CLASSIFICATION TO THE POSITIVE RADIAL SOLUTIONS WITH WEIGHTED BIHARMONIC EQUATION

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ABSTRACT. In this paper, we consider the weighted problem
\[ \Delta(|x|^{-\alpha} \Delta u) = |x|^\beta u^p, \quad u(x) > 0, \quad u(x) = u(|x|) \text{ in } \mathbb{R}^n \setminus \{0\}, \]
where \( n \geq 5, -n < \alpha < n - 4 \) and \((p, \alpha, \beta, n), p > 1\) belongs to the critical hyperbola
\[ \frac{n + \alpha}{2} + \frac{n + \beta}{p + 1} = n - 2. \]

We give two type-homoclinic functions \( v(t) := |x|^{\frac{n+\alpha+n}{2}} u(|x|), t = -\ln |x| \). On the other hand, for radial solution \( u \) with non-removable singularity at origin, \( v(t) \) is periodic and classification for all periodic functions are obtained with \(-2 < \alpha < n - 4\); while for \(-n < \alpha \leq -2\), there always exists a solution \( u(|x|) \) with non-removable singularity and the corresponding function \( v(t) \) is not periodic. It is also closely related to the Caffarelli-Kohn-Nirenberg inequality, and we get some results such as the best embedding constants and the existence in radial case. In particular, for \( \alpha = \beta = 0 \), it is related to the \( Q \)-curvature problem in conformal geometry.

1. Introduction. In this paper, we are interested in the positive radial solutions of the weighted equation
\[ \Delta(|x|^{-\alpha} \Delta u) = |x|^\beta u^p \text{ in } \mathbb{R}^n \setminus \{0\}, \] (1)
where \( n \geq 5, -n < \alpha < n - 4 \) and \((p, \alpha, \beta, n), p > 1\) belongs to the weighted critical hyperbola
\[ \frac{n + \alpha}{2} + \frac{n + \beta}{p + 1} = n - 2. \] (2)

When \( \alpha = \beta = 0 \), equation (1) can be reduced to \( \Delta^2 u = u^{\frac{n+4}{n-4}} \) in \( \mathbb{R}^n \setminus \{0\} \) which is conformal invariant and has a concrete meaning in the \( Q \)-curvature problem. In fundamental work [16], Lin has proved that all positive smooth solutions are

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raised from the smooth conformal metrics on $\mathbb{S}^n$ by stereograph projection, i.e., any positive smooth solution has the form (so-called spherical solution)

$$u_\lambda(x) = c_n \left( \frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n+4}{2}},$$

where $x_0 \in \mathbb{R}^n$, $c_n = n(n - 2)(n - 2)(n + 2)^{\frac{n+4}{2}}$ and $\lambda > 0$. Moreover, let

$$v_\lambda(t) := |x - x_0|^{\frac{n-4}{2}} u_\lambda(|x - x_0|), \quad t = -\ln(\lambda|x - x_0|),$$

directly computation yields that

$$v_\lambda(t) = c_n (2\cosh t)^{-\frac{n+4}{2}} \text{ in } \mathbb{R}.$$

Then $v_\lambda(\pm \infty) = 0$ and $v(t)$ has only one critical (maximum) point.

For the solution with non-removable singularity at origin, i.e. $u(0) = +\infty$, Lin [16] also showed that the positive solution $u$ of (1) with $\alpha = \beta = 0$ is radial. Passing to the Emden-Fowler coordinates and writing

$$u(x) = |x|^{-\frac{n-4}{2}} v(t), \quad t = -\ln |x|,$$

very recently, Guo-Huang-Wang-Wei [13] and Frank-König [11] have proved that the function $v(t)$ is periodic in $t$. Natural question: for general $(p,\alpha,\beta,n)$ satisfying the critical hyperbola, whether the homoclinic and periodic results still hold?

On the other hand, equation (1) is closely related to Caffarelli-Kohn-Nirenberg type inequalities

$$\int_{\mathbb{R}^n} |x|^{-\alpha} |\Delta u|^2 dx \geq C \left( \int_{\mathbb{R}^n} |x|^{\beta} |u|^{p+1} dx \right)^{\frac{2}{p+1}}, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n). \quad (3)$$

Define $\mathcal{N}(\mathbb{R}^n, \alpha)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\| u \|_\alpha^2 := \int_{\mathbb{R}^n} |x|^{-\alpha} |\Delta u|^2 dx.$$

Note that (3) holds for $u \in \mathcal{N}(\mathbb{R}^n, \alpha)$. Furthermore, the best constant in (3) is given by

$$S_p(\alpha, \beta) := \inf_{u \in \mathcal{N}(\mathbb{R}^n, \alpha) \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^n} |x|^{-\alpha} |\Delta u|^2 dx}{\left( \int_{\mathbb{R}^n} |x|^{\beta} |u|^{p+1} dx \right)^{\frac{2}{p+1}}} \right),$$

and the extremal functions for $S_p(\alpha, \beta)$ are ground state solutions of equation (1). A simple rescaling argument shows that $S_p(\alpha, \beta)$ vanishes unless $(p,\alpha,\beta,n)$ satisfies (2). Thus, from now on we always assume (2) holds.

Meanwhile, the equation (1) is also related to Hénon-Lane-Emden system

$$\begin{cases}
-\Delta u = |x|^\alpha \phi^q, \\
-\Delta \phi = |x|^{\beta} u^p
\end{cases} \quad (4)$$

with $q = 1$. Existence and nonexistence of equation (4) have been studied by many authors, see[6, 8, 9, 12, 15, 17, 18, 19, 21] and the references therein. It is well-known that the following critical hyperbola plays an important role in existence results

$$\frac{n+\alpha}{q+1} + \frac{n+\beta}{p+1} = n - 2.$$

More precisely, Bidaut-Veron and Giacomini [2] have shown that if $n \geq 3$, $\alpha, \beta > -2$, then the system (4) admits a positive classical radial solution $(u, \phi)$ continuous at the origin if and only if $(p, q)$ is above or on the critical hyperbola.
Consider the second order equation

\[- \text{div}(|x|^{-2a}\nabla u) = |x|^{-b_p}u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad n \geq 3\]  \hspace{1cm} (5)

with (non) removable singularity at 0. The case of \(a = b = 0, p = \frac{2n}{n-2}\), i.e. \(-\Delta u = |x|^\frac{n+2}{n-2}\), is related to the Yamabe problem. It is well known by Chen and Li [7], Caffarelli, Gidas and Spruck [3] that if \(u\) is a regular solution, then \(u\) must be the spherical solution. If 0 is non-removable, then \(|x|^{-\frac{2}{p-2}}u(x)\) is a periodic function in \(-\ln |x|\). For more general case, the periodic results were proved by Hsia, Lin and Wang [14]. On the other hand, for

\[0 \leq a < \frac{n-2}{2}, \quad a \leq b \leq a+1, \quad p = \frac{2n}{n-2 + 2(b-a)},\]

the least energy solution \(u\) of Euler-Lagrange equation (5) is also a minimizer for

\[S(a, b, \mathbb{R}^n) := \inf_{u \in D^2_{rad}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |x|^{-b_p}|u|^p \, dx\right)^\frac{1}{p}}\]

where \(S(a, b, \mathbb{R}^n)\) is the best constant for the classical Caffarelli-Kohn-Nirenberg inequality established in [4].

