Position-dependent mass Lagrangians: nonlocal transformations, Euler–Lagrange invariance and exact solvability

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Received 17 November 2014, revised 25 March 2015
Accepted for publication 22 April 2015
Published 15 May 2015

Abstract
A general nonlocal point transformation for position-dependent mass (PDM) Lagrangians and their mapping into a ‘constant unit-mass’ Lagrangians in the generalized coordinates is introduced. The conditions on the invariance of the related Euler–Lagrange equations are reported. The harmonic oscillator linearization of the PDM Euler–Lagrange equations is discussed through some illustrative examples including harmonic oscillators, shifted harmonic oscillators, a quadratic nonlinear oscillator, and a Morse-type oscillator. The Mathews–Lakshmanan nonlinear oscillators are reproduced and some ‘shifted’ Mathews–Lakshmanan nonlinear oscillators are reported. The mapping of an isotonic nonlinear oscillator into a PDM deformed isotonic oscillator is also discussed.

Keywords: classical position-dependent mass, nonlocal point transformation, Euler–Lagrange equations invariance

1. Introduction

The position-dependent mass (PDM) concept in quantum mechanics has attracted researchers’ attention ever since the introduction of the PDM von Roos Hamiltonian [1] (see a sample of references in [2–12] and related references cited therein). This research attention is manifested and inspired not only by the PDM feasible applicability in various fields of physics (e.g., many-body systems, semiconductors, quantum dots, quantum liquids, etc), but also by the mathematical challenge indulged in the von Roos Hamiltonian. Such a PDM setting have invigorated a relatively recent and rapid research attention on the PDM concept for classical mechanical systems (see, e.g., [12–25]) which was readily introduced by Mathews and Lakshmanan [18] back in 1974. Unlike the ordering ambiguity that arises in the kinetic energy term of the PDM von Roos quantum mechanical Hamiltonian, no such ordering ambiguities arise in the classical
mechanical PDM Hamiltonian. Yet, it has been asserted that the resolution of the ordering ambiguity conflict in the PDM quantum Hamiltonian may be sought in the classical and quantum mechanical correspondence (see, e.g., [12, 17] for more details on this issue).

Very recently [22] we have advocated that the equivalence between the Euler–Lagrange’s and Newton’s equations of motion is secured through some ‘good’ invertible coordinate transformation (i.e., $\frac{dx}{dq} \neq 0 \neq \frac{dq}{dx}$) and the introduction of the new PDM-byproduced reaction-type force $R_{\text{PDM}}(x, \dot{x}) = m'(x)\dot{x}^2/2$ into Newton’s law of motion (with the overhead dot representing time derivative and the prime denoting coordinate derivative). Hereby, we have shown that while the quasi-linear momentum is a conserved quantity (i.e., $\Pi = m_0(x, \dot{x}) = m_0(x, \dot{x}) = m', \dot{x}_0, \dot{x}$) and not the linear momentum (i.e., $p(\dot{x}) \neq p_0(\dot{x}, \dot{x})$, and $\dot{p}(\dot{x}) \neq 0$), the total energy remains conservative. Therein, we have conclude that the PDM setting is nothings but a manifestation of some ‘good’ invertible coordinate transformation that leaves the corresponding Euler–Lagrange equation invariant.

That is, for the PDM-Lagrangian

\[ L = T - V = \frac{1}{2} m(x)\dot{x}^2 - V(x), \]

the corresponding Euler–Lagrange equation

\[ \ddot{x} + \frac{1}{2} m'(x)\dot{x}^2 + \frac{d}{dx} \frac{V'(x)}{m(x)} = 0, \]

is invariant under invertible coordinate transformations. For more details on this issue the reader may refer to [17, 22]. Obviously, moreover, equation (2) is a quadratic Liénard-type differential equation (quadratic in terms of $\dot{x}^2$ in (2)) which serves as a very interesting model in both physics and mathematics (see, e.g., the sample of references [18–37] and related references cited therein).

