Model Specification Test with Unlabeled Data: Approach from Covariate Shift

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Abstract

We propose a novel framework of the model specification test in regression using unlabeled test data. In many cases, we have conducted statistical inferences based on the assumption that we can correctly specify a model. However, it is difficult to confirm whether a model is correctly specified. To overcome this problem, existing works have devised statistical tests for model specification. Existing works have defined a correctly specified model in regression as a model with zero conditional mean of the error term over train data only. Extending the definition in conventional statistical tests, we define a correctly specified model as a model with zero conditional mean of the error term over any distribution of the explanatory variable. This definition is a natural consequence of the orthogonality of the explanatory variable and the error term. If a model does not satisfy this condition, the model might lack robustness with regards to the distribution shift. The proposed method would enable us to reject a misspecified model under our definition. By applying the proposed method, we can obtain a model that predicts the label for the unlabeled test data well without losing the interpretability of the model. In experiments, we show how the proposed method works for synthetic and real-world datasets.

1 Introduction

While statistical methods have achieved considerable successes in recent years, we also need to consider the interpretability of a model to fulfill accountability. In several frameworks of the classical statistical inference in regression, we assume that the model is ‘correctly specified’, which refers to the case where the explanatory variable and the error term are uncorrelated. However, in some circumstances, we cannot be confident that the model specification is correct. To address this problem, existing works have developed statistical tests to detect misspecified models [Hausman, 1978, Davidson and MacKinnon, 1981, Vuong, 1983, Wooldridge, 1990, 1992, Smith, 1992, Sugiyama et al., 2008, Athey and Imbens, 2017, Komiyama and Shimao, 2018, Cox, 1961, 1962, Pesaran and Deaton, 1978, Smith, 1992, Victoria-Feser, 1997] proposed using the likelihood ratio to
compare non-nested alternatives. Godfrey [1984], Wooldridge [1990] proposed methods based on the Lagrange multiplier tests. Another conventional approach is to compare the residuals of two models. Durbin [1954], Wu [1973], Hausman [1978] are the most common methods following this approach. Wooldridge [1992] proposed a comparison between a parametric model and a non-parametric model to detect misspecification of the parametric model. Durbin [1954], Wu [1973], Hausman [1978] also proposed a test to detect the existence of endogeneity, i.e., the correlation between explanatory variable and the error term. Most existing works defined a correctly specified model as a model with the zero conditional mean of the error term and the expectation is taken over a distribution of train data. Therefore, for data with a different distribution from that of train data, the performance of the model will drop. On the other hand, if we define a correctly specified model as a model that has orthogonality between error term and explanatory variable, a correctly specified model can be a model minimizing the risk for any distribution of the explanatory variable. However, it is difficult to find such a model. Hence, in our paper, we define a correctly specified model as a model with the zero conditional mean of error term over distributions of train and test data.

In this paper, we consider a situation where we have labeled train data and unlabeled test data. We assume that the probability density function of the explanatory variable of the train data is different from that of the test data. Shimodaira [2000] termed this situation covariate shift and proposed an algorithm to construct an unbiased risk estimator for unlabeled test data. Under model misspecification, we cannot predict the behavior of a model trained using labeled train data when we use the model for unlabeled test data with a different distribution. However, Shimodaira [2000] proposed to weigh the loss function using the density ratio between train data and test data. The weighted average of the loss functions is an unbiased estimator of the risk for the test data. Thus, the method enables us to minimize an unbiased risk for the test data directly. This minimization theoretically guarantees the performance of prediction even if the model is misspecified. Therefore, we can regard the Shimodaira [2000]’s method as a solution for model misspecification.

On the other hand, in the case where the model is specified correctly, we can ignore the covariate shift. This is because we can obtain the same estimator by minimizing the mean squared error with the expectation over different probability density functions of the explanatory variable. Sugiyama [2006], Sugiyama et al. [2007] explained this property and showed the necessity of using the covariate shift in active learning because active learning distorts the distribution of original data in the process of data correction. Based on this property, we propose a novel framework of model specification test in regression with a linear model using unlabeled test data. We can obtain two estimators by minimizing unbiased risk estimators for distributions of train and test data. We set the null hypothesis as the case where the two estimators are the same; and the alternative hypothesis as the case where the two estimators are different. Because of the property of a correctly specified model, if the null hypothesis is rejected, we can conclude that the model is misspecified. On the other hand, even if we
accept the null hypothesis, we cannot conclude that the model is correctly specified. In some cases, we can obtain the same estimator by minimizing the mean squared error with the expectation over different probability density functions of the explanatory variable. However, when we accept the null hypothesis, the model can predict the label for the train and the test data with a theoretical guarantee. In conclusion, the proposed method enables us to find misspecified models and construct a model that can make statistical inference for train and test data. The remainder of the paper is organized as follows. In Section 2, we define the problem-setting and the framework of model specification. Section 3 shows our algorithm for model specification, while Section 4 discusses the source of the power of the statistical test. In Section 5, we show the performance of the proposed method using real datasets. Section 6 concludes the paper.

2 Problem-Setting

Our goal is to identify a misspecified model in regression. In this section, we describe the data-generating process and the framework of the model specification test.

2.1 Data Generating Process

Let \( X \) and \( Y \) be the feature and label spaces with an unknown distribution over \( X \times Y \). We assume access to train data \( \{y_i, x_i\}_{i=1}^{n_{tr}} \) and test data \( \{x_i\}_{i=1}^{n_{te}} \), where \( x_i \in X \) and \( y_i \in Y \). We also assume the data generating process as follows:

\[
\{y_i, x_i\}_{i=1}^{n_{tr}} \overset{i.i.d.}{\sim} p(y, x), \quad \{\tilde{y}_i, x_i\}_{i=1}^{n_{te}} \overset{i.i.d.}{\sim} q(y, x),
\]

where \( p(y, x) \) is a distribution of train data, \( q(y, x) \) is a distribution of train data, and \( \tilde{y}_i \in Y \) is an unobservable variable. In this paper, we assume that train and test data has the same support of \( x \), i.e., \( 0 < \frac{q(x)}{p(x)} < \infty \) for any \( x \in X \). As Shimodaira [2000], we assume that the probability \( y \) conditioned on \( x \) is invariant between the distributions \( p(y | x) \) and \( q(y | x) \), i.e.,

\[
p(y, x) = p(y | x)p(x), \quad q(y, x) = p(y | x)q(x).
\]

