An intrinsic proof of Gromoll-Grove diameter rigidity theorem

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Dedicated to Professor Karsten Grove on his sixtieth birthday

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1 Introduction

We will present a new proof of the following Gromoll-Grove diameter rigidity theorem.

Theorem A Let $M^n$ be a simply connected Riemannian manifold with sectional curvature $K \geq 1$. Suppose that $\text{Diam}(M^n) = \frac{\pi}{2}$ and $M^n$ is not homeomorphic to a sphere $S^n$. Then $M^n$ is isometric to one of $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{C}aP^2$, i.e., $M^n$ is isometric to a projective symmetric space over complex numbers, or quaternion numbers or Calay numbers.

Our proof does not use any loop spaces, which is totally different from [Wil]. Among other things, we use the Hessian comparison theorem for distance functions and the spherical metric on the tangent space instead, see Section 3 below. Although our new proof is longer than its earlier version, the most of arguments below remain to be elementary and self-contained.

2 The Gromoll-Grove fibration

We need to recall some known results from [GG1], in order to complete the proof of Theorem A. The results of [GG1] are related to the following example.

Example 2.0. (1) Let $M^n = \mathbb{C}P^n$ with the classical Fubini-Study metric and diameter $\frac{\pi}{2}$. Let $B_r(p)$ be the metric ball of radius $r$ and center $p$ in $\mathbb{C}P^n$, and let $S_r(p) = \partial B_r(p)$ be the metric sphere of radius $r$ centered at $p$. It is well-known that $S_{\frac{\pi}{2}}(p)$ is isometric to $\mathbb{C}P^{n-1}$.

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For each \( p \in \mathbb{C}P^2 \), we consider a polar coordinate \( \{(r, \Theta)\} \) of the tangent space \( T_p(\mathbb{C}P^2) \) and the exponential map \( \text{Exp}_p : T_p(\mathbb{C}P^2) \to \mathbb{C}P^2 \).

Let us choose a spherical metric
\[
g_1 = dr^2 + (\sin r)^2 d\Theta^2
\]
on the set \( B_r(0) = \{(r, \Theta)\mid 0 \leq r < \pi, \Theta \in S^{n-1}\} \), where \( d\Theta^2 \) is the canonical metric of constant curvature 1 on the unit sphere \( S^n \). With respect to the spherical metric \( g_1 \), the exponential map
\[
\text{Exp}_p : S^2(0) \to S^2(p)
\]
\[
\frac{\pi}{2} \Theta \to \text{Exp}_p \left( \frac{\pi}{2} \Theta \right)
\]
is a Hopf fibration.

Furthermore, for each \( q \in S^2(p) \), the fiber \( \text{Exp}_p^{-1}(q) \) is a great circle in the equator \( (S^2(0), g_1) \) of the unit sphere \( S^n = (B_\pi(0), g_1) \).

(2) We are going to elaborate the above construction by replacing the point \( p \) by a totally geodesic submanifold \( \mathbb{C}P^m \subset \mathbb{C}P^2 \) with \( 1 \leq m < \frac{n}{2} - 1 \), for the case \( \frac{n}{2} \geq 3 \). We let \( U_r(\mathbb{C}P^m) = \{z \in \mathbb{C}P^2 \mid d(z, \mathbb{C}P^m) < r\} \) be the tubular neighborhood and \( \partial[U_r(\mathbb{C}P^m)] \) its boundary.

Then \( \partial[U_r(\mathbb{C}P^m)] \) is isometric to a totally geodesic \( \mathbb{C}P^{m'} \subset \mathbb{C}P^2 \) with \( m' = \frac{n}{2} - m - 1 \). In this case, for each pair \( p \in \mathbb{C}P^m \) and \( q \in \mathbb{C}P^{m'} \) with distance \( d(p, q) = \frac{\pi}{2} \), we still have that the fiber \( \text{Exp}_p^{-1}(q) \) is a great circle in the equator \( (S^2(0), g_1) \) of the unit sphere \( S^n = (B_\pi(0), g_1) \), where \( \bar{B}_r(0) \subset T_p(\mathbb{C}P^2) \).

In fact, \( \mathbb{C}P^{m+1} \) has a decomposition \( \mathbb{C}P^{m+1} = \mathbb{C}^m \times \mathbb{C}^{m'+1} \). Such a decomposition induces a spherical join of \( S^{2m+1} \) and \( S^{2m'+1} \). More precisely, for each unit vector \( \bar{u} \in S^{m+1} \subset \mathbb{C}^{m+1} \), there are \( \bar{v} \in S^{2m+1} \) and \( \bar{w} \in S^{2m'+1} \)
\[
\bar{u} = (\cos r) \bar{v} + (\sin r) \bar{w}
\]
for some \( r \in [0, \frac{\pi}{2}] \). One can write \( S^{n+1} = S^{2m+1} \ast S^{2m'+1} \), where \( \frac{n}{2} = m + m' + 1 \). It follows that \( \mathbb{C}P^2 \) can be viewed as the “projective join” of \( \mathbb{C}P^m \) and \( \mathbb{C}P^{m'} \).

The pair of sub-manifolds \( \{\mathbb{C}P^m, \mathbb{C}P^{m'}\} \) with \( d(\mathbb{C}P^m, \mathbb{C}P^{m'}) = \frac{\pi}{2} \) above is called a dual pair of convex subsets of \( \mathbb{C}P^2 \) in \([GG1]\).

(3) When \( M^n \) is isometric to either \( \mathbb{H}P^2 \) or \( \mathbb{C}aP^2 \), there are similar decompositions. Q.E.D.

Inspired by Example 2.0, we consider the convexity of subset \( [M - B_r(p)] \), without the assumption \( \text{Diam}(M) = \frac{\pi}{2} \). Let \( \text{Inj}_M(x) \) denote the injectivity radius of \( M \) at \( x \).

**Proposition 2.1** Let \( M \) be a complete smooth Riemannian manifold with sectional curvature \( \geq 1 \) and \( \text{Diam}(M) \geq \frac{\pi}{2} \). Suppose that \( \sigma : [0, \ell] \to M \) is a length-minimizing geodesic of unit speed from \( x \). Then, for any \( 0 < r < \ell \), the second fundamental form of \( S_r(x) \) at \( \sigma(r) \) with
respect to the normal vector $\sigma'(r)$ is less than or equal to $\cot(r) I$ at $\sigma(r)$ in the barrier sense, where $I$ is the identity matrix.

Consequently, if $\text{Inj}_M(x) \geq \frac{\pi}{2}$, then $[M - B_{\frac{\pi}{2}}(x)]$ is a convex subset of $M$. In addition, if $\text{Diam}(M) \geq \ell > \frac{\pi}{2}$, then $[M - B_{\ell}(x)]$ is strictly convex.

**Proof.** This is a direct consequence of the Hessian comparison (see [Pe, p145]) for the distance function. Q.E.D.

**Proposition 2.2** ([GG1]) Let $M$ be a complete smooth Riemannian manifold with sectional curvature $\geq 1$ and $\text{Diam}(M) = \frac{\pi}{2}$. If $M$ is simply-connected and if $M$ has the integral cohomology ring of either $\mathbb{C}P^2$, $\mathbb{H}P^2$ or the Cayley plane $\mathbb{C}aP^2$, then there exists at least one point $p \in M$ with injectivity radius $\text{Inj}_M(p) \geq \frac{\pi}{2}$.

When $d(p,q) = \text{Diam}(M) = \frac{\pi}{2}$, the subset $S_{\frac{\pi}{2}}(p)$ is a critical sub-manifold of the distance function $f(x) = d(x,p)$. If $\text{Inj}_M(p) \geq \frac{\pi}{2}$, by Proposition 2.2 above, $S_{\frac{\pi}{2}}(p)$ is a totally geodesic submanifold. In fact, the dual convex subset $S_{\frac{\pi}{2}}(p)$ has some extra properties (cf. [GG1]), which we recall in the sequel.

Following [GG1], for $A \subset M$ we let $A' = \{ y \in M \mid d(y,A) = \frac{\pi}{2} \}$.

The following result was also stated in [GG1].

**Proposition 2.3** ([GG1]) Let $M$ be a simply connected Riemannian manifold with sectional curvature $\geq 1$ and $\text{Diam}(M) = \frac{\pi}{2}$. Suppose that the injectivity radius $\text{Inj}_M(p)$ of $M$ at $p$ is equal to $\frac{\pi}{2}$ and that $M$ is not homeomorphic to a sphere. Then

1. $M$ has integral cohomology ring of either $\mathbb{C}P^2$, $\mathbb{H}P^2$ or the Cayley plane $\mathbb{C}aP^2$;
2. if $A = \{p\}$, then $A' = \{ y \mid d(y,p) = \frac{\pi}{2} \}$ is a closed totally geodesic submanifold of positive dimension;
3. if $A = \{p\}$, then $(A')' = A$ and the cut radius $\text{Cut}_M(A')$ is equal to $\frac{\pi}{2}$ as well;
4. if $S_{\pi}(M) = \{ \tilde{v} \in T_pM \mid \|\tilde{v}\| = 1 \}$ then
   $$\pi_p = \tilde{\text{Exp}}_p : S_{\pi}(M) \rightarrow A', \quad \tilde{v} \mapsto \text{Exp}_p(\frac{\pi}{2} \tilde{v})$$
   is a Riemannian submersion;
5. if the Riemannian submersion $\pi_p : S_{\pi}(M) \rightarrow A'$ is a great circle fibration, then $M$ is isometric to either $\mathbb{C}P^2$, $\mathbb{H}P^2$ or the Cayley plane $\mathbb{C}aP^2$.

