Constraining theories of gravity by fundamental plane of elliptical galaxies

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Abstract

We show that fundamental plane of elliptical galaxies can be used to obtain observational constraints on metric theories of gravity. Being it connected to global properties of ellipticals, it can fix parameters of modified gravity. Specifically, we use fundamental plane to constrain modified theories of gravity with Yukawa-like corrections which commonly emerge in the post-Newtonian limit. After giving examples on how these corrections are derived, we first analyze the velocity distribution of elliptical galaxies comparing theoretical results of modified gravity with Yukawa-like corrections with astronomical data. According to these results, it is possible to constrain the parameters of the corrections discriminating among classes of models compatible with astronomical observations. We conclude that fundamental plane can be used as a standard tool to probe different theories of gravity in the weak field limit.

Keywords: Modified theories of gravity, experimental tests of gravitational theories, elliptical galaxies, fundamental plane, luminosity and mass functions.

1. Introduction

The need for dark matter emerged to describe dynamics of self-gravitating systems like stellar clusters, galaxies, groups and clusters of galaxies since the 30’s of last century\textsuperscript{[1, 2]}. In all these cases, there is more matter than that accounted for by luminous components assuming the validity of Newton potential at all astrophysical scales.

In order to explain such observational results, the first considered possibility was assuming the existence of subluminous components dubbed dark matter. Many candidates have been proposed to supplement the missing matter but, up to now, there is no final indication for their existence both at fundamental and astrophysical level\textsuperscript{[3]}. Furthermore, cosmological observations require another unknown dynamical component, the so-called dark energy, to account for the current accelerated expansion of the universe\textsuperscript{[4, 5, 6]}. Also in this case, addressing the phenomenon at particle level is revealing a severe challenge. Hence, the need of imposing unknown dark components could be nothing else but the signal of the breakdown of General Relativity at galactic, extragalactic and cosmological scales. In this context, modified theories of gravity, like $f(R)$, Brans-Dicke, and Gauss-Bonnet gravity, could be a way to explain cosmic accelerated expansion, large scale structure and galactic dynamics\textsuperscript{[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]}.

In the weak-field limit, alternative theories of gravity are expected to reproduce General Relativity which is successfully tested at Solar System scales\textsuperscript{[18]}. Several proposals have been formulated to go beyond $\Lambda$CDM in view of possible alternatives to explain astrophysical and cosmological observations. In particular, the flat rotation curves of galaxies, can be addressed by MOdified Newtonian Dynamics (MOND)\textsuperscript{[19]} which adopts an acceleration scale to
account for high velocities without dark matter. Besides phenomenology, this acceleration scale could be some new fundamental parameter of nature [20, 21].

MOND has been relativistically improved by TeVeS [22], an approach including additional vector and scalar gravitational degrees of freedom to the standard tensor field of General Relativity. MOND and TeVeS have been tested by gravitational lensing concluding that non-trivial dark matter components has to be added to match exactly astrophysical observations [23, 24, 25, 26].

MOND and TeVeS are two prototypes of several extensions and modifications of General Relativity proposed to work at infrared scales. For example, $f(R)$ gravity is a natural extension of Einstein’s theory: it does not fix a priori the form of the gravitational action, like the Einstein-Hilbet one, but assumes that it can be reconstructed by observations [9]. In this picture, dark components are a sort of curvature fluid acting as an interaction field at astrophysical and cosmological scales [27, 28].

Besides $f(R)$, more detailed theories have been proposed to address the phenomenology. However, some common features can be put in evidence for any proposal aimed to explain dynamics without dark component: i) General Relativity has to be reproduced at certain scale (e.g. at Solar System); ii) further degrees of freedom reduce to corrections of the Newtonian potential [29]; iii) these corrections are often Yukawa-like terms characterizing the scale where the behavior begins to stand out with respect to the Newtonian dynamics.

As reported in [30], several metric theories shows Yukawa-like corrections. Clearly, as we will show below, their parameters strictly depend on the specific gravitational theory worked out in the weak field limit. In some sense, fixing them means selecting the theory of gravity. However, the limitation of this paradigm is that the given theory of gravity have to be developed in the post-Newtonian regime and strong field effects are not considered.

