On a quadratic nonlinear Schrödinger equation: sharp well-posedness and ill-posedness

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Abstract We study the initial value problem of the quadratic nonlinear Schrödinger equation

\[ iu_t + u_{xx} = u\bar{u}, \]

where \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \). We prove that it’s locally well-posed in \( H^s(\mathbb{R}) \) when \( s \geq -\frac{1}{4} \) and ill-posed when \( s < -\frac{1}{4} \), which improve the previous work in [7]. Moreover, we consider the problem in the following space,

\[ H^{s,a}(\mathbb{R}) = \left\{ u : \|u\|_{H^{s,a}} \triangleq \left( \int \left( |\xi|^s \chi_{\{|\xi|>1\}} + |\xi|^a \chi_{\{|\xi|\leq 1\}} \right)^2 |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\} \]

for \( s \leq 0, a \geq 0 \). We establish the local well-posedness in \( H^{s,a}(\mathbb{R}) \) when \( s \geq -\frac{1}{4} - \frac{1}{2}a \) and \( a < \frac{1}{2} \). Also we prove that it’s ill-posed in \( H^{s,a}(\mathbb{R}) \) when \( s < -\frac{1}{4} - \frac{1}{2}a \) or \( a > \frac{1}{2} \).

It remains the cases on the line segment: \( a = \frac{1}{2}, -\frac{1}{2} \leq s \leq 0 \) open in this paper.

Keywords: nonlinear Schrödinger equation, local well-posedness, ill-posedness, Bourgain space

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1 Introduction

This paper is concerned with the low regularity behavior of the initial value problem (IVP) for 1-D quadratic nonlinear Schrödinger equations

\[ iu_t + u_{xx} = Q(u, \bar{u}), \quad x, t \in \mathbb{R}, \]  \hspace{1cm} (1.1)
\[ u(x, 0) = u_0(x), \]  \hspace{1cm} (1.2)

where \( Q : \mathbb{C}^2 \to \mathbb{C} \) is a quadratic polynomial. This particular problem as well as its higher dimensional version, has been extensively studied. Here, we refer some of them, which are closely related to our topic. As it’s well-known, the IVP of (1.1) is locally well-posed in \( H^s(\mathbb{R}) \) when \( s \geq 0 \) for any type quadratic nonlinearity, see [3] and [13]. The results were proved by the Strichartz estimates. It’s sharp in some sense, because the IVP (1.1) is ill-posed when \( s < 0 \) if the nonlinearity is \( |u|u \) (power type) (see [9] for focusing case, and [4] for defocusing case), by Gallilean invariance. However, it’s shown by Kenig, Ponce and Vega in [7] that, one can lower the regularity below \( s = 0 \) if the nonlinearity is not Gallilean invariance. Three typical nonlinearities of this type are

\[ Q(u, \bar{u}) = u^2, uu, \bar{u}^2. \]  \hspace{1cm} (1.3)

In [7], the authors established the local well-posedness for \( s > -\frac{3}{4} \) if the nonlinearity is of \( u^2 \) or \( \bar{u}^2 \) type, and for \( s > -\frac{1}{4} \) if it is \( uu \). The results were proved by the Bourgain argument (see [2] and [8]), which were mainly based on a bilinear estimate in Bourgain space \( X_{s,b} \). On the other hand, there are counterexamples shown in [7] and [12] that the key bilinear estimates in [7] fail to hold in \( X_{s,b} \), when \( s \leq -\frac{3}{4} \) for \( u^2, \bar{u}^2 \), and \( s \leq -\frac{1}{4} \) for \( uu \). It suggests that the common Bourgain space is not sufficient to study (1.1)(1.3) in a lower regular space. However, it doesn’t mean that it’s not well-posed in \( H^s(\mathbb{R}) \) of some lower indices. Indeed, in [1], Bejenaru and Tao pushed the threshold to \( s \geq -1 \) when the nonlinearity is \( u^2 \). The authors observed that the solution of (1.1) with \( Q(u, \bar{u}) = u^2 \) could be almost entirely supported in the spacetime-frequency domain \( \{ (\tau, \xi) : \tau > 0 \} \).

Combining this with some other observations (which we will try to describe below), they
introduced a modified Bourgian space as working space to avoid the failure in $X_{s,b}$ when $s \leq -\frac{3}{4}$. Further, they showed that the threshold $s \geq -1$ is sharp, that is, (1.1) is ill-posed when $s < -1$, for the nonlinearity $u^2$. Recently, in [10], the author showed that (1.1) is well-posed in $H^s(\mathbb{R})$ when $s \geq -1$ and is ill-posed when $s < -1$, for the nonlinearity $\bar{u}^2$.

In this paper, we are interested in

$$iu_t + u_{xx} = u\bar{u}.$$  \hspace{1cm} (1.4)

We strongly believe that the equation with the nonlinearity $u\bar{u}$ must behave differently from the two others, as what presented in [7]. One may not expect that the solution in this case can be almost supported in the region $\{\tau > 0\}$. We believe that the construction of the working space in [1] is heavily rely on the nonlinearity $u^2$, and is not well suitable in this situation. Therefore, we claim that the local result must be different from [1], and we wonder what the differences are. Indeed, applying the abstract and general theory in [1], we get our first result that the IVP of (1.4) is ill-posed in $H^s(\mathbb{R})$ when $s < -\frac{1}{4}$. That is,

**Theorem 1.1** (Ill-posedness below $H^{-\frac{1}{4}}(\mathbb{R})$). The IVP of (1.4) is not locally well-posed in $H^s(\mathbb{R})$ for any $s < -\frac{1}{4}$; more precisely, the solution operator fails to be uniformly continuous with respect to the $H^s(\mathbb{R})$ norm.

Therefore, we show that the local result related to (1.4) in [7] is sharp except the endpoint case when $s = -\frac{1}{4}$, which is one of the aim in this paper.

**Theorem 1.2** The IVP of (1.4) (1.2) is locally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ when $s = -\frac{1}{4}$. Moreover, the lifetime $\delta$ satisfies

$$\delta \sim \|u_0\|_{H^s}^\mu, \quad \text{for some} \quad \mu < 0.$$  

On the other hand, we observe that the ill-posedness of (1.4) is caused by the high-high interaction which cascades down into a very low frequency in the nonlinearity (see the computation in the proof of Theorem 1.1 in Section 5), while the low-frequency in $H^s(\mathbb{R})$
behaves as $L^2$. It implies that one may expect to lower regularity of the solution in high frequency by working it in another space which is based on a lower regular space in the low frequency than $L^2$-norm. For this purpose, we introduce a modification of Sobolev space $H^s(\mathbb{R})$. Define $H^{s,a}(\mathbb{R}) = \{ u : \| u \|_{H^{s,a}} < \infty \}$, where

$$\| u \|_{H^{s,a}} \triangleq \left\| (|\xi|^a \chi_{\{|\xi| \leq 1\}} + |\xi|^s \chi_{\{|\xi| > 1\}}) \hat{u}(\xi) \right\|_{L^2},$$

where $\chi_A$ is the characteristic function of the set $A$. It’s obvious that Schwartz space is dense in $H^{s,a}(\mathbb{R})$ when $a > -\frac{1}{2}$. When $a \geq 0$, then $H^s(\mathbb{R}) \hookrightarrow H^{s,a}(\mathbb{R})$ (particularly, they are equal when $a = 0$). In this paper, we always restrict that $s \leq 0$ and $a \geq 0$. We then turn our attention to study (1.4) in $H^{s,a}(\mathbb{R})$ and obtain

**Theorem 1.3 (Ill-posedness in $H^{s,a}(\mathbb{R})$).** The IVP of (1.4) is not locally well-posed in $H^{s,a}(\mathbb{R})$ for any $a > \frac{1}{2}$, $s \leq 0$ or $s < -\frac{1}{4} - \frac{1}{2}a$, $a \geq 0$; more precisely, the solution operator fails to be uniformly continuous with respect to the $H^{s,a}(\mathbb{R})$ norm.

**Theorem 1.4** Let $0 \leq a < \frac{1}{2}$, $-\frac{1}{4} - \frac{1}{2}a \leq s \leq 0$. Then the IVP of (1.4) (1.2) is locally well-posed for the initial data $u_0 \in H^{s,a}(\mathbb{R})$. Moreover, the lifetime $\delta$ satisfies

$$\delta \sim \| u_0 \|_{H^{s,a}}^{\mu'}, \quad \text{for some } \mu' < 0.$$

In fact, Theorems 1.3 and 1.4 extend the results of Theorems 1.1 and 1.2 respectively, by considering the well-posedness and ill-posedness theories in the modification Sobolev spaces.

