THE STRONG MAXIMAL RANK CONJECTURE AND MODULI SPACES OF CURVES

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Abstract. Building on recent work of the authors, we use degenerations to chains of elliptic curves to prove two cases of the Aprodu-Farkas strong maximal rank conjecture, in genus 22 and 23. This constitutes a major step forward in Farkas’ program to prove that the moduli spaces of curves of genus 22 and 23 are of general type. Our techniques involve a combination of the Eisenbud-Harris theory of limit linear series, and the notion of linked linear series developed by the second author.

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1. INTRODUCTION

The study of the moduli space $\mathcal{M}_g$ of curves of fixed genus $g$ is one of the most classical in algebraic geometry. Going back to Severi, based on examples in low genus there was a general expectation that these moduli spaces ought to be unirational. However, groundbreaking work of Harris and Mumford and Eisenbud [HM82] [Har84] [EH87] in the 1980’s showed that not only is $\mathcal{M}_g$ not unirational for large $g$, but it is in fact of general type for $g \geq 24$. Their fundamental technique was to compute the classes of certain explicit effective divisors on $\mathcal{M}_g$ arising from Brill-Noether theory, and use this to show that the canonical class of $\mathcal{M}_g$ can be written as the sum of an ample and an effective divisor. The particular families of divisors they considered were computable in all applicable genera, but did not

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suffice to prove that $M_g$ is of general type for $g \leq 23$. For the last 30 years, no new cases have been proved of $M_g$ being of general type. Roughly a decade ago, Farkas proposed (see [Far09c], [Far09b, §4], [Far09a, §7]) new families of expected divisors on $M_g$ as an approach to showing that $M_{22}$ and $M_{23}$ are of general type. He computed ‘virtual classes’ for these expected divisors in [Far09a] for genus 22 and in the new paper [Far18] for genus 23, and in both cases found that the classes satisfy the necessary inequalities to conclude that $M_{22}$ and $M_{23}$ are of general type, provided that they are indeed represented by effective divisors.\footnote{[Far09a] also includes an announcement that $M_{22}$ is of general type, but in [Far18] Farkas states that the intended proof “has not materialized.”}

The remaining steps of Farkas’ program can be described in terms of the following definition.

**Definition 1.1.** Given $g \geq 21$, let $d = 3 + g$. Let $D_g \subseteq M_g$ consist of curves $X$ of genus $g$ which admit a $g_d^1$ such that the resulting image of $X$ in $P^6$ lies on a quadric hypersurface.

For $g = 22$ or 23, in order to conclude that $M_g$ is of general type, one has to check two statements: first, that $D_g$ yields an effective divisor, or equivalently, that $D_g \subseteq M_g$; and second, that the class induced by $D_g$ agrees with the class computed by Farkas, or equivalently, that the subset of $D_g$ consisting of curves carrying infinitely many $g_d^1$s whose image lie on a quadric occurs in codimension strictly higher than 1. In this paper, we prove the first of these two statements, for both $g = 22$ and $g = 23$. An independent proof of this result has been obtained by Jensen and Payne in [JP18], using their tropical approach. Our main theorem is thus the following:

**Theorem 1.2.** In characteristic 0, the loci $D_{22}$ and $D_{23}$ are proper subsets of $M_{22}$ and $M_{23}$ respectively.

In fact, we show further that the closure of $D_{22}$ does not contain the locus of chains of genus-1 curves; see Theorem 8.4 below. In addition, our proof goes through unmodified for characteristic $p \geq 29$, and our techniques can in principle be applied to lower characteristics as well, but due to characteristic restrictions on the application to the geometry of $M_g$, we have not pursued this. See Remark 1.4 below.

The genus 21 case of Theorem 1.2 is a special case of the classical “maximal rank conjecture,” and was proved by Farkas in [Far09c] as part of an infinite family of divisors of small slopes. However, in this case the divisor in question does not quite satisfy the necessary inequality to obtain that $M_{21}$ is of general type. With applications to moduli spaces of curves in mind, Aprodu and Farkas proposed in Conjecture 5.4 of [AF11] a “strong maximal rank conjecture,” of which Theorem 1.2 constitutes two cases. These conjectures study ranks of multiplication maps. Specifically, given a linear series $(L, V)$ on a curve $X$, we have the multiplication map

\[
\text{Sym}^2 V \to \Gamma(X, L^\otimes 2).
\]

Note that the source has dimension $r(r+2)/2$, and assuming $X$ is Petri-general, the target has dimension $2d + 1 - g$. The image of $X$ under the linear series lies on a quadric if and only if (1.1) has a nonzero kernel. The classical maximal rank conjecture asserts that if $r \geq 3$, for a general $X$ and a general $g_d^1$ on $X$, the map
(1.1) should always be injective or surjective (and similarly for the higher-order multiplication maps). Many special cases of this were proved by various people; we omit discussion of most of these, but mention that the case of quadrics was first proved by Ballico [Bal12], and subsequent proofs were given by Jensen and Payne using a tropical approach [JP16], and by the present authors [LOTZ17] using a degeneration to a chain of genus-1 curves. Very recently, Larson has proved the full classical maximal rank conjecture [Lar17].

In contrast, the strong maximal rank conjecture remains wide open, even in the case of quadrics. Since the failure of (1.1) to have maximal rank is a determinantal condition, the strong maximal rank conjecture of Aprodu and Farkas (Conjecture 5.4 of [AF11]) is the following:

**Conjecture 1.3** (Aprodu-Farkas). Set $\rho := g - (r + 1)(g + r - d)$.

On a general curve of genus $g$, if $\rho < r - 2$, the locus of $g^r_d$s for which (1.1) fails to have maximal rank is equal to the expected determinantal codimension, which is $1 + \lceil \binom{r+2}{2} - (2d + 1 - g) \rceil$. In particular, when this expected codimension exceeds $\rho$, every linear series on $X$ should have maximal rank.\(^2\)

For the family of cases considered in Definition 1.1, we compute that $\rho = g - 21 = (2d + 1 - g) - \binom{r+2}{2}$, so in this case Conjecture 1.3 predicts that every linear series should yield (1.1) of maximal rank, and more specifically, should have injective multiplication map, just as we prove in Theorem 1.2 for the cases $g = 22, 23$.

Our proof builds on the ideas introduced in [LOTZ17], which combine the Eisenbud-Harris theory of limit linear series with ideas from the theory of linked linear series introduced by the second author in [Oss06] and [Oss14]. The idea is to start with a limit linear series on a chain $X_0$ of genus-1 curves, and describe a collection of global sections living in different multidegrees on $X_0$. We then take tensors of these sections and consider their image in a carefully chosen multidegree, showing that they have the correct-dimensional span. The first major difficulty in moving from the classical maximal rank conjecture to the strong maximal rank conjecture is that instead of being able to work with a single limit linear series in each case, we have to consider all possible limit linear series. We are able to overcome this for the full infinite family of cases described in Definition 1.1, and we expect that these ideas should extend to cover an infinite sequences of similarly constructed (infinite) families.

The more serious difficulty is that for certain degenerate limit linear series (which occur already in codimension 1), we do not have very good control over global sections occurring in the expected multidegrees when we have a family of linear series on smooth curves specializing to the given limit linear series. This can be expressed in terms of trying to understand the possible linked linear series lying over the given limit linear series. Even for these limit linear series, a nonempty open subset of the possible linked linear series will always behave well, but there will be some cases which are more slippery. To overcome this, we systematically use ideas from linked linear series to prove that when $\rho = 1$ or $\rho = 2$ we can always produce global sections of certain prescribed forms which must lie in the specialization of the family of linear series. For the case $\rho = 1$ (i.e., genus 22),

\(^2\)In fact, Aprodu and Farkas also include higher-degree multiplication maps in their conjecture. Farkas and Ortega [FO11] subsequently relax the $\rho < r - 2$ hypothesis in cases such as ours, where $\rho$ is less than the expected codimension.
this leads to a relatively brief proof of Theorem 1.2. However, already for $\rho = 2$
the situation is quite a bit more complicated. To handle the degenerate cases, we
consider variant multidegrees which depend more tightly on the limit linear series
in question, and (partially inspired by the earlier work [JP16] of Jensen and Payne
on a tropical approach to the classical maximal rank conjecture) we also consider
families of curves with highly specialized directions of approach, which gives us
further control over the behavior of the global sections in different multidegrees.

We expect that the tools we develop here will lead to proofs of infinite families
of the strong maximal rank conjecture, and have written the different parts of
the argument to be independent of $r$ and/or $\rho$ wherever this does not lead to
unnecessary complication. The nature of our approach also allows for proving cases
of the maximal rank conjecture where the expected codimension does not exceed $\rho$,
so that the locus of linear series which do not have maximal rank is nonempty. Our
approach should also be useful in other questions involving multiplication maps for
linear series, such as the conjecture of Bakker and Farkas (Remark 14 of [BF]),
which was motivated by connections to higher-rank Brill-Noether theory. Their
conjecture treats a certain specific family of cases, but with products of distinct
linear series in place of symmetric squares of a fixed one. In addition, our work in
§2 on nondegeneracy of certain morphisms from genus-1 curves to projective spaces
and in §4 on the structure of exact linked linear series is likely to be useful in other
settings as well.

The structure of the paper is as follows. In §2 we analyze certain maps from
genus-1 curves to projective spaces which arise naturally from tensor squares of
linear series, and show that these are nondegenerate morphisms in cases of interest.
In §3, we review the Eisenbud-Harris theory of limit linear series, and the related
theory of linked linear series introduced by the second author. In §4, we analyze
the possible structures of linked linear series lying over a given limit linear series in
the cases that can arise when $\rho \leq 2$. In §5 and §6, we analyze a certain collection of
sections which arise from taking the tensor square of a limit linear series, and give
an elementary criterion for them to be linearly independent. In §7, we apply this
criterion to a family of examples with $r = 6$, which include the genus-22 and genus-
23 cases of interest for the proof of Theorem 1.2. Finally, in §8, we put together
the analysis of the structure of linked linear series with the independence results of
§7 to complete the proof of Theorem 1.2.

Remark 1.4. We mention that although we impose characteristic-0 hypotheses in
our main theorem, these do not appear to be essential. Nearly everything we do is
characteristic-independent, but we use a characteristic-dependent result (Theorem
3.4 below) of Eisenbud and Harris to simplify the situation slightly by restricting
our attention to “refined” limit linear series (Definition 3.1 below). In fact, the only
characteristic dependence in Theorem 3.4 is the use of the Plücker inequality, which
still holds in characteristic $p$ and degree $d$ when $p > d$; see for instance Proposition
2.4 and Corollary 2.5 of [Oss06]. Thus, our proof of Theorem 1.2 extends as written
to characteristic $p > 25$ for $g = 22$ and $p > 26$ for $g = 23$.

Moreover, since our key specialization result (Proposition 3.10 below) on linked
linear series applies in arbitrary characteristic, there is no visible obstruction to
extending our proof to lower characteristics as well. However, key portions of the
argument for the implications for the geometry of $\mathcal{M}_g$ were written using character-
istic 0, and as far as we are aware no one has carefully analyzed which positive
characteristics they may apply to, so for the present paper it seems preferable to work in the simpler setting.

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2. Nondegeneracy calculations

In this section, we study maps from elliptic curves to projective space determined by comparing values of tensor products of certain tuples of sections at two points $P$ and $Q$. We will need two distinct results in this direction: first, we consider the situation that we let the point $Q$ vary. This is already considered in [LOTZ17], where we showed that these maps are morphisms, described them explicitly, and gave partial criteria for nondegeneracy. Here we extend the nondegeneracy criterion to a sharp statement for the case of tensor pairs. This is used to show that if we vary the location of the nodes on individual components, we can get possible linear dependencies to vary sufficiently nontrivially. Next, we will consider a new case, where $Q$ is fixed, but we have a separate varying parameter. This situation was not considered in [LOTZ17], but will be important to us in dealing with situations where the discrete data of the limit linear series does not fix the underlying line bundle in some components.

First, given a genus-1 curve $C$ and distinct $P, Q$ on $C$, and $c, d \geq 0$, let $\mathcal{L} = \mathcal{O}_C(cP + (d - c)Q)$. Then for any $a, b \geq 0$ with $a + b = d - 1$, there is a unique section (up to scaling by $k^\times$) of $\mathcal{L}$ vanishing to order at least $a$ at $P$ and at least $b$ at $Q$. Thus, we have a uniquely determined point $R$ such that the divisor of the aforementioned section is $aP + bQ + R$; explicitly, $R$ is determined by $aP + bQ + R \sim cP + (d - c)Q$, or

$$R \sim (c - a)P + (d - c - b)Q = (c - a)P + (1 + a - c)Q = P + (a + 1 - c)(Q - P) = Q + (a - c)(Q - P).$$

Thinking of $C$ as a torsor over $\text{Pic}^0(C)$, we see that $R = P$ if and only if $Q - P$ is $[a + c - 1]$-torsion, and $R = Q$ if and only if $Q - P$ is $[a - c]$-torsion. Note that (2.1) makes sense even when $Q = P$ (in which case $R = Q = P$), so we will use the formula for all $P, Q$, understanding that it has the initial interpretation as long as $Q \neq P$. To avoid trivial cases, we will assume that $a \neq c - 1$, and $b \neq d - c - 1$, or equivalently, $a + 1 - c \neq 0$, and $a - c \neq 0$.

Situation 2.1. Fix $\ell \geq 1$, and for $j = 0, \ldots, \ell$, set numbers $a_i^j, a_2^j, b_1^j, b_2^j$ satisfying:

- $a_i^j + b_i^j = d - 1$ for all $i, j$;
- $a_i^j - c \neq 0, -1$ for all $i, j$;
- $a_1^j + a_2^j$ is independent of $j$.

We now have sections $s_i^j$ with divisors $a_i^jP + b_i^jQ + R_i^j$, and forming tensor products yields sections $s_i^j = s_1^j \otimes s_2^j \in \Gamma(C, \mathcal{L}^{\otimes 2})$, with divisors

$$\left(a_1^j + a_2^j\right)P + \left(b_1^j + b_2^j\right)Q + R_1^j + R_2^j,$$

having the property that any two $R_1^j + R_2^j$ are linearly equivalent. Now, if $Q - P$ is not $|a_i^j + 1 - c|$-torsion for any $i, j$, we can normalize the $s_i^j$, uniquely up to simultaneous scalar, so that their values at $P$ are all the same. Then provided
that there is some \( j \) such that \( Q - P \) is not \( |a_i^j - c| \)-torsion for any \( i \), considering \( (s^0(Q), \ldots, s^\ell(Q)) \) gives a well-defined point of \( \mathbb{P}^\ell \).

**Notation 2.2.** With discrete data as in Situation 2.1, suppose \( P \) is fixed. For a given \( Q \in C \), denote by \( R_{ij}^Q \) the point determined as above by \( P \) and \( Q \), and by \( f_Q \) the point of \( \mathbb{P}^\ell \) determined by \( (s^0(Q), \ldots, s^\ell(Q)) \). Let \( U \) be the open subset of \( C \) consisting of all \( Q \) such that \( Q - P \) is not \( |a_i^j - c| \)-torsion or \( |a_i^j + 1 - c| \)-torsion for any \( i, j \).

In [LOTZ17] we showed that the map \( U \to \mathbb{P}^\ell \) given by \( Q \mapsto f_Q \) extends to a morphism \( f : C \to \mathbb{P}^\ell \).

Our main result is then the following, extending Corollary 2.7 of [LOTZ17].

**Proposition 2.3.** If \( C \) is not supersingular, all the \( a_i^j \) are distinct, and \( a_1^j + a_2^j \neq 2c - 1 \), then \( f \) is nondegenerate.

The proof relies on reduction to a good understanding of the \( \ell = 1 \) case. Indeed, we can view our map as being given by \( (1, f_1, \ldots, f_\ell) \), where \( f_j \) is the rational function constructed from the sections \( s^j, s^0 \) (note that we are switching the order of 0 and \( j \), because we are dividing through all terms by \( s^0 \)). Thus, nondegeneracy is equivalent to linear independence of the rational functions \( 1, f_1, \ldots, f_\ell \), whose zeroes and poles are described explicitly by the following result, which combines Lemma 2.3 and Corollary 2.4 of [LOTZ17].

**Lemma 2.4.** In the \( \ell = 1 \) case of Situation 2.1, the function \( f : U \to k^\times \) given by \( Q \mapsto (s^0/s^1)(Q) \) determines a rational function on \( C \). We then have

\[
\text{div } f = \sum_{i=1}^{2} ((P + \text{Pic}^0(C)[a_i^0] - c]) - (P + \text{Pic}^0(C)[a_i^1 - c])
- (P + \text{Pic}^0(C)[a_i^0 + 1 - c]) + (P + \text{Pic}^0(C)[a_i^1 + 1 - c])),
\]

where for a divisor \( D = \sum_j c_j P_j \) on \( \text{Pic}^0(C) \), the notation \( P + D \) indicates the divisor \( \sum_j c_j (P + P_j) \) on \( C \), using the \( \text{Pic}^0(C) \)-torsor structure on \( C \).

Moreover, if \( C \) is not supersingular, \( f \) is nonconstant if and only if \( a_1^1 + a_2^1 \neq 2c - 1 \).

In the above, the torsion subgroups \( \text{Pic}^0(C)[n] \) should be equipped with the multiplicities arising from the inseparable degree of the appropriate multiplication map. Thus, if \( k \) has characteristic 0 or characteristic \( p \) not dividing \( n \), then \( \text{Pic}^0(C)[n] \) is a reduced divisor, but otherwise all the points of \( \text{Pic}^0(C)[n] \) have coefficients given by the appropriate power of \( p \).

**Proof of Proposition 2.3.** By Lemma 2.4, \( f_1, \ldots, f_\ell \) are all non-constant. By re-indexing the pairs we may further assume

\[
a_1^1 < a_1^{\ell - 1} < \cdots < a_1^0 < a_2^0 < a_1^1 < \cdots < a_2^\ell.
\]

Let \( n_2^j := |a_i^j - c + 1| \), \( n_1^j := |a_i^j - c| \), and \( n^j = \max\{n_1^j, n_2^j\} \).

A first observation is that \( n_j^2 > n_j^{j - 1} \) for all \( j \): if \( a_i^{j-1} < c \) (respectively, \( a_i^{-1} > c \)), then \( n_i^{j-1} < n_i^j \) (respectively, \( n_i^{j-1} < n_i^2 \)), and thus \( n_i^{j-1} < n^j \); by a similar calculation, \( n_i^{j-1} < n^j \); thus, \( n_i^{j-1} < n^j \).
A second observation is that \( n^j \geq \max\{m_0^j, m_0^j\} \) for all \( j \geq 1 \), and if equality is attained, \( j \) must be 1. Indeed, when \( c < a_0^j \), we have \( m_0^j < m_2^j < n_2^j \) for all \( j \geq 1 \); meanwhile, either \( m_0^j \leq m_0^j \) if \( c < a_0^j \) or \( m_0^j \leq n_1^j \) if \( c > a_0^j \) for all \( j > 1 \); thus, \( \max\{m_0^j, m_0^j\} \leq n^j \) for all \( j \geq 1 \). When \( c > a_0^j \), \( m_2^j \leq m_1^j \leq n_1^j < n_1^j \) for all \( j > 1 \), and hence the same conclusion holds.

Now, we claim that \( f_j \) has poles at the strict \( n_j \)-torsion points. Recalling from Lemma 2.4 that the poles of \( f_j \) are supported among the \( m_1^j \) and \( n_1^j \)-torsion points for \( i = 1, 2 \), the above two observations show that \( 1, \ldots, f_{j-1} \) cannot have any poles at strict \( n^j \)-torsion points, which immediately implies that \( 1, f_1, \ldots, f_\ell \) are \( k \)-linearly independent. Thus, it suffices to prove the claim. Since the potential zeroes of \( f_j \) are supported among the \( m_1^j \) and \( n_1^j \)-torsion points, we just need to show that \( n^j \) does not divide \( m_1^j \) or \( n_0^j \) for \( i = 1, 2 \) and any \( j \geq 1 \). Moreover, we already know that \( n^j > n^0 \geq n_0^j \), so it is enough to consider the \( m_1^j \). We consider two cases.

**Case 1:** \( c < a_0^j \), so that also \( c < a_0^j \) for all \( j \). In this case,
\[
m_0^j < n_0^j \leq m_2^j < n_1^j \leq \cdots \leq m_2^j < n_2^j.
\]
In particular, we have \( n^j > m_2^j \), so it remains to compare \( n^j \) against \( m_1^j \). If \( n^j = m_1^j \), since \( n_1^j \) is always coprime to \( m_1^j \), the claim follows instantly. If \( n^j > m_1^j \), since \( |n_1^j - m_1^j| = 1 \), we have \( n^j \geq m_1^j \). But equality cannot hold as it would imply that \( a_1^j + a_2^j = 2c - 1 \), which is ruled out by our assumption. So we conclude the claim in this case.

**Case 2:** \( c > a_0^j \), so that \( c > a_0^j \) for all \( j \). If \( a_2^j > c \), \( n_2^j = m_2^j + 1 \) and hence \( n^j > m_2^j \). Meanwhile, \( n_1^j = m_1^j - 1 \). Similarly to the previous case, either \( n^j > m_1^j \) or \( n^j \) is coprime to \( m_1^j \), and the claim follows. If \( a_2^j < c \), \( n_2^j = m_2^j - 1 \). Under our assumption, \( n^j = n_1^j \) so is coprime to \( m_1^j \). But because \( j \geq 1 \), we have \( n_1^j \geq n_2^j + 2 = m_2^j + 1 \), so \( n_1^j > m_2^j \) and the claim follows.

We now move on to the new situation, where our point \( Q \) will be fixed, but our line bundle \( \mathcal{L} \) is allowed to vary. If we have \( a, b \) with \( a + b = d - 1 \), then the isomorphism class of a line bundle \( \mathcal{L} \) of degree \( d \) can be uniquely determined by a point \( R \) by setting \( \mathcal{L} \cong \mathcal{O}_C(aP + bQ + R) \). If we have \( a', b' \) with \( a' + b' = d - 1 \) also, and \( \mathcal{O}_C(aP + bQ + R) \cong \mathcal{O}_C(a'P + b'Q + R') \), then we find that \( R' = R + (a - a')(P - Q) \).

We fix discrete data as before, except that since \( \mathcal{L} \) will vary, we do not have any \( c \).

**Situation 2.5.** Fix \( \ell \geq 1 \), and for \( j = 0, \ldots, \ell \), set nonnegative integers \( a_1^j, a_2^j, b_1^j, b_2^j \) satisfying:

- \( a_1^j + b_1^j = d - 1 \) for all \( i, j \);
- \( a_1^j + a_2^j \) is independent of \( j \).

First suppose we fix \( \mathcal{L} \). As before, we have sections \( s_i^j \) of \( \mathcal{L} \) with divisors \( a_1^jP + b_1^jQ + R_i^j \), and we can take tensor products to obtain \( s_i^j = s_i^j \otimes s_i^j \) having divisors \( (a_1^j + a_2^j)P + (b_1^j + b_2^j)Q + R_i^j + R_i^j \). Note that the divisors \( R_i^j + R_i^j \) will all be linearly equivalent to one another by construction. If we assume that none of the \( R_i^j \) are equal to \( P \) (which will be the case provided that \( \mathcal{L} \) and \( Q \) are general
relative to \( P \)), we can normalize the \( s_j \) to have the same value at \( P \), and then we obtain a well-defined point \((s^0(Q), \ldots, s^r(Q)) \in \mathbb{P}^d\). But because we have said that \( \mathcal{L} \) is uniquely determined by \( R_1^0 \), we can view this procedure as giving a rational map from \( C \) to \( \mathbb{P}^d \), which we will now study. The argument will be similar to that of Lemma 2.3 and Corollary 2.7 of [LOTZ17], but a bit simpler.

**Proposition 2.6.** Suppose that \( P - Q \) is not \( m \)-torsion for any \( m \leq d \), and let \( U \subseteq C \) consist of the open subset of points not differing from \( P \) by \( m \)-torsion for any \( m \leq d \). Let \( \varphi : U \to \mathbb{P}^d \) be the map which uses the above construction with \( R_1^0 = R \) to send \( R \in U \) to \((s^0(Q), \ldots, s^r(Q)) \). Then \( \varphi \) extends to a nondegenerate morphism \( C \to \mathbb{P}^d \).

**Proof.** We first consider the case \( \ell = 1 \), proving that we obtain a nonconstant rational function, and showing further that the divisor of this function is equal to

\[
Q + (Q - (a_0^0 - a_0^1)(P - Q)) + (P - (a_0^0 - a_1^1)(P - Q)) + (P - (a_1^0 - a_1^2)(P - Q))
\]

\[- (Q - (a_0^0 - a_1^1)(P - Q)) - (Q - (a_0^0 - a_2^1)(P - Q)) - P - (P - (a_1^0 - a_2^2)(P - Q)).
\]

Consider \( R_1^0 \) as a divisor on \( C \times C \) by setting \( R_1^0 \) to be the graph of the morphism \( R \mapsto R + (a_0^0 - a_1^1)(P - Q) \) (so that \( R_1^0 \) is simply the diagonal, and in each fiber over \( R \) we obtain our original point \( R_1^0 \) for the case that \( \mathcal{L} \) is determined by setting \( R_1^0 = R \). Set \( D^j = R_1^0 + R_2^0 + (P - (a_0^0 - a_1^1 - 1)(P - Q)) \times C + (P - (a_1^0 - a_2^1 - 1)(P - Q)) \times C \)

for \( j = 0, 1 \). Then we claim that \( D^0 \) and \( D^1 \) are linearly equivalent. By construction, if we restrict to \( \{R\} \times C \) for any \( R \) not among the \( P - (a_0^0 - a_1^1)(P - Q) \), we get that \( D^0 \) and \( D^1 \) are linearly equivalent, so \( D^0 - D^1 \sim D \times C \) for some divisor \( D \) on \( C \). But if we restrict to \( C \times \{P\} \), we see that \( R_1^0 + R_2^0 \) restricts to \( (P - (a_0^0 - a_1^1)(P - Q)) + (P - (a_1^0 - a_2^1)(P - Q)) \), so the restrictions of \( D^0 \) and \( D^1 \) are linearly equivalent on \( C \times \{P\} \), and hence on \( C \times C \), as desired. Moreover, this shows that if \( t_0 \) and \( t_1 \) are sections of the resulting line bundle having \( D^0 \) and \( D^1 \) as divisors, then \( t_0|_{C \times \{P\}} \) has the same divisor as \( t_1|_{C \times \{P\}} \), so we can scale so that \( t_0 \) and \( t_1 \) are equal on \( C \times \{P\} \). Then we see that our map \( U \to \mathbb{P}^1 \) is given by composing \( R \mapsto (R, Q) \) with the rational function induced by our normalized choice of \( t_0, t_1 \). Thus, it is a rational function, as desired. We compute its divisor simply by looking at the restrictions of \( D^0 \) and \( D^1 \) to \( C \times \{Q\} \), which gives the claimed formula.

Now, for the case of arbitrary \( \ell \), we can consider the map to \( \mathbb{P}^d \) to be given by a tuple of rational functions induced from the \( \ell = 1 \) case, specifically by \((f_0, \ldots, f_{\ell - 1}, 1)\), where \( f_j \) comes from looking at \( s^j \) and \( s^\ell \). To show nondegeneracy, it suffices to show that the \( f_j \) are linearly independent, which we do by showing that each of them (other than \( f_\ell = 1 \)) has a pole which none of the others have. If we order so that \( a_0^0 < a_1^1 < \cdots < a_\ell^1 < a_\ell^\ell < a_\ell^{\ell - 1} < \cdots < a_0^2 \), we see that \( P - (a_\ell^1 - a_\ell^2)(P - Q) \) occurs among the poles of \( f_j \); indeed, given our non-torsion hypothesis on \( P - Q \), the only positive term in the divisor which could possibly cancel it is \( Q \), which would require \( a_1^1 - a_2^1 = 1 \), which is not possible with

\[3\text{In fact, we see that the divisors } R_1^1 + R_1^2 \text{ are already linearly equivalent, but we need the given definition of the } D^j \text{ precisely so that } t_0 \text{ and } t_1 \text{ can be normalized as desired along } P.\]
our above ordering. But again using our nontorsion hypothesis, and the fact that $a_{j}^{2} - a_{j}^{1}$ strictly decreases as $j$ increases, we see that we obtain the desired distinct poles.

3. Background on limit linear series and linked linear series

In this section we review background on limit linear series, as introduced by Eisenbud and Harris in [EH86], and on linked linear series, introduced by the second author in [Oss06] for two-component curves and generalized to arbitrary curves of compact type in [Oss14].

Recall that a curve of compact type is a projective nodal curve such that every node is disconnecting, or equivalently, the dual graph is a tree. To streamline our presentation, we will largely restrict our attention to the situation of curves of compact type together with one-parameter smoothings.

**Definition 3.1.** Let $X_{0}$ be a curve of compact type, with dual graph $\Gamma$. Given $r, d \geq 0$, a limit linear series on $X_{0}$ of dimension $r$ and degree $d$ is a tuple $(L_{v}, V_{v})_{v \in V(\Gamma)}$, where each $(L_{v}, V_{v})$ is a linear series of dimension $r$ and degree $d$ on the component $Z_{v}$ of $X_{0}$ corresponding to $v$. This tuple is further required to satisfy the following condition: if $Z_{v}$ and $Z_{v'}$ meet at a node $P_{e}$, and $a^{(v,e)}$ and $a^{(v',e)}$ are the vanishing sequences at $P_{e}$ of $(L_{v}, V_{v})$ and $(L_{v'}, V_{v'})$ respectively, then

$$a^{(v,e)}_{j} + a^{(v',e)}_{r-j} \geq d \quad \text{for } j = 0, \ldots, r.$$

A limit linear series is said to be refined if the above inequalities are equalities for all $e$ and $j$.

