WELL-POSEDNESS AND REGULARITY OF GENERALIZED
NAVIER-STOKES EQUATIONS IN SOME CRITICAL Q–SPACES

PENGTAO LI AND ZHICHUN Zhai

Abstract. We study the well-posedness and regularity of the generalized
Navier-Stokes equations with initial data in a new critical space $Q_{\alpha,\infty}^{\beta-1}(\mathbb{R}^n) = \nabla \cdot (Q_{\alpha,\infty}^\beta(\mathbb{R}^n))$, $\alpha, \beta \in \left(\frac{1}{2}, 1\right)$ which is larger than some known critical homogeneous Besov spaces. Here $Q_{\alpha}^\beta(\mathbb{R}^n)$ is a space defined as the set of all measurable functions with
\[
\sup(l(I))^{2(\alpha+\beta-1)-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} \, dx \, dy < \infty
\]
where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. In order to study the well-posedness and regularity, we give a Carleson measure characterization of $Q_{\alpha}^\beta(\mathbb{R}^n)$ by investigating a new type of tent spaces and an atomic decomposition of the predual for $Q_{\alpha}^\beta(\mathbb{R}^n)$. In addition, our regularity results apply to the incompressible Navier-Stokes equations with initial data in $Q_{\alpha,\infty}^{\beta-1}(\mathbb{R}^n)$.

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1. Introduction

This paper considers the well-posedness and regularity of the generalized Navier-Stokes equations on the half-space $\mathbb{R}^{1+n}_+ = (0, \infty) \times \mathbb{R}^n$, $n \geq 2$:

$$
\begin{align*}
\begin{cases}
\partial_t u + (-\Delta)^\beta u + (u \cdot \nabla) u - \nabla p = 0, & \text{in } \mathbb{R}^{1+n}_+; \\
\nabla \cdot u = 0, & \text{in } \mathbb{R}^{1+n}_+; \\
u|_{t=0} = a, & \text{in } \mathbb{R}^n
\end{cases}
\end{align*}
$$

(1.1)

with $\beta \in (1/2, 1)$. Here $(-\Delta)^\beta$ is the fractional Laplacian with respect to $x$ defined by

$$
(-\Delta)^\beta u(t, \xi) = |\xi|^{2\beta} \hat{u}(t, \xi).
$$

Note that the following scaling

(1.2)

$$
u_{\lambda}(t, x) = \lambda^{2\beta-1} u(\lambda^{2\beta} t, \lambda x), \quad p_{\lambda}(t, x) = \lambda^{4\beta-2} p(\lambda^{2\beta} t, \lambda x), \quad a_{\lambda}(x) = \lambda^{2\beta-1} a(\lambda x)
$$

is particularly significant for equations (1.1).

The fractional Laplacian operator appears in a wide class of physical systems and engineering problems, including Lévy flights, stochastic interfaces and anomalous diffusion problems. In fluid mechanics, the fractional Laplacian is often applied to describe many complicated phenomenons via partial differential equations.

When $\beta = 1$, equations (1.1) become the classical Navier-Stokes equations. A natural approach in studying the solutions is to iterate the corresponding operator $v \rightarrow e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} P(u \otimes v) ds$ and to find a fixed point. This solution is called mild solution. For the classical Navier-Stokes equations, this approach was pioneered by Kato and Fujita. The existence of mild solutions and their regularity have been established locally in time and global for small initial data in various functional spaces, for example, see Kato [K], Cannone [C], Giga-Miyakawa [GM], Koch-Tataru [KT], Germain-Pavlovic-Staffilani [GPS] and the references therein.

For equations (1.1), J. L. Lions [L] proved the global existence of the classical solutions when $\beta \geq \frac{1}{2}$ in dimensional 3. Similar result holds for general dimension $n$ if $\beta \geq \frac{1}{2} + \frac{n}{4}$, see Wu [W1]. For the important case $\beta < \frac{1}{2} + \frac{n}{4}$, Wu in [W2]-[W3] established the global existence for equations (1.1) in the homogeneous Besov spaces $\dot{B}_{p,q}^{1+\beta-2\beta}$ for $1 \leq q \leq \infty$ and for either $1/2 < \beta$ and $p = 2$ or $1/2 < \beta \leq 1$ and $2 < p < \infty$ and in $\dot{B}_{2,\infty}^r$ with $r > \max\{1, 1 + \frac{n}{2} - 2\beta\}$. For the corresponding regularity criteria, we refer the readers to Wu [W4].

Our work originates mainly from two observations in mathematics. At first, it is worth pointing out that most of the function spaces listed above are critical spaces. A space is called critical for equations (1.1) if it is invariant under the scaling

(1.3)

$$f_{\lambda}(x) = \lambda^{2\beta-1} f(\lambda x).$$

For example, the spaces $L^2_{\frac{n}{2}-1}, L^n, \dot{B}_{p,q}^{1+\frac{n}{p}-2\beta}$ and $BMO^{-1}$ are critical spaces for $\beta = 1$. For general positive $\beta$, $\dot{B}_{2,1}^{1+\frac{n}{2}-2\beta}$ and $\dot{B}_{2,\infty}^{1+\frac{n}{2}-2\beta}$ are critical spaces. We see that $L^2_{\frac{n}{2}-1} \hookrightarrow L^n \hookrightarrow \dot{B}_{p,q}^{1+\frac{n}{p}-2\beta} \hookrightarrow BMO^{-1}$. $BMO^{-1}$ is the largest critical space for $\beta = 1$ among the spaces listed above such where existence results are available. This fact inspires us to find a larger critical space for general $\beta > 0$ which includes $\dot{B}_{2,1}^{1+\frac{n}{2}-2\beta}$ and has a structure similar to $BMO^{-1}$. With the small initial value in the space, the corresponding global existence result holds.
On the other hand, in [X], Xiao introduced a new space $Q_{α,∞}^{-1}(\mathbb{R}^n)$ to replace $BMO^{-1}$ in [KT] and generalized Koch-Tataru’s global existence result for the classical Navier-Stokes equations. Here $Q_{α,∞}^{-1}(\mathbb{R}^n)$ is $\nabla \cdot (Q_α(\mathbb{R}^n))^n$ and $Q_α(\mathbb{R}^n)$ is the space of all measurable complex-valued functions $f$ on $\mathbb{R}^n$ satisfying

$$\|f\|_{Q_α(\mathbb{R}^n)} = \sup_I (l(I))^{2α-n} \left( \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2α}} dx dy \right)^{1/2} < \infty$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. It is easy to see that $Q_{α,∞}^{-1}$ and $BMO^{-1}$ are invariant under the scaling $f(x) \mapsto λf(λx)$ which is corresponding to the classical Navier-Stokes equations. If we want to generalize the results of Koch-Tataru [KT] and Xiao [X] to the fractional case, we should find a class of spaces such that their derivative spaces are invariant under the scaling (1.3).

The above two observations suggest us introducing the following spaces.

**Definition 1.1.** For $α \in (-∞, β)$ and $β \in (1/2, 1)$, we define $Q_α^β(\mathbb{R}^n)$ be the set of all measurable complex-valued functions $f$ on $\mathbb{R}^n$ satisfying

$$\|f\|_{Q_α^β(\mathbb{R}^n)} = \sup_I (1(l(I))^{2(α+β-1)n} \left( \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2α+2β}} dx dy \right)^{1/2} < \infty$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

**Remark 2.** Obviously, when $β = 1$ and $α < 0$, $Q_α^β(\mathbb{R}^n) = BMO$. Thus our well-posedness result generalizes that of Koch-Tataru [KT] and Xiao [X] to the fractional equations. It is easy to show that $Q_α^β(\mathbb{R}^n)$ is invariant in the following sense: for any $f \in Q_α^β(\mathbb{R}^n)$,

$$\|f(\cdot + x_0)\|_{Q_α^β(\mathbb{R}^n)} = \|\lambda^{2β+2} f(\cdot + x_0)\|_{Q_α^β(\mathbb{R}^n)}, \quad λ > 0 \text{ and } x_0 \in \mathbb{R}^n.$$

Xiao in [X] characterized $Q_α(\mathbb{R}^n)$ equivalently as

$$\|f\|^2_{Q_α(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2α-n} \int_0^r \int_{|y-x| < r} |\nabla e^{t\Delta} f(y)|^2 t^{-α} dy \, dt < \infty.$$

The advantage of this equivalent characterization is the occurrence of $e^{t\Delta}$ which generates the mild solutions for the classical Navier-Stokes equations. Note that the mild solutions for equations (1.1) can be represented as

$$u(t, x) = e^{-t(-\Delta)^{β}} a(x) - \int_0^t e^{-(t-s)(-\Delta)^{β}} P \nabla (u \otimes u) ds,$$

where

$$e^{-t(-\Delta)^{β}} f(x) := K_t^β(x) * f(x) \quad \text{with} \quad K_t^β(ξ) = e^{-t|ξ|^{2β}}$$

and $P$ is the Helmholtz-Weyl projection:

$$P = \{P_{j,k}\}_{j,k=1,\ldots,n} = \{δ_{j,k} + R_j R_k\}_{j,k=1,\ldots,n}$$

with $δ_{j,k}$ being the Kronecker symbol and $R_j = δ_j (\nabla)^{-1/2}$ being the Riesz transform. Thus, we should characterize $Q_α^β(\mathbb{R}^n)$ by the semigroup $e^{-t(-\Delta)^{β}}$. In fact, we prove that $f \in Q_α^β$ if and only if

$$\sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2α-n+2β-2} \int_0^r \int_{|y-x| < r} |\nabla e^{-(t-s)^{β}} f(y)|^2 t^{-2β} dy \, dt < \infty.$$
It is easy to check that for each $j = 1, \cdots, n$, $\partial_j K_t^\beta(x) := \phi_j(x)$ is a $C^\infty$ real-valued function on $\mathbb{R}^n$ satisfying the properties:

$$\phi_j \in L^1(\mathbb{R}^n), \quad |\phi_j(x)| \lesssim (1+|x|)^{-(n+1)}, \quad \int_{\mathbb{R}^n} \phi_j(x)dx = 0 \quad \text{and} \quad (\phi_j)_t(x) = t^{-n}\phi_j\left(\frac{x}{t}\right).$$

This observation leads us to characterize $Q^\beta_\alpha(\mathbb{R}^n)$ by a general $C^\infty$ real-valued function $\phi$ on $\mathbb{R}^n$ with the properties (1.8) as

$$f \in Q^\beta_\alpha(\mathbb{R}^n) \iff \sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^r \int_{|y-x|<r} |f*\phi_t(y)|^2 t^{-(1+2(\alpha-\beta+1))} dt dy < \infty,$$

i.e., $d\mu_{f,\phi,\alpha,\beta}(t,x) = |f*\phi_t(x)|^2 t^{-(1+2(\alpha-\beta+1))} dt dy$ is a $1-2(\alpha+\beta-1)/n$ Carleson measure.

In order to get (1.9), inspired by Coifman-Meyer-Stein [CMS] and Dafni-Xiao [DX1], we introduce new tent spaces $T^1_{\alpha, \beta}$ and $T^{\infty}_{\alpha, \beta}$, then define a space $HH^1_{\alpha, \beta}(\mathbb{R}^n)$ as a subspace of distributions in $\dot{L}^{2,-2(\beta-1)}$. Finally, we identify $Q^\beta_\alpha(\mathbb{R}^n)$ with the dual space of $HH^1_{\alpha, \beta}(\mathbb{R}^n)$.

In order to establish the equivalent norm (1.7) of the space $Q^\beta_\alpha(\mathbb{R}^n)$ we need the notation of Hausdorff capacity (see [A] and [YY]). By the definition of Hausdorff capacity, we should assume $\alpha+\beta-1 \geq 0$ to guarantee $\Lambda_{n-2(\alpha+\beta-1)}$ is meaningful. However if we define the space $Q^\beta_\alpha$ by (1.7) directly, the proofs for the well-posedness and regularity still holds without the restriction that $\alpha+\beta-1 \geq 0$.

We now give the organization of this paper. In Section 2, we introduce some notation and some facts about homogeneous Besov spaces, Hausdorff capacity and Carleson measures. In Section 3, in order to establish (1.9), we introduce a new type of tent spaces, their atomic decompositions and the predual space of $Q^\beta_\alpha(\mathbb{R}^n)$. The proofs of the main theorems in this section are similar to that of Dafni-Xiao [DX1]. For the completeness, we provide the details. In Section 4, we establish the well-posedness of the generalized Navier-Stokes equations in a new critical space $Q^\beta_{\alpha, \infty}(\mathbb{R}^n)$ which is the derivative spaces of $Q^\beta_\alpha(\mathbb{R}^n)$ and contains all known critical homogeneous Besov spaces for equations (1.1). In Section 5, we establish the regularity of the global solutions to equations (1.1) with the initial value in $Q^\beta_{\alpha, \infty}(\mathbb{R}^n)$ for $\beta \in (1, 2, 1]$.

In [YY], D. Yang and W. Yuan introduced two new classes of spaces, i.e. $F^{p',q}_{p',q'}$ and $FH^{p,q}_{p,q}$, and studied their dual relation. It’s worth mentioning that when dealing with the duality relation our method is different from that in [YY]. Because when $p \neq q$, the atomic decomposition of the tent space $FT_{p,q}^{-s,\tau/q}$ is not completely known, the authors do not invoke it in [YY]. For our case, because $Q^\beta_\alpha = F_{2,2}^{\alpha-\beta+1,\frac{1}{2}-\frac{2\alpha+\beta-1}{n}}$, we can apply the atomic decompositions of the tent space $T^1_{\alpha, \beta}$ and Hardy-Hausdorff space $HH^1_{\alpha, \beta}$ to prove the dual relation between $Q^\beta_\alpha$ and $HH^1_{\alpha, \beta}$. See also [YY], Remark 5.2, Page 2080.

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2. Notation and Preliminaries

In this paper the symbols \( \mathbb{C}, \mathbb{Z} \) and \( \mathbb{N} \) denote the sets of all complex numbers, integers and natural numbers, respectively. For \( n \in \mathbb{N}, \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space, with Euclidean norm denoted by \( |x| \) and the Lebesgue measure by \( dx \). \( \mathbb{R}^{1+n}_+ \) is the upper half-space \( \{(t,x) \in \mathbb{R}^{1+n}_+: t > 0, x \in \mathbb{R}^n \} \) with Lebesgue measure denoted by \( dt \, dx \).

A ball in \( \mathbb{R}^n \) with center \( x \) and radius \( r \) will be denoted by \( B = B(x,r) \), its Lebesgue measure by \( |B| \). A cube in \( \mathbb{R}^n \) will always mean a cube in \( \mathbb{R}^n \) with side parallel to the coordinate axes. The sidelength of a cube \( I \) will be denoted by \( l(I) \). Similarly, its volume will be denoted by \( |I| \).

The symbol \( U \lesssim V \) means that there exists a positive constant \( C \) such that \( U \leq CV, U \approx V \) means \( U \lesssim V \) and \( V \lesssim U \). For convenience, the positive constants \( C \) may change from one line to another and usually depend on the dimension \( n, \alpha, \beta \) and other fixed parameters.

The characteristic function of a set \( A \) will be denoted by \( 1_A \). For \( \Omega \subset \mathbb{R}^n \), the space \( C^\infty_0(\Omega) \) consists of all smooth functions with compact support in \( \Omega \). The Schwartz class of rapidly decreasing functions and its dual will be denoted by \( \mathscr{S}(\mathbb{R}^n) \) and \( \mathscr{S}'(\mathbb{R}^n) \), respectively. For a function \( f \in \mathscr{S}(\mathbb{R}^n) \), \( \hat{f} \) means the Fourier transform of \( f \).

We use \( \mathscr{S}_0 \) to denote the following subset of \( \mathscr{S} \),

\[
\mathscr{S}_0 = \{ \phi \in \mathscr{S} : \int_{\mathbb{R}^n} \psi(x) x^\gamma \, dx = 0, |\gamma| = 0, 1, 2, \ldots \},
\]

where \( x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n} \), \( |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n \). Its dual \( \mathscr{S}'_0 = \mathscr{S}' / \mathscr{S}'_0 = \mathscr{S}' / \mathcal{P} \), where \( \mathcal{P} \) is the space of multiforms.

We introduce a dyadic partition of \( \mathbb{R}^n \). For each \( j \in \mathbb{Z} \), we let

\[
D_j = \{ \xi \in \mathbb{R}^n : 2^{j-1} < |\xi| \leq 2^{j+1} \}.
\]

We choose \( \phi_0 \in \mathscr{S}(\mathbb{R}^n) \) such that \( \text{supp}(\phi_0) = \{ \xi : 2^{-1} < |\xi| \leq 2 \} \) and \( \phi_0 > 0 \) on \( D_0 \). Let

\[
\phi_j(\xi) = \phi_0(2^{-j} \xi) \quad \text{and} \quad \hat{\psi}_j(\xi) = \frac{\phi_j(\xi)}{\sum_j \phi_j(\xi)}.
\]

Then \( \Psi_j \in \mathscr{S} \) and \( \hat{\Psi}_j(\xi) = \hat{\psi}_0(2^{-j} \xi), \text{ supp}(\hat{\psi}_j) \subset D_j, \Psi_j(x) = 2^{jn} \Psi_0(2^j x) \).

Moreover,

\[
\sum_{k=-\infty}^{\infty} \hat{\psi}_k(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}
\]

To define the homogeneous Besov spaces, we let

\[
\Delta_j f = \Psi_j * f, j = 0, \pm 1, \pm 2, \ldots.
\]

For \( s \in \mathbb{R}^n \) and \( 1 \leq p, q \leq \infty \), we define homogeneous Besov spaces \( \dot{B}^s_{p,q} \) be the set of all \( f \in \mathscr{S}'_0 \) with

\[
\|f\|_{\dot{B}^s_{p,q}} = \left( \sum_{j=-\infty}^{\infty} (2^j \|\Delta_j f\|_{L^p})^q \right)^{1/q} < \infty \quad \text{for } q < \infty,
\]

\[
\|f\|_{\dot{B}^s_{p,q}} = \sup_{-\infty < j < \infty} 2^j \|\Delta_j f\|_{L^p} < \infty \quad \text{for } q = \infty.
\]
When $0 < s < 1$, we have the following equivalent characterization. If $1 \leq p, q < \infty$, then $f \in \dot{B}^s_{p,q}$ is equivalent to

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \, dx \right)^{q/p} \frac{dy}{|y|^{n+qs}} < \infty;$$

if $0 < s < 1$ and $1 \leq p < q = \infty$, $f \in \dot{B}^s_{p,q}$ amounts to

$$\sup_{y \in \mathbb{R}^n} |y|^{-s} \left( \int_{\mathbb{R}^n} |f(x+y) - f(x)|^p \right)^{1/p} < \infty.$$

We refer to Peetre [P] and Triebel [T] for more information. The usual homogeneous Sobolev space $\dot{L}^2_s$ defined by $\dot{L}^2_s = (-\Delta)^{-s/2} L^2$ is a special type of the homogeneous Besov space. That is, $\dot{B}^s_{2,2} = \dot{L}^2_s$.

The homogenous Besov spaces obey the following inclusion relations (see [BL]).

**Theorem 2.1.** Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.

(i) If $1 \leq q_1 \leq q_2 \leq \infty$, then $\dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n) \subseteq \dot{B}^{s_2}_{p,q_2}(\mathbb{R}^n)$;

(ii) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s_1 = s_2 + n \left( \frac{1}{p_1} - \frac{1}{p_2} \right)$, then $\dot{B}^{s_1}_{p_1,q} \subseteq \dot{B}^{s_2}_{p_2,q}$.

We recall the definition of fractional Carleson measures (see Essen-Janson-Peng-Xiao [EJPX]) and their connection with Hausdorff capacity established by Dafni-Xiao in [DX1].

**Definition 2.2.** For $p > 0$, we say that a Borel measure $\mu$ on $\mathbb{R}^{1+n}$ is a $p$–Carleson measure provided that

$$|||\mu|||_p = \sup \frac{\mu(S(I))}{l(I)^{np}} < \infty$$

where the supremum is taken over all Carleson boxes $S(I) = \{(t, x) : x \in I, t \in (0, l(I))\}$.

Obviously, the $1$–Carleson measures are the usual Carleson measures. On the other hand, similar to the case $p = 1$, if we denote by

$$T(E) = \{(t, x) \in \mathbb{R}^{1+n} : B(x, t) \subset E\}$$

the tent based on the set $E \subset \mathbb{R}^n$, then a Borel measure $\mu$ on $\mathbb{R}^{1+n}$ is a $p$–Carleson measure if and only if $||\mu||(T(E)) \leq C|B|^p$ holds for all balls $B \subset \mathbb{R}^n$. That is to say $p$–Carleson measures can be equivalently defined in terms of tents over balls.

We recall some definitions and properties about Hausdorff capacity (see Adams [A], Dafni-Xiao [DX1] and Yang-Yuan [YY]).

**Definition 2.3.** Let $d \in (0, n]$ and $E \subset \mathbb{R}^n$.

(i) The $d$–dimensional Hausdorff capacity of $E$ is defined by

$$\Lambda^{(\infty)}_d(E) = \inf \left\{ \sum_j r_j^d : E \subset \bigcup_{j=1}^\infty B(x_j, r_j) \right\},$$

where the infimum is taken over all covers of $E$ by countable families of open (closed) balls with radii $r_j$. 

(ii) The capacity $\tilde{\Lambda}_d^{(\infty)}(E)$ in the sense of Choquet is defined by

$$\tilde{\Lambda}_d^{(\infty)}(E) := \inf \left\{ \sum_j l(I_j)^d : E \subset (\cup_{j=1}^{\infty} I_j)^{0} \right\},$$

where $B^o$ denotes the interior of $B$ and the infimum ranges only over covers of $E$ by dyadic cubes.

(iii) For a function $f : \mathbb{R}^n \rightarrow [0, \infty]$, we define

$$\int_{\mathbb{R}^n} f d\tilde{\Lambda}_d^{(\infty)} := \int_0^\infty \Lambda_d^{(\infty)}(\{ x \in \mathbb{R}^n : f(x) > \lambda \}) d\lambda.$$

**Remark 2.4.** (i) $\Lambda_d^{(\infty)}$ is not a capacity in the sense of Choquet. But, its dyadic counterpart $\tilde{\Lambda}_d^{(\infty)}$ is a capacity since it is monotone, vanishes on the empty set, and satisfies the strong subadditivity condition

$$\tilde{\Lambda}_d^{(\infty)}(E_1 \cup E_2) + \tilde{\Lambda}_d^{(\infty)}(E_1 \cap E_2) \leq \tilde{\Lambda}_d^{(\infty)}(E_1) + \tilde{\Lambda}_d^{(\infty)}(E_2),$$

as well as the continuity conditions (see Adams [A]):

$$\tilde{\Lambda}_d^{(\infty)}(\cap_i K_i) = \lim_{i \to \infty} \tilde{\Lambda}_d^{(\infty)}(K_i), \{ K_i \} \text{ a decreasing sequence of compact sets},$$

$$\tilde{\Lambda}_d^{(\infty)}(\cup_i K_i) = \lim_{i \to \infty} \tilde{\Lambda}_d^{(\infty)}(K_i), \{ K_i \} \text{ an increasing sequence of sets.}$$

(ii) There exist positive constants $C_1(n, d)$ and $C_2(n, d)$ such that

$$C_1(n, d)\Lambda_d^{(\infty)}(E) \leq \tilde{\Lambda}_d^{(\infty)}(E) \leq C_2(n, d)\Lambda_d^{(\infty)}(E) \text{ for all } E \subset \mathbb{R}^n.$$

(iii) The integral with respect to $\tilde{\Lambda}_d^{(\infty)}(E)$ satisfies Fatou’s lemma

$$\int_{\mathbb{R}^n} \liminf f_n d\tilde{\Lambda}_d^{(\infty)} \leq \liminf \int_{\mathbb{R}^n} f_n d\tilde{\Lambda}_d^{(\infty)}.$$

For $x \in \mathbb{R}^n$, let $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} : |y - x| < t \}$ be the cone at $x$. Define the nontangential maximal function $N(f)$ of a measurable function on $\mathbb{R}^{n+1}$ by

$$N(f)(x) := \sup_{(y, t) \in \Gamma(x)} |f(y, t)|.$$

In [DX1], Dafni-Xiao characterized the fractional Carleson measures as follows.

