Shuffles and Concatenations in Constructing of Graphs

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Abstract

This is an investigation of the role of shuffling and concatenating in the theory of graph drawing. A simple syntactic description of these and related operations is proved complete in the context of finite partial orders, as general as possible. An explanation based on that is given for a previously investigated collapse of the permutohedron into the associahedron, and for collapses into other less familiar polyhedra, including the cyclohedron. Such polyhedra have been considered recently in connection with the notion of tubing, which is closely related to tree-like finite partial orders defined simply and investigated here in detail. Like the associahedron, some of these other polyhedra are involved in categorial coherence questions, which will be treated in a sequel to this paper.

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1 Introduction

Shuffles and concatenations, which are usually considered only for finite linear orders, are here defined for arbitrary binary relations (see Section 4). Shuffles serve to define on sets of relations an associative and commutative partial operation, which we call shuffle sum; concatenations serve analogously to define on sets of relations an associative partial operation, which we call concatenation product.

Shuffle sum and concatenation product are interesting to us for the following reason. The one-one map \( L \), which assigns to a partial order all its linear extensions, maps disjoint union and concatenation of partial orders into shuffle
sum and concatenation product respectively (see Section 4). And here is why
disjoint union and concatenation of partial orders are interesting to us.

We associate with a given graph $\Gamma$ a set of terms representing tree-like finite
partial orders $T(\Gamma)$, each of which may be understood as a possible history of the
constructing, or, in reverse order, destructing, of $\Gamma$. The set $T(\Gamma)$ determines
the graph $\Gamma$ uniquely, i.e. the map $T$ is one-one. The tree-like partial orders of
$T(\Gamma)$ are closely related to the *tubings* of [3], but they are defined more simply.

The members of $T(\Gamma)$ are built inductively in a simple manner with the help
of two operations corresponding to disjoint union and concatenation. These
operations correspond via the map $L$ mentioned above to shuffle sum and con-
catenation product.

The members of $T(\Gamma)$ label vertices of polyhedra that are obtained from per-
mutohedra by collapsing connected families of vertices into a single vertex. We
use the map $L$ to assign to a member of $T(\Gamma)$ the permutations in a connected
family of vertices of the permutohedron, which are collapsed into a single vertex.
The collapsing in question that produces associahedra has been studied previ-
ously in [18]. For a suitable choice of $\Gamma$, we obtain a collapsing that produces
cyclohedra, and other choices yield less familiar polyhedra.

These polyhedra stand for commuting diagrams that arise in various coher-
ence questions in category theory. It is shown in [6] how Mac Lane’s pentagon
of monoidal coherence arises by a collapsing of the same kind we have here from
a hexagon involved in symmetric monoidal coherence, and this matter is related
to the collapsing investigated in [18].

Some similar coherence questions based on the conceptual apparatus intro-
duced in [6], which we intend to treat in the future, involve some of the less
familiar polyhedra that occur as examples in the present paper. The *hemi-
associahedron* of Example 5.14 arises in the definition of a coherent notion of
weak Cat-operad. A Cat-operad is an operad enriched over the category $\text{Cat}$
of all small categories, as a 2-category with small hom-categories is a category
enriched over $\text{Cat}$ (for the notion of operad see [14]). The notion of weak Cat-
operad is to the notion of Cat-operad what the notion of bicategory is to the
notion of 2-category. Our notion of weak Cat-operad is coherent in the sense
that all the diagrams of canonical arrows commute, as in Mac Lane’s notion of
monoidal category. The commuting diagrams assumed for this notion may be
pasted to make the hemiassociahedron, besides making the three-dimensional
associahedron and permutohedron. We demonstrate that in [8], for which the
present paper lays the ground.

Our examples of collapsing depend on specific graphs $\Gamma$, but we show that
we have a general phenomenon, not to be found only in our examples. The maps
$T$ and $L$ for a given graph $\Gamma$ with $n$ vertices induce an equivalence relation on
the set of vertices of the $n-1$-dimensional permutohedron (see Section 5).

Our tree-like partial orders are easily described syntactically with two partial
binary operations, one, corresponding to disjoint union, associative and com-
mutative, and the other, corresponding to concatenation, just associative. This
syntactic description covers a wider class of finite partial orders, with a property more general than difunctionality, which we call trifunctionality. We obtain an isomorphism result concerning this matter (see Sections 2 and 3).

Our treatment of shuffles and concatenations may be connected to the algebras studied in [13] and elsewhere. The connection is however not clear.

After completing this paper, we learned that shuffles and concatenations have been studied previously in a manner related to ours in a number of papers cited in [1] (Section 3). In particular, the results of our Sections 2 and 3 were anticipated in [10] and [20] (Theorem 1; see also [19], Lemma 1, and [9], Theorem 3.1). Further references to previous related work may be found in [8] and [7], which develop the research we have started here, and give motivation for it.

2 Disjoint union and concatenation of relations

In this section we study preliminary matters concerning the partial operations of disjoint union and concatenation of binary relations. These operations are partial because we require disjointness of domains. We are interested in particular in applying these operations to partial orders that satisfy a property we call trifunctionality, which generalizes difunctionality (see references below). The results of this section prepare the ground for the isomorphism result of the next section.

A relation on a set \( X \) is, as usual, an ordered pair \( \langle R, X \rangle \) such that \( R \subseteq X^2 \). (We deal only with binary relations in this paper.) The set \( X \) is the domain of \( \langle R, X \rangle \).

For the relations \( \langle R, X \rangle \) and \( \langle S, Y \rangle \) such that \( X \cap Y = \emptyset \) we have

\[
\langle R, X \rangle + \langle S, Y \rangle = \text{df} \langle R \cup S, X \cup Y \rangle,
\]

\[
\langle R, X \rangle \cdot \langle S, Y \rangle = \text{df} \langle R \cup S \cup (X \times Y), X \cup Y \rangle.
\]

The operation \(+\) is disjoint union, while \(\cdot\) could be called concatenation, because this is what it is when \( \langle R, X \rangle \) and \( \langle S, Y \rangle \) are linear orders on finite domains, i.e. finite sequences. It is clear that \(+\) is associative and commutative, while \(\cdot\) is associative without being commutative for \( X \) and \( Y \) nonempty (for \( X \) or \( Y \) empty, \(+\) and \(\cdot\) coincide). It is easy to verify the following.

**Remark** +. If \( \langle R, X \rangle \) is such that \( X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset \) and for every \( x_1 \) in \( X_1 \) and every \( x_2 \) in \( X_2 \) we have \( (x_1, x_2) \notin R \) and \( (x_2, x_1) \notin R \), then there are relations \( \langle R_1, X_1 \rangle \) and \( \langle R_2, X_2 \rangle \) such that \( \langle R, X \rangle = \langle R_1, X_1 \rangle + \langle R_2, X_2 \rangle \).

**Remark** ·. If \( \langle R, X \rangle \) is such that \( X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset \) and for every \( x_1 \) in \( X_1 \) and every \( x_2 \) in \( X_2 \) we have \( (x_1, x_2) \in R \) and \( (x_2, x_1) \notin R \), then there are relations \( \langle R_1, X_1 \rangle \) and \( \langle R_2, X_2 \rangle \) such that \( \langle R, X \rangle = \langle R_1, X_1 \rangle \cdot \langle R_2, X_2 \rangle \).

Partial orders in this paper will be strict partial orders—i.e. relations that are irreflexive and transitive. Note that if \( \langle R, X \rangle \) is a partial order, then in Remark
we may omit the conjunct \((x_2, x_1) \notin R\), which follows from \((x_1, x_2) \in R\). We can trivially prove the following.

**Proposition 2.1.** If \(X \cap Y = \emptyset\), then \(\langle R, X \rangle\) and \(\langle S, Y \rangle\) are partial orders iff \(\langle R, X \rangle + \langle S, Y \rangle\) is a partial order, and the same with \(\cdot\) instead of +.

