Two-Loop Corrections for Nuclear Matter in a Covariant Effective Field Theory

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Abstract

Although one-loop calculations provide a realistic description of bulk and single-particle nuclear properties, it is necessary to examine loop corrections to develop a systematic finite-density power-counting scheme for the nuclear many-body problem when loops are included. Moreover, it is imperative to study exchange and correlation corrections systematically to make reliable predictions for other nuclear observables. One must also verify that the natural sizes of the one-loop parameters are not destroyed by explicit inclusion of many-body corrections. The loop expansion is applied to a chiral effective hadronic lagrangian; with the techniques of Infrared Regularization, it is possible to separate out the short-range contributions and to write them as local products of fields that are already present in our lagrangian. (The appropriate field variables must be re-defined at each order in loops.) The corresponding parameters implicitly include short-range effects to all orders in the interaction, so these effects need not be calculated explicitly. The remaining (long-range) contributions that must be calculated are nonlocal and resemble those in conventional nuclear-structure calculations. Calculations at the two-loop level are carried out to illustrate these techniques at finite densities and to verify that the coupling parameters remain natural when fitted to the empirical properties of equilibrium nuclear matter.

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Quantum Hadrodynamics (QHD) is a low-energy effective theory of the strong interaction that describes the strong nuclear force by the exchange of mesons between nucleons (the observed degrees of freedom at this energy scale). Initially proposed by Walecka [1], it has evolved over the years into a framework based on effective field theory (EFT) and density functional theory (DFT) [2–8]. EFT embodies basic principles that are common to many areas of physics, such as the separation of length scales in the description of natural phenomena. In EFT, the long-range (nonlocal) dynamics is included explicitly, while the short-range (local) dynamics is parametrized generically; all of the dynamics is constrained by the symmetries of the interaction. DFT tells us that the nuclear many-body system can be described by a universal energy functional that depends on nuclear densities and four-vector currents. With knowledge of the energy functional, one can calculate any observable for the (zero-temperature) many-body system. Moreover, a simplified treatment of the functional based on quasi-particle orbitals still provides an exact description of the bulk properties and some single-particle observables. Thus knowledge of the many-particle wave function is not needed to calculate this subset of observables [6]. The energy functional is constructed as an expansion in small parameters (the mean meson fields, which are in fact Kohn–Sham potentials, divided by a heavy mass, which could be the nucleon mass or the chiral symmetry breaking scale) and includes all possible terms consistent with the underlying symmetries of the system.

In the current formalism, field redefinitions are employed to place the complexity of the problem in the meson field self-interactions. Each term is characterized by an unknown coefficient which, once all the dimensional and combinatorial factors have been removed, is a dimensionless constant of order unity, an assumption known as naturalness [9, 10]. The natural separation of length scales of the system is embodied in this theory: the long-range dynamics is included explicitly and the short-range physics is contained in the parametrization. While this theory in principle contains all possible terms consistent with the underlying symmetries of QCD, in practice this is a perturbative expansion for the energy functional that can be truncated at a manageable level. Once it has been truncated, the now finite number of coefficients are fixed by experimental data; this theory can then be used for predictive purposes [11–14].

QHD is a strong-coupling theory. Unlike QCD, QHD is not asymptotically free, and currently no lattice version exists. Moreover, unlike QED, the QHD couplings are large, and there is no obvious asymptotic expansion to use to obtain results and refine them systematically. It is thus unknown whether QHD permits any expansion for systematic computation and refinement of theoretical results. One possibility is the loop expansion, which was partially explored in [15, 16]. In that work, explicit calculation of the short-range dynamics in terms of nucleons and heavy mesons produced enormous contributions to the energy, which rendered the loop expansion useless. This problem was only partially avoided by the inclusion of vertex corrections in the loops. In contrast, with the ideas of EFT, we can now provide a straightforward, physically motivated discussion of the short-range dynamics.

In the present work, the loop expansion for QHD is constructed in the usual manner [15, 17–19]. First we define the action and the exact ground-state generating functional through a path integral. This generating functional contains all possible diagrams. The path integral is used to define the effective action, which is then expanded in powers of \( \hbar \) (where \( \hbar \) acts as a bookkeeping parameter and is not necessarily small [20–22]). An equivalent way
to state this is that the effective action is expanded around its classical value by grouping terms according to the number of quantum loops in their corresponding diagrams. Then we perform the functional derivatives and acquire the loop integrals. For the purposes of this work, we are interested in the expansion only up to the two-loop level. Consideration of the effects of contributions from the three-loop level and higher will be considered in future investigations \[23, 24\]. All of the integrals that represent tadpole and disconnected diagrams cancel out in the two-loop effective action, and we are left with only the fully connected diagrams. In the nuclear matter limit, the effective action is proportional to the energy density. For the cases considered here, there are three integrals of interest. These three integrals each have two factors of the nucleon propagator and one meson propagator (either scalar, vector, or pion).

Why a loop expansion? The loop expansion is a simple and well-developed expansion scheme in powers of $\hbar$ that is derived from the path integral. The mean meson fields are included non-perturbatively and the correlations are included perturbatively. Therefore, one can analyze the many-body effects order by order. Indeed, previous work has shown the importance of “Hartree dominance” \[25\]: the mean-field terms dominate the nuclear energy, and exchange and correlation effects do not significantly modify the energy or nucleon self-energies, at least for states in the Fermi sea. We stress that we are not certain that the loop expansion is practical; the answer to this question is left for future consideration. However, the loop expansion has the advantage that it is fairly easy to separate the short-range and long-range dynamics and to analyze their structures.

The nucleon propagator can be separated into two components, known as the Feynman and Density parts \[19\]. This is accomplished by taking into account the proper pole structure of the propagator. The Feynman part describes the propagation of a baryon or antibaryon; the Density part involves only on-shell propagation in the Fermi sea and incorporates the exclusion principle.

As a result, the two-loop integrals can each be separated into three distinct parts, which we refer to as exchange, Lamb-shift, and vacuum-fluctuation contributions \[15\]. The exchange term has two factors of the Density portion of the nucleon propagator. Thus both momentum integrals are entirely within the Fermi surface and the exchange term is finite. This represents a contribution from long-range (nonlocal) physics and must be calculated explicitly. The Lamb-shift term contains both Feynman and Density parts of the nucleon propagator. This term is so named because it is analogous to the Lamb shift in atomic physics, where a particle in an occupied state interacts with a virtual particle in an unoccupied state that shifts its spectrum. This contribution is short-range, and we will show that it can be expressed as a sum of terms that are already present in the QHD EFT lagrangian. As this is an effective theory, the coefficients of these terms are determined by matching to empirical data; thus, these short-range terms are just absorbed into local terms already present in the lagrangian and should not be calculated explicitly. The vacuum fluctuation term contains two factors of the Feynman propagator and both momentum integrals extend outside the Fermi sphere. This involves the excitation of $NN$ pairs and is therefore also short-range physics. We will show that it can also be expressed as a sum of terms which exist in the EFT lagrangian. As a result, it can be removed in the same manner as the Lamb shift. In addition, if nonlinear meson self-interactions are included, a number of pure meson loops arise. These terms, however, can be expressed as a power series in the meson fields with undetermined coefficients. As before, these terms are just absorbed and do not need to be calculated explicitly. The result is that, for the cases considered here, only three finite
integrals representing the two-loop contributions from the scalar and vector meson and the pion need to be calculated.

It is interesting to note that the procedure described above is similar to Infrared Regularization. In Infrared Regularization, the one-loop self-energy contribution can be separated into soft and hard parts [26–30] (they are sometimes referred to as the infrared singular and regular parts [26, 27]). The hard (or infrared regular) parts arise from large momentum, or high-energy, contributions and are expressible as a power series of terms already contained in the underlying lagrangian; as in our case, they are just absorbed into the coefficients. The soft (or infrared singular) parts, in the notation of Ellis and Tang, contain both analytic and nonanalytic terms. The analytic portion results from high-energy dynamics and is therefore treated in the same manner as the hard contributions. The nonanalytic portion develops from low momentum, or low-energy physics, and is essentially long-range dynamics. This is the nonlocal contribution that must be calculated explicitly. The regularization procedure for the (closed) energy loops in this work is significantly simpler than the procedure for diagrams with external momenta, like scattering amplitudes or self-energies, because the momenta carried by the boson propagators are always spacelike.