In fact, the PDM concept may very well represent a position-dependent deformation of the mass. Which would, in turn, manifest some deformation in the potential force field the mass is moving within and may inspire nonlocal space-time point transformations. That is, if the PDM $m(x) = m, M(x)$ ($m$ is the standard constant mass and is taken as a unit mass throughout this work) is moving in a harmonic oscillator potential $V(x) = m(x)\omega^2\dot{x}^2/2$, then one may rewrite $V(x) = m, V(x)\omega^2M(x)\dot{x}^2/2 \Rightarrow V(u) = m, m, \omega^2u^2/2; u = \sqrt{M(x)}x$ to retain the standard format for the constant mass settings. In the process, one may need to use some position-dependent deformed/rescaled time as well. This is the focal point of the current methodical proposal.

This paper is organized as follows. In section 2, we introduce a generalized PDM nonlocal point transformation that maps a PDM-Lagrangian into a ‘constant unit-mass’ Lagrangian in the generalized coordinates and report on the conditions that secure the invariance of the related Euler–Lagrange equations. The harmonic oscillator linearization of the PDM Euler–Lagrange equations is discussed in section 3. To illustrate our methodical proposal, we consider (in the same section) some PDM Lagrangians for PDM particles moving in potential force fields (i) of a harmonic oscillator nature (i.e., $V(x) \sim m(x)x^2$), (ii) of only PDM-dependent nature (i.e., $V(x) \sim m(x)$), (iii) of a shifted harmonic oscillator nature, (i.e., $V(x) \sim m(x)(x + \xi)^2$), (iv) of a quadratic nonlinear oscillator nature (i.e., $V(x) \sim m(x)(1 + 2\alpha x)(1 + \lambda x)^2$), and (v) of a Morse-type oscillator nature (i.e., $V(x) \sim m(x)(1 - e^{-\eta x})^2$). We observe that while the Mathews–Lakshmanan nonlinear oscillators [18–24, 29–31] are reproduced in case (i) and (ii), some ‘shifted’ Mathews–Lakshmanan nonlinear oscillators are obtained in case (iii) for the same PDM $m(x) = 1/\left(1 + \lambda x^2\right); \lambda \geq 0$. Moreover, to show that the usage of the current methodical proposal is not only limited to oscillator linearization, we discuss (in section 4) the mapping of an isotonic nonlinear oscillator into a PDM deformed isotonic oscillator. We conclude in section 5.
2. PDM Lagrangians; nonlocal PDM-point transformation and invariance

Consider a classical particle with a constant ‘unit mass’ moving in the generalized coordinate $q = q(x)$, a potential force field $V(q)$, and a deformed/rescaled time $\tau$. In this case, the Lagrangian for such a system is given by

$$L(q, \dot{q}, \tau) = \frac{1}{2}\dot{q}^2 - V(q); \quad \dot{q} = \frac{dq}{d\tau},$$

and the corresponding Euler–Lagrange’s equation, therefore, reads

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \implies \frac{d^2q(x)}{d\tau^2} + \frac{\partial V(q)}{\partial q} = 0. \quad (4)$$

The introduction of an invertible nonlocal point transformation of the form

$$q \equiv q(x) = \int \sqrt{g(x)} \, dx, \quad \tau = \int f(x) \, dt \implies \frac{dr}{dt} = f(x) \neq 0; \quad x \equiv x(t). \quad (5)$$

could prove to be quite handy in the process. Under such nonlocal transformation settings, one may, in a straightforward manner, show that

$$\frac{dq}{d\tau} = \dot{q} = \dot{x} \sqrt{g(x)} = \dot{x} \sqrt{m(x)}, \quad \frac{d^2q}{d\tau^2} = \frac{\sqrt{g(x)}}{f(x)^2}\left[\ddot{x} + \frac{1}{2}\left(\frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}\right)\dot{x}^2\right]$$

and, in turn, equation (4) reads

$$\left[\ddot{x} + \frac{1}{2}\left(\frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)}\right)\dot{x}^2\right] + \frac{f(x)^2}{g(x)} \frac{dV(x)}{dx} = 0. \quad (7)$$

