The Sample Complexity of Forecast Aggregation

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Abstract

We consider a Bayesian forecast aggregation model where \( n \) experts, after observing private signals about an unknown binary event, report their posterior beliefs about the event to a principal, who then aggregates the reports into a single prediction for the event. The signals of the experts and the outcome of the event follow a joint distribution that is unknown to the principal, but the principal has access to i.i.d. “samples” from the distribution, where each sample is a tuple of experts’ reports (not signals) and the realization of the event. Using these samples, the principal aims to find an \( \varepsilon \)-approximately optimal (Bayesian) aggregator. We study the sample complexity of this problem. We show that, for arbitrary discrete distributions, the number of samples must be at least \( \tilde{\Omega}(m^{n-2}/\varepsilon) \), where \( m \) is the size of each expert’s signal space. This sample complexity grows exponentially in the number of experts \( n \). But if experts’ signals are independent conditioned on the realization of the event, then the sample complexity is significantly reduced, to \( O(1/\varepsilon^2) \), which does not depend on \( n \).
1 Introduction

Suppose you want to know whether it will rain tomorrow. A Google search on “weather” returns 40% probability of raining for tomorrow. The weather forecasting app on your phone shows 85%. And one of your friends, who is an expert in meteorology, predicts 65%. How do you aggregate these different predictions into a single, accurate prediction? This problem is called forecast aggregation or forecast combination [1, 2, 3]. It has innumerable applications in fields ranging from economics, statistics, operations research, to machine learning, decision theory, and of course, climate science.

A straightforward solution to forecast aggregation is to simply take the (unweighted) average of all experts’ forecasts. Simple as it is, unweighted average performs surprisingly well in practice, as observed by many forecast aggregation works from 1960s to 2020s (e.g., [1, 4, 5, 2, 3, 7]). Naturally, when past data of expert forecasts and event outcomes (e.g., historical weather forecasts and outcomes) are available, one may hope to leverage on such data to learn more accurate aggregators. While adjusting weights of experts in averaging according to their individual historical accuracy has led to improved accuracy for the aggregated prediction [8], interestingly, more sophisticated data-driven methods like adjusting the weights using ordinary least squared regression [9] and covariance estimation [5, 8] were often outperformed by the unweighted average. The dominance of simple aggregators over more sophisticated data-driven methods is observed so often in empirical applications that it is termed “the forecast combination puzzle” [10, p.428].

There are many potential explanations for the forecast combination puzzle [11, 9, 12]. In some scenarios, the past events are different from the future events in fundamental ways (e.g., geopolitical forecasting) and hence past data may not be informative in predicting the future. Another widely accepted conjecture is that the amount of past data is not large enough for a data-intensive method to perform well. Indeed, the sample sizes in many empirical forecast aggregation works are small under today’s “big-data” standard (24 in [13], 69 in [14], 87 in [15], 100 in [6]). However, there are aggregation settings where it is reasonable to think that future events are similar to past events and we do have abundant data — for instance, the forecasting of weather, stock prices [16], and some forecasting competitions on Kaggle [1]. Such settings are well-suited for data-driven methods. In fact, when given enough data, neural network based nonlinear aggregators can perform better than traditional linear aggregators [16, 17, 18]. It’s hence tempting to ask how many data are needed for data-driven aggregators to achieve high accuracy in these settings.

In this paper, we initiate the study of sample complexity of forecast aggregation, building upon a standard Bayesian forecasting model [19, 20, 21]. In this model, the optimal aggregator (which is Bayesian) depends on an underlying unknown joint distribution of the experts’ reports and the event outcome, from which we obtain samples. We ask:

How many samples do we need to approximately learn the optimal Bayesian aggregator?

We show that the sample complexity of forecast aggregation can be extremely large in the worst case. But it is significantly lower for some special families of distributions. Our results provide partial explanation for the observed forecast combination puzzle.

Main results. In our model, there are $n$ experts and one principal. Each expert $i$ observes a private signal $s_i \in S_i$ about an unknown binary event (state of the world) $\omega \in \{0, 1\}$. Experts’ signals and the event jointly follow an underlying distribution $P$. Each expert reports their posterior belief $r_i = P(\omega = 1 | s_i)$ about $\omega$ to the principal, who then aggregates all the reports $r_1, \ldots, r_n$ into a single prediction for $\omega$. The principal does not know $P$ but has i.i.d. “samples” from the

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[1] https://www.kaggle.com/c/m5-forecasting-accuracy
distribution, where each sample is a tuple of experts’ reports (not signals) and the realization of the event. Using these samples, the principal aims to find an aggregator \( \hat{f} : [0, 1]^n \rightarrow [0, 1] \) that is “\( \varepsilon \)-close” to the optimal Bayesian aggregator \( f^*(r_1, \ldots, r_n) = P(\omega = 1 | r_1, \ldots, r_n) \), where “\( \varepsilon \)-closeness” is measured in the mean squared difference: \( E_P[|\hat{f}(r_1, \ldots, r_n) - f^*(r_1, \ldots, r_n)|^2] \leq \varepsilon \).

Assuming a finite signal space of size \(|S_i| = m\) for each expert, we show that:

- To find an aggregator \( \hat{f} \) that is \( \varepsilon \)-close to the optimal aggregator \( f^* \) with high probability for any distribution \( P \), at most \( \tilde{O}(m^n \varepsilon^2) \) samples are sufficient, and at least \( \tilde{\Omega}(m^{n-2}/\varepsilon) \) samples are necessary (Theorem 3.1).

The exponential sample complexity lower bound \( \tilde{\Omega}(m^{n-2}/\varepsilon) \) means that the optimal aggregator \( f^* \) is rather difficult to learn, and significantly more difficult when the number of experts increases. As we show in the proof (which is a reduction from the distribution learning problem), this difficulty stems from the fact that we need to estimate the correlation between experts, which requires a large number of samples. This motivates us to study the special case where (the signals of) experts are conditionally independent, namely, \( s_i \)'s are independent conditioned on \( \omega \). The sample complexity is significantly reduced in this case:

- If experts are conditionally independent, then an \( \varepsilon \)-optimal aggregator \( \hat{f} \) can be found with high probability using only \( \tilde{O}(1/\varepsilon^2) \) samples. \( \tilde{\Omega}(1/\varepsilon^2) \) samples are necessary. (Theorem 4.2)

The sample complexity upper bound \( \tilde{O}(1/\varepsilon^2) \) in the conditional independence case does not depend on the number of agents \( n \) and the signal space size \( m \). We then further investigate two special conditional independence cases where the experts are either “very informative” or “very non-informative”. In both cases the sample complexity can be \( \tilde{O}(1/\varepsilon^2) \) or even smaller (Theorem 4.4 and 4.5).

In the remainder of the paper, we first discuss related works in Section 1.1. Section 2 introduces our model and preliminaries. Section 3 studies the sample complexity for general distributions. Section 4 focuses on the conditional independence case. We conclude and discuss future directions in Section 5.

1.1 Related Works

**Data-driven aggregation** Data-driven approaches to forecast aggregation date back to perhaps the first paper on forecast aggregation (Bates and Granger, 1969, [1]) and have been standard in the literature (see, e.g., surveys [2, 3] and more recent works [22, 14, 23, 24, 25, 26]). Many of these works focus on specific weighted average aggregators like *linear pooling* (weighted arithmetic mean) [22] and *logarithmic pooling* (normalized weighted geometric mean) [14, 26], and the goal is to estimate the optimal weights from data, which can usually be solved by standard statistical tools. However, those weighted average aggregators are not necessarily the optimal *Bayesian* aggregator unless the underlying distribution \( P \) of signals and event satisfies some strict conditions (e.g., experts’ forecasts are equal to the true probability of the event plus normally distributed noises [20]). For some distributions the performance of the Bayesian aggregator can be much better than the optimal weighted average. Our work aims to understand the sample complexity of Bayesian aggregation, namely how many samples are needed to approximate the Bayesian aggregator.

Our work is closely related to a recent work [23] on Bayesian aggregation via online learning. For the case of conditionally independent experts, [23] shows that the Bayesian aggregator can be approximated with \( \varepsilon = \tilde{O}(n\sqrt{T}) \) regret. By online-to-batch conversion (e.g., [27], Corollary 5.2),

\[2\] The \( \tilde{O}(\cdot) \) and \( \tilde{\Omega}(\cdot) \) notations omit logarithmic factors.
this implies a $T = \tilde{O}(n^2)$ sample complexity upper bound for the batch learning setting. Our paper studies the batch learning setting. For the conditional independence case, we obtain an improved bound of $\tilde{O}(1/\epsilon^2)$. However, the loss function considered by [23] is different from ours: theirs is the logarithmic loss while ours is the squared loss. Since the logarithmic loss is larger than the squared loss, their sample complexity bound should be greater than or equal to our bound. We do not know whether there is a gap between the tight sample complexity bounds for the two loss functions.

**Robust forecast aggregation**  Some recent works study the Bayesian aggregation problem not from a data-driven perspective [28, 29, 30]. In particular, [29] and [30] study a “robust forecast aggregation” problem where the principal does not know and does not have samples from the prior distribution over experts’ forecasts and the event. They study the worst-case approximation ratio obtainable by a prior-free aggregator compared to the prior-aware Bayesian aggregator. They obtain several negative results. We are able to give positive results ($\epsilon$-approximation) because of two reasons: (1) we have sample access to the prior, and (2) the Bayesian aggregator benchmark they choose is in fact stronger than ours; see Section 2 for a detailed discussion.

**Sample complexity of mechanism design**  Our work may remind the reader of a long line of research on the sample complexity of mechanism design, in particular revenue maximization (e.g., 31, 32, 33, 34, 35, 36, 37, 38, 39, 40). A key observation from these works is that, the number of samples needed to learn an $\epsilon$-optimal auction for $n \geq 1$ independent bidders is increasing in $n$, because when there are more bidders, although we can obtain higher revenue, the optimal auction benchmark is also improved. We see a similar phenomenon that the sample complexity increases with the number of experts in the forecast aggregation problem in the general case, but not in the case where experts are conditionally independent.

## 2 Model and Preliminaries

We first introduce our model of forecast aggregation, the optimal aggregator, and the formal definition of the sample complexity problem in Section 2.1. Because our analysis of the sample complexity of forecast aggregation will leverage on results from distribution learning, we review the related problem and results in Section 2.2.

### 2.1 Model: Sample Complexity of Forecast Aggregation

**Forecast aggregation**  There are $n$ experts and one principal. The principal wants to predict the probability that a binary event $\omega \in \Omega = \{0, 1\}$ happens ($\omega = 1$), based on information provided by the experts. For example, $\omega$ may represent whether it will rain tomorrow. We also refer to $\omega$ as “the state of the world”. Each expert $i = 1, \ldots, n$ observes some private signal $s_i \in S_i$ that is correlated with $\omega$, where $S_i$ denotes the space of all possible signals of expert $i$. We assume for now that $S_i$ is finite, and without loss of generality that all $S_i$’s have the same size $|S_i| = m$. We relax this assumption in Section 3 where we consider conditionally independent signals. Let $S = S_1 \times \cdots \times S_n$ be the joint signal space of all experts; $|S| = m^n$. $P$ is the joint distribution of signals $s = (s_1, \ldots, s_n)$ and event $\omega$, namely, $P$ is a distribution over $S \times \Omega$. Since the space $S \times \Omega$ is discrete, we can use $P(\cdot)$ to denote the probability: $P(s, \omega) = \Pr_P[s, \omega]$. Signals of different experts can be correlated conditioning on $\omega$. We assume that each expert $i$ knows the marginal joint distribution of their own signal $s_i$ and $\omega$, $P(s_i, \omega)$. Neither any expert nor the principal knows the entire distribution $P$. Each expert $i$ reports to the principal a forecast (or prediction) $r_i$ for
the event \( \omega \), which is equal to the conditional probability of \( \omega = 1 \) given their signal \( s_i \):

\[
    r_i = P(\omega = 1 \mid s_i) = \frac{P(\omega = 1)P(s_i \mid \omega = 1)}{P(\omega = 1)P(s_i \mid \omega = 1) + P(\omega = 0)P(s_i \mid \omega = 0)}. \tag{1}
\]

We note that \( r_i \) is a function of \( s_i \) and \( P \), but not \( \omega \). Let \( r = (r_1, \ldots, r_n) \in [0, 1]^n \) denote all experts’ reports. We sometimes use \( r_{-i} \) to denote the reports of all experts except \( i \).

The principal aggregates the experts’ reports \( r \) into a single forecast \( f(r) \) using some aggregation function, or aggregator, \( f : [0, 1]^n \rightarrow [0, 1] \). We measure the performance of an aggregator by the mean squared loss:

\[
    L_P(f) = E_P[|f(r) - \omega|^2]. \tag{2}
\]

The notation \( E_P[\cdot] \) makes it explicit that the expectation is over the random draw of \((s, \omega) \sim P \) followed by letting \( r_i = P(\omega = 1 \mid s_i) \). We omit \( P \) and write \( E[] \) when it is clear from the context.

Let \( f^* \) denote the optimal aggregator with respect to \( P \), which minimizes \( L_P(f) \):

\[
    f^* = \arg \min_{f : [0, 1]^n \rightarrow [0, 1]} L_P(f) = \arg \min_{f : [0, 1]^n \rightarrow [0, 1]} E_P[|f(r) - \omega|^2]. \tag{3}
\]

We have the following characterization of \( f^* \) and \( L_P(f) \): \( f^* \) is equal to the Bayesian aggregator, which computes the posterior probability of \( \omega = 1 \) given all the reports \( r = (r_1, \ldots, r_n) \). And the difference between the loss of \( f \) and the loss of \( f^* \) is equal to their expected squared difference.

**Lemma 2.1.** The optimal aggregator \( f^* \) and any aggregator \( f \) satisfy:

- \( f^*(r) = P(\omega = 1 \mid r) \), for almost every \( r \).
- \( L_P(f) - L_P(f^*) = E_P[|f(r) - f^*(r)|^2] \).

The proof of this lemma is in Appendix B.

We say an aggregator \( f \) is \( \varepsilon \)-optimal (with respect to \( P \)) if \( L_P(f) \leq L_P(f^*) + \varepsilon \). By Lemma 2.1, this is equivalent to \( E_P[|f(r) - f^*(r)|^2] \leq \varepsilon \). For an \( \varepsilon \)-optimal \( f \), we also say it \( \varepsilon \)-approximates \( f^* \).

**Definition 2.2.** An aggregator \( f \) is \( \varepsilon \)-optimal (with respect to \( P \)) if

\[
    E_P[|f(r) - f^*(r)|^2] \leq \varepsilon.
\]

**Discussion of the benchmark** \( f^* \) Our benchmark aggregator, the Bayesian aggregator \( f^* \), is common in the forecast aggregation literature (e.g., [13, 42, 3]). It is stronger than the typical “best expert” benchmark in no-regret learning (e.g., [43, 25]), but weaker than the “omniscient” aggregator that has access to the experts’ signals: \( f_{\text{omni}}(s) = P(\omega = 1 \mid s) \). If there is a one-to-one mapping between signals \( s_i \) and reports \( r_i \), then there is no difference between \( f^* \) and \( f_{\text{omni}} \). Otherwise, \( f_{\text{omni}} \) could be much stronger than \( f^* \) and an \( \varepsilon \)-approximation to \( f_{\text{omni}} \) using experts’ reports only is not always achievable, as noted by [29, 23, 30]. In contrast, an \( \varepsilon \)-approximation to \( f^* \) is always achievable (in fact, achieved by \( f^* \) itself). The difference between \( f^* \) and \( f_{\text{omni}} \) is known as the difference between “aggregating forecasts” and “aggregating information sets” in the forecast aggregation literature [13, p.198-199], [42, p.168-169], [3, p.143].

\(^4\) One may wonder whether the experts are willing to report \( r_i = P(\omega = 1 \mid s_i) \) truthfully. This can be guaranteed by a proper scoring rule. For example, we can reward each expert the Brier score \( C - \varepsilon |r - \omega|^2 \) after seeing the realization of \( \omega \). Each expert maximizes its expected score \( C - \varepsilon |r - \omega|^2 \) by reporting its belief of \( \omega \) truthfully.

\(^5\) These works give an XOR example where \( \omega = s_1 \oplus s_2 \), \( s_1 \) and \( s_2 \) are i.i.d. uniform \([0, 1]\) distributed. In this example, experts always report \( r_1 = 0.5, f_{\text{omni}}(s_1, s_2) = \omega, f^*(r_1, r_2) = 0.5, \) so \( L_P(f_{\text{omni}}) = 0 \) but \( L_P(f^*) = 0.25 > 0 \). No aggregator \( f \) that depends on experts’ reports only can do better than \( f^* \).
**Sample complexity** The principal has access to $T$ i.i.d. samples of forecasts and event realizations drawn from the underlying unknown distribution $P$:

$$S_T = \{(r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)})\}, \quad (s^{(t)}, \omega^{(t)}) \sim P, \quad r^{(t)}_i = P(\omega = 1 | s^{(t)}_i).$$

Here, we implicitly regard $P$ as a distribution over $(r, \omega)$ instead of $(s, \omega)$. The principal uses samples $S_T$ to learn an aggregator $\hat{f} = \hat{f}_{S_T}$, in order to approximate $f^*$. So, our sample complexity question becomes:

*How many samples are necessary and sufficient for finding an $\epsilon$-optimal aggregator $\hat{f}$ (with probability at least $1 - \delta$ over the random draw of samples)?*

The answer to the above question depends on the family of distributions we are interested in. Let $\mathcal{P}$ denote a family of distributions over $\mathcal{S} \times \Omega$. It could be the set of all distributions over $\mathcal{S} \times \Omega$, or in Section 4 we will only consider distributions where the signals are independent conditioned on $\omega$. We define the sample complexity of forecast aggregation, with respect to $\mathcal{P}$, formally:

**Definition 2.3.** The sample complexity of forecast aggregation (with respect to $\mathcal{P}$) is the minimum function $T_{\mathcal{P}}(\cdot, \cdot)$ of $\varepsilon \in (0, 1), \delta \in (0, 1)$, such that: if $T \geq T_{\mathcal{P}}(\varepsilon, \delta)$, then for any distribution $P \in \mathcal{P}$, with probability at least $1 - \delta$ over the random draw of $T$ samples $S_T$ from $P$ (and over the randomness of the learning procedure if it is randomized), we can obtain an aggregator $\hat{f} = \hat{f}_{S_T}$ such that $\mathbb{E}[|\hat{f}(r) - f^*(r)|^2] \leq \varepsilon$.

The principal is assumed to know the family of distributions $\mathcal{P}$ but does not know which distribution $P \in \mathcal{P}$ it is facing. There should be at least two different distributions in $\mathcal{P}$. Otherwise, the principal knows what the distribution is and there is no need for learning.

We do not consider computational complexity in Definition 2.3, namely, the principal has infinite power to compute $\hat{f}$. Yet, in all our upper bound results the learning algorithm is efficient in the number of samples. For lower bounds, this assumption only makes our results stronger.

### 2.2 Preliminaries: Distribution Learning in Total Variation Distance

Our analysis of the sample complexity of forecast aggregation, for both the upper bound and the lower bound, will use the sample complexity of another learning problem: *learning discrete distributions in total variation distance*. We review this problem in this subsection.

**Definition 2.4.** Let $D_1, D_2$ be two distributions on a discrete space $\mathcal{X}$. Denote by $D_i(x) = \Pr_{D_i}[x]$ the probability of $x \in \mathcal{X}$ in $D_i$. The total variation distance between $D_1$ and $D_2$ is $d_{TV}(D_1, D_2) = \frac{1}{2} \sum_{x \in \mathcal{X}} |D_1(x) - D_2(x)|$.

Let $\mathcal{D}$ be a family of distributions over $\mathcal{X}$. We define the sample complexity of learning distributions in $\mathcal{D}$ within total variation distance $\varepsilon$ to be the minimum function $T_{\mathcal{D}}^{TV}(\varepsilon, \delta)$, such that: if $T \geq T_{\mathcal{D}}^{TV}(\varepsilon, \delta)$, then for any distribution $D \in \mathcal{D}$, with probability at least $1 - \delta$ over the random draw of $T$ samples from $D$, we can obtain (from the $T$ samples) a distribution $\hat{D}$ such that $d_{TV}(\hat{D}, D) \leq \varepsilon$.

**Proposition 2.5** (e.g., [44, 45]). Let $\mathcal{D}$ be the set of all distributions over $\mathcal{X}$. Then, $T_{\mathcal{D}}^{TV}(\varepsilon, \delta) = \Theta\left(\frac{|\mathcal{X}| + \log(1/\delta)}{\varepsilon^2}\right)$. In particular, the upper bound can be achieved by using the empirical estimate $\hat{D}_{\text{emp}}$ (which is the uniform distribution over the $T$ samples). The lower bound holds regardless of what learning algorithm is used.

Total variation distance has a nice well-known property that we will use to show a sample complexity upper bound for our forecast aggregation problem:

**Fact 2.6.** For any function $f : \mathcal{X} \to [0, 1]$, $|\mathbb{E}_{x \sim D} f(x) - \mathbb{E}_{x \sim \hat{D}} f(x)| \leq d_{TV}(\hat{D}, D)$. 