Notice that $\{q\}'' = \{q\}'$ holds for all $q \in M$, when $\text{Diam}(M) = \frac{\pi}{2}$. We choose $A' = \{q\}'$ and $A = A''$. It is possible that $\min\{\dim A, \dim A'\} > 0$, see Example 2.0 above. If $M$ is allowed
to be non-simply-connected, and if $M^3$ is a lens space, then $\min\{\dim A, \dim A'\} = 1$. In other words, it might be difficult to find a point $p$ with $\text{Inj}_M(p) = \frac{n}{2}$ when $\text{Diam}(M) = \frac{n}{2}$. We can not choose $A$ with $\dim A = 0$ at the first place.

Thus, we need to describe the remaining case of $\min\{\dim A, \dim A'\} > 0$, where $A'' = A$ and $\{A, A'\}$ is a pair of dual convex subsets. It was shown in [GG1] that both $A$ and $A'$ are connected totally geodesic submanifolds without boundaries.

In what follows, we always let

$$S_q^\perp(B, M) = \{\vec{v} \in T_p M \mid \vec{v} \perp T_q(B), |\vec{v}| = 1\}$$

be the unit normal bundle of $B$ in $M$, when $A'$ is a submanifold of $M$.

**Proposition 2.4 ([GG1])** Let $M$ be a simply-connected Riemannian manifold with sectional curvature $\geq 1$ and diameter $\text{Diam}(M^n) = \frac{n}{2}$. For any $z \in M$ with $S^\perp_{\vec{v}}(z) \neq \emptyset$, we let $A' = S^\perp_{\vec{v}}(z)$ and $A = A''$. Suppose that $M^n$ is not homeomorphic to $S^n$. Then

1. both $A$ and $A'$ are simply-connected;
2. the cut radius $\text{Cut}_M(A)$ of $A$ in $M$ is equal to $\frac{n}{2}$ and $\text{Cut}(A) = A'$;
3. the cut radius $\text{Cut}_M(A')$ of $A'$ in $M$ is equal to $\frac{n}{2}$ and $\text{Cut}(A') = A$;
4. if $\dim(A') > 0$, then
   $$\pi_p = \widetilde{\text{Exp}}_p : S^\perp_p(A, M) \to A'$$
   $$\vec{v} \to \text{Exp}_p(\frac{n}{2}\vec{v})$$
   is a Riemannian submersion; similarly, if $\dim A > 0$ then $\pi_q : S^\perp_q(A', M) \to A$ is a Riemannian submersion for all $q \in A'$; furthermore, $\dim[\pi_p^{-1}(q)]$ is equal to one of $\{1, 3, 7\}$; $\dim M$, $\dim A$ and $\dim A'$ are even integers;
5. if the Riemannian submersion $\pi_p : S^\perp_p(A, M) \to A'$ with $\dim A' > 0$ is a great circle fibration for all $p \in A$ and if the Riemannian submersion $\pi_q : S^\perp_q(A', M) \to A$ is a great circle fibration for all $q \in A'$ whenever $\dim A > 0$, then $M^n$ is isometric to one of symmetric spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{C}A\mathbb{P}^2$.

**Definition 2.5** Let $M^n$ be a simply connected Riemannian manifold with sectional curvature $\geq 1$. Suppose that $\text{Diam}(M^n) = \frac{n}{2}$, $A'' = A$ and $(p, q) \in A \times A'$. When $\dim A' > 0$, the Riemannian submersion

$$\pi_p : S^\perp_p(A, M) \to A'$$
$$\vec{v} \to \text{Exp}_p(\frac{n}{2}\vec{v})$$

is called the Gromoll-Grove fibration with the total space $S^\perp_p(A, M)$.

Similarly, when $\dim A > 0$, the fibration $\pi_q : S^\perp_q(A', M) \to A$ is called the Gromoll-Grove fibration as well.
In next section, we will show that the Gromoll-Grove fibration
\[ S^k \rightarrow S^\perp_p(A, M) \rightarrow A' \]
is a great circle fibration for some \( k \in \{1, 3, 7\} \) whenever \( \dim(A') > 0 \); and hence \( M \) must be isometric to a symmetric space by [Ran].

### 3 The Gromoll-Grove fibration is isometrically congruent to a Hopf fibration

In this section, we will use a new method to show that the Gromoll-Grove fibration is isometrically congruent to a great circle fibration.

Throughout this section, the origin of \( T_pM \approx \mathbb{R}^n \) is denoted by \( 0_p \). We will always use a spherical metric \( g_1 \) on a ball \( B_\pi(0_p) \subset T_pM \):
\[ g_1 = dr^2 + (\sin r)^2 d\Theta^2 \]
where \( \{(r, \Theta)\} \) is the polar coordinate system of \( T_pM \approx \mathbb{R}^n \).

We consider the possibly tear-drop shaped fibres in the manifold \( M \), see Section 2 above. For each pair of \( p \in A \) and \( q \in A' \), we let
\[ \Sigma_{p,q} = \{\text{Exp}_p(t\vec{v}) | \vec{v} \in \pi_{p}^{-1}(q), 0 \leq t \leq \frac{\pi}{2}\} \]
and
\[ \tilde{\Sigma}_{p,q} = \{\tilde{w} \in \text{Exp}_{\pi_{p}}^{-1}(\Sigma_{p,q}) | ||\tilde{w}|| \leq \frac{\pi}{2}\} \]
be the truncated tangential cone of \( \Sigma_{p,q} \) at \( p \).

Our goal is to show that \( \tilde{\Sigma}_{p,q} \) is totally geodesic in \((B_\perp^\pi(0_p), g_1) \subset S^n\) and hence \( \partial \tilde{\Sigma}_{p,q} \) is totally geodesic in \( S^{n-1} \). Consequently, \( \pi_{p}^{-1}(q) \) is a \( k \)-dimensional circle in \( S^{n-1} \), where \( k \) is one of \( \{1, 3, 7\} \).

There are three elementary steps to show that \( \pi_{p}^{-1}(q) \) is a \( k \)-dimensional circle in \( S^{n-1} \).

**Step 1.** We will show that “if \( \tilde{\Sigma}_{p,q} \) has the first focal radius \( \geq \frac{\pi}{2} \) in \( S^n \) at all \( z \in \tilde{\Sigma}_{p,q} \) with \( 0 < |z| < \frac{\pi}{2} \), then \( \tilde{\Sigma}_{p,q} \) is a smooth totally geodesic submanifold of \( S^n \).”

**Step 2.** We will make the following elementary observation. Suppose contrary, \( \tilde{\Sigma}_{p,q} \) had the first focal radius \( 0 < t_0 < \frac{\pi}{2} \) in \((B_\perp^\pi(0_p), g_1) \subset S^n\) at some \( z \in \tilde{\Sigma}_{p,q} \) with \( 0 < |z| < \frac{\pi}{2} \). Then there would be a Jacobi field \( \{J(t)\} \) along a normal geodesic \( \sigma_{z,\vec{h}}(t) = \text{Exp}^S_{z}(t\vec{h}) \) such that \( \vec{h} \perp T_z(\tilde{\Sigma}_{p,q}), |\vec{h}| = 1 \) and \( J'(0) \in T_z(\tilde{\Sigma}_{p,q}). \)

Thus, we consider a special class of Jacobi fields with extra initial conditions on \( J'(0) \):
\[ \Gamma_{\sigma_{z,\vec{h}},\tilde{\Sigma}_{p,q}} = \{J | J'' + R(\sigma', J)\sigma' = 0, \langle J'(0), X \rangle = -\langle \vec{h}, \nabla_{J'(0)}X \rangle, \text{for all } X \in T_z(\tilde{\Sigma}_{p,q})\} \]
and
\[ \Gamma^0_{\sigma,z,\hat{\Sigma}} = \{ J \in \Gamma_{\sigma,z,\hat{\Sigma}} | J'(0) \in T_z(\hat{\Sigma}) \}. \] (3.1)

It will be shown
\[ \dim[\Gamma^0_{\sigma,z,\hat{\Sigma}}] = \dim[\hat{\Sigma}] = k + 1. \]

This step is applicable to all \((k + 1)\)-dimensional submanifold \(\hat{\Sigma} \subset S^n\), which is elementary.

Step 3. In this final step, we use Hessian comparison theorem to show that, “if \(\pi_p : S^1_p(A,M) \to A'\) is a Riemannian submersion, then, for all non-trivial Jacobi field \(J \in \Gamma^0_{\sigma,z,\hat{\Sigma}}\), we have \(J(t) \neq 0\) for all \(t \in (0, \frac{\pi}{2})\).” It follows that \(\hat{\Sigma}_{p,q}\) has the first focal radius \(\geq \frac{\pi}{2}\) and hence totally geodesic in \(S^n\). This completes the proof of Grove-Gromoll diameter rigidity Theorem.

