SOME UNIQUENESS RESULTS IN QUASILINEAR SUBHOMOGENEOUS PROBLEMS

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Abstract. We establish uniqueness results for quasilinear elliptic problems through the criterion recently provided in [6]. We apply it to generalized \( p \)-Laplacian subhomogeneous problems that may admit multiple nontrivial nonnegative solutions. Based on a generalized hidden convexity result, we show that uniqueness holds among strongly positive solutions and nonnegative global minimizers. Problems involving nonhomogeneous operators as the so-called \((p,r)\)-Laplacian are also treated.

1. Introduction and main results

A rather general criterion has been recently formulated in [6] to prove uniqueness results for positive critical points of a given functional. Roughly speaking, the authors take advantage of a basic convexity principle, namely, the fact that if \( f : [0,1] \to \mathbb{R} \) is differentiable and strictly convex then it has at most one critical point. This principle is then applied to \( f := I \circ \gamma \), where \( I \) is a given functional and \( \gamma \) is a path connecting two hypothetical critical points of \( I \). This uniqueness criterion finds one of its several applications in the generalized \( p \)-Laplacian problem

\[
\begin{aligned}
- \text{div} \left( h(|\nabla u|^p) |\nabla u|^{p-2} \nabla u \right) &= g(x, u) \quad \text{in } \Omega, \\
\frac{1}{p} H(|\nabla u|^p) - G(x, u) &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

which is the Euler-Lagrange equation associated to the functional

\[
I(u) := \int_\Omega \left( \frac{1}{p} H(|\nabla u|^p) - G(x, u) \right), \tag{1.1}
\]

acting in the Sobolev space \( W^{1,p}_0(\Omega) \). Here \( p \in (1,\infty) \), \( \Omega \) is a bounded and smooth domain of \( \mathbb{R}^N \) \( (N \geq 1) \), \( H(t) := \int_0^t h(s)ds \), and \( G(x, t) := \int_0^t g(x, s)ds \), where

(H1) \( h : [0, \infty) \to [0, \infty) \) is continuous, bounded and nondecreasing,

and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous and satisfies

\[
\begin{align*}
(A1) \quad |g(x, t)| &\leq C(1 + |t|^\sigma) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R} \quad \text{and some } C, \sigma > 0 \quad \text{with} \\
\quad \sigma(N - p) &\leq (p - 1)N + p, \\
(A2) \quad \text{For every } x \in \Omega \text{ the map } t \mapsto \frac{g(x, t)}{|t|^{p-1}} \text{ is nonincreasing in } (0, \infty).
\end{align*}
\]
Moreover, it is assumed that

(A3) Any nonnegative critical point of $I$ is continuous on $\overline{\Omega}$, and any two positive critical points $u, v$ of $I$ satisfy

$$\delta^{-1}v \leq u \leq \delta v \quad \text{in } \Omega,$$

for some $\delta \in (0, 1)$.

The next result follows from [6, Theorem 4.1]:

**Theorem 1.1.** Under the above conditions on $h$ and $g$, let $A$ be the set of positive critical points of $I$.

1. If $h > 0$ in $(0, \infty)$ then $A \subset \{\alpha u_0 : \alpha \geq 0\}$ for some $u_0 \in A$.

2. If either $h$ is increasing or the map in (A2) is decreasing, then $A$ is at most a singleton.

**Notation and terminology.** Before proceeding, let us fix some terminology. We say that $f$ is a nondecreasing (respect. increasing) map on $\mathbb{R}$ if $f(t) \leq f(s)$ (respect. $f(t) < f(s)$) for any $t < s$. We say that $u \in C(\overline{\Omega})$ is nonnegative (respect. positive) if $u(x) \geq 0$ (respect. $u(x) > 0$) for all $x \in \Omega$. If $u$ is a measurable function, inequalities involving $u$ are understood holding a.e., and integrals involving $u$ are considered with respect to the Lebesgue measure. A critical point is said nontrivial if it is not identically zero.

Let us comment on the assumptions of Theorem 1.1. First of all, (H1) and (A1) ensure that $I$ is a $C^1$ functional whose critical points are weak solutions of $(P)$.

