EXISTENCE OF TRAVELLING PULSES IN A NEURAL MODEL

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ABSTRACT. In 1992 G. B. Ermentrout and J. B. McLeod published a landmark study of travelling wave fronts for a differential-integral equation modeling a neural network. Since then a number of authors have extended the model by adding an additional equation for a “recovery variable”, thus allowing the possibility of travelling pulse type solutions. In a recent paper G. Faye gave perhaps the first rigorous proof of the existence (and stability) of a travelling pulse solution for such a model. The excitatory weight function $J$ used in this work allowed the system to be reduced to a set of four coupled ODEs, and a specific firing rate function $S$, with parameters, was considered. The method of geometric singular perturbation was employed, together with blow-ups. In this paper, while keeping the same $J$, we consider a more general class of functions $S$. We also remove a significant assumption used by Faye. We obtain travelling pulse solutions at two different speeds. The proofs are classical, and self-contained apart from standard ode material.

In 1992 G. B. Ermentrout and J. B. McLeod published a landmark study of travelling wave fronts for a differential-integral equation modeling a neural network. Since then a number of authors have extended the model by adding an additional equation for a “recovery variable”, thus allowing the possibility of travelling pulse type solutions. These differ from wave fronts because they begin and end at the same equilibrium point. Perhaps the first mathematically rigorous study of travelling pulses for these models was by G. Faye [1], who considered the following system of equations:

\[ \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{\mathbb{R}} J(x-y) q(y,t) S(u(y,t)) \, dy \]

\[ \frac{1}{\varepsilon} \frac{\partial q(x,t)}{\partial t} = 1 - q(x,t) - \beta q(x,t) S(u(x,t)) \]

where $J$ is a normalized exponential

\[ J(x) = \frac{b}{2} e^{-b|x|} \]

and the “firing rate” function $S$ is given by

\[ S(u) = \frac{1}{1 + e^{\lambda(u-\kappa)}} \]

for certain positive parameters $\varepsilon$, $\lambda$, $b$, $\kappa$, and $\beta$. See [1] for the neurobiological background and motivation for this model.

In [1] the author proves two interesting results about this system of equations, namely the existence of a “travelling pulse” solution and the stability of this solution. A travelling pulse solution of \[ (u(x,ct), q(x,ct)) \]
such that \( \lim_{s \to \infty} (u(s), q(s)) \) and \( \lim_{s \to -\infty} (u(s), q(s)) \) both exist and these limits are equal. In this paper we are interested in the existence of values of $c$ for
which (0.1) has such a solution. As we describe briefly below, using (0.2) leads to a set of four ode’s in which $c$ is a parameter. To show that a travelling pulse exists for some $c > 0$, Faye uses the theory of geometric singular perturbation initiated by Fenichel in [6] and extended by Jones and Kopell in [4]. The blowup method is also employed [5].

Here we extend the existence result from [1] in three ways. First, our proof is for a general function $S$ with certain properties. Second, we remove an important hypothesis used in [1], and third, we show that there are at least two travelling pulses, hence a “fast” pulse and a “slow” pulse. Our proofs use purely classical ode methods, following techniques from [3].

In particular, we use methods in Chapter 6 of [3], where travelling pulse solutions of the well-known FitzHugh-Nagumo equations are discussed. The FitzHugh-Nagumo system consists of a parabolic pde coupled to an ode, and the travelling wave substitution $s = x + ct$ leads to a system of three ode’s, instead of four as in the present case. While the setting here in $R^4$ presents new challenges, one aspect of the proof turns out to be simpler than the comparable step in [3] for FitzHugh-Nagumo. In each case the existence of a homoclinic orbit requires showing that there is not a very small periodic solution. For the FitzHugh-Nagumo equation this step was done by C. Conley, who found a functional in the relevant variables which was monotone in a half space $S$ which included the equilibrium point of the system, the origin in that case. This monotonicity implies that any solution which is bounded and restricted to the half space $S$ for large $s$ approaches equilibrium as $s \rightarrow \infty$. Thus the problem was reduced to finding such solutions, which was accomplished by the methods also used here. Their extension to a physically interesting system in $R^4$ is the main contribution of this paper. As will be seen, this extension is non-trivial. However the technique of Conley turns out to be unnecessary here. We show that in this problem the bounded solutions lie in a region where one of the original variables is monotonic, eliminating the need for a functional.

As further context, we refer to our proof in [10] that the well-known Hodgkin-Huxley system has a homoclinic orbit. The same problem arose there, namely to eliminate small periodic solutions. Not finding an appropriate Conley-type functional, we had to allow for solutions which oscillate around equilibrium. The method then was to construct a decreasing sequence of boxes and show that solutions eventually had to lie in each box. The technique was used later, and independently, by Rauch and Smoller in [9], on a different class of problems. It has not been established that for the Hodgkin-Huxley system there is a second homoclinic orbit.

Finally we mention another well-known biological model related to the one studied here. This is the differential-integral system proposed by Pinto and Ermentrout in [11]. The equations look quite similar, and again a fourth order system of ode’s is found. But no complete existence proof has been given for this system. Once difference between that ode system and the one studied here is that in [11] there can be complex eigenvalues for the linearization around equilibrium. So far, we have not been able to apply the “shrinking box” technique to the problem in [11], nor has an appropriate functional been found.

Travelling pulse solutions of (0.1) with (0.2) are shown to satisfy a system of ode’s by letting $v(s) = \int_{-\infty}^{\infty} b e^{-b s - r} q(\tau) J(u(\tau)) d\tau$ and computing $w = v'$ and

1These properties are satisfied by the function given in (0.3) for a range of parameter values.
We find that
\begin{equation}
\begin{aligned}
u' &= \frac{v-u}{v} \\
w' &= b^2(v - qS(u)) \\
q' &= \frac{\zeta}{c}(1 - q - \beta qS(u)).
\end{aligned}
\end{equation}

We will denote solutions of this system by \( p = (u, v, w, q) \), and we look for values of \( c \) for which there is a non-constant solution such that \( p(\infty) \) and \( p(-\infty) \) both exist and are equal. In the language of dynamical systems, \((u, q)\) is a pulse solution of (0.1) if and only if the orbit of \( p \) is a homoclinic orbit of (0.4).

We make the following assumptions on \( S \).

**Condition 1.** The function \( S \) is positive, increasing, bounded, and has a continuous first derivative \( S' \).

**Condition 2.** The function \( h(u) = \frac{u}{S(u)} \) has one local maximum followed by one local minimum, and no other critical points.

**Condition 3.** \( S \) is such that the system (0.4) has exactly one equilibrium point, say \( p_0 = (u_0, u_0, 0, q_0) \).

**Condition 4.** The function \( S \) is also such that the “fast” system
\begin{equation}
\begin{aligned}
u' &= \frac{v-u}{v} \\
w' &= b^2(v - q_0S(u))
\end{aligned}
\end{equation}

has three equilibrium points, \((u_0, u_0, 0)\), \((u_m, u_m, 0)\), and \((u_+, u_+, 0)\), with \( u_0 < u_m < u_+ \).

**Condition 5.**
\[
\int_{u_0}^{u_+} (q_0S(u) - u) \, du > 0.
\]

For convenience we will assume that \( 0 < S < 1 \) on \((-\infty, \infty)\). Then Conditions 1-4 imply that \( 0 < q_0 < 1 \), \( u_0 > 0 \), and \( u_+ < 1 \).

We will denote solutions of (0.5) by \( r = (u, v, w) \). The local minimum of \( h \) will be denoted by \( u_{\text{loc}} \). In [1] specific ranges of \( \kappa \) and \( \lambda \) are given so that these conditions are satisfied by the function given in (0.3). In Figure 1 we show the graphs of \( h, \frac{1}{1+\beta S} \) (the \( q \) nullcline), and \( q = q_0 \), when \( S \) is given by (0.3). We use the same parameter values as were chosen for illustration in [1].

We need two simple results about the behavior of solutions.

**Proposition 1.** The regions \( \left\{ v \leq 0, w \leq 0, \frac{1}{1+\beta} < q < \infty \right\} \) and \( \left\{ v > 1, w \geq 0, \frac{1}{1+\beta} < q < 1 \right\} \) are positively invariant for the system (0.4).

**Proof.** We are assuming that \( 0 < S(u) < 1 \) for all \( u \). Hence, \( q' > 0 \) if \( q \leq \frac{1}{1+\beta} \) and \( q' < 0 \) if \( q \geq 1 \). Therefore \( \left\{ \frac{1}{1+\beta} < q < 1 \right\} \) is positively invariant. Further, if \( \frac{1}{1+\beta} < q < 1 \) then \( v'' = w' < 0 \) if \( v \leq 0 \) and \( w' > 0 \) if \( v \geq 1 \). The result follows. \( \square \)

Note as well that because \( S \) is bounded, all solutions of (0.4) exist on \( R = (-\infty, \infty) \).

