Quantum Newtonian cosmology and the biconfluent Heun functions

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Abstract. Using the Lagrangian formulation of the Newtonian cosmology we write the Hamiltonian operator of the Schrödinger equation for a particle (galaxy) moving in a universe constituted of incoherent matter, that is, in a cosmological substratum which is comprised of dust. In the phase space defined by the fixed rectangular coordinate (comoving), the exact solution for wave function of the Newtonian Universe is obtained in terms of the biconfluent Heun functions. We obtain the first few Heun polynomials of the biconfluent case. In this scenario, the cosmological constant is taken into account and the role of their presence in this solution is emphasized as well as an exact expression for the energy levels is obtained.

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1. Introduction

The modern cosmology (or Einsteinian cosmology) is described using the theory of general relativity, whose formulation is geometric, so that the descriptions of the cosmological effects are associated to spacetime geometry. This formulation makes use of the spacetime concept which is worked out through the differential manifold concept. Others themes of modern mathematics, of complex nature, such as tensor algebra and continuous groups, are also used in this formulation [1].

In Newtonian cosmology the cosmological equation is obtained from the equation of motion for particles (galaxies) submitted to gravitational forces [2]. This leads us immediately to an idea of use the formalism developed by Lagrange and Hamilton, since such it provides us the equation and integral of motion, respectively.

The study of the interaction of quantum systems with gravitational fields goes back to the beginning of the last century, motivated by the idea of constructing a theory combining quantum physics and general relativity [3, 4, 5, 6].

Quantum Newtonian cosmology was proposed by Freedman et al. [7]. They construct a wave function for Newtonian cosmology in the framework of non-relativistic quantum mechanics through an elementary quantization to obtain a solution of the multiparticle Schrödinger equation. In this scenario, they use the WKB and Hartree-Fock approximations.

The search for a consistent and empirically established quantum theory of gravity is among the biggest open problems of fundamental physics. In the connection between quantum theory and gravity, at the level where gravity is treated as a classical interaction, the lowest level is quantum mechanics plus Newtonian gravity [8, 9, 10, 11].

The Heun equations [12] is widely involved in different domains of theoretical and experimental physics such as quantum mechanics, quantum gravity and general relativity, solid state physics and many others. A confluent form of the general Heun equation, derived via coalescence of its two finite regular singularities with the one located at the infinity, is the biconfluent Heun equation, that is a second order linear differential equation which has one regular singularity and an irregular singularity of rank 2 [13, 14].

This paper is organized as follows. In section 2 we obtain the Hamiltonian operator for quantum Newtonian cosmology. In section 3 we solve the Schrödinger equation in this background. In section 4 we present the biconfluent Heun equation, and then we obtain the Heun polynomials. In section 5 we obtain the energy spectrum for a particle (galaxy) moving in a Newtonian Universe. In section 6 we write the general and exact expression of the wave function. Finally, in section 7 we present our conclusions.

2. Hamiltonian operator

In our previous paper [15], using some assumptions about the structure of the universe and the Newtonian version of the cosmological principle, we have written the Lagrangian,
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$L$, for the motion of a particle (galaxy) of mass $m$ in the expansion as

$$L(R, \dot{R}) = \frac{1}{2} m \dot{R}^2 + \frac{GMm}{R} + \frac{1}{6} \Lambda m R^2,$$

(1)

where $\Lambda$ is the cosmological constant. From this Lagrangian, we obtained the analogous of the Einstein equation for the scale parameter, namely

$$\ddot{R} = -\frac{4}{3} \pi G \rho R + \frac{1}{3} \Lambda R ,$$

(2)

where $M = 4 \pi R^3 \rho / 3$ is the mass of the sphere (total mass of the Newtonian Universe). It is worth calling attention to the fact that we are considering the case $p = 0$, that is, dust cloud.

