I review the recent progress in understanding the complete gauge invariant decomposition of the nucleon spin with particular emphasis on its twist structure.

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1. Introduction

In recent years there has been renewed interest in the proper decomposition of the nucleon spin into the quark and gluon degrees of freedom (see Ref. 1 and references therein). The problem dates back to the classic paper 2 by Jaffe and Manohar in 1990 written shortly after the EMC discovery of the ‘proton spin crisis’ 3 and has remained elusive since then. The recent surge of interest was triggered by a controversial paper by Chen et al. 4 Their original proposal seemed somewhat cryptic, and the connection to observables in high energy experiments was unclear. However, it can be reinterpreted and revamped into a well-defined framework of spin decomposition consistent with perturbative QCD in which one can speak of familiar technical language like ‘twist’. In this short note I summarize the present understanding of the problem from my perspective. The details can be found in Refs. 5, 6, 7, 8.

2. Twist–two decomposition a.k.a. Ji decomposition

Let me begin with the Ji sum rule 9

\[ J^q = \frac{1}{2} \int_{-1}^{1} dx \left( H_q(x) + E_q(x) \right), \quad J^g = \frac{1}{4} \int_{-1}^{1} dx \left( H_g(x) + E_g(x) \right), \quad (1) \]

which relates the quark/gluon contribution to the nucleon spin \((J^q + J^g = \frac{1}{2})\) to the quark/gluon generalized parton distribution (GPD). I call this twist–two decomposition because the relevant GPDs, \(H_{q,g}\) and \(E_{q,g}\), are twist–two. The decomposition is based on the (improved) energy momentum tensor of QCD, and as such, all the operators involved are local and gauge invariant. Their matrix elements (i.e., GPDs) are measurable experimentally from deeply–virtual Compton scattering (DVCS) and also by lattice QCD simulations. It is thus a perfectly well-defined decomposition of the nucleon spin based on a firm theoretical background.
However, this is not the end of the story. There are several important questions which come to mind.

- What happened to $\Delta G$, the gluon polarization? There has been so much effort, both experimentally and theoretically, to extract this quantity in the QCD spin physics community. But it doesn’t even exist in the above decomposition.
- Can one interpret the integrand in (1) as a sort of ‘angular momentum density’, with $x$ being the longitudinal momentum fraction of quarks and gluons?
- $J^q$ (but not $J^g$) can be further decomposed, gauge invariantly, into the helicity and the orbital angular momentum (OAM) parts: $J^q = \frac{1}{2} \Delta \Sigma + L^q$. However, this kinetic OAM $L^q$ does not satisfy the canonical commutation relation because it involves the covariant derivative $L^q \sim \vec{x} \times D^q$, and $[D^i, D^j] = igF^{ij} \neq 0$. There is an opinion\cite{10} that each element of a given decomposition need not (and actually cannot, if one considers the quantum evolution) satisfy the commutation relation. But it would be nice to have the canonical OAM which satisfies the commutation relation at least at the tree level.
- Is the decomposition (1) relevant to the longitudinal polarization, or the transverse polarization, or both? Is it frame–independent?

Actually, these questions remain unanswered within the twist–two decomposition. In order to answer them, one has to go to twist–three.

3. Complete gauge invariant decomposition

Here is the complete decomposition originally proposed by Chen et al.\cite{11,11} and written in this ‘covariant’ form by Wakamatsu\cite{12}:

\[
M_{\mu \nu \lambda}^{\text{quark-spin}} = -\frac{1}{2} \epsilon^{\mu \nu \lambda \sigma} \bar{\psi} \gamma_5 \gamma_\sigma \psi, \quad (2)
\]

\[
M_{\mu \nu \lambda}^{\text{quark-orbit}} = \bar{\psi} \gamma_\mu (x^\nu iD^\lambda_{\text{pure}} - x^\lambda iD^\nu_{\text{pure}}) \psi, \quad (3)
\]

\[
M_{\mu \nu \lambda}^{\text{gluon-spin}} = F_{a}^{\mu \lambda} A^a_{\text{phys}} - F_{a}^{\mu \nu} A^a_{\text{phys}}, \quad (4)
\]

\[
M_{\mu \nu \lambda}^{\text{gluon-orbit}} = -F_{a}^{\mu \alpha} (x^\nu (D^\lambda_{\text{pure}} A^a_{\text{phys}})_a - x^\lambda (D^\nu_{\text{pure}} A^a_{\text{phys}})_a). \quad (5)
\]

