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Evidence for the continuum in 2D causal set quantum gravity

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Abstract
We present evidence for a continuum phase in a theory of 2D causal set quantum gravity which contains a dimensionless non-locality parameter $\epsilon \in (0, 1]$. We also find a phase transition between this continuum phase and a new crystalline phase which is characterized by a set of covariant observables. For a fixed size of the causal set, the transition temperature $\beta^{-1}$ decreases monotonically with $\epsilon$. The locus of the transition in the $\beta^2$ versus $\epsilon$ plane asymptotically approaches to the infinite temperature axis, suggesting that the continuum phase survives the analytic continuation.

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Causal set theory (CST) is an approach to quantum gravity in which the spacetime continuum is replaced by a locally finite poset or causal set, with the order relation $\prec$ being the analogue of the spacetime causal order [1]. The relation $\prec$ is therefore (i) acyclic, i.e. there is no $y$ such that $x \prec y$ and $y \prec x$, and (ii) transitive, i.e.

$$x \prec y, y \prec z \Rightarrow x \prec z.$$  

CST assumes a fundamental spacetime discreteness with the continuum arising as an approximation, much like the apparent continuity of a fluid whose underlying structure is discrete. This translates into the condition of local finiteness in the poset, which encodes the fact that a finite spacetime volume should contain a finite number of elements of the causal set. Thus, while the order relation in the causal set corresponds to the causality relations of the spacetime, the cardinality of an ‘interval’ or number in a causal set corresponds to the spacetime volume. In the continuum, the causal structure and the spacetime volume element together suffice to specify a Lorentzian spacetime [2–4], and hence the causal set captures the essence of a discrete Lorentzian geometry. Importantly, in order to preserve the number to spacetime volume correspondence, the continuum approximation occurs via a random Poisson process described for example in [5].

The recently constructed CST version of the Einstein–Hilbert action $S_{\text{CST}}$ [6] has made it possible for the first time to begin a serious study of the CST partition function:
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Figure 1. The 2D order from $^2\mathbb{M}$.

$$Z_{\text{CST}} = \sum_{C \in \Omega} \exp^{iS_{\text{CST}}/\hbar},$$

where $\Omega$ is a sample space of causal sets. A natural starting choice for $\Omega$ is the collection of countable causal sets, but further restrictions may arise from physical considerations.

In this work, we consider a particular restriction of the sample space to $\Omega^2_2$, the space of $N$-element ‘2D orders’, which along with the choice of the 2D CST action $S_{2\text{d}}$ gives rise to a 2D theory of causal sets. An $N$-element of 2D order is defined as follows. Let $U = (u_1, u_2, \ldots, u_N)$ and $V = (v_1, v_2, \ldots, v_N)$, with $u_i, v_j \in \mathbb{R}, u_i \neq u_j, v_i \neq v_j$ for $i \neq j$. $U$ and $V$ are total orders: for each pair $i \neq j$ either $u_i < u_j$ or $u_j < u_i$, and similarly for $V$. The 2D order $C = U \cap V$ is the intersection of $U$ and $V$: $e_i < e_j$ in $C$ iff $u_i < u_j$ and $v_i < v_j$. Each element of this poset $e_i$ thus comes with a label $i \in \{1, \ldots, N\}$, much like the coordinate system of a spacetime; label invariance is therefore the discrete analogue of coordinate invariance in the continuum. An important example of a 2D order is constructed from a set of $N$ distinct labelled events $\{e_1, \ldots, e_N\}$ in 2D Minkowski spacetime $^2\mathbb{M}$, with non-intersecting light cone coordinates, i.e. if $e_i = (u_i, v_i)$ in light cone coordinates, then $u_i \neq u_j$ and $v_i \neq v_j$ for $i \neq j$. The sets $U, V$ of $u$ and $v$ coordinates, respectively, are each of total orders; the 2D order $U \cap V$ is then the partial order obtained from the causal ordering in $^2\mathbb{M}$ of the $e_i$ (see figure 1).