Finally, this kind of corrections has been often investigated at Solar System, local and microscopic scales (the so called "fifth-force" issue [33]). On the other hand, a systematic investigation at galactic and extragalactic scales has never been performed, as far as we know.

In this paper, we propose a new approach by which the weak field limit of metric theories can be systematically investigated at extragalactic scales. To this aim, we shall adopt the fundamental plane (FP) of galaxies. The final goal is fixing the values of correction parameters with respect to the standard Newtonian potential, in order to select the corresponding field theory. Here, we perform the analysis assuming Yukawa-like corrections because wide classes of theories show them in the weak-field limit. In any case, the protocol can be adopted also for other gravitational corrections.

Let us start from the empirical fact that some global properties of normal elliptical galaxies are correlated. It is well known that there are three main global observables: the central projected velocity dispersion $\sigma_0$, the effective radius $r_e$, and the mean effective surface brightness (within $r_e$) $I_e$ [34, 35]. Any of the three parameters may be estimated from the other two, and, varying them, a plane is described within a more general three-dimensional configuration space. This correlated plane is referred to as the FP. It is defined and discussed in detail in several papers, see e.g [36, 37, 38, 40, 41] and references therein.

This important empirical relation [38]:

\[ \log(r_e) = a \times \log(\sigma_0) + b \times \log(I_e) + c, \]  

(1)
gives us the possibility to obtain observational constraints on the structure, formation, and evolution of early-type galaxies. Here we shall adopt Eq.(1) to constrain parameters of gravity theories. It is worth noticing that a FP can be defined for several self-gravitating systems ranging from stellar clusters up to gamma ray bursts [39].

The content of this paper is as follows. In Section 2, we discuss Yukawa-like corrections emerging from modified theories of gravity. In particular, we report the standard case of $f(R)$ gravity where Yukawa-like corrections naturally emerge. In Table 1, we show several examples of theories where Yukawa corrections are derived. In Section 3, we recover the FP of elliptical galaxies by theories of gravity with Yukawa-like corrections, describing the observations and giving details on our approach. Constraints on Yukawa parameters by FP are considered in Section 4. Section 5 is devoted to draw conclusions.

2. Yukawa-like corrections in the weak field limit of metric theories

As shown in several studies, Yukawa-like corrections emerges as common features for several metric theories of gravity. In particular for theories containing higher-order curvature invariants or scalar-tensor theories. See [14].
for a comprehensive review. As an example, we will show here the well-known case of \( f(R) \) gravity which can be considered as a sort of paradigm in this sense.

Let us start from the Einstein-Hilbert action improved with a generic analytic function of the Ricci scalar \( R \), that is \( f(R) \) gravity. The field equations are

\[
f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)g_{\mu\nu} + g_{\mu\nu}\Box f(R) = \kappa T_{\mu\nu},
\]

with the trace

\[
3\Box f'(R) + f'(R)R - 2f(R) = \kappa T.
\]

Here \( \kappa \) is the gravitational coupling and \( T_{\mu\nu} \) is the energy-momentum tensor of matter. We are interested in external solutions so we can ignore the matter contribution.

In the weak field limit, we can perturb the metric tensor with respect to the Minkowski background, i.e. \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) with \( |\eta_{\mu\nu}| \ll |h_{\mu\nu}| \). Let us assume an analytic \( f(R) \) Lagrangian expandable in Taylor series

\[
f(R) = \sum_{n} \frac{f^{n}(R)}{n!}(R - R_{0})^{n} \approx f_{0} + f'_{0}R + f''_{0}R^{2} + f'''_{0}R^{3} + \ldots
\]

where the prime indicates derivatives with respect to \( R \). The field equations, in the post-Newtonian limit up to \( O(4) \) order, are