Obviously, the comparison between equation (7) and (2) suggests that the Euler–Lagrange equations of motion (2) and (7) are identical if and only if $f(x)$ and $g(x)$ satisfy the condition

$$g(x) = mx f(x)^2 \iff \frac{g'(x)}{g(x)} - 2\frac{f'(x)}{f(x)} = \frac{m'(x)}{m(x)}. \quad (8)$$

One may now conclude that under such conditionally, equation (8), invertible (i.e., the Jacobian determinant $\det \left(\frac{dx}{d\dot{q}}\right) \neq 0$) nonlocal point transformation, the PDM Euler–Lagrangeian equations’ invariance is secured. That is

$$L(q, \dot{q}, \tau) = \frac{1}{2}\dot{q}^2 - V(q) \iff \left\{ \begin{array}{l} q(x) = \int \sqrt{m(x)} f(x) \, dx \\ \tau = \int f(x) \, dt \\ g(x) = m(x) f(x)^2 \\ \dot{q} = \dot{x} \sqrt{m(x)} \end{array} \right\} \iff L(x, \dot{x}, t)$$

$$= \frac{1}{2}m(x)\dot{x}^2 - V(x), \quad (9)$$

and hence the Euler–Lagrange equations of motion (4) and (7) are invariant under the nonlocal point transformation (9), therefore. In fact, this is a documentation that the Euler–Lagrange equation remains invariant under some local and nonlocal invertible point transformation (see, e.g., Muriel and Romero [32] and Pradeep et al [33] and related references cited therein).
At this very point, one should notice that our invertible nonlocal point transformation (9) is just a subset of the well known generalized Sundman transformations (see, e.g., [34–37] and related references cited therein)

\[ X = K(x, t), \; dT = N(x, t)\,dt, \; \frac{\partial K(x, t)}{\partial x} \neq 0, \]

that are used to define Sundman symmetries (see, e.g., [32, 34, 37]). Here

\[ T = \tau, \; N(x, t) = f(x), \; X = q(x), \; K(x, t) = \int \sqrt{m(x)}f(x)\,dx, \]

\[ N(x, t)\frac{\partial K(x, t)}{\partial x} = \sqrt{m(x)}f(x)^2 \neq 0. \]

The connection between our nonlocal transformation (9) and the generalized Sundman transformation [32] is therefore clear. Of course, the nonlocality is an obvious manifestation of the nonlocal term

\[ T = \int N(x, t)\,dt. \]

Nevertheless, such nonlocal transformations are usually used in the linearization of a class of nonlinear ordinary differential equations (ODEs) to transform them into solvable linear ODEs (an interesting issue for nonlinear physical problems). However, we shall use our nonlocal point transformation (9) beyond the linearization of ODEs. Namely, and in a more simplistic language, our nonlocal point transformation (9) also offers some mapping between Euler–Lagrange equations (4) and (7) where the solution of one of them (hence, \( L(q, \dot{q}, \tau) \)) or \( L(x, \dot{x}, t) \) is the reference Lagrangian) would lead to the solution of the other (hence, \( L(q, \dot{q}, \tau) \) becomes the target Lagrangian). This shall be illustrated in the forthcoming examples/models below.

