THE FUNCTOR $A_{\text{min}}$ FOR $(p-1)$-CELL COMPLEXES AND EHP SEQUENCES

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Abstract. Let $X$ be a co-$H$-space of $(p-1)$-cell complex with all cells in even dimensions. Then the loop space $\Omega X$ admits a retract $\bar{A}_{\text{min}}(X)$ that is the evaluation of the functor $A_{\text{min}}$ on $X$. In this paper, we determine the homology $H_\ast(\bar{A}_{\text{min}}(X))$ and give the EHP sequence for the spaces $\bar{A}_{\text{min}}(X)$.

1. Introduction

The algebraic functor $A_{\text{min}}$ was introduced in [10] arising from the question on the naturality of the classical Poincaré-Birkhoff-Witt isomorphism. For any ungraded module $V$, $A_{\text{min}}(V)$ is defined to be the smallest functorial coalgebra retract of $T(V)$ containing $V$. Then the functor $A_{\text{min}}$ extends canonically to the cases when $V$ is any graded module. (See [10] for details.) The functor $A_{\text{min}}$ admits the tensor-length decomposition with

$$A_{\text{min}}(V) = \bigoplus_{n=0}^{\infty} A_{\text{min}}^n(V),$$

where $A_{\text{min}}^n(V) = A_{\text{min}}(V) \cap T_n(V)$ is the homogenous component of $A_{\text{min}}(V)$. By the Functorial Poincaré-Birkhoff-Witt Theorem [10, Theorem 6.5], there exists a functor $B_{\text{max}}$ from (graded) modules to Hopf algebras with the functorial coalgebra decomposition

$$T(V) \cong A_{\text{min}}(V) \otimes B_{\text{max}}(V)$$

for any graded module $V$. The determination of $A_{\text{min}}(V)$ for general $V$ is equivalent to an open problem in the modular representation theory of the symmetric groups according to [10, Theorem 7.4], which seems beyond the reach of current techniques. Some properties of the algebraic functor $A_{\text{min}}$ have been studied in [7, 23] with applications in homotopy theory [8, 22]. In this article we determine $A_{\text{min}}(V)$ in the special cases when $V_{\text{even}} = 0$ and $\dim V = p-1$.

Denote by $L(V)$ the free Lie algebra generated by $V$. Write $L_n(V)$ for the $n$-th homogeneous component of $L(V)$. Observe that $[L_s(V), L_l(V)]$ is a submodule of $L_{s+l}(V)$ under the Lie bracket of $L(V)$. Let

$$\bar{L}_n(V) = L_n(V) / \sum_{i=2}^{n-2} [L_i(V), L_{n-i}(V)].$$

Define $\bar{L}_n^k(V)$ recursively by $\bar{L}_n^1(V) = \bar{L}_n(V)$ and $\bar{L}_n^{k+1}(V) = \bar{L}_n(\bar{L}_n^k(V))$.

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Roughly speaking, a corollary of theorems 1.1 and 1.2 is the following homological information:

\[
A^\min(V) \cong \bigotimes_{k=0}^{\infty} E(\tilde{L}_p^k(V)),
\]

where \(E(W)\) is the exterior algebra generated by \(W\).

An important observation is that

\[
\tilde{L}_p^k(V)_{\text{even}} = 0 \text{ and } \dim \tilde{L}_p^k(V) = p - 1
\]

for each \(k \geq 0\) provided that \(V_{\text{even}} = 0\) and \(\dim V = p - 1\). Replace \(V\) by \(\tilde{L}_p(V)\) in the above theorem. Then \(A^\min(\tilde{L}_p(V)) \cong \bigotimes_{k=1}^{\infty} E(\tilde{L}_p^k(V))\) with a coalgebra decomposition

\[
A^\min(V) \cong E(V) \otimes A^\min(\tilde{L}_p(V)),
\]

which indicates the existence of the EHP fibrations by taking the geometric realization of the functor \(A^\min\) and the Hopf invariants on the functor \(A^\min\).

The geometric realization of the (algebraic) functor \(A^\min\) was studied in [9, 14, 16, 17, 19] for giving the decompositions of the loop spaces of double suspensions of torsion spaces. The most general result so far is given as follows:

**Theorem 1.1.** Let the ground field \(k\) be of characteristic \(p > 2\). Let \(V\) be a graded module such that \(V_{\text{even}} = 0\) and \(\dim V = p - 1\). Then there is an isomorphism of coalgebras

\[
A^\min(V) \cong \bigotimes_{k=0}^{\infty} E(\tilde{L}_p^k(V)),
\]

where \(E(W)\) is the exterior algebra generated by \(W\).

Let \(X\) be a path-connected finite complex. Define \(b_X = \sum_{q=1}^{\infty} q \dim \tilde{H}_q(X; \mathbb{Z}/p)\). Roughly speaking, \(b_X\) is the summation of the dimensions of the cells in \(X\). A direct consequence of theorems 1.1 and 1.2 is the following homological information:

**Corollary 1.3.** Let \(p > 2\) and let \(Y\) be any \(p\)-local simply connected co-\(H\) space. Suppose that \(H_{\text{odd}}(Y) = 0\) and \(\dim \tilde{H}_s(Y) = p - 1\). Then there is an isomorphism of coalgebras

\[
H_*(\tilde{A}^\min(Y)) \cong E(\Sigma^{-1}\tilde{H}_s(Y)) \otimes \bigotimes_{k=1}^{\infty} E(\Sigma^\frac{k-1}{2^k} b^k - p^k \tilde{H}_s(Y)),
\]

where \(\Sigma^\frac{k-1}{2^k} b^k - p^k \tilde{H}_s(Y) = \tilde{L}_p^k(\Sigma^{-1}\tilde{H}_s(Y))\).

Recall that the classical Hopf invariants can be obtained from the suspension splitting of the loop suspensions. For the geometric functor \(A^\min\), the following suspension splitting theorem is a special case of Theorem 1.4.
From the above theorem, one gets the Hopf invariant $H_n$ generated by
for any simply connected co-

Theorem 1.5

Theorem 1.4. Let $Y$ be any $p$-local simply connected co-$H$ space. Then there is a suspension splitting

$$\Sigma \tilde{A}^\min(Y) \simeq \bigvee_{n=1}^{\infty} \tilde{A}^\min_n(Y)$$

such that

$$\Sigma^{-1} \tilde{H}_*(\tilde{A}^\min_n(Y)) \cong A^\min_n(\Sigma^{-1} \tilde{H}_*(Y))$$

for each $n \geq 1$. □

Note that each $\tilde{A}^\min_n(Y)$ is a co-$H$ space because it is a retract of $\Sigma \tilde{A}^\min(Y)$. From the above theorem, one gets the Hopf invariant $H_n$ defined as the composite

$$\tilde{A}^\min(Y) \xrightarrow{\Omega} \Omega \Sigma \tilde{A}^\min(Y) \simeq \bigvee_{n=1}^{\infty} \tilde{A}^\min_n(Y) \xrightarrow{\text{proj}} \Omega \tilde{A}^\min(Y) \xrightarrow{r} \tilde{A}^\min(\tilde{A}^\min_n(Y)),$$

where $r: \Omega Z \to \tilde{A}^\min(Z)$ is the functorial retraction for any simply connected co-$H$-space $Z$. In particular, there is a Hopf invariant

$$H_p: \tilde{A}^\min(Y) \to \tilde{A}^\min(\tilde{A}^\min_p(Y))$$

for any simply connected co-$H$ space $Y$. By Lemma [5,3]

$$\tilde{A}^\min_p(Y) \simeq \Sigma^{by}_p - p + 1 Y$$

provided that $\tilde{H}_{odd}(Y) = 0$ and $\dim \tilde{H}_*(Y) = p - 1$.

Theorem 1.5 (EHP Fibration). Let $p > 2$ and let $Y$ be any $p$-local simply connected co-$H$ space. Suppose that $\tilde{H}_{odd}(Y) = 0$ and $\dim \tilde{H}_*(Y) = p - 1$. Then there is a fibre sequence

$$\Omega \tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y) \xrightarrow{P} \tilde{E}(Y) \xrightarrow{E} \tilde{A}^{\min}(Y) \xrightarrow{H_p} \tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y).$$

with the following properties:

1) On mod $p$ homology $H_*(\tilde{E}(Y)) \cong E(\Sigma^{-1} \tilde{H}_*(Y))$ as coalgebras.
2) $H_*(\tilde{A}^{\min}(Y)) \cong H_*(\tilde{E}(Y)) \otimes H_*(\tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y))$ as coalgebras.
3) If $f: S^n \to Y$ is a co-$H$ map such that $f_* \neq 0; \tilde{H}_*(S^n) \to \tilde{H}_*(Y)$. Then there is a commutative diagram of fibre sequences

$$\Omega \tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y) \xrightarrow{P_f} \tilde{E}(Y) \xrightarrow{E} \tilde{A}^{\min}(Y) \xrightarrow{H_p} \tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y)$$

In particular, the map $P: \Omega \tilde{A}^{\min}(\Sigma^{by}_p - p + 1 Y) \to \tilde{E}(Y)$ factors through the bottom cell of $E(Y)$.

The following theorem gives a general criterion when the EHP fibration splits off. A space $X$ is called to have a retractile generating complex $C$ if $C$ is a retract of $\Sigma X$ with a retraction $r: \Sigma X \to C$ such that the mod $p$ cohomology $H^*(X)$ is generated by

$$M = \text{Im}(\Sigma^{-1} r^*: \Sigma^{-1} \tilde{H}^*(C) \to \tilde{H}^*(X))$$
and $M \cong QH^*(X)$, the set of indecomposables. Recall that a space $X$ is called (stably) atomic if any self (stable) map of $X$ inducing isomorphism on the bottom homology is a (stable) homotopy equivalence.

**Theorem 1.6.** Let $p > 2$ and let $Y$ be any $p$-local simply connected co-$H$ space such that $Y$ is stably atomic. Suppose that $\overline{H}_{\text{odd}}(Y) = 0$ and $\dim \overline{H}_*(Y) = p - 1$. Then the following statements are equivalent to each other:

1. The EHP fibration
   
   $\overline{E}(Y) \xrightarrow{E} \overline{A}^{\min}(Y) \xrightarrow{H_p} \overline{A}^{\min}(\Sigma^{by-p+1}Y)$ splits off.

2. $\overline{E}(Y)$ is an $H$-space.

3. There exists an $H$-space $X$ having $Y$ as a retractile generating complex.

4. The map $P : \Omega \overline{A}^{\min}(\Sigma^{by-p+1}Y) \rightarrow \overline{E}(Y)$ is null homotopic.

5. The composite
   
   $\Sigma^{by-p+1}Y \hookrightarrow \Omega \overline{A}^{\min}(\Sigma^{by-p+1}Y) \xrightarrow{P} \overline{E}(Y)$

   is null homotopic.

6. There exists a map $g : \Sigma^{by-p}Y \rightarrow \overline{A}^{\min}(Y)$ such that
   
   $g_* : H_*(\Sigma^{by-p}Y) \rightarrow H_*(\overline{A}^{\min}(Y))$

   is a monomorphism on the bottom cells of $\Sigma^{by-p}Y$.

It is a classical question whether the total space of a spherical fibration over a sphere is an $H$-space localized at an odd prime $p$. For having a possible $H$-space structure, the base space and the fibre must be odd dimensional spheres. For $p > 3$, it is well-known that the answer is positive [3, 4]. For $p = 3$, there are examples that do not admit $H$-space structure. The above theorem gives some properties for studying this question.

The article is organized as follows. In section 2, we study the representation theory on natural coalgebra decompositions of tensor algebras. The proof of Theorem 1.1 is given in Section 3. The geometry of natural coalgebra decompositions of tensor algebras is investigated in Section 4, where Theorem 1.4 is Theorem 4.3. In Section 5, we give the proof of Theorem 1.5. The proof of Theorem 1.6 is given in Section 6.

2. **The Functor $A^{\min}$ and the Symmetric Group Module $\text{Lie}(n)$**

2.1. **The Functor $A^{\min}$ on Ungraded Modules.** In this subsection, the ground ring is a field $k$. A coalgebra means a pointed coassociative cocommutative coalgebra. For any module $V$, the tensor algebra $T(V)$ is a Hopf algebra by requiring $V$ to be primitive. This defines $T : V \mapsto T(V)$ as a functor from ungraded modules to coalgebras. The functor $A^{\min} : V \mapsto A^{\min}(V)$ is defined to be the smallest coalgebra retract of the functor $T$ with the property that $V \subseteq A^{\min}(V)$. More precisely the functor $A^{\min}$ is defined by the following property:

1. $A^{\min}$ is a functor from $k$-modules to coalgebras with a natural linear inclusion $V \hookrightarrow A^{\min}(V)$. 


(2). \( A_{\text{min}}(V) \) is a natural coalgebra retract of \( T(V) \) over \( V \). Namely there exist natural coalgebra transformations \( s: A_{\text{min}} \to T \) and \( r: T \to A_{\text{min}} \) such that the diagram

\[
\begin{array}{ccc}
A_{\text{min}}(V) & \xrightarrow{SV} & T(V) & \xrightarrow{rV} & A_{\text{min}}(V) \\
V & = & V & = & V \\
\end{array}
\]

commutes for any \( V \) and \( r \circ s = \text{id}_{A_{\text{min}}} \).

