Flat bands in lattices with non-Hermitian coupling

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We study non-Hermitian photonic lattices that exhibit competition between conservative and non-Hermitian (gain/loss) couplings. A bipartite sublattice symmetry enforces the existence of non-Hermitian flat bands, which are typically embedded in an auxiliary dispersive band and give rise to non-diffracting “compact localized states”. Band crossings take the form of non-Hermitian degeneracies known as exceptional points. Excitations of the lattice can produce either diffracting or amplifying behaviors. If the non-Hermitian coupling is fine-tuned to generate an effective π flux, the lattice spectrum becomes completely flat, a non-Hermitian analogue of Aharonov-Bohm caging in which the magnetic field is replaced by balanced gain and loss. When the effective flux is zero, the non-Hermitian band crossing points give rise to asymmetric diffraction and anomalous linear amplification.

I. INTRODUCTION

Flat bands are one of a handful of band structure features, alongside band gaps and band-crossing points, that are of key interest in photonic, electronic, and cold atom lattices.12 In a flat band, diffraction is suppressed due to destructive interference, in a manner analogous to geometric frustration, giving rise to eigenmodes that are compactly localized in space. Flat bands have been observed with cold atom lattices,1 coupled laser arrays,2 superconducting wire networks,3,7,8, and photonic lattices.9,12 Lattices containing flat bands provide a unique setting for exploring unconventional localization, anomalous magnetic phases, and strongly-correlated states of matter such as fractional quantum Hall effects and spin liquids.9,13

Photonic lattices have the distinction of easily supporting structured gain (amplification) and loss (dissipation), which can be used to study interesting non-Hermitian wave effects.13,15 For instance, non-Hermitian lattices typically have non-orthogonal eigenstates and complex band energy; but when gain and loss are distributed so that the lattice is invariant under a combination of parity and time-reversal (P T symmetry), it is possible to have bands that are purely real.16,17 Moreover, P T symmetric band structures can exhibit non-Hermitian degeneracies known as exceptional points (EPs), where a pair of eigenstates coalesce and the Hamiltonian becomes defective.22,24 EPs can give rise to a wide variety of interesting phenomena, including unidirectional invisibility, chiral lasing, and enhanced spontaneous emission.25,32

An important question is whether the existence of flat bands is compatible with non-Hermiticity, and if so whether the presence of EPs might alter the behavior of the flat band states. Previous studies have analyzed how Hermitian flat bands are changed by the application of non-Hermitian perturbations,13,15 finding either that the symmetries protecting the flat band states are spoilt, or that the flat bands simply acquire nonzero gain or loss (breaking P T symmetry). Recently, Ramezani has shown that a flat band with completely real eigenvalues can exist in a quasi-1D P T photonic lattice.36 This P T symmetric flat band was realized by fine-tuning gain/loss levels, and it is not obvious what features of this model, compared to earlier models,33,35 make it possible.

Here, we show that P T symmetric flat bands generically occur in non-Hermitian lattices with a bipartite sublattice symmetry hosting a differing number of sites per sublattice.37,38 The flat bands emerge from competition between incompatible Hermitian and non-Hermitian (gain/loss) couplings, and possess properties that are qualitatively different from their Hermitian counterparts. We identify two novel types of P T symmetric flat band:

(i) A non-Hermitian variant of an “Aharonov-Bohm cage” (wave localization induced by a fine-tuned magnetic flux)37,38, with gain/loss couplings playing the role of a complex magnetic vector potential. The Bloch Hamiltonian is defective for every wavevector, and transport is entirely suppressed. Certain states experience anomalous sub-exponential (quadratic) amplification.

(ii) A flat band embedded within another dispersive band, where the band crossing points form isolated EPs embedded in a continuum. Unlike previously-studied “embedded EPs” created by localized defects,31 the EPs in this case occur in a translationally invariant lattice, corresponding to unstable Bloch waves displaying sub-exponential amplification.

In both cases, we find numerically that the flat band-induced wave localization is robust to weak disorder. The flat band states remain strongly localized, but the eigenvalue spectrum becomes complex because the disorder breaks the P T symmetry. As a result, some of the flat band states become preferentially amplified.