\[
\begin{align*}
&f'_{0}R^{(2)} - 2f'_{0}g_{tt}^{(2)} + 8f''_{0}R_{tt}^{(2)} - f'_{0}g_{rr}^{(2)} + 4f''_{0}R_{rr}^{(2)} = 0, \\
&f'_{0}R^{(2)} - 2f'_{0}g_{rr}^{(2)} + 8f''_{0}R_{rr}^{(2)} - f'_{0}g_{tt}^{(2)} = 0, \\
&2f'_{0}g_{tt}^{(2)} - r\left[ f'_{0}R^{(2)} - f'_{0}g_{tt}^{(2)} + 4f''_{0}R_{tt}^{(2)} - f'_{0}g_{rr}^{(2)} + 4f''_{0}R_{rr}^{(2)} \right] = 0, \\
&f'_{0}R^{(2)} + 6f''_{0}\left[ 2R_{tt}^{(2)} + 2R^{(2)}_{rr} \right] = 0, \\
&2g_{tt}^{(2)} + r\left[ 2g_{tt}^{(2)} - R^{(2)} - 2g_{rr}^{(2)} + 2g_{rr}^{(2)} + rR_{tt}^{(2)} \right] = 0.
\end{align*}
\]

These equations can be integrated giving the solution

\[
\begin{align*}
g_{tt}(r) &= 1 - \frac{GM}{f'_{0}r} + \frac{\delta_{1}L^{2}e^{-r/L}}{3}, \\
g_{rr}(r) &= 1 + \frac{GM}{f'_{0}r} + \frac{\delta_{1}L^{2}(1 + r/L)e^{-r/L}}{3r}, \\
R &= \frac{\delta_{1}e^{-r/L}}{r}.
\end{align*}
\]

where \( g_{tt} \) and \( g_{rr} \) are the time and radial metric coefficient respectively. \( R \) is Ricci scalar derived for this problem. We are assuming the spherical symmetry. Here \( L = \sqrt{-6f''_{0}/f'_{0}} \) is a scale length. The constants \( \delta_{1} \) and \( f'_{0} \) can be combined into another constant \( \delta \) giving the strength of the correction \([27, 42, 43, 44, 46, 47]\). Being

The functions \( g_{tt} \) and \( g_{rr} \) give two gravitational potentials

\[
\begin{align*}
\Phi(r) &= -\frac{2GM(1 + \delta e^{-r/L})}{rc^{2}(\delta + 1)}, \\
\Psi(r) &= \frac{2GM}{rc^{2}} \left[ \left( 1 + \delta e^{-r/L} \right) + \frac{(\delta c^{2} - L^{2})}{\delta(\delta + 1)} \right],
\end{align*}
\]

being, in general,

\[
g_{tt} = 1 - \frac{\Phi(r)}{c^{2}}, \quad g_{rr} = 1 + \frac{\Psi(r)}{c^{2}}.
\]
Eqs. (9) and (10) coincide as soon as \( r \to \infty \), i.e. asymptotically they give \( \Phi \approx \Psi \) that is the Newtonian potential is recovered. With a suitable readjustment of parameters,

\[
G \to \frac{2G}{1 + \delta}, \quad \alpha \to \frac{\delta}{\delta + 1}, \quad L \to \frac{1}{\lambda},
\]

and for \( c = 1 \), Eqs. (9) and (10) can be adapted to the standard form

\[
\Phi(r) = -\frac{GM(r)}{r} \left[ 1 + \alpha \exp(-\lambda r) \right].
\]

Again \( \lambda = L^{-1} \) is a scale length and \( \alpha \) gives the strength of the correction \([30, 45, 48]\). It is worth noticing that \( L = \lambda^{-1} = \frac{h}{m_R c} \) is a Compton length which can be related to an effective mass \( m_R \) coming from curvature. In early universe cosmology, it is the so-called Starobinsky scalaron \([49]\).

A potential like Eq. (13) is very generic for a large class of theories. Various combinations of Yukawa-like terms intervene in several metric theories of gravity. In Table 1, we report some examples where higher-order curvature terms or scalar fields give this kind of corrections in the weak field limit. The paradigm is that, adding further degrees of freedom in the Einstein-Hilbert action, the typical outcome is one or more Yukawa-like corrections in the post-Newtonian limit. In some sense, the exception is General Relativity where only the standard Newtonian gravitational potential is recovered.