In the QHD lagrangian, a well-developed mean-field power-counting scheme has been devised [2, 3]. As noted, the energy functional is an expansion in small parameters (the meson fields divided by the nucleon mass). In addition, the Fermi wave number, which is related to the size of the derivatives, is also small when divided by the nucleon mass. If naturalness holds, then adding up the powers of these ratios (and some counting factors) yields an accurate estimate of the size of a given term. This paper will seek to investigate the relationship between the two-loop integrals and this underlying power counting scheme.

The purpose of this work is to illustrate these techniques by working to the two-loop level in a loop expansion; the resulting two-loop integrals are separated into long-range and short-range physics. The short-range contributions are expressed in forms that already appear at the mean field level. Since the coefficients of these terms have yet to be determined, they are just redefined, thereby incorporating these new contributions into the mean field lagrangian. As a result, they are already present in the one-loop level QHD calculation. The long-range, nonlocal physics must be explicitly calculated. In this work, we fit the two-loop energy to the equilibrium point of nuclear matter. Then we compare sets constructed at the two-loop level with those of previous work developed at the mean field level and consider the naturalness of the parameters. We also examine how these new contributions fit into the power counting scheme. Some of this work has been discussed previously in an unpublished Ph.D. dissertation [31].

II. THEORY

In this section, we follow the usual procedure for constructing the loop expansion [19] and in the following subsections examine the one- and two-loop contributions, as in [15].

A. Loop Expansion—Background

Consider the following nonrenormalizable effective lagrangian, which extends the Walecka model to include the nonrenormalizable \( \pi N \) coupling and nonlinear scalar and vector meson interactions. This lagrangian can also be obtained from the chirally invariant lagrangian
in [2] by retaining only the lowest-order terms in the pion fields—sufficient for the two-loop calculations in this work—and a subset of the meson nonlinearities:\(^1\)

\[
\mathcal{L} = -\bar{\psi} \left[ \gamma_\mu (\partial_\mu - igV_\mu) - \frac{g_A}{f_\pi} \gamma_5 \partial_\mu \pi + (M - g_s \phi) \right] \psi \\
- \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m_\pi^2 \phi^2 - \frac{1}{4} V_{\mu\nu} V^{\mu\nu} - \frac{1}{2} m_\pi^2 V_\mu V_\mu - \frac{1}{2} (\partial_\mu \pi_a)^2 - \frac{1}{2} m_\pi^2 \pi_a^2 \\
+ \mathcal{L}_{NL} + \delta \mathcal{L} ,
\]

where \( V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \) and \( \pi = \frac{1}{2} \tau_a \). Here \( \psi \) are the fermion fields and \( \phi, V_\mu, \) and \( \pi_a \) are the meson fields (isoscalar-scalar, isoscalar-vector, and isovector-pseudoscalar, respectively) and the heavy meson fields are also chiral scalars. For the purposes of this work, we will not include effects from the isovector-vector channel or the electromagnetic field, but to do so is straightforward [2, 32]. The numerically small tensor coupling between the omega meson and the nucleon is not included, and nonlinearities in the meson sector of the lagrangian \( (\mathcal{L}_{NL}) \) will be considered later at the mean field level; the study of their effects at two-loop order is left for future work [23]. \( \delta \mathcal{L} \) contains all of the counterterms. Note that in this work, the conventions of [8] are used.

The action is defined as

\[
S[\phi, V_\mu] \equiv \int d^4x \mathcal{L}(x) ,
\]

and the exact ground-state to ground-state generating functional is defined through the following path integral [15]

\[
Z[j, J_\mu] \equiv \exp \left\{ iW[j, J_\mu]/\hbar \right\} \\
= \mathcal{N}^{-1} \int D(\bar{\psi}) D(\psi) D(\phi) D(V_\mu) D(\pi_a) \\
\times \exp \left\{ \frac{i}{\hbar} \int d^4x \left[ \mathcal{L}(x) + j(x) \phi(x) + J_\mu(x) V_\mu(x) \right] \right\} ,
\]

where

\[
\mathcal{N} \equiv \int D(\bar{\psi}) D(\psi) D(\phi) D(V_\mu) D(\pi_a) \exp \left\{ \frac{i}{\hbar} \int d^4x \mathcal{L}(x) \right\} .
\]

Here \( \mathcal{N} \) is the normalization factor (in effect, the vacuum subtraction), and \( j(x) \) and \( J_\mu(x) \) are the external sources corresponding to the meson fields \( \phi \) and \( V_\mu \), respectively. The connected generating functional \( W[j, J_\mu] \) contains all of the connected diagrams, and \( Z[j, J_\mu] \) contains all possible diagrams without "vacuum bubbles".

The classical values of the meson fields are determined by extremizing the action:

\[
\left[ \frac{\delta S}{\delta \phi(x)} \right]_{\phi = \phi_0} = (\partial^2 - m_\pi^2) \phi_0 = -j(x) ,
\]

\[
\left[ \frac{\delta S}{\delta V_\mu(x)} \right]_{V_\mu = V_\mu^0} = -\partial_\nu V_\mu^0 - m_\pi^2 V_\mu^0 = -J_\mu(x) .
\]

\(^1\) We are not making a chiral expansion in powers of the pion mass. We have simply included a pion mass for kinematical purposes in Eq. (1).
Next, we replace the fields by quantum fluctuations around their classical fields: $$\bar{\psi}(x) \to h^{1/2} \bar{\psi}(x), \psi(x) \to h^{1/2} \psi(x), \phi(x) \to \phi_0(x) + h^{1/2} \sigma(x), V_\mu(x) \to V^0_\mu(x) + h^{1/2} \bar{\eta}_\mu(x),$$ and $$\pi_a(x) \to h^{1/2} \Omega_a(x).$$ Notice that the fermion fields have no classical limit (and the pion has no mean field if one assumes no pion condensate). Observe that a factor of $$h^{1/2}$$ is associated with each quantum fluctuation; the numerical value of $$h$$ is immaterial, as it is just a bookkeeping parameter [20–22]. Next, a number of extra sources are included that are set equal to zero at the end ($u, U_\mu, \zeta_a, \xi$, and $$\bar{\xi}$$ corresponding to the quantum fluctuations $$\sigma, \bar{\eta}_\mu, \Omega_a, \bar{\psi},$$ and $$\psi$$, respectively). The path integral is then rewritten in terms of functional derivatives with respect to these new sources. The final result is [15]

$$Z[j, J_\mu] = \mathcal{N}^{-1} \exp \left\{ \frac{i}{\hbar} \int d^4 x \left[ \mathcal{L}_0(x) + j(x) \phi_0(x) + J_\mu(x) V^0_\mu(x) \right] \right\}$$

$$\times \exp \left\{ \frac{g A}{f_\pi} \left( \gamma_\mu \gamma_5 \partial_\mu \left[ \frac{-i\delta}{\delta \zeta_\alpha(x)} \right] \frac{\tau_a}{2} \right) + g_s \left[ \frac{-i\delta}{\delta u(x)} \right] \right\}$$

$$\times \exp \left\{ -i \int \int d^4 x d^4 y \bar{\epsilon}(x) G_H(x-y) \bar{\xi}(y) \right\}$$

$$\times \exp \left\{ \frac{i}{2} \int \int d^4 x d^4 y \left[ u(x) \Delta^0_S(x-y) u(y) \right] \right\}$$

$$+ U_\mu(x) D^0_{\mu\nu}(x-y) U_\nu(y) + \zeta_a(x) \Delta^{ab}_\pi(x-y) \zeta_b(y) \right\} \right\}_{\text{sources} = 0}, \quad (7)$$

where $$\mathcal{N}$$ is equal to the portion of $$Z$$ involving the variational derivatives, but with free propagators. $$\mathcal{L}_0$$ represents the lagrangian at the mean field level. The fermion propagators in momentum space are

