THE UNCROSSING PARTIAL ORDER ON MATCHINGS IS
EULERIAN

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Abstract. We prove that the partial order on the set of matchings of 2n points on
a circle, given by resolving crossings, is an Eulerian poset.

1. The Theorem

Let $P_n$ denote the set of matchings on 2n points, labeled 1, 2, ..., 2n in cyclic order on
a circle. Each $\tau \in P_n$ can be represented by (usually many) medial graphs, or strand di-
agrams. We say that a medial graph is lensless if any two strands intersect at most once. The following lensless medial graph represents \{\((1, 7), (2, 9), (3, 8), (4, 10), (5, 6)\)\} $\in P_5$.

For $\tau \in P_n$ we let $c(\tau)$ denote the number of crossings in a lensless medial graph
representing $\tau$. The set $P_n$ can be equipped with a partial order obtained by resolving
crossings:

$\tau' \preceq \tau$ if there is a lensless medial graph $G$ representing $\tau$ such that resolving
a crossing in $G$ gives a lensless medial graph $G'$ representing $\tau'$. The partial order $\leq$
on $P_n$ is the transitive closure of these cover relations.

The poset $P_n$ is graded with rank function given by $c(\tau)$, and we refer the reader
to [ALT, Ken, Lam] for alternative descriptions of $P_n$. The poset $P_n$ has a unique
maximum element, and Catalan number $C_n$ of minimum elements. Let $\hat{P}_n$ denote $P_n$
with a minimum $\hat{0}$ adjoined, where we declare that $c(\hat{0}) = -1$. Recall that a graded
poset $P$ with a unique minimum and a unique maximum, is Eulerian if, for every
interval $[x, y] \subset P$ where $x < y$, the number of elements of odd rank in $[x, y]$ is equal
to the number of elements of even rank in $[x, y]$. The following result was expected by
many experts.

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**Theorem 1.** \( \hat{P}_n \) is an Eulerian poset.

Here is a picture of \( \hat{P}_3 \):

![Diagram of \( \hat{P}_3 \)](image)

The same result for the Bruhat order of a Weyl group was established by Verma [Ver], and our approach is similar. Indeed, we rely on an embedding [Lam] of \( P_n \) into the (dual of the) Bruhat order of an affine symmetric group. Theorem 1 further amplifies the analogy between Bruhat order and the uncrossing partial order on \( P_n \).

Like Bruhat order, the poset \( P_n \) has a topological interpretation. In [Lam] a compactification \( E_n \) of the space of circular planar electrical networks with \( n \) boundary points was constructed. We have a stratification \( E_n = \bigsqcup_{\tau \in P_n} E_{\tau} \) by electroid cells, and \( E_{\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'} \). The same partial order occurs in the study of the positive orthogonal Grassmannian and scattering amplitudes for ABJM [HW, HWX]. The partial order \( P_n \) is also discussed in this context by Kim and Lee [KL] who observed a special case of Theorem 1. It seems to be expected by experts that these spaces are homeomorphic to balls.

We remark that in [ALT], it is shown that \( P_n \) is thin (intervals of length two are diamonds). The following conjecture is probably widely expected (see the related [ALT Conjecture 3.3]):

**Conjecture 1.** \( \hat{P}_n \) is lexicographically shellable.

An \( EL \)-shelling of \( \hat{P}_3 \) is given as follows. Denote the five non-crossing matchings by \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \), in order from left to right, as shown in the diagram above. We label each edge by one of the symbols \( \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3\} \). Label the covers \( \hat{0} \prec \tau \) by \( \tau \). Label the cover \( \eta \prec \tau \) where \( c(\tau) = 1 \), by \( \eta' \) where \( \eta' \neq \eta \) is uniquely defined by \( \hat{0} \prec \eta' < \tau \). Label the cover \( \eta \prec \tau \) where \( c(\tau) = 2 \), by \( \beta_i \), where \( \beta_i \) is uniquely determined by the conditions \( \beta_i < \tau \) and \( \beta_i \not\subseteq \eta \). Label the cover \( \eta \prec \hat{1} = \tau_{\text{top}} \) by \( \beta_i \) where \( \beta_i \) is uniquely determined by the condition \( \beta_i \not\subseteq \eta \).
Finally, order the symbols by \( \alpha_1 < \beta_1 < \beta_2 < \beta_3 < \alpha_2 \).