We call a relation \(\langle R, X \rangle\) **trifunctional** when for every \(x, y, z\) and \(u\) in \(X\) we have that if \((x, z)\) and \((y, z)\) and \((y, u)\) are in \(R\), then either \((x, u)\) or \((y, x)\) or \((u, z)\) is in \(R\). The following picture helps to grasp this implication:

\[
\begin{array}{c}
\text{z} \\
\text{x} \\
\text{u} \\
\text{y}
\end{array}
\]

If in this implication we omit the disjuncts \((y, x) \in R\) and \((u, z) \in R\) from the consequent, then we obtain the implication that defines difunctional relations (see [15] and [16], Section 4.4; our term “trifunctional” is motivated by “difunctional”, and by the fact that we have three conjuncts in the antecedent and three disjuncts in the consequent). We can prove the following.

**Proposition 2.2.** If \(X \cap Y = \emptyset\), then \(\langle R, X \rangle\) and \(\langle S, Y \rangle\) are trifunctional relations iff \(\langle R, X \rangle + \langle S, Y \rangle\) is trifunctional, and the same with \(\cdot\) instead of +.

**Proof.** For + the proof is trivial, and for \(\cdot\) the direction from right to left is trivial. It remains to prove that if \(\langle R, X \rangle\) and \(\langle S, Y \rangle\) are trifunctional, then \(\langle R, X \rangle \cdot \langle S, Y \rangle\) is trifunctional. So suppose that \(\langle R, X \rangle\) and \(\langle S, Y \rangle\) are trifunctional, and suppose that for \(x, y, z\) and \(u\) in \(X \cup Y\) we have \((x, z)\), \((y, z)\) and \((y, u)\) in \(R \cup S \cup (X \times Y)\). We have the following cases:

1) \(z \in X\); then \(x, y \in X\), and we have the subcases:
   1.1) \(u \in X\); then we appeal to the trifunctionality of \(\langle R, X \rangle\);
   1.2) \(u \in Y\); then we have \((x, u) \in X \times Y\);
2) \(z \in Y\); then we have the subcases:
   2.1) \(u \in X\); then \((u, z) \in X \times Y\);
   2.2) \(u \in Y\); then we have the subcases:
      2.21) \(x \in X\); then \((x, u) \in X \times Y\);
      2.22) \(x \in Y\); then we have the subcases:
          2.221) \(y \in X\); then \((y, x) \in X \times Y\);
          2.222) \(y \in Y\); then we appeal to the trifunctionality of \(\langle S, Y \rangle\).

For a relation \(\langle R, X \rangle\) and \(x_1\) and \(x_n\), where \(n \geq 2\), distinct elements of \(X\) we write \(x_1 \sim_R x_n\) when there is a sequence \(x_1 \ldots x_n\) such that for every \(i \in \{1, \ldots, n-1\}\) we have \((x_i, x_{i+1}) \in R\) or \((x_{i+1}, x_i) \in R\). We say that \(\langle R, X \rangle\) is **connected** when for every two distinct \(x\) and \(y\) in \(X\) we have \(x \sim_R y\). It is trivial to prove the following proposition.
Proposition 2.3. If for the relations \( \langle R, X \rangle \) and \( \langle S, Y \rangle \) we have that \( X \cap Y = \emptyset \), \( X \neq \emptyset \) and \( Y \neq \emptyset \), then \( \langle R, X \rangle + \langle S, Y \rangle \) is not connected and \( \langle R, X \rangle \cdot \langle S, Y \rangle \) is connected.

By relying on this proposition we obtain easily the following proposition, which will be applied in the proof of the completeness Proposition 3.1 in the next section.

Proposition 2.4. If \( \langle R_1, X_1 \rangle + \ldots + \langle R_n, X_n \rangle = \langle S_1, Y_1 \rangle + \ldots + \langle S_m, Y_m \rangle \), and for every \( i \in \{1, \ldots, n\} \) and every \( j \in \{1, \ldots, m\} \) we have that \( R_i \) and \( S_j \) are connected, while \( X_i \) and \( Y_j \) are not empty, then \( n = m \) and there is a bijection \( \pi : \{1, \ldots, n\} \to \{1, \ldots, m\} \) such that for every \( i \in \{1, \ldots, n\} \) we have \( \langle R_i, X_i \rangle = \langle S_{\pi(i)}, Y_{\pi(i)} \rangle \).

We also have the following proposition.

Proposition 2.5. If for the relations \( \langle R_1, X \rangle \), \( \langle R_2, X \rangle \) and \( \langle S, Y \rangle \) we have that \( X \cap Y = \emptyset \), then \( \langle R_1, X \rangle \cdot \langle S, Y \rangle = \langle R_2, X \rangle \cdot \langle S, Y \rangle \) or \( \langle S, Y \rangle \cdot \langle R_1, X \rangle = \langle S, Y \rangle \cdot \langle R_2, X \rangle \) implies \( R_1 = R_2 \).

Proof. Suppose \( \langle R_1, X \rangle \cdot \langle S, Y \rangle = \langle R_2, X \rangle \cdot \langle S, Y \rangle \). If \( (x, y) \in R_1 \), then \( (x, y) \in R_1 \cup S \cup (X \times Y) \), and hence \( (x, y) \in R_2 \cup S \cup (X \times Y) \); but then \( (x, y) \in R_2 \), because \( x, y \in X \). So \( R_1 \subseteq R_2 \), and we demonstrate in the same manner \( R_2 \subseteq R_1 \). That \( \langle S, Y \rangle \cdot \langle R_1, X \rangle = \langle S, Y \rangle \cdot \langle R_2, X \rangle \) implies \( R_1 = R_2 \) is demonstrated analogously.

We use this proposition to establish the following proposition, which will be applied in the proof of the completeness Proposition 3.1 in the next section.

Proposition 2.6. If \( \langle R_1, X_1 \rangle \cdot \langle S_1, Y_1 \rangle = \langle R_2, X_2 \rangle \cdot \langle S_2, Y_2 \rangle \), where \( \langle S_1, Y_1 \rangle \) and \( \langle S_2, Y_2 \rangle \) are either not connected or their domains are singletons, then \( \langle R_1, X_1 \rangle = \langle R_2, X_2 \rangle \) and \( \langle S_1, Y_1 \rangle = \langle S_2, Y_2 \rangle \).

Proof. Let \( \langle T, Z \rangle = \langle R_1, X_1 \rangle \cdot \langle S_1, Y_1 \rangle = \langle R_2, X_2 \rangle \cdot \langle S_2, Y_2 \rangle \). It is clear from the definition of \( \cdot \) that we have

1. \( (x \in Y_1 \& (x, y) \in T) \Rightarrow y \in Y_1 \),
2. \( (x \in X_2 \& y \in Y_2) \Rightarrow (x, y) \in T \).

Note also that from the assumption that \( \langle S_1, Y_1 \rangle \) and \( \langle S_2, Y_2 \rangle \) are either not connected or their domains are singletons it follows that \( Y_1 \) and \( Y_2 \) are not empty.

We show by reductio ad absurdum that \( Y_1 \subseteq Y_2 \). Suppose that not \( Y_1 \subseteq Y_2 \); so there is an \( x \) in \( Y_1 \) such that \( x \notin Y_2 \), which implies that \( x \in X_2 \). Then we show that \( Y_2 \subseteq Y_1 \):
$y \in Y_2 \Rightarrow (x, y) \in T$, by (2),
\[
\Rightarrow y \in Y_1, \text{ by (1)}.
\]
The set $Y_1$ cannot be a singleton, because if it were that, then $Y_2$, which is not empty, would be the same singleton, and we supposed that we do not have $Y_1 \subseteq Y_2$. So $Y_1$ is not a singleton, and hence $\langle S_1, Y_1 \rangle$ is not connected.

