Massive cosmological scalar perturbations

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We study the cosmological perturbations of the new bi-metric gravity proposed by Hassan and Rosen as a representation of massive gravity. The mass term in the model, in addition of ensuring ghost freedom for both metrics, causes the two scale factors to mix at the cosmological level and this affects the cosmological perturbation of the model. We find two combinations corresponding to the entropy and adiabatic perturbations of the theory. In this sense we show that the adiabatic perturbations could be a source for the entropy perturbations. So in addition to the adiabatic perturbations, entropy perturbations can also be present in this theory. We also show that the adiabatic perturbations are not constant at the super horizon scales, implying that the theory could not be used to describe the inflationary epoch, even if it can impose some corrections to the standard inflationary scenarios.

PACS numbers: 98.80.-k,04.50.Kd

I. INTRODUCTION

The current problems facing standard cosmology are of such breadth and depth that any fixation to these problems requires a deeper understanding of the prevailing theory that has been and is still in use, namely the general theory of relativity. Novel ideas, both old and new, have been suggested over its rather long history in the hope of a partial remedy to some of the most pressing problems we are facing today. Two interesting observations made over the period of the past two decades, namely, the accelerated expansion of the universe and galaxy rotation curves, amongst others, have stirred a plethora of activities aimed at modifying the standard Einstein-Hilbert (EH) action and hence to offer a solution to some of these problems. With partial success, one may consider various modifications to the standard EH action in the form of modified theories of gravity with and without torsion, gravity in extra dimensions, and more recently, massive gravity, to name but a few. The latter has been attracting the attention of many experts in the field in the past few years. It is based on an old idea by Fierz and Pauli (FP)\textsuperscript{2} where an effective field theory with a massive graviton was proposed. This theory was, however, problematic at the linearized level since the corresponding Newtonian potential was discontinuous for a vanishing mass, $m^2$, resulting, for example, in a large correction to the deflection of light around the Sun compared to the experimentally accepted value\textsuperscript{3} predicted by General Relativity (GR). This is referred to as the vDVZ discontinuity. The source of the discrepancy was traced to the degrees of freedom of the graviton, being two for a massless and five for a massive graviton. Another setback was discovered later on when it was demonstrated that when self interacting terms are added to the action, ghosts would appear in the theory\textsuperscript{4}. It has only been in the past few years that a method for fixing the above problems has been realized\textsuperscript{5}. Theories rooted in the FP action are collectively known as massive gravity.

Perturbation methods are and have been an integral part of attempts to find solutions to complicated problems. Massive gravity is therefore no exception in this regard. In non-linear theories such as GR, perturbation methods can be particularly useful when one seeks the effects of small changes in the metric. Such methods have been exploited in recent years to study, for example, the structure formations in the universe. It then seems only natural to conduct such a study when dealing with massive gravity. Such a study becomes even more attractive if one considers the fact that massive gravity is inherently a bi-metric theory. This comes about since any modification to the EH action in the form of a self-coupling term involving no derivative whose definition is based on one particular metric requires an additional metric which may be dynamical or fixed\textsuperscript{6,7}. The appearance of a second metric in the theory can be understood on general grounds. If the second metric is non-singular, spherically symmetric, and both are diagonal in some coordinate system, a Killing horizon for one metric must be a Killing horizon for the other\textsuperscript{8}. Interestingly, it has been shown that the off diagonal elements undergo no modification at large distances\textsuperscript{9}. From a cosmological perspective, in a bi-metric massive gravity theory with the second metric static, there is no spatially flat FRW solutions\textsuperscript{10}, contrary to the bi-gravity formulation of the FRW cosmology for which homogeneous solutions exist\textsuperscript{11,12}.

It has increasingly been realized that modern observational data can be explained by perturbation methods which have become an integral part of any study dealing with cosmological fluctuations. On the other hand,
from a theoretical point of view, the inflationary scenario can also be supported by the modern observational data rather accurately. This makes the studies of the cosmological perturbations in massive gravity [14] all the more interesting since its predictions are gradually becoming known and are still not on firm grounds. As was mentioned above, one realization of massive gravity is in the language of bi-metric theories. Consequently one may be interested in a cosmological perturbation theory in such models, for the existence of two metrics in these theories may result in non-trivial features, specially in the behavior of relative metric perturbations. In this paper we will consider scalar perturbations for both metrics in a massive gravity model. We will obtain two gauge invariant combinations of the perturbed functions which can be responsible for the adiabatic and entropy perturbations. As a result we will obtain the super horizon limit of the equations of perturbations and show that the adiabatic perturbation is not constant in this scale.

The scope of the paper is as follows: in the next section we present the bi-metric model studied here. Section II deals with the the cosmological perturbations of the model and definition of the gauge invariant quantities. In section III we obtain the equations of motion for the adiabatic perturbations and discuss the super horizon limit. Conclusions are drawn in section IV.

