Learning-Based Synthesis of Safety Controllers

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Abstract. We propose a machine learning framework to synthesize reactive controllers for systems whose interactions with their adversarial environment are modeled by infinite-duration, two-player games over (potentially) infinite graphs. Our framework targets safety games with infinitely many vertices, but it is also applicable to safety games over finite graphs whose size is too prohibitive for conventional synthesis techniques. The learning takes place in a feedback loop between a teacher component, which can reason symbolically about the safety game, and a learning algorithm, which successively learns an overapproximation of the winning region from various kinds of examples provided by the teacher. We develop a novel decision tree learning algorithm for this setting and show that our algorithm is guaranteed to converge to a reactive safety controller if a suitable overapproximation of the winning region can be expressed as a decision tree. Finally, we evaluate the performance of our framework on examples motivated by robotic motion planning.

Keywords: Reactive Synthesis · Safety Games · Machine Learning.

1 Introduction

Reactive synthesis offers an effective and promising way to solve a crucial practical problem: constructing correct and verified controllers for safety-critical systems. Rather than designing and implementing controllers by hand, reactive synthesis techniques construct controllers in an automatic fashion, thus, freeing engineers from this complex and error-prone task. In addition to being fully automatic, synthesis techniques produce correct-by-construction controllers that guarantee to satisfy the given specification, or they report that no such controller exist.

Typically, reactive synthesis is modeled as an infinite-duration game on a graph that is played by two antagonistic players: the system, which seeks to satisfy the specification, and the environment, which wants to violate it. More precisely, the specification and a model of the environment are in a first step converted into an infinite game. Then, one computes a winning strategy for the system, which prescribes how the system needs to play in order to win against every move of the environment. Finally, the winning strategy is translated into hard- or software, resulting in a reactive controller that satisfies the given specification.

In this paper, we focus on safety games, a class of infinite games that arises from safety specifications. Such specifications are in fact among the most important in practice (e.g., see Dwyer, Avrunin, and Corbett [5] for a survey of specification patterns) and capture many other interesting properties, including bounded-horizon reachability. In contrast to the classical setting, however, we consider safety games not only over finite graphs but also over graphs with infinitely many (even uncountably many) vertices. Such games arise naturally, for instance, when the interaction between the controlled system and its environment is too complex to be modeled by finite graphs (e.g., in motion planning over unbounded environments). Infinite game graphs also occur in the presence of data and in scenarios in which the environment has access to dynamic data structures, such as lists, stacks, or queues.

When the number of vertices of a game graph is infinite, traditional methods, which typically rely on an exploration of the whole graph, are no longer applicable. To enable the computation of winning strategies for such games, our first contribution is a machine learning framework that learns winning strategies rather than compute them explicitly. Inspired by earlier work in the context of automata learning [17], our framework defines a feedback loop akin to counterexample-guided inductive synthesis (CEGIS) [19] consisting of two entities: a teacher, who can reason symbolically about the game, and a learning algorithm,
whose goal is to learn a winning strategy from information provided by the teacher. In every iteration of the loop, the learning algorithm constructs a strategy, encoded as a proxy object called winning set, and proposes this strategy to the teacher. The teacher then checks whether the proposed strategy is in fact a winning strategy and stops the learning process if so. If this is not the case, however, the teacher returns a counterexample, which guides the learning algorithm further towards a winning strategy. Upon receiving a counterexample, the learner refines its conjecture and proceeds with the next iteration.

Motivated by recent success in using decision trees as a concise and effective representation of strategies in infinite games [2], our second contribution is a new learning algorithm for decision trees, which is tailored specifically to our framework sketched above. Our algorithm builds upon a recent learning algorithm for decision trees that has been proposed in the context of software verification [6] and that puts a strong emphasis on learning “small” trees. As a consequence of the latter, our algorithm can in many situations guarantee to learn a winning strategy if one can be expressed as a decision tree.

Even though a game graph is infinite or prohibitively large, a reactive controller with a compact representation might already realize the specification. In a motion planning scenario, for instance, the system often only needs to consider a small subset of possible interactions with the environment to satisfy the specification. Based on this observation, our learning-based approach possesses various desirable properties: (i) it leverages machine learning as an effective means to focus on the important parts of a game, (ii) it often learns “small” strategies in a rule-like format, which tend to be relatively easy for humans to understand (cf. Brázdil et al. [2]), and (iii) besides being mainly designed for games on infinite graphs, it also performs well on games over finite graphs. We demonstrate these properties empirically on a set of examples taken from robotic motion planning.

**Related Work.** Games over various types of infinite graphs have been studied, predominantly in the context of pushdown graphs [11]. For more general classes of graphs, a constraint-based approach [1], relying on constraint solvers such as Z3 [14], and various learning-based approaches have been proposed [15,17]. In the context of safety games over finite graphs, recent work [16] has demonstrated the ability of learning-based techniques to extract small reactive controllers from precomputed controllers with a potentially large number of states.

Our learning framework is an extension of an earlier framework by Neider and Topcu [17]. Their work considers so-called rational safety games (defined in terms of finite automata) and proposes an automaton learning approach to infer winning strategies. By contrast, we do not fix a specific representation of the game graph and only require that certain operations can be performed symbolically. Many common formalisms such as finite automata, various types of decision diagrams, as well as formulas in the first-order theories of linear integer and real arithmetic satisfy these requirements. However, we consider only finitely-branching game graphs, whereas Neider and Topcu also consider graphs with infinite branching.

The algorithm we design for learning decision trees builds on top of a learning algorithm recently proposed by Ezudheen et al. [6], which learns from data in form of Horn clauses. For this setting, other learning algorithms have been developed as well [3,21]. We have chosen Ezudheen et al.’s algorithm specifically for its property to guarantee convergence to a solution in many practical scenarios.

