A High Resolution Optimum 2D Coprime Planar Array

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Abstract—Designing a new class of rectangular two-dimensional sparse array to enhance the signal resolving capabilities with a limited number of sensors has always been a challenge. In order to estimate both azimuth and elevation angle of arrival of the signal we propose a new class of rectangular sparse array with sensors on a 2D plane, whose difference co-array produces enhanced virtual aperture which leads to an increased uniform degree of freedom. The proposed design generates large contiguous hole free region which shows a significant reduction in holes as compared to the existing array geometries. To show the practical realization of the proposed array, optimization is performed on it to maximize the directivity in the array steering look-up direction by suppressing the sidelobe levels to a greater extent.

Index Terms—Coprime planar array, direction-of-arrival estimation, multiple signal classification (MUSIC), Virtual array, Coarray, Optimization.

I. INTRODUCTION

Numerous applications of array signal processing exist in satellite communication, wireless communication, MIMO, the medical field, the military, radar, and sonar, among others. DOA calculation is extremely useful in underwater acoustic networks [1], wireless networking [2], localization techniques, tracking, navigation, etc. To improve DOA resolution, a critical method currently being used is to use an array of antennas rather than a single antenna, as was previously done, due to the sensor array’s superior performance. The sparse array’s objective is to find the array configuration that produces the desired co-array with the fewest possible sensors.

Various classifications of non-uniform arrays include Minimum Redundancy Array (MRA) [3], Minimum Hole Array (MHA) [4], Nested Array [5], Coprime Array [6], etc. Conventional Coprime array consists of two subarrays having 2M and N elements respectively with inter-element spacing between them as Nd and Md where d is less than λ/2 where M and N are coprime numbers and d is inter-element spacing between the sensor elements and can resolve up to (2MN) sources with (2M+N-1) sensors. To extend the research span from azimuth to elevation angle, a two-dimensional sparse array was investigated in [7]. One category of 2D sparse array is non-parallel arrays which are L shape [8], cross-shaped and V shape antenna array. V shape array [9] is a generalization of L shape array in which L shape antenna array is tilted by θ degrees to form V shape but the problem in L and V shape geometry is that it returns coupled estimation results.

Another category of the 2D sparse array is the planar array that can be a coprime planar array, i.e. CPA [10] or generalized coprime array GCPA [11]. In CPA two planes are constructed of M × M and N × N where M and N are coprime numbers. In GCPA two rectangular planar subarrays are created with the first subarray as subarray1 having N1 × M1 sensors and subarray2 having N2 × M2 sensors, where N1, N2 are the numbers of sensors on the x-axis and M1, M2 are the numbers of sensors on the y-axis. Both Ni and Mi are coprime numbers. The inter-element spacings among the elements of subarray1 are dx1 = N2 · λ / 2 in the x-axis direction and dy1 = M2 · λ / 2 in the y-axis direction and the inter-element spacing among the elements of subarray2 is dx2 = N1 · λ / 2 and dy2 = M1 · λ / 2, where λ represents the wavelength. But CPA and GCPA show poor performance due to a lack of array aperture. Hence to overcome the problem of phase coupled estimation results and to overcome mutual coupling we propose a planar sparse array which is a phased antenna array. Although the array-based methods for CPA [12] maintain increased DOFs, however, they do not fully exploit the advantage of the non-uniform geometry in terms of high resolution. In this paper, we present a novel computationally efficient method for 2-D DOA estimation where the non-uniform property of coprime sparse array is maintained based which not only provides ease in direction of arrival calculation but also helps in attaining a high degree of freedom with less number of holes as compared to other planar array geometries discovered so far.

II. PRELIMINARIES

A. Proposed Rectangular Array Geometry

The novel rectangular coprime array (RCPA) is constructed by making a planar geometry of two subarrays, where each subarray consists of a complete conventional coprime array consisting of 2M+N-1 elements where M and N are coprime numbers. In RCPA we take a complete conventional coprime array on both the y-axis and z-axis and then construct a plane from it such that the unique coprime structure of the sparse array is maintained at each point in the lattice, whereas CPA [13] constructs an M × M plane and then N × N plane, so it seems to be uniform rectangular array rather
than a coprime array in its respective planes. Moreover, in the proposed geometry, there is no need of finding out the superposition of the peaks in the pseudo spectrum like [14], and hence the proposed geometry is computationally efficient. The RCPA is shown in Fig.1 where N=3, M=2 coprime pair is taken having a total of 2M+N-1=6 elements on each axis thereby creating a plane of (2M+N-1) ×(2M+N-1) elements i.e 6×6=36 elements. First, we create the physical coprime planar array, where a coprime array is given as:

