A geometric construction of types for the smooth representations of PGL(2) of a local field

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March 2012

Abstract

We show that almost all (Bushnell and Kutzko) types of PGL(2, F), F a non-Archimedean locally compact field of odd residue characteristic, naturally appear in the cohomology of finite graphs.

Introduction

Let F be a non-Archimedean locally compact field and G be the group PGL(2, F). We assume that the residue characteristic of F is not 2. In previous works (2, 3) we defined a tower of directed graphs (X_n)_{n⩾0} lying G-equivariantly over the Bruhat-Tits tree X of G. We proved the two following facts:

Theorem 1 (2, Theorem (3.2.4), page 502). Let (π, V) be a non-spherical generic smooth irreducible representation. Then (π, V) is a quotient of the cohomology space with compact support H^1_c(X_{n(π)}, C), where n(π) is the conductor of π.

Theorem 2 (3, Theorem (5.3.2), page 512). If (π, V) is supercuspidal smooth irreducible representation of G, then we have:

\[ \dim_C \text{Hom}_G [H^1_c(X_{n(π)}, C), V] = 1. \]

In this paper we make the G-module structure of H^1_c(X_n, C) more explicit for all n ⩾ 0, and draw some interesting consequences.

Let us fix an edge [s_0, s_1] of X and denote by X_0 and X_1 the stabilizers in G of s_0 and [s_0, s_1] respectively. Then X_0 and X_1 form a set of representatives of the
two conjugacy classes of maximal compact subgroups in $G$. If $n$ is even, we have a $G$-equivariant mapping $p_n : \tilde{X}_n \to X$ which respects the graph structures. We denote by $\Sigma_n$ the subgraph $p_n^{-1}([s_0, s_1])$. If $n$ is odd, then after passing to the first barycentric subdivisions, we have a $G$-equivariant mapping $p_n : \tilde{X}_n \to X$ which respects the graph structures. We denote by $\Sigma_n$ the subgraph $p_n^{-1}(S(s_0, 1/2))$, where $S(s_0, 1/2)$ denotes the set of points $x$ in $X$ such that $d(x, s_0) \leq 1/2$ (here $d$ is the natural distance on the standard geometric realization of $X$, normalized in such a way that $d(s_0, s_1) = 1$).

Then for all $n$, $\Sigma_n$ is a finite graph, equipped with an action of $K_1$ if $n$ is even, and $K_0$ if $n$ is odd. So the cohomology spaces $H^1(\Sigma_n, \mathbb{C})$ provide finite dimensional smooth representations of $K_1$ or $K_0$, according to the parity of $n$.

For each $n \geq 0$, we define an finite set $P_n$ of pairs $(\mathcal{K}, \lambda)$ formed of a maximal compact subgroup $\mathcal{K} \in \{K_0, K_1\}$ and of an irreducible smooth representation of $\mathcal{K}$. By definition we have $(\mathcal{K}, \lambda) \in P_n$ if and only if there exists $k \in \{0, 1, \ldots, n\}$ such that $(\mathcal{K}, \lambda)$ is an irreducible constituent of the representation $H^1(\Sigma_k, \mathbb{C})$. For $(\mathcal{K}, \lambda) \in P_n$ and $k \leq n$, we denote by $m^k_\lambda$ the multiplicity of $\lambda$ in $H^1(\Sigma_k, \mathbb{C})$ and we set $m_{n, \lambda} = m_\lambda = n^0_\lambda + \cdots + n^n_\lambda$. Note that $n_\lambda$ depends on $(\mathcal{K}, \lambda)$ and $n$.

The main results of this article are the following.

**Theorem A.** For all $n \geq 0$, we have the direct sum decomposition :

$$H^1_c(\tilde{X}_n, \mathbb{C}) = \text{St}_G \oplus \bigoplus_{(\mathcal{K}, \lambda) \in P_n} (c\text{-ind}^G_{\mathcal{K}} \lambda)^{m_\lambda}.$$ 

(Here $\text{St}_G$ denotes the Steinberg representation of $G$).

**Theorem B.** For all $n \geq 0$, any element of $P_n$ is

a) either a type in the sense of Bushnell and Kutzko’s type theory [2], which is not a type for the unramified principal series

b) or a pair of the form $(\mathcal{K}_0, \chi \circ \det \otimes \text{St}_{\mathcal{K}_0})$, where $\chi$ is a smooth character of $F^\times$ of order 2, trivial on the group of $1$-units in $F^\times$, and $\text{St}_{\mathcal{K}_0}$ is the representation inflated from the Steinberg representation of PGL(2) of the residue field of $F$,

c) or the pair $(\mathcal{K}_1, 1_{\mathcal{K}_1})$, where $1$ denotes a trivial character.

**Corollary C.** Let $n \geq 0$. If $(\mathcal{K}, \lambda) \in P_n$ is a cuspidal type, then $m_{\lambda, n} = 1$.

Indeed this follows from Theorems 2 and A using Frobenius reciprocity for compact induction.

By Theorem 1, any Bernstein component of $G$, different from the unramified principal series component, must have a type in $P_n$ for $n$ large enough. Hence
the graphs $\tilde{X}_n$, $n \geq 0$, provide a geometric construction of types for almost all Bernstein components of $G$.

We conjecture that if $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ is a type of $G$, then $n\lambda = 1$.

Finally let us observe that this construction gives a new proof that the irreducible supercuspidal representations of $G$ are obtained by compact induction. Our proof differs from original Kutzko’s proof ([9], also see [4]) only at the exhaustion steps. Indeed our “supercuspidal” types are the same as Kutzko’s, but we prove that any irreducible supercuspidal representation contains such a type by using an argument based on [2] and [3], that is mainly on the existence of the new vector.

The article is organized as follows. The proof of Theorem A relies first on combinatorial properties of the graphs $\tilde{X}_n$ that are stated and proved in §2. Using this combinatorial properties and some homological arguments, we show in §3 how to relate the cohomology of $\tilde{X}_n$ to that of $\tilde{X}_{n-1}$. The irreducible components of $H^1(\Sigma_n)$ are determined in §4 when $n$ is even, and in §5 and §6 when $n$ is odd. A synthesis of the arguments of paragraphs 2 to 6, leading to a proof a theorem A and B, is given in §7.

We shall assume that the reader is familiar with the language of Bushnell and Kutzko’s type theory [5] and with the language of strata ([6], [4]).

1 Notation

We shall denote by

- $F$ a non-Archimedean non-discrete locally compact field,
- $\mathfrak{o}$ its valuation ring,
- $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$,
- $\varpi$ the choice of a uniformizer of $\mathfrak{o}$,
- $k = \mathfrak{o}/\mathfrak{p}$ the residue field of $F$,
- $p$ the characteristic of $k$,
- $q = p^f$ the cardinal of $k$,
- $G$ the group $\text{PGL}(2, F)$.
- $t_\varpi$ the image of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$ in $G$.

The results of this article are obtained under the

Hypothesis. The characteristic of $k$ is not 2

We shall often define an element, a subset, or a subgroup of $G$ by giving a (set of) representative(s) in $\text{GL}(2, F)$. 

3
We write $T$ for the diagonal torus of $G$ and $B \supset T$ for the upper standard Borel subgroup. We denote by $T^0$ the maximal compact subgroup of $T$, i.e. the set of matrices with coefficients in $\mathfrak{o}^\times$, and by $T^n$ the subgroup of matrices with coefficients in $1 + p^n$, $n > 0$.

Let $k$, $l$ be integers satisfying $k + l \geq 0$. Then $A(k,l) = \begin{pmatrix} \mathfrak{o} & p^l \\ p^k & \mathfrak{o} \end{pmatrix}$ is an $\mathfrak{o}$-order of $M(2,F)$. We denote by $\Gamma_0(k,l)$ the image in $G$ of its group of units.