$$G^0_F(k) = \frac{-1}{i \vec{k} + M - i\epsilon}, \quad (8)$$

$$G^0_H(k) = \frac{-1}{i \vec{k} - ig \gamma^\mu V^0_\mu + (M - g_s \phi_0)} \quad (9)$$

and the free meson propagators in momentum space are

$$\Delta^0_S(k) = \frac{1}{k^2 + m^2_S - i\epsilon}, \quad (10)$$

$$D^0_{\mu\nu}(k) = \frac{1}{k^2 + m^2_V - i\epsilon} \left( \delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2_V} \right), \quad (11)$$

$$\Delta^{ab}_\pi(k) = \frac{1}{k^2 + m^2 - i\epsilon} \delta^{ab} \quad (12)$$

It is assumed that all of the divergent integrals are regularized in some fashion that preserves the symmetries of the theory, for example, dimensional regularization. The longitudinal term in the vector meson propagator vanishes in the following analysis, as the vector meson couples to the conserved baryon current [19]. As usual, the infinitesimal $$\epsilon$$ is introduced to generate the proper pole structure.
The expectation values of the meson fields in the presence of external sources are

\[ \phi_e(x) = \frac{\langle 0^+ | \phi(x)|0^- \rangle}{\langle 0^+ |0^- \rangle} = -i\hbar \frac{\delta \ln Z[j, J_\mu]}{\delta j(x)} = \frac{\delta W[j, J_\mu]}{\delta j(x)} , \]  
\[ (13) \]

\[ V_\mu^e(x) = \frac{\langle 0^+ | \hat{V}_\mu(x)|0^- \rangle}{\langle 0^+ |0^- \rangle} = -i\hbar \frac{\delta \ln Z[j, J_\mu]}{\delta J_\mu(x)} = \frac{\delta W[j, J_\mu]}{\delta J_\mu(x)} . \]
\[ (14) \]

In the limit of vanishing sources, these expectation values become

\[ \lim_{j \to 0} \phi_e = \bar{\phi} = \text{constant} , \]
\[ \lim_{J_\mu \to 0} V_\mu^e = \nabla_\mu = \text{constant} . \]
\[ (15) \]
\[ (16) \]

We now define the effective action \( \Gamma \) by a functional Legendre transformation:

\[ \Gamma[\phi_e, V_\mu^e] \equiv W[j, J_\mu] - \int d^4x \left[ j(x)\phi_e(x) + J_\mu(x)V_\mu^e(x) \right] . \]
\[ (17) \]

In uniform nuclear matter, this effective action is related to the energy density \( \mathcal{E} \) by

\[ \lim_{j, J_\mu \to 0} \Gamma[\phi_e, V_\mu^e] = - \int d^4x \mathcal{E}[\bar{\phi}, \nabla_\mu] . \]
\[ (18) \]

### B. One-Loop Calculation

In this section, we consider the lowest-order terms in the loop expansion, the one-loop contributions; these are the terms that correspond to \( O(\hbar^0) \) in the expansion. The generating functional at one-loop order is [15]

\[ Z^{(1)}[j, J_\mu] \equiv \exp \left\{ iW^{(1)}[j, J_\mu]/\hbar \right\} = \exp \left\{ \frac{i}{\hbar} \int d^4x \left[ \mathcal{L}_0(x) + j(x)\phi_0 + J_\mu(x)V_\mu^0 \right] \right\} \times \exp \left\{ \text{tr} \ln \left[ G_0^0 G_\mathcal{H}_1 \right] \right\} , \]
\[ (19) \]

where we have assumed uniform classical fields. Therefore, the connected generating functional is (for now we have dropped the nonlinear meson self-interactions)

\[ W^{(1)}[j, J_\mu] = \int d^4x \left\{ -\frac{1}{2} m_S^2 \phi_0^2 - \frac{1}{2} m_V^2 V_\mu^0 V_\mu^0 + j_\phi + J_\mu V_\mu^0 \right. \]
\[ -i\hbar \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \left[ 1 - \frac{ig_V \gamma_\mu V_\mu^0 + g_s \phi_0}{i \not{k} + M - i\epsilon} \right] \} . \]
\[ (20) \]

Here “\( \text{tr} \)” denotes the summation over both spin and isospin. Following from the previous section, Eqs. (13) and (14) become \( \phi^{(1)}(x) = \phi_0 \) and \( V^{(1)}_\mu(x) = V_\mu^0 \), respectively. For a spatially uniform system, the classical vector field is \( V_\mu^0 = iV_0 \delta_{\mu 4} \). To calculate the momentum integral, we use the relation [15]

\[ -i\hbar \int \frac{d^4k}{(2\pi)^4} \text{tr} \ln \left[ 1 + \frac{g_V \gamma_4 V_0 - g_s \phi_0}{i \not{k} + M} \right] \]
\[ = i\hbar \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \gamma_4 k_4 \left( G_\mathcal{H}(k) - G_\mathcal{P}(k) \right) \right] , \]
\[ (21) \]
where \( G_F^0 \) comes from the vacuum subtraction \( \mathcal{N} \), which determines the proper boundary conditions. Then, using dimensional regularization to eliminate the vector field in the momentum integrals, the effective action becomes

\[
\Gamma^{(1)}[\phi_e, V_e^0] = \int d^4x \left\{ \frac{1}{2} m_V^2 V_0^2 - \frac{1}{2} m_3^2 \phi_0^2 - h g_V V_0 \rho_B \right. \\
- \frac{\hbar}{(2\pi)^3} \int d^3k E^*(k) \theta(k_F - |\vec{k}|) \\
+ i\hbar \int \frac{d^4k}{(2\pi)^4} k_4 \text{tr} \left[ \gamma_4 G_F^*(k) \right] - i\hbar \int \frac{d^4k}{(2\pi)^4} k_4 \text{tr} \left[ \gamma_4 G_F^0(k) \right] \right\}, \tag{22}
\]

where \( E^*(k) \equiv (\vec{k}^2 + M^*{}^2)^{1/2} \), the effective nucleon mass \( M^* \equiv M - g_S \phi_0 \), and \( \gamma \) is the spin–isospin degeneracy. Here we have separated the nucleon propagator into two parts \([19]\) (known as the Feynman and Density contributions), as shown below:

\[
G^*(k) = \frac{-1}{i\bar{k} + M - g_S \phi_0} \\
= (i\bar{k} - M^*) \left[ \frac{1}{\bar{k}^2 + M^*{}^2 - i\epsilon} - \frac{i\pi}{E^*(k)} \delta[k_4 - E^*(k)] \theta(k_F - |\vec{k}|) \right] \\
\equiv G_F^*(k) + G_D^*(k). \tag{23}
\]

This is done by taking the proper pole structure into account. The Feynman part describes the propagation of baryons and antibaryons; the Density part involves only on-shell propagation in the Fermi sea and corrects the propagation of positive-energy baryons for the Pauli exclusion principle.

To remove the divergences in the momentum integral in Eq. (22), we expand \( G_F^*(k) \) as a polynomial in \( g_S \phi_0 \) (the Furry expansion \([15]\))

\[
G_F^*(k) = \sum_{n=0}^{m} (M^* - M)^n \left[ G_F^0(k) \right]^{n+1} + (M^* - M)^{m+1} \left[ G_F^0(k) \right]^{m+1} G_F^*(k), \tag{24}
\]

which is valid for any \( m \geq 0 \). If we take \( m = 4 \) and insert into Eq. (22), we notice that the first four terms are divergent. The counterterms introduced to absorb these divergences are

\[
\delta \mathcal{L}^{(1)} = \hbar \sum_{n=1}^{4} \alpha_n \phi_0^n, \tag{25}
\]

where \( \alpha_i \) are pure numbers. The first term in the Furry expansion cancels the last term in Eq. (22); the next four terms cancel with the counterterms in Eq. (25). Thus, the one-loop effective action is just

\[
\Gamma^{(1)}[\phi_e, V_e^0] = \int d^4x \left\{ \frac{1}{2} m_V^2 V_0^2 - \frac{1}{2} m_3^2 \phi_0^2 - g_V V_0 \rho_B \\
- \frac{\gamma}{(2\pi)^3} \int d^3k E^*(k) \theta(k_F - |\vec{k}|) - \Delta \mathcal{E}_{VF}(M^*) \right\}, \tag{26}
\]

