane tangent to \( Y \) in \( y \). In particular, \( u^{-1}(j(Y)) \subset M_3(Y) \) is the locus of non-Cohen–Macaulay curves on \( Y \). If instead we consider an element \( C \in M_3(Y) \) such that \( u(C) \notin j(Y) \), let \( C := \langle C \rangle \cong \mathbb{P}^3 \) be the convex hull of the curve \( C \) in \( \mathbb{P}^5 \). Then, the fibre \( u^{-1}(u(C)) \) is one of the 72 distinct linear systems of aCM twisted cubics on the smooth integral cubic surface \( S_C := \Gamma_C \cap Y \) (cf. [11]). This two-dimensional family of curves, whose general element is smooth, is determined by the choice of a linear determinantal representation \([A]\) of the surface \( S_C \) (see [17, Chapter 9]). This means that \( A \) is a \( 3 \times 3 \)-matrix with entries in \( H^0(O_{\Gamma_C}(1)) \) and such that \( \det(A) = 0 \) is an equation for \( S_C \) in \( \Gamma_C \). The orbit \([A]\) is taken with respect to the action of \((\text{GL}_3(\mathbb{C}) \times \text{GL}_3(\mathbb{C}))/\Delta \), where \( \Delta = \{(tI_3, tI_3) \mid t \in \mathbb{C}^*\} \). Moreover, any curve \( C' \in u^{-1}(u(C)) \) is such that \( I_{C'/S_C} \) is generated by the three minors of a \( 3 \times 2 \)-matrix \( A_0 \) of rank two, whose columns are in the \( \mathbb{C} \)-linear span of the columns of \( A \).

2.3. Voisin’s rational map and the involution \( \iota \)

**Proposition 2.4 (Voisin [59])**. Let \( Y \subset \mathbb{P}^5 \) be a smooth cubic fourfold not containing a plane. Let \( F = F(Y) \) be the Fano variety of lines and let \( Z = Z(Y) \) be the LLSS eightfold of \( Y \). There exists a degree 6 dominant rational map

\[
\psi : F \times F \dashrightarrow Z.
\]

Moreover, let \( \omega_F, \omega_Z \) denote the symplectic forms of \( F \), respectively \( Z \). Then we have

\[
\psi^*(\omega_Z) = (p_1)^*(\omega_F) - (p_2)^*(\omega_F) \quad \text{in} \quad H^{2,0}(F \times F)
\]

(where \( p_j \) denotes projection on the \( j \)th factor).

**Proof.** This is [59, Proposition 4.8]. Let us briefly recall the geometric construction of the rational map \( \psi \). Let \((l, l') \in F \times F \) be a generic point, so that the two lines \( l, l' \) on \( Y \) span a three-dimensional linear space in \( \mathbb{P}^5 \). For any point \( x \in l \), the plane \( \langle x, l' \rangle \) intersects the smooth cubic surface \( S = \langle l, l' \rangle \cap Y \) along the union of \( l' \) and a residual conic \( Q_{x}' \), which passes through \( x \). Then, \( \psi(l, l') \in Z \) is the point corresponding to the two-dimensional linear system of twisted cubics on \( S \) linearly equivalent to the rational curve \( l \cup_x Q_{x}' \) (this linear system actually contains the \( \mathbb{P}^1 \) of curves \( \{l \cup_x Q_{x}' \mid x \in l\} \)).

**Remark 2.5.** The indeterminacy locus of the rational map \( \psi \) is the codimension 2 locus \( I \subset F \times F \) of intersecting lines (see [40, Theorem 1.2]). As proven in [15, Theorem...
1.1], the indeterminacy of the rational map \( \psi \) is resolved by a blow-up with centre \( I \subset F \times F \).

**Remark 2.6.** As is known from unpublished work of Lehn, Lehn, Sorger and van Straten, the LLSS eightfold \( Z \) is equipped with a biholomorphic anti-symplectic involution \( \iota \in \text{Aut}(Z) \) (cf. [35, 37]) which is induced, via the rational map \( \psi : F \times F \dashrightarrow Z \), by the involution of \( F \times F \) exchanging the two factors. Indeed, by [28, Remark 3.8], the involution \( \iota \) acts on the subset \( Z \setminus j(Y) \) as follows. A point \( p \in Z \setminus j(Y) \) corresponds to a linear determinantal representation \( [A] \) of a smooth cubic surface \( S = Y \cap \mathbb{P}^3 \), or equivalently to a six of lines \( \langle e_1, \ldots, e_6 \rangle \) on \( S \) (see [17, Section 9.1.2, 11]). Then, \( \iota(p) \) is the point of \( Z \setminus j(Y) \) associated with the (unique) six \( \langle e_1', \ldots, e_6' \rangle \) on the same surface \( S \) which forms a double-six together with \( \langle e_1, \ldots, e_6 \rangle \). In terms of linear determinantal representations, \( \iota(p) \) corresponds to the transposed representation \( [A'] \) of the surface \( S \) by [11, Proposition 3.2]. It can be checked that, if \( C \) is a twisted cubic curve on \( S \) in the linear system parametrized by the point \( p \), then, for any quadric \( Q \subset \langle C \rangle \) containing \( C \), the residual intersection \( C' \) of \( S \) with \( Q \) belongs to the fibre \( u^{-1}(\iota(p)) \) (see for instance [16, Footnote 47]). By the geometric description of the rational map \( \psi \), we conclude that \( \iota(\psi(l, l')) = \psi(l', l) \) : in fact, for \( x \in l \) and \( x' \in l' \), the reducible quadric \( Q = \langle x, l' \rangle \cup \langle x', l \rangle \) intersects the surface \( S \) along the union of \( l \cup_{x} Q_{x} \) and \( l' \cup_{x'} Q_{x'} \).

### 2.4. Voisin’s orbit filtration

**Definition 2.7 (Voisin [59]).** Let \( X \) be a variety of dimension \( n \). One defines \( S_{j}X \subset X \) as the set of points whose orbit under rational equivalence has dimension \( \geq j \). The descending filtration \( S_{*} \) on \( A^{n}(X) \) is defined by letting \( S_{j}A^{n}(X) \) be the subgroup generated by points \( x \in S_{j}X \).

**Remark 2.8.** Let \( X \) be a hyperkähler variety. As explained in [59], the expectation is that the filtration \( S_{*} \) is opposite to the (conjectural) Bloch–Beilinson filtration, and thus provides a splitting of the Bloch–Beilinson filtration on \( A^{n}(X) \) (this is motivated by Beauville’s famous speculations in [4]). This splitting should be ‘natural’, in the sense that it can be defined by correspondences and that it induces a bigrading on the Chow ring of \( X \).

**Definition 2.9.** Let \( Z \) be an LLSS eightfold, and let \( D \subset Z \) be the uniruled divisor obtained by resolving the indeterminacy locus of the rational map \( \psi \) of Proposition 2.4. One defines \( S_{j}^{\text{mob}}A^{8}(Z) \) to be the subgroup generated by points in \( S_{j}(Z \setminus D) \).

**Remark 2.10.** There is an inclusion

\[
S_{j}^{\text{mob}}A^{8}(Z) \subset S_{j}A^{8}(Z),
\]

which possibly is an equality (this is true when \( Z \) is birational to a Hilbert scheme of points on a K3 surface, cf. the proof of Lemma 3.10). As explained in [59], the piece \( S_{3}A^{8}(Z) \cap A^{8}_{\text{hom}}(Z) \) is supposed to be isomorphic to the graded \( \text{Gr}_{2}^{*}A^{8}(Z) \), where \( F_{BB}^{*} \) is the (conjectural) Bloch–Beilinson filtration. On the other hand, the graded \( \text{Gr}_{2}^{*}A^{8}(Z) \) should be defined by the projector \( \pi_{Z}^{14} \) of some Chow–Küneth decomposition for \( Z \). In
particular, the expectation is that $S_3^{\mathrm{mob}} A^8(Z)$ is somehow related to $(H^{14}(Z)$ and hence $H^{2,0}(Z)$.