II. THE MODEL

We begin with the bimetric action [1]

\[ S = -M_g^2 \int d^4x \sqrt{-g} R(g) - M_f^2 \int d^4x \sqrt{-f} R(f) + 2M_{eff}^2m^2 \int d^4x \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n (\sqrt{g^{-1}f}) , \]

(1)

where \( R(g) \) and \( R(f) \) are the Ricci scalars corresponding to metrics \( g_{\mu\nu} \) and \( f_{\mu\nu} \) respectively, and \( \beta_i \) are some arbitrary constants. We define the polynomials \( e_n \) as

\[
\begin{align*}
  e_0(X) &= 1, \\
  e_1(X) &= [X], \\
  e_2(X) &= \frac{1}{2} ([X]^2 - [X^2]), \\
  e_3(X) &= \frac{1}{6} ([X]^3 - 3[X][X^2] + 2[X^3]), \\
  e_4(X) &= \text{det}X.
\end{align*}
\]

(2)

with \( X = \sqrt{g^{-1}f} \). We also define

\[
\frac{1}{M_{eff}^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2}.
\]

(3)

Without the kinetic term for metric \( f_{\mu\nu} \) the action describes a theory of massive gravity which is ghost free to all orders in the decoupling limit [2]. The cosmology of such a theory was studied in [10].

The main point of the mass term is that it is symmetric in the metrics \( f \) and \( g \) in the sense that

\[
\int d^4x \sqrt{-g} \sum_{n=0}^{4} \beta_n e_n (\sqrt{g^{-1}f}) = \int d^4x \sqrt{-f} \sum_{n=0}^{4} \beta_n e_{4-n} (\sqrt{f^{-1}g}).
\]

(4)

So one can read off the equation of motion for \( f_{\mu\nu} \) by having the equation of motion for \( g_{\mu\nu} \). The equation of motion for \( g_{\mu\nu} \) can be easily obtained, resulting in [12]

\[
M_g^2 G_{\mu\nu} + m^2 M_{eff}^2 f_{\mu\nu} = 0,
\]

(5)

with

\[
J_{\mu\nu} = \beta_0 g_{\mu\nu} - \beta_1 (X_{\mu\nu} - g_{\mu\nu}e_1(X)) + \beta_2 (X^2_{\mu\nu} - X_{\mu\nu}e_1(X) + g_{\mu\nu}e_2(X)) - \beta_3 (X^3_{\mu\nu} - X^2_{\mu\nu}e_1(X) + X_{\mu\nu}e_2(X) - g_{\mu\nu}e_3(X)),
\]

(6)

where \( G_{\mu\nu} \) is the Einstein tensor for metric \( g_{\mu\nu} \). The equation of motion for \( f_{\mu\nu} \) is

\[
M_f^2 F_{\mu\nu} + m^2 M_{eff}^2 K_{\mu\nu} = 0,
\]

(7)

with

\[
K_{\mu\nu} = \alpha_0 f_{\mu\nu} - \alpha_1 (Y_{\mu\nu} - f_{\mu\nu}e_1(1)) + \alpha_2 (Y^2_{\mu\nu} - Y_{\mu\nu}e_1(1) + f_{\mu\nu}e_2(1)) - \alpha_3 (Y^3_{\mu\nu} - Y^2_{\mu\nu}e_1(1) + Y_{\mu\nu}e_2(1) - f_{\mu\nu}e_3(1)),
\]

(8)

where \( F_{\mu\nu} \) is the Einstein tensor for metric \( f_{\mu\nu} \) and we have defined \( Y = \sqrt{f^{-1}g} \) and \( \alpha_n = \beta_{4-n} \). The cosmological solutions of this bi-metric theory has been considered in [11].

III. COSMOLOGICAL PERTURBATIONS

In order to study the cosmological perturbations of the model, we consider the scalar metric perturbations of the two metrics (our notation is compatible with that in [16])

\[
ds^2_s = a^2(t) \left[ -(1 + 2\phi_1) dt^2 + 2\theta_1 B_1 dx^i dx^j + [(1 - 2\psi_1) \delta_{ij} + 2\theta_2 \partial_i \partial_j E_1] dx^i dx^j \right],
\]

(9)

and

\[
ds^2_f = a^2(t) \left[ -(1 + 2\phi_2) dt^2 + 2\theta_3 B_2 dx^i dx^j + [(1 - 2\psi_2) \delta_{ij} + 2\theta_4 \partial_i \partial_j E_2] dx^i dx^j \right].
\]

(10)

Let us define four zero order quantities for metrics \( g_{\mu\nu} \)

\[
X_1 = \beta_0 a_1^2 a_2 + 2\theta_2 a_1 a_2^2 + \beta_1 a_2^2,
\]

(11a)

\[
X_2 = 2\beta_0 a_1^3 + 7\beta_1 a_1^2 a_2 + 8\theta_2 a_1 a_2^2 + 3\beta_3 a_2^3,
\]

(11b)

\[
X_3 = \beta_0 a_1^3 + 2\beta_1 a_1^2 a_2 + \beta_2 a_1 a_2^2,
\]

(11c)

\[
X_4 = \beta_0 a_1^3 + 3\beta_1 a_1^2 a_2 + 3\beta_2 a_1 a_2^2 + \beta_3 a_2^3.
\]

(11d)
and \(f_{\mu
u}\)

\[
Y_1 = \beta_3 a^2_2 a_1 + 2\beta_2 a_2 a_1^2 + \beta_1 a_1^3,
\]

(12a)

\[
Y_2 = 2\beta_3 a^2_2 a_1 + 7\beta_3 a_2 a_1 + 3\beta_2 a_2 a_1^2 + 3\beta_1 a_1^3,
\]

(12b)

\[
Y_3 = \beta_1 a^2_2 + 2\beta_2 a_2 a_1 + \beta_3 a_2 a_1^2,
\]

(12c)

\[
Y_4 = \beta_4 a^2_3 + 3\beta_3 a_2 a_1 + 3\beta_2 a_2 a_1^2 + \beta_1 a_1^3.
\]