Given $u, v \in W^{1,p}_0(\Omega)$ positive, set

$$\gamma_p(t) := ((1 - t)u^p + tv^p)^{\frac{1}{p}}, \quad t \in [0, 1].$$

Since $h$ is nondecreasing, the map $t \mapsto \int_{\Omega} H(|\nabla \gamma_p(t)|^p)$ is convex on $[0, 1]$. Moreover, it is strictly convex if $u \neq v$ and $h$ is increasing or the map in (A2) is decreasing, cf. [6, Lemma 4.3]. Still by (A2), the map $t \mapsto G(x, t^p)$ is concave on $[0, \infty)$ and consequently $I$ is strictly convex along $\gamma_p$. Finally, (A3) guarantees that $\gamma_p$ is Lipschitz continuous at $t = 0$ and $t = 1$. It follows that if $I'(u) = I'(v) = 0$ then $t \mapsto I(\gamma_p(t))$ is differentiable on $[0, 1]$, with $t = 0$ and $t = 1$ as critical points, which is impossible for a strictly convex map. The same contradiction arises if $u, v$ are linearly independent and $h > 0$ in $(0, \infty)$, as in this case $I$ is still strictly convex along $\gamma_p$.

**Remark 1.2.**

1. If $h \equiv 1$ then (A3) is a consequence of (A1) and (A2). Indeed, first note that $g(x, 0) \geq 0$ in $\Omega$, since otherwise for some $x \in \Omega$ we have $\frac{g(x, t)}{t^p} \to -\infty$ as $t \to 0^+$, which is clearly impossible by (A2). We deduce that for any $t_0 > 0$ there exists $M > 0$ such that $g(x, t) + Mt^{p-1} \geq 0$ in $\Omega \times [0, t_0]$, in which case the strong maximum principle and Hopf Lemma apply, cf. [28]. In particular, the second assertion in (A3) is satisfied for any nonnegative critical points $u, v \neq 0$, so that in this case Theorem 1.1 holds with $A$ redefined as the set of nontrivial nonnegative critical points of $I$. Note also that by (A1) any such point belongs to $C^1(\overline{\Omega})$, cf. [13, 21].

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1. This property, known as (generalized) hidden convexity, plays an important role in several uniqueness results, and has connections with Hardy and Picone inequalities, as shown in [7].
(2) When dealing with $I$ defined in $W^{1,p}(\Omega)$, Theorem 1.1 needs the following modification:

(a) If the map in (A2) is decreasing then $A$ is at most a singleton.
(b) If $h$ is increasing and $A$ contains a nonconstant element, then it is a singleton.

As a matter of fact, one may easily find some $g$ and $h$ satisfying (H1) and (A1)-(A3) with $h$ increasing and $g(x,u) \equiv 0$ for $u$ in some interval of positive constants, in which case $A$ has infinitely many elements. The uniqueness result ‘modulo positive constants’ in (b) is due to the fact that the map $t \mapsto \int_{\Omega} H(|\nabla \gamma_{p}(t)|^{p})$ is identically zero and therefore not strictly convex if $u \not\equiv v$ are positive constants, see Remark 2.2 below.

Theorem 1.1 provides an alternative proof of some classical results for

$$\begin{cases}
-\Delta_{p}u = g(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$  \hspace{1cm} (1.2)

which corresponds to (P) with $h \equiv 1$. Indeed, in [6, Examples 4.4 and 4.6] it is shown that

- $A$ is a singleton if $g(x,u) = u^{p-1}$ and $1 < q < p$ (the subhomogeneous problem [12], or sublinear if $p = 2$ [8]).
- $A$ is one-dimensional if $g(x,u) = \lambda_{1}u^{p-1}$ (the homogeneous or eigenvalue problem [2]), where

$$\lambda_{1} = \lambda_{1}(p) := \inf \left\{ \int_{\Omega} |\nabla u|^{p} : u \in X, \int_{\Omega} |u|^{p} = 1 \right\}$$  \hspace{1cm} (1.3)

is the first eigenvalue of the $p$-Laplacian on $X$. In this case $A$ is the set of positive eigenfunctions associated to $\lambda_{1}$.

Theorem 1.1 also applies to (1.2) if

$$g(x,u) = a(x)u^{q-1} \quad \text{with } 1 < q < p. \hspace{1cm} (1.4)$$

and $a \in C(\overline{\Omega})$, $a \geq 0$, $\not\equiv 0$. On the other hand, it is clear that (A2) fails if $a$ is negative in some part of $\Omega$. In this case it is known that (P) may have dead core solutions, i.e. solutions vanishing in some open subset of $\Omega$ [11, Proposition 1.11], as well as solutions reaching some part of the boundary with null normal derivative [15, Proposition 2.9], which shows that (A3) is not satisfied either.

Our first purpose is to deal with a class of nonlinearities including (1.4) with a sign-changing. More precisely, instead of (A2) we shall assume that

(A2') There exists $q \in (1,p)$ such that for a.e. $x \in \Omega$ the map $t \mapsto \frac{g(x,t)}{t^{q-1}}$ is nonincreasing in $(0,\infty)$.