2.5. MCK decomposition

**Definition 2.11 (Murre [41]).** Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_0 + \pi_1 + \cdots + \pi_{2n} \quad \text{in} \ A^n(X \times X),$$

such that the $\pi_i$ are mutually orthogonal idempotents and $(\pi_i)_* H^*(X) = H^i(X)$.

(NB: ‘CK decomposition’ is shorthand for ‘Chow–Künneth decomposition’.)

**Remark 2.12.** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [25, 41].

**Definition 2.13 (Shen–Vial [49]).** Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \{(x, x, x) | x \in X\} \subset X \times X \times X.$$

An *MCK decomposition* is a CK decomposition $\{\pi_i^X\}$ of $X$ that is *multiplicative*, i.e. it satisfies

$$\pi_k^X \circ \Delta_X^{sm} \circ (\pi_i^X \times \pi_j^X) = 0 \quad \text{in} \ A^{2n}(X \times X \times X) \quad \text{for all} \ i + j \neq k.$$

(NB: ‘MCK decomposition’ is shorthand for ‘multiplicative Chow–Künneth decomposition’.)

A *weak MCK decomposition* is a CK decomposition $\{\pi_i^X\}$ of $X$ that satisfies

$$\left(\pi_k^X \circ \Delta_X^{sm} \circ (\pi_i^X \times \pi_j^X)\right)_*(a \times b) = 0 \quad \text{for all} \ a, b \in A^*(X).$$

**Remark 2.14.** The small diagonal (seen as a correspondence from $X \times X$ to $X$) induces the *multiplication morphism*

$$\Delta_X^{sm} : h(X) \otimes h(X) \to h(X) \quad \text{in} \ M_{\text{rat}}.$$

Suppose $X$ has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in} \ M_{\text{rat}}.$$

By definition, this decomposition is multiplicative if for any $i, j$ the composition

$$h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in} \ M_{\text{rat}}$$

factors through $h^{i+j}(X)$. 
If \( X \) has a weak MCK decomposition, then setting
\[
A^i_{(j)}(X) := \left( \pi^{X}_{2i-j}, \right)_* A^i(X),
\]
one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends \( A^i_{(j)}(X) \otimes A^{i'}_{(j')} (X) \) to \( A^{i+i'}_{(j+j')}(X) \).

It is expected that for any \( X \) with a weak MCK decomposition, one has
\[
A^i_{(j)}(X) \overset{?}{=} 0 \quad \text{for } j < 0, \quad A^i_{(0)}(X) \cap A^i_{\text{hom}}(X) \overset{?}{=} 0;
\]
this is related to Murre’s conjectures B and D that have been formulated for any CK decomposition [41].

The property of having an MCK decomposition is severely restrictive and is closely related to Beauville’s ‘(weak) splitting property’ [4]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [49, Section 8], as well as [21, 32, 50, 54].

In what follows, we will make use of the following:

**Theorem 2.15 (Vial [54])**. Let \( X \) be a Hilbert scheme \( S^{[m]} \), where \( S \) is a K3 surface. Then \( X \) has an MCK decomposition. Let \( A^*_s(X) \) denote the resulting bigrading. One has
\[
\bigoplus_{j=0}^{r} A^{2m}_{(2j)}(X) = S^{m-r} A^{2m}(X),
\]
where \( S_s \) is Voisin’s orbit filtration (Definition 2.7).

The pieces \( A^{2m}_s(X) \) can be described as follows: \( A^{2m}_{(0)}(X) \) is generated by \([o, o, \ldots, o]\) (where \( o \in A^2(S) \) is the distinguished 0-cycle of [7]). \( A^{2m}_{(2)}(X) \) is generated by expressions
\[
[x, o, \ldots, o] - [y, o, \ldots, o]
\]
(where \( x, y \) range over all points of \( S \)). \( A^{2m}_{(4)}(X) \) is generated by expressions
\[
[x, y, o, \ldots, o] - [x, o, \ldots, o] - [y, o, \ldots, o] + [o, o, \ldots, o].
\]
\( A^{2m}_{(6)}(X) \) is generated by expressions
\[
[x, y, z, o, \ldots, o] - [x, y, o, \ldots, o] - [x, z, o, \ldots, o] - [y, z, o, \ldots, o] + [x, o, \ldots, o] + [y, o, \ldots, o] + [z, o, \ldots, o] - [o, o, \ldots, o].
\](And so on...)

**Proof.** The MCK decomposition is constructed in [54, Theorem 1] (an alternative construction is given in [43]). The compatibility with Voisin’s filtration \( S_s \) is stated in [59, End of Section 4.1]. The formulae for the generators follow from [59, Theorem 2.5], where it is proven that the filtration \( S_s \) coincides with the filtration \( N_s \) defined in [59, Proposition 2.2].

(The \( m = 2 \) case also follows from [49].) \( \square \)
2.6. Multiplicative behaviour of the Chow ring of $S^{[m]}$

Let $X$ be a Hilbert scheme of points on a K3 surface. The following proposition shows that the Chow group of 0-cycles on $X$ is completely determined by the piece $A^2(2)(X)$. (This can be interpreted as a motivic manifestation of the fact that $H^*(X, \mathcal{O}_X)$ is determined by $H^2(X, \mathcal{O}_X)$, since $A^2(2)(X)$ is defined by the projector $\pi^2_X$).

**Proposition 2.16.** Let $X$ be a Hilbert scheme $X = S^{[m]}$ where $S$ is a K3 surface, and let $A^*(X)$ refer to the bigrading of Theorem 2.15.

(i) For any $j \geq 1$, intersection product induces surjections

$$A^2(2)(X)^{\otimes j} \to A^2_{(2j)}(X).$$

(ii) Let $h \in A^1(X)$ be a big divisor. For any $j \geq 0$ there are isomorphisms

$$h^{2m-j} : A^j_{(j)}(X) \rightarrow A^{2m}_{(j)}(X).$$

**Proof.** (i) Let us just treat the case $j = 2$ (the other cases are only notationally more complicated). Let us write $A^*(S^m)_{\mathfrak{S}_m}$ for the $\mathfrak{S}_m$-invariant part, where the symmetric group on $m$ elements $\mathfrak{S}_m$ acts on $S^m$ by permutation of the factors. We need to show that the map induced by intersection product

$$\left( A^2_{(2)}(S^m)_{\mathfrak{S}_m} \right)^{\otimes 2} \rightarrow A^4_{(4)}(S^m)_{\mathfrak{S}_m}$$

is surjective. By definition, an element $a$ on the right-hand side is a symmetric cycle of the form

$$(\pi_S^2 \times \pi_S^2 \times \pi_S^0 \times \cdots \times \pi_S^0 + \text{permutations})_*(d)$$

(where $d \in A^4(S^m)$). That is,

$$a = \sum_{\sigma \in \mathfrak{S}_m} \sigma_*(a_1 \times a_2 \times S \times \cdots \times S) \text{ in } A^4(S^m),$$

where $a_1, a_2 \in A^2(S)$ are degree zero 0-cycles.