**Theorem 2.5.** ([DX1] Theorem 4.2) Let $d \in (0, n]$ and $\mu$ be a Borel measure on $\mathbb{R}_+^{1+n}$. Then $\mu$ is a $d/n$–Carleson measure if and only if the inequality

$$\int_{\mathbb{R}_+^{1+n}} |f(t, y)|d|\mu| \leq A \int_{\mathbb{R}^n} N(f)d\Lambda_d^{(\infty)}$$

holds for all Borel measurable functions $f$ on $\mathbb{R}_+^{1+n}$. If this is the case then in (2.8) the constant $A \approx |||\mu|||_{d/n}$ which is defined by (2.7).

3. **Carleson Measure Characterization of $Q_\alpha^\beta(\mathbb{R}^n)$**

In this section, we establish the equivalent characterization (1.1). We first give some basic properties of $Q_\alpha^\beta(\mathbb{R}^n)$. Then inspired by Coifman-Meyer-Stein [CMS] and Dafni-Xiao [DX1], we introduce new tent spaces $T^1_{\alpha, \beta}$ and $T^\infty_{\alpha, \beta}$. Finally, we obtain the predual space of $Q_\alpha^\beta(\mathbb{R}^n)$. 

...
3.1. Basic Properties of $Q^\beta_\alpha (\mathbb{R}^n)$.

Lemma 3.1. Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then $f \in Q^\beta_\alpha (\mathbb{R}^n)$ if and only if

\[
\sup_I (l(I))^{-n+2(\alpha+\beta-1)} \int_{|y|<l(I)} \int_I |f(x+y) - f(x)|^2 \frac{dxdy}{|y|^{2(n-2(\alpha-\beta)+1)}} < \infty.
\]

Proof. If the double integrals (1.5) and (3.1) are denoted by $U_1(I)$ and $U_2(I)$, respectively, then by the change of variable $y \to x+y$ and simple geometry one obtains $U_1(I) \leq U_2(\sqrt{n}I)$ and $U_2(I) \leq U_1(3I)$. \hfill \Box

Theorem 3.2. Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then

(i) $Q^\beta_\alpha (\mathbb{R}^n)$ is decreasing in $\alpha$ for a fixed $\beta$, i.e.

\[
Q^\beta_{\alpha_1} (\mathbb{R}^n) \subset Q^\beta_{\alpha_2} (\mathbb{R}^n), \text{ if } \alpha_2 \leq \alpha_1;
\]

(ii) If $\alpha \in (-\infty, \beta - 1)$, then

\[
Q^\beta_\alpha (\mathbb{R}^n) = Q^\beta_{-\frac{\alpha}{2} + \beta - 1} (\mathbb{R}^n) := BMO^\beta (\mathbb{R}^n).
\]

Proof. (i) Suppose $\alpha_1 \leq \alpha_2$. If $f \in Q^\beta_{\alpha_2} (\mathbb{R}^n)$, then for any cube $I$ we have

\[
\int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha - 2\beta + 2}} dxdy \lesssim \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha - 2\beta + 2}} dxdy 
\]

\[
\lesssim \int_I \int_I |f(x) - f(2I)|^2 \lesssim |I|^{-\frac{4(\beta-1)}{n}} \|f\|^2_{BMO^\beta (\mathbb{R}^n)},
\]

since we can get easily

\[
\|f\|^2_{Q^\beta_{-\frac{\alpha}{2} + \beta - 1}} \approx \sup_I |I|^{1+\frac{4(\beta-1)}{n}} \int_I |f(x) - f(2I)|^2 dx.
\]

with $f(2I) = |I|^{-1} \int_I f(x)dx$ being the mean value of $f$ over the cube $I$. Hence

\[
\int_{|y|<l(I)} \int_I \frac{|f(x+y) - f(x)|^2}{|y|^{n+2\alpha - 2\beta + 2}} dxdy 
\]

\[
\lesssim \int_I \int_I |f(x) - f(2I)|^2 \lesssim |I|^{-\frac{4(\beta-1)}{n}} \|f\|^2_{BMO^\beta (\mathbb{R}^n)}.
\]

This tells us $BMO^\beta (\mathbb{R}^n) \subset Q^\beta_\alpha (\mathbb{R}^n)$.

Case II: $\alpha \in (-\infty, -\frac{\alpha}{2} + \beta - 1)$. In this case, $BMO^\beta (\mathbb{R}^n) \subset Q^\beta_\alpha (\mathbb{R}^n)$. If $f \in Q^\beta_\alpha (\mathbb{R}^n)$, let $I$ be a cube. If $x, y \in I$, then the set $\{ z \in I : \min(|x-z|, |y-z|) > \frac{1}{4} |I| \}$ has measure at least $\frac{1}{2} |I|$ and thus for $-2\alpha - n + 2\beta - 2 > 0$,

\[
\int_I \min \{ |x-z|^{-2\alpha-n+2\beta-2}, |y-z|^{-2\alpha-n+2\beta-2} \} dz \geq C |l(I)|^{-2\alpha-n+2\beta-2} |l(I)|^n \geq C |l(I)|^{-2\alpha+2\beta-2}.
\]
Hence we can get

$$\|f(I)\|_{2n+4\beta-4} \int I \int f(x) - F(y) dx dy \leq \|f(I)\|_{2n-2\alpha+2\beta-2} \int I \int f(x) - F(y) dx dy \min \{|x-z|^{-2\alpha-n+2\beta-2}, |y-z|^{-2\alpha-n+2\beta-2}\} dx dy dz$$

$$\lesssim \|f(I)\|_{-2n+2\beta-4} \int I \int \frac{|f(x) - F(y)|^2}{|x-y|^{n+2(\alpha-\beta+1)}} dx dy \lesssim \|f\|_{Q^\beta_{n\alpha}(\mathbb{R}^n)}.$$ 

Thus $Q^\beta_{n\alpha}(\mathbb{R}^n) \subseteq BMO^\beta(\mathbb{R}^n)$. This completes the proof of Lemma 3.2. 

In the following, we establish the connection between $Q^\beta_{n\alpha}(\mathbb{R}^n)$ and homogeneous Besov spaces.

**Theorem 3.3.** Let $n \geq 2$ and $\max\{1/2, \alpha\} < \beta < 1$.

(i) If $1 \leq q \leq 2$ and $\alpha + \beta - 1 > 0$, then $\dot{B}^{\alpha+\beta+1}_{q\alpha}(\mathbb{R}^n) \subseteq Q^\beta_{n\alpha}(\mathbb{R}^n)$.

(ii) Let $1 \leq q \leq \infty$, $\gamma_1 > (\alpha - \beta + 1)$ and $\gamma_2 > 0$. If $\gamma_1 - \gamma_2 = 2 - 2\beta$, then $\dot{B}^{\gamma_1}_{\gamma_2}(\mathbb{R}^n) \subseteq Q^\beta_{n\alpha}(\mathbb{R}^n)$.

**Remark 3.4.** Similar results hold for $\beta = 1$, see Essen-Janson-Peng-Xiao [EJPX] Theorem 2.7.

**Proof.** (i) It follows form (i) of Theorem 2.7 that $\dot{B}^{\alpha+\beta+1}_{q\alpha}(\mathbb{R}^n) \subseteq \dot{B}^{\alpha+\beta+1}_{q\alpha}(\mathbb{R}^n)$, so we can assume $q = 2$. For $f \in \dot{B}^{\alpha+\beta+1}_{q\alpha}(\mathbb{R}^n)$, Hölder’s inequality implies that for any cube $I$ in $\mathbb{R}^n$,

$$\int_{|y|<l(I)} I \int \frac{|f(x+y) - f(x)|^2}{|y|^{2n+2(\alpha-\beta+1)}} dy \lesssim |I|^{(n-2(\alpha-\beta-1))/n} \int_{\mathbb{R}^n} \left( \int I \frac{|f(x+y) - f(x)|^{(\alpha+\beta-1)}}{|x|^{n+2(\alpha-\beta+1)}} dx \right)^{(2\alpha+\beta-1)/n} \frac{dy}{|y|^{n+2(\alpha-\beta+1)}}.$$

This estimate, Lemma 3.1 and (2.2) imply that $f \in Q^\beta_{n\alpha}(\mathbb{R}^n)$.

(ii) According to (ii) of Theorem 2.7, we have $\dot{B}^{\gamma_1}_{\gamma_2}(\mathbb{R}^n) \subseteq \dot{B}^{\gamma_1}_{\gamma_2}(\mathbb{R}^n)$ for $\gamma_1 > \theta_1$, $\gamma_2 > \theta_2$ and $\gamma_1 - \gamma_2 = \theta_1 - \theta_2$. Thus we can suppose that $\gamma_1 < 1$ and $\gamma_2 < 1$. Assume that $f \in \dot{B}^{\gamma_1}_{\gamma_2}(\mathbb{R}^n)$. For any cube $I$ in $\mathbb{R}^n$, Hölder’s inequality and (2.3) tell us

$$\int_{|y|<l(I)} I \int |f(x+y) - f(x)|^2 dx \frac{dy}{|y|^{n+2(\alpha-\beta+1)}} \lesssim |I|^{(n-2\gamma_2)/n} \int_{|y|<l(I)} \left( \int I |f(x+y) - f(x)|^{\gamma_2/|x|^{\gamma_2}} dx \right)^{2\gamma_2/n} \frac{dy}{|y|^{n+2(\alpha-\beta+1)}} \lesssim \|f\|_{\dot{B}^{\gamma_1}_{\gamma_2}(\mathbb{R}^n)}^2.$$

Thus $f \in Q^\beta_{n\alpha}(\mathbb{R}^n)$. 

**3.2. New Tent Spaces.** We introduce new tent spaces motivated by similar arguments in Dafni-Xiao [DX1].
Definition 3.5. For \( \alpha > 0 \) and \( \max \{1/2, \alpha \} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \), we define \( T_{\alpha, \beta}^\infty \) be the class of all Lebesgue measurable functions \( f \) on \( \mathbb{R}_{+}^{1+n} \) with

\[
\|f\|_{T_{\alpha, \beta}^\infty} = \sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}} \int_{T(B)} |f(t,y)|^2 \frac{dt\,dy}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2} < \infty,
\]

where \( B \) runs over all balls in \( \mathbb{R}^n \).

Definition 3.6. For \( \alpha > 0 \) and \( \max \{1/2, \alpha \} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \), a function \( a \) on \( \mathbb{R}_{+}^{1+n} \) is said to be a \( T_{\alpha, \beta}^1 \)-atom provided there exists a ball \( B \subset \mathbb{R}^n \) such that \( a \) is supported in the tent \( T(B) \) and satisfies

\[
\int_{T(B)} |a(t,y)|^2 \frac{dt\,dy}{t^{1-2(\alpha-\beta+1)}} \leq \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.
\]

Definition 3.7. For \( \alpha > 0 \) and \( \max \{1/2, \alpha \} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \) the space \( T_{\alpha, \beta}^1 \) consists of all measurable functions \( f \) on \( \mathbb{R}_{+}^{1+n} \) with

\[
\|f\|_{T_{\alpha, \beta}^1} = \inf_{\omega} \left( \int_{\mathbb{R}_{+}^{1+n}} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dt\,dx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} < \infty,
\]

where the infimum is taken over all nonnegative Borel measurable functions \( \omega \) on \( \mathbb{R}_{+}^{1+n} \) with

\[
\int_{\mathbb{R}^n} N\omega d\Lambda_{n-2(\alpha+\beta-1)} \leq 1
\]

and with the restriction that \( \omega \) is allowed to vanish only where \( f \) vanishes.

Lemma 3.8. If \( \sum_j \|g_j\|_{T_{\alpha, \beta}^1} < \infty \), then \( g = \sum_j g_j \in T_{\alpha, \beta}^1 \) with

\[
\|g\|_{T_{\alpha, \beta}^1} \leq \sqrt{C_1^{-1}(n,d)C_2(n,d)} \sum_j \|g_j\|_{T_{\alpha, \beta}^1},
\]

where \( C_1(n,d), C_2(n,d) \) are the constants in (2.6).

Proof. The proof of this lemma is similar to that of Dafni-Xiao [DX1, Lemma 5.3].

Theorem 3.9. Let \( \alpha > 0 \) and \( \max \{1/2, \alpha \} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \), then

(i) \( f \in T_{\alpha, \beta}^1 \) if and only if there is a sequence of \( T_{\alpha, \beta}^1 \)-atoms \( a_j \) and an \( l^1 \)-sequence \( \{\lambda_j\} \) such that \( f = \sum_j \lambda_j a_j \). Moreover

\[
\|f\|_{T_{\alpha, \beta}^1} \approx \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}
\]

where the infimum is taken over all possible atomic decompositions of \( f \in T_{\alpha, \beta}^1 \). The right-hand side thus defines a norm on \( T_{\alpha, \beta}^1 \) which makes it into a Banach space.

(ii) The inequality

\[
\int_{\mathbb{R}_{+}^{1+n}} |f(t,y)g(t,y)| \frac{dt\,dy}{t} \leq C \|f\|_{T_{\alpha, \beta}^1} \|g\|_{T_{\alpha, \beta}^\infty}
\]

(3.2)
holds for all \( f \in T_{\alpha,\beta}^1 \) and \( g \in T_{\alpha,\beta}^\infty \).

(iii) The Banach space dual of \( T_{\alpha,\beta}^1 \) can be identified with \( T_{\alpha,\beta}^\infty \) under the following pairing

\[
\langle f, g \rangle = \int_{\mathbb{R}^{1+n}} f(t,y)g(t,y) \frac{dt dy}{t}.
\]

Proof. (i) Let \( a \) be a \( T_{\alpha,\beta}^1 \) atom. Then we can find a ball \( B = B(x_B,r) \subset \mathbb{R}^n \) such that \( \text{supp}(a) \subset T(B) \) and

\[
\int_{T(B)} |a(t,y)|^2 \frac{dt dy}{t^{1-2(\alpha-\beta+1)/n}} \leq \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.
\]

Fix \( \varepsilon > 0 \) and define

\[
\omega(t,x) = \kappa r^{-n+2(\alpha+\beta-1)} \min \left\{ 1, \left( \frac{r}{\sqrt{|x-x_B|^2+t^2}} \right)^{n-2(\alpha+\beta-1)+\varepsilon} \right\},
\]

where \( \sqrt{|x-x_B|^2+t^2} \) is the distance between \( (t,x) \) and \( (0,x_B) \). For \( x \in \mathbb{R}^n \), the distance in \( \mathbb{R}_+^{1+n} \) from the cone \( \Gamma(x) \) to \( (0,x_B) \) is \( \frac{|x-x_B|}{\sqrt{r}} \). So

\[
N\omega(x) = \sup_{(t,y) \in \Gamma(x)} \left| \kappa r^{-n+2(\alpha+\beta-1)} \min \left\{ 1, \left( \frac{r}{\sqrt{|x-x_B|^2+t^2}} \right)^{n-2(\alpha+\beta-1)+\varepsilon} \right\} \right| 
\]

\[
\leq \kappa r^{-n+2(\alpha+\beta-1)} \min \left\{ 1, \left( \frac{\sqrt{2}r}{|x-x_B|} \right)^{n-2(\alpha+\beta-1)+\varepsilon} \right\}.
\]

Thus

\[
\kappa^{-1} \int_{\mathbb{R}^n} N\omega d\Lambda^\infty_{n-2(\alpha+\beta-1)} \leq \int_0^\infty \Lambda^\infty_{n-2(\alpha+\beta-1)} \left( \{ x : N\omega(x) > \lambda \} \right) d\lambda.
\]

If \( \lambda < N\omega(x) \), then \( |x-x_B| \leq \sqrt{2} \left( \frac{\lambda}{\kappa r^{-n+2(\alpha+\beta-1)}} \right) \). Meanwhile, \( \lambda < N\omega(x) \leq \kappa r^{-n+2(\alpha+\beta-1)} \), so we obtain

\[
\kappa^{-1} \int_{\mathbb{R}^n} N\omega d\Lambda^\infty_{n-2(\alpha+\beta-1)} \leq \int_0^\infty \left( \frac{r^\alpha}{\kappa} \right)^{n-2(\alpha+\beta-1)+\varepsilon} d\lambda \lesssim 1.
\]

Moreover, on \( T(B) \) we have \( \omega^{-1}(t,x) = r^{n-2(\alpha+\beta-1)} \). By the definition of \( T_{\alpha,\beta}^1 \)-atom, we get

\[
\int_{T(B)} |a(t,y)|^2 \omega^{-1}(t,x) \frac{dt dx}{t^{1-2(\alpha-\beta+1)}} \lesssim 1.
\]

Thus \( a \in T_{\alpha,\beta}^1 \) with \( \|a\|_{T_{\alpha,\beta}^1} \lesssim 1 \). For any sum \( \sum_j \lambda_j a_j \) with \( \|\{\lambda_j\}\|_t = \sum |\lambda_j| < \infty \) and \( T_{\alpha,\beta}^1 \)-atoms \( a_j \), Lemma \( \ref{lemma} \) implies that the sum converges in the quasi-norm to \( f \in T_{\alpha,\beta}^1 \) with \( \|f\|_{T_{\alpha,\beta}^1} \lesssim \sum_j |\lambda_j| \).

Conversely, suppose that \( f \in T_{\alpha,\beta}^1 \). There exists a Borel measurable function \( \omega \geq 0 \) on \( \mathbb{R}_+^{1+n} \) such that

\[
\int_{\mathbb{R}_+^{1+n}} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dt dx}{t^{1-2(\alpha-\beta+1)}} \leq 2\|f\|_{T_{\alpha,\beta}^1}^2.
\]
For each $k \in \mathbb{Z}$, let $E_k = \{x \in \mathbb{R}^n : N\omega(x) > 2^k\}$. According to Dafni-Xiao [DX1, Lemma 4.1], there exists a sequence of dyadic cubes $\{I_{j,k}\}$ with disjoint interiors such that

$$\sum_j l(I_{j,k})^{n-2(\alpha+\beta-1)} \leq 2\Lambda_{n-2(\alpha+\beta-1)}^{(\infty)}(E_k)$$

and $T(E_k) \subset \cup_j S^*(I_{j,k})$.

Here we have used a Carleson box: $S^*(I_{j,k}) = \{(t,y) : y \in I_{j,k}, t < 2\text{diam}(I_{j,k})\}$ to replace the tent $T(I_{j,k})$ over the dilated cube $I_{j,k} = 5\sqrt{n}I_{j,k}$. Consequently, if we define $T_{j,k} = S^*(I_{j,k}) \cup_{m > k}\cup_l S^*(I_{l,m})$, these will have disjoint interiors for different values of $j$ or $k$. Now

$$\cup_{k=-K}^K \cup_j T_{j,k} = \cup_j S^*(I_{j,-K}) \cup_{m > K} \cup_l S^*(I_{l,m}) \supseteq T(E_{-K}) \cup_{m > K} \cup_l S^*(I_{l,m}).$$

Similar to the discussion in the proof of Dafni-Xiao [DX1, Theorem 5.4], we have

$$\cup_k \cup_j T_{j,k} \supseteq \cup_k T(E_k) \cap_k \cup_{m > k} \cup_l S^*(I_{l,m}) = \{(t,x) \in \mathbb{R}^{1+n} : \omega(t,x) > 0\} \setminus T_\infty$$

with $\Lambda_{n-2(\alpha+\beta-1)}^{(\infty)}(T_\infty) = |T_\infty| = 0$. Since $\omega$ is allowed to vanish only where $f$ vanishes, $f = \sum f1_{T_{j,k}}$ a.e. on $\mathbb{R}^{1+n}$. Defining $a_{j,k} = f1_{T_{j,k}}(\lambda_{j,k})^{-1}$ and

$$\lambda_{j,k} = \left(\left(l(I_{j,k})\right)^{n-2(\alpha+\beta-1)} \int_{T_{j,k}} |f(t,x)|^2 \frac{dt\,dx}{t^{1-2(\alpha+\beta+1)}}\right)^{1/2},$$

we get $f = \sum a_{j,k} \lambda_{j,k} a_{j,k}$ almost everywhere. Since $S^*(I_{j,k}) \subset T(B_{j,k})$ where $B_{j,k}$ is the ball with the same center as $I_{j,k}$ and radius $l(I_{j,k})/2$. $a_{j,k}$ is supported in $T(B_{j,k})$ and

$$\int_{T(B_{j,k})} |a_{j,k}(t,y)|^2 \frac{dt\,dy}{t^{1-2(\alpha+\beta+1)}}$$

$$\leq \left(l(I_{j,k})\right)^{-n+2(\alpha+\beta-1)} \left(\int_{T_{j,k}} |f(t,x)|^2 \frac{dt\,dx}{t^{1-2(\alpha+\beta+1)}}\right)^{-1} \left(\int_{T(B_{j,k})} |f(t,x)|^2 \frac{dt\,dx}{t^{1-2(\alpha+\beta+1)}}\right)^{-1}$$

$$\leq \left(l(I_{j,k})\right)^{-n+2(\alpha+\beta-1)} \leq |B_{j,k}|^{-1+2(\alpha+\beta-1)/n}.$$

Thus each $a_{j,k}$ is a $T_{\alpha,\beta}^1$-atom.
Thus if \( f \) is \( l^1 \)-summable. Noting that \( \omega \leq 2^{k+1} \) on \( T_{j,k} \subset (\cup_j S^+(I_{k+1}))^c \subset (T(E_{k+1}))^c \) and applying the Cauchy-Schwarz inequality, we obtain

\[
\sum_{j,k} |\lambda_{j,k}| \leq \sum_{j,k} (l(I_{j,k}))^{\frac{1}{2}-(\alpha+\beta-1)} \left( \int_{T_{j,k}} |f(t,x)|^2 \frac{dt \, dx}{t^{1-(\alpha+\beta+1)}} \right)^{1/2} \\
\leq \sum_{j,k} \sup_{T_{j,k}} \omega^{1/2} (l(I_{j,k}))^{\frac{1}{2}-(\alpha+\beta-1)} \left( \int_{T_{j,k}} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-(\alpha+\beta+1)}} \right)^{1/2} \\
\leq \left( \sum_{j,k} 2^{(k+1)} (l(I_{j,k}))^{n-2(\alpha+\beta-1)} \right)^{1/2} \left( \sum_{j,k} \int_{T_{j,k}} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-(\alpha+\beta+1)}} \right)^{1/2} \\
\lesssim \|f\|_{T_{\alpha,\beta}^1} \left( \sum_{k} 2^{k} \sum_{j} (l(I_{j,k}))^{n-2(\alpha+\beta-1)} \right)^{1/2} \\
\lesssim \|f\|_{T_{\alpha,\beta}^1} \left( \sum_{k} 2^{k} \Lambda_n^{(\infty)}(E_k) \right)^{1/2} \\
\lesssim \|f\|_{T_{\alpha,\beta}^1} \left( \int_{\mathbb{R}^n} N\omega d\Lambda_n^{(\infty)} \right)^{1/2} \lesssim \|f\|_{T_{\alpha,\beta}^1}.
\]

Thus \( T_{\alpha,\beta}^1 \) is a Banach space since it is complete in the quasi-norm (Lemma 3.3) and

\[
\|f\|_{T_{\alpha,\beta}^1} \approx \|f\|_{T_{\alpha,\beta}^1} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}
\]

where the infimum is taken over all possible atomic decompositions of \( f \in T_{\alpha,\beta}^1 \) and \( \|\cdot\|_{T_{\alpha,\beta}^1} \) is a norm.