(12d)

These terms can simplify the mass parts of the equations of motion. The background equations for metric \(g_{\mu\nu}\) is

\[
H^2 = \frac{1}{3} \frac{m^2 M^2_F}{M^2} \frac{1}{a_1} X_4,
\]

(13a)

\[
H^2 + 2H' = \frac{m^2 M^2}{M^2} \frac{1}{a_1} X_4,
\]

(13b)

where \(t \equiv d/dt\) and \(H_1 \equiv a'_1/a_1\). Similarly for \(f_{\mu\nu}\) we have

\[
H^2 = \frac{1}{3} \frac{m^2 M^2_M}{M^2} \frac{1}{a_2} Y_4,
\]

(14a)

\[
H^2 + 2H'_1 = \frac{m^2 M^2_M}{M^2} \frac{1}{a_2} Y_4
\]

(14b)

where \(H_2 \equiv a'_2/a_2\). For the kinetic term of the metric \(g_{\mu\nu}\), we obtain, to first order \[24\]

\[
G^{(1)}_{\mu\nu} = 2\nabla^2 \left( \psi_1 + H_1(E'_1 - B_1) \right) - 6H_1 \psi_1', \quad (15a)
\]

\[
G^{(1)}_{\mu i} = 2\partial_i \left( \phi_1 - \frac{1}{2} B_1 H_1 \right) H_1 - B_1 H'_1 + \psi_1' \quad (15b)
\]

\[
G^{(1)}_{ij} = \partial_i \partial_j \left( \psi_1 - \phi_1 - B'_1 - 4E_1 H'_1 + 2H_1 E'_1 + E''_1 \right)
- 2H_1 (B_1 + E_1 H_1), \quad (15c)
\]

and

\[
\sum_{i=1}^{3} G^{(1)}_{ii} = 6(\phi_1 + \psi_1) (H^2 + 2H'_1) + 6H_1(2\psi_1' + \phi_1')
+ 6\psi_1'' + \nabla^2 D, \quad (15d)
\]

with

\[
D = 4H_1 (B_1 - E'_1) - 2 (H^2 + 2H'_1) E_1
- 2 (\psi_1 - \phi_1 - B'_1 + E''_1). \quad (16)
\]

The mass term then becomes, to the first order (see the Appendix)

\[
J^{(1)}_{\mu 0} = \frac{1}{a_1} \left[ X_1 \left( 3(\psi_2 - \psi_1) + \nabla^2 (E_1 - E_2) \right) - 2X_3 \phi_1 \right], \quad (17a)
\]

\[
J^{(1)}_{\mu i} = \frac{1}{2a_1} \partial_i \left[ X_2 B_1 - X_1 B_2 \right], \quad (17b)
\]

\[
J^{(1)}_{ij} = \frac{1}{a_1} \partial_i \partial_j \left[ X_2 E_1 - X_1 E_2 \right], \quad i \neq j \quad (17c)
\]

and

\[
\sum_{i=1}^{3} J^{(1)}_{ii} = \frac{1}{a_1} \left[ 2X_3 (\nabla^2 E_1 - 3\psi_1)
+ X_1 (2\nabla^2 E_2 - 6\psi_2 + 3\phi_2 - 3\phi_1) \right]. \quad (17d)
\]

The kinetic term for the metric \(f_{\mu\nu}\) is represented by its corresponding Einstein tensor \(F_{\mu\nu}\) which can be deduced for the first order perturbations by making the transformation \(1 \rightarrow 2\) in equations \[15a\]-\[15d\]. The mass term can be obtained easily by transformations \(1 \leftrightarrow 2\) and \(\beta_n \rightarrow \beta_{4-n}\) in equations \[17a\]-\[17d\]. The result is

\[
K^{(1)}_{00} = \frac{1}{a_2} \left[ Y_1 \left( 3(\psi_1 - \psi_2) + \nabla^2 (E_2 - E_1) \right) - 2Y_4 \phi_2 \right], \quad (18a)
\]

\[
K^{(1)}_{0i} = \frac{1}{2a_2} \partial_i \left[ Y_2 B_2 - Y_1 B_1 \right], \quad (18b)
\]

\[
K^{(1)}_{ij} = \frac{1}{a_2} \partial_i \partial_j \left[ Y_2 E_2 - Y_1 E_1 \right], \quad i \neq j \quad (18c)
\]

and

\[
\sum_{i=1}^{3} K^{(1)}_{ii} = \frac{1}{a_2} \left[ 2Y_3 (\nabla^2 E_2 - 3\psi_2)
+ Y_1 (2\nabla^2 E_1 - 6\psi_1 + 3\phi_2 - 3\phi_1) \right]. \quad (18d)
\]