Let us define

$$b_1 := \sum_{\sigma \in \mathfrak{S}_m} \sigma_*(a_1 \times S \times S \times \cdots \times S) \in A^2_{(2)}(S^m)_{\mathfrak{S}_m},$$

$$b_2 := \sum_{\sigma \in \mathfrak{S}_m} \sigma_*(S \times a_2 \times S \times \cdots \times S) \in A^2_{(2)}(S^m)_{\mathfrak{S}_m}.$$

We have that

$$b_1 \cdot b_2 = \sum_{\sigma \in \mathfrak{S}_m} \sum_{\sigma' \in \mathfrak{S}_m} (\sigma \circ \sigma')_*(a_1 \times a_2 \times S \times S \times \cdots \times S)$$

is a non-zero multiple of $a$, and so we have proven surjectivity.
(ii) Again, one reduces to the Cartesian product $S^m$. That is, one considers
\[
\begin{array}{ccc}
X & \xrightarrow{f} & S^m_p \\
\downarrow & & \downarrow \quad p \\
S^m & \xrightarrow{p} & S^{(m)} = S^m/\mathfrak{S}_m
\end{array}
\]
where $f$ is the Hilbert–Chow morphism and $p$ is the quotient morphism.

There is a commutative diagram
\[
\begin{array}{ccc}
A^j_{(j)}(X) & \xrightarrow{h^m_{2m-j}} & A^{2m}_{(j)}(X) \\
\downarrow f^* & & \downarrow f^* \\
A^j(S^{(m)}) & \xrightarrow{f^* (h)^{2m-j}} & A^{2m}(S^{(m)}) \\
\downarrow p^* & & \downarrow p^* \\
A^j_{(j)}(S^m)\mathfrak{S}_m & \xrightarrow{p^* f^*(h)^{2m-j}} & A^{2m}_{(j)}(S^m)\mathfrak{S}_m
\end{array}
\]
in which composition of two vertical arrows on the left respectively on the right is an isomorphism (NB: the commutativity of the upper square follows from the projection formula).

Thus, it will suffice to prove that for a big $\mathfrak{S}_m$-invariant divisor $h$ on $S^m$, there are isomorphisms
\[
h^m_{2m-j} : A^j_{(j)}(S^m)\mathfrak{S}_m \xrightarrow{\cong} A^{2m}_{(j)}(S^m)\mathfrak{S}_m.
\]
The case $j = 2$ is proven in [31, Theorem 3.1]; the general case is only notationally more complicated. □

**Remark 2.17.** The $m = 2$ case of Proposition 2.16 is already proven in [49].

2.7. Birational hyperkähler varieties

**Theorem 2.18 (Rieß [47]).** Let $\phi : X \dashrightarrow X'$ be a birational map between two hyperKähler varieties of dimension $n$.

(i) There exists a correspondence $R_\phi \in A^n(X \times X')$ such that

$$(R_\phi)_* : A^*(X) \to A^*(X')$$

is a graded ring isomorphism.

(ii) For any $j$, there is equality

$$(R_\phi)_* c_j(X) = c_j(X') \quad \text{in } A^j(X').$$

(iii) There is equality

$$(\Gamma_{R})_* = (\bar{\Gamma}_\phi)_* : A^j(X) \to A^j(X') \quad \text{for } j = 0, 1, n - 1, n$$

(where $\bar{\Gamma}_\phi$ denotes the closure of the graph of $\phi$).
Proof. Item (i) is [47, Theorem 3.2], while item (ii) is [47, Lemma 4.4].

In a nutshell, the construction of the correspondence $R_\phi$ is as follows: [47, Section 2] provides a diagram

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow \Psi \\
\mathcal{X}'
\end{array}
\begin{array}{c}
\leftarrow C
\end{array}
$$

where $\mathcal{X}, \mathcal{X}'$ are algebraic spaces over a quasi-projective curve $C$ such that the fibres $\mathcal{X}_0, \mathcal{X}'_0$ are isomorphic to $X$ respectively $X'$, and where $\Psi: \mathcal{X} \dashrightarrow \mathcal{X}'$ is a birational map inducing an isomorphism $\mathcal{X}_{C \setminus \mathcal{I}} = \mathcal{X}'_{C \setminus \mathcal{I}}$ and whose restriction $\Psi|_\mathcal{X}_0$ coincides with $\phi$. The correspondence $R_\phi$ is then defined as the specialization (in the sense of [22], extended to algebraic spaces) of the graphs of the isomorphisms $\mathcal{X}_c \cong \mathcal{X}'_c$, $c \neq 0$.

Point (iii) is not stated explicitly in [47], and can be seen as follows. Let $U \subset X$ be the locus on which $\phi$ induces an isomorphism. The complement $T := X \setminus U$ has codimension $\geq 2$. Using complete intersections of hypersurfaces, one can find a closed subset $T \subset \mathcal{X}$ of codimension 2 such that $T_0$ contains $T$, i.e. $\Psi$ restricts to an isomorphism on $V := \mathcal{X}_0 \setminus T_0$. Since specialization commutes with pullback, the restriction of $R_\phi$ to $V \times X'$ is the specialization of the graphs of the morphisms $\mathcal{X}_c \setminus T_c \rightarrow \mathcal{X}'_c$, $c \neq 0$ to $V \times X'$, which is exactly the graph of the morphism $\phi|_V$, i.e.

$$
R_\phi|_{V \times X'} = \Gamma_{\phi|_V} = \bar{\Gamma}_\phi|_{V \times X'} \text{ in } A^n(V \times X').
$$

It follows that one has

$$
R_\phi = \bar{\Gamma}_\phi + \gamma \text{ in } A^n(X \times X'),
$$

where $\gamma$ is some cycle supported on $T_0 \times X'$. This proves point (iii): 0-cycles and 1-cycles on $X$ can be moved to be disjoint of $T_0$, and so $\gamma_j$ acts as zero on $A^j(X)$ for $j \geq n-1$. Likewise, $\gamma_j^*$ acts as zero on $A^j(X')$ for $j \leq 1$ for dimension reasons, and so (by inverting the roles of $X$ and $X'$) statement (iii) is proven.

Remark 2.19. As observed in [54, Introduction], the correspondence $R_\phi$ of Theorem 2.18 actually induces an isomorphism of Chow motives as $\mathbb{Q}$-algebra objects. This implies that the property ‘having a MCK decomposition’ is birationally invariant among hyperkähler varieties.

Remark 2.20. Let $Z$ be a hyperkähler variety birational to a Hilbert scheme $Z' = S^{[m]}$. It follows from Theorem 2.18 that Proposition 2.16 is valid for $Z$.

2.8. The Franchetta property

An important role in this paper is played by the Franchetta property. In a nutshell, a family of varieties has the Franchetta property if algebraic cycles that exist for all members of the family are well behaved. This is relevant here, because the graph of the anti-symplectic involution of an LLSS eightfold is such a cycle: it exists for all members of the locally complete family of LLSS eightfolds.
Definition 2.21. Let $\mathcal{X} \to B$ be some family of smooth projective varieties $X_b, b \in B$. We say that $\mathcal{X} \to B$ has the Franchetta property if for any $a \in A^i(\mathcal{X})$ and for any $b \in B$, there is equivalence

$$a|_{X_b} = 0 \quad \text{in} \quad H^{2i}(X_b) \iff a|_{X_b} = 0 \quad \text{in} \quad A^i(X_b).$$

This property is studied in [44, 46] for the universal family of $K3$ surfaces of low genus. This is extended to certain families of hyperkähler varieties in [18, 20].

