Neutrino oscillations in nuclear media

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Abstract

On basis of effective interactions of charged lepton and hadron currents, we obtain an effective interacting Hamiltonian of neutrinos in nuclear media up to the leading order. Using this effective Hamiltonian, we study neutrino mixing and oscillations in nuclear media and strong magnetic fields. We compute neutrino mixing angle and mass squared difference, and find the pattern of vacuum neutrino oscillations is modified in magnetized nuclear media. Comparing with the vacuum neutrino oscillation, we find that for high-energy neutrinos, neutrino oscillations are suppressed in the presence of nuclear media. In the general case of neutral nuclear media with the presence of electrons, we calculate the mixing angle and mass squared difference, and discuss the resonance and level-crossing in neutrino oscillations.

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1 Introduction

In the standard model (SM) for fundamental particle physics, neutrinos are massless and left-handed. They are produced via neutral and charge current interactions. In charge current interaction, each flavor of physical neutrino is produced together with a corresponding charged lepton. It was proposed by Pontecorvo [1] that each physical neutrino state could be a superposition of different mass eigenstates, i.e., neutrino mixing and oscillations. This implies that neutrinos are massive. The recent experiments provide strong evidences that confirm neutrino flavor oscillations [2]–[5].

Recently, there are many experimental and theoretical studies dedicating to neutrino physics, in particular neutrino oscillations. These studies to know neutrino masses and mixing angles can be one of the best way for understanding new physics beyond the Standard Model. This new physics can be related to fundamental symmetries of theories, fundamental dynamics of quantum gravity and other dynamics in extensions of the standard model. In addition, the knowledges about neutrino masses, mixing angles and oscillations are important for studying astrophysics and cosmology.

On the other hand, neutrino oscillations can significantly be modified, when neutrinos travel through media rather than the vacuum. This effect occurs when neutrinos under consideration experience different interactions by passing through media. Neutrino oscillations in media can be large, even although neutrino oscillations in the vacuum is small. This medium effect was studied by Wolfenstein [6], Mikheyev and Smirnov [7] (MSW), which we briefly discuss in Sec. (3).

In this article, by considering the neutrino-lepton current interacting hadronic current, we study the nuclear effect on neutrino oscillations while neutrinos are propagating in nuclear media and strong magnetic fields. In the system of two flavor neutrinos, we compute the neutrino mass squared difference, mixing angle and neutrino oscillation probability, and we show the pattern of neutrino vacuum oscillations is modified by effects of nuclear media and strong magnetic fields. The resonance of neutrino oscillations probability and the inversion of neutrino flavors are discussed. We show that for high-energy neutrinos, the effect of nuclear media on neutrino oscillations is important, neutrino flavor oscillations are suppressed. This should be considered in studying properties of neutrinos produced inside neutron stars, quark stars and magnetars in astrophysics.

This article is arranged as follows. In Secs. 2 and 3 we briefly discuss neutrino oscillations in the vacuum and normal media. In Sec. 4 we describe neutrino scattering in nuclear media by effective current-current interactions in the SM. In Sec. 5 we present
our results of neutrino oscillations in nuclear media. The case of strongly magnetized nuclear media is considered in Sec. 6. In Sec. 7 we consider neutrino oscillation in the general case of neutral nuclear media with the presence of electron and strong magnetic fields. The neutrino oscillation resonance and flavor inversion are studied in Sec. 8. The last section, summary and remarks are given.

2 Neutrino oscillation in vacuum

Flavor neutrinos ($\nu_e, \nu_\mu, \nu_\tau$) are always produced and detected in flavor eigenstates via their interacting with intermediate gauge bosons $W_\mu^{(\pm)}$ and $Z^0_\mu$ in the SM. Flavor neutrinos are weak interaction eigenstates and in principle they are superpositions of the mass eigenstates $|\nu_i\rangle$

$$\mathcal{H}|\nu_i\rangle = E_i|\nu_i\rangle, \quad i = 1, 2, 3,$$

where $E_i$ are the energy eigenvalues of the type-$i$ neutrino. For ultra-relativistic neutrinos, neutrino energies can be approximately written as

$$E_i \approx p_i + \frac{m_i^2}{2p_i},$$

where $m_i$ and $p_i$ are the type-$i$ neutrino mass and momentum respectively, and $p_i \gg m_i$.

Flavor eigenstates and Hamiltonian eigenstates (mass eigenstates) are related by an unitary transformation represented by a matrix $U$,

$$|\nu_l\rangle = \sum_{i=1}^{3} U_{li}|\nu_i\rangle,$$

where the flavor index $l = e, \mu, \tau$. This shows that flavor eigenstates is mixing of Hamiltonian eigenstate $|\nu_i\rangle$ and vice versa. Time evolution of flavor neutrino states are given by

$$|\nu_l(t)\rangle = e^{-i\mathcal{H}t}|\nu_l\rangle = \sum_{i=1}^{3} e^{-iE_i t} U_{li}|\nu_i\rangle,$$

indicating, after some time $t$, the evolution of these flavor states leads to flavor neutrino oscillations. The probability of such neutrino oscillations is given by

$$P_{\nu_l \to \nu_{l'}} = \langle\nu_{l'}|\nu_l\rangle^2 = \sum_{i,j} |U_{li}^* U_{l'i}^* U_{lj}^* U_{l'j}|^2 \cos\left(\frac{\Delta m^2_{ij} t}{2E} + \varphi_{ij}\right),$$

indicating, after some time $t$, the evolution of these flavor states leads to flavor neutrino oscillations.
where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $\varphi_{ij} = \arg(U_{li}U^*_{lj}U_{li}U^*_{lj})$.