Sensor locations in 1D coprime array are given by the set S with the total number of sensors as T= 2M+N-1 where M and N are coprime numbers and d is the inter-element spacing between the sensors in an array.

\[
S = \{Mnd \mid 1 \leq n \leq N - 1\} \cup \{Nmd \mid 0 \leq m \leq 2M - 1\}
\]

(1)

For example, lets take M=2, N=3 and, suppose d=1.

\[
S = \{2n \mid 1 \leq n \leq 2\} \cup \{3m \mid 0 \leq m \leq 3\}
\]

(2)

\[
S = \{0, 2, 3, 4, 6, 9\}
\]

(3)

The novel rectangular coprime array (RCPA) is constructed by making a planar geometry of two subarrays, where each subarray consists of a complete conventional coprime array consisting of 2M+N-1 elements where M and N are coprime numbers. In RCPA we take a complete coprime array on both the y-axis and z-axis and then construct a plane from it such that the unique coprime structure of the sparse array is maintained at each point in the lattice, whereas CPA [13] constructs an M × M plane and then N × N plane, so it seems to be uniform rectangular array rather than a coprime array in its respective planes. Moreover, in the proposed geometry, there is no need of finding out the superposition of the peaks in the pseudo spectrum, and hence the proposed geometry is computationally efficient. The RCPA is shown in Fig.1 where N=3, M=2 coprime pair is taken having a total of 2M+N-1=6 elements on each axis thereby creating a plane of (2M+N-1) ×(2M+N-1) elements i.e 6×6=36 elements.

Now we define the set L corresponding to the whose each entry corresponds to the normalized sensor locations in 2D coprime array. It is given by:

\[
L = \{(u, v)\forall u, v \in S\}
\]

(4)

B. 2D Array Data Model

Let M narrowband, uncorrelated, and far-field real sources impinge on the antenna array whose elements are located at 
\((u_x, v_x)\) where \((u_x, v_x) \in L\) defined in the next section. The \(i^{th}\) source have both azimuth and elevation angle as \(\theta, \phi\) respectively from the direction \((\theta_1, \phi_1), (\theta_2, \phi_2), (\theta_3, \phi_3), \ldots, (\theta_M, \phi_M)\) with respect to the normal to the array.

The array output can be expressed as

\[
Y(t) = \sum_{i=1}^{M} A(\theta_i, \phi_i) \cdot S_i(t) + N(t)
\]

(5)

where \(\theta_i, \phi_i\) are normalized DOA such that \(\theta_i = (d/\lambda)\sin(\theta)\cos(\phi_i)\) and \(\phi_i = (d/\lambda)\sin(\theta)\sin(\phi_i)\) and d=\(\lambda/2\) is the inter \(i\) sensor spacing. Each element of the steering vec-

Fig. 1: Proposed Rectangular Coprime Array(RCPA)

tor \(A(\theta_i, \phi_i) = [a(\theta_1, \phi_1), \ldots, a(\theta_M, \phi_M)]\) corresponding to the sensors at the location \((u_x, v_x) \in L\) is defined as \(e^{2\pi j(\theta u_x + \phi v_x)}\). Signal \(S_i(t) = [s_i(t), \ldots, s_M(t)]\) is the source signal vector and N(t) is the white Gaussian noise vector with zero mean and variance \(\sigma^2\), \(t = 1, 2, \ldots, T\) refers to the sampling time, where T is the total number of snapshots.

The covariance matrix of \(Y(t)\) can be written as:

\[
R = E[Y(t)Y(t)^H] = \sum_{i=1}^{M} \sigma_i^2 \cdot A(\theta_i, \phi_i) \cdot A^H(\theta_i, \phi_i) + \sigma^2 I
\]

(6)

where \(\sigma_i^2\) is the \(i^{th}\) source power and \(\sigma^2\) is the noise power. Note that the entity \(A(\theta_i, \phi_i) \cdot A^H(\theta_i, \phi_i)\) in the covariance matrix defined in eq.6 is of the form \(e^{2\pi j(\theta \cdot \phi)}(s_i - s_j)\) where \((s_i - s_j) \in D\) i.e the difference coarray having \((s_i - s_j)\) as the difference between \(i^{th}\) and \(j^{th}\) sensor location. Applying vectorization operation on eq.6 and reshaping it we get the autocorrelation vector defined on the difference coarray. Doa estimation using finite snapshots can be carried out by calculating the finite snapshot version of \(Y(t)\) and \(R\) as \(Y'(k)\) and \(R'\) respectively where \(k = 1, 2, 3 \ldots K\) be K realizations of eq.5. K are the total number of snapshots. The covariance matrix for which can be estimated as \(R' = \sum_{k=1}^{K} Y'(k)Y'(k)^H/K\)

Moreover, for sparse array, the covariance matrix \(R\) is not hermitian Toeplitz in general but for coarray we can construct a hermitian Toeplitz matrix by the method explained in [6] Now partitioning the signal subspace and noise subspace of this matrix and then applying 2D MUSIC on the hermitian Toeplitz matrix as in [15] will give the desired spectrum of the signal which helps in DOA estimation.

