PROPER HOLOMORPHIC POLYNOMIAL MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS OF CLASSICAL TYPE

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Abstract. We prove that two proper holomorphic polynomial maps between bounded symmetric domains of classical type which preserve the origin are equivalent if and only if they are isotropically equivalent. Using this property we show that each member of a one-parameter family of maps from [19] is inequivalent.

1. Introduction

Let $\Omega_1, \Omega_2$ be domains in $\mathbb{C}^n$ and $\mathbb{C}^N$ and $f, g : \Omega_1 \to \Omega_2$ be holomorphic maps. We say that $f$ is proper if $f^{-1}(K)$ is compact for every compact subset $K \subset \Omega_2$. We say that $f$ and $g$ are equivalent if and only if $f = A \circ g \circ B$ for some $B \in \text{Aut}(\Omega_1)$ and $A \in \text{Aut}(\Omega_2)$. For a domain $\Omega$, denote the group of automorphisms fixing $p \in \Omega$ by $\text{Isot}_p(\Omega)$. Suppose that for fixed $p \in \Omega_1$, $f(p) = g(p)$. Then we say that $f$ and $g$ are isotropically equivalent at $p$ if there are $U \in \text{Isot}_p(\Omega_1)$ and $V \in \text{Isot}_{g(p)}(\Omega_2)$ such that $f = V \circ g \circ U$. The notion of isotropic equivalence coincides with that of unitary equivalence of [3] defined when $\Omega_1$ and $\Omega_2$ are balls. The following domains are called bounded symmetric domains of classical type:

1. $\Omega_{I}^{r,s} = \{ Z \in M_{r,s}^\mathbb{C} : I_r - ZZ^* > 0 \}$, where $s \geq r = \text{rank}(\Omega_{I}^{r,s})$.
2. $\Omega_{II}^{r} = \{ Z \in M_{n,n}^\mathbb{C} : I_n - ZZ^* > 0, Z^t = -Z \}$, rank$(\Omega_{II}^{r}) = \lfloor \frac{r}{2} \rfloor$.
3. $\Omega_{III}^{n} = \{ Z \in M_{n,n}^\mathbb{C} : I_n - ZZ^* > 0, Z^t = Z \}$, rank$(\Omega_{III}^{n}) = n$.
4. $\Omega_{IV}^{r} = \{ Z = (z_1, \ldots, z_n) \in \mathbb{C}^n : ZZ^* < 1, 0 < 1 - 2ZZ^* + |ZZ|^2 \}$, rank$(\Omega_{IV}^{r}) = 2$.

Here we denote by $M > 0$ positive definiteness of square matrix $M$, by $M_{r,s}^\mathbb{C}$ the set of $r \times s$ complex matrices and by $I_r$ the $r \times r$ identity matrix.

The aim of this paper is to prove the following theorems as a generalization of the results in [3] which are concerned with the proper holomorphic polynomial maps between balls.

Theorem 1.1. Let $\Omega_1, \Omega_2$ be bounded symmetric domains of classical type and $f, g : \Omega_1 \to \Omega_2$ proper holomorphic polynomial maps such that $f(0) = g(0) = 0$. Then $f$ and $g$ are equivalent if and only if they are isotropically equivalent at $0$.

Theorem 1.2. There are uncountably many inequivalent proper holomorphic maps from $\Omega_{I}^{r,s}$ to $\Omega_{II}^{r-1,2s}$ for $r \geq 2, s \geq 2$.

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The motivation of this paper comes from generalizing the study on proper holomorphic maps between balls to bounded symmetric domains of rank greater than or equal to two. Proper holomorphic maps between balls have been studied for a long time since Alexander ([1]) proved that every proper holomorphic self-map of the $n$-dimensional unit ball $\mathbb{B}^n$ with $n \geq 2$ is a holomorphic automorphism. For proper holomorphic maps between balls with different dimensions, much work has been done, relating the maximum degree of proper holomorphic maps to the difference of dimensions between the domain ball and the target ball. As the first work along these lines, Webster ([22]) proved that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{n+1}$ with $n \geq 3$, $C^3$-smooth up to the boundary, is equivalent to the embedding

$$f_s : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0).$$

Given a proper holomorphic map $f$ from $\mathbb{B}^n$ to $\mathbb{B}^N$, we consider a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^{N+k}$ defined by $z \mapsto (f(z), 0, \ldots, 0)$ with $k$-zeros for which we will use the same notation $f$ if there is no confusion. When $n \geq 3$ and $N \leq 2n - 2$, Faran ([9]) showed that it is equivalent to $f_s$ if it is extended holomorphically over the boundary. Furthermore he precisely classified the equivalence classes of proper holomorphic maps from $\mathbb{B}^2$ to $\mathbb{B}^3$ which is $C^3$-smooth up to the boundary ([8]). In [10], Forstnerič proved that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ which is $C^{N-n+1}$-smooth up to the boundary is a rational map $(p_1, \ldots, p_N)/q$ where $p_j$ and $q$ are holomorphic polynomials of degree at most $N^2(N - n + 1)$. Since this work has been done, much results fit into the framework of providing sharp bounds in special situations. See [5, 7, 16], for more details. If $N = 2n - 1$, there is a proper holomorphic map which is called the Whitney map $f_w : \mathbb{B}^n \to \mathbb{B}^{2n-1}$ defined by

$$f_w(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_n z_1, z_n z_2, \ldots, z_n^2).$$