As we know, general relativity is invariant under coordinate transformations \[24\]. Hence we restrict our model to scalar perturbations so we only consider scalar coordinate transformations as follow

\[
t \rightarrow t + \delta t, \quad (19a)
\]

\[
x^i \rightarrow x^i + \delta^i \partial_j \delta x, \quad (19b)
\]

and consequently the scalar metric perturbations behave like \[25\]

\[
\phi_i \rightarrow \phi_i - \frac{1}{a_i} (a_i \delta t)' , \quad (20a)
\]

\[
B_i \rightarrow B_i + \delta t - \delta x', \quad (20b)
\]

\[
E_i \rightarrow E_i - \delta x, \quad (20c)
\]

\[
\psi_i \rightarrow \psi_i + H_i \delta t, \quad (20d)
\]

where \(i = 1, 2\). We then define six independent gauge invariant quantities as follows

\[
\Psi_i = \psi_i + H_i (E'_i - B_i), \quad (21a)
\]

\[
\Phi_i = \phi_i - \frac{1}{a_i} (a_i (E'_i - B_i))', \quad (21b)
\]

\[
\mathcal{E} = E_1 - E_2, \quad (21c)
\]

\[
B = B_1 - B_2, \quad (21d)
\]
with \( i = 1, 2 \). Using (21c) and (21d), one can define a gauge invariant quantity

\[
\Lambda = \mathcal{E}' - \mathcal{B},
\]

which simplifies the calculations that follow. In order to write the field equations in the gauge invariant form we can fix two scalar gauge freedoms by the following conditions

\[
E_1 + E_2 = 0, \quad B_1 + B_2 = 0.
\]

This gauge fixing has the advantage that the symmetry between the two metrics remains true, even after the gauge fixing. We then rewrite the kinetic term to first order for metric \( g_{\mu\nu} \) i.e. relations (15a)-(15d) in terms of six gauge invariant quantities \( \Psi_{1,2}, \Phi_{1,2}, \Lambda \) and \( \mathcal{E} \) as

\[
G_{0i}^{(1)} = 2\nabla^2 \Psi_1 - 3H_1(2\Psi_1 - H_1\Lambda)', 
\]

\[
G_{0i}^{(1)} = \partial_i \left[ 2(\Psi_1' + H_1\Phi_1) + \frac{3}{2} H_1^2 \Lambda - \frac{1}{2} (H_1^2 + 2H_1')\mathcal{E}' \right],
\]

\[
G_{ij}^{(1)} = \partial_i \partial_j \left[ \Psi_1 - \Phi_1 - (H_1^2 + 2H_1')\mathcal{E} \right],
\]

\[
\sum_{i=1}^{3} G_{ii}^{(1)} = \nabla^2 \left[ 2(\Psi_1 - \Phi_1) - (H_1^2 + 2H_1')\mathcal{E} \right] 
+ 6\Psi_1'' + 6H_1(\Phi_1 + 2\Psi_1)' - 3(H_1'' + H_1H_1')\Lambda 
+ 6(H_1^2 + 2H_1')(\Phi_1 + \Psi_1),
\]

and similarly for metric \( f_{\mu\nu} \) we obtain (26)

\[
F_{00}^{(1)} = 2\nabla^2 \Psi_2 - 3H_2(2\Psi_2 + H_2\Lambda)', 
\]

\[
F_{0i}^{(1)} = \partial_i \left[ 2(\Psi_2 + H_2\Phi_2) - \frac{3}{2} H_2^2 \Lambda + \frac{1}{2} (H_2^2 + 2H_2')\mathcal{E}' \right],
\]

\[
F_{ij}^{(1)} = \partial_i \partial_j \left[ \Psi_2 - \Phi_2 + (H_2^2 + 2H_2')\mathcal{E} \right],
\]

and

\[
\sum_{i=1}^{3} F_{ii}^{(1)} = \nabla^2 \left[ 2(\Psi_2 - \Phi_2) + (H_2^2 + 2H_2')\mathcal{E} \right] 
+ 6\Psi_2'' + 6H_2(\Phi_2 + 2\Psi_2)' + 3(H_2'' + H_2H_2')\Lambda 
+ 6(H_2^2 + 2H_2')(\Phi_2 + \Psi_2).
\]

In terms of gauge invariant quantities, the mass terms (17a)-(17d) can be rewritten as

\[
J_{00}^{(1)} = \frac{1}{a_1} \left[ \frac{3}{2} X_1 \left[ 2(\Psi_1 + \Psi_2) + (H_2 + H_1)\Lambda \right] + \frac{2}{3} \nabla^2 \mathcal{E} \right] 
- X_4 \left[ 2\Phi_1 + H_1\Lambda + \Lambda' \right],
\]

\[
J_{ij}^{(1)} = \frac{X_1 + X_2}{2a_1} \partial_i \partial_j \mathcal{E}, \quad i \neq j
\]

\[
\sum_{i=1}^{3} J_{ii}^{(1)} = \frac{1}{a_1} \left[ (X_3 - X_1)\nabla^2 \mathcal{E} - 6(X_3\Psi_1 + X_1\Psi_2) 
+ 3(H_1X_3 - H_2X_1)\Lambda 
+ \frac{3}{2} X_1 \left[ 2(\Phi_2 - \Phi_1) - (H_1 + H_2)\Lambda - 2\Lambda' \right] \right],
\]

and similarly for (18a)-(18d)

\[
K_{00}^{(1)} = \frac{1}{a_2} \left[ \frac{3}{2} Y_1 \left[ 2(\Psi_2 + \Psi_1) - (H_2 + H_1)\Lambda - \frac{2}{3} \nabla^2 \mathcal{E} \right] 
- Y_4 \left[ 2\Phi_2 - H_2\Lambda - \Lambda' \right] \right],
\]

\[
K_{0i}^{(1)} = - \frac{Y_1 + Y_2}{a_0} \partial_i \left( \mathcal{E}' - \Lambda \right),
\]

\[
K_{ij}^{(1)} = \frac{Y_1 + Y_2}{2a_0} \partial_i \partial_j \mathcal{E}, \quad i \neq j
\]

\[
\sum_{i=1}^{3} K_{ii}^{(1)} = \frac{1}{a_2} \left[ (Y_1 - Y_3)\nabla^2 \mathcal{E} - 6(Y_3\Psi_2 
+ Y_1\Psi_1) - 3(H_2Y_3 - H_1Y_1)\Lambda 
+ \frac{3}{2} Y_1 \left[ 2(\Phi_2 - \Phi_1) + (H_1 + H_2)\Lambda + 2\Lambda' \right] \right].
\]