6
3 General Distributions

In this section we characterize the sample complexity of forecast aggregation for general distributions $P$. We give an exponential (in the number of experts, $n$) upper bound and an exponential lower bound on the sample complexity, as follows:

**Theorem 3.1.** Let $P$ be the set of all distributions over $\mathcal{S} \times \Omega$, with $|\mathcal{S}| = m^n$. Suppose $n \geq 2$. The sample complexity of forecast aggregation with respect to $P$ is

$$O\left(\frac{m^n + \log(1/\delta)}{\varepsilon^2} \right) \geq T_P(\varepsilon, \delta) \geq \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon} \right).$$

We make two remarks. First, this theorem is for $n \geq 2$. When there is only one expert ($n = 1$), there is no need for aggregation, because the optimal “aggregator” $f^*$ simply outputs the forecast given by the only expert: $f^*(r_1) = P(\omega = 1 \mid r_1) = P(\omega = 1 \mid s_1) = r_1$. No learning is needed in this case, so the sample complexity is 0. Second, the upper bound and the lower bound given above have a multiplicative gap of $\frac{m^2}{\varepsilon}$, ignoring logarithmic terms. We do not know what the tight bound is; we conjecture it is $\Theta\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon}\right)$.

3.1 Proof of the Upper Bound

In this subsection we prove the $O\left(\frac{m^n + \log(1/\delta)}{\varepsilon^2}\right)$ upper bound in Theorem 3.1. This is a direct corollary of the distribution learning result introduced in Section 2.2.

We regard $P$ as a distribution over $r$ and $\omega$ instead of over $s$ and $\omega$. Since each possible report $r_i \in [0, 1]$ corresponds to some signal $s_i$ in $\mathcal{S}_i$, $P$ is a discrete distribution with support size at most $2m^n$. Let $\hat{P}_{\text{emp}}$ be the empirical distributions of reports and event realizations: $\hat{P}_{\text{emp}} = \text{Uniform}\{(r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)})\}$. By Proposition 2.5, with probability at least 1 $- \delta$ over the random draw of $T = O\left(\frac{2m^n + \log(1/\delta)}{\varepsilon^2}\right)$ samples, we have $d_{TV}(\hat{P}_{\text{emp}}, P) \leq \varepsilon$. According to Fact 2.6, and by the definition of $L_P(f)$, we have: for any aggregator $f : [0, 1]^n \rightarrow [0, 1]$,

$$|L_{\hat{P}_{\text{emp}}}(f) - L_P(f)| = \left|\mathbb{E}_{\hat{P}_{\text{emp}}} [f(r) - \omega] - \mathbb{E}_P [f(r) - \omega] \right| \leq d_{TV}(\hat{P}_{\text{emp}}, P) \leq \varepsilon.$$

Therefore, if we pick the empirically optimal aggregator $\hat{f}_{\text{emp}} = \arg\min_f L_{\hat{P}_{\text{emp}}}(f)$, we get

$$L_P(\hat{f}_{\text{emp}}) \leq L_{\hat{P}_{\text{emp}}}(\hat{f}_{\text{emp}}) + \varepsilon \leq L_{\hat{P}_{\text{emp}}}(f^*) + \varepsilon \leq L_P(f^*) + 2\varepsilon,$$

which means that $\hat{f}_{\text{emp}}$ is a $2\varepsilon$-optimal aggregator for $P$.

3.2 Proof (Sketch) of the Lower Bound

In this subsection we provide a proof sketch for the $\Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon}\right)$ lower bound in Theorem 3.1. The full proof is in Appendix C.

The main idea is a reduction from the distribution learning problem (defined in Section 2.2). Let $\mathcal{D} = \{D\}$ be a family of distributions over the joint signal space $\mathcal{S}$ to be learned in the distribution learning problem. We construct a corresponding family of distributions $P = \{P\}$ for the forecast aggregation problem, such that, if we can obtain an $\varepsilon$-optimal aggregator $\hat{f}$ for $P$, then we can convert $\hat{f}$ into a distribution $\hat{D}$ such that $d_{TV}(\hat{D}, D) \leq O(\sqrt{\varepsilon})$. Then, a sample complexity lower bound for the distribution learning problem for $\mathcal{D}$ gives a lower bound for the forecast aggregation problem for $\mathcal{P}$, which will be $\Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon^2}\right)$. 
How do we construct $P$ from $D$? We let $P$ be the following: first, $\omega$ follows Uniform\{0,1\}, namely, $P(\omega = 0) = P(\omega = 1) = \frac{1}{2}$; if $\omega = 0$, then the conditional distribution of signals is the uniform distribution over the entire signal space $S$, so $P(s \mid \omega = 0) = \frac{1}{m^n}$; if $\omega = 1$, then the conditional distribution of signals is equal to $D$, so $P(s \mid \omega = 1) = D(s)$. By Bayes’ rule, we have

$$P(\omega = 1 \mid s) = \frac{\frac{1}{2}P(s \mid \omega = 1)}{\frac{1}{2}P(s \mid \omega = 0) + \frac{1}{2}P(s \mid \omega = 1)} = \frac{D(s)}{\frac{1}{m^n} + D(s)} \approx O(m^n D(s))$$

(5)

if we choose a family of distributions $\mathcal{D} = \{D\}$ that are close to the uniform distribution over $S$, so that $D(s) \approx \frac{1}{m^n}$.

Then, we consider the Bayesian optimal aggregator $f^*(r) = P(\omega = 1 \mid r)$. Suppose there is a one-to-one mapping between signals $s$ and reports $r$, so $P(\omega = 1 \mid r)$ is the same as $P(\omega = 1 \mid s)$. Then,

$$f^*(r) = P(\omega = 1 \mid r) = P(\omega = 1 \mid s) \overset{6}{\approx} O(m^n D(s)) \implies D(s) \approx O\left(\frac{f^*(r)}{m^n}\right).$$

(6)

In words, $D(s)$ can be approximated by $f^*(r)$ divided by $m^n$, where $r$ is the report corresponding to $s$. Now, suppose we have obtained an $\varepsilon$-optimal aggregator $\hat{f}$, $\mathbb{E}[|\hat{f}(r) - f^*(r)|^2] \leq \varepsilon$. Using the equation $\hat{f}(r) = \frac{\hat{D}(s)}{1/m^n + \hat{D}(s)}$ we can compute $\hat{D}(s)$ such that:

$$\mathbb{E}\left[|\hat{D}(s) - D(s)|^2\right] \approx O\left(\mathbb{E}\left[\left|\frac{\hat{f}(r)}{m^n} - f^*(r)\right|^2\right]\right) = \frac{1}{(m^n)^2}O\left(\mathbb{E}\left[|\hat{f}(r) - f^*(r)|^2\right]\right) \leq \frac{1}{(m^n)^2}O(\varepsilon).$$

(7)

The left hand side is related to the total variation distance $d_{TV}(\hat{D}, D) = \frac{1}{2} \sum_{s \in S} |\hat{D}(s) - D(s)|$ as follows:

$$\mathbb{E}\left[|\hat{D}(s) - D(s)|^2\right] \geq \left(\mathbb{E}\left[|\hat{D}(s) - D(s)|\right]\right)^2 = \left(\sum_{s \in S} \frac{\Pr[D \mid s]}{D} \cdot |\hat{D}(s) - D(s)|\right)^2 \approx \left(\sum_{s \in S} \frac{1}{m^n} \cdot |\hat{D}(s) - D(s)|\right)^2 = \frac{1}{(m^n)^2}O(d_{TV}(\hat{D}, D))^2$$

(8)

(recall that we choose $D$ to be close to the uniform distribution, so $\Pr[D] \approx \frac{1}{m^n}$). Equations (7) and (8) give $d_{TV}(\hat{D}, D) \leq O(\sqrt{\varepsilon})$, which is what we want.

The remaining part of the proof is to find a family of distributions $\mathcal{D} = \{D\}$ that require many samples to learn and satisfy three conditions: (1) each $D$ is close to the uniform distribution over $S$; (2) for each $D$, the corresponding $P$ admits a one-to-one mapping between signals and reports; (3) given samples from $D$, which are signals $s^{(t)}$, we can compute the corresponding reports $r^{(t)}$ as the samples for our forecast aggregation problem, without knowing what $D$ is. To do (3) we will construct a family of distributions that have the same marginals $D(s_i) = D'(s_i), \forall i, \forall D, D' \in \mathcal{D}$, but jointly they are different, $D(s) \neq D'(s)$. Under these three conditions, we find a family $\mathcal{D}$ that requires $\Omega\left(\frac{m^n - 2 + \log(1/\delta)}{\varepsilon^2}\right)$ number of samples to learn (within total variation distance $\varepsilon$). Because we are restricting to a smaller set of distributions than the entire set of distributions over $S$, this number is smaller than the lower bound in Proposition 2.3 which is $\Omega(\frac{|S| + \log(1/\delta)}{\varepsilon^2})$.

4 Conditionally Independent Distributions

We have shown in Section 3 that, in order to learn approximately optimal aggregators for all (discrete) distributions, an exponential number of samples are needed. As shown in the proof, this
difficulty stems from the fact that the experts’ signals can be arbitrarily correlated conditioned on the event \( \omega \). Accurate estimation of such correlation requires many samples. In this section, we restrict attentions to the case where the experts’ signals are conditionally independent. It turns out that an \( \varepsilon \)-optimal aggregator can be learned using only \( O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}) \) samples in this case, which does not depend on \( n \). The assumption of discrete signal space can be relaxed here. We then further investigate two special and interesting families of conditionally independent distributions where the signals of experts are either “very informative” or “very non-informative”, both of which admit an even smaller sample complexity of \( O(\frac{1}{\varepsilon} \log \frac{1}{\delta}) \).

Let \( P \) be a conditionally independent distributions over \( \mathcal{S} \times \Omega \), namely, \( P(s \mid \omega) = \prod_{i=1}^{n} P(s_i \mid \omega) \) for all \( s \in \mathcal{S} \), for \( \omega \in \{0, 1\} \). Here, \( \mathcal{S}_i \) can be a continuous space, in which case, \( P(\cdot \mid \omega) \) represents a density function. We introduce some additional notations. Let \( p = P(\omega = 1) \) be the prior probability of \( \omega = 1 \). For technical convenience we assume \( p \in (0, 1) \). Define

\[
\rho = \frac{p}{1-p} = \frac{P(\omega = 1)}{P(\omega = 0)} \in (0, +\infty).
\]

We will be working with ratios like “\( \frac{r_i}{1-r_i} \)” and “\( \frac{p}{1-p} \)” a lot in this section. We will use the following characterization of the optimal aggregator \( f^* \) in the conditional independence case:

**Lemma 4.1** (e.g., \([46, 23]\)). Let \( P \) be a conditionally independent distribution. Given signals \( s = (s_1, \ldots, s_n) \) or the corresponding reports \( r = (r_1, \ldots, r_n) \), the posterior probability of \( \omega = 1 \) is:

\[
f^*(r) = P(\omega = 1 \mid r) = P(\omega = 1 \mid s) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{1-r_i}{r_i}}.
\]

(Define \( f^*(r) = 0 \) when \( \rho^{n-1} \prod_{i=1}^{n} \frac{1-r_i}{r_i} = +\infty \).)

We provide a proof of this lemma in Appendix D for completeness.

### 4.1 General Conditionally Independent Distributions

In this subsection, we give an \( O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}) \) upper bound and an \( \Omega(\frac{1}{\varepsilon} \log \frac{1}{\delta}) \) lower bound on the sample complexity of forecast aggregation with respect to all conditionally independent distributions.

**Theorem 4.2.** Let \( \mathcal{P}_{\text{all-ind}} \) be the set of all conditionally independent distributions over \( \mathcal{S} \times \Omega \). The sample complexity of forecast aggregation with respect to \( \mathcal{P}_{\text{all-ind}} \) is

\[
O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right) \geq T_{\mathcal{P}_{\text{all-ind}}} (\varepsilon, \delta) \geq \Omega\left(\frac{1}{\varepsilon} \log \frac{1}{\delta}\right).
\]

We provide the main ideas of the proof of Theorem 4.2 here. To prove the \( O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}) \) upper bound, we note that, according to Lemma 4.1 the optimal aggregator has the form \( f^*(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{1-r_i}{r_i}} \). We consider the class of aggregators \( \mathcal{F} = \{ f_\sigma : f_\sigma(r) = \frac{1}{1 + \sigma^{n-1} \prod_{i=1}^{n} \frac{1-r_i}{r_i}} \} \) parameterized by \( \sigma \in (0, +\infty) \). We show that the class of loss functions \( \mathcal{G} = \{ g_\sigma : g_\sigma(r, \omega) = |f_\sigma(r) - \omega|^2 \} \) associated with \( \mathcal{F} \) has pseudo-dimension \( \text{Pdim}(\mathcal{G}) = O(1) \). By the known result (e.g., \([47]\)) that the pseudo-dimension gives a sample complexity upper bound on the uniform convergence of a class of functions, we have that the empirically optimal aggregator in \( \mathcal{F} \) must be \( O(\varepsilon) \)-optimal on the true distribution (with probability at least \( 1 - \delta \)), given \( O\left(\frac{1}{\varepsilon} \left(\text{Pdim}(\mathcal{G}) \log \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right)\right) = O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right) \) samples. See details in Appendix D.2.

To prove the \( \Omega(\frac{1}{\varepsilon} \log \frac{1}{\delta}) \) lower bound, we use a reduction from the distinguishing distributions problem (defined in Appendix A). We construct two conditionally independent distributions \( P^1, P^2 \)
over the space $\Omega \times S$ that differ by $d_H^2(P^1, P^2) = O(\varepsilon)$ in squared Hellinger distance. Specifically, $P^1$ has prior $P^1(\omega = 1) = 0.5 - O(\frac{1}{n}) + O(\frac{\sqrt{2}}{n})$ and $P^2$ has prior $P^2(\omega = 1) = 0.5 - O(\frac{1}{n}) - O(\frac{\sqrt{2}}{n})$; the conditional distributions of each signal, $P^1(s_i | \omega)$ and $P^2(s_i | \omega)$, differ by $O(\frac{\sqrt{2}}{n})$ in squared Hellinger distance; taking the product of $n$ signals, $P^1(s | \omega)$ and $P^2(s | \omega)$ differ by $O(\varepsilon)$. The distinguishing distributions problem asks the following: given $T$ samples from either $P^1$ or $P^2$, we want to tell which distribution the samples are from. We show that, if we can solve the forecast aggregation problem, namely, $\varepsilon$-approximating $f^*(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{r_i}{1 - r_i}}$, then we can estimate $\rho$ with accuracy $O(\frac{\sqrt{2}}{n})$, and hence distinguish $P^1$ and $P^2$. But it is known that distinguishing $P^1$ and $P^2$ requires at least $\Omega(\frac{1}{\varepsilon} \log \frac{1}{\delta})$ samples. This gives the lower bound we claimed. See details in Appendix [D.3].

4.2 Strongly and Weakly Informative Experts

In this subsection we investigate two special families of conditionally independent distributions that admit $O(\frac{\varepsilon}{n} \log \frac{1}{\delta})$ sample complexity, which is smaller than the $O(\frac{\sqrt{2}}{\varepsilon} \log \frac{1}{\delta})$ bound for general conditionally independent distributions. In these two cases, the signals of experts are either “very informative” or “very non-informative”.

**Definition 4.3.** Let $\gamma \in [0, \infty]$ be a parameter. For an expert $i$, we say its signal $s_i \in S_i$ is

- $\gamma$-strongly informative if either $\frac{P(s_i | \omega = 1)}{P(s_i | \omega = 0)} \geq 1 + \gamma$ or $\frac{P(s_i | \omega = 1)}{P(s_i | \omega = 0)} \leq \frac{1}{1 + \gamma}$ holds.

- $\gamma$-weakly informative if $\frac{1}{1 + \gamma} \leq \frac{P(s_i | \omega = 1)}{P(s_i | \omega = 0)} \leq 1 + \gamma$.

An expert $i$ is $\gamma$-strongly (or $\gamma$-weakly) informative if all of its signals in $S_i$ are $\gamma$-strongly (or $\gamma$-weakly) informative.

A signal $s_i$ being $\gamma$-strongly (or $\gamma$-weakly) informative implies that its corresponding report $r_i$ will be “$\gamma$-away from” (or “$\gamma$-close to”) the prior $p = P(\omega = 1)$, in the “$\frac{r_i}{1 - r_i}$ and $\frac{p}{1 - p}$” ratio form. Specifically, if $s_i$ is $\gamma$-strongly informative, then from Equation (1) we have

$$\frac{r_i}{1 - r_i} = \frac{P(s_i | \omega = 1)}{P(s_i | \omega = 0)} \frac{p}{1 - p} \geq (1 + \gamma)\rho \quad \text{or} \quad \leq \frac{1}{1 + \gamma}\rho$$

(11)

where $\rho = \frac{p}{1 - p}$. As $\gamma$ gets larger, a $\gamma$-strongly informative signal (expert) is more informative in terms of predicting whether $\omega = 1$ or 0. This would make aggregation easier. If $s_i$ is $\gamma$-weakly informative, then:

$$\frac{1}{1 + \gamma}\rho \leq \frac{r_i}{1 - r_i} \leq (1 + \gamma)\rho.$$ 

(12)

As $\gamma$ gets smaller, a $\gamma$-weakly informative signal (expert) is less informative in predicting $\omega$, but in this case their report $r_i$ will be close to the prior $p$, which allows us to estimate the $\rho^{n-1}$ term in the optimal aggregator $f^*(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{r_i}{1 - r_i}}$ better. Those are some intuitions why both strongly and weakly informative signals can reduce the sample complexity of aggregation.

Formally, for $\gamma$-strongly informative experts with not-too-small $\gamma$, we have the following result:

**Theorem 4.4.** If $n \geq 32 \log \frac{2}{\varepsilon}$ and all experts are $\gamma$-strongly informative with $\frac{\gamma}{1 + \gamma} \geq 8\sqrt{\frac{2}{n} \log \frac{2}{\varepsilon}}$, then the sample complexity of forecast aggregation is at most $O(\frac{1}{\varepsilon n(1 + \gamma)^2} \log \frac{n}{\delta} + \frac{1}{\varepsilon} \log \frac{1}{\delta}) = O(\frac{1}{\varepsilon} \log \frac{1}{\delta})$.

\footnote{We note that an expert can be neither $\gamma$-strongly informative nor $\gamma$-weakly informative for any $\gamma$.}
We remark that the conditions of the theorem, \( n \geq 32 \log \frac{2}{\varepsilon} \) and \( \frac{n}{1+\gamma} \geq 8 \sqrt{n} \log \frac{2}{\varepsilon} \), are easier to be satisfied when the number of experts \( n \) increases, if the informativeness parameter \( \gamma \) of each expert is a constant or does not decrease with \( n \) (which we believe is a reasonable assumption given that experts are independent of each other). Also, if \( \gamma \) is fixed or increasing, then as \( n \) increases the sample complexity decreases.

The proof of Theorem 4.3 is in Appendix D.4. Roughly speaking, we divide each expert’s signal into two sets, \( S^1_i \) and \( S^0_i \): signals that are more likely to occur under \( \omega = 1 \) (i.e., \( \frac{P(s_i|\omega=1)}{P(s_i|\omega=0)} \geq 1+\gamma \)) and under \( \omega = 0 \) (i.e., \( \frac{P(s_i|\omega=1)}{P(s_i|\omega=0)} \leq \frac{1}{1+\gamma} \)). If the realized \( \omega \) is 1, then one may expect to see \( \Omega(\frac{\gamma}{1+\gamma} n) \) more \( S^1_i \) signals than the \( S^0_i \) signals from the \( n \) experts, because the probabilities of these two types of signals differ by \( \Omega(\frac{\gamma}{1+\gamma}) \) for each expert. If \( \omega \) is 0 then one may expect to see more \( S^0_i \) signals. So, by checking which type of signals are more we can tell whether \( \omega = 0 \) or 1. To tell whether a signal belongs to \( S^1_i \) or \( S^0_i \), we compare the corresponding report \( \frac{r_i}{1-\gamma_i} \) with \( \rho \) (namely, \( \frac{r_i}{1-\gamma_i} \geq (1+\gamma)\rho \iff s_i \in S^1_i \) and \( \frac{r_i}{1-\gamma_i} \leq \frac{1}{1+\gamma} \rho \iff s_i \in S^0_i \)), where \( \rho \) is estimated with accuracy \( \frac{1}{1+\gamma} \) using \( O(\frac{1}{\rho^3} \log \frac{1}{\varepsilon}) \) samples.

For \( \gamma \)-weakly informative experts with small \( \gamma \), we have another result:

**Theorem 4.5.** If all experts are \( \gamma \)-weakly informative with \( \gamma \leq 1 \), then the sample complexity of forecast aggregation is at most \( \min \{ O(\frac{\gamma}{1+\gamma} n \log \frac{1}{\varepsilon}), O(\frac{\gamma}{1+\gamma} \log \frac{1}{\varepsilon}) \} \). In particular, if \( \gamma = O(\frac{n}{\varepsilon}) \) then we have \( O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) \).

