SUBREPRESENTATION THEOREM FOR
p-ADIC SYMMETRIC SPACES

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Abstract. The notion of relative cuspidality for distinguished representations attached to p-adic symmetric spaces is introduced. A characterization of relative cuspidality in terms of Jacquet modules is given and a generalization of Jacquet’s subrepresentation theorem to the relative case (symmetric space case) is established.

Introduction

Let $G$ be a reductive $p$-adic group, $\sigma$ an involution on $G$ and $H$ the subgroup of all $\sigma$-fixed points in $G$. An admissible representation $(\pi, V)$ of $G$ is said to be $H$-distinguished if the space $(V^*)^H$ of all $H$-invariant linear forms on $V$ is non-zero. Such a representation arises as a local component of automorphic representations which are of particular interest (see [J] for example). From representation theoretic point of view, distinguished representations are the basic object of harmonic analysis on the $p$-adic symmetric space $G/H$. By the Frobenius reciprocity, these representations can be realized in the space of smooth functions on $G/H$. The classification or parametrization of such representations would be a fundamental problem. Over the real field, harmonic analysis on semisimple or reductive symmetric spaces has been fully developed since 1980’s (see [O] and [D]). By contrast, the theory over $p$-adic fields has not been developed yet.

In this paper, we suggest a new basic tool for the study of harmonic analysis on $p$-adic symmetric spaces, and establish the relative version (for general $p$-adic symmetric spaces) of Jacquet’s subrepresentation theorem (for general $p$-adic groups).

Jacquet’s subrepresentation theorem asserts that any irreducible admissible representation of a reductive $p$-adic group can be embedded in a parabolically induced representation. The inducing representation of a Levi subgroup can be taken as an irreducible cuspidal one, that

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is, a representation whose matrix coefficients are compactly supported modulo the center. To establish the relative version of this theory, first we introduce the notion of relative cuspidality as follows. An $H$-distinguished representation of $G$ is said to be $H$-relatively cuspidal if all the generalized matrix coefficients defined by $H$-invariant linear forms (which we shall call $H$-matrix coefficients) are compactly supported modulo the product of $H$ and the center of $G$. We use only such ones as the inducing representations. Besides, as the inducing subgroups, we use only a particular class of parabolic subgroups. A parabolic subgroup $P$ is said to be $\sigma$-split if $P$ and $\sigma(P)$ are opposite. In this case, $M = P \cap \sigma(P)$ is a $\sigma$-stable Levi subgroup of $P$ and the quotient $M/(M \cap H)$ gives a symmetric space of lower rank. Now we state the main theorem of this paper.

**Theorem A** (Theorem 7.1). Let $\pi$ be an irreducible $H$-distinguished representation of $G$. Then there exists a $\sigma$-split parabolic subgroup $P$ of $G$ and an irreducible $M \cap H$-relatively cuspidal representation $\rho$ of $M = P \cap \sigma(P)$ such that $\pi$ can be embedded in the induced representation $\text{Ind}_P^G(\rho)$.

This gives a generalization of Jacquet’s subrepresentation theorem in the following sense. Take an arbitrary reductive $p$-adic group $G_1$ and form the direct product $G = G_1 \times G_1$. Let $\sigma$ be the involution on $G$ which permutes factors. Then the $\sigma$-fixed point subgroup in $G$ is the diagonally embedded subgroup $\Delta G_1$. An irreducible $\Delta G_1$-distinguished representation of $G_1 \times G_1$ is of the form $\pi_1 \otimes \tilde{\pi}_1$ where $\pi_1$ is an irreducible admissible representation of $G_1$ and $\tilde{\pi}_1$ is the contragredient of $\pi_1$. It is $\Delta G_1$-relatively cuspidal if and only if $\pi_1$ is cuspidal in the usual sense. A $\sigma$-split parabolic subgroup of $G_1 \times G_1$ is of the form $P_1 \times P_1^-$ where $P_1$ and $P_1^-$ are opposite parabolics of $G_1$. Now, Theorem A applied to this situation will give an embedding of any irreducible admissible representation $\pi_1 \otimes \tilde{\pi}_1$ into

$$\text{Ind}_{P_1 \times P_1^-}^{G_1 \times G_1}(\rho_1 \otimes \tilde{\rho}_1) \simeq \text{Ind}_{P_1}^{G_1}(\rho_1) \otimes \text{Ind}_{P_1^-}^{G_1}(\tilde{\rho}_1)$$

where $\rho_1$ is an irreducible cuspidal representation of a Levi subgroup of $P_1$. In this way, Jacquet’s subrepresentation theorem can be recovered from our theorem. We have an embedding $\pi_1 \hookrightarrow \text{Ind}_{P_1}^{G_1}(\rho_1)$ on the first factor, and also $\tilde{\pi}_1 \hookrightarrow \text{Ind}_{P_1^-}^{G_1}(\tilde{\rho}_1)$ on the second factor at the same time.

In the process of obtaining Theorem A, we refer to Casselman’s proof of Jacquet’s subrepresentation theorem as a prototype. In [C], a canonical pairing of Jacquet modules is constructed. It provides a relation of the asymptotic behaviors between matrix coefficients for a given
representation and those for its Jacquet modules. By this relation, the well-known characterization of cuspidality is shown: An admissible representation is cuspidal if and only if the Jacquet modules along all proper parabolics vanish. After this characterization, Jacquet’s subrepresentation theorem readily follows from the inductive argument using the Frobenius reciprocity.

We consider a relative version of Casselman’s canonical pairing, which is expected to be a basic tool for harmonic analysis of $G/H$. Let $P$ be a $\sigma$-split parabolic subgroup with the Levi subgroup $M = P \cap \sigma(P)$. Let $(\pi_P, V_P)$ denote the Jacquet module of $(\pi, V)$ along $P$. We shall construct a mapping

$$r_P : (V^*)^H \to (V_P^*)^{M \cap H}$$

of invariant linear forms. This will provide a relation of the asymptotic behaviors between $H$-matrix coefficients for $\pi$ and $M \cap H$-matrix coefficients for $\pi_P$. Using this relation, we can deduce another main theorem which gives a characterization of relative cuspidality:

**Theorem B** (Theorem 6.9). An $H$-distinguished representation $(\pi, V)$ of $G$ is $H$-relatively cuspidal if and only if $r_P((V^*)^H) = 0$ for any proper $\sigma$-split parabolic subgroup $P$ of $G$.

Theorem A is a natural consequence of this characterization.

This paper is organized as follows. Section 1 is devoted to definitions, notation and some elementary properties of relatively cuspidal representations. In section 2, basic notation and properties concerning with tori, roots and parabolic subgroups associated to the involution are prepared. Section 3 deals with the analogue of Cartan decomposition for $p$-adic symmetric spaces given independently by Benoist-Oh [BO] and Delorme-Sécherre [DS]. This will be used in an important step of the analysis of $H$-matrix coefficients. In section 4, certain families of open compact subgroups are introduced. Lemma 4.6 on a property of the subgroups in such a family will be a key to the construction of the mapping $r_P$ in section 5. After that, we give a result Proposition 5.5 on an asymptotic behavior of $H$-matrix coefficients. This will be used repeatedly in the proof of various properties of $r_P$ such as the transitivity (Proposition 5.9). Section 6 is devoted to the proof of Theorem B (Theorem 6.9). The proof of Theorem A (Theorem 7.1) is given shortly in section 7. Finally, we give several examples of relatively cuspidal representations in section 8.
After this paper was completed, we learned that Nathalie Lagier has obtained independently the same result as our Proposition 5.5 and Proposition 5.6 (1), which has appeared in the announcement [L]. We are grateful to Patrick Delorme for pointing this out.

1. RELATIVELY CUSSIDAL REPRESENTATIONS

In this section, we shall introduce the notion of relative cuspidality for distinguished representations and give some elementary properties of them.

1.1. Let $F$ be a non-archimedean local field with the normalized absolute value $|·|_F$ and the valuation ring $\mathcal{O}_F$. We assume that the residual characteristic of $F$ is not equal to 2 throughout this paper. Let $G$ be a connected reductive group over $F$ and $\sigma$ an $F$-involution on $G$. The $F$-subgroup $\{h \in G | \sigma(h) = h\}$ of all $\sigma$-fixed points of $G$ is denoted by $H$. Let $Z$ be the $F$-split component of $G$, that is, the largest $F$-split torus lying in the center of $G$. Note that $Z$ is $\sigma$-stable. The group $G(F)$ of $F$-points of $G$ is denoted by $G_F$. Similarly, for any $F$-subgroup $R$ of $G$, we shall write $R = R(F)$.

1.2. Let $(\pi, V)$ be an admissible representation of $G$ over $\mathbb{C}$. It is said to be $H$-distinguished if the space $(V^*)^H$ of all $H$-invariant linear forms on $V$ is non-zero. For $\lambda \in (V^*)^H$ and $v \in V$, let $\varphi_{\lambda, v}$ denote the function on $G$ defined by

$$\varphi_{\lambda, v}(x) = \langle \lambda, \pi(x^{-1})v \rangle \quad (x \in G).$$

We call such functions $(H, \lambda)$-matrix coefficients of $\pi$. These are not the usual matrix coefficients of $\pi$, since $H$-invariant linear forms are not smooth in general. Let $C^\infty(G/H)$ denote the space of all right $H$-invariant locally constant $\mathbb{C}$-valued functions on $G$, on which $G$ acts by the left translation. Any subrepresentation of $C^\infty(G/H)$ is $H$-distinguished by the linear form $\varphi \mapsto \varphi(e)$. All the $(H, \lambda)$-matrix coefficients belong to $C^\infty(G/H)$ with the obvious $G$-equivariance

$$\varphi_{\lambda, \pi(g)v}(x) = \varphi_{\lambda, v}(g^{-1}x)$$

for $x, g \in G$. Set $T_\lambda(v) = \varphi_{\lambda, v}$ for $\lambda \in (V^*)^H$ and $v \in V$. Then the mapping $T_\lambda : V \to C^\infty(G/H)$ is a $G$-morphism, which is non-zero if and only if $\lambda \neq 0$.

1.3. Definition. An $H$-distinguished representation $(\pi, V)$ is said to be $(H, \lambda)$-relatively cuspidal (for $\lambda \in (V^*)^H$) if all the $(H, \lambda)$-matrix coefficients of $\pi$ are compactly supported modulo $ZH$. A representation
(π, V) is said to be \( H \)-relatively cuspidal if it is \((H, \lambda)\)-relatively cuspidal for all \( \lambda \in (V^*)^H \).

1.4. Let \( C_0^\infty(G/H) \) denote the subspace of \( C^\infty(G/H) \) consisting of all the functions \( \varphi \in C^\infty(G/H) \) which are compactly supported modulo \( ZH \). It is stable under \( G \) from the left. An \( H \)-distinguished representation \((π, V)\) is \((H, \lambda)\)-relatively cuspidal for \( \lambda \in (V^*)^H \) if and only if the image \( T_\lambda(V) \) is contained in \( C_0^\infty(G/H) \).

1.5. The group case. Here we see that the notion of relative cuspidality introduced above includes the usual cuspidality as a special case. Take a connected reductive \( F \)-group \( G_1 \) and let \( G \) be the direct product
\[
G = G_1 \times G_1
\]
equipped with the involution
\[
\sigma(g_1, g_2) = (g_2, g_1).
\]
The \( \sigma \)-fixed point subgroup \( H \) is the diagonally embedded subgroup
\[
\Delta G_1 = \{(g, g) \in G_1 \times G_1 \mid g \in G_1 \}.
\]
The mapping \((g_1, g_2) \mapsto g_1g_2^{-1}\) induces a \( G \)-equivariant isomorphism of the quotient \( G/H = (G_1 \times G_1)/\Delta G_1 \) onto the underlying variety of \( G_1 \) on which \( G = G_1 \times G_1 \) acts by multiplication from both sides. The \( F \)-split component \( Z \) of \( G \) is of the form \( Z_1 \times Z_1 \) where \( Z_1 \) is the \( F \)-split component of \( G_1 \). Henceforth, this situation is referred to as the group case.

Any irreducible admissible representation of \( G = G_1 \times G_1 \) is of the form \( \pi_1 \otimes \pi'_1 \) where \( \pi_1 \) and \( \pi'_1 \) are irreducible admissible representations of \( G_1 \). It is \( \Delta G_1 \)-distinguished if and only if \( \pi'_1 \) is isomorphic to the contragredient \( \widetilde{\pi}_1 \) of \( \pi_1 \). The natural pairing \( \langle \, \rangle_{\widetilde{\pi}_1 \times \pi_1} \) of \( \widetilde{\pi}_1 \) and \( \pi_1 \) defines a non-zero \( \Delta G_1 \)-invariant linear form, say \( \lambda \in ((\pi_1 \otimes \widetilde{\pi}_1)^*)^{\Delta G_1} \), which is unique up to constant. It is given by
\[
\langle \lambda, v_1 \otimes \widetilde{v}_1 \rangle = \langle \widetilde{v}_1, v_1 \rangle_{\widetilde{\pi}_1 \times \pi_1}
\]
for \( v_1 \in V_1 \) and \( \widetilde{v}_1 \in \widetilde{V}_1 \). Observe the relation
\[
\langle \lambda, (\pi_1 \otimes \widetilde{\pi}_1)(g_1, g_2)^{-1}(v_1 \otimes \widetilde{v}_1) \rangle = \langle \lambda, \pi_1(g_1^{-1})v_1 \otimes \widetilde{\pi}_1(g_2^{-1})\widetilde{v}_1 \rangle
\]
\[
= \langle \widetilde{\pi}_1(g_2^{-1})\widetilde{v}_1, \pi_1(g_1^{-1})v_1 \rangle_{\widetilde{\pi}_1 \times \pi_1} = \langle \widetilde{v}_1, \pi_1((g_1g_2^{-1})^{-1})v_1 \rangle_{\widetilde{\pi}_1 \times \pi_1}
\]
for \( g_1, g_2 \in G_1 \) and \( v_1 \in V_1, \widetilde{v}_1 \in \widetilde{V}_1 \). As is seen from this, the \( (\Delta G_1, \lambda) \)-matrix coefficients are identified with the usual matrix coefficients of \( \pi_1 \) through the mapping \((g_1, g_2) \mapsto g_1g_2^{-1}\).