By transforming to a spatial Fourier space, \( \nabla^2 \) is replaced by \(-k^2\), where \( k \) is the wave number. Using the above equations and after some algebra we arrive at writing \( \mathcal{E} \) and \( \Lambda \) as functions of \( \Phi_i \) and \( \Psi_i \) and deduce the following constraints

\[
\mathcal{E} = A_1(\Psi_1 - \Phi_1) = -A_2(\Psi_2 - \Phi_2),
\]

and

\[
\Lambda = -C_1 \left[ 2(H_1\Phi_1 + \Psi_1') - \frac{1}{2A_1} (A_1(\Psi_1 - \Phi_1))' \right] 
= C_2 \left[ 2H_2\Phi_2 + \Psi_2' - \frac{1}{2A_2} (A_2(\Psi_2 - \Phi_2))' \right],
\]

where

\[
\frac{1}{A_1} = H_1^2 + 2H_1' - m_g \frac{X_1 + X_2}{2a_1},
\]

\[
\frac{1}{A_2} = H_2^2 + 2H_2' - m_f \frac{Y_1 + Y_2}{2a_2}.
\]
\[
\frac{1}{C_1} = -m_g \frac{X_1 + X_2}{4a_1} + \frac{3}{2} H_1^2, \quad (32)
\]
\[
\frac{1}{C_2} = -m_f \frac{Y_1 + Y_2}{4a_2} + \frac{3}{2} H_2^2, \quad (33)
\]
and
\[
m_g = \frac{m^2 M_{ff}}{M_g^2}, \quad m_f = \frac{m^2 M_{ff}}{M_f^2}. \quad (34)
\]

Using the background equations (13a), (13b), (14a) and (14b), one can simplify the quantities \(A_i\) and \(C_i\) as follows
\[
A_1 = \frac{1}{2} C_1 = \frac{2a_1}{m_g} (2X_4 - X_1 - X_2)^{-1}, \quad (35)
\]
\[
A_2 = \frac{1}{2} C_2 = \frac{2a_2}{m_f} (2Y_4 - Y_1 - Y_2)^{-1}, \quad (36)
\]
which can be used to simplify the calculations. The dynamical equations then become

\[
-2k^2 \Psi_1 - 3H_1(2\Psi_1 - H_1\Lambda)' = m_g \left[ -\frac{3}{2a_1} X_1 \left( 2(\Psi_2 - \Psi_1) + (H_1 + H_2)\Lambda - \frac{2}{3} k^2 \mathcal{E} \right) + \frac{X_4}{a_1}(2\Phi_1 + H_1\Lambda + \Lambda') \right] \quad (38)
\]
\[
-2k^2 \Psi_2 - 3H_2(2\Psi_2 + H_2\Lambda)' = m_f \left[ -\frac{3}{2a_2} Y_1 \left( 2(\Psi_1 - \Psi_2) - (H_1 + H_2)\Lambda + \frac{2}{3} k^2 \mathcal{E} \right) + \frac{Y_4}{a_2}(2\Phi_2 - H_2\Lambda - \Lambda') \right] \quad (39)
\]
and
\[
- k^2 \left[ 2(\Phi_1 - \Psi_1) - (H_1^2 + 2H_1')\mathcal{E} \right] + 6H_1(\Phi_1 + 2\Psi_1)' + 6\Psi_1'' - 3(H_1'' + H_1H_1')\Lambda + 6(H_1^2 + 2H_1') (\Phi_1 + \Psi_1) = -\frac{m^2}{a_1} \left[ - k^2(X_3 - X_1)\mathcal{E} - 6(X_3\Psi_1 + X_1\Psi_2) + 3(H_1X_3 - H_2X_1)\Lambda + \frac{3}{2} X_1 \left[ 2(\Phi_2 - \Phi_1) - (H_1 + H_2)\Lambda - 2\Lambda' \right] \right] \quad (40)
\]
\[
- k^2 \left[ 2(\Phi_2 - \Psi_2) + (H_2^2 + 2H_2')\mathcal{E} \right] + 6H_2(\Phi_2 + 2\Psi_2)' + 6\Psi_2'' + 3(H_2'' + H_2H_2')\Lambda + 6(H_2^2 + 2H_2') (\Phi_2 + \Psi_2) = -\frac{m_f}{a_2} \left[ k^2(Y_3 - Y_1)\mathcal{E} - 6(Y_3\Psi_2 + Y_1\Psi_1) - 3(H_2Y_3 - H_1Y_1)\Lambda + \frac{3}{2} Y_1 \left[ 2(\Phi_2 - \Phi_1) + (H_1 + H_2)\Lambda + 2\Lambda' \right] \right]. \quad (41)
\]

where we keep in mind that the quantities \(\Lambda\) and \(\mathcal{E}\) must be replaced by (28) and (35). In this sense we obtain four equations for four dynamical variables.