(ii) Let \( \omega \) be a nonnegative Borel measurable function on \( \mathbb{R}^{1+n} \) satisfying \( \int_{\mathbb{R}^n} N\omega d\Lambda_n^{(\infty)} \leq 1 \). For \( g \in T_{\alpha,\beta}^\infty \), \( \mu_{g,n-2(\alpha+\beta-1)}(t,x) = |g(t,x)|^2 t^{-1-(\alpha+\beta+1)} \, dt \, dx \) is a \( 1-(\alpha+\beta-1)/n \)-Carleson measure. Then (2.3) tells us, with \( A \approx \|\mu_{g,n-2(\alpha+\beta-1)}\|_{n/2(\alpha+\beta-1)}/n \approx \|g\|_{T_{\alpha,\beta}^\infty}^2 \),

\[
\int_{\mathbb{R}^{1+n}_+} \omega(t,x) |g(t,x)|^2 \frac{dt \, dx}{t^{1+(\alpha+\beta+1)}} \lesssim \|g\|_{T_{\alpha,\beta}^\infty}^2 \int_{\mathbb{R}^n} N\omega d\Lambda_n^{(\infty)}(E_k) \lesssim \|g\|_{T_{\alpha,\beta}^\infty}^2.
\]

Thus if \( f \in T_{\alpha,\beta}^1 \), then

\[
\int_{\mathbb{R}^{1+n}_+} |f(t,x)|^2 \frac{dt \, dx}{t} \leq \left( \int_{\mathbb{R}^{1+n}_+} |f(t,x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-(\alpha+\beta+1)}} \right)^{1/2} \|g\|_{T_{\alpha,\beta}^\infty}.
\]

Hence we finish the proof of (ii) by taking the infimum on the right over all admissible \( \omega \).

(iii) Form (ii), we know that for every \( g \in T_{\alpha,\beta}^\infty \), the pairing

\[
\langle f, g \rangle = \int_{\mathbb{R}^{1+n}_+} f(t,x) g(t,x) \frac{dt \, dx}{t}.
\]
defines a bounded linear functional on $T^1_{\alpha,\beta}$. Now we prove the converse. Let $L$ be a bounded linear functional on $T^1_{\alpha,\beta}$. Fix a ball $B = B(x_B, r) \subset \mathbb{R}^n$. If $f$ is supported on $T(B)$ with $f \in L^2(T(B), t^{-1} \, dt \, dx)$ then

$$\int_{T(B)} |f(t, x)|^2 \frac{dt \, dx}{t^{1-2(\alpha-\beta)+1}} \leq r^{2(\alpha-\beta+1)} \int_{T(B)} |f(t, x)|^2 \frac{dt \, dx}{t} \leq \frac{1}{\|B\|^{1-2(\alpha+\beta-1)/n}} \rho^{n-2(\alpha+\beta-1)+2(\alpha-\beta+1)} \int_{T(B)} |f(t, x)|^2 \frac{dt \, dx}{t} \leq \frac{1}{\|B\|^{1-2(\alpha+\beta-1)/n}} \rho^{n-4\beta+4} \|f\|_{L^2(T(B), t^{-1} \, dt \, dx)}^2.$$ 

This tells us that $f(t, x)$ is a multiple of a $T^1_{\alpha,\beta}$-atom and $L$ is a bounded linear functional on $L^2(T(B), t^{-1} \, dt \, dx)$ which can be represented by the inner-product with some function $g_B \in L^2(T(B), t^{-1} \, dt \, dx)$. Taking $B_j = B(0, j)$, $j \in \mathbb{N}$, then $g_{B_j} = g_{B_{j+1}}$ on $T(B_j)$. So we get a single function $g$ on $\mathbb{R}^{1+n}_+$ that is locally in $L^2(t^{-1} \, dt \, dx)$ such that

$$L(f) = \int_{\mathbb{R}^{1+n}_+} f(t, x) g(t, x) \frac{dt \, dx}{t}$$

whenever $f \in T^1_{\alpha,\beta}$ is supported in some tent $T(B)$. By the atomic decomposition, the subset of such $f$ is dense in $T^4_{\alpha,\beta}$. We only need to prove $g \in T^\infty_{\alpha,\beta}$ with $\|g\|_{T^\infty_{\alpha,\beta}} \lesssim \|L\|$. 

For a ball $B \subset \mathbb{R}^n$ and every $\varepsilon > 0$, we set

$$f_\varepsilon(t, x) = t^{-2(\alpha-\beta+1)} g(t, x) 1_{T^\varepsilon(B)}(t, x)$$

where $T^\varepsilon(B)$ is the truncated tent $T(B) \cap \{(t, x) : t > \varepsilon\}$. Since $g \in L^2(T(B))$, we have

$$\int_{T(B)} |f_\varepsilon(t, x)|^2 \frac{dt \, dx}{t^{1-2(\alpha-\beta+1)}} = \int_{T^\varepsilon(B)} |g(t, x)|^2 \frac{dt \, dx}{t^{1+2(\alpha-\beta+1)}} \lesssim \infty.$$ 

Hence we can obtain that $f_\varepsilon$ is a multiple of a $T^1_{\alpha,\beta}$-atom with

$$\|f_\varepsilon\|_{T^1_{\alpha,\beta}} \lesssim \rho^{n-2(\alpha+\beta-1)} \int_{T^\varepsilon(B)} |g(t, x)|^2 \frac{dt \, dx}{t^{1+2(\alpha-\beta+1)}}.$$ 

According to the representation above, we also get

$$\int_{T^\varepsilon(B)} |g(t, x)|^2 \frac{dt \, dx}{t^{1+2(\alpha-\beta+1)}} \lesssim \|L\| \left( \rho^{-n+2(\alpha+\beta-1)} \int_{T^\varepsilon(B)} |g(t, x)|^2 \frac{dt \, dx}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2}.$$ 

This gives us

$$\left( \rho^{-n+2(\alpha+\beta-1)} \int_{T^\varepsilon(B)} |g(t, x)|^2 \frac{dt \, dx}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2} \lesssim \|L\|,$$

that is, $g \in T^\infty_{\alpha,\beta}$ with $\|g\|_{T^\infty_{\alpha,\beta}} \lesssim \|L\|$. This completes the proof of Theorem 3.9. \qed
3.3. The Preduality of $Q_\alpha^2(\mathbb{R}^n)$. In this subsection, we introduce a new space which can be viewed as the predual of $Q_\alpha^2(\mathbb{R}^n)$. Then, we give an atomic decomposition for this space. For this purpose we need the following lemma which is Lemma 1.1 in [FJW].

**Lemma 3.10.** Fix $N \in \mathbb{N}$. Then there exists a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that
(1) $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$;
(2) $\phi$ is radial;
(3) $\phi \in C^\infty(\mathbb{R}^n)$;
(4) $\int_{\mathbb{R}^n} x^\gamma \phi(x)dx = 0$ if $\gamma \in \mathbb{N}^n$, $x^\gamma = x_1^\gamma_1 x_2^\gamma_2 \cdots x_n^\gamma_n$, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$;
(5) $\int_0^\infty (\hat{\phi}(t\xi))^2 \frac{dt}{t} = 1$ if $\xi \in \mathbb{R}^n \setminus \{0\}$.

For $\phi$ satisfying the conditions of Lemma 3.10 and any $f \in \mathcal{S}(\mathbb{R}^n)$, we have the well known Calderón reproducing formula

\[ f = \int_0^\infty f \ast \phi_t \ast \phi_t \frac{dt}{t} = \lim_{\epsilon \rightarrow 0, N \rightarrow \infty} \int_{\epsilon}^N f \ast \phi_t \ast \phi_t \frac{dt}{t}. \]

We introduce the notation of $HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$ in the sense of distributions.

**Definition 3.11.** For $\phi$ as in above lemma, $\alpha > 0$ and $\max\{1/2, \alpha\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we define the Hardy-Hausdorff space $HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$ to be the class of all distributions $f \in L^2_2 + 2(\beta-1)(\mathbb{R}^n)$ with

\[ \|f\|_{HH_{1-\alpha, \beta}^1(\mathbb{R}^n)} := \|f \ast \phi_t\|_{T_{\alpha, \beta}} < \infty. \]

**Theorem 3.12.** $\| \cdot \|_{HH_{1-\alpha, \beta}^1(\mathbb{R}^n)}$ is a quasi-norm. Furthermore, $HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$ is complete under this quasi-norm.

**Proof.** Obviously, $\| \cdot \|_{HH_{1-\alpha, \beta}^1(\mathbb{R}^n)}$ is a quasi-norm according to the linearity of $\rho_\phi(t, x) = f \ast \phi_t(x)$ and the corresponding property of $\| \cdot \|_{T_{\alpha, \beta}}$. Suppose that $\{f_j\}$ is a Cauchy sequence. By the Calderón reproducing formula and Theorem 3.3 we get $L^2_2 + 2(\beta-1) \rightarrow Q_\alpha^2$ and for every $\psi \in \mathcal{S}(\mathbb{R}^n)$

\[ |(f_j - f_k, \psi)| \lesssim \|\rho_\phi(f_j - f_k)\|_{T_{\alpha, \beta}} \|\phi_t \ast \psi\|_{T_{\alpha, \beta}} \lesssim \|\rho_\phi(f_j - f_k)\|_{T_{\alpha, \beta}} \|\psi\|_{Q_\alpha^2} \lesssim \|\rho_\phi(f_j - f_k)\|_{T_{\alpha, \beta}} |\psi|_{L^2_2 + 2(\beta-1)}. \]

This deduces that $\{f_j\}$ is a Cauchy sequence in $L^2_2 + 2(\beta-1)$. By completeness, $f = \lim f_n$ exists in $L^2_2 + 2(\beta-1)$. Thus there exists a subsequence such that $f = f_1 + \sum_{j \geq 1} (f_{j+1} - f_j)$ in $\mathcal{S}'(\mathbb{R}^n)$ with $\sum \|f_{j+1} - f_j\|_{HH_{1-\alpha, \beta}^1(\mathbb{R}^n)} < \infty$. Then we have

\[ \|\rho_\phi(f)\|_{T_{\alpha, \beta}} \lesssim (\|\rho_\phi(f_1)\|_{T_{\alpha, \beta}} + \sum \|\rho_\phi(f_{j+1} - f_j)\|_{T_{\alpha, \beta}}) < \infty \]

and so $f \in HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$. Similarly we can prove $f_j \rightarrow f$ in $HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$. \qed

**Definition 3.13.** Let $\alpha > 0$ and $\max\{1/2, \alpha\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. A tempered distribution $a$ is called an $HH_{1-\alpha, \beta}^1(\mathbb{R}^n)$ atom if $a$ is supported in a cube $I$ and satisfies the following two conditions:

(i) a *local Sobolev* $- (\alpha - \beta + 1)$ condition: for all $\psi \in \mathcal{S}$

\[ |\langle a, \psi \rangle| \leq \text{diam}(I)^{-\frac{\alpha}{2} + \alpha + \beta - 1} \left( \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{2(\alpha - \beta + 1)}} dx dy \right)^{1/2}; \]
(ii) a cancelation condition: \( \langle a, \psi \rangle = 0 \) for any \( \psi \in \mathcal{S} \) which coincides with a polynomial of degree \( \leq \frac{n}{2} + 1 \) in a neighborhood of \( I \).

In [DX], Dafni-Xiao established the following fractional Poincaré inequality which will help us to understand the previous definition.

**Lemma 3.14.** Let \( \psi \in C^\infty(\mathbb{R}^n) \) and \( I \) be a cube. Denote by \( \psi(I) \) the average of \( \psi \) over \( I \). If \( 0 \leq \alpha_1, \alpha_2 < \beta \) for a fixed \( \beta \in (1/2, 1) \), then

\[
\| \psi - \psi(I) \|_{L^2} \leq n^{n/4} diam(I)^{\alpha_1 - \beta + 1} \left( \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2(\alpha_1-\beta+1)}} \, dx \, dy \right)^{1/2} \\
\leq n^{n/4} diam(I)^{\alpha_2 - \beta + 1} \left( \int_I \int_I \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2(\alpha_2-\beta+1)}} \, dx \, dy \right)^{1/2} \\
\leq C \, diam(I) \| \nabla \psi \|_{L^2(I)}
\]

with \( C \) depending only on the dimension and \( \alpha_2 \). If in addition \( \int \frac{\partial \psi}{\partial x_k} \, dx = 0 \) for all \( k = 1, \cdots, n \), then the quantities above are also bounded by

\[
C \, diam(I) \| \nabla \psi - (\nabla \psi)_I \|_{L^2(I)} \leq C n^{n/4} diam(I)^{\alpha_1 - \beta + 2} \left( \int_I \int_I \frac{|
abla \psi(x) - \nabla \psi(y)|^2}{|x-y|^{n+2(\alpha_1-\beta+1)}} \, dx \, dy \right)^{1/2}.
\]

Here \( (\nabla \psi)_I \) denotes the vector whose coordinates are the means \( (\frac{\partial \psi}{\partial x_k})(I) \), \( k = 1, \cdots, n \).

**Remark 3.15.** Similar to Remark (2) after Lemma 6.2 of Dafni-Xiao [DX], we can prove that an \( HH_{-\alpha, \beta} \)-atom \( a \) belongs to the homogeneous Sobolev spaces \( \dot{L}^2_s \) with \( \alpha + \beta - 1 \leq s \leq \frac{n}{2} + 1 \). Particularly, we have

\[
|\langle a, \psi \rangle| \lesssim (diam(I))^{-\frac{n}{2} + \alpha + \beta - 1} \| \psi \|_{L^2_{-\beta+1}}.
\]

This deduces \( \|a\|_{\dot{L}^2_{-(\alpha-1)}} \lesssim (diam(I))^{-\frac{n}{2} + \alpha + \beta - 1} \). Meanwhile, \( |\langle a, \psi \rangle| \lesssim (diam(I)) \| \nabla \psi \|_{L^2_2} \) and so \( \|a\|_{\dot{L}^2_{-(\alpha-2\beta+3)}} \lesssim diam(I) \).

We can obtain the atomic decomposition of \( HH_{-\alpha, \beta} \) as follows.

**Theorem 3.16.** Let \( \alpha > 0 \) and \( \max\{1/2, \alpha\} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \). A tempered distribution \( f \) on \( \mathbb{R}^n \) belongs to \( HH_{-\alpha, \beta} \) if and only if there exist \( HH_{-\alpha, \beta} \)-atoms \( \{a_j\} \) and an \( l^1 \)-summable sequence \( \{\lambda_j\} \) such that \( f = \sum \lambda_j a_j \) in the sense of distributions. Moreover,

\[
\|f\|_{HH_{-\alpha, \beta}} \approx \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}.
\]

**Proof.** Part 1. “\( \leftarrow \)” By the completeness of \( HH_{-\alpha, \beta} \), we only need to prove that if \( a \) is an \( HH_{-\alpha, \beta} \)-atom then \( a \) is in \( HH_{-\alpha, \beta} \) with the quasinorm bounded by a constant. Since \( a \) is an \( HH_{-\alpha, \beta} \)-atom and \( \alpha + \beta - 1 \leq \frac{n}{2} - 2(\beta - 1) \leq \frac{n}{2} + 1 \), Remark 3.13 implies that \( a \in \dot{L}^2_{\frac{n}{2} + 2(\beta - 1)} \) with norm bounded by a constant. On the other hand, assume that \( I \) is the support of \( a \) and \( x_I \) represents its center. For \( \varepsilon \in (0, 2) \), let

\[
\omega(t, x) = \kappa(I(I))^{-n+2(\alpha+\beta-1)} \min \left\{ 1, \left( \frac{l(I)}{\sqrt{|x-x_I|^2 + l^2}} \right)^{n-2(\alpha+\beta-1)+\varepsilon} \right\}
\]
where \( \kappa \) is a constant to be chosen later. Similar to the proof of Theorem 3.9 we have

\[
N \omega(x) \leq \kappa(l(I))^{-n+2\alpha+\beta-1} \min\left\{ 1, \left( \frac{\sqrt{2(1)} |x-x_I|}{|x-x_I|} \right)^{n-2(\alpha+\beta-1)+\varepsilon} \right\}
\]

and so \( \int_{\mathbb{R}^n} N \omega dA^{(\infty)}_{n-2(\alpha+\beta-1)} \lesssim \kappa \leq 1 \) by choosing \( \kappa \) small enough.

Now, let \( B_I = B(x_I, \text{diam}(I)) \), \( E_I = (0, \text{diam}(I)) \times B_I \) and \( E_I^c = \mathbb{R}_{+1}^n \setminus E_I \). Suppose \( S_a \) is the support of \( a \ast \phi_I(x) \) in \( \mathbb{R}_{+1}^n \). We have

\[
\int_{\mathbb{R}_{+1}^n} |a \ast \phi_I(x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-2(\alpha+\beta+1)}} = \left( \int_{E_I} + \int_{E_I^c \cap S_a} \right) |a \ast \phi_I(x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-2(\alpha+\beta+1)}}.
\]

By the definition of the cylinder \( E_I \) in \( \mathbb{R}_{+1}^n \), we can find a half-ball centered at \((0, x_I)\) to cover \( E_I \). Thus we have \( \omega^{-1} \lesssim (l(I))^{n-2(\alpha+\beta-1)} \) on \( E_I \). This fact implies that

\[
\int_{E_I} |a \ast \phi_I(x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-2(\alpha+\beta+1)}} \leq (l(I))^{n-2(\alpha+\beta-1)} \int_{\mathbb{R}^n} |\hat{\alpha}(\xi)|^2 |\hat{\phi}(t\xi)|^2 d\xi dt \int_{t=0}^{\infty} \left( \frac{1}{|l(l(I))^{n-2(\alpha+\beta+1)}|} \right) \lesssim (l(I))^{n-2(\alpha+\beta-1)} \|a\|_{L^2_{\alpha+\beta-3}} \lesssim 1.
\]

For the integral on \( E_I^c \cap S_a \). If \( z \in I, x \notin B_I \) and \( t \leq |x-x_I|/2 \), then

\[
|x-z| \geq |x-x_I| - \text{diam}(I)/2 \geq |x-x_I|/2 \geq t,
\]

and \( a \ast \phi_I(x) = \int a(z) \phi_I(x-z) \, dz = 0 \). Otherwise, we have

\[
|a \ast \phi_I(x)| \leq \|a\|_{L^2_{\alpha+\beta-3}} \|\phi_I\|_{L^2_{\alpha+\beta-3}} \leq \text{diam}(I)t^{-(n-2(\alpha+\beta-3))} \int_{\mathbb{R}^n} |\hat{\phi}(t\xi)|^2 |\xi|^{-4\beta+6} d\xi \leq \text{diam}(I)t^{-(n-2(\alpha+\beta+3))}.
\]

It is easy to check \( \int \approx \sqrt{|x-x_I|^2 + t^2} := r(t, x) > \text{diam}I \). This implies that

\[
\omega^{-1}(t,x) \approx \kappa^{-1}(l(I))^{n-2(\alpha+\beta-1)} f^{n-2(\alpha+\beta-1)+\varepsilon} \lesssim (l(I))^{-\varepsilon} t^{n-2(\alpha+\beta-1)+\varepsilon}.
\]

Then we can get

\[
\int_{E_I^c \cap S_a} |a \ast \phi_I(x)|^2 \omega^{-1}(t,x) \frac{dt \, dx}{t^{1-2(\alpha+\beta+1)}} \lesssim (l(I))^{2(-\varepsilon)} \int_{E_I^c \cap S_a} t^{\varepsilon-n-3} dt \, dx \lesssim (l(I))^{2(-\varepsilon)} \int_{(r(x) \geq \text{diam}(I))} t^{\varepsilon-n-3} \, dt \lesssim (l(I))^{2(-\varepsilon-2)} \lesssim 1.
\]

