Nonergodic dynamics of dipolar lattice bosons

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Non-ergodic dynamics in many-body interacting systems has recently attracted much interest. Polar lattice gases present, even in absence of disorder, an intriguing dynamics, which has been mostly discussed in the hard-core regime. We show that the dynamics of soft-core dipolar lattice bosons may be remarkably different. We find that contrary to hard-core dipoles, bosons tend to delocalize for large inter-site interaction strengths due to the crucial role played by interaction-induced hopping. Interestingly, the interplay between bare and interaction-induced tunneling may lead, near a lattice-depth-dependent value of the dipole strength, to an exact decoupling of the Hilbert space between ergodic states with at most singly-occupied sites, and the remaining strongly non-ergodic states. Our results highlight the key role that density-dependent hopping may play in future experiments on out-of-equilibrium systems with long-range interactions.

Many-body localization (MBL) has attracted in recent years a major attention as a paradigmatic manifestation of nonergodic dynamics in the presence of disorder [1–3]. While the very existence of MBL in the thermodynamic limit remains a controversial and extensively discussed topic [4–14], experimental signatures of nonergodic dynamics in finite systems on a time scale of several hundreds of tunneling times have been clearly observed [15–20]. Recent years have brought also a number of examples of nonergodic dynamics in disorder-free systems, ranging from implementations of lattice gauge theories [21–23], to tilted lattices and smooth potentials [24–29]. A prominent example, related to an approximate global constraint and an appropriate choice of the initial state, is given by the so-called quantum scars [22, 30–32]. Approximate global constraints result often in Hilbert-space fragmentation [33].

A particularly interesting example of Hilbert-space fragmentation and disorder-free nonergodic dynamics is provided by polar gases in optical lattices [34]. A sufficiently-large dipole strength results in an emerging dynamical constraint given by the approximate conservation of the number of pairs of nearest-neighbor (NN) particles. This, combined with the eventual conservation of the number of next-to-NN pairs, results in Hilbert space shattering [34] and strongly nonergodic dynamics in hard-core systems. In those systems, on-site interactions are assumed large-enough to prevent more than one particle per lattice site.

In this Letter, we show that the dynamics of soft-core dipolar bosons, with possibly multiply-occupied sites, may be radically different than their hard-core counterparts. This marked difference results from the crucial role played by the density-dependent tunneling (DDT), and its interplay with the bare hopping. Although DDT may be generally relevant in Hubbard models with strong-enough on-site interactions [35, 36], it is particularly relevant in polar lattice gases due to the long-range dipole-dipole interactions, as shown by recent studies of their ground-state properties [37–40]. Our results show that due to DDT, a growing dipole strength results in enhanced particle delocalization, in stark contrast to the hard-core case. Moreover, DDT induces for a particular dipole strength a peculiar exact decoupling of the Hilbert space into ergodic and strongly non-ergodic states.

Model. – We consider externally oriented dipolar bosons in a deep one-dimensional optical lattice, assuming a strong confinement in the transversal directions. The system is well described by the extended Bose-Hubbard (EBH) model ($\hbar = 1$):

$$\hat{H}_{\text{EBH}} = -t \sum_{j=1}^{L-1} (\hat{a}_j^\dagger \hat{a}_{j+1} + \text{H.c.}) + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1) + \frac{V}{2} \sum_{i\neq j} \frac{1}{|i-j|^2} \hat{n}_i \hat{n}_j - T \sum_{j=1}^{L-1} \left( \hat{a}_j^\dagger (\hat{n}_j + \hat{n}_{j+1}) \hat{a}_{j+1} + \text{H.c.} \right).$$