**IV. EQUATIONS OF MOTION AND THE SUPER HORIZON LIMIT**

From (13) we know that a non-trivial classical solution for the scale factors can be deduced by assuming \(a_1 = a_2\). With this assumption the solution will be homogeneous de-Sitter solution. In this case the adiabatic direction which should be along the homogeneous solution is clearly along the straight line, \(a_1 = a_2\) in the phase space. It means that the adiabatic fluctuations should have the same contribution from both metrics (9) and (10). Or in other words, one expects that a suitable combination of the variables \(\Psi = \Psi_1 + \Psi_2\) and \(\Phi = \Phi_1 + \Phi_2\) should be responsible for the adiabatic perturbations. Similarly, a suitable combination of the variables \(\psi = \Psi_1 - \Psi_2\) and \(\phi = \Phi_1 - \Phi_2\) which is orthogonal to the adiabatic one should be responsible for the entropy perturbations.

At this stage a discussion on the relation of curvature perturbation to \(\Psi_1\) and \(\Psi_2\) would be in order. As is well known (16), curvature perturbation is defined in terms of matter and metric perturbations. However the definition in a bi-metric model needs some care. As has been mentioned previously (13), a convenient physical quantity for coupling to matter is the average of the two metrics. This claim is based on the fact that by assuming a exchangeable role for both metrics, the only combination which is
allowed at the linear level is the sum of the first order perturbations in the tensor mode [13] or in the scalar mode which is the purpose of the present work. Therefore, if we were to add matter to our formulation, a convenient combination representing curvature perturbation would be that constructed from \( \psi_1 + \psi_2 \). The absence of matter in the present discussion means that a suitable gauge invariant quantity constructed out of \( \psi_1 + \psi_2 \) would be \( \Psi_1 + \Psi_2 \), which is the same as the adiabatic perturbation defined above.

To solve equations (38)-(41), we assume \( a_1 = a_2 = a \) and define constants \( x_i \) and \( y_i \) as

\[
X_i = a^3 x_i, \quad Y_i = a^3 y_i, \tag{42}
\]

which, using equations (13a) and (14a), imply that

\[
m_g x_4 = m_f y_4. \tag{43}
\]

In order to write the equations in terms of the new variables \( \Psi, \Phi, \psi \) and \( \phi \), we define

\[
\alpha = \frac{1}{2} \left( \frac{x_4 + y_4}{x_1 y_1} \right), \tag{44}
\]

\[
\beta = \frac{1}{2} \left( \frac{x_4 - y_4}{x_1 y_1} \right). \tag{45}
\]

The constraint equations (28) and (37) can then be written as

\[
\alpha (\Psi - \Phi) + \beta (\psi - \phi) = 0, \tag{46}
\]

\[
\alpha (H \Phi + \psi') + \beta (H \phi + \psi') = 0. \tag{47}
\]

Equations (38) and (39) take the form

\[
6H \Psi' + (2k^2 + \frac{\beta^2}{\alpha^2 - \beta^2} k^2) \Psi + (6H^2 - \frac{\beta^2}{\alpha^2 - \beta^2} k^2 \phi + 18\beta \frac{\alpha^2}{\alpha^2 - \beta^2} H^3 \Lambda) = \frac{\beta \alpha}{\alpha^2 - \beta^2} k^2 (\phi - \psi) + \frac{18\beta}{\alpha^2 - \beta^2} H^2 \psi, \tag{48}
\]

and

\[
6H \psi' + (2k^2 - \frac{\alpha^2}{\alpha^2 - \beta^2} k^2 + 18\alpha \frac{\beta^2}{\alpha^2 - \beta^2} H^2 \phi) + (6H^2 - \frac{\alpha^2}{\alpha^2 - \beta^2} k^2 + 6H^2) \phi - \frac{18\alpha}{\beta^2 - \alpha^2} H^3 \Lambda \psi = \frac{\alpha \beta}{\alpha^2 - \beta^2} k^2 (\Psi - \Phi). \tag{49}
\]

Also one can rewrite the relation (29) as

\[
\Lambda = \frac{\alpha^2 - \beta^2}{3\alpha H^2} \left[ \frac{3}{2} \Psi' + \frac{1}{2} \psi' + H \Psi + H \phi \right] = \frac{\beta^2 - \alpha^2}{3\beta H^2} \left[ \frac{3}{2} \Psi' + \frac{1}{2} \psi' + H \Psi + H \phi \right]. \tag{50}
\]

Upon substituting \( \Lambda \) we obtain

\[
\Psi' + \Phi' = \frac{k^2}{3H} \Psi + \left( 2H - \frac{k^2}{3H} \right) \Psi = \frac{-6\beta}{\alpha^2 - \beta^2} H \psi, \tag{51}
\]

\[
\psi' + \phi' = \frac{k^2}{3H} \phi + \left( 2H - \frac{6\alpha}{\alpha^2 - \beta^2} H - \frac{k^2}{3H} \right) \psi = 0. \tag{52}
\]

Equation (51) has an important property, showing that the entropy perturbation can be a source for the adiabatic perturbation.

