On dlog image of $K_2$ of elliptic surface minus singular fibers

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Abstract

Let $\pi : X \to C$ be an elliptic surface over an algebraically closed field of characteristic zero and $D_i = \pi^{-1}(P_i)$ the singular fibers. Put $U = X - \cup_i D_i$. Our objective in this paper is the image of the dlog map $\Gamma(U, K_2) \to \Gamma(X, \Omega^2_X(\log \sum D_i))$. In particular, we give an upper bound of the rank of the dlog image, which is computable in many cases. This also allows us to construct indecomposable parts of Bloch’s higher Chow groups $\text{CH}^2(X, 1)$ in special examples.

1 Introduction

Let $U$ be a nonsingular variety over a field of characteristic zero. Let $X$ be a smooth compactification of $U$ such that $D := X - U$ is a normal crossing divisor. Then there is the dlog map

$$\text{dlog} : \Gamma(U, K_2) \to \Gamma(X, \Omega^2_X(\log D))$$

from the $K$-cohomology group to the space of the algebraic differential 2-forms on $X$ with logarithmic poles along $D$. When the base field is $\mathbb{C}$, the dlog image is contained in the Betti cohomology group $H^2_B(U, \mathbb{Q}(2))$:

$$\text{dlog} \Gamma(U, K_2) \otimes \mathbb{Q} \subseteq \Gamma(X, \Omega^2_X(\log D)) \cap H^2_B(U, \mathbb{Q}(2)). \quad (1.1)$$

In the paper [2] Beilinson conjectured that the equality always holds true in (1.1) (see also [10] 5.18, 5.20). However very little is known about it.

In the present paper we study the dlog image in a different way from the above when $U$ is the complement of the singular fibers in an elliptic surface. The main results are Theorems A and B.

Theorem A concerns $K_2$ of Tate curves. Let $K$ be a finite unramified extension of $\mathbb{Q}_p$ and $R$ its integer ring. Let $E_q$ be the Tate curve over the ring $R((q)) := R[[q]][q^{-1}]$ of formal power series with coefficient in $R$. Then Theorem A gives a criterion for 2-forms on $E_q$ to be contained in dlog image of $K_2(E_q)$. It also gives a bounding space for $K_2$ of elliptic surface over $R$. Namely let $\pi : X_R \to C_R$ be an elliptic surface over $R$ which has $R$-rational multiplicative fibers $D_R$ (see [23] for the precise definition). Put $X_K := X_R \times_R K$, $X_K := X_R \times_R \mathbb{K}^*$ etc. Let $U_K$ be the complement of the all singular fibers. Then we introduce a $\mathbb{Z}_p$-submodule $\Phi(X_R, D_R)_{\mathbb{Z}_p} \subseteq \Gamma(X_R, \Omega^2_{X_R/R}(\log D_R))$ and show

$$\text{dlog} \Gamma(U_K, K_2) \otimes \mathbb{Z}_p \subseteq \Phi(X_R, D_R)_{\mathbb{Z}_p} \otimes \mathbb{Z}_p \mathbb{Q}_p \quad (1.2)$$
under a mild condition on $p$ (Theorem 3.6).

Theorem B concerns an upper bound of the rank of the dlog image of $K_2$ of elliptic surface minus singular fibers. Due to Theorem A we get $\text{rank}_{\mathbb{Z}} \text{dlog} \Gamma(U_K, K_2) \leq \text{rank}_{\mathbb{Z}_p} \Phi(X_R, D_R)_{\mathbb{Z}_p}$. However this is not enough to compute an explicit bound as it is usually difficult to compute the $\mathbb{Z}_p$-rank of $\Phi(X_R, D_R)_{\mathbb{Z}_p}$. Theorem B enables us to replace $\Phi(X_R, D_R)_{\mathbb{Z}_p}$ with a finite dimensional $\mathbb{F}_p$-space $\Phi(X_R, D_R)_{\mathbb{F}_p}$, which will be introduced in [6.1]. Theorems A and B imply that

$$\text{rank}_{\mathbb{Z}} \text{dlog} \Gamma(U_K, K_2) \leq \dim_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p} \tag{1.3}$$

under some conditions on $p$ and $X_R$ (Theorem 6.2).

One can compare (1.2) or (1.3) with (1.1). I expect that the equality also holds in (1.2) (Conjecture 3.7). If it is true, then the equality in (1.3) also holds (see Conjecture 6.3 and the remark after it). Conjecture 3.7 is true in the modular case (Theorem 8.1). However I have no idea how to attack this problem in general.

Our $\Phi(X_R, D_R)_{\mathbb{Z}_p}$ and $\Phi(X_R, D_R)_{\mathbb{F}_p}$ are defined from the Fourier expansions of 2-forms at the neighborhoods of multiplicative fibers. They are completely different from the right hand side of (1.1). One of the advantage is that $\Phi(X_R, D_R)_{\mathbb{F}_p}$ is computable in many examples. In fact we will give the following examples in [4]

**Corollary 1.1 (Theorem 9.1)** Let $\pi : X \to \mathbb{P}^1$ be the minimal elliptic surface over $\mathbb{C}$ such that the general fiber $\pi^{-1}(t)$ is the elliptic curve defined by $Y^2 = X^3 + X^2 + t^n$. Let $U$ be the complement of the all singular fibers in $X$. Then we have

$$\text{rank} \text{dlog} \Gamma(U, K_2) = 2 \quad \text{for } n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.$$

The dlog image is generated by

$$\text{dlog} \left\{ \frac{Y - X}{Y + X}, -\frac{t^n}{X^3} \right\}, \quad \text{dlog} \left\{ \frac{iY - (X + 2/3)}{iY + (X + 2/3)}, -\frac{t^n + 4/27}{(X + 2/3)^3} \right\}.$$ 

Since $\pi : X \to \mathbb{P}^1$ is defined over $\mathbb{Q}$, it seems difficult to obtain the above result without using $\Phi(X_R, D_R)_{\mathbb{F}_p}$. For example I do not know whether $\dim_{\mathbb{Q}} F^2 \cap H_B^2(U, H^2(\mathbb{Q}(2))) = 2$.

Computations of the rank of dlog image can be applied to the constructions of the indecomposable parts of the Adams weight piece $K_1(X)^{(2)}$ of $X$ (which is isomorphic to Bloch’s higher Chow group $\text{CH}^2(X, 1) \otimes \mathbb{Q}$) in special examples. In fact using Corollary 1.1 together with Stiller’s computations on Néron-Severi groups ([25], [26]), we can obtain the following:

**Corollary 1.2 (Theorem 9.5)** Let $\pi : X \to \mathbb{P}^1$ be as above and $D_i$ $(1 \leq i \leq n + 1)$ the multiplicative fibers. Let $K_1^{\text{ind}}(X)^{(2)}$ denotes the indecomposable $K_1$

$$K_1^{\text{ind}}(X)^{(2)} \overset{\text{def}}{=} K_1(X)^{(2)}/(\mathbb{C}^* \otimes \text{NS}(X))$$

where $\text{NS}(X)$ denotes the Néron-Severi group. Then we have

$$\dim \text{Image}(\bigoplus_{i=1}^{n+1} K_1'(D_i) \to K_1^{\text{ind}}(X)^{(2)}) = n - 1 \quad \text{for } n = 7, 11, 13, 17, 19, 23, 29.$$
I do not know how to show the non-vanishing of the regulator image of $\oplus_i K'_1(D_i)$ in the indecomposable part $H^3_D(X, \mathbb{Q}(2))/\left(\mathbb{C}^* \otimes \text{NS}(X)\right)$ of the Deligne-Beilinson cohomology group.

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2 Preliminaries

For an abelian group $M$, we denote by $M[n]$ (resp. $M/n$) the kernel (resp. cokernel) of multiplication by $n$. $M_{\text{tor}}$ denotes the torsion subgroup of $M$.

2.1 Algebraic $K$-theory and $K$-cohomology

Let $X$ be a separated noetherian scheme. Let $P(X)$ be the exact category of locally free sheaves, and $BQP(X)$ the simplicial set attached to $P(X)$ by Quillen ([17], [24]). Quillen’s higher $K$-groups of $X$ are defined as the homotopy groups of $BQP(X)$:

$$K_i(X) \overset{\text{def}}{=} \pi_{i+1} BQP(X), \quad i \geq 0.$$ 

We refer [24] for the general properties of higher $K$-theory such as, products, localization exact sequences, norm maps (also called transfer maps) etc.

Let $K_i$ denote the Zariski sheaf on $X$ associated to the presheaf

$$U \mapsto K_i(U) \quad (U \subset X).$$

The Zariski cohomology groups $H^i_{\text{Zar}}(X, K_i)$ are called the $K$-cohomology groups ([24] §5 etc.). Assume that $X$ is a regular irreducible scheme of dimension $d$. We denote by $X^j$ the set of points of height $j$. We write by $\kappa(x)$ the residue field of a point $x \in X$ and $i_x : \{x\} \rightarrow X$ the inclusion. Then we have the complex

$$0 \rightarrow K_i \rightarrow K_i(\kappa(\eta)) \rightarrow \bigoplus_{x \in X^1} i_x K_{i-1}(\kappa(x)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^d} i_x K_{i-d}(\kappa(x)) \rightarrow 0 \quad (2.1)$$

of Zariski sheaves where $\eta$ is the generic point.

Theorem 2.1 (Gersten conjecture) The complex (2.1) is exact in either of the following cases.

1. $i \geq 0$ and $X$ is a nonsingular variety over a field,

2. $i = 2$ and $X$ is smooth over a Dedekind domain.

Proof. The former is due to Quillen [17] Thm. 5.11, and the latter is due to Bloch [4]. Q.E.D.
Suppose that $X$ is either of the cases in Theorem 2.1. Then (2.1) gives the flasque resolution of the sheaf $\mathcal{K}_i$ so that we have the isomorphism

$$H^j_{\text{Zar}}(X, \mathcal{K}_i) \cong \frac{\ker(\bigoplus_{x \in X^1} K_{i-j}(\kappa(x)) \to \bigoplus_{x \in X^1} K_{i-j-1}(\kappa(x)))}{\text{Image}(\bigoplus_{x \in X^1} K_{i-j+1}(\kappa(x)) \to \bigoplus_{x \in X^1} K_{i-j}(\kappa(x)))}$$  \tag{2.2}

Hereafter, we often use the identification (2.2). In particular, we identify $\Gamma(X, \mathcal{K}_2) = H^0_{\text{Zar}}(X, \mathcal{K}_2)$ with the kernel of the tame symbol

$$\tau = \bigoplus_{x \in X^1} \tau_x : K^M_2(\kappa(\eta)) \to \bigoplus_{x \in X^1} \kappa(x)^*, \quad \{f, g\} \mapsto \sum_{x \in X^1} (-1)^{\text{ord}_x(f)\text{ord}_x(g)} \frac{\text{ord}_x(g)}{\text{ord}_x(f)}. \tag{2.3}$$

Here $K^M_2$ denotes Milnor’s $K_2$.

Let $X$ be a regular scheme of dimension $d$. There is the natural map $K_2(X) \to \ker \tau$, which is surjective if tensoring with $\mathbb{Z}[1/(d+1)!]$ ([23 Théorème 4 iv]). In particular, if $X$ is either of the cases in Theorem 2.1 we have a natural surjection

$$K_2(X) \otimes \mathbb{Z}[\frac{1}{(d+1)!}] \to \Gamma(X, \mathcal{K}_2) \otimes \mathbb{Z}[\frac{1}{(d+1)!}]. \tag{2.4}$$

### 2.2 Regulator and dlog maps on $\Gamma(X, \mathcal{K}_2)$

Let $W_k$ be a nonsingular variety over a field $k$. There are the regulator maps (also called the Chern class maps)

$$c_B : \Gamma(W_k, \mathcal{K}_2) \to H^2_B(W_k, \mathbb{Z}(2)) \tag{2.5}$$

to the Betti cohomology group when $k = \mathbb{C},$

$$c_{\text{ét}} : \Gamma(W_k, \mathcal{K}_2) \to H^2_{\text{ét}}(W_k, \mathbb{Z}/n(2)) \tag{2.6}$$

to the étale cohomology group when $n$ is invertible in $k,$ and

$$c_{\text{dR}} : \Gamma(W_k, \mathcal{K}_2) \to H^2_{\text{dR}}(W_k/k) \tag{2.7}$$

to the de Rham cohomology group when $k$ is of characteristic zero (cf. [27 §23. See also [3, 13] for the general Chern class maps on Quillen’s $K$-groups).

The following theorem will play an essential role in the proof of Theorem B.

**Theorem 2.2 (Suslin’s exact sequence ; [27 Corollary 23.4])** Suppose that $n$ is invertible in $k$. There is the natural exact sequence

$$0 \to \Gamma(W_k, \mathcal{K}_2)/n \xrightarrow{c_{\text{dR}}} H^2_{\text{ét}}(W_k, \mathbb{Z}/n(2)) \to H^1_{\text{Zar}}(W_k, \mathcal{K}_2)[n] \to 0. \tag{2.8}$$

Suppose that $k$ is of characteristic zero. The de Rham regulator $c_{\text{dR}}$ factors through the Hodge filtration $F^2 H^2_{\text{dR}}(W_k/k) = \Gamma(\overline{W}_k, \Omega^2_{\overline{W}_k}(\log D_k))$ where $\overline{W}_k \supset W_k$ is a smooth compactification such that $D_k := \overline{W}_k - W_k$ is a normal crossing divisor. Thus it gives rise to the dlog map

$$\text{dlog} : \Gamma(W_k, \mathcal{K}_2) \to \Gamma(\overline{W}_k, \Omega^2_{\overline{W}_k}(\log D_k)), \tag{2.9}$$
which is written as $\sum\{f, g\} \mapsto \sum df/f \cdot dg/g$ under the identification \ref{eq2.2}.

Let $A$ be a discrete valuation ring with a uniformizer $\pi$. Suppose that the quotient field $K$ is of characteristic zero. Let $W_A$ be a smooth scheme over $A$ and $\overline{W}_A \supset W_A$ smooth over $A$ such that $D_A = \sum D_{i,A} := \overline{W}_A - W_A$ is a relative normal crossing divisor over $A$, which means that all $D_{i,A}$ are smooth over $A$ and any intersection locus of $D_A$ are also smooth. Put $W_K := W_A \times_A K$. Then the dlog map \ref{eq2.9} induces

$$\text{dlog} : \Gamma(W_K, \mathcal{K}_2) \longrightarrow \Gamma(\overline{W}_A, \Omega^2_{\overline{W}_A/A}(\log D_A)).$$  \tag{2.10}

We can see it in the following way. Let $Ω^j_{W_A/A}(\log) := \lim_{\rightarrow} E \Gamma(W_A - E_{\text{sing}}, Ω^j_{W_A/A}(\log E)) \subset Ω^j_K(W)/K$

where $E$ runs over all divisors on $\overline{W}_A$ such that each irreducible component is flat over $A$ and $E_{\text{sing}}$ denotes the singular locus. We claim that the image of the dlog map $K^M(\mathcal{K}(W)) \longrightarrow Ω^2_{\mathcal{K}(W)/K}$

is contained in $Ω^2_{\overline{W}_A/A}(\log)$. In fact let $f \in K(W)^*$. Since $d(cf)/(cf) = df/f$ for $c \in K^*$ and $\overline{W}_A$ is smooth over $A$, we may replace $f$ with $\pi^\alpha f$ so that all irreducible components of the divisor of $f$ are flat over $A$. This shows $df/f \in Ω^2_{\overline{W}_A/A}(\log)$ for all $f \in K(W)^*$ and hence $df/f \cdot dg/g \in Ω^2_{\overline{W}_A/A}(\log)$. Let

$$\text{Res}_x : Ω^j_{\overline{W}_A/A}(\log) \longrightarrow Ω^{j-1}_{\kappa(x)/K}$$

be the residue map at $x$. Then we have a commutative diagram

$$\begin{array}{ccc}
K^M(\mathcal{K}(W)) & \longrightarrow & \bigoplus_{x \in W_K^1} \kappa(x)^* \\
\text{dlog} & & \text{dlog} \\
Ω^2_{\overline{W}_A/A}(\log) & \xrightarrow{\oplus \text{Res}_x} & \bigoplus_{x \in W_K^1} Ω^1_{\kappa(x)/K}.
\end{array}$$ \tag{2.11}

The kernel of the bottom arrow is

$$\lim_{\rightarrow} E \Gamma(\overline{W}_A - E_{\text{sing}}, Ω^2_{\overline{W}_A/A}(\log D_A)) \subset \lim_{\text{codim}Z \geq 2} \Gamma(\overline{W}_A - Z, Ω^2_{\overline{W}_A/A}(\log D_A))$$

$$= \Gamma(\overline{W}_A, Ω^2_{\overline{W}_A/A}(\log D_A)).$$

The last equality follows from the fact that $Ω^2_{\overline{W}_A/A}(\log D_A)$ is locally free of finite rank. Thus the diagram \ref{eq2.11} together with the identification \ref{eq2.2} gives rise to \ref{eq2.10}.

### 2.3 Elliptic surface and multiplicative fiber

Let $A$ be a field or a discrete valuation ring. In this paper, we mean by an elliptic surface over $A$ a projective flat morphism $π_A : X_A \rightarrow C_A$ of $A$-schemes such that

(i). $X_A$ and $C_A$ are projective and smooth schemes over $A$ of relative dimension 2 and 1 respectively,
(ii). The general fiber of \( \pi_A \) is an elliptic curve,

(iii). \( \pi_A \) has a section \( e_A : C_A \to X_A \).

Let \( K \) be the quotient field of \( A \) and \( \overline{K} \) the algebraic closure. Put \( X_K := X_A \times_A K \) and \( X_{\overline{K}} := X_A \times_A \overline{K} \) etc. If all fibers of \( \pi_K : X_K \to C_K \) are free from \((-1)\)-curves, we call it \textit{minimal}. It is well-known that there is a unique minimal elliptic surface \( \pi_K' : V_K \to C_K \) over \( K \) with a surjective map \( \rho_K : X_K \to V_K \) such that \( \pi_K' \rho_K = \pi_K \):

\[
\begin{array}{ccc}
X_K & \xrightarrow{\rho_K} & V_K \\
\pi_K & \downarrow & \pi_K' \\
C_K & \xrightarrow{} & \\
\end{array}
\]

The classification of the singular fibers of minimal elliptic surface is well-known thanks to works of Kodaira and Néron. I refer the reader to the book [21] IV §8. Note that if \( \pi_K \) has a section, then there is no multiple fiber (often denoted by \( \pi I_m \)).

Let \( \tilde{C} := \mathbb{P}^1 \times \mathbb{Z}/m \) be the disjoint union of copies of \( \mathbb{P}^1 \), indexed by \( \mathbb{Z}/m \). Attach the point 0 of \( i \)-th \( \mathbb{P}^1 \) to the point \( \infty \) of \( (i+1) \)-th \( \mathbb{P}^1 \). Then we have a connected proper curve \( C_\mathbb{Z} \) over Spec\( \mathbb{Z} \) with the normalization \( \tilde{C} \to C_\mathbb{Z} \). We call \( C_\mathbb{Z} \times S \) the \textit{standard Néron polygon} (or \textit{standard m-gon}) over a scheme \( S \) (cf. [6] II. 1.1).

Let \( D_{\overline{K}} \) be a singular fiber of \( \pi_{\overline{K}} \). If there is a closed subscheme \( D_\mathbb{Z}^+ \subset D_{\overline{K}} \) which is isomorphic to a standard \( m \)-gon, then we call \( D_{\overline{K}} \) a \textit{multiplicative fiber} or type \( I_m \). This is equivalent to say that \( \rho_{\overline{K}}(D_{\overline{K}}) \) is a standard \( m \)-gon.

If the elliptic surface \( \pi_A : X_A \to C_A \) satisfies the following condition, we say that it has \textit{A-rational} multiplicative fibers:

**\textbf{(Rat)}** There is a closed subscheme \( \Sigma_A \subset C_A \) which is a disjoint union of finite copies of Spec\( A \) such that

(i). a singular fiber \( \pi_{\overline{K}}^{-1}(P) \) over \( P \in C_A(\overline{K}) \) is multiplicative if and only if \( P \in \Sigma_A(\overline{K}) \),

(ii). for each \( A \)-rational point \( P \) of \( \Sigma_A \), there is a closed subscheme \( D_A^+ \subset \pi_A^{-1}(P) \) which is isomorphic to a standard Néron polygon over \( A \).

When \( A = \overline{K} \), the above is automatically satisfied.

\textbf{Notation.} Let \( \pi_A : X_A \to C_A \) be an elliptic surface over a discrete valuation ring or a field \( A \) satisfying \textbf{(Rat)}. Let \( \Sigma_A = \{ P_1, \cdots, P_s \} \) with \( P_i \cong \text{Spec} A \). We put \( S_A := C_A - \Sigma_A \), \( D_{i,A} := \pi_A^{-1}(P_i) \), \( D_A := \sum D_{i,A} \) and \( U_A := X_A - D_A \):

\[
\begin{array}{ccc}
U_A & \xrightarrow{\pi_A} & X_A \\
\downarrow & & \downarrow \\
S_A & \xrightarrow{} & C_A.
\end{array}
\]

We put by \( r_i \geq 1 \) the number of the irreducible components of the standard Néron polygon \( D_{i,A}^+ \subset D_{i,A} \) (i.e. \( D_{i,A}^+ \) is a standard \( r_i \)-gon). Equivalently, \( r_i \) is the pole order of the functional \( j \)-invariant of \( X_{\overline{K}} \to C_{\overline{K}} \) at \( P_i \). Moreover let \( S_A^0 \subset S_A \) be an arbitrary
open subscheme such that $T_A := (S_A - S_A^0)_{\text{red}}$ is flat over $A$ where ‘red’ denotes the reduced subscheme. We put $C_A := C_A - T_A$, $Y_A := \pi_A^{-1}(T_A)_{\text{red}}$, $U_A^0 := X_A - D_A - Y_A$ and $X_A^0 := X_A - Y_A$:

$$
\begin{align*}
U_A^0 & \longrightarrow X_A^0 \\
\pi_A & \downarrow \quad \downarrow \pi_A \\
S_A^0 & \longrightarrow C_A^0.
\end{align*}
$$

### 2.4 Boundary maps

Let $\pi_A : X_A \to C_A$ be an elliptic surface over a discrete valuation ring or a field $A$ satisfying (Rat). For each multiplicative fiber $D_{i,A}$, we fix an irreducible component $Z_{i,A} \subset D_{i,A}^\dagger$ of the standard Néron polygon and a singular point $Q_{i,A}$ of $D_{i,A}^\dagger$ such that $Q_{i,A} \in Z_{i,A}$. Put $Z_{i,A}^* = Z_{i,A} \cap D_{i,A}^{\dagger,\text{reg}}$ where $D_{i,A}^{\dagger,\text{reg}}$ denotes the regular locus. Then $Z_{i,A}^*$ is isomorphic to $\mathbb{G}_m,A$. Let $\tau_i : \Gamma(U_A^0, \mathcal{K}_2) \to \Gamma(Z_{i,A}^*, \mathcal{K}_1)$ be the tame symbol (2.3) at $Z_{i,A}^*$ and $\text{ord}_i : \Gamma(Z_{i,A}^*, \mathcal{K}_1) \to \mathbb{Z}$ the map of order at $Q_{i,A}$. Put $\partial_i := \text{ord}_i \cdot \tau_i$. We call the following map the **boundary map** in algebraic $K$-theory:

$$
\partial = \bigoplus_{i=1}^s \partial_i : \Gamma(U_A^0, \mathcal{K}_2) \longrightarrow \mathbb{Z}^{\oplus s}.
$$

(2.12)

It is easy to see that $\partial_i$ does not depend on the choices of $Z_{i,A}$ nor $Q_{i,A}$ up to sign (cf. [19] 1.5 or [42, I] below). Since our objective is the image of the boundary, the sign ambiguity does not matter. We write the composition $K_2(U_A^0) \to \Gamma(U_A^0, \mathcal{K}_2) \xrightarrow{\partial} \mathbb{Z}^{\oplus s}$ by the same notation $\partial$.

There are the corresponding maps (which we also call the **boundary maps**)

$$
\partial_B : H_B^2(U_C^0, \mathbb{Z}(2)) \longrightarrow \mathbb{Z}^{\oplus s}
$$

(2.13)

on the Betti cohomology when $A = \mathbb{C}$ and

$$
\partial_{et} : H_{et}^2(U_{\overline{K}}^0, \mathbb{Z}/n(2)) \longrightarrow \mathbb{Z}/n^{\oplus s}
$$

(2.14)

on the étale cohomology when $A = \overline{K}$ is an algebraically closed field in which $n$ is invertible. They are compatible with $\partial$ under the regulator maps $c_B$ and $c_{et}$. The map $\partial_B$ is defined as the composition of the following maps

$$
H_B^2(U_C^0, \mathbb{Z}(2)) \longrightarrow \bigoplus_{i=1}^s H_B^1(Z_{i,C}^*, \mathbb{Z}(1)) \xrightarrow{\cong} \bigoplus_{i=1}^s \mathbb{Z}.
$$

The definition of $\partial_{et}$ is similar. Note that $\partial_B$ (or $\partial_{et}$) is also defined as the composition of $H_B^2(U_C^0, \mathbb{Z}(2)) \to H_{D,Y,B}^2(X_C, \mathbb{Z}(2))$ and the Poincare-Lefschetz duality $H_{D,B}^2(X_C, \mathbb{Z}(2)) \cong H_1(D_C, \mathbb{Z}) = \mathbb{Z}^{\oplus s}$. In particular, $\partial_B$ is a homomorphism of mixed Hodge structure (we put the trivial Hodge structure of the right hand side of (2.13), and $\partial_{et}$ is compatible in $G_k$-action.

Let $K$ be the quotient field of $A$. Suppose that the characteristic of $K$ is zero. Put $X_K := X_A \times_A K$ etc. Then the dlog map (2.10) induces

$$
dlog : \Gamma(U_K^0, \mathcal{K}_2) \longrightarrow \Gamma(X_A, \Omega_{X_A/K}^2(\log D_A)).
$$

(2.15)
We can see it in the following way. A priori the dlog image is contained in \( \Gamma(U^0_K, \Omega^2_{X_A/K}) \).

It follows from (2.10) that we have \( \text{dlog} \Gamma(U^0_K, \mathcal{K}_2) \subset \Gamma(X^0_A, \Omega^2_{X_A/A}(\log D_A)) \). Moreover we claim that \( \text{dlog} \Gamma(U^0_K, \mathcal{K}_2) \subset \Gamma(X_K, \Omega^2_{X_K/K}(\log D_K)) \). To see this, we may replace \( K \) with \( \mathbb{C} \). Then it is enough to show \( F^2H^2_{\text{dr}}(U^0_C/\mathbb{C}) \cap H^2_B(U^0_C, \mathbb{Q}(2)) \subset F^2H^2_{\text{dr}}(U^0_C/\mathbb{C}) = \Gamma(X_C, \Omega^2_{X_C/\mathbb{C}}(\log D_C)) \). However it follows from the exact sequence

\[
H^2_{Y,B}(U_C, \mathbb{Q}) \longrightarrow H^2_B(U_C, \mathbb{Q}) \longrightarrow H^2_B(U^0_C, \mathbb{Q}) \longrightarrow H^3_{Y,B}(U_C, \mathbb{Q})
\]
of mixed Hodge structures and the fact that \( H^2_{Y,B}(U_C, \mathbb{Q}) \cong \mathbb{Q}(-1)^\oplus \) and \( H^3_{Y,B}(U_C, \mathbb{Q}) \) is of type \( (1,2) + (2,1) \) (cf. [5] III). Thus we get

\[
\text{dlog} : \Gamma(U^0_K, \mathcal{K}_2) \longrightarrow \Gamma(X^0_A, \Omega^2_{X_A/A}(\log D_A)) \cap \Gamma(X_K, \Omega^2_{X_K/K}(\log D_K)) = \Gamma(X_A, \Omega^2_{X_A/A}(\log D_A))
\]

where the equality follows from the fact that \( \Omega^2_{X_A/A}(\log D_A) \) is a locally free sheaf.