The main technique to prove Theorems 1.2 and 1.4 (together) is via a fixed point argument in some modified Bourgain spaces ($S^{-\rho,a}$, see below). We are indebted in [1] for the stimulating arguments. These results do not conclude anything about the case in $H^{s,a}(\mathbb{R})$ when $a = \frac{1}{2}, -\frac{1}{2} \leq s \leq 0$.

**Some notations.** We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \leq CB$ for some large constant $C$ which may vary from line to line. We use $A \ll B$ to denote the statement $A \leq C^{-1}B$, and use $A \sim B$ to mean $A \lesssim B \lesssim A$. The notation $a+$ denotes $a + \epsilon$ for any small $\epsilon$, and $a-$ for $a - \epsilon$. $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We use $\| f \|_{L^q_t L^r_x}$ to denote
the mixed norm \( \left( \int \| f(x, \cdot) \|_{L^q}^p \, dx \right)^{\frac{1}{p}} \). Moreover, we denote \( \tilde{u}(\xi) \) and \( \tilde{u}(\xi, \tau) \) to be the spatial and spacetime Fourier transform of \( u \) respectively, and use \( \hat{f} \) or \( \mathcal{F}^{-1}_{\xi \tau} \) to denote the inverse Fourier transform of \( f \).

The rest of this article is organized as follows. In Section 2, we construct the working space. In Section 3, we derive some preliminary estimates. In Section 4, we recall some general well-posedness and ill-posedness theories and give the frames of the proof of the main theorems. In Section 5, we prove Theorems 1.1 and 1.3. In Section 6, we establish the key bilinear estimates to prove Theorems 1.2 and 1.4.

## 2 Construction of working space

In this section, we will construct the working space in building on Theorems 1.2 and 1.4. As what implied in [12], the standard Bourgain space \( X_{s,b} \) is not sufficient to handle the well-posedness in the critical case \( H^{-\frac{1}{4}}(\mathbb{R}) \) or some lower regularity spaces. Moreover, observing the counterexamples in [7] and [12], the failure of \( X_{s,b} \) in the bilinear estimate is caused when the \( \tau - \xi^2 \) is far away from \( \xi^2 \) in the spacetime-frequency domain \((\xi, \tau)\). For this reason, if one enhances some force in the working space to control the behavior of the equation when \( \tau - \xi^2 \) is large and \( \xi \) is small, then one may avoid those counterexamples.

We use the spirit of [1] to realize it. For constructing a proper working space, we need some sum spaces. First, we define some Bourgian-type spaces by the Fourier transform.

We will take \( \hat{X}_{s,b} \) and \( \hat{X}^{s,b} \) to be the closure of the Schwartz functions under the norms\(^1\):

\[
\| f \|_{\hat{X}_{s,b}} \triangleq \| \langle \xi \rangle^s \langle \tau - \xi^2 \rangle^b f \|_{L^2_{\xi \tau}}; \quad (2.1)
\]

\[
\| f \|_{\hat{X}^{s,b}} \triangleq \left( \sum_j 2^{2sj} \left( \sum_d 2^{bd} \| f \|_{L^2_{\xi} L^2_{\tau}(A_j \cap B_d)} \right)^2 \right)^{\frac{1}{2}}, \quad (2.2)
\]

where

\[
A_j \triangleq \{ (\xi, \tau) \in \mathbb{R}^2 : 2^j \leq \langle \xi \rangle < 2^{j+1} \};
\]

\[
B_d \triangleq \{ (\xi, \tau) \in \mathbb{R}^2 : 2^d \leq \langle \tau - \xi^2 \rangle < 2^{d+1} \}.
\]

\(^1\text{All sums and unions involving } j \text{ and } d \text{ shall be over the nonnegative unless otherwise mentioned.}\)
Remark. \( \hat{X}_{s,b} \) are the Fourier transforms of the standard Bourgain spaces \( X_{s,b} \). That is, 
\[
\| u \|_{X_{s,b}} = \| \hat{u} \|_{\hat{X}_{s,b}} \quad \text{for} \quad u \in X_{s,b}.
\]
Further, we note the relationship that, for any \( s \in \mathbb{R}, b' < b \),
\[
\hat{X}_{s,b} \hookrightarrow \hat{X}_{s,b} \hookrightarrow \hat{X}_{s,b}'.
\]
(2.3)
Define the functions \( m_{s,a} \) as
\[
m_{s,a} = |\xi|^a \chi_{\{|\xi|\leq 1\}} + |\xi|^s \chi_{\{|\xi|> 1\}}.
\]
Let \( X \) and \( Y \) are spaces under the norms:
\[
\| f \|_X \equiv \| m_{-\rho,a} f \|_{\hat{X}^0, \frac{1}{2}}; \quad (2.4)
\]
\[
\| f \|_Y \equiv \| m_{\alpha,a} f \|_{\hat{X}_0, \beta} + \| m_{-\rho,a} f \|_{L^2_\xi L^1_\tau}; \quad (2.5)
\]
where \( 0 \leq a < \frac{1}{2}, 0 \leq \rho \leq \frac{1}{4} + \frac{1}{2} a \) with the parameters \( \alpha = \left( \frac{1}{4} - \frac{1}{2} \rho \right) +, \beta = 0+ \).

Now, we define our first important space.
\[
Z \equiv X + Y \quad \text{(2.6)}
\]
with the norm
\[
\| f \|_Z = \inf \left\{ \| f_1 \|_X + \| f_2 \|_Y : f_1 \in X; f_2 \in Y; f = f_1 + f_2 \right\}.
\]
We give some properties on these spaces.

**Lemma 2.1** \( \hat{X}^{0,\frac{1}{2}} \subset L^2_\xi L^1_\tau \).

*Proof.* By a dyadic decomposition on \( \xi \), it suffices to show
\[
\| f \|_{L^2_\xi L^1_\tau(A_j)} \lesssim \sum_d 2^d \| f \|_{L^2_\xi L^1_\tau(B_d \cap A_j)}.
\]
It follows easily from the triangle inequality and Hölder’s inequality. \( \Box \)

Next, we give a pasting lemma between \( X \) and \( Y \). We define the set
\[
B \geq d \equiv \bigcup_{d' \geq d} B_{d'}; \quad B \leq d \equiv \bigcup_{d' \leq d} B_{d'}.
\]
Lemma 2.2 Let $f$ be a reasonable function and $k_0 \triangleq \frac{2\rho + 2\alpha}{1 - 2\beta}$.

(1) If $\text{supp } f \subset \bigcup_j (A_j \cap B_{\geq k_0 - 5})$, then

$$\|f\|_Y = \|f\|_Z;$$

(2) If $\text{supp } f \subset \bigcup_j (A_j \cap B_{\leq k_0 + 5})$, then

$$\|f\|_X = \|f\|_Z.$$

Proof. It’s trivial when $j = 0$, so we just consider $j \geq 1$. For (1), we only need to show

$$\|f\|_{\tilde{X}_{\alpha, \beta}} \lesssim \|f\|_{\tilde{X} - \rho, \frac{1}{2}},$$

when $\text{supp } f \subset \bigcup_j (A_j \cap B_{\geq k_0 - 5})$. Indeed, we have

$$\|\xi^\alpha \langle \tau - \xi \rangle^\beta f\|_{L^2_\tau}^2 = \sum_j \sum_{d \geq k_0 - 5} 2^{2\alpha j} 2^{3d} \|f\|_{L^2_\xi, (A_j \cap B_d)}^2 \\
\lesssim \sum_j \sum_{d \geq k_0 - 5} 2^{(2\rho + 2\alpha + k_0(2\beta - 1))j} 2^{-2\rho j} 2^d \|f\|_{L^2_\xi, (A_j \cap B_d)}^2 \\
= \sum_j \sum_{d \geq k_0 - 5} 2^{-2\rho j} 2^d \|f\|_{L^2_\xi, (A_j \cap B_d)}^2 \\
\lesssim \sum_j 2^{-2\rho j} \left( \sum_{d \geq k_0 - 5} 2^d \|f\|_{L^2_\xi, (A_j \cap B_d)} \right)^2.$$

For (2), it suffices to show that

$$\|f\|_{\tilde{X} - \rho, \frac{1}{2}} \lesssim \|f\|_{\tilde{X}_{\alpha, \beta}}.$$

This follows from

$$\sum_{d \leq k_0 + 5} 2^d \|f\|_{L^2_\xi, (A_j \cap B_d)} \lesssim 2^{(1 - \beta)k_0} \left( \sum_{d \leq k_0 + 5} 2^{2\beta d} \|f\|_{L^2_\xi, (A_j \cap B_d)} \right)^{\frac{1}{2}} \\
\leq 2^{(\rho + \alpha)j} \|\langle \tau - \xi \rangle^\beta f\|_{L^2_\xi, (A_j)},$$

where we used the Cauchy-Schwartz’s inequality in the first step.

