Generalized IBL models for gravity-driven flow over inclined surfaces

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Abstract. In this investigation we propose several generalized first-order integral-boundary-layer (IBL) models to simulate the two-dimensional gravity-driven flow of a thin fluid layer down an incline. Various cases are considered and include: isothermal and non-isothermal flows, flat and wavy bottoms, porous and non-porous surfaces, constant and variable fluid properties, and Newtonian and non-Newtonian fluids. A numerical solution procedure is also proposed to solve the various model equations. Presented here are some results from our numerical experiments. To validate the generalized IBL models comparisons were made with existing results and the agreement was found to be reasonable.

1. Introduction
Thin liquid films with a free surface flowing down an inclined plane are prone to interfacial instabilities triggered by long-wave instabilities brought about by inertial effects which generate waves propagating along the surface. An illustration of this occurrence can be observed when rain water drains along a sloped pavement. This phenomenon is actually encountered in many different settings where it may have important consequences [1,2]. In industrial processes interfacial instability is a crucial factor, for example, in coating applications and in the effective operation of various equipment involving heat and mass transfer. In the environment, surface waves in lava and debris flows can coalesce into large bores with enough force to cause serious damage.

Due to the prevalence and significance of the instability of gravity-driven flows, researchers have shown considerable interest in predicting the conditions for the onset of instability, and determining the structure of the resulting waves. Groundbreaking experimental and theoretical investigations have been carried out by the father and son team Kapitza and Kapitza [3] during their house arrest in the late 1940s. They established a dimensionless group, now known as the “Kapitza number”, combining the surface tension and the viscosity of the fluid. They obtained a relationship between this quantity and the inertial effects, specified by the Reynolds number, describing the conditions for the onset of instability. A more systematic theoretical study, based on a linear stability analysis of the Navier-Stokes equations, was later undertaken by both Benjamin [4] and Yih [5] who found that the critical Reynolds number is actually proportional to the cotangent of the angle of inclination. Although instabilities are triggered by inertial effects, they can be enhanced by thermocapillarity effects if the liquid film is heated from below or above.
Several different types of models have been developed to model the flow down an incline and most of them assume that the thickness of the fluid layer is much smaller than the characteristic length in the direction of the flow. Here, we report on three commonly used models. The first is the shallow-water model (SWM) which is founded on shallow-water theory, and hence assumes that the fluid is incompressible and inviscid. This model is limited to gentle inclines and incorporates modifications to make it more realistic such as adding terms to partially account for viscosity and bottom friction. Two different formulations exist: one pertaining to laminar flow and another for turbulent flow. A thorough description of the two versions of the model can be found in [6]. Another model is known as the integral-boundary-layer (IBL) model and is derived more rigorously from the Navier-Stokes equations. These equations are first simplified by discarding negligible terms based on a scaling argument. These simplified equations are then integrated across the fluid layer to eliminate the cross-flow variation. An assumed velocity profile is inserted when necessary in order to perform the integration along with appropriate boundary conditions along the incline and free surface. The IBL model was originally developed by Shkadov [7]. The third model is referred to as a weighted residual model (WRM) and is derived following a similar procedure to that used for the IBL model. The main difference is that before integrating in the cross-stream direction, the equations are multiplied by carefully selected weight functions. This method was originally proposed by Ruyer-Quil and Manneville [8] and is fully described in [9].

In this study we have adopted the IBL model and have generalized it to handle isothermal and non-isothermal flows, flat and wavy bottoms, porous and non-porous surfaces, constant and variable fluid properties, and Newtonian and non-Newtonian fluids. Although IBL models fail to precisely estimate the critical Reynolds number for the onset of instability, they adequately predict the size and the propagation speed of the waves that form on the surface under unstable flow conditions, and produce results that are comparable with observations and experiments. Indeed, this approach continues to be used for more complicated flows because they are easier to implement. Although we will focus on first-order IBL models in this investigation, extensions to higher-order IBL models are possible. Higher-order models, in general, become rather complicated and at times not much is gained. For example, a second-order IBL model will yield the same critical Reynolds number as a first-order IBL model.

This paper is organized as follows. In the next section we introduce a convenient coordinate system and present the governing equations and boundary conditions for the case of isothermal flow of a Newtonian fluid down a flat inclined surface, which we denote as the base case. The IBL model for the base case is then formulated and a numerical solution procedure is outlined. The subsequent section is devoted to extensions of the base case to model more complicated flows and following that various results are presented. The final section summarizes the key findings.

2. The Base Case

We first consider the case of isothermal flow of a Newtonian fluid down a flat non-porous surface that is inclined at an angle \( \beta \) with respect to the horizontal as shown in Figure 1. An \((x, z)\) rectangular coordinate system is chosen with the \(x\)-axis pointing down the incline and the \(z\)-axis pointing into the liquid layer. In the diagram \(z = h(x, t)\) refers to the position of the free surface.

The governing Navier-Stokes equations for two-dimensional, viscous, incompressible flow cast in dimensionless form are as follows

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

\[\text{Re} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\text{Re} \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + 3,
\]
Here, $u, w$ are the velocities in the $x, z$ directions, respectively, and $p$ is the pressure. The dimensionless parameter appearing is the Reynolds number $Re = \frac{\rho Q}{\mu}$; $Q$ denotes the prescribed flow rate, $\rho$ is the density, and $\mu$ is the dynamic viscosity. The scales used to cast the equations in dimensionless form are as follows: the Nusselt thickness given by

$$H = \left(\frac{3\mu Q}{g\rho \sin \beta}\right)^{1/3},$$

was the length scale where $g$ is the acceleration due to gravity, the pressure was scaled using $\rho Q^2$, velocities were scaled according to $Q H$, and time was scaled using $\frac{H^2}{Q^2}$.

