A log PSS morphism with applications to Lagrangian embeddings

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Abstract

Let $M$ be a smooth projective variety and $D$ an ample normal crossings divisor. From topological data associated to the pair $(M, D)$, we construct, under assumptions on Gromov–Witten invariants, a series of distinguished classes in symplectic cohomology of the complement $X = M \setminus D$. Under further ‘topological’ assumptions on the pair, these classes can be organized into a log(arithmetic) PSS morphism, from a vector space which we term the logarithmic cohomology of $(M, D)$ to symplectic cohomology. Turning to applications, we show that these methods and some knowledge of Gromov–Witten invariants can be used to produce dilations and quasi-dilations (in the sense of Seidel–Solomon [Geom. Funct. Anal. 22 (2012) 443–477]) in examples such as conic bundles. In turn, the existence of such elements imposes strong restrictions on exact Lagrangian embeddings, especially in dimension 3. For instance, we prove that any exact Lagrangian in any complex three-dimensional conic bundle must be diffeomorphic to a product $S^1 \times \Sigma_g$ or a connect sum $\#^n S^1 \times S^2$.

1. Introduction

Let $M$ be a smooth projective variety, $D$ an ample strict (or simple) normal crossings divisor and $X = M \setminus D$ its complement. The affine variety $X$ has the structure of an exact finite-type convex symplectic manifold and hence one can associate to $X$ its symplectic cohomology, $SH^*(X)$ [15, 24, 85], a version of classical Hamiltonian Floer cohomology defined for such manifolds. Symplectic cohomology has a rich topological quantum field theory (TQFT) structure and plays a central role in understanding the Fukaya categories of such $X$ [3, 71] as well as various approaches to mirror symmetry [33].

While there are very few complete computations of symplectic cohomology in the literature, Seidel [71, 72] has suggested that symplectic cohomology for $X$ as above should be computable in terms of topological data associated to the compactification $(M, D)$ and (relative) Gromov–Witten invariants of $(M, D)$. In the case when $D$ is smooth, this proposal has been explored in some depth by Diogo and Diogo–Lisi [20, 21].

The first part of this work introduces an approach to studying the symplectic cohomology $SH^*(X)$ in terms of the relative geometry of the compactification $(M, D)$ (when $D$ is normal crossings). In §3.2 we introduce an abelian group which captures the combinatorics and topology of the normal crossings compactification, called the log(arithmetic) cohomology of $(M, D)$:

$$H^*_{log}(M, D).$$ (1.1)
In the special case that $D = D$ is a smooth divisor, the log cohomology has a simple form:

$$H^*_\log(M, D) = H^*(M \setminus D) \oplus tH^*(S_D)[t],$$

(1.2)

where $S_D$ denotes the unit normal bundle to $D$. In general, $H^*_\log(M, D)$ is generated by classes of the form $\alpha t^\vec{v}$, where $\alpha$ lies in the cohomology of certain torus bundles over open strata of $D$ and $\vec{v}$ is a multiplicity vector.

Our approach is inspired by a map introduced by Piunikhin, Salamon and Schwarz [63] relating the (quantum) cohomology of $M$ and the Hamiltonian Floer cohomology of a non-degenerate Hamiltonian. Namely, we define a linear subspace of admissible classes $H^*_\log(M, D)^{ad} \subset H^*_\log(M, D)$ together with a map

$$\text{PSS}^+_\log : H^*_\log(M, D)^{ad} \to SH^+(X),$$

(1.3)

where $SH^+_*$ is the positive or high-energy part of symplectic cohomology, a canonically defined quotient of the complex defining $SH^*$ by ‘low energy (cohomological) generators’.

There is a wide class of topological pairs $(M, D)$ for which the map (1.3) is particularly well behaved (for example, take $M$ any projective variety and $D$ the union of at least $\dim C M + 1$ generic hyperplane sections). For a topological pair, $H^*_\log(M, D)^{ad} = H^*_\log(M, D)$, and there is a canonical lifting of (1.3) to a map we term the log PSS morphism

$$\text{PSS}^\log_\log : H^*_\log(M, D) \to SH^*(X).$$

(1.4)

The key feature of topological pairs is that relevant moduli spaces of relative pseudo-holomorphic curves are all generically empty; for example, there should be no holomorphic spheres in $M$ which intersect a single component of $D$ in one point; see §3.3 for the precise definition of a topological pair.

**Remark 1.1.** In a sequel to this paper [30], we define a natural ring structure on $H^*_\log(M, D)$ and show that the log PSS morphism (1.4) is a ring isomorphism for all topological pairs $(M, D)$ (rather those pairs satisfying a strengthening of the topological condition called being multiplicatively topological), leading to many explicit new computations of symplectic cohomology rings.

When the pair $(M, D)$ is not topological and $\sigma \in H^*_\log(M, D)^{ad}$ is an admissible class, we formulate the obstruction to lifting $\text{PSS}^+_\log(\sigma)$ from $SH^+(X)$ to $SH^*(X)$ in terms of a Gromov–Witten invariant associated to $\sigma$. When this obstruction vanishes, this provides a way to produce distinguished classes in $SH^*(X)$ which we will show may be applied to study the symplectic topology of $X$.

It should be noted that the construction of the log PSS morphism (1.4) and the proof of the chain equation are considerably more involved than for its classical analog. We conjecture that for general $(M, D)$, a further analysis of the moduli spaces appearing in (1.4) could produce an isomorphism between $SH^*(X)$ and the cohomology of a cochain model of log cohomology $C^\text{log}_\log(M, D)$ equipped with an extra differential accounting for non-vanishing relative Gromov–Witten moduli spaces. Such analysis would require stronger analytic results to deal with the subtle compactness and transversality questions which arise (related issues already arise in the study of relative Gromov–Witten invariants relative a normal crossings divisor). Our obstruction analysis is indeed inspired by this conjectural theory, and can be viewed as an elementary special case.

The rest of this paper explores applications of our construction to Lagrangian embeddings. An interesting class of examples can be constructed as follows. Let $X^o$ be an affine variety with trivial canonical bundle and consider a regular function $f : X^o \to \mathbb{C}$ whose zero set $Z^o$ is
smooth. We define the affine conic bundle to be the affine variety $X$ defined by the equation:

$$X = \{(u, v, \bar{x}) \in \mathbb{C}^2 \times X' \mid uv = f(\bar{x})\}. \tag{1.5}$$

The symplectic topology of these varieties is very rich and can be approached from different perspectives; see for instance [7, 49, 75] (these references generally take the base to be either $(\mathbb{C}^*)^{n-1}$ or $\mathbb{C}^{n-1}$). For example, there is a standard construction of Lagrangian spheres in $X$ given by taking a suitable Lagrangian disc $j : D^{n-1} \to (\mathbb{C}^*)^{n-1}$ with boundary on the discriminant locus and ‘suspending’ it to a Lagrangian $S^o \hookrightarrow X$ [34, 40, 73].

It is natural to ask: what are the possible topologies of exact Lagrangians in these conic bundles? The suspension construction typically provides a large collection of Lagrangian spheres and Seidel [75, Chapter 11] has provided constructions of exact Lagrangian tori in certain examples. Our first application is the following classification result concerning the spheres and Seidel bundles.

**Theorem 1.2** (See Theorem 6.22). Let $X$ be a three-dimensional affine conic bundle of the form (1.5) over an affine surface $X'$ with trivial canonical bundle. Then, any closed, oriented, exact Lagrangian submanifold $Q \hookrightarrow X$ must be diffeomorphic to one of the following 3-manifolds:

- a product $S^1 \times B$ for $B$ a Riemann surface of genus at least 1; or
- a connected sum $\#_n S^1 \times S^2$ for some $n \geq 0$

(by convention, the case $n = 0$ corresponds to $S^3$).

Theorem 6.22 represents a relatively sharp classification: following Proposition 6.25, we provide examples of exact Lagrangian embeddings $\#_n S^1 \times S^2 \to X$ for $n \geq 2$ (these arise by essentially replacing the disc in the suspension construction by a suitable exact embedding of $C_n$, a sphere minus $n + 1$-discs). It also suggests a close connection between exact Lagrangian surfaces in $X'$ (with boundary on $Z'$) and exact Lagrangian 3-folds in the affine conic bundle $X$. To illustrate this, it is well known that there are no exact Lagrangian embeddings of closed surfaces of genus $g \geq 2$ in $(\mathbb{C}^*)^2$. Motivated by this, we prove that when $X' = (\mathbb{C}^*)^2$, one can strengthen Theorem 1.2 by excluding exact Lagrangian products of surfaces of genus $g \geq 2$ with $S^1$.

**Proposition 1.3** (Proposition 6.25). Let $X$ be a three-dimensional conic bundle over $(\mathbb{C}^*)^2$ of the form (1.5) and such that the discriminant locus $Z'$ is connected. Let $j : Q \hookrightarrow X$ be a closed, oriented, exact Lagrangian submanifold of $X$. Then $Q$ is diffeomorphic to either $T^3$ or $\#_n S^1 \times S^2$.

By combining our methods with those from [76], we also prove the following result concerning disjoinability of Lagrangian spheres:

**Theorem 1.4** (see Theorem 6.20). Let $k$ be a field and $n \geq 3$ be an odd integer. Suppose that $X$ is a conic bundle of the form (1.5) of total dimension $n$ over an affine variety $X'$ with trivial canonical bundle and that $Q_1, \ldots, Q_r$ is a collection of embedded Lagrangian spheres which are pairwise disjoinable. Then the classes $[Q_1], \ldots, [Q_r]$ span a subspace of $H_n(X, k)$ which has rank at least $r/2$.

An important special case is when $r = 1$, in which case the theorem immediately implies that for any embedded Lagrangian sphere $Q \hookrightarrow X$, the class $[Q] \in H_n(X, \mathbb{Z})$ is non-zero and primitive (Corollary 6.21).
For our final application, we study a question posed by Smith concerning the ‘persistence (or rigidity) of unknottedness of Lagrangian intersections’.

**Question 1.5 (Smith).** In a symplectic manifold $X^6$, if a pair of Lagrangian 3-spheres $S_1$ and $S_2$ meet cleanly along a circle, can the isotopy class of this knot (in $S_1$ and $S_2$) change under (nearby) Hamiltonian isotopy?

Note that there is no smooth obstruction to changing the knot types. In a forthcoming paper, Evans–Smith–Wemyss study the case where $S^1 = S_1 \cap S_2$ is unknotted in both factors under an additional assumption on the identification of normal bundles

$$\eta : \nu_{S_1}S_1 \cong \nu_{S_2}S_2$$

induced by the symplectic form on $X$. They prove in this setting that if there is a nearby Hamiltonian isotopy $\hat{S}_1$ of $S_1$ and $\hat{S}_2$ of $S_2$ so that $\hat{S}_1$ continues to meet $\hat{S}_2$ cleanly along a knot, then this knot must be the unknot in one component and the unknot or trefoil in the other; see Proposition 6.28 for a precise statement. In §6.4, under the same assumptions on the intersection of $S_1$ and $S_2$, we rule out the remaining trefoil case, while also giving an alternative proof of Proposition 6.28, thereby answering Question 1.5 negatively in this case. Our main result, Proposition 6.29, is somewhat stronger and rules out any pair of Lagrangian 3-spheres meeting cleanly in a non-trivial knot in a standard ‘plumbing’ neighborhood of $S_1 \cup S_2$.

To explain how the results about conic bundles and knottedness above connect to our discussion of log PSS, note that the PSS$_{\text{log}}$ morphism gives a method for producing distinguished classes in $SH^\ast(X)$, and **algebraic relations between them**$^1$. It is well understood, going back to ideas of Viterbo, that the existence of solutions in $SH^\ast(X)$ satisfying certain relations often produce strong restrictions on Lagrangian embeddings [76, 78, 84]. Concretely, let $Q \hookrightarrow X$ denote an exact Lagrangian embedding of a closed Spin manifold. The well-known **Viterbo transfer map** [65, 84] gives a unital Batalin–Vilkovisky (BV) morphism from $SH^\ast(X)$ to the symplectic cohomology of a neighborhood of $Q$. $SH^\ast(T^*Q)$, which in turn can be identified [1, 2, 6, 69, 84] with the string topology BV algebra of $Q$, $H_{n-1}(LQ)$, a topological invariant [14]. Thus, one can transfer distinguished elements in $SH^\ast(X)$ to distinguished elements in $H_{n-1}(LQ)$, the existence of which in turn restricts the topology of $Q$, especially in dimension 3. The broad idea of producing classes/relations in loop homology in order to constrain the 3-manifold topology of a Lagrangian embedding first appears in Fukaya’s pioneering work [26, 27] on Lagrangian embeddings in $\mathbb{C}^3$; in contrast the types of distinguished classes we produce are different, as are both the means by which we obtain them and our method of deducing topological restrictions from them; see Remark 5.20.

As an example, a **dilation** [78] is an element $B \in SH^1(X)$ satisfying $\Delta B = 1$, where $\Delta$ denotes the BV operator. A straightforward application of Viterbo’s principle (along with a string topology computation) implies that if $SH^\ast(X)$ admits a dilation, $X$ does not contain an exact Lagrangian $K(\pi, 1)$ [78, Corollary 6.3]. It should be noted that the existence of dilations has other implications for Lagrangian embeddings as well, for example, [76] has used dilations to study disjointability questions for Lagrangian spheres. Weakening the above condition, a **quasi-dilation** ([75, p. 194], where the definition is attributed to Seidel–Solomon) is a pair $(\Psi_s, \alpha) \in SH^1(X) \times SH^0(X)$ where $\alpha$ is invertible and $\Delta(\alpha \Psi_s) = \alpha$. The existence of quasi-dilations imposes slightly weaker, but nevertheless quite strong constraints on the topology of $Q$. We make these constraints very explicit when $\dim Q = 3$, using techniques from 3-manifold topology:

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$^1$Much like the classical PSS morphism, PSS$_{\text{log}}$ in many cases intertwines topological/GW-type relations in $H^\ast_{\text{log}}(M, D)$ with TQFT-structure relations in $SH^\ast(X)$. This is explored further in the sequel [30].
Proposition 1.6 (Corollary 5.18). Let $Q^3 \hookrightarrow X$ be an exact Lagrangian embedding of a closed oriented 3-manifold. If $SH^*(X,\mathbb{Z})$ admits a quasi-dilation, then $Q$ is either diffeomorphic to $S^1 \times B$ with $B$ a Riemann surface or diffeomorphic to a connected sum $\#_n S^1 \times S^2$.

There are variants of Proposition 1.6 for quasi-dilations in $SH^*(X,\mathbb{Q})$; see Corollary 5.18. As usual, the chief difficulty in applying Proposition 1.6 and its variants to restrict exact Lagrangian embeddings is performing partial computations of the BV algebra structure in symplectic cohomology in order to produce quasi-dilations. The log PSS morphism developed in this paper gives a method of performing such computations, which we use to prove the following result, suggested by Seidel using mirror symmetric considerations [75, first paragraph of Lecture 19]:

**Theorem 1.7** (Theorem 6.10). Let $X$ be the conic bundle appearing in (1.5). Then $SH^*(X,\mathbb{Z})$ admits a quasi-dilation.

When combined with Proposition 1.6, this immediately implies Theorem 1.2. Proposition 1.3 is then proven by combining Theorem 1.7 with some additional algebraic relations which exist in the zeroth symplectic cohomology group $SH^0(X)$ when $X^0 = (\mathbb{C}^*)^2$, which enable us to further rule out Lagrangians in $X$ of the form $S^1 \times B$ for $g(B) > 1$.

Seidel [75, Lecture 19] has shown that quasi-dilations have similar implications for disjointability questions (in the sense of [76]) as dilations. Theorem 1.4 therefore follows by combining Seidel’s results from [76] with Theorem 1.7. An essential ingredient in Theorem 1.4 is the specific geometry of the construction of the quasi-dilation, which is what enables us to interpret the constraints imposed by disjointability in terms of the topology of $X$.

We expect that our techniques apply in many other geometric situations as well. For example, we also study the case where $M$ is a Fano variety of dimension $n \geq 3$ and $D$ is a smooth very ample divisor satisfying $H^2(M,k) = k(PD(D))$ and $K_M^{-1} = O(mD)$ with $m \geq \frac{n}{2}$. Under hypotheses on the cohomology and Gromov–Witten invariants of such $(M,D)$, we show using PSS\text{log} that $SH^*(M\setminus D)$ admits a dilation; see Theorem 6.2. The theorem is closely related to a question of Seidel [75, Conjecture 18.6]; see Remark 6.5.

**Related work**

We note that there are several independent recent and/or ongoing works which have non-trivial conceptual overlap with the present paper. We have already mentioned the work Diogo–Lisi [20, 21], who use techniques from symplectic field theory to give a formula for the abelian group $SH^*(X)$ in terms of topological data and relative Gromov–Witten invariants for a large class of pairs $(M,D)$ with $D = D$ smooth. Closer in spirit to the present work, a forthcoming paper of Borman and Sheridan constructs a particular distinguished class in $SH^*(M\setminus D)$ for suitable pairs $(M,D)$ (see Remark 4.26 for more details about how their class fits into our framework) and uses this to develop a relationship between $SH^*(X)$ and $QH^*(M)$, the quantum cohomology of the compactification $M$. A modified version of their construction in the setting of Lefschetz pencils also appears in a recent preprint of Seidel [77].

**Conventions**

Throughout this paper the ring $k$ will be either $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$ unless explicitly stated otherwise (see Remark 4.9). All grading conventions in this paper are cohomological, and our conventions for grading symplectic cohomology follows [3].
2. Symplectic cohomology

2.1. The definition

Let \((\hat{X}, \theta)\) be a Liouville domain; that is, a 2n-dimensional manifold \(\hat{X}\) with boundary \(\partial \hat{X}\) with 1-form \(\theta\) such that \(\omega := d\theta\) is symplectic and the Liouville vector field \(Z\) (defined as the \(\omega\) dual of \(\theta\)) is outward pointing along \(\partial \hat{X}\). More precisely, flowing along \(-Z\) is defined for all times, giving rise to a canonical embedding of the negative half of the symplectization

\[
j : (0, 1]_R \times \partial \hat{X} \to \hat{X}
\]

\[
j^*(\omega) = d(R\theta|_{\partial \hat{X}}).
\]

The (Liouville) completion of \(\hat{X}\) is formed by attaching (using the aforementioned identification) a cylindrical end to the boundary of \(\hat{X}\):

\[
\hat{X} := \hat{X} \cup \partial \hat{X} [1, \infty)_R \times \partial \hat{X}.
\]

The Liouville completion \(\hat{X}\) is also exact symplectic, equipped with 1-form \(\hat{\theta}\) with \(\hat{\omega} = d\hat{\theta}\) symplectic. Explicitly, \(\hat{\theta}\) is equal to \(\theta\) on \(\hat{X}\) and \(R\theta\) on the cylindrical end, where \(R\) denotes the canonical \([1, \infty)\) coordinate. Recall that in the above situation, \((\partial \hat{X}, \alpha := \theta|_{\partial \hat{X}})\) is a contact manifold; it possesses a canonical Reeb vector field \(X_{\text{Reeb}}\) determined by \(\alpha(X_{\text{Reeb}}) = 1\), \(d\alpha(X_{\text{Reeb}}, -) = 0\). The spectrum of \(\partial \hat{X}\), denoted \(\text{Spec}(\partial \hat{X}, \theta)\) is the set of real numbers which are lengths of Reeb orbits in \(\partial \hat{X}\). We will assume throughout that this spectrum is discrete (which follows from choosing \(\alpha\) sufficiently generic).

Observe that after choosing a compatible almost-complex structure then (negative) the first Chern class \(-c_1(\hat{X})\) is represented by the complex line bundle \(\mathcal{K} = \text{det}_{\mathbb{C}}(T\hat{X})^{\vee}\). For simplicity (and in order to set up \(\mathbb{Z}\)-gradings), we will assume that \(c_1(\hat{X}) = 0\) and moreover that we have fixed a choice of trivialization of \(\mathcal{K}\).

**Definition 2.1.** A (time-dependent) Hamiltonian \(H : S^1 \times \hat{X} \to \mathbb{R}\) is admissible with slope \(\lambda\) if \(H(t, x) = \lambda R + C_0\) near infinity, where \(R\) is the cylindrical coordinate near \(\infty\) (note \(R\) only depends on \(x \in \hat{X}\)) and \(C_0\) is an arbitrary constant.

For each positive real number \(\lambda\) which is not in the spectrum of \(\partial \hat{X}\), we choose an admissible Hamiltonian of slope \(\lambda\), denoted \(H^\lambda\). Each Hamiltonian \(H\) determines an action functional on the free loop space of \(\hat{X}\), whose value on a free loop \(x : S^1 \to \hat{X}\) is given by

\[
\mathcal{A}_H(x) := -\int_{S^1} x^*\hat{\theta} + \int_0^1 H(t, x(t))dt.
\]

By definition (and partly convention), the Hamiltonian vector field of \(H\) with respect to \(\hat{\omega}\) is the unique \((S^1\)-dependent\) vector field \(X_H\) on \(\hat{X}\) such that \(\hat{\omega}(-, X_H) = dH\). Critical points of \(\mathcal{A}_H\) are time-1 orbits of the Hamiltonian vector field \(X_H\) with respect to the symplectic form \(\hat{\omega} = d\hat{\theta}\). It is well known that we may perturb the functions \(H^\lambda\) so that all 1-periodic orbits are non-degenerate, while keeping the functions admissible (for a nice reference directly applicable to our setting see [55, Lemma 2.2]).

Let \(\mathcal{X}(H^\lambda)\) denote the set of time-1 orbits of \(X_H^\lambda\). To each orbit \(x \in \mathcal{X}(H^\lambda)\), there is an associated one-dimensional real vector space \(\mathfrak{a}_x\) called the orientation line, defined via index theory as the determinant line of a local operator \(D_x\) associated to \(x\) (this goes back to [25] but our treatment follows, for example, [3, §C.6]; note particularly that the choice of local
operator is fixed by the choice of trivialization of $\mathcal{K}$ chosen above). Over a ring $k$, the Floer cochain complex is, as a vector space

$$CF^*(\hat{X}, H^\lambda) := \bigoplus_{x \in \mathcal{X}(H^\lambda)} \omega_x|_k,$$

where for any real one-dimensional vector space $V$, its $k$-normalization

$$|V|_k$$

is the rank one free $k$-module generated by possible orientation of $V$, modulo the relation that the sum of opposite orientations vanishes. To define a differential on (2.4), pick a compatible (potentially $t \in S^1$-dependent) almost-complex structure $J_t$ which is of contact type, meaning on the conical end, it satisfies

$$\hat{\theta} \circ J_t = dR. \tag{2.6}$$

Given a pair of orbits $x_1$ and $x_0$, a Floer trajectory between $x_1$ and $x_0$ is formally a gradient flowline for $A_H$ between $x_1$ and $x_0$ using the metric on $\hat{\mathcal{L}}\hat{X}$ induced by $J_t$, or equivalently a map $u : \mathbb{R} \times S^1 \to \hat{X}$, asymptotic to $x_{\pm}$ at $\pm \infty$ satisfying a PDE called Floer’s equation:

$$\begin{align*}
\begin{cases}
u : \mathbb{R} \times S^1 \to \hat{X}, \\
\lim_{s \to -\infty} u(s, -) = x_0 \\
\lim_{s \to +\infty} u(s, -) = x_1 \\
\partial_s u + J_t(\partial_t u - X_{H^\lambda}) = 0.
\end{cases}
\end{align*} \tag{2.7}$$

Denote by

$$\tilde{M}(x_0, x_1) \tag{2.8}$$

the moduli space of Floer trajectories between $x_1$ and $x_0$, or solutions to (2.7). For generic $J_t$, (2.8) is a manifold of dimension

$$|x_0| - |x_1|,$$

where $|x|$ (or equivalently $\text{deg}(x)$), the index of the local operator $D_x$ associated to $x$, is equal to $n - CZ(x)$, where $CZ(x)$ is the Conley–Zehnder index of $x$ (with respect to the symplectic trivialization of $x^*T\hat{X}$ induced by the trivialization of $\mathcal{K}$) [15, 25]. There is an induced $\mathbb{R}$-action on the moduli space $\tilde{M}(x_0, x_1)$ given by translation in the $s$-direction, which is free for non-constant solutions. Whenever $|x_0| - |x_1| \geq 1$, for a generic $J_t$ the quotient space

$$M(x_0, x_1) := \tilde{M}(x_0, x_1)/\mathbb{R} \tag{2.9}$$

is a manifold of dimension $|x_0| - |x_1| - 1$.

The most delicate part of setting up such a theory for exact (non-compact) symplectic manifolds, where much of the analysis is otherwise simplified, is an a priori compactness result ensuring that solutions to the PDE cannot escape to $\infty$ in the target (this is vital input into the usual Gromov compactness theorems for moduli spaces, which apply to maps into a compact target). Technically, one can ensure a priori compactness by carefully choosing the behavior of $J_t$ and $H$ near $\infty$ so that a sort of maximum principle holds for solutions to Floer’s equation (at least outside a compact set); see, for example, [72]. The maximum principle ensures that any solution $u$ to Floer’s equation remains in some compact subset of the target $Z \subset \hat{X}$, which depends only on the asymptotics of $u$ (and possibly $\hat{X}$, $H$, $J_t$).
Given that input, a version of Gromov compactness ensures that for generic choices, whenever \(|x_0| - |x_1| = 1\), the moduli space (2.9) is compact of dimension 0. Moreover, orientation theory associates, to every rigid element \(u \in \mathcal{M}(x_0, x_1)\) an isomorphism of orientation lines \(\mu_u : o_{x_1} \cong o_{x_0}\), and hence an induced map \(\mu_u : |o_{x_1}|_k \rightarrow |o_{x_0}|_k\). Using this, one defines the \(|o_{x_1}|_k - |o_{x_0}|_k\) component of the differential

\[
(\partial_{C,F})_{x_1,x_0} = \sum_{u \in \mathcal{M}(x_0, x_1)} \mu_u
\]

whenever \(|x_0| = |x_1| + 1\) (and 0 otherwise). When applied to two-dimensional moduli spaces (which are one-dimensional after quotienting by \(\mathbb{R}\)), Gromov compactness and gluing analysis imply that

**Lemma 2.2.** Given \(x_0, x_1 \in \mathcal{X}(H^\lambda)\) such that \(|x_0| = |x_1| + 2\) for generic \(J_t\), \(\mathcal{M}(x_0, x_1)\) admits a compactification \(\overline{\mathcal{M}}(x_0, x_1)\) with

\[
\partial \overline{\mathcal{M}}(x_0, x_1) := \bigcup_{y \in \mathcal{X}(H^\lambda), |y| = |x_1| + 1} \mathcal{M}(x_0, y) \times \mathcal{M}(y, x_1).
\]

By a standard argument this implies that \(\partial_{C,F}^2 = 0\) and hence that the cohomology of \(\partial_{C,F}\) is well defined. We will denote by

\[
HF^*(\hat{\mathcal{X}}, H^\lambda) := H^*(CF^*(\hat{\mathcal{X}}, H^\lambda), \partial_{C,F});
\]

standard techniques involving continuation maps shows that this group only depends on \(\lambda \in \mathbb{R}\). Furthermore, whenever \(\lambda_2 \geq \lambda_1\) there are continuation maps \([72]\)

\[
\epsilon_{\lambda_1, \lambda_2} : HF^*(\hat{\mathcal{X}}, H^{\lambda_1}) \rightarrow HF^*(\hat{\mathcal{X}}, H^{\lambda_2})
\]

defined as follows: Let \(H_{s,t}\) be a map from \(\mathbb{R} \times S^1 \rightarrow C^\infty(\hat{\mathcal{X}}; \mathbb{R})\) which agrees with \(H^{\lambda_1}\) near \(+\infty\) and \(H^{\lambda_2}\) near \(-\infty\) and which is monotonic, meaning (at least away from a compact set in \(X\)) \(\partial_s H \leq 0\). Let \(J_{s,t}\) be a compatible \(\mathbb{R} \times S^1\) dependent almost-complex structure agreeing with a choice of \(J_{\lambda_1}\) used to define \(CF^*(\hat{\mathcal{X}}, H^{\lambda_1})\) near \(+\infty\) and a choice of \(J_{\lambda_2}\) used to define \(CF^*(\hat{\mathcal{X}}, H^{\lambda_2})\) near \(-\infty\). Then, if \(x_1\) is an orbit of \(H^{\lambda_1}\) and \(x_2\) is an orbit of \(H^{\lambda_2}\), the \(o_{x_1} - o_{x_2}\) component of the map (2.12) is defined on the chain level by counting maps \(u : \mathbb{R} \times S^1 \rightarrow \hat{\mathcal{X}}\) satisfying Floer’s equation for \(H_{s,t}\) and \(J_{s,t}\):

\[
\partial_t u + J_{s,t}(\partial_s u - X_{H_{s,t}}) = 0
\]

which in addition satisfy requisite asymptotics:

\[
\left\{ \begin{array}{l}
\lim_{s \to -\infty} u(s, -) = x_2 \\
\lim_{s \to +\infty} u(s, -) = x_1.
\end{array} \right.
\]

(A standard transversality, compactness, and gluing argument ensures that the chain-level map is in fact a chain map, as long as a maximum principle holds for elements of the moduli space. This maximum principle is where one requires \(\lambda_2 \geq \lambda_1\).)

Define symplectic cohomology to be the colimit of this directed system

\[
SH^*(\hat{\mathcal{X}}) := \lim_{\lambda \rightarrow \lambda^+} HF^*(\hat{\mathcal{X}}, H^\lambda).
\]

The symplectic cohomology of a Liouville domain is by definition the symplectic cohomology of its completion

\[
SH^*(\hat{\mathcal{X}}) := SH^*(\hat{\mathcal{X}}).
\]
It is not hard to prove that these definitions are independent of the various choices made along the way \[72\]. By construction there are canonical morphisms, for each \(\lambda\)

\[c_{\lambda,\infty} : HF^*(\hat{X}, H^\lambda) \to SH^*(\hat{X}).\] (2.16)

In particular, given that there is a canonical isomorphism \(HF^*(\hat{X}, H^\lambda) \cong H^*(\hat{X})\) when \(0 < \lambda \ll 1\) is smaller than the period of any Reeb orbit on \(\partial \hat{X}\) (see, for example, \[65, \S 5\]), there is a canonical map \(H^*(\hat{X}) \to SH^*(\hat{X})\) \[85\]. The cone of (a chain-level version of) this map is often called high-energy (or positive) symplectic cohomology and denoted \(SH^*_+(\hat{X})\) (Recall from the introduction that the most canonical formulation of the log PSS map in the presence of holomorphic spheres is in terms of \(SH^*_+(\hat{X})\)).

Let us review a convenient alternate construction of high-energy symplectic cohomology \(SH^*_+(\hat{X})\), which uses a slightly more restrictive choice of Hamiltonians (but manages to avoid using chain-level colimit constructions). Consider Hamiltonians \(H^\lambda\) on \(\hat{X}\) for which

- \(H^\lambda = 0\) on \(\hat{X}\);
- over the collar region satisfy \(H^\lambda = h_\lambda(r)\), with \(h'_\lambda(r) \geq 0\) and \(h''_\lambda(r) \geq 0\);
- for some \(R_H\) near 1, we have that \(h_\lambda(r) = \lambda(r - 1)\) for \(r \geq R_H\).

After taking a suitable small perturbation, which for simplicity we may assume is time independent in the interior of \(\hat{X}\), action considerations show that the (orientation lines associated to) orbits in the interior generate a subcomplex \(CF^*_-(\hat{X}, H^\lambda) \subset CF^*(\hat{X}, H^\lambda)\). Set

\[CF^*_+(\hat{X}, H^\lambda) := \frac{CF^*(\hat{X}, H^\lambda)}{CF^*_-(\hat{X}, H^\lambda)}.\] (2.17)

This construction passes to direct limits, giving rise to

\[SH^*_+(\hat{X}) := \lim_{\lambda} HF^*_+(\hat{X}, H^\lambda).\] (2.18)

It is not difficult to see that there is a long exact sequence

\[\cdots \to H^*(\hat{X}) \to SH^*(\hat{X}) \to SH^*_+(\hat{X}) \to H^{*+1}(\hat{X}) \to \cdots\] (2.19)

Finally, we observe that for such Hamiltonians \(H^\lambda\), the integrated maximum principle of \[8, \text{Lemma 7.2}\] implies that all Floer trajectories and continuation maps (for \(\lambda \leq \lambda'\)) between orbits actually lie in the compact region \(\hat{X} \cup \{R \leq R_H\}\) (seeing as all orbits of \(H^\lambda\) and \(H^{\lambda'}\) themselves lie in this region).

### 2.2. Algebraic structures

Among its many other TQFT structures, symplectic cohomology is a BV-algebra; in particular it has a pair of pants product and a BV operator which we now define.

Recall that a negative cylindrical end, respectively, a positive cylindrical end near a puncture \(z\) of a Riemann surface \(\Sigma\) consists of a proper holomorphic embedding

\[\epsilon_- : (-\infty, 0] \times S^1 \to \Sigma,\] (2.20)

respectively, a proper holomorphic embedding

\[\epsilon_+ : [0, \infty) \times S^1 \to \Sigma\] (2.21)

asymptotic to \(z\). We use standard product coordinates \((s, t)\) on both these semi-infinite cylinders inherited from their embedding in \(\mathbb{R}_s \times S^1_t = \mathbb{R} \times (\mathbb{R}_t/\mathbb{Z})\), with associated standard
complex structure $j\partial_s = \partial_t$. Let $\Sigma$ be a Riemann surface equipped with suitable cylindrical ends $\epsilon_i$. Suppose to each cylindrical end we have associated a time-dependent Hamiltonian $H_i$. Let $K \in \Omega^1(\Sigma, C^\infty(\hat{\mathcal{X}}))$ be a 1-form on $\Sigma$ with values in smooth functions on $\hat{\mathcal{X}}$ which, along the cylindrical ends, satisfies:

$$
\epsilon_i^*(K) = H_i dt \text{ for } |s| \gg 0
$$

for some functions $H_i$. To such a $K$, which we call a perturbation 1-form, we may associate a Hamiltonian vector field valued 1-form $X_K \in \Omega^1(\Sigma, C^\infty(T\hat{\mathcal{X}}))$, characterized by the property that for any tangent vector $\vec{r}_z \in T_z \Sigma$, $X_K(\vec{r}_z)$ is the Hamiltonian vector field of the Hamiltonian function $K(\vec{r}_z)$ (on the ends, $\epsilon_i^*(X_K) = X_H_i \otimes dt$).

In order to define Floer-theoretic operations, we fix the following additional data on $\Sigma$.

- A (surface-dependent) family of admissible $J$, meaning $J$ should be of contact type. Further, when restricted to cylindrical ends $J$ should depend only on $t$.
- A subclosed 1-form $\beta$ (meaning $d\beta \leq 0$ pointwise, where positivity, respectively, negativity is detected by comparison with the standard complex orientation), which restricts to $d\iota \cdot dt$, for $d_i \in \mathbb{Z}^+$, when restricted to the cylindrical ends.
- A perturbation 1-form $K$ (as defined above) restricting to $H_\lambda \cdot dt$ on the cylindrical ends (in other words, in terms of the discussion above, we are now associating the Hamiltonian $H_i := H_\lambda$ to the $i$th cylindrical end). We further require that outside of a compact set on $\hat{\mathcal{X}}$,

$$
K = H^\lambda \beta, \tag{2.22}
$$

where $H^\lambda$ is the choice of admissible Hamiltonian (in the sense of Definition 2.1) used to define $HF^*(\hat{\mathcal{X}}; H^\lambda)$.