The choice of $\Omega^2_2$ is motivated by the fact that a causal set underlying any conformally flat, topologically trivial 2D spacetime is a 2D order [7]. On the other hand, not all 2D orders are approximated by continuum spacetimes, and hence $\Omega^2_2$ is strictly a restriction to poset rather than spacetime dimension. Only in a limited sense do these dimensions coincide: every 2D order $C$ admits an order preserving embedding $\Phi$ into $^2\mathbb{M}$, i.e. for every $e_i < e_j$ in $C$, $\Phi(e_i)$ causally precedes $\Phi(e_j)$ in $^2\mathbb{M}$. Such an embedding, though necessary, is not sufficient to ensure a continuum approximation for $C$. A striking feature of $\Omega^2_2$ is that in the limit $N \to \infty$ it is dominated by ‘random’ 2D orders, namely those approximated by $^2\mathbb{M}$ [7–9]. A random 2D order is the intersection of two total orders $U$ and $V$ whose elements $u_i$ and $v_i$, respectively, are chosen randomly and independently from $\{1, \ldots, N\}$. Thus, causal sets approximated by $^2\mathbb{M}$ dominate the uniform measure on $\Omega^2_2$, and it is of obvious interest to study the effect of the CST action $S_{2\text{d}}$ on this entropic feature of $\Omega^2_2$.

While there is no natural Planck scale in 2D gravity, CST requires a volume cut-off $V_p = l_p^2$ for the continuum approximation. In addition, the CST action includes a free ‘non-locality’ scale $l_k > l_p$ [6, 10] and is given by

$$S_{2\text{d}}[C, \epsilon]/\hbar = 4\epsilon \left( N - 2\epsilon \sum_{n=0}^{N-2} N_n f(n, \epsilon) \right),$$

where $\Omega$ is a sample space of causal sets. A natural starting choice for $\Omega$ is the collection of countable causal sets, but further restrictions may arise from physical considerations.
Figure 2. The function $f(n, \epsilon)$ for $\epsilon = 0.12$.

where $\epsilon = \frac{l_p^2}{\ell^2}$, $N_n$ is the number of cardinality $n$ intervals $I[i, j] \equiv \{ k | i < k < j \}$ in $C$ and $f(n, \epsilon) = (1 - \epsilon)^n (1 - 2\epsilon n (1 - \epsilon)^{-1} + \epsilon^2 n(n - 1)(1 - \epsilon)^{-2}/2)$. The figure 2 shows the typical profile of $f(n, \epsilon)$, with $f(n, \epsilon) \sim 0$ for $n > 1/\epsilon$. To avoid infrared errors in $N_n$ for finite $N$, $\epsilon$ is in addition required to be bounded by $N\epsilon > 1$.

As discussed above, the analogue of covariance in CST is label independence and our goal is to calculate expectation values of label-independent observables from the 2D partition function. An example of such an observable is the action $S_{2d}/\hbar$ itself, while another is the Myrheim–Myer dimension $d_{MM}$, equal to the spacetime dimension when the causal set is approximated by $^nM$ [11]. However, the sample space $\Omega_{2D}$ is itself the collection of labelled 2D orders. Now, the automorphisms Aut($C$) of an $N$-element-labelled causal set $C$ are the set of relabellings of $C$ that do not produce a distinct labelled causal set. An extreme example is the antichain or completely unordered set—each relabelling produces the same labelled causal set. Since every 2D order admits $N!$ worth of relabellings each unlabelled 2D order appears $N!/|\text{Aut}(C)|$ times in our 2D partition function. While it is possible to view this weight as a quantisation ambiguity, it is important to stress that the observables we construct are nevertheless strictly label independent.