where the factor \( \hbar \) has been omitted and

\[
\Delta \mathcal{E}_{VF}(M^*) \equiv -i (M^* - M)^5 \int \frac{d^4k}{(2\pi)^4} k_4 \text{tr} \left\{ \gamma_4 \left[ G_F^0(k) \right]^5 G_F^*(k) \right\}. \tag{27}
\]
As before, the mean fields are determined by extremizing the effective action. At the one-loop level, the energy density is [using Eq. (18)]

\[ E^{(1)}[M^*, \rho_B] = g_V V_0 \rho_B - \frac{1}{2} m_V^2 V_0^2 + \frac{m_S^2}{2 g_S^2} (M - M^*)^2 \]

\[ + \frac{\gamma}{(2\pi)^3} \int d^3 k E^*(k) \theta(k_F - |\vec{k}|) + \Delta E_{V_F}(M^*), \] (28)

which is the relativistic Hartree approximation in the original Walecka model [1]. The final term in the energy density is written as [15]

\[ \Delta E_{V_F}(M^*) = \frac{M^4}{4\pi^2} \left\{ \frac{(g_S \phi_0)^5}{5 M^5} + \frac{(g_S \phi_0)^6}{30 M^6} + \frac{(g_S \phi_0)^7}{105 M^7} + \cdots \right. 

\[ \left. + \frac{4!(n-5)!}{n!} \frac{(g_S \phi_0)^n}{M^n} + \cdots \right\} \] (29)

Note that this term has no explicit density dependence.

As discussed in [3, 33], the general form of each term in \( \Delta E_{V_F} \) shows that this vacuum contribution is unnaturally large. If we accept the naturalness assumption, the conclusion is that the vacuum contribution is not well described by \( \Delta E_{V_F} \) at the one-baryon-loop level. Baryons are incorrect degrees of freedom for computing short-range loops. Furthermore, to make these terms natural, it is certain that there must be large cancellations from higher-order loops, which therefore must be calculated. Yet, we know that in principle the polynomial terms in \( \phi \) should appear in an effective lagrangian, since they satisfy all the symmetry requirements; moreover, there will always be short-range contributions to these terms that we cannot calculate. So the proposal is that we need not work hard to get the vacuum contributions—instead, we can adjust the unknown natural coefficients to include them!

C. Two-Loop Calculation

In this section, we present the corrections to the theory arising from the two-loop contributions. We define the connected generating functional at the two-loop level as

\[ W^{(2)} = W^{(1)} + W_2. \] (30)

Keeping only the terms of \( O(\hbar) \) in the loop expansion (which is essentially an expansion in the coupling constants \( g_S, g_V \), and \( g_A \)), the connected generating functional is

\[ W_2 = \frac{i\hbar}{2} \int \int d^4 x d^4 y \left\{ g_S^2 \delta_{\alpha\beta} \delta_{\alpha'\beta'} \left[ \frac{-i \delta}{\delta u(x)} \right] \left[ \frac{-i \delta}{\delta u(y)} \right] - g_V \gamma_{\mu} \alpha \beta \left[ \frac{-i \delta}{\delta U_{\mu}(x)} \right] \left[ \frac{-i \delta}{\delta U_{\mu}(y)} \right] - g_A \frac{f^2}{f^2} \left[ \frac{-i \delta}{\delta \zeta_{\alpha}(x)} \cdot \frac{\tau_3}{2} \right]_{\alpha\beta} \left[ \frac{-i \delta}{\delta \zeta_{\beta}(y)} \cdot \frac{\tau_3}{2} \right]_{\alpha'\beta'} \right\} \]
at the two-loop level are \[15\] diagrams, we must consider the effective action. The expectation values of the meson fields \(\phi\) contain tadpole contributions.

\[
\begin{align*}
&\text{The effective action at the two-loop level is} \\
&W_2 = \frac{\hbar^2}{2} \int d^4x \int d^4k \int d^4q \\
&\times \left( g_S^2 \left\{ \Delta^0_S(k - q) \text{tr} [G_H(k)G_H(q)] - \Delta^0_S(0) \text{tr} [G_H(k)G_H(q)] \right\} \\
&- g_V^2 \left\{ \mathcal{D}^0_{\mu
u}(k - q) \text{tr} [\gamma_\mu G_H(k)\gamma_\nu G_H(q)] - \mathcal{D}^0_{\mu
u}(0) \text{tr} [\gamma_\mu G_H(k)\gamma_\nu G_H(q)] \right\} \\
&- \frac{g_A^2}{f^2} \Delta^a_{\pi}(k - q) \text{tr} \left[ (k - q)\gamma_5 \tau^a \frac{f}{2} G_H(k)(k - q)\gamma_5 \tau^b \frac{f}{2} G_H(q) \right] \right) \\
&- \text{VEV} \\
&\text{where VEV is the vacuum subtraction, which is just equivalent to the rest of } W_2 \text{ with free propagators. After working out the variational derivatives, the full expression becomes}
\end{align*}
\]

\[
W_2 = \frac{\hbar^2}{2} \int d^4x \int d^4k \int d^4q \\
\times \left( g_S^2 \left\{ \Delta^0_S(k - q) \text{tr} [G_H(k)G_H(q)] - \Delta^0_S(0) \text{tr} [G_H(k)G_H(q)] \right\} \\
- g_V^2 \left\{ \mathcal{D}^0_{\mu
u}(k - q) \text{tr} [\gamma_\mu G_H(k)\gamma_\nu G_H(q)] - \mathcal{D}^0_{\mu
u}(0) \text{tr} [\gamma_\mu G_H(k)\gamma_\nu G_H(q)] \right\} \\
- \frac{g_A^2}{f^2} \Delta^a_{\pi}(k - q) \text{tr} \left[ (k - q)\gamma_5 \tau^a \frac{f}{2} G_H(k)(k - q)\gamma_5 \tau^b \frac{f}{2} G_H(q) \right] \right) \\
- \text{VEV} .
\]

\[
W_2 \text{ contains all the connected two-loop diagrams. To isolate and remove the tadpole diagrams, we must consider the effective action. The expectation values of the meson fields at the two-loop level are [15]}
\]

\[
\phi^{(2)}_e(x) = \frac{\delta W^{(1)}}{\delta j(x)} = \phi_0(x) - \hbar g_S \int d^4y \int \frac{d^4k}{(2\pi)^4} \Delta^0_S(x - y) \text{tr} [G_H(k)] , \\
V^{(2)}_\mu(x) = \frac{\delta W^{(1)}}{\delta J_\mu(x)} = V^0_\mu(x) - i\hbar g_V \int d^4y \int \frac{d^4k}{(2\pi)^4} \mathcal{D}^0_{\mu
u}(x - y) \text{tr} [\gamma_\mu G_H(k)] ,
\]

where \(W^{(1)} = S_0 + W_1\), and there are no pion tadpoles. Thus, the effective meson fields contain tadpole contributions.