At this stage it is appropriate to take the following steps to make the equations less complicated and therefore easier to solve. From the first condition (46) we have

\[
\alpha \Psi + \beta \psi = \alpha \Phi + \beta \phi, \tag{53}
\]

and by the weighted addition of equations in (51) and (52) one finds

\[
\left( \alpha \Psi + \beta \psi \right)' - \frac{k^2}{3H} \left( \alpha \Phi + \beta \phi \right) = 0, \tag{54}
\]

\[
\left( 2H - \frac{k^2}{3H} \right) \left( \alpha \Psi + \beta \psi \right) = 0. \tag{55}
\]

Now by plugging (53) into the above equation we have

\[
\left( \alpha \Psi + \beta \psi \right)' \left( H - \frac{k^2}{3H} \right) \left( \alpha \Psi + \beta \psi \right) = 0. \tag{56}
\]

The second condition (47) results in

\[
\left( \alpha \Psi + \beta \psi \right)' = -H \left( \alpha \Phi + \beta \phi \right), \tag{57}
\]

and reduces to

\[
\left( \alpha \Psi + \beta \psi \right)' + H \left( \alpha \Psi + \beta \psi \right) = 0, \tag{58}
\]

due to (53). Comparison of (56) with (58) shows

\[
\frac{k^2}{3H} \left( \alpha \Psi + \beta \psi \right) = 0, \tag{59}
\]

which leads to

\[
\alpha \Psi = -\beta \psi, \tag{60}
\]

and consequently from (53)

\[
\alpha \Phi = -\beta \phi. \tag{61}
\]

We note that (50) and (51) render equations (51) and (52) equivalent. So by plugging the above relations into (51) one finds

\[
\left( \Psi + \Phi \right)' = \frac{k^2}{3H} \left( \Psi + \Phi \right) + 2\gamma H \Psi = 0, \tag{62}
\]

with

\[
\gamma = 1 - \frac{3\alpha}{\alpha^2 - \beta^2}. \tag{63}
\]
The same procedure as above can be employed for the remaining equations \([60]\) and \([61]\). They, in conjunction with equations \([60]\) and \([61]\), can be used to arrive at an equation of the form

\[
\psi'' + \Phi'' + H(\psi' + \Phi') + 2\gamma H^2(2\psi - \Phi) = 0,
\]

which is independent of \([62]\). So for four variables \(\psi, \phi, \Psi, \Phi\) we have four independent equations \([60]\), \([61]\), \([62]\), and \([64]\). The two latter equations can be converted to a first order differential equation for \(\Psi\) and \(\Phi\) by differentiating equation \([62]\) and subtract the result from \([64]\). The result is

\[
\left(\frac{k^2}{3H} + H\right)(\psi + \Phi') - \left(\frac{k^2}{3} + 2\gamma H^2\right)(\psi + \Phi) - 2\gamma H\psi' + 4\gamma H^2\psi = 0.
\]

Equations \([62]\) and \([65]\) constitute our final resulting equations for \(\Psi\) and \(\Phi\). Now, combining the above equation with \([62]\) results in

\[
(\Psi + \Phi)'' - 2H(\psi + \Phi') + \left(\frac{k^2}{3} - 2\gamma H^2\right)(\psi + \Phi) = 0.
\]

Since \(H = -1/t\) as a consequence of the background equations, the solution of the above equation is

\[
\Psi + \Phi = c_1 j_n\left(\frac{kt}{\sqrt{3}}\right) + c_2 y_n\left(\frac{kt}{\sqrt{3}}\right)
\]

where \(c_1\) and \(c_2\) are integration constants, \(j_n(x)\) and \(y_n(x)\) are spherical Bessel functions and

\[
n = \frac{-1 + \sqrt{1 + 8\gamma}}{2}.
\]

Consequently, using the above solutions and either of equations \([62]\) or \([65]\), one can find \(\Psi(t)\) and \(\Phi(t)\) separately. This means that by employing equations \([60]\) and \([61]\), the functions \(\psi(t)\) and \(\phi(t)\) and eventually \(\psi_1, \psi_2, \phi_1\) and \(\phi_2\) can be found as functions of time.

For super-horizon modes, i.e. \(k \sim 0\), equation \([66]\) results in

\[
\Psi + \Phi = c_1 t + c_2 t^{-2},
\]

and by considering the fact that \(t\) is the conformal time and \(t \in \{0, \infty\}\), the term \(\sim c_1 t\) can be considered as the damping term. The growing mode, i.e. \(\sim c_2 t^{-2}\) term, results in

\[
\Psi = -c_2 \frac{1}{\gamma} t^{-2}, \quad \Phi = c_2 \frac{1 + \gamma}{\gamma} t^{-2}
\]

This result suggests that in the theory of bi-metric gravity that we have studied in this paper, the curvature perturbation is not constant at the super horizon limit. This result is compatible with \([18]\). However in \([18]\) the authors have studied an old bi-metric model in the presence of the cosmological constant and the Fierz-Pauli mass term. In contrast, we have studied a ghost free bi-metric model without any cosmological constant and with a non-linear generalization of the Fierz-Pauli mass term. The mass term in such a theory cannot then be used as a manifestation of inflation. At the background level there is no way to a graceful exit from inflation. In order to achieve such an exit, one should not consider the mass as a constant, rather, it should be considered as a function of some other fields \([14]\). Now, if one assumes the mass to be a function of the inflaton field then the possibility of having a reduced mass at the end of inflation would not be far-fetched and therefore a graceful exit from de-Sitter phase would become a reality \([15]\).

