How to project onto the monotone nonnegative cone using Pool Adjacent Violators type algorithms

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Abstract

The metric projection onto an order nonnegative cone from the metric projection onto the corresponding order cone is derived. Particularly, we can use Pool Adjacent Violators-type algorithms developed for projecting onto the monotone cone for projecting onto the monotone nonnegative cone too.

1 Introduction

The metric projection onto convex cones is an important tool in solving problems in metric geometry, statistics, image reconstruction etc. In almost all applications the projection onto a convex cone is part of an iterative process, hence its efficiency is of crucial importance.

Projecting onto an order cone [4] is a fundamental tool for solving isotonic regression problems (see [7]). We will call an order nonnegative cone the intersection of an order cone with the nonnegative orthant. An order nonnegative cone correspond to an isotonic nonnegative regression problem which we will define from an isotonic regression problem by superimposing the nonnegativity of variables.

* 1991 A M S Subject Classification. Primary 90C33; Secondary 15A48, Key words and phrases. Metric projection onto the monotone nonnegative cone
An order nonnegative cone is a pointed cone which is the subcone of the corresponding order cone which is not pointed.

A special case of the isotone regression problem is the case of the regression with respect to a complete order ([7], [1]). The corresponding cone is called monotone cone. It turns out that the corresponding monotone nonnegative cone is important in the metric geometry and the image reconstruction. Whereby the importance of projecting onto this cone.

A simple finite method of projection onto the so-called isotone projection cones (cones having the property that the metric projection onto them is isotone with respect to the order relation defined by these cones) proposed by us (see [6]) has become important in the effective handling of all the problems involving projection onto these cones. The monotone nonnegative convex cone used in the Euclidean distance geometry (see [2]) is an isotone projection one. Our method has become important in the effective handling of the problem of map-making from relative distance information e.g., stellar cartography (see www.convexoptimization.com/wikimization/index.php/Projection_on_Polyhedral_Cone and Section 5.13.2.4 in [2]).

Due to the importance for the regression theory of the projection onto the monotone cone, there are various efficient methods of projection onto it. These methods emerge from the so-called Pool Adjacent Violators (PAV) algorithm, which has nothing to do with the geometric approach of [6]. The PAV algorithm exploits the specific feature of the monotone cone, and due to its efficiency it would be desirable its adaptation for the monotone nonnegative cone too. Our note aims to do this by joining PAV and the geometric approach specific for [6].

In this note we show that the projection of a point onto an order nonnegative cone is the positive part (with respect to the lattice structure of the nonnegative orthant in \(\mathbb{R}^m\)) of the projection of the point onto the corresponding order cone.

### 2 Preliminaries

Let \(W\) be a convex cone in \(\mathbb{R}^m\), i.e., a nonempty set with (i) \(W + W \subset W\) and (ii) \(tW \subset W\), \(\forall t \in \mathbb{R}_+ = [0, +\infty)\).

The cone \(K \subset \mathbb{R}^m\) is said pointed, if \(K \cap (-K) = \{0\}\).

The polar of the convex cone \(W\) is the set

\[
W^\perp := \{y \in \mathbb{R}^m : \langle x, y \rangle \leq 0, \forall x \in K\},
\]

where \(\langle \cdot, \cdot \rangle\) is a scalar product in \(\mathbb{R}^m\).

If \(W\) is a closed convex cone, then from the extended Farkas lemma (or bipolar theorem, see e.g. Theorem 14.1 in [8] p. 121) is

\[
W^{\perp \perp} = (W^\perp)^\perp = W.
\]
If $Z$ is another closed convex cone, then $W$ and $Z$ are called mutually polar if $Z = W^\perp$ (and hence $W = Z^\perp$ by the lemma of Farkas).

The scalar product $\langle \cdot, \cdot \rangle$ defines a metric $d$ on $\mathbb{R}^m$ by setting $d(x, y) = \langle x - y, x - y \rangle^{1/2}$ for any $x, y \in \mathbb{R}^m$. Denote by $P_W : \mathbb{R}^m \to W$ the projection onto the closed convex cone $W$ (or the nearest point mapping), which associates to $x \in \mathbb{R}^m$ its (unique with respect to the metric defined by the scalar product $\langle \cdot, \cdot \rangle$) nearest point $P_W x \in W$.