Here are the details for each step.

Step 1. We present a sufficient condition for totally geodesic property.

A subset \(C \subset M\) is \(\alpha\)-convex in \(M\) if, for all geodesic segments \(\sigma : [0, \ell] \to M\) of length \(\ell < a\) with endpoints in \(C\), one has \(\sigma([0, \ell]) \subset C\).

Proposition 3.1 If \(\hat{\Sigma}_{p,q}\) has the first focal radius \(\geq \frac{\pi}{2}\) in \(S^n = (B_\pi(0_p), g_1)\) at all \(z \in \hat{\Sigma}_{p,q}\) with \(0 < |z| < \frac{\pi}{2}\), then \(\hat{\Sigma}_{p,q}\) is a smooth totally geodesic submanifold with boundary in \(S^n = (B_\pi(0_p), g_1)\). Moreover, \(\pi_{p}^{-1}(q) \approx [\partial \hat{\Sigma}_{p,q}]\) is a totally geodesic great \(k\)-dimensional circle in \(S^n\).

Proof. (1) Let \(\tilde{h}_0 \in S^1(\hat{\Sigma}_{p,q}, S^n)\) be a unit norm vector of \(\hat{\Sigma}_{p,q}\) at \(z_0\) and \(\sigma_0(t) = \text{Exp}_{z_0}(t\tilde{h}_0)\). Let \(\text{Exp} : S^1(\hat{\Sigma}_{p,q}, S^n) \times [0, \infty) \to S^n\) be the exponential map along the normal bundle near \((z_0, \tilde{h}_0, t)\) for \(t \geq 0\). Suppose that \(\zeta : (-\delta, \delta) \to S^1(\hat{\Sigma}_{p,q}, S^n)\) is a curve with \(\zeta(0) = (z_0, \tilde{h}_0)\) and \(\zeta(s) = (z(s), \tilde{h}(s))\). Then \(F(t, s) = \text{Exp}_{z(s)}[t\tilde{h}(s)]\) gives rise to a Jacobi field \(\{J(t)\}\) defined by
\[ J(t) = \frac{\partial F}{\partial s}(t, 0) \]
along \(\sigma_0\).

Our goal is to show that, under our assumption, we have
\[ \langle J(0), \nabla_{J(0)}\tilde{h}(s) \rangle = \langle J(0), J'(0) \rangle \geq 0. \] (3.2)

Since \(\hat{\Sigma}_{p,q}\) is not a hypersurface, we consider a tubular neighborhood of \(\hat{\Sigma}_{p,q}\). Choose \(\varepsilon_1\) sufficiently small so that \(B_{\varepsilon_1}(z_0) \cap \hat{\Sigma}_{p,q}\) is an embedded \((k + 1)\)-dimensional ball. Let \(\varepsilon_0\) be the
cut-radius of $\Sigma_{p,q}\cap B_{\varepsilon_1}(z_0)$. Choose $\varepsilon < \frac{1}{8} \min\{\varepsilon_0, \varepsilon_1\}$. Then there is a nearest point projection from $B_{\varepsilon}(z_0) \to \Sigma_{p,q}$. If \{G(\cdot, s)\} is an 1-family variation of $\sigma_0|_{[\varepsilon, \varepsilon]}$, which are orthogonal to $\partial[U_\varepsilon(\Sigma_{p,q})]$, then by using the nearest point projection, such a family \{G(\cdot, s)\} can be extended as an 1-family of normal geodesics \{F(\cdot, s)\} from $\Sigma_{p,q}$ with $F(\cdot, 0) = \sigma_0(\cdot)$.

Hence we see that the hypersurface $\partial[U_\varepsilon(\Sigma_{p,q})]$ has focal radius $\geq \frac{\pi}{2} - \varepsilon$ along $\sigma_0$, where $U_\varepsilon(C) = \{y \in S^n | d(y, C) < \varepsilon\}$.

To prove (3.2), it is sufficient to show that $\sigma_0(\cdot)$ is totally geodesic at $z$ with $0 < d(z, 0_p) < \frac{\pi}{2}$. It remains to show that $\partial[\Sigma_{p,q}]$ is a k-dimensional great circle in $S^n$.

For this purpose, we let $S^{n-1} = \{(\tilde{v}, 0) | (\tilde{v}, 0) \in S^n \subset \mathbb{R}^{n+1}\}$ be the equator of $S^n$. Let $\Psi : S^n - \{\pm \varepsilon_{n+1}\} \to S^{n-1}$ be the nearest point projection to the equator $S^{n-1}$ given by $\Psi(\varepsilon) = \frac{\varepsilon - (\varepsilon, \varepsilon_{n+1})\varepsilon_{n+1}}{|\varepsilon - (\varepsilon, \varepsilon_{n+1})\varepsilon_{n+1}|}$. It is easy to see that $\Psi$ takes a geodesic segment in $S^n$ to a arc of a great circle in $S^{n-1}$. Since $\Sigma_{p,q}$ is totally geodesic at $z$ with $0 < d(z, 0_p) < \frac{\pi}{2}$, $\Psi(\Sigma_{p,q})$ must be contained in a totally geodesic subset in the equator $S^{n-1}$. However, it is easy to see that $\Psi(\Sigma_{p,q}) = \partial[\Sigma_{p,q}]$. It follows that $\partial[\Sigma_{p,q}]$ is totally geodesic in $S^n$. Consequently, $\pi_{p,j}(q)$ is a great circle in $S^n$.

(2) We first observe $\text{Inj}_{A'}(q) = \frac{\pi}{2}$, due to [Ran]. Here is a direct proof of $\text{Inj}_{A'}(q) = \frac{\pi}{2}$ without using results of [Ran].

We now consider the Riemannian submersion $\pi_p : S_p^+(A, M) \to A'$, where $S^{n-1} = (\partial B_{\frac{\pi}{2}}(0), g_1)$ is the equator of $S^n \subset \mathbb{R}^{n+1}$. Since $\pi^{-1}(q)$ is totally geodesic, it is a great circle. We still isometrically embed $S^n$ into $\mathbb{R}^{n+1}$ as above. It follows that the linear sub-space $\text{Span}\{\pi^{-1}(q)\}$ spanned by $\pi^{-1}(q)$ is isometric to a $(k + 1)$-dimensional $\mathbb{R}^{k+1}$.

For each geodesic segment of unit speed $\hat{\sigma} : [0, \frac{\pi}{2}] \to A'$ from $q$ to $y = \hat{\sigma}(\frac{\pi}{2})$, we will show that $d_{A'}(q, y) = \frac{\pi}{2}$.
Let \( \tilde{\sigma} : [0, \frac{\pi}{2}] \to S^{n-1} \) be a horizontal lift of \( \tilde{\sigma} \). As we pointed out above, we can write \( \tilde{\sigma}(t) = \cos t \tilde{q} + \sin t \tilde{\sigma}'(0) \), where \( \tilde{\sigma}'(0) = \tilde{y} \perp \tilde{q} \). At time \( t = \frac{\pi}{2} \), the vector \( \tilde{\sigma}'(\frac{\pi}{2}) \) becomes horizontal. Thus, \( \tilde{q} = \tilde{\sigma}'(\frac{\pi}{2}) \) is orthogonal to \( T_y(\pi^{-1}(y)) \subset R_y^{k+1} \), where \( \tilde{y} = \tilde{\sigma}(\frac{\pi}{2}) \in \pi^{-1}(y) \). Recall that \( \tilde{y} = \tilde{\sigma}'(0) \perp \tilde{q} \). Hence, \( \tilde{q} \perp R_y^{k+1} \).

Suppose contrary, if \( d_A(q, y) = \alpha < \frac{\pi}{2} \). Then there would be another length-minimizing geodesic \( \hat{\sigma}_2 : [0, \alpha] \to A' \) from \( q \) to \( y \). Using the horizontal lift \( \hat{\sigma}_2 \) of \( \hat{\sigma}_2 \) with the initial point \( \tilde{q} \), we would be able to find \( \tilde{z} = \hat{\sigma}_2(\alpha) \in \pi^{-1}(y) \). It would follow that the angle between \( \tilde{q} \) and \( \tilde{z} \) is equal to \( \alpha < \frac{\pi}{2} \), which contradicts to the fact \( \tilde{q} \perp R_y^{k+1} \). Thus, any geodesic segment \( \hat{\sigma} : [0, \frac{\pi}{2}] \to A' \) of unit speed is length-minimizing, and hence \( \text{Inj}_{A'}(q) = \frac{\pi}{2} \).

Let us now further prove \( \text{Inj}_M(q) = \frac{\pi}{2} \). Let \( \sigma : [0, \frac{\pi}{2}] \to M \) be any geodesic segment with \( \sigma(0) = q \) and unit speed. If \( \sigma'(0) \perp A' \), then by Proposition 2.1 \( z = \sigma(\frac{\pi}{2}) \in A \) and hence \( d(q, \sigma(\frac{\pi}{2})) = \frac{\pi}{2} \). Thus, \( \sigma : [0, \frac{\pi}{2}] \to M \) is length-minimizing in this case.