3. Oscillator-linearization of the PDM Euler–Lagrange equation

We now consider the classical particle with a constant `unit mass` moving in the above conditionally time-rescaled generalized coordinate (3) under the influence of the potential force field \( V(q) = \omega^2 q^2/2 \). Then, the oscillator Lagrangian

\[ L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2, \tag{10} \]

would yield a linear oscillator-type Euler–Lagrange equation in the form of

\[ \frac{d^2q(x)}{d\tau^2} + \omega^2 q = 0. \tag{11} \]

With some suitable initial conditions (\( q(0) = A \), and \( \dot{q}(0) = 0 \), say), it admits the periodic solution

\[ q(x) = A \cos(\omega \tau + \phi). \]

In this case, equations (4) and (6) would imply

\[ \ddot{x} + \frac{1}{2} \frac{m'(x)}{m(x)} \dot{x}^2 + \omega^2 \frac{f(x)}{\sqrt{m(x)}} q(x) = 0. \tag{12} \]
Which, when compared with (2) results in the linear-oscillator mapping condition
\[
\frac{1}{m(x)} \frac{\partial V(x)}{\partial x} = \omega^2 \sqrt{\frac{f(x)}{m(x)}} q(x) \iff q(x) = \frac{1}{\omega^2 \sqrt{m(x)f(x)}} \frac{\partial V(x)}{\partial x}
\] (13)

Therefore, the mapping between the conditionally time-rescaled generalized coordinate system and the Cartesian one-dimensional \(x\)-coordinate is now clear. We illustrate the applicability of the above oscillation applicability of the above oscillator-linearization procedure through the following examples.

3.1. Mathews–Lakshmanan PDM-nonlinear oscillators I; an \(f(x) = m(x)\) case

Let us consider the PDM particle moving in the harmonic oscillator force field \(V(x) = m(x)\omega^2 x^2/2\) with the corresponding Lagrangian
\[
L(x, \dot{x}, t) = \frac{1}{2} m(x)\dot{x}^2 - \frac{1}{2} m(x)\omega^2 x^2.
\] (14)

Then, the linear-oscillator mapping condition (13) would suggest
\[
q(x) = x\sqrt{m(x)} = \int \sqrt{m(x)f(x)}\,dx,
\] (15)

provided that
\[
f(x) = \left(1 + \frac{1}{2} \frac{m'(x)}{m(x)}\right).
\] (16)

Moreover, the assumption that \(f(x) = m(x)\) would result in a specific PDM-function. That is
\[
f(x) = 1 + \frac{1}{2} \frac{m'(x)}{m(x)} = m(x) \iff f(x) = m(x) = \frac{1}{\pm \lambda x^2}; \lambda \geq 0.
\] (17)

Therefore, the nonlocal transformation (9) now (with \(f(x) = m(x)\)) reads
\[
L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \iff \begin{cases} q(x) = x\sqrt{m(x)} \\ \dot{q} = \dot{x}\sqrt{m(x)} \\ \frac{d\tau}{dt} = m(x) \\ g(x) = m(x)^3 \end{cases} \iff L(x, \dot{x}, t)
\]
\[
\frac{1}{2} \left(\dot{x}^2 - \omega^2 x^2\right) = \frac{1}{\pm \lambda x^2}; \lambda \geq 0,
\] (18)

and, in turn, yields the Mathews–Lakshmanan oscillators’ equations [23–25]
\[
\dot{x} = \pm \frac{\lambda x}{1 \pm \lambda x^2} + \frac{\omega^2 x}{1 \pm \lambda x^2} = 0.
\] (19)

With the solutions
\[
x(t) = A \cos(\Omega t + \phi); \Omega^2 = \frac{\omega^2}{1 \pm \lambda A^2}.
\] (20)

Hereby, one may mention that the PDM-quantum mechanical versions of such nonlinear oscillators are studied by Lakshmanan and Chandrasekar [24] and by Cariñena et al [30, 31], where the transition from the PDM-classical system to the PDM-quantum one is analyzed.
through some differential geometric factorization recipe. For more details on this issue the reader may refer to [24, 30, 31] and related references cited therein.