Assume that there are only two flavor neutrino species, for example $\nu_e$ and $\nu_\mu$. The unitary matrix $U$ is explicitly given by

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

where $\theta \equiv \theta_{12}$ presents a mixing angle. Eq. (3) becomes

$$|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle,$$
$$|\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle.$$

The Hamiltonian (11) in the base of mass eigenstates is

$$\hat{H}_v = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \simeq E + \begin{pmatrix} m_1^2/2E & 0 \\ 0 & m_2^2/2E \end{pmatrix},$$

where because of $p_i \gg m_i$, based on Eq. (2) the leading contribution to the neutrino energy $E_i$ is obtained by assuming $p_1 \approx p_2 = E$. By using Eqs. (6,8) the Hamiltonian in the base of flavor eigenstates is given by

$$\hat{H}_v = U\hat{H}U^\dagger = E + \frac{\Delta m^2}{4E} + \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

where $\Delta m^2 \equiv \Delta m^2_{12} = m_2^2 - m_1^2 (m_2 > m_1)$ and the mixing angle $\theta$ is given by

$$\tan 2\theta = \frac{2\hat{H}_{12}}{\hat{H}_{22} - \hat{H}_{11}},$$

and the probability of neutrino oscillation is

$$P_{\nu_\mu \leftrightarrow \nu_e} = \sin^2 2\theta \sin^2 \left[ (E_2 - E_1)t \right] = \sin^2 2\theta \sin^2 \left[ (E_2 - E_1)L \right],$$

where $E_2 - E_1 = \Delta m^2/(2E)$ and the second line is for relativistic neutrinos $L \approx ct$. This is the description of neutrino flavor mixing and oscillation in the vacuum. The discussions and calculations are applied for two-level systems ($\theta_{23}, \Delta m^2_{23}$) and ($\theta_{13}, \Delta m^2_{13}$). About this section, readers are referred to Refs. [8]-[10] for more details.
3 Neutrino oscillations in normal media

In the presence of a normal medium, effective neutrino mixing angles and mass squared differences can be modified by their interacting with particles in medium. Although interacting cross-sections are very small, modifications can be important if particle densities are very large.

Suppose that neutrinos travel through a normal neutral medium, where charged leptons are mainly electrons, the description of neutrino oscillations is modified due to the fact that electron neutrino scattering with electrons in medium are affected by both charged and neutral current interactions, while muon and tau neutrinos scatterings with medium are affected by the neutral current interaction only. This difference modifies the neutrino flavor mixing and oscillation in the vacuum as described in Eqs. (6-11). In the base of flavor eigenstates for two neutrino flavors $\nu_e$ and $\nu_\mu$, the charged and neutral current interactions in normal media induce an effective potential [6, 7]

\[
V_C = \sqrt{2} G_F n_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V_N = -\frac{1}{\sqrt{2}} G_F n_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(12)

where $n_e$ and $n_n$ are the number densities of electrons and nucleons. The potential $V_N$ from the neutral-current interaction is the same for all of neutrino flavors, and it only shifts the neutrino energy by a negligible small amount and does not affect neutrino oscillations. Due to the additional effective potentials $V_C$ and $V_N$ (12), the Hamiltonian (9) is changed to

\[
\hat{H}_m = E + \frac{m_1^2 + m_2^2}{4E} - \frac{1}{\sqrt{2}} G_F n_n
\]

\[
+ \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2} G_F n_e & 0 \\ 0 & 0 \end{pmatrix}.
\]

(13)

Using the unitary matrix (6) characterized by $\theta_m$ the mixing angle in normal media to diagonalize this Hamiltonian, one obtains the effective mass squared difference $\Delta m_m^2$ and the mixing angle $\theta_m$ [6, 7] (see also [8])

\[
\Delta m_m^2 = \sqrt{(\Delta m^2 \cos 2\theta - 2\sqrt{2} G_F n_e E)^2 + (\Delta m^2 \sin 2\theta)^2},
\]

(14)

\[
\tan 2\theta_m = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2} G_F n_e E}.
\]

(15)

In this base of Hamiltonian eigenstates, flavor neutrino states are expressed as

\[
|\nu_e\rangle = \cos \theta_m |\nu_1\rangle_m + \sin \theta_m |\nu_2\rangle_m,
\]

\[
|\nu_\mu\rangle = -\sin \theta_m |\nu_1\rangle_m + \cos \theta_m |\nu_2\rangle_m,
\]

(16)
and the probability of neutrino oscillations in normal media is given,
\[ P_{\mu \leftrightarrow e}^m(t) = \sin^2 2\theta_m \sin^2 \left(\frac{E_2^m - E_1^m}{E} t\right), \]  
(17)
where \( E_2^m \) and \( E_1^m \) are energy eigenvalues of the Hamiltonian (13) and
\[ E_2^m - E_1^m = \frac{\Delta m^2}{2E}. \]  
(18)
In the absence of a normal medium, \( V_C = 0 \), \( \Delta m^2 = \Delta m^2 \) and \( \theta_m = \theta \), one obtains neutrino flavors mixing and oscillations (11) in the vacuum.

