A short derivation of the structure theorem for graphs with excluded topological minors

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Abstract

As a major step in their proof of Wagner’s conjecture, Robertson and Seymour showed that every graph not containing a fixed graph \( H \) as a minor has a tree-decomposition in which each torso is almost embeddable in a surface of bounded genus. Recently, Grohe and Marx proved a similar result for graphs not containing \( H \) as a topological minor. They showed that every graph which does not contain \( H \) as a topological minor has a tree-decomposition in which every torso is either almost embeddable in a surface of bounded genus, or has a bounded number of vertices of high degree. We give a short proof of the theorem of Grohe and Marx, improving their bounds on a number of the parameters involved.

1 Introduction

A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) by contracting edges. In a series of 23 papers, published between 1983 and 2012, Robertson and Seymour developed a deep theory of graph minors which culminated in the proof of Wagner’s Conjecture \cite{18}, which asserts that in any infinite set of finite graphs there is one which is a minor of another. One of the landmark results proved along the way, and indeed a fundamental step in resolving Wagner’s Conjecture, is a structure theorem for graphs excluding a fixed graph as a minor \cite{17}. It is easy to see that \( G \) cannot contain \( H \) as a minor if there is a surface into which \( G \) can be embedded but \( H \) cannot. Loosely speaking, the structure theorem of Robertson and Seymour asserts an approximate converse to this, thereby revealing the deep connection between topological graph theory and the theory of graph minors:

\textbf{Theorem 1} \cite{17} (informal). For any \( n \in \mathbb{N} \), every graph excluding the complete graph \( K_n \) as a minor has a tree-decomposition in which every torso is almost embeddable into a surface into which \( K_n \) is not embeddable.

A graph \( H \) is a topological minor of a graph \( G \) if \( G \) contains a subdivision of \( H \) as a subgraph. It is easy to see that \( G \) then also contains \( H \) as a minor. The converse is not true, as there exist cubic graphs with arbitrarily large complete minors. For topological minors, we thus have an additional degree-based obstruction, which is fundamentally different from the topological obstruction of surface-embeddings for graph minors. Grohe and Marx \cite{10} proved a result in a similar spirit to Theorem 1 for graphs excluding a fixed graph as a topological minor:
Theorem 2 ([10] (informal)). For any \( n \in \mathbb{N} \), every graph excluding \( K_n \) as a
topological minor has a tree-decomposition in which every torso either

(i) has a bounded number of vertices of high degree, or

(ii) is almost embeddable into a surface of bounded genus.

More recently, Dvořák [8] refined the embeddability condition of this theorem
to reflect more closely the topology of embeddings of an arbitrary graph \( H \) which
is to be excluded as a topological minor.

The proof given in [10], which uses Theorem 2 as a block-box, is algorithmic
and explicitly provides a construction of the desired tree-decomposition, however
as a result the proof is quite technical in parts. In this paper, we give a short
proof of Theorem 2 which also provides a good heuristic for the structure of
graphs without a large complete topological minor, as well as improving the
implicit bounds given in [10] on many of the parameters in their theorem. Our
proof is non-constructive, but we note that it can easily be adapted to give an
algorithm to find either a subdivision of \( K_r \) or an appropriate tree-decomposi-
tion. However, the run time of this algorithm will be much slower than that of
the algorithm given in [10].

One of the fundamental structures we consider are \( k \)-blocks. A \( k \)-block in
a graph \( G \) is a set \( B \) of at least \( k \) vertices which is inclusion-maximal with the
property that for every separation \( (U, W) \) of order \( < k \), we either have \( B \subseteq U \)
or \( B \subseteq W \). The notion of a \( k \)-block, which was first studied by Mader [14, 13],
has previously been considered in the study of graph decompositions [4, 3, 5].

It is clear that a subdivision of a clique on \( k+1 \) vertices yields a \( k \)-block. The
converse is not true for any \( k \geq 4 \), as there exist planar graphs with arbitrarily
large blocks. The second author [20] proved a structure theorem for graphs
without a \( k \)-block:

Theorem 3 ([20]). Let \( G \) be a graph and \( k \geq 2 \). If \( G \) has no \( (k+1) \)-block then
\( G \) has a tree-decomposition in which every torso has at most \( k \) vertices of degree
at least \( 2k(k-1) \).