We now consider a one-parameter smoothing of $X_{0}$, as follows.

**Situation 3.2.** Suppose $B$ is the spectrum of a discrete valuation ring with algebraically closed residue field, and $\pi : X \to B$ is flat and proper, with special fiber $X_{0}$ a curve of compact type, and smooth generic fiber $X_{\eta}$. Suppose further that the total space $X$ is regular, that $\pi$ admits a section.

Now, suppose we have a line bundle $L_{\eta}$ generically – more precisely, we allow for the possibility that $L_{\eta}$ is only defined after a finite extension of the base field of $X_{\eta}$. We can then take a finite base change $B' \to B$ so that $L_{\eta}$ is defined over $X'_{\eta}$, and then $X'$ may not be regular, but the line bundle $L_{\eta}$ will still extend over $X_{0}$ because $X_{0}$ is of compact type. Moreover, there is a unique extension of $L_{\eta}$ having any specified multidegree (i.e., tuple of degrees on each component) adding up to $d$: because $X$ was regular each component $Z_{v}$ of $X_{0}$ is a Cartier divisor in $X$, and twisting by the $\mathcal{O}_{X}(Z_{v})$ (or more precisely, their pullbacks to $X'$) will increase the degree by 1 on each component meeting $Z_{v}$, and decrease the degree on $Z_{v}$ correspondingly. For a multidegree $\omega$, we denote this unique extension by $\tilde{L}_{\omega}$. In particular, for each $Z_{v}$, we can consider the multidegree $\omega^{v}$ which concentrates degree $d$ on $Z_{v}$, and has degree 0 elsewhere.

Eisenbud and Harris (Proposition 2.1 of [EH86]) show the following specialization result:

---

4In [Oss06], linked linear series were called ‘limit linear series,’ but the name was changed subsequently to reduce confusion.
Proposition 3.3 (Eisenbud-Harris). Given a linear series \((\mathcal{L}_n, V^n)\) on \(X_n\) of dimension \(r\) and degree \(d\), if we set \(\mathcal{L}^e := (\tilde{\mathcal{L}}_\omega)|_{Z_\omega}\), and \(V^v := (V_n \cap \Gamma(X', \tilde{\mathcal{L}}_v'))|_{Z_v}\), then the resulting tuple \((\mathcal{L}^e, V^v)_v\) is a limit linear series on \(X_0\).

They also show (Theorem 2.6 of [EH86]) the following:

Theorem 3.4 (Eisenbud-Harris). In characteristic 0, after finite base change and blowing up nodes in the special fiber, we may assume that the specialized limit linear series constructed by Proposition 3.3 is refined.

Note that the only effect on \(X_0\) of the base change and blowup is that chains of genus-0 curves are introduced at the nodes. Assuming we blow up to fully resolve the singularities resulting from the base change, these chain of curves have length equal to one less than the ramification index of the base change, so in particular they are the same at every node.

We now move on to linked linear series. The first observation is that if we have two multidegrees \(\omega\) and \(\omega'\), then there is a unique collection of nonnegative coefficients \(c_v \in \mathbb{Z}\), not all positive, such that \(\tilde{\mathcal{L}}_\omega \cong \tilde{\mathcal{L}}_{\omega'}(-\sum c_v Z_v)\). In this way, we obtain an inclusion \(\tilde{\mathcal{L}}_{\omega} \hookrightarrow \tilde{\mathcal{L}}_{\omega'}\) which is defined uniquely up to scaling. If we define \(\mathcal{L}_\omega := \tilde{\mathcal{L}}_\omega|_{X_0}\), we get induced maps \(\mathcal{L}_\omega \to \mathcal{L}_{\omega'}\) which are no longer injective, as they vanish identically on the components \(Z_v\) with \(c_v > 0\). However, they are injective on the remaining components. Passing to global sections we obtain maps

\[ f_{\omega, \omega'} : \Gamma(X_0, \mathcal{L}_\omega) \to \Gamma(X_0, \mathcal{L}_{\omega'}) \]

From the construction we see that \(f_{\omega, \omega'} \circ f_{\omega', \omega}\) always vanishes identically. Although the twisted line bundles \(\mathcal{L}_\omega\) can be described intrinsically on the special fiber, the maps \(f_{\omega, \omega'}\) depend on the smoothing of \(X_0\) whenever the locus on which they are nonvanishing is disconnected.

To minimize notation, we will define linked linear series only in the above specialization context.

Definition 3.5. Given \(\mathcal{L}_0\) of degree \(d\) and the induced tuple \((\mathcal{L}_0)_\omega\) of line bundles, a linked linear series of dimension \(r\) (and degree \(d\)) on the \(\mathcal{L}_\omega\) is a tuple \((V_\omega)_\omega\) for all multidegrees of total degree \(d\) where each \(V_\omega \subseteq \Gamma(X_0, \mathcal{L}_\omega)\) is an \((r+1)\)-dimensional space of global sections, and for every \(\omega, \omega'\), we have

\[ f_{\omega, \omega'}(V_\omega) \subseteq V_{\omega'} \]

We then see easily from the definitions that we have:

Proposition 3.6. If we have \((\mathcal{L}_n, V_n)\) generically, and for all \(\omega\) we set \(V_\omega = (V_n \cap \Gamma(X', \tilde{\mathcal{L}}_v'))|_{X_0}\), we obtain a linked linear series.

Moreover, this process is visibly compatible with the Eisenbud-Harris specialization process, and we have a forgetful map which visibly commutes with specialization:

Theorem 3.7. If \((V_\omega)_\omega\) is a linked linear series on \(\mathcal{L}_\omega\), and we set \(\mathcal{L}^v = \mathcal{L}_{\omega'}|_{Z_v}\) and \(V^v = V_{\omega'}|_{Z_v}\) for all \(v \in V(\Gamma)\), then \((\mathcal{L}^v, V^v)\) is a limit linear series.

This is explicitly stated (in the generality of higher-rank vector bundles) as part of Theorem 4.3.4 of [Oss14], but is primarily a consequence of Lemma 4.1.6 of loc. cit.

In [Oss14], the following notion is introduced:
Definition 3.8. A linked linear series is simple if there exist multidegrees \( \omega_0, \ldots, \omega_r \) and sections \( s_j \in \Gamma(X_0, L_{\omega_j}) \) such that for every \( \omega \), the \( f_{\omega, \omega}(s_j) \) form a basis of \( V_\omega \).

The simple linked linear series form an open subset, and are particularly easy to understand (hence the name). However, we will be forced to consider more general linked linear series arising under specialization. We therefore introduce the following open subset, originally introduced in [Oss06] in the two-component case.

Definition 3.9. A linked linear series is exact if for every multidegree \( \omega \), and every proper subset \( S \subseteq V(\Gamma) \), if \( L_{\omega'} \cong L_\omega(-\sum_{v \in S} Z_v) \), then
\[
f_{\omega, \omega'}(V_\omega) = V_{\omega'} \cap \ker f_{\omega', \omega}.
\]

An important special case in the definition, and the only one which we will use in the present paper, is that \( \omega' \) is obtained from \( \omega \) by decreasing the degree by 1 on a single component and increasing it correspondingly on an adjacent component.

While we cannot always ensure our linked linear series are simple, we can ensure they are exact:

Proposition 3.10. If \((\mathcal{L}_\eta, V_\eta)\) is defined over \(X_\eta\) itself, then the resulting linked linear series is exact.

The proof is exactly the same as in the two-component case, which is explained immediately before the statement of Theorem 5.2 of [EO13]. Thus, even if \((\mathcal{L}_\eta, V_\eta)\) is not defined over \(X_\eta\), we can take a finite base change to make it defined, and blow up the resulting singularities of the total space to put ourselves into position to apply Proposition 3.10.

4. Degenerate linked linear series

The purpose of this section is to analyze the structures of the possible exact linked linear series lying over limit linear series in the situations that can arise when \( \rho \leq 2 \). We will henceforth restrict our attention to the case that our reducible curve \( X_0 \) is a chain, although for the moment we don’t have to place any restrictions on the genus of the components.

Situation 4.1. Suppose that \( X_0 \) is obtained by starting with smooth curves \( Z_1, \ldots, Z_N \), with each \( Z_i \) having distinct marked points \( P_i, Q_i \), and gluing \( Q_i \) to \( P_{i+1} \) for each \( i = 1, \ldots, N - 1 \).

In this situation, we can encode a multidegree \( w \) as follows:

Notation 4.2. If we have fixed a total degree \( d \), if we write \( w = (c_2, \ldots, c_N) \), we let \( \text{md}_d(w) \) be the multidegree which has degree equal to \( c_2 \) on \( Z_1 \), to \( c_{i+1} - c_i \) on \( Z_i \) for \( i = 2, \ldots, N - 1 \), and to \( d - C_N \) on \( Z_N \).

\( w \) is bounded if \( 0 \leq c_i \leq d \) for all \( i \).

To avoid notational clutter, we will frequently write simply \( \text{md}(w) \) when the total degree is clear, and we will write abbreviate \( \mathcal{L}_{\text{md}(w)} \) by \( \mathcal{L}_w \), \( f_{\text{md}(w), \text{md}(w')} \) by \( f_w, w' \), and so forth. Note that \( \text{md} \) is invertible: in any fixed total degree, any multidegree \( \omega \) has a unique \( w \) such that \( \omega = \text{md}(w) \). The total degrees will always be equal to \( d \) for the remainder of this section.

We will assume without further comment that all \( w \) are bounded. The point of this is that if \( w \) is bounded, then for all \( i \), the map \( f_w, w' \) will be injective on
understand sections in multidegree
the component
obtained from
describe the behavior of the maps
the way in which we encode the combinatorial data of a limit linear series. We first
distinguish sections as special cases.

Notation 4.2 is that for \(1 < i < N\), the line bundle \(\mathcal{L}_w|_{Z_i}\) is
obtained from \(\mathcal{L}^w\) by twisting down by \(c_iP_i\) and by \((d - c_{i+1})Q_i\), leaving degree
\(d - c_i - (d - c_{i+1}) = c_{i+1} - c_i\). This notation is very helpful in connection with
the way in which we encode the combinatorial data of a limit linear series. We first
describe the behavior of the maps \(f_{w,w'}\) under the above encoding.

**Proposition 4.3.** Given \(w = (c_2, \ldots, c_N), w' = (c'_2, \ldots, c'_N)\) and total degree \(d,\)
the map \(\mathcal{L}_w \to \mathcal{L}_{w'}\) vanishes identically on the component \(Z_i\) if and only if
\[
\sum_{j=i+1}^{N} (c'_j - c_j) > \min_{1 \leq i' \leq N} \sum_{j=i'+1}^{N} (c'_j - c_j).
\]
In particular, if \(c'_i < c_i\) or \(c'_{i+1} > c_{i+1}\) then the map vanishes identically on \(Z_i\),
and if \(c'_i = c_i\) for \(i > 1\), then the map vanishes identically on \(Z_i\) if and only if it
vanishes identically on \(Z_{i-1}\).

See Proposition 4.6 and Remark 3.14 of [LOTZ17].

We now move onto how we encode the discrete data of limit linear series. First,
the following is easy to check via an elimination argument.

**Proposition 4.4.** Let \(Z\) be a smooth projective curve, and \(P, Q \in Z\) distinct. Let
\((\mathcal{L}, V)\) be a \(g^r_d\) on \(Z\). Then there is a unique (unordered) set of pairs \((a_0, b_0), \ldots, (a_r, b_r)\)
with all \(a_j\) distinct and all \(b_j\) distinct such that there exists a basis \(s_0, \ldots, s_r\) of \(V\)
with \(\text{ord}_P s_j = a_j\) and \(\text{ord}_Q s_j = b_j\) for \(j = 0, \ldots, r\).

Note that the \(s_j\) themselves are not unique, although a given \(s_j\) can be modified
only by adding multiples of \(s_{j'}\) which simultaneously satisfy \(\text{ord}_P s_{j'} > \text{ord}_P s_j\) and
\(\text{ord}_Q s_{j'} > \text{ord}_Q s_j\). We then can introduce a table of numbers to a refined limit
linear series as follows.

**Notation 4.5.** Let \((\mathcal{L}^i, V^i)\) be a refined limit \(g^r_d\) on \(X_0\), and for each \(i\) let \((a_j^i, b_j^i)\)
be the set of pairs given by Proposition 4.4.

Construct the \((r + 1) \times N\) table \(T'\) from left to right, with the \(i\)th column of
\(T'\) consisting of the pairs \((a_j^i, b_j^i)\) for \(j = 0, \ldots, r\), and the ordering of each column
determined as follows: \(a_j^i\) should be strictly increasing, and for \(i > 1\) and each \(j\),
we require \(a_j^i = d - b_{j-1}^i\). For fixed \(i\), we refer to the \(a_j^i\) and the \(b_j^i\) as making up the
**subcolumns** of the \(i\)th column of \(T'\).

For each \(j\), let \(w_j = (a_j^0, \ldots, a_j^N)\), and set \(\omega_j = \text{md}(w_j)\).

Note that the set of pairs of Proposition 4.4 is giving a relative ordering of the
vanishing sequences at \(P\) and \(Q\), so the condition that the limit linear series is
refined means that we can always impose that \(a_j^i = d - b_{j-1}^i\). The reason for
arranging our table ordering in this way is that we can always choose sections
\(s_j^i \in V^i\) such that \(\text{ord}_P s_j^i = a_j^i\) and \(\text{ord}_Q s_j^i = b_j^i\), and then in multidegree \(\omega_j\)
there is a unique section \(s_j\) obtained from gluing together the \(s_j^i\) (although as
noted above, the choices of \(s_j^i\) are not unique in general).
Definition 4.6. We say that a **swap** occurs in column $i$ between rows $j, j'$ if $a^i_j < a^i_{j'}$ and $b^i_j < b^i_{j'}$ or if $a^i_j > a^i_{j'}$ and $b^i_j > b^i_{j'}$. A swap is **minimal** if further $|a^i_j - a^i_{j'}| = |b^i_j - b^i_{j'}| = 1$ and either $a^i_j + b^i_j = d$ or $a^i_{j'} + b^i_{j'} = d$.

Now, given a limit linear series on our chain of curves, there may be more than one linked linear series lying over it. If the limit linear series is “chain-adaptable” in the sense of [Oss14] (i.e., if there are no swaps in the table $T'$), the linked linear series is unique, and simple, generated by $s_j$ described above. However, in the non-chain-adaptable case it is not unique. A nonempty open subset of the set of possible linked linear series will always be simple, generated by sections similar to the $s_j$ described above.\(^5\) From the point of view of proving Theorem 1.2, these simple cases behave essentially as if they contained all the $s_j$, and are much more straightforward to handle. However, even among the exact linked linear series, not all of them are necessarily simple. We can nonetheless use exactness to obtain fairly good control over what these linked linear series look like. We address all the cases that can arise for $\rho \leq 2$ below.

For the rest of this section, we suppose we have fixed a refined limit linear series along with the resulting table $T'$ as described above, as well as a choice of all the $s^i_j$.

The starting point of our analysis is that for any $w = (c_2, \ldots, c_N)$ (always implicitly assumed bounded), the linkage condition implies that the $(r+1)$-dimensional space $V_w$ in our linked linear series must consist of sections which are obtained by linear combinations of sections obtained by gluing, for a fixed $j$, the sections $s^i_j$ to one another as $i$ varies, where each $s^i_j$ that appears must satisfy $a^i_j \geq c_i$ and $b^i_j \geq d - c_{i+1}$, and if the first (respectively, second) inequality is an equality we must also have $s^{i-1}_j$ (respectively, $s^{i+1}_j$) included in the gluing. Indeed, a section in $V_w$ must be a linear combination of such $s^i_j$, and since the $a^i_j$ and $b^i_j$ are all distinct for fixed $i$, at most one can have equality on each side, leading to the desired form for the gluing.

Proposition 4.7. Suppose that the $j_0$th row of $T'$ has the property that for all $j < j_0$ we have $b^i_j > b^i_{j_0}$ for $i = 1, \ldots, N - 1$, and for all $j > j_0$ we have $a^i_j > a^i_{j_0}$ for $i = 2, \ldots, N$. Then any linked linear series lying over the given limit linear series contains the expected section $s_{j_0}$.

**Proof.** We just have to see that the space of global sections in multidegree $\omega_{j_0}$ obtained from all possible gluings of the $s^i_j$ has dimension exactly $r + 1$, so that any linked linear series must contain the whole space, including $s_{j_0}$. But for $j < j_0$ since $b^i_j > b^i_{j_0}$ for $i < N$, we have $a^{-1}_{j_0} < a^{-1}_j$, so $s^{i+1}_{j_0}$ cannot appear at all in multidegree $\omega_{j_0}$. Thus, only $s^i_j$ can appear, glued to the zero section on every other component. Similarly, for $j > j_0$ only $s^N_j$ can appear. And since each $s^i_{j_0}$ has precisely the desired vanishing at the nodes, $s_{j_0}$ is the unique way to glue them together, so we obtain an $(r + 1)$-dimensional space in total, as desired. \(\square\)

When the hypotheses of Proposition 4.7 are not satisfied for every $j_0$, then we can have linked linear series – even exact ones – which do not contain all of the $s_{j_0}$, and are not even simple. Moreover, we expect that these actually occur as

\(^5\)For instance, in the case of a single swap they may differ from the $s_j$ by adding multiples of certain other sections in the first and/or last columns. This results in distinct possibilities for the simple linked linear series; see Example 4.3.5 of [Oss14].
specializations of linear series on the generic fiber. This leads us to introduce the following notion:

**Definition 4.8.** For $\ell > 1$, let $\mathcal{S} = (S_1, \ldots, S_{\ell})$ be a tuple of subsets of $\{1, \ldots, N\}$ such that for all pairs $i < i'$, every element of $S_i$ is less than or equal to every element of $S_{i'}$, and such that every element of $\{1, \ldots, g\}$ is contained in some $S_i$. Let $\vec{j} = (j_1, \ldots, j_\ell)$ be a tuple of elements of $\{0, \ldots, r\}$, possibly with repetitions.

Then given a fixed limit linear series and corresponding choices of the $s_j^i$, a **mixed section** of type $(\vec{\mathcal{S}}, \vec{j})$ is a $w$ and a section $s$ in multidegree $\text{md}(w)$ which is a sum from $i = 1$ to $\ell$ of sections obtained by gluing $s_j^i$, for all $i' \in S_i$ to the zero section on other components.

In the above definition, it is convenient to allow the possibility that some of the $S_i$ are empty. The choice of $w$ is not always determined uniquely by the type of a mixed section when there are sufficiently large gaps between the relevant values of the $a_j^i$, but in our arguments the particular value of $w$ will never arise. In cases where the $s_j^i$ are not uniquely determined, the type of a mixed section may depend on these choices. However, this dependence will be irrelevant to our independence arguments.

We will show that in the cases of interest, even when a given $s_j$ is not in our linked linear series, we can ensure that there are mixed sections of rather precise forms, which can in some sense take the place of the missing $s_j$.

The following single swap between a pair of rows is the only form of degeneracy that can occur in the $\rho = 1$ case; in the below lemma, we also allow for the possibility that there could be other swaps occurring in other rows.

**Proposition 4.9.** Suppose that our limit linear series has a single swap between the $j_0$th and $(j_0 - 1)$st rows, occurring in the $i_0$th column, and for all $j < j_0 - 1$ we have $b_j^i > b_{j-1}^i$, $b_{j_0}^i$ for $i = 1, \ldots, N - 1$, and for all $j > j_0$ we have $a_j^i > a_{j_0}^i$, $a_{j_0 - 1}^i$ for $i = 2, \ldots, N$. Then any linked linear series lying over the given limit linear series contains the expected section $s_{j_0 - 1}$, and the multidegrees associated to $(a_{j_0}^{i_0 - 1}, \ldots, a_{j_0}^{i_0}, a_{j_0}^{i_0 + 1}, \ldots, a_{j_0}^N)$ and $(a_{j_0}^{i_0 - 1}, a_{j_0}^{i_0 - 1}, a_{j_0}^{i_0 + 1}, \ldots, a_{j_0}^N)$ must contain the respective images of the section $s_{j_0}$. These images consist respectively of 0 on the first $i_0 - 1$ components and $s_{j_0}^i$ for $i = i_0, \ldots, N$, and of $s_{j_0}^i$ for $i = 1, \ldots, i_0$, and 0 on the last $N - i_0$ components.

Let $w = (c_2, \ldots, c_N)$. If $c_i < a_{j_0}^{i_0 - 1}, a_{j_0}^{i_0}$ for all $i$, the linked linear series contains $s_{j_0}^i$ in multidegree $\text{md}(w)$, and if $c_i > a_{j_0}^{i_0 - 1}, a_{j_0}^{i_0}$ for all $i$, the linked linear series contains $s_{j_0}^N$ in multidegree $\text{md}(w)$ (in both cases, glued to 0 on the other components).

**Proof.** First, in the multidegree $\omega_{j_0 - 1}$, as in the proof of Proposition 4.7, the $s_j^i$ for $j \neq j_0 - 1, j_0$ can only contribute for $i = 1$ (if $j < j_0 - 1$) or $i = N$ (if $j > j_0$), and the $s_{j_0 - 1}$ glue uniquely to give $s_{j_0 - 1}$. Finally, the $s_{j_0}^i$ can only contribute at $i = i_0$, so we find that the space obtained from all the $s_j^i$ is $(r + 1)$-dimensional, and $s_{j_0 - 1}$ must be in the linked linear series, as desired.

Next, consider $w' = (a_{j_0 - 1}^{i_0 - 1}, a_{j_0 - 1}^{i_0}, a_{j_0}^{i_0 + 1}, \ldots, a_N^N)$. Note that $f_{w_{j_0}, w}(s_{j_0})$ is equal to $s_{j_0}$ from $i_0$ to $N$ (inclusive), and 0 strictly before $i_0$. We claim that the space of possible sections from the $s_j^i$ in multidegree $\text{md}(w')$ is precisely $(r + 1)$-dimensional, so the linked linear series is uniquely determined in this multidegree.
By hypothesis, the $s_j^i$ for $j < j_0 - 1$ can only contribute for $i = 1$, and the $s_j^i$ for $j > j_0$ can only contribute for $i = N$. The $s_{j_0-1}^i$ could in principle contribute for $i < i_0$ and $i = N$, but if the $s_{j_0-1}^i$ appeared for $i < i_0$, they all would be nonvanishing at the relevant nodes, and they would have to glue to something nonvanishing in the $i_0$th column. But this would have to be $s_{i_0-1}^1$, which does not have enough vanishing on the right to appear in multidegree $md(w')$. Thus, we conclude that the $s_{j_0-1}^i$ can only appear for $i = N$ (where it is glued to the zero section on all other columns). Finally, the $s_j^i$ can only appear for $i > i_0$, where they are nonzero at all interior nodes, and therefore have a unique gluing, which must yield $f_{w_{j_0},w'}(s_{j_0})$. Thus we get the claimed dimension $r + 1$, and conclude that $f_{w_{j_0},w'}(s_{j_0})$ is contained in the linked linear series.

Similarly, if $w'' = (a_{j_0}^2, \ldots, a_{j_0}^i, a_{j_0+1}^j, \ldots, a_N^j)$, we find that space of possible sections is $(r + 1)$-dimensional, and contains $f_{w_{j_0},w''}(s_{j_0})$.

Now, suppose we are given $w$ with $c_i < a_{j_0-1}^i$, $a_{j_0}^i$ for all $i$. Then Proposition 4.3 implies that $f_{w'',w}$ is nonzero precisely on the 1st component, so $f_{w'',w}(f_{w_{j_0},w''}(s_{j_0}))$ is equal to $s_{j_0}^i$ glued to 0, as desired. The situation with $c_i > a_{j_0-1}^i$, $a_{j_0}^i$ is similar, but with $w''$ in place of $w''$. □

Now we will systematically consider cases where the limit linear series has only one or two swaps, with no swaps in any other rows. We start with the case of a single swap.

**Proposition 4.10.** Suppose that our limit linear series has precisely one swap, between the $j_0$th and $(j_0 - 1)$st rows, and occurring in the $i_0$th column.

Then any linked linear series lying over the given limit linear series contains the expected sections $s_j$ for all $j \neq j_0$. If the linked linear series is exact, then it must contain mixed sections $s_{j_0}^i$ and $s_{j_0}^0$ of type $((S_1^i, S_2^i), (j_0-1, j_0))$ and $((S_1^0, S_2^0), (j_0, j_0-1))$ respectively, with $S_1^i$ supported strictly left of $i_0$, and $S_2^0$ supported strictly right of $i_0$.

Note that the possibility that the linked linear series contains the section $s_{j_0}$ itself is contained in the proposition by allowing $S_1^i$ and $S_2^0$ both to be empty.

**Proof.** Start with the $w'$ from the proof of Proposition 4.9. Note that if $i_0 = 1$, then the proposition says that $s_{j_0}$ itself is in our linked linear series, consistent with the stated form for $s_{j_0}^i$. Otherwise, if $i_0 > 1$ we consider iteratively changing $w'$ by increasing the twists by 1 for $i' \leq i_0$ (starting at $i_0$) until they each agree with $a_{j_0}^i$. We note that every such modified $w'$ has an $(r + 2)$-dimensional space of global sections obtained from the $s_{j_0}^i$, described explicitly as follows: $s_{j_0}^i$ for $j < j_0 - 1$; $s_{j_0}^N$ for $j > j_0$; $s_{j_0-1}^N$; a section obtained by gluing the $s_{j_0-1}^i$ for $i$ from 1 to $i' - 1$ (which is the last column in which $w'$ agrees with $a_{j_0-1}^i$); and a section obtained by gluing the $s_{j_0}^i$ from either $i' - 1$ or $i'$ to $N$, beginning with the last column in which $w'$ has coefficient strictly less than $a_{j_0}^i$. For each $j \neq j_0 - 1$, since there is a unique section constructed from the $s_{j_0}^i$, it is necessarily equal to $f_{w, w'}(s_j)$. In addition, since we know $s_j$ is in our linked linear series for $j \neq j_0$, we have that $f_{w, w'}(s_j)$ is necessarily contained in our linked linear series for $j \neq j_0 - 1, j_0$.

Now, suppose that our linked linear series contained $f_{w_{j_0},w'}(s_{j_0})$ for the old $w'$; we claim that it either also contains it for the new $w'$, or contains a section of the form desired for $s_{j_0}^i$. Indeed, increasing the twist in the $i$th column corresponding
to twisting once by every component from $i$ to $N$. We observe that $f_{w_{j_i}, w}(s_{j_0})$ is in the kernel of the map from the old $w'$ to the new one, so by the definition of exactness, the linked linear series must contain some $s$ in the new multidegree mapping to $f_{w_{j_i}, w}(s_{j_0})$ in the old one. Using the above description of the space of global sections, this is necessarily a combination of the $f_{w_{j_i}, w}(s_j)$ for $j < j_0 - 1$ and $j = j_0$, together with the section from the $s_{j_0}^i$ for $i = 1$ to $i' - 1$. Moreover, since we observed above that $f_{w_{j_i}, w}(s_j)$ is contained in our linked linear series for $j < j_0 - 1$, we can subtract these off to obtain a combination of the sections from the $j_0 - 1$ and $j_0$ rows. If the $j_0 - 1$ term vanishes, we have that $f_{w_{j_i}, w}(s_{j_0})$ is contained in our linked linear series for the new $w'$, and if the $j_0 - 1$ term is nonzero, we have something of the desired form for $s_{j_0}^i$ (with the minimal element of $S'_2$ being either $i'$ or $i' - 1$ according to where $f_{w_{j_i}, w}(s_{j_0})$ begins), as claimed. Iterating this process, we either obtain the desired $s_{j_0}^i$, or we eventually reach $w' = w_{j_0}$ and find that the linked linear series actually contains $s_{j_0}$ itself.

As the situation is completely symmetric, the construction of $s_{j_0}''$ is similar, starting from the multidegree $w''$ from the proof of Proposition 4.9. □

**Example 4.11.** We will use the below as a running example, showing a possible table $T'$ associated to a limit linear series in the case $r = 6$, $g = 22$, $d = 25$ (here we assume every component is of genus 1).

| 0  | 25 | 4  | 21 | 12 | 13 | 23 | 10 | 11 | 20 | 15 | 16 | 19 | 14 | 18 | 17 | 9  | 22 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1  | 20 | 6  | 22 | 14 | 13 | 16 | 19 | 11 | 10 | 17 | 8  | 21 | 7  | 12 | 15 | 24 |
| 2  | 11 | 13 | 21 | 15 | 14 | 17 | 20 | 12 | 11 | 18 | 9  | 23 | 8  | 16 | 19 | 25 |
| 3  | 0  | 14 | 16 | 18 | 17 | 20 | 23 | 15 | 14 | 21 | 9  | 24 | 8  | 22 | 25 | 0  |
| 4  | 21 | 15 | 17 | 20 | 18 | 21 | 24 | 16 | 15 | 22 | 10 | 23 | 9  | 25 | 0  | 21 |
| 5  | 16 | 17 | 20 | 23 | 18 | 21 | 24 | 16 | 17 | 22 | 11 | 24 | 10 | 25 | 0  | 22 |
| 6  | 18 | 17 | 20 | 23 | 18 | 21 | 24 | 16 | 17 | 22 | 11 | 24 | 10 | 25 | 0  | 23 |

Since there is no ramification at $P_1$, the first entries of the table agree with the row labels, so we have not shown the labels separately.