The \( O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) \) term in the sample complexity follows from the result for general conditionally independent distributions (Theorem 4.1). The \( O(\frac{\gamma n}{1+\gamma} \log \frac{1}{\varepsilon}) \) term is proved in Appendix D.5. We give the rough idea here using \( \gamma = \frac{1}{n} \) as an example. The proof relies on the observation that \( \mathbb{E} \left[ \prod_{i=1}^{n} \frac{r_i}{1-\gamma_i} \mid \omega = 0 \right] = \rho^n \). Since experts are weakly informative, each of their reports \( \frac{r_i}{1-\gamma_i} \) is around \( \rho \) in the range \( \left[ \frac{1}{1+\gamma} \rho, (1+\gamma)\rho \right] \subseteq \left[ \frac{1}{1+\gamma} \rho, (1+\frac{1}{n})\rho \right] \). Taking the product, we have \( \prod_{i=1}^{n} \frac{r_i}{1-\gamma_i} \in \left[ \frac{\rho^n}{e^\gamma}, eho^n \right] \), which is in a bounded range. This allows us to use Chernoff bound to argue that the \( \rho^n \) (or \( \rho^{n-1} \)) term in the optimal aggregator \( f^*(r) = \frac{1}{1+\rho^{n-1}} \prod_{i=1}^{n} \frac{r_i}{1-\gamma_i} \) can be estimated, with \( O(\sqrt{\varepsilon}) \) accuracy, using only \( O(\frac{e^\gamma}{\varepsilon} \log \frac{1}{\varepsilon}) = O(\frac{\gamma}{\varepsilon} \log \frac{1}{\varepsilon}) \) samples of \( \prod_{i=1}^{n} \frac{r_i}{1-\gamma_i} \). The aggregator using the estimate \( \hat{f} = \frac{1}{1+\hat{\rho}^{n-1}} \prod_{i=1}^{n} \frac{r_i}{1-\gamma_i} \), turns out to be \( O(\varepsilon) \)-optimal.

## 5 Conclusions and Future Directions

In this work, we show an exponential gap between the sample complexity of the forecast aggregation problem for general distributions, \( \tilde{O}(\frac{m^{n-2}}{\varepsilon}) \), and conditionally independent distributions, \( \tilde{O}(\frac{1}{\varepsilon}) \). This gap is due to the need of estimating the (conditional) correlation between experts in the general case, which is not needed when experts are conditionally independent. Notably, the bound \( \tilde{O}(\frac{1}{\varepsilon}) \) for conditionally independent distributions does not depend on the number of experts.

One interesting future direction is to find the tight sample complexity for the conditional independence case, in particular, to determine its dependence on \( \varepsilon \). One possible approach is to explore the case where an expert has a mixture of strongly and weakly informative signals, and the case where some experts are strongly informative and some are weakly informative. Other interesting directions include, for example, to explore the middle ground between fully correlated experts and conditionally independent experts (e.g., the partial evidence model in [23]), and to find a meaningful weaker benchmark than the Bayesian aggregator that can always be approximated with a small sample complexity.
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A Additional Definitions and Lemmas

A.1 Hellinger Distance

In the proof of sample complexity lower bound results we will use another notion of distance between distributions: the Hellinger distance:

Definition A.1. The (squared) Hellinger distance between two distributions $D_1$ and $D_2$ on a discrete space $\mathcal{X}$ is

$$d_H^2(D_1, D_2) = \frac{1}{2} \sum_{x \in \mathcal{X}} \left( \sqrt{D_1(x)} - \sqrt{D_2(x)} \right)^2 = 1 - \sum_{x \in \mathcal{X}} \sqrt{D_1(x)D_2(x)}.$$  

It is known that the Hellinger distance gives upper bounds on the total variation distance:

Fact A.2 (e.g., Lemma 2.3 in [48]).

- $d_{TV}(D_1, D_2) \leq \sqrt{2}d_H(D_1, D_2)$.

- More strongly, $1 - d_{TV}^2(D_1, D_2) \geq (1 - d_H^2(D_1, D_2))^2$.

We will use the following lemma to bound the squared Hellinger distance between two distributions that are close to each other:

Lemma A.3. Let $D_1$ and $D_2$ be two distributions on $\mathcal{X}$ such that, for all $x \in \mathcal{X}$, $1 - \varepsilon \leq \frac{D_2(x)}{D_1(x)} \leq 1 + \varepsilon$. Then, $d_H^2(D_1, D_2) \leq \frac{1}{2}\varepsilon^2$.

Proof. By definition,

$$d_H^2(D_1, D_2) = \frac{1}{2} \sum_{x \in \mathcal{X}} \left( \sqrt{D_1(x)} - \sqrt{D_2(x)} \right)^2 = \frac{1}{2} \sum_{x \in \mathcal{X}} D_1(x) \left( 1 - \sqrt{\frac{D_2(x)}{D_1(x)}} \right)^2.$$ 

If $1 - \varepsilon \leq \frac{D_2(x)}{D_1(x)} < 1$, then we have $(1 - \sqrt{\frac{D_2(x)}{D_1(x)}})^2 \leq (1 - \sqrt{1 - \varepsilon})^2 \leq (1 - (1 - \varepsilon))^2 = \varepsilon^2$. If $1 + \varepsilon \geq \frac{D_2(x)}{D_1(x)} > 1$, then we have $(1 - \sqrt{\frac{D_2(x)}{D_1(x)}})^2 \leq (\sqrt{1 + \varepsilon} - 1)^2 \leq ((1 + \varepsilon) - 1)^2 = \varepsilon^2$. These two cases together imply

$$d_H^2(D_1, D_2) \leq \frac{1}{2} \sum_{x \in \mathcal{X}} D_1(x) \cdot \varepsilon^2 = \frac{1}{2}\varepsilon^2. \quad \Box$$

We will use a property of the Hellinger distance between distributions defined on a product space. Suppose $D_1$ and $D_2$ are distributions over a product space $\mathcal{X} \times \mathcal{Y}$, so they can be decomposed into the marginal distribution of $x \in \mathcal{X}$ and the conditional distribution of $y \in \mathcal{Y}$ given $x$, namely, $D_1(x, y) = D_{1,x}(x) \cdot D_{1,y|x}(y|x)$ and $D_2(x, y) = D_{2,x}(x) \cdot D_{2,y|x}(y|x)$. Then, the squared Hellinger distance between $D_1$ and $D_2$ satisfies the following:

Lemma A.4.

$$d_H^2(D_1, D_2) \leq d_H^2(D_{1,x}, D_{2,x}) + \max_{x \in \mathcal{X}} d_H^2(D_{1,y|x}, D_{2,y|x}).$$

In particular, if $x$ and $y$ are independent, then $d_H^2(D_1, D_2) \leq d_H^2(D_{1,x}, D_{2,x}) + d_H^2(D_{1,y}, D_{2,y})$. 

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Proof. By definition,

\[ d^2_H(D_1, D_2) = 1 - \sum_{x \in X} \sum_{y \in Y} \sqrt{D_1(x, y)D_2(x, y)} \]

\[ = 1 - \sum_{x \in X} \sqrt{D_{1,x}(x)D_{2,x}(x)} + \sum_{x \in X} \sqrt{D_{1,x}(x)D_{2,x}(x)}(1 - \sum_{y \in Y} \sqrt{D_{1,y|x}(y|x)D_{2,y|x}(y|x)}) \]

\[ = d^2_H(D_{1,x}, D_{2,x}) + \sum_{x \in X} \sqrt{D_{1,x}(x)D_{2,x}(x)} \cdot \max_{x \in X} d^2_H(D_{1,y|x}, D_{2,y|x}) \]

\[ \leq d^2_H(D_{1,x}, D_{2,x}) + \sum_{x \in X} \sqrt{D_{1,x}(x)D_{2,x}(x)} \cdot \max_{x \in X} d^2_H(D_{1,y|x}, D_{2,y|x}) \]

where the last inequality is because \( \sum_{x \in X} \sqrt{D_{1,x}(x)D_{2,x}(x)} = 1 - d^2_H(D_{1,x}, D_{2,x}) \leq 1. \)

Let \( D^{\otimes T} \) denote the distribution of \( T \) i.i.d. samples from \( D \), namely, the product of \( T \) independent \( D \)’s. We have the following lemma that relates \( d^2_H(D^{\otimes T}_1, D^{\otimes T}_2) \) with \( d^2_H(D_1, D_2) \).

**Lemma A.5** (e.g., [49]). \( d^2_H(D^{\otimes T}_1, D^{\otimes T}_2) = 1 - (1 - d^2_H(D_1, D_2))^T \leq T d^2_H(D_1, D_2). \)

### A.2 Distinguishing Distributions

The squared Hellinger distance characterizes how many samples are needed to distinguish two distributions: namely, given samples from a distribution randomly chosen from \( \{D_1, D_2\} \), we are to guess whether the samples are from \( D_1 \) or \( D_2 \). It is known that at least \( T = \Omega(\frac{1}{d^2_H(D_1, D_2)} \log \frac{1}{\delta}) \) samples are needed to distinguish two distributions with probability at least \( 1 - \delta \). Formally, let \( i \sim \text{Uniform}\{1, 2\} \) denote the index of the randomly chosen distribution from which samples are drawn, and let \( j \in \{1, 2\} \) be the index of the distribution we guess based on samples. Then, no matter how we guess, the probability of making a mistake is lower bounded as follows:

**Lemma A.6** (e.g., [49]). The probability of making a mistake when distinguishing two distributions \( D_1 \) and \( D_2 \) using \( T \) samples, \( \Pr[j \neq i] = \frac{1}{2} \Pr[j \neq i \mid i = 1] + \frac{1}{2} \Pr[j \neq i \mid i = 2] \), is at least:

- \( \Pr[j \neq i] \geq \frac{1}{2} - \sqrt{\frac{T}{2} d_H(D_1, D_2)}. \)

- \( \Pr[j \neq i] \geq \frac{1}{4}(1 - d^2_H(D_1, D_2))^2 T \geq \frac{1}{4} e^{-4T d^2_H(D_1, D_2)}, \text{ assuming } d^2_H(D_1, D_2) \leq \frac{1}{2}. \)

The second item implies that, in order to achieve \( \Pr[j \neq i] \leq \delta \), we must have \( T \geq \frac{1}{4d^2_H(D_1, D_2)} \log \frac{1}{4\delta} \).

**Proof.** We provide a proof for this lemma for completeness. Let \( D^{\otimes T}_1 \) and \( D^{\otimes T}_2 \) denote the distributions of \( T \) i.i.d. samples from \( D_1 \) and \( D_2 \), respectively. The draw of \( T \) samples from \( D_1 \) or \( D_2 \) is equivalent to the draw of one sample from \( D^{\otimes T}_1 \) or \( D^{\otimes T}_2 \). Given one sample from \( D^{\otimes T}_1 \) or \( D^{\otimes T}_2 \),
the probability of making a mistake when guessing the distribution is at least:

\[
\Pr[j \neq i] = \frac{1}{2} \left( \Pr[j = 2 \mid i = 1] + \Pr[j = 1 \mid i = 2] \right)
\]

\[
= \frac{1}{2} \left( 1 - \Pr[j = 1 \mid i = 1] + \Pr[j = 1 \mid i = 2] \right)
\]

\[
= \frac{1}{2} \left( \Pr[j = 1 \mid i = 1] - \Pr[j = 1 \mid i = 2] \right)
\]

\[
= \frac{1}{2} \left( \mathbb{E}_{D_1^2} \mathbb{1} \{ j = 1 \} - \mathbb{E}_{D_2^2} \mathbb{1} \{ j = 1 \} \right)
\]

by Fact 2.6 \geq \frac{1}{2} - \frac{1}{2} d_{TV}(D_1^2, D_2^2). \tag{13}

We then upper bound \( d_{TV}(D_1^2, D_2^2) \) in two ways, which will prove the two items of the lemma, respectively. First, according to first item of Fact A.2, we have

\[
d_{TV}(D_1^2, D_2^2) \leq \sqrt{2} d_H(D_1^2, D_2^2).
\]

By Lemma A.5

\[
d_H(D_1^2, D_2^2) \leq \sqrt{T} d_H(D_1, D_2).
\]

The above two inequalities give \( d_{TV}(D_1^2, D_2^2) \leq \sqrt{2T} d_H(D_1, D_2) \). This proves the first item of the lemma.

Second, according to the second item of Fact A.2 and Lemma A.5, we have

\[
1 - d_{TV}^2(D_1^2, D_2^2) \geq (1 - d_H^2(D_1^2, D_2^2))^2 = (1 - d_H^2(D_1, D_2))^2T
\]

Since \( 1 - d_{TV}^2(D_1^2, D_2^2) = (1 + d_{TV}(D_1^2, D_2^2))(1 - d_{TV}(D_1^2, D_2^2)) \leq 2(1 - d_{TV}(D_1^2, D_2^2)) \), we have

\[
1 - d_{TV}(D_1^2, D_2^2) \geq \frac{1}{2} (1 - d_H^2(D_1, D_2))^2T.
\]

Plugging into (13), we get

\[
\Pr[j \neq i] \geq \frac{1}{4} (1 - d_H^2(D_1, D_2))^2T.
\]

When \( d_H^2(D_1, D_2) < \frac{1}{2} \), we use the inequality \( 1 - x \geq e^{-2x} \) for \( 0 < x < \frac{1}{2} \) to conclude that

\[
\Pr[j \neq i] \geq \frac{1}{4} (e^{-2d_H^2(D_1, D_2)})^2T = \frac{1}{4} e^{-4T d_H^2(D_1, D_2)}.
\]

\[\square\]

# B Missing Proofs from Section 2

## B.1 Proof of Lemma 2.1

To prove the first item \( f^*(r) = P(\omega = 1 \mid r) \), we simply note that

\[
f^* = \arg\min_f \mathbb{E}_r \left[ \mathbb{E} \left[ |f(r) - \omega|^2 \mid r \right] \right]
\]

should minimize \( \mathbb{E} \left[ |f(r) - \omega|^2 \mid r \right] \) for almost every \( r \), which gives \( f^*(r) = \mathbb{E}[\omega \mid r] = P(\omega = 1 \mid r) \).
To prove the second item $L_P(f) - L_P(f^*) = \mathbb{E}_P[|f(r) - f^*(r)|^2]$, we note that, by the definition of $L_P(\cdot)$ and the fact that $f^*(r) = \mathbb{E}[\omega | r]$ proven above,

$$
L_P(f) - L_P(f^*) = \mathbb{E}[|f(r) - \omega|^2] - \mathbb{E}[|f^*(r) - \omega|^2] \\
= \mathbb{E}[f(r)^2] - 2\mathbb{E}[f(r)\omega] - \mathbb{E}[f^*(r)^2] + 2\mathbb{E}[f^*(r)\omega] \\
= \mathbb{E}[f(r)^2] - 2\mathbb{E}_r[f(r)\mathbb{E}[\omega | r]] - \mathbb{E}[f^*(r)^2] + 2\mathbb{E}_r[f^*(r)\mathbb{E}[\omega | r]] \\
= \mathbb{E}[f(r)^2] - 2\mathbb{E}_r[f(r)f^*(r)] - \mathbb{E}[f^*(r)^2] + 2\mathbb{E}_r[f^*(r)^2] \\
= \mathbb{E}[f(r)^2] - 2f(r)f^*(r) + f^*(r)^2 \\
= \mathbb{E}[|f(r) - f^*(r)|^2].
$$

\section{Missing Proofs from Section 3}

\subsection{Proof of Theorem 3.1: the Lower Bound $\Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon} \right)$}

The formal proof consists of two parts: (1) reducing the distribution learning problem with respect to a family of distributions $\mathcal{D} = \{D\}$ that satisfy some properties to the forecast aggregation problem; (2) finding such a family of distributions $\mathcal{D}$ that require $T_{TV}^D(\varepsilon, \delta) = \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon} \right)$ samples to learn. They are presented in Section \ref{C.1.1} and Section \ref{C.1.2} respectively.

\subsubsection{Part 1: Reducing Distribution Learning to Forecast Aggregation}

Let $\mathcal{D}$ be a family of distributions over $\mathcal{S} = S_1 \times \cdots \times S_n$. For reasons to be clear later, we index the distributions in $\mathcal{D}$ by $z$, so $\mathcal{D} = \{D_z\}_z$. We define three properties of $\mathcal{D}$ to be used later.

\begin{definition}
We say a family of distributions $\mathcal{D} = \{D_z\}_z$ satisfies

1. \textit{B-uniformly bounded}, if: there exists constant $B \geq 1$, such that $D_z(s) \leq \frac{B}{|\mathcal{S}|} = \frac{B}{m^n}, \forall s \in \mathcal{S}, D_z \in \mathcal{D}$.

2. \textit{same marginal}, if: for any $D_z, D_{z'} \in \mathcal{D}$, any $i \in \{1, \ldots, n\}$, any $s_i \in S_i$, $D_z(s_i) = D_{z'}(s_i)$.

Hence, we can denote $D_z(s_i) = D(s_i)$.

3. \textit{distinct marginal probabilities}, if: for any $D_z \in \mathcal{D}$, any $s_i \neq s'_i \in S_i$, $D_z(s_i) \neq D_z(s'_i)$.

\end{definition}

Given a family of distributions $\mathcal{D} = \{D_z\}_z$ satisfying the above three properties, we construct a family of distributions $\mathcal{P} = \{P_z\}_z$ for the forecast aggregation problem, such that the $\varepsilon$-optimal aggregation with respect to $\mathcal{P}$ will imply $O(\sqrt{\varepsilon})$ total variation distance learning with respect to $\mathcal{D}$. The construction of $\mathcal{P}$ is as follows: for any distribution $D_z \in \mathcal{D}$, we define a corresponding distribution $P_z \in \mathcal{P}$ as follows:

\begin{equation}
\begin{cases}
P_z(\omega = 0) = P_z(\omega = 1) = \frac{1}{2}, \\
P_z(\cdot | \omega = 0) = \text{Uniform}(\mathcal{S}), \text{ namely, } P_z(s | \omega = 0) = \frac{1}{|\mathcal{S}|} = \frac{1}{m^n}, \\
P_z(\cdot | \omega = 1) = D_z, \text{ namely, } P_z(s | \omega = 1) = D_z(s).
\end{cases}
\end{equation}

We have the following observations about $\mathcal{D} = \{D_z\}_z$ and $\mathcal{P} = \{P_z\}_z$:

\begin{fact}
If $\mathcal{D} = \{D_z\}_z$ satisfies the three properties in Definition \ref{C.1}, then:
\end{fact}
1. Given signal \( s_i \), expert \( i \)'s report is \( r_i = P_z(\omega = 1 \mid s_i) = \frac{D_z(s_i)}{\frac{1}{m} + D_z(s_i)} = \frac{D(s_i)}{\frac{1}{m} + D(s_i)} \), which is the same for all distributions \( D_z \in \mathcal{D} \).

2. For every expert \( i \), given different signals \( s_i \neq s'_i \), its reports \( r_i \neq r'_i \). So, there is a one-to-one mapping between \( s_i \) and \( r_i \), for every \( i \in \{1, \dots, n\} \) and also a one-to-one mapping between the joint signal \( s = (s_1, \dots, s_n) \) and the joint report \( r = (r_1, \dots, r_n) \).

3. For any joint signal \( s \) and the corresponding joint report \( r \), we have \( D_z(s) = \frac{f^*(r)}{m^n(1-f^*(r))} \).

Proof. We prove the three items one by one:

1. By definition, the marginal distribution of signals, \( P_z(s) \), is

\[
P_z(s) = P_z(\omega = 0)P_z(s \mid \omega = 0) + P_z(\omega = 1)P_z(s \mid \omega = 1) = \frac{1}{2} \left( \frac{1}{m} + D_z(s) \right).
\]

Fixing \( s_i \), summing over \( s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in \mathcal{S}_{-i} \), we get

\[
P_z(s_i) = \frac{1}{2} \left( \frac{|\mathcal{S}_{-i}|}{m^n} + \sum_{s_{-i} \in \mathcal{S}_{-i}} D_z(s) \right) = \frac{1}{2} \left( \frac{1}{m} + D_z(s_i) \right).
\]

So, given signal \( s_i \), expert \( i \) should report

\[
r_i = P_z(\omega = 1 \mid s_i) = \frac{P_z(\omega = 1)P_z(s_i \mid \omega = 1)}{P_z(s_i)} = \frac{D_z(s_i)}{\frac{1}{m} + D_z(s_i)},
\]

and this is equal to \( \frac{D(s_i)}{\frac{1}{m} + D(s_i)} \) by the "same marginal" property.

2. Given \( s_i \neq s'_i \), by the "distinct marginal probabilities" property we have \( D_z(s_i) \neq D_z(s'_i) \). Since \( r_i = \frac{D_z(s_i)}{\frac{1}{m} + D_z(s_i)} \) and \( \frac{1}{m} + x \) is a strictly increasing function of \( x \), it follows that \( r_i \neq r'_i \).

3. According to item 2, there is a one-to-one mapping between \( s = (s_1, \dots, s_n) \) and \( r = (r_1, \dots, r_n) \); in other words, observing signals \( s_1, \dots, s_n \) is equivalent to observing reports \( r_1, \dots, r_n \). Therefore, by Bayes' rule we have

\[
f^*(r) = P_z(\omega = 1 \mid r) = P_z(\omega = 1 \mid s) = \frac{P_z(\omega = 1)P_z(s \mid \omega = 1)}{P_z(s)}
\]

by (14) and (15) = \( \frac{D_z(s)}{\frac{1}{m^n} + D_z(s)} \).

Rearranging, we obtain \( D_z(s) = \frac{f^*(r)}{m^n(1-f^*(r))} \). \( \square \)

Lemma C.3. Let \( P_z \) be defined above. If we have an aggregator \( \hat{f} \) that is \( \varepsilon \)-optimal with respect to \( P_z \), then we can find a distribution \( \hat{D} \) such that \( d_{TV}(\hat{D}, D_z) \leq (1 + B)^2 \sqrt{\varepsilon} \).