Proposition 2.22. Let $\mathcal{Y} \to B$ denote the universal family of smooth cubic fourfolds, where $B$ is a Zariski open in $\mathbb{P}^{55}$. The Franchetta property holds for $A^4(\mathcal{Y} \times_B \mathcal{Y}).$

Proof. Let $\Gamma \in A^4(\mathcal{Y} \times_B \mathcal{Y})$ be a correspondence that is homologically trivial when restricted to the very general fibre. According to [58, Proposition 1.6] (cf. also [34, Proposition 5.1], where the precise form of Voisin’s argument applied here is detailed), there exists a cycle $\gamma \in A^4(\mathbb{P}^5 \times \mathbb{P}^5)$ such that

$$\Gamma|_{b} = \gamma|_{b} \quad \text{in} \quad A^4(Y_b \times Y_b) \quad \forall \quad b \in B.$$

The cycle $\gamma$ is of the form $\sum_{j=0}^{4} c_j h^j \times h^{4-j}$, where $h \in A^1(\mathbb{P}^5)$ is a hyperplane section and $c_j \in \mathbb{Q}$. The assumptions imply that $\gamma|_{b}$ is homologically trivial for $b \in B$ very general. Since the summands $(h^j \times h^{4-j})|_b$ live in different summands of the Künneth decomposition of $H^8(Y_b \times Y_b)$, each $c_j(h^j \times h^{4-j})|_b$ must be homologically trivial, which is only possible when $c_j = 0$, and so $\gamma = 0$. □

Remark 2.23. Using the Franchetta property for $F \times_B F$ (where $F$ is the universal Fano variety of lines in a cubic fourfold) [18], it is possible to prove that $\mathcal{Y} \times_B \mathcal{Y} \to B$ (and also $\mathcal{Y}^{(2)} \times_B \mathcal{Y}^{(2)} \to B$) has the Franchetta property in all codimensions. We will not need this generality here.

2.9. The transcendental motive of a cubic fourfold

Proposition 2.24 (Bolognesi and Pedrini [10]). Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. There exists a Chow motive $t(Y) \in \mathcal{M}_{rat}$, with the property that

$$A^*(t(Y)) = A^3_{hom}(Y),$$

$$H^*(t(Y)) = H^4_{tr}(Y)$$

(where $H^4_{tr}(Y)$ denotes the orthogonal complement of the algebraic classes with respect to cup-product).

Proof. The construction, which is a variant of the construction of the transcendental motive of a surface [26], is done in [10, Section 2]. Alternatively, one can use the projectors on the algebraic part of cohomology constructed for any variety satisfying the standard conjectures in [53]. □
3. Main results

For any LLSS eightfold $Z$, we construct (Theorem 3.1) a Chow motive $t(Z)$ which is responsible for one piece of Voisin’s orbit filtration on the Chow group of zero-cycles, i.e. the motive $t(Z)$ determines $S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z)$. Using this construction, we can show (Theorem 3.7) that the anti-symplectic involution $\iota$ acts as $-\text{id}$ on this piece $S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z)$. Next, we restrict to LLSS eightfolds birational to Hilbert schemes, where we have an MCK decomposition. Using the good properties of the MCK decomposition, we deduce (Theorems 3.9 and 3.11) that the involution $\iota$ acts on zero-cycles in the expected way.

3.1. An isomorphism of motives

Theorem 3.1. Let $Z = Z(Y)$ be an LLSS eightfold, where $Y \subset \mathbb{P}^5$ is any smooth cubic not containing a plane. There exists a motive $t(Z) \in \mathcal{M}_{\text{rat}}$ with the property that

$$A^*(t(Z)) = S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z).$$

There is an isomorphism of motives

$$t(Z) \cong t(Y)(-5) \quad \text{in } \mathcal{M}_{\text{rat}},$$

inducing, in particular, an isomorphism of $\mathbb{Q}$-vector spaces

$$S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \cong A^3_{\text{hom}}(Y).$$

Proof. Let us write $F := F(Y)$ for the Fano variety of lines, and $P \in A^3(F \times Y)$ for the universal line. We define a correspondence

$$Q := \hat{\Gamma}_{\psi} \circ (t \Gamma_{p_1} - t \Gamma_{p_2}) \circ t \pi \in A^3(Y \times Z),$$

where $\psi: F \times F \dashrightarrow Z$ is Voisin’s rational map, and $p_j: F \times F \to F$ is projection to the $j$th factor. The correspondence $Q$ has the following property: \hfill \Box

Claim 3.2. Let $h \in A^1(Z)$ be the polarization. There exists a constant $c_h \in \mathbb{Q}^*$ such that

$$c_h t^* Q \circ \Gamma_{h^6} \circ Q \circ \pi^t_Y = \pi_Y^t \quad \text{in } A^4(Y \times Y).$$

(Here $\Gamma_{h^6} := \Gamma_{\tau} \circ t \Gamma_{\tau} \in A^{14}(Z \times Z)$, where $\tau: S \to Z$ is the inclusion of a smooth complete intersection surface $S \subset Z$ cut out by $h$.)

In particular, the composition

$$A^3_{\text{hom}}(Y) \xrightarrow{Q^*} A^2_{\text{hom}}(Z) \xrightarrow{h^6} A^8_{\text{hom}}(Z) \xrightarrow{(c_h Q)^*} A^3_{\text{hom}}(Y)$$

is the identity.
Admitting the claim, the theorem is readily proven: one defines the motive $t(Z)$ by the projector
\[ \pi_{Z}^{tr} := c_{h} \Gamma_{h_{6}} \circ Q \circ \pi_{Z}^{tr} \circ t Q \in A^{8}(Z \times Z). \]
The claim implies that $\pi_{Z}^{tr}$ is idempotent, and that
\[ t Q : t(Z) := (Z, \pi_{Z}^{tr}, 5) \to t(Y) := (Y, \pi_{Y}^{tr}, 0) \quad \text{in} \ M_{\text{rat}} \]
is an isomorphism, with inverse
\[ c_{h} \Gamma_{h_{6}} \circ Q \in A^{9}(Y \times Z). \]
The isomorphism of motives $t(Z) \cong t(Y)(-5)$ implies that $A^{*}(t(Z)) = A^{3}(t(Z))$. To
determine $A^{3}(t(Z))$, we use the relation with the Fano variety of lines $F := F(Y)$ via
Voisin’s rational map $\psi$. Indeed, by construction we have
\[ (\pi_{Z}^{tr})_{*}A^{8}(Z) = \text{Im} \left( Q_{*}A_{\text{hom}}^{3}(Y) \xrightarrow{h_{6}} A^{8}(Z) \right). \]
Now, there are equalities
\[
\begin{align*}
\text{Im} \left( Q_{*}A_{\text{hom}}^{3}(Y) \xrightarrow{h_{6}} A^{8}(Z) \right) & = \psi_{*} \text{Im} \left( (p_{1})^{*} - (p_{2})^{*} \right) P^{*}A_{\text{hom}}^{3}(Y) \xrightarrow{\psi^{*}(h_{6})} A^{8}(F \times F) \\
& = \psi_{*} \text{Im} \left( A_{(2)}^{2}(F) \otimes A_{0}(F) \oplus A_{0}(F) \otimes A_{(2)}^{2}(F) \right) \xrightarrow{\psi^{*}(h_{6})} A^{8}(F \times F) \\
& = \psi_{*} \left( A_{(2)}^{4}(F) \otimes A_{(0)}^{4}(F) \oplus A_{(0)}^{4}(F) \otimes A_{(2)}^{4}(F) \right) \\
& = \psi_{*} \left( S_{1}A^{4}(F) \otimes S_{2}A^{4}(F) \oplus S_{2}A^{4}(F) \otimes S_{1}A^{4}(F) \right) \cap A_{\text{hom}}^{8}(F \times F) \\
& = S_{3}^{\text{mob}}A^{8}(Z) \cap A_{\text{hom}}^{8}(Z).
\end{align*}
\]
Here, the first equality is by definition of $Q$. The second equality is the fact that
$P^{*}A_{\text{hom}}^{3}(Y) = A_{(2)}^{2}(F)$ (indeed, $A_{(2)}^{2}(F) = P^{*}P_{*}A_{\text{hom}}^{4}(F)$ [49, Proof of Proposition 21.10]),
and $P_{*}A_{\text{hom}}^{3}(F) = A_{\text{hom}}^{3}(Y)$, since $A^{3}(Y)$ is generated by lines [45]. For the third equality,
one observes that $H^{1}(F, \mathbb{Q}) = 0$ and so the big line bundle $\psi^{*}(h) \in A^{1}(F \times F)$ can be
written $\psi^{*}(h) = (p_{1})^{*}(\ell_{1}) + (p_{2})^{*}(\ell_{2})$ with $\ell_{j} \in A^{1}(F)$ big and universally defined. Since
the very general $F$ has Picard number 1, the $\ell_{j}$ must be non-zero multiples of the polar-
ization class of $F$. Hence, one has isomorphisms $A_{(2)}^{2}(F) \xrightarrow{\ell_{j}} A_{(2)}^{4}(F)$ [49, Proposition
22.2] and surjections
\[ A_{(0)}^{2}(F) \xrightarrow{\ell_{j}} A_{(0)}^{4}(F) \cong \mathbb{Q}[\ell_{j}] \]
(hard Lefschetz), which proves the third equality. The fourth equality is an application of
[59, Proposition 4.5]. For the last equality, one notes that $S_{1}A^{4}(F)$ is generated by points
lying on a uniruled divisor, and $S_{2}A^{4}(F)$ is generated by points on the constant cycle
surfaces constructed by Voisin [59, Proposition 4.4]; since one can find a uniruled divisor
$D$ and a constant cycle surface $S$ such that $D \times S$ is in general position with respect to the indeterminacy locus of $\psi$, this gives the last equality.