2.2 Model Specification

Next, we consider specifying a model of the conditional mean function \( E[y|x] \). Let us define a basis of \( x \) as \( z \in \mathbb{R}^k \), i.e., \( z \) can be defined as a function such that \( z : \mathbb{R}^d \to \mathbb{R}^k \). To identify the parameter of the model, let us assume the situation where the dimension of the model is less than or equal to the number of samples, i.e., \( k \leq n_{tr} \) and \( k \leq n_{te} \), and \( \mathbb{E}_{p(x)}[z(x_i)z^T(x_i)] \) is finite and invertible. When we emphasize \( z \) as a function of \( x \), we denote \( z \) as \( z(x) \). For example, for \( x \in \mathbb{R} \), a basis \( z \in \mathbb{R}^3 \) can be \( z = (x, x^2, x^3)^T \). Then, a linear model based on a basis is defined as follows:

\[
y_i = z^T(x_i)\beta + \varepsilon_i,
\]
where $\beta \in \mathbb{R}^k$ is a $k$ dimensional coefficient of $z$ and $\varepsilon$ is the error term. The mean and the variance of the error term $\varepsilon$ is zero and $\sigma^2$, respectively. We define a correctly specified model as a model with zero conditional mean of the error term as follows:

**Definition 1.** If a model is correctly specified, the conditional mean of the error term is zero for any distribution over $\mathcal{X}$, i.e.,

$$E_{p(y|x)w(x)}[\varepsilon_i | z(x_i)] = E_{p(y|x)w(x)}[\varepsilon_i | x_i] = 0,$$  \hspace{1cm} (2)

where $w(x)$ denotes any distribution of the explanatory variable over $\mathcal{X}$.

Most frameworks of model specification tests define a correctly specified model as a model with the zero conditional mean of the error term only over train data [Hausman, 1978], i.e.,

$$E_{p(y,x)}[\varepsilon_i | z(x_i)] = 0.$$

In our problem setting, we consider a stronger condition. If a model is correctly specified, the conditional mean of the error term over any distribution over $\mathcal{X}$ is zero. If we assume an orthogonality between the explanatory variable and the error term on a correctly specified model, we can derive this condition straightforwardly. If a model satisfies this condition, the model becomes robust to distribution shift, i.e., the model can be a minimizer of the squared mean error over any distribution over $\mathcal{X}$.

### 2.3 Estimating the Parameters

We consider the following two least squares to estimate the parameter of model (1) using labeled train data and unlabeled test data as follows:

$$\alpha^* = \arg \min_{b \in \mathbb{R}^k} E_{p(y,x)}[(y_i - z^\top (x_i) b)^2],$$

$$\gamma^* = \arg \min_{b \in \mathbb{R}^k} E_{q(y,x)}[(\tilde{y}_i - z^\top (x_i) b)^2].$$

Finding the parameters $\alpha^*$ and $\gamma^*$ based on the above equations is equivalent to finding the parameters $\alpha^*$ and $\gamma^*$ that satisfy the following equations:

$$E_{p(y,x)}[z(x_i) (y_i - z^\top (x_i) \alpha^*)] = 0,$$

$$E_{q(y,x)}[z(x_i) (\tilde{y}_i - z^\top (x_i) \gamma^*)] = 0.$$

If a model is correctly specified, $\alpha^* = \gamma^* = \beta$ holds. We use this property to construct a model specification test.

### 2.4 Framework of Model Specification Test

As discussed above, if a model is correctly specified, $\alpha^* = \gamma^* = \beta$ holds. Based on this fact, we construct the following null hypothesis $\mathcal{H}_0$ and the alternative hypothesis $\mathcal{H}_1$:

$$\mathcal{H}_0 : \alpha^* = \gamma^*, \quad \mathcal{H}_1 : \alpha^* \neq \gamma^*.$$  \hspace{1cm} (3)
When we reject the null hypothesis, we can conclude that the model is not correctly specified. On the other hand, even if we accept the null hypothesis, we cannot conclude with certainty that the model is correctly specified. There are some cases where \( \alpha^* = \gamma^* \) even if the condition of a correctly specified model (2) does not hold. However, if \( \alpha^* = \gamma^* \) holds, we can use the model for both train and test data for prediction.

3 Algorithm

Following the problem-setting, we introduce an algorithm for detecting a mis-specified model. To estimate the parameter \( \gamma \), we face the problem that we do not have the label of the test data and cannot approximate \( \mathbb{E}_{q(y,x)} \left[ (y_i - z(x_i)^T b)^2 \right] \).

Shimodaira [2000] showed that the same estimator can be obtained by minimizing the following objective function:

\[
\mathbb{E}_{p(y,x)} \left[ (y_i - z(x_i)^T b)^2 \frac{q(x_i)}{p(x_i)} \right],
\]

where \( \frac{q(x_i)}{p(x_i)} \) is called the density ratio. We replace the above objective function using samples moments. Then, given \( \frac{q(x_i)}{p(x_i)} \) for \( i = 1, \ldots, n \), the parameter is estimated as

\[
\tilde{\gamma} = \left( \frac{1}{n} \sum_{i=1}^{n} z(x_i) z(x_i)^T \frac{q(x_i)}{p(x_i)} \right)^{-1}
\]

\[
\times \frac{1}{n} \sum_{i=1}^{n} z(x_i) y_i \left( \frac{q(x_i)}{p(x_i)} \right)
\]

\[
= \left( \frac{1}{n} \sum_{i=1}^{n} z(x_i) z(x_i)^T \frac{q(x_i)}{p(x_i)} \right)^{-1}
\]

\[
\times \frac{1}{n} \sum_{i=1}^{n} z(x_i) \varepsilon_i \left( \frac{q(x_i)}{p(x_i)} \right)^{\frac{1}{2}} + \beta,
\]

where \( y_i = z(x_i) \beta + \left( \frac{q(x_i)}{p(x_i)} \right)^{\frac{1}{2}} \varepsilon_i \). In our algorithm, we estimate the density ratio \( \frac{q(x_i)}{p(x_i)} \) first. Then, we estimate the parameters \( \gamma^* \) and \( \alpha^* \) using the system of equations with a estimator of the density ratio. After obtaining the estimators and the asymptotic distributions, we conduct a statistical test based on the hypotheses defined in (3). We demonstrate the pseudo code in Algorithm 1.

