Transient generalized Taylor-Couette flow: a semi-analytical approach

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ABSTRACT
Semi-analytical solution of transient generalized Taylor-Couette flow of a viscous, incompressible fluid between the gaps of concentric rotating cylinders under applied azimuthal pressure gradient is presented. The dimensionless governing equations are transformed into standard Bessel equation with the aid of Laplace transformation technique and by a suitable transformation. Analytical solution of the Bessel equation is obtained and the Riemann-sum-approximation method of Laplace inversion is utilized. The solution obtained is validated by comparing the Riemann-sum-approximation solution with the exact steady-state solutions obtained separately. The velocity profile and skin frictions on both surfaces of cylinders are depicted graphically and discussed. The present study reveals that the velocity profile of the fluid is enhanced with increase in time and angular velocity. Furthermore, back flow occurs for adverse pressure gradient.

1. Introduction
In the early days [1], fluid flow between rotating concentric cylinders has been fascinating scientists for millennia with its remarkably varied patterns and its chaotic and turbulent behaviour. In the past years, Taylor-Couette flow problem has attracted great attention because of its importance in flow stability. For the instability of Taylor-Couette flow between concentric rotating cylinders [2], uses energy gradient theory to analyse the instability of the fluid flow. They opined that the energy gradient method which is a semi-empirical theory is effective for rotating fluid flows. Stability and instability of Taylor-Couette flow and Couette flow analysis have been investigated thoroughly numerically and experimentally over the past years found in [3–9]. Hristova et al. [10] stated that transient growth is enhanced by the curvature of the rotating cylinders [11]. In a report, studies numerically the non-normal energy transient growth in a Taylor-Couette system. Their study focused mostly on the linear stable regime of counter-rotation of the cylinders.

In addition, numerical study of the flow of non-isothermal fluid through a rotating curved duct with square cross-section is carried out by [12] using spectral method to analyse two-dimensional flow of viscous incompressible fluid. They revealed that there is an existence of two and four vortex solutions on the various branches and that these vortices are produced due to the centrifugal force and as well as the Coriolis force or by the combinations of both forces.

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In the aspect of heat and mass transfer, Aberkane et al. [13] carried out the effect of an axial magnetic field on the heat and mass transfer in rotating annulus and [14] studied numerically radiating flow and heat transfer in a mixed convection vertical channel where the channel is partially filled with a Darcy-Forchheimer porous substrate [13] found out that increase in Hartmann number decreases Nusselt number as well as Sherwood number. More works in the area of Taylor-Couette flows are found in [15–19].

Effect of channel walls rotation on micromixer devices was carried out by Kawabe et al. [20]. In their study, the flow is dominated by two parameters, Dean and Taylor number and four effects of moving walls were investigated numerically and experimentally. LIF method is used to measure and examine the secondary flow characteristics and OpenFOAM (CFD Solver) is used to perform three-dimensional numerical simulation to investigate the chaotic mixing mechanism. From review of some of the literature, it is obvious that Taylor-Couette flow problem is lacking complete analytical and/or semi-analytical solution to study stability and instability as well as velocity profile of the problem [21] considered the solution of unsteady MHD Couette flow problem in a permeable surface formed by two concentric porous cylinders which length are infinite and the fluid is induced either by impulsive or the accelerated movements of the outer cylinders. They observed that accelerations of the flow are caused by suctions, whereas injection causes retardations.

Analytical solutions of equations of motion of a Newtonian fluid for the fully developed laminar flow between the gap of concentric cylinders and the fluid flow is due to the application of oscillating circumferential pressure gradient called finite-gap oscillating Dean flow was considered by Tsangaris et al. [22]. Tsangaris and Vlachakis [23] extended the problem to the case where the cylinders are porous. In the presents study, we extend the work of [22,23] to the case when the two cylinders rotates and the fluid is induced with the application of azimuthal pressure gradient (generalized Taylor-Couette problem), and to the best of authors’ findings, there was none that considered the fully developed semi-analytical solution of transient generalized Taylor-Couette flow. Hence the motivation of this present study.

This research paper presents a semi-analytical solution of the transient generalized Taylor-Couette flow of an incompressible fluid between the gaps of porous concentric rotating cylinders due to an azimuthal pressure gradient taking into consideration radial flow. The fluid is assumed to be Newtonian with constant viscosity coefficient. The governing equations of the problem are solved by the uses of Laplace transformation technique and Laplace inversion method called Riemann-sum approximation. Effect of dimensionless pressure gradient parameter, dimensionless time, angular velocity and the ratio of the radii on the velocity profile and skin friction are discussed. Analytical solution of the steady state for velocity profile and skin friction has been obtained and numerical results are tabulated for comparisons, which is used to validate the results. The diagrammatic representation of the problem is depicted in Figure 1.