It is inequivalent to the embedding $f_s$. Moreover Huang and Ji ([13]) proved that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$ with $n \geq 3$ is equivalent to $f_s$ or $f_w$ and any proper holomorphic embedding which is $C^2$-smooth up to the boundary is equivalent to $f_s$. If the dimension of the target domain is larger than $2n$, there are infinitely many inequivalent proper holomorphic maps. For example, $f_\theta : \mathbb{B}^n \to \mathbb{B}^{2n}$ given by

$$f_\theta(z) = (z_1, \ldots, z_{n-1}, \cos \theta z_n, \sin \theta z_1 z_n, \ldots, \sin \theta z_n z_n)$$

with $0 \leq \theta \leq \pi/2$ are found by D’Angelo ([3]). In [3], D’Angelo showed that any two proper holomorphic polynomial map from $\mathbb{B}^n$ to $\mathbb{B}^N$ preserving the origin are equivalent if and only if they are isotropically equivalent at the origin and as a consequence ([11]) are inequivalent for all $0 \leq \theta < \pi/2$. Interestingly, Hamada ([11]) showed that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{2n}$ with $n \geq 4$ is equivalent to $f_\theta$ for some $\theta$, $0 \leq \theta \leq \pi/2$ and Huang, Ji and Xu ([15]) showed that any proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $4 \leq n \leq N \leq 3n - 4$ which is $C^3$-smooth up to the boundary should be equivalent to $f_\theta$ for some $\theta$, $0 \leq \theta \leq \pi/2$. Recently Huang-Ji-Yin ([14]) proved that any proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^N$ with $n \geq 8$ and $3n + 1 \leq N \leq 4n - 7$ should be equivalent to proper rational map from $\mathbb{B}^n$ to $\mathbb{B}^{3n}$.

As a one generalization of the unit ball, one consider bounded symmetric domains which are Hermitian symmetric spaces of non-compact type with non-smooth boundaries. There are several rigidity theorems on proper holomorphic maps between bounded symmetric domains of rank greater than or equal two. In contrast
with the case of the unit balls, the difference of the ranks between the domains is more crucial than that of the dimensions. The first result on bounded symmetric domains along those lines is the following which is due to Tsai. Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic map between irreducible bounded symmetric domains $\Omega_1$ and $\Omega_2$. If $\text{rank}(\Omega_1) \geq \text{rank}(\Omega_2)$, then $\text{rank}(\Omega_1) = \text{rank}(\Omega_2)$ and $f$ should be a totally geodesic isometric embedding with respect to the Bergman metrics on the domains (see [20]). If $\Omega_1 = \Omega^I_{r,r-1}$ and $\Omega_2 = \Omega^I_{r,r}$, then $f$ is also a totally geodesic isometric embedding (see [21]), although the rank of $\Omega_2$ is larger than that of $\Omega_1$. Furthermore, Ng ([18]) showed that for $f : \Omega^I_{r,s} \to \Omega^I_{r',s'}$, if $s \geq 2$, $s \geq r' \geq r$ and $r' \leq 2r - 1$, then $f$ is equivalent to the embedding, $Z \mapsto \left( \begin{array}{cc} Z & 0 \\ 0 & 0 \end{array} \right)$. If the difference of the ranks of the domains gets bigger, then it is expected that there are lots of inequivalent proper holomorphic maps. In [19], one way of finding proper holomorphic maps between bounded symmetric domains of type $I$ is suggested and several proper holomorphic maps are constructed. For example, for $r' = 2r - 1$ and $s' = 2s - 1$, there is a generalized Whitney map defined by

$$
\begin{pmatrix}
z_{11} & \cdots & z_{1s} \\
\vdots & \ddots & \vdots \\
z_{r1} & \cdots & z_{rs}
\end{pmatrix}
\mapsto
\begin{pmatrix}
z_{11}^2 & z_{11}z_{12} & \cdots & z_{11}z_{1s} & z_{12} & \cdots & z_{1s} \\
z_{11}z_{21} & z_{21}z_{12} & \cdots & z_{21}z_{1s} & z_{22} & \cdots & z_{2s} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{11}z_{r1} & z_{r1}z_{12} & \cdots & z_{r1}z_{1s} & z_{r2} & \cdots & z_{rs} \\
z_{21} & z_{22} & \cdots & z_{2s} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & \cdots & 0
\end{pmatrix}.
$$

In this paper, as a one step to observe analogous phenomenon on proper holomorphic maps between bounded symmetric domains of rank greater than or equal to two, we generalize the result of D’Angelo in [3] to the domains of classical type.

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### 2. Preliminaries

In this section, we introduce terminology and some basic background. A bounded domain $\Omega$ is called *symmetric* if for each $p \in \Omega$, there is a holomorphic automorphism $t_p$ such that $t_p^2$ is the identity map of $\Omega$ which has $p$ as an isolated fixed point. All bounded symmetric domains are homogeneous domain, i.e. the automorphism group acts transitively on the domain. In 1920’s, Cartan classified all irreducible bounded symmetric domains. There are 4 classical types and 2 exceptional types. The four classical types are given by (1), (2), (3) and (4) in the introduction. Note that $\Omega^I_{m,1}$ is the m-dimensional unit ball and $\Omega^H_{1,1}$ is the unit disc.