Note that we have a single swap, occurring in the 9th column between the $j = 2$ and $j = 3$ rows. This leads to having an extra dimension of possibilities in the multidegree obtained from the $j = 3$ row, as the $j = 2$ row can appear in the first or last columns. Consequently, it is possible that an exact linked linear series lying over this limit linear series might not contain $s_3$, but might only contain mixed sections $s'_2$ and $s''_2$ as in Proposition 4.10, with $s'_3$ agreeing with $s_3$ for $i > 9$, but switching to $s_2$ at some $i < 9$, and $s''_3$ agreeing with $s_3$ for $i \leq 9$, but switching to $s_2$ at some $i > 9$. In both cases, the switch occurs in a column mixing $s'_2$ and $s'_3$ unless, the column in question has a gap of at least 2 between the $j = 2$ and $j = 3$ rows. Since this doesn’t occur for $i < 9$, we see that $s'_2$ always has a mixed column, while $s''_2$ may not.

When $\rho = 2$, there are four additional forms of degeneracy that can occur, which we consider one by one. They all involve having exactly two swaps, occurring in distinct columns. The first case is when the swaps occur in disjoint pairs of rows.

**Proposition 4.12.** (“Disjoint swap”) Suppose that our limit linear series contains precisely two swaps, and these occur in disjoint pairs of rows, say $j_0 - 1, j_0$ and $j_1 - 1, j_1$. Then any linked linear series lying over the given limit linear series contains the expected sections $s_j$ for all $j \neq j_0, j_1$. If the linked linear series is exact, then for $\ell = 0, 1$ it must contain mixed sections $s'_{j_0}$ and $s''_{j_0}$ of type $(S'_{1+2\ell}, S_{2+2\ell})$ and $(S''_{1+2\ell}, S''_{2+2\ell}, (j_0 - 1, j_0), (j_0, j_0 - 1))$ respectively, with $S'_{1+2\ell}$ supported strictly left of $i_\ell$ and $S''_{2+2\ell}$ supported strictly right of $i_\ell$. 
Proof. This is essentially identical to the proof of Proposition 4.10. The only new point which arises is that in constructing the sections $s'_j_{j_0}$, $s''_{j_0}$ we need to know that we can always subtract off any $s_j$ part which arises in the iterative procedure, and similarly with $j_0$ and $j_1$ switched. But this follows from the last assertion of Proposition 4.9.

The next case is that a single pair of rows can undergo two swaps in different columns.

**Proposition 4.13.** ("Repeated swap") Suppose that our limit linear series has precisely two swaps, both between the $j_0$th and $(j_0-1)$st rows, with the first occurring in the $i_0$th column, and the second in the $i_1$st column for some $i_1 > i_0$.

Then any exact linked linear series lying over the given limit linear series contains mixed sections $s'_{j_0-1}$ and $s''_{j_0-1}$ of type $((S'_1, S'_2), (j_0 - 1, j_0, j_0 - 1))$ and $((S''_1, S''_2), (j_0 - 1, j_0))$ respectively, with $S'_1$ supported strictly right of $i_1$, and $S''_1$ supported strictly right of $i_1$, and it contains mixed sections $s'_{j_0}$ and $s''_{j_0}$ of type $((S'_1, S'_2), (j_0 - 1, j_0))$ and $((S''_1, S''_2), (j_0, j_0 - 1, j_0))$ respectively, with $S'_4$ supported strictly left of $i_0$ and $S''_4$ supported strictly right of $i_0$.

Proof. The proof is similar to the proof of Proposition 4.10. For $s'_{j_0-1}$, we first consider $w' = (a^2_{j_0-1}, \ldots, a_{j_0-1}, a^{i_1+1}_{j_0-1}, a_{j_0-1}, \ldots, a^N_{j_0-1})$. Note that $f_{w_{j_0-1}, w}(s_{j_0-1})$ is equal to $s_{j_0-1}$ from $i_1$ to $N$ (inclusive), and 0 elsewhere. Indeed, these are the only columns in which the $s_{j_0-1}$ can be supported, since they do not satisfy the correct inequalities from $i_0$ to $i_1 - 1$, and for $i < i_0$ they satisfy them with equality, so would have to be glued to a nonzero element in the $i_0$th column. As in the proof of Proposition 4.7, we check that we have dimension exactly $r + 1$ in multidegree $\text{md}(w')$, with the unique contribution from the $j_0$ row coming from $s^N_{j_0}$. Thus, we find that $f_{w_{j_0-1}, w}(s_{j_0-1})$ is necessarily contained in multidegree $\text{md}(w')$.

We then iterate changing $w'$ by 1, increasing the twist by 1 in the $i'$th column for $i' \leq i_1$ to change them from $a_{j_0-1}$ to $a_{j_0-1}^i$. Using exactness, at each stage we either find the linked linear series still contains $f_{w_{j_0-1}, w}(s_{j_0-1})$ for the new value of $w'$, or it contains the sum of $f_{w_{j_0-1}, w}(s_{j_0-1})$ with a section obtained by gluing the $s_{j_0}$ for $i = i_0, \ldots, i' - 1$. In the first case, we continue to iterate the process of changing $w'$, and if we do not ever get the second case, we end up with $s_{j_0-1}$ itself in our linked linear series. On the other hand, once the second case occurs, we begin to iteratively change $w'$ by increasing the twist by 1 in the $i'$th column for $i' \leq i_0$ to change them from $a_{j_0-1}^i$ to $a_{j_0}^i$. Each time the twist increases above $a_{j_0-1}^i$, we could obtain a contribution obtained from gluing $s_{j_0-1}^i$ from $i = 1$ to $i' - 1$, and if this occurs, we get our desired $s'_{j_0-1}$. Otherwise, we keep iterating,

TABLE 1. A typical example of the “repeated swap” situation. In general, there may be larger gaps between the rows, although when $\rho = 2$ the gaps at the $i_0$ and $i_1$ columns must both be equal to 1.
and each time the twist at $i'$ reaches $a_i'$, the portion of the section obtained from the $s_{j_0}'$ extends to include $i' - 1$. Again, if we never get a contribution from the $s_{j_0-1}'$ for $i \leq i'$, we will end up with a section as required for $s_{j_0-1}'$, having $S_i' = \emptyset$.

The construction of $s_{j_0-1}''$ is similar, but simpler: we set our initial $w'' = (a_{j_0-1}^2, a_{j_0-1}^3, \ldots, a_{j_0-1}^{i_0+1}, a_{j_0-1}^{i_1}, \ldots, a_{j_0-1}^N)$, and then we iteratively decrease the twists for $i' > i_0$ by 1 to change them from $a_i'$ to $a_i''$, until we obtain the desired result.

The construction of $s_{j_0}'$ and $s_{j_0}''$ follows the same process. For $s_{j_0}'$, we start with $w' = (a_{j_0-1}^2, a_{j_0-1}^3, \ldots, a_{j_0-1}^{i_0+1}, a_{j_0-1}^{i_1}, \ldots, a_{j_0-1}^N)$, and we iteratively increase the twists for $i' \leq i_0$ by 1 to change them from $a_i'$ to $a_i''$. Finally, for $s_{j_0}''$, we start with $w'' = (a_{j_0-1}^2, a_{j_0-1}^3, \ldots, a_{j_0-1}^{i_0+1}, a_{j_0-1}^{i_1}, \ldots, a_{j_0-1}^N)$, obtaining a section glued from the $s_i$ for $i \leq i_0$. We iteratively decrease the twists for $i' > i_0$ by 1 to change them from $a_i'$ to $a_i''$, until we obtain a contribution from the $a_{j_0-1}'$ (necessarily ending at $i_1$), and then we iteratively decrease the twists for $i' > i_1$ by 1 to change them from $a_i'$ to $a_i''$, eventually obtaining either $s_{j_0}$ itself, or the desired $s_{j_0}''$.

The last cases involve three consecutive rows undergoing two swaps. There are only two different ways this can occur for $\rho = 2$, but it turns out that these two ways behave quite differently. The two cases can be understood as either having one row which sums to $d$ in both of the relevant columns, or one row which sums to $d - 2$ in both of the relevant columns. The latter turns out to be more degenerate.

**Proposition 4.14.** (“First 3-cycle”) Suppose that our limit linear series has one swap between the $j_0$th and $(j_0+1)$st rows occurring in the $i_0$th column, and a second swap between the $(j_0-1)$st and $(j_0+1)$st rows in the $i_1$st column for some $i_1 > i_0$, and no other swaps.

Then any linked linear series lying over the given limit linear series contains $s_{j_0-1}$ and $s_{j_0}$. If further the linked linear series is exact, then it contains mixed sections $s_{j_0+1}'$, $s_{j_0+1}''$ and $s_{j_0+1}'$ of type $((S_1', S_2', S_3'), (j_0-1, j_0, j_0+1))$, $((S_1'', S_2'', S_3''), (j_0+1, j_0-1, j_0))$ and $((S_1'''', S_2'''', S_3''''), (j_0, j_0+1, j_0-1))$, respectively, with $S_i'$ supported strictly left of $i_1$, $S_2'$ supported strictly left of $i_0$, $S_3'$ supported strictly right of $i_0$, $S_1'''$ supported strictly left of $i_0$, and $S_3'''$ supported strictly right of $i_1$.

Note that if $S_2' = \emptyset$, then $S_1'$ may contain elements greater than $i_0$, and similarly if $S_2'' = \emptyset$, then $S_3''$ may contain elements less than $i_1$.

**Proof.** First, it is routine to check that the multidegrees $\omega_{j_0-1}$ and $\omega_{j_0}$ both have only $(r+1)$-dimensional spaces of possible sections, so that $s_{j_0-1}$ and $s_{j_0}$ must both

| $j_0-1$ | $j_0$ | $j_0+1$ |
|--------|-------|---------|
| \( i_0 \) |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |

| $i_0$ |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |

| $i_1$ |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |
| $a_{j_0-1}$ |
| $a_{j_0}$ |
| $a_{j_0+1}$ |
lie in any linked linear series. Indeed, for the former, the $s_{j_0}^i$ can contribute only for $i = N$, while the $s_{j_0 + 1}^i$ can contribute only for $i = i_1$, while for the latter, the $s_{j_0 - 1}^i$ can contribute only for $i = 1$, and the $s_{j_0 + 1}^i$ can contribute only for $i = i_0$.

Now, to construct the sections $s_{j_0 + 1}^i$, $s_{j_0 + 1}''$, and $s_{j_0 + 1}'''$ we proceed as in the previous propositions. For $s_{j_0 + 1}^i$, we start with $w' = (a_{j_0 - 1}^2, \ldots, a_{j_0 - 1}^{i_0}, a_{j_0 + 1}^{i_0 + 1}, \ldots, a_{j_0 + 1}^N)$, and then iteratively increase the twist by 1 at a time for $i' \leq i_1$, initially increasing it from $a_{j_0 - 1}$ to $a_{j_0 + 1}$. For $i' > i_0$, this process behaves as before, either extending the contribution from the $a_{j_0}^i$ iteratively to the left without introducing any other nonzero contributions, or producing a section $s_{j_0 + 1}^i$ as desired, having $S_2 = \emptyset$.

Once $i' \leq i_0$, we still iteratively increase the twist from $a_{j_0 - 1}^i$ to $a_{j_0 + 1}^i$, but we are required to pass $a_{j_0}^i$ in the process. This introduces a third possibility: once the twist at $i'$ is strictly greater than $a_{j_0}^i$, we could obtain a contribution from $s_{j_0}^{i' - 1}$.

Also, for $i' < i_0$, once the twist at $i'$ is equal to $a_{j_0}^i$, we could obtain a contribution from both $s_{j_0}^{i' - 1}$ and $s_{j_0}^i$. If either of these occurs, we move to the next $i'$, and for the remaining $i'$, instead of increasing the twist from $a_{j_0 - 1}^i$ to $a_{j_0 + 1}^i$, we only increase to $a_{j_0}^i$. Note that we may obtain contributions from the $s_{j_0}^i$ (for $i = i' - 1$ or $i = i' - 1, i'$) and $s_{j_0 - 1}^i$ (for $i = 1, \ldots, i' - 1$) simultaneously at some point, which still gives an $s_{j_0 + 1}^i$ of the desired form. On the other hand, if we never obtain a contribution from the $s_{j_0}^i$, then the resulting $s_{j_0 + 1}^i$ simply has $S_2 = \emptyset$.

For $s_{j_0 + 1}''$, we start with $w'' = (a_{j_0 - 1}^2, \ldots, a_{j_0}^{i_0}, a_{j_0 + 1}^{i_0 + 1}, \ldots, a_{j_0 + 1}^N)$, and then follow the same procedure as for $s_{j_0 + 1}^i$, iteratively decreasing the twist at $i' > i_0$ from $a_{j_0}^i$ to $a_{j_0}^i$ with the possibility of a contribution from the $s_{j_0 - 1}^i$ once $i'$ passes $i_1$.

Finally, for $s_{j_0 + 1}'''$ set $w''' = (a_{j_0 - 1}^2, \ldots, a_{j_0}^{i_0}, a_{j_0 + 1}^{i_0 + 1}, \ldots, a_{j_0 + 1}^{i_1 + 1}, \ldots, a_{j_0 + 1}^N)$ initially. We then iteratively increase the twist at $i' \leq i_0$ from $a_{j_0}^i$ to $a_{j_0 + 1}^i$, and iteratively decrease the twist at $i' > i_1$ from $a_{j_0 - 1}^i$ to $a_{j_0 + 1}^i$ to construct $s_{j_0 + 1}'''$. \(\square\)

**Proposition 4.15.** ("Second 3-cycle") Suppose that our limit linear series has one swap between the $(j_0 - 1)$st and $j_0$th rows occurring in the $i_0$th column, and a second swap between the $(j_0 - 1)$st and $(j_0 + 1)$st rows in the $i_1$st column for some $i_1 > i_0$, and no other swaps.

Then any linked linear series lying over the given limit linear series contains $s_{j_0 - 1}^i$. If further the linked linear series is exact, then it contains mixed sections $s_{j_0}^i$ and $s_{j_0}''$ of type $(S_1', S_2')$, $(j_0 - 1, j_0)$ and $(S_1'', S_2'', S_3'')$, $(j_0, j_0 + 1, j_0 - 1)$ respectively, with $S_1'$ supported strictly left of $i_0$, $S_2''$ supported at or right of $i_1$, and $S_3''$ supported

| $j_0 - 1$ | $j_0$ | $j_0 + 1$ |
|------------|-------|-----------|
| $a - 1$    | $d - a$ | $a + 2$   |
| $d - a$    | $a + 1$ | $a + 1$   |
| $a + 2$    | $a + 1$ | $d - a' - 1$ |
| $a + 3$    | $d - a' - 1$ | $a' + 1$ |
| $d - a' - 3$ | $a' + 1$ | $d - a' + 4$ |
| $a' + 2$    | $d - a' + 4$ | $d' + 2$ |

Table 3. A typical example of the "second 3-cycle" situation. In general, there may be larger gaps between the rows, although when $\rho = 2$ the gaps between the $j_0 - 1$ and $j_0$ rows at $i_0$ and between the $j_0 - 1$ and $j_0 + 1$ rows at $i_1$ must both be equal to 1.
strictly right of $i_0$. Similarly, it contains mixed sections $s'_{j_0+1}$ and $s''_{j_0+1}$ of type
\((S'_4, S'_5, S'_5), (j_0 - 1, j_0, j_0 + 1)\) and \((S''_3, S''_3), (j_0 + 1, j_0 - 1)\) respectively, with $S'_3$
supported strictly left of $i_1$, $S'_4$ supported at or left of $i_0$, and $S''_3$ supported strictly
right of $i_1$. Moreover, if $i_1 \in S''_3$ then also $i_1 \in S''_3$, and if $i_0 \in S'_4$, then also
$i_0 \in S'_5$. Finally, either we can have $S'_2 = S''_3 = \{1, \ldots, N\}$, or it also contains a
mixed section $s'''$ of type \((S''_3, S''_3, S''_3; (j_0, j_0 - 1, j_0 + 1))\), where every element of
$S''_3$ is strictly between $i_0$ and $i_1$.

**Proof.** For the most part, this is straightforward and similar to the previous
propositions, but there is one new subtlety to address, and the idea for the construction of $s'''$ is new. We first construct $s'_{j_0}$, starting with $w' = (a^2_{j_0-1}, \ldots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \ldots, a^N_{j_0})$.

We then iteratively increase the twist from $a^{i_0}_{j_0-1}$ to $a^{i_0}_{j_0}$ for $i' \leq i_0$, and
obtain our $s'_{j_0}$ as usual. We then do the same procedure for $s''_{j_0+1}$, starting with
\[ w'' = (a^2_{j_0+1}, \ldots, a^{i_0}_{j_0+1}, a^{i_0+1}_{j_0+1}, \ldots, a^N_{j_0+1}). \]

Next, we construct $s'''_{j_0+1}$ starting with $w''' = (a^2_{j_0+1}, \ldots, a^{i_0}_{j_0+1}, a^{i_0+1}_{j_0}, \ldots, a^N_{j_0-1})$. We
then iteratively decrease the twist at $i' > i_0$ from $a^{i_0}_{j_0}$ to $a^{i_0}_{j_0}$. For $i' \leq i_0$, this
behaves as in the previous propositions, with one new subtlety: for each intermediate
value of $w'$, the $s^i_{j_0+1}$ can contribute only in the $i_1$ column, but because we do not
know that $s_{j_0+1}$ is contained in our linked linear series, we also do not know
a priori that this contribution from $s^i_{j_0+1}$ in multidegree $md(w')$ is contained in
our linked linear series. However, since we have already constructed $s''_{j_0+1}$, we can
use its image in $md(w')$. One checks that its only possible support in $md(w')$ is
in the $i_1$ column, so that in fact the multidegree-$md(w')$ part of our linked linear
series necessarily contains the section given by $s^0_{j_0+1}$, and we can subtract it off as
necessary from the section we are constructing. Thus, for $i' \leq i_1$, we can iterate as
before, and will either obtain an $s^i_{j_0}$ as desired (with $S'_2 = \emptyset$), or we will obtain a
section made up of the $s^i_{j_0}$ for $i \leq i_1$, and vanishing identically for $i > i_1$. In
the latter case, we continue to iteratively decrease the twists defining $w'$ for $i > i_1$, but
as in the construction of $s'_{j_0+1}$ in the proof of Proposition 4.14, to get from $a_{j_0}$ to
$w'$ we need to pass $a_{j_0+1}$, which is where the possible contribution from the $j_0 + 1$ may occur.

The construction of $s'_{j_0+1}$ follows the same pattern as that of $s''_{j_0}$, but starting
with $w' = (a^2_{j_0-1}, \ldots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \ldots, a^N_{j_0})$. Here we use the image of $s'_{j_0}$ in order
to subtract off any contributions of $s^0_{j_0}$ which occur.

Finally, for $s'''$, we start with $w' = w_{j_0}$. We observe that there is an \((r + 2)\)-
dimensional space of potential sections in multidegree $\omega_{j_0}$, with the $s^i_j$ for $j < j_0 - 1$
contributing only for $i = 1$, the $s^i_j$ for $j \geq j_0 + 1$ contributing only for $i = N$, the $s^i_{j_0}$
contributing only with $s_{j_0}$ itself, and the $s^i_{j_0-1}$ contributing separately for $i = 1$ and
$i = N$. We must therefore have a three-dimensional space of combinations of the
four sections $s^1_{j_0-1}, s^N_{j_0}, s^N_{j_0+1}, s_{j_0}$ and $s_{j_0}$. It follows by elimination that this space
must contain (at least) one of the following: $s_{j_0}$ plus a (possibly zero) multiple of
$s^1_{j_0-1}$; $s_{j_0}$ plus a (possibly zero) multiple of $s^N_{j_0+1}$; $s^1_{j_0}$ and $s^N_{j_0+1}$. The first case
means that we can take $S''_2 = \{1, \ldots, N\}$, while in the second we get a valid choice of $s'''$. In the third case, we begin with $s^N_{j_0+1}$, and iteratively twist the multidegree
as before. For $i' > i_1$, we change $w'$ from twisting down by $a_{j_0}^{i_0}$ to $a_{j_0}^{i_0+1}$, and at
each stage, we must either obtain the desired $s'''$, or a section made up purely of the
$s'_{j_0+1}$, in which case we continue to iterate. Note that in these multidegrees,
we continue to have that the only possible contributions of the $s_{j'}$ (for $j \neq j_0$) supported strictly left of $i'$ come for $j \leq j_0 - 1$, and we can take the image of $s_{j_0-1}$ from multidegree $\omega_{j_0}$, so all these can be subtracted off as necessary. When $i' \leq i_1$, we will have $a_{j_0-1}'$ between $a_{j_0}'$ and $a_{j_0+1}'$: we still iteratively increase the twist, but a new possibility occurs: once we are twisting down by strictly more than $a_{j_0-1}'$, we could obtain a contribution from $a_{j_0-1}'$. If this occurs, we will continue to iterate, but stopping after increasing the twist from $a_{j_0}'$ to $a_{j_0-1}'$ for each smaller $i'$.

If we have continued with contributions from $s_{j_0+1}'$ for each $i'$, then once we reach $i_0$, we will again have no other $a_{j_0}'$ between $a_{j_0}'$ and $a_{j_0+1}'$, so we will ultimately obtain an $s''$ of the desired form, with $S_{j_0}'' = \emptyset$. On the other hand, if we have switched from the $s_{j_0+1}'$ to the $s_{j_0-1}'$, then we see that this must terminate (necessarily with an $s'''$ of the desired form) before we reach $i' = i_0$, because there is no section in column $i_0$ which can glue to $s_{j_0+1}'$.

Now, if the above construction did not give $s'''$ because we had $S_{j_0}'' = \{1, \ldots, N\}$, we apply precisely the same process starting in multidegree $\omega_{j_0+1}$, and we find that unless we also have $S_{j_0}''' = \{1, \ldots, N\}$, we end up with the desired $s'''$. \hfill $\square$

Up until now, everything we have done has been insensitive to insertion of genus-0 components. However, to handle the genus-23 case, we will need to impose restrictions on direction of approach; more precisely, we will require that the genus-1 components be separated by exponentially increasing numbers of genus-0 components (going from right to left). The reason for doing this is that, if our limit linear series has all changes to the $s_j$ for $j \neq j_0$ come for $j \leq j_0 - 1$, and we can take the image of $s_{j_0-1}$ from multidegree $\omega_{j_0}$, so all these can be subtracted off as necessary. When $i' \leq i_1$, we will have $a_{j_0-1}'$ between $a_{j_0}'$ and $a_{j_0+1}'$: we still iteratively increase the twist, but a new possibility occurs: once we are twisting down by strictly more than $a_{j_0-1}'$, we could obtain a contribution from $a_{j_0-1}'$. If this occurs, we will continue to iterate, but stopping after increasing the twist from $a_{j_0}'$ to $a_{j_0-1}'$ for each smaller $i'$.

If we have continued with contributions from $s_{j_0+1}'$ for each $i'$, then once we reach $i_0$, we will again have no other $a_{j_0}'$ between $a_{j_0}'$ and $a_{j_0+1}'$, so we will ultimately obtain an $s''$ of the desired form, with $S_{j_0}'' = \emptyset$. On the other hand, if we have switched from the $s_{j_0+1}'$ to the $s_{j_0-1}'$, then we see that this must terminate (necessarily with an $s'''$ of the desired form) before we reach $i' = i_0$, because there is no section in column $i_0$ which can glue to $s_{j_0+1}'$.

Now, if the above construction did not give $s'''$ because we had $S_{j_0}'' = \{1, \ldots, N\}$, we apply precisely the same process starting in multidegree $\omega_{j_0+1}$, and we find that unless we also have $S_{j_0}''' = \{1, \ldots, N\}$, we end up with the desired $s'''$. \hfill $\square$

It turns out that it is convenient to count not the number of genus-0 components between a pair of genus-1 components, but rather the number of nodes. When we consider restricted directions, we will assume that the first and last components have genus 1, and we will denote by $\ell_1, \ldots, \ell_{N-1}$ the number of nodes between each consecutive pair of genus-1 components, so that $\ell_i - 1$ is the number of genus-0 components. The reason why this is more convenient is that if we take a ramified base change with ramification index $e$, and then blow up to resolve the resulting singularities, we will insert $e - 1$ new genus-0 components at every node, which has the effect of multiplying all the $\ell_i$ by $e$. Thus, the ratios of the $\ell_i$ are invariant under this operation.
Definition 4.16. We say that \( X_0 \) is left-weighted if we have
\[
\ell_i \geq 4d \sum_{i'=i+1}^{N-1} \ell_i'.
\]

Definition 4.17. Given \( \vec{S} = (S_1, \ldots, S_t) \) and \( \vec{j} = (j_1, \ldots, j_t) \), a mixed section of type \((\vec{S}, \vec{j})\) is said to be controlled if for every \( i = 2, \ldots, t \) with \( S_i \neq \emptyset \), the minimal element of \( S_i \) is either a genus-1 component or strictly closer to the next genus-1 component to the right than to the previous one on the left.

Proposition 4.18. Suppose that \( X_0 \) is left-weighted. Then:

1. In the situation of Proposition 4.12, if we assume further that \( i_0 \) and \( i_1 \) have genus 1, then we may require that \( s_{j_0}' \) and \( s_{j_1}' \) are controlled, that \( S_1' \) does not contain any \( i < i_0 \) which has genus 1, and that \( S_1' \) does not contain any \( i < i_1 \) which has genus 1.

2. In the situation of Proposition 4.15, if we assume further that \( i_0 \) and \( i_1 \) have genus 1, then we may require that \( s_{j_0}' \) is controlled, and that \( S_1' \) does not contain any \( i < i_0 \) which has genus 1.

Proof. (1) For \( s = 0, 1 \), in the proof of Proposition 4.12, every value of \( w' \) arising in the iterative procedure will have that the potential support of the \( s_{j_s}' \) in multidegree \( m(d(w')) \) has two connected components: one extending from \( i = 1 \) to \( i = i' - 1 \), and the other supported at \( i = N \). The reason that we cannot continue our iterative procedure indefinitely is that we may have that \( f_{w_{j_s-1}, w'}(s_{j_s-1}) \) is supported (partially or entirely) at \( i = N \). If we write \( w' = (c_2, \ldots, c_N) \), we will have that \( a_{j_s-1}^i - c_i' > 0 \) for \( i > i_s \), and \( a_{j_s-1}^i - c_i' < 0 \) for \( i' \leq i \leq i_s \), and \( a_{j_s-1}^i - c_i' = 0 \) for \( i < i' \). Suppose \( i_s \) is the \( m_s \)th genus-1 component. Then in the notation used in Definition 4.16 above, we can say (extremely conservatively) that
\[
\sum_{i=i_s+1}^{N} (a_{j_s-1}^i - c_i') \leq d \sum_{i=m_s}^{N-1} \ell_i \leq \frac{\ell_{m_s-1}}{4}.
\]
Thus, if \( i_s - i' > \frac{\ell_{m_s-1}}{4d} \), then \( \sum_{i=i'}^{N} (a_{j_s-1}^i - c_i') < 0 \), so we certainly have that \( f_{w_{j_s-1}, w'}(s_{j_s-1}) \) is supported entirely on the left, so that we can subtract off any contribution from the \( s_{j_s}' \) and continue our iterative procedure. The desired conditions on \( s_{j_s}' \) follow.

(2) This is essentially the same as (1) (the analogous statement for \( s_{j_0}'+1 \) is a bit more complicated, but we don’t need it).

5. General setup

We now describe the basic situation for taking the tensor square of a limit linear series, and considering images in a fixed multidegree of total degree \( 2d \). We will specialize to the case that the components \( Z_i \) all have genus at most 1, but we begin by extending our discrete data from the base limit linear series to its tensor square.

Notation 5.1. In the situation of Notation 4.5, let \( T \) be the \( \binom{r+2}{2} \times N \) table with rows indexed by unordered pairs \((j, j')\) with \( j, j' \in \{0, \ldots, r\} \), having entries \((a_{(j,j')}', b_{(j,j')}')\) defined by
\[
a_{(j,j')}' = a_j + a_{j'}, \quad \text{and} \quad b_{(j,j')}' = b_j + b_{j'}.
\]
The following definition controls when sections can have nonzero image in a given multidegree. Note that Proposition 4.3 involved only multidegrees and not limit linear series, so we can apply it equally well with $2d$ in place of $d$. This then motivates the following definition.

Definition 5.2. In the situation of Notation 5.1, fix total degree $2d$, and $w = (c_2, \ldots, c_N)$. We say that the $(j, j')$ row is potentially appearing (respectively potentially starting) in column $i$ and multidegree $md_{2d}(w)$ if $a_{(j, j')}^i \geq c_i$ and $b_{(j, j')}^i \geq 2d - c_{i+1}$ (respectively $a_{(j, j')}^i > c_i$ and $b_{(j, j')}^i \geq 2d - c_{i+1}$, respectively $a_{(j, j')}^i \geq c_i$ and $b_{(j, j')}^i > 2d - c_{i+1}$).

Now, we will specialize to the genus-1 case:

Situation 5.3. In Situation 4.1, suppose further that every $Z_i$ has genus at most 1. Fix $d \geq 0$, and suppose also that all the $P_i$ and $Q_i$ are general (and in particular each $P_i - Q_i$ is not $\ell$-torsion for any $\ell \leq d$).

Note that our generality condition cannot be imposed component by component, but also involves interaction between components; this arises in the proof of Lemma 6.3 below.