$A^a_{\text{phys}}$ is the ‘physical part’ of the gauge field which transforms homogeneously under gauge rotations $A^{\text{phys}} \rightarrow U A^{\text{phys}} U^\dagger$. The difference $A^{\text{pure}} = A - A^{\text{phys}}$ is pure gauge (i.e., it is a gauge rotation of the vacuum configuration), and appears in the modified covariant derivative $D^\mu_{\text{pure}} \equiv \partial^\mu + igA^\mu_{\text{pure}} = D^\mu - igA^\mu_{\text{phys}}$. There has been a lot of controversy as to what exactly $A^{\text{phys}}$ is.\cite{13,14} Also, the seeming ‘covariance’ has to be taken with great care. One can avoid these subtleties by working in the infinite momentum frame which is the only frame where the partonic picture makes sense.
and connections to high energy experiments can be established. My choice is

\[ A_{\text{phys}}^i(x) = - \int dy^- K(y^- - x^-) W_{xy} F^{+\mu}(y^-, \vec{x}) W_{yx}, \]  

(6)

where \( W_{xy} \) is the Wilson line from \( y^- \) to \( x^- \), and \( K(y^-) \) is either \( \frac{1}{2} \epsilon(y^-), \theta(y^-) \) or \( -\theta(-y^-) \). (6) obviously transforms homogeneously under gauge rotation \( s \), and it can be shown \(^5\) that the difference \( A - A_{\text{phys}} \) is pure gauge. With this definition, I can write

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \Delta G + L_{\text{can}}^q + L_{\text{can}}^g, \]  

(7)

where each term on the right–hand–side is the appropriate matrix element of the corresponding operator in \(^2\)–\(^4\). Note that this is a complete, gauge invariant decomposition which features \( \Delta G \) as the gluon helicity part. It is actually the gauge invariant completion of Jaffe–Manohar \(^2\). The quark OAM \( L_{\text{can}}^q \) is different from \( L_q \), and the gluon OAM \( L_{\text{can}}^g \) can be defined. They are the canonical OAMs which satisfy the commutation relation because of the property \([D_{\text{pure}}^i, D_{\text{pure}}^j] = igF_{\mu\nu}^{ij} = 0\). The expression \(^7\) itself quite often appears in the literature and in presentations, but the precise gauge invariant definitions of \( L_{\text{can}}^q,g \) are seldom articulated or glossed over. I wish to stress that, by showing the relation \(^7\) with the experimentally measurable \( \Delta G \), one is implicitly accepting the above decomposition \(^2\)–\(^5\) with \( A_{\text{phys}} \) as given by \(^6\).

4. Orbital angular momentum

The difference between the kinetic OAM \( L^q \) and the canonical \( L_{\text{can}}^q \) is called the potential OAM \( L_{\text{pot}}^q \). Its operator definition is

\[ \bar{\psi} \gamma^+ \left( x^i A_{\text{phys}}^i - x^j A_{\text{phys}}^j \right) \psi. \]  

(8)