The system of equations (1)-(3) are to be solved subject to the following boundary conditions. Along the free surface $z = h(x, t)$ we impose the dynamic conditions

$$p = \frac{2}{Re \left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right)} \left[\frac{\partial u}{\partial x} \left(\left(\frac{\partial h}{\partial x}\right)^2 - 1\right) - \frac{\partial h}{\partial x} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] - \frac{We \frac{\partial^2 h}{\partial x^2}}{\left(1 + \left(\frac{\partial h}{\partial x}\right)^2\right)^{3/2}}, \quad (4)$$

$$0 = \left(1 - \left(\frac{\partial h}{\partial x}\right)^2\right) \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - 4 \frac{\partial h}{\partial x} \frac{\partial u}{\partial x} \frac{\partial x}{\partial x}. \quad (5)$$

Conditions (4) and (5) arise from a balance between the ambient pressure and surface tension with $We = \frac{\sigma Q^2}{\mu^2}$ denoting the Weber number and $\sigma$ is surface tension. The kinematic condition along the free surface is given by

$$w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{at} \quad z = h(x, t). \quad (6)$$

Lastly, along the incline $z = 0$ we apply the no-slip and impermeability conditions

$$u = w = 0. \quad (7)$$
Next we derive the long-wave equations which are best suited to capture the long-wave instability and known to be the most unstable. This is accomplished by introducing slow time (τ) and space (X) variables and rescaling the vertical velocity (W) according to

$$\tau = \varepsilon t, \ X = \varepsilon x, \ w = \varepsilon W,$$

where $0 < \varepsilon \ll 1$ is a small parameter which measures the ratio of the thickness of the fluid layer to some characteristic length in the $x$ direction. Then the scaled governing equations (1)-(3) to first-order in $\varepsilon$ become

$$\frac{\partial u}{\partial X} + \frac{\partial W}{\partial z} = 0,$$  

$$\varepsilon Re \left( \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial X} + W \frac{\partial u}{\partial z} \right) = -\varepsilon Re \frac{\partial p}{\partial X} + 3 + \frac{\partial^2 u}{\partial z^2},$$

$$0 = -Re \frac{\partial p}{\partial z} - 3 \cot \beta + \varepsilon \frac{\partial^2 W}{\partial z^2}.$$

These can be viewed as the first-order long-wave equations. Likewise, the transformed boundary conditions at $z = h(X, \tau)$ to first-order in $\varepsilon$ are

$$p = \frac{2\varepsilon}{Re} \frac{\partial W}{\partial z},$$

$$W = \frac{\partial h}{\partial \tau} + u \frac{\partial h}{\partial X},$$

$$0 = \frac{\partial u}{\partial z},$$

while the bottom conditions at $z = 0$ are

$$u = W = 0.$$

We note that to first order the surface tension term disappears from the problem.

Next, we define the flow rate, $q(X, \tau)$, and prescribe a profile for $u$ as follows

$$q = \int_0^h udz, \ u = \frac{3gz(2h - z)}{2h^3}.$$

We note that the parabolic velocity profile is based on the steady-state unidirectional solution for a constant fluid thickness, $h$. Integrating the continuity equation across the fluid layer and imposing the kinematic condition yields

$$\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0.$$

Integrating the hydrostatic equation (10) subject to condition (11) produces an expression for the pressure which can then be substituted into the momentum equation (9). Then integrating the momentum equation across the fluid layer using condition (13) leads to the relation

$$\varepsilon Re \left( \frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \int_0^h u^2 dz \right) = 3h \left( 1 - \varepsilon \cot \beta \frac{\partial h}{\partial X} \right) - \frac{\partial u}{\partial z} \bigg|_{z=0}.$$
In order to evaluate the integral and the last term in the above equation we make use of the prescribed velocity profile. Doing this then brings us to the following equation for the flow rate
\[
\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left( \frac{6q^2}{5h} + \frac{3 \cot \beta}{2Re} h^2 \right) = \frac{3}{\varepsilon Re} \left( h - \frac{q}{h^2} \right) .
\] (16)

Hence, our IBL model for the base case consists of equations (15) and (16) governing the flow variables \( h \) and \( q \). We note that a consequence of integrating across the fluid is that the \( z \) dependence has now been eliminated.

To numerically solve the system of equations (15) and (16) we implement the fractional-step splitting technique [10]; that is, we first solve
\[
\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0 ,
\]
\[
\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left( \frac{6q^2}{5h} + \frac{3 \cot \beta}{2Re} h^2 \right) = 0 ,
\]
over a time step \( \Delta \tau \), and then solve
\[
\frac{\partial q}{\partial \tau} = \frac{3}{\varepsilon Re} \left( h - \frac{q}{h^2} \right) ,
\] (17)
using the solution obtained from the first step as an initial condition for the second step. The second step then returns the solution for \( q \) at the new time \( \tau + \Delta \tau \) with the understanding that \( h \) is constant during the second step.