We consider the functional $I$ defined either in $W^{1,p}_{0}(\Omega)$ or $W^{1,p}(\Omega)$. To overcome (A3) we restrict ourselves to strongly positive critical points, i.e. critical points lying in

$$\mathcal{P}^{o} := \begin{cases}
\{ u \in C^{1}_{0}(\Omega) : u > 0 \text{ in } \Omega, \partial_{\nu}u < 0 \text{ on } \partial\Omega \} & \text{if } X = W^{1,p}_{0}(\Omega), \\
\{ u \in C^{1}(\overline{\Omega}) : u > 0 \text{ on } \overline{\Omega} \} & \text{if } X = W^{1,p}(\Omega),
\end{cases}$$

as a matter of fact, one may easily find some $g$ and $h$ satisfying (H1) and (A1)-(A3) with $h$ increasing and $g(x,u) \equiv 0$ for $u$ in some interval of positive constants, in which case $A$ has infinitely many elements. The uniqueness result ‘modulo positive constants’ in (b) is due to the fact that the map $t \mapsto \int_{\Omega} H(|\nabla \gamma_{p}(t)|^{p})$ is identically zero and therefore not strictly convex if $u \not\equiv v$ are positive constants, see Remark 2.2 below.
where $\nu$ is the outward unit normal to $\partial \Omega$. Such restriction is not technical, as $I$ may have multiple nonnegative critical points if $h \equiv 1$ and $g$ is given by $[1,4]$ with a sign-changing, cf. [19] Remark 1.3(3)]. See also [33, 4] and [10] Theorem 1.4(ii) and [17] Proposition 5.1] for $p = 2$. On the other hand, $I$ has at most one positive critical point for such $h$ and $g$, cf. [3, 12] for $p = 2$ and [18] for $p > 1$. Let us note that a condition similar to (A2') with $p = 2$ appears in [12].

Let $X = W^{1, p}_0(\Omega)$. In association with (A2'), we shall consider the path

$$\gamma_q(t) := ((1 - t)u^q + tv^q)^{\frac{1}{q}}, \quad t \in [0, 1],$$

(1.5) to connect two critical points $u, v \in P^\circ$. This path has been mostly used when $q = p$, but for $q < p$ one can find the expression $\gamma_q(1/2) = (u^{q/p} + v^{q/p})^{\frac{1}{q}}$ in uniqueness arguments for sublinear or subhomogeneous type problems [18, 19, 20, 25]. Note that $\gamma_q$ is Lipschitz continuous at $t = 0$ for such $u, v$ [6, Corollary 3.3]. By Lemma 2.1 below, if $h > 0$ in $(0, \infty)$ and $u \not\equiv v$ then $t \mapsto \int_\Omega |\nabla \gamma_q(t)|^p$ is strictly convex on $[0, 1]$. (A2') provides then the strict convexity of $t \mapsto I(\gamma_q(t))$. Furthermore, when dealing with global minimizers of $I$, uniqueness holds not only within $P^\circ$, but more generally among nonnegative functions. For such $u, v$ the path $\gamma_q$ is not necessarily Lipschitz continuous at $t = 0$, yet $I$ is strictly convex along $\gamma_q$ for $u \not\equiv v$.

In particular, the inequality

$$I(\gamma_q(t)) < (1 - t)I(u) + tI(v)$$

holds for any $t \in (0, 1)$, and the global minimality of $I(u) = I(v)$ eventually yields a contradiction. Lastly, let us mention that if $X = W^{1, p}(\Omega)$ then the above arguments hold assuming moreover that either $u$ or $v$ is nonconstant.

Our second purpose is to deal with problems involving nonhomogeneous operators, as $-\Delta_p - \Delta_r$ with $1 < p < r$, in which case the associated functional is

$$I(u) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p + \frac{1}{r} |\nabla u|^r - G(x, u) \right),$$

defined on $W^{1, r}(\Omega)$ or $W^{1, r}_0(\Omega)$. Some uniqueness results have been proved in [3, 14, 23, 27] for this class of problems under Dirichlet boundary conditions. Note that this functional corresponds to (1.1) with $h(t) = 1 + t^{\frac{r}{p} - 1}$.

