CLOSED REEB ORBITS ON THE SPHERE AND SYMPLECTICALLY DEGENERATE MAXIMA

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Abstract. We show that the existence of one simple closed Reeb orbit of a particular type (a symplectically degenerate maximum) forces the Reeb flow to have infinitely many periodic orbits. We use this result to give a different proof of a recent theorem of Cristofaro-Gardiner and Hutchings asserting that every Reeb flow on the standard contact three-sphere has at least two periodic orbits. Our methods are based on adapting the machinery originally developed for proving the Hamiltonian Conley conjecture to the contact setting.

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1. Introduction and main results

1.1. Introduction. In this paper, we use the techniques originally developed for proving the Hamiltonian Conley conjecture to show that the existence of one simple closed Reeb orbit of a particular type – a symplectically degenerate maximum (SDM) – forces the Reeb flow to have infinitely many periodic orbits. We apply this result to give a different proof of a recent theorem of Cristofaro-Gardiner...
and Hutchings, [CGH], asserting that every contact form supporting the standard contact structure on the three-sphere has at least two closed Reeb orbits.

The proof of our Conley conjecture type result closely follows its Hamiltonian counterpart from [GG2] (see also [Gi2, He2, Hi3, SZ] for other relevant results) with local contact homology, introduced in [HM], used in place of local Floer homology.

The idea connecting the Conley conjecture and the existence of at least two closed Reeb orbits on $S^3$ is, of course, that if there were a contact form (giving rise to the standard contact structure) with only one simple closed Reeb orbit, this orbit would be an SDM, which in turn would ensure the existence of infinitely many periodic orbits. This is established by a straightforward index analysis with two non-trivial inputs. One comes from the results in [HWZ1, HWZ2] eliminating certain index patterns. The other one is a technical theorem relating the local contact homology of an isolated ($\kappa$-iterated) closed Reeb orbit with the ($\mathbb{Z}_\kappa$-equivariant) local Floer homology of its Poincaré return map (cf. [Gi2, HM]). This is Theorem 2.1, which enables us to apply the results from [GG3] concerning local Floer homology in the contact context, and which we feel may be of use for a variety of other questions.

1.2. Main results. Let us now state the two principal results of the paper.

**Theorem 1.1** ([CGH]). Let $\alpha$ be a contact form on $S^3$ such that $\ker \alpha$ is the standard contact structure. Then the Reeb flow of $\alpha$ has at least two closed orbits.

To put this theorem in context, let us recall some relevant results. First, note that the existence of at least one closed Reeb orbit in this case is a theorem of Viterbo; [Vi1]. Furthermore, any strictly convex hypersurface in $\mathbb{R}^4$ carries either two or infinitely many closed characteristics; [HWZ3]. The same holds for any non-degenerate contact form on $S^3$ supporting the standard contact structure, provided that all stable and unstable manifolds of the hyperbolic periodic orbits intersect transversally; [HWZ4]. In fact, any non-degenerate contact form on a closed three-manifold has at least two simple closed Reeb orbits, and, if the manifold is not a lens space, there are at least three such orbits; [HT]. Moreover, any symmetric, compact star-shaped hypersurface in $\mathbb{R}^4$ has at least two closed characteristics regardless of whether the resulting contact form is non-degenerate or not; [Lo2]. Finally, we refer the reader to, e.g., [EH, HZ, Lo1, Wa] and references therein for results and conjectures concerning closed Reeb orbits in higher dimensions.

As has been pointed out above, we establish Theorem 1.1 as a consequence of the following

**Theorem 1.2.** Let $(M^{2n-1}, \ker \alpha)$ be a contact manifold admitting a strong symplectically aspherical filling $(W, \omega)$. Assume that the Reeb flow of $\alpha$ has an isolated simple closed Reeb orbit which is a symplectically degenerate maximum (SDM) with respect to some symplectic trivialization of $\ker \alpha|_x$. Furthermore, when $x$ is not contractible in $W$, we require $(W, \omega)$ to be symplectically atoroidal. Then the Reeb flow of $\alpha$ has infinitely many periodic orbits.

Recall that $(W, \omega)$ is called symplectically aspherical when $[\omega]|_{\pi_2(W)} = 0$ and $c_1(TW)|_{\pi_2(W)} = 0$, and $(W, \omega)$ is symplectically atoroidal when $[\omega]$ and $c_1(TW)$ vanish on all toroidal classes, i.e., $\langle [\omega], v \rangle = 0$ and $\langle c_1(TW), v \rangle = 0$ for every map $v: T^2 \to W$. We refer the reader to Section 2.2 for the definition and detailed discussion of symplectically degenerate maxima. The proof of Theorem 1.2 draws heavily from recent work on the Conley conjecture in the Hamiltonian setting; see
[Gi2, GG2, He2, Hi3]. In particular, similarly to the Hamiltonian case, Theorem 1.2 follows from a result (Theorem 4.1) asserting that the contact homology of α does not vanish for certain action intervals. It is very likely that the requirement that W is symplectically aspherical can be relaxed, with minimal modifications to the proof, and replaced by the assumption that $c_1(TW)|_{\pi_2(W)} = 0$ as in [GG2, He2] when x is contractible in W or, in general, by the assumption that $c_1(TW)$ is atoroidal. Let us also call the reader’s attention to the condition that the SDM orbit is simple. Although this assumption appears to be of a technical nature (cf. [Hi1, Hi2]), it does play a crucial role in our proof and eliminating it would probably require developing a different approach to the problem.

As an immediate application of Theorem 1.2, we observe that a Riemannian or Finsler manifold has infinitely many closed geodesics whenever it has one prime closed SDM geodesic. This is the case, for instance, for the closed geodesic considered in [Hi1, Proposition I] whenever this geodesic is prime (and isolated), and hence the proposition follows, under these hypotheses, from our Theorem 1.2. Furthermore, note that Theorem 1.2 completes a “mostly Floer–theoretic” proof of the existence of infinitely many closed geodesics on $S^2$; see [Ba, Fr] and also [Hi1, Hi2] and references therein for the original argument. Indeed, as is observed in [Hi2], the curve shortening method from [Gr] yields a Lusternik–Schnirelmann closed geodesic x with non-trivial local (Morse) homology in degree three, which without loss of generality can be assumed to be isolated. If x is non-rotating (i.e., its mean index is equal to two), x is an SDM and Theorem 1.2 applies. When x is not non-rotating, the existence of infinitely many closed geodesics is a consequence of the main theorem in [HMS]. (See also [BH, CKRTZ, Ke] for other relevant results including a symplectic proof of Franks’ theorem.)

**Remark 1.3.** As is clear from the proof, a result similar to Theorem 1.2 holds when an SDM orbit is replaced by a symplectically degenerate minimum (SDMin); see Remark 2.10. For instance, the closed geodesic x from the main theorem in [Hi2] is an SDMin, whenever x is isolated. Hence, in particular, combining Theorems 1.2 and 4.1 with Remark 4.4, we reprove the main result of [Hi2] under the additional assumption that the geodesic in question is isolated and prime.

A word is due on the degree of rigor in this paper, which varies considerably between its different parts. First of all, it should be noted that the paper heavily relies on the machinery of contact homology (see, e.g., [Bou, EGH] and references therein), which is yet to be fully put on a rigorous basis (see [HWZ5, HWZ6]). With this reservation in mind, the proof of Theorem 1.2 is essentially complete. Here the usage of contact homology is rather formal and the argument follows closely the proof of the Conley conjecture as given in [GG2], omitting for the sake of brevity some details which are straightforward to fill in. Theorem 1.1 is rigorously established as a consequence of Theorem 1.2 and, at least on the expository level, of a technical result (Theorem 2.1), mentioned above, relating local Floer and local contact homology. The proof of Theorem 2.1 is only sketched in this paper. Although some of the details can be filled in as in [HM], the argument must ultimately utilize the machinery of multivalued perturbations. The difficulty arising here is illustrated in Section 5, where we analyze the Morse theoretic counterpart of the problem by discussing an isomorphism between equivariant and invariant Morse homology for manifolds equipped with a finite group action.
However, it is worth pointing out that, in fact, the proof of Theorem 1.1 depends only on a particular case of Theorem 2.1 and on some results from [HWZ1, HWZ2]. This particular case is Corollary 2.2, originally proved in [HM] and ultimately independent of the theorem. (An alternative approach is also outlined in Section 3).

Finally, note that the foundational difficulties discussed above can be circumvented by using equivariant symplectic homology (see [BO1, BO2, Vi2]) in place of contact homology; cf. [HM] vs. [McL]. Alternatively, one may utilize equivariant Rabinowitz Floer homology, which appears to carry essentially the same information as contact or equivariant symplectic homology; see [AF, CFO]. However, in either case, this approach is likely to require geometrically much less transparent, perhaps cumbersome, arguments than those given in the present paper.

Notation and conventions. Throughout the paper, we use the conventions and notation of [GG3] when working with notions from Hamiltonian dynamics, including Floer homology and the Conley–Zehnder index, and of [HM] in the contact geometry setting. In particular, we utilize the normalization of the Conley–Zehnder index in which the index of a non-degenerate maximum with small eigenvalues in dimension \(2n\) is equal to \(n\). All homology groups are taken with rational coefficients unless stated otherwise. The global contact homology is always understood to be the linearized contact homology. The Floer homology, local or global, is graded by the Conley–Zehnder index, while the contact homology in dimension \(2n - 1\) is, as is customary, graded by the Conley–Zehnder index of the return map plus \(n - 3\); the Morse homology is graded by the Morse index. We use the notation \(\mu_{\text{CZ}}(x)\) for the Conley–Zehnder index of a periodic orbit \(x\) of a Hamiltonian diffeomorphism and \(|x|\) for the degree of a closed Reeb orbit \(x\). The mean index of \(x\) is denoted by \(\Delta_H(x)\), where \(H\) is a Hamiltonian, or by \(\Delta(x)\). (See, e.g., [Lo1, SZ], for miscellaneous properties of the Conley–Zehnder and mean indexes used throughout the paper.)

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2. Local analysis

2.1. Local contact and Floer homology. Consider a closed orbit \(x\) of the Reeb flow for a contact form \(\alpha\) on a manifold \(M^{2n-1}\). We do not require \(x\) to be simple, i.e., \(x\) can be a multiply-covered, iterated orbit. However, we do require \(x\) to be isolated in the loop space. In this setting, the local contact homology \(\text{HC}_* (\alpha, \kappa[S^1])\) at \(x\) is defined. This is the homology of a complex \(\text{CC}_*(\alpha', \kappa[S^1])\) generated by some of the closed Reeb orbits which \(x\) splits into under a non-degenerate perturbation \(\alpha'\) of \(\alpha\).