(3). \( A_{\text{min}} \) is minimal with respect to the above two conditions: if \( A \) is any functor from \( k \)-modules to coalgebras with natural linear inclusion \( V \subset A(V) \) such that \( A(V) \) is a natural coalgebra retract of \( T(V) \) over \( V \), then \( A_{\text{min}}(V) \) is a natural coalgebra retract of \( A(V) \) over \( V \).

By the minimal assumption, the functor \( A_{\text{min}} \) is unique up to natural equivalence if it exists. The existence of the functor \( A_{\text{min}} \) follows from \([16, \text{Theorem 4.12}]\).

There is a multiplication on \( A_{\text{min}} \) given by the composite

\[
A_{\text{min}} \otimes A_{\text{min}} \xrightarrow{\mu} T \otimes T \xrightarrow{r} A_{\text{min}}
\]

and so \( A_{\text{min}} \) is a functor from modules to quasi-Hopf algebras, where a quasi-Hopf algebra means a coassociative and cocommutative bi-algebra without assuming the associativity of the multiplication. The multiplication on \( A_{\text{min}} \) induces a new natural coalgebra transformation \( r'_{V}: T(V) \to A_{\text{min}}(V) \) given by

\[
r'_{V}(x_1 \otimes \cdots \otimes x_n) = \left( \left( \cdots \left( (x_1 \cdot x_2) \cdot x_3 \right) \cdots \right) \cdot x_n \right)
\]

for any \( x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \). By the minimal assumption, the composite

\[
A_{\text{min}} \xrightarrow{s} T \xrightarrow{r'} A_{\text{min}}
\]

is a natural equivalence. Consider \( T(V) \) as an \( A_{\text{min}}(V) \)-comodule via the map \( r'_{V} \).

According to \([16, \text{Proposition 6.1}]\), the cotensor product

\[
B_{\text{max}}(V) = k \square A_{\text{min}}(V) T(V)
\]

is natural sub Hopf algebra of \( T(V) \) with a natural coalgebra equivalence

\[
(2.1) \quad k \otimes_{B_{\text{max}}(V)} T(V) \cong A_{\text{min}}(V).
\]

Together with \([16, \text{Lemmas 5.3}]\), there is a natural coalgebra decomposition

\[
(2.2) \quad T(V) \cong B_{\text{max}}(V) \otimes A_{\text{min}}(V).
\]

By taking tensor length decomposition, we have \( B_{\text{max}}(V) = \bigoplus_{n=0}^{\infty} B_{n}^{\text{max}}(V) \) with \( B_{n}^{\text{max}}(V) = B_{\text{max}}(V) \cap T_{n}(V) \). Let

\[
Q_{\text{max}}(V) = Q B_{\text{max}}(V)
\]

be the indecomposables of \( B_{\text{max}}(V) \) with tensor length decomposition \( Q_{\text{max}}(V) = \bigoplus_{n=2}^{\infty} Q_{n}^{\text{max}}(V) \). (Note that \( B_{1}^{\text{max}}(V) = 0 \) and so \( B_{\text{max}}(V) \) has no nontrivial indecomposable elements of tensor length 1.) According to \([16, \text{Section 2}]\), all of the functors \( A_{\text{min}}, A_{n}^{\text{max}}, B_{\text{max}}, Q_{\text{max}}^{n}, Q_{n}^{\text{max}} \) extend canonically for graded modules.
2.2. The Symmetric Group Module \( \text{Lie}(n) \). Let the ground ring \( R \) be any commutative ring with identity. The module \( \text{Lie}^R(n) \), which is simply denoted as \( \text{Lie}(n) \) if the ground ring is clear, is defined as follows. Let \( V \) be a free \( R \)-module of rank \( n \) with a basis \( \{ e_i \mid 1 \leq i \leq n \} \). The module \( \gamma_n \) is defined to be the submodule of \( V^\otimes n \) spanned by

\[
e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)}
\]

for \( \sigma \in \Sigma_n \). The module \( \text{Lie}(n) \) is defined by

\[
\text{Lie}(n) = \gamma_n \cap L_n(V) \subseteq V^\otimes n.
\]

The \( \Sigma_n \)-action on \( \gamma_n \) is given by permuting letters \( e_1, e_2, \ldots, e_n \). Since both \( \gamma_n \) and \( L_n(V) \) are invariant under permutations of the letters, \( \text{Lie}(n) \) is an \( R(\Sigma_n) \)-submodule of \( \gamma_n \).

Note that \( \text{Lie}(n) \) is the submodule of \( L_n(V) \) spanned by the homogenous Lie elements in which each \( e_i \) occurs exactly once. By the Witt formula, \( \text{Lie}(n) \) is a free \( R \)-module of rank \( (n-1)! \). Following from the antisymmetry and the Jacobi identity, \( \text{Lie}(n) \) has a basis given by the elements

\[
[[e_1, e_{\sigma(2)}], e_{\sigma(3)}], \ldots, e_{\sigma(n)}]
\]

for \( \sigma \in \Sigma_{n-1} \). (See [11].) Observe that \( \text{Lie}(n) \) is the image of the \( R(\Sigma_n) \)-map

\[
\beta_n: \gamma_n \to \gamma_n \quad \beta_n(a_1 \otimes \cdots \otimes a_n) = [[a_1, a_2], \ldots, a_n].
\]

Thus \( \text{Lie}(n) \) can be also regarded as the quotient \( R(\Sigma_n) \)-module of \( \gamma_n \) with the projection \( \beta_n: \gamma_n \to \text{Lie}(n) \).

**Proposition 2.1.** Let \( V \) be any graded projective module and let \( \Sigma_n \) act on \( V^\otimes n \) by permuting factors in graded sense. Then there is a functorial isomorphism

\[
\text{Lie}(n) \otimes_{R(\Sigma_n)} V^\otimes n \cong L_n(V)
\]

for any graded module \( V \).

**Proof.** Clearly the quotient map \( \beta_n: V^\otimes n = \gamma_n \otimes_{R(\Sigma_n)} V^\otimes n \to L_n(V) \), \( x_1 \otimes \cdots \otimes x_n \mapsto [[x_1, x_2], \ldots, x_n] \), factors through the quotient \( \text{Lie}(n) \otimes_{R(\Sigma_n)} V^\otimes n \) and so there is an epimorphism \( \text{Lie}(n) \otimes_{R(\Sigma_n)} V^\otimes n \twoheadrightarrow L_n(V) \). On the other hand note that \( L = \bigoplus_{n=1}^{\infty} \text{Lie}(n) \otimes_{R(\Sigma_n)} V^\otimes n \) has the canonical graded Lie algebra structure generated by \( V \). So the inclusion \( V \hookrightarrow L \) induces an epimorphism of graded Lie algebras \( L(V) \twoheadrightarrow L \). The assertion follows. \( \square \)

Consider \( T_n: V \mapsto V^\otimes n \) as a functor from projective (ungraded) modules to projective (ungraded) modules. Denote by \( \text{Hom}(F,F') \) the set of natural linear transformations from a functor \( F \) to a functor \( F' \) provided that \( F \) preserves direct limits. By [7] Lemma 3.8, \( \text{Hom}(T_n, T_m) = 0 \) if \( n \neq m \) and there is an isomorphism of rings

\[
\theta: \text{End}_{R(\Sigma_n)}(\gamma_n) \to \text{End}(T_n, T_n)
\]

given by

\[
\theta(\phi) = \phi \otimes \text{id}_{V^\otimes n}: \gamma_n \otimes_{R(\Sigma_n)} V^\otimes n = V^\otimes n \to \gamma_n \otimes_{R(\Sigma_n)} V^\otimes n = V^\otimes n
\]

for \( \phi \in \text{End}_{R(\Sigma_n)}(\gamma_n) \). Replacing \( \gamma_n \) by \( \text{Lie}(n) \), we have the morphism of rings

\[
\theta: \text{End}_{R(\Sigma_n)}(\text{Lie}(n)) \to \text{End}(L_n).
\]
Proposition 2.2. If \( n \neq m \), then \( \text{Hom}(L_n, L_m) = 0 \). Moreover the map
\[
\theta : \text{End}_{R(\Sigma_n)}(\text{Lie}(n)) \longrightarrow \text{End}(L_n)
\]
is an isomorphism.

Proof. Let \( \phi : L_n \rightarrow L_m \) be a natural transformation. Then the composite
\[
T_n \longrightarrow L_n \xrightarrow{\phi} L_m \longrightarrow T_m
\]
is a natural transformation, which is zero as \( \text{Hom}(T_n, T_m) = 0 \). Thus \( \phi = 0 \). For
the second statement, clearly \( \theta \) is a monomorphism. Let \( \phi_V : L_n(V) \rightarrow L_n(V) \) be
any natural transformation. Let \( \bar{V} \) be the free \( R \)-module of rank \( n \), which defines
\( \gamma_n \). Consider the commutative diagram
\[
\begin{array}{c}
\gamma_n \downarrow \quad \downarrow \quad \downarrow \\
\text{Lie}(n) \quad \quad \text{Lie}(n) \quad \quad \text{Lie}(n) \quad \quad \text{Lie}(n)
\end{array}
\]
where the existence of \( \phi' \) follows from the fact that the composite of the maps in
the top row maps \( \gamma_n \) into \( \gamma_n \) and \( \text{Lie}(n) = \gamma_n \cap L_n(\bar{V}) \). Let \( V \) be any ungraded
module and let \( a_1 \otimes \cdots \otimes a_n \) be any homogenous element in \( V^{\otimes n} \). Let \( f : \bar{V} \rightarrow V \)
be \( R \)-linear map such that \( f(e_i) = a_i \). By the naturality, there is a commutative
diagram
\[
\begin{array}{cccc}
\text{Lie}(n) & \longrightarrow & L_n(V) & \longrightarrow & L_n(V) & \longrightarrow & T_n(V) \\
\phi' & \downarrow \phi_V & \downarrow \phi_V & \downarrow \phi_V & \downarrow & \\
\text{Lie}(n) & \longrightarrow & L_n(V) & \longrightarrow & L_n(V) & \longrightarrow & T_n(V)
\end{array}
\]
Thus
\[
\theta(\phi')([a_1, a_2], \ldots, a_n]) = \phi([[a_1, a_2], \ldots, a_n])
\]
and hence the result. \( \square \)

Corollary 2.3. There is a one-to-one correspondence between the decompositions
of the functor \( L_n \) and the decompositions of \( \text{Lie}(n) \) over \( R(\Sigma_n) \). \( \square \)

A functor from modules to modules \( Q \) is called \( T_n \)-projective if \( Q \) is naturally
equivalent to a retract of the functor \( T_n \).

Proposition 2.4. Let \( Q \) be a \( T_n \)-projective functor and let \( \phi : Q \rightarrow L_n \) be a natural
linear transformation. Then there exists a natural linear transformation \( \tilde{\phi} : Q \rightarrow T_n \)
such that \( \phi = \beta_n \circ \tilde{\phi} \).

Proof. Let \( \bar{V} \) be the free \( R \)-module of rank \( n \), which defines \( \gamma_n \). Let \( \bar{Q} = \gamma_n \cap Q(\bar{V}) \).
Since \( Q \) is a retract of the functor \( T_n \), \( \bar{Q} \) is a summand of \( \gamma_n \) over \( R(\Sigma_n) \) and so \( \bar{Q} \)
is a projective \( R(\Sigma_n) \)-module. From the fact that \( \text{End}_{R(\Sigma_n)}(\gamma_n) \cong \text{End}(T_n) \),
\[
Q(V) = \bar{Q} \otimes_{R(\Sigma_n)} V^{\otimes n}
\]
for any module $V$. By the proof of Proposition 2.2, the map
\[
\theta: \text{Hom}_{R(\Sigma_n)}(\bar{Q}, \text{Lie}(n)) \to \text{Hom}(Q, L_n), \quad f \mapsto f \otimes \text{id}_{V \otimes n}
\]
is an isomorphism. Thus there exists a unique $R(\Sigma_n)$-map
\[
\phi': \bar{Q} \to \text{Lie}(n)
\]
such that $\theta(\phi') = \phi$. Since $\bar{Q}$ is projective, the lifting problem
\[
\begin{array}{ccc}
\gamma_n & \downarrow & \beta_n \\
\bar{Q} & \phi' & \text{Lie}(n)
\end{array}
\]
has a solution. The assertion follows by tensoring with $V \otimes n$ over $R(\Sigma_n)$. □

By inspecting the proof, each $T_n$-projective subfunctor $Q$ of $L_n$ induces a $R(\Sigma_n)$-projective submodule $\bar{Q}$ of $\text{Lie}(n)$. Conversely, each $R(\Sigma_n)$-projective submodule $\bar{Q}$ of $\text{Lie}(n)$ induces a $T_n$-projective subfunctor $Q$, $V \mapsto \bar{Q} \otimes R(\Sigma_n) V \otimes n$, of $L_n$. Thus we have the following:

**Proposition 2.5.** There is a one-to-one correspondence between $T_n$-projective subfunctors of $L_n$ and $R(\Sigma_n)$-projective submodules of $\text{Lie}(n)$. □

### 3. Proof of Theorem 1.1

In this section, the ground field is of characteristic $p > 2$. The notation $V$ means any fixed connected graded module such that $V_{\text{even}} = 0$ and $\dim V = p - 1$. The general graded or ungraded module is then denoted by $W$.

Let $W$ be any module and let $T: W \to T(W)$ be the functor from modules to Hopf algebras, where the tensor algebra $T(W)$ is Hopf by saying $W$ primitive.