Recall that an admissible representation is said to be cuspidal if all the usual matrix coefficients are compactly supported modulo the
center. Now it is obvious from the above identification that $\pi_1 \otimes \tilde{\pi}_1$ is $(\Delta G_1, \lambda)$-relatively (actually $\Delta G_1$-relatively) cuspidal if and only if $\pi_1$ is cuspidal as a representation of $G_1$.

1.6. For an admissible representation $(\pi, V)$ of $G$ and a quasi-character $\omega$ of $Z$, put

\[ V_\omega = \{ v \in V \mid \pi(z)v = \omega(z)v \text{ for all } z \in Z \}. \]

We call a subrepresentation of $V$ an $\omega$-subrepresentation if it is contained in $V_\omega$, and call $V$ an $\omega$-representation if $V = V_\omega$.

1.7. Lemma. Let $(\pi, V)$ be an admissible representation of $G$ of finite length. Then there exists a non-trivial quotient representation of $V$ which is isomorphic to an $\omega$-subrepresentation of $V$ for some quasi-character $\omega$ of $Z$.

Proof. Recall from [C, 2.1.9] the direct sum decomposition $V = \bigoplus_\omega V_{\omega,\infty}$ where $\omega$ runs over a finite set of quasi-characters of $Z$.

\[ V_{\omega,n} = \{ v \in V \mid (\pi(z) - \omega(z))^n v = 0 \text{ for all } z \in Z \} \]

and

\[ V_{\omega,\infty} = \bigcup_{n} V_{\omega,n}. \]

It is enough to consider the case $V = V_{\omega,\infty}$ for some quasi-character $\omega$ of $Z$. If $V = V_{\omega,1} = V_\omega$, there is nothing to prove. If not, there exist elements $z_0 \in Z$ and $v_0 \in V$ such that

\[ \pi(z_0)v_0 - \omega(z_0)v_0 \neq 0. \]

Set $\phi_1 = \pi(z_0) - \omega(z_0)\text{id}_V : V \to V$. This is a non-zero $G$-morphism with non-trivial kernel. Let $V_1$ be the image of $\phi_1$. It is a proper $G$-submodule of $V$. If $V_1 \subset V_\omega$, then $V/\text{Ker}(\phi_1) \simeq V_1$ will give the desired quotient. If $V_1 \not\subset V_\omega$, there are $z_1 \in Z$ and $v_1 \in V_1$ such that

\[ \pi(z_1)v_1 - \omega(z_1)v_1 \neq 0. \]

Set $\phi_2 = (\pi(z_1) - \omega(z_1)\text{id}_V)|_{V_1} : V_1 \to V_1$. Again this is a non-zero $G$-morphism with non-trivial kernel. Let $V_2$ be the image of $\phi_2$, which is a proper $G$-submodule of $V_1$. In this way, we can construct a non-zero proper $G$-submodule $V_k$ of $V_{k-1}$ with a surjective $G$-morphism $V_{k-1} \to V_k$ inductively as long as $V_{k-1}$ is not contained in $V_\omega$. We obtain a decreasing sequence

\[ V \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_k \neq 0, \]

of $G$-submodules, together with surjective $G$-morphisms

\[ V \xrightarrow{\phi_1} V_1 \xrightarrow{\phi_2} V_2 \xrightarrow{\cdots} V_k. \]
We must have $V_k \subset V_\omega$ for some $k$ by the finiteness of length of $V$. Now $V/\ker(\phi_k \circ \cdots \circ \phi_1) \simeq V_k$ will give the desired quotient. \qed

1.8. For a quasi-character $\omega$ of $Z$ which is trivial on $Z \cap H$, let $C^\infty_{0,\omega}(G/H)$ denote the space of all $\varphi \in C^\infty_0(G/H)$ satisfying

$$\varphi(zg) = \omega^{-1}(z)\varphi(g)$$

for $g \in G$ and $z \in Z$. If an $\omega$-representation is $H$-distinguished, then we must have $\omega|_{Z\cap H} \equiv 1$. An $H$-distinguished $\omega$-representation $(\pi, V)$ is $(H, \lambda)$-relatively cuspidal for $\lambda \in (V^*)^H$ if and only if the image $T_\lambda(V)$ is contained in $C^\infty_{0,\omega}(G/H)$.

If $\omega$ is unitary, then any subrepresentation of $C^\infty_{0,\omega}(G/H)$ is pre-unitary by the $G$-invariant hermitian inner product

$$\langle \langle \varphi, \psi \rangle \rangle = \int_{G/ZH} \varphi(\tilde{g})\overline{\psi(\tilde{g})}d\tilde{g} \quad (\varphi, \psi \in C^\infty_{0,\omega}(G/H))$$

where $d\tilde{g}$ denotes a $G$-invariant measure on the quotient $G/ZH$.

1.9. Let $X(G, \mathbb{R}^\times_{>0})$ be the set of all positive real valued quasi-characters of $G$ and put

$$X(G/H, \mathbb{R}^\times_{>0}) = \{ \chi \in X(G, \mathbb{R}^\times_{>0}) \mid \chi|_H \equiv 1 \}.$$ 

We define $X(Z, \mathbb{R}^\times_{>0})$ and $X(Z/Z \cap H, \mathbb{R}^\times_{>0})$ similarly.

1.10. **Lemma.** (1) The restriction map

$$X(G, \mathbb{R}^\times_{>0}) \overset{\text{res}}{\rightarrow} X(Z, \mathbb{R}^\times_{>0}), \quad \text{res}(\chi) = \chi|_Z \quad (\chi \in X(G, \mathbb{R}^\times_{>0}))$$

is bijective.

(2) The above restriction map sends $X(G/H, \mathbb{R}^\times_{>0})$ bijectively onto $X(Z/Z \cap H, \mathbb{R}^\times_{>0})$.

**Proof.** (1) is well-known (e.g., [C 5.2.5]). For (2), first it is obvious that $\chi|_Z \in X(Z/Z \cap H, \mathbb{R}^\times_{>0})$ for all $\chi \in X(G/H, \mathbb{R}^\times_{>0})$. On the other hand, for a given $\omega \in X(Z/Z \cap H, \mathbb{R}^\times_{>0})$, let us take the element $\chi \in X(G, \mathbb{R}^\times_{>0})$ such that $\chi|_Z = \omega$ by (1). Since $z\sigma(z) \in Z \cap H$ for all $z \in Z$, we have $\omega(z\sigma(z)) = 1$ so that $\omega \circ \sigma = \omega^{-1}$. Thus we have $$(\chi \circ \sigma)|_Z = (\chi|_Z) \circ \sigma = (\chi|_Z)^{-1} = \chi^{-1}|_Z,$$ which shows that $\chi \circ \sigma = \chi^{-1}$ on $G$ by (1). This proves that $\chi|_H \equiv 1$ since $\chi(h) = \chi^{-1}(h) \in \mathbb{R}^\times_{>0}$ for all $h \in H$. \qed

1.11. **Proposition.** Any finitely generated $(H, \lambda)$-relatively cuspidal representation of $G$ has a non-trivial $H$-distinguished irreducible quotient.
Proof. Let \((\pi, V)\) be a finitely generated admissible representation with a non-zero \(\lambda \in (V^*)^H\) such that \(T_\lambda(V) \subset C_0^\infty(G/H)\). The quotient \(V/\text{Ker}(T_\lambda)\) is finitely generated, hence is of finite length by [C, 6.3.10]. Apply Lemma 1.7 to obtain a quotient \((\pi, \overline{V})\) of \(V/\text{Ker}(T_\lambda)\), hence of \(V\), which is isomorphic to an \(\omega\)-subrepresentation of \(V/\text{Ker}(T_\lambda)\) for some \(\omega\). Then \((\pi, \overline{V})\) is regarded as a subrepresentation of \(C_0^\infty(G/H)\) through \(\overline{V} \cong T_\lambda(V) \subset C_0^\infty(G/H)\).

Let \(\Re(\omega)\) be the quasi-character of \(Z\) defined by \(\Re(\omega)(z) = (\omega(z)\overline{\omega(z)})^{1/2}\) for \(z \in Z\). This belongs to \(X(Z/Z \cap H; \mathbb{R}_{>0})\) and \(\Im(\omega) := \omega\Re(\omega)^{-1}\) is unitary. Now, take the element \(\chi \in X(G/H; \mathbb{R}_{>0})\) such that \(\chi|_Z = \Re(\omega)^{-1}\) by Lemma 1.10 (2) and consider the representation \(\chi \cdot \overline{\pi}\) on \(\overline{V}\) for \(g \in G\). It is regarded as a subrepresentation of \(C_0^\infty(G/H)\), which is pre-unitary by 1.8. Thus \((\chi \cdot \overline{\pi}, \overline{V})\) is decomposed into a direct sum of finitely many irreducible subrepresentations (see [C 2.1.14]). The decomposition for the action \(\chi \cdot \overline{\pi}\) also yields that for the action \(\overline{\pi}\), hence the claim follows. \(\square\)

2. \(\sigma\)-split parabolic subgroups

In this section, we shall prepare basic notation and properties concerning with tori, roots and parabolic subgroups associated to \(p\)-adic symmetric spaces.

2.1. An \(F\)-split torus \(S\) in \(G\) is said to be \((\sigma, F)\)-split if \(\sigma(s) = s^{-1}\) for all \(s \in S\). Take a maximal \((\sigma, F)\)-split torus \(S_0\) of \(G\) and a maximal \(F\)-split torus \(A_0\) of \(G\) containing \(S_0\). Then \(A_0\) is \(\sigma\)-stable by [HW, 4.5]. Put \(M_0 = Z_G(S_0)\) and \(M_0 = Z_G(A_0)\). Clearly these are also \(\sigma\)-stable.

Let \(X_F^+(A_0)\) be the free \(\mathbb{Z}\)-module of all \(F\)-rational characters of \(A_0\) on which \(\sigma\) acts naturally. Let \(\Phi = \Phi(G, A_0)\) be the root system of \((G, A_0)\). Then \(\sigma\) leaves \(\Phi\) stable. Set \(\Phi_\sigma = \{ \alpha \in \Phi \mid \sigma(\alpha) = \alpha \}\).

This is a closed subsystem of \(\Phi\). As in [HH, 1.6], we choose a \(\sigma\)-basis \(\Delta\) (and the corresponding set \(\Phi^+\) of positive roots) of \(\Phi\) so that \(\sigma(\alpha) \notin \Phi^+\) for all \(\alpha \in \Phi^+ \setminus \Phi_\sigma\). Put \(\Delta_\sigma = \Delta \cap \Phi_\sigma\). This gives a basis of the subsystem \(\Phi_\sigma\). Let \(w_{\Delta_\sigma}\) be the longest element of the Weyl group of \(\Phi_\sigma\) with respect to the basis \(\Delta_\sigma\). As in [HH 1.7], let \(\sigma^*\) be the involution on \(X_F^+(A_0)\) given by \(\sigma^* = -\sigma \circ w_{\Delta_\sigma}\). It is known that \(\sigma^*\) leaves \(\Delta\) stable.
2.2. Let $S_0 \subset A_\emptyset$ and $\Delta$ be as above. Let $P_\emptyset (\supset A_\emptyset)$ be the minimal parabolic $F$-subgroup of $G$ corresponding to $\Delta$. In this paper, parabolic $F$-subgroups containing such $P_\emptyset$ are said to be standard with respect to $(S_0, A_\emptyset, \Delta)$. They correspond to subsets of $\Delta$. Let us fix the notation. For a subset $I \subset \Delta$, let $P_I$ be the standard parabolic $F$-subgroup of $G$ corresponding to $I$ with the unipotent radical $U_I$. Let $A_I$ denote the identity component of the intersection of all the kernels of $\alpha : A_\emptyset \to G_m, \alpha \in I$, and set $M_I = Z_G(A_I)$. We have a Levi decomposition $P_I = M_I \ltimes U_I$. The torus $A_I$ is the $F$-split component of $M_I$. Let $P_I^-$ be the unique parabolic $F$-subgroup of $G$ such that $P_I \cap P_I^- = M_I$ and $U_I^-$ the unipotent radical of $P_I^-$. Later we shall drop the subscript $I$ if there is no fear of confusion.

