INTRODUCTION

Recent progress in quantum communications has caused the great interest to the problems connected with divisions of quantum systems into subsystems and reunifications of subsystems into a joint system.

Although general theory of such processes was proposed in 1927 [von Neumann 1927], so far, a division of a quantum system into subsystems is usually described in a fictitious manner. As an example, here we quote the classical paper on the photon teleportation [Bouwmeester et al 1997]. Describing the photon teleportation experiment they write:

The entangled state contains no information on the individual particles; it only indicates that two particles will be in the opposite states. The important property of an entangled pair that as soon as a measurement on one particles projects it, say, onto $| \leftrightarrow \rangle$ the state of the other one is determined to be $| \uparrow \downarrow \rangle$, and vice versa. How could a measurement on one of the particles instantaneously influence the state of the other particle, which can be arbitrary far away? Einstein, among many other distinguished phyicsists, could simply not accept this ”spooky action at a distance”. But this property of entangled states has been demonstrated by numerous experiments.

Nevertheless Einstein was quite right in his non-acceptance of such point of view. In this paper we develope the correct approach to describe the phenomena completely adequate to the physical problem. The basic notion of our approach is Conditional Density Matrix.
CONDITIONAL DENSITY MATRIX

Consider two systems $S_1$ and $S_2$. The joint system is denoted as $S_{12}$.

The principal question that we want to answer here is how the states of the subsystems are related to the state of the joint system, and vice versa.

Let $\rho_1$ and $\rho_2$ be the density matrices of the systems $S_1$ and $S_2$.

If at least one of the states $\rho_1$ or $\rho_2$ is pure (i.e. $\rho_i^2 = \rho_i$) then these states determine the state of the compound system $S_{12}$ uniquely:

$$\rho = \rho_1 \otimes \rho_2$$

If the state of the system $S_{12}$ is $\rho_{12}$ then the state of the system $S_1$ is determined by the following equation:

$$\rho_1 = Tr_2(\rho_{12}).$$

Now we can define the conditional density matrix.

If the state of the system $S_{12}$ is $\rho_{12}$ then the state of the system $S_1$ upon the condition that the system $S_2$ is in the pure state $\rho_2$, $\rho_2^2 = \rho_2$ is

$$\rho_{1/2} = \frac{Tr_2(\rho_2 \rho_{12})}{Tr(\rho_2 \rho_{12})}.$$

Example: Parapositronium

As an example we consider parapositronium - the system consisting of an electron and a positron. The total spin of the system is equal to zero. In this case the nonrelativistic approximation is valid and the state vector of the system is represented in the form of the product

$$\Psi(\vec{r}_e, \sigma_e; \vec{r}_p, \sigma_p) = \Phi(\vec{r}_e, \vec{r}_p)\chi(\sigma_e, \sigma_p).$$

The spin wave function is equal to

$$\chi(\sigma_e, \sigma_p) = \frac{1}{\sqrt{2}} (\chi_{\vec{n}}(\sigma_e)\chi_{-\vec{n}}(\sigma_p) - \chi_{\vec{n}}(\sigma_p)\chi_{-\vec{n}}(\sigma_e)).$$

Here $\chi_{\vec{n}}(\sigma)$ and $\chi_{-\vec{n}}(\sigma)$ are the eigenvectors of the operator that projects spin onto the vector $\vec{n}$:

$$(\vec{\sigma}\vec{n}) \chi_{\vec{n}}(\sigma) = \chi_{\vec{n}}(\sigma),$$

$$(\vec{\sigma}\vec{n}) \chi_{-\vec{n}}(\sigma) = -\chi_{-\vec{n}}(\sigma).$$

The spin density matrix of the system is determined by the operator with the kernel

$$\rho(\sigma; \sigma') = \chi(\sigma_e, \sigma_p) \chi^*(\sigma_e', \sigma_p').$$
The spin density matrix of the electron is
\[
\rho_e(\sigma, \sigma') = \sum_\xi \chi(\sigma, \xi) \chi^*(\sigma', \xi) =
\frac{1}{2} \left( \chi_{\vec{n}}(\sigma) \chi_{\vec{n}}(\sigma') + \chi_{\vec{n}}(\sigma) \chi_{\vec{n}}(\sigma') \right) = \frac{1}{2} E(\sigma, \sigma').
\]
In this state the electron is completely unpolarized.