3.1 Density Ratio Estimation

To estimate the density ratio \( \frac{q(x_i)}{p(x_i)} \), we can estimate the probability density functions of the numerator \( q(x_i) \) and the denominator \( p(x_i) \), separately. However,
Algorithm 1

Input: Train data \( \{y_i, x_i\}_{i=1}^{n_{tr}} \) and test data \( \{x_i\}_{i=1}^{n_{te}} \).

Estimate the density ratio \( r(x) = \frac{q(x)}{p(x)} \).

Specify a model \( z(\cdot) \).

Estimate \( \theta \) as (8).

Estimate \( \sigma^2 \) as (9).

Construct a Wald statistic \( W \) by (10).

The Wald statistic \( W \) converges to chi-squared distribution with degree of freedom \( k \) in distribution. Based on this fact, conduct a statistical test on the hypothesises defined by (3).

due to Vapnik’s principle, we should avoid solving more difficult intermediate problems than the target problem. Sugiyama et al. [2012] summarized methods estimating the density ratio directly. Among the existing methods, we employ Least-Squares Importance Fitting (LSIF), which uses the squared loss to fit the density-ratio function. The reason for this choice is that there is an algorithm called unconstrained Least-Squares Importance Fitting (uLSIF) with a computational advantage. We can obtain the closed-form solution just by solving the linear equations. Thus, uLSIF is numerically stable when it is regularized properly. Moreover, the leave-one-out cross-validation score for uLSIF can also be computed analytically, which significantly improves the computational efficiency in model selection.

Here, we introduce the formulation of LSIF. In LSIF, we estimate the density ratio \( r(x) = \frac{q(x)}{p(x)} \) directly. Let \( S \) be the class of non-negative measurable functions \( s : \mathcal{X} \rightarrow \mathbb{R}^+ \). We consider minimizing the following squared error between \( s \) and \( r \):

\[
R_{DR}(s) := \mathbb{E}_{p(y,x)}[(s(x_i) - r(x_i))^2] \\
= \mathbb{E}_{p(y,x)}[(r(x_i))^2] - 2\mathbb{E}_{q(y,x)}[s(x_i)] + \mathbb{E}_{p(y,x)}[(s(x_i))^2].
\]

The first term of the last equation does not affect the result of minimization and we can ignore the term, i.e., the density ratio is estimated through the following minimization problem:

\[
s^* = \arg\min_{s \in S} R_{DR}(s) \\
= \arg\min_{s \in S} \left[ \frac{1}{2} \mathbb{E}_{p(y,x)}[(s(x_i))^2] - \mathbb{E}_{q(y,x)}[s(x_i)] \right].
\]

As mentioned above, to minimize the empirical version of (5), we use uLSIF [Kanamori et al., 2009]. Given a hypothesis class \( \mathcal{H} \), we obtain \( \hat{r} \) by

\[
\hat{r}(x_i) = \arg\min_{s \in \mathcal{H}} \left[ \frac{1}{2} \mathbb{E}_{p(y,x)}[(s(x_i))^2] - \mathbb{E}_{q(y,x)}[s(x_i)] + \mathcal{R}(s) \right].
\]
where \( \hat{E}_{w(y,x)} \) denotes the averaging operator over a joint distribution \( w(y,x) \) and \( R \) is a regularization term. The convergence rate is shown by Sugiyama et al. [2012]. For simplicity, we assume that \( n^{tr} = n^{te} = N \). Then, the convergence rate is given as follows:

\[
\mathbb{E}_{p(y,x)} \left[ \left( \hat{r}(x_i) - \left( \frac{q(x_i)}{p(x_i)} \right) \right)^2 \right] = O_p \left( N^{-\frac{1}{1+\gamma}} \right), \tag{7}
\]

where \( 0 < \gamma < 1 \) is a constant depending on the functional form of the density ratio function.

### 3.2 Estimator and Asymptotic Distribution

For simplicity, we assume that \( n^{tr} = n^{te} = N \). Consider replacing the density ratio in (4) with an estimator. We construct an estimator of \( \gamma^* \) as follows:

\[
\hat{\gamma} = \left( \frac{1}{N} \sum_{i=1}^{N} z(x_i) z(x_i)^\top \left( \hat{r}(x_i) \right) \right)^{-1} \times \frac{1}{N} \sum_{i=1}^{n} z(x_i) \varepsilon_i \hat{r}(x_i) + \beta.
\]

The asymptotic distribution of \( \hat{\gamma} \) is given by the following theorem:

**Theorem 1.**

\[
\sqrt{N} (\hat{\gamma} - \beta) \stackrel{d}{=} \sqrt{N} (\hat{\gamma} - \beta) + o_p(1) \xrightarrow{d} \mathcal{N}(0, \sigma^2 V^{-1}),
\]

where \( V = \mathbb{E}_{p(y,x)} \left[ z(x_i) z(x_i)^\top \left( \frac{q(x_i)}{p(x_i)} \right) \right] \) and \( \sigma^2 \) is the variance of the error term.

