CONCENTRATIONS IN KINETIC TRANSPORT EQUATIONS AND HYPOELLIPTICITY

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ABSTRACT. We establish improved hypoelliptic estimates on the solutions of kinetic transport equations, using a suitable decomposition of the phase space. Our main result shows that the relative compactness in all variables of a bounded family \( f_\lambda(x,v) \in L^p \) satisfying some appropriate transport relation

\[ v \cdot \nabla_x f_\lambda = (1 - \Delta_x)^{\frac{\beta}{2}} (1 - \Delta_v)^{\frac{\alpha}{2}} g_\lambda \]

may be inferred solely from its compactness in \( v \).

This method is introduced as an alternative to the lack of known suitable averaging lemmas in \( L^1 \) when the right-hand side of the transport equation has very low regularity, due to an external force field for instance. In a forthcoming work, the authors make a crucial application of this new approach to the study of the hydrodynamic limit of the Boltzmann equation with a rough force field [4].

Our main objective in this paper is to obtain strong compactness on the moments of the solutions to some kinetic transport equation with rough source terms:

\[ \partial_t f + v \cdot \nabla_x f = g. \]

Typically we are interested in the transport in some \( L^p \) force field \( F \) such as gravity or electrostatic force, for which characteristics are not even defined

\[ \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q. \]

This problem has been investigated by a number of authors: the first section of the present paper attempts to present the different methods which have been used. The specificity of our approach is to deal with distributions \( f \) having little integrability, ideally we would like to consider \( L^1 \) functions. Our main results are given in the second section, they however do not recover the \( L^1 \) limiting case. Actually, in that case, we are not even able to state a conjecture regarding strong compactness. Examples and counterexamples are given in the last section to enlighten the difficulty.

The results we establish here are particularly convenient for the study of hydrodynamic limits of the Boltzmann equation, insofar as the density fluctuation is known to be a little bit regular with respect to the velocity variable \( v \), while the source term \( Q \) in the transport equation is not better than \( L^1 \) and the force field \( F \) is typically in \( L^2 \). We will therefore give in a forthcoming paper [4] a derivation of the Navier-Stokes equations with external (possibly self-induced) force field from the Boltzmann equation. We further point out that our results include the averaging lemmas that are needed both for cutoff and non-cutoff cross sections.

\textbf{Date:} May 11, 2010.

D. Arsénilo would like to gracefully acknowledge the support from the foundation Sciences Mathématiques de Paris during the genesis of this work.
1. STATE OF THE ART

1.1. Classical averaging theory. The main idea behind velocity averaging lemma is that the symbol of the free transport is essentially elliptic, and more precisely that it is elliptic outside from a small zone of the velocity space which provides small contributions to averages.

Basic results due to Agoshkov [1] and Golse, Lions, Perthame and Sentis [13] are then proved using the Fourier transform in $L^2$ and some classical interpolation arguments.

**Theorem 1.1 (13).** Let $1 < p < +\infty$, and $f, g \in L^p(dxdv)$ be such that

$$v \cdot \nabla_x f = g.$$  

Then, for all $\varphi \in C^\infty_c(\mathbb{R}^D)$,

$$\left\| \int f \varphi \, dv \right\|_{W^{s,p}(dx)} \leq C(\varphi) \left( \left\| f \right\|_{L^p(dx,dv)} + \left\| g \right\|_{L^p(dx,dv)} \right),$$  

with $s = \inf(1/p, 1/p')$.

A first extension can be obtained using some generalization of the Fourier transform, namely some dyadic decomposition, referred to as Littlewood-Paley decomposition. Besov spaces are then the natural functional spaces to express the gain of regularity:

$$B^{s,p}_{q} = \left\{ f \in \mathcal{S}' : \left(2^j \left\| \Delta_j f \right\|_{L^p} \right) \in l^q(\mathbb{N}) \right\}$$  

where $\Delta_j$ is some localization in Fourier space at frequencies of order $2^j$.

These techniques, going back to DiPerna, Lions and Meyer [11] and refined by Bézard [7], allow to account for source terms involving derivatives in $v$ and fractional derivatives in $x$.

**Theorem 1.2 (11).** Let $1 < p \leq 2$ and $f, g \in L^p(dxdv)$ be such that

$$v \cdot \nabla_x f = (1 - \Delta_x)^{\tau/2}(1 - \Delta_v)^{m/2} g$$  

for some $m \in \mathbb{R}^+$, $\tau \in [0,1]$.

Then, for all $\varphi \in C^\infty_c(\mathbb{R}^D)$,

$$\left\| \int f \varphi \, dv \right\|_{B^{s,p}_{2}(dx)} \leq C(\varphi) \left( \left\| f \right\|_{L^p(dx,dv)} + \left\| g \right\|_{L^p(dx,dv)} \right),$$  

with $s = (1 - \tau)/ p(1 + m)$.

Actually one can prove a slightly more precise result with Sobolev spaces instead of Besov spaces (see [7]) : the method remains essentially the same but uses a refined interpolation argument involving Hardy spaces instead of $L^1$.

1.2. Characteristics and dispersion. Another crucial property of the free transport is the propagation along characteristics

$$X(t,x,v) = x - tv, \quad V(t,x,v) = v,$$

responsible in particular for the dispersive behavior of the solutions. By “dispersive”, we mean here
(i) a global effect, related to the fact that mass goes to infinity, and leading to some Strichartz estimates;
(ii) a local effect giving some gain of integrability which can be seen as a local smoothing property;
(iii) a mixing property, expressing a kind of duality between $x$ and $v$ variables.

All these properties are of course connected, but they do not have the same stability with respect to perturbations (change of spatial domain, introduction of force fields, . . . ). For instance (i) cannot hold in bounded domains.

A systematic study of these dispersive behaviors in non flat geometries has been carried out in [21]. For our purpose, we will retain only two important features.

The mixing property (iii) is the crucial tool which allows to establish $L^1$ averaging lemma under the one and only assumption of integrability with respect to $v$ variables.

**Theorem 1.3** ([14]). Let $(f_\lambda)$ and $(g_\lambda)$ be bounded families of $L^1(dxdv)$ such that $(f_\lambda)$ is locally equiintegrable with respect to $v$ and
\[ v \cdot \nabla_x f_\lambda = g_\lambda. \]

Then, for all $\varphi \in C_c^\infty (\mathbb{R}^D)$,
\[ \left( \int f_\lambda \varphi \, dv \right) \quad \text{is relatively strongly compact in } L^1_{loc}(dx). \]

The proof is based on averaging lemma in weakly compact subsets of $L^1$ [13], together with the Dunford-Pettis criterion giving the weak compactness of equiintegrable families. Getting this equiintegrability relies on some duality method, coupled with the dispersion inequality established by Castella and Perthame [10]
\[ \|f_0(X,V)\|_{L^\infty(dx,L^1(dv))} \leq t^{-D} \|f_0\|_{L^1(dx,L^\infty(dv))}. \]

(1.2)

A generalization can be obtained for transport in some smooth force field, provided that characteristics can be locally defined and have the same local mixing property.

**Theorem 1.4** ([15]). Let $F \in L^2 \cap W^{1,\infty}(dx)$ be some given force field. Let $(f_\lambda)$ and $(g_\lambda)$ be bounded families of $L^1(dxdv)$ such that $(f_\lambda)$ is locally equiintegrable with respect to $v$ and
\[ v \cdot \nabla_x f_\lambda + F \cdot \nabla_v f_\lambda = g_\lambda. \]

Then, for all $\varphi \in C_c^\infty (\mathbb{R}^D)$,
\[ \left( \int f_\lambda \varphi \, dv \right) \quad \text{is relatively strongly compact in } L^1_{loc}(dx). \]

In the case when there is no regularity on the force field, combining the mixing property and some suitable dyadic decomposition, Nader Masmoudi and the first author have obtained in [2, 3] an averaging result assuming that the equiintegrability in $v$ can be quantified in some Besov space (which, in particular, is a subspace of the local Hardy space).
Theorem 1.5 ([2, 3]). Let \((f_\lambda)\) and \((g_\lambda)\) be bounded families of \(L^1(\text{d}x\text{d}v)\) such that \((f_\lambda)\) is nonnegative, uniformly bounded in \(L^1(\text{d}x, B_1^0(\text{d}v))\) and
\[
v \cdot \nabla_x f_\lambda = (1 - \Delta_v)\gamma g_\lambda \quad \text{for some } \gamma \in \mathbb{R}.
\]
Then \((f_\lambda)\) is equiintegrable (with respect to all variables). In particular, if \(v \cdot \nabla_x f_\lambda + F \cdot \nabla_v f_\lambda = g_\lambda\) for some force field \(F \in L^\infty(\text{d}x)\), then, for all \(\varphi \in C^\infty_0(\mathbb{R}^D)\),
\[
\left( \int f_\lambda \varphi \, \text{d}v \right) \text{ is relatively strongly compact in } L^1_{\text{loc}}(\text{d}x).
\]

Integration along characteristics has been also used in other ways, for instance - by Lions and Perthame [19] to obtain a suitable representation of \(\rho(t, x) = \int f(t, x, v) \, \text{d}v\) and deduce some regularity on the electric field of the Vlasov-Poisson equation; - by Jabin and Vega [18]; - by Berthelin and Junca [6] to get \(L^2\) averaging lemma with optimal regularity in smooth force fields.

1.3. Hypoellipticity and global regularity. Another way to express the duality between \(x\) and \(v\) variables is referred to as hypoellipticity, and can be formulated in terms of commutators
\[
[v \cdot \nabla_x, \nabla_v] = \nabla_x.
\]
We then expect the regularity with respect to \(v\) to be transferred on \(x\).

The systematic study of hypoelliptic operators goes back to Hörmander [17], who considered in particular the kinetic Fokker-Planck equation. The approach based on commutator identities has then been developed by Rotschild and Stein [20].

Theorem 1.6 ([20]). Let \(f, g \in L^2(\text{d}t\text{d}x\text{d}v)\) be such that
\[
\partial_t f + v \cdot \nabla_x f - \Delta_v f = g.
\]
Then,
\[
\|\Delta_v f\|_{L^2(\text{d}t\text{d}x\text{d}v)} + \left\|(-\Delta_x)^{1/3} f\right\|_{L^2(\text{d}t\text{d}x\text{d}v)} \leq C \left( \|f\|_{L^2(\text{d}t\text{d}x\text{d}v)} + \|g\|_{L^2(\text{d}t\text{d}x\text{d}v)} \right).
\]

A more general statement regarding the global regularity of the solutions to transport equations has been obtained by Bouchut [8].

Theorem 1.7 ([8]). Let \(f, g \in L^2(\text{d}x\text{d}v)\) be such that
\[
v \cdot \nabla_x f - \Delta_v f = g
\]
and
\[
(-\Delta_v)^{\max(\beta, \gamma)/2} f, (-\Delta_v)^{\gamma/2} g \in L^2(\text{d}x\text{d}v)
\]
with \(\gamma \geq 0\) and \(0 \leq 1 - \gamma \leq \beta\).

Then,
\[
\left\|(-\Delta_x)^{s/2} f\right\|_{L^2} \leq C \left( \left\|(-\Delta_v)^{\max(\beta, \gamma)/2} f\right\|_{L^2} + \left\|(-\Delta_v)^{\gamma/2} g\right\|_{L^2} \right),
\]
with \(s = \beta / (1 - \gamma + \beta)\).
Such techniques are not easy to extend to the $L^1$ context because of the lack of $L^1$ continuity of pseudodifferential operators of order 0. Furthermore, they do not allow to account for derivatives in $v$, except for elliptic terms such as $\Delta_v f$.

2. Main results

In this work, we intend to use a combination of the various techniques presented above in order to improve the compactness statement for rough source terms and distributions with little integrability.