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2 It is not necessary to discuss Fourier transforms, as is usually done here.
3 \( \lambda = 20, \kappa = 0.22, \beta = 5, b = 4.5 \)
Proposition 2. If $p = (u, v, w, q)$ is a solution of (0.4), and $u(t) \geq u_{\text{knee}}$ for some $t$, then either $q'(t) < 0$ or $q(t) < h(u_{\text{knee}})$.

Proof. This follows from Condition 3, which implies that the graph of the decreasing function $q = \frac{1}{1 + \beta S(u)}$ in the $(q,u)$ plane, where $q' = 0$, passes under the point $(u_{\text{knee}}, h(u_{\text{knee}}))$. (See Figure 1.)

Here is our main Theorem:

Theorem 1. If Conditions 1-5 are satisfied, and $\varepsilon$ is positive and sufficiently small, then there are at least two positive values of $c$ such that (0.4) has a non-constant solution $p$ satisfying

$$\lim_{t \to -\infty} p(t) = \lim_{t \to \infty} p(t) = (u_0, u_0, 0, q_0).$$

Remark 1. In [1], only one such value of $c$ is found, and there is an extra hypothesis about the system (0.5). (Hypotheses 3.1) As far as we know, this hypothesis can only be checked by numerically solving the system (0.5). We discuss this further in Section 2.

1. Proof of Theorem 1

In the first, and longest, part of the proof we show that there is a “fast” pulse, with speed $c^*(\varepsilon)$ which tends to a positive number as $\varepsilon$ tends to zero. In the second part we look for a “slow” pulse, with a speed $c_*(\varepsilon)$ which tends to zero as $\varepsilon$ tends to zero.

We will show that for any possible homoclinic orbit, $u > 0$. We look for homoclinic orbits such that, as well, $q < q_0$ in $(-\infty, \infty)$. In searching for the fast solution we will consider for each $c > 0$ a certain uniquely defined solution $p_c = (u_c, v_c, w_c, q_c)$ such that $p_c(-\infty) = p_0$. We will show that there is a nonempty bounded set of positive values of $c$, called $\Lambda(\varepsilon)$, such that either $q_c$ exceeds $q_0$ at some point, or $u_c$ becomes negative. We then examine the behavior of $p_{c^*}(\varepsilon)$ where $c^*(\varepsilon) = \sup \Lambda(\varepsilon)$. The goal is to show that $p_{c^*}(\varepsilon)(\infty) = p_0$. This is done by eliminating all the other possible behaviors of $p_{c^*}(\varepsilon)$, often by showing that a particular behavior implies that all values of $c$ close to $c^*(\varepsilon)$ are not in $\Lambda(\varepsilon)$.

We start this process by analyzing the system (0.5). We need the following results about this system.
Lemma 1. If Conditions 1 and 2 are satisfied, then for each \( c > 0 \) the equilibrium point \((u_0, u_0, 0)\) of (0.1) is a saddle point, with a one dimensional unstable manifold \( \mathcal{U}_{0,c}^u \) and a two dimensional stable manifold \( \mathcal{S}_{0,c}^s \). There is, for each \( c > 0 \), a unique solution \( r_c = (u_{0,c}, v_{0,c}, w_{0,c}) \) of (0.5) with \( r_c(t) \in \mathcal{U}_{0,c}^u \) for all \( t \) and satisfying the conditions

\[
\begin{align*}
  u_{0,c}(0) &= u_m, \\
  w_{0,c} &> 0 \text{ on } (-\infty, 0].
\end{align*}
\]

Further, there is a unique \( c = c_0^* > 0 \) such that

\[
\lim_{t \to \infty} r_c^*(t) = (u_+, u_+, 0).
\]

In other words, the branch \( \mathcal{U}_{0,c_0^*}^u \) of \( \mathcal{U}_{0,c}^u \) pointing into the positive octant \( u > u_0, v > v_0, w > 0 \) is a heteroclinic orbit connecting \((u_0, u_0, 0)\) to \((u_+, u_+, 0)\). Also, \( w_{0,c_0^*} > 0 \) on \( R \). This implies that on \( R \), \( v_{0,c_0^*}^0 > 0 \) and \( u_{0,c_0^*}^0 > 0 \).

Lemma 2. If \( c > c_0^* \) then \( w_{0,c} > 0 \) on \( R \), and both \( u_{0,c} \) and \( v_{0,c} \) tend to \( \infty \). If \( 0 < c < c_0^* \), then \( v_{0,c} \) increases to a unique maximum, after which \( v_{0,c}^0 < 0 \) and both \( u_{0,c} \) and \( v_{0,c} \) tend to \(-\infty\). Further, if \( t_1(c) \) is the zero of \( v_{0,c}^0 \), where \( v_{0,c}^0 \) is a maximum, then \( v_{0,c}^0(t_1(c)) < 0 \), and \( \lim_{c \to c_0^*} r_c(t_1(c)) = (u_+, u_+, 0) \).

In [1], results about system (0.5), including parts of these two lemmas, are proved by reference to the integral equation for \( u \) in (0.1) with \( q = q_0 \). This is the equation which was studied by G. B. Ermentrout and J. B. McLeod in [2], and it is stated in [1] that adapting their methods leads to the existence of the heteroclinic orbit \( \mathcal{U}_{c_0}^* \). Alternatively, Lemmas 1 and 2 can be proved with purely ode methods. We have included such proofs in an appendix.

Turning to the full system (0.4), the following result is basic. The proof is routine and also left to the appendix.

Lemma 3. Suppose that Conditions 3 and 4 hold, and let \( p_0 = (u_0, u_0, 0, q_0) \) be the unique equilibrium point of (0.4). Then for any \( \varepsilon \geq 0 \) and \( c > 0 \) the system (0.4) has a one dimensional unstable manifold at \( p_0 \), say \( \mathcal{U}_{c_0}^u \), with branch \( \mathcal{U}_{c_0}^u \) starting in the region \( \{u > u_0, v > u_0, w > 0\} \). If \( p = (u, v, w, q) \) is a solution lying on this manifold, then for large negative \( t, u_0 < u(t) < v(t) \) and \( w(t) > 0 \). Also, \( q_{c_0} \equiv q_0, \) while if \( \varepsilon > 0 \) then \( q_{c, \varepsilon}(t) < q_0 \) for large negative \( t \). The invariant manifold \( \mathcal{U}_{c, \varepsilon}^u \) depends continuously on \( (\varepsilon, c) \) in \( \varepsilon \geq 0, c > 0 \). (The meaning of continuity here is made clear in the text below.) Finally, if \( \lambda_1(c, \varepsilon) \) is the positive eigenvalue of the linearization of (0.4) around \( p_0 \), then \( \lambda_1(c, \varepsilon) > \lambda_1(c, 0) \) for each \( c > 0 \) and \( \varepsilon > 0 \).

The following proposition follows trivially from (0.4) and will be used a number of times, often without specific mention.
Proposition 3.

If \( u' = 0 \) then \( u'' = \frac{v'}{c} \).

If \( u' = u'' = 0 \) then \( u''' = \frac{v''}{c} \).

If \( u' = u'' = w''' = 0 \) then \( w''' = -\frac{b^2}{c} q'S(u) \).

If \( q'' = 0 \) then \( q'''' = -\frac{\varepsilon}{c} \beta qS'(u) u' \).

If \( w' = 0 \) then \( v'' = w'' = b^2 (v' - q'S(u) - qS'(u) w') \).

If \( q' = 0 \) then \( q'' = 0 \) and \( q''' = \frac{\varepsilon}{c} \beta qS'(u) u'' = -\frac{\varepsilon}{c^2} \beta qS'(u) v' \).

If \( q' = v' = 0 \) then \( q'' = -\frac{\varepsilon}{c} \beta qS'(u) w' = -\frac{\varepsilon}{c^2} \beta qS'(u) v'' = -\frac{\varepsilon}{c} \beta qS'(u) u''' \).

We use the fourth of these to prove

Lemma 4. For any \( \varepsilon > 0 \) and \( c > 0 \), if \( p \) is a solution on \( \mathcal{U}_{\varepsilon,c}^+ \) and \( u' \geq 0 \) on an interval \((-\infty, \tau)\), then \( q' < 0 \) on \((-\infty, \tau)\).

Proof. If \( u' \) never changes sign, let \( \sigma \) denote \( \infty \). Otherwise, suppose that \( u' \) first changes sign at \( \sigma \). If \( q'(\tau) = 0 \) for some \( \tau < \sigma \), then \( u''(\tau) \geq 0 \), and by Proposition 3, \( u'(\tau) \leq 0 \). From the definitions of \( \sigma \) and \( \tau \), \( u'(\tau) = 0 \) and so \( q''(\tau) = 0 \). Since \( u' \) does not change sign at \( \tau \), \( u''(\tau) = 0 \) and so \( q'''(\tau) = 0 \). Hence at \( \tau \),

\[
  u' = u'' = q' = q'' = q''' = 0.
\]

If \( u'''(\tau) = 0 \) then \( p(\tau) \) is an equilibrium point, a contradiction. If \( u'''(\tau) < 0 \) then \( \tau \) is a local maximum of \( u' \), which is inconsistent with the assumption that \( u' \geq 0 \) on \((-\infty, \sigma)\). Hence \( u'''(\tau) > 0 \). But then \( q''''(\tau) < 0 \). This again implies that \( q' \) does not change sign to the left of \( \tau \), contradicting the definition of \( \tau \). This completes the proof of Lemma 4.