In the paper we just consider positive radial solutions to equation (1) with (non) removable singularity at origin. Denote \(N^2_{rad}(\mathbb{R}^n, \alpha) = \{u|u(x) = u(|x|), u(\cdot) \in N^2(\mathbb{R}^n, \alpha)\}\) and

\[S^p_{rad}(\alpha, \beta) := \inf_{u \in N^2_{rad}(\mathbb{R}^n, \alpha) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |x|^{-\alpha}|\Delta u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |x|^\beta|u|^{p+1} \, dx\right)^\frac{2}{p+1}}.\]

It is well known that the equation (1) is invariant with respect to the weighted dilation

\[u_\lambda(x) \mapsto \lambda^{-\frac{n-4-\alpha}{4}}u(\lambda x), \quad \lambda > 0.\]

From now on we will identify solutions up to the above dilation. Our main results are as following.

**Theorem 1.1.** Let \(-n < \alpha < n - 4, \quad p > 1\) and \((p, \alpha, \beta, n)\) satisfies (2). Then in \(N^2_{rad}(\mathbb{R}^n, \alpha)\), the problem (1) admits a unique positive solution \(u\). Moreover,

i) If \(\alpha = \beta > -2\), then

\[u_1(x) = \frac{C_1}{2m_1} \left(1 + |x|^{2+\alpha}\right)^{-\frac{n-4-\alpha}{2m_1}},\]

with

\[m_1 = -\frac{n-4-\alpha}{2+\alpha}, \quad C_1^{p-1} = m_1(m_1-1)(m_1-2)(m_1-3)^\nu_1^4, \quad \nu_1 = \frac{2 + \alpha}{2}.\]

ii) If \((n+\alpha)(n+\beta) = (n-4-\alpha)^2\) with \(\alpha < -2\), then

\[u_2(x) = \frac{C_2}{2m_2} |x|^{2+\alpha} \left(1 + |x|^{-(2+\alpha)}\right)^\frac{n+\alpha}{2m_2},\]

with

\[m_2 = \frac{n+\alpha}{2+\alpha}, \quad C_2^{p-1} = m_2(m_2-1)(m_2-2)(m_2-3)^\nu_2^4, \quad \nu_2 = -\frac{2 + \alpha}{2}.\]
The explicit solution $u_2$ in the second case is a new type singular solution which exhibits a new phenomenon comparing to the second order equation. More precisely, $v(t) = |x|^{-\frac{n\alpha}{n-\alpha}}u_2(x), t = -\ln |x|$ goes to zero as $t \to \infty$ which implies that $v(t)$ is not periodic while in the second order case, the function $|x|^{-\frac{n\alpha}{n-\alpha}}u(x)$ must be periodic in $-\ln |x|$ as we stated before.

**Theorem 1.2.** Under the conditions of Theorem 1.1, the following hold

i) $S_p^{rad}(\alpha, \beta) > 0$ and $S_p^{rad}(\alpha, \beta)$ is achieved in $\mathcal{N}_p^{rad}(\mathbb{R}^n, \alpha)$;

ii) For $\alpha = \beta > -2$, then $S_p^{rad}(\alpha, \alpha) = \omega_n^{\frac{2(2+\alpha)}{n+\alpha}}\mu_{1,p}(\alpha)$ and $S_p^{rad}(\alpha, \alpha)$ is achieved by $u_1(x)$, where

$$
\mu_{1,p}(\alpha) = \nu_1^{\frac{2(2n+2)}{n+\alpha}} m_1(m_1 - 1)(m_1 - 2)(m_1 - 3) 
\times \left[ \frac{4m_1(m_1 - 1)}{(2m_1 - 1)(2m_1 - 3)} B(-m_1, \frac{1}{2}) \right]^{\frac{2(2+\alpha)}{n+\alpha}};
$$

iii) For $(n + \alpha)(n + \beta) = (n - 4 - \alpha)^2$ with $\alpha < -2$, then $S_p^{rad}(\alpha, \beta) = \omega_n^{\frac{-2(2+\alpha)}{n+\alpha}}\mu_{2,p}(\alpha)$, and $S_p^{rad}(\alpha, \beta)$ is achieved by $u_2(x)$, where

$$
\mu_{2,p}(\alpha) = \nu_2^{\frac{2(2n-6)}{n+\alpha}} m_2(m_2 - 1)(m_2 - 2)(m_2 - 3) 
\times \left[ \frac{4m_2(m_2 - 1)}{(2m_2 - 1)(2m_2 - 3)} B(-m_2, \frac{1}{2}) \right]^{\frac{-2(2+\alpha)}{n+\alpha}}.
$$

Here $u_i(x), \nu_i$ and $m_i, i = 1, 2$ are given in Theorem 1.1.

Next we consider that 0 is a non-removable singular point. Let $v, \psi \in C^2(\mathbb{R})$ be defined by

$$
u(|x|) = |x|^{-\lambda_1} v(t), \phantom{|x|^{-\lambda_1}} \phi(|x|) = |x|^{-\lambda_2} \psi(t), \phantom{|x|^{-\lambda_1}} \text{with} \phantom{|x|^{-\lambda_1}} t = -\ln |x|
$$

where

$$
\lambda_1 = \frac{n + \beta}{p + 1} = \frac{n - 4 - \alpha}{2}, \phantom{|x|^{-\lambda_1}} \lambda_2 = \frac{n + \alpha}{2}.
$$

Due to $-n < \alpha < n - 4$, then $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = n - 2$.

Denote the constants

$$
A := \frac{n - 2}{2} - \lambda_1 = -\frac{n - 2}{2} + \frac{2 + \alpha}{2}, \phantom{|x|^{-\lambda_1}} B := \frac{n + \beta n + \alpha}{p + 1} = \lambda_1 \lambda_2.
$$

Notice that $A^2 + B = \left(\frac{n - 2}{2}\right)^2$.

Direct computation shows that a radial pair $(u, \phi)$ solves (4) with $q = 1$ in $\mathbb{R}^n \setminus \{0\}$ if and only if the functions $v, \psi$ satisfy

$$
\begin{align*}
-v'' + 2Av' + Bv &= \psi \quad \text{in} \ \mathbb{R}, \\
-\psi'' - 2A\psi' + B\psi &= v^p \quad \text{in} \ \mathbb{R},
\end{align*}
$$

i.e. $v(t)$ satisfies the following fourth-order ordinary differential equation

$$
v^{(4)}(t) - K_2 v''(t) + K_0 v(t) = v^p(t) \quad \text{in} \ \mathbb{R} \quad (6)
$$

with the constants

$$
K_2 = \frac{(n - 2)^2 + (\alpha + 2)^2}{2} = \lambda_1^2 + \lambda_2^2, \phantom{|x|^{-\lambda_1}} K_0 = \frac{(n - 4 - \alpha)^2(n + \alpha)^2}{16} = \lambda_1^2 \lambda_2^2.
$$

Clearly, $K_2^2 \geq 4K_0$. 
Obviously, equation (6) has two constant solutions \( v \equiv 0, \ell \) where \( \ell = K_0^{-1/4} \). Moreover, we can build the following result.