Let $y_1$ and $y_2$ be two distinct elements of $Y_1$ such that we do not have $y_1 \sim_{S_1} y_2$. The following three cases exhaust all the possibilities for $y_1$ and $y_2$ as elements of $X_2 \cup Y_2$.

The first case is when one of $y_1$ and $y_2$ is in $X_2$ and the other is in $Y_2$. Let $y_1$ be in $X_2$ and $y_2$ in $Y_2$. Then by (2) we obtain $(y_1, y_2) \in T$, and since $y_1$ and $y_2$ are in $Y_1$, we have $(y_1, y_2) \in S_1$, which is a contradiction.

The second case is when $y_1$ and $y_2$ are both in $X_2$. Since $Y_2$ is not empty, for some $y$ in $Y_2$ we have that $(y_1, y)$ and $(y_2, y)$ are in $T$. Since $Y_2 \subseteq Y_1$, we have $y \in Y_1$, from which we infer that $(y_1, y)$ and $(y_2, y)$ are in $S_1$; this is a contradiction.

The third case is when $y_1$ and $y_2$ are both in $Y_2$. Since $x \in X_2$, we have that $(x, y_1)$ and $(x, y_2)$ are in $T$, and since $x \in Y_1$, we have that $(x, y_1)$ and $(x, y_2)$ are in $S_1$, which is a contradiction.

So we have established that $Y_1 \subseteq Y_2$, and we establish in an analogous manner that $Y_2 \subseteq Y_1$. Hence we have $Y_1 = Y_2$.

We show now that $S_1 = S_2$. We have first that
\[
(x, y) \in S_1 \Rightarrow (x, y) \in T \quad & \quad x, y \in Y_1,
\]
\[
\Rightarrow (x, y) \in T \quad & \quad x, y \in Y_2,
\]
\[
\Rightarrow (x, y) \in S_2.
\]
So $S_1 \subseteq S_2$, and we show analogously that $S_2 \subseteq S_1$.

Let $Y = Y_1 = Y_2$. Since $X_1$ and $Y$ are disjoint, and $X_2$ and $Y$ are disjoint too, from $X_1 \cup Y = X_2 \cup Y$ we infer $X_1 = X_2$, and then by Proposition 2.5 we conclude that $\langle R_1, X_1 \rangle = \langle R_2, X_2 \rangle$.

\[\square\]

### 3 Diversified S-terms and relations in FTP

In this section we characterize syntactically in a very simple manner trifunctional partial orders on finite sets. This is a freely generated structure, i.e. algebra, with two partial operations, one associative and commutative, corresponding to disjoint union, and the other associative, corresponding to concatenation. The operations are partial because we require that every free generator occurs just once in an element of our structure. We prove that this syntactically defined structure is isomorphic to the structure of trifunctional partial orders on finite sets with the operations of disjoint union and concatenation.
Consider terms built out of an infinite set of variables, which we denote by $x, y, z, \ldots, x_1, \ldots$ with the binary operations $+$ and $\cdot$, which we call sum and product. Consider structures, i.e. algebras, with two binary operations $+$ and $\cdot$ such that $+$ is associative and commutative, while $\cdot$ is associative. Let $S$ be the structure of this kind freely generated by infinitely many generators. We may take that the elements of $S$ are equivalence classes of the terms introduced above, which hence we call $S$-terms, while the variables $x, y, z, \ldots$ are $S$-variables. On these equivalence classes we define the operations $+$ and $\cdot$ by $[t] + [s] = [t + s]$ and $[t] \cdot [s] = [t \cdot s]$.

An $S$-term is called diversified when no $S$-variable occurs in it more than once. Since associativity and commutativity preserve diversification, it is clear that if the equivalence class $[t]$ is an element of $S$ for $t$ a diversified $S$-term, then every element of $[t]$ is diversified. We say that the element $[t]$ of $S$ is diversified when $t$ is a diversified $S$-term.

Let FTP be the set of trifunctional partial orders on nonempty finite sets of $S$-variables. We define by induction on complexity a map $\kappa$ from the set of diversified $S$-terms to the set FTP:

$\kappa(x) = \langle \emptyset, \{x\} \rangle$,
$\kappa(t + s) = \kappa(t) + \kappa(s)$,
$\kappa(t \cdot s) = \kappa(t) \cdot \kappa(s)$.

That $\kappa(t)$ is indeed a member of FTP for every diversified $S$-term $t$ follows from the fact that the relation $\langle \emptyset, \{x\} \rangle$ is in FTP, and from Propositions 2.1 and 2.2.

Since the operation $+$ on relations is associative and commutative, while $\cdot$ is associative, the map $\kappa$ induces a map $K$ from the set of diversified elements of $S$ to FTP, which is defined by:

$K[t] = \kappa(t)$.

We use $K[t]$ as an abbreviation for $K([t])$. We can prove the following completeness proposition.

**Proposition 3.1.** The map $K$ is one-one.

**Proof.** Suppose $\kappa(t) = \kappa(s)$. We proceed by induction on the number $k$ of $S$-variables in $t$. Since the domains of the relations $\kappa(t)$ and $\kappa(s)$ are the same, the same $S$-variables occur in $t$ and $s$, and hence $k$ is also the number of $S$-variables in $s$.

If $k = 1$, then $t$ and $s$ are the same $S$-variable. If $k > 1$, let $t$ be of the form $t_1 + \ldots + t_n$ and $s$ of the form $s_1 + \ldots + s_m$, for $n \geq 1$ and $m \geq 1$, with $t_i$, for $i \in \{1, \ldots, n\}$, and $s_j$, for $j \in \{1, \ldots, m\}$, $S$-variables or products. (Since $k > 1$, it is impossible that $n = 1$ and $t_1$ is an $S$-variable.) Since we have

$\kappa(t) = \kappa(t_1) + \ldots + \kappa(t_n) = \kappa(s_1) + \ldots + \kappa(s_m) = \kappa(s)$,
by Proposition 2.4 we conclude that \( n = m \), and that there is a bijection \( \pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\} \) such that for every \( i \in \{1, \ldots, n\} \) we have \( \kappa(t_i) = \kappa(s_{\pi(i)}) \).

If \( n = m > 1 \), then by the induction hypothesis we have \([t_i] = [s_{\pi(i)}] \) for every \( i \in \{1, \ldots, n\} \), and hence \([t] = [s]\), by the associativity and commutativity of + in \( S \).

If \( n = m = 1 \), then \( t \) and \( s \) are products, and by the associativity of \( \cdot \) in \( S \) we have \([t] = [t_1 \cdot t_2]\) and \([s] = [s_1 \cdot s_2]\) for \( t_2 \) and \( s_2 \) either sums or \( S \)-variables.

Since we have \( \kappa(t) = \kappa(t_1) \cdot \kappa(t_2) = \kappa(s_1) \cdot \kappa(s_2) = \kappa(s) \),

by Proposition 2.6 and the induction hypothesis we obtain \([t] = [s]\). \( \dashv \)

For the proof of Proposition 3.2 below, which will help us to establish that \( K \), besides being one-one, is alsoonto, we need the notion of inner element of \( X \) for a relation \( \langle R, X \rangle \): this is an element \( y \) of \( X \) such that for some \( x \) and \( z \) in \( X \) we have \((x, y) \in R \) and \((y, z) \in R \).

For a relation \( \langle R, X \rangle \) and \( y \) an element of \( X \), let the relation \( \langle R - y, X - \{y\}\rangle \) be defined by

\[ R - y = \{(u, v) \in R \mid u \neq y \land v \neq y \} \]

Then we can formulate the following, which is easy to establish.

**Remark on Inner Elements.** If \( y \) is an inner element of \( X \) for \( \langle R, X \rangle \) in FTP and connected, then \( \langle R - y, X - \{y\}\rangle \) is in FTP and connected.