Lemma 5. If \( p = (u, v, w, q) \) is a solution on \( \mathcal{U}_{\varepsilon,c}^+ \) then \( w > 0 \) on an interval \((-\infty, \tau)\) with \( u(\tau) = u_m \).

Proof. Observe that \( h(u) > q_0 \) for \( u_0 < u < u_m \). It follows that if \( u_0 < u < u_m \) and \( q < q_0 \) on an interval \((-\infty, \tau)\), then \( w' > 0 \) on this interval. Hence \( w > 0 \) as long as \( u_0 < u \leq u_m \) and \( q < q_0 \). (That is, if \( u_0 < u \leq u_m \) and \( q < q_0 \) on \((-\infty, \tau)\), then \( w > 0 \) on this interval.) Since \( u' > 0 \) as long as \( w = v' \geq 0 \), Lemma 4 implies that \( w > 0 \) as long as \( u_0 < u \leq u_m \), proving Lemma 5.

Hence the conditions \( u(0) = u_m \) and \( w > 0 \) on \((-\infty, 0]\) determine a unique solution \( p_{\varepsilon,c} = (u_{\varepsilon,c}, v_{\varepsilon,c}, w_{\varepsilon,c}, q_{\varepsilon,c}) \) on \( \mathcal{U}_{\varepsilon,c}^+ \).

Let

\[
  \Omega = \left\{ (u, v, w, q) \mid 0 < u < 1, 0 < v < 1, \frac{1}{1 + \beta} < q < 1 \right\}.
\]

Since \( (u_0, v_0, w_0, q_0) \in \Omega \), it follows from Proposition 3 that if \( \mathcal{U}_{\varepsilon,c}^+ \) is a homoclinic orbit, then it lies entirely in \( \Omega \).
Using the last sentence of Lemma 6, it follows that \( c_1 \) can be chosen in \((0, c_0')\) such that if \( c_1 \leq c < c_0' \), then \( u_{0,c}(t_1(c)) > \frac{u_{knee} + u_+}{2} \), where \( t_1(c) \) is the unique zero of \( u'_{0,c} \) and so the point of absolute maximum of \( u_{0,c} \). Let \( I = [c_1, c_0' + 1] \). We note that the unstable manifold \( U^+_{\varepsilon,c} \) varies continuously with \((\varepsilon, c)\) for \( \varepsilon \geq 0 \) and \( c \in I \). To be more precise, for each \( T \) and each \( \varepsilon_0 > 0 \), \( p_{\varepsilon,c}(t) \) is continuous in \((\varepsilon, c, t)\) uniformly for \(-\infty < t \leq T, 0 \leq \varepsilon \leq \varepsilon_0 \), and \( c \in I \).

Henceforth in this paper we will assume that \( c_1 \) is chosen as just described. One consequence for the full system (0.4) is the following.

**Lemma 6.** There is an \( \varepsilon_0 > 0 \) and a \( T_1 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \) and \( c \in I = [c_1, c_0' + 1] \), then \( u_{\varepsilon,c}(\tau) = u_{knee} \) for some \( \tau \leq T_1 \), and \( w_{\varepsilon,c} > 0 \) on \((-\infty, \tau] \). Further, \( \varepsilon_0 \) can be chosen so that \( u_{\varepsilon,c} \) has a unique zero, say at \( t = t_1(c_1) \), \( v_{\varepsilon,c} < 0 \) on \((t_1, \infty)\), and both \( v_{\varepsilon,c} \) and \( u_{\varepsilon,c} \) tend to \(-\infty \) as \( t \to \infty \). Finally, \( u'_{\varepsilon,c} \) has a unique zero, say \( \sigma \in (t_1(c), \infty) \).

**Proof.** From the choice of \( c_1 \) there is a \( \delta > 0 \) such that if \( c \in I \) then \( v_{\varepsilon,c} = u_{0,c} \geq \delta \) in the interval \([0, \tau]\) where \( u_m \leq u \leq \frac{2u_{knee} + u_+}{3} \). From \( cu' = v - u \) it follows that for some \( T_1 > 0 \), if \( c_1 \leq c \leq c_0' \), then \( u_{0,c} = \frac{2u_{knee} + u_+}{3} \) before \( t = T_1 \). The uniform continuity of \( u_{\varepsilon,c}(t) \) in \((\varepsilon, c, t)\), for \(-\infty < t \leq T_1 \) and \( d \in I \) in any compact interval \([0, \varepsilon]\) with \( \varepsilon > 0 \), then implies the first conclusion of the Lemma. The remaining assertions of Lemma 6 follow by similar arguments and use again of the first equation in (0.4).

**Lemma 7.** If \( \varepsilon > 0 \) and \( c > c_0' \), then \( u_{\varepsilon,c} > 0, w_{\varepsilon,c} > 0, u'_{\varepsilon,c} > 0, \) and \( u_{\varepsilon,c} \to \infty \) as \( t \to \infty \).

**Proof.** Let \( p = (u, v, w, q) = p_{\varepsilon,c} \). Lemma 5 implies that if \( \varepsilon > 0 \) then \( q < q_0 \) on any interval \((-\infty, \ell)\) where \( w > 0 \), since in such an interval \( u' > 0 \). Also, as long as \( w > 0 \) we can consider \( u, w, \) and \( q \) as functions of \( v \). Say that \( u = U(v), \) \( w = W(v), \) and \( q = Q(v) \). Then

\[
U'(v) = \frac{v - U(v)}{cW(v)} \quad \text{and} \quad W'(v) = \frac{\nu'(v - Q(v))S(U(v))}{W(v)}
\]

For a given \( c > c_0' \) we compare \( w = W(v) \) with the solution when \( \varepsilon = 0 \). Thus, consider the solution \( p_{0,c} \), noting that by Lemma 2, \( u_{0,c} > 0 \) on \( R \), and \( v_{0,c} \to \infty \).

Let \( p_{1} = p_{0,c} \). Then we can write \( u_1 = U_1(v_1), w_1 = W_1(v_1), \) and \( q = q_0 \). The equations become

\[
U'_1(v) = \frac{v - U_1(v)}{cW_1(v)} \quad \text{and} \quad W'_1(v) = \frac{\nu'(v - q_0S(U_1(v)))}{W_1(v)}
\]

Since \( \lambda_1(c, \varepsilon) > \lambda_1(c, 0) \) (Lemma 3), it is seen by considering eigenvectors of the linearization of (0.4) around \( p_0 \) that for \( v \) sufficiently close to \( u_0 \) (i.e. for large negative \( t \)), \( U(v) < U_1(v) \) and \( W(v) > W_1(v) \). If, at some first \( \hat{v} \), one of these inequalities should fail while the other still holds, then a contradiction results from comparing (1.2) and (1.3), because \( q < q_0 \) and \( S \) is increasing. For example, if

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4See the footnote at the end of the appendix for a further discussion of this point.

5The relevant matrix \( B \) is given in Appendix B.
Lemma 8. \( \Lambda \) is an open subset of the half line \( c \geq c_1 \).

Proof. Suppose that \( c \in \Lambda \), and choose \( t_3 = t_3(c) \) as in the definition of \( \Lambda \). Note from (0.14) that if \( u_c(t_3) = 0 \) then there is a \( \tau < t_3 \) such that \( v(\tau) = 0, v'(\tau) \leq 0 \). Also, \( v'' < 0 \) if \( v \leq 0 \). Hence \( v(t_3) < 0 \) and \( u'(t_3) < 0 \).

Also, (0.4) implies that if \( q(t_3) = q_0 \) and \( u(t_3) < u_0 \) then \( q'(t_3) > 0 \). Since \( p_c(t) \) is a smooth function of \( c \), uniformly for \( t \) in, say, \((-\infty, t_3(c^*) + 1\) ), it follows that for \( c \) in some neighborhood of \( c^* \), \( t_i(c) \) is defined for \( i = 1, \ldots, 3 \) and all the inequalities in the definition of \( \Lambda \) continue to hold, so that this neighborhood lies in \( \Lambda \). \qed

Lemma 6 and the definition of \( c_1 \) imply that \( c_1 \in \Lambda(\varepsilon) \) if \( 0 < \varepsilon < \varepsilon_0 \), while by Lemmas 7 and 8 if \( c > c_0^* \), then \( c \notin \Lambda \). The numbers \( t_i \) depend on \( c \), and when we need to emphasize this we will denote them by \( t_i(c) \), for \( i = 1, 2, 3 \).

We now let \( c^*(\varepsilon) = \sup \Lambda(\varepsilon) \). (This is finite, by Lemma 7)

Lemma 9. Choose \( \varepsilon_0 \) as in Lemma 6. Suppose that \( 0 < \varepsilon < \varepsilon_0 \). Then \( U_{\varepsilon, c^*(\varepsilon)} \) is a homoclinic orbit of \( \Lambda(\varepsilon) \).
Proof. We need several additional lemmas. Fixing \( \varepsilon \) in \((0, \varepsilon_0)\), we will now drop the \( \varepsilon \)-dependence of \( p_{c, \varepsilon} \) and its components, and of \( \Lambda (\varepsilon) \), from our notation, writing \( p_c \) and \( \Lambda \). When the dependence of \( p_c \) on \( c \) is not crucial to an argument we will use \( u, v, w, q \) for its components.