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## 2. Medial graphs with lenses

The notion of a medial graph representing a matching requires some care in the case that lenses are present.

Suppose \( G \) is any medial graph. A **lens** in \( G \) is a pair of strands that intersect twice or more. By repeatedly applying the moves in Figures 1–3, we can reduce \( G \) to a lensless medial graph \( G' \). If \( G' \) represents a matching \( \tau \in P_n \), we shall say that \( G \) represents \( \tau \in P_n \) as well. The following result follows from the interpretation of medial graphs in terms of electrical networks [ALT, Ken, Lam].

**Lemma 1.** Suppose \( G \) is a medial graph with lenses and \( G', G'' \) are two lensless graphs obtained from \( G \) by the moves Figures 1–3. Then \( G' \) and \( G'' \) represent the same matching \( \tau \in P_n \).

**Lemma 2.** Suppose \( G \) is a medial graph, possibly with lenses, that represents \( \tau \). Suppose \( G' \) is obtained from \( G \) by resolving any number of crossings in any way, and removing all closed interior loops that result. If \( G' \) represents \( \tau' \), then \( \tau \geq \tau' \).

**Proof.** In the language of [Lam], \( G \) is the medial graph of some electrical network \( \Gamma \), such that \( \mathcal{L}(\Gamma) \in E_\tau \). Then \( G' \) is the medial graph of \( \Gamma' \), where \( \Gamma' \) is obtained from \( \Gamma \) by either removing some edges, or contracting some edges. (Interior loops correspond to isolated interior components of \( \Gamma' \).) Thus we must have \( \mathcal{L}(\Gamma') \in E_{\tau'}, \) and the result follows from [Lam, Theorem 5.8], which states that \( E_{\tau'} = \bigsqcup_{\eta \leq \tau} E_{\eta} \). \( \square \)
3. The Proof

For a subset $S \subset \hat{P}_n$, write $\chi(S) = \sum_{\tau \in S} (-1)^{c(\tau)}$. We need to show that $\chi([\tau, \eta]) = 0$ whenever $\tau < \eta$.

Let $\tilde{S}_{2n}$ denote the poset (under Bruhat order) of affine permutations with period $2n$. These are bijections $g : \mathbb{Z} \to \mathbb{Z}$ such that $g(i + 2n) = g(i) + 2n$ for all $i \in \mathbb{Z}$. Define an injection $\iota : P_n \hookrightarrow \tilde{S}_{2n}$ by $\tau \mapsto g_\tau$, where

$$g_\tau(i) := \begin{cases} 
\tau(i) & \text{if } i < \tau(i), \\
\tau(i) + 2n & \text{if } i > \tau(i), 
\end{cases}$$

for $i \in \{1, 2, \ldots, 2n\}$. Here $\tau(i)$ is the element matched to $i$ in $\tau$. In [Lam] it is shown that $\iota$ expresses $P_n$ as dual to an induced subposet of $\tilde{S}_{2n}$. We refer the reader to [Lam] for full details.

For $f \in \tilde{S}_{2n}$, we let

$$D_L(f) := \{i \in \mathbb{Z}/2n\mathbb{Z} \mid s_i f < f\} \quad \text{and} \quad D_R(f) := \{i \in \mathbb{Z}/2n\mathbb{Z} \mid f s_i < f\}$$

be the left and right descent sets of $f$. The following result is standard, see [BB, Proposition 2.2.7].

**Lemma 3.** Suppose $f \leq g$ in $\tilde{S}_{2n}$. If $i \in D_L(g) \setminus D_L(f)$ then $f \leq s_i g$ and $s_i f \leq g$. If $i \in D_R(g) \setminus D_R(f)$ then $f \leq g s_i$ and $f s_i \leq g$.