where $a_j^\dagger (a_j)$ denotes the bosonic creation (annihilation) operator, and $t$ is the bare tunneling amplitude. The first line of Eq. (1) is the standard Bose-Hubbard model, in which the on-site interaction strength $U$ results from both contact-like and dipole-dipole interactions. We fix $U/t = 3$ below. The second line describes the intersite dipolar interactions [41], characterized by the dipolar strength $V$, which may be tuned by changing the dipole orientation with respect to the axis of the optical lattice. We note in passing, that keeping $U$ fixed while changing $V$ requires an appropriate tuning of the contact interaction, via e.g. a Feshbach resonance. The last line corresponds to the DDT, with amplitude $T$. For the moderate $U/t$ value we consider, $T$ only depends significantly on the dipolar interactions. For a given lattice depth $V_0 = s E_R$, with $E_R$ the recoil energy, and employing the appropriate form of the on-site functions, one finds that
$T$ is proportional to $V$ in a broad range of $V$ values [42]. Interestingly, the DDT amplitude $T$ is negative, a key feature in our discussion below.

We consider in the following a half-filled lattice with open boundary conditions. Since the maximal occupation per site equals the total particle number we are limited, in an exact diagonalization study, to small system sizes, typically of $N = 6$ (7) bosons on $L = 12$ (14) sites. In studying such small systems we are encouraged by recent experiments [19, 20].

As recently shown in Ref. [34], hard-core dipolar lattice bosons undergo Hilbert-space shattering for large-enough $V/t$ due to the emergent conservation of the number of NN and next-to-NN links. A similar behavior is shared by soft-core bosons in the absence of DDT, $T = 0$ (Fig. 1 (a)). In the presence of DDT, the fragmentation largely disappears, even for large $V/t$ values (Fig. 1 (b)). This has profound consequences for the dynamics, as already anticipated by the behavior of the mean gap ratio and the mean fractal dimensions.

**Gap ratio.**—The gap ratio is defined by the minimum of two consecutive spacings between eigenenergies [43]. The mean gap ratio in a given energy range, which we denote as $\bar{\tau}$, provides important spectral information on the dynamics (see Fig. 1 (c)). Integrable systems with Poissonian level statistics are characterized by $\bar{\tau} \approx 0.383$, while for ergodic time-reversal invariant system one expects $\bar{\tau} \approx 0.53$. We observe the latter behavior only for small $V/t$. For increasing $V$, a general decrease of $\bar{\tau}$ is observed. While for $T = 0$ the gap ratio reaches a Poissonian value for large $V/t$ manifesting quasi integrability, the presence of DDT ($T \neq 0$) leads to mixed dynamics with $\bar{\tau}$ in between Poissonian and GOE statistics. Note that in the latter case there is a pronounced sharp minimum around $V/t = 8.8$. We discuss this minimum below.

**Fractal dimensions.**—The behavior of $\bar{\tau}$ is further confirmed by that of the mean fractal dimensions of eigenstates $D_\alpha$. The participation entropy $S_\alpha$ for a given eigenstate $|\Psi\rangle$ is

$$S_\alpha = \frac{1}{1-\alpha} \ln \left( \sum_{i=1}^{N} |\langle i | \Psi \rangle|^{2\alpha} \right).$$

The corresponding fractal dimensions are defined as $D_\alpha = S_\alpha/\mathcal{N}$, where $\mathcal{N}$ is the Hilbert space dimension.

The Shannon entropy is simply $S_1$, while $S_2 = -\ln(\text{IPR})$, where $\text{IPR} = \sum_i |\langle i | \Psi \rangle|^4$ is the inverse participation ratio (IPR) in the basis $\{|i\rangle\}$. Averaging over all eigenstates we obtain $D_2$. Note that for large $V/t$, the values of $D_2$ are much smaller for $T = 0$ than in the presence of DDT, showing that DDT enhances delocalization. While we show data for $N = 7$, a similar behavior is obtained for smaller $N$ (also at half filling), with $D_1 \neq D_2$ indicating the multifractal character of the eigenstates both for $T = 0$ and in the presence of DDT.