The first step involves solving a nonlinear system of hyperbolic conservation laws which, when expressed in vector form, can be written compactly as
\[
\frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial X} = 0 ,
\]
where
\[
\mathbf{U} = \left[ \begin{array}{c} h \\ q \end{array} \right] , \quad \mathbf{F}(\mathbf{U}) = \left[ \begin{array}{c} \frac{6q^2}{5h} + \frac{3 \cot \beta}{2Re} h^2 \\ q \end{array} \right] .
\]

While there are several schemes available to solve such a system, we chose to adopt MacCormack’s method [11] because of its versatility and simplicity. It can handle all the flow configurations considered in this study. For some of the flows considered eigen-based methods will not be practical because of the complicated eigenstructure of the system. MacCormack’s method is a conservative second-order accurate finite-difference scheme which correctly captures discontinuities and converges to the physical weak solution of the problem via the explicit predictor-corrector scheme
\[
\mathbf{U}^* = \mathbf{U}^n - \frac{\Delta \tau}{\Delta X} \left[ \mathbf{F}(\mathbf{U}^n_{j+1}) - \mathbf{F}(\mathbf{U}^n_{j}) \right] ,
\]
\[
\mathbf{U}^{n+1} = \frac{1}{2} \left( \mathbf{U}^n_j + \mathbf{U}^* \right) - \frac{\Delta \tau}{2\Delta X} \left[ \mathbf{F}(\mathbf{U}^*_{j+1}) - \mathbf{F}(\mathbf{U}^*_{j-1}) \right] ,
\]
where the notation \( \mathbf{U}^n_j \equiv \mathbf{U}(X_j, \tau_n) \) is utilized with \( \Delta X \) denoting the uniform grid spacing and \( \Delta \tau \) is the time step.

Since \( h \) remains constant during the second step, the second step amounts to solving a linear ordinary differential equation. The exact solution to (17) is easily found to be
\[
q(X, \tau + \Delta \tau) = h_0^3 + (q_0 - h_0^3) e^{-c_1 \Delta \tau} \quad \text{where} \quad c_1 = \frac{3}{\varepsilon Re h_0^2} .
\] (18)

In the above \( h_0 \) and \( q_0 \) denote the solutions emerging from the first step of the procedure.
3. Extensions

In this section we will entertain several extensions of the base case.

3.1. Wavy Bottom

The first extension involves an uneven incline. The bottom over which the fluid flows is no longer \( z = 0 \) but rather \( z = \zeta(X) \) with \( \zeta(X) \) denoting the bottom topography. Although we will present the model equations for arbitrary \( \zeta(X) \), the numerical results to be discussed later will focus on a surface having periodic undulations characterized by the amplitude and wavelength of the undulations. Here, the wavelength of the undulations is taken to be the length scale in the \( x \) direction. The governing equations are identical to the base case given by (8)-(10). The only boundary condition that changes is the kinematic condition along the free surface \( z = h(X, \tau) + \zeta(X) \) which now becomes

\[
W = \frac{\partial h}{\partial \tau} + u \frac{\partial h}{\partial X} + u\zeta',
\]

where the prime denotes differentiation with respect to \( X \). We also note that the no-slip and impermeability conditions (14) are now applied along the bottom \( z = \zeta(X) \) instead of \( z = 0 \).

The prescribed velocity profile must also change to account for the variable bottom; it now takes the form

\[
u = \frac{3q}{2h^3} \left[ 2(h + \zeta)z - z^2 - \zeta^2 - 2h\zeta \right].
\]

Following the procedure outlined in the previous section yields the following equations

\[
\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0,
\]

\[
\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left( \frac{6q^2}{5h} + 3\cot\beta h^2 \right) = -\frac{3\cot\beta}{Re} \zeta' h + \frac{3}{\varepsilon Re} \left( h - \frac{q}{h^2} \right).
\]

The numerical solution procedure for solving the system (20)-(21) will be very similar to the base case. The only difference will be in the second step where equation (17) now gets replaced by

\[
\frac{\partial q}{\partial \tau} = -\frac{3\cot\beta}{Re} \zeta' h + \frac{3}{\varepsilon Re} \left( h - \frac{q}{h^2} \right),
\]

and has the solution

\[
q(X, \tau + \Delta \tau) = h_0^0(1 - \frac{h}{h_0^0} \cot \beta' \zeta') + [q_0 - h_0^0(1 - \frac{h}{h_0^0} \cot \beta' \zeta')]e^{-\varepsilon \Delta \tau}.
\]

As a final note we add that if we set \( \zeta = 0 \) then we recover the base case.

3.2. Porous Bottom

This case corresponds to an incline that is porous. According to Beavers & Joseph [12] this alters the bottom boundary condition along \( z = 0 \) to be a slip condition given by

\[
u = \delta \frac{\partial u}{\partial z},
\]

where the dimensionless parameter \( \delta > 0 \) is the scaled slip length; when \( \delta = 0 \) this reduces to the familiar no-slip condition. The velocity profile for this case becomes

\[
u = \frac{3q(2hz - z^2 + 2\delta h)}{2h^2(h + 3\delta)}.
\]
Again, following the procedure outlined in the previous section we obtain

\[ \frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0, \quad (24) \]

\[ \frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left( \frac{3q^2(2h^2 + 10\delta h + 15\delta^2)}{5h(h + 3\delta)^2} + \frac{3\cot \beta}{2Re} h^2 \right) = \frac{3}{\varepsilon Re} \left( h - \frac{q}{h(h + 3\delta)} \right). \quad (25) \]

Observe that setting \( \delta = 0 \) again recovers the base case.