2.1. Transfer of compactness in $L^p$. Recall that, for any $1 \leq p < \infty$, the Riesz-Frèchet-Kolmogorov compactness criterion (see [9, 26]) asserts that a family $\{f_\lambda(x,v)\}_{\lambda \in \Lambda} \subset L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ is relatively compact in the strong topology of $L^p$ if and only if

(i) it is uniformly bounded, i.e. $\sup_{\lambda \in \Lambda} \|f_\lambda\|_{L^p} < \infty$,
(ii) it is uniformly norm-continuous, i.e.

$$\limsup_{\delta \to 0} \sup_{\lambda \in \Lambda} \|f_\lambda(x+h,v+l) - f_\lambda(x,v)\|_{L^p} = 0,$$

(iii) it is tight, i.e. $\lim_{R \to \infty} \sup_{\lambda \in \Lambda} \left\| f_\lambda 1_{\{|x|+|v|>R\}} \right\|_{L^p} = 0$.

Furthermore, if the tightness of the bounded family is not known to hold, it is still possible to deduce its local relative compactness, i.e. the relative compactness of the family in $L^p(K)$ for any compact subset $K \subset \mathbb{R}_x^D \times \mathbb{R}_v^D$, from the mere knowledge of (2.1). Thus, it is precisely condition (2.1) that guarantees that oscillations and concentrations cannot happen and we say in this case that the $f_\lambda$’s are locally relatively compact. This motivates the following definition.

Definition. For any $1 \leq p < \infty$, a bounded family $\{f_\lambda(x,v)\}_{\lambda \in \Lambda} \subset L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ is said to be locally relatively compact in $v$ if and only if it satisfies

$$\limsup_{\delta \to 0} \sup_{\lambda \in \Lambda} |x|<\delta \|f_\lambda(x,v+l) - f_\lambda(x,v)\|_{L^p} = 0.$$

Our main result is the following transfer of compactness in $L^p$ for $1 < p < +\infty$.

Theorem 2.1. Let the bounded family $\{f_\lambda(x,v)\}_{\lambda \in \Lambda} \subset L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, for some $1 < p < \infty$, be locally relatively compact in $v$ and such that

$$v \cdot \nabla_x f_\lambda(x,v) = (1 - \Delta_x)^{\frac{\beta}{2}} (1 - \Delta_v)^{\frac{\alpha}{2}} g_\lambda(x,v),$$

for all $\lambda \in \Lambda$ and for some bounded family $\{g_\lambda(x,v)\}_{\lambda \in \Lambda} \subset L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Then, the collection $\{f_\lambda(x,v)\}_{\lambda \in \Lambda}$ is locally relatively compact in $L^p(\mathbb{R}_x^D \times \mathbb{R}_v^D)$ (in all variables).

As an easy consequence of this result, we obtain a criterion of non-concentration in $L^p$.

Corollary 2.2. Let $\{|f_\lambda(x,v)|^p\}_{\lambda \in \Lambda}$ (with $1 < p < \infty$) be a bounded family of $L^1(\mathbb{R}_x^D \times \mathbb{R}_v^D)$, locally relatively compact in $v$ and such that

$$v \cdot \nabla_x f_\lambda(x,v)$$

is uniformly bounded in $L^p(\mathbb{R}_x^D, W^{-k,p}(\mathbb{R}_v^D))$ for some $k \geq 0$.

Then, the collection $\{|f_\lambda(x,v)|^p\}_{\lambda \in \Lambda}$ is locally uniformly equiintegrable (with respect to all variables).
Note that this is not really the transfer of equiintegrability we would like to establish by analogy with Theorem 1.3. As we will see in the last section, there is actually an obstruction for such a transfer of weak \( L^1 \) compactness.

Regarding the transfer of strong \( L^1 \) compactness, we have no result at the present time: our proof which is based on Fourier multipliers estimates fails in \( L^1 \), and we have not found - for the moment - any alternative to investigate that case.

2.2. Strategy of the proof. The proof of theorem 2.1 is contained in section 3.5. The argument we will use to establish \( L^p \) compactness is rather simple insofar as it does not involve neither dyadic decompositions and complicated functional spaces, such as Besov spaces defined with such a decomposition, nor interpolation between functional spaces.

It is based on an explicit decomposition which relies on a good understanding of the properties of the transport operator at the microlocal level. More precisely, we will identify two different behaviors of the transport operator, namely an elliptic component with regularizing properties, and an hypoelliptic component with mixing properties.

The main technical tool is the theory of Fourier multipliers that we will recall briefly in the next section.

3. Hypoellipticity and ellipticity in \( L^p \)

We will employ the standard notation \( \langle z \rangle = \left(1 + |z|^2\right)^{\frac{1}{2}} \), valid for any vector \( z \) in any Euclidean space of any dimension. As usual, we will use indices to emphasize specific dependences of constants and will denote generic constants that depend solely on fixed parameters by \( C \).

Sections 3.1, 3.2 and 3.3 are a preparation to the main theorems presented in sections 3.4 and 3.5.

3.1. Fourier multipliers. Since we are going to deal with functions of two \( D \)-dimensional variables, namely the space \( x \in \mathbb{R}^D \) and velocity \( v \in \mathbb{R}^D \) variables, we wish to clarify now the notation on Fourier analysis that we will employ. Firstly, we will systematically denote by \( \eta \in \mathbb{R}^D \) and \( \xi \in \mathbb{R}^D \) the respective dual variables of \( x \in \mathbb{R}^D \) and \( v \in \mathbb{R}^D \). The Fourier transform in all variables of \( f \in \mathcal{S}(\mathbb{R}^D \times \mathbb{R}^D) \) (where \( \mathcal{S} \) denotes the Schwartz space of rapidly decaying functions) is defined by

\[
\mathcal{F} f(\eta, \xi) = \int_{\mathbb{R}^D \times \mathbb{R}^D} e^{-i(x \cdot \eta + v \cdot \xi)} f(x, v) \, dx \, dv
\]

and its inverse is given by

\[
f(x, v) = \mathcal{F}^{-1} f(\eta, \xi) = \frac{1}{(2\pi)^{2D}} \int_{\mathbb{R}^D \times \mathbb{R}^D} e^{i(x \cdot \eta + v \cdot \xi)} \mathcal{F} f(\eta, \xi) \, d\eta \, d\xi.
\]

We will sometimes use the Fourier transforms with respect to \( x \) or \( v \) only. It this case, we will utilize the obvious notations \( \mathcal{F}_x, \mathcal{F}^{-1}_x, \mathcal{F}_v \) and \( \mathcal{F}^{-1}_v \). For convenience, we will also employ \( \hat{f} \) to denote the Fourier transform when no ambiguity with respect to the use of variables is to be feared.

We recall now the definition of Fourier multipliers and some of their basic facts. The reader may also consult [5, 16] for a clear treatment of the subject. A
tempered distribution \( \rho(\eta, \xi) \in S' (\mathbb{R}^D \times \mathbb{R}^D) \) is called a Fourier multiplier on \( L^p (\mathbb{R}^D \times \mathbb{R}^D) \), for some \( 1 \leq p \leq \infty \), if the mapping

\[
(3.3) \quad f(x, v) \mapsto \left( \mathcal{F}^{-1} \rho \right) * f(x, v) = \mathcal{F}^{-1} \rho \mathcal{F} f(x, v),
\]

well-defined for all \( f \in S (\mathbb{R}^D \times \mathbb{R}^D) \), can be extended into a bounded operator over \( L^p (\mathbb{R}^D \times \mathbb{R}^D) \). The Banach space of Fourier multipliers is denoted by \( M^p (\mathbb{R}^D \times \mathbb{R}^D) \) and is endowed with the norm

\[
(3.4) \quad \| \rho \|_{M^p} = \sup_{\| f \|_{L^p} = 1} \left\| \left( \mathcal{F}^{-1} \rho \right) * f \right\|_{L^p},
\]

which is merely the operator norm of the mapping (3.3).

It is well-known (see [5]) that the Fourier multiplier norms are invariant with respect to bijective affine transformations. That is, considering any affine transformation \( a : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}^D \times \mathbb{R}^D \), it holds that

\[
\| \rho( a(\eta, \xi)) \|_{M^p} = \| \rho(\eta, \xi) \|_{M^p}
\]

provided \( a \) is bijective. In particular, this includes all the dilations, even the coordinatewise dilations, and the rotations. This invariance proves to be a very handy tool when dealing with Fourier multipliers.

It turns out that it is fairly easy to obtain the relations

\[
(3.5) \quad M^p = M^p' \quad \text{and} \quad \mathcal{F} M = M^1 \subset M^p \subset M^2 = L^\infty,
\]

where \( \mathcal{M} \) denotes the space of bounded Borel regular measures. However, further characterizing \( M^p \) for \( p \neq 1, 2, \infty \) proves to be a much more difficult task. Fortunately, we have the following theorem, which comprises important tools from Fourier analysis that are commonly used to establish the boundedness of multipliers.

**Theorem 3.1 (Multiplier theorem).** Let \( m(\eta) : \mathbb{R}^D \to \mathbb{R} \) be such that the Hörmander-Mikhlin criterion

\[
(3.6) \quad \sum_{|\alpha| \leq D + 1} \sup_{0 < R < \infty} R^{-D} \int_{R < |\eta| < 2R} \left| |\eta|^{-\alpha} \partial_\eta^\alpha m(\eta) \right|^2 d\eta \leq A
\]

is satisfied, or such that the Marcinkiewicz-Mikhlin criterion

\[
(3.7) \quad \sum_{\alpha \in \{0, 1\}^D} \sup_{\eta \in \mathbb{R}^D} \left| \eta^\alpha \partial_\eta^\alpha m(\eta) \right| \leq A
\]

is satisfied.

Then, for all \( 1 < p < \infty \), the function \( m(\eta) \) is a Fourier multiplier over \( L^p (\mathbb{R}^D) \), i.e. \( m(\eta) \in M^p (\mathbb{R}^D) \), and satisfies

\[
(3.8) \quad \| m(\eta) \|_{M^p} \leq C_p A,
\]

where \( C_p > 0 \) is a finite constant that only depends on \( p \).

The sufficiency of the Hörmander-Mikhlin condition (3.6) was shown in [16], while an even weaker version of the Marcinkiewicz-Mikhlin condition (3.7) is to be found in [22, p. 109, Theorem 6'].

Notice that the Hörmander-Mikhlin and the Marcinkiewicz-Mikhlin criteria in the above theorem are neither inclusive nor disjoint. In the sequel, we will make
critical use of the Marcinkiewicz-Mikhlin condition because it allows for relatively nasty behavior of the multiplier near the coordinate axes.

3.2. Relative strong compactness in Lebesgue spaces. We have the following convenient characterization of local relative compactness in the Fourier space.

**Proposition 3.2.** Let $1 \leq p < \infty$ and consider a fixed truncation $\chi(r) \in C^\infty_c(\mathbb{R})$, such that $\mathbb{1}_{\{|r| \leq \frac{1}{2}\}} \leq \chi(r) \leq \mathbb{1}_{\{|r| \leq 1\}}$, say.

Then, a bounded family $\{f_{\lambda}(x, v)\}_{\lambda \in \Lambda} \subset L^p(\mathbb{R}^D \times \mathbb{R}^D)$ is locally relatively compact if and only if

$$\lim_{R \to \infty, \lambda \in \Lambda} \left\| F^{-1} (1 - \chi) \left( \frac{\sqrt{|\eta|^2 + |\xi|^2}}{R} \right) Ff_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} = 0. \tag{3.9}$$

Furthermore, it is locally relatively compact in $v$ if and only if

$$\lim_{R \to 0, \lambda \in \Lambda} \left\| F^{-1}_v (1 - \chi) \left( \frac{|\xi|}{R} \right) F_v f_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} = 0. \tag{3.10}$$

**Proof.** We will only show that the local relative compactness in $v$ is equivalent to \(3.10\). It will be clear from our rather standard arguments that the equivalence between local relative compactness in all variables and \(3.9\) holds true.