More generally, we shall assume that

$$(H1') \quad h : [0, \infty) \to [0, \infty) \text{ is continuous, nondecreasing, and } h(t) \leq C(1 + t^{r-1})$$

for some $C > 0$, $r > p$, and any $t \geq 0$,

and, instead of (A1),

$$(A1') \quad |g(x, t)| \leq C(1 + |t|^{\sigma}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R} \text{ and some } C, \sigma > 0 \text{ with } \sigma(N - r) \leq (r - 1)N + r.$$

It is clear that $I$ given by (E4) is a $C^1$ functional on $W^{1, r}(\Omega)$ under (H1') and (A1'), and these conditions are weaker than (H1) and (A1), respectively. Finally, in view of Remark 12(1) we strengthen (A3) as follows:
(A3') Any nonnegative critical point of $I$ is continuous on $\overline{\Omega}$, and any two such points $u, v \neq 0$ satisfy $0 < \delta^{-1}v \leq u \leq \delta v$ in $\Omega$, for some $\delta \in (0, 1)$.

The next subsection contains the statement of our results, which are proved in section 2. We apply our results to a few particular problems and recall some known uniqueness results in section 3.

1.1. Statement of our results. From now on $I$ is the functional given by (1.1), and $A$ and $B$ are the sets of nontrivial nonnegative critical points and nonnegative global minimizers of $I$, respectively.

Theorem 1.3. Let $h$ satisfy (H1) and $h > 0$ in $(0, \infty)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (A1) and (A2').

(1) If $X = W^{1,p}_0(\Omega)$ then $A \cap P^o$ and $B$ are at most a singleton.

(2) If $X = W^{1,p}(\Omega)$ and $A \cap P^o$ contains a nonconstant element then it is a singleton. The same conclusion holds for $B$.

Remark 1.4.

(1) As already observed, the assertions on $B$ rely essentially on (1.6), rather than the strict convexity of $I$ along $\gamma$. Even more, (1.6) satisfied at some $t \in (0, 1)$ suffices to reach a contradiction. This inequality with $t = 1/2$ has been used in [13, 15] to show that $B$ is a singleton if $h \equiv 1$ and $g(x, u) = \lambda u^{p-1} + a(x)u^{q-1}$, with $q \in (1, p)$, $\lambda \leq 0$ and $a$ sign-changing.

(2) One can extend the results on $B$ to deal with local minimizers as follows: assume that $C := \{w \in X : I'(w) = 0, I(w) = c\}$ contains a local minimizer $u$ (nonconstant if $X = W^{1,p}(\Omega)$) for some $c \in \mathbb{R}$. If for any $v \in C$ with $u \neq v$ the inequality (1.6) holds for every $t \in (0, 1)$, then $C = \{u\}$. This can be seen as a particular case of [6, Theorem 2.2], without requiring $\gamma$ to be Lipschitz continuous at $t = 0$.

(3) As discussed in Remark 1.2(2), the assumption that $A \cap P^o$ contains a nonconstant element is natural when $X = W^{1,p}(\Omega)$. One can remove this condition assuming in addition that the map in (A2') is decreasing, or that $g(x, c) \equiv 0$ for any $c > 0$. Note also that if $X = W^{1,p}(\Omega)$ and $g$ satisfies (A1), (A2'), and $g(x, c) \equiv 0$ for exactly one constant $c > 0$, then $A \cap P^o = \{c\}$. Indeed, in such case there is no other positive constant solving (P), and by Theorem 1.3 we deduce that $I$ has no further critical point in $A \cap P^o$. This situation occurs for instance if $g(x, u) = u^{q-1} - u^{r-1}$ with $q \in (1, p)$ and $r > q$, in which case $A \cap P^o = \{1\}$.

(4) Under the assumptions of Theorem 1.3 the sets $A \cap P^o$ and $B$ may be empty, as we shall see in section 3. However, when $X = W^{1,p}_0(\Omega)$ the set $B$ is nonempty if we assume moreover that $h$ is bounded away from zero and $g(x, t)$ is odd with respect to $t$ or $g(x, t) = 0$ for $t < 0$. Indeed, from (A2') we deduce that $G(x, t) \leq C_2(1 + |t|^p)$ for some $C_2 > 0$ and every $t \in \mathbb{R}$, so that $I$ is coercive, weakly lower-semicontinuous and therefore has a global minimum on $W^{1,p}_0(\Omega)$. Finding sufficient conditions to have $A \cap P^o$
nonempty is a more delicate issue. Such conditions have been provided for $g(x, u) = \lambda u^{p-1} + a(x)u^{q-1}$, with $1 < q < p$, $\lambda \leq 0$ and $a$ sign-changing in [8, 19], see section 3.

Next we consider $h$ and $g$ under (H1') and (A1'). First we observe that Theorem 4.1 can be easily extended to this situation. Let us recall that $I$ given by (1.1) acts now on $W^{1,r}(\Omega)$ or $W^{1,r}_0(\Omega)$.

