The effect of convective boundary condition on MHD mixed convection boundary layer flow over an exponentially stretching vertical sheet

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Abstract. A theoretical study that describes the magnetohydrodynamic mixed convection boundary layer flow with heat transfer over an exponentially stretching sheet with an exponential temperature distribution has been presented herein. This study is conducted in the presence of convective heat exchange at the surface and its surroundings. The system is controlled by viscous dissipation and internal heat generation effects. The governing nonlinear partial differential equations are converted into ordinary differential equations by a similarity transformation. The converted equations are then solved numerically using the shooting method. The results related to skin friction coefficient, local Nusselt number, velocity and temperature profiles are presented for several sets of values of the parameters. The effects of the governing parameters on the features of the flow and heat transfer are examined in detail in this study.

1. Introduction

Mixed convection flows, or combination of forced and natural convection, are often found in very high power output devices in which the forced convection does not dissipate the necessary heat. In such cases, combining natural convection with forced convection produces the desired results. Nuclear reactor technology and electronic cooling [1] are some of the examples of the engineering applications of mixed convection. In addition, the flow and thermal boundary layers of a continuous stretching surface have wide applications in metallurgical processes (drawing of continuous filaments using quiescent fluids, annealing and tinning of copper wires, glass blowing, manufacturing of plastic and rubber sheets, crystal growing, and continuous cooling and fibre spinning [2]). Many researchers
considered linear, nonlinear, quadratic, and exponential variations of stretching velocity and temperature distributions. However, Magyari and Keller [3] were the first who investigated the flow and thermal boundary layers on a stretching sheet, which has exponential variations of velocity and temperature. As a result, the findings due to the problems of flow induced by an exponential variation of stretching surface have been produced [4–10]. Furthermore, the cases of the mixed convection fluid flow over an exponentially stretching sheet have been reported [11–18]. The convective heat transfer involves in engineering procedures, namely, gas turbines, nuclear plants, storage of thermal energy etc [19]. These processes obtain high temperature, which the flow is subjected to the convective boundary condition. Therefore, a number of studies on boundary layer flows with a convective boundary condition, including mixed convection boundary layer flow [20–27] and a stretching sheet [28–35], have been reviewed since then.

Motivated by the above researchers, the purpose of this research paper is to develop the system of MHD mixed convection heat transfer of an electrically conducting fluid over an exponentially stretching continuous surface with an exponential temperature distribution. The system is maintained using applied convective boundary conditions, exponential magnetic field, viscous dissipation, buoyancy force, and internal heat generation. The mathematical formulation reported by Pal [13] is followed, which the impermeable surface is assumed to be stretched with an exponential velocity $U_w = U_0 e^{y/L}$ and the surface is maintained at a temperature $T_w(x) = T_w + (T_0 - T_w) e^{x/L}$. The nonlinear boundary value problems are solved numerically using the Runge–Kutta method with a shooting technique. Numerical solutions are obtained by the form of tables and graphs, to illustrate the characteristics of the boundary layer flow and heat transfer.

2. Mathematical formulation

Consider the two-dimensional flow of an incompressible, viscous, and electrically conducting fluid over an exponentially impermeable stretching surface (see Figure 1) with a magnetic field $B(x)$. The $x$-axis is parallel to the stretching surface in the direction of the movement of the sheet and the $y$-axis is perpendicular to it. A transverse magnetic field is applied in the $y$-axis. The induced magnetic field in equation (2) is assumed to be negligible since the magnetic Reynolds number for the flow is very small. The impermeable surface is assumed to be stretched with an exponential velocity $U_w = U_0 e^{y/L}$ over an inquiescent fluid and the surface is maintained at a temperature $T_w(x) = T_w + (T_0 - T_w) e^{x/L}$, where $T_w$ is the temperature of the ambient fluid and $T_0$ is the reference temperature. We have followed the formulation of $U_w$ and $T_w(x)$ by Pal [13]. Considering the assumptions of Boussinesq and boundary layer approximations, the flow and heat transfer problems are governed by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + g \beta (T - T_w) - \frac{\sigma B^2(x)}{\rho} u$$

(2)

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\sigma}{\rho c_p} B^2(x) u^2 + \frac{\mu}{\rho c_p} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{Q}{\rho c_p} (T - T_w)$$

(3)

where $u$ and $v$ are the components of velocity in the $x$ and $y$ directions, respectively, $\nu = \mu \rho$ is the kinematic viscosity, $\mu$ is the viscosity, $\rho$ is the fluid density, $g$ is acceleration due to gravity, $\beta$ is the thermal expansion coefficient, $\sigma$ is the electrical conductivity, $\alpha$ is the thermal diffusivity, $T$ is the temperature of the fluid, $c_p$ is the specific heat of the fluid at a constant pressure, and $Q > 0$ is the
internal heat generation coefficient. In Equation (2), we consider the special form of magnetic field, $B(x) = B_0 e^{x/L}$. 