There are two conjugacy classes of maximal compact subgroups of $G$. The first one has representative $K = \Gamma_0(0,0)$. A representative $\tilde{I}$ of the second one is the semidirect product of the cyclic group generated by $\Pi = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ with the Iwahori subgroup $I = \Gamma_0(1,0)$.

The group $K$ is filtered by the normal compact open subgroups $K_n = \begin{pmatrix} 1 + p^n & p^n \\ p^n & 1 + p^n \end{pmatrix}$, $n \geq 1$.

The group $I$ is filtered by the normal compact subgroups $I_n$, $n \geq 1$, defined by:

$I_{2k+2} = \begin{pmatrix} 1 + p^{k+1} & p^{k+1} \\ p^{k+2} & 1 + p^{k+1} \end{pmatrix}$, $I_{2k+1} = \begin{pmatrix} 1 + p^{k+1} & p^k \\ p^{k+1} & 1 + p^{k+1} \end{pmatrix}$, $k \geq 0$.

The subgroups $I_n$, $n \geq 1$, are normalized by $\Pi$.

We denote by $X$ the Bruhat-Tits building of $G$. This is a uniform tree with valency $q + 1$. As a $G$-set and as a simplicial complex $X$ identifies with the following complex. Its vertices are the homothety classes $[L]$ of full $\mathfrak{o}$-lattices $L$ in the vector space $V = F^2$. Two vertices $[L]$ and $[M]$ define an edge is and only if there exists a basis $(e_1, e_2)$ of $V$ such that, up to homothety, we have $L = \mathfrak{o}e_1 \oplus \mathfrak{o}e_2$ and $M = \mathfrak{o}e_1 \oplus pe_2$.

The vertices of the standard apartment (i.e. the apartment stabilized by $T$) are the $s_k = [\mathfrak{o} \oplus p^k]$, $k \in \mathbb{Z}$. The element $t_\infty$ acts as $t_\infty s_k = s_{k+1}$, $k \in \mathbb{Z}$. The maximal compact subgroup $K$ is the stabilizer of $s_0$ and $\tilde{I}$ (resp. $I$) is the global stabilizer (resp. pointwise stabilizer) of the edge $[s_0, s_1]$. If $l \geq k$, the pointwise stabilizer of the segment $[s_k, s_l]$ is $\Gamma_0(l, -k)$.

2 Combinatorics of $\tilde{X}_n$

We recall the construction of the directed graphs $\tilde{X}_n$, $n \geq 1$.

For any integer $k \geq 1$, an oriented $k$-path in $X$ is an injective sequence of vertices $(s_i)_{i=0,\ldots,k}$ in $X$ such that, for $k = 0, \ldots, k - 1$, $\{s_i, s_{i+1}\}$ is an edge in $X$. We shall allow the index $i$ to run over any interval of integers of length
\(k + 1\). Let us fix an integer \(n \geq 1\). The directed graph \(\hat{X}_n\) is constructed as follows. Its edge set (resp. vertex set) is the set of oriented \((n + 1)\)-paths (resp. \(n\)-paths) in \(X\). If \(a = \{s_0, s_1, ..., s_n\} \) is an edge of \(\hat{X}_n\), its head (resp. tail) is \(a^+ = \{s_1, s_2, ..., s_n\}\) (resp. \(a^- = \{s_0, s_1, ..., s_n\}\)). The graphs we obtain this way are actually simplicial complexes. The group \(G\) acts on \(\hat{X}_n\) is an obvious way; the action preserves the structure of directed graph.

When \(n = 2m\) is even, we have a natural simplicial projection \(p = p_n : \hat{X}_n \to X\) given on vertices by \(p(s_{m}, ..., s_0, ..., s_m) = s_0\). It is \(G\)-equivariant. Let \(e = \{s_0, t_0\}\) be an edge of \(X\). We are going to describe the finite simplicial complex \(p^{-1}(e)\). An edge in \(\hat{X}_n\) above the edge \(e\) corresponds to an oriented \((2m + 1)\)-path of one of the following forms:

i) \((s_{m}, s_{m+1}, ..., s_0, t_0, ..., t_{m-1}, t_m)\)

ii) \((t_{-1}, t_0, ..., s_m, s_0, ..., s_{m-1}, s_m)\)

Let \(C_{2m-1}(e)\) the set of \((2m - 1)\)-paths \(c = (s_{m+1}, ..., s_0, t_0, ..., t_{m-1})\). We say that \(c \in C_{2m-1}(e)\) lies above \(e\). Fix \(c \in C_{2m-1}(e)\) and consider the simplicial sub-complex \(\hat{X}_{2m}[e, c]\) of \(\hat{X}_{2m}\) whose edges correspond to the \((2m + 1)\)-paths of the form

\[(a, s_{m+1}, ..., s_0, t_0, ..., t_{m-1}, b)\]

So \(a\) (resp. \(b\)) can be any neighbour of \(s_{m+1}\) (resp. \(t_{m-1}\)) different from \(s_{m+2}\) (resp. \(t_{m-1}\)), with the convention that \(s_1 = t_0\) and \(t_{-1} = s_0\). The simplicial complex \(\hat{X}_{2m}[e, c]\) is connected. It is indeed isomorphic to the complete bipartite graph with sets of vertices:

\[\{a : \text{a neighbour of } s_{m+1}, a \neq s_{m+2}\} \text{ and } \{b : \text{b neighbour of } t_{m-1}, b \neq t_{m-2}\}\]

**Lemma 2.1.** Let \(e \text{ and } e'\) be two edges of \(X\) and \(c \in C_{2m-1}(e), c' \in C_{2m-1}(e')\). Then \(\hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c'] \neq \emptyset\) if and only if we are in one of the following cases:

i) \(e = e'\) and \(c = c'\) (so that \(\hat{X}_{2m}[e, c] = \hat{X}_{2m}[e', c']\))

ii) \(e \cap c'\) is reduced to one vertex of \(X\) and \(e \cup c'\) is an oriented \(2m\)-path in \(X\).

In that case \(\hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c']\) is reduced to the vertex of \(\hat{X}_{2m}\) corresponding to the \(2m\)-path \(e \cup c'\).

**Proof.** If \(\hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c']\neq \emptyset\), then \(e \cap c' = p(\hat{X}_{2m}[e, c]) \cap p(\hat{X}_{2m}[e', c'])\neq \emptyset\). Assume first that \(e = e'\). Then \(c = c'\) for if \(c \neq c'\), then \(\hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c'] = \emptyset\); indeed if \(\hat{s}\) is a vertex of \(\hat{X}_{2m}[e, c]\) then it determines \(c\) uniquely. Now assume that \(e \cap c'\) is a vertex. Let \(\hat{s} \in \hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c']\). Then \(\hat{s}\) contains \(c\) and \(c'\) as subsequences, with \(c \neq c'\). So by a length argument \(s = c \cup c'\). Conversely if \(c \cup c'\) is an oriented \(2m\)-path then \(c \cup c'\) is a vertex of \(X\) lying in \(\hat{X}_{2m}[e, c] \cap \hat{X}_{2m}[e', c']\).

**Corollary 2.2.** For any edge \(e\) of \(X\), the connected components of \(p^{-1}(e)\) are the \(\hat{X}_{2m}[e, c]\), where \(e\) runs over \(C_{2m-1}(e)\).

Define a 1-dimensional simplicial complex \(Y_{2m-1}\) in the following way. Its vertices are the connected components \(\hat{X}_{2m}[e, c]\), where \(e\) runs over the edges of
X and c over $C_{2m-1}(e)$, and two vertices $\tilde{X}_{2m}[e, c]$ and $\tilde{X}_{2m}[e', c']$ are linked by an edge if they intersect. Note that $Y_{2m-1}$ is naturally a $G$-simplicial complex.