Altogether, this proves that

$$(\pi_Z^{\tr})_* A^8(Z) = S^\text{mob}_3 A^8(Z) \cap A^8_{\text{hom}}(Z).$$

It only remains to prove Claim 3.2, i.e. we need to prove the vanishing

$$(c_h \cdot Q \circ \Gamma_{h^6} \circ Q - \Delta_Y) \circ \pi_Y^{\tr} = 0 \quad \text{in} \quad A^4(Y \times Y).$$

Let $\pi_Y^{\prim} \in A^4(Y \times Y)$ be the projector defined as

$$\pi_Y^{\prim} := \Delta_Y - \frac{1}{3} \sum_j h^j \times h^{4-j} \in A^4(Y \times Y).$$

Since $H^{2,2}(Y, \mathbb{Q})$ is one-dimensional for a very general cubic $Y$, there is equality $\pi_Y^{\tr} = \pi_Y^{\prim}$ for very general $Y$. Since $(Y, \pi_Y^{\tr}, 0)$ is a submotive of $(Y, \pi_Y^{\prim}, 0)$, it suffices to prove the vanishing

$$(c_h \cdot Q \circ \Gamma_{h^6} \circ Q - \Delta_Y) \circ \pi_Y^{\prim} = 0 \quad \text{in} \quad A^4(Y \times Y),$$

for all smooth cubic fourfolds $Y$. But this cycle is universally defined (i.e., it is the restriction of a cycle in $A^4(Y \times_B Y)$, where $Y \to B$ is the universal family of cubic fourfolds as in Proposition 2.22). Hence, thanks to the Franchetta property (Proposition 2.22) one is reduced to showing the vanishing

$$(c_h \cdot Q \circ \Gamma_{h^6} \circ Q - \Delta_Y) \circ \pi_Y^{\prim} = 0 \quad \text{in} \quad H^8(Y \times Y),$$

for the very general cubic $Y$.

To prove the vanishing (3), we proceed as follows:

**Lemma 3.3.** Let $Y \subset \mathbb{P}^5$ be any smooth cubic. There exists a non-zero constant $c_h$ such that

$$(c_h \cdot Q \circ \Gamma_{h^6} \circ Q - \Delta_Y)_* = 0: H^4_{\tr}(Y) \to H^4_{\tr}(Y).$$

**Proof.** Since $H^4_{\tr}(Y)$ is the smallest Hodge structure for which the complexification contains $H^{3,1}(Y) \cong \mathbb{C}$, it suffices to prove there is a non-zero constant $c_h$ such that

$$(c_h \cdot Q \circ \Gamma_{h^6} \circ Q - \Delta_Y)_* = 0: H^{3,1}(Y) \to H^{3,1}(Y).$$

By definition of $Q$, it suffices to prove there is a non-zero constant $c_h$ making the following diagram commute:

$$
\begin{array}{cccccc}
H^{3,1}(Y) & \xrightarrow{P_*} & H^{2,0}(F) & \xrightarrow{p_1^* - p_2^*} & H^{2,0}(F \times F) & \xrightarrow{\psi^*} & H^{2,0}(Z) \\
\downarrow \frac{1}{c_h} \text{id} & & \downarrow \cdot (h_F)^2 & & \downarrow \cdot h^6 & & \\
H^{3,1}(Y) & \xrightarrow{P_*} & H^{4,2}(F) & \xrightarrow{(p_1)_* - (p_2)_*} & H^{8,6}(F \times F) & \xrightarrow{\psi^*} & H^{8,6}(Z)
\end{array}
$$

(Here $h_F \in A^1(F)$ denotes a polarization.)
The left square of this diagram is well known to be commutative (essentially this is the Abel–Jacobi isomorphism established in [6]). To see that the right-hand square of this diagram commutes, we observe that thanks to Proposition 2.4, we have a relation \( \tilde{\psi}^*(\omega_Z) = \tilde{p}_1^*(\omega_F) - \tilde{p}_2^*(\omega_F) \). This implies that the top horizontal line is an isomorphism: more precisely, the composition \( \psi_* (p_1^* - p_2^*) \) sends \( \omega_F \) to \( 6\omega_Z \in H^{2,0}(Z) \). Let \( \omega_Y^* \in H^{6,8}(Z) \) be the class dual to \( \omega_Z \) under cup product, and let \( \omega_Y^* \) be its complex conjugate. By hard Lefschetz, we know that \( \omega_Z : h^6 \in H^{8,6}(Z) \) is a rational multiple of the class \( \omega_Y^* \). By duality, the composition \( ((p_1)_* - (p_2)_*)^* \) sends \( \omega_Y^* \) to \( \frac{1}{6}\omega_Y^* \in H^{0,2}(F) \), and so \( \omega_Y^* \) is sent to \( \frac{1}{6}\omega_Y^* \in H^{0,2}(F) \). Using hard Lefschetz on \( F \), we can rescale \( h_F \) by some rational multiple to make the right-hand square commutative. This proves Lemma 3.3.

Lemma 3.3 implies that one can write

\[
\left( c_h t^* Q \circ \Gamma_{h^6} \circ Q - \Delta_Y \right) = \gamma \quad \text{in} \quad H^8(Y \times Y),
\]

where \( \gamma \) is a completely decomposed cycle, i.e. a cycle supported on a finite union of subvarieties \( V_i \times W_i \subset Y \times Y \), with \( \dim V_i + \dim W_i = 4 \). Since the composition of the completely decomposed cycle \( \gamma \) with \( \pi^{\text{prim}}_Y \) is zero (for any smooth cubic \( Y \)), this implies the necessary vanishing (3), and so the theorem is proven. \( \square \)

**Remark 3.4.** For cyclic cubic fourfolds (i.e. cubic fourfolds of the form \( f(x_0, \ldots, x_4) + x_5^3 = 0 \)), a statement similar to Theorem 3.1 was proven in [13, Proposition 3.7].

We also define the corresponding piece in the codimension 2 Chow group of \( Z \):

**Definition 3.5.** For any LLSS eightfold \( Z \), let

\[
A^2_{[2]}(Z) := Q_* A^3_{\text{hom}}(Y) \subset A^2_{\text{hom}}(Z)
\]

(where \( Q \) is as in (2)).