To be more precise, assume that \(x = y^c\), where \(y\) is a simple closed Reeb orbit of \(\alpha\). Fix a small tubular isolating neighborhood \(U = B \times S^1\) of \(x\) (or, from a geometrical perspective, of \(y\)) and let \(\alpha'\) be a \(C^\infty\)-small non-degenerate perturbation of \(\alpha\). Consider closed Reeb orbits of \(\alpha'\) in the homotopy class \(\kappa[S^1]\), contained in \(U\). Note that some of these orbits can be iterated. We call an orbit bad if it is an even iteration of an orbit with an odd number of Floquet multipliers in \((-1, 0)\). Otherwise, an orbit is said to be good. (Note that all orbits in question are non-contractible in \(U\).) The complex \(\text{CC}_*(\alpha', \kappa[S^1])\) is generated by the good closed orbits of \(\alpha'\) in the free homotopy class \(\kappa[S^1]\). (It is not hard to show that the
solutions of the Cauchy–Riemann equation in the symplectization of \( U \), asymptotic to such orbits, stay away from the boundary region in \( U \); hence, compactness. See [HM] for more details.) When \( \kappa > 1 \), making this construction rigorous encounters fundamental difficulties inherent in the definition of the contact homology: the symplectization of \( U \) need not admit an almost complex structure \( J \) meeting the regularity requirements, see [Bou, EGH, HM]. We proceed assuming that such an almost complex structure \( J \) exists.

On the other hand, consider the Poincaré return map \( \varphi_x \) of \( x \). This is the germ of a Hamiltonian diffeomorphism with isolated fixed point, which we still denote by \( x \). Then we also have the local Floer homology of \( \varphi_x \) at \( x \) defined; see [Gi2, GG3, McL]. We denote this homology by \( HF_\ast(x) \) or \( HF_\ast(\varphi_x) \), depending on the context. When \( x \) is simple, the groups \( HC_\ast(x) \) and \( HF_\ast(x) \) are isomorphic up to a shift of degree; see [HM, Proposition 5.1]. (This can also be established by repeating word-for-word the proof of [EKP, Proposition 4.30] – the fact that the Floer homology of a stable Hamiltonian structure is independent of a framing and of an adjusted almost complex structure.) Our next goal is to show that this is still true in general, once the ordinary local Floer homology group \( HF_\ast(x) \) is replaced by its equivariant counterpart.

To describe the setting more accurately, let us first focus on the case of global Floer homology. Thus let \( W \) be a closed symplectic manifold, which for our purposes can be required to be symplectically aspherical. Consider a one-periodic in time Hamiltonian \( H : W \times S^1 \to \mathbb{R} \), where \( S^1 = \mathbb{R} / \mathbb{Z} \). For any positive integer \( \kappa \), the Hamiltonian \( H \) can also be treated as \( \kappa \)-periodic. In this case, we will use the notation \( H^{\kappa} : W \times S^1_\kappa \to \mathbb{R} \), where \( S^1_\kappa = \mathbb{R} / \kappa \mathbb{Z} \), and, abusing terminology, call \( H^{\kappa} \) the \( \kappa \)th iteration of \( H \). The critical points of the action functional for \( H^{\kappa} \) are precisely the contractible \( \kappa \)-periodic orbits of \( H \) and \( \mathbb{Z}_\kappa \) acts on the set of \( \kappa \)-periodic orbits by time-shift. Let us first assume that \( H \) is strongly non-degenerate, i.e., all Hamiltonians \( H^{\kappa} \) are non-degenerate. The definition of the \( \mathbb{Z}_\kappa \)-equivariant Floer homology \( HF^{\mathbb{Z}_\kappa}_{\ast}(H^{\kappa}) \) is identical, essentially word-for-word with obvious modifications, to the construction of the \( S^1 \)-equivariant Floer homology for autonomous Hamiltonians; see [Vi2, Section 5] and [BO1, BO2]. (In Section 5 we illustrate the construction by the “elementary” case of the equivariant Morse homology.) The resulting homology has the expected properties similar to the standard Floer homology. In particular, we also have the filtered equivariant Floer homology, continuation maps, and, by continuity, the construction extends to all, not necessarily non-degenerate, Hamiltonians. The total \( \mathbb{Z}_\kappa \)-equivariant Floer homology is independent of \( H \) and does not appear to carry much new information when \( W \) is closed. Over \( \mathbb{Q} \), we have \( HF^{\mathbb{Z}_\kappa}_{\ast}(H^{\kappa}) \cong H_\ast(W) \) since the group homology of \( \mathbb{Z}_\kappa \) vanishes, i.e., \( H_\ast > 0(\mathbb{Z}_\kappa) = 0 \). Note however that the filtered \( \mathbb{Z}_\kappa \)-equivariant Floer homology can differ from its non-equivariant counterpart.

Coming back to our discussion of the local Floer and contact homology, observe that the construction of \( \mathbb{Z}_\kappa \)-equivariant Floer homology localizes in a straightforward way. Namely, assume that \( x = y^\kappa \), i.e., \( x \) is the \( \kappa \)th iteration of a simple orbit \( y \). Then \( \varphi_x = \varphi_y^\kappa \), and the \( \mathbb{Z}_\kappa \)-equivariant local Floer homology \( HF^{\mathbb{Z}_\kappa}_{\ast}(\varphi_x) = HF^{\mathbb{Z}_\kappa}_{\ast}(\varphi_y)^\kappa \) is defined. In general, \( HF^{\mathbb{Z}_\kappa}_{\ast}(x) \neq HF_{\ast}(x) \) even over \( \mathbb{Q} \).

Finally, to explicitly relate the local contact and Floer homology, we need to fix the gradings of these groups. To this end, we pick a symplectic trivialization of \( \ker \alpha|_y \). Such a trivialization determines the grading of the local contact homology...
and also enables us to view $\varphi_x$ and $\varphi_y$ as elements of the universal covering of the group of the germs of Hamiltonian diffeomorphisms, which in turn determines the grading of the local Floer homology. We have $|x| = \mu_{cz}(x) + (n - 3)$.

**Theorem 2.1.** Let $x = y^n$ be an isolated closed Reeb orbit. Then we have $\text{HC}_*(x) = \text{HF}^{\mathbb{Z}_n}_{* + n - 3}(x)$.

As readily follows from the construction of the equivariant Floer homology (see [Vi2, Section 5] and [BO2, Section 5] and also Section 5), there exists a spectral sequence with $E^2_{*,*} = \text{HF}_*(x) \otimes \mathbb{Z}_n = \text{HF}_*(x)$ converging to $\text{HF}^{\mathbb{Z}_n}_*(x)$. As a consequence, we obtain

**Corollary 2.2 ([HM]).** The total dimension, over $\mathbb{Q}$, of the local contact homology does not exceed the total dimension of the local Floer homology: $\dim_\mathbb{Q} \text{HC}_*(x) \leq \dim_\mathbb{Q} \text{HF}_*(x)$. Furthermore, $\text{HC}_*(x) = \text{HF}_{* + n - 3}(x)$ when the orbit $x$ is simple, i.e., $\kappa = 1$.

**Remark 2.3.** It is worth keeping in mind that even though we establish here Corollary 2.2 as a consequence of Theorem 2.1, the corollary can be proved directly; see [HM] for a very detailed argument.

**Outline of the proof of Theorem 2.1.** As above, let $U = B \times S^1$ be a small isolating tubular neighborhood of $y$ and let $\alpha'$ be a $C^\infty$-small non-degenerate perturbation of $\alpha$. Recall that $\text{HC}_*(x)$ is, by definition, the homology of the complex $C_*([\alpha', \kappa[S^1]])$ generated by the good orbits of $\alpha'$ in $U$ in the homotopy class $\kappa[S^1]$ and that we assume that the symplectization of $U$ admits an almost complex structure $J$ such that the necessary regularity requirements are satisfied.

Consider the $k$-fold covering $\bar{U} = B \times S^1_k$ of $U$. We denote by $\bar{\alpha}'$ and $\bar{J}$ the lifts of $\alpha'$ and $J$ to $\bar{U}$. The group of deck transformations $\mathbb{Z}_k$ acts on the contact complex $C_*([\bar{\alpha}', [S^1_k]])$ for the homotopy class $[S^1_k]$. Namely, the shift $g: t \mapsto (t + g)$ on $S^1$ acts by sending a periodic orbit $z$ of $\bar{\alpha}'$ to the orbit $\pm g(z)$; see [HM, Section 6.3]. The regularity of $\bar{J}$ guarantees that this action commutes with the differential and, again essentially by definition, the contact complex $C_*([\alpha', \kappa[S^1]])$ agrees with the invariant part $C_*([\bar{\alpha}', [S^1_k]])^{\mathbb{Z}_n}$ of the complex for $\bar{\alpha}'$. Hence

$$\text{HC}_*(x) = \text{HC}_*([\alpha', \kappa[S^1]]) = \text{HC}_*([\bar{\alpha}', [S^1_k]])^{\mathbb{Z}_n} =: \text{HC}_*(\bar{x})^{\mathbb{Z}_n},$$

(2.1)

where $\bar{x}$ is the inverse image of $x$ in $\bar{U}$; cf. [HM, Section 6].

**Example 2.4 (Good vs. bad).** Assume that $x = y^2$ and $x$ is non-degenerate. Let $g$ be the non-trivial element in $\mathbb{Z}_2$. Then $g: \bar{x} \mapsto \bar{x}$ when $x$ is good and $g: \bar{x} \mapsto -\bar{x}$ when $x$ is bad. Thus, in the former case, we have $\text{HC}_*(x) = \text{HC}_*(\bar{x}) = \mathbb{Q}$ (supported in degree $\mu_{cz}(x) + n - 3$). In the latter case, $\text{HC}_*(x) = 0$, but $\text{HC}_*(\bar{x}) = \mathbb{Q}$.