### 3.2. Mathews–Lakshmanan PDM-nonlinear oscillators II; an $f(x) = \beta \, m'(x) / 2m(x)$ case

Consider the PDM particle moving in a force field of the form $V(x) = \beta^2 \omega^2 m(x)/2$, where $\beta$ is a non-zero constant introduced for the convenience of calculations. Then the corresponding Lagrangian is given by

$$L(x, \dot{x}, t) = \frac{1}{2} m(x) \dot{x}^2 - \frac{\alpha^2}{2} \beta^2 m(x).$$

With the choice that

$$f(x) = \beta \frac{m'(x)}{2m(x)} \implies q(x) = \beta \int \frac{1}{2 \sqrt{m(x)}} \dot{x} \, dx = \beta \sqrt{m(x)} = m(x) = \exp \left( \frac{2}{\beta} \int f(x) \, dx \right),$$

one may immediately show that the nonlocal transformation (9) reads

$$L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \alpha^2 \dot{q}^2 \implies \left\{ \begin{array}{l}
q(x) = \beta \sqrt{m(x)} \\
\dot{q} = \dot{x} \sqrt{m(x)} \\
d\tau/dt = \beta m'(x)/2m(x) \\
g(x) = \beta m'(x)^2/4m(x)
\end{array} \right\} \implies L(x, \dot{x}, t)
$$

$$= \frac{1}{2} m(x) \left( \dot{x}^2 - \beta^2 \omega^2 \right).$$

(23)

To imply the Euler–Lagrange equation

$$\ddot{x} + \frac{1}{2} \frac{m'(x)}{m(x)} \dot{x}^2 + \beta^2 \omega^2 \frac{m'(x)}{2m(x)} = 0.$$

(24)

In this case, moreover, the Mathews–Lakshmanan oscillators’ equations [23–25]

$$\ddot{x} \mp \frac{\lambda x}{1 \pm \lambda x^2} \dot{x}^2 + \frac{\alpha^2 x}{1 \pm \lambda x^2} = 0; \lambda = \pm \frac{1}{\beta^2}; \lambda \geq 0,$$

(25)

are obtained using

$$m(x) = \frac{1}{1 \pm \lambda x^2}.$$

(26)

and admit solutions in the form of

$$x(t) = A \cos (\Omega t + \phi); \quad \Omega^2 = \frac{\alpha^2}{1 \pm \lambda A^2}.$$

(27)

Nevertheless, one should also notice that the Mathews–Lakshmanan oscillators’ equations are obtained by direct substitutions of the set of four potential force fields.
\[
V(x) = \begin{cases} 
-m(x) \omega^2 / 2 \lambda; & \text{for } m(x) = 1 / (1 + \lambda x^2) \\
+m(x) \omega^2 / 2 \lambda; & \text{for } m(x) = 1 / (1 - \lambda x^2) \\
\frac{1}{2} m(x) \omega^2 x^2; & \text{for } m(x) = 1 / (1 \pm \lambda x^2)
\end{cases}
\] (28)

in (2) (each potential at a time, of course). This is very much related to the nature of the given PDM function.

3.3. Shifted Mathews–Lakshmanan PDM-nonlinear oscillators III; an \( f(x) = m(x) \) case

Let us consider the PDM particle moving in the shifted harmonic oscillator force field

\[
V(x) = m(x) \omega^2 (x + \xi)^2 / 2.
\]

Then the corresponding Lagrangian is given by

\[
L(x, \dot{x}, t) = \frac{1}{2} m(x) \dot{x}^2 - \frac{1}{2} m(x) \omega^2 (x + \xi)^2.
\] (29)

With the choice that

\[
q(x) = (x + \xi) \sqrt{m(x)} \implies \frac{dq(x)}{dx} = \sqrt{m(x)} f(x) \implies f(x) = \left(1 + \frac{1}{2} \frac{m'(x)}{m(x)} (x + \xi)\right).
\] (30)

the assumption \( f(x) = m(x) \) would yield the specific PDM-function of the form

\[
f(x) = m(x) = \frac{1}{1 \pm \lambda (x + \xi)^2}; \quad \lambda \geq 0.
\] (31)