4 Neutrino scattering in nuclear media

In the SM of fundamental particle physics, the effective interacting Hamiltonian of neutrinos (\( \nu_e, \nu_\mu \)) interacting with leptons (\( e, \mu \)) and quarks (\( u, d \)) is given by the V-A current-current interactions (see Fig.1-a), that is mediated by a massive charged gauge boson \( W^\pm \),
\[ \mathcal{H}_w = \frac{G_F}{\sqrt{2}} V_{ud} \left[ \bar{d} \gamma^\lambda (1 - \gamma_5) u \right] \left[ \bar{\nu}_e \gamma^\lambda (1 - \gamma_5) e + \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \mu \right] + \text{h.c.}, \]  
(19)
where \( G_F \) is the Fermi coupling constant and \( V_{ud} \) is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element. Introducing the axial current \( J_\pi^- \) for a charged pion, one has
\[ q_\mu F_\pi = \langle 0 | J_\pi^- (0) | \pi^- (q) \rangle, \]  
(20)
where the pion decay constant \( F_\pi \) encodes strong interaction effects [11]. The matrix element of the interacting Hamiltonian (19) can be written as an effective interacting vertex in energy-momentum space
\[ \mathcal{V}_{\nu l, \pi} = \frac{G_F}{\sqrt{2}} V_{ud} \langle 0 | J_\pi^- (0) | \pi^- (q) \rangle \bar{\nu}_l (k) \gamma^\lambda (1 - \gamma_5) l (p) + \text{h.c.} \]  
= \[ \bar{G}_F q_\lambda [\bar{\nu}_l (k) \gamma^\lambda (1 - \gamma_5) l (p) ] + \text{h.c.}, \]  
(21)
where \( l = e, \mu \), \( \bar{G}_F \equiv G_F V_{ud} F_\pi / \sqrt{2} \) and the energy-momentum conservation \( q = k + p \). Actually, the vertex (21) describes an effective interacting of neutrino \( \bar{\nu}_l (k) \) and lepton \( l^- (p) \) with nuclear matter via virtual pion fields \( \pi^- (q) \), as schematically shown in Fig.1-b by a dashed line ending with a cross.

Suppose that a neutrino travels through nuclear media, and interacts with quarks via V-A interactions (19), which is described as an effectively interacting vertex (21). By
Figure 1: Figure (a) indicates an effective interacting vertex \((19)\). Figure (b) indicates an effective interacting vertex \((21)\) of neutrino and lepton scattering with nuclear media represented by “×”, and the dashed line represents this effective interaction mediated by charged virtual particles of \(W^\pm\) and \(\pi^\pm\) fields.

Figure 2: The Figure (a) indicates the effective amplitude \((22)\) of a neutrino scattering with nuclear media (×) by exchanging two virtual particle fields. As indicated by Fig. 1a, Figure (b) indicates the contributions of the effective interaction \((19)\) to the amplitude \((22)\) of neutrino scattering with nuclear media (×).

Exchanging virtual particles of energy-momentum \(q\), a neutrino interacts with nuclear media and converts into a lepton, which in turn converts back to a neutrino, as shown in Fig. (2). This scattering amplitude receives all contributions by exchanging even number of virtual charged particles. At the leading order (by exchanging two virtual charged particles), we compute the leading order amplitude \(\mathcal{M}_{\nu_i\nu'_i}\) of neutrino scattering with nuclear media,

\[
\mathcal{M}_{\nu_i\nu'_i}(k, k', q, q', p) = \bar{G}_F^2 q \lambda q' \bar{\nu}_i(k) \gamma^\lambda (1 - \gamma_5) [l(p)\bar{l}(p)] \gamma^\rho (1 - \gamma_5) \nu_i(k')
\]

\[
= \bar{G}_F^2 m_l^2 \bar{\nu}_i(k) (1 + \gamma_5) [l(p)\bar{l}(p)] (1 - \gamma_5) \nu_i(k'), \tag{22}
\]

where \(q = k + p, q' = p + k'\), and in the second line we use Dirac equations for lepton and massless neutrino, i.e., \(\gamma^\rho l(p) = m_l l(p)\) and \(\bar{\nu}(k)\bar{k} \simeq 0\). Moreover, using the spinor wave-functions \(u_i^{(a)}(p)\) of leptons \([12]\)

\[
[l(p)\bar{l}(p)] = \sum_{a=1,2} u_i^{(a)}(p) \bar{u}_i^{(a)}(p) = \frac{\gamma^\rho + m_l}{2m_l}, \tag{23}
\]
we obtain
\[ M_{\nu_l,\nu'_l}(k, k', q, q', p) = \frac{1}{2} \bar{G}_F^2 m_t \bar{v}_l(k)(1 + \gamma_5) p (1 - \gamma_5) \nu_l(k') \]
\[ = \frac{1}{2} \bar{G}_F^2 m_t \bar{v}_l(k)(1 + \gamma_5)(q' - k')(1 - \gamma_5) \nu_l(k'). \]  
(24)

In order to obtain the total amplitude, we need to integrate over phase spaces of \( q \) and \( q' \) (see Fig. 2),
\[ M_{\nu_l,\nu'_l}(k, k') = \frac{1}{2} \bar{G}_F^2 m_t \int_q \int_{q'} \bar{v}_l(k)(1 + \gamma_5)(q' - k')(1 - \gamma_5) \nu_l(k') \]
\[ = -\frac{1}{2} \bar{G}_F^2 m_t \int_q \int_{q'} \bar{v}_l(k)(1 + \gamma_5) k'(1 - \gamma_5) \nu_l(k'), \]  
(25)

where
\[ \int_q \equiv \int_q \frac{d^4 q}{(2\pi)^4} = \int_q \frac{d^3 q}{2q_0 (2\pi)^3}, \]  
(26)

and \( q_0 \simeq q'_0 \approx m_\pi \) pion mass. Because we study the modification of neutrino energy-spectrum by this scattering, we only consider the case \( k = k' \), which induces \((2\pi)^3 \delta^3(q - q')\), as a result we approximately obtain
\[ M_{\nu_l,\nu'_l}(k, k) \approx -\frac{1}{8} \bar{G}_F^2 m_t m_\pi n_\pi \bar{v}_l(k)(1 + \gamma_5) k'(1 - \gamma_5) \nu_l(k), \]  
(27)

where the nucleon number-density
\[ n_\pi = \int \frac{d^3 q}{(2\pi)^3}. \]  
(28)

