Feynman integral for functional Schrödinger equations

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This paper is dedicated to M. I. Vishik and his Seminar at the Moscow State University.

ABSTRACT. We consider functional Schrödinger equations associated with a wide class of Hamiltonians in all Fock representations of the bosonic canonical commutation relations, in particular the Cook-Fock, Friedrichs-Fock, and Bargmann-Fock models. An infinite-dimensional symbolic calculus allows to prove the convergence of the corresponding Hamiltonian Feynman integrals for propagators of coherent states.

DEDICATION

The roots of my appreciation of M. I. Vishik and his PDE Seminar go back to the golden 1960’s of Moscow mathematics (cf. 33).

In the Spring of 1960 A. Volpert gave a lecture at the Moscow Gelfand seminar about his index formula of elliptic boundary problems for PDE systems in two variables. His approach was based on the Muskhelishvili index formula of one-dimensional singular integral operators.

I. M. Gelfand interpreted the Volpert index formula in terms of characteristic cohomological classes and suggested to generalize it to elliptic boundary problems in higher dimensions and on compact manifolds using a homotopy of the elliptic data.

The Vishik seminar began in the Spring of 1961 with a response to the Gelfand challenge. It started with a review of still recent Calderon-Zygmund algebra of multivariable singular integral operators and their applications to the Cauchy problem for elliptic PDE’s. Soon the Calderon-Zygmund algebra was extended to the algebra of singular integro-differential operators (see 10). The elliptic homotopy of the extended algebra is much simpler because its symbols are continuous rather than polynomial in cotangent directions. Indeed, this extension was essential for the first general solution of the Gelfand index problem, given by M. Atiyah and I. Singer [1].

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As stated in [21], the Kohn-Nirenberg pseudodifferential operator algebra is a
refinement of the algebra of singular integro-differential operators.
Later M. I. Vishik initiated the infinite-dimensional pseudodifferential operator
theory (cf. [29]).
This article is a contribution in this direction.

1. Introduction

1.1. Analytical aspects of quantum field theory. We begin with E.Witten’s
remarks ([31], p.346)
Mathematically, quantum field theory involves integration, and el-
liptic operators, on infinite-dimensional spaces. Naive attempts
to formulate such notions in infinite dimensions lead to all sorts
of trouble. To get somewhere, one needs the very delicate con-
structions considered in physics, constructions that at first sight
look rather specialized to many mathematicians. For this rea-
son, together with inherent analytical difficulties that the subject
presents, rigorous understanding has tended to lag behind devel-
opment of physics.

There are three basic analytical formulations of quantum field theory [8]. They
describe either a classical evolution of quantum fields, or a quantum evolution of
classical fields.

We sketch the case of neutral bosons of positive mass μ where the classical
fields are represented by real functions φt(x) on the Euclidean space Rd.
The classical Lagrangean is a real function L(φt, ∇φt, ˙φt) of φt and its space
and time derivatives. E.g., for self-interacting bosons
\[ L = \frac{1}{2}\left[\dot{\varphi}^2_t - (\nabla \varphi_t)^2 - \mu^2 \varphi^2_t\right] - V(\varphi_t), \]
where V is a scalar potential. (If V = 0 then the boson field is free.)
The classical Hamiltonian is a real function H(φt, ∇φt, πt), where πt = ∂/∂\dot{φ}_t.
In our example
\[ H = \frac{1}{2}\left[\dot{\varphi}^2_t + (\nabla \varphi_t)^2 + \mu^2 \varphi^2_t\right] + V(\varphi_t). \]

1. Operator formulation.
The operator formulation was introduced by W. Heisenberg and W. Pauli in
1929. They defined quantum fields as unbounded selfadjoint operator-valued fields
\( \hat{\varphi}_t \) on the space Rd that satisfy (in the modernized form) the canonical commutation
relations
\[ [\hat{\varphi}_t, \pi_t] = i \int dx \varphi_t(x) \pi_t(x), \]
and the classical partial differential Euler-Lagrange equations on Rd × R associated
with the Lagrangean L
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} + \nabla \cdot \frac{\partial L}{\partial \nabla \phi} - \frac{\partial L}{\partial \phi} = 0. \]
In our example
\[ \partial_t^2 \hat{\varphi}_t - \nabla^2 \hat{\varphi}_t + \mu^2 \hat{\varphi}_t + V'(\hat{\varphi}_t) = 0. \]
If \emph{non-linear}, the Euler-Lagrange equations are ill suited for operator valued solutions. Yet they can be solved explicitly when $V = 0$ (i.e. for free fields). For non-zero $V$ physicists approximate solutions using the perturbation series expansion or lattice discretization. The numerical results are in amazing agreement with experiments. However both approximations diverge in the limit. It is suggestive that relativistic theory is impossible if $V \neq 0$ and $d > 3$ (cf. [5]).

2. Functional differential formulation.

The functional differential formulation was introduced by S. Tomonaga in 1946 and by J. Schwinger in 1951. We present it in a non-relativistic form.

The quantum states $\Psi_t(\phi)$ are time dependent functionals of classical fields satisfying the \emph{linear} functional differential Schrödinger equation (in appropriate units)

\begin{equation}
\frac{\partial}{\partial t} \Psi_t + iH(\hat{\phi}, \hat{\pi}) \Psi_t = 0.
\end{equation}

For comparison, in quantum mechanics the quantum states are functions $\phi_t(q)$ of the position vectors $q \in \mathbb{R}^d$. The conjugate momentum vectors are $p \in \mathbb{R}^d$, so that the Schrödinger equation is

\begin{equation}
\frac{\partial}{\partial t} \psi_t(q) + iH(\hat{q}, \hat{p}) \psi_t(q) = 0,
\end{equation}

where $\hat{q}$ and $\hat{p}$ satisfy the canonical commutation relations $[\hat{q}, \hat{p}] = i(q \cdot p)$. Because canonically conjugate $\hat{q}$ and $\hat{p}$ do not commute, the meaning of the Hamiltonian operator $H(\hat{\phi}, \hat{\pi})$ is ambiguous.

In 1927 H. Weyl proposed a definition of $H(\hat{q}, \hat{p})$ based on the group-theoretical approach to canonical commutation relations (cf. [30]). That was the first appearance of a pseudodifferential operator. A formal calculus of Weyl operators was developed by J. Moyal [24]. In the 60's mathematical physicists found many other heuristic symbolic calculi of Weyl operators, the most general by C. Agarwal and E. Wolf [2]. Rigorous mathematics was established by F. Berezin [7] for Wick and anti Wick symbols, by L. Hörmander [16] for the Weyl-Moyal calculus, and in [11] for the Agarwal-Wolf calculus. (See also Chapter 4 of [26].)