In any genus, we always have $a_j^i + b_j^i \leq d$ for all $j$; if $Z_i$ has genus 0, there are no further restrictions, but under the genericity hypothesis of Situation 5.3 we have that if the genus of $Z_i$ is equal to 1, we can have $a_j^i + b_j^i = d$ for exactly one value of $j$, and in this case the underlying line bundle is uniquely determined as $\mathcal{O}(a_j^i P_i + b_j^i Q_i)$. The generic situation is that $a_j^i + b_j^i = d - 1$ for all other $j$, but in positive codimension we can have strictly smaller sums as well – see the proof of Proposition 2.1 of [Oss] for an analysis of these codimensions. As compared to [LOTZ17], we have to consider arbitrary refined limit linear series, allowing columns to sum to $d - 2$ or less, and to switch orders. Summing to $d - 2$ or less means that where sections actually appear in a given multidegree can be more complicated to analyze, even with well-behaved (i.e., ‘unimaginative’) multidegrees. However, the key point is that the natural necessary and sufficient conditions for a row to appear in a given column in a given multidegree in the case that all sums are at least $d - 1$ still gives a necessary condition in full generality. When we have swaps, the limit linear series are not chain-adaptable, so the linked linear series living over them can be non-simple, and in fact quite degenerate.

Definition 5.4. For a given limit linear series, we say that the $j$th row is exceptional in column $i$ if $a_j^i + b_j^i < d - 1$ when $Z_i$ has genus 1, or if $a_j^i + b_j^i < d$ when $Z_i$ has genus 0.

While we imagine starting from a chain of genus-1 curves, we allow for inserting any number of rational components at nodes, so that we will have $N$ components, of which exactly $g$ will have genus 1 (including the first and last components) and $N - g$ will have genus 0. Given $i$ between 1 and $N$, we will denote by $g(i)$ the number of genus-1 components between 1 and $i$, inclusive.

It will be convenient to package our sequences $a_j^i$ of vanishing sequences in a slightly different form, as follows.

Notation 5.5. For $j = 0, \ldots, r$ and $i = 0, \ldots, N$, we can determine an integer (possibly negative) $\lambda_{i,j}$ by

$$a_j^{i+1} = g(i) + j - \lambda_{i,j}.$$
If $\lambda_{i,j} > \lambda_{i-1,j}$, we say $\delta_i = j$; otherwise, we say there is no $\delta_i$.

Here $g(0) = 0$ by convention, and we also set the convention that $a^{N+1}_j = b^{N}_{r-j}$ for all $j$.

Intuitively, we think of the $\lambda_{i,j}$ as forming a sequence $\lambda_i$ of generalized ‘shapes’ (not necessarily skew, or connected), behaving as follows: $\lambda_{0,j} \leq 0$ for all $j$; any number of ‘squares’ can be removed from the righthand side in each step, and at most one ‘square’ is added at each stage, with the possibility of adding a ‘square’ only in the genus-1 components. It is however possible for the $\lambda_{i,j}$ to be negative, either because we are starting with nontrivial ramification sequences, or because we remove too many squares early on. Because the $a^j_i$ are always distinct for a fixed $i$, we see that we will always have the set $j - \lambda_{i,j}$ consisting of $r + 1$ distinct integers.

Remark 5.6. Before discussing tensor squares, we briefly recall the significance of $\rho$ in this setup. We need to have $b^{N}_j$ nonnegative (and distinct) for all $j$, or equivalently $a^{N+1}_j$ bounded by $d - r, d - r + 1, \ldots, d$. In particular, $\sum_{j=0}^{r} a^{N+1}_j \leq (r + 1)d - \left(\frac{r + 1}{2}\right)$.

Since $\sum_{j} \lambda_{i,j}$ can increase by at most 1 as $i$ increases (and only on genus-1 components), and $\lambda_{0,j} \leq 0$ for all $j$, we see that for $\rho = 0$, we must have $\lambda_{0,j} = 0$ for all $j$ (i.e., minimal initial vanishing sequence at $P_1$), no places where $\lambda_{i,j}$ decreases (i.e., no exceptional columns for any row), and a $\delta_i$ for every genus-1 column $i$. When $\rho > 0$, the total amount of initial ramification, exceptional columns, and genus-1 columns without $\delta_i$ is bounded by $\rho$. A swap is necessarily a case of an exceptional column, and can contribute exactly 1 to $\rho$ precisely when it is minimal and occurs in a genus-1 column. Note also that a minimal swap can occur at most once in a given (genus-1) column.

Moving on to tensor squares, we now recall the following definition from [LOTZ17], updated to allow for genus-0 components:

Definition 5.7. We say a multidegree of total degree $2d$ is unimaginative if it assigns degree 0 to every genus-0 component, and degree 2 or 3 to every genus-1 component. By extension, we will say that $\nu$ is unimaginative if $md_{2d}(\nu)$ is. Given a fixed unimaginative multidegree, we will let $\gamma_i$ be the number of 3s in the first $i$ columns.

We will work throughout only with unimaginative multidegrees. Thus, the multidegree is encoded by twisting down by $2d - 2g(i) - \gamma_i$ on the righthand of the $i$th column, and by twisting down by $2g(i) + \gamma_i$ on the lefthand side of the $(i + 1)$st column, for all $i < N$.

Then the following is straightforward from the definitions:

Proposition 5.8. A row $(j_1, j_2)$ is potentially appearing in the $i$th and $(i + 1)$st columns only if

$$j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} = \gamma_i.$$

A row $(j_1, j_2)$ is potentially appearing in the $i$th column only if

$$j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} \geq \gamma_{i-1} \quad \text{and} \quad j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} \leq \gamma_i.$$
A row \((j_1, j_2)\) is potentially starting in the \(i\)th column only if
\[ j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} > \gamma_{i-1} \quad \text{and} \quad j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} \leq \gamma_i. \]
A row \((j_1, j_2)\) is potentially ending in the \(i\)th column only if
\[ j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} \geq \gamma_{i-1} \quad \text{and} \quad j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} < \gamma_i. \]
A row \((j_1, j_2)\) is potentially starting and ending in the \(i\)th column only if
\[ j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} > \gamma_{i-1} \quad \text{and} \quad j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} < \gamma_i. \]

Note that the sequence \(j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2}\) decreases by at most 1 each time \(i\) increases, unless \(j_1 = j_2\), when it can decrease by 2. Similarly, \(\gamma_i\) is nondecreasing, and increases by at most 1 each time \(i\) increases. As mentioned previously, for fixed \(i\), the \(j - \lambda_{i,j}\) consists of \(r + 1\) distinct values.

**Corollary 5.9.** If the multidegree has a 2 in the \(i\)th column, then there can be at most one row potentially starting in it, and at most one row potentially ending in it.

There can be a row potentially starting in the \(i\)th column only if \(\delta_i\) exists and either there exists \(j\) such that
\[ \delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = \gamma_i \]
or
\[ 2\delta_i - 2\lambda_{i,\delta_i} = \gamma_i - 1. \]
In these cases, the potentially starting rows are \((\delta_i, j)\) or \((\delta_i, \delta_i)\), respectively.

There can be a row potentially ending in the \(i\)th column only if \(\delta_i\) exists and either there exists \(j\) such that
\[ \delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = \gamma_i - 1 \]
or
\[ 2\delta_i - 2\lambda_{i,\delta_i} = \gamma_i - 2. \]
In these cases, the potentially ending rows are \((\delta_i, j)\) or \((\delta_i, \delta_i)\), respectively.

**Proof.** Since in this case \(\gamma_i = \gamma_{i-1}\), Proposition 5.8 implies that the \((j_1, j_2)\) row can be potentially starting in the \(i\)th column only if \(\lambda_{i,j_1} > \lambda_{i-1,j_1}\) or \(\lambda_{i,j_2} > \lambda_{i-1,j_2}\), which is to say if \(\delta_i\) exists and \(j_1\) or \(j_2\) is equal to \(\delta_i\). Moreover, in this case \(\lambda_{i,\delta_i} = \lambda_{i-1,\delta_i} + 1\), so we conclude that the two stated cases are the only possibilities for having
\[ j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} > \gamma_{i-1} = \gamma_i \geq j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2}, \]
and that moreover in the first case we must also have \(\lambda_{i-1,j} = \lambda_{i,j}\) unless \(j = \delta_i\).

Now, there is at most one \(j\) satisfying the first identity of the corollary, since the \(j - \lambda_{i,j}\) are all distinct. Moreover, if there is some \(j\) satisfying the first, then the second one cannot hold, since this would force
\[ \delta_i - \lambda_{i-1,\delta_i} = \delta_i - \lambda_{i,\delta_i} + 1 = j - \lambda_{i,j} = j - \lambda_{i-1,j}, \]
which is not allowed. This completes the proof of the assertions on rows potentially starting in the \(i\)th column, and the assertion on rows potentially ending in the \(i\)th column is proved similarly.

The following corollary has a similar proof, which we omit.
Corollary 5.10. If the multidegree has a 3 in the $i$th column, then there can be at most one row potentially starting and ending in the $i$th column, and this occurs only if $\delta_i$ exists and either there exists $j$ such that

$$\delta_i + j - \lambda_i,\delta_i - \lambda_i,j = \gamma_i - 1$$

or

$$2\delta_i - 2\lambda_i,\delta_i = \gamma_i - 2.$$

In addition, for a fixed $j \neq \delta_i$, there is at most one value of $j'$ such that the $(j,j')$ row is potentially starting in column $i$, and at most one value of $j'$ such that the $(j,j')$ row is potentially ending in column $i$.

Now, given a refined limit linear series, we can also construct a second table $\bar{T}$ of vanishing numbers which is obtained from the first simply by reordering each subcolumn into strict increasing (respectively, decreasing) order. Put differently, $\bar{T}$ is obtained from the limit linear series simply by taking vanishing sequences at each point, and ignoring the interplay between the pair of points. We will denote the $\lambda$ sequence obtained from $\bar{T}$ by $\bar{\lambda}$, and the entries of the table $\bar{T}$ by $(\bar{a}_i^j, \bar{b}_j^i)$.

Here, if we picture skewing the rows of the $\bar{\lambda}$ according to the initial ramification sequence $a_1^j - j$, the sequence $\bar{\lambda}$ will give a genuine sequence of skew shapes, terminating with a skew shape containing the one obtained by starting from the usual $(r+1) \times (r+g-d)$ center rectangle, and adding squares on the left determined by the initial ramification sequence.

For $\ell \geq 1$, we denote by $\bar{\lambda}_\ell^j$ the number of $j$ such that $\bar{\lambda}_{\ell,j} \geq \ell$, which we can visualize as the number of squares in the $\ell$th column of $\bar{\lambda}$, numbered so that the “first column” is the first column of the main $(r+1) \times (r+g-d)$ rectangle.

The following lemma is the key to our analysis, showing in particular that if we place multidegree 3 in the correct places, we can obtain fine control over what happens with the rows involving $\delta_i+1$.

Lemma 5.11. Given $1 \leq \ell_1 < \ell_2$ and $n > 0$, suppose that $\bar{\lambda}_{\ell_1}^\ell_1 + \bar{\lambda}_{\ell_2}^\ell_2 = n$, but $\bar{\lambda}_{\ell_1-1}^\ell_1 + \bar{\lambda}_{\ell_2-1}^{\ell_2} < n$. Then there does not exist a $j$ such that

$$\delta_i + j - \lambda_i,\delta_i - \lambda_i,j = n - 1 - \ell_1 - \ell_2,$$

and we do not have

$$2\delta_i - 2\lambda_i,\delta_i = n - 2 - \ell_1 - \ell_2.$$

Moreover, if we have some $j$ with $\lambda_i,j < \lambda_{i-1,j}$, then we cannot have

$$\delta_i + j - \lambda_i,\delta_i - \lambda_i,j = n - \ell_1 - \ell_2$$

or

$$\delta_i + j - \lambda_{i-1,\delta_i} - \lambda_{i-1,j} = n - 1 - \ell_1 - \ell_2.$$

Proof. We first prove the case that $\bar{\lambda}_j' = \lambda_i'$ for all $j'$. Note that we necessarily have a $\delta_i$, and it must be the row of the lowest square in either the $\ell_1$th or $\ell_2$th column of $\lambda_i$. Note also that if $\lambda_{i,j} < \lambda_{i-1,j}$ for some $j$, then since we assumed that $\bar{\lambda}_{\ell_1}^{\ell_1} + \bar{\lambda}_{\ell_2}^{\ell_2} < n$, we cannot have $\lambda_i,j$ equal to $\ell_1 - 1$ or $\ell_2 - 1$. Thus, we will prove the desired statement on (5.3) by proving that if (5.3) is satisfied for any $j$, then we must have $\lambda_i,j$ equal to $\ell_1 - 1$ or $\ell_2 - 1$.

\footnote{Although this definition could in principle be applied also to $\lambda_i$, it does not seem to have any particular significance when $\lambda_i \neq \lambda_i$.}
First consider the case that $\lambda^{\ell_1}_i, \lambda^{\ell_2}_i$ are distinct and positive, and set $j_s = \lambda^{\ell_1}_i - 1$ for $s = 1, 2$. Thus, we necessarily have $\delta_i = j_1$ or $j_2$. Note that $(j_1 + 1) + (j_2 + 1) = n$ by hypothesis, and for $s = 1, 2$ write $m_s = \lambda_{i,j_s} - \ell_s$, so that necessarily $m_s \geq 0$ for $s = 1, 2$, with equality for (at least) one $s$. It follows that we have

$$j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} = (j_1 + 1) + (j_2 + 1) - 2 - (m_1 + \ell_1) - (m_2 + \ell_2) = n - 2 - \ell_1 - \ell_2 - m_1 - m_2 < n - 1 - \ell_1 - \ell_2.$$ 

Thus, the only way to get (5.1) would be to set $\lambda_{i,j_1} = \ell_s$, so if as above we have $j > j_s$ in place of $j_s$, the value of the above expression jumps by at least $2 + m_s$. Moreover, we can only use $j$ in place of $j_1$ if $\delta_i = j_2$, in which case we must have $m_2 = 0$, and similarly if we use $j$ in place of $j_2$, so we conclude that (5.1) is not possible. We also see that if we have (5.3), then necessarily $j = j_s + 1$ and $\lambda_{i,j} = \ell_s$, as asserted. By the same reasoning, if $\delta_i = j > j_2$, then (5.2) is also impossible, because $m_1 = 0$ replacing $j_2$ by $\delta_i$ increases the lefthand side by at least $2 + m_2$. On the other hand, if $\delta_i = j_2$ then replacing $j_1$ by $\lambda_{i,j}$ decreases the lefthand side, making it too small to satisfy (5.2).

Finally, suppose we have some $j$ such that $\lambda_{i,j} < \lambda_{i-1,j}$; say $\lambda_{i,j} = \lambda_{i-1,j} - p$ for some $p > 0$. Then (5.4) is equivalent to

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = n - 2 - \ell_1 - \ell_2 + p,$$

so as above we have $j_s \neq \delta_i$, then necessarily $j > j_s$, so that by definition of $j_s$ we must have $\lambda_{i,j} < \lambda_{i,j_s}$. On the other hand, since we have assumed that $\lambda^{\ell_1}_{i-1} + \lambda^{\ell_2}_{i-1} < n$, we must have that $\lambda_{i,j}, \lambda_{i,j} + p$ does not contain $\ell_s$, so it follows that $\lambda_{i,j} + p < \ell_s = \lambda_{i,j_s} - m_s$. We conclude that $j - \lambda_{i,j} > 1 + j_s - \lambda_{i,j_s} + p + m_s$, so

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} > (n - 2 - \ell_1 - \ell_2 - m_s) + 1 + p + m_s = n - 1 - \ell_1 - \ell_2 + p,$$

proving the desired impossibility of (5.4).

The next case is that $\lambda_{i}$ has no entries in the $\ell_2$th column, so that $\delta_i + 1 = n$, and $\lambda_{i,\delta_i} = \ell_1$. In this case, we have

$$\delta_i - \lambda_{i,\delta_i} = (\delta_i + 1) - 1 - \ell_1 = n - 1 - \ell_1.$$

But since the $\ell_2$th column is empty, for all $j$ we have $\lambda_{i,j} < \ell_2$, so we find that

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} > n - 1 - \ell_1 + j - \ell_2 \geq n - 1 - \ell_1 - \ell_2,$$

showing that (5.1) cannot hold, and that (5.3) can hold only if $\lambda_{i,j} = \ell_2 - 1$. We also see that

$$2\delta_i - 2\lambda_{i,\delta_i} = 2n - 2 - 2\ell_1 > 2n - 2 - \ell_1 - \ell_2 > n - 2 - \ell_1 - \ell_2,$$

so (5.2) does not hold either. Finally, because $\lambda^{\ell_1}_{i-1} + \lambda^{\ell_2}_{i-1} < n$ we necessarily have also $\lambda_{i-1,j} < \ell_2$, so

$$\delta_i + j - \lambda_{i-1,\delta_i} - \lambda_{i-1,j} = \delta_i + j - \lambda_{i,\delta_i} - \lambda_{i-1,j} + 1 > n - 1 - \ell_1 - \ell_2 + 1 = n - \ell_1 - \ell_2,$$

proving that (5.4) also cannot hold.
The final case is that \( \lambda_i \) has the same number of entries in the \( \ell_1 \)th and \( \ell_2 \)th columns, so that we must have \( n \) even, with \( \delta_i + 1 = n/2 \), and also \( \lambda_i, \delta_i = \ell_2 \). In this case, we have

\[
\delta_i - \lambda_i, \delta_i = (\delta_i + 1) - 1 - \ell_2 = n/2 - 1 - \ell_2.
\]

Thus if we set \( j_1 = j_2 = \delta_i \), we find that

\[
j_1 + j_2 - \lambda_i, j_1 - \lambda_i, j_2 = n - 2 - 2\ell_2 < n - 2 - \ell_1 - \ell_2,
\]
so (5.2) does not hold. But because the \( \ell_1 \)th column has exactly \( \delta_i + 1 \) entries, leaving \( j_2 = \delta_i \) and using \( j_1 > \delta_i \) results in an increase of at least \( 2 + \ell_2 - \ell_1 \), yielding

\[
j_1 + j_2 - \lambda_i, j_1 - \lambda_i, j_2 \geq n - \ell_2 - \ell_1,
\]
so we see that (5.1) also cannot be satisfied. Moreover, we can have (5.3) only if

\[
j = \delta_i + 1 \quad \text{and} \quad \lambda_i, j = \ell_1 - 1.
\]
Finally, as in the first case considered, if \( \lambda_i, j = \lambda_i, 1 - p \) for \( p > 0 \), then in order to have (5.4) we would need to have \( j > j_2 \), which then implies that \( \lambda_i, j + p < \ell_1 = \lambda_i, \delta_i - \ell_2 + \ell_1 \), so

\[
\delta_i + j - \lambda_i, \delta_i - \lambda_i, j > (n - 2 - 2\ell_2) + 1 + (\ell_2 - \ell_1 + p) = n - 1 - \ell_1 - \ell_2 + p,
\]
again yielding that (5.4) is not possible.

This completes the proof of the lemma in the case that \( \bar{\lambda}_i = \lambda_i \) for all \( i' \). We will see that the general case follows. The main observation is the following: if \( \bar{\lambda}_i, j = \bar{\lambda}_i, 1 - j + 1 \), and we let \( j' \) be such that \( \bar{a}_{j', 1} = a_{j', 1} + 1 \), then we necessarily have \( \lambda_i, j' = \lambda_i, 1 - j' + 1 \), and we cannot have any swaps in the \( i \)th column involving the \( j' \)th row. Indeed, the identity \( \bar{\lambda}_i, j = \bar{\lambda}_i, 1 + 1 \) means that we have \( \bar{a}_{j, i} = \bar{a}_{j', i} \), which means that \( a_{j', 1} = a_{j', i} \) for the \( j' \) such that exactly \( j \) values of \( a_i \) are less than \( a_{j', i} \). We also have exactly \( j \) values of \( a_{j', 1} \) less than \( a_{j', 1} \). It then follows that we must have \( j'' = j' \): we cannot have \( a_{j', 1} = a_{j', i} \), since then we would have \( a_{j', 1} = a_{j', i} \). But if \( a_{j', 1} = a_{j', i} \), then \( j'' \) occurs among the values of \( m \) with \( a_{j', 1} < a_{j', i} \), so there is necessarily some \( m \) with \( a_{j', 1} < a_{j', i} \) but \( a_{i, j''} \geq a_{j', i} \), again leading to a contradiction. This proves the observation, noting that the fact that \( j' = j'' \) rules out any swaps involving the \( j' \)th row.

We then conclude that the impossibility of (5.1) and (5.2) reduces to the case that \( \lambda_i = \lambda_i \), since both equations can be phrased in terms of the values of \( j - \lambda_i, j = a_{j', i} = a_{j', 1} - g(i') \), and our above observation implies that we have \( a_{j', i} = a_{j', 1} = \bar{a}_{j, i} = \bar{a}_{j', 1} \) (here we use \( \delta_i \) to denote the values of \( \delta \) coming from \( \bar{T} \)). Next, suppose that we have some \( j \) with \( \lambda_i, j < \lambda_i, 1 - j \): we claim that if \( j' \) is such that \( a_{j', 1} = a_{j', 1} = a_{j', 1} = a_{j', 1} = a_{j', 1} = a_{j', 1} = a_{j', 1} \), and \( j'' \) is such that \( a_{j' = j'', 1} = a_{j', 1} \), then we necessarily also have that \( \bar{\lambda}_i, j' = \bar{\lambda}_i, 1 - j' \) and \( \bar{\lambda}_i, j'' = \bar{\lambda}_i, 1 - j'' \). Given this claim, the impossibility of (5.3) and (5.4) follows from the case that \( \lambda_i = \lambda_i \) for all \( i' \). By our above observations on the case \( \bar{\lambda}_i = \bar{\lambda}_i, 1 + 1 \), it suffices to prove that \( \bar{\lambda}_i, j' \neq \bar{\lambda}_i, 1 - j' \) and \( \bar{\lambda}_i, j'' \neq \bar{\lambda}_i, 1 - j'' \), or equivalently, that \( \bar{a}_{j', 1} \neq \bar{a}_{j', i} \). But in order to have \( a_{j', 1} \neq \bar{a}_{j', 1} + 1 \), we would need to have \( a_{j', 1} + 1 \) occurring among the \( a_{j', i} \), with precisely \( j' \) strictly smaller values also occurring. But by definition we have \( j' \) values strictly smaller than \( a_{j', 1} \) occurring in \( a_{j', 1} \), and using our observation on lack of swaps when \( \bar{\lambda}_i, j = \bar{\lambda}_i, 1 + 1 \) we see that every one of these also must yield a value of \( a_{j', i} \) strictly smaller than \( a_{j', 1} + 1 \). But we have in addition that
\[ a_j^i < a_j^{i+1} - 1 \], so we conclude that there are at least \( j' + 1 \) values in \( a_j^i \) strictly less than \( a_j^{i+1} - 1 \), proving the desired inequality by contradiction.

Similarly, in order to have \( \bar{a}_{j''}^{i+1} = \bar{a}_{j''}^i + 1 = a_j^i + 1 \), we would need to have \( a_j^i + 1 \) occurring among the \( a_j^i \), with precisely \( j'' \) strictly smaller values also occurring. By definition, we have only \( j'' \) values among the \( a_j^i \) strictly smaller than \( a_j^i \), and every value of \( a_j^{i+1} \) which is strictly smaller than \( a_j^i \) + 1 must come from one of these. But again using our observation on the lack of swaps when \( \lambda_{i,j} = \lambda_{i-1,j} + 1 \), we see that the value \( a_j^i + 1 \) in \( a_j^{i+1} \) must itself come from a row in \( a_j^{i+1} \) with value strictly smaller than \( a_j^i \), so we conclude that if \( a_j^i + 1 \) occurs in \( a_j^{i+1} \), there must be strictly fewer than \( j'' \) entries in \( a_j^{i+1} \) which are strictly smaller than it. This proves the claim, and the lemma.

\[ \square \]

6. An Independence Criterion

Suppose we have a limit linear series, and fix choices of sections \( s_j^i \) matching the vanishing orders in our table. We make the following definition:

**Definition 6.1.** Given an unimaginative multidegree \( \omega \), for all \((j_1,j_2)\), let \( n(j_1,j_2) \) be the number of places (i.e., collections of contiguous columns) where the \((j_1,j_2)\) row could potentially appear in the multidegree \( \omega \). Let \( s_{(j_1,j_2),i} \) for \( i = 1, \ldots, n(j_1,j_2) \) be the induced sections in multidegree \( \omega \) with precisely the given support. Then the full collection of \( s_{(j_1,j_2),i} \) are the **potentially appearing** sections in multidegree \( \omega \), and their span in \( \Gamma(X_0, (\mathcal{L}^\otimes 2)_{\omega}) \) is the **potential ambient space**.

Note that in the above, we require that each \( s_{(j_1,j_2),i} \) be potentially starting in its first column of support and potentially ending in its last column of support. Thus, there may be individual columns in which the \((j,j')\) row satisfies the inequalities to potentially appear in that column, but which does not occur in any of the \( s_{(j_1,j_2),i} \) because it fails necessarily inequalities in other columns.

The \( s_{(j_1,j_2),i} \) are each unique up to scaling given a choice of the \( s_j^i \). The \( s_j^i \) are not unique, but they can differ only by multiples of \( s_j^i \), with strictly higher vanishing at both points. Then if \( s_j^i \) has potential support (in the \( i \)th column), necessarily \( s_j^i \) has a connected component of potential support consisting precisely of the \( i \)th column. We conclude that the potential ambient space is independent of the choice of the \( s_j^i \). Consequently, the dimension of the span – and in particular the linear independence – of the potentially appearing sections is likewise independent of choices.

As in [LOTZ17], we will give an elementary independence criterion in given multidegrees, stated in terms of iterated dropping of sections. However, while in [LOTZ17] we determined the image of each \( s_j \otimes s_j' \) in multidegree \( \omega \) and phrased our criterion for linear independence in terms of dropping rows, in order for us to handle degenerate cases it will be important to shift our attention from rows to potentially appearing sections. The below definition is to be applied during the iterative procedure, so refers to “remaining” sections (i.e., those which have not yet been dropped).

**Definition 6.2.** We say that the \( i \)th column of \( T \) is **semicritical** in multidegree \( \omega \) if it satisfies the following conditions:

- it has a value of \( \delta_i \) (in particular, it has genus 1);
• the minimal values among the potentially appearing sections remaining in the two subcolumns of column $i$ add to at least $2d - 2$;

• if the $(j, \delta_i)$ row remains in the $i$th column for some $j \neq \delta_i$, then the $j$th row is not exceptional.

If further the minimal values among the remaining potentially appearing sections are not both one less than the values in the $(\delta_i, \delta_i)$ row, we say that the $i$th column is critical.

The following is our criterion for checking that the potentially appearing sections are linearly independent in a given multidegree.

**Lemma 6.3.** For a given limit linear series, and given unimaginative multidegree $\omega$, suppose that we can drop all potentially appearing sections by iterative application of the following rules:

(i) if in some column $i$, there is a unique remaining potentially appearing section supported in that column having minimal $a_i^{(j,j')}$ value, or a unique one having minimal $b_i^{(j,j')}$, then the one achieving the minimum may be dropped;

(ii) if there are at most two remaining potentially appearing sections with support in some genus-1 column $i$, and neither of them involves an exceptional row, then they can both be dropped;

(iii) if there are $i < i'$ such that the block of columns from $i$ to $i'$ has the following properties, then all the remaining potentially appearing sections supported in this block can be dropped:

- there are at most 3 remaining potentially appearing sections supported in each of the $i$th and $i'$th columns;

- within the block, there are at most three potentially appearing sections continuing from any column to the next;

- every column strictly between $i$ and $i'$ has degree 2;

- both the $i$th column and the $i'$th column are semicritical, and either $i$ is critical with no remaining potentially appearing section ending in the $i$th column, or $i'$ is critical with no remaining potentially appearing section starting in the $i'$th column.

Then the potentially appearing sections in multidegree $\omega$ are linearly independent.

**Proof.** Suppose we had a hypothetical linear dependence among the potentially appearing sections. We claim that in each case (i), (ii), (iii), the coefficients of the relevant potentially appearing sections would be forced to vanish. In case (i), this is clear: the uniqueness of the minimal value of $a_i^{(j,j')}$ means that $s_i^{(j,j')}$ vanishes to strictly smaller order at $P_i$ than any other remaining potentially appearing section, and similarly for $b_i^{(j,j')}$. In both cases, the coefficient would have to vanish in any linear dependence.

In case (ii), we need to see that for a fixed column $i$, any two $s_i^{(j,j')}$ have to be linearly independent provided that they do not involve any exceptional rows. If either of them involves $\delta_i$, this is automatic, since either the $a_i^{(j,j')}$ or $b_i^{(j,j')}$ values are forced to be distinct. On the other hand, if neither involves $\delta_i$, we claim that the sections in question must have distinct zeroes on $Z_i$ away from $P_i$ and $Q_i$. Indeed, if we have $a, b, a', b'$ with $a + b = d - 1 = a' + b'$, then the unique sections $s, s'$ of our given line bundle vanishing to order at least $a$ at $P_i$ and $b$ at $Q_i$ (respectively, $a'$ at $P_i$ and $b'$ at $Q_i$) have $\text{div } s = aP_i + bQ_i + R$ and $\text{div } s' = a' P_i + b' Q_i + R'$ for some
We see that we have a linear equivalence $R - R' \sim (a' - a)P_i + (b' - b)Q_i$, and if $0 \leq a, a' \leq d$, we see that $R \neq R'$ because of our running generality hypothesis on $P_i, Q_i$. Thus, tensors of different sections of this form always have zeroes in distinct places on $Z_i$, and must be linearly independent.