**Theorem 1.5.** Let $h$ satisfy (H1'), and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (A1'), (A2), and (A3').

1. If $h > 0$ in $(0, \infty)$ then $A \subset \{\alpha u_0 : \alpha \geq 0\}$ for some $u_0 \in A$.
2. If the map in (A2) is decreasing then $A$ is at most a singleton.
3. If $h$ is increasing and $A$ contains a nonconstant element, then it is a singleton.

The extension of Theorem 1.3 reads as follows:

**Theorem 1.6.** Let $h$ satisfy (H1') and $h > 0$ in $(0, \infty)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (A1') and (A2').

1. If $X = W^{1,r}_0(\Omega)$ then $A \cap \mathcal{P}^o$ and $B$ are at most a singleton.
2. If $X = W^{1,r}(\Omega)$ and $A \cap \mathcal{P}^o$ contains a nonconstant element then it is a singleton. The same conclusion holds for $B$.

**Remark 1.7.** In the same way as in Theorem 1.3, the condition that $A$ contains a nonconstant element can be removed from Theorem 1.5(3) if the map in (A2) is decreasing or $g(x, c) \not\equiv 0$ for any $c > 0$. A similar remark applies to $A \cap \mathcal{P}^o$ and $B$ in Theorem 1.6.

Finally, let us mention that in view of their variational nature, Theorems 1.3, 1.5, and 1.6 can be adapted to obtain uniqueness results for the equation

$$-\text{div}(h(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = g(x, u) \quad \text{in} \quad \Omega$$

under nonlinear or mixed boundary conditions (as shown in [7, Examples 4.7 and 4.8]). We refer to [24] for similar results in the semilinear case $h \equiv 1$ and $p = 2$.

2. **Proofs**

The proofs are based on [6, Theorem 1.1] and the following lemma, which sharpens the general hidden convexity established in [7, Proposition 2.6] by analyzing the strict convexity of the map $t \mapsto \int_{\Omega} H(|\nabla u|^p(t))$. Moreover $H$ is not assumed to be homogeneous:

**Lemma 2.1.** Let $1 < q < p$, and $u, v \in W^{1,p}(\Omega)$ with $u, v \geq 0$ in $\Omega$.

1. Let $\gamma_q$ be given by (1.3). Then

$$|\nabla \gamma_q(t)|^p \leq (1 - t)|\nabla u|^p + t|\nabla v|^p \quad \forall t \in [0, 1].$$

Moreover, for $t \in (0, 1)$ the strict inequality holds in the set

$$\tilde{Z}(u, v) := \{x \in \Omega : u(x) \neq v(x), |\nabla u(x)| + |\nabla v(x)| > 0\}.$$
Remark 2.2.

Proof.

(1) It is enough to prove \[22\] in the set

\[ Z = Z(u, v) := \{ x \in \Omega : |\nabla u(x)| + |\nabla v(x)| > 0 \}, \]

since outside \( Z \) one has \( \nabla \gamma_q(t) = 0 \). We claim that [1] Lemma 3.5] applies with \( Q(t) = t^q \) and \( M(t) = t^p \). Indeed, in this case \( F_1(t) = qt^{1-\frac{q}{p}} \) and \( F_2(t) = t^p \). One may easily check that \( F(z_1, z_2) := F_1(z_1)F_2(z_2) \) is concave in \( [0, \infty) \times [0, \infty) \) and strictly concave in \( (0, \infty) \times (0, \infty) \). Arguing as in the proof of (3.9) in [1] Lemma 3.5] we find that

\[ F((1-t)z + t\bar{z}) > (1-t)F(z) + tF(\bar{z}) \]

if \( t \in (0,1), z \in [0, \infty) \times [0, \infty), \) and \( \bar{z} \in (0, \infty) \times (0, \infty) \) with \( z \neq \bar{z} \). Applying this inequality to \( z = (Q(u), M(|\nabla u|)) \) and \( \bar{z} = (Q(v), M(|\nabla v|)) \) we infer that [22] holds with strict inequality in \( \tilde{Z}(u, v) \).