If \( \sigma'(0) = (\cos \beta) \tilde{v} + (\sin \beta) \tilde{h} \) for some \( \tilde{v} \perp A' \), \( \tilde{h} \in T_q(A') \) and \( 0 < \beta < \frac{\pi}{2} \), we let \( \Psi_{A'} : [M - A'] \to A \) be the nearest point projection, and let \( \Psi_{A'} : [M - A] \to A' \) the nearest point projection. Let

\[ z = \sigma\left(\frac{\pi}{2}\right) \]

For \( y = \Psi_{A'}(z) = \Psi_{A'}(\sigma(\frac{\pi}{2})) \), by Lemma 3.1 of [GG1], \( \{q, y, z\} \) and \( \Psi_{A'}(\sigma(\mathbb{R})) \) are contained in a totally geodesic 2-sphere. Moreover, \( \Psi_{A'}(\sigma([0, \frac{\pi}{2}])) \) is a geodesic segment of length \( \frac{\pi}{2} \). Thus, since the injectivity radius of \( A' \) is equal to \( \frac{\pi}{2} \), one has that \( d_{A'}(y, q) \) is equal to the length of \( \Psi_{A'}(\sigma([0, \frac{\pi}{2}])) \), which is \( \frac{\pi}{2} \). Let \( x = \Psi_A(z) \in A \). It is clear that \( d(x, q) \geq d(A, q) = \frac{\pi}{2} \). Hence, we have \( \{x, y\} \subset \partial B_{\frac{\pi}{2}}(q) \). \{q\}'.

We already showed that \( \{x, y\} \subset \{q\}' \) holds. It now follows from Proposition 1.3 of [GG1] that \( \{q\}' \) is \( \pi \)-convex. Because \( z \) lies on a geodesic segment of length \( \frac{\pi}{2} < \pi \) from \( x \) to \( y \) and \( \{x, y\} \subset \{q\}' \), by the \( \pi \)-convexity of \( \{q\}' \) we obtain that \( z \in \{q\}' \). Therefore, any geodesic segment \( \sigma : [0, \frac{\pi}{2}] \to M \) of unit speed from \( q \) is length-minimizing for all cases. The assertion of \( \text{Inj}_M(q) = \frac{\pi}{2} \) is proved.

Q.E.D.

**Step 2.** For the convenience to the reader, we include the detailed proof of the following elementary result.

**Proposition 3.2** Let \( \tilde{\Sigma} \subset S^n \) be a \((k+1)\)-dimensional submanifold which is smooth at \( z \in \tilde{\Sigma} \). Suppose that \( \tilde{h} \in T_z(S^n) \) is a unit normal vector of \( \tilde{\Sigma} \) at \( z \), \( \sigma_{z, \tilde{h}}(t) = \text{Exp}_{z}^{S^n}(t \tilde{h}) \) and let \( \Gamma^0_{\sigma_{z, \tilde{h}}} : \tilde{\Sigma} \) be as in \( \text{S1} \) above. Then

1. \( \dim[\Gamma^0_{\sigma_{z, \tilde{h}}} : \tilde{\Sigma} p, q] = k + 1; \)

2. If \( \tilde{\Sigma} \) has the first focal radius \( t_0 < \frac{\pi}{2} \) along \( \sigma_{z, \tilde{h}} \), then there must be a non-trivial Jacobi field \( \{J(t)\} \) along \( \sigma_{z, \tilde{h}} \) with \( J'(0) = (-\cot t_0)J(0) \in T_z(\tilde{\Sigma}) \), and hence \( J \in \Gamma^0_{\sigma_{z, \tilde{h}}} \).
Proof. (1) Let $N(\tilde{\Sigma})|_{B_\varepsilon(z)} = \{(y, w) | y \in \tilde{\Sigma}, d(y, z) < \varepsilon, w \perp T_y(\tilde{\Sigma})\}$ be the normal bundle of $\tilde{\Sigma}$ near $z$. Suppose that $G = \text{Exp}^{\mathbb{S}^n} : N(\tilde{\Sigma})|_{B_\varepsilon(z)} \rightarrow \mathbb{S}^n$ is the exponential map of $\mathbb{S}^n$ restricted to the normal bundle of $\tilde{\Sigma}$. For any curve $\zeta : (-\delta, \delta) \rightarrow N(\tilde{\Sigma})$ with $\zeta(0) = (z, \tilde{h})$ with $\zeta(s) = (y(s), \tilde{w}(s))$, there is an 1-family of geodesics given by $F(t, s) = G(y(s), t\tilde{w}(s)) = \text{Exp}_{y(s)}[t\tilde{w}(s)]$.

Let $J(t) = \frac{\partial F}{\partial t}(t, 0) = G_y|_{(z, \tilde{h})} \zeta'(0)$. Since $\frac{\partial F}{\partial t}(0, s) = \tilde{w}(s) \perp \tilde{\Sigma}$ for all $s \in (-\delta, \delta)$, by the Gauss-Codazzi equation we obtain that, for all $\tilde{X} \in T_z(\tilde{\Sigma})$,

$$\langle J'(0), X \rangle = \langle \tilde{w}'(0), X \rangle = -\langle \tilde{w}(0), \nabla_{y'(0)}X \rangle = -\langle \tilde{h}, \nabla_{J(0)}X \rangle$$

holds. Hence, the tangential component of $J'(0)$ is uniquely determined by the second fundamental form of $\tilde{\Sigma}$:

$$\langle J'(0), X \rangle = -II_{\tilde{h}}(J(0), X) = -\langle \tilde{h}, \nabla_{J(0)}X \rangle$$ (3.4)

for all $X \in T_z(\tilde{\Sigma})$. Let us consider the classical Weingarten map $W^\tilde{h} : T_z(\tilde{\Sigma}) \rightarrow T_z(\tilde{\Sigma})$, where $W^\tilde{h}(Y)$ is given by the second fundamental form associated with $\tilde{h}$:

$$\langle W^\tilde{h}(Y), X \rangle = -II_{\tilde{h}}(Y, X) = -\langle \tilde{h}, \nabla_X Y \rangle$$ (3.5)

for all $X \in T_z(\tilde{\Sigma})$. Hence, our Jacobi field $J$ satisfies the Codazzi equation

$$[J'(0)]^\top = W^\tilde{h} J(0),$$ (3.6)

where $[\tilde{\eta}]^\top$ denotes the tangential component of $\tilde{\eta} \in T_z(\mathbb{S}^n)$. It follows that

$$J \in \Gamma_{\sigma_z, \tilde{\Sigma}}.$$

For $J \in \Gamma^0_{\sigma_z, \tilde{\Sigma}}$, we further require that $J'(0) \in T_z(\mathbb{S}^n)$. Hence, it follows from (3.6) that

$$J'(0) = W^\tilde{h} J(0)$$ (3.7)

Because $J(0) \in T_z(\tilde{\Sigma})$ and $\dim(\tilde{\Sigma}) = k + 1$, by (3.7) one has $\dim[\Gamma^0_{\sigma_z, \tilde{\Sigma}}] \leq (k + 1)$.

We now prove that $\dim[\Gamma^0_{\sigma_z, \tilde{\Sigma}}] \geq (k + 1)$. Let $\{v_1, ..., v_{k+1}\}$ be an orthogonal basis of $T_z(\tilde{\Sigma})$.

It is well-known that, on each curve $s \rightarrow y_i(s)$ with $y_i(0) = z$ and $y_i'(0) = \tilde{v}_i$, there is a unique a vector field $\{\tilde{w}_i(s)\}$ satisfying $\tilde{w}_i(s) \perp T_{y_i(s)}(\tilde{\Sigma})$ and

$$[\nabla_{y_i'(s)} \tilde{w}_i(s)]^\perp = 0$$ (3.8)

with $\tilde{w}_i(0) = \tilde{h}$, where $[\tilde{\eta}]^\perp$ denotes the normal component of $\tilde{\eta}$. The linear system (3.8) has $(n - k - 1)$-unknowns and $(n - k - 1)$-equations. Thus, the system (3.8) has a unique solution $\tilde{w}_i(s)$ with $\tilde{w}_i(0) = \tilde{h}$.  

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Let \( F_i(t, s) = \text{Exp}_{y_i(s)}[t \vec{w}_i(s)] \) and \( J_i(t) = \frac{\partial F_i}{\partial s}(t, 0) \). Then \( J_i \in \Gamma^0_{\sigma, \tilde{\Sigma}} \) for \( i = 1, 2, \ldots, k + 1 \). Clearly, \( \{ J_1(0), \ldots, J_{k+1}(0) \} = \{ \vec{v}_1, \ldots, \vec{v}_{k+1} \} \) are linearly independent. Thus, \( \dim[\Gamma^0_{\sigma, \tilde{\Sigma}}] \geq (k+1) \).