**Corollary 2.3.** As a $G$-simplicial complex $Y_{2m-1}$ is canonically isomorphic to the complex $\tilde{X}_{2m-1}$.

Assume that $m \geq 1$. We say that an edge of $\tilde{X}_{2m-1}$ lies above a vertex $s_0$ of $X$ if as an oriented $2m$-path it has the form $(s_1, \ldots, s_n, s_m)$. For any vertex $s_0$ of $X$ we write $\tilde{X}_{2m-1}[s_0]$ for the subsimplicial complex of $Y$ formed of the edges lying above $s_0$.

**Lemma 2.5** When $m = 1$ the simplicial complexes $\tilde{X}_{2m-1}[s_0] = \tilde{X}_1[s_0]$ are connected.

**Proof.** We may identify the neighbour vertices of $s_0$ in $X$ with the points of the projective line $\mathbb{P}^1(\tilde{M}) \simeq \mathbb{P}^1(k)$, where $s_0 = [M]$ and $\tilde{M} = M/p_KM$. The vertices of $\tilde{X}_1[s_0]$ are the oriented 1-paths $(s_0, x)$, $(y, s_0), x, y \in \mathbb{P}^1(\tilde{M})$. Two oriented 1-paths of the form $(x, s_0)$ and $(s_0, y)$ are linked by the edge $(x, s_0, y)$.

Let $(x, s_0), (y, s_0)$ be two oriented 1-paths with $x \neq y$. Since $|\mathbb{P}^1(k)| \geq 3$, there exists $z \in \mathbb{P}^1(\tilde{M})$ distinct from $x$ and $y$. Then $(x, s_0)$ is linked to $(s_0, z)$ via the path $(x, s_0, z)$ and $(s_0, z)$ is linked to $(y, s_0)$ via the path $(y, s_0, z)$. For vertices of the form $(s_0, x)$, $(s_0, y)$ the proof is similar.

We now assume that $m > 1$. We write $C_{2m-2}(s_0)$ for the set $(2m - 2)$-paths of the form $(s, s_{m+1}, \ldots, s_0, \ldots, s_{m-1})$. For any $c \in C_{2m-2}(s_0)$, we consider the subsimplicial complex $\tilde{X}_{2m-1}[s_0, c]$ of $\tilde{X}_{2m-1}$ whose edges corresponds to the $2m$-paths of the form $(a, s, s_{n+1}, \ldots, s_0, s_{n-1}, b)$. We have results similar to lemma 1.2, corollaries 2.2 and 2.3.

**Lemma 2.6.** i) For any vertex $s_0$ of $X$ and for $c \in C_{2m-2}(s_0)$, $\tilde{X}_{2m-1}[s_0, c]$ is connected. It is indeed isomorphic to a complete bipartite graph constructed on two sets of $q = |k|$ elements.

ii) Let $s$ and $s'$ be vertices of $X$, $c \in C_{2m-2}(s)$ and $c' \in C_{2m-2}(s')$. Then $\tilde{X}_{2m-1}[s, c] \cap \tilde{X}_{2m-1}[s', c'] \neq \emptyset$ if and only if $s = s'$ and $c = c'$, or $\{s, s'\}$ is an edge in $X$ and $c \cup c'$ is an oriented $2m-1$-path. In this last case $\tilde{X}_{2m-1}[s, c] \cap \tilde{X}_{2m-1}[s', c'] = \{\tilde{s}\}$, where the vertex $\tilde{s}$ of $\tilde{X}_{2m-1}$ corresponds to the $(2m - 1)$-path $c \cup c'$.

iii) For any vertex $s$ of $X$, the connected components of $\tilde{X}_{2m-1}[s]$ are the $\tilde{X}_{2m-1}[s, c], c$ running over $C_{2m-2}(s)$.

We can consider the 1-dimensional simplicial complex $Z_{2m-2}$ whose vertices are the connected components $\tilde{X}_{2m-1}[s, c], s$ running over the vertices of $X$ and $c$ over $C_{2m-2}(s)$, and where two connected components define an edge if and only if they intersect. Note that $Z_m$ is naturally a $G$-simplicial complex.

**Corollary 2.7.** As a $G$-simplicial complex $Z_{2m-2}$ is isomorphic to $X_{2m-2}$.
3 The cohomology of $\tilde{X}_n$: first reductions

If $\Sigma$ is a locally finite 1-dimensional simplicial complex, we write $\Sigma^0$ (resp. $\Sigma^{(1)}, \Sigma^1$) for its set of vertices (resp. non-oriented edges, oriented edges). We let $C_0(\Sigma)$ (resp. $C_1(\Sigma)$) denote the $\mathbb{C}$-vector space with basis $\Sigma^0$ (resp. $\Sigma^{(1)}, \Sigma^1$). We define the space $C^0_c(\Sigma, \mathbb{C}) = C^0(\Sigma)$ (resp. $C^1_c(\Sigma, \mathbb{C}) = C^1(\Sigma)$) of oriented simplicial 0-cochains (resp. 1-cochains) with compact support by:

$$C^0_c(\Sigma) = \text{space of all linear forms } f : C_0(\mathbb{Z}) \to \mathbb{C} \text{ such that } f(s) = 0 \text{ except for a finite number of vertices } s;$$

$$C^1_c(\Sigma) = \text{space of all linear forms } \omega : C_1(\mathbb{C}) \to \mathbb{C} \text{ such that } f([a,b]) = 0 \text{ except for a finite number of oriented edges } [a,b] \text{ and } f([a,b]) = -f([b,a]).$$

We set $C^k_c(\Sigma) = 0$ for $k \in \mathbb{Z} \setminus \{0,1\}$ and define a coboundary map $d : C^0_c(\Sigma) \to C^1(\Sigma)$ by $df([a,b]) = f(b) - f(a)$. The cohomology of the cochain complex $(C^*(\Sigma), d)$ computes the cohomology with compact support $H^*_c(\Sigma, \mathbb{C}) = H^*_c(\Sigma)$ of (the standard geometric realization of) $\Sigma$. If $\Sigma$ is acted upon by a group $H$ whose action is simplicial then $(C^*_c(\Sigma), d)$ is in a straightforward way a complex of $H$-modules and its cohomology computes $H^1_c(\Sigma)$ as a $H$-module. When $T$ is finite we drop the subscripts $c$.

Since the stabilizer of a finite number of vertices of $X$ is open in $G$, we see that for $n \geq 1$, the $G$-modules $C^0_c(\tilde{X}_n), C^1_c(\tilde{X}_n)$ and therefore $H^1_c(\tilde{X}_n)$ are smooth.

In the sequel we fix $m \geq 1$ and we abbreviate $\tilde{X}_{2m} = \tilde{X}$. The disjoint union $\tilde{X} = \bigsqcup_{e \in X^{(1)}} \tilde{X}_e$, where $\tilde{X}_e = p^{-1}(e)$, induces an isomorphism:

$$C^1_c(\tilde{X}) \simeq \bigoplus_{e \in X^{(1)}} C^1_c(\tilde{X}_e)$$

Similarly the non-disjoint union $\tilde{X}^0 = \bigsqcup_{e \in X^{(1)}} \tilde{X}^0_e$ induces an injection:

$$j : C^0_c(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} C^0_c(\tilde{X}_e)$$
We have the following commutative diagram of \(G\)-modules:

\[
\begin{array}{cccccccc}
H^0_c(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} H^0_c(\tilde{X}_e) & \longrightarrow & \text{coker} j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0_c(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} C^0_c(\tilde{X}_e) & \longrightarrow & \text{coker} j & \longrightarrow & 0 \\
\downarrow d & & \downarrow \oplus d_e & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^1_c(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} C^1_c(\tilde{X}_e) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1_c(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} H^1_c(\tilde{X}_e) & \longrightarrow & 0
\end{array}
\]

Here, for \(e \in X^{(1)}\), \(d_e\) denote the coboundary map \(C^0_c(\tilde{X}_e) \to C^1_c(\tilde{X}_e)\). Since \(\tilde{X}\) is connected ([2] Lemma 4.1) and non compact, we have \(H^0_c(\tilde{X}) = 0\). So the snake lemma gives the kernel-cokernel exact sequence:

\[
0 \to \bigoplus_{e \in X^{(1)}} H^0_c(\tilde{X}_e) \to \text{coker} j \to H^1_c(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1_c(\tilde{X}_e) \to 0
\]

that is

\[
(3.3) \quad 0 \to \text{coker} j / \varphi \left( \bigoplus_{e \in X^{(1)}} H^0_c(\tilde{X}_e) \right) \to H^1_c(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1_c(\tilde{X}_e) \to 0
\]

Abbreviate \(Y = Y_{2m-1}\).