This is gauge invariant by itself. Inserting \(^5\) and noticing that \( F^{+i} = E^i + (\vec{v} \times \vec{B})^i \) is basically the color Lorentz force, one sees that the above operator is physically interpreted as torque experienced by a quark as it propagates through the nucleon wavefunction. \(^15\) By taking the nonforward matrix element of \(^5\), I can eliminate the transverse coordinate \( x^i \) and replace it with the derivative with respect to the momentum transfer \( \Delta^i \). The remaining quark–gluon operator resembles the one familiar in the collinear twist–three mechanism of single–spin asymmetry (SSA). Guided by this analogy, I define doubly–unintegrated densities

\[
\int \frac{dy^- dz^-}{(2\pi)^2} e^{\frac{i}{2}(x_1 + x_2) \vec{P}^+ z^- + i(x_2 - x_1) \vec{P}^+ y^-} \\
\times \langle P^+ S' \bar{\psi} (-z^- / 2) \gamma^+ W_{xy} \gamma^+ W_{y^z} \psi (z^- / 2) | PS \rangle \\
= \frac{1}{P^+} e^{\mu\rho\sigma} \bar{S}_\rho \bar{P}_\sigma \Psi (x_1, x_2) + \frac{1}{P^+} e^{\nu\rho\sigma} \bar{S}_\rho \Delta_{\sigma} \Phi (x_1, x_2) + \cdots ,
\]  

(9)
where $\Delta = P' - P$, $x_1$ and $x_2 - x_1$ are the momentum fractions assigned to the outgoing quark and gluon, respectively. The first term is relevant to SSA (in the transversely polarized case). The second term is relevant to the longitudinally polarized case, and is related to the potential OAM $L_{pot} = \int dx_1 dx_2 P \Phi_F(x_1, x_2)$.

(10)

Speaking of the collinear twist–three approach to SSA, I recall that SSA has an alternative description in terms of the transverse momentum dependent distribution (TMD). This motivates me to define the nonforward generalization of TMD

$$\int dz^- d^2z_T e^{i(x\cdot\bar{P} + z^- - iq_T \cdot z_T)} \langle P' S' | \bar{\psi}(-z^-/2, -z_T/2) \gamma^+ W_{-it} \gamma_5 \psi(z^-/2, z_T/2) | PS \rangle \sim i \bar{P}^+ \epsilon^{+ij} S^+ \eta_{T}, \Delta, \tilde{f}(x, q_T^2, \xi, \Delta_T, q_T^2),$$

(11)

where the Wilson line is U–shaped along the light–cone direction extending to $z^- = \pm \infty$. The matrix elements like (11) were previously defined and classified in Ref. 17 where they were called ‘generalized parton correlation functions’. It can be shown that the canonical OAM is given by the following moment of $\tilde{f}$ (called $F_{1,4}$ in Ref. 17).

$$L^q_{can} = \frac{1}{2} \int dx d^2q_T q_T^2 \tilde{f}(x, q_T^2).$$

(12)

Actually, the equation (12) was first derived by Lorce and Pasquini18 using the Wigner distribution neglecting the Wilson line. Since the Wigner distribution describes the phase space (position and momentum) density of partons, it can be naturally used to define an OAM which is the cross product of the position and the momentum. The question is which OAM one gets in this way. Interestingly, this is determined by the choice of the contour in the Wilson line. As stated above, the U–shaped Wilson line along the light cone leads to the canonical OAM. However, the straight Wilson line leads instead to the kinetic OAM.

5. Twist analysis

Now I come to the issue of ‘twist’ (Ref. 7, see also Ref. 21). The two decompositions discussed so far, the Ji decomposition and the complete decomposition, are related as follows

$$J^q = \frac{1}{2} \Delta \Sigma + L^q_{can} + L_{pot},$$

(13)

$$J^g + L_{pot} = \Delta \Sigma + L^g_{can}.$$

(14)

Remarkably, these relations can be understood at the density level. Actually, it is possible to uniquely (in a certain sense) define the density of the canonical OAM $L^q_{can} = \int dx L^q_{can}(x)$. This allows me to analyze the twist structure of the complete decomposition, and in particular, its relevance to twist–three GPDs.
Let me begin with the relation $L^q = L^q_{\text{can}} + L^q_{\text{pot}}$. $L^q$ involves the ‘D–type’ correlator $\bar{\psi}\gamma D\psi$ and $L^q_{\text{pot}}$ involves the ‘F–type’ correlator $\bar{\psi}\gamma F^+i\psi$. It is known that these two types of correlators are related in terms of the doubly–unintegrated densities defined similarly to the second term of (9)