The most general form of Floer’s equation that we will be studying in this paper is

$$
\begin{cases}
  u: \Sigma \to \hat{\mathcal{X}}, \\
  (du - X_K)^{0,1} = 0.
\end{cases} \tag{2.23}
$$

To such $u$ we can associate the geometric energy

$$
E_{geo}(u) := \frac{1}{2} \int_\Sigma ||du - X_K||^2 \tag{2.24}
$$

as well as the topological energy

$$
E_{top}(u) = \int_\Sigma u^*\omega - d(u^*K). \tag{2.25}
$$

For solutions $u$ of (2.23), we have a relationship

$$
E_{geo}(u) = E_{top}(u) + \int_\Sigma u^*\Omega_K, \tag{2.26}
$$

where the curvature $\Omega_K$ of a perturbation 1-form $K$ is the exterior derivative of $K$ in the $\Sigma$ direction; this is a 2-form on $\Sigma$ with values in $C^\infty(\hat{\mathcal{X}})$ or equivalently a section of $\pi^*(\Lambda^2 T^*\Sigma) \to \Sigma \times \hat{\mathcal{X}}$, where $\pi: \Sigma \times \hat{\mathcal{X}} \to \Sigma$ is the projection (note $u^*\Omega_K$ above denotes pull-back by the graph of $u$). If we momentarily assume that $K$ is of the form (2.22) and $H^\lambda \geq 0$ on all of $\hat{\mathcal{X}}$, then $\Omega_K = H^\lambda d\beta$. The non-negativity of $H^\lambda$ and subclosedness of $\beta$ would therefore imply that $\int_\Sigma \Omega_K \leq 0$, giving an inequality (under these assumptions)

$$
E_{geo}(u) \leq E_{top}(u). \tag{2.27}
$$
In particular, if \( E_{\text{top}}(u) \leq 0 \), it would follow that \( du = X_K \) everywhere. Now under our assumptions on \( K \) and \( H^\lambda \) (see third bullet point above), (2.22) and \( H^\lambda > 0 \) are only required to hold outside of some compact set \( R \) of \( \hat{X} \). This nevertheless implies one can a priori control geometric energy in terms of topological energy (which in turn can be computed in terms of the actions of asymptotics via Stokes’ theorem), as we now recall (compare, for example, [74, §5c]). First for any solution \( u \) of (2.23), if \( \Sigma \) is the subset of the domain of \( u \) mapping outside \( R \), then \( \int_{\Sigma} u^* \Omega_K \leq 0 \), hence \( u|_{\Sigma_1} \) satisfies (2.27). For \( u \) itself, we note first that \( \Omega_K = 0 \) near the cylindrical ends (where \( K \) is closed in the \( \Sigma \) direction), hence its support lies in \( \Sigma \times \hat{X} \) where \( \Sigma \subset \Sigma \) is some compact subset independent of \( u \). Also, the restriction of \( \Omega_K \) to 2-forms on \( \Sigma \) with values in \( C^\infty(R) \) is (by compactness of \( R \) and \( \Sigma \)) necessarily bounded above (by say \( \text{Advol}_{\Sigma} \) for some \( A > 0 \) and a nowhere vanishing volume form \( d\text{vol}_{\Sigma} \)). It follows that, using the notation \( \hat{\Sigma}^c := \Sigma \setminus \Sigma_1 \) for the region mapping to \( R \), the curvature satisfies

\[
\int_{\Sigma} u^* \Omega_K = \int_{\Sigma} u^* \Omega_K + \int_{\Sigma^c} u^* \Omega_K \leq \int_{\Sigma^c} u^* \Omega_K = \int_{\Sigma^c \cap \hat{\Sigma}^c} u^* \Omega_K \leq \int_{\Sigma^c \cap \hat{\Sigma}^c} \text{Advol}_{\Sigma} \quad \text{as } \Omega_K = 0 \text{ outside } \Sigma
\]

is bounded above independently of \( u \) by \( C := A \cdot \text{vol}(\Sigma) \) (note that while \( \hat{\Sigma}^c \) depends on \( u \), \( \Sigma \) is independent of \( u \) and hence so is this bound). Thus we obtain a bound on the geometric energy of a solution to Floer’s equation in terms of the topological energy and a constant \( C \) depending only on the perturbation 1-form \( K \):

\[
E_{\text{geo}}(u) \leq E_{\text{top}}(u) + C. \tag{2.28}
\]

We begin by defining the pair of pants product as an operation:

\[
(- \cdot -) : HF^* (H^\lambda) \otimes HF^* (H^\lambda) \to HF^* (H^{2\lambda}); \tag{2.29}
\]

to do so we specialize to the case where \( \Sigma \) is the pair of pants, viewed as a sphere minus three points. Labeling the punctures of \( \Sigma \) by \( z_1 \), \( z_2 \) and \( z_0 \), we equip \( \Sigma \) with positive cylindrical ends \( c^1_+, c^2_+ \) around \( z_1 \) and \( z_2 \) and a negative end \( c^0_- \) around \( z_0 \). We count solutions to Floer’s equation (2.23) such that

\[
\begin{align*}
&u : \Sigma \to \hat{X}, \\
&\lim_{s \to -\infty} u(c^0_+(s,-)) = x_0 \\
&\lim_{s \to -\infty} u(c^1_+(s,-)) = x_1 \\
&\lim_{s \to -\infty} u(c^2_+(s,-)) = x_2.
\end{align*} \tag{2.30}
\]

The operation (2.29) is compatible with the continuation maps \( c_{\lambda_1, \lambda_2} \), and hence induces a product \( SH^* (\hat{X}) \otimes SH^* (\hat{X}) \to SH^* (\hat{X}) \) which we call the pair of pants product.

We will need to consider certain parameterized operations on symplectic cohomology as well. In the simplest case, consider the cylinder \( \mathbb{R}_s \times S^1_t \). We will consider operations parameterized by a value \( r \in S^1 \). Set

\[
H^r_t (x) = H(t-r, x), \quad J^r_{t,x} = J_{t-r}.
\]
Choose a pair \((J_{s,t,r}, H_{s,t,r})\) such that along the cylindrical ends:

\[
\begin{align*}
(J_{s,t,r}, H_{s,t,r}) &= (J^{(r)}, H^{(r)}) & \text{for all } s \ll 0 \\
(J_{s,t,r}, H_{s,t,r}) &= (J, H) & \text{for all } s \gg 0.
\end{align*}
\] (2.31)

We will denote the rotated Hamiltonian orbits of \(H^{(r)}_{t}\) by \(x^{(r)}\). Note that these orbits are bijection with those of \(H_{t}\). Under this correspondence, Floer trajectories of \((H^{(r)}_{t}, J^{(r)}_{s,t,r})x_{0}, x_{1}\) with asymptotics \(x^{(r)}_{0}, x^{(r)}_{1}\) are naturally in bijection with trajectories of \((H_{t}, J_{s,t})x_{0}, x_{1}\); that is, there is a canonical isomorphism between the Floer complexes of \(H^{(r)}_{t}\) and \(H_{t}\). For later use, we denote this former moduli space, the solutions to Floer’s equation for \((H^{(r)}_{t}, J^{(r)}_{s,t,r})\) with asymptotics \(x^{(r)}_{0}, x^{(r)}_{1}\), by \(M_{r}(x_{0}, x_{1})\). For generic \((J_{s,t,r}, H_{s,t,r})\), the BV operator defines an operation

\[
\Delta : HF^{*}(\hat{X}, H^{\lambda}) \to HF^{*-1}(\hat{X}, H^{\lambda})
\]

by counting solutions to the parameterized Floer equation:

\[
\begin{align*}
& r \in S^{1}, \\
& u : \Sigma \to \hat{X}, \\
& \lim_{s \to -\infty} u(s, -) = x^{(r)}_{0} \\
& \lim_{s \to \infty} u(s, -) = x_{1} \\
& \partial_{s}u + J_{s,t,r}(\partial_{t}u - X_{H_{s,t,r}}) = 0.
\end{align*}
\] (2.32)

Furthermore this operation is compatible with the continuation maps \(\epsilon_{\lambda_1, \lambda_2}\), and hence induces an operator \(\Delta : SH^{*}(\hat{X}) \to SH^{*-1}(\hat{X})\) which we also call the BV operator. The BV operator and the product together give \(SH^{*}(\hat{X})\) the structure of a unital BV-algebra (meaning that there is a unit for the algebra structure, \(\Delta^{2} = 0\), and lastly there is a suitable compatibility relation between \(\Delta\) and the product which we omit; see, for example, \([31]\) for a precise formula, which is not directly relevant to this paper).

The final general property of symplectic cohomology that we need concerns its functoriality. Namely, let \(j : \hat{W} \subset \hat{X}\) be a sub-Liouville domain. We have a Viterbo functoriality map

\[
j^{!} : SH^{*}(\hat{X}) \to SH^{*}(\hat{W})
\] (2.33)

which respects the BV structures on both sides:

**Lemma 2.3.** The Viterbo functoriality map \(j^{!}\) is a morphism of unital BV algebras (and in particular preserves the BV operator and the product).

### 3. Complements of normal crossings divisors

#### 3.1. Nice symplectic and almost-complex structures

**Definition 3.1.** A log-smooth compactification of a smooth complex \(n\)-dimensional affine variety \(X\) is a pair \((M, D)\) with \(M\) a smooth, projective \(n\)-dimensional variety and \(D \subset M\) a divisor satisfying

\[
X = M \setminus D.
\] (3.1)

The divisor \(D\) is normal crossings in the strict sense, for example,
\[ \mathbf{D} := D_1 \cup \cdots \cup D_i \cup \cdots \cup D_k, \text{ where } D_i \text{ are smooth components of } \mathbf{D}; \text{ and } \] (3.2)

There is an ample line bundle \( \mathcal{L} \) on \( M \) together with a section \( s \in H^0(\mathcal{L}) \) whose divisor of zeroes is \( \sum_i \kappa_i D_i \) with \( \kappa_i > 0 \).

(3.3)

It follows from Hironaka’s embedded resolution of singularities— see [9, Theorem 1.6; 39, Main Theorem I], for a constructive proof and [36] for an expository reference — that any smooth affine variety admits a log-smooth compactification. While this is well known (see [72, Lemma 4.4]), we briefly recall the argument as we will need similar arguments in the proof of Lemma 6.12. The precise version of Hironaka’s result that we will use is as follows:

**Theorem 3.2 (Embedded resolution of singularities).** Let \( Y \to W \) be a closed embedding of a reduced variety (=finite-type scheme over \( \mathbb{C} \)) into a smooth variety \( W \). Then there exists a birational morphism \( \pi_r : W' \to W \) given by a sequence of blow-ups at smooth centers:

\[ W^r := W_r \to \cdots \to W_j \to \cdots \to W_1 \to W \]

such that letting \( E_j \) denote the exceptional divisors of \( \pi_j : W_j \to W \) (so \( E_r \) is the exceptional locus of \( W' \to W \)):

- (1) the proper transform \( Y' \) of \( Y \) in \( W' \) is smooth;
- (2) \( Y' \) and \( E_r \) simultaneously have only normal crossings.

- For each \( j \), letting \( Y_j \subset W_j \) denote the proper transform of \( Y \) to \( W_j \), we have that either
  1. the blow-up center \( N_j \) lies in the singular locus of \( Y_j \)
  2. \( Y_j \) is smooth and \( N_j \subset E_j \cap Y_j \).

We will also need the following statement (both for the argument below and in § 6).

**Lemma 3.3** [35, Exercise 7.14(b) of Chapter 2]. Let \( W \) be a smooth variety with ample line bundle \( \mathcal{L} \). Let \( \pi : W' \to W \) denote the blow-up of \( W \) along a smooth subvariety \( N \) and let \( E \) denote the exceptional divisor. Then there exists an \( n_0 \) such that \( \pi^*(\mathcal{L}) \otimes n \otimes \mathcal{O}(-E) \) is (very) ample for all \( n > n_0 \).

We are now in a position to prove

**Lemma 3.4.** Any smooth affine variety \( X \) admits a log-smooth compactification \((M, \mathbf{D})\).

**Proof.** Compactify \( X \subset \mathbb{C}^n \) inside of \( \mathbb{P}^n \) to a variety \( \tilde{M} \). Note that \( \tilde{M} \setminus X \) supports an ample divisor \( \tilde{D} \) (the scheme-theoretic intersection of \( \tilde{M} \) with the hyperplane at infinity). Apply embedded resolution of singularities (first to the variety \( \tilde{M} \) to obtain a smooth \( \tilde{M} \) and then to the compactifying divisors inside of \( \tilde{M} \)) to obtain a normal crossings compactification \((M, \mathbf{D})\) of \( X \).

It remains to show that the pair \((M, \mathbf{D})\) satisfies (3.3). Let \( M_1 \) denote the first stage of the embedded resolution of singularities (that is, the proper transform of \( \tilde{M} \) inside of the resolution \( W_1 \)). Then by Lemma 3.3, for \( n \) sufficiently large, \( n \pi^*(\tilde{D}) - E_1 \) defines an effective ample divisor on \( W_1 \) (it is effective because the pull-back of the section defining \( \tilde{D} \) vanishes along \( E_1 \)). The restriction to \( M_1 \) therefore defines an effective ample divisor supported on \( M_1 \setminus X \). Continuing in this way, we see inductively that \( \mathbf{D} \) supports an effective ample divisor

\[ \dagger \text{This means that } E_r \text{ is normal crossings and meets } Y' \text{ transversely or equivalently that } E_r \cup Y' \text{ is a (not necessarily equidimensional) normal crossings scheme (see also the fourth paragraph of [36, p. 335]).} \]
To get one which has positive coefficients on each component $D_i$, one can take $\sum_i^k D_i + mF$ which is ample for $m$ sufficiently large by [35, Exercise 7.5(b) of Chapter 2]. □

For all pairs $(M, D)$ in this paper, we will assume that the canonical bundle is supported on $D$

$$\wedge^n M^* \cong \mathcal{O}\left(\sum_{i=1}^k -a_i D_i\right)$$

and choose a meromorphic volume form $\Omega_{M,D}$ on $M$ which is non-vanishing on $X$ and has poles of order $a_i$ along $D_i$ as in (3.5).

Given a subset $I \in \{1, \ldots, k\}$ define

$$D_I := \bigcap_{i \in I} D_i; \quad (3.6)$$

we refer to the associated open parts of the stratification induced by $D$ as

$$\hat{D}_I = D_I \setminus \bigcup_{j \not\in I} D_j.$$  

(3.7)

By convention, we set $D_\emptyset := M$ and $\hat{D}_\emptyset = X$. Denote by $i : X \hookrightarrow M$ the natural inclusion map.

DEFINITION 3.5. Let $X$ be a symplectic manifold equipped with a 1-form $\theta_X$ such that $\omega_X = d\theta_X$. Let $Y$ denote the $\omega_X$-dual of $\theta_X$. We say that $(X, \theta_X)$ is a finite-type convex symplectic manifold if there exists an exhausting function $f_X : X \to [0, \infty)$ together with a $c_0 \in \mathbb{R}^{\geq 0}$ such that $df_X(Y) > 0$ over all of $f_X^{-1}[c_0, \infty)$.

See [55, §A] for a comprehensive survey of these structures including the notion of deformation of these structures that we will use. Note in particular that any finite-type convex symplectic manifold $X$ has a well-defined symplectic cohomology group (in the sense of the previous section) defined as follows: pick an $f_X$ as above, note that the sublevel set $\bar{X} := f_X^{-1}([N, \infty))$ is a Liouville domain for any $N > c_0$ (the completion $\hat{X}$ of which is convex deformation equivalent to $X$ and in particular independent of choices up to deformation equivalence), and define

$$SH^*(X) := SH^*(\bar{X})$$

for any such $N > c_0$. Any affine variety has a canonical (up to deformation equivalence) structure of a finite-type convex symplectic manifold constructed as follows.

EXAMPLE 3.1 (Stein symplectic structure). Pick a holomorphic embedding $i : X \to \mathbb{C}^N$ and equip $X$ with

- the 1-form $\theta = i^* (\sum_{k=1}^N \frac{r_k^2}{2} d\theta_k)$ (in terms of polar coordinates $(r_k, \theta_k)$ on $\mathbb{C}$); and
- the exhausting function $f_X = i^* (\sum_k |z_k|^2)$.

Up to deformation equivalence, the resulting convex symplectic structure is independent of choice of $i : X \hookrightarrow \mathbb{C}^N$.

There is a different construction of a (deformation equivalent) convex symplectic structure which is more natural from the point of view of normal crossings compactifications.

EXAMPLE 3.2 (Logarithmic symplectic structure). Let $I$ be a log-smooth compactification of $X$ as above, and $s \in H^0(\mathcal{L})$ a section of a line bundle cutting out $D$ as in Definition 3.1, we can equip $X$ with
• the 1-form $\theta = d^c \log |s|$;
• the exhausting function is given by $f_X = -\log |s|$,

where $|\cdot|$ is any choice of positive Hermitian metric on the line bundle $L$ (once more, the result is independent of $|\cdot|$ up to deformation equivalence).

The deformation equivalence of the above two convex symplectic structures for a given $X$ is proven in [55, Lemma 5.18] (compare [72, Lemma 4.4]). This implies in particular that the deformation class of convex symplectic structure on $X$ induced by a log-smooth compactification is independent of $(M, D)$. Here we recall a further deformation of the convex symplectic structure on $X$, due to McLean [55] (see [72] for the case $\dim_C X = 2$), with ‘nice’ properties at infinity with respect to a given compactification $(M, D)$. To begin, after a deformation (as symplectic submanifolds) we assume that the smooth components of $D$ intersect orthogonally in $M$:

**Theorem 3.6** [55, Lemma 5.3, 5.15]. There exists a deformation of the divisors $D$ (through symplectic divisors) such that they intersect orthogonally with respect to the symplectic structure on $M$. This does not change the symplectomorphism type of the complement $X = M \setminus D$, or the deformation class of its convex symplectic structure.

**Definition 3.7** [55, Lemma 5.14]. A convex symplectic structure $(W, \theta)$ constructed from a log-smooth compactification $(M, D)$ is nice if (after first implicitly applying a deformation to make the divisors symplectically orthogonal as in Theorem 3.6) there exist tubular neighborhoods $U_i$ of $D_i$ with symplectic projection maps

$$\pi_i : U_i \to D_i$$

such that on a $|I|$-fold intersection of tubular neighborhoods

$$U_I := \cap_{i \in I} U_i = U_{i_1} \cap \cdots \cap U_{i_{|I|}},$$

iterated projection $|I|$ times

$$\pi_I := \pi_{i_1} \circ \pi_{i_2} \circ \cdots \circ \pi_{i_{|I|}} : U_I \to D_I$$

is a symplectic fibration with structure group $U(1)^{|I|}$ and with fibers symplectomorphic to a product of standard symplectic discs $\prod_{i \in I} D_\epsilon$ of some radius $\epsilon$ and such that $\theta$ restricts to

$$\sum_{i \in I} \left( \frac{1}{2} r_i^2 - \kappa_i \right) d\varphi_i,$$

on each fiber, where $(r_i, \varphi_i)$ are standard polar coordinates for $D_\epsilon$ and the $\kappa_i$ are as in (3.3). Moreover, each $\pi_i$ for $i \in I$ is fiber-preserving, sending

$$\prod_{j \in I} D_\epsilon \to \prod_{j \in I \setminus \{i\}} D_\epsilon.$$ 

**Theorem 3.8** [55, Theorem 5.20]. Given an affine variety $X$ equipped with a log-smooth compactification $(M, D)$ as above, there exists a convex symplectic structure $(W, \tilde{\theta})$ deformation equivalent to the canonical (up to deformation equivalence) convex symplectic structure $(X, \theta)$ which is nice.

Henceforth, we replace $(X, \theta)$ by the corresponding nice structure. By shrinking the tubular neighborhoods in Definition 3.7 if necessary, we suppose their size $\epsilon > 0$ is sufficiently small so that $\epsilon^2 \ll \kappa_i$ for each $i \in \{1, \ldots, k\}$. In this setting, there is a nice Liouville domain $\bar{X} \subset X$, 

obtained by smoothing the complement $M \setminus \cup_i U_i$ of the union of the tubular neighborhoods around smooth divisors. Next, we describe an explicit model of $\hat X$, after first recalling some additional relevant notation and consequences of having a nice structure.

Observe that the coordinates $r_i$ in Definition 3.7 induce smooth functions

$$r_i^2 : U_i \to \mathbb{R}$$

(3.10)

on the neighborhoods $U_i$. On the intersections $U_I$, these functions give rise to commuting Hamiltonian $S^1$ actions. Denote by $UD$ the union

$$UD := \cup U_i.$$ 

A basic important consequence of the ‘nice’ property is

**Lemma 3.9.** (1) The symplectic orthogonal to the tangent space of any fiber $\pi_i^{-1}(p)$ is contained in a level set of the radial function $r_i^2$.

(2) In particular, if $I = \{i_1, \ldots, i_s\}$, any smooth function $f$ of the corresponding radial functions $f(r_{i_1}^2, \ldots, r_{i_s}^2) : U_I \to \mathbb{R}$ has Hamiltonian vector field $X_f$ tangent to the fibers of $\pi_I$ at points $p \in U_I \setminus \cup_{j \notin I} U_j$ in $U_I$ away from a deeper stratum, with Hamiltonian vector field of following the form: for $F = \pi_I^{-1}(p)$ with $p \in U_I \setminus \cup_{j \notin I} U_j$,

$$(X_f)|_F = X_f|_F = \sum_{i \in I} 2 \frac{\partial f}{\partial (r_i)} \partial_{r_i}.$$ 

(3.11)

(3) Let $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$ be any subset. In the associated neighborhood $U_I$ of $D_I$ away from points in $U_j$ for $j \notin I$, any two functions of the radial coordinates $f(r_{i_1}^2, \ldots, r_{i_s}^2)$, $g(r_{i_1}^2, \ldots, r_{i_s}^2)$ have commuting Hamiltonian vector fields: $\omega(X_f, X_g) = df(X_g) = -dg(X_f) = 0$. In particular, $d(r_i^2)(X_f) = 0$ for any $i$ and $f$ as above.

**Proof.** By definition any vector field $Y$ symplectically orthogonal to the fibers of $\pi_i$ preserves the function $r_i^2$, that is, $d(r_i^2)(Y) = \omega(X_i^2, Y) = 0$. It follows that $X_i^2$ is tangent to the fibers of $\pi_i$, which implies (1) (as the symplectic orthogonal to the tangent space of a fiber is contained in the symplectic orthogonal to $X_i^2$, which is the kernel of $d(r_i^2)$). For (2), note that (1) implies that, on $U_I \setminus \cup_{j \notin I} U_j$ where $f$ depends only on $r_i$, for $i \in I$, $df = \omega(X_f, -)$ is zero on any vector simultaneously tangent to the level sets of all radial functions $r_i^2$ for $i \in I$, in particular on any vector that is symplectically orthogonal to all fiber directions in $U_I$, hence $X_f$ is tangent to the fibers. It follows also that $X_f|_F = X_f|_F$, from which the computation (3.11), and hence the remainder of (2) is immediate (given our knowledge of the fiberwise symplectic form as $d$ of (3.9)). (3) can be deduced from (3.11) and the fact that $d(r_i^2)(\partial_{r_i}) = 0$. □

**Definition 3.10.** Define

$$S_I$$

to be the unit $T^{[I]}$ bundle around each $D_I$, for example, the set of points in $U_I$ where $r_i = \delta$ for some small $\delta < \epsilon$ and $i \in I$. We will use the notation $S_I := S_{\{i\}}$ for the one-dimensional torus bundle over $D_I$. Let

$$S_I$$

denote the restriction of the $T^{[I]}$ bundle to $\hat D_I$.

Let $J$ denote a subset of $\{1, \ldots, n\} \setminus I$. There is a codimension $|J|$ stratum of $S_I$ corresponding to the restricted torus bundle $S_I|_{D_I \cup J}$. Near these strata, we have radial coordinates $r_j$. We let $UD_{I,j}$ denote set of points in $S_I$, where $r_j \leq \epsilon$. We let $\overline{S}_I$ denote the natural closure of
the open manifold \( S_I \setminus \{ \cup_{j \not\in I} UD_{I,j} \} \). This is a manifold-with-corners whose boundary is the region

\[
\partial \mathcal{S}_I = \bigcup_{j \not\in I} (\{ r_j = \epsilon \} \setminus \cup_{k \neq j} \{ r_k < \epsilon \}).
\]

We adopt the convention that \( S_\emptyset = M \) and

\[
S_\emptyset := M \setminus UD.
\]

For each \( I \), let \( N_I \) denote the normal bundle to the stratum \( D_I \) and \( \hat{N}_I \) the restriction of this bundle to \( \hat{D}_I \). These two bundles have been equipped with canonical \( U(1)^{|I|} \) structures and there is a canonical identification of the associated torus bundles with \( S_I \) and \( \hat{S}_I \), respectively.

In order to define \( \hat{X} \), we first choose a smooth function \( q : [0, \epsilon^2) \to \mathbb{R} \) satisfying:

1. there exists \( \epsilon_q \in (0, \epsilon) \) such that \( q(s) = 0 \) if and only if \( s \in [\epsilon_q^2, \epsilon^2) \);
2. the derivative of \( q(s) \) is strictly negative when \( q(s) \neq 0 \);
3. \( q(s) = 1 - s^2 \) near \( s = 0 \);
4. there is a unique \( \hat{s} \in [0, \epsilon^2) \) with \( q''(\hat{s}) = 0 \) and \( q(\hat{s}) \neq 0 \).

(Compare [56, proof of Theorem 5.16].) Define

\[
F : UD \to \mathbb{R}
\]

\[
F(x) = \sum_{i=1}^{k} q(r_i^2),
\]

where we implicitly smoothly extend \( q(r_i^2) \) to be 0 outside of the region \( U_i \) where \( r_i^2 \) is defined.

**Lemma 3.11.** For \( \delta > 0 \) sufficiently small, the Liouville vector field associated to \( \theta \) is strictly outward pointing along \( F^{-1}(\delta) \).

**Proof.** This computation is very close to that in [55, Lemma 5.17] (also, compare [56, Theorem 5.16] for a related calculation in the case of concave boundaries); we include a sketch for convenience. First, note that if \( \delta \) is small, then \( F^{-1}(\delta) \subset UD \) and \( F^{-1}(\delta) \cap D = \emptyset \); the latter condition implies that the Liouville vector field is defined at all points of \( F^{-1}(\delta) \). We need to show that for any \( x \in F^{-1}(\delta) \), \( dF(Z) = \omega(X_F, Z) = -\theta(X_F) > 0 \), where \( X_F \) denotes the Hamiltonian vector field of \( F \). Let \( X_{q(r_i^2)} \) denote the Hamiltonian vector field of \( q(r_i^2) \), so \( X_F = \sum_{i=1}^{k} X_{q(r_i^2)} \). The condition that \( (X, \theta) \) is nice implies (see Lemma 3.9) that \( X_{q(r_i^2)} \) is tangent to the fibers \( D_i \) of \( \pi \), with the following form:

\[
(X_{q(r_i^2)})(D_i) = X_{q(r_i^2)}|_{D_i} = -\frac{1}{2} q'(r_i^2) \frac{\partial}{\partial \varphi_i}.
\]

Applying \( \theta \), we have that

\[
\theta(X_{q(r_i^2)}) = \left( \frac{1}{2} r_i^2 - \kappa_i \right) d\varphi_i(X_{q(r_i^2)}) = \frac{1}{2} q'(r_i^2) \left( \kappa_i - \frac{1}{2} r_i^2 \right)
\]

(3.14)

(this extends smoothly by zero outside \( U_i \)). Note that \( q'(r_i^2) \leq 0 \) with equality exactly when \( r_i^2 \geq \kappa_i^2 \), and also note that \( \kappa_i > \frac{1}{2} r_i^2 \) for \( r_i^2 \leq \epsilon_i^2 \) (recall that \( \epsilon < \kappa_i \) for all \( i \)). It follows that (3.14) is non-positive, and strictly negative if \( r_i^2 \) is sufficiently small. Hence \( \theta(X_F) = \sum_{i=1}^{k} \theta(X_{q(r_i^2)}) = \sum_{i=1}^{k} q'(r_i^2)(\kappa_i - \frac{1}{2} r_i^2) \leq 0 \) as well. Also condition (2) implies that at least one \( q'(r_i^2) < 0 \) for \( x \in F^{-1}(\delta) \) as \( \delta \neq 0 \), so we are done.

Fixing such a \( \delta \), let \( \partial X = F^{-1}(\delta) \) and
Definition 3.12. Define $\tilde{X}$ to be the compact region in $X$ bounded by $\partial \tilde{X}$ defined above.

Remark 3.13. Conditions (1)–(4) for the function $q$ are stronger than required for this paper, but are helpful for further analysis of $\partial \tilde{X}$ in the sequel [30].

Flowing by $Z$ for some small negative time $t_0$ defines a collar neighborhood of the boundary $\partial \tilde{X} \times (-t_0,0] \cong C(\partial \tilde{X}) \subset \tilde{X}$. Letting $X^o$ denote the complement of this collar in $\tilde{X}$

$$X^o = \tilde{X} \setminus C(\partial \tilde{X}),$$

we obtain a Liouville coordinate on points in $X$ outside $X^o$

$$R : X \setminus X^o \to \mathbb{R},$$

defined as usual by $R(x) = e^t$, where $t$ is the time required to flow by $-Z$ from $x$ to the hypersurface $\partial \tilde{X}$.

Setting

$$\rho_i := \frac{1}{2} r_i^2,$$  \hspace{1cm} (3.15)

$F$ can be written as $\sum_{i=1}^k q(2\rho_i)$. Our first claim is that

Lemma 3.14. The Liouville coordinate $R$ defined above is a function of $\rho_1, \ldots, \rho_k$ on $X \setminus \dot{\tilde{X}} = \{ R \geq 1 \}$, that is, $R = R(\rho_1, \rho_2, \ldots, \rho_k)$ (or rather $R$ depends smoothly on whichever subset of $\{\rho_1, \ldots, \rho_k\}$ is defined near a given point in $(X \setminus \dot{\tilde{X}}) \subset UD$).

Proof. Let $H_I = F^{-1}(\delta) \cap (U_I \cup \bigcup_{J \notin I} U_J)$ be the portion of the hypersurface $\partial \tilde{X}$ which lies in $U_j \setminus \bigcup_{J \notin I} U_j$. We denote by $V_I \cup U_I \cap (X \setminus \dot{\tilde{X}})$ the subset of points $x \in U_I \cap X$ for which the time $-\log R(x)$ flow by $Z$ from $x$ lands in the subset $H_I \subset \partial X$. In $U_I \cap X$, the vertical component of $Z$ with respect to the symplectic fibration $\pi_I : U_I \to D_I$ is given by

$$Z_{vert} = \sum_{i \in I} (\rho_i - \kappa_i) \partial_{\rho_i};$$  \hspace{1cm} (3.16)

as $\kappa_i > \rho_i$ in $U_I$, each $\rho_i$ (where defined) decreases, respectively, increases along positive, respectively, negative flowlines of $Z$ (and $Z_{vert}$). It follows that the positive flowlines of $Z$ (and $Z_{vert}$) from $H_I$ continue to lie in $U_I$ and negative flowlines from $x \in U_I \cap X$ can only leave $U_I$ for $i \in I$ but cannot enter $U_J$ for $j \notin I$. Hence any point $x \in \{ R \geq 1 \}$ which flows by $-Z$ to $H_I$ must be contained in $V_I \subset U_I$, and every point $x \in U_I \cap \{ R \geq 1 \}$ is contained in $V_J \subset U_J$ for some $J \subseteq I$, that is, $\{ R \geq 1 \} = \cup_J V_J$. It therefore suffices to show that in a neighborhood of $x \in V_I \subset U_I$, $R$ depends smoothly on $\rho_i$ for $i \in I$ (note if $x \in V_I \cap U_J$ for $J \subseteq I$, it will follow that $R$ depends smoothly on $\{\rho_i|j \in J\}$, hence $\{\rho_i|i \in I\}$ if $J \subseteq I$).

Let $F_I : U_I \to \mathbb{R}$ denote the function $\sum_{i \in I} q(2\rho_i)$ and we view $F_I^{-1}(\delta)$ as defining a hypersurface in $U_I$. Note that since $F_I = F$ on (a small neighborhood in $U_I$ containing) $(U_I \setminus \bigcup_{J \notin I} U_J)$, it follows that $H_I = F_I^{-1}(\delta) \setminus \bigcup_{J \notin I} U_J \subset F_I^{-1}(\delta)$. We define $V_{I,vert}$ to be the locus of points in $U_I \cap \{R \geq 1\}$ which flow negatively by $Z_{vert}$ to $F_I^{-1}(\delta)$. Define $R_{I,vert}(p)$ for $p \in V_{I,vert}$ to be $e^{t_{vert}}$, where $t_{vert}$ is the time it takes to flow along $-Z_{vert}$ from the point $p$ to $F_I^{-1}(\delta)$. Note evidently that $R_{I,vert}$ is a function of $\rho_{i_1}, \ldots, \rho_{i_{|I|}}$ (where $I = \{i_1, \ldots, i_{|I|}\}$).

Now, observe that $Y = Z - Z_{vert}$ is by definition orthogonal to the fibers $\prod_{I \in I} D_I$ of $\pi_I$, hence (by niceness) tangent to the level set of each $r_i$ (and thus $\rho_i$), for $i \in I$. In particular, if $\phi_X$ denotes the time $t$ flow of a vector field $X$ then for any $i \in I$ and $p \in U_I$ such that $\phi^i_X(p) \in U_I$, $\phi^i_Z(t) \in U_I$ too and $\rho_i(\phi^i_X(p)) = \rho_i(\phi^i_{Z_{vert}}(p))$. Seeing as $H_I \subset F_I^{-1}(\delta)$, it therefore follows that $V_I \subset V_{I,vert}$ and $R = R_{I,vert}$ on $V_I$. Since $R_{I,vert}$ is a smooth function of $(\rho_{i_1}, \ldots, \rho_{i_{|I|}})$, we are done. \qed
Although $Z$ does not extend across $D$, we observe that $R$ does:

**Lemma 3.15.** The function $R$ extends smoothly to a function $R_M : M \setminus X^\circ \to \mathbb{R}$.

**Proof.** Letting $\rho_I := (\rho_{i_1}, \ldots, \rho_{i_{|I|}}) : U_I \to \mathbb{R}^{|I|}$, it suffices to check the assertion for the functions $R_{I,vert}(\rho_{i_1}, \ldots, \rho_{i_{|I|}})$ defined in the previous lemma, which we view as functions defined on the domain $\Omega := \rho_I(V_I^{vert}) \subset \{0 < \rho_I < \frac{1}{2}Z^2\} = \rho_I(U_I \cap X)$ in $\mathbb{R}^{|I|} \setminus \cup_{i_j \in I}(\rho_{i_j} = 0)$; we would like to show this functions extends smoothly across $\rho_{i_j} = 0$ for $i_j \in I$ (or at least over the portion of these planes meeting the closure of $\Omega$). Let $H = F_I^{-1}(\delta) \subset \rho_I(U_I)$ and on $\rho_I(U_I \cap X)$ define $Z_{vert} = \sum_{j=1}^{|I|}(\rho_{i_j} - \kappa_{i_j})\partial_{\rho_{i_j}}$ (implicitly this is $d\rho_I(Z_{vert})$); the function $R_{I,vert} : \Omega \to \mathbb{R}$ is by definition the exponential of the time it takes to flow by $-Z_{vert}$ to $H$. Observe that $-Z_{vert}$ is naturally defined (not just on $\Omega$ as specified but) on all of $\mathbb{R}^{|I|}$; it is a radial vector field centered at $\rho_I = 重载 := (\kappa_{i_1}, \ldots, \kappa_{i_{|I|}})$ (a point outside $\rho_I(U_I)$), whose flow evidently exists for all time and converges at time $+\infty$ to the point $\bar{\rho}$; its flowlines are (reparametrizations of) straight lines from $\bar{\rho}$. Similarly, $H$ extends to the global hypersurface $F_I^{-1}(\delta)$ where $F_I = \sum_{i_j \in I} q(2\rho_{i_j}) : \mathbb{R}^{|I|} \to \mathbb{R}$. Observe that $F_I$ is weakly decreasing along positive flowlines of $-Z_{vert}$ and in fact strictly decreasing at points $x$ with $F_I(x) = \delta$, as

\[
dF_I(-Z_{vert}) = \sum_{i \in I} 2(\kappa_i - \rho_i)q'(\rho_i),
\]

each $q'(\rho_i) \leq 0$ can only be strictly negative when $(\kappa_i - \rho_i) > 0$, and there is at least one $\rho_i$ with $q'(\rho_i) < 0$ at all points of $F_I^{-1}(\delta)$. By continuity and monotonicity, the flowline (that is, straight line) from any $x$ with $F_I(x) > \delta$ to $\bar{\rho}$ (noting $F_I(\bar{\rho}) = 0$) must cross $F_I^{-1}(\delta)$ at a unique point. In particular, we see that the function $R_{I,vert}$ extends smoothly across (at least) all such points with $F_I \geq \delta$. Note that if some $\rho_i = 0$, $F_I \geq 1$, so it follows that $R_{I,vert}$ extends smoothly across the coordinate hyperplanes as desired. \hfill \Box

### 3.2. Logarithmic cohomology

We now define the logarithmic cohomology group of $(M, D = D_1 \cup \cdots \cup D_k)$

\[
H_{log}^*(M, D)
\]

which will play a key role in this paper. Recall the notation from the previous section $D_I, D_I, S_I, \bar{S}_I$ for the stratified components of $D$, their open parts, and their unit torus normal bundles. To start, let $C_*^{BM}(S_I, k)$ denote the chain complex of smooth Borel–Moore chains of $\bar{S}_I$ with $k$-coefficients. The manifolds $\bar{S}_I$ have canonical orientations (induced by the natural ordering of the set $I$) and we will pass freely between cohomology classes $\alpha \in H^*(S_I)$ and the corresponding Poincare dual Borel–Moore homology classes. We will use multi-index notation, meaning we fix variables $t_1, \ldots, t_k$, and for a vector $\vec{v} = (v_1, \ldots, v_k) \in \mathbb{Z}^k_{\geq 0}$, denote

\[t^\vec{v} := t_1^{v_1} \cdots t_k^{v_k}.
\]

Consider the cochain complex $C_{log}^*(M, D)$ generated by elements of the form $\alpha \cdot t^\vec{v}$, where $\alpha$ is a (co)-chain in $C_*^{BM}(\bar{S}_I, k)$ for some subset $I \subset \{1, \ldots, k\}$, and $\vec{v} = (v_1, \ldots, v_k)$ is a vector of non-negative integer multiplicities strictly supported on $I$, meaning $v_i > 0$ if and only if $i \in I$.