Various versions of \emph{infinite-dimensional pseudodifferential operators} have been introduced by M. Vishik (cf. [29]), O. Smolyanov and A. Khrennikov [27] (cf. also [18]), and in a greater generality by P. Krée (cf. [22]). However their symbolic calculus is not sufficient for a full fledged theory of ellipticity, with the notable exception of B.Lascar work [23]. The latter is an infinite-dimensional version of Berezin theory [7]. Significantly, the Berezin-Lascar definition is not based on Fourier transform.

3. Functional integral formulation.

The functional integral formulation of quantum mechanics was introduced by R. Feynman in 1942 and of quantum field theory in 1950.

Formally independent of the differential Schrödinger equation the formulation starts with the solution $\Psi_t$ of (1.6) in the functional integral form:

\begin{equation}
\Phi_t(\phi_t) = \int d\phi_0 \langle \phi_t | \phi_0 \rangle \Phi_0(\phi_0),
\end{equation}

where the integral kernel $\langle \phi_t | \phi_0 \rangle$ is considered to be the \emph{quantum propagator of classical fields} from $\phi_0$ to $\phi_t$. 
In quantum mechanics the integral is finite-dimensional:

\[
\psi_t(q_t) = \int dq_0 \langle q_t | q_0 \rangle \psi_0(q_0),
\]

so that \( \langle q_t | q_0 \rangle \) is a Green function for the Cauchy problem for the Schrödinger equation. Actually the Green function is an oscillatory distribution. Therefore the integral (1.9) has to be understood in a distributional sense.

Starting with a Dirac idea, Feynman came up with the remarkable representation of the quantum propagator \( \langle \phi_t | \phi_0 \rangle \) as a “sum over classical histories” \( \phi_\tau, 0 \leq \tau \leq t \), on the space \( \mathbb{R}^d \):

\[
\langle \phi_t | \phi_0 \rangle = \sum_{\phi_0} \exp i \int_0^t d\tau \int_{\mathbb{R}^d} dx L(\phi_\tau, \nabla \phi_\tau, \dot{\phi}_\tau).
\]

Due to “interference” of the oscillating exponentials, the relevant histories are in the vicinity of the classical one, i.e., the solution of the Euler-Lagrange equation. So, in principle, this “sum” would be semi-classical.

In the first publication \[12\] Feynman gave only a quantum mechanical version of such “sum” based on a heuristic physical picture of multiple scattering of free particles on the potential. His derivation was based on a semi-classical postulate for short time quantum propagators.

A shorthand notation for the “sum” is the Lagrangean Feynman integral

\[
\int_{\phi_0}^{\phi_t} D[\phi] e^{i \int_0^t d\tau \int_{\mathbb{R}^d} dx L(\phi_\tau, \nabla \phi_\tau, \dot{\phi}_\tau)}.
\]

This compact notation suggests iterated integration, integration by parts, substitution rule, WKB approximation, Gaussian integrals, etc.

Most of mathematical pursuit of a rigorous Lagrangean Feynman integral is within the frame of the quantum mechanical Schrödinger equation (cf. \[17\]), usually for Hamiltonians of the form “Minus Laplacian plus scalar potential”.

Another form of the Feynman integral is the Hamiltonian Feynman integral

\[
\int_{\phi_0}^{\phi_t} D[\phi_\tau, \pi_\tau] e^{i \int_0^t d\tau \int_{\mathbb{R}^d} dx \{[\pi_\tau, \dot{\phi}_\tau] - H(\phi_\tau, \pi_\tau)\}}.
\]

The Hamiltonian version for quantum field theory was proposed by R. Feynman \[13\] in 1951. W. Tobocman \[28\] derived in 1956 its quantum mechanical version modifying Feynman’s heuristic semi-classical postulate.

In 1960 J. Klauder \[19\] introduced the formal Hamiltonian Feynman integral over the coherent state histories for the propagator of the coherent states, in the neat agreement with Feynman equation (1.10) since the quantum coherent states are labeled by the classical fields.

In 1984 J. Klauder and I. Daubechies \[20\] gave a rigorous definition of the Klauder-Feynman integral in quantum mechanics with the Hamiltonian function \( H \) replaced by the antinormal (i.e., antiwick) symbol of the Hamiltonian operator.

Their definition, unlike the original sequential Feynman’s construction, is a limit of Wiener integrals over the phase space. It is applicable, e.g., to selfadjoint differential operators. (However the non-trivial selfadjointness property is only postulated.)
A sequential Hamiltonian quantum-mechanical Feynman type integral is proposed in [11]. There the convergence proof is based on the pseudodifferential calculus.

A rigorous construction of Lagrangean Feynman type integral for self-interacting boson fields can be found in [15], mostly for $d < 3$.

1.2. Content of the paper. Section 2 sketches an infinite-dimensional pseudodifferential symbolic calculus and an associated ellipticity theory for functional quasi differential operators. (The complete details are in a forthcoming paper.)

The exposition is based on B. Lascar’s infinite-dimensional extension [23] to functional differential operators of Berezin’s theory [7] of antiwick symbols on $\mathbb{R}^n$.

Our theory is for quasi differential operators in all Fock representations of the bosonic canonical commutation relations, in particular in the Cook-Fock, Friedrichs-Fock, and Bargmann-Fock models. Appropriately we do not use the infinite-dimensional Fourier transform.

At the end of the section we give rather general sufficient conditions for essential selfadjointness of functional quasi differential operators.

The theorem 3.1 of Section 3 presents a rigorous Hamiltonian Klauder-Feynman sequential integral for the quantum propagators of classical fields generated by elliptic functional quasi differential operators. It is based on a suitable version of the Feynman-Tobocman semi-classical postulate. Being a limit of multiple functional integrals over the phase space, the integral satisfies standard rules of the elementary integral calculus including the substitution rule for linear canonical transformations. The convergence proof is based on the antiwick symbolic calculus of Section 2.

1.3. Notational convention. Throughout the paper the expression $A \prec B$ for variable $A$ and $B$ means that there exists a constant $C > 0$ such that $|A| \leq C|B|$.

2. Quasi differential operators over Hilbert spaces

2.1. Hilbert spaces with conjugation. Let $\mathcal{H}$ be a complex separable Hilbert space with a conjugation $\psi \mapsto \psi^*$, an antilinear norm-preserving involution on $\mathcal{H}$.

As in [6], $\psi_1 \psi_2$ denotes the Hermitean inner product of $\psi_1$ and $\psi_2$.

The real part of $\mathcal{H}$ is the real Hilbert subspace $\mathcal{R}\mathcal{H} = \{ \phi \in \mathcal{H} : \phi^* = \phi \}$ with the inner product $\phi_1 \phi_2$. In quantum theory $\mathcal{R}\mathcal{H}$ represents the configuration space.