The rest of the proof relies heavily, at least on the conceptual level, on the constructions of the invariant and equivariant Morse homology outlined in Section 5 and on the proof of Proposition 5.1. Namely, similarly to the definition of the invariant Morse homology, one can define the $\mathbb{Z}_n$-invariant (local) Floer homology for an iterated Hamiltonian or an iterated orbit, provided, of course, that there exists a one-periodic almost complex structure satisfying the regularity requirements. Hence we have the $\mathbb{Z}_n$-invariant local Floer homology $\text{HF}_*(\varphi_{\bar{x}})^{\mathbb{Z}_n}$ of the Poincaré return map of $\bar{x}$. Clearly, $\varphi_{\bar{x}} = \varphi_x = \varphi_y^n$ and $\text{HF}_*(\varphi_{\bar{x}})^{\mathbb{Z}_n} = \text{HF}_*(\varphi_x)^{\mathbb{Z}_n}$. A homotopy between a contact framing in $\bar{U}$ and a Hamiltonian framing (see [HM, Section 5]
and \([\text{EKP}, \text{Proposition 4.30}]\) induces an isomorphism \(\text{HC}_*(\tilde{x}) = \text{HF}_*(\varphi_{\tilde{x}})\). When
the homotopy is regular and \(\mathbb{Z}_\kappa\)-invariant, the continuation map commutes with
the \(\mathbb{Z}_\kappa\)-action, and we obtain an isomorphism
\[
\text{HC}_*(\tilde{x})^{\mathbb{Z}_\kappa} = \text{HF}_{*+n-3}(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa} =: \text{HF}_{*+n-3}(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa}.
\]
Furthermore, Proposition 5.1 establishes an isomorphism between invariant and
equivariant Morse homology. With suitable modifications, the proposition and its
proof carry over to the realm of Floer homology, again provided that the required
regularity conditions are met. As a consequence, we have
\[
\text{HF}_*(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa} = \text{HF}_{*+n-3}(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa}.
\]
(2.3)
Combining the isomorphisms (2.1), (2.2), and (2.3), we conclude that \(\text{HC}_*(\tilde{x}) = \text{HF}_*(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa}\).
\(\blacksquare\)

Remark 2.5. Eliminating the assumption that the symplectization of
\(U\) admits a
regular almost complex structure \(J\), crucial for the proof of Theorem 2.1, requires
the machinery of multi-valued perturbations in the context of contact homology.
Developing such a machinery is currently work in progress by Hofer, Wysocki and
Zehnder; see [HWZ5, HWZ6] and references therein, and also Remark 5.4.

2.2. Symplectically degenerate maxima. In this section, we recall several rele-
vant results concerning symplectically degenerate maxima, following mainly
[GG3, Section 5.1]. Let \(\varphi = \varphi_H\) be an element of the universal cover of the group of germs of
Hamiltonian diffeomorphisms of \(\mathbb{R}^2\) at a point \(x\). Throughout this section, it
will be convenient to assume that \(x\) is an isolated fixed point for all iterations \(\varphi^\kappa\).
(Treating \(x\) as a one-periodic orbit, we will sometimes write \(x^\kappa\) to indicate the order
of iteration.) Recall that \(\Delta_H(x)\) is the mean index of \(x\).

Definition 2.6. The point \(x\) is said to be a \textit{symplectically degenerate maximum}
(or an SDM) if \(\Delta(x) = 0\) and \(\text{HF}_{*+n-3}(\varphi_{\tilde{x}})^{\mathbb{Z}_\kappa} \neq 0\).

Note that requiring \(\varphi\) to be an element of the universal cover rather than just
a Hamiltonian diffeomorphism makes the mean index of \(x\) and the grading of the
local Floer homology well defined. (Otherwise, \(\Delta(x)\) and the grading would
be defined only modulo \(2\mathbb{Z}\).) A germ of a Hamiltonian diffeomorphism \(\varphi\) is called
an SDM if it admits an SDM lift to the universal cover. This is equivalent to that
\(\Delta_{H}(x) \in 2\mathbb{Z}\) and \(\text{HF}_{*+\Delta}(\varphi_{\tilde{x}}) \neq 0\) for any (or, equivalently, some) Hamiltonian
\(H\) generating \(\varphi\).

Example 2.7. Let \(H\) be an autonomous Hamiltonian attaining a strict local max-
imum at \(x\). Assume in addition that \(d^2H_x = 0\) or, more generally, that all eigen-
values of \(d^2H_x\) are zero. Then, as is easy to see, \(x\) is an SDM, cf. Proposition
2.9. (Here, as is customary in Hamiltonian dynamics, the eigenvalues of a qua-
dratic form on a symplectic vector space are, by definition, the eigenvalues of the
linear symplectic vector field it generates. Thus, for instance, the quadratic form
\(Q(p, q) = q^2\) on the standard \((\mathbb{R}^2, dp \wedge dq)\) has zero eigenvalues.)

Recall that \(\varphi\) (or \(x\)) is said to be totally degenerate if all eigenvalues of \(D\varphi\) are
equal to 1. An iteration \(\kappa\) is called admissible if the generalized eigenvalue 1 has
the same multiplicity for \(D\varphi\) and \(D\varphi^\kappa\). For instance, all \(\kappa\) are admissible when
\(\varphi\) is totally degenerate or when none of the eigenvalues of \(D\varphi\) is a root of unity.
Although this fact is not explicitly used in what follows, it is useful to keep in mind
that an admissible iteration of an isolated orbit is automatically isolated; see [GG3, Proposition 1.1].

**Proposition 2.8** ([GG3]). The following conditions are equivalent:

(a) the point \( x \) is an SDM;
(b) \( HF_d(\varphi_H^i) \neq 0 \) for some sequence of admissible iterations \( \kappa_i \to \infty \);
(c) the point \( x \) is totally degenerate, \( HF_d(\varphi_H) \neq 0 \) and \( HF_d(\varphi_H^i) \neq 0 \) for at least one admissible \( \kappa \geq d+1 \);
(d) \( HF_d(\varphi_H^\kappa) = \mathbb{Q} \) and \( HF_{\neq d}(\varphi_H^\kappa) = 0 \) for all \( \kappa \).

Equivalence of (a) and (b) and (c) is proved in [GG3]. Clearly (d) implies (b). The fact that (d) also follows from (b) can be easily extracted from the proof of [GG3, Theorem 1.1] combined with [Gi2, Section 3.1].

We observe that, as a consequence of the proposition and of the persistence of local Floer homology (see [GG3, Theorem 1.1]), the point \( x^\kappa \) is an SDM for any admissible iteration \( \kappa \) if and only if \( x \) is an SDM. The next proposition shows that the behavior of \( \varphi_H \) near an SDM is similar to that described in Example 2.7. The essence of this result is that \( x \) is an SDM if and only if \( \varphi \) can be generated by a Hamiltonian \( K \) with local maximum at \( x \) and arbitrarily small Hessian.

**Proposition 2.9** ([Gi2, H3]). The point \( x \) is an SDM if and only if for every \( \eta > 0 \) there exists a Hamiltonian \( K \) near \( x \) such that \( \varphi_K = \varphi_H \) in the universal covering of the group of germs of Hamiltonian diffeomorphisms at \( x \) and

(i) \( x \) is a strict local maximum of \( K_t \) for all \( t \in S^1 \);
(ii) \( d^2(K_t)_x \) has zero eigenvalues and \( \|d^2(K_t)_x\|_{\Xi} < \eta \) for all \( t \in S^1 \) and some symplectic basis \( \Xi \) in \( T_xM \).

Here \( \|d^2(K_t)_x\|_{\Xi} \) stands for the norm of \( d^2(K_t)_x \) with respect to the Euclidean inner product on \( T_x\mathbb{R}^{2n} \) for which \( \Xi \) is an orthonormal basis; see [Gi2, Section 2.1.3].

**Remark 2.10.** Replacing \( d \) by \( -d \) in the above discussion, we arrive at the notion of a symplectically degenerate minimum (SDMin); see [He1]. It has properties similar to those of an SDM but with a maximum replaced by a minimum. Note that \( x \) is an SDM for \( \varphi \) if and only if \( x \) is an SDMin for \( \varphi^{-1} \).

An isolated closed Reeb orbit \( x \) of \( \alpha \) on \( M^{2n-1} \) is said to be an SDM if it is an SDM for the Poincaré return map \( \varphi \). Note that this notion depends on the choice of a trivialization of \( \ker \alpha |_x \). For this trivialization gives rise to a lift of \( \varphi \) to the universal cover of the group of germs of Hamiltonian diffeomorphism. Applying Proposition 2.8, and keeping in mind the difference in gradings and that \( d = n-1 \), we obtain

**Proposition 2.11.** The following conditions are equivalent:

(a) the orbit \( x \) is an SDM with respect to a trivialization such that \( \Delta(x) = 0 \);
(b) \( HC_{2n-4}(x^{\kappa_i}) \neq 0 \) for some sequence of admissible iterations \( \kappa_i \to \infty \);
(c) the orbit \( x \) is totally degenerate, \( HC_{2n-4}(x) \neq 0 \) and \( HC_{2n-4}(x^{\kappa_i}) \neq 0 \) for at least one admissible iteration \( \kappa \geq n \);
(d) \( HC_{2n-4}(x^{\kappa}) = \mathbb{Q} \) and \( HC_{\neq 2n-4}(x^{\kappa}) = 0 \) for all \( \kappa \).
2.3. Local Floer and contact homology in low dimensions. In this section, we state some simple properties of local Floer homology in dimension two and local contact homology in dimension three to be used in what follows. These properties are easy consequences of the results from [GG3] and Theorem 2.1.

Let \( x \) be a closed Reeb orbit in dimension three (with a fixed trivialization of \( \ker \alpha \) along \( x \)) and let \( \varphi \) be an element of the universal cover of the group of germs of Hamiltonian diffeomorphisms of \( \mathbb{R}^2 \) at a point or more generally a periodic orbit, which we also denote by \( x \). (For instance, we can have \( \varphi = \varphi_x \) in the setting of Section 2.1.) We always assume that \( x^k \) is isolated for all \( \kappa \). As in Section 2.1, we will use the notation \( HF_*(x) \) or \( HF_*(\varphi) \) and \( HC_*(x) \) for the local Floer and local contact homology of \( x \).

We start by recalling the following particular case of [GG3, Theorem 1.1] mentioned in Section 2.2 and repeatedly used in this paper:

**Proposition 2.12** (Persistence of Local Floer Homology, [GG3]). Let, as above, \( x \) be a periodic orbit (simple or iterated) of \( \varphi \) in dimension two. Assume that \( x \) is degenerate. Then

\[
HF_*(\varphi^k) = HF_{*+(\kappa-1)\Delta(x)}(\varphi).
\]

Furthermore, local Floer and local contact homology in low dimensions have the following features:

(LF1) The local homology groups \( HF_*(\varphi) \) are concentrated in at most one degree \( k \in [\Delta(x)-1, \Delta(x)+1] \), i.e., \( HF_k(\varphi) \neq 0 \) for at most one \( k \) and \( |k-\Delta(x)| \leq 1 \). When \( x \) is non-degenerate \( k = \mu_{cz}(x) \). In general, we will call \( k \) the generalized Conley–Zehnder index of \( x \) and keep the notation \( k = \mu_{cz}(x) \). Likewise, \( HC_*(x) \) is concentrated in at most one degree \( k \in [\Delta(x)-2, \Delta(x)] \). When \( x \) is non-degenerate and the local homology does not vanish, \( k = |x| \).

In general, we keep the notation \( |x| \) for this degree, when \( HC_*(x^k) \neq 0 \).

When \( x \) is non-degenerate this is obvious. In the degenerate case, the result in the case of Floer homology can be easily extracted from the proof of [GG3, Theorem 1.1] and the fact that the local Morse homology of a function in two variables is concentrated in at most one degree. The assertion for contact homology now readily follows from Theorem 2.1 or even from Corollary 2.2, cf. Remark 2.3.