**Particle dynamics**—As expected from the previous results, the dynamics with DDT is markedly different than that obtained for $T = 0$. We consider in the following the time evolution when starting with initial Fock states, $|\varphi\rangle$. We choose carefully the initial-state energy, such that it lies at the center of an energy regime with a large density

![Figure 1](image1.png)

**Figure 1.** Probability distribution of the energy eigenvalues for $T = 0$ (a) and in the presence of DDT (b) for $V/t = 30$ and $U/t = 3$, for $N = 7$ bosons in $L = 14$ sites. (c) Mean gap ratio, $\bar{\tau}$ and (d) mean fractal dimensions $D_\alpha$ (see text) of the eigenstates as a function of $V$ with (circles) and without (triangles) DDT.

![Figure 2](image2.png)

**Figure 2.** Inhomogeneity $I(\tau_f)$ at time $\tau_f = 500/t$, as a function of $V/t$, for $N = 6$, $L = 12$, and $U/t = 3$. In the absence of DDT (dot-dashed: red (black) curve for the $N_{NN} = L/6$ ($L/4$) sector), the dynamics becomes steadily more non-ergodic with increasing $V/t$. The presence of DDT strongly modifies the dynamics. The blue (black) dashed line show our results for $s = 8$ for the $N_{NN} = L/6$ ($L/4$) sector. The green dashed line depict the case of a deeper $s = 10$ lattice (for the $N_{NN} = L/6$ sector). Observe that the peak of enhanced inhomogeneity depends on the lattice depth, being at $V/t = 8.8$ for $s = 8$ and at $V/t \approx 13$ for $s = 10$. The inset shows the dependence of $T/t$ on $V/t$ for $s = 10$. The confuence of tunneelings $T/t = -1$ occurs for $V/t \approx 13$. 


of states, avoiding areas of low density of states occurring due to a possible Hilbert-space fragmentation. For the study of the dynamics, it is customary to consider an initial density wave, with every second state occupied and every second state empty. For a polar lattice gas, with strong NN interactions, this is not a good choice, as then the number of occupied adjacent pairs is zero and the density wave lies at the extremes of the spectrum. Instead, we consider a manifold of initial Fock states, with a given number of NN pairs, \( N_{NN} = \sum \langle \phi | \hat{n}_i \hat{n}_{i+1} | \phi \rangle \). We take either \( N_{NN} = L/6 \) or \( N_{NN} = L/4 \). As discussed in Ref. [34], the latter is the most populated sector of states in the possibly fragmented Hilbert space. We are particularly interested in how the originally very inhomogeneous population in the lattice redistributes amongst the sites, and in particular whether it becomes eventually homogeneous. For the soft-core case under consideration, the density homogenization is best analyzed by monitoring the inhomogeneity parameter at a given time \( \tau \):

\[
\mathcal{I}(\tau) = \frac{\sum^{L}_{i=1} \langle \hat{n}_i(\tau) \rangle - \rho}{\sum^{L}_{i=1} \langle \hat{n}_i(0) \rangle - \rho},
\]

with \( \rho = N/L \) the overall particle density (\( \rho = 1/2 \) in our simulations). Note that \( 0 < \mathcal{I}(\tau) < 1 \), with 0 (1) indicating a fully homogeneous (inhomogeneous) distribution.

Figure 2 shows \( \mathcal{I}(\tau_f) \), after an experimentally accessible time \( \tau_f = 500/t \) [24]. For each value of \( V/t \) (which results in a given value of \( T/t \) that depends on the lattice depth \( s \)), we obtain \( \mathcal{I}(\tau_f) \) after averaging over initial Fock states with \( N_{NN} = L/4 \) (in Fig. 2 we depict also the results for \( N_{NN} = L/6 \) for comparative purposes).