For $\tau \in P_n$ and $i \in \mathbb{Z}/2n\mathbb{Z}$, we let

$$i \in \begin{cases} 
A(\tau) & \text{if the strands labeled } i \text{ and } i + 1 \text{ do not cross,} \\
B(\tau) & \text{if the strands labeled } i \text{ and } i + 1 \text{ cross,} \\
C(\tau) & \text{if } i \text{ is matched with } i + 1,
\end{cases}$$

considered in a lensless strand diagram $G$ representing $\tau$. We thus have a disjoint union $\mathbb{Z}/2n\mathbb{Z} = A(\tau) \cup B(\tau) \cup C(\tau)$. Define $s_i \cdot \tau$ by

$$g_{s_i \cdot \tau} = s_i g_\tau s_i.$$

For example, if $i \in A(\tau)$ then $s_i \cdot \tau$ is obtained from $\tau$ by adding a crossing between strands $i$ and $i + 1$, close to the boundary at $i$ and $i + 1$. Note that we have

$$i \in \begin{cases} 
A(\tau) & \text{if } s_i g_\tau s_i < g_\tau \text{ or, equivalently, } s_i \cdot \tau \leq \tau, \\
B(\tau) & \text{if } s_i g_\tau s_i > g_\tau \text{ or, equivalently, } s_i \cdot \tau \geq \tau, \\
C(\tau) & \text{if } s_i g_\tau s_i = g_\tau \text{ or, equivalently, } s_i \cdot \tau = \tau.
\end{cases}$$

**Lemma 4.** Suppose $\tau \leq \eta$ in $P_n$. If $i \in A(\tau) \cap B(\eta)$ then $\tau \leq s_i \cdot \eta$ and $s_i \cdot \tau \leq \eta$.

**Proof.** Since $i \in A(\tau)$ we have $i \in D_L(g_\tau) \cap D_R(g_\eta)$. Since $i \in B(\eta)$ we have $i \notin D_L(g_\eta) \cup D_R(g_\eta)$. By Lemma 3 we have $s_i g_\eta \leq g_\tau$. We also have, $i \notin D_R(s_i g_\eta)$ since $g_{s_i \cdot \eta} = s_i g_\eta s_i > s_i g_\eta$. So by Lemma 3 again, we have $s_i g_\eta s_i \leq g_\tau$, or equivalently, $g_{s_i \cdot \eta} \leq g_\tau$, or equivalently, $\tau \leq s_i \cdot \eta$. The proof of $s_i \cdot \tau \leq \eta$ is similar. \qed

The following result follows easily from the “uncrossing” definition.

**Lemma 5.** Suppose $\tau \leq \eta$ and $i \in A(\tau)$. Then $i \notin C(\eta)$.
Proof of Theorem \[7\] We shall first prove the theorem for intervals \([\tau, \eta] \subset P_n\) where \(\tau < \eta\); that is, intervals not involving \(\hat{0}\). We proceed by descending induction on \(c(\tau) + c(\eta)\). The base case where \(\eta\) is the maximal element of \(P_n\) and \(c(\tau) = \binom{n}{2} - 1\) is clear. Also if \(c(\eta) - c(\tau) = 1\) the result is clear, so we may assume that \(c(\eta) - c(\tau) \geq 2\).

Since \(\tau\) is not the maximal element, \(D_L(g_{\tau})\) and \(D_R(g_{\tau})\) are non-empty, so \(A(\tau)\) is non-empty (see also the proof of [Lam] Lemma 4.14]). Let \(i \in A(\tau)\).

Case 1: \(i \in A(\eta)\). Then

\[\tau, \eta = [\tau, s_i \cdot \eta] \setminus \{\sigma \mid \tau \leq \sigma \leq s_i \cdot \eta, \sigma \not\in \eta\}.\]

We claim that

\[\{\sigma \mid \tau \leq \sigma \leq s_i \cdot \eta, \sigma \not\in \eta\} = \{\sigma \mid s_i \cdot \tau \leq \sigma \leq s_i \cdot \eta, \sigma \not\in \eta\}\]

Suppose \(\tau \leq \sigma \leq s_i \cdot \eta\) and \(\sigma \not\in \eta\). If \(i \in A(\sigma)\), then since \(i \in B(s_i \cdot \eta)\), applying Lemma 4 to \(\sigma < s_i \cdot \eta\) we would obtain \(\sigma \leq \eta\), contradicting our assumption. If \(i \in C(\sigma)\), then Lemma 5 would be violated. Thus \(i \in B(\sigma)\), or \(s_i \cdot \sigma < \sigma\). Now apply Lemma 4 to \(\tau \leq \sigma\) to obtain \(s_i \cdot \tau \leq \sigma\). This proves (2).