Putting Eq. (27) together with free kinetic term of neutrinos propagating through vacuum, we obtain the effective bilinear Lagrangian for neutrinos,
\[ L_{\bar{\nu}_e, \nu} = \bar{v}_e(k)(1 + \gamma_5) k'[1 - A_e](1 - \gamma_5) \nu_e(k), \]  
(29)
\[ L_{\bar{\nu}_\mu, \nu} = \bar{v}_\mu(k)(1 + \gamma_5) k'[1 - A_\mu](1 - \gamma_5) \nu_\mu(k), \]  
(30)

where
\[ A_e \approx \left[ \frac{G_F}{\sqrt{2}} V_{ud} F_\pi \right]^2 \left[ \frac{m_\pi}{8m_\pi^2} \right] n_\pi, \quad A_\mu \approx \left[ \frac{G_F}{\sqrt{2}} V_{ud} F_\pi \right]^2 \left[ \frac{m_\pi}{8m_\pi^2} \right] n_\pi. \]  
(31)

This implies that neutrinos receive wave-function renormalization when they propagate through nuclear media.
The same discussions and calculations are applied for considering tau neutrino scattering with nuclear media,

\[ \mathcal{L}_{\bar{\nu}_\tau, \nu} = \bar{\nu}_\tau(k)(1 + \gamma_5) \gamma \nu(k) \] (1 - \gamma_3) \nu_\tau(k), \quad (32)

\[ \mathcal{A}_\tau \approx \left( \frac{G_F \sqrt{2}}{V_{ud}} \right)^2 \frac{m_\tau}{m_\pi^2} n_\pi. \quad (33) \]

We clearly have \( \mathcal{A}_\tau \gg \mathcal{A}_\mu \gg \mathcal{A}_e \). It has to be pointed out that we only consider interactions of charged currents (19) mediated by virtual charged particles, and disregard interactions of neutral currents. Because the neutral current interaction is universal for all of neutrino and lepton flavors, as will be discussed in Sec. 3 for the MSW case.

### 5 Neutrino oscillations in nuclear media

In this section, we turn to study the effects of an uniform nuclear medium on neutrino oscillations. Analogously to the MSW approach, using Eqs. (9,29), we approximately obtain the effective Hamiltonian in the base of neutrino flavor states,

\[ \mathcal{H} = E + \frac{m_1^2 + m_2^2}{4E} + \frac{\Delta m^2}{4E} \left( \begin{array}{cc} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{array} \right) + \left( \begin{array}{cc} E \mathcal{A}_e & 0 \\ 0 & E \mathcal{A}_\mu \end{array} \right), \quad (34) \]

where the last term is due to the contributions of neutrinos scattering with nuclear media. The same procedure (13,15) described in Sec. 3 leads to the results of the mass squared difference \( \Delta m_n^2 \) is,

\[ \Delta m_n^2 = 2 \sqrt{(\mathcal{A}_e - \mathcal{A}_\mu)^2 E^4 + \Delta m^2 (\mathcal{A}_e - \mathcal{A}_\mu) E^2 \cos 2\theta + \left( \Delta m^2 / 2 \right)^2}, \quad (35) \]

and the mixing angle is

\[ \tan 2\theta_n = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta + 2(\mathcal{A}_e - \mathcal{A}_\mu) E^2}. \quad (36) \]

In the base of eigenstates of Hamiltonian (34), neutrino flavor states are represented by

\[ |\nu_e\rangle = \cos \theta_n |\nu_1\rangle_n + \sin \theta_n |\nu_2\rangle_n, \]

\[ |\nu_\mu\rangle = -\sin \theta_n |\nu_1\rangle_n + \cos \theta_n |\nu_2\rangle_n. \quad (37) \]

Based on these results (35,36) and using Eq. (11), we obtain the probability of neutrino oscillations in nuclear medium

\[ P_{\nu_\mu \leftrightarrow \nu_e}^n(t) = \sin^2 2\theta_n \sin^2 \left( (E_2^n - E_1^n)t \right), \quad (38) \]
where

\[ E_2^n - E_1^n = \frac{\Delta m^2_{21}}{2E}. \]

(39)

When \( A_e = A_\mu = 0 \), it reduces to neutrino oscillation in the vacuum.

From Eq. (31) for the Fermi coupling constant \( G_F = 1.16 \times 10^{-5} \text{ GeV}^{-2} \), the pion decay constant \( F_\pi = 130.41 \text{ MeV} \) \[13\], \( m_\mu/m_\pi \approx 0.5 \) and \( n_\pi \sim 10^{38} \text{ cm}^{-3} \) in nuclear media, we have

\[ A_e \approx 3 \times 10^{-16}, \quad A_\mu \approx 3 \times 10^{-14}. \]

(40)

In this case we approximately have

\[ \Delta m^2_n \approx 2\sqrt{A_\mu^2 E^4 + \Delta m^2 A_\mu E^2 \cos 2\theta + (\Delta m^2/2)^2}, \]

(41)

\[ \tan 2\theta_n \approx \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta + 2A_\mu E^2}. \]

(42)