Let $B(W)$ be the sub Hopf algebra generated by $L_n(W)$ for $n$ not a power of $p$. By [16 Theorem 1.5], $B(W) \subseteq B^{\text{max}}(W)$ for any ungraded module $W$ and so for any graded or ungraded module $W$. It follows that there is an epimorphism
\[
q: k \otimes B(W) T(W) \to A^\text{min}(W)
\]
for any ungraded or graded module $W$.

We are going to determine $k \otimes B(V) T(V)$ and to show that the map $q$ is in fact an isomorphism when $W = V$.

#### 3.1. Determination of $k \otimes B(V) T(V)$

Let $B^{[s]}(W) = \langle B(W), L_p(W) \mid s \geq k \rangle$ be the sub Hopf algebra of $T(W)$ generated by $B(W)$ and $L_p(W)$ for $s \geq k$. Then there is a tower of sub Hopf algebras
\[
\cdots \subseteq B^{[k+1]}(W) \subseteq B^{[k]}(W) \subseteq \cdots \subseteq B^{[1]}(W) \subseteq B^{[0]}(W) = T(W)
\]
with the intersection
\[
B(W) = \bigcap_{k=0}^{\infty} B^{[k]}(W).
\]

Define the functor $\tilde{L}_n$ by:
\[
\tilde{L}_n(W) = L_n(W) / \left( \sum_{i=2}^{n-2} [L_i(L_i), L_{n-i}(W)] \right)
\]
Lemma 3.1. For all Hopf algebra $L$, the proof is given by induction. First consider the short exact sequence of $L$

$$0 \longrightarrow \Phi$$

where $B$ is the Hopf algebra of $L$.

For each $k \geq 0$, the periodicity information:

$$(3.2) \quad (\bar{\Lambda}(k))_{\text{even}} = 0 \text{ and } \dim L_p(V) = p - 1$$

for all $k \geq 0$.

Lemma 3.2. Let $A$ be a connected Hopf algebra of finite type and let $B$ be a sub Hopf algebra of $A$. Suppose that $A$ is a tensor algebra as an algebra with a choice of inclusion $QA \rightarrow A$. Then there is a short exact sequence

$$0 \longrightarrow Q(B) \longrightarrow (k \otimes B A) \otimes Q(A) \longrightarrow I(k \otimes B A) \longrightarrow 0.$$

Now we determine the sub Hopf algebra $B[k](V)$:

Lemma 3.2. For each $k \geq 0$, there is a short exact sequence of Hopf algebras

$$B[k+1](V) \longrightarrow B[k](V) \longrightarrow E(\bar{L}_p(V)).$$

Moreover

1. $B[k](V)$ is the sub Hopf algebra of $T(V)$ generated by $L_p(V)$ and $Q_j B(V)$ for $2 \leq j < p^k$.

2. For any possible Steenrod module structure on $V$, $\bar{L}_p(V)$ is a suspension of $V$.

Proof. The proof is given by induction. First consider the short exact sequence of Hopf algebra

$$B'(V) \longrightarrow T(V) \longrightarrow \Lambda(V),$$

where $B'(V) = k \square \Lambda(V) T(V)$. Then $B[1](V) \subseteq B'(V)$. There is a short exact sequence

$$0 \longrightarrow Q(B'(V)) \longrightarrow \Lambda(V) \otimes V \longrightarrow I \Lambda(V) \longrightarrow 0.$$  

Since $V_{\text{even}} = 0$, $E(V) = \Lambda(V)$. By the assumption of $\dim V = p - 1$, we have $E(V)_j = 0$ for $j \geq p$. Thus $Q_j (B'[V]) = 0$ for $j > p$ and so $B'(V) \subseteq B[1](V)$.

Hence $B'(V) = B[1](V)$ From the above exact sequence,

$$Q_p(B'[V]) \cong E_{p-1}(V) \otimes V \cong Q_p(B[1](V)) = \bar{L}_p(V)$$

is of dimension $p - 1$. Since $\dim E_{p-1}(V) = 1$, $\bar{L}_p(V) = E_{p-1}(V) \otimes V$ is a suspension of $V$ for any possible Steenrod module structure on $V$. Thus the assertions hold for $k = 0$. Suppose that the assertion holds for $k$. Consider the short exact sequence of Hopf algebras

$$B''(V) \longrightarrow B[k](V) \longrightarrow \Lambda(\bar{L}_p(V)),$$

where $B''(V) = k \square \Lambda(\bar{L}_p(V)) B[k](V)$. Then $B[k+1](V) \subseteq B''(V)$ because $\phi$ is a Hopf map which sends the generators for $B[k+1](V)$ to zero. Since $\bar{L}_p(V)_{\text{even}} = 0$ with $\dim \bar{L}_p(V) = p - 1$, $\Lambda(\bar{L}_p(V)) = E(\bar{L}_p(V))$ and so there is short exact sequence

$$0 \longrightarrow Q(B''(V)) \longrightarrow E(\bar{L}_p(V)) \otimes Q(B[k](V)) \longrightarrow I E(\bar{L}_p(V)) \longrightarrow 0.$$  

It follows that $Q(B''(V)) = 0$ for $j > p^{k+1}$ and so $B''(V) \subseteq B[k+1](V)$. By the Lie action of $E(\bar{L}_p(V))$ on $Q(B''(V))$,

$$Q_{p^{k+1}} B[k+1](V) = Q_{p^{k+1}} B''(V) = \bar{L}_p(V) \cong E_{p-1}(\bar{L}_p(V)) \otimes \bar{L}_p(V).$$
Since \( \dim E_{p-1}(\bar{L}_p^k(V)) = 1 \), \( \bar{L}_p^{k+1}(V) \) is a suspension of \( \bar{L}_p^k(V) \). The induction is finished and hence the result.

\[\Box\]

**Proposition 3.3.** There is a (non-functorial) coalgebra decomposition

\[
\mathbf{k} \otimes_{B(V)} T(V) \cong \bigotimes_{k=0}^\infty E(\bar{L}_p^k(V))
\]

over any possible Steenrod algebra structure on \( V \).

**Proof.** Consider the short exact sequence of coalgebras

\[
\begin{align*}
\mathbf{k} \otimes_{B(V)} B^{[k+1]}(V) & \xhookleftarrow{i} \mathbf{k} \otimes_{B(V)} B^{[k]}(V) \twoheadrightarrow E(\bar{L}_p^k(V)).
\end{align*}
\]

Observe that up to tensor length \( p^{k+1} - 1 \)

\[
(\mathbf{k} \otimes_{B(V)} B^{[k]}(V))_j \cong E(\bar{L}_p^k(V))_j
\]

for \( j \leq p^{k+1} \). Thus there is a coalgebra cross-section

\[
s : E(\bar{L}_p^k(V)) \twoheadrightarrow \mathbf{k} \otimes_{B(V)} B^{[k]}(V).
\]

The cross-section \( s \) is a morphism over the Steenrod algebra for all possible Steenrod algebra structure on \( V \) and it is also a morphism over \( \text{GL}(V) \) when \( \text{GL}(V) \) acts on \( V \) by forgetting the grading of \( V \).

According to [11, Theorem 1.1], \( B(W) \) is a functorial coalgebra retract of \( T(W) \) for any graded module \( W \). Let \( r_W : T(W) \rightarrow B(W) \) be a functorial coalgebra retraction. From \( B(W) \subseteq B^{[k]}(W) \subseteq T(W) \), the restriction

\[
r_V|_{B^{[k]}(W)} : B^{[k]}(V) \twoheadrightarrow B(W)
\]

is a functorial coalgebra retraction. It follows that the short exact sequence of coalgebras

\[
B(W) \xhookleftarrow{i} B^{[k]}(W) \twoheadrightarrow \mathbf{k} \otimes_{B(W)} B^{[k]}(W)
\]

splits off. This gives a functorial coalgebra decomposition

\[
B^{[k]}(W) \cong B(W) \otimes (\mathbf{k} \otimes_{B(W)} B^{[k]}(W))
\]

for any graded module \( W \). Thus the functor

\[
W \mapsto \mathbf{k} \otimes_{B(W)} B^{[k]}(W)
\]

is a functorial from modules to quasi-Hopf algebra because \( \mathbf{k} \otimes_{B(W)} B^{[k]}(W) \) is a functorial coalgebra retract of the Hopf algebra functor \( B^{[k]}(W) \).

Evaluating \( W = V \), \( \mathbf{k} \otimes_{B(V)} B^{[k]}(V) \) is a quasi-Hopf algebra over the Steenrod algebra for all possible Steenrod algebra structure on \( V \). By [10, Lemma 5.3], the map

\[
(\mathbf{k} \otimes_{B(V)} B^{[k+1]}(V)) \otimes E(\bar{L}_p^k(V)) \xrightarrow{\mu(j \otimes s)} \mathbf{k} \otimes_{B(V)} B^{[k]}(V)
\]

is an isomorphism and hence the result.

\[\Box\]
3.2. **Proof of Theorem 1.1** Consider the functorial short exact sequence of algebras

\[ B^{\text{max}}(W) \rightarrow T(W) \xrightarrow{r} A^{\text{min}}(W) = k \otimes B^{\text{max}} T(W). \]

Define

\[ A^{\text{min}}_n(W) = \text{Im}(T_n(W)) \subseteq T(W) \xrightarrow{r} A^{\text{min}}(W). \]

Then

\[ A^{\text{min}}(W) = \bigoplus_{n=0}^{\infty} A^{\text{min}}_n(W) \]

is a bigraded coalgebra and the decomposition

\[ T(W) \cong B^{\text{max}}(W) \otimes A^{\text{min}}(W) \]

is a functorial coalgebra decomposition of bi-graded coalgebras, where the second grading on \( B^{\text{max}}(W) \) is given by \( B^{\text{max}}_n(W) = T_n(W) \cap B^{\text{max}}(W) \). Note that

\[ L_n(W) = \text{Im}(\beta_n: W^\otimes n \rightarrow W^\otimes n). \]

Denote by \( \beta_n \) the epimorphism \( W^\otimes n \rightarrow L_n(W) \) if there are no confusions. The algebraic version of the James-Hopf map \( H_n: T(W) \rightarrow T(W^\otimes n) \) is defined in \([7, 16, 17]\).

**Lemma 3.4.** There is a commutative diagram of natural transformations of functors

\[
\begin{array}{cccccc}
T & \xrightarrow{H_p^k} & T(T_p^k) & \xrightarrow{T(r)} & T(A^{\text{min}}_p^k) & \xrightarrow{A^{\text{min}}(A^{\text{min}}_p^k)} \\
\downarrow & & \downarrow & & \downarrow & \\
Q^{\text{max}}_{p^k+1} & \xrightarrow{L_p(T_p^k)} & L_p(T_p^k) & \xrightarrow{L_p(r)} & L_p(A^{\text{min}}_p^k) & \xrightarrow{\bar{L}_p(A^{\text{min}}_p^k)} \\
\downarrow & & \downarrow & & \downarrow & \\
Q^{\text{max}}_{p^k+1} & \xrightarrow{\phi'} & T_{p^k+1} & \xrightarrow{\phi} & T_p(T_p^k) & \xrightarrow{T_p(r)} & T_p(A^{\text{min}}_p^k) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\beta_{p^k} & \xrightarrow{\beta_p} & \beta_p & \xrightarrow{\beta_p} & \beta_p & & \\
\end{array}
\]

for some \( \phi \) and \( \phi' \).

**Proof.** Since all functors in the diagram are well-defined over \( \mathbb{Z}(p) \), it suffices to show that the assertion holds when the ground ring is \( \mathbb{Z}(p) \). The top three squares commute because the maps in the top row are coalgebra maps, which send the primitives to the primitives. It is clear that the bottom right square commutes. By Proposition 2.4, there exist lifting \( \phi' \) and \( \phi \) such that the bottom left and middle squares commute. \( \square \)

**Lemma 3.5.** Let \( n \geq 2 \) and let \( W \) be any graded module. Then the composite

\[ W^\otimes n = W^\otimes 2 \otimes W^\otimes n-2 \xrightarrow{\beta_n} L_n(W) \rightarrow \bar{L}_n(W) \]

factors through the quotient

\[ \left( W^\otimes 2 / \langle a_1 \otimes a_2 + (-1)^{|a_1||a_2|} a_2 \otimes a_1 \rangle \right) \otimes \Lambda_{n-2}(W) \]

of \( W^\otimes n \).
Proof. By the skew-symmetric property on the first two factors, the iterated Lie operad $\beta_n: W^\otimes n \to L_n(W)$ factors through the quotient
\[
\left( W^\otimes 2/(a_1 \otimes a_2 + (-1)^{|a_1||a_2|}a_2 \otimes a_1) \right) \otimes W^\otimes n-2.
\]
Consider the short exact sequence of Hopf algebras
\[
B[1](W) \hookrightarrow T(W) \rightarrow \Lambda(W).
\]
Note that there is a commutative diagram
\[
\bigoplus_{n=2}^\infty L_n(W) \hookrightarrow IB[1](W) \quad \text{and} \quad \bigoplus_{n=2}^\infty \bar{L}_n(W) \twoheadrightarrow QB[1](W).
\]
The assertion follows from [2, Lemma 3.12] that the Lie bracket in $T(W)$ induces an action of $\Lambda(W)$ on $QB[1](W) = \bigoplus_{n=2}^\infty \bar{L}_n(W)$.