From now on, we will use the notation $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(r+s, \mathbb{C})$ to split $M$ into 4 block matrices with $A \in M^C_{r,s}$, $B \in M^C_{r,s}$, $C \in M^C_{s,r}$ and $D \in M^C_{s,r}$. We will denote by $ASM^C_{n,n}$ the set of anti-symmetric complex $n \times n$ matrices and by $SM^C_{n,n}$ the set of symmetric complex $n \times n$ matrices.
Let $U(r, s)$ be the subgroup of $GL(r + s, \mathbb{C})$ satisfying

$$M\begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix} = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix}$$

for all $M \in U(r, s)$. Let $SU(r, s)$ be the subset of $U(r, s)$ which consists of the matrices with determinant one. Explicitly,

$$SU(r, s) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(r + s, \mathbb{C}) : AA^* - BB^* = I_r, AC^* = BD^*, CC^* - DD^* = -I_s \right\}.$$  

(2.1)

Let $O(n + 2, \mathbb{C})$ be the complex orthogonal group of $(n + 2) \times (n + 2)$ matrices.

Since every bounded symmetric domain is Hermitian symmetric space of non-compact type, the domain can be canonically embedded into the corresponding compact dual (Borel embedding) and every holomorphic automorphism of the domain can be extended to the automorphism of its compact dual. For example, let $G_{r,s}$ be the Grassmannian of $r$-planes in $r + s$ dimensional complex vector space $\mathbb{C}^{r+s}$ which is the compact dual of $\Omega^I_{r,s}$. For $X \in M(r+s)^r_\mathbb{C}$ of rank $r$, denote $[X]$ the $r$-plane in $\mathbb{C}^{r+s}$ which is generated by row vectors of $X$. For $M \in GL(r + s, \mathbb{C})$, $M$ acts on $G_{r,s}$ by $[X] \mapsto [XM]$. Then the Borel embedding $\xi^I$ of $\Omega^I_{r,s}$ is given by

$$\xi^I(Z) = [I_r, Z] \in G_{r,s}$$

and $M \in U(r,s)$ acts on $\Omega^I_{r,s}$ by $g_M(Z) := \xi^I^{-1}(\xi(Z)M)$. Explicitly, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(r,s)$, $M$ acts on $\Omega^I_{r,s}$ by

$$Z \mapsto (A + ZC)^{-1}(B + ZD).$$  

(2.2)

Similarly, for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$ acts on $\Omega^H_n$ and $\Omega^{{II}'}_n$ by (2.2).

In case of $\Omega^{{IV}'}_n$, the explicit expression of the holomorphic automorphism is little more messy. The compact dual of $\Omega^{{IV}'}_n$ is the hyperquadric $H_n$ in $\mathbb{P}_{n+1}$ given by $H_n := \{ [z_1, \ldots, z_{n+2}] \in \mathbb{P}_{n+1} : z_1^2 + \cdots + z_{n+1}^2 - z_{n+2}^2 = 0 \}$. Then the Borel embedding $\xi^{{IV}'}$ is given by

$$\xi^{{IV}'}(Z) = [-2iZ, 1 + ZZ^t, i(1 - ZZ^t)] \in H_n$$

(2.3)

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n + 2, \mathbb{C}) \cap U(n,2)$, then $M$ acts on $\Omega^{{IV}'}_n$ by

$$Z = (z_1, \ldots, z_n) \mapsto \frac{1}{(-2iZB + Z'D)(i,1)}(2iZA - Z'C)$$

where $Z' = (1 + ZZ^t, i - iZZ^t)$.

The automorphism groups of classical domains and their isotropy groups at the origin are given by the following:

1. $\text{Aut} (\Omega^I_{r,s}) = U(r, s)$, $\text{Isot} (\Omega^I_{r,s}) = \left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} : U \in U(r), V \in U(s) \right\}$
In this section, we will prove Theorem 1.1. Define a polynomial function \( S_{r,s}^I : \Omega_{r,n} \to \mathbb{R} \) by

\[
S_{r,s}^I(Z) = \det(I_r - ZZ^*)
\]

for \( Z = (z_{ij}) \in M_{r,s}^\mathbb{C} \), as a real polynomial in \( \text{Re}(z_{ij}), \text{Im}(z_{ij}) \) where \( 1 \leq i \leq r, 1 \leq j \leq s \). \( S_{r,s}^I \) is a polynomial of degree 2 in each \( \text{Re}(z_{ij}), \text{Im}(z_{ij}) \). In case of \( \Omega_{r,s}^{I^I} \), it is known that \( \det(I_r - ZZ^*) = s_{r,s}^{I^I}(Z)^2 \) for some polynomial \( s_{r,s}^{I^I}(Z) \) (cf. [17]). Define \( S_{r,s}^{I^I} : \Omega_{r,s}^{I^I} \to \mathbb{R} \) and \( S_{r,s}^{I^I^I} : \Omega_{r,s}^{I^I^I} \to \mathbb{R} \) by

\[
S_{r,s}^{I^I}(Z) = s_{r,s}^{I^I}(Z) \quad \text{for} \quad Z \in ASM_{n,n}^\mathbb{C},
\]

\[
S_{r,s}^{I^I^I}(Z) = \det(I_r - ZZ^*) \quad \text{for} \quad Z \in SM_{n,n}^\mathbb{C},
\]

and

\[
S_{r,s}^{IV}(Z) = 1 - 2ZZ^* + |ZZ^*|^2 \quad \text{for} \quad Z \in \mathbb{C}^n
\]

\( S_{r,s}^I(Z), S_{r,s}^{I^I}(Z), S_{r,s}^{I^I^I}(Z) \) and \( S_{r,s}^{IV}(Z) \) are called the \textit{generic norm} of the corresponding domains cf. [17]. Then \( S_{r,s}^{I^I}(Z) \) is a polynomial of degree 2 in each \( \text{Re} \, z_{ij}, \text{Im} \, z_{ij} \) for \( 1 \leq i < j \leq n \) and \( S_{r,s}^{I^I^I}(Z) \) is a polynomial of degree 4 in each \( \text{Re} \, z_{ij}, \text{Im} \, z_{ii} \) for \( 1 \leq i \leq n \).