The Lagrangian given by Eq. (1) is independent of the time, because the system is under the action of a uniform force field, so that the constant quantity of the motion is $H_{cl}$, called classical Hamiltonian of the system, which can be defined as

$$H_{cl} = \left( \frac{\partial L}{\partial \dot{R}} \right) \dot{R} - L ,$$

(3)

from which we obtained the analogous of the Friedmann equation, namely

$$\dot{R}^2 = \frac{C}{R} + \frac{1}{3} \Lambda R^2 - k ,$$

(4)

where $C = 8 \pi G \rho R^3 / 3$ and $k = -2E / m$ being constants, and $E = H_{cl}$ is the total energy, which is the constant of the motion for this case.

Thus, from Eq. (3) we can write the classical Hamiltonian as

$$H_{cl}(R, \dot{R}) = \frac{1}{2} m \dot{R}^2 - \frac{GMm}{R} - \frac{1}{6} \Lambda m R^2 .$$

(5)

In the fixed rectangular coordinate (comoving), we can define the phase space as $(R, P_R)$ with

$$P_R = m \dot{R}$$

(6)

being the linear momentum. Then, substituting Eq. (6) into Eq. (5), the classical Hamiltonian takes the form

$$H_{cl}(R, P_R) = \frac{P_R^2}{2m} - \frac{GMm}{R} - \frac{1}{6} \Lambda m R^2 .$$

(7)

Now, performing the canonical quantization

$$P_R \rightarrow P_{Rop} = -i \hbar \frac{d}{dR} ,$$

(8)

we obtain the Hamiltonian operator of a particle (galaxy) moving in a cosmologic substratum which is comprised of dust. It is given by

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dR^2} - \frac{GMm}{R} - \frac{1}{6} \Lambda m R^2 .$$

(9)
3. Schrödinger equation

In this section we will solve the Schrödinger equation for a particle (galaxy) in the Newtonian Universe.

We may write the Schrödinger equation in the form of the eigenvalue equation

$$H\psi(R) = E\psi(R), \quad (10)$$

where it is understood that the eigenfunction $\psi(R)$ corresponds to the eigenvalue $E$,
with $\Psi(R,t) = \psi(R)e^{-iEt/\hbar}$ being the general wave solution of the time-dependent Schrödinger equation.

Substituting Eq. (9) into Eq. (10), we obtain

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(R)}{dR^2} + \left( -\frac{GMm}{R} - \frac{1}{6}\Lambda mR^2 \right) \psi(R) = E\psi(R). \quad (11)$$

Note that all eigenfunctions correspond to bound states of positive energy. It is more suitable to rewrite Eq. (11) in terms of “dimensionless” quantities. To do this, we first introduce the “dimensionless” eigenvalues

$$\gamma = \frac{2E}{\hbar\Omega}, \quad (12)$$

where

$$\Omega = \left( -\frac{\Lambda}{3} \right)^{1/2}. \quad (13)$$

We will also use the “dimensionless” variable

$$x = \tau R, \quad (14)$$

where the coefficient $\tau$ is given by

$$\tau = \left( \frac{m\Omega}{\hbar} \right)^{1/2}. \quad (15)$$

Then, the Schrödinger equation (11) can be rewritten as

$$\frac{d^2\psi(x)}{dx^2} + \left( \gamma - x^2 - \frac{1}{2}\frac{\delta}{x} \right) \psi(x) = 0, \quad (16)$$

where the parameter $\delta$ is given by

$$\delta = -\frac{4GMm^2}{\tau \hbar^2}. \quad (17)$$

Now, let us analyse the asymptotic behaviour of $\psi$ when $|x| \to \infty$. For any finite value of the total energy $E$ the quantity $\gamma$ and the term in $x^{-1}$ become negligible with respect to $x^2$ in the limit $|x| \to \infty$, so that in this limit Eq. (16) reduces to

$$\left( \frac{d^2}{dx^2} - x^2 \right) \psi(x) = 0. \quad (18)$$

The functions which satisfy this equation for large enough $|x|$ are

$$\psi(x) = x^r e^{\pm x^2/2}. \quad (19)$$
The wave function $\psi$ must be bounded everywhere, including at $x = \pm \infty$. Therefore, the physically acceptable solution must contain only the minus sign in the exponent. The asymptotic analysis suggests looking for solutions to Eq. (16) which are valid for all $x$ having the form