As a real Hilbert space, $\mathcal{H}$ is a symplectic vector space with the symplectic form $\Im(\alpha \beta^*)$, the imaginary part of the Hermitean product. As such, $\mathcal{H}$ is the phase space of quantum field theory.

The antidual $\mathcal{H}^*$ is the Hilbert space of continuous antilinear functionals on $\mathcal{H}$. By Riesz theorem, they may be represented as $\psi^*$.

The Hilbert space $\mathcal{H}^* \times \mathcal{H}$ carries the conjugation $(\alpha^*, \beta) = (\beta^*, \alpha)$. The corresponding real part $\mathcal{R}$ is the antidiagonal $\{(\psi^*, \psi) : \psi \in \mathcal{H}\}$. The isometry $\psi \mapsto (1/\sqrt{2})(\psi^*, \psi)$ is a representation of $\mathcal{H}$ as a real Hilbert space.

2.2. Fock quantization. A Fock quantization is a pair of linear representations of $\mathcal{H}$ by closable operators $q^+(\psi)$ and $q^-(\psi)$ on a dense subspace $\mathcal{F}_0$ of another Hilbert space $\mathcal{F}$ such that:

(1) On $\mathcal{F}_0$ the Hermitean adjoint of $q^-(\psi^*)$ is $q^+(\psi)$.
(2) The commutators satisfy the Fock commutation relations
\[
[q^{-}(\alpha^{*}), q^{+}(\beta)] = \alpha^{*}\beta, \\
[q^{+}(\alpha), q^{+}(\beta)] = 0 = [q^{-}(\alpha), q^{-}(\beta)].
\] (2.1)

(3) There is a distinguished unit element \(\Omega_{0} \in \mathcal{F}_{0}\) such that \(q^{-}(\psi)\Omega_{0} = 0\) for all \(\psi\).

(4) The subspace \(\mathcal{F}_{0}\) is the linear span of \(q^{+}(\psi)^{n}\Omega_{0}, \psi \in \mathcal{H}, \ n \in \mathbb{N}\).

Note that \(\mathcal{F}_{0}\) is invariant for both \(q^{+}(\psi)\) and \(q^{-}(\psi)\).

The quadruple \((\mathcal{F}, \Omega_{0}, q^{+}, q^{-})\) is defined by these axioms uniquely up to unitary equivalence.

The Hilbert space \(\mathcal{F}\) is called a Fock space over \(\mathcal{H}\). Its elements \(\Psi\) are quantum states. The Hermitean inner product of \(\Xi\) and \(\Upsilon\) in \(\mathcal{F}\) is denoted \(<\Xi|\Upsilon>\) with Dirac assumption that it is antilinear on the left and linear on right. The state \(\Omega_{0}\) is called the vacuum state. The operators \(q^{+}(\psi)\) are creation operators and \(q^{-}(\psi)\) are annihilation operators.

2.3. Functional integral. An increasing sequence of \(n\)-dimensional subspaces \(\mathcal{H}^{(n)}\) in \(\mathcal{H}\) is said to be complete if their union is dense in \(\mathcal{H}\).

Let \(\Psi = \Psi(\phi)\) be a (non-linear) functional on \(\mathbb{R}\mathcal{H}\), and \(\Psi^{(n)} = \Psi^{(n)}(\phi^{(n)})\) be the restrictions of \(\Psi\) on \(\mathbb{R}\mathcal{H}^{(n)}\).

The functional integral of \(\Psi\) over \(\mathbb{R}\mathcal{H}\) is defined (cf. (2.2)) as the limit of the Lebesgue integrals over \(\mathbb{R}\mathcal{H}^{(n)}\)

\[
\int d\phi \Psi(\phi) = \lim_{n \to \infty} \frac{1}{(\pi)^{n/2}} \int d\phi^{(n)} \Psi^{(n)}(\phi^{(n)}),
\]

if the limit is the same for all complete sequences of \(n\)-dimensional subspaces in \(\mathcal{H}\).

The finite-dimensional renormalizations are chosen so that the Gaussian functional integral

\[
\int d\phi \ e^{-\phi\phi} = 1.
\]

The functional integral is a positive linear functional on the space of integrable functionals. The integral over a product Hilbert space is equal to the iterated integrals. The integral is invariant under translations by all elements of \(\mathbb{R}\mathcal{H}\).

The integration by parts involves the derivatives in the directions of \(\alpha \in \mathbb{R}\mathcal{H}\)

\[
\int d\phi \ \Xi \frac{\partial\Upsilon}{\partial\alpha} = - \int d\phi \ \Upsilon \frac{\partial\Xi}{\partial\alpha}
\]

if \(\Xi \Upsilon \to 0\) as \(\phi \to \infty\) and both integrals exist.

For functionals \(\Psi = \Psi(\psi^{*}, \psi)\) on \(\mathbb{R}\) the functional integral is the limit of the integrals over \(\mathbb{R}^{(n)} = \mathbb{R}(\mathcal{H}^{(n)} \times \mathcal{H}^{(n)})\).

\[
\int d\psi^{*}d\psi \Psi(\psi^{*}, \psi) = \lim_{n \to \infty} \frac{1}{(2\pi)^{n}} \int d\psi^{*(n)}d\psi^{(n)} \Psi^{(n)}(\psi^{*(n)}, \psi^{(n)})
\]
2.4. Examples of Fock quantizations.

2.4.1. Cook tensor quantization. The Cook space $F_0$ is the space of symmetric tensors on $\mathcal{H}$ (cf. [9] and [23], Section 2.4). The Fock space is its Hilbert space completion. The creation operators $q^+(\psi)$ are symmetrized exterior products with $\psi$ on the right. The vacuum state is $\Omega_0 = 1$. The annihilation operators $q^-(\psi)$ are symmetrized contractions with $\psi$ on the left.

2.4.2. Friedrichs functional quantization. The Friedrichs Fock space is the Gaussian Hilbert space $L^2(\mathbb{R}\mathcal{H}, e^{-\phi\phi} d\phi)$ (cf. [15]). The space $F_0$ is the space of continuous polynomials on $\mathbb{R}\mathcal{H}$. The vacuum state is $\Omega_0 = e^{-\phi\phi/2}$. For $\phi \in \mathbb{R}\mathcal{H}$, the annihilation and creation operators are the functional differential operators $q^-(\phi) = \partial/\partial\phi - \phi$ and $q^+(\phi) = \partial/\partial\phi + \phi$. To all $\psi \in \mathcal{H}$ they are extended by the complex linearity.

2.4.3. Bargmann functional quantization. The Bargmann space $F_0$ is the space of continuous analytic polynomials $\Pi = \Pi(\psi^*)$ on $\mathcal{H}^*$. The annihilation operators $q^-(\psi)$ are the directional derivatives $\partial/\partial\psi^*$. The creation operators $q^+(\psi)$ are multiplications with the linear forms $\psi$ on $\mathcal{H}^*$. The vacuum state is $\Omega_0 = 1$.