(2) If \( \tilde{\Sigma} \) has the first focal point \( \sigma(t_0) \) along \( \sigma \) with \( 0 < t_0 < \frac{\pi}{2} \), then there must be an orthogonal Jacobi field \( \{ J(t) \} \) along \( \sigma \) in \( S^n \) with \( J(0) \in T_z(\tilde{\Sigma}) \) and \( J(t_0) = 0 \). It is well-known that, in \( S^n \), any Jacobi field \( \{ J(t) \} \) with \( J(t_0) = 0 \) can be expressed as \( J(t) = \sin(t - t_0)cE(t) \), where \( c \) is a non-zero constant and \( \{ E(t) \} \) is a unit parallel vector field along \( \sigma \).

Because \( 0 < t_0 < \frac{\pi}{2} \), we obtain that \( J(0) = -\sin(t_0)cE(0) \neq 0 \). Since \( J(0) \in T_z(\tilde{\Sigma}) \), we see that \( E(0) = -\frac{1}{\sin t_0}J(0) \in T_z(\tilde{\Sigma}) \). It follows that \( J'(0) = \cot(t_0)cE(t) \in \tilde{\Sigma} \) and hence \( J \in \Gamma^0_{\sigma, \tilde{\Sigma}} \). Q.E.D.

**Step 3.** We will use the Hessian comparison theorem to show that if \( \pi_p : S^2_p(A, M) \to A' \) is a Riemannian submersion, then \( \tilde{\Sigma}_{p,q} \) has the first focal radius \( \geq \frac{\pi}{2} \) in \( S^n \), and hence \( \pi_p^{-1}(q) \) is a great circle by Step 1.

We will divide it into two sub-steps:

**Step 3.1.** Using the Hessian comparison theorem, we study the decomposition of \( T_p(M^n) \) associated with \( \{ A, A' \} \) and parallel transports. As an application, we will show that the covariant derivatives of horizontal lifting vector fields along each fiber \( \tilde{\Sigma}_{p,q} \) must be vertical.

**Step 3.2.** By Step 3.1, we will construct all Jacobi fields \( J \in \Gamma^0_{\sigma, \tilde{\Sigma}} \) with vertical initial derivatives explicitly. A simple calculation will show that any non-trivial element \( J \in \Gamma^0_{\sigma, \tilde{\Sigma}} \) has the non-vanishing property \( J(t) \neq 0 \) for \( 0 \leq t < \frac{\pi}{2} \). This will complete the proof of Theorem A.

**Step 3.1.** Hessian comparison and the covariant derivatives of horizontal lifting vector fields along each fiber.

We will frequently use the following result.

**Proposition 3.3 (cf. [GG1])** Let \( A, A' \) and \( M \) be as in Proposition 2.4. Suppose that \( (p, q) \in A \times A' \). Then

1. Whenever \( \dim(A') > 0 \), for any unit tangent vector \( \vec{v}_0 \in T_q(A') \) and a unit normal vector \( \vec{v} \in S^1_\vec{v}(A', M) \), the image \( \text{Exp}_q(\mathbb{R}^2_{\vec{v}_0, \vec{v}}) \) is a totally geodesic immersed \( 2 \)-sphere of constant sectional curvature \( 1 \), where \( \mathbb{R}^2_{\vec{v}_0, \vec{v}} = \text{Span}_\mathbb{R}\{\vec{v}_0, \vec{v}\} \) is a real \( 2 \)-dimensional tangent subspace spanned by \( \{\vec{v}_0, \vec{v}\} \).

2. Let \( \Psi_{A'} : [M - A] \to A' \) be the nearest point projection, \( \Phi_{A'} : ([B^+_{\vec{v}}(0_p) - \{0_p\}], g_t) \to A' \) be given by \( \Phi_{A'}(z) = \Psi_{A'}(\text{Exp}_p(z)) \) for \( z \in B^+_{\vec{v}}(0_p) = \{z \in T_p(M) | z \perp T_p(A), |z| < \frac{\pi}{2} \} \) and
Suppose that \( S_{r_0}^{1}(0_p) = \partial B_{r_0}^{1}(0_p) \). Then for \( r_0 \in (0, \frac{\pi}{2}) \rightarrow A' \), the map
\[
\Phi_{A'}|_{S_{r_0}^{1}(0_p)} : S_{r_0}^{1}(0_p) \rightarrow A'
\]
is a Riemannian submersion up to a constant factor \( c = \frac{1}{\sin r_0} \) with respect to the spherical metric \( g_1 \) on \( S_{r_0}^{1}(0_p) \subset B_{\pi}(0_p) \).

The similar conclusions hold at \( p \in A \) if \( \dim(A) > 0 \).

Let us consider the normal bundle of \( \tilde{\Sigma}_{p,q} \) at \( z \) with \( 0 < |z| \leq \frac{\pi}{2} \).

**Definition 3.4** Let \( (p,q) \in A \times A' \) be as above. If \( z \in \tilde{\Sigma}_{p,q} \subset B_{\pi}(0) \subset T_p(M^n) \) with \( 0 < |z| \leq \frac{\pi}{2} \) and if \( \tilde{h} \perp T_z(\tilde{\Sigma}_{p,q}) \) then the vector \( \tilde{h} \) is called a horizontal vector.

Similarly, if \( \hat{z} \in M^n \) with \( 0 < d(p, \hat{z}) \leq \frac{\pi}{2} \) and \( \hat{h} \perp T_{\hat{z}}(\Sigma_{p,q}) \) then the vector \( \hat{h} \) is called a horizontal vector.

The horizontal subspace at \( \hat{z} \) is denoted by \( H_{\hat{z}} \).

We will use the Hessian comparison theorem show that the horizontal subspaces is invariant under the parallel translation along radial geodesics from \( A' \) to \( p \). If \( c : [a, b] \rightarrow M \) is a curve, we let \( \tau_{c(t_1)}^{c(t_2)} \) be the parallel translation along the curve \( c \).

**Theorem 3.5** Suppose that \( (p,q) \in A \times A' \) and \( M^n \) are as in Proposition 2.4 and suppose that \( \sigma : [0, \frac{\pi}{2}] \rightarrow M^n \) be a geodesic of unit speed from \( q \) to \( p \). Then the tangent space \( T_{\sigma(t)}(M^n) \) has the following orthogonal decomposition:
\[
T_{\sigma(t)}(M^n) = \tau_{\sigma(0)}^{\sigma(t)}[T_q(A')] \bigoplus \tau_{\sigma(q)}^{\sigma(t)}[T_q(A)] \bigoplus T_{\sigma(t)}(\Sigma_{p,q}).
\]
Hence, \( \tau_{\sigma(0)}^{\sigma(t)}[T_q(A')] \bigoplus \tau_{\sigma(q)}^{\sigma(t)}[T_q(A)] \) is equal to the horizontal subspace \( H_{\sigma(t)} \) at \( \sigma(t) \) for \( t \in [0, \frac{\pi}{2}] \).

**Proof.** By Lemma 3.1 of [GG1], \( \sigma'(t) \perp H_{\sigma(t)} \). We need to show that \( T_{\sigma(t)}(\Sigma_{p,q}) \perp H_{\sigma(t)} \). For this purpose, we use the sharp version of Hessian comparison.

Let \( m = \dim A, m' = \dim A' \) and \( k + 1 = \dim(\Sigma_{p,q}) \). We will also see that \( \dim M^n = m + m' + (k + 1) \).

Let \( f(x) = d(x, A') \). Because \( A' \) is totally geodesic, there are \( m' \) Jacobi fields \( \{J_1(t), J_2(t), \ldots, J_{m'}(t)\} \) along \( \sigma \) such that \( \{J_1(0), J_2(0), \ldots, J_{m'}(0)\} \) is an orthonormal basis of \( T_q(A') \) and \( J_i'(0) = 0 \) for \( i = 1, 2, \ldots, m' \).

Similarly, if \( \dim A > 0 \) there are \( m \) Jacobi fields \( \{J_{m'+1}(t), J_{m'+2}(t), \ldots, J_{m'+m}(t)\} \) along \( \sigma \) such that \( \{J_{m'+1}(0), J_{m'+2}(0), \ldots, J_{m'+m}(0)\} \) is an orthonormal basis of \( T_q(A') \) and \( J_{m'+i}'(0) = 0 \) for \( i = 1, 2, \ldots, m \).