Thus, we first suppose that the $f_{\lambda}$’s are locally relatively compact in $v$ and we estimate, for any approximate identity $\rho_\delta(l) = \frac{1}{\delta^p} \rho \left( \frac{l}{\delta} \right)$, where $\delta > 0$ and $\rho \in C^\infty_c(\mathbb{R}^D)$ is non-negative, supported in the unit ball, and such that $\int_{\mathbb{R}^D} \rho \, dl = 1$,

$$|\rho_\delta \ast_v f_{\lambda}(x, v) - f_{\lambda}(x, v)|^p = \left| \int_{\mathbb{R}^D} (f_{\lambda}(x, v-l) - f_{\lambda}(x, v)) \rho_\delta(l) \, dl \right|^p \leq \int_{\mathbb{R}^D} |f_{\lambda}(x, v-l) - f_{\lambda}(x, v)|^p \rho_\delta(l) \, dl. \tag{3.11}$$

Consequently, integrating in all variables, we deduce

$$\lim_{\delta \to 0, \lambda \in \Lambda} \|\rho_\delta \ast_v f_{\lambda} - f_{\lambda}\|_{L^p} \leq \limsup_{\delta \to 0, \lambda \in \Lambda} \|f_{\lambda}(x, v+l) - f_{\lambda}(x, v)\|_{L^p} = 0. \tag{3.12}$$

Thus, we only have to show that \(3.10\) holds for the velocity regularizations $\rho_\delta \ast_v f_{\lambda}$ for fixed $\delta > 0$, which is easily performed recalling that $F_v(\rho_\delta \ast_v f_{\lambda})(\xi) = \hat{\rho}(\delta \xi) F_v f_{\lambda}(\xi)$ and then noticing that the multiplier norm satisfies

$$\left\| (1 - \chi)( \frac{|\xi|}{R} ) \hat{\rho}(\delta \xi) \right\|_{M^p(\mathbb{R}^D)} = \left\| (1 - \chi)( |\xi| ) \hat{\rho}(R \delta \xi) \right\|_{M^p(\mathbb{R}^D)} \leq \left\| F^{-1}_v (1 - \chi)( |\xi| ) \hat{\rho}(R \delta \xi) \right\|_{L^1(\mathbb{R}^D)} \to 0, \quad \text{as } R \to 0, \tag{3.13}$$

for $\delta$ is fixed and $\hat{\rho}$ decays rapidly. This conclude the proof of the necessity of \(3.10\).

In order to demonstrate its sufficiency, we suppose that the $f_{\lambda}$’s satisfy \(3.10\), from which we infer that the mollifications in velocity $\rho_\delta \ast_v f_{\lambda}$, where $\hat{\rho}(\xi) = \chi(|\xi|)$, constitute uniform approximations of the $f_{\lambda}$’s for small values of $\delta > 0$. It is then enough to notice, for any fixed $\delta > 0$, that these velocity regularizations are locally relatively compact in $v$ because $\rho$ is rapidly decaying. This concludes the justification of the proposition. \(\square\)
Notice that, when \( p = 2 \) in the preceding proposition, one may replace the
cutoff function \( \chi(r) \) by the characteristic function \( \mathbb{1}_{[-1,1]}(r) \) without changing its
statement. This is a straightforward consequence of the equivalence \( M^2 = L^\infty \),
which follows from Plancherel’s theorem. When \( p \neq 2 \), this is plainly wrong in view of Fefferman’s multiplier result for the ball [12] (even though it would still
be true for certain classes of characteristic functions). Thus, in the \( L^2 \) setting of
section 3.4, we will utilize proposition 3.2 with characteristic functions for sim-
licity, while we will stick to mollified characteristic functions when dealing with
the general \( L^p \) setting of section 3.5.

3.3. Hypoellipticity via transport of frequencies. We wish now to explain how
the mechanism of hypoellipticity in the transport equation can be interpreted via
the transport of frequencies. Even though this remains rather elementary, its un-
derstanding provides the key to the proofs of the main results below and gives a
very explicit picture of the transfer mechanisms driving hypoellipticity in kinetic
transport equations.

Thus, let us suppose that, for suitable collections \( \{f_\lambda\}_{\lambda \in \Lambda} \) and \( \{g_\lambda\}_{\lambda \in \Lambda'} \),

\[
\eta \cdot \nabla_x f_\lambda(x, v) = (1 - \Delta_x)_{\beta}^\Lambda (1 - \Delta_v)^{\frac{3}{2}} g_\lambda(x, v). 
\]

Then, introducing an interpolation parameter \( t \geq 0 \), it trivially holds that

\[
\begin{aligned}
\left\{ (\partial_t + \eta \cdot \nabla_x) f_\lambda \right\} & = (1 - \Delta_x)_{\beta}^\Lambda (1 - \Delta_v)^{\frac{3}{2}} g_\lambda, \\
\left\{ f_\lambda(t = 0) \right\} & = f_\lambda.
\end{aligned}
\]

Hence the interpolation formula, for any \( t \geq 0 \),

\[
f_\lambda(x, v) = f_\lambda(x - tv, v) + \int_0^t (1 - \Delta_x)_{\beta}^\Lambda (1 - \Delta_v)^{\frac{3}{2}} g_\lambda(x - sv, v) \, ds,
\]

which is merely Duhamel’s representation formula for the linear equation (3.15). The use of this representation formula is a key idea in order to understand the
transfer phenomena in kinetic transport equations (see [2, 13, 14]).

It is the analysis of an analog representation formula in Fourier variables that
provides us with the main tool for understanding hypoellipticity. Thus, it is readily seen that, when expressing the transport equation (3.14) in Fourier variables,
one obtains the dual transport equation

\[
\eta \cdot \nabla_\xi \mathcal{F} f_\lambda(\eta, \xi) = -\langle \eta \rangle^\beta \langle \xi \rangle^\alpha \mathcal{F} g_\lambda(\eta, \xi).
\]

In particular, the interpolation formula (3.16) is still valid for the above transport
relation, which yields, for any \( t \geq 0 \),

\[
\mathcal{F} f_\lambda(\eta, \xi) = \mathcal{F} f_\lambda(\eta, \xi - t\eta) - \int_0^t \langle \eta \rangle^\beta \langle \xi - s\eta \rangle^\alpha \mathcal{F} g_\lambda(\eta, \xi - s\eta) \, ds.
\]

Notice that it is possible to let the parameter \( t \) depend on the Fourier variables
\( \eta \) and \( \xi \), and even on other parameters such as the physical variables \( x \) and \( v \),
since these are fixed for the moment. Consequently, applying the inverse Fourier
transform and using elementary changes of variables, we deduce the general rep-
resentation formula

\[
\mathcal{F}^{-1} p(x, v, \eta, \xi) \mathcal{F} f_\lambda(x, v) = \mathcal{F}^{-1} e^{itv \cdot \eta} p(x, v, \eta, \xi + t\eta) \mathcal{F} f_\lambda(x, v)
\]

\[
- \int_0^1 \mathcal{F}^{-1} e^{istv \cdot \eta} p(x, v, \eta, \xi + st\eta) t \langle \eta \rangle^\beta \langle \xi \rangle^\alpha \mathcal{F} g_\lambda(x, v) \, ds,
\]

which follows from Plancherel’s theorem. When \( p \neq 2 \), this is plainly wrong in view of Fefferman’s multiplier result for the ball [12] (even though it would still
be true for certain classes of characteristic functions). Thus, in the \( L^2 \) setting of
section 3.4, we will utilize proposition 3.2 with characteristic functions for sim-
licity, while we will stick to mollified characteristic functions when dealing with
the general \( L^p \) setting of section 3.5.
for any appropriate symbol $p(x,v,\eta,\zeta)$. Again, we insist that, in the above formula, the parameter $t$ may be a function of all the physical and the Fourier variables, i.e. $t = t(x,v,\eta,\zeta)$. This suggests that the use of the theory of pseudo-differential operators (see [23, 24, 25], for instance) will be necessary. Fortunately, the specific structure of the symbols we will utilize will allow us to treat them more simply as multipliers with the standard theorems from Fourier analysis, already presented in section 3.1, which will yield sharper results.

The above formula (3.19) is at the heart of our results as it efficiently embodies hypoellipticity. Indeed, supposing for simplicity that the symbol $p(\eta,\zeta)$ only depends on the Fourier variables, and that it is supported inside $\{|\eta| > R, |\zeta| \leq K\}$, for large values of $R, K > 0$, then it is obvious that $p(\eta,\zeta + t\eta)$ is supported inside $\{|\eta| > R, |\zeta + t\eta| \leq K\}$. Since $|\zeta| \geq t|\eta| - |\zeta + t\eta|$, we deduce, as illustrated by the figure 1, that $p(\eta,\zeta + t\eta)$ is then supported inside $\{|\zeta| > tR - K\}$, which includes solely large values of $\zeta$ for suitable choices of parameters $t$, $R$ and $K$.

Consequently, we see that the control of large space frequencies, i.e. large values of $\eta$, may be deduced from the behavior of large velocity frequencies, i.e. large values of $\zeta$. This transfer is precisely the expression of hypoellipticity in the kinetic transport operator.

3.4. Transfer of compactness in $L^2$. Before presenting our general result on the transfer of compactness in the $L^p$ setting, we analyse it in the much simpler and insightful $L^2$ setting. Indeed, we find its proof quite illuminating on the transfer of compactness.

**Theorem 3.3.** Let the bounded family $\{f_\lambda(x,v)\}_{\lambda \in \Lambda} \subset L^2(\mathbb{R}^D_x \times \mathbb{R}^D_v)$ be locally relatively compact in $v$ and such that

(3.20) \[ v \cdot \nabla_x f_\lambda(x,v) = (1 - \Delta_x)^{\frac{\alpha}{2}} (1 - \Delta_v)^{\frac{\beta}{2}} g_\lambda(x,v), \]
for all \( \lambda \in \Lambda \) and for some bounded family \( \{ g_\lambda(x,v) \} \) in \( L^2(\mathbb{R}^D_x \times \mathbb{R}^D_v) \), where \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \). Then, the collection \( \{ f_\lambda(x,v) \} \) is locally relatively compact in \( L^2(\mathbb{R}^D_x \times \mathbb{R}^D_v) \) (in all variables).

**Proof.** In view of proposition 3.2, it is sufficient, in order to obtain the local relative strong compactness of the \( f_\lambda \)'s, to show that the \( L^2 \) norm of

\[
F^{-1}\chi_{\{|\eta| > R\text{ or } |\xi| > K\}} Ff_\lambda,
\]

i.e. the high frequencies of \( f_\lambda \), can be made arbitrarily small, uniformly in \( \lambda \in \Lambda \), for suitably chosen generally large parameters \( R, K > 0 \). Since one clearly has the disjoint decomposition

\[
\{ |\eta| > R \text{ or } |\xi| > K \} = \{ |\xi| > K \} \cup \{ |\eta| > R \text{ and } |\xi| \leq K \},
\]

it is then enough to exhibit a uniformly small control, for large values of \( R > 1 \), of the \( L^2 \) norm of

\[
F^{-1}\chi_{\{|\eta| > R \text{ and } |\xi| \leq K(R)\}} Ff_\lambda,
\]

where \( \lim_{R \to \infty} K(R) = \infty \), for the local relative compactness in \( v \) of the \( f_\lambda \)'s guarantees, according to proposition 3.2, the uniform smallness of \( F^{-1}\chi_{\{|\eta| > K(R)\}} Ff_\lambda \) provided \( K(R) > 0 \) is large enough. Specifically, we will set

\[
K(R) = \frac{1}{2} \delta R^{-\frac{1}{(1-\alpha/2)}},
\]

where \( \delta > 0 \) is a small fixed parameter to be determined. Notice that \( K(R) \) is increasing since \( \alpha \geq 0 \) and \( \beta < 1 \). In particular, for any given \( \delta > 0 \), we may always assume that \( R > 1 \) is so large that \( K(R) > 1 \).