The piece \( A^2_{[2]}(Z) \) is likely to coincide with \( A^2_{\text{hom}}(Z) \), although this seems hard to prove (just as it is hard to prove that the inclusion \( A^2_{(2)}(F(Y)) \subset A^2_{\text{hom}}(F(Y)) \) is an equality, cf. [49]).

**Proposition 3.6.** Let \( Y \subset \mathbb{P}^5 \) be a smooth cubic not containing a plane, and let \( F := F(Y) \) and \( Z := Z(Y) \) be the Fano variety of lines, respectively the associated LLSS eightfold.

(i) The projector \( ^t\pi^\text{tr}_Z \in A^8(Z \times Z) \) has the property that

\[
( ^t\pi^\text{tr}_Z)_* A^*(Z) = A^2_{[2]}(Z).
\]

(ii) There is an isomorphism

\[
\cdot h^6 : A^2_{[2]}(Z) \xrightarrow{\cong} S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z)
\]

(In particular, there is a correspondence-induced isomorphism \( A^2_{[2]}(Z) \cong A^3_{\text{hom}}(Y) \)).
Proof. (i) Claim 3.2 guarantees that
\[ t^! \pi^tr_Z = c_h Q \circ \pi^tr_Y \circ t^! Q \circ \Gamma_{h^6} \in A^8(Z \times Z) \]
is idempotent, and that
\[ Q: t(Y) := (Y, \pi^tr_Y, 0) \to (Z, t^! \pi^tr_Z, -1) \text{ in } \mathcal{M}_{\text{rat}} \]
is an isomorphism, with inverse \( c_{h^6} t^! Q \circ \Gamma_{h^6} \). It follows that
\[ (t^! \pi^tr_Z)_* A_j(Z) = 0 \quad \forall j \neq 2, \]
and
\[ (t^! \pi^tr_Z)_* A^2(Z) = Q_* A^3_{\text{hom}}(Y) =: A^2_{[2]}(Z). \]
(ii) This is immediate from the proof of Theorem 3.1, where it is shown that
\[ A^3_{\text{hom}}(Y) \cong S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) = (\pi^tr_Z)_* A^8(Z) = \text{Im} \left( Q_* A^3_{\text{hom}}(Y) \xrightarrow{h^6} A^8(Z) \right). \]

3.2. The anti-symplectic involution and 0-cycles

Theorem 3.7. Let \( Z = Z(Y) \) be an LLSS eightfold, where \( Y \subset \mathbb{P}^5 \) is any smooth cubic not containing a plane. Let \( \iota \in \text{Aut}(Z) \) be the anti-symplectic involution; Then
\[ \iota_* = \text{id}: S_4 A^8(Z) \to A^8(Z), \]
\[ \iota_* = -\text{id}: S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \to A^8(Z). \]

Proof. Let \( F = F(Y) \) be the Fano variety of lines in \( Y \). Via Voisin’s rational map, one sees as in [59] that \( S_4 A^8(Z) \) is one-dimensional with generator \( \psi_*(o_F \times o_F) \), where \( o_F \in A^4_{(0)}(F) = S_2 A^4(F) \) is the distinguished zero-cycle of [49]. There is a commutative diagram
\[ A^8(F \times F) \xrightarrow{\psi_*} A^8(Z) \]
\[ \downarrow (\iota_F)_* \quad \quad \quad \downarrow \iota_* \]
\[ A^8(F \times F) \xrightarrow{\psi_*} A^8(Z), \]
where \( \iota_F \) is the map exchanging the two factors [13, Remark 2.19]. Since \( o_F \times o_F \) is invariant under \( \iota_F \), it follows that \( \iota \) acts as the identity on \( S_4 A^8(Z) \).

To prove the statement for \( S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \), let us first ensure that \( \iota_* \) preserves this subgroup:

Lemma 3.8. We have
\[ \iota_* \left( S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z) \right) \subset S^3_{\text{mob}} A^8(Z) \cap A^8_{\text{hom}}(Z). \]
It follows from the above considerations that the relative correspondence $R. Laterveer$

having a $j$-dimensional orbit. The inverse image $\psi^{-1}(x) \subset F \times F$ consists of points outside the indeterminacy locus $I$ of $\psi$. Let $i \in \text{Aut}(F \times F)$ be the map exchanging the two factors. Since $I$ is symmetric, $i_*(\psi^{-1}(x))$ is disjoint from $I$, and so $\iota(x)$, which is equal to the support of $\psi_*\iota_*(\psi^{-1}(x))$ (Remark 2.6) lies outside of $D$, i.e. $\iota_*(x) \in S^3_{\text{mob}} A^8(Z)$. □

Next, we use the fact that

$$S^3_{\text{mob}} \cap A^6_{\text{hom}}(Z) = (\Gamma_{h^6} \circ Q)_* A^3_{\text{hom}}(Y)$$

(where $Q$ is the correspondence of (2), and $\pi_Y^{tr}$ is the projector on $H^4_{tr}(Y)$ as in Proposition 2.24). The involution $\iota$, being anti-symplectic, acts as $-\text{id}$ on $H^2_{tr}(Z) = (Q \circ \pi_Y^{tr})_* H^4(Y)$, and so

$$c_h t Q \circ \Gamma_{6^c} \circ \Gamma_{h^6} \circ Q \circ \pi_Y^{tr} + \pi_Y^{tr} = 0 \quad \text{in } H^8(Y \times Y)$$

(where $c_h \in \mathbb{Q}^*$ and $h$ are as in Claim 3.2).

Let us now consider things family-wise. As in Proposition 2.22, let $Y \to B$ and $Z \to B$ denote the universal family of smooth cubic fourfolds not containing a plane, respectively LLSS eightfolds. Let $Q \in A^3(Y \times_B Z)$ be the relative correspondence as in the proof of Claim 3.2, and let $\Gamma_{h^6} \in A^1(Z \times_B Z)$ be the relative correspondence induced by some ample element $h \in A^1(Z)$. Let $h_Y \in A^1(Y)$ be a relative hyperplane section. The correspondence

$$\pi_Y^{\text{prim}} := \Delta_Y - \frac{1}{3} \sum_{j=0}^4 (pr_1)^*(h_Y)^j \cdot (pr_2)^*(h_Y)^4-j \in A^4(Y \times_B Y)$$

is such that $\pi_Y^{\text{prim}}|_b = \pi_Y^{tr}|_b$ for $b \in B$ very general, and

$$(\pi_Y^{\text{prim}}|_b)_* = (\pi_Y^{tr}|_b)_* = \text{id} : A^3_{\text{hom}}(Y_b) \to A^3(Y_b) \quad \forall b \in B. \quad (4)$$

It follows from the above considerations that the relative correspondence

$$c_h t Q \circ \Gamma_{t} \circ \Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}} + \pi_Y^{\text{prim}} \in A^4(Y \times_B Y)$$

is homologically trivial when restricted to the very general fibre. But then, the Franchetta property (Proposition 2.22) ensures that this relative correspondence is fibrewise zero, i.e.

$$\left( c_h t Q \circ \Gamma_{t} \circ \Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}} \right)|_b = -\pi_Y^{\text{prim}}|_b \in A^4(Y_b \times Y_b) \quad \forall b \in B.$$

Composing on both sides with $Q|_b$ on the right and with $(c_h \Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}})|_b$ on the left, it follows that

$$\left( c_h^2 (\Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}} \circ t Q \circ \Gamma_{t} \circ \Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}} \circ t Q) \right)|_b$$

$$= -c_h \left( \Gamma_{h^6} \circ Q \circ \pi_Y^{\text{prim}} \circ t Q \right)|_b \in A^8(Z_b \times Z_b),$$
for all \( b \in B \). Applying this equality of correspondences to \( A^8_{\text{hom}}(Z_b) \), and using (4), we obtain the equality of actions

\[
(\pi^*_b \circ \Gamma_b \circ \pi^*_b)_* = -(\pi^*_b)_* : A^8_{\text{hom}}(Z_b) \to A^8(Z_b) \quad \forall b \in B,
\]

where \( \pi^*_b := c_h \Gamma_b \circ Q_b \circ \pi_Y^* Q_b \) is the projector on \( S_3^{\text{mob}} \cap A^8_{\text{hom}}(Z_b) \) as in Theorem 3.1. Combined with Lemma 3.8, this means that

\[
(t_b)_* = - \text{id} : S_3^{\text{mob}} A^8(Z_b) \cap A^8_{\text{hom}}(Z_b) \to A^8(Z_b) \quad \forall b \in B,
\]

and so the theorem is proven.