$$\psi(x) = x \, e^{-x^2/2} P(x),$$

where $P(x)$ are functions which must not affect the asymptotic behaviour of $\psi$. Substituting Eq. (20) into Eq. (16), we obtain the following differential equation for $P(x)$

$$\frac{d^2 P}{dx^2} + \left( \frac{2}{x} - 2x \right) \frac{dP}{dx} + \left[ (\gamma - 3) - \frac{1}{2} \frac{\delta}{x} \right] P = 0. \quad (21)$$

Next, we will find analytical solutions for Eq. (21).

### 4. Biconfluent Heun equation

Equation (21) is a particular case of the biconfluent Heun equation [16], which in the canonical form is given by

$$\frac{d^2 y}{dx^2} + \left( \frac{1 + \alpha}{x} - \beta - 2x \right) \frac{dy}{dx} + \left\{ (\gamma - \alpha - 2) - \frac{1}{2} (\delta + (1 + \alpha) \beta) \frac{1}{x} \right\} y = 0, \quad (22)$$

where $y(x) = \text{HeunB}(\alpha, \beta, \gamma, \delta; x)$ are the biconfluent Heun functions.

Let us present solutions of Eq. (22) as a expansion

$$y(x) = \sum_{s=0}^{\infty} c_s x^s, \quad (23)$$

$$\frac{d}{dx} y(x) = \sum_{s=1}^{\infty} sc_s x^{s-1}, \quad (24)$$

$$\frac{d^2}{dx^2} y(x) = \sum_{s=2}^{\infty} s(s-1)c_s x^{s-2}. \quad (25)$$

Substituting Eqs. (23), (24) and (25) into Eq. (22), leads to

$$\sum_{s=2}^{\infty} s(s-1)c_s x^{s-2} + \left( \frac{1 + \alpha}{x} - \beta - 2x \right) \sum_{s=1}^{\infty} sc_s x^{s-1} + \left\{ (\gamma - \alpha - 2) - \frac{1}{2} (\delta + (1 + \alpha) \beta) \frac{1}{x} \right\} \sum_{s=0}^{\infty} c_s x^s = 0. \quad (26)$$

From Eq. (26) it follows

$$\sum_{s=2}^{\infty} s(s-1)c_s x^{s-2} + \sum_{s=1}^{\infty} (1 + \alpha) sc_s x^{s-2} - \sum_{s=1}^{\infty} \beta sc_s x^{s-1}$$
\[-\sum_{s=1}^{\infty} 2sc_s x^s + \sum_{s=0}^{\infty} (\gamma - \alpha - 2)c_s x^s - \sum_{s=0}^{\infty} \frac{1}{2}[\delta + (1 + \alpha)\beta]c_s x^{s-1} = 0,\]

(27)

or, equivalently

\[\sum_{S=0}^{\infty} (S + 2)(S + 1)c_{S+2} x^S + \sum_{S=-1}^{\infty} (1 + \alpha)(S + 2)c_{S+2} x^S
- \sum_{S=0}^{\infty} \beta(S + 1)c_{S+1} x^S - \sum_{S=1}^{\infty} 2Sc_s x^S + \sum_{S=0}^{\infty} (\gamma - \alpha - 2)c_s x^S
- \sum_{S=0}^{\infty} \frac{1}{2}[\delta + (1 + \alpha)\beta]c_{S+1} x^S = 0.\]

(28)

Collecting terms of same order in \(x\), we get

\[\left\{ (1 + \alpha)c_1 - \frac{1}{2}[\delta + (1 + \alpha)\beta]c_0 \right\} x^{-1}
+ \left[ 2(2 + \alpha)c_2 - \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + \beta \right\} c_1 + (\gamma - \alpha - 2)c_0 \right]\]
+ \sum_{S=1}^{\infty} \left[ (S + 2)(S + 2 + \alpha)c_{S+2} - \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + (S + 1)\beta \right\} c_{S+1}
+ (\gamma - \alpha - 2 - 2S)c_S \right] x^S = 0.\]