The differential on $C_{log}^*(M, D)$ is induced from the differential on $C_*^{BM}(\bar{S}_I, k)$.

\[
\uparrow\text{In contrast, the corresponding vector field } -Z_{vert} \text{ on } U_I \cap X \text{ does not, of course, extend across any } D_i = \{\rho_i = 0\} \text{ for } i \in I.
\]
**Definition 3.16.** We will denote by
\[ \vec{v}_I \]
the vector \((v_1, \ldots, v_k)\) strictly supported on \(I\) with entries,
\[ v_i := \begin{cases} 1 & i \in I \\ 0 & \text{otherwise} \end{cases} \]
In the case that \(I\) consists of a single element \(\{i\}\), we will use the notation \(\vec{v}_i\). We refer to the vectors \(\vec{v}_I\) as *primitive vectors*.

We can give an efficient description of \(C^*_\log(M, D)\) as follows:
\[ C^*_\log(M, D) := \bigoplus_{I \subset \{1, \ldots, k\}} t^{\vec{v}_I} C_{2n-|I|-1}(\hat{S}_I, k)[t_i \mid i \in I] \]  
(3.18)
where \(S_\emptyset = X\), and \(S_I = \emptyset\) if the intersection \(\cap_{i \in I} D_i\) is empty. We set
\[ H^*_\log(M, D) := H^*(C^*_\log(M, D)). \]  
(3.19)
The cohomology \(H^*_\log(M, D)\) is generated by classes of the form \(\alpha t^{\vec{v}}\). The logarithmic cohomology of \((M, D)\) of slope \(< \lambda\), denoted \(H^*_\log(M, D)_{|\lambda]}\), is the sub \(k\)-module generated by those elements of the form \(\alpha t^{\vec{v}}\) for some subset \(I \subset \{1, \ldots, k\}\), such that
\[ \sum_i v_i \kappa_i < \lambda \]
(recall the definition of \(\kappa_i\) in (3.3)).

If \(\lambda_1 \leq \lambda_2\), we get an inclusion:
\[ i_{\lambda_1, \lambda_2} : H^*_\log(M, D)_{|\lambda_1]} \to H^*_\log(M, D)_{|\lambda_2]} \]

The volume form \(\Omega_{M, D}\) from (3.5) gives rise to a cohomological \(\mathbb{Z}\)-grading on \(H^*_\log(M, D)\) via the following rule:
\[ |\alpha t^{\vec{v}}| = |\alpha| + 2 \sum_i (1 - a_i)v_i. \]  
(3.20)

**Remark 3.17.** For a general pair \((M, D)\), an elaboration of standard Morse–Bott arguments allow one to produce a spectral sequence
\[ H^*_\log(M, D) \Rightarrow SH^*(X) \]  
(3.21)
from logarithmic cohomology converging to symplectic cohomology [30, 57]. It is likely that the differential on the higher pages can be described in terms of certain relative or log Gromov–Witten invariants. Often (for instance in the examples of topological pairs discussed in §3.3), one can use elementary considerations involving homotopy classes of generators or indices to conclude that (3.21) degenerates. However, this method does not enable one to compute the ring structure on \(SH^*(X)\), whereas the log PSS morphism does [30].

**Definition 3.18.** Given a class \(A \in H_2(M, \mathbb{Z})\), we will denote by \(A \cdot D\) the multivector \([A \cdot D_I] \in \mathbb{Z}^k\).

We finish this section by defining a BV-type operator on logarithmic cohomology. Note that for any stratum \(D_I\), there is a natural \(T^{\vec{v}_I}\) action on \(\hat{S}_I\). Any \(\vec{v}\) which is strictly supported
on I gives rise to a natural homomorphism $\phi_\varphi : S^1 \to T^{|I|}$ defined by $\phi_\varphi (e^{i\theta}) = e^{i\varphi \theta}$. The homomorphism $\phi_\varphi$ in turn induces an action

$$\Gamma_\varphi : S^1 \times \hat{S}_I \to \hat{S}_I$$

such that $\Gamma_\varphi$ is proper. So we may define the pushforward on BM homology

$$\Gamma_\varphi_\ast : H^{BM}_\ast (S^1 \times \hat{S}_I) \to H^{BM}_\ast (\hat{S}_I).$$

**Definition 3.19.** Denote the natural fundamental class on $S^1$ by $\epsilon^\vee$. Given a class $\alpha t^\varphi \in H^\ast_{\log} (M, D)$, define

$$\Delta(\alpha t^\varphi) = \Gamma_\varphi_\ast (\epsilon^\vee \otimes \alpha) t^\varphi.$$  \hspace{1cm} (3.23)

### 3.3. Topological pairs and tautologically admissible classes

We say that a class $A \in H_2(M)_\omega$ is spherical if it is in the image of $\pi_2(M) \to H_2(M, \mathbb{Z})$. Let

$$H_2(M)_\omega$$

denote the classes $A \in H_2(M, \mathbb{Z})$ such that $\omega(A) > 0$.

**Definition 3.20.** We say that a class $\alpha t^\varphi \in H^\ast_{\log} (M, D)$ is tautologically admissible if for every spherical class $A \in H_2(M)_\omega$,

$$A \cdot D_i > v_i$$

for some $i \in \{1, \ldots, k\}$.

**Definition 3.21.** A pair $(M, D)$ is topological if every class $\alpha t^\varphi$ is tautologically admissible in the sense of Definition 3.20.

There are numerous examples of topological pairs. We list some examples.

1. If $\pi_2(M) = 0$, any pair $(M, D)$ will be topological.
2. Whenever each smooth component $D_i$ of $D$ corresponds to powers of the same line bundle and the number of components of $D$ satisfies $k \geq \dim C M + 1$, then $(M, D)$ is topological. (Such so-called Kähler pairs were also considered in [79] to simplify holomorphic curve theory). For example $(\mathbb{P}^n, D = \{ \geq n + 1 \text{ generic planes} \})$ is such an example.

**Remark 3.22.** Definition 3.21 can be relaxed to a (substantially) broader class of $(M, D)$ for whom suitable 1-point relative Gromov–Witten moduli spaces(in $M \text{ rel } D$) are empty for certain complex structures preserving $D$ (see [30, §5]). The key compactness result (Lemma 4.14) about log PSS moduli spaces continues to hold for this more general notion of topological pair; we have opted for technical simplicity to use Definition 3.21 to avoid discussion of bubbling in the relative setting.

### 3.4. Admissible classes

In this subsection, we define a subspace of admissible classes in logarithmic cohomology described in the introduction.

We will sometimes (but not always) impose the following assumption (on specific subsets $I$ which will be clear from context).

**Assumption (A1):** All spherical classes $A \in H_2(M)_\omega$ satisfying

$$A \cdot D = \bar{v}_I$$

are indecomposable.
**Definition 3.23.** We say that a class $\alpha t^\mathbf{v} \in H^*_\log(M, D)$ is admissible if either

- $\alpha t^\mathbf{v} \in H^*_\log(M, D)$ is tautologically admissible; or
- $\mathbf{v} = \mathbf{v}_I$ is primitive, Assumption (A1) holds and for any spherical class $A \in H_2(M, \omega)$ such that $A \cdot D \neq \mathbf{v}_I$, there exists an $i \in \{1, \ldots, k\}$ such that $A \cdot D_i > v_i$.

We will similarly refer to (tautologically) admissible cocycles $\alpha c t^\mathbf{v} \in C^*_\log(M, D)$ which are cocycles which represent (tautologically) admissible classes. We define

$$H^*_\log(M, D)^{ad} \subset H^*_\log(M, D)$$

(3.27)
to be the $k$-module generated by admissible classes.

**Remark 3.24.** For a topological pair $(M, D)$, of course $H^*_\log(M, D)^{ad} = H^*_\log(M, D)$. Also, in a slight abuse of terminology, we will use the terminology ‘primitive admissible’ to refer exclusively to classes satisfying the second bullet point of Definition 3.23 (even though the vectors $\mathbf{v}$ associated to tautologically admissible classes can of course also be primitive).

### 3.5. A Gromov–Witten invariant

In this section, we define an element we call the obstruction class

$$GW_\varphi(\alpha) \in H^*(X)$$

(3.28)

associated to any admissible class $\alpha t^\mathbf{v}$. The element $GW_\varphi(\alpha)$ can only be non-zero when $\alpha t^\mathbf{v}$ is not tautologically admissible, which by hypotheses can only happen when $\mathbf{v} = \mathbf{v}_I$ is primitive.

**Remark 3.25.** The setup we choose here is by no means the most general possible, but is the simplest setup which avoids technical issues like multiply-covered curves and gluing curves with non-constant components in the divisor $D$ while still allowing for non-trivial applications. In particular, we expect there should be a more general notion of admissible class for non-primitive vectors for which one can define obstruction classes.

In what follows, we frequently pass between cohomology $H^*(X)$ and the Poincaré dual Borel–Moore homology $H^{2n-\ast}_{BM}(X)$. We begin by defining a class of almost-complex structures adapted to $(M, D)$.

**Definition 3.26.** Let $\mathcal{J}(M, D)$

(3.29)

be the Banach manifold of compatible $C^\infty$ almost-complex structures which preserve $D$ and which are $C^\epsilon$ over (or rather, exponentials of $C^\epsilon$ infinitesimal deformations of) some fixed almost-complex structure, where $C^\epsilon$ is Floer’s $C^\epsilon$ norm [23] on sections of a vector bundle for some $\epsilon = (\epsilon_i)_{i \in \mathbb{N}}$. Similarly, we let $\mathcal{J}(D_I, \cup_{j \notin I} D_j)$ be the space of compatible complex structures on $D_I$ which preserve all divisors $D_j$ for $j \notin I$ (as usual, taking $I = \emptyset$ gives back $\mathcal{J}(M, D)$).

All complex structures in this paper will be of class $\mathcal{J}(M, D)$.

**Definition 3.27.** Given a Riemann surface $C$ and a marked point $p \in C$, we define the projectivized tangent space

$$S_pC := (T_pC \setminus \{0\})/\mathbb{R}^\ast.$$
We let \((C, z_0, z_1, \ldots, z_n)\) denote the Riemann sphere with \(n + 1\) fixed marked points. Each marked point \(z_i\) will be enhanced with a choice of element \(\tilde{r}_{z_i} \in S_{z_i} C\). For this paper, the most important cases will be \(n = 1\), when we may take \(z_0 = \{0\}\) and \(z_1 = \{\infty\}\) and \(n = 2\), when we may take \(z_0 = \{0\}, z_1 = \{1\}, z_2 = \{\infty\}\). In the case \(n = 1\) or \(n = 2\), we may assume the elements \(\tilde{r}_{z_i}\) correspond to the positive real direction. These additional trivializations will be used to place jet conditions on holomorphic maps as we now explain.

**Definition 3.28.** We say that a \(J\)-holomorphic map \(u : C \to M\) is \(D_I\)-regular if \(u(C) \not\subseteq \bar{D}_i\) for \(i \in I\).

Let \(u : C \to M\) be a \(D_I\)-regular \(J\)-holomorphic map such that for some \(p \in C\), \(u(p) \in \bar{D}_I\). Fix a local complex coordinate \(z\) around \(p\) and an element \(i \in I\). In a small neighborhood \(W\) about \(u(p)\), choose a system of smooth complex coordinates \(y_1 = u_1 + \sqrt{-1}v_1, \ldots, y_n = u_1 + \sqrt{-1}v_n\) such that \(u(p) = \{y_1 = 0, \ldots, y_n = 0\}\), \(W \cap D_i = \{y_1 = 0\}\), and \(J(\partial y_i) = \partial y_i\) along \(W \cap D_i\). Inside of \(W\), write \(u(z) = (f_i(z), \hat{u}(z))\), where \(f_i(z)\) denotes the projection to the first factor (and \(\hat{u}(z)\) denotes the projection to the remaining factors). By [43, Lemma 3.4] (see also [81, equation (6.1) and Remark 6.1]) \(f_i(z)\) (which along \(W \cap D_i\) is the locally defined normal component of \(u\) to \(D_i\)) has an expansion

\[
f_i(z) = a_i z^{v_i} + O(|z|^{v_i + 1}),
\]

where \(v_i\), the **multiplicity** is a positive integer and the leading coefficient \(a_i \in \mathbb{C}^*\). As explained in [16, §6; 44, §6], the coefficient \(a_i\) is the \(v_i\)-jet of the component of \(u\) normal to \(D_i\) at \(p\) modulo higher order terms and so intrinsically

\[
a_i \in (T^*_p \mathbb{C}P^1)^{v_i} \otimes (N_i)_{u(p)}. \tag{3.30}
\]

In our case, \(C = \mathbb{C}P^1\) and fixing a non-zero unit vector (or real positive ray) in \(T^*_p (\mathbb{C}P^1)\), we can think of \(a_i \in (N_i)_{u(p)} \setminus \{0\}\); define \([a_i] \in (N_i)_{u(p)}/\mathbb{R}^+\) to be the real projectivization. Then, the direct sum \([\oplus_{i \in I} a_i] \in S_I u(p), \text{ lives in a fiber of the torus bundle } S_I \text{ to } D_I\). Thus, if \(u\) satisfies an incidence condition as in Definition 3.28 with multiplicities \((v_i)_{i \in I}\), we can define the enhanced evaluation of \(u\) at \(z_0\) (following [16, 42]), taking values in \(S_I\) as

\[
 Ev_{z_0}^I (u) := (u(z_0), [\oplus_{i \in I} a_i]). \tag{3.31}
\]

**Definition 3.29.** For any \(A \in H_2(M, \mathbb{Z})\) satisfying (3.26), set \(\tilde{\mathcal{M}}_{0,2}(M, D, \tilde{\nu}_I, A)\) to be the moduli space of maps \(u : (C, z_0, z_1) \to (M, D)\) with

\[
u_*([\mathbb{C}P^1]) = A \tag{3.32}
\]

\[
u^{-1}(D_I) = z_0. \tag{3.33}
\]

**Definition 3.30.** From the moduli spaces in Definition 3.29 form:

\[
\tilde{\mathcal{M}}_{0,2}(M, D, \tilde{\nu}_I) := \bigsqcup_{A, A \cdot D = \tilde{\nu}_I} \tilde{\mathcal{M}}_{0,2}(M, D, \tilde{\nu}_I, A) \tag{3.34}
\]

\[
\mathcal{M}_{0,2}(M, D, \tilde{\nu}_I) := \tilde{\mathcal{M}}_{0,2}(M, D, \tilde{\nu}_I)/\mathbb{R}, \tag{3.35}
\]

where the quotient in the latter equation is by \(\mathbb{R}\)-translations.

We have an evaluation map at \(z_1 = \infty\) which sends \(u\) to \(u(\infty)\):

\[
ev_{\infty} : \mathcal{M}_{0,2}(M, D, \tilde{\nu}_I) \to M. \tag{3.36}
\]

Let

\[
\mathcal{M}_{0,2}(M, D, \tilde{\nu}_I)^0 \tag{3.36}
\]
denote the preimage $ev_\infty^{-1}(X)$. We will frequently use the following topological assumption to simplify compactness and transversality arguments.

**Lemma 3.31.** Assume that Assumption (A1) holds. Then the moduli space $\mathcal{M}_{0,2}(M, D, \vec{v}_I)^o$ is a smooth oriented manifold for generic $J \in \mathcal{J}(M, D)$. Moreover the map 

$$ev_\infty : \mathcal{M}_{0,2}(M, D, \vec{v}_I)^o \to X$$

is proper.

**Proof.** Consider a sequence of curves with $ev_\infty \subset K$ for some compact set $K \subset X$. There is no sphere bubbling in non-trivial classes because the classes $A$ are indecomposable. As the evaluation $ev_\infty$ lies in $X$, it follows that the non-trivial component of the Gromov limit $u$ is $D_I$ regular and thus carries both distinguished marked points. Thus no bubbling can occur and it therefore follows that the map $ev_\infty$ is proper. The fact that the moduli space is smooth for generic $J$ follows from [16, Lemma 6.7](we will review related arguments in §4.4). The orientation comes from the usual orientation on the moduli space $\mathcal{M}_{0,2}(M, D, \vec{v}_I)$ arising from the fact that the Fredholm operator $D_{\bar{\partial}}$ is homotopic to the operator $\bar{\partial}$ together with the fact that the $\mathbb{R}$ translations preserve this orientation.

If $Ev_{\vec{v}_0}(u)$ is contained in $S_I \setminus \hat{S}_I$, then because $u$ has intersection number zero with all of the divisors $D_j$ for $j \notin I$ then $u$ must be completely contained in $D$ and hence $ev_\infty(u) \in D$ as well. Therefore we have an evaluation

$$Ev_{\vec{v}_0} : \mathcal{M}_{0,2}(M, D, \vec{v}_I)^o \to \hat{S}_I.$$  

(3.37)

For any class $\alpha$ in $H^*(\hat{S}_I)$, we may define a BM homology class in $H^{BM}_*(X)$ via

$$GW_{\vec{v}_I}(\alpha) = [ev_{\infty,*}(Ev_{\vec{v}_0,*,*}(\alpha))] \in H^{BM}_*(X).$$  

(3.38)

**Definition 3.32.** For any primitive admissible class $\alpha t^{\vec{v}_I} \in \mathcal{H}^*_0(M, D)$, we refer to the class defined in equation (3.38) as the obstruction class associated to $\alpha t^{\vec{v}_I}$. We extend this to non-primitive admissible classes by setting $GW_{\vec{v}_I}(\alpha) = 0$.

**Lemma 3.33.** For any stratum $D_I$, let $\alpha = 1 \in H^0(\hat{S}_I)$ and suppose that $\alpha t^{\vec{v}_I}$ is admissible. Then $GW_{\vec{v}_I}(\alpha) = 0$.

**Proof.** In this case there are no constraints along $\hat{S}_I$ so by definition our invariant is simply the pushforward

$$ev_{\infty,*}[\mathcal{M}_{0,2}(M, D, \vec{v}_I)^o].$$

But, since the evaluation map $ev_\infty$ factors through the quotient $\mathcal{M}_{0,2}(M, D, \vec{v}_I)^o/S^1$ it follows that

$$ev_{\infty,*}[\mathcal{M}_{0,2}(M, D, \vec{v}_I)^o] = 0.$$  

4. The log PSS morphism

4.1. Symplectic cohomology from Hamiltonians on the compactification

We continue with the setup and notation described in §3.1, with $X = M \setminus D$ (with its nice convex symplectic structure in the sense of Definition 3.7) equipped with the (holomorphic on $X$) restriction of a fixed meromorphic volume form $\Omega_{M,D}$ as in (3.5). In this section, we
Recast the construction of the symplectic cohomology \( SH^*(X) \) in terms of the geometry and Floer theory of \((M, D)\), a point of view that will prove technically useful in constructing the log PSS map.

Recall from \( \S\) 3.1 the construction of the function \( R_M : M \setminus X^o \to \mathbb{R} : R_M \) is the smooth extension to \( M \) (guaranteed to exist by Lemma 3.15) of the Liouville coordinate on \( X \) induced by (the exponential of the time it takes to flow by \( Z \) from) the hypersurface \( \partial X \) constructed above Definition 3.12. We set \( R_D = \min_D R_M \), and now consider Hamiltonians \( h^\lambda_M : M \to \mathbb{R}^{>0} \) (for \( \lambda > 0 \)) such that

- \( h^\lambda_M \) vanishes on \( X \);
- on \( M \setminus X^o \), \( h^\lambda_M = h^\lambda_M(R_M) \) is a function of \( R_M \); Moreover, for some \( R_H \in (1, R_D) \) which is much closer to 1 than \( R_D \) (so \( \log R_H \ll \log R_D \)) we have that
  \[
  h^\lambda_M = \lambda (R_M - 1) \quad \forall R_M \geq R_H;
  \]
- on \( M \setminus X^o \), \( (h^\lambda_M)'(R_M) \geq 0 \) and \( (h^\lambda_M)''(R_M) \geq 0 \).

Choose a \( \mu \) such that

\[
R_H < R_D - \mu. \tag{4.1}
\]

Let \( V_0 \subset UD \) be the open subset containing \( D \) defined by \( V_0 := (R_M)^{-1}(R_D - \mu, \infty) \). Let \( V \supset V_0 \) denote the slightly larger open set \( V := (R_M)^{-1}(R_H, \infty) \).

Note that since (by Lemma 3.14) \( R_M \) is a function of the \( \rho_1, \ldots, \rho_k \) on \( M \setminus \hat{X} \) (which contains \( D \)) , Lemma 3.9 implies that the Hamiltonian flow of \( h^\lambda_M \) is tangent to the level sets of each \( \rho_i \) in this region and hence (as the flow is zero outside this region) everywhere \( \rho_i \) is defined; in particular the Hamiltonian flow preserves the divisor \( D \). It follows that time-1 orbits of this Hamiltonian are either completely contained in \( D \) or completely contained in \( X \) (and in fact \( X \setminus V \), as by construction there are no closed orbits in \( V \setminus X \)). We will refer to the orbits contained in \( D \) as divisorial orbits and denote them by \( X(D; h^\lambda_M) \) and all other orbits by \( X(X; h^\lambda_M) \). We can make all of the orbits non-degenerate by a \( C^2 \) small time-dependent perturbation \( H^\lambda_M : M \to \mathbb{R} \) supported in small neighborhoods of the orbit sets (in \( X \) and near \( D \)). We can moreover ensure this \( C^2 \) small perturbation \( H^\lambda_M \) satisfies the following key properties:

- the perturbation is disjoint from \( V \setminus V_0 \);  
- the Hamiltonian flow of \( H^\lambda_M \) continues to preserve each divisor \( D_i \).

We now give details of this construction, using a refined (to satisfy the above properties) variant of the perturbation appearing in [55, Proof of Lemma 6.8] (though note that [55] only perturbed the orbits appearing in \( X \)). For every \( I \) (including \( I = \emptyset \), recalling the convention that \( D_\emptyset = M \) and \( \hat{D}_\emptyset = X \)), Lemma 3.9 (in particular part (2)) implies that the orbits of \( X_{h^\lambda_M} \), that lie in the open locus \( \hat{D}_I \) are disjoint unions of manifolds-with-corners, corresponding to constant orbits living in \( D_I \) away from any deeper stratum as well as orbits which wind a certain number of times around (and live in a neighborhood of) some deeper stratum \( D_J \) for \( I \subset J \). Let us use the notation \( Y_\alpha \) to refer to the connected component of orbits in \( \hat{D}_I \) winding \( w_j \) times around \( D_J \) for \( j \in J \setminus I \), where \( \alpha = (I, J, w) \) is a tuple with \( \emptyset \subset I \subset J \subset \{1, \ldots, k\} \) and \( w \in \mathbb{Z}_\geq 0 \setminus J \) is a tuple of positive integers (by convention \( \mathbb{Z}_0^J = \{0\} \) so we allow the case \( \alpha = (I, I, 0) \), corresponding to the constant orbits in \( D_I \)). Concretely, \( Y_\alpha \) can be explicitly described as the locus \( (\hat{D}_I \cap U_J) \cap \bigcap_{j \in J \setminus I} \left\{ \frac{\partial}{\partial \rho_j} = -w_j \right\} \setminus \bigcup_{I \subsetneq J} \{ x \in U_I : \frac{\partial}{\partial \rho_I} \neq 0 \} \). Each \( Y_\alpha \) is a \( T^{J_\alpha} \) torus bundle (of the form \( \{ \rho_j = d_j, \alpha : j \in J \setminus I \} \) for some small constants \( d_j, \alpha \)) over a closed submanifold-with-corners of \( \hat{D}_I \) which we call \( \overline{Y}_\alpha \); if \( J = I \) then \( \overline{Y}_\alpha = Y_\alpha \). In turn, \( \overline{Y}_\alpha \) is a compact manifold-with-corners obtained by removing from \( D_J \) subsets of the form \( \{ \rho_t < c_{t, \alpha} \} \) for \( t \notin J \) and some small constants \( c_{t, \alpha} \); in particular its boundary and corner strata are the points of the form \( \{ \rho_t = c_{t, \alpha} \} \) for any \( t \in \{1, \ldots, k\} \setminus J \). The Hamiltonian vector
field $X_{h^\lambda}$ restricted to the fiber of $U_I$ over any point $x \in Y_\alpha \subset D_I$ is of the form $\sum_{i \in I} \lambda_i \frac{d}{d\pi_i}$ with $\lambda_i > 0$, which infinitesimally generates a non-trivial rotation of the fibers fixing the points where $\rho_i = 0$. It follows that these orbits are non-degenerate in the normal (transversal to $D_I$) directions in $M$ over the open parts $Y_\alpha$. The open locus $Y_\alpha$ is moreover Morse–Bott as an orbit set in $D_I$ (for $I = J$ this is elementary and for the case $J \supset I$ we refer to Step 2 of [56, Proof of Theorem 5.16], which we note relies on the special form of the $q$ function chosen in §3.1); hence we conclude that $Y_\alpha$ is Morse–Bott in $M$; see Figure 1 for a schematic.

We now choose open submanifolds $Y'_\alpha$ of $D_I$ containing $Y_\alpha$ which have manifold-with-corners closures (by, for example, removing regions where $\rho_i \leq c_{\alpha i} - \delta_{\alpha i}$ for very small $\delta_{\alpha i}$). Fix disjoint isolating neighborhoods $U_\alpha$ of $Y_\alpha$ in the open subset $U_J \setminus \bigcup_{j \in J \setminus I} D_j \cup \bigcup_{j \in J \setminus I} U_I$ of $M$ lying in $V_0$ if $I \neq \emptyset$ and outside $V$ if $I = \emptyset$, which are $T^J$-equivariant subbundles of $(U_j)|_{Y'_\alpha}$ for some (open in $D_I$) $Y'_\alpha \supset Y_\alpha$, whose fibers with respect to $\pi_I$ a product of discs (the fibers of $\pi_j$ with $j \in I$) and annuli (the fibers of $\pi_j$ for $j \in J \setminus I$). Let $Y'_\alpha \subset D_I$ denote the natural $T^{J \setminus I}$ bundle over $Y'_\alpha$ such that $Y_\alpha \subset Y'_\alpha$. The Hamiltonian flow of $X_{h^\lambda}$ along $Y_\alpha$ generates a local circle action which extends to $U_I$ as a one parameter subgroup of the $T^{J \setminus I}$ symmetries. Let $X_\alpha$ denote the Hamiltonian vector field of the inverse circle action and $\Delta_\alpha$ denote the time $t$-flow of $X_\alpha$. Let $\hat{h}_\alpha$ be an outward pointing (on the boundary components of the closure) Morse function on $Y'_\alpha$ and let $h_\alpha : S^3 \times U_\alpha \to \mathbb{R}$ denote the pull-back of the time-dependent function $\hat{h}_\alpha \circ \Delta_\alpha(x)$. Note that $h_\alpha$ is $T^J$-equivariant (where $\alpha = (I, J, w)$) but by construction not $T^{J \setminus I}$-equivariant.

Fix a smaller $T^J$-equivariant isolating neighborhood of $Y_\alpha$ in $M$, $U'_\alpha \subset U_\alpha$ which fibers over some smaller $Y''_\alpha \subset Y'_\alpha$ containing $Y_\alpha$. Choose a $T^J$-equivariant cutoff function $f_\alpha$ supported in $U_\alpha$ with $f_\alpha = 1$ on $U'_\alpha$. Then for some $\delta$ sufficiently small set

$$H^\lambda_M := h^\lambda_M + \sum_\alpha \delta f_\alpha h_\alpha. \quad (4.2)$$
Note that all of the additions to \( H^\lambda_M \) are supported on disjoint neighborhoods which are disjoint from \( V \setminus V_0 \), implying the first desired property of this perturbation. It follows from the \( T^I \) equivariance of the perturbation (in every \( U_I \)) that for every \( x \in D_i \)
\[(X_{H^\lambda_M})_x \in T_x(D_i), \quad (4.3)\]
which of course implies the second desired property of the Hamiltonian flow of \( H^\lambda_M \) preserving \( D \). It remains to verify non-degeneracy of \( H^\lambda_M \). First, a compactness argument shows that for \( \delta \) sufficiently small, all of the orbits of \( X_{H^\lambda_M} \) lie in \( U_\alpha' \) and away from the boundary and corner strata (because in the limit as \( \delta \to 0 \), there are no orbits along those strata, compare \([55, \text{Proof of Lemma 6.8}]\)). Then standard Morse–Bott theory (compare \([51, \text{Proposition B.4}]\)) implies that for sufficiently small \( \delta \) the set of orbits is (finite and) non-degenerate as desired.

Fix a small constant \( \hbar > 0 \). Note that by taking \( \delta \) sufficiently small in the above perturbation, we can (and henceforth will) assume that \( H^\lambda_M > -\hbar \) everywhere.

As before we let \( \mathcal{X}(D; H^\lambda_M) \) denote the time-1 orbits of \( H^\lambda_M \) contained in \( D \) (which we also call divisorial orbits) and we denote all other orbits by \( \mathcal{X}(X; H^\lambda_M) \).

For what follows, recall the definition of \( J(M, D) \) in Definition 3.26.

**Definition 4.1.** Define \( \mathcal{J}(V) \subset \mathcal{J}(M, D) \) to be the subspace of compatible almost-complex structures which
\[\begin{align*}
&\bullet \text{ preserve } D \text{ (this is in fact automatic from saying } \mathcal{J}(V) \subset \mathcal{J}(M, D)), \text{ and} \\
&\bullet \text{ are of contact type on the closure of } V \setminus V_0.
\end{align*}\]

Define
\[CF^*(X \subset M; H^\lambda_M) := \bigoplus_{x \in \mathcal{X}(X; H^\lambda_M)} |o_x|, \quad (4.5)\]
where \( |o_x| \) is the \( k \)-normalization of the orientation line \( o_x \) as in (2.5).

**Definition 4.2.** Let \( \mathcal{J}_F(V) \) denote the space of \( S^1 \) dependent complex structures, \( C^\infty(S^1; \mathcal{J}(V)) \).

The differential on (4.5) is defined by counting solutions to (2.7) in \( M \) (instead of \( X \)) with respect to a generic time-dependent \( J_t \in \mathcal{J}_F(V) \) which additionally satisfy the topological constraint
\[u \cdot D = 0. \quad (4.6)\]

In view of the following key *positivity of intersection property*, the condition (4.6) actually implies \( u \cap D = \emptyset \).

**Lemma 4.3.** Let \( H := H_{s,t} \) be any Hamiltonian preserving \( D \), and \( J := J_{s,t} \) an almost-complex structure (both possibly cylinder dependent) for which \( D \) is a \( J \)-holomorphic divisor (with normal crossings). Then, for any Floer trajectory \( u \) of \( (H, J) \) with asymptotics outside \( D \), \( u \cdot D = \sum_{z \in u \cap D} (u \cdot D)_z \), where each local intersection multiplicity \((u \cdot D)_z \) is \( \geq 1 \).

The above lemma is a consequence of positivity of intersection of \( J \)-holomorphic curves \( u \) with \( J \)-holomorphic divisors \( D \), along with Gromov’s trick, which allows us to import results from the analysis of \( J \)-holomorphic curves to the case of Floer curves.
Proposition 4.4 (Gromov’s trick; see, for example, [54, §8.1]). Let \((\Sigma, j)\) be a Riemann surface equipped with a 1-form \(\gamma\), surface dependent Hamiltonian \(H_\Sigma\) and almost-complex structure \(J_\Sigma\). Then, \(u : \Sigma \to M\) solves Floer’s equation \((du - X_{H_\Sigma} \otimes \gamma)^{0,1} = 0\) if and only if \(\tilde{u} = (id, u) : \Sigma \to \Sigma \times M\) is a \(\tilde{J}\)-holomorphic section with respect to the almost-complex structure on \(\Sigma \times M\)

\[
\tilde{J} = \begin{pmatrix}
J \circ (X_{H_\Sigma} \otimes \gamma) - (X_{H_\Sigma} \otimes \gamma) \circ j & 0 \\
0 & J
\end{pmatrix}. \tag{4.7}
\]

Proof of Lemma 4.3. If \(J\) and \(H\) preserve \(D\), then the almost-complex structure \(\tilde{J}\) appearing in Proposition 4.4 preserves the divisor \(\Sigma \times D \subset \Sigma \times M\). Hence, we can apply positivity of intersection for ordinary \(J\)-holomorphic curves.

Applying Lemma 4.3, we see that Floer curves in \(M\) which are counted in the definition of \(CF^*(X \subset M; H^\lambda_M)\) actually lie in \(X\). Then [8, Lemma 7.2] implies that such curves must in fact stay away from an open neighborhood of \(D\). Hence, the compactness theorem (Lemma 2.2) transfers over to this setup. In fact, more is true.

Proposition 4.5. There is a canonical isomorphism of chain complexes

\[
CF^*(X \subset M; H^\lambda_M) \cong CF^*(\hat{X}; H^\lambda), \tag{4.8}
\]

where \(\hat{X}\) (compare (2.3)) is the completion of the Liouville domain \(\hat{X} \subset X\) from Definition 3.12 and \(H^\lambda\) is the family of Hamiltonians on \(\hat{X}\) used to define \(CF^*(\hat{X}; H^\lambda)\) and \(CF^+_+(\hat{X}; H^\lambda)\) in (2.17).