Thus, in virtue of the representation formula (3.19), we first obtain

\[
F^{-1}\chi_{\{|\eta| > R \text{ and } |\xi| \leq K(R)\}} Ff_\lambda(x,v) = F^{-1}e^{it\xi \cdot \eta}\chi_{\{|\eta| > R \text{ and } |\xi + t\eta| \leq K(R)\}} Ff_\lambda(x,v)
\]

\[-\int_0^t F^{-1}e^{is\xi \cdot \eta}\chi_{\{|\eta| > R \text{ and } |\xi + s\eta| \leq K(R)\}} t \langle \eta \rangle^{\beta} \langle \xi \rangle^\alpha \mathcal{F}g_\lambda(x,v) \, ds,
\]

where we set \( t = t(|\eta|) = \frac{2K(R)}{|\eta|} = \frac{1}{2} \delta R^{-\frac{1}{(1-\alpha/2)}} \). Notice

\[
\{ |\eta| > R \text{ and } |\xi + t\eta| \leq K(R) \} \subset \{ K(R) \leq |\xi| \leq 3K(R) \},
\]

so that, in particular,

\[
\chi_{\{|\eta| > R \text{ and } |\xi + t\eta| \leq K(R)\}} = \chi_{\{|\eta| > R \text{ and } |\xi + t\eta| \leq K(R)\}} \chi_{\{|\xi| \geq K(R)\}}.
\]

Furthermore, recalling \( \alpha \geq 0 \), \( \beta - 1 \leq 0 \) and the definition (3.24) of \( K(R) \), we see that

\[
\chi_{\{|\eta| > R \text{ and } |\xi + s\eta| \leq K(R)\}} t \langle \eta \rangle^{\beta} \langle \xi \rangle^\alpha \leq \chi_{\{|\eta| > R \text{ and } |\xi| \leq 3K(R)\}} 2K(R) \frac{\langle \eta \rangle^{\beta}}{|\eta|} \langle \xi \rangle^\alpha
\]

\[
\leq 2K(R) \frac{1 + R^2}{R} \left( 1 + 9K(R)^2 \right)^\frac{s}{2} \leq 2\delta^{a-10\delta}. 
\]
Consequently, utilizing that $M^2 = L^\infty$ and incorporating (3.27) and (3.28) into (3.25), we deduce
\[
\left\| F^{-1} 1_{\{ |\eta| > R \text{ and } |\xi| \leq K(R) \}} F f_{\lambda}(x, v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)} \leq \\
2 \left\| F^{-1} 1_{\{ |\xi| \geq K(R) \}} F f_{\lambda}(x, v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)} + 2 \delta^{-a} 10^2 \delta \| g_{\lambda} \|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)}.
\]
Therefore, recalling the disjoint decomposition (3.22),
\[
\left\| F^{-1} 1_{\{ |\eta| > R \text{ or } |\xi| > K(R) \}} F f_{\lambda}(x, v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)} \leq \\
2 \left\| F^{-1} 1_{\{ |\xi| \geq K(R) \}} F f_{\lambda}(x, v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)} + 2 \delta^{-a} 10^2 \delta \| g_{\lambda} \|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)}.
\]
Hence, in virtue of the local relative compactness in velocity of the $f_{\lambda}$'s and since
\[
\lim_{R \to \infty} K(R) = \infty,
\]
we infer
\[
\limsup_{R \to \infty} \sup_{\lambda \in \Lambda} \left\| F^{-1} 1_{\{ |\eta| > R \text{ or } |\xi| > K(R) \}} F f_{\lambda}(x, v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)} \leq 2 \delta^{-a} 10^2 \delta \sup_{\lambda \in \Lambda} \| g_{\lambda} \|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)},
\]
which, by the boundedness of the $g_{\lambda}$'s, the arbitrariness of $\delta > 0$ and proposition 3.2 implies the relative compactness of the $f_{\lambda}$'s in all variables, and thus concludes the proof of the theorem. \hfill \square

3.5. **Proof of Theorem 2.1** We provide now a proof of the main theorem 2.1 by extending the above demonstration of theorem 3.3 to the $L^p$ setting, for any $1 < p < \infty$.

**Proof.** In view of proposition 3.2 it is sufficient, in order to obtain the local relative strong compactness of the $f_{\lambda}$'s, to show that the $L^p$ norm of
\[
\mathcal{F}^{-1} \left[ 1 - \chi \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K} \right) \right] F f_{\lambda},
\]
i.e. the high frequencies of $f_{\lambda}$, can be made arbitrarily small, uniformly in $\lambda \in \Lambda$, for suitably chosen generally large parameters $R, K > 0$. Here, $\chi(r) \in C^\infty_c(\mathbb{R})$ is a fixed truncation as used in the statement of proposition 3.2, i.e. it satisfies $1_{\{|r| \leq 4\}} \leq \chi(r) \leq 1_{\{|r| \leq 1\}}$, say. Since one clearly has the decomposition
\[
1 - \chi \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K} \right) = (1 - \chi) \left( \frac{|\xi|}{K} \right) + (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K} \right),
\]
it is then enough to exhibit a uniformly small control, for large values of $R > 1$, of the $L^p$ norm of
\[
\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) F f_{\lambda},
\]
where $\lim_{R \to \infty} K(R) = \infty$, for the local relative compactness in $v$ of the $f_{\lambda}$'s guarantees, according to proposition 3.2, the uniform smallness of $\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\xi|}{K(R)} \right) F f_{\lambda}$ provided $K(R) > 0$ is large enough. As previously, we will set
\[
K(R) = \frac{1}{2} \delta^{-a} \frac{R}{R^{1+\beta}},
\]
where $\delta > 0$ is a small fixed parameter to be determined, and $R > 1$ is so large that $K(R) > 1$.

We wish now to pursue the present proof by mimicking the arguments found in the demonstration of theorem 3.3 in the $L^2$ setting, that is, using the hypoellipticity of the transport operator $v \cdot \nabla_x$, which, we feel, is best expressed through the transport of frequencies as explained in section 3.3 and, in particular, through the representation formula (3.19). The difficulty here lies in the fact that, when $p \neq 2$, the regularity of Fourier multipliers over $L^p$ becomes significant, as explained in section 3.1, and this is problematic in the treatment of formula (3.19). Indeed, the terms $e^{-itv \cdot \eta}$ and $e^{-itv \cdot \xi}$ in (3.19) may be rapidly oscillating, and thus have uncontrolled derivatives, when $iv \cdot \eta$ is large. In order to circumvent this difficulty, we introduce now, for any fixed velocity $v \in \mathbb{R}^D$, a further decomposition of space frequencies into

$$R^D = \left\{ \eta \in \mathbb{R}^D : t|v \cdot \eta| < L \right\} \bigcup \left\{ \eta \in \mathbb{R}^D : t|v \cdot \eta| \geq L \right\},$$

where $L > 1$ is a typically large parameter to be chosen later on. Thus, on the first domain of the above splitting, we will be able to carry out our arguments using the hypoellipticity in space and velocity, while, on the second domain, we will prefer to use the elliptic properties in space of the transport operator, as seen in standard velocity averaging lemmas. Eventually, the contribution from the elliptic part will be treated as an arbitrarily small perturbation for large $L$. As usual, the use of a smooth partition of the unity will be more appropriate than characteristic functions of the sets found in (3.36).

**Hypoelliptic control.** We first deal with the term stemming from the hypoelliptic part in the decomposition (3.36).

Thus, we want to use the representation formula (3.19) with

$$(3.37) \quad p(x, v, \eta, \xi) = \chi \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right),$$

and $t = t(|\eta|) = \frac{2K(R)}{|\eta|}$,

where $K(R)$ has been defined in (3.35). With this specific choice, the representation formula (3.19) may be recast as

$$\mathcal{F}^{-1}p(x, v, \eta, \xi)\mathcal{F}f_{\lambda}(x, v) = \mathcal{F}^{-1}m_1(v, \eta) m_2(\eta, \xi) \mathcal{F}f_{\lambda}(x, v)$$

$$- \int_0^1 \mathcal{F}^{-1}m_3(v, \eta) m_4(\eta, \xi) \mathcal{F}g_\lambda(x, v) \, ds,$$

where $\mathcal{F}^{-1}m_1$ and $\mathcal{F}^{-1}m_2$ are weighted by characteristic functions that control the contribution from the hypoelliptic part. This allows us to handle the oscillatory behavior of the terms involving $iv \cdot \eta$ more effectively.
where the multipliers are defined by

\[
\begin{align*}
m_1(v, \eta) &= e^{i2K(R)v \cdot \frac{\eta}{|\eta|}} \chi \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right), \\
m_2(\eta, \xi) &= (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \left| \frac{\xi}{K(R)} \right| + 2 \frac{\eta}{|\eta|} \right), \\
m_3(v, \eta) &= e^{i2K(R)v \cdot \frac{\eta}{|\eta|}} \chi \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right), \\
m_4(\eta, \xi) &= (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \left| \frac{\xi}{K(R)} \right| + s2 \frac{\eta}{|\eta|} \right) 2K(R) \left( \frac{\eta}{|\eta|} \right)^{\delta} \left( \xi \right)^{\alpha}.
\end{align*}
\]

(3.39)

It should be emphasized that, considering the velocity \( v \) as a fixed parameter, \( m_1 \) and \( m_3 \) may indeed be regarded as well-defined multipliers on the space variable \( x \) only, rather than symbols of more general pseudo-differential operators. This will allow us to control the induced operators with standard Fourier multiplier theorems, thus yielding a sharper control on their operator norms. In particular, we will obtain bounds that do not depend on \( v \), which would not be possible with the standard theorems from the theory of pseudo-differential operators.

It is shown in lemmas 3.4 and 3.5 that the operators generated by the above Fourier multipliers are bounded with a uniform control on the norms. Furthermore, notice that, on the support of \( m_2(\eta, \xi) \), it holds

\[
|\xi + 2K(R) \frac{\eta}{|\eta|}| \leq K(R) \text{ so that } |\xi| \geq K(R),
\]

hence

\[
m_2(\eta, \xi) = m_2(\eta, \xi) (1 - \chi) \left( \frac{|\xi|}{K(R)} \right).
\]

(3.41)

On the whole, incorporating (3.41) into (3.38) and exploiting the uniform bounds from lemmas 3.4 and 3.5, we arrive at the estimate

\[
\left\| \mathcal{F}^{-1} \chi \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \left| \frac{\xi}{K(R)} \right| \right) \mathcal{F} f_\lambda(x, v) \right\|_{L^p(\mathbb{R}^D)} \leq C_p \left( L^D + 1 \right) \left\| \mathcal{F}^{-1} (1 - \chi) \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda \right\|_{L^p} + \delta \| G_\lambda \|^2_{L^p},
\]

(3.42)

for some finite constant \( C_p > 0 \) depending on \( p \) but independent of \( R, K(R), \delta \) and \( L \).

