I. INTRODUCTION

Understanding operator spreading in quantum many-body systems poses several intriguing challenges. Given an initially local-in-space operator $O$, its dynamics under a many-body Hamiltonian $H$ is $O(t) = e^{iHt}Oe^{-iHt}$. The support of the operator increases with time, and the initially local information spreads within an emerging light-cone. The most urgent question is as to whether a generic local operator admits an efficient representation as a Matrix Product Operator [1–6] (MPO). An affirmative answer would suggest that it is possible to simulate operator spreading with classical computers, with tremendous implications for Noisy Intermediate-Scale Quantum [7] (NISQ) computing technologies. A figure of merit for the MPO-simulability is the so-called Operator Space Entanglement Entropy (OSEE), which is the entanglement entropy in operator space.

Since its inception [8], the OSEE is attracting flourishing interest [5, 6, 8–11]. It has been suggested in Ref. 4 that in integrable systems the OSEE grows at most logarithmically with time, as it was found for free fermions [10]. Very recently, a logarithmic bound has been derived for the so-called rule 54 chain [12], which is believed to be representative of generic integrable systems. This has been checked in spin chains [12]. Oppositely, it has been argued that the OSEE grows linearly [10] in generic systems. Interestingly, this linear growth is predicted by the random unitary scenario, which posits that universal out-of-equilibrium features of the OSEE can be captured by replacing the evolution operator $e^{iHt}$ with random unitary gates [13–17]. Despite all these efforts, however, the general mechanism behind the dynamics of the OSEE is yet to be unveiled, even for integrable systems. This is in contrast with the entanglement of a state, for which a powerful quasiparticle picture [18–21] explains the entanglement dynamics in terms of the ballistic motion of entangled quasiparticles.

One goal of this paper is to show that for generic integrable systems the OSEE reflects the diffusion of the operator front. Here, building on Ref. 12 we provide a tight logarithmic bound for the OSEE of some simple operators in the rule 54 chain. Remarkably, the same bound is saturated in the spin-1/2 Heisenberg XXZ chain, at least away from the free-fermion point and the isotropic XX point. This suggests a universal relation between diffusive and OSEE dynamics. Finally, we numerically investigate how this scenario is affected by integrability-breaking interactions.

To define the OSEE $S(O)$ we bipartite the system as $A \cup B$, and consider the Schmidt decomposition of $O$ as $O/\sqrt{\text{Tr}(O^\dagger O)} = \sum_i \sqrt{\lambda_i} O_{A,i} \otimes O_{B,i}$, with $O_{A/B,i}$ two orthonormal bases for the operators with support in $A$ and $B$, and $\lambda_i > 0$ the so-called Schmidt coefficients. The operator entanglement is $S(O) = -\sum_i \lambda_i \ln \lambda_i$.

II. OSEE IN THE RULE 54 CHAIN

Here we focus on the OSEE spreading in the rule 54 chain [22]. The Hilbert space is that of a system of qubits $s_x = 0, 1$. The dynamics is generated by a three-site unitary gate $U_z$ acting as
FIG. 2. (a) Dynamics of a diagonal operator \( O \) in the rule 54. (a) Double lightcone. \( O(0) \) creates a pair of scattering left/right movers \( x = 0 \). They scatter with the background solitons. The upper and the lower half-lightcones coincide. (b) Typical evolution. Solitons positions \( x_1, x_2 \) are measured from the left edge of the light-cone (dashed line). Three different regions appear. Region 2 is the “reduced lightcone”. (c) MPO representation for \( O(t) \). Index \( a = 1, 2, 3 \) denotes the three regions in (b). The composite index \((\hat{ji}, j_r)\) with \( j_i, j_r \in [1, t] \) tracks the positions of the two solitons, and \( \beta, \beta' \) are as in Fig. 1. In 1 and 3, \( O \) is the identity, and \( a = a' = 1, 3 \) and \( j_i = j_i' = j_r = j_r' = 0 \). An example of MPO contraction is shown. (d) MPO in region 2. All nonzero tensor elements are shown. (e) Tensors at the interface between different regions. At 1, 2 one has a left mover and \((j_i, j_r) = (2t - x - 1, [x/2])\). At 1, 3 the right mover that emerged at the center is found. (f) Cartoon for OSEE spreading in integrable systems (top). The operator front spreads with the dressed velocity \( v_d \), implying that a number \( \propto t \) of left and right moving solitons are present in the lightcone. The bipartition as \( A \cup B \) with \( A = [−t, x] \) is shown. The OSEE reflects the number of ways of distributing between \( A \) and \( B \) the solitons that are present in the lightcone. The effective MPO describing \( O \) is reported. The virtual indices of the grouped tensors for \( A \) and \( B \) take values \( \min(t − x, t + x) \), corresponding to the maximum number of solitons that can be stored in the smaller of the two subsystems. As a result the entanglement profile as a function of the cut position exhibits a “pancake” structure (see bottom), as opposed with the random-unitary scenario, which gives a “pyramid” profile (dotted profile).