Therefore, the nonlocal transformation (9) now reads

\[
L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \iff \begin{aligned}
q(x) &= (x + \xi) \sqrt{m(x)} \\
\dot{q} &= \dot{x} \sqrt{m(x)} \\
d\tau / dt &= m(x) \\
g(x) &= m(x)^3
\end{aligned} \iff L(x, \dot{x}, t)
\]

\[
= \frac{1}{2} \left[ \dot{x}^2 - \omega^2 (x + \xi)^2 \right]; \quad \lambda \geq 0,
\] (32)

to imply the shifted Mathews–Lakshmanan oscillators’ equations

\[
\ddot{x} \mp \frac{\lambda (x + \xi)}{1 \pm \lambda (x + \xi)^2} \dot{x}^2 + \frac{\omega^2 (x + \xi)}{1 \pm \lambda (x + \xi)^2} = 0.
\] (33)

with the solutions

\[
x(t) = A \cos (\Omega t + \phi) - \xi; \quad \Omega^2 = \frac{\omega^2}{1 \pm \lambda A^2}.
\] (34)

Yet, one can follow (step-by-step) the nonlocal transformation recipe in (23) to show that if the PDM in (31) is moving in the potential force field \( V(x) = \beta^2 \omega^2 m(x)/2 \), it would admit similar Euler–Lagrange dynamical equations of motion as those in (33) and exactly follow the same trajectory as that in (34). Under such PDM settings, one would observe that the shifted Mathews–Lakshmanan oscillators’ equations (33) are obtained by direct substitutions of the set of four potential force fields
\[ V(x) = \begin{cases} 
-\frac{m(x)\omega^2}{2\lambda}; & \text{for } m(x) = 1 + \lambda(x + \xi)^2 \\
+\frac{m(x)\omega^2}{2\lambda}; & \text{for } m(x) = 1 - \lambda(x + \xi)^2 \\
\frac{1}{2}m(x)\omega^2(x + \xi)^2; & \text{for } m(x) = 1(1 \pm \lambda(x + \xi)^2) 
\end{cases} \] (35)

in (2) (each potential at a time, of course).

### 3.4. A quadratic nonlinear PDM Oscillator; an \( f(x) = 1 \) case

Consider a PDM particle moving in the potential force field

\[ V(x) = -\frac{a^2}{2\lambda^2}m(x)(1 + 2\lambda x)(1 + \lambda x)^2 \] (36)

with the corresponding Lagrangian

\[ L(x, \dot{x}, t) = \frac{1}{2}m(x)\dot{x}^2 + \frac{a^2}{2\lambda^2}m(x)(1 + 2\lambda x)(1 + \lambda x)^2. \] (37)

Let us now defined a point transformation (with \( f(x) = 1 \)) of the form

\[ q(x) = \frac{x}{1 + \lambda x}. \] (38)

In this case, one obtains

\[ 1 + \frac{1}{4} \frac{m'(x)}{m(x)} \dot{x}^2 = m(x)^{1/4} \implies m(x) = \left(1 + \lambda x\right)^4 \implies q(x) = \frac{x}{1 + \lambda x}. \] (39)

Under such settings, the nonlocal transformation (9) reads

\[ L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \alpha^2 q^2 \implies \begin{cases} q(x) = x m(x)^{1/4} \\
\dot{q} = \dot{x} \sqrt{m(x)} \\
d\tau/dt = 1 \\
g(x) = m(x) \end{cases} \implies L(x, \dot{x}, t) \]

\[ = \frac{1}{2} m(x) \left( \dot{x}^2 + \frac{\alpha^2}{\lambda^2}(1 + 2\lambda x)(1 + \lambda x)^2 \right). \] (40)

and yields the nonlinear quadratic oscillator equation [23, 24]

\[ \ddot{x} - \frac{2\lambda}{1 + \lambda x} \dot{x}^2 + \alpha^2 x(1 + \lambda x) = 0; \quad \alpha^2 = \omega^2, \] (41)

that admits a solution of the form

\[ x(t) = \frac{A \cos(at + \phi)}{1 - \lambda A \cos(at + \phi)}; \quad 0 \leq A < \frac{1}{\lambda}. \] (42)