We already knew that the cut-radii of \( A' \) and \( A \) are equal to the diameter of \( M^n \), which is \( \frac{\pi}{2} \). Thus, \( J_i(t) \neq 0 \) for \( i = 1, 2, \ldots, 8 \) and \( t \in [0, \frac{\pi}{2}] \).
≥ 1, by Berger comparison theorem (or the 2nd Rauch comparison theorem), we can find a 
parallel vector field \{E_i(t)\} along \sigma such that \(J_i(t) = \cos tE_i(t)\) for \(i = 1, 2, \ldots, m\) and \(J_{m+j}(t) = \sin tE_{m+j}(t)\) for \(j = 1, \ldots, m\) if \(m = \dim A > 0\). It is clear that

\[
\text{Hess}(f)(J, J) = \langle J(t), J'(t) \rangle.
\]

It also is well-known that the Hessian of distance function \(f\) satisfies the so-called Riccati equation:

\[
\nabla_{\sigma'(t)}[\text{Hess}(f)] + [\text{Hess}(f)]^2 + R = 0.
\]

More precisely, we let \(\{E_i(t)\}_{1 \leq i \leq n}\) be a parallel orthonormal base along the geodesic segment \(\varphi_v\) with \(E_n(t) = \sigma'(t)\), \(H_{ij}(t) = \text{Hess}(f)(E_i(t), E_j(t))\) and \(R_{ij}(t) = \langle R(\sigma(t), E_i(t))\sigma'(t), E_j(t)\rangle\), where \(R(X, Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z - \nabla_{[X, Y]} Z\) is the curvature tensor. Thus, we have

\[
H' + H^2 + R = 0.
\]

Let

\[
W_{A'}(t) = \{Y(t) \mid H(., Y(t))|_{\sigma(t)} = \tan(t)\langle ., Y(t) \rangle\}
\]

and

\[
W_A(t) = \{Y(t) \mid H(., Y(t))|_{\sigma(t)} = \cot(t)\langle ., Y(t) \rangle\}.
\]

We have shown that the eigenspace \(\{W_{A'}(t)\}\) is invariant under parallel translation along \(\sigma\). Similarly, if \(\dim A > 0\), then \(\{W_A(t)\}\) is invariant under parallel translation along \(\sigma\).

Choose \(t_0 = \frac{\pi}{2}\). It is clear \(\cot \frac{\pi}{2} \neq \tan \frac{\pi}{2}\). Thus,

\[
W_{A'}(t) \perp W_A(t)
\]

whenever \(\dim A > 0\).

In what follows, we prove that

\[
T_{\sigma(t)}(\Sigma_{p,q}) \perp [W_{A'}(t) \bigoplus W_A(t)].
\]

We already showed that \(E_j(t) \in W(t)\) for \(j = 1, 2, \ldots, m'\). Notice that \(H_{jj}(t)\) blows up as \(t \to 0^+\) for \(j > (m' + m)\). If \(\{\lambda_{m+m'+1}(t), \lambda_{m+m'+2}, \ldots, \lambda_{m+m'+k}(t)\}\) are other eigenvalues of \(H\), then \(\lambda_j(t) \to +\infty\) as \(t \to 0\) for \(j \leq (m + m')\). Thus, the corresponding eigenvectors are orthogonal to \(W_{A'}(t)\), because eigenvalues are different.

Similarly, if \(\dim A > 0\), we consider \(t \to \frac{\pi}{2}\), then \(\lambda_j(t) \to +\infty\) as \(t \to \frac{\pi}{2}\) for \(j \leq (m + m')\). For the same reason, the corresponding eigenvectors are orthogonal to \(W_A(t)\), because eigenvalues are different.

Therefore, we proved

\[
H_{ij}(t) = 0
\]

for \(i = 1, \ldots, (m + m')\) and \(j > (m + m')\).
Let \( \{x_1, ..., x_{m'}\} \) be a geodesic normal coordinate system of \( A' \) at \( q \) given by \( G : \mathbb{R}^{m'} \to A' \) with \( G(x_1, ..., x_{m'}) = \text{Exp}_q(\sum_1^{m'} x_i E_i(0)) \). Recall that \( \dim\{[T_q(A')]^\perp\} = m + k + 1 \). Thus there exists an orthonomal basis \( \{E_{m'+1}, ..., E_{k-1}, E_n\} \) of \([T_q(A')]^\perp\) such that \( E_n = \sigma'(0) \). Let \( \vec{\theta} = (\theta_{m'+1}, ..., \theta_n) \) with \(|\vec{\theta}| \leq 1\). Then \( \Psi : B_1(0) \to S^{n-m'-1} = S_q^\perp(A', M^n) \) given by

\[
\Psi(\theta_{m'+1}, ..., \theta_{n-1}) = \sum_{j=m'+1}^n \theta_j E_j + \sqrt{1 - |\vec{\theta}|^2} \sigma'(0)
\]
gives rise to a local coordinate system of \( S^{n-m'-1} = S_q^\perp(A', M^n) \) around \( \sigma'(0) \). Using the parallel transport \( \tau_{G(x)}^{G(0)} \) from \( q = G(0) \) to \( G(x) \) we have a local coordinate system given by

\[
(\theta_{m'+1}, ..., \theta_{n-1}) \to \tau_{G(x)}^{G(0)}(\Psi(\vec{\theta})) \text{ for } S_{G(x)}^\perp(A', M^n).
\]

Therefore, \( \{(x_1, ..., x_{m'}; \theta_{m'+1}, ..., \theta_{n-1}, t)\} \) gives rise to a local coordinate for normal bundle of \( A' \) in \( M^n \) near \((x, t, \sigma'(0))\). In fact, the \( F(x_1, ..., x_{m'}; \theta_{m'+1}, ..., \theta_{n-1}, t) = \text{Exp}_{G(x)}(\tau_{G(x)}^{G(0)}(\Psi(\vec{\theta}))) \) does the job. Finally we let \( C_{ji}(t) = (\frac{\partial F}{\partial \theta_j})(E_i)_{\sigma(t)} \). It is well-known that \( H(t) = C'(t)C(t)^{-1} \). We already showed that \( W_{A'}(0) = T_q(A') \) and \( W(t) \) is parallel along \( \sigma \). Using \( C_{ji}(0) = 0 \) and the fact \( H_{ij}(t) = 0 \) for \( i = 1, ..., m' \) and \( j > (m'+1) \), by the integration of \( C'(t) = H(t)C(t) \) from \( \frac{\vec{\theta}}{2} \) to \( t \) we conclude that

\[
C_{ij}(t) = 0
\]

for \( i = 1, ..., m' \) and \( j > (m'+1) \). Thus, we see that \( \frac{\partial F}{\partial \theta_j} \in [W_{A'}(t)]^\perp \) for \( j > (m'+1) \).

Therefore, both tangential subspace \( T_{\sigma(t)}(\Sigma_{p,q}) \bigoplus W_{A'}(t) \) and sub-space \( W_{A'}(t) \) at \( \sigma(t) \) are invariant under parallel translation along \( \sigma \). It follows that

\[
T_{\sigma(t)}(\Sigma_{p,q}) \perp W_{A'}(t).
\]

For the same reason, if \( \dim A > 0 \), one has

\[
T_{\sigma(t)}(\Sigma_{p,q}) \perp W_{A}(t)
\]
as well. We already proved \( W_{A}(t) \perp W_{A'}(t) \). This completes the proof. Q.E.D.

Theorem 3.5 indicates that there is a non-trivial relation between the exponential map \( \text{Exp}_A \) along the normal bundle of \( A \) and the exponential map \( \text{Exp}_{A'} \) along the normal bundle of \( A' \). As an application of Theorem 3.5, we draw some conclusions.

**Corollary 3.6** Let \((p, q) \in A \times A'\), \( A \) and \( A' \) be as in Proposition 2.4 and \( \dim A' > 0 \). Suppose that \( \vec{\eta} \in T_q(A') \), \( \hat{\zeta} \in \Sigma_{p,q} \) with \( 0 < d(\hat{\zeta}, A') < \frac{\pi}{2} \), \( \hat{h}_q(\hat{\zeta}) \) is the parallel transport of \( \vec{\eta} \) along the unique length-minimizing geodesic segment from \( q \) to \( \hat{\zeta} \), \( z = (\text{Exp}_p)^{-1}(\hat{z}) \in \Sigma_{p,q} \) and

\[
\vec{h}_q(z) = ([\text{Exp}_p]^{-1})_* \hat{h}_q(\hat{\zeta})
\]
Then the horizontal lifting vector field $\{\vec{h}_\eta\}_{z\in \tilde{\Sigma}_{p,q}}$ of $\eta$ has the property

$$\nabla_X \vec{h}_\eta \in T_z(\tilde{\Sigma}_{p,q})$$

for all $X \in T_z(\tilde{\Sigma}_{p,q})$.

**Proof.** We first consider the case $X = \nabla r$, where $r(z) = |z| = d(0_p, z)$. By our assumption, there is a unique geodesic segment of unit speed from $q$ to $\hat{z}$, say $\sigma_{q,\hat{z}}$. Let $\vec{v} = \sigma'_{q,\hat{z}}(0)$. By Proposition 3.3 if we let $\mathbb{R}^2_{\vec{q},\vec{v}}$ be the subspace spanned by $\{\vec{h}, \vec{v}\}$, then $\hat{S}^2 = \text{Exp}_q(\mathbb{R}^2_{\vec{q},\vec{v}})$ is a totally geodesic immersed 2-sphere $S^2_{\vec{q},\vec{v}}$ of constant curvature 1, which passes both $p$ and $q$. It follows that, on the unit 2-sphere $S^2_{\vec{h},\vec{v}}$, one has

$$\nabla_{\nabla r} \vec{h}_\eta |_{\hat{z}} = 0 \quad (3.10)$$

We now consider the remaining case $X \in T_z(\tilde{\Sigma}_{p,q})$ but $X \perp \nabla r$. Let

$$B_{0_p,\pi} = \{z \in T_p(M) \mid z \perp T_p(A), |z| \leq \pi\}.$$

In terms of the spherical metric $g_1$, the sub-manifold $(B_{0_p,\pi}, g_1)$ is a totally geodesic $(n - m)$-dimensional sphere $S^{n-m}$, where $m = \dim A$, $m' = \dim A'$, $k + 1 = \dim \Sigma_{p,q}$ and $n = \dim M = m + m' + k + 1$.