**Theorem 1.3.** Assume \(-2 < \alpha < n - 4\) and \( u \in C^4(\mathbb{R}^n \setminus \{0\}) \) be any positive radial solution of (1) with non-removable singularity at origin. Define \( v(t) = |x|^{-2/n} u(|x|) \) with \( t = -\ln |x| \), then

i) \( v \) is periodic;

ii) Let \( a \in (0, \ell) \), then there admits a unique (up to translations) periodic solution \( v \) of (6) with minimal value \( a \).

**Remark 1.4.** For the case \(-n < \alpha \leq -2\), Proposition 4.3 means that there always exists a solution \( u(|x|) \) with non-removable singularity at zero and the corresponding function \( v(t) \) is even and strictly decreasing in \( \mathbb{R}_+ \). A special example is given by part (ii) of Theorem 1.1. Hence in this case, we can’t expect each function \( v(t) \) is always periodic.

In section 2, we will give the qualitative properties for general linear equation. Theorem 1.1 and Theorem 1.2 will be proved in section 3. In section 4, we will give the proof of Theorem 1.3.

2. Qualitative properties for general linear equation. In this section, we will give general results for the linear equation.

**Lemma 2.1.** Let \( g \in C(\mathbb{R}) \) and assume that there exists \( q > 1 \) such that \( |g(s)| \geq |s|^q \) for large \( |s| \). Let \( v \) be a solution to \( \sum_{1 \leq k \leq m} a_k v^{(k)}(t) = g(v) \), \( m \geq 1 \) in \((t_0, \infty)\) where \( a_k \in \mathbb{R} \) and \( a_m \neq 0 \). Assume \( \lim_{t \to \infty} v = \theta \in \mathbb{R} \cup \{\pm \infty\} \), then \( \theta \in \mathbb{R} \) satisfying \( g(\theta) = 0 \).

**Proof.** By contradiction, we may assume \( \lim_{t \to +\infty} v(t) = +\infty \) and \( g(s) > s^q \) for large \( s \). Then there exists \( t_1 > 0 \) such that

\[
\sum_{k=1}^m a_k v^{(k)}(t) = g(v) \geq v^q, \quad \forall \ t > t_1. \tag{7}
\]

Now we choose a nonnegative function \( \phi_0 \in C^\infty([0, \infty)) \) satisfying \( \phi_0 > 0 \) in \([0, 2)\), \( \phi_0(t) = 1 \) in \([0, 1]\), \( \text{supp}(\phi_0) = [0, 2] \) and

\[
\int_0^2 |\phi_0^{(i)}(s)| \frac{s^q}{s^q + 1} \frac{1}{s^q} (s)ds =: A_i < \infty, \quad \forall \ i \in \mathbb{N}. \tag{8}
\]

Let \( T > t_1 \), multiplying inequality (7) by \( \phi(t) := \phi_0(\frac{t - t_1}{T - t_1}) \) and integrating by parts, we obtain

\[
\int_{t_1}^\infty \gamma(s)v(s)ds \geq \int_{t_1}^\infty v^q(s)\phi(s)ds - C,
\]

where

\[
\gamma(s) = \sum_{k=1}^m (-1)^k a_k \phi^{(k)}(s).
\]

By Young’s inequality, there exists \( C_1 > 0 \) such that

\[
v(s)\gamma(s) \leq \frac{1}{2} v^q(s)\phi(s) + C_1 \phi^{-\frac{1}{q}}(s) \sum_{i=1}^m |\phi^{(i)}(s)| \frac{s^q}{s^q}, \quad \forall \ s \in [t_1, 2T - t_1].
\]
Therefore
\[ \frac{1}{2} \int_{t_1}^{2T-t_1} v^q \phi ds \leq C_1 \sum_{i=1}^{m} \int_{t_1}^{2T-t_1} |\phi^{(i)}(s)|^{\frac{q}{i+1}} \phi^{\frac{1}{i+1}}(s) ds + C. \]

Using (8), we get
\[ \int_{t_1}^{T-t_1} v^q ds \leq \int_{t_1}^{2T-t_1} v^q \phi ds \leq C_1 m \sum_{i=1}^{m} A_i (T-t_1)^{1-\frac{q}{i+1}} + C. \tag{9} \]

Sending \( T \) to \( \infty \), we observe that the right-hand side of (9) remains bounded, while the left-hand side goes to \( +\infty \) since \( v(t) \to +\infty \) as \( t \to \infty \). Hence \( \theta \in \mathbb{R} \).

On the other hand, if \( g(\theta) \neq 0 \), without loss of generality, assume \( g(\theta) > 0 \). Then for any \( \epsilon > 0 \), there exists \( T \gg 1 \) such that
\[ g(\theta) - \epsilon \leq \sum_{1 \leq k \leq m} a_k v^{(k)}(t) \leq g(\theta) + \epsilon, \quad \forall \ t > T. \]

Taking \( \epsilon \) small such that \( g(\theta) - \epsilon > 0 \), integrating over \( [T, t] \), since \( v(t) \to \theta \in \mathbb{R} \) as \( t \to \infty \), for \( t > T \gg 1 \) there holds
\[ \sum_{k=1}^{m} a_k v^{(k)}(t) \geq \tilde{c}_1 t, \]
where \( \tilde{c}_1 \) is a positive number. So we get
\[ \sum_{k=2}^{m} a_k v^{(k)}(t) \geq c_1 t, \quad \text{for} \ t > T \gg 1 \]
with \( c_1 > 0 \).

As the above procedures, we can get
\[ v(t) > c_m t^m, \quad \text{as} \ t \to \infty, \]
which leads to a contradiction. Hence \( g(\theta) = 0 \). \( \square \)

Now we consider the general fourth-order equation
\[ v^{(4)}(t) - \bar{A} v^{''}(t) + \bar{B} v = f(v), \tag{10} \]
where \( \bar{A}, \bar{B} \in \mathbb{R} \) and
\[ |f(s)| \geq |s|^q \quad \text{for} |s| \gg 1 \quad \text{and some} \ q > 1. \tag{11} \]

Here we will give the qualitative properties of solutions to equation (10).

**Lemma 2.2.** Assume that \( f \) satisfies (11). Let \( v \) be a global solution of (10) with eventual monotonicity, then for \( 1 \leq k \leq 4 \), \( \lim_{t \to \pm \infty} v^{(k)}(t) = 0 \).

**Proof.** By Lemma 2.1, we have that \( \lim_{t \to \pm \infty} v = \theta \in \mathbb{R} \), so \( v \) is bounded. Then for \( |t| \) large enough, there holds
\[ H''(t) \geq 0 \text{ or } \leq 0, \quad \text{where} \ H(t) := v'' - \bar{A}v. \]

Without loss of generality, we assume \( H''(t) \geq 0 \), i.e \( H(t) \) is convex near \( \infty \), which yields
\[ \lim_{t \to \pm \infty} H(t) = \gamma_1 \in \mathbb{R} \cup \{+\infty\}. \]
Hence there holds
\[ \gamma_2 := \lim_{t \to \pm \infty} v''(t) = \gamma_1 + \bar{A}\theta. \]
Now we claim $\gamma_2 = 0$.