**Elliptic control.** Now, we treat the term stemming from the elliptic part in the decomposition (3.36).

We will show that the \( L^p \) norm of

\[
\mathcal{F}^{-1} (1 - \chi) \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \left| \frac{\xi}{K(R)} \right| \right) \mathcal{F} f_\lambda
\]

is uniformly small provided \( L \) is sufficiently large, independently of \( R \) and \( \delta \). This part of the proof does not use the assumption that the \( f_\lambda \)'s are relatively compact in \( v \). In fact, the above expression may be regarded as a remainder. The main idea consists in inverting the transport operator \( v \cdot \nabla_x \), which is only possible because we have restricted the space frequencies to where \( v \cdot \eta \) is not too small.
Thus, expressing $v \cdot \eta e^{i(x \cdot \eta + v \cdot \xi)}$, we obtain

$$\mathcal{F}^{-1}(1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda =$$

$$-i \left( \frac{2\pi}{2D} \right) \int_{R^D} \int_{R^D} \eta \cdot \nabla \chi e^{i(x \cdot \eta + v \cdot \xi)} \frac{1}{v \cdot \eta} (1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right)$$

$$\left( 1 - \chi \right) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda (\eta, \xi) \, d\eta d\xi.$$

Therefore, a mere integration by parts combined with the transport relation \((3.17)\) yields

$$\mathcal{F}^{-1}(1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda =$$

$$i \mathcal{F}^{-1} \left( \frac{L}{K(R)} \frac{|\eta|}{v \cdot \eta} \right) (1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right)$$

$$\left( 1 - \chi \right) \left( \frac{|\eta|}{R} \right) \frac{\eta \cdot \xi}{|\xi|} \chi' \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda$$

$$- i \mathcal{F}^{-1} \left( \frac{L}{K(R)} \frac{|\eta|}{v \cdot \eta} \right) (1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right)$$

$$\left( 1 - \chi \right) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) K(R) \frac{\langle \eta \rangle^\beta}{|\eta|} \langle \zeta \rangle^\alpha \mathcal{F} g_\lambda,$$

which may be recast as

$$\mathcal{F}^{-1}(1 - \chi) \left( \frac{K(R)}{L} \frac{v \cdot \eta}{|\eta|} \right) (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F} f_\lambda =$$

$$i \frac{L}{D} \mathcal{F}^{-1} m_9(v, \eta) \left( m_5(\eta) \cdot m_6(\xi) \right) \mathcal{F} f_\lambda$$

$$- i \frac{L}{D} \frac{K(R)^{1+\alpha}}{R^{1-\beta}} \mathcal{F}^{-1} m_9(v, \eta) m_7(\eta) m_8(\xi) \mathcal{F} g_\lambda,$$

where the multipliers are defined by

$$m_5(\eta) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \frac{\eta}{|\eta|},$$

$$m_6(\xi) = \chi' \left( \frac{|\xi|}{K(R)} \right) \frac{\xi}{|\xi|},$$

$$m_7(\eta) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \frac{R^{1-\beta} \langle \eta \rangle^\beta}{|\eta|},$$

$$m_8(\xi) = \chi \left( \frac{|\xi|}{K(R)} \right) \frac{\langle \zeta \rangle^\alpha}{K(R)^{a}},$$

$$m_9(v, \eta) = \frac{L}{K(R) \cdot \eta} (1 - \chi) \left( \frac{K(R) \cdot v \cdot \eta}{L \cdot |\eta|} \right).$$

Again, considering the velocity $v$ as a fixed parameter, $m_9$ may indeed be regarded as a well-defined multiplier on the space variable $x$ only, rather than a symbol of
a more general pseudo-differential operator. This will allow us to control the induced operator with standard Fourier multiplier theorems, thus yielding a sharper control (independent of $v$) on its operator norm.

It is shown in lemmas 3.4, 3.6 and 3.7 that the operators generated by the above Fourier multipliers are bounded with a uniform control on the norms. Therefore, exploiting these bounds and recalling from (3.35) the definition of $K(R)$, we deduce from (3.46) that

$$(3.48)$$

$$\left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{K(R) \nu \cdot \eta}{L} \right) (1 - \chi) \left( \frac{\eta}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_p \frac{1}{L} \left( \|g\lambda\|_{L^p} + \delta \|g\lambda\|_{L^p} \right),$$

for some finite constant $C_p > 0$ depending on $p$ but independent of $R, K(R), \delta$ and $L$.

**Conclusion of the proof.** We are now ready to deduce the local relative compactness of the $f_{\lambda}$.'s.

Combining the hypoelliptic estimate (3.42) with the elliptic estimate (3.48), we easily deduce

$$(3.49)$$

$$\left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{\eta}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_p \left( L^D + 1 \right) \left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda} \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} + \frac{C_p}{L} \|f\lambda\|_{L^p},$$

for some finite constant $C_p > 0$ depending on $p$ but independent of $R, K(R), \delta$ and $L$. Therefore, recalling the decomposition (3.33),

$$(3.50)$$

$$\left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_p \left( L^D + 1 \right) \left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda} \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} + \frac{C_p}{L} \|f\lambda\|_{L^p}.$$

Hence, in virtue of the local relative compactness in velocity of the $f_{\lambda}$'s and since $\lim_{R \to \infty} K(R) = \infty$, we infer

$$(3.51)$$

$$\lim_{R \to \infty} \sup_{\lambda \in \Lambda} \left\|\mathcal{F}^{-1} (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{|\xi|}{K(R)} \right) \mathcal{F}f_{\lambda}(x, v) \right\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_p \left( L^D + 1 \right) \delta \sup_{\lambda \in \Lambda} \|g\lambda\|_{L^p} + \frac{C_p}{L} \sup_{\lambda \in \Lambda} \|f\lambda\|_{L^p},$$

which, by the boundedness of the $g\lambda$'s, the arbitrariness of $\delta > 0$ and $L > 1$, and proposition 3.2 implies the relative compactness of the $f_{\lambda}$'s in all variables, and thus concludes the proof of the theorem. \qed
Lemma 3.4. The Fourier multipliers

\[
m_1(v, \eta) = e^{i2K(R)v \cdot \frac{\eta}{|\eta|}} \chi \left( \frac{K(R) v \cdot \eta}{|\eta|} \right),
\]

(3.52)

\[
m_3(v, \eta) = e^{i2K(R)v \cdot \frac{\eta}{|\eta|}} \chi \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right),
\]

\[
m_9(v, \eta) = \frac{L |\eta|}{K(R) v \cdot \eta} (1 - \chi) \left( \frac{K(R) v \cdot \eta}{L |\eta|} \right).
\]

belong to \( M^p \left( \mathbb{R}^D \right) \), for every \( 1 < p < \infty \). In particular, their multiplier norms satisfy

\[
\|m_1\|_{M^p} \leq C_p \left( L^D + 1 \right),
\]

(3.53)

\[
\|m_3\|_{M^p} \leq C_p \left( L^D + 1 \right),
\]

\[
\|m_9\|_{M^p} \leq C_p,
\]

for some finite constant \( C_p > 0 \) depending on \( p \) but independent of \( v, K(R), L \) and \( s \).

Notice that, in the above lemma, the Hörmander-Mikhlin criterion \( \text{(3.6)} \) is not satisfied by the multipliers.

Proof. It is to be emphasized \( v \in \mathbb{R}^D \) is regarded as a fixed parameter here. Notice that, in virtue of the invariance of multiplier norms with respect to surjective affine transformations (see \( \text{(3.5)} \)), which includes in particular rotations and dilations, we only have to consider the multipliers

\[
\bar{m}_1(v, \eta) = e^{i2K(R)v \cdot \frac{\eta_1}{|\eta_1|}} \chi \left( \frac{K(R)|v| \eta_1}{L |\eta_1|} \right),
\]

(3.54)

\[
\bar{m}_3(v, \eta) = e^{i2K(R)v \cdot \frac{\eta_1}{|\eta_1|}} \chi \left( \frac{K(R)|v| \eta_1}{L |\eta_1|} \right),
\]

\[
\bar{m}_9(v, \eta) = \frac{L |\eta|}{K(R)|v| \eta_1} (1 - \chi) \left( \frac{K(R)|v| \eta_1}{L |\eta_1|} \right).
\]

We treat now the multiplier \( \bar{m}_1(v, \eta) \) with an application of the multiplier theorem \( \text{(3.1)} \). Thus, we are merely going to verify that it satisfies the Marcinkiewicz-Mikhlin criterion \( \text{(3.7)} \). To this end, we first notice that the pointwise boundedness of the multiplier is obvious. Next, we easily compute its first derivatives

\[
\eta_i \partial_{\eta_i} \bar{m}_1(v, \eta) = \left( \delta_{ij} - \frac{\eta_i^2}{|\eta|^2} \right) e^{i2K(R)v \cdot \frac{\eta_1}{|\eta_1|}} \varphi \left( \frac{K(R)|v| \eta_1}{L |\eta_1|} \right),
\]

(3.55)

where \( \delta_{ij} \) denotes Kronecker’s delta and \( \varphi(r) = r (i2L \chi(r) + \chi'(r)) \), from which we also deduce the pointwise boundedness of \( \eta_i \partial_{\eta_i} \bar{m}_1(v, \eta) \) uniformly in \( K(R) \) and \( |v| \). However, we emphasize that this bound is not uniform with respect to \( L \) since it is proportional to \( L^D + 1 \). Next, noticing that \( \eta_i \partial_{\eta_i} \bar{m}_1(v, \eta) \) has a similar format as \( \bar{m}_1(v, \eta) \) with \( \varphi(r) \) instead of \( \chi(r) \) and up to multiplication of the well-behaved function \( \left( \delta_{ij} - \frac{\eta_i^2}{|\eta|^2} \right) \), we conclude that the Marcinkiewicz-Mikhlin criterion \( \text{(3.7)} \) is satisfied by \( \bar{m}_1(v, \eta) \) with a bound that is independent of \( v \) and \( K(R) \) and is proportional to \( L^D + 1 \). It follows that \( \text{(3.4)} \) holds for \( m_1(v, \eta) \).
As to the multiplier \( m_3(v, \eta) \), recalling that \( 0 \leq s \leq 1 \), it is clear that the same reasoning holds true and thus, that (3.4) is verified for \( m_3(v, \eta) \).

Finally, regarding \( m_9(v, \eta) \), we may simply rewrite it as \( m_9(v, \eta) = \psi \left( \frac{K(R)|v|}{\delta |\eta|} \right) \), where \( \psi(r) = \frac{1}{r} (1 - \chi) (r) \). Since \( \psi(r) \) and all its derivatives are bounded pointwise, it is readily seen with a straightforward calculation that the Marcinkiewicz-Mikhlin condition (3.7) is satisfied by \( m_9(v, \eta) \). We deduce that (3.4) holds for \( m_9(v, \eta) \), which concludes the proof of the lemma. \( \square \)

**Lemma 3.5.** The Fourier multipliers

\[
m_2(\eta, \xi) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + 2 \eta}{|\eta|} \right),
\]

(3.56)

\[
m_4(\eta, \xi) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + s2 \eta}{|\eta|} \right) 2K(R)^{a+1}R^{a-1} \]

\[
\quad = (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + s2 \eta}{|\eta|} \right) 2^{-a} \delta.
\]

(3.58)

belong to \( M^p \left( \mathbb{R}^D_{\eta} \times \mathbb{R}^D_\xi \right) \), for every \( 1 < p < \infty \). In particular, their multiplier norms satisfy

\[
\|m_2\|_{M^p} \leq C_p,
\]

(3.57)

\[
\|m_4\|_{M^p} \leq C_p \delta,
\]

for some finite constant \( C_p > 0 \) depending on \( p \) but independent of \( R, K(R) \) and \( s \).

*Proof.* Notice that, in virtue of the invariance of multiplier norms with respect to surjective affine transformations (see [5]), which includes in particular the dilations, we only have to consider the multipliers

\[
m_2(\eta, \xi) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + 2 \eta}{|\eta|} \right),
\]

(3.58)

\[
m_4(\eta, \xi) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + s2 \eta}{|\eta|} \right) 2K(R)^{a+1}R^{a-1} \]

\[
= (1 - \chi) \left( \frac{|\eta|}{R} \right) \chi \left( \frac{\xi + s2 \eta}{|\eta|} \right) 2^{-a} \delta.
\]

In particular, notice that on the support of \( m_2(\eta, \xi) \) and \( m_4(\eta, \xi) \) it holds

\[
|\xi| \leq \left| \xi + s2 \frac{\eta}{|\eta|} \right| + \left| s2 \frac{\eta}{|\eta|} \right| \leq 1 + s2 \leq 3.
\]

(3.59)

The multipliers \( m_2(\eta, \xi) \) and \( \delta^{-1} m_4(\eta, \xi) \) are then treated with an application of the multiplier theorem [3.1]. Indeed, one may check with straightforward calculations that the Marcinkiewicz-Mikhlin criterion (3.7) is satisfied, with bounds independent of \( \delta \). This concludes the proof of the lemma. \( \square \)

**Lemma 3.6.** The Fourier Multipliers

\[
m_5(\eta) = (1 - \chi) \left( \frac{|\eta|}{R} \right) \frac{\eta}{|\eta|},
\]

(3.60)

\[
m_6(\xi) = \chi' \left( \frac{\xi}{K(R)} \right) \frac{\xi}{|\xi|}
\]
belong to $M^p \left( R_D^\eta \times R_D^\xi \right)$, for every $1 < p < \infty$. In particular, their multiplier norms satisfy
\begin{align}
\|m_5\|_{M^p} &\leq C_p, \\
\|m_6\|_{M^p} &\leq C_p,
\end{align}
for some finite constant $C_p > 0$ depending on $p$ but independent of $R$ and $K(R)$.