Before proving Theorem 1, we show the following lemmas. Let us denote \( a(f) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} z(x_i) \varepsilon_i \hat{r}(x_i) \) and \( A(f) = \frac{1}{N} \sum_{i=1}^{N} z(x_i) z(x_i)^\top f(x_i) \) for a function \( f : \mathcal{X} \to \mathbb{R} \). For \( a(f) \) and \( A(f) \), the following lemmas hold:

**Lemma 1.**

\[
a(\hat{r}) - a \left( \frac{q}{p} \right) \xrightarrow{p} 0.
\]

**Lemma 2.**

\[
A(\hat{r}) - A \left( \frac{q}{p} \right) \xrightarrow{p} 0.
\]

We show the proofs in Appendix A and Appendix B. Using the above two lemmas, we prove Theorem 1 as follows.
Proof. We have
\[
\sqrt{N} (\hat{\gamma} - \beta) = A \left( \frac{q}{p} \right)^{-1} a \left( \frac{q}{p} \right),
\]
\[
\sqrt{N} (\tilde{\gamma} - \beta) = A (\hat{r})^{-1} a (\hat{r}).
\]
Therefore,
\[
\sqrt{N} (\hat{\gamma} - \beta) = \sqrt{N} (\tilde{\gamma} - \beta) + A (\hat{r})^{-1} \left( a (\hat{r}) - a \left( \frac{q}{p} \right) \right)
\]
\[
- A \left( \frac{q}{p} \right)^{-1} \left( A (\hat{r}) - A \left( \frac{q}{p} \right) \right) A (\hat{r})^{-1} a \left( \frac{q}{p} \right).
\]
To prove the theorem it suffices to show
\[
a (\hat{r}) - a \left( \frac{q}{p} \right) \overset{p}{\to} 0, \quad A (\hat{r}) - A \left( \frac{q}{p} \right) \overset{p}{\to} 0.
\]
From Lemma 1 and 2 the above relationships hold.
From the law of large numbers,
\[
\left( \frac{1}{N} \sum_{i=1}^{n} z (x_i) z (x_i)^\top \left( \frac{q(x_i)}{p(x_i)} \right) \right)^{-1} \overset{p}{\to} V^{-1}.
\]
Here, \(V^{-1}\) exists because we assumed that \(0 < \frac{q(x)}{p(x)} < \infty\) for any \(x \in \mathcal{X}\) and \(E_p[x_i]z(x_i)z(x_i)^\top\) is finite and invertible. From the central limit theorem,
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{n} z (x_i) \varepsilon_i \left( \frac{q(x_i)}{p(x_i)} \right)^{\frac{1}{2}} \overset{d}{\to} \mathcal{N}(0, \sigma^2 V).
\]
From Eq. (4), the asymptotic variance is \(\sigma^2 V^{-1}\).

3.3 Statistical Test for Model Specification

Based on the above results, we show a procedure of the statistical test. First, we estimate parameters of the following system of equations. Second, we construct a Wald statistic that converges to the chi-squared distribution under the null hypothesis.
System of Equations: Let $\dot{Y}$, $Z$, $\dot{Z}$, $\Sigma$, and $\dot{\Sigma}$ be

$$\dot{Y} = (y_1 \ y_2 \ \ldots \ y_N \ y_1 \ y_2 \ \ldots \ y_N)^{\top},$$

$$Z = (z(x_1)^{\top} \ z(x_2)^{\top} \ \ldots \ z(x_N)^{\top})^{\top},$$

$$\dot{Z} = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} q(x_1) & 0 \\ p(x_1) \ldots & 0 \\ 0 & q(x_N) \\ p(x_N) \end{pmatrix},$$

$$\dot{\Sigma} = \begin{pmatrix} \Sigma & 0 \\ 0 & I_N \end{pmatrix},$$

where $I_N$ denotes the $N$-dimensional identity matrix. Let us define a vector of parameters as $\theta = \begin{pmatrix} \gamma \\ \alpha \end{pmatrix}$ and an estimator of $\theta$ as $\hat{\theta}$. We construct an estimator $\hat{\theta}$ as follows:

$$\hat{\theta} = \left( \dot{Z}^{\top} \dot{\Sigma} \dot{Z} \right)^{-1} \dot{Z}^{\top} \dot{\Sigma} \dot{Y}. \quad (8)$$

Hypothesis Test with A Wald Statistic: Let $\hat{\sigma}^2$ be an estimator of the variance of the error term. We construct an estimator as follows:

$$\hat{\sigma}^2 = \frac{(Y - Z\hat{\alpha})^{\top} (Y - Z\hat{\alpha})}{n^{\text{tr}} - k}. \quad (9)$$

Let us define a Wald statistic $W$ as

$$W = (R\hat{\theta})^{\top} \left( R\hat{\sigma}^2 (\dot{Z}^{\top} \dot{\Sigma} \dot{Z})^{-1} R \right)^{-1} (R\hat{\theta}), \quad (10)$$

where $R$ is a $k \times 2k$ matrix such that

$$R = (I_k - I_k)$$

i.e, the $j$th element of $R\hat{\theta}$ is $\hat{\gamma}_j - \hat{\alpha}_j$, where $\alpha_j$ and $\gamma_j$ are the $j$th elements of the vector $\alpha$ and $\gamma$.

Theorem 2. Under the null hypothesis, a Wald statistic $W$ converges in distribution to the chi-squared distribution with $k$ degrees of freedom, i.e., $W \overset{d}{\rightarrow} \chi^2(k)$.

We show the proof in Appendix C.

4 Discussion

In this section, we discuss three issues on the proposed method; statistical test with any weighted functions, model specification from accepted models, and distributionally robust learning.
4.1 Statistical Test with Any Weighted Functions

In our framework, the power of a test comes from the difference of distributions, i.e., the existence of covariate shift enables us to conduct a statistical test. Therefore, if the train data and the test data follow the same distribution, we cannot detect a misspecified model with the proposed method. On the other hand, we can also propose a similar statistical test without using unlabeled test data. If a model is correctly specified, for any function $h(x)$ with the domain included by that of the train data, we can obtain the true parameter $\beta$ by minimizing the following risk:

$$\mathcal{L}(b) = \mathbb{E}_{p(y,x)} \left[ (y_i - z(x_i)^\top b)^2 h(x) \right].$$

If a model is misspecified, we can reject the model by finding a distribution $h(x)$ under which $\hat{\beta} = \arg\min_{b \in \mathbb{R}^k} \mathcal{L}(b)$ is different from $\hat{\alpha}$.

However, this strategy will cause a similar problem as the multiple testing problem. If we search for $h(x)$ that can reject the null hypothesis, we increase a probability that the true null hypothesis is rejected. We might solve this problem by using existing methods of the multiple testing problem. However, in this paper, we take another strategy that avoids this problem by using unlabeled test data.