## 2 Controller Synthesis and Safety Games

We follow the game-theoretic approach to controller synthesis as popularized by McNaughton [13] and view the problem as an infinite-duration, two-player game on a directed graph. Such games are played by two antagonistic players: Player 0, who embodies the system, and Player 1, who embodies the environment. In this setting, the type of specification dictates the type of game, and a winning strategy for Player 0 translates immediately into a controller that satisfies the given specification (we refer the reader to Grädel, Thomas, and Wilke [10] for a comprehensive discussion of this connection). Since we are interested in synthesizing controllers from safety specifications, the remainder of this paper is concerned with so-called safety games. However, before we introduce these types of games formally, let us first fix basic notations.
Basic Notations. Let $\mathbb{B} = \{0, 1\}$ denote the set of Boolean values (0 representing $false$ and 1 representing $true$), $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{R}$ the set of real numbers. Given a set $A$, we denote the set of all finite sequences of elements of $A$ by $A^*$ and the set of all infinite sequences by $A^\omega$. Moreover, for a binary relation $R \subseteq X \times X$ and two sets $A, B \subseteq X$, the image of $A$ under $R$ is the set $R(A) = \{b \in B \mid \exists a \in A: (a, b) \in R\}$ and the preimage of $B$ under $R$ is the set $R^{-1}(B) = \{a \in A \mid \exists b \in B: (a, b) \in R\}$.

Safety Games. Our definitions and notations mainly follow those of Grädel, Thomas, and Wilke [10], and we refer the reader to this textbook for further details on infinite-duration games. Formally, a safety game is a five-tuple $G = (V_0, V_1, E, I, F)$ consisting of two disjoint sets $V_0, V_1$ of vertices controlled by Player 0 and Player 1, respectively (we denote their union by $V = V_0 \cup V_1$ and assume $V \neq \emptyset$), a directed edge relation $E \subseteq V \times V$, a nonempty set $I \subseteq V$ of initial vertices, and a set $F \subseteq V$ of safe vertices. The directed graph $(V, E)$ is typically called game graph. In contrast to the classical setting, we do not restrict $V$ to be finite but allow even uncountable sets. However, we do make the following two restrictions to the edge relation: we assume that (1) every vertex has at least one outgoing edge (i.e., $E(\{v\}) \neq \emptyset$ for each $v \in V$), and (2) $E(\{v\})$ is finite for every $v \in V$, though not necessarily bounded. Note that the first restriction is standard and simply avoids situations in which the game gets stuck. The second restriction, on the other hand, is required by our learning framework and ensures that the data to learn from is always a finite object.

A safety game is played in rounds: initially, a token is placed on one of the initial vertices $v_0 \in I$; in each round, the player controlling the current vertex then moves the token to the next vertex along one of the outgoing edges. This process of moving the token is repeated ad infinitum and results in an infinite sequence $\pi = v_0 v_1 \ldots \in V^\omega$ with $v_0 \in I$ and $(v_i, v_{i+1}) \in E$ for every $i \in \mathbb{N}$, which is called a play. The winner of a play is determined by the winning condition $F$ in that a play $\pi = v_0 v_1 \ldots$ is winning for Player 0 if $v_i \in F$ for every $i \in \mathbb{N}$—otherwise it is winning for Player 1.

In the framework of infinite games, synthesizing a controller amounts to computing a so-called winning strategy for Player 0, which prescribes how Player 0 needs to move in order to win a play. Formally, a strategy for Player 0 is a function $\sigma: V^* \times V_0 \to V$ such that $(v_n, \sigma(v_0 \ldots v_n)) \in E$ for every $v_0 \ldots v_n \in V^*V_0$. A strategy is called winning if every play that is played according to $\sigma$ (i.e., that satisfies $v_{n+1} = \sigma(v_0 \ldots v_n)$ for all $n \in \mathbb{N}$ with $v_n \in V_0$) is winning for Player 0. It is well known that safety games permit memoryless winning strategies where the choice of the next vertex depends only on the vertex the play has currently reached. Such a strategy can then easily be implemented as a controller: the controller tracks the current vertex of a play and chooses the next move according to the strategy. Hence, the objective in the remainder of this paper is to compute a memoryless winning strategy for Player 0. We refer to this as solving a game.

If the game graph underlying a safety game is finite, memoryless winning strategies can be computed in linear time using a simple fixed-point computation [10]. For infinite game graphs, on the other hand, this is no longer an option as a fixed-point computation might not converge in finite time. Thus, we propose a novel learning-based algorithm that learns a (memoryless) winning strategy rather than computing one iteratively. However, as strategies are often complex objects (e.g., large if-then-else tables), we aim for a proxy object, named winning set, that is often structurally simpler, yet carries sufficient information to extract a (memoryless) winning strategy.

Winning Sets. Intuitively, a winning set is a subset of the safe vertices that contains all initial vertices and is a trap for Player 1 (i.e., Player 0 can force any play to stay inside this set regardless of how Player 1 plays). Formally, we define winning sets as follows.

Definition 1 (Winning set). Let $G = (V_0, V_1, E, I, F)$ be a safety game. A winning set is a set $W \subseteq V$ satisfying (1) $I \subseteq W$, (2) $W \subseteq F$, (3) $E(\{v\}) \cap W \neq \emptyset$ for all $v \in W \cap V_0$ (existential closedness), and (4) $E(\{v\}) \subseteq W$ for all $v \in W \cap V_1$ (universal closedness).

A winning set $W$ immediately provides a winning strategy for Player 0: starting in $I \subseteq W$, Player 0 simply moves to a successor vertex inside $W$ whenever it is his turn (note that this is possible since $W$ is existentially closed). As $W$ is also universally closed, a straightforward induction over the length of plays
We now propose a machine learning framework for learning winning sets in safety games and, thus, reactive safety controllers. Our framework is a generalization of earlier work by Neider and Topcu [17], which encodes (countably) infinite game graphs using finite automata and uses automaton learning to learn winning sets. By contrast, the framework proposed here allows for game graphs with uncountably many vertices and is not restricted to a symbolic representation in terms of finite automata.