We are ready to define our working space. Let $S^{-\rho, \alpha}, N^{-\rho, \alpha}$ be the closure of the Schwartz functions under the norms

$$\|u\|_{S^{-\rho, \alpha}} \triangleq \|\tilde{u}\|_Z; \quad (2.7)$$

$$\|u\|_{N^{-\rho, \alpha}} \triangleq \left\| \frac{\tilde{u}(\xi, \tau)}{\langle \tau - \xi \rangle^\beta} \right\|_Z, \quad (2.8)$$
where we write \( s = -\rho \), and \( S^{-\rho,a} \) is our working space, \( N^{-\rho,a} \) is the space related to Duhamel term. It’s easy to see that they are both Banach spaces.

**Remark.** The space \( X^{s,b} \) is a stronger space than \( X_{\rho,a}^{s,b} \) and can be regarded as a refined space of \( X_{\rho,a}^{s,b} \) in some situations. However, the space \( X^{s,b} \) is seemingly still insufficient to handle the critical case \( s = -\frac{1}{4} \) in \( H^s(\mathbb{R}) \), because of the weight of \( l^1 \)-norm in (2.2). For this reason, we shall add the weaker space (for the same exponents) \( X_{s,b}^{-1} \) to deal with the high-to-low frequency cascade case.

### 3 Some Preliminary Estimates

We will denote by \( \{ S(t) \}_{t \in \mathbb{R}} \) to be the unitary group generated by the corresponding linear equation of (1.4)

\[
v_t + v_{xx} = 0, \quad x, t \in \mathbb{R},
\]

such that \( v = S(t)u_0 \) solves (3.1)(1.2). It is also defined explicitly by spatial Fourier transform as

\[
\hat{S(t)u_0}(\xi) \equiv e^{-it\xi^2}\hat{u_0}(\xi).
\]

First, we present a well-known Stricharz estimate due to Bourgain space (see [6] for example). Recall that \( X_{s,b}^{0,1} \) is the standard Bourgain space, then

**Lemma 3.1** For \( u \in X_{0,\frac{1}{2}+} \), we have

\[
\| u \|_{L^6_{xt}} \lesssim \| u \|_{X_{0,\frac{1}{2}+}}. \tag{3.2}
\]

By interpolating between (3.2) and the following equality

\[
\| F \|_{L^2_{xt}} = \| F \|_{X_{0,0}}, \tag{3.3}
\]

we can generalize (3.2) as below.

**Lemma 3.2** For \( \theta \geq \frac{3}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \), \( q \in [2, 6] \) and \( F \in X_{0,\theta+} \), we have

\[
\| F \|_{L^q_{xt}} \lesssim \| F \|_{X_{0,\theta+}}. \tag{3.4}
\]
Next, we introduce some multiplier operators (appeared in [6], but with another versions). For nonnegative functions \( f, g, h \), define

\[
I_k^s(f, g, h) = \int m_k(\xi, \xi_1, \xi_2)^s f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) h(\xi, \tau)
\]

for \( k = 1, 2, 3 \), where \( \int = \int _{\xi_1+\xi_2=\tau} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \), and the multipliers \( m_k \) are defined as

\[
m_1 = |\xi|; \quad m_2 = |\xi + \xi_2|; \quad m_3 = |\xi_2|.
\]

Then we have

**Lemma 3.3** Let \( f, g, h \) are reasonable functions, then

\[
I_{\frac{1}{1}}(f, g, h) \lesssim \| h \|_{L^2} \| f \|_{\mathcal{X}_{0, \frac{1}{1}}} \| g \|_{\mathcal{X}_{0, \frac{1}{1}}}; \tag{3.6}
\]

\[
I_{\frac{1}{2}}(f, g, h) \lesssim \| f \|_{L^2} \| g \|_{\mathcal{X}_{0, \frac{1}{1}}} \| h \|_{\mathcal{X}_{0, \frac{1}{1}}}; \tag{3.7}
\]

**Proof.** We use the argument in [5] to prove the lemma. For \( I_{\frac{1}{1}} \), we change variables by setting

\[
\tau = \lambda + \xi^2, \quad \tau_1 = \lambda_1 + \xi_1^2, \quad \tau_2 = \lambda_2 - \xi_2^2, \quad (3.8)
\]

then, \( I_{\frac{1}{1}}(f, g, h) \) is changed into

\[
\int m_{\frac{1}{1}}^s f(\xi_1, \lambda_1 + \xi_1^2) g(-\xi_2, -\lambda_2 + \xi_2^2) h(\xi_1 + \xi_2, \lambda_1 + \lambda_2 + \xi_1^2 - \xi_2^2) d\xi_1 d\xi_2 d\lambda_1 d\lambda_2. \tag{3.9}
\]

We change variables again as follows. Let

\[
(\eta, \omega) = T(\xi_1, \xi_2), \quad (3.10)
\]

where

\[
\eta = T_1(\xi_1, \xi_2) = \xi_1 + \xi_2,
\]

\[
\omega = T_2(\xi_1, \xi_2) = \lambda_1 + \lambda_2 + \xi_1^2 - \xi_2^2.
\]

Then the Jacobian \( J \) of this transform satisfies

\[
|J| = 2|\xi_1 + \xi_2|.
\]
Define
\[ H(\eta, \omega, \lambda_1, \lambda_2) = f g \circ T^{-1}(\eta, \omega, \lambda_1, \lambda_2), \]
then, by eliminating \(|J|^\frac{1}{2}\) with \(m^\frac{1}{2}_1\), (3.9) has a bound of
\[
\int h(\eta, \omega) \cdot \frac{H(\eta, \omega, \lambda_1, \lambda_2)}{|J|^\frac{1}{2}} d\eta d\omega d\lambda_1 d\lambda_2.
\] (3.11)

By Hölder' inequality, we have
\[
(3.11) \leq \|h\|_{L^2_{\eta\omega}} \cdot \left( \int \frac{|H(\eta, \omega, \lambda_1, \lambda_2)|^2}{|J|} d\eta \omega \right)^\frac{1}{2} d\lambda_1 d\lambda_2
\]
\[
\lesssim \|h\|_{L^2} \int \|f(\xi_1, \lambda_1 + \xi_2^2)\|_{L^2_{\xi_1}} d\lambda_1 \cdot \int \|g(-\xi_2, -\lambda_2 + \xi_2^2)\|_{L^2_{\xi_2}} d\lambda_2
\]
\[
\lesssim \|h\|_{L^2} \|f\|_{\dot{X}_{0, \frac{1}{4}+}} \|g\|_{\dot{X}_{0, \frac{1}{4}+}},
\]
where we employed the inverse transform of (3.10) in the second step and Hölder' inequality in the third step.

For \(I_{1}^\frac{1}{2}\), the modification of the proof is replacing the variable transform \((\eta, \omega)\) by
\[
\eta = T_1(\xi, \xi_2) = \xi - \xi_2,
\]
\[
\omega = T_2(\xi, \xi_2) = \lambda - \lambda_2 + \xi_2^2 + \xi_2^2.
\]
Then the Jacobian \(J\) in this situation satisfies
\[
|J| = 2|\xi + \xi_2|.
\]
Therefore, we have the claim by the same argument as above.

For \(I_{3}^\frac{1}{2}\), we take
\[
\eta = T_1(\xi, \xi_1) = -\xi + \xi_1,
\]
\[
\omega = T_2(\xi, \xi_1) = -\lambda + \lambda_1 - \xi_1^2 + \xi_1^2.
\]
in this time. Then the Jacobian \(J\) in this situation satisfies
\[
|J| = 2|\xi_2|.
\]
So the claim follows again.

When \(s = 0\), by (3.10) we have
\[
I_1^0(f, g, h) \leq \|h\|_{L^2} \|\tilde{f}\|_{L^p} \|\tilde{g}\|_{L^q}
\]
\[
\lesssim \|h\|_{L^2} \|f\|_{\dot{X}_{0, b+}} \|g\|_{\dot{X}_{0, a'+}},
\] (3.12)
where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $b = 3 \left( \frac{1}{2} - \frac{1}{p} \right)$, $b' = 3 \left( \frac{1}{2} - \frac{1}{q} \right)$, that is $b + b' = \frac{3}{4}$, and $b, b' \in \left[ \frac{1}{4}, \frac{1}{2} \right]$.