III. LAGS CALCULATION IN 2D RCPA

Lags or the difference coarray of any array is calculated by taking the pair wise difference of the sensor coordinates. The difference coarray \(D\) for the physical array \(L\) is defined as the set of differences between the sensor locations given in set \(L\) i.e. \(D = \{(s_i - s_j) : (s_i, s_j) \in L\}\) i.e the difference between \(i^{th}\) and \(j^{th}\) sensor location. Let \(D_s\) represent the contiguous range of the sensors in the difference coarray \(D\) which is
hole free and the most useful when comes to the application of estimation algorithms. The virtual sensor positions in the difference coarray are also called lags. The total lags are contiguous in the range \(- (MN + M - 1) \leq i, j \leq (MN + M - 1)\) as shown in Fig.2. The proposed RCPA requires \((3M)^2\) total number of sensors to generate the covariance matrix of the dimensions \((M(N + 1))^2 \times (M(N + 1))^2\) whereas in the conventional CPA, the critical holes in the difference coarray are filled by the proposed holes-filling geometry thus enlarging the contiguous range of the difference coarray and increasing the effective DOF. As shown in Table.I there has been 10.88% reduction in holes in our proposed geometry as compared to CPA [16] as shown in Fig.3. Moreover, in CPA there is the presence of holes in between the virtual array structure which is also reduced to a far extent in the proposed geometry as we can see the holes are absent only at \(i,j= \pm 8\) thereby giving symmetry in the structure of the hole also and this leads to increase the degree of freedom.

![Fig. 3: Virtual array of Coprime Planar Array with M1=3, M2=4](image)

**IV. WEIGHT FUNCTION**

The weight function can be defined as the count of repetitions of occurrence of virtual sensor positions in the lag matrix or difference coarray. Let \(g(n)\) be an integer-valued function that counts the periodicity of \(n^{th}\) sensor element in the virtual array \(\Phi_q\) which can be written as

\[
G_n = \{v_{j1}, v_{j2}, \ldots, v_{jq}\} \forall \{v_{j1}, v_{j2}, \ldots, v_{jq}\} \in S \\
g(n) = |G_n| (7)
\]

where \(|k|\) is the cardinality of the set \(k\) and \(v_{jq}\) represents unique lags in a coarray defined by eq.10.

**V. SIMULATION RESULTS**

In order to prove the validation of proposed geometry we have simulated it by taking various sparse arrays. First we took nested array for \(N=6\) whose physical sensors are found located at \(P=\{1,2,3,4,8,12\}\). Taking physical positions given in set \(P\) on the z-axis and coprime array positions derived in set \(S\) from Eq.3 on the y-axis, we get a rectangular nested coprime array(RNCPA). Similarly, forming Cadis for \(N=5,M=2,P=2\) we get physical array positions as \(P=\{0,1,2,3,4,10\}\) on z-axis thus forming Rectangular Cadis Coprime Array(RCdCPA) and for CACIS we have \(P=\{0,1,2,3,4,5\}\) on z-axis and thus leads to Rectangular CACIS Coprime Array(RCsCPA) respectively. TableII shows the summarized results of all the configurations and increased degree of freedom with just 36 physical sensors. 2D MUSIC algorithm is applied to the proposed RCPA geometry. Fig.4 gives the 2D MUSIC spectrum of the virtual array when 8 signals impinges upon it having azimuth angle in the span of -50° to 50° and elevation angle in the span of-90° to 90° with the step of 0.5°.