**Part 2. “\( \Rightarrow \)”** Suppose \( f \in HH_{1,\alpha,\beta}^{1}(\mathbb{R}^n) \). Note that the Calderón reproducing formula \( (3.3) \) holds in the sense of distributions. Since the support of \( \phi \) is the unit ball, we can denote

\[
f^{\varepsilon,N}(x) = \int_{S^{n-1}} F(t,y) \phi_I(x-y) \frac{dt \, dy}{t^{n}}
\]
where \( F(t, y) = f \ast \phi(t) \) and \( S^{\varepsilon,N} \) is the strip \( \{(t, x) \in \mathbb{R}^{1+n}_+ : \varepsilon \leq t \leq N\} \). Similar to the proof of Theorem 3.9, there is an \( \omega \geq 0 \) on \( \mathbb{R}^{1+n}_+ \) such that
\[
\int_{\mathbb{R}^{1+n}_+} |F(t, x)|^2 \omega^{-1}(t, x) \frac{dtdx}{t^{1-2(\alpha+\beta+1)}} \leq 2\|F\|_{T^{1}_{\alpha,\beta}}.
\]
Let \( T_{j,k} \) be the corresponding structures over the set \( E_k = \{N\omega > 2^k\} \) as those in Theorem 3.9 (i). Noting that \( T_{j,k} \) have mutually disjoint interiors and \( F = \sum F\chi_{T_{j,k}} \) a.e. on \( \mathbb{R}^{1+n}_+ \), we let
\[
g^{\varepsilon,N}_{j,k}(x) = \int_{S^{\varepsilon,N} \cap T_{j,k}} F(t, y)\phi_t(x-y) \frac{dtdy}{t}.
\]
Since \( T_{j,k} \subset T(I_{j,k}^*) \), these smooth functions in \( x \) is supported in \( \{x : \Gamma(x) \cap T_{j,k} \neq \emptyset\} \subset I_{j,k}^* \) and have the same number moments as \( \phi \). We want to verify that there are distributions \( g_{j,k} \) such that \( g^{\varepsilon,N}_{j,k} \rightarrow g_{j,k} \) as \( \varepsilon \rightarrow 0 \) and \( N \rightarrow \infty \) with \( f = \sum_{j,k} g_{j,k} \) in \( \mathcal{S}'(\mathbb{R}^n) \). To see this, noting that \( \omega \leq 2^{k+1} \) on \( T_{j,k} \), we have
\[
|\langle g^{\varepsilon,N}_{j,k}, \psi \rangle| = \left| \int_{\mathbb{R}^n} \left( \int_{S^{\varepsilon,N} \cap T_{j,k}} F(t, y)\phi_t(x-y) \frac{dtdy}{t} \right) \psi(x) dx \right|
\leq 2^{(k+1)/2} \left( \int_{S^{\varepsilon,N} \cap T_{j,k}} |F(t, y)|^2 \omega^{-1}(t, y) \frac{dtdy}{t^{1-2(\alpha+\beta+1)}} \right)^{1/2} \left( \int_{S^{\varepsilon,N} \cap T_{j,k}} |\psi \ast \phi_t(y)|^2 \frac{dtdy}{t^{1+2(\alpha+\beta+1)}} \right)^{1/2}
\leq 2^{(k+1)/2} \left( \int_{S^{\varepsilon,N} \cap T_{j,k}} |F(t, y)|^2 \omega^{-1}(t, y) \frac{dtdy}{t^{1-2(\alpha+\beta+1)}} \right)^{1/2} \left( \int_{|\cdot|_{T_{j,k}} \leq T} \frac{\|\psi(x) - \psi(y)\|^2}{|x-y|^{n+2(\alpha+\beta+1)} dtdy} \right)^{1/2}.
\]
Similarly, we obtain that for \( \varepsilon_1 < \varepsilon_2 \) and \( N_1 > N_2 \),
\[
|\langle g^{\varepsilon_1,N_1}_{j,k} - g^{\varepsilon_2,N_2}_{j,k}, \psi \rangle| \leq C_k \left( \int_{(S^{\varepsilon_1,N_1-1} \cap S^{\varepsilon_2,N_2}) \cap T_{j,k}} \left| F(t, y) \right|^2 \omega^{-1}(t, y) \frac{dtdy}{t^{1-2(\alpha+\beta+1)}} \right)^{1/2} \left\| \psi \right\|_{L^2_{\alpha+\beta+1}}.
\]
This gives us that \( \|g^{\varepsilon_1,N_1}_{j,k} - g^{\varepsilon_2,N_2}_{j,k}\|_{L^2_{\alpha+\beta+1}} \rightarrow 0 \) as \( \varepsilon_1, \varepsilon_2 \rightarrow 0 \) and \( N_1, N_2 \rightarrow \infty \). Thus, \( g^{\varepsilon,N}_{j,k} \rightarrow g_{j,k} \in L^2_{\alpha+\beta+1} \) in the sense of distributions and \( g_{j,k} \) is supported in \( I_{j,k}^* \) with
\[
\|g_{j,k}\|_{L^2_{\alpha+\beta+1}(3I_{j,k}^*)} \lesssim 2^{(k+1)/2} \left( \int_{T_{j,k}} |F(t, y)|^2 \omega^{-1}(t, y) \frac{dtdy}{t^{1-2(\alpha+\beta+1)}} \right)^{1/2}.
\]
Let \( a_{j,k} = g_{j,k}\|L^2_{\alpha+\beta+1}(3I_{j,k}^*)^{-1} (1(3I_{j,k}^*))^{(\alpha+\beta+1)-\frac{\alpha}{2}} \) and \( \lambda_{j,k} = \|g_{j,k}\|_{L^2_{\alpha+\beta+1}(3I_{j,k}^*)} (1(3I_{j,k}^*))^{\frac{\alpha}{2}-(\alpha+\beta+1)} \).
Then
\[
|\langle a_{j,k}, \psi \rangle| \leq \frac{1}{\|g_{j,k}\|_{L^2_{\alpha+\beta+1}(3I_{j,k}^*)}} (1(3I_{j,k}^*))^{(\alpha+\beta+1)-\frac{\alpha}{2}} \left( \int_{S^{\varepsilon,N} \cap T_{j,k}} |F(t, y)|^2 w(t, y)^{-1} \frac{dtdy}{t^{1-2(\alpha+\beta+1)}} \right)^{\frac{1}{2}} \left( \int_{3I_{j,k}^*} \int_{3I_{j,k}^*} |\psi(x) - \psi(y)|^2 \frac{dtdy}{|x-y|^{n+2(\alpha+\beta+1)}} \right)^{\frac{1}{2}}.
\]
Lemma 3.17. (i) If \( \omega \) is an \( HH^{1}_{1,\alpha,\beta} \)-atom, then there exists a nonnegative function \( \omega \) on \( \mathbb{R}^{1+n} \) with \( \int_{\mathbb{R}^{n}} N_\omega d\Lambda_n^{-1}(x) \leq 1 \) and

\[
\sigma(\alpha, \omega) = \sup_{|\eta| \leq \delta} \left( \int_{\mathbb{R}^{1+n}} |a * \psi_t(x-y) - a * \psi_t(x)|^2 \omega(t,x)^{-1} \frac{dt dx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \to 0.
\]

(ii) \( HH^{1}_{1,\alpha,\beta} \cap C^0(\mathbb{R}^n) \) is dense in \( HH^{1}_{1,\alpha,\beta} \).

Proof. (i) For a fixed \( \varepsilon \in (0,2) \), the same \( \omega \) defined in the proof of Theorem 3.14.

This means that \( a_{j,k} \) are \( HH^{1}_{1,\alpha,\beta} \)-atoms. On the other hand, the Cauchy-Schwarz inequality implies that

\[
\sum_{j,k} |\lambda_{j,k}| \lesssim \left( \sum_{j,k} 2^{k+1}(l(3I_{j,k}^*))^{n-2(\alpha-\beta+1)} \right)^{1/2} \left( \sum_{j,k} \int_{T_{j,k}} |F(t,y)|^2 \omega^{-1}(t,y) \frac{dt dy}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2}
\]

\[
\lesssim \left( \sum_{j,k} 2^{k+1} \Lambda_{n-2(\alpha-\beta+1)}(3I_{j,k}^*) \right)^{1/2} \left( \sum_{j,k} \int_{\mathbb{R}^{1+n}} |F(t,y)|^2 \omega^{-1}(t,y) \frac{dt dy}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2}
\]

\[
\lesssim \left( \sum_{j,k} \int_{E_k} 2^{k+1} d\Lambda_{n-2(\alpha-\beta+1)}(E_k) \right)^{1/2} \|f\|_{HH^{1}_{1,\alpha,\beta}}
\]

\[
\lesssim \left( \sum_{j,k} \int_{E_k} N_\omega(x) d\Lambda_{n-2(\alpha-\beta+1)} \right)^{1/2} \|f\|_{HH^{1}_{1,\alpha,\beta}} \lesssim \|f\|_{HH^{1}_{1,\alpha,\beta}}.
\]

The above estimates tell us that \( \sum_{j,k} g_{j,k} = \sum_{j,k} \lambda_{j,k} a_{j,k} \) converges to a distribution \( g \) in \( HH^{1}_{1,\alpha,\beta}(\mathbb{R}^n) \). We need to verify that \( g = f \). Since for a fix \( \psi \in \mathcal{S}(\mathbb{R}^n) \), every \( 0 < \varepsilon < N \),

\[
|\langle g_{j,k}, \psi \rangle| \lesssim 2^{(k+1)/2} \left( \int_{T_{j,k}} |F(t,y)|^2 w(t,y)^{-1} \frac{dt dy}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \left( \int_{3I_{j,k}^*} |\psi(x) - \psi(y)|^2 \frac{dt dy}{|x-y|^{n+2(\alpha-\beta+1)}} \right)^{1/2}
\]

\[
\lesssim 2^{(k+1)/2} (l(3I_{j,k}^*))^{n-2(\alpha-\beta+1)} \left( \int_{T_{j,k}} |F(t,y)|^2 w(t,y)^{-1} \frac{dt dy}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \|\psi\|_{Q_{\alpha,\beta}(\mathbb{R}^n)}
\]

\[
\lesssim \|f\|_{HH^{1}_{1,\alpha,\beta}} \|\psi\|_{Q_{\alpha,\beta}(\mathbb{R}^n)}.
\]

Then, \( \lim_{\varepsilon \to 0, N \to \infty} \sum_{j,k} g_{j,k} = \sum_{j,k} g_{j,k} = g \). Meanwhile, we can also obtain that

\[
\sum_{j,k} \int_{\mathbb{R}^{1+n}} 1_{S_{\varepsilon,N} \cap T_{j,k}^*} F(t,y) \phi_t * \psi(y) \frac{dt dy}{t} = \int_{S_{\varepsilon,N}} F(t,y) \phi_t * \psi(y) \frac{dt dy}{t} = \langle f, \psi \rangle.
\]

This tells us \( \sum_{j,k} g_{j,k} = \varepsilon \to f \) in \( \mathcal{S}'(\mathbb{R}^n) \). Therefore \( g = f \) in \( \mathcal{S}'(\mathbb{R}^n) \).
Note that
\[
\sup_{|y|<\delta} \left( \int_{B_{t,x}^{1+n}} |a \ast \phi_t(x-y) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \\
\lesssim \left( \left( \sup_{|y|<\delta} \int_{E_I} + \sup_{|y|<\delta} \int_{E_I \cap S_{a,\delta}} \right) |a \ast \phi_t(x-y) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2},
\]
where $B_I$ is the ball $B(x_I,2diam(I))$, and $E_I = (0, 2diam(I)) \times B_I$. By Fourier transforms, we can estimate the first term as
\[
\sup_{|y|<\delta} \int_{E_I} |a \ast \phi_t(x-y) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}}
\lesssim (l(I))^{n-2(\alpha+\beta-1)} \sup_{|y|<\delta} \int_{\mathbb{R}^n} |a \ast (\hat{\phi}_t^y - \hat{\phi}_t)(x)|^2 \frac{dtdx}{t^{1-2(\alpha-\beta+1)}}
\lesssim (l(I))^{n-2(\alpha+\beta-1)} \sup_{|y|<\delta} \int_{\mathbb{R}^n} |\hat{\phi}(t\xi)|^2 \min\{2, \delta |\xi|\}^2 \int_0^\infty |\hat{\phi}(t\xi)|^2 \frac{dt}{t^{1-2(\alpha-\beta+1)}} d\xi
\lesssim (l(I))^{n-2(\alpha+\beta-1)} \sup_{|y|<\delta} \int_{\mathbb{R}^n} |\hat{\phi}(t\xi)|^2 |\xi|^2 |\xi|^{-2(\alpha+\beta-1)} \int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t^{1-2(\alpha-\beta+1)}} d\xi \to 0
\]
as $\delta \to 0$ according to the dominated convergence theorem.

For the second term. Since $\text{supp}(a) = I$, when $x \notin B_I$ and $t \leq |x-x_I|/4$, we obtain $|y| < \text{diam}(I) < \frac{1}{2}|x-x_I|$ for $y \in B(0, \delta)$ with $\delta < \text{diam}(I)$. Therefore
\[
|x-y-z| > |x-x_I| - |z-x_I| - |y| \geq \frac{3}{4}|x-x_I| \geq t.
\]
On the other hand $|x-z| \geq \frac{3}{4}|x-x_I| > t$. These estimates imply that $a \ast [\phi_t(x-y) - \phi_t(x)] = 0$. Otherwise, we have
\[
|a \ast \phi_t(x-y) - a \ast \phi_t(x)| \lesssim \|a\|_{L^2(\mathbb{R}^{n+1})} \|\hat{\phi}_t^y - \hat{\phi}_t^x\|_{L^2(\mathbb{R}^{n+1})}
\lesssim \text{diam}(I) \left( \int_{\mathbb{R}^n} |\phi_t^x - \phi_t^y(\xi) - \phi_t^x(\xi)|^2 |\xi|^{-4\beta+6} d\xi \right)^{1/2}
\lesssim \text{diam}(I) \left( \int_{\mathbb{R}^n} \min\{2, \delta |\xi|\}^2 |\phi_t^x(\xi)|^2 |\xi|^{-4\beta+6} d\xi \right)^{1/2}
\lesssim \text{diam}(I) \delta \left( \int_{\mathbb{R}^n} |\phi_t^x(\xi)|^2 |\xi|^{-4\beta+6} d\xi \right)^{1/2}
\lesssim \text{diam}(I) \delta t^{2\beta-4-n}.
\]
Using the above estimates and the fact $\omega^{-1} \lesssim t^{n-2(\alpha-\beta+1)+\varepsilon}$, we have
\[
\int_{E_I \cap S_{a,\delta}} |a \ast \phi_t(x-y) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta+1)}} \\
\lesssim \delta^2 (l(I))^{2-\varepsilon} \int_{E_I \cap S_{a,\delta}} t^{-n-5+\varepsilon} dtdx
\lesssim \delta^2 (l(I))^{2-\varepsilon} \int_0^\infty \lambda^{5-\varepsilon} d\lambda \to 0
\]
as $\delta \to 0$. Thus $\sigma_3(a, \omega) \to 0$ as $\delta \to 0$. 

(ii) For an $HH^1_{\alpha,\beta}$-atom $a$, take $\eta \in C^\infty(\mathbb{R}^n)$ with support in $B(0,1)$ and $\int \eta = 1$. Then $a \ast \eta_j \in C^\infty(\mathbb{R}^n)$ and $\eta_j = j^n\eta(jx)$ form an approximate identity, $a \ast \eta_j \rightarrow a$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \rightarrow \infty$. For any nonnegative function $\omega$ on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} N\omega d\Lambda^{(\infty)}_{n-2(\alpha+\beta-1)} \leq 1$, we have

\[
\left( \int_{\mathbb{R}^n} |a \ast \eta_j \ast \phi_t(x) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta)+1}} \right)^{1/2} \leq \int_{\mathbb{R}^n} |\eta_j(y)| \left( \int_{\mathbb{R}^n} |a \ast \phi_t(x-y) - a \ast \phi_t(x)|^2 \omega^{-1}(t,x) \frac{dtdx}{t^{1-2(\alpha-\beta)+1}} \right)^{1/2} \leq \sigma_\omega(a,\omega).
\]

From (i), we know that for every $\varepsilon > 0$ there exists an $\omega$ such that $\sigma_\omega(a,\omega) < \varepsilon$ with $j$ large enough. Taking the infimum over all $\omega$ induces

\[\|a \ast \eta_j - a\|_{HH^1_{\alpha,\beta}} < \varepsilon\text{ for large }j,\]

that is, $a \ast \eta_j \rightarrow a$ in $HH^1_{\alpha,\beta}$. Hence, we can get the desired density from the fact that every $f \in HH^1_{\alpha,\beta}$ can be approximated by finite sums of atoms. \(\square\)

**Lemma 3.18.** For $\alpha > 0$, $\max\{\alpha,1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x) dx = 0$, let

\[d\mu_{f,\phi,\alpha,\beta}(t,x) = |(f \ast \phi_t)(y)|^2 t^{-1-2(\alpha-\beta)+1} dt dy.
\]

Then there is a constant $C$ such that for any cubes $I$ and $J$ in $\mathbb{R}^n$ with center $x_0$ and $l(J) \geq 3l(I)$,

(i) [expression]

(ii) If in addition $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ then

\[\mu_{f,\phi,\alpha,\beta}(S(I)) \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2(\alpha-\beta)+1}} \frac{dx dy}{|l(I)|^{n-2(\alpha-\beta)}} \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-x_0|^{n+1}} \frac{dx}{|l(I)|^{n-2(\alpha-\beta)}} \right)^2.
\]

**Proof.** This lemma is a special case of Dafini-Xiao [DX1] Lemma 3.2. \(\square\)

**Theorem 3.19.** Let $\phi$ be a function as in Lemma 3.18, $\alpha > 0$ and $\max\{\alpha,1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. If $f \in Q^2_{\alpha}(\mathbb{R}^n)$ then $d\mu_{f,\phi,\alpha,\beta}(t,x) = |(f \ast \phi_t)(x)|^2 t^{-1-2(\alpha-\beta)+1} dt dx$ is a $1-2(\alpha + \beta - 1)/n$-Carleson measure.

**Proof.** The proof follows from (ii) Lemma 3.18 by taking $J = 3I$. \(\square\)

To establish the equivalent \(3.19\) we need another theorem which contains the converse of Theorem 3.19.

**Theorem 3.20.** Consider the operator $\pi_\phi$ defined by

\[\pi_\phi(F) = \int_0^\infty F(t,\cdot) \ast \phi_t \frac{dt}{t}.
\]

\[\pi_\phi(F) = \int_0^\infty F(t,\cdot) \ast \phi_t \frac{dt}{t}.
\]
(i) The operator $\pi_\phi$ is a bounded and surjective operator form $\mathcal{T}^\infty_{\alpha,\beta}$ to $Q^\beta_{\alpha}$. More precisely, if $F \in \mathcal{T}^\infty_{\alpha,\beta}$ then the right-hand side of the above integral converges to a function $f \in Q^\beta_{\alpha}$ and

$$\|f\|_{Q^\beta_{\alpha}} \lesssim \|F\|_{\mathcal{T}^\infty_{\alpha,\beta}}$$

and any $f \in Q_{\alpha,\beta}$ can be thus represented.

(ii) The operator $\pi_\psi$ initially defined on $F \in \mathcal{T}^1_{\alpha,\beta}$ with compact support in $\mathbb{R}^{1+n}$ extends to a bounded and surjective operator form $\mathcal{T}^1_{\alpha,\beta}$ to $HH^1_{\alpha,\beta}$.

Proof. (i) Taking $f = \pi_\phi(F)$, we only need to prove $\sup_I D_{f,\alpha,\beta}(I) < \infty$ where

$$D_{f,\alpha,\beta}(I) = \int_{[t(I)]}^\infty |F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)|^2 dt.$$ 

Denote the function $x \mapsto f(x + y)$ by $f_y$ and note that the integral in (5.5) is valid in $\mathcal{S}'(\mathbb{R}^n)$ modulo constants, that is, when it acts on test functions of integration zero, we obtain

$$f_y - f = \int_0^\infty \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)) g(x) \right| \frac{dtdx}{t}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Fix a cube $I$ and $y \in B(0, l(I))$. For any $g \in C_0^\infty(I)$, we write

$$|f_y - f| \leq \int_0^{|y|} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)) g(x) \right| \frac{dtdx}{t}$$

+ \int_{|y|}^{l(I)} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)) g(x) \right| \frac{dtdx}{t}

+ \int_0^\infty \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)) g(x) \right| \frac{dtdx}{t}

:= |A_1(g, y) + A_2(g, y) + A_3(g, y)|.

For $A_1(g, y), |y| < l(I)$ verifies that $g - g_0$ is supported in the dilated cube $3I$. Also if $t \leq |y|$ we have that $\phi_\epsilon (g - g_0)$ is supported in the large cube $J = 5I$. Then we can get

$$A_1(g, y) \leq \int_0^{|y|} \left( \int_J |F(t,\cdot)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |g(x)|^2 dx \right)^{1/2} \frac{dtdx}{t}$$

$$\lesssim \int_0^{|y|} \left( \int_J |F(t,\cdot)|^2 dx \right)^{1/2} \frac{dtdx}{t}$$

For $A_2$, if $|y| < t$, by changing variable $z - y = z$, we get

$$|F(t,\cdot)^\alpha (\cdot) \star \phi(x) - (F(t,\cdot)^\alpha (\cdot) \star \phi(x)) |

\leq \int_{\mathbb{R}^n} |\phi(t^{-1}y + z) - \phi(z)| |F(t, x - tz)| dz \lesssim t^{-1} |y| \sup_{|\xi| \leq 1} |\nabla \phi(\xi)| \int_{|z| \leq 1} |F(t, x - tz)| dz \lesssim C_\phi t^{-1} |y| \int_{|z| \leq 1} |F(t, x - tz)| dz$$
with $C_\phi = \sup |\nabla \phi| < \infty$. Fubini’s theorem and the fact that $g$ is supported in $I$

\begin{align*}
A_2(g, y) & \leq C_\phi |y| \int_{|y|}^{\frac{t(I)}{|y|}} \int_{|z| \leq 2} |F(t, x - tz)||g(x)| dx dz \frac{dt}{t^2} \\
& \lesssim C_\phi \|g\|_{L^2} |y| \int_{|y|}^{\frac{t(I)}{|y|}} \left( \int_{|z| \leq 2} |F(t, x - tz)|^2 dx \right)^{1/2} \frac{dt}{t^2}
\end{align*}

where $|z_i| \leq 2$ and $C = Vol(B(0, 2))$.

For $A_3$, let $G_y(t, x) = \phi_t * (g_y - g)(x)1_{\{(t, x); t \geq |y|\}}$. Then the inequality (3.17) implies that

\begin{align*}
A_3 &= \int_{R_+^{n+1}} |G_y(t, x)|^2 \omega^{-1}(t, x) \frac{dtdt}{t^{1-2(\alpha - \beta + 1)}} \\
& \leq |I|^{-\varepsilon} \int_{l(I)}^{\infty} \int_{R^n} |\phi_t * (g_y - g)(x)|^2 dx \frac{dtdt}{t^{2n-2(\alpha + \beta - 1)+\varepsilon}} \\
& \leq |I|^{-\varepsilon} \|g\|_{L^2}^2 \int_{l(I)}^{\infty} \|\phi_t^y - \phi_t\|^2_{L^2} t^{n-4\beta + 3 + \varepsilon} dt \\
& \leq |I|^{-\varepsilon} \|g\|_{L^2}^2 \int_{l(I)}^{\infty} \int_{R^n} |(\widehat{\phi_t^y} - \widehat{\phi_t})(\xi)|^2 d\xi t^{n-4\beta + 3 + \varepsilon} dt \\
& \leq |I|^{-\varepsilon} \|g\|_{L^2}^2 \int_{l(I)}^{\infty} \int_{R^n} \left|1 - e^{2\pi iy \xi} \right|^2 |\phi(t\xi)|^2 d\xi t^{n-4\beta + 4 + \varepsilon} dt \\
& \leq C_\phi |I|^{-\varepsilon} \|g\|_{L^2}^2 |y|^{-4\beta + 4 + \varepsilon}.
\end{align*}

In the last inequality we have used the fact:

\begin{align*}
\int_{R^n} \left|1 - e^{2\pi iy \xi} \right|^2 d\xi \lesssim |y|^{-4\beta + 4}.
\end{align*}

In fact, we can write

\begin{align*}
\int_{R^n} \left|1 - e^{2\pi iy \xi} \right|^2 d\xi & \lesssim \int_{R^n} \left|1 - e^{2\pi iy \xi} \right|^2 |y|^{n-4\beta + 4 + \varepsilon} d(y \xi) |y|^n \\
& \lesssim |y|^{-4\beta + 4} \left( \int_{|z| \leq 1} + \int_{|z| > 1} \right) \left|1 - e^{2\pi i z} \right|^2 |z|^{n-4\beta + 4 + \varepsilon} dz \\
& := |y|^{-4\beta + 4}(I_1 + I_2).
\end{align*}
It is easy to see that
\[ I_2 = \int_{|z| \geq 1} \frac{1 - e^{2\pi i z}}{|z|^{n+\varepsilon-4\beta+4}} \, dz \lesssim \int_{|z| \geq 1} \frac{|z|^{-1}}{|z|^{n+\varepsilon-4\beta+4}} \, dz \lesssim 1, \]
\[ I_1 = \int_{|z| < 1} \frac{1 - e^{2\pi i z}}{|z|^{n+\varepsilon-4\beta+4}} \, dz \lesssim \int_{|z| < 1} \sum_{k=1}^{\infty} \frac{(2\pi i z)^{k-1}}{k!} \frac{|z|^2}{|z|^{n+\varepsilon-4\beta+4}} \, dz \]
\[ \lesssim \int_{|z| < 1} |z|^{1-\varepsilon+4\beta-4} \, dz \lesssim 1. \]
Then \( \|G_y\|_{T_{a,\beta}^2} \leq \|g\|_{L^2(I)^{n-\varepsilon}} |y|^{-4\beta+4}. \) Thus we get
\[ \|f_y - f\|_{L^2(I)} \leq \sup_{g \in C_0(I), \|g\| \leq 1} |(f_y - f, g)| \]
\[ \lesssim \int_{|y| < l(I)} \left( \int_{J \cap I} |F(t, x)|^2 \, dx \right)^{1/2} \frac{dt}{t} + \|y\| \int_{|y| < l(I)} \left( \int_{J \cap I} |F(t, x - tz)|^2 \, dx \right)^{1/2} \frac{dt}{t^2} \]
\[ + \|F\|_{T_{a,\beta}^2} l(I)^{n-\varepsilon/2} |y|^{\varepsilon/2-2\beta+2}. \]
Then, by Hardy’s inequality (see Stein [5]), we have
\[ \int_{|y| < l(I)} \int_{J} |f(x + y) - f(x)|^2 \frac{dx \, dy}{|y|^{n+2(\alpha-\beta+1)}} \]
\[ \lesssim \int_{0}^{l(I)} \left( \int_{0}^{s} \left( \int_{J} |F(t, x)|^2 \, dx \right)^{1/2} \frac{dt}{t} \right)^2 \frac{ds}{s^{1+2(\alpha-\beta+1)}} \]
\[ + \int_{0}^{l(I)} \left( \int_{J} \left( \int_{J} |F(t, x - tz)|^2 \, dx \right)^{1/2} \frac{dt}{t^2} \right)^2 \frac{ds}{s^{2(\alpha-\beta+1)-1}} \]
\[ + \|F\|_{T_{a,\beta}^2} l(I)^{n-\varepsilon} \int_{0}^{l(I)} \frac{s^{n-1} s^{-4\beta+4}}{s^{n+2(\alpha-\beta+1)}} \, ds \]
\[ \lesssim \int_{0}^{l(I)} \int_{J} \frac{|F(t, x)|^2}{t^{1-2(\alpha-\beta+1)}} \, dx \, dt \]
\[ + \int_{0}^{l(I)} \int_{J} |F(t, x - tz)|^2 t^{1-2(\alpha-\beta+1)} \, dt \, dx \]
\[ + \|F\|_{T_{a,\beta}^2} l(I)^{n-\varepsilon} l(I)^{\varepsilon-2(\alpha-\beta+1)} \]
\[ \lesssim l(I)^{n-2(\alpha+\beta-1)} \|F\|_{T_{a,\beta}^2} \]
since for each \( t \leq l(I), |z| \leq 2 \) implies \( I-tz \subset J = 5I \). Then we get \( \sup_{I} D_{f,a,\beta}(I) \lesssim \|F\|_{T_{a,\beta}^2}^2 < \infty \), that is \( f \in Q_{a,\beta}^2 \) and \( \|f\|_{Q_{a,\beta}^2} \lesssim \|F\|_{T_{a,\beta}^2}. \)