Lemma 10. Suppose that \( p = (u, v, w, q) \) is a non-constant solution of (0.4) satisfying one of the following sets of conditions at some \( \tau \):

1. \( u'(\tau) = 0, \quad v'(\tau) = 0, \quad w'(\tau) \leq 0, \quad q'(\tau) \leq 0, \quad u(\tau) \geq u_0 \)
2. \( u'(\tau) = 0, \quad v'(\tau) < 0, \quad w'(\tau) = 0, \quad q'(\tau) = 0, \quad u(\tau) \geq u_0 \)
3. \( u'(\tau) = 0, \quad v'(\tau) \leq 0, \quad q'(\tau) > 0, \quad u(\tau) = u_0 \)
4. \( u'(\tau) = 0, \quad v'(\tau) \leq 0, \quad w'(\tau) > 0, \quad q'(\tau) \geq 0, \quad u(\tau) \geq u_0 \)

Then \( p(\tau) \notin U_{\varepsilon,c,\varepsilon}(\varepsilon) \).

Proof. Suppose that (i) holds. Then \( w'(\tau) \) and \( q'(\tau) \) cannot both vanish. If \( w'(\tau) = 0 \) and \( q'(\tau) < 0 \) then \( w''(\tau) = -b^2 q'(\tau) S(u(\tau)) > 0 \). Hence in some interval \((\tau - \delta, \tau)\),

\[
(1.4) \quad w' < 0 \quad \text{and} \quad q' < 0.
\]

If \( q'(\tau) = 0 \) and \( w'(\tau) < 0 \) then

\[
q''(\tau) = -\frac{\varepsilon}{c^2} q(\tau) S'(u(\tau)) w'(\tau) = 0
\]

\[
q'''(\tau) = -\frac{\varepsilon}{c^2} q(\tau) S'(u(\tau)) w'(\tau) = 0
\]

\[
q^{(iv)}(\tau) = -\frac{\varepsilon}{c^2} q(\tau) S'(u(\tau)) w'(\tau) > 0.
\]

Once again we see that (1.4) holds on some interval \((\tau - \delta, \tau)\).

Consider the “backward” system satisfied by \( P(s) = p(\tau - s) \). If \( P = (U, V, W, Q) \) then

\[
U' = \frac{U - V}{\varepsilon}, \quad V' = -W, \quad W' = b^2 (QS(U) - V), \quad Q' = \frac{\varepsilon}{c} (Q + \beta QS(U) - 1)
\]

Also,

\[
U'(0) = 0, \quad V'(0) = 0, \quad W'(0) \geq 0, \quad \text{and} \quad Q'(0) \geq 0.
\]

From (1.4) and (1.6) it follows that on some interval \( 0 < s < \delta, (1.7) \)

\[
U' > 0, \quad V' < 0, \quad W' > 0, \quad \text{and} \quad Q' > 0.
\]

We claim that these inequalities hold for all \( s > 0 \). If, on the contrary, one of them fails at a first \( s_0 > 0 \), then

\[
(1.8) \quad U(s_0) > U(0), \quad V(s_0) < V(0), \quad W(s_0) > W(0), \quad \text{and} \quad Q(s_0) > Q(0).
\]

But (1.8), (1.5), and (1.6) imply that at \( s_0 \), all of the inequalities in (1.7) still hold, because \( S' > 0 \). This contradiction implies that \( U, \quad W, \quad \text{and} \quad Q \) continue to increase, and \( V \) continues decrease on \( 0 < s < \infty \), and in particular, \( U \) does not tend to \( u_0 \) as \( s \to \infty \). Thus, \( p(\tau) \notin U_{\varepsilon,c,\varepsilon}(\varepsilon) \).

The proofs in cases (ii), (iii) and (iv) are similar and left to the reader.  \( \square \)
We now begin our study of the properties of \( p_{c^*} \). Lemma 11 will assist us in proving the following result.

**Lemma 11.** The number \( t_1(c^*) \) is still defined, as the first zero of \( u'_{c^*} \), and \( u''_{c^*}(t_1(c^*)) < 0 \). Either \( U^+_{c^*}(c) \) is homoclinic or \( t_2(c^*) \) is still defined, as the first zero of \( u - u_0 \). Also, if \( U^+_{c^*}(c) \) is not homoclinic then \( u'_{c^*} < 0 \) on \((t_1, t_2)\).

**Proof.** Suppose that \( t_1(c^*) \) is not defined. Then \( u'_{c^*} > 0 \) on \(({-\infty, \infty})\). Since \( p_0 \) is the only equilibrium point of \((0, 4)\), this implies that for some \( \tau \), \( v_{c^*}(\tau) > u_{c^*} (\tau) > 1 \) and \( v'_{c^*}(\tau) > 0 \). Then these inequalities hold at \( \tau \) for nearby \( c \), and by Proposition 1 \( v_c(t) > 1 \) for \( t > \tau \). Hence \( u_c > 1 \) on \([\tau, \infty)\) and so \( c \notin \Lambda \), contradicting the definition of \( c^* \). Therefore \( t_1(c^*) \) is defined.

We now show that \( u''_{c^*}(t_1) < 0 \). Again assume that \( p = p_{c^*} \), and suppose that \( u''(t_1) = 0 \). If \( u''(t_1) < 0 \) then \( t_1 \) is a local maximum of \( u' \), which is not possible because \( t_1 \) is the first zero of \( u' \). Hence at \( t_1 \), \( u'' = \frac{b^2}{c} (v - qS(u)) \geq 0 \). If \( u''(t_1) = 0 \), then at \( t_1 \),

\[
u''_{c^*} = \frac{b^2}{c} (v - qS(u)) > 0,\]

by Lemma 4. This implies that \( u' \) changes sign from negative to positive at \( t_1 \), again a contradiction of the definition of \( t_1 \). Hence at \( t_1 \),

\[
u''_{c^*} = \frac{b^2}{c} (v - qS(u)) > 0,\]

or \( q(\tau_1) < \frac{u(\tau_1)}{S(u)} = \frac{u(\tau_1)}{S(u)} \), since \( u'_{c^*}(\tau_1) = 0 \). Also, \( u'' \geq 0 \) in some interval \((t_1, t_1 + \delta)\). However \( u \) is bounded by 1 and does not tend to a limit above \( u_0 \). Therefore \( u' \) changes sign at some \( \tau > t_1(c^*) \). Since \( u' \geq 0 \) on \(({-\infty, \tau}] \), \( v \geq u \) on this interval. At \( \tau \), \( u'' \leq 0 \), and so there is a point \( \sigma \) in \((t_1, \tau) \) such that \( u'' = 0 \) and \( u'' = \frac{u''(\tau)}{c} \leq 0 \). Hence at \( \sigma \), \( q \geq \frac{u}{S(u)} \geq \frac{u}{S(u)} \). But \( u(\tau_1) \geq u_{knee} \), and \( \frac{u}{S(u)} \) is increasing in \((u_{knee}, \infty) \), so \( q(\sigma) > q(\tau_1) \), again a contradiction of Lemma 4. We have therefore proved the first sentence of Lemma 11.

**Lemma 12.** If \( p = p_{c^*} \), then \( u' < 0 \) as long after \( t_1 \) as \( q' \leq 0 \).

**Proof.** Suppose instead that there is a first \( \tau > t_1 \) such that \( q' \leq 0 \) on \(({-\infty, \tau}] \) but \( u'_{c^*}(\tau) = 0 \). Then \( u''_{c^*}(\tau) \geq 0 \). First consider the case \( q' < 0 \) on \(({-\infty, \tau}] \).

If \( u''_{c^*}(\tau) > 0 \) then \( u'_{c^*} \) changes from negative to positive before \( q' = 0 \), and this will be true as well for \( c \) close to \( c^* \), contradicting the definition of \( c^* \). Hence suppose that \( u''_{c^*}(\tau) = \frac{u'_{c^*}(\tau)}{c} = 0 \). If \( u''_{c^*} > 0 \), then \( u' \) has a local minimum at \( \tau \), contradicting the definition of \( \tau \). Hence

\[
u''_{c^*}(\tau) = \frac{u'_{c^*}(\tau)}{c} \leq 0.\]

But now the conditions in (i) of Lemma 10 are satisfied, giving a contradiction.