The error bars indicate the result from the bootstrap estimate based on data for about 200 initial conditions in the sector. Observe that in absence of DDT, the inhomogeneity increases steadily for growing \( V \), resembling the case of hard-core bosons [34]). The inhomogeneity is markedly different in the presence of DDT, as shown for two different lattice depths \( s = 8 \) and 10. Note that, in agreement with the spectral analysis above, the DDT results in a lower inhomogeneity at large \( V/t \), i.e. it favors delocalization, although full ergodicity, i.e. \( \mathcal{I} = 0 \), is not reached, arguably due to the finite size of the systems. Our results show that the DDT plays a crucial role in the dynamics of soft-core polar lattice gases, especially for shallow lattices (low \( s \)). For a larger \( s \), the value of \( T \) decreases, and the delocalizing effect of the DDT becomes less relevant compared to the localizing role of inter-site interactions, as illustrated in Fig. 2.

The dependence on the lattice depth is further analyzed in Fig. 3. For a fixed strong dipolar interaction, \( V/t = 50 \), we evaluate \( \mathcal{I}(\tau_f = 500/t) \), for different lattice depths \( s \). We consider separately initial states with \( N_{NN} = L/6 \) and \( L/4 \). For low-enough \( s \), the system reaches homogeneity, despite the large \( V/t \) value. In contrast, for deeper lattices, \( s > 10 \), \( \mathcal{I}(\tau_f) \) reaches large values indicating non-ergodic dynamics and strong memory of the initial conditions. This is due to the fact that DDT strength \( T \) rapidly decays with \( s \), and hence DDT may be safely neglected for deep lattices.

**Mean-field analysis.**— The previous results may be qualitatively understood using mean-field decoupling [39]:

\[
-\hat{a}_j^\dagger [t + T (\hat{n}_i + \hat{n}_j)] \hat{a}_j \simeq -(t + 2 \rho T) \hat{a}_i^\dagger \hat{a}_j.
\]

For our case, \( \rho = 1/2 \), the effective mean-field hopping is \( t_{eff} = t + T \). For \( T/t = -1 \), the mean-field hopping vanishes, explaining the strongly strengthened non-ergodicity observed in that case. We discuss this case in more detail below. Moreover, non-ergodicity is regulated by the ratio \( V/t_{eff} = V/\rho T \). Since \( T \langle V \rangle = \alpha(s) + \beta(s) V \), for a sufficiently large \( V/t \), the ratio approaches \( V/t_{eff} \simeq -1/\beta(s) \).

Hence, increasing the dipolar strength, does not result (as in the absence of DDT) in a diverging ratio between inter-site interactions and hopping, which leads necessarily to localization, but rather in a saturated ratio, \( |V/t_{eff}|_{\text{max}} \). This maximal ratio depends on the lattice depth, increasing with growing \( s \). This explains two relevant qualitative features in Fig. 2 and Fig. 3: the independence of \( \mathcal{I} \) of \( V/t \) for large-enough \( V/t \), and the very low inhomogeneity observed even for large \( V/t \) for shallow lattices (low \( s \) values). The latter results from the low value of \( |V/t_{eff}|_{\text{max}} \).

**Critical dipole strength.**— Interestingly, in the presence of DDT, the inhomogeneity \( \mathcal{I}(\tau_f) \) presents a marked maximum at \( V/t = 8.8 \) for \( s = 8 \) shifting to \( V/t = 13 \) for \( s = 10 \). The observed maximum in the dynamics matches the sharp minimum in \( \tau \) and \( \mathcal{D}_b \) in Fig. 1. This feature corresponds to the case with \( T/t = -1 \) [42], for
Hilbert space shows an almost Poissonian statistics with \( \approx 0.527 \), the remaining part of the Hilbert space shows an almost Poissonian statistics with \( \approx 0.399 \). We note that the splitting of the Hilbert space into decoupled sectors results solely from the \( T/t = -1 \) condition, being independent of the character of interactions. In particular, it is independent of the considered \( 1/r^3 \) tail, and hence a similar effect should occur for any long-range potential.