Interpolation between (3.6) and (3.12) twice, we have

**Corollary 3.4** Let $I_1^s$ be defined by (3.5), then for any $s \in [0, \frac{1}{2}]$,

$$I_1^s(f, g, h) \lesssim \|h\|_{L^2} \|f\|_{\dot{X}_{0, b_1}} \|g\|_{\dot{X}_{0, b_2}}$$

(3.13)

where $b_1 = \frac{1}{2}(1 - s' + s)$, $b_2 = \frac{1}{4}(2s' + 1)$ for any $s' \in [s, \frac{1}{2}]$.

For $I_2^s$ and $I_3^s$, the similar estimates hold too. But in this paper, we just need the following crude estimates. For any $s < \frac{1}{2}$,

$$I_1^s(f, g, h) \lesssim \|h\|_{L^2} \|f\|_{\dot{X}_{0, \frac{1}{4}}} \|g\|_{\dot{X}_{0, \frac{1}{4}}}$$

(3.14)

$$I_2^s(f, g, h) \lesssim \|f\|_{L^2} \|g\|_{\dot{X}_{0, \frac{1}{4}}} \|h\|_{\dot{X}_{0, \frac{1}{4}}}$$

(3.15)

$$I_3^s(f, g, h) \lesssim \|g\|_{L^2} \|f\|_{\dot{X}_{0, \frac{1}{4}}} \|h\|_{\dot{X}_{0, \frac{1}{4}}}$$

(3.16)

**Remark.** Sometimes, one may interested in some critical estimates in Lemma 3.2 by replace $b = 1/2$ by $b = 1/2$ which may be useful to deal with some limiting case. In general, one may not have such critical estimates in $\dot{X}^{s,b}$. However, in some especial case, for example, when $|\xi| \ll |\xi_1|$ in $I_1^s(f, g, h)$ (similar for $I_2^s, I_3^s$), they hold in $\dot{X}^{s,b}$. Particularly, if we consider another Bourgain spaces $\dot{X}^{s,b}$, defined by the norm

$$\|f\|_{\dot{X}^{s,b}} \triangleq \sum_d 2^{bd} \|\langle \xi \rangle^s f\|_{L^2(B_d)}$$

then the critical estimates hold in these spaces. In fact, the spaces $\dot{X}^{s,b}$ are stronger than $\dot{X}^{s,b}$ in the sense that

$$\|f\|_{\dot{X}^{s,b}} \leq \|f\|_{\dot{X}^{s,b}}$$

(3.17)

which easily follows by the triangle inequality of $l^2$-norm.

### 4 Preparatory Theory

Recall the scale invariance that, if $u(x, t)$ is a solution of IVP (1.4) (1.2), then for any $\lambda > 0$,

$$u_{\lambda}(x, t) = \lambda^{-2} u(x/\lambda, t/\lambda^2)$$

(4.1)
is also a solution of (1.4) with the initial data replaced by

\[ u_{0,\lambda}(x) = \lambda^{-2}u_0(x/\lambda). \]

Note that for \( \lambda > 1 \),

\[ \|u_{0,\lambda}\|_{H^{s,a}} \lesssim \lambda^{-\frac{3}{2}-s}\|u_0\|_{H^{s,a}}, \text{ when } a \geq s. \]

So we can scale the initial data to be a small size in \( H^{s,a}(\mathbb{R}) \) (\( H^s(\mathbb{R}) \) when \( a = 0 \)) for \( s > -\frac{3}{2} \). It thus suffices to prove Theorems 1.2 and 1.4 for small initial data in \( H^{-\frac{2}{7}}(\mathbb{R}) \) and \( H^{s,a}(\mathbb{R}) \) respectively, with the lifetime \( \delta = 1 \).

Next, by the Duhamel’s formula, we can rewrite (1.4) (1.2) in integral form as

\[ u(t) = S(t)u_0 + \int_0^t S(t-t')u(t')\bar{u}(t') \, dt'. \]

If we are interested in (locally) solving the IVP up to time \( \delta = 1 \), then it can be replaced by

\[ u(t) = \eta(t)S(t)u_0 + \eta(t)\int_0^t S(t-t')u(t')\bar{u}(t') \, dt' \]

\[ \triangleq L(u_0) + N(u, u), \tag{4.2} \]

where

\[ L(u_0) \triangleq \eta(t)S(t)u_0; \quad N(u, v) \triangleq \eta(t)\int_0^t S(t-t')u(t')\bar{v}(t') \, dt', \]

and \( \eta(t) \) is a smooth bump function supported in the interval \([-2, 2]\) such that \( \eta(t) = 1 \) on \([-1, 1]\).

Now we recall some well-posedness and ill-posedness theories established in [1] (a little general in this paper). We shall be somewhat brief here and refer the reader to [1] for more details. Let \((D, \| \cdot \|_D)\) and \((S, \| \cdot \|_S)\) are Banach spaces, such that \( L : D \to S \) and \( N : S \times S \to S \) are densely defined. If

\[ (1) \|L(u_0)\|_S \lesssim \|u_0\|_D; \]

\[ (2) \|N(u, v)\|_S \lesssim \|u\|_S\|v\|_S. \]
Then we say the equation (4.2) is quantitatively well-posed in $D, S$. By a standard fixed point argument, if (4.2) is quantitatively well-posed in $D, S$, then it has local existence, continuity, uniqueness in $S$, when $\|u_0\|_D$ is small enough. If one still has the energy estimate

\[(3) \quad \|u\|_{C^0_0([0,1]:D)} \lesssim \|u_0\|_S,\]

then (4.2) is locally well-posed for the data in $D$ with small norm.

Concretely, we consider $D$ to be a weighted $L^2$ space. Define

$$
\|u_0\|_D \triangleq \|m(\xi)\hat{u}(\xi)\|_{L^2_\xi}; \quad \|u\|_S \triangleq \|\tilde{u}\|_{\tilde{S}} \triangleq \|m(\xi)\tilde{u}\|_{\tilde{S}_0}
$$

for some function $m$ and Hilbert space $\tilde{S}^0$. Since

$$
\tilde{u}(\xi, \tau) = \tilde{u}(-\xi, -\tau),
$$

by the argument in [1] (with a bit modification), (1)–(3) can be replaced by

(i) $|f| \leq |g|$, then $\|f\|_{\tilde{S}} \leq \|g\|_{\tilde{S}}$;

(ii) $\|f\|_{L^2_\xi L^1_\tau} \lesssim \|f\|_{\tilde{S}^0}$;

(iii) $\|f\|_{\tilde{S}^0} \lesssim \|f\|_{\tilde{X}_{0,100}}$;

(iv) $\|\langle \tau - \xi^2 \rangle^{-1}(f * g^*)\|_S \lesssim \|f\|_S \|g\|_S$, where $g^*(\xi, \tau) = g(-\xi, -\tau)$.

It easy to see that $(D, S) = (H^{-\rho,a}, S^{-\rho,a})$ satisfies the condition (i)–(iii). So we can see that the only work left to finish the proof of Theorems 1.2 and 1.4 (together) is the bilinear estimate (iv) in the solution space. More precisely, to prove Theorems 1.2 and 1.4, (iv) is equivalent to

$$
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} (\tilde{u} * \tilde{v}^*) \right\|_Z \lesssim \|\tilde{u}\|_Z \|\tilde{v}\|_Z,
$$

where $\tilde{v}^*(\xi, \tau) = \tilde{v}(-\xi, -\tau)$. It will be established in Section 6.
Next, suppose that (4.2) is quantitatively well-posed in $D, S$. If we define the nonlinear map $A_n : D \to S$ for $n = 1, 2, \cdots$ as

$$A_1(u_0) \triangleq L(u_0);$$
$$A_n(u_0) \triangleq \sum_{n_1, n_2 \geq 1; n_1 + n_2 = n} N(A_{n_1}(u_0), A_{n_2}(u_0)) \quad \text{for } n > 1,$$
then the solution map

$$u[u_0] = \sum_{n=1}^{\infty} A_n(u_0) \quad \text{in } S \quad \text{(absolutely convergent)}$$

for small data $u_0 \in D$. Moreover, we have

**Proposition 4.1** ([1]) Suppose that (4.2) is quantitatively well-posed in $D, S$, with a solution map $u_0 \mapsto u[u_0]$ from a ball $B_D$ in $D$ to a ball $B_S$ in $S$. Suppose that these spaces are then given other norms $D'$ and $S'$, which are weaker than $D$ and $S$ in the sense that

$$\|u_0\|_{D'} \lesssim \|u_0\|_D, \quad \|u\|_{S'} \lesssim \|u\|_S.$$

Suppose that the solution map $u_0 \mapsto u[u_0]$ is continuous from $(B_D, \| \cdot \|_{D'})$ to $(B_S, \| \cdot \|_{S'})$. Then for each $n$, the nonlinear operator $A_n : D \to S$ is continuous from $(B_D, \| \cdot \|_{D'})$ to $(B_S, \| \cdot \|_{S'})$.