With this considerations in mind, we will study the FP and constrain \( \lambda \) and \( \alpha \) by astronomical data. These parameters can suitably reproduce dark matter effects \([27]\). According to this procedure, one can, in principle, "reconstruct" the class of theories of gravity "compatible" with observations. In some sense, this is an "inverse scattering" procedure which could reveal useful to probe theories by galactic data. It is a sort of "blind" approach in which we are not requesting a priori the validity of a given model but we are asking for classes of compatible theories fixed by the range of parameters.

3. Fundamental plane of modified gravity with Yukawa-like corrections

As said above, we are going to constrain the parameters \( \alpha \) and \( \lambda \) considering a sample of ellipticals that distribute along the FP. We shall describe the observations and the method that we are going to adopt.

3.1. Observations and method

We use the observational data for physical properties of stellar systems given in the paper by Burstein et al. (1997) \([50]\). The table "Global Relationships for Physical Properties of Stellar Systems", given in ASCII format and labeled 'metaplanetab1', is available within the arXiv version of \([50]\): https://arxiv.org/e-print/astro-ph/9707037. It summarizes data, like the self-consistent effective radii, effective luminosities, characteristic dynamic velocities, and some other related data. In our previous paper \([34]\), we already described the columns from this table which are interesting in the present case. We also emphasize, that from the sample of 1150 observed galaxies, we selected and studied 401 ellipticals.

Let us take into account the relation for circular velocity, consisting of the Newtonian contribution, and the correction term due to modified gravity. Then, for the given Yukawa parameters \((\alpha, \lambda)\), we calculate the theoretical values of velocity dispersion (see the next Subsection 3.2).

3.2. Velocity dispersion and the singular isothermal sphere model

In the case of Newtonian potential, it is: \( \Phi_N(r) = -\frac{GM(r)}{r} \) and circular velocity \( v_N^2(r) = r \cdot \Phi_N'(r) \) \([30]\). In the case of modified potential, supposing the spherically distributed mass in elliptical galaxies, we have \( v_c^2(r) = r \cdot \Phi'(r) \)
| Modified Gravity Model | Corrected Newtonian potential | Yukawa parameters |
|------------------------|-------------------------------|------------------|
| $f(R)$                 | $\Phi(r) = -\frac{GM}{r} \left[ 1 + \alpha e^{-mr} \right]$ | $m^2 = -\frac{c_{0,1}}{f_{0,0}(r)}$ |
| $f(R, □R) = R + a_0 R^2 + a_1 R □ R$ | $\Phi(r) = -\frac{GM}{r} \left[ 1 + c_0 e^{-r^2/h_0} + c_1 e^{-r^2/h_1} \right]$ | $c_{0,1} = \frac{1}{a_1} + \frac{b_0}{2 \sqrt{a_0^2 + b_0}}$ |
| $f(R, □R, □□R) = R + \sum_{i=0}^P a_i R □ □ R$ | $\Phi(r) = -\frac{GM}{r} \left[ 1 + \sum_{i=0}^P c_i \exp(-r/l_i) \right]$ | $l_{0,1} = \sqrt{-3a_0 \pm \sqrt{9a_0^2 + 6a_1}}$ |
| $f(R, R_{\alpha \beta} R_{\alpha \beta})$ | $\Phi(r) = -\frac{GM}{r} \left[ 1 + \frac{1}{3} e^{-m_1 r} - \frac{4}{3} e^{-m_2 r} \right]$ | $m_{1,2} = -\frac{1}{f_{0,0}(r)} \cdot Y = R_{\mu \nu} R_{\mu \nu}$ |
| $f(R, \phi) + \omega(\phi) \phi_{,\alpha} \phi^{,\alpha}$ | $\Phi(r) = -\frac{GM}{r} \left[ 1 + g(\xi, \eta) e^{-m_{1,2} r} + \frac{[1/3 - g(\xi, \eta)] e^{-m_{1,2} r}}{3} \right]$ | $m_{1,2} = -\frac{1}{f_{0,0}(r)} \cdot Z = R_{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}$ |
| $f(R, R_{\alpha \beta} R_{\alpha \beta}, \phi) + \omega(\phi) \phi_{,\alpha} \phi^{,\alpha}$ | $\Phi(r) = -\frac{GM}{r} \left[ 1 + g(\xi, \eta) e^{-m_{1,2} r} + \frac{[1/3 - g(\xi, \eta)] e^{-m_{1,2} r}}{3} \right]$ | $m_{1,2} = -\frac{1}{f_{0,0}(r)} \cdot Z = R_{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}$ |