\[
U_x = |s_{x-1}, s'_x, s_{x+1}\rangle \langle s_{x-1}|s_x|s_{x+1}\rangle,
\]

where \( s'_x = s_{x-1} + s_{x+1} - s_{x-1}s_{x+1} \). \( U_x \) flips the qubit at \( x \) if one of the neighbouring qubits is 1. Any qubit configuration is evolved as \( U = \prod_{x=\text{even}}^t \prod_{x=\text{odd}} U_x \). The rule 54 chain possesses well-defined quasiparticles, which is the key property of integrable systems. Quasiparticles are emergent left/right moving solitons. They correspond to pairs of adjacent qubits that are in the 1 state (more details are reported in Appendix C). Crucially, solitons undergo pairwise elastic scattering, which is implemented as a Wigner time delay [23] (cf. Fig. 1). Again, this is also generic for integrable models (see Ref. 25). Two solitons that are scattering correspond to the qubit configuration 010. The mapping between qubits and left/right movers is encoded as an MPO with bond dimension \( \chi = 4 \). Here we work directly in soliton space. As it is shown in Figure 1, a site \( x \) can be empty (empty box), or occupied by a left (right) mover (boxes with slanted lines in the Figure) if \((-1)^{x+t} = -1\), or by two scattering solitons (vertical lines). If \((-1)^{x+t} = 1\) the two (“emitting”) solitons will reappear at time \( t + 1 \), whereas if \((-1)^{x+t} = 1\) the (“merging”) solitons will reappear at \( t + 1 \), reflecting the Wigner delay. We are interested in the Heisenberg dynamics of local operators. Let us first consider the identity operator \( 1 = \prod_{x=0}^{t-1} \sum_{s_x} |s_x\rangle \langle s_x| \) in soliton space. As for all diagonal operators, one can consider the evolution of the ket or bra separately, because they evolve in the same way under application of \( U \) and \( U^\dagger \). One now has the evolution of the “flat” superposition \( \prod_x \sum_{s_x} |s_x\rangle \). In soliton space this maps to the flat superposition of all allowed soliton configurations. This is efficiently encoded as an MPO (see Appendix C) as

\[
1 = \sum_{\{\beta_x\}} \prod_{x} A_{\beta_x, \beta_{x+1}}^{x} \langle \tau_x| \tau_x \rangle.
\]

Here \( A_{\beta_x, \beta_{x+1}}^{x} \) is a tensor living on site \( x \). The index \( \tau_x \) labels the soliton configuration, \( \beta_x \in [0, \chi] \) are the virtual indices, with \( \chi \) the bond dimension. Here \( A_{\beta_x, \beta_{x+1}}^{x} = 1 \) only for the cases shown in Fig. 1 (c-e), and it is zero otherwise. The role of \( \beta_x \) is to enforce some kinematic constraints, for instance, that a left mover is followed only by a right mover or by an empty site (see Fig. 10 in Appendix C). Since \( \chi \) is small and the identity operator does not evolve, one has that \( S(1) \) is constant in time.