The numerical solution procedure for solving the system (24)-(25) will also be similar to that of the base case. There are two changes that need to be made. Here, the first step involves solving

\[ \frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left( \frac{3q^2(2h^2 + 10\delta h + 15\delta^2)}{5h(h + 3\delta)^2} + \frac{3\cot \beta}{2Re} h^2 \right) = 0, \]

while the second step solves

\[ \frac{\partial q}{\partial \tau} = \frac{3}{\varepsilon Re} \left( h - \frac{q}{h(h + 3\delta)} \right). \quad (26) \]

The exact solution to (26) is easily found to be

\[ q(X, \tau + \Delta \tau) = h_0^2(\Delta \tau) + [q_0 - h_0^2(\Delta \tau)]e^{-c_2\Delta \tau} \text{ where } c_2 = \frac{3}{\varepsilon Re h_0(\Delta \tau)}. \quad (27) \]

### 3.3. Non-Newtonian Fluid

The previous two cases dealt with extensions regarding the surface of the incline. We next consider the gravity-driven flow of a non-Newtonian power-law fluid. The first-order long-wave dimensionless equations are as follows \([13,14]\)

\[ \frac{\partial u}{\partial X} + \frac{\partial W}{\partial z} = 0, \quad (28) \]

\[ \varepsilon Re \left( \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial X} + W \frac{\partial u}{\partial z} \right) = -\varepsilon Re \frac{\partial p}{\partial X} + \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right)^n + \left( \frac{2n + 1}{n} \right)^n, \quad (29) \]

\[ 0 = -Re \frac{\partial p}{\partial z} + 2\varepsilon \frac{\partial}{\partial z} \left[ \frac{\partial W}{\partial z} \left( \frac{\partial u}{\partial z} \right)^{n-1} \right] + \varepsilon \frac{\partial}{\partial X} \left( \frac{\partial u}{\partial z} \right)^n - \left( \frac{2n + 1}{n} \right)^n \cot \beta, \quad (30) \]

where \( n \) is the power-law index. Positive values of \( n \) less than unity corresponds to a shear-thinning fluid while \( n > 1 \) corresponds to a shear-thickening fluid. When \( n = 1 \) we obtain a Newtonian fluid and equations (28)-(30) simplify to equations (8)-(10). The corresponding boundary conditions are the same as those for the base case given by (11)-(14) with the exception of (11) which now becomes

\[ p = \frac{2\varepsilon}{Re} \frac{\partial W}{\partial z} \left( \frac{\partial u}{\partial z} \right)^{n-1}. \quad (31) \]

For a power-law fluid the velocity profile is given by

\[ u = \frac{(1 + 2n)q}{(1 + n)h} \left[ 1 - \left( \frac{1 - \frac{z}{h}}{n} \right)^{\frac{1 + n}{n}} \right]. \]

Again, when \( n = 1 \) this reproduces the parabolic velocity profile for the base case.
In terms of the flow rate, $q$, and the fluid thickness, $h$, the governing equations take the form

$$\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0,$$

(32)

$$\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{2(2n+1)q^2}{(3n+2)h} + \left( \frac{2n+1}{n} \right)^n \cot \beta \frac{h^2}{2Re} \right] = \left( \frac{2n+1}{n} \right)^n \frac{1}{\varepsilon Re} \left[ h - \left( \frac{q}{h^2} \right)^n \right].$$

(33)

To solve the system (32)-(33) we proceed by first solving

$$\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0,$$

$$\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{2(2n+1)q^2}{(3n+2)h} + \left( \frac{2n+1}{n} \right)^n \cot \beta \frac{h^2}{2Re} \right] = 0,$$

using MacCormack’s method and then solve

$$\frac{\partial q}{\partial \tau} = \left( \frac{2n+1}{n} \right)^n \frac{1}{\varepsilon Re} \left[ h - \left( \frac{q}{h^2} \right)^n \right].$$

(34)

Exact solutions to (34) for specific values of $n$ have been found. For the shear-thickening case $n = 2$ the solution to (34) is given by

$$\frac{h_0^{5/2} + q(X, \tau + \Delta \tau)}{h_0^{5/2} - q(X, \tau + \Delta \tau)} = \left( \frac{h_0^{5/2} + q_0}{h_0^{5/2} - q_0} \right) e^{c_3 \Delta \tau} \quad \text{where} \quad c_3 = \frac{25}{2\varepsilon Re h_0^{3/2}},$$

(35)

while for the shearing-thinning case $n = \frac{1}{2}$ the exact solution is

$$\sqrt{q(X, \tau + \Delta \tau) + h_0^2 \ln(\sqrt{q(X, \tau + \Delta \tau) - h_0^2})} = \sqrt{q_0 + h_0^2 \ln(\sqrt{q_0} - h_0^2)} - \frac{\Delta \tau}{\varepsilon Re h_0}. \quad (36)$$

We will focus our results on these values. We note that equation (36) requires solving a nonlinear algebraic equation for $q(X, \tau + \Delta \tau)$ at each time step.

### 3.4. Non-isothermal Flow

The last extension concerns non-isothermal flows. This situation arises when the incline is maintained at a different temperature than that of the surrounding ambient fluid. Here, we will take the temperature of the incline, $T_0$, to exceed the ambient temperature, $T_a$, and both $T_0$ and $T_a$ will be taken to be constant. We will also allow the fluid properties to vary with temperature.

A standard approach is to let the fluid properties vary linearly with temperature as follows

$$\rho = \rho_a - \hat{\alpha} (T - T_a), \quad \mu = \mu_a - \hat{\lambda} (T - T_a), \quad \kappa = \kappa_a + \hat{\Lambda} (T - T_a), \quad \sigma = \sigma_a - \gamma (T - T_a),$$

where $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\Lambda}$ and $\gamma$ are positive dimensional parameters measuring the rate of change with respect to temperature, and $\rho_a$, $\mu_a$, $\kappa_a$ and $\sigma_a$ are reference values at $T = T_a$ of the density, viscosity, thermal conductivity and surface tension, respectively.