We will consider a photonic lattice of coupled optical waveguides. Under the tight-binding approximation, the evolution of the optical field within the lattice can be de-
The simplest examples of geometrical frustration involve three modes, e.g. trimers of antiferromagnetic spins. Similarly, we begin our study of competing non-Hermitian couplings by analyzing a $\mathcal{PT}$-symmetric trimer, described by the Hamiltonian matrix

$$
\hat{H} = \begin{pmatrix}
\gamma + iV & C + i\gamma & 0 \\
C + i\gamma & \Delta & C - i\gamma \\
0 & C - i\gamma & -iV
\end{pmatrix}.
$$

When the central site detuning $\Delta$ is real, this is the most general Hamiltonian describing a $\mathcal{PT}$ symmetric trimer that is reflection symmetric ($\mathcal{P}$) about the central site and has only nearest-neighbor coupling. The real parameters $V$ and $\gamma$ are small, all the eigenvalues are real. Complex eigenvalues emerge abruptly as we tune the system through an EP by increasing the magnitude of $V$ and/or $\gamma$. The eigenstates of a $\mathcal{PT}$-symmetric system form complex conjugate pairs, and the $\mathcal{PT}$-breaking transition involves pairs of eigenstates. In a system with an odd number of eigenstates, there are two distinct ways for $\mathcal{PT}$-breaking to occur, which we illustrate in Fig. 1 using the limiting cases $\gamma = 0$ and $V = 0$. In the first case [Fig. 1(a)], where the gain/loss is purely on-site, a pair of neighboring eigenmodes coalesce to produce a “conventional” $\mathcal{PT}$-breaking transition at an EP, at $V = V_c \approx 0.42C$. The remaining mode always has a real eigenvalue, and does not play any important role in the transition. In this case, the $\mathcal{PT}$-breaking can be captured entirely by a two-mode approximation (see below).

A different kind of $\mathcal{PT}$-breaking transition is shown in Fig. 1(b). Here, the gain/loss is purely off-diagonal, and the system enters the $\mathcal{PT}$-broken phase at $\mathcal{E}P_2$ when $\gamma = \gamma_c = \sqrt{C^2 + \Delta^2/8}$, where the first and third eigenmodes coalesce. Before reaching $\mathcal{E}P_2$, however, the second and third eigenmodes must first cross at the point labelled $\mathcal{E}P_1$ ($\gamma = C$). This degeneracy point is an EP, because the two participating eigenmodes coalesce to $|\Psi_2\rangle = |\Psi_3\rangle = (e^{-i\pi/4}, 0, -e^{i\pi/4})$, and $\hat{H}$ is defective. However, $\mathcal{E}P_1$ does not correspond to a $\mathcal{PT}$-breaking transition; all the eigenvalues in its vicinity are real.

To gain a better understanding of $\mathcal{E}P_1$, note that $\hat{H}$ has a “dark eigenstate” $|\Psi_2\rangle = (C - i\gamma, 0, -C - i\gamma)$ in which the central site is unexcited. Its eigenvalue $E_2 = 0$ is independent of $C, \gamma$, and $\Delta$; hence it cannot belong to a pair of $\mathcal{PT}$-broken eigenvalues. In the Hermitian context, this kind of “flat” eigenvalue is characteristic of systems with a bipartite symmetry, where pairs of identical states are only coupled via an intermediate state. In the present system, $\gamma/C$ fixes the relative phase of the two end-sites, which is reminiscent of the role of a vector potential. The relative phase is required to maintain a
balanced flow of energy between the gain and loss media. But in contrast to Hermitian vector potentials in 1D arrays, this relative phase cannot simply be removed by a gauge transformation; it has the observable effect of inducing eigenstate non-orthogonality. The dark state is $\mathcal{PT}$-antisymmetric in the Hermitian limit ($\gamma = 0$) and $\mathcal{PT}$-symmetric in the limit of completely non-Hermitian couplings ($C = 0$).

In the strongly-detuned limit, $\Delta \gg C, V, \gamma$, the central site may be eliminated from the eigenvalue problem. For small energies $\delta E \ll \Delta$, Eq. (1) reduces to an effective $2 \times 2$ Hamiltonian:

$$
\hat{H}_{\text{eff}} = \begin{pmatrix} -(C + i\gamma)^2/\Delta + iV & -(C^2 + \gamma^2)/\Delta \\ -(C^2 + \gamma^2)/\Delta & -(C - i\gamma)^2/\Delta - iV \end{pmatrix}.
$$

For $\gamma = 0$, $\hat{H}_{\text{eff}}$ is the Hamiltonian that describes a $\mathcal{PT}$-symmetric dimer with reduced effective coupling strength $|C_{\text{eff}}| = C^2/\Delta$. This has a $\mathcal{PT}$-breaking transition at $|V| = |C_{\text{eff}}|$, corresponding to the point labelled "EP" in Fig. 1(a). On the other hand, the “frustrated" trimer ($V = 0$) has an effective coupling strength $|C_{\text{eff}}| = (C^2 + \gamma^2)/\Delta$ that increases with the non-Hermiticity $\gamma$, in competition with the effective non-Hermitian potential $V_{\text{eff}} = -C\gamma/\Delta$. This induces an EP at $|\gamma| = |C|$, corresponding to EP1 in Fig. 1(b), but the perturbative eigenvalues $\delta E = 0, 2(\gamma^2 - C^2)/\Delta$ remain real; $\mathcal{PT}$-breaking cannot occur within the two mode approximation.