**Lemma 3.4.** We have a canonical isomorphism of \(G\)-modules

\[
\text{coker} j / \varphi \left( \bigoplus_{e \in X^{(1)}} H^0_c(\tilde{X}_e) \right) \cong H^1_c(Y).
\]

**Proof.** From corollary 2.2 we have

\[
\bigoplus_{e \in X^{(1)}} C^0_c(\tilde{X}_e) = \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C(e)} C^0_c(\tilde{X}_{e,c}).
\]

So the map \(j\) is given by \(f \mapsto \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C(e)} f_{e,c}\), where \(f_{e,c} = f|_{C_0(\tilde{X}_{e,c})}\). Consider the \(G\)-equivariant morphism of vector spaces

\[
\psi : \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C(e)} C^0_c(\tilde{X}_{e,c}) \to C^1_c(Y)
\]
given as follows. If \( \sigma \) is an oriented edge of \( Y \) then there exist uniquely determined edges \( e_o, e'_o \) of \( X \), \( c_o \in C(e_o), c'_o \in C(e'_o) \), such that \( \sigma \) corresponds to the intersection \( \tilde{X}_{e_o,c_o} \cap \tilde{X}_{e'_o,c'_o} = \{ s_o \} \), \( s_o \in \tilde{X}^0 \). We then set

\[
\psi((f_{e,c})_{e,c})(\sigma) = f_{e'_o,c'_o}(s_o) - f_{e_o,c_o}(s_o).
\]

Then \( \psi \) is surjective and its kernel is precisely \( j(C^0_\varnothing(\tilde{X})) \). So we may identify \( \text{coker} \ j \) with \( C^1_c(Y) \). From corollary 2.2, we have

\[
\bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) = \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C(e)} H^0(\tilde{X}_{e,c})
\]

so that we may identify \( \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \) with \( C^0_\varnothing(\tilde{Y}) \). Under our identifications the map \( \varphi : \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \to \text{coker} \ j \) corresponds to the coboundary map \( d : C^0_\varnothing(\tilde{Y}) \to C^1_c(Y) \), and we are done since all our identifications are \( G \)-equivariant.

**Proposition 3.5.** For \( m \geq 1 \), we have an isomorphism of \( G \)-modules:

\[
H^1_c(\tilde{X}_n) \cong H^1_c(\tilde{X}_{2m-1}) \oplus c\text{-ind}_{K_{e_o}} G H^1_c(\tilde{X}_{e_o})
\]

for any edge \( e_o \) of \( x \) and where \( K_{e_o} \) denotes the stabilizer of \( e_o \) in \( G \).

**Proof.** From the short exact sequence (3.3) and lemma 3.4, we have the exact sequence of \( G \)-modules:

\[
0 \to H^1_c(Y) \to H^1_c(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1(\tilde{X}_e) \to 0
\]

Since \( G \) acts transitively on the edges of \( X \), \( \bigoplus_{e \in X^{(1)}} H^1_c(\tilde{X}_e) \) identifies with the compactly induced representation \( c\text{-ind}_{\tilde{X}_e}^G H^1_c(\tilde{X}_{e_o}) \). Moreover by [Vign ??](Trouver la bonne référence) this induced representation is projective in the category of smooth complex representations of \( G \). So the sequence (3.7) splits.

We assume that \( m \geq 1 \) and we abbreviate \( \tilde{X} = \tilde{X}_{2m-1} \). The disjoint union \( \tilde{X}^1 = \bigsqcup_{s \in X^0} \tilde{X}^1_s \) induces an isomorphism:

\[
C^1_c(\tilde{X}) \cong \bigoplus_{s \in X^0} C^1_c(\tilde{X}_s)
\]

\[
\omega \mapsto (\omega_{|C_c(\tilde{X}_s)})_{s \in X^0}
\]

Similarly the non-disjoint union \( \tilde{X}^0 = \bigsqcup_{s \in X^0} \tilde{X}^0_s \) induces an injection:

\[
j : C^0_\varnothing(\tilde{X}) \hookrightarrow \bigoplus_{s \in X^0} C^0_\varnothing(\tilde{X}_s)
\]

\[
f \mapsto (f_{|C_\varnothing(\tilde{X}_s)})_{s \in X^0}
\]
We have the following commutative diagram of $G$-modules:

\[
\begin{array}{ccccccccc}
H^0_c(\tilde{X}) & \rightarrow & \bigoplus_{s \in X^0} H^0(\tilde{X}_s) & \xrightarrow{\varphi} & \text{coker} j \\
\downarrow & & \downarrow & & \\
0 & \rightarrow & C^0_c(\tilde{X}) & \xrightarrow{j} & \bigoplus_{s \in X^0} C^0(\tilde{X}_s) & \xrightarrow{\text{coker} j} & 0 \\
\downarrow & & \oplus d_s & & \downarrow & & \\
0 & \rightarrow & C^1_c(\tilde{X}) & \rightarrow & \bigoplus_{s \in X^0} C^1(\tilde{X}_s) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1_c(\tilde{X}) & \rightarrow & \bigoplus_{s \in X^0} H^1(\tilde{X}_s) & \rightarrow & 0 \\
\end{array}
\]

Here, for $s \in X^0$, $d_s$ denote the coboundary map $C^0_c(\tilde{X}_s) \rightarrow C^1_c(\tilde{X}_s)$. By Lemma 2.4, $\tilde{X}$ is connected. So we have $H^0_c(\tilde{X}) = 0$ since $\tilde{X}$ is non-compact. The snake lemma gives the kernel-cokernel exact sequence:

\[(3.9) \quad 0 \rightarrow \text{coker} j / \varphi \left( \bigoplus_{s \in X^0} H^0(\tilde{X}_s) \right) \rightarrow H^1_c(\tilde{X}) \rightarrow \bigoplus_{s \in X^0} H^1(\tilde{X}_s) \rightarrow 0\]

Lemma 3.10. We have a canonical isomorphism of $G$-modules

\[\text{coker} j / \varphi \left( \bigoplus_{s \in X^0} H^0(\tilde{X}_s) \right) \simeq H^1_c(\tilde{X}_{2m-2}).\]

Proof. It is similar to the proof of lemma 3.4 and relies on lemma 2.6 and corollary 2.7.

Proposition 3.11 For $m \geq 1$, we have an isomorphism of $G$-modules:

\[H^1_c(\tilde{X}_{2m-1}) \simeq H^1_c(\tilde{X}_{2m-2}) \oplus \text{c-ind}^G_{\hat{K}_{s_o}} H^1(\tilde{X}_{s_o})\]

for any vertex $s_o$ and where $\hat{K}_{s_o}$ denotes the stabilizer of $s_o$ in $G$.

Proof. Similar to the proof of proposition 3.5.

Recall [3] that $\tilde{X}_0$ is different from $X$. This is a directed graph whose set of vertices is isomorphic to $X^0$ as a $G$-set and whose set of edges is isomorphic to the $G$-set of oriented edges of $X$.