Proof. The generators for each complex are identical by construction. Furthermore in the common subregion of \(X\) and \(\hat{X}\) where Floer curves for \(CF^*(X \subset M, H^\lambda_M)\) and \(CF^*(\hat{X}, H^\lambda)\), respectively, are each forced to stay by [8, Lemma 7.2] (and also Lemma 4.3 for the former complex), \((H^\lambda_M, J^\lambda)\) is exactly equal to \((H^\lambda, J_1)\). Thus, the Floer trajectories are in bijection too.

As usual, we may consider a subcomplex of \(CF^*(X \subset M, H^\lambda_M)\) generated by constant orbits in \(\mathcal{A}(X; H^\lambda_M)\), which we denote by \(CF^*_{\text{cont}}(X \subset M, H^\lambda_M)\) as well as the quotient complex \(CF^+_{\text{cont}}(X \subset M, H^\lambda_M)\). The bijection from equation (4.8) gives rise to an identification:

\[
CF^+_+(X \subset M; H^\lambda_M) \cong CF^+_{\text{cont}}(\hat{X}; H^\lambda). \tag{4.9}
\]

Remark 4.6. To orient the reader with the several closely related Floer complexes that we have introduced, it may be helpful to note that the complex \(CF^*(X \subset M; H^\lambda_M)\) is neither a subcomplex nor a quotient complex of the Floer chain complex of \(M; CF^*(M; H^\lambda_M)\) because we have avoided counting Floer trajectories which intersect \(D\) to define our differential. In general \(CF^*(M; H^\lambda_M)\) is only defined (potentially using virtual techniques) over a suitable choice of Novikov ring.

One can similarly exhibit the continuation maps (2.12) in terms of Floer theory in \((M, D)\), which we now briefly sketch (as the setup adapts straightforwardly). Given a pair \(\lambda_1 \leq \lambda_2\), one considers an \(s\)-dependent family of time-dependent Hamiltonians \(H^{s,t}_{M,\lambda}\) and almost-complex structures \(J^{s,t} \in \mathcal{J}(V)\) on \(M\) for \((s, t) \in \mathbb{R} \times S^1\), such that each \(J^{s,t}\) and \(X_{H^{s,t}_{M,\lambda}}\) preserves \(D\) and such that \((H^{s,t}_{M,\lambda}, J^{s,t})\) coincides with the previously made choices of \((H^\lambda_M, J^\lambda)\) (made for \(CF^*(X \subset M; H^\lambda_M)\)) when \(s \gg 0\) and \((H^\lambda_M, J^\lambda)\) (made for \(CF^*(X \subset M; H^\lambda_M)\)) when \(s \ll 0\).
We further require these choices to agree with the Hamiltonians and almost-complex structures appearing in the continuation map (2.12) on the subregion common to $X$ and $\hat{X}$, and more generally for $H^s_t$ to be a monotone decreasing in $s$ function of $R_M$ on $M \setminus X^o$ (or rather a small perturbation thereof, where the perturbation once more avoids $V \setminus V_0$). Counting Floer solutions in $M$ (associated to $(H^s_t, J^s_t)$) with 0 intersection number with $D$ then defines the chain-level continuation map
\[
c_{\lambda_1, \lambda_2} : CF^*(X \subset M; H^\lambda_M) \to CF^*(X \subset M; H^\lambda_M). \tag{4.10}
\]
An argument identical to Proposition 4.5 implies that, under the identification (4.8), this map (4.10) coincides with the previously defined continuation map (2.12) on the chain level (similarly for the $CF^+$ variant of the continuation map). As desired this implies that the symplectic cohomology $SH^*(X)$ (and $SH^*_{+}(X)$) can be computed as
\[
SH^*(X) \cong \lim_{\lambda} HF^*(X \subset M; H^\lambda_M) \tag{4.11}
\]
and
\[
SH^*_{+}(X) \cong \lim_{\lambda} HF^*_{+}(X \subset M; H^\lambda_M). \tag{4.12}
\]

4.2. The classical PSS morphism

Next, recall the classical PSS map [63]. Consider the domain
\[
S = \mathbb{C}P^1 \setminus \{0\},
\]
thought of as a punctured sphere, with a distinguished marked point $z_0 = \infty$ and a negative cylindrical end (2.20) near $z = 0$ which for concreteness we take to be given by
\[
(s, t) \to e^{2\pi(s+it)}.
\]
The coordinates $(s, t)$ extend to all of $S \setminus z_0$. Fix a subclosed 1-form $\beta$ which restricts to $dt$ on the cylindrical end and which restricts to zero in a neighborhood of $z_0$. To be explicit, we consider a non-negative, monotone non-increasing cutoff function $\rho(s)$ such that
\[
\rho(s) = \begin{cases} 
0 & s \gg 0 \\
1 & s \ll 0
\end{cases} \tag{4.13}
\]
and let
\[
\beta = \rho(s)dt. \tag{4.14}
\]

Near $z_0$, we also fix a distinguished tangent vector which points in the positive real direction.

**Definition 4.7.** Let $J_S(V)$ denote the subspace of (smoothly) domain-dependent complex structures on $S$, $J_S = \{J_z\}_{z \in S}$ with $J_z \in J(V)$, such that
\begin{itemize}
  \item $J_S$ is independent of $z \in S$ in $V_0$ and in a neighborhood of $z_0$;
  \item along the cylindrical end, the complex structure only depends on the *time coordinate* in a neighborhood of 0, for example, $(J_S)_{s,t} = J_t$ for all $s < -K$ for some $K$.
\end{itemize}

**Definition 4.8.** Fix $J_S \in J_S(V)$. For any orbit $x_0 \in \mathcal{X}(H^\lambda_M)$, we define $M(x_0)$ as the space of solutions to
\[
u : S \to M
\]
satisfying
\[
(du - X_{H^\lambda_M} \otimes \beta)^{0,1} = 0 \tag{4.15}
\]
with asymptotic condition
\[
\lim_{s \to -\infty} u(\epsilon(s,t)) = x_0. \tag{4.16}
\]
For each $\alpha \in H^*_{BM}(X)$, fix a chain-level representative $\alpha_c \in C^*_{BM}(X)$. For any such representative $\alpha_c \in C^*_{BM}(X)$, and orbit $x_0$ in $\mathcal{X}(X; H^*_M)$, choose a generic surface dependent almost-complex structure $J_s$ as above which agrees with some $J_i$ used to define the Floer complex $CF^*(X \subset M, H^*_M)$ along the cylindrical end. Consider those $u \in \mathcal{M}_M(x_0)$ such that $u(S) \subset X$ and denote this moduli space by $\tilde{\mathcal{M}}_M(x_0)$. Next form
\[\mathcal{M}(\alpha_c, x_0) := \mathcal{M}_M(x_0) \times_{ev_{x_0}} \alpha_c.\] (4.17)

For generic choices, this is a manifold of dimension $|x_0| - |\alpha|$ provided that $|x_0| - |\alpha| \leq 1$. A standard orientation analysis shows that whenever $|x_0| - |\alpha| = 0$, (rigid) elements $u \in \mathcal{M}(\alpha_c, x_0)$ induce isomorphisms of orientation lines
\[\mu_u : \mathbb{R} \to o_{x_0}.\] (4.18)

Thus, we can define
\[\text{PSS}(\alpha_c) = \sum_{x_0, |x_0| - |\alpha| = 0, u \in \mathcal{M}(\alpha_c, x_0)} \mu_u.\] (4.19)

It is a classical fact that this count gives rise to a well-defined map:
\[\text{PSS} : H^*(X) \to HF^*(X \subset M; H^*_M).\] (4.20)

In fact, we have a factorization:
\[\begin{array}{ccc}
H^*(X) & \xrightarrow{\text{PSS}} & HF^*(X \subset M; H^*_M) \\
\leftarrow & & \leftarrow
\end{array}\]

To see this, note that by Stokes’ theorem, the topological energy $E_{top}(u)$ to any solution $u \in \mathcal{M}(\alpha_c, x_0)$ (in the sense of (2.25)) is simply the action $\mathcal{A}(x_0)$; on the other hand by the discussion in §2.2 (specifically (2.28)) $E_{geo}(u)$ is an upper bound for geometric energy $E_{geo}(u)$ up to a very small error (because the perturbation 1-form in this case is $K = H^\lambda \beta$, where $\beta$ is subclosed and $H^\lambda_M$ is $C^2$-close to a non-negative function). On the other hand one can directly compute that the action of any non-constant orbit in $\mathcal{X}(X; H^\lambda_M)$ is negative (and less than a fixed $-\delta$, again using the fact that $H^\lambda_M$ is sufficiently close to $h^\lambda_M$). Hence for $H^\lambda_M$ close to $h^\lambda_M$, any element $u \in \mathcal{M}(\alpha_c, x_0)$ with $x_0$ non-constant would have to have $E_{geo}(u) < 0$, meaning $u$ cannot exist. Hence $\mathcal{M}(\alpha_c, x_0)$ is only possibly non-empty for constant orbits $x_0$, producing the desired factorization.

It will be technically convenient to distinguish the variant map
\[\text{PSS}_{\varnothing} : H_{2n-*}(\bar{X}, \partial\bar{X}) \to HF^*_\varnothing(X \subset M; H^*_M),\] (4.21)

which is defined by representing elements of $H_{2n-*}(\bar{X}, \partial\bar{X})$ by relative pseudo-cycles $[47]$ $P$ such that $\partial P \subset \partial\bar{X}$ and counting elements of
\[\mathcal{M}(P, x_0) = \mathcal{M}_M(x_0) \times_{ev} P\]

for varying $x_0$ as in (4.19). On homology this map agrees with the classical PSS map (4.20) defined using singular cochains under the isomorphism $H^*(X) \cong H_{2n-*}(\bar{X}, \partial\bar{X})$. Furthermore, given a generic relative null-bordism $Z_b$ for a pseudo-manifold $P$, we have that
\[\partial_{CF} \circ \text{PSS}_{\varnothing}(Z_b) = \text{PSS}_{\varnothing}(P).\] (4.22)

**Remark 4.9.** As stated in our conventions (see §1), we restrict to $k = \mathbb{Z}, \mathbb{Q},$ or $\mathbb{C}$ in order to make the standard technically simplifying use of the theory of pseudo-cycles up to bordism $[47]$ as a model for homology with $k$ coefficients. To model general homology classes over arbitrary
it seems better to pass to a different model for homology (which still interacts well with Floer theory, such as Morse homology as implemented in the sequel article \([30]\)). On the other hand sometimes a given homology class over \(k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}\) can be represented by a pseudo-chain (for example, in characteristic \(p\) a pseudo-chain whose boundary is a \(p\)-fold cover), in which case the simplified Floer-theoretic arguments given here work without further modification.

4.3. **Adding multiplicity and jet constraints near \(\infty\)**

One of the main objectives of the present paper is to describe a version of the map (4.20) which takes into account tangency conditions along the divisor \(D\). As before let \(\vec{v}\) be a vector strictly supported on a subset \(I\) with \(D_I\) a non-empty stratum. The map \(\text{PSS}^\lambda_{\log}((-t)^{\vec{v}})\) is an enhanced version of the classical PSS morphism, involving curves passing through cycles not necessarily in \(X\) but at \(\infty\) \((D_i)\), with various incidence and multiplicity conditions. The following moduli space therefore plays a central role in this paper.

**Definition 4.10.** Fix \(J \in \mathcal{J}_S(V)\). For every orbit \(x_0 \in \mathcal{X}(X; H^\lambda_M)\) and \(\vec{v}\) as above, define a moduli space

\[ \mathcal{M}(\vec{v}, x_0) \]

as follows: consider the space of maps

\[ u : S \to M \]

satisfying Floer’s equation

\[ (du - X_{H^\lambda_M} \otimes \beta)^{0,1} = 0 \]

with asymptotics and tangency/intersection conditions

\[ \lim_{s \to -\infty} u(\epsilon(s,t)) = x_0 \]

\[ u(z) \notin D \text{ for } z \neq z_0; \]

\[ u(z_0) \text{ intersects } D_i \text{ with multiplicity exactly } v_i \text{ for all } i \text{ (so } u(z_0) \notin D_i \text{ if } i \notin I). \]

In the setting of (4.26), as in §3.5 (and specifically (3.31)), the real-oriented projectivized \(v_i\) normal jets of the map \(u\) (with respect to a fixed real tangent ray in \(T_{z_0}C\)) give an **enhanced evaluation map**

\[ \text{Ev}^\vec{v}_{z_0} : \mathcal{M}(\vec{v}, x_0) \to \mathcal{S}_I. \]

**Definition 4.11.** The moduli space \(\mathcal{M}(\vec{v}, \alpha_c, x_0)\) is defined to be the moduli space

\[ \mathcal{M}(\vec{v}, x_0) \times_{\text{Ev}^\vec{v}_{z_0}} \alpha_c, \]

**Lemma 4.12.** The virtual dimension of \(\mathcal{M}(\vec{v}, \alpha_c, x_0)\) is given by

\[ \text{vdim}(\mathcal{M}(\vec{v}, \alpha_c, x_0)) = |x_0| - |\alpha t^{\vec{v}}|, \]

where \(|\alpha t^{\vec{v}}|\) is the grading on log cohomology induced by the holomorphic volume form \(\Omega_{M,D}\) in (3.20) (and similarly \(|x_0|\) is the grading on orbits induced by \(\Omega_{M,D}\)).

**Proof.** This is standard: one notes that at a point \(u \in \mathcal{M}(\vec{v}, \alpha_c, x_0)\), the tangent space to \(\mathcal{M}(\vec{v}, \alpha_c, x_0)\) is the kernel of the linearized operator \(\tilde{D}^\prime_u\) corresponding to Floer’s equation...
intersected with the linearization of constraints (4.27) and (4.29). Hence, the dimension of
the tangent space is the index of $D_0$ (which is $|x_0| + 2c_1(u) = |x_0| + \sum_i 2a_i v_i$), minus
the codimension of the constraints imposed ($|\alpha| + \sum_i 2v_i$).

For a generic choice of almost-complex structure and when $\text{vdim}(\mathcal{M}(\bar{\nu}, \alpha_c, x_0)) \leq 1$, the
moduli space is a manifold of the expected dimension, by a standard Sard–Smale argument
applied to Lemmas 4.15 and 4.18 (which are deferred to the next subsection). We next show
that the PSS moduli space has a suitable compactification provided that the value of the
Hamiltonian $H^{\lambda}_M$ is sufficiently big along the symplectization region. We start with some
preparatory lemmas.

**Lemma 4.13.** Given a solution $u \in \mathcal{M}(\bar{\nu}, \alpha_c, x_0)$, its topological energy (as defined in (2.25))
satisfies

$$E_{\text{top}}(u) = \sum_i v_i \kappa_i - \int_{x_0(t)} x^* \theta - \int_S d(\omega^* H^{\lambda}_M \beta). \tag{4.31}$$

**Proof.** Recall from (2.25) (with $K = H^{\lambda}_M \beta$) that

$$(u) = \int_S u^* \omega - \int_S d(\omega^* H^{\lambda}_M \beta).$$

The intersection of $u$ with $D$ is an isolated point, so we can removed a small ball $B(z_0)$ around
that point and as the size of the ball goes to zero, we have that

$$\int_S u^* \omega = -\int_{S^1} x_0^* \theta - \int_{\partial B(z_0)} (-\kappa_i d\varphi_i) \tag{4.32}$$

$$= \sum_i v_i \kappa_i - \int_{S^1} x_0^* \theta. \tag{4.33}$$

Therefore

$$(u) = \sum_i v_i \kappa_i - \int_{x_0(t)} x^* \theta - \int_S d(\omega^* H^{\lambda}_M \beta)$$

as claimed. □

To state our key compactness result, note that given a primitive admissible class $\nu_0$ and a generic complex structure, we defined in (3.36) a Gromov–Witten type moduli space
$\mathcal{M}_{0,2}(M, D, \bar{\nu}_I)_{\nu}$ along with in (3.37) an evaluation map $E_{\nu_0}$ from it to $\tilde{S}_I$. Restricting the
fiber product

$$\mathcal{M}_{0,2}(M, D, \bar{\nu}_I)_{\nu} \times_{E_{\nu_0}} \alpha_c \tag{4.34}$$

to $\tilde{X} \subset X$ defines a pseudo-cycle (rel boundary) $\text{GW}_{\bar{\nu}_I}(\alpha_c)$ representing $\text{GW}_{\bar{\nu}_I}(\alpha)$ (which was
defined in (3.38) using $\mathcal{M}_{0,2}(M, D, \bar{\nu}_I)_{\nu}$). We suppress the choice of complex structure from
our notation, though they are of course necessary for chain-level constructions.

**Lemma 4.14.** Suppose that $\sum_i v_i \kappa_i + h - \lambda < 0$, where $h > 0$ is the previously chosen small constant satisfying (4.4). Let $\alpha_c t^\lambda$ be an admissible cocycle and $x_0$ be a Hamiltonian orbit in
$\mathcal{X}(X; H^\lambda_M)$. Then, for generic $J_S \in J_S(V)$, (i) if $|x_0| - |\alpha t^\lambda| = 0$, the moduli space $\mathcal{M}(\bar{\nu}, \alpha_c, x_0)$ is a compact 0-manifold, and (ii) if $|x_0| - |\alpha t^\lambda| = 1$, the moduli space $\mathcal{M}(\bar{\nu}, \alpha_c, x_0)$ admits
a compactification (in the sense of Gromov–Floer convergence) \( \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0) \) to a compact 1-manifold-with-boundary such that

1. if \( \vec{v} \) is not primitive,
   \[
   \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0) = \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0)_F := \bigcup_{x', |x_0| = |x'| = 1} \mathcal{M}(\vec{v}, \alpha_c, x') \times \mathcal{M}(x_0, x');
   \]
   (4.35)

2. if \( \vec{v} = \vec{v}_I \),
   \[
   \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0) = \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0)_F \bigcup \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0)_S,
   \]
   where
   \[
   \partial \overline{\mathcal{M}}(\vec{v}, \alpha_c, x_0)_S := GW_{\vec{v}_I}(\alpha_c) \times_{ev} \mathcal{M}(\vec{v}_0, x_0).
   \]
   (4.36)

(This latter space is empty if \( \alpha_c \vec{v} \) is tautologically admissible.)

**Proof.** The space \( M \) is compact and so standard Gromov–Floer compactness results imply that the only possible accumulation points of a sequence of elements in \( \mathcal{M}(\vec{v}, \alpha_c, x_0) \) are a PSS solution followed by a sequence of broken Floer cylinders with trees of sphere bubbles glued on at any point (where intermediate Floer cylinder breaking a priori occurs along any orbit of \( H^\lambda_M \)). The lemma follows from (applying transversality — discussed above Lemma 4.13 and more in §4.4 — and standard gluing results to) the below sequence of assertions regarding a given limiting stable curve.

(i) There are no cylinder breakings along orbits of \( H^\lambda_M \) contained in the divisor \( D \).

(ii) There are no sphere bubbles unless \( \vec{v} \) is primitive (that is, \( \vec{v} = v_I \) for some \( I \)), in which case sphere bubbling occurs in codimension 1 and contributes boundary strata in (2) above.

(iii) Any cylinder component has 0 intersection number with \( D \). (Hence, by positivity of intersection Lemma 4.3, is completely contained in \( X = M \setminus D \)).

Beginning with assertion (i), we proceed by contradiction and assume that \( y \) is the last orbit of \( \mathcal{X}(D; H^\lambda_M) \) which arises as an asymptote of such a degeneration. More precisely, suppose \( y \) is in \( \mathcal{X}(D; H^\lambda_M) \), and that the limiting stable curve \( u_\infty \) contains \( (u_1, u_2) \) with \( u_1 \) a broken PSS solution asymptotic to \( y \) at its output and and \( u_2 \) is a broken Floer trajectory from \( y \) to \( x_0 \) (there may be sphere bubbles as well attached to \( u_1 \) and/or \( u_2 \), but for now we ignore them). Abbreviating \( H := H^\lambda_M \), we have by (4.4) (as \((-d\beta) \geq 0 \) and \( \int (-d\beta) = 1 \)) that

\[
E_{\text{top}}(u_1) = E_{\text{geo}}(u_1) - \int H d\beta \geq -\hbar.
\]

(4.38)

We now show that \( u_2 \) cannot exist by energy considerations. Namely, let \( R \) be the Liouville coordinate and consider the slice \( \{ R = R_H \} \). Along this slice,

\[
H = \lambda(R - 1)
\]

(4.39)

and hence (as \( \theta(X_H) = \omega(Z, X_H) = dH(Z) = dH(R\partial R) = RH'(R) = \lambda R \) on this slice)

\[
\theta(X_H) = H + \lambda.
\]

(4.40)

Consider the portion \( \tilde{S} = u_2^{-1}(R^{-1}([R_H, \infty))) \) of the domain of \( u_2 \) mapping to the region above this slice. Let \( \tilde{S} \) denote the remainder of the domain of \( u_2 \), \( \tilde{S} := \text{dom}(u_2) \setminus \tilde{S} \) and let \( \tilde{S}_\infty \) denote the domain of \( u_\infty \). The geometric energy of \( u_2 \) restricted to \( \tilde{S} \) can be estimated by

\[
E_{\text{geo}}((u_2)|_{\tilde{S}}) \leq E_{\text{top}}((u_2)|_{\tilde{S}})
\]

(4.41)

\[
\leq E_{\text{top}}(u_\infty) - E_{\text{top}}(u_1) - E_{\text{top}}((u_2)|_{\tilde{S}})
\]

(by (2.27))
\( \leq E_{\text{top}}(u_\infty) + h - E_{\text{top}}((u_2)|_S) \) \hspace{1cm} \text{(by (4.38))}

\[ = \sum_i v_i \kappa_i - \int_{x_0(t)} x^* \theta - \int_{S_\infty} d(H\beta) + h - E_{\text{top}}((u_2)|_S). \] \hspace{1cm} \text{(by Lemma 4.13)}

In the equations above, the definition of any integral (such as topological energy) over a broken curve is by definition the sum of the integrals over the components. The inequality in the 2nd equation comes from the fact that the topological energy over sphere bubbles is strictly positive. In applying Lemma 4.13, we use the fact that \( E_{\text{top}} \) is a topological quantity and hence is preserved under limits in the Gromov topology. Let \( \partial S = u_2^{-1}(R^{-1}(R_H)) \) (with its induced orientation). Combining the fact that \( E_{\text{top}}((u_2)|_S) = -\int_{x_0(t)} x^* \theta - \int_{\partial S} u^* \theta - \int_S d(H\beta) \) with the equality \( -\int_{S_\infty} d(H\beta) + \int_S d(H\beta) = -\int_{\partial S} u^*(H\beta) \) (by Stokes’ theorem), gives us that

\[ \sum_i v_i \kappa_i - \int_{x_0(t)} x^* \theta - \int_{S_\infty} d(H\beta) + h - E_{\text{top}}((u_2)|_S) = \sum_i v_i \kappa_i + h + \int_{\partial S} u^* \theta - u^*(H\beta) \]

\[ = \sum_i v_i \kappa_i + h + \int_{\partial S} u^* \theta - u^*(\theta(X_H))\beta + \int_{\partial S} \lambda \beta. \] \hspace{1cm} \text{(by (4.40))}

Now because \( \beta = dt \) on the domain of \( u_2 \), Stokes’ theorem implies that

\[ \int_{\partial S} \lambda \beta = -\int_y \lambda \beta = -\lambda. \] \hspace{1cm} \text{(4.41)}

Therefore, by our assumption that \( \sum_i v_i \kappa_i + h - \lambda < 0 \),

\[ \sum_i v_i \kappa_i + h + \int_{\partial S} u^* \theta - u^*(\theta(X_H))\beta + \int_{\partial S} \lambda \beta \leq \int_{\partial S} u^* \theta - u^*(\theta(X_H))\beta. \] \hspace{1cm} \text{(4.42)}

The rest proceeds as in [8, Lemma 7.2], but we recall the details for completeness. Setting \( X_K = X_H \otimes dt \) we have that the right-hand side of (4.42)

\[ = \int_{\partial S} \theta \circ (du - X_K) \] \hspace{1cm} \text{(4.43)}

\[ = -\int_{\partial S} \theta \circ J \circ (du - X_H \otimes dt) \circ j \] \hspace{1cm} \text{(4.44)}

\[ = -\int_{\partial S} dR \circ du \circ j, \] \hspace{1cm} \text{(4.45)}

as \( dR(X_H) = 0 \). Finally, letting \( \hat{n} \) denote the outward normal along \( \partial \tilde{S} \), observe that \( \partial \tilde{S} \) is oriented by the vector \( j\hat{n} \). Now we calculate that \( -dR(du)j(j\hat{n}) = -dR(du)(-\hat{n}) \leq 0 \), which implies that the final integral, hence \( E_{\text{geo}}((u_2)|_S) \) is non-positive. It follows that \( du = X_K \) everywhere and in particular, \( R \) must be constant on \( u_2(S) \) which is impossible. Hence, such a breaking cannot occur.

Since cylinder components have asymptotics all contained in \( X = M \setminus D \), it follows that the entire broken curve and each of its components define classes in \( H_2(M, M \setminus D) \), hence have well-defined topological intersection number with each of the components of \( D \), which is additive over the components of the broken curve. The total topological intersection number of the broken curve with a given \( D_i \) is equal to \( v_i \), and for any components of the stable curve not completely contained in \( D_i \), the intersection number with \( D_i \) must be positive, by Lemma 4.3.

Turning to assertion (ii), note that \( M \setminus D \) is exact so every non-constant sphere bubble in our stable curve must intersect \( D \). Since \( \alpha_c \beta^q \) is admissible, we see that
• if $\bar{v}$ is not primitive then there are no sphere bubbles at all. Indeed, denoting by $A$ the
sum of the homology classes of all non-constant sphere bubbles in the given stable curve,
admissibility implies $A \cdot D_i > v_i$ for some $i$. But additivity of intersection numbers implies
that $A \cdot D_i + (\text{remaining curve}) \cdot D_i = v_i$, a contradiction as the latter intersection number
is non-negative;
• if $\bar{v} = v_i$ for some $i$, the same arguments imply that a sphere bubble can only possibly
appear in an indecomposable homology class $A$ with $A \cdot D = \bar{v}_i$. If this happens, there
must be a unique sphere bubble $u_i$ with $[u_i] \cdot D = \bar{v}_i$. The remaining curve $u_2$, which does
not have any components contained in $D$, must have 0 intersection number with $D$ and hence
(by positivity of intersection) be completely contained in $X = M \setminus D$ (in particular,$u_1$ must contain the point $z_0$). Furthermore $u_1$, as it meets $u_2$ at $u_1(\infty) \in X$, must lie in
$\mathcal{M}_{0,2}(M, D, \bar{v}_i)$ (in particular is not completely contained in $D$ either).

Finally, let us turn to (iii). We have shown that all components of our stable curve are not
completely contained in $D$, hence intersect $D$ positively. Also, we have shown that the total
intersection number of any components which are not Floer cylinders (for example, the unique
element of $\mathcal{M}(\bar{v}, \alpha_c, x')$ along with any sphere bubbles) with $D$ is $\bar{v}$. It follows that cylinder
components each have 0 intersection number with $D$ as desired.

With assertions (i) and (iii) established, standard gluing analysis then shows that the moduli
spaces (4.35) and (4.36) are the only ones arising in codimension 1. To justify the appearance
of the sphere bubble in codimension 1 (rather than 2), note that we are studying the moduli
space of maps from a plane with fixed real tangent ray at $0$ (and by abuse of notation, their
orientations) by the gluing theory.

Suppose for the remainder of the subsection that $\sum_{i} v_i k_i + h - \lambda < 0$. For any admissible
cocycle $\alpha_c t^q$, we define $\text{PSS}_{\log}^\lambda(\alpha_c t^q) \in CF_+(X \subset M; H_M^\lambda)$ (again, the complex structure $J_i$
needed to define $CF_+(X \subset M; H_M^\lambda)$ should agree with $J_S$ along the cylindrical end) by the formula

$$\text{PSS}_{\log}^\lambda(\alpha_c t^q) = \sum_{x_0, \text{vdim}(\mathcal{M}(\bar{v}, \alpha_c, x_0)) = 0} \sum_{u \in \mathcal{M}(\bar{v}, \alpha_c, x_0)} \mu_u, \quad (4.46)$$

where once more, for a rigid element $u \in \mathcal{M}(\bar{v}, \alpha_c, x_0)$, $\mu_u : \mathbb{R} \to o_{x_0}$ is the isomorphism induced on
orientation lines (and by abuse of notation, their $k$-normalizations) by the gluing theory.

Let $\text{PSS}_{\log}^{\lambda,+}(\alpha_c t^q)$ be the image:

$$\text{PSS}_{\log}^{\lambda,+}(\alpha_c t^q) = \overline{\text{PSS}_{\log}^\lambda(\alpha_c t^q)} \in CF_+(X \subset M; H_M^\lambda). \quad (4.47)$$

It follows from Lemma 4.14 that $\text{PSS}_{\log}^{\lambda,+}(\alpha_c t^q)$ defines a cocycle:

$$\partial_{CF_+}(\text{PSS}_{\log}^{\lambda,+}(\alpha_c t^q)) = 0. \quad (4.48)$$

If $\alpha_c t^q$ is tautologically admissible, then Lemma 4.14 implies that $\text{PSS}_{\log}^\lambda(\alpha_c t^q)$ defines a cocycle
as well:

$$\partial_{CF}(\text{PSS}_{\log}^\lambda(\alpha_c t^q)) = 0. \quad (4.49)$$

4.4. Transversality

Fix a multiplicity vector $\bar{v} = (v_1, \ldots, v_k)$ corresponding to our collection of divisors $D_1, \ldots, D_k$.
The purpose of this section is to prove that we can achieve transversality for $\text{PSS}_{\log}^\lambda((-)t^q)$
within the class of complex structures $J_S(V)$. We view $J_S(V)$ as a subspace of surface
dependent complex structures in $J(M, D)$. Concretely, fixing a pair $(H_M, J_i)$ used to define
$CF^*(X \subset M; H_{SM}^n)$ (that is, achieving transversality), we will show transversality can be achieved within the subspace $\tilde{J} \subset \mathcal{J}_S(V)$ of $S$-dependent almost-complex structures which agree with this particular $J_1$ in the cylindrical end for $s \ll 0$.

Let $m$ be an integer larger than $|\mathcal{V}| = \sup_i v_i + 1$, and for some (real) $p > 1$ define

$$B_{S,x}^{m,p}$$

(4.50) to be the Banach manifold of maps $u : S \to M$ which are locally in $W^{m,p}$ and which converge to some orbit $x$ in the $W^{m,p}$ sense with respect to the fixed cylindrical end (see, for example, [5, Definition 4.1; 25, Proof of Theorem 5.1; 28, Definition 7.1.3; 60, (C.1.8)] for precise articulations of this condition). When $S$ and $x$ are implicit we will just use the notation $B^{m,p}$. Over (4.50) there is a Banach bundle

$$\mathcal{E} := \mathcal{E}_{S,x}^{m-1,p}$$

(4.51) whose fiber over $f : S \to M$ consists of the space of maps

$$\mathcal{E}_f := W^{m-1,p}(S, \Omega^{0,1}_S \otimes f^*TM),$$

(4.52) where on the right-hand side, the Sobolev space of sections $S \to \Omega^{0,1}_S \otimes f^*TM$ is defined using the cylindrical ends on $S$ (to fix a measure on $S$), an almost-complex structure on $S$, and a metric and connection on $M$ (though it is independent of the particular choice thereof) — see for instance [70] or the references cited above in the definition of $B^{m,p}$. Again, we leave $S$ and $x$ implicit when they are fixed, and simply use the notation $\mathcal{E}$. For any $H$ with time-1 orbit $x$ and any $J$, the Cauchy–Riemann operator

$$\tilde{\partial}_{H,J} := (d(-) - X_H \otimes \gamma)^{0,1}$$

(4.53) induces a section $\tilde{\partial}_{H,J} : B^{m,p} \to \mathcal{E}_{S,x}^{m-1,p}$ which is Fredholm whenever $x$ is non-degenerate [22].

There is an associated universal moduli space to our problem, the space of solutions to Floer’s equation over $S$ for varying $J$, which have $v_i$ multiplicity intersection with $D_i$ at $z_0$:

$$\mathcal{M}_{univ}(\mathcal{V}, x_0) := \{(f, J) \in B^{m,p} \times \tilde{J}, \tilde{\partial}_{H,J} f = 0, d^{\psi_i-1}(f)_{z_0} \in T_f(z_0)D_i \text{ for } 1 \leq i \leq k\}$$

(4.54) (the notation $d^{\psi_i-1}(f)$ means the $v_i - 1$-jet of $f$, in the sense described in [16, §6]). Let $T_J(\tilde{J})$ denote the space of infinitesimal deformations of our (domain-dependent) $J$ within the class $\tilde{J}$, which necessarily vanish in a neighborhood of $0$ on the cylindrical end of $S$. We recall that our almost-complex structures are all $C^\infty$, and moreover $C^\epsilon$ deformations of a fixed almost-complex structure in the sense of Definition 3.26. The tangent space to $\mathcal{M}_{univ}(\mathcal{V}, x_0)$ at a point $(f, J)$, using a local chart around $f(z_0)$ in $X$ in which $D_i = \{y_i = 0\} \subset \mathbb{C}^n$ (where $y_1, \ldots, y_n$ are the standard coordinates), so that points $f(z)$ for $z$ near $z_0$ are thought of as living in $\mathbb{C}^n$:

$$T_{(f,J)} \mathcal{M}_{univ}(\mathcal{V}, x_0) = \{(\psi, Y) \in T_fB^{m,p} \times T_J(\tilde{J})d^{\psi_i-1}(\psi_{z_0}) \in \{y_i = 0\} \text{ for } 1 \leq i \leq k\}.$$  

(4.55)

As before there is an enhanced evaluation map $\text{Ev}_{\mathcal{V}} : \mathcal{M}_{univ}(\mathcal{V}, x_0) \to \tilde{S}_I$ and with respect to it, we define for any cocycle $\beta \in C^\ast(\tilde{S}_I) := C_{BM}^{BM}(\mathbb{S}^{2n-1})_{-1}(\tilde{S}_I)$, the constrained moduli space

$$\mathcal{M}_{univ}(\beta, \mathcal{V}, x_0) := \mathcal{M}_{univ}(\mathcal{V}, x_0) \times_{\text{Ev}_{\mathcal{V}}} \beta.$$  

(4.56)

The following two key lemmas are the main results of this section.

**Lemma 4.15.** For $m - 2/p > |\mathcal{V}|$, the space $\mathcal{M}_{univ}(\mathcal{V}, x_0)$ is a smooth Banach manifold.

**Lemma 4.16.** For $m - 2/p > |\mathcal{V}|$, and $\beta \in C^\ast(\tilde{S}_I)$, where $I = \text{supp}(\mathcal{V})$, the space $\mathcal{M}_{univ}(\beta, \mathcal{V}, x_0)$ is a smooth Banach manifold.
The proofs of Lemmas 4.15 and 4.16 follow closely [16, Lemma 6.5 and 6.6]. Fixing an $\ell \geq 0$ with $\ell < m - 2/p$, we define a subspace of the tangent spaces to $B^{m,p}$ of tangent vectors with vanishing $\ell$ jet:

$$B^{m,p}_0 = \{ \psi \in T_f B^{m,p}, d^\ell \psi_{z_0} = 0 \}. \quad (4.57)$$

Similarly, we define a subspace of (the vertical tangent space to) $E^{m-1,p}_f$ of sections with vanishing $(\ell - 1)$ jet (a condition which is by convention vacuous if $\ell = -1$):

$$E^{m-1,p}_0 = \{ \eta \in E^{m-1,p}_f, d^{\ell-1} \eta_{z_0} = 0 \}. \quad (4.58)$$

The first assertion is that

**Lemma 4.17.** The linearized operator

$$F_0 : B^{m,p}_0 \oplus T_J(\tilde{f}) \rightarrow E^{m-1,p}_0$$

which sends

$$(\psi, Y) \rightarrow D_f \psi + \frac{1}{2} Y_{\tilde{z}, f(z)} \circ (df_{\tilde{z}} - X_{f(z)} \otimes \gamma_{\tilde{z}}) \circ j_z \quad (4.60)$$

is surjective.