2. Mathematical analysis

The motion of a viscous, incompressible fluid found between the gaps of two horizontal cylinders with infinite length is considered. The inner and outer cylinders rotate having radii $r_1$ and $r_2$, and angular velocities of $\omega_1$ and $\omega_2$ respectively. At time $t \leq 0$, the fluids are assumed to be at rest, when $t > 0$, and moves with constant velocity and the fluid exists in the region $r_2 - r_1 (r_2 > r_1)$. The time dependent flow formation in the region is due to sudden application of an azimuthal pressure gradient $\left( \frac{\partial p}{\partial \phi} \right)$.

Since the length of the cylinders are infinite and the fluid flow is assumed to be fully developed ($v_\phi = 0$) and continuity equation, $v_\phi$ is proved to be a function of the radial coordinate $r$ and time $t$ only, $v_\phi = v(r,t)$, and the momentum equations reduces to the following forms

$$
\rho \frac{\gamma^2}{r} = \frac{\partial p}{\partial r}
$$

Figure 1. Geometry of the problem.
\[
\rho \frac{\partial v}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right)
\]

(2)

the initial and boundary conditions are as given below

\[
t \leq 0 : \ v = 0 \text{ for } r_1 \leq r \leq r_2
\]

\[
t > 0 \begin{cases} v = \omega_1 r_1 \text{ at } t = r_1 \\ v = \omega_2 r_2 \text{ at } t = r_2 \end{cases}
\]

(3)

where \( v \) is the circumferential velocity component, \( \rho \) is the fluid density, \( p \) is the static pressure and \( \mu \) is the dynamic viscosity of the fluid.

3. Dimensionless analysis

By introducing the following dimensionless variables

\[
R = \frac{r}{r_1}, \lambda = \frac{r_2}{r_1}, \omega = \frac{\omega_2}{\omega_1}, T = \frac{vt}{r_1^2}
\]

\[
V = \frac{vr_1}{v}, P = -\frac{r_1^2}{\mu} \frac{\partial p}{\partial \varphi}
\]

(4)

Equations (2) and (3) reduces to

\[
\frac{\partial V}{\partial T} = \frac{P}{R} + \left( \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} - \frac{V}{R^2} \right)
\]

(5)

the initial and boundary conditions

\[
T \leq 0 : V = 0 \text{ for } 1 \leq R \leq \lambda
\]

\[
T > 0 \begin{cases} V = 1 \text{ at } R = 1 \\ V = \lambda \omega \text{ at } R = \lambda \end{cases}
\]

(6)

4. Solution to the problem

4.1. Velocity

The solution of Equation (5) can be obtained using the Laplace transforms technique. Defined as:

\[
\tilde{V}(R, s) = \int_0^\infty V(R, T)e^{-st}dT, (s > 0)
\]

where \( s \) is the Laplace parameter. Taking the Laplace transformation of Equations (5) and (6), the following equation is obtained

\[
\frac{d^2 \tilde{V}}{dR^2} + \frac{1}{R} \frac{d\tilde{V}}{dR} - \left( 1 + sR^2 \right) \frac{\tilde{V}}{R^2} = -\frac{P}{Rs}
\]

(7)

with the boundary conditions

\[
R = 1 \text{ at } \tilde{V} = \frac{1}{s} \\
R = \lambda \text{ at } \tilde{V} = \frac{\lambda \omega}{s}
\]

(8)

Using the approach of Tsangaris and Vlachakis [22], Equation (7) which is a linear non-homogeneous differential equation can be reduced by using the transformation given below:

\[
\tilde{V}(R, s) = V(R, s) + \frac{P}{Rs^2}
\]

(9)

Substituting Equation (9) on Equation (7) with boundary conditions (8), the velocity profile in Laplace domain is given as

\[
\tilde{V}(R, s) = C_1 I_1 (R\sqrt{s}) + C_2 K_1 (R\sqrt{s}) + \frac{P}{Rs^2}
\]

(10)

where \( C_1 \) and \( C_2 \) are constants and is given in Appendix and \( I_1 \) and \( K_1 \) are the modified Bessel functions of first and second kind and first order, respectively.