4 Determination of the inducing representations – I

Let $m \geq 0$ be a fixed integer and $e_0 = [s_0, s_1]$ be the standard edge. The aim of this section is to determine the $\mathcal{X}_{e_0}$-module $H^1(\tilde{X}_{2m}(e_0))$. Here we have $\mathcal{X}_{e_0} = \tilde{I}$,
the normalizer in $G$ of the standard Iwahori subgroup. We have the semidirect products:

$$\tilde{I} = \langle \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} \rangle \rtimes I = E^\times I$$

for any totally ramified subfield extension $E/F \subset M(2, F)$ such that $E^\times$ normalizes $I$.

We first assume that $m \geq 1$. By Corollary (2.2), we have the disjoint union:

$$\tilde{X}_{2m}[e_0] = \coprod_{c \in C_{2m-1}(e_0)} \tilde{X}_{2m}[e_0, c].$$

The group $\tilde{I}$ acts transitively on $C_{2m-1}(e_0)$. This comes from the standard fact that $I$, the pointwise stabilizer of $e_0$ acts transitively on the apartments of $X$ containing $e_0$.

Let $c_0 \in C_{2m-1}(e_0)$ be the path

$$s_{-m+1}, \ldots, s_0, s_1, \ldots, s_m.$$

The global stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ in $\tilde{I}$ is the pointwise stabilizer of $c_0$ in $\tilde{I}$, that is

$$\Gamma_0(m, m-1) = \begin{pmatrix} \sigma^x & p^{m-1} \\ p^m & \sigma^x \end{pmatrix} = T^0 I_{2m-1}.$$

It follows that

$$(4.1) \quad H^1(\tilde{X}_{2m}[e_0]) = \text{ind}_{T^0 I_{2m-1}}^{\tilde{I}} H^1(\tilde{X}_{2m}[e_0, c_0]).$$

On the other hand, an easy calculation shows that the pointwise stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ is $T^1 I_{2m}$, where $T^1$ is the congruence subgroup of $T$ given by

$$T^1 = \begin{pmatrix} 1 + p & 0 \\ 0 & 1 + p \end{pmatrix}.$$

So the $T^0 I_{2m-1}$-module $H^1(\tilde{X}_{2m}[e_0, c_0])$ may be viewed as a representation of the finite group $T^0 I_{2m-1}/T^1 I_{2m}$, that is a semidirect product of the cyclic group $k^\times$ with the abelian group $I_{2m-1}/I_{2m} \simeq k \oplus k$.

Set $\Gamma = \tilde{X}_{2m}[e_0, c_0]$. This is a finite directed graph. Let $\Sigma_m$ (resp. $\Sigma_{m+1}$) denote the set of verticed of $X$ that are neighbours of $s_{-m+1}$ and different from $s_{-m+2}$ resp. neighbours of $s_m$ and different from $s_{m-1}$. Then the vertex set of $\Gamma$ is

$$\Gamma^0 = \{(a, s_{-m+1}, \ldots, s_0, \ldots, s_m) : a \in \Sigma_m \} \coprod \{(s_{-m+1}, \ldots, s_0, \ldots, s_m, b) : b \in \Sigma_{m+1} \} \simeq \Sigma_m \coprod \Sigma_{m+1}$$

and its edge set is
\[ \Gamma^1 = \{(a, s_{-m+1}, \ldots, s_0, \ldots, s_m, b) : a \in \Sigma_{-m}, b \in \Sigma_{m+1}\} \cong \Sigma_{-m} \times \Sigma_{m+1}. \]

In particular \( \Gamma \) is a bipartite graph based on two sets of \( q \) elements. In particular, its Euler character is given by

\[ \chi(\Gamma) = 1 - \dim \mathbb{C}H^1(\Gamma) = 2q - q^2, \]

so that

\[ \dim \mathbb{C}H^1(\Gamma) = q^2 - 2q + 1 = (q - 1)^2. \tag{4.2} \]

Let \( \mathbb{C}[\Gamma^1] \) be the space of complex function on \( \Gamma^1 \) and \( \mathcal{H}(\Gamma) \) be the space of harmonic 1-cochains on \( \Gamma \):

\[ \mathcal{H}(\Gamma) = \{ f \in \mathbb{C}[\Gamma] ; \sum_{a \in \Gamma^1, s \in a} [a : s]f(a) = 0, \text{ all } s \in \Gamma^0 \}. \]

Here \( [a : s] \) denote an incidence number. In our case:

\[ \begin{align*}
\text{(Harm)} \quad f \in \mathcal{H}(\Gamma) \text{ iff } \\
&\sum_{a \in \Sigma_{-m}} f(a, s_{-m+1}, \ldots, s_m, b) = 0, \text{ all } b \\
&\sum_{b \in \Sigma_{m+1}} f(a, s_{-m+1}, \ldots, s_m, b) = 0, \text{ all } a
\end{align*} \]

This is a standard result (see e.g. [3] Lemma (1.3.2)), that, as a \( T^0/I_{2m-1}/T^1I_{2m} \)-module, \( H^1(\Gamma) \) is isomorphic to the contragredient module of \( \mathcal{H}(\Gamma) \).

An easy computation shows that we may identify \( \Gamma^1 \) with \( k \times k \) in such a way that:

1) an element of \( I_{2m-1} = \left( \begin{array}{cc} 1 + p^m & p^{m-1} \\ p^m & 1 + p^m \end{array} \right) \) acts as

\[ (1 + \left( \begin{array}{cc} \omega^m a & \omega^{m-1} b \\ \omega^m c & \omega^m d \end{array} \right)) (x, y) = (x + \bar{b}, y + \bar{c}) \]

for \( a, b, c, d \in \mathfrak{o}, x, y \in \mathbf{k} \), and

2) an element of \( T^0 \) acts as

\[ \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) (x, y) = (\bar{a} \bar{d}^{-1} x, \bar{d}^{-1} y) \]

and the condition (Harm) writes:

\[ f \in \mathcal{H}(\Gamma) \text{ iff } \begin{cases} \\
\sum_{x \in \mathbf{k}} f(x, y) = 0, \text{ all } y \in \mathbf{k} \\
\sum_{y \in \mathbf{k}} f(x, y) = 0, \text{ all } x \in \mathbf{k}
\end{cases} \]
A basis of $\mathbb{C}[\Gamma]$ is formed of the functions $\chi_1 \otimes \chi_2(x, y) = \chi_1(x)\chi_1(y)$, where, for $i = 1, 2$, $\chi_i$ runs over the characters of $(k, +)$. It is clear that the $(q - 1)^2$ dimensional subspace of $\mathbb{C}[\Gamma]$ generated by the $\chi_1 \otimes \chi_2$, $\chi_1 \neq 1$, $\chi_2 \neq 1$, is contained in $\mathcal{H}(\Gamma)$. So using (4.2), we obtain:

\begin{equation}
\mathcal{H}(\Gamma) = \text{Span}\{\chi_1 \otimes \chi_2 \mid \chi_i \in \tilde{k}^\times, \chi_i \neq 1, i = 1, 2\}.
\end{equation}

It follows from (4.3) that as an $I_{2m-1}/I_{2m}$-module, the space $\mathcal{H}(\Gamma)$ is the direct sum of $1$-dimensional representations corresponding to the characters $\alpha = \alpha(\chi_1, \chi_2)$ given by

$$\alpha(1 + \begin{pmatrix} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{pmatrix}) = \chi_1(a)\chi_2(b).$$

In particular $\mathcal{H}(\Gamma)$ is isomorphic to its contragredient and therefore isomorphic to $H^1(\Gamma)$ as an $I_{2m-1}/I_{2m}$-module. In the language of strata (the reader may refer to [4]§4), for $\chi_i \neq 1$, $i = 1, 2$, the character $\alpha(\chi_1, \chi_2)$ corresponds to a stratum of the form $[3, 2m, 2m - 1, \beta]$, where $\beta$ is the standard Iwahori order and $\beta \in M(2, F)$ is an element of the form $\Pi^{2m-1}(u \begin{pmatrix} 0 \\ 0 \end{pmatrix} v)$, $u, v \in \mathfrak{o}^\times$. In the terminology of [3]§4, page 98, this stratum is a ramified simple stratum.