(2) Let \( t, s, \alpha \in [0,1] \). Note that

\[ \gamma_q((1-\alpha)t + \alpha s) = ((1-\alpha)\gamma_q(t)^q + \alpha\gamma_q(s)^q)^{\frac{1}{q}} \]

so applying [22] with \( u, v \) replaced by \( \gamma_q(t), \gamma_q(s) \) respectively, we find that

\[ |\nabla \gamma_q((1-\alpha)t + \alpha s)|^p \leq (1-\alpha)|\nabla \gamma_q(t)|^p + \alpha|\nabla \gamma_q(s)|^p, \]

with strict inequality in \( \tilde{Z}(\gamma_q(t), \gamma_q(s)) \). Since \( H \) is nonincreasing and convex, we have

\[ \int_{\Omega} H(|\nabla \gamma_q((1-\alpha)t + \alpha s)|^p) \leq (1-\alpha) \int_{\Omega} H(|\nabla \gamma_q(t)|^p) + \alpha \int_{\Omega} H(|\nabla \gamma_q(s)|^p). \]

Assume by contradiction that equality holds for some \( t \neq s \) and \( \alpha \in (0,1) \). Note that \( h > 0 \) yields that \( H \) is increasing in \( (0, \infty), \) so that such equality holds only if \( \tilde{Z}(\gamma_q(t), \gamma_q(s)) \) is null. It follows that \( \nabla \gamma_q(t) \equiv \nabla \gamma_q(s), \) so \( \gamma_q(t) \equiv \gamma_q(s) + c \) for some \( c \in \mathbb{R}. \) If \( X = W_0^{1,p}(\Omega) \) then \( c = 0 \) and \( u \equiv v. \) If \( X = W^{1,p}(\Omega) \) then we can assume that \( c > 0, \) i.e. \( \gamma_q(t) > \gamma_q(s). \) It follows that \( |\nabla \gamma_q(t)| + |\nabla \gamma_q(s)| = 0 \) a.e., so that \( \gamma_q(t), \gamma_q(s) \) are constant functions. Consequently \( u, v \) are constant, and we reach a contradiction.

\[ \square \]

Remark 2.2.

(1) Lemma [22] has been proved for \( t = 1/2 \) in [20]. Using this inequality the authors show that the set of nonnegative minimizers for the Rayleigh quotient \( \frac{\int_{\Omega} |\nabla u|^p}{(\int_{\Omega} a(x)|u|^q)^{\frac{p}{q}}} \) in \( W_0^{1,p}(\Omega) \) is one-dimensional if \( 1 < q \leq p, \) with \( a \)
allowed to change sign.

(2) As shown in [6, Lemmas 3.9 and 4.3], the inequality (2.1) and the convexity of $t \mapsto \int_\Omega H(|\nabla \gamma_q(t)|^p)$ still hold for $q = p$. However, this map is strictly convex if either $h > 0$ and $u, v$ are linearly independent or $h$ is increasing and $u \neq v$, with one of them nonconstant when $X = W^{1,p}(\Omega)$. Indeed, as shown in the proof of [6, Lemma 4.3](2), if $t \mapsto \int_\Omega H(|\nabla \gamma_p(t)|^p)$ is not strictly convex then $u, v$ are linearly dependent, and for $h$ increasing we deduce that $|\nabla \gamma_p(t_1)| = |\nabla \gamma_p(t_2)|$ for some $t_1 \neq t_2$. It follows that $u \equiv v$ or $\nabla u \equiv \nabla v \equiv 0$, i.e. $u \equiv v$ or $u, v$ are positive constants.

**Proof of Theorem 1.3.** Let us first prove the assertions on $A \cap P^\circ$. We adapt the proof of [6, Theorem 4.1], which is based on [6, Theorem 1.1]. Assume by contradiction that $u, v \in P^\circ$ are critical points of $I$ with $u \neq v$. First note that $\gamma_q$ is locally Lipschitz at $t = 0$ [6, Corollary 3.3] since $u, v$ are comparable in the sense of (3.2), as shown in [6, Lemma 3.4]. Assuming that $u$ is nonconstant if $X = W^{1,p}(\Omega)$, Lemma 2.1 yields that $t \mapsto \int_\Omega H(|\nabla \gamma_q(t)|^p)$ is strictly convex on $[0, 1]$. (A2') yields that $t \mapsto G(x, t\hat{x})$ is concave on $[0, \infty)$, so $t \mapsto I(\gamma_q(t))$ is strictly convex, and its derivative vanishes at $t = 0$ and $t = 1$, which is impossible.