2.3. A parabolic $F$-subgroup $P$ of $G$ is said to be $\sigma$-split if $P$ and $\sigma(P)$ are opposite. In such a case, $P \cap \sigma(P)$ gives a $\sigma$-stable Levi subgroup of $P$. Let us fix $S_0, A_\emptyset$ and $\Delta$ as above. If $P_I$ is $\sigma$-split for a subset $I \subset \Delta$, then we must have $\sigma(P_I) = P_I^-$, that is, $P_I \cap \sigma(P_I) = M_I$ (see [HW] 4.6). Hence we also have $\sigma(U_I) = U_I^-$. Further, recall from [HH] 2.6 that $P_I$ is $\sigma$-split if and only if $\Delta_\sigma \subset I$ and the subsystem $\Phi_I$ of $\Phi$ generated by $I$ is $\sigma$-stable. There is an alternative description for $P_I$ to be $\sigma$-split as follows: If $I$ contains $\Delta_\sigma$, then $w_{\Delta_\sigma}$ leaves $\Phi_I$ stable so that

$$\Phi_I \text{ is } \sigma\text{-stable } \iff \Phi_I \text{ is } \sigma^* (\text{or } w_{\Delta_\sigma})\text{-stable}$$

$$\iff I \text{ is } \sigma^\ast\text{-stable}$$

(since $\sigma^\ast(\Delta) = \Delta$ and $\Delta \cap \Phi_I = I$).

We say that a subset of $\Delta$ is $\sigma$-split if it contains $\Delta_\sigma$ and is $\sigma^\ast$-stable. Thus $P_I$ is $\sigma$-split if and only if $I$ is a $\sigma$-split subset of $\Delta$.

Note that every $\sigma$-split parabolic $F$-subgroup arises as a standard $\sigma$-split one for a suitable choice of $S_0, A_\emptyset$ and $\Delta$. See [HW] 4.6, 4.7 and also Lemma 2.5 (1) below.

2.4. The subset $I = \Delta_\sigma$ is a minimal $\sigma$-split subset of $\Delta$. The corresponding parabolic $F$-subgroup $P_{\Delta_\sigma}$ is denoted by $P_\emptyset$. This is a minimal $\sigma$-split parabolic $F$-subgroup of $G$ and the $\sigma$-stable Levi subgroup $M_{\Delta_\sigma} = P_\emptyset \cap \sigma(P_\emptyset)$ coincides with $M_0 = Z_G(S_0)$ (see [HW] 4.7)). Put $P_\emptyset^0 = P_{\Delta_\sigma}^0, U_0 = U_{\Delta_\sigma}$ and $U_0^- = U_{\Delta_\sigma}^-$.  

2.5. Lemma. Let us fix $S_0, A_\emptyset$ and $\Delta$ as above.

(1) Any $\sigma$-split parabolic $F$-subgroup of $G$ is of the form $\gamma^{-1}P_I\gamma$ for some $\sigma$-split subset $I \subset \Delta$ and $\gamma \in (M_0H)(F)$.

(2) If $(M_0H)(F) = M_0H$, then any $\sigma$-split parabolic $F$-subgroup of $G$ is $H$-conjugate to a standard $\sigma$-split one.
Proof. (2) follows directly from (1). For (1), let $P \subseteq G$ be any $\sigma$-split parabolic $F$-subgroup. As in [HW] 4.9, 4.11, we can take an element $\gamma \in (P_0H)(F)$ such that $\gamma P \gamma^{-1}$ contains $P_0$ and is $\sigma$-split. It is enough to see that $p^{-1}\gamma \in (M_0H)(F)$ for some $p \in P_0$. Since $P_0H = P_0H$ by [HW] 4.8, we have

$$x := \gamma \sigma(\gamma)^{-1} \in (U_0M_0U_0)(F) = U_0M_0U_0^{-}.$$ 

Express $x$ as $x = u_0m_0u_0^{-}$ with $u_0 \in U_0$, $m_0 \in M_0$ and $u_0^{-} \in U_0^{-}$. Since $x = \sigma(x)^{-1}$, we have

$$u_0m_0u_0^{-} = \sigma(u_0)^{-1}\sigma(m_0)^{-1}\sigma(u_0)^{-1}.$$ 

By the uniqueness of the expression of elements of $U_0M_0U_0^{-}$, we must have

$$m_0 = \sigma(m_0)^{-1}, \quad u_0^{-} = \sigma(u_0)^{-1}.$$ 

Now it is seen that

$$u_0^{-1}\gamma \sigma(u_0^{-1})^{-1} = u_0^{-1}x\sigma(u_0) = u_0^{-1}x(u_0^{-1})^{-1} = m_0 \in M_0.$$ 

This shows that $u_0^{-1}\gamma \in (M_0H)(F)$. \qed

2.6. For each $\alpha \in \Delta \setminus \Delta_\sigma$, we put $I_\alpha = \Delta \setminus \{\alpha, \sigma^*(\alpha)\}$. (It may happen that $\alpha = \sigma^*(\alpha).$) This is a maximal proper $\sigma$-split subset of $\Delta$. Hence all the maximal $\sigma$-split parabolic $F$-subgroups of $G$ are of the form $\gamma^{-1}P_{I_\alpha}\gamma$ for some $\alpha \in \Delta \setminus \Delta_\sigma$ and $\gamma \in (M_0H)(F)$.

2.7. For a $\sigma$-split subset $I \subseteq \Delta$, the identity component of $A_I \cap S_0$ is denoted by $S_I$. We shall call $S_I$ the $(\sigma, F)$-split component of $M_I$ (or of $P_I$). Note that the $(\sigma, F)$-split component $S_{\Delta_\sigma}$ of the minimal $\sigma$-split parabolic $F$-subgroup $P_{\Delta_\sigma} = P_0$ coincides with $S_0$, the maximal $(\sigma, F)$-split torus we choose.

For a positive real number $\varepsilon \leq 1$, set

$$A_I^- (\varepsilon) = \{ a \in A_I \mid |a^\alpha|_F \leq \varepsilon \ (\alpha \in \Delta \setminus I) \}$$

and

$$S_I^- (\varepsilon) = S_I \cap A_I^- (\varepsilon) = \{ s \in S_I \mid |s^\alpha|_F \leq \varepsilon \ (\alpha \in \Delta \setminus I) \}.$$ 

Also put

$$S_I^+ (\varepsilon) = \{ s \in S_I \mid s^{-1} \in S_I^- (\varepsilon) \}$$

$$= \{ s \in S_I \mid |s^\alpha|_F \geq \varepsilon \ (\alpha \in \Delta \setminus I) \}.$$ 

Since $S_I^- (\varepsilon) \subset A_I^- (\varepsilon)$, the following lemma is apparent (see [C] 1.4.3).

2.8.Lemma. Let $I \subseteq \Delta$ be a $\sigma$-split subset. For any two open compact subgroups $U_1$, $U_2$ of $U_I$, there exists a positive real number $\varepsilon \leq 1$ such that $sU_1s^{-1} \subset U_2$ for all $s \in S_I^- (\varepsilon)$. 

Next we shall give a lemma on the modulus character of \(\sigma\)-split parabolic subgroups.

2.9. **Lemma.** Let \(P = M \ltimes U\) be a \(\sigma\)-split parabolic \(F\)-subgroup of \(G\). Then the modulus character \(\delta_P\) of \(P\) is trivial on \(M \cap H\).

**Proof.** Let \(A\) be the \(F\)-split component of \(M\). Since \(\delta_P\) is positive real valued, it is determined by the values on \(A\) according to Lemma 1.10 (1). It is enough to show that \(\delta_P\) is trivial on \(A \cap H\) by (2) of Lemma 1.10. We may suppose that \(P = P_I\) for a \(\sigma\)-split subset \(I \subset \Delta\). Then we have

\[
\delta_P(a) = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^\alpha|^m_{F} \quad (a \in A)
\]

where \(m_\alpha\) denotes the dimension of the root space attached to \(\alpha \in \Phi\). Note that \(m_\alpha = m_{\sigma(\alpha)}\) since \(\sigma\) maps the root space attached to \(\alpha\) isomorphically onto the one attached to \(\sigma(\alpha)\). If further \(a \in A \cap H\), we have

\[
\delta^2(a) = \delta(a)\delta(\sigma(a)) = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^\alpha|^m_{F} \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^\sigma(\alpha)|^{m_{\sigma(\alpha)}}
\]

\[
eq \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^\alpha|^m_{F} \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^{\sigma(\alpha)}|^{|m_{\sigma(\alpha)}|_{F}}
\]

\[
eq \prod_{\alpha \in \Phi^+ \setminus \Phi_I} |a^\alpha|^m_{F} \prod_{\beta \in \Phi^+ \setminus \Phi_I} |a^{-\beta}|^{m_{\beta}} = 1
\]

and the claim follows. \(\square\)

3. **The analogue of Cartan decomposition**

We shall recall from [BO] and [DS] a certain decomposition theorem related to \(p\)-adic symmetric spaces and give a variant of it for our later use. It will play an important role in the study of the support of \(H\)-matrix coefficients.

3.1. Let us fix a triple \((S_0, A_0, \Delta)\) as in 2.1. For simplicity, put \(S_0^+ = S_{\Delta_0}^+(1)\) so that

\[
S_0^+ = \{ s \in S_0(F) \mid |s^\alpha|_F \geq 1 (\alpha \in \Delta \Delta_0) \}.
\]

Let \(\{S_0^{(j)}\}_{j \in J}\) be a set of representatives for \(H\)-conjugacy classes of maximal \((\sigma, F)\)-split tori of \(G\). In this case, the index set \(J\) is finite by [HH 2.12]. For each \(j \in J\), there exists an element \(y_j \in (M_0 H)(F)\) such that \(y_j^{-1} S_0 y_j = S_0^{(j)}\) by [HW 10.3]. Set

\[
W_G(S_0^{(j)}) = N_G(S_0^{(j)})/Z_G(S_0^{(j)})
\]
and

\[ W_H(S_0^{(j)}) = N_H(S_0^{(j)})/Z_H(S_0^{(j)}). \]

Naturally \( W_H(S_0^{(j)}) \) is regarded as a subgroup of the finite (Weyl) group \( W_G(S_0^{(j)}) \). Let \( N_j' \subset N_G(S_0^{(j)}) \) be a complete set of representatives of \( W_G(S_0^{(j)})/W_H(S_0^{(j)}) \) in \( N_G(S_0^{(j)}) \).

3.2. Theorem ([DS, Theorem 3.10]). There exists a compact subset \( \Omega \) of \( G \) such that

\[ G = \bigcup_{j \in J} \bigcup_{n \in N_j} \Omega S_0^{+} y_j nH. \]

An essentially equivalent assertion is given also in [BO, Theorem 1.1].

3.3. Now for each \( j \in J \), we have

\[ N_G(S_0^{(j)}) = N_G(y_j^{-1} S_0 y_j) = y_j^{-1} N_G(S_0) y_j. \]

Put

\[ N_j' = \{ y_j n y_j^{-1} \mid n \in N_j \}. \]

We have \( N_j' \subset N_G(S_0) \) for all \( j \in J \). The above decomposition is written as

\[ G = \bigcup_{j \in J} \bigcup_{n \in N_j'} \Omega S_0^{+} y_j nH. \]

Furthermore, we have \( n \sigma(n)^{-1} \in M_0(F) \) for any \( n \in N_G(S_0) \) since \( n \sigma(n)^{-1} \) centralizes \( S_0 \). Consequently, for any \( n \in N_G(S_0) \) and \( y \in (M_0 H)(F) \), we have

\[ ny \sigma(ny)^{-1} = n(y \sigma(y)^{-1}) \sigma(n)^{-1} = n(y \sigma(y)^{-1}) n^{-1} \cdot n \sigma(n)^{-1} \in M_0. \]

This means that \( ny \in (M_0 H)(F) \). As a result, we have a variant of Theorem 3.2 in the following form.

3.4. Corollary. There exists a compact subset \( \Omega \) of \( G \) and a finite subset \( \Gamma \) of \( (M_0 H)(F) \) such that

\[ G = \Omega S_0^{+} \Gamma H. \]

4. Preliminaries on open compact subgroups

For a later use on the study of invariant linear forms on Jacquet modules, we shall construct a particular family of open compact subgroups adapted to the involution.
4.1. In this section, all parabolic $F$-subgroups are standard ones with respect to some fixed data $(S_0, A_0, \Delta)$ as in 2.3, but the subscripts $I \subset \Delta$ will be omitted. For a parabolic $F$-subgroup $P = M \ltimes U$ of $G$ and an open compact subgroup $K$ of $G$, we set

$$U_K = U \cap K, \quad M_K = M \cap K, \quad U_K^- = U^- \cap K.$$ 

Note that $\sigma(M_K) = M_K$ and $\sigma(U_K) = U_K^-$ if $K$ is $\sigma$-stable and $P$ is $\sigma$-split.

4.2. For each choice of a maximal $F$-split torus $A_0$ and a basis $\Delta$ of the root system of $(G, A_0)$, there is a decreasing sequence $\{K'_n\}_{n \geq 0}$ of open compact subgroups of $G$ satisfying the following properties as in [C, 1.4.4]:

1. It gives a fundamental system of open neighborhoods of the identity $e$ in $G$.
2. For each $n \geq 1$, the subgroup $K'_n$ is normal in $K'_0$ and the quotient $K'_n/K'_{n+1}$ is a finite abelian $p$-group where $p$ denotes the residual characteristic of $F$.
3. For each $K' = K'_n$ ($n \geq 1$) and each standard parabolic $F$-subgroup $P = M \ltimes U$ of $G$ (corresponding to $(A_0, \Delta)$) with the $F$-split component $A$, the product map

$$U_K^- \times M_K' \times U_K' \to K'$$

is bijective and

$$a \cdot U_K' \cdot a^{-1} \subset U_K', \quad a^{-1} \cdot U_K^- \cdot a \subset U_K^-$$

for all $a \in A^{-1}(1)$.
4. For each standard parabolic subgroup $P = M \ltimes U$ of $G$, the family $\{M \cap K'_n\}_{n \geq 0}$ enjoys the same properties as (1)-(3) above for the group $M$.