We discuss the following two cases. The case (i) small vacuum mixing angle \( \theta \sim 0 \) and from Eq. (39), we approximately have

\[ E_2^n - E_1^n \approx \frac{\Delta m^2}{2E} + A_\mu E, \quad \tan 2\theta_n \approx 0. \]

(43)

The case (ii) large vacuum mixing angle \( \theta \sim \pi/4 \) and Eq. (39) is approximated to be

\[ E_2^n - E_1^n \approx \sqrt{\left(\frac{\Delta m^2}{2E}\right)^2 + (A_\mu E)^2}, \quad \tan 2\theta_n \approx \frac{\Delta m^2}{2A_\mu E^2}. \]

(44)

The first case implies that the nuclear medium effect could possibly be relevant for neutrino oscillations when neutrino energy is very large. In the second case for large neutrino energy, however, the mass squared difference is dominated by the nuclear effect \((A_\mu E)^2\) and mixing angle \( \theta_n \) becomes smaller.

In order to see the nuclear medium effect on neutrino oscillations, we consider the case of neutron stars or quark stars. Suppose that electron neutrinos \( \nu_e \) are produced in the core of neutron stars and their average energy is in the \( 1 \text{MeV} \lesssim \bar{E} \lesssim 10 \text{MeV} \) \[14\]. Based on Eqs. (39)(42) for neutrino neutrino energy \( E = 1 \text{MeV} \), we calculate the probability (38) of electron neutrinos converting to muon neutrinos when they travel from the center to surface of stars. In calculations, we use the vacuum oscillation parameters: the squared mass difference \( \Delta m^2_{\text{sun}} \approx 7.59 \times 10^{-5} \text{eV}^2 \) and \( \tan^2 \theta_{\text{sun}} \approx 0.47 \) from the Solar neutrino experiment \[2\]. The result is plotted in Fig. 3.
Figure 3: The probability of electron neutrinos converting to muon neutrinos in the presence of nuclear medium is plotted as a function of traveling distance $L \simeq ct$. Neutrino energy $E = 1$MeV.

In comparison, we make the same calculation in absence of nuclear matter (the vacuum oscillation) and result is plotted in Fig. 4. These plots show that in the absence of nuclear medium (vacuum oscillation), the probability of electron neutrino converting to muon neutrino at the surface of stars is about 5%. While in the presence of nuclear medium, the probability of electron neutrino converting to muon neutrino at the surface of stars is about $7 \times 10^{-7}$, averaged over traveling distance. This means that most electron neutrinos remain electron neutrinos. By this comparison, it implies that the effect of nuclear medium on neutrino oscillations should be considered when one studies properties of neutrinos created inside neutron stars and quark stars, in particular for high-energy neutrinos. When neutrino energy $E \sim 0.1$MeV, $E\mathcal{A}_\mu \sim (\delta m^2/2E)$, the effect of nuclear media $E\mathcal{A}_\mu$ is comparable with $(\delta m^2/2E)$ [see Eqs. (40,44)]. Therefore low-energy neutrinos $E < 0.1$MeV, the effect of nuclear media on neutrino oscillations is not important, and neutrino oscillation pattern is slightly deviated from the pattern of vacuum neutrino oscillations.
probability

0.01 0.02 0.03 0.04

20 000 40 000 60 000 80 000 100 000

Figure 4: The probability (38) of electron neutrinos converting to muon neutrinos in the absence of nuclear medium (the vacuum oscillation) is plotted as a function of traveling distance \( L \simeq c t \). Neutrino energy \( E = 1 \text{MeV} \).

6 Neutrino oscillations in strongly magnetized nuclear media

In the SM for particle physics, if neutrinos are massive, they have electric dipole and magnetic moment due to quantum corrections. Therefore neutrinos interact with electromagnetic fields, although they are electrically neutral. Neutrino magnetic moment is given by

\[
\mu_i = \frac{3G_F e}{8\sqrt{2}/\pi^2} m_i, \quad i = 1, 2, \tag{45}
\]

which is proportional to neutrino masses \( m_i \) [15]. In the unit of the Bohr magneton \( \mu_B = e/(2m_e) \), \( \mu_i = 3.1 \times 10^{-19} \mu_B (m_i/1\text{eV}) \). The interacting Hamiltonian is diagonal in the base of mass eigenstates and given by

\[
\mathcal{H}_b = -\mu_i B = -\frac{3G_F m_e^2}{8\sqrt{2}/\pi^2} \left( \frac{B}{B_c} \right) m_i, \tag{46}
\]

where the critical field \( B_c \equiv m_e^2/e = 4.3 \times 10^{13} \text{ Gauss} \). The interacting Hamiltonian (46) is very small, \( \mathcal{H}_b \sim 10^{-2}(m_e/m_W)^2 m_i \ll m_i \) for \( B = B_c \), where \( m_W \simeq 80 \text{GeV} \) is the \( W \)-gauge boson mass [13]. Therefore this interaction could be relevant only for very strong magnetic fields \( B \).
Using Eqs. (34) and (45), the effective Hamiltonian in the base of flavor eigenstates is given by

\[
\hat{H}_{nb} = \hat{H}_n + \hat{H}_b = E + \frac{M_1 + M_2}{2} + \frac{\Delta M^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} E_A e_0 & 0 \\ 0 & E A_\mu \end{pmatrix},
\]

(47)

where

\[
M_i \equiv \frac{m_i^2}{2E} - \mu_i B, \quad \frac{\Delta M^2}{2E} \equiv \frac{\Delta m^2}{2E} - (\mu_2 - \mu_1)B.
\]

(48)