(29)

Thus, Eq. (29) gives the recursion relation for expansion coefficients

\[(1 + \alpha)c_1 = \frac{1}{2}[\delta + (1 + \alpha)\beta]c_0,\]

\[2(2 + \alpha)c_2 = \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + \beta \right\} c_1 - (\gamma - \alpha - 2)c_0,\]

\[(S + 2)(S + 2 + \alpha)c_{S+2} = \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + (S + 1)\beta \right\} c_{S+1}
- (\gamma - \alpha - 2 - 2S)c_S, \quad S \geq 0.\]

(30)

From this, after simple change in notation, we obtain

\[c_0 = 1,\]

\[(1 + \alpha)c_1 = \frac{1}{2}[\delta + (1 + \alpha)\beta],\]

\[2(1 + \alpha)(2 + \alpha)c_2 = \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + \beta \right\} \frac{1}{2}[\delta + (1 + \alpha)\beta]
- (1 + \alpha)(\gamma - \alpha - 2),\]

\[6(1 + \alpha)(2 + \alpha)(3 + \alpha)c_3 = \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + 2\beta \right\}
\times \left\{ \frac{1}{2}[\delta + (1 + \alpha)\beta] + \beta \right\} \frac{1}{2}[\delta + (1 + \alpha)\beta] - (1 + \alpha)(\gamma - \alpha - 2)]
- 2(2 + \alpha)(\gamma - \alpha - 4)\frac{1}{2}[\delta + (1 + \alpha)\beta],\]

(31)
which permits us to define
\[ A_0 = c_0 = 1 , \]
\[ A_1 = (1 + \alpha)c_1 = \frac{1}{2}[\delta + (1 + \alpha)\beta] , \]
\[ A_2 = 2(1 + \alpha)(2 + \alpha)c_2 , \]
\[ A_3 = 6(1 + \alpha)(2 + \alpha)(3 + \alpha)c_3 , \]
\[ \vdots \]
\[ A_s = s!(1 + \alpha)s c_s , \quad s \geq 0 , \quad (32) \]

where
\[ (1 + \alpha)_s = \frac{\Gamma(s + 1 + \alpha)}{\Gamma(1 + \alpha)} \]
\[ = \begin{cases} 
(1 + \alpha)(2 + \alpha)\cdots(s + \alpha) & s = 1, 2, 3, \ldots \\
1 & s = 0 \end{cases} \quad (33) \]

Then, when \( \alpha \) is not a negative integer, the biconfluent Heun functions can be written as [17]
\[ \text{HeunB}(\alpha, \beta, \gamma, \delta; x) = \sum_{s \geq 0} \frac{A_s}{(1 + \alpha)_s s!} x^s , \quad (34) \]

where
\[ A_{s+2} = \left\{ (s + 1)\beta + \frac{1}{2}[\delta + (1 + \alpha)\beta] \right\} A_{s+1} \]
\[ - (s + 1)(s + 1 + \alpha)(\gamma - \alpha - 2 - 2s)A_s , \quad s \geq 0 . \quad (35) \]

4.1. Heun polynomials

From the recursion relation given by Eq. (35), the function \( \text{HeunB}(\alpha, \beta, \gamma, \delta; x) \) becomes a polynomial of degree \( n \) if and only if the two following conditions are fulfilled [18]:
\[ \begin{align*}
(i) & \quad \gamma - \alpha - 2 = 2n, \quad n = 0, 1, 2, \ldots \\
(ii) & \quad A_{n+1} = 0
\end{align*} \quad (36) \]

where \( A_{n+1} \) is a polynomial of degree \( n + 1 \) in \( \delta \); there are at most \( n + 1 \) suitable values of \( \delta \), labeled
\[ \delta^\mu, \quad 0 \leq \mu \leq n . \quad (37) \]