Proof. Because \( D_z \) is \( B \)-uniformly bounded, from (17) we can verify that \( f^*(r) \) satisfies \( f^*(r) \leq \frac{B}{1+B} \). So, we can assume \( \hat{f}(r) \leq \frac{B}{1+B} \) as well (if \( \hat{f}(r) > \frac{B}{1+B} \), we can let \( \hat{f}(r) = \frac{B}{1+B} \); this only reduces \( \mathbb{E}[|\hat{f}(r) - f^*(r)|^2] \)). Define \( \hat{D} \) by letting \( \hat{D}(s) = \frac{\hat{f}(r)}{m^n(1-\hat{f}(r))}, \forall s \in \mathcal{S} \), where \( r \) is the reports corresponding to \( s \) (cf., Fact C.2). Then, \( d_{TV}(\hat{D}, D_z) \)

\[
d_{TV}(\hat{D}, D_z) = \frac{1}{2} \sum_{s \in \mathcal{S}} |\hat{D}(s) - D_z(s)| = \frac{1}{2} \sum_{s \in \mathcal{S}} \left| \frac{\hat{f}(r)}{m^n(1-\hat{f}(r))} - \frac{f^*(r)}{m^n(1-f^*(r))} \right|
\]

\[
= \frac{1}{2m^n} \sum_{s \in \mathcal{S}} \left| \frac{\hat{f}(r)}{1-\hat{f}(r)} - \frac{f^*(r)}{1-f^*(r)} \right|.
\]
Because \( \hat{f}(r), f^*(r) \leq \frac{B}{1+B} \) and the function \( \frac{x}{1+x} \) has derivative \( \frac{1}{(1+x)^2} \leq (1+B)^2 \) when \( x \leq \frac{B}{1+B} \), we have

\[
d_{TV}(\hat{D}, D_z) \leq \frac{1}{2m^n}(1+B)^2 \sum_{s \in \mathcal{S}} |\hat{f}(r) - f^*(r)|. \tag{18}
\]

Then, we upper bound \( \sum_{s \in \mathcal{S}} |\hat{f}(r) - f^*(r)| \). By the definition that \( \hat{f} \) is an \( \varepsilon \)-optimal aggregator and the Jensen’s inequality \( \mathbb{E}[X^2] \geq \mathbb{E}[X]^2 \), we have

\[
\varepsilon \geq \mathbb{E}_{P_z} \left[ |\hat{f}(r) - f^*(r)|^2 \right] \geq \mathbb{E}_{P_z} \left[ |\hat{f}(r) - f^*(r)| \right]^2.
\]

Taking the square root,

\[
\sqrt{\varepsilon} \geq \mathbb{E}_{P_z} \left[ |\hat{f}(r) - f^*(r)| \right] = \sum_{s \in \mathcal{S}} P_z(s) |\hat{f}(r) - f^*(r)|
\]

by (15) \( = \sum_{s \in \mathcal{S}} \frac{1}{2} \left( \frac{1}{m^n} + D_z(s) \right) |\hat{f}(r) - f^*(r)| \geq \sum_{s \in \mathcal{S}} \frac{1}{2m^n} |\hat{f}(r) - f^*(r)|. \]

Multiplying by \( 2m^n \), we get \( \sum_{s \in \mathcal{S}} |\hat{f}(r) - f^*(r)| \leq 2m^n \sqrt{\varepsilon} \). Plugging this into (18), we obtain

\[
d_{TV}(\hat{D}, D_z) \leq \frac{1}{2m^n}(1+B)^2 2m^n \sqrt{\varepsilon} = (1+B)^2 \sqrt{\varepsilon},
\]

which concludes the proof.

Now, we present the reduction from learning \( \mathcal{D} = \{ D_z \}_z \) in total variation distance to forecast aggregation for \( \mathcal{P} = \{ P_z \}_z \). We use notations \( x^{(1)}, \ldots, x^{(T)} \in \mathcal{S} \) to represent the samples from \( D_z \). From \( x^{(1)}, \ldots, x^{(T)} \) we construct samples \( (r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)}) \) for the forecast aggregation problem. After obtaining a solution \( \hat{f} \) to the latter problem, we convert it into a solution \( \hat{D} \) to the former. Details are as follows:

**Input:** \( T \) i.i.d. samples \( x^{(1)}, \ldots, x^{(T)} \) from an unknown distribution \( D_z \in \mathcal{D} \).

**Reduction:**

1. Draw \( T \) samples \( \omega^{(1)}, \ldots, \omega^{(T)} \sim \text{Uniform}\{0,1\} \).

2. For each \( t = 1, \ldots, T \), do the following:
   - If \( \omega^{(t)} = 0 \), draw \( s^{(t)} \sim \text{Uniform}(\mathcal{S}) \).
   - If \( \omega^{(t)} = 1 \), let \( s^{(t)} = x^{(t)} \).
   - For each \( i \), compute \( r_i^{(t)} = \frac{D(s_i^{(t)})}{m + D(s_i^{(t)})} \). Let \( r^{(t)} = (r_1^{(t)}, \ldots, r_n^{(t)}) \).

3. Feed \( S_T = \{ (r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)}) \} \) to the forecast aggregation problem. Obtain solution \( \hat{f} \).

4. Convert \( \hat{f} \) into \( \hat{D} \) according to Lemma \( \text{C.3} \).

**Output:** \( \hat{D} \).

We remark that, in the second step of the reduction, the report \( r_i^{(t)} \) can be computed even if \( D_z \) is unknown, because the \( D_z(s_i^{(t)}) = D(s_i^{(t)}) \) is common for all \( D_z \)'s and hence known.
Using the above reduction, we show that the sample complexity of \( \varepsilon \)-optimal forecast aggregation for \( \mathcal{P} \) cannot be smaller than the sample complexity of learning \( \mathcal{D} \) within total variation distance \( O(\sqrt{\varepsilon}) \).

**Lemma C.4.** Let \( \mathcal{D} = \{D_z\}_z \) be a family of distributions that satisfies the three properties in Definition C.1. Let \( \mathcal{P} = \{P_z\}_z \) be defined above. Then, \( T_{\mathcal{P}}(\varepsilon, \delta) \geq T_{\mathcal{D}}^{TV}((1 + B)^2 \sqrt{\varepsilon}, \delta) \).

**Proof.** First, we verify that the distribution of samples \( S_T \) in the above reduction is exactly the distribution of \( T \) samples \( \{(r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)})\} \) from \( P_z \). This is because: (1) the distribution of \( \omega^{(t)} \) is \( \text{Uniform}\{0, 1\} \), as defined in \( P_z \); (2) given \( \omega^{(t)} = 0 \), the distribution of \( s^{(t)} \) is \( \text{Uniform}(\mathcal{S}) \), as defined in \( P_z \); (3) given \( \omega^{(t)} = 1 \), the distribution of \( s^{(t)} \) is the same as the distribution of \( x^{(t)} \), which is \( D_z \), because the random draws of \( \omega^{(t)} \) and \( x^{(t)} \) are independent; (4) according to Fact C.2, the report \( r^{(t)}_i = \frac{D(s^{(t)}_i)}{\frac{1}{m} + D(s^{(t)}_i)} = \frac{D_x(s^{(t)}_i)}{\frac{1}{m} + D_x(s^{(t)}_i)} = P_z(\omega = 1 \mid s^{(t)}_i) \), as desired.

Then, by the definition of sample complexity of forecast aggregation, if we are given \( T = T_{\mathcal{P}}(\varepsilon, \delta) \) samples \( S_T \) for the forecast aggregation problem, then with probability at least \( 1 - \delta \) we should be able to find an \( \varepsilon \)-optimal aggregator \( \hat{f} \) with respect to \( P_z \). According to Lemma C.3, we can convert \( \hat{f} \) into a \( \hat{D} \) such that

\[
d_{TV}(\hat{D}, D_z) \leq (1 + B)^2 \sqrt{\varepsilon}.
\]

By the definition of sample complexity \( T_{\mathcal{D}}^{TV}(\cdot, \delta) \) of distribution learning, \( T \) must be at least

\[
T \geq T_{\mathcal{D}}^{TV}((1 + B)^2 \sqrt{\varepsilon}, \delta),
\]

which proves the lemma. \( \square \)

**C.1.2 Part 2: Constructing a Family of Difficult-to-Learn Distributions**

In Lemma C.4, we showed that the sample complexity of \( \varepsilon \)-optimal forecast aggregation is lower bounded by the sample complexity of learning a family of distributions that satisfies the three properties in Definition C.1 within total variation distance \( O(\sqrt{\varepsilon}) \). In this section, we construct such a family of distributions \( \mathcal{D} = \{D_z\}_z \) that requires \( T_{\mathcal{D}}^{TV}(\varepsilon_{TV}, \delta) = \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon_{TV}}\right) \) samples to learn. This will prove the \( \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon_{TV}}\right) \) lower bound in Theorem 3.3. For technical convenience, we assume \( \varepsilon_{TV} < \frac{1}{10}, \delta < 0.01 \).

**Construction of \( \mathcal{D} = \{D_z\}_z \).** Without loss of generality, we assume \( |\mathcal{S}_i| = m \) to be an even integer, and let each signal space be \( \mathcal{S}_i = \{1, \ldots, m\} =: S_i \). Recall that \( D_z \) is a distribution over the joint signal space \( \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n = S^n = \{1, \ldots, m\}^n \). We write a signal \( s \in S^n \) as \( s = (b, x, y) \), where \( b \in S^{n-2} \) and \( x, y \in S \). We sort all the \( m^n \) signals in \( S^n \) by the lexicographical order, from \((1, \ldots, 1, 1), (1, \ldots, 1, 2), \ldots, (m, \ldots, m, m)\). We number the signals from 1 to \( m^n \), using

\[
\text{num}(s) = \text{num}(b, x, y) \in \{1, \ldots, m^n\},
\]

(19)

to denote the number of each signal \( s \). We divide the whole signal space \( S^n \) into \( |S^{n-2}| = m^{n-2} \) “buckets”, each of size \( m^2 \) and denoted by \( B_b \):

\[
B_b = \{(b, x, y) : x, y \in S\}, \quad b \in S^{n-2}.
\]

(20)

We first define a “base” distribution \( D_{\text{base}}(\cdot) \), then we construct the distributions \( D_z \)’s by modifying the probabilities of the base distribution within each bucket \( B_b \). Let \( \gamma = 1 + \frac{1}{m^n} \). The
base distribution is defined as follows:

\[ D_{\text{base}}(s) = \frac{\gamma^{\text{num}(s)}}{W}, \quad W = \sum_{s \in S} \gamma^{\text{num}(s)} = \sum_{\ell=1}^{m^n} \gamma^\ell. \]  

(21)

Because \( 1 \leq \gamma^\ell \leq \gamma^{m^n} \leq (1 + \frac{1}{m^n})^{m^n} \leq e \), we have

\[ m^n \leq W \leq em^n. \]  

(22)

We assign a sign \( z_b \in \{+1, -1\} \) to each bucket \( B_b \), and let \( z \) be a vector of length \( m^{n-2} \) that includes the signs of all buckets:

\[ z = (z_b)_{b \in S^{n-2}}, \quad z_b \in \{+1, -1\}. \]

We have \( 2^{m^{n-2}} \) different \( z \)'s in total, and hence \( 2^{m^{n-2}} \) different distributions \( D_z \)'s in \( \mathcal{D} \) in total. Let \( c = 20 \), so that \( c\varepsilon \text{TV} \leq 1/2 \). For each \( z \), we define \( D_z \) as follows: within each bucket \( B_b \), for each element \((b, x, y) \in B_b\) let

\[ D_z(b, x, y) = \begin{cases} 
D_{\text{base}}(b, x, y) + \frac{\varepsilon \gamma^{m^n}}{W} & \text{if } x \leq \frac{m}{2}, \; y \leq \frac{m}{2} \\
D_{\text{base}}(b, x, y) - \frac{\varepsilon \gamma^{m^n}}{W} & \text{if } x \leq \frac{m}{2}, \; y > \frac{m}{2} \\
D_{\text{base}}(b, x, y) - \frac{\varepsilon \gamma^{m^n}}{W} & \text{if } x > \frac{m}{2}, \; y \leq \frac{m}{2} \\
D_{\text{base}}(b, x, y) + \frac{\varepsilon \gamma^{m^n}}{W} & \text{if } x > \frac{m}{2}, \; y > \frac{m}{2} 
\end{cases} \]

(23)

**Claim C.5.** The family \( \mathcal{D} = \{ D_z \}_z \) defined above satisfies the three properties in Definition C.1: B-uniformly bounded with \( B = e + 1/2 \), same marginal, distinct marginal probabilities.

**Proof.** **B-uniformly bounded:** For any \( s \), any \( D_z \), by definition,

\[ D_z(s) \leq D_{\text{base}}(s) + \frac{c\varepsilon \gamma^{m^n}}{W} \leq \frac{\gamma^{m^n}}{W} + \frac{c\varepsilon \gamma^{m^n}}{W} \leq \frac{e}{m^n} + \frac{1/2}{m^n}. \]

So, the distribution is uniformly bounded with \( B = e + 1/2 \).

**Same marginal:** Consider each \( D_z(s_i) \). We want to show that \( D_z(s_i) \) does not depend on \( z \), and in fact, \( D_z(s_i) = D_{\text{base}}(s_i) \). If \( i \in \{1, \ldots, n - 2\} \), namely, \( s_i \) is a component of the vector \( b \), then we have

\[ D_z(s_i) = \sum_{s_{-i} \in S_{-i}} D_z(s_i, s_{-i}) = \sum_{b \in S^{n-2}; b_i = s_i} \sum_{x=1}^{m} \sum_{y=1}^{m} D_z(b, x, y). \]

We notice that, fixing any \( b \), the numbers of \( \pm \frac{\varepsilon \gamma^{m^n}}{W} \) and \( \mp \frac{\varepsilon \gamma^{m^n}}{W} \) in the summation \( \sum_{x=1}^{m} \sum_{y=1}^{m} D_z(b, x, y) \) are the same. So, they cancel out, and we obtain

\[ D_z(s_i) = \sum_{b \in S^{n-2}; b_i = s_i} \sum_{x=1}^{m} \sum_{y=1}^{m} D_{\text{base}}(b, x, y) = D_{\text{base}}(s_i). \]

If \( i = n - 1 \), namely \( s_i = x \), then, we have:

\[ D_z(s_i) = \sum_{s_{-i} \in S_{-i}} D_z(s_i, s_{-i}) = \sum_{b \in S^{n-2}} \sum_{y=1}^{m} D_z(b, x, y). \]
Fixing any \( b \), the numbers of \( \frac{+2\varepsilon TV}{W} \) and \( \frac{-2\varepsilon TV}{W} \) in the summation \( \sum_{y=1}^{m} D_{z}(b, x, y) \) are the same. So, they cancel out, and we obtain

\[
D_{z}(s) = \sum_{b \in S^{n-2}} \sum_{y=1}^{m} D_{\text{base}}(b, x, y) = D_{\text{base}}(s).
\]

Finally, if \( i = n \), namely \( s_{i} = y \), then by a similar argument as above we have

\[
D_{z}(s_{i}) = \sum_{b \in S^{n-2}} \sum_{x=1}^{m} D_{\text{base}}(b, x, y) = D_{\text{base}}(s_{i}).
\]

**Distinct marginal probabilities:** By the “same marginal” property above we know that \( D_{z}(s) = D_{\text{base}}(s) \). So, to prove “distinct marginal probabilities” we only need to prove \( D_{\text{base}}(s) \neq D_{\text{base}}(s') \) for \( s \neq s' \). Without loss of generality assume \( s_{i} < s'_{i} \). By the definition \( D_{\text{base}}(s) = \frac{\text{num}(s)}{W} \) and the fact that \( \text{num}(s_{i}, s_{-i}) < \text{num}(s'_{i}, s_{-i}) \) for any \( s_{-i} \in S_{-i} \), we have

\[
D_{\text{base}}(s) = \sum_{s_{-i} \in S_{-i}} D_{\text{base}}(s) < \sum_{s_{-i} \in S_{-i}} D_{\text{base}}(s') = D_{\text{base}}(s'),
\]

so \( D_{\text{base}}(s) \neq D_{\text{base}}(s') \).

**Proposition C.6.** For the family \( \mathcal{D} = \{D_{z}\}_{z} \) constructed above, for \( \varepsilon_{TV} < \frac{1}{m} \) and \( \delta < 0.01 \), we have \( T_{D}^{TV}(\varepsilon_{TV}, \delta) = \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon_{TV}}\right) \).

The proof of Proposition C.6 is analogous to a textbook proof of Proposition 2.5 (the sample complexity lower bound for learning all distributions), which uses reductions from the distinguishing distributions problem. Roughly speaking, if one can learn the unknown distribution \( D_{z} \) well then one must be able to guess most of the components of the sign vector \( z = (z_{b})_{b \in S^{n-2}} \) correctly, meaning that one can distinguish whether the distribution \( D_{z} \) on bucket \( B_{b} \) is \( D_{b_{+1}} \) or \( D_{b_{-1}} \). However, since \( D_{b_{+1}} \) and \( D_{b_{-1}} \) are “\( O(\varepsilon_{TV}) \)-close” to each other, to distinguish them requires \( \Omega\left(\frac{1}{\varepsilon_{TV}}\right) \) samples. In average, there are only \( O\left(\frac{T}{m^{n-2}}\right) \) samples falling into a bucket (because there are \( m^{n-2} \) buckets in total and the distribution \( D_{z} \) is close to uniform). We thus need \( O\left(\frac{T}{m^{n-2}}\right) = \Omega\left(\frac{1}{\varepsilon_{TV}}\right) \), which gives \( T = \Omega\left(\frac{m^{n-2}}{\varepsilon_{TV}}\right) \). Ignoring logarithmic terms, this proves the proposition. See Appendix C.2 for a full proof.

**Finishing the proof:** By Lemma C.4 and Proposition C.6, plugging in \( \varepsilon_{TV} = (1 + B)^{2}\sqrt{\varepsilon} \) with \( B = e + 1/2 \), we have the lower bound on the sample complexity of forecast aggregation:

\[
T_{p}(\varepsilon, \delta) \geq T_{D}^{TV}((1 + B)^{2}\sqrt{\varepsilon}, \delta) = \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{(1 + B)^{2}\sqrt{\varepsilon}}\right) = \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon}\right).
\]

**C.2 Proof of Proposition C.6**

To simplify notations we write \( \varepsilon_{TV} = \varepsilon \). We want to prove that \( \Omega\left(\frac{m^{n-2} + \log(1/\delta)}{\varepsilon_{TV}}\right) \) samples are needed to learn within total variation distance \( \varepsilon \) the family of distributions \( \mathcal{D} = \{D_{z}\}_{z} \) defined by:
∀ b ∈ S^{n-2}, ∀ (b, x, y) ∈ B_b,

\[ D_z(b, x, y) = \begin{cases} 
D_{base}(b, x, y) + \frac{2cε}{W} & \text{if } x \leq \frac{m}{2}, \ y \leq \frac{m}{2} \\
D_{base}(b, x, y) - \frac{2cε}{W} & \text{if } x \leq \frac{m}{2}, \ y > \frac{m}{2} \\
D_{base}(b, x, y) - \frac{2cε}{W} & \text{if } x > \frac{m}{2}, \ y \leq \frac{m}{2} \\
D_{base}(b, x, y) + \frac{2cε}{W} & \text{if } x > \frac{m}{2}, \ y > \frac{m}{2}
\end{cases}, \quad D_{base}(b, x, y) = \frac{\gamma \text{num}(b, x, y)}{W}, \]

with constants $c = 20$, $γ = 1 + \frac{1}{m^3}$, and index vector $z = (z_b)_{b \in S^{n-2}} \in \{-1, 1\}^{mn-2}$.

First, we note that if we can learn $D_z$ very well, then we can guess the vector $z$ correctly for a large fraction of its $m^{n-2}$ components. Formally, suppose we obtain from $T$ samples a distribution $\hat{D}$ such that $d_{TV}(\hat{D}, D_z) \leq ε$, then we find the distribution $D_w$ in $\mathcal{D}$, $w = (w_b)_{b \in S^{n-2}} \in \{-1, 1\}^{mn-2}$, that is closest to $\hat{D}$ in total variation distance. By definition we have $d_{TV}(D_w, \hat{D}) \leq d_{TV}(D_z, \hat{D}) \leq ε$. Hence, by triangle inequality, $d_{TV}(D_w, D_z) \leq d_{TV}(D_w, \hat{D}) + d_{TV}(\hat{D}, D_z) \leq 2ε$. Let

\[ \# = |\{ b \in S^{n-2} | w_b \neq z_b\}| \]

be the number of different components of $w$ and $z$. We claim that:

**Claim C.7.** $d_{TV}(D_w, D_z) \leq 2ε$ implies $\# \leq \frac{2W}{cm^2}$.

**Proof.** Whenever we have a different component $w_b \neq z_b$, this different component contributes the following to the total variation distance between $D_w$ and $D_z$:

\[ \frac{1}{2} \sum_{(b, x, y) \in B_b} |D_w(b, x, y) - D_z(b, x, y)| = \frac{1}{2} \sum_{(b, x, y) \in B_b} \frac{2cε}{W} = \frac{m^2cε}{W}. \]  

(24)

So, the number of different components of $w$ and $z$ is at most $\frac{d_{TV}(D_w, D_z)}{m^2cε} \leq \frac{2W}{cm^2}$.