The projection mapping $P_W$ onto $W$ is characterized by the following theorem of Moreau [5].

Theorem [Moreau] Let $W, Z \subset \mathbb{R}^m$ be two mutually polar convex cones in $H$. Then, the following statements are equivalent:

(i) $z = x + y$, $x \in W$, $y \in Z$ and $\langle x, y \rangle = 0$,

(ii) $x = P_W z$ and $y = P_Z z$.

3 Order cones and order nonnegative cones

Suppose that $\mathbb{R}^m$ is endowed with a Cartesian coordinate system, and $x \in \mathbb{R}^m$, $x = (x^1, \ldots, x^m)$, where $x^i$ are the coordinates of $x$ with respect to this reference system.

Endow the index set $\{1, \ldots, m\}$ with a partial order $\preceq$. The order cone [4] with respect to the partial order $\preceq$ is defined by

$$W_{\preceq} = \{x \in \mathbb{R}^m : x^i \leq x^j \text{ whenever } i \preceq j\}.$$ 

The order nonnegative cone corresponding to the order cone $W_{\preceq}$ is defined by

$$K_{\preceq} = W_{\preceq} \cap \mathbb{R}^m_+.$$ 

We obviously have

$$K_{\preceq} \subset W_{\preceq}. \quad (1)$$

4 Projection onto an order nonnegative cone via the projection onto the corresponding order cone

Let $b = (b^1, \ldots, b^m) \in \mathbb{R}^m$ and $w = (w_1, \ldots, w_m) \in \mathbb{R}_+^m$ a vector of positive weights. The partial order $\preceq$ defines a directed acyclic graph over the nodes $\{1, \ldots, m\}$ such that $(i, j) \in E_{\preceq}$ whenever $i \preceq j$, where $E_{\preceq}$ is the set of edges of the graph. Recall that the isotonic regression problem is the following minimization problem:

Minimize $\sum_{i=1}^m w_i (x_i - b_i)^2$ subject to $x_i \leq x_j$, $\forall (i, j) \in E_{\preceq}$. 

\[\text{3}\]
We define the \textit{isotonic nonnegative regression} problem as the following minimization problem:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^n w_i (x_i - b_i)^2 \\
\text{subject to} & \quad x_i \leq x_j, \quad \forall (i, j) \in E, \\
& \quad x_k \geq 0 \quad \text{for } k = 1, \ldots, m.
\end{align*}
\]

If we define the scalar product \(\langle \cdot, \cdot \rangle\) by \(\langle x, y \rangle = w_1 x_1 y_1 + \cdots + w_m x_m y_m\) for any \(x, y \in \mathbb{R}^m\), then the isotonic regression problem is equivalent to projecting \(b\) onto \(W_\leq\). Similarly, the isotonic nonnegative regression problem is equivalent to projecting \(b\) onto \(K_\leq\). However, the following results hold for any scalar product \(\langle \cdot, \cdot \rangle\) in \(\mathbb{R}^m\).

For any vector \(a = (a^1, \ldots, a^m) \in \mathbb{R}^m\) denote by \(a^+ = \sup \{a, 0\}\) its positive part with respect to the lattice structure of \(\mathbb{R}^+_m\). Thus, if \(m = 1\), then \(a^+ = \max \{a, 0\}\) and if \(m > 1\), then \(a^+ = (a^1^+, \ldots, a^m^+)\), where \(a^i^+ = (a^i)^+\) for any \(i \in \{1, \ldots, m\}\). Similarly denote by \(a^- = \sup \{-a, 0\}\) its negative part with respect to the lattice structure of \(\mathbb{R}^+_m\). Thus, if \(m = 1\), then \(a^- = \max \{-a, 0\}\) and if \(m > 1\), then \(a^- = (a^1^-, \ldots, a^m^-)\), where \(a^i^- = (a^i)^-\) for any \(i \in \{1, \ldots, m\}\). It is easy to see that \(a = a^+ - a^-\).