As illustrated in Figure 2, the learning takes place in a counterexample-guided feedback loop (CEGIS) [19] with two entities: a teacher, who has knowledge about the safety game, and a learner (or learning algorithm), whose objective is to learn a winning set, but who is agnostic to the game. In every iteration of the loop, the learner conjectures a set $H \subseteq V$, called hypothesis, based on the information about the game it has accumulated so far. Then, the teacher checks whether this set $H$ is in fact a winning set—queries of this type are often called equivalence or correctness queries. Although the teacher does not know a winning set
(the task is to learn one after all), it can verify whether the hypothesis is one by checking Conditions (1) to (4) of Definition 1. If the hypothesis satisfies these conditions, then $H$ is a winning set and the learning stops. If this is not the case, the teacher replies with a counterexample that witnesses the violation of one of these conditions. Then, the feedback loop continues until a winning set has been found. The definition below formalizes the concept of counterexamples and fixes the communication between the teacher and the learner.

**Definition 2 (Teacher for Safety Games).** Let $G = (V_0, V_1, E, I, F)$ be a safety game. Given a hypothesis $H \subseteq V$, the teacher replies as follows (whereby the order in which the checks are performed is arbitrary):

1. If $I \not\subseteq H$, then the teacher returns a positive counterexample $v \in I \setminus H$.
2. If $H \not\subseteq F$, then the teacher returns a negative counterexample $v \in H \setminus F$.
3. If there exists a $v \in H \cap V_0$ with $E(\{v\}) \cap H = \emptyset$, then the teacher returns an existential counterexample $v \rightarrow (v_1 \lor \ldots \lor v_n)$ with $\{v_1, \ldots, v_n\} = E(\{v\})$.
4. If there exists a $v \in H \cap V_1$ with $E(\{v\}) \not\subseteq H$, then the teacher returns a universal counterexample $v \rightarrow (v_1 \land \ldots \land v_n)$ with $\{v_1, \ldots, v_n\} = E(\{v\})$.

If $H$ passes all four checks, then the teacher returns “yes”.

Each case of Definition 2 corresponds to one condition of Definition 1. Thus, it is not hard to verify that a given hypothesis is in fact a winning set if the teacher replies “yes” (as it satisfies Definition 1). Counterexamples, on the other hand, witnesses the violation of one of these conditions and guide the learner towards a winning set by communicating exactly why the hypothesis is incorrect. For instance, the meaning of a positive counterexample is that any future hypothesis needs to include this vertex (as it is initial), whereas a negative counterexample must be excluded (as it is not a safe vertex). An existential counterexample $v \rightarrow (v_1 \lor \ldots \lor v_n)$ signals that the hypothesis is not existentially closed and requires that if a future hypothesis contains $v$, it also needs to contains an least one vertex of the vertices $v_1, \ldots, v_n$. Similarly, a universal counterexample $v \rightarrow (v_1 \land \ldots \land v_n)$ signals that the hypothesis is not universally closed and requires that if a future hypothesis contains $v$, it needs to contains all vertices $v_1, \ldots, v_n$. Note that existential and universal counterexamples are always finite objects since we assume $E(\{v\})$ to be finite for every $v \in V$. Let us illustrate this learning process with an example motivated by motion planning.

**Example 2.** We continue Example 1. All hypotheses produced in the course of the learning process are depicted in Figure 3. Gray shaded areas indicate vertices in the hypothesis, safe vertices are surrounded by a bold line, and initial vertices are indicated by diagonal lines.

Let us assume that the learner proposes the hypothesis $H_1 = \emptyset$ in the first iteration of the loop, which is shown in Figure 3a. Since this hypothesis does not include any initial vertex, it does not satisfy Condition (1) of Definition 1. Thus, the teacher returns a positive counterexample, say $0 \in I \setminus H$.

Next, suppose that the learner proposes the hypothesis $H_2 = V$, shown in Figure 3b. This hypothesis is consistent with the positive counterexample. However, it also includes unsafe vertices and, thus, does not satisfy Condition (2) of Definition 1. Hence, the teacher replies with a negative counterexample, say $-1 \in H \setminus F$.

Let us now assume that the learner conjectures $H_3 = [0, 1]$, depicted in Figure 3c. This conjecture is consistent with both the positive and the negative counterexample. However, it is not existentially closed because Player 0 has no choice but to move the robot outside of $H_3$. Therefore, the teacher replies with an existential counterexample, say $0 \rightarrow (1 \lor 1)$.
Suppose now that the learner proposes the hypothesis $H_4 = [0, 2)$ in the fourth iteration, shown in Figure 3d. Although this conjecture is consistent with all counterexamples received so far, it is not universally closed because Player 1 can move the robot into the interval $[2, 3)$, which is not included in $H_4$. Thus, the teacher returns a universal counterexample, say $2 \rightarrow (1 \land 3)$.

Finally, the learner proposes the hypothesis $H_5 = [0, 3)$, depicted in Figure 3e. This hypothesis satisfies all conditions of Definition 2. Thus, $H_5$ is a winning set and the learning terminates.

Our learning framework is straightforward to implement if the underlying game graph is finite, in which case the teacher can be build on top of an explicit representation of the game. However, if the underlying game graph becomes too large or is infinite, one has to choose a suitable representation for sets of vertices and the edge relation that allows performing operations on the graph symbolically. More precisely, the chosen symbolic representation must feature Boolean operations (i.e., union, intersection, and complementation), and the image $E(A)$ and preimage $E^{-1}(A)$ of symbolically represented sets $A \subseteq V$ need to be computable. Moreover, the emptiness problem (i.e., “given a set $A$, decide whether $A = \emptyset$”) needs to be decidable, and it must be possible to extract an element from $A$ if it is nonempty. Examples of such symbolic representations include many common formalisms such as finite automata, various types of decision diagrams, and first-order formulas in linear integer and real arithmetic.