Now, since a subdivision of a complete graph gives rise to both a complete
minor and a block, there are two obvious obstructions to the existence of a
large topological minor, the absence of a complete minor or the absence
of a large block. The upshot of Theorem 2 is that in a local sense these are
the only obstructions, any graph without a large topological minor has a tree-
decomposition into parts whose torsos either don’t contain a large minor, or
don’t contain a large block. Furthermore, by Theorem 1 and Theorem 3, the
converse should also be true: if we can decompose the graph into parts whose
torsos either don’t contain a large minor or don’t contain a large block, then
we can refine this tree-decomposition into one satisfying the requirements of
Theorem 2.

The idea of our proof is as follows. Both large minors and large blocks point
towards a ‘big side’ of every separation of low order. A subdivision of a clique
simultaneously gives rise to both a complete minor and a block and, what’s
more, the two are hard to separate in that they choose the same ‘big side’ for
every low-order separation. A qualitative converse to this is already implicit
in previous work on graph minors and linkage problems: if a graph contains a
large complete minor and a large block which cannot be separated from that minor, then the graph contains a subdivision of a complete graph.

Therefore, if we assume our graph does not contain a subdivision of \( K_r \), then we can separate any large minor from every large block. It then follows from the \textit{tangle tree theorem} of Robertson and Seymour \cite{robertson1986graph} – or rather its extension to \textit{profiles} \cite{robertson1995graphs,robertson1995graph,robertson1995graph} – that there exists a tree-decomposition which separates the blocks from the minors. Hence each part is either free of large minors or of large blocks.

However, in order to apply Theorems \ref{thm:1} and \ref{thm:3}, we need to have control over the torsos, and not every tree-decomposition will provide that: it might be, for example, that separating some set of blocks created a large minor in one of the torsos. We therefore contract some parts of our tree-decomposition and use the minimality of the remaining separations to prove that this does not happen.

A second nice feature of our proof is that we avoid the difficulty of constructing such a tree-decomposition by choosing initially a tree-decomposition with certain connectivity properties, the proof of whose existence already exists in the literature, and then simply \textit{deducing} that this tree-decomposition has the required properties.

We are going to prove the following:

\textbf{Theorem 4.} Let \( r \) be a positive integer and let \( G \) be a graph containing no subdivision of \( K_r \). Then \( G \) has a tree-decomposition of adhesion \(< r^2\) such that every torso either

- (i) has fewer than \( r^2 \) vertices of degree at least \( 2r^4 \), or
- (ii) has no \( K_{2r,2} \)-minor.

Combining Theorems \ref{thm:1} and \ref{thm:3} then yields Theorem \ref{thm:2}.

Let us briefly compare the bounds we get to the result of Grohe and Marx \cite{grohe2007eulerian}. It is implicit in their results that if \( G \) contains no subdivision of \( K_r \), then \( G \) has a tree-decomposition of adhesion \( O(r^6) \) such that every torso either has \( O(r^6) \) vertices of degree \( \Omega(r^7) \), has no \( K_{\Omega(r^6)} \) minor or has size at most \( O(r^6) \). In this way, Theorem \ref{thm:4} gives an improvement on the bounds for each of the parameters. Recently Liu and Thomas \cite{liu2018approximate} also proved an extension of the work of Dvořák \cite{dvorak2012adhesion}, with the aim to more closely control the bound on the degrees of the vertices in (i). Their results, however, only give this structure ‘relative’ to some tangle.

\section{Notation and background material}

All graphs considered here are finite and undirected and contain neither loops nor parallel edges. Our notation and terminology mostly follow that of \cite{dvorak2012adhesion}.

Given a tree \( T \) and \( s,t \in V(T) \), we write \( sTt \) for the unique \( s-t \)-path in \( T \). A \textit{separation} of a graph \( G = (V, E) \) is a pair \((A, B)\) with \( V = A \cup B \) such that there are no edges between \( A \setminus B \) and \( B \setminus A \). The \textit{order} of \((A, B)\) is the number of vertices in \( A \cap B \). We call the separation \((A, B)\) \textit{tight} if for all \( x, y \in A \cap B \), both \( G[A] \) and \( G[B] \) contain an \( x-y \)-path with no internal vertices in \( A \cap B \).