This changes dramatically for the OSEE of a local operator. By adapting a remarkable result of Ref. 26 it has been shown [12] that the dynamics of operators is described by an MPO with \( \chi \propto t^2 \). This implies the “naive” bound \( S(O) \leq 2 \ln(t) \) for the OSEE. Here we show that the growth of the OSEE reflects the fluctuations of the number of solitons between \( A \) and its complement. This allows us to derive a tighter bound for the OSEE spreading. To derive our result, we review the construction of the MPO for the diagonal operator that inserts two scattering solitons at \( L/2 \), i.e., \( O = |010\rangle\langle 010| \). This is illustrated in Fig. 2. \( O \) is diagonal, implying that the upper and the lower lightcones coincide. At \( t > 0 \) a left and right movers are emitted from \( L/2 \). They play a crucial role in the MPO contraction. Indeed, \( O(t) \) corresponds to the flat superposition of all the possible soliton configurations that contain the left and right movers that were inserted at the origin at \( t = 0 \). This simple constraint on the soliton configurations implies that the OSEE grows logarithmically.

We note that as the solitons emitted from the center scatter with the background solitons, they undergo two
biased random walks. Their positions $x_1, x_2 = 0, 1, \ldots$ at time $t$, which are measured from the left edge of the lightcone (dashed lines in Fig. 2 (b)), are determined by the scatterings. The crucial observation is that all the background solitons that scattered with the two solitons emitted from the center are contained in the “reduced lightcone” within them (region 2 in Fig. 2 (b)). Outside of the reduced lightcone $O(t)$ is the identity. To construct the MPO for $O(t)$ we complement the MPO for the identity in Fig. 1 with some extra indices. First, we introduce an index $a = 1, 2, 3$ to keep track of the different regions. The number of left/right movers in region 2 is tracked by two extra indices $j_l, j_r$. Finally, the index $\beta$ is as in Fig. 1. The structure of the MPO is summarised in Fig. 2 (c). Physically, $j_r$ at $(x, t)$ counts the number of right movers in region 2, whereas $j_l$ is the expected distance between $x$ and the right mover that emerged from the center, assuming that there are no left movers in the remaining interval $[x + 1, t]$ of the lightcone. In regions 1, 3 we set $a = a' = 1, 3$, and $j_l = j_r = j_l = j_r = 0$. The allowed values of $j_l, j_r$ in region 2 for which the MPO is nonzero are reported in Fig. 2 (d). The interpretation is straightforward. For instance, if at site $x$ there is no soliton, one has $j_r' = j_r - 1$. If a left mover is present, one has that $j_l' = j_l - 3$, because the left mover shifts the right mover emerging from the center by two sites to the left (see Fig. 2 (b)). Finally, Fig. 2 (e) shows the tensors at the interface between regions 1, 2 and 2, 3. At the boundary 1, 2 a left mover is present, and $j_l, j_r$ is initialized as $j_r = [x_1/2]$ and $j_l = 2t - x_1 - 1$. At the boundary 2, 3 one has $j_l = j_r = 0$, ensuring that all the background solitons expected within the reduced lightcone have been found and the right mover that emerged from the center is on that site. Notice that there is a subtlety due to the kinematics of solitons if two scattering solitons are met at 2, 3 (see Fig. 2 (e)). This, however, does not affect the leading logarithmic growth of the OSEE. Now, since $0 \leq j_l, j_r \leq t$, the bond dimension of the MPO that describes $O(t)$ is clearly $\mathcal{O}(t^2)$, implying that $S \leq 2 \ln(t)$. To proceed, we observe that due to the scatterings, the left and right movers that emerged from the center move with a “dressed” velocity $v_d = 1/2$ (the bare velocity is $v_b = 1$). Crucially, their trajectories, and the operator front, exhibit diffusion [28, 29]. This diffusion is essential to have nonzero entanglement. Indeed, the dressed solitons behave as free particles, their trajectories cross each other. This implies that a flat superposition of dressed solitons is mapped onto itself by the dynamics, which implies the absence of entanglement production.