(ii) Firstly, we verify that for a \( T_{a,\beta}^- \) atom \( a \), the integral in (5.5) converges in \( L^2_{\nu^{-2}}(\mathbb{R}^2) \) to a distribution which is a multiple of an \( HH^1_{-\alpha,\beta} \)-atom. Assume \( \pi_{\alpha,\beta}^a(x, t) \) is supported in \( T(B) \) for some \( B \). For \( \varepsilon > 0 \), let
\[ \pi_{\alpha,\beta}^a(x, t) = \int_{\varepsilon}^{\infty} a(t, \cdot) \ast \phi_t(x) \frac{dx \, dt}{t} \]
and \( T^\varepsilon(B) \) be the truncated tent \( T(B) \cap \{ (t,x) : t > \varepsilon \} \). The Cauchy-Schwarz inequality and (ii) of Lemma 3.18 imply that
\[
|\langle \pi_\alpha^1(a), \psi \rangle| \leq \left( \int_{T^\varepsilon(B)} |a(t,x)|^2 \frac{dxdt}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \left( \int_{T^\varepsilon(B)} |\psi \ast \phi_\varepsilon(x)|^2 \frac{dxdt}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2}
\]
hold for any \( \psi \in \mathcal{H}(\mathbb{R}^n) \), where \( \tilde{B} \) is some fixed dilate of the ball \( B \). Since the right-hand side is dominated by \( \|\psi\|_{Q^\alpha_1} \leq \|\psi\|_{L^2_{n/2+2-2\beta}} \), the same argument also gives, for \( 0 < \varepsilon_1 < \varepsilon_2 \),
\[
|\langle \pi_\alpha^1(a) - \pi_\alpha^2(a), \psi \rangle| \leq \left( \int_{T^\varepsilon(B) \setminus T^{\varepsilon^2}(B)} |a(t,x)|^2 \frac{dxdt}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} \|\psi\|_{L^2_{n/2+2-2\beta}}.
\]
Thus \( \pi_\alpha(a) = \lim_{\varepsilon \to 0} \pi_\alpha^\varepsilon(a) \) exists in \( \hat{L}^2_{n/2-2+2\beta} \). This distribution is supported in \( \tilde{B} \) and satisfies condition (i) of Definition 3.12 since \( \phi \) satisfies the same condition. Therefore \( \pi_\alpha(a) \) is a multiple of an \( HH^{-1}_{-\alpha,\beta} \) atom. For a function \( F = \sum_j \lambda_j a_j \) in \( T^1_{\alpha,\beta} \) and a test function \( \psi \in \mathcal{H}(\mathbb{R}^n) \), by Theorem 3.20 we have
\[
\int_{\mathbb{R}^n} (F(t, \cdot) \ast \phi_\varepsilon)(x) \psi(x) \frac{dxdt}{t} = \sum_j \lambda_j \langle \pi_\alpha a_j, \psi \rangle = \left( \sum_j \lambda_j \pi_\alpha a_j, \psi \right),
\]
since \( \langle \phi \ast \psi \rangle(t,x) \) is a function in \( T^\infty_{\alpha,\beta} \). So \( \pi_\alpha(F) = \sum_j \lambda_j \pi_\alpha a_j \in \mathcal{H}^\prime(\mathbb{R}^n) \) and
\[
\|\pi_\alpha(F)\|_{HH^{-1}_{-\alpha,\beta}} \leq \inf \sum_j |\lambda_j| \approx \|F\|_{T^1_{\alpha,\beta}}
\]
the infimum being taken over all possible atomic decompositions of \( F \) in \( T^1_{\alpha,\beta} \). This finishes the proof of Theorem 3.20.

By Theorem 3.19, Lemma 3.17 and Theorem 3.20, using a similar argument of Dafni-Xiao [DX, Theorem 7.1], we can prove the following duality theorem.

**Theorem 3.21.** The duality of \( HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n) \) is \( Q^\alpha_0(\mathbb{R}^n) \) in the following sense: if \( g \in Q^\alpha_0 \) then the linear functional
\[
L(f) = \int_{\mathbb{R}^n} f(x)g(x)dx,
\]
defined initially for \( f \in HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n) \), has a bounded extension to all elements of \( HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n) \) with \( \|L\| \leq C\|g\|_{Q^\alpha_0(\mathbb{R}^n)} \). Conversely, if \( L \) is a bounded linear functional on \( HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n) \) then there is a function \( g \in Q^\alpha_0(\mathbb{R}^n) \) so that \( \|g\|_{Q^\alpha_0(\mathbb{R}^n)} \leq C\|L\| \) and \( L \) can be written in the above form for every \( f \in HH^{-1}_{-\alpha,\beta}(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n) \).

4. **Well-Posedness of Generalized Navier-Stokes Equations**

In this section, we deal with the well-posedness for the generalized Navier-Stokes system in the setting of \( Q^\alpha_0(\mathbb{R}^n) \). Before stating our main result, we first introduce a new critical spaces, i.e. the derivative spaces of \( Q^\alpha_0(\mathbb{R}^n) \). Then we establish some theorems and lemmas which will be used in the proof of the well-posedness.
4.1. Some properties of $Q^{\beta,-1}_{\alpha,\infty}$.

**Definition 4.1.** For $\alpha > 0$ and $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we say that a tempered distribution $f \in Q^{\beta,-1}_{\alpha,\infty}$ if and only if
\[
\sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^{r^2} \int_{|y-x| < r} |R_i^\beta * f(y)|^2 t^{-\frac{\alpha}{\beta}} dy dt < \infty.
\]

**Remark 4.2.** In Definition 4.1, if we take $\beta = 1$, the space $Q^{1,-1}_{\alpha,\infty}$ becomes the space $Q^{-1}_{\alpha,\infty}$ introduced by Xiao in [X].

In the next theorem, we prove an useful characterization of $Q^{\beta,-1}_{\alpha,\infty}$. For this purpose, we need the following lemma.

**Lemma 4.3.** For $\alpha > 0$ and $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, let $f_{j,k} = \partial_j \partial_k (-\Delta)^{-1} f(j,k = 1,2,\cdots,n)$. If $f \in Q^{\beta,-1}_{\alpha,\infty}$ for $\beta \in (\frac{1}{2},1)$, then $f_{j,k} \in Q^{\beta,-1}_{\alpha,\infty}$.

**Proof.** Take $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(\phi) \subset B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$. Write $\phi_r(x) = r^{-n} \phi(\frac{x}{r})$ and define $g_r(t,x) = \phi_r * \partial_j \partial_k (-\Delta)^{-1} e^{-t(-\Delta)^{\beta}} f(x)$.

Then
\[
e^{-t(-\Delta)^{\beta}} f_{j,k}(x) = \partial_j \partial_k (-\Delta)^{-1} e^{-t(-\Delta)^{\beta}} f(x) = f_r(t,x) + g_r(t,x).
\]

Since $B^{2\beta-1}_{1,1}$ is the predual of the homogeneous Besov space $B^{1-2\beta}_{1,\infty}$ and $Q^{\beta,-1}_{\alpha,\infty} \hookrightarrow B^{1-2\beta}_{1,\infty}$ (see Remark 4.5 and Theorem 4.6 below), we have
\[
\|g_r(t,)\|_{L^\infty} \leq \|\phi\|_{B^{2\beta-1}_{1,1}} \|\partial_j \partial_k (-\Delta)^{-1} e^{-t(-\Delta)^{\beta}} f\|_{B^{1-2\beta}_{1,\infty}} \lesssim C\|f\|_{B^{1-2\beta}_{1,\infty}}.
\]

Therefore
\[
\int_0^{r^2} \int_{|y-x| < r} |g_r(t,y)|^2 t^{-\alpha/\beta} dy dt \lesssim r^{n-2\alpha-2\beta+2} \|f\|_{B^{1-2\beta}_{1,\infty}}^2 \|f\|_{Q^{\beta,-1}_{\alpha,\infty}}^2.
\]

To estimate $f_r$ we take $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B(0,10) = \{x \in \mathbb{R}^n : |x| < 10\}$ and define $\varphi_{r,x} = \varphi(\frac{x}{r})$. Then $f_r = F_{r,x} + G_{r,x}$ with
\[
G_{r,x} = \partial_j \partial_k (-\Delta)^{-1} \varphi_{r,x} e^{-t(-\Delta)^{\beta}} f - \varphi_r * \partial_j \partial_k (-\Delta)^{-1} \varphi_{r,x} e^{-t(-\Delta)^{\beta}} f.
\]

Using Plancherel’s identity, we have
\[
\int_0^{r^2} \|\partial_j \partial_k (-\Delta)^{-1} \varphi_{r,x} e^{-t(-\Delta)^{\beta}} f\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^{r^2} \left( \int_{\mathbb{R}^n} |\xi_j \xi_k \xi|^{-2} (\varphi_{r,x} e^{-t(-\Delta)^{\beta}} f)(\xi)|^2 d\xi \right) \frac{dt}{t^{\alpha/\beta}}.
\]

Similarly we can prove
\[
\int_0^{r^2} \|\varphi_r * \partial_j \partial_k (-\Delta)^{-1} \varphi_{r,x} e^{-t(-\Delta)^{\beta}} f\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^{r^2} \|\varphi_{r,x} e^{-t(-\Delta)^{\beta}} f\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}.
\]
Thus, we obtain
\[ \int_0^r \|G_{r,\cdot}(t, \cdot)\|^2_{L^2} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^r \| \varphi_{r,\cdot} e^{-t(-\triangle)^\beta} f \|^2_{L^2} \frac{dt}{t^{\alpha/\beta}}. \]

To bound \( F_{r,x} \), noting that
\[ \int_{|y-x|<r} |F_{r,x}(t, y)|^2 dy \lesssim r^{n+1} \int_{|w-x|\geq r} |e^{-t(-\triangle)^\beta} f(w)|^2 |x-w|^{-(n+1)} dw, \]
we establish
\[ \int_0^r \left( \int_{|y-x|<r} |F_{r,x}(t, y)|^2 dy \right) \frac{dt}{t^{\alpha/\beta}} \lesssim r^{n+1} \int_{|w-x|\geq r} \left( \int_0^r |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \]
\[ \lesssim \sum_{k=1}^{\infty} 2^{-k(n+1)} \int_{|w-x|\leq 2^{k+1} r} \left( \int_0^r |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \]
\[ \lesssim \sum_{k=1}^{\infty} 2^{-k(n+1)} \left( \int_0^{2^{k+1} r} \int_{|w-x|\leq 2^{k+1} r} |e^{-t(-\triangle)^\beta} f(w)|^2 \frac{dt}{t^{\alpha/\beta}} \right) dw \]
\[ \lesssim r^{n-2\alpha-2\beta+2} \| f \|_{Q_{\alpha,\infty}^{\beta,-1}} \sum_{k=1}^{\infty} 2^{-k(2\alpha+2\beta-1)} \lesssim r^{n-2\alpha-2\beta+2} \| f \|_{Q_{\alpha,\infty}^{\beta,-1}}. \]

Now we have proved that
\[ \int_0^r \left( \int_{|y-x|<r} |f_{r,\cdot}(t, y)|^2 dy dt \right) \frac{dt}{t^{\alpha/\beta}} \lesssim r^{n-2\alpha-2\beta+2} \| f \|_{Q_{\alpha,\infty}^{\beta,-1}}, \]
that is, \( f_{j,k} \in Q_{\alpha,\infty}^{\beta,-1} \). \( \square \)

Using Lemma 4.2, we can prove the following theorem. By this theorem, we can regard \( Q_{\alpha,\infty}^{\beta,-1} \) as derivatives of \( Q_{\alpha}^{\beta} \).

**Theorem 4.4.** For \( \alpha > 0 \) and \( \max\{\alpha, 1/2\} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \), \( Q_{\alpha,\infty}^{\beta,-1} = \nabla \cdot (Q_{\alpha}^{\beta})^n \), where a tempered distribution \( f \in \mathbb{R}^n \) belongs to \( \nabla \cdot (Q_{\alpha}^{\beta})^n \) if and only if there are \( f_j \in Q_{\alpha}^{\beta} \) such that \( f = \sum_{j=1}^{\infty} \partial_j f_j \).

**Proof.** For any \( f \in \nabla \cdot (Q_{\alpha}^{\beta})^n \), there exist \( f_1, f_2, \ldots, f_n \in Q_{\alpha}^{\beta} \) such that \( f = \sum_{j=1}^{\infty} \partial_j f_j \). We have
\[ \| f \|_{Q_{\alpha,\infty}^{\beta,-1}} \leq \sum_{j=1}^{\infty} \| \partial_j f_j \|_{Q_{\alpha,\infty}^{\beta,-1}} \leq \sum_{j=1}^{\infty} \| f_j \|_{Q_{\alpha}^{\beta}}. \]

On the other hand, if \( f \in Q_{\alpha,\infty}^{\beta,-1} \) and \( f_{j,k} = \partial_j \partial_k (-\Delta)^{-1} f \), then \( f_{j,k} \in Q_{\alpha,\infty}^{\beta,-1} \) according to Lemma 4.3. Thus we have \( f_k = -\partial_k (-\Delta)^{-1} f \in Q_{\alpha}^{\beta} \) and
\[ \left( \sum_{k=1}^{\infty} \partial_k f_k \right)(\xi) = -\sum_{k=1}^{\infty} i\xi_k \hat{f}_k(\xi) = -\sum_{k=1}^{\infty} i\xi_k \times i\xi_k |\xi|^{-2} \hat{f}(\xi) = \hat{f}(\xi). \]
\( \square \)
Remark 4.5. \( Q^{\beta,-1}_{\alpha}(\mathbb{R}^n) \) is critical for equations (1.1) since \( Q^{\beta}_{\alpha}(\mathbb{R}^n) \) is the derivative space of \( Q^{\beta}_{\alpha}(\mathbb{R}^n) \) and \( Q^{\beta}(\mathbb{R}^n) \) is invariant under the scaling \( f(x) \rightarrow \lambda^{2\beta - 2} f(\lambda x) \).

In the following theorem we apply the arguments in the proof of the “minimality of \( \dot{B}^{0}_{1,1} \)” used by Frazier-Jaweth-Weiss in [FJW] to prove that \( \dot{B}^{1-2\beta}_{\infty,\infty} \) contains all critical spaces for equations (1.1). The special case \( \beta = 1 \) of this theorem was proved by Cannone in [C2].

Theorem 4.6. If a translation invariant Banach space of tempered distributions \( X \) is a critical space of the generalized Navier-Stokes equations (1.1). Then \( X \) is continuously embedded in the Besov space \( \dot{B}^{1-2\beta}_{\infty,\infty} \).

Proof. It follows from the assumption that \( X \hookrightarrow \mathcal{S}' \) and for any \( f \in X \)

\[
(4.1) \quad \| f(\cdot) \|_X = \| \lambda^{2\beta-1} f(\lambda \cdot - x_0) \|_X, \lambda > 0, x_0 \in \mathbb{R}^n.
\]

\( X \hookrightarrow \mathcal{S}' \) implies that there exists a constant \( C \) such that

\[
| \langle K^{2\beta}_{1}, f \rangle | \leq C \| f \|_X, \forall f \in X.
\]

According to the transformation invariant of \( X \), we have

\[
\| e^{-(\Delta)^\beta} f \|_{L^\infty} = \| K^{2\beta} f \|_{L^\infty} \leq C \| f \|_X \quad \text{for} \forall f \in X.
\]

Using the fact \( \hat{f}(\lambda x)(\xi) = \lambda^{-n} \hat{f}(\xi/\lambda) \), the definition of \( e^{-(\Delta)^\beta} f(x) \) and the scaling property (1.1), we obtain that

\[
\lambda^{2\beta-1} \| e^{-(\Delta)^\beta} f \|_{L^\infty} \leq C \| f \|_X.
\]

It follows from Miao-Yuan-Zhang [MYZ Proposition 2.1] that for \( s < 0, f \in \dot{B}^{s}_{\infty,\infty} \) if and only if

\[
\sup_{r>0} r^{-s} \| e^{-r(\Delta)^\beta} f \|_{L^\infty} < \infty.
\]

Thus \( X \hookrightarrow \dot{B}^{1-2\beta}_{\infty,\infty} \).

\[\square\]

Theorem 4.7. Let \( \alpha > 0 \) and \( \max\{\alpha, \frac{1}{2}\} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \). If \( 1 \leq q \leq \infty, 2 < p < \infty \) and \( \alpha + \beta < 1 + \frac{\beta}{p} < 2\beta \), then \( \dot{B}^{1-\frac{\beta}{p} - 2\beta}_{p,q} \) and \( \dot{B}^{1+\frac{\beta}{p} - 2\beta}_{2,q} \) are continuously embedded in \( Q^{\beta,-1}_{\alpha+1,\infty} \).

Proof. We first prove \( \dot{B}^{1-\frac{\beta}{p} - 2\beta}_{p,q} \hookrightarrow Q^{\beta,-1}_{\alpha+1,\infty} \). Since \( \dot{B}^{1+\frac{\beta}{p} - 2\beta}_{p,q} \subset \dot{B}^{1+\frac{\beta}{p} - 2\beta}_{2,q} \). Assume that \( q = \infty \), it follows form 1 + \( \frac{\beta}{p} \) - 2\( \beta \) < 0 and Proposition 2.1 of [MYZ] that for any \( f \in \dot{B}^{1+\frac{\beta}{p} - 2\beta}_{2,q} \),

\[
\sup_{r>0} r^{-1+\frac{\beta}{p} - 2\beta} \| e^{-r(\Delta)^\beta} f \|_{L^p} < \infty.
\]
Then we have

\[
\int_0^1 \int_{|y-x|<\tau} |e^{-t(-\Delta)\beta} f(y)|^2 t^{-\alpha/\beta} dy dt
\]

\[
\lesssim r^{n(p-2)/p} \int_0^1 \|e^{-t(-\Delta)\beta} f\|_{L_p}^2 t^{-\alpha/\beta} dt
\]

\[
\lesssim r^{n(p-2)/p} \int_0^1 \left( \sup_{t>0} t^{-(1+\frac{n}{p}-2\beta)/2\beta} \|e^{-t(-\Delta)\beta} f\|_{L_p} \right)^2 t^{(1+\frac{n}{p}-2\beta)/\beta} t^{-\alpha/\beta} dt
\]

\[
\lesssim r^{n(p-2)/p} \int_0^1 t^{(1+\frac{n}{p}-2\beta)/\beta} t^{-\alpha/\beta} dt
\]

\[
\lesssim r^{-n(\alpha-1-\beta)}.
\]

Thus \( f \in Q^\beta_{\alpha,\infty} \). Now we prove \( B_2^{1+\frac{n}{p}-2\beta} \hookrightarrow Q^\beta_{\alpha,\infty} \). Since \( 0 < \alpha < \beta \) and \( 1/2 < \beta < 1 \), we can find \( p \in (2, \infty) \) large enough such that \( \alpha + \beta < 1 + \frac{n}{p} < 2\beta \) and \( 1 + \frac{n}{p} - 2\beta = 1 + \frac{n}{p} - 2\beta + n \left( \frac{1}{2} - \frac{1}{p} \right) \). Then (ii) of Theorem 2.11 implies

\( B_2^{1+\frac{n}{p}-2\beta} \hookrightarrow B_p,q \hookrightarrow Q^\beta_{\alpha,\infty} \).