The Bargmann Fock space $\mathcal{B}$ is the closed subspace of continuous antiholomorphic functionals $\Psi(\psi^*)$ in $L^2(\mathcal{H}^* \times \mathcal{H}, e^{-\psi^*\psi} d\psi^* d\psi)$ (cf. [4] and [6]).

2.4.4. Geometric quantizations. There is a functional Fock quantization associated with every Lagrangean subspace of the symplectic space $\mathcal{H}^* \times \mathcal{H}$ (cf. [32]). E.g., the Friedrichs quantization is associated with $\mathcal{R}$ and the Bargmann quantization with $\mathcal{H}^* \times \{0\}$.

2.5. Hermite polynomials and coherent states. The quantum states

$$(2.6) \quad \prod_{j=1}^n q^+(\alpha_j)\Omega_0 \in \mathcal{F}_0, \alpha_1, \ldots, \alpha_n \in \mathcal{H},$$

are called Hermite polynomials of order $n$.

E.g., in the Bargmann space they are $\prod_{j=1}^n (\psi^*\alpha_j)$.

By polarization, the Hermite polynomials are linear combinations of the Hermite monomials $q^+(\alpha)^n\Omega_0$.

We have (cf. (6.3.12) of [13])

$$(2.7) \quad \langle q^+(\alpha)^n\Omega_0 | q^+(\beta)^n\Omega_0 \rangle = \delta_{mn} n!(\alpha^*\beta)^n.$$  

This shows that the correspondence between $\alpha$ and $q^+(\alpha)^n\Omega_0$ is one to one. It follows also that if $\{\epsilon_j\} \subset \mathcal{H}_+$ is an orthonormal basis in $\mathcal{H}$, then the Hermite monomials $q^+(\epsilon_j)^n\Omega_0$ form an orthogonal basis in $\mathcal{F}$ with $|q^+(\epsilon_j)^n\Omega_0| = \sqrt{n!}$.

The quantum states

$$(2.8) \quad \Omega_\alpha = \sum_{n=1}^{\infty} \frac{1}{n!} q^+(\alpha)^n\Omega_0, \alpha \in \mathcal{H},$$

are called (non-normalized) coherent states.

In the Bargmann space, $\Omega_\alpha(\psi^*) = e^{\psi^*\alpha}$.

In view of (2.7), we have

$$(2.9) \quad \langle \Omega_\alpha | \Omega_\beta \rangle = e^{\alpha^*\beta}.$$  

This equality implies that $\Omega_\alpha$ belong to $\mathcal{F}$ and that the correspondence between $\alpha$ and $\Omega_\alpha$ is one to one.
We have, by (2.7),

\begin{equation}
q^+(\beta)\Omega_\alpha(\psi^*) = (\beta\psi^*)\Omega_\alpha(\psi^*), \quad q^-(\beta^*)\Omega_\alpha(\psi^*) = (\beta^*\alpha)\Omega_\alpha(\psi^*).
\end{equation}

The coherent states form an overcomplete orthogonal basis, i.e., if \( \Xi, \Upsilon \in F \) then we have the analogue of the Plancherel equality

\begin{equation}
\langle \Xi | \Upsilon \rangle = \int d\psi^* d\psi e^{-\psi^* \psi} \langle \Xi | \Omega_\psi \rangle \langle \Omega_\psi | \Upsilon \rangle.
\end{equation}

To verify the equation it suffices to check it on the total set of Hermite monomials. For them the equation holds because of (2.7).

In other terms, every \( \Psi \in F \) has an analogue of the Fourier expansion

\begin{equation}
\langle \Omega_\alpha | \Psi \rangle = \int d\psi^* d\psi e^{-\psi^* \psi} e^{\alpha^* \psi} \langle \Omega_\psi | \Psi \rangle.
\end{equation}

### 2.6. Sobolev Fock spaces \( F^s \)

Suppose \( h \) is a real non-negative selfadjoint operator in \( H \) such that \((1 + h)^{-1}\) is a Hilbert-Schmidt operator. Let

\[ \|\psi\|^2_+ = \psi^* (1 + h) \psi, \quad \|\psi\|^2_- = \psi^* (1 + h)^{-1} \psi. \]

Then we have the Hilbert space \( \mathcal{H}_+ \subset \mathcal{H} \) of all \( \psi \) with finite norm \( \|\psi\|_+ \) and the Hilbert space \( \mathcal{H}_- \), that is the completion of \( \mathcal{H} \) with respect to the norm \( \|\psi\|_- \).

These spaces inherit the conjugation from \( \mathcal{H}_- \). We got the nested triple

\[ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-, \]

where both inclusions are Hilbert-Schmidt operators with dense ranges.

The Sobolev space \( B^s \), \( s \in \mathbb{R} \), is the Hilbert space of continuous holomorphic functionals \( \Psi(\psi^*) = \langle \Omega_\psi | \Psi \rangle \) on \( \mathcal{H}^* \) with finite norm \( \|\Psi\|_s \) defined by (cf. [23], Definition 2.2)

\begin{equation}
\|\Psi\|_s^2 = \int d\psi^* d\psi e^{-\psi^* \psi} (1 + |\psi|_-)^s |\Psi(\psi^*)|^2.
\end{equation}

Every \( B^s \) contains all continuous holomorphic polynomials on \( \mathcal{H}^* \).

The topological intersection \( B^\infty = \cap_s B^s \) is a Frechet space. Its topological dual is \( B^{-\infty} = \cup_s B^s \).

By definition, the Hilbert space \( A^s \), \( -\infty < s < \infty \), consists of the functionals \( \Psi(\psi^*, \psi) \) on \( \mathcal{R} \) with finite norm defined as in (2.13).

**Remark 2.1** (Proposition 2.4 of [23]). The operator

\begin{equation}
\Psi(\psi^*, \psi) \mapsto \int d\xi^* d\xi e^{-\xi^* \xi} e^{\psi^* \xi} \Psi(\xi^*, \xi)
\end{equation}

is the orthogonal projector of \( A^s \) onto \( B^s \).

Since the Fock quantization is unique up to unitary equivalence, we have counterparts \( F^s \) of \( B^s \) in any representation of Fock commutation relations.