Lemma 3.1.

1. For \( Z = (z_{ij}) \in M_{r,s}^\mathbb{C} \), the coefficient of \( (\text{Re} \, z_{ij})^2 \) in \( S_{r,s}^I(Z) \) is

\[
-\det(I_{r-1} - Z'Z'^*)
\]

where \( Z' \) is the \((i,j)\) minor of \( Z \).

2. For \( Z = (z_{ij}) \in ASM_{n,n}^\mathbb{C} \), the coefficient of \( (\text{Re} \, z_{ij})^4, 1 \leq i < j \leq n \) in \( S_{r,s}^{I^I}(Z) \) is

\[
\det(I_{n-2} - Z''Z'')
\]

where \( Z'' \) is \((n-2) \times (n-2)\) matrix obtained by removing \( i,j \)-th rows and \( i,j \)-th columns in \( Z \).

3. For \( Z = (z_{ij}) \in SM_{n,n}^\mathbb{C} \), the coefficient of \( (\text{Re} \, z_{ij})^4, 1 \leq i < j \leq n \) and \( (\text{Re} \, z_{ij})^2 \) for \( 1 \leq i \leq n \) in \( S_{r,s}^{I^I^I}(Z) \) are

\[
\det(I_{n-2} - Z''Z'') \quad \text{and} \quad -\det(I_{n-1} - Z'Z'^*)
\]

respectively where \( Z'' \) is \((n-2) \times (n-2)\) matrix removing \( i,j \)-th rows and \( i,j \)-th columns in \( Z \) and \( Z' \) is the \((i,i)\) minor of \( Z \).
Proof.

1. For $Z \in M^C_{r,s}$, denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ with $X = (X', X'') = (x_{ij}) \in M^C_{r-1,s}$ where $X' \in M^C_{r-1,1}$, $X'' \in M^C_{r-1,s-1}$ and $Y = (y_1, \ldots, y_s) \in M^C_{1,s}$. We only consider the coefficient of $(\text{Re} y_1)^2$. The coefficient of $(\text{Re} y_1)^2$ in $S^I_{r,s}(Z)$ is $\frac{\partial^2}{\partial y_1 \partial y_1} S^I_{r,s}(Z)$. Since

$$\frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \frac{\partial}{\partial y_1} \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ -YX^* & 1 - YY^* \end{pmatrix}$$

and $YX^* = (y_1 x_{11} + \cdots + y_s x_{1s}, \ldots, y_1 x_{(r-1)1} + \cdots + y_s x_{(r-1)s})$, we obtain

$$\frac{\partial}{\partial y_1} \det(I_r - ZZ^*) = \det \begin{pmatrix} I_{r-1} - XX^* & -XY^* \\ (-x_{11}, -x_{21}, \ldots, -x_{(r-1)1}) & -y_1 \end{pmatrix}$$

and

$$\frac{\partial^2}{\partial y_1 \partial y_1} \det(I_r - ZZ^*)$$

$$= \det \begin{pmatrix} I_{r-1} - XX^* & \begin{pmatrix} -x_{11} \\ \vdots \\ -x_{(r-1)1} \end{pmatrix} \\ (-x_{11}, -x_{21}, \ldots, -x_{(r-1)1}) & -1 \end{pmatrix}$$

$$= \det \begin{pmatrix} I_{r-1} - X''X''^* & 0 \\ (-x_{11}, -x_{21}, \ldots, -x_{(r-1)1}) & -1 \end{pmatrix}$$

(3.4)

The second equation comes from subtracting $j$-th row of (3.3) by $x_{j1}$ times the $r$-th row of (3.3).

2. We will only consider the coefficient of $(\text{Re} z_{12})^4$. By Lemma 3.1, for $W = (w_{ij}) \in M^C_{r,n}$, det$(I_n - WW^*) = a_2(\text{Re} w_{12})^2 + a_1(\text{Re} w_{12}) + a_0$ where $a_i$ are polynomials in $\text{Re} w_{12}$, $\text{Im} w_{12}$, $i \neq 1$, $j \neq 2$ and $\text{Im} w_{12}$. Since the coefficient of $(\text{Re} z_{12})^4$ is the coefficient of $(\text{Re} w_{12})^2(\text{Re} w_{21})^2$ substituted $w_{ij} = z_{ij}$ for $1 \leq i < j < n$, $w_{ji} = -z_{ij}$ for $1 \leq i < j \leq n$ and $w_{ii} = 0$ for $1 \leq i \leq n$, the coefficient of $(\text{Re} z_{12})^4$ is $\det(I_{n-2} - Z''Z''^*)$ where $Z'' = (z_{ij})_{3 \leq i \leq n, 3 \leq j \leq n}$.

By Proposition 3.2. $S^I_{r,s}(Z)$, $S^{II}_{n}(Z)$, $S^{III}_{n}(Z)$ and $S^{IV}_{n}(Z)$ are irreducible.

Proof. In case of $S^I_{r,s}(Z)$, $S^{II}_{n}(Z)$, $S^{III}_{n}(Z)$ and $S^{IV}_{n}(Z)$ are irreducible.