Let $W$ be a graded module such that $W_j = 0$ for $j \neq 1$. Let $\Sigma_n$ act on $W^\otimes n$ by permuting positions in graded sense. Let $\text{GL}(W)$ be the general linear group of the vector space $W_1$. The Schur algebra [13] is defined by
\[
S(W) = \text{Im}(\phi: k(\text{GL}(W)) \to \text{End}(W^\otimes n)),
\]
where $\phi(f) = f^\otimes n$. By [5, Theorem 4.1],
\[
(3.4) \quad \text{Im}(k(\Sigma_n) \to \text{End}(W^\otimes n)) = \text{End}_{S(W)}(W^\otimes n)
\]
if $k$ is algebraically closed. Thus if $k$ is algebraically closed, then for any $k(\text{GL}(W))$-map
\[
f: W^\otimes n \to W^\otimes n
\]
there exists an element $\alpha \in k(\Sigma_n)$ such that
\[
f(a_1 \otimes \cdots \otimes a_n) = \alpha \cdot (a_1 \otimes \cdots \otimes a_n).
\]

Lemma 3.6. Let $\bar{V}$ be a graded module such that $\dim \bar{V}_1 = p - 1$ and $\bar{V}_j = 0$ for $j \neq 1$. Let $\text{GL}(\bar{V})$ be the general linear group with the action on $\bar{V}$. Assume that the ground field $k$ is algebraically closed. Then, as a morphism over $k(\text{GL}(\bar{V}))$, the composite
\[
q: T_p(\bar{V}) = \bar{V}^\otimes p \xrightarrow{\beta_p} L_p(\bar{V}) \xrightarrow{\bar{L}_p(\bar{V})} \bar{V} \otimes E_{p-1}(\bar{V})
\]
does NOT have a cross-section.

Proof. Suppose that there exists a $k(\text{GL}(\bar{V}))$-map $\phi: \bar{V} \otimes E_{p-1}(\bar{V}) \longrightarrow \bar{V}^\otimes p$ such that the composite $q \circ \phi$ is the identity map of $\bar{V} \otimes E_{p-1}(\bar{V})$. We are going to find a contradiction.

Let $\{x_1, x_2, \ldots, x_{p-1}\}$ be a basis for $\bar{V}$. Note that
\[
\dim E_{p-1}(\bar{V}) = 1
\]
with a basis $\{(x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1})\}$. The set
\[
\{x_i \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1}) \mid 1 \leq i \leq p-1\}
\]
is an endomorphism over $k(\text{GL}(\bar{V}))$, from equation (3.3) there exists an element $\alpha \in k(\Sigma_p)$ such that
\[ \phi \circ q(w) = \alpha \cdot w \]
for $w \in \bar{V}^\otimes p$. By counting the occurrence of $x_j$'s, $\phi(x_1 \otimes (x_1 \wedge \cdots \wedge x_{p-1}))$ is a linear combination of monomials $x_i x_{i+1} \cdots x_k$ in which $x_i$ occurs twice and $x_j$ occurs once for each $j \neq i$. By Lemma 3.5 there is a commutative diagram
\[
\begin{array}{ccc}
\bar{V}^\otimes p & \xrightarrow{\beta_p} & L_p(\bar{V}) \\
\pi \downarrow & & \downarrow \bar{\pi} \\
(\bar{V}^\otimes 2/(x_i x_j - x_j x_i)) \otimes E_{p-2}(\bar{V}) & \xrightarrow{g} & \check{L}_p(\bar{V}) = \bar{V} \otimes E_{p-1}(\bar{V}).
\end{array}
\]
By counting the occurrence of $x_j$'s,
\[ \pi \circ \phi(x_1 \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1})) = k_1 x_1^2 \otimes (x_2 \wedge x_3 \wedge \cdots \wedge x_{p-1}) + \sum_{i=1}^{p-1} k_i x_1 x_i \otimes (x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{p-1}). \]
for some $k_i \in k$. Note that $x_1 \wedge x_j = -x_j \wedge x_i$. By interchanging $x_i$ and $x_{i+1}$, there is an equation
\[ k_i = -k_{i+1} \]
for $i \geq 2$. By equation (3.3), the map
\[ \pi \circ \phi: \bar{V} \otimes E_{p-1}(\bar{V}) \rightarrow (\bar{V}^\otimes 2/(x_i x_j - x_j x_i)) \otimes E_{p-2}(\bar{V}) \]
is a morphism over the Steenrod algebra by regarding $\bar{V}$ as any graded module over the Steenrod algebra with $V_{\text{even}} = 0$. Take the Steenrod module structure on $\bar{V}$ by setting $P^1 x_1 = x_2$, $P^1 x_1 = 0$ for $i > 1$ and $P^k x_j = 0$ for $k \geq 1$ and $j > 1$. Then
\[
P^2 x_1 x_i \otimes (x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{p-1}) = x_2 x_i \otimes (x_2 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{p-1}) = 0 \]
for $i \geq 3$. Note that
\[
P^2 x_1 \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1}) = x_2 \otimes (x_2 \wedge x_3 \wedge \cdots \wedge x_{p-1}) = 0. \]
By applying $P^2$ to equation (3.6),
\[
0 = P^2 (k_1 x_1^2 \otimes (x_2 \wedge x_3 \wedge \cdots \wedge x_{p-1}) + k_2 x_1 x_2 \otimes (x_1 \wedge x_3 \wedge \cdots \wedge x_{p-1})) = k_1 x_1^2 \otimes (x_2 \wedge x_3 \wedge \cdots \wedge x_{p-1}) + k_2 x_2^2 \otimes (x_2 \wedge x_3 \wedge \cdots \wedge x_{p-1}).
\]
Thus
\[ k_1 = -k_2 \]
and so
\[ \pi \circ \phi(x_1 \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1})) = k_1 \sum_{i=1}^{p-1} (-1)^{i-1} x_1 x_i \otimes (x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{p-1}). \]
Define
\[ \alpha_1 = x_1 \otimes x_1 \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_{p-1} \in \bar{V}^\otimes p. \]
Proof of Theorem 1.1. By Proposition 3.3, it suffices to show that the epimorphism 
\[ \pi \hat{\beta}_p(\alpha_1) = \pi([x_1, x_1, \ldots, x_{p-1}]) \]
\[ = 2x_1 \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{p-1}). \]
for each \( k \), we may assume that the ground field \( k \) is an isomorphism. It follows that the retraction map \( q \circ \phi \) depends only on the characteristic of the ground field, we may assume that the ground field \( k \) is algebraically closed. Denote by \( \phi_k \) the epimorphism \( k \otimes B(V) T(V) \rightarrow A_{\text{min}}(V) \). By Proposition 3.3
\[ \tilde{L}^k_p(V) = P_{p^k}(k \otimes B(V) T(V)) \]
for each \( k \geq 0 \). By equation (3.1), \( A_{\text{min}}(V) \) is functorial retract of \( k \otimes B(V) T(V) \). It suffices to show by induction that the retraction map
\[ \phi_k|\tilde{L}^k_p(V): \tilde{L}^k_p(V) = P_{p^k}(k \otimes B(V) T(V)) \rightarrow P_{p^k}A_{\text{min}}(V) \]
is an isomorphism for each \( k \geq 0 \). The statement holds for \( k = 0 \). Suppose that the statement holds for \( s \leq k \) with \( k > 0 \) and consider
\[ \phi_{k+1}|\tilde{L}^{k+1}_p(V): \tilde{L}^{k+1}_p(V) = P_{p^{k+1}}(k \otimes B(V) T(V)) \rightarrow P_{p^{k+1}}A_{\text{min}}(V). \]
By choosing \( V \) to a module over the Steenrod algebra with a basis given by \( \{v, P^1v, P^2v, \ldots, P^{p-2}v\} \), then \( V \) is an indecomposable module over the Steenrod algebra. By Lemma 3.2, \( \tilde{L}^{k+1}_p(V) \) is a suspension of \( V \) as a module over the Steenrod algebra. Thus \( \tilde{L}^{k+1}_p(V) \) is an indecomposable module over the Steenrod algebra. It follows that the retraction
\[ \phi_{k+1}|\tilde{L}^{k+1}_p(V): \tilde{L}^{k+1}_p(V) = P_{p^{k+1}}(k \otimes B(V) T(V)) \rightarrow P_{p^{k+1}}A_{\text{min}}(V) \]
is either identically zero or an isomorphism.

Suppose that \( \phi_{k+1} |_{L_p^{k+1}(V)} = 0 \). Then

\[
P_{p^{k+1}} A_{\min}^*(V) = 0
\]

and so

\[
L_{p^{k+1}}(V) \subseteq B_{\max}^*(V).
\]

By Lemma 3.2, \( B^{[k+1]}(V) \) is the sub Hopf algebra of \( T(V) \) generated by \( L_p^{k+1}(V) \) and \( L_n(V) \) for \( n \) not a power of \( p \). It follows that

\[
B^{[k+1]}(V) \subseteq B_{\max}^*(V)
\]

and so

\[
PA_{\min}^*(V) \subseteq \bigoplus_{s=1}^k \tilde{L}_p^s(V).
\]

From the induction, \( \tilde{L}_p^s(V) \subseteq A_{\min}^*(V) \) for each \( s \leq k \). Thus

\[
PA_{\min}^*(V) = \bigoplus_{s=1}^k \tilde{L}_p^s(V).
\]

It follows that there is a \((\text{non-functorial})\) isomorphism of coalgebra

\[
A_{\min}^*(V) \cong \bigotimes_{s=0}^k E(\tilde{L}_p^s(V)).
\]

By computing Poincaré series, the inequality (3.7) becomes the equality

\[
B^{[k+1]}(V) = B_{\max}^*(V).
\]

Thus

\[
Q_{p^{k+1}}^\max(V) = Q_{p^{k+1}} B_{\max}^*(V) = Q_{p^{k+1}} B^{[k+1]}(V) \cong \tilde{L}_p^{k+1}(V).
\]

Observe that the inclusion

\[
\tilde{L}_p^k(V) = P_{p^k} A_{\min}^*(V) \hookrightarrow A_{p^k}^*(V)
\]

is an isomorphism and \( \tilde{L}_p^{k+1}(V) = \tilde{L}_p(\tilde{L}_p^k(V)) \). The composite of natural transformations

\[
(3.8) \quad Q_{p^{k+1}}^\max \hookrightarrow L_{p^{k+1}} \xrightarrow{L_p(T_{p^k})} L_p(A_{p^k}^\min) \xrightarrow{L_p(A_{p^k}^\min)} \tilde{L}_p(A_{p^k}^\min)
\]

becomes an isomorphism by evaluating on the graded module \( V \). Write \( \tilde{V} \) for \( \tilde{L}_p^k(V) = P_{p^k} A_{\min}^*(V) = A_{p^k}^\min(V) \). By Lemma 3.3, the composite of the first three natural transformations in equation (3.8) admits a lifting of natural transformation into \( T_p(A_{p^k}^\min) \) via the epimorphism \( \beta_p : T_p(A_{p^k}^\min) \twoheadrightarrow L_p(A_{p^k}^\min) \) and so the composite

\[
q : T_p(\tilde{V}) = \tilde{V} \otimes_{p^k} L_p(\tilde{V}) \xrightarrow{\beta_p} L_p(\tilde{V}) \xrightarrow{\cong} \tilde{V} \otimes E_{p-1}(\tilde{V})
\]

admits a cross-section over the general linear group algebra \( k(\text{GL}(\tilde{V})) \) by forgetting the grading of \( \tilde{V} \). This is impossible according to Lemma 3.6.

Thus \( \phi_{k+1} |_{L_p^{k+1}(V)} \neq 0 \) and so \( \phi_{k+1} |_{L_p^{k+1}(V)} \) must be an isomorphism. The induction is finished and hence the result.
4. The Geometry of Natural Coalgebra Decompositions

4.1. Geometric Realizations. Denote by $\text{CoH}$ the category of $p$-local simply connected co-$H$-spaces of finite type and co-$H$-maps. Let $[\Omega, \Omega]_{\text{CoH}}$ be the group of natural transformations of the functor $\Omega$ from $\text{CoH}$ to the homotopy category of spaces.

**Theorem 4.1** (Geometric Realization Theorem). Let $Y$ be any simply connected co-$H$-space of finite type and let

$$T(V) \cong A(V) \otimes B(V)$$

any natural coalgebra decomposition for ungraded modules over $\mathbb{Z}/p$. Then there exist homotopy functors $\bar{A}$ and $\bar{B}$ from $\text{CoH}$ to spaces such that

1. there is a functorial decomposition

$$\Omega Y \simeq \bar{A}(Y) \times \bar{B}(Y)$$

2. On mod $p$ homology the decomposition

$$H_*(\Omega Y) \cong H_*(\bar{A}(Y)) \otimes H_*(\bar{B}(Y))$$

is with respect to the augmentation ideal filtration

3. On mod $p$ homology

$$E^0_*(\bar{A}(Y)) = A(\Sigma^{-1}\bar{H}_*(Y)) \text{ and } E^0_*(\bar{B}(Y)) = B(\Sigma^{-1}\bar{H}_*(Y)),$$

where $A$ and $B$ are the canonical extensions of the functors $A$ and $B$ for graded modules

For the functors $\bar{A}_{\text{min}}$ and $\bar{B}_{\text{max}}$, we have the following geometric realization theorem.

**Theorem 4.2.** There exist homotopy functors $\bar{A}_{\text{min}}$, $\bar{Q}_{\text{max}}$, $n \geq 2$, from $\text{CoH}$ to spaces such that for any $p$-local simply connected co-$H$ space $Y$ of finite type the following hold:

1) $\bar{Q}_{\text{max}}(Y) = \Omega\left(\bigvee_{n=2}^{\infty} \bar{Q}_n^{\text{max}}(Y)\right)$.