### 2.7. Functional quasi differential operators

Consider an antwivck monomial operator

\begin{equation}
Q_{\{\alpha_j\}}^{k,l} = \prod_{j=1}^k q^-(\alpha^*_j) \prod_{j=k+1}^{k+l} q^+(\alpha_j),
\end{equation}

where \( \alpha_j \in \mathcal{H}_+, \quad j = 1, 2, \ldots, k + l \). Note that this is a functional differential operator in any functional Fock space (2.4).
In the Bargmann space $\mathcal{B}$ we have, by (2.4.3) and (2.12),

$$
Q_{\{\alpha_j\}}^{k,l} \Psi(\psi^*) = \prod_{j=1}^{k} \partial_{\alpha_j} \left[ \prod_{j=k+1}^{k+l} (\psi^* \alpha_j) \Psi(\psi^*) \right]
$$

(2.16)

$$
\int d\xi^* d\xi \ e^{-\xi^* \xi} e^{\phi^* \xi} P_{\{\alpha_j\}}^{k,l}(\xi^*, \xi) \Psi(\xi^*),
$$

where

$$
P_{\{\alpha_j\}}^{k,l}(\xi^*, \xi) = \prod_{j=1}^{k} \alpha_j^* \xi \prod_{j=k+1}^{k+l} \xi^* \alpha_j, \ \alpha_j \in \mathcal{H}_+,
$$

(2.17)

is the Hermite polynomial on $\mathcal{R}$ that is continuous in the norm $\| \cdot \|_\infty$. In view of (2.14), the monomial operator $Q_{\{\alpha_j\}}^{k,l}$ is bounded from $\mathcal{F}^s$ to $\mathcal{F}^{s-k-l}$ for all $s$.

The same holds for the general antwick operator $Q(A)$ of order $\rho \in \mathbb{R}$

$$
Q(A) \Psi(\psi^*) = \int d\xi^* d\xi \ e^{-\xi^* \xi} e^{\phi^* \xi} A(\xi^*, \xi) \Psi(\xi^*),
$$

(2.18)

where $A(\xi^*, \xi) < 1 + \| \xi \|_\infty \rho$ and is continuous in the norm $\| \cdot \|_\infty$.

Let $\{e_j\} \subset \mathcal{H}_+$ be an orthonormal basis in $\mathcal{H}$. Then we have the Hermite orthogonal expansion of $A$ with constant coefficients $c_{\{j_1, \ldots, j_{k+l}\}}^{k,l}$$

$$
A = \sum_{\{j_1, \ldots, j_{k+l}\}} c_{\{j_1, \ldots, j_{k+l}\}}^{k,l} P_{\{\alpha_j\}}^{k,l}(e_{j_1}, \ldots, e_{j_{k+l}}), \sum (k + l)! \|c_{\{j_1, \ldots, j_{k+l}\}}^{k,l}\| < \infty.
$$

(2.19)

Now (2.18) and (2.19) imply the expansion of the antwick operator $Q(A)$ into antwick monomial operators:

$$
Q(A) = \sum c_{\{j_1, \ldots, j_{k+l}\}}^{k,l} Q(P_{\{\alpha_j\}}^{k,l}(e_{j_1}, \ldots, e_{j_{k+l}})), \sum (k + l)! \|c_{\{j_1, \ldots, j_{k+l}\}}^{k,l}\| < \infty.
$$

(2.20)

As in [9], the correspondence between $A$ and $Q(A)$ is one to one. Accordingly, $A$ is called the antwick symbol of $Q(A)$. In view of (2.14) the operator norm of $Q(A)$ in $\mathcal{F}$

$$
\|Q(A)\| \leq \sup |A(\xi^*, \xi)|
$$

(2.21)

and, if $A(\xi^*, \xi)$ is bounded from below by a constant $c$, then

$$
\langle \Psi | Q(A) | \Psi \rangle \geq c \langle \Psi | \Psi \rangle.
$$

(2.22)

By the definition (2.13) and Remark 2.1, it follows that $Q(A)$ of order $\rho$ is a bounded operator from $\mathcal{F}^s$ to $\mathcal{F}^{s-\rho}$ with the operator norm

$$
\|Q(A)\|_{s,s-\rho} < \sup |A(\psi^*, \psi)|/(1 + \|\psi\|_s)^\rho.
$$

(2.23)

In consequence, $Q(A)$ are continuous operators in $\mathcal{F}^\infty$ and $\mathcal{F}^{-\infty}$.

In view of (2.9) and (2.11), the coherent states matrix element of $Q(A)$ is

$$
\langle \Omega_\alpha | Q(A) | \Omega_\beta \rangle = \int d\xi^* d\xi \ e^{-\xi^* \xi} A(\xi^*, \xi) e^{\alpha^* \xi + \xi^* \beta}.
$$

(2.24)

This is an entire function in $(\alpha^*, \beta)$ on $\mathcal{H}_+ \times \mathcal{H}$. As such it is uniquely defined by its restriction $\langle \Omega_\psi | Q(A) | \Omega_\psi \rangle$ to the antidiagonal $\mathcal{R}$.

The wick (i.e., normal) symbol, $A_{\{\psi\}}^{\{\psi\}}(\psi^*, \psi)$ of $Q(A)$, is defined by the coherent matrix element (cf. [9]) as

$$
A_{\{\psi\}}^{\{\psi\}}(\psi^*, \psi) = \langle \Omega_\psi | Q(A) | \Omega_\psi \rangle e^{-\psi^* \psi}.
$$

(2.25)
Actually the right side of (2.25) defines the wick symbol $A(Q)\omega_1(\psi^*, \psi)$ for any continuous operator $Q$ from $\mathcal{F}^\infty$ to $\mathcal{F}^{-\infty}$ even when $Q$ is not an antiwick operator.

An antiwick operator $Q(A)$ is called quasi differential of order $\rho \in \mathbb{R}$ if its antiwick symbol $A(\psi^*, \psi)$ is a quasi polynomial of order $\rho$, i.e., $A(\psi^*, \psi)$ is $C^\infty$ Frechet differentiable, the $j$-th differentials $D^j A(\psi^*, \psi; d\psi^*, d\psi)$ are continuous polynomials in $(d\psi^*, d\psi)$ with respect to the weaker norm $\|d\psi\|_-$, and for every $j$ the norm

$$\|D^j A(\psi^*, \psi)\|_+ = \sup_{0 \neq d\psi \in \mathcal{H}_-} |D^j A(\psi^*, \psi; d\psi^*, d\psi)|/[\|d\psi\|^2]$$

satisfies

$$\|D^j A(\psi^*, \psi)\|_+ \sim (1 + \|\psi\|_-)^{\rho-j}.$$  

In particular, $A(\psi^*, \psi)$ is continuous with respect to the norm $\| \cdot \|_-$ and

$$A(\psi^*, \psi) \sim (1 + \|\psi\|_-)^\rho.$$  

The linear spaces of the quasi polynomials of the order $\rho$ is denoted as $Q^\rho$, their intersection as $Q^{-\infty}$, and their union as $Q^\infty$.

### 2.8. Agarwal-Wolf symbols.

Quasi differential operators $Q(A)$ over $\mathbb{R}^n$ are pseudodifferential operators. However, as in \[7\] and \[23\], they are defined without the Fourier transform. Yet starting from the antiwick symbol one may define all its Agarwal-Wolf pseudodifferential symbols including the most common.