(LF2) Assume that \( x \) is degenerate and \( HF_*(x) \neq 0 \) (or \( HC_*(x) \neq 0 \) in the contact case). Then the mean index \( \Delta(x) \) is an even integer and, in the notation of (LF1), \( \mu_{cz}(x^\kappa) = 1 + \kappa \Delta(x) + \epsilon \) (or \( |x^\kappa| = \kappa \Delta(x) + \epsilon \)), where \( \epsilon \) is either \( \pm 1 \) or 0, and is independent of \( \kappa \).

Here only the assertion that \( \epsilon \) is independent of \( \kappa \) requires a proof. In the case of Floer homology, this fact can be readily extracted from the proof of the persistence of local Floer homology; see [GG3, Theorem 1.1]. The assertion in the contact case, follows now from Theorem 2.1 or Corollary 2.2. Note that at this point we can only claim that \( HC_*(x^\kappa) \) must vanish for \( \ast \neq |x^\kappa| \), but not that \( HC_{|x^\kappa|}(x^\kappa) \neq 0 \); see however (3.3).

Let us now take a closer look at the situation when the local Floer or contact homology of \( x \) or its iterations vanish. Assume first that \( x \) is degenerate. Then, as is easy to see, we can have \( HF_*(x) = 0 \) in all degrees. (In all examples known to us, such homologically trivial orbits are simple.) By Corollary 2.2, \( HC_*(x) = 0 \), when \( x \) is a closed Reeb orbit with \( HF_*(x) = 0 \). Moreover, \( HF_*(x^\kappa) = 0 \) and \( HC_*(x^\kappa) = 0 \) for all \( \kappa \). Hypothetically, it is also possible that \( HC_*(x) = 0 \) while \( HF_*(x) \neq 0 \), but
we do not have any examples of degenerate orbits for which this is the case. When
$x$ is non-degenerate, clearly $\text{HF}_{\mu_{cz}(x)}(x) \neq 0$. However, $\text{HC}_x(x)$ can vanish even
when $x$ is non-degenerate. This is the case if and only if $x$ is an even iteration of a
hyperbolic orbit with negative Floquet multipliers; see Example 2.4.

Note that when the local Floer or local contact homology is trivial, the definition
of $\mu_{cz}(x)$ or of $|x|$ from (LF1) does not apply. When $\text{HF}_x(x) \neq 0$, e.g., in the non-degenerate case, we still set $|x| = \mu_{cz}(x) - 1$. Alternatively, in Section 3, we use,
as a matter of convenience, a suitable ad hoc definition of $|x|$.

The next fact, stated for the sake of brevity only in the contact case, is an
immediate consequence of Proposition 2.8 (and Theorem 2.1 applied to an SDM).

(LF3) A simple orbit $x$ is an SDM (with respect to some trivialization of $\ker \alpha|_x$)
if and only if one of the following conditions is satisfied:

(i) $\text{HC}_{2m_\kappa}(x^{\kappa^i}) \neq 0$ (or equivalently $\text{HC}_{2m_\kappa}(x^{\kappa^i}) = \mathbb{Q}$) for some $m \in \mathbb{Z}$
and some sequence of admissible iterations $\kappa_i \to \infty$;

(ii) $x$ is degenerate and $\text{HC}_{2m}(x) \neq 0$ (or equivalently $\text{HC}_{2m}(x) = \mathbb{Q}$),
where $\Delta(x) = 2m$ and $m \in \mathbb{Z}$.

3. PROOF OF THEOREM 1.1

Our goal in this section is to derive Theorem 1.1 from Theorem 1.2. Thus let
$\alpha$ be a contact form supporting the standard contact structure on $S^3$. The key
property of $\alpha$ we need is that $\text{HC}_x(\alpha)$ is concentrated in even degrees $2, 4, \ldots$ and
that $\text{HC}_x(\alpha) = \mathbb{Q}$ in these degrees; see, e.g., [Bou] and references therein. (Here
the indices are evaluated with respect to a global symplectic trivialization of $\ker \alpha$
on $S^3$; such a trivialization is unique up to homotopy.)

Arguing by contradiction, let us assume that $\alpha$ has only one simple closed Reeb
orbit $x$. We will show that $x$ is then an SDM, and hence, by Theorem 1.2, the Reeb
flow of $\alpha$ has infinitely many periodic orbits.

First, observe that, by (LF1), every iteration $x^\kappa$ can contribute to the homology
in at most one degree $|x^\kappa| = \mu_{cz}(x^\kappa) - 1$, even when $x^\kappa$ is degenerate. In particular,
for some values of $\kappa$ we must have $|x^\kappa| = 2, 4, \ldots$ or equivalently $\mu_{cz}(x^\kappa) = 3, 5, \ldots$

As an immediate consequence, we see that $x$ cannot be hyperbolic. Indeed, if $x$
is hyperbolic and its Floquet multipliers are positive, $\mu_{cz}(x^\kappa)$ is even for all $\kappa$. If
the Floquet multipliers are negative, we have $\mu_{cz}(x^\kappa) = \kappa(2\ell + 1)$, for some integer $\ell$, and $\dim \text{HF}_{\kappa(2\ell+1)}(x^\kappa) = 1$, which is also impossible. (In fact, as is easy to see,
we would have $\dim \text{HC}_{\kappa(2\ell+1)}(x^\kappa) = 1$ if $\kappa$ is odd and $\text{HC}_{\kappa(2\ell+1)}(x^\kappa) = 0$ if $\kappa$ is
even.)

Thus $x$ is elliptic, i.e., its Floquet multipliers have absolute value one. Let
$\Delta = 2m + 2\lambda$, where $m \in \mathbb{Z}$ and $\lambda \in [0, 1)$, be the mean index of $x$. Then
$\Delta(x^\kappa) = \kappa \Delta$ and $0 < \Delta < 3$.

Next let us show that $x$ cannot be strongly non-degenerate. (An orbit is strongly
non-degenerate when all its iterations are non-degenerate.) First, observe $x^\kappa$ is
strongly non-degenerate if and only if $\lambda \notin \mathbb{Q}$. Furthermore, in this case $\mu_{cz}(x^\kappa) = 1 + 2m\kappa + 2[\kappa \lambda]$ (see, e.g., [Lo1]), and hence

$|x^\kappa| = 2m\kappa + 2[\kappa \lambda]$. 

As a consequence, $|x^\kappa|$ assumes only even values starting with $|x| = 2$. (Hence
$m = 1$.) However, some arbitrarily large even degrees are skipped when $|\kappa \lambda|$ jumps from one integer value to the next. This is impossible. Alternatively, the
fact that \( x \) cannot be strongly non-degenerate, including the hyperbolic case, also follows immediately from the results of [HWZ4].

Next, recall that in general, when \( \HC_\alpha(x^\kappa) \neq 0 \), we denote by \( |x^\kappa| \) the degree in which this space does not vanish. (By (LF1), this degree is unique.) We have

\[
|x^\kappa| = 2m\kappa + 2[\kappa\lambda] + \epsilon_\kappa, \tag{3.1}
\]

where \( \epsilon_\kappa \) is either \(-2\) or \(-1\) or \(0\). (When \( x^\kappa \) is non-degenerate, \( \epsilon_\kappa = 0 \).) For all \( \kappa \) such that \( x^\kappa \) is degenerate, \( \epsilon_\kappa \) takes the same value due to (LF2).

When \( x^\kappa \) is degenerate, we can have \( \HC_\alpha(x^\kappa) = 0 \) in all degrees; see Section 2.3. In this case, we use (3.1) with, say, \( \epsilon_\kappa = 0 \) to define \( |x^\kappa| \). Note that with this convention \( \epsilon_\kappa \) is still independent of \( \kappa \) when \( x^\kappa \) is degenerate.

Observe now that necessarily \( m = 0 \) or \( m = 1 \).

Assume first that \( m = 1 \). Then either \( \lambda = 0 \) or \( \lambda > 0 \). In the former case, by (LF3), \( x \) is an SDM (with respect to a twisted trivialization along \( x \)) as required. In the latter case, we can assume that \( \lambda \) is rational. Then \( \lambda = 1/\ell \), for some integer \( \ell > 1 \). (Otherwise, \( |x^\kappa| \) would skip an even value when \( [\kappa\lambda] \) jumps from 0 to 1.) Thus \( |x^\kappa| \) is the sequence

\[
2, \ldots, 2(\ell - 1), 2\ell + \epsilon_\ell, 2(\ell + 1) + 2, \ldots, 2(2\ell - 1) + 2, 4\ell + 2 + \epsilon_{2\ell}, 2(2\ell + 1) + 4, \ldots,
\]

which is clearly impossible.

It remains to examine the case \( m = 0 \). Then \( \Delta = \lambda \in (0, 1) \) and again \( \lambda \in \mathbb{Q} \). As in the case \( m = 1 \), it is easy to see that \( \lambda = 1/\ell \) for some integer \( \ell > 1 \). As a consequence, \( x^\kappa \) degenerates when \( \kappa \) is divisible by \( \ell \), and

\[
|x^\kappa| = 0, 1, 2, \ldots, 2, 3, \ldots \tag{3.2}
\]

There are two ways to rule out this possibility. (We are not aware of any example of the germ of a Hamiltonian diffeomorphism with this behavior, but we do not have a “local” proof that this is impossible.)

The first one is to invoke some of the results of Hofer–Wysocki–Zehnder; see [HWZ1, HWZ2]. Namely, by [HWZ2, Theorems 1.7 and 1.8], there exists a non-constant finite energy plane in the symplectization of \( S^3 \), injective in \( S^3 \) and asymptotic to a simple closed Reeb orbit \( y \). By [HWZ1, Theorem 4.3], \( \mu_{CZ}(y) \geq 2 \). However, in our setting, the only simple orbit is \( x \), which is non-degenerate, and \( \mu_{CZ}(x) = 1 \).

Alternatively, using the full strength of Theorem 2.1 and more, one can argue as follows. First, one shows that in dimension three

\[
\dim \HC_\alpha(z^\kappa) = \dim \HC_\alpha(z) \tag{3.3}
\]

up to a shift of degree, when \( z \) is degenerate. Taking into account Theorem 2.1, we can express (3.3) as \( \dim \HF_\alpha^z(z^\kappa) = \dim \HF_\alpha^z(z) \), again up to a shift, where \( z = y^\nu \) and \( y \) is simple. In fact, (3.3) is a contact (or equivariant) analogue of Proposition 2.12. Using (3.2) and (3.3) for \( z = x^\ell \) and setting \( r = \dim \HC_\alpha(z) \), we see that the spaces \( \HC_\alpha(x^\kappa) \) fit together to form a complex

\[
0 \leftarrow \mathbb{Q}^{\ell - 1} \leftarrow \mathbb{Q}^r \leftarrow \mathbb{Q}^{\ell - 1} \leftarrow \mathbb{Q}^r \leftarrow \cdots
\]

with the first non-zero term \( \mathbb{Q}^{\ell - 1} \) occurring in degree zero. The homology of this complex is \( \HC_\alpha(\alpha) \), i.e., it is \( \mathbb{Q} \) in every even degree starting with two and zero otherwise. We claim that this is impossible. Indeed, let \( p_j \geq 0 \) be the rank of the differential \( \mathbb{Q}^r \leftarrow \mathbb{Q}^{\ell - 1} \) from degree \( 2j \) to degree \( 2j - 1 \). Thus, for instance, \( p_1 =\)
$r - (\ell - 1)$. Now, a simple inductive argument shows that $p_j = j p_1 + (j - 1) \to \infty$, which contradicts the obvious restriction $p_j \leq r$.