A standard route to evaluating the path integral is via a Euclideanization of the partition function, which renders the quantum system into a thermodynamic one. Replacing the set of Lorentzian metrics with Euclidean ones, however, makes little of the importance of causal structure, and moreover, can lead to highly fractal, non-manifold like behaviour [12]. Instead, as suggested in [13] we introduce a new parameter $\beta$ into the 2D partition function $Z_{2D}[\beta] = \sum_{C \in \Omega_{2D}} \exp(iS_{2d}/\hbar)$; when $\beta \to i\beta$, this gives the thermodynamic partition function

$$Z_{2D} = \sum_{C \in \Omega_{2D}} \exp^{-\beta S_{2d}[C]/\hbar}.$$  (3)

Note that $\Omega_{2D}$ is unchanged in the process, leaving no ambiguity in the interpretation of the covariant observables. In particular, these observables remain Lorentzian even after analytical continuation. For $\beta \to 0$, one recovers the uniform distribution dominated by 2D random orders, but as $\beta \to \infty$, since the action is not positive definite, the largest negative values of $S_{2d}/\hbar$ should dominate, modulo entropic effects. Thus, one expects a cross-over at finite $\beta$.

We use Markov chain Monte Carlo (MCMC) methods to study (3). Our results are the first larger effort to study causal set quantum dynamics using MCMC techniques. The restriction to two dimensions leads to a substantial simplification which translates into rapid mixing or thermalization of the Markov chain. Our simulations are carried out for relatively small causal sets ($N = 50$), but this is sufficient to show emergent continuum behaviour. One of the main observations of our present work is the evidence for a continuum phase, in which
the expectation value of the observables matches those of Minkowski spacetime. In addition, we show evidence for a phase transition at finite $\beta$, rather than the cross-over suggested above.

We define the exchange move $\mu_\chi : C = U \cap V \rightarrow C' = U' \cap V'$ in $\Omega_{2D}$ as follows. Pick $U$ or $V$ at random. Without loss of generality, let this be $U$. Next, pick a pair of elements $u_i, u_j$ in $U$ at random and perform the exchange $u_i \leftrightarrow u_j$, while leaving $V$ unchanged. The new 2D order $C' = U' \cap V'$ has two elements $e'_i = (u'_i, v_i) = (u_j, v_i)$ and $e'_j = (u'_j, v'_j) = (u_i, v_j)$ which differ from $e_i, e_j$ in $C$, while all other elements remain the same. Consider the three possible cases as follows. (a) If $e_i < e_j$, then $u_i < u_j$ and $v_i < v_j$, so that $u'_i > u'_j$, while $v'_i < v'_j$. Thus, $e'_i$ and $e'_j$ are unrelated in $C'$. (b) Similarly, if $e_i > e_j$, then after the exchange the two elements are unrelated in $C'$. (c) If $e_i$ and $e_j$ are unrelated either (i) $u_i < u_j$ and $v_i > v_j$, so that $e'_i < e'_j$ or (ii) $u_i > u_j$ and $v_i < v_j$, so that $e'_i > e'_j$, i.e. $e'_i$ and $e'_j$ are related in $C'$. Thus in all three cases, $C'$ is distinct from $C$ in $\Omega_{2D}$, i.e. $\mu_\chi$ has no fixed points in $\Omega_{2D}$. Since the exchange move does not change the sets $U$ or $V$, for convenience, instead of $u_i, v_i \in \mathbb{R}$, we choose $u_i, v_i \in [1, 2, \ldots, N]$.

We employ a Metropolis–Hastings algorithm, accepting a move if the difference in the action $\Delta S_{2d}$ is negative and rejecting a move only if $\exp^{-\Delta S_{2d}/\hbar} < r$, where $r$ is a random number in $[0, 1)$. Since each move depends on a pair of elements, we define a sweep to be $N(N - 1)/2$ moves. The observables calculated per sweep are as follows. (i) The ordering fraction $\chi = 2r/N(N - 1)$, where $r$ is the number of relations in $C$. $\chi$ is analogous to the filling fraction in an Ising model and in $2\mathbb{M}$, $\chi = \langle (d_{MM}^{-1}) \rangle \sim 0.5$. (ii) The action $S_{2d}[C]/\hbar$. In $2\mathbb{M}$ ($S_{2d}/\hbar$) has a residual or boundary value of 4 [14]. (iii) Time asymmetry, $t_{\text{so}}$. Since the action is time-symmetric, one expects a time-asymmetry or symmetry breaking to set in only at low temperatures. A rough measure of this is the difference in the number of maximal and minimal elements in $C$. In $2\mathbb{M}$, $\langle t_{\text{so}} \rangle \sim 0$. (iv) The height $h$ of $C$ is the length of the longest chain. In $2\mathbb{M}$, $\langle h \rangle$ is the maximum proper time [15]. (v) $N_p$ for $n \in [0, N - 2]$. In $2\mathbb{M}$, $N_p$ has a specific monotonic fall-off with $n$, as shown in figure 6.