**Proof.** Just as in the proof of the preceding lemma 3.5 by the invariance of multiplier norms with respect to dilations, we only have to consider
\begin{align}
m_5(\eta) &= (1 - \chi) (|\eta|) \frac{\eta}{|\eta|}, \\
m_6(\xi) &= \chi' (|\xi|) \frac{\xi}{|\xi|}.
\end{align}
The boundedness of both multipliers, with norms independent of any parameter, easily follows from the facts that $\frac{\eta}{|\eta|}$ and $\frac{\xi}{|\xi|}$ induce the Riesz transforms, which are bounded over $L^p$, and that $\chi (|\eta|)$ and $\chi' (|\xi|)$ are smooth and compactly supported. This concludes the proof.

\[\square\]

**Lemma 3.7.** The Fourier multipliers
\begin{align}
m_7(\eta) &= (1 - \chi) \left( \frac{|\eta|}{R} \right)^{R^{1-\beta} (|\eta|)^\beta} \\
m_8(\xi) &= \chi \left( \frac{|\xi|}{K(R)} \right) \frac{\langle \xi \rangle^a}{K(R)^a},
\end{align}
belong to $M^p \left( R_D^\eta \times R_D^\xi \right)$, for every $1 < p < \infty$. In particular, their multiplier norms satisfy
\begin{align}
\|m_7\|_{M^p} &\leq C_p, \\
\|m_8\|_{M^p} &\leq C_p,
\end{align}
for some finite constant $C_p > 0$ depending on $p$ but independent of $R$ and $K(R)$.

**Proof.** Just as in the proof of lemmas 3.5 and 3.6 by the invariance of multiplier norms with respect to dilations, we only have to consider
\begin{align}
m_7(\eta) &= (1 - \chi) (|\eta|) \frac{(R\eta)^\beta}{R^\beta} \frac{1}{|\eta|}, \\
m_8(\xi) &= \chi (|\xi|) \frac{\langle K(R)\xi \rangle^a}{K(R)^a}.
\end{align}
The boundedness of both multipliers, with norms independent of any parameter, follows then directly from an easy application of the multiplier theorem 3.1 which concludes the proof.

\[\square\]
4. Transfer of Regularity

Here, we show how our methods can be used to obtain results on the transfer of regularity in the spirit of Bouchut [8]. We focus on the time independent kinetic transport equation, since it is more adapted to our strategy of proof, even though the results are not limited to that case.

4.1. Transfer of Regularity in $L^2$. As in the treatment of the transfer of compactness, we first provide a demonstration in the much simpler and insightful $L^2$ setting.

We will employ the Sobolev spaces $H^r_x \left( \mathbb{R}^D \times \mathbb{R}^D \right)$ and $H^r_v \left( \mathbb{R}^D \times \mathbb{R}^D \right)$ defined, for any $r \in \mathbb{R}$, as the subspaces of tempered distributions endowed with the respective norms

$$\| f(x,v) \|_{H^r_x \left( \mathbb{R}^D \times \mathbb{R}^D \right)} = \left\| (1 - \Delta_x)^{\frac{r}{2}} f(x,v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)}$$

and

$$\| f(x,v) \|_{H^r_v \left( \mathbb{R}^D \times \mathbb{R}^D \right)} = \left\| (1 - \Delta_v)^{\frac{r}{2}} f(x,v) \right\|_{L^2(\mathbb{R}^D \times \mathbb{R}^D)}$$

Theorem 4.1. Let $f(x,v) \in H^r_v \left( \mathbb{R}^D \times \mathbb{R}^D \right)$, where $r \geq 0$, be such that

$$v \cdot \nabla_x f(x,v) = (1 - \Delta_x)^{\frac{\beta}{2}} (1 - \Delta_v)^{\frac{\alpha}{2}} g(x,v),$$

for some $g(x,v) \in L^2(\mathbb{R}^D \times \mathbb{R}^D)$ and $\alpha \geq 0, 0 \leq \beta \leq 1$.

Then, $f$ belongs to $H^\sigma_x \left( \mathbb{R}^D \times \mathbb{R}^D \right)$, where

$$\sigma = (1 - \beta) \frac{r}{1 + r + \alpha},$$

and the following estimate holds

$$\| f \|_{H^\sigma_x} \leq C \| f \|_{H^r_v} + C \| g \|_{L^2},$$

for some finite constant $C > 0$ that only depends on fixed parameters.

Proof. Notice first that the $L^2$ norm of

$$\mathcal{F}^{-1} \{ (\eta)^\nu \leq (\xi)^\nu \} \langle \eta \rangle^\nu \mathcal{F} f(x,v)$$

is clearly bounded since $f$ lies in $H^r_v$. Thus, we merely have to estimate the $L^2$ norm of

$$\mathcal{F}^{-1} \{ (\eta)^\nu > (\xi)^\nu \} \langle \eta \rangle^\nu \mathcal{F} f(x,v)$$

in order to conclude.

To this end, note first that clearly

$$\left\| \mathcal{F}^{-1} \{ (\eta)^\nu > (\xi)^\nu \text{ and } |\eta| \leq 1 \} \langle \eta \rangle^\nu \mathcal{F} f \right\|_{L^2} \leq C \| f \|_{H^r_v}.$$
where \( C > 0 \) only depends on fixed parameters. Furthermore, in virtue of the representation formula (3.19), we obtain
\[
\mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \mathcal{F} f(x, v) =
\]
\[
\mathcal{F}^{-1} e^{i t v \cdot \eta} \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi + t\eta)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \mathcal{F} f(x, v) =
\]
\[- \int_0^1 \mathcal{F}^{-1} e^{i s t v \cdot \eta} \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi + s t\eta)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \langle t \rangle^\beta \langle \xi \rangle^a \mathcal{F} g(x, v) \, ds,
\]
where we set \( t = t(|\eta|) \), for any \( \eta \neq 0 \), as being the unique solution to the identity \( \langle t\eta \rangle = 2 \langle \eta \rangle^{\frac{r}{2}} \).

In particular, it holds on \( \{ (\eta)^\sigma > (\xi + t\eta)^r \} \) that
\[
\langle \xi \rangle \geq \langle t\eta \rangle - \langle \xi + t\eta \rangle > \langle \eta \rangle^{\frac{r}{2}},
\]
which implies
\[
\left| e^{i s t v \cdot \eta} \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi + s t\eta)^r \} \langle \eta \rangle^\sigma \right| \leq \langle \xi \rangle^\sigma.
\]
Thus, we easily conclude, since \( M^2 = L^\infty \), that
\[
\left\| \mathcal{F}^{-1} e^{i t v \cdot \eta} \{ (\eta)^\sigma > (\xi + t\eta)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \mathcal{F} f \right\|_{L^2} \leq \| f \|_{H^\sigma},
\]
which takes care of the first term in the right-hand side of the representation formula (4.9).

As to the remaining expression in (4.9), we first notice, on the set \( \{ (\eta)^\sigma > (\xi + s t\eta)^r \} \), that it holds
\[
\langle \xi \rangle \leq \langle s t\eta \rangle + \langle \xi + s t\eta \rangle < 3 \langle \eta \rangle^{\frac{r}{2}}.
\]
It follows that, utilizing relation (4.4),
\[
\left| e^{i s t v \cdot \eta} \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi + s t\eta)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \langle t \rangle^\beta \langle \xi \rangle^a \right| \leq 3^a \left| \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi + s t\eta)^r \text{ and } |\eta| > 1 \} \langle t \rangle^1 - \langle \eta \rangle^{\frac{r}{2}} \right| \leq 2 \sqrt{2} \cdot 3^a.
\]
Hence, since \( M^2 = L^\infty \),
\[
\left\| \mathcal{F}^{-1} e^{i t v \cdot \eta} \{ (\eta)^\sigma > (\xi + s t\eta)^r \text{ and } |\eta| > 1 \} \langle \eta \rangle^\sigma \langle t \rangle^\beta \langle \xi \rangle^a \mathcal{F} g \right\|_{L^2} \leq 2 \sqrt{2} \cdot 3^a \| g \|_{L^2}.
\]

On the whole, combining estimates (4.8), (4.12) and (4.15) with the interpolation formula (4.9), we infer
\[
\| f \|_{H^\sigma} \leq \left\| \mathcal{F}^{-1} \{ (\eta)^\sigma \leq (\xi)^r \} \langle \eta \rangle^\sigma \mathcal{F} f \right\|_{L^2} + \left\| \mathcal{F}^{-1} \{ (\eta)^\sigma > (\xi)^r \} \langle \eta \rangle^\sigma \mathcal{F} f \right\|_{L^2} \leq C \| f \|_{H^\sigma} + C \| g \|_{L^2},
\]
where \( C > 0 \) only depends on fixed parameters, which concludes the proof of the theorem. \( \square \)
4.2. Transfer of regularity in $L^p$. We extend now the result on the transfer of regularity in $L^2$ from the previous section 4.1 to the $L^p$ setting, for any $1 < p < \infty$.

We will employ the Sobolev spaces $W^{\sigma\nu}_r (\mathbb{R}^D \times \mathbb{R}^D)$ and $W^{\sigma\nu}_c (\mathbb{R}^D \times \mathbb{R}^D)$ defined, for any $r \in \mathbb{R}$ and $1 < p < \infty$, as the subspaces of tempered distributions endowed with the respective norms

$$
\| f(x,v) \|_{W^{\sigma\nu}_r(\mathbb{R}^D \times \mathbb{R}^D)} = \left\| (1 - \Delta_x)^{\frac{\sigma}{2}} f(x,v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)}
$$

(4.17)

and

$$
\| f(x,v) \|_{W^{\sigma\nu}_c(\mathbb{R}^D \times \mathbb{R}^D)} = \left\| (1 - \Delta_v)^{\frac{\nu}{2}} f(x,v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)}
$$

(4.18)

Theorem 4.2. Let $f(x,v) \in W^{\sigma\nu}_c (\mathbb{R}^D \times \mathbb{R}^D)$, where $r \geq 0$ and $1 < p < \infty$, be such that

$$
v \cdot \nabla_x f(x,v) = (1 - \Delta_x)^{\frac{\sigma}{2}} (1 - \Delta_v)^{\frac{\nu}{2}} g(x,v),
$$

(4.19)

for some $g(x,v) \in L^p (\mathbb{R}^D \times \mathbb{R}^D)$ and $\alpha \geq 0, 0 \leq \beta \leq 1$.

Then, $f$ belongs to $W^{\sigma\nu}_c (\mathbb{R}^D \times \mathbb{R}^D)$, where

$$
\sigma = (1 - \beta) \frac{r}{1 + \tau + \alpha},
$$

(4.20)

and the following estimate holds

$$
\| f \|_{W^{\sigma\nu}_r} \leq C \| f \|_{W^{\sigma\nu}_c} + C \| g \|_{L^p},
$$

(4.21)

for some finite constant $C > 0$ that only depends on fixed parameters.

Proof. Consider a fixed truncation $\chi(r) \in \mathcal{C}^\infty(\mathbb{R})$, such that $1_{\{ |r| \leq 1 \}} \leq \chi(r) \leq 1_{\{ |r| \leq 2 \}}$, say. We begin with the simple decomposition

$$
\mathcal{F}^{-1} \langle \eta \rangle^\sigma \mathcal{F} f = \mathcal{F}^{-1} m_1(\eta, \xi) \langle \xi \rangle^\tau \mathcal{F} f + \mathcal{F}^{-1} (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-\tau} \right) \langle \eta \rangle^\sigma \mathcal{F} f,
$$

(4.22)

where the Fourier multiplier $m_1(\eta, \xi)$ is defined by

$$
m_1(\eta, \xi) = \chi \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \langle \eta \rangle^\sigma \langle \xi \rangle^{-\tau}
$$

(4.23)

+ $\chi (|\eta|) (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \langle \eta \rangle^\sigma \langle \xi \rangle^{-\tau}$.

