Abstract. A transformation of gamma max-infinitely divisible laws viz. geometric gamma max-infinitely divisible laws is considered in this paper. Some of its distributional and divisibility properties are discussed and a random time changed extremal process corresponding to this distribution is presented. A new kind of invariance (stability) under geometric maxima is proved and a max-AR(1) model corresponding to it is also discussed.

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1. Introduction. Parallel to the classical notions of infinitely divisible (ID) laws and its subclass geometrically ID (GID) laws we have max-infinitely divisible (MID) and geometric-MID (G-MID) laws in the maximum setup which are discussed in Balkema and Resnick (1977), Rachev and Resnick (1991), Mohan (1998) and Satheesh (2002). For d.f.s

\[ F(x) = e^{-\psi(x)} \]

that are MID (which is always true in \( \mathbb{R} \)) distributions with d.f.s of the form

\[ \frac{1}{(1 + \psi(x))} \]

are referred to as G-MID laws. Processes related to these distributions are extremal processes, Rachev and Resnick (1991), Pancheva, et al. (200) and max-AR(1) processes, Satheesh and Sandhya (2006). Satheesh (2002) introduced \( \varphi \)-MID laws with d.f \( \varphi\{-\log F(x)\} \) for a Laplace transform (LT) \( \varphi \) and a d.f. \( F \). \( \varphi \)-MID laws can also be seen as the d.f obtained by randomizing the parameter \( c>0 \) in the first Lehman alternative \( F^c \) obtained from \( F \), by a distribution with LT \( \varphi \). Setting \( -\log F(x) = \psi(x) \) and taking \( \varphi \) to be the LT of a gamma(\( \beta \)) law we get the d.f

\[ \frac{1}{(1 + \psi(x))^{\beta}} \]

which we will refer to as the d.f of a gamma-MID law. When \( F \) is max-semi-stable and \( \varphi \) is exponential we get exponential max-semi-stable laws characterized in a max-AR(1) set up in Satheesh and Sandhya (2006). A gamma-max-semi-stable law was also discussed therein to illustrate the derivation of max-semi-selfdecomposable laws.

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Pillai (1990) had introduced geometric exponential (G-exponential) laws having Laplace transform (LT) \[
\frac{1}{1+\log(1+\lambda)}.\]
Similarly, the LT \[
\frac{1}{1+\log(1+\lambda)^\beta}, \quad \beta > 0
\]
will be called as that of geometric gamma(\(\beta\)) (G-gamma(\(\beta\))) law. Motivated by this construction, and writing the d.f of gamma-MID(\(\beta\)) laws as \(e^{-\beta \log(1+\psi(x))}\), \(\beta > 0\), we get the d.f of geometric-gamma-MID (G-gamma-MID(\(\beta\))) laws as \[
\frac{1}{1+\beta \log(1+\psi(x))}, \quad \beta > 0.
\]
Geometric generalized gamma laws were introduced and studied in Sandhya and Satheesh (2007).

The purpose of this note is to discuss certain properties of gamma-MID and G-gamma-MID models. Potential applications of these models are in finance, insurance and stock market; see Kaufman (2001), Rachev (1993) and Mittnick and Rachev (1993). Distributional properties of these models including a new kind of invariance (stability) under geometric(\(p\))-maximum, are presented in section 2. In section 3 we discuss extremal processes and a max-AR(1) model related to them. The support of the distributions discussed here is \(\mathbb{R}\), and by a geometric(\(p\)) law we mean a geometric law on \(\{1,2,\ldots\}\) with mean \(a = \frac{1}{p}\).

2. Divisibility properties of gamma-MID and G-gamma-MID laws.

With the above terminologies we have:

**Theorem 2.1** G-gamma-MID laws are G-MID.

**Proof.** We know that a d.f \(F(x)\) is G-MID iff \(e^{-\frac{x}{1-(1-x^{-1})}}\) is MID, Rachev and Resnick (1991). From our construction we have the d.f \(e^{-\beta \log(1+\psi(x))}\), \(\beta > 0\) that is always MID.