Note that $r \leq s$. We use induction with respect to $k$ on $S^I_{r,s-r+k}$. For $Z \in M^C_{r,s-r+k}$, $1 - ZZ^*$ is irreducible. Suppose that $S^I_{r-1,s-1}(Z)$ is irreducible and $S^I_{r,s}(Z) = AB$ for some polynomial $A$ and $B$. Denote $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ as in the proof of Lemma 3.1. The degree of $\text{Re} y_1 = 2$.

Step 1: Suppose that there is a nonzero monomial of $(\text{Re} y_1)^2$ in $A$. Then $S^I_{r,s}(Z)$ should be in the form

$$S^I_{r,s}(Z) = (\mu(\text{Re} y_1)^2 + \sigma \text{Re} y_1 + \nu) B$$

where $\mu, \sigma, \nu$ and $B$ are polynomials without $\text{Re} y_1$ variable. Without loss of generality, we may assume that $B$ is not a constant. Then by Lemma 3.1, $\mu B$ is $\det(I_{r-1} - X''X''^*)$ which is irreducible by the induction hypothesis. Hence $B$
should be $\det(I_{r-1} - X^n X''^*)$ (up to constants). This implies that it consists of the monomials of variable $X'$. Then every variable of $X'$ should be in $A$. Consider the coefficient of $\left(\text{Re } x_{11}\right)^2$ which is irreducible and does not contain $\text{Re } x_{1j}$, $\text{Im } x_{1j}$ variables. However the coefficient of $\left(\text{Re } x_{11}\right)^2$ should contain $B$ and this induces a contradiction. Therefore there is no second order term in each $A$ and $B$.

**Step 2** Suppose that $S_{r,s}^I(Z) = (\mu \text{Re } y_1 + \sigma)(\nu \text{Re } y_1 + \rho)$ where $\mu, \sigma, \nu, \rho$ has no Re $y_1$ variable. Then $\mu \nu$ is irreducible by Lemma 3.1, and hence $\mu$ or $\nu$ is a constant. If $\mu$ is a constant, $\nu = \det(I_{r-1} - X^n X''^*)$ up to constant. Note that $\nu$ contains second order terms of variables in $X''$. Since there is no $\text{Re } y_1$ term in $\rho$, $\nu \text{Re } y_1 + \rho$ should have second order term. But by Step 1, $\nu \text{Re } y_1 + \rho$ cannot have second order term. Hence $S_{r,s}^I(Z)$ is irreducible.

**In case of $S_{n}^{II}$, $S_{n}^{III}$, $S_{n}^{IV}$:** Since for $n = 1$, $S_{1}^{III}(z) = 1 - |z|^2$ for $z \in \Delta$ which is irreducible, use the same method (induction) as in the proof of the case $S_{r,s}^I$ considering factorization with respect to $\text{Re } z_{11}$ variable. In case of $S_{n}^{IV}$, we can easily show that it is irreducible. We omit the proof.

We only need to prove that $S_{n}^{III}(Z)$ is irreducible. For $n = 2$, let $Z = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Then $S_{2}^{II}(Z) = 1 - |a|^2$ which is irreducible. For $n = 3$, let $Z = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Then $S_{3}^{II}(Z) = 1 - |a|^2 - |b|^2 - |c|^2$ which is also irreducible. Assume that $S_{2n}^{II}(Z)$ and $S_{2n-1}^{II}(Z)$ are irreducible. Since the even dimensional case is similar to the odd dimensional case, we will only consider the odd dimensional case. Since the coefficient of $(\text{Re } z_{ij})^2$ in $S_{2n+1}^{II}(Z)^2$ is $\det(I_{2n-1} - Z^n Z''^*)$ as in Lemma 3.1 [2], the coefficient of $(\text{Re } z_{ij})^2$ in $S_{2n+1}^{II}(Z)$ is irreducible. Hence similar proof of the case $S_{r,s}^I$ can be applied.

Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic polynomial map where $\Omega_1, \Omega_2$ are irreducible bounded symmetric domains of classical type. Let $S_1, S_2$ be the corresponding generic norms. Since $f$ is proper, by Proposition 3.2,

$$S_2(f(Z)) = 0 \text{ whenever } S_1(Z) = 0.$$ 

(Note that if $Z \in \partial \Omega_{IV}^n$, $S_{n}^{IV}(Z) = 0$ since if $ZZ^* = 1$, $S_{n}^{IV}(Z) < 0$. Hence we do not need to consider $ZZ^* - 1$ term in the definition of $\Omega_{IV}^n$.) Hence there is a real analytic map $F_f$ such that

$$S_2(f(Z)) = S_1(Z) F_f(Z).$$

We can polarize this equation by

$$S_2(f(Z), f(W)) = S_1(Z, W) F_f(Z, W).$$

**Example 3.3.**

1. Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$\det(I_{r'} - f(Z)f(W)^*) = \det(I_{r} - ZW^*) F_f(Z, W).$$

2. Let $f : \Omega_{IV}^n \to \Omega_{IV}^n$ be a proper holomorphic polynomial map. Then (3.6) is expressed by

$$1 - 2f(Z)f(W)^* + (f(Z)f(Z)^*) (f(W)f(W)^*) = \left(1 - 2W^* + (ZZ^*)(WW^*)\right) F_f(Z, W).$$
Lemma 3.4. Let $g : \Omega_1 \to \Omega_2$ and $f : \Omega_2 \to \Omega_3$ be proper holomorphic polynomial maps. Then

$$F_{f \circ g}(Z, W) = F_g(Z, W) \cdot F_f(g(Z), g(W)).$$

Proof.