The Poincare residue gives rise to the boundary map

\[
\partial_{\text{DR}} : \Gamma(X_A, \Omega^2_{X_A/A}(\log D_A)) \longrightarrow A^{\oplus s}
\]

on the de Rham cohomology group. It is compatible with \( \partial \) under the dlog map (2.15):

\[
\begin{align*}
\Gamma(X_A, \Omega^2_{X_A/A}(\log D_A)) & \xrightarrow{\text{dlog}} A^{\oplus s} \\
& \xrightarrow{\partial_{\text{DR}}} A^{\oplus s} \\
& \xrightarrow{\partial} \mathbb{Z}^{\oplus s}.
\end{align*}
\]

Lemma 2.3 \( \text{dlog} \Gamma(U^0_A, \mathcal{K}_2) \cong \partial \Gamma(U^0_A, \mathcal{K}_2) \). In particular, \( 0 \leq \text{rank}_\mathbb{Z}\text{dlog} \Gamma(U^0_A, \mathcal{K}_2) \leq s \).

Proof. It is enough to show that \( \partial_{\text{DR}} \) is injective on the image \( \text{dlog} \Gamma(U^0_A, \mathcal{K}_2) \). To do this, we may replace \( A \) with \( \overline{K} \). Moreover it follows from a standard argument that we may assume \( \overline{K} = \mathbb{C} \). \( \partial_{\text{DR}} \) is compatible with \( \partial_B \) and the dlog image is contained in \( H^2_B(U^0_C, \mathbb{Z}(2))/(\text{torsion}) \). Therefore it is enough to show that the following map

\[
F^2H^2_{\text{dr}}(U^0_C/\mathbb{C}) \cap \left(H^2_B(U^0_C, \mathbb{Z}(2))/(\text{torsion}) \right) \xrightarrow{\partial_B} \mathbb{Z}^{\oplus s}
\]
is injective. Since \( \partial_B \) is a homomorphism of mixed Hodge structures, the injectivity of (2.18) follows from the Hodge symmetry. Q.E.D.

Lemma 2.4

(1) \( \partial \Gamma(U^0_A, \mathcal{K}_2) = \partial \Gamma(U^0_K, \mathcal{K}_2) \).

(2) Suppose \( A = K \) (i.e. \( A \) is a field). Then we have \( \partial \Gamma(U_K, \mathcal{K}_2) = \partial \Gamma(U^0_K, \mathcal{K}_2) \) and \( \partial \Gamma(U^0_K, \mathcal{K}_2) \otimes \mathbb{Q} = \partial \Gamma(U^0_K, \mathcal{K}_2) \otimes \mathbb{Q} \).
Proof. (2). Let $k$ be the residue field of $K$ and $U_k^0$ and $S_k$ the special fibers over Spec$(k)$. It follows from a commutative diagram

$$
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
k(U_k^0)^* & \bigoplus_{x \in (U_k^0)^1} \mathbb{Z} \\
\downarrow & \downarrow \\
K_2^M(K(U^0)) & \bigoplus_{x \in (U_k^0)^1} \kappa(x)^* & \bigoplus_{x \in (U_k^0)^2} \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
K_2^M(K(U^0)) & \bigoplus_{x \in (U_k^0)^1} \kappa(x)^* & \bigoplus_{x \in (U_k^0)^2} \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

and the Gersten conjecture (Theorem 2.1) that we have an exact sequence

$$
\Gamma(U_A^0, K_2) \to \Gamma(U_k^0, K_2) \to \Gamma(U_k^0, K_1). \tag{2.19}
$$

Since $\Gamma(U_k^0, K_1) \cong \Gamma(S_k^0, K_1)$, we have a commutative diagram

$$
\begin{array}{ccc}
\Gamma(S_A^0, K_2) & \to & \Gamma(S_k^0, K_2) \\
\downarrow & & \downarrow \\
\Gamma(U_A^0, K_2) & \to & \Gamma(U_k^0, K_2) \\
\downarrow & & \downarrow \\
\Gamma(S_A^0, K_2) & \to & \Gamma(S_k^0, K_2) \\
\end{array}
$$

with exact rows. A diagram chase yields that the map

$$
\Gamma(U_A^0, K_2) \oplus \pi_k^* \Gamma(S_k^0, K_2) \to \Gamma(U_k^0, K_2)
$$

is surjective. It is clear from the definition that $\partial \kappa_k^* \Gamma(S_k^0, K_2) = 0$. Thus we have $\partial \Gamma(U_A^0, K_2) = \partial \Gamma(U_k^0, K_2)$.

(2). We first show that $\partial \Gamma(U_K, K_2) = \partial \Gamma(U_k^0, K_2)$. It follows from a commutative diagram

$$
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\bigoplus_{\eta \in Y_K^0} \kappa(\eta)^* & \bigoplus_{x \in Y_K^1} \mathbb{Z} \\
\downarrow & \downarrow \\
K_2^M(K(U)) & \bigoplus_{x \in U_K^1} \kappa(x)^* & \bigoplus_{x \in U_K^2} \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
K_2^M(K(U)) & \bigoplus_{x \in (U_K^0)^1} \kappa(x)^* & \bigoplus_{x \in (U_K^0)^2} \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

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and the Gersten conjecture (Theorem 2.1) that we have an exact sequence
\[ \Gamma(U_K, \mathcal{K}_2) \rightarrow \Gamma(U^0_K, \mathcal{K}_2) \rightarrow \ker c. \] (2.20)

We claim \( \ker c \cong \Gamma(T_K, \mathcal{K}_1) \). In fact, let \( \mu : \tilde{Y}_K \rightarrow Y_K \) be the normalization. Then we have a commutative diagram
\[
\begin{array}{cccc}
0 & \rightarrow & \ker \mu_* & \rightarrow \\
\downarrow & & \downarrow & \\
\bigoplus_{\eta \in \tilde{Y}_K^0} \kappa(\eta)^* & \xrightarrow{\tilde{c}} & \bigoplus_{x \in \tilde{Y}_K} \mathbb{Z} & \xrightarrow{\mu_*} \\
\downarrow & & \downarrow & \\
\bigoplus_{\eta \in Y_K^0} \kappa(\eta)^* & \xrightarrow{c} & \bigoplus_{x \in Y_K} \mathbb{Z} & \\
\downarrow & & \downarrow & \\
0. & & & \\
\end{array}
\]

Since \( \tilde{Y}_K \) is regular, we have \( \ker \tilde{c} \cong \Gamma(T_K, \mathcal{K}_1) \) and \( \text{Coker} \tilde{c} \cong \text{Pic}(\tilde{Y}_K) \). Therefore it is enough to show that the map \( \ker \mu_* \rightarrow \text{Pic}(\tilde{Y}_K) \) is injective. To do this, we may replace \( K \) with \( \mathfrak{K} \). Let \( \text{Pic}(\tilde{Y}_K) \rightarrow \mathbb{Z}^\oplus \) be the degree map. Then the kernel of the composition \( \ker \mu_* \rightarrow \text{Pic}(\tilde{Y}_K) \rightarrow \mathbb{Z}^\oplus \) is isomorphic to \( H_1(\Gamma(\mathfrak{Y}_K), \mathbb{Z}) \) where \( \Gamma(\mathfrak{Y}_K) \) denotes the dual graph (cf. [6] I 3.5). Since all fibers over \( T_{\mathfrak{K}} \) are not multiplicative, it is zero.

Now we have a commutative diagram
\[
\begin{array}{cccc}
\Gamma(S_K, \mathcal{K}_2) & \rightarrow & \Gamma(S^0_K, \mathcal{K}_2) & \rightarrow & \Gamma(T_K, \mathcal{K}_1) \\
\pi_K \downarrow & & \pi_K \downarrow & & \pi_K \downarrow \\
\Gamma(U_K, \mathcal{K}_2) & \rightarrow & \Gamma(U^0_K, \mathcal{K}_2) & \rightarrow & \ker c \\
e_k \downarrow & & e_k \downarrow & & e_k \downarrow \\
\Gamma(S_K, \mathcal{K}_2) & \rightarrow & \Gamma(S^0_K, \mathcal{K}_2) & \rightarrow & \Gamma(T_K, \mathcal{K}_1) \\
\end{array}
\]

with exact rows. A diagram chase yields that the map
\[ \Gamma(U_K, \mathcal{K}_2) \oplus \pi_K^* \Gamma(S^0_K, \mathcal{K}_2) \rightarrow \Gamma(U^0_K, \mathcal{K}_2) \]
is surjective. Since \( \partial \pi_K^* \Gamma(S^0_K, \mathcal{K}_2) = 0 \), we have \( \partial \Gamma(U_K, \mathcal{K}_2) = \partial \Gamma(U^0_K, \mathcal{K}_2) \).

Next we show that \( \partial \Gamma(U^0_K, \mathcal{K}_2) \otimes \mathbb{Q} = \partial \Gamma(U^0_K, \mathcal{K}_2) \otimes \mathbb{Q} \). There is a finite extension \( L/K \)
such that \( \partial \Gamma(U^0_L, \mathcal{K}_2) = \partial \Gamma(U^0_K, \mathcal{K}_2) \). We want to show \( \partial \Gamma(U^0_K, \mathcal{K}_2) \otimes \mathbb{Q} \cong \partial \Gamma(U^0_L, \mathcal{K}_2) \otimes \mathbb{Q} \). Recall that the elliptic surface \( X_K \rightarrow C_K \) has \( K \)-rational multiplicative fibers (i.e. (Rat)). Therefore the norm map \( N_{L/K} : \Gamma(U^0_L, \mathcal{K}_2) \rightarrow \Gamma(U^0_K, \mathcal{K}_2) \) on \( K \)-cohomology induces a commutative diagram
\[
\begin{array}{cccc}
\Gamma(U^0_L, \mathcal{K}_2) & \xrightarrow{\partial} & \mathbb{Z}^\oplus \\
N_{L/K} \downarrow & & \downarrow \text{mult. by } [L:K] \\
\Gamma(U^0_K, \mathcal{K}_2) & \xrightarrow{\partial} & \mathbb{Z}^\oplus. \\
\end{array}
\]

Thus the assertion follows. Q.E.D.
Due to Lemma 2.4 (2), the quotient \(\frac{\partial \Gamma(U, K_2)}{\partial \Gamma(U_K, K_2)}\) is finite. We will later discuss when it is \(p\)-torsion free in case that \(K\) is a local field (Theorem B (1)).

### 3 Theorem A and \(\Phi(X_R, D_R)\mathbb{Z}_p\)

#### 3.1 Tate curves over complete rings

Let \(A\) be a noetherian local ring which is complete with respect to an ideal \(qA\). Then the Tate curve \(E_q = E_{q, A[q^{-1}]}\) is defined to be the proper smooth scheme over \(A[q^{-1}]\) defined by the equation

\[
y^2 + xy = x^3 + a_4(q)x + a_6(q)
\]  

with

\[
a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad a_6(q) = -\sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5)q^n}{12(1 - q^n)}. \tag{3.2}
\]

Let \(O\) be the infinity point of \(E_q\). The series

\[
x(u) = \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n u)^2} - 2 \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \tag{3.3}
\]

\[
y(u) = \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \tag{3.4}
\]

converge for all \(u \in A[q^{-1}]^* - q\mathbb{Z}\). They induce a homomorphism

\[
A[q^{-1}]^*/q\mathbb{Z} \longrightarrow E_q(A[q^{-1}]), \quad u \longmapsto \begin{cases} (x(u), y(u)) & \text{if } u \notin q\mathbb{Z} \\ O & \text{if } u \in q\mathbb{Z}, \end{cases} \tag{3.5}
\]

which is bijective if \(A\) is a complete discrete valuation ring. We often use the uniformization \((3.5)\). In particular, we have

\[
\frac{dx}{2y + x} = \frac{du}{u} \tag{3.6}
\]

which we call the canonical invariant 1-form.

**Definition 3.1 (Theta function)**

\[
\theta(u) = \theta(u, q) \overset{\text{def}}{=} (1 - u) \prod_{n=1}^{\infty} (1 - q^n u)(1 - q^n u^{-1})
\]

\(\theta(u)\) converges for all \(u \in A[q^{-1}]^*\) and satisfy

\[
\theta(qu) = \theta(u^{-1}) = -u^{-1}\theta(u). \tag{3.7}
\]

In particular a function

\[
f(u) = c \prod_{i} \frac{\theta(\alpha_i u)}{\theta(\beta_i u)} \tag{3.8}
\]
is $q$-periodic if $\prod_i \alpha_i/\beta_i = 1$. Suppose that $A = R$ is a complete discrete valuation ring. We denote by $K$ its quotient field. Then, for any rational function $f(u)$ on $E_q(K)$, one can find $c$, $\alpha_i$, $\beta_i \in K^*$ such that $f(u)$ is given as in (3.8). Thus we have the one-one correspondence

$$K(E_q)^* \xrightarrow{\sim} \left\{ c \prod_i \frac{\theta(\alpha_i u)}{\theta(\beta_i u)} : c, \alpha_i, \beta_i \in K^* \text{ with } \prod_i \alpha_i/\beta_i = 1 \right\},$$

by which we often identify the both sides.

### 3.2 Statement of Theorem A

Let $p \geq 3$ be a prime number, and $r \geq 1$ an integer. Let $R_0$ be the integer ring of a finite unramified extension $K_0$ of $\mathbb{Q}_p$ of degree $d$. Since $K_0/\mathbb{Q}_p$ is unramified, there is a cyclotomic basis $\mu = \{\zeta_1, \ldots, \zeta_d\}$ of $R_i$, namely $\zeta_i$ are roots of unity such that $R_0 = \bigoplus_{i=1}^d \mathbb{Z}_p \zeta_i$. Letting $q_0$ be an indeterminate we put $q := q_0^r$. Let $E_q = E_{q,R_0((q_0))}$ be the Tate curve over $R_0((q_0)) := R_0[[q_0]][q_0^{-1}]$.

The $d$log map $K_2 \to \Omega^2_{E_q/R_0}$ of Zariski sheaves on $E_q$ gives rise to

$$dlog : \Gamma(E_q, K_2) \longrightarrow \Gamma(E_q, \Omega^2_{E_q/R_0}).$$

We also write by $d$log the composite map

$$K_2(E_q) \longrightarrow \Gamma(E_q, K_2) \xrightarrow{\text{dlog}} \Gamma(E_q, \Omega^2_{E_q/R_0}) \longrightarrow R_0((q_0)) \frac{dq_0}{q_0} \frac{du}{u},$$

where the last map is the natural map. Define a map

$$\phi : R_0((q_0)) \frac{dq_0}{q_0} \frac{du}{u} \longrightarrow \prod_{k \geq 1} (\mathbb{Z}_p/k^2\mathbb{Z}_p)^{\oplus d}$$

in the following way. (Note that the inner component of the right hand side is zero unless $p|k$.) Express $f(q_0) \frac{dq_0}{q_0} \frac{du}{u} = \left( \sum_{j \in \mathbb{Z}} c_j q_0^j \right) \frac{dq_0}{q_0} \frac{du}{u}$

$$= \left( \sum_{j \leq 0} c_j q_0^j + \sum_{k=1}^{\infty} \sum_{i=1}^d a_k^{(i)} \frac{\zeta_i q_0^k}{1 - \zeta_i q_0^k} \right) \frac{dq_0}{q_0} \frac{du}{u}$$

with $a_k^{(i)} \in \mathbb{Z}_p$, which are uniquely determined. Then we define

$$\phi \cdot f(q_0) \frac{dq_0}{q_0} \frac{du}{u} \longmapsto (\bar{a}_k^{(1)}, \ldots, \bar{a}_k^{(d)})_{k \geq 1}$$

where bars denote modulo $k^2\mathbb{Z}_p$.

**Theorem A.** Suppose that $p$ is prime to $2r$. Then the composition

$$K_2(E_q) \xrightarrow{\text{dlog}} R_0((q_0)) \frac{dq_0}{q_0} \frac{du}{u} \xrightarrow{\phi} \prod_{k \geq 1} (\mathbb{Z}_p/k^2\mathbb{Z}_p)^{\oplus d}$$

is zero.
Remark 3.2 We can replace $K_2(E_q)$ with $\Gamma(E_q, K_2)$ in Theorem A if $p \geq 5$. In fact the map $K_2(E_q) \otimes \mathbb{Z}[1/6] \to \Gamma(E_q, K_2) \otimes \mathbb{Z}[1/6]$ is surjective as $\dim E_q = 2$ (Théorème 4 iv)).

Theorem A is the most important result in this paper. It gives a criterion for 2-forms to be contained in the dlog image.

We will prove Theorem A in §4 and §5. The outline is as follows. We first construct the following commutative diagram

\[
\begin{array}{ccc}
K_2(E_q) & \xrightarrow{\tau_\infty} & \tau_{\text{et}}^\infty \\
\text{dlog} \downarrow & & \downarrow \text{dlog} \\
R_0((q_0)) \frac{dq_0}{q_0} \frac{du}{u} \xleftarrow{\iota} R_0((q_0))^* \otimes \mathbb{Z}[r^{-1}] \xrightarrow{h} (R_0((q_0))[p^{-1}])^*/p^\nu
\end{array}
\]

for $\nu \geq 1$ where $h$ is the natural map from the ring homomorphism $R_0((q_0)) \to R_0((q_0))[p^{-1}]$ and $\iota$ is given by $h \mapsto \frac{dh}{h} \frac{du}{u}$. The map $\tau_{\text{et}}^\infty$ is defined from the étale regulator and the weight exact sequence on the étale $H^1$ of Tate curve (cf. (4.43)). The map $\tau_\infty$ is defined in the way of algebraic $K$-theory, which turns out to be an analogue of the Beilinson regulator, though our construction is completely different from Beilinson’s one (and it is rather simple). Although the three maps dlog, $\tau_\infty$ and $\tau_{\text{et}}^\infty$ are defined in different ways, it turns out that they are compatible. In particular we can reduce the problem to the characterization of the image of $\tau_{\text{et}}^\infty$. It follows from the definition that the image of $\tau_{\infty}$ is contained in the kernel of the map $x \mapsto x \cup q$ of cup-product in étale cohomology groups of $R_0((q_0))[p^{-1}]$. Thus we get a vanishing

\[
\{\tau_\infty(\xi), q\} = 0 \text{ in } K_2^M(R_0(q_0))/p^\nu \tag{3.15}
\]

for all $\xi \in K_2(E_q)$ where $R_0(q_0)$ denotes the $p$-adic completion of $R_0((q_0))$ (cf. §4.1.1). Finally we apply Kato’s explicit reciprocity law to (3.15) and obtain the desired result.

Remark 3.3 In an earlier version, I proved Theorem A only in the case of $R_0 = \mathbb{Z}_p$ and only for $k \leq p(2p - 2)$. The outline of the proof was the same, but I used the Artin-Hasse formula instead of Kato’s explicit reciprocity law. Dr. Yasuda informed me Kato’s work and pointed out that we can get more general statement.

The map $\phi_\mu$ depends on the basis $\mu$. However we have

Lemma 3.4 The kernel of $\phi_\mu$ does not depend on the choice of $\mu$.

Proof. Let $f(q_0) \in R_0[[q_0]]$ be expressed as in (3.13). Then

\[
ed_k^{(i)} \in k\mathbb{Z}_p \text{ for } \forall k, \ i \iff f(q_0) = q_0^k \frac{d}{dq_0} \log h(q_0), \ \exists h(q_0) \in R_0((q_0))^*
\tag{3.16}
\]

where $\log(1 - x) = -\sum_{i \geq 1} x^i/i$. Note that $h(q_0)$ is uniquely determined up to constant. Let $\sigma : R_0 \to R_0$ be the Frobenius automorphism and $\varphi : R_0((q_0)) \to R_0((q_0))$ the endomorphism such that $\varphi(aq_0^i) = \sigma(a)q_0^{ip}$ for $a \in R_0$ (which is also called the Frobenius).
Put $l_{\varphi}(g) = p^{-1} \log(\varphi(g)/g^p)$ for $g \in R_0((q_0))^*$ (cf. [5.3]). Let $\zeta \in R_0$ be an arbitrary root of unity. Since $\sigma(\zeta) = \zeta^p$, we have

$$l_{\varphi}(1 - \zeta q_0^j) = \sum_{m \geq 1, (m,p) = 1} \frac{(\zeta q_0^j)^m}{m}, \quad j \geq 1.$$ 

Therefore if we express

$$h(q_0) = c(-q_0)^m \prod_{k=1}^d \prod_{i=1}^{d} (1 - \zeta_i q_0^k)^{b_k(i)}$$

with $c \in R_0^*$, $m \in \mathbb{Z}$ and $-kb_k(i) = a_k(i)$, then we have

$$l_{\varphi}(h) = \sum_{k=1}^\infty \sum_{i=1}^d \sum_{m \geq 1, (m,p) = 1} b_k(i)(\zeta q_0^k)^m/m.$$ 

One can easily show that

$$a_k^{(i)} \in k^2 \mathbb{Z}_p \text{ for } \forall k, i$$

and

$$l_{\varphi}(h) = q_0 \frac{d}{dq_0} g(q_0), \quad \exists g(q_0) \in R_0((q_0))$$

by the induction on $k$. Since the right hand side of (3.16) and (3.18) do not depend on $\mu$, we see that so does not the kernel of $\phi_{\mu}$. Q.E.D.

### 3.3 $\Phi(X_R, D_R)_{\mathbb{Z}_p}$: a bounding space of dlog image

Let $p \geq 3$ be a prime number. Let $K$ be a finite unramified extension of $\mathbb{Q}_p$ and $R$ its integer ring. Let $\pi_R : X_R \to C_R$ be an elliptic surface over $R$ satisfying (Rat). We use the notations in [2,3]. Assume that $p$ is prime to $r_1 \cdots r_s$. Let $t_i \in \mathcal{O}_{C_R}$ be the uniformizer of $P_{i,R}$ for $1 \leq i \leq s$. Let $i_t : \text{Spec} R((t_i)) \longrightarrow S_R$ be the punctured neighborhood of $P_{i,R}$ and $X_i$ the fiber product:

$$X_i \longrightarrow U_R \quad \text{Spec} R((t_i)) \quad \overset{i_t}{\longrightarrow} \quad S_R.$$ 

Then $X_i$ is isomorphic to the Tate curve (e.g. [6] VII Cor. 2.6). More precisely let $q \in R((t_i))$ be the unique power series such that $\text{ord}_{t_i}(q) = r_i$ and

$$j(X_i) = \frac{1}{q} + 744 + 196884q + \cdots.$$ 

Then there is an isomorphism

$$X_i \cong E_{q,R((t_i))}$$

of $R((t_i))$-schemes (e.g. [6] VII. 2.6). The isomorphism (3.20) is unique up to the translation and the involution $u \mapsto u^{-1}$. Put $a_i := t_i^*/j(X_i)|_{t_i=0} = t_i^*/q|_{t_i=0} \in R^*$. Let
$K_i = K(a_i^{1/r_i})$ and $R_i$ its integer ring. Since $p$ is prime to $r_i$, $K_i$ is also unramified over $\mathbb{Q}_p$. There is $q_i \in R_i((t_i))$ such that $q_i^{r_i} = q$ and we have $R_i((t_i)) = R_i((q_i))$. Put $\Gamma(X, \Omega^2_{X/R}(\log D_R))_{z_p} := \partial_{dR}^{-1}(\mathbb{Z}^{\oplus s})$ where $\partial_{dR}$ is the boundary map (2.16). We have the composition of natural maps

$$
\Gamma(X, \Omega^2_{X/R}(\log D_R))_{z_p} \longrightarrow \Gamma(X, \Omega^2_{X,i/R}) \cong \Gamma(E_{q,R_i((t_i))}, \Omega^2_{E_{q,R_i((t_i))}/R})
\longrightarrow R_i((q_i)) \frac{dq_i}{q_i} \frac{du}{u} \quad (3.21)
$$

where the isomorphism in the middle is due to (3.20). Let $d_i = [K_i : \mathbb{Q}_p]$ and fix a cyclotomic basis $\mu_i$ of $R_i$. There are the corresponding maps on $K$-cohomology to (3.21) which give rise to a commutative diagram

$$
\begin{array}{c}
\Gamma(U, K_2) \otimes \mathbb{Z}_p \xrightarrow{\text{dlog}} \Gamma(X, \Omega^2_{X/R}(\log D_R))_{z_p} \\
\downarrow \\
\Gamma(E_{q,R_i((q_i))}, K_2) \otimes \mathbb{Z}_p \xrightarrow{\text{dlog}} \Phi((q_i)) \frac{dq_i}{q_i} \frac{du}{u} \quad (3.22) \\
\downarrow \phi_{\mu_i} \\
\prod_{k \geq 1}(\mathbb{Z}_p/k^2\mathbb{Z}_p)^{\oplus d_i}.
\end{array}
$$

Let $\phi_i$ be the composition of the right vertical arrows.

**Definition 3.5** $\Phi(X, D_R)_{z_p} := \ker(\oplus \phi_i) = \bigcap_{i=1}^{d} \ker \phi_i \subset \Gamma(X, \Omega^2_{X/R}(\log D_R))_{z_p}$.

It follows from Lemma 3.4 that $\Phi(X, D_R)_{z_p}$ does not depend on the choice of the cyclotomic basis $\mu_i$ and hence depends only on $(X, D_R)$. By Theorem A and Remark 3.2 we have $\phi_{\mu_i}(\text{dlog}) = 0$ in (3.22) if $p \geq 5$ and $(\phi_{\mu_i}(\text{dlog})) \otimes \mathbb{Q} = 0$ if $p \geq 3$. Thus we have $\text{dlog}(U, K_2) \otimes \mathbb{Z}_p \subseteq \Phi(X, D_R)_{z_p}$ if $p \geq 5$ and $\text{dlog} \Gamma(U, K_2) \otimes \mathbb{Q}_p \subseteq \Phi(X, D_R)_{z_p} \otimes \mathbb{Q}_p$ if $p \geq 3$.

Summarizing above together with Lemmas 2.3 and 2.4 we have

**Theorem 3.6** Let $R$ be the ring of integer in an unramified extension $K$ of $\mathbb{Q}_p$, and $\pi_R : X_R \rightarrow C_R$ an elliptic surface over $R$ satisfying (Rat). Then we have

$$
\text{dlog} \Gamma(U^0_R, K_2) \otimes \mathbb{Z}_p \subseteq \Phi(X, D_R)_{z_p} \quad \text{if } p \nmid r_1 \cdots r_s, \quad (3.23)
$$

$$
\text{dlog} \Gamma(U^0_R, K_2) \otimes \mathbb{Q}_p \subseteq \Phi(X, D_R)_{z_p} \otimes \mathbb{Q}_p \quad \text{if } p \nmid 2r_1 \cdots r_s. \quad (3.24)
$$

Although the sequence (3.14) is far from exact, I expect that the equality in (3.24) holds, namely the dlog image is characterized by the numerical conditions on the Fourier coefficients:

**Conjecture 3.7** (characterization of dlog image)

$$
\text{dlog} \Gamma(U^0_R, K_2) \otimes \mathbb{Q}_p = \Phi(X, D_R)_{z_p} \otimes \mathbb{Q}_p.
$$

We will see in §8 that it is true for modular elliptic surfaces.
4 Construction of $\langle \; , \; \rangle$ and $\tau_\infty$

In this section we construct maps

$$\langle \; , \; \rangle : K^M_2(\mathcal{K}(u)^b) \rightarrow \mathcal{K}^*, \quad (4.1)$$

$$\tau_\infty : K_2(E_{q,A[q^{-1}]} \otimes \mathbb{Z}[\frac{1}{r}]) \rightarrow A[q^{-1}]^* \otimes \mathbb{Z}[\frac{1}{r}], \quad (4.2)$$

where $\mathcal{K}$ is a complete discrete valuation field (see §4.1.1 below for the definition of $\mathcal{K}(u)^b$), $(A, q, r)$ is as in the beginning of §4.2, and $E_{q,A[q^{-1}]}$ is the Tate curve over $A[q^{-1}]$. Both maps play important roles in the proof of Theorem A. I keep in mind the cases of $K = L$ a finite extension of $\mathbb{Q}_p$ or $K = k((q_0))$ for (4.1) and the case that $A = R_0((q))$ or $R_0$ for (4.2).

4.1 Construction of the pairing $\langle \; , \; \rangle : K^M_2(\mathcal{K}(u)^b) \rightarrow \mathcal{K}^*$

Let $\mathcal{R}$ be a complete discrete valuation ring with a uniformizer $\pi_\mathcal{K}$ and $\mathcal{K}$ the quotient field of $\mathcal{R}$. Let ord$_{\mathcal{K}} : \mathcal{K} \rightarrow \mathbb{Z} \cup \{+\infty\}$ be the valuation such that ord$_{\mathcal{K}}(\pi_\mathcal{K}) = 1$. By convention, ord$_{\mathcal{K}}(0) = +\infty$. This map is uniquely extended on $\overline{\mathcal{K}}$ the algebraic closure of $\mathcal{K}$, which we also write by ord$_{\mathcal{K}}$. We put $\mathcal{R}_n = \mathcal{R}/\pi_\mathcal{K}^n$, $U_n = 1 + \pi_\mathcal{K}^n \mathcal{R}$ ($n \geq 1$) and $k = \mathcal{R}/\pi_\mathcal{K} \mathcal{R}$ the residue field.