We now observe that in the reduced lightcone there are $\mathcal{O}(t)$ left/right movers. Let us consider the bipartition $A \cup B = [-t, x] \cup [x, t]$, with $x \leq 0$. A crude approximation for $O(t)$ gives

$$O(t) = \sum_{k=0}^{t-|x|} \frac{\sqrt{\binom{t-|x|}{k} \binom{t+|x|}{t-k}}}{\sqrt{2^k t}} O^A_k \otimes O^B_{t-k}. \quad (2)$$

Here $O^A_k$ and $O^B_{t-k}$ are normalised operators in $A$ and $B$ constructed with $k$ and $t - k$ solitons. In (2) we assume that $O^A_k$ and $O^B_{t-k}$ are some “flat” superpositions of all the configurations with $k$ and $t - k$ solitons, i.e., we assume that the positions of the background solitons are maximally “scrambled” within the reduced lightcone. This is not true in general because solitons scatter locally. We also assume that $O^A_0$ and $O^B_{t-k}$ form orthonormal bases for $A$ and $B$. The two binomials in the sum in (2) give the number of ways of arranging the solitons in the two subsystems. Note that for large $t$ the behavior of (2) is dominated by the configurations with $k = (t - |x|)/2$, showing a spreading $\sqrt{t}$. This reflects that there is an average number $(t - |x|)/2$ of solitons in subsystem $A$. The number of solitons in $A$ fluctuates, the fluctuations being $\propto \sqrt{t}$. We anticipate that these fluctuations are responsible for the growth of the OSEE. Crucially, this mechanism is different from the spreading of the state entanglement after a global quantum quench, where entanglement is produced locally at each point in space and it is transported by entangled multiplets of quasiparticles [18–20]. This is also different from the random-unitary scenario. The main assumption of this scenario is that the entanglement profile $S(x, t)$ satisfies the equation $\partial_t S = \Gamma \partial_x S$. Here $\Gamma$ is the entropy production rate, which depends on the spatial variation of the entropy profile, and it is nonzero at any point in space. This implies that the entanglement profile has the typical “pyramid” shape (Fig. 2 (f)). In contrast, the logarithmic growth in integrable systems is reflected in a “pancake” structure in the entanglement profile (Fig. 2 (f), see also Appendix B).

We can now derive a bound on the OSEE growth from (2). The bond dimension of the decomposition from (2) is $t - |x| + 1$. Note that $t - |x|$ is the largest number of solitons that can be accommodated within $A$. The eigenvalues of the reduced density matrix for $A$ are simply $\lambda_k = \binom{t-|x|}{k} \binom{t+|x|}{t-k} / \binom{2t}{t}$. Notice that the fact that there are only $\propto t$ eigenvalues is an approximation. In the rule 54 chain one should expect $\propto t^2$ nonzero eigenvalues, instead of the $\propto t$ predicted by the argument above. On the other hand, the number $\propto t^2$ does not imply the scaling $2 \ln(t)$ for the OSEE because the eigenvalues are not equal but exhibit a nontrivial distribution. By using the explicit form of $\lambda_k$ one obtains the analytical bound for the OSEE as (see Ref. 30 for a similar calculation)

$$S_{\text{max}} = \frac{1}{2} \ln(t). \quad (3)$$

Crucially, the prefactor $1/2$ in (3) is reminiscent of the $\sqrt{t}$ fluctuations in the number of solitons in the subsystems $A$ and $B$. Eq. (3) is expected to hold for the
simple, i.e., low-rank, diagonal operator. We should remark that the prefactor of the OSEE growth should depend on the structure of the operator. For instance, the identity operator, for which the OSEE is constant in time, is \( P_1 = 1 \). On the other hand, the OSEE of \( S_z = P_2 - P_1 \) grows logarithmically. Moreover, (see Ref. 16) for traceful operators, the prefactor of the OSEE growth depends on the trace. Also, for off-diagonal operators (see Fig. 2 (a)) the upper and lower lightcones do not coincide, suggesting a faster growth of the OSEE.