Our focus in this work is to study these observables as a function of the inverse temperature $\beta$ as well as the non-locality parameter $\epsilon$. Our simulations are carried out for $N = 50$, with $\epsilon$ ranging between 0.1 and 1. Each Markov chain consists of 10 000 sweeps which translates to 12.5 million attempted moves. We test for thermalization starting from eight different types of initial 2D orders, including the random 2D order, the chain or total order, and the antichain or totally unordered set. We find rapid mixing starting from each of these 2D orders, an example of which is shown in figure 3. The expectation values for the observables $O$ are calculated from data taken every autocorrelation time $\tau_O$.

For fixed $\epsilon$, we find a rapid change in $\langle O \rangle$ around $\beta = \beta_c$, as shown in figure 4, which strongly suggests a phase transition. The error bars are shown, but these are typically very small.

That there is an actual differentiation into two phases becomes explicit on examining the 2D orders themselves. We record the 2D orders in the Markov chain every 100 sweeps. For $\beta < \beta_c$, we find a ‘continuum’ phase (phase I), while for $\beta > \beta_c$, we find a new ‘crystalline’ phase (phase II). We show examples of configurations in these two phases in figure 5, where the light cone coordinates have been turned clockwise by $\pi/4$ for the ease of plotting, with each element $e_i = (u_i, v_i)$.

A glance at a typical causal set $C_I$ from phase I shows that it bears a strong resemblance to a random discretization of $2\mathbb{M}$. Indeed, the expectation values for a fixed $\beta < \beta_c$ corroborate this ‘visual’: for $\epsilon = 0.12$ and $\beta = 0.1$, for example, (a) $\langle \chi \rangle = 0.499$ with an error less than $10^{-8}$, which means that the Myrheim–Myer dimension $\langle d_{MM} \rangle \sim 2.004$, (b) $\langle h \rangle = 10.232 \pm 0.014$, which is comparable to the height $10l_p$ of a volume $V = 50l_p^3$ interval.
in $2^M$, (c) $\langle t_{\text{as}} \rangle = 0.027 \pm 0.024$ which is close to zero, and (d) $\langle S \rangle / \hbar = 3.846 \pm 0.013$ which is comparable to the residual or boundary value of 4 for $2^M$ [14]. In addition, if we plot the abundance $N_n$ of the intervals of cardinality $n$ in $C_I$ and contrast it with that for a random 2D order (which is a discretization of $2^M$), we find that these match very closely as shown in figure 6. As $\beta$ nears the transition, the expectation values of these observables gradually change. However, the typical causal set continues to retain the features of a random 2D order.

A typical causal set $C_{II}$ in phase II, on the other hand, has a most unexpected character as shown in figure 5. It has a regularity, or crystalline nature, and a limited time extent, suggesting that it does not have a continuum approximation. The values of most observables differ considerably from those in phase I. For example, for $\epsilon = 0.12$, $\beta = 3.5$ (a) $\langle \chi \rangle = 0.579$ which gives $\langle d_{\text{BM}} \rangle \sim 1.727$, (b) $\langle h \rangle = 4.180 \pm 0.018$ (c) $\langle t_{\text{as}} \rangle = -10.292 \pm 0.102$, which is a large deviation from the expected value of zero. This may be the result of a spontaneous breaking of the time symmetry, similar to that in the low temperature phase of the Ising model, but may also be the result of poorer statistics at larger $\beta$. (d) $\langle S \rangle / \hbar = -41.367 \pm 0.054$, i.e. the action tries to take on the lowest possible (negative) value. In contrast to phase I, the abundances of intervals as shown in figure 7 are very different from that of a causal set discretization of $2^M$. In particular, $N_n$ align themselves with the positive part of $f(n, \epsilon)$ and vanish for those $n$ for which $f(n, \epsilon) < 0$, thus minimizing the action.