For the starter, consider a formal power series on $\mathcal{H}^* \times \mathcal{H}$

$$\omega(\xi^*, \xi) = 1 + \sum_{n=1}^{\infty} \omega_n(\xi^*, \xi),$$

where $\omega_n$ are continuous polynomials of degree $n$ on $\mathcal{H}^*_+ \times \mathcal{H}_-$. The formal $\omega$-symbol of the operator $Q(A)$ is defined by

$$A^\omega(\psi^*, \psi) = \omega(\partial_{\psi^*}, -\partial_{\psi})A(\psi^*, \psi) = 1 + \sum_{n=1}^{\infty} \omega_n(\partial_{\psi^*}, -\partial_{\psi})A(\psi^*, \psi).$$

Applying the classical Borel-Hörmander construction (cf. in \[24\], Proposition 3.5) to $A \in Q^\rho$ we get a quasi polynomial $A^{[1]} \in Q^\rho$ such that for $N = 1, 2, \ldots$

$$A^{[1]} = -\sum_{n<N} \omega_n(\partial_{\psi^*}, -\partial_{\psi})A(\psi^*, \psi) \in Q^{\rho-N}.$$  

The differences between such $A^{[1]}$ belong to $Q^\infty$.

In particular, the formal antiwick symbol $A^1$ corresponds to $\omega = 1$. Among other formal symbols we have (cf. \[2\])

1. Wick (or normal) symbol $A^\rho$ with $\omega(\psi^*, \psi) = e^{-\psi^* \psi}$.
2. Weyl symbol $A^\gamma$ with $\omega(\psi^*, \psi) = e^{-\frac{1}{2} \psi^* \psi}$.
3. Left symbol $A^\lambda$ with $\omega(\psi^*, \psi) = e^{\frac{1}{2} (\psi^* \psi - \psi^* \psi)} e^{-\frac{1}{2} \psi^* \psi}$.
4. Right symbol $A^\rho$ with $\omega(\psi^*, \psi) = e^{\frac{1}{2} (\psi^* \psi - \psi^* \psi)} e^{-\frac{1}{2} \psi^* \psi}$.
5. $\sigma$-symbol $A^\sigma$ with $\omega(\psi^*, \psi) = e^{\sigma \psi^* \psi} e^{-\frac{1}{2} \psi^* \psi}$, $\sigma \in \mathbb{C}$.
6. $\tau$-symbol $A^\tau$ with $\omega(\psi^*, \psi) = e^{\tau (\psi^* \psi - \psi^* \psi)} e^{-\frac{1}{2} \psi^* \psi}$, $\tau \in \mathbb{C}$.

In mathematical literature, where $\mathcal{H} = \mathbb{R}^n$, the left and right symbols are called Kohn-Nirenberg symbols.

For every $\omega$ the symbols $A^{[\omega]}$ satisfy the inequality (2.27).
2.9. Symbolic calculus. Since the quasi differential operators act on $F_\infty$ (and $F_{-\infty}$), the product $Q(B)Q(C)$ of two such operators with $B \in \mathcal{Q}^\rho$ and $C \in \mathcal{Q}^\tau$ is well defined. In fact, the product is a quasi differential operator $Q(A)$ with the formal antiwick symbol $A^1$ (cf. Proposition 3.6 of [23] for differential operators)

\begin{equation}
A^1(\psi^*, \psi) = \sum_{n=0}^{\infty} (-1)^n (n!)^{-1} \partial^n B(\psi^*, \psi) \cdot \partial^* C(\psi^*, \psi),
\end{equation}

where the “dot” product denotes the contraction of the polynomial analytic and antianalytic differentials:

\begin{equation}
\int d\xi d\xi e^{-\xi^* \xi} \partial^n B(\psi^*, \psi)(\xi^n) \partial^* C(\psi^*, \psi)(\xi^n).
\end{equation}

Moreover, for every antiwick symbol $A^{[1]}(\psi^*, \psi)$,

\begin{equation}
A^{[1]}(\psi^*, \psi) = \sum_{n=0}^{N} (-1)^n (n!)^{-1} \partial^n B(\psi^*, \psi) \cdot \partial^* C(\psi^*, \psi) \in \mathcal{Q}^{\rho+\tau-N}.
\end{equation}

**Remark 2.2.** Using the relations between Agarwal-Wolf symbols and the antiwick symbols it is possible to derive the asymptotic expansions of the Agarwal-Wolf symbols for the products of functional pseudodifferential operators.

**Remark 2.3.** An operator $Q(A)$ is symmetric if and only if $A$ is real.

Using the Plancherel identity (2.11) one gets the coherent state matrix element of the product $Q(B)Q(C)$

\begin{equation}
\langle \Omega_\alpha | Q(B)Q(C) | \Omega_\beta \rangle = \int d\psi^* d\psi e^{-\psi^* \psi} \langle \Omega_\alpha | Q(B) | \Omega_\psi \rangle \langle \Omega_\psi | Q(C) | \Omega_\alpha \rangle
\end{equation}

\begin{align*}
&= \int d\psi^* d\psi e^{-\psi^* \psi} \\
&\times \int d\xi_2 d\xi_2 e^{-\xi_2^* \xi_2} B(\xi_2^*, \xi_2) \int d\xi_1^* d\xi_1 e^{-\xi_1^* \xi_1} e^{\xi_1^* \beta} C(\xi_1^*, \xi_1) \\
&= \int d\xi_2 d\xi_2 d\xi_1^* d\xi_1 e^{-\xi_2^* \xi_2} e^{-\xi_1^* \xi_1} e^{\alpha^* \xi_2} e^{\xi_1^* \beta} B(\xi_2^*, \xi_2) C(\xi_1^*, \xi_1),
\end{align*}

so that the coherent state kernel of the product $Q(B)Q(C)$

\begin{equation}
\langle \Omega_\alpha | Q(B)Q(C) | \Omega_\beta \rangle e^{-\beta^* \beta}
\end{equation}

\begin{align*}
&= \int d\xi_2 d\xi_2 d\xi_1^* d\xi_1 B(\xi_2^*, \xi_2) C(\xi_1^*, \xi_1) e^{(\alpha^* - \xi_2^*) \xi_2 + (\xi_1^* - \xi_2) \xi_1 + (\xi_1^* - \beta^*) \beta}.
\end{align*}

2.10. Families of quasi differential operators. Consider a parametrized family $Q(A_\epsilon)$, $0 < \epsilon \leq \epsilon_0$, of quasi differential operators in $\mathcal{Q}^\rho$ such that for some real number $\mu$

\begin{equation}
\| D^k A_\epsilon(\psi^*, \psi) \|_+ \prec e^{\mu + k} (1 + \| \psi \|_\rho)^{-k}
\end{equation}

(cf. Section 29 of [26]).