That \( \langle R - y, X - \{y\}\rangle \) is connected is clear from the following pictures concerning chains that ensure connectedness:

![Diagram](image)

For every such chain connecting \( u \) and \( v \) in \( \langle R, X \rangle \) that involves the inner element \( y \) there is a substitute chain connecting \( u \) and \( v \) in \( \langle R - y, X - \{y\}\rangle \), which does not involve \( y \). Then we have the following.

**Proposition 3.2.** If \( \langle R, X \rangle \) is in FTP and connected, and there are at least two elements in \( X \), then for some relations \( \langle R_1, X_1 \rangle \) and \( \langle R_2, X_2 \rangle \) with \( X_1 \) and \( X_2 \) nonempty \( \langle R, X \rangle = \langle R_1, X_1 \rangle \cdot \langle R_2, X_2 \rangle \).
Proof. We proceed by induction on the number \( k \) of inner elements of \( X \) for \( \langle R, X \rangle \). If \( k = 0 \), let

\[
\begin{align*}
X_1 &= \{ x \in X \mid (\exists y \in X) (x, y) \in R \}, \\
X_2 &= \{ x \in X \mid (\exists y \in X) (y, x) \in R \}.
\end{align*}
\]

Then \( X = X_1 \cup X_2 \), since \( \langle R, X \rangle \) is transitive and connected, and there are at least two elements in \( X \). We have \( X_1 \cap X_2 = \emptyset \), since there are no inner elements in \( X \). We also have \( X_1 \neq \emptyset \) and \( X_2 \neq \emptyset \), since \( \langle R, X \rangle \) is connected and there are at least two elements in \( X \). We can conclude that

\[
(\varepsilon) \quad (\forall x_1 \in X_1)(\forall x_2 \in X_2) (x_1, x_2) \in R,
\]

because trifunctionality here implies difunctionality,\(^1\) and by difunctionality we may, to put it roughly, shorten chains that ensure connectedness. This is clear from the following picture:

![Diagram](image)

We can conclude also that

\[
(\notin) \quad (\forall x_1 \in X_1)(\forall x_2 \in X_2) (x_2, x_1) \notin R.
\]

Otherwise, \( \langle R, X \rangle \) would not be irreflexive. It remains only to apply Remark \( \cdot \) of the preceding section to establish the basis of our induction.

Suppose the number \( k \) of inner elements of \( X \) for \( \langle R, X \rangle \) is greater than 0. If \( x \) is such an element, then, by the Remark on Inner Elements, we have for \( \langle R - x, X - \{x\} \rangle \) too that it is in FTP and connected, and \( X - \{x\} \) has \( k - 1 \) inner elements for \( \langle R - x, X - \{x\} \rangle \). So, by the induction hypothesis, there are relations \( \langle R_1', X_1' \rangle \) and \( \langle R_2', X_2' \rangle \) with \( X_1' \) and \( X_2' \) nonempty such that \( \langle R - x, X - \{x\} \rangle = \langle R_1', X_1' \rangle \cdot \langle R_2', X_2' \rangle \). We may assume that \( \langle R_2', X_2' \rangle \) is prime with respect to \( \cdot \), in the sense that there are no relations \( \langle S, Y \rangle \) and \( \langle T, Z \rangle \) with \( Y \) and \( Z \) nonempty such that \( \langle R_2', X_2' \rangle = \langle S, Y \rangle \cdot \langle T, Z \rangle \). (If there were such relations, then we would pass to \( \langle T, Z \rangle \) instead of \( \langle R_2', X_2' \rangle \), and rely on the associativity of \( \cdot \); we may iterate that.)

Since \( x \) is an inner element of \( X \) for \( \langle R, X \rangle \), there is a \( w \) in \( X \) such that \((x, w) \in R \).

1) If \( w \in X_1' \), then we take

\[
(x1) \quad X_1 = X_1' \cup \{x\}, \quad X_2 = X_2',
\]

and we can conclude that \((\varepsilon)\) and \((\notin)\) hold. It remains to apply Remark \( \cdot \).

2) If \( w \in X_2' \), then we have the following subcases.

2.1) For every \( y \) in \( X_1' \) we have \((y, x) \in R \). Then we take

\[\text{9}\]

---

\(^1\)As a matter of fact, a relation \( \langle R, X \rangle \) in FTP is difunctional iff there are no inner elements in \( X \) for \( \langle R, X \rangle \).
\[(x2)\] \[X_1 = X'_1, \quad X_2 = X'_2 \cup \{x\},\]
and we can conclude that \((\in)\) and \(\not\in\) hold. It remains to apply Remark \(\cdot\).

2.2) For some element \(y\) in \(X'_1\) we have \((y,x) \notin R\). Then we take \((x1)\), and to conclude that \((\in)\) and \(\not\in\) hold it is enough to establish that

\[(*)\] \[(\forall x_2 \in X'_2) (x,x_2) \in R.\]

Suppose we do not have \((*)\); i.e., for some \(v\) in \(X'_2\) we have \((x,v) \notin R\). Let

\[Y = \{x_2 \in X'_2 \mid (x,x_2) \notin R\},\]
\[Z = \{x_2 \in X'_2 \mid (x,x_2) \in R\}.\]

The sets \(Y\) and \(Z\) are not empty, since \(v \in Y\) and \(w \in Z\). Take an arbitrary \(u\) from \(Y\) and an arbitrary \(z\) from \(Z\). We have that \((y,x) \in R\) by the definition of \(Z\), and \((x,u) \notin R\) by assumption, and \((y,u) \notin R\) by the definition of \(Y\). So, by trifunctionality, we may conclude that \((u,z) \in R\), which implies \((u,z) \in R - x\).

This implies that for every \(u\) in \(Y\) and every \(z\) in \(Z\) we have \((u,z) \in R'_2\), which, by Remark \(\cdot\), contradicts the assumption that \(\langle R'_2, X'_2 \rangle\) is prime with respect to \(\cdot\).

So we have \((*)\), and hence \((\in)\) and \(\not\in\) hold. It remains to apply Remark \(\cdot\).

\(\dash\)

Then we can prove the following.

**Proposition 3.3.** The map \(K\) is onto.

*Proof.* We want to show that for \(\langle R, X \rangle\) in FTP there is a diversified \(S\)-term \(t\) such that \(\kappa(t) = \langle R, X \rangle\). We proceed by induction on the number of \(S\)-variables in \(X\). For the basis, if \(X = \{x\}\), then \(R = \emptyset\), and \(t\) is \(x\). Suppose for the induction step that there are at least two \(S\)-variables in \(X\).

If \(\langle R, X \rangle\) is not connected, then, by Remark \(+\) of the preceding section, for some relations \(\langle R_1, X_1 \rangle\) and \(\langle R_2, X_2 \rangle\) with \(X_1\) and \(X_2\) nonempty \(\langle R, X \rangle = \langle R_1, X_1 \rangle + \langle R_2, X_2 \rangle\). So the cardinality of \(X_1\) and \(X_2\) is strictly smaller than the cardinality of \(X\). By Propositions 2.1 and 2.2, we can conclude that \(\langle R_1, X_1 \rangle\) and \(\langle R_2, X_2 \rangle\) are in FTP, and then we apply the induction hypothesis.

If \(\langle R, X \rangle\) is connected, then we apply Proposition 3.2, and reason as in the preceding paragraph. \(\dash\)

So, by the definition of \(K\) and by Propositions 3.1 and 3.3, we can conclude that \(K\) is an isomorphism between a substructure of \(S\) made of diversified elements and a structure on FTP. This is an isomorphism of two algebras with partial operations \(+\) and \(\cdot\).
4 Shuffle sums and concatenation products on relationships

In this and in the next section we obtain the main results of the paper, which are summarized in the Introduction (see Section 1). In this section we consider shuffles of arbitrary binary relations, and with their help we define two partial operations on sets of relations with the same domain. These operations, which we call shuffle sum and concatenation product, are partial because we require again disjointness of domains. The one-one map $L$, which assigns to a partial order all its linear extensions, maps disjoint union and concatenation of partial orders into shuffle sum and concatenation product. With $L$, and with two other related one-one maps, which are more general, we obtain other isomorphic representations of the partial algebras of Section 3.