Assuming for a moment the proof of Lemma 4.17, we may now give the proof of Lemma 4.15.

**Proof of Lemma 4.15.** Without loss of generality by reordering the divisors we can assume that the support of $\bar{v}$ is $\{ 1, \ldots, |I| \}$ and that the vectors $v_i$ are ordered so that $v_i \leq v_{i+1}$. Let $\bar{v}^{(i)}$ denote the vector $(v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \ldots, v_0, 0, 0, \ldots, 0)$ where the final $v_i$ occurs in position $|I|$. We will prove by induction on $i$ that the moduli spaces $M_{\text{univ}}(\bar{v}^{(i)}, x_0)$ are smooth Banach manifolds, where as in (4.54)

$$M_{\text{univ}}(\bar{v}^{(i)}, x_0) = \{ (f, J) \in B^{m,p} \times \tilde{T}_J, \partial_H, f = 0, d^\ell \eta_{z_0} = 0 \}. \quad (4.61)$$

The base case $i = 0$ is completely standard, so assume it is true for $i$. If $v_i = v_{i+1}$ then $\bar{v}^{(i)} = \bar{v}^{(i+1)}$ completing the induction, so let us assume $v_{i+1} > v_i$. Let $\bar{v}^{(i,j)}$ denote the vectors $(v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1} + j, \ldots, v_0, 0, \ldots, 0)$ for $0 \leq j \leq v_{i+1} - v_i$ and define $M_{\text{univ}}(\bar{v}^{(i,j)}, x_0)$ in the same way as above. The inductive step is itself proven by induction, this time on $j$. For every point $(f, J) \in M_{\text{univ}}(\bar{v}^{(i,j)}, x_0)$ pick a chart near $p = f(z_0)$ which we identify with (an open subset of) $\mathbb{C}^n$ so that $p$ corresponds to the origin. Under this identification, we assume that each $D_j$ is identified with the hyperplane given by $y_i = 0$, that the chart is small enough so that (on the preimage of this chart under $f$) the inhomogeneous term in (4.15) is zero and that after a linear change of coordinates $J(0) = i$, where $i$ is the standard almost-complex structure on $\mathbb{C}^n$. It suffices to show that for every such $(f, J)$ the $v_i + j + 1$-linearized jet map normal to $\mathbb{C}^n/\mathbb{C}^{|I|+i}$ is surjective. This is proven exactly as in part (b) of [16, Lemma 6.5].

Continuing to assume the proof of Lemma 4.17, the proof of Lemma 4.16 requires one further lemma.

**Lemma 4.18.** Assume that in the notation of Lemma 4.15, $m - 2/p > |\bar{v}| + 1$. Then the evaluation map

$$\text{Ev}_{z_0} : M_{\text{univ}}(\bar{v}, x_0) \rightarrow S_J$$

is a submersion.
Proof. As before, let \((f, J)\) be a point in \(\mathcal{M}_{\text{univ}}(\vec{v}, x_0)\) and \(ev_{z_0} : \mathcal{M}_{\text{univ}}(\vec{v}, x_0) \to \hat{D}_I\) the ordinary evaluation map (that is, \(ev_{z_0}(f, J) = f(z_0)\)). There is an exact sequence

\[ 0 \to T_{ev_{z_0}(f, J)} T' \to T_{ev_{z_0}(f, J)} \hat{D}_I \to T_{ev_{z_0}(f, J)} \hat{D}_I \to 0. \]  

We first show that \(d(ev_{z_0})\) surjects onto \(T_{ev_{z_0}} \hat{D}_I\). To prove this, we let \(U\) be a sufficiently small open neighborhood of \(ev_{z_0}(f, J) = f(z_0)\) in \(\hat{D}_I\), over which the normal bundle of each \(D_i\) is trivial for \(i \in I\). We may consider an open neighborhood of \(f(z_0)\) in \(V \subset M\) of the form \(U = \bar{U} \times D^{\|f\|}_I \subset V\) (as usual we abuse notation and use \(\subset\) for the trivializing diffeomorphism of \(U_I\) followed by the tubular neighborhood map), where \(\epsilon' \ll \epsilon/2\) (recall \(\epsilon\) is the size of the tubular neighborhoods specified in Definition 3.7). We assume that \(f^{-1}(U) = U_{z_0}\) is a connected neighborhood of \(z_0\) on which the inhomogeneous term in (4.15) vanishes. Let \(v\) be a tangent vector to \(D_I\) at \(f(z_0)\) and consider a Hamiltonian \(H : D_I \to \mathbb{R}\) supported in \(U\) and for which

\[ X_H(f(z_0)) = v. \]  

(4.64)

We next consider a cutoff function \(\eta(x) : \mathbb{R} \to \mathbb{R}^{\geq 0}\) which is 1 for \(x \leq \epsilon'\) and 0 for \(x \geq \epsilon/2\) (note by hypotheses \(\epsilon' < \epsilon/2\)). Denote by

\[ H = \left( \prod_{j \in I} \eta(r_j) \right) \pi_I^*(\hat{H}). \]  

(4.65)

Let \(L_{X_H} J\) denote the Lie derivative of \(J\) in the direction of \(X_H\). The flow of the Hamiltonian vector field preserves all divisors \(D_i\). Then by [54, Exercise 3.1.3], discussion on page 60 together with the invariance of intersection number with the divisors \(D_i\) under homotopies, the tuple \((-X_H, L_{X_H} J)\) defines a tangent vector to the moduli space \(\mathcal{M}_{\text{univ}}(\vec{v}, x_0)\) at the point \((f, J)\). Applying the differential of \(ev_{z_0}\) at the point \((f, J)\) gives:

\[ d(ev_{z_0})_{(f, J)}(-X_H, L_{X_H} J) = v. \]  

(4.66)

It therefore suffices to prove surjectivity of the linearized evaluation map to the fibers. But this is accomplished in the same way by taking for any \(i \in I\) the Hamiltonians \(H_i = \frac{1}{2}(\prod_{j \in I} \eta(r_j)) r_i^2\). Let \(\phi_{i,t}\) denote Hamiltonian flow of this Hamiltonian. Then

\[ d(Ev_{z_0}^\vec{v}(X_{H_j}, L_{X_H} J) = \left. \frac{d}{dt} \right|_{t=0} (Ev_{z_0}^\vec{v}(\phi_{i,t}^{-1}(f), \phi_{i,t}^*(J))) = \partial_{b_it}, \]  

(4.67)

where \(\partial_{b_it}\) is the generator of the circle action in the \(i\)th fiber and hence this proves surjectivity onto the fibers. \(\square\)

Proof of Lemma 4.16. This is an immediate consequence of Lemmas 4.15 and 4.18. \(\square\)

Proof of Lemma 4.17. This proof is a direct adaptation of [16, Lemma 6.6]; as in [16], the presence of the \(\ell\) jet condition forces one to work directly with distributions instead of the more typical argument of reducing to the \(W^{1,p}\) case where one can use the adjoint equation (as in [25; 54, Proposition 3.2.1]). As in [16, Lemma 6.6], standard functional analytic properties of \(E^{m-1,p'}_f\) (see [16]) reduce the proof of this lemma to verifying that \((\text{Im} F_0)^\perp \subset (E^{m-1,p'}_0)^\perp\) in the dual space \((E^{m-1,p'}_f)^*\) (where \(\perp\) denotes annihilator). Suppose that \(\Lambda \in (E^{m-1,p'}_f)^*\) is contained in \((\text{Im} F_0)^\perp\), that is, it vanishes on \(\text{Im}(F_0)\). Then elliptic regularity for distributions implies that the restriction of \(\Lambda\) to \(S^\ast := S \setminus z_0\) is represented by some smooth section \(\eta : S^\ast \to E\), for example,

\[ \Lambda(\phi) = \langle \phi, \eta \rangle_{L^2}, \]
with \( D^*_\eta = 0 \) and \( \langle Y \circ (df - X \otimes \gamma) \circ j, \eta \rangle = 0 \) for all \( Y \in T\tilde{J} \). To complete the proof (or rather reduce it to arguments in [16]) it suffices to show that \( \eta \) vanishes on \( S^* \); here we shall deviate slightly from the argument of [16] (which closely emulate the standard transversality argument for simple \( J \)-holomorphic curves c.f., [54, Proposition 3.2.1]), turning instead to the corresponding standard argument for solutions to Floer-type equations (compare [25]).

To prove \( \eta = 0 \), first let \( R(f) \) denote the set of points \( z \) which are regular, meaning that

\[
du_z - X_{u(z)} \otimes \gamma_z \neq 0. \tag{4.68}
\]

A standard argument (see [8, §8.2; 25, Theorem 4.3] for a version adapted to general Floer curves) establishes that regular points are open and dense in any open set \( U \subset S \) where \( \partial\gamma|_U = 0 \). For any regular point \( z \in R(f) \) with \( \eta(z) \neq 0 \) and \( f(z) \notin V \), one may use the usual argument of [25] to produce a \( Y \) in \( T\tilde{J} \) (in particular \( Y = 0 \) sufficiently near 0) such that \( \langle Y(f) \circ df(z') \circ j, \eta(z') \rangle \geq 0 \) and \( > 0 \) at \( z \), a contradiction to \( \eta(z) \neq 0 \) (note we require \( f(z) \notin V \) because our almost-complex structures are assumed to be of a restricted form within \( V \); of course such \( z \) always exists because the asymptotics of \( f \) lie outside \( V \), compare [8, Lemma 8.7]). This implies that \( \eta = 0 \) on a neighborhood in \( S^* \) of 0; hence \( \eta = 0 \) on all of \( S^* \) by unique continuation. \( \square \)

4.5. (In)dependence of various choices

We now investigate to what extent the \( PSS^+_{log} \) map depends on various choices.

**Lemma 4.19.** For any admissible \( \alpha_c t^\varphi \) and \( \lambda_1 \leq \lambda_2 \) we have an equality

\[
PSS^+_{log} (\lambda_2, \alpha_c t^\varphi) = c_{\lambda_1, \lambda_2} \circ PSS^+_{log} (\lambda_1, \alpha_c t^\varphi) \tag{4.69}
\]

on the level of cohomology.

**Proof.** The proof is by a degeneration of domain (as usual equipped with suitable Floer data) argument. We consider a non-negative, monotone non-increasing, function \( f(s) \) such that

\[
\begin{align*}
&f(s) = \lambda_1, \quad s \gg 0; \\
&f(s) = \lambda_2, \quad s \ll 0.
\end{align*}
\]

The 1-form

\[
\beta = f(s) \rho(s) dt,
\]

where \( \rho \) is the cutoff function described in (4.13) and in the lines above it, is subclosed. Now let \( q \in \mathbb{R} \) be a parameter and consider the 1-parameter family of subclosed 1-forms:

\[
\beta^q = \rho(s - q) f(s) dt.
\]

We will count solutions to the perturbed pseudo-holomorphic curve equation with input \( \alpha_c \) and output \( x_0 \). Denote the resulting parametrized moduli space (for varying \( q \in \mathbb{R} \)) by

\[
P(\nu, \alpha_c, x_0) = \prod_{q \in \mathbb{R}} P_q(\nu, \alpha_c, x_0). \tag{4.70}
\]

When \( |x_0| - |\alpha t^\varphi| = 0 \), \( \text{vdim}(P(\nu, \alpha_c, x_0)) = 1 \). The proof of Lemma 4.14 extends verbatim to exclude a priori a number of components from the Gromov–Floer compactification of the moduli space (4.70), such as Floer breaking along orbits in \( D \), and most sphere bubbles. The end result is that the moduli space (4.70) has a compactification whose boundary includes three types of components:

\[
\begin{align*}
&\text{the limit as } q \to \infty; \\
&q \to -\infty; \\
&\text{broken curves at finitely many } q \text{ values.}
\end{align*}
\]
The first limit corresponds to the composition $c_{\lambda_1, \lambda_2} \circ \text{PSS}^\lambda_{\log}$. The second limit corresponds to taking $\text{PSS}^\lambda_{\log}$. The third type of boundary component consists of two different strata:

$$\prod_{y, |y|-|\alpha|^2|=-1} \mathcal{P}(\vec{\alpha}, \alpha_c, y) \times \mathcal{M}(x_0, y)$$

(4.71)

$$\mathcal{P}(\vec{\alpha}_0, \text{GW}(\alpha_c), x_0)$$

(4.72)

(where by our hypothesis of admissibility this latter moduli space (4.72) is empty unless $\vec{\alpha}$ is primitive, that is, equal to $\vec{\alpha}_I$ for some $I$).

Consider the assignment $T$ of degree $-1$ given by counting solutions to (4.70) in the usual fashion, namely

$$T(\alpha_c t^\vec{v}) := \sum_{y, |y|-|\alpha|^2|=-1} \sum_{u \in \mathcal{P}(\vec{\alpha}, \alpha_c, y)} \mu_u,$$

(4.73)

where $\mu_u : \mathbb{R} \to \mathfrak{a}_y$ is the induced isomorphism on orientation lines given by a gluing analysis. Then, by studying the boundary strata of one-dimensional components of (4.70) and noting that the stratum (4.72) contributes zero in $CF^+_\lambda$, we obtain that $T$ defines a chain homotopy between $\text{PSS}^\lambda_{\log} + c_{\lambda_1, \lambda_2} \circ \text{PSS}^\lambda_{\log}$, for example, we have that

$$\text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v}) - c_{\lambda_1, \lambda_2} \circ \text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v}) = \partial_{CF^+_\lambda} \circ T(\alpha_c t^\vec{v}).$$

(4.74)

It follows from the lemma that for any such chain $\alpha_c t^\vec{v}$, we may obtain an element

$$\text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v}) := c_{\lambda, \infty} \circ \text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v}) \in SH^*_\lambda(X)$$

(4.75)

which is independent of $\lambda$.

**Lemma 4.20.** For any admissible class $\alpha t^\vec{v}$, and given two cycles $\alpha_c$ and $\alpha_c'$ representing $\alpha$, we have that

$$[\text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v})] = [\text{PSS}^\lambda_{\log}(\alpha_c' t^\vec{v})] \in HF^+_\lambda(X \subset M, H^\lambda_M).$$

(4.76)

**Proof.** Let $\alpha_0$ be a chain such that $\partial(\alpha_0) = \alpha_c - \alpha_c'$. Then after choosing complex structures generically, consider the moduli space of elements of (4.23) whose enhanced evaluation lies along $\alpha_0$, which we denote by $\mathcal{M}(\vec{\alpha}, \alpha_0, x_0)$. When this moduli space is one-dimensional, it admits a compactification with boundary (ignoring strata which contribute zero in the quotient complex $CF^+_\lambda$) $\mathcal{M}(\vec{\alpha}, \alpha_c, x_0), \mathcal{M}(\vec{\alpha}, \alpha_c', x_0)$ and

$$\bigsqcup_{y, |y|-|\alpha|^2|=-1} \mathcal{P}(\vec{\alpha}_0, y) \times \mathcal{M}(x_0, y).$$

(4.77)

Thus $\text{PSS}^\lambda_{\log}(\alpha_c t^\vec{v})$ and $\text{PSS}^\lambda_{\log}(\alpha_c' t^\vec{v})$ are cohomologous as claimed.

As a consequence of Lemma 4.20, the PSS construction gives rise to a well-defined map on cohomology

$$\text{PSS}^\lambda_{\log} : H^*_\log(M, D)_{[\lambda]} \rightarrow HF^*_\lambda(X \subset M; H^\lambda_M)$$

(4.78)
for \( \lambda' + \hbar < \lambda \). Furthermore, by Lemma 4.19, we have a commutative diagram:

\[
\begin{array}{ccc}
H^*_{\log}(M; D)^{ad} & \xrightarrow{\iota^\lambda_1, \lambda_2} & H^*_{\log}(M; D)^{ad} \\
\downarrow \text{PSS}^{\lambda_1, \lambda_2} & & \downarrow \text{PSS}^{\lambda_1, \lambda_2} \\
HF_+^*(X \subset M; H^*_M) & \xrightarrow{\varphi^\lambda_1, \lambda_2} & HF_+^*(X \subset M; H^*_M)
\end{array}
\]

(4.79)

As a consequence, we obtain a well-defined map

\[
\text{PSS}^+_\log : H^*_{\log}(M; D)^{ad} \to SH^+_*(X).
\]

(4.80)

A similar further argument shows that this map is indeed canonical, that is, independent of choices made (of complex structures and Hamiltonians in the class specified).

4.6. Lifting to symplectic cohomology

We will need one more choice to define classes in symplectic cohomology:

**Definition 4.21.** Fix \( \lambda \). Given an admissible cocycle \( \alpha_c t^\varphi \) with \( \lambda \geq \sum \kappa_i v_i + \hbar \) corresponding to a primitive vector, we say that a pseudo-manifold with boundary \( Z_b \) is a bounding cochain for \( \alpha_c t^\varphi \) if

\[
\partial_{CF} \circ \text{PSS}_\log^\lambda(\alpha_c t^\varphi) - \partial_{CF} \circ \text{PSS}_\log^\lambda(Z_b) = 0.
\]

(4.81)

We formally extend this definition to non-primitive \( \vec{v} \) by only allowing \( Z_b = 0 \). We denote such a pair by \( (\alpha_c t^\varphi, Z_b) \).

We suppress \( \lambda \) in the notation that follows.

**Lemma 4.22.** If \( \alpha_c t^\varphi \) is an admissible cocycle with vanishing obstruction class (3.38), then there exists \( Z_b \) such that (4.81) holds.

**Proof.** Assume first that \( \vec{v} \) is not primitive, or more generally that \( \alpha t^\varphi \) is tautologically admissible. Given \( \alpha_c t^\varphi \), consider a given \( x_0 \) such that \( \text{vdim}(\mathcal{M}(\vec{v}, \alpha_c, x_0)) = 1 \). The boundary of the compactification \( \overline{\mathcal{M}(\vec{v}, \alpha_c, x_0)} \) has one component

\[
\bigcup_{|x_0| = |x'| + 1} \mathcal{M}(\vec{v}, \alpha_c, x') \times \mathcal{M}(x_0, x')
\]

(4.82)

which represents the coefficient of \( |\varphi_{x_0}| \) in the composition \( \partial_{CF} \circ \text{PSS}(\alpha_c t^{\vec{v}}) \); in particular \( \partial_{CF} \circ \text{PSS}(\alpha_c t^{\vec{v}}) = 0 \), and \( \alpha_c t^{\vec{v}} \) satisfies (4.81) with \( Z_b = 0 \) as desired. In the primitive non-tautologically admissible case, choose generic null-bordism \( Z_b \) for \( GW_{\vec{v}}(\alpha_c) \). There is one extra component to the boundary of \( \mathcal{M}(\vec{v}, \alpha_c, x_0) \) in this case, given by

\[
GW_{\vec{v}}(\alpha_c) \times_{ev} \mathcal{M}(\vec{v}_0, x_0).
\]

(4.83)

In this case, we have that

\[
\partial_{CF}(\text{PSS}_\log^\lambda(\alpha_c t^\varphi)) = \text{PSS}_\log^\lambda(GW_{\vec{v}}(\alpha_c)).
\]

(4.84)

In view of (4.22), this can be canceled out by \( \partial_{CF} \circ \text{PSS}_\log^\lambda(Z_b) \) and thus (4.81) holds. \( \square \)

For any pair \( (\alpha_c t^\varphi, Z_b) \), define \( \text{PSS}_\log^\lambda(\alpha_c t^\varphi, Z_b) \in HF^*(X \subset M; H^*_M) \) via the formula

\[
\text{PSS}_\log^\lambda(\alpha_c t^\varphi, Z_b) = [\text{PSS}_\log^\lambda(\alpha_c t^\varphi) - \text{PSS}_\log^\lambda(Z_b)].
\]

(4.85)
For the rest of this subsection, we exclusively consider admissible cocycles \( \alpha_c t^{\varphi} \) for which we may take \( Z_0 = 0 \) for generic \( J \in \mathcal{J}_S(V) \). This applies for all tautologically admissible classes, but also in other situations. For example:

**Definition 4.23.** After choosing a triangulation \( T \) of \( \hat{S}_I \), we can represent the fundamental class \( \alpha \) on \( \hat{S}_I \) by a cycle given by taking the sum of all \( 2n - |I| \)-dimensional simplices, \( \alpha_c \). We refer to this as the fundamental cycle associated to \( T \).

**Lemma 4.24.** Fix a triangulation \( T \) of \( \hat{S}_I \). If \( \alpha = 1 \in H^0(\hat{S}_I) \), and \( \alpha_c \) is the fundamental cycle associated to \( T \), then equation (4.81) holds with \( Z_0 = 0 \).

**Proof.** As in the proof of Lemma 3.33, the evaluation \( ev_{\infty,*} \) map factors through a lower dimensional chain \( [\mathcal{M}_{0,2}(M, D, \tilde{v}_I)^c / S^1] \). For generic \( J \), the PSS moduli space misses this cycle completely, implying that

\[
PSS_{\varphi}(GW_{\varphi}(\alpha_c)) = 0
\]  

at the chain level. \( \square \)

In view of Lemma 4.24, we will always set \( Z_0 = 0 \) and will drop it from the notation when working with classes of the form \( \alpha_c t^{\varphi} \), with \( \alpha_c \) a fundamental cycle. We now state analogs of Lemmas 4.19 and 4.20.

**Lemma 4.25.** Fix an admissible cocycle \( \alpha_c t^{\varphi} \) for which we may take \( Z_0 = 0 \) for generic \( J \in \mathcal{J}_S(V) \). For any \( \lambda_1 \leq \lambda_2 \), we have an equality

\[
PSS^\lambda_{\varphi}(\alpha_c t^{\varphi}) = c_{\lambda_1, \lambda_2} \circ PSS^\lambda_{\varphi}(\alpha_c t^{\varphi}).
\]  

**Proof.** This is proven exactly as in Lemma 4.19. \( \square \)

It follows from the lemma that for any such cocycle \( \alpha_c t^{\varphi} \), we may obtain an element

\[
PSS^\lambda_{\varphi}(\alpha_c t^{\varphi}) := c_{\lambda, \infty} \circ PSS^\lambda_{\varphi}(\alpha_c t^{\varphi}) \in SH^*(X)
\]  

which is independent of \( \lambda \).

**Remark 4.26.** Let \((M, D)\) be any pair where all \( \kappa_i \) are equal to some \( \kappa > 0 \). For every component \( D_i \), consider a fundamental chain \( \alpha_i \) on the corresponding circle bundle \( \hat{S}_i \). We leave it as an exercise in this case to verify that \( \alpha_i t^{\varphi_i} \) is admissible for each \( i \) in the sense of Definition 3.23 (second bullet point): for instance to verify Assumption (A1), suppose that \( A \in H_2(M)_\omega \) satisfied \( A = A_1 + A_2 \) with \( A_1, A_2 \in H_2(M)_\omega \). Then \( A_1 \cdot D + A_2 \cdot D = \tilde{v}_i \) so \( \sum_j (A_1 \cdot D_j + A_2 \cdot D_j) = 1 \); on the other hand since \( [\omega] = \kappa (\sum_j [D_j]) \), we have \( \omega(A_1) = \kappa \sum_j (A_1 \cdot D_j) > 0 \) and similarly \( \omega(A_2) = \kappa \sum_j (A_2 \cdot D_j) > 0 \), which is impossible. Now Lemma 4.24 further implies that \( \alpha_c t^{\varphi} \) is an admissible cocycle with \( Z_0 = 0 \), hence we obtain via (4.88) canonical classes \( \theta_i = PSS^\lambda_{\varphi}(\alpha_i t^{\varphi_i}) \in SH^*(X) \). It is the classes \( \theta_{BS} = \sum_i \theta_i \) which are constructed in a forthcoming paper by Borman and Sheridan.

**Lemma 4.27.** For any tautologically admissible class \( \alpha t^{\varphi} \), and given two cycles \( \alpha_c \) and \( \alpha_c' \) representing \( \alpha \), we have that

\[
[PSS^\lambda_{\varphi}(\alpha_c t^{\varphi})] = [PSS^\lambda_{\varphi}(\alpha_c' t^{\varphi})] \in HF^*(X \subset M, H^\lambda_M).
\]  

**Proof.** This is proven exactly as in Lemma 4.20. \( \square \)
For topological pairs \((M,D)\), Lemma 4.27 implies that there is a well-defined map:

\[
PSS^\lambda_{\log} : H^*_\log(M,D)_{[\lambda]} \to HF^*(X \subset M; H^*_M)
\]  
(4.90)

for \(\lambda' + \hbar < \lambda\). The natural analog of the diagram (4.79) (without ‘+’) also holds, giving rise to a map:

\[
PSS^\lambda_{\log} : H^*_\log(M,D) \to SH^*(X).
\]  
(4.91)

4.7. Interplay with the BV operator

In this subsection, we show that the log PSS map intertwines the BV operator \(\Delta\) on symplectic cohomology with the map \(\Delta\) on \(H^*_\log(M,D)\) defined in (3.23). Rather than prove a maximally general statement, we content ourself with proving such compatibility for log cohomology classes which have primitive multiplicity and are representable by (pushforwards of) fundamental chains (along proper maps), or more generally pseudo-cycles (see Remark 4.32); variations on these arguments would handle more general log cohomology classes.

Let \(S = \mathbb{C}P^1 \setminus \{0\}\), thought of as a punctured sphere, with a distinguished marked point \(z_0 = \{\infty\}\) and a cylindrical end around \(\{0\}\) as before. At \(z_0\) we consider an \(S^1\)-family of domains by adding a unit-tangent vector (or equivalently, a real tangent ray) at \(z_0\), \(\mathcal{T}^1 \in S_{z_0} S\) which we allow to take on arbitrary values (previously a single such vector was fixed and used in our definition of the enhanced evaluation map; see discussion above (4.28)). We may, in complete analogy with log PSS, define the moduli spaces \(\mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)\) of maps from this varying family of domains with intersection multiplicity, enhanced evaluation constraint (with respect to \(\mathcal{T}^1\)), and Floer asymptotic conditions specified by \(\vec{v}, \alpha_c\), and \(x_0\) respectively, satisfying Floer’s equation (for some generic \(J\)). We can (also as before) define an operation \(PSS^\lambda_{\log,S^1}\) by the formula

\[
PSS^\lambda_{\log,S^1}(\alpha_c t^\varphi) = \sum_{x_0} \sum_{u \in \mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)} \mu_u \in CF^*(X \subset M, H^*_M),
\]  
(4.92)

where \(\mu_u : \mathbb{R} \to \sigma_{x_0}\) is the isomorphism of orientation lines (and their \(k\)-normalizations) induced by rigid elements \(u\) (an analog of Lemma 4.14 implies that the above count is well defined for generic \(J\)). Furthermore,

**Lemma 4.28.** If \([\alpha_c t^\varphi] \in H^*_\log(M,D)\) is an admissible class, then the element \(PSS^\lambda_{\log,S^1}(\alpha_c t^\varphi)\) is a cocycle for the Floer differential, that is, \(\partial_{CF}(PSS^\lambda_{\log,S^1}(\alpha_c t^\varphi)) = 0\).

**Proof.** The same analysis performed in Lemma 4.14 tells us that, in the case \(|x_0| - |\alpha t^\varphi| = 0\), the Gromov–Floer compactification \(\mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)\) is a compact 1-manifold-with-boundary

\[
\partial \mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0) = \partial \mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)_F := \bigcup_{x' | x_0 - |x'| = 1} \mathcal{M}_{S^1}(\vec{v}, \alpha_c, x') \times \mathcal{M}(x_0, x');
\]  
(4.93)

in particular while as before there could be a sphere bubble appearing the limit of a sequence of maps if \(\vec{v} = \vec{v}_j\), such a bubble now appears in the (more usual) codimension 2 and hence does not appear in (4.93) (note that the last paragraph in the proof of Lemma 4.14, which explained why sphere bubbling occurred in codimension 1 in that case, no longer applies as we allow the tangent ray at \(z_0\) to vary in \(\mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)\)). The desired statement then follows as usual by counting elements of \(\partial \mathcal{M}_{S^1}(\vec{v}, \alpha_c, x_0)\).
Note that \( \alpha_c t^{\bar{r}} \) is not necessarily tautologically admissible in the above Lemma. For simplicity, we now consider a special class of such \( \alpha_c t^{\bar{r}} \), those with \( \alpha_c \) represented by a submanifold and with primitive multiplicity \( \bar{v} := \vec{v} \). Let \( \vec{v}_I \) be a primitive vector and

\[
\Gamma_{\vec{v}_I} : S^1 \times \hat{S}_I \to \hat{S}_I
\]

the corresponding \( S^1 \) action (see (3.22)). Let \( P \) be an oriented (possibly non-compact) manifold and \( p : P \to \hat{S}_I \) be a proper map. Choose a triangulation on \( P \) and let \( \alpha_c \) denote the pushforward of the fundamental chain along the map \( p \), with \( \alpha = [\alpha_c] \) the corresponding cohomology class. We then have a map

\[
\Gamma_p : S^1 \times P \to \hat{S}_I
\]

which is defined as the composition of maps \( \Gamma_p := \Gamma_{\vec{v}} \circ (\text{id} \times p) \). Let \( \sigma_c \) denote the pushforward of the fundamental chain (defined, for example, as the Eilenberg-Zilber product of the fundamental chain on \( S^1 \) and the fundamental chain on \( P \)) under \( \Gamma_p \) and let \( \sigma := [\sigma_c] \) the corresponding cohomology class.

**Lemma 4.29.** Let \( P, \alpha \) be as above and assume that \( \alpha t^{\bar{r}} \) is an admissible class in \( H^*_S(M,D) \). Then, for any bounding cochain \( Z_b \) for \( \alpha_c t^{\bar{r}} \) as in §4.6, there is a cohomological equality

\[
[\Delta \circ \text{PSS}^\lambda_{S_\log}(\alpha_c t^{\bar{r}}, Z_b)] = [\text{PSS}^\lambda_{S_\log, \langle \sigma \rangle}(\alpha_c t^{\bar{r}})].
\]

**Proof.** The proof involves studying a version of the log PSS moduli space for a family of domains which, along with their Floer data, are parametrized by \((r,q) \in S^1 \times [0,\infty)\). To simplify the notation, we will denote \( H^*_S(M) \) by \( H_t \) (as usual \( J_t \) will be an almost-complex structure needed to define the Floer complex). We fix an extension of \((H_t, J_t)\) to a family of surface dependent Hamiltonians/almost-complex structures \((H_{s,t,r}, J_{s,t,r})\) used to define the BV operator as in (2.31). Finally, over \( S \), we fix the subclosed 1-form \( \beta \) as in (4.14) as well as \( J_S \in \mathcal{J}_S(V) \).

For each point \( \nu = (r,q) \in S^1 \times [0,\infty) \), we associate the domain \( S = \mathbb{C}P^1 \setminus \{0\} \), thought of as a punctured sphere with negative end along with

- a distinguished marked point \( z_0 \) at \( \infty \) along with a distinguished tangent direction at \( z_0 \) pointing in the positive real direction;
- a surface dependent almost-complex structure \( J_\nu \) which agrees with some surface independent \( J_0 \) in a neighborhood of \( z_0 \) and agrees with \( J_{t-r} \) along the cylindrical end;
- a perturbation 1-form \( K_\nu \) which vanishes in a neighborhood of \( z_0 \) and which along the cylindrical end, is equal, in some neighborhood (which will depend on \( q \)) of \(-\infty\), to

\[
K_\nu = X_{H(t-r,x)} \otimes dt.
\]

Abbreviating \( X_r = X_{H_{s,t,r}} \) and \( J_r = J_{s,t,r} \), we assume that the \( \nu \)-dependent choice of Floer data satisfies the following consistency conditions.

- For \( q \gg 0 \), the datum \((K_\nu, J_\nu)\) is obtained by taking finite connected sum (by an amount depending on \( q \) and limiting as \( q \to \infty \) to a nodal connect sum) of the Floer data \((X_{H_t} \otimes \beta, J_S)\) and \((X_r \otimes dt, J_r)\) along the cylindrical ends. That is, for \( q \gg 0 \), the Floer data coincides at a point \((s,t)\) in (cylindrical coordinates) with \((X_{s+N+q,t,r} \otimes dt, J_{s+N+q,t,r})\) for some sufficiently large fixed \( N \) (chosen so that \( s \geq N \) lies in the region where \((H_{s,t,r}, J_{s,t,r})\) coincides with \((H_t, J_t))\).

---

1Strictly speaking, to make the discussion compatible with the general framework outlined in §2, we should make the negative cylindrical end \( \epsilon_r \) around \( z = 0 \) depend on \( r \in S^1 \). However, this would make the notation more cumbersome.
Over \( q = 0 \), the datum on the cylindrical end agrees with the \( r \)-clockwise rotation of the data \((H^\lambda_M, J_S)\), that is, \((K_\nu, J_\nu) = (X_{H^\lambda_M(t-r)} \otimes \beta, J_S(s, t-r))\).

For any \( \vec{v} \) and \( x_0 \), let \( \mathcal{P}^{S^1}(\vec{v}, x_0) \) denote the moduli space of maps \( u : S \to M \) which solve the equation

\[
(du - X_{K_\nu})^0 = 0
\]

and which satisfy (4.27) as well as the asymptotic condition

\[
\lim_{s \to -\infty} u(\epsilon(s, t)) = x_0^{(r)}.
\]

(For the present argument, we will only need to consider \( \vec{v} = \vec{v}_I \) or \( \vec{v} = \vec{v}_\emptyset \)) Form the moduli spaces

\[
\mathcal{P}^{S^1}(\vec{v}_I, \alpha_c, x_0) := \mathcal{P}^{S^1}(\vec{v}_I, x_0) \times_{\text{Ev}_{z_0}} \alpha_c.
\]

(4.100)

\[
\mathcal{P}^{S^1}(\vec{v}_\emptyset, Z_b, x_0) := \mathcal{P}^{S^1}(\vec{v}_\emptyset, x_0) \times_{\text{ev}} Z_b.
\]

(4.101)

and suppose \( |\alpha_c t^{\vec{v}_I} - |x_0| = 1 \), so that both moduli spaces above (recalling that \( Z_b \) is a bounding cochain for \( \alpha_c t^{\vec{v}_I} \)) are one-dimensional for generic choices of Floer data satisfying the above conditions (by a standard variation of the arguments in §4.4). The key step in the proof is to analyze the boundary strata of their Gromov compactifications; we will summarize the result of appealing to Gromov compactness as well as the relevant variant of Lemma 4.14.

First, in the compactification of (4.100), four types of boundary strata occur:

**Type I:** In the limit as \( q \to \infty \), when solutions converge to elements of the fiber product

\[
\bigcup_{|x_1| - |x_0| = 1} \mathcal{M}(\vec{v}_I, \alpha_c, x_1) \times \mathcal{M}_{S^1}(x_0, x_1),
\]

(4.102)

where \( \mathcal{M}_{S^1}(x_0, x_1) \) denotes the space of solutions to (2.31). The signed count of elements in this fiber product defines the composition \( \Delta \circ \text{PSS}^\lambda_{S^1}(\alpha_c t^{\vec{v}_I}) \).