If $\gamma_2 \neq 0$, then for $t$ large enough, $v''(t) \geq c_0 > 0$ for $\gamma_2 > 0$ or $v''(t) \leq -c_0 < 0$ for $\gamma_2 < 0$. Hence, for $t \gg 1$ there holds $v(t) \geq c_0 t^2$ or $v(t) \leq -c_0 t^2$, which is impossible since $v$ is bounded. Therefore,

$$\lim_{t \to \pm\infty} v''(t) = 0, \quad \lim_{t \to \pm\infty} H(t) = -\bar{A}\theta.$$ 

Next we show that $\lim_{t \to \pm\infty} v'(t) = 0$. Indeed, for $t$ large enough, there exists $\xi \in (0, t, t + 1)$ such that

$$v(t) + v(t) = v'(t) + \frac{1}{2} v''(\xi).$$

Obviously $\lim_{t \to \pm\infty} v'(t) = 0$ thanks to $v(t) \to \theta$, $v''(\xi) \to 0$ as $t \to \infty$. $\lim_{t \to \pm\infty} H(t) = -\bar{A}\theta$ and the convexity of $H$ imply that $\lim_{t \to \pm\infty} H'(t) = 0$, i.e.

$$\lim_{t \to \pm\infty} (v''(t) - \bar{A}v'(t)) = 0.$$ 

Thus, $\lim_{t \to \pm\infty} v'''(t) = 0$. By equation (10), we have $\lim_{t \to \pm\infty} v^{(4)}(t) = 0$. This completes the proof.

By Lemma 2.1 and Lemma 2.2, we see that if $v$ is eventually monotone, then $v$ is bounded. In fact, we can prove that any solution to equation (10) is always bounded. Our result is as follows.

**Lemma 2.3.** Assume that $f$ satisfies (11). Let $v \in C^4(\mathbb{R})$ be a global solution to (10) with $\bar{A}, \bar{B} \in \mathbb{R}$, then $v$ is bounded.

**Proof.** As equation (10) is autonomous, we only need to show that $v$ is bounded in $[0, +\infty)$. Otherwise, we can consider $v(-t)$ replacing $v(t)$.

Define the energy function

$$E_v(t) := -v'''v' + \frac{1}{2}(v'')^2 + \frac{\bar{A}}{2}v^2 + G(v)(t),$$

where $G(v) := \int_0^v f(s) ds - \frac{\bar{B}}{2} v^2$. By direct computation, we get

$$\frac{d}{dt} E_v(t) = 0, \quad \text{i.e. } E_v \equiv C.$$ 

Thus $G(v)$ hence $v$ is bounded for all critical points of $v$. Therefore either $v$ is bounded or $v$ is eventually monotone at infinity. If $v$ is eventually monotone, by Lemma 2.1, we can get the bound of $v$ immediately.

For $\bar{A}, \bar{B} > 0$, $\bar{A}^2 \geq 4\bar{B}$, there exist two positive constants $\lambda, \mu$ such that

$$\lambda + \mu = \bar{A}, \quad \lambda\mu = \bar{B}.$$ 

In addition, if $f(s)$ is increasing in $s$ and $f(0) \geq 0$, then we can obtain the following general comparison principle for fourth-order equation (10).

**Lemma 2.4.** Assume that $f'(t) > 0, t \in (0, +\infty)$ and $f(0) \geq 0$, $\bar{A}, \bar{B} > 0$ with $\bar{A}^2 \geq 4\bar{B}$. Let $v, w$ be two global solutions to (10) and suppose $v(0) = w(0), v'(0) = w'(0)$, then $v \equiv w$. 

Proof. Denote $z = v - w$, then $z(0) = z'(0) = 0$ and $z$ satisfies the following system:

$$
\begin{cases}
    z'' - \lambda z = \rho, \\
    \rho'' - \mu \rho = f(v) - f(w).
\end{cases}
$$

Without loss of generality, assume $z''(0) \geq 0$ and $z''(0) \geq 0$, otherwise, we can consider $z(-t)$. Then $\rho(0) \geq 0$ and $\rho'(0) \geq 0$. Suppose by contradiction $z \neq 0$. With the uniqueness of ODE system, $(z''(0))^2 + (\rho''(0))^2 \neq 0$. Then we can deduce that $z > 0$ in $(0, \sigma)$ for $\sigma \ll 1$. As $f$ is strictly increasing, then $f(v) - f(w) > 0$ in $(0, \sigma)$. It yields

$$
\rho'' - \mu \rho > 0, \quad \forall t \in (0, \sigma)
$$

i.e.

$$
\left(\frac{d}{dt} - \sqrt{\mu}\right)\left(\frac{d}{dt} + \sqrt{\mu}\right)\rho > 0.
$$

By the fact of $\rho(0) \geq 0$, $\rho'(0) \geq 0$, let $\chi := \rho' + \sqrt{\mu\rho}$, then $\chi(0) \geq 0$ and $(e^{-\sqrt{\mu}t}\chi')' > 0$, which yields that $\chi > 0$ for $t \in (0, \sigma)$. As the above procedures, we can deduce that $\rho > 0$ in $(0, \sigma)$. So $z'' - \lambda z > 0$ for $t \in (0, \sigma)$, which leads to $z'' > 0$. Because of $z'(0) = 0$, then $z'(t) > 0$ in $(0, \sigma)$, i.e. $z$ is strictly increasing on $(0, \sigma)$. Since $\sigma > 0$ is arbitrary with the property that $z > 0$ in $(0, \sigma)$, we infer that $z > 0$ for all time and get $z'(t) > \chi'(\sigma)$ in $(\sigma, +\infty)$. Contradiction to the bound of $z$. Hence $z \equiv 0$ that is $v \equiv w$. \hfill \Box

As the direct consequence of Lemma 2.4, we have the following symmetric result which will play an important role in proving periodic results.

**Lemma 2.5.** Under the conditions of Lemma 2.4. For any global solution $v$ to equation (10),

i) suppose that $v'(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then $v$ is symmetric with respect to $t_0$, i.e., for all $t \in \mathbb{R}$, $v(t_0 + t) = v(t_0 - t)$.

ii) suppose that the function $f$ is odd and $v(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Then $v$ is anti-symmetric with respect to $t_0$, i.e., for all $t \in \mathbb{R}$, $v(t_0 + t) = -v(t_0 - t)$.

3. **Proof of Theorem 1.1 and Theorem 1.2.** In this section, we will prove Theorem 1.1 and Theorem 1.2. Recall our equation

$$
(v^{(4)} - K_2 v'' + K_0 v) = v^p \quad \text{in} \quad \mathbb{R} \tag{12}
$$

where $-n < \alpha < n - 4$, $K_2 = \frac{(n-2)^2 + (\alpha+2)^2}{2}$, $K_0 = \frac{(n-4-\alpha)^2(n+\alpha)^2}{16}$ and $p > 1$ satisfies (2). Note that $K_2^2 \geq 4K_0 > 0$.

Applying Theorem 2.2 of [1] directly, we could get the existence and uniqueness of smooth solution to equation (12).