4.1.1 $\mathcal{K}(u)$ and $\mathcal{K}(u)^b$

Letting $u$ be an indeterminate, we define a ring $\mathcal{R}\langle u \rangle$ to be the $\pi_\mathcal{K}$-adic completion of $\mathcal{R}(\langle u \rangle) := \mathcal{R}[[u]][u^{-1}]$:

$$\mathcal{R}\langle u \rangle \overset{\text{def}}{=} \lim_{\leftarrow i} \left( \mathcal{R}/\pi_\mathcal{K}^i(\langle u \rangle) \right).$$

$\mathcal{R}\langle u \rangle$ is the subring of the ring $\mathcal{R}[[u, u^{-1}]]$ of formal Laurent series which is generated by

$$\sum_{i=-\infty}^{+\infty} a_i u^i \quad (a_i \in \mathcal{R})$$

such that ord$_{\mathcal{K}}(a_i) \rightarrow +\infty$ as $i \rightarrow -\infty$. Since $\mathcal{R}(\langle u \rangle)$ is a PID, $\mathcal{R}\langle u \rangle$ is a discrete valuation ring with a uniformizer $\pi_\mathcal{K}$. We denote by $\mathcal{K}(u)$ the fractional field : $\mathcal{K}(u) = \mathcal{K} \otimes_{\mathcal{R}} \mathcal{R}(u) = \mathcal{R}(u)[\pi_\mathcal{K}^{-1}]$. The valuation on $\mathcal{K}(u)$ is given by

$$\text{ord}_{\mathcal{K}(u)} \left( \sum_{i=-\infty}^{+\infty} a_i u^i \right) = \min(\text{ord}_{\mathcal{K}}(a_i); i \in \mathbb{Z}).$$

Any $f \in \mathcal{K}(u)$ has the following expression

$$f = a_0 u^a \prod_{i=1}^{\infty} (1 - a_{-i} u^{-i})(1 - a_i u^i),$$

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with
\[
\begin{aligned}
a_0 &\in K^* \\
a_i &\in R \\
\text{ord}_K(a_{-i}) &> 0 & i > 0 \\
\text{ord}_K(a_{-i}) &\to +\infty & i \to +\infty.
\end{aligned}
\tag{4.3}
\]

Note \(\text{ord}_{K(u)}(f) = \text{ord}_K(a_0)\). The residue field of \(K(u)\) is the field \(k((u))\) of formal power series with coefficient in \(k\). Note that the field \(K(u)\) contains \(K(u)\), however neither \(K(u)\) nor \(K(u)\) contains \(K((u))\).

**Lemma 4.1** Let \(f \in K(u)\). The following are equivalent.

1. For any \(n \geq 1\), there are \(f_n \in K(u)\) and \(\phi_n \in \pi_K^R K(u)\) such that \(f = f_n(1 + \phi_n)\).
2. For any \(n \geq 1\), there are \(f'_n \in K(u)\) and \(\phi'_n \in \pi_K^R K(u)\) such that \(f = f'_n + \phi'_n\).

**Proof.** Assume that (1) holds. Put \(N = \text{ord}_{K(u)}(f)\). We have \(N = \text{ord}_{K(u)}(f_n)\) for all \(n \geq 1\). Let \(n_1 = \max(n - N, 1)\). Put \(f'_n = f_n\) and \(\phi'_n = f_n, \phi_n\). Since \(\text{ord}_{K(u)}(\phi_n) \geq n_1 + N\), we have (2). Conversely, assume that (2) holds. Then for all \(n > N\) we have \(N = \text{ord}_{K(u)}(f'_n)\). Put \(f_n = f'_{n+N}\) and \(\phi_n = \phi'_{n+N}/f'_{n+N}\). Since \(\text{ord}_{K(u)}(\phi_n) \geq (n + N) - N = n\), we have (1).

Q.E.D.

**Definition 4.2** We define
\[
K(u)^b \subset K(u)
\]
as the subset of the elements such that the equivalent conditions in Lemma 4.1 hold. Moreover, we define
\[
R(u)^b \overset{\text{def}}{=} \{ f \in K(u)^b; \text{ord}_{K(u)}(f) \geq 0 \}.
\]

**Lemma 4.3** \(K(u)^b\) is a complete discrete valuation field with \(\pi_K\) a uniformizer, and \(R(u)^b\) is its valuation ring. The residue field is \(k(u)\).

**Proof.** If \(f \in K(u)^b\) and \(g \in (K(u)^b)^*\) then \(fg^{-1} \in K(u)^b\) by Lemma 4.1 (1) and \(f - g \in K(u)^b\) by Lemma 4.1 (2). Therefore \(K(u)^b\) is a field. The map \(\text{ord}_{K(u)}\) gives the discrete valuation on \(K(u)^b\). Then \(R(u)^b\) is its valuation ring by definition. It follows from Lemma 4.1 (2) that any Cauchy sequence in \(R(u)^b\) converges to an element in \(R(u)^b\). Thus \(R(u)^b\) is complete. The last assertion is clear. Q.E.D.

**Example.** Let \(q \in K^*\) such that \(\text{ord}_K(q) > 0\). The typical example of elements of \(K(u)^b\) is the theta function \(\theta(u, q) = (1 - u) \prod_{n \geq 1} (1 - q^n u)(1 - q^n u^{-1})\). On the other hand, let \(f = a_0 + a_1 u + a_2 u^2 + \cdots \in R[[u]]\) such that \(f \mod(\pi_K) \not\in k(u)\). Then \(f\) is contained in \(R(u)\) but not in \(R(u)^b\).
For \( \alpha \in \overline{K} \) let \( \tau_\alpha \) denote the tame symbol at \( u = \alpha \):

\[
\tau_\alpha : K^2_M(\mathcal{K}(u)) \rightarrow \mathcal{K}^*, \quad \{f, g\} \mapsto (-1)^{\text{ord}_\alpha(f)\text{ord}_\alpha(g)} \left( \frac{f^{\text{ord}_\alpha(g)}}{g^{\text{ord}_\alpha(f)}} \right) \mid u = \alpha.
\]

Define a pairing

\[
(\mathcal{K}(u)^{\dagger})^* \times (\mathcal{K}(u)^{\dagger})^* \rightarrow \mathcal{K}^*/U_n, \quad (f, g) \mapsto (f, g)_n \overset{\text{def}}{=} \sum_{\text{ord}_\alpha(\alpha) > 0} \tau_\alpha \{f_n, g_n\} \quad (4.4)
\]

where \( f = f_n(1 + \phi_n) \) and \( g = g_n(1 + \psi_n) \) are as in Lemma 4.1 (1) and \( \alpha \) runs over all \( \alpha \in \overline{K} \) such that \( \text{ord}_\mathcal{K}(\alpha) > 0 \) (including \( \alpha = 0 \)). Since the choices of \( f_n \) or \( g_n \) are not unique, we need to show that the above pairing does not depend on them:

**Lemma 4.4** The pairing \( (4.4) \) is well-defined.

**Proof.** We want to show that if \( h \in \mathcal{K}(u)^* \cap (1 + \pi_\mathcal{K}^n\mathcal{R}(u)) \) and \( h' \in \mathcal{K}(u)^* \) then we have

\[
\sum_{\text{ord}_\mathcal{K}(\alpha) > 0} \tau_\alpha \{h, h'\} \in U_n. \quad (4.5)
\]

More generally, we show

\[
\sum_{\text{ord}_\mathcal{K}(\alpha) > 0} \tau_\alpha \{h, a\} \in U_n := 1 + \pi_\mathcal{K}^n\overline{\mathcal{K}} \quad (4.6)
\]

\[
\sum_{\text{ord}_\mathcal{K}(\alpha) > 0} \tau_\alpha \{h, b - u\} \in U_n \quad (4.7)
\]

for \( a, b \in \overline{K} \) where \( \overline{\mathcal{K}} \) denotes the integer ring of \( \overline{K} \).

For any \( h \in \mathcal{K}(u)^* \) there is a decomposition

\[
h = cu^n \prod_{\text{ord}_\mathcal{K}(\alpha_i) > 0} (1 - \alpha_i u^{-1})^{r_i} \prod_{\text{ord}_\mathcal{K}(\beta_j) \geq 0} (1 - \beta_j u)^{s_j} \quad (4.8)
\]

with \( n, r_i, s_j \in \mathbb{Z} \) and \( c, \alpha_i, \beta_j \in \overline{K}^* \). Since \( h \in \mathcal{K}(u)^* \) and \( \mathcal{K} \) is complete, both of

\[
\prod_{\text{ord}_\mathcal{K}(\alpha_i) > 0} (1 - \alpha_i u^{-1})^{r_i}, \quad \prod_{\text{ord}_\mathcal{K}(\beta_j) \geq 0} (1 - \beta_j u)^{s_j}
\]

are in \( \mathcal{K}(u)^* \) and hence \( c \in \mathcal{K}^* \). Assume \( h \in 1 + \pi_\mathcal{K}^n\mathcal{R}(u) \). Then reduction mod \( \pi_\mathcal{K} \) shows \( n = 0 \). Moreover since

\[
c \prod_{\text{ord}_\mathcal{K}(\alpha_i) > 0} (1 - \alpha_i u^{-1})^{r_i} = c(1 + a_1 u^{-1} + a_2 u^{-2} + a_3 u^{-3} + \cdots)
\]

\[
\equiv \prod_{\text{ord}_\mathcal{K}(\beta_j) \geq 0} (1 - \beta_j u)^{-s_j} \mod \pi_\mathcal{K}^n\mathcal{R}(u)
\]

\[
= 1 + b_1 u + b_2 u^2 + b_3 u^3 + \cdots,
\]

we have \( c - 1 \equiv a_i \equiv b_j \equiv 0 \mod \pi_\mathcal{K}^n \). Therefore it is enough to show \( (4.6) \) and \( (4.7) \) in the following cases:
Case (i) \( h = c \) where \( c \equiv 1 \mod \pi_K^n \).

Case (ii) \( h = \prod_{\text{ord}_K(\alpha_i) > 0} (1 - \alpha_i u^{-1})^{r_i} = 1 + a_1 u^{-1} + a_2 u^{-2} + \cdots \) with \( \forall a_i \equiv 0 \mod \pi_K^n \).

Case (iii) \( h = \prod_{\text{ord}_K(\beta_j) \geq 0} (1 - \beta_j u)^{s_j} = 1 + b_1 u + b_2 u^2 + \cdots \) with \( \forall b_i \equiv 0 \mod \pi_K^n \).

Proof: Case (i). Easy.

Proof: Case (ii). (4.6) is clear. We show (4.7). If \( \text{ord}_K(b) \leq 0 \), then
\[
\sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \{ h, b - u \} = \prod_i \left( \frac{b}{b - \alpha_i} \right)^{r_i} = (1 + a_1 b^{-1} + a_2 b^{-2} + \cdots)^{-1} \equiv 1 \mod \pi_K^n.
\]

If \( \text{ord}_K(b) > 0 \) and \( b \neq 0 \), then
\[
\sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \{ h, b - u \} = \sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \left\{ \prod_{\alpha_i \neq b} (1 - \alpha_i u^{-1})^{r_i}, b - u \right\} + \left\{ (1 - bu^{-1})^r, b - u \right\} = \prod_{\alpha_i \neq b} \left( \frac{b}{b - \alpha_i} \right)^{r_i} \cdot \frac{b^r}{b^r} = 1.
\]

If \( b = 0 \), then
\[
\sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \{ h, -u \} = \sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \left\{ \prod_{\alpha_i \neq 0} (1 - \alpha_i u^{-1})^{r_i}, -u \right\} = \prod_{\alpha_i \neq 0} \left( \frac{-\alpha_i}{-\alpha_i} \right)^{r_i} = 1.
\]

Proof: Case (iii). (4.6) is clear. (4.7) is also clear if \( \text{ord}_K(b) \leq 0 \). Suppose \( \text{ord}_K(b) > 0 \). Then
\[
\sum_{\text{ord}_K(\alpha) > 0} \tau_{\alpha} \{ h, b - u \} = \prod_{\text{ord}_K(\beta_j) \geq 0} (1 - \beta_j b)^{s_j} = 1 + b_1 b + b_2 b^2 + \cdots \equiv 1 \mod \pi_K^n.
\]

This completes the proof. Q.E.D.

Lemma 4.5 (1) \( \langle \ , \ \rangle_n \) is biadditive.

(2) Let \( f, g \in (\mathcal{K}(u)^\flat)^* \) and \( f \in 1 + \pi_K^n \mathcal{R} \langle u \rangle \). Then \( \langle f, g \rangle_n = 1 \).

(3) For \( m \geq n \) the pairings \( \langle \ , \ \rangle_m \) and \( \langle \ , \ \rangle_n \) are compatible:
\[
\begin{array}{ccc}
(\mathcal{K}(u)^\flat)^* \times (\mathcal{K}(u)^\flat)^* & \xrightarrow{\langle \ , \ \rangle_m} & \mathcal{K}^*/U_m \\
\| & & \downarrow \\
(\mathcal{K}(u)^\flat)^* \times (\mathcal{K}(u)^\flat)^* & \xrightarrow{\langle \ , \ \rangle_n} & \mathcal{K}^*/U_n
\end{array}
\]

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(4) $\langle f, 1 - f \rangle_n = 1$ for any $f \in (\mathcal{K}(u)^\flat)^*$ such that $f \neq 1$.

Proof. (1), (2) and (3) are obvious. To show (4), let $f = f_m(1 + \phi_m)$ ($m \geq n$) be the decomposition as in Lemma 4.1 (1). Put $N = \text{ord}_{\mathcal{K}(u)}(f) = \text{ord}_{\mathcal{K}(u)}(f_m)$ and $M = \text{ord}_{\mathcal{K}(u)}(1 - f)$. Then for $m > M - N$ the order of $1 - f_m = 1 - f + f_m\phi_m$ is equal to $M$. Therefore we have

$$1 - f = 1 - f_m - f_m\phi_m = (1 - f_m)
\frac{1}{1 - f_m}
(1 - f_m)(1 - \psi_m)$$

with $\text{ord}_{\mathcal{K}(u)}(\psi_m) \geq m + N - M$. Take $m \geq \max(n, n + M - N)$. Thus we have

$$\langle f, 1 - f \rangle_n = \sum_{\text{ord}_\mathcal{K}(\alpha) > 0} \tau_\alpha\{f_m, 1 - f_m\} = 1.$$ 

This completes the proof. Q.E.D.

Due to Lemma 4.5 (2) it forms the projective system. We define

$$\text{Lemma 4.6 (Explicit description of the pairing)}$$

Let $f, g \in \mathcal{K}(u)^\flat$ be expressed as

$$f = a_0 u^n \prod_{i=1}^\infty (1 - a_i u^{-i})(1 - a_i u^i), \quad g = b_0 u^m \prod_{i=1}^\infty (1 - b_i u^{-i})(1 - b_i u^i),$$

where $a_i$ and $b_i$ are as in (4.3). Then we have

$$\langle f, g \rangle = (-1)^{nm} \frac{a_0^m}{b_0^m} \prod_{i,j \geq 1} \frac{(1 - a_i^{j/(i,j)} b_j^{i/(i,j)})^{(i,j)}}{(1 - a_i^{j/(i,j)} b_j^{i/(i,j)})^{(i,j)}}$$

where $(i, j)$ denotes the greatest common divisor of $i$ and $j$.

Proof. Note that $\langle , \rangle_n$ annihilates the subgroup $\{1 + \pi_k^* \mathcal{R}(u) \cap \mathcal{K}(u)^\flat, -\}$ by Lemma 4.5 (2). Now (4.11) is straightforward from the definition. Q.E.D.

Let $\mathcal{R}(u)^\flat = \mathcal{R}(u)^\flat/\pi_k^* \mathcal{R}(u)^\flat \hookrightarrow \mathcal{R}(u)/\pi_k^* \mathcal{R}(u) = \mathcal{R}_n((u))$ be the subring. Then due to Lemma 4.5 (2) (and Lemma 4.6), our $\langle , \rangle$ also induces the following map

$$\langle , \rangle : K_2^M(\mathcal{R}(u)^\flat)_n \longrightarrow \mathcal{R}_n^*.$$ 

(4.12)

Remark 4.7 We will see that the pairing (4.12) can be extended on $K_2^M(\mathcal{R}_n((u)))$ and the same formula as (4.11) holds (§4.2.3). However I do not know how to show it directly from (4.11).
4.1.3 Definition of $\tau_\infty: K_2(E_q, \mathcal{K}) \to \mathcal{K}^*$

Let $E_q = E_{q, \mathcal{K}}$ be the Tate curve over $\mathcal{K}$. The uniformization (3.5) induces a natural inclusion $\mathcal{K}(E_q) \otimes \mathcal{K}(u) \to \mathcal{K}(u)^b$ as $\theta(u) \in \mathcal{K}(u)^b$. We define $\tau_\infty: K_2(E_q, \mathcal{K}) \to \mathcal{K}^*$ as the composition of the maps

$$K_2(E_q, \mathcal{K}) \to K_2^M(\mathcal{K}(E_q)) \to K_2^M(\mathcal{K}(u)^b) \xrightarrow{\langle \cdot, 1 \rangle} \mathcal{K}^*.$$ 

This gives the definition of (4.2) when $A$ is a complete discrete valuation ring.

**Remark 4.8 (Beilinson’s regulator)** Suppose $\mathcal{K} = \mathbb{C}((q))$ and $E_q = \mathbb{C}^*/q^\mathbb{Z}$ with $|q| \ll 1$. Then one can easily see that our $\tau_\infty$ is equal to Beilinson’s regulator:

$$\tau_\infty(\xi) = \exp \left( \sum \frac{1}{2\pi i} \int_\delta \log f \frac{dg}{g} - \log g(o) \frac{df}{f} \right), \quad \xi = \sum \{f, g\} \in K_2(E_q)$$

where $o$ is an initial point and $\delta \in \pi_1(E_q, o)$ is the vanishing cycle.

4.2 Construction of $\tau_\infty$ for arbitrary $A$

Let $A$ be a regular local ring which is complete with respect to the maximal ideal $\mathfrak{m}$. Let $q \in \mathfrak{m}$ and $q \neq 0$. We assume that $\mathcal{O} := A/\sqrt{qA}$ is also regular. Since $A$ is a UFD, the prime ideal $\sqrt{qA}$ is generated by a prime element $\pi$. Let $r \geq 1$ be the integer such that $\pi^r A = qA$. Let $\hat{A}$ be the completion of $A_{\pi}$ with respect to $\pi$. Then $\hat{A}$ is a complete discrete valuation ring with the uniformizer $\pi$. We have already constructed the map

$$\tau_\infty: K_2(E_q, \hat{A}[q^{-1}]) \to \hat{A}[q^{-1}]^*$$

in (4.13). There is the natural map $K_2(E_q, A[q^{-1}]) \to K_2(E_q, \hat{A}[q^{-1}])$. In this section we show $\tau_\infty(K_2(E_q, A[q^{-1}]) \otimes \mathbb{Z}[1/r]) \subset A[q^{-1}]^* \otimes \mathbb{Z}[1/r]$, which allows us to define

$$\tau_\infty: K_2(E_q, A[q^{-1}]) \otimes \mathbb{Z}[\frac{1}{r}] \to A[q^{-1}]^* \otimes \mathbb{Z}[\frac{1}{r}].$$

(Unfortunately I could not remove “$\otimes \mathbb{Z}[1/r]$”. However it does not matter for our purpose because we only handle $K_2(-) \otimes \mathbb{Z}_p$ and the prime number $p$ is supposed to be prime to $r$.)

4.2.1 Regular model $\mathcal{E}_{q,A}$

Let $X$ be the proper scheme over $A$ defined by the equation (3.1). Then $X \times_A \text{Spec} A[q^{-1}]$ is isomorphic to $E_{q, A[q^{-1}]}$, and $X$ is regular except at the locus $x = y = \pi = 0$. Taking $(r - 1)$-times blowing ups along the locus, we get a regular scheme $\mathcal{E}_{q,A}$ over $A$. It is proper over $A$. Moreover $\mathcal{E}_{q,A} \times_A \text{Spec} A[q^{-1}]$ is isomorphic to $E_{q, A[q^{-1}]}$ and the special fiber $D_{\mathcal{O}} := \mathcal{E}_{q,A} \times_A \text{Spec} \mathcal{O}$ is the standard Néron $r$-gon over $\mathcal{O}$. Note that $\mathcal{E}_{q,A}$ is the generalized elliptic curve in the sense of [6] II 1.12 (see also VII. 1 for rigid geometric construction of $\mathcal{E}_{q,A}$).

Quillen’s localization exact sequence (cf. [24] Prop.(5.15)) yields an exact sequence

$$K_2(\mathcal{E}_{q,A}) \to K_2(E_q, A[q^{-1}]) \to K_1'(D_{\mathcal{O}}) \xrightarrow{i_*} K_1(\mathcal{E}_{q,A})$$

(4.15)
where \( i : D_O \hookrightarrow \mathcal{E}_{q,A} \). We first calculate \( K'_1(D_O) \). Let \( D_{O}^{\text{reg}} \) be the regular locus of \( D_O \). It is the disjoint union of \( r \)-copies of \( G_{m,O} \). Then we have an exact sequence

\[
K_2(G_{m,O})_{\oplus r} \xrightarrow{\delta} K_1(O)_{\oplus r} \xrightarrow{i} K'_1(D_O) \xrightarrow{\delta'} K_1(G_{m,O})_{\oplus r} \rightarrow \mathbb{Z}^r. \tag{4.16}
\]

Note \( K_1(G_{m,O}) \cong \mathbb{Z} \oplus O^* \) and \( K_2(G_{m,O}) \cong K_2(O) \oplus O^* \) (loc.cit. Cor.(5.5)). Then \( \delta(K_2(O)_{\oplus r}) = 0 \) and the induced map

\[
(K_2(G_{m,O})/K_2(O))_{\oplus r} \xrightarrow{\delta'} (O^*)_{\oplus r} \rightarrow K_1(O)_{\oplus r} \cong (O^*)_{\oplus r}
\]

is given by the matrix

\[
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
& \ddots \\
& & 1 \\
& & -1 & 1
\end{pmatrix}
\] \tag{4.17}

Therefore the cokernel of \( \delta \) is isomorphic to \( O^* \). Similarly, \( \delta'(O^*)_{\oplus r} = 0 \) and the induced map \( K_1(G_{m,O})/K_1(O)_{\oplus r} = \mathbb{Z}^r \rightarrow \mathbb{Z}^r \) is also given by (4.17). Thus we have \( \ker \delta' \cong \mathbb{Z} \oplus O^*_{\oplus r} \):

\[
0 \rightarrow O^* \rightarrow K'_1(D_O) \rightarrow \mathbb{Z} \oplus O^*_{\oplus r} \rightarrow 0. \tag{4.18}
\]

Since the first map in (4.18) has the splitting by the map \( K'_1(D_O) \rightarrow K_1(O) \) induced from the structure morphism, we have

\[
K'_1(D_O) \cong \mathbb{Z} \oplus O^*_{\oplus r} \oplus O^*. \tag{4.19}
\]

This also implies

\[
\bigoplus_{k=1}^r K_1(D_O^{(k)}) \rightarrow K'_1(D_O) \rightarrow \mathbb{Z} \rightarrow 0 \tag{4.20}
\]

where \( D_O^{(k)} \) are the irreducible components of \( D_O \).

Next we study the kernel of \( i_* \). Let us consider the composition of the following 4 maps

\[
\bigoplus_{k=1}^r K_1(D_O^{(k)}) \rightarrow K'_1(D_O) \xrightarrow{i_*} K_1(\mathcal{E}_{q,A}) \xrightarrow{i^*} K_1(D_O) \rightarrow \bigoplus_{k=1}^r K_1(D_O^{(k)}). \tag{4.21}
\]

Note \( D_O^{(k)} \cong \mathbb{P}^1_O \) and \( K_1(\mathbb{P}^1_O) \cong O^*_{\oplus 2} \). The composition (4.21) is given by the matrix

\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
& \ddots & 1 \\
& & -2 & 1 \\
& & & 1 & -2
\end{pmatrix}
\] \tag{4.22}
Therefore we have

$$\text{kernel of } \mathbf{1.21} = \{ (\eta_1, \ldots, \eta_r) \in r K_1(D^k_\mathcal{O}) \mid r \eta_1 = \cdots = r \eta_r \}.$$ 

This shows

$$\text{ker } i_* \subset \mathbb{Z} \oplus M \oplus \mathcal{O}^*, \quad M := \{ (c_1, \ldots, c_r) \in \mathcal{O}^{* \oplus r} \mid c_1^i = \cdots = c_r^i \}.$$ (4.23)

under the identification (4.19).

We see that the inclusion in (4.28) is equal if we invert $r$. Let us see the last component $\mathcal{O}^*$. Let $p : \text{Spec}A[q^{-1}] \to E_q A[q^{-1}]$ be a rational point. Then $p, K_2(A[q^{-1}]) \subset K_2(E_q A[q^{-1}])$ is onto the component $\mathcal{O}^*$. Next, we see the second component. Let $f_E : E_q A[q^{-1}] \to \text{Spec} A[q^{-1}]$ be the structure morphism. Then $f_E K_2(A[q^{-1}]) \subset K_2(E_q A[q^{-1}])$ is onto the diagonal component of $\mathcal{O}^{* \oplus r}$, which is equal to $M$ if we invert $r$.

Finally we see the first component $\mathbb{Z}$. Note that the composition $K_2(E_q A[q^{-1}]) \to K_1'(D_\mathcal{O}) \to \mathbb{Z}$ is the boundary map $\partial$ in (2.24). We show that the image of $\partial$ contains $r \mathbb{Z}$. Since $A$ is complete with respect to $q$, there is the homomorphism $\mathbb{Z}[[q]] \to A$ and it gives rise to a commutative diagram

$$
\begin{align*}
K_2(E_q, \mathbb{Z}((q))) & \xrightarrow{\partial} \mathbb{Z} \\
\downarrow & \\
K_2(E_q, A[q^{-1}]) & \xrightarrow{\partial} \mathbb{Z}.
\end{align*}
$$

Therefore it is enough to show that the top arrow is surjective. Let $0 < a < b < c$ be integers. Put $q_0 = q^{1/c}$ and consider the embedding $\mathbb{Z}((q)) \hookrightarrow \mathbb{Z}((q_0))$. Let

$$f(u) := \frac{\theta(q_0^a u)^c}{\theta(u)^{c-a} \theta(qu)^a} = (-u)^a \left( \frac{\theta(q_0^a u)}{\theta(u)} \right)^c$$

and

$$g(u) := \frac{\theta(q_0^b u)^c}{\theta(u)^{c-b} \theta(qu)^b} = (-u)^b \left( \frac{\theta(q_0^b u)}{\theta(u)} \right)^c$$

the rational functions on $E_q, \mathbb{Z}((q_0))$. More precisely, let $Q_i$ be the $\mathbb{Z}((q_0))$-rational point of $E_q, \mathbb{Z}((q_0))$ corresponding to $u = q_0^{-i}$. Then $f$ (resp. $g$) has a zero at $Q_a$ (resp. $Q_b$) and a pole at $Q_0$ (resp. $Q_0$). Put

$$\xi^M_{(a,b,c)} := \left\{ \begin{array}{ll} f(u) & g(u) \\ f(q_0^{-b}) & g(q_0^{-a}) \end{array} \right\} \in K^2_2(\mathcal{O}(E_q, \mathbb{Z}((q_0)) - Q_*),$$ (4.24)

where $Q_* = Q_a + Q_b + Q_0$. The Milnor $K_2$-symbol $\xi^M_{(a,b,c)}$ defines a Quillen $K_2$-symbol $\xi^Q_{(a,b,c)} \in K_2(E_q, \mathbb{Z}((q_0)) - Q_*).$ Recall the localization exact sequence

$$K_2(E_q, \mathbb{Z}((q_0))) \longrightarrow K_2(E_q, \mathbb{Z}((q_0)) - Q_*) \overset{\tau}{\longrightarrow} K_1(Q_*) = K_1(\mathbb{Z}((q_0)) \oplus 3$$

where $\tau$ is the tame symbol. It is easy to see $\tau(\xi^Q_{(a,b,c)}) = 0$. Therefore there is a lifting $\xi'_{(a,b,c)} \in K_2(E_q, \mathbb{Z}((q_0)))$. We put

$$\xi_{(a,b,c)} := N(\xi'_{(a,b,c)}) \in K_2(E_q, \mathbb{Z}((q)))$$ (4.25)
where $N : K_2(E_{q,Z((q_0))}) \to K_2(E_{q,Z((q))})$ denotes the norm map for $\mathbb{Z}((q_0))/\mathbb{Z}((q))$. A direct calculation yields that $\partial(\xi_{(a,b,c)}) = a(b-a)(b-c)$ (cf. [13] Rem.5.5). Since the diagram

$$
\begin{array}{ccc}
K_2(E_{q,Z((q_0))}) & \xrightarrow{\partial} & \mathbb{Z} \\
N & \downarrow & \downarrow \\
K_2(E_{q,Z((q))}) & \xrightarrow{\partial} & \mathbb{Z}
\end{array}
$$

is commutative, we have $\partial(\xi_{(a,b,c)}) = a(b-a)(b-c)$. Choose $(a,b,c) = (1,2,3)$. Thus we have the surjectivity of $\partial : K_2(E_{q,Z((q))}) \to \mathbb{Z}$.