**Type II:** The moduli space restricted to \( q = 0 \) appears as part of the boundary, and can be identified with \( \mathcal{M}_{S^1}(\vec{v}_I, \alpha_c, x_0) \). To see this, note that in the definition of \( \mathcal{M}_{S^1}(\vec{v}_I, \alpha_c, x_0) \), if we want to fix the vector \( \vec{r}_0 \) to point in the positive real direction, we can rotate the surface \( S \) by \( (s, t) \to (s, t - r) \). This is equivalent to solving Floer’s equation for the rotated inhomogeneous term \((X_{H^\lambda_M(t-r)} \otimes \beta)\) with rotated almost-complex structure \( J_S(s, t-r) \) such that the solutions are asymptotic to the output orbit \( x_0^{(r)} \). This boundary accounts for (on the chain level) the right-hand side of (4.96).

**Type III:** This is the stratum where sphere bubbles form as in the proof of Lemma 4.14. It is given by

\[
\mathcal{P}^{S^1}(\vec{v}_\emptyset, \text{GW}_{\vec{v}_I}(\alpha_c), x_0) := \mathcal{P}^{S^1}(\vec{v}_\emptyset, x_0) \times_{\text{ev}} \text{GW}_{\vec{v}_I}(\alpha_c).
\]

(4.103)

**Type IV:** Finally, we have ordinary cylindrical breaking at intermediate values of \( (r, q) \) as well, which gives rise to

\[
\bigcup_{|x_0| - |x_1| = 1} \mathcal{P}^{S^1}(\vec{v}_I, \alpha_c, x_1) \times \mathcal{M}_r(x_0, x_1) \cong \bigcup_{|x_0| - |x_1| = 1} \mathcal{P}^{S^1}(\vec{v}_I, \alpha_c, x_1) \times \mathcal{M}(x_0, x_1),
\]

(4.104)

where \( \mathcal{M}_r(x_0, x_1) \) is the moduli space of twisted Floer trajectories introduced just before (2.32) and \( \mathcal{P}^{S^1}(\vec{v}_I, \alpha_c, x_1) \) denotes the moduli space over some fixed \( r \in S^1 \). Since \( \mathcal{M}_r(x_0, x_1) \) can be canonically identified with \( \mathcal{M}(x_0, x_1) \), we obtain the second identification, and conclude that this contribution is exact.
Next, the one-dimensional moduli spaces (4.101) again have (for generic choices of data) a compactification involving strata over \( q = \infty \) which give strata of the same type as (Type I) above as well as strata with cylindrical breaking as in (Type IV) above.\(^1\) There is also a third boundary stratum isomorphic to (4.103) (though it is important to note that it arises from curves whose evaluation at \( z_0 \) lies on the boundary of \( Z_b \), which by definition is a null-bordism for \( GW_{\vec{v}_1}(\alpha_c) \), rather than from sphere bubbling).

With this analysis in place, we complete the proof of (4.96). As in the proof of Lemma 4.19, we may use the count of rigid elements of \( \mathcal{P}^{S^1}(\vec{v}_I, \alpha_c, y) \) and \( \mathcal{P}^{S^1}(\vec{v}_0, Z_b, y) \) to define an operator \( T^{S^1} \) of degree \(-2\) via the formulae:

\[
T^{S^1}(\alpha_c t^{\vec{v}_i}) := \sum_{y,|y|=-|\alpha_c t^{\vec{v}_i}|} \sum_{u \in \mathcal{P}^{S^1}(\vec{v},\alpha_c,y)} \mu_u \tag{4.105}
\]

\[
T^{S^1}(Z_b) := \sum_{y,|y|=-|Z_b|} \sum_{u \in \mathcal{P}^{S^1}(\vec{v}_0,Z_b,y)} \mu_u. \tag{4.106}
\]

The above analysis of the boundary of the compactified one-dimensional moduli spaces then implies that

\[
\Delta \circ \text{PSS}_{\text{log}}^{\lambda}(\alpha_c t^{\vec{v}_i}) - \Delta \circ \text{PSS}_{\text{log}}(Z_b) - \text{PSS}_{\text{log},S^1}^{\lambda}(\alpha_c t^{\vec{v}_i}) = \partial_{CF} \circ (T^{S^1}(\alpha_c t^{\vec{v}_i}) - T^{S^1}(Z_b)). \tag{4.107}
\]

In deducing this equation, we have used the fact that the contributions of the two boundary strata isomorphic to (4.103) cancel each other out. Equation (4.107) descends to (4.96) on the level of cohomology, completing the proof.

**Lemma 4.30.** Let \( p : P \to \tilde{S}_I \) be as in Lemma 4.29. We have an equality

\[
\text{PSS}_{\text{log},S^1}^{\lambda}(\alpha_c t^{\vec{v}_i}) = \text{PSS}_{\text{log}}^{\lambda}(\sigma_c t^{\vec{v}_i}). \tag{4.108}
\]

**Proof.** By definition, the moduli space \( \mathcal{M}(\vec{v}_I, \sigma_c, x_0) \) is a subspace of \( \mathcal{M}(\vec{v}_I, x_0) \times (S^1 \times P) \). Denoting points in this product by \((u, (\theta, x))\), then \( \mathcal{M}(\vec{v}_I, \sigma_c, x_0) \) is the subset of maps \( u \in \mathcal{M}(\vec{v}_I, x_0) \) where \( \text{Ev}_{z_0}^{\vec{v}_I} = \theta \cdot p(x) \). Meanwhile, \( \mathcal{M}_{S^1}(\vec{v}_I, \alpha_c, x_0) \) is similarly a subspace of \( (\mathcal{M}(\vec{v}_I, x_0) \times S_{z_0} \mathbb{C}) \times P \). We identify \( S_{z_0} \mathbb{C} \cong S^1 \) by viewing \( r_{z_0}^\theta \) as the rotation of the positive real ray by some \( \theta^{-1} \) for \( \theta \in S^1 \). If the enhanced evaluation with respect to \( r_{z_0}^\theta \) is \( p(x) \), then the enhanced evaluation with respect to the positive real ray is \( \theta \cdot p(x) \) (and vice-versa). Thus, the map \( ((u, \theta), x) \mapsto (u, (\theta, x)) \) defines a canonical bijection of moduli spaces of curves used to define the left and right-hand sides of equation (4.108). Moreover, it is easy to check that the conditions for these moduli spaces to be cut out transversely are equivalent and that the bijection preserves (relative) orientations.

Altogether, Lemmas 4.29 and 4.30 along with the definition (3.23) of the BV operator on log cohomology immediately imply

**Corollary 4.31.** Let \( \alpha t^{\vec{v}_i} \) be a primitive admissible class with \( \alpha = [\alpha_c] \) representable by the fundamental chain \( p_*[P] \) associated to a proper (not necessarily compact) map \( P \to \tilde{S}_I \), and let \( Z_b \) be a bounding cochain for \( \alpha_c t^{\vec{v}_i} \). Then, on the level of cohomology, there is an equality:

\[
[\text{PSS}_{\text{log}}^{\lambda}(\Delta(\alpha_c t^{\vec{v}_i}))] = [\Delta \circ \text{PSS}_{\text{log}}^{\lambda}(\alpha_c t^{\vec{v}_i}, Z_b)]. \tag{4.109}
\]

\(^1\)Unlike the above case, the boundary over \( q = 0 \) is generically empty because \( \mathcal{M}(\vec{v}_0, Z_b, x_0) \) is empty; this is one explanation for why there is no bounding cochain on the right-hand side of (4.96) even though there is one on the left.
Remark 4.32. Corollary 4.31 (and in particular Lemmas 4.29 and 4.30) immediately generalize (with the same proof) to the case that $\alpha$ is representable by a pseudo-cycle rather than the fundamental chain of a smooth map. When our divisor $D = D$ consists of a single smooth component, $SD$ is compact and we can represent every cohomology class by a pseudo-cycle, allowing us to apply this variant of Corollary 4.31 to all primitive log cohomology classes. We will make use of this in the proof of Theorem 6.2.

5. Quasi-dilations in string topology

For simplicity, we assume† all manifolds $Q$ appearing in this section are connected, closed, oriented and Spin. (5.1)

The main goal of this section is to explain how the existence of quasi-dilations in the string topology of such $Q$ imply topological constraints on the manifold, particularly in dimension 3.

5.1. Review of string topology

Throughout this subsection let $n = \dim(Q)$ and $k$ will be an arbitrary commutative ring. For any such $Q$, Chas and Sullivan [14, 17] constructed a BV algebra structure on the homology of the free loop space of $Q$, $\mathbb{H}^*(LQ,k) = H_{n-*}(LQ,k)$, called the loop homology algebra (over $k$). The basic manifestations of this structure most relevant to us are

- $H^*(LQ,k)$ has a unit element $1 \in H^0(LQ) \cong H_n(LQ,k)$ corresponding to a choice of fundamental class $[Q]$ via the inclusion of constant loops $Q \rightarrow LQ$; and
- $H^*(LQ,k)$ has a BV-operator, which is especially easy to describe. Denote the canonical rotation action on the loop space by
  $$\Gamma : S^1 \times LQ \rightarrow LQ.$$ (5.2)

The BV-operator is given by
  $$\Delta(\alpha) = \Gamma_*(\epsilon \otimes \alpha),$$ (5.3)

where $\epsilon \in H_1(S^1)$ is a fixed choice of fundamental class.

There is an isomorphism of groups [1, 6, 69, 84]
  $$\mathcal{AS} : SH^*(T^*Q,k) \rightarrow \mathbb{H}^*(LQ,k)$$ (5.4)

which intertwines algebra structures [2]. Although it is not explicitly mentioned, the particular isomorphism (5.4) given in [1] can be straightforwardly shown to intertwine BV operators as well, hence is an isomorphism of BV-algebras (alternatively, see the more recent [6]). Fixing the base-point $1 \in S^1$, evaluating a loop at 1 produces an evaluation morphism
  $$ev : LQ \rightarrow Q.$$ (5.5)

Let $\mathbb{H}^*(Q)$ denote the intersection ring of $Q$ (meaning the homology $H_{n-*}(Q,k)$ equipped with the intersection product). The induced map
  $$ev_* : \mathbb{H}^*(LQ,k) \rightarrow \mathbb{H}^*(Q,k)$$

is a unital ring homomorphism.

Fix a BV-algebra $(A,\Delta)$ over $k$. Let $\alpha \in A^0$ and $\Psi \in A^1$ be two elements satisfying
  $$\Delta(\Psi) = \alpha.$$ (5.5)
We say that \((\Psi, \alpha)\) is
- a *quasi-dilation* if \(\alpha\) is a unit with respect to the algebra structure in \(A\);
- a *dilation* if \(\alpha = 1\).

Whenever the string topology BV-algebra \(\mathbb{H}^*(LQ, \mathbb{k})\) of a manifold \(Q\) carries a pair \((\Psi, \alpha)\) which define a (quasi-) dilation, we will say that \(Q\) admits a (quasi-) dilation over \(\mathbb{k}\).

**Remark 5.1.** The notion of quasi-dilation is due to Seidel [75], who introduced it in the equivalent form
\[
\Delta(\alpha \Psi S) = \alpha. \tag{5.6}
\]
Equation (5.6), which is related to the dilation condition (5.5) by replacing \(\alpha \Psi S\) with \(\Psi\) (which is an invertible operation), has a natural geometric interpretation. Let \(Z\) be a smooth algebraic variety equipped with an invertible function \(\alpha\), a holomorphic volume form \(\omega\), and a vector field \(\Psi S\). The condition that the vector field \(\Psi S\) dilates \(\alpha \omega\) may be written as
\[
L(\Psi S)(\alpha \omega) = \alpha \omega.
\]
Under a standard dictionary between commutative and non-commutative geometry (see, for example, [75, Lecture 19]), this translates to the equation (5.6).

We will also consider one further definition, which is specific to the string topology setup, but useful for formulating Lemmas 5.13 and 5.15 and in the proof of Corollary 5.18.

**Definition 5.2.** For any (closed, oriented, spin) manifold \(Q\), let \(\alpha \in \mathbb{H}^0(LQ, \mathbb{k})\), \(\Psi \in \mathbb{H}^1(LQ, \mathbb{k})\) be two elements satisfying (5.5). We say that \((\Psi, \alpha)\) is a *pseudo-dilation* if \(\text{ev}_*(\alpha)\) lies in \(\mathbb{k} \times \) inside \(\mathbb{H}^0(Q, \mathbb{k}) = \mathbb{k}\). Similar to the (quasi-) dilation case, whenever such \((\Psi, \alpha)\) exist, we will say that \(Q\) admits a pseudo-dilation (over \(\mathbb{k}\)).

Evidently if \((\Psi, \alpha)\) is a quasi-dilation, then it is a pseudo-dilation.

It will be helpful at several points later on to discuss the case where \(Q\) is a \(K(G, 1)\) space in slightly more detail. Let \(C(G)\) denote the set of conjugacy classes of \(G\) and for any element \(g \in G\), let \(C_g\) denote the centralizer of \(g\). Following, for example, [14, §10.1], we have that, up to homotopy, the loop space decomposes as a disjoint union of connected components
\[
LQ \cong \bigsqcup_{[g] \in C(G)} K(C_g, 1), \tag{5.7}
\]
where \(C_g\) denotes the centralizer of a representative \(g\) of the given conjugacy class \([g]\). Note that each of these components \(K(C_g, 1)\) is a covering space of the original \(Q\) and thus we immediately see that \(H_*(LQ)\) vanishes for \(* > n\) or equivalently that \(\mathbb{H}^*(LQ)\) is concentrated in non-negative degree. The decomposition (5.7) also easily yields the following well-known description of \(\mathbb{H}^0(LQ)\) (which is additively \(\mathbb{H}^n(LQ)\)):

**Lemma 5.3.** There is a ring isomorphism \(\mathbb{H}^0(LQ) \cong \mathbb{Z}_k(G)\), the center of the group algebra \(k[G]\).

**Proof sketch.** \(\mathbb{Z}_k(G)\) can be presented more explicitly as follows: for any group \(G\), and any commutative ring \(k\), the center \(\mathbb{Z}_k(G)\) has a free \(k\)-basis \(\{\sum_{g \in C} g\}\) indexed by finite conjugacy classes \(C\) of \(G\). Similarly, a given connected component of the loop space has homological dimension \(n\) precisely when, using the same notation as in (5.7), \(C_g\) has finite index in \(G\) (so that \(K(C_g, 1)\) is a *finite cover* of \(K(G, 1)\)). Hence, \(\mathbb{H}^0(LQ)\) also has a canonical basis indexed
by finite conjugacy classes. It is not difficult to check that this isomorphism preserves ring structures on both sides.

We also have the following useful observation of Seidel–Solomon [78, Example 6.2]:

**Lemma 5.4.** Suppose $Q$ is a $K(G,1)$ space. Then $Q$ does not admit a dilation over any $k$.

**Proof.** The BV operator respects the decomposition of the loop space into connected components. So, for the purpose ruling out dilations it suffices to analyze the component of contractible loops which, as can be seen from (5.7), retracts onto the space of constant loops. The BV-operator therefore acts trivially on homology classes arising from this component and there are no dilations.

Let us determine which (as usual closed, oriented) surfaces admit quasi-dilations. It follows immediately from the main calculation in [58] that $H^*(LS^2, \mathbb{Z})$ admits a quasi-dilation over $\mathbb{Z}$ which becomes a dilation after tensoring with any field $k$ of characteristic not equal to 2. The remaining surfaces are aspherical and hence can be analyzed using (5.7) (together with the related Lemmas 5.3 and 5.4). When the genus $g = 1$, $H^*(LT^2, \mathbb{Z})$ admits a quasi-dilation as can be seen by explicit computation using (5.7); $E_k(G) = k[G]$, which is isomorphic to a Laurent polynomial ring in two variables and one may take $\alpha$ to be one of the generators. On the other hand if $g \geq 2$, it is well known that the center of the group algebra $k[\pi_1(\Sigma_g)]$ is trivial (see, for example, [18, p. 564]) so $H^0(L\Sigma_g) = k$. It follows that if $g \geq 2$, there are no quasi-dilations in $H^*(L\Sigma_g, k)$, because there are no dilations.

Our description of 2-manifolds admitting quasi-dilations uses the classification of surfaces. To obtain a similar description of 3-manifolds admitting quasi-dilations, we need to import some tools from 3-manifold topology. The next subsection recalls the necessary background and §5.3 carries out the classification.

5.2. Some 3-manifold topology

We begin with the following definition:

**Definition 5.5.** We say that a closed, oriented manifold $Q$ is dominated by a closed oriented manifold $\hat{Q}$ if there is a non-zero degree map $f: \hat{Q} \to Q$. We say that $Q$ is 1-dominated if the map $f$ has degree 1.

In this subsection, we will study the following question:

**Question 5.6.** Which 3-manifolds are dominated (respectively, 1-dominated) by products $S^1 \times \hat{B}$ where $\hat{B}$ is an oriented Riemann surface?

Let us first recall the notion of a Seifert manifold, which will arise at several points in our discussion. Let $Q_f$ denote the mapping torus of the rotation $f: D^2 \cong D^2$ by $2\pi p/q$ with $p/q \in \mathbb{Q}$ and $p,q$ relatively prime. $Q_f$ is homeomorphic to $S^1 \times D^2$ and is decomposed into disjoint circles: a special fiber which is the image in the quotient space $[0,1] \times \{0\}$ and the regular fibers which are the images of the union of $q$ segments of the form $[0,1] \times \{x_i\}$ where the $x_i \in D^2$ form an orbit under rotation. A Seifert fibering of a manifold $Q$ is a decomposition of $Q$ into disjoint circles (‘fibers’) such that each fiber has a neighborhood which is fiber-preserving diffeomorphic to a neighborhood of a fiber in one of the above standard fiberings of $S^1 \times D^2$ (for some choice of $p/q$).
A Seifert manifold $Q$ is a 3-manifold which admits a Seifert fibration (see, for example, [46] for a nice introduction to these manifolds). The orbit space given by collapsing each of the fibers to a point can be given the structure of a two-dimensional orbifold $B_Q$. Seifert manifolds are easily classified by their (un-normalized) Seifert invariants $(g, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$, where $g$ is the genus\footnote{For non-orientable surfaces, one regards the genus as being minus the number of $\mathbb{R}P^2$ connect summands needed to construct $B_Q$.} of the surface (underlying) $B_Q$ and the $(\alpha_i, \beta_i)$ are relatively prime integers. As we will only need these invariants in the proof of Lemma 5.10, we do not recall their definition here but refer to [46, p. 34].

Let $h \in \pi_1(Q)$ denote the class of a regular fiber of the Seifert fibering. Then it follows from the standard presentation of the fundamental group of a Seifert manifold [46, p. 34] that the subgroup $\langle h \rangle$ generated by $h$ is a normal subgroup. The following celebrated result shows that this property characterizes Seifert fibered 3-manifolds:

**Theorem 5.7** [12, 29]. Let $Q$ be a closed oriented 3-manifold such that $\pi_1(Q)$ admits a non-trivial cyclic normal subgroup. Then $Q$ is Seifert fibered. If the center of $\pi_1(Q)$ is non-trivial, then $Q$ is Seifert fibered over an oriented base.

**Remark 5.8.** Building on this result, reference [18] shows something slightly stronger — that a closed, oriented 3-manifold $Q$ with $\mathbb{Z}_k(\pi_1(Q)) \neq k$ is Seifert-fibered. Parallel to the discussion for 2-manifolds, Lemmas 5.3 and 5.4 then immediately show that any aspherical 3-manifold which admits a quasi-dilation is Seifert-fibered. Our approach in §5.3 handles all cases simultaneously but also relies on Theorem 5.7 (via Theorem 5.9).

The starting point for our analysis will be the following Theorem which relies on Theorem 5.7 as well as many other important developments in 3-manifold topology (notably [32, 38, 61, 62]).

**Theorem 5.9** [48, Theorems 1 and 3]. Let $\hat{B}$ be an oriented Riemann surface and let $Q$ be a closed, oriented 3-manifold dominated by $S^1 \times \hat{B}$. Then, either

- $Q$ is finitely covered by $S^1 \times B$ for $B$ closed and oriented of genus at least 1 (and in particular aspherical), or
- $Q$ admits a metric of positive scalar curvature. Equivalently (see the two paragraphs preceding Remark 1 on [48, p. 24]), $Q$ is finitely covered by a connected sum $\#_n S^1 \times S^2$ (by convention, the case $n = 0$ corresponds to $S^3$).

In order to classify 3-manifolds admitting quasi-dilations over $\mathbb{Z}$ (see Lemma 5.15), we will need the following refinement of Theorem 5.9 which concerns 1-domination by $S^1 \times \hat{B}$.

**Lemma 5.10.** Let $Q$ be a closed, oriented aspherical manifold which is 1-dominated by $S^1 \times \hat{B}$. Then $Q \cong S^1 \times B$ for $B$ closed and oriented of genus at least 1.

**Proof.** Because the map $f : S^1 \times \hat{B} \to Q$ has degree 1, it is surjective on $\pi_1$. Let $h \in \pi_1(S^1)$ denote a generator of $\pi_1(S^1)$. As the map does not factor through $\hat{B}$, we must have that $f_*(h) \neq 0 \in \pi_1(Q)$ and is central by surjectivity. By Theorem 5.7, this implies that $Q$ is Seifert fibered.

We may therefore apply the following simplified version of a result of Rong ([67, Theorem 3.2]) which states that if $f : Q' \to Q$ is a map of degree 1 between Seifert manifolds and $Q'$ has Seifert invariants $(g', (\alpha_1', \beta_1'), \ldots, (\alpha_l', \beta_l'))$ then $Q$ has Seifert invariants $(g, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m))$ with $l \geq m$ (and $g' \geq g$ but this plays no role).
our present (very simple) case, the Seifert invariants of $S^1 \times \hat{B}$ are empty and so $Q$ also admits a Seifert fibration with empty Seifert invariants. This immediately implies that $Q \cong S^1 \times B$. □

Finally, in the proofs of Lemmas 5.15 and 5.17, we will also require the following two lemmas concerning spherical space forms (that is, a quotient of $S^3$ by a finite subgroup of $SO(4)$ acting freely on $S^3$ by rotations) and their fundamental groups.

Lemma 5.11. Let $Q$ be a spherical space form. For any prime $p$ which divides $|\pi_1(Q)|$ and any map $f : S^1 \times B \to Q$, $p$ divides $\deg(f)$.

Proof. We first note the following fact: Let $\pi : C \to S^1 \times B$ be a finite covering where $B$ is an oriented Riemann surface. Then $C \cong S^1 \times B_C$ for some Riemann surface $B_C$. To see this, observe that the manifold $C$ inherits a canonical Seifert fibration structure $C \to B_C$ by [46, Lemma 8.1] and there is an induced covering map $B_C \to B$ (see [10, Lemma 4.4] for a detailed proof of this fact). It follows that $B_C$ is an ordinary Riemann surface (without orbifold structure) and the Euler number of the circle bundle $C \to B_C$ is zero. Therefore $C \cong S^1 \times B_C$.

With this established, let $p$ be a prime as in the statement of the lemma. We first note that it suffices to assume that the map $f$ in the statement of the lemma is surjective on $\pi_1$. To see this, suppose that it is not surjective on $\pi_1$, then the map $f$ factors through a covering space $\tilde{Q} \to Q$ so that $\tilde{f}$ is $\pi_1$-surjective. Then, if $p$ no longer divides $\pi_1(\tilde{Q})$, this means that the degree of $h$ must be divisible by $p$ which implies that the degree of $f$ is divisible by $p$ and we are done. Otherwise, replacing $Q$ by $\tilde{Q}$ gives the desired reduction.

Having reduced to this case, we assume for the rest of the proof that $f$ is surjective on $\pi_1$. The next step is to reduce to the case where $Q$ is a lens space. To do this, note that by Cauchy’s theorem, we may find a cyclic subgroup $\langle g \rangle$ of order $p$ in $\pi_1(Q)$ and we let $\tilde{Q}$ be the covering space of $Q$ corresponding to this subgroup. Using the fact noted at the beginning of this proof that any finite covering space of $S^1 \times B$ is of the same form, we may, by passing to covering spaces in both the source and the target of the map, obtain a Cartesian square:

$$
\begin{array}{ccc}
S^1 \times B & \xrightarrow{\tilde{f}} & \tilde{Q} \\
\downarrow{f} & & \downarrow{h}
\end{array}
$$

so that $\tilde{f}$ is $\pi_1$-surjective. Then, if $p$ no longer divides $\pi_1(\tilde{Q})$, this means that the degree of $h$ must be divisible by $p$ which implies that the degree of $f$ is divisible by $p$ and we are done. Otherwise, replacing $Q$ by $\tilde{Q}$ gives the desired reduction.

Finally we treat the case where $Q$ is a lens space. Let $B$ denote the Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$. The Bockstein long exact sequence shows that $H^1(Q, \mathbb{Z}/p\mathbb{Z}) \to H^2(Q, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism. This (together with the cup-product structure on the cohomology of lens spaces) shows that there is a generator $\alpha \in H^1(Q, \mathbb{Z}/p\mathbb{Z})$ such that $\alpha \cup B(\alpha)$ is a generator of $H^3(Q, \mathbb{Z}/p\mathbb{Z})$. Since the Bockstein homomorphisms are zero for $S^1 \times B$, the lemma follows from the naturality of the cup products and Bockstein homomorphisms. □
Lemma 5.12. For any finite group $\pi$ which acts freely on $S^3$,

$$H_3(B\pi, \mathbb{Z}) \cong \mathbb{Z}/|\pi|, \quad (5.8)$$

where $B\pi$ denotes the classifying space of $\pi$ and $|\pi|$ denotes its cardinality.

Proof. By Lemma 3.1 and the discussion after the first Definition on [80, p. 268] (see [11, § 6.1 and Example 6.3] for a nice exposition) the Tate cohomology groups $\hat{H}^*(\pi, \mathbb{Z})$ are 4-periodic for any finite group which acts freely on $S^3$. In particular, we have

$$\hat{H}^{-4}(\pi, \mathbb{Z}) \cong \hat{H}^0(\pi, \mathbb{Z}) \cong \mathbb{Z}/|\pi|, \quad (5.9)$$

where the second isomorphism in (5.9) is [11, equation (6.2)]. But by definition of Tate cohomology, $\hat{H}^{-4}(\pi, \mathbb{Z}) := H_3(B\pi, \mathbb{Z})$ and so (5.9) immediately implies (5.8). □

5.3. Classifying 3-manifolds which admit quasi-dilations

We now turn to describing how the results of the previous section constrain the topology of 3-manifolds admitting quasi-dilations.

Lemma 5.13. Suppose that a closed oriented 3-manifold $Q$ admits a pseudo-dilation over $\mathbb{Z}$. Then $Q$ is dominated by $S^1 \times B$ for some closed, orientable Riemann surface $B$.

Proof. By definition the element $\Psi$ is a class $H^1(LQ) = H_2(LQ)$. Without loss of generality assume that $\Psi$ is concentrated in the homology of a single connected component of the loop space $H_2(L_cQ)$. By [82] (see also [54, Ex. 6.5.4] for a simple argument in degree 2), we may represent a non-zero multiple of this class (that is, $N\Psi$ for some non-zero $N \in \mathbb{Z}$) by a map from an orientable, connected 2-manifold

$$\Sigma : \hat{B} \to L_cQ.$$

It follows that the map

$$\text{ev} \circ \Gamma \circ (\text{id} \times \Sigma) : S^1 \times \hat{B} \to Q$$

has non-zero degree, in particular that $Q$ is dominated by this manifold. □

It follows from Theorem 5.9 that we have the following result:

Theorem 5.14. Let $Q$ be a closed, oriented 3-manifold which admits a pseudo-dilation over $\mathbb{Z}$. Then, either

- $Q$ is finitely covered by $S^1 \times B$ for $B$ closed and oriented of genus at least 1 (and in particular aspherical), or
- $Q$ admits a metric of positive scalar curvature. Equivalently, $Q$ is finitely covered by a connected sum $\#_n S^1 \times S^2$ (by convention, the case $n = 0$ corresponds to $S^3$).

One can improve this result when the pseudo-dilation $(\Psi, \alpha)$ has an integral lift:

Lemma 5.15. Suppose $Q$ is a closed, oriented 3-manifold.

(1) If $Q$ admits a metric of positive scalar curvature and a pseudo-dilation over $\mathbb{Z}$ then $Q \cong \#_n S^1 \times S^2$, $n \geq 0$.

(2) If $Q$ is aspherical and admits a quasi-dilation over $\mathbb{Z}$, then $Q \cong S^1 \times B$ where $B$ is a Riemann surface of genus at least 1.
Proof. To prove (1) observe that if \(Q\) admits a pseudo-dilation over \(\mathbb{Z}\), the fundamental class can be written as the sum of pushforwards of the fundamental class along \(f_j : S^1 \times B_j \to Q\) (the \(j\) range over different components of \(\mathcal{L}Q\)). In particular, we must have that the integers \(\text{deg}(f_j)\) are relatively prime. On the other hand, as was shown in \([32, 38, 61, 62]\) and reviewed in \([48, \S2]\), \(Q\) admits a prime decomposition with primes \(S^1 \times S^2\) or spherical space forms \(Q_i\). For every spherical space form which appears, we obtain a degree 1 map \(Q \to Q_i\). This implies by Lemma 5.11 that for each map \(f_j : S^1 \times B_j \to Q\), and every \(p\) which divides \(|\pi_1(Q_i)|\), \(p\) divides \(\text{deg}(f_j)\). Therefore, every \(Q_i\) must be \(S^3\) and the lemma is proved.

To prove (2) note that \(Q\) is aspherical and hence the fundamental group of \(Q\) is torsion free.\(^{1}\)

Remark 5.16. Lemma 5.15 is sharp in the sense that examples of exact Lagrangian embeddings in Appendix A show that there exist quasi-dilations in the loop homology algebra over \(\mathbb{Z}\) of \(\#_nS^1 \times S^2\) for all \(n\) (the Künneth formula produces an integral quasi-dilation when \(Q \cong S^1 \times B\)).

Finally, we examine the implications of the existence of dilations over \(k = \mathbb{Z}\) or a field of characteristic \(p > 0\) leading to a curious characterization of the 3-sphere in terms of dilations:

Lemma 5.17. Suppose that \(Q\) is a closed, oriented 3-manifold.

1. For any \(Q\) such that \(\mathbb{H}^*(\mathcal{L}Q, \mathbb{Z})\) admits a dilation, then \(Q \cong S^3\).
2. Suppose that \(\mathbb{H}^*(\mathcal{L}Q, k)\) admits a dilation where \(k\) is a field of characteristic \(p > 0\). Then \(Q\) admits a prime decomposition with primes \(S^1 \times S^2\) or spherical space forms \(Q_i\) such that \(p\) does not divide \(\pi_1(Q_i)\).

Proof. For (1), by Lemma 5.15 we have \(Q\) is an aspherical \(S^1 \times B\) or \(Q = Q_n := \#_nS^1 \times S^2\) for \(n \geq 1\). It is impossible that \(Q\) is an aspherical \(S^1 \times B\) by Lemma 5.4 and so we only need to treat the second case. We have a degree 1 map \(f : Q_n \to S^1 \times S^2\), which induces a map \(f_L : \mathcal{L}Q_n \to \mathcal{L}(S^1 \times S^2)\) on loop spaces. The map \(f_L\) has the following two properties:

\[
f_{L,*}([Q_n]) = [S^1 \times S^2] \quad \text{and} \quad \Delta(f_{L,*}(\Psi)) = f_{L,*}(\Delta(\Psi)),
\]

where \([Q_n]\) and \([S^1 \times S^2]\) denote fundamental cycles of constant loops and \(\Psi\) is an arbitrary class in \(\mathbb{H}^*(\mathcal{L}Q_n)\). If \(Q_n\) had an integral dilation \((\Psi, [Q_n])\), then the two equations from (5.10) show that

\[
\Delta(f_{L,*}(\Psi)) = f_{L,*}(\Delta(\Psi)) = [S^1 \times S^2]
\]

and hence that \(S^1 \times S^2\) would have an integral dilation, which is known not to be true from the calculation of \([58]\) mentioned above together with an elementary argument using the Künneth formula.

To check (2), note that for any prime summand we get a degree 1 map \(f : Q \to Q_i\) again inducing a map \(f_L\) on loop spaces satisfying the properties (5.10). As in the proof of (1), this

\(^1\)To see this, suppose there is finite cyclic subgroup \(C \subset \pi_1(Q)\) which induces a covering space \(BC \to Q\). As \(BC\) has infinite homological dimension, this is impossible.
means that if \( Q \) had a dilation \((\Psi, [Q])\) over \( k \), then
\[
\Delta(f_{L,*}(\Psi)) = f_{L,*}(\Delta(\Psi)) = [Q].
\]  
Equation (5.12) together with Lemma 5.4 implies that none of the \( Q_i \) can be aspherical.

So, again in view of equation (5.12), the claim (2) reduces to showing that for any \( Q = S^3/\pi \), the dilation \( Q \) does not admit a dilation over a field of characteristic \( p \) which divides \(|\pi|\). As the classifying space \( B\pi \) can be obtained from \( S^3/\pi \) by attaching the higher cells, the classifying map induces a surjection:
\[
p_* : H_3(S^3/\pi) \to H_3(B\pi).
\]  
which by equation (5.8) is an isomorphism in characteristic \( p \) (it is here that we use that \( p \) divides \(|\pi|\)). Suppose we have \( \Psi \in H_2(LQ, k) \) for \( k \) a field of characteristic \( p \) with \( \Delta(\Psi) = [Q] \). Then by transferring, we obtain a class \( p_*(\Psi) \in H_2(L_{[ud]} B\pi, k) \) with \( \Delta(p_*(\Psi)) \neq 0 \). This is a contradiction, since for any aspherical space, the space of contractible loops retracts onto the constant loops and hence \( \Delta = 0 \).

These classification results have the following application to Lagrangian embeddings:

**Corollary 5.18.** Suppose that \( X \) is a six-dimensional Liouville domain such that \( SH^*(X, \mathbb{Q}) \) admits a quasi-dilation. Suppose that \( Q \hookrightarrow X \) is an exact Lagrangian embedding. Then \( Q \) is one of the two types of manifolds listed in Theorem 5.14. If the quasi-dilation lifts to \( SH^*(X, \mathbb{Z}) \), then \( Q \cong \#_n S^1 \times S^2 \) for some \( n \geq 0 \) or \( Q \cong S^1 \times B \) with \( g(B) \geq 1 \).

**Proof.** Assume \( SH^*(X, \mathbb{Q}) \) admits a quasi-dilation \((\Psi, \alpha)\) and consider the Viterbo restriction (recall (2.33)) of these elements \((j^*\Psi, j^*\alpha)\) to a Weinstein neighborhood \( D^*Q \). Because we make no assumption on the Maslov class of our Lagrangians (see [65] for other applications of Viterbo functoriality in the absence of Maslov class assumptions), \( \mathcal{A}S \circ j^i \) no longer necessarily preserves \( \mathbb{Z} \)-gradings (only \( \mathbb{Z}/2\mathbb{Z} \)-gradings) and so \( \mathcal{A}S \circ j^i(\alpha) \) and \( \mathcal{A}S \circ j^i(\Psi) \) may not be concentrated in a single degree. Nevertheless, because \( \mathcal{A}S \) and \( j^i \) are both BV-algebra maps and \( ev_* \) is an algebra map, \( ev_*(\mathcal{A}S \circ j^i(\alpha)) \) must still be a unit in \( \mathbb{H}^*(Q, k) \). It follows that the degree 0 piece of \( ev_*(\mathcal{A}S \circ j^i(\alpha)) \) is a unit in \( \mathbb{H}^0(Q, k) \) and consequently that \( Q \) admits a rational pseudo-dilation by taking the degree 1 component of \( \mathcal{A}S \circ j^i(\Psi) \) and the degree 0 component of \( \mathcal{A}S \circ j^i(\alpha) \). The result then follows from Theorem 5.14.