Using the first law of thermodynamics and standard thermodynamic relationships along with the Boussinesq approximation the first-order long-wave Navier-Stokes and energy equations in dimensionless form are as follows [15,16]

$$\frac{\partial u}{\partial X} + \frac{\partial W}{\partial z} = 0,$$

(37)
\[ \varepsilon \text{Re} \left( \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial X} + W \frac{\partial u}{\partial z} \right) = -\varepsilon \text{Re} \frac{\partial p}{\partial X} + 3(1 - \alpha T) + \frac{\partial}{\partial z} \left[ (1 - \lambda T) \frac{\partial u}{\partial z} \right], \]  
\( (38) \)

\[ \text{Re} \frac{\partial p}{\partial z} = -3(1 - \alpha T) \cot \beta + \varepsilon \frac{\partial}{\partial z} \left[ (1 - \lambda T) \frac{\partial W}{\partial z} \right] - \varepsilon \lambda \frac{\partial T}{\partial X} \frac{\partial u}{\partial z} - \varepsilon \lambda \frac{\partial T}{\partial z} \frac{\partial W}{\partial z}, \]  
\( (39) \)

\[ \varepsilon \text{Pr} \text{Re} \left( \frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial X} + W \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left[ (1 + \Lambda T) \frac{\partial T}{\partial z} \right]. \]  
\( (40) \)

In the above \( P_r = \frac{\mu \alpha \kappa_p}{k_a} \) is the Prandtl number with \( \kappa_p \) denoting the specific heat, and \( \alpha = \tilde{\alpha} \Delta T/\rho_a \), \( \lambda = \tilde{\lambda} \Delta T/\mu_a \), \( \Lambda = \tilde{\Lambda} \Delta T/\kappa_a \) are scaled rate of change parameters, and \( \Delta T = (T_0 - T_a) \) is the temperature change between the incline and the ambient fluid which was used to scale the temperature difference \((T - T_a)\). The case with constant fluid properties can easily be recovered by setting \( \alpha = \lambda = \Lambda = \gamma = 0 \).

The corresponding first-order dimensionless boundary conditions along the free surface \( z = h(X, \tau) \) are

\[ p = \frac{2 \varepsilon (1 - \lambda T)}{\text{Re}} \left( \frac{\partial W}{\partial z} + \frac{\partial h}{\partial X} \frac{\partial u}{\partial z} \right), \]  
\( (41) \)

\[ -\varepsilon \text{Ma} \text{Re} \left( \frac{\partial T}{\partial X} + \frac{\partial h}{\partial X} \frac{\partial T}{\partial z} \right) = (1 - \lambda T) \frac{\partial u}{\partial z}, \]  
\( (42) \)

\[ -\text{Bi} T = (1 + \Lambda T) \frac{\partial T}{\partial z}, \]  
\( (43) \)

\[ W = \frac{\partial h}{\partial \tau} + u \frac{\partial h}{\partial X}, \]  
\( (44) \)

where \( \text{Bi} = \alpha g H/k_a \) is the Biot number with \( \alpha \) referring to the heat transfer coefficient across the liquid-air interface and \( \text{Ma} = \gamma H \Delta T/(\rho_a Q^2) \) is the Marangoni number. Condition (43) is essentially a statement of Newton’s Law of Cooling. We note that although the surface tension term does not appear at first order, the effect of variations in surface tension, as measured by the parameter \( \text{Ma} \), does make an appearance. On the incline \( z = 0 \) the no-slip, impermeability and temperature conditions are

\[ u = W = 0, \quad T = 1. \]  
\( (45) \)

Next, we introduce the flow rate, \( q(X, \tau) \), the interfacial temperature, \( \theta(X, \tau) = T(X, z = h, \tau) \), and the change of variable \( \phi(X, \tau) = h(X, \tau)(\theta(X, \tau) - 1) \) along with the assumed profiles for \( T \) and \( u \) given by

\[ T = 1 + \frac{(\theta - 1)z}{h} = 1 + \frac{\phi z}{h^2}, \quad u = \frac{3qz(2h - z)}{2h^3}. \]  

The profiles for \( T \) and \( u \) are based on the steady-state unidirectional solutions to the Navier-Stokes and energy equations for the case with constant fluid properties and fluid thickness while \( \phi \) is related to the linear heat content in the fluid layer. The velocity profile is identical to that used in the base case. From equations (37) - (40) it follows that the steady-state profiles for \( u \) and \( T \), denoted by \( u_s \) and \( T_s \), respectively, are given by

\[ u_s(z) = \frac{3z(2h - z)}{2h^3}, \quad T_s(z) = 1 - \frac{\text{Bi} z}{(1 + \Lambda + \text{Bi} h)}. \]

As previously noted, the velocity profile is parabolic whereas the temperature profile is linear. Exact steady-state profiles for \( T \) and \( u \) have been derived in [15] for the case with variable fluid properties, but the profiles are very complicated and as shown in Figures 2 and 3 they agree closely with the much simpler parabolic and linear profiles listed above.
Integrating the system (37)-(40) across the fluid layer using the procedure outlined in the base case along with the above prescribed profiles for the velocity and temperature together with the boundary conditions (41)-(45) yields

\[
\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0.
\]  (46)
\[
\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{6q^2}{5h} + \frac{Maq}{h} + \frac{\cot \beta}{Re} \left( \frac{3(1 - \alpha)}{2} - \frac{\alpha q}{h^2} \right) h^2 \right] = \frac{3}{\varepsilon Re} \left[ (1 - \alpha)h - \frac{\alpha q}{2} - \frac{(1 - \lambda)q}{h^2} \right],
\]
(47)

\[
\frac{\partial \phi}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{5q \phi}{4h} \right] = - \frac{2}{\varepsilon Pr Re} \left[ Bi \left( 1 + \frac{\phi}{h} \right) + \frac{(1 + \Lambda) \phi}{h^2} \right].
\]
(48)

Hence, our IBL model consists of equations (46)-(48) in terms of the flow variables \( h, q \) and \( \phi \). The base case is recovered by simply setting \( \alpha = \lambda = \Lambda = Bi = Ma = \phi = 0 \).