**Proposition 4.9** Under the isomorphism $K'_1(D_{\mathcal{O}}) \cong \mathbb{Z} \oplus \mathcal{O}^{\oplus r} \oplus \mathcal{O}^r$ where $M := \{(c_1, \cdots, c_r) \in \mathcal{O}^{\oplus r} | c_1^r = \cdots = c_r^r\}$. The inclusion is equal if we tensor $\mathbb{Z}[1/r]$.

More precisely, we can see from the above that $(\mathbb{Z} \oplus M \oplus \mathcal{O}^r)/\ker i_*$ is a subquotient of $(\mathbb{Z}/r\mathbb{Z})^{\oplus r+1}$.

**Corollary 4.10** Let $p : \text{Spec}A[q^{-1}] \to E_{q,A[q^{-1}]}$ be a rational point and $f_E : E_{q,A[q^{-1}]} \to \text{Spec}A[q^{-1}]$ the structure morphism. Let $\xi_{(a,b,c)}$ be as in (4.25) and also think of it being a $K_2$-symbol of $E_{q,A[q^{-1}]}$ by the natural homomorphism $\mathbb{Z}[[q]] \to A$. Then

$$
K_2(E_{q,A}) \oplus \mathbb{Z} \cdot (1,2,3) \oplus p_! K_2(A[q^{-1}]) \oplus f_E^* K_2(A[q^{-1}]) \longrightarrow K_2(E_{q,A[q^{-1}]})
$$

(4.26)
is surjective tensoring with $\mathbb{Z}[1/r]$.

### 4.2.2 Boundary map in algebraic $K$-theory

Let $S$ be a noetherian ring. We recall the boundary map

$$
\partial : K_{i+1}(S((u))) \longrightarrow K_i(S)
$$

(4.27)

from K. Kato [13] §2. Let

- $\mathcal{P}_S$ : the category of projective $S$-modules of finite rank
- $\mathcal{M}_S$ : the category of finite $S$-modules
- $\mathcal{M}_S^f$ : the category of finite $S$-modules which have finite projective resolutions.

Moreover let $\mathcal{H}_{S[[u]]}$ be the category of finite $S[[u]]$-modules $M$ such that $M \otimes_{S[[u]]} S((u)) = 0$ and there is a resolution

$$
0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow M \longrightarrow 0
$$

(4.28)

with $E_i \in \mathcal{P}_{S[[u]]}$.

By Quillen’s resolution theorem ([24] Theorem (4.6)) $K_i(S) \overset{\text{def}}{=} K_i(\mathcal{P}_S) \xrightarrow{\cong} K_i(\mathcal{M}_S^f)$. By Quillen’s localization theorem ([9], [24] Theorem (9.1)) we have the exact sequence

$$
\cdots \longrightarrow K_{i+1}(S[[u]]) \longrightarrow K_{i+1}(S((u))) \overset{\partial}{\longrightarrow} K_i(\mathcal{H}_{S[[u]]}) \longrightarrow K_i(S[[u]]) \longrightarrow \cdots
$$

(4.29)
Let $M \in \mathcal{H}_S[[u]]$. Then we can think of $M$ being a finite $S$-module. Since $S[[u]]$ is a flat $S$-module, (4.28) gives a flat resolution of $S$-module. Therefore $M$ has a finite projective resolution of $S$-modules. Thus we can think of $M$ being an object of $\mathcal{M}_S^f$. Let

$$\iota : \mathcal{H}_S[[u]] \to \mathcal{M}_S^f$$

be the functor which sends $M$ to $M$ (as $S$-module). It gives rise to a map $\iota_* : K^1(\mathcal{H}_S[[u]]) \to K^1(S)$. Then we define $\partial = \iota_* \partial'$.

Let us consider the Tate curve $E_{q,A}[[q^{-1}]]$. Let $E_{q,A}$ be as in §4.2.1. Put $A_n = A/q^nA$ for $n \geq 1$. Then there are the morphisms of schemes

$$\mathbb{G}_{m,A_n} \ni E_{q,A} \times_A \text{Spec} A_n \to E_{q,A} \leftarrow E_{q,A}[[q^{-1}]].$$

This give rise to

$$K^2(E_{q,A}) \to K^2(A_n[u,u^{-1}]) \to K^2(A_n((u))) \xrightarrow{\partial} K^1(A_n) = A^*_n.$$

Passing to the projective limit we have

$$\tau^\dagger : K^2(E_{q,A}) \to \lim_n A^*_n = A^*.$$

**Proposition 4.11** Suppose $A = \mathcal{R}$ a complete discrete valuation ring. Then the following diagram

$$\begin{array}{ccc}
K^2(E_{q,\mathcal{R}}) & \xrightarrow{\tau^\dagger} & \mathcal{R}^* \\
\downarrow & & \downarrow \cap \\
K^2(E_{q,K}) & \xrightarrow{\tau\infty} & K^* \\
\end{array}$$

is commutative. Here the bottom map $\tau\infty$ is as in §4.1.3.

**Corollary 4.12** $\tau\infty(K^2(E_{q,A})) \subset A^*$.

**Proof.** Apply Proposition 4.11 for $\mathcal{R} = \hat{A}$. Q.E.D.

We prove Proposition 4.11. Note that the map $K^2(\mathcal{R}_n[u,u^{-1}]) \to K^2(\mathcal{R}_n((u)))$ factors through $K^2(\mathcal{R}(u)_n)$ where $\mathcal{R}(u)_n = \mathcal{R}(u)^{\mathfrak{p}}/\mathcal{R}(u)^{\mathfrak{p}}$. By a theorem of van der Kallen $K^2(\mathcal{R}(u)_n)$ is isomorphic to Milnor’s $K^2_M(\mathcal{R}(u)_n)$ as the residue field is infinite ([11], [12]). Therefore, to show Proposition 4.11 it is enough to show that the following diagram

$$\begin{array}{ccc}
K^2_M(\mathcal{R}(u)_n) & \xrightarrow{(,)} & \mathcal{R}_n \\
\downarrow & & \downarrow = \\
K^2(\mathcal{R}_n((u))) & \xrightarrow{\partial} & \mathcal{R}_n \\
\end{array}$$

is commutative.

**Lemma 4.13** Let $f \in \mathcal{R}_n[[u]]^*$. Then $\partial\{f,u\} = f(0)$ where $\partial$ is as in (4.27).
Proof. There are commutative diagrams

\[
\begin{array}{c}
K_2(\mathcal{R}_n((u))) \xrightarrow{\partial} \mathcal{R}_n^* \\
\uparrow \hspace{2cm} \uparrow \\
K_2(\mathcal{R}(u)) \xrightarrow{\partial} \mathcal{R}^* \\
\downarrow \hspace{2cm} \downarrow \\
K_2(K((u))) \xrightarrow{\partial} K^*.
\end{array}
\]

Since \( f \) comes from \( \mathcal{R}[[u]]^* \) it is enough to show \( \partial\{g, u\} = g(0) \) for any \( g \in K[[u]]^* \). This is well-known (e.g. [24] Lemma (9.12)). Q.E.D.

**Lemma 4.14** Let \( f \in \mathcal{R}_n[[u]]^* \). Let \( a \in \mathcal{R}_n \) be a nilpotent element (i.e. \( a \in \pi_K \mathcal{R}_n \)). Then \( \partial\{f, u - a\} = f(a) \).

**Proof.** Since \( a \) is nilpotent we have an isomorphism

\[
h : \mathcal{R}_n[[u]] \rightarrow \mathcal{R}_n[[v]], \quad u \mapsto v + a
\]

of \( \mathcal{R}_n \)-algebras. The functor \( \Phi_h : M \mapsto M \otimes_{\mathcal{R}_n[[u]]} \mathcal{R}_n[[v]] \) induces a commutative diagram

\[
\begin{array}{c}
\mathcal{H}_{\mathcal{R}_n[[u]]} \\
\downarrow \Phi_h \\
\mathcal{M}^f_{\mathcal{R}_n} \\
\downarrow \iota \\
\mathcal{H}_{\mathcal{R}_n[[v]]}
\end{array}
\]

where \( \iota \) is as in (4.30). Thus we get a commutative diagram

\[
\begin{array}{c}
K_2(\mathcal{R}_n((u))) \xrightarrow{\partial} K_2(\mathcal{H}_{\mathcal{R}_n[[u]]}) \\
\downarrow h^* \hspace{2cm} \downarrow \Phi_{h^*} \\
K_2(\mathcal{R}_n((u))) \xrightarrow{\partial} K_2(\mathcal{H}_{\mathcal{R}_n[[v]]}) \\
\downarrow \iota^* \hspace{2cm} \downarrow h^*
\end{array}
\]

and hence we have \( \partial h^* = \partial \). Now we show \( \partial\{f, u - a\} = f(a) \). Due to the commutative diagram (4.35) we have \( \partial\{f, u - a\} = \partial\{f(v + a), v\} \). By Lemma 4.13 we have \( \partial\{f(v + a), v\} = f(a) \). This completes the proof. Q.E.D.

**Lemma 4.15** Let \( a \in \mathcal{R}_n \) be a nilpotent element. Then we have \( \partial\{a - u, u\} = 1 \).
Proof. Let $k = \mathcal{R}/\pi_K$ be the residue field. It follows from a commutative diagram

$$
\begin{align*}
K_2(\mathcal{R}_n((u))) & \xrightarrow{\partial} \mathcal{R}_n^* \\
\downarrow & \\
K_2(k((u))) & \xrightarrow{\partial} k^*
\end{align*}
$$

that we have $\partial\{a - u, u\} \equiv \partial\{-u, u\} \equiv 1 \mod q$. This means $\partial\{a - u, u\} = 1 + \pi_K f$ for some $f \in \mathcal{R}_n$. Therefore it is enough to show that there is $N \geq 1$ which is prime to $\text{char}(k)$ such that $\partial\{a - u, u\}^N = 1$.

Let $N \geq 1$ be an integer which is prime to $\text{char}(k)$ and such that $a^N = 0$. Let $\zeta_N$ be a primitive $N$-th root of unity. We may suppose that $\mathcal{R}$ contains $\zeta_N$ by replacing $\mathcal{R}$ with $\mathcal{R}[\zeta_N]$. In the same way as the diagram (4.35), we can see that the isomorphism

$$
v : \mathcal{R}_n[[u]] \rightarrow \mathcal{R}_n[[u]], \quad u \mapsto \zeta_N u
$$

induces a commutative diagram

![Diagram](4.36)

Thus we have

$$
\partial\{a - u, u\}^N = \prod_{i=0}^{N-1} \partial\{a - \zeta_N^i u, \zeta_N^i u\}^N
$$

$$
= \prod_{i=0}^{N-1} \partial\{a - \zeta_N^i u, u\}^N
$$

$$
= \partial\{a^N - u^N, u^N\}
$$

$$
= \partial\{-u^N, u^N\}
$$

$$
= 1.
$$

This completes the proof. Q.E.D.

Now we show that the diagram (4.33) is commutative. Due to Lemma 4.6 it is enough to show

$$
\partial\{f, g\} = (-1)^{nm} a_0^n b_0^n \prod_{i,j \geq 1} \frac{(1 - a_{i/(i,j)} b_{i/(i,j)^{-1}})}{(1 - a_{-i/(i,j)} b_{i/(i,j)^{-1}})}
$$

for

$$
f = a_0 u^n \prod_{i=1}^{\infty} (1 - a_{-i} u^{-i})(1 - a_i u^i), \quad g = b_0 u^n \prod_{i=1}^{\infty} (1 - b_{-i} u^{-i})(1 - b_i u^i),
$$

27
with
\[
\begin{cases}
  a_i, b_i \in \mathcal{R}_n^* & i \neq 0 \\
  a_{-i}, b_{-i} \in \pi_K \mathcal{R}_n & i > 0 \\
  a_{-i} = b_{-i} = 0 & i \gg 0.
\end{cases}
\]

We can reduce it to check (4.37) in the following cases:
\[
(f, g) = \begin{cases}
  (u, h) & h \in \mathcal{R}_n[[u]]^* \\
  (u, 1 - au^{-i}) & a \in \pi_K \mathcal{R}_n, \ i > 0 \\
  (1 - au^{-i}, h) & a \in \pi_K \mathcal{R}_n, \ i > 0, \ h \in \mathcal{R}_n[[u]]^* \\
  (1 - au^{-i}, 1 - bu^{-j}) & a, b \in \pi_K \mathcal{R}_n, \ i, j > 0.
\end{cases}
\]

The first case immediately follows from Lemma 4.13. To show the rest, we may replace \( \mathcal{R}_n \) with \( \mathcal{R}[a^{1/m}]/(\pi_K^n) \). Thus we can assume that \( a^{1/i}, b^{1/j} \in \mathcal{R}_n \) and hence may assume \( i = j = 1 \). Then the assertion follows from Lemmas 4.13 4.14 and 4.15. This completes the proof of the commutativity of (4.33) and hence Proposition 4.11.

4.2.3 Proof of \( \tau_\infty(K_2(E_q,A[q^{-1}]) \otimes \mathbb{Z}[1/r]) \subset A[q^{-1}]^* \otimes \mathbb{Z}[1/r]. \)

Due to Corollary 4.10, it is enough to show the assertion for each component in 4.20. Corollary 4.12 asserts \( \tau_\infty(K_2(\mathcal{E}_{q,A})) \subset A^*. \) Moreover it is easy to see \( \tau_\infty(p_1 K_2(A[q^{-1}])) = 0 \) and \( \tau_\infty(f_E^* K_2(A[q^{-1}])) = 0. \) Therefore the rest of the proof is to show \( \tau_\infty(\xi_{(1,2,3)}) \in A[q^{-1}]^*. \) However it follows from Lemma 4.6 that we have
\[
\tau_\infty(\xi_{(a,b,c)}) = N_{A' / A} \tau_\infty(\xi_{(a,b,c)}') = N_{A' / A} \left( (-q_0)^{a(b-a)(b-c)} \left( \frac{M_b}{M_{b-a} M_a} \right)^c \right)
\]
where
\[
M_i := \prod_{n=1}^\infty \frac{(1 - q_0^nc-c+i)nc-c+i}{(1 - q_0^nc-c-i)}, \quad (1 \leq i < c)
\]
and \( N_{A' / A} \) is the norm map (cf. [1] Cor.5.3). Thus it is clearly contained in \( A[q^{-1}]^* \). This completes the proof.

4.2.4 Functoriality of \( \tau_\infty \)

Let \( (A_i, q_i, \pi_i) \ (i = 1,2) \) be as in the beginning of 4.12. Let \( \phi : A_1 \to A_2 \) be a homomorphism of local rings such that \( \phi(q_1) = q_2 \). Let \( q_i A_i = \pi_i^r A_i \) (then \( r_1 | r_2 \)).

**Proposition 4.16** The following diagram
\[
\begin{align*}
K_2(E_{q_1,A_1[q_1^{-1}]}) \otimes \mathbb{Z}[1/r_1] & \xrightarrow{\tau_\infty} A_1[q_1^{-1}]^* \otimes \mathbb{Z}[1/r_1] \\
\phi^* \downarrow & \quad \quad \phi^* \downarrow \\
K_2(E_{q_2,A_2[q_2^{-1}]}) \otimes \mathbb{Z}[1/r_2] & \xrightarrow{\tau_\infty} A_2[q_2^{-1}]^* \otimes \mathbb{Z}[1/r_2]
\end{align*}
\]
is commutative.

**Proof.** Due to Corollary 4.10 it is enough to show the assertion for \( \xi_{(a,b,c)} \) and \( K_2(\mathcal{E}_{q_1,A_1}) \). As for \( \xi_{(a,b,c)} \), it directly follows from (4.39). As for \( K_2(\mathcal{E}_{q_1,A_1}) \), it follows from the functoriality of \( \tau_\infty \). Q.E.D.
Remark 4.17 Since $\langle , \rangle$ is clearly functorial, Proposition 4.16 is trivial when $A_i$ are complete discrete valuation rings. However the ring homomorphism $\phi$ does not necessarily induce $\hat{A}_1 \to \hat{A}_2$ in general. Hence we cannot reduce the proof to the functoriality of $\langle , \rangle$.

4.3 Compatibility with dlog map

Let $\mathcal{O}$ be a regular local ring and $A = \mathcal{O}[[t]]$ the ring of formal power series with coefficients in $\mathcal{O}$. Let $r \geq 1$ and $a \in A^*$ and put $q = at^r$. We have constructed the map

$$\tau_\infty : K_2(E_q) \to \mathcal{O}((t))^* \otimes \mathbb{Z}[1/r]$$

for the Tate curve $E_q = E_{q, \mathcal{O}((t))}$ over $\mathcal{O}((t))$.

Proposition 4.18 Suppose that $r$ is invertible in the quotient field of $\mathcal{O}$. Then the dlog map factors through $\tau_\infty$ as follows

$$\begin{array}{ccc}
\mathcal{O}((t))^* \otimes \mathbb{Z}[1/r] & \to & \mathcal{O}((t)) \frac{dt}{t} \otimes \mathbb{Z}[1/r] \\
\downarrow \tau_\infty & & \downarrow \iota \\
K_2(E_q) \otimes \mathbb{Z}[1/r] & \to & \mathcal{O}((t)) \frac{dt}{t} \otimes \mathbb{Z}[1/r]
\end{array}$$

where the map $\iota$ is given as follows

$$\iota : \mathcal{O}((t))^* \to \mathcal{O}((t)) \frac{dt}{t} \otimes \mathbb{Z}[1/r], \quad h \mapsto \frac{dh}{h} \frac{du}{u}.$$ 

Proof. We may replace $\mathcal{O}$ with its quotient field $F$. Then $\mathcal{K} = F((t))$ is a complete discrete valuation field with the valuation ring $\mathcal{R} = F[[t]]$. It is enough to show that the following diagram

$$\begin{array}{ccc}
F((t))^* & \to & F((t)) \frac{dt}{t} \otimes \mathbb{Z}[1/r] \\
\downarrow \langle , \rangle & & \downarrow \text{dlog} \\
K_2^M(\mathcal{K}(u)^\flat) & \to & F((t)) \frac{dt}{t} \otimes \mathbb{Z}[1/r]
\end{array}$$

is commutative where the right vertical arrow is given by

$$h \mapsto \frac{dh}{h} \frac{du}{u}.$$ 

Put

$$\hat{\Omega}_{K/F}^1 := \left( \varinjlim_{\nu} \Omega_{R/F}^1 / t^{\nu} \Omega_{R/F}^1 \right) \otimes_{\mathcal{R}} \mathcal{K}, \quad \hat{\Omega}_{\mathcal{K}(u)^\flat}/\mathcal{K} := \left( \varinjlim_{\nu} \Omega_{R(\mathcal{K}(u)^\flat)/R}^1 / t^{\nu} \Omega_{R(\mathcal{K}(u)^\flat)/R}^1 \right) \otimes_{\mathcal{R}} \mathcal{K}.$$ 

Note

$$\frac{dx}{2y + x} = \frac{du}{u} \quad \text{in} \quad \hat{\Omega}_{\mathcal{K}(u)^\flat}/\mathcal{K}.$$ 

Let

$$\text{Res} : \hat{\Omega}_{\mathcal{K}(u)^\flat}/\mathcal{K} \to \mathcal{K}$$
be the residue map at $u = 0$, namely if we express $\omega = \sum_{n \in \mathbb{Z}} a_n u^n du$ in the unique way then $\text{Res}(\omega) = a_{-1}$. It is extended to a map

$$\Omega^2_{K(u)/F} \longrightarrow \hat{\Omega}^1_{K/F} \otimes \hat{\Omega}^1_{K(u)/K} \xrightarrow{\text{id} \otimes \text{Res}} \hat{\Omega}^1_{K/F}.$$ (4.41)

Then the commutativity of (4.40) is equivalent to that the following diagram

$$\xymatrix{ K^M_2(K(u)) \ar[d]_{\text{dlog}} & K^* \ar[d]_{\text{dlog}} \\
\Omega^2_{K(u)/\mathbb{Q}_p} & \hat{\Omega}^1_{K/\mathbb{Q}_p}.}
$$ (4.42)

is commutative. However this follows from Lemma 4.6 and the fact that

$$(u - \alpha)^{-1} = \begin{cases} u^{-1} \sum_{n=0}^{\infty} (\alpha u^{-1})^n & \text{ord}_K(\alpha) > 0 \\
-\alpha^{-1} \sum_{n=0}^{\infty} (\alpha^{-1} u)^n & \text{ord}_K(\alpha) \leq 0
\end{cases}
$$

in $K$. This completes the proof of Proposition 4.18 Q.E.D.

### 4.4 Compatibility with étale regulator

Let $A$ be a complete regular local ring and $q \in A$ a nonzero and not invertible element. Let $n$ be a nonzero integer. Let $f_E : E_{q,A[q^{-1}, n^{-1}]} \to A[q^{-1}, n^{-1}]$ be the Tate curve. Then there is the weight exact sequence

$$0 \longrightarrow \mathbb{Z}/n(j) \longrightarrow R^1 f_* \mathbb{Z}/n(j) \longrightarrow \mathbb{Z}/n(j - 1) \longrightarrow 0$$ (4.43)

of etale sheaves on Spec$A[q^{-1}, n^{-1}]$ (cf. [6] VII 1.13). Recall the etale regulator map (also called the Chern class map)

$$c_{2,2}^{\text{ét}} : K_2(E_{q,A[q^{-1}, n^{-1}]}) \longrightarrow H^2_{\text{ét}}(E_{q,A[q^{-1}, n^{-1}]}, \mathbb{Z}/n(2))$$ (4.44)

to the étale cohomology group ([8, 22]). The regulator (4.44) together with the Leray spectral sequence gives rise to a map

$$\rho : K_2(E_{q,A[q^{-1}, n^{-1}]}) \longrightarrow H^1_{\text{ét}}(A[q^{-1}, n^{-1}], R^1 f_* \mathbb{Z}/n(2)).$$ (4.45)

(cf. [21] Lem.2.1). We have from (4.43) and (4.45)

$$\tau_{\infty} : K_2(E_{q,A[q^{-1}, n^{-1}]}) \longrightarrow H^1_{\text{ét}}(A[q^{-1}, n^{-1}], \mathbb{Z}/n(1)) \cong A[q^{-1}, n^{-1}]^*/n,$$ (4.46)

where “$\cong$” follows from the fact that $A[q^{-1}, n^{-1}]$ is a UFD.

**Proposition 4.19** Let $(A, q, \pi, r)$ be as in the beginning of §4.2 Suppose that $n$ is prime to $r$. Then the following diagram

$$\xymatrix{ K_2(E_{q,A[q^{-1}]}) \otimes \mathbb{Z}[1/r] \ar[d] & A[q^{-1}]^* \otimes \mathbb{Z}[1/r] \ar[d] \\
K_2(E_{q,A[q^{-1}, n^{-1}]})/n & A[q^{-1}, n^{-1}]^*/n \ar[l]_{\tau_{\infty}}}
$$

is commutative in the following cases.

(i). $A$ is a complete discrete valuation ring.

(ii). $A = R_0[[t]]$ where $R_0$ is an integer ring of a finite extension $K_0$ of $\mathbb{Q}_p$ and $q = at^r$ with $a \in A^*$.  

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4.4.1 Proof of (i)

Let \( A = \mathcal{R} \) be a complete discrete valuation ring. In this case we do not need \( \otimes \mathbb{Z}[1/r] \):

\[
\begin{align*}
K_2(E_{q,K}) & \xrightarrow{\tau_\infty} \mathcal{K}^* \\
\downarrow & \\
K_2(E_{q,K})/n & \xrightarrow{\tau^\text{et}} \mathcal{K}^*/n
\end{align*}
\]

(4.47)

where \( n \) is supposed to be invertible in \( \mathcal{K} \).

To prove the commutativity of the diagram (4.47) we first extend \( \tau^\text{et} \) to a pairing \( \langle , \rangle^\text{et} \). Let

\[
K_2^M(\mathcal{K}(u)) \to H_\text{et}^2(\mathcal{K}(u), \mathbb{Z}/n(2))
\]

be the Galois symbol map. Let \( (K_2^M(\mathcal{K}(u)))' \) be defined as

\[
0 \to (K_2^M(\mathcal{K}(u)))' \to K_2^M(\mathcal{K}(u)) \to H_\text{et}^2(\mathcal{K}(u) \otimes_K \overline{\mathcal{K}}, \mathbb{Z}/n(2)).
\]

Then the Leray spectral sequence yields

\[
\rho : (K_2^M(\mathcal{K}(u)))' \to H^1(\mathcal{K}, H_\text{et}^1(\mathcal{K}(u) \otimes_K \overline{\mathcal{K}}, \mathbb{Z}/n(2))).
\]

(4.49)

We define the natural map

\[
H_\text{et}^1(\mathcal{K}(u) \otimes_K \overline{\mathcal{K}}, \mathbb{Z}/n(j)) \cong (\mathcal{K}(u) \otimes_K \overline{\mathcal{K}})^* \to \mathbb{Z}/n(j - 1)
\]

as

\[
f \mapsto \text{Res} \frac{df}{f}
\]

where Res denotes the residue map at \( u = 0 \), namely if we express \( \omega = \sum_{n \in \mathbb{Z}} a_n u^n du \) in the unique way then \( \text{Res}(\omega) = a_{-1} \). The maps (4.49) and (4.50) give rise to a pairing

\[
\langle , \rangle^\text{et} : (K_2^M(\mathcal{K}(u)))' \to \mathcal{K}^*/n.
\]

(4.51)

(In \([1]\), the pairing \( \langle , \rangle^\text{et} \) is written as \( \hat{\tau}^\text{et}_\infty \).) By definition of \( \mathcal{K}(u)^\flat \) there is a natural map \( K_2^M(\mathcal{K}(u)^\flat) \to (K_2^M(\mathcal{K}(u)))' \). We also write the composition \( K_2^M(\mathcal{K}(u)^\flat) \to (K_2^M(\mathcal{K}(u)))' \to \mathcal{K}^*/n \) by \( \langle , \rangle^\text{et} \).

It is clear from the construction that \( \langle , \rangle^\text{et} \) is compatible with \( \tau^\text{et} \). On the other hand \( \langle , \rangle \) is compatible with \( \tau_\infty \). Therefore it is enough to show the following:

**Lemma 4.20** The following diagram

\[
\begin{align*}
K_2^M(\mathcal{K}(u)^\flat) & \xrightarrow{\langle , \rangle} \mathcal{K}^* \\
\downarrow & \\
K_2^M(\mathcal{K}(u)^\flat)/n & \xrightarrow{(\langle , \rangle)^\text{et}} \mathcal{K}^*/n
\end{align*}
\]

is commutative.
Proof. The pairing $\langle \ , \ \rangle$ is characterized by the following commutative diagram

$$
\begin{array}{ccc}
K_2^M(K(u)) & \stackrel{\sum \tau_\alpha}{\longrightarrow} & K^* \\
\downarrow & & \downarrow \\
K_2^M(K(u)^b) & \longrightarrow & K^*. \\
\end{array}
$$

The same commutative diagram holds for $\langle \ , \ \rangle^{\text{ét}}$ (\textit{H} Thm.4.4). Thus the assertion follows.

Q.E.D.

This completes the proof of Proposition 4.19 (i).