In virtue of lemma 4.2 below, $m_1$ belongs to $M^p$ for every $1 < p < \infty$ so that

$$
\left\| \mathcal{F}^{-1} m_1(\eta, \xi) \langle \xi \rangle^\tau \mathcal{F} f \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} \leq \| f \|_{W^{\sigma\nu}_c(\mathbb{R}^D \times \mathbb{R}^D)}
$$

(4.24)

and we merely have to estimate the $L^p$ norm of the last term in the above decomposition 4.22 in order to conclude.
We introduce now, for any fixed velocity \( v \in \mathbb{R}^D \) and for a suitable parameter \( t = t (|\eta|) \), a further decomposition of space frequencies into
\[
R_\eta^D = \left\{ \eta \in \mathbb{R}^D : \left| v \cdot \eta \right| < 1 \right\} \cup \left\{ \eta \in \mathbb{R}^D : \left| v \cdot \eta \right| \geq 1 \right\}.
\]

Specifically, just as in proof of theorem 4.1, we set \( t = t (|\eta|) \) as being the unique solution to \( \langle t \eta \rangle = \frac{2}{5} \langle \eta \rangle^5 \), for any \( \eta \neq 0 \). Thus, on the first domain of the above splitting, we will be able to carry out our arguments using the hypoelliptic property of the transport operator, as seen in standard velocity averaging lemmas. As usual, the use of a smooth partition of the unity will be more appropriate than characteristic functions of the sets found in (4.25).

**Hypoelliptic control.** We first deal with the term stemming from the hypoelliptic part in the decomposition (4.25).

Thus, we use the representation formula (3.19) with
\[
p(x, v, \eta, \xi) = \chi (tv \cdot \eta) (1 - \chi) \langle |\eta| \rangle (1 - \chi) \left( \langle \eta \rangle^{\sigma} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{\sigma},
\]
which may be recast, with this specific choice, as
\[
\mathcal{F}^{-1} p(x, v, \eta, \xi) \mathcal{F} f(x, v) = \mathcal{F}^{-1} m_2 (v, \eta) m_3 (v, \eta, \xi) m_4 (v, \eta) m_5 (v, \eta, \xi) \mathcal{F} g(x, v) ds,
\]
where the multipliers are defined, using relation (4.20), by
\[
m_2 (v, \eta) = e^{itv \cdot \eta} \chi (tv \cdot \eta) (1 - \chi) \langle |\eta| \rangle,
\]
\[
m_3 (v, \eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^{\sigma} \langle \xi + t\eta \rangle^{-r} \right) \langle \eta \rangle^{\sigma} \langle \xi \rangle^{-r},
\]
\[
m_4 (v, \eta) = e^{itv \cdot \eta} \chi (tv \cdot \eta) (1 - \chi) \langle |\eta| \rangle t \langle \eta \rangle^{1 - \frac{\sigma}{r}},
\]
\[
m_5 (v, \eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^{\sigma} \langle \xi + st\eta \rangle^{-r} \right) \langle \eta \rangle^{-\frac{\sigma}{a}} \langle \xi \rangle^{\sigma}.
\]

It is shown in lemmas 4.3 and 4.4 that the operators generated by the above Fourier multipliers are bounded with a uniform control on the norms. Therefore, we deduce from (4.27) that
\[
\left\| \mathcal{F}^{-1} \chi (tv \cdot \eta) (1 - \chi) \langle |\eta| \rangle (1 - \chi) \left( \langle \eta \rangle^{\sigma} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{\sigma} \mathcal{F} f \right\|_{L^p (\mathbb{R}^D \times \mathbb{R}^D)} \leq C \left( \| f \|_{W_{t, v}^{r,p}} + \| g \|_{L^p} \right),
\]
for some finite constant \( C > 0 \) depending only on fixed parameters.

**Elliptic control.** Now, we treat the term stemming from the elliptic part in the decomposition (4.25).

We will show that the \( L^p \) norm of
\[
\mathcal{F}^{-1} (1 - \chi) (tv \cdot \eta) (1 - \chi) \langle |\eta| \rangle (1 - \chi) \left( \langle \eta \rangle^{\sigma} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{\sigma} \mathcal{F} f
\]
enjoys a control similar to (4.29). The main idea is the same that was employed in the proof of theorem 2.3 to obtain the elliptic control, it consists in inverting the transport operator \( v \cdot \nabla_x \), which is only possible because we have restricted the space frequencies to where \( v \cdot \eta \) is not too small.
Thus, expressing \( v \cdot \eta e^{i(x \cdot \eta + \varphi \xi)} = -i \eta \cdot \nabla_{\xi} e^{i(x \cdot \eta + \varphi \xi)} \), we obtain

\[
\mathcal{F}^{-1} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \mathcal{F} f =
\]

\[
\frac{-i}{(2\pi)^{2D}} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \eta \cdot \nabla_{\xi} e^{i(x \cdot \eta + \varphi \xi)} \frac{1}{v \cdot \eta} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \mathcal{F} f \, d\eta d\xi.
\]

(4.31)

Therefore, a mere integration by parts combined with the transport relation (3.17) yields

\[
\mathcal{F}^{-1} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \mathcal{F} f =
\]

\[
i \mathcal{F}^{-1} m_8(v, \eta) m_6(\eta, \xi) \langle \xi \rangle^r \mathcal{F} f - i \mathcal{F}^{-1} m_8(v, \eta) m_7(\eta, \xi) \mathcal{F} g,
\]

where the multipliers are defined by, using relation (4.20),

\[
m_6(\eta, \xi) = \left( \frac{\eta \cdot \xi}{|\eta|} \right) \chi' \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \langle \xi \rangle^{-1} (\xi)^{-(2r+1)}
\]

\[
(4.32)
\]

\[
m_7(\eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{-r} \eta \langle \xi \rangle^r,
\]

\[
m_8(v, \eta) = \frac{1}{tv \cdot \eta} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) t \langle \eta \rangle^{1-r}.
\]

\[
(4.33)
\]

It is shown in lemmas 4.3 and 4.4 that the operators generated by the above Fourier multipliers are bounded with a uniform control on the norms. Therefore, exploiting these bounds, we deduce from (4.32) that

\[
\left\| \mathcal{F}^{-1} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \mathcal{F} f \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} 
\]

\[
\leq C \left( \|f\|_{W^{q,p}_v} + \|g\|_{L^p} \right),
\]

for some finite constant \( C > 0 \) depending only on fixed parameters.

**Conclusion of the proof.** We are now ready to deduce the sought regularity estimate on \( f \).

Thus, combining the hypoelliptic estimate (4.29) with the elliptic estimate (4.34), we easily deduce

\[
\left\| \mathcal{F}^{-1} (1 - \chi) (|\eta|) (1 - \chi) \left( \langle \eta \rangle^{q'} \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{q} \mathcal{F} f(x, v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} 
\]

\[
\leq C \left( \|f\|_{W^{q,p}_v} + \|g\|_{L^p} \right),
\]

for some finite constant \( C > 0 \) depending only on fixed parameters. Therefore, recalling the decomposition (4.22) and the estimate (4.24), we infer

\[
\left\| \mathcal{F}^{-1} \langle \eta \rangle^{q'} \mathcal{F} f(x, v) \right\|_{L^p(\mathbb{R}^D \times \mathbb{R}^D)} \leq C \left( \|f\|_{W^{q,p}_v} + \|g\|_{L^p} \right),
\]

which concludes the proof of the theorem. \( \square \)
Lemma 4.3. The Fourier multipliers

\[
m_1(\eta, \xi) = \chi \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} + \chi (|\eta|) (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \langle \eta \rangle^\sigma \langle \xi \rangle^{-r}
\]
\[
m_3(\eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi + t\eta \rangle^{-r} \right) \langle \eta \rangle^\sigma \langle \xi \rangle^{-r},
\]
\[
m_5(\eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi + st\eta \rangle^{-r} \right) \langle \eta \rangle^{-\frac{\sigma}{\xi}} \langle \xi \rangle^a
\]
\[
m_6(\eta, \xi) = r \frac{\eta \cdot \xi}{\langle \eta \rangle \langle \xi \rangle} \chi'(\langle \eta \rangle^\sigma \langle \xi \rangle^{-r}) \langle \eta \rangle^{(2r+1)} \langle \xi \rangle^{-(2r+1)}
\]
\[
m_7(\eta, \xi) = (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \langle \eta \rangle^{-\frac{\sigma}{\xi}} \langle \xi \rangle^a
\]

belong to \( M^p \left( R^D_y \times R^D_\xi \right) \), for every \( 1 < p < \infty \).

Proof. It turns out that the boundedness of each of the above multipliers follows from a direct application of the multiplier theorem [3.1]. Indeed, one may check that the Marcinkiewicz-Mikhlin criterion [3.7] is verified in each case, and so we merely give now a few hints aimed at justifying the validity of the computations involved.

First, the treatment of the first multiplier \( m_1(\eta, \xi) \) is completely straightforward once we notice that \( \chi (|\eta|) (1 - \chi) \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \) is compactly supported, recalling that \( (1 - \chi) (r) \) is supported on \( \{|r| \geq 1\} \).

Next, notice that on the support of \( m_3(\eta, \xi) \) it holds

\[
\langle \xi \rangle \geq \langle t\eta \rangle - \langle \xi + t\eta \rangle \geq 2 \langle \eta \rangle^\frac{\sigma}{\xi} - \langle \eta \rangle^\frac{\sigma}{\xi} = \langle \eta \rangle^\frac{\sigma}{\xi},
\]
hence \( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \leq 1 \), while on the support of \( m_5(\eta, \xi) \) it holds

\[
\langle \xi \rangle \leq \langle t\eta \rangle + \langle \xi + t\eta \rangle \leq 2 \langle \eta \rangle^\frac{\sigma}{\xi} + \langle \eta \rangle^\frac{\sigma}{\xi} = 3 \langle \eta \rangle^\frac{\sigma}{\xi},
\]
hence \( \langle \eta \rangle^{-\frac{\sigma}{\xi}} \langle \xi \rangle^a \leq 3^a \), which clearly implies the pointwise boundedness of the multipliers. Their derivatives are treated similarly.

As to \( m_6(\eta, \xi) \), its boundedness follows from the facts that \( \frac{\eta}{|\eta|} \) and \( \frac{\xi}{|\xi|} \) induce the Riesz transforms, which are bounded over \( L^p \), and that it may be recast as

\[
m_6(\eta, \xi) = r \frac{\eta \cdot \xi}{|\eta| |\xi|} \frac{|\eta|}{|\xi|} \chi'(\langle \eta \rangle^\sigma \langle \xi \rangle^{-r}),
\]
where \( \varphi(r) = \chi'(r)r^{2+\frac{1}{2}} \) is smooth and compactly supported.

Finally, regarding \( m_7(\eta, \xi) \), simply notice that it may be rewritten as \( m_7(\eta, \xi) = \varphi \left( \langle \eta \rangle^\sigma \langle \xi \rangle^{-r} \right) \), where \( \varphi(r) = (1 - \chi)(r)^{\frac{1}{2}} \) is smooth, bounded and has bounded derivatives.

This concludes the justification of the lemma. \( \square \)

Lemma 4.4. The Fourier multipliers

\[
m_2(v, \eta) = e^{iv \cdot \eta} \chi (tv \cdot \eta) (1 - \chi) (|\eta|),
\]
\[
m_4(v, \eta) = e^{istv \cdot \eta} \chi (tv \cdot \eta) (1 - \chi) (|\eta|) t (\eta)^{1-\frac{\sigma}{\xi}},
\]
\[
m_8(v, \eta) = \frac{1}{tv \cdot \eta} (1 - \chi) (tv \cdot \eta) (1 - \chi) (|\eta|) t (\eta)^{1-\frac{\sigma}{\xi}}
\]
belong to $M^p \left( \mathbb{R}^D_\eta \right)$, for every $1 < p < \infty$. In particular, their multiplier norms are independent of $v$.