We now have enough material to prove the following result.

**Proposition (4.4).** Let $\lambda$ be an irreducible constituent of $H^1(\bar{X}_{2m}[e_0])$. Then the compactly induced representation $c - \text{Ind}_I \lambda$ is irreducible, whence supercuspidal.

**Proof.** It is a standard result that an irreducible compactly induced representation is supercuspidal (see [10] or [8], page 194).

The proof of the irreducibility is also standard by an argument due to Kutzko. But we repeat it for convenience. By Frobenius reciprocity, the restriction of $\lambda$ to $I_{2m-1}$ contains a character $\alpha$ corresponding to a (ramified) simple stratum. Since $\lambda$ is irreducible and since $\tilde{I}$ normalizes $I_{2m-1}$, the restriction $\lambda|_{I_{2m-1}}$ is a direct sum $\alpha_1 \oplus \cdots \alpha_r$ of $\tilde{I}$-conjugates of $\alpha(\chi_1, \chi_2)$. They all correspond to simple strata. Let $g \in G$ be an element intertwining $\lambda$ with itself. Then by restriction it intertwines a character $\alpha_i$ with a character $\alpha_j$ for some $j = 1, \ldots, r$. By [3] Lemma (16.1), page 111, such an element $G$ must belong to $\tilde{I}$. It follows that the $G$-intertwining of $\lambda$ is equal to $\tilde{I}$ and that the representation $c - \text{Ind}_I^G \lambda$ is irreducible according to Mackey’s irreducibility criterion ([8] Proposition (1.5), page 195).

We finally consider the case $m = 0$. The directed graph $\bar{X}_0$ has $X^0$ as vertex set. An edge $\{t, s\}$ in $X$ gives rise to two edges $[s, t]$ and $[t, s]$ in $\bar{X}_0$. Since the
action of $G$ on $\tilde{X}_0$ preserves the structure of digraph, the $G$-module $H^1_c(\tilde{X}_0)$ may be computed using the following complex:

$$0 \rightarrow C^0_c(\tilde{X}_0) \rightarrow C^{(1)}_c(\tilde{X}_0) \rightarrow \ldots$$

where $C^{(1)}_c(\tilde{X}_0)$ is the space of (unoriented) 1-cochains, that is the space of maps from $\tilde{X}_0^{(1)}$ (unoriented edges) to $C$ with finite support. The coboundary map is here given by $df[s,t] = f(t) - f(s)$. Consider the $G$-equivariant injection $j : C^{(1)}_c(X) \rightarrow C^{(1)}_c(\tilde{X}_0)$ given by $j(\omega) : [s,t] \mapsto \omega([s,t])$. We have the commutative diagram of $G$-modules:

\[
\begin{array}{ccccccc}
0 & \rightarrow & C^0_c(X) & \xrightarrow{\text{id}} & C^0_c(\tilde{X}_0) & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C^1_c(X) & \xrightarrow{j} & C^1_c(\tilde{X}_0) & \rightarrow & C^{(1)}_c(\tilde{X}_0)/\text{Im}j & \rightarrow & 0
\end{array}
\]

The quotient $C^{(1)}_c(\tilde{X}_0)/\text{Im}j$ identifies with the subspace of $C^{(1)}_c(\tilde{X}_0)$ formed of those functions $f$ satisfying $f([s,t]) = f([t,s])$ for all edges $\{s,t\}$ of $X$. This subspace is nothing other than the compactly induced representation $c - \text{Ind}^G_f 1_f$. The cokernel exact sequence writes:

$$0 \rightarrow H^1_c(X) \rightarrow H^1_c(\tilde{X}_0) \rightarrow c\text{-ind}^G_f 1_f \rightarrow 0$$

Now we use the following two facts:

- the representation $c - \text{Ind}^G_f 1_f$ is a projective object of the category of smooth representations of $G$,
- the $G$-module $H^1_c(X)$ is isomorphic to the Steinberg representation $\text{St}_G$ of $G$ (\cite{7})

to obtain:

**Proposition (4.5).** The $G$-module $H^1_c(\tilde{X}_0)$ is isomorphic to $\text{St}_G \oplus c - \text{Ind}^G_f 1_f$.

## 5 The inducing representations – II

We now determine the $\mathcal{K}_{s_0}$-module $H^1(\tilde{X}_{2m+1}[s_0])$. The arguments are very often similar to those of the previous section and we will not give all details. Since the case $m = 0$ requires slightly different techniques we postpone it to the end of the section and assume first that $m > 0$.

Recall that the stabilizer $\mathcal{K}_{s_0}$ of $s_0$ in $G$ is the image $K$ of $\text{GL}(2,\mathfrak{o})$ in $G$.

Let $c_0 \in C_{2m}(s_0)$ be the path $(s_{-m}, ..., s_0, ..., s_m)$. Its pointwise stabilizer is $\Gamma_0(m, m) = T^0 K_m$. So as a $K$-module, $H^1(\tilde{X}_{2m+1}[s_0])$ is isomorphic to the
induced representation $\text{Ind}_{T_0 K_m}^{K_m} H^1(\tilde{X}_{2m} \{ s_0, c_0 \})$. Moreover the pointwise stabilizer of $\tilde{X}_{2m+1} \{ s_0, c_0 \}$ is $T^1 K_{m+1}$ and $H^1(\tilde{X} \{ s_0, c_0 \})$ may be viewed as a representation of $T^0 K_m / T^1 K_{m+1}$.

As in the previous section, one may consider the bipartite graph $\Omega$ whose both vertex sets identify with $K_m$ equipped with an action of $K_m$ on $\Omega^1$ given by

$$[I_2 + w^m \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)](x, y) = (x + \tilde{b}, y + \tilde{c}) ,$$

the action of $T^0$ being given by

$$\left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right).(x, y) = (\tilde{a}d^{-1}x, \tilde{d}a^{-1}y) .$$

Then the contragredient of the $T^0 K_m / T^1 K_{m+1}$-module $H^1(\tilde{X} \{ s_0, c_0 \})$ is isomorphic to the space $\mathcal{H}(\Omega)$ of harmonic cochains on $\Omega$. As in the previous section this later space is generated by the functions $\chi_1 \otimes \chi_2$, where $\chi_i, i = 1, 2$, runs over the non trivial characters of $(k, +)$. The line $\mathbb{C}\chi_1 \otimes \chi_2$ is acted upon by $K_m$ via the character $\alpha(\chi_1, \chi_2)$ given by

$$\alpha(\chi_1, \chi_2) [I_2 + w^m \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)] = \chi_1(b)\chi_2(a) .$$

It follows that $\mathcal{H}(\Omega)$ is isomorphic to its contragredient and that $H^1(\tilde{X}_{2m} \{ s_0, c_0 \})$ is the direct sum of the characters $\alpha(\chi_1, \chi_2), \chi_i \neq 1, i = 1, 2$.

For $\chi_i \neq 1, i = 1, 2$, the character $\alpha(\chi_1, \chi_2)$ corresponds to a stratum of the form $[M(2, \sigma), m, m - 1, \beta]$, where $\beta \in M(2, F)$ is given by $w^{-m} \left( \begin{array}{cc} 0 & u \\ v & 0 \end{array} \right)$, $u, v \in \tilde{k}^\times$. This stratum is either simple and non-scalar or split fundamental according to whether $uv \mod p$ is a square in $k^\times$ or not (here we have used the fact that $\text{Char}(k) \neq 2$.