2) There is a functorial decomposition

$$\Omega Y \simeq \bar{A}_{\text{min}}(Y) \times \bar{B}_{\text{max}}(Y).$$

3) On mod $p$ homology the decomposition

$$H_*(\Omega Y) \cong H_*(\bar{A}_{\text{min}}(Y)) \otimes H_*(\bar{B}_{\text{max}}(Y))$$

is with respect to the augmentation ideal filtration.

4) On mod $p$ homology

$$E^0_*(\bar{A}_{\text{min}}(Y)) = A_{\text{min}}(\Sigma^{-1}\bar{H}_*(Y)),$$
$$E^0_*(\bar{B}_{\text{max}}(Y)) = B_{\text{max}}(\Sigma^{-1}\bar{H}_*(Y)),$$
$$E^0_*(\bar{Q}_{\text{max}}(Y)) = Q_{\text{max}}(\Sigma^{-1}\bar{H}_*(Y)).$$
4.2. **Suspension Splitting Theorems.** In this subsection, we review the suspension splitting theorems.

A *graded space* means a space $W$ with a homotopy decomposition

$$
\phi_W : W \xrightarrow{\sim} \bigvee_{n=1}^\infty W_n.
$$

For any graded space $W$, the homology $\hat{H}_*(W)$ is filtered by

$$
I^t \hat{H}_*(W) = \phi_*^{-1}(\hat{H}_*(\bigvee_{n=t}^\infty W_n))
$$

for $t \geq 1$. A *graded co-$H$ space* means a graded space $W$ such that $W$ is a co-$H$ space. Thus each factor $W_n$ is also a co-$H$ space. The following lemma gives a general criterion for decomposing the retracts of graded co-$H$ spaces in terms of grading factors.

**Lemma 4.3.** \([6]\) Let $W$ be a simply connected $p$-local graded co-$H$ space of finite type. Let $f : W \to W$ be a self-map such that on mod $p$ homology

1) $f_* : \hat{H}_*(W) \to \hat{H}_*(W)$ preserves the filtration.
2) The induced bigraded map $E^0 f_* : E^0 \hat{H}_*(W) \to E^0 \hat{H}_*(W)$ is an idempotent:

$$
E^0 f_* \circ E^0 f_* = E^0 f_* : E^0 \hat{H}_*(W) \to E^0 \hat{H}_*(W).
$$

Let $A(f) = \text{hocolim}_f W$ be the homotopy colimit and let $A_n(f) = \text{hocolim}_g W_n$, where $g_n$ is the composite

$$
g_n : W_n \hookrightarrow \bigvee_{k=1}^\infty W_k \xrightarrow{\phi_W^{-1}} W \xrightarrow{f} W \xrightarrow{W_k} \bigvee_{k=1}^\infty W_k \longrightarrow W_n.
$$

Then there is a canonical homotopy decomposition of the homotopy colimit

$$
A(f) \simeq \bigvee_{n=1}^\infty A_n(f)
$$

such that

$$
\hat{H}_*(A_n(f)) \cong \text{Im}(E^0_n f_* : E^0 \hat{H}_*(W) \cong \hat{H}_*(W_n) \to E^0 \hat{H}_*(W) \cong \hat{H}_*(W_n)).
$$

Let $X$ be any path-connected space. Let $\hat{H}_*(\Sigma \Omega X)$ be filtered by the powers of the augmentation ideal filtration. From the classical suspension splitting Theorem \([10]\)

$$
\phi : \Sigma \Omega \Sigma X \xrightarrow{\sim} \bigvee_{n=1}^\infty \Sigma X^{(n)},
$$

$\Sigma \Omega \Sigma X$ is a simply graded co-$H$ space and the filtration

$$
I^t \hat{H}_*(\Sigma \Omega \Sigma X) = \phi_*^{-1}(\hat{H}_*(\bigvee_{n=t}^\infty \Sigma X^{(n)}))
$$

coincide with (the suspension) of the augmentation ideal filtration of $H_*(\Sigma \Omega X)$.

Let $W$ and $W'$ be graded spaces. Then $W \wedge W'$ is a graded space with homotopy equivalence

$$
W \wedge W' \xrightarrow{\sim} \bigvee_{n=1}^\infty W_n \wedge \bigvee_{n=1}^\infty W'_n = \bigvee_{n=1}^{n-1} \bigvee_{i=1}^n W_i \wedge W'_{n-i}.
$$
and the homology $\bar{H}_*(W \wedge W') = \bar{H}_*(W) \otimes \bar{H}_*(W')$ is the tensor product of filtered modules. Moreover if $W$ is a graded co-$H$ space, then $W \wedge W'$ is a graded co-$H$ space.

Recall that any natural coalgebra retract $A(V)$ of $T(V)$ for ungraded modules admits the tensor length decomposition

$$A(W) = \bigoplus_{n=0}^{\infty} A_n(W)$$

for any graded or ungraded module $W$.

**Theorem 4.4** (Suspension Splitting Theorem [6]). Let $A(V)$ be any natural coalgebra retract of $T(V)$ for any ungraded modules $V$ and let $\bar{A}$ be the geometric realization of $A$. Then for any $p$-local simply connected co-$H$ space $Y$ of finite type and any $p$-local path-connected co-$H$-space $Z$, there is a functorial splitting

$$Z \wedge \bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} [Z \wedge \bar{A}(Y)]_n$$

such that

$$\bar{H}_*([Z \wedge \bar{A}(Y)]_n) \cong \bar{H}_*(Z) \otimes A_n(\Sigma^{-1} \bar{H}_* (Y))$$

for each $n \geq 1$. In particular, for a $p$-local simply connected co-$H$-space $Y$ there is a functorial suspension splitting

$$\Sigma \bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} \bar{A}_n(Y)$$

such that

$$\Sigma^{-1} \bar{H}_*(\bar{A}_n(Y)) \cong A_n(\Sigma^{-1} \bar{H}_*(Y))$$

for each $n \geq 1$. □

### 4.3. Hopf Invariants

In this section, we review the results in Hopf invariants from [6]. Let $A(V)$ be a natural coalgebra retract of $T(V)$ for ungraded modules. From the suspension splitting theorem (Theorem 4.4), there is a decomposition

$$\Sigma \bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} \bar{A}_n(Y)$$

and so it induces Hopf invariants

$$H_n: \bar{A}(Y) \longrightarrow \Omega(\bar{A}_n(Y)).$$

For computational purpose on homology, it is useful to make a particular choice of Hopf invariants $H_n$. Let

$$\mathcal{H}: \Sigma \Omega \Sigma X \simeq \Sigma J(X) \longrightarrow \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$$

be the fat James-Hopf invariants. Let $H_*(\Omega \Sigma X)$ be filtered by the products of the augmentation ideal and let $H_*(\bigvee_{n=1}^{\infty} X^{(n)})$ be filtered by

$$\bigoplus_{t \geq n} H_*(X^{(t)}).$$
By \[17\), Proposition 3.7, the isomorphism
\[
\mathcal{H}_* : H_*(\Sigma\Omega\Sigma X) \xrightarrow{\sim} H_*(\bigvee_{n=1}^{\infty} \Sigma X^{(n)})
\]
preserves the filtration. Note that the composite
\[
\Sigma\Omega\Sigma X \xrightarrow{\mathcal{H}} \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \xrightarrow{\text{proj}} \Sigma X^{(n)}
\]
is the James-Hopf invariant \(H_n\). Let \(Y \in \text{CoH}\) be a simply connected \(p\)-local co-
\(H\)-space of finite type with the cross-section map \(s_{\mu'} : Y \to \Sigma\Omega Y\). Let \(f\) be the
composite
\[
\Sigma\Omega\Sigma\Omega Y \xrightarrow{\Sigma\Omega\Sigma Y} \Sigma\Omega Y \xrightarrow{\Sigma\Omega Y} \Sigma\Omega\Sigma Y.
\]
By Lemma 4.3, the space \([\Sigma\Omega Y]_n\) is defined to be the homotopy colimit of the self
map \(g_n(Y)\) given by the composite
\[
\Sigma(\Omega Y)^{(n)} \xrightarrow{\Sigma\Omega(s_{\mu'} \circ \sigma)} \bigvee_{k=1}^{\infty} \Sigma(\Omega Y)^{(k)} \xrightarrow{\mathcal{H}^{-1}} \Sigma\Omega\Sigma\Omega Y \xrightarrow{\mathcal{H}} \bigvee_{k=1}^{\infty} \Sigma(\Omega Y)^{(k)}
\]
\text{Proposition 4.5.} For any \(Y \in \text{CoH}\), the composite
\[
\mathcal{H}^Y : \Sigma\Omega Y \xrightarrow{\Sigma\Omega Y} \Sigma\Omega\Sigma Y \xrightarrow{\Sigma\Omega Y} \bigvee_{n=1}^{\infty} \text{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)}
\]
is a homotopy equivalence. \(\square\)

Now define the \(n\)th Hopf invariant \(H_n^Y\) to be the adjoint map to the composite
\[
\Sigma\Omega Y \xrightarrow{\mathcal{H}^Y} \bigvee_{n=1}^{\infty} \text{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)} \xrightarrow{\text{proj}} \text{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)} = [\Sigma\Omega Y]_n.
\]
\text{Theorem 4.6.} \(\square\) For any \(Y \in \text{CoH}\), there is a commutative diagram
\[
\begin{array}{ccc}
E^0H_*([\Omega Y]) & \xrightarrow{E^0H_*^Y} & E^0H_*([\Sigma\Omega Y]_n) \\
\text{proj} & \cong & \text{proj} \\
T((\Sigma^{-1}H)_*(Y)) & \xrightarrow{H_n} & T((\Sigma^{-1}H)_*(Y))^n \\
\end{array}
\]
where \(H_n : T(V) \to T(V^{\otimes n})\) is the algebraic James-Hopf map. \(\square\)

5. Proof of Theorem \[13\]

5.1. Proof of Theorem \[13\] Now we give the proof of Theorem \[13\]. All spaces
are localized at \(p > 2\). For a natural coalgebra retract \(A\) of \(T\), let \(\tilde{A}\) be its geo-
metric realization. According to the suspension splitting theorem \[13\], there is a decomposition
\[
\Sigma\tilde{A}(Y) \simeq \bigvee_{n=1}^{\infty} \tilde{A}_n(Y).
\]
In particular, we have notations $\bar{A}^{\text{min}}$ and $\bar{A}^n_{\text{min}}$. Let $X$ be a path-connected finite complex. Define

$$b_X = \sum_{q=1}^{\infty} q \dim \bar{H}_q(X; \mathbb{Z}/p).$$

**Lemma 5.1.** Let $Y$ be a simply connected co-$H$-space such that $\bar{H}_{\text{odd}}(Y) = 0$ and $\dim \bar{H}_s(Y) = p - 1$. Then

$$\bar{A}^{\text{min}}_{p^k-1}(Y) \simeq S^{(b_Y - p + 1)k^{k+1}} + 1.$$

**Proof.** Let $V = \Sigma \bar{A}^{\text{min}}(Y)$. Then $V_{\text{even}} = 0$ and $\dim V = p - 1$. By Theorem 1.1

$$\bar{A}^{\text{min}}(V) = \bigotimes_{k=0}^{\infty} E(\bar{L}_p^k(V)).$$

By considering tensor length $p^k - 1$, we have

$$\bar{A}^{\text{min}}_{p^k-1}(V) = E_{p-1}(V) \otimes E_{p-1}(\bar{L}_p(V)) \otimes \cdots \otimes E_{p-1}(\bar{L}_p^{k-1}(V)),$$

which is a one-dimensional module. The assertion follows from Theorem 1.1. □

Let $\tilde{Q}_n^{\text{max}}$ the geometric realization of the indecomposables of $B^{\text{max}}(V)$ as given in Theorem 4.2. Note that $Y$ is a retract of $\Sigma \bar{A}^{\text{min}}(Y)$ by Theorem 4.4.

**Lemma 5.2.** Let $Y$ be any $p$-local simply connected co-$H$-space of finite type. Then

1. there is a splitting cofibre sequence

$$\tilde{Q}_n^{\text{max}}(Y) \to \bar{A}^{\text{min}}(Y) \wedge Y \to (\Sigma \bar{A}^{\text{min}}(Y))/Y.$$

Thus there is a decomposition $Y \wedge \bar{A}^{\text{min}}(Y) \simeq \tilde{Q}_n^{\text{max}}(Y) \vee (\Sigma \bar{A}^{\text{min}}(Y))/Y$.

2. There is a decomposition

$$[Y \wedge \bar{A}^{\text{min}}(Y)]_n \simeq \tilde{Q}_n^{\text{max}}(Y) \vee \bar{A}^n_{\text{min}}(Y).$$

3. If $Y = \Sigma X$, then

$$[Y \wedge \bar{A}^{\text{min}}(Y)]_n \simeq X \wedge \bar{A}^n_{\text{min}}(Y).$$

4. If $\bar{A}^{\text{min}}_{n-1}(Y) \simeq \Sigma Z$ for some $Z$, then

$$[Y \wedge \bar{A}^{\text{min}}(Y)]_n \simeq Y \wedge Z.$$

5. $\bar{A}^{\text{min}}_p(Y) \simeq [Y \wedge \bar{A}^{\text{min}}(Y)]_p$.