The polynomial \( A_{n+1} \), which has \( n + 1 \) real roots when \( 1 + \alpha > 0 \) and \( \beta \in \mathbb{R} \), is the determinant of dimension \( n + 1 \) given by
\[
\begin{vmatrix}
\delta' & 1 & 0 & 0 & \cdots & \cdots & 0 \\
2(1 + \alpha)n & \delta' - \beta & 1 & 0 & \cdots & \cdots & 0 \\
0 & 4(2 + \alpha)(n - 1) & \delta' - 2\beta & 1 & 0 & \cdots & 0 \\
0 & 0 & \gamma_2 & \delta' - 3\beta & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \gamma_{s-1} & \delta'_{s-1} \\
0 & 0 & 0 & 0 & 0 & \gamma_s & \delta_s \\
\end{vmatrix}, \quad (38)
\]
where
\[
\delta' \equiv -\frac{1}{2}[\delta + (1 + \alpha)\beta] , \tag{39}
\]
\[
\delta'_s = \delta' - (s + 1)\beta , \tag{40}
\]
\[
\gamma_s = 2(s + 1)(s + 1 + \alpha)(n - s) . \tag{41}
\]

Now, let us return to the wave functions \(\psi(x)\). Using Eq. (20), we see that the physically acceptable solutions of Eq. (21) are given by
\[
\psi_n(x) = x e^{-x^2/2} P_{n,\mu}^{(\alpha,\beta)}(x) , \tag{42}
\]
where the functions \(P_{n,\mu}^{(\alpha,\beta)}(x)\) are polynomials of degree \(n\), with \(0 \leq \mu \leq n\). Moreover, the polynomials \(P_{n,\mu}^{(\alpha,\beta)}(x)\) satisfy the biconfluent Heun equation (22) with
\[
\gamma = \alpha + 2(n + 1) \quad \text{and} \quad A_{n+1} = 0 ,
\]
\[
\frac{d^2 P_{n,\mu}}{dx^2} + \left(\frac{1 + \alpha}{x} - \beta - 2x\right) \frac{d P_{n,\mu}}{dx} + \left\{2n - \frac{1}{2}[\delta^\prime \mu + (1 + \alpha)\beta] \frac{1}{x}\right\} P_{n,\mu} = 0 . \tag{43}
\]

The polynomials \(P_{n,\mu}^{(\alpha,\beta)}(x) = \text{HeunB}(\alpha,\beta,\alpha + 2(n + 1),\delta^\prime_n; x)\) are called Heun polynomials of the biconfluent case. It is clear from the foregoing discussion that they are uniquely defined, except for an arbitrary multiplicative constant. Computing the highest terms in this polynomial gives the formula
\[
x^n + \frac{1}{(1 + \alpha)_1} \delta x^{n-1} + \frac{1}{(1 + \alpha)_2} \left[\delta'(\delta' - \beta) - 2(1 + \alpha)n\right] \frac{x^{n-2}}{2!} + \frac{1}{(1 + \alpha)_3} \left[\delta'(\delta' - \beta)(\delta' - 2\beta) - 4\delta'(2 + \alpha)(n - 1)\right] - 2(\delta' - 2\beta)(1 + \alpha)n \frac{x^{n-3}}{3!} + Q(x) , \tag{44}
\]
with \(d^n Q \leq n - 4\).

In our case, the parameters \(\alpha\) and \(\beta\) are given by
\[
\alpha = 1 , \tag{45}
\]
\[
\beta = 0 , \tag{46}
\]
so that, one immediately deduces from this the results for small \(n\), the explicit form of the Heun polynomials \(P_{n,\mu}(1,0; x)\) as:
\[
n = 0: \quad \gamma = 3, \quad \delta^0_0 = 0
\]
\[
P_{0,0}(1,0; x) = 1 ; \tag{47}
\]
\[
n = 1: \quad \gamma = 5, \quad \delta^1_0 - 16 = 0
\]
\[
P_{1,\mu}(1,0; x) = x - \frac{1}{2} \delta^1_\mu , \tag{48}
\]
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\[ n = 2: \gamma = 7, (\delta_{\mu}^2)^3 - 80\delta_{\mu}^2 = 0 \]

\[ P_{2,\mu}(1,0; x) = x^2 - \frac{1}{2} \frac{\delta_{\mu}^2}{2} x + \frac{1}{6} \left[ (\delta_{\mu}^2)^2 - 8 \right] \frac{1}{2}. \]  \hspace{1cm} (49)