Here the latter half of (2) is not apparent in [C, 1.4.4]. However, in the $F$-split case, the argument in [G, 2.2.11] shows that each quotient $K'_n/K'_{n+1}$ is isomorphic to the additive group of a Lie algebra over the residue field of $F$. The general case is reduced to the $F$-split case as in [C, 1.4.4].

4.3. Lemma. Fix a data $(S_0, A_0, \Delta)$ as in 2.7 and let $\{K'_n\}_{n \geq 0}$ be as above. Put $K'_n = K'_n \cap \sigma(K'_n)$ for each $n$. Then the family $\{K'_n\}_{n \geq 0}$ is a decreasing sequence of $\sigma$-stable open compact subgroups of $G$ satisfying the following properties:

1. It gives a fundamental system of open neighborhoods of the identity $e$ in $G$. 

(2) For each $n \geq 1$, the subgroup $K_n$ is normal in $K_0$ and the quotient $K_n/K_{n+1}$ is a finite abelian $p$-group.

(3) For each $K = K_n$ ($n \geq 1$) and each $\sigma$-split parabolic subgroup $P = M \rtimes U$ of $G$ (standard with respect to $(S_0, A_\emptyset, \Delta)$) with the $(\sigma, F)$-split component $S$, the product map

$$U_K^{-1} \times M_K \times U_K \to K$$

is bijective and

$$s \cdot U_K \cdot s^{-1} \subset U_K, \quad s^{-1} \cdot U_K^{-1} \cdot s \subset U_K$$

for all $s \in S^-(1)$.

(4) For each $\sigma$-split parabolic subgroup $P = M \rtimes U$ of $G$ standard with respect to $(S_0, A_\emptyset, \Delta)$, the family $\{M \cap K_n\}_{n \geq 1}$ enjoys the same properties as (1)–(3) above for the group $M$.

Proof. These are derived directly from the corresponding properties of $K'_n$ in (1.2). First, note that $k$ belongs to $K_n$ if and only if both $k$ and $\sigma(k)$ belong to $K'_n$. Now, (1) is obvious. For (2), take any $k_1, k_2 \in K_n$ and consider their commutator. By (1.2 (2)), we have $k_1k_2k_1^{-1}k_2^{-1} \in K'_n$ and

$$\sigma(k_1)\sigma(k_2)\sigma(k_1)^{-1}\sigma(k_2)^{-1} = \sigma(k_1k_2k_1^{-1}k_2^{-1}) \in K'_n,$$

hence $k_1k_2k_1^{-1}k_2^{-1} \in K'_{n+1}$. For (3), it is sufficient to show the surjectivity of the product map. Given $k \in K = K_n$, decompose $k$ and $\sigma(k)^{-1} \in K' = K'_n$ as

$$k = u^-_1m_1u_1, \quad \sigma(k)^{-1} = u^-_2m_2u_2$$

where $u^-_i \in U^-_K$, $m_i \in M_{K'}$, $u_i \in U_{K'}$ ($i = 1, 2$) by (1.2 (3)). Then we have

$$\sigma(u_2)u^-_1 = \sigma(m_2)^{-1}\sigma(u_2)^{-1}u_1^{-1}m_1^{-1} \in U^- \cap P = \{e\},$$

which shows that $u^-_1 = \sigma(u_2)^{-1}$ and in turn, $m_1 = \sigma(m_2)^{-1}$, $u_1 = \sigma(u^-_2)^{-1}$. Now we have

$$u^-_1 = \sigma(u_2)^{-1} \in (U^- \cap K') \cap (\sigma(U) \cap \sigma(K')) = U^- \cap K$$

and similarly $m_1 \in M \cap K$, $u_1 \in U \cap K$. Finally, (4) follows once (1)–(3) are verified.

We say that a family $\{K_n\}_{n \geq 1}$ of $\sigma$-stable open compact subgroups of $G$ is adapted to $(S_0, A_\emptyset, \Delta)$ if it satisfies the above properties (1)–(4). By the bijectivity of the product map of (3), any element $k \in K = K_n$ can be written uniquely as

$$k = u^- \cdot m \cdot u^+, \quad u^- \in U_K^{-1}, m \in M_K, u^+ \in U_K.$$
Such an expression is called the Iwahori factorization with respect to $P$.

4.4. Let $p$ be an odd prime and $C$ a finite abelian $p$-group. The homomorphism $a \mapsto a^2$ of $C$ into itself is bijective. The inverse map is denoted by $a \mapsto a^{1/2}$. Let $\sigma$ be an involution on $C$. If $a \in C$ satisfies the condition $\sigma(a)^{-1} = a$, then $b = a^{1/2}$ is an element of $C$ such that $a = b\sigma(b)^{-1}$.

4.5. Lemma. Let $p$ be an odd prime, $K$ a totally disconnected compact group and $\sigma$ an involution on $K$. Suppose that $K$ has a decreasing sequence

$$K = K_1 \supset K_2 \supset K_3 \supset \cdots \supset K_n \supset \cdots$$

of $\sigma$-stable open normal subgroups such that

(i) $K_n/K_{n+1}$ is a finite abelian $p$-group for each $n \geq 1$, and

(ii) $\bigcap_{n \geq 1} K_n = \{e\}$.

Then, for any $y \in K$ satisfying the condition $\sigma(y)^{-1} = y$, there exists an element $k' \in K$ such that $k = k'\sigma(k')^{-1}$.

Proof. Our discussion below is similar to that of [PR, Theorem 6.8]. Look at the involution on the quotient $K_1/K_2$ induced by $\sigma$. If a given $k \in K = K_1$ satisfies $\sigma(k)^{-1} = k$, then by 4.4 there exists an element $y_1 \in K_1$ such that

$$y_1\sigma(y_1)^{-1} \equiv k \pmod{K_2}.$$

Set $k_2 = y_1^{-1}k\sigma(y_1)$. It is an element of $K_2$ and satisfies

$$\sigma(k_2)^{-1} = y_1^{-1}\sigma(k)^{-1}\sigma(y_1) = y_1^{-1}k\sigma(y_1) = k_2.$$

Looking at the involution on $K_2/K_3$ induced by $\sigma$, there exists an element $y_2 \in K_2$ such that

$$y_2\sigma(y_2)^{-1} \equiv k_2 \pmod{K_3}.$$

Set $k_3 = y_2^{-1}k_2\sigma(y_2) \in K_3$. We have $\sigma(k_3)^{-1} = k_3$. In this way, we can take $k_{n+1}, y_{n+1} \in K_{n+1}$ from $k_n, y_n \in K_n$ by the rules

$$k_{n+1} = y_n^{-1}k_n\sigma(y_n), \quad y_{n+1}\sigma(y_{n+1})^{-1} \equiv k_{n+1} \pmod{K_{n+2}}.$$

Consider the sequence $\{z_n\}$ in $K$ defined by $z_n = y_1y_2\cdots y_n$. It has a subsequence $\{z_{n_\nu}\}$ which converges to an element, say $k'$, of $K$. Note that

$$k_{n+1} = y_n^{-1}\cdots y_1^{-1}k\sigma(y_1)\cdots\sigma(y_n) = z_n^{-1}k\sigma(z_n) \in K_{n+1}.$$

Thus

$$(k')^{-1}k\sigma(k') \in \bigcap_{\nu} K_{n_{\nu}+1} = \{e\}.$$
which shows the claim. □

4.6. Lemma. Let $P = M \ltimes U$ be a $\sigma$-split parabolic $F$-subgroup which is standard with respect to $(S_0, A_\emptyset, \Delta)$ and $K = K_n$ ($n \geq 1$) a $\sigma$-stable open compact subgroup of $G$ from the family $\{K_n\}$ adapted to $(S_0, A_\emptyset, \Delta)$. Then,

$$U_K \subset H M_K U_K^-.\]$$

Proof. For a given $u \in U_K$, consider the element $k := u^{-1} \sigma(u) \in K$. By using the Iwahori factorization with respect to $P$, express $k$ as

$$k = u^- \cdot m \cdot u^+, \quad u^- \in U_K^-, \quad m \in M_K, \quad u^+ \in U_K.$$

Since $\sigma(k)^{-1} = k$, we have

$$\sigma(u^+)^{-1} \cdot \sigma(m)^{-1} \cdot \sigma(u^-)^{-1} = u^- \cdot m \cdot u^+$$

so that $u^+ = \sigma(u^-)^{-1}$ and $\sigma(m)^{-1} = m$. Here note that the group $M_K$ satisfies the assumption of Lemma 4.5. Thus we can take an element $m' \in M_K$ such that $m = m' \sigma(m')^{-1}$. As a result, we have

$$u^{-1} \sigma(u) = k = u^- \cdot m' \sigma(m')^{-1} \cdot \sigma(u^-)^{-1} = u^- m' \sigma(u^- m')^{-1}.$$

This shows that $uu^- m' \in H$, hence $u \in H M_K U_K^-$. □

5. Invariant linear forms on Jacquet modules

For a given $H$-invariant linear form $\lambda$ on an $H$-distinguished representation, we shall construct a canonical linear form $r_P(\lambda)$ on the Jacquet module along each $\sigma$-split parabolic $F$-subgroup $P$ by using Casselman's canonical lifting. It turns out that $r_P(\lambda)$ is $M \cap H$-invariant where $M = P \cap \sigma(P)$. We have a useful relation between $(H, \lambda)$-matrix coefficients and $(M \cap H, r_P(\lambda))$-matrix coefficients on the $(\sigma, F)$-split component of $P$.

5.1. From now on, we say briefly that $P$ is a $\sigma$-split parabolic subgroup of $G$ if it is the group of $F$-points of a $\sigma$-split parabolic $F$-subgroup $P$ of $G$. Also we say that $S$ is the $(\sigma, F)$-split component of $P$ if it is the group of $F$-points of the $(\sigma, F)$-split component $S = S_I$ of $P = P_I$, and so on. As a Levi subgroup of a $\sigma$-split parabolic subgroup, we always take the $\sigma$-stable one as in 2.3.

Let $(\pi, V)$ be an admissible representation of $G$ and $P = M \ltimes U$ a $\sigma$-split parabolic subgroup with the $(\sigma, F)$-split component $S$. We regard $P$ as a standard one with respect to a suitable choice of $(S_0, A_\emptyset, \Delta)$. Let $(\pi_P, V_P)$ denote the normalized Jacquet module of $(\pi, V)$ along $P$: The space $V_P$ is given as the quotient $V/V(U)$ where $V(U)$ denotes
the subspace of $V$ spanned by all the elements of the form $\pi(u)v - v$, $v \in V$, $u \in U$. The action $\pi_P$ of $M$ on $V_P$ is normalized so that

$$\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi(m)v)$$

for $m \in M$ and $v \in V$. Here $j_P : V \to V_P$ denotes the canonical projection and $\delta_P$ the modulus of $P$.

5.2. Let us recall the construction of Casselman’s canonical lifting. For a compact subgroup $K$ of $G$, let $V^K$ be the subspace of $V$ of all $K$-fixed vectors and $P_K : V \to V^K$ the projection operator given by

$$P_K(v) = \frac{1}{\text{vol}(K)} \int_K \pi(k)vdk \quad (v \in V).$$

For a compact subgroup $U_1$ of $U$, set

$$V(U_1) = \left\{ v \in V \mid \int_{U_1} \pi(u)vdu = 0 \right\}.$$

It is known that $V(U)$ is the union of all $V(U_1)$ where $U_1$ ranges over all compact subgroups of $U$. Now, for a given $\varpi \in V_P$, take an open compact subgroup $K = K_n$ from the family adapted to $(S_0, A_\emptyset, \Delta)$ such that $\varpi \in (V_P)^{M_K}$. Next choose an open compact subgroup $U_1$ of $U$ with $V^K \cap V(U) \subset V(U_1)$. Finally we take a positive real number $\varepsilon \leq 1$ such that $sU_1s^{-1} \subset U_K$ for all $s \in S^-(\varepsilon)$ by Lemma 2.8. Since $S^-(\varepsilon)$ is contained in $A^-(\varepsilon)$, we may replace $A^-(\varepsilon)$ in the argument of \cite[§4]{C} by $S^-(\varepsilon)$. We have an isomorphism

$$P_K(\pi(s)V^K) \simeq (V_P)^{M_K}$$

by the restriction of the canonical projection $j_P : V \to V_P$ as in \cite[4.1.4]{C}. The element $v \in P_K(\pi(s)V^K)$ satisfying $j_P(v) = \varpi$ is called the canonical lift of $\varpi \in V_P$ with respect to $K$. It depends on the choice of $K$, but not on $U_0$ and $\varepsilon$. If $v'$ is another canonical lift of $\varpi$ with respect to $K' \subset K$, then we have

$$v' \in V^{MKU_K^-}, \quad v = P_K(v') = P_K(v')$$

by \cite[4.1.8]{C}.