![Fig. 4: 2D Music Spectrum of RCPA](image)

**TABLE I: Percentage of hole in proposed RCPA**

| Antenna Array | Percentage of holes |
|---------------|---------------------|
| CPA           | 34.4 %              |
| Proposed RCPA | 23.52%              |

**A. RMSE with changing SNR**

RMSE (root mean square error) for the DOA can be defined as

\[
RMSE = \sqrt{\frac{1}{LQ} \sum_{i=1}^{L} \sum_{j=1}^{Q} \{ (\theta_q(i) - \theta_q)^2 + (\phi_q(i) - \phi_q)^2 \} } (8)
\]

where \(\theta_q(i)\) and \(\phi_q(i)\) are the estimates of \(\theta_q\) and \(\phi_q\) for the \(i^{th}\) Monte Carlo trial, \(i = 1, \ldots, L\). In the first set of simulation,
TABLE II: Comparison Of Various Rectangular Sparse Arrays.

| Rectangular Sparse Array | Number Of Sensors In Subarrays | Contiguous Range Of Lags | Fully Augmented Range Of Lags | DOF |
|--------------------------|--------------------------------|-------------------------|-------------------------------|-----|
| Rectangular coprime array (RCPA) | N=3,M=2 | -7<i<7 ; -7<j<7 | -9<i<9 ; -9<j<9 | 133 |
| Rectangular Nested Coprime Array (RNCPA) | N=6 | -11<i<11 ; -7<j<7 | -11<i<11 ; -10<j<10 | 173 |
| Rectangular Cadis Coprime Array (RCdCPA) | N=5,M=2,p=2 | -7<i<7 ; -5<j<5 | -10<i<10 ; -10<j<10 | 83 |
| Rectangular CACIS Coprime Array (RCsCPA) | N=5,M=2,p=2 | -7<i<7 ; -5<j<5 | -7<i<7 ; -5<j<5 | 83 |

we consider the case where we change the SNR and see the effect of increasing SNR on the proposed geometry. To enable a feasible comparison we consider a planar array composed of a coprime pair as (M=2, N=3). Let us take assume 49 uncorrelated sources falling on it whose normalized DOA are picked up randomly from \(( \theta', \phi' ) = [ -0.5, 0.5 ] \). Let the number of snapshots is fixed at 500 and SNR varied from 0dB to 15dB. We use \( L = 500 \) independent trials in the Monte carlo simulations. Fig.5 shows the RMSE performance comparison among the proposed geometry and existing planar geometries like CPA and GCPA (Generalized coprime array) as a function of input signal-to-noise ratio (SNR), where RMSE between azimuth and elevation angle decreases as signal to noise ratio increases. It can be observed that RMSE of proposed geometry is of the order of \( 10^{-3} \) which shows significant improvement in DOA estimation.

B. RMSE with changing Snapshot.

Considering same virtual planar array composed of a coprime pair as (M=2, N=3). Let us take assume 49 uncorrelated sources falling on it whose normalized DOA are picked up randomly from \(( \theta', \phi' ) = [ -0.5, 0.5 ] \). Let the number of SNR is fixed to 0dB and the number of snapshots varies from 200 to 1000. Fig.6, shows the RMSE performance comparison among the proposed geometry and existing planar geometries like CPA and GCPA (Generalized coprime array) as a function of number of snapshots, where RMSE between azimuth and elevation angle decreases as signal to noise ratio increases. It can be observed that RMSE of proposed geometry is of the order of \( 10^{-3} \) which shows significant improvement in DOA estimation.

VI. OPTIMIZATION

Optimization is used to derive the pattern synthesis weights. We tried to optimize the pattern whose main lobe is along azimuth and elevation 0 degrees. The pattern should satisfy the following constraints like Maximize the directivity, Suppress interferences 30 dB below main lobe, side lobe levels should be 17 dB below main lobe and within -20 and 20 degrees azimuth or elevation. To improve directivity needs to minimize the total radiated power, which is given by \( w^*Rn*w \) which is our objective function. A second-order cone programming solver is used to derive the array weights that can provide us with the desired pattern as shown in Fig.7. The table shows that we can meet the calculated requirements and the weight computation time is 13.99 seconds.

VII. CONCLUSION

In this paper, we have proposed a new class of rectangular arrays which is a combination of sensors distributed over a plane, whose difference co-array generates a much wider spectrum giving rise to the large fully augmentable range. This proposed planar array geometry leads to a more flexible array layout which leads to significant reduction in number of holes i.e.10% as compared to CPA whose virtual array shows holes within the contiguous range also. The optimized RCPA
geometry shows significant performance improvement in terms of achieving directivity and interference suppression in the array look-up direction. The advantage of the proposed antenna array geometry is that it provides higher DOF and larger array aperture as seen in case of Rectangular CACIS Coprime Array (RCsCPA), Rectangular Cadis Coprime Array (RCdCPA) and Rectangular Nested Coprime Array (RNCPA) with 83,173 and 133 sources can be estimated with just 36 physical sensors.

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