**Proposition 3.1.** In $H^2(\mathbb{R})$, the equation (12) has a unique nontrivial solution $v$ vanishing at infinity. Moreover, $v$ can be taken to be even, positive and strictly decreasing on $\mathbb{R}_+$ achieving

$$
\inf_{v \in H^2(\mathbb{R}) \setminus \{0\}} \frac{\int_{-\infty}^{\infty} |v''|^2 + K_2 |v'|^2 + K_0 v^2 dt}{\left(\int_{-\infty}^{\infty} |v|^{p+1} dt\right)^{\frac{2}{p+1}}}
$$

Based on Proposition 3.1, for the special cases we can give out the explicit form of the solution to (12). Indeed, we can set

$$
v(t) = C(cosh \nu t)^m,
$$
where $C$, $\nu$, $m$ are to be determined later. Assume that $\nu > 0$ as $\cosh(\cdot)$ is even. By complex and tedious calculation, as $K_2 \geq 2\sqrt{K_0}$ where equality holds if and only if $\alpha = -2$, there holds
\[ v''(t) = C\nu^2m^2(\cosh \nu t)^m - Cm(m-1)\nu^2(\cosh \nu t)^{m-2} \]
and
\[ v^{(4)}(t) = C\nu^4m^4(\cosh \nu t)^m - Cm(m-1)\left[m^2 + (m-2)^2\right]\nu^4(\cosh \nu t)^{m-4} + Cm(m-1)(m-2)(m-3)\nu^4(\cosh \nu t)^{m-4}. \]

Hence,
\[ v^{(4)}(t) - K_2v''(t) + K_0v(t) = C\left[\nu^4m^4 - K_2m^2\nu^2 + K_0\right](\cosh \nu t)^m - Cm(m-1)(m-2)(m-3)\nu^4(\cosh \nu t)^{m-4} = C^p(\cosh \nu t)^{mp}. \]

Take $\nu$ such that
\[ \nu^2 = \frac{K_2}{m^2 + (m-2)^2}. \]
To make sure that such $\nu$ is the solution, one needs
\[ \nu^4m^4 - K_2m^2\nu^2 + K_0 = 0, \quad Cm(m-1)(m-2)(m-3)\nu^4(\cosh \nu t)^{m-4} = C^p(\cosh \nu t)^{mp}. \]

Hence
\[ m = -\frac{4}{p-1} < 0, \quad C^{p-1} = m(m-1)(m-2)(m-3)\nu^4. \]
Under the condition
\[ \alpha = \beta \text{ or } (n+\alpha)(n+\beta) = (n-4-\alpha)^2, \]
one can obtain that
\[ \frac{K_2}{2\sqrt{K_0}} = \frac{(n-2)^2 + (\alpha + 2)^2}{(n-4-\alpha)(n+\alpha)} = \frac{(p+1)^2 + 4}{4(p+1)}, \]
which means
\[ \frac{K_2}{m^2 + (m-2)^2} = \frac{K_2 - \sqrt{K_0^2 - 4K_0}}{2m^2}. \]
So $\nu^4m^4 - K_2m^2\nu^2 + K_0 = 0$.

**Proof of Theorem 1.1.** In $\mathcal{N}^2_{\text{rad}}(\mathbb{R}^n, \alpha)$, Proposition 3.1 yields the existence and uniqueness of the positive radial solution $u$ to equation (1). Moreover, for such radial solution we have
\[ |x|^{\frac{n-4-\alpha}{2}}u(|x|) = C(\cosh \nu t)^m, \quad t = -\ln |x|, \]
i.e.
\[ u(|x|) = \frac{C}{2^m}|x|^{-\gamma}(1 + |x|^{2\nu})^m \quad \text{with } \gamma = \frac{n-4-\alpha}{2} + \nu m. \]
It is easy to see that
\[ \nu = \sqrt{\frac{K_2}{m^2 + (m-2)^2}} = \frac{p-1}{2}\sqrt{\frac{K_2}{(p+1)^2 + 4}}. \]
Recall $\frac{n+\beta}{2} + \frac{n+\beta}{p+1} = n-2$, equivalently $p+1 = \frac{2(n+\beta)}{n-4-\alpha}$ and $m = -\frac{4}{p-1}$.

**Case 1.** $\alpha = \beta > -2$, then
\[ m_1 = -\frac{n-4-\alpha}{2+\alpha}, \quad \nu_1 = \frac{2+\alpha}{2}, \quad \gamma_1 = 0, \quad C_1^{p-1} = m_1(m_1-1)(m_1-2)(m_1-3)\nu_1^4. \]
Thus, we get
\[ u_1(x) = \frac{C_1}{2m_1} \left(1 + |x|^{2+\alpha}\right)^{-\frac{n+\alpha}{2+\alpha}}. \]

**Case 2.** \((n+\alpha)(n+\beta) = (n-4-\alpha)^2\) with \(\alpha < -2\), then \(p + 1 = \frac{2(n+\beta)}{n-4-\alpha} > \frac{2(n-4-\alpha)}{n+\alpha} > 2\). So we have
\[
m_2 = \frac{n+\alpha}{2+\alpha}, \quad \nu_2 = \sqrt{\frac{K_2}{m_2^2 + (m_2 - 2)^2}} = -\frac{2 + \alpha}{2}, \quad \gamma_2 = -(2 + \alpha).
\]

Thus, we obtain
\[
u_2(x) = \frac{C_2}{2m_2} |x|^{\frac{2+\alpha}{2}} \left(1 + |x|^{-\frac{2+\alpha}{2}}\right)^{\frac{n+\alpha}{n+\alpha}}
\]
with \(C_2^{\alpha-1} = m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)\nu_2^4\).

Note that for \(\alpha = \beta = 0\), there holds
\[
p = \frac{n + 4}{n - 4}, \quad m = -\frac{n - 4}{2}, \quad \nu = 1, \quad C_n = \left(\frac{n(n - 4)(n^2 - 4)}{16}\right)^{\frac{n-4}{n-4}},
\]
from which we obtain Lin’s result
\[
u_\lambda(x) = C_n \left(\frac{\lambda}{1 + \lambda^2 |x|^{-2}}\right)^{\frac{n-4}{n-4}}, \quad x_0 \in \mathbb{R}^n, \quad \lambda > 0.
\]

**Proof of Theorem 1.2.** To any radial function \(u \in \mathcal{N}^{\alpha}_{\text{rad}}(\mathbb{R}^n, \alpha)\), we associate a function \(v \in H^2(\mathbb{R})\) via the Emden-Fowler transform \(v(t) = |t|^{\frac{\alpha+1}{2}} u(|t|), t = -\ln |x|\).

Then by direct calculation, there holds
\[
\int_{\mathbb{R}^n} |x|^{\beta} |u|^{p+1} dx = \omega_n \int_{-\infty}^{\infty} |v|^{p+1} dt,
\]
and
\[
\int_{\mathbb{R}^n} |x|^{-\alpha} |\Delta u|^{2} dx = \omega_n \int_{-\infty}^{\infty} |v''|^2 + K_2 |v'|^2 + K_0 v^2 dt,
\]
where \(\omega_n\) is the measure of \(S_n^{-1}\). Thus, \(S_{p,\text{rad}}^{\alpha,\beta}(\alpha, \beta) = \omega_n^{\frac{1}{p+1}} \mu_p(\alpha)\), where
\[
\mu_p(\alpha) = \inf_{v \in H^2(\mathbb{R}) \setminus \{0\}} \frac{\int_{-\infty}^{\infty} |v''|^2 + K_2 |v'|^2 + K_0 v^2 dt}{\left(\int_{-\infty}^{\infty} |v|^{p+1} dt\right)^{\frac{1}{p+1}}}.
\]

By Proposition 3.1, \(\mu_p(\alpha)\) hence \(S_{p,\text{rad}}^{\alpha,\beta}(\alpha, \beta)\) is achieved.