For case (iii), note that the condition that the degree is 2 on every column between $i$ and $i'$ means by Corollary 5.9 that there is at most one potentially appearing section starting and at most one ending in each of these columns. Noting that the situation is fully symmetric, suppose without loss of generality that $i'$ is critical, with no remaining potentially appearing sections starting in it. If $i$ or $i'$ has fewer than three remaining potentially appearing sections, we may use (ii) to drop these, and then move iteratively through the rest of the block, using that at most one potentially appearing section starts or ends in each column to repeatedly use (i) to drop the remaining sections from the block. Thus, suppose that $i$ and $i'$ both have three remaining potentially appearing sections. Note also that if any column $i''$ has only one potentially appearing section spanning $i''$ and $i'' + 1$, then the minimal value in the right subcolumn of $i''$ is necessarily unique, so we can use case (i) to drop the section in question. Moreover, there can be at most one other potentially appearing section supported in column $i''$ (the one ending there), so we can drop this one as well, and then we can move iteratively left and right to drop the entire block. Thus, we may further suppose that every column in the block has at least two potentially appearing sections spanning it and the next column.

Next, normalize our sections as follows: scale all sections spanning the $i' - 1$ and $i'$ column so they agree at $Q_{i'-1}$, and then go back one column at a time, scaling any newly appearing section so that its value at the previous node agrees with the value of a section which has already been fixed. In this way, we will fix a normalization of all our sections except for those which are supported in only one column. Although the normalization depends on some choices, they are of a discrete nature, and can be fixed based purely on the discrete data of the limit linear series.

Now, consider a hypothetical nonzero linear dependence involving the rows in our block. First, the coefficients of the linear dependence cannot vanish identically in the remaining potential sections of any column, since otherwise the condition that at most one potentially appearing section ends in each column would imply that there was a column with exactly one nonzero coefficient among its remaining potentially appearing sections. Next, we see that the coefficients are unique up to simultaneous scaling for the three potentially appearing sections in column $i$. Indeed, since we have assumed that $i$ is semicritical, its three potentially appearing sections must be pairwise independent.

Since we have at most one new potentially appearing section in each column, we find that the coefficient for any new one is always uniquely determined by the previous ones. Since there are no new potentially appearing sections in column $i'$, we find that even before considering this column, we have already uniquely determined all of the coefficients (up to simultaneous scaling) of all of the potentially appearing sections remaining in the block. Moreover, we claim that these coefficients (excluding the ones for potentially appearing sections supported only in a single column) are uniquely determined up to finite indeterminacy by the marked curves $Z_i, \ldots, Z_{i'-1}$ together with the discrete data of the limit linear series. Indeed, there are only two ways in which nontrivial moduli can enter the picture:
if there are columns $i^\prime\prime$ between $i$ and $i' - 1$ either having no $\delta_{i'}$, or having some sections $s_{j}^{\prime\prime}$ which are not uniquely determined up to scalar. This becomes slightly delicate, since in both these cases, varying the moduli could affect both the normalization we have chosen and the linear dependence. However, we will show that in both cases, there will in fact be only finitely many possibilities which still preserve the linear dependence. Note that by hypothesis, neither of these nontrivial moduli occurs in the $i$th column. Note also that we cannot have both occurring at once, as the $s_{j}^{\prime\prime}$ can only fail to be determined up to scalar if they involve an exceptional row, and since we have assumed we have degree 2 between $i$ and $i'$, these can only appear if paired with the $\delta_{i'}$ row.

First consider the case that we have no $\delta_{i'}$. Then since we have degree 2, every potentially appearing section in column $i^\prime\prime$ must extend to both the preceding and subsequent columns; in particular, there can be at most three such sections. If there are fewer than three, they cannot be independent, leading to an immediate contradiction. If there are three, say $s_{0}^{\prime\prime}, s_{1}^{\prime\prime}, s_{2}^{\prime\prime}$, then they are necessarily dependent with a unique dependence $c_{0}s_{0}^{\prime\prime} + c_{1}s_{1}^{\prime\prime} + c_{2}s_{2}^{\prime\prime} = 0$ which can be determined by requiring that it holds at both $P_{i'}$ and $Q_{i'}$. We claim that for any fixed choice of $c_{0}, c_{1}, c_{2}$ (not all zero), there can be only finitely many choices of the line bundle $\mathcal{L}^{\prime\prime}$ such that the resulting cancellation holds at both points. For this claim, we can renormalize our sections so that the values of the $s_{j}^{\prime\prime}$ agree at $P_{i'}$, and we just want to see that the values at $Q_{i'}$ must move nondegenerately in $\mathbb{P}^{2}$ as $\mathcal{L}^{\prime\prime}$ varies. But this is precisely the content of Proposition 2.6.

Next, suppose that we have an exceptional row $j$ involved in column $i^\prime\prime$, necessarily paired with the $\delta_{i'}$ row. As before, a linear dependence in the $i^\prime\prime$ necessarily has to give cancellation at both $P_{i'}$ and $Q_{i'}$. Suppose that the $j$th row and the $\delta_{i'}$th row have entries $a, b$ and $a', b'$ respectively, so that $a + b = d - 2$ and $a' + b' = d$. There are two cases: if $a = a' - 1$, so that also $b = b' - 1$ (and $i^\prime\prime$ has a swap in it), then the moduli for the section $s_{j}^{\prime\prime}$ consists simply of adding multiples of the section $s_{j}^{\prime\prime}$, which doesn’t affect the value at either $P_{i'}$ or $Q_{i'}$, and only affects the coefficient of the $(\delta_{i'}, \delta_{i'})$ row, which in this case is supported purely in the $i^\prime\prime$ column.\(^{7}\) On the other hand, if $a \neq a' - 1$, observe that since the degree is 2 in this column, we cannot have any other sections involving $\delta_{i'}$ starting or ending in the column, and therefore we have no sections starting or ending in the column. Thus, there are at most three potentially appearing sections in column $i^\prime\prime$, and the other ones can’t involve any exceptional row, and must therefore be linearly independent. It follows that in our linear dependence, the coefficient of $s_{(j, \delta_{i'})}$ must be nonzero. Now, varying $s_{j}$ will change the relationship between the values at $P_{i'}$ and $Q_{i'}$ (we can view the moduli for $s_{j}$ as adding multiples of a section vanishing to order $a + 1$ at $P_{i'}$ and order $b$ at $Q_{i'}$). Since this variation of moduli affects only a single potentially appearing section, and we know it must have nonzero coefficient in our linear dependence, there is only one choice of $s_{j}^{\prime\prime}$ compatible with the previously determined linear dependence, and we have no nontrivial moduli in this case.

Finally, note that although our normalization was not determined for potentially appearing sections supported in a single column, scaling these does not affect the coefficients of any of the sections spanning the $i' - 1$ and $i'$ column, so we have that...

\(^{7}\)In this situation, varying $s_{j}^{\prime\prime}$ doesn’t even change the limit linear series, but insofar as we made a choice in our setup, we have to consider its possible effects.
the possible coefficients of these sections are determined up to finitely many possibilities. It thus suffices to show that if we vary the gluing points on the component corresponding to the final column, the (unique, if it exists) linear independence on the three potentially appearing sections varies nontrivially.

Now, necessarily the last column has the same value $a$ in all three rows in its left subcolumn. On the right subcolumn, the criticality condition rules out that there is a unique minimum value, although if there were the situation would be even simpler, since we could just drop the potentially appearing sections in this column right away. If $b$ is the minimal value for the right subcolumn, we similarly see that we must have $a + b = 2d - 2$, or we could not have three (or even two) remaining potentially appearing sections. Thus, the only two cases to consider are that $b$ is attained twice, or in all three rows. The last condition in the definition of criticality implies that none of the $(a, b)$ rows are obtained by adding the $\delta_i$ row to an exceptional row. Now, if all three rows are $(a, b)$ rows, we can directly apply Proposition 2.3 to conclude that the linear dependence in the $i$th column varies nontrivially with $P_i$, $Q_i$, as desired. On the other hand, if two rows are $(a, b)$ rows, we again apply Proposition 2.3 to these two rows, and since we have normalized all three rows so that the values at $P_i$ agree, we again see that the linear dependence among the three has to vary nontrivially with $P_i$, $Q_i$, as desired. \hfill \Box

7. The $r = 6$ case

We now specialize to $r = 6$, and suppose we have $g = 21 + \varepsilon$ and $d = 24 + \varepsilon$ for some $\varepsilon \geq 0$, so that $\rho = \varepsilon$. Then our multidegree has total degree $2d = 2g + 6$, so it is determined by placing 3s in six columns, and 2s in the rest.

Although it turns out we will have flexibility in which multidegree to consider, for the purposes of classifying cases, it is helpful to introduce the following.

Definition 7.1. Given a limit linear series, the default multidegree $\omega_{\text{def}}$ is determined by placing a 3:

1. in the first column;
2. in the first column with $\lambda_1^1 + \lambda_1^2 = 5$;
3. in the first column with $\lambda_1^1 + \lambda_3^3 = 7$;
4. in the column immediately after the last column with $\lambda_1^1 + \lambda_3^3 = 7$;
5. in the column immediately after the last column with $\lambda_2^2 + \lambda_3^3 = 9$;
6. in the last column.

Note that $\lambda_i^j$ can only increase in a genus-1 column, so the default multidegree is unimaginative.

Proposition 7.2. Fix an unimaginative multidegree. Then for a column $i$, there can be at most three rows spanning columns $i$ and $i+1$ except in the following circumstances:

1. $\gamma_i = 0$ and $\lambda_1^1 + \lambda_1^2 \geq 8$;
2. $\gamma_i = 2$ and $\lambda_1^1 + \lambda_3^3 \geq 7$;
3. $\gamma_i = 4$ and $\lambda_2^2 + \lambda_3^3 \leq 7$;
4. $\gamma_i = 6$ and $\lambda_1^1 + \lambda_3^3 \leq 6$.

In particular, in the default multidegree there are never more than three rows spanning a given pair of columns.
Proof. We will use the criterion from Proposition 5.8; since this only involves the values of \( j - \lambda_{i,j} = a_j^{i+1} - g(i) \), the general case reduces immediately to the notationally simpler situation that \( \lambda_i = \bar{\lambda}_i \) for all \( i \). We thus assume that we are in this situation. Then, because the sequence \( j - \lambda_{i,j} \) is strictly increasing in \( j \), we see that pairs \((j_1, j_2)\) satisfying the identity for appearing in the \( i \)th and \((i + 1)\)st columns from Proposition 5.8 must be strictly nested, so we can have at most \( r/2 + 1 = 4 \) of them, and we can only have all of these if \( \lambda_{i,j} + \lambda_{i,r-j} \) is constant for all \( j \), so that in particular

\[
2\lambda_{i,r/2} = \lambda_{i,0} + \lambda_{i,r}.
\]

We also see that we have to have

\[
\gamma_i = r - 2\lambda_{i,r/2} = 6 - 2\lambda_{i,r/2},
\]

(so in particular \( \gamma_i \) has to be even) and more generally for \( j = 0, \ldots, r \) we have

\[
\gamma_i = r - \lambda_{i,j} - \lambda_{i,r-j}.
\]

Summing, we find that

\[
\sum_{j=0}^{r} \lambda_{i,j} = \frac{(r + 1)(r - \gamma_i)}{2} = 7(3 - \frac{\gamma_i}{2}),
\]

so \( \lambda_{i,r/2} = 3 - \frac{\gamma_i}{2} \).

If \( \gamma_i = 0 \) we must have \( \lambda_{i,r/2} = 3 \) and we conclude that we would have to have \( \bar{\lambda}_i^1 + \bar{\lambda}_i^3 \geq 8 \).

If \( \gamma_i = 2 \), we need \( \lambda_{i,r/2} = 2 \). Let \( n \) be the number of values of \( j \) with \( \lambda_{i,j} \leq 0 \); then we must have \( \lambda_{i,r-j} \geq 4 \) for the same \( n \) values of \( j \), so \( \bar{\lambda}_i^1 + \bar{\lambda}_i^3 \geq (r + 1 - n) + n = 7 \), as desired.

Similarly, if \( \gamma_i = 4 \) then \( \lambda_{i,r/2} = 1 \), so if we have \( n \) values of \( j \) with \( \lambda_{i,j} \geq 3 \), then we also have \( \lambda_{i,r-j} \leq -1 \), so as before we find \( \bar{\lambda}_i^1 + \bar{\lambda}_i^3 \leq (r + 1 - n) + n = 7 \).

Finally, if \( \gamma_i = 6 \) we have \( \lambda_{i,r/2} = 0 \) and we conclude that we would have to have \( \bar{\lambda}_i^1 + \bar{\lambda}_i^3 \leq 6 \), as claimed. \( \square \)

We can now prove the following theorem, which will in particular prove the desired maximal rank statement in all sufficiently nondegenerate cases for all \( \epsilon \) in our family of cases. It will also suffice to prove the genus-22 case of our main theorem.

**Theorem 7.3.** In the default multidegree, we can always drop all potentially appearing sections using the rules from Lemma 6.3, so the potentially appearing sections are all linearly independent.

Proof. In the first column, unless there is a swap with \( \delta_1 = 1 \) we will have at most the rows \((0,0), (0,1) \) and \((0,2)\) among the potentially appearing sections, while if there is a swap with \( \delta_1 = 1 \) we will have at most the rows \((0,1) \) and \((1,1)\) potentially appearing. In either case, these must all have distinct orders of vanishing, so can all be dropped. According to Corollary 5.9, we will have at most one new row with a potentially appearing section in each column until we get to the next column of degree 3, so these can all be dropped.

Now, suppose that \( i \) is minimal such that \( \bar{\lambda}_i^1 + \bar{\lambda}_i^2 = 5 \). Then we are looking at \( \ell_1 = 1 \) and \( \ell_2 = 2 \), so \( \gamma_i - 1 = 1 = 5 - 1 = \ell_1 - \ell_2 \), and according to Corollary 5.10 and Lemma 5.11, we have no potentially appearing sections supported entirely in the \( i \)th column. Any other new potentially appearing sections would have to be
supported in the $i$th and $(i+1)$st columns, so by Proposition 7.2, we have at most three of these. Note that if we choose $i'$ minimal so that $\bar{\lambda}_i + \bar{\lambda}_{i'} = 6$, then in this case $\gamma_{i'} = 2 = 6 - 1 - \ell_1 - \ell_2$, so according to Corollary 5.9 and Lemma 5.11, there is no row starting in the $i'$th column. We then see that the $i$th (respectively, $i'$th) columns are critical: if $a, b$ are the minimum values in the subcolumns, they have to add to at least $2d - 2$ or the rows would not be potentially starting in the $i$th column (respectively, potentially supported in the $i'$th column). The last condition of semicriticality and the condition for criticality then follow from the second and first parts of Lemma 5.11, respectively. It follows that the hypotheses of Lemma 6.3 (iii) are satisfied, so we can drop all rows occurring in this block. We can then again handle any additional columns before the next degree-3 one.

The setup being symmetric, we can also go from right to left in the same manner, eliminating all potentially appearing sections occurring in any columns outside the middle two degree-3 columns. For these columns, we are considering $\ell_1 = 1$ and $\ell_2 = 3$, so we have $\gamma_i - 1 = 2 = 7 - 1 - \ell_1 - \ell_2$ and $\gamma_i - 1 = 3 = 8 - 1 - \ell_1 - \ell_2$ respectively, and according to Corollary 5.10 and Lemma 5.11, neither column has any potentially appearing section supported entirely in it. As before, we find we must have a block satisfying the hypotheses of Lemma 6.3 (iii), which we can then eliminate.

If the specialization of our linear series contains the “expected” sections $s_j$ for every $j = 0, \ldots, r$ in the expected multidegrees $\omega_j$ (as in Proposition 4.7), then Theorem 7.3 implies that the images of each $s_j \otimes s_{j'}$ in the default multidegree are linearly independent, so the multiplication map has the desired rank $(r + 2)^2 = 28$. However, some linear series may have more degenerate specializations, and the remainder of the paper will be devoted to applying Theorem 7.3 (and variants thereof) to handle these situations as well. For this, the statement in terms of potentially appearing sections (as opposed to the separate rows considered in [LOTZ17]) is crucial. In interesting cases, we can have strictly more than 28 potentially appearing sections. This does not contradict the fact that we know the multiplication map can have rank at most 28, because these do not occur separately in the linked linear series coming as the specialization of any fixed family of linear series on the smooth fibers. In most limits, for every $(j_1, j_2)$ we will have a unique linear combination of the potentially appearing sections in the $(j_1, j_2)$ row which actually arise in the specialization. What makes the degenerate cases more interesting is that in these cases, we may have more than one linear combination occurring from a given row, precisely in situations where the specialization fails to contain any potentially appearing sections from some other row – see Example 8.5 below.

**Example 7.4.** We continue with the running example of Example 4.11 in Table 4. Ultimately, the default multidegree used in Theorem 7.3 will be sufficient to handle the genus-22 case, and most of the genus-23 cases. However, for certain degenerate cases we will need to consider other multidegrees instead.

We will thus want to develop the following results describing the flexibility we have in choosing the multidegree while maintaining linear independence.

**Proposition 7.5.** Suppose we have an unimaginative multidegree $\omega$ determined by placing degree 3 in genus-1 columns as follows:

1. in one column which is either the first, or a column with no exceptional rows and satisfying $\bar{\lambda}_1 + \bar{\lambda}_2 \leq 4$ and $\bar{\lambda}_{i,0} \leq 2$;
eliminated on both the left and right. The only other block that requires rule (iii) contains the 17th and 18th columns, the 16th column, which is again eliminated using rule (iii), after all other potentially appearing sections have been appearing in the 12th column, which is part of another block, extending from the 7th column to 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47.

| 47 | 46 | 45 | 44 | 43 | 42 | 41 | 40 | 39 | 38 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 |

Table 4. The above table is the $T$ obtained from the tensor square of the limit linear series considered in Example 4.11, which has $r = 6$, $g = 22$, and $d = 25$. We have also included the $ω$ corresponding to the default multidegree $ωdef$; for ease of reading, we place the multidegree at both the top and bottom of the table, and include not only the values $c_i$ for $i = 2, \ldots, 22$, but also $2d - c_i$ in the preceding subcolumns. We have highlighted the potentially appearing sections; note that the (2, 2) row contains two, while the rest all have a unique one. These two potentially appearing sections are thus treated separately in Theorem 7.3; the first appears as part of a block in the 5th and 6th columns which is eliminated using rule (iii) of Lemma 6.3, while the second occurs as the only new potentially appearing sections appearing in the 12th column, which is part of another block, extending from the 7th column to the 16th column, which is again eliminated using rule (iii), after all other potentially appearing sections have been eliminated on both the left and right. The only other block that requires rule (iii) contains the 17th and 18th columns, and is eliminated after the potentially appearing sections appearing to the right have all been dropped. Following the proof of Theorem 7.3, we see that we can eliminate all sections outside the aforementioned three blocks going inward from both the left and right ends, using only iterated applications of rule (i).
(2) in one column with $\lambda_1 + \lambda_2 = 5$ but $\lambda_{1-1} + \lambda_{2-1} = 4$;
(3) in one column between the first column with $\lambda_1 + \lambda_2 = 6$ and the first column with $\lambda_1 + \lambda_2 = 7$ (inclusive);
(4) in one column between the column immediately after the last column with $\lambda_1 + \lambda_3 = 7$ and the column immediately after the last column with $\lambda_2 + \lambda_3 = 8$ (inclusive);
(5) in one column with $\lambda_1^2 + \lambda_1^3 = 10$ but $\lambda_{1-1}^2 + \lambda_{1-1}^3 = 9$;
(6) in one column which is either the last, or a column with no exceptional rows and satisfying $\lambda_{2-1}^2 + \lambda_{2-1}^3 \geq 10$ and $\lambda_{1-3} \geq 1$.

Then the potentially appearing sections in multidegree $\omega$ are still linearly independent.

Proof. The main new ingredient is verifying that if we place the first degree $3$ in a (genus-$1$) column after the first, but still satisfying $\bar{\lambda}_1 = 4$, then provided we also have no exceptional rows, we will in fact obtain at most two potentially appearing sections starting in the $i$th column. Note that in this case, in particular there are no swaps in the $i$th column. By Proposition 5.8, for the $(j, j')$ row to have a potentially appearing section starting in the $i$th column, we will need $j + j' - \lambda_{i-1,j} - \lambda_{i-1,j'} > \gamma_{i-1} = 0$ and $j + j' - \lambda_{i,j} - \lambda_{i,j'} \leq \gamma_{i} = 1$, or equivalently

$$\lambda_{i-1,j} + \lambda_{i-1,j'} < j + j' \leq 1 + \lambda_{i,j} + \lambda_{i,j'}.$$  

(7.1)

For this assertion, since we are assuming no swaps occur in the $i$th column, it suffices to check the case with $\bar{\lambda}_i = \lambda_i$ for all $i$, which simplifies notation. Now, since $\lambda_1 + \lambda_2 \leq 4$, we must have $\lambda_{i,j} \leq 0$ for $j \geq 4$ and $\lambda_{i,j} \leq 1$ for $j = 2, 3$. It follows that to satisfy the righthand inequality above, we must have at least one of $j, j'$ equal to $0$ or $1$. Moreover, by Corollary 5.10 we have that for $j = 0, 1$, if $j \neq \delta_i$, then there is at most one value of $j'$ satisfying the above inequalities. In particular, we conclude that if $\delta_i \neq 0, 1$, we have at most two potentially appearing sections, as claimed.

Now, if $\delta_i = 0$, we need to see that we can have at most two rows of the form $(0, j')$ appearing in the $i$th column, and if two appear, then none of the form $(1, j')$ can appear for $j' > 0$. Suppose first $(0, 0)$ is potentially starting in the $i$th column. By (7.1) this could only happen if $\lambda_{i-1,0} < 0$, so $\lambda_{i,j'} = \lambda_{i-1,j'} < 0$ for all $j' > 0$, and then $(0, j')$ cannot satisfy the righthand side of (7.1) for any $j' > 0$. On the other hand, if we have $j'' > j' > 0$ such that $(0, j')$ and $(0, j'')$ both appear, then we have

$$\lambda_{i-1,0} + \lambda_{i-1,j'} < j' < j'' \leq 1 + \lambda_{i,0} + \lambda_{i,j''} \leq 2 + \lambda_{i-1,0} + \lambda_{i-1,j'},$$

so the only possibility is that $j' = 1 + \lambda_{i-1,0} + \lambda_{i-1,j'}$ and $j'' = j' + 1$, with $\lambda_{i,j''} = \lambda_{i,j'}$. It immediately follows that we could not have $(0, j'')$ appearing for any $j'' \neq 0, j', j''$. We also check that $(1, j''')$ cannot be potentially appearing for any $j''' > 0$ in this situation. Indeed, $1 + j'''$ will be too large if $j''' \geq j'$. The $(1, 1)$ row cannot satisfy (7.1) for parity reasons, and in order to have $(1, 2)$ appearing we would need $j'' > 4$, but we note that in this case $1 + \lambda_{i,0} + \lambda_{i,j''} \leq 3$, contradicting (7.1).

Finally, consider the case that $\delta_i = 1$. If the $(1, 1)$ row is potentially starting in the $i$th column, by parity we have to have $1 = \lambda_{i,1}$, so for all $j > 1$ we have $\lambda_{i,j} = \lambda_{i-1,j} \leq \lambda_{i-1,1} = 0$. Then we cannot have $(1, j')$ potentially starting for any $j' > 1$, so we have at most two rows potentially starting. On the other hand, if we
have \( j'' > j' > 1 \) potentially starting in the \( i \)th column, we are just as above forced to have \( j' = \lambda_{i-1,1} + \lambda_{i-1,j'} \) and \( j'' = j' + 1 \), with \( \lambda_{i,j''} = \lambda_{i,j'} \), and we claim we cannot have \((0,j''')\) potentially starting for any \( j''' \). Indeed, if \( j''' \leq j'' \), then we have \( j''' - \lambda_{i-1,j'''} \leq j'' - \lambda_{i-1,j''} \), so

\[
\begin{align*}
\lambda_{i-1,j'''} &\leq \lambda_{i-1,j'''} + \lambda_{i-1,j''} \\
&= 1 + \lambda_{i-1,1} + \lambda_{i-1,j''} - \lambda_{i-1,j'''} + \lambda_{i-1,j''}
\end{align*}
\]

violating (7.1). But \( j'' \geq 3 \), so if \( j''' \geq 4 \) we cannot satisfy (7.1) without violating our hypothesis that \( \lambda_{i-1,0} \leq 2 \). We thus conclude the desired statement on the number of potentially appearing sections starting in column \( i \).

Now, since we have assumed that our first column with degree 3 has no exceptional rows, the fact that it has at most two potentially appearing sections starting in it means that we can still eliminate sections from left to right until we reach the second column of degree 3, just as in the proof of Theorem 7.3. We also see that the second column of degree 3 will still be critical, with at most three potentially appearing sections starting in it. The next step depends on the location of the third column of degree 3. If the first column with \( \lambda^1_i + \lambda^2_i = 6 \) still has degree 2, we will eliminate this block from left to right, as before. On the other hand, if the first column with \( \lambda^1_i + \lambda^3_i = 7 \) has degree 2, we do not need to have eliminated everything from the left in order to eliminate the central block, since the potentially supported rows in multidegree \( \omega \) will be precisely the same as the potentially starting rows in \( \omega_{\text{def}} \). Thus, if the third column of degree 3 is strictly between these, we can eliminate both adjacent blocks first, and then eliminate all potentially appearing sections one by one from both sides until we reach this final column, which can have at most one remaining potentially appearing section by Corollary 5.10. However, if the third column of degree 3 is the first column with \( \lambda^1_i + \lambda^3_i = 6 \), we see that this will be critical with at most three potentially appearing sections ending in it, and we will instead eliminate the central block first, and then eliminate the block between the second and third columns of degree 3 last.

The situation is symmetric on the right, so we see that in all cases we will be able to eliminate all potentially appearing sections in a suitable order. \( \square \)

It will also be important to consider moving degree 3 into a column with a swap, which we analyze with the below lemma.

**Lemma 7.6.** Suppose that in the \( i \)th column, we have a \( \delta_i \), and for some \( j_0 \) we have \( a_{\delta_i}^j = a_{j_0}^1 + 1 \), but \( a_{\delta_i}^{j+1} = a_{j_0}^1 - 1 \). Suppose further that our multidegree assigns degree 3 to the \( i \)th column, and write it as usual as \( \text{md}_3(w) \) for \( w = (c_2, \ldots, c_3) \).

Then we can have at most four potentially appearing sections start in the \( i \)th column. Moreover, we can only have four if either we have

\[
2\bar{a}_{i}^3 = c_i + 1,
\]

or if \((\delta_i, \delta_i)\) is potentially starting, and one of the following three possibilities holds:

1. \( a_{(\delta_i, \delta_i)}^i = c_i + 1 \);
2. \( a_{(\delta_i, \delta_i)}^{i+1} = c_i + 2 \), and the \((j_0, \delta_i)\) row is potentially starting;
3. \( a_{(\delta_i, \delta_i)}^{i+1} = c_i + 3 \), with \( a_{\delta_i}^i = \bar{a}_i^i \).
We can also have at most four potentially appearing sections end in the \( i \)th column, with four of them ending only if either we have
\[
2a^i_3 = c_i,
\]
or if \((\delta_i, \delta_i)\) is potentially ending, and one the following three possibilities holds:
\[
\begin{align*}
(1) & \quad a^i_{(\delta_i, \delta_i)} = c_i, \text{ with } a^i_{b_i} = a^i_3; \\
(2) & \quad a^i_{(\delta_i, \delta_i)} = c_i + 1, \text{ and the } (j_0, \delta_i) \text{ row is potentially ending;}
\end{align*}
\]
(3) \( a^i_{(\delta_i, \delta_i)} = c_i + 2; \)

We have written the above to allow for swaps having occurred prior to the \( i \)th column. If no swaps have occurred, the \( j_0 \) in the lemma statement is necessarily \( \delta_i - 1 \), and the third exceptional case would require \( \delta_i = 4 \) (respectively, \( \delta_i = 2 \)) in the statement on potential support starting (respectively, ending).

**Proof.** In order to have a potentially appearing section start in the \((j, j')\) row, we must have \( a^i_{(j, j')} > c_i \) and \( a^{i+1}_{(j, j')} \leq c_{i+1} = c_i + 3 \). It immediately follows that if \( j, j' \neq \delta_i \), then we must have \( a^i_{(j, j')} = c_i + 1 \), and neither \( j \) nor \( j' \) equal to \( j_0 \). If \( j' = \delta_i \) for \( j \neq \delta_i \), we could also have \( a^i_{(j, \delta_i)} = c_i + 2 \), provided that \( j \neq j_0 \). And \( j_0 \) can occur only if \( a^i_{(j_0, \delta_i)} = c_i + 1 \), or equivalently if \( a^i_{(\delta_i, \delta_i)} = c_i + 2 \). Recall that Corollary 5.10 says that if \((j, \delta_i)\) occurs for some \( j \neq \delta_i \), then there is no \( j' \neq \delta_i \) with \((j, j')\) also occurring. Next, we note that we can have at most two rows of the form \((j, \delta_i)\) occurring. Indeed, if \( a^i_{(\delta_i, \delta_i)} = c_i + 3 \), then \( a^i_{(j, \delta_i)} = c_i + 2 \) only for \( j = j_0 \), so the \((j_0, \delta_i)\) row does not occur, and we can have at most one additional row, having \( a^i_{(j, \delta_i)} = c_i + 1 \). On the other hand, if \( a^i_{(\delta_i, \delta_i)} \neq c_i + 3 \), then we have at most two rows, because they have to satisfy \( a^i_{(j, \delta_i)} = c_i + 1 \) or \( c_i + 2 \). We also observe that we can have a row of the form \((j, j)\) for \( j \neq \delta_i \) only for a unique choice of \( j \), necessarily with \( 2a^i_j = c_i + 1 \) and \( j \neq j_0 \), and then we cannot have \((\delta_i, \delta_i)\) occurring, since \( a^i_j \neq a^i_3 - 1 \) for \( j \neq j_0 \).