Following the same calculations (35-38), we obtain the probability of neutrino oscillations in strongly magnetized nuclear media

\[
P_{\nu_{\mu} \leftrightarrow \nu_e}(t) = \sin^2 2\theta_{nb} \sin^2 \left[ (E_{2n}^{nb} - E_{1n}^{nb})t \right],
\]

(49)

where the mass squared difference is given by,

\[
\Delta m^2_{nb} = 2 \sqrt{(A_\mu - A_e)^2 E^4 + \Delta M^2 (A_\mu - A_e) E^2 \cos 2\theta + (\Delta M^2/2)^2},
\]

(50)

and the mixing angle is

\[
\tan 2\theta_{nb} = \frac{\Delta M^2 \sin 2\theta}{\Delta M^2 \cos 2\theta + 2(A_\mu - A_e) E^2}.
\]

(51)

Analogously to the previous section, considering the case (i) small vacuum mixing angle \( \theta \sim 0 \) we approximately have

\[
E_{2n}^{nb} - E_{1n}^{nb} \approx \frac{\Delta m^2}{2E} - (\mu_2 - \mu_1)B + A_\mu E, \quad \tan 2\theta_{nb} \approx 0.
\]

(52)

Whereas the case (ii) large vacuum mixing angle \( \theta \sim \pi/4 \) and Eq. (39) is approximated to be

\[
E_{2n}^{nb} - E_{1n}^{nb} \approx \sqrt{\left[ \frac{\Delta m^2}{2E} - (\mu_2 - \mu_1)B \right]^2 + (A_\mu E)^2},
\]

\[
\tan 2\theta_{nb} \approx \frac{\Delta m^2}{2E} - (\mu_2 - \mu_1)B}{A_\mu E}.
\]

(53)

The first case implies that the nuclear medium effect could possibly be relevant for neutrino oscillations when neutrino energy is very large.
7 Neutrino oscillations in general case

In this sections we give a general discussion of neutrino oscillations in neutral nuclear media with the presence of both electrons and strong magnetic fields. If electron presents in nuclear media, the $V_C$-term (12) should be added into the component $\hat{H}_{11}$, as discussed in Sec. 3. For this general case the Hamiltonian in the base of neutrino flavor eigenstates is given by

$$\hat{H}_{nmb} = \hat{H}_{nb} + \hat{H}_m = E + \frac{M_1 + M_2}{2} - \frac{1}{\sqrt{2}} G_F n_n + \frac{\Delta M^2}{4E} \left( \begin{array}{cc} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{array} \right) \left( \begin{array}{cc} E A_e + \sqrt{2} G_F n_e & 0 \\ 0 & E A_\mu \end{array} \right),$$

Diagonalizing Eq. (54), we obtain the eigenvalues of $\hat{H}_{nmb}$ in mass eigenstates

$$\hat{E}_i = E - \frac{1}{\sqrt{2}} G_F n_n + \tilde{m}_i^2 / 2E,$$

where $\tilde{m}_i$ are effective neutrino masses in the presence of electrons, nuclear media and magnetic fields. As a result, the effective neutrino mass squared difference and mixing angles are given by where the mass squared difference is given by,

$$\Delta \tilde{m}^2 = 2 \sqrt{(A_\mu - A_e)^2 E^4 + \Delta M^2 (A_\mu - A_e) E^2 \cos 2\theta + (\Delta M^2 / 2)^2},$$

and the mixing angle is

$$\tan 2\tilde{\theta} = \frac{\Delta M^2 \sin 2\theta}{\Delta M^2 \cos 2\theta + 2(A_\mu - A_e) E^2},$$

where $A_e \equiv A_e + \sqrt{2} G_F n_e / E$.

Neutrino flavor states are expressed by

$$|\nu_e\rangle = \cos \tilde{\theta} |\tilde{\nu}_1\rangle + \sin \tilde{\theta} |\tilde{\nu}_2\rangle,$$
$$|\nu_\mu\rangle = -\sin \tilde{\theta} |\tilde{\nu}_1\rangle + \cos \tilde{\theta} |\tilde{\nu}_2\rangle,$$

in terms of eigenstates $|\tilde{\nu}_i\rangle$ of Hamiltonian (54). The probability of neutrino oscillations is given by

$$P_{\nu_{\mu}\leftrightarrow\nu_e}(t) = \sin^2 2\tilde{\theta} \sin^2 \left[ (E_{2}^{nmb} - E_{1}^{nmb}) t \right].$$
where the vacuum mixing angle \( \theta \) is changed to \( \tilde{\theta} \) (57), and the neutrino energy difference is changed to

\[
E_{2}^{\text{m}} - E_{1}^{\text{m}} = \frac{\Delta \tilde{m}^2}{2E}. \tag{60}
\]

In the absence of nuclear media and magnetic fields, Eqs. (59) and (60) reduces to the MSW case Eqs. (17) and (18).

Neglecting the term \( A_e \) [see Eq. (40] and the effect of magnetic fields, we approximately have

\[
\Delta \tilde{m}^2 \approx 2\sqrt{\chi^2 E^4 + \Delta m^2 \chi E^2 \cos 2\theta + (\Delta m^2 / 2)^2}, \tag{61}
\]

\[
\tan 2\tilde{\theta} \approx \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta + 2\chi E^2}, \tag{62}
\]

\[
\chi \equiv A_\mu - \sqrt{2}G_F n_e / E. \tag{63}
\]

Assuming that in neutral nuclear media the electron density is about three orders of magnitude smaller than nuclear density \( n_e \sim 10^{-3} n_\pi \), as the case of neutron stars, in Eq. (63), we compare the nuclear effect \( A_\mu \) [see Eq. (40] with the MSW effect (12),

\[
\frac{\sqrt{2}G_F n_e}{E} \sim 10^{-9} m_\pi / E, \tag{64}
\]

both effects are comparable for very high neutrino energy \( E \sim 10^5 \times m_\pi \). However, if neutral nuclear media have very small densities of electrons and protons, the nuclear effect (\( A_\mu \)) is more important than the MSW effect.