Setting \[\frac{1}{F(x)} - 1 = \beta \log \{1+\psi(x)\}\] we have \[
\frac{1}{1+\beta \log(1+\psi(x))} = F(x) \text{ is G-MID.}
\]

**Theorem 2.2** A d.f \(F(x) = \frac{1}{1+\log(1+\psi(x))^{\beta}}, \beta > 0\) is G-gamma-MID iff \(\frac{1}{(1+\psi(x))^{\beta}}, \beta > 0\) is a d.f. This is clear from their construction above.

**Remark 2.1** It is interesting to note that \[\frac{1}{(1-\log F(x))} = G(x)\] is a d.f for a given d.f \(F(x)\) and in turn \[\frac{1}{(1-\log G(x))}\] is also a d.f and this operation can be repeated to obtain new d.f.s.

**Theorem 2.3** Every G-gamma-MID(\(\beta\)) distribution is the limit distribution of geometric(\(\frac{1}{\beta}\))-max of i.i.d gamma-MID(\(\frac{\beta}{n}\)) variables as \(n \to \infty\).
Proof. Let $F_n(x)$ denote the d.f of a geometric ($\frac{1}{n}$)-max of i.i.d gamma-MID ($\frac{\beta}{n}$) variables.

Thus, 

$$F_n(x) = \frac{\frac{1}{n}(1 + \psi(x))^\frac{\beta}{n}}{1 - \frac{n-1}{n}(1 + \psi(x))^{-\frac{\beta}{n}}}$$

$$= \frac{1}{n(1 + \psi(x))^{\frac{\beta}{n}} - (n-1)}$$

$$= \frac{1}{1 + n\{(1 + \psi(x))^{\frac{\beta}{n}} - 1\}}.$$ 

Hence, 

$$\lim_{n \to \infty} F_n(x) = \frac{1}{1 + \log(1 + \psi(x))^{\beta}} = \frac{1}{1 + \beta \log(1 + \psi(x))},$$

proving the assertion.

Theorem 2.4 The limit of $n$-max of G-gamma-MID ($\frac{\beta}{n}$) laws is gamma-MID($\beta$) as $n \to \infty$.

Proof. Since 

$$\lim_{n \to \infty} \left\{\frac{1}{1 + \frac{\beta}{n} \log(1 + \psi(x))} \right\}^n = e^{-\beta \log(1 + \psi(x))} = \frac{1}{(1 + \psi(x))^{\beta}},$$

the claim is proved.

Theorem 2.5 A distribution is invariant under geometric($p$)-max up to a scale-change iff it is geometric max-semi-stable with d.f 

$$\frac{1}{1 + \psi(x)} = \frac{1}{1 + a \psi(bx)},$$

$a > 1$ and $b \in (0,1) \cup (1,\infty)$.

Proof. See theorem 3.2 in Satheesh and Sandhya (2006) and its proof which is formulated in the max-AR(1) set up and under the terminology of exponential-max-semi-stable law. If $b > 1$ the geometric-max-semi-stable law is Frechet type and if $b < 1$ it is Weibull type, see Satheesh and Sandhya (2006). More generally we also have:

Theorem 2.6 For a d.f of the form $H(x) = \frac{1}{1 + \psi(x)}$, $\frac{1}{1 + a \psi(x)}$ is a d.f of a geometric($p$)-max for any $a > 0$.

Proof. 

$$\frac{1}{1 + a \psi(x)} = \frac{1}{a} \left(\frac{1}{a} + \psi(x)\right)^{-1} = \frac{1}{a} \left[\frac{1}{a} + \left(\frac{1}{H(x)} - 1\right)\right]^{-1}$$

$$= \frac{1}{a} H(x) \left[\frac{H(x) + a - aH(x)}{a}\right]^{-1}$$

$$= \frac{1}{a} H(x) \left[1 - \left(1 - \frac{1}{a}\right) H(x)\right]^{-1}$$
\[\sum_{k=0}^{\infty} \frac{1}{d} \left(1 - \frac{1}{a}\right)^k [H(x)]^k\]

\[= P\left(\text{Max}(X_1, X_2, \ldots, X_{N(p)}) \leq x\right), \quad p = \frac{1}{a},\]

where \(N(p)\) is a geometric\((p)\) r.v and \(X_i\)'s are i.i.d with d.f \(H(x)\) proving the assertion.