Now, if we do not have any \((j, \delta_i)\) occurring, then \( j_0 \) also cannot occur, and we are left with only 5 values of \( j \) from which to choose distinct pairs, so we can obtain at most three pairs (allowing one of them to have repeated entries). Similarly, if we have exactly one \((j, \delta_i)\), then necessarily \( j \neq j_0 \), or we would be in the 2nd exceptional case with also \((\delta_i, \delta_i)\) occurring, so we have that for the remaining pairs we must choose from values not equal to \( \delta_i, j_0, j \), leaving four values, and at most two pairs. We therefore see that in order to have four rows potentially starting, two of them need to involve \( \delta_i \).

Next, if we have \((j_1, \delta_i)\) and \((j_2, \delta_i)\) occurring, with neither \( j_1, j_2 \) equal to \( \delta_i \) (and hence also neither equal to \( j_0 \)), then any remaining rows have to be chosen as distinct pairs from the remaining \((r + 1) - 4 = 3\) indices, with at most one pair having repeated value. We thus obtain at most four rows, with four occurring only if \( 2a^i_j = a^i_{j_3} + a^i_{j_4} = c_i + 1 \) for some \( j, j_3, j_4 \neq \delta_i, j_0, j_1, j_2 \). Moreover, we see that there must be exactly three values of \( j' \) with \( a^i_{j'} < a^i_j \) in this case: if \( a^i_{j_3} < a^i_j \), then these are \( \delta_i, j_0, j_1 \), and exactly one of \( j_3, j_4 \), with necessarily \( j_1, j_2 \) and the other of \( j_3, j_4 \) having \( a^i_{j'} > a^i_j \). If \( a^i_{j_3} > a^i_j \), then \( a^i_{j_0} \) must also be greater than \( a^i_j \), so we similarly find exactly three values are smaller. Thus \((7.2)\) must hold.

It remains to consider the case that \((\delta_i, \delta_i)\) row is potentially starting, and the only thing left to prove is the description of case (3), where \( a^i_{(\delta_i, \delta_i)} = c_i + 3 \). Here, we
must also have a $j$ with $a_{i,j}^1 = c_i + 1$, and if we have two additional rows appearing, these must come from two additional pairs nested around $a_{i,j}^1$ in value, so since $a_j^i < a_{j,0}^i < a_{j,1}^i$, in this case, we obtain the desired statement. The statement on rows ending is symmetric.

We next give two background propositions which do not require that $r = 6$.

**Proposition 7.7.** For a fixed limit linear series, $w$, and column $i$, if $a_{i,j}^1 > c_i$, then $(j, j')$ has a component of potential support strictly right of $i - 1$, and if if $a_{i,j}^1 < c_i$, then $(j, j')$ has a component of potential support strictly left of $i$.

Conversely, suppose further that $w$ is unimaginative. If $(j, j')$ has a component of potential support strictly right of $i - 1$, and if neither $j$ nor $j'$ is exceptional in any column strictly right of $i - 1$, then $a_{i,j}^1 > c_i$. Similarly, if $(j, j')$ has a component of potential support strictly left of $i$, and if neither $j$ nor $j'$ is exceptional in any column strictly left of $i$, then $a_{i,j}^1 < c_i$.

In particular, in the unimaginative case, if the potential support of $(j, j')$ is disconnected, then at least one of $j, j'$ must be exceptional somewhere.

**Proof.** The first part is straightforward, and we omit the proof. For the second part, the point is that the unimaginative hypothesis together with the non-exceptional hypothesis together imply that the relevant portion of the sequence $a_{i,j}^1 - c_i$ is nondecreasing in the relevant range as $i'$ decreases, so in the first case if its positivity for some $i' \geq i$ implies it remains positive at $i' = i$, while in the second case its negativity for some $i' \leq i$ implies it remains negative at $i' = i$.

**Proposition 7.8.** The number of swaps in a given limit linear series is bounded by $p$.

Suppose that we have $p$ swaps. Then the swaps must all be minimal, and occur in genus-$1$ columns, and we cannot have any exceptional behavior other than what is needed for the swaps. Moreover, for any unimaginative $w$, the potential support of the $(j, j')$ row is connected unless the sum of the number of swaps for which the $j$th row is exceptional and the number of swaps for which the $j'$th row is exceptional is at least $2$.

**Proof.** The first assertions follow immediately from Remark 5.6. For the last, we can have disconnected potential support in the $(j, j')$ row only if the sequence $a_{i,j}^1 - c_i$ goes from positive to negative as $i$ decreases, possibly over multiple columns. But we observe that if only one of $j$ and $j'$ are exceptional at a swap, which is moreover minimal and in a genus-$1$ column, then the sequence $a_{i,j}^1 - c_i$ can decrease only by $1$ as $i$ decreases. Thus, if this occurs only once, it cannot go from positive to negative, and we cannot have disconnected potential support.

Using Lemma 7.6, we can prove the following.

**Corollary 7.9.** Suppose that $p = 2$ and $r = 6$. Then:

1. if we are in the “first 3-cycle” situation of Proposition 4.14, there exists an unimaginative multidegree $\omega'$ such that the $(j_0 - 1, j_0)$ row has a unique potentially appearing section in multidegree $\omega'$, whose support does not contain $i_0$ or $i_1$, and such that all the potentially appearing sections are linearly independent.
(2) if we are in the “second 3-cycle” situation of Proposition 4.15, suppose that in the default multidegree \( \omega_{\text{def}} \), we have the inequalities
\[
2a_{j_0-1}^{i_0} \leq c_{i_0} - 1, \quad \text{and} \quad 2a_{j_0+1}^{i_1+1} \geq c_{i_1+1} + 1,
\]
with exactly one of the two inequalities satisfied with equality. Then there exists an unimaginative multidegree \( \omega' \) such that the \((j_0 - 1, j_0 - 1)\) row does not have potentially appearing sections both left of \(i_0\) and right of \(i_1\) in multidegree \( \omega' \), and such that all the potentially appearing sections are linearly independent.

Proof. (1) In this situation, the \(j_0\) and \(j_0 - 1\) rows each have only one column adding to \(d - 2\), so the potential support of \((j_0 - 1, j_0)\) can be disconnected only if \(a_{j_0-1}^{i_0} + a_{j_0}^{i_0} = c_{i_0} - 1\) and \(a_{j_0+1}^{i_1+1} + a_{j_0+1}^{i_1+1} = c_{i_1+1} + 1\), we have degree 2 in every column from \(i_0\) to \(i_1\), inclusive, and we do not have \(\delta_i\) equal to \(j_0 - 1\) or \(j_0\) for any \(i\) between \(i_0\) and \(i_1\). It then follows in particular that \(a_{j_0+1,j_0+1}^{i_0} \geq c_{i_0} + 2\) and \(a_{j_0+1,j_0+1}^{i_1+1} \leq c_{i_1+1} - 2\), or equivalently, \(a_{j_0+1,j_0+1}^{i_1} \leq c_{i_1}\).

Consider the default multidegree \( \omega_{\text{def}} \). If the \((j_0 - 1, j_0)\) row has connected potential support, we are done: since \( \omega_{\text{def}} \) has degree 2 in both the \(i_0\) and \(i_1\) columns, the \((j_0 - 1, j_0)\) row cannot have any support in either of these. On the other hand, if the \((j_0 - 1, j_0)\) row has disconnected potential support, then we will use Lemma 7.6 to verify that we can move one degree 3 into either the \(i_0\) or \(i_1\) column while maintaining the independence conclusion of Theorem 7.3. We will then obtain the desired statement: certainly, the \((j_0 - 1, j_0)\) row will have connected potential support. If the 3 was moved to the \(i_0\) column, then the \((j_0 - 1, j_0)\) still cannot have any potential support at \(i_1\). If the 3 was moved from the right, we still have \(a_{j_0-1}^{i_0} + a_{j_0}^{i_0} = c_{i_0} - 1\), ruling out potential support at \(i_0\), but if it was moved from the left, then this will decrease \(c_{i_0}\) by 1, and we will then have \(a_{j_0-1}^{i_0} + a_{j_0}^{i_0} = c_i\) for \(i_0 \leq i \leq i_1\), meaning that any potential support at \(i_0\) would have to continue right to \(i_1\), but we will still have \(a_{j_0+1,j_0+1}^{i_1+1} + a_{j_0+1,j_0+1}^{i_1+1} = c_{i_1+1} + 1\), so there cannot be any potential support at \(i_1\). A similar analysis holds if we moved the 3 to \(i_1\), proving the desired result.

To prove that we can always move a 3 as desired, we first make some general observations regarding when we will be able to move degree 3 from the left or right onto \(i_0\) or \(i_1\). Recall that \(\delta_i = 0 = j_0 + 1\). Since \(a_{j_0+1,j_0+1}^{i_0} \leq c_{i_1}\), moving a degree 3 to \(i_1\) from the right will always lead to at most 3 rows starting in the \(i_1\) column, unless \(2a_{j_0}^{i_1} = c_{i_1} + 1\), or equivalently,
\[
5 - \gamma_{i_1-1} = 2\lambda_{i_1-1,3}.
\]
In addition, \(a_{j_0-1,j_0+1}^{i_1} < c_{i_1}\), so the \((j_0 - 1, j_0 + 1)\) row will not be among the appearing rows.

We next consider what happens if we move a degree 3 to \(i_0\) from the left. This will decrease \(c_{i_0}\) by 1, so we have to rule out that in multidegree \( \omega_{\text{def}} \) we have \(2a_{j_0}^{i_0} = c_{i_0}\), or equivalently,
\[
6 - \gamma_{i_0-1} = 2\lambda_{i_0-1,3}.
\]
Additionally, if \(a_{j_0+1,j_0+1}^{i_0} \geq c_{i_0} + 3\) in \( \omega_{\text{def}} \), then after moving the degree 3 to \(i_0\), none of the other exceptional cases of Lemma 7.6 can occur, so as long as we do not have (7.5), we will have at most three rows with potential support starting at \(i_0\). The only other possibility is that \(a_{j_0+1,j_0+1}^{i_0} = c_{i_0} + 2\), which is equivalent to
2j_0 - \gamma_{i_0} - 1 = 2\lambda_{i_0 - 1,j_0 + 1}; moreover, after moving a 3 from the left to \(i_0\) we will have \(a^{i_0}_{(j_0 + 1,j_0 + 1)} = c_{i_0} + 3\), so we could potentially be only in the third exceptional case in Lemma 7.6. Thus, the only case for concern is that \(j_0 + 1 = 4\), so we simply need to check that in cases where we wish to move a 3 from the left, we never have

\[(7.6) \quad 6 - \gamma_{i_0 - 1} = 2\lambda_{i_0 - 1,4}.
\]

Finally, in either case after the move we will have \(a^{i_0}_{(j_0,j_0 + 1)} = a^{i_0}_{(j_0 + 1,j_0 + 1)} - 1 \geq c_{i_0} + 2\), so the \((j_0,j_0 + 1)\) row cannot be among the rows starting at \(i_0\).

We now describe how to modify our default multidegree, depending on the location of \(i_0\) and \(i_1\). If we have \(\gamma_{i_0} = \gamma_{i_1} = 1\), then we will move the next 3 from the right to column \(i_1\), and we will obtain at most three rows with potential support starting in \(i_1\); by the above observation, it suffices to rule out (7.4), but we have \(5 - \gamma_{i_1 - 1} = 4\). To have equality we would need \(\lambda_{i_1 - 1,3} = 2\), which would imply \(\lambda_{i_1 - 1}^1 + \lambda_{i_1 - 1}^2 \geq 8\), in which case we would not have had \(\gamma_{i_1} = 1\) in \(\omega_{\text{def}}\).

Next, suppose \(\gamma_{i_0} = \gamma_{i_1} = 2\), and we have \(\lambda_{i_1}^1 + \lambda_{i_1}^2 < 6\). In this case, we will move the 3 to \(i_0\) from the left, and \(\gamma_{i_0} - 1 = 2\) in \(\omega_{\text{def}}\), so if either (7.6) or (7.5) is satisfied, we must have \(\lambda_{i_0 - 1,3} \geq 2\). But this would force

\[\lambda_{i_1}^1 + \lambda_{i_1}^2 \geq \lambda_{i_0 - 1}^1 + \lambda_{i_0 - 1}^2 \geq 8,
\]

contradicting the hypothesis for the case in question. We again conclude that we have at most 3 rows starting, and again the \((j_0,j_0 + 1)\) row is not among them.

On the other hand, if \(\gamma_{i_0} = \gamma_{i_1} = 2\), and we have \(\lambda_{i_1}^1 + \lambda_{i_1}^2 \geq 6\), then we will move a 3 from the right, and (7.4) is not satisfied for parity reasons, so we will have at most three new rows starting. Finally, if \(\gamma_{i_0} = \gamma_{i_1} = 3\), neither (7.6) nor (7.5) can be satisfied for parity reasons, so we can move a 3 from the left to \(i_0\), and have at most three starting rows.

The remaining cases are treated symmetrically, with rows starting replaced by rows ending. In each case, we see that the basic structure of the proof of Theorem 7.3 is preserved by our change of multidegree, so our linear independence is likewise preserved, yielding the desired statement.

(2) Suppose that in multidegree \(\omega_{\text{def}}\), we have

\[2a^{i_0}_{j_0 - 1} = c_{i_0} - 1, \text{ but } 2a^{j_0 + 1}_{j_0 - 1} > c_{i_1 + 1} + 1.
\]

We will show that we can always move a 3 from the left to a genus-1 column on or right of \(i_0\), while preserving linear independence. This will eliminate potential support in the \((j_0 - 1,j_0 - 1)\) row left of \(i_0\), as desired. Now, in this situation, we necessarily have

\[2(j_0 - 1) + 1 - \gamma_{i_0 - 1} = 2\lambda_{i_0 - 1,j_0 - 1},
\]

so in particular \(\gamma_{i_0 - 1}\) must be odd.

**Case** \(\gamma_{i_0 - 1} = 1\). We have \(j_0 - 1 = \lambda_{i_0 - 1,j_0 - 1}\). Because \(\lambda_{i_0 - 1}^1 + \lambda_{i_0 - 1}^2 < 5\) necessarily, this forces \(j_0 - 1 = 1\), so we have \(\lambda_{i_0 - 1,j_0 - 1} = 1\).

First, if \(i_1\) is the genus-1 column immediately following \(i_0\), we observe that if we move the first 3 to \(i_0\), considering only the inequalities at \(i_0\), there can be at most three rows with potential support starting at \(i_0\): \((1,2), (2,2)\) and \((0,j)\) for a unique \(j > 2\). But in this case the actual potential support of \((1,2)\) is connected and supported strictly to the right of \(i_1\). Thus, there are in fact at most two rows with potential support starting at \(i_0\), and neither of them involves the exceptional
row (specifically, \( j = 1 \)), so even after moving the first 3 to \( i_0 \) we will be able to eliminate potentially appearing sections from left to right as before.

Next, suppose that \( i_1 \) is not the genus-1 column immediately following \( i_0 \), and denote this column by \( i \). Suppose also that there is no degree-3 column between \( i_0 \) and \( i_1 \), so that in particular \( \lambda_{i_0}^1 + \lambda_{i_1}^2 \leq 4 \). We observe that we must also have \( \lambda_{i_0,0}^1 = 1 \), since we have \( \lambda_{i_0,1} = \lambda_{i_0,2} = 1 \), and we must have \( \lambda_{i_1,1-2} = \lambda_{i_1,1-3} \geq 1 \), so the only way we can avoid having a column of degree 3 before \( i_1 \) is if also \( \lambda_{i_1-1,0} = 1 \). We can then apply Proposition 7.5 to move the first 3 to column \( i \), and we will still obtain linear independence.

Finally, if we have a column of degree 3 between \( i_0 \) and \( i_1 \), say in column \( i \), so that \( \lambda_{i_0}^1 + \lambda_{i_1}^2 = 5 \), then we claim that if we move the first 3 from the left to \( i_0 \), we will have at most two potentially appearing sections ending in column \( i \), and at most two potentially appearing sections supported in the first column with \( \lambda_{i_0}^1 + \lambda_{i_1}^2 = 6 \). This will prove the desired statement, since we can then eliminate the potentially appearing sections starting from \( i' \) and moving both left and right from there. For checking the possible inequalities in column \( i' \), moving the 3 from the left to \( i_0 \) won’t affect anything, so the argument for Theorem 7.3 implies a priori that there are at most three rows satisfying the inequalities at \( i' \) for potentially appearing sections to be supported there. We will check that there is always only one such row which satisfies the inequalities at \( i' \), but does not in fact have potential support there. Because we have a 3 between \( i_0 \) and \( i_1 \), we must have \( 2a_{i_0,0}^{i_0} = c_{i_1} + 1 \) + 2. If \( i' < i_1 \), the row in question is \( (1,1) = (j_0 - 1, j_0 - 1) \): indeed, in this situation we will have \( a_{i_0,0}^{i_0} = c_{i_1} \) for all \( i'' \) with \( i < i'' \leq i_1 \). In this case (recalling that \( j_0 - 1, j_0 - 1 \) does satisfy the necessary inequalities at \( i' \), but its actual potential support (after moving the 3 to \( i_0 \)) is strictly to the right of \( i_1 \). On the other hand, if \( i' > i_1 \), the row in question will be \( (1,2) = (j_0 - 1, j_0) \); we have \( a_{i_0,0}^{i_0} + a_{i_0}^{i_0} = c_{i_1} + 1 \), so \( a_{i_0,0}^{i_0} < c_{i_1} \), and because the potential support is connected, it must be strictly left of \( i_1 \). However, we claim that we must have \( a_{i_0,0}^{i_0} + a_{i_0}^{i_0} = c_{i_1} + 1 \), and that this must extend through the column \( i' \), so that the inequalities for potential support are satisfied at \( i' \). Indeed, the only way this could fail is if \( i'' = j_0 \) for some \( i'' \) with \( i_0 < i'' < i' \). But we know that \( \lambda_{i_0-1,j_0-1} = \lambda_{i_0-1,j_0} = 1 \), so if \( i'' = j_0 \) anywhere after \( i_0 \), it increases \( \lambda_{i''}^1 + \lambda_{i''}^2 \) to at least 5. Thus, this could only happen for \( i'' = i' \), which then forces us to have \( \lambda_{i''}^1 = 3 \) and \( \lambda_{i''}^2 = 1 \). However, in this case, because we cannot have a gap between the \( j_0 - 1 \) and \( j_0 + 1 \) column at \( i_0 \), this would force us to also increase \( \lambda_{i_0}^1 \) to 4 before \( i_1 \), which violates our hypothesis that \( i'' > i_1 \). Thus, in either situation we have shown that the column \( i' \) has at most two potentially appearing sections supported on it, and it remains to check that the column \( i \) has at most two potentially appearing sections ending in it. But we either have \( \lambda_{i-1}^1 = 4 \) and \( \lambda_{i-1}^2 = 0 \) or \( \lambda_{i-1}^1 = 3 \) and \( \lambda_{i-1}^2 = 1 \), and one can calculate directly that because we cannot have \( i_1 = 4 \) or 4 in the second case, \( \delta_i = 1 \) in either case (recalling that by column \( i \) we have had a swap between rows 1 and 2), or \( \delta_i = 3 \) in the first case, the only rows with potential support ending in column \( i \) are \( (1,2) \) and \( (0,j) \) for a unique value of \( j \), yielding the desired statement.

Case \( \gamma_{i_0-1} = 3 \). We can have either \( j_0 - 1 = 2 \) and \( \lambda_{i_0-1,j_0-1} = 1 \), or \( j_0 - 1 = 3 \) and \( \lambda_{i_0-1,j_0-1} = 2 \). First, suppose that \( (j_0 - 1, j_0) \) has potential support strictly to the right of \( i_1 \), or equivalently, that there are no columns between \( i_0 \) and \( i_1 \) having degree 3, or with \( \delta_i = j_0 - 1 \) or \( j_0 \). In this case, if we move a 3 from the left to \( i_0 \), by Lemma 7.6 at most four rows satisfy the inequalities at \( i_0 \) to have potentially
appearing sections starting in $i_0$, and we see that these include $(j_0 - 1, j_0)$. But $(j_0 - 1, j_0)$ does not actually have potential support at $i_0$, so in this case we have at most three rows starting at $i_0$, and none of them involve the exceptional row (specifically, $j_0 - 1$), so we can eliminate this central block just as in Theorem 7.3, and we conclude we still have linear independence.

Now, the possibility that we have $\delta_i = j_0 - 1$ in between $i_0$ and $i_1$ is ruled out by the inequality $2c_{j_0-1}^{i_1+1} > c_{i_1+1} + 1$. If there is a column with $\delta_i = j_0$, but no column having degree 3 between $i_0$ and $i_1$, we will move the third degree-3 from the left to $i_1$, and the $(j_0 + 1, j_0 + 1) = (\delta_{i_1}, \delta_{i_1})$ row is supported strictly to the right of $i_1$.

In addition (7.2) is ruled out by parity reasons, so by Lemma 7.6 we have at most three rows starting at $i_1$, and we also see that $(j_0 - 1, j_0 + 1)$ is not among them, as it will have potential support strictly to the right of $i_1$. Thus, no row involving $j_0 - 1$ (the exceptional row) has potential support starting at $i_1$, and in this case we can eliminate all potentially appearing sections just as in Theorem 7.3.

Next, suppose there is some column with degree 3 between $i_0$ and $i_1$, but no column with $\delta_i = j_0$. In this case, we will move the 4th 3 to the first column $i$ with $\lambda_i^2 + \lambda_i^3 = 9$, and the 3rd 3 to $i_0$. If $\lambda_i^2 + \lambda_i^3 < 9$, then according to Proposition 7.5, moving the 4th 3 doesn’t disrupt linear independence, and then we are in exactly the same situation as the first case considered above, with $(j_0 - 1, j_0)$ having potential support strictly to the right of $i_1$. On the other hand, if $\lambda_i^2 + \lambda_i^3 = 9$, we will still maintain linear independence, but for different reasons: we claim that will have at most three rows ending in the $i$th column, no row ending in the first column $i'$ with $\lambda_{i'}^2 + \lambda_{i'}^3 = 8$, and only two rows ending in the first column $i''$ with $\lambda_{i''}^2 + \lambda_{i''}^3 = 10$.

Thus, we will be able to eliminate potentially appearing sections from the right, treating the columns from $i'$ to $i$ as a block to which to apply Lemma 6.3 (3), and we will in this way eliminate all potentially appearing sections supported on either side of $i_0$. This leaves at most one potentially appearing section, which can then be eliminated. Thus, it suffices to prove the above claim. By the argument for Proposition 7.5, we have no potentially appearing section supported only in the $i$th column, and at most three continuing from the previous column, so there are at most three ending in the $i$th column, as claimed. The fact that there are no rows ending in the $i'$th column is immediate from Corollary 5.9 and Lemma 5.11. Finally, we know from the proof of Theorem 7.3 that there at most three rows satisfying the inequalities in column $i''$ to have potential support ending there. Moreover, we see that $(j_0 - 1, j_0)$ is necessarily one of them. Indeed, since we have one column with degree 3 and none with $\delta_i = j_0 - 1$ or $j_0$ between $i_0$ and $i_1$, we see that we necessarily have $a_{(j_0-1, j_0)}^{i_1+1} = c_{i_1+1}$ even after changing the multidegree. But after $i_1$, any column with $\delta_i = j_0 - 1$ or $j_0$ will increase $\lambda_i^2 + \lambda_i^3$, so this cannot occur strictly between $i_1$ and $i''$, and we conclude that $a_{(j_0-1, j_0)}^{i_1+1} = c_{i_1+1}$ as well. Since column $i''$ has degree 3, this means that $(j_0 - 1, j_0)$ satisfies the inequalities to have potential support ending at $i''$. But again using that the 4th 3 is still left of $i_1$, the actual potential support of $(j_0 - 1, j_0)$ is contained to the left of $i_1$, so we conclude that column $i''$ has at most two rows with potential support ending there, completing the proof of the claim.

It remains to analyze the possibility that we have a column of degree 3 and a column with $\delta_i = j_0$ in between $i_0$ and $i_1$. Recall that we have either $j_0 - 1 = 2$ and $\lambda_{j_0-1, j_0-1} = 1$, or $j_0 - 1 = 3$ and $\lambda_{j_0-1, j_0-1} = 2$. We first claim that in the latter case, we cannot have $\delta_i = j_0$ in between $i_0$ and $i_1$ without forcing there to
be two columns of degree 3 in between, or equivalently, forcing $\bar{\lambda}_{i_1}^2 + \bar{\lambda}_{i_1}^3 \geq 10$. Indeed, since we cannot have a gap between $a_{j_0}^{i_0}$ and $a_{j_0}^{i_0 - 1}$ for the swap, we must have $\bar{\lambda}_{i_0}^2 \geq 5$, and then for the same reason at $i_1$ we must have $\bar{\lambda}_{i_1}^2 \geq 6$. But having some $\delta_1 = j_0$ also requires $\bar{\lambda}_{i_1}^3 \geq 4$, so we conclude that we would necessarily have $\bar{\lambda}_{i_1}^2 + \bar{\lambda}_{i_1}^3 \geq 10$, as claimed. Thus, it suffices to treat the situation that $\lambda_{i_0-1,j_0-1} = 1$.

In this situation, we have $\bar{\lambda}_{i_0}^2 + \bar{\lambda}_{i_0}^3 \geq 6$, and we will move the third 3 to column $i_0$ and the fourth 3 to column $i_1$. We claim that we will have at most two rows with potentially appearing sections ending in $i_1$, and neither involves the exceptional row (specifically, $j_0 - 1$, which is 2). Thus, we will be able to eliminate all potentially appearing sections from the left and from the right of $i_0$, and finally eliminate the at most one potentially appearing section supported only at $i_0$. To verify the claim, we see that we necessarily have

\[ 5 \leq \bar{\lambda}_{i_1}^1 \leq 7, \quad \bar{\lambda}_{i_1}^2 = 3, \quad \text{and} \quad 1 \leq \bar{\lambda}_{i_1}^3 \leq 3. \]

We compute that the only rows satisfying the inequalities to potentially end at $i_1$ are $(3, 4)$, $(0, 6)$, $(1, 5)$, $(1, 6)$ and $(3, 5)$, but by the uniqueness part of Corollary 5.10, we see that the only way we can have three of these occurring at once is if we have $(3, 4)$, $(0, 6)$ and $(1, 5)$. However, we also have that $(3, 4)$ can only end if $\bar{\lambda}_{i_1}^3 \leq 2$, $(0, 6)$ can only end if $\bar{\lambda}_{i_1}^2 \leq 6$, and $(1, 5)$ can only end if one of the preceding two inequalities is strict. But together these imply that $\bar{\lambda}_{i_1}^1 + \bar{\lambda}_{i_1}^3 \leq 7$, meaning that we cannot have all the rows ending at $i_1$ under our hypothesis that the 4th 3 comes before $i_1$.

This concludes the case $\gamma_{i_0 - 1} = 3$.

**Case** $\gamma_{i_0 - 1} = 5$. We necessarily have $j_0 - 1 = \lambda_{i_0 - 1,j_0 - 1} + 2$, and since $\bar{\lambda}_{i_0 - 1}^2 + \bar{\lambda}_{i_0 - 1}^3 \geq 10$, we find that $j_0 - 1 = 4$ is the only possibility. But then if we move the fifth 3 to $i_0$, even if we obtain two rows involving $\delta_{i_0} = 5$ with potential support ending at $i_0$, we can have at most one more (necessarily of the form $(j, 6)$ for some $j$). Moreover, the $(4, 5)$ row is not one of these, as it will have potential support starting, not ending, at $i_0$. We can therefore still eliminate the block spanning from the first column with $\bar{\lambda}_{i_1}^2 + \bar{\lambda}_{i_1}^3 = 9$ to column $i_0$ just as before.

The case that $2\delta_{j_0 - 1} = c_{i_1 + 1} + 1$ but $2\delta_{j_0 - 1} < c_{i_0} - 1$ is handled completely symmetrically, completing the proof. \[\square\]

**8. Proofs in the degenerate case**

To conclude the proof of our main theorem, we show that there are always multidegrees such that on the one hand, the potentially appearing sections are still linearly independent, and on the other hand, tensors coming from any exact linked linear series must generate at least $\binom{r+2}{2} = 28$ linearly independent combinations of the potentially appearing sections. The key point is that even though there are cases where some row may not have any potentially appearing section occurring in our linked linear series in the chosen multidegree, in those cases we have to have more than one combination of sections from some other row. In fact, the arguments of this section are independent of $r$.

So far, in §4 we proved statements on existence of mixed sections, while in the following sections, we proved statements on linear independence of potentially appearing sections. These threads are related by the following.
Lemma 8.1. Suppose $s, s'$ are mixed sections of multidegrees $\text{md}_d(w)$ and $\text{md}_d(w')$, and let $\text{md}_d(w'')$ be another multidegree. Then $f_{w+w',w''}(s \otimes s')$ lies in the potential ambient space in multidegree $\text{md}(w'')$.