While a relation on $X$ is an ordered pair $(R, X)$ such that $R \subseteq X^2$, i.e., $R \in \mathcal{P}(X^2)$, let a relationship on $X$ be an ordered pair $[U, X]$ such that $U \subseteq \mathcal{P}(X^2)$, i.e., $U \in \mathcal{P}(\mathcal{P}(X^2))$. In a relationship $[U, X]$ the set $U$ is a family of the form $\{R_i \mid i \in I \& R_i \subseteq X^2\}$. The set $X$ is the domain of $[U, X]$.

For the relationships $[U, X]$ and $[V, Y]$ such that $X \cap Y = \emptyset$ we have

$$[U, X] + [V, Y] =_{df} \{(Q \subseteq (X \cup Y)^2 \mid (\exists R \in U)(\exists S \in V) (Q \cap X^2 = R \& Q \cap Y^2 = S)\}, X \cup Y],$$

$$[U, X] \cdot [V, Y] =_{df} \{(Q \subseteq (X \cup Y)^2 \mid (\exists R \in U)(\exists S \in V) (Q, X \cup Y) = (R, X) \cdot (S, Y)\}, X \cup Y],$$

where $\cdot$ in $(R, X) \cdot (S, Y)$ is the concatenation introduced in Section 2. We call $\cdot$ in $[U, X] \cdot [V, Y]$, which we have just defined, concatenation product.

When for $(R, X)$, $(S, Y)$ and $(Q, X \cup Y)$ such that $X \cap Y = \emptyset$ we have $Q \cap X^2 = R$ and $Q \cap Y^2 = S$ we say that $(Q, X \cup Y)$ is a shuffle of $(R, X)$ and $(S, Y)$, because this is what it is when $(R, X)$, $(S, Y)$ and $(Q, X \cup Y)$ are finite linear orders. We call $+$ in $[U, X] + [V, Y]$, defined above, shuffle sum.

The disjoint union $(R, X) + (S, Y)$ and the concatenation $(R, X) \cdot (S, Y)$ of $(R, X)$ and $(S, Y)$ are shuffles of $(R, X)$ and $(S, Y)$; they are limit cases of shuffles. The disjoint union is a shuffle $(Q, X \cup Y)$ such that for every $x$ in $X$ and every $y$ in $Y$ we have $(x, y) \notin Q$ and $(y, x) \notin Q$, while the concatenation is a shuffle $(Q, X \cup Y)$ such that for every $x$ in $X$ and every $y$ in $Y$ we have $(x, y) \in Q$ and $(y, x) \notin Q$ (see the Remarks + and $\cdot$ in Section 2).

Consider the map $E$ from the set of relations on $X$ to the set of relationships on $X$ defined by:

$$E(R, X) =_{df} \{(R' \subseteq X^2 \mid R \subseteq R')\}, X].$$

We use $E(R, X)$ as an abbreviation for $E((R, X))$, and omit parentheses in the same way in analogous situations below.
It is trivial to show that $E$ is one-one, because $\bigcap \{ R' \subseteq X^2 \mid R \subseteq R' \} = R$. We can also show that the image by $E$ of disjoint union is shuffle sum; namely, we have the following.

**Proposition 4.1.** $E((R, X) + (S, Y)) = E(R, X) + E(S, Y)$.

**Proof.** We have to prove that $R \cup S \subseteq Q \subseteq (X \cup Y)^2$ iff

$$\exists R' \exists S' (R \subseteq R' \subseteq X^2 \& S \subseteq S' \subseteq Y^2 \& Q \cap X^2 = R' \& Q \cap Y^2 = S')$$

From left to right, it is enough to remark that from the left-hand side we can infer that $R \subseteq Q \cap X^2 \subseteq X^2$ and $S \subseteq Q \cap Y^2 \subseteq Y^2$. From right to left the inference is trivial. $\dashv$

On the other hand, we cannot show that $E((R, X) \cdot (S, Y))$ is the concatenation product $E(R, X) \cdot E(S, Y)$. This is because

$$(Q1) \quad R \cup S \cup (X \times Y) \subseteq Q \subseteq (X \cup Y)^2$$

need not imply

$$(Q2) \quad \exists R' \exists S' (R \subseteq R' \subseteq X^2 \& S \subseteq S' \subseteq Y^2 \& Q = R' \cup S' \cup (X \times Y))$$

though it is implied by it. There are sets $Q$ that satisfy $(Q1)$ and have in them a pair $(y, x)$ for some $x \in X$ and some $y \in Y$.

Consider the map $P$ from the set of partial orders on $X$ to the set of relationships on $X$ defined by replacing $R' \subseteq X^2$ in the definition of $E(R, X)$ by $R' \subseteq X^2$ and $R'$ is a partial order. It is again trivial to show that $P$ is one-one (for the same reason why $E$ is one-one).

Let the definitions of shuffle sum $+$ and concatenation product $\cdot$ on relationships be modified by replacing $Q \subseteq (X \cup Y)^2$ by $Q \subseteq (X \cup Y)^2$ and $Q$ is a partial order. A shuffle of two partial orders need not be a partial order, but the concatenation of two partial orders is a partial order (see Proposition 2.1); so the modified definition of concatenation product amounts to the old definition for relationships $[U, X]$ such that $U$ is a set of partial orders on $X$. We can prove the following.

**Proposition 4.2.** $P((R, X) + (S, Y)) = P(R, X) + P(S, Y)$.

For that we proceed as for Proposition 4.1. Now however we also have the following.

**Proposition 4.3.** $P((R, X) \cdot (S, Y)) = P(R, X) \cdot P(S, Y)$.

**Proof.** It is enough to prove that for partial orders $Q$ the condition $(Q1)$ implies $(Q2)$ (the converse is trivial). Suppose $(Q1)$, and let $R' = Q \cap X^2$ and $S' = Q \cap Y^2$. To show $(Q2)$ it is enough to show

$$Q = (Q \cap X^2) \cup (Q \cap Y^2) \cup (X \times Y).$$
To show that the right-hand side of this equation is indeed a subset of $Q$ follows easily from $(Q1)$. For the converse inclusion it is enough to verify that for every $x$ in $X$ and every $y$ in $Y$ we cannot have $(y, x)$ in $Q$. This follows from $X \times Y \subseteq Q$ together with the transitivity and irreflexivity of $Q$. ⊢

A relation $(R, X)$ is a linear order when it is a partial order (as in Section 2) and for every distinct $x$ and $y$ in $X$ either $(x, y) \in R$ or $(y, x) \in R$. Consider now the map $L$ from the set of partial orders on $X$ to the set of relationships on $X$ defined by replacing $R' \subseteq X^2$ in the definition of $E\langle R, X \rangle$ by $R' \subseteq X^2$ and $R'$ is a linear order. To prove that $L$ is one-one is now not so trivial, and we need some preparation for that.

**Proposition 4.4.** For a partial order $\langle R, X \rangle$ such that for some distinct $x$ and $y$ in $X$ we have $(y, x) \notin R$, the transitive closure $\langle Tr(R \cup \{(x, y)\}), X \rangle$ is a partial order.

*Proof.* We show that this transitive closure is irreflexive. If for some $z$ in $X$ we had $(z, z) \in Tr(R \cup \{(x, y)\})$, then there would be a chain $u_1, \ldots, u_n$ such that $u_1 = u_n = z$, and either $(u_i, u_{i+1}) \in R$ or $(u_i, u_{i+1}) = (x, y)$. For some $i$ we must have $(u_i, u_{i+1}) = (x, y)$; otherwise $R$ would not be irreflexive. Let $u_k$ be the leftmost $x$ in the chain, and let $u_l$ be the rightmost $y$ in the chain. Then we must have $(u_l, u_k) \in R$, which contradicts $(y, x) \notin R$. ⊢

One can show by elementary means that every finite partial order on $X$ can be extended to a linear order on $X$. (This is related to what is called topological sorting in algorithmic graph theory.) With less elementary means one can show the same thing for any partial order, not necessarily finite (see [12], p. 19). So, by combining this with Proposition 4.4, we obtain the following.