If an electron passes through a polarization filter then the pass probability is independent of the filter orientation. The same fact is valid for the positron if its spin state is measured independently of the electron.

Now let us consider quite different experiment. Namely, the positron passes through the polarization filter and the electron polarization is simultaneously measured. The operator that projects the positron spin onto the vector \(\vec{m}\) (determined by the filter) is given by the kernel
\[
P(\sigma, \sigma') = \chi_{\vec{m}}(\sigma) \chi_{\vec{m}}^*(\sigma').
\]
Now the conditional density matrix of the electron equals to
\[
\rho_{e/p}(\sigma, \sigma') = \frac{\sum_{(\sigma, \sigma')} \chi_{\vec{m}}(\sigma) \chi_{\vec{m}}^*(\sigma') \chi(\sigma_e, \sigma') \chi^*(\sigma_e', \sigma)}{\sum_{(\xi, \sigma, \sigma')} \chi_{\vec{m}}(\sigma) \chi_{\vec{m}}^*(\sigma') \chi(\xi, \sigma') \chi^*(\xi, \sigma)}.
\]
The result of the summation is
\[
\rho_{e/p}(\sigma, \sigma') = \chi_{\vec{m}}(\sigma) \chi_{\vec{m}}^*(\sigma').
\]
Thus, if the polarization of the positron is well defined then the electron appears to be polarized in the opposite direction.

**TELEPORTATION**

In the Innsbruck experiment on a photon state teleportation, the initial state of the system is the result of the unification of the pair of photons 1 and 2 being in the antisymmetric state \(\chi(\sigma_1, \sigma_2)\) with summary angular momentum equal to zero and the photon 3 being in the state \(\chi_{\vec{m}}(\sigma_3)\) (that is, being polarized along the vector \(\vec{m}\)). The joint system state is given by the density matrix
\[
\rho(\sigma, \sigma') = \Psi(\sigma)\Psi^*(\sigma'),
\]
where the wave function of the joint system is the product
\[
\Psi(\sigma) = \chi(\sigma_1, \sigma_2) \chi_{\vec{m}}(\sigma_3).
\]
Considering then the photon 2 only (without fixing the states of the photons 1 and 3) we find the photon 2 to be completely unpolarized with the density matrix
\[
\rho_{\sigma_2, \sigma_2'} = Tr_{(1, 3)} \rho(\sigma_1, \sigma_2; \sigma_3, \sigma_1, \sigma_2', \sigma_3) = \frac{1}{2} E(\sigma_2, \sigma_2').
\]
However, if the photon 2 is registered when the state of the photons 1 and 3 has been determined to be $\chi(\sigma_1, \sigma_3)$ then the state of the photon 2 is given by the conditional density matrix

$$\rho_{2/\{1,3\}} = \frac{Tr_{\{1,3\}}(P_{1,3} \rho_{1,2,3})}{Tr(P_{1,3} \rho_{1,2,3})}.$$ 

Here $P_{1,3}$ is the projection operator

$$P_{1,3} = \chi(\sigma_1, \sigma_3) \chi^*(\sigma_1, \sigma_3).$$

To evaluate the conditional density matrix it is convenient to preliminary find the vectors

$$\phi(\sigma_1) = \sum_3 \chi_m^*(\sigma_3) \chi(\sigma_1, \sigma_3)$$

and

$$\theta(\sigma_2) = \sum_1 \phi^*(\sigma_1) \chi(\sigma_1, \sigma_2).$$

The vector $\theta$ equals to

$$\theta(\sigma_2) = -\frac{1}{2} \chi_m(\sigma_2)$$

and the conditional density matrix of the photon 2 appears to be equal to

$$\rho_{2/\{1,3\}} = \chi_m(\sigma_2) \chi_m^*(\sigma_2').$$

Thus, if the subsystem consisting of the photons 1 and 3 is forced to be in the antisymmetric state $\chi(\sigma_1, \sigma_3)$ (with total angular momentum equal to zero) then the photon 2 appears to be polarized along the vector $\vec{m}$.