Proof. By definition of mixed sections as sums, it suffices to treat the case that $s$ is obtained purely from gluing together $s_j^1$ for fixed $j$, and $s'$ is obtained from gluing together $s_j'$ for fixed $j'$. But in this case the result is clear, since $f_{w+w',w''}(s \otimes s')$ must be a combination of potentially appearing sections from the $(j, j')$ row. □

The following lemma is convenient for cutting down the number of possibilities to consider.

Lemma 8.2. Let $s, s'$ be mixed sections of multidegrees $\text{md}(w)$ and $\text{md}(w')$ and types $(\vec{S}, \vec{j})$ and $(\vec{S}', \vec{j}')$ respectively. Suppose that for some $i$ with $1 < i < N$, we have $\ell_i \neq \ell_i'$ and $\ell_i' \neq \ell_i''$ such that $i \in S_{\ell_i} \cap S_{\ell_i'} \cap S'_{\ell_i''}$. Then for any unimaginative $w''$, the map $f_{w+w',w''}$ vanishes identically on $Z_i$.

If further either $\{j_{\ell_i}, j_{\ell_i'}\} = \{j'_{\ell_i}, j'_{\ell_i'}\}$ or $\{j_{\ell_i}, j_{\ell_i'}\} \cap \{j'_{\ell_i}, j'_{\ell_i'}\} = \emptyset$, then the same conclusion holds when $i = 1$ or $i = N$.

Proof. First consider the case $1 < i < N$, and write $w = (c_2, \ldots, c_N)$ and $w' = (c'_2, \ldots, c'_N)$. The hypotheses mean that $w$ allows for support of both $s_j^1$ and $s_j'$, for some distinct $j, j'$, so we need to have $a_j^1, a_{j'}^1 \geq c_i$ and $b_j^1, b_{j'}^1 \geq d - c_{i+1}$. Without loss of generality, suppose $a_j^1 < a_{j'}^1$. We must have either $a_j^1 + b_{j'}^1 < d$ or $a_j^1 + b_j^1 < d$. Then if $b_{j'}^1 > b_j^1$, we have either $c_{i+1} \geq d - b_j^1 > d - b_{j'}^1 > a_j^1 \geq c_i$ or $c_{i+1} \geq d - b_j^1 > a_j^1 > a_{j'}^1 \geq c_i$, so in either case we have $c_{i+1} \geq c_i + 2$. On the other hand, if $b_j^1 < b_{j'}^1$, we have $c_{i+1} \geq d - b_j^1 > d - b_{j'}^1 > a_j^1 \geq c_i$, so we again have $c_{i+1} \geq c_i + 2$. The same argument holds for $w'$, so we conclude that $c_{i+1} + c_{i+1}' \geq c_i + c_i' + 4$, which implies that $f_{w+w',w''}$ vanishes on $Z_i$, since if we write $w'' = (c''_2, \ldots, c''_N)$, the unimaginative hypothesis means that $c''_{i+1} \leq c_i + 3$.

Next, if $i = 1$, the unimaginative hypothesis means that $c_2$ is equal to 2 or 3, and it follows (see the proof of Theorem 7.3) that only the rows $(0, 0), (0, 1), (1, 1)$ and $(0, 2)$ can have potential support in the column, with not both $(0, 0)$ and $(1, 1)$ occurring. If $f_{w+w',w''}$ is nonzero on $Z_1$, then $f_{w+w',w''}(s \otimes s')$ must have $(j_{\ell_1}, j'_{\ell_1'})$ parts with potential support at $i = 1$ for $u = 1, 2$ and $v = 1, 2$, and this isn’t possible if either $\{j_{\ell_1}, j_{\ell_1'}\} = \{j'_{\ell_1}, j'_{\ell_1'}\}$ or $\{j_{\ell_1}, j_{\ell_1'}\} \cap \{j'_{\ell_1}, j'_{\ell_1'}\} = \emptyset$. The case $i = N$ is symmetric. □

We treat the case of a single swap as follows.

Proposition 8.3. Suppose a limit linear series contains precisely one swap, occurring between the rows $j_0, j_0 - 1$ in column $i_0$. Then for any multidegree $\omega$, with notation as in Proposition 4.10, the images in multidegree $\omega$ of the tensors of pairs of the $s_j$ for $j \neq j_0$, and $s_{j_0}^i, s_{j_0}'$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially appearing sections.

Note that in the proposition statement, we are not asserting that the actual global sections in multidegree $\omega$ are linearly independent, merely that the relevant vectors of coefficients (expressing the sections in question as combinations of the potentially appearing sections) are linearly independent.

Proof. For any row $(j, j')$ with neither $j, j'$ equal to $j_0$, since we have $s_j$ and $s_{j'}$ in our linked linear series, we obtain a nonzero contribution from an $s_{(j, j'), i}$. In
particular, considering all \(j, j' \neq j_0 - 1, j_0\) we obtain \(\binom{r}{2} + 2(r - 1) = \binom{r+2}{2} - 3\) independent combinations of potentially appearing sections, supported among the rows \((j, j')\) with \(j \neq j_0 - 1, j_0\). Finally, we consider the tensors of \(s_{j_0 - 1}, s'_{j_0}, s''_{j_0}\), and claim we obtain three distinct linear combinations, necessarily supported among the rows \((j_0 - 1, j_0 - 1), (j_0 - 1, j_0), (j_0, j_0)\). Consider the images of \(s'_{j_0} \otimes s''_{j_0}, s''_{j_0} \otimes s'_{j_0}, s'_{j_0} \otimes s''_{j_0}\). If any of their images contain any portion of the \((j_0, j_0)\) row, then considering \(s_{j_0 - 1} \otimes s_{j_0 - 1}, s_{j_0 - 1} \otimes s'_{j_0}, s_{j_0 - 1} \otimes s''_{j_0}\), the same argument as above shows we obtain two distinct combinations of type \((j_0 - 1, j_0 - 1)\) and/or \((j_0 - 1, j_0)\), so we are done. The only alternative is that the first three tensors come from the \((j_0 - 1, j_0 - 1)\), \((j_0 - 1, j_0)\) and \((j_0 - 1, j_0 - 1)\) rows respectively, with the first and last having disjoint support. Thus, in this case these three are all linearly independent, and we again obtain the desired conclusion.

We are now ready to prove the genus 22 case of Theorem 1.2; in fact, we will prove a more general statement for \(\rho = 1\) cases of the strong maximal rank conjecture.

The main point is that if we have a smoothing family \(\pi : X \to B\) as in Situation 3.2, and a generic linear series \((\mathcal{L}_B^g, V_B)\), which after base change and blowup we may assume is rational on the generic fiber, we can apply the linked linear series construction both to \((\mathcal{L}_B^g, V_B)\) and to \((W_B, \mathcal{L}_B^g \otimes \mathcal{O}_D)\), where \(W_B\) is the image of the multiplication map (1.1). Then we will have that for any multidegree \(\omega\) of total degree 2\(d\), and any multidegrees \(\omega', \omega''\) of total degree \(d\), and any sections \(s' \in V_{\omega'}\) and \(s'' \in V_{\omega''}\), then necessarily \(f_{\omega' + \omega''} \omega(s' \otimes s'')\) lies in \(W_B\); see the discussion following Situation 4.10 of [LOTZ17]. Thus, in order to give a lower bound on the rank of (1.1), we can choose many different \(\omega', \omega''\) and \(s', s''\), and show that they span a certain-dimensional subspace of \((\mathcal{L}_B^g \otimes \mathcal{O}_D)\).
lying over a refined limit linear series admits some multidegree $\omega$ such that the combined images $f_{\omega,+\omega''}(s' \otimes s'')$ span an $\binom{r+2}{2}$-dimensional space. For the $\omega$ in the statement, it then suffices to show that these sections give $\binom{r+2}{2}$ independent combinations of the potentially appearing sections. In this case, since $\rho = 1$, we can have at most one swap; see Remark 5.6. If we have no swaps, we obtained the desired independence directly from the independence of the potentially appearing sections, using Proposition 4.7. On the other hand, if we have a single swap, the desired result follows from Proposition 8.3.

Because we have proved the statement for all $X_0$ at once, we conclude the stronger assertion on the closure of $\mathcal{D}_{(g,r,d)}$ (see the proof of the last part of Corollary 4.11 of [LOTZ17] for details of a similar argument).

In particular, the genus-22 case of Theorem 1.2 follows immediately from Theorem 8.4 together with Theorem 7.3.

**Example 8.5.** We continue with the running genus-22 example of Examples 4.11 and 7.4. Observe that in the default multidegree, the (unique) potentially appearing section in row $(2,3)$ extends from the 7th column to the 11th column. This means that if $s'_3$ and $s''_3$ have the smallest possible portions coming from the $j = 3$ row, so that $s'_3$ only has nonzero $s'_1$ parts for $i \geq 8$ and $s''_3$ for $i \leq 10$, then the potentially appearing section for the $(2,3)$ row cannot come from either $s_2 \otimes s'_3$ or $s_2 \otimes s''_3$. This means that these sections (or more precisely, their images in multidegree $\omega_{\text{def}}$) are forced to yield potentially appearing sections from the $(2,2)$ row, with $s_2 \otimes s'_3$ necessarily yielding the one supported from columns 5 through 7, and $s_2 \otimes s''_3$ necessarily yielding the one supported in column 12. Thus, we explicitly see the lack of a $(2,3)$ section being offset by the inclusion of two independent $(2,2)$ sections.

We now move on to the $\rho = 2$ case, as needed for the genus-23 case of Theorem 1.2. Propositions 4.7 and 8.3 will still suffice to handle the cases that we have fewer than two swaps, so what remains is to analyze the four cases with two swaps, which we treat one by one. In all four cases, we will have swaps occurring in distinct columns $i_0 < i_1$, and we will find convenient to introduce shorthand notation as follows: we will write for instance

$$s'_{j_0} \otimes s''_{j_0+1} = (j_0 - 1, j_0 + 1)_L + (j_0 - 1, j_0)_R + (j_0, j_0 + 1)_L$$

to indicate that the image of $s'_{j_0} \otimes s''_{j_0+1}$ in the relevant multidegree is a combination of potentially appearing sections from the $(j_0 - 1, j_0 + 1)$, $(j_0 - 1, j_0)$ and $(j_0, j_0 + 1)$ rows, where the first must be supported strictly left of $i_0$, and the second strictly right of $i_1$, and the third has no restrictions on its support. We will also use subscripts $C$ to denote support strictly between $i_0$ and $i_1$, $LC$ to denote support strictly left of $i_1$, and $CR$ to denote support strictly right of $i_0$.

The first case to address is the following.

**Proposition 8.6.** Suppose that we are in the “repeated swap” case described in Proposition 4.13, so that our limit linear series contains precisely two swaps, and these both occur in the same pair of rows, say $j_0, j_0 - 1$. Then for any unimaginative multidegree $\omega$, with notation as in Proposition 4.13, the images in multidegree $\omega$ of the tensors of pairs of the $s_j$ for $j \neq j_0, j_0 - 1$, and $s'_{j_0 - 1}, s''_{j_0 - 1}, s'_1, s''_1$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially appearing sections.
Proof: Just as in the proof of Proposition 8.3, for \( j, j' \neq j_0, j_0 - 1 \), our linked linear series contains \( s_j \) and \( s_{j'} \), so the image of \( s_j \otimes s_{j'} \) always gives a potentially appearing section from row \((j, j')\).

Now consider \( j \neq j_0, j_0 - 1 \); we claim that \( s_j \otimes s_{j_0}^{s_1}, s_j \otimes s_{j_0}'' \) cannot all coincide, and hence have a two-dimensional span. Indeed, if \( s_j \otimes s_{j_0}^{s_1} \) coincides with \( s_j \otimes s_{j_0}' \), they must be of the form \((j, j_0 - 1_l) + (j, j_0)_R\). But the former cannot occur in \( s_j \otimes s_{j_0}'' \), and the latter cannot occur in \( s_j \otimes s_{j_0}^{s_1} \), so we obtain the desired independence for these sections.

It remains to show that we have at least three independent sections among all tensors of the \( s_{j_0}^{s_1}, s_{j_0}'' \). We first consider the four tensor squares; according to Lemma 8.2, these can only contain types \((j_0 - 1, j_0 - 1)\) and \((j_0, j_0)\), with no type \((j_0 - 1, j_0)\) appearing. Now, the possible \((j_0, j_0)\) parts of \( s_{j_0}^{s_1} \) and \( s_{j_0}'' \) are disjoint, so we conclude that either these two are distinct, or they are of pure type \((j_0 - 1, j_0 - 1)\). Similarly, the sections \( s_{j_0}^{s_1} \) and \( s_{j_0}'' \) are either distinct or of pure type \((j_0, j_0)\). Thus, it suffices to show that we cannot have all of our tensors in the span of a single pair of sections, each of pure type \((j_0 - 1, j_0 - 1)\) or \((j_0, j_0)\). Now, \( s_{j_0}^{s_1} \otimes s_{j_0}' \) cannot have a \((j_0 - 1, j_0 - 1)\) part, and \( s_{j_0}'' \otimes s_{j_0}'' \) cannot have a \((j_0, j_0)\) part, so the only possibility to consider is that one of our sections is purely of type \((j_0 - 1, j_0 - 1)\), and the other is purely of type \((j_0, j_0)\).

If the \((j_0 - 1, j_0 - 1)\) part occurs in \( s_{j_0}'' \otimes s_{j_0}' \), it must be supported strictly to the left of \( i_0 \). Then \( s_{j_0}'' \otimes s_{j_0}' \) cannot have a \((j_0 - 1, j_0 - 1)\) part, so must be of type \((j_0, j_0)\), and the support must be strictly to the right of \( i_1 \). On the other hand, if the \((j_0, j_0)\) part occurs in \( s_{j_0}'' \otimes s_{j_0}' \), it must again be supported strictly to the right of \( i_1 \), and then \( s_{j_0}'' \otimes s_{j_0}' \) cannot have a \((j_0, j_0)\) part, so must be of type \((j_0 - 1, j_0 - 1)\), again supported to the left of \( i_0 \). But in either case, \( s_{j_0}'' \otimes s_{j_0}' \) cannot be a linear combination of these two sections, as desired.

We now start imposing that \( \rho = 2 \) and that \( X_0 \) is left-weighted. Roughly speaking, the first gives us control over potential support of sections, while the second ensures that the actual support occurs where we want it to.

**Proposition 8.7.** Suppose that \( \rho = 2 \), and \( \omega \) is an unimaginative multidegree. Then:

1. In the “disjoint swap” situation treated in Proposition 4.12, suppose without loss of generality that \( i_0 < i_1 \). Then we necessarily have that \( i_0 \) and \( i_1 \) both have genus 1, the two swaps are minimal, and no rows are exceptional except \( r_0 = 1 \) at \( i_0 \) and \( r_1 = 1 \) at \( i_1 \). In multidegree \( \omega \), the potential support of every \((j, j')\) is connected except possibly for \((j_0 - 1, j_0 - 1)\), \((j_1 - 1, j_1 - 1)\), and \((j_0 - 1, j_1 - 1)\). Moreover, if \((j_0 - 1, j_1)\) has disconnected potential support in multidegree \( \omega \), the potential support must be made up of two components, one contained strictly to the right of \( i_1 \), and one contained strictly to the left of \( i_0 \), and the potential support of \((j_0 - 1, j_1)\) is contained strictly right of \( i_1 - 1 \), and the potential support of \((j_0, j_1 - 1)\) is contained strictly left of \( i_0 + 1 \). Finally, if the potential support of \((j_0 - 1, j_1)\) is contained strictly left of \( i_0 \), then \((j_0 - 1, j_1 - 1)\) must also have a component of potential support contained strictly left of \( i_0 \), and if the potential support of \((j_0, j_1 - 1)\) is contained strictly right of \( i_1 \), then \((j_0 - 1, j_1 - 1)\) must also have a component of potential support contained strictly right of \( i_1 \).
(2) In the “first 3-cycle” situation described in Proposition 4.14, we necessarily have that \( i_0 \) and \( i_1 \) both have genus 1, the two swaps are minimal, and no rows are exceptional except row \( j_0 \) at \( i_0 \) and row \( j_0 - 1 \) at \( i_1 \). In multidegree \( \omega \), the potential support of every In multidegree \( \omega \), the potential support of every \( (j, j') \) is contained except possibly for \( (j_0 - 1, j_0 - 1) \), \( (j_0 - 1, j_0) \), and \( (j_0, j_0) \). Moreover, if for some \( j \), the potential support of \( (j, j_0) \) has a component strictly to the left of \( i_0 \), then the potential support of \( (j, j_0 - 1) \) is contained to the left of \( i_0 \), and if the potential support of \( (j, j_0 - 1) \) has a component strictly to the right of \( i_1 \), then the potential support of \( (j, j_0) \) is entirely contained strictly to the right of \( i_1 \).

Finally, if \( (j_0 - 1, j_0) \) has potential support contained entirely strictly to the left of \( i_1 \), then the potential support of \( (j_0 - 1, j_0 + 1) \) cannot be contained to the right of \( i_1 \); if it has potential support contained entirely strictly to the right of \( i_0 \), then the potential support of \( (j_0, j_0 + 1) \) cannot be contained to the left of \( i_0 \); and if it has potential support contained entirely strictly between \( i_0 \) and \( i_1 \), then \( (j_0 - 1, j_0 - 1) \) has potential support contained entirely strictly to the left of \( i_1 \), and \( (j_0, j_0) \) has potential support contained entirely strictly to the right of \( i_0 \).

**Proof.** We write as usual \( \omega = \text{md}(w) \) with \( w = (c_2, \ldots, c_g) \).

(1) The first assertions follow from Proposition 7.8, and following the proof we see further in order for \( (j_0 - 1, j_1 - 1) \) to have disconnected support, the support must be split between strictly right of \( i_1 \) and strictly left of \( i_0 \), as claimed. Next, if the potential support of \( (j_0 - 1, j_1 - 1) \) has a component lying strictly right of \( i_1 \), then we have

\[
a^{i_1}_{(j_0 - 1, j_1)} = a^{i_1 + 1}_{(j_0 - 1, j_1)} - 1 = a^{i_1 + 1}_{(j_0 - 1, j_1 - 1)} - 2 > c_{i_1 + 1} - 2 \geq c_{i_1},
\]

which implies (using our previous connectedness statement) that the potential support of \( (j_0 - 1, j_1) \) is supported strictly to the right of \( i_1 - 1 \), as desired. The corresponding statement on support left of \( i_0 \) and \( i_0 + 1 \) follows similarly. Finally, if the potential support of \( (j_0 - 1, j_1) \) is contained strictly left of \( i_0 \), then we have

\[
a^{i_0}_{(j_0 - 1, j_1 - 1)} < a^{i_0}_{(j_0 - 1, j_1)} \leq c_{i_0},
\]

so \( (j_0 - 1, j_1 - 1) \) must also have a component of potential support strictly left of \( i_0 \), as desired. The last statement on support strictly right of \( i_1 \) follows similarly.

(2) Most of the argument is similar to (1). For the support of \( (j, j_0) \) to have a component strictly to the left of \( i_0 \) and \( i_0 + 1 \) follows similarly. Finally, if the potential support of \( (j_0, j_0) \) is connected and strictly to the left of \( i_0 \). The statement on support to the right of \( i_1 \) is proved in exactly the same way. For the last assertion, note that the \( (j_0 - 1, j_0 - 1) \) row has no support at \( i_1 \), and the \( (j_0, j_0) \) has no support at \( i_0 \), since both sum to \( 2d - 4 \) in the relevant columns. \( \square \)

**Proposition 8.8.** Suppose that \( X_0 \) is left-weighted, and that the rows \( j, j' \) have no exceptional behavior in any genus-0 columns. Then the image of \( s_j \otimes s_{j'} \) in any unimaginative multidegree \( \omega \) is equal to the leftmost potentially appearing section in the \( (j, j') \) row.
Proof. The lack of exceptional behavior away from genus-1 components means that the $a_{i,j}^{(j,j')}_{0}$ are constant on the genus-0 components. The idea is then that the left-weighting means that the leftmost negative value of $a_{i,j}^{(j,j')}_{0} - c_{i}$ is repeated so many times that it must lead to a strict minimum of the partial sums. Compare the proof of Proposition 4.18, where in (4.1) we now replace $d$ by $2d$ due to having passed to the tensor square. □

Proposition 8.9. Suppose that we are in the “disjoint swap” case described in Proposition 4.12, so that our limit linear series contains precisely two swaps, and these occur in disjoint pairs of rows, say $j_{0}, j_{0} - 1$ and $j_{1}, j_{1} - 1$. Suppose further that $\rho = 2$, and that $X_{0}$ is left-weighted. Then for any unimaginative multidegree $\omega$, with notation as in Proposition 4.12, if we suppose that we have chosen $s_{j_{0}}^{t}$ and $s_{j_{1}}^{t}$, as allowed by Proposition 4.18, then the images in multidegree $\omega$ of the tensors of pairs of the $s_{j}$ for $j \neq j_{0}, j_{1}$, and $s_{j_{0}}^{t}, s_{j_{0}}^{t}, s_{j_{1}}^{t}, s_{j_{1}}^{t}$ contain $(\tau + 2)^{3}$ independent linear combinations of the potentially appearing sections.

Proof. Without loss of generality, assume that $i_{0} < i_{1}$. First note that by Proposition 8.7 (1), the hypothesis that $\rho = 2$ means that in order to have two swaps, they both must occur at genus-1 components. Then by Proposition 4.18, we may assume that $s_{j_{1}}^{t}$ is controlled, and that the $j_{1}$-part of $s_{j}^{t}$ does not contain any genus-1 component left of $i_{1}$. We also have that every $(j, j')$ has connected potential support unless $j, j' \in \{j_{0} - 1, j_{1} - 1\}$.

Now, if we have $j, j' \neq j_{0}, j_{0} - 1, j_{1}, j_{1} - 1$, then we know that $f_{w_{j} + w_{j'}, w}(s_{j} \otimes s_{j'})$ is nonzero and composed of $s_{j}^{t}$. Now, suppose $j \neq j_{0}, j_{0} - 1, j_{1}, j_{1} - 1$. Then the same argument as in Proposition 8.3 also shows that if we consider the images in multidegree $\omega$ of $s_{j} \otimes s_{j_{0} - 1}$, $s_{j} \otimes s_{j_{0}}$, and $s_{j} \otimes s_{j_{0}}'$, we either obtain one section of type $(j, j_{0} - 1)$ and one with a contribution of type $(j, j_{0})$, or two sections of type $(j, j_{0} - 1)$, but having disjoint support. The same holds with $j_{1}$ in place of $j_{0}$. Together, these produce $(\tau + 2)^{3} + 4(r - 3) = (\tau + 2)^{3} - 10$ linearly independent combinations. It thus suffices to show that we have 10 linearly independent combinations coming from tensor products of pairs of the sections $s_{j_{0} - 1}, s_{j_{0}}^{t}, s_{j_{0}}^{t}, s_{j_{1} - 1}, s_{j_{1}}^{t}, s_{j_{1}}^{t}$. Just as in the proof of Proposition 8.3, tensor products of the first three sections yield three independent combinations, with contributions contained among the types $(j_{0} - 1, j_{0} - 1), (j_{0} - 1, j_{1}),$ and $(j_{0}, j_{0})$. Tensor products of the last three sections likewise yield three combinations, with $j_{1}$ replacing $j_{0}$ in the types.

It remains to consider the tensors with types contained among $(j_{0} - 1, j_{1} - 1), (j_{0} - 1, j_{1}), (j_{0}, j_{1} - 1)$ and $(j_{0}, j_{1})$. First suppose that $(j_{0} - 1, j_{1} - 1)$ has connected potential support in multidegree $\omega$. Then just as in the single-swap case, at least one of $s_{j_{0} - 1} \otimes s_{j_{0}}^{t}, s_{j_{0} - 1} \otimes s_{j_{0}}^{t}$ must involve a $(j_{0} - 1, j_{1})$ part, and at least one of $s_{j_{0}}^{t} \otimes s_{j_{1} - 1}, s_{j_{0}}^{t} \otimes s_{j_{1} - 1}$ must involve a $(j_{0}, j_{1} - 1)$ part. Since $s_{j_{0} - 1} \otimes s_{j_{1} - 1}$ is pure of type $(j_{0} - 1, j_{1} - 1)$, and all of these have unique potential support, we find that the span of these sections contains the (unique) pure types of each of $(j_{0} - 1, j_{1} - 1), (j_{0}, j_{1} - 1)$ and $(j_{0}, j_{1})$. Thus, if we have anything with a nonzero part of type $(j_{0}, j_{1})$, this gives a fourth independent combination. On the other hand, if nothing has a $(j_{0}, j_{1})$ part, then we must have the following:

\[s_{j_{0}}^{t} \otimes s_{j_{1}}^{t} = (j_{0} - 1, j_{1})_{L} + (j_{0}, j_{1} - 1)_{R},\]

\[s_{j_{0}}^{t} \otimes s_{j_{1}}^{t} = (j_{0} - 1, j_{1} - 1)_{L} + (j_{0}, j_{1} - 1)_{LC} + (j_{0} - 1, j_{1})_{L},\]

\[s_{j_{0}}^{t} \otimes s_{j_{1}}^{t} = (j_{0} - 1, j_{1})_{CR} + (j_{0} - 1, j_{1} - 1)_{R} + (j_{0}, j_{1} - 1)_{R}.\]
First consider the possibility that the \((j_0 - 1, j_1)\) part of \(s_{j_0}^{i_0} \otimes s_{j_1}^{i_1}\) is nonzero. Then by Proposition 8.7 (1), we have that \((j_0, j_1 - 1)\) has support strictly left of \(i_0\) too, which in turn means that \((j_0, j_1 - 1)\) can’t have support strictly right of \(i_1\). But this leaves no possibility for \(s_{j_0}^{i_0} \otimes s_{j_1}^{i_1}\). On the other hand, if the \((j_0, j_1 - 1)\) part of \(s_{j_0}^{i_0} \otimes s_{j_1}^{i_1}\) is nonzero, we have that \((j_0 - 1, j_1 - 1)\) must have support strictly right of \(i_1\), and hence that \((j_0 - 1, j_1)\) can’t have support strictly left of \(i_0\), leaving no possibility for \(s_{j_0}^{i_0} \otimes s_{j_1}^{i_1}\). We conclude that it is not possible for these tensors not to have some \((j_0, j_1)\) part, giving the desired four independent combinations when \((j_0 - 1, j_1 - 1)\) has connected potential support.

It remains to treat the case that \((j_0 - 1, j_1 - 1)\) has disconnected potential support in multidegree \(\omega\). Then Proposition 8.7 tells us that this potential support has two parts, contained strictly left of \(i_0\) and right of \(i_1\) respectively. Moreover, it says that the potential support of \((j_0 - 1, j_1)\) is contained strictly right of \(i_1 - 1\) and the potential support of \((j_0, j_1 - 1)\) is contained strictly left of \(i_0 + 1\). This forces \(s_{j_0}^{i_0} \otimes s_{j_1}^{i_1}\) to be of pure \((j_0, j_1)\) type. Now, we observe that two of the sections \(s_{j_0}^{i_0 - 1} \otimes s_{j_1 - 1}^{i_1}, s_{j_0}^{i_0 - 1} \otimes s_{j_1}^{i_1}, s_{j_0 - 1}^{i_0} \otimes s_{j_1}^{i_1}\) must be independent, either involving a \((j_0 - 1, j_1)\) part and a \((j_0 - 1, j_1 - 1)\) part, or two \((j_0 - 1, j_1 - 1)\) parts. Similarly, \(s_{j_0}^{i_0} \otimes s_{j_1 - 1}\) and \(s_{j_0}^{i_0 - 1} \otimes s_{j_1 - 1}\) must either involve a \((j_0, j_1 - 1)\) part or two \((j_0 - 1, j_1 - 1)\) parts.

We see that the only way to avoid having four independent combinations would be if these five tensors are all of pure type \((j_0 - 1, j_1 - 1)\), necessarily achieving support independently both on the left and right. But we note that because the potential support of \((j_0, j_1 - 1)\) is contained strictly left of \(i_0 + 1\), and because (in the disconnected support case) we must have \(a_{(j_0 - 1, j_1 - 1)}^{i_0 - 1, i_1 - 1} = c_i\) for \(i_0 < i \leq i_1\), the only way that \(s_{j_0}^{i_0 - 1} \otimes s_{j_1 - 1}\) can fail to have a \((j_0, j_1 - 1)\) part is if \(s_{j_0}^{i_0 - 1}\) is not controlled, and more specifically if its \(j_0\) portion does not extend more than halfway to the next genus-1 component after \(i_0\). On the other hand, \(s_{j_1}^i\) is controlled and has \(j_1\) part not containing any genus-1 component smaller than \(i_1\), so we conclude that in this situation its \(j_1\) part is disjoint from the \(j_0\) part of \(s_{j_0}^{i_0 - 1}\), and then \(s_{j_0}^{i_0 - 1} \otimes s_{j_1}^{i_1} = (j_0, j_1 - 1) + (j_0 - 1, j_1)\), and gives a fourth independent combination. This completes the proof of the proposition.

We now move on to consider the two remaining cases, both involving a pair of swaps in overlapping columns.