![Figure 1. Physical model and coordinate system.](image_url)

The appropriate boundary conditions are as follows:

$$
u = U_w(x) = U_0 e^{x/L}, \quad v = 0, \quad -k \frac{\partial T}{\partial y} = h_0 (T_0 - T) \quad \text{at} \quad y = 0$$

$$u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty$$

(4)

where $k$ is the thermal conductivity of the fluid and $h_0$ is the convective heat transfer coefficient.

The new similarity variables that were introduced are as follows:

$$\eta = \frac{y}{L} \left( \frac{Re}{2} \right)^{\frac{1}{2}} e^{-\frac{L}{\mu L}}, \quad \psi(x,y) = \sqrt{2Re} \nu e^{(y/L)} f(\eta), \quad T(x,y) = T_w + (T_\infty - T_w) e^{\alpha x/L} \theta(\eta)$$

(5)

where $\psi$ is the stream function which is defined as $u = \partial \psi / \partial y$ and $v = -\partial \psi / \partial x$, and $a$ as the similarity variable for temperature $T(x,y)$ is a parameter of the temperature distribution in the stretching surface.

Substituting similarity variables (5) into equations (1)–(3), we obtain

$$f''' + ff'' - 2f'(f')^2 - \frac{2Ha^2}{Re} f' + 2\lambda \left(e^{x/L}\right) \left(e^{-2x}\right) \theta = 0$$

(6)
\[
\frac{1}{\Pr} \theta'' + f \theta' - a f' \theta + Ec \left(e^{2X - (aX/2)} \right) \left[2 \frac{Ha^2}{Re} \left(f' (\eta) \right)^2 + (f^* (\eta))^2 \right] + 2A^* e^{-X} \theta = 0 \quad (7)
\]

where the prime denotes differentiation with respect to \( \eta \), and the boundary conditions reduce to

\[
f'(\eta) = 1, \quad f(\eta) = 0, \quad \theta(\eta) = \left(1/|Bi|\right) \theta'(\eta) + \left(1/e^{aX/2}\right) \quad \text{at} \quad \eta = 0
\]
\[
f'(\eta) \to 0 \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \quad (8)
\]

where \( X = x/L \) is the dimensionless coordinate along the plate; \( Ha = \sqrt{\sigma B_o^2 L^2 / \rho \nu} \) is the Hartmann number; \( Re = U_o L / \nu \) is the Reynolds number; \( \lambda = Gr / Re^2 \) is the constant mixed convection parameter, where \( Gr = \left(g \beta (T_0 - T_w) L^3 / \nu^2 \right) \) is the Grashof number; \( Pr = \nu / \alpha \) is the Prandtl number; \( Bi = h_0 y / \eta k \) is the Biot number; and \( A^* = Q L^2 / (\mu c_p \cdot Re) \) is the dimensionless heat generation parameter.

In the above similarity equations, the effect of the magnetic field is considered as a ratio of the Hartmann number to the Reynolds number. Note that \( \lambda > 0, \lambda < 0 \), and \( \lambda = 0 \) correspond to assisting flow, opposing flow, and forced convection flow, respectively. The value \( \lambda > 0 \) refers to the assisting flow parameter, whereas the opposing flow parameter, denoted as \( \lambda < 0 \), refers to the value of mixed convection parameter.

The skin friction coefficient \( C_f \) and the Nusselt number \( Nu_x \), which are the physical parameters of interest in the present problem, are given by

\[
C_f = \frac{2 \rho \nu}{\rho U_w^2 (x)} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad Nu_x = -\frac{x}{T_w - T_w} \left( \frac{\partial T}{\partial y} \right)_{y=0}. \quad (9)
\]

Substituting Equation (5) into Equation (9), we obtain

\[
\frac{C_f}{\sqrt{Re_x}} \frac{\sqrt{2X}}{2} = f''(0), \quad \frac{Nu_x}{\sqrt{Re_x} \left( \sqrt{X/2} \right)} = -\theta'(0). \quad (10)
\]

where \( Re_x = x U_w (x) / \nu \) is the Reynolds number.