4.4.2 Proof of (ii)

We can reduce the proof of (ii) to the commutativity of (4.47). In fact, we may assume that $n = \ell^r$ with $\ell$ a prime number (possibly $\ell = p$). Let $\pi_0$ be a uniformizer of $R_0$ and $s_m : R_0[[t]] \to R_0$ the homomorphism given by $t \mapsto \pi^m$. Then the map

$$
\prod_m s_m : \lim_{\nu} A[q^{-1}]^*/\ell^{\nu} \longrightarrow \prod_m \lim_{\nu} K_0^*/\ell^{\nu}
$$

is injective where $m$ runs over all positive integers which are prime to $\ell$. Due to the functoriality of $\tau_\infty$ which follows from Proposition 4.16 and the functoriality of $\tau_\infty^{\text{ét}}$ which follows from the functoriality of etale regulator, we can reduce the proof to the case $A = R_0$.

This completes the proof of Proposition 4.19 (ii).

5 Proof of Theorem A

We are now in a position to prove Theorem A.

Let $K_0$, $R_0 = \bigoplus_{i=1}^d \mathbb{Z}_p \zeta_i$, $d = [K_0 : \mathbb{Q}_p]$ and $R_0((q_0))$ with $q = q_0^r$ be as in \textit{3.2}. Let $E_q = E_{q,R_0((q_0))}$ be the Tate curve and

$$
dlog : K_2(E_q) \otimes \mathbb{Z}_p \longrightarrow R_0((q_0))\frac{dq_0}{q_0} \frac{du}{u}
$$

(5.1)

the dlog map. Since $p$ is prime to $r$, it follows from Proposition \textit{4.18} that it factors through $\tau_\infty$:

$$
\begin{array}{ccc}
R_0((q_0))^* \otimes \mathbb{Z}_p & \stackrel{\tau_\infty}{\longrightarrow} & \iota \\
\downarrow & & \downarrow \\
K_2(E_q) \otimes \mathbb{Z}_p & \longrightarrow & R_0((q_0))\frac{dt}{t} \frac{du}{u} \\
\end{array}
$$

(5.2)

where

$$
\iota : R_0((q_0))^* \otimes \mathbb{Z}_p \longrightarrow R_0((q_0))\frac{dq_0 du}{q_0} \frac{du}{u}, \quad h \otimes a \longmapsto a\frac{dh}{h} \frac{du}{u}.
$$
Let $\xi \in K_2(E_q)$ be an arbitrary element. Express $h(q_0) := \tau_\infty(\xi)$ in the following way.

$$h(q_0) = c(-q_0)^m \prod_{k=1}^\infty \prod_{i=1}^d (1 - \zeta q_0^k)^{b_k^{(i)}}$$  \hspace{1cm} (5.3)

with $c \in R_0^*$, $m \in \mathbb{Z}$ and $b_k^{(i)} \in \mathbb{Z}_p$. Then

$$d\log(\xi) = q_0 \frac{d \log h(q_0)}{dq_0} \cdot \frac{dq_0}{q_0} \frac{du}{u}$$

$$= \left( m + \sum_{k=1}^\infty \sum_{i=1}^d (-k b_k^{(i)}) \frac{\zeta q_0^k}{1 - \zeta q_0^k} \right) \frac{dq_0}{q_0} \frac{du}{u}$$

due to the commutativity of (5.2). Therefore Theorem A is equivalent to say $b_k^{(i)} \in k\mathbb{Z}_p$ for all $k$ and $i$. We will show it in the following steps:

(a0) $c^r = 1$.

(a1) Let $R_0(q_0)$ be the $p$-adic completion of $R_0((q_0))$. Then

$$\{h(q_0), q_0\} = 0 \text{ in } K_2^M(R_0(q_0))/p^\nu, \quad \nu \geq 1.$$

(a2) Apply Kato's explicit reciprocity law for $K_2^M(R_0(q_0))/p^\nu$.

5.1 Proof of (a0)

Embed $R_0((q_0)) \hookrightarrow K_0((q_0))$. Then $\mathcal{K} = K_0((q_0))$ is a complete discrete valuation field with the valuation ring $\mathcal{R} = K_0[[q_0]]$. We constructed the map $\tau_\infty : K_2(E_q, \mathcal{K}) \to \mathcal{K}^*$ in §4.1.3 which is compatible with the étale regulator (see (4.47)). We thus have a commutative diagram

$$\begin{array}{ccc}
K_2(E_q, \mathcal{K}) & \xrightarrow{\tau_\infty} & \mathcal{K}^* \\
\rho \downarrow & & \downarrow \rho_{et} \\
H^1_{et}(\mathcal{K}, R^1 f_{E_q} \mathbb{Z}/n(2)) & \xrightarrow{s_q} & \mathcal{K}^*/n \xrightarrow{K^M(\mathcal{K})/n}
\end{array}$$  \hspace{1cm} (5.4)

for $n \geq 1$ where the bottom is the exact sequence arising from the weight exact sequence (4.43) and the isomorphisms $H^1_{et}(\mathcal{K}, \mathbb{Z}/n(1)) \cong \mathcal{K}^*/n$ and $H^2_{et}(\mathcal{K}, \mathbb{Z}/n(2)) \cong K_2^M(\mathcal{K})/n$ (4.35). Since the extension datum of (4.43) is $q$, $s_q$ is the map given by $x \mapsto \{x, q\}$. Therefore we have

$$\{h(q_0), q\} = r\{h(q_0), q_0\} = 0 \text{ in } K_2^M(\mathcal{K})/n.$$

Applying the tame symbol $K_2^M(\mathcal{K})/n \to K^*_0/n$ we have $c^r = 1$ in $K_0^*/n$. Since $n \geq 1$ is arbitrary, we have $c^r = 1$ in $K_0$. This completes the proof.
5.2 Proof of (a1)

Let $K_0(q_0) = R_0(q_0)[p^{-1}]$ be the quotient field. In the same way as (5.4) we have a commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
K_2(E_q) \otimes \mathbb{Z}_p \\
\rho^\infty
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
R_0((q_0))^* \otimes \mathbb{Z}_p \\
\tau^\infty
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
K_0(q_0)^*/p^\nu \\
\delta_q
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
K_2^M(K_0(q_0))/p^\nu \\
\end{array}
\end{array}
\end{array}
\quad (5.5)

and hence

$$
\{h(q_0), q\} = r\{h(q_0), q_0\} = 0 \quad \text{in } K_2^M(K_0(q_0))/p^\nu. \quad (5.6)
$$

Claim 5.1 The map $K_2^M(R_0(q_0))/p^\nu \rightarrow K_2^M(K_0(q_0))/p^\nu$ is injective.

Proof. By a theorem of van der Kallen $K_2^M(R_0(q_0)) = K_2(R_0(q_0))$ as the residue field is infinite (11, 12). Recall the localization exact sequence

$$
K_2(\mathbb{F}_q((q_0))) \rightarrow K_2(R_0(q_0)) \rightarrow K_2(K_0(q_0)) \rightarrow K_1(\mathbb{F}_q((q_0))) = \mathbb{F}_q((q_0))^*. \quad (5.7)
$$

The right arrow is surjective as $R_0(q_0)^* \rightarrow \mathbb{F}_q((q_0))^*$ is surjective. Separate (5.7) as follows:

- $K_2(\mathbb{F}_q((q_0))) \rightarrow K_2(R_0(q_0)) \rightarrow M \rightarrow 0,$
- $0 \rightarrow M \rightarrow K_2(K_0(q_0)) \rightarrow \mathbb{F}_q((q_0))^* \rightarrow 0.$

Since the multiplication by $p^\nu$ on $\mathbb{F}_q((q_0))^*$ is injective, we have $M/p^\nu \hookrightarrow K_2(K_0(q_0))/p^\nu.$ On the other hand, since $K_2(\mathbb{F}_q((q_0)))/p^\nu = K_2^M(\mathbb{F}_q((q_0)))/p^\nu = 0$ (13 Lemma 5.6), we have $K_2(R_0(q_0))/p^\nu \hookrightarrow M/p^\nu.$ Thus we get $K_2^M(R_0(q_0))/p^\nu = K_2(R_0(q_0))/p^\nu \hookrightarrow K_2(K_0(q_0))/p^\nu = K_2^M(K_0(q_0))/p^\nu$ as required. Q.E.D.

Now (a1) follows from (5.6) together with Claim 5.1.

5.3 Proof of (a2) : Kato’s explicit reciprocity law

Since we work on $R_0((q_0)) \otimes \mathbb{Z}_p$, we may neglect c due to (a0).

Let $\varphi : R_0(q_0) \rightarrow R_0(q_0)$ be the Frobenius such that $\varphi(a_\zeta^i q_0^j) = a_\zeta^i p^j q_0^j$ for $a \in \mathbb{Z}_p$ and $i, j \in \mathbb{Z}$. Let $l_\varphi(f) := p^{-1} \log(\varphi(f)/f^p)$. Then Kato’s explicit reciprocity law describes the syntomic regulator

$$
\theta_\varphi : K_2^M(R_0(q_0))/p^\nu \rightarrow \Omega^1_{R_0(q_0)/dR_0(q_0)} \otimes \mathbb{Z}/p^\nu, \quad \nu \geq 1
$$

for $p \geq 3$ explicitly (14 Cor. 2.9) as follows:

$$
\theta_\varphi(a, b) = l_\varphi(a) \frac{1}{p} \varphi \left( \frac{db}{b} \right) - l_\varphi(b) \frac{da}{a}.
$$

Due to (a1), we have

$$
\theta_\varphi(h, q_0) = l_\varphi(h) \frac{dq_0}{q_0} = 0, \quad (5.8)
$$

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and hence
\[ l_{\varphi}(h) = q_0 \frac{dg}{dq_0}, \quad \exists g = \sum_{k} c_k q_0^k \in R_0(q_0) \] (5.9)
in \( R_0(q_0)/p\nu = R_0((q_0))/p\nu \) for all \( \nu \geq 1 \). We have from (5.3) and (5.9) that
\[
\begin{align*}
l_{\varphi}(h) &= \sum_{k=1}^{\infty} \sum_{i=1}^{d} \sum_{m \geq 1, (m,p)=1} b_k^{(i)}(\zeta q_0^k)^m \frac{m}{m} \\
&= \sum_{k} kc_k q_0^k
\end{align*}
\]
in \( R_0(q_0)/p\nu = R_0((q_0))/p\nu \) for all \( \nu \geq 1 \). One can easily show \( b_k^{(i)} \in k\mathbb{Z}_p \) for all \( k \) and \( i \) by the induction on \( k \). This completes the proof.

6 Theorem B and \( \Phi(X_R, D_R)_{\mathbb{F}_p} \)

We use the notations in \( \S 2.3 \).

**Theorem B.** Let \( F \) be a field of characteristic zero and and \( \pi_F : X_F \to C_F \) an elliptic surface over \( F \) satisfying \( (\text{Rat}) \). Let \( p \) be a prime number. Put \( X_{\overline{F}} = X_F \times_F \overline{F} \) etc.

1. Assume that for any finite \( p\)-torsion \( G_F \)-module \( M \), \( H^i_{\text{ét}}(F,M) \) is finite for all \( i \geq 0 \) where \( G_F \) denotes the absolute Galois group of \( F \) (e.g. \( F \) is a local field). Then we have

\[
\partial \Gamma(U^0_F, \mathcal{K}_2) \otimes \mathbb{Z}_p = \partial \Gamma(U^0_{\overline{F}}, \mathcal{K}_2) \otimes \mathbb{Z}_p \subset \mathbb{Z}_p^{\oplus s}
\]
if \( p \) is prime to \( 6r_1 \cdots r_s \) and \( H^3_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}_p) \) is torsion free.

2. The quotient

\[
\mathbb{Z}_p^{\oplus s}/(\partial \Gamma(U^0_{\overline{F}}, \mathcal{K}_2) \otimes \mathbb{Z}_p)
\]
is torsion free if \( p \) is prime to \( 6r_1 \cdots r_s \), \( H^3_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}_p) \) is torsion free and

\[
H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p(2))^G = 0.
\]

Note that there is the natural isomorphism \( H^3_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}_p) \cong H^3_B(X_C, \mathbb{Z}) \otimes \mathbb{Z}_p \) for a fixed embedding \( \overline{F} \hookrightarrow \overline{C} \) due to a theorem of M. Artin and the universal coefficient theorem. Therefore \( H^3_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}_p) \) is torsion free for almost all \( p \).

The proof of Theorem B will be given in \( \S 7 \) where Suslin’s exact sequence (Theorem 2.2) plays an essential role (for example Claim 7.9 below is one of the key step).

6.1 \( \Phi(X_R, D_R)_{\mathbb{F}_p} \): an upper bound of the rank of the dlog image

Admitting Theorems A and B, we give an upper bound of the rank of the dlog image of \( K_2 \) of elliptic surface minus singular fibers over an algebraically closed field of characteristic zero.

Let \( R \) be the ring of integer in an unramified extension \( K \) of \( \mathbb{Q}_p \), and \( \pi_R : X_R \to C_R \) an elliptic surface over \( R \) satisfying \( (\text{Rat}) \).
Definition 6.1 Let \( f : \mathbb{Z}_p^{\oplus s} \to \mathbb{F}_p^{\oplus s} \) be the reduction modulo \( p \). Then we put
\[
\Phi(X_R, D_R)_{\mathbb{F}_p} := f(\partial_{\text{dR}}(\Phi(X_R, D_R)_{\mathbb{Z}_p})) \subset \mathbb{F}_p^{\oplus s}.
\]

Theorem 6.2 Put \( X_K := X_R \times_R K \), \( X_K := X_R \times_R \overline{K} \) etc. Assume the following conditions (1) and (2).

(1) \( p \not| 6r_1 \cdots r_s \) and \( H^3_{\text{et}}(X_K, \mathbb{Z}_p) \) is torsion free.

(2) There is a finitely generated subfield \( F \subset K \) such that \( \pi_{K} : X_K \to C_K \) is defined over \( F \), which we write \( \pi_{F} : X_F \to C_F \), and it satisfies \((\text{Rat})\) and \( H^2_{\text{et}}(X_F, \mathbb{Z}/p(2))^{G_F} = 0 \).

Then we have
\[
d\log \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p \subset \Phi(X_R, D_R)_{\mathbb{Z}_p}, \tag{6.1}
\]
\[
\text{rank}_{\mathbb{Z}_p} \text{dlog} \Gamma(U_0^R, K_2) \leq \text{dim}_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p}. \tag{6.2}
\]

Proof. (6.1) follows from Theorem 3.6 (3.23) and Theorem B (1). We show (6.2). Applying \( \partial_{\text{dR}} \) on (6.1) we have
\[
\partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p \subset \partial_{\text{dR}}(\Phi(X_R, D_R)_{\mathbb{Z}_p}). \tag{6.3}
\]
Applying \( f \) to (6.3), we have
\[
\text{dim}_{\mathbb{Z}_p} f(\partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p) \leq \text{dim}_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p}. \tag{6.4}
\]
Due to Theorem B (2), \( \partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p \) is a direct summand of \( \mathbb{Z}_p^{\oplus s} \). Therefore
\[
\text{rank}_{\mathbb{Z}_p} \partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p = \text{dim}_{\mathbb{F}_p} f(\partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p). \tag{6.5}
\]
Then (6.2) follows from (6.4) and (6.5). Q.E.D.

\( \text{dim}_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p} \) is the desired bound. It is computable in many cases. See §9.

Conjecture 6.3 Under the assumptions in Theorem 6.2,
\[
\text{rank}_{\mathbb{Z}_p} \text{dlog} \Gamma(U_0^R, K_2) = \text{dim}_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p}.
\]

We show that Conjecture 3.7 implies the above. In fact, Conjecture 3.7 implies \( \partial \Gamma(U_0^R, K_2) \otimes \mathbb{Q}_p = \partial_{\text{dR}} \Phi(X_R, D_R)_{\mathbb{Z}_p} \otimes \mathbb{Q}_p \) and hence \( \partial_{\text{dR}} \Phi(X_R, D_R)_{\mathbb{Z}_p} / \partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p \) is finite. However due to Theorem B (2), it must be torsion free, which means \( \partial \Gamma(U_0^R, K_2) \otimes \mathbb{Z}_p = \partial_{\text{dR}} \Phi(X_R, D_R)_{\mathbb{Z}_p} \). Due to Theorem B (2) we have Conjecture 6.3.
7 Proof of Theorem B

7.1 Some Computations of cohomology groups

Let $\pi : V \to C$ be a minimal elliptic surface over $\mathbb{C}$ with a section $e : C \to V$. We omit to write the subscript “$\mathbb{C}$” for the simplicity of the notations. Let $T = \{P_i\} \subset C$ be the set of points such that $\pi^{-1}(P)$ is a singular fiber if and only if $P \in T$. Put $Y_i := \pi^{-1}(P_i), Y = \sum Y_i, S^0 = C - T$ and $U^0 = V - Y$. We assume that there is at least one multiplicative fiber. $H^\bullet(V)$ (resp. $H_\bullet(V)$) denotes the Betti (singular) cohomology group (resp. homology group).

Let $n \geq 1$ be an integer. There is the long exact sequence

$$\cdots \to H^2_\ast(V, \mathbb{Z}/n(\langle \rangle)) \to H^2(V, \mathbb{Z}/n(j)) \to H^2(U^0, \mathbb{Z}/n(\langle \rangle)) \to \cdots. \tag{7.1}$$

Let $Y_i = \bigcup_k Y_i^{(k)}$ be the irreducible decomposition. There is the isomorphism $H^2_\ast(V, \mathbb{Z}/n(1)) \cong \bigoplus_{i,k} \mathbb{Z}/n \cdot y_i^{(k)}$ in which a base $y_i^{(k)}$ corresponds to the component $Y_i^{(k)}$. Denote by $[-]$ the cycle class in the cohomology group of $V$. Then the map $H^2_\ast(V, \mathbb{Z}/n(1)) \to H^2(V, \mathbb{Z}/n(1))$ is given by $y_i^{(k)} \mapsto [Y_i^{(k)}]$. Let $H^2_\ast(V, \mathbb{Z}/n(1)) \to \bigoplus_{i,k} H^2(Y_i^{(k)}, \mathbb{Z}/n(1))$ and $H^2_\ast(V, \mathbb{Z}/n(1)) \to H^2(e(C), \mathbb{Z}/n(1))$ be the pull-back maps. Then the composition

$$\bigoplus_{i,k} \mathbb{Z}/n \cdot y_i^{(k)} = H^2_\ast(V, \mathbb{Z}/n(1)) \longrightarrow H^2(V, \mathbb{Z}/n(1)) \longrightarrow \bigoplus_{i,k} H^2(Y_i^{(k)}, \mathbb{Z}/n(1)) \oplus H^2(e(C), \mathbb{Z}/n(1)) = \bigoplus_{i,k} \mathbb{Z}/n \cdot y_i^{(k)} \oplus \mathbb{Z}/n \cdot e \tag{7.2}$$

is given as follows

$$y_i^{(k)} \mapsto \sum_{i',k'} (Y_i^{(k)}, Y_{i'}^{(k')}) \cdot y_{i'}^{(k')} + (Y_i^{(k)}, e(C)) \cdot e \tag{7.3}$$

where $(-, -)$ denotes the intersection numbers.

Lemma 7.1 Let $Y_i = \pi^{-1}(P_i) = \sum_{k \geq 1} m_{i,k}^{(k)} \cdot Y_i^{(k)}$ be the scheme theoretic fiber. We put

$$[Y_i] := \sum_{k} m_{i,k}^{(k)} \cdot y_i^{(k)}.$$ 

Suppose that $n$ is prime to $6r_1 \cdots r_s$. Then the composition induces an isomorphism

$$\bigoplus_{i,k \geq 1} \mathbb{Z}/n \cdot y_i^{(k)}/\langle [Y_i] - [Y_j]; i < j \rangle \xrightarrow{n} \bigoplus_{i \geq 1, k \geq 2} \mathbb{Z}/n \cdot y_i^{(k)} \oplus \mathbb{Z}/n \cdot e$$

where $\langle [Y_i] - [Y_j]; i < j \rangle$ is the subgroup generated by $[Y_i] - [Y_j]$ for all $i < j$. \hfill 37
Proof. Since the cardinality of the both sides is the same, it is enough to show the surjectivity. To do this, it is enough to see that the map

$$\bigoplus_{k \geq 1} \mathbb{Z}/n \cdot y_i^{(k)} \longrightarrow \bigoplus_{k \geq 2} \mathbb{Z}/n \cdot y_i^{(k)} \oplus \mathbb{Z}/n \cdot e$$  (7.4)

is surjective (and hence bijective) for each $i$. We can check it as the case may be. For example, if $Y_i$ is of type $I_{r_i}$, then the matrix of (7.4) is given by

$$
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & -2 & \\
& \ddots & \ddots \\
& & -2 & 1 & 0 \\
& & 1 & -2 & 0 \\
& & 1 & 0 & 1 \\
& & 0 & 1 & 1
\end{pmatrix}
$$

The determinant of it is equal to $(-1)^{r_i-1} r_i$. It is prime to $n$. Therefore (7.4) is bijective. The proofs for the other types are similar. Q.E.D.

Lemma 7.2 Let $V_t = \pi^{-1}(t)$ be a smooth fiber. Then $H_1(V_t, \mathbb{Q}) \to H_1(V, \mathbb{Q})$ is zero.

Proof. Recall the assumption that there is a multiplicative fiber, say $Y_1$. Let $\sigma$ be the local monodromy around $P_1$. Then the action of $\sigma$ on $H^1(V_t, \mathbb{Z})$ is given by the matrix

$$
\begin{pmatrix}
1 & r \\
0 & 1
\end{pmatrix}, \quad r \geq 1.
$$

Therefore the image of $H^1(V, \mathbb{Q}) \to H^1(V_t, \mathbb{Q})$ has dimension $\leq 1$. On the other hand it constitutes a sub Hodge structure of $H^1(V_t, \mathbb{Q})$ (7 II). Thus it is zero. Q.E.D.

Proposition 7.3 Suppose that $n$ is prime to $6r_1 \cdots r_s$. Then the following are equivalent.

1. $n$ is prime to $\sharp H_1(V, \mathbb{Z})_{\text{tor}}$.
2. $H^1(V_t, \mathbb{Z}/n)^{\pi_1(S^0, t)} = 0$.
3. $H^1(S^0, \mathbb{Z}/n) \cong H^1(U^0, \mathbb{Z}/n)$
4. $H^1(C, \mathbb{Z}/n) \cong H^1(V, \mathbb{Z}/n)$
5. $H_1(V_t, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n)$ is zero.

Proof. Note that $H_1(V, \mathbb{Z}/n) = H_1(V, \mathbb{Z})/n$ for any CW complex $V$. By the universal coefficient theorem and the fact that $H_1(S^0, \mathbb{Z})$ and $H_1(C, \mathbb{Z})$ are torsion free, we have

$$3 \iff H_1(S^0, \mathbb{Z}/n) \cong H_1(U^0, \mathbb{Z}/n)$$

$$11 \iff H_1(C, \mathbb{Z}/n) \cong H_1(V, \mathbb{Z}/n).$$
Let \( E_{pq}^n = H^p(S^0, R^q \pi_* Z/n) \Rightarrow H^p + q(U^0, Z/n) \) be the Leray spectral sequence. Due to the section \( e \), the maps \( E_{20}^p \rightarrow E^p \) are injective. Therefore we have an exact sequence

\[
0 \rightarrow H^1(S^0, Z/n) \rightarrow H^1(U^0, Z/n) \rightarrow H^1(V_t, Z/n)_{\pi_1(S^0, t)} \rightarrow 0
\]

and hence the desired equivalence.

In the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(S^0, Z/n) & \rightarrow & H^1(U^0, Z/n) & \rightarrow & H^1(V_t, Z/n)_{\pi_1(S^0, t)} & \rightarrow & 0 \\
| & f_1 & | & f_2 & | & f_3 & | \\
0 & \rightarrow & H^1(C, Z/n) & \rightarrow & H^1(V, Z/n) & \rightarrow & \text{Coker} & \rightarrow & 0
\end{array}
\]

the rows are split exact sequences. We want to show that \( f_3 \) is bijective. To do this, it is enough to show that \( \text{Coker } f_1 \rightarrow \text{Coker } f_2 \) is surjective as \( f_1 \) and \( f_2 \) are injective. Let the notations be as in Lemma 7.1. There are the isomorphisms

\[
\text{Coker } f_1 \cong \ker \left( \bigoplus_i \frac{Z}{n} \cdot p_i \rightarrow H^2(C, Z/n) \right),
\]

\[
\text{Coker } f_2 \cong \ker \left( \bigoplus_{i,k} \frac{Z}{n} \cdot y_i^{(k)} \rightarrow H^2(V, Z/n) \right)
\]

where \( p_i \) corresponds to \( P_i \). The map \( \text{Coker } f_1 \rightarrow \text{Coker } f_2 \) is given by

\[
p_i \mapsto [Y_i] := \sum_k m_i^{(k)} \cdot y_i^{(k)}.
\]

By Lemma 7.1 \( \text{Coker } f_2 \) is generated by \([Y_i] - [Y_j] \) (\( i < j \)). Therefore \( \text{Coker } f_1 \rightarrow \text{Coker } f_2 \) is surjective.

Due to Lemma 7.2 we have a split exact sequence

\[
0 \rightarrow H_1(V, Z)_{\text{tor}} \rightarrow H_1(V, Z) \rightarrow H_1(C, Z) \rightarrow 0.
\]

By the universal coefficients theorem we have

\[
0 \rightarrow H^1(C, Z/n) \rightarrow H^1(V, Z/n) \rightarrow \text{Hom}(H_1(V, Z)_{\text{tor}}, Z/n) \rightarrow 0
\]

and hence the desired equivalence.

It follows from Lemma 7.2 that we have

\[
H_1(V, Z/n) = H_1(V, Z)/n = (H_1(V, Z)/H_1(V, Z)_{\text{tor}}) \otimes Z/n.
\]

Therefore it is enough to show that the map

\[
H_1(V_t, Z) \rightarrow H_1(V, Z)/H_1(V, Z)_{\text{tor}} \rightarrow H_1(V, Q)
\]

is zero. This follows from Lemma 7.2.
We want to show that the map \( e_s : H_1(C, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \) is surjective. To do this, it is enough to show that \( H_1(S^0, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \) is surjective. There is the split exact sequence

\[
1 \longrightarrow \pi_1(V) \longrightarrow \pi_1(U^0) \longrightarrow \pi_1(S^0) \longrightarrow 1.
\]

In particular we have that \( \pi_1(V) \oplus \pi_1(S^0) \to \pi_1(U^0) \) is surjective and hence so is \( H_1(V, \mathbb{Z}/n) \oplus H_1(S^0, \mathbb{Z}/n) \to H_1(U^0, \mathbb{Z}/n) \). On the other hand since \( H_1(U^0, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \) is surjective, we have the surjectivity of \( H_1(V, \mathbb{Z}/n) \oplus H_1(S^0, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \).

However by \( \ref{eq:7.5} \), \( H_1(V, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \) is zero. Thus we have the surjectivity of \( H_1(S^0, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \). Q.E.D.

**Lemma 7.4** If there is a singular fiber which is not a multiplicative one, then the equivalent conditions in Proposition \( \ref{prop:7.3} \) are satisfied for all \( n \) such that \( (n, 6r_1 \cdots r_s) = 1 \).

**Proof.** We see that \( \ref{eq:7.4} \) is satisfied. Let \( Y_j \) be the singular fiber which is not multiplicative. Then it is simply connected. Since \( H_1(V, \mathbb{Z}/n) \to H_1(V, \mathbb{Z}/n) \) factors through \( H_1(Y_j, \mathbb{Z}/n) \) it is zero. Q.E.D.

**Lemma 7.5** Assume that \( n \) is prime to \( 6r_1 \cdots r_s \) and satisfies the equivalent conditions in Proposition \( \ref{prop:7.3} \). Then \( n \) is prime to the cardinality of the torsion part of \( H^2(V, \mathbb{Z}) \).

**Proof.** This follows from a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(V, \mathbb{Z})/n & \longrightarrow & H^1(V, \mathbb{Z}/n) & \longrightarrow & H^2(V, \mathbb{Z})[n] & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^1(C, \mathbb{Z})/n & \xrightarrow{\cong} & H^1(C, \mathbb{Z}/n)
\end{array}
\]

and \( \ref{eq:7.1} \). Q.E.D.