We should also stress that the behavior of the OSEE in free-fermion systems is different from (3). For instance, the OSEE of \( S_z \) saturates, whereas that of \( S_x \) increases as \( 1/3 \ln(t) \). Interestingly, the prefactor \( 1/3 \) could reflect the absence of diffusion for free fermions, suggesting that the OSEE could be potentially useful to distinguish interacting integrable from free systems [31].

III. INTEGRABLE DYNAMICS: RULE 54 AND XXZ SPIN CHAIN

To benchmark our main result (3), in Fig. 3 we discuss the case of the rule 54 chain. We focus on the projector operator \( P_\downarrow \equiv (1/2 - S_z) \), the raising operator \( S^+ \), and \( S_- \), all inserted at the center of the chain. The symbols are tDMRG data [32−34]. For \( S^+ \), we report the bond dimension \( \chi \). The full line is Eq. (3), whereas the dashed-dotted line is \( 2S_{\text{max}} \). The agreement between (3) and the data is excellent for \( P_\downarrow \), signalling that the bound (3) is saturated. For \( S^\pm \) one should also expect \( S = 2S_{\text{max}} \) (see Ref. 16). A fit to \( \kappa \ln(t) + a \) gives \( \kappa \approx 0.9 \). For \( S^+ \), we observe a reasonable agreement with \( 2S_{\text{max}} \), although finite-time effects seem larger.

We now discuss the universality of (3). We consider a generalisation of the spin-1/2 XXZ chain defined by the Hamiltonian

\[
H = \sum_{i=1}^{L} \frac{1}{2}(S^+_i S^-_{i+1} + S^-_i S^+_i) + \Delta \sum_{i=1}^{L} S^z_i S^z_{i+1} + \Delta' \sum_{i=1}^{L} S^+_i S^-_{i+2}
\]

where \( \Delta, \Delta' \) are real parameters. For \( \Delta' = 0 \) the model is integrable for any \( \Delta \), whereas \( \Delta' \neq 0 \) breaks integrability (see Appendix A). Let us consider the integrable case, i.e., \( \Delta' = 0 \). We discuss the OSEE of \( P_\downarrow \) in Fig. 4 (a) and that of \( S^+ \) and \( S^z \) in Fig. 4 (b). The tDMRG data for \( P_\downarrow \) exhibit a clear logarithmic increase. For \( \Delta \neq 1 \) they are compatible with \( S_{\text{max}} + c(\Delta) \), suggesting universality of the prefactor 1/2 of the OSEE. Interestingly, \( c(\Delta) \) reflects the behavior of the diffusion constant [35, 36], i.e., it increases with \( \Delta \) for \( 0 \leq \Delta \leq 1 \), then it decreases for \( \Delta > 1 \), saturating for \( \Delta \to \infty \). For \( \Delta \to 1 \) the diffusion constant diverges [35, 36], which signals superdiffusive behavior, suggesting violations of (3) for \( \Delta = 1 \). The data in the inset of Fig. 4 might suggest the behavior \( \kappa \ln(t) \) with \( \kappa > 1/2 \), although they could just signal large finite-time corrections. It has been proposed in Ref. 37 that the superdiffusive behavior as \( t^{2/3} \) arises at \( \Delta = 1 \), suggesting \( S \propto 2/2 \ln(t) \) (reported for comparison in Fig. 4). Finally, in Fig. 4 (b) we discuss \( S^\pm \). The OSEE increases faster. Finite-time effects are large, and the evidence for the behavior \( S \propto 2S_{\text{max}} \) is weak.