5.3. Proposition. Let $\lambda$ be an $H$-invariant linear form on an admissible representation $(\pi, V)$ of $G$ and $P$ a $\sigma$-split parabolic subgroup standard with respect to $(S_0, A_\emptyset, \Delta)$.

(1) For $K = K_n$ ($n \geq 1$) in the family $\{K_n\}$ adapted to $(S_0, A_\emptyset, \Delta)$ and $v \in V^{MKU_K^-}$, one has

$$\langle \lambda, v \rangle = \langle \lambda, P_{U_K}(v) \rangle.$$
(2) For two canonical lifts \( v, v' \in V \) of a given \( \pi \in V_P \), one has
\[
\langle \lambda, v \rangle = \langle \lambda, v' \rangle.
\]

**Proof.** (1) Let \( U_1 \) be an open compact subgroup of \( U_K \) which fixes \( v \). Then we have
\[
\langle \lambda, \mathcal{P}_{U_K}(v) \rangle = \langle \lambda, \frac{1}{\text{vol}(U_K)} \int_{U_K} \pi(u)vdu \rangle
= \langle \lambda, \frac{\text{vol}(U_1)}{\text{vol}(U_K)} \sum_{u_i \in U_K/U_1} \pi(h_i m_i u_i^-)v \rangle.
\]
Let us express each \( u_i \in U_K \) as \( u_i = h_i m_i u_i^- \) where \( h_i \in H, m_i \in M_K \) and \( u_i^- \in U_K^- \) by Lemma 4.6. Then this is equal to
\[
\langle \lambda, \frac{\text{vol}(U_1)}{\text{vol}(U_K)} \sum_i \pi(h_i m_i u_i^-)v \rangle = \langle \lambda, v \rangle
\]
since \( \lambda \) is \( H \)-invariant and \( v \in V^{M_K U_K} \).

(2) Assume that \( v \) (resp. \( v' \)) is the canonical lift of \( \pi \in V_P \) with respect to \( K \) (resp. \( K' \)). It is enough to consider the case where \( K' \subset K \). By the remark preceding this proposition, we have
\[
v' \in V^{M_K U_K}, \quad v = \mathcal{P}_{U_K}(v').
\]
It follows from (1) that
\[
\langle \lambda, v \rangle = \langle \lambda, \mathcal{P}_{U_K}(v') \rangle = \langle \lambda, v' \rangle.
\]
\[\square\]

5.4. After the above proposition, we can define a linear form \( r_P(\lambda) \) on the Jacquet module \( V_P \) along a \( \sigma \)-split parabolic subgroup \( P \) by
\[
\langle r_P(\lambda), \pi \rangle = \langle \lambda, v \rangle
\]
where \( v \in V \) is a canonical lift of \( \pi \in V_P \).

5.5. **Proposition.** Let \( \lambda \) be an \( H \)-invariant linear form on an admissible representation \((\pi, V)\) of \( G \) and \( P = M \ltimes U \) a \( \sigma \)-split parabolic subgroup of \( G \) with the \((\sigma, F)\)-split component \( S \).

(1) For each \( v \in V \), there exists a positive real number \( \varepsilon \leq 1 \) such that
\[
\langle \lambda, \pi(s)v \rangle = \delta_P^{1/2}(s) \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle
\]
for any \( s \in S^-(\varepsilon) \).

(2) Assume that \( \overline{\lambda} \) is a linear form on \( V_P \) having the following property: For each \( v \in V \), there exists a positive real number \( \varepsilon \leq 1 \) such that
\[
\langle \lambda, \pi(s)v \rangle = \delta_P^{1/2}(s) \langle \overline{\lambda}, \pi_P(s)j_P(v) \rangle
\]
for any $s \in S^-(\varepsilon)$. Then $\lambda$ coincides with $r_P(\lambda)$.

Proof. (1) For a given $v \in V$, choose an open compact subgroup $K = K_n$ from the adapted family such that $v \in V^K$. Take an open compact subgroup $U_1$ of $U_K$ with $V^K \cap V(U) \subset V(U_1)$. Let $\varepsilon \leq 1$ be a positive real number such that $sU_1s^{-1}$ is contained in $U_K$ for all $s \in S^-(\varepsilon)$. Then, by the Iwahori factorization with respect to $P$, we have $\pi(s)v \in V^{MKU_K}$ so that

$$j_P(\pi(s)v) = j_P(P_{U_K}(\pi(s)v)) = j_P(P_K(\pi(s)v))$$

for all $s \in S^-(\varepsilon)$. On the other hand, since $j_P(v) \in (V_P)^{MK}$ and $s$ is central in $M$, we have $\pi_P(s)j_P(v) \in (V_P)^{MK}$ so that

$$j_P(\pi(s)v) = \delta_P^{1/2}(s)\pi_P(s)j_P(v) \in (V_P)^{MK}.$$ 

These relations show that $\mathcal{P}_K(\pi(s)v) = \mathcal{P}_{U_K}(\pi(s)v)$ is a canonical lift of $\delta_P^{1/2}(s)\pi_P(s)j_P(v)$. Thus, by definition we have

$$\langle r_P(\lambda), \delta_P(s)^{1/2}\pi_P(s)j_P(v) \rangle = \langle \lambda, \mathcal{P}_{U_K}(\pi(s)v) \rangle$$

and the right hand side is equal to $\langle \lambda, \pi(s)v \rangle$ by Lemma 5.6.3 (1).

(2) Take an open compact subgroup $K = K_n$ from the adapted family. Let $\varepsilon$ be a positive real number such that

$$\langle \lambda, \pi(s)v \rangle = \delta_P(s)^{1/2}\langle \lambda, \pi_P(s)j_P(v) \rangle$$

for all $s \in S^-(\varepsilon)$ and all $v \in V^K$. This is possible since $V^K$ is finite dimensional. We may choose this $\varepsilon$ so that the space $\mathcal{P}_K(\pi(s)V^K)$ is independent of $s \in S^-(\varepsilon)$ by [C 4.1.6]. Since $\pi(s)v \in V^{MKU_K}$, we have

$$\langle \lambda, \pi(s)v \rangle = \langle \lambda, \mathcal{P}_{U_K}(\pi(s)v) \rangle = \langle \lambda, \mathcal{P}_K(\pi(s)v) \rangle$$

for all $s \in S^-(\varepsilon)$. On the other hand, $\mathcal{P}_K(\pi(s)v)$ is a canonical lift of $\delta_P^{1/2}(s)\pi_P(s)j_P(v)$ again, so we have

$$\langle \lambda, \mathcal{P}_K(\pi(s)v) \rangle = \delta_P^{1/2}(s)\langle r_P(\lambda), \pi_P(s)j_P(v) \rangle.$$ 

As a result, we have

$$\langle \lambda, \pi_P(s)j_P(v) \rangle = \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle$$

for any $s \in S^-(\varepsilon)$. Since $j_P(V^K) = (V_P)^{MK}$ (see [C 3.3.3]), this shows that $\lambda = r_P(\lambda)$ on $(V_P)^{MK}$. Letting $K$ vary in the adapted family, we conclude that $\lambda = r_P(\lambda)$ on $V_P$. \qed

5.6. Proposition. Let $\lambda$ be an $H$-invariant linear form on an admissible representation $(\pi, V)$ of $G$ and $P = M \ltimes U$ a $\sigma$-split parabolic subgroup of $G$.

(1) The linear form $r_P(\lambda)$ on $V_P$ is $M \cap H$-invariant.

(2) The mapping $r_P : (V^*)^H \rightarrow (V_P)^{M\cap H}$ is linear.
Proof. (1) For a given $m \in M \cap H$, set $\lambda = r_{P}(\lambda) \circ \pi_{P}(m)$. By (1) of Proposition 5.5 for the vector $\pi(m)v$, we can take $0 < \varepsilon \leq 1$ such that

$$\langle \lambda, \pi(s)\pi(m)v \rangle = \delta_{P}^{1/2}(s)\langle r_{P}(\lambda), \pi_{P}(s)j_{P}(\pi(m)v) \rangle$$

for all $s \in S^{-}(\varepsilon)$. Since $s$ is central in $M$, the left hand side is equal to

$$\langle \lambda, \pi(m) \pi(s)v \rangle = \langle \lambda, \pi(s)v \rangle,$$

while the right hand side is equal to

$$\delta_{P}^{1/2}(s)\delta_{P}^{1/2}(m)\langle r_{P}(\lambda), \pi_{P}(s)\pi_{P}(m)j_{P}(v) \rangle = \delta_{P}^{1/2}(s)\langle r_{P}(\lambda), \pi_{P}(m)\pi_{P}(s)j_{P}(v) \rangle = \delta_{P}^{1/2}(s)\langle \lambda, \pi_{P}(s)j_{P}(v) \rangle$$

by Lemma 2.9. Thus $\lambda$ has the property of Proposition 5.5 (2), which implies that $r_{P}(\lambda)$ coincides with $\lambda = r_{P}(\lambda) \circ \pi_{P}(m)$.

(2) For any $\lambda_{1}, \lambda_{2} \in (V^{*})^{H}$ and $c_{1}, c_{2} \in \mathbb{C}$, it is easy to see that $c_{1}r_{P}(\lambda_{1}) + c_{2}r_{P}(\lambda_{2})$ satisfies the unique property that $r_{P}(c_{1}\lambda_{1} + c_{2}\lambda_{2})$ must have in Proposition 5.5 (2).

5.7. The group case. Here we note that the mapping $r_{P}$ in the situation of 1.5 is the well-known one constructed by Casselman. Let $G$ be the group $G_{1} \times G_{1}$ with the involution $\sigma$ which permutes factors. Then $\sigma$-split parabolic subgroups are those of the form $P_{1} \times P_{1}^{-}$ where $P_{1}$ and $P_{1}^{-}$ are mutually opposite parabolic subgroups of $G_{1}$. Set $M_{1} = P_{1} \cap P_{1}^{-}$. For an irreducible $\Delta G_{1}$-distinguished representation $\pi_{1} \otimes \pi_{1}$ of $G_{1} \times G_{1}$, let $\lambda$ be the $\Delta G_{1}$-invariant linear form on $\pi_{1} \otimes \pi_{1}$ defined by the canonical pairing of $\pi_{1}$ and $\pi_{1}$ as in 1.5. Then $r_{P_{1} \times P_{1}^{-}}(\lambda)$ is an $(M_{1} \times M_{1}) \cap \Delta G_{1} = \Delta M_{1}$-invariant linear form on the Jacquet module $(\pi_{1} \otimes \pi_{1})_{P_{1} \times P_{1}^{-}} \simeq (\pi_{1})_{P_{1}} \otimes (\pi_{1})_{P_{1}^{-}}$. It is exactly the one given by the pairing of the Jacquet modules $(\pi_{1})_{P_{1}}$ and $(\pi_{1})_{P_{1}^{-}}$ constructed in [C, 4.2.2].

5.8. We study the transitivity of the mappings $r_{P}$ with respect to the inclusion of $\sigma$-split parabolic subgroups.

Let $P = M \ltimes U$ be a $\sigma$-split parabolic subgroup of $G$. It is obvious that $\sigma$-split parabolic subgroups of $M$ are of the form $M \cap Q$ where $Q$ is a $\sigma$-split parabolic subgroup of $G$ contained in $P$. Let $L$ be the $\sigma$-stable Levi subgroup of $Q$. It is also the $\sigma$-stable Levi subgroup of $M \cap Q$. As is well-known, $(V_{P})_{M \cap Q}$ is isomorphic to $V_{Q}$ as an $L$-module. Fix an isomorphism and identify $(V_{P})_{M \cap Q}$ with $V_{Q}$ from now on. There are induced mappings

$$r_{P} : (V^{*})^{H} \to (V_{P})^{M \cap H}, \quad r_{M \cap Q} : (V_{P})^{M \cap H} \to ((V_{P})^{*})_{M \cap Q}^{L \cap H}$$
and
\[ r_Q : (V^*)^H \to (V_Q^*)^{L \cap H} = ((V_P^*)_{M \cap Q})^{L \cap H} \]
of invariant linear forms.

5.9. **Proposition.** For \( P \) and \( Q \) as above, one has
\[ r_{M \cap Q} \circ r_P = r_Q. \]
In other words, the diagram
\[
\begin{array}{ccc}
(V^*)^H & \xrightarrow{r_P} & (V_P^*)^{M \cap H} \\
\downarrow r_Q & & \downarrow r_{M \cap Q} \\
(V_Q^*)^{L \cap H} & \xrightarrow{r_{M \cap Q}} & ((V_P^*)_{M \cap Q})^{L \cap H}
\end{array}
\]
is commutative.