$$F.T.\langle P'S'\bar{\psi}\gamma^+ F^+ i\psi | PS \rangle \sim \Phi_F(x_1, x_2),$$
$$F.T.\langle P'S'\bar{\psi}\gamma^+ \gamma_5 F^+ i\psi | PS \rangle \sim \Phi_F(x_1, x_2),$$
$$F.T.\langle P'S'\bar{\psi}\gamma^+ D^i \psi | PS \rangle \sim \Phi_D(x_1, x_2),$$
$$F.T.\langle P'S'\bar{\psi}\gamma^+ \gamma_5 D^i \psi | PS \rangle \sim \Phi_D(x_1, x_2),$$

the relation reads

$$\Phi_D(x_1, x_2) = \delta(x_1 - x_2)L^q_{\text{can}}(x_1) + \mathcal{P} \frac{1}{x_1 - x_2} \Phi_F(x_1, x_2),$$

which is the doubly–unintegrated version of $L^q = L^q_{\text{can}} + L^q_{\text{pot}}$ (cf. (10)). Eq. (16) naturally defines the canonical OAM density $L^q_{\text{can}} = \int dx L^q_{\text{can}}(x)$. The delta function $\delta(x_1 - x_2)$ ensures that, in the quark–gluon–quark system described by the operator $\bar{\psi}D\psi$, the gluon carries zero longitudinal momentum $x_2 - x_1 = 0$. Thus the variable $x$ in $L^q_{\text{can}}(x)$ is indeed the longitudinal momentum fraction assigned to the quark, which makes its density interpretation preferable. In contrast, there is an ambiguity in defining a ‘density of the kinetic OAM’ $L^q = \int dx L^q(x)$. For instance, one can define either $L^q(x) = \int dx' \Phi_D(x, x')$ or $L^q(x) = \int dx' \Phi_F(x + x'/2, x - x'/2)$.

The expression of $L^q_{\text{can}}(x)$ is complicated, but owing to the equation of motion it can be written as

$$L^q_{\text{can}}(x) = x(H_q(x) + E_q(x) + G_3(x)) - \Delta q(x)$$
$$- \int dx' \mathcal{P} \frac{1}{x - x'} \left( \Phi_F(x, x') + \Phi_F(x, x') \right),$$

where $G_3(x)$ is one of the twist–three GPDs defined as

$$F.T.\langle P'S'\bar{\psi}(-z/2)\gamma_5 \gamma_\perp \psi(z/2) | PS \rangle = G_3(x)\bar{u}(P'S')\gamma_\perp u(PS) + \cdots.$$  

Integrating (18) over $x$, I get

$$\int dx x G_3(x) = -L^q.$$  

This identity was first derived in Ref. [23] However, there the authors worked in the parton model where there is no distinction between $L^q$ and $L^q_{\text{can}}$. (17) and (19) show that, while the integral of $G_3$ is related to the kinetic OAM, $G_3(x)$ itself is rather related to the canonical OAM.

Furthermore, $G_3(x)$ can be eliminated from (17) due again to the equation of motion.
motion. The result is

$$L^q_{\text{can}}(x) = x \int_x^{x(x)} \frac{dx'}{x'} (H_q(x') + E_q(x')) - x \int_x^{x(x)} \frac{dx'}{x'^2} \Delta q(x')$$

$$- x \int_x^{x(x)} dx_1 \int_{-1}^1 dx_2 \Phi_F(x_1, x_2) P \frac{3x_1 - x_2}{x_1^2 (x_1 - x_2)^2}$$

$$- x \int_x^{x(x)} dx_1 \int_{-1}^1 dx_2 \tilde{\Phi}_F(x_1, x_2) P \frac{1}{x_1^2 (x_1 - x_2)},$$

where $\Delta q$ is the usual polarized quark distribution. Eq. (20) completely reveals the twist structure of $L^q_{\text{can}}(x)$. It can be decomposed into the 'Wandzura–Wilczek' part which is related to the twist–two GPDs, and the 'genuine twist–three' part. Taking the first moment of (20), I get

$$L^q_{\text{can}} = J^q - \frac{1}{2} \Delta \Sigma - L_{\text{pot}},$$

which is precisely (13).