When \( Q \) admits a metric of positive scalar curvature, the second statement is proven in exactly the same way after invoking Part (1) of Lemma 5.15. When \( Q \) is aspherical note that we have that \( \mathbb{H}^*(LQ) \) is concentrated in non-negative degree which implies that the degree 0 piece of \( \mathcal{A}S \circ j^i(\alpha) \) is also invertible. We therefore conclude using part (2) of Lemma 5.15. \( \square \)

**Remark 5.19.** Corollary 5.18 has implications in symplectic topology outside of the main cases we consider in this paper. For example, let \( X \) be a three-dimensional \( A_n \) Milnor fiber which we recall admits a Lefschetz fibration with general fiber \( T^* S^2 \). Using Lefschetz fibrations techniques (as in [78, §7]), \( SH^*(X, \mathbb{Z}) \) can be seen to admit a quasi-dilation which becomes a dilation after tensoring with any field \( k \) with characteristic not equal to 2 (because as noted at the end of § 5.1, \( \mathbb{H}^*(LS^2, \mathbb{Z}) \) has a quasi-dilation with this property). This implies that if \( Q \) is a rational homology sphere which admits an exact Lagrangian embedding, \( j : Q \hookrightarrow X \), then \( Q \cong S^3 \).

**Remark 5.20.** In pioneering work, Fukaya [26, 27] produces for (any) Lagrangian embedding \( Q \subset \mathbb{C}^3 \), a Maurer–Cartan element of the (chain-level \( L_\infty \) structure on the) (equivariant) chains on the free loop space \( C_{n-*}(LQ) \) whose associated deformation is trivial. In the spirit of the present section, it may be interesting to study the following homological
(rather than chain level) version of Fukaya’s condition: Let \( \alpha \in H^0(\mathcal{L}Q, k) \) and \( \Psi \in H^1(\mathcal{L}Q, k) \) be two loop space classes. Then \((\Psi, \alpha)\) define a coordinate function if

\[ [\alpha, \Psi] = 1. \]

We conjecture that if \( Q \) is a closed, orientable 3-manifold with a coordinate function \((\Psi, \alpha)\), then \( Q \) is diffeomorphic to \( S^1 \times B \), where \( B \) is an orientable Riemann surface. While discussing Fukaya’s work, it also seems important to note that in the present paper, we are able to obtain much sharper results when our quasi-dilations are defined integrally (see Lemma 5.15) — something which is not possible (or relevant) in Fukaya’s setup.

6. Dilation and applications

6.1. Dilation

Let \( M \) be a Fano variety of dimension \( n \geq 3 \) and \( D \) a very ample smooth divisor such that \( H^2(M, k) = k(\mathrm{PD}(D)) \), where \( \mathrm{PD} \) denotes the Poincaré dual, and

\[ K_M^{-1} = O(mD) \]

for \( m > n/2 \). Let \( j_D : D \to M \) denote the inclusion.

**Lemma 6.1.** As usual, let \( X = M \setminus D \). Then, under the preceding assumptions,

\[ H^2(X, k) = 0. \]

**Proof.** Consider the long exact sequence (all homologies are taken with \( k \)-coefficients):

\[ \cdots \to H_{2(n-2)}(D) \cong H_{2(n-2)}(M) \to H_{2n-2}(M, D) \to H_{2n-3}(D) \to \cdots \]

By the adjunction formula, \( D \) is a Fano manifold. Hence, we have that \( D \) is simply connected (see, for example, [19, §5.1]) and in particular \( H_{2n-3}(D, k) \cong H^1(D, k) = 0 \). We therefore have that \( H_{2n-2}(M, D) = 0 \). Using excision and Poincaré duality, this implies that \( H^2(X, k) = 0 \) as claimed. \( \square \)

For any three cohomology classes \( \beta_i \in H^*(M) \), let \( GW_M(\beta_1, \beta_2) \) and \( GW_M(\beta_1, \beta_2, \beta_3) \) denote the 2-point and 3-point Gromov–Witten invariants on \( M \) [54]. Throughout this subsection, since we are in the case of a smooth divisor, we will use the more suggestive notation \( S_D \) for \( S_1 \), the unit normal bundle around \( D \). Similarly, for a class/chain in log cohomology, we will use the notation \( \alpha_D \) for the class \( \alpha \) with primitive multiplicity vector on the single smooth component \( D_1 = D \).

**Theorem 6.2.** Let \( \alpha_0 \) be a class in \( H^{2m-4}(M) \) and set \( \alpha_i = \mathrm{PD}(D)^i \cup \alpha_0 \) for \( i \in \{1, 2\} \). Suppose that

\[ j_D^*(\alpha_2) = 0 \in H^*(D) \] (6.1)

\[ GW_M(\mathrm{PD}(D), \alpha_1, pt) \in k^\times. \] (6.2)

Then \( SH^*(X, k) \) admits a dilation.

The proof of Theorem 6.2 requires two lemmas. To state the first, we introduce some notation. Choose two generic smooth very ample divisors \( D_N, D'_N \subset D \) representing the normal bundle \( ND \) (these exist by Bertini’s theorem). Recall that given such a \( D_N \), one may form the real-oriented blow-up \( B_D(D_N) \) of \( D \) along \( D_N \), which is a submanifold with boundary inside of \( S_D \).
(see e.g. [68, §8.2]). Locally, given a holomorphic function \(f : U \to \mathbb{C}\), the real-oriented blow-up along \(f^{-1}(0)\) is constructed as closure of the graph of

\[
\frac{f}{||f||} : U \setminus \{f^{-1}(0)\} \to S^1
\]

inside of \(U \times S^1\). For a general smooth divisor, it is constructed by patching together these local models and has boundary \(\pi^{-1}(D_N) \subset S_D\).

Pick a generic pseudo-cycle \(P\) representing \(j^*_D(\alpha_0)\) in \(H^*(D)\) (the condition (6.2) implies that \(j^*_D(\alpha_0)\) is a non-trivial class by the adjunction formula). Recall that by definition of a pseudo-cycle, the limiting set of \(P\) is covered by a smooth manifold \(W_P\) of dimension at most \(\dim(P) - 2\). By perturbing \(P\) (or \(D_N\)), we can assume that \(D_N\) intersects \(P\) and \(W_P\) transversely. Under this transversality assumption, \(L := P \times_D Bl_{D_N}(D) \to S_D\) is a pseudo-cycle (with boundary) whose boundary \(\partial L\) is the pull-back of \(S_D\) along \(k : P \times_D D_N \to D\).

We next set \(K_0 := k \times_D Bl_{D_N}(D)\), which after a possible further perturbation of either \(P\) or \(D_N\), will be a pseudo-cycle with boundary the pull-back of \(S_D\) along

\[
P \times_D (D_N \cap D_N') \to D.
\]

Under assumption (6.1), \(P \times_D (D_N \cap D_N') \to D\) represents a trivial (pseudo) homology class, that is, there exists a pseudo-cycle \(K_1\) which bounds it. We may therefore may glue \(K_0\) with \(\pi^{-1}(K_1)\) (and smoothly approximate) to form a pseudo-cycle \(K\) mapping to \(S_D\). We have that \(H^2(X) = 0\) by Lemma 6.1 and so

\[
GW_{\varphi_1}([K]) = 0 \in H^2(X), \tag{6.3}
\]

where \(GW_{\varphi_1}([K]) = 0\) is the obstruction class defined as in equation (3.38). We can therefore choose a Hamiltonian \(H^\lambda_M\) with \(\lambda > m + h\) and pick a bounding relative pseudo-cycle \(Z_\delta\) so \(PSS^\lambda_{\log}(Kt, Z_\delta)\) defines a class in \(SH^*(X, k)\). As before consider the fiber product

\[
ev_{\infty} : M_{0,2}(M, D, \vec{v}_1)^\circ \times_{\ev_0} L \to X. \tag{6.4}
\]

This defines a pseudo-cycle \(GW_{\varphi_1}(L)\) when restricted to \(\tilde{X}\). To see this, note that \(\partial L\) is a union of circle fibers and so there is an rotational \(S^1\)-symmetry on \(M_{0,2}(M, D, \vec{v}_1)^\circ \times_{\ev_0} \partial L\). Hence, as in the proof of Lemma 3.33, the map from the fiber product

\[
ev_{\infty} : M_{0,2}(M, D, \vec{v}_1)^\circ \times_{\ev_0} \partial L \to X \tag{6.5}
\]

factors through the quotient \((M_{0,2}(M, D, \vec{v}_1)^\circ \times_{\ev_0} \partial L)/S^1\). We may now state our first lemma:

**Lemma 6.3.** There is a (cohomological) equality

\[
\Delta(PSS^\lambda_{\log}(Kt, Z_\delta)) = -PSS^\lambda_{\varphi_1}(GW_{\varphi_1}(L)). \tag{6.6}
\]

**Proof.** The argument of Corollary 4.31 and Lemmas 4.29 and 4.30 (c.f. Remark 4.32) shows that there is a cohomological equality

\[
\Delta(PSS^\lambda_{\log}(Kt, Z_\delta)) = PSS^\lambda_{\log}(\Gamma_K t), \tag{6.7}
\]

where \(\Gamma_K : S^1 \times K \to SD\) denotes the spinning (with speed 1) from (4.95). It is useful at this stage to recall that \(K\) was defined by gluing \(K_0\) and \(K_1\) above. Next, we make two observations: the first is that the spinning \(\Gamma_K : S^1 \times K_1 \to SD\) factors through \(K_1\). Thus, for any \(x_0\) such that \(|x_0| = |\Gamma_K t|\), we can choose complex structures so that for any \(u \in M(\vec{v}_1, x_0)\), \(\ev_{\varphi_{\tilde{x}_0}}(u)\) is disjoint from \(\Gamma_K t\). The second observation is that the pseudo-cycles \(\Gamma_K\) and \(\partial L\) are identified

\[^1\text{This class a priori depends on } \lambda\text{ since we did not address invariance issues when } Z_\delta \neq 0.\]
away from $S^1 \times \partial K_0$ viewed as a submanifold of (the domain of) $\Gamma_{K_0}$ and the pull-back of $SD$ to $P \times_D (D_N \cap D'_N)$ viewed as a submanifold of (the domain of) $\partial L$ (the images of these two pseudo-cycles coincide exactly).

Combining these two observations shows that

$$PSS^\lambda_{\log}(\Gamma_{Kt}) = PSS^\lambda_{\log}(\partial Lt).$$

(6.8)

We therefore have that

$$\Delta(PSS^\lambda_{\log}(Kt, Z_b)) = PSS^\lambda_{\log}(\partial Lt).$$

(6.9)

Meanwhile we have that

$$\partial_C \cdot PSS^\lambda_{\log}(Lt) = PSS^\lambda_{\log}(\partial Lt) + PSS^\lambda_{\log}(GW_{\vec{v}_1}(L)).$$

(6.10)

This means that on the level of cohomology, $PSS^\lambda_{\log}(\partial Lt) = -PSS^\lambda_{\log}(GW_{\vec{v}_1}(L))$. Combining this with (6.9) we see that as desired, on the level of cohomology

$$\Delta(PSS^\lambda_{\log}(Kt, Z_b)) = -PSS^\lambda_{\log}(GW_{\vec{v}_1}(L)).$$

(6.11)

□

Lemma 6.4. We have an equality

$$GW_{\vec{v}_1}(L) \cdot [pt] = GW(PD(D), \alpha_1, pt),$$

(6.12)

where $\cdot$ on the left-hand side denotes the intersection product (recall that $GW_{\vec{v}_1}(L) \in H^{BM}_{2n}(X) \cong H^0(X)$).

Proof. Fix a generic point $pt$ in the interior of $\bar{X}$. Let $S^1$ act on $M_{0,2}(M, D, \vec{v}_1)$ by rotation in the domain. Observe that for generic $J \in \mathcal{J}(M, D)$, we have an orientation preserving bijection of moduli spaces

$$GW_{\vec{v}_1}(L) \times_{ev_\infty} pt = M_{0,2}(M, D, \vec{v}_1)/S^1 \times_{ev_0} P \times_{ev_\infty} pt.$$ (6.13)

Moreover all such curves passing through $pt$ are $D_u$ regular for generic $J$ even before taking fiber products with $L$ and $P$, respectively. It follows that $GW_{\vec{v}_1}(L) \cdot pt = GW_M(\alpha_1, pt)$. To conclude, observe that the divisor equation implies that $GW_M(\alpha_1, pt) = GW(PD(D), \alpha_1, pt)$.

□

Proof of Theorem 6.2. Set $\beta = PSS^\lambda_{\log}(Kt, Z_b)$. Lemmas 6.3 and 6.4 imply that $\Delta\beta = -GW(PD(D), \alpha_1, pt) \cdot PSS_{\vec{v}_1}(1) = -GW(PD(D), \alpha_1, pt) \cdot 1$, a non-zero multiple of the unit by hypothesis. Now normalize. □

Remark 6.5. Theorem 6.2 is closely related to [75, Conjecture 18.6]. In the situation of Theorem 6.2, Seidel proposes an explicit formula for the Hamiltonian Floer cohomology (with its BV operator) of a Hamiltonian of a specific slope $\lambda$ and suggests applying this to produce dilations (see [75, Examples 18.8 and 18.9]). Theorem 6.2 uses the log PSS map to bypass the computation of Hamiltonian Floer cohomology.

Remark 6.6. It would be interesting to apply Theorem 6.2 to cases where the Gromov–Witten invariants are well understood (such as homogeneous varieties). A sample case where this should be possible is where $M$ is a generic hyperplane generating the divisor class group in $Gr(2, 2n + 2)$ and $D$ is a generic hyperplane section.
6.2. Quasi-dilations

For $J \in \mathcal{J}(M, D)$ and $A \neq 0 \in H_2(M, \mathbb{Z})$, let $\mathcal{M}_{0,0}(M, A, J)$ denote the moduli space of $J$-holomorphic spheres

$$u : \mathbb{C}P^1 \to M$$

such that $u_*([\mathbb{C}P^1]) = A$. Let $\tilde{\mathcal{M}}_{0,0}(M, A, J)$ denote the moduli space

$$\tilde{\mathcal{M}}_{0,0}(M, A, J) := \bigcup_{A_i \neq 0, \sum A_i = A} \prod_{i} \mathcal{M}_{0,0}(M, A_i, J).$$

(6.14)

For the remainder of this section, assume $(M, D)$ is a pair and $J := \{2, \ldots, |J| + 1\}$ is a subset of $\{1, \ldots, k\}$ such that $D_J = \cap_{j \in J} D_j$ is connected and non-empty. We assume that there exists a volume form $\Omega$ on $M$ whose divisor of zeroes is as in (3.5) with $a_i = 1$ for $i \in \{1 \cup J\}$. We will also assume that $D_1$ is a divisor which satisfies $D_1 \cap D_J = \emptyset$ as well as the following conditions (B1)–(B4).

(B1) The normal bundle to $D_1$ is trivial when restricted to $D_1 \setminus \cup_{i \neq 1} D_i$.

Because the torus bundle $\hat{S}_1$ is trivial, the cohomology splits as the cohomology $H^*(\hat{S}_1) \cong H^*(S^1) \otimes H^*(D_1)$. In particular, if we fix an isomorphism $k[\epsilon]/\epsilon^2 \cong H^*(S^1)$, there is a corresponding cohomology class $\beta_1 := \epsilon \otimes 1 \in H^*(\hat{S}_1)$. This cohomology class in turn gives rise to a generator $\beta_1 t^{\vec{v}_1} \in H^1_{\log}(M, D)$. We fix a section $\beta_{1,c}$ inducing the trivialization of $\hat{S}_1$ (which gives a cycle representing $\beta_1$). Let $\alpha_1 t^{\vec{v}_1}$ denote the generator in $H^1_{\log}(M, D)$ corresponding to the fundamental class on $\hat{S}_1$ and let $\alpha_{1,c}$ be a representative fundamental cycle constructed via the Eilenberg-Zilber product of fundamental cycles (and the isomorphism $\hat{S}_1 \cong S^1 \times D_1$ induced by $\beta_{1,c}$). Let $\alpha_j t^{\vec{v}_j}$ denote the generator in $H^1_{\log}(M, D)$ corresponding to the fundamental class in $\hat{S}_j$ and choose a fundamental cycle representative $\alpha_{j,c}$. We require

(B2) The class $\beta_1 t^{\vec{v}_1}$ is admissible with vanishing obstruction class.

(B3) There exists a $J_0 \in \mathcal{J}(V)$ so that

$$\tilde{\mathcal{M}}_{0,0}(M, B_1, J_0) = \mathcal{M}_{0,0}(M, B_1, J_0)$$

(6.15)

for any class $B_1 \in H_2(M)_\omega$ which satisfies $B_1 \cdot D = \vec{v}_j$.

(B4)

(a) Let $B$ be a spherical class with $B \cdot D = \vec{v}_1 + \vec{v}_j$. Given a decomposition of $B = B' + B''$ with $B' \cdot D_i \geq 0$ for all $i$ and such that $\tilde{\mathcal{M}}_{0,0}(M, B', J_0)$ and $\tilde{\mathcal{M}}_{0,0}(M, B'', J_0)$ are non-empty, we have that $B' \cdot D = \vec{v}_1$ or $B' \cdot D = \vec{v}_j$.

(b) Set $\vec{v} = \vec{v}_1 + \vec{v}_j$. Then for any $B$ with $\tilde{\mathcal{M}}_{0,0}(M, B, J_0) \neq \emptyset$ either

$$B \cdot D \in \{\vec{v}, \vec{v}_1, \vec{v}_j\}$$

(6.16)

or there exists $i \in \{1, \ldots, k\}$ with $B \cdot D_i > v_i$.

Because $\vec{v}_1$ is admissible, Lemma 3.31 shows that for generic $J$, the moduli space $\mathcal{M}_{0,2}(M, D, \vec{v}_1)$ is smooth and maps properly to $X$. Note that by Gromov compactness, for an almost-complex structure $J$ sufficiently close to $J_0$, conditions (B3) and (B4) continue to hold with $J_0$ replaced by $J$. In what follows, we will assume that all complex structures are taken sufficiently close to $J_0$ so that these conditions still hold. In particular, we have the following lemma:

**Lemma 6.7.** For generic $J$ sufficiently close to $J_0$, the moduli spaces $\mathcal{M}_{0,2}(M, D, \vec{v}_j)$ are smooth and map properly to $X$ as well.
Proof. Consider a sequence of curves with \( ev_\infty \subset K \) for some compact set \( K \subset X \). As stated above, for \( J \) sufficiently close to \( J_0 \), condition (B3) continues to hold and hence no bubbling can occur in this sequence. The rest of the argument proceeds exactly as in the proof of Lemma 3.31. \( \square \)

In the proof of Lemma 6.11 (which concerns product structures), we will also need to consider 3-pointed versions of these moduli spaces:

**Definition 6.8.** Let \( B \in H_2(M)_0 \) be a spherical class such that \( B \cdot D = \vec{v}_1 + \vec{v}_J \) and let \( \mathcal{M}_{0,3}(M, D, \vec{v}_1, \vec{v}_J, B) \) denote the space of \( D_1 \) and \( D_J \)-regular maps (Definition 3.28) \( u : (C, z_0, z_1, z_\infty) \to (M, D) \) such that

\[
\begin{align*}
&u_*([\mathbb{C}P^1]) = B \\
u^{-1}(D_1) = z_0 \\
u^{-1}(D_j) = z_1 \text{ for } j \in J.
\end{align*}
\]

We let

\[
\mathcal{M}_{0,3}(M, D, \vec{v}_1, \vec{v}_J) := \bigsqcup_B \mathcal{M}_{0,3}(M, D, \vec{v}_1, \vec{v}_J, B). \tag{6.17}
\]

As the classes are primitive, all such curves \( u \) are automatically somewhere injective and thus we may achieve transversality for such maps. Consider \( ev_\infty^{-1}(X) = \mathcal{M}_{0,3}(M, D, \vec{v}_1, \vec{v}_J)^o \). We have a partial compactification of this moduli space \( \overline{\mathcal{M}}_{0,3}(M, D, \vec{v}_1, \vec{v}_J)^o \) given by incorporating nodal curves with the following properties:

- \( ev_\infty \in X \).
- There are exactly two non-constant components \( u_0 \) and \( u_1 \) in classes \( B_0 \) and \( B_1 \) with \( B_0 \cdot D = \vec{v}_1 \) and \( B_1 \cdot D = \vec{v}_J \). Furthermore, each of these components is \( D \)-regular and \( z_0 \) lies in \( u_0 \) while \( z_1 \) lies in \( u_1 \).
- There are either no constant components, or one constant component containing the marked point \( z_\infty \).

**Lemma 6.9.** For \( J \) sufficiently near \( J_0 \), the evaluation map

\[
ev_\infty : \overline{\mathcal{M}}_{0,3}(M, D, \vec{v}_1, \vec{v}_J)^o \to X
\]

is proper.

Proof. Consider a sequence of curves in \( \mathcal{M}_{0,3}(M, D, \vec{v}_1, \vec{v}_J)^o \) such that \( ev_\infty \) lies in some fixed compact set \( K \). Then by (B4) (a), there are two possible cases for bubbling:

**Case I:** There is only one non-constant component \( u \) with some constant bubbles attached. Then two of the marked points would have to lie on the same constant component. But this is impossible because \( z_0, z_1, z_\infty \) all map to pairwise disjoint subsets of \( M \) (recall that by assumption \( D_1 \cap D_J = \emptyset \)).

**Case II:** There are two non-constant components \( u_0 \) and \( u_1 \) with homology classes \( B_0 \cdot D = \vec{v}_1 \) and \( B_1 \cdot D = \vec{v}_J \). Because \( ev_\infty \) lies in \( K \) and hence in \( X \), we know that one of the components must be \( D \)-regular and from intersection considerations, we see that \( u_0 \) and \( u_1 \) can only meet in \( X \). It follows that both components are \( D \)-regular. As in Case I, constant components can only contain (exactly) one of the marked points and hence there is at most one constant component which is glued to \( u_0 \) at one nodal point and glued to \( u_1 \) at another nodal point (the constant component has a total of three special points). This potential constant component
must therefore map to a point where \( u_0 \) and \( u_1 \) intersect, which must lie in \( X \). It follows that the third special point (the marked point) on the constant component must be \( z_∞ \).

For generic \( J \) near \( J_0 \), we therefore have that \( \overline{M}_{0,3}(M, D, \bar{v}_1, \bar{v}_j) \) defines a relative pseudo-cycle of codimension 0

\[
GW(\bar{v}_1, \bar{v}_j)
\]

when intersected with \( \bar{X} \).

**Theorem 6.10.** Let \((M, D)\) be a pair as above which satisfies \((B1)-(B4)\). Choose a Hamiltonian \( H_\beta^\lambda \) with \( \lambda > \kappa_1 + \hbar \) (as in Lemma 4.14) and choose a bounding cochain \( Z_b \) for \( \beta_1 t^{\bar{v}_1} \). Suppose further that the cohomology class \( GW(\bar{v}_1, \bar{v}_j) \in H^0(X)^\times \), and let \( \beta_{1,c}, \alpha_{1,c} \) be the chain-level representatives of \( \beta_1 \) and \( \alpha_1 \) chosen above. Then the pair

\[
(\epsilon_{\lambda, \infty} \circ PSS^\lambda_{log}(\beta_{1,c} t^{\bar{v}_1}, Z_b), PSS^\lambda_{log}(\alpha_{1,c} t^{\bar{v}_1}))
\]

defines a quasi-dilation.

**Proof.** This follows immediately from the fact that on cohomology \( \Delta(PSS^\lambda_{log}(\beta_{1,c} t^{\bar{v}_1}, Z_b)) = PSS^\lambda_{log}(\alpha_{1,c} t^{\bar{v}_1}) \) (by Corollary 4.31, noting that \( \beta_{1,c} \) and \( \alpha_{1,c} \) as chosen tautologically satisfy \( [\Delta(\beta_{1,c} t^{\bar{v}_1})] = [\alpha_{1,c} t^{\bar{v}_1}] \)); see (3.23) for the definition of \( \Delta \) on log cohomology), together with the identity proven in Lemma 6.11 (which by the hypothesis \( GW(\bar{v}_1, \bar{v}_j) \in H^0(X)^\times \) implies that \( PSS^\lambda_{log}(\alpha_{1,c} t^{\bar{v}_1}) \) is a unit). \( \square \)

**Lemma 6.11.** Under the assumptions of Theorem 6.10, let \( \alpha_{1,c} \) and \( \alpha_{j,c} \) denote the chain-level representatives of \( \alpha_1 \) and \( \alpha_j \) chosen above. Then there is a cohomological equality

\[
PSS^\lambda_{log}(\alpha_{1,c} t^{\bar{v}_1}) \cdot PSS^\lambda_{log}(\alpha_{j,c} t^{\bar{v}_j}) = PSS(GW(\bar{v}_1, \bar{v}_j)).
\]

**Proof.** We define an auxiliary moduli space which will allow us to prove that the PSS map defined above preserves the ring structure. We work over a parameter space \( q \in [b, \infty) \) with \( b \gg 0 \). We consider the surfaces \( S_{q,2} \) whose underlying domain is \( CP^1 \setminus \{0\} \) with a negative cylindrical end as before, but this time with two distinguished marked points at \( z_1 = q \) and \( z_2 = -q \). Let \( \beta \) be a subclosed 1-form on \( S_{q,2} \) satisfying

- the form \( \beta \) restricts to \( 2dt \) on the cylindrical end;
- \( \beta = 0 \) in neighborhoods of \( z_1 = -q \) and \( z_2 = q \).

Define

\[
\mathcal{M}_q(M, \bar{v}_1, \bar{v}_j; x_0)
\]

to be the moduli space of pairs \((q, u)\), with \( q \in [b, \infty) \) and

\[
u : S_{q,2} \to M
\]
as usual satisfying

\[
(du - X_{H_\beta^\lambda} \otimes \beta)^{0,1} = 0
\]

with asymptotic condition

\[
\lim_{s \to -\infty} u(\epsilon(s, t)) = x_0
\]

and tangency/intersection conditions

\[
u(x) \notin D \text{ for } x \neq z_i;
\]

\[
u(z_1) \text{ intersects } D_i \text{ with multiplicity } \bar{v}_1;
\]

\[
u(z_2) \text{ intersects } D_i \text{ with multiplicity } \bar{v}_j.
\]
When $|x_0| = 0$, $\mathcal{M}_q(M, \vec{v}_1, \vec{v}_J; x_0)$ has dimension 1 for generic choices. These moduli spaces then admit a Gromov compactification, which has two distinct types of boundary strata.

**Type I:** This case corresponds to the usual cylindrical breaking and sphere bubbling at some finite $q \in [b, \infty)$. Cylindrical breaking gives rise to strata:

$$
\bigsqcup_{|y| = -1} \mathcal{M}_q(M, \vec{v}_1, \vec{v}_J; y) \times \mathcal{M}(x_0, y).
$$

(6.25)

On the other hand, sphere bubbling does not arise at finite $q$. To see why this is the case, suppose that we are given a configuration consisting of a collection sphere bubbles attached to a solution $u_{\text{thimble}} : S_{q,2} \to M$. Note that by (B4)(b) the total homology class of all sphere bubbles must lie in a class $B$ which satisfies (6.16) and that there can be at most two non-constant bubble components. Suppose there is a sphere bubble $u_{\text{sphere}}$ which attaches at a point in $S_{q,2}$ which is not $z_1$ or $z_2$. We have

$$
u_{\text{thimble}} \cdot \mathbf{D} + u_{\text{sphere}} \cdot \mathbf{D} \leq \vec{v}_1 + \vec{v}_J.
$$

(6.26)

Then because there can be at most two non-constant bubble components, there must be at least one marked point ($z_1$ or $z_2$) where no sphere bubble attaches. Assume without loss of generality that this point is $z_1$, in which case we have that $u_{\text{thimble}}$ intersects $\mathbf{D}$ with multiplicity $\vec{v}_1$ at $z_1$. By (6.26) (and (6.16)), it follows that $u_{\text{sphere}}$ is the only sphere bubble, which means there is no sphere bubbling at $z_2$ either. Hence we have

$$
u_{\text{thimble}} \cdot \mathbf{D} \geq \vec{v}_1 + \vec{v}_J.
$$

(6.27)

We then have that $u_{\text{thimble}} \cdot \mathbf{D} + u_{\text{sphere}} \cdot \mathbf{D} > \vec{v}_1 + \vec{v}_J$, which is a contradiction. It follows that there is no sphere bubbling away from $z_1$ and $z_2$. Moreover, the only sphere bubbles which occur at finite $q$ are $\mathbf{D}$-regular sphere bubbles in classes which intersect $\mathbf{D}$ with multiplicity $\vec{v}_1$ and $\vec{v}_J$. However, these bubble configurations are of codimension at least 2 and hence do not arise in our one-dimensional moduli space.

**Type II:** The compactification also incorporates limits as the parameter $q \to \infty$. This stratum is given by the fiber product:

$$
\mathcal{M}_{0,3}(M, \mathbf{D}, \vec{v}_1, \vec{v}_J)^{\circ} \times_{x \in \infty} \mathcal{M}(\vec{v}_0, x_0).
$$

(6.28)

Similar arguments to those given in the (Type I) case explain why there are no configurations with spherical components mapped entirely into $\mathbf{D}$. The operation associated to counting rigid configurations of (6.28) for varying $x_0$ is by definition the composition

$$
PSS(GW(\vec{v}_1, \vec{v}_J)).
$$

(6.29)

Next consider the moduli space $\mathcal{M}_b(M, \vec{v}_1, \vec{v}_J; x_0)$ which is the restriction of the above moduli space to domains $S_{b,2}$. We have that the operation defined by $\mathcal{M}_b(M, \vec{v}_1, \vec{v}_J; x_0)$ is homotopic to (6.29) (as usual, counting configurations of the form (6.25) shows that the difference between these two operations is a Floer coboundary). Let $\Sigma$ denote the pair of pants equipped with its three standard cylindrical ends as in (2.30). We consider the nodal domain $\bar{S}_n$ of the form

$$
S \cup_e \Sigma \cup_e \bar{S},
$$

where the negative cylindrical ends of $S$ are glued to the positive cylindrical ends of $\Sigma$, a pair of pants. Maps from $\bar{S}_n \to M$ are given by the fiber product of moduli spaces given by

$$
\prod_{x_1, x_2} \mathcal{M}(\vec{v}_1, \alpha_1, x_1) \times \mathcal{M}(\Sigma, x_0, x_1, x_2) \times \mathcal{M}(\vec{v}_J, \alpha_J, x_2).
$$

We construct a homotopy between this moduli space and $\mathcal{M}_b(M, \vec{v}_1, \vec{v}_J; x_0)$ in two steps. First, we perform a finite connect sum along the cylindrical ends. Then, we can further
We now turn to constructing a wide class of examples. Let $X^o$ be a smooth affine variety with trivial canonical bundle $\Lambda^{\text{top}} T^* X^o \cong \mathcal{O}_{X^o}$ and $Z^o \hookrightarrow X^o$ be a principal, smooth hypersurface.

**Definition 6.12.** We say that a pair $(\bar{M}, \bar{D})$ with $\bar{D} = \sum_i \bar{D}_i$ such that each $\kappa_i > 0$;

1. $M$ is equipped with an ample line bundle $\mathcal{O}(\sum_i \kappa_i i)$ such that each $\kappa_i > 0$;
2. the closure of $Z^o$ in $\bar{M}$, $Z$, is a smooth hypersurface which intersects each stratum $\bar{D}_i = \cap_{j \in I} \bar{D}_j$, transversely; and
3. $\bar{D}$ supports a canonical divisor, that is, $\Lambda^{\text{top}} \bar{M} \cong \mathcal{O}_{\bar{M}}(\sum_i -a_i D_i)$ for some $a_i \in \mathbb{Z}$.

**Lemma 6.13.** Let $Z^o \hookrightarrow X^o$ be a smooth hypersurface in a smooth affine variety with trivial canonical bundle. Then there is always a good compactification $(\bar{M}, \bar{D})$ of $Z^o \hookrightarrow X^o$.

**Proof.** Because $X^o$ is a smooth affine variety, we can use Lemma 3.4 to find a simple-normal crossings compactification $(\bar{M}_o, \bar{D}_o)$ of $X^o$ such that $\bar{D}_o$ supports an effective ample divisor. Let $Z'$ be the compactification of $Z^o$ in $\bar{M}_o$. Next, we can do an embedded resolution of singularities (Theorem 3.2) to the divisor $\bar{D}_o \cup Z'$ to obtain a pair $(\bar{M}, \bar{D})$ so that the proper transform of $Z'$, $Z$, is smooth and such that the divisor $\bar{D} \cup Z$ is simple-normal crossings. The smooth centers of the sequence of blow-ups involved in the resolution will all lie over $\bar{D}_o$. As in the proof of Lemma 3.4, we can therefore use Lemma 3.3 to show that $\bar{D}$ supports an effective ample divisor $F$. Again, to get one which has positive coefficients on each component $\bar{D}_i$, one can take $\sum_i \bar{D}_i + mF$ which is ample for $m$ sufficiently large by [35, Exercise 7.5(b) of Chapter 2].

Finally, to address (3), note that by applying [35, Proposition 6.5 of Chapter 2] inductively, one obtains an exact sequence

$$\bigoplus_{i=3,\ldots,r+2} \mathbb{Z} \to \text{Pic}(\bar{M}) \to \text{Pic}(X^o) \to 0,$$

where the first map denotes the inclusion of the divisors $\mathcal{O}(\bar{D}_i)$ into the Picard group of $\bar{M}$ and the second is restricting the line bundle to $X^o$. As the canonical bundle restricted to $X^o$ is trivial, it follows that $\Lambda^{\text{top}} \bar{M} \cong \mathcal{O}_{\bar{M}}(\sum_i -a_i \bar{D}_i)$ for $a_i \in \mathbb{Z}$. \hfill \Box

As $Z^o \hookrightarrow X^o$ is principal, it is the zero set of a regular function $f : X^o \to \mathbb{C}$. We define the **affine conic bundle** to be the smooth affine variety $X$ defined by the equation

$$X = \{(u, v, \bar{x}) \in \mathbb{C}^2 \times X^o | uv = f(\bar{x})\}. \quad (6.30)$$

This variety depends only on the hypersurface $Z^o$. We will use a good compactification $(\bar{M}, \bar{D})$ to produce a compactification $(\hat{M}, \hat{D})$ of the affine conic bundle $(6.30)$. Let $\mathbb{C}P^1$ be equipped with its standard toric polarization $H_i$, $i = \{1, 2\}$. Consider the blow-up of $M := \bar{M} \times \mathbb{C}P^1$ at $Z \times H_1$ which we denote by

$$\pi : \hat{M} \to M.$$

On $M$, consider divisors $D_i = \bar{M} \times H_i$ for $i = \{1, 2\}$ and $D_i = \bar{D}_i \times \mathbb{C}P^1$ for $i = \{3, \ldots, r+2\}$ and let $\mathcal{D}$ denote the normal crossings collection of divisors given by

$$\mathcal{D} = D_1, D_2, D_3, \ldots, D_{r+2}.$$
Let \( \hat{D}_1 \) denote the proper transform of all of the divisors \( D_i \). Then the divisors \( \hat{D} \) are normal crossings and \( \hat{M} \) has a canonical divisor with \( a_1 = a_2 = 1 \). We choose a sufficiently small rational number \( \epsilon_Z = 1/m \) for \( m \gg 0 \) and equip the blow-up with the \( \mathbb{Q} \)-divisor given by

\[
\epsilon_Z \hat{D}_1 + (1 - \epsilon_Z) \hat{D}_2 + \sum_{i=3}^{r+2} a_i \hat{D}_i.
\]  

(6.31)

Note that the exceptional divisor \( E \) of the blow-up is linearly equivalent to \( \hat{D}_2 - \hat{D}_1 \) and hence (6.31) is linearly equivalent to \(-\epsilon_Z E + \hat{D}_2 + \sum_{i=3}^{r+2} a_i \hat{D}_i \). Therefore (6.31) defines a rational ample line bundle for sufficiently small \( \epsilon_Z \) again using Lemma 3.3. We say that \((\hat{M}, \hat{D})\) is the conic bundle associated to the compactification \((\hat{M}, \hat{D})\). The affine conic bundle from (6.30) is then the complement \( X = \hat{M} \setminus \hat{D} \). We may perturb the exceptional divisor \( E \) so that it is orthogonal to all of the \( D \) and take the complex structure in \( \mathcal{J}(V) \) which preserves \( E \).