4. Proof of Theorem 1.2

The argument is a rather faithful adaptation of the proof of the Conley conjecture given in [GG2] to the contact case. Hence we only briefly outline the proof skipping numerous technical details.

4.1. Filtered contact homology in the presence of an SDM. As in the proof of the degenerate case of the Hamiltonian Conley conjecture (see, e.g., [Gi2, GG2, HI3, He2]), Theorem 1.2 is a consequence of the fact that the presence of an SDM forces the filtered contact homology to be non-zero for certain arbitrarily small action intervals located right above the action of the SDM iterations.

To be more precise, let $(M^{2n-1}, \ker \alpha)$ be a contact manifold admitting a strong symplectically aspherical filling $(W, \omega)$. As in Theorem 1.2, assume that the Reeb flow of $\alpha$ has a simple closed Reeb orbit $x$ which is a symplectically degenerate maximum with respect to some symplectic trivialization of $\ker \alpha |_x$. Denote by $\text{HC}_*(M, \alpha)$ the (filtered) contact homology of $(M, \alpha)$, linearized by means of $W$, for the collection of free homotopy classes $\{[x^\kappa] | \kappa \in \mathbb{N}\}$. When $x$ is contractible in $W$, the condition that $c_1(TW) |_{\pi_2(W)} = 0$ ensures that $\text{HC}_*(M, \alpha)$ has a well-defined grading. If $x$ is not contractible, we require, in addition, $(W, \omega)$ to be atoroidal, and fix a symplectic trivialization of $TW |_{x^\kappa}$ to have the grading well defined. (In fact, it would be sufficient to fix, a trivialization of the complex line bundle $(\wedge^n W |_{x^\kappa})^{\otimes 2}$; see, e.g., [Bou, EGH, Es] for more details.) To be more specific, in this case, we pick a trivialization of $TW |_x$ and equip $TW |_{x^\kappa}$, for $\kappa \geq 2$, with “iterated trivializations”. (When $x$ is contractible, it is equipped with a trivialization which extends to a disk bounded by $x$.) From now on, all degrees and (mean) indices are taken with respect to such fixed reference trivializations unless explicitly stated otherwise.

Assume that all iterations $x^\kappa$ are isolated. Let $2m$, where $m \in \mathbb{Z}$, be the Maslov index of the trivialization with respect to which $x$ is an SDM. Thus, by Proposition 2.11, $\text{HC}_*(x^\kappa)$ is concentrated and equal to $\mathbb{Q}$ in degree $d_\kappa = 2m\kappa + 2n - 4$. Denote by $c$ the action of $x$. Then the action of $x^\kappa$ is $\kappa c$.

**Theorem 4.1.** For any sufficiently small $\epsilon > 0$, 

$$\text{HC}_{d_\kappa+1}^{(\kappa c, \kappa c+\epsilon)}(M, \alpha) \neq 0$$

for all large iterations $\kappa > \kappa_0(\epsilon)$.

**Remark 4.2.** Strictly speaking here, as in similar results in the Hamiltonian setting (see, e.g., [Gi2, Proposition 4.7], [GG2, Theorem 1.7] and [He2, Theorem 1.5]), one has to replace the action interval $(\kappa c, \kappa c + \epsilon)$ by $(\kappa c + \delta, \kappa c + \epsilon)$ for some arbitrarily small $\delta \in (0, \epsilon)$, and assume that $\epsilon$ is not in a certain zero measure set, to comply with the official definition of the filtered contact or Floer homology requiring the end points of the interval to be outside the action spectrum. Furthermore, in $(4.1)$, we in fact have $\text{HC}_{d_\kappa+1}^{(\kappa c, \kappa c+\epsilon)}(M, \alpha) \cong \mathbb{Q}$ as in the Hamiltonian case.

Theorem 1.2 follows immediately from Theorem 4.1.

**Remark 4.3.** Note that in Theorem 1.2 we make no claim concerning the growth rate of the number of closed Reeb orbits. Recall in this connection that, in the settings
of the Conley conjecture, for any Hamiltonian diffeomorphism with isolated fixed points, every sufficiently large prime occurs as a period of a simple periodic orbit; [Gi2, GG2, He2, Hi3, SZ]. In a similar vein, the number of closed geodesics on $S^2$ with length less than or equal to $\ell$ grows at least as prime numbers, i.e., it is bounded from below by $\text{const} \cdot \ell / \log \ell$; [Hi1].

Remark 4.4. An assertion similar to Theorem 4.1 holds when an SDM orbit is replaced by an SDMin Reeb orbit. In this case we have $HC_{2m-3}^{(\kappa_+\cdot \kappa_-)} (M, \alpha) \neq 0$. As has been mentioned in Remark 1.3, this result partially generalizes the main theorem of [Hi2] asserting, in the presence of SDMin geodesic, the existence of infinitely many prime closed geodesics on $S^2$ with length in certain intervals.

4.2. Proof of Theorem 4.1.

4.2.1. Local model and its extension. Let a simple Reeb orbit $x$ be an SDM of the Reeb flow of $\alpha$ on $M^{2n-1}$ with respect to a trivialization of $\ker \alpha |_x$. There exists a tubular neighborhood $U = B \times S^1$ (where $B$ is the ball $B_R$ of radius $R$ in $\mathbb{R}^{2n-2}$), adapted to the trivialization in the obvious sense, and a Hamiltonian $H: B \times S^1 \to (0, \infty)$ such that

(H1) $\alpha = \lambda + H dt$, where $\lambda = (p dq - q dp)/2$ is the $U(n-1)$-invariant primitive of the standard symplectic structure on $B$;

(H2) $H_t(0) = c$ is a strict local maximum of $H_t$ for all $t \in S^1$, and $c$ is independent of $t$;

(H3) $d^2 H_t$ at 0 has zero eigenvalues.

This fact readily follows from Proposition 2.9; see the proof of [HM, Lemma 5.2]. Note that for any $\eta_0 > 0$ this can be done so that $\|d^2 H_t\| < \eta_0$ on $B$ for all $t \in S^1$.

(Recall also that the eigenvalues of a quadratic form on a symplectic vector space are, by definition, the eigenvalues of the linear symplectic vector field it generates; see Example 2.7.)

Next, as in [Gi2, GG2], let us consider functions $H_\pm : B \to (0, \infty)$ such that

$$H_+ \geq H_t \geq H_- \quad \text{for all } t \in S^1,$$  

(4.2) with equality attained only at $0 \in B$, and $H_+ \equiv c_+ > 0$ and $H_- \equiv c_- > 0$ on the region $Y = B_R \setminus B_{R/2}$ near the boundary of $B$. These functions are also required to be $C^0$-close to $H$ and satisfy some additional conditions.

Namely, set $\rho = \|z\|$ on $B$. Then $H_+$ is a bump–function on $B$, constant near 0 and on $Y$, and depending only on $\rho$. In other words, we have

(H$^+$) $H_+$ is a decreasing function of $\rho$, equal to $c$ for $\rho$ close to 0 and equal to $c_+$ for $\rho \in [R/2, R]$.

The function $H_-$ is obtained by composing a certain function $F$ on $B$ with a (linear) symplectic transformation $\Psi: \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$, i.e., $H_- = F \circ \Psi$. The function $F$ is also a bump function on $B$ depending only on $\rho$, but it has a non-degenerate maximum at $0 \in B$. To be more specific, we have

(H$^-$) $F$ is a decreasing function of $\rho$, equal to $c$ at 0 and equal to $c_- > 0$ on $Y$;

(H$^-$) $F$ has a non-degenerate maximum at 0 and the eigenvalues of the quadratic form $d^2 F(0)$ on the symplectic vector space $T_0B$ are small;

(H$^-$) there exists a family of linear symplectic transformations $\Psi_s, s \in [0, 1]$, with $\Psi_0 = id$ and $\Psi_1 = \Psi$ such that $F_s := F \circ \Psi_s \leq H_+$ and $F_s \equiv c_-$ on $Y$ for all $s$. 

Closed Reeb Orbits on the Sphere
The existence of functions $H_{\pm}$ with these properties is guaranteed by (H2) and (H3). (Note that the main point of (H2) is that for any $\eta_1 > 0$ one can construct $F$ such that the eigenvalues of $d^2F(0)$ are smaller than $\eta_1$.) In addition, the functions $H_{\pm}$ and $F$ are required to satisfy some other natural conditions such as those for the bump functions in [Gi2, Section 7.2]. These conditions are not explicitly used in the argument below.

Remark 4.5. It is worth pointing out that this construction of $H_-$ differs slightly from the one used in [Gi2], where $H_-\equiv$ is obtained from $F$ by applying a loop of Hamiltonian diffeomorphisms to the flow of $F$ rather than composing $F$ with $\Psi$. Both, here and in the Hamiltonian setting, either construction can be utilized, for only the existence of an isospectral deformation from $F$ to $H_-$ majorated by $H_+$ is essential; see (H_3). However, we find our present approach somewhat simpler than the original one. Furthermore, in both settings, it would be convenient to replace the condition that $H_\equiv c$ near zero in $(H_+)$ by that $d^2H(0) = 0$. (Then, for instance, $H_+$ could be taken independent of $c$.) However, we keep $(H_+)$ in its present form to simplify references to the arguments in [Gi2, GG1, GG2].

Consider now the symplectic manifold $\Pi = U \times L = B \times S^1 \times L$ with the symplectic form $\tilde{\omega} = \omega + dh \wedge dt$, where $L \subset \mathbb{R}$ is some interval containing $c$, and $h$ is the coordinate on $L$. We can choose $L$ to be small and such that $\Pi$ contains the graphs of $H_{\pm}$, $F$ and $H$. (Recall that the functions $H_{\pm}$ are $C^0$-close to $H$, and clearly $H$ has small variation on $U$.) There exists a symplectic embedding of $\Pi$ into the symplectization $\hat{M} = M \times (0, \infty) \subset T^*M$ sending the graph of $H$ to $U$ and such that the pull-back of $\tau\alpha$, where $\tau$ is the coordinate on $(0, \infty)$, is the form $\lambda + hdt$. (Here we have identified $M$ and the graph of $\alpha$ in $\hat{M}$ and $M \times \{1\} \subset \hat{M}$.)