This simple model demonstrates that a bipartite $\mathcal{PT}$-symmetric system, with competing Hermitian and non-Hermitian couplings, produces a dark state with energy pinned to $E_0 = 0$. Even when the bipartite symmetry is broken (i.e., $V$ and $\gamma$ both nonzero), the $\mathcal{PT}$-breaking is limited to $C - \sqrt{|\Delta V|} < |\gamma| < C + \sqrt{|\Delta V|}$. For sufficiently large $\gamma$, the system returns to the unbroken phase.

### III. FRUSTRATED LATTICES

We now generalize the trimer to a $\mathcal{PT}$-symmetric lattice with bipartite sublattice symmetry, where the dark state becomes a flat band of zero-energy Bloch states. Consider a 1D lattice obeying tight-binding equations:

$$
\begin{align*}
Ea_n &= \kappa_1 b_n + \kappa_2 b_{n+1}, \\
Eb_n &= \Delta b_n + \kappa_1 a_n + \kappa_2 a_{n-1} + \kappa_3 c_n + \kappa_4 c_{n-1}, \\
Ec_n &= \kappa_3 b_n + \kappa_4 b_{n-1},
\end{align*}
$$

where $n = 1, \ldots, N; \Delta$ is the detuning of the central sites; and $\kappa_j \equiv C_j + i\gamma_j$ are complex couplings. Fourier transforming gives $\hat{H}(k)\Psi(k) = E(k)\Psi(k)$, where

$$
\hat{H}(k) = \begin{pmatrix} 0 & \kappa_1 + \kappa_2 e^{ik} & 0 \\
\kappa_1 + \kappa_2 e^{ik} & \Delta & \kappa_3 + \kappa_4 e^{-ik} \\
0 & \kappa_3 + \kappa_4 e^{-ik} & 0 \end{pmatrix}.
$$

The energy eigenvalues are

$$
E(k) = 0, \frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + 2(\kappa_1\kappa_2 + \kappa_3\kappa_4) \cos k + \sum_j \kappa_j^2},
$$

where $C_{\text{eff}} = \kappa_1\kappa_3 + \kappa_2\kappa_4$ is an effective coupling strength for two dispersive bands, and $\sum_j \kappa_j^2$ is an effective energy detuning. The $E = 0$ flat band is a generalization of the dark state of the trimer model from Section II. The Bloch states in this band all have $b_n = 0$, regardless of the values of $\Delta$, $\kappa_j$, and $k$. This is due to the bipartite sublattice symmetry in Eqs. (3a) and (3c); since Eq. (3b) only provides $N$ constraints to the $2N$ remaining degrees of freedom $(a_n, c_n)$, there are $N$ degenerate solutions.

The flat band allows for the construction of compact localized eigenstates, which are non-diffracting and vanish identically outside of a few sites. To find these solutions, we assume that the $a_n$ and $c_n$ amplitudes are only nonzero at sites $p, p + 1$, and solve Eq. (3b):

$$
\begin{align*}
\kappa_1 a_p + \kappa_3 c_p &= 0, \\
\kappa_2 a_{p+1} + \kappa_4 c_{p+1} &= 0, \\
\kappa_2 a_p + \kappa_4 c_p + \kappa_1 a_{p+1} + \kappa_3 c_{p+1} &= 0.
\end{align*}
$$

This yields the eigenmode amplitudes

$$
\begin{align*}
a_p &= 1, & a_{p+1} &= \kappa_4/\kappa_3, \\
c_p &= -\kappa_1/\kappa_3, & c_{p+1} &= -\kappa_2/\kappa_3,
\end{align*}
$$

with all other $a_n, c_n$ vanishing.

Similar to the dark mode of the trimer model, the flat band exists independent of whether the $\kappa$ parameters are real (Hermitian) or complex (non-Hermitian). In the latter case, however, the flat band states have a nontrivial phase profile, and there can be two qualitatively different types of crossings between the flat and dispersive bands, depending on the lattice geometry.