The set of all separations of \( G \) of order \(< k \) will be denoted by \( S_k(G) \). An \textit{orientation} of \( S_k(G) \) is a subset of \( S_k(G) \) containing precisely one element from each pair \( \{(A, B), (B, A)\} \subseteq S_k(G) \). The orientation is \textit{consistent} if it
does not contain two separations \((A, B), (C, D)\) with \(B \subseteq C\) and \(D \subseteq A\). A separation distinguishes two orientations \(O_1, O_2\) of \(S_k(G)\) if precisely one of \(O_1, O_2\) contains it. It does so efficiently if it has minimum order among all separations distinguishing them.

Recall that, given an integer \(k\), a set \(B\) of at least \(k\) vertices of \(G\) is a \(k\)-block if it is inclusion-maximal with the property that for every separation \((U, W)\) of order \(< k\), either \(B \subseteq U\) or \(B \subseteq W\). Observe that \(B\) induces a consistent orientation \(O_B := \{(U, W): B \subseteq W\}\) of \(S_k(G)\).

Given an integer \(m\), a model of \(K_m\) is a family \(\mathcal{X}\) of \(m\) pairwise disjoint sets of vertices of \(G\) such that \(G[\mathcal{X}]\) is connected for every \(X \in \mathcal{X}\) and \(G\) has an edge between \(X\) and \(Y\) for any two \(X, Y \in \mathcal{X}\). The elements of \(\mathcal{X}\) are called branch sets. Note that, if \((U, W)\) is a separation of order \(< m\), then exactly one of \(U \setminus W\) and \(W \setminus U\) contains some branch set. In this way, \(\mathcal{X}\) induces a consistent orientation \(O_X\) of \(S_k(G)\), where \((U, W) \in O_X\) if and only if some branch set of \(\mathcal{X}\) is contained in \(W\).

A tree-decomposition of \(G\) is a pair \((T, \mathcal{V})\), where \(T\) is a tree and \(\mathcal{V} = (V_i)_{\in T}\) is a family of sets of vertices of \(G\) such that:

- for every \(v \in V(G)\), the set of \(t \in V(T)\) with \(v \in V_t\) induces a non-empty subtree of \(T\);
- for every edge \(vw \in E(G)\) there is a \(t \in V(T)\) with \(v, w \in V_t\).

If \((T, \mathcal{V})\) is a tree-decomposition of \(G\), then every \(st \in E(T)\) induces a separation

\[
(U_s, W_t) := \left( \bigcup_{u \in T_s} V_u, \bigcup_{v \in T_t} V_v \right).
\]

Note that \(U_s \cap W_t = V_s \cap V_t\). In this way, every edge \(e \in E(T)\) has an order given by the order of the separation it induces, which we will write as \(|e|\). Similarly, an edge of \(T\) (efficiently) distinguishes two orientations if the separation it induces does. We say that \((T, \mathcal{V})\) (efficiently) distinguishes two orientations \(O\) and \(P\) if some edge of \(T\) does. We call \((T, \mathcal{V})\) tight if every separation induced by an edge of \(T\) is tight.

The adhesion of \((T, \mathcal{V})\) is the maximum order of an edge. If the adhesion of \((T, \mathcal{V})\) is less than \(k\) and \(O\) is an orientation of \(S_k(G)\), then \(O\) induces an orientation of the edges of \(T\) by orienting an edge \(st\) towards \(t\) if \((U_s, W_t) \in O\). If \(O\) is consistent, then all edges will be directed towards some node \(t \in V(T)\), which we denote by \(t_O\) and call the home node of \(O\). When \(O\) is induced by a block \(B\) or model \(\mathcal{X}\), we abbreviate \(t_B := t_{O_B}\) and \(t_X := t_{O_X}\), respectively.

Observe that every edge \(e \in E(T)\) distinguishes two orientations \(O\) and \(P\) if and only if \(e \in E(t_{OTP})\).