4.2. Skin friction

The skin friction or shear stress between the fluid layers due to azimuthal pressure gradient or in a rotating cylinders (i.e. flow along azimuthal direction \( (0, \nu_0, 0) \)) \( \tau \) is defined as \( \tau = \frac{R \partial \tilde{V}}{\partial R} \) as used in [24–26]. It can also be written as \( \tau = \frac{\partial \tilde{V}}{\partial R} \). Differentiating and simplifying gives the first expression for \( \tau \). Differentiating Equation (10) with respect to \( R \), the skin friction at \( R = 1 \) and \( R = \lambda \) are given in Equation (11) and Equation (12) respectively.

\[
\tilde{\tau}_1 = R \left. \frac{d}{dR} \left( \frac{\tilde{V}}{R} \right) \right|_{R=1} = \sqrt{s} \left[ C_1 I_2 (\sqrt{s}) - C_2 K_2 (\sqrt{s}) \right] - \frac{2P}{s^2}
\]

(11)

\[
\tilde{\tau}_\lambda = R \left. \frac{d}{dR} \left( \frac{\tilde{V}}{R} \right) \right|_{R=\lambda} = \sqrt{s} \left[ C_1 I_2 (\lambda \sqrt{s}) - C_2 K_2 (\lambda \sqrt{s}) \right] - \frac{2P}{\lambda^2 s^2}
\]

(12)

where \( I_2 \) is the second order modified Bessel function of the first kind and \( K_2 \) is the second order modified Bessel function of the second kind.

In order to determine the velocity and skin friction in the time domain, due to the complexity of Equations (10), (11) and (12), a numerical approach based on the method of Laplace inversion called Riemann-sum approximation used by [27–30] has been found to be very efficient and promising tool in transforming from Laplace domain to time domain. In this numerical method of inversion, any function in the \( s \) domain can be inverted to the time \( t \) domain by using the following equation:

\[
\nu(R, T) = \frac{e^{\omega T}}{T} \sum_{k=1}^{n} \left( \frac{n}{\omega} \right) R e^{i\frac{\omega \pi}{T}} (-1)^k
\]

(13)

where \( \omega \) is the "real part of" \( \sqrt{-1} \) and \( i \) is the imaginary number, \( n \) is the number of terms used in the Riemann-sum approximation and \( \epsilon \) is the real part of the Bronwich contour that is used in inverting Laplace transforms. The Laplace inversion method called Riemann-sum approximation involves a single summation for the numerical process. Its accuracy depends on the value of \( \epsilon \) and the truncation error dictated by \( n \). Tzou [31], stated that the value of \( \epsilon \) must be selected so that the Bronwich contour contains all the branch points. The
number of terms \((n)\) used is 1000. For faster convergence, the quantity \(\epsilon T = 4.7\) gives the most satisfactory and acceptable results.

### 4.3. Steady state solutions

The steady state velocity distribution \(V_s\) is obtained by setting \(\frac{\partial V_s}{\partial T}\) in Equation (5) to zero, thereby, the following differential equation is obtained

\[
\frac{d^2 V_s}{dr^2} + \frac{1}{R} \frac{dV_s}{dr} - \frac{V_s}{R^2} = -\frac{P}{R}
\]

(14)

By using the transformation variable as follows

\[
R = e^t
\]

(15)

Equation (14) is transformed into the following ordinary differential equation

\[
\frac{d^2 V_s}{dt^2} - V_s = -Pe^t
\]

(16)

Using Equation (15) and Equation (16), the solution of Equation (14) is given as

\[
V_s(R) = C_3 R + C_4 \frac{P}{2} R \ln R
\]

(17)

where \(C_3\) and \(C_4\) are constants given in Appendix.

The steady state skin friction \(\tau_{1s}\) and \(\tau_{2s}\) is obtained similarly as follows

\[
\tau_{1s} = R \left. \frac{d}{dR} \left( \frac{V_s}{R} \right) \right|_{R=1} = \left[ \frac{\lambda^2 (P \ln \lambda + 2(\omega - 1))}{(1 - \lambda^2)} \right] - \frac{P}{2}
\]

(18)

\[
\tau_{2s} = R \left. \frac{d}{dR} \left( \frac{V_s}{R} \right) \right|_{R=\lambda} = \left[ \frac{P \ln \lambda + 2(\omega - 1)}{(1 - \lambda^2)} \right] - \frac{P}{2}
\]

(19)