We first consider the lattice in Fig. 2(a), which contains two legs with opposite non-Hermitian coupling strengths: $\kappa_1 = C + i\gamma_1$, $\kappa_2 = \alpha \kappa_1$, $\kappa_3 = \kappa_1^*$, and $\kappa_4 = \kappa_2^*$ (where $\alpha$ is a real parameter). Between adjacent central ($b_n$) sites, coupling via the upper leg accumulates a phase $\theta = \text{Arg}(\kappa_1\kappa_2) = 2\tan(\gamma/C)$, while coupling via the lower leg accumulates an equal and opposite phase $-\theta$. This resembles a net magnetic flux of $\Delta\theta = 2\theta$. As we increase $\gamma$ from zero to $C$, the effective coupling $C_{\text{eff}} = \alpha(C^2 - \gamma^2)$ vanishes, and the dispersive bands consequently all flatten out. This corresponds to the case $\Delta\theta = \pi$, and is thus analogous to the formation of an “Aharonov-Bohm cage” in a Hermitian lattice [19,20]. But unlike those Hermitian models, one of the dispersive bands coalesces onto the $E = 0$ flat band, forming an entire band of $N$ non-Hermitian degeneracies, as shown in Fig. 2(b). At the point EP1 ($\gamma = C$), $\hat{H}(k)$ is not diagonalizable for any $k$; it only has two eigenvectors,

$$
|\Psi_1\rangle = \left(i, \frac{\Delta}{C} e^{i\pi/4}, 1\right), \quad |\Psi_2\rangle = (i, 0, 1),
$$

that correspond to the flat bands.
where $|\Psi_2\rangle$ is doubly degenerate. Upon increasing $\gamma$ past $EP_1$, the degeneracy is lifted, but the spectrum remains purely real until the two dispersive bands coalesce at $EP_2$ ($\gamma/C \approx 1.1$).

An interesting property of Aharonov-Bohm cages is that despite the nonzero couplings between lattice sites, localized excitations only spread a finite distance; propagation over longer distances is completely suppressed (not just exponentially suppressed) by destructive interference between the two legs formed by the $a$ and $c$ sites. To test whether our non-Hermitian model has this property, we simulate the evolution $i\partial_t \Psi = \hat{H}(\Psi)$ for an initial excitation localized to either an $a$ or $b$ site of the lattice. The results, shown in Fig. 2(c),(d), show a complete suppression of diffraction when $\gamma = C$. The initial excitation spreads to the neighboring unit cells, but no further; the intensity beyond the neighbouring cells vanishes to within numerical precision. When eigenstates of $\hat{H}$ are excited, there are periodic oscillations, but otherwise the EP results in quadratic amplification of the total power. By contrast, if we excite four sites according to Eq. (5), we can form a compact localized state with $z$-invariant intensity.

Qualitatively different behavior occurs when each leg has balanced non-Hermitian couplings, such that the effective flux always vanishes. As an example, we consider the "stub" lattice shown in Fig. 3(c) where coupling occurs via a single leg with balanced gain and loss: $\kappa_1 = C$, $\kappa_2 = 0$, and $\kappa_3 = \kappa_4^* = C + i\gamma$. In this case, the non-Hermiticity parameter $\gamma$ has the effect of increasing the effective coupling $C_{\text{eff}} = C^2 + \gamma^2$, broadening the two dispersive bands. This can be understood as different wavevectors $k$ experiencing different effective amounts of gain/loss. By increasing $\gamma$, we can continuously tune the lower dispersive band eigenvalues in Fig. 3 through $E = 0$ flat band to generate a succession of EP's at critical momenta $k = k_c$, where

$$\cos(k_c) = 1 - 5/(2 + 2\gamma^2/C^2).$$

(8)

The EP's form isolated exceptional points in the continuum: while the Hamiltonian becomes defective at $k_c$, the spectrum in the vicinity of the EP remains purely real. Previously, similar EP's in the continuum were obtained using a specially-tailored defect potential. These EPs,
IV. DISORDER

In practical implementations, fabrication disorder may broaden the flat band to some nonzero width, lifting its degeneracy. We now study how the flat band states behave in this situation. With a random onsite potential, the tight binding equations (3) become

\[ Ea_n = \epsilon'_a a_n + \kappa_1 b_n + \kappa_2 b_{n+1}, \]
\[ Eb_n = (\epsilon'_b + \Delta) b_n + \kappa_1 a_n + \kappa_2 a_{n-1} + \kappa_3 c_n + \kappa_4 c_{n-1}, \]
\[ Ec_n = \epsilon'_c c_n + \kappa_3 b_n + \kappa_4 b_{n+1}. \] (9)

The uniformly-distributed independent random variables \( \epsilon'_a \in [-W/2, W/2] \) describe uncorrelated on-site disorder in the waveguide depths, or in the resonant frequencies of ring resonators. The disorder breaks the sublattice and \( PT \) symmetries of the non-Hermitian flat band lattice, so the energy eigenvalues \( E \) all become non-real.