It is clear that $T^0$ leaves the set of characters corresponding to simple strata (resp. split fundamental strata) stable. So we may write

$$H^1(\tilde{X}_{2m} \{ s_0, c_0 \}) = H^1(\tilde{X}_{2m} \{ s_0, c_0 \})_{\text{simple}} \oplus H^1(\tilde{X}_{2m} \{ s_0, c_0 \})_{\text{split}}$$

where $H^1(\tilde{X}_{2m} \{ s_0, c_0 \})_{\text{simple}}$ (resp. $H^1(\tilde{X}_{2m} \{ s_0, c_0 \})_{\text{split}}$) is the sub-$T^0 K_m$-module which decomposes as a $K_m / K_{m+1}$-module as a direct sum of (characters corresponding to) simple non-scalar strata (resp. split fundamental strata).

We have a result similar to proposition (4.4), whose proof uses the same arguments.

**Proposition (5.1).** Let $\lambda$ be an irreducible constituent of

$$\text{Ind}_{T_0 K_m}^{K_m} H^1(\tilde{X}_{2m+1} \{ s_0, c_0 \})_{\text{simple}} \subset H^1(\tilde{X}_{2m+1} \{ s_0 \}) .$$
Then the compactly induced representation $c\text{-ind}_{K}^{G} \lambda$ is irreducible, whence supercuspidal.

The study of $\text{Ind}_{K_{0}K_{m}}^{K} H^{1}(\tilde{X}_{2m+1}[s_{0}, c_{0}])_{\text{split}}$ is the aim of the next section.

We are now going to determine the $K$-module structure of $H^{1}(\tilde{X}_{1}[s_{0}])$. Set $G = \text{PGL}(2, k) \simeq K/K^{1}$ and write $B$ and $T$ for the upper Borel subgroup and diagonal torus of $G$ respectively. Let $U$ be the unipotent radical of $B$. As a $K$-set the set of neighbour vertices of $s_{0}$ is isomorphic to $P_{1}(k) = G/B$.

The graph $\Omega = \tilde{X}_{1}[s_{0}]$ has for vertex set the set of paths of the form $(s, s_{0})$ or $(s_{0}, s)$ where $s$ runs over the neighbour vertices of $s_{0}$ in $X$. So the space $C^{0}(\Omega)$ of 0-cochains identifies with the space $F(P_{1}(k) \bigsqcup P_{1}(k))$ of complex valued functions on the disjoint union $P_{1}(k) \bigsqcup P_{1}(k)$. So has a $G$-module $C^{0}(\Omega)$ is isomorphic to $1_{G} \oplus \text{St}_{G} \oplus 1_{G} \oplus \text{St}_{G}$, where $1$ denotes a trivial representation and $\text{St}$ a Steinberg representation.

The $G$-set $\Omega^{1}$ is the set of paths of the form $(s, s_{0}, t)$, where $s$ and $t$ are two different neighbour vertices of $s_{0}$. This $G$-set is isomorphic to the quotient $G/T$. The space $C^{1}(\Omega)$ of unoriented 1-cochains identifies as $G$-module with the space $F(G/T)$.

Fix a non-trivial character $\psi$ of $U$. It is well knows that the induced representation $\text{Ind}_{U}^{G} \psi$ is multiplicity free. Its irreducible constituent form by definition the generic (irreducible) representations of $G$. Moreover an irreducible representation is generic if and only if it is not a character.

We have a natural $G$-equivariant map $\Phi : F(G/T) \longrightarrow \text{Ind}_{U}^{G} \psi$, given by

$$\Phi(f)(g) = \sum_{u \in U} f(gu) \psi(u), \quad f \in F(G/T), \quad g \in G.$$  

If a function $f$ lies in the kernel of $\Phi$, then we have $\sum_{u \in U} f(\theta(u) = 0$, for all $g \in G$ an all non-trivial character $\theta$ of $U$. Indeed it suffices to use the fact that the action of $T$ on $U$ by conjugation acts transitively on the non-trivial characters of $U$ and the right invariance of $f$ under the action of $T$. So the kernel of $\Phi$ consists of the function $f$ such that $u \mapsto f(\theta(u)$ is constant function on $U$, for all $g \in G$.

In other words $\text{Ker} \Phi = F(G/B) \simeq 1_{G} \oplus \text{St}_{G}$. By a dimension argument, we see that $\Phi$ is surjective. It follows that

$$C^{1}(\Omega) \simeq \text{Ind}_{U}^{G} \psi \oplus 1_{G} \oplus \text{St}_{G}.$$  

We have the cochain complex of $G$-modules:

$$0 \longrightarrow C^{0}(\Omega) \longrightarrow C^{1}(\Omega) \longrightarrow 0$$  

16
Since $\Omega$ is connected the kernel of the coboundary operator is the trivial module $\mathbb{C}$. Hence in the Grothendieck groups of $G$-modules, we have: $dC^0(\Omega) \simeq 2.1_G + 2.\text{St}_G - 1_G = 1_G + 2.\text{St}_G$. Therefore

$$H^1(\Omega) = C^1(\Omega)/dC^0(\Omega) \simeq \text{Ind}_U^G\psi + 1_G + \text{St}_G - 1_G - 2.\text{St}_G = \text{Ind}_U^G\psi - \text{St}_G.$$ 

Since $q = |k|$ is odd, there exists a unique non-trivial character of $k^\times/(k^\times)^2$, that we denote by $\chi_0$. The irreducible constituents of the Gelfand-Graev representation $\text{Ind}_U^G\psi$ are the following:

- the irreducible cuspidal representations of $G$,
- the principal series $\text{Ind}_B^G\chi \otimes \chi^{-1}$, where $\chi : k^\times \to \mathbb{C}^\times$ is a character such that $\chi^2 \neq 1$ (i.e. $\chi \notin \{1, \chi_0\}$).
- the steinberg representation $\text{St}_G$,
- (when $q$ is odd) the twisted representation $\text{St}_G \otimes \chi_0$.

If $\sigma$ is a cuspidal representation of $G = K/K^1$, then the induced representation $c\text{-ind}_K^K\sigma$ is irreducible and supercuspidal ([4], (11.5), page 81). Such a representation of $G$ is called a level 0 supercuspidal representation.

A principal series of $G = K/K^1$ may be written as $\text{Ind}_I^K\rho$, where $\rho$ is a character of $I/I^1$. The pair $(I, \rho)$ is actually a type in the sense of Bushnell and Kutzko’s type theory. For technical reason we postpone definitions and references to the next section. Since the representation $\text{Ind}_I^K\rho$ is irreducible, it is a type for the same constituent as $(I, \rho)$.

To sum up, we have proved the following.

**Proposition (5.2).** An irreducible constituent $\lambda$ of $H^1(\tilde{X}_1[\mathfrak{s}_0])$ is of one of the following forms

(i) the inflation of a cuspidal representation of $G$; in that case $c\text{-ind}_K^K\lambda$ is a level 0 irreducible supercuspidal representation of $G$.

(ii) the inflation to $K$ of the representation $\text{St}_G \otimes \chi_0$,

(iii) a type of the form $\text{Ind}_I^K\rho$, where the $\rho$ is inflated from a character of $I/I^1 \simeq (k^\times \times k^\times)/k^\times$ of the form $\chi \otimes \chi^{-1}$, $\chi^2 \neq 1$.

Note that in (iii), the pair $(K, \text{Ind}_I^K\rho)$ is a principal series type.