**Proof.** Let \( \lambda \) be an element of \((V^*)^H\). We put \( \overline{\lambda} = r_{M \cap Q}(r_P(\lambda)) \). It is regarded as an \( L \cap H \)-invariant linear form on \( V_Q \). Let \( S_L \) be the \((\sigma,F)\)-split component of \( Q \). By Proposition 5.5 (1), we can choose a positive real number \( \varepsilon \leq 1 \) for each \( v \in V \) such that both
\[ \langle \lambda, \pi(s)v \rangle = \delta_{P}^{1/2}(s) \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle \]
and
\[ \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle = \delta_{M \cap Q}^{1/2}(s) \langle r_{M \cap Q}(r_P(\lambda)), \pi_{M \cap Q}(s)j_{M \cap Q}(j_P(v)) \rangle \]
hold for any \( s \in S_L^{-}(\varepsilon) \). Identifying \( j_{M \cap Q}(j_P(v)) \) with \( j_Q(v) \), we have
\[ \langle \lambda, \pi(s)v \rangle = \delta_{P}^{1/2}(s) \delta_{M \cap Q}^{1/2}(s) \langle \overline{\lambda}, \pi_Q(s)j_Q(v) \rangle \]
for all \( s \in S_L^{-}(\varepsilon) \). This means that \( \overline{\lambda} \) satisfies the unique property that \( r_Q(\lambda) \) must have in (2) of Proposition 5.5. \( \square \)

6. **Characterization of relative cuspidality**

We shall give a characterization of relative cuspidality in terms of Jacquet modules along \( \sigma \)-split parabolic subgroups.
6.1. Let \((\pi, V)\) be an \(H\)-distinguished admissible representation of \(G\). For a given non-zero \(H\)-invariant linear form \(\lambda \in (V^*)^H\) on \(V\), we have defined the \((H, \lambda)\)-matrix coefficient \(\varphi_{\lambda, v} (v \in V)\) by
\[
\varphi_{\lambda, v}(g) = \langle \lambda, \pi(g^{-1})v \rangle.
\]
Recall that \((\pi, V)\) is said to be \((H, \lambda)\)-relatively cuspidal if the support of \(\varphi_{\lambda, v}\) is compact modulo \(ZH\) for all \(v \in V\).

Let \(Z_{\sigma}\) denote the \((\sigma, F)\)-split component of \(Z\). We have an almost direct product decomposition \(Z = Z_{\sigma} \cdot (Z \cap H)\). This implies that 
\[
Z/Z_{\sigma}(Z \cap H)
\]
finite. Hence \((\pi, V)\) is \((H, \lambda)\)-relatively cuspidal if and only if the support of \(\varphi_{\lambda, v}\) is compact modulo \(Z_{\sigma}H\) for all \(v \in V\).

A large part of this section is devoted to the proof of the following theorem.

6.2. Theorem. Let \((\pi, V)\) be an \(H\)-distinguished admissible representation of \(G\) and \(\lambda\) a non-zero \(H\)-invariant linear form on \(V\). Then, \((\pi, V)\) is \((H, \lambda)\)-relatively cuspidal if and only if \(r_P(\lambda) = 0\) for any proper \(\sigma\)-split parabolic subgroup \(P\) of \(G\).

First we shall prove the only if part. The proof of the if part will be given in \(\S 8\) after several preparatory lemmas.

6.3. Proof of the only if part. Let \(P = M \times U\) be a proper \(\sigma\)-split parabolic subgroup of \(G\) with the \((\sigma, F)\)-split component \(S\). We regard \(P\) as a standard one \(P_1\) with respect to a suitable choice of \((S_0, A_0, \Delta)\) and a \(\sigma\)-split subset \(I \subset \Delta\) as in \(\S 3\). Assume that \((\pi, V)\) is \((H, \lambda)\)-relatively cuspidal. Let \(K = K_n\) be any member of the family \(\{K_n\}\) of \(\S 3\) adapted to \((S_0, A_0, \Delta)\). Since \(V^K\) is finite dimensional, we can take a compact subset \(C\) of \(G\) so that the support of \(\varphi_{\lambda, v}\) is contained in \(Z_{\sigma}CH\) for all \(v \in V^K\). Here let us observe that \(Z_{\sigma}CH \cap S^+(1)\) is contained in some subset of \(S^+(1)\) which is compact modulo \(Z_{\sigma}^+\). For a given \(s \in Z_{\sigma}CH \cap S^+(1)\), write \(s = zch\) with \(z \in Z_{\sigma}^+, c \in C\) and \(h \in H\). Since \(h = c^{-1}z^{-1}s\) is fixed by \(\sigma\), we have
\[
\sigma(c)^{-1}zs^{-1} = c^{-1}zs^{-1},
\]
thus
\[
(z^{-1}s)^2 = c\sigma(c)^{-1}.
\]
This shows that \((z^{-1}s)^2\), hence also \(z^{-1}s\), stays in a compact subset of \(S^+(1)\). Now we may choose a positive real number \(\varepsilon < 1\) such that
\[
Z_{\sigma}CH \cap S^+(1) \subset \{ s \in S \mid 1 \leq |s^n|_F < \varepsilon \} (\alpha \in \Delta \setminus I)\}
\]
Since \(s \in S^{-}(\varepsilon)\) implies that \(s^{-1} \notin Z_{\sigma}CH \cap S^+(1)\), we have
\[
\langle \lambda, \pi(s)v \rangle = \varphi_{\lambda, v}(s^{-1}) = 0
\]
for all $s \in S^-(\varepsilon)$ and $v \in V^K$. On the other hand, by (1) of Proposition 5.5, we may choose $\varepsilon'$ such that the relation

$$\langle \lambda, \pi(s)v \rangle = \frac{1}{2} \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle$$

holds for any $s \in S^-(\varepsilon')$ and $v \in V^K$. Putting $\varepsilon'' = \min(\varepsilon, \varepsilon')$, we have

$$\langle r_P(\lambda), \pi_P(s)j_P(v) \rangle = 0$$

for all $s \in S^-(\varepsilon'')$ and $v \in V^K$. This shows that $r_P(\lambda)$ vanishes on $j_P(V^K) = (V_P)^M$. Letting $K$ vary in the adapted family $\{K_n\}$, we conclude that $r_P(\lambda) = 0$ on $V_P$. □

6.4. Now we turn to the converse direction. Let $\lambda$ be a non-zero $H$-invariant linear form on an $H$-distinguished admissible representation $(\pi, V)$ of $G$. Assuming that $r_P(\lambda) = 0$ for proper $\sigma$-split parabolic subgroups $P$ of $G$, we shall investigate the support of $(H, \lambda)$-matrix coefficients of $\pi$. First we observe a relation between $r_P$ and $r_Q$ when $P$ and $Q$ are conjugate.

6.5. Fix a data $(S_0, A_\emptyset, \Delta)$ as in 2.1 and let $P_0 = M_0 \ltimes U_0$ be the corresponding minimal $\sigma$-split parabolic $F$-subgroup as in 2.4. For an element $\gamma \in (M_0H)(F)$, we put $m_\gamma = \gamma \sigma(\gamma)^{-1} \in M_0$. Define the $F$-involution $\sigma_\gamma$ on $G$ by

$$\sigma_\gamma(g) = m_\gamma \sigma(g) m_\gamma^{-1} = \gamma \sigma(\gamma^{-1}g \gamma)^{-1}.$$

The $\sigma_\gamma$-fixed point subgroup in $G$ coincides with $\gamma H \gamma^{-1}$, which is denoted by $H_\gamma$. Note that $S_0$ is also a maximal $(\sigma_\gamma, F)$-split torus. Moreover, any $\sigma$-split parabolic subgroup $P$ standard with respect to $(S_0, A_\emptyset, \Delta)$ is also $\sigma_\gamma$-split. The $(\sigma, F)$-split component and the $(\sigma_\gamma, F)$-split component coincide for such a parabolic subgroup $P$. Now we can define the mapping

$$(6.5.1) \quad r_P : (V^*)^{H_\gamma} \rightarrow (V_P^*)^{M \cap H_\gamma}$$

for an $H_\gamma$-distinguished representation $(\pi, V)$ by changing $\sigma$ to $\sigma_\gamma$ and $H$ to $H_\gamma$.

On the other hand, for $\gamma$ and $P$ as above, put $Q = \gamma^{-1}P \gamma$. It is $\sigma$-split, but possibly non-standard. If $S$ denotes the $(\sigma, F)$-split component of $P$, then the conjugate $\gamma^{-1}S \gamma$ gives that of $Q$. We can define the mapping

$$(6.5.2) \quad r_Q : (V^*)^H \rightarrow (V_Q^*)^{\gamma^{-1}M \gamma \cap H}$$

for an $H$-distinguished representation $(\pi, V)$. Two mappings (6.5.1) and (6.5.2) are related by the isomorphism $\pi^*(\gamma) : (V^*)^H \rightarrow (V^*)^{H_\gamma}$ as follows.
6.6. Lemma. For $\gamma$, $P$ and $Q$ as above, the relation
\[ \langle r_P(\pi^*(\gamma)\lambda), j_P(v) \rangle = \langle r_Q(\lambda), j_Q(\pi(\gamma^{-1})v) \rangle \]
holds for every $\lambda \in (V^*)^H$ and $v \in V$.

Proof. The mapping $\pi(\gamma^{-1}) : V \to V$ sends $V(U)$ isomorphically onto $V(\gamma^{-1}U\gamma)$, hence induces an isomorphism
\[ \pi(\gamma^{-1}) : V_P = V/V(U) \to V/V(\gamma^{-1}U\gamma) = V_Q \]
with the relation
\[ \pi(\gamma^{-1})(j_P(v)) = j_Q(\pi(\gamma^{-1})v) \]
Under this notation, the right hand side of the lemma is written as $\langle r_Q(\lambda), \pi(\gamma^{-1})(j_P(v)) \rangle$. Set $\lambda = r_Q(\lambda) \circ \pi(\gamma^{-1})$. We show that $\lambda$ satisfies the unique property in (2) of Proposition 5.5 that $r_P(\pi^*(\gamma)\lambda)$ must have. For $s$ in the $(\sigma, F)$-split component $S$ of $P$, we have
\[ \delta_P^{1/2}(s)(\lambda, \pi_P(s)j_P(v)) = \langle r_Q(\lambda), \pi(\gamma^{-1}) \left( \delta_P^{1/2}(s)\pi_P(s)j_P(v) \right) \rangle \]
\[ = \langle r_Q(\lambda), \pi(\gamma^{-1}) (j_P(\pi(s)v)) = \langle r_Q(\lambda), j_Q(\pi(\gamma^{-1})\pi(s)v) \rangle \]
\[ = \langle r_Q(\lambda), \delta_Q^{1/2}(\gamma^{-1}s\gamma)\pi_Q(\gamma^{-1}s\gamma)j_Q(\pi(\gamma^{-1})v) \rangle. \]
By Proposition 5.5 (1) applied to $Q$, there exists a positive real number $\varepsilon \leq 1$ such that the last quantity is equal to
\[ \langle \lambda, \pi(\gamma^{-1}s\gamma)\pi(\gamma^{-1})v \rangle = \langle \pi^*(\gamma)\lambda, \pi(s)v \rangle \]
for all $s \in S^{-}(\varepsilon)$. This shows the claim. \hfill \Box

Next we shall see what happens if $r_P(\lambda)$ vanishes for a single $P$.

6.7. Lemma. Fix a data $(S_0, A_0, \Delta)$ and let $I \subset \Delta$ be a $\sigma$-split subset, $\gamma$ an element of $(M_\delta H)(F)$ and $\Omega$ a compact subset of $G$. Let $\lambda$ be a non-zero $H$-invariant linear form on an admissible representation $(\pi, V)$ of $G$. Suppose that $r_{\gamma^{-1}P\gamma}(\lambda) = 0$. Then for each $v \in V$, there exists a positive real number $\varepsilon = \varepsilon_{\lambda, v} \leq 1$ such that $\varphi_{\lambda, v}$ vanishes identically on $\Omega S_I^+(\varepsilon)\gamma H$.

Proof. We abbreviate $P_I$ and $S_I^-(\varepsilon)$ respectively as $P$ and $S^{-}(\varepsilon)$. For $k \in \Omega$, $s \in S^{-}(\varepsilon)$ and $h \in H$, we have
\[ \varphi_{\lambda, v}(ks^{-1}\gamma h) = \langle \lambda, \pi(\gamma^{-1})\pi(s)\pi(k^{-1})v \rangle = \langle \pi^*(\gamma)\lambda, \pi(s)\pi(k^{-1})v \rangle. \]
Note that $\pi(k^{-1})v$ stays in a finite dimensional subspace for any $k \in \Omega$. By applying Proposition 5.5 (1) to the linear form $\pi^*(\gamma)(\lambda) \in (V^*)^H$,
we can take a positive real number $\varepsilon \leq 1$ such that
\[
\langle \pi^* (\gamma) \lambda, \pi (s) \pi (k^{-1}) v \rangle = \delta^1_{p^2} (s) \langle r_P (\pi^* (\gamma) \lambda), \pi_P (s) j_P (\pi (k^{-1}) v) \rangle
\]
for any $s \in S^- (\varepsilon)$ and $k \in \Omega$. By Lemma 6.6, the right hand side is equal to
\[
\langle r_{\gamma^{-1} P \gamma} (\lambda), j_{\gamma^{-1} P \gamma} (\pi (k^{-1}) \pi (s) \pi (k^{-1}) v) \rangle,
\]
which is zero by assumption.  \hfill \Box

Now we give the rest of the proof of 6.2.