We first show the $Ω\left(\frac{\log(1/δ)}{ε}\right)$ part in the sample complexity lower bound, and then show the $Ω\left(\frac{m^{n-2}}{ε}\right)$ part. Together, they give the lower bound $\max\{Ω\left(\frac{\log(1/δ)}{ε}\right), Ω\left(\frac{m^{n-2}}{ε}\right)\} = Ω\left(\frac{m^{n-2} + \log(1/δ)}{ε}\right)$. Consider the distribution $D_{+1} \in \mathcal{D}$ whose index is the all “+1” vector and the distribution $D_{-1} \in \mathcal{D}$ whose index is the all “−1” vector. According to (24), the total variation distance between $D_{+1}$ and $D_{-1}$ is $d_{TV}(D_{+1}, D_{-1}) = m^{n-2} \cdot \frac{m^2cε}{W} = \frac{m^{n-2}cε}{W}$ because $+1$ and $-1$ have $m^{n-2}$ different components. Since $W \leq em^n$ (22), we have $d_{TV}(D_{+1}, D_{-1}) \geq \frac{m^n}{e} > 2ε$. Consider the distinguishing distributions problem (defined in Appendix A) where we want to distinguish $D_{+1}$ and $D_{-1}$. If we can learn from samples a distribution $\hat{D}$ that is $ε$-close in total variation distance to the unknown distribution $D_{+1}$ or $D_{-1}$, then we can perfectly tell whether the unknown distribution is $D_{+1}$ or $D_{-1}$ because the two distributions are more than $2ε$-away from each other. Lemma [A.6] implies that, to distinguish $D_{+1}$ and $D_{-1}$ with probability $1 - δ$, the number of samples must be at least $Ω\left(\frac{\log(1/δ)}{d_{TV}(D_{+1}, D_{-1})}\right) = Ω\left(\frac{\log(1/δ)}{ε}\right)$. This proves the $Ω\left(\frac{\log(1/δ)}{ε}\right)$ part.

We then prove the $Ω\left(\frac{m^{n-2}}{ε}\right)$ part. Suppose we first draw the vector $z$ from $\{-1, 1\}^{mn-2}$ uniformly at random, then draw $T$ samples from $D_z$. We obtain the $D_w$ as above. Let’s consider the expected number of different components of $w$ and $z$ in this two-step random draw procedure:

\[ z, T \text{ samples} \quad \mathbb{E}[\#] = \mathbb{E}\left[ \sum_{b \in S^{n-2}} 1\{z_b \neq w_b\}\right] = \sum_{b \in S^{n-2}} \mathbb{E}[1\{z_b \neq w_b\}]. \]  

(25)
We consider each component $E \left[ 1 \{ z_b \neq w_b \} \right]$ in the above summation. Suppose that, within the $T$ samples drawn from $D_z$, $T_b$ of them fall into the bucket $B_b$. So, $T_b$ follows the Binomial($T, D(B_b)$) distribution with

$$D(B_b) = \sum_{(b, x, y) \in B_b} D_z(b, x, y) = \frac{1}{W} \sum_{(b, x, y) \in B_b} \gamma^{num(b, x, y)}.$$  

(Notice that the $+\frac{z_b c\varepsilon}{W}$ and $-\frac{z_b c\varepsilon}{W}$ cancel out in the summation and hence $D(B_b)$ does not depend on $z$.) Let $D_{z_b}$ denote the “$B_b$-part” of distribution $D_z$, namely, $D_z$ conditioning on $B_b$: 

$$D_{z_b}(s) = \frac{D_z(s)}{D(B_b)}, \quad \forall s \in B_b.$$  

We think of the random draw of the vector $z$ and the $T$ samples as follows: first, we draw $T_b$ from Binomial($T, D(B_b)$); second, we draw $z_b \in \{+1, -1\}$ uniformly at random; third, we draw $T_b$ samples from the conditional distribution $D_{z_b}$; forth, we draw the remaining vector $z_{-b}$ and the remaining $T - T_b$ samples (which are samples outside of $B_b$). Only writing the first two steps explicitly, we have

$$\mathbb{E}\left[ 1 \{ z_b \neq w_b \} \right] = \mathbb{E}_{T_b, z_b} \left[ \mathbb{E}_{T_b, z_b} \left[ 1 \{ z_b \neq w_b \} \mid z_b, T_b \right] \right]$$

$$= \mathbb{E}_{T_b} \left[ \mathbb{E}_{T_b, z_b} \left[ 1 \{ z_b \neq w_b \} \mid z_b \in \{ +1, T_b \} \right] + \frac{1}{2} \mathbb{E}_{T_b, z_b} \left[ 1 \{ z_b \neq w_b \} \mid z_b = -1, T_b \right] \right]$$

$$= \mathbb{E}_{T_b} \left[ \frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = +1, T_b \right] + \frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = -1, T_b \right] \right].$$  \hspace{1cm} (26) 

**Claim C.8.** For any $T_b$, $\frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = +1, T_b \right] + \frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = -1, T_b \right] \geq \frac{1}{2} - 2c\varepsilon \sqrt{T_b}.$

**Proof.** We notice that $\frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = +1, T_b \right] + \frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = -1, T_b \right]$ is the probability that we make a mistake when guessing the sign $z_b$ using $w_b$, if (1) $z_b$ is chosen from $\{ -1, +1 \}$ uniformly at random; (2) we are given $T_b$ samples from $D_{z_b}$; (3) we then draw the remaining vector $z_{-b}$ and the remaining samples; (4) finally, we use the $D_w$ computed from all samples to get $w_b$. The steps (3) and (4) define a randomized function that maps the $T_b$ samples of $D_{z_b}$ to $w_b \in \{ 0, 1 \}$, and therefore, according to the first item of Lemma A.6, we have

$$\frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = +1, T_b \right] + \frac{1}{2} \Pr \left[ z_b \neq w_b \mid z_b = -1, T_b \right] \geq \frac{1}{2} - \sqrt{T_b} \frac{d_H(D_{z_b=+1}, D_{z_b=-1})}{2}.  \hspace{1cm} (27)$$

Then we consider $d_H(D_{z_b=+1}, D_{z_b=-1})$. We use Lemma A.3 to do so. For any $s \in B_b$, we have, on the one hand,

$$\frac{D_{z_b=+1}(s)}{D_{z_b=-1}(s)} = \frac{\gamma^{num(s)} + 2c\varepsilon}{\gamma^{num(s)} - 2c\varepsilon} \geq \frac{\gamma^{num(s)} - c\varepsilon}{\gamma^{num(s)} + c\varepsilon} \geq 1 - 2c\varepsilon,$$

because $\frac{a - b}{a + b} = 1 - \frac{2b}{a + b} \geq 1 - 2b$ for $a = \gamma^{num(s)} \geq 1$. On the other hand,

$$\frac{D_{z_b=+1}(s)}{D_{z_b=-1}(s)} \leq \frac{\gamma^{num(s)} + c\varepsilon}{\gamma^{num(s)} - c\varepsilon} \leq 1 + 4c\varepsilon,$$

because $\frac{a + b}{a - b} = 1 + \frac{2b}{a - b} \leq 1 + 4b$ for $a = \gamma^{num(s)} \geq 1$ and $b = c\varepsilon \leq \frac{1}{2}$. Therefore, by Lemma A.3 we have

$$d_H^2(D_{z_b=+1}, D_{z_b=-1}) \leq \frac{1}{2} (4c\varepsilon)^2 = 8c^2\varepsilon^2.  \hspace{1cm} (28)$$

Combining (27) and (28) proves our claim. $\square$
By (20) and Claim (C.8), we have $\mathbb{E}[\mathbf{1}\{z_b \neq w_b\}] \geq \mathbb{E}\left[\frac{1}{2} - 2\varepsilon \sqrt{T_b}\right]$. Summing over $b \in S^{n-2}$, (25) becomes

$$
\sum_{b \in S^{n-2}} \mathbb{E}[\mathbf{1}\{z_b \neq w_b\}] \geq \sum_{b \in S^{n-2}} \mathbb{E}\left[\frac{1}{2} - 2\varepsilon \sqrt{T_b}\right] = \frac{m^{n-2}}{2} - 2\varepsilon \sum_{b \in S^{n-2}} \mathbb{E}[\sqrt{T_b}]
$$

(by Jensen’s inequality $\mathbb{E}[\sqrt{X}] \leq \sqrt{\mathbb{E}[X]} \geq \frac{m^{n-2}}{2} - 2\varepsilon \sum_{b \in S^{n-2}} \mathbb{E}[\sqrt{T_b}]$).

Because $\mathbb{E}[T_b] = T \cdot D(B_b) = \frac{T}{W} \sum_{(b,x,y) \in B_b} \gamma^{\text{num}(b,x,y)} \leq \frac{T}{W} m^2 \gamma^m$, we have

$$
\sum_{b \in S^{n-2}} \mathbb{E}[\mathbf{1}\{z_b \neq w_b\}] \geq \frac{m^{n-2}}{2} - 2\varepsilon \sum_{b \in S^{n-2}} \sqrt{\frac{T}{W} m^2 \gamma^m} = \frac{m^{n-2}}{2} - 2\varepsilon \sum_{b \in S^{n-2}} \sqrt{\frac{T}{W} m^2 \gamma^m}.
$$

Now, let’s consider the probability with which we can obtain $\hat{D}$ such that $D_{TV}(\hat{D}, D_2) \leq \varepsilon$. We want to show that this probability is at most $0.99 < 1 - \delta$ if $T$ is less than $10^{-5} \cdot \frac{m^{n-2}}{\varepsilon^2}$. Recall from Claim (C.7) that $D_{TV}(\hat{D}, D_2) \leq \varepsilon$ implies $\# = \sum_{b \in S^{n-2}} \mathbf{1}\{z_b \neq w_b\} \leq \frac{2W}{cm^2}$. So, the probability is at most

$$
\Pr\left[ \sum_{b \in S^{n-2}} \mathbf{1}\{z_b \neq w_b\} \leq \frac{2W}{cm^2} \right] = \Pr\left[ \sum_{b \in S^{n-2}} \mathbf{1}\{z_b = w_b\} \geq m^{n-2} - \frac{2W}{cm^2} \right]
$$

(by Markov’s inequality) \leq \frac{\mathbb{E}\left[\sum_{b \in S^{n-2}} \mathbf{1}\{z_b = w_b\}\right]}{m^{n-2} - \frac{2W}{cm^2}} \leq \frac{m^{n-2}}{2} + 2\varepsilon \varepsilon m^{n-2} \sqrt{\frac{T}{W} m^2 \gamma^m}
$$

(by (20)) \leq \frac{m^{n-2}}{2} + 2\varepsilon \varepsilon m^{n-2} \sqrt{\frac{\epsilon T}{m^{n-2}}} \leq \frac{m^{n-2}}{2} + 2\varepsilon \varepsilon m^{n-2} \sqrt{\frac{\epsilon T}{m^{n-2}}} \leq \frac{1}{2} + 2\varepsilon \sqrt{\frac{\epsilon T}{m^{n-2}}} \leq 0.99 < 1 - \delta,

when $c = 20$, $T < 10^{-5} \cdot \frac{m^{n-2}}{\varepsilon^2}$, and $\delta < 0.01$. This means that, in order to obtain such $\hat{D}$ with probability at least $1 - \delta$, at least $10^{-5} \cdot \frac{m^{n-2}}{\varepsilon^2}$ samples are needed.
D Missing Proofs from Section 4

D.1 Proof of Lemma 4.1

By Bayes’ rule and the fact that \( s_1, \ldots, s_n \) are independent conditioning on \( \omega \),

\[
P(\omega = 1 \mid s) = \frac{P(\omega = 1) \prod_{i=1}^{n} P(s_i \mid \omega = 1)}{P(\omega = 1) \prod_{i=1}^{n} P(s_i \mid \omega = 1) + P(\omega = 0) \prod_{i=1}^{n} P(s_i \mid \omega = 0)}
\]

\[
= \frac{\prod_{i=1}^{n} P(\omega = 1) P(s_i \mid \omega = 1)}{\prod_{i=1}^{n} P(\omega = 1) P(s_i \mid \omega = 1) + \prod_{i=1}^{n} P(\omega = 0) P(s_i \mid \omega = 0)}
\]

\[
= \frac{\prod_{i=1}^{n} \frac{P(\omega = 1) P(s_i \mid \omega = 1)}{P(s_i)}}{\prod_{i=1}^{n} \frac{P(\omega = 1) P(s_i \mid \omega = 1)}{P(s_i)} + \prod_{i=1}^{n} \frac{P(\omega = 0) P(s_i \mid \omega = 0)}{P(s_i)}}
\]

\[
= \frac{\prod_{i=1}^{n} r_i}{\prod_{i=1}^{n} r_i + \rho^{-1} \prod_{i=1}^{n} (1 - r_i)}
\]

Since the expression \( \frac{1}{1 + \rho^{-1} \prod_{i=1}^{n} (1 - r_i)} \) only depends \( r_i \)'s but not \( s_i \)'s, we have:

\[
f^*(r) = P(\omega = 1 \mid r) = P(\omega = 1 \mid s) = \frac{1}{1 + \rho^{-1} \prod_{i=1}^{n} (1 - r_i)}.
\]

D.2 Proof of Theorem 4.2: the \( O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right) \) Upper Bound

We prove the sample complexity upper bound \( O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right) \) in Theorem 4.2 using the tool of pseudo-dimension.

**Definition D.1** (e.g., [47]). Let \( \mathcal{F} \) be a class of functions mapping from input space \( X \) to real numbers \( \mathbb{R} \). A set of inputs \( \{x_1, \ldots, x_d\} \subseteq X \) is said to be pseudo-shattered by \( \mathcal{F} \) if there exist thresholds \( t_1, \ldots, t_d \in \mathbb{R} \) such that for any label vector \( b \in \{-1, +1\}^d \) there exists a function \( f_b \in \mathcal{F} \) such that \( f_b(x_i) > t_i \) if \( b_i = 1 \) and \( f_b(x_i) < t_i \) if \( b_i = -1 \), for every \( i \in \{1, \ldots, d\} \).

The pseudo-dimension of \( \mathcal{F} \), \( \text{Pdim}(\mathcal{F}) \), is the size of the largest set of inputs that can be pseudo-shattered by \( \mathcal{F} \).

Pseudo-dimension gives a sample complexity upper bound for the uniform convergence of the empirical means of a class of functions to their true means:

**Theorem D.2** (e.g., [47]). If \( \mathcal{F} \) is a class of functions mapping from input space \( X \) to \([0, 1]\) with pseudo-dimension \( \text{Pdim}(\mathcal{F}) = d \), then, for any distribution \( \mathcal{D} \) on \( X \), with probability at least \( 1 - \delta \) over the random draw of \( T = O\left(\frac{1}{\varepsilon^2}\left(d \cdot \log \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right)\right) \) samples \( x^{(1)}, \ldots, x^{(T)} \) from \( \mathcal{D} \), we have: for every \( f \in \mathcal{F} \), \( |\mathbb{E}_{x \sim \mathcal{D}}[f(x)] - \frac{1}{T} \sum_{t=1}^{T} f(x^{(t)})| \leq \varepsilon \).

According to Lemma D.1 for conditionally independent distributions the Bayesian aggregator \( f^* \) has the form \( f^*(r) = f_\rho(r) = \frac{1}{1 + \rho^{-1} \prod_{i=1}^{n} r_i} \) for some \( \rho \in (0, +\infty) \) (in particular, \( \rho = \frac{1}{\varepsilon^2} \)).

In the following proof, we let \( \sigma \in (0, +\infty) \) be a parameter and consider the class of aggregators
\[ \mathcal{F} = \{ f_\sigma : f_\sigma(r) = \frac{1}{1 + \sigma^n - 1} \prod_{i=1}^{n} \frac{1 - r_i}{r_i}, \forall r \}. \]

We rename \( \prod_{i=1}^{n} \frac{1 - r_i}{r_i} \) as \( x \) and consider the class \( \mathcal{G} \) of loss functions associated with \( \mathcal{F} \), \( \mathcal{G} = \{ g_\sigma : g_\sigma(x, \omega) = |f_\sigma(r) - \omega|^2 = |\frac{1}{1 + \sigma^n - 1} - \omega|^2 \} \), which is also parameterized by \( \sigma \in (0, +\infty) \).

**Lemma D.3.** \( \text{Pdim}(\mathcal{G}) \leq 1. \)

**Proof.** Suppose \( \text{Pdim}(\mathcal{G}) = d \). By definition, there exists a set of inputs \( \{(x_1, \omega_1), \ldots, (x_d, \omega_d)\} \) that can be pseudo-shattered by \( \mathcal{G} \). Suppose \( t_1, \ldots, t_d \in \mathbb{R} \) such that for each of the \( 2^d \) label vectors \( b \in \{-1, +1\}^d \), there exists a corresponding \( \sigma_b \in (0, +\infty) \) such that \( g_{\sigma_b}(x_i, \omega_i) > t_i \) if \( b_i = +1 \) and \( g_{\sigma_b}(x_i, \omega_i) < t_i \) if \( b_i = -1 \), for every \( i \in \{1, \ldots, d\} \).

Fix such a set of inputs \( \{(x_1, \omega_1), \ldots, (x_d, \omega_d)\} \). Consider each \( (x_i, \omega_i) \). Because \( g_\sigma(x_i, \omega_i) = |\frac{1}{1 + \sigma^n - 1} - \omega_i|^2 \) is a monotone function of \( \sigma \) (monotonically decreasing if \( \omega_i = 0 \) and monotonically increasing if \( \omega_i = 1 \)), the set of \( \sigma \in (0, +\infty) \) with \( g_\sigma(x_i, \omega_i) > t_i \) and the set of \( \sigma \in (0, +\infty) \) with \( g_\sigma(x_i, \omega_i) < t_i \) divide the entire interval \((0, +\infty)\) into two sub-intervals \((0, a_i)\) and \((a_i, +\infty)\), at some dividing point \( a_i \). Enumerating all inputs \( (x_1, \omega_1), \ldots, (x_d, \omega_d) \), they correspond to \( d \) dividing points, dividing the entire interval into \( d + 1 \) sub-intervals. For each sub-interval, all the \( \sigma \)'s within that sub-interval give the same label vector \( b \in \{-1, +1\}^d \) for the set of inputs. Considering all sub-intervals, at most \( d + 1 \) distinct label vectors can be generated. So, to generate all the \( 2^d \) label vectors we must have \( d + 1 \geq 2^d \), which gives \( d \leq 1. \)

By Lemma D.3 and Theorem D.2, plugging in \( d = \text{Pdim}(\mathcal{G}) \leq 1 \), we obtain the following: with probability at least \( 1 - \delta \) over the random draw of \( T = O\left(\frac{1}{\epsilon^2}(d \cdot \log \frac{1}{\delta} + \log \frac{1}{\epsilon})\right) = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right) \) samples \((r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)})\) from \( P \), letting \( x^{(t)} = \prod_{i=1}^{n} \frac{1 - r_i^{(t)}}{r_i^{(t)}} \), we have: for any \( \sigma \in (0, +\infty) \),

\[
\left| \mathbb{E}_P[g_\sigma(x, \omega)] - \frac{1}{T} \sum_{t=1}^{T} g_\sigma(x^{(t)}, \omega^{(t)}) \right| \leq \epsilon.
\]

This is equivalent to

\[
\left| \mathbb{E}_P[|f_\sigma(r) - \omega|^2] - \frac{1}{T} \sum_{t=1}^{T} |f_\sigma(r^{(t)}) - \omega^{(t)}|^2 \right| \leq \epsilon \iff |L_P(f_\sigma) - L_{\hat{P}_{\text{emp}}}(f_\sigma)| \leq \epsilon,
\]

where \( \hat{P}_{\text{emp}} \) is the empirical distribution Uniform\( \{(r^{(1)}, \omega^{(1)}), \ldots, (r^{(T)}, \omega^{(T)})\} \). By the same logic as the proof of the upper bound in Theorem 3.1 (Section 3.1), the empirically optimal aggregator \( f_{\hat{\rho}} \), where \( \hat{\rho} = \arg\min_{\sigma \in (0, +\infty)} L_{\hat{P}_{\text{emp}}}(f_\sigma) \), is \( 2\epsilon \)-optimal.

**D.3 Proof of Theorem 4.2: the \( \Omega\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right) \) Lower Bound**

The proof uses a reduction from the distinguishing distributions problem (defined in Appendix A). We construct two conditionally independent distributions \( P^1, P^2 \) over the space \( \Omega \times S_1 \times \cdots \times S_n \) with each \( |S_i| = 2, S_i = \{a, b\} \). Given \( T \) samples from either \( P^1 \) or \( P^2 \), we want to tell which distribution the samples are coming from. We will show that, if we can solve the forecast aggregation problem, then we can distinguish the two distributions (with high probability), which requires \( T = \Omega\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right) \) samples according to Lemma A.6.