The effective action at the two-loop level is

\[
\Gamma^{(2)}[\phi_e, V^{e}_\mu] = S[\phi_0, V^{0}_\mu] + W_1[\phi_0, V^{0}_\mu] + W_2[\phi_0, V^{0}_\mu] \\
+ \int d^4x \left\{ j(x) [\phi_0(x) - \phi_e(x)] + J_\mu(x) [V^0_\mu(x) - V^e_\mu(x)] \right\} \\
+ O(\hbar^3) .
\]

Next, we change variables to \(\phi_0 = \phi_e + \phi_1\) and \(V^{0}_\mu = V^e_\mu + V^1_\mu\), where \(\phi_1\) and \(V^1_\mu\) cancel quantum corrections contained in \(\phi_e\) and \(V^e_\mu\) respectively. Then, we expand this expression as a power series in \(\phi_1\) and \(V^1_\mu\) about \(\phi_e\) and \(V^e_\mu\). Using Eqs. (5) and (6), we get

\[
\Gamma^{(2)}[\phi_e, V^{e}_\mu] = \Gamma^{(1)}[\phi_e, V^{e}_\mu] + W_2[\phi_e, V^{e}_\mu]
\]
\[ + \frac{\hbar^2}{2} g_S^2 \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \Delta^0_S(0) \text{tr} [G_H(k)] \text{tr} [G_H(q)] \]

\[- \frac{\hbar^2}{2} g_V^2 \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \Delta^0(0) \text{tr} [\gamma_\mu G_H(k)] \text{tr} [\gamma_\nu G_H(q)] , \quad (35)\]

where the last two terms cancel with terms in \( W_2 \) (the tadpole diagrams drop out). Now we use dimensional regularization to eliminate the dependence on \( V_0^0 \). Lastly, we write the energy density as \( \mathcal{E}^{(2)} = \mathcal{E}^{(1)} + \mathcal{E}_2 \), where [15]

\[ \mathcal{E}_2 = - \int \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left[ \frac{g_S^2}{2} \Delta^0_S(k - q) \left\{ \text{tr} [G^*_s(k)G^*_s(q)] - \text{tr} [G^0_F(k)G^0_F(q)] \right\} \right. \]

\[ - \frac{g_V^2}{2} \Delta^0(0) \left\{ \text{tr} [\gamma_\mu G^*_s(k)\gamma_\nu G^*_s(q)] - \text{tr} [\gamma_\mu G^0_F(k)\gamma_\nu G_F(q)] \right\} \]

\[ - \frac{g_A^2}{2f^2} \Delta^0(k - q) \left\{ \text{tr} \left[ (k - q)\gamma_\delta \frac{\tau_a}{2} G^*_s(k) \right] (k - q) \gamma_\delta \frac{\tau_b}{2} G^*_s(q) \right\} \]

\[ - \text{tr} \left[ (k - q)\gamma_\delta \frac{\tau_a}{2} G^0_F(k) \right] (k - q) \gamma_\delta \frac{\tau_b}{2} G^0_F(q) \right\} \right] , \quad (36)\]

and the factors of \( \hbar \) have been suppressed. The terms involving the free nucleon propagators come from the VEV subtraction. The corresponding two-loop diagrams are shown in Fig. 1.

Since each \( G^* \) can be separated into Feynman and Density parts [see Eq.(23)], we can rewrite the first term in \( \mathcal{E}_2 \) as the following sum [15]

\[ \mathcal{E}_\phi^{(2)} = \mathcal{E}_{\phi-EX}^{(2)} + \mathcal{E}_{\phi-LS}^{(2)} + \mathcal{E}_{\phi-VF}^{(2)} , \quad (37)\]

FIG. 1: Two-loop diagrams. The double lines represent baryon propagators [Eq. (23)]. The dashed, wiggly, and dotted lines represent scalar, vector, and pion propagators, respectively [Eqs. (10) to (12)].
It turns out this is also true for the vector and pion terms, so that we may write the energy density at the two-loop level as

\[ E^{(2)} = E^{(1)} + E^{(2)}_{EX} + E^{(2)}_{LS} + E^{(2)}_{VF}. \]  

(39)

\( E^{(2)}_{EX} \) is called the "exchange term". It has two factors of \( G^*D \), which restricts the double integral to the inside of the Fermi sphere. It is finite and explicitly density-dependent; therefore, we regard it as a purely many-body effect. It can be calculated straightforwardly. This corresponds to the exchange of identical fermions in occupied states.

\( E^{(2)}_{LS} \) is analogous to the Lamb shift in atomic physics, since it involves particles in occupied states whose energies are shifted by interactions with the fluctuating meson fields at finite density. These fluctuations modify the short-range structure of the baryon due to the existence of the background meson fields.

\( E^{(2)}_{VF} \) is a true vacuum fluctuation, since it involves both meson and baryon virtual excitations. This involves the excitations of \( NN \) pairs, which is also short-range physics.

The \( E^{(2)}_{EX} \) contributions contain long-range dynamics [they are characterized by the length scale of \( O(k_F^{-1}) \approx 0.8 \text{ fm} \)] and are nonlocal (they involve logarithmic functions of momenta); thus, we must calculate them explicitly. The contributions \( E^{(2)}_{LS} \) and \( E^{(2)}_{VF} \) correspond to short-range physics [they are characterized by length scales of \( O(M^{-1}) \approx 0.2 \text{ fm} \)]. So we expect that the latter two can be absorbed into the coefficients of the local terms in the lagrangian.

Normally, certain counterterms are introduced at the two-loop level to deal with the divergences in \( E^{(2)}_{LS} \) and \( E^{(2)}_{VF} \), and the remaining finite contributions, which depend on the renormalization conditions, can be calculated numerically [15]. However, for a nonrenormalizable effective lagrangian, an infinite number of counterterms are required in principle. Here we will argue that both \( E^{(2)}_{LS} \) and \( E^{(2)}_{VF} \) can be written in forms that are already present in our lagrangian (before truncation). Therefore, by adjusting the coefficients of these terms, these contributions are completely absorbed.

\[ E^{(2)}_{LS} = g^2 \int \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \Delta^0_S(k-q) \text{tr} \left[ G^*_F(k)G^*_D(q) \right] \]

(40)

where the self-energy is

\[ \Sigma^*_{\phi}(k) = -\frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} \Delta^0_S(k-q)G^*_F(q). \]  

(41)

Note that this expression contains no explicit density dependence. We substitute the Furry expansion [see Eq. (24)] into the self-energy:

\[ \Sigma^*_{\phi}(k) = \frac{g^2}{2} \sum_{n=0}^{\infty} (M^* - M)^n \int \frac{d^4q}{(2\pi)^4} \Delta^0_S(k-q) \left[ G^0_F(q) \right]^{n+1}, \]

(42)
but now we have let the sum go to infinity and not cut it off as before. This can be rewritten as a Taylor series
\[\Sigma^*_F(k) = \left. \frac{1}{n!} \frac{d^n \Sigma^*_F(k)}{d M^*} \right|_{M^* = M} (M^* - M)^n. \quad (43)\]

We then expand \(\Sigma^*_F(k)\) as a Taylor series around \(i\vec{k} = M\) to separate the momentum dependence, or
\[\Sigma^*_F(k) = \Sigma^*_F(i\vec{k} = M) + \left. \frac{d \Sigma^*_F(k)}{d (i\vec{k})} \right|_{i\vec{k} = M} (i\vec{k} - M) + \frac{1}{2} \left. \frac{d^2 \Sigma^*_F(k)}{d (i\vec{k})^2} \right|_{i\vec{k} = M} (i\vec{k} - M)^2 + \ldots \quad (44)\]

The self-energy now becomes
\[\Sigma^*_F(k) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{m! n!} \left. \frac{d^{m+n} \Sigma^*_F(k)}{d (i\vec{k})^m d M^*} \right|_{i\vec{k} = M, M^* = M} (i\vec{k} - M)^m (M^* - M)^n \quad (45)\]

where \(a^\phi_{mn}\) are just pure numbers, some of which are finite and others infinite (these correspond to divergent diagrams). This consequence of Lorentz covariance allows one to rewrite the Lamb-shift contribution to the energy density as
\[E^{(2)}_{\phi-LS} = 2 \sum_{m=0}^\infty \sum_{n=0}^\infty a^\phi_{mn} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ G_D^*(k) (i\vec{k} - M)^m (M^* - M)^n \right] \]
\[= 2 \sum_{m=0}^\infty \sum_{n=0}^\infty a^\phi_{mn} (M^* - M)^n \int \frac{d^4 k}{(2\pi)^4} \frac{i\pi}{E^*(k)} \delta [k_4 - E^*(k)] \theta \left( k_F - |\vec{k}| \right) \text{tr} [(i\vec{k} - M^*) (i\vec{k} - M)^m]. \quad (46)\]