Furthermore, hypotheses must also be expressible in the chosen symbolic formalism. However, to build efficient learning algorithms, it is often necessary to restrict the class of hypotheses—typically called the *hypothesis space* and denoted by $\mathcal{H}$—even further. Examples of such restricted hypothesis spaces are conjunctive formulas (e.g., as used by the popular Houdini algorithm [7]) or decision trees, which are common, for instance, in learning-based software verification [3, 8, 9, 21] and also used in this work. Note that it might happen that a winning set exist, though it cannot be expressed as a hypothesis in the hypothesis space. Thus, the choice of the hypothesis space and, hence, the learning algorithm needs to be made carefully in order for the learning to succeed.

A second important property of learning algorithms is what we call “consistency”. To make this notion precise, let us assume that the learner accumulates counterexamples in a so-called *game sample* $S_G = (\text{Pos}, \text{Neg}, \text{Ex}, \text{Un})$ consisting of a finite set $\text{Pos}$ of positive counterexamples, a finite set $\text{Neg}$ of negative counterexample, a finite set $\text{Ex}$ of existential counterexamples, and a finite set $\text{Un}$ of universal counterexamples. Then, we say that a hypothesis $H \subseteq V$ is consistent with a game sample $S_G = (\text{Pos}, \text{Neg}, \text{Ex}, \text{Un})$ if (1) $v \in H$ for each $v \in \text{Pos}$, (2) $v \notin H$ for each $v \in \text{Neg}$, (3) $v \in H$ implies $\{v_1, \ldots, v_n\} \cap H \neq \emptyset$ for each $v \rightarrow (v_1 \lor \ldots \lor v_n) \in \text{Ex}$, and (4) $v \in H$ implies $\{v_1, \ldots, v_n\} \subseteq H$ for each $v \rightarrow (v_1 \land \ldots \land v_n) \in \text{Un}$. Moreover, we call a learner consistent if it always produces a consistent hypothesis. Consistency is an important property as it prevents the learner from making the same mistake twice and ensures progress towards a winning set.
In fact, the notion of consistency allows us to show that our framework is sound in the sense that any consistent learner learns a winning set in the limit if one exists in the chosen hypothesis space. This is formalized in the next theorem.

**Theorem 1.** Let a teacher for a safety game (as described in Definition 2) and a consistent learner over an hypothesis space $\mathcal{H}$ be given. If there exists a winning set in $\mathcal{H}$, then there exists an ordinal $\alpha \in \Omega$, where $\Omega$ denotes the class of all ordinals, such that the learner proposes a winning set after at most $\alpha$ iterations.

Theorem 1 is a consequence of the fact that our learning framework is in instance of an abstract learning framework for synthesis (ALF), as introduced by Löding et al. [12]. The proof roughly proceeds as follows. Since the teacher of Definition 2 allows “progress” (i.e., every counterexample refutes the current hypothesis) and we assume the learner to be consistent, the learner never conjectures the same hypothesis twice. In the worst case, the learner will have exhausted all incorrect hypotheses after $\alpha$ iterations for an ordinal $\alpha \in \Omega$ with cardinality less or equal to $|\mathcal{H}|$. Since the teacher is also “honest” (i.e., it does not return spurious counterexamples), the learner necessarily produces a winning set in the subsequent iteration if one exists. For further details, we refer the reader to Löding et al. [12], who prove the convergence of ALFs in a general setting.

Finally, let us point out that the safety games as defined in Section 2 are very general and even allow encoding computations of Turing machines. Consequently, determining the winner of such safety games is undecidable in general, and any algorithm for computing winning sets can be a semi-algorithm at best (i.e., an algorithm that, on termination, gives the correct answer but does not guarantee to halt). The algorithm we design in the next section is of this kind.

## 4 Learning Decision Trees from Game Samples

We now fix the hypothesis space $\mathcal{H}$ to be the class of all decision trees (as defined shortly) and describe an algorithm to learn consistent decision trees from game samples. Our algorithm builds on top of a learning algorithm recently proposed by Ezudheen et al. [6] in the context of software verification. To ease the presentation in this section, we abstract from the setting of infinite games and assume that the data to learn from is taken from an abstract domain $\mathcal{D}$, whose elements we call data points. We encourage the reader to think of data points and vertices as synonyms and define concepts such as game samples and consistency analogously for data points.

An example of a decision tree is shown in Figure 4. In general, decision trees are binary trees whose inner nodes are labeled with predicates from an a priori fixed set $\mathcal{P}$ and whose leaves are labeled with Boolean values. In this context, each predicate is a function $p: \mathcal{D} \rightarrow \mathbb{B}$ that maps data points to a Boolean values and corresponds to a property of interest. Typically, the set $\mathcal{P}$ is finite, but Ezudheen et al.’s algorithm can build decision trees even from infinite sets of predicates if the underlying domain is numeric. The algorithm we design in this section retains this feature.

In the remainder of this section, we view a decision tree $t$ as a representation of an (infinite) set $D(t) \subseteq \mathcal{D}$ of data points. Whether a data point $d \in \mathcal{D}$ belongs to this set depends on its valuation $t(d) \in \mathbb{B}$, which is defined as follows: starting at the root node, we recursively descend left (right) if $d$ satisfies (does not satisfy) the predicate at the current node and define $t(d)$ to be the label of the leaf node that is ultimately reached by this procedure. The set of data points represented by $t$ is then simply the set $D(t) = \{ d \in \mathcal{D} \mid t(d) = 1 \}$. Since the learner proposes this set as a hypothesis to the teacher, we call a decision tree $t$ consistent with a game sample $S_G$ if $D(t)$ is consistent with $S_G$.