Let \( c = 32 \). We assume \( \epsilon < 2^{-18} \), so that \( c\sqrt{\epsilon} < \frac{1}{16} \). For \( P^1 \), we let

\[
P^1(\omega = 1) = 0.5 - \frac{1}{16n} + \frac{c\sqrt{\epsilon}}{n} =: p^1.
\]
For $P^2$, we let
$$P^2(\omega = 1) = 0.5 - \frac{1}{16n} - \frac{c\sqrt{\varepsilon}}{n} =: p^2.$$ 

We require that, in the forecast aggregation problem under both distributions $P^1$ and $P^2$, whenever expert $i$ sees signal $a$, $b$, she reports
$$r_a = 0.5, \quad r_b = 0,$$
respectively. This gives the following conditional probabilities $P^1(\cdot \mid \omega), P^2(\cdot \mid \omega)$:

$$
\begin{align*}
\frac{P^1(a \mid \omega = 0)}{P^1(a \mid \omega = 1)} &= \frac{p^1 - r_b}{r_a - r_b} \left[1 - \frac{1}{1 - p^1} \right] = \left[1 + \frac{1 - r_b}{1 + \frac{1}{8n} - \frac{2c\sqrt{\varepsilon}}{n}} \right]^{-1}, \quad \frac{P^1(b \mid \omega = 0)}{P^1(b \mid \omega = 1)} = \left[1 + \frac{1}{8n} - \frac{2c\sqrt{\varepsilon}}{n} \right]^{-1}. \\
\frac{P^2(a \mid \omega = 0)}{P^2(a \mid \omega = 1)} &= \frac{p^2 - r_b}{r_a - r_b} \left[1 - \frac{1}{1 - p^2} \right] = \left[1 + \frac{1 - r_b}{1 + \frac{1}{8n} - \frac{2c\sqrt{\varepsilon}}{n}} \right], \quad \frac{P^2(b \mid \omega = 0)}{P^2(b \mid \omega = 1)} = \left[1 + \frac{1}{8n} - \frac{2c\sqrt{\varepsilon}}{n} \right].
\end{align*}
$$

(30)

Given $T$ samples from the unknown distribution $P \in \{P^1, P^2\}$, each of which is a vector of $\omega^{(t)}$ and all experts' signals $s_i^{(t)} \in \{a, b\}$, we feed the corresponding reports $r_i^{(t)} \in \{r_a, r_b\}$ and $\omega^{(t)}$ to the forecast aggregation problem and obtain a solution $\hat{f}$, which is an $\varepsilon$-optimal aggregator. We want to use $\hat{f}$ to estimate the prior $p = P(\omega = 1) \in \{p^1, p^2\}$ so that we can tell apart $P^1$ and $P^2$. Recall from Lemma 4.11 that $f^*(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{r_i}{1 - r_i}}$, where $\rho = \frac{p}{1 - p}$. Writing $\rho$ in terms of $f^*(r)$, we have

$$\rho = n^{-1} \sqrt{\frac{1}{f^*(r)} - 1} \prod_{i=1}^{n} \frac{r_i}{1 - r_i}.$$

In particular, when $r_i = r_a = 0.5$ for all $i \in \{1, \ldots, n\}$, we have:

$$\rho = n^{-1} \sqrt{\frac{1}{f^*(r_{0.5})} - 1}, \quad r_{0.5} = (0.5, \ldots, 0.5).$$

So, we estimate $\rho$ by:

$$\hat{\rho} = n^{-1} \sqrt{\frac{1}{f(r_{0.5})} - 1}.$$

Now, we want to argue that, if $\hat{f}$ is $\varepsilon$-optimal, then $|\hat{\rho} - \rho|$ is at most $O(\sqrt{\varepsilon}/n)$. Consider the function:

$$h(x) = n^{-1} \sqrt{\frac{1}{x} - 1},$$

whose derivative is

$$h'(x) = -\frac{1}{n - 1} \left( \frac{x}{1 - x} \right)^{1 - \frac{1}{n-1}} \frac{1}{x^2}.$$

By definition, we have

$$|\hat{\rho} - \rho| = |h(\hat{f}(r_{0.5})) - h(f^*(r_{0.5}))|.$$

(32)

Claim D.4. $\frac{1}{2} \leq f^*(r_{0.5}) \leq \frac{2}{3}$. 

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Proof. For \( P \in \{P^1, P^2\} \), its \( \rho = \frac{p}{1-p} \) satisfies
\[
1 \geq \rho \geq \rho^{n-1} \geq \rho^n = \left( \frac{0.5 - \frac{1}{16n} - \frac{\sqrt{c}}{cn}}{0.5 + \frac{1}{16n} + \frac{\sqrt{c}}{n}} \right)^n = \left( \frac{1 - \frac{1}{8n} - \frac{2\sqrt{c}}{n}}{1 + \frac{1}{8n} + \frac{2\sqrt{c}}{n}} \right)^n > \left( 1 - 2\left( \frac{1}{8n} + \frac{2\sqrt{c}}{n} \right) \right)^n \geq 1 - 2\left( \frac{1}{8} + 2\sqrt{c} \right) > \frac{1}{2}, \quad (33)
\]
where in the last three transitions we used the inequalities \( \frac{1-x}{1+x} > 1 - 2x \) and \( (1 - x/n)^n \geq 1 - x \) for \( x \in (0, 1) \) and the fact that \( c\sqrt{c} < \frac{1}{16} \). So,
\[
f^*(r_{0.5}) = \frac{1}{1 + \rho^{n-1}} \in \left[ \frac{1}{1 + 1 + 1 + \frac{1}{2}} \right] = \left[ \frac{1}{2}, \frac{2}{3} \right],
\]
which proves the claim. \( \square \)

With Claim \([D.4]\), we can without loss of generality assume \( \frac{1}{2} \leq \hat{f}(r_{0.5}) \leq 2 \) as well (otherwise, we can truncate \( \hat{f}(r_{0.5}) \) to this range; this only reduces the approximation error \( E[(\hat{f}(r) - f^*(r))^2] \)).

**Claim D.5.** For \( \frac{1}{2} \leq x \leq \frac{2}{3} \), \( |h'(x)| \leq \frac{8}{n-1} \).

**Proof.**\[
|h'(x)| = \frac{1}{n-1} \left( \frac{x}{1-x} \right)^{1-n-1} \cdot \frac{1}{x^2} \leq \frac{1}{n-1} \left( \frac{\frac{2}{3}}{1 - \frac{2}{3}} \right)^{1-n-1} \frac{1}{(\frac{1}{2})^2} = \frac{4}{n-1} \cdot 2^{1-n-1} \leq \frac{8}{n-1}. \quad \square
\]

**Claim D.6.** If \( \hat{f} \) is \( \varepsilon \)-optimal, then \( |\hat{f}(r_{0.5}) - f^*(r_{0.5})| < 2\sqrt{\varepsilon} \).

**Proof.** If \( \hat{f} \) is \( \varepsilon \)-optimal, i.e., \( E[(\hat{f}(r) - f^*(r))^2] \leq \varepsilon \), then, by Jensen’s inequality \( E[X^2] \geq E[X]^2 \), we have
\[
\sqrt{\varepsilon} \geq E[|\hat{f}(r) - f^*(r)|] = \sum_r P(r) |\hat{f}(r) - f^*(r)| \geq P(r_{0.5}) |\hat{f}(r_{0.5}) - f^*(r_{0.5})|. \quad (34)
\]
For both \( P \in \{P^1, P^2\} \), we have
\[
P(r_{0.5}) = p \cdot P(r_{0.5} \mid \omega = 1) + (1 - p) \cdot P(r_{0.5} \mid \omega = 0) = p \cdot P(a \mid \omega = 1)^n + (1 - p) \cdot P(a \mid \omega = 0)^n \geq p \cdot 1 + (1 - p) \cdot \left( \frac{1 - \frac{1}{8n} - \frac{2\sqrt{c}}{n}}{1 + \frac{1}{8n} + \frac{2\sqrt{c}}{n}} \right)^n \geq \frac{1}{2}.
\]
Plugging \( P(r_{0.5}) > \frac{1}{2} \) into \((34)\), we get \( |\hat{f}(r_{0.5}) - f^*(r_{0.5})| < 2\sqrt{\varepsilon} \). \( \square \)

From \((32)\), Claim \([D.4]\), Claim \([D.5]\) and Claim \([D.6]\) we get
\[
|\hat{p} - \rho| = |h(\hat{f}(r_{0.5})) - h(f^*(r_{0.5}))| \leq \frac{8}{n-1} |\hat{f}(r_{0.5}) - f^*(r_{0.5})| < \frac{8}{n-1} \cdot 2\sqrt{\varepsilon} = \frac{16}{n-1} \sqrt{\varepsilon}. \quad (35)
\]
Since \( p = \frac{\rho}{1+\rho} \) as a function of \( \rho \) has a bounded derivative \( \frac{\partial p}{\partial \rho} = \frac{1}{(1+\rho)^2} \leq 1 \), Equation \((35)\) implies
\[
|\hat{p} - p| < \frac{16}{n-1} \sqrt{\varepsilon}
\]
if we use \( \hat{p} = \frac{\hat{p}}{1 + \hat{p}} \) as an estimate of \( p \). This allows us to tell part \( P^1 \) and \( P^2 \) because the difference between \( p^1 \) and \( p^2 \) is greater than twice of our estimation error \( |\hat{p} - p| \):

\[
|p^1 - p^2| = \frac{2c\sqrt{\varepsilon}}{n} = \frac{64\sqrt{\varepsilon}}{n} \geq \frac{64\sqrt{\varepsilon}}{2(n-1)} = \frac{32\sqrt{\varepsilon}}{n-1} > 2|\hat{p} - p|.
\]

Therefore, we can tell part \( P^1 \) and \( P^2 \) by checking whether \( p^1 \) or \( p^2 \) is closer to \( \hat{p} \).

Finally, we upper bound the squared Hellinger distance between \( P^1 \) and \( P^2 \). This will give the sample complexity lower bound we want.

**Claim D.7.** \( d^2_H(P^1, P^2) \leq O(c^2 \varepsilon) \).

**Proof.** For the marginal distributions of \( \omega \), \( P^1_\omega \) and \( P^2_\omega \), according to Lemma \[A.3\] and the fact that

\[
1 \geq \frac{P^2(\omega)}{P^1(\omega)} = \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 + \frac{2c\sqrt{\varepsilon}}{n}} = 1 + \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 + \frac{2c\sqrt{\varepsilon}}{n}} \geq 1 + \frac{1 - 8n}{1 + 8n} = 1 - O\left(\frac{c\sqrt{\varepsilon}}{n}\right),
\]

we have

\[
d^2_H(P^1_\omega, P^2_\omega) \leq O\left(\frac{(c\sqrt{\varepsilon})^2}{n}\right) = O\left(\frac{c^2 \varepsilon}{n^2}\right).
\]

Given \( \omega = 0 \) or \( 1 \), we consider the conditional distributions of each \( s_i \), \( P^1_{s_i|\omega} \) and \( P^2_{s_i|\omega} \). For \( s_i = a \), we have

\[
1 \geq \frac{P^2(a \mid \omega)}{P^1(a \mid \omega)} = \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 + \frac{2c\sqrt{\varepsilon}}{n}} \cdot \frac{1 + \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 - \frac{1 - 8n}{1 + 8n}}}{1 - \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 - \frac{1 - 8n}{1 + 8n}}} = 1 - \frac{4c\sqrt{\varepsilon}}{1 + 16c\sqrt{\varepsilon}} = 1 - O\left(\frac{c\sqrt{\varepsilon}}{n}\right).
\]

For \( s_i = b \), we have

\[
1 \geq \frac{P^1(b \mid \omega)}{P^2(b \mid \omega)} = \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 + \frac{2c\sqrt{\varepsilon}}{n}} \cdot \frac{1 + \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 - \frac{4c\sqrt{\varepsilon}}{n}}}{1 + \frac{4c\sqrt{\varepsilon}}{n}} = 1 - \frac{4c\sqrt{\varepsilon}}{1 + 16c\sqrt{\varepsilon}} = 1 - O\left(\frac{c\sqrt{\varepsilon}}{n}\right).
\]

So, \( d^2_H(P^1_{s_i|\omega}, P^2_{s_i|\omega}) \) can be upper bounded as follows:

\[
d^2_H(P^1_{s_i|\omega}, P^2_{s_i|\omega}) = \frac{1}{2} \left[ \left( \sqrt{P^1(a \mid \omega)} - \sqrt{P^2(a \mid \omega)} \right)^2 + \left( \sqrt{P^1(b \mid \omega)} - \sqrt{P^2(b \mid \omega)} \right)^2 \right] 
\]

\[
= \frac{1}{2} \left[ \frac{P^1(a \mid \omega)(1 - \sqrt{P^2(a \mid \omega)})^2 + P^2(b \mid \omega)(1 - \sqrt{P^1(b \mid \omega)})^2}{1 + \frac{1 - \frac{4c\sqrt{\varepsilon}}{n}}{1 + \frac{4c\sqrt{\varepsilon}}{n}}} \cdot \left( \frac{1}{4n} + \frac{4c\sqrt{\varepsilon}}{n} \right) \cdot \left( 1 - \sqrt{1 - O\left(\frac{c\sqrt{\varepsilon}}{n}\right)} \right)^2 \right] 
\]

(since \( 1 - \sqrt{1 - x} \leq x \))

\[
\leq \frac{1}{2} \left[ O\left(\frac{c\sqrt{\varepsilon}}{n}\right)^2 + O\left(\frac{1}{n}\right) \cdot O\left(c\sqrt{\varepsilon}\right)^2 \right] 
\]

\[
= O\left(\frac{c^2 \varepsilon}{n}\right).
\]
Since $P^1 = P^1 \omega \prod_{i=1}^n P^1_{s_i | \omega}$ and $P^2 = P^2 \omega \prod_{i=1}^n P^2_{s_i | \omega}$, we have, by Lemma [A.4] and Lemma [A.5]

$$d_H^2(P^1, P^2) \leq d_H^2(P^1, P^2) + \max_{\omega \in \{0, 1\}} \left\{ d_H^2 \left( \prod_{i=1}^n P^1_{s_i | \omega}, \prod_{i=1}^n P^2_{s_i | \omega} \right) \right\}$$

$$\leq d_H^2(P^1, P^2) + \max_{\omega \in \{0, 1\}} \left\{ n \cdot d_H^2(P^1_{s_i | \omega}, P^2_{s_i | \omega}) \right\}$$

and

$$\leq O\left( \frac{c^2 \varepsilon}{n^2} \right) + \max_{\omega \in \{0, 1\}} \left\{ n \cdot O\left( \frac{c^2 \varepsilon}{n} \right) \right\} = O(c^2 \varepsilon).$$

Therefore, according to Lemma [A.6] to tell apart $P^1$ and $P^2$ with probability at least $1 - \delta$ we need at least

$$T = \Omega\left( \frac{1}{d_H^2(P^1, P^2)} \log \frac{1}{\delta} \right) = \Omega\left( \frac{1}{c^2 \varepsilon} \log \frac{1}{\delta} \right)$$

samples. This concludes the proof.

### D.4 Proof of Theorem 4.4

#### D.4.1 Additional Notations and Lemmas

We introduce some additional notations and lemmas for the proof. Let $\mu_0$ be the expected average report of all experts conditioning on $\omega = 0$:

$$\mu_0 = \frac{1}{n} \sum_{i=1}^n E[r_i | \omega = 0] = \frac{1}{n} \sum_{i=1}^n E_{s_i | \omega = 0}\left[ P(\omega = 1 | s_i) | \omega = 0 \right],$$

which is equal to the expected prediction of $\omega$ given expert $i$’s signal $s_i$ where $s_i$ is distributed conditioning on $\omega = 0$, averaged over all experts. Symmetrically, let

$$\mu_1 = \frac{1}{n} \sum_{i=1}^n E[1 - r_i | \omega = 1] = \frac{1}{n} \sum_{i=1}^n E_{s_i | \omega = 1}\left[ P(\omega = 0 | s_i) | \omega = 1 \right].$$

Recall that $p = P(\omega = 1)$.

**Fact D.8.** $(1 - p) \mu_0 = p \mu_1$.

**Proof.** For each expert $i$, by the law of total expectation and the fact that $r_i = P(\omega = 1 | s_i)$, we have the following equations:

$$(1 - p) \cdot E[r_i | \omega = 0] + p \cdot E[r_i | \omega = 1] = P(\omega = 0) \cdot E[r_i | \omega = 0] + P(\omega = 1) \cdot E[r_i | \omega = 1]$$

$$= E[r_i]$$

$$= E_{s_i} \left[ P(\omega = 1 | s_i) \right] = P(\omega = 1) = p.$$

Subtracting $p$ from both sides, we get

$$(1 - p) \cdot E[r_i | \omega = 0] - p \cdot E[1 - r_i | \omega = 1] = 0.$$

Averaging over all experts $i \in \{1, \ldots, n\}$, we conclude that

$$(1 - p) \cdot \frac{1}{n} \sum_{i=1}^n E[r_i | \omega = 0] - p \cdot \frac{1}{n} \sum_{i=1}^n E[1 - r_i | \omega = 1] = 0.$$

\[\square\]
Lemma D.9. If \((1 - p)\mu_0 = p \mu_1 < \frac{\varepsilon}{\delta}\), then the averaging aggregator \(f_{\text{avg}}(r) = \frac{1}{n} \sum_{i=1}^{n} r_i\) is \(\varepsilon\)-optimal.

Proof. If \((1 - p)\mu_0 = p \mu_1 < \frac{\varepsilon}{\delta}\), then the expected loss of \(f_{\text{avg}}\) is at most

\[
L_P(f_{\text{avg}}) = \mathbb{E}[(f_{\text{avg}} - \omega)^2] \\
= p \mathbb{E}[(f_{\text{avg}}(r) - 1)^2 \mid \omega = 1] + (1 - p) \mathbb{E}[(f_{\text{avg}}(r) - 0)^2 \mid \omega = 0] \\
\leq p \mathbb{E}[1 - f_{\text{avg}}(r) \mid \omega = 1] + (1 - p) \mathbb{E}[f_{\text{avg}}(r) \mid \omega = 0] \\
= p \mathbb{E}[1 - \frac{1}{n} \sum_{i=1}^{n} r_i \mid \omega = 1] + (1 - p) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} r_i \mid \omega = 0\right] \\
= p \mu_1 + (1 - p) \mu_0 \\
< \varepsilon + \varepsilon = \varepsilon,
\]

which implies that \(f_{\text{avg}}\) is \(\varepsilon\)-optimal.

The following lemma says that \(O\left(\frac{1}{\varepsilon} \log \frac{1}{\delta}\right)\) samples are sufficient to tell whether the mean of a random variable is below \(\varepsilon\) or above \(\frac{\varepsilon}{2}\):

Lemma D.10. Given \(T = \frac{40}{\varepsilon} \log \frac{2}{\delta}\) i.i.d. samples \(X^{(1)}, \ldots, X^{(T)}\) of a random variable \(X \in [0, 1]\) with unknown mean \(\mathbb{E}[X] = \mu\), with probability at least 1 - \(\delta\) we can tell whether \(\mu < \varepsilon\) or \(\mu \geq \frac{\varepsilon}{2}\). This can be done by checking whether the empirical mean \(\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} X^{(t)}\) is < \(\frac{3}{4}\varepsilon\) or \(\geq \frac{3}{4}\varepsilon\).

Proof. If \(\mu \geq \varepsilon\), using the multiplicative version of Chernoff bound we have

\[
\Pr\left[\hat{\mu} < \frac{3}{4}\varepsilon\right] \leq \Pr\left[\hat{\mu} < \frac{3}{4}\mu\right] \leq e^{-\frac{(\frac{3}{4}\mu - \frac{3}{4}\varepsilon)^2}{2\varepsilon}} \leq e^{-\frac{\varepsilon^2}{8\varepsilon}} \leq \delta.
\]

Namely, with probability at least 1 - \(\delta\), it holds that

\[
\hat{\mu} \geq \frac{3}{4}\varepsilon.
\]

If \(\mu < \varepsilon\), then using the additive version of Chernoff–Hoeffding theorem, we have

\[
\Pr\left[\hat{\mu} < \mu - \frac{\varepsilon}{4}\right] \leq e^{-D(\mu - \frac{\varepsilon}{4} || \mu)T}, \\
\Pr\left[\hat{\mu} > \mu + \frac{\varepsilon}{4}\right] \leq e^{-D(\mu + \frac{\varepsilon}{4} || \mu)T},
\]

where \(D(x||y) = x \ln \frac{x}{y} + (1 - x) \ln \frac{1 - x}{1 - y}\). Using the inequality \(D(x||y) \geq \frac{(x-y)^2}{2y}\) for \(x \leq y\) and \(D(x||y) \geq \frac{(x-y)^2}{2x}\) for \(x \geq y\), we obtain:

\[
\Pr\left[\hat{\mu} < \mu - \frac{\varepsilon}{4}\right] \leq e^{-\frac{(\frac{\varepsilon}{4})^2}{2\mu} T} \leq e^{-\frac{\varepsilon^2}{16\mu} T} = e^{-\frac{\varepsilon^2}{16\mu} T} \leq \frac{\delta}{2}, \\
\Pr\left[\hat{\mu} > \mu + \frac{\varepsilon}{4}\right] \leq e^{-\frac{(\frac{\varepsilon}{4})^2}{2(\mu + \frac{\varepsilon}{4})} T} \leq e^{-\frac{\varepsilon^2}{32(\mu + \frac{\varepsilon}{4})} T} = e^{-\frac{\varepsilon^2}{32(\mu + \frac{\varepsilon}{4})} T} \leq \frac{\delta}{2}.
\]

By a union bound, with probability at least 1 - \(\delta\), we have

\[
|\hat{\mu} - \mu| \leq \frac{\varepsilon}{4}.
\]

Combining the case of \(\mu \geq \varepsilon\) and \(\mu < \varepsilon\), we conclude that: with probability at least 1 - \(\delta\),
\begin{itemize}
  \item If $\hat{\mu} < \frac{3}{4} \varepsilon$, then we must have $\mu < \varepsilon$.
  \item If $\hat{\mu} \geq \frac{3}{4} \varepsilon$, then we have $\mu \geq \varepsilon$ or $\varepsilon \geq \hat{\mu} - \frac{\varepsilon}{4} \geq \frac{\varepsilon}{2}$. In either case, we have $\mu \geq \frac{\varepsilon}{2}$.
\end{itemize}

The last lemma we will use shows how to estimate the unknown value of $\rho = \frac{p}{1-p} = \frac{P(\omega=1)}{P(\omega=0)}$ with accuracy $\Delta$ using $T = O\left(\frac{1}{n \Delta^2} \log \frac{1}{\delta}\right)$ samples. Notice that, if one simply uses the empirical value $\hat{\rho} = \frac{\sum_{t=1}^{T} (\omega^{(t)}=1)}{\sum_{t=1}^{T} (\omega^{(t)}=0)}$ to estimate $\rho$, then by Chernoff bound this needs $T = O\left(\frac{1}{\Delta^2} \log \frac{1}{\delta}\right)$ samples, which is larger than what we claim by a factor of $n$. This sub-optimality is because one only uses the $\omega^{(t)}$'s in the samples to estimate $\rho$, wasting the reports $r^{(t)}_i$'s. By using $r^{(t)}_i$'s to estimate $\rho$, we can reduce the number of samples by a factor of $n$. The basic idea is the following: According to Fact D.8 we have $\rho = \frac{p}{1-p} = \frac{\mu_0}{\mu_1} = \frac{\mathbb{E}[\sum_{i=1}^{n} r_i | \omega=0]}{\mathbb{E}[\sum_{i=1}^{n} (1-r_i) | \omega=1]}$. The numerator $\mathbb{E}[\sum_{i=1}^{n} r_i | \omega=0]$ and the denominator $\mathbb{E}[\sum_{i=1}^{n} (1-r_i) | \omega=1]$ can be estimated from samples of $r^{(t)}_i$'s where $\omega^{(t)} = 0$ and 1 respectively. The total number of $r^{(t)}_i$'s is $Tn$, because we have $n$ experts per sample. This reduces the needed number of samples by a factor of $n$. Formally:

**Lemma D.11.** For conditionally independent distribution $P$, we can estimate $\rho = \frac{P(\omega=1)}{P(\omega=0)}$ with accuracy $\frac{\hat{\rho} - \rho}{\rho} \leq \Delta < 1$ (equivalently, $\hat{\rho} \in 1 \pm \Delta$) with probability at least $1 - \delta$ using

$$T = O\left(\frac{1}{(1-p)\mu_0 \cdot n \cdot \Delta^2} \log \frac{1}{\delta} + \frac{1}{\min\{p, 1-p\}} \log \frac{1}{\delta}\right)$$

samples of $(\omega^{(t)}, r^{(t)})$'s, by letting $\hat{\rho} = \frac{\frac{1}{\#0} \sum_{t: \omega^{(t)}=0} \sum_{i=1}^{n} r^{(t)}_i}{\frac{1}{\#1} \sum_{t: \omega^{(t)}=1} \sum_{i=1}^{n} (1-r^{(t)}_i)}$, where $\#0$ and $\#1$ are the numbers of samples with $\omega^{(t)} = 0$ and 1 respectively.