2.1. Numerical method for solution

The boundary value problems in equations (6)–(8) are solved by converting them into an initial value problem using the shooting method. The fourth-order Runge–Kutta integration scheme is applied to solve the initial value problem. We set

\[
f' = f p,
\]
\[
f p' = f p p,
\]
\[
f p p' = -f (f p p) + 2 (f p) + 2 \left(\frac{Ha^2}{Re} \right) f p - 2 \lambda \left(e^{aX/2}\right) \left(e^{-2X}\right) \theta.
\]
\[
\theta' = \theta p,
\]
\[
\theta p' = -Pr (f \theta p) + a Pr (f p \theta) - Pr Ec \left(e^{2X - (aX/2)} \right) \left[2 \left(\frac{Ha^2}{Re} \right) (f p)^2 + (f p p)^2 \right] \quad (12)
\]

with the boundary conditions.
\[ f = 0, \quad fp = 1, \quad \theta = (1/Bi) \theta p + (1/e^{x/2}) \quad \text{at } \eta = 0 \]
\[ fp \to 0, \quad \theta \to 0 \quad \text{as } \eta \to \infty \]  

(13)

In order to carry out the integration in equations (11) and (12) as an initial value problem, we assume initial values for \( fpp(0) \), i.e. \( f''(0) \) and \( \theta p(0) \), i.e. \( \theta'(0) \). We also assume a suitable finite value for \( \eta \to \infty \), say \( \eta_\infty \). Then, we calculate the values of \( f'(\eta_\infty) \) and \( \theta(\eta_\infty) \), by adjusting the assumed values of \( f''(0) \) and \( \theta'(0) \) to give a better approximation of the solution. We consider that the assumed values of \( \eta_\infty \), \( f''(0) \) and \( \theta'(0) \) satisfy the boundary conditions in Equation (13) \( \left( f'(\eta_\infty) = 0 \right) \) and \( \theta(\eta_\infty) = 0 \). This steps is repeated with higher value of \( \eta_\infty \) until two consecutive values of \( f''(0) \) and \( \theta'(0) \) vary significantly by a specified value. Finally, the most appropriate value of \( \eta_\infty \) is selected, which is the largest \( \eta_\infty \) that satisfies the solution.

3. Results and discussion

Numerical computations have been performed for the velocity profile \( f'(\eta) \), the temperature profile \( \theta(\eta) \), the skin friction coefficient \( f''(0) \) and the local Nusselt number \( \theta'(0) \) for various values of physical parameters, such as mixed convection parameter \( \lambda \), parameter of the temperature distribution in the stretching surface \( a \), Hartmann number \( Ha^2/Re \), Biot number \( Bi \), Eckert number \( Ec \), dimensionless heat generation parameter \( A' \) and dimensionless coordinate along the plate \( X \). To examine the accuracy of the numerical method, we compare our results with those obtained by Magyari and Keller [3], Al-Odat et al. [5], and Pal [13] in Table 1. In this table, the comparison is made for the nonmagnetic case \( Ha^2/Re = 0 \), non-buoyant flow \( \lambda = 0 \), and when \( Bi = Ec = X = A' = 0 \). This table also shows the comparison of the values of \( -\theta'(0) \) for different values of \( Pr \) and \( a \). From this table, it can be observed that the present results are in very good agreement with the previous results, which proves our theoretical study and numerical computation.

On the other hand, multiple solutions are obtained from the numerical shooting technique. Stability analysis determines which solutions correspond to physically meaningful solution. Merkin [36] have performed a stability analysis of dual solutions and stated that upper branch (or namely as the first solution) is stable and owns physical meaning in real situation. Otherwise, lower branch (second and the subsequent solutions) is unstable.