Applying the fractional-step splitting technique to the system (46)-(48), we first solve

\[
\frac{\partial h}{\partial \tau} + \frac{\partial q}{\partial X} = 0,
\]

\[
\frac{\partial q}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{6q^2}{5h} + \frac{Maq}{h} + \frac{\cot \beta}{Re} \left( \frac{3(1 - \alpha)}{2} - \frac{\alpha q}{h^2} \right) h^2 \right] = 0,
\]

\[
\frac{\partial \phi}{\partial \tau} + \frac{\partial}{\partial X} \left[ \frac{5q \phi}{4h} \right] = 0,
\]

over a time step \( \Delta \tau \), and then solve

\[
\frac{\partial q}{\partial \tau} = \frac{3}{\varepsilon Re} \left[ (1 - \alpha)h - \frac{\alpha q}{2} - \frac{(1 - \lambda)q}{h^2} \right],
\]

\[
\frac{\partial \phi}{\partial \tau} = - \frac{2}{\varepsilon Pr Re} \left[ Bi \left( 1 + \frac{\phi}{h} \right) + \frac{(1 + \Lambda) \phi}{h^2} \right],
\]

using the solution obtained from the first step as an initial condition for the second step. The second step then returns the solution for \( q \) and \( \phi \) at the new time \( \tau + \Delta \tau \) with the understanding that \( h \) is constant during the second step. In vector form the first step can be expressed in matrix form as

\[
\frac{\partial U}{\partial \tau} + \frac{\partial F(U)}{\partial X} = 0,
\]

where

\[
U = \begin{bmatrix} h \\ q \end{bmatrix}, \quad F(U) = \begin{bmatrix} \frac{6q^2}{5h} + \frac{Maq}{h} + \frac{\cot \beta}{Re} \left( \frac{3(1 - \alpha)}{2} - \frac{\alpha q}{h^2} \right) h^2 \\ \frac{5q \phi}{4h} \end{bmatrix},
\]

and is solved using MacCormack's method as previously explained.

Since \( h \) remains constant during the second step, the second step represents a coupled system of linear ordinary differential equations which can be solved exactly. We first solve the equation for \( \phi \) which gives

\[
\phi(X, \tau + \Delta \tau) = \left( \phi_0 + \frac{c_4}{c_5} \right) e^{-c_6 \Delta \tau} - \frac{c_4}{c_5} \quad \text{where} \quad c_4 = \frac{2Bi}{\varepsilon Pr Re}, \quad c_5 = \frac{2(1 + \Lambda + Bi h_0)}{\varepsilon Pr Re h_0^2}.
\]
(49)

Substituting this solution into the equation for \( q \) we obtain the following

\[
q(X, \tau + \Delta \tau) = q_0 e^{c_6 \Delta \tau} + \frac{c_7}{c_6} \left( e^{c_7 \Delta \tau} - 1 \right) - \frac{c_8}{(c_5 + c_6)} \left( e^{c_8 \Delta \tau} - e^{-c_6 \Delta \tau} \right),
\]
(50)

where

\[
c_6 = - \frac{3(1 - \lambda)}{\varepsilon Re h_0^2}, \quad c_7 = \frac{3(1 - \alpha)h_0}{\varepsilon Re} + \frac{3\alpha c_4}{2\varepsilon c_5 Re}, \quad c_8 = \frac{3\alpha}{2\varepsilon Re} \left( \phi_0 + \frac{c_4}{c_5} \right).
\]

In the above \( \phi_0, q_0 \) and \( h_0 \) denote the solutions emerging from the first step of the procedure.
4. Results

The generalized IBL models presented in the previous section are completely characterized by the following dimensionless parameters: $Re, \delta, Pr, Ma, Bi, \alpha, \lambda, \Lambda, \beta, \varepsilon$, the function $\zeta(X)$ and the power-law index $n$. Simulations were conducted on a computational domain of length $L$ subject to periodic boundary conditions. The evolution of the unsteady flow was computed by imposing small perturbations on the initial steady-state solutions. The perturbations were taken to have a wavelength equal to $L$ since long wavelengths are expected to be most unstable. By monitoring the growth of the disturbances as the perturbed solutions were marched in time we were able to estimate the onset of instability by incrementing the Reynolds number and noticing the first instance when the flow develops a permanent series of waves on the free surface. In our numerical experiments the length of the computational domain was assigned the value $L = 2$, the grid spacing and time step were $\Delta X = 0.01, \Delta \tau = 0.001$, respectively. Unless otherwise stated the value of the parameter $\varepsilon$ was taken to be 0.05. Although the domain length is arbitrary, we observed that $L = 2$ was sufficiently large to trigger the long wavelength instability. As a numerical check the volume of fluid was computed at each time step and was observed to remain constant to within several decimal places. Lastly, we emphasize that no numerical difficulties were encountered and the proposed algorithm proved to be fast and robust.

We begin with the base case. Here, the initial steady-state solutions are simply $h_s = q_s = 1$. Before presenting results from our nonlinear simulations we first performed a linear stability analysis on the system (15)-(16). Setting $\tilde{h} = 1 + \hat{h}$ and $\tilde{q} = 1 + \hat{q}$ and substituting into the system (15)-(16) and linearizing yields

$$\frac{\partial \hat{h}}{\partial \tau} + \frac{\partial \tilde{q}}{\partial X} = 0,$$

$$\frac{\partial \hat{q}}{\partial \tau} + 3 \cot \beta \frac{\partial \tilde{h}}{\partial X} + \frac{12}{5} \frac{\partial \tilde{q}}{\partial X} = \frac{3}{\varepsilon Re} \left(3\tilde{h} - \tilde{q}\right).$$

Next, we set $(\hat{h}, \hat{q}) = (\tilde{h}, \tilde{q})e^{ik(x-ct)}$ where the perturbation wavenumber, $k$, is taken to be real and positive, while $c$ is a complex quantity with the real part denoting the phase speed and the imaginary part, $\Im(c)$, is related to the growth rate. For neutral stability we find that the phase speed is $c = 3$ and the critical Reynolds number, $Re_{\text{crit}}^{\text{IBL}}$, is given by

$$Re_{\text{crit}}^{\text{IBL}} = \cot \beta,$$

which is in full agreement with the result obtained by Shkadov [7] and agrees reasonably well with the prediction from the full Navier-Stokes equations

$$Re_{\text{crit}} = \frac{5}{6} \cot \beta,$$

obtained by Benjamin [4] and Yih [5].