6.8. Proof of the if part. Recall from Corollary 3.4 that $G$ is decomposed as
\[
G = \Omega S^+_0 \Gamma H
\]
for a suitable compact subset $\Omega$ of $G$ and a finite subset $\Gamma$ of $(M_0 H)(F)$.
Assume that $r_P (\lambda) = 0$ for all proper $\sigma$-split parabolic subgroup $P$.
For a given $v \in V$, let $\varepsilon$ be the minimum of $\varepsilon_{I, \gamma}$ in Lemma 6.7 where $I$ runs over all proper $\sigma$-split subsets of $\Delta$ and $\gamma$ runs over $\Gamma$. Then, $\varphi_{\lambda,v}|_{\Omega s^{-1} \Gamma H} \equiv 0$ if $s \in S_0^- (1) \cap S^- (\varepsilon)$ for some $\sigma$-split subset $I \subset \Delta$. In particular, $s \in S_0^- (1)$ cannot be in $S^-_{I,\alpha} (\varepsilon)$ for all $\alpha \in \Delta \setminus \Delta_{\sigma}$ if $\varphi_{\lambda,v}$ is not identically zero on $\Omega s^{-1} \Gamma H$. Here $I_\alpha$ is the maximal $\sigma$-split subset of $\Delta$ as in 2.6. Note that $s^\alpha = s^{\sigma^* (\alpha)}$ for all $s \in S_0$ if $\alpha \in \Delta \setminus \Delta_{\sigma}$. As a result, the support of $\varphi_{\lambda,v}$ is contained in the union of $\Omega s^{-1} \Gamma H$ where $s \in S_0^- (1)$ satisfies $\varepsilon < |s^\alpha|_F \leq 1$ for all $\alpha \in \Delta \setminus \Delta_{\sigma}$. However, the set $\{ s \in S_0 \mid \varepsilon < |s^\alpha|_F \leq 1 (\alpha \in \Delta \setminus \Delta_{\sigma}) \}$ is finite modulo $Z^{-}_\sigma \cdot S_0 (O_F)$. Hence the support of $\varphi_{\lambda,v}$ is compact modulo $Z^{-}_\sigma H$. This completes the proof of Theorem 6.2. \hfill \Box

Recall that $(\pi, V)$ is said to be $H$-relatively cuspidal if it is $(H, \lambda)$-relatively cuspidal for all $\lambda \in (V^*)^H$. Now we have obtained one of our main theorem.

6.9. Theorem. An $H$-distinguished admissible representation $(\pi, V)$ of $G$ is $H$-relatively cuspidal if and only if $r_P ((V^*)^H) = 0$ for any proper $\sigma$-split parabolic subgroup $P$ of $G$.

6.10. Remark. If all $\sigma$-split parabolic $F$-subgroups are $H$-conjugate to one in a fixed standard class (see Lemma 2.5 (2) for example), then by Lemma 6.6, it turns out that $r_P ((V^*)^H)$ vanishes for all proper $\sigma$-split $P$ if and only if it does for all standard proper $P$. Thus, in such a case, the relative cuspidality is characterized by the vanishing of $r_P$ only for all standard proper $\sigma$-split $P$. 

6.11. The group case. The above characterization of the relative cuspidality gives the well-known theorem due to Jacquet in the group case. Let \( \pi_1 \otimes \tilde{\pi}_1 \) be an irreducible \( \Delta G_1 \)-distinguished representation of \( G_1 \times G_1 \) with the canonical invariant linear form \( \lambda \) (see 1.5). For a \( \sigma \)-split parabolic \( P_1 \times P_1^- \), the linear form \( r_{P_1 \times P_1^-}(\lambda) \) is regarded as the Casselman’s pairing of \( (\pi_1)_{P_1} \) and \( (\tilde{\pi}_1)_{P_1^-} \) (see 5.7). It vanishes if and only if \( (\pi_1)_{P_1} \) vanishes since Casselman’s pairing is non-degenerate \[C, 4.2.4\]. Thus, Theorem 6.9 in the group case asserts that \( \pi_1 \) is cuspidal if and only if \( (\pi_1)_{P_1} = 0 \) for all proper parabolic subgroup \( P_1 \) of \( G_1 \).

7. Relative subrepresentation theorem

Here we give a proof of the relative version of Jacquet’s subrepresentation theorem. From an inductive argument using the Frobenius reciprocity and the transitivity of the mapping \( r_P \), it is a natural consequence of the characterization theorem in the previous section.

7.1. Theorem. Let \( \pi \) be an irreducible \( H \)-distinguished admissible representation of \( G \). Then there exists a \( \sigma \)-split parabolic subgroup \( P = M \ltimes U \) of \( G \) and an irreducible \( M \cap H \)-relatively cuspidal representation \( \rho \) of \( M \) such that \( \pi \) is a subrepresentation of \( \text{Ind}_P^G(\rho) \).

Proof. We show this by induction on the dimension \( r \) of the maximal \( (\sigma, F) \)-split tori of \( G/Z \). If \( r = 0 \), then \( G/ZH \) is compact by \[HW, 4.3\]. Hence every \( H \)-distinguished representation is \( H \)-relatively cuspidal. Assume \( r > 0 \). If \( (\pi, V) \) is \( H \)-relatively cuspidal, there is nothing to prove. If not, then there exists a non-zero \( H \)-invariant linear form \( \lambda \in (V^*)^H \) such that \( \pi \) is not \( (H, \lambda) \)-relatively cuspidal. It follows from Theorem 6.2 that there exists a proper \( \sigma \)-split parabolic subgroup \( P \) of \( G \) such that \( r_P(\lambda) \neq 0 \). Let \( Q = M_Q \ltimes U_Q \) be minimal among such. Then by Proposition 5.9 and Theorem 6.2 it is seen that the Jacquet module \( \pi_Q \) is \( (M_Q \cap H, r_Q(\lambda)) \)-relatively cuspidal. Apply Proposition 1.11 and take an irreducible \( M_Q \cap H \)-distinguished quotient \( \rho' \) of \( \pi_Q \). By the Frobenius reciprocity asserting that

\[
\text{Hom}_G(\pi, \text{Ind}_Q^G(\rho')) \simeq \text{Hom}_{M_Q}(\pi_Q, \rho') \neq 0,
\]

we have an embedding of \( \pi \) into \( \text{Ind}_Q^G(\rho') \). Now the dimension of maximal \( (\sigma, F) \)-split tori in \( M_Q/A_Q \) (where \( A_Q \) denotes the \( F \)-split component of \( M_Q \)) is strictly less than \( r \). By the induction hypothesis applied to \( \rho' \), there exists a \( \sigma \)-split parabolic subgroup \( M_Q \cap P = M_P \ltimes U_P \) of \( M_Q \) and an irreducible \( M_P \cap H \)-relatively cuspidal representation \( \rho \) of \( M_P \) such that \( \rho' \) is a subrepresentation of \( \text{Ind}_{M_Q \cap P}^{M_Q}(\rho) \). Here \( P \) is a \( \sigma \)-split parabolic subgroup of \( G \) contained in \( Q \) and \( M_P \) is the \( \sigma \)-stable
Levi subgroup of $P$ (see 5.8). As a consequence, $\pi$ is a subrepresentation of $\text{Ind}_Q^G (\text{Ind}_{M_Q \cap P}^G (\rho)) \simeq \text{Ind}_P^G (\rho)$. □

7.2. The group case. We shall apply the above theorem to the group case. For any irreducible admissible $\Delta G_1$-distinguished representation $\pi_1 \otimes \tilde{\pi}_1$ of $G_1 \times G_1$, there exist a $\sigma$-split parabolic subgroup $P_1 \times P_1^-$ and an irreducible $\Delta M_1$-relatively cuspidal representation $\rho_1 \otimes \tilde{\rho}_1$ of $M_1 \times M_1$ such that $\pi_1 \otimes \tilde{\pi}_1$ can be embedded in

$$\text{Ind}_{P_1 \times P_1^-}^{G_1 \times G_1} (\rho_1 \otimes \tilde{\rho}_1) \simeq \text{Ind}_{P_1}^{G_1} (\rho_1) \otimes \text{Ind}_{P_1^-}^{G_1} (\tilde{\rho}_1).$$

Here $\rho_1$ is an irreducible cuspidal representation of $M_1$. The embedding $\pi_1 \hookrightarrow \text{Ind}_{P_1}^{G_1} (\rho_1)$ on the first factor asserts nothing but Jacquet’s subrepresentation theorem (see [C, 5.1.2]). On the second factor we have the embedding $\tilde{\pi}_1 \hookrightarrow \text{Ind}_{P_1^-}^{G_1} (\tilde{\rho}_1)$ at the same time (see [S, 3.3.1]).

8. EXAMPLES OF RELATIVELY CUSPIDAL REPRESENTATIONS

This section is devoted to several examples of relatively cuspidal distinguished representations.

8.1. Proposition. Cuspidal $H$-distinguished representations are $H$-relatively cuspidal.

Proof. This is immediate from Theorem 6.9. □

Certain examples of cuspidal distinguished representations were constructed by Hakim and Mao [HM1, HM2] for the symmetric pairs $(GL_n(E), U_n(E/F))$ and $(GL_n(F), O_n(F))$.

In the rest of this section, we shall give two examples of non-cuspidal but relatively cuspidal distinguished representations.

8.2. The symmetric pair $(GL_n, GL_{n-1} \times GL_1)$. For $n \geq 3$, it is known that irreducible cuspidal representations of $GL_n(F)$ do not have any non-zero $GL_{n-1}(F)$-invariant linear form (see [P, Proof of Theorem 2]). Therefore irreducible cuspidals cannot be $GL_{n-1}(F) \times GL_1(F)$-distinguished. We shall construct a class of relatively cuspidal representations below. For $n = 3$ similar representations were treated by D. Prasad in [P]. Representations attached to finite symmetric pair of this type was studied in [vD].
8.2.1. Let $G$ be the group $GL_n/F$. Put
\[ \epsilon = \begin{pmatrix} 1 & 1 \\ 1_{n-2} & 1 \end{pmatrix}, \quad \epsilon' = \begin{pmatrix} 1_{n-1} & 1 \\ 1 & 1_{n-2} \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & 1 \\ 1_{n-2} & 1 \end{pmatrix}. \]

We consider the inner involution $\sigma = \text{Int}(\epsilon)$ on $G$. Let $H$ be the $\sigma$-fixed point subgroup in $G$ and $H'$ the fixed point subgroup of the other involution $\text{Int}(\epsilon')$. We have
\[ H' = \left\{ \begin{pmatrix} A & 0 \\ 0 & u \end{pmatrix} \middle| A \in GL_{n-1}, u \in GL_1 \right\}. \]

By the relation $\epsilon = \eta \epsilon' \eta^{-1}$, it is seen that $H = \eta^{-1} H' \eta$. As a maximal $(\sigma,F)$-split torus, we shall take $S_0 = \{ \text{diag}(s, 1, \cdots, 1, s^{-1}) \mid s \in GL_1 \}$.

Let $A_0$ be the maximal $F$-split torus consisting of all the diagonal matrices and $\Delta$ the set of simple roots of $(G, A_0)$ corresponding to the Borel subgroup of upper triangular matrices. If $e_i \in X^*(A_0)$ (for $1 \leq i \leq n$) denotes the $F$-rational character of $A_0$ given by $e_i(\text{diag}(a_1, a_2, \cdots, a_n)) = a_i$, then $\Delta$ is described as
\[ \Delta = \{ e_i - e_{i+1} \mid 1 \leq i \leq n - 1 \}. \]

It is easy to see that $\Delta$ is a $\sigma$-basis. We have
\[ \Delta_\sigma = \{ e_2 - e_3, e_3 - e_4, \cdots, e_{n-2} - e_{n-1} \} \]
and
\[ \sigma(e_1 - e_2) = -(e_2 - e_n), \quad \sigma(e_{n-1} - e_n) = -(e_1 - e_{n-1}). \]

There is only one proper standard $\sigma$-split parabolic subgroup $P_0$, the minimal one corresponding to $\Delta_\sigma$. It is the standard parabolic subgroup of type $(1, n-2, 1)$, i.e.,
\[ P_0 = P_{1,n-2,1} = \left\{ \begin{pmatrix} a & A^* \\ 0 & b \end{pmatrix} \in G \mid a, b \in GL_1, A \in GL_{n-2} \right\}. \]

The $\sigma$-stable Levi subgroup $M_0$ of $P_0$ is given by
\[ M_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A_b \end{pmatrix} \in G \mid a, b \in GL_1, A \in GL_{n-2} \right\}. \]

Note that $\sigma$ acts on $M_0$ as
\[ \sigma \left( \begin{pmatrix} a & 0 \\ 0 & A_b \end{pmatrix} \right) = \begin{pmatrix} b & 0 \\ 0 & A_a \end{pmatrix}, \]
hence $M_0 \cap H$ consists of all the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & A_a \end{pmatrix}$. This implies that the Galois cohomology of $M_0 \cap H$ over $F$ is trivial. So we have $(M_0 \cap H)(F) = M_0 H$ and thus every proper $\sigma$-split parabolic subgroup of $G$ is $H$-conjugate to $P_0$ in this case by Lemma 2.5 (2).
8.2.2. Let \( Q = P_{n-2,2} \) be the standard parabolic subgroup of \( G \) of type \((n-2,2)\), i.e.,

\[
Q = \{ \begin{pmatrix} A^* & 0 \\ 0 & g \end{pmatrix} \mid A \in GL_{n-2}, \ g \in GL_2 \}.
\]

This is not \( \sigma \)-split nor \( \sigma \)-stable, but the conjugate \( \eta Q \eta^{-1} \) is \( \sigma \)-stable. Take a representation \((\rho, W_\rho)\) of \( GL_2(F) \) and consider the normalized induction

\[
\pi = \text{Ind}^G_Q(1_{GL_{n-2}(F)} \otimes \rho).
\]

8.2.3. \textbf{Proposition.} If \( \rho \) is an irreducible cuspidal representation of \( GL_2(F) \) with trivial central character, then the induced representation

\[
\pi = \text{Ind}^G_Q(1_{GL_{n-2}(F)} \otimes \rho)
\]

is irreducible, \( H \)-distinguished and \( H \)-relatively cuspidal.