By Theorem 1.1, we have that for \(\alpha = \beta > -2\), or \((n+\alpha)(n+\beta) = (n-4-\alpha)^2\) with \(\alpha < -2\), \(v = C(\cosh \nu t)^m, m < 0\). Direct calculation shows that for any number \(\gamma < 0\),
\[
\int_{0}^{\infty} \cosh \nu t dt = \nu^{-1} \int_{0}^{1} \frac{x^\gamma}{\sqrt{x^2 - 1}} dx
\]
\[
= \nu^{-1} \int_{0}^{1} \frac{y^{-\gamma} - 1}{\sqrt{1 - y^2}} dy = \nu^{-1} \frac{1}{2} \int_{0}^{1} x^{-\frac{\gamma+1}{2}} (1 - x)^{-\frac{1}{2}} dx
\]
\[
= \frac{\nu^{-1}}{2} B\left(-\frac{\gamma}{2}, \frac{1}{2}\right) = \frac{\nu^{-1}}{2} \frac{\Gamma(-\frac{\gamma}{2}) \Gamma(\frac{1}{2})}{\Gamma\left(\frac{\gamma}{2}\right)}.
\]
In conclusion, 
\[
\int_{-\infty}^{\infty} |v|^{p+1} dt = C^{p+1} 2 \int_{0}^{\infty} (\cosh \nu t)^{m(p+1)} dt = C^{p+1} \nu^{-1} \Gamma \left( \frac{1}{2} \right) \frac{\Gamma \left( \frac{1}{2} - m \right)}{\Gamma \left( \frac{1}{2} - m - \frac{1}{2} \right)}
\]

On the other hand, 
\[
|v''|^2 + K_2 |v'|^2 + K_0 v^2 = 2 K_2 C^2 \nu^2 m^2 (\cosh \nu t)^{2m} - 2 m^2 C^2 \nu^4 [2m^2 - 3m + 2] (\cosh \nu t)^{2m-2} + C^2 m^2 (m-1)^2 \nu^4 (\cosh \nu t)^{2m-4}.
\]

Thus we get 
\[
\int_{-\infty}^{\infty} |v''|^2 + K_2 |v'|^2 + K_0 v^2 dt = \int_{-\infty}^{\infty} 2 K_2 C^2 \nu^2 m^2 (\cosh \nu t)^{2m} - 2 m^2 C^2 \nu^4 [2m^2 - 3m + 2] (\cosh \nu t)^{2m-2} dt + \int_{-\infty}^{\infty} C^2 m^2 (m-1)^2 \nu^4 (\cosh \nu t)^{2m-4} dt
\]
\[
= C^2 \nu m^2 \left[ 2 K_2 \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - m \right)} - 2 \nu^2 (2m^2 - 3m + 2) \frac{\Gamma (1-m) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - 2m \right)} \right]
\]
\[
+ C^2 \nu m^2 (m-1)^2 \nu^2 \frac{\Gamma (2-m) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - 2m \right)}
\]
\[
= C^2 \nu m^2 \left[ 2 K_2 - 2 \nu^2 (2m^2 - 3m + 2) \frac{2m^2 - 1}{2m} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} - m \right)} \frac{\Gamma (1-m) \Gamma \left( \frac{1}{2} \right)}{\Gamma (2-m) \Gamma \left( \frac{3}{2} - 2m \right)} \right]
\]
\[
+ C^2 \nu m^2 (m-1)^2 \nu^2 \frac{4m(m-1)}{3(2m)(1-2m)} \frac{\Gamma (1-m) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - 2m \right)}
\]
\[
= C^2 \nu^3 m^2 \frac{4(m-1)^2(m-2)(m-3)}{(2m-1)(2m-3)} \frac{\Gamma (1-m) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - 2m \right)}.\]

In conclusion, 
\[
\mu_p (\alpha) = \nu^{1+\frac{2}{p+1}} m(m-1)(m-2)(m-3) \left[ \frac{4m(m-1)}{(2m-1)(2m-3)} B \left( -m, \frac{1}{2} \right) \right]^{\frac{2}{p+1}}.
\]

**Case 1.** \( \alpha = \beta > -2 \). Recall 
\[
\nu_1 = \frac{2 + \alpha}{2}, \quad m_1 = \frac{n - 4 - \alpha}{2 + \alpha}, \quad \frac{2}{p+1} = \frac{n - 4 - \alpha}{n + \alpha},
\]
then 
\[
\mu_{1,p} (\alpha) = \nu_1^{\frac{2(2m_1+2)}{n+\alpha}} m_1(m_1-1)(m_1-2)(m_1-3) \times \left[ \frac{4m_1(m_1-1)}{(2m_1-1)(2m_1-3)} B \left( -m_1, \frac{1}{2} \right) \right]^{\frac{2(2m_1+2)}{n+\alpha}},
\]
and 
\[
S_p^{rad} (\alpha, \alpha) = \omega_n^{\frac{2(2m+\alpha)}{n+\alpha}} \mu_{1,p} (\alpha).
\]
achieved by
\[ u_1(x) = \frac{C_1}{2m_1} (1 + |x|^{2+\alpha})^{-\frac{n-4-\alpha}{2+\alpha}}, \]
where \( C_1^{n-1} = m_1(m_1 - 1)m_1 - 2(m_1 - 3)\nu_4^2. \)

**Case 2.** \((n + \alpha)(n + \beta) = (n - 4 - \alpha)^2\) with \(\alpha < -2\). Using
\[ \nu_2 = -\frac{2 + \alpha}{2}, \quad m_2 = \frac{n + \alpha}{2 + \alpha}, \quad \frac{2}{p + 1} = \frac{n + \alpha}{n - \alpha - 4}, \]
we get
\[ \mu_{2,p}(\alpha) = \nu_2 \frac{2(2+\alpha-6)}{n-4-\alpha}m_2(m_2 - 1)(m_2 - 2)(m_2 - 3) \times \left[ \frac{4m_2(m_2 - 1)}{(2m_2 - 1)(2m_2 - 3)} B(m_2,\frac{1}{2}) \right]^{-\frac{2(2+\alpha)}{n-4-\alpha}} \]
and
\[ S_p^\alpha(\alpha, \beta) = \omega_n \mu_{2,p}(\alpha) \]
achieved by
\[ u_2(x) = \frac{C_2}{2m_2} |x|^{2+\alpha} \left( 1 + |x|^{-2-\alpha} \right)^{-\frac{n}{2+\alpha}}, \]
with \( C_2^{n-1} = m_2(m_2 - 1)(m_2 - 2)(m_2 - 3)\nu_4^2. \)

**Proof.** Recall the energy function
\[ E_v(t) := -v'(t)v''(t) + \frac{1}{2}v'^2(t) + \frac{K_2}{2} v^2(t) - \frac{K_0}{2} v^2 + \frac{1}{p + 1} v^{p+1}, \]
satisfying
\[ \frac{d}{dt} E_v(t) \equiv 0, \quad \text{i.e.} \quad E_v(t) \equiv \mu. \tag{13} \]
Assume that \( v'(t) = 0 \) has finite roots in \( \mathbb{R} \), thus \( v \) is monotone for large \( t \). By Lemma 2.1,
\[ v \to 0 \quad \text{or} \quad v \to \ell := K_0^{\frac{1}{2+\alpha}} \quad \text{as} \quad t \to \pm \infty. \]
Denote \( E_v(\pm \infty) := \lim_{t \to \pm \infty} E_v(t) \).