**Proof.** (1) and (2). Since $B^{\text{max}}(Y) \simeq \Omega \tilde{Q}^{\text{max}}(Y)$, there is a fibre sequence

$$\Omega Y \to \bar{A}^{\text{min}}(Y) \to \tilde{Q}^{\text{max}}(Y) \to Y.$$

There is a (right) action of $\Omega Y$ on $\bar{A}^{\text{min}}(Y)$ such that the diagram

$$\begin{array}{ccc}
\Omega Y \times \Omega Y & \to & \Omega Y \\
\partial \times \text{id}_{\Omega Y} & & \\
\downarrow & & \\
\bar{A}^{\text{min}}(Y) \times \Omega Y & \to & \bar{A}^{\text{min}}(Y)
\end{array}$$

splits. □
commutes. Consider the commutative diagram

\[
\begin{array}{cccccc}
\Sigma \Omega \bar{Q}^\text{max}(Y) & \xrightarrow{\sigma} & \bar{Q}^\text{max}(Y) \\
(\Omega Y) \wedge Y & \xrightarrow{id \wedge s} & \Sigma \Omega Y \wedge Y & \xrightarrow{H} & \Sigma \Omega Y & \xrightarrow{\Sigma \Omega \bar{A}^\text{min}(Y)) / Y}
\end{array}
\]

\[
\begin{array}{cccccc}
\bar{A}^\text{min}(Y) \wedge Y & \xrightarrow{id \bar{A}^\text{min}(Y) \wedge s} & \Sigma \bar{A}^\text{min}(Y) & \wedge \Sigma \Omega Y & \xrightarrow{H} & \Sigma \bar{A}^\text{min}(Y) & \xrightarrow{\Sigma \bar{A}^\text{min}(Y)) / Y}
\end{array}
\]

where \( H \) is the Hopf construction. Observe that the composite of the middle row is a homotopy equivalence. It follows that there is a sequence

\[\bar{Q}^\text{max}(Y) \xrightarrow{f} \bar{A}^\text{min}(Y) \wedge Y \xrightarrow{g} (\Sigma \bar{A}^\text{min}(Y)) / Y,\]

such that \( g \circ f \simeq * \), where \( g \) is the composite of the bottom rows and \( f \) is the composite of the maps from top row down to the middle right composing with the homotopy inverse from \( (\Sigma \Omega Y) / Y \) to \( \Omega Y \wedge Y \) and the left column map. Let \( C_f \) be the homotopy cofibre of the map \( f \). By Lemma 3.1, the resulting map \( C_f \rightarrow (\Sigma \bar{A}^\text{min}(Y)) / Y \) is a homotopy equivalence. Thus Sequence (5.1) is a cofibre sequence. Let \( r \) be the composite

\[r: \bar{A}^\text{min}(Y) \wedge Y \xrightarrow{\xi^\text{min} \wedge id_Y} \Omega Y \wedge Y \xrightarrow{H_0(id_Y \wedge s)} \Sigma \Omega Y \rightarrow \Sigma \Omega \bar{Q}^\text{max}(Y) \rightarrow \bar{Q}^\text{max}(Y).\]

By the proof of [22, Lemma 2.37], on homology

\[(r \circ f)_*: H_*(\bar{Q}^\text{max}(Y)) \longrightarrow H_*(\bar{Q}^\text{max}(Y))\]

is an isomorphism. Thus \( f: \bar{Q}^\text{max}(Y) \rightarrow \bar{A}^\text{min}(Y) \wedge Y \) admits a retraction and hence assertion (1). Let \( H_*(\Omega Y) \) be filtered by the products of the augmentation ideal. Then \( H_*(\bar{A}^\text{min}(Y)) \) and \( H_*(\bar{Q}^\text{max}(Y)) \) have the induced filtration. Observe that the maps \( f, g \) and \( r \) induces filtration preserving maps on homology. Thus the decomposition

\[Y \wedge \bar{A}^\text{min}(Y) \simeq \bar{Q}^\text{max}(Y) \lor (\Sigma \bar{A}^\text{min}(Y)) / Y\]

induces a graded decomposition and hence assertion (2).

(3) By Theorem 4.4

\[\Sigma \bar{A}^\text{min}(Y) \simeq \bigvee_{k=1}^{\infty} \bar{A}^\text{min}(Y).\]

Let \( s: \bar{A}^\text{min}_{n-1}(Y) \rightarrow \Sigma \bar{A}^\text{min}(Y) \) be the inclusion. Then composite

\[X \wedge \bar{A}^\text{min}_{n-1}(Y) \xrightarrow{id_X \wedge s} X \wedge \Sigma \bar{A}^\text{min}(Y) \simeq Y \wedge \bar{A}^\text{min}(Y) \simeq \bigvee_{k=2}^{\infty} [Y \wedge \bar{A}^\text{min}(Y)]_k \xrightarrow{\text{proj}} [Y \wedge \bar{A}^\text{min}(Y)]_n\]

induces an isomorphism on homology and hence assertion (3).
(4) Let \( s: Y \to \Sigma\Omega Y \) be a cross-section to the evaluation map \( \sigma: \Sigma\Omega Y \to Y \). Let \( f \) be the idempotent
\[
\Sigma\Omega Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{\cdot \land \text{id}_{\bar{A}^{\text{min}}(Y)}} Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{\Sigma\Omega Y \land \bar{A}^{\text{min}}(Y)} \Sigma\Omega Y \land \bar{A}^{\text{min}}(Y)
\]
and let \( g_n \) be the composite
\[
\Omega Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{f} \Sigma\Omega Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{\Sigma\Omega Y \land \bar{A}^{\text{min}}(Y)} \Omega Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{\text{proj}} \Omega Y \land \bar{A}^{\text{min}}(Y),
\]
where the top homotopy equivalence follows from assertion (3) with \( X = \Omega Y \). According to Lemma 4.3,
\[
[Y \land \bar{A}^{\text{min}}(Y)]_n \simeq \text{hocolim}_{g_n} \Omega Y \land \bar{A}^{\text{min}}(Y)
\]
with the canonical retraction
\[
r: \Omega Y \land \bar{A}^{\text{min}}(Y) \longrightarrow [Y \land \bar{A}^{\text{min}}(Y)]_n.
\]
Now the composite
\[
Y \land Z \xrightarrow{\cdot \land \text{id}_Z} \Sigma\Omega Y \land Z \simeq \Omega Y \land \bar{A}^{\text{min}}(Y) \xrightarrow{r} [Y \land \bar{A}^{\text{min}}(Y)]_n
\]
induces an isomorphism on homology and so it is a homotopy equivalence. Assertion (4) follows.

(5). According to [16, Section 11.2, p.97], \( Q^\text{max}_p(V) = 0 \). Thus
\[
\bar{H}_s(Q^\text{max}_p(Y)) = 0
\]
and so \( \bar{Q}^\text{max}_p(Y) \simeq * \). By assertion (2),
\[
\bar{A}^{\text{min}}_p(Y) \simeq [Y \land \bar{A}^{\text{min}}(Y)]_p
\]
and hence the result. □

Lemma 5.3. Let \( Y \) be a simply connected co-H-space such that \( \bar{H}_\text{odd}(Y) = 0 \) and \( \dim \bar{H}_s(Y) = p - 1 \). Then
\[
\bar{A}^{\text{min}}_p(Y) \simeq \Sigma^{bv-p+1} Y.
\]
Proof. By Lemma 5.1, \( \bar{A}^{\text{min}}_{p-1} = S^{bv-p+2} \). By Lemma 5.2 (4),
\[
[Y \land \bar{A}^{\text{min}}(Y)]_p \simeq \Sigma^{bv-p+1} Y.
\]
The result then follows from Lemma 5.2 (5). □

Proof of Theorem 1.5 (1) and (2). Let
\[
H^Y: \Omega Y \longrightarrow \Omega(\Sigma\Omega Y)
\]
be the James-Hopf invariant. By Lemma 5.3
\[
\bar{A}^{\text{min}}_p(Y) \simeq \Sigma^{bv-p+1} Y.
\]
Define the map \( H_p \) by the composite
\[
\xymatrix{\bar{A}^{\text{min}}(Y) \ar[r] & \Omega(\bar{A}^{\text{min}}_p(Y)) \ar[r]^{H^Y_p} & \Omega(\Sigma\Omega Y) \ar[r] & \bar{A}^{\text{min}}(\Sigma\bar{A}^{\text{min}}_p(Y)) = \bar{A}^{\text{min}}_p(\Sigma^{bv-p+1} Y).}
\]
Let \( \tilde{E}(Y) \) be the homotopy fibre of the map \( H_p \). Then there is a fibre sequence

\[
\tilde{E}(Y) \longrightarrow A^{\min}(Y) \xrightarrow{H_p} A^{\min}(\tilde{A}_p(Y)).
\]

We compute the homology of \( \tilde{E}(Y) \). Let \( V = \Sigma^{-1} \tilde{H}_*(Y) \). According to Theorem 4.6

\[
\tilde{E}_0^0 H_p \ast = H_p : T(V) = H_0(\Omega V) \longrightarrow T(V^\otimes p) = E^0 H_*(\Omega(\Sigma \Omega Y_p))
\]

is the usual algebraic James-Hopf map. Thus there is a commutative diagram

\[
\begin{array}{cccc}
T(V) & \xrightarrow{H_p} & T(V^\otimes p) \\
\downarrow & & \downarrow \\
E^0 H_*(A^{\min}(Y)) & \xrightarrow{E^0 H_p \ast} & E^0 H_*(\tilde{A}_p^{\min}(Y))
\end{array}
\]

where \( H_p \) is the algebraic James-Hopf map. By Theorem 1.1 and Lemma 3.2 the primitives of \( A^{\min}_p(V) \) is given by

\[
PA^{\min}_p(V) = \bigoplus_{k=0}^{\infty} \tilde{L}_p^k(V).
\]

Let \( \tilde{L} \) be the sub Lie algebra of \( L(V) \) generated by \( L_p(V) \). Note that

\[
\tilde{L}_p^k(V) \subseteq \tilde{L}
\]

for \( k \geq 1 \). By [21, Theorem 1.1], the James-Hopf map

\[
H_p|_{\tilde{L}_p} : \tilde{L}_p \longrightarrow PT(V^\otimes p)
\]

is a morphism of Lie algebras. Since

\[
H_p|_{L_p} : L_p \rightarrow V^\otimes p
\]

is the canonical inclusion,

\[
H_p(\tilde{L}_p^{k+1}(V)) = \tilde{L}_p^k(L_p(V))
\]

for each \( k \geq 0 \). Note that \( \tilde{L}_p(V) \subseteq A^{\min}_p(V) \) and, since \( \dim V = p - 1 \),

\[
\dim \tilde{L}_p(V) = \dim A^{\min}_p(V) = p - 1.
\]

Thus \( \tilde{L}_p(V) = A^{\min}_p(V) \). It follows that

(5.3) \[ PE^0 H_p : PE^0 H_*(A^{\min}_p(Y)) \longrightarrow PE^0 H_*(\tilde{A}_p^{\min}(Y)) \]

is onto with the kernel \( L^0_p(V) = V \).

For any graded Hopf algebra \( A \), let \( A^{ab} \) be the abelianization of \( A \) and let

\[
\phi : A \longrightarrow A^{ab}
\]

be the quotient map. Define \( \tilde{\phi} \) to be the composite

\[
H_*(A^{\min}_p(Y)) \xrightarrow{\phi} H_*(\Omega Y) \xrightarrow{\phi} H_*(\Omega Y)^{ab}.
\]

Since \( H_*(\Omega Y) \) is generated by odd dimensional elements, \( H_*(\Omega Y)^{ab} \) is an exterior algebra. Moreover \( E^0 H_*(\Omega Y)^{ab} \) is primitively generated exterior algebra. Thus

(5.4) \[ E^0 H_*(\Omega Y)^{ab} = E(V). \]
Consider the map
\[ \theta: H_*(\tilde{A}^{\min}(Y)) \xrightarrow{\psi} H_*(\tilde{A}^{\min}(Y)) \otimes H_*(\tilde{A}^{\min}(Y)) \]
(5.5)

From equations (5.3) and (5.4),
\[ E^0\theta: PE^0H_*(\tilde{A}^{\min}(Y)) \rightarrow PE^0(H_*(\Omega Y)^{ab} \otimes H_*(\tilde{A}^{\min}(A^p_Y(Y)))) \]
is an isomorphism. Thus
\[ E^0\phi: E^0H_*(\tilde{A}^{\min}(Y)) \rightarrow E^0(H_*(\Omega Y)^{ab} \otimes H_*(\tilde{A}^{\min}(A^p_Y(Y)))) \]
is a monomorphism. Since both sides have the same Poincaré series, \( E^0\phi \) is an isomorphism and so \( \phi \) is an isomorphism. Thus there is an algebra isomorphism on cohomology
\[ H^*(\tilde{A}^{\min}(Y)) \cong H^*(\tilde{A}^{\min}(A^p_Y(Y))) \otimes (H_*(\Omega Y)^{ab})^*. \]

By the Eilenberg-Moore spectral sequence, there is an epimorphism
\[ H_*(\Omega Y)^{ab} \rightarrow H_*(\tilde{E}(Y)) \]
and so the Poincaré series
\[ \chi(H_*(\tilde{E}(Y))) \leq \frac{\chi(H_*(\tilde{A}^{\min}(Y)))}{\chi(H_*(A^{\min}(A^p_Y(Y))))}. \]
By applying the Serre spectral sequence to the fibre sequence
\[ \tilde{E}(Y) \rightarrow \tilde{A}^{\min}(Y) \rightarrow \tilde{A}^{\min}(A^p_Y(Y)), \]
we have the \( E^2 \)-terms given by
\[ H_*(\tilde{E}(Y)) \otimes H_*(\tilde{A}^{\min}(A^p_Y(Y))) \]
and so
\[ \chi(H_*(\tilde{A}^{\min}(Y))) \leq \chi(H_*(\tilde{E}(Y))) \chi(H_*(\tilde{A}^{\min}(A^p_Y(Y)))) . \]
It follows that
\[ \chi(H_*(\tilde{E}(Y))) = \frac{\chi(H_*(\tilde{A}^{\min}(Y)))}{\chi(H_*(A^{\min}(A^p_Y(Y))))} \]
and so
\[ H_*(\Omega Y)^{ab} \cong H_*(\tilde{E}(Y)). \]
This finishes the proof of assertion (1) and (2).