4.2. Several Technical Lemmas. We prove several technical lemmas used in the proof of our well-posedness result.

**Lemma 4.8.** Given \( \alpha \in (0, 1) \). For a fixed \( T \in (0, \infty) \) and a function \( f(\cdot, \cdot) \) on \( \mathbb{R}^{1+n}_+ \), let \( A(t) = \int_0^t e^{-(t-s)(-\Delta)\beta} (\Delta) f(s, x) ds \). Then

\[
\int_0^T \| A(t, \cdot) \|^2_{L^2} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^T \| f(t, \cdot) \|^2_{L^2} \frac{dt}{t^{\alpha/\beta}}.
\]

**Proof.** According to the definition of \( e^{-(t-s)(-\Delta)\beta} \), by Fubini’s and Plancherel’s theorem, we have

\[
I_A = \int_0^\infty \| A(t, \cdot) \|^2_{L^2} \frac{dt}{t^{\alpha/\beta}}
\]

\[
= \int_0^\infty \left( \int_0^t \left| \xi \right|^{2\beta} e^{-(t-s)(-\Delta)\beta} \hat{f}(s, \xi) \hat{\xi} \right)^2 \frac{dt}{t^{\alpha/\beta}}
\]

\[
\lesssim \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \int_0^t \frac{|\xi|^{2\beta}}{\exp (t-s)|\xi|^{2\beta}} |\hat{f}(s, \xi)|^2 ds \right)^2 \frac{dt}{t^{\alpha/\beta}} \right) \hat{\xi} \frac{d\xi}{\tilde{\xi}}
\]

\[
\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^t 1_{0 \leq s \leq t} \frac{|\xi|^{2\beta}}{\exp (t-s)|\xi|^{2\beta}} |\hat{f}(s, \xi)|^2 ds \right)^2 \frac{dt}{t^{\alpha/\beta}} \right) \hat{\xi} \frac{d\xi}{\tilde{\xi}}
\]

\[
\lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^t 1_{0 \leq s \leq t} \frac{|\xi|^{2\beta}}{e^{(t-s)|\xi|^{2\beta}}} ds \right) \left( \int_0^t \frac{|\xi|^{2\beta}}{e^{(t-s)|\xi|^{2\beta}}} |\hat{f}(s, \xi)|^2 ds \right) \frac{dt}{t^{\alpha/\beta}} \right) \hat{\xi} \frac{d\xi}{\tilde{\xi}}.
\]
Since \( \int_0^t |\xi|^{2\beta} e^{-(t-s)|\xi|^{2\beta}} ds \leq e^{-t|\xi|^{2\beta}} (e^{|\xi|^{2\beta}} - 1) \leq 1 \), we have

\[
I_A \lesssim \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_0^t |\xi|^{2\beta} e^{-(t-s)|\xi|^{2\beta}} |\widehat{f(s, \xi)}|^2 ds \right) dt \right)^{\alpha/\beta} d\xi
\]

Using the estimates from Lemma 4.9.

\[
\|f(t, \cdot)(x)\|_{L^2}^{\alpha/\beta} \geq R(t) \lesssim \int_0^\infty \|f(t, \cdot)(x)\|_{L^2}^{\alpha/\beta} dt.
\]

Lemma 4.9. For \( \beta \in (1/2, 1) \) and \( N(t, x) \) defined on \((0, 1) \times \mathbb{R}^n\), let \( A(N) \) be the quantity

\[
A(\alpha, \beta, N) = \sup_{x \in \mathbb{R}^n, t \in (0, 1)} t^{2\alpha-n+2\beta-2} \int_0^t \int_{|y-x|<r} |f(t, x)| dx ds.
\]

Then for each \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) there exists a constant \( b(k) \) such that the following inequality holds:

\[(4.3) \int_0^1 \left\| t^{\frac{k}{2}} (\Delta)^{\frac{k-1}{2}} e^{-\frac{t}{2} (\Delta)^{\frac{k-1}{2}}} \int_0^t N(s, \cdot) ds \right\|_{L^2}^{\alpha/\beta} dt \leq b(k) A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s, x)| dx ds.
\]

Proof. Using the inner-product \((\cdot, \cdot)\) in \( L^2 \) with respect to the spatial variable \( x \in \mathbb{R}^n \), we obtain

\[
I = \int_0^1 \left\| t^{\frac{k}{2}} (\Delta)^{\frac{k-1}{2}} e^{-\frac{t}{2} (\Delta)^{\frac{k-1}{2}}} \int_0^t N(s, \cdot) ds \right\|_{L^2}^{\alpha/\beta} dt
\]

\[
= \int_0^1 \left( \int_0^t t^{\frac{k}{2}} (\Delta)^{\frac{k-1}{2}} e^{-\frac{t}{2} (\Delta)^{\frac{k-1}{2}}} N(s, \cdot) ds \right) dt
\]

\[
\lesssim 2Re \left( \int_0^1 \int_{0<h<s<1} (N(s, \cdot), (\Delta)^{\frac{k-1}{2}} t^{-(\Delta)^{\frac{k-1}{2}}} e^{-t(\Delta)^{\frac{k-1}{2}}} N(h, \cdot) dh) L_2 L^2 dt \right)
\]

\[
\lesssim 2Re \left( \int_0^1 \int_{0<h<s<1} (N(s, \cdot), (\Delta)^{\frac{k-1}{2}} t^{-(\Delta)^{\frac{k-1}{2}}} e^{-t(\Delta)^{\frac{k-1}{2}}} N(h, \cdot) dh) L_2 L^2 dt \right)
\]

where \( L_k(t) = \sum_{m=0}^k b_m(k) t^{m(\Delta)^{\frac{k-1}{2}}} e^{-t(\Delta)^{\frac{k-1}{2}}} \).

We consider the \( \nu \)-th derivative of the kernel \( K_{i\beta}^x(x) \) and let

\[
(K_{i\beta})^\nu(x) = (-\Delta)^{\nu/2} K_{i\beta}^x(x) \quad \text{and} \quad (K_{i\beta}^x)^\nu(x) = (-\Delta)^{\nu/2} K_{i\beta}^x(x).
\]

Using the estimates

\[
(K_{i\beta})^\nu(x) \lesssim \frac{1}{(1 + |x|)^{\alpha + \nu}} \quad \text{and} \quad (K_{i\beta}^x)^\nu(x) = t^\nu t^{-\frac{\nu}{2}} (K_{1\beta}^x)^\nu (\frac{x}{t^{1/2\beta}}).
\]
(see Miao-Yuan-Zhang [MYZ, Lemma 2.2 and Remark 2.1]), we get the kernel of the above operator satisfies the estimate:

\[ (-\Delta)^{1-\beta} L_k(t)(x,y) \lesssim \sum_{m=0}^{k} t^{-\frac{n+2-2\beta}{2\beta}} \frac{b_m(k)}{(1+t^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}}, \]

we have

\[
\left| \int_0^s (-\Delta)^{1-\beta} L_k(s) N(h,x) dh \right| \lesssim s^{-\frac{n+2-2\beta}{2\beta}} \int_{\mathbb{R}^n} \sum_{m=0}^{k} b_m(k) \frac{|N(h,y)| dy dh}{(1+s^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}}.
\]

\[
\lesssim s^{-\frac{n+2-2\beta}{2\beta}} \sum_{m=0}^{k} b_m(k) \int_{\mathbb{R}^n} \int_0^s \frac{|N(h,y)| dy dh}{(1+s^{-1/2\beta}|x-y|)^{n+2m\beta+2-2\beta}}
\]

\[
\lesssim b(k) \sup_{x \in \mathbb{R}^n} \sup_{0 < t < 1} t^{-\frac{n+2-2\beta}{2\beta}} \int_0^t \int_{|x-y| < t^{1/2\beta}} |N(h,y)| dy dh
\]

\[
\lesssim b(k) \sup_{x \in \mathbb{R}^n} \sup_{0 < \rho < 1} \rho^{2\alpha-n+2\beta-2} \int_0^1 \int_{|x-y| < \rho} |N(h,y)| dy dh.
\]

Hence we can get

\[
I \lesssim b(k) \left( \int_0^1 \int_{\mathbb{R}^n} |N(s,x)| ds dx \right)^{\frac{1}{\alpha/\beta}} A(\alpha, \beta, N).
\]

This completes the proof. \(\square\)

**Remark 4.10.** Similarly when \(k = 0\), we can prove the following inequality:

\[
\int_0^1 \left\| (-\Delta)^{\frac{1}{2}} e^{-t(-\Delta)^{\frac{1}{2}}} \int_0^t N(s,\cdot) ds \right\|_{L^p}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s,x)| ds dx \frac{ds dx}{s^{\alpha/\beta}}.
\]

**Lemma 4.11.** For \(1 \leq j, k \leq n\) and \(t > 0\), the operator \(Q^\beta_{j,k,t} = \frac{1}{2\pi} \partial_j \partial_k e^{-t(-\Delta)^{\beta}}\) is a convolution operator with the kernel \(K^\beta_{j,k}(x) = \frac{1}{t^{n+\beta}} K^\beta_{j,k}(\frac{x}{t^{\frac{n+\beta}{\beta}}})\) for a smooth function \(K^\beta_{j,k}\) such that for all \(\alpha \in \mathbb{N}^n\)

\[
(1+|x|)^{n+|\alpha|} \partial^\alpha K^\beta_{j,k} \in L^\infty(\mathbb{R}^n).
\]

**Proof.** Since \(K^\beta_{j,k}(\xi) = \frac{\xi_j \xi_k}{|\xi|^{n+2\beta}} e^{-|\xi|^{2\beta}}\), we have \(\partial^\alpha K^\beta_{j,k}(\xi) \lesssim |\xi|^{\alpha} \frac{\xi_j \xi_k}{|\xi|^{n+2\beta}} e^{-|\xi|^{2\beta}}\) and \(\int |\partial^\alpha K^\beta_{j,k}(\xi)| d\xi < \infty\). Thus \(\partial^\alpha K^\beta_{j,k}(x) \in L^\infty(\mathbb{R}^n)\).

For \(|x| \leq 1\), we have

\[
|(1+|x|)^{n+|\alpha|} \partial^\alpha K^\beta_{j,k}(x)| \lesssim |\partial^\alpha K^\beta_{j,k}(x)| \lesssim 1.
\]

For \(|x| > 1\), we write \(K^\beta_{j,k} = (I - S_0) K^\beta_{j,k} + \sum_{l<0} \Delta_l K^\beta_{j,k}\) where \((I - S_0) K^\beta_{j,k} \in S\) and \(\Delta_l K^\beta_{j,k} = 2^l \omega^\beta_{j,k,l}(2^l x)\) with \(\omega^\beta_{j,k,l}(\xi) = \psi(\xi) \frac{\xi_j \xi_k}{|\xi|^{n+2\beta}} e^{-|\xi|^{2\beta}} \in L^1\). Then the set
\{\omega_{j,k,l}^\beta : l < 0\} is bounded in \(S\) and there exists an uniform constant \(C_N\) such that
\[
(1 + 2^l|x|)^{4l(n+|\alpha|)}|\partial^\alpha \triangle_j K_{j,k}^\beta (x)| \leq C_N.
\]
Thus
\[
|\partial^\alpha S_0 K_{j,k}(x)| \lesssim \sum_{2^l|x| \leq 1} 2^l(2l(\alpha+1)) + \sum_{2^l|x| > 1} 2^l(2l(\alpha+1)-4) |x|^{-4} \lesssim |x|^{-n-1}.
\]

\[\square\]

4.3. Well-Posedness. In this subsection, we establish the well-posedness result for the solutions to the equations (1.1). Throughout this subsection, we always assume \(\beta \in (\frac{1}{2}, 1).\) In fact our results can be also applied to the case \(\beta = 1,\) that is, the classical Navier-Stokes equations. Hence our results can be regarded as a generalization of the result of Koch-Tataru \([KT]\) when \(\alpha = 0, \beta = 1\) and that of Xiao \([X]\) when \(\alpha \in (0, 1), \beta = 1.\)

**Definition 4.12.** Let \(\alpha > 0\) and \(\max\{1/2, \alpha\} < \beta < 1\) with \(\alpha + \beta - 1 \geq 0.\)

(i) A tempered distribution \(f\) on \(R^n\) belongs to \(Q^{\beta-1}_{\alpha,T}(R^n)\) provided
\[
\|f\|_{Q^{\beta-1}_{\alpha,T}(R^n)} = \sup_{x \in R^n, r \in (0, T)} \left(2^{\alpha-n+2\beta-2} \int_0^T \int_{|y-x| < r} |K_1^\beta \ast f(y)|^{2t^{-\beta}} dy dt \right)^{1/2} < \infty;
\]

(ii) A tempered distribution \(f\) on \(R^n\) belongs to \(VQ^{\beta-1}_{\alpha,T}(R^n)\) provided \(\lim_{T \rightarrow 0} \|f\|_{Q^{\beta-1}_{\alpha,T}(R^n)} = 0;\)

(iii) A function \(g\) on \(R_1^{1+n}\) belongs to the space \(X_{\alpha,T}^\beta(R^n)\) provided
\[
\|g\|_{X_{\alpha,T}^\beta(R^n)} = \sup_{t \in (0, T)} t^{1-\frac{\alpha}{\beta}} \|g(t, \cdot)\|_{L^\infty(R^n)}
\]
\[
+ \sup_{x \in R^n, r \in (0, T)} \left(2^{\alpha-n+2\beta-2} \int_0^T \int_{|y-x| < r} |g(t, y)|^{2t^{-\alpha/\beta}} dy dt \right)^{1/2} < \infty.
\]

**Theorem 4.13.** Let \(n \geq 2, \alpha > 0\) and \(\max\{\alpha, 1/2\} < \beta < 1\) with \(\alpha + \beta - 1 \geq 0.\) Then

(i) The fractional Navier-Stokes system (1.1) has a unique small global mild solution in \((X_{\alpha,\infty}^\beta)^n\) for all initial data \(a\) with \(\nabla \cdot a = 0\) and \(\|a\|_{(Q^{\beta-1}_{\alpha,T})^n}\) being small.

(ii) For any \(T \in (0, \infty)\) there is an \(\varepsilon > 0\) such that the fractional Navier-Stokes system (1.1) has a unique small mild solution in \((X_{\alpha,T}^\beta)^n\) on \((0, T) \times R^n\) when the initial data \(a\) satisfies \(\nabla \cdot a = 0\) and \(\|a\|_{(Q^{\beta-1}_{\alpha,T})^n} \leq \varepsilon.\) In particular for all \(a \in (VQ^{\beta-1}_{\alpha,T})^n\) with \(\nabla \cdot a = 0\) there exists a unique small local mild solution in \((X_{\alpha,T}^\beta)^n\) on \((0, T) \times R^n.\)

**Proof.** By Picard's contraction principle, it suffices to verify the bilinear operator
\[
B(u, v) = \int_0^T e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v) ds
\]
is bounded from \((X_{\alpha,T}^\beta)^n \times (X_{\alpha,T}^\beta)^n\) to \((X_{\alpha,T}^\beta)^n.\)

**Part 1.** \(L^2\) bound. We want to establish that if \(x \in R^n\) and \(r^{2\beta} \in (0, T)\) then
\[
\int_0^{r^{2\beta}} \int_{|y-x| < r} |B(u, v)|^2 dy ds \lesssim \|u\|^2_{(X_{\alpha,T}^\beta)^n} \|v\|^2_{(X_{\alpha,T}^\beta)^n}.
\]
To this aim, define \( 1_{r,x}(y) = 1_{|y-x|<10r}(y) \), i.e., the indicate function on the ball \( \{ y \in \mathbb{R}^n : |y-x|<10r \} \). We divide \( B(u,v) \) into three parts: \( B(u,v) = B_1(u,v) + B_2(u,v) + B_3(u,v) \), where

\[
B_1(u,v) = \int_0^s e^{-(s-h)(-\Delta)^{\beta}/(1-r,x)u \otimes v)dh,
\]
\[
B_2(u,v) = \int_0^s e^{-(s-h)(-\Delta)^{\beta}}(\Delta)^{-1/2}(1-r,x)u \otimes v)dh,
\]
\[
B_3(u,v) = \int_0^s e^{-(s-h)(-\Delta)^{\beta}}(-\Delta)^{1/2}(1-r,x)u \otimes v)dh.
\]

At first, we estimate \( B_2(u,v) \) as

\[
I = \int_0^s \| B_2(u,v) \|_L^2 \frac{dt}{t^{\alpha/\beta}} \\
\lesssim \int_0^s \| \int_0^s e^{-(s-h)(-\Delta)^{\beta}}(\Delta)^{-1/2}(1-r,x)u \otimes v)dh \|_L^2 \frac{dt}{t^{\alpha/\beta}} \\
\lesssim \int_0^s \| \int_0^s e^{-(s-h)(-\Delta)^{\beta}}(-\Delta)^{1/2}(1-r,x)u \otimes v)dh \|_L^2 \frac{dt}{t^{\alpha/\beta}} \\
\lesssim \int_0^s \| \int_0^s e^{-(s-h)(-\Delta)^{\beta}}(-\Delta)^{1/2}(1-r,x)u \otimes v)dh \|_L^2 \frac{dt}{t^{\alpha/\beta}}.
\]

Since \( \sup_{s \in (0,\infty)} s^{1-2\beta}(1-e^{-s^2\beta}) \) is bounded on \( L^2 \) with operator norm \( \lesssim s^{1-\beta} \). Write \( (1-r,x)u(s,x) \otimes v(s,x) = M(s,x) \). Thus, using the Cauchy-Schwarz inequality, we have

\[
I \lesssim \int_0^s s^{2-\frac{\alpha}{\beta}} \| M(s,\cdot) \|_L^2 \frac{ds}{s^{\alpha/\beta}} \\
\lesssim \int_0^s s^{2-\frac{\alpha}{\beta}} \int_{|y-x|<r} |u(s,y)v(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}} \\
\lesssim \left( \sup_{s \in (0,\infty)} s^{1-\frac{1}{\beta}} \| u(s,y) \|_\infty \right) \left( \sup_{s \in (0,\infty)} s^{1-\frac{1}{\beta}} \| v(s,y) \|_\infty \right) \\
\times \left( \int_0^s \int_{|y-x|<r} |u(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}} \right) \left( \int_0^s \int_{|y-x|<r} |v(s,y)|^2 dy \frac{ds}{s^{\alpha/\beta}} \right) \\
\lesssim r^{n-2\alpha-2(\beta-1)} \| u \|_L^2 (X_{\alpha,T})^\alpha \| v \|_L^2 (X_{\alpha,T})^\alpha.
\]
Now by Lemma 4.3 with \( k = 0 \), we estimate the term \( B_3 \) as follows.

\[
\int_0^{r^{2\beta}} \| B_3(u, v) \|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^{r^{2\beta}} \left\| \frac{1}{t^{\alpha/\beta}} \int_0^t M(s, \cdot) \, ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \\
\lesssim r^{n-2\alpha+6\beta-2} \int_0^{r^{2\beta}} \left\| \frac{1}{t^{\alpha/\beta}} \int_0^t M(s, \cdot) \, ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \\
\lesssim \int_0^{r^{2\beta}} \| M(r^{2\beta} s, r) \|_{L^1} \frac{ds}{s^{\alpha/\beta}} C(\alpha, \beta; f) \\
= r^{n-2\alpha+6\beta-2} \times II \times A(\alpha, \beta; M(r^{2\beta} s, ry)).
\]

For \( II \), we have

\[
II = r^{2\alpha-n-2\beta} \int_0^{r^{2\beta}} \int_{|z-x| < r} |M(t, z)| \frac{dzt}{t^{\alpha/\beta}} \\
\lesssim r^{2-4\beta} \| u \|_{(X^\alpha_{\alpha,T})^n} \| v \|_{(X^\beta_{\alpha,T})^n}.
\]

For \( C(\alpha, \beta; M(r^{2\beta} s, ry)) \), we have

\[
C(\alpha, \beta; M(r^{2\beta} s, ry)) \lesssim \rho^{2\alpha-n+2(\beta-1)} \int_0^{r^{2\beta}} \int_{|y-x| < \rho} |M(r^{2\beta} s, ry)| \frac{dys}{s^{\alpha/\beta}} \\
\lesssim \rho^{2\alpha-n+2(\beta-1)} r^{2\alpha-n-2\beta} \int_0^{r^{2\beta}} \int_{|z-x| < r^{\rho}} |M(t, z)| \frac{dzt}{t^{\alpha/\beta}} \\
\lesssim r^{2-4\beta} (r^{\rho})^{2\alpha-n+2(\beta-1)} \int_0^{r^{2\beta}} \int_{|z-x| < r^{\rho}} |M(t, z)| \frac{dzt}{t^{\alpha/\beta}} \\
\lesssim r^{2-4\beta} \| u \|_{(X^\beta_{\alpha,T})^n} \| v \|_{(X^\beta_{\alpha,T})^n}.
\]

Therefore we get

\[
\int_0^{r^{2\beta}} \| B_3(u, v) \|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim r^{n-2\alpha+6\beta-2} r^{2-4\beta} r^{2-4\beta} \| u \|_{(X^\alpha_{\alpha,T})^n} \| v \|_{(X^\beta_{\alpha,T})^n}^2 = r^{n-2\alpha-2\beta+2} \| u \|_{(X^\alpha_{\alpha,T})^n} \| v \|_{(X^\beta_{\alpha,T})^n}^2,
\]

that is,

\[
r^{2\alpha-n+2(\beta-2)} \int_0^{r^{2\beta}} \| B_3(u, v) \|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \lesssim \| u \|_{(X^\alpha_{\alpha,T})^n} \| v \|_{(X^\beta_{\alpha,T})^n}^2.
\]

For the estimate of \( B_1 \), According to Lemma 4.11, we have

\[
e^{-t(-\Delta)^\beta} P \nabla \cdot f(x) = \int \nabla K^\beta_{j,k,t}(x-y)f(y) dy
\]

and

\[
\nabla K^\beta_{j,k,t}(x-y) \lesssim \frac{1}{t^{\beta+\frac{1}{2}}} \left( 1 + t^{-1/2} |x-y| \right)^{n+1} \lesssim \left( t^{1/2} + |x-y| \right)^{n+1}.
\]
Thus
\[
|B_1(u, v)| \leq \left| \int_0^s e^{-(s-h)(-\Delta)_{\beta}^{n}} P \nabla \cdot ((1 - 1_{r,x})u \otimes v) dh \right|
\]
\[
\lesssim \int_0^s \int_{|z-x| \geq 10r} \frac{|u(h, z)||v(h, z)|}{(s-h)^{1/2 \beta} + |z-y|^{n+1}} dz dh.
\]
When \(|z-x| \geq 10r, 0 < s < r^{2\beta}\) and \(|y-x| < r\), we have \(|y-z| \geq |z-x| - |y-x| \geq 9r > 9|y-x|\). Thus \(|x-z| \leq |x-y| + |y-z| \leq \frac{3}{2}|y-z| + |y-z| = \frac{9}{2}|y-z|\). This gives us
\[
|B_1(u, v)| \lesssim \int_0^{r^{2\beta}} \int_{|z-x| \geq 10r} \frac{|u(h, z)||v(h, z)|}{|x-z|^{n+1}} dz dh = I_1 \times I_2.
\]
where
\[
I_1 = \left( \int_0^{r^{2\beta}} \int_{|z-x| \geq 10r} \frac{|u(h, z)|^2}{|x-z|^{n+1}} dz dh \right)^{1/2}
\]
\[
\lesssim \left( \sum_{j=3}^{\infty} \int_0^{r^{2\beta}} \int_{2^{j}r \leq |z-x| \leq 2^{j+1}r} \frac{|u(h, z)|^2}{(2^{j}r)^{n+1}} dz dh \right)^{1/2}
\]
\[
\lesssim \left( \sum_{j=3}^{\infty} \frac{1}{(2^{j}r)^{n+1}} \left( r^{2\beta} \right)^{n/\beta} (2^{j}r)^{2\beta - 2} (2^{j}r)^{2\beta - 2} \right)^{1/2}
\]
\[
\lesssim \left( \sum_{j=3}^{\infty} (2^{j}r)^{2\alpha - n} (2^{j}r)^{-1} (2^{j}r)^{2\beta - 2} (2^{j}r)^{2\beta - 2} \right) \int_0^{r^{2\beta}} \int_{|z-x| \leq 2^{j+1}r} |u(h, z)|^2 dz dh \right)^{1/2}
\]
\[
\lesssim \left( \frac{1}{r^{2\beta - 1}} \right)^{1/2} \|u\|_{X_0^{\beta}}^n.
\]
Similarly, we obtain \(I_2 \lesssim \left( \frac{1}{r^{2\beta - 1}} \right)^{1/2} \|v\|_{X_0^{\beta}}^n\). Thus \(|B_1(u, v)| \lesssim \frac{1}{r^{2\beta - 1}} \|u\|_{X_0^{\beta}}^n \|v\|_{X_0^{\beta}}^n\).
When \(0 < \alpha < \beta\), we have
\[
\int_0^{r^{2\beta}} \int_{|y-x| < r} |B_1(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \frac{1}{r^{2\beta - 1}} \int_0^{r^{2\beta}} \frac{dt}{t^{\alpha/\beta}} \|u\|_{X_0^{\beta}}^2 \|v\|_{X_0^{\beta}}^2
\]
\[
\lesssim \frac{r^{n - 2\alpha - 2\beta + 2}}{r^{\alpha/\beta}} \|u\|_{X_0^{\beta}}^2 \|v\|_{X_0^{\beta}}^2.
\]
This implies that
\[
r^{2\alpha - n + 2(\beta - 1)} \int_0^{r^{2\beta}} \int_{|y-x| < r} |B_1(u, v)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_0^{\beta}}^2 \|v\|_{X_0^{\beta}}^2.
\]
Part 2. \(L^\infty\) bound. The aim of this part is to prove
\[
\|B(u, v)\|_{L^\infty} \lesssim t^{\frac{n-1}{2}} \|u\|_{X_0^{\beta}} \|v\|_{X_0^{\beta}}, \forall t \in (0, T).
\]
If \(\frac{s}{2} \leq s < t\) then
\[
\|e^{-(t-s)(-\Delta)^{\beta}} P \nabla \cdot (u \otimes v)\|_{L^\infty} \lesssim \|u\|_{L^\infty} \|v\|_{L^\infty} \lesssim (t-s)^{\frac{n-1}{2}} s^\frac{\beta}{2} \|u\|_{X_0^{\beta}} \|v\|_{X_0^{\beta}}.
\]
If $0 < s < \frac{t}{4}$ then $t - s \approx t$ and so
\[
|e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v)| \lesssim \int_{\mathbb{R}^n} \left( \frac{|u(s, y)||v(s, y)|}{(t-s)\frac{1}{s^\beta} + |x-y|} \right)^{n+1} dy
\]
\[
\lesssim \int_{\mathbb{R}^n} \left( \frac{|u(s, y)||v(s, y)|}{t\frac{1}{s^\beta} + |x-y|} \right)^{n+1} dy
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^n} \int_{x-y \in t\frac{1}{s^\beta} (k+[0,1]^n)} \frac{|u(s, y)||v(s, y)|}{(t\frac{1}{s^\beta} (1+|k|))(n+1)^{n+1}} dy ds.
\]
This gives us
\[
|B(u, v)| \lesssim \int_0^{t/2} |e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v)| ds + \int_{t/2}^t |e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v)| ds
\]
\[
\lesssim \sum_{k \in \mathbb{Z}^n} \int_0^{t/2} \int_{x-y \in t\frac{1}{s^\beta} (k+[0,1]^n)} |u(s, y)||v(s, y)| dy ds
\]
\[
+ \int_{t/2}^t (t-s)^{-\frac{1}{s^\beta}} s^{\frac{1}{s^\beta} - 2} ds \|u\|_{(X^\alpha_{\cdot, \cdot, \cdot})^n} \|v\|_{(X^\beta_{\cdot, \cdot, \cdot})^n}
\]
\[
:= I_3 + I_4.
\]
Here,
\[
I_4 \lesssim \int_{t/2}^t (t-s)^{-\frac{1}{s^\beta}} s^{\frac{1}{s^\beta} - 2} ds \|u\|_{(X^\alpha_{\cdot, \cdot, \cdot})^n} \|v\|_{(X^\beta_{\cdot, \cdot, \cdot})^n}
\]
\[
\lesssim t^{\frac{1}{s^\beta} - 2} t^{\frac{1}{s^\beta} - \frac{1}{2}} \|u\|_{(X^\alpha_{\cdot, \cdot, \cdot})^n} \|v\|_{(X^\beta_{\cdot, \cdot, \cdot})^n}
\]
\[
\lesssim t^{\frac{1}{s^\beta} - 1} \|u\|_{(X^\alpha_{\cdot, \cdot, \cdot})^n} \|v\|_{(X^\beta_{\cdot, \cdot, \cdot})^n}.
\]
On the other hand, we have
\[
I_3 \lesssim \sum_{k \in \mathbb{Z}^n} \left( \int_0^{t/2} \int_{x-y \leq t\frac{1}{s^\beta}} |u(s, y)|^2 dy ds \right)^{1/2}
\]
\[
\times \left( \int_0^{t/2} \int_{x-y \leq t\frac{1}{s^\beta}} |v(s, y)|^2 dy ds \right)^{1/2}
\]
\[
:= \sum_{k \in \mathbb{Z}^n} (t\frac{1}{s^\beta} (1+|k|))^{-(n+1)} I_{3,1} \times I_{3,2}.
\]
Here,
\[
I_{3,1} = \left( \int_0^{t/2} \int_{x-y \leq t\frac{1}{s^\beta}} |u(s, y)|^2 dy ds \right)^{1/2}
\]
\[
= \left( t\frac{1}{s^\beta} (n-2\beta + 2) t\frac{1}{s^\beta} (2\alpha - n + 2\beta - 2) \int_0^{t/2} \int_{x-y \leq t\frac{1}{s^\beta}} |u(s, y)|^2 dy ds \right)^{1/2}
\]
\[
\lesssim t^{n-2\beta + 2} \|u\|_{(X^\alpha_{\cdot, \cdot, \cdot})^n}.
\]
Similarly, we get $I_{3,2} \lesssim t^{\frac{1}{3}(n-2\beta+2)}\|v\|_{(X_{\alpha,T}^\beta)^n}$. These estimates about $I_{3,1}$ and $I_{3,2}$ imply that