To each value \( n = 0, 1, 2, \ldots \) will then correspond an function \( \text{HeunB}(\alpha, \beta, \gamma, \delta; x) \) which is a polynomial of degree \( n \) in \( x \), and a physically acceptable wave function \( \psi(x) \) given by Eq. (42).

5. Energy levels

We see from Eq. (36) that the eigenvalue \( \gamma \) must take on one of the discrete values

\[ \gamma - \alpha - 2 = 2n, \quad n = 0, 1, 2, \ldots, \]  \hspace{1cm} (50)

where the quantum number \( n \) is a positive integer or zero.

Using Eq. (12) we therefore find that the energy spectrum of a particle (galaxy) in the Newtonian Universe is given by

\[ E_n = \left( n + \frac{3}{2} \right) \hbar \Omega = \left( n + \frac{3}{2} \right) \hbar \left( -\frac{\Lambda}{3} \right)^{\frac{1}{2}}, \quad n = 0, 1, 2, \ldots. \]  \hspace{1cm} (51)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential_energy_newtonian_universe.png}
\caption{The potential energy of the Newtonian Universe \( U_{\text{eff}}(R) = -\frac{GMm}{R} - \frac{1}{2} \Lambda m R^2 \). Also shown the first few energy eigenvalues.}
\end{figure}

We see from Eq. (51) that this quantum mechanical energy spectrum consist of an infinite sequence of discrete levels (see Fig. 1), which are equally spaced and are similar to those obtained for the three-dimensional isotropic oscillator. We remark that the eigenvalues given by Eq. (51) are non-degenerate, since for each value of the quantum number \( n \) there exists only one eigenfunction.
If we consider a scenario in which matter is dominated by a negative cosmological constant, that is, $\Lambda = -|\Lambda|$, the energy spectrum is given by

$$E_n = \left( n + \frac{3}{2} \right) \hbar \left( \frac{|\Lambda|}{3} \right)^{1/2}, \quad n = 0, 1, 2, \ldots . \quad (52)$$

6. The wave function for the Newtonian Universe

Using Eq. (42), we see that to each of the discrete values $E_n$ of the energy, given by Eq. (51), there corresponds one, and only one, physically acceptable eigenfunction, namely

$$\psi_n(R) = N_n \tau R e^{-\tau^2 R^2/2} P_{n,\mu}(\alpha, \beta; \tau R) = N_n \tau R e^{-\tau^2 R^2/2} \text{HeunB}(\alpha, \beta, \alpha + 2(n + 1), \delta_n; \tau R), \quad (53)$$

where we have returned to our original variable $R$, and the full expression of the parameters $\alpha, \beta, \gamma$, and $\delta$ are given by:

$$\alpha = 1 ; \quad (54)$$

$$\beta = 0 ; \quad (55)$$

$$\gamma = \frac{2E}{\hbar} \left( -\frac{3}{\Lambda} \right)^{1/2} ; \quad (56)$$

$$\delta = -4GM \left( \frac{m}{\hbar} \right)^{3/2} \left( -\frac{3}{\Lambda} \right)^{1/4}. \quad (57)$$

The quantity $N_n$, written on the right of Eq. (53) is a constant which (apart from an arbitrary phase factor) can be determined by requiring that the wave function given by Eq. (53) be normalised to unity. That is

$$\int_0^\infty |\psi_n(R)|^2 dR = \left| N_n \right|^2 \int_0^\infty x^2 e^{-x^2} P^2_{n,\mu}(\alpha, \beta; x) dx = 1. \quad (58)$$