**Case 1.** \( v \to 0 \) as \( t \to -\infty \), \( v \to \ell \) as \( t \to +\infty \). Using Lemma 2.2, we get that
\[ E_v(-\infty) = 0, \quad \text{while} \quad E_v(+\infty) = -\frac{K_0}{2} \ell^2 + \frac{1}{p + 1} \ell^{p+1} = \ell^2 K_0 \left( \frac{1}{p + 1} - \frac{1}{2} \right) < 0. \]
This contradicts to (13).

**Case 2.** \( v \to \ell \) as \( t \to -\infty \), \( v \to 0 \) as \( t \to +\infty \). It is similar to Case 1. Indeed, we can consider \( v(-t) \).

**Case 3.** \( v \to \ell \) as \( t \to \pm \infty \). As \( v \neq \ell \), there exists a global maximal point or minimal point \( t_0 \). By (13), we see that \( E_v(+\infty) = E_v(t_0) \).
On the other hand, \( E_v(+\infty) = G(\ell) := \frac{1}{p+1} \ell^{p+1} - \frac{K_0}{2} \ell^2 \), where
\[
G(s) := \frac{1}{p+1} s^{p+1} - \frac{K_0}{2} s^2.
\]
Obviously \( G'(s) < 0 \) for \( s < \ell \) and \( G'(s) > 0 \) for \( s > \ell \). Thus
\[
E_v(t_0) = \frac{1}{2} v''(t_0) + G(v(t_0)) > \frac{1}{2} v''(t_0) + G(\ell) \geq G(\ell).
\]
Contradiction.

**Case 4.** \( v \to 0 \) as \( t \to \pm \infty \). By direct calculation, we have that the equation (12) has four eigenvalues
\[
\lambda_1 = \frac{n + \alpha}{2}, \quad \lambda_2 = \frac{n - \alpha - 4}{2}, \quad \lambda_3 = -\frac{n + \alpha}{2}, \quad \lambda_4 = -\frac{n - 4 - \alpha}{2}.
\]
As \(-2 < \alpha < n - 4\), we see that \( \lambda_1 > \lambda_2 > 0, \lambda_3 < \lambda_4 < 0 \). By the variation of parameters method, the solution \( v(t) \) to (12) is given by
\[
v(t) = \sum_{i=1}^{4} A_i e^{\lambda_i t} + \sum_{i=1}^{4} B_i \int_{0}^{t} e^{\lambda_i (t-s)} v^p(s) ds
\]
\[= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \sum_{i=3}^{4} A_i e^{\lambda_i t} - \sum_{i=1}^{2} B_i \int_{t}^{\infty} e^{\lambda_i (t-s)} v^p(s) ds \]
\[+ \sum_{i=3}^{4} B_i \int_{0}^{t} e^{\lambda_i (t-s)} v^p(s) ds
\]
where we used the fact \( e^{-\lambda_1 s} v^p(s), e^{-\lambda_2 s} v^p(s) \in L^1(\mathbb{R}_+) \). As \( \lim_{t \to \infty} v(t) = 0 \) and \( \lambda_1, \lambda_2 > 0 \), there holds \( A_1' = 0 = A_2' \). Therefore, there exists \( C > 0 \) such that for any \( t \geq 0 \), there holds
\[
|v(t)| \leq C e^{\lambda_1 t} + C \int_{t}^{\infty} e^{\lambda_2 (t-s)} |v^p(s)| ds + C \int_{0}^{t} e^{\lambda_4 (t-s)} |v^p(s)| ds.
\]
Moreover, for any \( \delta > 0 \), there exists \( M > 0 \) such that \( |v^p(s)| \leq \delta |v(s)| \) if \( s \geq M \). Then for \( t \geq M \),
\[
|v(t)| \leq O(e^{\lambda_1 t}) + C \delta \int_{t}^{\infty} e^{\lambda_2 (t-s)} |v(s)| ds + C \delta \int_{M}^{t} e^{\lambda_4 (t-s)} |v(s)| ds,
\]
\[= O(e^{\lambda_1 t}) + C \delta K_1(t) + C \delta K_2(t)
\]
with
\[
K_1(t) := \int_{t}^{\infty} e^{\lambda_4 (t-s)} |v(s)| ds, \quad K_2(t) := \int_{t}^{\infty} e^{\lambda_2 (t-s)} |v(s)| ds.
\]
Using (14), if we fix \( \delta > 0 \) small enough such that \( 2C \delta \leq \lambda_2 = -\lambda_4 \), there holds
\[
(K_1 - K_2)'(t) = 2|v(t)| + \lambda_4 K_1(t) - \lambda_2 K_2(t)
\]
\[\leq 2C \delta (K_1 + K_2) + \lambda_4 K_1(t) - \lambda_2 K_2 + O(e^{\lambda_4 t})
\]
\[\leq O(e^{\lambda_4 t}).
\]
As \( \lim_{t \to \infty} v(t) = 0 \), we have readily
\[
\lim_{t \to \infty} K_1(t) = \lim_{t \to \infty} K_2(t) = 0.
\]
Hence, \((K_2 - K_1)(t) \leq O(e^{\lambda t})\) as \(t \to \infty\). Going back to (14),
\[ |v(t)| \leq O(e^{\lambda t}) + 2C\delta K_1(t). \]

Consequently,
\[ K_1'(t) = |v(t)| + \lambda_4 K_1(t) \leq O(e^{\lambda t}) + (2C\delta + \lambda_4)K_1(t). \]
So \(K_1(t) = O(e^{(\lambda_4 + 2C\delta)t})\). By ODE theory we have \(|v(t)| = O(e^{(\lambda_4 + 2C\delta)t})\). Again back to just before (14), we have
\[ v(t) = O(e^{\lambda_4 t}). \]
Hence, for \(r \to 0^+\), there holds \(u = O(1)\) which contradicts to the non-removable singularity.

**Remark 4.2.** By Proposition 4.1, we have that there are two real numbers \(t_0 < t_1\) such that \(v'(t_0) = v'(t_1) = 0\), \(v'(t) \neq 0\) in \((t_0, t_1)\). By Lemma 2.5, \(v\) must be periodic. In other words, for any radial solution \(u\) with non-removable singularity at origin, the function \(v(t) := |x|^{-\frac{n-1}{2}} u(x), t = -\ln |x|\) is periodic.

**Proposition 4.3.** Let \(v(t)\) be the solution in Proposition 3.1, \(u(|x|) = |x|^{-\frac{n-1}{2}} v(t), t = -\ln |x|\). If \(-n < \alpha < -2\). For \(\alpha = -2\), one just needs to modify the proof with minor changes.

Note that \(\lambda_2 = \frac{n-1}{2} - \lambda_1 = \frac{n-4}{2} > 0, \lambda_4 = -\lambda_2, \lambda_3 = -\lambda_1\). According to the variation of parameters method,
\[ v(t) = \sum_{i=1}^{4} A_i e^{\lambda_i t} + \sum_{i=1}^{2} B_i \int_{t}^{\infty} e^{\lambda_i(t-s)} v^{p}(s) ds + \sum_{i=3}^{4} B_i \int_{-\infty}^{t} e^{\lambda_i(t-s)} v^{p}(s) ds. \]
Using \(\lim_{t \to \pm \infty} v(t) = 0\), obviously \(A_1 = A_2 = A_3 = A_4 = 0\). The fact \(v(t) = v(-t)\) leads to \(B_1 = B_3, B_2 = B_4\).