This proposition gives us a way to disprove well-posedness in coarse topologies, simply by establishing that at least one of the operators $A_n$ is discontinuous.

### 5 Some Ill-posedness Analysis

In this section, we concentrate our attention on the consequences which are derived from the application on Proposition 4.1. We expect to obtain some necessary restriction on the regularity exponents for well-posedness theory. Roughly speaking, by Proposition 4.1, if the solution map is continuous from $D$ to $S$, then so is the quadratic

$$A_2 : u_0 \mapsto N(Lu_0, Lu_0).$$
We consider $D = H^{s,a}(\mathbb{R})$, $S = C_0^0([0,1]; H^{s,a}(\mathbb{R}))$, in which we set the lifetime $\delta = 1$ by scale invariance. Fix $N \gg 1$ and $\varepsilon_0 \ll 1$, set
\[
\hat{u}_0(\xi) = \varepsilon_0 N^{-s} \chi_{[-10,10]}(|\xi| - N),
\]
then $\|u_0\|_{H^{s,a}} \sim \varepsilon_0$, for any $a \in \mathbb{R}^+.

First, we consider the case $D = H^s(\mathbb{R})$ and $S = C_0^0([0,1]; H^s(\mathbb{R}))$, then
\[
\|A_2\|_{C_0^0([0,1]; H^s)}\text{ is equal to }
\sup_{0 \leq t \leq 1} \left\| \int_0^t S(t-t') \left( S(t')u_0 \cdot \overline{S(t')u_0} \right) dt' \right\|_{H^s}.
\]

Further, (5.1) has a lower bound of
\[
\sup_{0 \leq t \leq 1} \left\| \int_0^t \exp(-it\xi^2) \exp(2it'\xi(\xi - \xi_1)) \hat{u}_0(\xi_1) \hat{u}_0(-\xi + \xi_1) \ d\xi_1 dt' \right\|_{L^2_\xi}. \tag{5.2}
\]

Note that, for $\xi \in \left[ \frac{1}{100N}, \frac{1}{10N} \right]$,\n\[
\text{Re} \left( \exp(-it\xi^2) \exp(2it'\xi(\xi - \xi_1)) \right) > \frac{1}{2}, \tag{5.3}
\]
whenever $0 \leq t' \leq t \leq 1$ and $\xi_1$ resides in the support of $u_0$. Hence, we have
\[
(5.2) \gtrsim N^{-2s} \left\| \right\|_{L^2_\xi(\frac{1}{100N}, \frac{1}{10N})} \sim N^{-2s - \frac{1}{2}}.
\]

For the continuity of $A_2$, it’s necessary that $s \geq -\frac{1}{4}$. This proves Theorem 1.1.

From the computation above, the threshold is much restricted by the $L^2$–norm in the low frequency in $H^s(\mathbb{R})$. It’s a reason that we consider the modification spaces $H^{s,a}(\mathbb{R})$.
to lower the regularity in low frequency. A similar computation (but replaces $H^s(\mathbb{R})$ by $H^{s,a}(\mathbb{R})$) shows that the necessary condition on $s$ is changed into

$$s \geq -\frac{1}{4} - \frac{1}{2}a.$$  \hfill (5.4)

One may thus expect to lower the exponent $s$ by setting $a > 0$.

On the other hand, the exponent $s$ can’t lower to $-\infty$ by choosing various $a$ in (5.4). Indeed, if we localize $\xi$ to the region $(1, 2)$, then similarly, $\|A_2\|_{C^0_{\xi}([0,1];H^{s,a})}$ has a lower bound of

$$\sup_{0 \leq t \leq 1} \left\| \int_0^t \exp \left( -it\xi^2 \right) \exp(2it'\xi(\xi - \xi_1)) \hat{u}_0(\xi_1) \hat{u}_0(-\xi + \xi_1) d\xi_1 dt' \right\|_{L^2_{\xi}(1,2)}. \hfill (5.5)$$

Set $t = \frac{1}{100}N^{-1}$ now, then again we have (5.3), and (5.5) has a lower bound of $N^{-2s-1}$, which implies another restriction that $s \geq -\frac{1}{2}$ for each $a \in \mathbb{R}^+$. Moreover, set a new data

$$\hat{u}_0(\xi) = \varepsilon_0 N^{a + \frac{1}{2}} \chi_{[N^{-1}, 2N^{-1}]}(\|\xi\|),$$

then $\|u_0\|_{H^{s,a}} \sim \varepsilon_0$, for any $s \in \mathbb{R}$. On the other hand, $\|A_2\|_{C^0_{\xi}([0,1];H^{s,a})}$ is equal to

$$\sup_{0 \leq t \leq 1} \left\| \int_0^t \exp \left( -it\xi^2 \right) \exp(2it'\xi(\xi - \xi_1)) \hat{u}_0(\xi_1) \hat{u}_0(-\xi + \xi_1) d\xi_1 dt' \right\|_{L^2_{\xi}(1/100N^{-1}, 1/100N^{-1})} \nabla \geq N^{2(a + \frac{1}{2})} N^{-1} N^{-a} N^{-\frac{1}{2}} \nabla = N^{a - \frac{1}{2}},$$

which implies the necessary condition on the exponent $a$ of $a \leq \frac{1}{2}$ for each $s \in \mathbb{R}$. Thus proves Theorem 1.3.

6 Bilinear Estimates

As discussing above, in order to prove Theorem 1.4, we just need (4.4). By the pasting Lemma 2.3, we divide the proof of (4.4) into four cases. It will be very convenient to using the estimate (3.17) in the following precess.
Lemma 6.1 When $\text{supp} \, \hat{u}, \hat{v} \subset \bigcup_{j} (A_j \cap B_{\geq k_0j})$, then
\[
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} (\hat{u} \ast \hat{v}^*) \right\|_X \lesssim \| \hat{u} \|_Y \| \hat{v} \|_Y. \tag{6.1}
\]

Proof. By (3.17), it suffices to show that
\[
\sum_d 2^{-\frac{d}{2}} \left\| m_{-\rho,a}(\xi)(\hat{u} \ast \hat{v}^*) \right\|_{L^2_{\xi^2}(B_d)} \lesssim \| m_{\alpha,a} \hat{u} \|_{X_{0,a}} \| m_{\alpha,a} \hat{v} \|_{X_{0,a}},
\]
which is equivalent to show
\[
\sum_d 2^{-\frac{d}{2}} \left\| m_{-\rho,a}(\xi) \int \frac{f(\xi_1, \tau_1)}{m_{\alpha,a}(\xi_1) \langle \tau_1 - \xi_1^2 \rangle^\beta} \frac{g(-\xi_2, -\tau_2)}{m_{\alpha,a}(\xi_2) \langle \tau_2 + \xi_2^2 \rangle^\beta} \right\|_{L^2_{\xi^2}(B_d)} \lesssim \| f \|_{L^2_{\xi^2}} \| g \|_{L^2_{\xi^2}} \tag{6.2}
\]
for any $f, g \in L^2(\mathbb{R}^2)$, where $\int_{\tau} = \int_{\tau_1 = \xi_2 = \xi} d\xi_1 d\tau_1$. We may only consider the integration over the region of $|\xi_1| \geq |\xi_2|$ (it’s similar for $|\xi_1| \leq |\xi_2|$). Then we divide (6.2) into two parts to analyze.

Part 1. $|\xi_2| \lesssim 1$; Part 2. $|\xi_2| \gg 1$.

Part 1. $|\xi_2| \lesssim 1$. Note that $|\xi| \lesssim |\xi_1|$, so we always have
\[
m_{-\rho,a}(\xi) \cdot m_{\alpha,a}(\xi)^{-1} \lesssim 1,
\]
no matter when $|\xi_1| \leq 1$ or $|\xi_1| \geq 1$. Therefore, we have
\[
\sum_d 2^{-\frac{d}{2}} \left\| m_{-\rho,a}(\xi) \int \frac{f(\xi_1, \tau_1)}{m_{\alpha,a}(\xi_1) \langle \tau_1 - \xi_1^2 \rangle^\beta} \frac{g(-\xi_2, -\tau_2)}{m_{\alpha,a}(\xi_2) \langle \tau_2 + \xi_2^2 \rangle^\beta} \right\|_{L^2_{\xi^2}(B_d)} \lesssim \| f \|_{L^2_{\xi^2}} \| g \|_{L^2_{\xi^2}},
\]
where
\[
\frac{1}{q} + \frac{1}{q_1} = \frac{1}{2}; \quad \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \leq 1; \quad \frac{1}{q_2} = \frac{1}{2} + \frac{1}{q_4}; \quad \frac{1}{q_3} = \frac{1}{2} + \frac{1}{q_5}
\]
with \( q > 2, \beta q_4 > 1, \beta q_5 > 1, a < \frac{1}{2} \). By an elementary computation, we see that \( q, q_i, i = 1, \ldots, 5 \) are reasonable when \( \beta > 0, a < \frac{1}{2} \).