In Fig. 4, we show the effect of disorder on the spectum of the non-Hermitian Aharonov-Bohm cage from Fig. 2. These results are obtained by numerically diagonalizing Eq. (9) for various disorder strengths \( W \); the other lattice parameters are chosen so that the lattice is tuned to \( EP_1 \) in the disorder-free limit (\( W = 0 \)). As described in Section III, the ordered lattice has flat bands at energies \( E_1 = 0 \) (a flat band of non-Hermitian degeneracies), and \( E_2 = \Delta = 2C \) (a Hermitian-like flat band). In Fig. 4(a), we plot the mean disorder-induced shifts \( \langle \delta E \rangle \equiv \langle E - E_{1,2} \rangle / C \) to the energy eigenvalues as a function of the disorder strength \( W \). The shifts grow as \( \delta E \sim \sqrt{W} \) in the non-Hermitian flat band, and \( \delta E \sim W \) in the Hermitian-like flat band. In both cases, \( \text{Re}(\delta E) \) and \( \text{Im}(\delta E) \) have similar magnitudes.

To characterize the degree of localization of the eigenstates \( \Psi^{(j)} \) in the disordered lattice, we compute the participation numbers \( P_j = 1 / \sum_n |\Psi^{(j)}_n|^2 \). The results are shown in Fig. 4(b). For the compact localized states of the non-Hermitian flat band, \( P_{CLS} = 2 \) (in comparison, a delocalized state has \( P \sim N \), where \( N \) is the lattice size). For both the bands, we find \( P_j \sim P_{CLS} \) regardless of the value of \( W \). This indicates that the eigenstates of a weakly-disordered lattice behave much more like compact localized states than a delocalized Bloch wave basis. The disorder simply causes the states to be exponentially localized, rather than vanishing identically outside the localization region.

This behavior is consistent with previous studies on the effect of diagonal disorder on isolated one-dimensional Hermitian flat bands. There, the vanishing of the band velocity \( (v_G = 0) \) implies that the disorder strength \( W \) sets the only relevant energy scale. Applying degenerate perturbation theory, one can show that the energy bandwidth is set by \( W \), while the eigenstates themselves are insensitive to \( W \) (always being in the “strong disorder” regime \( W \gg v_G \)). Our present results show that non-Hermitian flat bands exhibit similar behaviour. Interestingly, however, the presence of non-Hermitian degeneracies results in a square root sensitivity of the energy eigenvalues to perturbations, \( \delta E \sim \sqrt{W} \). This holds as long as \( W \) is smaller than the band gap; larger \( W \) will induce hybridization between the bands, affecting the mode profiles. For example, in the strong disorder limit \( W \gg C \) the inter-waveguide coupling will become negligible, resulting in \( P_j \approx 1 \).

Fig. 5 shows the effects of disorder on the “embedded” non-Hermitian flat band from Fig. 2. In Hermitian systems, the compact localized states of embedded flat bands are known to be unstable against disorder:
We have shown how to generalize flat bands to non-Hermitian systems by combining a bipartite symmetry with frustration between energy conserving (Hermitian) and non-conserving (non-Hermitian) couplings. Such systems are “frustrated” because Hermitian couplings favour eigenstates with neighboring sites either in-phase or \( \pi \) out-of-phase (to ensure zero net power flow and a stationary intensity distribution), whereas the non-Hermitian couplings favor \( \pm \pi/2 \) phase differences (to balance power flow between media with gain and loss). At first glance, the energy spectra and compact localized states of the non-Hermitian flat bands resemble that of their Hermitian counterparts. However, deeper analysis of the eigenstates and propagation dynamics reveals important differences. For example, the non-Hermitian flat bands can host isolated exceptional points which are unaccompanied by a \( \mathcal{PT} \)-breaking transition. The compact localized states may be subexponentially amplified, and robust to weak disorder.

In the future, it would be interesting to further study, both theoretically and in experiments, the peculiar beam dynamics associated with the non-Hermitian flat band states, along with their interplay with disorder and interactions. In settings where Aharonov-Bohm caging induced by real or synthetic magnetic fields is difficult to implement, the non-Hermitian flat bands may provide a useful alternative method to design flat dispersion and achieve transverse confinement of light.

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