$$I_3 \lesssim t^{\frac{1}{3}(n+1)}t^{\frac{1}{3}(n-2\beta+2)}\|u\|_{(X_{\alpha,T}^\beta)^n}\|v\|_{(X_{\alpha,T}^\beta)^n} \lesssim t^{\frac{1}{3}-1}\|u\|_{(X_{\alpha,T}^\beta)^n}\|v\|_{(X_{\alpha,T}^\beta)^n}.$$

Thus $t^{1-\frac{1}{3}}\|B(u, v)\|_{L^\infty} \lesssim \|u\|_{(X_{0,T}^\beta)^n}\|v\|_{(X_{0,T}^\beta)^n}$.

Therefore, we establish the boundedness of $B(u, v)$ and finish the proof of (i) and (ii) by taking $T = \infty$ and $T \in (0, \infty)$, respectively.

\[\square\]

5. Regularity of Generalized Navier-Stokes equations

In this section, we study the regularity of the solutions to the equations (1.1) with $\beta \in (1/2, 1)$. For $\beta = 1$, that is, the classical Navier-Stokes equations, the regularity has been studied by several authors. In [GPS], Germain-Pavlović-Staffilani analyzed the regularity properties of the solutions constructed by Koch-Tataru. More precisely, they showed that under certain smallness condition of initial data, the solution $u$ to the classical Navier-Stokes equations constructed in [KT] satisfies the following regularity property:

$$t^{\frac{1}{2}}\nabla^k u \in X^0, \text{ for all } k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

where $X^0$ denotes the space where the solution constructed by Koch and Tataru belongs.

In this section, we establish a similar result for the solutions of the equations (1.1) evolving initial data in $Q_{\alpha,\infty}^{\beta,-1}$ with $\beta \in (1/2, 1]$. In fact we get the solution $u$ to the equations (1.1) satisfies:

$$t^{\frac{1}{2}}\nabla^k u \in X_{\alpha}^{\beta,0} \text{ for all } k$$

where $X_{\alpha}^{\beta,0}$ is the space $X_{\alpha,\infty}^{\beta}$ constructed in (iii) of Definition 4.12 for $\beta \in (1/2, 1)$ and $X_{\alpha,\infty}^{1}$ in Xiao [X] for $\beta = 1$. For convenience of the study, we introduce a class of spaces $X_{\alpha}^{\beta,k}$ as follows.

**Definition 5.1.** For a nonnegative integer $k$ and $\beta \in (1/2, 1]$, we introduce the space $X_{\alpha}^{\beta,k}$ which is equipped with the following norm:

$$\|u\|_{X_{\alpha}^{\beta,k}} = \|u\|_{X_{\alpha,\infty}^{\beta,k}} + \|u\|_{X_{\alpha,C}^{\beta,k}}$$

where

$$\|u\|_{X_{\alpha,\infty}^{\beta,k}} = \sup_{\alpha_1 + \cdots + \alpha_n = k} \sup_{t} t^{\frac{2\alpha - n + 2\beta - 2}{2}} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(\cdot, t)\|_{L^\infty},$$

$$\|u\|_{X_{\alpha,C}^{\beta,k}} = \sup_{\alpha_1 + \cdots + \alpha_n = k} \sup_{x_0, \tau} \left( \int_{\tau^{2\beta}}^{\tau^{2\beta}} \int_{|y-x_0|<r} \|t^{\frac{1}{3}} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(t, y)\|_{L^\infty} dy dt \right)^{1/2}.$$

In the following, we will denote $\nabla^k u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u$ with $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $k = \alpha_1 + \cdots + \alpha_n$. 
5.1. **Several Technical Lemmas.** Before stating the main result of this section, we prove several preliminary lemmas associated with the fractional heat semigroup $e^{-t(-\Delta)^{\beta}}$. Recall that $e^{-t(-\Delta)^{\beta}} f(x) = K_t^\beta * f(x)$ where $K_t^\beta$ is defined by $(K_t^\beta)(\xi) = e^{-|\xi|^{2\beta}}$ and $P$ is the Helmholtz-Weyl projection.

**Lemma 5.2.** Let $\beta \in (1/2, 1)$. There exists a constant $C > 0$ depending only on $\alpha$ such that
\[
|\partial_x^k P\nabla K_t^\beta(x)| \leq C^k k^{k/2} t^{-k/2} (k^{-\beta / 2} t^{1/2 beta} + |x|)^{-n-1}
\]
for all $t > 0, x \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

**Proof.** By a dilation argument, we have
\[
\partial_x^k P\nabla K_t^\beta(x) = t^{-\frac{k}{2\beta}} \partial_x^k P\nabla K_t^\beta(x/t^{1/\beta}).
\]
If we could prove $|\partial_x^k P\nabla K_t^\beta(x)| \leq C^k k^{k/2} (k^{-\beta / 2} t^{1/2 beta} + |x|)^{-n-1}$, then we have
\[
|\partial_x^k P\nabla K_t^\beta(x)| \leq t^{-\frac{k}{2\beta}} \partial_x^k P\nabla K_t^\beta(x/t^{1/\beta}) \leq C^k k^{k/2} t^{1/2 beta} (k^{-\beta / 2} t^{1/2 beta} + |x|)^{-n-1}.
\]
Hence we obtain the desired.

By the semigroup property, it is easy to see that $\partial_x^k P\nabla K_t^\beta = P\nabla K_t^{\beta} \ast \partial_x^k K_t^{\beta}$.

So we need to prove the following two estimates:
\[
(5.1) \quad |P\nabla K_t^{\beta}(x)| \leq C(1 + |x|)^{-n-1}
\]
\[
(5.2) \quad |\partial_x^k K_t^{\beta}(x)| \leq C^{k-1} k^{k+1} (k^{-\beta / 2} t^{1/2 beta} + |x|)^{-n-1}.
\]

For (5.1). Taking $\alpha = 1$ in the Lemma 4.11 we have
\[
(1 + |x|)^{n+1} |P\nabla K_t^{\beta}(x)| \leq C,
\]
that is, (5.1) is obvious.

For (5.2), we claim that $|\partial_x K_t^{\beta}(x)| \leq C(1 + |x|)^{-n-1}$. In fact when $|x| < 1$,
\[
(1 + |x|)^{n+1} |\partial_x K_t^{\beta}(x)| \leq 2^{n+1} \int_{\mathbb{R}^n} |i\xi| e^{-|\xi|^{2\beta} / 2} d\xi \lesssim C.
\]

When $|x| > 1$, we define the operator
\[
L(x, D) = \frac{x \cdot \nabla \xi}{i|x|^2}, \quad \text{that is} \quad L(x, D)e^{ix \cdot \xi} = e^{ix \cdot \xi}
\]
and choose a $C^\infty_c(\mathbb{R}^n)$–function $\rho(x)$ satisfying:
\[
\rho(\xi) = \begin{cases}
1, & |\xi| \leq 1, \\
0, & |\xi| > 2,
\end{cases}
\]
we have
\[
|\partial_x K_t^{\beta}(x)| \leq \left| \int_{\mathbb{R}^n} \rho \left( \frac{\xi}{\beta} \right) i\xi e^{-|\xi|^{2\beta} / 2} e^{ix \cdot \xi} d\xi \right| \\
+ \left| \int_{\mathbb{R}^n} [1 - \rho \left( \frac{\xi}{\beta} \right)] i\xi e^{-|\xi|^{2\beta} / 2} e^{ix \cdot \xi} d\xi \right| \\
:= I_3 + I_4.
\]
For $I_3$, we have

$$I_3 \lesssim \int_{\mathbb{R}^n} \rho \left( \frac{\xi}{\delta} \right) |\xi|e^{-|\xi|^{2\beta}/2} d\xi \lesssim \int_{|\xi| < 2\delta} \delta d\xi \lesssim \delta^{n+1}.$$ 

For $I_4$, using the integration by parts and $L^* = \frac{-x\nabla}{|x|^2}$, we have

$$I_4 = \left| \int_{\mathbb{R}^n} (L^*)^N \left( 1 - \rho \left( \frac{\xi}{\delta} \right) i\xi e^{-|\xi|^{2\beta}/2} \right) e^{ix\cdot\xi} d\xi \right|$$

$$\lesssim C_N |x|^{-N} \int_{\mathbb{R}^n} \sum_{k=0}^{N} C_k^{N} \nabla^k \left[ 1 - \rho \left( \frac{\xi}{\delta} \right) \right] \nabla^{N-k}(i\xi e^{-|x|^{2\beta}/2}) d\xi$$

$$\lesssim C_N |x|^{-N} \int_{|\xi| > \delta} |\xi|^{2\beta-2N+1} e^{-|\xi|^{2\beta}/2} d\xi$$

$$+ C_N |x|^{-N} \int_{\delta < |\xi| < 2\delta} |\xi|^{2\beta-2N+1} e^{-|\xi|^{2\beta}/2} d\xi$$

$$\lesssim C_N |x|^{-N} \int_{|\xi| > \delta} |\xi|^{1-N} d\xi + C_N |x|^{-N} \int_{\delta < |\xi| < 2\delta} \delta^{-k} |\xi|^{1-N+k} d\xi$$

$$\lesssim C_N |x|^{-N} \delta^{n+1} - N.$$ 

So we get, taking $\delta = |x|^{-1}$,

$$|\partial_t K_{1/2}^\beta(x)| \lesssim \delta^{n+1} + C_N |x|^{-N} \delta^{n+1} - N \lesssim C_N |x|^{-(n+1)} \lesssim C_N (1 + |x|)^{-(n+1)}.$$ 

Then we have

$$|\partial_t K_{1/2}^\beta(x)| \lesssim k \frac{\delta}{|x|} k \frac{\delta}{|x|} |\partial_t K_{1/2}^\beta(k^{1/2}x)| \lesssim C(k^{-1} + |x|)^{-n-1}.$$ 

Because the following integral inequality (see [MS]):

$$\int_{\mathbb{R}^n} (a + |x-y|)^{-n-1}(b + |y|)^{-n-1} dy \leq ca^{-1}(a + |x|)^{-n-1} \text{ for } 0 < a \leq b,$$

we have

$$|\partial_{i,j}^2 K_{1/2}^\beta(x)| = \left| \int_{\mathbb{R}^n} \partial_i K_{1/2}^\beta(x) \partial_k K_{1/2}^\beta(y) dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} (k^{-1/\beta} + |x-y|)^{-n-1}(k^{-1/\beta} + |y|)^{-n-1} dy$$

$$\lesssim k \frac{\delta}{|x|} (k^{-1/\beta} + |x|)^{-n-1}.$$ 

Operating the process above $k-1$ times, we get $|\partial_x^k P \nabla K_{1}^\beta(x)| \lesssim k \frac{\delta}{|x|} (k^{-1/\beta} + |x|)^{-n-1}$ and

$$|\partial_x^k P \nabla K_{1}^\beta(x)| = \left| \int_{\mathbb{R}^n} P \nabla K_{1/2}(x-y) \partial_x^k K_{1/2}(y) dy \right|$$

$$\lesssim \int_{\mathbb{R}^n} c'(1 + |x-y|)^{-n-1} e^{k-1} k^{-1} (k^{-1/\beta} + |y|)^{-n-1} dy$$

$$\lesssim C_k k \frac{\delta}{|x|} (k^{-1/\beta} + |x|)^{-n-1} k^{-1/\beta}$$

$$\lesssim C_k k \frac{\delta}{|x|} (k^{-1/\beta} + |x|)^{-n-1}.$$ 

This completes the proof of this lemma. ☐
The following lemma can be regarded as a generalization of Proposition 3.2 of [GPS].

**Lemma 5.3.** If $r$ is a natural number, $\alpha \in (0, 1)$ and $\max\{\alpha, 1/2\} < \beta \leq 1$, the operator

$$P^\beta_x f(t, x) = \int_0^t e^{-(t-s)(-\Delta)^\beta} (t - s + \frac{1}{2})^\beta \nabla^{r+2\beta} f(s, x) ds$$

is bounded on $L^2([0, T], L^2(\mathbb{R}^n, dx), df / \mu^\beta)$ for any $T \in [0, \infty]$ with constants $p(r)$ and $q(r)$.

**Proof.** By Plancherel’s theorem and Hölder’s inequality, we have

$$\int_0^\infty \|P^\beta_x f(t, \cdot)\|^2_{L^2} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^t e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} f(s, \xi) ds \right)^2 dt d\xi$$

$$\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left( \int_0^t e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} ds \right) \left( \int_0^t e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} ds \right) dt d\xi.$$

Because $t^{1/2\beta} - s^{1/2\beta} \leq (t-s)^{1/2\beta}$ for $2\beta > 1$ and $0 < s < t$, it is easy to see that

$$\int_0^t e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} ds \lesssim \int_0^t e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} ds \lesssim \int_0^\infty e^{-v^{1/2\beta}} dv \lesssim 1.$$

Then we have, by $t^{1/2\beta} - s^{1/2\beta} \leq (t-s)^{1/2\beta}$ for $2\beta > 1$ and $0 < s < t$ again,

$$\int_0^\infty \|P^\beta_x f(t, \cdot)\|^2_{L^2} \frac{dt}{t^{\alpha/\beta}} \lesssim \int_{\mathbb{R}^n} |\hat{f}(s, \xi)|^2 \left( \int_s^\infty e^{-(t-s)|\xi|^2} (t - s + \frac{1}{2})^\beta |\xi|^{r+2\beta} dt \right) ds d\xi \lesssim \int_{\mathbb{R}^n} |\hat{f}(s, \xi)|^2 \left( \int_0^\infty e^{-u|\xi|^2} u^{\alpha/\beta} |\xi|^{r+2\beta} du \right) ds d\xi \lesssim \int_{\mathbb{R}^n} |\hat{f}(s, \xi)|^2 ds d\xi.$$

This completes the proof of this lemma. \hfill \square

**Lemma 5.4.** For any $k \geq 0$, $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, there exists a constant $C(k)$ such that

$$\|e^{-t(-\Delta)^\beta} u\|_{X^{\alpha, k}_t} \leq C(k) \|u\|_{Q^{\alpha, -1}_t}.$$

**Proof.** Because $\|u\|_{X^{\alpha, k}_t} = \|u\|_{H^{\alpha, k}_t} + \|u\|_{N^{\alpha, 0, C}_t}$, we split the proof into two parts.

For the $L^\infty_t$ part of the norm. Because $Q^{\alpha, -1}_t \hookrightarrow B^{1-2\beta}_t$ and $\nabla^k : B^{1-2\beta}_t \hookrightarrow B^{1-2\beta-k}_t$, we have

$$\|\nabla^k e^{-t(-\Delta)^\beta} u\|_{L^\infty_t} \leq t^{1-\frac{2\beta-k}{2\beta}} \|\nabla^k u\|_{B^{1-2\beta-k}_t} \leq t^{1-\frac{2\beta-k}{2\beta}} \|u\|_{B^{1-2\beta}_t} \leq t^{1-\frac{2\beta-k}{2\beta}} \|u\|_{Q^{\alpha, -1}_t},$$
Then we can get \( t^{\frac{2\beta+1}{2\alpha-2}} \| \nabla^k e^{-t(-\Delta)^{\beta}} u \|_\infty \leq \| u \|_{Q^{\alpha,-1}_\alpha} \).

For the Carleson part. Because \( u \in Q^{\alpha,-1}_\alpha = \nabla \cdot (Q^{\alpha}_\alpha)^n \), there exists a sequence \( \{f_j\} \subset Q^{\alpha}_\alpha \) such that \( u = \sum_j \partial_j f_j \). We only need to prove (5.3)
\[
\sup_{x \in \mathbb{R}^n, r > 0} r^{2\alpha-n+2\beta-2} \int_0^r \int_{|y-x| < t} \left| \nabla^k e^{-t(-\Delta)^{\beta}} \partial_j f_j \right|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim C(k)\| \partial_j f_j \|_{Q^{\alpha,-1}_\alpha}^2 = C(k)\| f_j \|_{Q^{\alpha}_\alpha}^2.
\]

Taking \( \psi(x) = \nabla^k \partial_j e^{-t(-\Delta)^{\beta}} (x) = \int_{\mathbb{R}^n} (i\xi)^k \partial_j e^{-\xi^2 t^\beta} e^{2\pi i x \cdot \xi} d\xi \), we can justify the function \( \psi(x) \) with \( \psi(u) = (it\xi)^k (it\xi e^{-2\pi i t^\beta} \xi) \) satisfying the conditions in (1.8):
\[
|\psi(x)| \lesssim (1 + |x|)^{-n-1}, \quad \psi(x) \in L^1 \quad \text{and} \quad \int_{\mathbb{R}^n} \psi(x) dx = 0.
\]

By the equivalent characterization of \( Q^{\alpha}_\alpha \) (see (1.9)), we have
\[
\sup_{x,r} r^{2\alpha-n+2\beta-2} \int_0^r \int_{|y-x| < t} \left| s^k \nabla^k e^{-s2(-\Delta)^{\beta}} (s\partial_j f_j) \right|^2 \frac{dyds}{s^{1+2a-2(\beta-1)}} \lesssim C(k)\| f_j \|_{Q^{\alpha}_\alpha}^2.
\]

By a change of variable: \( t = s^{2\beta} \), we get the desired result (5.3). \( \square \)

5.2. Regularity. Now we state the main theorem of this section.

**Theorem 5.5.** Let \( \alpha > 0 \) and \( \max\{\alpha, 1/2\} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \). There exists an \( \varepsilon = \varepsilon(n) \) such that if \( \| u_0 \|_{Q^{\alpha,-1}_\alpha} < \varepsilon \), the solution \( u \) to equations (1.1) verifies:
\[
t^{\frac{k}{\alpha}} \nabla^k u \in X^{\alpha,0}_\alpha \quad \text{for any} \quad k \geq 0.
\]

**Proof.** We can see that the solution to the equations (1.1) can be represented as
\[
u(t,x) = e^{-t(-\Delta)^{\beta}} u(0,x) - B(u,v)(t,x),
\]
where
\[
B(u,v)(t,x) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}} P \nabla \cdot (u(s,x) \otimes v(s,x)) ds.
\]
Here \( u \otimes v \) denotes the tensor product of \( u \) and \( v \). For the linear term \( e^{-t(-\Delta)^{\beta}} u(0,x) := e^{-t(-\Delta)^{\beta}} u_0 \), by Proposition [5.4] we have \( \| e^{-t(-\Delta)^{\beta}} u_0 \|_{X^{\alpha,k}_\alpha} \leq C(k)\| u_0 \|_{Q^{\alpha,-1}_\alpha} \). Now we estimate the nonlinear term. We write \( \tilde{X}^{\alpha,k}_\alpha = \cap_{l=0}^k X^{\alpha,k}_\alpha \) equipped with the norm \( \sum_{l=0}^k \| \cdot \|_{X^{\alpha,k}_\alpha} \). We shall prove that the bilinear operator maps
\[
B(u,v) : \tilde{X}^{\alpha,k}_\alpha \times \tilde{X}^{\alpha,k}_\alpha \rightarrow \tilde{X}^{\alpha,k}_\alpha.
\]

**Part 1** \( N^{\beta,k}_{\alpha,\infty} \) norm. Here we shall prove that
\[
\| B(u,v) \|_{N^{\beta,k}_{\alpha,\infty}} \lesssim C_0(k)\| u \|_{X^{\beta,0}_\alpha} \| v \|_{X^{\beta,0}_\alpha} + C(k) \sum_{l=1}^{k-1} \| u \|_{X^{\beta,k}_{\alpha,\infty}} \| v \|_{X^{\beta,k-1}_{\alpha,\infty}}
+ C_1\| u \|_{X^{\beta,0}_\alpha} \| v \|_{X^{\beta,k}_\alpha} + C_1\| u \|_{X^{\beta,0}_\alpha} \| v \|_{X^{\beta,k}_\alpha}.
\]
If \( 0 < s < t(1 - \frac{1}{m}) \), \( \frac{t}{m} < t - s < t \). By Lemma [5.2] we have

\[
I = \int_0^{t(1 - \frac{1}{m})} \left| \nabla^k e^{-(t-s)(-\Delta)^{\beta}} P \nabla \cdot (u(s, x) \otimes v(s, x)) \right| \, ds
\]

\[
= C^k \int_0^{t(1 - \frac{1}{m})} \int_{\mathbb{R}^n} \frac{|u(s, y)||v(s, y)|}{(t-s)^{\frac{n+1}{2}}} \, dyds
\]

\[
\leq C^k \sum_{q \in \mathbb{Z}^n} \int_0^{t(1 - \frac{1}{m})} \int_{1 < t < 1/2^n} \frac{|u(s, y)||v(s, y)|}{|k^{-\beta/2} + |q||^{n+1}} \, dyds.
\]

Because \( \sum_{q \in \mathbb{Z}^n} \frac{1}{|k^{-\beta/2} + |q||^{n+1}} \approx k^{1/2^\beta} \), we have

\[
I \leq C^k k^{1/2^\beta} \left( \frac{m}{t} \right)^{(n+k+1)/2^\beta} \int_0^{t(1 - \frac{1}{m})} \int_{x-y \in t^{1/2^n} \mathbb{Q}_0} \, dyds
\]

\[
\leq C^k k^{1/2^\beta} \left( \frac{m}{t} \right)^{(n+k+1)/2^\beta} \int_0^t \int_{|z-y| < t^{1/2^\beta}} \, dyds
\]

\[
\leq C_0(k) t^{1/2^\beta} m^{-\frac{n+k+1}{2}} \|u\|_{X_0^\alpha,0} \|v\|_{X_0^\beta,0}
\]

where \( C_0(k) = C^k k^{1/2^\beta} m^{-\frac{n+k+1}{2}} \).