Ezudheen et al.’s algorithm has been developed in the context of software verification and expects so-called Horn samples as input. Formally, a Horn sample is a finite set $S_H$ containing Horn constraints of the form $(d_1 \land \ldots \land d_n) \rightarrow d$ or $(d_1 \land \ldots \land d_n) \rightarrow false$ where $i \in \mathbb{N}$ and $d,d_1,\ldots,d_n \in \mathcal{D}$ are data points. Note that the left-hand-side of a Horn constraint might be empty (i.e., $i = 0$), in which case it is interpreted as true.

\[
\begin{align*}
\text{conditions:} & \quad y < 2 \\
\text{labels:} & \quad 1 \\
\text{tree representation:} & \quad \begin{array}{c}
1 \quad x \geq 0 \\
\quad \begin{array}{c}
0 \quad 1
\end{array}
\end{array}
\end{align*}
\]

**Fig. 4:** A decision tree over $\mathcal{D} = \mathbb{R}$ and $\mathcal{P} = \{ x \geq 0, y < 2 \}$.
Algorithm 1: Learning decision trees from game samples

Input: A game sample $S_G = (\text{Pos}, \text{Neg}, \text{Ex}, \text{Un})$ and a (finite) set $\mathcal{P}$ of predicates, both over the domain $D$

1. Construct a Horn sample $S_H$ as follows:
   - for each positive example $d \in \text{Pos}$, add the Horn constraint $d \rightarrow \text{false}$;
   - for each negative example $d \in \text{Neg}$, add the Horn constraint $\text{true} \rightarrow d$;
   - for each existential implication $d \rightarrow (d_1 \lor \cdots \lor d_n) \in \text{Ex}$, add the Horn constraint $(d_1 \land \cdots \land d_n) \rightarrow d$; and
   - for each universal implication $d \rightarrow (d_1 \land \cdots \land d_n) \in \text{Un}$, add the Horn constraints $d_1 \rightarrow d, \ldots, d_n \rightarrow d$ to $S_H$.
2. Apply Ezudheen et al.’s learning algorithm to learn a decision tree $t_H$ over $\mathcal{P}$ that is consistent with $S_H$.
3. Swap the Boolean labels of all leaves of $t_H$ to obtain a new decision tree $t_G$ satisfying $D(t_G) = D \setminus D(t_H)$ (i.e., the decision trees $t_H$ and $t_G$ are structurally identical but every label $b \in \mathbb{B}$ in $t_H$ is replaced with $1 - b$).

Given a Horn sample $S_H$, Ezudheen et al.’s algorithm learns a decision tree $t$ that is consistent with $S_H$ in the sense that (1) $\{d_1, \ldots, d_n\} \subseteq D(t)$ implies $d \in D(t)$ for each Horn constraint of the form $(d_1 \land \ldots \land d_n) \rightarrow d$ in $S_H$ and (2) $\{d_1, \ldots, d_n\} \subseteq D(t)$ for each Horn constraint of the form $d_1 \land \ldots \land d_n \rightarrow \text{false}$ in $S_H$; by extension, we then also say that $D(t)$ is consistent with $S_H$. Should no consistent decision tree exist (e.g., because the set $\mathcal{P}$ does not contain sufficient predicates to separate positive and negative examples), the algorithm aborts with an error. The theorem below summarizes these key properties.

Theorem 2 (Ezudheen et al. [6]). Let $S_H$ be a Horn sample and $\mathcal{P}$ a finite set of predicates, both over the domain $D$. Moreover, let $n$ be the number of data points and $k$ the number of Horn constraints in $S_H$. If a decision tree $t$ over $\mathcal{P}$ exists that is consistent with $S_H$, then Ezudheen et al.’s algorithm learns one in time $O(n|\mathcal{P}| + n^2k)$, assuming that predicates can be evaluated in constant time.

We are now ready to present our algorithm for learning decision trees from game samples, which is shown in pseudo code as Algorithm 1. It builds on top of Ezudheen et al.’s algorithm and proceeds in three steps: first, it translates a game sample $S_G$ into an “equivalent” Horn sample $S_H$; then, it applies Ezudheen et al.’s learning algorithm to obtain a decision tree $t_H$ that is consistent with $S_H$; finally, it translates $t_H$ into a decision tree $t_G$, which is consistent with $S_G$. More precisely, the Horn sample $S_H$ constructed in Step 1 has the property that for every decision tree $t_H$ the set $D(t_H)$ is consistent with $S_H$ if and only if its complement $D \setminus D(t_H)$ is consistent with the game sample $S_G$. Since the decision tree $t_G$ obtained in Step 3 of our algorithm satisfies $D(t_G) = D \setminus D(t_H)$, it is thus consistent with $S_G$. This property is formalized next and proven in Appendix A.

Lemma 1. Let $S_G$ be a game sample and $\mathcal{P}$ a finite set of predicates, both over the domain $D$. Moreover, let $S_H, t_H, t_G$ be as in Algorithm 1. Then, $D(t_H)$ is consistent with $S_H$ if and only if $D(t_G)$ is consistent with $S_G$.

The correctness of Algorithm 1, stated in Theorem 3 below, is now a direct consequence of Lemma 1 and Theorem 2. Note that the term $n|\text{Un}|$ in the runtime estimation stems from the fact that each universal counterexample might have $n$ data points occurring on its right-hand-side, resulting in $n$ Horn constraints.

Theorem 3. Let $S_G = (\text{Pos}, \text{Neg}, \text{Ex}, \text{Un})$ be a game sample and $\mathcal{P}$ a finite set of predicates, both over the domain $D$. Moreover, let $n$ be the number of data points in $S_G$. If a decision tree $t_G$ exists that is consistent with $S_G$, then Algorithm 1 learns one in time $O(n|\mathcal{P}| + n^2(|\text{Pos}| + |\text{Neg}| + |\text{Ex}| + n|\text{Un}|))$, assuming that predicates can be evaluated in constant time.