We have left to consider the case that \( q'_{c^*}(\tau) = u'_{c^*}(\tau) = 0 \). Then \( q''_{c^*}(\tau) = 0 \). If \( u''_{c^*}(\tau) > 0 \) then \( q'''_{c^*}(\tau) < 0 \) so \( q' \) changes sign in an interval \((\tau, \tau + \delta) \). Hence in this case, \( u' \) changes sign (from negative to positive) before \( q' > 0 \). For \( c \) close to \( c^* \) there are two possibilities: either \( u'_{c^*} \) changes sign from negative to positive before \( q' > 0 \), and so before \( u = u_0 \), or else \( u'_{c^*} < 0 \) in a neighborhood of \( \tau \). We then have \( \tau + \frac{1}{2} \delta \), \( u'_{c^*} > 0 \) and \( u > u_0 \). (See Figure 2) Neither of these possibilities occurs if \( c \in \Lambda \), so once again, \( c^* \notin \partial \Lambda \), a contradiction. □
It follows that there is a first \( \tau_1 > t_1 \) such that \( q_{\text{c}_*}^{(\prime)} (\tau_1) = 0 \). Also, \( u_{\text{c}_*}^{(\prime)} (\tau_1) < 0 \), and (equivalently by Proposition \[3\]) \( q_{\text{c}_*}^{(\prime\prime)} (\tau_1) > 0 \).

**Lemma 13.** \( q_{\text{c}_*}^{(\prime)} > 0 \) and \( u_{\text{c}_*}^{(\prime)} < 0 \) as long after \( \tau_1 \) as \( u_{\text{c}_*} \geq u_0 \).

**Proof.** Let \( p = p_{\text{c}_*} \). Since \( q''(\tau_1) > 0 \), \( q' > 0 \) and \( u' < 0 \) on some interval \( (\tau_1, \tau_1 + \delta) \) with \( \delta > 0 \). We claim that \( q' > 0 \) on any such half-closed interval in which \( u' < 0 \). This follows because, by Proposition \[3\], \( q'' > 0 \) at any point where \( q' = 0 \) and \( u' < 0 \).

We next show that \( u' < 0 \) on any interval \( (\tau_1, \tau_1 + \delta) \) in which \( q' > 0 \) and \( u \geq u_0 \). If not, then there is a first \( \sigma > \tau_1 \) with \( u'(\sigma) = 0 \), \( q'(\sigma) > 0 \) and \( u \geq u_0 \) on \( (-\infty, \sigma) \). Then \( u''(\sigma) > 0 \). If \( u''(\sigma) > 0 \), then \( u'_{\text{c}_*} > 0 \) in some interval \( (\sigma, \sigma + \delta) \). In this case, for \( c \) close enough to \( c_* \), \( u'_{\text{c}_*} \) changes sign after \( t_1 \) but before \( u_{\text{c}_*} < u_0 \) or else \( u_{\text{c}_*} \) crosses \( u_0 \) and back again, and such \( c \) cannot lie in \( \Lambda \), a contradiction.

Hence, \( u''(\sigma) = 0 \). But then, because \( u(\sigma) = v(\sigma) \),

\[
    u'''(\sigma) = \frac{b^2}{c^2} (v - qS(u)) = \frac{b^2}{c^2} (u - qS(u)).
\]

In the region where \( q' > 0 \) and \( u \geq u_0 \), \( q < \frac{q_S(u)}{u(\sigma)} \). Hence, \( u'''(\sigma) > 0 \) and again \( u'_{\text{c}_*} > 0 \) in an interval to the right of \( \sigma \) but before \( u_{\text{c}_*} < u_0 \), a contradiction as before.

The only other possibility contradicting Lemma \[14\] is that there is a first \( \tau > \tau_1 (c_*) \) where \( u_{\text{c}_*} (\tau) \geq u_0 \) and \( q_{\text{c}_*}^{(\prime)} (\tau) = u_{\text{c}_*}^{(\prime)} (\tau) = 0 \). We consider two cases: (a) \( q_{\text{c}_*} (\tau) < q_0 \) and \( u_{\text{c}_*} (\tau) > u_0 \), and (b) \( q_{\text{c}_*} (\tau) = q_0 \), \( u_{\text{c}_*} (\tau) = u_0 \). First consider (a). In an interval \( (\tau - \delta, \tau) \), \( q_{\text{c}_*}^{(\prime)} > 0 \), \( u_{\text{c}_*}^{(\prime)} < 0 \), and \( u_{\text{c}_*} > u_0 \), and so at \( \tau \), if \( p = p_{\text{c}_*} \), then

\[
    u' = 0, \quad u'' \geq 0, \quad q' = 0, \quad q'' = 0.
\]

Also,

\[
    q''' = -\frac{\epsilon}{c} \beta qS''(u) u'' \leq 0.
\]

But \( u_{\text{c}_*}''(\tau) > 0 \) is impossible because it means that even for nearby \( c \), \( u_{\text{c}_*} > 0 \) after \( t_1 \) but before \( u = u_0 \). Therefore at \( \tau \), \( q''' = 0 \) and \( u'' = 0 \). Then

\[
    q^{(iv)} = -\frac{\epsilon}{c} qS'(u) u'''.
\]
Lemma 14. If \( \varepsilon < c < c_1 \) \( v > u \) in the region \((t_1, \tau)\). This proves Lemma 9.

Proof. \((U, R)\) is only possible if \( u > u_0 \) homoclinic (with \( u \) close to \( c^* \) then either \( u \) crosses \( u_0 \) twice, or \( p_c \) doesn’t reach the region \( u < u_0 \) before \( u' > 0 \), both of which mean that \( c \notin \Lambda \).

Thus \( u' (\tau) = 0 \). If \( u'' (\tau) > 0 \) then \( u' > 0 \) to the right of \( \tau \). As before, if \( c \) is close to \( c^* \) then either \( u \) crosses \( u_0 \) twice, or \( p_c \) doesn’t reach the region \( u < u_0 \) before \( u' > 0 \), both of which mean that \( c \notin \Lambda \).

If \( u'' (\tau) = \frac{w''(\tau)}{c} = 0 \) then \( p (\tau) \) is again an equilibrium point. The third possibility, \( u'' (\tau) = \frac{w''(\tau)}{c} < 0 \) implies that (ii) of Lemma 10 is satisfied, and thus again gives a contradiction. This completes the proof of Lemma 13.

If \( u > u_0 \) on \( R \) then \( u'_c < 0 \) and \( q'_c > 0 \) on \( (\tau_1, \infty) \), and \( U_{c, c^*}^+ \) is homoclinic. This proves Lemma 11.

Thus, for \( p = p_c \). If \( U_{c, c^*}^+ \) is not homoclinic then \( t_2 \) exists with \( u (t_2) = u_0 \) and \( u' < 0 \) on \((t_1, t_2)\). However, there is no \( t_3 \) such that \( u < u_0 \) on \((t_2, t_3)\) and either \( u (t_3) = 0 \) or \( q(t_3) = 0 \), and this has already been ruled out. Therefore if \( t > t_2 \) \( c^* \) then \( 0 < u_c < u_0 \) and \( q_c > 0 \), for otherwise nearby values of \( c \) are once again not in \( \Lambda \). If \( U_{c, c^*}^+ \) is not homoclinic and \( p = p_c \), there must be a first \( \tau > t_2 \) \( c^* \) with \( u (\tau) = u_0 \), \( u' (\tau) = 0 \), \( q (\tau) \leq q_0 \), and \( u'' (\tau) \leq 0 \).

Suppose that this is the case and also \( q (\tau) < q_0 \). Then \( q' (\tau) > 0 \) if \( u'' (\tau) \leq 0 \), then (iii) of Lemma 10 applies and gives a contradiction. Hence \( q (\tau) = q_0 \). Then at \( \tau \), \( q' = u' = u'' = 0 \), and \( u'' = \frac{w''}{c} < 0 \). But this is case (ii) of Lemma 10 and so also impossible.

This completes the proof of Lemma 13. We have established that if \( U_{c, c^*}^+ \) is not homoclinic (with \( u > u_0 \) on \( R \) then for large \( t \), \( 0 < u_c (t) < u_0 \) and \( q_c > 0 \). This is only possible if \( U_{c, c^*}^+ \) is homoclinic (with \( q_c < 0 \) and \( u_c > 0 \) for large \( t \)). This proves Lemma 9.

To complete the proof of Theorem 1 we look for a second homoclinic orbit, with \( c < c_1 \).

1.1. The slow pulse. Again we adapt the method in [3]. With \( c_1 \in (0, c_0^*) \) and \( \varepsilon_0 \) as in Lemma 6, choose \( \hat{c} \in (0, c) \).

Lemma 14. If \( c \) is positive and sufficiently small, then the solution \( p_{\varepsilon, c} \) remains in the region \( v > u \) on \( R \), and \( u \) crosses \( u = 1 \).

Proof. Since the proof uses some of the easier parts of the proof of Lemma 2, it is included in the appendix.

Now let
\[
\Sigma = \{ c > 0 \mid \text{There is a } \tau_1 > 0 \text{ such that } q'_c < 0 \text{ on } (\infty, \tau_1) \},
\]
\[
q'_c (\tau_1) = 0, \text{ and } u'_c (\tau_1) < 0.
\]

and
\[
\Sigma_1 = \{ c \in \Sigma \mid q'_c > 0 \text{ on any interval } (\tau_1, T) \text{ in which } u_c > 0 \text{ and } q_c < q_0 \}.
\]
Lemma 6 implies that $u'_{c_1}$ has a unique zero. As in the proof of Lemma 13 this implies that $q'_c$ has a unique zero, and so $c_1 \in \Sigma_1$. Also, Lemma 14 shows that there is an interval $(0, c_2(\dot{\varepsilon}))$ which contains no points of $\Sigma$.