In order to evaluate the integral on the right of Eq. (58), we consider the power series expansion of $P_{n,\mu}(\alpha, \beta; x)$ in $x$ given by Eq. (34) as well as the second power series expansion

$$P_{m,\nu}(\alpha, \beta; x) = \sum_{l=0}^m \frac{A_l}{(1 + \alpha)l!} x^l. \quad (59)$$

Using Eqs. (34) and (59), we may then write

$$N_n = \left( \frac{T}{I} \right)^{1/2}, \quad (60)$$

where

$$I = \int_0^\infty x^2 e^{-x^2} P_{n,\mu}(\alpha, \beta; x) P_{m,\nu}(\alpha, \beta; x) dx$$

$$= \sum_{k=0}^n \sum_{l=0}^m \frac{A_k A_l}{(1 + \alpha)_k(1 + \alpha)_l k!l!} \int_0^\infty x^{2+k+l} e^{-x^2} dx. \quad (61)$$
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For \( n = m, \mu = \nu \) and \( k = l \), we have

\[
I = \int_0^\infty x^2 e^{-x^2} P_{n,\mu}^2(\alpha, \beta; x) dx
\]

\[
= \sum_{k=0}^n \frac{A_k^2}{[(1 + \alpha)k!]^2} \int_0^\infty x^{2+2k} e^{-x^2} dx .
\]

(62)

Since

\[
\int_0^\infty y^{\lambda-1} e^{-y} dy = \frac{1}{\eta^{\lambda/u}} \frac{\Gamma(\lambda)}{\Gamma(\lambda u)} ,
\]

(63)

the integral on the right-hand side of Eq. (62) is simply

\[
I = \sum_{k=0}^n \frac{A_k^2}{[(1 + \alpha)k!]^2} \frac{1}{2} \frac{\Gamma\left(\frac{3}{2} + k\right)}{\Gamma\left(\frac{3}{2} + k\right)} .
\]

(64)

From Eqs. (60) and (64) we see that apart from an arbitrary complex multiplicative factor of modulus one the normalisation constant \( N_n \) is given by

\[
N_n = \left[ \sum_{k=0}^n \frac{A_k^2}{[(1 + \alpha)k!]^2} \frac{1}{2} \frac{\Gamma\left(\frac{3}{2} + k\right)}{\Gamma\left(\frac{3}{2} + k\right)} \right]^{-1/2} ,
\]

(65)

so that the normalised Newtonian Universe eigenfunctions are given by

\[
\psi_n(R) = \left[ \sum_{k=0}^n \frac{A_k^2}{[(1 + \alpha)k!]^2} \frac{1}{2} \frac{\Gamma\left(\frac{3}{2} + k\right)}{\Gamma\left(\frac{3}{2} + k\right)} \right]^{-1/2}
\times \tau R e^{-\tau^2 R^2/2} \text{HeunB}(\alpha, \beta, \alpha + 2(n + 1), \delta_{\mu}; \tau R) .
\]

(66)

7. Conclusions

We have presented exact and general solution of the Schrödinger equation for a particle (galaxy) moving in a universe constituted of incoherent matter and in the presence of the cosmological constant.

This complete set of solutions, given in terms of the biconfluent Heun functions, satisfies the appropriate boundary conditions \( \psi(0) = 0 \) and \( \psi(\infty) = 0 \), and is valid over the range \( 0 \leq x < \infty \).

In order to get a well-behaved solution, we impose the polynomial condition for these analytic solutions and we obtain the Heun polynomials of the biconfluent case. We written the first three polynomials for this scenario.

We also obtain the exact expression for the energy spectrum, which is proportional to the cosmological constant. These energy levels are similar to those obtained for the three-dimensional isotropic oscillator. This can be physically explained as a consequence of the fact that the fixed rectangular coordinate (comoving) to be the scale factor, that is, the radius of the spherical Newtonian Universe. From a mathematical point of a view, this is expected because the term proportional to \( R^2 \) in the equation of motion dominates the function behaviour.

Indeed, all solutions presented here depend on the cosmological constant.
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