### 3.5. A Morse-oscillator; an \( f(x) = n \) case

Consider a PDM particle moving in a Morse-type oscillator force field with the corresponding Lagrangian
\[ L(x, \dot{x}, t) = \frac{1}{2} m(x) \dot{x}^2 - V(x); \quad V(x) = \frac{1}{2} m(x) \alpha^2 (1 - e^{-\eta x})^2 \] (43)

Let us define a nonlocal transformation of the form

\[ q(x) = \sqrt{m(x)} - 1 = \int \sqrt{m(x)} f(x) \, dx. \] (44)

Then one obtains

\[ \sqrt{m(x)} f(x) = \frac{m'(x)}{2\sqrt{m(x)}} \implies f(x) = \frac{m'(x)}{2m(x)} = \eta \implies m(x) = \exp(2\eta x) \implies \tau = \eta t. \] (45)

Under such settings, the nonlocal transformation (9) reads

\[ L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \alpha^2 \dot{q}^2 \leftrightarrow \begin{cases} q(x) = \sqrt{m(x)} - 1 \\ \dot{q} = \dot{x}\sqrt{m(x)} \\ d\tau/dt = \eta \\ g(x) = \eta^2 m(x) \end{cases} \implies L(x, \dot{x}, t) \]

\[ = \frac{1}{2} m(x) \left( \dot{x}^2 - \alpha^2 (1 - e^{-\eta x})^2 \right). \] (46)

and, in turn, implies the Morse-type PDM oscillator’s equation [23, 24]

\[ \ddot{x} + \eta \dot{x}^2 + \frac{\alpha^2}{\eta} (1 - e^{-\eta x}) = 0; \quad \alpha^2 = \omega^2 \eta^2 \] (47)

with the solution

\[ x(t) = \frac{1}{\eta} \ln (1 + A \cos(\omega t + \phi)); \quad 0 \leq A < 1. \] (48)

4. Mapping an isotonic nonlinear-oscillator into a PDM-deformed isotonic nonlinear-oscillator

In this section we consider a unit mass particle moving in an isotonic oscillator force field in the above conditionally time-rescaled generalized coordinate system. Then the corresponding isotonic oscillator Lagrangian

\[ L(q, \dot{q}, \tau) = \frac{1}{2} \dot{q}^2 - V(q); \quad V(q) = \frac{1}{2} \omega^2 q^2 + \frac{\beta}{q^2}. \] (49)

would imply the Euler–Lagrange equation

\[ \frac{d^2 q(x)}{d\tau^2} + \omega^2 q - \frac{2\beta}{q^3} = 0, \] (50)

which is known as the Ermakov–Pinney’s equation. It admits a general solution [29] of the form

\[ q = \frac{1}{\omega A} \sqrt{\left( \omega^2 A^4 - 2\beta \right)} \sin^2 (\omega \tau + \delta) + 2\beta. \] (51)

This model follows the conditional nonlocal transformation used for the Mathews–Lakshmanan PDM-oscillators I (i.e., the condition that \( f(x) = m(x) = 1/\left(1 + \lambda x^2\right); \lambda \geq 0 \)). In this case, the nonlocal transformation (9) reads
\[ L(q, \dot{q}, \tau) = \frac{1}{2} q^2 - \frac{1}{2} \omega^2 q^2 - \frac{\beta}{q^2} \begin{cases} q(x) = x \sqrt{m(x)} \\ \dot{q} = \dot{x} \sqrt{m(x)} \\ \frac{\partial}{\partial t} = m(x) \\ g(x) = m(x)^3 \end{cases} \Leftrightarrow L(x, \dot{x}, t) = \frac{1}{2} \left( \dot{x}^2 + \omega^2 x^2 \right) - \frac{\beta}{x^2} \left( 1 \pm \lambda x^2 \right) \]