Based on our numerical simulations using $\cot \beta = 1$ the critical Reynolds number was estimated to lie in the range $0.87 < Re_{\text{crit}}^{\text{IBL}} < 0.88$ which lies between the values predicted by (53) and (54). Figure 4 illustrates a snapshot of the fluid thickness at $\tau = 25$ for $Re = 0.9$ and $Re = 1$ which are both supercritical cases. The plot shows the formation of large amplitude waves as a result of the instability. The actual number of waves in the interval $L$ depends on several factors including the value of $Re$. We see that for $Re = 0.9$ two waves emerge while for $Re = 1$ we obtain four waves. The only difference that we noticed when a larger computational domain was used is in the number of waves appearing over the domain. Although not shown here, the flow rate, $q$, produces a similar plot but with different amplitudes. Because viscous terms are absent in our first-order IBL model the waves appear as tall sharp spikes. A second-order IBL model would include viscous terms and would yield smaller amplitude waves which
are more consistent with experimental observations. These smaller amplitude waves are the result of the smearing effect brought on by viscosity.

Next we discuss some results for the uneven bottom case. We take the scaled bottom topography to be the wavy function given by \( \zeta(X) = a_b \cos(2\pi X) \) with scaled amplitude \( a_b \). The wavelength of the undulations does not appear since it was taken to be the length scale in the \( x \) direction. The initial steady-state solutions are more complicated due to the wavy bottom and are given by \( q_s = 1 \) and \( h_s(X) \) where \( h_s(X) \) satisfies the differential equation

\[
\varepsilon(5 \cot \beta h_s^3 - 2Re)h_s' = 5\left(1 - \varepsilon \cot \beta \zeta'\right)h_s^3 - 1,
\]

which was solved numerically subject to periodic boundary conditions. Here, the prime denotes differentiation with respect to \( X \). An approximate analytical solution was also constructed in the form of a series

\[
h_s(X) = 1 + \varepsilon h_1(X) + O(\varepsilon^2),
\]

where \( h_1(X) \) was easily found to be

\[
h_1(X) = \frac{1}{3} \cot \beta \zeta'.
\]

We note that an analytic expression for \( \text{Re}_{IBL}^{\text{crit}} \) is not possible since the linearized equations contain coefficients that are periodic functions and hence require the application of Floquet-Bloch theory. We report on a particular case having \( \cot \beta = 1 \) and \( a_b = 0.25 \). Here, it was found that the critical Reynolds number was in the range \( 0.87 < \text{Re}_{IBL}^{\text{crit}} < 0.88 \) which is identical to the base case having a flat bottom. Previous studies [17,18] have shown that for weak surface tension a wavy bottom tends to stabilize the flow. However, since our first-order model does not include surface tension it makes sense that the wavy bottom case predicts the same critical Reynolds number as the flat bottom case. Shown in Figure 5 is the free surface at times \( t = 1 \) and \( t = 5 \). The steady-state solution and bottom contour are also plotted.

For the porous bottom case a linear stability analysis of the system (24)-(25) yields

\[
\text{Re}_{IBL}^{\text{crit}} = \frac{\cot \beta}{(1 + 3\delta)(1 + \delta)},
\]
Figure 5. The free surface and bottom contour, $\zeta(X) = 0.25 \cos(2\pi X)$, for the case $Re = 0.88$ and $\cot \beta = 1$.

and the corresponding wave speed is $c = 3(1 + 2\delta)$. Comparing (55) with (53) we observe that a porous bottom leads to an increased phase speed and destabilizes the flow: this was shown to occur with other models [19,20] as well. Also, when $\delta = 0$ equations (53) and (55) are identical. Based on the full Navier-Stokes equations [19,20] the critical Reynolds number was found to be

$$Re_{crit} = \frac{5}{6} \cot \beta \left( \frac{1 + 3\delta}{1 + 6\delta + \frac{25}{2}\delta^2} \right).$$  \hspace{1cm} (56)

From our numerical simulations using $\cot \beta = 1$ and $\delta = 0.1$ the critical Reynolds number was estimated to lie in the range $0.65 < Re_{crit} < 0.66$ which is less than the value obtained from the base case with $\cot \beta = 1$ and agrees closely with the value predicted by (55).