Let $\Psi_{A'} : [M - A] \to A'$ be the nearest point projection, $S^1_{p,r} = \{z \in T_p(M) \mid z \perp T_p(A), |z| = r\}$. By Proposition 3.3 and Theorem 3.5 in terms of the spherical metric $g_1$ on $B_{\pi}(0_p)$, the map

$$\tilde{\Psi}_{A'} : S^1_{p,r} \to A'$$

$$z \to \Psi_{A'}[\text{Exp}_p(z)]$$

is a Riemannian submersion up to a constant factor $\frac{1}{\sin r}$. Consequently, if $\{\vec{h}_{\vec{h}_1}, \ldots, \vec{h}_{\vec{h}_{m'}}\}$ is an orthonormal basis of $T_q(A')$, then by Theorem 3.5 and its proof, the set of vectors

$$\{\vec{h}_{\vec{h}_1}, \ldots, \vec{h}_{\vec{h}_{m'}}\}$$

form a basis of the normal bundle $N(\tilde{\Sigma}_{p,q}, B_{0_p,\pi})$ of $\tilde{\Sigma}_{p,q}$ at $z$ in $S^{n-m} = (B_{0_p,\pi}, g_1)$.

Let $\{x_1, \ldots, x_{m'}\}$ be the geodesic normal coordinate of $A'$ at $q$. We choose $\vec{h}_{\vec{h}_1} = \frac{\partial}{\partial x_1}$ at $0_q$, i.e., we use the map $(x_1, \ldots, x_{m'}) \to \text{Exp}_q(x_1 \vec{h}_1 + \ldots x_{m'} \vec{h}_{m'})$ as the geodesic coordinate system of $A'$ at $q$.

Suppose that $G(x) = \text{Exp}_q(x_1 \vec{h}_1 + \ldots x_{m'} \vec{h}_{m'})$ and recall that $\vec{v} = \sigma'_{q,\hat{z}}(0)$. Let us consider the Fermi coordinate system (the exponential map) along $A'$:

$$F(x, \rho \vec{v}) = \text{Exp}_{G(x)}[\tau_{q}^{G(x)}(\rho \vec{v})].$$

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By the proof of Theorem 3.5, we have
\[ h_i = \frac{1}{\sin |z|} \left| \frac{\partial (\text{Exp}^{-1}_p \circ F)}{\partial x_i} \right|_{(0, \rho_0 \bar{v})}, \]
where \[ \rho_0 = \frac{\pi}{2} - |z|. \]

For simplicity, we denote \( \text{Exp}^{-1}_p \circ F \) by \( \tilde{F} \). We now choose \( X = \frac{\partial \tilde{F}}{\partial v_i} \mid_{(0, \rho_0 \bar{v})} \) for \( \bar{v} = (v_1, ..., v_k) \in [T_q(A')]^\perp \) and \( |\bar{v}| = 1 \), where we only allow \( \bar{v} \in S_q^\perp (A', M) \). It is easy to see (cf. [CE, page2]) that, if \( [X, Y] = 0 = [Y, Z] = [X, Z] \), then
\[ \langle \nabla_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle. \]

Therefore, setting \( \rho_0 = \frac{\pi}{2} - |z| \) and by a direct calculation, one has that, if \( X = \frac{\partial \tilde{F}}{\partial v_i} \mid_{(0, \rho_0 \bar{v})} \) then
\[ \langle \nabla_X \bar{h}_\eta_i, \bar{h}_\eta_j \rangle = 0 + 0 - 0 = 0 \]
for all \( i, j = 1, ..., m' \). This completes the proof. Q.E.D.

A direct consequence of the above corollary is the following result.

**Corollary 3.7** Let \( (p, q) \in A \times A' \), \( A \) and \( A' \) be as in Proposition 2.4 and \( \dim A' > 0 \). \( \bar{\eta} \in T_q(A'), \hat{z} \in \Sigma_{p,q} \) with \( 0 < d(\hat{z}, A') < \frac{\pi}{2} \), \( \bar{h}_\eta(\hat{z}) \) is the parallel transport of \( \bar{\eta} \) along the unique length-minimizing geodesic segment from \( q \) to \( \hat{z} \), \( z = (\text{Exp}_p)^{-1}(\hat{z}) \in \tilde{\Sigma}_{p,q} \),
\[ \bar{h}_\eta(z) = [(\text{Exp}_p)^{-1}]_\hat{z} \bar{h}_\eta(\hat{z}) \]

and
\[ F_{\bar{\eta}}(t, z) = \text{Exp}_z^\text{Sn} \left[ t \bar{h}_\eta(z) \right]. \] (3.12)

Then the corresponding Jacobi fields
\[ J_i(t) = \frac{\partial F_{\bar{\eta}}}{\partial z_i}(t, z) \] (3.13)
along the geodesic \( \{ F_{\bar{\eta}}(., z) \} \) has the property
\[ J_i'(0) \in T_z(\tilde{\Sigma}_{p,q}) \] (3.14)
for \( i = 1, ..., k + 1 \), where \( \{ z_1, ..., z_{k+1} \} \) is any local coordinate system of \( \tilde{\Sigma}_{p,q} \) around \( z \).
Step 3.2. Proof of Theorem A.

We recall some elementary facts about the geodesic triangles in a unit 2-sphere \( S^2 \), which are isometrically immersed in \( M \), see Proposition 3.3 (1) above.

Lemma 3.1 Let \( \hat{z} \in \Sigma_{p,q} \) with \( 0 < r_0 = d(\hat{z}, A) < \frac{\pi}{2} \) and \( S^2 = S^2_{q,(z,h)} \) be a totally geodesic immersed 2-sphere in \( M \) given by \( S^2_{q,(z,h)} = \text{Exp}_z(\mathbb{R}^2_{q,(z,h)}) \), where \( \hat{h} \) is a unit horizontal vector and \( \mathbb{R}^2_{q,(z,h)} = \text{Span}\{\hat{h}, \text{Exp}_z^{-1}(q)\} \) described as in Proposition 3.3(1).

(1) If \( \varphi_{\hat{h}}(t) = \text{Exp}_z(t\hat{h}) \) for some unit horizontal vector at \( \hat{z} \) with \( 0 < r_0 = d(\hat{z}, A) < \frac{\pi}{2} \), then \( \varphi_{\hat{h}}(\frac{\pi}{2}) \in A' \); 
(2) For \( 0 < t < \frac{\pi}{2} \), the distance function satisfies \( r(t) = d(\varphi_{\hat{h}}(t), A) = \arccos[\cos r_0 \cos t] \); Consequently, \( d(\varphi_{\hat{h}}(t), A') = \arcsin[\cos r_0 \cos t] \), where \( r_0 = d(\hat{z}, A) \).
(3) Let \( \Psi_{A'} : [M - A] \to A' \) be the nearest point projection and \( \ell(t) \) be the length of \( \Psi_{A'}[\varphi_{\hat{h}}([0,t])] \). Then
\[
\ell(t) = \arccos[\frac{\sin r_0 \cos t}{\sqrt{1 - (\cos r_0 \cos t)^2}}].
\]
(4) The vector \( [\varphi'_{\hat{h}}(t) - \langle \varphi'_{\hat{h}}(t), \nabla r \rangle \nabla r] \) remains to be horizontal, where \( r(x) = d(A, x) \).

The lemma above can be proved by the law of cosine in \( S^2 \), see [Pe, page 314]. Finally, we can now show that \( \tilde{\Sigma}_{p,q} \) has focal radius \( \geq \frac{\pi}{2} \) in \( S^n \).

Lemma 3.2 Let \( z \in \tilde{\Sigma}_{p,q} \subset S^n \) and \( J_i(t) \) be as in Corollary 3.7 above. Then
\[
J_i(t) \neq 0
\]
for all \( t \in (0, \frac{\pi}{2}) \). Consequently, \( \tilde{\Sigma}_{p,q} \) has focal radius \( \geq \frac{\pi}{2} \) in \( S^n \).

Proof. We choose a special local coordinate system of \( \tilde{\Sigma}_{p,q} \) at \( z \) as follows. By Corollary 3.7, \( J'_i(0) \in T_z(\tilde{\Sigma}_{p,q}) \) for all \( i = 1, \ldots, k + 1 \). We can choose \( (k + 1)\)-principal directions \( \{e_1, \ldots, e_{k+1}\} \) of the Weingart map \( W\tilde{h} : X \to (\nabla_X \tilde{h}(z))^\top = \nabla_X \tilde{h}(z) \) for all \( X \in T_z(M) \), where
\[
\langle W\tilde{h}X, Y \rangle = \langle \nabla_X \tilde{h}(z), Y \rangle
\]
for all \( X, Y \in T_z(M) \) and \( (\tilde{w})^\top \) is the tangential component of \( \tilde{w} \).