**PAIRS OF POLARIZED PHOTONS**

Now consider a modification of the Innsbruck experiment. Let there be two pairs of photons (1, 2) and (3, 4). Suppose that each pair is in the pure antisymmetric state $\chi$. The spin part of the density matrix of the total system is given by the equation

$$\rho(\sigma, \sigma') = \Psi(\sigma) \Psi^*(\sigma').$$

The wave function of the total system is the product of the wave functions of the subsystems

$$\Psi(\sigma) = \chi(\sigma_1, \sigma_2) \chi(\sigma_3, \sigma_4).$$

If the photons 2 and 4 are polarised along $\chi_{\vec{m}}(\sigma_2)$ and $\chi_{\vec{s}}(\sigma_4)$ then the wave function of the system is transformed into

$$\Phi(\sigma) = \chi_{\vec{m}}(\sigma_1) \chi_{\vec{m}}(\sigma_2) \chi_{\vec{r}}(\sigma_3) \chi_{\vec{s}}(\sigma_4).$$
Here $\vec{n}$, $\vec{m}$ and $\vec{r}$, $\vec{s}$ are pairs of mutually orthogonal vectors.

Now the conditional density matrix of the pair of photons 1 and 3 is

$$\rho_{(1,3)/(2,4)}(\sigma, \sigma') = \Psi(\sigma_1, \sigma_3) \Psi^*(\sigma'_1, \sigma'_3).$$

The wave function of the pair is the product of wave functions of each photon with definite polarization

$$\Psi(\sigma_1, \sigma_3) = \chi_{\vec{n}}(\sigma_1) \chi_{\vec{r}}(\sigma_3).$$

Pairs of polarized photons appear to be very useful in quantum communication.

**QUANTUM REALIZATION OF VERNAM COMMUNICATION SCHEME**

Let us recall the main idea of Vernam communication scheme [Vernam 1926]. In this scheme, Alice encrypts her message (a string of bits denoted by the binary number $m_1$) using a randomly generated key $k$. She simply adds each bit of the message with the corresponding bit of the key to obtain the scrambled text ($s = m_1 \oplus k$, where $\oplus$ denotes the binary addition modulo 2 without carry). It is then sent to Bob, who decrypts the message by subtracting the key ($s \ominus k = m_1 \oplus k \ominus k = m_1$). Because the bits of the scrambled text are as random as those of the key, they do not contain any information. This cryptosystem is thus provably secure in sense of information theory. Actually, today this is the only provably secure cryptosystem!

Although perfectly secure, the problem with this security is that it is essential that Alice and Bob possess a common secret key, which must be at least as long as the message itself. They can only use the key for a single encryption. If they used the key more than once, Eve could record all of the scrambled messages and start to build up a picture of the plain texts and thus also of the key. (If Eve recorded two different messages encrypted with the same key, she could add the scrambled text to obtain the sum of the plain texts: $s_1 \oplus s_2 = m_1 \oplus k \oplus m_2 \oplus k = m_1 \oplus m_2 \oplus k \oplus k = m_1 \oplus m_2$, where we used the fact that $\oplus$ is commutative.) Furthermore, the key has to be transmitted by some trusted means, such as a courier, or through a personal meeting between Alice and Bob. This procedure may be complex and expensive, and even may lead to a loophole in the system.

With the help of pairs of polarized photons we can overcome the shortcomings of the classical realization of Vernam scheme. Suppose Alice sends to Bob pairs of polarized photons obtained according to the rules described in the previous section. Note that the concrete photons’ polarizations are set up in Alice’s laboratory and Eve does not know them. If the polarization of the photon 1 is set up by a random binary number $p_i$ and the polarization of the photon 3 is set up by a number $m_i \oplus p_i$ then each photon (when considered separately) does not carry any information. However, Bob after obtaining these photons can add corresponding binary numbers and get the number $m_i$ containing the information ($m_i \oplus p_i \oplus p_i = m_i$).

In this scheme, a secret code is created during the process of sending and is transferred to Bob together with the information. It makes the usage of the scheme completely secure.
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Vernam 1926 — G. Vernam, Cipher printing telegraph systems for secret wire and radio telegraph communications, J. Am. Inst. of Electrical Engineers, 45, pp. 109-115, 1926.
In this paper we develop the Conditional Density Matrix formalism for adequate description of division and unification of quantum systems. Applications of this approach to the descriptions of parapositronium, quantum teleportation and others examples are discussed.

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The result of the summation is
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