If \( t(1 - \frac{1}{m}) \leq s \leq t \), by Young’s inequality, we have

\[
\left| \nabla^k e^{-(t-s)(-\Delta)^{\beta}} P \nabla \cdot (u(s, x) \otimes v(s, x)) \right| = \left| P \nabla e^{-(t-s)(-\Delta)^{\beta}} \nabla^k (u(s, x) \otimes v(s, x)) \right|
\]

\[
\leq \|P \nabla e^{-(t-s)(-\Delta)^{\beta}}(x)\|_{L^1} \|\nabla^k (u(s, x) \otimes v(s, x))\|_{L^\infty}.
\]

By the estimate for the generalized Oseen kernel:

\[
|P \nabla^{l+1} e^{-(\Delta)^{\beta}}(x)| \leq \frac{1}{(1 + |x|)^{n+1+l}} \quad \text{and} \quad P \nabla^{l+1} e^{-(\Delta)^{\beta}} = u^{-\frac{n+1}{2}} \nabla^{l+1} e^{-(\Delta)^{\beta}}(\frac{x}{u^{1/2^\beta}}),
\]

we have

\[
|P \nabla^{l+1} e^{-u(\Delta)^{\beta}}| \leq u^{-\frac{n+1}{2}} \left( 1 + \frac{|x|}{u^{1/2^\beta}} \right)^{l+n+1} \leq \frac{1}{(u^{1/2^\beta} + |x|)^{l+n+1}}.
\]

Then we take \( l = 0 \) and have

\[
\|P \nabla e^{-u(\Delta)^{\beta}}\|_{L^1} \leq \int_0^\infty \frac{|x|^{n-1}}{(u^{1/2^\beta} + |x|)^{n+1}} \, dx \leq \frac{1}{u^{1/2^\beta}}.
\]

Hence we can get

\[
|\nabla^k e^{-(t-s)(-\Delta)^{\beta}} P \nabla (u(s, x) \otimes v(s, x))| \leq \frac{1}{(t-s)^{1/2^\beta}} \sum_{l=0}^k \binom{k}{l} \|\nabla^l u(s, .)\|_{L^\infty} \|\nabla^{k-l} v(s, .)\|_{L^\infty}
\]

\[
\leq \frac{1}{(t-s)^{1/2^\beta}} \sum_{l=0}^k \binom{k}{l} \|u\|_{N_0^\alpha,0} \|v\|_{N_0^\beta,0} \frac{1}{s^{(2^\beta-1+l)/2^\beta} s^{(2^\beta-1+k-1)/2^\beta}}.
\]
So we have

\[ \left| \int_{t(1-\frac{1}{m})}^{t} \nabla e^{-(t-s)(-\Delta)^{\beta}} P\nabla^{k+1}(u(s, x) \otimes v(s, x)) ds \right| \]

\[ \leq \sum_{l=0}^{k} \left( \frac{k}{l} \right) \|u\|_{N^{\beta,1}_{\alpha,\infty}} \|v\|_{N^{\beta,k-1}_{\alpha,\infty}} \int_{t(1-\frac{1}{m})}^{t} \frac{1}{(t-s)^{1/2\beta}} \frac{1}{s^{(4\beta-2k)/2\beta}} ds. \]

For the integral in the last inequality, we make the change of variable: \( s = zt. \) Because \( t \left( 1 - \frac{1}{m} \right) < s < t \) implies \( \left( 1 - \frac{1}{m} \right) < z < 1, \) we have

\[ II = \int_{t(1-\frac{1}{m})}^{t} \frac{1}{s^{(4\beta-2k)/2\beta}} ds \]

\[ \leq t^{1-\frac{2\beta-k}{2\beta}} \left( 1 - \frac{1}{m} \right)^{-4\beta-2k} \int_{1-\frac{1}{m}}^{1} (1-z)^{-\frac{1}{2\beta}} dz \]

\[ = t^{1-\frac{2\beta-k}{2\beta}} \left( 1 - \frac{1}{m} \right)^{-4\beta-2k} \left( \frac{1}{m} \right)^{1-\frac{1}{2\beta}}. \]

Denote \( g(m) = \left( 1 - \frac{1}{m} \right)^{-4\beta-2k} \left( \frac{1}{m} \right)^{1-\frac{1}{2\beta}} \) and take \( m = m(k) = k^{\frac{k-3}{2+2\beta}}. \) We can prove that \( g(m) \to 0 \) as \( k \to \infty. \) Then we have \( II \lesssim Ct^{-\frac{2\beta-k}{2\beta}} \) for \( k \geq 1. \)

Therefore we have

\[ \left| \int_{t(1-\frac{1}{m})}^{t} \nabla e^{-(t-s)(-\Delta)^{\beta}} P\nabla^{k+1}(u(s, x) \otimes v(s, x)) ds \right| \]

\[ \lesssim Ct^{\frac{2\beta-k}{2\beta}} \left[ \sum_{l=1}^{k} \left( \frac{k}{l} \right) \|u\|_{N^{\beta,l}_{\alpha,\infty}} \|v\|_{N^{\beta,k-l}_{\alpha,\infty}} + \|u\|_{N^{\beta,k}_{\alpha,\infty}} \|v\|_{N^{\beta,k}_{\alpha,\infty}} + \|u\|_{N^{\beta,k}_{\alpha,\infty}} \|v\|_{N^{\beta,k}_{\alpha,\infty}} \right]. \]

**Part 2**: \( N^{\beta,k}_{\alpha,C} \) norm. We split \( B(u, v) \) as follows: \( B(u, v) = B_1(u, v) + B_2(u, v) \) with

\[ B_1(u, v)(t, x) = \int_{0}^{t} e^{-(t-s)(-\Delta)^{\beta}} P\nabla \left[ 1 - \phi \left( \frac{x - x_0}{R^{1/2\beta}} \right) \right] u(s, x) \otimes v(s, x) ds \]

\[ B_2(u, v)(t, x) = \int_{0}^{t} e^{-(t-s)(-\Delta)^{\beta}} P\nabla \phi \left( \frac{x - x_0}{R^{1/2\beta}} \right) u(s, x) \otimes v(s, x) ds \]

where \( \phi \left( R^{1/2\beta} x_0 \right) = \phi((x - x_0)/R^{1/2\beta}) \) for a smooth function \( \phi \) supported in \( B(0, 15) \) and equals to 1 on \( B(0, 10). \)
For the estimate for $B_1$. Because $|P\nabla^{k+1} e^{-(t-s)(-\Delta)^{\beta}}(x)| \lesssim \frac{K(k)}{[(t-s)^{1/2\beta} + |x-y|]^{n+\beta}}$
and $0 < t < R$, we have

\[
|t^{\frac{1}{2\beta}} \nabla^k B_1(u, v)(t, x)| \\
\lesssim t^{\frac{1}{2\beta}} \left| \nabla^k \int_0^t \int_{|y-x_0| \geq 10R^{1/2\beta}} e^{-(t-s)(-\Delta)^{\beta}} P\nabla(x - y)u(s, y)v(s, y)dyds \right| \\
\lesssim K(k) t^{\frac{1}{2\beta}} \int_0^t \int_{|y-x_0| \geq 10R^{1/2\beta}} \frac{|u(s, y)||v(s, y)| dyds}{[(t-s)^{1/2\beta} + |x-y|]^{n+\beta}} \\
\lesssim K(k) t^{\frac{1}{2\beta}} R^{1/2\beta} \int_0^t \int_{|y-x_0| \geq 10R^{1/2\beta}} \frac{|u(s, y)||v(s, y)| dyds}{R^{n+\beta}} \\
\lesssim K(k) D(k) R^{1-2\beta} \|u\|_{X_{\alpha,0}^\beta} \|v\|_{X_{\alpha,0}^\beta}.
\]

Then we have, taking $R = r^{2\beta}$,

\[
r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} |t^{\frac{1}{2\beta}} \nabla^k B_1(u, v)(t, y)|^2 dydt \lesssim (K(k) D(k))^2 r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x| < r} r^{2-4\beta} \|u\|^2_{X_{\alpha,0}^\beta} \|v\|^2_{X_{\alpha,0}^\beta} dydt \\
\lesssim (K(k) D(k))^2 \|u\|^2_{X_{\alpha,0}^\beta} \|v\|^2_{X_{\alpha,0}^\beta}.
\]

For the estimate for $B_2$. We further split $B_2$ as $B_2 = B_2^1 + B_2^2$ with

\[
B_2^1 = \frac{1}{\sqrt{-\Delta}} P\nabla \int_0^t e^{-(t-s)(-\Delta)^{\beta}} \frac{\Delta}{\sqrt{-\Delta}} \left( I - e^{-s(-\Delta)} \right) \left( \phi_R \frac{1}{R^{1-\beta}} u(s, x_0) \otimes v(s, x_0) \right) ds, \\
B_2^2 = \frac{1}{\sqrt{-\Delta}} P\nabla e^{-(t-s)^{\beta}} \int_0^t \phi_R \frac{1}{R^{1-\beta}} u(s, x_0) \otimes v(s, x_0) ds.
\]

At first we estimate the term $t^{\frac{1}{2\beta}} \nabla^k B_2^1$. Without loss of the generalization, we assume $k$ is odd. The proof of the case that $k$ is even is similar. If $k$ is odd, we have $k = 2K + 1$ for $K \in \mathbb{Z}_+$. Because $\frac{1}{2} < \beta < 1$, we have

\[
t^{\frac{1}{2\beta}} = \left( t^{\frac{2K}{\beta}} - s^{\frac{2K}{\beta}} + s^{\frac{2K}{\beta}} \right)^{2K} \left( t^{\frac{2K}{\beta}} - s^{\frac{2K}{\beta}} + s^{\frac{2K}{\beta}} \right) \\
= \sum_{l=0}^{2K} \binom{2K}{l} \left( t^{\frac{2K-l}{\beta}} - s^{\frac{2K-l}{\beta}} \right)^{2K-l} s^{\frac{2K-l}{\beta}} \left( t^{\frac{2K-l}{\beta}} - s^{\frac{2K-l}{\beta}} + s^{\frac{2K-l}{\beta}} \right) \\
= \sum_{l=0}^{2K} \binom{2K}{l} \left( t^{\frac{2K}{\beta}} - s^{\frac{2K}{\beta}} \right)^{2K-l+1} s^{\frac{2K-l}{\beta}} + \sum_{l=0}^{2K} \binom{2K}{l} \left( t^{\frac{2K}{\beta}} - s^{\frac{2K}{\beta}} \right)^{2K-l} s^{\frac{2K-l+1}{\beta}}.
\]
Then we have, setting $M(s, x) = \phi_{R^{\frac{1}{2^\beta}, x_0}}(s)u(s, x) \otimes v(s, x)$,

\[
t^{\frac{2}{\beta}} \nabla^k B_2^1 = \sum_{l=0}^{2K-1} \left( \frac{2K}{l} \right) \frac{P^1}{\sqrt{-\Delta}} P_{2K-l+1}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{1}{2^\beta}} \nabla^l M(s, x) \right) + \frac{P^1}{\sqrt{-\Delta}} P_{2K}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{1}{2^\beta}} \nabla^{2K} M(s, x) \right) + \sum_{l=0}^{2K-1} \left( \frac{2K}{l} \right) \frac{P^1}{\sqrt{-\Delta}} P_{2K-l}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) \right) + \frac{P^1}{\sqrt{-\Delta}} P_{2K}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{2K+1}{2^\beta}} \nabla^{2K+1} M(s, x) \right).
\]

Since $\sup_{s \in (0, \infty)} s^{1-2\beta}(1-e^{-s^{2\beta}}) < \infty$ for $\frac{1}{2} < \beta < 1$, we can obtain that $(\Delta)^{1/2-\beta}(I - e^{-s(\Delta)^{\beta}})$ is bounded on $L^2$ with operator norm $\lesssim s^{1-\frac{1}{2^\beta}}$. By Lemma 5.3 and the $L^2$-boundedness of Riesz transform, we have

\[
r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| \frac{P^1}{\sqrt{-\Delta}} P_{2K-l}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) \right) \right|^2 \frac{dxds}{s^{a/\beta}} \leq p(2K-l)r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{\mathbb{R}^{n}} \left| (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) \right|^2 \frac{dxds}{s^{a/\beta}} \leq p(2K-l)r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| s^{1-\frac{1}{2^\beta}} s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) \right|^2 \frac{dxds}{s^{a/\beta}}.
\]

Because $0 < s < r^{2\beta}$ and

\[
s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) = s^{\frac{1}{2^\beta}} \nabla^{l+1} \left( \phi_{R^{\frac{1}{2^\beta}, x_0}}(s) u(s, x) \otimes v(s, x) \right) = \sum_{m+\eta \leq l+1} \left[ s^{\frac{1}{2^\beta}+\frac{m}{2^\beta}} \nabla^m u(s, x) \right] \left[ s^{\frac{1}{2^\beta}} \nabla^\eta v(s, x) \right] \left[ s^{\frac{1}{2^\beta}+\frac{m}{2^\beta}} \nabla^{l+1-m-\eta} \phi_{R^{\frac{1}{2^\beta}, x_0}} \right],
\]

then we can get, taking $R = r^{2\beta}$,

\[
r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| s^{1-\frac{1}{2^\beta}} s^{\frac{1}{2^\beta}} \nabla^{l+1} M(s, x) \right|^2 \frac{dxds}{s^{a/\beta}} \leq \sum_{m+\eta \leq l+1} \| u \|^2_{L^2_{a, \infty}} \| v \|^2_{L^2_{a, \infty}}.
\]

In a similar way, we have

\[
r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| \frac{P^1}{\sqrt{-\Delta}} P_{2K-l}^\beta \left( (\Delta)^{\frac{1}{2}-\beta} (I - e^{-s(\Delta)^{\beta}}) s^{\frac{2K+1}{2^\beta}} \nabla^{2K+1} M(s, x) \right) \right|^2 \frac{dxds}{s^{a/\beta}} \leq p(0)r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{\mathbb{R}^{n}} \left| s^{1-\frac{1}{2^\beta}} s^{\frac{2K+1}{2^\beta}} \nabla^{2K+1} M(s, x) \right|^2 \frac{dxds}{s^{a/\beta}} \leq p(0)r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| s^{\frac{2K+1}{2^\beta}} \nabla^{2K+1} M(s, x) \right|^2 \frac{dxds}{s^{a/\beta}}.
\]

\[
r^{2a-n+2\beta-2} \int_0^{2\beta} \int_{|x-x_0| < r} \left| \frac{2K+1}{2^\beta} \sum_{m+\eta \leq 2K+1} \nabla^m u \nabla^\eta v \nabla^{2K+1-m-\eta} \phi_{R^{\frac{1}{2^\beta}, x_0}} \right|^2 \frac{dxds}{s^{a/\beta}} \leq p(0) \left( \| u \|^2_{L^2_{a, \infty}} \| v \|^2_{L^2_{a, \infty}} + \| u \|^2_{L^2_{a, \infty}} \| u \|^2_{L^2_{a, \infty}} + r(K) \| u \|^2_{L^2_{a, \infty}} \| v \|^2_{L^2_{a, \infty}} \right).
\]
Similarly we can estimate the terms associated with $P_{1}^{\beta}$ and $P_{2K-l+1}^{\beta}$. Combining all the estimates together, we can prove

$$
\int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left| \frac{t}{\alpha^2} \nabla^{k} B_{2}^{\beta}(u,v)(t,x) \right|^{2} \frac{dxdt}{t_{1/\beta}} \right)^{1/2}
\leq C_{1} \|u\|_{X_{\alpha}^{\beta,0}} \|v\|_{X_{\alpha}^{\beta,k}} + C_{2} \|v\|_{X_{\alpha}^{\beta,0}} \|u\|_{X_{\alpha}^{\beta,k-1}} + C(k) \|u\|_{X_{\alpha}^{\beta,k-1}} \|u\|_{X_{\alpha}^{\beta,k-1}}.
$$

Now we estimate the term $B_{2}^{\beta}$. Taking the change of variables: $s = r^{2\beta} \theta$, $x = rz$ and $t = r^{2\beta} \tau$, we have

$$
I = \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left| \frac{t}{\alpha^2} \nabla^{k} B_{2}^{\beta}(u,v)(t,x) \right|^{2} \frac{dxdt}{t_{1/\beta}}
= \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left| \frac{t}{\alpha^2} \nabla^{k} \frac{P\nabla}{\sqrt{-\Delta}} \sqrt{-\Delta} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(s,x) ds \right|^{2} \frac{dxdt}{t_{1/\beta}}
\leq \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left| \frac{t}{\alpha^2} \nabla^{k+1} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(s,x) ds \right|^{2} \frac{dxdt}{t_{1/\beta}}
= \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left( \frac{1}{\alpha^2} \nabla^{k+1} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(r^{2\beta} \theta, rz) r^{2\beta} d\theta \right) \frac{dxdt}{t_{1/\beta}}
= \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left( \nabla^{k+1} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(r^{2\beta} \theta, rz) r^{2\beta} d\theta \right) \frac{dxdt}{t_{1/\beta}}.
$$

Denote by $\nabla_{\nu}^{\nu} e^{-t(-\Delta)^{\beta}/2}(x,y)$ the kernel of the operator $\nabla_{\nu}^{\nu} e^{-t(-\Delta)^{\beta}/2} \nu > 0$. Because $\frac{1}{2} < \beta \leq 1$, we have

$$
\left| \frac{r^{(1-\beta)}}{r^{2\beta}} \nabla_{\nu}^{k(1-\beta)} e^{-t(-\Delta)^{\beta}/2}(x,y) \right| \lesssim \frac{r^{(1-\beta)}}{r^{2\beta}} \frac{1}{(\tau/2)^{\frac{k(1-\beta)}{2}} (1 + \frac{\tau-x_{0}}{\tau/2})^{n+k(1-\beta)}} \in L^{1}(\mathbb{R}^{n})
$$

uniformly in $\tau$. By Young’s inequality and Lemma 4.9 we have

$$
I = \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left( \frac{1}{\alpha^2} \nabla^{k+1} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(r^{2\beta} \theta, rz) r^{2\beta} d\theta \right) \frac{dxdt}{t_{1/\beta}}
\leq \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} \left( \frac{1}{\alpha^2} \nabla^{k+1} e^{-t(-\Delta)^{\beta}} \int_{0}^{t} M(r^{2\beta} \theta, rz) r^{2\beta} d\theta \right) \frac{dzd\theta}{t_{1/\beta}}
\leq r^{\beta-4} b(k) A(\alpha, \beta, M) \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} |M(r^{2\beta} \theta, rz)| \frac{dzd\theta}{t_{1/\beta}}
:= r^{\beta-4} b(k) A(\alpha, \beta, M) I_{M}.
$$

For $A(\alpha, \beta, M)$, we have

$$
A(\alpha, \beta, M) = \rho^{2\alpha-n+2\beta-2} \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} |M(r^{2\beta} s, ry)| \frac{dsdy}{s^{\alpha}/\beta}
\leq \rho^{2-4\beta} (r^{2\alpha-n+2\beta-2} \int_{0}^{r_{1}^{\alpha-n+2\beta-2}} \int_{|y-x_{0}|<r} |M(t, z)| \frac{dzdt}{t^{\alpha}/\beta}
\leq \rho^{2-4\beta} \|u\|_{X_{\alpha}^{\beta,0}} \|v\|_{X_{\alpha}^{\beta,0}}.
$$
For $I_M$, we have
\[ \int_0^1 \int_{\mathbb{R}^n} |M(r^{2\beta} \theta, rz)|^\frac{d\theta dz}{\theta^{1/\beta}} \leq \int_0^1 \int_{\mathbb{R}^n} |M(t, z)|^{\frac{r^{2\beta-n} dt dz}{r^{-2\alpha t^{\alpha/\beta}}}} \leq r^{2-4\beta} \|u\|_{X_0^{\alpha,0}} \|v\|_{X_0^{\beta,0}}. \]
Then we get
\[ r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| t^{\frac{2\beta}{\beta-k}} \nabla^k B_2^2(u, v)(t, x) \right|^2 \frac{dx dt}{t^\alpha t^{\alpha/\beta}} \leq b(k) \|u\|_{X_0^{\beta,0}} \|v\|_{X_0^{\beta,0}}. \]
Now we have proved that
\[ \|B(u, v)\|_{X_0^{\beta,k}} \leq C_0(k) \|u\|_{X_0^{\alpha,0}} \|v\|_{X_0^{\beta,0}} + C(k) \|u\|_{X_0^{\beta,k-1}} \|v\|_{X_0^{\beta,k-1}} + C_1 \|u\|_{X_0^{\beta,k}} \|v\|_{X_0^{\beta,k}} + C_1 \|u\|_{X_0^{\beta,k}} \|v\|_{X_0^{\beta,0}}. \]

Similar to the method applied in Lemma 4.3 of [GPS], if we construct the approximating sequence $u^j$ by
\[ u^{-1} = 0, \quad u^0 = e^{-t(-\Delta)^{\beta}} u_0, \quad u^{j+1} = u^j + B(u^j, u^j), \]
we can get the following lemma and hence complete the proof of Theorem 5.5.

**Lemma 5.6.** Let $\alpha > 0$ and $\max\left\{\frac{1}{2}, \alpha\right\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. Suppose $u_0$ be small enough in $Q_{10}^{\beta,-1}$. Then for any $k \geq 0$, there exist constants $D_k$ and $E_k$ such that
\[ \|u\|_{\tilde{X}_0^{\beta,k}} \lesssim D_k \quad \text{and} \quad \|u^{j+1} - u^j\|_{\tilde{X}_0^{\beta,k}} \lesssim E_k(\frac{2}{3})^j. \]
In particular, for any $k \geq 0$, $u^j$ converges in $\tilde{X}_0^{\beta,k}$. \qed

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