It was proved \( \nabla r|_z = \frac{\tilde{w}}{|\tilde{w}|} \) and \( \tilde{h}(z) \) span a totally geodesic 2-sphere of constant curvature 1, see Proposition 3.3 (1) above. Thus, \( \nabla r|_z \) is an eigenvector of \( W\tilde{h} \). We choose \( e_{k+1} = \nabla r|_z \). Furthermore, in \( S^2 \), the corresponding Jacobi field can be written as \( J_{k+1}(t) = (\cos t)E_{k+1}(t) \), where \( \{E(t)\} \) is a parallel vector along \( \sigma_{z,h}(t) = \text{Exp}_z(t\tilde{h}) \) with \( E(0) = \nabla r|_z \). Hence, \( J_{k+1}(t) \neq 0 \) for all \( t \in (0, \frac{\pi}{2}) \).
We now consider the remaining \( \{J_1, \ldots, J_k\} \). Let
\[
v_i(s) = |z|[(\cos s)\frac{z}{|z|} + (\sin s)e_i]
\]
and
\[
F_i(t, s) = E_{v_i(s)}^S(t\hat{F}_i(v_i(s)))
\]
for \( i = 1, \ldots, k \). Finally, we set
\[
J_i(t) = \frac{\partial F}{\partial s}(t, 0)
\]
for \( i = 1, \ldots, k \).

In order to prove that \( J_i(t) \neq 0 \) for \( t \in (0, \frac{\pi}{2}) \), we use
\[
\hat{F}_i(t, s) = \text{Exp}_p^M[(\text{Exp}^{S^n}_p)^{-1}(F_i(t, s))]
\]
for \( i = 1, \ldots, k \). By Lemma 3.1 one has
\[
0 < \rho(t) = d(A', \hat{F}_i(t, s)) = \arcsin[(\cos |z|)\cos t] < \frac{\pi}{2}
\]
for \( 0 \leq t < \frac{\pi}{2} \). Let \( G = \text{Exp}_p^M[(\text{Exp}^{S^n}_p)^{-1}] \) and
\[
\hat{J}_i(t) = \frac{\partial \hat{F}_i}{\partial s}(t, 0) = G_sJ_i(t).
\]
Because \( G \) is a local diffeomorphism at all \( x \in B_{\frac{\pi}{2}}(0_p) \) with \( 0 < |x| < \frac{\pi}{2} \), using (3.15) one concludes the following is true: "\( J_i(t) \neq 0 \) holds for \( t \in (0, \frac{\pi}{2}) \) if and only if \( \hat{J}_i(t) \neq 0 \) holds for \( t \in (0, \frac{\pi}{2}) \)."

It remains to verify that \( \hat{J}_i(t) \neq 0 \) for \( t \in (0, \frac{\pi}{2}) \). For this purpose, we express \( \hat{J}_i(t) \) in terms of the Fermi coordinates along \( A' \) instead. In terms of the Fermi coordinates along \( A' \), we will clearly see that \( \hat{J}_i(t) \neq 0 \) for \( t \in (0, \frac{\pi}{2}) \). The detail for the new expressions of \( \hat{J}_i(t) \) and \( \hat{F}_i(t, s) \) can be given as follows:

Notice that \( \{h_{\hat{q}}(v_i(s)), \text{Exp}_{v_i(s)}^{-1}(q)\} \) span a totally geodesic immersed 2-sphere \( S^2_{v_i(s), \hat{h}_{\hat{q}}} \) of constant curvature 1. Such a 2-sphere \( S^2_{v_i(s), \hat{h}_{\hat{q}}} \) passes through the geodesic \( \hat{\sigma}_{\hat{q}}(\ell) = \text{Exp}_p(\ell\hat{q}) \).

Let
\[
\hat{\psi}_i(s) = \text{Exp}_q^{-1}[\hat{F}_i(0, s)],
\]
(3.16)
\( \tau_q^{\hat{\psi}(\ell)} \) be the parallel translation along \( \hat{\sigma}_{\hat{q}} \) and let \( \Psi_{A'} : [M - A] \rightarrow A' \) be the nearest point projection. Then, by Lemma 3.1 one has
\[
\ell(t) = d(\Psi_{A'}(\hat{\varphi}_{\hat{q}}(t), q), q) = \arccos\left(\frac{\sin r_0 \cos t}{\sqrt{1 - (\cos r_0 \cos t)^2}}\right).
\]
A direct calculation shows that if \( q(t) = \hat{\sigma}_q(\ell(t)) \) then

\[
\hat{F}_i(t, s) = \operatorname{Exp}_{q(t)}[\tau_q^{\ell(t)}(\rho(t)\hat{\psi}_i(s))]
\]

for \( i = 1, \ldots, k \). It follows that

\[
\hat{J}_i(t) = [\operatorname{Exp}_{q(t)}]_{*}[\tau_q^{\ell(t)}(\rho(t)\hat{\psi}_i(0))] = 0
\]

for all \( t \). Recall that the parallel transport \( \tau_q^{\ell(t)} : T_q(M) \to T_{q(t)}(M) \) is an isometry. Since the cut radius of \( A' \) is equal to \( \frac{\pi}{2} \), it follows equations (3.15) - (3.19) that

\[
\hat{J}_i(t) = [\operatorname{Exp}_{q(t)}]_{*}[\tau_q^{\ell(t)}(\rho(t)\hat{\psi}_i(0))] \neq 0
\]

for \( i = 1, \ldots, k \) and \( t \in (0, \frac{\pi}{2}) \), as long as \( \hat{\psi}_i(0) \neq 0 \) and \( \rho(t) \neq 0 \). Recall that \( J_i(0) \neq 0 \) and \( \rho(t) \neq 0 \) for \( t \in (0, \frac{\pi}{2}) \). Hence \( \hat{\psi}_i(0) \neq 0 \) and \( \hat{J}_i(t) \neq 0 \) holds for \( i = 1, \ldots, k \) and \( t \in (0, \frac{\pi}{2}) \). This completes the proof.

The end of the proof of Theorem A. By Steps 1-3 above, we proved that \( \pi_{p}^{-1}(q) \) is a great circle for each \( (p, q) \in A \times A' \). Furthermore, it follows from Proposition 3.1(2) that \( \operatorname{Diam}(A') = \frac{\pi}{2} \). We can also choose a point \( y \in A' \) with \( d(y, q) = \frac{\pi}{2} \). Using Proposition 3.1(2) again, we see that \( \operatorname{Inj}_y(y) = \frac{\pi}{2} \). By replacing \( p \) by \( y \) if needed, we may always assume that \( \operatorname{Inj}_y(p) = \frac{\pi}{2} \) and \( \dim A = 0 \). Hence, by [Ran], \( \pi_y : S_y(M) \to A' \) is isometric to the classical Hopf fibration and \( M \) is isometric to one of \( \{CP^n, HP^n, CaP^2\} \).

Professor Grove kindly pointed out that “if \( \pi_y : S_y(M) \to A' \) is a great circle fibration then one can show that \( M^n \) is isometric to one of \( \{CP^n, HP^n, CaP^2\} \) directly without using [Ran].” The following argument is an outline of a direct proof inspired by Professor Grove, but authors are responsible for all possible errors.

Let \( y \in A' \) with \( d(y, q) = \frac{\pi}{2} \) be as above and \( M' \) be the convex hull of \( \{y\} \cup \Sigma_{p,q} \) in \( M \). Then, by the \( \pi \)-convexity of \( S_{\frac{\pi}{2}}(y) \) described in [GG1], one has \( \pi_y : S_y(M') \to \Sigma_{p,q} \) a Riemannian submersion as well. Steps 1-3 above implies that \( \pi_y : S_y(M') \to \Sigma_{p,q} \) is a great circle fibration. For any great circle fibration \( \pi_y : S_y(M') \to \Sigma_{p,q} \), using O’Neill formula, one can easily show that \( \Sigma_{p,q} \) is isometric to a round sphere of constant curvature \( 4 \). Thus, each fiber \( \Sigma_{p,q} \) is isometric to \( S^{k+1} \) up to a factor \( \frac{1}{2} \).

The metric of \( (M^n, g) \) can now be explicitly expressed as follows.

Recall that \( M = \cup_{q \in A'} \Sigma_{p,q} \). For each \( \hat{z} \in M^n \) and \( \xi \in T_{\hat{z}}(M) \) with \( r(\hat{z}) = d(p, \hat{z}) \), we let \( \xi^H \) denote the horizontal component of \( \xi \) and we let \( \xi^v \) denote the vertical component of \( \xi \).
Since each $\Sigma_{p,q}$ is isometric to $S^{k+1}$ up to a factor $\frac{1}{2}$ and $\pi_p : S_p(M) \to A'$ is a great circle fibration, we have

$$|\xi|^2_g = (\sin r)^2|\xi^H|^2 + \left[ \frac{1}{2} \sin(2r) \right]^2 |\xi^v|^2.$$  \hspace{1cm} (3.20)

Using (3.20) and an induction method on $\frac{\dim M}{k}$, one can show that $(M, g)$ is isometric to one of $\{\mathbb{C}P^n, \mathbb{H}P^n, \mathbb{C}aP^2\}$. Q.E.D.

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