8 Neutrino oscillation resonance and level-crossing

In this section, we turn to consider effects of non-uniform media on neutrino oscillations. As discussed neutrino oscillation in media depends on electron and baryon densities. Thus the variation of these densities changes the oscillation parameters, mass squared difference (14, 35) and mixing angle (15, 36). Therefore, on the way of neutrino traveling through non-uniform media, the probability of neutrino oscillations is changing and reaches a resonance at which neutrino oscillation is maximal even if the vacuum mixing angle is small.

8.1 MSW resonance and level-crossing

First we recall discussions on neutrino oscillations in a normal medium and resonance of maximal neutrino mixing and oscillations (6, 7). When neutrinos propagate through
media where electron density $n_e$ varies, in consequence the probability of neutrino oscillations changes, as the mass squared difference $\Delta m^2$ \((14)\) and mixing angle $\theta_m$ \((15)\) vary. Then the flavor composition of the neutrinos $(\nu_e, \nu_\mu)$ along their traveling path is a function of the electron density profile, and the phenomena of the MSW resonance and level-crossing may occur.

For simplicity, we discuss these phenomena for a small vacuum mixing angle $(\theta \ll 1)$. In this case, Eq. \((7)\) shows that the electron neutrino $|\nu_e\rangle$ is mainly in the mass eigenstate $|\nu_1\rangle$ and the muon neutrino $|\nu_\mu\rangle$ mainly $|\nu_2\rangle$. If the electron density is so small $n_e \simeq 0$ that $2\sqrt{2}G_F n_e E \ll \Delta m^2 \cos 2\theta$ in Eqs. \((14, 15)\), then we have $\theta_m \simeq \theta \sim 0$, and

$$|\nu_1\rangle_m \simeq |\nu_1\rangle, \quad |\nu_2\rangle_m \simeq |\nu_2\rangle.$$  \((65)\)

Eq. \((16)\) tells us that the electron neutrino $|\nu_e\rangle$ is mainly in the state $|\nu_1\rangle_m$ and the muon neutrino is $|\nu_\mu\rangle$ mainly $|\nu_2\rangle_m$, i.e., the flavor composition $(\nu_e, \nu_\mu)$ of the neutrinos is

$$|\nu_e\rangle \approx |\nu_1\rangle_m, \quad |\nu_\mu\rangle \approx |\nu_2\rangle_m.$$  \((66)\)

If the electron density is a such critical value

$$n_e^c = \frac{\Delta m^2}{\sqrt{2}G_F E} \cos 2\theta,$$  \((67)\)

that $2\sqrt{2}G_F n_e E = \Delta m^2 \cos 2\theta$, Eq. \((15)\) gives rise a maximal mixing angle $\theta_m \approx \pi/4$, as a result, maximal probability of the neutrino oscillation occurs, the MSW resonance. The flavor composition $(\nu_e, \nu_\mu)$ of the neutrinos is

$$|\nu_e\rangle = \frac{1}{\sqrt{2}} (|\nu_1\rangle_m + |\nu_2\rangle_m),$$

$$|\nu_\mu\rangle = \frac{1}{\sqrt{2}} (|\nu_2\rangle_m - |\nu_1\rangle_m).$$  \((68)\)

If the electron density is so large that $2\sqrt{2}G_F n_e E \gg \Delta m^2 \cos 2\theta$, i.e., $n_e \gg n_e^c$, Eq. \((15)\) shows that tan $2\theta_m \to 0^-$, $\theta_m \to \pi/2$. Eq. \((16)\) gives the flavor composition

$$|\nu_e\rangle \approx |\nu_2\rangle_m, \quad |\nu_\mu\rangle \approx -|\nu_1\rangle_m.$$  \((69)\)

Comparison between the flavor composition \((66)\) for small electron density with the one \((69)\) for large electron density shows an inversion of neutrino flavors. This neutrino flavor inversion is known as the MSW level-crossing of neutrino flavors.
8.2 Resonance and level-crossing in the general case

As discussed in the previous section for the MSW resonance, when the dominator of Eq. (57) vanishes
\[
[\Delta m^2 - 2(\mu_2 - \mu_1)BE] \cos 2\theta + 2(A_\mu - A_e)E^2 - 2\sqrt{2}GFn_eE = 0,
\]
the effective neutrino mixing angle is maximal ($\tilde{\theta} \sim \pi/4$), leading to a resonance of the neutrino oscillation probability in the general case. The condition (70) for the resonance gives a critical electron density
\[
n_e^c \simeq \left( \frac{1}{\sqrt{2}G_F} \right) \left[ \left( \frac{\Delta m^2}{2E} - (\mu_2 - \mu_1)B \right) \cos 2\theta + (A_\mu - A_e)E \right],
\]
as a function of neutrino energy, nuclear density $n_\pi$ and magnetic field $B$, in addition to vacuum mixing angle $\theta$, and neutrino masses $m_{1,2}$. Note that the first term in Eq. (71) corresponds the MSW critical density, and other terms come from the effects of magnetized nuclear media. When $n_\pi = 0$ and $B = 0$, critical density (71) becomes to the MSW one (67).