### 5. Results and discussion

In the present study, MATLAB program is written to compute and plot the graphs of the velocity profile and skin friction and also to study the effect of \(T\), \(\omega\), \(\lambda\) and \(P\) on the velocity profile and skin friction at the outer surface of the inner cylinder and at the inner surface of the outer cylinder. Figures 2–4 shows the effect of time \(T\), angular velocity, \(\omega\), and dimensionless pressure gradient, \(P\). Figure 2 reveals that the dimensionless velocity profile increases as time, \(T\) increases. The velocity approaches a steady state which is shown in Table 1 as well as skin friction at \(R=1\) and \(R=\lambda\) as indicated in Tables 2 and 3. It is clear from Figure 3 that the velocity profile increases as the angular velocity, \(\omega\) increases. An increase in angular velocity increases the fluid motion since it is applied in the direction of the pressure gradient. Since angular velocity signifies the ratio of the angular velocity of the outer cylinder to the angular velocity of the inner cylinder, \(\omega = 1.0\) implies that the cylinders rotate at the same rate, \(\omega = 1.5\) and \(\omega = 2.0\) implies that the outer cylinder rotates faster than the inner cylinder which leads to higher
### Table 1. Numerical values obtained for the velocity using Riemann-sum approximation method and those obtained analytically at steady state for different values of $T$, $P$ and $\omega$.

| $\omega$ | $T$ | $P$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ | $R$ | $R.S$ | $E.S$ |
|---------|-----|-----|-----|------|------|-----|------|------|-----|------|------|-----|------|------|-----|------|------|-----|------|------|-----|------|------|
| $0.0$   |     |     |     |      |      |     |      |      |     |      |      |     |      |      |     |      |      |     |      |      |     |      |      |
| $0.2$   | $1.2$ | $0.6159$ | $0.6511$ | $0.6688$ | $0.7111$ | $0.7218$ | $0.7712$ | $0.9842$ | $1.1400$ | $1.0371$ | $1.2000$ | $1.0901$ | $1.3844$ | $1.1683$ | $1.3844$ | $1.2212$ | $1.4444$ | $1.2742$ | $1.5045$ |
| $0.4$   | $1.2$ | $0.6466$ | $0.6511$ | $0.7057$ | $0.7111$ | $0.7648$ | $0.7712$ | $1.1198$ | $1.1400$ | $1.1790$ | $1.2000$ | $1.2381$ | $1.3644$ | $1.3565$ | $1.3844$ | $1.4156$ | $1.4444$ | $1.4747$ | $1.5045$ |

Note: $R.S$: Riemann-sum; $E.S$: Exact Solution.

### Table 2. Numerical values obtained for skin friction at $R = 1$ using Riemann-sum approximation method and those obtained analytically at steady state for different values of $T$, $P$ and $\omega$.

| $\omega$ | $T$ | $\lambda$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ | $R.S$ | $E.S$ |
|---------|-----|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $0.0$   |     | $0.00$ |     |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| $0.2$   | $1.2$ | $-5.0905$ | $-6.6421$ | $-4.9397$ | $-6.5455$ | $-4.8969$ | $-6.4488$ | $1.4555$ | $-0.0967$ | $1.5533$ | $0.0010$ | $1.6491$ | $0.0967$ | $4.7285$ | $3.1760$ | $4.8253$ | $3.2727$ | $4.9221$ | $3.3694$ |