This is the max-analogue of lemma.3.1 in Pillai (1990). In particular we have:

**Theorem 2.7** Geometric\((p)\)-max of i.i.d G-gamma-MID\((\beta)\) variables is G-gamma-MID\((\frac{\beta}{p})\) for any \(p \in (0,1)\) where the geometric\((p)\) r.v is independent of the components.

**Proof.** The d.f of geometric\((p)\)-max of i.i.d G-gamma-MID\((\beta)\) variables is given by;

\[
\frac{p}{1 + \beta \log(1 + \psi(x))} \quad \frac{1 - (1 - p)/\{1 + \beta \log(1 + \psi(x))\}}{1 - \log(1 + \psi(x))}
\]

which is the d.f of a G-gamma-MID\((\frac{\beta}{p})\) law.

**Remark 2.2** The invariance property under geometric\((p)\)-max described above is new and is different from the invariance property under geometric\((p)\)-max up-to a scale-change in theorem.2.5 characterizing geometric-max-semi-stable laws. In theorem.2.7 it is invariance up-to a change of shape parameter.

**3. Processes related to gamma-MID and G-gamma-MID laws.**

**3.1 Random Time Changed Extremal Processes.** Extremal processes (EP) are processes with increasing right continuous sample paths and independent max-increments. The univariate marginals of an EP determine its finite dimensional distributions. Also, the max-increments of an EP are MID. The EPs will be referred to by the distribution of their max-increments. Pancheva, et al. (2006) has discussed random time changed or compound EPs and their theorem.3.1 together with property.3.2 reads: Let \(\{Y(t), t \geq 0\}\) be an EP having homogeneous max-increments with d.f \(F_i(y) = \exp \{-t\mu([\lambda, y])\}, y \geq \lambda, \lambda \) being the bottom of the rectangle \(\{F>0\}\) and \(\mu\) the exponential measure of \(Y(1)\), that is, \(\mu([\lambda, y]) = -\log F(y)\).

Let \(\{T(t), t \geq 0\}\) be a non-negative process independent of \(Y(t)\) having stationary, independent and additive increments with Laplace transform (LT) \(\phi\). If \(\{X(t), t \geq 0\}\) is the compound EP obtained by randomizing the time parameter of \(Y(t)\) by \(T(t)\), then \(X(t) = Y(T(t))\) and its d.f is:
\[ P\{X(t)<x\} = \{\varphi(\mu([\lambda,x]))\} \].

Pancheva, et al. (2006) also showed that in the above setup \(Y(T(t))\) is also an EP. Extending the discussion in Pancheva, et al. (2006) Satheesh and Sandhya (2006) showed that the EP obtained from a random time changed (by compounding a) max-semi-stable EP is max-semi-selfdecomposable(b) if the compounding process is semi-selfdecomposable. Here we have some more results in this direction.

**Theorem 3.1** The EP obtained by compounding a gamma-MID(\(\beta\)) EP having homogeneous max-increments is G-gamma-MID(\(\beta\)) if the compounding process is unit exponential.

**Proof.** If \(\{Y(t)\}\) is a gamma-MID(\(\beta\)) EP and \(\{T(t)\}\) is unit exponential with \(d.f. G\) then:

\[
\frac{1}{1 + \beta \log(1 + \psi(x))} = \int_{0}^{\infty} e^{-\beta \log(1 + \psi(x))} dG(t),
\]

which proves the assertion.

**Theorem 3.2** The EP \(\{Y(T(t))\}\) obtained from a random time changed EP \(\{Y(t)\}\) having homogeneous max-increments with \(d.f. e^{-\beta x}\) is G-gamma-MID(\(\beta\)) if the compounding process \(\{T(t)\}\) is G-gamma(\(\beta\)).