Next we present some results for the non-Newtonian case. First, a linear stability analysis of equations (32)-(33) gives the following relation for the critical Reynolds number

$$Re_{crit}^{IBL} = \frac{n^{2-n} - n}{(1 + 2n)1-n} \cot \beta,$$

which is in full agreement with the results reported in [13,14]. The phase speed is given by $c = (1 + 2n)/n$ which simplifies to $c = 3$ for the Newtonian case. Also, when $n = 1$ equation (57) reproduces (53). For the shear-thinning case with $n = \frac{1}{2}$ the numerical scheme can be accelerated by rewriting equation (36) in terms of the Lambert W function, $W$, [21] as follows. First we write (36) in the form $\xi = ne^\eta$ where

$$\xi = \frac{1}{h_0} e^\chi, \quad \eta = \sqrt{\frac{q}{h_0^2}} - 1, \quad \chi = \frac{\sqrt{q_0}}{h_0^2} + \ln(\sqrt{q_0} - h_0^2) - \frac{\Delta \tau}{\varepsilon Re h_0^3}.$$  \hspace{1cm} (58)

Then the solution is $\eta = W(\xi)$. The advantage of this is that $W(\xi)$ can be approximated by an accurate formula [22] which allows $q$ to be determined without having to iteratively solve the nonlinear equation (36).
Lastly, we discuss the non-isothermal case. Here, the initial steady-state solutions are given by

\[ h_s = 1, \quad q_s = \frac{1}{(1 - \lambda)} \left[ 1 - \frac{\alpha}{2} (\phi_s + 2) \right], \quad \phi_s = -\frac{Bi}{(1 + \Lambda + Bi)} . \]

Although the expression obtained for the critical Reynolds number, \( Re_{IBL}^{crit} \), is too lengthy and not worth presenting, setting \( \alpha = \lambda = \Lambda = 0 \) (i.e. constant fluid properties) simplified the expression significantly and yields

\[ Re_{IBL}^{crit} = \frac{3(1 + Bi)^2 \cot \beta}{3(1 + Bi)^2 + (5 + 2Bi)BiMa} , \quad (58) \]

which is in reasonable agreement with the expression obtained from the full Navier-Stokes (and energy) equations [23] given by

\[ Re^{crit} = \frac{10(1 + Bi)^2 \cot \beta}{12(1 + Bi)^2 + 5BiMa} . \quad (59) \]

Note that in the isothermal limit (58) reduces to (53).

Next, we consider a more general case having the following parameter values: \( Re = 0.8, \ Pr = 7, \ Ma = Bi = \cot \beta = 1, \alpha = \lambda = \Lambda = 0.1 \) and \( \varepsilon = 0.1 \). For this choice of parameter values the critical Reynolds number was estimated to lie in the range \( 0.70 < Re_{crit} < 0.71 \) based on our nonlinear numerical simulations, and thus, this case corresponds to an unstable configuration. The estimate of the critical Reynolds number comes in close agreement with that of a linear stability analysis of the full Navier-Stokes equations [23] which predicts \( Re_{crit} = 0.67 \). Further, the linear stability analysis applied to our IBL model predicts \( Re_{IBL}^{crit} = 0.81 \) which agrees reasonably well with our nonlinear simulations.

Plotted in Figure 6 is the time evolution of the fluid thickness, \( h \). The diagram illustrates the formation and development of large amplitude waves. At \( \tau = 1 \) three small waves are seen and with the passage of time these waves amplify and move closer together. The pattern shown at \( \tau = 25 \) was observed to persist for large times thus forming a permanent wave structure. Shown in Figure 7 is a snap shot of the fluid thickness, \( h \), and the surface temperature, \( \theta = 1 + \phi/h \), for the configuration shown in Figure 6 at \( \tau = 50 \). As previously mentioned the variations in \( h \) align with the peaks in \( q \); however, the surface temperature is out of phase with the fluid thickness, that is, the surface temperature is smallest when \( h \) is largest. This is because the steady-state temperature given by

\[ T_s = 1 - \frac{Bi z}{(1 + \Lambda + Bi)}, \]

decreases linearly with \( z \). Hence, the smallest value of \( T_s \) occurs when \( z \) is largest. Contrasted in Figure 8 are \( Re_{crit} \) values based on linear stability analyses of our IBL model and the full Navier-Stokes equations as well as our nonlinear simulations for the case having the following parameter values: \( Pr = 7, \cot \beta = 1, \ Ma = 0.5 \) and \( \alpha = \lambda = \Lambda = 0 \) over the interval \( 0 \leq Bi \leq 10 \). The plot reveals that for most of the values of \( Bi \) the nonlinear simulations predict values in between those of the linearized IBL model and the linearized Navier-Stokes equations. Also, it is worth noting that the Navier-Stokes equations predicts a minimum in \( Re_{crit} \) at \( Bi = 1 \) with a minimum value of

\[ Re_{crit,min} = \frac{40 \cot \beta}{48 + 5Ma} , \]

while the IBL model predicts a minimum value of

\[ Re_{IBL}^{crit,min} = \frac{36 \cot \beta}{36 + 25Ma} , \]

at \( Bi = 5 \).
Figure 6. Time evolution of the fluid thickness, $h$.

Figure 7. The fluid thickness, $h$, and the surface temperature, $\theta$, at $\tau = 50$.

5. Summary
Presented in this study were generalized first-order IBL models for various gravity-driven flow configurations down an incline. The proposed models have the capability of handling both Newtonian and non-Newtonian fluids, isothermal and non-isothermal flows, fluids possessing constant and variable fluid properties, a porous and non-porous incline, and even flat and wavy inclined surfaces. A numerical solution procedure to integrate the equations was also proposed and was successful in faithfully simulating the flow as it transitioned from a stable stationary state to an unstable state characterized by a permanent wave structure propagating along the free
surface. The models were validated by making comparisons with known established results, and in all cases considered the agreement was reasonable. The advantages offered by the proposed IBL models are:

1) the speed, accuracy and robustness of the analytical-numerical solution procedure,
2) the simplicity and versatility of the models, and
3) the predictions made by the models agreed reasonably well with those obtained using more sophisticated models.

A disadvantage of these first-order IBL models is that the simulated waves tend to be larger than those typically observed in an experiment. Although this can be corrected by extending the models to second-order, it would lead to more complicated equations which would require a different numerical solution procedure than that presented in this study. The goal here was to present simplistic models that yield reasonable accuracy and that are easy to solve numerically.

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