3.1. Mixed convection

The illustrations of \( f''(0) \) and \( -\theta'(0) \) versus mixed convection parameter \( \lambda \) are shown in Figures 2 and 3, respectively. For the case of the assisting flow \( (\lambda > 0) \), the occurrence of three solutions can be seen when the value of mixed convection parameter \( \lambda \geq 0.01234 \). The point at \( \lambda = 0.01234 \) is the minimum point of \( f''(0) \) and \( -\theta'(0) \) for the third solution. Thus, it has only two solutions for the values of \( \lambda \) between 0 and 0.01234 for the assisting flow. For the case of the opposing flow \( (\lambda < 0) \), two solutions are obtained in the range \(-0.24616 < \lambda < 0 \). The first and second solutions are combined at the critical point \( \lambda_c = -0.24616 \). A unique solution is obtained at the critical point \( \lambda_c = -0.24616 \), whereas no solutions exist for a small value of mixed convection parameter \( \lambda < -0.24616 \). Finally, three solutions are obtained for a large value of assisting flow parameter. However, two solutions are obtained for a small value of assisting and opposing flow parameters. When a very small value of mixed convection parameter is considered (large opposing parameter), no solutions are found. In Figure 2, it is shown that the values of \( f''(0) \) increase with an increase in \( \lambda \) for all solution profiles.
The values of \( f''(0) \) are close to each other for all solution profiles for a larger \( \lambda \). The values of \( -\theta'(0) \) decrease when the value of \( \lambda \) approaches 0 for the third solution profile.

**Table 1.** Comparison of local Nusselt number \( -\theta'(0) \) calculated by (...) Magyari and Keller [3], <...> Al-odat et al. [5], {...} Pal [13] and * * present method.

| \( \alpha \) | Pr | \( a = -1.5 \) | \( a = -0.5 \) | \( a = 0.0 \) | \( a = 1.0 \) | \( a = 3.0 \) |
|---|---|---|---|---|---|---|
| \( -0.20405 \) | \( 0.17582 \) | \( 0.33049 \) | \( 0.59434 \) | \( 0.97666 \) | \( 1.00841 \) |
| \( -0.37741 \) | \( 0.29988 \) | \( 0.54964 \) | \( 0.95478 \) | \( 1.56029 \) | \( 1.56030 \) |
| \( -0.92386 \) | \( 0.63411 \) | \( 1.12209 \) | \( 1.86907 \) | \( 2.93854 \) | \( 2.93853 \) |
| \( -1.35324 \) | \( 0.87043 \) | \( 1.52124 \) | \( 2.50013 \) | \( 3.88656 \) | \( 3.88656 \) |
| \( -1.88850 \) | \( 1.15032 \) | \( 1.99184 \) | \( 3.24212 \) | \( 5.00047 \) | * | * |
| \( -2.20000 \) | \( 1.30861 \) | \( 2.25742 \) | \( 3.66037 \) | \( 5.62820 \) | * | * |

**Figure 2.** Variation of \( f''(0) \) with mixed convection parameter \( \lambda \).
Figure 3. Variation of $-\theta'(0)$ with mixed convection parameter $\lambda$.

3.2 The assisting flow case, $\lambda > 0$

To understand the characteristics of the flow and heat transfer in the assisting flow case, the variations in skin friction coefficient $f''(0)$, local Nusselt number $-\theta'(0)$, velocity profile $f'(\eta)$, and temperature profile $\theta(\eta)$ are presented in figures and tables. The velocity profiles $f'(\eta)$ for $\lambda = 2$ are shown in Figure 4. The velocity of the first solution declines continuously with an extension in the distance of the point from the stretching sheet. In the second and third solution profiles, the velocity $f'(\eta)$ initially decreases at a point and reaches to a certain negative value, then for higher values of $\eta$ it starts to increase. The velocity of the second and third solution profiles reaches a very high value in the boundary layer with a negative fluid velocity. In other words, reversal flow arises at the fluid near the edge of the stretching sheet. Figure 5 shows the variations in temperature profiles when $\lambda = 2$. In this figure, the temperature is found to increase initially at a point for all profiles, but it drops for a large value of $\eta$. The occurrence of a peak for a small value of $\eta$ indicates that there is the highest temperature value at the point of the fluid near the stretched wall. The increment in the temperature difference between the shrieked wall and the adjacent fluid is indicated by the increase in the temperature peak value. As a result, the heat transfer process from the stretching sheet to the ambient fluid is improved.