4.2 Model Selection from Accepted Models

The proposed method can be used as a method for model selection. Even if we cannot narrow down the candidates of true models to one by a statistical test, we can select a model based on the value $W$ as heuristics. As we show in the following experimental section, a model with the lowest value $W$ showed the lowest squared error in many cases. Therefore, the proposed method can be considered as a method for model selection.

4.3 Distributionally Robust Learning

Distributionally Robust Supervised Learning (DRSL) is a paradigm to tackle a distribution shift problem, which does not assume that a distribution of test data is the same as that of train data [Quionero-Candela et al., 2009, Bagnell, 2005, Ben-Tal et al., 2013, Duchi et al., 2014, Wen et al., 2014, Namkoong and Duchi, 2016, 2017, Hu et al., 2018, Subbaswamy et al., 2018]. If a model is correctly specified in the sense of our definition, the performance of the model is robust to distribution shift. Therefore, we can consider that the proposed method rejects a model that is not robust to distribution shift.

5 Experiments

In this section, we report experimental results which were conducted using synthetic data and real-world datasets.
Table 1: Results of the experiments using the synthetic datasets. We show the sample size (Sample Size), the value of the Wald statistic $W$ that converges to the chi-squared distribution (Chi2) and the averaged squared error for test data (Error). We denote the significant level of the Wald statistic using * on the right top of the value.

| Sample Size | Dataset 1A | Chi2 Error | Model 1B | Chi2 Error | Model 1C | Chi2 Error | Model 1D | Chi2 Error | Model 1E | Chi2 Error |
|-------------|------------|------------|----------|------------|----------|------------|----------|------------|----------|------------|
| 100         | 0.171      | 0.992      | 0.302    | 3.114      | 0.366    | 2.742      | 0.306    | 2.143      | 0.008    | 0.728      |
| 250         | 0.178      | 0.803      | 0.451    | 2.737      | 1.464    | 2.075      | 0.610    | 1.772      | 0.230    | 0.459      |
| 500         | 0.271      | 0.967      | 2.641    | 3.691      | 2.653    | 2.317      | 4.494    | 2.278      | 2.091    | 4.551      |
| 750         | 0.073      | 1.160      | 4.936    | 3.953      | 4.279    | 2.626      | 0.439    | 2.314      | 1.957    | 5.103      |
| 1,000       | 1.389      | 1.053      | 5.506    | 3.584      | 5.608    | 2.334      | 5.276    | 2.488      | 2.382    | 3.388      |
| 1,250       | 0.356      | 0.957      | 4.298    | 3.476      | 4.126    | 2.267      | 2.832    | 2.308      | 3.809    | 2.920      |
| 1,500       | 0.050      | 1.001      | 4.308    | 3.344      | 3.794    | 2.352      | 4.329    | 1.940      | 4.761    | 4.940      |
| 1,750       | 0.297      | 1.058      | 4.660    | 3.103      | 3.009    | 2.149      | 5.173    | 2.246      | 6.890    | 4.676      |
| 2,000       | 0.002      | 0.981      | 0.057    | 2.956      | 0.066    | 1.989      | 0.032    | 1.216      | 0.032    | 1.455      |
| 2,250       | 0.003      | 1.011      | 0.044    | 3.538      | 0.060    | 2.390      | 0.048    | 2.289      | 0.071    | 3.704      |
| 2,500       | 0.319      | 1.071      | 12.036   | 3.305      | 11.674   | 2.294      | 8.533    | 2.272      | 12.564   | 4.704      |
| 2,750       | 0.224      | 0.980      | 9.095    | 3.344      | 8.285    | 2.002      | 7.111    | 2.386      | 3.702    | 4.716      |
| 3,000       | 0.132      | 1.015      | 5.603    | 3.307      | 6.064    | 2.195      | 3.469    | 2.521      | 8.641    | 5.407      |

| Sample Size | Dataset 2A | Chi2 Error | Model 2B | Chi2 Error | Model 2C | Chi2 Error | Model 2D | Chi2 Error | Model 2E | Chi2 Error |
|-------------|------------|------------|----------|------------|----------|------------|----------|------------|----------|------------|
| 100         | 2.543      | 0.984      | 0.059    | 73.369     | 2.579    | 9.293      | 0.359    | 28.407     | 0.008    | 0.728      |
| 250         | 0.003      | 1.473      | 0.022    | 65.588     | 0.025    | 7.501      | 0.024    | 29.984     | 0.011    | 0.913      |
| 500         | 0.022      | 0.959      | 9.144    | 120.478    | 3.310    | 6.423      | 2.382    | 25.557     | 2.746    | 196.982    |
| 750         | 0.536      | 0.982      | 12.521   | 102.066    | 2.964    | 6.766      | 1.816    | 25.139     | 3.497    | 99.952     |
| 1,000       | 0.304      | 0.998      | 10.270   | 87.096     | 6.675    | 6.775      | 3.714    | 30.117     | 2.752    | 119.114    |
| 1,250       | 0.203      | 0.958      | 14.608*  | 86.294     | 9.161    | 6.888      | 5.836    | 28.346     | 2.587    | 117.378    |
| 1,500       | 0.374      | 0.994      | 13.626*  | 54.489     | 8.669    | 6.660      | 12.768*  | 32.068     | 2.267    | 92.562     |
| 1,750       | 0.003      | 1.042      | 21.917** | 79.487     | 9.036    | 7.075      | 9.125    | 39.917     | 4.792    | 167.933    |
| 2,000       | 0.191      | 1.042      | 27.064***| 105.968    | 9.305    | 6.649      | 9.393    | 29.481     | 6.974    | 124.671    |
| 2,250       | 0.316      | 1.039      | 26.074***| 95.968     | 9.305    | 6.649      | 9.393    | 29.481     | 6.974    | 124.671    |
| 2,500       | 0.351      | 1.000      | 20.433***| 83.588     | 14.224*  | 6.908      | 10.181*  | 30.141     | 7.042    | 118.696    |
| 2,750       | 0.381      | 1.022      | 25.715***| 82.055     | 8.012    | 6.064      | 4.800    | 29.555     | 12.392** | 121.747    |
| 3,000       | 0.228      | 1.027      | 34.333***| 82.572     | 17.009** | 6.418      | 12.668*  | 28.846     | 10.025** | 117.167    |