Table 1: Yukawa-like corrections are a general feature of several modified gravity models. In particular, they emerge in Extended Theories of Gravity which are natural extension of General Relativity [9]. In some sense, further degrees of freedom, related to higher-order terms or scalar fields, give rise to these corrections in the weak field limit. This is a general result as discussed in [29]. In the Table, we report examples of modified gravity models showing Yukawa-like corrections in the post-Newtonian limit. Detailed discussions of these results are reported in [15, 29, 30, 31, 32].
Here we start from the Yukawa-like gravitational potential \( \Phi(r) \) and derive the connection between \( v^2_c(r) \) and parameters of this potential. Let us start from Eq. (13) and its derivative. We have:

\[
\Phi(r) = -\frac{GM(r)}{r} - a\frac{GM(r)}{r} e^{-\lambda r}, \\
\Phi'(r) = \left(-\frac{GM(r)}{r}\right)' + a\left(-\frac{GM(r)}{r}\right)' e^{-\lambda r} + a\left(-\frac{GM(r)}{r}\right)(e^{-\lambda r})', \\
= \Phi_N'(r) + ae^{-\lambda r}\Phi_N'(r) + a\Phi_N(r)(-\lambda)e^{-\lambda r} \\
= \Phi_N'(r) + ae^{-\lambda r}\Phi_N'(r) - \alpha\lambda e^{-\lambda r}\Phi_N(r) \\
v^2_c(r) = v^2_N(r) + ae^{-\lambda r}v^2_N(r) + \alpha\lambda e^{-\lambda r}v^2_N(r) \\
= \frac{GM(r)}{r} + \frac{GM(r)}{r} \alpha(1 + \lambda r) e^{-\lambda r}. 
\]

Therefore the squared circular velocity \( v^2_c(r) = r \cdot \Phi'(r) \) takes the form: \( v^2_c(r) = \frac{GM(r)}{r} \left(1 + \alpha (1 + \lambda r) e^{-\lambda r}\right) \). Now, we can write this expression as a sum of the Newtonian contribution \( v^2_N(r) \) and the correction term due to modified gravity \( v^2_{corr}(r) \):

\[
v^2_c(r) = v^2_N(r) + v^2_{corr}(r), \quad (15)
\]

where:

\[
v^2_N(r) = \frac{GM(r)}{r}, \\
v^2_{corr}(r) = \frac{\alpha GM(r) \cdot (1 + \lambda r)}{r} e^{-\lambda r}. \quad (16)
\]

Here, it is important to stress that the corrective velocity term can be expressed also via Newtonian velocity dispersion as \( v^2_{corr}(r) = \alpha v^2_N(r)(1 + \lambda r)e^{-\lambda r} \).

For the mass distribution in elliptical galaxies, we can assume the singular isothermal sphere (SIS) model. Then, the density profile is \( \rho_{SIS}(r) = \frac{\sigma^2_{SIS}}{2\pi G r^2} \), and the corresponding mass, included within a radius \( r \), grows linearly with \( r \) as:

\[
M_{SIS}(r) = \frac{2\sigma^2_{SIS}}{G} \cdot r. \quad (17)
\]

Using Eq. (17), for the circular velocity, we get:

\[
v^2_c(r) = 2\sigma^2_{SIS} \left(1 + \alpha (1 + \lambda r) e^{-\lambda r}\right), \quad (18)
\]

and then, taking into account Eq. (16), we have:

\[
v^2_N(r) = 2\sigma^2_{SIS}, \\
v^2_{corr}(r) = 2\alpha \sigma^2_{SIS} (1 + \lambda r) e^{-\lambda r}. \quad (19)
\]