**Proposition 4.5.** For a partial order $\langle R, X \rangle$ such that for some distinct $x$ and $y$ in $X$ we have $(y, x) \notin R$, there is a linear order $\langle R', X \rangle$ such that $R \subseteq R'$ and $(x, y) \in R'$.

We can now prove that $L$ is one-one, which amounts to the following.

**Proposition 4.6.** For the partial orders $\langle R, X \rangle$ and $\langle S, X \rangle$ we have that $L\langle R, X \rangle = L\langle S, X \rangle$ implies $R = S$.

*Proof.* Suppose $L\langle R, X \rangle = L\langle S, X \rangle$ and suppose $(u, v) \in R$. We infer that for every linear order $S' \subseteq X^2$ such that $S \subseteq S'$ we have $(u, v) \in S'$. If $(u, v) \notin S$, then we obtain a contradiction with the help of Proposition 4.5. ⊢

Let the definitions of shuffle sum $+$ and concatenation product $\cdot$ on relationships be now modified by replacing $Q \subseteq (X \cup Y)^2$ by $Q \subseteq (X \cup Y)^2$ and $Q$ is a linear order. A shuffle of two linear orders need not be a linear order, but the concatenation of two linear orders is a linear order, and so the definition of con-
catenation product just modified amounts to the old definition for relationships $[U, X]$ such that $U$ is a set of linear orders on $X$.

We can now prove the following by proceeding as for Propositions 4.1 and 4.3.

**Proposition 4.7.** $L(\langle R, X \rangle + \langle S, Y \rangle) = L\langle R, X \rangle + L\langle S, Y \rangle$.

**Proposition 4.8.** $L(\langle R, X \rangle \cdot \langle S, Y \rangle) = L\langle R, X \rangle \cdot L\langle S, Y \rangle$.

By combining Proposition 3.1 with the facts that the maps $E$, $P$ and $L$ are one-one, we obtain new isomorphic representations of the structure made of the diversified elements of $S$ (see Section 3).

## 5 S-forests of graphs

In this section we deal with the matters concerning the constructing of graphs, which we summarized in the Introduction (see Section 1). This is the main and concluding section of our paper. We define first tree-like elements of the structure $S$ of Section 3, and we show that what corresponds to these elements by the isomorphism $K$ are indeed tree-like relations in FTP.

Consider the set $C$ of elements of $S$ (see Section 3) defined inductively as follows:

- for every $S$-variable $x$, we have $[x] \in C$;
- if $[t], [s] \in C$, then $[t + s] \in C$;
- if $[t] \in C$ and $x$ is an $S$-variable, then $[x \cdot t] \in C$.

An alternative definition of $C$ is obtained by replacing the third clause with:

- if $[t] \in C$ and $+$ does not occur in the $S$-term $s$, then $[s \cdot t] \in C$.

Let an $S$-*forest* be a diversified element of $C$. An $S$-*tree* is, for example, $[((x \cdot y) \cdot z) + u]$. An $S$-*tree* is an $S$-forest that is not of the form $[t + s]$; for example, $[w \cdot (((x \cdot y) \cdot z) + (u + v))]$. Since $+$ and $\cdot$ are associative, there are in these examples superfluous parentheses, which we omit later. Note that $[x] = \{x\}$, and that every member of $[t_1 + t_2]$ is of the form $t'_1 + t'_2$, while every member of $[t_1 \cdot t_2]$ is of the form $t'_1 \cdot t'_2$.

Let us call a partial order $(R, X)$ for $X$ a finite set of $S$-variables an FTP-*forest* when for every $x, y, z \in X$

$((x, z) \in R \land (y, z) \in R) \Rightarrow (x = y \lor (x, y) \in R \lor (y, x) \in R)$.

It is easy to see that FTP-forests are trifunctional, and hence they are in FTP (see Section 3). We say that an FTP-forest $(R, X)$ is an FTP-*tree* when there is an $x \in X$, called root, such that for every $y \in X$ different from $x$ we have $(x, y) \in R$. The root is unique. (Usually, our FTP-forests are called trees in set
theory, and a tree, which need not be finite, is defined as a partial order such that for every element the set of its predecessors is well-ordered.)

The following four propositions are about the map $K$ of Section 3.

**Proposition 5.1.** For $[t]$ an $S$-forest, $K[t]$ is an FTP-forest.

*Proof.* We proceed by induction on the length of $t$. If $t$ is an $S$-variable, this is trivial, because $K[t]$ is the empty relation. If $t$ is $t_1 + t_2$, this is trivial, by the induction hypothesis.

Suppose $t$ is $u \cdot t_1$, $(x, z) \in K[u \cdot t_1]$ and $(y, z) \in K[u \cdot t_1]$. Then if $x$ and $y$ are $u$, then $x = y$. If $x$ is $u$ and $y$ is in $t_1$, then $(x, y) \in K[u \cdot t_1]$. If $y$ is $u$ and $x$ is in $t_1$, then $(y, x) \in K[u \cdot t_1]$. If both $x$ and $y$ are in $t_1$, then we apply the induction hypothesis. ⊣

**Proposition 5.2.** For $[t]$ an $S$-tree, $K[t]$ is an FTP-tree.

*Proof.* If $t$ is the $S$-variable $x$, then $K[t]$ is $\langle \emptyset, \{x\} \rangle$, which is an FTP-tree. If $t$ is $[x \cdot t']$, then $x$ is the root of $K[t]$. ⊣

**Proposition 5.3.** For every FTP-forest $\langle R, X \rangle$ there is an $S$-forest $[t]$ such that $K[t] = \langle R, X \rangle$.

*Proof.* By Proposition 3.3 there is a diversified $S$-term $t$ such that $K[t] = \langle R, X \rangle$. If $t$ has a subterm of the form $(s + r) \cdot w$, then for an $S$-variable $x$ in $s$, an $S$-variable $y$ in $r$ and an $S$-variable $z$ in $w$, we have $(x, z) \in R$, $(y, z) \in R$, but neither $x = y$, nor $(x, y) \in R$, nor $(y, x) \in R$. So $\langle R, X \rangle$ is not an FTP-forest. ⊣

**Proposition 5.4.** For every FTP-tree $\langle R, X \rangle$ there is an $S$-tree $[t]$ such that $K[t] = \langle R, X \rangle$.

*Proof.* Just note that $t$ of the preceding proof cannot be of the form $t_1 + t_2$. Otherwise $\langle R, X \rangle$ would not be an FTP-tree. ⊣

So $K$ establishes an isomorphism between $S$-forests and FTP-forests on the one hand, and $S$-trees and FTP-trees on the other hand.

We pass now to graphs and their constructing. After the following definitions, we will give a series of examples.

A graph is a symmetric and irreflexive relation $\langle G, X \rangle$ whose domain $X$ is finite and nonempty (see [11], Chapter 2). We will now define inductively a map $T$ from the set of graphs $\langle G, X \rangle$ such that $X$ is a set of $S$-variables to the power set of the set of $S$-forests; i.e. $T(\langle G, X \rangle)$, which abbreviates $T'(\langle G, X \rangle)$, is a set of $S$-forests:

if $X = \{x\}$, then $T(\langle G, X \rangle) = \{[x]\}$;

supposing for the following two clauses that there are at least two $S$-variables
in $X$:

if $\langle G, X \rangle$ is connected, then

$$T(G, X) = \{ [x \cdot t] \mid x \in X \ & \ [t] \in T(G-x, X-\{x\}) \};$$

if $\langle G, X \rangle$ is not connected, and it is of the form $\langle G_1, X_1 \rangle + \langle G_2, X_2 \rangle$ for $\langle G_1, X_1 \rangle$ and $\langle G_2, X_2 \rangle$ graphs (i.e. for $X_1$ and $X_2$ nonempty), then

$$T(G, X) = \{ [t_1 + t_2] \mid [t_1] \in T(G_1, X_1) \ & \ [t_2] \in T(G_2, X_2) \}. $$

It is not difficult to prove that for every graph $\langle G, X \rangle$, and $x$ and $y$ distinct elements of $X$, we have $(x, y) \in G$ iff for every $[t]$ in $T(G, X)$ the S-term $t$ has no subterm $t_1 + t_2$ with $x$ in one of $t_1$ and $t_2$, and $y$ in the other. From that we infer immediately that the map $T$ is one-one.