**Proof.** We know that the LT of a G-gamma(\(\beta\)) law is \(\frac{1}{1 + \log(1 + \lambda)^{\beta}}\). If \(G\) denote its \(d.f.\) then;

\[
\frac{1}{1 + \beta \log(1 + \psi(x))} = \int_{0}^{\infty} e^{-\psi(x)} dG(t),
\]

proving the assertion.

**3.2. An Auto-regressive Model.** Now consider a first order max-autoregressive (max-AR(1)) model described as below. Here a sequence of r.vs \(\{X_n, n>0\} \) defines the max-AR(1) scheme if for some 0<p<1 there exists an innovation sequence \(\{\epsilon_n\}\) of i.i.d r.vs such that;

\[
\begin{align*}
X_n &= X_{n-1}, \text{ with probability } p \\
 &= X_{n-1} \lor \epsilon_n, \text{ with probability } (1-p).
\end{align*}
\]

In terms of \(d.f.s\) and assuming stationarity this is equivalent to;

\[
F(x) = F(x)\{p + (1-p) F_\epsilon(x)\}. \text{ That is;}
\]

\[
F(x) = \frac{pF_\epsilon(x)}{1 - (1-p)F_\epsilon(x)}
\]
Hence \( \{X_n\} \) is a geometric\((p)\)-max of the innovation sequence \( \{\varepsilon_n\} \). Invoking theorem 2.7 we have proved;

**Theorem 3.3** In the max-AR(1) structure (1) the sequence \( \{X_n\} \) and the innovation sequence \( \{\varepsilon_n\} \) are related as follows for any \( p \in (0,1) \). \( \{X_n\} \) is G-gamma-MID(\( \beta \)) variables iff \( \{\varepsilon_n\} \) is G-gamma-MID(\( \frac{\beta}{p^p} \)). Equivalently,

**Theorem 3.4** A necessary and sufficient condition for an max-AR(1) process \( \{X_n\} \) with the structure in (1) is stationary Markovian for any \( p \in (0,1) \) with G-gamma-MID(\( \beta \)) distribution is that the innovation’s are G-gamma-MID(\( \frac{\beta}{p^p} \)) distributed.

**References.**

Balkema, A. A and Resnick, S. (1977). Max-infinite divisibility, *J. Appl. Probab.* **14**, 309-319.

Kaufman, E (2001). *Statistical Analysis of Extreme Values, From Insurance, Finance, Hydrology and Other Fields*, Extended 2nd Edition, Birkhauser.

Mittnick, S. and Rachev, S. T (1993). Modelling asset returns with alternative stable distributions, *Econ. Rev.*, **12**, 161-330.

Mohan, N. R (1998). On geometrically max-infinitely divisible laws, *J. Ind. Statist. Assoc.* **36**, 1-12.

Pancheva, E; Kolkovska, E. T and Jordanova, P. K (2006). Random time changed external processes, *Probab. Theory Appl.* **51**, 752-772.

Pillai, R. N (1990). Harmonic mixtures and geometric infinite divisibility, *J. Ind. Statist. Assoc.* **28**, 87-98.

Rachev, S. T (1993). Rate of convergence for maxima of random arrays with applications to stock returns, *Statist. and Decisions*, **11**, 279-288.

Rachev, S. T and Resnick, S (1991). Max-geometric infinite divisibility and stability, *Comm. Statist. – Stoch. Mod.* **7**, 191-218.

Sandhya, E and Satheesh, S (2007). Geometric generalized gamma distributions and related processes, *to appear in Appl. Math. Sci.*, 2008.

Satheesh, S (2002). Aspects of randomization in infinitely divisible and max-infinitely divisible laws, *ProbStat Models* **1**, 7-16.

Satheesh, S and Sandhya, E (2006b). Max-semi-selfdecomposable laws and related processes, *Statist. Probab. Lett.* **76**, 1435-1440.