Notice that, in the above lemma, the Hörmander-Mikhlin criterion (3.6) is not satisfied by the multipliers.

**Proof.** It is to be emphasized that $v \in \mathbb{R}^D$ is regarded as a fixed parameter here. Notice that, in virtue of the invariance of multiplier norms with respect to surjective affine transformations (see [5]), which includes in particular rotations and dilations, we only have to consider the multipliers

\[(4.41)\]
\[m_2(v, \eta) = e^{itv|\eta_1} \chi(t|v|\eta_1) (1 - \chi)(|\eta|),\]
\[m_4(v, \eta) = e^{ist|\eta_1} \chi(t|v|\eta_1) (1 - \chi)(|\eta|) t \langle \eta \rangle^{1-\frac{\sigma}{\gamma}},\]
\[m_8(v, \eta) = \frac{1}{t|v|\eta_1} (1 - \chi)(t|v|\eta_1) (1 - \chi)(|\eta|) t \langle \eta \rangle^{1-\frac{\sigma}{\gamma}}.\]

Then, noticing that

\[(4.42)\]
\[t \langle |\eta| \rangle |\eta| = \left( 4 \langle \eta \rangle^2 - 1 \right)^{\frac{1}{2}}\]

is smooth and that $(1 - \chi)(|\eta|) t \langle |\eta| \rangle \langle \eta \rangle^{1-\frac{\sigma}{\gamma}}$ is uniformly bounded, straightforward calculations show that the Marcinkiewicz-Mikhlin criterion (3.7) is satisfied by each of the above multipliers, thereby showing their boundedness thanks to the multiplier theorem 3.1. □

5. **About the $L^1$ Case**

5.1. **Counterexamples to dispersion and mixing.**

- A first situation where there is equiintegrability with respect to $v$ but not with respect to $x$ for the solutions to the transport equation

\[\nu \partial_x f_\epsilon = \partial_v g_\epsilon\]

is the oscillating case (in any dimension):

\[f_\epsilon(x, v) = \frac{1}{\epsilon} \chi \left( \frac{x}{\epsilon} \right) \cos \left( \frac{v}{\epsilon} \right),\]

where $\chi$ is any nonnegative smooth function. In that case, we indeed have that both $(f_\epsilon)$ and $(g_\epsilon)$ are uniformly bounded in $L^1$, and that $(f_\epsilon)$ is equiintegrable with respect to $v$, but $(f_\epsilon)$ concentrates on $x = 0$.

Note however that this example does not provide any obstruction to $L^1$ averaging lemma since the moments converge strongly to $0 : \forall \varphi \in C_c^\infty(\mathbb{R})$,

\[\int f_\epsilon(x, v) \varphi(v) dv = \chi \left( \frac{x}{\epsilon} \right) \int \cos \left( \frac{v}{\epsilon} \right) \varphi(v) dv \frac{dv}{\epsilon}\]

\[= -\chi \left( \frac{x}{\epsilon} \right) \int \sin \left( \frac{v}{\epsilon} \right) \varphi(v) dv \rightarrow 0 \text{ in } L^1(\mathbb{R}).\]

- The second counterexample to equiintegrability we would like to discuss is related to some spreading of mass at infinity (in dimension higher than 2). We indeed consider the solutions to the transport equation

\[(v_1 \partial_{x_1} + v_2 \partial_{x_2}) f_n = \partial_{v_1} g_{1,n} + \partial_{v_2} g_{2,n}\]
from which we deduce that
\[ \Psi \]
for some \( \chi \in C^\infty_c(\mathbb{R}^2) \) and some even function \( \psi \in C^\infty_c(\mathbb{R}^2) \), and
\[ \Phi_{1,n}(x,v) = n\chi \left( n^2 x_1, \frac{x_2}{n} \right) \Psi_1 \left( nv_1, \frac{v_2}{n} \right) \]
\[ \Phi_{2,n}(x,v) = \frac{1}{n} \chi \left( n^2 x_1, \frac{x_2}{n} \right) \Psi_2 \left( nv_1, \frac{v_2}{n} \right) \]
where \( \Psi_j \) is the function with compact support defined by \( \partial_{v_j} \Psi_j(v_1, v_2) = \nu \psi(v_1, v_2) \).

Note however that this example does not yet provide any obstruction to \( L^1 \) averaging lemma since the moments also converge strongly to 0:
\[ \int f_n(x,v) \varphi(v) dv = n\chi \left( n^2 x_1, \frac{x_2}{n} \right) \int \psi \left( nv_1, \frac{v_2}{n} \right) \varphi(v) dv \to 0 \in L^1_{loc}(\mathbb{R}^2). \]

5.2. The one-dimensional case.
Although, there are counterexamples to dispersion, we are able in 1D to prove the strong compactness of the moments for a kinetic equation of the following form
\[ v\partial_x f + F\partial_v f = g \]
for some \( L^\infty \) force field \( F \), provided that \( f \) is equiintegrable with respect to \( v \). More precisely, we will establish the following

**Theorem 5.1.** Let \( F \in L^\infty(dx) \). Let \( (f_\lambda) \) and \( (g_\lambda) \) be bounded families of \( L^1(dx dv) \) such that \( (f_\lambda) \) is locally equiintegrable with respect to \( v \) and
\[ v\partial_x f_\lambda + F\partial_v f_\lambda = g_\lambda. \]

Then, for all \( \varphi \in C^\infty_c(\mathbb{R}) \),
\[ \left( \int f_\lambda \varphi dv \right) \]
is relatively strongly compact in \( L^1_{loc}(dx) \).

**Proof.** Without loss of generality, we can assume that \( f_\lambda \) is non negative (replacing \( f_\lambda \) by its absolute value if necessary). As usual, we will show that \( (f_\lambda) \) is locally equiintegrable with respect to \( x \) and \( v \), then conclude using a suitable truncation together with standard \( L^p \) averaging lemma.

- The proof of equiintegrability with respect to \( x \) relies on the usual mixing formula
\[ \int \varphi(v)f_\lambda(x,v)dv = \int \varphi(v)f_\lambda(x-tv,v)dv + \int_0^t \int \varphi(v)(-\partial_v(Ff_\lambda) + g_\lambda)(x-sv,v)dsdv, \]
where \( \varphi \in C^\infty_c(\mathbb{R}) \). Since \( f_\lambda \) is equiintegrable with respect to \( v \), we can assume without loss of generality that the support of \( \varphi \) is bounded away from zero.

Then, in order to handle the first term, we use the dispersion estimate [1.2] which provides
\[ \int 1_A(x+vt) dv \leq \frac{|A|}{t^\beta}, \]
from which we deduce that
\[ \int dx 1_A(x) \int \varphi(v)f_\lambda(x-tv,v)dv = \int d\varphi(v)1_A(x+vt)f_\lambda(x,v)dvdx \]
tends uniformly to 0 as \( |A| \to 0 \), for any fixed \( t > 0 \).
To deal with the second term, we use suitable integration by parts
\[ \int_0^t \int \varphi(v) \partial_v (F \lambda)(x - sv, v) ds dv = - \int_0^t \int \partial_v \varphi(v) F \lambda(x - sv, v) ds dv + \int_0^t \int \varphi(v) s \partial_v (F \lambda)(x - sv, v) ds dv \]
\[ = - \int_0^t \int \partial_v \varphi(v) F \lambda(x - sv, v) ds dv - \int_0^t \int \varphi(v) \frac{s}{v} \partial_v (F \lambda)(x - sv, v) ds dv \]
\[ = - \int_0^t \int \partial_v \varphi(v) F \lambda(x - sv, v) ds dv + \int_0^t \int \varphi(v) \frac{1}{v} F \lambda(x - sv, v) ds dv \]
\[ - \left[ \int \varphi(v) \frac{s}{v} F \lambda(x - sv, v) dv \right]_0^t \]
from which we deduce that
\[ \int_0^t \int \varphi(v) \partial_v (F \lambda)(x - sv, v) ds dv = O(t) \]
where the bound depends on the Lipschitz norm of \( \varphi \) as well as of the distance of its support to 0.

We obtain directly a similar estimate for the last term
\[ \int_0^t \int \varphi(v) g \lambda(x - sv, v) ds dv = O(t) \]

- The local equiintegrability of \( \lambda \) (with respect to all variables) is then obtained by a simple interpolation argument.

Fix \( R > 0 \) and let \( \mathcal{A} \) be any borelian subset of the ball \( B(0, R) \subset \mathbb{R}^2 \). We define
\[ \mathcal{A} = \left\{ x \in \mathbb{R} : \int 1_{\mathcal{A}}(x, v) dv \geq |\mathcal{A}|^{1/2} \right\} . \]

We then have
(5.1)
\[ \int 1_{\mathcal{A}}(x, v) f \lambda(x, v) dv = \int 1_{\mathcal{A}}(x, v) 1_{\mathcal{A}}(x) f \lambda(x, v) dv + \int 1_{\mathcal{A}}(x, v)(1 - 1_{\mathcal{A}}(x)) f \lambda(x, v) dv . \]

Because of the equiintegrability in \( v \), the second term in the right-hand side of (5.1) tends to 0 as \( |\mathcal{A}| \to 0 \).

Since \( \lambda \) is nonnegative and \( 1_{\mathcal{A}} \) is supported in the ball of radius \( R \), we have
\[ \int 1_{\mathcal{A}}(x, v) 1_{\mathcal{A}}(x) f \lambda(x, v) dv \leq \int 1_{\mathcal{A}}(x) \varphi(v) f \lambda(x, v) dv \]
for any nonnegative test function \( \varphi \) such that \( \varphi(v) = 1 \) if \( |v| \leq R \). In particular, the first term in the right-hand side of (5.1) tends to 0 as \( |\mathcal{A}| \to 0 \).

- For the strong compactness of the moments, we then use the following decomposition
\[ \lambda = \gamma \left( \frac{\lambda}{\Lambda} \right) + \left( \lambda - \gamma \left( \frac{\lambda}{\Lambda} \right) \right) , \]
where \( \gamma \in C^\infty_0(\mathbb{R}) \) is some smooth truncation such that
\[ \gamma(x) = x \text{ if } |x| \leq 1. \]
The first term $f_{1,\lambda}$ is bounded in $L^p$ for any $p \in [1, \infty]$ and satisfies
\[v \partial_x f_{1,\lambda} + F \partial_v f_{1,\lambda} = \frac{1}{\Lambda} g_{\lambda} \gamma' \left( \frac{f_{1,\lambda}}{\Lambda} \right)\]

From Sobolev’s embeddings, we deduce that the source term - which is bounded in $L^1$ - can be written in the form
\[(1 - \Delta_x)^s (1 - \Delta_v)^s h_{A,\lambda}\]
with $s < \frac{1}{2}$ and $(h_{A,\lambda})$ bounded in some $L^p(dx dv)$ for $p > 1$ uniformly in $\lambda$.

Theorem 1.2 then implies that the moments of $f_{1,\lambda}$ are strongly compact in $L^1$:
for any $\varphi \in C_c^\infty(\mathbb{R})$, any $\lambda > 0$ and each compact $K \subset \mathbb{R}$,
\[\left\| \int f_{1,\lambda}(x + y, v) \varphi(v) dv - \int f_{1,\lambda}(x, v) \varphi(v) dv \right\|_{L^1(K)} \to 0 \text{ as } |y| \to 0\]
uniformly in $\lambda$.

The second term $f_{2,\lambda}$ can be made small by choosing $\Lambda$ sufficiently large. In particular, for any $\varphi \in C_c^\infty(\mathbb{R})$ and each $K \subset \mathbb{R}$
\[\left\| \int f_{2,\lambda}(x, v) \varphi(v) dv \right\|_{L^1(K)} \to 0 \text{ as } \lambda \to 0 .\]

Combining both results, we get the expected strong compactness. □

A challenging open question is to determine whether or not this result can be extended in the multidimensional case.
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