We put

\[
C_{\mathbb{Z}/n}(j) := \text{Coker}(H_2^V(V, \mathbb{Z}/n(j)) \longrightarrow H^2(V, \mathbb{Z}/n(j))),
\]

\[
I_{\mathbb{Z}/n}(j) := \text{Image}(H_2^V(V, \mathbb{Z}/n(j)) \longrightarrow H^2(V, \mathbb{Z}/n(j))).
\]

From \( \ref{eq:7.1} \) we have exact sequences

\[
0 \longrightarrow I_{\mathbb{Z}/n}(j) \longrightarrow H^2(V, \mathbb{Z}/n(j)) \longrightarrow C_{\mathbb{Z}/n}(j) \longrightarrow 0,
\]

(7.5)

\[
0 \longrightarrow C_{\mathbb{Z}/n}(j) \longrightarrow H^2(U^0, \mathbb{Z}/n(j)) \xrightarrow{\delta} H^3_Y(V, \mathbb{Z}/n(j)) \longrightarrow H^3(V, \mathbb{Z}/n(j)).
\]

(7.6)

Note that \( \delta \) in \( \ref{eq:7.6} \) gives rise to the boundary map (cf. \( \ref{eq:2.3.4} \))

\[
\partial_B : H^2(U^0, \mathbb{Z}/n(j)) \longrightarrow H^3_Y(V, \mathbb{Z}/n(j)) \xrightarrow{\cong} \mathbb{Z}/n(j - 2)^{\oplus s}.
\]

(7.7)
Proposition 7.6 Assume that \( n \) is prime to \( 6r_1 \cdots r_s \) and satisfies the equivalent conditions in Proposition 7.3.

1. The exact sequence (7.6) has a splitting induced from (7.3). In particular \( C_{\mathbb{Z}/n}(j) \) is a direct summand of \( H^2(V, \mathbb{Z}/n(j)) \). Moreover we have \( I_{\mathbb{Z}/n}(j) \cong \mathbb{Z}/n(j-1)^\oplus \).

2. The boundary map \( \partial_B \) is surjective. Hence we have an exact sequence

\[
0 \longrightarrow C_{\mathbb{Z}/n}(j) \longrightarrow H^2(U^0, \mathbb{Z}/n(j)) \overset{\partial_B}{\longrightarrow} \mathbb{Z}/n(j - 2)^\oplus \longrightarrow 0.
\]  
(7.8)

3. \( H^2(V, \mathbb{Z}/n) \cong (H^2(V, \mathbb{Z})/H^2(V, \mathbb{Z})_{\text{tor}}) \otimes \mathbb{Z}/n. \)

Proof. (1). This follows from Lemma 7.1.

(2). It is enough to show that the map \( H^3(V, \mathbb{Z}/n(j)) \rightarrow H^3(V, \mathbb{Z}/n(j)) \) in (7.6) is zero. Due to the Poincare-Lefschetz duality we have

\[
\begin{array}{ccc}
H^3(V, \mathbb{Z}/n(j)) & \xrightarrow{\cong} & H^3(V, \mathbb{Z}/n(j)) \\
\oplus_{i=1} H_1(Y, \mathbb{Z}/n(j-2)) & \xrightarrow{\cong} & H_1(V, \mathbb{Z}/n(j-2)).
\end{array}
\]

We see that \( H_1(Y, \mathbb{Z}/n) \rightarrow H_1(V, \mathbb{Z}/n) \) is zero for each \( i \). Since there is a surjective map \( H_1(V_i, \mathbb{Z}/n) \rightarrow H_1(Y_i, \mathbb{Z}/n) \), it is enough to see that \( H_1(V_i, \mathbb{Z}/n) \rightarrow H_1(V, \mathbb{Z}/n) \) is zero. However this follows from Proposition 7.3 (3).

(3). This follows from Proposition 7.3 (1), Lemma 7.5 and the exact sequence

\[
\begin{array}{cccc}
H^2(V, \mathbb{Z}) & \overset{n}{\longrightarrow} & H^2(V, \mathbb{Z}) & \longrightarrow H^2(V, \mathbb{Z}/n) & \longrightarrow H^3(V, \mathbb{Z})[n] & \longrightarrow 0 \\
\downarrow & & & & & \\
H_1(V, \mathbb{Z})[n].
\end{array}
\]

Q.E.D.

7.2 Proof of Theorem B

Due to Lemma 7.4 the assertions in Theorem B do not depend on the choice of \( U^0_F \). Thus we may assume that \( U^0_F = X_F \) (all singular fibers). Let \( \pi'_F : V_F \rightarrow C_F \) be the minimal elliptic surface associated to \( X_F \). Then \( H^2_{\text{et}}(X_{\mathcal{F}}, \mathbb{Z}/n) \cong H^2_{\text{et}}(V_{\mathcal{F}}, \mathbb{Z}/n) \oplus \mathbb{Z}/n^{\oplus} \) and \( H^1_{\text{et}}(X_{\mathcal{F}}, \mathbb{Z}/n) \cong H^1_{\text{et}}(V_{\mathcal{F}}, \mathbb{Z}/n) \) for \( i \neq 2 \). Therefore if we replace \( X_F \) with \( V_F \), the assumptions in Theorem B hold. Thus we may replace \( X_F \) with \( V_F \). We put

\[
C^\text{et}_{\mathbb{Z}/p'}(j) := \text{Coker}(H^2_{\text{et},Y}(V_{\mathcal{F}}, \mathbb{Z}/p')(j)) \longrightarrow H^2_{\text{et}}(V_{\mathcal{F}}, \mathbb{Z}/p'(j)),
\]

\[
I^\text{et}_{\mathbb{Z}/p'}(j) := \text{Image}(H^2_{\text{et},Y}(V_{\mathcal{F}}, \mathbb{Z}/p'(j)) \longrightarrow H^2_{\text{et}}(V_{\mathcal{F}}, \mathbb{Z}/p'(j)),
\]

and \( C^\text{et} := \lim C^\text{et}_{\mathbb{Z}/p'}, I^\text{et}_{\mathbb{Z}/p'} := \lim I^\text{et}_{\mathbb{Z}/p'} \). Thanks to a theorem of M. Artin, there are the natural isomorphisms

\[
H^\bullet_{\text{et}}(W_{\mathcal{F}}, \mathbb{Z}/n(j)) \cong H^\bullet_B(W_{\mathcal{C}}, \mathbb{Z}/n(j))
\]
for any separated scheme $W$ of finite type over $\bar{\mathbb{F}}$ with an embedding $\bar{\mathbb{F}} \hookrightarrow \mathbb{C}$. Note that $H^\ast_{\text{et}}(W_{\mathbb{F}}, \mathbb{Z}_p)$ is torsion free if and only if $p$ is prime to the cardinality of $H^\ast_{\text{et}}(W_{\mathbb{C}}, \mathbb{Z})_{\text{tor}}$. Therefore we can apply Proposition \ref{prop_10} for the étale cohomology groups of $V_{\mathbb{F}}$. Thus we have the exact sequence

$$0 \rightarrow C^\ast_{\mathbb{Z}_p}(j) \rightarrow H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}_p(j)) \xrightarrow{\partial} \mathbb{Z}_p(j - 2)^{\otimes s} \rightarrow 0 \quad (7.9)$$

of $G_F$-modules from Proposition \ref{prop_10} (2). $C^\ast_{\mathbb{Z}_p}(j)$ is a direct summand of $H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}_p(j))$ by Proposition \ref{prop_10} (1). Moreover $H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}_p(j))$ is torsion free and there is an exact sequence

$$0 \rightarrow H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}_p(j)) \xrightarrow{\nu} H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}_p(j)) \rightarrow H^2_{\text{et}}(V_{\mathbb{F}}, \mathbb{Z}/p^\nu(j)) \rightarrow 0 \quad (7.10)$$

by Proposition \ref{prop_10} (3).

### 7.2.1 Proof of Theorem B (1)

For a scheme $U'$ with $U'_0 \rightarrow U'$ we put

$$\Gamma(U')_{\text{et}} := \text{Image}(\Gamma(U', \mathcal{K}_2) \otimes \mathbb{Z}_p \rightarrow H^2_{\text{et}}(U'_0, \mathbb{Z}_p(2))).$$

Then we show

$$\Gamma(U'_0)_{\text{et}} = \Gamma(U'_0)_{\text{et}}. \quad (7.11)$$

**Claim 7.7** $\Gamma(U'_0)_{\text{et}} \otimes \mathbb{Q}_p = \Gamma(U'_0)_{\text{et}} \otimes \mathbb{Q}_p$.

Note that $\Gamma(U'_0)_{\text{et}} \cap C^\ast_{\mathbb{Z}_p}(2)$ is torsion so that we have $\Gamma(U'_0)_{\text{et}} \otimes \mathbb{Q}_p \cong \partial_{\text{et}}(\Gamma(U'_0)_{\text{et}}) \otimes \mathbb{Q}_p = \partial \Gamma(U'_0, \mathcal{K}_2) \otimes \mathbb{Q}_p$ for $F' = F$ or $\bar{\mathbb{F}}$. Therefore the assertion follows from Lemma \ref{lem_24} (2).

**Claim 7.8** $\Gamma(U'_0)_{\text{et}} \subset H^2_{\text{et}}(U'_0, \mathbb{Z}_p(2))^{G_F}$.

In fact, it follows from Claim \ref{claim_7.7} that we have $\Gamma(U'_0)_{\text{et}} \otimes \mathbb{Q}_p = \Gamma(U'_0)_{\text{et}} \otimes \mathbb{Q}_p \subset H^2_{\text{et}}(U'_0, \mathbb{Q}_p(2))^{G_F}$. Then the assertion follows from the fact that $H^2_{\text{et}}(U'_0, \mathbb{Z}_p(2))$ is torsion free.

**Claim 7.9** $H^2_{\text{et}}(U'_0, \mathbb{Z}_p(2))/\Gamma(U'_0)_{\text{et}}$ is torsion free.

Put $M := H^2_{\text{et}}(U'_0, \mathbb{Z}_p(2))/\Gamma(U'_0)_{\text{et}}$. Due to the finiteness of the Galois cohomology groups of $F$, $M$ is finitely generated over $\mathbb{Z}_p$. Therefore it is enough to show that

$$M/p^\nu \rightarrow M/p^{\nu+1}, \quad x \mapsto p \cdot x$$

is injective. Due to Suslin’s exact sequence (Theorem \ref{thm_22}) we have an injective map $M/p^\nu \hookrightarrow H^1_{\text{Zar}}(U'_0, \mathcal{K}_2)[p^\nu]$ and a commutative diagram

$$\begin{array}{ccc}
M/p^\nu & \longrightarrow & H^1_{\text{Zar}}(U'_0, \mathcal{K}_2)[p^\nu] \\
p \downarrow & & \downarrow \cap \\
M/p^{\nu+1} & \longrightarrow & H^1_{\text{Zar}}(U'_0, \mathcal{K}_2)[p^{\nu+1}].
\end{array}$$

Thus Claim \ref{claim_7.9} follows.
Claim 7.10 There is an exact sequence

\[ 0 \longrightarrow H^2_{\text{ét}}(S^0_F, \mathbb{Z}_p(2)) \longrightarrow H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2)) \longrightarrow H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2))^G_F \longrightarrow 0. \]

This is split by the section \( e \).

Consider the Hochschild-Serre spectral sequences

\[ E^{ij}_{2, U^0} = H^i(F, H^j_{\text{ét}}(U^0_F, \mathbb{Z}/p^j(2))) \Rightarrow H^{i+j}_{\text{ét}}(U^0_F, \mathbb{Z}/p^j(2)), \]

\[ E^{ij}_{2, S^0} = H^i(F, H^j_{\text{ét}}(S^0_F, \mathbb{Z}/p^j(2))) \Rightarrow H^{i+j}_{\text{ét}}(S^0_F, \mathbb{Z}/p^j(2)). \]

By Proposition 7.3 (3), \( H^1_{\text{ét}}(S^0_F, \mathbb{Z}/p^j(2)) \cong H^1_{\text{ét}}(U^0_F, \mathbb{Z}/p^j(2)) \). Therefore we have \( E^{ij}_{2, S^0} \cong E^{ij}_{2, U^0} \) for \( j = 0, 1 \). Thus we have \( E^{11}_{2, S^0} \cong E^{11}_{2, U^0} \) and \( E^{20}_{2, S^0} \cong E^{20}_{2, U^0} \), and hence an exact sequence

\[ 0 \longrightarrow H^2_{\text{ét}}(S^0_F, \mathbb{Z}/p^j(2)) \longrightarrow H^2_{\text{ét}}(U^0_F, \mathbb{Z}/p^j(2)) \overset{v}{\longrightarrow} H^2_{\text{ét}}(U^0_F, \mathbb{Z}/p^j(2))^G_F. \]

The rest of the proof is to show that the right arrow \( v \) is surjective, which is equivalent to \( E^{20}_{2, U^0} = E^{20}_{2, U^0} \). To do this it is enough to show that the map \( E^{20}_{2, U^0} \rightarrow E^{21}_{2, U^0} \) and \( E^{20}_{3, U^0} \rightarrow E^{30}_{3, U^0} \) are zero. The latter follows from the injectivity of \( E^{20}_{2, U^0} \rightarrow E^{20}_{2, U^0} \). We show the former. There is a commutative diagram

\[
\begin{array}{cccccc}
E^{20}_{2, U^0} & \xrightarrow{a} & E^{21}_{2, U^0} & \xrightarrow{b} & E^{30}_{2, U^0} & \\
\uparrow & & \uparrow \cong & & \uparrow U & \\
\gamma_{2, S^0} & \xrightarrow{c} & E^{21}_{2, S^0} & \xrightarrow{d} & E^{30}_{2, S^0} & \\
\end{array}
\]

with exact rows. Therefore \( a = 0 \iff b \) is injective \( \iff d \) is injective \( \iff c = 0 \). However this is clear as \( E^{20}_{2, S^0} = 0 \).

Now we show (7.11). By Claim 7.10 we have a commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & H^2_{\text{ét}}(S^0_F, \mathbb{Z}_p(2)) & \longrightarrow & H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2)) & \longrightarrow & H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2))^G_F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Gamma(S^0_F, \mathcal{K}_2) \otimes \mathbb{Z}_p & \xrightarrow{\pi_F^*} & \Gamma(U^0_F, \mathcal{K}_2) \otimes \mathbb{Z}_p & \longrightarrow & \text{Coker } \pi_F^* & \longrightarrow & 0
\end{array}
\]

with split exact rows by the section \( e_F \). Together with Claim 7.9 we have that

\[ H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2))^G_F / \Gamma(U^0_F, \mathbb{Z}_p) \cong (7.12) \]

is torsion free. Therefore, in the following commutative diagram

\[
\begin{array}{ccc}
\Gamma(U^0_F, \mathbb{Z}_p) & \xrightarrow{c} & H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2))^G_F \\
\downarrow a & & \downarrow & \\
\Gamma(U^0_F, \mathbb{Z}_p) & \xrightarrow{c} & H^2_{\text{ét}}(U^0_F, \mathbb{Z}_p(2))^G_F,
\end{array}
\]

Coker \( a \) is torsion free. On the other hand Coker \( a \) is torsion by Claim 7.11. This means that \( a \) is bijective. This completes the proof of (7.11) and hence Theorem B (1).
7.2.2 Proof of Theorem B (2)

In the same way as Claim 7.8 we have

\[ \text{Image}(\Gamma(U^0_{/T}, K_2) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(U^0_{/T}, \mathbb{Z}_p(2))) \subset H^2_{\text{ét}}(U^0_{/T}, \mathbb{Z}_p(2))^{GF}. \] (7.13)

Applying \( \mathbb{R}\Gamma(G_F, -) \) on (7.9), we have

\[ 0 \longrightarrow H^2_{\text{ét}}(U^0_{/T}, \mathbb{Z}/p^\nu(2))^{GF} \longrightarrow (\mathbb{Z}/p^\nu)^{\oplus s} \longrightarrow H^1(G_F, C_{Z/p^\nu}^{\text{ét}}(2)) \]

\[ \uparrow \]

\[ \Gamma(U^0_{/T}, K_2)/p^\nu \]

with an exact row. Passing to the projective limit, we have

\[ 0 \longrightarrow H^2_{\text{ét}}(U^0_{/T}, \mathbb{Z}/p^\nu(2))^{GF} \longrightarrow (\mathbb{Z}/p^\nu)^{\oplus s} \longrightarrow \lim_{\nu} H^1(G_F, C_{Z/p^\nu}^{\text{ét}}(2)) \]

\[ \text{dlog} \uparrow \]

\[ \Gamma(U^0_{/T}, K_2) \otimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^{\oplus s}. \]

We want to show that the cokernel of \( \partial \) is torsion free. In the same way as the proof of Claim 7.9 we can show that the cokernel of \( \text{dlog} \) is torsion free. Since \( C_{Z/p^\nu}^{\text{ét}}(2) \) is a direct summand of \( H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2)) \), it is enough to show that

\[ \lim_{\nu} H^1(G_F, H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2))) \] (7.14)

is torsion free. We have an exact sequence

\[ 0 \longrightarrow H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu-1(2)) \longrightarrow H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2)) \longrightarrow H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p(2)) \longrightarrow 0 \]

from (7.10). Applying \( \mathbb{R}\Gamma(G_F, -) \) to the above, we see that the following map

\[ H^1(G_F, H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu-1(2))) \longrightarrow H^1(G_F, H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2))) \]

is injective for all \( \nu \geq 1 \) due to the vanishing \( H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p(2))^{GF} = 0 \). Passing to the projective limit, we have the injectivity of

\[ \lim_{\nu} H^1(G_F, H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2))) \longrightarrow \lim_{\nu} H^1(G_F, H^2_{\text{ét}}(V_{/T}, \mathbb{Z}/p^\nu(2))), \quad x \mapsto p \cdot x. \]

This means that (7.14) is torsion free. This completes the proof of Theorem B (2).

8 Modular elliptic surface

The purpose of this section is to prove Conjecture 3.7 for a universal elliptic curve \( E_N \to X(N) \) \( (N \geq 3) \) over a modular curve (Theorem 8.1).
8.1 Preliminaries and notations on modular curves

For a congruence subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$ (i.e. $\Gamma \supset \Gamma(N)$ for some $N \geq 1$), we denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight $k$ with respect to $\Gamma$. See [7] Def. 1.2.3, or [20] §2.1 for the definition ($M_k(\Gamma)$ is denoted by $A_k(\Gamma)$ in [20]).

8.1.1 Hecke operators

We focus on the special case $\Gamma = \Gamma(N)$ with $N \geq 3$. Let

$$\Gamma_0 = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \right| \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} * & 0 \\ 0 & * \end{array} \right) \mod N \right\}. $$

The group $\Gamma_0/\Gamma \cong (\mathbb{Z}/N)^*$ acts on $S_k(\Gamma)$ by

$$f[\gamma] := (c\tau + d)^{-k} f \left( \frac{a\tau + b}{c\tau + d} \right), \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0, \quad \tau \in \mathcal{H} $$

where $\mathcal{H}$ denotes the complex upper half plane. Since $f[\gamma]_k$ depends only on $d$ mod $N$, it is sometimes written as $\langle d \rangle f$, and $\langle d \rangle$ is called a diamond operator ([7] 5.2). Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}$ be a Dirichlet character. We put $\chi(a) = 0$ if $(a, N) \neq 1$. We define $S_k(\Gamma_0, \chi) \subset S_k(\Gamma)$ as the set of all $f \in S_k(\Gamma)$ satisfying

$$\langle d \rangle f = \chi(d) f \quad \text{for all } d \in (\mathbb{Z}/N)^*. $$

Then we have

$$S_k(\Gamma) = \bigoplus_\chi S_k(\Gamma_0, \chi) $$

where $\chi$ runs over all Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^*$. The decomposition (8.3) is mutually orthogonal with respect to the Petersson inner product ([20] (3.5.4)).

There are the Hecke operators $T_m$ on $S_k(\Gamma)$. See [20] §3.2 for the definition (where $T_m$ is denoted by $T(m)$). They satisfy $T_n T_m = T_{nm}$ for $(n, m) = 1$ and

$$T_{p^r+1} = \begin{cases} T_p T_{p^r} - p^{k-1} \langle p \rangle T_{p^{r-1}} & \text{if } p \nmid N \\ T_p T_{p^r} & \text{if } p | N \end{cases} $$

on $S_k(\Gamma)$ for $r \geq 1$ and any prime number $p$ (loc.cit. Thm. 3.24 and (3.5.8)). The Hecke operators and diamond operators are mutually commutative and normal with respect to the Petersson inner product (loc.cit. Thm. 3.41). Therefore they are simultaneously diagonalizable. A common eigen function for all $T_n$ is called a Hecke eigenform. If $f \in S_k(\Gamma)$ is a Hecke eigenform, then there is a unique Dirichlet character $\chi$ such that $f \in S_k(\Gamma_0, \chi)$ (loc.cit. Prop. 3.53).

Put $q_N = \exp(2\pi i \tau/N)$. Let $f = \sum_{n=1}^\infty c_n q_N^n$ be the Fourier expansion at $\infty$. Suppose that $f \in S_k(\Gamma_0, \chi)$. Then the Fourier expansion of $T_m f$ is given as follows (loc.cit. (3.5.12))

$$T_m f = \sum_{n=1}^\infty c_n^* q_N^n, \quad c_n^* = \sum_{a(n,m)} \chi(a) a^{k-1} c_{mn/a^2}. $$

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In particular, if $\langle p \rangle f = \sum_{n=1}^{\infty} c_n q_n^p$ for a prime number $p$, then we have
\[
T_p f = \begin{cases} 
\sum_{n=1}^{\infty} (c_{np} + p^{k-1} c_{n/p}) q_n^p & \text{if } p \mid N \\
\sum_{n=1}^{\infty} c_n q_n^p & \text{if } p \nmid N
\end{cases}
\] (8.6)
where we put $c_{n/p} = 0$ unless $p \mid n$. Suppose that $f \in S_k(\Gamma_0, \chi)$ is a Hecke eigenform. If $c_1 = 1$, then we say that $f$ is normalized. In this case we have $T_m f = c_m f$ and
\[
c_{nm} = c_n c_m \quad ((n, m) = 1), \quad c_{p^{r+1}} = cp^{r} c_p - \chi(p) p^{k-1} c_{p^{r-1}} \quad (r \geq 1)
\] (loc.cit. Thm. 3.43). It is well-known that the characteristic polynomial of $T_m$ has rational integer coefficients for all $m \geq 1$ (loc.cit. (3.5.20)). Hence all Fourier coefficients of a normalized Hecke eigenform are algebraic integers in a number field.

8.1.2 Modular curves: Deligne and Rapoport [6]

Let $\Gamma = \Gamma(N)$ with $N \geq 3$. Put $\mathcal{O}_N := \mathbb{Z}[\zeta_N, 1/N]$ where $\zeta_N$ is a primitive $N$-th root of unity. The main result of [6] tells that there are the algebraic curve
\[
X(N)_{\mathcal{O}_N} \rightarrow \text{Spec} \mathcal{O}_N
\] (8.8)
and the open subscheme $Y(N)_{\mathcal{O}_N} \subset X(N)_{\mathcal{O}_N}$ such that
\[
X(N)_{\mathcal{O}_N} \times_{\mathcal{O}_N} \mathbb{C} \cong (\mathfrak{g} \cup \mathbb{Q} \cup \{\infty\})/\Gamma(N)
\]
\[
Y(N)_{\mathcal{O}_N} \times_{\mathcal{O}_N} \text{Spec} \mathbb{C} \cong \mathfrak{g}/\Gamma(N).
\]
They are called the modular curves. The morphism (8.8) is projective and smooth (loc.cit. IV. Cor. 2.9), and the geometric fiber is connected (loc.cit. IV. Cor. 5.5). The complement $\Sigma_{\mathcal{O}_N} := X(N)_{\mathcal{O}_N} \setminus Y(N)_{\mathcal{O}_N}$ is the disjoint union of copies of $\text{Spec} \mathcal{O}_N$ (loc.cit. VII. Cor. 2.5), which we put $\Sigma_{\mathcal{O}_N} = \{P_1, \cdots, P_s\}$. We call the points $P_i$ the cusps.

Since $X(N)_{\mathcal{O}_N}$ is the fine moduli of generalized elliptic curves with level structure $N$ (loc.cit. IV. Def. 2.4), we have the universal elliptic curve
\[
\pi : \mathcal{E}_N \rightarrow X(N)_{\mathcal{O}_N}.
\]
The morphism $\pi$ is projective over $X(N)_{\mathcal{O}_N}$, and smooth over $Y(N)_{\mathcal{O}_N}$. The fiber $\mathcal{E}_{i, \mathcal{O}_N} := \pi^{-1}(P_i)$ over a cusp $P_i \in \Sigma_{\mathcal{O}_N}$ is a standard Néron $N$-gon. More precisely, let $\Delta_{P_i} : \text{Spec} \mathcal{O}_N[[q_N]] \rightarrow X(N)_{\mathcal{O}_N}$ be the formal neighborhood of $P_i$. Then the fiber $\pi^{-1}(\Delta_{P_i})$ is isomorphic to the regular model $\mathcal{E}_q, \mathcal{O}_N[[q_N]]$ in (4.2.1) (loc.cit. VII. Cor. 2.4). In particular, the elliptic surface $\pi : \mathcal{E}_N \rightarrow X(N)_{\mathcal{O}_N}$ has only singular fibers of type $I_N$ over the cusps and satisfies (Rat).

Let $D_{\mathcal{O}_N} := \sum_i D_{i, \mathcal{O}_N} = \pi^{-1}(\Sigma_{\mathcal{O}_N})$. Define an invertible sheaf $\Omega$ by the exact sequence
\[
0 \rightarrow \pi^* \Omega^1_{X(N)/\mathcal{O}_N}(\Sigma_{\mathcal{O}_N}) \rightarrow \Omega_{\mathcal{E}_N/\mathcal{O}_N}(\log D_{\mathcal{O}_N}) \rightarrow \Omega \rightarrow 0.
\]
The sheaf $\Omega$ is isomorphic to the dualizing sheaf $\mathcal{H}^{-1}(R\pi^* \mathcal{O})$. Put $\omega = \pi_* \Omega$. This is also an invertible sheaf which is generated by $du/u$ locally at the cusps where $du/u$ is the canonical invariant 1-form of the Tate curve (cf. [3.6]). Then there is the natural isomorphism
\[
\omega \otimes 2 \cong \Omega^1_{X(N)/\mathcal{O}_N}(\Sigma_{\mathcal{O}_N}), \quad \frac{du}{u} \otimes 2 \rightleftharpoons \frac{dq_N}{q_N}
\] (8.9)
8.2 Dlog image of $K_2$ of modular elliptic surface

Let $N \geq 3$ be an integer and $p$ a prime number such that $p \not| 2N$. Let $K$ be a finite unramified extension of $\mathbb{Q}_p$ of degree $d$. Let $R$ be the ring of integers in $K$. Suppose $\zeta_N \in R$. Let $X(N)_R := X(N)_{\mathcal{O}_N} \times_{\mathcal{O}_N} R$ be the modular curve over $R$ and

$$\pi_R : X_R = \mathcal{E}_N \times_{\mathcal{O}_N} R \rightarrow X(N)_R$$

the universal elliptic curve. Let $\Sigma_R = \{P_1, \cdots, P_s\} \subset X(N)_R$ be the cusps. We put $D_R = \sum_i \pi_R^{-1}(P_i)$ and $U_R = X_R - D_R$.

**Theorem 8.1** $\text{dlog} \Gamma(U_R, \mathcal{K}_2) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \Phi(X_R, D_R)_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

8.3 Proof of Theorem 8.1

We use the following theorem of A. Beilinson.

**Theorem 8.2 ([3] Cor. 3.1.8.)** The rank of $\text{dlog} \Gamma(U_R, \mathcal{K}_2)$ is equal to $s (=\text{the number of cusps of } X(N))$.

More precisely, Beilinson constructed the Eisenstein symbols in $\Gamma(U_R, \mathcal{K}_2)$, and showed that the boundary map $\partial$ induces the bijection on the space of the Eisenstein symbols. See [19] §7 for another proof.