Given \(t \in V(T)\), the torso at \(t\) is the graph obtained from \(G[V_t]\) by adding, for every neighbor \(s\) of \(t\), an edge between any two non-adjacent vertices in \(V_s \cap V_t\). More generally, given a subtree \(S \subseteq T\), the torso at \(S\) is the graph obtained from \(G \left[ \bigcup_{s \in S} V_s \right]\) by adding, for every edge \(st \in E(T)\) with \(S \cap \{s, t\} = \{s\}\), an edge between any two non-adjacent vertices in \(V_s \cap V_t\).

We also define contractions on tree-decompositions: Given \((T, \mathcal{V})\) and an edge \(st \in E(T)\), to contract the edge \(st\) we form a tree-decomposition \((T', \mathcal{V}')\) where

- \(T'\) is obtained by contracting \(st\) in \(T\) to a new vertex \(x\);
• Let \( V'_x := V_x \cup V_t \) and \( V'_u := V_u \) for all \( u \in V(T) \setminus \{s,t\} \).

It is simple to check that \((T', V')\) is a tree-decomposition. We note that the separations induced by an edge in \( E(T) \setminus \{st\} \) remain the same, as do the torsos of parts \( V_u \) for \( u \neq s, t \).

We say a tree-decomposition \((T,V)\) is \( k\)-lean if it has adhesion \(< k\) and the following holds for all \( p \in [k] \) and \( s, t \in T \): If \( st \) contains no edge of order \(< p \), then every separation \((A,B)\) with \(|A \cap V_s| \geq p\) and \(|B \cap V_t| \geq p\) has order at least \( p \).

Let \( n := |G| \). The fatness of \((T,V)\) is the sequence \((a_0, \ldots , a_n)\), where \( a_i \) denotes the number of parts of order \( n - i \). A tree-decomposition of lexicographically minimal fatness among all tree-decompositions of adhesion smaller than \( k \) is called \( k\)-atomic. These tree-decompositions play a pivotal role in our proof, but we actually only require two properties that follow from this definition. It was observed by Carmesin, Diestel, Hamann and Hundertmark \[2\] that the short proof of Thomas’ Theorem \[19\] given by Bellenbaum and Diestel in \[1\] also shows that \( k\)-atomic tree-decompositions are \( k\)-lean (see also \[9\]).

**Lemma 5** (\[1\]). Every \( k\)-atomic tree-decomposition is \( k\)-lean.

It is also not hard to see that \( k\)-atomic tree-decompositions are tight. In \[20\], the second author used \( k\)-atomic tree-decompositions to prove a structure theorem for graphs without a \( k\)-block. In fact, the proof given there yields the following:

**Lemma 6** (\[20\]). Let \( G \) be a graph and \( k \) a positive integer. Let \((T,V)\) be a \( k\)-atomic tree-decomposition of \( G \) and \( t \in V(T) \) such that \( V_t \) contains no \( k\)-block of \( G \). Then the torso at \( t \) contains fewer than \( k \) vertices of degree at least \( 2k^2 \).

Let \( G \) be a graph and \( Z \subseteq V(G) \). We denote by \( G^{Z} \) the graph obtained from \( G \) by making the vertices of \( Z \) pairwise adjacent. A \( Z\)-based model is a model \( \mathcal{X} \) of \( K_{|Z|} \) such that \( X \cap Z \) consists of a single vertex for every \( X \in \mathcal{X} \).

The following lemma of Robertson and Seymour \[16\] is crucial to our proof.

**Lemma 7** (\[16\]). Let \( G \) be a graph, \( Z \subseteq V(G) \) and \( p := |Z| \). Let \( q \geq 2p - 1 \) and let \( \mathcal{X} \) be a model of \( K_q \) in \( G^{Z} \). If \( \mathcal{X} \) and \( Z \) induce the same orientation of \( S_p(G^{Z}) \), then \( G \) has a \( Z\)-based model.

### 3 The proof

Let us fix throughout this section a graph \( G \) with no subdivision of \( K_r \), let \( k := r(r - 1), m := 2k \), and let \((T,V)\) be a \( k\)-atomic tree-decomposition of \( G \).