If the abstract domain $D$ is numeric, say $D \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$, a simple extension of Ezudheen et al.’s algorithm, and hence Algorithm 1, can be used to learn decision trees even over infinite sets of predicates. This extension was originally proposed by Quinlan [18] and assumes the set of predicates to be $\mathcal{P}^* = \{d[i] \leq c \mid c \in \mathbb{R}, i \in \{1, \ldots, n\}\}$ where $d[i]$ denotes the $i$-th component of the domain $D$. The key idea...
is in fact straightforward: to separate two data points \( d_1, d_2 \in \mathcal{D} \) (e.g., a positive and negative data point), it suffices to consider only such predicates \( d[i] \leq c \) for which the constant \( c \) occurs as an actual value in \( d_1 \) or \( d_2 \). Thus, Algorithm 1 can restrict itself to a finite subset of \( \mathcal{P}^\star \), which only depends on the values of the data points in the given sample.

In many situations, Algorithm 1 in fact guarantees that the overall feedback loop of Section 3 converges to a winning set in finite time if one can be expressed as a decision tree. For instance, if the set \( \mathcal{P} \) is finite, then there exist only finitely many semantically different decision trees (in terms of the set of data points they represent). Since Algorithm 1 always produces consistent decision trees, it will have exhausted all incorrect trees after a finite amount of time. If a winning set exists and can be expressed as a decision tree over \( \mathcal{P} \), then the subsequent hypothesis will necessarily be one. On the other hand, if the underlying domain is numeric, and Algorithm 1 operates over the infinite set \( \mathcal{P}^\star \), a technique proposed by Ezudheen et al. can be used to guarantee that a winning set will be learned in finite time (if one can be expressed as a decision tree over \( \mathcal{P}^\star \)). Due to the limited space, we have to refer the reader to Ezudheen et al. [6] for details and can here only state our main result.

**Theorem 4.** Let \( \mathcal{G} \) be a safety game. If the learner of Section 3 uses Algorithm 1 over a finite set \( \mathcal{P} \) of predicates, then the learner is guaranteed to learn a winning set after a finite number of iterations if there exists one that is expressible as a decision tree over \( \mathcal{P} \). The analogous statement is true for the set \( \mathcal{P}^\star \).

## 5 Experimental Evaluation

In this section, we evaluate the performance of our learning framework and the decision tree learning algorithm on benchmarks inspired by robotic motion planning. Most of these benchmarks involve one or more robots that interact in an unbounded grid-world and, hence, result in games over graphs with infinitely many vertices.

To encode our games symbolically, we resorted to formulas in the first-order theory of linear integer arithmetic. More precisely, we used variables \( x_1, \ldots, x_n \in \mathbb{N} \) to encode vertices of a game graph and represented (infinite) sets of vertices by formulas \( \varphi(x_1, \ldots, x_n) \). Moreover, we encoded the edge relation of a game graph by a formula \( \psi(x_1, \ldots, x_n, x'_1, \ldots, x'_n) \) where \( x_1, \ldots, x_n \) represents a source-vertex and \( x'_1, \ldots, x'_n \) a destination-vertex. Note that this encoding restricts our game graphs to countably many vertices. We have made this choice deliberately to simplify our implementation.

We have implemented a prototype\(^1\) of both the teacher and the decision tree learner, named \( DT-Synth \), as described next.

**Teacher:** We have implemented the teacher on top of the Z3 SMT solver [14]. Upon receiving a hypothesis in the form of a formula \( \varphi_H(x_1, \ldots, x_n) \), the teacher performs a series of satisfiability queries according to Definition 2 in order to check for counterexamples. If a satisfiability query succeeds, the teacher derives a counterexample from the model return by Z3 (potentially requiring a finite number of additional satisfiability checks to compute the successors of a vertex in the case of existential and universal counterexamples).

**Learner:** We have implemented the learner based on code provided by Ezudheen et al. [6]. In addition to the set \( \mathcal{P}^\star \) of predicates described in Section 4, we added octagonal constraints of the form \( x_i \pm x_j \leq c \) to be able to express distances between robots. After a consistent decision tree has been learned, the learner converts it into a formula \( \varphi_H := \bigvee_{\pi \in \Pi} \bigwedge_{\psi \in \pi} \psi \) where \( \Pi \) is the set of all paths from the root to a leaf labeled with \( \text{true} \) and \( \psi \in \pi \) denotes that the predicate \( \psi \) occurs on \( \pi \) (negated if the path descends to the right). Then, it hands \( \varphi_H \) over to the teacher. Following the description in Section 4, it is not hard to verify that a vertex satisfies the formula \( \varphi_H \) obtained from the tree \( t \) if and only if the vertex belongs to \( D(t) \).

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\(^1\) The code/benchmarks are available at https://github.com/OliverMa1/DT-Synth.
Table 1: Results of the first benchmark suite.

| Game                | DT-Synth |          | SAT-Synth |          | RPNI-Synth |          |
|---------------------|----------|----------|-----------|----------|------------|----------|
|                     | time in s | Iter. Size | Pos | Neg | Ex | Uni | time in s | Iter. Size | Pos | Neg | Ex | Uni | time in s | Iter. Size | Pos | Neg | Ex | Uni |
| Diagonal Limited    | 2.04     | 23       | 5  | 10  | 6  | 4  | 2  | 1.49     | 64        | 4  | 1   | 56 | 3  | 1.01     | 64        | 4  | 1   | 49 | 8  |
| Box                 | 0.77     | 9        | 5  | 2   | 3  | 0  | 3  | 0.77     | 9         | 5  | 2   | 3  | 0  | 0.77     | 9         | 5  | 2   | 3  | 0  |
| Box Limited         | 0.28     | 4        | 2  | 1   | 2  | 0  | 0  | 0.74     | 36        | 4  | 1   | 33 | 0  | 0.32     | 15        | 5  | 1   | 10 | 1  |
| Solitary Box        | 0.24     | 4        | 2  | 1   | 2  | 0  | 0  | 5.81     | 76        | 6  | 2   | 70 | 3  | 0.42     | 16        | 6  | 1   | 13 | 1  |
| Evasion             | 0.63     | 6        | 3  | 1   | 1  | 1  | 2  | 89.15    | 237       | 7  | 2   | 219| 7  | 1.40     | 82        | 11 | 1   | 62 | 9  |
| Follow              | 0.86     | 11       | 5  | 1   | 6  | 0  | 3  | 95.90    | 300       | 7  | 2   | 278| 7  | 12.36    | 352       | 16 | 1   | 273| 3  |
| Program-repair      | 0.99     | 14       | 11 | 1   | 2  | 10 | 0  | 1.42     | 69        | 3  | 2   | 64 | 2  | 0.15     | 7         | 3  | 1   | 2  | 3  |
| Square 5x5          | 7.77     | 61       | 12 | 1   | 16 | 28 | 15 | ——       | ——        | —— | ——  | —— | —— | 7.77     | 61        | 12 | 1   | 16 | 28 |