**Proof.** Recall that $p = P(\omega = 1)$, $1-p = P(\omega = 0)$, and $\rho = \frac{p}{1-p}$. According to Fact D.8 (which says $(1-p)\mu_0 = p\mu_1$), we have

$$\rho = \frac{p}{1-p} = \frac{\mu_0}{\mu_1} = \frac{\sum_{i=1}^{n} \mathbb{E}[r_i | \omega = 0]}{\sum_{i=1}^{n} \mathbb{E}[1-r_i | \omega = 1]} \quad (40)$$

Consider the following way of estimating $\rho$ from $T$ samples $(\omega^{(t)}, r^{(t)})_{t=1}^{T}$: Let $\#0$, $\#1$ be the numbers of samples where $\omega^{(t)} = 0$, 1, respectively:

$$\#0 = \sum_{t=1}^{T} \mathbb{1}\{\omega^{(t)} = 0\}, \quad \#1 = \sum_{t=1}^{T} \mathbb{1}\{\omega^{(t)} = 1\}.$$ 

We let

$$\hat{\rho} = \frac{\frac{1}{\#0} \sum_{t: \omega^{(t)}=0} \sum_{i=1}^{n} r^{(t)}_i}{\frac{1}{\#1} \sum_{t: \omega^{(t)}=1} \sum_{i=1}^{n} (1-r^{(t)}_i)} \quad (41)$$

Now, we compare the $\hat{\rho}$ in (41) and the $\rho$ in (40): we see that $\frac{1}{\#0} \sum_{t: \omega^{(t)}=0} \sum_{i=1}^{n} r^{(t)}_i$ is an (unbiased) estimate of the numerator $\sum_{i=1}^{n} \mathbb{E}[r_i | \omega = 0] = n\mu_0$ and that $\frac{1}{\#1} \sum_{t: \omega^{(t)}=1} \sum_{i=1}^{n} (1-r^{(t)}_i)$ is an (unbiased) estimate of the denominator $\sum_{i=1}^{n} \mathbb{E}[1-r_i | \omega = 1] = n\mu_1$. We use Chernoff bounds to argue that the accuracy of the two estimates is within $\Delta$ with high probability if $\#0$ and $\#1$ are big enough. Suppose that, when drawing the $T$ samples, we draw all the $\omega^{(t)}$'s first (and hence
#0, #1 are determined), and then draw all the \( r_i^{(t)} \)'s. After all the \( \omega^{(t)} \)'s are drawn, the \( r_i^{(t)} \)'s become independent, because the signals \( s_i^{(t)} \)'s are conditionally independent given \( \omega^{(t)} \). Therefore, we can use Chernoff bounds:

\[
\Pr \left[ \frac{1}{#_0} \sum_{t, \omega^{(t)} = 0} \sum_{i=1}^{n} r_i^{(t)} - n \mu_0 > \Delta n \mu_0 \right] \leq 2e^{-\frac{#_0 \mu_0 \Delta^2}{4}},
\]

\[
\Pr \left[ \frac{1}{#_1} \sum_{t, \omega^{(t)} = 1} \sum_{i=1}^{n} (1 - r_i^{(t)}) - n \mu_1 > \Delta n \mu_1 \right] \leq 2e^{-\frac{#_1 \mu_1 \Delta^2}{4}}.
\]

Requiring \( \delta \geq 2e^{-\frac{#_0 \mu_0 \Delta^2}{4}} \) and \( \delta \geq 2e^{-\frac{#_1 \mu_1 \Delta^2}{4}} \), namely,

\[
#_0 \geq \frac{3}{n \mu_0 \Delta^2} \log \frac{2}{\delta}, \quad #_1 \geq \frac{3}{n \mu_1 \Delta^2} \log \frac{2}{\delta},
\]

we have, with probability at least \( 1 - 2\delta \), both of the following hold:

\[
\frac{1}{#_0} \sum_{t, \omega^{(t)} = 0} \sum_{i=1}^{n} r_i^{(t)} \in (1 \pm \Delta) n \mu_0,
\]

\[
\frac{1}{#_1} \sum_{t, \omega^{(t)} = 1} \sum_{i=1}^{n} (1 - r_i^{(t)}) \in (1 \pm \Delta) n \mu_1,
\]

Then, we argue that (42) can be satisfied with high probability if \( T \) is large enough. This is again done by a Chernoff bound: since \( E[#_j] = E[\sum_{t=1}^{T} 1(\omega^{(t)} = j)] = T \cdot P(\omega = j) \), for \( j = 0, 1 \), we have

\[
\Pr \left[ \#_0 - T(1 - p) \geq \frac{1}{2} T(1 - p) \right] \leq 2e^{-\frac{T(1 - p)(\frac{1}{2} - p)}{4}}, \quad \Pr \left[ \#_1 - Tp \geq \frac{1}{2} Tp \right] \leq 2e^{-\frac{Tp(\frac{1}{2} - p)}{4}}. \]

So, if we are given

\[
T \geq \frac{12}{\min\{p, 1 - p\}} \log \frac{2}{\delta}
\]

samples, then we can ensure that with probability at least \( 1 - 2\delta \), it holds \( #_0 \geq \frac{1}{2} T(1 - p) \) and \( #_1 \geq \frac{1}{2} Tp \). Then, in order for (42) to be satisfied, we can let

\[
\frac{1}{2} T(1 - p) \geq \frac{3}{n \mu_0 \Delta^2} \log \frac{2}{\delta}, \quad \frac{1}{2} Tp \geq \frac{3}{n \mu_1 \Delta^2} \log \frac{2}{\delta}.
\]

This gives

\[
T \geq \max \left\{ \frac{6}{(1 - p) \mu_0 \cdot n \Delta^2} \log \frac{2}{\delta}, \frac{6}{p \mu_1 \cdot n \Delta^2} \log \frac{2}{\delta}, \frac{6}{(1 - p) \mu_0 \cdot n \Delta^2} \log \frac{2}{\delta} \right\} \quad (1 - p) \mu_0 = p \mu_1 \quad (1 - p) \mu_0 \cdot n \Delta^2 \log \frac{2}{\delta}. \]

Both (45) and (46) are satisfied when

\[
T \geq \frac{6}{(1 - p) \mu_0 \cdot n \Delta^2} \log \frac{2}{\delta} + \frac{12}{\min\{p, 1 - p\}} \log \frac{2}{\delta}.
\]

To conclude, if we are given \( T = \frac{6}{(1 - p) \mu_0 \cdot n \Delta^2} \log \frac{2}{\delta} + \frac{12}{\min\{p, 1 - p\}} \log \frac{2}{\delta} \) samples, then with probability at least \( 1 - 4\delta \), (43) holds, which implies

\[
\hat{\rho} \in \frac{(1 \pm \Delta) \mu_0}{(1 \pm \Delta) \mu_1} = \frac{(1 \pm \Delta)}{(1 \pm \Delta)} \rho \subseteq (1 \pm 4\Delta) \rho \quad \implies \quad |\hat{\rho} - \rho| \leq 4\Delta,
\]

for \( \Delta < \frac{1}{4} \).
D.4.2 The proof

We want to show the $O\left(\frac{1}{\epsilon n} \log \frac{1}{\epsilon n} + \frac{1}{\epsilon n} \log \frac{1}{\epsilon n}\right)$ sample complexity upper bound for the case where experts have $\gamma$-strongly informative signals.

We first use $O\left(\frac{1}{\epsilon n} \log \frac{1}{\epsilon n}\right)$ samples to tell whether $(1 - p)\mu_0 = p\mu_1 < \frac{\epsilon}{2}$ or $(1 - p)\mu_0 = p\mu_1 \geq \frac{\epsilon}{2}$. We note that

$$(1 - p)\mu_0 = P(\omega = 0) \cdot \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} r_i \mid \omega = 0\right] = \mathbb{E}\left[1\{\omega = 0\} \cdot \frac{1}{n} \sum_{i=1}^{n} r_i\right],$$

which is the expectation of the random variable $X = 1\{\omega = 0\} \cdot \frac{1}{n} \sum_{i=1}^{n} r_i$. So, according to Lemma D.10, we can tell whether $(1 - p)\mu_0 < \frac{\epsilon}{2}$ or $(1 - p)\mu_0 \geq \frac{\epsilon}{2}$ using $O\left(\frac{1}{\epsilon n} \log \frac{1}{\epsilon n}\right)$ samples of $X$. If $(1 - p)\mu_0 = p\mu_1 < \frac{\epsilon}{2}$, then according to Lemma D.9 the averaging aggregator $f_{\text{avg}}(r) = \frac{1}{n} \sum_{i=1}^{n} r_i$ is $\varepsilon$-optimal. We hence obtained an $\varepsilon$-optimal aggregator in this case. So, in the following proof, we assume $(1 - p)\mu_0 = p\mu_1 \geq \frac{\epsilon}{2}$.

For each expert $i$, let $S_i^1 = \{s_i \in S_i : \frac{P(s_i, \omega = 1)}{P(s_i, \omega = 0)} \geq 1 + \gamma\}$ be its set of $\gamma$-strongly informative signals that are more likely to be realized under $\omega = 1$ than under $\omega = 0$. Let $S_i^0 = S_i \setminus S_i^1 = \{s_i \in S_i : \frac{P(s_i, \omega = 1)}{P(s_i, \omega = 0)} \leq \frac{1}{1 + \gamma}\}$ be the set of signals that are more likely to be realized under $\omega = 0$. Since $\frac{r_i}{1 - r_i} = \frac{P(s_i, \omega = 1)}{P(s_i, \omega = 0)} \rho$ by Equation (1), whenever an expert receives a signal in $S_i^1$, its report satisfies

$$\frac{r_i}{1 - r_i} \geq (1 + \gamma)\rho, \quad \forall s_i \in S_i^1; \quad (47)$$

and whenever it receives a signal in $S_i^0$, its report satisfies

$$\frac{r_i}{1 - r_i} \leq \frac{1}{1 + \gamma} \rho, \quad \forall s_i \in S_i^0. \quad (48)$$

We will use the notation $P(S_i^u \mid \omega) = P(s_i \in S_i^u \mid \omega) = \sum_{s_i \in S_i^u} P(s_i \mid \omega)$, for $u \in \{0, 1\}$. Given a set of $n$ signals $s_1, \ldots, s_n$, one per expert, we let $X_1 = \sum_{i=1}^{n} 1\{s_i \in S_i^1\}$ be the total number of signals that belong to the $S_i^1$ sets, and similarly let $X_0 = \sum_{i=1}^{n} 1\{s_i \in S_i^0\}$. We have $X_0 + X_1 = n$, and by definition,

$$\mathbb{E}[X_1 \mid \omega = 1] = \sum_{i=1}^{n} P(S_i^1 \mid \omega = 1) \geq (1 + \gamma)P(S_i^1 \mid \omega = 0) = (1 + \gamma)\mathbb{E}[X_1 \mid \omega = 0]. \quad (49)$$

$$\mathbb{E}[X_0 \mid \omega = 0] = \sum_{i=1}^{n} P(S_i^0 \mid \omega = 0) \geq (1 + \gamma)P(S_i^0 \mid \omega = 1) = (1 + \gamma)\mathbb{E}[X_0 \mid \omega = 1]. \quad (50)$$

Claim D.12. At least one of $\mathbb{E}[X_1 \mid \omega = 1]$ and $\mathbb{E}[X_0 \mid \omega = 0]$ is $\geq \frac{n}{2}$.

Proof. Suppose on the contrary both $\mathbb{E}[X_1 \mid \omega = 1]$ and $\mathbb{E}[X_0 \mid \omega = 0]$ are $< \frac{n}{2}$. Then, from (50), we have

$$\mathbb{E}[X_0 \mid \omega = 1] \leq \frac{1}{1 + \gamma} \mathbb{E}[X_0 \mid \omega = 0] < \frac{n}{2},$$

This implies $n = \mathbb{E}[X_0 + X_1 \mid \omega = 1] < \frac{n}{2} + \frac{n}{2} = n$, a contradiction. \qed
Let $u \in \{0, 1\}$ be an index such that
\[
\mathbb{E}[X^u \mid \omega = u] \geq \frac{n}{4}.
\] (51)

Claim D.12 guarantees that such a $u$ exists. We construct a “hypothetical” aggregator $f_{\text{hypo}}$ that, having access to $\rho$ and $\mathbb{E}[X^u \mid \omega]$, predicts whether $\omega = 0$ or 1 by counting the number $X^u$ of signals that belong to the $S_i^u$ sets and comparing it with its expectations under $\omega = 0$ and 1, $\mathbb{E}[X^u \mid \omega = 0]$ and $\mathbb{E}[X^u \mid \omega = 1]$, respectively. Specifically, given reports $r = (r_1, \ldots, r_n)$ as input, with corresponding unobserved signals $s = (s_1, \ldots, s_n)$, $f_{\text{hypo}}$ does the following:

1. If $u = 1$, count how many reports $r_i$’s satisfy $\frac{r_i}{1 - r_i} \geq (1 + \gamma)\rho$; If $u = 0$, count how many reports $r_i$’s satisfy $\frac{r_i}{1 - r_i} \leq \frac{1}{1 + \gamma}\rho$. According to (47) and (48), this number is exactly equal to the number of signals that belong to the $S_i^u$ sets, $X^u$.

2. Then, check whether $X^u$ is closer (in terms of absolute difference) to $\mathbb{E}[X^u \mid \omega = u]$ or $\mathbb{E}[X^u \mid \omega = 1 - u]$. If $X^u$ is closer to $\mathbb{E}[X^u \mid \omega = u]$, output $f_{\text{hypo}}(r) = u$; otherwise, output $f_{\text{hypo}}(r) = 1 - u$.

We claim that $f_{\text{hypo}}$ is $\varepsilon$-optimal.

**Claim D.13.** Given $\frac{\beta}{1 + \gamma} \geq 8\sqrt{\frac{2}{n} \log \frac{2}{\varepsilon}}$ and $\mathbb{E}[X^u \mid \omega = u] \geq \frac{n}{4}$, $f_{\text{hypo}}$ is $\varepsilon$-optimal.

**Proof.** Given either $\omega = 0$ or 1, consider the conditional random draw of signals $s_1, \ldots, s_n$. Because $X^u = \sum_{i=1}^n 1\{s_i \in S_i^u\}$ and the random variables $1\{s_i \in S_i^u\}, i = 1, \ldots, n$, are $[0,1]$-bounded and independent conditioning $\omega$, by Hoeffding’s inequality we have
\[
\Pr\left[|X^u - \mathbb{E}[X^u \mid \omega]| \geq a \mid \omega\right] \leq 2e^{-\frac{2a^2}{n}}.
\]
Let
\[
a = \sqrt{\frac{n}{2} \log \frac{2}{\varepsilon}}.
\] (52)

Then with probability at least $1 - 2e^{-\frac{2a^2}{n}} = 1 - \varepsilon$, it holds
\[
|X^u - \mathbb{E}[X^u \mid \omega]| < a.
\] (53)

Consider the difference between $\mathbb{E}[X^u \mid \omega = u]$ and $\mathbb{E}[X^u \mid \omega = 1 - u]$. By (49) and (50), we have
\[
\mathbb{E}[X^u \mid \omega = 1 - u] \leq \frac{1}{1 + \gamma}\mathbb{E}[X^u \mid \omega = u] = \left(1 - \frac{\gamma}{1 + \gamma}\right)\mathbb{E}[X^u \mid \omega = u].
\]

By the assumption $\mathbb{E}[X^u \mid \omega = u] \geq \frac{n}{4}$,
\[
\mathbb{E}[X^u \mid \omega = 1 - u] \leq \mathbb{E}[X^u \mid \omega = u] - \frac{\gamma}{1 + \gamma} \cdot \frac{n}{4}.
\]

By the assumption $\frac{\gamma}{1 + \gamma} \geq 8\sqrt{\frac{2}{n} \log \frac{2}{\varepsilon}}$, we have $\frac{\gamma}{1 + \gamma} \cdot \frac{n}{4} \geq 8\sqrt{\frac{2}{n} \log \frac{2}{\varepsilon}} \cdot \frac{n}{4} = 4\sqrt{\frac{n}{2} \log \frac{2}{\varepsilon}} = 4a$. Therefore
\[
\mathbb{E}[X^u \mid \omega = u] - \mathbb{E}[X^u \mid \omega = 1 - u] \geq 4a.
\] (54)

Because we already had $|X^u - \mathbb{E}[X^u \mid \omega]| < a$ (which happened with probability at least $1 - \varepsilon$), if $X^u$ turns out to be closer to $\mathbb{E}[X^u \mid \omega = u]$ it must be that $\omega = u$; if $X^u$ turns out to be closer
to $\mathbb{E}[X^u \mid \omega = 1 - u]$ it must be that $\omega = 1 - u$. In either case, our output $f_{\text{hypo}}(r)$ is equal to $\omega$, having a loss 0. If $|X^u - \mathbb{E}[X^u \mid \omega]| < a$ did not happen, our loss is at most 1. Therefore, the expected loss of our aggregator $f_{\text{hypo}}$ is at most

$$L_P(f_{\text{hypo}}) = \mathbb{E}_\omega \left[ \mathbb{E}[|f_{\text{hypo}}(r) - \omega|^2 \mid \omega] \right] \leq \mathbb{E}_\omega \left[ (1 - \varepsilon) \cdot 0 + \varepsilon \cdot 1 \right] = \varepsilon.$$

Since the expected loss of the optimal aggregator $f^*$ is non-negative, $f_{\text{hypo}}$ is $\varepsilon$-optimal.

In the remaining proof, we show how to use samples to learn a “real” aggregator $\hat{f}$ that implements the same functionality as the hypothetical aggregator $f_{\text{hypo}}$ and hence is $\varepsilon$-optimal. We have two learning tasks: First, we need to estimate $\rho$, so that we can implement the step (1) of $f_{\text{hypo}}$ which tells apart $\hat{r}_i - \hat{r}_j \geq (1 + \gamma)\rho$ and $\hat{r}_i - \hat{r}_j \leq \frac{1}{1 + \gamma}\rho$. Second, we need to find an index $u \in \{0, 1\}$ such that $\mathbb{E}[X^u \mid \omega = u] \geq \frac{\varepsilon}{4}$ and estimate $\mathbb{E}[X^u \mid \omega = u]$, so that we can implement the step (2) of $f_{\text{hypo}}$ which tells whether $X^u$ is closer to $\mathbb{E}[X^u \mid \omega = u]$ or $\mathbb{E}[X^u \mid \omega = 1 - u]$. We show that these two tasks can be achieved using $O\left(\frac{1}{\varepsilon} \log \frac{1}{\delta} + \frac{1}{\varepsilon} \log \frac{1}{\delta_0} \right)$ samples, with probability at least $1 - O(\delta)$.