The linear space of the quasi polynomial families $A_\epsilon$, satisfying (2.36), is denoted $\mathcal{Q}^{\rho, \mu}$. 
Similarly to (2.34), the product $Q(B_\epsilon)Q(C_\epsilon)$ of quasi differential operator families with $B \in Q^\rho_\epsilon$ and $C \in Q^\nu_\epsilon$ is a quasi differential operator family $Q(A_\epsilon)$ such that

$$A_\epsilon(\psi^*, \psi) = -\sum_{n=0}^N (-1)^n(n!)^{-1} \partial^n B_\epsilon(\psi^*, \psi) \cdot \partial^n C_\epsilon(\psi^*, \psi) \in Q^{\rho+\tau-N, \mu+\nu+N}_\epsilon.$$  

(2.37)  

### 2.11. Ellipticity.

A quasi differential operator $Q(A)$ of a positive order $\rho$ is called **elliptic** if:

1. There exists positive $\sigma \leq \rho$ such that for sufficiently large $\|\psi\|$ we have $\|\psi\|_{-\sigma} \prec A(\psi^*, \psi)$.

2. For given non-negative integers $k$ and $l$, if $\|\psi\|_{-\sigma}$ is sufficiently large then $\|\partial^k \partial^l A(\psi^*, \psi)\|_{-\sigma} \prec A(\psi^*, \psi)\|\psi\|^{-(k+l)}$.

The standard application of the symbolic calculus (cf. [26]) provides a quasi differential parametrix $Q(P)$ of order $-\sigma$ such that $Q(A)Q(P) - 1 \in Q^{-\infty}$ and $Q(P)Q(A) - 1 \in Q^{-\infty}$. This entails that the zero set of an elliptic operator belongs to $F^{-\infty}$.

On $R^n$ the following remark is Shubin’s Theorem 26.2 in [26]. The proof is the same.

**Remark 2.4.** Any elliptic quasi differential operator $Q(A)$ with real $A$ is essentially selfadjoint on $F_0$. Its closure is $Q(A)$ on the domain $\Psi \in F : Q(A)\Psi \in F$.

**Proof.** The operator $Q(A)$ is symmetric on $F_0$.

Let $Q(A)_0$ be the restriction of $Q(A)$ on $F_0$, $Q(A)_0$ its closure, and $Q(A)_0^*$ its Hermitian adjoint in $F$. The domain of $Q(A)_0^*$ consists of $\Psi \in F$ such that $Q(A)\Psi \in F$, and $Q(A)_0^* = Q(A)$ on this domain.

An essential selfadjointness test of $Q(A)_0$ is that the zeros of two operators $Q(A)_0^* + i$ and $Q(A)_0^* - i$ belong to the domain of $Q(A)_0$ (cf. [26], Theorem 26.1).

Here both operators are elliptic quasi differential so the zero sets are within $F^{-\infty}$. By the closed graph theorem, $Q(A)$ is continuous in $F^{-\infty}$. Since $F_0$ is dense in $F^{-\infty}$, the test verifies the essential selfadjointness of $Q(A)_0$.

**Remark 2.5.** If $\mathcal{H}$ is infinite-dimensional then a generic elliptic quasi differential operator is not Fredholm. Correspondingly, the spectrum of an elliptic operator is almost never discrete.

### 3. Klauder-Feynman integral

Let a quasi differential operator $Q(A)$ be essentially selfadjoint on $F_0$. If $\Psi_t$ is a solution of the functional Schrödinger equation (1.6) in a Fock space $F$

$$\partial_t \Psi_t + iQ(A)\Psi_t = 0$$

then, by the coherent state Plancherel equality (2.11),

$$\langle \Omega_\alpha | e^{-iQ(A)t} | \Psi_0 \rangle = \int d\beta^* d\beta e^{-\beta^* \beta} \langle \Omega_\alpha | e^{-iQ(A)t} | \Omega_\beta \rangle \langle \Omega_\beta | \Psi_0 \rangle,$$

so that $\langle \Omega_\alpha | e^{-iQ(A)t} | \Omega_\beta \rangle e^{-\beta^* \beta}$, $\alpha, \beta \in \mathcal{H}$, is the quantum propagator of the coherent states (cf. (1.8)).
THEOREM 3.1. Suppose a quasi differential operator $Q(A)$ is elliptic with a real antswick symbol $A$. (By Remark 2.4, $Q(A)$ is essentially selfadjoint on $\mathcal{F}_0$.)

Then the coherent state propagator (3.1) is equal to the limit at $n = \infty$ of the $n$-multiple functional integrals over the phase space $\mathcal{H}$

\[ (3.2) \quad \int \prod_{j=1}^{n} d\psi_j^* d\psi_j \exp \sum_{j=0}^{n} [(\psi_{j+1} - \psi_j)^* \psi_j - iA(\psi_j^*, \psi_j)\frac{t}{n}], \]

where $\psi_{n+1} = \alpha, \; \psi_0 = \beta$.

**Proof. Step 1.**

By ellipticity of $Q(A)$, the quasi polynomial family

\[ (3.3) \quad A_{t/n} = (1 + iA t/n)^{-1}, \; n = 1, 2, \ldots, \]

parametrized by $\epsilon = t/n$, belongs to $Q_{t/n}^{0,0}$.

Applying (2.37) to the quasi differential operator families $1 + iQ(A)t/n$ and $Q(A_{t/n})$, we get

\[ (1 + iQ(A)t/n)Q(A_{t/n}) = Q(D_{t/n}), \]

where

\[ D_{t/n}(\psi^*, \psi) = (1 + iA(\psi^*, \psi)t/n)(1 + iA(\psi^*, \psi)t/n)^{-1} \]
\[ + \partial(1 + iA(\psi^*, \psi)t/n) \cdot \partial^*(1 + iA(\psi^*, \psi)t/n)^{-1} + (t/n)^2 R_{t/n} \]
\[ = 1 + (t/n)^2 S_{t/n}, \]

with the quasi polynomial family $S_{t/n}$ uniformly bounded with respect to $n$.

By (2.21), the operator norms of the quasi differential operators $Q(S_{t/n})$ in $\mathcal{F}$ are uniformly bounded with respect to $n$ as well. Therefore

\[ (3.4) \quad |Q(A_{t/n}) - (1 + iQ(A)t/n)^{-1}| \prec 1/n^2. \]

We have the telescopic equality

\[ Q(A_{t/n})^n - (1 + iQ(A)t/n)^{-n} \]
\[ = \sum_{m=0}^{n-1} Q(A_{t/n})^m [(Q(A_{t/n}) - (1 + iQ(A)t/n)^{-1})(1 + iQ(A)t/n)^{n-1-m}. \]

In view of (2.21), the operator norm of $Q(A_{t/n})$ is $\leq 1$, and the operator norm of $(1 + iQ(A)t/n)^{-1}$ is $\leq 1$ by the spectral theorem. So (3.5) entails that, uniformly in $n$,

\[ (3.6) \quad |Q(A_{t/n})^n - (1 + iQ(A)t/n)^{-n}| \prec n(1/n^2). \]

Along with the equality $\exp[-iQ(A)t] = \lim_{n \to \infty} (1 + iQ(A)t/n)^{-n}$ this implies the strong operator limit

\[ (3.7) \quad e^{-iQ(A)t} = \lim_{n \to \infty} Q(A_{t/n})^n. \]

This equation is a modification of the semi-classical Feynman-Tobocman postulate.