From now on we treat $\Pi$ as a subset of $\hat{M}$. It is important however to keep in mind that such an embedding does not in general send the fibers $L$ of the projection $\Pi \to U$ to the fibers $(0, \infty)$ of the projection $\pi: \hat{M} \to M$.

Let $U_{\pm}$ be the images of the graphs of $H_{\pm}$ in $\hat{M}$. These images are contact submanifolds in $\hat{M}$ (with the restriction of $\tau\alpha$ taken as a contact form) lying, in the obvious sense, above and below $U$. Furthermore, the contact submanifolds $U_{\pm}$ in $\hat{M}$ extend to contact submanifolds $M_{\pm}$, contactomorphic to $(M, \ker\alpha)$ and lying again above and below $M$ in $\hat{M}$. We use the notation $\alpha_{\pm}$ for the resulting contact forms obtained by restricting $\tau\alpha$ to $M_{\pm}$.

In other words, we have constructed two contact embeddings $M_{\pm}$ of $M$ into $\hat{M}$, lying above and below $M$ and $C^0$-close to $M$. The pull-backs $\alpha_{\pm}$ of $\tau\alpha$ to $M_{\pm}$ are contact forms such that $\alpha_{\pm} = f_{\pm}\alpha$, where $f_+ > 1$ and $f_- < 1$, on the complement of, say, $B_{R/2} \times S^1$. Moreover, it is not hard to show that the contact structures $\ker\alpha_{\pm}$ are isotopic to $\ker\alpha$ with support in $U$.

Remark 4.6. For a suitable embedding $\Pi \hookrightarrow \hat{M}$, it is not hard to guarantee that $\alpha_+$ corresponds to the differential form $f_+\alpha|_U$ with $f_+ \geq 1$. Whether the image of $\alpha_-$ corresponds to a differential form or not, (i.e., if it is transverse to the fibers of $\pi$) is less clear. (This would be the case if $H_-$ was rotationally symmetric, i.e., when $H_\equiv = F$.) However, the image of $\alpha_-$ corresponds to the differential form $f_-\alpha$ with $f_- < 1$ near the boundary of $U$.)

Finally, the part of $\hat{M}$ containing $M$ and $M_{\pm}$ can be symplectically embedded into the completion $\hat{W} = W \cup M \times [1, \infty)$. As a consequence, we obtain exact
symplectic fillings $W_{\pm}$ of $(M, \alpha_{\pm})$. Furthermore, the region $V \subseteq \hat{M}$ bounded by $M_-$ and $M_+$ can be treated as a symplectic cobordism from $M_-$ (the negative end) to $M_+$ (the positive end). (Strictly speaking $M_+$ and $M_-$ intersect along $x$.

For the sake of brevity, we ignore this issue; for the intersection can be eliminated by an arbitrarily small perturbation of $M_-$, which, in turn, would only have an arbitrarily small effect on the action filtration.) Clearly, $V$ factors as a composition of cobordisms from $(M, \alpha_-)$ to $(M, \alpha)$ and from $(M, \alpha)$ to $(M, \alpha_+)$. Thus, for any (generic) action interval $I$, we obtain the maps

$$
HC^I_\ast(M, \alpha_+) \to HC^I_\ast(M, \alpha) \to HC^I_\ast(M, \alpha_-),
$$

where the contact homology of $(M, \alpha_{\pm})$ is taken with respect to the filling $W_{\pm}$ introduced above. To prove the theorem, it now suffices to show that the map

$$
HC^I_\ast(\kappa, \kappa+\epsilon)(M, \alpha_+) \to HC^I_\ast(\kappa, \kappa+\epsilon)(M, \alpha_-)
$$

is non-zero in degree $d_\kappa + 1$.

Before turning to the proof of (4.3), let us elaborate on the inter-dependence of various ingredients of the above construction. The constants $c_{\pm}$, depending only on $U$ and $H$, are fixed first. The upper limit of $\epsilon$ and the lower bound $\kappa_0(\epsilon)$ for $\kappa$ depend on $U$, $H$ and these constants. Then we chose the function $H_+$, which depends on $H$ and $U$ and $\epsilon$ and, of course, $c_+$ . Finally, the function $H_-$ is then chosen individually for each $\kappa$. Specifics of extending $\alpha_{\pm}$ beyond II are immaterial.

4.2.2. Direct sum decomposition of contact homology. Our goal in this section is to describe a contact analogue of the direct sum decomposition of filtered Floer homology from [GG2, Section 5.1].

Consider the class of contact forms $\beta$ on $M$ satisfying the following conditions:

(B1) $\beta = \lambda + K dt$ on $U$, where $K > 0$ is a function on $U$ taking values in $L$, and $K$ is constant on $Y \times S^1$;

(B2) $\beta = f \alpha$ outside $U$.

Remark 4.7. The condition (B2) is not essential, but it is convenient to have it in what follows. Note that, as a consequence, $(M, \beta)$ is fillable by $W$ with a symplectic structure modified near the boundary. A variant of such a fillability requirement would be sufficient for our purposes. The assumption that $K$ takes values in $L$ can also be relaxed.

Among the forms in this class are $\alpha_{\pm}$ and the form $\alpha_F$ which is equal to $\alpha_-$ outside $U$ and to $\lambda + F dt$ in $U$.

Set $a := K |_{Y \times S^1}$. Let $I$ be a interval, not containing any of the points $\{ka | k \in \mathbb{N}\}$. We require this interval to be sufficiently short: $|I| < \epsilon_U$, where $|I|$ stands for the length of $I$ and the value of $\epsilon_U$ is to be specified shortly. Fix $\kappa$, and let $\gamma_\kappa$ be the free homology class of $\kappa S^1 \subset U$ in $W$. (Note that $\gamma_\kappa = 0$ when $S^1$ is contractible in $W$.)

Even though we are working with the linearized contact homology, we can view the filtered contact homology complex of $\beta$ as generated by the closed Reeb orbits of a small non-degenerate perturbation of $\beta$. Thus let us first assume that all closed Reeb orbits of $\beta$ except those in $Y \times S^1$ are non-degenerate. (It would be sufficient to require this only for the orbits with action in $I$ and in the class $\gamma_\kappa$.) Let finally $CC^I_\ast(U, \beta, \kappa[S^1])$ be the graded subspace of the complex $CC^I_\ast(\beta, \gamma_\kappa)$ generated by the orbits lying in $U$ in the homotopy class $\kappa[S^1]$. 
Lemma 4.8. There exists a constant $\epsilon_U > 0$, depending only on $U$ and $\lambda$ (or rather on $\lambda|_{Y \times S^1}$) such that $CC^i_c(U, \beta, \kappa[S^1])$ is a subcomplex and, in fact, a direct summand in the complex $CC^i_c(\beta, \gamma)$ whenever $|I| < \epsilon_U$. In other words, we have the direct sum decomposition

$$CC^i_c(\beta, \gamma) = CC^i_c(U, \beta, \kappa[S^1]) \oplus CC^i_c(M \setminus U, \beta, \gamma)$$

for some subcomplex $CC^i_c(M \setminus U, \beta, \gamma)$. As a consequence, we also obtain a direct sum decomposition

$$HC^i_c(\beta, \gamma) = HC^i_c(U, \beta, \kappa[S^1]) \oplus HC^i_c(M \setminus U, \beta, \gamma) \quad (4.4)$$

regardless of whether $\beta$ is non-degenerate or not, provided that the end points of $I$ are outside the action spectrum of $\beta$, the interval $I$ does not contain any of the points $\{ka \mid k \in \mathbb{N}\}$, and $|I| < \epsilon_U$.

Remark 4.9. By definition, the complex $CC^i_c(M \setminus U, \beta, \gamma)$ is generated by the orbits that are not in $CC^i_c(U, \beta, \kappa[S^1])$. Note that in spite of the notation these orbits can be contained in $U$. The exact nature of this complex and of its homology is inessential for our purposes.

This lemma is a contact analogue of [GG2, Lemma 5.1] and it can be proved exactly in the same fashion. (See also [He2].) The key observation is that for any holomorphic curve $u$ in $M$ its image in $M$ further projects to a holomorphic curve in $Y$, i.e., the map $\pi(u) \cap (Y \times S^1) \rightarrow Y$ is holomorphic. Since crossing the shell $Y$ requires for a holomorphic curve to have some positive minimal energy $\epsilon_U$, the lemma follows.

The important feature of the group $HC^i_c(U, \beta, \kappa[S^1])$ is that it can be evaluated by entirely local means, by working only with closed Reeb orbits in $U$ and holomorphic cylinders in the symplectization of $U$.

Let now $\beta_0$ and $\beta_1$ be two forms in the above class. Assume that these forms bound a symplectic cobordism from $\beta_1$ (the negative end) to $\beta_0$ (the positive end) in $M$. To be more specific, let us require (although these conditions can be considerably relaxed) that $\beta_1 = f_i \alpha$ outside $U$ with $i = 0, 1$, and $f_0 \geq f_1$ and that $\beta_1 = \lambda + K dt$ in $U$ with $K_0 \geq K_1$. Then the decomposition $(4.4)$ is preserved by the natural map from $HC^i_c(\beta_0, \gamma)$ to $HC^i_c(\beta_1, \gamma)$, provided that $I$ does not meet the sets $\{ka_0 \mid k \in \mathbb{N}\}$ and $\{ka_1 \mid k \in \mathbb{N}\}$, where $a_i = K_i |_{Y \times S^1}$. In other words, we have the commutative diagram:

$$\begin{array}{ccc}
HC^i_c(\beta_0, \gamma) & = & HC^i_c(U, \beta_0, \kappa[S^1]) \oplus HC^i_c(M \setminus U, \beta_0, \gamma) \\
\downarrow & & \downarrow \\
HC^i_c(\beta_1, \gamma) & = & HC^i_c(U, \beta_1, \kappa[S^1]) \oplus HC^i_c(M \setminus U, \beta_1, \gamma)
\end{array}$$

Applying this to $\alpha_+ = \beta_0$ and $\alpha_- = \beta_1$ with $I = (\kappa \alpha, \kappa \alpha + \epsilon)$, we see that it suffices to show that the map

$$HC^{(\kappa \alpha, \kappa \alpha + \epsilon)}_{d_{s+1}}(U, \alpha_+, \kappa[S^1]) \rightarrow HC^{(\kappa \alpha, \kappa \alpha + \epsilon)}_{d_{s+1}}(U, \alpha_-, \kappa[S^1]) \quad (4.5)$$

is non-zero when all of the above requirements are satisfied. Note that the map $(4.5)$ is completely independent of the behavior of $\alpha_0$ outside $U$ or of the topology of $(M, \ker \alpha)$, which is why the particulars of extending $\alpha_0$ beyond $U$ are not essential for the proof. We will examine this map more closely in the next section.
Remark 4.10. By the very definition, the complex $CC_* (U, \beta, \kappa[S^1])$ is generated by the $\kappa$-periodic orbits of $K$ (with the exception of the bad orbits). Hence, the first step in calculating the homology $HC_* (U, \beta, \kappa[S^1])$ is that of analyzing the periodic orbits of $K$, which is a problem in Hamiltonian dynamics; see Section 4.2.3. Taking this point one step further, one can show that $HC_1^I (U, \beta, \kappa[S^1]) = HF^Z_{\kappa \cdot I} (K)$, where the right-hand side is the $\mathbb{Z}_\kappa$-equivariant filtered Floer homology of $K$ on $B$, which is defined as long as $|I| < \varepsilon_U$. The proof of this fact is completely analogous to the proof of Theorem 2.1. This consideration applies to the cobordism maps as well.