Similarly, I can define the canonical gluon OAM density $L^g_{\text{can}}(x)$ and analyze its twist structure. Again, the definition is unique in the sense that $x$ is interpretable as the longitudinal momentum fraction of the outgoing gluon. As in (17), the density is related to one of the twist–three gluon GPDs. By eliminating the twist–three GPD using the equation of motion, I get the decomposition of $L^g_{\text{can}}(x)$ into the part related to the twist–two gluon GPDs and genuine twist–three, three gluon distributions. Its first moment of course coincides with (14).

### 6. Transverse spin decomposition

Actually, in the discussions so far I implicitly assumed that the spin is longitudinally polarized. In the transversely polarized case, the situation is a bit more subtle. Firstly, one has to use the Pauli–Lubanski vector

$$W^\mu = - \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} P^\nu \int d^3 x M^{+ \rho \sigma},$$

instead of the angular momentum tensor $J^{\mu \nu} = \int M^{+ \mu \nu}$ itself. The reason is that the latter cannot give a frame–independent decomposition because rotation and boost do not commute. The relevant component is

$$W^i = \epsilon^i (P^- \int d^3 x M^{++} - P^+ \int d^3 x M^{+-}),$$

where

$$M^{++} = x^+ T^{++} - x^j T^{+-}, \quad M^{+-} = x^- T^{++} - x^j T^{+-}.$$
frame–dependence completely. The matrix element of the twist–four operator $T^{++}_{q,g}$ contains a term proportional to the metric tensor:

$$\langle P'S'|T^{++}_{q,g}|PS \rangle \sim g^{++}\bar{C}_{q,g}\bar{u}(P'S')u(PS), \quad (25)$$

which has no counterpart in the matrix elements of $T^{++}_{q,g}$ and $T^{+i}_{q,g}$ (because $g^{++} = g^{+i} = 0$). Then the question is whether this term does any harm, and unfortunately the answer is yes. As observed in Ref. [26], the nonforward product of spinors $\bar{u}(P'S')u(PS)$ is not a Lorentz scalar. It contains a manifestly frame–dependent term

$$\bar{u}(P'S')u(PS) \approx 2M + i\frac{P^3}{M(P^0 + M)}\epsilon^{ij}\Delta S_j, \quad (26)$$

in the transversely polarized case (but not in the longitudinally polarized case). The linear term in $\Delta$ modifies the Ji sum rule as

$$J^{q,g} \rightarrow J^{q,g} + \frac{P^3}{2(P^0 + M)}\bar{C}_{q,g}, \quad (27)$$

keeping the sum $J^q + J^g$ unchanged because $\bar{C}^q + \bar{C}^g = 0$.

If the Ji decomposition has a problem, then what about the complete decomposition (2)–(5)? A careful analysis shows that the best one can achieve in the transversely polarized case is

$$\frac{1}{2} = \frac{1}{2}\Delta \Sigma + \Delta G + L_{can}, \quad (28)$$

where $\Delta \Sigma$ and $\Delta G$ are numerically the same as in the longitudinally polarized case and given by the matrix elements of (2) and (4). However, the canonical OAM $L_{can}$ cannot be separated into the quark and gluon contributions. Trying to do so will result in frame–dependent terms similar to that encountered in (27).

To conclude, I note that all the four questions that I posed in Section 2 have been answered. In the longitudinally polarized case, the complete gauge invariant decomposition of the nucleon spin—the twist–three decomposition—is now available even at the level of the density in $x$.

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*aThere is a mismatch in the coefficient of $\bar{C}$ between Ref. [8] and Ref. [24]. The difference comes from the meaning of $d^3x$ in [24]. The former used the light–front form $d^3x = dx^-d^2x_\perp$, while the latter used the instant form $d^3x = dx_1dx_2dx_3$.}
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