We continue to assume that $\varepsilon = \dot{\varepsilon}$. Let

$$c_3 = \sup \{ c < c_1 \mid c \notin \Sigma \}.$$ 

**Lemma 15.** There is a $\tau_1$ such that $q'_{c_3} < 0$ on $(-\infty, \tau_1)$, $q'_{c_3}(\tau_1) = 0$, $q''_{c_3}(\tau_1) = 0$, and $q'''_{c_3}(\tau_1) < 0$.

**Proof.** If $q'_{c_3} < 0$ on $R$ then there is a $\sigma > 0$ such that $u_{c_3}(\sigma) = 1$ and $u'_{c_3}(\sigma) > 0$.

From the continuity of $p_c(\dot{t})$ with respect to $c$, the same is true for $u_c$ if $c$ is sufficiently close to $c_3$. In particular, again $q'_c < 0$ on $(-\infty, \infty)$. But then $c \notin \Sigma$, contradicting the definition of $c_3$.

Therefore a first $\tau_1$ is defined such that $q'_{c_3}(\tau_1) = 0$. Then $q''_{c_3}(\tau_1) > 0$. Also, by Proposition 3 $q'''_{c_3}(\tau_1) = -\beta S'(u_{c_3}(\tau_1)) u''_{c_3}(\tau_1)$. If $q''_{c_3}(\tau_1) > 0$ then by the implicit function theorem, $\tau_1(c)$ is defined for nearby $c$ as the first zero of $q'_c$, with $q''_{c_3}(\tau_1(c)) > 0$ and $q'_c < 0$ on $(-\infty, \tau_1(c))$, contradicting the definition of $c_3$. Hence $q''_{c_3}(\tau_1) = u'_{c_3}(\tau_1) = 0$. If $q'''_{c_3}(\tau_1) > 0$ then $\tau_1$ is a local minimum of $q'_{c_3}$, contradicting the definition of $\tau_1$. If $q'''_{c_3}(\tau_1) = 0$ then $q''_{c_3}(\tau_1) = -\frac{3}{c_3} \beta q_{c_3}(\tau_1) S'(u_{c_3}(\tau_1)) w'_{c_3}(\tau_1)$, and since $q'_{c_3}(\tau_1) = 0$ and $q_{c_3}(\tau_1) < 0$, $w'_{c_3}(\tau_1) > 0$ and $q'''_{c_3}(\tau_1) < 0$. This implies that $q'_{c_3} > 0$ on an interval $(\tau_1 - \delta, \tau_1)$, again a contradiction. Hence $q'''_{c_3}(\tau_1) < 0$, completing the proof of Lemma 15.

Thus, $q'_{c_3} < 0$ in some interval $(\tau_1, \tau_1 + \delta)$. This result implies that $c_3 \notin \Sigma$. However the interval $(c_3, c_1) \subset \Sigma$. Lemma 15 also implies that points near to $c_3$ are not in $\Sigma_1$, since they must have a change of sign of $q'_c$ from positive to negative after $\tau_1(c)$. Let

$$c_* = \inf \{ c > c_3 \mid c \in \Sigma_1 \}.$$ 

We claim that $U^+_{\tau_1, c_*}$ is a homoclinic orbit.

The proof uses techniques very similar to those above. First observe that $c_* > c_3$ and $c_* \in \Sigma$. Therefore $\tau_1 = \tau_1(c_*)$ is defined as in the definition of $\Sigma$. Then use the following result.

**Lemma 16.** If $c \in \Sigma_1$, then $u'_c < 0$ on any interval $[\tau_1(c), \tau_1(c) + \delta]$ in which $u_c \geq u_0$.

**Proof.** If $u'_c = 0$ at some first $\sigma > \tau_1$ with $u_c(\sigma) \geq u_0$, then $u''_c(\sigma) \geq 0$. But in the region where $u \geq u_0$ and $q'' > 0$, $w'$ is positive, and this implies that $p_c$ crosses into $q'' < 0$, a contradiction of the definition of $\Sigma_1$. \hfill $\square$

**Corollary 1.** If $U^+_{\tau_1, c_3}$ is not homoclinic then there is a $t_2 > \tau_1$ such that $u_{c_*}(t_2) = u_0$ and $u'_{c_*} < 0$ on $[\tau_1, t_2]$. Further, $u_{c_*} < u_0$ on $(t_2, \infty)$.

**Proof.** Let $p = p_{c_*}$. Lemma 16 implies the existence of $t_2$. Suppose there is a first $\sigma > t_2$ with $u(\sigma) = u_0$. From the definitions of $\Sigma_1$ and $c_*$, $q' \geq 0$ on $[\sigma, \infty)$. Since $\frac{1}{1 + \beta S(u_0)}$ is decreasing and $q'' > 0$ if $q < \inf \{ \beta S(u_0) \geq 0 \}$, $u$ cannot increase indefinitely.

Hence there is a $\rho \geq \sigma$ with

$$u(\rho) \geq u_0, u'(\rho) = 0, u''(\rho) = \frac{w(\rho)}{c} \leq 0,$$

$$w'(\rho) > 0, q'(\rho) \geq 0.$$
A contradiction then results from (iv) of Lemma 10.

Now apply the technique of Lemma 8, including use of Proposition 3 and Lemma 10, to show that \( u^*_c > 0 \) and \( q^*_c < q_0 \) on \((t_2, \infty)\). In particular, Lemma 10 is used to show that there is no \( t > t_2 \) (in fact, no \( t \) at all) with \((u^*_c(t), q^*_c(t)) = (u_0, q_0)\). It follows that on \((t_2, \infty)\), \( q^*_c' > 0 \), and so indeed, \( U_{\varepsilon, c, \ell}^+ \) is homoclinic. \(\square\)

2. DISCUSSION

It is interesting to compare the solutions found above, and in [1], with well-known results for the FitzHugh Nagumo model [3], which is the system

\[
\begin{align*}
    u' &= v \\
    v' &= cv - f(u) + w \\
    w' &= \varepsilon (u - \gamma w),
\end{align*}
\]

where

\[ f(u) = u(1 - u)(u - a), \]

and \( a \), chosen in \((0, \frac{1}{2})\), represents the “threshold” in the model.

As stated earlier, there is an additional hypothesis in the existence result given in [1], namely Hypothesis 3.1 in that paper. Without taking time to state it precisely, this hypothesis implies that the homoclinic orbit of (0.4) passes “under the knee”. This refers to the projection of the orbit in \( \mathbb{R}^4 \) onto the \((u, q)\) plane, and means that \( \min_{s \in \mathbb{R}} q(s) < q_{\text{knee}} = h(u_{\text{knee}}) \). (See Figure 3)

![Figure 3. under the knee](image)

This is in contrast to the well-known behavior for the FitzHugh-Nagumo pulse, which for small \( \varepsilon \) can be described as having a jump up, during which \( u \) increases rapidly while \( w \) is nearly zero, followed by a slow increase in \( w \), and then a jump down, with \( w \) again nearly constant (but positive), and \( u \) decreasing rapidly. In this case, the jump down occurs before \((u, w)\) reaches the knee. (Figure 4).

We have done some preliminary numerical investigation to test whether it is possible, in the model studied in this paper and with the particular function \( S \) in (0.3), to adjust the parameters \( \lambda \) and \( \kappa \) so that the jump down occurs before \( w \) reaches the knee. We have not found such a pair \((\lambda, \kappa)\), but we cannot assert that none exists. For the FitzHugh-Nagumo model, however, with the particular function \( f \) given above, it is clear that the jump down is always before reaching the knee. This follows because the graph of \( f \) is symmetric around its inflection point. So we searched numerically for alternative functions to use for \( f \) which, while still “cubic like”, permit the down jump of the singular solution to be (in the
FitzHugh-Nagumo case) over the knee. We found such a function, as illustrated in Figure 5. We are not aware of a method which determines analytically where the downjump occurs, either for the model of Faye or that of FitzHugh-Nagumo when $f$ is assymmetric.

Finally, we remark that the existence proof in [3] for fast and slow homoclinic orbits of the FitzHugh-Nagumo system does apply to functions $f$ such as that pictured in Figure 5. Where the “downjump” occurs is of no concern. On the other hand it appears that the proof by geometric perturbation in this case, while probably basically still valid, requires a more complicated analysis because the downjump of the singular solution may occur at the knee.

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Lemma 1. \( tu \) is in (\( T \))

The characteristic polynomial of \( f(A) \)

Recall that \( h(u) = \frac{u}{s(u)} \). Condition \( 1 \) implies that the equation \( q_0 = h(w) \) has three solutions, \( u_0 < u_m < u_+ \), and by Condition \( 2 \) \( h' (u_0) > 0 \). It follows that \( F \) has one real positive eigenvalue. Also, \( f (-\frac{1}{c}) = \frac{b^2}{c} S'(u_0) q_0 > 0 \), which implies that \( F \) has two real negative eigenvalues.