In general, one can write
\[(i\vec{k} - M)^m = f_m(k^2, M) i\vec{k} + g_m(k^2, M), \quad (47)\]

where \(f_m\) and \(g_m\) are polynomials in \(k^2\) and \(M\). This lets one reduce the integral to
\[-2\gamma \int \frac{d^4 k}{(2\pi)^4} \left[ k^2 f_m(k^2, M) + M^* g_m(k^2, M) \right] \]
\[\times \frac{i\pi}{E^*(k)} \delta [k_4 - E^*(k)] \theta \left( k_F - |\vec{k}| \right) \]
\[= \frac{1}{2} F_m(M^*, M) \frac{\gamma}{(2\pi)^3} \int d^3 k \frac{M^*}{E^*(k)} \theta \left( k_F - |\vec{k}| \right) \]
\[= \frac{1}{2} F_m(M^*, M) \rho_S. \quad (48)\]
Here $\rho_s$ is the scalar baryon density and the on-shell condition has been used. Substituting this result into Eq. (46), we get

$$
\mathcal{E}_{\phi-LS}^{(2)} = \rho_s \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m+n}^\phi F_{m}(M^*, M) (M^* - M)^n
$$

$$
= \rho_s \sum_{m=0}^{\infty} a_m^\phi (gs\phi_0)^m. \tag{49}
$$

Before the field redefinitions were conducted on the effective lagrangian to put it in canonical form, terms like $\rho_s (gs\phi_0)^n = \langle \bar{\psi}\psi \rangle (gs\phi_0)^n$ appeared in the theory. Since the coefficients of these terms will be eliminated by field redefinitions, the two-loop contributions to these terms can just be absorbed, and there is no need to calculate them. These arguments rely only on the Lorentz structure of the self-energy and the on-shell condition imposed by the Density propagator, so similar conclusions follow for the contributions from the vector mesons and pions.

We treat the term $\mathcal{E}_{\phi-VF}^{(2)}$ in a similar manner:

$$
\mathcal{E}_{\phi-VF}^{(2)} = -\frac{g_s^2}{2} \int \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \Delta_s^0(k - q) \left\{ \text{tr}[G_F^*(k)G_F^*(q)] - \text{tr}[G_F^0(k)G_F^0(q)] \right\}. \tag{50}
$$

Note that this contribution has no explicit density dependence. Using Eq. (24), we can write

$$
\mathcal{E}_{\phi-VF}^{(2)} = -\frac{g_s^2}{2} \int \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \Delta_s^0(k - q) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (M^* - M)^{m+n} \times \left[ \frac{1}{k^2 + M^2} \right]^{m+1} \left[ \frac{1}{q^2 + M^2} \right]^{n+1} \text{tr} [(ik - M)^{m+1} (iq - M)^{n+1}], \tag{51}
$$

where $m + n \neq 0$. (The $m + n = 0$ term cancels the vacuum term.) We can use Eq. (47) to solve the trace, or

$$
\mathcal{E}_{\phi-VF}^{(2)} = -g_s^2 \gamma \int \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \Delta_s^0(k - q) \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (M^* - M)^{m+n} \left[ \frac{1}{k^2 + M^2} \right]^{m+1} \left[ \frac{1}{q^2 + M^2} \right]^{n+1} \times [g_{m+1}(k^2, M)g_{n+1}(q^2, M) - k \cdot q f_{m+1}(k^2, M)f_{n+1}(q^2, M)]. \tag{52}
$$

Now one can expand the baryon denominators around $k^2 = M^2$ and $q^2 = M^2$ and perform a Wick rotation to Euclidean space. What remains are some integrals over a complicated polynomial in $k^2$, $q^2$, $k \cdot q$, $M^*$, and $M$. However complicated, these four-dimensional integrals can be done, and we are left with

$$
\mathcal{E}_{\phi-VF}^{(2)} = \sum_{m=0}^{\infty} b_m^\phi (gs\phi_0)^m, \tag{53}
$$

where $b_m^\phi$ are constants that depend on $M$. These terms can also be absorbed into preexisting terms in the lagrangian, and hence there is no need to calculate them. Fortunately, only
a few coefficients are required for an accurate description of bulk nuclear properties \[34\]. Note that the forms in Eqs. \((49)\) and \((53)\) are consistent with the explicit results for these integrals given in \[15\].

### E. Long-range Physics

In this section, we consider the two-loop exchange contributions to the energy density in the presence of background mean fields. These terms are nonanalytic functions of the Fermi momentum and correspond to nonlocal contributions to the energy. The scalar meson contribution to the exchange term becomes

\[
E^{(2)}_{\phi - EX} = \frac{-g_{S}^{2}}{2} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}q}{(2\pi)^{4}} \Delta_{S}^{0}(k - q) \int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}q}{(2\pi)^{4}} \Delta_{S}^{0}(k - q) \frac{1}{(k - q)^{2} + m_{S}^{2}}
\]

where we have used Eq. \((23)\) and integrated out most of the angular dependence. The vector meson contribution is

\[
E^{(2)}_{V - EX} = \frac{g_{V}^{2}}{2} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}q}{(2\pi)^{4}} \Delta_{S}^{0}(k - q) \int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}q}{(2\pi)^{4}} \Delta_{S}^{0}(k - q) \frac{1}{(k - q)^{2} + m_{V}^{2}}
\]

where we integrated out most of the angular dependence. The contribution to the integral from the longitudinal portion of the vector propagator vanishes if one works out the trace. Finally, we consider the two-loop contribution arising from pion exchange. We remove most of the angular dependence and arrive at the result

\[
E_{\pi - EX}^{(2)} = \frac{g_{A}^{2}}{2f_{\pi}^{2}} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{d^{4}q}{(2\pi)^{4}} \Delta_{S}^{0}(k - q)
\]
that naturalness holds is also left for a subsequent investigation. We partially answer that question here, but a more complete proof is preserved; we include the two-loop effects also calls into question whether the naturalness assumption is preserved.

The exploration of the effects of these terms at the two-loop level is left for future work. The long-range dynamics is nonlocal and has to be calculated explicitly. For the purposes of this work, we are interested in applying the loop expansion only to ordinary, symmetric nuclear matter. For that reason, we neglected both electromagnetic effects and meson exchange, although they both appear in the mean field lagrangian [2, 3]. In addition, the physics (including all the divergences) was isolated and absorbed into the coefficients of the lagrangian. The long-range dynamics order by order and for analyzing their structures. The short-range, local lagrangian. This expansion provided a simple scheme for separating the short- and long-range dynamics at the two-loop level with a parameter set (M0A) that is natural ($g_A = 1.5876$ and $f_\pi = 93$ MeV [2]. In addition, both L2 and W1 sets lead to equilibrium at $k_F = 1.3$ fm$^{-1}$. The mean meson fields are determined by extremizing the meson field equations and are used as input to the exchange integrals. Then the full two-loop energy density is extremized with respect to the meson fields. The results for all three exchange terms are shown in Table II for both L2 and W1 parameter sets. The third set (M0A) listed in Table I was fit at the two-loop level to nuclear equilibrium ($E/M = -16.10$ MeV and $k_F = 1.3$ fm$^{-1}$) by adjusting $g_S$ and $g_V$ using a downhill simplex method to minimize a least-squares fit (with respective weights of 0.0015 and 0.002). The binding curves of all three sets are shown in Fig. 2. The curves for the L2 and W1 sets include the two-loop contributions and are compared to the saturation curve of the M0A set at the two-loop level. Fig. 3 shows results when the L2 and W1 sets are evaluated at the one-loop level, and the M0A set is evaluated at the two-loop level. First, we note that nuclear saturation can be reproduced at the two-loop level with a parameter set (M0A) that is natural ($g_{S,V} \approx 4\pi$). Second, Fig. 2 shows that while the two-loop contributions are not large, they are not negligible either.

Now we consider the numerical analysis of the three surviving two-loop integrals (the exchange terms). We use the parameter sets listed in Table I. For the pion term, we use $g_A^2 = 1.5876$ and $f_\pi = 93$ MeV [2]. In addition, both L2 and W1 sets lead to equilibrium at $k_F = 1.3$ fm$^{-1}$. The mean meson fields are determined by extremizing the meson field equations and are used as input to the exchange integrals. Then the full two-loop energy density is extremized with respect to the meson fields. The results for all three exchange terms are shown in Table II for both L2 and W1 parameter sets. The third set (M0A) listed in Table I was fit at the two-loop level to nuclear equilibrium ($E/M = -16.10$ MeV and $k_F = 1.3$ fm$^{-1}$) by adjusting $g_S$ and $g_V$ using a downhill simplex method to minimize a least-squares fit (with respective weights of 0.0015 and 0.002). The binding curves of all three sets are shown in Fig. 2. The curves for the L2 and W1 sets include the two-loop contributions and are compared to the saturation curve of the M0A set at the two-loop level. Fig. 3 shows results when the L2 and W1 sets are evaluated at the one-loop level, and the M0A set is evaluated at the two-loop level. First, we note that nuclear saturation can be reproduced at the two-loop level with a parameter set (M0A) that is natural ($g_{S,V} \approx 4\pi$). Second, Fig. 2 shows that while the two-loop contributions are not large, they are not negligible either.