Argue by contradiction. Assume \(u(0) = O(1)\), which means that \(v(t) = O(e^{\lambda_4 t})\).
Hence \(B_1 = B_3 = 0\). In a word, we get that
\[ v(t) = B \int_{t}^{\infty} e^{\lambda_2(t-s)} v^{p}(s) ds + B \int_{-\infty}^{t} e^{\lambda_4(t-s)} v^{p}(s) ds, \quad \text{for some } B > 0. \]

Direct computations give out that
\[ v'(t) = B\lambda_2 \left( \int_{t}^{\infty} e^{\lambda_2(t-s)} v^{p}(s) ds - \int_{-\infty}^{t} e^{\lambda_4(t-s)} v^{p}(s) ds \right); \]
\[ v''(t) = -2B\lambda_2 v^{p}(t) + \lambda_2^2 v(t); \]
\[ v'''(t) = -2B\lambda_2 p v^{p-1}(t) v'(t) + \lambda_2^3 v'(t); \]
\[ v^{(4)}(t) = -2B\lambda_2 p(p-1) v^{p-2}(t) [v'(t)]^2 - 2B\lambda_2 p v^{p-1}(t) v''(t) + \lambda_2^4 v''(t). \]

Using \(K_2 = \lambda_2^2 + \lambda_2^4, K_0 = \lambda_2^2 \lambda_2^2\) and equation (12), we can obtain that
\[-2B\lambda_2 p(p-1) \left( \frac{v'(t)}{v(t)} \right)^2 + 4B^2 \lambda_2^2 p v^{p-1}(t) - 2B\lambda_2^3 p + 2B\lambda_2^4 \lambda_2 = 1. \]

Consider \(t = 0, v'(0) = 0\) and \(t \to +\infty, v(t) \to 0\), then
\[ 4B^2 \lambda_2^2 p v^{p-1}(0) - 2B\lambda_2^3 p + 2B\lambda_2^4 \lambda_2 = 1, \quad -2B\lambda_2^3 p + 2B\lambda_2^4 \lambda_2 \geq 1, \]
from which we can deduce
\[ 4B^2 \lambda_2^2 p v^{p-1}(0) \leq 0. \]
Proposition 4.1 and Remark 4.2.

Proof Theorem 1.3. i) can be obtained by Proposition 4.1 and Remark 4.2.

Proof Theorem 1 in [11]. For the convenience of the readers, we give the details.

For any $a \in (0, \ell)$ and $\gamma \geq 0$, we denote $v_\gamma$ be the unique solution to
\begin{align*}
\begin{cases}
v^{(4)}(t) - K_2 v''(t) + K_0 v(t) = |v|^{p-1}v(t) & \text{in } [0, T_\gamma), \\
v(0) = a, \quad v'(0) = v''(0) = v'''(0) = 0, & \text{in } [0, T_\gamma),
\end{cases}
\end{align*}
where $T_\gamma \in (0, \infty)$ is the maximal existence interval. Also, let
\[ b := - \min_{v \in \mathbb{R}^n} f(v) > 0, \quad \text{where } f(v) = |v|^{p-1}v - K_0 v. \]
Suppose that $\gamma > \frac{b}{K_2} =: \gamma_0$, then we see that
\[ v^{(4)}(t) = K_2 v''(t) + f(v_\gamma), \]
and
\[ v^{(4)}(0) = K_2 \gamma + f(v_\gamma(0)) > 0. \]
Hence, $v''(t)$ is increasing initially. Moreover $v^{(4)}(t) > 0$ holds in $[0, T_\gamma)$. As $v''(0) = 0$, we have $v''(t) > 0$ in $(0, T_\gamma)$. If $T_\gamma = \infty$, then $v_\gamma \to \infty$ i.e. $v_\gamma$ is unbounded. If $T_\gamma < \infty$, then $v_\gamma \to \infty$ as $t \to T_\gamma$. In conclusion, $v_\gamma$ is unbounded for $t \to T_\gamma$ for $\gamma > \gamma_0$. So we can restrict $\gamma \leq \gamma_0$.

Denote $F(v) := \int_0^v f(s)ds$, then
\[ E_{v_\gamma}(0) = \frac{\gamma^2}{2} + F(a) \leq \frac{\gamma_0^2}{2} + F(a). \]
Since $F(v_\gamma) \to \infty$ as $v_\gamma \to \infty$ due to $p > 1$, there exists $M \gg 1$ such that
\[ F(v_\gamma) > \frac{\gamma_0}{2} + F(a) \quad \text{for all } v > M. \]
Moreover, assume $v_\gamma(t_0) > M$ for $\gamma \leq \gamma_0$, we must have $v'_\gamma(t_0) \neq 0$. Otherwise,
\[ E_{v_\gamma}(t_0) = \frac{v''(t_0)^2}{2} + F(v_\gamma(t_0)) \geq F(v_\gamma(t_0)) > \frac{\gamma_0^2}{2} + F(a) \geq E_{v_\gamma}(0), \]
which is a contradiction. It implies that $v_\gamma$ which once enters the interval $(M, +\infty)$ cannot leave it and hence it is certainly not the periodic solution we are looking for.

On the other hand, for $\gamma = 0$, then $v^{(4)}(0) = f(a) < 0$ by $a < \ell$. So $v'' < 0$ and $v' < 0$ in $(0, \sigma)$, $\sigma \ll 1$. As $f(v) < 0$ for $v \in (0, a)$, we can deduce $v^{(k)}(t) < 0$, $k = 1, 2, 3$ until $v_\gamma(t)$ reaches a negative value. In a word, there exists $t_0$ such that $v_\gamma(t_0) < 0$. Based on the above analysis, we can define the shooting set
\[ S := \{ \gamma \geq 0 : v_\gamma < 0 \quad \text{for some } t \in (0, T_\gamma) \}, \]
\[ T := \{ \gamma \geq 0 : v_\gamma > M \quad \text{for some } t \in (0, T_\gamma) \quad \text{and } v_\gamma > 0 \quad \text{on } (0, t) \}. \]
Then $S$ and $T$ are open sets in $[0, +\infty)$ because of the continuous dependence of the solution on the initial data. Obviously $0 \in S$, $(\gamma_0, +\infty) \subset T$ and $S \cap T = \emptyset$.

Furthermore, we can rewrite the equation as a system
\[ \begin{cases}
v''(t) - \lambda v = w, & \text{in } [0, T_\gamma) \\
w''(t) - \mu w = |v|^{p-1}v, & \text{in } [0, T_\gamma) \\
v(0) = a, \quad v'(0) = 0, \quad w(0) = \gamma - \lambda a, \quad w'(0) = 0,
\end{cases} \]
from which one can get that the solution $v_\gamma$ is strictly increasing in $\gamma$. Shooting parameter interval $[0, +\infty)$ is connected, but $S \cup T \neq [0, +\infty)$, i.e. there exists $\gamma^*$ such that $0 < v_{\gamma^*} \leq M$. So $v_{\gamma^*}$ is bounded. Since $f$ is locally Lipschitz, we have $T_{\gamma^*} = +\infty$. By even reflection, we can get a global solution. By the above argument, $v_{\gamma^*}$ must be periodic.

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