Further, it is easily seen that there is an eigenvector corresponding to the positive eigenvalue of \( F \) which points into the positive octant. If \( r = (u, v, w) \) is a solution lying on the branch \( \mathcal{U}_{0,c}^+ \) of the unstable manifold of \( F \) at \( r_0 \), then initially, \( u', v' = w, \) and \( w' \) are positive. It follows from the first two equations of \( F \) that \( v > u \) as long as \( w > 0 \). Also, \( u \geq q_0 S(u) \) for \( u_0 \leq u \leq u_m \), and so \( w' > 0 \). While \( u \) is in \( (u_0, u_m) \), \( u \) is also in \( (u_0, u_m) \). Hence there is a first \( t_0 \) such that \( u(t_0) = u_m \), and we can assume that \( t_0 = 0 \). We have now proved the assertions of the first and second sentences of Lemma \( 1 \).

For the third sentence we need a comparison lemma. For each \( c > 0 \), let \( r_c = (u_c, v_c, w_c) \) be the unique solution of \( F \) on \( \mathcal{U}_{0,c}^+ \) such that \( w_c > 0 \) on \( (-\infty, 0) \) and \( u_c(0) = u_m \). Suppose that \( w_c > 0 \) on a maximal interval \( (-\infty, T_c) \), where possibly \( T_c = \infty \). Then in \( (-\infty, T_c) \) we can consider \( u \) and \( w \) as functions of \( v \), letting \( u_c(t) = U_c(v_c(t)) \) and \( w_c(t) = W_c(v_c(t)) \). This defines the functions \( U_c \) and \( W_c \) on the interval \( I_c = (u_0, \lim_{t \to T_c} v_c(t)) \), and for \( v \) in this interval,

\[
U_c'(v) = v - U_c(v) c W_c(v), \quad W_c'(v) = \frac{v^2 (v - q_0 S(U_c(v)))}{W_c(v)}.
\]
Lemma 17. If $c_2 > c_1 > 0$ then $\lim_{t \to T_{c_2}} v_{c_2}(t) > \lim_{t \to T_{c_1}} v_{c_1}(t)$ if the second of these two limits is finite, or else each of these limits is infinite and $r_{c_1}$ and $r_{c_2}$ are both defined on $R = (-\infty, \infty)$. Further, in the interval $I_{c_1}$,

\[(A.4)\]

\[
\begin{align*}
U_{c_2} &< U_{c_1} \\
W_{c_2} &> W_{c_1}
\end{align*}
\]

Proof. We first show that \((A.4)\) holds on some initial interval $u_0 < v < u_0 + \delta$. This is seen by comparing unit eigenvectors corresponding to the positive eigenvalues $\lambda_1(c_1)$ and $\lambda_1(c_2)$ of the linearizations of \((1.3)\) around $r_0$. Suppose that for a particular $c$ the eigenvector corresponding to $\lambda_1(c)$ is $(n_1(c), n_2(c), n_3(c))$. Then

\[
\begin{align*}
n_1(c) &= \frac{n_2(c)}{(1 + \lambda_1(c))} \\
n_3(c) &= \lambda_1(c) n_2(c)
\end{align*}
\]

Inequalities \((A.4)\) follow near $r_0$ if $\lambda_1(c_2) > \lambda_1(c_1)$. For this we turn to the characteristic polynomial of $A$, given in \((A.1)\) but now denoted by $F(X,c)$.

It is easier to work with $F = cf$, noting that $c > 0$. The positive eigenvalue of $A$ is determined by the equation

\[F(\lambda_1(c), c) = 0\]

and the condition $\lambda_1(c) > 0$. Then

\[
\begin{align*}
\frac{\partial F}{\partial X}(\lambda_1(c), c) \frac{d\lambda_1(c)}{dc} &= -\frac{\partial F}{\partial c}(\lambda_1(c), c).
\end{align*}
\]

Since $F(0, c) < 0$, $\frac{\partial F}{\partial X}(0, c) < 0$ and $\frac{\partial^2 F}{\partial X^2}(X, c) > 0$ for $X \geq 0$, $\frac{\partial F}{\partial c}(\lambda_1(c), c) > 0$. Also, $\frac{\partial F}{\partial c}(\lambda_1, c) = \lambda_1(\lambda_1 - b^2)$. It follows that $\frac{d\lambda_1(c)}{dc} > 0$ if $\lambda_1 < b$. But

\[F(b, c) = \frac{b^2}{c} S'(u_0) q_0 > 0,
\]

so indeed, $\lambda_1(c) < b$.

Therefore \((A.4)\) holds on some interval $(u_0, u_0 + \delta)$. Suppose that the first inequality fails at a first $\hat{v} \in I_{c_1}$, while the second holds over $(u_0, \hat{v})$. Then at $\hat{v}$, $U_1 = U_2$, $W_2 > W_1$. But then, $U_3(\hat{v}) < U_3(\hat{v})$, a contradiction since $U_2 < U_1$ on $(0, \hat{v})$. A similar argument eliminates the other possibilities, using the fact that $S$ is increasing, and this completes the proof of the Lemma 17.

Now we wish to show that for small $c > 0$, $w_c = 0$ before $v_c = 1$. It is in this step that Condition 3 is used.

Lemma 18. There is a $\bar{w} > 0$ such that for any $c > 0$, if $|w_c(\tau)| > \bar{w}$ and $0 < v_c < 1$, then $|w_c| > \bar{w}$ for $t > \tau$ and $v_c$ leaves the interval $(0, 1)$. If $w_c(\tau) > \bar{w}$ then $v_c$ crosses 1, while if $w_c(\tau) < -\bar{w}$ then $v_c$ crosses 0.

Proof. Let $\bar{w} = \sqrt{2}b$. Since $|w'| \leq b^2$, if $w(\tau) = \sqrt{2}b$, then for $s > 0$, $w(\tau + s) \geq \sqrt{2}b - b^2s$, from which follows that $v$ must leave $(0, 1)$ before $s = \frac{\sqrt{2}}{b}$.

Lemma 19. If $0 < w_c \leq \bar{w}$ on $(-\infty, \tau)$ then $0 < v_c - u_c < c\bar{w}$ on this interval. If $|w_c| \leq \bar{w}$ on $(-\infty, \sigma)$, then $|v_c - u_c| < c\bar{w}$ on this interval.

Proof. With $r = r_c$, $(v - u)' = w - \frac{w_c}{r} \leq \bar{w} - \frac{w_c}{r}$, so if $v - u > c\bar{w}$ then $(v - u)' < 0$. Also, if $v - u = 0$ in $(-\infty, \tau)$ then $(v - u)' > 0$. Since $v - u \to 0^+$ as $t \to -\infty$, the first sentence of the lemma follows and the second is similar.
Based on this lemma, we consider, in addition to (0.5), the system

(A.5) \[ v' = w \]
\[ w' = b^2 (v - q_0 S(v)) \]

This system has equilibrium points at \((u_0, 0)\), \((u_m, 0)\), and \((u_+, 0)\), and a standard phase plane analysis, assuming Condition [3] shows that the positive branch \(U^+_0\) of unstable manifold of \(A.5\) at \((u_0, 0)\) is homoclinic. Also we consider the system

(A.6) \[ v' = w \]
\[ w' = b^2 (v - q_0 S(v - \hat{c})) \]

for small \(\hat{c}\). Choose \(\hat{c}\) so small that this system also has three equilibrium points, and a homoclinic orbit based at the left most of these. This orbit entirely encloses the homoclinic orbit of \(A.5\).

Finally we consider the system

(A.7) \[ v' = w \]
\[ w' = b^2 (v - q_0 S(v + \hat{c})) \]

For sufficiently small \(\hat{c}\) this system also has a homoclinic orbit. This orbit lies entirely inside the homoclinic orbit of \(A.5\). However, the lower left branch \(U^-_{0,0}\) of the unstable manifold of this system crosses the homoclinic orbits of \(A.5\) and \(A.6\), and this branch will play a role below. (See Figure 6.)

![Figure 6. homoclinic orbits of, from inner to outer, \(A.7\), \(A.5\), and \(A.6\), part of \(U^-_{0,0}\) for \(A.6\), and an orbit of \(0.5\) (dotted).](image)

From now on, \((v_1, w_1)\), \((v_2, w_2)\), and \((v_3, w_3)\) will denote the unique solutions of the systems \(A.5\), \(A.6\), and \(A.7\) respectively which lie on the homoclinic orbits of those systems and satisfy \(v_i(0) = u_m\). In each of these cases, if \((v, w)\) is homoclinic then \(|w|\) is bounded by \(\bar{w}\). This follows from the definition of \(\bar{w}\) in Lemma 18, the results of which also apply to \(A.6\) and \(A.7\), with the same proofs. If \(|w|\) exceeds \(\bar{w}\) then \(p\) is not bounded.