Next, we compare the magnitudes of these integrals to terms in the meson sector at

$$
\times \text{tr} \left[ (\not{k} - \not{q}) \frac{\gamma_5}{2} G_D(k)(\not{k} - \not{q}) \frac{\gamma_5}{2} G_D(q) \right]
$$

$$
= - \frac{g_A^2}{2f_\pi^2} \int \int \frac{d^4k}{(2\pi)^3} \frac{d^4q}{(2\pi)^3} \frac{\theta(k_F - |\not{k}|) \theta(k_F - |\not{q}|)}{2E^*(k) - 2E^*(q)} \frac{1}{(k - q)^2 + m_\pi^2} \delta_{ab}
$$

$$
\times \delta(k_4 - E^*(k)) \delta(q_4 - E^*(q)) \frac{1}{(k - q)^2 + m_\pi^2} \delta_{ab}
$$

$$
\times \text{tr} \left[ (\not{k} - \not{q}) \frac{\gamma_5}{2} (i\not{k} - M^*) (\not{k} - \not{q}) \frac{\gamma_5}{2} (i\not{q} - M^*) \right]
$$

$$
= \frac{\gamma g_A^2}{8\pi^4 f_\pi^2} \left( \frac{5\gamma - 8}{16} \right) M^{*2} \int_0^{k_F} \frac{|\vec{k}|^2 |\vec{k}|}{E^*(k)} \int_0^{k_F} \frac{|\vec{q}|^2 |\vec{q}|}{E^*(q)} \int_{-1}^{1} d\cos \theta \left[ \frac{E^*(k)E^*(q) - |\vec{k}||\vec{q}| \cos \theta - M^{*2}}{2E^*(k)E^*(q) - 2|\vec{k}||\vec{q}| \cos \theta - 2M^{*2} + m_\pi^2} \right].
$$

(56)

### III. DISCUSSION

In this work, we performed the loop expansion to the two-loop level for a model QHD lagrangian. This expansion provided a simple scheme for separating the short- and long-range dynamics order by order and for analyzing their structures. The short-range, local physics (including all the divergences) was isolated and absorbed into the coefficients of the lagrangian. The long-range dynamics is nonlocal and has to be calculated explicitly. For the purposes of this work, we are interested in applying the loop expansion only to ordinary, symmetric nuclear matter. For that reason, we neglected both electromagnetic effects and meson exchange, although they both appear in the mean field lagrangian [2, 3]. In addition, we exclude the tensor terms in the fermion sector and the nonlinear meson self-couplings. The exploration of the effects of these terms at the two-loop level is left for future work. The inclusion of the two-loop effects also calls into question whether the naturalness assumption [2, 10] is preserved; we partially answer that question here, but a more complete proof that naturalness holds is also left for a subsequent investigation.
### TABLE I: Parameter sets used in this work.

|        | L2 [3] | W1 [2] | M0A |
|--------|--------|--------|-----|
| $m_S/M$| 0.55378| 0.60305| 0.54|
| $m_V/M$| 0.83387| 0.83280| 0.83280|
| $g_s/4\pi$| 0.83321| 0.93797| 0.79361|
| $g_V/4\pi$| 1.09814| 1.13652| 0.96811|

### TABLE II: Size of two-loop integrals for the parameter sets in Table I. Values are in MeV.

|        | L2 [3] | W1 [2] | M0A  |
|--------|--------|--------|------|
| $\mathcal{E}_{\phi-EX}^{(2)}$| 42.59  | 46.65  | 40.30|
| $\mathcal{E}_{V-EX}^{(2)}$| -29.51 | -30.60 | -23.14|
| $\mathcal{E}_{\pi-EX}^{(2)}$| 12.39  | 12.21  | 12.44|

### FIG. 2: Comparison of the nuclear binding curves for the sets L2 and W1 (both with one-loop parameters but with the two-loop contributions included) and M0A (two-loop level).

the mean field level. However, neither set L2 nor W1 includes nonlinearities in the meson fields. Therefore, we will compare the exchange integrals to terms from the sets Q1 and Q2 [2]. Here Q1 and Q2 both include cubic and quartic scalar field self-couplings and Q2 also contains a quartic vector field self-coupling. These nonlinear terms are contained in

$$
\mathcal{L}_{NL} = -\left( \frac{\kappa_3 g_S \phi}{3! M} + \frac{\kappa_4 g_S^2 \phi^2}{4! M^2} \right) m_S^2 \phi^2 + \frac{1}{4!} \zeta_0 g_V^2 (V_\mu V^\mu)^2
$$

of Eq. (1). This does not imply that these nonlinearities were taken into account when the
two-loop integrals were calculated; it is simply instructive to compare the relative sizes of the mean field nonlinearities and the two-loop contributions. Note that the Q1 and Q2 mean fields were obtained by extremizing the one-loop energy, including $\mathcal{L}_{NL}$, while the $M^*$ in the exchange integrals was obtained by minimizing $\mathcal{E}^{(2)}$ of Eq. (39). While the inclusion of the nonlinear terms in the meson sector at the two-loop level has yet to be performed, a cursory glance tells us that they will affect only the meson propagators in two-loop integrals. Thus the overall magnitude of these terms is not expected to change much when these nonlinear effects are included.

Fig. 4 shows the results of this comparison. The crosses in Fig. 4 represent the expected magnitude per order of terms in the meson sector using the rules of naive dimensional analysis (NDA) [2] with the chiral symmetry breaking scale of $\Lambda = 650$ MeV. Thus, one observes that the two-loop integrals are roughly equivalent to $\nu = 3$ order in the power counting in the mean field lagrangian. While this is not large, they cannot be neglected in a description of nuclear matter saturation properties, particularly in view of the nearly complete cancellation of scalar and vector terms at order $\nu = 2$. We emphasize that the explicit computation of short-range, two-loop contributions leads to unnaturally large terms in the energy of nuclear matter [15]; in the EFT approach, these contributions are absorbed in the local counterterms and are either removed by field redefinitions or are determined by fitting to empirical nuclear data.

We stress that the results shown here are for the crudest truncation in the underlying lagrangian. It is well known that an accurate description of nuclear saturation (and the equation of state) requires the inclusion of nonlinear meson self-interactions, which is the subject of future work [23]. Furthermore, the question of whether or not the loop expansion is valid for QHD cannot be answered at the two-loop level. While two-loop calculations are a necessary step in that direction, an investigation into higher-loop effects is required; this is also the subject of future work [24].
FIG. 4: Comparison of the magnitudes of the mean field terms in the meson sector with the two-loop exchange integrals. The inverted triangles represent, from top to bottom, the scalar, vector, and pion two-loop integrals. The abscissa represents the order $\nu$ in the power counting at the mean field level [2]. Absolute values are shown.

### A. Infrared Regularization

The separation of the short-range and long-range dynamics, and the subsequent absorption of the short-range physics into the lagrangian are analogous to Infrared Regularization in Chiral Perturbation Theory [26–30]. In the language of Ellis and Tang [26, 27], a loop integral can be separated into hard and soft components

\[
G = (G - \hat{R}\hat{S}G) + \hat{R}\hat{S}G ,
\]

where $G$ is the loop integral, $\hat{S}$ projects out the soft part, and $\hat{R}$ renormalizes $\hat{S}G$ to remove the ultraviolet divergences. To extract the soft part, do the following: take the loop momentum to be of order $Q$, make a $Q/M$ expansion of the integrand, and interchange the order of integration and summation. Now consider a loop integral with momentum $q$, specifically the $q_4$ part of the integral. Closing the contour by a semicircle at infinity, we get the sum of three contributions: the semicircle, the soft poles, and the hard poles. The soft and hard poles are of order $Q$ and $M$ respectively. (It has been assumed that the hard and soft poles can be separated, as is the case in theories with Goldstone bosons and massive baryons.) The semicircle may produce divergences, but these can be removed by the usual renormalization. The soft poles cannot be expanded in $Q/M$, such as those in the pion propagator. However, a $Q/M$ expansion can be made around the hard poles since the loop momentum is of order $Q$. Finally, integrating term by term removes the hard contribution, and thus we are left with the unrenormalized soft part, in which ultraviolet divergences may still occur (which are removed by applying $\hat{R}$). Alternative implementations can be found in [28–30].