The existence of these distinct phases thus strongly suggests a phase transition. It is tempting to draw the obvious analogy with the Ising model: at high temperatures one has the disordered or random phase I, while at low temperatures, there is the highly ordered crystalline phase II which exhibits a spontaneous breaking of symmetry. Although it is difficult at this stage of our work to assess the order of this transition, there are hints that it may be of second order. To begin with, the autocorrelation time peaks at the phase transition temperature as do the fluctuations in the observables, while the transitions shown in figure 4 appear to be smooth. Moreover, the ‘specific heat’ per unit element

$$C = \frac{\beta^2}{N} \left( \langle S^2 / \hbar^2 \rangle - \langle S / \hbar \rangle^2 \right)$$

shows a characteristic peak, as shown in figure 8.

In the near future we hope to perform a detail analysis of the dependence on the cardinality $N$, which would provide more supporting evidence.
It is important however to stress that while the nature of this phase transition is of interest, it does not play a crucial role in determining continuum behaviour. In other lattice-based approaches in which discretization is used as a calculational tool, the appearance of a second-order phase transition implies the existence of the continuum limit. However, in a fundamentally discrete theory like CST, it is only the continuum approximation we seek, and as we have just seen from examining the observables in phase I, this does not depend on the existence and the nature of a phase transition.

More important to our discussion is then the question of what this thermodynamic calculation means for quantum gravity. How much of the above discussion, if any, survives
Figure 5. The continuum and the crystalline phases. Each element $e_i$ is represented by its $(U, V)$ coordinates $e_i = (u_i, v_i)$.

Figure 6. $N_n$ in $C_I$ is compared with that for a 2D random order.

Figure 7. The first figure is a comparison of $N_n$ in $C_{II}$ with the random 2D order for small $n$. It follows the positive part of $f(n, \epsilon)$.

the analytic continuation? As $\epsilon$ varies in $(0, 1]$, our simulations show that the phase transition survives, but the temperature of the transition increases monotonically with $\epsilon$. Using the maximal size of the fluctuations to estimate the temperature of the transition, we plot $\beta^2$ as a function of $\epsilon$. The negative $\beta^2$ axis corresponds to the region of interest, i.e. the quantum regime, while the positive $\beta^2$ axis corresponds to the thermodynamic regime to which our
above calculations belong. As shown in figure 9, the locus of $\beta_2^c$ which separates phases I from II asymptotes to the $\beta^2 = 0$ axis which itself belongs to phase I, i.e. the continuum. This strongly suggests that the non-continuum crystalline phase II is confined to the $\beta^2 > 0$ region and that the continuum phase survives the analytic continuation into the $\beta^2 < 0$ region. Moreover, smaller $\epsilon$ is favoured by phase I, with this being consistent with the fact that fluctuations in $S_{\text{CST}}/\hbar$ are better suppressed at small $\epsilon$, hence yielding a more reliable continuum approximation. A similar analysis has also been used in studying the phase structure of QCD and its analytic continuation [16].

A more careful study of the phase diagram is clearly in order. In a recent paper [17], the phase structure of the 4D causal dynamical triangulation model of quantum gravity was studied. The phase diagram of this theory includes three phases: two of which, phases C and B seem at least superficially to be analogous to our phases I and II, respectively. The phase transition B–C is argued in [17] to be second order, and it would be interesting to explore whether these two theories lie in the same universality class.

Since a more ambitious goal is to work with the unrestricted sample space $\Omega_1$ and an action with at least dimension 4 [14], it is not immediate that 2D CST can teach us straightforward lessons. Nevertheless, our analysis opens a new window into causal sets, and with it a host of questions that can finally begin to be addressed. One of the more interesting of these is whether there is a renormalization group-type analysis with stable fixed points for $\epsilon < 1$. This
would suggest that the non-locality scale is not a free parameter, but can be determined from the quantum dynamics.

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