5.1 Numerical Experiment

This experiment demonstrates results of model specification test using synthetic data. We generated a set of $n$ samples $\{x_i\}_{i=1}^n$ as follows:

$$\{x_i\}_{i=1}^n \sim \text{i.i.d. } N(\mu, \Sigma),$$

where $\mu = (0 \ 0 \ 0 \ 0)^T$ and

$$\Sigma = \begin{pmatrix}
1 & -0.2 & 0.2 & -0.2 \\
-0.2 & 1 & -0.2 & 0.2 \\
0.2 & -0.2 & 1 & -0.2 \\
-0.2 & 0.2 & -0.2 & 1
\end{pmatrix}.$$
where $g(x_i) = x_{1i} + x_{2i}^2 + x_{3i}^3 + x_{4i}^4$, $x_{ji}$ is the $j$th element of $x_i$, and $C$ is a parameter for normalizing the probability to be $\sum_{i=1}^n p(x_i)$ is train data) = 0.5.
Then, we generated $y_i$ by the following two models. The first model is

$$y_i = x_{1i} + x_{2i} + x_{3i} + x_{4i} + x_{1i}x_{2i} + x_{3i}x_{4i} + \varepsilon_i,$$

and the second model is

$$y_i = 8x_{1i} - 6x_{1i}^2 + 4x_{2i} - 2x_{3i} + x_{4i} + \varepsilon_i,$$

where $\varepsilon_i$ follows the standard normal distribution. Thus, we generated two datasets. We call the first dataset as Dataset 1 and the second dataset as Dataset 2. We generated $n = 100, 250, 500, 750, 1,000, 1,250, 1,500, 1,750, 2,000, 2,250, 2,500, 2,750$, and 3,000 samples for each dataset. For Dataset 1, we used the following six models:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{1i} x_{2i} + \beta_6 x_{3i} x_{4i} + \varepsilon_i,$$

where $\beta_j j \in \mathbb{Z}^+$, which indicates a set of non-negative integers, denotes an unknown parameter. We call these models Model 1A, Model 1B, Model 1C, Model 1D, Model 1E, and Model 1F, respectively. Model 1A denotes a correctly specified model. For Dataset 2, we used the following six models:

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{1i}^2 + \beta_3 x_{2i} + \beta_4 x_{3i} + \beta_5 x_{4i} + \varepsilon_i,$$

where $\beta_j j \in \mathbb{Z}^+$ denotes an unknown parameter. We call these models Model 2A, Model 2B, Model 2C, Model 2D, Model 2E, and Model 2F, respectively. Model 2A denotes a correctly specified model. For each model, we calculated the value of the Wald statistic $W$, which converges to chi-squared distribution in distribution under the null hypothesis, and the squared loss. We show the results in
Table 2: Results of the experiments using the real-world datasets. We show the value that decides the constant $C$ in (11) (Prob), the value of the Wald statistic $W$ that converges to the chi-squared distribution (Chi2) and the averaged squared error for test data (Error). We denote the significant level of the Wald statistic using ‘*’ on the right top of the value.

| Prob | Model 3A | Model 3B | Model 3C | Model 3D |
|------|----------|----------|----------|----------|
|      | Chi2     | Error    | Chi2     | Error    | Chi2     | Error    | Chi2     | Error    |
| 0.1  | 0.151    | 0.012    | 0.085    | 0.011    | 1.063    | 0.043    | 0.978    | 0.043    |
| 0.2  | 0.071    | 0.011    | 0.024    | 0.013    | 0.387    | 0.052    | 0.354    | 0.053    |
| 0.3  | 4.815    | 0.010    | 0.512    | 0.009    | 0.638    | 0.049    | 0.558    | 0.049    |
| 0.4  | 3.909    | 0.008    | 2.267    | 0.009    | 2.073    | 0.047    | 2.394    | 0.047    |
| 0.5  | 3.245    | 0.009    | 1.909    | 0.010    | 0.900    | 0.047    | 0.810    | 0.047    |
| 0.6  | 1.428    | 0.009    | 1.279    | 0.010    | 1.269    | 0.049    | 1.148    | 0.048    |
| 0.7  | 3.772    | 0.012    | 1.197    | 0.011    | 0.941    | 0.050    | 0.939    | 0.049    |

| Prob | Model 4A | Model 4B | Model 4C | Model 4D |
|------|----------|----------|----------|----------|
|      | Chi2     | Error    | Chi2     | Error    | Chi2     | Error    | Chi2     | Error    |
| 0.1  | 880.608*** | 4.951    | 35.001*** | 0.100    | 39.862*** | 0.101    | 75.465*** | 0.103    |
| 0.2  | 483.725*** | 5.263    | 7.722    | 0.105    | 5.675    | 0.107    | 5.409    | 0.111    |
| 0.3  | 262.925*** | 5.240    | 20.358**  | 0.102    | 4.287    | 0.103    | 5.494    | 0.107    |
| 0.4  | 358.909*** | 5.269    | 17.885*   | 0.103    | 5.552    | 0.103    | 12.289**  | 0.108    |
| 0.5  | 406.640*** | 5.269    | 92.220*** | 0.101    | 57.539*** | 0.101    | 93.076*** | 0.106    |
| 0.6  | 499.195*** | 5.321    | 353.502*** | 0.104    | 286.920*** | 0.104    | 354.594*** | 0.109    |
| 0.7  | 159.397*** | 5.263    | 8.950    | 0.106    | 5.607    | 0.105    | 1.149    | 0.109    |

Table 1: If the Wald statistic $W$ is 5%, 1%, or 0.1% significant, we put *, **, or *** on the right top of the value of $W$, respectively.

In the result of the experiment using Dataset 1, we could not reject the almost all null hypotheses of misspecified models. However, in all cases, Model 1A, which is a correctly specified model, showed the lowest Wald statistic $W$. In practice, we can consider that Model 1A is a model that is most likely to be a correctly specified model as heuristics. In the result of the experiment using Dataset 2, we can confirm that the proposed method could reject misspecified models. As the sample size increases, we could detect misspecified models more accurately.