For the considered sample of elliptical galaxies, the Newtonian circular velocity at the effective radius is [see the explanation of Table 1 in Ref. 51] \( v_N(r_e) = \sigma_0 \), where \( \sigma_0 \) is the observed velocity dispersion. Therefore,

\[
\sqrt{2}\sigma_{SIS} = \sigma_0. \quad (20)
\]
From Eqs. (18) and (20), we have:

\[ v^2_e(r) = \sigma_0^2 \left( 1 + \alpha (1 + \lambda r) e^{-\lambda r} \right), \]  

and, furthermore,

\[ v^2_N = \sigma_0^2 \]

\[ v^2_{corr} = \alpha \sigma_0^2 (1 + \lambda r) e^{-\lambda r}. \]  

(22)

The circular velocity at the effective radius, i.e. for \( r = r_e \), is:

\[ v^2_e(r_e) = \sigma_0^2 \left( 1 + \alpha (1 + \lambda r_e) e^{-\lambda r_e} \right). \]  

(23)

For the sake of simplicity, we introduce the new variable:

\[ w = \lambda r_e, \]  

(24)

and, from Eq. (23), it is:

\[ v^2_e(r_e) = \sigma_0^2 \left( 1 + \alpha (1 + w) e^{-w} \right). \]  

(25)

As in the Newtonian case where \( v_N(r_e) = \sigma_0 \) (see Eq. (22)), we can assume that the expression \( v_e(r_e) = \sigma^{\text{theor}} \) is valid also in the case of modified gravity with Yukawa-like corrections, where \( \sigma^{\text{theor}} \) is a theoretical velocity dispersion. Therefore:

\[ \sigma^{\text{theor}} = \sigma_0 \sqrt{1 + \alpha (1 + w) e^{-w}} \]  

(26)

For different combinations of \((\alpha, w)\), we calculate \( \sigma^{\text{theor}} \) and then use it for the FP fit. As it can be seen from Eq. (26), the Newtonian case can be recovered for \( \alpha = 0 \) or \( w = -1 \), and then \( \sigma^{\text{theor}} = \sigma_0 \). However, \( w \) has to be a positive number, and then \( \alpha = 0 \).

3.3. The 3D fit of fundamental plane to the observations

Considering Eq. (1), let us perform a 3D fit of the function \( \log(r_e) \), depending on the two independent variables \( \log(\sigma^{\text{theor}}) \) and \( \log(I_e) \) vs the observational data. We adopt a least-squares algorithm. See [34, 35] for more details. In this way, we obtain the best fit coefficients of the FP described by Eq. (1), that is \( a, b \) and \( c \).

4. Constraining the Yukawa parameters

Let us now vary the parameters \( \lambda \) and \( \alpha \) and discuss the matching between theoretical results and observations for velocity dispersion \( \sigma \) as a function of the effective radius \( r_e \) in the case of elliptical galaxies listed in Table 1 of Ref. [50]. First, we fix the value \( \lambda \cdot r_e = 1 \) and vary parameter \( \alpha \). Then, we try different combination of values \( \lambda \cdot r_e \) and \( \alpha \). Finally, we will discuss the results.

In Fig [1] we show velocity dispersion \( \sigma \) as a function of effective radius \( r_e \). Theoretical values of velocity dispersion \( \sigma^{\text{theor}} \) are presented for \( \lambda \cdot r_e = 1 \) and for the three values of Yukawa parameter \( \alpha \): 0.2, 0.5 and 0.8. From our results, we can see that the agreement between observed and theoretical values is relatively good only in case of small values of parameter \( \alpha \) (\( \alpha = 0.2 \)). For larger values of parameter \( \alpha \) (\( \alpha = 0.5 \) or 0.8) agreement is very poor.

In Fig [2] it is shown the velocity dispersion \( \sigma \) as a function of the effective radius \( r_e \), but for negative values of \( \alpha : -0.2, -0.5, -0.8 \) and for \( \lambda \cdot r_e = 1 \). Like in the previous case, we can conclude that the agreement between observed and theoretical values is relatively good only in case of small values of parameter \( \alpha \), that is \( \alpha = -0.2 \). For larger values of parameter \( \alpha \) (\( \alpha = -0.5 \) or -0.8) agreement is very poor.