The effects of this maximal interference between bare tunneling and DDT are best illustrated by considering the time evolution of different Fock state families. Figure 4(a) shows (for \( N = 6 \), \( L = 12 \), \( s = 8 \), and \( V/t = 8.8 \)) the long-time dynamics of different initial states (all belonging to the same \( N_{NN} = L/6 \) sector). The hard-core boson decoupled sector (family I) behaves ergodically with \( \tau \) practically vanishing for times \( \tau > 200/t \) (in agreement with [34]). States with a single double-occupancy (family II) decay much slower. Those with a single pair \( \cdots 12 \cdots \) (sub-family II A) present a rapid initial decay of \( I \) corresponding to the spreading of the remaining three bosons over the available space, resembling the family I. After a short time (of the order of 1), a slower decay of \( I \) takes place determined by the highly non-resonant mixing of the occupied pair with the rest. The rest of family II, with a single double-occupancy surrounded by empty sites, decays much slower already at short times and then presents a pronounced plateau (at this stage a single occupancy still survives the dynamics) finally reaching a nonzero value. The rest of the states (family III) is characterized by a single large occupancy, and presents a very slow dynamics. In absence of DDT (Fig. 4(b)) initial states belonging to families I or II undergo a rapid homogenization. Interestingly, only a partial homogenization occurs for a single high-occupancy initial states. This is due to the energy penalty induced by the on-site interaction [44]. The different properties of the ergodic (hard-core) and localized (other states) subspaces are also reflected in their entanglement-entropy properties. While the mean entanglement entropy for the hard-core sector follows a volume law increasing linearly with the system size, \( L \), for the other states the entanglement entropies are much lower and rather follow an area law [42].

**Conclusions**—We considered the dynamics of soft-core dipolar bosons in one-dimensional optical lattices, paying a particular attention to the role of density-dependent tunnelings. In stark contrast to the hard-core case, for large dipolar interactions, interaction-induced hopping may induce delocalization. Moreover, the destructive interplay between bare hopping and density-dependent hopping may result for a critical dipole strength in an exact separation of the Hilbert space between an ergodic hard-core subspace and strongly non-ergodic states. Our results show that interaction-induced hopping may play a crucial role in future experiments on the dynamics of dipolar gases in optical lattices.

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**SUPPLEMENTARY MATERIAL TO "NONERGODIC DYNAMICS OF INTERACTING BOSONIC DIPOLES IN OPTICAL LATTICE"**

**Parameters of the extended Bose-Hubbard model**

The calculation of the parameters of the extended Bose-Hubbard model closely parallels the technique described in detail in Ref. [40]. We assume a quasi one-dimensional model with an optical lattice along $x$, and a tight harmonic confinement in the transversal directions, leading to the single-particle trapping potential

$$V_t(r) = \frac{m\omega^2}{2} (y^2 + z^2) + V_0 \cos^2(kx),$$  \hspace{1cm} (S1)

where $m$ is the particle mass, $\omega$ is the harmonic trapping frequency along $y$ and $z$, and $k$ is the wavevector of the laser that forms the optical lattice.

The Hamiltonian of the system may be expressed as (see e.g. [36]):

$$\hat{H} = \int d^3r \hat{\Psi}^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_t(r) \right] \hat{\Psi}(r) \hspace{1cm} (S2)$$

\[+ \frac{1}{2} \int d^3r \int d^3r' \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r') V_{\text{int}}(r-r') \hat{\Psi}(r') \hat{\Psi}(r),\]

with the bosonic field operators $\hat{\Psi}(r)$ and $\hat{\Psi}^\dagger(r)$ that obey the commutation relation $[\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = \delta^3(r-r')$. $V_{\text{int}}(r-r')$ describes interactions between bosons that is conveniently split into the contact and dipole-dipole terms:

$$V_{\text{int}}(r) = V_c(r) + V_d(r).$$  \hspace{1cm} (S4)

The contact term is characterized by the s-wave scattering length, $a_s$. Using the customary notation, $V_c(r) = g_0^{(3)}(r)$, with $g = 4\pi\hbar^2a_s/m$. Dipole-dipole interactions give a second interaction term, $V_d(r)$. We consider dipoles polarized by an external field along the $z$ axis (perpendicular to the axis of the optical lattice) with

$$V_d(r) = C \frac{1 - 3 \cos^2(\theta)}{r^3},$$  \hspace{1cm} (S5)

where $\theta$ is the angle between the dipole and $r$. The dipole-dipole interaction is anisotropic in space since the force depends on the dipole orientation. The strength of the dipole-dipole interactions $C = \mu_0\mu^2/4\pi = (d^2/(4\pi\epsilon_0))$ for magnetic (electric) dipoles with moment $\mu$ (d) where $\mu_0$ ($\epsilon_0$) are the magnetic (electric) permeability, respectively.