**Part 2**. \(|\xi_2| \gg 1\). Then \(|\xi_1| \gg 1\), and we have

\[
\left\| m_{-\rho,a}(\xi) \int \frac{f(\xi_1, \tau_1)}{m_{a,a}(\xi_1)(\tau_1 - \xi_1^2)\beta} m_{a,a}(\xi_2)(\tau_2 + \xi_2^2)\beta \right\|_{L_{\xi}^2(B_d)}
= \left\| m_{-\rho,a}(\xi) \left( \frac{f}{|\xi|^\alpha(\tau - \xi^2)^\beta} \right) \right\|_{L_{\xi}^2(B_d)}
\lesssim \left\| m_{-\rho,a}(\xi) \right\|_{L_{\xi}^pL_{\tau}^4(B_d)} \left\| \frac{f}{|\xi|^\alpha(\tau - \xi^2)^\beta} \right\|_{L_{\xi}^pL_{\tau}^4(|\xi| \gg 1)}
\lesssim 2^{d/q} \left\| f \right\|_{L_{\xi}^2} \left\| \xi \right\|_{L_{\xi}^dL_{\tau}^4(|\xi| \gg 1)} \left\| g^* \right\|_{L_{\xi}^2} \left\| \xi \right\|_{L_{\xi}^dL_{\tau}^4(|\xi| \gg 1)}
\lesssim 2^{d/q} \left\| f \right\|_{L_{\xi}^2} \left\| g \right\|_{L_{\xi}^2},
\]

where \( q, q_i, i = 1, \ldots, 5 \) as Part 1, and

\[
\frac{1}{p} + \frac{1}{p_1} = \frac{1}{2}; \frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3} - 1; \frac{1}{p_2} = \frac{1}{2} + \frac{1}{p_4}; \frac{1}{p_3} = \frac{1}{2} + \frac{1}{p_5}
\]

with \( \rho p > 1, \alpha p_4 > 1, \alpha p_5 > 1 \). They are reasonable when

\[
2\alpha > \frac{1}{2} - \rho, \beta > 0.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 6.2** When \( \text{supp} \, \hat{u} \subset \bigcup_j (A_j \cap B_{\leq k_0j}) \), \( \text{supp} \, \hat{v} \subset \bigcup_j (A_j \cap B_{\geq k_0j}) \), then

\[
\left\| \frac{1}{(\tau - \xi^2)} (\hat{u} \ast \hat{v}^*) \right\|_{X} \lesssim \|\hat{u}\|_X \|\hat{v}\|_Y. \tag{6.3}
\]

**Proof.** By (3.17), it suffices to show that

\[
\sum_d 2^{-d} \left\| m_{-\rho,a}(\xi) \int \frac{f(\xi_1, \tau_1)}{m_{-\rho,a}(\xi_1)(\tau_1 - \xi_1^2)^{\frac{\alpha}{2}}} m_{a,a}(\xi_2)(\tau_2 + \xi_2^2)\beta \right\|_{L_{\xi}^2(B_d)}
\lesssim \left\| f \right\|_{L_{\xi}^2} \left\| g \right\|_{\dot{X}^{0,0}} \tag{6.4}
\]

for any \( f \in L^2(\mathbb{R}^2), g \in \dot{X}^{0,0}\). we divide (6.4) into two parts to analyze.

**Part 1**. \(|\xi_2| \lesssim 1\) or \(|\xi_1| \lesssim 1\); **Part 2**. \(|\xi_2| \gg 1\) and \(|\xi_1| \gg 1\).
Part 1. \(|\xi_2| \leq 1\) or \(|\xi_1| \leq 1\). It concludes that

\[|\xi|, |\xi_1|, |\xi_2| \leq 1; \text{ or } |\xi| \sim |\xi_1| \gg 1, |\xi_2| \leq 1; \text{ or } |\xi| \sim |\xi_2| \gg 1, |\xi_1| \leq 1.\]

But they all can be treated as Part 1 in the proof of Lemma 6.1. Indeed, when \(|\xi|, |\xi_1|, |\xi_2| \leq 1\), then \(|\xi| \leq |\xi_1|\) or \(|\xi| \leq |\xi_2|\). So we have

\[m_{-\rho,\alpha}(\xi) \cdot m_{-\rho,\alpha}(\xi_1)^{-1} \leq 1, \text{ or } m_{-\rho,\alpha}(\xi) \cdot m_{\alpha,\alpha}(\xi_2)^{-1} \leq 1.\]

When \(|\xi| \sim |\xi_1| \gg 1, |\xi_2| \leq 1\). Then

\[m_{-\rho,\alpha}(\xi) \cdot m_{-\rho,\alpha}(\xi_1)^{-1} \leq 1.\]

When \(|\xi| \sim |\xi_2| \gg 1, |\xi_1| \leq 1\). Then

\[m_{-\rho,\alpha}(\xi) \cdot m_{\alpha,\alpha}(\xi_2)^{-1} \leq 1.\]

Hence, the argument used in Part 1 in the proof of Lemma 6.1 follows \((6.4)\) in this part.

Part 2. \(|\xi_2| \gg 1\) and \(|\xi_1| \gg 1\). We further divide \((6.2)\) into two subparts to analyze.

Subpart 1. \(|\xi_1| \gg 1, |\xi_2| \gg 1, |\xi| \gtrsim |\xi_1|\). Then

\[m_{-\rho,\alpha}(\xi) \cdot m_{-\rho,\alpha}(\xi_1)^{-1}, m_{\alpha,\alpha}(\xi_2)^{-1} \lesssim 1.\]

Therefore, by \((3.4)\),

\[
\begin{align*}
&\left\| m_{-\rho,\alpha}(\xi) \int_{\star} \frac{f(\xi_1, \tau_1)}{m_{-\rho,\alpha}(\xi_1)(\tau_1 - \xi_1^2)^{1/2} m_{\alpha,\alpha}(\xi_2)(\tau_2 + \xi_2^2)^{1/2}} g(-\xi_2, -\tau_2) \right\|_{L^2_{\xi_2}(B_d)} \\
\lesssim &\left\| \int_{\star} \frac{f(\xi_1, \tau_1)}{(\tau_1 - \xi_1^2)^{1/2} (\tau_2 + \xi_2^2)^{1/2}} g(-\xi_2, -\tau_2) \right\|_{L^2_{\xi_2}(B_d)} \\
= &\sup \left\| h_d(\xi, \tau) \frac{f(\xi_1, \tau_1)}{(\tau_1 - \xi_1^2)^{1/2} (\tau_2 + \xi_2^2)^{1/2}} g \right\|_{L^2_{\xi_2}(B_d)} \\
\lesssim &\sup \left\| h_d \right\|_{L^4_{\xi_2}} \left\| \hat{\mathcal{F}}^{-1}_{\xi_2} \left( \frac{f}{(\tau - \xi^2)^{1/2}} \right) \right\|_{L^4_{\xi_2}} \left\| g \right\|_{L^2_{\xi_2}} \\
\lesssim &\sup \left\| h_d \right\|_{L^4_{\xi_2}} \left\| \hat{\mathcal{F}}^{-1}_{\xi_2} \left( \frac{f}{(\tau - \xi^2)^{1/2}} \right) \right\| \left\| g \right\|_{L^2_{\xi_2}} \\
\lesssim &\left( \frac{d}{d + \frac{d}{2}} \right)^d \left\| f \right\|_{L^2_{\xi_2}} \left\| g \right\|_{L^2_{\xi_2}},
\end{align*}
\]
where \( \int_* = \int_{\xi_1 + \xi_2 = \xi} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \), and the supremum is over the set \( H_d = \{ h_d(\xi, \tau) : \|h_d\|_{L^2} \leq 1, \text{supp} \ h_d \subset B_d \} \). Inserting it into (6.3) in the left-hand side, we get the estimate in this subpart.