**Theorem 3.9.** Let \( Z \) be an LLSS eightfold birational to a Hilbert scheme \( Z' = S^{[4]} \) where \( S \) is a K3 surface. Let \( \iota' \in \text{Bir}(Z') \) be the map induced by \( \iota \in \text{Aut}(Z) \). Then

\[
(i')_* = \text{id} : A^8_{(j)}(Z') \to A^8(Z') \quad \text{for } j \in \{0, 4, 8\},
\]

\[
(i')_* = - \text{id} : A^8_{(j)}(Z') \to A^8(Z') \quad \text{for } j \in \{2, 6\}.
\]

(Here the bigrading refers to the MCK decomposition of [54].)

**Proof.** The Hilbert scheme \( Z' \) has an MCK decomposition [54], and the induced decomposition of \( A^8(Z') \) is compatible with Voisin’s orbit filtration \( S_* \), in the sense that

\[
\bigoplus_{j=0}^r A^8_{(2j)}(Z') = \text{hom}_{\text{S}_4} A^8(Z') \quad (5)
\]

[59, End of section 4.1].

Using Rieß’ work (Theorem 2.18), the MCK decomposition of \( Z' \) induces an MCK decomposition for \( Z \), and there exists a correspondence \( R_\phi \in A^8(Z' \times Z) \) inducing an isomorphism of bigraded rings \( A^8_{(s)}(Z') \cong A^8_{(s)}(Z) \). In order to prove Theorem 3.9, we want to relate the orbit filtration \( S_* \) on \( A^8(Z) \) with the ‘motivic’ decomposition \( A^8_{(s)}(Z) \). This is the content of the following lemma. \( \square \)

**Lemma 3.10.** Let \( Z \) be as in Theorem 3.9, and let \( A^8_{(s)}(Z) \) refer to the bigrading induced by the bigrading on \( A^8(Z) \). There is an inclusion

\[
A^8_{(2)}(Z) \subset S_3^{\text{mob}} \cap A^8_{\text{hom}}(Z).
\]

**Proof.** Let \( \phi : Z' \dashrightarrow Z \) denote the birational map. By virtue of Rieß’ work (Theorem 2.18), there is equality

\[
A^8_{(2)}(Z) = (\tilde{\Gamma}_\phi)_* A^8_{(2)}(Z').
\]

Thanks to equality (5), it follows there is an inclusion

\[
A^8_{(2)}(Z) \subset (\tilde{\Gamma}_\phi)_* S_3 A^8(Z').
\]

(6)

Let \( I \subset Z' \) denote the indeterminacy locus of the map \( \phi \), and let \( D' \subset Z' \) denote the strict transform of the uniruled divisor \( D \subset Z \) coming from the rational map \( \psi \). For any
point \( z \in Z' \), the maximal-dimensional components of the orbit under rational equivalence \( O_z \) are Zariski dense in \( Z' \) ([57, Lemma 3.5], cf. also [51, Proof of Theorem 0.5] where this precise statement is obtained). Hence, the orbit of any point in \( Z' \) has a maximal-dimensional irreducible component not contained in \( I \cup D' \), and hence

\[
(\overline{\Gamma}_\phi)_* S_3 A^8(Z') \subset S_3^\text{mob} A^8(Z). \tag{7}
\]

Combining (6) and (7), one obtains

\[
A^8_{(2)}(Z) \subset S_3^\text{mob} A^8(Z).
\]

Since clearly \( A^8_{(2)}(Z) \subset A^8_{\text{hom}}(Z) \), this proves the lemma.

\[\square\]

Theorem 3.7, combined with Lemma 3.10, implies that

\[
\iota_* = -\text{id}: A^8_{(2)}(Z) \to A^8(Z). \tag{8}
\]

There is a commutative diagram

\[
\begin{array}{ccc}
A^8(Z') & \xrightarrow{(R_\phi)_*} & A^8(Z) \\
\downarrow (\iota')_* & & \downarrow \iota_* \\
A^8(Z') & \xrightarrow{(R_\phi)_*} & A^8(Z)
\end{array} \tag{9}
\]

(To check commutativity of this diagram, we note that (thanks to the moving lemma) any 0-cycle \( b \in A^8(Z') \) can be represented by a cycle \( \beta \) supported on an open \( U' \) such that the birational map \( \phi: Z' \to Z \) restricts to an isomorphism \( \phi|_{U'}: U' \cong U \) and \( U \subset Z \) is \( \iota \)-stable. As \( (R_\phi)_* \) and \( (\overline{\Gamma}_\phi)_* \) coincide on 0-cycles (Theorem 2.18), the image \( (R_\phi)_*(b) \in A^8(Z) \) is represented by the cycle \( (\phi|_{U'})_*(\beta) \) with support on \( U \). This shows commutativity of the diagram.)

This commutative diagram, plus the fact that \( \iota \) acts as the identity on \( S_4 A^8(Z) = A^8_{(0)}(Z) \) (Theorem 3.7), proves the \( j = 0 \) case of the theorem. Moreover, the commutative diagram, plus equality (8), gives

\[
(\iota')_* = -\text{id}: A^8_{(2)}(Z') \to A^8(Z'), \tag{10}
\]

which is the \( j = 2 \) case of the theorem.

Taking a \( \iota \)-invariant ample divisor \( h \in A^1(Z) \) and invoking the isomorphism

\[
\cdot h^6: A^2_{(2)}(Z) \cong A^8_{(2)}(Z) \tag{11}
\]

(Remark 2.20), it follows from (8) that also

\[
\iota_* = -\text{id}: A^2_{(2)}(Z) \to A^2(Z). \tag{12}
\]

The intersection product induces surjections

\[
A^2_{(2)}(Z)^{\otimes j} \to A^2_{(2j)}(Z) \quad (j = 1, 2, 3, 4)
\]
(Remark 2.20). Since intersection product and $\iota^*$ commute, equality (12) thus implies that
\[ \iota^* = (-1)^j \text{id}: A_{(2j)}^8(Z) \to A_{(2j)}^8(Z) \quad (j = 1, 2, 3, 4). \]
Taking once again a $\iota$-invariant ample divisor $h \in A^4(Z)$, and invoking the isomorphism
\[ h^{8-2j}: A_{(2j)}^2(Z) \cong A_{(2j)}^8(Z) \]
(Remark 2.20), it follows that
\[ \iota^* = (-1)^j \text{id}: A_{(2j)}^8(Z) \to A_{(2j)}^8(Z) \quad (j = 1, 2, 3, 4). \]
Combining (13) with the commutative diagram (9), we find that
\[ (\iota')^*[x, y, z, t] = [x, y, z, t] - 2([x, y, z, o] + [x, y, t, o] + [x, z, t, o] + [y, z, t, o]) \]
\[ + 4([x, y, o, o] + [x, z, o, o] + \cdots + [z, t, o]) \]
\[ - 8([x, o, o, o] + [y, o, o, o] + [z, o, o, o] + [t, o, o, o]) \]
\[ + 16[o, o, o, o]. \]
(Here, $x, y, z, t \in S$ and $o \in A^2(S)$ denotes the distinguished 0-cycle.)