IV. NON-INTEGRABLE DYNAMICS

The soliton picture should breakdown for generic models, because they do not possess quasiparticles. According to the random-unitary scenario, this would imply a linear growth of the OSEE. However, it has been suggested in Ref. 38 that if a conservation law is present, the Rényi operator entanglement \( S(2) \) of the associated local
operator exhibits logarithmic growth, even if the system is nonintegrable. Notice that for systems without conservation laws, for instance Floquet systems, the linear growth of operator entanglement is supported by exact calculations [39–41]. Our tDMRG results for $P_1$ are in Fig. 5. It is enlightening to first consider the integrable case for $\Delta' = 0$ and $\Delta = 0.4$. At very short times $t \approx 2$, the OSEE exhibits a jump, reflecting that at $\Delta = 0$ the OSEE saturates (see the result for $\Delta = 0$ in the Figure). Then, there is an intermediate regime, where a nearly-linear growth is present. The asymptotic behavior sets in at longer times. Upon breaking integrability tDMRG simulations become more challenging. At short times a linear increase is observed. However, this could be reminiscent of the transient regime also observed for $\Delta' = 0$. In fact, a change in behavior happens at $t^*(\Delta')$, with $t^*$ increasing with $\Delta'$. The data in Fig. 5 are compatible with two scenarios. In one scenario the OSEE increases linearly at asymptotically long times. The asymptotic regime sets in after a long transient in which the system behaves as if it was integrable. The prefactor of the linear growth should presumably increase with $\Delta'$. Alternatively, the breaking of integrability gives rise to a longer transient, as compared with the integrable case, before the logarithmic behavior sets in. Longer transients should be expected generically for nonintegrable systems because transport is dominated by diffusion.

\section{Conclusions}

We have shown that in integrable systems the growth of the OSEE of some simple operators exhibits a logarithmic increase. Our work opens several research avenues. First, it would important to derive \textit{ab initio} the behavior in (3), at least in the rule 54 chain, for instance, by using the recent developments in Ref. 42 and 43. It is also important to understand the OSEE for more complicated operators and systems. Our data for non-integrable systems do not allow to reach a conclusion on the behavior of the OSEE in generic systems, although they are compatible with Ref. 38. It is of fundamental importance to clarify this issue. Finally, the argument leading to (3) gives that $S_{\text{max}}$ is the same for all the Rényi entropies $S^{(\alpha)}$. However, we numerically checked that although $S^{(\alpha)}$ exhibit logarithmic growth, the prefactor is smaller than $1/2$ and it depends on $\alpha$. It would be interesting to clarify this issue by studying the Rényi entropies. Finally, it would be interesting to clarify the relationship between OSEE and anomalous transport, for instance superdiffusion [36, 44, 45].

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\textbf{Appendix A: Spectral diagnostic for the non-integrable case}

Here we address the integrability of the hamiltonian (A1). We consider the general hamiltonian

\begin{equation}
H = H_{XXZ} + \sum_i \frac{J_i}{2} (S^+_i S^-_{i+1} + S^-_i S^+_i) + \Delta' \sum_i S^z_i S^z_{i+2},
\end{equation}

where $H_{XXZ}$ is the standard Heisenberg $XXZ$ hamiltonian

\begin{equation}
H_{XXZ} = \sum_i \frac{1}{2} (S^+_i S^-_{i+1} + S^-_i S^+_i) + \Delta \sum_i S^z_i S^z_{i+1}.
\end{equation}

For $J' = \Delta' = 0$ one recovers the $XXZ$ chain, which is integrable by the Bethe ansatz for any $\Delta$. To understand the effect of the integrability breaking terms we study the gaps $\delta_n$ between adjacent levels of the energy spectrum of (A1). Here we define $\delta_n$ as

\begin{equation}
\delta_n \equiv E_{n+1} - E_n,
\end{equation}

with $E_n$ energy levels. For chaotic systems the behavior of $\delta_n$ should be described by an appropriate random matrix ensemble, provided that the contribution of the density of states, which is model dependent, is removed. An alternative solution is to focus on the ratio between consecutive gaps $r_n$ as [46].