Remark 4.11. The condition that the interval $I$ does not intersect the set $\{ka \mid k \in \mathbb{N}\}$ is purely technical and can be eliminated by slightly modifying our requirements on the form $\beta$. This can be done by constructing a contact analogue of the direct sum decomposition of Floer homology from [He2]; cf. [Us].

4.2.3. Homological calculation. Our goal is to show that the map (4.5) is non-zero, which implies the assertion of the theorem. In this section, we will prove that, for a suitable choice of the parameters of the construction, this map is in fact an isomorphism from $Q$ to $Q$:

$Q = HC^{(\kappa, \kappa + \varepsilon)}_{d_* + 1} (U, \alpha_+, \kappa[S^1]) \xrightarrow{\cong} HC^{(\kappa, \kappa + \varepsilon)}_{d_* + 1} (U, \alpha_-, \kappa[S^1]) = Q.$

The proof of (4.6) is based, in particular, on the following result applied to the contact homology localized to $U$ as above.

Proposition 4.12 (Invariance of the Filtered Contact Homology). Let $\beta_s$, $s \in [0, 1]$, be a family of contact forms on $M$ and $I_s$ a family of intervals such that for every $s$ the end points of $I_s$ are outside the action spectrum of $\beta_s$. Then the contact homology spaces $HC_*^F (\beta_s)$ are isomorphic. Furthermore, assume that the forms $\beta_s$ foliate, in the obvious sense, a cylindrical cobordism from $\beta_1$ to $\beta_0$ and the interval $I_s = I$ is fixed. Then the natural map $HC_*^F (\beta_0) \rightarrow HC_*^F (\beta_1)$ is an isomorphism. Finally, to have an isomorphism in a particular degree $* = k$, it is sufficient to assume only that the end points of $I_s$ are outside the action spectra corresponding to the orbits of degree $k$ and $k \pm 1$.

This proposition can be proved exactly in the same way as its Floer homological counterparts; see, e.g., [BPS, Gi1, Vi2].

As the first application, by (H–3), we have the following commutative diagram

$HC_*^F (U, \alpha_+, \kappa[S^1]) \xrightarrow{\cong} HC_*^F (U, \alpha_-, \kappa[S^1]) \xrightarrow{\cong} HC_*^F (U, \alpha_F, \kappa[S^1])$

where the form $\alpha_F$ is defined similarly to $\alpha_-$, but with $F$ in place of $H$; see Section 4.2.2. This is a contact analogue of, say, the results in [GG2, Sections 2.2.1 and 5.3.2]. As a consequence of (4.7), we may replace $\alpha_-$ by $\alpha_F$ in (4.6). Thus, to complete the proof of the theorem, we need to establish the isomorphism

$Q = HC^{(\kappa, \kappa + \varepsilon)}_{d_* + 1} (U, \alpha_+, \kappa[S^1]) \xrightarrow{\cong} HC^{(\kappa, \kappa + \varepsilon)}_{d_* + 1} (U, \alpha_F, \kappa[S^1]) = Q.$

Let us now specify the parameters of the construction, needed to ensure that (4.8) holds. As a starting point, the form $\alpha$ is given and the set $\Pi$ is fixed in advance, as are the function $H$ satisfying the requirements (H1-H3) and the interval $L$. Next
we pick \( c_{\pm} \). Here, in addition to \( c_{\pm} \in L \), we require the points \( c_{\pm} \) to be relatively close to \( c \), e.g., \( |c - c_{\pm}| < c/2 \). It is also convenient to take these points from the set \( Q_c \). Thus \( c_{\pm} = p_{\pm}c/q_{\pm} \) for some integers \( p_{\pm} \) and \( q_{\pm} \).

As the next step, we determine the upper bound on \( \epsilon > 0 \). Namely, we have \( \epsilon < c_{U} \) and \( \epsilon \) is small relative to \( |c - c_{\pm}| \) (e.g., \( \epsilon < |c - c_{\pm}|/2 \)) and, finally, \( \epsilon < c/\max\{q_{-}, q_{+}\} \). It is easy to see that under the latter requirement, for any sufficiently large \( \kappa \), the interval I = \((\kappa c, \kappa c + \epsilon)\) does not intersect the sets \( \{kc_{\pm} \mid k \in \mathbb{N}\} \). (Without the assumption that \( \epsilon < c/\max\{q_{-}, q_{+}\} \), this would be true only for an infinite sequence of iterations \( \kappa \).) Hence the machinery from Section 4.2.2 applies. We pick \( \kappa \) meeting the latter requirement and such that

\[
\kappa|c - c_{\pm}| > \pi r^2. \tag{4.9}
\]

The function \( H_{+} \) (with \( c_{+} \) as above) is required to meet the condition \((H_{+})\) and be such that \( \pi r^2 < \epsilon \), where \( r \) is the radius of the ball on which \( H \equiv c \). Note that this function can be chosen independently of \( \kappa \).

Finally, we fix \( H_{-} \) (with \( c_{-} \) as above) satisfying \((H_{-1-H}, 3)\). Here, \((H_{-2})\) reads specifically as that the eigenvalues of \( d^2F(0) \) are so small (while still positive) that the Conley–Zehnder index of \( \kappa d^2F(0) \) is \( n - 1 \), i.e., all eigenvalues of the latter quadratic form are smaller than \( \pi \).

With these conditions in mind, there are several (ultimately not so different) ways to establish (4.8) and thus finish the proof of the theorem.

First of all, note that the fact that both of the homology groups in (4.8) are indeed \( \mathbb{Q} \) (under the assumptions \((H_{-2})\) and (4.9)) can be proved by simple analysis of the \( \kappa \)-periodic orbits of \( H_{+} \) and \( F \), which is done in, e.g., [Gi2, GG1]. (Here the picture becomes particularly transparent once the Morse–Bott approach to contact homology is utilized; see [Bou] and references therein. Note also that the action of \( \alpha_{+} \) on the corresponding orbit is approximately equal to \( \kappa c + \pi r^2 \) for large \( \kappa \) and \( H_{+} \) constructed exactly as in [Gi2, Section 7.3].) Then to prove that the map (4.8) is an isomorphism, it suffices to analyze the behavior of \( \kappa \)-periodic orbits for a suitable homotopy from \( H_{+} \) to \( F \) and apply Proposition 4.12, cf. [GG1].

Alternatively, one can argue as follows (cf. [GG2]). Let us fix a sufficiently small parameter \( \delta > 0 \) and suppressing \( U \) and \( \kappa[S^1] \), which are fixed now, in the notation let us write \( HC_{d_{\alpha}}(\beta) \) for \( HC_{d_{\alpha}}(U, \beta, \kappa[S^1]) \). (Note that \( \delta \) depends on \( H_{+}, F, \epsilon, \) and \( \kappa \). In particular, \( \delta \ll \epsilon \).) Then, as is easy to see (cf. [GG1]),

\[
HC_{d_{\alpha}}(\kappa c - \delta, \kappa c + \delta)(\alpha_{+}) = \mathbb{Q} = HC_{d_{\alpha}}(\kappa c - \delta, \kappa c + \delta)(\alpha_{F})
\]

and, moreover, the natural continuation map between these spaces is an isomorphism. Here the main point is that the orbit \( x^\kappa \) is the only generator of degree \( d_{\kappa} \) for the contact complexes of \( \alpha_{+} \) and \( \alpha_{F} \), where for \( \alpha_{+} \) we slightly perturb \( H_{+} \) to make \( 0 \in B \) into a non-degenerate maximum; cf. [Gi2, Lemma 2.5].

Next, focusing first on \( \alpha_{+} \), we have

\[
HC_{d_{\alpha} + 1}(\kappa c - \delta, \kappa c + \epsilon)(\alpha_{+}) = 0 \quad \text{and} \quad HC_{d_{\alpha}}(\kappa c - \delta, \kappa c + \epsilon)(\alpha_{+}) = 0.
\]

These identities are proved, for instance, by deforming \( H_{+} \) to a Hamiltonian without periodic orbits of required index and applying Proposition 4.12. Thus, from the long exact sequence

\[
\cdots \rightarrow HC_{*}(\kappa c - \delta, \kappa c + \delta)(\alpha_{+}) \rightarrow HC_{*}(\kappa c - \delta, \kappa c + \epsilon)(\alpha_{+}) \rightarrow HC_{*}(\kappa c + \delta, \kappa c + \epsilon)(\alpha_{+}) \rightarrow \cdots
\]
we conclude that the connecting map
\[ \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d+1}(\alpha_+) \overset{\cong}{\longrightarrow} \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d}(\alpha_+) = \mathbb{Q} \]
is an isomorphism.

This argument applies to $\alpha F$ word-for-word and we also have the isomorphism
\[ \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d+1}(\alpha F) \overset{\cong}{\longrightarrow} \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d}(\alpha F) = \mathbb{Q} \]
Combining these facts, we arrive at the commutative diagram
\[ \begin{array}{ccc}
\text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d+1}(\alpha_+) & \overset{\cong}{\longrightarrow} & \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d}(\alpha_+) = \mathbb{Q} \\
\downarrow & & \downarrow \cong \\
\text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d+1}(\alpha F) & \overset{\cong}{\longrightarrow} & \text{HC}^{(\kappa \delta, \kappa \delta + \epsilon)}_{d}(\alpha F) = \mathbb{Q}
\end{array} \]
where the horizontal maps and the second vertical map are isomorphisms. Hence the first vertical map is also an isomorphism. This (together with Remark 4.2) implies (4.8).

Finally, one may take advantage of the isomorphism between $\text{HC}^{\kappa}(U, \beta, \kappa[S^1])$ and $\text{HF}_{-\infty}^{\kappa}(K)$ from Remark 4.10. It is not hard to see that for the Hamiltonians $H_{\pm}$ and $F$ the $\mathbb{Z}_\kappa$-action on the Floer complex is trivial, and hence proving (4.6) and (4.8) amounts to establishing isomorphisms in the ordinary Floer homology, which is done exactly in this setting in [GG1].