Analogously to the MSW level-crossing, we discuss the level-crossing of neutrino oscillations in the general case. Suppose that the vacuum mixing angle is small ($\theta \ll 1$), magnetic fields, nuclear media and electrons densities are small, i.e., $n_\pi \sim 0$, $n_e \sim 0$, $B \sim 0$. Then, from Eqs. (57,58) $\tilde{\theta} \simeq \theta \sim 0$, which implies that the electron neutrino $|\nu_e\rangle$ is mainly in the mass eigenstate $|\tilde{\nu}_1\rangle \simeq |\nu_1\rangle$ and the muon neutrino $|\nu_\mu\rangle$ is mainly $|\tilde{\nu}_2\rangle \simeq |\nu_2\rangle$, similarly to Eq. (65). As increasing magnetic field, nucleon and electron densities, from Eq. (57), the neutrino mixing angle increases, and the resonance takes place if the condition (70) is satisfied. In this case, Eqs. (57) and (71) give rise a maximal mixing $\tilde{\theta} \approx \pi/4$, and maximal probability of neutrino oscillations occurs. The flavor composition $(\nu_e, \nu_\mu)$ of the neutrinos is
\[
|\nu_e\rangle = \frac{1}{\sqrt{2}} (|\tilde{\nu}_1\rangle + |\tilde{\nu}_2\rangle),
\]
\[
|\nu_\mu\rangle = \frac{1}{\sqrt{2}} (|\tilde{\nu}_2\rangle - |\tilde{\nu}_1\rangle).
\]
As further increasing magnetic field $B$, electron and nucleon densities, from Eqs. (57,71), the condition for the MSW level-crossing to occur is that
\[
[\Delta m^2 - 2(\mu_2 - \mu_1)BE] \cos 2\theta + 2(A_\mu - A_e)E^2 - 2\sqrt{2}GFn_eE \to \infty^-,
\]
i.e., $\tan 2\tilde{\theta} \to 0^-$, $\tilde{\theta} \to \pi/2$, which is the same as the MSW result. The flavor composition is
\[
|\nu_e\rangle \approx |\tilde{\nu}_2\rangle, \quad |\nu_\mu\rangle \approx -|\tilde{\nu}_1\rangle.
\]
This shows an inversion of neutrino flavors, the MSW level-crossing of neutrino flavors, as discussed in the previous section.

9 Summary and remarks

Neutrino mass, mixing and oscillation are fundamental issues to be completely understood beyond the standard model of elementary particle physics. It still needs much effort to understand the origin of neutrino masses \((m_1, m_2, m_3)\) and mixing angles \((\theta_{12}, \theta_{23}, \theta_{13})\) and \(CP\) phase \(\delta\), and so far they can be treated as fundamental parameters, which are probably not independent [16]. In this article, we study the mixing and oscillation of electron and muon neutrinos, base on the assumption that the mixing angle \(\theta_{12}\) between the first and second generations is much larger than the mixing angles \(\theta_{23}\) and \(\theta_{13}\).

After a brief review of vacuum neutrino oscillations and matter neutrino oscillations (MSW solution), we study neutrino oscillations in strongly magnetized nuclear media. For this purpose, we first compute the effective Hamiltonian of neutrino interacting with nuclear media up to the leading order \(O(G_F^2)\). As shown in Fig. 2, this is in fact the process of neutrino absorption \((\nu + n \rightarrow p + e^-)\) then neutrino emission \((p + e^- \rightarrow n + \nu)\). The leading order contributions to both neutrino absorption and emission are \(O(G_F)\). In order to obtain the neutrino energy spectrum in nuclear media, we set neutrino incoming momentum “\(k\)” to be the same as neutrino outgoing momentum “\(k'\)” in Eq. (25), see Fig. 2 based on the assumption that nuclear medium is a very massive and rigid solid so that the recoiling effect is completely negligible, \(q = -q'\) in Eq. (25). According to Eqs. (20) and (21), this assumption can also be viewed as a scenario that a propagating neutrino \(\nu_{e,\mu,\tau}(k)\)” interacts with short ranged fields \(W^+(q)\) and \(\pi^+(q)\) in nuclear media, as a result its wave function receives the correction \((A_{e,\mu,\tau})\). This approximation should be further examined to see the effect of “incoherent” neutrino scattering with nuclear media for \(k \neq k'\) on the neutrino oscillation in nuclear media that we consider. With this Hamiltonian, we calculate the effective mixing angle and mass squared difference, as a result we obtain the probability of neutrino oscillations in nuclear media and strong magnetic fields. Moreover, we discuss the resonance and level-crossing of neutrino oscillations in magnetized nuclear media. It is shown that the effects due to magnetized nuclear media can modifies the pattern of vacuum neutrino oscillations. As example, we calculate the effect of nuclear media on neutrino oscillation, and find that neutrino oscillations are suppressed for high-energy neutrino \(E > 1\text{MeV}\). These effects on neutrino oscillations need to be considered in studying neutrino physics.
in supernovae, neutron stars, quark stars and magnetars, where neutrinos produced are very energetic $1\text{MeV} \lesssim E \lesssim 10\text{MeV}$ \cite{14}. It is also interesting to see whether or not the neutrino oscillation in nuclear media has any effect on the decay of heavy ions \cite{17}.

Moreover, it is interesting to study nuclear medium effects on neutrino mixing and oscillation in the three generation case, electron, muon and tau neutrinos ($\nu_e, \nu_\mu, \nu_\tau$) \cite{35} to find modifications of the vacuum mixing angles $\theta_{ij}$ and mass squared difference $\Delta m^2_{ij}$, and the $CP$ phase $\delta$ \cite{18}. In this case, the effect of tau neutrino scattering with nuclear media [see Eq. (32,33)] should be important.

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