Note: $R.S$: Riemann-sum; $E.S$: Exact Solution.
| \( T \) | \( \omega \) | \( \lambda \) | \( R.S \) | \( E.S \) | \( R.S \) | \( E.S \) | \( R.S \) | \( E.S \) | \( R.S \) | \( E.S \) | \( R.S \) | \( E.S \) | \( R.S \) | \( E.S \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 1.2 | -4.4601 | -4.4598 | -4.5458 | -4.5455 | -4.6316 | -4.6311 | -1.7770 | 0.0856 | -1.8636 | -0.0009 | -1.9484 | -0.0856 | -0.4354 | 2.3584 | -0.5212 | 2.2727 | -0.6069 | 2.1871 |
| 1.4 | -1.9339 | -1.9338 | -2.0835 | -2.0833 | -2.2331 | -2.2328 | -2.0236 | 0.1495 | -2.1746 | -0.0015 | -2.3227 | -0.1495 | -2.0684 | 1.1912 | -2.2180 | 1.0417 | -2.3676 | 0.8922 |
| 1.6 | -1.0742 | -1.0833 | -1.2723 | -1.2821 | -1.4705 | -1.4808 | -2.2557 | 0.1987 | -2.4558 | -0.0020 | -2.6520 | -0.1987 | -2.8464 | 0.8397 | -3.0446 | 0.6410 | -3.2427 | 0.4423 |
| 1.8 | -0.5870 | -0.6553 | -0.8162 | -0.8929 | -1.0454 | -1.1305 | -2.3014 | 0.2376 | -2.5329 | -0.0024 | -2.7598 | -0.2376 | -3.1586 | 0.6840 | -3.3878 | 0.4646 | -3.6169 | 0.2088 |
| 0.4 | 1.2 | -4.4601 | -4.4598 | -4.5458 | -4.5455 | -4.6316 | -4.6311 | -1.3180 | 0.0009 | -1.4029 | -0.0856 | 0.3830 | 2.3584 | 0.2973 | 2.2727 | 0.2115 | 2.1871 |
| 1.4 | -1.9339 | -1.9338 | -2.0835 | -2.0833 | -2.2331 | -2.2328 | -1.3870 | 0.1495 | -1.5381 | -0.0015 | -1.6863 | -0.1495 | -1.1336 | 1.1912 | -1.2632 | 1.0417 | -1.4128 | 0.8922 |
| 1.6 | -1.0833 | -1.0833 | -1.2821 | -1.2821 | -1.4809 | -1.4808 | -1.5573 | 0.1987 | -1.7581 | -0.0020 | -1.9549 | -0.1987 | -1.7942 | 0.8397 | -1.9930 | 0.6410 | -2.1919 | 0.4423 |
| 1.8 | -0.6524 | -0.6553 | -0.8929 | -0.8929 | -1.1270 | -1.1305 | -1.7272 | 0.2376 | -1.9669 | -0.0024 | -2.2019 | -0.2376 | -2.2647 | 0.6840 | -2.5020 | 0.4646 | -2.7394 | 0.2088 |
| Steady State | 1.2 | -4.4598 | -4.4598 | -4.5458 | -4.5455 | -4.6311 | -4.6311 | 0.0856 | 0.0856 | -0.0009 | -0.0009 | -0.0856 | -0.0856 | 2.3584 | 2.3584 | 2.2727 | 2.2727 | 2.1871 |
| 1.4 | -1.9338 | -1.9338 | -2.0833 | -2.0833 | -2.2328 | -2.2328 | 0.1495 | 0.1495 | -0.0015 | -0.0015 | -1.495 | -1.495 | 1.1912 | 1.1912 | 1.0417 | 1.0417 | 0.8922 |
| 1.6 | -1.0833 | -1.0833 | -1.2821 | -1.2821 | -1.4808 | -1.4808 | 0.1987 | 0.1987 | -0.0020 | -0.0020 | -1.987 | -1.987 | 0.8397 | 0.8397 | 0.6410 | 0.6410 | 0.4423 |

Note: R.S: Riemann-sum; E.S: Exact Solution.
6. Conclusion

Transient generalized Taylor-Couette flow of a viscous incompressible fluid in the annular gap between two rotating concentric cylinders due to an azimuthal pressure gradient has been studied. The governing momentum equation along the continuity equation is derived and solved semi-analytically. The Riemann-sum approximation method of Laplace inversion is used
to invert the Bessel equation obtained through the Laplace transform technique into the time domain. The effect of the governing parameters appearing in the present study has been depicted graphically. Specifically, the following conclusions are obtained:

- The velocity of the fluid increases with increase in time $T$ and reaches steady state at large value of time $T$.
- As the angular velocity $\omega$ increases, the velocity profile of the fluid appreciates.
- Skin friction at both surfaces (at $R = 1$ and $R = \lambda$) increases in its values with increase in time, $T$ and, pressure gradient as shown in Tables 2 and 3.
- Increase in the pressure gradient $P$, increases the velocity profile of the fluid. For $P < 0$, back flow occurs as shown numerically in Table 1.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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**Appendix**

\[
\begin{align*}
C_1 &= \frac{\lambda [p - s] K_1 (\lambda \sqrt{s}) + [p - 2\alpha s \lambda^2] K_1 (\sqrt{s})}{\lambda s^2 [K_1 (\sqrt{s}) h_1 (\lambda \sqrt{s}) - K_1 (\lambda \sqrt{s}) h_1 (\sqrt{s})]} \\
C_2 &= \frac{\lambda [s - p] h_1 (\lambda \sqrt{s}) + [p - 2\alpha s \lambda^2] h_1 (\sqrt{s})}{\lambda s^2 [K_1 (\sqrt{s}) h_1 (\lambda \sqrt{s}) - K_1 (\lambda \sqrt{s}) h_1 (\sqrt{s})]} \\
C_3 &= \frac{2[1 - \lambda^2 (\omega - P h \lambda)]}{1 - \lambda^2} \\
C_4 &= \frac{\lambda^2 (P h \lambda + 2(\omega - 1))}{2[1 - \lambda^2]}
\end{align*}
\]