Using the notation in the preceding section we have the following result:

\textbf{Theorem 1} For any \(y \in \mathbb{R}^m\) we have

\[P_{K_\leq} y = (P_{W_\leq} y)^+.\]

\textbf{Proof.} We have from Moreau’s theorem that

\[y = P_{W_\leq} y + P_{W_\leq}^+ y, \quad \langle P_{W_\leq} y, P_{W_\leq}^+ y \rangle = 0. \tag{2}\]

Denote \(v = (P_{W_\leq} y)^+\). Using the notations in the theorem, let \(u = P_{W_\leq} y - v\), that is,

\[u = -(P_{W_\leq} y)^-.\]

Denote \(z = P_{W_\leq}^+ y\). Then, (2) becomes

\[y = u + v + z \quad \text{with} \quad \langle u + v, z \rangle = 0. \tag{3}\]

First we show that from the special form of \(W_\leq\), we have \(u, v \in W_\leq\). Indeed, if we denote \(x = P_{W_\leq} y\), then

\[u = (-x^1^-, \ldots, -x^m^-)\]

and

\[v = (x^1^+, \ldots, x^m^+).\]

It is easy to see that the functions \(\mathbb{R} \ni t \mapsto -t^-\) and \(\mathbb{R} \ni t \mapsto t^+\) are monotone increasing. Since \(x \in W_\leq\) we have \(x^i \leq x^j\) whenever \(i \preceq j\). Hence, by using the monotonicity of \(\mathbb{R} \ni t \mapsto -t^-\), we get \(u^i \leq u^j\) whenever \(i \preceq j\). Similarly, by using the monotonicity of \(\mathbb{R} \ni t \mapsto t^+\), we get \(v^i \leq v^j\) whenever \(i \preceq j\). Thus, \(u, v \in W_\leq\) which together with (3) and \(z \in W_\leq^+\) yield

\[\langle u, z \rangle = \langle v, z \rangle = 0. \tag{4}\]
From $K_\preceq \subset W_\preceq$ (see (1)), it follows that
\[ W_\preceq \subset K_\preceq. \tag{5} \]

From the fact that all the coordinates of $u$ are nonpositive and the elements in $K_\preceq$ have nonnegative coordinates, it follows that
\[ u \in K_\preceq. \tag{6} \]

Let us write now
\[ y = v + (u + z) \]
and observe that $v \in K_\preceq$. Further $u + z \in K_\perp$ from (5) and (6). We also have
\[ \langle v, u + z \rangle = \langle v, u \rangle + \langle v, z \rangle = 0, \]
because of (4) and the forms of $u$ and $v$.

Using again the theorem of Moreau, it follows the conclusion of the theorem.

\[ \square \]

5 Projection onto the monotone nonnegative cone via the projection onto the monotone cone

There are various efficient methods for projecting onto the monotone cone emerging from the PAV algorithm (see e. g. [1]). These methods are intimately related to the special structure of the monotone cone, and their justification is more function theoretic than geometric. Due to their simplicity their usage is desirable for projection onto the monotone nonnegative cone too.

As a consequence of Theorem 1 we have the following corollary;

Corollary 1 Suppose that $W$ is the monotone cone, that is,
\[ W = \{ x \in \mathbb{R}^m : x^1 \leq x^2 \leq \ldots \leq x^m \}, \]
and
\[ K = \{ x \in \mathbb{R}^m : 0 \leq x^1 \leq x^2 \leq \ldots \leq x^m \} \]
is the monotone nonnegative cone. Then for an arbitrary $y \in \mathbb{R}^m$ it holds
\[ P_Ky = (P_Wy)^+, \]
where $^+$ stands for the lattice operation defined by the order induced by the nonnegative orthant in $\mathbb{R}^m$. 

To exploit the efficiency of PAV-type algorithms in projecting onto the monotone nonnegative cone, we can proceed as follows: For an arbitrary $y \in \mathbb{R}^m$ we can determine $P_W y = (x^1, \ldots, x^m)$, the projection of $y$ on the monotone cone $W$, by using a PAV-type algorithm (e.g. the algorithm in [3]). Then, take the vector $$v = (x^{1+}, \ldots, x^{m+}),$$ where $x^{i+}$ denotes the “positive part” of the coordinate $x^i$. Then, according the above corollary, the projection of $y$ on the monotone nonnegative cone $K \subset W$ engendering $W$ is given by $$P_K y = v.$$

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