and satisfies the Euler–Lagrange equation for a PDM-deformed isotonic nonlinear-oscillator

\[ \ddot{x} = \frac{\dot{x}}{1 \pm \lambda x^2} \left( \frac{\dot{x}}{1 \pm \lambda x^2} \right)^2 + \frac{\omega^2 x}{1 \pm \lambda x^2} - \frac{2\beta}{x^2} \left( 1 \pm \lambda x^2 \right) = 0; \quad \alpha = \mp \omega^2. \]  

The general solution of which is given by

\[ x(t) = \frac{1}{2\Omega^2} \sqrt{\left( \Omega^2 A^2 - 2\beta \right) \sin^2 (\Omega t + \delta) + 2\beta}; \quad \omega^2 = \left( 1 \pm \lambda A^2 \right) \left( A^2 \pm \frac{2\beta}{A^2} \right). \]  

Hereby, we observe that our nonlocal point transformation (9) has offered (in addition to the nonlinear-oscillators’ linearizations discussed above) a mapping (52) of a unit mass isotonic nonlinear-oscillator into a PDM-deformed isotonic nonlinear-oscillator. Hence, \( L(q, \dot{q}, \tau) \) plays the role as a reference Lagrangian and \( L(x, \dot{x}, t) \) as a target Lagrangian in (52).

5. Concluding remarks

In this work, we have introduced a general nonlocal point transformation for PDM Lagrangians and their mapping into a ‘constant unit-mass’ Lagrangians in the generalized coordinates. The conditions on the invariance of the related Euler–Lagrange equations are also reported. The harmonic oscillator linearization of the PDM Euler–Lagrange equations is discussed through some illustrative examples including harmonic oscillators, shifted harmonic oscillators, a quadratic nonlinear oscillator, and a Morse-type oscillator. The Mathews–Lakshmanan nonlinear oscillators are reproduced and some ‘shifted’ Mathews–Lakshmanan nonlinear oscillators are reported. We have also discussed the mapping of an isotonic nonlinear oscillator into a PDM deformed isotonic oscillator. In the light of the experiment above our observations are in order.

In connection with the Mathews–Lakshmanan nonlinear oscillators I and II, we observe that the PDM-function (26) subjected to move in the set of four potential force fields (28) (one at a time) admits/feels exactly similar dynamical effects as documented in the corresponding Euler–Lagrange equations of motion (19) and (25) and follows exactly similar trajectories. This tendency of similar dynamics, similar trajectories and similar total energies

\[ E = \frac{1}{2} \omega^2 \frac{A^2}{1 \pm \lambda A^2}, \]  

is attributed to the nature of the PDM-functional settings used. Similar dynamics, trajectories and total energies

\[ E = \frac{1}{2} \omega^2 \frac{(A - \xi)^2}{1 \pm \lambda (A - \xi)^2}, \]  

trends are also observed for the shifted Mathews–Lakshmanan nonlinear oscillators III.
The scope of the applicability of the current methodical proposal extends beyond the nonlinear-oscillators’ linearizations of the PDM Euler–Lagrange equations (documented through the illustrative examples in section 3 above) into the extraction of exact solutions of more complicated dynamical problems. The mapping of the isotonic nonlinear oscillator in the generalized coordinates (i.e., reference/target-Lagrangian) into a PDM-deformed isotonic oscillator on the x-coordinate (target/reference-Lagrangian) was just one of such exact-solution extractions through the nonlocal transformation (9). The extension of the applicability of our invertible nonlocal transformation (9) may include more than one-dimensional classical systems as well.

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