Let us now deal with $B$. Assume that $u, v \in B$ and $u \neq v$ (with at least one of them nonconstant when $X = W^{1,p}(\Omega)$). By Lemma 2.1 the map $t \mapsto I(\gamma_q(t))$ is strictly convex on $[0, 1]$, so (1.6) holds for any $t \in (0, 1)$, which is clearly impossible since $u, v$ minimize $I$ globally. \hfill \Box

**Proof of Theorem 1.5.** One can repeat the proof of [6, Theorem 4.1] with the same path $\gamma_p$, and replacing $W^{1,p}(\Omega)$ by $W^{1,\hat{r}}(\Omega)$. By (H1') and (A1') the functional $I$ is $C^1$ on $W^{1,\hat{r}}(\Omega)$. Note also that [6, Lemma 4.3(ii)] holds in $W^{1,\hat{r}}(\Omega)$ (or $W^{1,p}(\Omega)$) if we assume in addition that either $u$ or $v$ is nonconstant, cf. Remark 2.2. \hfill \Box

**Proof of Theorem 1.6.** By (H1’) we have $0 \leq H(t) \leq C(1 + t\hat{x})$ for some $C > 0$, and any $t \geq 0$. Note that (2.1) holds also with $p$ replaced by $r$, so that Lemma 2.1(2) holds in particular for $u, v \in X$, with $X = W^{1,\hat{r}}(\Omega)$ or $X = W^{\hat{r}}_0(\Omega)$. We can then repeat the proof of Theorem 1.3 to get the desired conclusions. \hfill \Box

3. Examples

Next we apply our results to some specific problems. Below we assume that $a, b \in L^{\infty}(\Omega)$, and $p^*$ is the critical Sobolev exponent, i.e. $p^* = Np/(N - p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

(1) First we apply Theorem 1.3 to $h \equiv 1$, i.e. to the problem (1.2). Let us observe that in the following cases the strong maximum principle and the Hopf lemma do not apply.

(a) $g(x, u) = a(x)u^{q-1}$ with $q \in (1, p)$ and $a$ sign-changing.

Note that $g(x, c) \neq 0$ for any constant $c > 0$, so we deduce that $A \cap P^\circ$ and $B$ are at most a singleton if $X = W^{1,p}(\Omega)$ or $X = W^{1,p}_0(\Omega)$.  


When $p = 2$ and $a$ is smooth, the fact that $A \cap \mathcal{P}^o$ is at most a singleton has been proved in [4] for $X = H^1(\Omega)$, and in [12] for $X = H^1_0(\Omega)$. These results hold for classical solutions, and are proved via a change of variables and the maximum principle. In [18] Theorem 1.1] it is shown, for $p > 1$, that $B$ is a singleton (assuming that $\int_\Omega a < 0$ if $X = W^{1,p}(\Omega)$) and $A \cap \mathcal{P}^o \subset B$. It is also shown in [18] Theorem 1.2 and Remark 2.10] that $A \cap \mathcal{P}^o$ is a singleton if either $q$ is close enough to $p$ or $X = W^{1,p}_0(\Omega)$ and $a^−$ is sufficiently small. When $a$ is too negative in some part of $\Omega$ every nonnegative critical point has a dead core [11] Proposition 1.11], i.e. a region where it vanishes. In this situation no positive solution exists, so that $A \cap \mathcal{P}^o$ is empty, but $B$ is a singleton. Finally, both $A \cap \mathcal{P}^o$ and $B$ are empty when $\int_\Omega a > 0$ and $X = W^{1,p}(\Omega)$. Indeed, in this case it is clear that the functional is unbounded from below (over the set of constant functions), and taking $u^{1−q}$ as test function we see that $\int_\Omega a < 0$ if $u \in A \cap \mathcal{P}^o$.

(b) $g(x, u) = a(x)u^{q−1} + b(x)u^{r−1}$ with $q \in (1, p)$, $r \in (q, p^\ast)$, a sign-changing and $b \leq 0$, $\neq 0$.

For such $g$ one may easily show that $I$ is coercive. Moreover $g(x, c) \neq 0$ for any constant $c > 0$, so that $B$ is a singleton and $A \cap \mathcal{P}^o$ is at most a singleton.

Following the approach of [4], the fact that $A \cap \mathcal{P}^o$ is at most a singleton has been proved in [1] for $X = H^1(\Omega)$, $r = p = 2$, and $b \equiv \lambda < 0$. In [10] the results of [18] were extended to any $r = p$ and $b \equiv \lambda < 0$. In this case $B$ is a singleton [18] Theorem 1.1], whereas $A \cap \mathcal{P}^o$ is a singleton if either $q$ is close enough to $p$ or $a^−$ is sufficiently small [18] Theorem 1.6 and Proposition 4.6], and it is empty if $a$ is negative enough in some part of $\Omega$ [18] Theorem 1.8]. When $X = W^{1,p}(\Omega)$, $\int_\Omega a < 0$, and $q$ is close to $p$, the condition $\lambda \leq 0$ is optimal for the uniqueness in $A \cap \mathcal{P}^o$, as $I$ has at least two critical points in $\mathcal{P}^o$ for $\lambda > 0$ small [18] Theorem 1.4]. Finally, it is not difficult to see that $A \cap \mathcal{P}^o$ and $B$ are empty if $\lambda > 0$ is large enough.