$$S_1(Z, W)F_{f \circ g}(Z, W) = S_3(f \circ g(Z), f \circ g(W)) = S_2(g(Z), g(W)) \cdot F_f(g(Z), g(W)) = S_1(Z, W)F_g(Z, W) \cdot F_f(g(Z), g(W)).$$

\[\square\]

Lemma 3.5. Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an automorphism of a domain $\Omega$ with appropriate block matrices $A, B, C, D$. Then $F_U$ is given as the following:

1. If $\Omega = \Omega_r \cup \Omega''$, then

$$F_U(Z, W) = \frac{1}{\det(A + ZC) \det(A + WC)}.$$

2. If $\Omega = \Omega''$, then

$$F_U(Z, W) = \frac{1}{f_U(z, w)} \text{ where } f_U(Z, W)^2 = \det(A + ZC) \det(A + WC).$$

3. If $\Omega = \Omega''', \text{ then } F_U(Z, W) = \frac{1}{2 \{(-2iZB + Z'D)(i, 1)^t\} \{(-2iWB + W'D)(i, 1)^t\}}$

Proof. (1 and 2) Since by (21),

$$U(Z)U(W)^* = (A + ZC)^{-1}(B + ZD)((A + WC)^{-1}(B + WD))^* = (A + ZC)^{-1}((A + ZC)(A + WC)^* + ZZ^* - I_r)((A + WC)^{-1})^* = I_r - (A + ZC)^{-1}(I_r - ZZ^*)((A + WC)^{-1})^*,$$

(3.7)

$$\det(I_r - U(Z)U(W)^*) = \frac{\det(I_r - ZW^*)}{\det(A + ZC) \det(A + WC)}.$$

(3) Note that for $Z = (z_1, \ldots, z_n)$, $S_n^{IV}(Z) = -\frac{1}{2}Q(\xi^{IV}(Z))$ where $Q(x_1, \ldots, x_{n+2}) = |x_1|^2 + \ldots + |x_n|^2 - |x_{n+1}|^2 - |x_{n+2}|^2$ and $\xi^{IV}(Z) = (-2iZ, 1 + ZZ^*, i(1 - ZZ^*))$. Then

$$S_n^{IV}(g_M(Z)) = -\frac{1}{2}Q\left(\xi^{IV}(\xi^{IV-1}(\xi^{IV}(Z)M))\right) = -\frac{1}{2}Q\left(\frac{-2iZA + Z'C}{(-2iZB + Z'D)(i, 1)^t} \cdot \frac{-2iZB + Z'D}{(-2iZB + Z'D)(i, 1)^t}\right) = -\frac{1}{2} \frac{S_n^{IV}(Z)}{|(-2iZB + Z'D)(i, 1)^t|^2}$$

Remark 3.6. In case of $\Omega'''$, since for $Z \in ASM_n^{C}$, $C^tA + C^tZC$ is anti-symmetric. This implies that

$$\det(A + ZC) = \frac{\det(C^tA + C^tZC)}{\det(C)} = \frac{(Pf(C^tA + C^tZC))^2}{\det(C)}$$
where $Pf(Y)$ is the Pfaffian polynomial of a matrix $Y$ and hence $F_U$ is a rational function.

**Proof of Theorem 1.1** Suppose that $g \circ U = V \circ f$ for some $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in \text{Aut}(\Omega_2)$ and $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in \text{Aut}(\Omega_1)$. Then since $F_{g\circ U} = F_{V \circ f}$, by Lemma 3.4

$$F_g(U(Z), U(W))F_U(Z, W) = F_V(f(Z), f(W))F_f(Z, W).$$

By multiplying $S^1(Z, W)$ to both side, we obtain

$$S^2(g \circ U(Z), g \circ U(W)) = F_V(f(Z), f(W))S^1(f(Z), f(W)).$$

For simplicity, we only consider that the case $\Omega_1$ and $\Omega_2$ are bounded symmetric domains of type I. (3.3) is

$$\det(I_{r^*} - f(Z))f(W)^* = \det(I_{r^*} - g \circ U(Z)(g \circ U(W))^*) \frac{\det(V_1 + f(Z) V_3)}{\det(V_1 + f(W) V_3)}.$$

Put $W = 0$ in (3.9). Then

$$1 = \det(I_{r^*} - g \circ U(Z)(g \circ U(0))^*) \frac{\det(V_1 + f(Z) V_3)}{\det(V_1)}.$$  

If $U_3 = 0$, then by $g(U_1^{-1} Z U_4) = V \circ f(Z)$, we obtain $0 = g(0) = V \circ f(0) = V_1^{-1} V_2$ and hence $V_2 = 0$. So assume $U_3 \neq 0$. Note that in this case, $\det(U_1 + Z U_3)$ is not a constant. Suppose that $g \circ U(0) \neq 0$. Then $\det(I_{r^*} - g \circ U(Z)(g \circ U(0))^*)$ is not a constant and hence it should be of the form $p/q$ where $p$ and $q$ are non-constant polynomial without common factors and $q = \det(U_1 + Z U_3)$ for some $l$. But since product of $p/q$ and polynomial cannot be a constant, (3.10) induces a contradiction. Hence $g \circ U(0)$ should be zero. This implies that $0 = g(U_1^{-1} U_3) = V \circ f(0) = V(0) = V_1^{-1} V_2$. Hence $V_2 = 0$ (and also $V_3 = 0$) and hence $1 = \det(V_1 + f(Z) V_3)$. Put this in (3.9).