**Task 1: estimate $\rho$, using $T_1 = O\left(\frac{1}{\varepsilon} \log \frac{1}{\delta} + \frac{1}{\varepsilon} \log \frac{1}{\delta_0} \right)$ samples.** We want to use samples to obtain an estimate $\hat{\rho}$ of $\rho$ such that $\frac{1}{1 + \gamma}\rho < \hat{\rho} < (1 + \gamma)\rho$. So, by checking whether $\hat{r}_{i} - \hat{r}_{j} > \hat{\rho}$ or $\hat{r}_{i} - \hat{r}_{j} < \hat{\rho}$ we can tell apart $\hat{r}_{i} - \hat{r}_{j} \geq (1 + \gamma)\rho$ and $\hat{r}_{i} - \hat{r}_{j} \leq \frac{1}{1 + \gamma}\rho$. Using Lemma [D.11] with $\Delta = \frac{\gamma}{1 + \gamma}$, we obtain a $\hat{\rho}$ such that

$$\hat{\rho} \in (1 \pm \Delta)\rho,$$

with probability at least $1 - \delta$ using

$$T_1 = O\left(\frac{1}{(1 - p)\mu_0 n\Delta^2} \log \frac{1}{\delta} + \frac{1}{\min\{p, 1 - p\}} \log \frac{1}{\delta_0} \right) \leq O\left(\frac{1}{\varepsilon n(\frac{\gamma}{1 + \gamma})^2} \log \frac{1}{\delta} + \frac{1}{\varepsilon} \log \frac{1}{\delta_0}\right)$$

samples (recall that we have $\min\{p, 1 - p\} \geq (1 - p)\mu_0 = p\mu_1 \geq \frac{\varepsilon_0}{4}$). The $\hat{\rho}$ then satisfies

$$\hat{\rho} < \left(1 + \frac{\gamma}{1 + \gamma}\right)\rho < (1 + \gamma)\rho \quad \text{and} \quad \hat{\rho} > \left(1 - \frac{\gamma}{1 + \gamma}\right)\rho = \frac{1}{1 + \gamma}\rho,$$

as desired.

**Task 2: find $u$ such that $\mathbb{E}[X^u \mid \omega = u] \geq \frac{\varepsilon}{4}$ and estimate $\mathbb{E}[X^u \mid \omega = u]$, using $T_2 = O\left(\frac{1}{\varepsilon} \log \frac{1}{\delta_0}\right)$ samples.** First, we show how to use $T_2 = O\left(\frac{1}{\varepsilon} \log \frac{1}{\delta_0}\right)$ samples to estimate both $\mathbb{E}[X^0 \mid \omega = 0]$ and $\mathbb{E}[X^1 \mid \omega = 1]$ with accuracy $a = \sqrt{\frac{2}{\varepsilon} \log \frac{2}{\delta_0}}$. By the same argument as in the proof of Lemma [D.11] (Equations [44] and [45]), we know that with probability at least $1 - 2\delta$ over the random draws of

$$T_2 \geq \frac{12}{\min\{p, 1 - p\}} \log \frac{2}{\delta}$$

samples, the numbers of samples $(\omega^{(t)}, r_1^{(t)}, \ldots, r_n^{(t)})$’s where $\omega^{(t)} = 0$ and $\omega^{(t)} = 1$, denoted by $\#_0$ and $\#_1$, must satisfy

$$\#_0 \geq \frac{1}{2}(1 - p)T_2, \quad \#_1 \geq \frac{1}{2}pT_2.$$
We consider the samples where $\omega^{(t)} = 0$. There are $\#_0 n$ total number of $r_i^{(t)}$’s. Suppose we have accomplished Task 1. Then, for each $r_i^{(t)}$, we can tell whether the corresponding signal $s_i^{(t)}$ belongs to $\mathcal{S}_i^{(t)}$ by checking whether $\hat{\rho} < 1 - r_i^{(t)}$. So, we can exactly compute the total number of such signals in the $t$-th sample, $X^{0(t)} = \sum_{i=1}^{n} \mathbb{1}\{s_i^{(t)} \in \mathcal{S}_i^{(0)}\}$, whose expected value is $\mathbb{E}[X^{0} \mid \omega = 0]$. Because signals are independent given $\omega^{(t)} = 0$, by Hoeffding’s inequality we have

$$\Pr \left[ \left| \sum_{t: \omega(t) = 0}^{n} \frac{1}{\sum_{t: \omega(t) = 0}} \mathbb{1}\{s_i^{(t)} \in \mathcal{S}_i^{(0)}\} - \#_0 \mathbb{E}[X^{0} \mid \omega = 0] \right| \geq \#_0 a \right] \leq 2e^{-\frac{2(\#_0 a)^2}{\#_0 \mathbb{E}[X^{0} \mid \omega = 0]}} = 2e^{-\frac{2\#_0 a^2}{n}}.$$  

Plugging in $a = \sqrt{\frac{2}{n} \log \frac{2}{\varepsilon}}$ and $\#_0 \geq \frac{1}{2}(1 - p)T_2$, we get

$$\Pr \left[ \left| \frac{1}{\#_0} \sum_{t: \omega(t) = 0} X^{0(t)} - \mathbb{E}[X^{0} \mid \omega = 0] \right| \geq a \right] \leq 2e^{-\frac{\#_0}{n} \log \frac{2}{\varepsilon}} \leq 2e^{-\frac{1}{2}(1 - p)T_2 \log \frac{2}{\varepsilon}}.$$  

Similarly, considering the samples where $\omega^{(t)} = 1$, we get

$$\Pr \left[ \left| \frac{1}{\#_1} \sum_{t: \omega(t) = 1} X^{1(t)} - \mathbb{E}[X^{1} \mid \omega = 1] \right| \geq a \right] \leq 2e^{-\frac{\#_1}{n} \log \frac{2}{\varepsilon}} \leq 2e^{-\frac{1}{2}pT_2 \log \frac{2}{\varepsilon}}.$$  

Therefore, if we require

$$T_2 \geq \frac{2 \log(2/\delta)}{\min\{p, 1 - p\} \log(2/\varepsilon)}$$  \hspace{1cm} (56)

then with probability at least $1 - 2\delta$, both

$$\left| \frac{1}{\#_0} \sum_{t: \omega(t) = 0} X^{0(t)} - \mathbb{E}[X^{0} \mid \omega = 0] \right| < a, \quad \left| \frac{1}{\#_1} \sum_{t: \omega(t) = 1} X^{1(t)} - \mathbb{E}[X^{1} \mid \omega = 1] \right| < a$$

hold. Namely, $\frac{1}{\#_0} \sum_{t: \omega(t) = 0} X^{0(t)}$ and $\frac{1}{\#_1} \sum_{t: \omega(t) = 1} X^{1(t)}$ are $a$-accurate estimates of $\mathbb{E}[X^{0} \mid \omega = 0]$ and $\mathbb{E}[X^{1} \mid \omega = 1]$. Equations (55) and (56) together imply that $T_2 = O \left( \frac{1}{\min\{p, 1 - p\} \log \frac{2}{\delta}} \right) \leq O \left( \frac{1}{\log \frac{2}{\varepsilon}} \right)$ samples suffice.

Then, we identify an index $v \in \{0, 1\}$ such that $\mathbb{E}[X^{u} \mid \omega = u] \geq \frac{n}{2}$. By Claim 112 there exists a $v \in \{0, 1\}$ with $\mathbb{E}[X^{v} \mid \omega = v] \geq \frac{n}{2}$. Since $\frac{1}{\#_v} \sum_{t: \omega(t) = v} X^{v(t)}$ is an $a$-accurate estimate of $\mathbb{E}[X^{v} \mid \omega = v]$, we must have

$$\frac{1}{\#_v} \sum_{t: \omega(t) = v} X^{v(t)} \geq \mathbb{E}[X^{v} \mid \omega = v] - a \geq \frac{n}{2} - a.$$  

So, at least one of $u \in \{0, 1\}$ must satisfy $\frac{1}{\#_u} \sum_{t: \omega(t) = u} X^{u(t)} \geq \frac{n}{2} - a$. By picking any such a $u$, we are guaranteed that $\mathbb{E}[X^{u} \mid \omega = u] \geq \frac{1}{\#_u} \sum_{t: \omega(t) = u} X^{u(t)} - a \geq \frac{n}{2} - 2a$. Given the assumption $n \geq 32 \log \frac{2}{\varepsilon}$ in the statement of the theorem, we have

$$\frac{a}{n} = \sqrt{\frac{1}{2n} \log \frac{2}{\varepsilon}} \leq \frac{1}{8}.$$  

Hence, $\mathbb{E}[X^{u} \mid \omega = u] \geq \frac{n}{2} - 2a \geq \frac{n}{2} - 2(\frac{n}{8}) = \frac{n}{4}$.

Finally, as argued above, an $a$-accurate estimate of $\mathbb{E}[X^{u} \mid \omega = u]$ is given by $\frac{1}{\#_u} \sum_{t: \omega(t) = u} X^{u(t)}$.  

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Constructing $\hat{f}$. After accomplishing Tasks 1 and 2 using $T_1 + T_2 = O\left(\frac{1}{\varepsilon n (1 + \gamma)} \log \frac{1}{\delta} + \frac{1}{\varepsilon} \log \frac{1}{\delta}\right)$ samples, we construct a $\hat{f}$ that implements the same functionality as $f_{\text{hypo}}$. Let

$$M = \frac{1}{\#u} \sum_{t, \omega(t) = u} X^{u(t)} - 2a,$$

where $\frac{1}{\#u} \sum_{t, \omega(t) = u} X^{u(t)}$ is our estimate of $\mathbb{E}[X^u \mid \omega = u]$ in Task 2 and $a = \sqrt{\frac{n}{2} \log \frac{2}{\delta}}$.

**Claim D.14.** $\mathbb{E}[X^u \mid \omega = u] - a > M > \mathbb{E}[X^u \mid \omega = 1 - u] + a$.

**Proof.** Because $\frac{1}{\#u} \sum_{t, \omega(t) = u} X^{u(t)}$ is an $a$-accurate estimate of $\mathbb{E}[X^u \mid \omega = u]$, we have

$$\mathbb{E}[X^u \mid \omega = u] > \frac{1}{\#u} \sum_{t, \omega(t) = u} X^{u(t)} - a = M + a.$$

Recall from Equation (54) that $\mathbb{E}[X^u \mid \omega = 1 - u] \leq \mathbb{E}[X^u \mid \omega = u] - 4a$. So,

$$\mathbb{E}[X^u \mid \omega = 1 - u] < \left(\frac{1}{\#u} \sum_{t, \omega(t) = u} X^{u(t)} + a\right) - 4a = M - a.$$

The above two inequalities prove the claim.

Given reports $r = (r_1, \ldots, r_n)$ as input, we let $\hat{f}$ do the following:

1. If $u = 1$, count how many reports $r_i's$ satisfy $\frac{r_i}{1 - r_i} > \hat{\rho}$; If $u = 0$, count how many reports $r_i's$ satisfy $\frac{r_i}{1 - r_i} < \hat{\rho}$. Let this number be $X$;

2. Then, check whether $X > M$ or $X \leq M$. If $X > M$, output $\hat{f}(r) = u$; otherwise, output $\hat{f}(r) = 1 - u$.

We argue that $\hat{f}$ implements the same functionality as $f_{\text{hypo}}$: (1) In Task 1 we got $\frac{1}{1 + \gamma} \rho < \hat{\rho} < (1 + \gamma)\rho$. So, by checking whether $\frac{r_i}{1 - r_i} > \hat{\rho}$ or $< \hat{\rho}$ we can exactly tell whether $\frac{r_i}{1 - r_i} \geq (1 + \gamma)\rho$ or $\leq \frac{1}{1 + \gamma} \rho$. Hence, we have $X = X^u$, the number of signals that belong to the $S_i^u$ sets. (2) Recall from Equation (53) that with probability at least $1 - \varepsilon$, $X$ is $\alpha$-close to its expectation $\mathbb{E}[X^u \mid \omega]$. Then, according to Claim D.14 $X > M$ implies that $X$ is closer to $\mathbb{E}[X^u \mid \omega = u]$; $X < M$ implies that $X$ is closer to $\mathbb{E}[X^u \mid \omega = 1 - u]$. So, $\hat{f}$ implements both of the two steps in $f_{\text{hypo}}$. Hence, according to Claim D.13 $\hat{f}$ is $\varepsilon$-optimal.

**D.5 Proof of Theorem 4.5**

According to Lemma 4.1, the optimal aggregator is

$$f^*(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{1 - r_i}{r_i}},$$

where $\rho = \frac{\rho}{\sum_{i=1}^{n} \frac{1 - r_i}{r_i}}$. We claim that an approximately optimal aggregator can be obtained by first estimating $\rho$ from samples and then use the aggregator with the estimate $\hat{\rho}$:

$$\hat{f}(r) = \frac{1}{1 + \hat{\rho}^{n-1} \prod_{i=1}^{n} \frac{1 - r_i}{r_i}}.$$
Lemma D.16. The following observation is the key:

\[ y = \frac{\sqrt{\varepsilon}}{\sqrt{n}} \]

As corollaries, for \( k \) conditionally independent reports \( r_1, \ldots, r_k \), we have \( \mathbb{E} \left[ \Pi_{i=1}^{k} \frac{r_i}{1-r_i} \mid \omega = 0 \right] = \rho^k \) and \( \mathbb{E} \left[ \Pi_{i=1}^{k} \frac{1-r_i}{r_i} \mid \omega = 1 \right] = \frac{1}{\rho^k} \).

Proof. Because \( \frac{r_i}{1-r_i} = \frac{P(s_i \mid \omega = 1)}{P(s_i \mid \omega = 0)} \rho \) (from (11)), we have

\[
\mathbb{E} \left[ \frac{r_i}{1-r_i} \mid \omega = 0 \right] = \sum_{s_i \in S_i} P(s_i \mid \omega = 0) \frac{P(s_i \mid \omega = 1)}{P(s_i \mid \omega = 0)} \rho = \sum_{s_i \in S_i} P(s_i \mid \omega = 1) \rho = \rho.
\]
For conditionally independent \( r_1, \ldots, r_k \), we have

\[
\mathbb{E}\left[ \prod_{i=1}^{k} \frac{r_i}{1 - r_i} \mid \omega = 0 \right] = \prod_{i=1}^{k} \mathbb{E}\left[ \frac{r_i}{1 - r_i} \mid \omega = 0 \right] = \rho^k.
\]

Similarly, we can prove \( \mathbb{E}\left[ \prod_{i=1}^{k} \frac{r_i}{1 - r_i} \mid \omega = 1 \right] = \frac{1}{\rho} \) and \( \mathbb{E}\left[ \prod_{i=1}^{k} \frac{1 - r_i}{r_i} \mid \omega = 1 \right] = \frac{1}{\rho^k} \).

Let \( \Delta = \frac{\sqrt{e}}{3\gamma n} \) and suppose we are given \( T = \frac{6e}{\gamma n \Delta^2} \log \frac{2}{\delta} = \frac{54e\gamma n}{\varepsilon} \log \frac{2}{\delta} = O\left( \frac{\gamma n}{\varepsilon} \log \frac{1}{\delta} \right) \) samples. Suppose when drawing the samples we first draw the events \( \omega^{(t)} \)'s, and then draw the reports \( r_i^{(t)} \)'s conditioning on \( \omega^{(t)} \) being 0 or 1. After the first step, the numbers of samples with \( \omega^{(t)} = 0 \) and \( \omega^{(t)} = 1 \) are determined, which we denote by \( \#_0 \) and \( \#_1 \). Since \( \#_0 + \#_1 = T \), one of them must be at least \( T/2 \). We argue that whether \( \#_0 \geq T/2 \) or \( \#_1 \geq T/2 \) we can estimate \( \rho^{1/\gamma} \) with accuracy 3\( \Delta \). (For simplicity, we assume that \( 1/\gamma \) is an integer.)

- If \( \#_0 \geq T/2 \), then we consider the \( \#_0 n \) reports \( r_i^{(t)} \)'s in the samples with \( \omega^{(t)} = 0 \). We divide these \( \#_0 n \) reports evenly into \( \#_0 n \gamma \) groups, each of size \( 1/\gamma \), denoted by \( G_1, \ldots, G_{\#_0 n \gamma} \).

  Consider the product of \( \frac{r_i^{(t)}}{1 - r_i^{(t)}} \)'s in a group \( G_j \); because \( r_i^{(t)} \)'s are independent given \( \omega = 0 \), by Lemma D.16 we have

  \[
  \mathbb{E}\left[ \prod_{r_i^{(t)} \in G_j} \frac{r_i^{(t)}}{1 - r_i^{(t)}} \mid \omega = 0 \right] = \rho^{1/\gamma}.
  \]

  Using \( \log \) and the inequality \((1 + \gamma)^{1/\gamma} \leq e\), we have

  \[
  \frac{1}{e} \rho^{1/\gamma} \leq \frac{1}{(1 + \gamma)^{1/\gamma}} \rho^{1/\gamma} \leq \prod_{r_i^{(t)} \in G_j} \frac{r_i^{(t)}}{1 - r_i^{(t)}} \leq (1 + \gamma)^{1/\gamma} \rho^{1/\gamma} \leq e \rho^{1/\gamma}.
  \]

  Let \( X_j \) be the random variable \( \frac{1}{e \rho^{1/\gamma}} \prod_{r_i^{(t)} \in G_j} \frac{r_i^{(t)}}{1 - r_i^{(t)}} \). From the above equation and inequality we have \( \mathbb{E}[X_j] = \frac{1}{e} \) and \( X_j \in \left[ \frac{1}{e \rho^{1/\gamma}}, 1 \right] \subseteq [0, 1] \). So, by Chernoff bound,

  \[
  \mathbb{P} \left[ \sum_{j=1}^{\#_0 n \gamma} X_j \in (1 - \Delta) \frac{1}{e} \mid \omega = 0 \right] \geq 1 - 2e^{-\frac{\#_0 n \gamma \Delta^2}{4e}} \geq 1 - 2e^{-\frac{T \omega \Delta^2}{4e}} = 1 - \delta,
  \]

  given our choice of \( T \). Multiplying \( \frac{1}{\#_0 n \gamma} \sum_{j=1}^{\#_0 n \gamma} X_j \) by \( e \rho^{1/\gamma} \), we obtain the following estimate of \( \rho^{1/\gamma} \):

  \[
  \rho_{0}^{1/\gamma} := \frac{1}{\#_0 n \gamma} \sum_{j=1}^{\#_0 n \gamma} \prod_{r_i^{(t)} \in G_j} \frac{r_i^{(t)}}{1 - r_i^{(t)}} \in (1 - \Delta) \rho^{1/\gamma}.
  \]

  Dividing by \( \rho^{1/\gamma} \), we get \( (\frac{\rho_{0}^{1/\gamma}}{\rho})^{1/\gamma} \in 1 \pm \Delta \).

- If \( \#_1 \geq T/2 \), then by considering the \( \#_1 n \) reports in the samples with \( \omega^{(t)} = 1 \), dividing them into \( \#_1 n \gamma \) groups of size \( 1/\gamma \), \( H_1, \ldots, H_{\#_1 n \gamma} \), and similarly defining random variable \( Y_j = \frac{1}{e} \prod_{r_i^{(t)} \in H_j} \frac{1 - r_i^{(t)}}{r_i^{(t)}} \), we obtain the following estimate of \( (\frac{1}{\rho_1})^{1/\gamma} \):

  \[
  (\frac{1}{\rho_1})^{1/\gamma} := \frac{1}{\#_1 n \gamma} \sum_{j=1}^{\#_1 n \gamma} \prod_{r_i^{(t)} \in H_j} \frac{1 - r_i^{(t)}}{r_i^{(t)}} \in (1 - \Delta)(\frac{1}{\rho})^{1/\gamma}.
  \]
Multiplying by \( \rho^{1/\gamma} \), we get \( \left( \frac{\rho}{\rho_1} \right)^{1/\gamma} \in 1 \pm \Delta \). Taking the reciprocal and noticing that \( \frac{1}{1 \pm 3\Delta} \subseteq 1 \pm 3\Delta \) when \( \Delta < \frac{1}{3} \), we obtain \( \left( \frac{\hat{\rho}}{\rho} \right)^{1/\gamma} \in 1 \pm 3\Delta \).

From the discussion above we obtained an estimate \( \hat{\rho} \in \{ \hat{\rho}_0, \hat{\rho}_1 \} \) of \( \rho \) such that \( \left( \frac{\hat{\rho}}{\rho} \right)^{1/\gamma} \in 1 \pm 3\Delta \). Raising to the power of \( \gamma \), and using the inequality \((1 - x)\gamma \geq 1 - x\gamma \) and \((1 + x)\gamma \leq e^{x\gamma} \leq 1 + 2x\gamma \) for \( x\gamma \leq 1 \), we get
\[
\frac{\hat{\rho}}{\rho} \in (1 \pm 3\Delta)^\gamma \subseteq [1 - 3\Delta\gamma, e^{3\Delta\gamma}] \subseteq [1 - 3\Delta\gamma, 1 + 6\Delta\gamma].
\]

In particular, this implies \( \left| \frac{\hat{\rho} - \rho}{\rho} \right| \leq 6\Delta\gamma = \frac{2\sqrt{\varepsilon}}{n} \). Then, according to Claim D.15, the aggregator \( \hat{f} \) defined by \( \hat{f}(r) = \frac{1}{1 + \rho^{n-1} \prod_{i=1}^{n} \frac{r_i}{r_i}} \) is \( \varepsilon \)-optimal. We hence obtained an \( \varepsilon \)-optimal aggregator.