Figure 4. Velocity profiles when $\lambda = 2$. 
Tables 2 and 3 show the values of skin friction coefficient $f''(0)$ and local Nusselt number $-\theta'(0)$ for different values of $X$, $a$, $Ha^2/Re$, $Bi$, $Ec$, and $A^*$ for the case of the assisting flow. From these tables, it can be seen that for the increasing values of $X$, the values of $f''(0)$ and $-\theta'(0)$ decrease for all solution profiles. Table 3 shows that increasing parameter $a$ increases the rate of local Nusselt number, whereas the effect of $a$ decreases the rate of skin friction coefficient. It is found that for the first solution profile the value of $f''(0)$ reduces with an increase in magnetic field, whereas for other solution profiles it shows a reverse trend. This table also shows that the effect of magnetic field reduces the rate of local Nusselt number. The results from Table 2 indicate that the skin friction coefficient decreases with an increment in Biot number, but an opposite effect has been observed in the rate of local Nusselt number. However, the effect of $Ec$ and $A^*$ on the values of $f''(0)$ and $-\theta'(0)$ shows reverse characteristics when compared with the effect of the increase in Biot number.

**Table 2.** Values of the skin friction coefficient $f''(0)$ for several sets of values of the parameters when $\lambda = 2$ and $Pr = 1$.

| Parameters | Values | First solution | Second solution | Third solution |
|------------|--------|----------------|----------------|---------------|
| $X$        | 1.47   | -0.8172        | -0.8199        | -0.8183       |
|            | 1.49   | -0.8306        | -0.8332        | -0.8317       |
| $a$        | -1.49  | -0.8486        | -0.8502        | -0.8481       |
|            | -1.47  | -0.8690        | -0.8691        | -0.8662       |
| $Ha^2/Re$  | 0.02   | -0.8375        | -0.8393        | -0.8379       |
|            | 0.04   | -0.8385        | -0.8376        | -0.8367       |
| $Bi$       | 1.6    | -0.8670        | -0.8701        | -0.8695       |
|            | 1.8    | -0.9092        | -0.9132        | -0.9139       |
| $Ec$       | 0.11   | -0.8166        | -0.8128        | -0.8071       |
|            | 0.13   | -0.7752        | -0.7545        | -0.7416       |
While studying the effect of individual parameters the following values are used:

\[ a = -1.5, \quad X = 1.5, \quad Bi = 1.5, \quad Ec = 0.1, \quad Ha^2/Re = 0.1, \quad A^* = 0.1. \]

### Table 3. Values of the local Nusselt number \(-\theta'(0)\) for several sets of values of the parameters when \(\lambda = 2\) and \(Pr = 1.\)

| Parameters | Values     | First solution | Second solution | Third solution |
|------------|------------|----------------|-----------------|---------------|
| \(X\)     | 1.47       | -8.8429        | -9.1517         | -9.4545       |
|           | 1.49       | -9.0823        | -9.4063         | -9.7290       |
| \(a\)     | -1.49      | -8.8106        | -9.1613         | -9.5034       |
|           | -1.47      | -8.1078        | -8.4835         | -8.8366       |
| \(Ha^2/Re\) | 0.02      | -9.4528        | -9.8197         | -10.1080      |
|           | 0.04       | -9.9440        | -10.4318        | -10.5848      |
| \(Bi\)    | 1.6        | -8.6862        | -9.0038         | -9.3208       |
|           | 1.8        | -7.9822        | -8.2798         | -8.5763       |
| \(Ec\)    | 0.11       | -9.8097        | -10.3287        | -10.7515      |
|           | 0.13       | -11.0357       | -12.0666        | -12.6371      |
| \(A^*\)   | 0.094      | -9.0167        | -9.3370         | -9.6766       |
|           | 0.098      | -9.1438        | -9.4717         | -9.8070       |

While studying the effect of individual parameters the following values are used:

\[ a = -1.5, \quad X = 1.5, \quad Bi = 1.5, \quad Ec = 0.1, \quad Ha^2/Re = 0.1, \quad A^* = 0.1. \]