Since the prescription for $\hat{R}\hat{S}G$ contains all the soft parts, the hard portion must be given by $(G - \hat{R}\hat{S}G)$; the hard part involves only the large momentum contributions, including
some ultraviolet divergences. We can write the hard part as a series of local counterterms—a well-known result that is fundamental to the idea of effective field theories. Indeed, large momenta correspond to short distances that are tiny compared with the wavelengths of the external particles, so the effects can be described by a local interaction. Thus, we can perform the extra renormalization of absorbing the hard parts into the parameters of the lagrangian.

In our case, we separate the baryon propagator into the Feynman and Density parts. The Feynman part contains the high-momentum contributions and also contains all the hard poles. The Density part contains all the soft poles describing the valence nucleons, which occur at lower momenta. (Note that the momenta in the meson propagators are always spacelike.) One can write the one-loop self-energy for the scalar two-loop integral as

\[ \Sigma^\phi(k) = -\frac{g_s^2}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_S^0(k-q)G^*(q) \]

\[ = -\frac{g_s^2}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_S^0(k-q)G_F^*(q) - \frac{g_s^2}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_S^0(k-q)G_D^*(q). \]  

The first term contains only the hard poles and can be absorbed by the processes outlined above: we expand either in powers of momentum or in powers of the mean fields (Furry’s theorem). Then interchange orders of summation and integration; what’s left are series in the mean fields. The second term contains the soft poles, but they are regularized by the theta function in the Density propagator and hence this contribution is finite. As a result, the procedure outlined in this work is analogous to Infrared Regularization.

**B. Power Counting**

To illustrate the order of the two-loop exchange integrals, we expand them in powers of momenta and pick out the dominant contributions. We can expand the isoscalar meson propagators as

\[ \frac{1}{x+1} = 1 - x + x^2 - x^3 + \ldots, \]  

where \( x = (k-q)/m_S \leq 1 \) for the scalar meson and \( x = (k-q)/m_V \leq 1 \) for the vector meson. This is essentially an expansion of meson exchange into contact and gradient interaction terms. Thus, one can pick out the leading (i.e., contact) terms

\[ \mathcal{E}^{(2)}_{\phi - EX} \approx \frac{g_s^2}{4\pi^4m_S^2} \int_0^{k_F} \frac{d^4k}{E^*(k)} \int_0^{k_F} \frac{|\vec{k}^2|d|\vec{k}|}{E^*(q)} (E^*(k)E^*(q) + M^{*2}) \]

\[ = \frac{g_s^2}{16m_S^2} (\rho_B^2 + \rho_S^2) \]  

\[ \mathcal{E}^{(2)}_{V - EX} \approx \frac{g_V^2}{2\pi^4m_V^2} \int_0^{k_F} \frac{d^4k}{E^*(k)} \int_0^{k_F} \frac{|\vec{q}^2|d|\vec{q}|}{E^*(q)} (E^*(k)E^*(q) - 2M^{*2}) \]

\[ = \frac{g_V^2}{8m_V^2} (\rho_B^2 - 2\rho_S^2) \]

In the case of the pion two-loop exchange integral, the pion mass is too small for Eq. (60)
to converge. Therefore, we instead take the chiral limit \((m_\pi \to 0)\) to get the dominant term

\[ E^{(2)}_{\pi} \approx \frac{3g^2_A}{32f^2_\pi} \rho_S^2 + O(m_\pi^2) . \]  

(63)

These are the terms which set the scale for the exchange contributions.

One can acquire the scale by comparison to terms at the mean field level; for instance, the terms in Eqs. (61) and (62) in ratio with the leading vector mean field term are (where we have used the vector meson field equation)

\[ \frac{g^2_S}{16m^2_S} (\rho^2_B + \rho^2_S) \approx \frac{2}{7} \]  

(64)

\[ \frac{g^2_V}{2m^2_V} \rho^2_B \approx -\frac{1}{5} \]  

(65)

Next, we consider the leading terms at normal density \((k_F = 1.3 \text{ fm}^{-1})\) using W1 [2]; we get \( E^{(2)}_{\phi} / \rho_B \approx 57 \text{ MeV}, E^{(2)}_{V} / \rho_B \approx -34 \text{ MeV}, \) and \( E^{(2)}_{\pi} / \rho_B \approx 17 \text{ MeV}. \) These terms are roughly third order \((\nu = 3)\) in the power counting. Observe that the self-energies are reproduced at the 10–40% level using only the leading-order contact term or with the chiral limit. To incorporate the corrections to these results, it is clearly most efficient to retain the full meson propagators in the exchange integrals.

Exchange interactions are two-body, so they go like \( O(\rho^2) \) in \( E \). This is the same dependence as the two-body mean field terms at \( O(\nu = 2) \). Because of the lack of a spin–isospin sum in the exchange terms, the exchange terms are numerically smaller and contribute at the same level as the \( O(\nu = 3) \) terms. A deeper understanding of how loops fit into the finite-density power-counting scheme will require a look at higher-order terms in the loop expansion, which is currently under investigation.

In Fig. 5, we show the individual and total two-loop contributions to the energy per particle plotted as a function of the density \( \rho = \rho_B \). Since the ratio \( \rho_S / \rho_B \) decreases as \( \rho_B \) increases, one sees that the qualitative behavior of the scalar and vector contributions agrees with the leading-order estimates in Eq. (61) and (62). This is not surprising, since the two-loop exchange integrals are dominated by the contact terms, which have a simple density dependence. The total two-loop contribution is almost linearly dependent on the density, which occurs “by construction” with the appropriately refit coupling parameters. While this implies that the two-loop terms are relatively short-ranged, they are nonetheless nonlocal and hence they cannot be absorbed directly into the parameterization via Infrared Regularization.

**IV. SUMMARY**

In this work, we studied two-loop corrections to symmetric nuclear matter in a covariant effective field theory. The loop expansion gives a straightforward way to separate the
short-distance physics from the long-distance physics. The former can be absorbed into counterterms already present in the effective lagrangian, and they are either removed by field redefinitions or fitted to empirical data. The remaining long-range exchange integrals are nonlocal and must be computed explicitly. They produce modest corrections to the nuclear binding curve and can be compensated by a small adjustment of the coupling parameters.

Since exchange integrals in effective hadronic field theories have been studied for more than 30 years [35], it is important to enumerate the new features of our calculations. First, the QHD model studied here has its basis in a Lorentz-covariant, chiral-invariant, hadronic effective field theory that is tailored to the nuclear many-body problem [2] and that successfully describes bulk and single-particle nuclear properties at the one-loop level [2–7, 11–14]. Second, using standard procedures of EFT, the loop expansion at finite density provides a systematic, well-defined treatment of the short- and long-range contributions to the integrals that can be extended to higher orders in loops [23, 31]. Third, when interpreted in the context of density functional theory, the exchange contributions to the energy introduce nonanalytic density dependence that is qualitatively different from that appearing in the mean-field theory. This should allow for an improved approximation to and parametrization of the exact energy functional. In addition, as a criterion for discussing the size of the exchange contributions, one can readjust the coupling parameters to reproduce the nuclear matter equilibrium point and see if they remain of natural size; this was indeed the case with the two couplings adjusted here. The naturalness of the remaining parameters in the underlying EFT will be studied in future work. Finally, the size of the two-loop integrals (roughly third order in the mean field power counting) was determined. Fuller consideration of how the loop expansion fits into the power counting is left for future investigation.

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