Let us note that in [10] it is claimed that $B$ is a singleton for $r = 2q$ and $b \equiv −1$, but we have not found a proof of this uniqueness result.

(c) $g(x, u) = a(x)u^{q−1} + b(x)u^{r−1}$ with $1 \leq r < q < p$, $a^− \neq 0$, and $b \geq 0$, $\neq 0$.

In this case $B$ is a singleton if we assume that $\int_\Omega a < 0$ when $X = W^{1,p}(\Omega)$. If $a$ is sign-changing then $A \cap \mathcal{P}^o$ is a singleton when $q$ is close enough to $p$ or $X = W^{1,p}_0(\Omega)$ and $a^−$ is sufficiently small. Indeed, as mentioned in (1), in this case the equation $−\Delta_p u = a(x)u^{q−1}$ has exactly one solution in $\mathcal{P}^o$, and since $b \geq 0$ this solution is a subsolution for $−\Delta_p u = g(x, u)$. Since this problem has arbitrarily large supersolutions, we obtain a solution in $\mathcal{P}^o$ by the sub-supersolutions method, so that $A \cap \mathcal{P}^o$ is a singleton.
We apply now Theorems 1.5 and 1.6 to \( h(t) = 1 + t^{\frac{r}{p}} - 1 \), with \( 1 < p < r \). Note that for such \( h \) and \( g \) satisfying \((A1')\), critical points of \( I \), defined on \( W^{1,r}(\Omega) \) or \( W^{1,r}_0(\Omega) \), belong to \( C^1(\Omega) \), cf. [22].

(a) \( g(x, u) = a(x) u^{q-1} - b(x) u^{p-1} \), with \( q > p \) and \( b \geq 0 \).

Let us first assume \( q < r^* \). The strong maximum principle and the Hopf lemma [26, Theorems 5.3.1 and 5.5.1] apply to \((P)\), so \((A3')\) is satisfied. Note also that \( g(x, c) \equiv 0 \) for at most one constant \( c > 0 \) (and in this case \( a, b \) are linearly dependent). By Theorem 1.5 we deduce that \( I \) has at most one nontrivial nonnegative critical point. Since \( p < r \) and \( b \geq 0 \), it follows that \( I \) is coercive, assuming if \( X = W^{1,r}(\Omega) \) that \( b \not\equiv 0 \) or \( \int_{\Omega} a < 0 \). It has a nontrivial global minimizer if \( a(x) \geq \lambda_1(p) \), where \( \lambda_1(p) \) is given by (2.1). More generally, this result holds if \( \inf_{u \in X} \int_{\Omega} (|\nabla u|^p - a(x)|u|^p) < 0 \), which is also a necessary condition to have \( A \neq \emptyset \). Now, for \( q \geq r^* \) one may argue as in [6, Example 4.9] to show that \( A \) is a singleton if \( a \equiv a_0 > \lambda_1(p) \) and \( b \equiv b_0 > 0 \). Note that if \( X = W^{1,r}(\Omega) \) then \( A = \{ (ab_0^{1-r})^{-\frac{1}{r^*}} \} \).

For \( a > \lambda_1(p) \), \( X = W^{1,r}_0(\Omega) \), and \( q = r \), the existence and uniqueness of a positive critical point has been observed in [5, 23, 27].

(b) \( g(x, u) = a(x) u^{q-1} \) with \( 1 < q < p \) and \( a^+ \neq 0 \).

If \( a \geq 0 \) then the strong maximum principle and Hopf Lemma apply, so \((A3')\) holds and we can apply Theorem 1.6 to deduce that \( A \) is at most a singleton. If \( X = W^{1,r}_0(\Omega) \) then \( I \) is clearly coercive and has a nontrivial global minimizer, so \( A \) is a singleton. Now, if \( X = W^{1,r}(\Omega) \) then integrating the equation we see that \( A \) is empty.

The existence and uniqueness of a positive critical point has been proved in [14] for the Dirichlet problem with \( N = 1 \) and \( a \) a positive constant.

Assume now that \( a \) changes sign. By Theorem 1.6 the sets \( A \cap P^0 \) and \( B \) are at most a singleton (note that no positive constant solves \((P)\)). Since \( q < p < r \), it is straightforward that \( I \) is coercive and has a nontrivial global minimizer (so \( B \) is a singleton), assuming in addition that \( \int_{\Omega} a < 0 \) if \( X = W^{1,r}(\Omega) \).

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SOME UNIQUENESS RESULTS

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