\begin{equation}
0 \leq r_n \equiv \min\{\delta_n, \delta_{n-1}\}/\max\{\delta_n, \delta_{n-1}\} \leq 1.
\end{equation}
For Poisson-distributed energy levels, i.e., for integrable systems, the average value of the ratio is $\langle r_n \rangle = 2\ln(2) - 1 \approx 0.386$. In the non-integrable case one should expect that energy levels are described by the Gaussian Orthogonal Ensemble (GOE). This gives $4 - 2\sqrt{3} \approx 0.535$. Our results are reported in Fig. 6. The data are obtained from exact diagonalisation of a chain with $L \leq 18$ sites. Periodic boundary conditions are used. In the Figure $N_\uparrow$ is the number of up spins, which fixes the magnetization sector. Most of the data are at half-filling $N_\uparrow = L/2$, although we consider also $N_\uparrow = L/2 - 1$. We denote with $p = \pm 1$ the eigenvalue of the parity under reflection with respect to the center of the chain. Here $p_z = \pm 1$ is the eigenvalue of the spin inversion operator. Empty symbols are for the integrable case, i.e., the XXZ chain with $\Delta = 0$ (cf. (A2)). The different symbols are for different symmetry sectors. In the legend we only report the quantum numbers that are fixed are reported in the legend. The full and dashed lines are the expected results for integrable and chaotic models.

FIG. 6. Spectral diagnostics of integrability-breaking. The figure shows the ratio of consecutive gaps $r_n$ (cf. (A3)) versus the system size $L$ for the Hamiltonian (A1). Here we focus on the case with $\Delta = 0.4$, and several $J'$ and $\Delta'$. The empty symbols are the data for the integrable case $J' = \Delta' = 0$. The different symbols correspond to different number of up spins (magnetization) $N_\uparrow$, spatial parity eigenvalue $p$, and spin inversion eigenvalue $p_z$. Only the quantum numbers that are fixed are reported in the legend. The full and dashed lines are the expected results for integrable and chaotic models.

For Poisson-distributed energy levels, i.e., for integrable systems, the average value of the ratio is $\langle r_n \rangle = 2\ln(2) - 1 \approx 0.386$. In the non-integrable case one should expect that energy levels are described by the Gaussian Orthogonal Ensemble (GOE). This gives $4 - 2\sqrt{3} \approx 0.535$. Our results are reported in Fig. 6. The data are obtained from exact diagonalisation of a chain with $L \leq 18$ sites. Periodic boundary conditions are used. In the Figure $N_\uparrow$ is the number of up spins, which fixes the magnetization sector. Most of the data are at half-filling $N_\uparrow = L/2$, although we consider also $N_\uparrow = L/2 - 1$. We denote with $p = \pm 1$ the eigenvalue of the parity under reflection with respect to the center of the chain. Here $p_z = \pm 1$ is the eigenvalue of the spin inversion operator. Empty symbols are for the integrable case, i.e., the XXZ chain with $\Delta = 0$ (cf. (A2)). The different symbols are for different symmetry sectors. In the legend we only report the quantum numbers that are fixed are reported in the legend. The full and dashed lines are the expected results for integrable and chaotic models.

This is different upon breaking integrability. The data are reported as full symbols in Fig. 6. First, one should stress that the Wigner-Dyson result $\langle r_n \rangle \approx 0.535$ is expected to hold in the limit $L \to \infty$ if one factors out all the conserved quantities. The down-triangle in the figure are the data for $J' = 0.1$ and $\Delta' = 0.04$. Clearly, finite-size corrections are present, although the data for the largest size $L = 18$ are converging to the expected result. The up triangles and the diamonds are the data for $J' = 0$ and $\Delta' = 0.1$ and $\Delta' = 0.2$, respectively. Upon increasing $\Delta'$, the data approach the Wigner-Dyson result faster, as expected. Still, in both cases there is reasonable agreement with the random matrix result for $L = 18$. However, we should remark that, although the analysis performed here suggests that for $\Delta' = 0.1, 0.2$ the Hamiltonian (A1) is not integrable, it does now give any information on the time-scale after which the effect of the integrability-breaking interactions start to appear.