**Proof.** This is a translation of Theorem 3.9, using the generators for the various pieces $A_{(j)}^8(Z')$ given in Theorem 2.15. We know that $\iota'$ preserves these pieces (Theorem 3.9), and so one can check the formula of Theorem 3.11 piece by piece. This is a direct computation. For example, Theorem 2.15 says that $A_{(8)}^8(Z')$ is generated by expressions of the form
\[ [x, y, z, t] - ([x, y, z, o] + [x, y, t, o] + \cdots) + ([x, y, o, o] + [x, z, o, o] + \cdots) \]
\[ - ([x, o, o, o] + [y, o, o, o] + \cdots) + [o, o, o, o], \]
and the formula of Theorem 3.11 leaves expressions of this form invariant, which is in agreement with the fact that $\iota'$ is the identity on $A_{(8)}^8(Z')$ (Theorem 3.9). The verification of the action of $\iota'$ on $A_{(j)}^8(Z')$, $j < 8$ is similar (but easier).

3.3. **A reformulation**

We can reformulate Theorem 3.9 into a simpler statement, that does not mention MCK decompositions or bigradings on the Chow ring:

**Theorem 3.11.** Let $Z$ be an LLSS eightfold birational to a Hilbert scheme $Z' = S^{[4]}$ where $S$ is a K3 surface. Let $\iota' \in \text{Bir}(Z')$ be the map induced by $\iota \in \text{Aut}(Z)$. Then
\[ (\iota')^*[x, y, z, t] = [x, y, z, t] - 2([x, y, z, o] + [x, y, t, o] + [x, z, t, o] + [y, z, t, o]) \]
\[ + 4([x, y, o, o] + [x, z, o, o] + \cdots + [z, t, o]) \]
\[ - 8([x, o, o, o] + [y, o, o, o] + [z, o, o, o] + [t, o, o, o]) \]
\[ + 16[o, o, o, o]. \]
(Here, $x, y, z, t \in S$ and $o \in A^2(S)$ denotes the distinguished 0-cycle.)

*Proof.* This is a translation of Theorem 3.9, using the generators for the various pieces $A_{(j)}^8(Z')$ given in Theorem 2.15. We know that $\iota'$ preserves these pieces (Theorem 3.9), and so one can check the formula of Theorem 3.11 piece by piece. This is a direct computation. For example, Theorem 2.15 says that $A_{(8)}^8(Z')$ is generated by expressions of the form
\[ [x, y, z, t] - ([x, y, z, o] + [x, y, t, o] + \cdots) + ([x, y, o, o] + [x, z, o, o] + \cdots) \]
\[ - ([x, o, o, o] + [y, o, o, o] + \cdots) + [o, o, o, o], \]
and the formula of Theorem 3.11 leaves expressions of this form invariant, which is in agreement with the fact that $\iota'$ is the identity on $A_{(8)}^8(Z')$ (Theorem 3.9). The verification of the action of $\iota'$ on $A_{(j)}^8(Z')$, $j < 8$ is similar (but easier).
3.4. Another reformulation

Theorem 3.9 also admits a reformulation that does not mention LLSS eightfolds:

**Theorem 3.12.** Let $S$ be a K3 surface of Picard number 1 and degree $d$ satisfying $d > 2$ and $d = (6n^2 - 6n + 2)/a^2$ for some $n, a \in \mathbb{Z}$. Let $X = S^{[4]}$ be the Hilbert scheme. Then $\text{Bir}(X) = \mathbb{Z}/2\mathbb{Z}$ and the non-trivial element $\iota \in \text{Bir}(X)$ acts on $A^8(X)$ as in Theorem 3.11.

**Proof.** Given a K3 surface $S$ of Picard number 1 and degree $d > 2$, the Hilbert scheme $X = S^{[4]}$ has $\text{Bir}(X)$ a group of order 1 or 2 [16, Section 4.3.1]. On the other hand, the condition $d = (6n^2 - 6n + 2)/a^2$ for some $n, a \in \mathbb{Z}$ implies (and actually is equivalent to the fact) that $X$ is birational to an LLSS eightfold [39, Proposition 5.3], and so $\text{Bir}(X)$ is non-trivial. It follows that $X$ as in the theorem must have $\text{Bir}(X) = \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element $\iota \in \text{Bir}(X)$ acts on $A^8(X)$ as in Theorem 3.11.

\[ \square \]

3.5. A consequence

**Corollary 3.13.** Let $Z$ be an LLSS eightfold birational to a Hilbert scheme $Z' = S^{[4]}$ where $S$ is a K3 surface. Let $X := Z/\langle \iota \rangle$ be the quotient under the anti-symplectic involution $\iota \in \text{Aut}(Z)$. Then

\[ \text{Im} \left( A^2(X)^\otimes 4 \to A^8(X) \right) = \text{Im} \left( A^3(X) \otimes A^2(X) \otimes A^2(X) \otimes A^1(X) \right) = \mathbb{Q}[h^8], \]

where $h \in A^1(X)$ is an ample divisor.

**Proof.** Recall that $X$ is a quotient variety, and so the Chow groups with $\mathbb{Q}$-coefficients of $X$ have a ring structure (cf. §2.1).

As noted before, the birationality $Z \to Z'$ induces an isomorphism of bigraded rings $A^*_{(4)}(Z) \cong A^*_{(4)}(Z')$ (Theorem 2.18). We have seen (equality (12)) that $\iota$ acts as $-\text{id}$ on $A^2_{(2)}(Z)$, and so

\[ A^2_{(2)}(Z)^\iota = 0. \]

Similar reasoning also shows that

\[ A^2(Z)^\iota \subset A^2_{(0)}(Z). \]

Indeed, let $b \in A^2(Z)^\iota$, and let $h \in A^1(Z)$ be once again a $g$-invariant ample class. Then $b \cdot h^2 \in A^8(Z)^\iota$ and so (in view of Theorem 3.9) we know that $b \cdot h^2 \in A^8_{(0)}(Z)$. Writing the decomposition $b = b_0 + b_2$, where $b_j \in A^2_{(j)}(Z)$, and invoking the hard Lefschetz type isomorphism (11), we see that $b_2 = 0$.

The corollary is now readily proven. For instance, the image of the intersection map

\[ \text{Im} \left( A^3(Z)^\iota \otimes A^2(Z)^\iota \otimes A^2(Z)^\iota \otimes A^1(Z)^\iota \to A^8(Z) \right) \]

is contained in

\[ \text{Im} \left( \bigoplus_{j \leq 2} A^3_{(j)}(Z) \otimes A^2_{(0)}(Z) \otimes A^2_{(0)}(Z) \otimes A^1(Z) \to A^8(Z) \right) \subset \bigoplus_{j \leq 2} A^8_{(j)}(Z). \]
On the other hand, the image is obviously $\iota$-invariant, and so
\[
\text{Im} \left( A^3(Z)^4 \otimes A^2(Z)^4 \otimes A^2(Z)^4 \otimes A^1(Z)^4 \to A^8(Z) \right) \\
\subset A^8(Z)^4 \subset A^8_{(0)}(Z) \oplus A^8_{(4)}(Z) \oplus A^8_{(8)}(Z)
\]
(where the second inclusion is equality (13)). It follows that
\[
\text{Im} \left( A^3(Z)^4 \otimes A^2(Z)^4 \otimes A^2(Z)^4 \otimes A^1(Z)^4 \to A^8(Z) \right) \subset A^8_{(0)}(Z) \cong \mathbb{Q}[h^8].
\]

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