In order to prove Theorem 8.1 it is enough to show $\text{rank}_{\mathbb{Z}_p} \Phi(X_R, D_R)_{\mathbb{Z}_p} \leq s$. Recall the definition (8.11) and (8.12). We have from (8.9) the natural isomorphisms

$$M_k(\Gamma) \xrightarrow{\cong} \Gamma(X(N)_{\mathcal{O}_N}, \omega^{\otimes k}) \otimes_{\mathcal{O}_N} \mathbb{C}, \quad f \mapsto f \frac{du^{\otimes k}}{u} \quad (8.10)$$

(loc.cit. VII. 4.6). We identify $M_k(\Gamma)$ with the set of sections of $\omega^{\otimes k}$ by (8.10). Then $S_k(\Gamma)$ can be identified with

$$\Gamma(X(N)_{\mathcal{O}_N}, \Omega^1_{X(N)/\mathcal{O}_N} \otimes \omega^{\otimes k-2}) \otimes_{\mathcal{O}_N} \mathbb{C}.$$ 

For an $\mathcal{O}_N$-algebra $R$, we put

$$M_k(\Gamma)_R \overset{\text{def}}{=} \Gamma(X(N)_{\mathcal{O}_N}, \omega^{\otimes k}) \otimes_{\mathcal{O}_N} R, \quad (8.11)$$

$$S_k(\Gamma)_R \overset{\text{def}}{=} \Gamma(X(N)_{\mathcal{O}_N}, \Omega^1_{X(N)/\mathcal{O}_N} \otimes \omega^{\otimes k-2}) \otimes_{\mathcal{O}_N} R. \quad (8.12)$$

Let $f_{i,\chi} \in S_k(\Gamma)$ be the normalized Hecke eigenforms and $f_{i,\chi} = \sum_{n \geq 1} c_{i,\chi}(n)q^n$ the Fourier expansions. Let $F_N = \mathbb{Q}(\zeta_N, c_{i,\chi}(n))_{i,\chi,n}$ which is a finite extension of $\mathbb{Q}$. Then $f_{i,\chi}$ is contained in $S_k(\Gamma)_F = S_k(\Gamma)_{\mathcal{O}_N} \otimes_{\mathcal{O}_N} F_N$ and form a basis of it:

$$S_k(\Gamma)_{F_N} = \bigoplus_{i,\chi} F_N \cdot f_{i,\chi}. \quad (8.13)$$

(see [19] §47). Moreover we have the natural isomorphism

$$M_3(\Gamma) \xrightarrow{\cong} \Gamma(X(R), \omega^{\otimes 2}) \otimes_{\mathcal{O}_N} \mathbb{C}, \quad f \mapsto f \frac{du \cdot d\log u}{u} \quad (8.14)$$

Therefore in order to show $\text{rank}_{\mathbb{Z}_p} \Phi(X_R, D_R)_{\mathbb{Z}_p} \leq s$, it is enough to show

$$(S_3(\Gamma) \cap \Phi(X_R, D_R)_{\mathbb{Z}_p}) \otimes \mathbb{Q}_p = 0. \quad (8.15)$$
8.3.1 Lemmas on the Fourier coefficients

Fix a cyclotomic basis \( \mu = \{ \zeta_1, \ldots, \zeta_d \} \) of \( R \). Let \( \Phi'_{z_p} \subset S_3(\Gamma)_R \) be the \( \mathbb{Z}_p \)-submodule generated by \( f \in S_3(\Gamma)_R \) such that if we express

\[
f = \sum_{k=1}^{\infty} \sum_{i=1}^{d} a_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}}, \quad a_k^{(i)} \in \mathbb{Z}_p
\]

then \( a_k^{(i)} \equiv 0 \mod p^{2k} \mathbb{Z}_p \) for all \( k \) and \( i \). Since \( \gamma \in \text{SL}_2(\mathbb{Z}) \) acts on the cusps transitively, we see

\[
S_3(\Gamma)_R \cap \Phi(X_R, D_R)_{\mathbb{Z}_p} = \bigcap_{\gamma \in \text{SL}_2(\mathbb{Z})} \gamma \Phi'_{z_p}.
\]

**Lemma 8.3** Let \( f \in S_3(\Gamma)_R \) be a cusp form of weight 3 and level \( N \). If \( f \in \Phi(X_R, D_R)_{\mathbb{Z}_p} \) then \( \langle d \rangle f \in \Phi(X_R, D_R)_{\mathbb{Z}_p} \) for all \( d \in (\mathbb{Z}/N)^* \). Moreover if \( f \in \Phi'_{z_p} \) then \( T_m f \in \Phi'_{z_p} \) for all \( m \geq 1 \).

**Proof.** An element \( \gamma \in \Gamma_0/\Gamma \cong (\mathbb{Z}/N\mathbb{Z})^* \) induces an automorphism of the universal elliptic curve \( X_R \to X(N)_R \) in a natural way, which we denote by \( \sigma_\gamma \). By the definition \( (8.1) \) we have

\[
\sigma_\gamma^*(f \frac{dq_N du}{q_N u}) = \langle d \rangle f \frac{dq_N du}{q_N u}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0.
\]

(8.16)

Therefore if \( f \in \Phi(X_R, D_R)_{\mathbb{Z}_p} \) then \( (8.16) \) is also contained in \( \Phi(X_R, D_R)_{\mathbb{Z}_p} \), namely \( \langle d \rangle f \in \Phi(X_R, D_R)_{\mathbb{Z}_p} \) for all \( d \in (\mathbb{Z}/N)^* \).

Next suppose that \( f \in \Phi'_{z_p} \). It is enough to show \( T_\ell f \in \Phi'_{z_p} \) for all prime \( \ell \). Express

\[
f = \sum_{k=1}^{\infty} \sum_{i=1}^{d} a_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}}, \quad \langle \ell \rangle f = \sum_{k=1}^{\infty} \sum_{i=1}^{d} b_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}}
\]

with \( a_k^{(i)}, b_k^{(i)} \in \mathbb{Z}_p \). Here we put \( \langle \ell \rangle f = 0 \) if \( \ell | N \). Since \( f \) and \( \langle \ell \rangle f \) are contained in \( \Phi'_{z_p} \), we have

\[
a_k^{(i)} \equiv b_k^{(i)} \equiv 0 \mod k^2 \mathbb{Z}_p \quad \text{for all } k, i
\]

(8.17)

by definition. Using \( (8.6) \), a direct calculation yields

\[
T_\ell(f) = \sum_{i=1}^{d} \left( \sum_{(\ell,k)=1} a_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}} + \sum_{\ell \neq k} a_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}} + \sum_{k=1}^{\infty} \ell b_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}} \right).
\]

In order to show \( T_\ell(f) \in \Phi'_{z_p} \), the only non trivial part is to show

\[
a_k^{(i)} \frac{\zeta_i q_N^{k}}{1 - \zeta_i q_N^{k}} \in \Phi'_{z_p}.
\]

However this can be proved by the same argument as in the proof of Lemma 3.4 Q.E.D.
Lemma 8.4 Let $A$ be a discrete valuation ring and $\pi$ a uniformizer. Let $\alpha \in A$ be a non-zero element such that $e := \text{ord}_\pi(\alpha) \geq 1$. Let $\{a_n\}_{n \geq 0}$ be a sequence in $A$ satisfying

$$a_0 = 1, \quad a_{n+2} = a_1a_{n+1} - \alpha a_n, \quad n \geq 0. \quad (8.18)$$

Suppose that there are integers $d \geq 1$ and $s \geq 0$ such that

$$a_{n+d} \equiv a_n \mod \pi^{en-s} \quad \text{for all } n \geq 0. \quad (8.19)$$

Then there is a $d$-th root of unity $\zeta$ such that

$$a_n = \frac{\zeta^{n+1} - (\zeta^{-1}\alpha)^{n+1}}{\zeta - \zeta^{-1}\alpha}, \quad n \geq 0. \quad (8.20)$$

Proof. We have from (8.18) that

$$a_{n+k+2} = a_{k+1}a_{n+1} - \alpha a_k a_n \quad (n, k \geq 0). \quad (8.21)$$

Put $k = d - 1, d - 2$ in (8.21) and apply (8.19):

$$-\alpha a_{d-1} a_n \equiv (1 - a_d) a_{n+1} \mod \pi^{en-e-s}, \quad n \geq 0, \quad (8.22)$$

$$(1 + \alpha a_{d-2}) a_n \equiv a_{d-1} a_{n+1} \mod \pi^{en-s}, \quad n \geq 0. \quad (8.23)$$

Repeating them, we have

$$(-\alpha a_{d-1})^d a_n \equiv (1 - a_d)^d a_{n+d} \equiv (1 - a_d)^d a_n \mod \pi^{en-s}, \quad n \geq 0, \quad (8.24)$$

$$(1 + \alpha a_{d-2})^d a_n \equiv a_{d-1}^d a_{n+d} \equiv a_{d-1}^d a_n \mod \pi^{en-s}, \quad n \geq 0. \quad (8.25)$$

We first claim that $a_1 \not\equiv 0 \mod \pi$. Suppose $a_1 \equiv 0 \mod \pi$. Then we have $a_n \equiv a_1^n \equiv 0 \mod \pi$ for all $n \geq 1$ by (8.18). (8.24) yields

$$a_n \equiv (-\alpha a_{d-1})^d a_n \equiv \cdots \equiv (-\alpha a_{d-1})^{dk} a_n \equiv 0 \mod \pi^{en-s}, \quad k \gg 1$$

as $\text{ord}_\pi(\alpha) = e > 0$. Suppose that there is an integer $m \geq 0$ such that $a_n \equiv 0 \mod \pi^{en+m-s}$ for all $n \geq 1$. Then we have

$$\alpha a_n = a_1 a_{n+1} - a_{n+2} \equiv 0 \mod \pi^{en+e+1+m-s}$$

and hence $a_n \equiv 0 \mod \pi^{en+m+1-s}$. Therefore the induction yields $a_n \equiv 0 \mod \pi^{en+m-s}$ for all $m \geq 0$ and $n \geq 1$. This means $a_n = 0$ for all $n \geq 1$, which contradicts (8.18).

We now have $a_1 \not\equiv 0 \mod \pi$. Then $a_n \equiv a_1^n \not\equiv 0 \mod \pi$ for all $n \geq 0$ by (8.18). Therefore (8.24) and (8.25) imply $(-\alpha a_{d-1})^d \equiv (1 - a_d)^d$ and $(1 + \alpha a_{d-2})^d \equiv a_{d-1}^d$ mod $\pi^{en-s}$ for all $n \geq 0$, which means $(-\alpha a_{d-1})^d = (1 - a_d)^d$ and $(1 + \alpha a_{d-2})^d = a_{d-1}^d$. Put

$$-\alpha a_{d-1} = \zeta(1 - a_d), \quad 1 + \alpha a_{d-2} = \zeta' a_{d-1} \quad (8.26)$$
where $\zeta, \zeta'$ are $d$-th roots of unity. We claim $\zeta = \zeta'$. In fact due to (8.22) and (8.28), we have

$$-\alpha a_{d-1}a_n \equiv -\zeta^{-1}\alpha a_{d-1}a_{n+1} \mod \pi^{en+e-s}, \quad n \geq 0,$$

$$a_{d-1}a_n \equiv \zeta^{-1}a_{d-1}a_{n+1} \mod \pi^{en-s}, \quad n \geq 0.$$  

The above implies $\zeta \equiv \zeta' \mod \pi^{en-s}$ for all $n \geq 0$, hence $\zeta = \zeta'$. Thus we have

$$0 \overset{(8.20)}{=} 1 - a_d + \zeta^{-1}a_{d-1} \overset{(8.18)}{=} 1 - (a_1a_{d-1} - \alpha a_{d-2}) + \zeta^{-1}a_{d-1} \overset{(8.20)}{=} a_{d-1} - a_1a_{d-1} + \zeta^{-1}a_{d-1}.$$

Since $a_{d-1} \neq 0$, we have $a_1 = \zeta + \zeta^{-1}\alpha$. This yields (8.20) by the induction on $n$. Q.E.D.

### 8.3.2 End of the proof

We finish the proof of Theorem 8.1. It is enough to show (8.15). Suppose that there is a non-zero $f = \sum_{n=1}^{\infty} c(n)q^n \in S_3(\Gamma)_{R}$ such that $f \in \Phi(X_R, D_R)_{Z_p}$. Let $f$ be expressed

$$f = \sum_{k=1}^{\infty} \sum_{i=1}^{d} a_i(k) \frac{\zeta_i q^k}{1 - \zeta_i q^k}$$

with $a_i(k) \in Z_p$. Let $\sigma : R \to R$ be the Frobenius automorphism. Then

$$c(p^r) = \sum_{k=0}^{r} \sum_{i=1}^{d} a_i(p^k) \zeta_i^{p^r-k} = \sigma(c(p^{r-1})) + \sum_{i=1}^{d} a_i(p^r)\zeta_i, \quad r \geq 1$$

as $\zeta_i^p = \zeta_i^p$. Since $f \in \Phi(X_R, D_R)_{Z_p} \subset \Phi'_{Z_p}$ we have

$$c(p^{r+1}) \equiv \sigma(c(p^r)) \mod p^{2r+2}R, \quad r \geq 0.$$  

Repeating it, we have

$$c(p^{r+d}) \equiv c(p^r) \mod p^{2r+2}R, \quad r \geq 0 \quad (8.27)$$

as $\sigma^d = 1$. Let $f_{i,\chi} = \sum_{n=1}^{\infty} c_{i,\chi}(n)q^n$ be the normalized Hecke eigenforms, and express $f = \sum_{i,\chi} \alpha_{i,\chi}f_{i,\chi}$ with $\alpha_{i,\chi} \in K(c_{i,\chi}(n))_{i,\chi,n}$ (cf. (8.13)). Then (8.27) is written as

$$\sum_{i,\chi} \alpha_{i,\chi}c_{i,\chi}(p^{r+d}) \equiv \sum_{i,\chi} \alpha_{i,\chi}c_{i,\chi}(p^r) \mod p^{2r+2}R, \quad r \geq 0. \quad (8.28)$$

On the other hand By Lemma 8.3 we have $T_m f = \sum_{i,\chi} c_{i,\chi}(m)\alpha_{i,\chi}f_{i,\chi} \in \Phi'_{Z_p}$ for all $m \geq 1$. Similarly to (8.28) we have

$$\sum_{i,\chi} c_{i,\chi}(m)\alpha_{i,\chi}c_{i,\chi}(p^{r+d}) \equiv \sum_{i,\chi} c_{i,\chi}(m)\alpha_{i,\chi}c_{i,\chi}(p^r) \mod p^{2r+2}R, \quad r \geq 0 \quad (8.29)$$

for all $m \geq 1$. Since $f_{i,\chi}$ are linearly independent, (8.28) yields that there is an integer $s \geq 0$ which does not depend on $r$ such that

$$\alpha_{i,\chi}c_{i,\chi}(p^{r+d}) \equiv \alpha_{i,\chi}c_{i,\chi}(p^r) \mod p^{2r+2-s}R', \quad r \geq 0 \quad (8.30)$$

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for all \(i\) and \(\chi\) where \(R'\) is the ring of integers in \(K(c_{i,\chi}(n))_{i,\chi,n}\). Take \(f_{i,\chi}\) such that \(\alpha_{i,\chi} \neq 0\). Due to (8.7) and (8.30), one can apply Lemma 8.4 for \(a_n = c_{i,\chi}(p^n)\). Then we have

\[
c_i,\chi(p^r) = \frac{\zeta^{r+1} - (\zeta^{-1} \chi(p^2))^{r+1}}{\zeta - \zeta^{-1} \chi(p^2)}, \quad r \geq 0.
\]

However, since \(f_{i,\chi}\) is a cusp form of weight 3, we have an estimate

\[
|c_i,\chi(n)| \leq Cn^{3/2}
\]

where \(|c_i,\chi(n)|\) denotes the complex absolute value (\([20]\) Lem. 3.62, \([7]\) Prop. 5.9.1). This is the contradiction. This completes the proof of (8.15) and hence Theorem 8.1.

9 Example : elliptic surfaces \(Y^2 = X^3 + X^2 + t^n\)

Let \(\pi : X \to C = \mathbb{P}^1_C\) be the minimal elliptic surface over \(C\) such that the general fiber \(\pi^{-1}(t)\) is the elliptic curve defined by

\[
Y^2 = X^3 + X^2 + t^n, \quad n \geq 1.
\]

The functional \(j\)-invariant is \(-64/(t^n(1 + 27t^n/4))\). Put \(n = 6k + l\) where \(k \geq 0\) and \(1 \leq l \leq 6\). Then the canonical bundle \(K_X\) is \(\pi^*\mathcal{O}(k - 1)\) and the Hodge numbers are as follows:

| 1  |
| --- |
| 0  |
| \(k\) | 10 + 10\(k\) | \(k\) |
| 0  |
| 1  |

In particular \(X\) is rational if \(1 \leq n \leq 6\), a K3 surface if \(7 \leq n \leq 12\) and \(\kappa(X) = 1\) if \(n \geq 13\). There are \((n+2)\) singular fibers, \(D_0 = \pi^{-1}(0), D_i = \pi^{-1}(\sqrt{-4/27\zeta_i^n}) (1 \leq i \leq n)\) and \(Y_\infty = \pi^{-1}(\infty)\). The multiplicative fibers are \(D = D_0 + \cdots + D_n\). \(D_0\) is of type \(I_n\) and \(D_i\) is of type \(I_1\) for \(1 \leq i \leq n\). The type of \(Y_\infty\) is as follows:

| \(l\) | 1 | 2 | 3 | 4 | 5 | 6 |
| --- | --- | --- | --- | --- | --- | --- |
| \(Y_\infty\) | II* | IV* | I_0 | IV | II | (smooth) |

The number of multiplicative fibers is \(s = n + 1\). We put \(\Sigma = \{0, \sqrt{-4/27\zeta_i^n} (1 \leq i \leq n)\} \subset C\) and \(U = X - D\).

The purpose of this section is to prove the following:

**Theorem 9.1** Suppose that \(n\) is a prime number with \(2 \leq n \leq 29\). Then we have

\[
\text{rank } \text{dlog } \Gamma(U, K_2) = 2.
\]

The dlog image is generated by

\[
\text{dlog } \left\{ \frac{Y - X}{Y + X} - \frac{t^n}{X^2} \right\}, \quad \text{dlog } \left\{ \frac{iY - (X + 2/3)}{iY + (X + 2/3)} \cdot \frac{t^n + 4/27}{(X + 2/3)^3} \right\}
\]

Note \(\text{dlog } \Gamma(U, K_2) = \text{dlog } \Gamma(U - Y_\infty, K_2)\) (Lemmas 2.3 and 2.4).
9.1 Proof of Theorem 9.1

Let \( \partial : \Gamma(U, \mathcal{K}_2) \to \mathbb{Z}^{b+1+n} \) be the boundary map where the base \((0, \cdots, b, \cdots, 0) \in \mathbb{Z}^{b+1+n} \) corresponds to the fiber \( D_b \) for \( 0 \leq b \leq n \). We want to show that the rank of \( \partial \Gamma(U, \mathcal{K}_2) \) is 2. A direct calculation yields

\[
\partial \left\{ \frac{Y - X}{Y + X}, -\frac{t^n}{X^3} \right\} = (n, 0, \cdots, 0),
\]

\[
\partial \left\{ \frac{iY - (X + 2/3)}{iY + (X + 2/3)}, -\frac{t^n + 4/27}{(X + 2/3)^3} \right\} = (0, 1, \cdots, 1).
\]

Therefore we have \( \text{rank} \, \partial \Gamma(U, \mathcal{K}_2) \geq 2 \) and hence it is enough to show \( \text{rank} \, \partial \Gamma(U, \mathcal{K}_2) \leq 2 \). To do this we use Theorem 6.2.

9.1.1 Step 1 : Choice of \( p \geq 5 \) and \( \pi_R : X_R \to C_R \)

Let \( p \not| 6n \) be a prime number. Let \( K = \mathbb{Q}_p(\sqrt{-1}, \sqrt{-4/27}, \zeta_n) \) be an unramified extension over \( \mathbb{Q}_p \) and \( R \) the ring of integers in \( K \). Let

\[
\pi_R : X_R \to C_R
\]

be the elliptic surface over \( R \) obtained from the defining equation \( Y^2 = X^3 + X^2 + t^n \) in a natural way such that \( X_K := X_R \times_R K \) is minimal. One can easily check that it is an elliptic surface in the sense of \( \text{(2.3)} \) and satisfies \( \text{(Rat)} \). Since \( 6 \not| n \), the fiber \( Y_\infty \) is not a multiplicative type. Therefore \( H^2_{\text{ét}}(X_K, \mathbb{Z}_p) \) is torsion free by Lemma 7.4 and Proposition 7.6 [3]. Thus the condition in Theorem 6.2 (1) is satisfied. Let us see the condition in Theorem 6.2 (2). Let \( F = \mathbb{Q}(\sqrt{-1}, \sqrt{-4/27}, \zeta_n) \). One can check that \( \pi_K : X_K \to C_K \) is defined over \( F \), which we write \( \pi_F : X_F \to C_F \). It is easy to see that \( \pi_F : X_F \to C_F \) satisfies \( \text{(Rat)} \). We discuss the condition

\[
H^2(X_F, \mathbb{Z}/p(2))^{GF} = 0. \tag{9.2}
\]

To do this, we see the Frobenius action. Suppose that \( X_F \) has a good reduction at a prime \( \mathfrak{p} \) of \( F \) which is prime to \( p \). Let \( \ell \) be a rational prime such that \( \mathfrak{p} \not| \ell \). Let \( \kappa \) be the residue field at \( \mathfrak{p} \) and put \( t = [\kappa : \mathbb{F}_\ell] \). Let \( \text{Frob}_t \in \text{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell) \) be the Frobenius automorphism. By the proper smooth base change theorem we have

\[
H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_\ell}, \mathbb{Z}/p(2)) \cong H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}/p(2)).
\]

By Proposition 7.6 [3] we have

\[
H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}/p(2)) \cong H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}_p(2)) \otimes \mathbb{Z}/p\mathbb{Z}
\]

and \( H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}_p(2)) \) is torsion free. Denote by \( \text{Frob}_t^{\text{arith}} \) the action of \( \text{Frob}_t \) on \( H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}_p(2)) \). If \( p \) satisfies

\[
p \not| \det(1 - \text{Frob}_t^{\text{arith}} : H^2_{\text{ét}}(X_{\mathbb{F}_\ell}, \mathbb{Z}_p(2))), \tag{9.3}
\]

then (9.2) holds. There is the geometric Frobenius endomorphism on \( X_{\overline{\mathbb{F}}_\ell} \). Denote by \( \text{Frob}_{t}^{\text{geo}} \) the action of the geometric Frobenius on \( H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_\ell}, \mathbb{Z}_p(2)) \). Then \( \text{Frob}_{t}^{\text{geo}} \).
\[ \text{Frob}^{\text{arith}} = \text{Frob}^{\text{arith}}. \text{Frob}^{\text{geo}} = \text{Frob}^{\text{geo}}. \text{Frob}^{\text{geo}} \text{ is the identity. By the Poincare duality theorem, the dual of } H^2_{\text{et}}(X_\pi, \mathbb{Z}_p(2)) \text{ is } H^2_{\text{et}}(X_\pi, \mathbb{Z}_p). \text{ Thus we have} \]

\[ \det(1 - \text{Frob}^{\text{arith}} : H^2_{\text{et}}(X_\pi, \mathbb{Z}_p(2))) = \det(1 - \text{Frob}^{\text{geo}} : H^2_{\text{et}}(X_\pi, \mathbb{Z}_p)). \quad (9.4) \]

\[ X_F \text{ is defined over } \mathbb{Q} \text{ and } X_\kappa \text{ is defined over the prime field } \mathbb{F}_\ell, \text{ which we denote by } X_\mathbb{Q} \text{ and } X_\mathbb{F}_\ell \text{ respectively. Let } \text{Frob}^{\text{geo}} \text{ be the geometric Frobenius action on } H^2_{\text{et}}(X_\mathbb{F}_\ell, \mathbb{Z}_p) = H^2_{\text{et}}(X_\kappa, \mathbb{Z}_p) \text{ and } \alpha_i \text{ its eigenvalues. Since } \text{Frob}^{\text{geo}} = (\text{Frob}^{\text{geo}})^t \text{ we have} \]

\[ \det(1 - \text{Frob}^{\text{geo}} : H^2_{\text{et}}(X_\kappa, \mathbb{Z}_p)) = \prod_{i=1}^{10+12k} (1 - \alpha_i^t). \quad (9.5) \]

The eigenvalues \( \alpha_i \) are computed from the zeta function of \( X_\mathbb{F}_\ell \). Namely letting \( \nu_m(X_\mathbb{F}_\ell) \) be the cardinality of the \( \mathbb{F}_\ell \)-rational points of \( X_\mathbb{F}_\ell \), the zeta function \( Z(X_\mathbb{F}_\ell, T) \in \mathbb{Z}[[T]] \) is defined as follows:

\[ Z(X_\mathbb{F}_\ell, T) \overset{\text{def}}{=} \exp \sum_{m=1}^{\infty} \frac{\nu_m(X_\mathbb{F}_\ell)}{m} T^m. \]

Due to the Lefschetz trace formula we have

\[ Z(X_\mathbb{F}_\ell, T) = \frac{1}{(1 - T)(1 - \ell^2T)} \det(1 - \text{Frob}^{\text{geo}} T : H^2_{\text{et}}(X_\mathbb{F}_\ell, \mathbb{Z}_p)) \]

Hence we have

\[ \nu_m(X_\mathbb{F}_\ell) = 1 + \ell^{2m} + \sum_{i=1}^{10+12k} \alpha_i^m. \quad (9.6) \]

Moreover it follows from the Poincare duality that we have

\[ \prod_{i=1}^{10+12k} \alpha_i = \pm \ell^{10+12k}. \quad (9.7) \]

Let us compute \( \nu_n(X_\mathbb{F}_\ell) \). Put

\[ X^o := \text{Spec} \mathbb{F}_\ell[X, Y, t]/(Y^2 - X^3 - X^2 - t^n) \rightarrow X_\mathbb{F}_\ell. \]

Since \( X_\mathbb{F}_\ell = (X^o \setminus \{t = 0\}) \cup D_0 \cup Y_\infty \cup e(\mathbb{P}^1), \) we have

\[ \nu_m(X_\mathbb{F}_\ell) = 1 + (12k - n + 11)\ell^m + \nu_m(X^o) \text{ unless } 6|n. \quad (9.8) \]

Suppose that \( n \) is a prime number and \((\ell \text{ mod } n)\) is a generator of \((\mathbb{Z}/n)^*\). Then \( \mathbb{F}_\ell \rightarrow \mathbb{F}_\ell, x \mapsto x^n \) is bijective unless \( n - 1|n \) and hence we have

\[ \nu_m(X^o) = \ell^{2m} \text{ unless } n - 1|m. \quad (9.9) \]

By (9.6), (9.8) and (9.9) we have

\[ \sum_{i=1}^{10+12k} \alpha_i^m = (12k - n + 11)\ell^m \text{ unless } n - 1|m. \quad (9.10) \]
Claim 9.2 Suppose \( n \geq 11 \). Then (9.11) and (9.10) yield
\[
\sum_{i=1}^{10+12k} \alpha_i^{n-1} = \begin{cases} 
(10 + 12k)^{\ell^{n-1}} & \text{if } \prod_i \alpha_i = -\ell^{10+12k} \\
10 & \text{if } \prod_i \alpha_i = \ell^{10+12k} 
\end{cases}
\]
\[
\sum_{i=1}^{10+12k} \alpha_i^{2n-2} = (10 + 12k)^{\ell^{n-1}} & \text{if } \prod_i \alpha_i = \ell^{10+12k}.
\]

Proof. Exercise on symmetric polynomials. Q.E.D.

By the Weil-Riemann conjecture we have
\[
|\alpha_i^\sigma| = \ell \quad \text{for any } \sigma : \overline{\mathbb{Q}} \to \mathbb{C}.
\] (9.11)

Therefore Claim 9.2 implies \( \alpha_i^{2n-2} = \ell^{2n-2} \) for all \( 1 \leq i \leq 10 + 12k \). Since \( 2n - 2 \) is divided by \( t = [\kappa : \mathbb{F}_\ell] \), (9.3) holds if \( p \not| (1 - \ell^{2n-2}) \) for \( n \geq 11 \). There exist infinitely many \( \ell \) such that \((\ell \mod p)\) is a generator of \((\mathbb{Z}/p)^*\) and \((\ell \mod n)\) is a generator of \((\mathbb{Z}/n)^*\). For such \( \ell \), if \( p - 1 \not| 2n - 2 \), then \( p \not| (1 - \ell^{2n-2}) \) and hence (9.3). Thus (9.2) holds if \( p - 1 \not| 2n - 2 \) for \( n \geq 11 \).

In the cases \( n = 2, 3, 5, 7 \), we check (9.3) as the case may be.

Case \( n = 2 \). We put \( \ell = 13 \). Then \( \kappa = \mathbb{F}_{13} \) \((t = 1)\). By a direct calculation we have
\[
\nu_1(X_{\mathbb{F}_{13}}) = 13^2 + 10 \cdot 13 + 1 \quad (\Longrightarrow \sum_{i=1}^{10} \alpha_i = 10 \cdot 13).
\]

Then it follows from the Weil-Riemann conjecture (9.11) that we have \( \alpha_i = 13 \) for all \( 1 \leq i \leq 10 \). Thus if \( p \not| 13 - 1 \) \((\Longleftrightarrow p \geq 5)\) then (9.3) and hence (9.2) hold.