But we have

\[\{\sigma \mid s_i \cdot \tau \leq \sigma \leq s_i \cdot \eta, \sigma \not\in \eta\} = [s_i \cdot \tau, s_i \cdot \eta] \setminus [s_i \cdot \tau, \eta]\]

so by induction \(\chi([\sigma \mid s_i \cdot \tau \leq \sigma \leq s_i \cdot \eta, \sigma \not\in \eta]) = \chi([s_i \cdot \tau, s_i \cdot \eta]) - \chi([s_i \cdot \tau, \eta]) = 0\).

By induction again, \(\chi([\tau, s_i \cdot \eta]) = 0\), so using (1) we obtain \(\chi([\tau, \eta]) = 0\).

Case 2: \(i \in B(\eta)\). We apply Lemma 4 to see that \(\sigma \in [\tau, \eta]\) implies \(s_i \cdot \sigma \in [\tau, \eta]\). By Lemma 5 it follows that \(\sigma \mapsto s_i \cdot \sigma\) is an involution which swaps elements of odd rank with elements of even rank. Thus \(\chi([\tau, \eta]) = 0\).

Case 3: \(i \in C(\eta)\). This is impossible by Lemma 5.

Thus we have shown that \(\chi([\tau, \eta]) = 0\) for intervals \(\tau < \eta\) not involving \(\hat{0}\).

Now suppose \(\tau = \hat{0}\) and \(0 < \eta\). We may suppose that \(c(\eta) \geq 1\). Then \(B(\eta) \neq 0\).

Let \(i \in B(\eta)\). By Lemma 4, the map \(\sigma \mapsto s_i \cdot \sigma\) establishes an involution on the set \(\{\sigma \in (0, \eta) \mid i \notin C(\sigma)\}\), and this involution swaps the parity of \(c(\sigma)\). Let

\[S = \{\sigma \in (0, \eta) \mid i \in C(\sigma)\text{ or } \sigma = \hat{0}\}\]

It thus suffices to show that \(\chi(S) = 0\). We claim that \(S\) has a unique maximal element \(\kappa\). This would complete the proof since then \(S = [0, \kappa]\) and we may proceed by induction.

We now construct \(\kappa \in P_n\). Let \(G\) be a medial graph representing \(\eta\). The strands \(p_i\) and \(p_{i+1}\) beginning at \(i\) and \(i + 1\) cross each other since \(i \in B(\eta)\). We shall assume \(p_i\) and \(p_{i+1}\) cross each other at \(q\) before intersecting any other strands. Let \(G'\) be obtained from \(G\) by uncrossing \(q\) so that \(i\) is matched directly with \(i + 1\). Let \(\kappa \in P_n\) be represented by \(G''\) (in the sense explained in Section 2).

Suppose \(\sigma \in S\). We need to show that \(\sigma \leq \kappa\). A (lensless) medial graph \(G''\) representing \(\sigma\) can be obtained from \(G\) by uncrossing some subset \(C\) of the crossings of \(G\) (see the comment after [Lam] Lemma 4.12]). Since \(i \in C(\sigma)\) we must have \(q \in C\).

If \(q\) is resolved in \(G''\) in the same way as in \(G'\) then we are done: \(G''\) can be obtained from \(G'\) by the same number of uncrossings, and Lemma 2 gives \(\sigma \leq \kappa\). Now suppose \(q\) is
resolved in $G''$ in the direction different to the one in $G'$. We then observe that the medial graph $G'''$ obtained from $G''$ by resolving the uncrossing $q$ in the other direction also represents the same matching $\sigma$ (a closed interior loop will appear in $G'''$, which can be removed). This completes the proof that $S = [0, \kappa]$, and thus the theorem. □

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