**Subpart 2.** \( |\xi_1| \gg 1, |\xi_2| \gg 1, |\xi| \ll |\xi_1| \). Then \( |\xi_1| \sim |\xi_2| \), recall \( \rho < \frac{1}{2} \) and by (3.16),

\[
\left\| m_{-\rho, a}(\xi) \int_* \frac{f(\xi_1, \tau_1)}{m_{-\rho, a}(\xi_1)(\tau_1 - \xi_1^2)} \frac{g(-\xi_2, -\tau_2)}{m_{\rho, a}(\xi_2)(\tau_2 + \xi_2^2)} \right\|_{L^2_{|t|}(B_d)} \\
\lesssim \left\| \int_* |\xi|^\rho f(\xi_1, \tau_1) \frac{g(-\xi_2, -\tau_2)}{\langle \tau_1 - \xi_1^2 \rangle^\frac{1}{2}} \right\|_{L^2_{|t|}(B_d)} \\
= \sup_{H_d} \int_* |\xi|^\rho h_d(\xi, \tau) \frac{f(\xi_1, \tau_1)}{\langle \tau_1 - \xi_1^2 \rangle^\frac{1}{2}} g(-\xi_2, -\tau_2) \\
\sim \sup_{H_d} I_3^\rho(h_d, f, g) \\
\lesssim \sup_{H_d} \|h_d\|_{\dot{X}_0^{\frac{1}{2}}} \left\| \frac{f}{\langle \tau - \xi^2 \rangle^\frac{1}{2}} \right\|_{\dot{X}_0^{\frac{1}{2}}} \|g\|_{L^2_{|t|}} \\
\lesssim 2^{(\frac{1}{2} - d)} \|f\|_{L^2_{|t|}} \|g\|_{L^2_{|t|}}.
\]

Inserting it into (6.3) in the left-hand side, we have the claim. \( \square \)

**Lemma 6.3** When \( \text{supp} \ \tilde{v} \subset \bigcup_j (A_j \cap B_{\leq k_0 j}) \), \( \text{supp} \ \tilde{u} \subset \bigcup_j (A_j \cap B_{\geq k_0 j}) \), then

\[
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} (\tilde{u} \ast \tilde{v}^*) \right\|_X \lesssim \|\tilde{u}\|_Y \|\tilde{v}\|_X. \tag{6.5}
\]

**Proof.** It’s much similar to the proofs of Lemma 6.2. But one shall using the estimate on \( I_2^\rho \) as a substitute of \( I_3^\rho \) in Subpart 2. We omit the details. \( \square \)

**Lemma 6.4** When \( \text{supp} \ \tilde{u}, \tilde{v} \subset \bigcup_j (A_j \cap B_{\leq k_0 j}) \), then

\[
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} (\tilde{u} \ast \tilde{v}^*) \right\|_Z \lesssim \|\tilde{u}\|_X \|\tilde{v}\|_X. \tag{6.6}
\]

**Proof.** It suffices to show that for any \( f, g \in \dot{X}^{0,0} \),

\[
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} \int_* \frac{f(\xi_1, \tau_1)}{m_{\rho, a}(\xi_1)(\tau_1 - \xi_1^2)} \frac{g(-\xi_2, -\tau_2)}{m_{-\rho, a}(\xi_2)(\tau_2 + \xi_2^2)} \right\|_Z \lesssim \|f\|_{\dot{X}^{0,0}} \|g\|_{\dot{X}^{0,0}}. \tag{6.7}
\]

We may assume that \( |\xi_1| \geq |\xi_2| \) in the integral domain (it’s similar for \( |\xi_1| \leq |\xi_2| \)). Then we divide (6.7) into three parts to analyze.

Part 1. \( |\xi_2| \lesssim 1 \); Part 2. \( |\xi_2| \gg 1, |\xi| \sim |\xi_1| \); Part 3. \( |\xi_2| \gg 1, |\xi| \ll |\xi_1| \).
Part 1. $|ξ_2| \lesssim 1$. By using the embedding $X \hookrightarrow Z$ in the left-hand side of (6.7), it can be treated as Part 1 in the proof in Lemma 6.1.

Part 2. $|ξ_2| \gg 1, |ξ| \sim |ξ_1|$. Then, $|ξ|, |ξ_1| \gg 1$. By $X \hookrightarrow Z$ and (3.17), the left-hand side of (6.7) is bounded by

$$
\sum_d 2^{-\frac{d}{2}} \left\| \int_\mathbb{R} \frac{f(ξ_1, τ_1)}{\langle τ_1 - ξ_1^2 \rangle^\frac{1}{2}} \frac{|ξ_2|^p g(-ξ_2, -τ_2)}{\langle τ_2 + ξ_2^2 \rangle^\frac{1}{2}} \right\|_{L_{ξ, τ}^2(B_d)}.
$$

(6.8)

Note that, by (3.16),

$$
\left\| \int_\mathbb{R} \frac{f(ξ_1, τ_1)}{\langle τ_1 - ξ_1^2 \rangle^\frac{1}{2}} \frac{|ξ_2|^p g(-ξ_2, -τ_2)}{\langle τ_2 + ξ_2^2 \rangle^\frac{1}{2}} \right\|_{L_{ξ, τ}^2(B_d)} = \sup_{H_d} \left\| \int_{\mathbb{R}} \frac{f}{\langle τ - ξ^2 \rangle^\frac{1}{2}} \frac{g^*}{\langle τ + ξ^2 \rangle^\frac{1}{2}} \right\|_{L_{ξ, τ}^2} \leq \sup_{H_d} \left\| h_d \right\|_{\hat{X}_{0, \frac{d}{2}}} \left\| \frac{f}{\langle τ - ξ^2 \rangle^\frac{1}{2}} \right\|_{\hat{X}_{0, \frac{d}{2}}} \left\| g \right\|_{L_{ξ, τ}^2} \leq 2^{\left(\frac{d}{2}\right)-d} \left\| f \right\|_{L_{ξ, τ}^2} \left\| g \right\|_{L_{ξ, τ}^2},
$$

where the set $H_d$ is defined in the proof of Lemma 6.2. Inserting it into (6.8), we have (6.7) in this part.

Part 3. $|ξ_2| \gg 1, |ξ| \ll |ξ_1|$. Then $|ξ_1| \sim |ξ_2|$. We further split it into two subparts to analyze.

Subpart 1. $|τ - ξ^2| \gtrsim \max\{|τ_1 - ξ_1^2|, |τ_2 + ξ_2^2|\}$; Subpart 2. $|τ - ξ^2| \ll \max\{|τ_1 - ξ_1^2|, |τ_2 + ξ_2^2|\}$.

The division is based on the following algebraic identity

$$
τ - ξ^2 = (τ_1 - ξ_1^2) + (τ_2 + ξ_2^2) - 2ξξ_2,
$$

which implies

$$
\max\{|τ - ξ^2|, |τ_1 - ξ_1^2|, |τ_2 + ξ_2^2|\} \gtrsim |ξ||ξ_2|.
$$

(6.9)

Subpart 1. $|τ - ξ^2| \gtrsim \max\{|τ_1 - ξ_1^2|, |τ_2 + ξ_2^2|\}$. Then by (6.9), we have $|τ - ξ^2| \gtrsim |ξ||ξ_2|$.

By the embedding $Y \hookrightarrow Z$ and Lemma 2.1, it suffices to show

$$
(1) \left\| \frac{m_{-p,a}(ξ_1)}{\langle τ - ξ^2 \rangle} \int_\mathbb{R} \frac{f(ξ_1, τ_1)}{m_{-p,a}(ξ_1)} \frac{g(-ξ_2, -τ_2)}{m_{-p,a}(ξ_2)} \right\|_{L_{ξ, τ}^2 L_{ξ, τ}^2} \lesssim \left\| f \right\|_{L_{ξ, τ}^2} \left\| g \right\|_{L_{ξ, τ}^2},
$$

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for reasonable functions $f, g$.

For (1), we need a further division to analyze.

(a). $|\xi| \geq 1$; (b). $1 \geq |\xi| \geq |\xi|^{-1}$; (c). $|\xi| \leq |\xi|^{-1}$.