First, we will show that \((T,V)\) efficiently distinguishes every \( k\)-block from every model of \( K_m \) in \( G \). This allows us to split \( T \) into two types of sub-trees, those containing a \( k\)-block and those containing a model of \( K_m \). **Lemma 3** allows us to bound the number of high degree vertices in the torsos in the latter components. We will then show that if we choose these sub-trees in a sensible way then we can also bound the order of a complete minor contained in the torsos of the former. Hence, by contracting each of these sub-trees in \((T,V)\) we will have our desired tree-decomposition.

To show that \((T,V)\) distinguishes every \( k\)-block from every model of \( K_m \) in \( G \), we must first show that they are distinguishable, that is, no \( k\)-block and \( K_m \)
induce the same orientation. The following lemma, as well as its proof, is similar to Lemma 6.11 in [10].

**Lemma 8.** Let $B$ be a $k$-block and $X$ a model of $K_m$ in $G$. If $B$ and $X$ induce the same orientation of $S_k$, then $G$ contains a subdivision of $K_r$ with arbitrarily prescribed branch vertices in $B$.

**Proof.** Suppose $B$ and $X$ induce the same orientation and let $B_0$ be an arbitrary subset of $B$ of size $r$. Let $H$ be the graph obtained from $G$ by replacing every $b \in B_0$ by an independent set $J_b$ of order $(r - 1)$, where every vertex of $J_b$ is adjacent to every neighbor of $b$ in $G$ and to every vertex of $J_c$ if $b, c$ are adjacent. Let $J := \bigcup J_b$ and note that $|J| = k$. We regard $G$ as a subgraph of $H$ by identifying each $b \in B$ with one arbitrary vertex in $J_b$. In this way we can regard $X$ as a model of $K_m$ in $H$.

Assume for a contradiction that there was a separation $(U, W)$ of $H$ such that $|U \cap W| < |J|$. Let $J \subseteq U$ and $X \subseteq W \setminus U$ for some $X \subseteq X$. We may assume without loss of generality that for every $b \in B_0$, either $J_b \subseteq U \cap W$ or $J_b \cap (U \cap W) = \emptyset$. Indeed, if there is a $z \in J_b \setminus (U \cap W)$, then $z \in U \setminus W$, and we can delete any $z' \in J_b \cap W$ from $W$ and maintain a separation (because $N(z) = N(z')$) with the desired properties. In particular, for every $b \in B_0$ we find $b \in W$ if and only if $J_b \subseteq W$. Since $|U \cap W| < |J|$, it follows that there is at least one $b_0 \in B_0$ with $J_{b_0} \subseteq (U \setminus W)$. Let $(U', W') := (U \cap V(G), W \cap V(G))$ be the induced separation of $G$. Then $X \subseteq W' \setminus U'$ and $b_0 \in U' \setminus W'$. Since $|U' \cap W'| \leq |U \cap W| < k$ and $B$ is a $k$-block, we have $B \subseteq U'$. But then $(U', W')$ distinguishes $B$ and $X$, which is a contradiction to our initial assumption.

We may now apply Lemma 7 to $H$ and find a $J$-based model $\mathcal{Y} = (Y_j)_{j \in J}$ in $H$. For each $b \in B_0$, label the vertices of $J_b$ as $(v_{b,j}^c)_{c \in B_0 \setminus \{b\}}$. For $b \neq c$, $H$ has a path $P_{b,c} \subseteq Y_{b,j} \cup Y_{c,j}$ and the paths obtained like this are pairwise disjoint, because the $Y_j$ are, and $P_{b,c} \cap J = \{v_{b,j}^c, v_{c,j}^b\}$. For each such path $P_{b,c}$, obtain $P_{b,c} \subseteq G$ by replacing $v_{b,j}^c$ by $b$ and $v_{c,j}^b$ by $c$. The collection of these paths $(P_{b,c})_{b,c \in B_0}$ gives a subdivision of $K_r$ with branch vertices in $B_0$. □

Now we can show that $(T, \mathcal{V})$ efficiently distinguishes every $k$-block from every model of $K_m$ in $G$.

**Lemma 9.** $(T, \mathcal{V})$ efficiently distinguishes all orientations of $S_k(G)$ induced by $k$-blocks or models of $K_m$.