(3) Let \( \tilde{Y} \) be the homotopy cofibre of the map \( f: S^n \rightarrow Y \). Since \( f \) is a co-\( H \)-map, \( \tilde{Y} \) is a co-\( H \)-space. By the naturality of \( \tilde{A}^{\min} \), the pinch map \( q: Y \rightarrow \tilde{Y} \) induces a map
\[ \tilde{A}^{\min}(Y) \rightarrow \tilde{A}^{\min}(\tilde{Y}). \]
Let \( V' = \Sigma^{-1} \text{Im}(f_\#: H_*(S^n) \rightarrow H_*(Y)) \) and let \( \tilde{V} = \Sigma^{-1} H_*(\tilde{Y}) \cong V/V' \). Since \( \dim \tilde{V} = p - 2 \),
\[ E^0H_*(\tilde{A}^{\min}(\tilde{Y})) = E(\tilde{V}) \]
by [16] Corollary 11.6. From the proof of assertions (1) and (2), the composite
\[ H_*(\tilde{E}(Y)) \xrightarrow{\psi} H_*(\tilde{A}^{\min}(Y)) \xrightarrow{\theta} H_*(\Omega Y) \rightarrow H_*(\Omega Y)^{ab} \]
is an isomorphism. Let \( g \) be the composite
\[ \tilde{E}(Y) \rightarrow \tilde{A}^{\min}(Y) \rightarrow \tilde{A}^{\min}(\tilde{Y}). \]
Consider the commutative diagram
\[
\begin{array}{c}
\cdots \rightarrow H_*(\bar{E}(Y)) \xrightarrow{g_*} H_*(\bar{A}_{\min}^*(Y)) \xrightarrow{\gamma} H_*(\bar{A}_p(Y)) \xrightarrow{\gamma} H_*(\Omega Y) \rightarrow \cdots
\end{array}
\]

where the composites in the top and bottom rows are isomorphisms. Thus
\[
g_* : H_*(\bar{E}(Y)) \rightarrow H_*(\bar{A}_{\min}^*(Y))
\]
is onto. By [12, Proposition 4.20], both \(H^*_*(\Omega Y)\) and \(H^*_*(\Omega \bar{Y})\) are primitively generated because they are commutative with trivial restricted maps. Thus
\[
H^*_*(\Omega \bar{Y}) \cong E(V) \text{ and } H^*_*(\Omega \bar{Y}) \cong E(V) \cong E(\bar{V}) \otimes E(V')
\]
as Hopf algebras. From diagram 5.8 the map \(g_*\) induces a coalgebra decomposition
\[
H_*(\bar{E}(Y)) \cong H_*(\bar{A}_{\min}^*(Y)) \otimes E(V').
\]
It follows that the homotopy fibre of the map \(g : \bar{E}(Y) \rightarrow \bar{A}_{\min}^*(Y)\) is \(S^{n-1}\) by using the Eilenberg-Moore spectral sequence. Thus there is a homotopy commutative diagram of fibre sequences
\[
\begin{array}{c}
\cdots \rightarrow \Omega \bar{A}_{\min}^*(\bar{A}_p^*(Y)) \xrightarrow{P_f} S^{n-1} \xrightarrow{B_f} \bar{A}_{\min}^*(\bar{A}_p^*(Y)) \rightarrow \cdots
\end{array}
\]
and hence the result.

5.2. **Self-maps of \(\bar{A}_{\min}^*(Y)\).** Let \(Y\) be a simply connected co-\(H\)-space of \((p-1)\)-cell complex with the cells in even dimensions. In this case, the integral homology of \(Y\) is torsion free and so we can take homology \(H_*(Y)\) over \(p\)-local integers. Let \(f \) be any self-map of \(\bar{A}_{\min}^*(Y)\). Let \(V = \Sigma^{-1} H_*(Y)\) with a basis \(\{x_{i_1}, \ldots, x_{i_{p-1}}\}\), where the degree \(|x_{i_i}| = l_i\) with \(l_1 \leq \cdots \leq l_{p-1}\). By Theorem 1.5 there is a coalgebra isomorphism
\[
H_*(\bar{A}_{\min}^*(Y)) \cong \bigotimes_{k=0}^\infty H_*(\bar{E}(\bar{A}_{p_p}^*(Y))) \cong \bigotimes_{k=0}^\infty E(\bar{A}_{p_p}^*(V)) \cong \bar{A}_{\min}^*(V).
\]
Thus the primitives
\[
PH_*(\bar{A}_{\min}^*(Y)) = \bigoplus_{k=0}^\infty \bar{A}_{p_p}^*(V)
\]
and so on cohomology

$$QH^*(\bar{A}^\text{min}(Y)) = \bigoplus_{k=0}^\infty A^\text{min}_{p^k}(V)^*.$$  

Note that $A^\text{min}_{p^k}(V)$ has a basis in the form

$$w_{p^k,i} = [(x_{l_1} \wedge \cdots \wedge x_{l_{p-1}})^{k-1} x_{l_i}]$$

for $1 \leq i \leq p-1$. Let $f^*_k$ be the composite

$$A^\text{min}_{p^k}(V)^* \xhookrightarrow{\sim} QH^*(\bar{A}^\text{min}(Y)) \xrightarrow{f^*} QH^*(\bar{A}^\text{min}(Y)) \xrightarrow{\text{proj}} A^\text{min}_{p^1}(V)^*.$$  

Note that $|w_{p^k,p-1}| < |w_{p^{k'},1}|$ for $k < k'$. Thus

\begin{equation}
(5.9) \quad f^*_k = 0 \text{ for } k \neq l
\end{equation}

and so

\begin{equation}
(5.10) \quad f^*(A^\text{min}_{p^k}(V)^*) \subseteq A^\text{min}_{p^k}(V)^*.
\end{equation}

It follows that

$$f^*: A^\text{min}(V)^* \cong H^*(\bar{A}^\text{min}(Y)) \xrightarrow{f^*} H^*(\bar{A}^\text{min}(Y)) = A^\text{min}(V)^*$$

is a graded map with respect to the tensor length. Since $\dim A^\text{min}_{p^k-1}(V) = 1$, let $\deg^k(f)$ be the degree of the map

$$f^*: A^\text{min}_{p^k-1}(V)^* \xrightarrow{\sim} A^\text{min}_{p^k-1}(V)^*.$$

**Proposition 5.4.** Let $f: \bar{A}^\text{min}(Y) \to \bar{A}^\text{min}(Y)$ be any map. Then $\deg^k(f)$ over $p$-local integers is invertible if and only if

$$f_*: H_j(\bar{A}^\text{min}(Y);\mathbb{Z}/p) \xrightarrow{\sim} H_j(\bar{A}^\text{min}(Y);\mathbb{Z}/p)$$

is an isomorphism for $j < (l_1 + \cdots + l_{p-1}) \cdot \frac{k-1}{p^k-1} + l_1$.

**Proof.** The assertion follows from that

$$\deg^k(f) = \prod_{j=0}^{k-1} \det \left( f^*: A^\text{min}_{p^j}(V)^* \xrightarrow{\sim} A^\text{min}_{p^j}(V)^* \right)$$

for cohomology with coefficients over $p$-local integers. \qed

Since $\bar{A}^\text{min}(Y)$ is an $H$-space, there is a product $f \ast g$ for any self maps $f$ and $g$ of $\bar{A}^\text{min}(Y)$.

**Proposition 5.5.** Let $f$ and $g$ be any self maps of $\bar{A}^\text{min}(Y)$. Then

$$\deg^k(f \ast g) = \prod_{j=0}^{k-1} \det \left( f_* + g_*: A^\text{min}_{p^j}(V) \xrightarrow{\sim} A^\text{min}_{p^j}(V) \right).$$

**Proof.** Note that

$$(f \ast g)_*(w) = f_*(w) + g_*(w)$$

for any primitive element $w$. The assertion follows. \qed
6. Proof of Theorem 1.6

Lemma 6.1. Let \( Y \) be a \( p \)-local simply connected co-H-space such that \( H_{\text{od}}(Y) = 0 \) and \( \dim H_s(Y) = p - 1 \).

(1) If \( Y \) admits a nontrivial decomposition, then the EHP fibration splits off.

(2) If \( Y \) is atomic, then \( \tilde{E}(Y) \) is also atomic.

Proof. (1). Suppose that \( Y \cong Y_1 \lor Y_2 \) be a nontrivial decomposition. Since \( \dim H_s(Y_i) < p - 1, H_*(\tilde{A}^\text{min}(Y_1)) = E(\Sigma^{-1}H_*(Y_i)) \). Let \( p_i \) be the composite

\[
\tilde{A}^\text{min}(Y) \to \Omega Y \xrightarrow{\text{proj}} \Omega Y_i \to \tilde{A}^\text{min}(Y_1).
\]

Then the composite

\[
\tilde{E}(Y) \xrightarrow{E} \tilde{A}^\text{min}(Y) \xrightarrow{(p_1,p_2)} \tilde{A}^\text{min}(Y_1) \times \tilde{A}^\text{min}(Y_2)
\]

is a homotopy equivalence as it induces an isomorphism on homology. Assertion (1) follows.

(2). Let \( f : \tilde{E}(Y) \to \tilde{E}(Y) \) be any self map inducing an isomorphism on the bottom cell. Let \( V = \Sigma^{-1}H_*(Y) \). By Theorem 1.3 \( H_*(\tilde{E}(Y)) = E(V) \) as coalgebras. Then

\[
Pf_* : V = PE(V) \longrightarrow V = PE(V)
\]

is an isomorphism on the bottom cell. From the suspension splitting theorem (Theorem 1.4),

\[
\Sigma \tilde{E}(Y) \cong \bigwedge_{n=1}^{p-1} \tilde{A}^\text{min}_n(Y)
\]

with \( \Sigma^{-1}H_*(\tilde{A}^\text{min}_n(Y)) = A^\text{min}_n(Y) \). Now the composite

\[
g : Y = \tilde{A}^\text{min}_1(Y) \to \Sigma \tilde{E}(Y) \xrightarrow{\Sigma f} \Sigma \tilde{E}(Y) \cong \bigwedge_{n=1}^{p-1} \tilde{A}^\text{min}_n(Y) \longrightarrow \tilde{A}^\text{min}_1(Y) = Y
\]

is a homotopy equivalence because on homology \( \Sigma^{-1}g_* = Pf_* : V \to V \) is an isomorphism on the bottom cell. It follows that \( Pf_* : V \to V \) is an isomorphism and so \( f_* : E(V) \to E(V) \) is an isomorphism. Thus \( f \) is a homotopy equivalence and hence the result. \( \square \)

Let \( H_*(\Omega Y) \) be filtered by the products of the augmentation ideal. For spaces \( X \) and \( Z \) such that \( H_*(X) \) and \( H_*(Z) \) are filtered, a map \( f : X \to Z \) is called filtered of \( f_* : H_*(X) \to H_*(Z) \) is a filtered map. A retract \( X \) of \( \Omega Y \) is called filtered if there is a self map \( f \) of \( \Omega Y \) such that \( X \cong \text{hocolim}_f \Omega Y \). Note that for each filtered retract \( X \) of \( \Omega Y \) there is a filtration on \( H_*(X) \) induced from the filtration of \( H_*(\Omega Y) \). Consider all possible filtered retracts of \( \Omega Y \). Divide them in two types: a filtered retract \( X \) of \( \Omega Y \) is called of type \( A \) if \( X \) contains the bottom cells of \( \Omega Y \), otherwise it is called of type \( B \).

Lemma 6.2. For any \( p \)-local simply connected co-H-space \( Y \), there exists a minimal filtered retract \( X^\text{min}(Y) \) of \( \Omega Y \) of type \( A \) such that: (1) \( X^\text{min}(Y) \) is a filtered retract of \( \Omega Y \) of type \( A \) and (2) any filtered retract of \( \Omega Y \) of type \( A \) is a filtered retract of \( X(Y) \). Moreover there is a filtered decomposition

\[
\Omega Y \cong X^\text{min}(Y) \times \Omega Q.
\]
Proof. Let $S$ be the set of all filtered retract of $\Omega Y$ of type $A$. Define a partial order on $S$ by setting $X \leq X'$ if $X$ is a filtered retract of $X'$. From $X_1, X_2 \in S$, let $f$ be the composite

$$\Omega Y \to X_1 \to \Omega Y \to X_2 \to \Omega Y.$$ 

Let $X_3 = \hocolim_f \Omega Y$. Then $X_3$ is common filtered retract of $X_1$ and $X_2$. Namely $X_3 \leq X_1$ and $X_3 \leq X_2$. By Zorn Lemma, $S$ has the minimal element $X^\min(Y)$. Consider the commutative diagram

$$\begin{array}{cccccc}
\Omega X^\min(Y) & \to & X^\min(Y) & \to & H \to & \Sigma X^\min(Y) \\
\Omega j & & \downarrow & & \downarrow & \\
\Omega Y & \to & X^\min(Y) & \to & Q \to & Y
\end{array}$$

where $j$ is the adjoint map of the retraction $\Omega Y \to X^\min(Y)$. Then the composite

$$X^\min(Y) \to \Omega Y \to X^\min(Y)$$

is a homotopy equivalence by the minimal assumption of $X^\min(Y)$. It follows that there is a filtered decomposition

$$\Omega Y \simeq X^\min(Y) \times \Omega Q$$

and hence the result. \hfill \Box

Proof of Theorem 1.6. (1) $\implies$ (2). If EHP fibration splits off, then $E(Y)$ is a retract of $A^\min(Y)$. Note that $A^\min(Y)$ is an $H$-space as it is a retract of $\Omega Y$. Thus $E(Y)$ is an $H$-space.