Recall that in Lemmas 1 and 2 \(r_{0,c} = (u_{0,c}, v_{0,c}, w_{0,c})\) denoted the unique solution on the unstable manifold \(U_{0,c}\) such that \(u_{0,c}(0) = 0\) and \(w_{0,c} > 0\) on \((-\infty, 0]\). In the rest of this proof we will denote this solution by \((u, v, w)\). By Lemma 19 we can choose \(c\) so small that if \(t_1\) is the first zero of \(w_c\), then

(A.8) \[ v - \hat{c} < u < v \]
on \((-\infty, t_1]\).

**Lemma 20.** Condition $A.8$ implies that if $w_c > 0$ on $(-\infty, t]$, then $(v_c(t), w_c(t))$ is in the annular region between the orbit of $(v_1, w_1)$ and the orbit of $(v_2, w_2)$.

**Proof.** The proof is similar to the proof of Lemma 17. As long as $w_c > 0$, $u_c$ and $w_c$ can be considered functions of $v_c$. Also,

\[
\frac{dw_c}{dv_c} = \frac{b^2 (v_c - q_0 S(u_c))}{w_c}.
\]

By considering the eigenvalues of the linearizations of $A.6$ as functions of $c$ we can show, using $A.2$, that for large negative $t$, $(v(t), w(t))$ lies in the claimed annular region. Suppose that for some first $\tau$, $(v_c(\tau), w_c(\tau))$ lies on the upper boundary of this region, that is, on the homoclinic orbit of $A.6$, at a point where $w > 0$. The slope of this homoclinic orbit at this point is

\[
\frac{dw_2}{dv_2} = \frac{b^2 (v_2 - q_0 S(v_2 - \hat{c}))}{w_2} = \frac{b^2 (v_c(\tau) - q_0 S(v_c(\tau) - \hat{c}))}{w_c(\tau)}.
\]

But $u_c(\tau) > v_c(\tau) - \hat{c}$ (since $u_c < v_c$ as long as $w_c \geq 0$), and since $S$ is increasing and $w_c(\tau) > 0$, it follows from $A.9$ that $\frac{dw_c}{dv_c} < \frac{dw_2}{dv_2}$, and so the curve $(v_c, w_c)$ arrives at this point from outside of the annular region, contradicting the definition of $\tau$. In a similar manner it is shown that $(v_c, w_c)$ lies above the orbit of $(v_1, w_1)$ as long as $w_c > 0$. This uses the bound $u_c < v_c$ as long as $w_c > 0$.

A similar comparison shows that if $t = t_1(c)$ is the first point where $w_c(t) = 0$ then $v_c(t_1(c))$ is an increasing function of $c$, for $0 < c < c_0^*$, and that for $c > c_0^*$, $w_c > 0$ on $R$. This shows the uniqueness of $c_0^*$. To complete the proof of Lemma 2 we show that from the first $t_1$ where $w_c(t_1) = 0$, the curve $(v_c, w_c)$ lies either to the right or below the orbit $(v_3, w_3)$, and also below the left branch of the unstable manifold of $A.7$. At least up to the point where $w_c = -\hat{w}$. (If $w_c = -\hat{w}$, then, as in Lemma 18, $v_c$ becomes negative, which is what we are trying to show. See Figure 8.) This follows by the same sort of comparison as above, now comparing $(v_c, w_c)$ with the lower half of the unstable manifold of $A.7$. This is possible because by Lemma 19, $u_c < v_c + \hat{c}$ as long as $-\hat{w} < w_c < 0$. We omit further details.

**A.1. Proof of Lemma 14**

**Proof.** The result follows from Lemma 18 if $w_c$ increases monotonically to above $\hat{w}$, so we can assume that if $t_1$ is the first zero, if any, of $w_c$, then $w_c < \hat{w}$ on $(0, t_1)$. As earlier in obtaining $A.8$, it follows that if $p = p_c$, then

\[
v > u > v - c\hat{w}
\]
on $(0, t_1)$.

Also, we consider the equation obtained from (0.4) by formally setting $c = 0$ in (0.4), namely

\[
v'' = b^2 (v - \frac{S(v)}{1 + \beta S(v)})
\]

This equation is easily seen to have only one equilibrium point, a saddle, and the branch of the unstable manifold is unbounded, with $w > 0$ on $(-\infty, -c)$. Assuming that $v < 1$, estimate (A.10) shows that $v - u$ tends to zero uniformly on the interval $(\infty, t_1)$, as $c$ approaches zero (even though $t_1$ may be unbounded as a function of $c$), as does $S(v) - S(u)$. The result follows easily.

\[\square\]
APPENDIX B. PROOF OF LEMMA 3

Proof. Suppose that the linearization of (0.4) around \( p_0 \) is \( P' = BP \). Then

\[
B = \begin{pmatrix}
-\frac{1}{c} & \frac{1}{c} & 0 & 0 \\
0 & 0 & 1 & 0 \\
-b^2 q_0 S' (u_0) & b^2 & 0 & -b^2 S (u_0) \\
-\frac{\varepsilon}{c^2} \beta q_0 S' (u_0) & 0 & 0 & -\frac{\varepsilon}{c^2} (1 + \beta S (u_0))
\end{pmatrix}
\]

The characteristic polynomial of \( B \) with respect to \( \varepsilon \) is

\[
g(X) = X^4 + \frac{1}{c} (1 + \varepsilon (\beta S (u_0) + 1)) X^3 + \left(-b^2 + \frac{1}{c^2} \varepsilon (\beta S (u_0) + 1)\right) X^2 + \frac{b^2}{c} (q_0 S' (u_0) - 1 - \varepsilon (\beta S (u_0) + 1)) X + \frac{b^2}{c^2} \varepsilon (q_0 S' (u_0) - 1 - \beta S (u_0))
\]

While proving Lemma 4 we showed that if \( \varepsilon = 0 \), then one of the non-zero eigenvalues of \( B \) is positive and two are real and negative. We also saw that \( q_0 S' (u_0) < 1 \), and therefore, \( \det B < 0 \) if \( \varepsilon > 0 \). Since the trace of \( B \) is also negative, if \( \varepsilon > 0 \) then \( B \) has either one or three eigenvalues with negative real parts, and for sufficiently small \( \varepsilon \) it has three, all of which are real. In fact, since \( g(0) < 0 \), \( g'(0) < 0 \), and \( g'''(X) > 0 \) if \( X > 0 \), \( B \) has exactly one real positive eigenvalue for every \( (\varepsilon, c) \) in the positive quadrant \( \varepsilon > 0, c > 0 \). For each \( c > 0 \), as \( \varepsilon \) increases the other roots of \( g \) remain in the left hand plane unless, for some \( \varepsilon \), two of them are pure imaginary. Consideration of the characteristic polynomial in this case (one negative, one positive, and twopure imaginary roots) shows that the coefficients of \( X \) and \( X^3 \) have the same sign. This is not the case with \( g \), because the coefficient of \( X^3 \) is positive and the coefficient of \( X \) is negative.

Hence, as asserted in Lemma 3, the unstable manifold \( U_{\varepsilon,c} \) of (0.4) at \( p_0 \) is one dimensional. Further, because \( q_0 S' (u_0) < 1 \), it follows from (B.1) that if \( \mu = (\mu_1, \mu_2, \mu_3, \mu_4) \) is the unit eigenvector of \( B \) with \( \mu_1 > 0 \), then \( \mu_2 > 0 \), and \( \mu_3 > 0 \). Also (B.1) implies that if \( \varepsilon = 0 \) then \( \mu_4 = 0 \) and if \( \varepsilon > 0 \) then \( \mu_4 < 0 \). The claimed behavior for large negative \( t \) of solutions on \( U_{\varepsilon,c} \) follows. The continuity of \( U_{\varepsilon,c} \) for \( \varepsilon \geq 0 \) follows from Theorem 6.1 in chapter 6 in the text of Hartman.

The final assertion of the lemma, that \( \lambda_1 (c, \varepsilon) > \lambda_1 (c, 0) \) if \( c > 0 \) and \( \varepsilon > 0 \) follows by writing the characteristic polynomial of \( B \) in the form

\[
g(x) = \lambda f(x) + \frac{\varepsilon}{c} (1 + \beta S (u_0)) f(x) - \frac{\varepsilon}{c^2} b^2 S (u_0) \beta S' (u_0) q_0.
\]

We see that since \( f(\lambda_1 (c, 0)) = 0 \), \( g(\lambda_1 (c, 0)) < 0 \). Hence \( \lambda_1 (c, \varepsilon) > \lambda_1 (c, 0) \), completing the proof of Lemma 3.

\[\square\]

\[\text{6Our terminology is different from Hartman's because we define stable and unstable manifolds even when } \varepsilon = 0. \text{ In this case the unstable manifold } U_{\varepsilon,c}^s, \text{ all we need, is the set of all solutions } p(t) \text{ which tend to } p_0 \text{ at an exponential rate as } t \to -\infty. \text{ To obtain the desired continuity of } U_{\varepsilon,c}^s \text{ with respect to } \varepsilon \text{ and } c, \text{ apply Hartman's theorem to (0.5) augmented with equations } \varepsilon' = 0 \text{ and } c' = 0. \text{ This is the closest we come to center manifolds in our approach.}\]