5.2 Experiment with Real-World Data

In this section, we applied the proposed method to two real-world datasets. The first dataset was created by randomly sampling 400 elementary schools from the California Department of Education’s API 2000 dataset [1] [Chen and Wells, 2003]. This data contains a measure of school academic performance as well as

1) We can download the data with Stata command “use https://stats.idre.ucla.edu/stat/stata/webbooks/reg/elemapi2.dta”
other attributes of the elementary schools, such as class size, enrollment, and poverty. The sample size of this dataset is 376. The second dataset consists of the result of a randomized experiment on class size conducted in the United States, the Tennessee Student/Teacher Achievement Ratio experiment, known as Project STAR\textsuperscript{2}\cite{Krueger1999}. Project STAR was a longitudinal study in which 11,600 kindergarten students and their teachers were randomly assigned to different size classes from kindergarten through third grade. The sample size of this dataset is 5,766. We artificially made labeled train data and unlabeled test data based on the following probability:

\[ p(x_i \text{ is train data}) = C \frac{1}{\exp(-g(x_i))}, \quad (11) \]

where \( g(x_i) = x_{i1} + x_{i2} + x_{i3} + x_{i4}, \) \( x_{ij} \) is the \( j \)th element of \( x_i \), \( C \) is a parameter for normalizing the probability to be \( \sum_{i=1}^{n} p(x_i \text{ is train data}) = \text{Prob} \), and \( \text{Prob} \) is a constant such that \( \text{Prob} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\} \). For both datasets and \( \text{Prob} \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\} \), we regress the score of the children’s classes on their features. For the first dataset, we extracted 22 features from the original dataset and used the following four models:

\[ y_i = \beta_0 + \sum_{j=1}^{20} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{6} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{2} \beta_j x_{ji} + \varepsilon_i, \]

where \( \beta_j \in \mathbb{Z}^+ \) denotes an unknown parameter. We call these models Model 3A, Model 3B, Model 3C, and Model 3D, respectively. For the second dataset, we extracted 10 features from the original dataset and used the following four models:

\[ y_i = \beta_0 + \sum_{j=1}^{20} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{6} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ji} + \varepsilon_i, \]
\[ y_i = \beta_0 + \sum_{j=1}^{2} \beta_j x_{ji} + \varepsilon_i, \]

\textsuperscript{2}We can download the data from https://economics.mit.edu/files/3827.
where $\beta_j \in \mathbb{Z}^+$ denotes an unknown parameter. We call these models Model 4A, Model 4B, Model 4C, and Model 4D. For each model, we calculated the value of $W$, which converges to chi-squared distribution in distribution under the null hypothesis, and the squared loss. We show the result in Table 2. If $W$ is 5%, 1%, or 0.1% significant, we put *, **, or *** on the right top of the value of $W$, respectively.

In the result of the experiment using the dataset of Chen and Wells [2003], we could not reject any models. We consider that it is because the sample size of the dataset is 376 and it is insufficient for the test. In the result of the experiment using the dataset of Krueger [1999], we could reject the model with the worst error successfully.

6 Conclusion

In this paper, we proposed a novel framework of model specification test in regression. Extending the conventional definition of a correctly specified model, we defined a correctly specified model more generally. By conducting a model specification test under this definition, we can reject a model that is not robust to the domain shift. Even though we cannot confirm whether a model is correctly specified or not in the case where the null hypothesis is accepted, we can regard the accepted model is robust to distribution shift between train and test data.

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A Proof of Lemma 1

Proof. For any vector $v$, denote $(v)_l$ as the $l$th element of $v$.

\[ a(\hat{r}) - a\left(\frac{q}{p}\right) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} z(x_i) \varepsilon_i \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right). \]  

(12)

For any $l = 1, ..., k$, from the Cauchy–Schwarz inequality,

\[ \left| a(\hat{r}) - a\left(\frac{q}{p}\right) \right| \leq \sqrt{\frac{1}{N} \sum_{i=1}^{N} (z(x_i))_l^2 \varepsilon_i^2 \sum_{i=1}^{N} \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2}. \]

From Markov’s inequality, for any $\epsilon > 0$,

\[ \mathbb{P}\left[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 \geq \epsilon \right] \leq \frac{\mathbb{E}_{p(y,x)}\left[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 \right]}{\epsilon}. \]

Therefore,

\[ \mathbb{P}\left[ \frac{\left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2}{\mathbb{E}_{p(y,x)}\left[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 \right]} \geq \frac{1}{\delta} \right] \leq \delta, \]

where $\delta = \mathbb{E}_{p(y,x)}\left[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 \right]$. Using the convergence rate of an estimator of the density ratio \[\text{(11)},\]

\[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 = \mathcal{O}_{p}\left(\mathbb{E}_{p(y,x)}\left[ \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 \right]\right) \]

\[ = \mathcal{O}_{p}\left(\mathcal{O}_{p}\left(N^{-\frac{\gamma}{2(1+\gamma)}}\right)\right). \]

Therefore,

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\hat{r}(x_i) - \left(\frac{q(x_i)}{p(x_i)}\right)\right)^2 = \mathcal{O}_{p}\left(\mathcal{O}_{p}\left(N^{-\frac{\gamma}{2(1+\gamma)}}\right)\right). \]
Because $0 < \gamma < 1$,

\[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \hat{r}(x_i) - \frac{q(x_i)}{p(x_i)} \right)^2 \overset{p}{\to} 0. \]

Because $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (z(x_i))^2 \epsilon_i^2$ is stochastically bounded by the central limit theorem, (12) converges to 0 in probability. \(\square\)

### B Proof of Lemma 2

**Proof.** For any matrix $M$, denote $(M)_{m,l}$ as the $(m, l)$ element of $M$.

\[ A(\hat{r}) - A \left( \frac{q}{p} \right) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} z(x_i) z(x_i)^\top \left( \hat{r}(x_i) - \left( \frac{q(x_i)}{p(x_i)} \right) \right). \]

For any $m, l = 1, \ldots, k$, from the Cauchy–Schwarz inequality,

\[ \left| \left( A(\hat{r}) - A \left( \frac{q}{p} \right) \right)_{m,l} \right| \]

\[ \leq \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( (z(x_i) z(x_i)^\top) \circ (z(x_i) z(x_i)^\top) \right)_{m,l}} \]

\[ \times \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \hat{r}(x_i) - \left( \frac{q(x_i)}{p(x_i)} \right) \right)^2}, \]

where $\circ$ is an operator of the Hadamard product.