We evaluated our prototype on two benchmark suits. The first suite contains eight games over infinite graphs taken from Neider and Topcu [17]. To assess the performance of our approach on these games, we compared our implementation to two of Neider and Topcu’s learning-based synthesis algorithms, for brevity here called RPNI-Synth and SAT-Synth. The second benchmark suite is designed to compare the performance of our approach to existing synthesis tools, namely GA VS+ [4], TuLiP [20], and a naive implementation of a symbolic fixed-point algorithm using automata as representation of the game graphs [17]. As these tools operate only on games over finite graphs, we considered a motion planning problem in a bounded 1-dimensional grid-world, similar to Examples 1 and 2, whose size can be scaled by a parameter \( m \in \mathbb{N} \).

We conducted all experiments on an Intel Xeon E7-8857 v2 CPU running a 64-bit Debian operating system. We limited the memory to 4 GB and used a timeout of 300 s.

First Benchmark Suite. Our first benchmark suit comprises the following eight games, taken from Neider and Topcu [17]. Most games are from the area of motion planning, and all are over infinite graphs.

**Diagonal game:** A robot moves in an infinite, discrete two-dimensional grid world. Player 0 controls the robot’s vertical movement, while Player 1 controls the horizontal. Player 0 wins if the robot stays within two cell around the diagonal.

**Box game:** A variation of the diagonal game. Both players can move the robot in an vertical, horizontal or diagonal direction by one cell. Player 0 wins if the robot stays within a horizontal stripe of width three.

**Limited Box game:** A variation of the box game. Player 0 can only control the robot’s vertical movement and Player 1 the horizontal.

**Solitary box game:** Another variation of the Box game in which only Player 0 is in control of the robot.

**Evasion game:** Two robots are moving in an infinite, discrete two-dimensional grid world. The robots take turns moving at most one cell in any direction. Each players controls one robot. Player 0’s objective is to avoid getting caught by Player 1’s robot.

**Follow game:** A version of the evasion game where Player 0’s goal is to keep its robot within a Manhattan distance of two cells to the environment’s robot.

**Program repair game:** The program-repair game by Beyene et al. [1].

**Square game:** A variation of the box game, where Player 0 wins if the robot stays within a fixed size square (here 5 \( \times \) 5).

Table 1 compares the runtimes of DT-Synth, SAT-Synth, and RPNI-Synth on the first benchmark suite. Since our framework produces the same types of counterexamples as Neider and Topcu’s original framework [17], Table 1 also lists the number of positive, negative, existential and universal counterexamples that are generated during learning. In addition, the table lists the size of the final decision tree learned by DT-Synth (in terms of the number of inner nodes) and the size of the final automata produced by RPNI-Synth and SAT-Synth (in terms of the number of states).

As Table 1 shows, both DT-Synth and RPNI-Synth outperformed SAT-Synth on all but one game, the latter even timing out on one. The performance of DT-Synth and RPNI-Synth was roughly equal and, except
Figure 5 compares the runtimes of the various techniques for increasing values of the parameter $m$ and, thus, increasing number of vertices (we also included RPNI-Synth and SAT-Synth in this comparison). DT-Synth performed best and solved games up to $m = 800$ before timing out. RPNI-Synth and SAT-Synth share the second place, both having solved games up to $m = 375$. All techniques that rely on a full exploration of the game graph, on the other hand, performed worse than the learning-based methods. While GAVS+ and the fixed-point algorithm were still able solve games up to $m = 325$ and $m = 275$, respectively, TuLiP performed much worse and could only solve games up to $m = 75$ before running out of memory. On this benchmark suite, DT-Synth was on average six times faster than the second best tool, RPNI-Synth. Although designed for games over infinite graphs, these results demonstrate clearly that our learning-based approach performs well even on games over large finite arenas.

### 6 Conclusion

We have developed a machine learning framework for synthesizing reactive safety controllers whose interaction with their environment is modeled by games over infinite graphs. For this framework, we have designed a learning algorithm for decision trees that learns winning strategies and is in many situations guaranteed to find a solution if one exists. Our experimental evaluation promises applicability of our approach to a wide range of interesting practical problems.

A promising direction for future work would be to apply our technique to distributed synthesis problems and other, more complex synthesis settings. Moreover, we plan to extend our learning-based framework to more general winning conditions, such as reachability and liveness.
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A Proof of Lemma 1

Proof (of Lemma 1). Since consistency is a conjunction of conditions for individual examples and Horn constraints, respectively, we show Lemma 1 for each element of a sample individually. Recall that \( D(t_G) = D \setminus D(t_H) \).