Case \( n = 3 \). We put \( \ell = 7 \). Then \( \kappa = \mathbb{F}_{7^6} \) \((t = 6)\).
\[
\nu_1(X_{\mathbb{F}_7}) = 7^2 + 10 \cdot 7 + 1.
\]

Therefore we have \( \alpha_i = 7 \) for all \( 1 \leq i \leq 10 \). Thus if \( p \not| 7^6 - 1 \) \((\Longleftrightarrow p \neq 2, 3, 19, 43)\) then (9.3) holds. On the other hand let \( \ell = 13 \). Then \( \kappa = \mathbb{F}_{13^3} \) \((t = 3)\) and
\[
\nu_1(X_{\mathbb{F}_{13}}) = 13^2 + 10 \cdot 13 + 1.
\]

Therefore we have \( \alpha_i = 13 \) for all \( 1 \leq i \leq 10 \) and if \( p \not| 13^3 - 1 \) \((\Longleftrightarrow p \neq 2, 3, 61)\) then (9.3) holds. As a result, if \( p \geq 5 \) then (9.2) holds.

Case \( n = 5 \). We put \( \ell = 19 \). Then \( \kappa = \mathbb{F}_{19^2} \) \((t = 2)\).
\[
\nu_m(X_{\mathbb{F}_{19}}) = \begin{cases} 
19^{2m} + 6 \cdot 19^m + 1 & 2 \not|m \\
19^4 + 10 \cdot 19^2 + 1 & m = 2.
\end{cases}
\]

Therefore we have \( \alpha_i^2 = 19^2 \) for all \( 1 \leq i \leq 10 \). Thus if \( p \geq 7 \) then (9.3) holds.

Case \( n = 7 \). We put \( \ell = 13 \). Then \( \kappa = \mathbb{F}_{13^2} \) \((t = 2)\).
\[
\nu_m(X_{\mathbb{F}_{13}}) = \begin{cases} 
13^{2m} + 16 \cdot 13^m + 1 & 2 \not|m \\
13^4 + 22 \cdot 13^2 + 1 & m = 2.
\end{cases}
\]

Therefore we have \( \alpha_i^2 = 13^2 \) for all \( 1 \leq i \leq 22 \). Thus if \( p \not| 2 \cdot 3 \cdot 7 \cdot 13 \) then (9.3) holds.

Summarizing the above, the elliptic surface \( \pi_R : X_R \to C_R \) over \( R \) satisfies the conditions in Theorem 6.2 in the following cases.

\[
\begin{array}{cccccc}
 n & 2 & 3 & 5 & 7 & n \geq 11 \\
p & p \geq 5 & p \geq 5 & p \geq 7, p \neq 19 & p \not| 2 \cdot 3 \cdot 7 \cdot 13 & p - 1 \not| 2(n - 1)
\end{array}
\]

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9.1.2 Step 2 : Computation of \( \dim_{\mathbb{F}_p} \Phi(X_R, D_R)_{\mathbb{F}_p} \)

Let \( p, K = \mathbb{Q}_p(\sqrt{-1}, \sqrt[3]{-4/27}, \zeta_n) \) and \( \pi_R : X_R \to C_R \) be as above. Put

\[
Y' = \sqrt{-1}Y, \quad X' = -X - \frac{2}{3}, \quad t' = \left( \sqrt[3]{-\frac{4}{27}} \right)^{-1} t, \quad t_i = \zeta_n^{-i}t' - 1 \quad (1 \leq i \leq n).
\]

Then

\[
Y'^2 = X'^3 + X'^2 + \frac{4}{27}(t'^n - 1) = X'^3 + X'^2 + \frac{4}{27}((t_i + 1)^n - 1). \quad (9.12)
\]

Let \( q_i \in t_i R[[t_i]] \) be the formal power series such that

\[
j = \frac{1}{q_i} + 744 + 196884q_i + \cdots = -\frac{432}{(t_i + 1)^n((t_i + 1)^n - 1)}. \quad (9.13)
\]

\( q_i \) is the period of the Tate curve in the neighborhood of the singular fiber \( D_i \) \( (1 \leq i \leq n) \). Conversely the \( q_i \)-expansion of \( t_i \) is as follows:

\[
t_i = -\frac{432}{n} q_i + \frac{41472n + 93312}{n^2} q_i^2 + \cdots. \quad (9.14)
\]

Then \( R((q_i)) = R((t_i)) \) and the isomorphism between the Tate curve \( E_{q_i R((q_i))} \) \( (3.1) \) and \( U_{R \times S_R} \) \( \text{Spec} R((q_i)) \) is given by

\[
X' = 4 \frac{x + (1 - E_4^{1/4})/12}{E_4^{1/2}}, \quad Y' = 4 \frac{2y + x}{E_4^{3/4}}
\]

where \( E_4 \) is the Eisenstein series:

\[
E_4 = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q_i^n}{1 - q_i^n}, \quad E_4^{1/4} = 1 + 60q_i - 4860q_i^2 + \cdots.
\]

In particular we have

\[
\frac{dX'}{Y'} = E_4^{1/4} \frac{dx}{2y + x} = \frac{E_4^{1/4} du}{u}. \quad (9.15)
\]

\( \Gamma(X_R, \Omega^{2}_{X_R/\mathbb{R}}(\log(D_R))) \) is generated by

\[
t'^a dt'^d X'/Y', \quad \frac{dt'}{t'} \frac{dX'}{Y'}, \quad \frac{dt'}{t' - \zeta_n^b Y'} \quad (0 \leq a \leq k - 1, \ 1 \leq b \leq n)
\]

where \( n = 6k + l \) with \( k \geq 0 \) and \( 1 \leq l \leq 6 \). Since

\[
\partial_{\text{DR}} \left( t'^a dt'^d \frac{dX'}{Y'} \right) = (0, \cdots, 0), \quad \partial_{\text{DR}} \left( \frac{dt'}{t' - \zeta_n^b Y'} \right) = (\sqrt{-1}, 0, \cdots, 0),
\]

\[
\partial_{\text{DR}} \left( \frac{dt'}{t' - \zeta_n^b Y'} \right) = (0, \cdots, 1, \cdots, 0),
\]

\[
\partial_{\text{DR}} \left( \frac{dt'}{t'} \frac{dX'}{Y'} \right) = (0, \cdots, b, \cdots, 0). \]
we have

\[
\Gamma(X_R, \Omega^2_{X_R/R}(\log D_R))_{\varphi} = \sum_{a=0}^{k-1} \frac{R_{i_a} dt_i}{Y^r} + \sum_{b=1}^n \frac{\frac{dt_i}{\nu - \zeta_{i_b}^n}}{Y^r} + \frac{dt_i}{t_i + 1} \frac{dX'}{Y^r} + \frac{dt_i}{t_i - 1} \frac{dX'}{Y^r},
\]

The other cases are similar (left to the reader).

Let us compute \(\Gamma(X_R, \Omega^2_{X_R/R}(\log D_R))_{\varphi} \leq 2\) as the case may be. We check it only in case \(n = 7\).

Let \(n = 7\). We take \(p = 11\). Then \(K = \mathbb{Q}_{11}(\sqrt{-1}, \zeta_7) = \mathbb{Q}_{11}(\zeta) (\zeta := \sqrt{-1}\zeta_7)\), \(R = \mathbb{Z}_{11}[\zeta]\) and \([K : \mathbb{Q}_{11}] = 6\). Fix a cyclotomic basis \(\mu = \{1, \zeta, \zeta^2, \ldots, \zeta^5\}\). Put

\[
a = \zeta_7 + \zeta_7^2 + \zeta_7^4 = 4 + 11 + 11^2 + 7 \cdot 11^3 + \cdots
\]

\[
b = \zeta_7^3 + \zeta_7^5 = 6 + 9 \cdot 11 + 9 \cdot 11^2 + 3 \cdot 11^3 + \cdots.
\]

The minimal polynomial of \(\zeta_7\) over \(\mathbb{Q}_{11}\) is \(x^3 - ax^2 + bx - 1\). \(\Gamma(X_R, \Omega^2_{X_R/R}(\log D_R))_{\varphi} \leq 6\) is generated by

\[
Rdt \frac{dX'}{Y^r}, \quad \mathbb{Z}_{11} \sqrt{-1} \frac{dt'}{t'} \frac{dX'}{Y^r}, \quad \mathbb{Z}_{11} \frac{dt}{t} \frac{dX'}{Y^r} \quad (1 \leq b \leq 7).
\]

Let us compute

\[
\phi_i : \Gamma(X_R, \Omega^2_{X_R/R}(\log D_R))_{\varphi} \longrightarrow \prod_{k \geq 1} (\mathbb{Z}_{11}/k^2 \mathbb{Z}_{11})^\oplus 6
\]

for \(1 \leq i \leq n\). Since

\[
d \log \left\{ \frac{Y - X}{Y + X}, -\frac{t^2}{X^3} \right\} = 7 \sqrt{-1} \frac{dt'}{t'} \frac{dX'}{Y^r},
\]

\[
d \log \left\{ \frac{iY - (X + 2/3)}{iY + (X + 2/3)}, -\frac{t^2 + 4/7}{(X + 2/3)^3} \right\} = \frac{d(t^2)}{t^2 - 1} \frac{dX'}{Y^r}
\]

we have

\[
\phi_i \left( \sqrt{-1} \frac{dt'}{t'} \frac{dX'}{Y^r} \right) = 0, \quad \phi_i \left( \frac{d(t^2)}{t^2 - 1} \frac{dX'}{Y^r} \right) = 0.
\]

Let

\[
\frac{dt'}{t'} \frac{dX'}{Y^r} = f_b(q_i) \frac{dq_i}{q_i} \frac{du}{u}.
\]

Express

\[
f_b(q_i) = E_4^{1/4} \frac{1}{t_i + 1 - \zeta_{7}^{b-1} q_i} \frac{dt_i}{dq_i},
\]

\[
= E_4^{1/4} \frac{1}{1 - \zeta_{7}^{b-1}} \sum_{m \geq 0} \left( \frac{-t_i}{1 - \zeta_{7}^{b-1}} \right)^m q_i \frac{dt_i}{dq_i},
\]

\[
= \sum_{k \geq 1} \frac{a_k q_i^k}{1 - q_i^k} + \frac{b_k \zeta q_i^k}{1 - \zeta q_i^k} + \cdots + \frac{f_k \zeta^5 q_i^k}{1 - \zeta^5 q_i^k}
\]
for $b \not\equiv i \mod 7$ and
\[
f_b(q_i) = E_4^{1/4} \frac{1}{t_i} \frac{dt_i}{dq_i} = 1 + \sum_{k \geq 1} \frac{a_k q_i^k}{1 - q_i^k}
\]
for $b = i$. By definition, $\phi_i(\frac{dt_i}{dq_i}, dX') = (\bar{a}_k, \ldots, \bar{f}_k)_{k \geq 1}$ where $\bar{a}_k = (a_k \mod k^2 \mathbb{Z}_{11})$. The following is the table for $k = 11, 22, 33, 44$ (don’t check it by hand).

| $b - i$ | $a_{11}$ | $b_{11}$ | $c_{11}$ | $d_{11}$ | $e_{11}$ | $f_{11}$ | $a_{22}$ | $b_{22}$ | $c_{22}$ | $d_{22}$ | $e_{22}$ | $f_{22}$ |
|---------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1       | 99       | 0        | 99       | 0        | 55       | 0        | 11       | 0        | 99       | 0        | 99       | 0        |
| 2       | 77       | 0        | 55       | 0        | 88       | 0        | 110      | 0        | 55       | 0        | 99       | 0        |
| 3       | 11       | 0        | 55       | 0        | 33       | 0        | 99       | 0        | 99       | 0        | 88       | 0        |
| 4       | 66       | 0        | 88       | 0        | 99       | 0        | 99       | 0        | 88       | 0        | 44       | 0        |
| 5       | 33       | 0        | 33       | 0        | 55       | 0        | 110      | 0        | 33       | 0        | 22       | 0        |
| 6       | 22       | 0        | 33       | 0        | 33       | 0        | 99       | 0        | 0        | 0        | 88       | 0        |
| 0       | 55       | 0        | 0        | 0        | 0        | 0        | 77       | 0        | 0        | 0        | 0        | 0        |

| $b - i$ | $a_{33}$ | $b_{33}$ | $c_{33}$ | $d_{33}$ | $e_{33}$ | $f_{33}$ | $a_{44}$ | $b_{44}$ | $c_{44}$ | $d_{44}$ | $e_{44}$ | $f_{44}$ |
|---------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1       | 99       | 0        | 88       | 0        | 99       | 0        | 33       | 0        | 55       | 0        | 55       | 0        |
| 2       | 0        | 0        | 99       | 0        | 55       | 0        | 77       | 0        | 11       | 0        | 66       | 0        |
| 3       | 88       | 0        | 11       | 0        | 44       | 0        | 11       | 0        | 0        | 0        | 11       | 0        |
| 4       | 33       | 0        | 55       | 0        | 88       | 0        | 55       | 0        | 55       | 0        | 0        | 0        |
| 5       | 55       | 0        | 66       | 0        | 11       | 0        | 11       | 0        | 55       | 0        | 55       | 0        |
| 6       | 88       | 0        | 44       | 0        | 66       | 0        | 33       | 0        | 66       | 0        | 55       | 0        |
| 0       | 0        | 0        | 0        | 0        | 0        | 0        | 22       | 0        | 0        | 0        | 0        | 0        |

Similarly, we put
\[
\zeta^b dt \frac{dX'}{Y'} = \zeta^{b+8i} dt \frac{dX'}{Y'} = g_b(q_i) \frac{dq_i}{q_i} \frac{du}{u}
\]
\[
g_b(q_i) = \zeta^{b+8i} E_4^{1/4} \frac{dt_i}{dq_i} = \sum_{k \geq 1} \frac{a_k q_i^k}{1 - q_i^k} + \frac{b_k q_i^k}{1 - \zeta q_i^k} + \cdots + \frac{f_k q_i^k}{1 - \zeta^5 q_i^k}.
\]

| $b + 8i$ | $a_{11}$ | $b_{11}$ | $c_{11}$ | $d_{11}$ | $e_{11}$ | $f_{11}$ | $a_{22}$ | $b_{22}$ | $c_{22}$ | $d_{22}$ | $e_{22}$ | $f_{22}$ |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0        | 2        | 0        | 0        | 0        | 0        | 0        | 93       | 0        | 0        | 0        | 0        | 0        |
| 1        | 0        | 2        | 0        | 42       | 0        | 54       | 109      | 95       | 106      | 92       | 8        | 101      |
| 2        | 25       | 0        | 86       | 0        | 79       | 0        | 49       | 0        | 79       | 0        | 22       | 0        |
| 3        | 0        | 0        | 0        | 44       | 0        | 79       | 86       | 0        | 101      | 66       | 1        | 29       |
| 4        | 0        | 0        | 42       | 0        | 44       | 0        | 66       | 0        | 48       | 0        | 68       | 0        |
| 5        | 0        | 54       | 0        | 25       | 0        | 86       | 60       | 101      | 116      | 49       | 88       | 37       |

| $b + 8i$ | $a_{33}$ | $b_{33}$ | $c_{33}$ | $d_{33}$ | $e_{33}$ | $f_{33}$ | $a_{44}$ | $b_{44}$ | $c_{44}$ | $d_{44}$ | $e_{44}$ | $f_{44}$ |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0        | 114      | 0        | 0        | 0        | 0        | 0        | 53       | 0        | 0        | 0        | 0        | 0        |
| 1        | 0        | 24       | 0        | 93       | 0        | 28       | 17       | 27       | 91       | 83       | 22       | 3        |
| 2        | 113      | 0        | 7        | 0        | 106      | 0        | 35       | 0        | 43       | 0        | 67       | 0        |
| 3        | 0        | 24       | 0        | 88       | 0        | 28       | 77       | 0        | 60       | 110      | 44       | 38       |
| 4        | 20       | 0        | 15       | 0        | 113      | 0        | 99       | 0        | 17       | 0        | 51       | 0        |
| 5        | 0        | 60       | 0        | 40       | 0        | 52       | 35       | 3        | 65       | 35       | 77       | 72       |

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The above tables yields that the kernel of $\phi_1$ is contained in

$$Z_{11} \sqrt{-1} \frac{dt}{t'} \frac{dX'}{Y'} + Z_{11} \frac{dt}{t'} \frac{dX'}{Y'} + 11 \cdot \Gamma(X_R, \Omega^2_{X/R} \log D_R)_{Z_{11}}.$$

Hence we have $\dim_{F_{11}} \Phi(X_R, D_R)_{F_{11}} \leq 2$. This completes the proof.

### 9.2 Indecomposable parts of $K_1(X)$ with higher rank

Let $X$ be a nonsingular variety over $\mathbb{C}$. Let $Y \subset X$ be a closed subscheme. The Adams operations on $K^Y(X)_{\mathbb{Q}} = K^Y_i(X) \otimes \mathbb{Q} \cong K^Y_i(Y) \otimes \mathbb{Q}$ give rise to the decomposition $\bigoplus_j K^Y_i(X)^{(j)}$ (23). In case $i = 1$ we have the decomposition $K_1(X)_{\mathbb{Q}} = \bigoplus_{j=1}^{\dim X+1} K_1(X)^{(j)}$. Of particular interest to us here is $K_1(X)^{(2)}$, which is isomorphic to Bloch’s higher Chow group $\text{CH}^2(X, 1) \otimes \mathbb{Q}$. Let $\text{NS}(X)$ be the Néron-Severi group of $X$. Then there is the natural map $\mathbb{C}^* \otimes \text{NS}(X)_{\mathbb{Q}} \rightarrow K_1(X)^{(2)}$. We denote its cokernel by $K_1^{\text{ind}}(X)^{(2)}$ and call the indecomposable $K_1$ of $X$. An element $x \in K_1(X)^{(2)}$ is called indecomposable if it is non trivial in $K_1^{\text{ind}}(X)^{(2)}$.

#### Lemma 9.3

Let $\pi : X \rightarrow C$ be a minimal elliptic surface over $\mathbb{C}$ and $U^0$ the complement of the all singular fibers. Let $\text{NS}(X)^{\prime} \subset \text{NS}(X)$ be the subgroup generated by the irreducible components of singular fibers and the section $e(C)$. Then there is an exact sequence

$$K_2(U^0)_{\mathbb{Q}} \xrightarrow{\partial} \mathbb{Q}^{\oplus s} \rightarrow K_1(X)^{(2)}/(\mathbb{C}^* \otimes \text{NS}(X)^{\prime}). \quad (9.16)$$

#### Proof.

Let $D_i$ be the multiplicative fibers and $Y_i$ the singular fibers of other types. Let $D_i = \sum_k D_i^{(k)}$ and $Y_i = \sum_k Y_i^{(k)}$ be the irreducible decompositions. Quillen’s localization exact sequence (24 Prop.(5.15)) yields an exact sequence

$$K_2(U^0)^{(2)} \rightarrow \bigoplus_i K_1(D_i)^{(1)} \oplus \bigoplus_j K_1(Y_j)^{(1)} \rightarrow K_1(X)^{(2)} \rightarrow K_2(U^0)^{(1)}. \quad (9.17)$$

On the other hand, there are exact sequences

$$\bigoplus_k K_1(D_i^{(k)}) \rightarrow K_1'(D_i) \rightarrow \mathbb{Z} \rightarrow 0, \quad (9.18)$$

$$\bigoplus_k K_1(Y_j^{(k)}) \rightarrow K_1'(Y_j) \rightarrow 0, \quad (9.19)$$

$$\bigoplus_k \mathbb{C}^* \cdot [D_i^{(k)}] \rightarrow K_1'(D_i)^{(1)} \rightarrow \mathbb{Q} \rightarrow 0, \quad (9.20)$$

$$\bigoplus_k \mathbb{C}^* \cdot [Y_j^{(k)}] \rightarrow K_1'(Y_j)^{(1)} \rightarrow 0. \quad (9.21)$$

(As for $D_i$, the above is shown in 4.2.1 However the same argument also works for $Y_j$).

Let $\text{NS}(X)^{\prime \prime} \subset \text{NS}(X)^{\prime}$ be the subgroup generated by all irreducible components of the singular fibers. Then $\text{NS}(X)^{\prime} = \text{NS}(X)^{\prime \prime} \oplus \mathbb{Z}[e(C)]$. We claim that the map

$$\mathbb{C}^* \otimes \text{NS}(X)^{\prime \prime}_{\mathbb{Q}} \rightarrow K_1(X)^{(2)}$$
is injective. In fact let \( K_1(X)^{(2)} \to K_1(D_i)^{(2)} \oplus K_1(Y_j)^{(2)} \cong \mathbb{C}^{\oplus n} \) be the pull-back. Then the composition \( \mathbb{C}^{\oplus n} \otimes \text{NS}(X)' \to K_1(X)^{(2)} \to \mathbb{C}^{\oplus n} \) is given by the intersection matrix on \( \text{NS}(X)' \). Since the intersection pairing on \( \text{NS}(X)' \) is non-degenerate (cf. proof of Lemma 7.1), it is bijective. Now a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Q}^{\oplus n+1} \\
\uparrow & & \uparrow \\
K_2(U^0)^{(2)} & \xrightarrow{\partial_2} & \bigoplus_i K'_1(D_i)^{(1)} \oplus \bigoplus_j K'_1(Y_j)^{(1)} \\
\uparrow & & \uparrow \\
\bigoplus_{i,k} \mathbb{C}^{\ast} \otimes [D_i^{(k)}] \oplus \bigoplus_{j,k} \mathbb{C}^{\ast} \otimes [Y_j^{(k)}] & \longrightarrow & \mathbb{C}^{\ast} \otimes \text{NS}(X)'' \\
\uparrow & & \uparrow \\
0 & \longrightarrow & K_1(X)^{(2)}/\mathbb{C}^{\ast} \otimes \text{NS}(X)''
\end{array}
\]

with \( \partial = \partial_1 \partial_2 \) yields an exact sequence

\[
K_2(U^0)_Q \xrightarrow{\partial} \mathbb{Q}^{\oplus n} \longrightarrow K_1(X)^{(2)}/\mathbb{C}^{\ast} \otimes \text{NS}(X)''.
\] (9.22)

The rest of the proof is to show

\[
\text{Image} \left( \bigoplus_i K'_1(D_i)^{(1)} \to K_1(X)^{(2)} \right) \cap (\mathbb{C}^{\ast} \otimes [e(C)]) = 0.
\] (9.23)

Let \( X_t \) be a smooth fiber. Let \( K_1(X)^{(2)} \to K_1(X_t)^{(2)} \) be the pull-back and \( K_1(X_t)^{(2)} \to \mathbb{C}^{\ast} \) the norm map (also called the transfer map) for \( X \to \text{Spec} \mathbb{C} \). Then the composition \( \mathbb{C}^{\ast} \otimes [e(C)] \to K_1(X)^{(2)} \to K_1(X_t)^{(2)} \to \mathbb{C}^{\ast} \) is bijective. On the other hand the composition \( K'_1(D_i) \to K_1(X)^{(2)} \to K_1(X_t)^{(2)} \) is clearly zero. This shows that the image of \( \bigoplus_i K'_1(D_i)^{(1)} \) does not intersect with \( \mathbb{C}^{\ast} \otimes [e(C)] \). Thus we have (9.23). Q.E.D.

**Corollary 9.4** Let \( \pi : X \to \mathbb{P}^1 \) be the minimal elliptic surface over \( \mathbb{C} \) defined by \( Y^2 = X^3 + X^2 + t^n \). Suppose that \( n \) is prime to 30. Then there is an exact sequence

\[
K_2(U^0)_Q \xrightarrow{\partial} \mathbb{Q}^{\oplus n+1} \longrightarrow K_1^{\text{ind}}(X)^{(2)}. \tag{9.24}
\]

**Proof.** In this case \( \text{NS}(X)'_Q = \text{NS}(X)'_Q \) (25) p.185 Example 7, (26) Example 4). Q.E.D.

By Theorem 9.4 and Corollary 9.4 we have

**Theorem 9.5** Let \( \pi : X \to \mathbb{P}^1 \) be the minimal elliptic surface over \( \mathbb{C} \) defined by \( Y^2 = X^3 + X^2 + t^n \) and \( D_i \) the multiplicative fibers. Suppose that \( n \) is a prime number with \( 7 \leq n \leq 29 \). Then we have

\[
\text{dim Image} \left( \bigoplus_{i=0}^n K'_1(D_i)_Q \to K_1^{\text{ind}}(X)^{(2)} \right) = n - 1.
\]

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References

[1] M. Asakura: Surjectivity of $p$-adic regulators on $K_2$ of Tate curves. to appear in Invent. Math.

[2] A. Beilinson: Higher regulators and values of $L$-functions. J. Soviet Math. 30 (1985), 2036–2070

[3] ———: Higher regulators of modular curves. In Applications of Algebraic $K$-theory to Algebraic Geometry and Number theory. Contemp. Math. 55 (1986), 1–34.

[4] S. Bloch: A note on Gersten’s conjecture in the mixed characteristic case. In Applications of algebraic $K$-theory to algebraic geometry and number theory. Contemp. Math. 55-1. (1986), 75–78.

[5] P. Deligne: Théorie de Hodge II, III. Publ. Math. IHES, 40 (1972) 5–57, 44 (1974) 5–77.

[6] P. Deligne and M. Rapoport: Les schémas de modules de courbes elliptiques. Modular functions of one variable, II, pp. 143–316. Lecture Notes in Math., Vol. 349, Springer, 1973.

[7] F. Diamond and J. Shurman: A first course in modular forms. Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005.

[8] H. Gillet: Riemann-Roch theorem for higher Algebraic $K$-theory. Adv. Math. 40 (1981), 203–289.

[9] D. Grayson: Higher algebraic $K$-theory. II (after D. Quillen). pp. 217–240. Lecture Notes in Math. Vol. 551, Springer, 1976.

[10] U. Jannsen: Mixed Motives and Algebraic $K$-theory. Lecture note in Math. 1400, Springer, 1990.

[11] van der Kallen: Generators and relations in algebraic $K$-theory. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305–310, Acad. Sci. Fennica, Helsinki, 1980.

[12] ———: The $K_2$ of rings with many units. Ann. Sci. ÉcoleNorm. Sup. (4), 10 (1977) No.4, 473–515.

[13] K. Kato: A generalization of local class field theory by using $K$-groups II. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 3, 603–683.

[14] ———: The explicit reciprocity law and the cohomology of Fontaine-Messing. Bull. Soc. Math. France 119 (1991), no. 4, 397–441.

[15] A.S. Merkur’ev and A.A. Suslin: $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism. Math. USSR Izv. 21 No.2 (1983), 307–340.
[16] J. Milnor: Introduction to Algebraic $K$-theory. Ann. of Math Studies 72, Princeton 1970.

[17] D. Quillen: Higher algebraic $K$-theory. I. pp. 85–147. Lecture Notes in Math. Vol. 341, Springer, Berlin 1973.

[18] P. Schneider: Introduction to the Beilinson conjectures. In Beilinson’s Conjectures on Special Values of $L$-Functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math. Vol. 4, 1–35, 1988.

[19] N. Schappacher and A. J. Scholl: The boundary of the Eisenstein symbol. Math. Ann. 290 (1991), 419–430.

[20] G. Shimura: Introduction to the arithmetic theory of automorphic functions. Publ. Math. Soc. Japan, 11, 1971. Princeton University Press, 1994.

[21] J. Silverman: Advanced Topics in the Arithmetic of Elliptic curves. GTM 151, Springer, 1994.

[22] C. Soulé: $K$-théorie des anneaux d’entiers de corps de nombres et cohomologie étale. Invent. Math. 55 (1979) pp. 251–295.

[23] ———: Opération en $K$-théorie algébrique. Canad. J. Math. 37 (1985) no. 3 488–550.

[24] V. Srinivas: Algebraic $K$-theory. Progress in Math. Vol 90 (1991), Birkhäuser.

[25] P. Stiller: Automorphic forms and the Picard number of an elliptic surface. Aspects of Mathematics, E5. (1984), Vieweg.

[26] ———: The Picard numbers of elliptic surfaces with many symmetries. Pacific J. Math. 128 (1987) no. 1 157–189.

[27] A. Suslin: Algebraic $K$-theory and the norm residue homomorphism. J. Soviet Math. 30 (1985), 2556–2611.

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