Note that if $\langle G, X \rangle$ is connected, then the S-forests in $T(G, X)$ are S-trees. These S-trees are in one-to-one correspondence with what in [4] (Section 2) is called maximal $(n-1)$-tubings of $\langle G, X \rangle$, where $n$ is the cardinality of $X$ (the notion of tubing is introduced in [3], Section 2, and modified in [5], Section 2). The tubings of graphs $\langle G, X \rangle$ that are not connected do not however correspond exactly to the S-forests in $T(G, X)$.

**Examples 5.1.** We give now examples of $T(G, X)$ for a number of connected graphs $\langle G, X \rangle$.

**Example 5.11.** If $\langle G, X \rangle$ is the connected graph

\[
\begin{array}{c}
\text{u} \\
/ \\
/ \\
x \quad z \\
/ \\
y
\end{array}
\]

then in $T(G, X)$ we find twenty four S-trees, which are obtained from the twenty four permutations of the four S-variables $x, y, z$ and $u$ by inserting $\cdot$. These S-trees naturally label the vertices of the three-dimensional permutohedron:
In this permutohedron, and in the other examples later, there is an edge between the vertices labelled by $[t]$ and $[s]$ when there is a linear order in $L(K[t])$ and another one in $L(K[s])$ that differ from each other just by a transposition of immediate neighbours. (We discuss this matter after the examples.)

**Example 5.12.** If $⟨G, X⟩$ is the connected graph

![Graph Image](image)

obtained from the graph in the preceding example by omitting the edge $\{y, u\}$, then in $T(⟨G, X⟩)$ we find twenty two $S$-trees, with which we label the vertices of the following polyhedron, obtained from the three-dimensional permutohedron by collapsing the two vertices $[x \cdot z \cdot y \cdot u]$ and $[x \cdot z \cdot u \cdot y]$ into the single vertex labelled $[x \cdot z \cdot (y + u)]$, and the two vertices $[z \cdot x \cdot y \cdot u]$ and $[z \cdot x \cdot u \cdot y]$ into the single vertex labelled $[z \cdot x \cdot (y + u)]$: 
We propose to call this polyhedron *hemicyclohedron*. This name will be explained in the next example. (We will not prove here that the hemicyclohedron, conceived as an abstract polytope, can be realized, and the same with other such polyhedra later.)

**Example 5.13.** If \((G, X)\) is the connected graph

![Graph](image)

obtained from the graph in the preceding example by omitting the edge \(\{x, z\}\), then in \(T(G, X)\) we find twenty \(S\)-trees, with which we label the vertices of the three-dimensional cyclohedron (see [17], Section 4, and [3], Corollary 2.7):
Something analogous to what happened in the lower left corner of our picture of the three-dimensional permutohedron in order to obtain the hemicyclohedron happened now in the upper right corner too. This explains the name of the hemicyclohedron.

Example 5.14. If \((G, X)\) is the connected graph

\[
\begin{align*}
&\begin{array}{c}
  \text{Example 5.14. If } (G, X) \text{ is the connected graph} \\
  \text{obtained from the graph in Example 5.12 by omitting the edge } \{x, u\} , \text{ then in } T(G, X) \text{ we find the eighteen } S\text{-trees that label the vertices of the following polyhedron:}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
  \text{obtained from the graph in Example 5.12 by omitting the edge } \{x, u\} , \text{ then in } T(G, X) \text{ we find the eighteen } S\text{-trees that label the vertices of the following polyhedron:}
\end{array}
\end{align*}
\]
We propose to call this polyhedron (which is called $X_4^2$ in [2], Figure 17) *hemiassociahedron*. This name will be explained in the next example.

**Example 5.15.** If $\langle G, X \rangle$ is the connected graph

![Graph](image)

obtained from the graph in the preceding example by omitting the edge $\{x, z\}$, then in $T(G, X)$ we find the fourteen $S$-trees that label the vertices of the three-dimensional associahedron:

![Diagram](image)

In [18] it is explained how this associahedron is obtained from the three-dimensional permutohedron by two perpendicular cuts. The previous polyhedron, the hemiassociahedron, is obtained by one such cut. This should be also clear from our picture of the associahedron, where one cut, which it shares with our hemiassociahedron, is at the basis, while the other is on the right-hand side. This explains the name of the hemiassociahedron.

**Example 5.16.** If $\langle G, X \rangle$ is the connected graph

![Graph](image)

obtained from the graph in Example 5.14 by omitting the edge $\{x, y\}$, then in $T(G, X)$ we find the sixteen $S$-trees that label the vertices of the following polyhedron:
Since it arises from a three-pointed star, we could perhaps call this polyhedron the three-dimensional astrohedron. (It is called $D_4$ in [2], Figure 17.)

To sum up the previous six examples, we have the following picture (with the number of vertices in parentheses):

- 5.15 associahedron (14)
- 5.16 astrohedron (16)
- 5.13 cyclohedron (20)
- 5.14 hemiassociahedron (18)
- 5.12 hemicyclohedron (22)
- 5.11 permutohedron (24)

We take now an example with a graph that is not connected.

**Example 5.2.** If $\langle G, X \rangle$ is the graph

```
\[ \begin{array}{c}
\bullet \\
\circ \\
\end{array} \]
```

obtained from the graph in Example 5.14 by omitting the edge $\{z, u\}$, then in $T(G, X)$ we find the six $S$-forests that label the vertices of the following hexagon:
The \(S\)-forests in \(T\langle G, X \rangle\) may be conceived as records of the history of the destruction of \(\langle G, X \rangle\), which is a history of the construction of \(\langle G, X \rangle\) in reverse order. This destruction of graphs is based on vertex removal (which one finds in Ulam’s Conjecture; see [11], Chapter 2). We read the \(S\)-forest from left to right, and we interpret the occurrence of an \(S\)-variable that we encounter in this reading as the record of the removal of the vertex made of this \(S\)-variable and of the edges involving this vertex. The removal of vertices joined by \(\cdot\) happened consecutively, while for those joined by \(+\) it happened simultaneously in time. The commutativity of \(+\) means that what is recorded on the two sides of \(+\) happened simultaneously.

For example, the \(S\)-forest \([x \cdot y \cdot z] + u\) from Example 5.2 may be taken as a record of a destruction where, simultaneously, one removes on the one side the vertices \(x, y\) and \(z\) and on the other side the vertex \(u\); the removal of \(x, y\) and \(z\) is done consecutively so as to produce the film:

\[
\begin{array}{c}
\circ\circ\circ \\
\circ\circ\circ \\
\circ\circ\circ \\
\circ\circ\circ \\
\circ\circ\circ \\
\circ\circ\circ \\
\end{array}
\]

Our examples of collapsing depend on specific graphs \(\langle G, X \rangle\), but we will show presently that we have here a general phenomenon, not to be found only in our examples. The maps \(T\) and \(L\) for a given graph \(\langle G, X \rangle\) with \(n\) vertices induce an equivalence relation on the set of vertices of the \(n-1\)-dimensional permutohedron, whose equivalence classes are described by \(L(K[t])\) for \([t]\) in \(T\langle G, X \rangle\). Moreover, the permutations corresponding to the members \(L(K[t])\) are those assigned to a connected family of vertices of the permutohedron. For example, the four permutations that correspond to \([x \cdot y \cdot z] + u\], \([x \cdot y \cdot u \cdot z]\], \([x \cdot u \cdot y \cdot z]\] and \([u \cdot x \cdot y \cdot z]\] are given by the linear orders in \(L(K[(x \cdot y \cdot z) + u])\). The four vertices of the permutohedron labelled by these permutations make a connected family (see Example 5.11). We first define precisely the required notions, and then prove three propositions, which establish all that.