The irreducibility follows from the general result of [Z, 3.2, 4.2]. Up to 8.2.5 we show that \( \pi \) carries a non-zero \( H' \)-invariant linear form (hence is \( H \)-distinguished by the relation \( H = \eta^{-1}H'\eta \)). Relative cuspidality will be seen in 8.2.6.

8.2.4. Set \( O = QH' \). This is a closed \((Q, H')\)-double coset of \( G \) since \( Q \) is \( \text{Int}(\epsilon') \)-stable (see [HW, 13.3]). Let \( I(\rho; O) \) be the space of all locally constant mappings \( \phi : O \to W_\rho \) satisfying

\[
\phi \left( \begin{pmatrix} A^* & 0 \\ 0 & g \end{pmatrix} x \right) = \delta_P \left( \begin{pmatrix} A^* & 0 \\ 0 & g \end{pmatrix} \right)^{1/2} \rho(g) \phi(x)
\]

\[
= |\det(A)| \cdot |\det(g)|^{-(n-2)/2} \cdot \rho(g) \phi(x)
\]

for all \( \begin{pmatrix} A^* & 0 \\ 0 & g \end{pmatrix} \in Q \) and \( x \in O \). This is a smooth \( H' \)-module by the right translation. The restriction map \( \text{Ind}^G_Q(1_{GL_{n-2}(F)} \otimes \rho) \to I(\rho; O) \) is a surjective \( H' \)-morphism. Taking \( H' \)-invariants in the dual mapping, we have an injection

\[
(I(\rho; O)^{\ast})^{H'} \hookrightarrow (\text{Ind}^G_Q(1_{GL_{n-2}(F)} \otimes \rho)^{\ast})^{H'}.
\]

So it is enough to construct a non-zero element of \((I(\rho; O)^{\ast})^{H'}\).

8.2.5. Set \( R = Q \cap H' \) and \( \rho_R = \left( (1_{GL_{n-2}(F)} \otimes \rho) \delta_Q^{1/2} \right) \mid_R \). Explicitly the subgroup \( R \) is given by

\[
R = \left\{ \begin{pmatrix} A & * & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a \\ 0 & \ldots & 0 & d \end{pmatrix} \mid A \in GL_{n-2}(F), \ a, d \in F^\times \right\}
\]
and the representation \( \rho_R \) of \( R \) on \( W_\rho \) is given by
\[
\rho_R \left( \begin{array}{cccc}
A & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & A & 0 \\
0 & \cdots & 0 & d
\end{array} \right) = |\det(A)| \cdot |a|^{-(n-2)/2} \cdot |d|^{(n-2)/2} \cdot \rho \left( \begin{array}{c}
0 \\
a \\
0 \\
d
\end{array} \right).
\]

It is seen that \( \phi|_{H'} \) belongs to the unnormalized induction \( \text{ind}_{R}^{H'}(\rho_R) \) for all \( \phi \in \text{I}(\rho; \mathcal{O}) \). Furthermore, the mapping \( \phi \mapsto \phi|_{H'} \) gives an isomorphism
\[
\text{I}(\rho; \mathcal{O}) \cong \text{ind}_{R}^{H'}(\rho_R)
\]
of \( H' \)-modules. By a version of Frobenius reciprocity [C, 2.4.3], we have isomorphisms
\[
(I(\rho; \mathcal{O})^*)^{H'} \cong \text{Hom}_{H'} \left( \text{ind}_{R}^{H'}(\rho_R), \mathbb{C} \right) \cong \text{Hom}_R (\rho_R, \mathbb{C}_{\delta_R})
\]
where \( \mathbb{C}_{\delta_R} \) is the one dimensional representation of \( R \) defined by the modulus character \( \delta_R \) of \( R \). Note that \( \delta_R \) is given by
\[
\delta_R \left( \begin{array}{cccc}
A & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & A & 0 \\
0 & \cdots & 0 & d
\end{array} \right) = |\det(A)| \cdot |a|^{-(n-2)}.
\]

So the space \( \text{Hom}_R (\rho_R, \mathbb{C}_{\delta_R}) \) consists of all the linear forms \( \nu : W_\rho \rightarrow \mathbb{C} \) satisfying
\[
\langle \nu, \rho \left( \begin{array}{c}
0 \\
a \\
0 \\
d
\end{array} \right) w \rangle = |a|^{-(n-2)/2} \cdot |d|^{(n-2)/2} \langle \nu, w \rangle
\]
for all \( w \in W_\rho \) and \( a, d \in F^\times \). Under the assumption that \( \rho \) is cuspidal with trivial central character, we can construct such a non-zero linear form \( \nu : W_\rho \rightarrow \mathbb{C} \) by using the Kirillov model \( \mathcal{K}_\rho \) of \( \rho \) (see [JL]). The space \( \mathcal{K}_\rho \) is \( \mathbb{C}_{\infty}^\infty(F^\times) \), on which the action of the diagonals is described as
\[
(\rho \left( \begin{array}{c}
0 \\
a \\
0 \\
d
\end{array} \right) \xi)(x) = \xi(ax)
\]
for \( \xi \in \mathcal{K}_\rho \), \( x \in F^\times \). Let \( \nu \) be the linear form on \( \mathcal{K}_\rho \) defined by
\[
\langle \nu, \xi \rangle = \int_{F^\times} |x|^{(n-2)/2} \xi(x) d^\times x
\]
where \( d^\times x \) denotes a Haar measure on the multiplicative group \( F^\times \). It behaves under the action of \( \left( \begin{array}{c}
0 \\
a \\
0 \\
d
\end{array} \right) = \left( \begin{array}{c}
0 \\
a d^{-1} \\
0 \\
d
\end{array} \right) \) as
\[
\langle \nu, \rho \left( \begin{array}{c}
0 \\
a \\
0 \\
d
\end{array} \right) \xi \rangle = \langle \nu, \rho \left( \begin{array}{c}
a d^{-1} \\
0 \\
0 \\
d
\end{array} \right) \xi \rangle = \int_{F^\times} |x|^{(n-2)/2} \xi(\rho(d^{-1})x) d^\times x
\]
\[
= \int_{F^\times} |xa^{-1}d|^{(n-2)/2} \xi(x) d^\times x = |a|^{-(n-2)/2} \cdot |d|^{(n-2)/2} \langle \nu, \xi \rangle.
\]
This is the desired property.
8.2.6. Finally we show that $\pi = \text{Ind}_{G}^{Q}(1_{GL_{n-2}(F)} \otimes \rho)$ is $H$-relatively cuspidal. Assume the contrary. Recall that any proper $\sigma$-split parabolic subgroup is $H$-conjugate to $P_0$. By Theorem 7.1 there exists an irreducible $M_0 \cap H$-distinguished (relatively cuspidal) representation, say $\chi_1 \otimes \theta \otimes \chi_2$, of $M_0 \simeq F^{\times} \times GL_{n-2}(F) \times F^{\times}$ such that $\pi$ can be embedded in $\text{Ind}_{P_0}^{G}(\chi_1 \otimes \theta \otimes \chi_2)$. We must have $\theta = 1_{GL_{n-2}(F)}$ and $\chi_2 = \chi_1^{-1}$ by the $M_0 \cap H$-distinguishedness. Now we have an embedding

$$\text{Ind}_{P_{0,2,2}}^{G}(1_{GL_{n-2}(F)} \otimes \rho) \hookrightarrow \text{Ind}_{P_{1,1,1}}^{G}(\chi_1 \otimes 1_{GL_{n-2}(F)} \otimes \chi_1^{-1})$$

which contradicts to the cuspidality of $\rho$.

8.3. The symmetric pair $(GL_{2n}, Sp_n)$. This kind of symmetric pair was studied by Heumos and Rallis in [HR] (cf. [BKS] for the finite field case). They have shown that $Sp_n(F)$-models and (non-degenerate) Whittaker models of $G = GL_{2n}(F)$ are disjoint. It turns out that there is no irreducible cuspidal $Sp_n(F)$-distinguished representation, since cuspidals always have Whittaker model. In the following, we shall see that irreducible $Sp_n(F)$-distinguished Langlands quotient representations of $GL_{2n}(F)$ constructed in [HR, 11.3] are $Sp_n(F)$-relatively cuspidal if the inducing representations of $GL_n(F)$ are cuspidal in the usual sense.

8.3.1. Let $G$ be the group $GL_{2n}/F$ and $J_n \in G$ the alternating matrix given by

$$J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ \vdots \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Consider the involution $\sigma$ on $G$ defined by

$$\sigma(g) = J_n^{t}g^{-1}J_n^{-1}.$$ 

The $\sigma$-fixed point subgroup $H$ of $G$ is the symplectic group with respect to $J_n$. As a maximal $(\sigma, F)$-split torus we take

$$S_0 = \{ \text{diag}(s_1, s_1, s_2, \cdots, s_n, s_n) \mid s_i \in GL_1 \}.$$ 

Let $A_\emptyset$ be the maximal $F$-split torus consisting of all the diagonal matrices and $\Delta$ the set of simple roots of $(G, A_\emptyset)$ corresponding to the Borel subgroup of upper triangular matrices. Then $\Delta$ is a $\sigma$-basis: We have

$$\sigma(e_i - e_{i+1}) = \begin{cases} e_i - e_{i+1} & \text{if } i \text{ is odd,} \\ -(e_{i-1} - e_{i+2}) & \text{if } i \text{ is even} \end{cases}$$ 

where the notation is as in 8.2.1. The minimal $\sigma$-split parabolic subgroup $P_0$ corresponding to $\Delta_\sigma$ is the standard one of type $(2, 2, \cdots, 2)$.
The $\sigma$-stable Levi subgroup $M_0$ of $P_0$ is isomorphic to the product of $n$ copies of $GL_2$. Looking at the action of $\sigma$ on $M_0$, it is seen that $M_0 \cap H$ is isomorphic to the product of $n$ copies of $SL_2$. Again by the triviality of the Galois cohomology of $M_0 \cap H$ and Lemma 2.5 (2), every $\sigma$-split parabolic subgroup is $H$-conjugate to a standard one. Maximal standard $\sigma$-split parabolics are ones of type $(m, 2n - m)$ where $m$ is even and any maximal $\sigma$-split parabolics are $H$-conjugate to such ones.

8.3.2. Let $J'_n$ be the alternating matrix given by

$$J'_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

and $H'$ the symplectic group with respect to $J'_n$. There is an element $\eta \in G$ such that $J_n = \eta J'_n \eta$, which gives the relation $H = \eta^{-1} H' \eta$. Thus an admissible representation $(\pi, V)$ of $G$ is $H$-distinguished if and only if it is $H'$-distinguished.

8.3.3. Let $Q = P_{n,n}$ be the standard parabolic subgroup of type $(n, n)$ given by

$$Q = \{(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}) \mid A, D \in GL_n\}.$$ 

The conjugate $\eta Q \eta^{-1}$ is $\sigma$-stable. Take an irreducible admissible representation $\rho$ of $GL_n(F)$ and form the normalized induction

$$I_{\rho} = \text{Ind}^G_Q \left( \rho \vert \text{det}(\cdot) \right)^{1/2} \otimes \rho \vert \text{det}(\cdot) \left|^{-1/2} \right).$$

In [HR, 11.3.1.2], it is shown that $I_{\rho}$ is $H'$-distinguished, reducible of length 2 and the unique irreducible quotient $\pi_{\rho}$ of $I_{\rho}$ also is $H'$-distinguished (hence $H$-distinguished).

8.3.4. Proposition. If $\rho$ is an irreducible cuspidal representation of $GL_n(F)$, then $\pi_{\rho}$ is $H$-relatively cuspidal.

Proof. By 6.10 and the remark at the end of 8.3.1, it is enough to see that $r_P ((\pi_{\rho})^H) = 0$ for all maximal standard $\sigma$-split parabolic subgroup $P$. We show that $(\pi_{\rho})_P$ cannot be $M \cap H$-distinguished for any maximal $\sigma$-split parabolic $P = P_{m,2n-m} = M \ltimes U$. Except for the case that $m = n$ and $n$ is even, $P$ is not associated to $Q$, hence $(I_{\rho})_P$ (and also $(\pi_{\rho})_P$) vanishes by the cuspidality of $\rho$ (see [BZ, 2.13(a)]). If $n$ is even and $m = n$ (namely $P = Q$), then $(I_{\rho})_P$ (and also $(\pi_{\rho})_P$) is cuspidal by [BZ, 2.13(c)]. In this case, $M = GL_n(F) \times GL_n(F)$ and $M \cap H = \hat{S}_{P_{n}}(F) \times \hat{S}_{P_{n}}(F)$, so the cuspidal $M$-module $(\pi_{\rho})_P$ cannot be $M \cap H$-distinguished by the remark at the beginning of 8.3.1. \qed
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