5. Appendix: Equivariant and Invariant Morse Homology

In this section, we illustrate the constructions from Section 2.1 by analyzing a finite-dimensional model in which the arguments are particularly transparent.

Consider a Morse function $f$ on a closed oriented manifold $M$. To define the Morse complex $\text{CM}_*(f)$ over $\mathbb{Z}$ or $\mathbb{Q}$, we fix a Riemannian metric such that the stable and unstable manifolds (for the anti-gradient flow) of the critical points of $f$ intersects transversely, and assign an orientation to every unstable manifold. The complex $\text{CM}_*(f)$ is then generated by the critical points of $f$ and graded by the Morse index. The differential $\partial x = \sum \mu(x, y)y$ is defined by counting anti-gradient trajectories connecting a point $x$ of index $k$ and with points $y$ of index $k-1$ with signs depending on whether the intersection orientation of a trajectory (induced by the orientation of $M$ and those of the stable/unstable manifolds) matches its orientation as a flow line. It is important that the differential depends on the metric even though the graded vector space $\text{CM}_*(f)$ is completely determined by the function $f$ only. Note also that the transversality requirement, often referred to as the Morse–Smale condition, is satisfied for a generic metric. The homology $\text{HM}_*(M)$ of the complex $(\text{CM}_*(f), \partial)$ is isomorphic to $\text{H}_*(M)$ and called the Morse homology of $f$. (See, e.g., [Jo, Chap. 7] for more details.)

Let now $G$ be a finite group acting on $M$ by orientation preserving diffeomorphisms. Assume that $f$ is $G$-invariant. Then the equivariant Morse complex $\text{CM}^G_*(f)$ of $f$ and the equivariant Morse homology $\text{HM}^G_*(f)$ are defined as follows. Fix a sequence of finite-dimensional smooth approximations $EG_N \to BG_N$ of the universal bundle $EG \to BG$. More specifically, this is a sequence of closed smooth manifolds $EG_N$ with free $G$-action, such that $\pi_k(EG_N) = 0$ for $k = 1, \ldots, k_N \to \infty$, and a sequence of $G$-equivariant embeddings $EG_N \hookrightarrow EG_{N+1}$. We set $BG_N =
EG_N/G. (Such approximations can be obtained, for instance, by taking a faithful representation G \to U(m) and letting EG_N be the Stiefel manifold of unitary m-frames in C^N with G acting via the standard U(m)-action. When G = \mathbb{Z}_n, we have EG_N = S^{2n-1} with \mathbb{Z}_n acting diagonally, and BG_N is a lens space.) Furthermore, pick a sequence of G-invariant Morse functions h_N on \text{EG}_N such that h_{n+1}|_{\text{EG}_N} = h_N and the critical points of h_N are necessarily critical points of h_{n+1} of the same index. Finally, we also require that all critical points of h_{n+1} outside \text{EG}_N have index grater than some \mu_N \to \infty. (This is an extra condition on both h_N and \text{EG}_N needed to ensure that the Morse complexes CM_\ast(f_N) of the functions f_N on the finite-dimensional approximations M_N \to (EG \times M)/G defined below stabilize as N \to \infty.)

The function f + h_N descends to a smooth Morse function f_N on the quotient M_N = (EG \times M)/G. (The quotient is smooth since G acts freely on \text{EG}_N \times M, and we should think of M_N as a finite-dimensional approximation to \text{(EG \times M)}/G.) Note that again f_{n+1}|_{M_N} = f_N, the critical points of f_N are necessarily critical points of f_{n+1} of the same index, and all critical points of f_{n+1} outside M_N have index greater than \mu_N \to \infty. Pick a sequence of Riemannian metrics on M_N such that M_N \subset M_{N+1} is invariant under the gradient flow of f_{N+1} and the Morse–Smale condition is satisfied for each f_N. Clearly, there are natural maps of complexes CM_\ast(f_N) \to CM_\ast(f_{N+1}) and, moreover, these complexes stabilize in every fixed range of degrees as N \to \infty. By definition, the equivariant Morse complex of f is CM_G(f) := \lim_{N \to \infty} CM_\ast(f_N) and its homology HM_G(f) = \lim_{N \to \infty} HM_\ast(f_N) is the equivariant Morse homology of f. Note that this complex does depend on the approximation scheme (EG_N, h_N), while its homology is independent of the approximation and isomorphic to the equivariant homology H_G^M(M) of M.

Next let us assume that M admits a G-invariant metric which satisfies the Morse–Smale condition. In contrast with the non-equivariant case, such metrics need not exist as simple examples show. Define an action of G on CM_\ast(f) by setting g \in G to send a critical point x, a generator of CM_\ast(f), to \pm g(x). Here g(x) \in M is the image of x under the map g and the sign is determined by whether or not g matches the orientations of the unstable manifold of x and the unstable manifold of g(x). This choice of signs guarantees that we indeed have a G-action on the complex CM_\ast(f), i.e., that the action and \partial commute. Hence we also obtain a G-action on HM_\ast(f). In particular, we have the invariant subcomplex CM_\ast(f)^G and the invariant part HM_\ast(f)^G of homology.

All these constructions extend in a straightforward way to the filtered Morse homology and to the local Morse homology (see, e.g., [Gi2, Section 3.1]).

Up to this point, the choice of the coefficient ring was immaterial: it could be \mathbb{Z} or any other ring or field. Moreover, our construction of the equivariant Morse homology would go through for any compact Lie group G. However, from this moment on, it becomes essential that all the complexes and homology groups are taken over \mathbb{Q}, or more generally over a field of zero characteristic, and that the group G is finite. First, note that under these conditions, HM_\ast(f)^G is the homology of the complex CM_\ast(f)^G and that HM_\ast(f)^G is isomorphic to H_\ast(M/G). More importantly, we have

**Proposition 5.1.** When the ground field is \mathbb{Q} and G is finite, the equivariant and invariant Morse homology groups are isomorphic: HM_G(f) = HM_\ast(f)^G. The same holds for filtered and local Morse homology.
Remark 5.2. Even for filtered or local Morse homology, this proposition is but a very particular case of the identification $H^G_\ast(M) = H_\ast(M/G)$, which holds over $\mathbb{Q}$ for any action with finite stabilizers of a compact Lie group $G$ on any finite CW-complex. (We refer the reader to, e.g., [GGK, Appendix C] and references therein for this and other standard results on equivariant (co)homology used in this section.) However, what is important for us is that Proposition 5.1 is essentially Morse-theoretic, and hence can be translated to the realm of Floer homology; cf. the proof of Theorem 2.1.

The following simple example illustrates the role of signs in the construction of the invariant Morse complex.

Example 5.3 (Signs). Let $f$ be a hyperbolic quadratic form on $M = \mathbb{R}^2$ and let $G = \mathbb{Z}_2$ act by central symmetry. Clearly, $f$ is a Morse function with only one critical point, the origin $0$. We denote by $x$ the corresponding generator of the local Morse homology $CM_\ast(f, 0)$. Thus $CM_\ast(f, 0)$ and $HM_\ast(f, 0)$ are both equal to $\mathbb{Q}$ and concentrated in degree zero. The central symmetry inverses the orientation of the unstable manifold of $0$, and hence sends $x$ to $-x$. As a consequence, $CM_\ast(f, 0)^G = 0$ and $HM_\ast(f, 0)^G = 0$. Likewise, it is not hard to see (for instance, using the Morse–Bott construction of the equivariant Morse homology, cf. [Bot, Lecture 3]) that $HM_\ast^G(f, 0) = 0$. Indeed, $HM_\ast^G(f, 0)$ is isomorphic to the homology of $\mathbb{R}P^\infty$ with twisted rational coefficients corresponding to the double cover of $\mathbb{R}P^\infty$. This homology is zero.

Proof of Proposition 5.1. We will prove the proposition for the ground field $\mathbb{C}$. Then, as is easy to show, the result for $\mathbb{Q}$ follows.

Under our assumptions on $f$, we can equip $EG_N \times M$ with a product metric satisfying the Morse–Smale condition. Then $CM_\ast(f + h_N) = CM_\ast(f) \otimes CM_\ast(h_N)$, and hence

$$CM_\ast^G(f) = (CM_\ast(f) \otimes CM_\ast(h))^G,$$

where we set $CM_\ast(h) := \lim_{N \to \infty} CM_\ast(h_N)$.

Let us decompose the complexes $CM_\ast(f)$ and $CM_\ast(h)$, viewed as representations of $G$, into the sum of isotypical components:

$$CM_\ast(f) = CM_\ast(f)^G \oplus \bigoplus_\sigma V_\sigma$$

and

$$CM_\ast(h) = CM_\ast(h)^G \oplus \bigoplus_\eta W_\eta,$$

where direct sums range over all non-trivial irreducible complex representations $\sigma$ and $\eta$ of $G$. It readily follows from Schur’s lemma that these are indeed direct sums of complexes.

We have

$$\left( CM_\ast(f) \otimes CM_\ast(h) \right)^G = \left( CM_\ast(f)^G \otimes CM_\ast(h)^G \right) \oplus \bigoplus_{\sigma, \eta} (V_\sigma \otimes W_\eta)^G.$$ \hspace{1cm} (5.1)

For a product of an isotypical component corresponding to the trivial representation and the one corresponding to $\sigma$ or $\eta$ (e.g., $CM_\ast(f)^G \otimes W_\eta$) is again $\sigma$- or $\eta$-isotypical, and hence its invariant part is zero.

The homology of the complex $CM_\ast(h)^G$ is the group homology $H_\ast(G)$ and the complex $CM_\ast(h)$ is acyclic (except degree zero). Since $G$ is finite, $H_\ast(G; \mathbb{C}) = \mathbb{C}$, concentrated in degree zero. It follows that all complexes $W_\eta$ are acyclic. Thus only the first term in (5.1) makes a non-trivial contribution to the homology and $HM_\ast^G(f) = HM_\ast(f)^G \otimes \mathbb{C} = HM_\ast(f)^G$ as required.
Remark 5.4 (Multi-valued perturbations). A generalization of the proposition to the case where there is no $G$-invariant metric satisfying the Morse–Smale condition would have to rely on a variant of the machinery of multi-valued perturbations. (See [FO, FOOO, LT, HWZ5, HWZ6] and references therein, and also Remark 2.5.) This machinery should enable one to construct a version of the Morse complex with a natural $G$-action even when a $G$-invariant Morse–Smale metric does not exist. There is a good conceptual understanding of how this could be done, but this is still a non-trivial task and, to the best of our knowledge, there are no published accounts of such a construction. (Once this is accomplished, the proof of Proposition 5.1 should go through word-for-word.) This problem can be thought of as a Morse theoretic analogue of the transversality problem arising in the definition of the contact homology, cf. Section 2.1 and Theorem 2.1.

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