(2) $\implies$ (3). From the suspension splitting theorem (Theorem 1.4),

$$\Sigma A^\min(Y) \simeq \bigvee_{n=1}^{\infty} A^\min_n(Y)$$

with $A^\min_1(Y) = Y$ and $\Sigma^{-1} \bar{H}_s(A^\min_n(Y)) = A^\min_n(\Sigma^{-1} \bar{H}_s(Y))$. The composite

$$\Sigma E(Y) \to \Sigma A^\min(Y) \overset{\text{proj}}{\to} \bigvee_{n=1}^{p-1} A^\min_n(Y)$$

is a homotopy equivalence because $H_*(\bar{E}(Y)) \to H_*(A^\min(Y))$ is a monomorphism and the connectivity of $\bigvee_{n=p}^{\infty} A^\min_n(Y)$ is greater than the dimension of $\Sigma E(Y)$. Thus $Y$ is a retract of $\Sigma E(Y)$. By Theorem 1.5 $H^*(E(Y)) = E(\Sigma^{-1} \bar{H}^*(Y))$ and so $E(Y)$ is an $H$-space with $Y$ as a retractile generating complex.

(3) $\implies$ (1). Let $X$ be any $H$-space having $Y$ as a retractile generating complex. Since $Y$ is a retract of $X$, let $s: Y \to \Sigma X$ be an inclusion and let $r: \Sigma X \to Y$ be the retraction such that $H^*(X)$ is generated by $M = r^*(\Sigma^{-1} \bar{H}_s(Y))$ with $M \cong QH^*(X)$. Let $V = \Sigma^{-1} \bar{H}_s(Y)$. Since $H_{\text{odd}}(Y) = 0$, $V_{\text{even}} = 0$. Note that $H_s(X)$ is a quasi-Hopf algebra as $X$ is an $H$-space. Thus $H^*(X)$ is the exterior algebra generated by $V^*$ by the Borel Theorem [12 Theorem 7.11]. There is a homotopy
commutative diagram of fibre sequences

\[
\begin{array}{ccccccc}
\Omega \Sigma X & \xrightarrow{\tilde{\partial}} & X & \xrightarrow{\partial} & X \wedge X & \xrightarrow{H} & \Sigma X \\
\Omega s & & \downarrow & & \downarrow & & \text{pull} s \\
\Omega Y & \xrightarrow{\partial} & X & \xrightarrow{\cdot} & B & \xrightarrow{\cdot} & Y,
\end{array}
\]

where \(H: \Sigma X \wedge X \to \Sigma X\) is the Hopf fibration for the \(H\)-space \(X\). Let \(r': X \to \Omega Y\) be the adjoint map of \(r\). Since the dimension of \(X\) less than the connectivity of \(A^{\text{min}}(\Sigma^{b_Y-p+1}Y)\), the composite

\[
X \xrightarrow{r'} \Omega Y \longrightarrow A^{\text{min}}(Y)
\]

lifts to the fibre \(\tilde{E}(Y)\) of the EHP-fibration. Let \(r'': X \to \tilde{E}(Y)\) be a lifting of the above composite. Consider the composite

\[
\begin{array}{ccccccc}
\phi: \Omega Y & \xrightarrow{\partial} & X & \xrightarrow{r''} & \tilde{E}(Y) & \xrightarrow{E} & A^{\text{min}}(Y) & \subset & \Omega Y \\
j & & \downarrow j & & \downarrow j & & \downarrow j & & \downarrow j \\
\bigvee_{\alpha} S^n & \xrightarrow{\cdot} & \bigvee_{\alpha} S^n & \xrightarrow{j} & \bigvee_{\alpha} S^n & \xrightarrow{j} & \bigvee_{\alpha} S^n & \xrightarrow{j} & \bigvee_{\alpha} S^n,
\end{array}
\]

where \(j\) is the inclusion of the bottom cells. Let

\[
\tilde{X} = \text{hocolim}_\phi \Omega Y.
\]

Then \(\tilde{X}\) is a common retract of \(X, \tilde{E}(Y), A^{\text{min}}(Y)\) and \(\Omega Y\), which contains the bottom cell. Since \(Y\) is atomic, \(\tilde{E}(Y)\) is also atomic by Lemma 6.1. Thus the retraction

\[
\tilde{E}(Y) \xrightarrow{E} A^{\text{min}}(Y) \xrightarrow{j} \Omega Y \xrightarrow{\text{hocolim}_\phi \Omega Y} \text{hocolim}_\phi \Omega Y = \tilde{X}
\]

is a homotopy equivalence. It follows that the EHP-sequence

\[
\tilde{E}(Y) \xrightarrow{E} A^{\text{min}}(Y) \xrightarrow{H_p} A^{\text{min}}(\Sigma^{b_Y-p+1}Y)
\]

admits a retraction \(A^{\text{min}}(Y) \to \tilde{E}(Y)\) and so the EHP fibration splits off.

Now \((1) \implies (4)\) and \((4) \implies (5)\) are obvious. \((5) \implies (6)\) follows from the homotopy commutative diagram

\[
\begin{array}{ccccccc}
\Omega A^{\text{min}}(\Sigma^{b_Y-p+1}Y) & \xrightarrow{P} & \tilde{E}(Y) & \xrightarrow{E} & A^{\text{min}}(Y) & \xrightarrow{H_p} & A^{\text{min}}(\Sigma^{b_Y-p+1}Y) \\
& & \downarrow g & & \downarrow g & & \downarrow g \\
\Sigma^{b_Y-p+1}Y & \xrightarrow{\cdot} & \Sigma^{b_Y-p}Y & \xrightarrow{\cdot} & \Sigma^{b_Y-p}Y & \xrightarrow{\cdot} & \Sigma^{b_Y-p}Y,
\end{array}
\]

where the bottom row is the cofibre sequence.
(6) \implies (3). Consider the commutative diagram

\[
\begin{array}{ccc}
\bar{A}^\text{min}(\Sigma^{by-p+1}Y) & \xrightarrow{g} & \Omega \Sigma^{by-p+1}Y \\
\downarrow & & \downarrow \\
\Sigma^{by-p}Y & \xrightarrow{g} & \bar{A}^\text{min}(Y),
\end{array}
\]

where \(g\) is the H-map induced by \(g\). Let \(\bigvee_\alpha S^q\) be the bottom cells of \(\Sigma^{by-p}Y\). Observe that

\[
H_p: H_q(\bar{A}^\text{min}(Y)) \xrightarrow{\sim} H_q(\bar{A}^\text{min}(\Sigma^{by-p+1}Y)).
\]

Thus there is a commutative diagram

\[
\begin{array}{ccc}
\theta: \Omega Y & \xrightarrow{H_p} & \bar{A}^\text{min}(Y) \\
\downarrow & & \downarrow \\
\bigvee_\alpha S^q & \xrightarrow{j} & \bigvee_\alpha S^q,
\end{array}
\]

where \(j\) is the inclusion of the bottom cells.

Let \(H_*(\Omega Y)\) be filtered by the products of the augmentation ideal. Then \(H_*(\bar{A}(Y))\) has the induced filtration. Observe that

\[
g_*(\tilde{H}_*(\Sigma^{by-p}Y)) \subseteq I^pH_*(\bar{A}^\text{min}(Y)) \subseteq I^pH_*(\Omega Y)
\]

because the top dimension of \(H_*(\tilde{E}(Y)) = \oplus_{n=1}^{p-1} A^\text{min}(\Sigma^{-1}Y)\) is less than the connectivity of \(\Sigma^{by-p}Y\). It follows that

\[
\tilde{g}_*: H_*(\Sigma^{by-p+1}Y) = H_*(J(\Sigma^{by-p}Y)) \longrightarrow H_*(\Omega Y)
\]

preserves the filtration. Thus \(\theta\) is a filtered map because the other factors of \(\theta\) are filtration-preserving maps on homology. Let

\[Z = \text{hocolim}_\theta \Omega Y\]

be the homotopy colimit. Then \(Z\) is a common filtered retract of \(\Omega Y\), \(\bar{A}^\text{min}(Y)\), \(\bar{A}^\text{min}(\Sigma^{by-p+1}Y)\), and \(\Sigma \Omega^{by-p+1}Y\). By Lemma 4.3, \(\Sigma Z\) is a common filtered retract of \(\Sigma \Omega Y\), \(\Sigma \bar{A}^\text{min}(Y)\), \(\Sigma \bar{A}^\text{min}(\Sigma^{by-p+1}Y)\), and \(\Sigma \Omega \Sigma^{by-p+1}Y\). Thus \([\Sigma Z]_p\) is a retract of

\[
[\Sigma \bar{A}^\text{min}(\Sigma^{by-p+1}Y)]_p = \Sigma^{by-p+1}Y
\]

containing the bottom cells by diagram (6.4). It follows that

\[
[\Sigma Z]_p = [\Sigma \bar{A}^\text{min}(\Sigma^{by-p+1}Y)]_p = \Sigma^{by-p+1}Y
\]

because \(\Sigma^{by-p+1}Y\) is atomic. Thus \(Z\) contains \(\Sigma^{by-p}Y\) as the bottom piece.

Consider the filtered decomposition

\[\Omega Y \simeq \bar{B}^\text{max}(Y) \times \bar{A}^\text{min}(Y)\]

Since \(Z\) is a filtered retract of \(\bar{A}^\text{min}(Y)\), there is a further decomposition

\[\Omega Y \simeq \bar{B}^\text{max}(Y) \times Z \times A',\]
where $A'$ is a filtered retract of $\Omega Y$ of type $A$. By Lemma 6.2, $A'$ admits a further filtered decomposition

$$A' \simeq B' \times X^{\text{min}}(Y).$$

This gives a filtered decomposition

$$(6.6) \quad \Omega Y \simeq \tilde{B}^{\text{max}}(Y) \times Z \times B' \times X^{\text{min}}(Y)$$

and so the space $Q$ given in Lemma 6.2 has the property that

$$(6.7) \quad \Omega Q \simeq \tilde{B}^{\text{max}}(Y) \times Z \times B'.$$

On homology

$$(6.8) \quad P^E_0 H_*(Z) \oplus P^E_0 H_*(\tilde{B}^{\text{max}}(Y)) \subseteq P^E_0 H_*(\Omega Q).$$

Let $V = \Sigma^{-1} H_*(Y)$. From equation (6.5),

$$(6.9) \quad \tilde{L}_p(V) = \Sigma_p^{\text{min}}(V) = \Sigma^{-1} H_*(\Sigma_p^{\text{min}}(\Sigma Y - p + 1 Y)) \subseteq P^E_0(\Omega Q).$$

By equation (6.8),

$$p-1 \bigoplus_{n=2}^\infty Q_n^{\text{max}}(V) \oplus \tilde{L}_p(V) \subseteq P H_*(\Omega Q).$$

Since $H_*(\Omega Q)$ is a Hopf algebra, the sub Hopf algebra

$$(6.10) \quad B = \bigoplus_{n=2}^{p-1} Q_n^{\text{max}}(V) \oplus \tilde{L}_p(V) \subseteq H_*(\Omega Q).$$

By Lemma 3.2, $B = B^{[1]}(V)$ with a short exact sequence of Hopf algebras

$$B \longrightarrow T(V) \longrightarrow E(V).$$

It follows that the Poincaré series

$$\chi(H_*(\Omega Q)) \geq \chi(B) = \frac{\chi(T(V))}{\chi(E(V))}.$$ 

From the decomposition

$$\Omega Y \simeq \Omega Q \times X^{\text{min}}(Y),$$

the Poincaré series

$$\chi(H_*(X^{\text{min}}(Y))) \leq \chi(E(V)).$$

Since $X^{\text{min}}(Y)$ is filtered retract of $\Omega Y$, $[\Sigma X^{\text{min}}(Y)]_1$ is a retract of $[\Sigma \Omega Y]_1 = Y$. Because $Y$ is atomic, $[\Sigma X^{\text{min}}(Y)]_1 \simeq Y$. It follows that

$$V \subseteq E^0 H_*(X^{\text{min}}(Y)).$$

Since $X^{\text{min}}(Y)$ is an $H$-space, $E^0 H_*(X^{\text{min}}(Y))$ is a Hopf algebra and so

$$\chi(H_*(X^{\text{min}}(Y))) = \chi(E^0 H_*(X^{\text{min}}(Y))) \geq \chi(E(V)).$$

Thus $\chi(H_*(X^{\text{min}}(Y))) = \chi(E(V))$ and so $H_*(X^{\text{min}}(Y)) = E(V)$ as coalgebras. It follows that $X^{\text{min}}(Y)$ is an $H$-space having $Y$ as a retractile generating complex and hence statement (3). The proof is finished. □
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