For a linear order \(\langle L, X \rangle\) of a finite set \(X\) we call \(L\) a permutation of \(X\). Let \(\Lambda\) be a set of permutations of \(X\). For \(L_1\) and \(L_n\), where \(n \geq 2\), distinct members of \(\Lambda\), we write \(L_1 \sim_\Lambda L_n\) when there is a sequence \(L_1 \ldots L_n\) such that \(L_1, \ldots, L_n \in \Lambda\) and for every \(i \in \{1, \ldots, n-1\}\) we have that for two distinct \(x\) and \(y\) in \(X\)
\[ L_{i+1} = (L_i - \{(x, y)\}) \cup \{(y, x)\}. \]

In other words, \( L_{i+1} \) differs from \( L_i \) just by a transposition of immediate neighbours. We say that \( \Lambda \) is connected when for every two distinct \( L \) and \( L' \) in \( \Lambda \) we have \( L \sim_\Lambda L' \). Here are the three propositions we announced above.

**Proposition 5.5.** For every partial order \( \langle R, X \rangle \) with \( X \) finite, and \( L\langle R, X \rangle = [A, X] \), the set of permutations \( \Lambda \) is connected.

**Proof.** If \( X \) is \( \emptyset \) or a singleton, then \( R = \emptyset \), and \( \Lambda = \{\emptyset\} \), which is connected by our definition. If the cardinality \( |X| \) of \( X \) is at least \( 2 \), we proceed by induction on \( |X| \).

For the basis, if \( |X| = 2 \), then the only interesting case is when \( X = \{x, y\} \) and \( R = \emptyset \). In that case \( \Lambda = \{(x, y), (y, x)\} \), which is clearly connected.

If \( |X| > 2 \), then let \( x \) be an element of \( X \) such that for every \( y \) in \( X \) we have \( (y, x) \notin R \). Since \( X \) is finite, there must be such an \( x \). Let \( L \) and \( L' \) be two different elements of \( \Lambda \). We want to show that \( L \sim_\Lambda L' \). Let

\[ S^x = \{(y, x) \mid (y, x) \in S\}, \quad S_x = \{(x, y) \mid (y, x) \in S\}, \]
\[ M = (L - L^x) \cup L_x, \quad M' = (L' - L'^x) \cup L'_x. \]

It is clear that the finite sequences that correspond to the permutations \( M \) and \( M' \) begin with \( x \). We conclude that \( L \sim_\Lambda M \) or \( L = M \), and \( L' \sim_\Lambda M' \) or \( L' = M' \). By the induction hypothesis, we have \( M - x \sim_\Lambda M' - x \) or \( M - x = M' - x \).

From all that \( L \sim_\Lambda L' \) follows. \( \dashv \)

**Proposition 5.6.** For every graph \( \langle G, X \rangle \) with \( X \) a finite nonempty set of \( S \)-variables, and every permutation \( L \) of \( X \), there is an \( S \)-forest \( [t] \) in \( T\langle G, X \rangle \) such that \( L \in L(K[t]) \).

**Proof.** We proceed by induction on the cardinality of \( X \). If \( X \) is a singleton, then we just follow the definitions. Suppose for the induction step that \( X \) has at least two \( S \)-variables.

If \( \langle G, X \rangle \) is connected, then let the sequence corresponding to the permutation \( L \) be \( x y_1 \ldots y_n \) for \( n \geq 1 \). By the induction hypothesis, there is an \( S \)-forest \( s \in T\langle G-x, X-\{x\} \rangle \) such that the permutation \( L' \) corresponding to \( y_1 \ldots y_n \) belongs to \( L(K[s]) \). Then we have that \( L \in L(K[x \cdot s]) \).

Suppose \( \langle G, X \rangle \) is not connected, and is of the form \( \langle G_1, X_1 \rangle + \langle G_2, X_2 \rangle \) for \( \langle G_1, X_1 \rangle \) and \( \langle G_2, X_2 \rangle \) graphs (i.e. for \( X_1 \) and \( X_2 \) nonempty). By the induction hypothesis, there are \( S \)-forests \( s_1 \in T\langle G_1, X_1 \rangle \) and \( s_2 \in T\langle G_2, X_2 \rangle \) such that for a permutation \( L_1 \) of \( X_1 \) and a permutation \( L_2 \) of \( X_2 \) we have \( L_1 \in L(K[s_1]) \), \( L_2 \in L(K[s_2]) \), and \( \langle L, X \rangle \) is a shuffle of \( \langle L_1, X_1 \rangle \) and \( \langle L_2, X_2 \rangle \). Then we have that \( L \in L(K[s_1 + s_2]) \). \( \dashv \)

**Proposition 5.7.** For every graph \( \langle G, X \rangle \) with \( X \) a finite nonempty set of \( S \)-variables, and every \([t] \) and \([t'] \) in \( T\langle G, X \rangle \), if \( L(K[t]) \) and \( L(K[t']) \) are not disjoint, then \([t] = [t'] \).
Proof. We proceed by induction on the cardinality of $X$. If $X$ is a singleton \{x\}, then $T(\emptyset, \{x\}) = \{\{x\}\}$, and $L(K[x]) = \{\langle 0, \{x\}\rangle\}$; the proposition holds trivially. Suppose for the induction step that $X$ has at least two $S$-variables.

If $\langle G, X \rangle$ is connected, then every element of $T(G, X)$ is of the form $[x \cdot s]$ for some $x$. Suppose that for some $[x \cdot s], [x' \cdot s'] \in T(G, X)$ we have $L(K[x \cdot s]) \cap L(K[x' \cdot s']) \neq \emptyset$. It follows that $x$ is $x'$, and since $[s], [s'] \in T(G-x, X-\{x\})$ and $L(K[s]) \cap L(K[s']) \neq \emptyset$, by the induction hypothesis we obtain that $[s] = [s']$. Hence $[x \cdot s] = [x' \cdot s']$.

Suppose $\langle G, X \rangle$ is not connected, and is of the form $\langle G_1, X_1 \rangle + \langle G_2, X_2 \rangle$ for $\langle G_1, X_1 \rangle$ and $\langle G_2, X_2 \rangle$ graphs; suppose also that $[t], [t'] \in T(G, X)$. Then, by relying on the associativity and commutativity of $+$, we may infer that $[t] = [t_1 + t_2]$ and $[t'] = [t'_1 + t'_2]$ for $[t_i], [t'_i] \in T(G_i, X_i)$ and $i \in \{1, 2\}$. Suppose for some $\langle L, X \rangle$ we have $\langle L, X \rangle \in L(K[t]) \cap L(K[t'])$. We infer that $\langle L \cap X^2, X_i \rangle \in L(K[t_i]) \cap L(K[t'_i])$, and hence by the induction hypothesis we obtain that $[t_i] = [t'_i]$. Hence $[t] = [t']$.

For every graph $\langle G, X \rangle$ with $X$ a finite nonempty set of $S$-variables, from Propositions 5.6 and 5.7 we infer that $\{L(K[t]) \mid [t] \in T(G, X)\}$ is a partition of $\{\langle L, X \rangle \mid L$ is a permutation of $X\}$. We know moreover by Proposition 5.5 that every member $L(K[t])$ of this partition is a connected set of permutations. Hence what we had in the examples of this section is a general phenomenon.

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