For sufficiently deep optical lattice ($s = V_0/E_R > 3$, where $E_R = \hbar^2k^2/(2m)$) we may expand the field operator as

$$\hat{\Psi}(r) = \sum_{j=1}^{L} V_j(r) \hat{a}_j = \sum_{j=1}^{L} \phi_0(y) \phi_0(z) W_j(x) \hat{a}_j,$$  \hspace{1cm} (S6)
where \( j = 1, \ldots, L \) denotes the site index, and the operator \( \hat{a}_j \) annihilates boson at site \( j \). The corresponding basis function \( W_j(\mathbf{r}) \) is the product of the ground states of the harmonic oscillators along \( y, z \), and the Wannier function (of the lowest band) along the lattice (shallower lattices may implicate the necessity of taking higher bands into account). Plugging Eq. (S6) into Eq. (S3), one expresses the Hamiltonian in as a polynomial of the annihilation and creation operators. Employing the orthogonality of the Wannier functions one arrives at the form:

\[
\hat{H} = -t \sum_{j=1}^{L-1} \left( \hat{a}_j^\dagger \hat{a}_{j+1} + H.c. \right) + \frac{1}{2} \sum_{i,j,k,l} V_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l, \tag{S7}
\]

where the integrals \( V_{ijkl} \) are explicitly given as

\[
V_{ijkl} = \int d^3 \mathbf{r} d^3 \mathbf{r}' W_i(\mathbf{r}) W_j(\mathbf{r}') W_k(\mathbf{r} - \mathbf{r}') W_l(\mathbf{r}). \tag{S8}
\]

The single particle tunneling amplitude is obtained from the single particle part of the Hamiltonian

\[
t = - \int d^3 \mathbf{r} W_i(\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_i(\mathbf{r}) \right] W_{i+1}(\mathbf{r}) \nonumber
\]

\[
= \int dx W_i(x) \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V_0 \cos^2(kx) \right] W_{i+1}(x), \tag{S9}
\]

where we limit ourselves to NN tunneling only assuming a sufficiently deep optical lattice (for shallow lattices, with \( s = V_0/E_R < 4 \), one might need to include next-to-NN tunnelings into the picture, see e.g. Ref. [46].)

As it turns out the integrals over perpendicular directions can be explicitly carried out \([47–50]\) yielding

\[
V_{ijkl} = \int dx dx' W_i(x) W_j(x') V_{1D}(x - x') W_k(x) W_l(x'). \tag{S10}
\]

where the effective quasi-one-dimensional potential is given by

\[
V_{1D} = \left( g_{1D} - \frac{2C}{3l^2} \right) \delta(|x - x'|) + \frac{C}{l^3} \left\lvert \frac{\pi}{8} \right\rvert \frac{1 + (x - x')^2}{l^2} \text{Erfc} \left( \frac{|x - x'|}{l\sqrt{2}} \right) - \frac{|x - x'|}{2l}, \tag{S11}
\]

in terms of the harmonic oscillator length in the perpendicular direction, \( l = (\hbar/m\omega)^{1/2} \) and the effective 1D contact interaction strength \( g_{1D} = g/(2\pi \ell^2) \). In Eq. (S11) \( \delta \) stands for Dirac delta function and Erfc for the complementary error function. Note that the term proportional to the delta function contains contributions from both the contact and dipole-dipole interactions.