**Proof.** Let us call a consistent orientation $O$ of $S_k(G)$ anchored if for every $(U, W) \in O$, there are at least $k$ vertices in $W \cap V_{t_O}$.

Note that every orientation $O = O_B$ induced by a $k$-block $B$ is trivially anchored, since $B \subseteq V_{t_B}$. But the same is true for the orientation $O = O_X$ induced by a model $X$ of $K_m$. Indeed, let $(U, W) \in O_X$. Then every set in $X$ meets $V_{t_X}$. At least $k$ branch sets of $X$ are disjoint from $U \cap W$, say $X_1, \ldots, X_k$, and they all lie in $W \setminus U$. For $1 \leq i \leq k$, let $x_i \in X_i \cap V_{t_X}$ and note that $R := \{x_1, \ldots, x_k\} \subseteq W \cap V_{t_X}$.

We may now show that $(T, \mathcal{V})$ efficiently distinguishes all anchored orientations of $S_k(G)$. Let $O_1, O_2$ be anchored orientations of $S_k(G)$ and let their home nodes be $t_1$ and $t_2$ respectively. If $t_1 \neq t_2$, let $p$ be the minimum order of an edge along $t_1 T t_2$, and put $p := k$ otherwise. Choose some $(U, W) \in O_2 \setminus O_1$ of minimum order. Since $O_1$ and $O_2$ are anchored, we have $|U \cap V_{t_1}| \geq k$ and
\(|W \cap V_\ell| \geq k\). As \((T, V)\) is \(k\)-lean, it follows that \(|U \cap W| \geq p\). Hence \(t_1 \neq t_2\) and \((T, \mathcal{V})\) efficiently distinguishes \(O_1\) and \(O_2\).

Let us call a node \(t \in V(T)\) a block-node if it is the home node of some \(k\)-block and model-node if it is the home node of a model of \(K_m\).

Let \(F \subseteq E(T)\) be inclusion-minimal such that every \(k\)-block is efficiently distinguished from every model of \(K_m\) by some separation induced by an edge in \(F\). We now define a red/blue colouring \(c : V(T) \to \{r, b\}\) by letting \(c(t) = b\) if the component of \(T - F\) containing \(t\) contains a block-node and letting \(c(t) = r\) if it contains a model-node. Let us first show that this is in fact a colouring of \(V(T)\).

**Lemma 10.** Every node receives exactly one colour.

**Proof.** Suppose first that \(t \in V(T)\) is such that the component of \(T - F\) containing \(t\) contains both a block-node and a model-node. Then there is a \(k\)-block \(B\) and a \(K_m\)-minor \(X\) such that \(t_B Tt\) and \(t_X Tt\) both contain no edges of \(F\). But then \(B\) and \(X\) are not separated by the separations induced by \(F\), a contradiction.

Suppose now that \(t \in V(T)\) is such that the component \(S\) of \(T - F\) containing \(t\) contains neither a block nor a minor. Let \(f_1, \ldots, f_n\) be the edges of \(T\) between \(S\) and \(T\setminus S\), ordered such that \(|f_i| \geq |f_j|\) for all \(i \leq n\). By minimality of \(F\), there is a block-node \(t_B\) and a model-node \(t_X\) such that \(f_1\) is the only edge of \(F\) that efficiently distinguishes \(B\) and \(X\). Since \(t_B \cap X \notin S\), there is a \(j \geq 2\) such that \(f_j \in E(t_B Tt_X)\), and so \(f_j\) distinguishes \(B\) and \(X\) as well, and since \(|f_1| \geq |f_j|\), it does so efficiently, contradicting our choice of \(B\) and \(X\)

**Lemma 11.** Let \(s \in E(T)\) and suppose \(s\) is blue and \(t\) is red. Then \(G[W_t]\) has a \((V_s \cap V_t)\)-based model.

**Proof.** Let \(Q := V_s \cap V_t\). Let \(t_B\) be a block-node in the same component of \(T - F\) as \(s\) and let \(t_X\) be a model-node in the same component as \(t\). Since the separations induced by \(F\) efficiently distinguish \(B\) and \(X\), it must be that \(st \in F\) and \((U_s, W_t)\) efficiently distinguishes \(B\) and \(X\).