$$\det(I_{r^*} - f(Z))f(W)^*) = \det(I_{r^*} - g \circ U(Z)(g \circ U(W))^*).$$

Since right side of (3.10) is singular on $\{ Z \in \Omega_{r,s} : \det(U_1 + Z U_3) = 0 \}$, $U_3$ should be zero. \qed

4. Application

In this section, we suggest examples which are 1-parameter family of inequivalent proper holomorphic maps between bounded symmetric domains of classical type. We use Theorem 1.1 to prove that proper holomorphic maps $f_t : \Omega_1 \to \Omega_2$ are inequivalent for each $0 \leq t \leq 1$. Every example in this section is obtained in [19]. As in [6], we define the following equivalence relation:

Let $\Omega_1, \Omega_2$ be domains. Consider a continuous map $H : \Omega_1 \times [0, 1] \to \Omega_2$. Denote $H_t(z) = H(z, t)$. Suppose that $H_t : \Omega_1 \to \Omega_2$ is holomorphic for each $t \in [0, 1]$. Then we will say $H_t$ is a continuous family of holomorphic maps from $\Omega_1$ to $\Omega_2$.

**Definition 4.1.** Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_1 \to \Omega_3$ be proper holomorphic maps. Then $f$ and $g$ are homotopic in the target domain $\Omega$ if for each $t \in [0, 1]$, there is a proper holomorphic maps $H_t : \Omega_1 \to \Omega$ such that

- there are totally geodesic embeddings $e_k : \Omega_k \to \Omega$ for $k = 2, 3$, with respect to their Bergman metrics,
- $H_0 = e_2 \circ f$ and $H_1 = e_3 \circ g$,
• \( H_t \) is a continuous family of holomorphic maps from \( \Omega_1 \) to \( \Omega \).

4.1. 1-parameter family of proper holomorphic maps among \( \Omega_{1,s}^I \).

Consider proper holomorphic maps \( f, g : \Omega_{2,2}^I \rightarrow \Omega_{3,3}^I \) which are defined by
\[
f \left( \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & z_1z_2 & z_2 \\ z_3z_1 & z_2z_3 & z_4 \\ z_4 & z_3 & 0 \end{pmatrix}, \quad \text{for} \quad \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}^I
\]
\[
g \left( \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & \sqrt{2}z_1z_2 & z_2 \\ \sqrt{2}z_2z_3 & z_1z_4 + z_2z_3 & \sqrt{2}z_2z_4 \\ z_3^2 & \sqrt{2}z_3z_4 & z_4^2 \end{pmatrix}
\]
and \( f_t : \Omega_{2,2}^I \rightarrow \Omega_{4,4}^I \) be proper holomorphic maps for \( 0 \leq t \leq 1 \) defined by
\[
(4.1) \quad \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} \frac{z_1^2}{\sqrt{2} - t} & \frac{2(1 - t)}{2 - t} z_1z_2 + z_2z_3 & \frac{\sqrt{1 - t} z_2}{\sqrt{2} - t} \\ \frac{\sqrt{1 - t} z_3}{\sqrt{2} - t} & 2 \frac{1 - t}{2 - t} z_2z_4 & z_4^2 \\ \frac{\sqrt{1 - t} z_4}{\sqrt{2} - t} & \frac{1 - t}{2 - t} z_2z_4 & 0 \end{pmatrix}.
\]

Then it is easily observed that \( f \) and \( g \) are homotopic in the target domain \( \Omega_{4,4}^I \) through \( f_t \).

**Corollary 4.2.** \( f_t \) are inequivalent for different \( t, 0 \leq t \leq 1 \).

**Proof.** Suppose that \( f_t \circ A = B \circ f_s \) for some \( A \in U(2, 2) \) and \( B \in U(4, 4) \). Without loss of generality, we may assume that \( t \neq 0 \). Then by Theorem \( \ref{thm:1} \), \( Lf_s(Z)M \) for some \( U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in U(2), V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \in U(2) \) and \( L = (L_{1,1}), M = (M_{ij}) \in U(4) \). Denote \( f_t = \sum f_{t,j} \), where \( f_{t,j} \) is a homogeneous polynomial of degree \( j \). Then \( f_{t,j}(UZV) = Lf_{s,j}(Z)M \) for each \( j \). Consider linear part
\[
f_{t,1} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{1 - t} z_2 \\ 0 & 0 & 0 & \sqrt{1 - t} z_4 \\ \sqrt{1 - t} z_3 & 0 & 0 & 0 \end{pmatrix}.
\]

At first, at \( Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \),
\[
f_{t,1}(UZ_1V) = f_{t,1} \left( \begin{pmatrix} U_1V_1 & U_1V_2 \\ U_3V_1 & U_3V_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 & \sqrt{1 - t} U_1V_2 \\ 0 & 0 & 0 & \sqrt{1 - t} U_3V_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
and \( Lf_{s,1}(Z_1)M = 0 \). This implies that \( U_1V_2 = U_3V_2 = U_3V_1 = U_3V_2 = 0 \). Suppose that \( U_3 \neq 0 \). Then \( V_1 = V_2 = 0 \). This is a contradiction to \( V \in U(2) \). Hence \( U_3 = 0 \). This implies that \( U_2 = 0 \) and \( U_1 \neq 0 \), hence \( V_2 = V_3 = 0 \).
This implies that $L \sqrt{0} \leq 0$

Proof of Theorem 1.2. The same method of the proof in Corollary 4.2 can be applied. We omit the proof.
4.2. 1-parameter family of proper holomorphic maps among $\Omega^{III}$.