When (a), $|\xi| \geq 1$. Recall that $\rho < \frac{1}{2}$, then the left-hand side of (1) is controlled by

\[
\| \frac{|\xi|^{-\rho}}{\langle \tau - \xi^2 \rangle} \int \frac{|\xi|^2}{2^\rho} f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \|_{L^2 L^1_t} \\
\lesssim \| |\xi|^{-\rho-1} \int |\xi|^2 f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \|_{L^2 L^1_t} \\
\lesssim \| |\xi|^{-\rho-1}(f \ast g^\ast) \|_{L^2 L^1_t} \\
\lesssim \| |\xi|^{-\rho-1} \|_{L^2 L^\infty(|\xi| \geq 1)} \| f \ast g^\ast \|_{L^\infty L^1_t} \\
\lesssim \| f \|_{L^2 L^1_t} \| g \|_{L^2 L^1_t}.
\]

When (b), $1 \geq |\xi| \geq |\xi|^{-1}$. Note that $2\rho - a \leq \frac{1}{2}$, the left-hand side of (1) is controlled by

\[
\| \frac{|\xi|^a}{\langle \tau - \xi^2 \rangle} \int |\xi|^2 f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \|_{L^2 L^1_t} \\
\lesssim \| \frac{1}{\langle \tau - \xi^2 \rangle^{1-a}} \int |\xi|^2 f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \|_{L^2 L^1_t} \\
\lesssim \sum_{j_1} \sum_{0 \geq j \geq -j_1} \sum_{d \geq j + j_1} 2^{(a-1)d} 2^{\frac{j_1}{2}} \|(f_j \ast g_j^\ast) \|_{L^2 L^1_t(\hat{A}_j \cap B_d)}, \tag{6.10}
\]

where

\[
f_{j_1}(\xi, \tau) = f(\xi, \tau) \chi_{A_{j_1}}(\xi, \tau), \quad g_{j_1}(\xi, \tau) = g(\xi, \tau) \chi_{A_{j_1}}(\xi, \tau),
\]

and

\[
\hat{A}_j = \{ (\xi, \tau) \in \mathbb{R}^2 : 2^j \leq |\xi| \leq 2^{j+1} \}.
\]
Further, recall that $a < \frac{1}{2}$, we have

\[
(6.10) \lesssim \sum_{j_1} \sum_{0 \geq j} \sum_{d \geq j + j_1} 2^{(a-1)d} 2^{\frac{d}{2z}} \|1\|_{L^\infty_t L^\infty_\xi(A_j \cap B_d)} \| (f_{j_1} * g_{j_1}^*) \|_{L^\infty_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} \sum_{0 \geq j} \sum_{d \geq j + j_1} 2^{(a-1)d} 2^{\frac{d}{2z}} \| f_{j_1} \|_{L^\infty_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} \sum_{0 \geq j} \sum_{d \geq j + j_1} 2^{(a-1)d} 2^{\frac{d}{2z}} \| f_{j_1} \|_{L^2_\xi L^1_t} \| g_{j_1} \|_{L^2_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} 2^{(a-\frac{1}{2})j_1} 2^{(a-\frac{1}{2})j} \| f_{j_1} \|_{L^2_\xi L^1_t} \| g_{j_1} \|_{L^2_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} \| f_{j_1} \|_{L^2_\xi L^1_t} \| g_{j_1} \|_{L^2_\xi L^1_t}
\]

\[
\lesssim \| f \|_{L^2_\xi L^1_t} \| g \|_{L^2_\xi L^1_t},
\]

where we use the Cauchy-Schwarz inequality in the last step.

When (c), $|\xi| \leq |\xi_1|^{-1}$. Again, the left-hand side of (1) is controlled by

\[
\left\| \frac{1}{\langle \tau - \xi^2 \rangle} \int \left| \xi_1 \right|^{2\rho-a} f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \right\|_{L^2_\xi L^1_t}
\]

\[
\lesssim \left\| \int \left| \xi_1 \right|^{\frac{3}{2}} f(\xi_1, \tau_1) g(-\xi_2, -\tau_2) \right\|_{L^2_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} 2^{\frac{a}{2}} \left\| \int \left| f_{j_1}(\xi_1, \tau_1) g_{j_1}(\xi_2, -\tau_2) \right| \right\|_{L^2_\xi L^1_t(\xi \leq 2^{-j_1})}
\]

\[
\lesssim \sum_{j_1} 2^{\frac{a}{2}} \|1\|_{L^2_\xi L^{\infty}(\xi \leq 2^{-j_1})} \| f_{j_1} * g_{j_1}^* \|_{L^\infty_\xi L^1_t}
\]

\[
\lesssim \sum_{j_1} \| f_{j_1} \|_{L^2_\xi L^1_t} \| g_{j_1} \|_{L^2_\xi L^1_t}
\]

\[
\lesssim \| f \|_{L^2_\xi L^1_t} \| g \|_{L^2_\xi L^1_t}.
\]

For (2). When $|\xi| \leq 1$, then the left-hand side of (2) is dominated by

\[
\left\| \frac{|\xi|^a}{\langle \tau - \xi^2 \rangle^{1-\beta}} \int \left| \xi_1 \right|^{2\rho-a} \frac{f(\xi_1, \tau_1) g(-\xi_2, -\tau_2)}{\langle \tau_1 - \xi_1^2 \rangle^{\frac{3}{2}} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}}} \right\|_{L^2_\xi}
\]

\[
\lesssim \left\| \frac{1}{\langle \tau - \xi^2 \rangle^{1-a-\beta}} \int \left| \xi_1 \right|^{\frac{1}{2}} \frac{f(\xi_1, \tau_1) g(-\xi_2, -\tau_2)}{\langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}}} \right\|_{L^2_\xi}.
\]  

(6.11)
Choosing $\beta$ small enough, such that $1 - a - \beta > \frac{1}{2}$. Remember that $|\tau - \xi^2| \gtrsim \max\{|\tau_1 - \xi_1^2|, |\tau_2 + \xi_2^2|\}$, so we have a crude bound of (6.11) that

$$\left\| \frac{1}{(\tau - \xi^2)^{\frac{1}{2}+}} \int \frac{1}{(\tau - \xi^2)^{\frac{1}{2}+}} \frac{f(\xi_1, \tau_1) g(-\xi_2, -\tau_2)}{\langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+}} \right\|_{L^2_{\xi^2}} \approx \sup_{\|a\|_{L^2_{\xi^2}} \leq 1} f_{\beta} \left( \frac{h}{(\tau - \xi^2)^{\frac{1}{2}+}}, \frac{f}{(\tau - \xi^2)^{\frac{1}{2}+}}, \frac{g^*}{(\tau - \xi^2)^{\frac{1}{2}+}} \right)$$

where we use (3.7) in the third step.

When $|\xi| \geq 1$, then the left-hand side of (2) is dominated by

$$\left\| \xi^{a-1+\beta} \int_{\mathbb{R}} |\xi_1|^{2\rho-1+\beta} \frac{f(\xi_1, \tau_1) g(-\xi_2, -\tau_2)}{\langle \tau_1 - \xi_1^2 \rangle^{\frac{1}{2}+} \langle \tau_2 + \xi_2^2 \rangle^{\frac{1}{2}+}} \right\|_{L^2_{\xi^2}}$$

where we choosing $\beta$ small enough again, such that $2\rho - 1 + \beta \leq 0$.

**Subpart 2.** $|\tau - \xi^2| \ll \max\{|\tau_1 - \xi_1^2|, |\tau_2 + \xi_2^2|\}$. Then by (6.9), we have

$$|\tau_1 - \xi_1^2| = \max\{|\tau_1 - \xi_1^2|, |\tau_2 + \xi_2^2|\} \gtrsim |\xi||\xi_2| \quad (6.12)$$

or

$$|\tau_2 + \xi_2^2| = \max\{|\tau_1 - \xi_1^2|, |\tau_2 + \xi_2^2|\} \gtrsim |\xi||\xi_2|.$$
We just consider the case (6.12) (the other is similar). By the embedding $X \hookrightarrow Z$ and (3.17), the left-hand side of (6.7) is dominated by

$$
\sum_d 2^{-\frac{d}{2}} \left\| m_{\rho,a}(\xi) \int_\mathbb{R} |\xi_2|^2 \left(\frac{f(\xi_1, \tau_1) g(-\xi_2, -\rho, a)}{(\tau_1 - \xi_1^2)^{\frac{d}{2}} (\tau_2 + \xi_2^2)^{\frac{d}{2}}} \right) \right\|_{L^2_{\xi} (B_d)}
$$

by the fact $a < \frac{1}{2}$ and $|\tau_1 - \xi_1^2| \geq \max \{|\tau_2|, |\tau_2 + \xi_2^2|\}$. Since

$$
\left\| \int_\mathbb{R} |\xi_2|^{\frac{1}{2}} f(\xi_1, \tau) \left(\frac{g(-\xi_2, -\tau_2)}{(\tau_2 + \xi_2^2)^{\frac{d}{2}+}} \right) \right\|_{L^2_{\xi\tau} (B_d)}
$$

$$
\approx \sup_{H_d} \left\| \frac{h_d \cdot f}{(\tau_2 + \xi_2^2)^{\frac{d}{2}+}} \right\|_{L^2_{\xi\tau} (B_d)}
$$

$$
\lesssim \sup_{H_d} \|h_d\|_{\hat{X}_{0,\frac{d}{2}+}} \|f\|_{L^2_{\xi\tau}} \|g\|_{L^2_{\xi\tau}}
$$

where the set $H_d$ is defined in the proof of Lemma 6.2. Inserting it into (6.13), we obtain (6.13). This completes the proof of the lemma.

Combining (6.1), (6.3), (6.5) and (6.6), we establish (4.4), and hence finish the proof of Theorems 1.2 and 1.4.

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