Let \(Y := (X \cap W_t)_{X \in X}\). Since \((U_s, W_t) \in O_X\), \(Y\) is a model of \(K_m\) in \(G[W_t]^Q\). We wish to apply Lemma 7 to \(Q\) and \(Y\) in the graph \(G[W_t]\). Suppose \(Q\) and \(Y\) do not induce the same orientation of \(S_Q^G(G[W_t]^Q)\). That is, there is a separation \((U, W)\) of \(G[W_t]^Q\) with \(|U \cap W| < |Q|\) and \(Q \subseteq U\) such that \(Y \cap U = \emptyset\) for some \(Y \in Y\). There is an \(X \in X\) so that \(Y = X \cap G[W_t]\). Note that \(X \cap U\) is empty as well. Now \((U', W') := (U \cup U_s, W)\) is a separation of \(G\). Note that

\[X \cap U' = X \cap U_s = \emptyset,\]

because \(X\) is connected, meets \(W_t\) and does not meet \(Q\). Therefore \(X \subseteq W' \setminus U'\) and \(B \subseteq U_s \subseteq U'\). But \(|U' \cap W'| = |U \cap W| < |Q|\), which contradicts the fact that \((U_s, W_t)\) efficiently distinguishes \(B\) and \(X\). Therefore, by Lemma 7, \(G[W_t]\) has a \(Q\)-based model. \(\square\)

Using the above we can bound the size of a complete minor in the torso of a blue component. The next lemma plays a similar role to Lemma 6.9 in [10].

**Lemma 12.** Let \(S \subseteq T\) be a maximal subtree consisting of blue nodes. Then the torso of \(S\) has no \(K_m\)-minor.
Proof. Let $F_S := \{(s,t) : st \in E(T), s \in S, t \notin S\}$. For every $(s,t) \in F_S$, the node $s$ is blue and $t$ is red. By Lemma 11 $G_t$ has a $(V_s \cap V_t)$-based complete minor $Y^{s,t}$. Contract each of its branch sets onto the single vertex of $V_s \cap V_t$ that it contains. Do this for every $(s,t) \in F_S$. After deleting any vertices outside of $V_S := \bigcup_{s \in S} V_s$, we obtain the torso of $S$ as a minor of the graph $G$.

Suppose the torso of $S$ contained a $K_m$-minor. Then $G$ has a $K_m$-minor $\mathcal{X}$ such that every $X \in \mathcal{X}$ meets $V_S$. Therefore $\mathcal{X}$ orients every edge $st \in E(T)$ with $(s,t) \in F_S$ towards $s$. But then $t_X \in S$, contradicting the assumption that $S$ contains no red nodes.

We can now finish the proof. Let $(T', V')$ be obtained from $(T, V)$ by contracting every maximal subtree consisting of blue nodes and let the vertices of $T'$ inherit the colouring from $V(T)$. We claim that $(T', V')$ satisfies the conditions of Theorem 4.

Indeed, firstly, the adhesion of $(T', V')$ is at most that of $(T, V)$, and hence is at most $k$. Secondly, the torso of every red node in $(T', V')$ is the torso of some red node in $(T, V)$, which by Lemma 6 has fewer than $k$ vertices of degree at least $2k^2$. Finally, by Lemma 12 the torso of every blue node in $(T', V')$ has no $K_m$ minor. Since $k = r(r-1)$ and $m = 2k$, the theorem follows.

As claimed in the introduction, it is not hard to turn this proof into an algorithm to find either a subdivision of $K_r$ or an appropriate tree-decomposition. Indeed, the proof of Lemma 5 can easily be adapted to give an algorithm to find a tight $k$-lean tree-decomposition. Similarly, in order to colour the vertices of the tree red or blue we must check for the existence of a $K_m$ minor or a $k$-block having this vertex as a home node, both of which can be done algorithmically (see [10] and [2]). However, we note that the running time of such an algorithm, or at least a naive implementation of one, would have run time $\sim |V(G)|^{f(r)}$ for some function of the size of the topological minor $K_r$ we are excluding, whereas the algorithm of Grohe and Marx has run time $g(r)|V(G)|^{O(1)}$, which should be much better for large values of $r$.

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