- Let \( d \in Pos \) be a positive example and \( d \rightarrow false \) the corresponding Horn constraint generated in Step 1. Then,

\[
   t_H \text{ is consistent with } d \rightarrow false \iff d \not\in D(t_H) \\
   \iff d \in D(t_G) \\
   \iff t_G \text{ is consistent with } d \in Pos.
\]

- Let \( d \in Neg \) be a negative example and \( true \rightarrow d \) the corresponding Horn constraint generated in Step 1. Then,

\[
   t_H \text{ is consistent with } true \rightarrow d \iff d \in D(t_H) \\
   \iff d \not\in D(t_G) \\
   \iff t_G \text{ is consistent with } d \in Neg.
\]

- Let \( d \rightarrow (d_1 \lor \ldots \lor d_n) \in Ex \) be an existential counterexample and \((d_1 \land \ldots \land d_n) \rightarrow d \) the corresponding Horn constraint generated in Step 1. Then,

\[
   t_H \text{ is consistent with } (d_1 \land \ldots \land d_n) \rightarrow d \\
   \iff \{d_1, \ldots, d_n\} \subseteq D(t_H) \text{ implies } d \in D(t_H) \\
   \iff \text{either there exists an } i \in \{1, \ldots, n\} \text{ with } d_i \not\in D(t_H) \text{ or } d \in D(t_H) \\
   \iff \text{either there exists an } i \in \{1, \ldots, n\} \text{ with } d_i \in D(t_G) \text{ or } d \not\in D(t_G) \\
   \iff t_G \text{ is consistent with } d \rightarrow (d_1 \lor \ldots \lor d_n).
\]

- Let \( d \rightarrow (d_1 \land \ldots \land d_n) \in Un \) be a universal implication and \( d_1 \rightarrow d, \ldots, d_n \rightarrow d \) the corresponding Horn constraints generated in Step 1. Then,

\[
   t_H \text{ is consistent with } d_1 \rightarrow d, \ldots, d_n \rightarrow d \\
   \iff \text{either } d_i \not\in D(t_H) \text{ for each } i \in \{1, \ldots, n\} \text{ or } d \in D(t_H) \\
   \iff \text{either } d_i \in D(t_G) \text{ for each } i \in \{1, \ldots, n\} \text{ or } d \not\in D(t_G) \\
   \iff t_G \text{ is consistent with } d \rightarrow (d_1 \land \ldots \land d_n).
\]

In total, we obtain that \( D(t_G) \) is consistent with \( S_G \) if and only if \( D(t_H) \) is consistent with \( S_H \). \(\square\)

B Illustration of Decision Tree

Reconsider the box game mentioned in Section 5, in which a robot moves in an infinite, discrete two-dimensional grid world. Both players can move the robot in an vertical, horizontal or diagonal direction.
We have encoded the box game using three variables $x, y, z \in \mathbb{Z}$, as described next. The variables $x$ and $y$ correspond to the $x$-coordinate and the $y$-coordinate of the robot, respectively. Moreover, we used the variable $z$ to indicate which player is currently in control of the robot. In the case of $z = 0$, Player 0 is in control, thus $V_0 := \{(x, y, z) \in \mathbb{Z}^3 \mid z = 0\}$. On the other hand, if $z = 1$, then Player 1 is in control of the robot, thus $V_1 := \{(x, y, z) \in \mathbb{Z}^3 \mid z = 1\}$. We allowed for an arbitrary starting position as long as $y = 0$, which means that the set of initial vertices is $I := \{(x, y, z) \in \mathbb{Z}^3 \mid (z = 1 \lor z = 0) \land y = 0\}$. Furthermore, we fixed the set of safe vertices to be the horizontal stripe within $-1 \leq y \leq 1$, thus $F := \{(x, y, z) \in \mathbb{Z}^3 \mid (z = 1 \lor z = 0) \land -1 \leq y \leq 1\}$. Finally, it is left to define the edge relation. The robots movement to the right, for instance, can be defined by the relation $R := \{(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \mid z_1, z_2 \in \{0, 1\}, z_2 = 1 - z_1, x_2 = x_1 + 1 \land y_2 = y_1\}$. The remaining directions can be encoded analogously, and the edge relation $E$ is the union of all directions.

The decision tree (representing a winning set) learned by our prototype is depicted in Figure 7. We first observe that the predicate at the root node is $z \leq 0$. Since $z$ determines the player who is currently in control of the robot, the left subtree (i.e., where $z \leq 0$ is true) encodes vertices owned by Player 0 in the winning set, while the right subtree (i.e., where $z \leq 0$ is false) encodes vertices owned by Player 1 in the winning set.

Let us first consider the left subtree. Only vertices of the game that satisfy $\neg(y \leq -2)$ and $y \leq 1$ lead to the unique leaf node labeled with 1. All of these vertices are safe and correspond to the gray-shaded area in Figure 8a. Conversely, only the vertices satisfying $\neg(y \leq -1)$ and $y \leq 0$ lead to the unique leaf node labeled with 1 in the right subtree. This corresponds to the gray-shaded area in Figure 8b. Again all of these vertices are safe. It is not hard to verify that Player 0 can stay within these sets, whereas Player 1 cannot force a play to an outside vertex. Moreover, the set of initial vertices is included. Thus, the learned decision tree in fact encodes a winning set.

Note that in the definition of the winning set requires $W \subseteq V$ to hold. However, this is not true in this example since the decision tree evaluates to 1 even in situations where $z \notin \{0, 1\}$, while $V$ contains only vertices with $z \in \{0, 1\}$. Note that this is not a problem in our example since the only checks that require $W \subseteq V$ to hold true are the checks for existential and universal closedness. Those checks require the winning
set $W$ to be intersected with either $V_0$ or $V_1$, which effectively ignores all vertices that violate $z \in \{0,1\}$. Furthermore, the safe set $F$ allows any value for $z$.

In conclusion, the winning set depicted in Figure 8 distinguishes between the player having control of the robot. Depending on that, the winning set is a horizontal stripe as depicted in Figure 8a and Figure 8b, respectively. The decision tree of Figure 7 describes these sets in an easy manner.