The largest contribution is given by the on-site interaction term (diagonal in \( ijk\ell \) indices), traditionally denoted as \( U \equiv V_{iii} \). For contact interactions this is the dominant term. For long-range dipolar interactions the next important term has the form of a density-density interaction \( V_{ijij} + V_{ijji} \hat{n}_i \hat{n}_j \) (for \( i \neq j \)) where often only the NN term for \( j = i \pm 1 \) is taken into account. We mention parenthetically that while for contact interactions \( V_{ijij} \) and \( V_{ijji} \) are identical, for a dipolar potential one finds that \( |V_{ijij}| \ll |V_{ijji}| \). Since Wannier functions are well localized on sites, for tight perpendicular binding and \( 1/r^3 \) potential one may approximate \( V_{ijij} = V/|i - j|^3 \) recovering the typical dipolar tail; here \( V = V_{0110} \) is the value of the integral for the NNs. A standard extended Bose-Hubbard model (see e.g. [51]) considers just terms involving \( U \) and \( V \) coefficients and neglects the dipolar tail. The latter may play an important role in the dynamics of the system \([34, 45]\), and may differ from the standard \( 1/r^3 \) decay if the transversal confinement is not sufficiently strong \([45]\).

\[\begin{figure}
\centering
\includegraphics[width=\textwidth]{FigureS1.png}
\caption{Relation between the DDT rate \( T \) and \( V \) assuming a constant \( U/t = 3 \). The dash-dotted line shows the almost perfect correspondence with a linear function. The dashed lines emphasize that at \( V/t = 8.8 \), \( T/t = -1 \).}
\end{figure}\]

Other important terms, introduced by Hirsch [35] for strongly-correlated spinful fermions, are density-dependent tunnelings (DDT), also called correlated hoppings, coming from \( V_{ijkl} \) terms with three equal indices. The most important corresponds to NN correlated tunneling, e.g. \( V_{i(i+1)j} \hat{a}_i^\dagger \hat{n}_j \hat{a}_{i+1} \). For shortness of notation
the corresponding amplitude is denoted by $T$ (or rather, due to some historical reasons $-T$ [36]). Taking together contributions containing $U$, $V$, and $T$ terms, one arrives at the extended Bose-Hubbard Hamiltonian of Eq. (1).

Figure S1 shows that $T$ varies linearly with $V$ for a broad range of value of $V/t$ (this is so for the chosen geometry, but does not need to be the case in general). For a lattice depth $s = 8$, which we employ at many points in the main text, one obtains $T/t = -1$ at $V/t = 8.8$. Changing $s$, displaces the critical value of $V/t$ to achieve this condition, since the bare hopping changes exponentially with the lattice depth.

**Further spectral analysis**

Additional information about the eigenstates can be obtained from the evaluation of their half-chain entanglement entropy, defined as the von Neumann entropy of the reduced density matrix $\rho_{L/2}$, $S = -Tr[\rho_{L/2} \ln \rho_{L/2}]$. where $\rho_{L/2} = Tr_{1,\ldots, L/2}|\psi\rangle \langle \psi|$ is obtained after tracing out half of the system for a given eigenstate $|\psi\rangle$. Under the condition $T/t = -1$, we may separately analyze the ergodic (hard-core) and nonergodic sectors. As shown in Fig. S2, the hard-core sector is characterized by large entropies, with the average growing linearly with the system size (volume law - compare inset). The characteristic finger-like structure is due to a partial Hilbert-space fragmentation. By comparison, the mean entropy of the states belonging to the nonergodic subspace seems to grow much more weakly with the system size, and the distribution of $S$ is broad with many low entanglement states.

Figure S2. (a) Entanglement entropy $S$ for the hard-core sector at $T/t = -1 (V/t = 8.8)$ for $N/L = 9/18$. The dashed horizontal line corresponds to the random-matrix theory value, vertical lines indicate a single sector taken to calculate the average entropy (shown in the inset for different system sizes). Panel (b) shows the entanglement entropy for the remaining decoupled sector for $N/L = 7/14$. The average entropy does not seem to depend on the system size for this nonergodic sector.