**Proposition 8.10.** Suppose that we are in the “first 3-cycle” situation described in Proposition 4.14, so that our limit linear series contains precisely two swaps, with one swap between the \(j_0\)th and \((j_0 + 1)\)st rows occurring in the \(i_0\)th column, and a second swap between the \((j_0 - 1)\)st and \((j_0 + 1)\)st rows in the \(i_1\)st column for some \(i_1 > i_0\). Suppose further that \(\rho = 2\), and that we have an unimaginary multidegree \(\omega\) such that the \((j_0 - 1, j_0)\) row has a unique potentially appearing section in multidegree \(\omega\), whose support does not contain \(i_0\) or \(i_1\). Then with notation as in Proposition 4.14, the images in multidegree \(\omega\) of the tensors of pairs of the \(s_j\) for \(j \neq j_0 + 1\), and \(s_{j_0 + 1}^j, s_{j_0 + 1}^{j''}, s_{j_0 + 1}^{j'''}\) contain \(\binom{\rho + 2}{2}\) independent linear combinations of the potentially appearing sections.

**Proof.** We first show that for \(j \neq j_0 - 1, j_0, j_0 + 1\), the sections

\[ s_j \otimes s_{j_0}^{i_0 - 1}, s_j \otimes s_{j_0}^{i_0}, s_j \otimes s_{j_0 + 1}^{i_0 - 1}, s_j \otimes s_{j_0 + 1}^{i_0}, s_j \otimes s_{j_0 + 1}^{i_0 + 1} \]

must yield at least three independent combinations. But the first two tensors yield \((j, j_0 - 1)\) and \((j, j_0)\) parts, so if any of the last three have any \((j, j_0 + 1)\) part, we
obtain the desired independence. On the other hand, if not we find that

\[ s_j \otimes s'_{j_0+1} = (j, j_0 - 1)L_C + (j, j_0)L; \]
\[ s_j \otimes s''_{j_0+1} = (j, j_0 - 1)R + (j, j_0)C_R; \]
\[ s_j \otimes s'''_{j_0+1} = (j, j_0)L + (j, j_0 - 1)R. \]

If the \((j, j_0)L\) part of the last tensor is nonzero, then by Proposition 8.7, the potential support of both the \((j, j_0 - 1)\) and \((j, j_0)\) rows are connected and contained strictly to the left of \(i_0\), leaving no possibility for the second tensor. But if the \((j, j_0 - 1)R\) part of the last tensor is nonzero, then we similarly have that the potential support of both the \((j, j_0 - 1)\) and \((j, j_0)\) rows are contained strictly to the right of \(i_1\), leaving no possibility for the first tensor. Thus, we reach a contradiction, and conclude that we must obtain a \((j, j_0 + 1)\) part, giving the desired three independent combinations.

Next, we consider the 15 tensors arising from

\[ s_{j_0-1}, s_{j_0}, s'_{j_0+1}, s''_{j_0+1}, s'''_{j_0+1}; \]

we need to show that these yield 6 independent linear combinations. By hypothesis, we have that the potential support of the \((j_0 - 1, j_0)\) row is connected and does not contain \(i_0\) or \(i_1\), so we organize cases according to its support. First suppose that the support of the \((j_0 - 1, j_0)\) row is entirely to the left of \(i_0\); then according to Proposition 8.7, the same holds for the \((j_0 - 1, j_0 - 1)\) row, and the \((j_0 - 1, j_0 + 1)\) row cannot have its support to the right of \(i_1\). We then see that \(s_{j_0-1} \otimes s''_{j_0+1}\) cannot have any \((j_0 - 1, j_0 - 1)\) or \((j_0 - 1, j_0)\) parts, so must be of \((j_0 - 1, j_0 + 1)\) type. Similarly, \(s''_{j_0+1} \otimes s'''_{j_0+1}\) cannot have any \((j_0 - 1, j_0 - 1), (j_0 - 1, j_0), (j_0 - 1, j_0 + 1)\) parts, so it must contain \((j_0, j_0 + 1)\) or \((j_0 + 1, j_0 + 1)\) parts. In addition, the pair \(s_{j_0} \otimes s''_{j_0+1}\) and \(s_{j_0} \otimes s'''_{j_0+1}\) must contain either a \((j_0, j_0 + 1)\) part, or two distinct \((j_0, j_0)\) parts, supported left and right of \(i_0\), respectively. Given that we always have \((j_0 - 1, j_0 - 1), (j_0 - 1, j_0)\) and \((j_0, j_0)\) parts, the only way we could fail to have produced six independent combinations is if \(s''_{j_0+1} \otimes s'''_{j_0+1}\) has type \((j_0, j_0 + 1)\), and we have only one \((j_0, j_0)\) part. But then considering \(s''_{j_0+1} \otimes s'''_{j_0+1}\) and using Lemma 8.2, we find that we must produce a \((j_0 + 1, j_0 + 1)\) part or two distinct \((j_0, j_0)\) parts, so we necessarily obtain the sixth combination.

Similarly, if the potential support of the \((j_0 - 1, j_0)\) row is entirely to the right of \(i_1\), then Proposition 8.7 tells us that the same holds for \((j_0, j_0)\), and that the potential support of the \((j_0, j_0 + 1)\) row cannot be to the left of \(i_0\). Then \(s_{j_0} \otimes s'_{j_0+1}\) must be of \((j_0, j_0 + 1)\) type, and \(s'_{j_0+1} \otimes s''_{j_0+1}\) must have \((j_0, j_0 + 1)\) or \((j_0 + 1, j_0 + 1)\) parts. The pair \(s_{j_0-1} \otimes s'_{j_0+1}\) and \(s_{j_0-1} \otimes s''_{j_0+1}\) must contain either a \((j_0 - 1, j_0 + 1)\) part, or two distinct \((j_0 - 1, j_0 - 1)\) parts, and in either case the tensors \(s''_{j_0+1} \otimes s'''_{j_0+1}\) and \(s''_{j_0+1} \otimes s'''_{j_0+1}\) together with the usual tensors of \(s_{j_0-1}\) and \(s_{j_0}\) must complete the six independent combinations.

Finally, if the potential support of the \((j_0 - 1, j_0)\) row is between the \(i_0\) and \(i_1\) columns, then by Proposition 8.7, we know that the potential support of \((j_0 - 1, j_0 - 1)\) is left of \(i_1\) and the potential support of \((j_0, j_0)\) is right of \(i_0\). We then see that the tensors \(s_{j_0-1} \otimes s''_{j_0+1}, s_{j_0} \otimes s''_{j_0+1}\), and \(s''_{j_0+1} \otimes s'''_{j_0+1}\) must be pure of types \((j_0 - 1, j_0 + 1), (j_0, j_0 + 1), (j_0 + 1, j_0 + 1)\) respectively, yielding the desired six combinations. □
Proposition 8.11. Suppose that we are in the “second 3-cycle” situation described in Proposition 4.15, so that our limit linear series contains precisely two swaps, with one swap between the \((j_0 - 1)st\) and \(j_0 th\) rows occurring in the \(i_0 th\) column, and a second swap between the \((j_0 - 1)st\) and \((j_0 + 1)st\) rows in the \(i_1 st\) column for some \(i_1 > i_0\). Suppose further that \(\rho = 2\), that \(X_0\) is left-weighted, and that we have an unimaginative \(w = (c_2, \ldots, c_N)\) satisfying one of the following three conditions:

1. the \((j_0 - 1), j_0 - 1)\) row does not have potentially appearing sections both left of \(i_0\) and right of \(i_1\); or
2. \(2a_{j_0-1}^0 = c_{j_0} - 1\), and \(2a_{j_0+1}^0 = c_{j_0+1} + 1\); or
3. \(2a_{j_0-1}^0 = c_{j_0} - 2\), and \(2a_{j_0+1}^0 = c_{j_0+1} + 2\), and \(w\) has degree 2 in both \(i_0\) and \(i_1\).

Then with notation as in Proposition 4.15, the images in multidegree \(\text{md}(w)\) of the tensors of pairs of the \(s_j\) for \(j \neq j_0, j_0 + 1\), and \(s_j', s_j'', s_{j_0+1}', s_{j_0+1}'', s''\) contain \(\binom{t+2}{2}\) independent linear combinations of the potentially appearing sections.

Proof. First suppose \(j \neq j_0 - 1, j_0, j_0 + 1\): we show that we can always obtain three linearly independent combinations of potential appearing sections from the rows \((j, j_0 - 1)\), \((j, j_0)\) and \((j, j_0 + 1)\). \(s_j \otimes s_{j_0-1}\) always yields a pure \((j, j_0 - 1)\) part. If \(S_2' = S_2'' = \{1, \ldots, N\}\), then \(s_j \otimes s_{j_0}'\) has a nonzero \((j, j_0)\) part and no \((j, j_0 + 1)\) part, while \(s_j \otimes s_{j_0+1}'\) has a nonzero \((j, j_0 + 1)\) part, so we get the desired three combinations. Otherwise, we have

\[
\begin{align*}
    s_j \otimes s_{j_0}' &= (j, j_0 - 1)L + (j, j_0) \\
    s_j \otimes s_{j_0}' &= (j, j_0) + (j, j_0 + 1)R + (j, j_0 - 1)CR \\
    s_j \otimes s_{j_0+1}' &= (j, j_0 - 1)L + (j, j_0)LC + (j, j_0 + 1) \\
    s_j \otimes s_{j_0+1}' &= (j, j_0 + 1) + (j, j_0 - 1)R \\
    s_j \otimes s_{j_0+1}' &= (j, j_0) + (j, j_0 - 1)C + (j, j_0 + 1),
\end{align*}
\]

where \(R'\) and \(L'\) denote possible support at and right of \(i_1\) and at and left of \(i_0\), respectively, and if \(s_j \otimes s_{j_0}'\) has a nonzero \((j, j_0 + 1)\) part with support containing \(i_1\), its \((j, j_0)\) part must be nonzero, and similarly for the \((j, j_0)\) and \((j, j_0 + 1)\) parts of \(s_j \otimes s_{j_0+1}'\). Now, suppose that \((j, j_0 - 1)\) has connected potential support which is not contained strictly right of \(i_0\). Then \((j, j_0 + 1)\) cannot have any potential support strictly right of \(i_1\) without also forcing \((j, j_0 - 1)\) to have potential support strictly right of \(i_1\), so the \((j, j_0)\) part of \(s_j \otimes s_{j_0}'\) must be nonzero. But then adding \(s_j \otimes s_{j_0+1}' = (j, j_0 + 1)\) and \(s_j \otimes s_{j_0-1}'\) yields three independent sections. Similarly, if \((j, j_0 - 1)\) has connected potential support not contained strictly left of \(i_1\), then \((j, j_0)\) cannot have potential support strictly left of \(i_0\), so \(s_j \otimes s_{j_0+1}'\) has nonzero \((j, j_0 + 1)\) part, and adding \(s_j \otimes s_{j_0}' = (j, j_0)\) and \(s_j \otimes s_{j_0-1}'\) yields the desired combinations. For connected potential support, the only remaining possibility is that \((j, j_0 - 1)\) has potential support strictly between \(i_0\) and \(i_1\), in which case \(s_j \otimes s_{j_0}' = (j, j_0)\) and \(s_j \otimes s_{j_0+1}' = (j, j_0 + 1)\).

Finally, since \(\rho = 2\), the only remaining possibility is that \((j, j_0 - 1)\) has potential support both left of \(i_0\) and right of \(i_1\), and in this case we must have \(a_{(j, j_0-1)}^0 = c_{j_0} - 1\) and \(a_{(j, j_0+1)}^0 = c_{j_0+1} + 1\). Then \((j, j_0 + 1)\) cannot have potential support strictly right of \(i_1\), and \((j, j_0)\) cannot have potential support strictly left of \(i_0\), so as above we find that if the \((j, j_0 + 1)\) part of \(s_j \otimes s_{j_0}'\) is nonzero (necessarily with support at \(i_1\)), then
the \((j, j_0)\) part must also be nonzero, and if the \((j, j_0)\) part of \(s_j \otimes s'_{j_0+1}\) is nonzero, then the \((j, j_0 + 1)\) part must also be nonzero. Now, we have \(s_j \otimes s'_{j_0}\) and \(s_j \otimes s''_{j_0+1}\) linearly independent always, and the only way they could fail to be independent from \(s_j \otimes s''_1\) is either \(s_j \otimes s'_{j_0} = (j, j_0)\) or \(s_j \otimes s''_{j_0+1} = (j, j_0+1)\), while the only way they could fail to be independent from \(s_j \otimes s'_{j_0-1}\) if is either \(s_j \otimes s'_{j_0} = (j, j_0 - 1)\) or \(s_j \otimes s''_{j_0+1} = (j, j_0)\). If \(s_j \otimes s''_{j_0} = (j, j_0)\) and \(s_j \otimes s''_{j_0+1} = (j, j_0 - 1)\), we see that \(s_j \otimes s''_{j_0+1}\) necessarily gives a third independent combination, while if \(s_j \otimes s'_{j_0} = (j, j_0 - 1)\) and \(s_j \otimes s''_{j_0+1} = (j, j_0 + 1)\), we see that \(s_j \otimes s''_{j_0}\) necessarily gives a third independent combination.

It remains to show that we can get six independent combinations from the rows \((j_0 - 1, j_0 - 1), (j_0 - 1, j_0), (j_0 - 1, j_0 + 1), (j_0, j_0), (j_0, j_0 + 1),\) and \((j_0 + 1, j_0 + 1)\). If \(S'_2 = S''_2 = \{1, \ldots, N\}\), then we immediately get the six tensors coming from \(s_{j_0-1} \otimes s'_{j_0}, s''_{j_0}, s''_{j_0+1}\). Note that we are making use of Lemma 8.2 in the case of self-tensors.

\[
\begin{align*}
  s_{j_0-1} \otimes s'_{j_0} &= (j_0 - 1, j_0 - 1)_L + (j_0 - 1, j_0) \\
  s_{j_0-1} \otimes s''_{j_0} &= (j_0 - 1, j_0) + (j_0 - 1, j_0 + 1)_R + (j_0 - 1, j_0 - 1)_CR \\
  s_{j_0-1} \otimes s''_{j_0+1} &= (j_0 - 1, j_0 - 1)L_C + (j_0 - 1, j_0)_L + (j_0 - 1, j_0 + 1) \\
  s_{j_0} \otimes s'_{j_0} &= (j_0 - 1, j_0) + (j_0, j_0) \\
  s''_{j_0} \otimes s'_{j_0} &= (j_0, j_0) + (j_0 + 1, j_0 + 1)_R + (j_0 - 1, j_0 - 1)_CR \\
  s''_{j_0} \otimes s''_{j_0} &= (j_0 - 1, j_0) + (j_0, j_0) + (j_0, j_0 + 1)_R \\
  s''_{j_0} \otimes s''_{j_0+1} &= (j_0 - 1, j_0 - 1)L_C + (j_0, j_0)_L + (j_0 + 1, j_0 + 1) \\
  s'_j \otimes s'_{j_0+1} &= (j_0 - 1, j_0 - 1)_L + (j_0 - 1, j_0)_L + (j_0 - 1, j_0 + 1)_L + (j_0, j_0)_L + (j_0, j_0 + 1)_R \\
  s''_{j_0+1} \otimes s''_{j_0+1} &= (j_0 - 1, j_0 + 1)_L + (j_0 - 1, j_0 - 1)_R + (j_0, j_0 + 1) \\
  s''_{j_0} \otimes s''_{j_0+1} &= (j_0 - 1, j_0 - 1)_R + (j_0 - 1, j_0)_R + (j_0 - 1, j_0 + 1)_CR \\
  &\quad + (j_0, j_0 + 1) + (j_0 + 1, j_0 + 1)_R.
\end{align*}
\]

As above, we separate out cases by the potential support of the \((j_0 - 1, j_0 - 1)\) row. Note that because the entries sum up to \(2d - 4\) in both the \(i_0\) and \(i_1\) columns, the \((j_0 - 1, j_0 - 1)\) row cannot have any potential support in either of these columns in any unimaginative multidegree. First suppose the potential support is strictly to the left of \(i_0\). In this case none of the relevant rows can have potential support extending right of \(i_1\), so we get \(s_{j_0-1} \otimes s''_{j_0+1} = (j_0 - 1, j_0 + 1), s''_{j_0} \otimes s''_{j_0} = (j_0, j_0),\) and \(s''_{j_0+1} \otimes s''_{j_0+1} = (j_0 + 1, j_0 + 1)\), and the \((j_0 - 1, j_0)\) part of \(s_{j_0-1} \otimes s''_{j_0}\) must be nonzero. We also have \(s'_{j_0} \otimes s''_{j_0+1} = (j_0 - 1, j_0 + 1)_L + (j_0, j_0 + 1)\) and \(s''_{j_0} \otimes s''_{j_0+1} = (j_0 - 1, j_0 + 1)_CR + (j_0, j_0 + 1) + (j_0 + 1, j_0 + 1)_R\).
(which won’t happen when $\rho = 2$), and in either case together with $s_{j_0-1} \otimes s_{j_0-1}$ we get the desired six independent combinations.

Similarly, if the potential support of the $(j_0-1, j_0-1)$ row is strictly to the right of $i_1$, we will have $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0)$, $s'_{j_0} \otimes s'_{j_0} = (j_0, j_0)$, $s'_{j_0+1} \otimes s'_{j_0+1} = (j_0+1, j_0+1)$, with $s_{j_0-1} \otimes s'_{j_0+1}$ having a nonzero $(j_0-1, j_0+1)$ part, and $s_{j_0} \otimes s''_{j_0+1} = (j_0-1, j_0)R + (j_0, j_0+1)$ and $s_{j_0} \otimes s''_{j_0+1} = (j_0-1, j_0)LC + (j_0, j_0+1) + (j_0, j_0)L'$, and we again obtain six independent combinations in the same manner.

If the potential support of the $(j_0-1, j_0-1)$ row is strictly between $i_0$ and $i_1$, then none of the relevant rows can have support either left of $i_0$ or right of $i_1$, and we get $s_{j_0-1} \otimes s''_{j_0} = (j_0-1, j_0)$, $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0+1)$, $s'_{j_0} \otimes s'_{j_0} = (j_0, j_0)$, $s'_{j_0+1} \otimes s''_{j_0+1} = (j_0+1, j_0+1)$, and $s'_{j_0} \otimes s''_{j_0+1} = (j_0, j_0+1)$.

If the $(j_0-1, j_0-1)$ row has disconnected potential support to the left of $i_0$ and strictly between $i_0$ and $i_1$, then once again none of the relevant rows can have potential support extending right of $i_1$, and because $\rho = 2$ we must have $a_{j_0-1, j_0-1} = c_{i_0-1}$, so none of the other relevant rows can have their potential support contained strictly left of $i_0$, either. Moreover, the $(j_0-1, j_0)$ row must have potential support containing $i_0$, so $s'_{j_0} \otimes s''_{j_0}$ cannot have any $(j_0-1, j_0)$ part, and its $(j_0, j_0)$ part must be nonzero. We then find that $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0+1)$, $s'_{j_0+1} \otimes s''_{j_0+1} = (j_0+1, j_0+1)$, and $s_{j_0} \otimes s''_{j_0+1} = (j_0, j_0+1)$. If the $(j_0-1, j_0)$ part of $s_{j_0-1} \otimes s''_{j_0}$ is nonzero, then these together with $s_{j_0-1} \otimes s_{j_0-1}$ give six independent combinations. Otherwise, we must have $s_{j_0-1} \otimes s''_{j_0} = (j_0-1, j_0-1)$, and we see that $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0-1)C$, and we see that $s_{j_0-1} \otimes s''_{j_0} = (j_0-1, j_0-1) + (j_0-1, j_0-1)R$ respectively.

If $(j_0-1, j_0-1)$ has three components of potential support, necessarily left of $i_0$, strictly between $i_0$ and $i_1$, and right of $i_1$, then none of the relevant rows other than $(j_0-1, j_0-1)$ can have potential support contained strictly left of $i_0$ or strictly right of $i_1$, and we also know that the potential support of the $(j_0-1, j_0)$ (respectively, $(j_0-1, j_0-1)$) row contains $i_0$ (respectively, $i_1$). We then have that $s'_{j_0} \otimes s''_{j_0+1} = (j_0, j_0+1)$, and that $s_{j_0} \otimes s''_{j_0}$ and $s_{j_0+1} \otimes s''_{j_0+1}$ have nonzero $(j_0, j_0)$ and $(j_0+1, j_0+1)$ parts, respectively. We also have $s_{j_0-1} \otimes s''_{j_0} = (j_0-1, j_0-1)L + (j_0-1, j_0), s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0-1)R + (j_0-1, j_0+1)$, and $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0-1)C + (j_0-1, j_0+1)$. To have a dependence between these, we need (at least one of) $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0)$ or $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0+1)$. On the other hand, to have a dependence between the first five and $s_{j_0-1} \otimes s_{j_0-1}$, we need $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0-1)L$ or $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0-1)R$. If $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0)$ and $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0-1)R$, we see that $s_{j_0-1} \otimes s''_{j_0+1}$ must have a nonzero $(j_0-1, j_0-1)C$ or $(j_0-1, j_0+1)$ part, and thus gives a sixth independent combination. On the other hand, if $s_{j_0-1} \otimes s'_{j_0} = (j_0-1, j_0-1)L$
and $s_{j_0-1} \otimes s''_{j_0+1} = (j_0 - 1, j_0 + 1)$, we see that $s_{j_0-1} \otimes s''_{j_0}$ must have a nonzero $(j_0-1, j_0-1)_{CR}$ or $(j_0-1, j_0)$ part, and again gives a sixth independent combination.

It remains to analyze the case that $(j_0-1, j_0-1)$ has two components of potential support, one left of $i_0$, and the other right of $i_1$. By hypothesis, we only have to address the case that $a_{(j_0-1, j_0-1)} = c_{i_0} - 2$ and $a_{(j_0-1, j_0-1)} = c_{i_0+1} + 2$, and that we have degree 2 in both $i_0$ and $i_1$. In this situation, the $(j_0-1, j_0)$ row has potential support strictly left of $i_0$, but none of the other relevant rows do, and the $(j_0, j_0)$ row must have support containing $i_0$ and extending left to at least the previous genus-1 component. Similarly, the $(j_0-1, j_0+1)$ row has potential support strictly right of $i_1$, but none of the other relevant rows do, and the $(j_0 + 1, j_0 + 1)$ row has support containing $i_1$ and extending to the right to at least the next genus-1 component. We also see that the potential support of $(j_0, j_0+1)$ must be contained between $i_0$ and $i_1$ inclusive, and cannot be equal solely to $i_0$ or to $i_1$. In particular, $s'_{j_0} \otimes s''_{j_0+1}$ cannot have a $(j_0-1, j_0)$ or $(j_0-1, j_0+1)$ part, so must be equal to $(j_0, j_0+1)$.

Now, $s_{j_0-1} \otimes s_{j_0-1} = (j_0-1, j_0-1)_L$ because $X_0$ is left-weighted, and we begin by considering the case that no tensor has a $(j_0-1, j_0-1)_R$ part. Then we must have $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1, j_0+1)$, $s'_{j_0+1} \otimes s''_{j_0+1} = (j_0+1, j_0+1)$, $s''_{j_0} \otimes s''_{j_0} = (j_0, j_0)$, and we also see that $s_{j_0-1} \otimes s''_{j_0}$ must be $(j_0-1, j_0)$, because it could only have a $(j_0-1, j_0+1)_R$ part if the $j_0$ part of $s''_{j_0}$ extends through $i_1$, and in this case the fact that $X_0$ is left-weighted gives us that $s_{j_0-1} \otimes s''_{j_0} = (j_0-1, j_0)$ regardless. Thus, we obtain the desired six independent combinations in this case.

On the other hand, if any tensor has a $(j_0-1, j_0-1)_R$ part, we need to produce only three more independent combinations, and we consider the four tensors $s'_{j_0} \otimes s''_{j_0}$, $s'_{j_0+1} \otimes s''_{j_0+1}$, $s_{j_0-1} \otimes s''_{j_0}$, and $s_{j_0-1} \otimes s''_{j_0+1}$. These must have at least a three-dimensional span unless they collapse into equal pairs, and there are two possibilities for this: either $s'_{j_0} \otimes s''_{j_0} = s_{j_0-1} \otimes s''_{j_0}$ and $s'_{j_0+1} \otimes s''_{j_0+1} = s''_{j_0} \otimes s''_{j_0+1} = (j_0+1, j_0+1)$, or $s'_{j_0} \otimes s''_{j_0} = s''_{j_0} \otimes s''_{j_0+1} = (j_0, j_0)$ and $s'_{j_0+1} \otimes s''_{j_0+1} = s_{j_0-1} \otimes s''_{j_0+1} = (j_0, j_0+1)$. Moreover, Proposition 4.18 implies that the $j_0$-part of $s''_{j_0}$ doesn’t contain any genus-1 components left of $i_0$. Then we necessarily have $s'_{j_0} \otimes s''_{j_0} = (j_0-1, j_0)$, so only the first possibility above can occur. Now, in general we have $s''_{j_0} \otimes s''_{j_0} = (j_0, j_0) + (j_0-1, j_0+1)_R + (j_0-1, j_0-1)c_{CR} - (j_0+1, j_0+1)_R$, which in our case simplifies to $s''_{j_0} \otimes s''_{j_0} = (j_0, j_0) + (j_0-1, j_0+1)_R + (j_0, j_0+1)_R$.

If this has nonzero $(j_0, j_0)$ or $(j_0, j_0+1)$ term, we have our sixth independent combination. On the other hand, if the $(j_0+1, j_0+1)$ term is nonzero, the $(j_0, j_0+1)$ term must also be. Because the potential support of $(j_0, j_0+1)$ must end no later than $i_1$ and cannot be supported solely at $i_1$, if the $(j_0, j_0+1)$ term of $s''_{j_0} \otimes s''_{j_0}$ is nonzero, this means that the $j_0$ part of $s''_{j_0}$ must extend to cover all of $(j_0, j_0+1)$ (note that the proof of Lemma 8.1 indicates that a $(j_0, j_0+1)$ part has to come from either a $j_0$ part of $s''_{j_0}$ and a $(j_0+1)$ part of $s''_{j_0}$ or vice versa, but not some mixture of the two). But we know that this contains at least one genus-1 component strictly right of $i_0$, so since the support of $(j_0, j_0)$ ends at $i_0$, and $X_0$ is left-weighted, we conclude that we would have to have $s''_{j_0} \otimes s''_{j_0} = (j_0, j_0)$ in this case. Thus, in all cases we obtain the desired six independent combinations.

\[\square\]
We can now prove the genus-23 case of our main theorem. As with the genus-22 case, we phrase the result more generally to apply to other $\rho = 2$ cases in the future.

**Theorem 8.12.** Fix $g, r, d$ with $r \geq 3$ and $\rho = 2$. In characteristic 0, suppose that for every left-weighted $X_0$ of genus $g$ as in Situation 5.3, and every refined limit $g'_d$ on $X_0$, there is an unimaginative $w = (c_2, \ldots, c_N)$ such that the potentially appearing sections in multidegree $md(w)$ are linearly independent, and satisfying the following additional conditions:

(i) if the limit $g'_d$ falls into the “first 3-cycle” situation described in Proposition 4.14, we require that the $(j_0 - 1, j_0)$ row has a unique potentially appearing section in multidegree $md(w)$, whose support does not contain $i_0$ or $i_1$;

(ii) if the limit $g'_d$ falls into the “second 3-cycle” situation described in Proposition 4.15, we require that one of the following three conditions is satisfied:

1. the $(j_0 - 1, j_0 - 1)$ row does not have potentially appearing sections both left of $i_0$ and right of $i_1$; or
2. $2a_{j_0-1}^{i_0} = c_{i_0} - 1$, and $2a_{j_0-1}^{i_1+1} = c_{i_1+1} + 1$; or
3. $2a_{j_0-1}^{i_0} = c_{i_0} - 2$, and $2a_{j_0-1}^{i_1+1} = c_{i_1+1} + 2$, and $w$ has degree 2 in both $i_0$ and $i_1$.

Then the strong maximal rank conjecture holds for $(g, r, d)$, and more specifically, a general curve of genus $g$ does not have any $g'_d$ for which (1.1) is not injective.

**Proof.** The proof is essentially the same as that of Theorem 8.4, still using Propositions 4.7 and 8.3 to treat the cases that our refined limit linear series has no swaps or one swap, respectively, and adding Propositions 8.6, 8.9, 8.10 and 8.11 to address the cases with two swaps. Using Remark 5.6, these are the only possibilities, since for $\rho = 2$ we cannot have swaps involved more than two rows in a single column. The only other difference is that because we assume $X_0$ is left-weighted, we are forced to consider only special directions of approach to $X_0$ in $\mathcal{M}_g$. Recalling that being left-weighted is preserved under the insertions of genus-0 chains which occur when we base change and then blow up to resolve the resulting singularities, we do however conclude that for suitable smoothing families, the generic fiber cannot carry a $g'_d$ for which (1.1) is not injective. □

Putting Theorem 8.12 together with Theorem 7.3 and Corollary 7.9, we immediately conclude the genus-23 case of Theorem 1.2.

**Remark 8.13.** In our arguments for the $g = 23$ case, we used the $\rho = 2$ hypothesis in two distinct ways: first, to limit the number of swaps occurring to two, but then also to control the behavior of the rest of the limit linear series when two swaps did occur, for instance limiting the number of possibilities for rows having disconnected potential support. This may appear discouraging from the point of view of generalizing to cases with higher $\rho$, but as $\rho$ increases, one also obtains more flexibility in choosing multidegrees while still maintaining linear independence of the potentially appearing sections. Indeed, we are taking advantage of this phenomenon already in the $\rho = 2$ case with Corollary 7.9.

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