In Fig [3] velocity dispersion \( \sigma \) is shown as a function of effective radius \( r_e \), but for four different values of the \( \lambda \cdot r_e \) products: 0.01, 0.1, 1 and 10 and for three different values of parameter \( \alpha \): 0.01, 0.1 and 1. We can see that, for
The observed values (blue full circles) and the Newtonian velocity dispersion at the effective radius $\sigma_0$ are taken from [50]. Theoretical values of velocity dispersion $\sigma_{\text{theor}}$ are calculated for the following product of Yukawa parameter $\lambda$ and effective radius $r_e$: $\lambda \cdot r_e = 1$ and for the three values of Yukawa parameter $\alpha$: 0.2, 0.5 and 0.8.

In Fig. 1 we show velocity dispersion $\sigma$ as a function of effective radius $r_e$, for a sample of elliptical galaxies listed in Table 1 of [50]. The observed values (blue full circles) and the Newtonian velocity dispersion at the effective radius $\sigma_0$ are taken from [50]. Theoretical values of velocity dispersion $\sigma_{\text{theor}}$ are calculated for the following product of Yukawa parameter $\lambda$ and effective radius $r_e$: $\lambda \cdot r_e = 1$ and for the three values of Yukawa parameter $\alpha$: 0.2, 0.5 and 0.8.

In Fig. 2 we show the same as in Fig. 1 but for negative values of $\alpha$: $-0.2$, $-0.5$, $-0.8$. If $\lambda \cdot r_e = 10$, the agreement between observed and theoretical values is excellent for all the three studied values of $\alpha$. If we look at Eq. (26), $w = \lambda \cdot r_e$, $e^{-w}$ is a very small number and, even if $\alpha$ is close to 1, $\sigma_{\text{theor}}$ is close to $\sigma_0$. This is the reason why when $\lambda \cdot r_e = 10$, the agreement between observed and theoretical values is excellent for all the three studied values of $\alpha$. In the case when $\alpha = 1$, agreement for $\lambda \cdot r_e = 0.01$, 0.1, 1 is poor, but it is better in the case of smaller values of $\lambda \cdot r_e$ ($\lambda \cdot r_e = 0.01$, 0.1). In cases of small values of parameter $\alpha = 0.01$ and 0.1, agreement is very good for all the three different values of $\lambda \cdot r_e$. ($\lambda \cdot r_e = 0.01$, 0.1, 1).

In Fig. 4, we show velocity dispersion $\sigma$ as a function of effective radius $r_e$, for the same values of the $\lambda \cdot r_e$ product: 0.01, 0.1, 1 and 10, like in Fig. 3, but for the three different negative values of parameter $\alpha$: $-0.01$, $-0.1$ and -1. We notice that $\alpha$ can take positive and negative values and the dependence of results is not symmetric with respect to the sign of $\alpha$. From Figs. 3 and 4, we can conclude that, in both cases, if value of $\alpha$ is small enough ($\lesssim 0.1$) agreement between observed and theoretical values is excellent.

Figs. 5 and 6 show the same like Figs. 3 and 4 but for the following four values of the $\lambda \cdot r_e$ product: 2, 4, 6 and 8 (now, scale is much more narrow than in Figs. 3 and 4). Figs. 5 and 6 help us to find when the $\lambda \cdot r_e$ product is enough large to give satisfactory agreement between observed and theoretical values. We can conclude that it happens when the value of the product $\lambda \cdot r_e$ is $\gtrsim 6$.

In Fig. 7, we present the FP of ellipticals with velocity dispersion $\sigma_{\text{theor}}$, observed effective radius $r_e$ and observed mean surface brightness $I_e$. For a given pair of Yukawa parameters ($\lambda, \alpha$) (we choose $\lambda \cdot r_e = 10$ and $\alpha = 0.01$), using
the same procedure described in \[34\], we calculate FP coefficients \((a, b)\), and obtain the values \(a = 1.62, b = -0.64\). We perform now the fitting procedure (see \[34\] for details) and obtain the black solid line which represents the best 3D fit of FP. Our calculated FP coefficients \((a, b)\) are in good agreement with FP coefficients obtained by Bender et al. using observational data \[36\]. In other words, the thickness and the tilt of FP, derived from observations, fix the values of \(\alpha\) and \(\lambda\) in Eq.(13). For the specific case of \(f(R)\) gravity, we can infer the values of \(f'(0)\) and \(f''(0)\), that is, we can reconstruct, up to the second order, the polynomial in Eq.(4). Analogue procedures can be developed for any model in Table 1.