**Step 2.**

The normal symbol of $Q(A_{t/n})$ is given by

\[ \langle \Omega_{\psi_2}|Q(A_{t/n})|\Omega_{\psi_0}\rangle e^{-\psi_0^* \psi_0} = \int d\psi_1^* d\psi_1 e^{(\psi_2 - \psi_1)^* \psi_1 + (\psi_1 - \psi_0)^* \psi_0} A_{t/n}(\psi^*, \psi), \]
so that, by (2.35), the normal symbol of $Q(A_{t/n})^n$ is

$$\langle \Omega_\alpha | Q(A_{t/n})^n | \Omega_\beta \rangle e^{-\beta \tau} = \int \prod_{j=1}^n d\psi_j^{*} d\psi_j A_{t/n}(\psi_j^{*}, \psi_j) e^{\sum_{j=0}^n (\psi_{j+1} - \psi_j)^{*} \psi_j}$$

with $\psi_{n+1} = \alpha$, $\psi_0 = \beta$.

**Step 3.**

Consider yet another telescopic equality

$$Q(A_{t/n})^n - Q(e^{-iAt/n})^n$$

$$= \sum_{m=0}^{n-1} Q(A_{t/n})^m [Q(A_{t/n}) - Q(e^{-iAt/n})] Q(e^{-iAt/n})^{n-1-m}.$$  

The middle factor is the (not a quasi differential) operator with the antiwick symbol $A_{t/n} - e^{-iAt/n}$.

Using the second Taylor approximation in the $t$-variable (with Lagrange integral remainder) at $t = 0$, we get

$$A_{t/n} - e^{-iAt/n} = (iA/n)^2 \int_0^t d\tau (t - \tau) [-2(1 + iA\tau)^{-3} - e^{-iA\tau}] \propto (At/n)^2.$$ 

Suppose $\rho$ is the order of $A$.

Then, by (2.23), $Q(A_{t/n} - e^{-iAt/n})$ is a bounded operator from $\mathcal{F}^\rho$ to $\mathcal{F}$ with the operator norm

$$\|Q(A_{t/n}) - Q(e^{-iAt/n})\|_{\rho, 0} \approx (t/n)^{2} \sup |A(\psi^*, \psi)|/(1 + |\psi^-|^\rho).$$

Similarly, the 0-order operator $Q(e^{-iAt/n})$ is bounded in $\mathcal{F}$ with the operator norm $\leq 1$ and the 0-order operator $Q(A_{t/n})$ is bounded in $\mathcal{F}^\rho$ also with the operator norm $\leq 1$.

Now telescopic inequality (3.9) and these norm estimates imply

$$|Q(A_{t/n})^n - Q(e^{-iAt/n})^n|_{\rho, 0} \approx n(t/n)^2.$$ 

Therefore, for given $\Omega_\alpha$ and $\Omega_\beta$,

$$\langle \Omega_\alpha | Q(A_{t/n})^n - Q(e^{-iAt/n})^n | \Omega_\beta \rangle e^{-\beta \tau} \propto n(t/n)^2.$$ 

By (2.35),

$$\langle \Omega_\alpha | Q(e^{-iAt/n})^n | \Omega_\beta \rangle$$

$$= \int \prod_{j=1}^n d\psi_j^{*} d\psi_j \exp \sum_{j=0}^n [(\psi_{j+1} - \psi_j)^{*} \psi_j - iA(\psi_j^{*}, \psi_j)t/n].$$

The theorem follows from (3.7), (3.12), and (3.13). \hfill \Box

**Remark 3.2.** In the notation $\tau_j = jt/n$, $\psi_{t_j} = \psi_j$, $j = 0, 1, 2, \ldots, n$, and $\Delta \tau_j = \tau_{j+1} - \tau_j$, $\Delta \psi_j = \psi_{j+1} - \psi_j$, the multiple integral (3.2) is

$$\int \prod_{j=1}^n d\psi_{t_j}^{*} d\psi_t \exp i \sum_{j=0}^n \Delta \tau_j [-i(\Delta \psi_j/\Delta \tau_j|\psi_{t_j}) - A(\psi_{t_j}^{*}, \psi_{t_j})].$$
Its limit at \( n = \infty \) (if exists) is the Klauder-Feynman integral \([19]\) for the coherent states propagator

\[
\langle \Omega_\alpha | e^{-iQ(A)t} | \Omega_\beta \rangle e^{-\beta^\ast \beta} = \int_\alpha^\beta \prod_{0 < \tau < t} d\psi^\ast_{\tau} d\psi_\tau \exp i \int_0^t d\tau \left[ -i \langle \dot{\psi}_\tau | \psi_\tau \rangle - A(\psi^\ast_\tau, \psi_\tau) \right],
\]

where, similarly to \([20]\), the Hamiltonian functional is replaced with the antiwick symbol \( A \).

Theorem 3.2 gives sufficient conditions for the limit existence, even in the case of infinite degrees of freedom.

As the limit of the multiple functional integrals, the Feynman integral over a product Hilbert space is equal to the iterated integrals. Let \( c \) be a \( \cdot \mid \cdot \) -continuous linear canonical transformation of \( \mathcal{H} \) as a real symplectic space. (Unlike \([3]\) we do not assume that the canonical transformation is proper.) Then we have the substitution rule

\[
\int_{c\alpha}^{c\beta} \prod_{0 < \tau < t} d\psi^\ast_{\tau} d\psi_\tau \exp i \int_0^t d\tau \left[ -i \langle c^{-1} \dot{\psi}_\tau | c^{-1} \psi_\tau \rangle - A(c^{-1} \psi^\ast_\tau, c^{-1} \psi_\tau) \right]
\]

\[
= \int_{\alpha}^{\beta} \prod_{0 < \tau < t} d\psi^\ast_{\tau} d\psi_\tau \exp i \int_0^t d\tau \left[ -i \langle \dot{\psi}_\tau | \psi_\tau \rangle - A(\psi^\ast_\tau, \psi_\tau) \right].
\]

This follows from Theorem 3.1 since the product \( \prod_{j=1}^n d\psi^\ast_j d\psi_j \) is a symplectic invariant.

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