Consider the proper holomorphic maps $h_0 : \Omega^{III}_2 \to \Omega^{III}_3$ and $h_1 : \Omega^{III}_2 \to \Omega^{III}_3$ defined by

$$h_0 \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) = \left( \begin{array}{c} z_1^2 \\ \sqrt{2} \sqrt{z_1} z_2 \\ \sqrt{2} z_1 z_3 + z_2^2 \end{array} \right)$$

$$h_1 \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) = \left( \begin{array}{c} z_1^2 \\ \frac{2}{2-t} z_1 z_2 + z_3^2 \\ \sqrt{\frac{2-t}{2}} z_3 \end{array} \right), \quad \text{for} \quad \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) \in \Omega^{III}_2$$

which are the restriction of $f_0$ and $f_1$ to $\Omega^{III}_2$. These are homotopically equivalent in the target domain $\Omega^{III}_4$ by $h_t : \Omega^{III}_2 \to \Omega^{III}_4$ which are proper holomorphic maps given by $h_t(Z) = f_t|_{\Omega^{III}_2}(Z)$. Explicitly, (4.4)

$$h_t : \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) \mapsto \left( \begin{array}{c} z_1^2 \\ \frac{2}{2-t} z_1 z_2 + z_3^2 \\ 2 \sqrt{\frac{2-t}{2}} z_3 \end{array} \right)$$

We can apply similar method to $h_t$ and a higher version of $h_t$, we obtain the following:

**Corollary 4.4.** There are uncountably many inequivalent proper holomorphic maps from $\Omega^{III}_n$ to $\Omega^{III}_n$ with $n' = 1(n^2 + 3n - 2)$.

**References**

[1] Alexander, H., *Holomorphic mappings from the ball and polydisc*, Math. Ann. 209 (1974), 249-256.
[2] D’Angelo, John P., *Polynomial proper maps between balls*, Duke Math. J. 57 (1988), no. 1, 211-219.
[3] D’Angelo, John P., *Proper holomorphic maps between balls of different dimensions*, Michigan Math. J. 35 (1988), no. 1, 83-90.
[4] D’Angelo, John P., *Polynomial proper holomorphic mappings between balls*, II. Michigan Math. J. 38 (1991), no. 1, 53-65.
[5] D’Angelo, John P.; Lebl, Jiří, *On the complexity of proper holomorphic mappings between balls*, Complex Var. Elliptic Equ. 54 (2009), no. 3-4, 187-204.
[6] D’Angelo, John P.; Lebl, Jiří, *Homotopy equivalence for proper holomorphic mappings*, arXiv:1408.1104 (2014).
[7] D’Angelo, John P.; Lebl, Jiří; Peters, Han, *Degree estimates for polynomials constant on a hyperplane*, Michigan Math. J. 55 (2007), no. 3, 693-713.
[8] Faran, James J., *Maps from the two-ball to the three-ball*, Invent. Math. 68 (1982), no. 3, 441-475.
[9] Faran, James J., *The linearity of proper holomorphic maps between balls in the low codimension case*, J. Differential Geom. 24 (1986), no. 1, 15-17.
[10] Forstneric, Franc., *Extending proper holomorphic mappings of positive codimension*, Invent. Math. 95 (1989), no. 1, 31-61.
[11] Hamada, Hidetaka, *Rational proper holomorphic maps from $B^n$ into $B^{2n}$*, Math. Ann. 331 (2005), no. 3, 693-711.
[12] Huang, Xiaojun, *On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions*, J. Differential Geom. 51 (1999), no. 1, 13-33.
[13] Huang, Xiaojun; Ji, Shanyu, *Mapping $B^n$ into $B^{2n-1}$*, Invent. Math. 145 (2001), no. 2, 219-250.
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[14] Huang, Xiaojun; Ji, Shanyu; Yin, Wanke, *On the third gap for proper holomorphic maps between balls*, Math. Ann. **358** (2014), no. 1-2, 115-142.

[15] Huang, Xiaojun; Ji, Shanyu; Xu, Dekang, *A new gap phenomenon for proper holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$*, Math. Res. Lett. **13** (2006), no. 4, 515-529.

[16] Lebl, Jiří; Peters, Han, *Polynomials constant on a hyperplane and CR maps of spheres*, Illinois J. Math. **56** (2012), no. 1, 155-175 (2013).

[17] Loos, Ottmar, *Jordan pairs*. Lecture Notes in Mathematics, Vol. **460**, Springer-Verlag, Berlin-New York, 1975. xvi+218 pp.

[18] Ng, Sui-Chung, *Holomorphic double fibration and the mapping problems of classical domains*, Int Math Res Notices first published online September 18, 2013 doi:10.1093/imrn/rnt200.

[19] Seo, Aeryeong, *New examples of proper holomorphic maps among symmetric domains*, to be appeared in Michigan Math. J on 2015.

[20] Tsai, I Hsun, *Rigidity of proper holomorphic maps between symmetric domains*, J. Differential Geom. **37** (1993), no. 1, 123-160.

[21] Tu, Zhen-Han, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains*, Math. Z. **240** (2002), no. 1, 13-35.

[22] Webster, S. M., *The rigidity of CR hypersurfaces in a sphere*, Indiana Univ. Math. J. **28** (1979), no. 3, 405-416.

[23] Wolf, Joseph A., *Fine structure of Hermitian symmetric spaces*. Symmetric spaces, (Short Courses, Washington Univ., St. Louis, Mo., 1969-1970), pp. 271-357. Pure and App. Math., Vol. **8**, Dekker, New York, 1972.

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