5. Discussion and conclusions

Considering results in Figs[17] we conclude that the Yukawa-like correction can have an important influence on stellar dynamics of ellipticals, and hence on their FP. These results also show that some of the theories of gravity reported in Table 1 are applicable only at some specific domains for the scale distance \(L\). For example, the \(f(R, R_s \beta R^2 \delta)\) theory (4th case in Table 1) should be more suitable for shorter scales (i.e. for smaller values of \(L\)), since the coefficients in its potential \((1/4\) and \(-3/4\)) are not sufficiently small in magnitude, as it is expected for such scales. A similar conclusion holds for the modified Gauss-Bonnet gravity \[17\] (see the 5th case from Table 1). This result indicates that characteristic scale lengths of these two theories are possibly related to the effective radii \(r_e\) of the ellipticals, as it is for \(f(R) = R^n\) gravity (see \[14\]). Due to opposite signs of the two exponential terms in the potential, this is valid only if their scale lengths are approximately the same. However, if this is not the case, by a suitable choice of scale lengths, it is possible to obtain a good agreement with observations at different astronomical and cosmological scales. In this sense, FP can be a useful tool to select reliable theories.

Specifically, we can compare the Yukawa correction of the present investigation with corrections in our previous paper \[34\], where we recovered the FP using a power-law \(f(R) = R^n\) gravity. There, the best agreement with the astronomical observations is obtained for values \(r_c/r_e = 0.001 - 0.01\). That is why we can say that our further gravitational corrections
Figure 4: The same as in Fig. [3] but for negative values of $\alpha$: $-0.01$, $-0.1$, $-1$.

Figure 5: The same as in Fig. [3] but for the following four $\lambda \cdot r_e$ products: 2, 4, 6 and 8.
radius $r_e$ is more important than $r_c$ to model stellar dynamics. In that case, the gravity parameter $\beta$ (values for $\beta$ that we analyzed: 0.02; 0.04; 0.06 and 0.08) gave relatively good results. There $\beta$ is a parameter related to the correction of Newtonian potential; it is function of the index $n$ of a given $f(R)$ gravity model. We obtained a strong dependence of results on values $r_c/r_e$ and a weak dependence on the values of $\beta$. In the case of the Yukawa correction, the best agreement is obtained for $w = \lambda \cdot r_e \gtrsim 6$ (for all the three studied values of $\alpha$). This is the same tendency like in our previous study when we obtained the best agreement for values $r_c/r_e = 0.001 - 0.01$. However, in this case, we can get also very good agreement even for much smaller values of $w = \lambda \cdot r_e$, but only if $\alpha$ is small enough in magnitude (i.e. $\lesssim 0.1$). We can conclude that $\alpha$ can take positive and negative values and the dependence of results is not symmetric with respect of sign of $\alpha$, but in both cases, if value of $\alpha$ is small enough in magnitude, agreement is very good. The
interpretation of this result is that we have to expect that Yukawa-like correction is relatively small with respect to the Newtonian leading term but, at long distances with respect to \( r_c \), corrections can become relevant. In other words, for scale distances \( L = \lambda^{-1} \approx 10 r_c \), Yukawa corrections become important and fit the distribution of ellipticals on the FP.

As a final consideration, it is worth saying that \( \alpha \) and \( \lambda \) are related to the given class of theories. As discussed above, they are related to \( f'_{\alpha} \) and \( f''_{\alpha} \) for \( f(R) \) gravity. This means that the shape of \( f(R) \) function is constrained by the FP parameters. The same can be done for the other theories in Table 1. This will be the argument of a forthcoming paper.

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