Then, because $\hat{r}(x_i) \overset{p}{\to} \frac{q(x_i)}{p(x_i)}$ and $\frac{1}{N} \sum_{i=1}^{N} \left( (z(x_i) z(x_i)^\top) \circ (z(x_i) z(x_i)^\top) \right)_{m,l} \overset{p}{\to} E \left[ \left( (z(x_i) z(x_i)^\top) \circ (z(X) z(x_i)^\top) \right)_{m,l} \right]$,

\[ A(\hat{r}) - A \left( \frac{q}{p} \right) \overset{p}{\to} 0. \]

\(\square\)

### C Proof of Theorem 2

**Proof.** Define $\mathcal{E}, \Sigma^{1/2},$ and $\hat{\Sigma}^{1/2}$ as

\[ \mathcal{E} = (\epsilon_1 \epsilon_2 \ldots \epsilon_N \epsilon_1 \epsilon_2 \ldots \epsilon_N)^\top, \]

20
\[
\Sigma^{1/2} = \begin{pmatrix}
\sqrt{\frac{q(x_1)}{p(x_1)}} & & \\
& \ddots & & \\
& & 0 & \\
0 & & & \sqrt{\frac{q(x_N)}{p(x_N)}}
\end{pmatrix},
\]

and
\[
\tilde{\Sigma}^{1/2} = \begin{pmatrix}
\Sigma^{1/2} & 0 \\
0 & I_N
\end{pmatrix}
\]

so that we have
\[
\begin{align*}
\left(\Sigma^{1/2}\right)^T &= \Sigma^{1/2}, \\
\Sigma^{1/2}\Sigma^{1/2} &= \Sigma, \\
\left(\Sigma^{1/2}\right)^T &= \tilde{\Sigma}^{1/2}, \text{ and} \\
\tilde{\Sigma}^{1/2}\tilde{\Sigma}^{1/2} &= \tilde{\Sigma}.
\end{align*}
\]

Note that if \(\Sigma\) is invertible, so is \(\Sigma^{1/2}\).

Note also that \(\hat{\theta}\) is obtained from
\[
\hat{\theta} = \left(\tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right)^{-1} \tilde{Z}^\top \tilde{\Sigma} \hat{Y},
\]

which means the linear regression model of interest is in the following form:
\[
\tilde{Y} = \tilde{Z}\theta + \tilde{\Sigma}^{-1/2} \tilde{\epsilon}.
\]

The asymptotic distribution of \(\hat{\theta}\) is
\[
\sqrt{2N} \left(\hat{\theta} - \theta\right) = \sqrt{2N} \left(\left(\tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right)^{-1} \tilde{Z}^\top \tilde{\Sigma} \hat{Y} - \theta\right)
= \sqrt{2N} \left(\left(\tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right)^{-1} \tilde{Z}^\top \tilde{\Sigma} \left(\tilde{Z}\theta + \tilde{\Sigma}^{-1/2} \tilde{\epsilon}\right) - \theta\right)
= \sqrt{2N} \left(\left(\tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right)^{-1} \tilde{Z}^\top \tilde{\Sigma} \tilde{\Sigma}^{-1/2} \tilde{\epsilon}\right)
= \sqrt{2N} \left(\left(\tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right)^{-1} \tilde{Z}^\top \tilde{\Sigma} \tilde{\Sigma}^{-1/2} \tilde{\epsilon}\right)
= \frac{1}{2N} \tilde{Z}^\top \tilde{\Sigma} \tilde{\Sigma}^{-1/2} \tilde{\epsilon}.
\]

By the law of large numbers, \(\frac{1}{N} \tilde{Z}^\top \tilde{\Sigma} \tilde{Z}^{-1} \xrightarrow{p} \mathbb{E}\left[\frac{1}{N} \tilde{Z}^\top \tilde{\Sigma} \tilde{Z}\right] := Q\), which is invertible, because \(\frac{q(x)}{p(x)}\) is strictly positive and \(\mathbb{E}[z(x_i)z^\top(x_j)]\) is invertible.
Besides, \( \frac{1}{\sqrt{2N}} \hat{Z}^\top \Sigma^{1/2} \mathcal{E} \overset{d}{\rightarrow} \mathcal{N}(0, \sigma^2 Q) \) by the central limit theorem, since the observations are i.i.d.

Therefore, by Slutsky’s theorem,

\[
\sqrt{2N} (\hat{\theta} - \theta) \overset{d}{\rightarrow} \mathcal{N}(0, \sigma^2 Q^{-1}).
\]  

(13)

If we multiply (13) by \( R \), we have the following:

\[
R \sqrt{2N} (\hat{\theta} - \theta) \overset{d}{\rightarrow} \mathcal{N}(0, R\sigma^2 Q^{-1} R^\top).
\]

Under the null hypothesis, \( R\theta = 0 \). Therefore,

\[
R\sqrt{2N}\hat{\theta} \overset{d}{\rightarrow} \mathcal{N}(0, R\sigma^2 Q^{-1} R^\top).
\]

Therefore,

\[
(R\sqrt{2N}\hat{\theta})^\top (R\sigma^2 Q^{-1} R^\top)^{-1} (R\sqrt{2N}\hat{\theta}) \overset{d}{\rightarrow} \chi^2(k).
\]

Note that \( \left( \frac{1}{2N} \hat{Z}^\top \Sigma \hat{Z} \right)^{-1} \overset{P}{\rightarrow} Q^{-1} \) and \( \hat{\sigma}^2 \overset{P}{\rightarrow} \sigma^2 \). Therefore, by the continuous mapping theorem and Slutsky’s theorem, we have

\[
(R\hat{\theta})^\top \left( R\hat{\sigma}^2 (\hat{Z}^\top \Sigma \hat{Z})^{-1} R^\top \right)^{-1} (R\hat{\theta}) \overset{d}{\rightarrow} \chi^2(k).
\]