On the other hand it seems that the observational data, e.g. the cosmic microwave background radiation, are consistent with an inflationary scenario with nearly adiabatic and Gaussian fluctuations. So the results of this paper are not compatible with observation except by imposing some constraints on the entropy mode’s effects. Therefore, the consideration of a constant mass and the addition of some sort of an inflationary scenario to the theory should result in imposing some corrections to the curvature perturbations at the super horizon limit which restrict the parameters of the theory to that of the slow roll parameters. Eventually we expect that the observational data constrains the mass of the graviton in this scenario. However, this result may be acceptable because massive gravity is responsible for the late time accelerated expansion of the universe \([21]\).

V. CONCLUSIONS

Modern observational data have become so accurate that meaningful comparisons with theoretical predictions are now a reality. This has led to the introduction of various calculational techniques, including metric perturbation methods, which have been playing an increasingly important role. In this paper we have employed such a method to study a bi-metric massive gravity theory. The present work proposes a full discussion of the scalar perturbations of the bi-metric gravity, with all the potential terms at our disposal. Because the mass term mixes the scale factors of the two metrics, we expect both the entropy and adiabatic perturbations to be non-zero and depend upon each other. This can also be seen from the dynamical equations of the model. As a result, similar to the double scalar field models \([21]\), one may expect that the adiabatic perturbation should be along the path of the classical solutions in the phase-space and the entropy perturbation to be orthogonal to it. According to \([13]\), a classical solution in the framework used here is \(a_1 = a_2\). So it seems natural to think that the sum of two scalar perturbations corresponds to the adiabatic perturbation and the entropy perturbation corresponds to the difference of the two scalar fields. Since we have four scalar perturbations, the suitable combinations are the terms \(\Psi_1 \pm \Psi_2\) and \(\Phi_1 \pm \Phi_2\). From equation \([51]\), one can see
explicitly that the entropy perturbations are a source of the adiabatic perturbations which is in agreement with double scalar field models [21].

Interestingly, equation (68) shows that the super horizon adiabatic perturbations are not constant for the bimetric model discussed in this paper. We may expect that the mass term for graviton can only impose some corrections to the results of the inflationary scenario which we must add to the theory using other methods. In this scenario, the mass term in the model becomes restricted by the slow roll parameters. This can be the subject of a future work.

Note added

During the completion of the present work, a study of the same subject appeared in [22].

Acknowledgments

The authors would like to thank Hassan Firouzjahi and Zabra Haghani for useful comments. We are also grateful to the anonymous referee for constructive comments which have improved the quality of the paper.

VI. APPENDIX

In this appendix we derive the first order approximations to matrix $\mathcal{X} = \sqrt{g} f^{-1}$. Using equations (69) and (70) we obtain, to first order in perturbation, for matrix $\mathcal{G} = g^{-1} f$

$$\mathcal{G} = \mathcal{G}^{(0)} + \epsilon \mathcal{G}^{(1)} + \mathcal{O}(\epsilon^2),$$

(70)

where

$$(\mathcal{G}^{(0)})_{\mu}^{\nu} = \left(\frac{a_2}{a_1}\right)^2 \delta_{\mu}^{\nu},$$

(71)

and

$$(\mathcal{G}^{(1)} \mathcal{G}^{-1})_{\mu}^{\nu} = -2 \left(\frac{a_2}{a_1}\right)^2 (\phi_1 - \phi_2),$$

(72)

$$(\mathcal{G}^{(1)})_{0}^{i} = \left(\frac{a_2}{a_1}\right)^2 \partial_i (B_1 - B_2),$$

(73)

$$(\mathcal{G}^{(1)})_{i}^{j} = -2 \left(\frac{a_2}{a_1}\right)^2 [(\psi_1 - \psi_2) \delta_i^j + \partial_i \partial_j (E_1 - E_2)].$$

(74)

We finally expand the matrix $\mathcal{X} = \sqrt{g}$

$$\mathcal{X} = \mathcal{X}^{(0)} + \epsilon \mathcal{X}^{(1)} + \mathcal{O}(\epsilon^2),$$

(75)

where

$$(\mathcal{X}^{(0)})_{\mu}^{\nu} = \frac{a_2}{a_1} \delta_{\mu}^{\nu},$$

(76)

and

$$(\mathcal{X}^{(1)})_{\mu}^{\nu} = \frac{a_2}{2a_1} (\mathcal{G}^{(1)})_{\mu}^{\nu}.$$  

(77)

The expansion of matrix $\mathcal{Y} = \sqrt{f^{-1}g}$ can be easily obtained by observing that metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ can be transformed to each other by the transformation $1 \rightarrow 2$. 

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[23] Note that in [16] the Einstein tensor has been written with an upper and a lower index but we write it with both lower indices.

[24] It is important to mention that a general bi-metric model enjoys a diff$^2$-invariance. But as it is shown in [12], in the presence of an interaction term this symmetry reduces to diff-invariance which is exactly the GR symmetry.

[25] Note that we employ the scalar coordinate transformations for both metrics. It implicitly means that we assume both metrics live on the same manifold (this assumption may cause some ambiguities in the definition of the physical distance and parallel transportation which is out of the scope of the present paper. However, as was mentioned in [13], a natural way of resolving this problem is by considering the average of both metrics as the physical metric at least in the linear regime). However Arkani-Hamed et al. [17] have assumed that each metric belongs to a certain manifold and the mass term effectively stitches these manifolds together. In this point of view each metric is invariant under its corresponding coordinate transformation.

[26] We note that in order to read the equations of metric $f_{\mu\nu}$ from the equations of metric $g_{\mu\nu}$, one may make the transformations $\Lambda \to -\Lambda$ and $\mathcal{E} \to -\mathcal{E}$. 