Appendix B: Entanglement profiles

Here we discuss the behavior of the spatial profile of the $OSEE$ of the projector operator $P_z \equiv 1/2 - S^z$ in both integrable and non-integrable systems. The operator is inserted at the center of the chain. Our results are presented in Fig. 7. We consider the deformed XXZ chain hamiltonian in (A1). We fix $\Delta = 0.4$. In Fig. 7 (a) we focus on the integrable case $J' = 0$ and $\Delta' = 0$. The figure shows the $OSEE$ plotted as a function of $x/t$, with $x$ measured from the chain center and $t$ the time. (a) shows the integrable case, i.e., the XXZ chain with $\Delta = 0.4$. In (b) and (c) we consider the nonintegrable deformation of the XXZ.

FIG. 7. Profile of the operator entanglement. The results are for $P_z \equiv 1/2 - S^z$ inserted at the center of the chain. The operator entanglement is plotted as a function of the rescaled position $x/t$, with $x$ measured from the chain center and $t$ the time. (a) shows the integrable case, i.e., the XXZ chain with $\Delta = 0.4$. In (b) and (c) we consider the nonintegrable deformation of the XXZ.
Appendix C: Solitonic machines

The mapping between the computational basis and the soliton basis for the rule 54 chain (see Fig. 1) is reported in Fig. 8 in the framework of finite-state machines. The possible states of the machine are \( s = 0, 1, 2, 3 \). These are the states that are explored by a machine that scans a bit configuration site by site proceeding from left to right. The internal states of the machine are determined by the bit configurations on nearest-neighbour sites. The goal of the machine is to identify pairs of consecutive 11, which correspond to left/right movers, and the configuration 010, which corresponds to two scattering solitons. Let us assume that the machine is at site \( x \) and that \( s_x = s_{x-1} = 0 \). This defines the internal state 0 of the machine. State 1 means that \( s_x = 1 \) and \( s_{x-1} = 0 \). State 2 is defined by the condition that on \( x \) there is no solitons and \( s_{x-1} = 1 \). Finally state 3 means that on site \( x \) there is a pair of scattering solitons (vertical line). Note that the presence of state 2 imposes some kinematic constraint for the solitons, i.e., that a left and right mover has to be followed by at least two empty boxes. To illustrate the solitonic patterns that correspond to the identity, in Fig. 10 we report all the solitonic configurations that are allowed on three sites.

Now state 0 means that at site \( x \) there is no solitons and on \( x - 1 \) there were no free left and right movers (slanted lines). State 1 means that on \( x \) there is a left/right mover. State 2 is defined by the condition that on \( x \) there is no soliton and a left/right mover is present at \( x - 1 \). Finally, state 3 means that on site \( x \) there is a pair of scattering solitons (vertical line). Note that the presence of state 2 imposes some kinematic constraint for the solitons, i.e., that a left and right mover has to be followed by at least two empty boxes. To illustrate the solitonic patterns that correspond to the identity, in Fig. 10 we report all the solitonic configurations that are allowed on three sites.

FIG. 8. MPO representation of the mapping between computational basis and the soliton basis in the rule 54 chain. The diagram shows the finite-state machine encoding the mapping. The possible states of the machine are labeled as \( s = 0, 1, 2, 3 \). The arrows denote transitions between different states. (a-c) Tensors forming the MPO. The lower indices take values 0, 1, 2, 3. The upper index can be the empty box (no solitons), slanted lines denoting left and right movers, and the vertical line, which corresponds to a pair of scattering solitons. The presence of the left and right mover depends on the combined parity of spatial position and time. The virtual indices \( \beta, \beta' \) for which the tensor is nonzero are the states of the machine connected by the tensor.

FIG. 9. Soliton machine that generates the MPO representation of the identity operator (infinite-temperature state). In (a-c) we report the tensors forming the MPO representation. The virtual indices of the tensor have values in the space of the machine states \( \beta, \beta' = 0, 1, 2, 3 \).

FIG. 10. All the possible solitonic configurations on a system with \( L = 3 \) sites. The configurations are obtained by using the MPO representation of the identity in Fig. 9.

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