## 6 The inducing representations – III

We keep the notation as in the previous section. To determine the structure of $\text{Ind}_{K^0 K^m}^K H^1(\tilde{X}_{2m+1}[\mathfrak{s}_0, c_0])$ split, we first recall crucial facts on split strata and types for principal series representations. The basic reference for type theory is [5].
Let $\chi$ be a character of $T$, that we view as a character of $T^0$ by restriction. Assume that the conductor of $\chi$ is $n > 0 : T^n \subset \ker \chi$ and $n$ is minimal for this property. Set

$$J_\chi = \begin{pmatrix} o^x & 0 \\ p^n & o \times \end{pmatrix} = \Gamma_0(p^n).$$

If $U$ and $\bar{U}$ denotes the groups of upper and lower unipotent matrices respectively, then $J_\chi$ has an Iwahori decomposition:

$$J_\chi = (J_\chi \cap \bar{U}).(J_\chi \cap T).(J_\chi \cap U)$$

and one may define a character $\rho_\chi$ of $J_\chi$ by

$$\rho_\chi(\bar{u}t^0u) = \chi(t^0), \quad \bar{u} \in J_\chi \cap \bar{U}, \quad u \in J_\chi \cap U, \quad t^0 \in T^0.$$ 

Let $R_{[T,T]}$ be the Bernstein component of the category of smooth representations of $G$ whose objects are the representations $V$ satisfying the following property: any irreducible subquotient of $V$ occurs in a parabolically induced representation $\text{Ind}_G^T(\chi \otimes \chi_0)$, where $B$ is a Borel subgroup with Levi component $T$ and $\chi_0$ an unramified character of $T$. We then have.

**Theorem (6.1) (A. Roche)** The pair $(J_\chi, \rho_\chi)$ is a type for $R_{[T,T]}$.

This is indeed Theorem (7.7) of [11]. Note that our $J_\chi$ is not exactly Roche’s one, but a conjugate under an element of $T$ (see [11], Example (3.5)).

**Proposition (6.2).** With the notation as before, assume that $\chi|_{T^0}$ is not of the form $\alpha \circ \text{Det}$, where $\alpha$ is a character of $\mathfrak{o}^\times$ (necessarily of order 2). Then the induced representation $\text{Ind}_{J_\chi}^G(\rho_\chi)$ is irreducible. In particular it is a type for $R_{[T,T]}$.

**Proof.** Let $W$ be the extended affine Weyl group of $G$ w.r.t. $T$ and set $W_\chi = \{w \in W ; \; w\chi = \chi\}$. Then by Theorem (4.14) of [11], the $G$-intertwining of $\rho_\chi$ is $J_\chi W_\chi J_\chi$. The hypothesis on $\chi$ forces $W_\chi = T/T^0$. So $(J_\chi W_\chi J_\chi) \cap K = J_\chi T^0 J_\chi = J_\chi$, and we may apply Mackey’s criterion of irreducibility.

For $n > 0$ and $q \in \{0, ..., n\}$, define compact open subgroups of $G$ as follows:

$$q_{\mathfrak{h}}_1 = \begin{pmatrix} 1 + p^n & p^q \\ p^{n+1} & 1 + p^n \end{pmatrix} \text{ and } q_{\mathfrak{h}}_2 = \begin{pmatrix} 1 + p^{n+1} & p^q \\ p^{n+1} & 1 + p^{n+1} \end{pmatrix}.$$ 

These groups are particular cases of groups considered in [1], §(2.3). The quotients $q_{\mathfrak{h}}_1/q_{\mathfrak{h}}_2$, $q = 0, ..., n$, are abelian, and for $\alpha \in \mathfrak{k}^\times$, one may define a character $\psi_\alpha$ of $q_{\mathfrak{h}}_1/q_{\mathfrak{h}}_2$ by the formula:

$$\psi_\alpha(I_2 + \begin{pmatrix} \varpi^n a & \varpi^q b \\ \varpi^{n+1} c & \varpi^n d \end{pmatrix}) = \psi(\alpha(a - d)).$$
where \( \psi \) is a fixed non-trivial character of \((k, +)\). In fact, \((\psi_\alpha)|_{n\mathfrak{h}_1}\) is the restriction to \(n\mathfrak{h}_1\) of a split fundamental stratum of \(K_n/K_{n+1}\). We shall need the following result.

**Lemma (6.3).** If a smooth representation of \(K\) contains \((\psi_\alpha)|_{n\mathfrak{h}_1}\) by restriction, then it contains the character \((\psi_\alpha)|_{n\mathfrak{h}_1}\).

**Proof.** Since the characteristic of \(k\) is not 2, then \(\alpha \neq -\alpha (\psi_\alpha)|_{n\mathfrak{h}_1}\) is the restriction to \(n\mathfrak{h}_1\) of a split fundamental stratum of \(K_n/K_{n+1}\). Our lemma is then a particular case of [12], Lemma (2.4.5).

**Proposition (6.4).** Let \(\lambda\) be an irreducible constituent of \(\text{Ind}_{\Gamma_0K_m}^{K} H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{split}}\). Then with the notation as above, \(\lambda\) is of the form \(\text{Ind}_{\rho, \chi}^{J_1, \rho_\chi}\), for some principal series type \((J_\chi, \rho_\chi)\) with \(\chi\) of conductor \(m + 1\).

**Proof.** We know that such a \(\lambda\) contains a split fundamental stratum of the form \([M(2, \sigma), m, m - 1, b]\), where \(b = \varpi^{-m} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\), \(u, v \in \sigma^\times\), and \(uv\) is a square modulo \(p\). If \(\alpha \in \sigma\) is such that \(\alpha^2 \equiv uv \mod p\), then the stratum is equivalent to a \(K\)-conjugate of \([M(2, \sigma), m, m - 1, b']\), where \(b' = \varpi^{-m} \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\). So we deduce that \(\lambda\) contains this latter stratum by restriction. Now consider the group \(n\mathfrak{h}_1\) for \(n = m\). The representation \(\lambda\) contains the character \((\psi_\alpha)|_{n\mathfrak{h}_1}\) by restriction. By applying Lemma (6.3) we obtain that it contains the character \((\psi_\alpha)|_{n\mathfrak{h}_1}\). This character clearly extends to \(T_0K_1 = \Gamma_0(m + 1, 0)\) and the quotient \(T_0K_1/\Gamma_0\) is abelian. It follows that \(\lambda\) contains and extension of \(\psi_\alpha\) to \(\Gamma_0(m + 1, 0)\). Such an extension is of the form \((J_\chi, \rho_\chi)\), for some character \(\chi\) of \(T\) of conductor \(m + 1\). The fact that \(\lambda\) is induced from \((J_\chi, \rho_\chi)\) follows from Proposition (6.2).

## 7 Synthesis

We now prove Theorems A and B of the introduction.

By Proposition (3.5) and (3.11), we have isomorphisms of \(G\) modules:

\[
H^1_c(\tilde{X}_{2m}) \simeq H^1_c(\tilde{X}_{2m-1}) \oplus c\text{-ind}_{\chi_1}^{\chi_2} H^1(\Sigma_{2m}), \ m \geq 1. 
\]

(1)

\[
H^1_c(\tilde{X}_{2m+1}) \simeq H^1_c(\tilde{X}_{2m}) \oplus c\text{-ind}_{\chi_0}^{\chi_2} H^1(\Sigma_{2m+1}), \ m \geq 0. 
\]

(2)

Recall that with the notation of the introduction, we have:

- \(\Sigma_{2m} = (\tilde{X}_{2m})_{c_0}, \Sigma_{2m+1} = (\tilde{X}_{2m+1})_{s_0}\),
- \(\chi_0 = \chi_{s_0}, \chi_1 = \chi_{c_0}\).

19
Moreover, by Proposition (4.5), we have

\[ H^1_c(\tilde{X}_0) \simeq St_G \oplus c\text{-}ind^G_{\tilde{X}_1} H^1(\Sigma_0) \]  

so that (1) holds for \( m = 0 \). Hence Theorem A follows from (1) and (2) by a straightforward inductive argument.

Theorem B follows from the discription of the irreducible components of \( H^1(\Sigma_n) \) given in Proposition (4.4) \( (n \text{ even and } n > 0) \), Proposition (4.5) \( (n = 0) \), and Propositions (5.1) and (6.4) \( (n \text{ odd}) \).

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