### 3.3 The opposing flow case, \(\lambda < 0\)

To compare the findings of the previous section, the numerical calculations of \(f''(0), \quad -\theta'(0), \quad f'(\eta)\) and \(\theta(\eta)\) for \(\lambda < 0\) are shown in this section. Figure 6 and 7 show the velocity \(f'(\eta)\) and temperature \(\theta(\eta)\) profiles when \(\lambda = -0.1\). Velocity profiles for \(\lambda < 0\) show the same characteristics as for \(\lambda > 0\), except the nonexistence of the third solution profile. In Figure 7, the temperature is found to enhance initially at a point, but it reduces for a large value of \(\eta\). The temperature of the first solution profile near the stretching sheet is higher than that of the second solution profile; however, it shows a reverse behaviour for the distance at an adjacent fluid.
4. Figure 6. Velocity profiles when \( \lambda = -0.1 \)

5. Figure 7. Temperature profiles when \( \lambda = -0.1 \)

To study the effect of several physical parameters on the rate of skin friction coefficient and local Nusselt number, their values are presented in Table 4 and 5. In Table 5, the rate of local Nusselt number reduces due to the effect of parameter \( X \), whereas this parameter increases the values of skin friction coefficient. From this, it is inferred that the effects of \( a \) and \( Bi \) increase the skin friction coefficient and local Nusselt number. It is evident from these two figures that the values of both \( f''(0) \) and \( -\theta'(0) \) reduce by enhancing the values of \( Ha^2/Re \), \( Ec \), and \( A^* \).
Table 4. Values of the skin friction coefficient $f''(0)$ for several sets of values of the parameters when $\lambda = -0.2$ and $Pr = 1$.

| Parameters | Values | First solution | Second solution |
|------------|--------|----------------|-----------------|
| $\lambda$  | 1.47   | 1.3459         | 1.3527          |
|            | 1.49   | 1.3444         | 1.3517          |
| $a$        | -1.49  | 1.3431         | 1.3507          |
|            | -1.47  | 1.3418         | 1.3499          |
| $Ha^2/Re$  | 0.02   | 1.3516         | 1.3598          |
|            | 0.04   | 1.3673         | 1.3768          |
| $Bi$       | 1.6    | 1.3407         | 1.3489          |
|            | 1.8    | 1.3361         | 1.3452          |
| $Ec$       | 0.11   | 1.3471         | 1.3537          |
|            | 0.13   | 1.3544         | 1.3583          |
| $A^*$      | 0.04   | 1.3362         | 1.3463          |
|            | 0.08   | 1.3408         | 1.3494          |

*While studying the effect of individual parameters the following values are used $a = -1.5$, $\lambda = 1.5$, $Bi = 1.5$, $Ec = 0.1$, $Ha^2/Re = 0.01$, and $A^* = 0.1$.

4. Conclusion

The problems of mixed convection heat transfer in a steady flow with the effect of convective surface boundary condition, magnetic field, viscous dissipation, and internal heat generation are theoretically discussed in this study. We focus on the flow of a viscous fluid, which passes an exponentially stretching continuous surface with an exponential temperature variation. It is observed that the three solution profiles exist for both skin friction coefficient and local Nusselt number when the flow is assisting, i.e. with the large assisting flow parameter, and only two solution profiles exist for the small values of opposing and assisting flow parameters. In order to show the solution profiles for mixed convection, curves are drawn for velocity profile, temperature profile, skin friction coefficient and local Nusselt number in the figures. As discussed in the previous section, it is found that the characteristics of the boundary layer flow and heat transfer are significantly affected by the physical parameters.
Table 5. Values of the local Nusselt number $-\theta'(0)$ for several sets of values of the parameters when $\lambda = -0.2$ and $Pr = 1$.

| Parameters | Values | $-\theta'(0)$ |
|-----------|--------|---------------|
| $X$       | 1.47   | -8.2950       |
|           | 1.49   | -8.6336       |
| $a$       | -1.49  | -8.6229       |
|           | -1.47  | -8.2623       |
| $Ha^2/Re$ | 0.02   | -8.9018       |
|           | 0.04   | -9.0826       |
| $Bi$      | 1.6    | -8.5655       |
|           | 1.8    | -8.1798       |
| $Ec$      | 0.11   | -9.4742       |
|           | 0.13   | -10.8043      |
| $A^*$     | 0.04   | -7.6429       |
|           | 0.08   | -8.3795       |

*a*While studying the effect of individual parameters the following values are used $a = -1.5$, $X = 1.5$, $Bi = 1.5$, $Ec = 0.1$, $Ha^2/Re = 0.01$, and $A^* = 0.1$.

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