**Lemma 6.14.** Let \((\hat{M}, \hat{D})\) be a conic bundle associated to a good compactification \((\hat{M}, \hat{D})\). Set \( J = \{ 2 \} \). Then the pair \((\hat{M}, D)\) (together with the above choice of complex structure \( J_0 \)) satisfy (B1)-(B4) above. Moreover \( GW(\vec{v}_1, \vec{v}_j) = 1 \).

An immediate Corollary of the above lemma and Theorem 6.10 is

**Corollary 6.15.** If \( X \) is any conic bundle over an affine variety \( X^o \) with trivial canonical bundle as in (6.30), then \( X \) admits an integral quasi-dilation.

**Proof of Lemma 6.14.** To prove (B1), note that in the complement of \( \hat{D}_2 \), the divisor \( E + \hat{D}_1 \) is a principal divisor. In particular, restricting to \( \hat{D}_1 \), we have that \( ND_1 \simeq \mathcal{O}(-E)_{|\hat{D}_1} \).

\( E \cap \hat{D}_1 \) can be identified with a copy of \( Z^o \) inside of \( X^o \). Because \( Z^o \) is principal we have that \( \mathcal{O}(-E)_{|\hat{D}_1} \) is trivial and (B1) follows. To prove (B2), we check that \( \beta_1 t^\vec{v}_1 \) is admissible with vanishing obstruction class. First note that the class \( B_0 \) of a \( J_0 \)-holomorphic sphere with \( u \cdot D = \vec{v}_1 \) has minimal symplectic area (\( \epsilon_Z = 1/m \)) and thus cannot bubble. Next, observe that via the argument of Lemma 6.4 the obstruction class can be calculated using a Gromov–Witten invariant. To calculate this, we may use the standard integrable complex structure where the moduli space of spheres is empty except in the class \( B_0 = [E_{\mathbb{C}P^1}] \) of the fibers of the projection \( E \rightarrow \hat{M} \) (exceptional spheres). Moreover, the spheres of class \( [E_{\mathbb{C}P^1}] \) are precisely the collection of exceptional spheres. These spheres are checked to be regular by applying [54, Lemma 3.3.1]. As the restriction of the exceptional divisor \( E \) to \( X \) is principal, it follows that \([E] = 0 \in H^2(X)\).

We take \( J_0 \in \mathcal{J}(V) \) to be any almost-complex structure preserving \( E \). To prove (B3) and in particular equation (6.15), let \( B_1 \) be such a homology class. Note that any sphere meeting \( \hat{D}_2 \) with multiplicity 1 must intersect \( E \) non-negatively. As a consequence, it can meet \( \hat{D}_1 \) at most once. It must therefore have non-negative weighted intersection with the remaining divisors and thus the total area of this sphere is either \( 1 - \epsilon_Z \) or 1. The latter case is impossible as spheres in class \( B_1 \) have area \( 1 - \epsilon_Z \).

To prove (B4)(a), assume \( B' \cdot D \neq \vec{v}_1 \) or \( \vec{v}_j \). Observe that it cannot intersect some \( \hat{D}_i \) for \( i \geq 3 \) with positive multiplicity. Otherwise, its symplectic area (with respect to the rational polarization) would be at least 1, which is the same as the area of \( B \). It follows that \( B' \cdot D = d \vec{v}_1 \) with \( d > 1 \). However, then the spheres in class \( B'' \) must have a component which meets \( \hat{D}_2 \) with multiplicity one, which by the preceding paragraph must have area at least \( 1 - \epsilon_Z \). Hence, there can be no such bubbling. Condition (B4)(b) is immediate from simple area considerations — if the class \( B \) had positive area and strictly negative intersection with one of the divisors...
Let \( \hat{D}_i, i \geq 3 \), it would need to have intersection strictly bigger than one with either \( \hat{D}_1 \) or \( \hat{D}_2 \) to compensate. The final claim about the Gromov–Witten invariant can again be checked using the integrable complex structure. All spheres in the class \( B \) lie in the fiber of the conic fibration \( \pi_{\hat{M}}: \hat{M} \to \bar{M} \) and there is a unique such sphere (bubbled into two components in classes \( B_0 \) and \( B_1 \) where the conic fibration is singular) passing through every point. \( \square \)

**Remark 6.16.** One may generalize this example somewhat by considering the blow-up of \( M := \bar{M} \times \mathbb{C}P^s \) at \( Z \times \bigcap_{i=2}^{s+1} H_i \).

### 6.3. Applications to embedding problems

We finally turn to the proofs of our last two applications. We first need a lemma (Lemma 6.19) which is a variant of [76, Theorem 1.7] for quasi-dilations. We will now summarize some of the ingredients from [76, 78] which will be used in the proof of Lemma 6.19 and the remaining facts required from these papers are reviewed in Appendix B. Let \((M, D)\) be a pair and \( \Psi \in HF^1(X \subset M; H_{m\lambda}^2) \) (the motivation for using Hamiltonians of slope 2\( \lambda \) will be made clear momentarily). Then [76, § 4(e)] introduced certain \( \Psi \)-twisted Floer cohomology groups \( \tilde{H}^*(X) \) which fit into a long exact sequence

\[
\cdots \tilde{H}^{s-1}(X) \to HF^{s-1}(X \subset M; H_{m\lambda}^{-\lambda}) \xrightarrow{-\Psi} HF^s(X \subset M; H_{m\lambda}^\lambda) \to \tilde{H}^s(X) \to \cdots ,
\]

(6.32)

where \( \Psi \cdot \) denotes multiplication with \( \Psi \).

Suppose \( Q \) is an exact (closed, oriented) Lagrangian submanifold in \( X \). We assume that \( Q \) is equipped with the additional choices needed to make it an object of the \( \mathbb{Z} \)-graded Fukaya category of \( X \) and let \( CF^\ast(Q, Q) \) denote the Lagrangian Floer cochain complex of \( Q \). For any \( \lambda > 0 \), there is a closed-open map (see [78, § 2] for a review)

\[
\phi_Q^0: CF^\ast(X \subset M; H_{m\lambda}^{2\lambda}) \to CF^\ast(Q, Q).
\]

(6.33)

**Definition 6.17.** Let \( \Psi \) be a cocycle in \( CF^1(X \subset M; H_{m\lambda}^{2\lambda}) \). A \( \Psi \)-equivariant structure on \( Q \) is a choice of \( c_Q \in CF^0(Q, Q) \) such that \( \partial(c_Q) = \phi_Q^0(\Psi) \). Two \( c_Q \) are equivalent if their difference is a coboundary.

Given a cohomology class \( \Psi \in HF^1(X \subset M; H_{m\lambda}^{2\lambda}) \), a \( \Psi \)-equivariant structure on \( Q \) is the choice of a cochain-level lift \( \Psi' \) of \( \Psi \) together with a \( \Psi' \)-equivariant structure on \( Q \). Note that if \( \Psi = \Psi' + \partial_{CF}(a) \) the space of \( \Psi \) and \( \Psi' \) equivariant structures up to equivalence are canonically identified.

In what follows, we will sometimes use \( \hat{Q} \) to denote a Lagrangian with an equivariant structure when we wish to suppress the particular choice of \( c_Q \). Given a pair of equivariant Lagrangians \( \hat{Q}_0, \hat{Q}_1 \), we can introduce certain canonical operators (see (B.2) for more details)

\[
\Phi_{\hat{Q}_0, \hat{Q}_1}: HF^\ast(\hat{Q}_0, \hat{Q}_1) \to HF^\ast(\hat{Q}_0, \hat{Q}_1).
\]

(6.34)

Let us assume that the coefficient ring \( k \) is a field and let \( \bar{k} \) denote the algebraic closure of \( k \). Then we can decompose \( HF^\ast(\hat{Q}_0, \hat{Q}_1) \otimes_k \bar{k} \) into a direct sum (over \( \sigma \in \bar{k} \)) of its generalized \( \sigma \)-eigenspaces \( HF^\ast(\hat{Q}_0, \hat{Q}_1)_\sigma \) for \( \Phi_{\hat{Q}_0, \hat{Q}_1} \). We can define the Lefschetz super-trace of \( \hat{Q}_0, \hat{Q}_1 \) by the formula

\[
\hat{Q}_0 \cdot \hat{Q}_1 = \sum_{\sigma} \sigma \cdot \chi(HF^\ast(\hat{Q}_0, \hat{Q}_1)_\sigma) \in k,
\]

(6.35)

where \( \sigma \) runs over elements of \( \bar{k} \) and \( \chi(HF^\ast(\hat{Q}_0, \hat{Q}_1)_\sigma) \) is the Euler characteristic of the \( \sigma \)-eigenspace.
The following is the main Floer-theoretic result of [76].

**Theorem 6.18.** The groups \( \bar{H}^* \) have a canonical non-degenerate pairing \( I \). Furthermore, for any equivariant Lagrangian \( \bar{Q} \), there is a canonical element \( [[\bar{Q}]] \in \bar{H}^* \) such that for any pair of equivariant Lagrangians \( \bar{Q}_0, \bar{Q}_1 \), we have that

\[
I([[\bar{Q}_0]], [[\bar{Q}_1]]) = (-1)^{(n+1)/2} \bar{Q}_0 \cdot \bar{Q}_1.
\]

This should be thought of as a kind of Cardy relation in the (infinitesimally) equivariant context. We are now in a position to prove

**Lemma 6.19.** Let \((M, D)\) be a pair with \( n = \dim_{\mathbb{C}} M \) odd. Suppose that there exists a slope \( \lambda \) such that

- (a) there exists an element \( \Psi \in HF^1(X \subset M; H_m^{2\lambda}) \) such that \( c_{\lambda, \infty}(\Psi) \in SH^1(X, \mathbb{Z}) \) defines the degree 1 piece of an integral quasi-dilation;
- (b) the natural map \( H^* (X, \mathbb{Z}) \to HF^* (X \subset M; H_m^{\lambda}) \) is an isomorphism.

Suppose further that \( Q_1, \ldots, Q_r \) is a collection of embedded Lagrangian spheres which are pairwise disjoint. Then for any field \( k \), the classes \([Q_1], \ldots, [Q_r]\) span a subspace of \( H_n(X, k) \) which has rank at least \( r/2 \).

**Proof.** The proof follows the proof of [76, Theorem 1.7] almost verbatim. (The only modification we must make is that in our situation, there are non-trivial signs appearing in the computation [76, (3.17)].) More precisely, fix a field \( k \). We can transfer the element \( \Psi \) to obtain a quasi-dilation over \( k \). For the remainder of the proof all (Floer) cohomology groups will be taken to have \( k \)-coefficients. Because the \( Q_i \) are spheres and \( H^1(Q_i, k) = 0 \), all \( Q_i \) can be given a \( \Psi \)-equivariant structure \( \bar{Q}_i \). We use these \( \Psi \)-equivariant structures to obtain endomorphisms \( \Phi_{Q_i, Q_j} \) on \( HF^*(Q_i, Q_i) \) according to the formula from (B.2). By equation (B.3), we have that this operation acts by \( \pm 1 \) on the degree \( n \) piece (here we use assumption (a) and in particular that our quasi-dilation is defined over \( \mathbb{Z} \)). By (B.4), we therefore have that the Lefschetz traces from (6.35) are given by (compare with [76, equation (3.17)])

\[
\bar{Q}_i \cdot \bar{Q}_j = \begin{cases} 
\pm 1 & \text{for } i = j \\
0 & \text{for } i \neq j.
\end{cases}
\]

(6.36)

Theorem (6.18) then implies that the elements \( [[\bar{Q}_i]] \in \bar{H}^*(X) \) are linearly independent and generate a subspace of dimension \( r \).

The crucial point in Seidel’s proof is that the image of \( HF^*(X \subset M; H_m^{\lambda}) \to \bar{H}^*(X) \) from (6.32) is isotropic for the pairing \( I \).\(^1\) Hence, the intersection of the subspace generated by \( [[\bar{Q}_i]] \) with this subspace has at most rank \( r/2 \). Using the exact sequence (6.32), it therefore follows that if we let \( c \) denote the natural map \( c: \bar{H}^*(X) \to HF^*(X \subset M; H_m^{-\lambda}) \), \( c(\text{Span}([[\bar{Q}_i]])) \) has dimension at least \( r/2 \). Now by dualizing assumption (b), we have that the natural map \( \text{PSS}^\vee: HF^*(X \subset M; H_m^{-\lambda}) \to H_n(X, k) \) is an isomorphism. Lastly, by combining [76, equation (3.11)] with the discussion below [76, (2.39)], \( \text{PSS}^\vee \circ ([[\bar{Q}_i]]) = [Q_i] \). It follows that \( \text{Span}([[Q_i]]) \in H_n(X, k) \) has dimension at least \( r/2 \) as desired. \( \square \)

**Theorem 6.20.** Let \( k \) be a field and \( n \geq 3 \) be an odd integer. Suppose that \( X \) is an affine conic bundle of total dimension \( n \) over an affine variety \( X^o \) with trivial canonical bundle and that \( Q_1, \ldots, Q_r \) is a collection of embedded Lagrangian spheres in \( X \) which are pairwise

\(^1\)This is true for any \( \Psi \in HF^1(X \subset M; H_m^{2\lambda}) \), not just dilations or quasi-dilations.
disjoinable. Then the classes \([Q_1], \ldots, [Q_r]\) span a subspace of \(H_n(X, k)\) which has rank at least \(r/2\).

Proof. Choose a compactification \((\hat{M}, \hat{D})\) associated to a good compactification of \(Z^o \hookrightarrow X^o\). Choose the parameters \(\epsilon, \delta, \eta, \delta_0\) appearing in the definition of \(X\) to be sufficiently small. Then because the coefficient of \(\hat{D}_1\) in (6.31) is the smallest, the period computations in [56, Theorem 5.16] show that the Reeb orbit which winds once around \(\hat{D}_1\) is the Reeb orbit of lowest period \(\lambda_0\). Choose a generic slope \(\lambda\) such that \(2\lambda > \lambda_0 > \lambda\). Then \(\Psi := \text{PSS}^{2\lambda}(\beta_{1,\lambda}, H^1, Z_\beta) \in HF^*(X \subset \hat{M}; H^{\lambda}_m)\) defines the degree 1 part of a quasi-dilation by combining Theorem 6.10 and Lemma 6.14. Moreover, we have an isomorphism \(H^*(X) \cong HF^*(X \subset \hat{M}; H^{\lambda}_m)\). Now apply Lemma 6.19.

\[\text{Corollary 6.21.} \quad \text{For any embedded Lagrangian sphere } Q \hookrightarrow X, \text{ the class } [Q] \in H_n(X, \mathbb{Z}) \text{ is non-zero and primitive.}\]

Proof. Letting \(r = 1\) in Theorem 6.20, we see that the class \([Q] \in H_n(X, k)\) is non-zero for every field \(k\). This implies the statement of the corollary.

6.3.1. Lagrangians in three-dimensional affine conic bundles. We now turn to our other main application involving three-dimensional conic bundles over affine surfaces:

\[\text{Theorem 6.22.} \quad \text{Let } X^o \text{ be an affine surface with trivial canonical bundle and let } X \text{ be a three-dimensional conic bundle over } X^o \text{ of the form (6.30). Suppose that } j : Q \hookrightarrow X \text{ is a closed, oriented, exact Lagrangian submanifold of } X. \text{ Then } Q \text{ is diffeomorphic to } S^1 \times B \text{ for } B \text{ of genus at least 1 or } \#_n S^1 \times S^2 \text{ (as usual } n = 0 \text{ corresponds to } S^3 \text{ by convention).}\]

Proof. This follows immediately by combining Corollary 5.18 and Lemma 6.14.

The classification of Theorem 6.22 is in a sense optimal. To explain this, note first that if \(B\) is an exact embedded genus \(g\) surface in some \(X^o\), then \(S^1 \times B\) is exactly embedded in \(\mathbb{C}^* \times X^o\) (one can construct similar examples when \(Z^o\) is non-empty). To describe how one constructs exact Lagrangian \(#_n S^1 \times S^2\) in conic bundles, it is useful to recall that a semi-free group action on a manifold \(Q\) is a non-trivial \(S^1\) action such that the isotropy subgroup of every point is either \(\{id\}\) or all of \(S^1\). Let \(Q\) be a closed, oriented 3-manifold with a semi-free \(S^1\) action such that the quotient space \(Q/S^1 =: \Sigma_{g,h}\) is an oriented surface of genus \(g\) with \(h\) boundary components. Whenever \(h > 0\), we have that \(Q \cong \#_n S^1 \times S^2\) where \(n = 2g + h - 1\) (see [64, Theorem 1]). Turning to symplectic topology, one can arrange that the Kahler form on \(X\) is \(S^1\)-invariant under the action

\[e^{i\theta} \cdot (u, v, \bar{v}) = (e^{i\theta} u, e^{-i\theta} v, \bar{v}).\]

The fixed point locus of this action occurs where \(u = v = 0\), which we will assume lies at moment level set \(\mu = \epsilon Z\). Given an embedded exact Lagrangian surface \(j_o : \Sigma_{g,h} \rightarrow X^o\) with \(j_o^-(Z^o) = \partial \Sigma_{g,h}\), we let \(Q\) denote the set of points

\[\{x \subset X, \pi_{X^o}(x) \subset j_o(\Sigma_{g,h}), \mu(x) = \epsilon Z\}, \quad (6.37)\]

where \(\pi_{X^o}\) is the natural projection \(X \rightarrow X^o\). In nice examples, \(Q\) is an embedded exact Lagrangian 3-fold which inherits a semi-free action of \(S^1\) from the action on \(X\) (the action has fixed points which lie over \(\partial \Sigma_{g,h}\)). We give some specific examples in Appendix A.

For a given affine base \(X^o\), it may be possible to improve this classification. For example, in view of the explicit constructions of exact Lagrangians outlined above, it is natural to ask:
Question 6.23. Does there exist an exact embedding of $S^1 \times B$, for $B$ of genus at least 2 inside of a conic bundle over $(\mathbb{C}^*)^2$?

We will now explain, using ideas from mirror symmetry, that the answer to this question is 'no.' To recall how the mirror to $X$ is constructed following [7, 13], let $\mathcal{P}$ denote the Newton polytope of $f(x)$ in $M_\mathbb{R}$ and let $\Sigma$ be the fan in $M_\mathbb{R} \oplus \mathbb{R}$ associated with a coherent unimodular triangulation of $\mathcal{P}$. Set $X^\vee$ to be the associated toric variety which by definition carries a dense algebraic torus $(\mathbb{C}^*)^3$.

A function on $X^\vee$ is determined by its restriction to $(\mathbb{C}^*)^3$ and hence can be described by a unique character $(n,k) \in N \oplus \mathbb{Z}$ (here $N$ denotes the dual lattice to $M_\mathbb{Z} \subset M_\mathbb{R}$ as is standard in the literature on toric varieties). We set $\chi_{n,k} : X^\vee \to \mathbb{C}$ to be the function associated to a given character. Of course not every function on $(\mathbb{C}^*)^3$ extends to a function on $X^\vee$, but the definition of the toric variety $X^\vee$ implies that functions which do extend are determined in a straightforward way by the Newton polytope. Namely if $A$ denotes the set of lattice points inside the polytope, we have that $\Gamma(O_{X^\vee}) = \bigoplus_{n,k \in \mathcal{C}} \mathbb{C} \cdot \chi_{n,k}$, where

$$\mathcal{C} := \{(n,k) \in N \oplus \mathbb{Z} \text{ such that } n(m) + k \geq 0 \text{ for all } m \in A\}. \quad (6.38)$$

In particular, for any $n$, there is always a $k_0$ for which $\chi_{n,k_0}$ defines a regular function on $X^\vee$ whenever $k \geq k_0$. Note that multiplication of functions in this ring is just given by addition in the character lattice. Set $p = \chi_{0,1} - 1$ and let $H$ be the (anticanonical) divisor in $X^\vee$ defined by $p^{-1}(0)$. Then motivated by the geometry of Lagrangian torus fibrations, [7] have predicted that $X^\vee := X^\vee \setminus H$ should be a mirror to $X$. The assertion that these two spaces are mirror implies a number of predictions concerning the symplectic topology of $X$. Perhaps the simplest such prediction was recently confirmed by the following calculation from [13]:

Lemma 6.24 [13, Theorem 8.2]. There is an isomorphism of rings:

$$\Gamma(O_{X^\vee}) \cong \text{SH}^0(X, \mathbb{C}). \quad (6.39)$$

Under this isomorphism, the weight $n$ above is identified with the homology class of a Hamiltonian orbit in $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2$.

In the proof of Proposition 6.25, we will only use the following two properties/consequences of this isomorphism.

1. Pick any non-zero vector $n$ and let $k$ be a positive integer so that $\chi_{n,k}, \chi_{-n,k}$ both extend to functions on $X^\vee$ and hence $X^\vee$. Then we have that

$$\chi_{n,k} \cdot \chi_{-n,k} = \chi_{0,2k} = (1 + p)^{2k}. \quad (6.40)$$

We may use (6.39) to obtain classes in $\text{SH}^0(X, \mathbb{C})$ (which we give the same name) satisfying the same equation.

2. The only units corresponding to contractible orbits under (6.39) are all of the form $a \cdot p^d$ for some $d$ and $a \in \mathbb{C}^*$.

Proposition 6.25. Let $X$ be a three-dimensional conic bundle over $(\mathbb{C}^*)^2$ of the form (6.30) and such that the discriminant locus $Z^\vee$ is connected. Let $j : Q \hookrightarrow X$ be a closed, oriented, exact Lagrangian submanifold of $X$. Then $Q$ is diffeomorphic to either $T^3$ or $#_n S^1 \times S^2$.

Proof. In view of Theorem 6.22, it remains to rule out those manifolds $S^1 \times B$ with genus of $B \geq 2$. Let $j : Q \hookrightarrow X$ be a putative exact embedding of $S^1 \times B$ with $g \geq 2$. Recall from (5.4) that we have a canonical isomorphism:

$$\text{SH}^*(T^*Q) \cong \mathbb{H}^*(\mathcal{L}Q). \quad (6.41)$$
By (6.41) and Lemma 5.3 (which applies because \( Q \) is aspherical) we have that
\[
SH^0(T^*Q) \cong \mathbb{H}^0(LQ) \cong Z(\mathbb{C}[\pi_1(Q)]). \tag{6.42}
\]
Again using the fact that \( Z(\mathbb{C}[\pi_1(B)]) = \mathbb{C} \) (see, for example, [18, p. 564]), the center of \( \mathbb{C}[\pi_1(Q)] \) is of the form
\[
Z(\mathbb{C}[\pi_1(Q)]) \cong \mathbb{C}[h, h^{-1}], \tag{6.43}
\]
where \( h \) is a generator of \( \pi_1(S^1) \). For a possibly inhomogeneous element \( z \in SH^*(T^*Q) \), let
\[
[z]_0 \in SH^0(T^*Q)
\]
denote its degree 0 piece. As noted in the discussion following (5.7), because \( Q \) is aspherical, we have that \( \mathbb{H}^*(LQ) \) is concentrated in non-negative degree (and hence the same is true for \( SH^*(T^*Q) \) in view of (6.41)). This implies that if \( z \) is an invertible (possibly inhomogeneous) element of \( SH^*(T^*Q) \), \([z]_0 \) is invertible in \( SH^0(T^*Q) \).

For the remainder of the proof, we identify elements of \( SH^0(X) \) with functions on \( X^\vee \) using (6.39) and \( SH^0(T^*Q) \cong \mathbb{C}[h, h^{-1}] \) using the composition of (6.42) and (6.43). Since \( p \in SH^0(X) \) is invertible, then as noted above, we have that \([j^!(p)]_0 \) is also invertible, where \( j^! \) is the Viterbo restriction map (2.33) from \( SH^*(X) \) to \( SH^*(T^*Q) \). Then
\[
[j^!(p)]_0 = b \cdot h^s \tag{6.44}
\]
for some integer \( s \) and \( b \in \mathbb{C}^* \) because these are the only units in a Laurent polynomial ring. Next, recall the quasi-dilation
\[
(\xi_{\lambda, \infty} \circ \text{PSS}_{\log}(\beta_1, c \cdot \tilde{v}_1, Z_b), \text{PSS}_{\log}(\alpha_1, c \cdot \tilde{v}_1))
\]
constructed in Theorem 6.10. We also have that
\[
\Delta(\xi_{\lambda, \infty} \circ \text{PSS}_{\log}(\beta_1, c \cdot \tilde{v}_1, Z_b)) = \text{PSS}_{\log}(\alpha_1, c \cdot \tilde{v}_1) = a \cdot p^d \tag{6.45}
\]
for some \( d \) and \( a \in \mathbb{C}^* \) because the only units corresponding to contractible orbits under (6.39) are all of this form.

This implies that \( s \) from (6.44) is non-zero because \([j^!(p^d)]_0 = \left(\left([j^!(p)]_0\right)^d\right)_0 \) is in the image of the BV-operator\(^1\) and hence must lie in (a summand of symplectic cohomology corresponding to) a non-zero free homotopy class. The fact that \( s \neq 0 \) in turn implies that \( j_*(h) = 0 \in H_1(X) \) because \( p \) corresponds to the trivial free homotopy class. Pick any non-zero vector \( n \) and let \( k \) be a positive integer so that \( \chi_{n,k}, \chi_{-n,k} \) both extend to functions on \( X^\vee \) and hence \( X^\vee \). Then \( s \neq 0 \) also implies that
\[
[j^!(1 + p)^{2k}]_0 = (1 + b \cdot h^s)^{2k} \neq 0. \tag{6.46}
\]
Hence, by equation (6.40), \([j^!(\chi_{n,k})]_0 \neq 0 \in SH^0(T^*Q) \cong \mathbb{C}[h, h^{-1}] \). However, this is a contradiction as \([j^!(\chi_{n,k})]_0 \) cannot be a Laurent polynomial in \( h \) because \( n \neq 0 \) but \( j_*(h) = 0 \).

**Remark 6.26.** The key equation (6.40) could also potentially be proven using the methods in this paper. Namely, consider the simplest case of conic bundles over \((\mathbb{C}^*)^2\) which compactify to conic bundles over \(\mathbb{CP}^2\) (in general \( M \) above can be taken to be a blow-up of \( \mathbb{CP}^2 \)) with its three toric divisors \( D_j \). Next, let \( \tilde{D}_j \) be the corresponding divisors in \( X \). Then the classes \( \text{PSS}_{\log}(\alpha_{\tilde{D}_j}, \tilde{v}_1) \) and \( \text{PSS}_{\log}(\alpha_{\tilde{D}_j \cup \tilde{D}_j}, \tilde{v}_1) \) corresponding to the fundamental chains on \( \tilde{S}_1 \) and \( \tilde{S}_{[2,3]} \) can be shown to satisfy (6.40) using a more complicated variant of Lemma 6.11.

\(^1\) We have \( \Delta(\frac{1}{n}[j^!(\xi_{\lambda, \infty} \circ \text{PSS}_{\log}(\beta_1, c \cdot \tilde{v}_1, Z_b)])_0 = [j^!(p^d)]_0 \) by (6.45) and Lemma 2.3.
Remark 6.27. It is reasonable to expect that for a given Laurent polynomial \( f \), there exists only finitely many \( n \) for which \( \#_n S^1 \times S^2 \) embeds as an exact Lagrangian in the conic bundle \( X \) determined by \( f \). Proving this, however, seems to require new ideas.

6.4. Knottedness of Lagrangian intersections

The main result of this section is Proposition 6.29, concerning the rigidity of the knot types of Lagrangian 3-manifolds meeting cleanly in a knot. This result is in the vein of Question 1.5 stated in the introduction and builds upon a forthcoming paper by Evans-Smith–Wemyss. First, some preliminaries: Let \( S_1 \) and \( S_2 \) be two 3-spheres, and \( S^1 \hookrightarrow S_i \) a pair of knots together with an identification of normal bundles

\[ \eta : \nu_{S^1} S_1 \cong \nu_{S^1} S_2. \]

It is well known that the above data determines a Stein manifold,

\[ W_{\eta}(\kappa_1, \kappa_2) = T^* S_1 \#_{S^1, \eta} T^* S_2, \]

the result of plumbing the cotangent bundles of \( S_i \) along the knots \( \kappa_i \), using the identification \( \eta \) of normal bundles of \( \kappa_1 \) with \( \kappa_2 \). We briefly recall the construction of \( W_{\eta}(\kappa_1, \kappa_2) \), referring the reader to \([4, \S A]\) for complete details. Choose Riemannian metrics on \( S_i \) (assume for simplicity they induce the same metric on \( S^1 \)) and let \( D^* S_i \) denote the disc cotangent bundle with respect to this metric. Then by Weinstein’s isotropic embedding theorem there are open neighborhoods \( US_i^1 \subset D^* S_i \) which are symplectomorphic to a disc subbundle of the symplectic vector bundle \( \nu_{S^1} S_i \oplus \mathbb{C} = \nu_{S^1} S_i \oplus \sqrt{-1} \nu_{S^1} S_i \) pulled back to \( D^* S^1 \) (of small radius). The Weinstein neighborhoods can be chosen so that the images of the real factors \( \nu_{S^1} S_i \) over the zero section of \( D^* S^1 \) are precisely \( US_i^1 \cap S_i \). \( W_{\eta} \) is obtained by gluing \( US_1^1 \) to \( US_2^1 \) by the map \( \eta \otimes \sqrt{-1} \) obtained from composing the given isomorphism of normal bundles with multiplication by \( \sqrt{-1} \) in the complexified bundle.

These manifolds \( W_{\eta}(\kappa_1, \kappa_2) \) retract onto \( S_1 \cup S_1 S_2 \), from which it follows using Mayer–Vietoris that they all satisfy

\[ H^2(W_{\eta}) \cong H_2(W_{\eta}) \cong \mathbb{Z} \]  \hspace{1cm} (6.47)

\[ H^3(W_{\eta}) \cong H_3(W_{\eta}) \cong \mathbb{Z} \oplus \mathbb{Z} \]  \hspace{1cm} (6.48)

and all other (co)homology groups in positive degree vanish.

These Stein manifolds are the local models for spheres intersecting cleanly in a circle. More precisely, if \((X, \omega)\) is any symplectic manifold which contains a pair of Lagrangian spheres \( S_i \) meeting cleanly along a circle, then \( \omega \) induces an isomorphism \( \eta \) of the underlying (unoriented) normal bundles. An open neighborhood of \( S_1 \cup S_1 S_2 \) is then symplectically equivalent to \( W_{\eta}(\kappa_1, \kappa_2) \). In the case that \( X \) is itself a Liouville domain, the inclusion of a small closed tubular neighborhood of \( S_1 \cup S_1 S_2 \) will be a Liouville embedding (this follows from the vanishing of \( H^1(W_{\eta}) \)).

The normal bundles \( \nu_{S^1} S_i \) are trivial and the space of identifications (up to homotopy) form a torsor over \( \mathbb{Z} \). We next observe that it is possible to fix a canonical base-point for this torsor. To do this, note that there are two possible fibered (Polterovich) Lagrangian surgeries of the zero sections \( S_1 \) and \( S_2 \). Observe that because \( S^1 \subset W_{\eta} \) is isotropic, a neighborhood \( U(S^1) \subset W_{\eta} \) may be identified with the total space of a disc subbundle of a rank four symplectic vector bundle \( E \) over \( D^* S^1 \). The normal bundles \( \nu_{S^1} S_i \) give rise to transverse Lagrangian subbundles of \( E \), showing that \( E \) is in fact a trivial symplectic bundle. In view of this, we may perform the Polterovich surgery construction in each fiber, where one performs one of the two possible surgeries of 2-planes \( \mathbb{R}^2 \cup i\mathbb{R}^2 \subset \mathbb{C}^2 \). The result are exact Maslov index 0 Lagrangian submanifolds \( K, K' \) (see, for example, the discussion in \([52, \S 2.2.2]\) for more details).
In the case $\kappa_i$ are both linearly embedded unknots, the resulting manifolds $K$ and $K'$ are given by Dehn surgery of integer slope (depending on the identification of the normal bundles) and hence give rise to a $S^1 \times S^2$, $S^3$ or a lens space. Let $W_n$ denote the unique plumbing for which $H_1(K, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. In their study of $W_1$ (and related plumbings $W_n$), Evans–Smith–Wemyss proved a result in the direction of a negative answer to Question 1.5:

PROPOSITION 6.28 (forthcoming work of Evans–Smith–Wemyss). If there is a Hamiltonian isotopy $\tilde{S}_1$ of $S_1$ and $\tilde{S}_2$ of $S_2$ in $W_1$ so that $\tilde{S}_1$ meets $\tilde{S}_2$ cleanly along a knot, then this knot must be the unknot in one component and the unknot or trefoil in the other component.

The goal of this section is to prove the following:

PROPOSITION 6.29. Given two Lagrangians spheres $\tilde{S}_1$ and $\tilde{S}_2$ in $W_1$ so that $\tilde{S}_1$ meets $\tilde{S}_2$ cleanly along a knot, then this knot must be the unknot in both components.

In particular, our main result rules out the remaining trefoil possibility (and along the way gives an alternate proof of Proposition 6.28). The key starting point for us is that, as observed by Evans–Smith–Wemyss, $W_1$ embeds into an explicit affine variety. Namely, let $M$ be a flag 3-fold, realized as a hypersurface of bidegree (1,1) in $\mathbb{P}^2 \times \mathbb{P}^2$ and $D$ be a smooth hyperplane $(1,1)$ section.

LEMMA 6.30 (forthcoming work of Evans–Smith–Wemyss). There is an embedding $W_1 \hookrightarrow M \setminus D$.

Proof. $M$ is explicitly the hypersurface given by
\[ M := \{xx' + yy' + zz' = 0 \subset \mathbb{P}^2 \times \mathbb{P}^2 \}\]
Consider the pencil of (1,1) hypersurfaces in $M$ given by
\[ D_{\mu_1, \mu_2} := \{\mu_1(xx' - zz') + \mu_2(yy') = 0\} \cap M, \quad [\mu_1 : \mu_2] \subset \mathbb{P}^1.\]
The fiber $D_{1,0}$ is a smooth complex surface and contains the base locus of the pencil which is the locus $\{xx' = zz' = yy' = 0\} \cap M$. Thus, setting $X = M \setminus D_{1,0}$, we obtain a fibration $W : X \to \mathbb{C}$ with general fiber $(\mathbb{C}^*)^2$. Furthermore, it is not difficult to check that this fibration is of Morse–Bott type with critical values $\{-1,0,1\}$ and critical fibers isomorphic to $(\mathbb{C}^*) \times (\mathbb{C} \cup \mathbb{C})$. The map $W$ is $T^2$-equivariant with respect to the torus which acts on the ambient $\mathbb{P}^2 \times \mathbb{P}^2$ by the formula
\[ (\theta, \phi) \cdot ([x : y : z], [x' : y' : z']) = ([e^{i\theta}x : y : e^{i\theta}z], [e^{-i\phi}x' : y' : e^{-i\phi}z']).\]
Consider the paths $\gamma_1 := [-1, 0] \subset \mathbb{C}$ and $\gamma_2 := [0, 1] \subset \mathbb{C}$. For each of the paths $\gamma_1$, there is a $T^2$ equivariant Lagrangian 3-sphere $S_i$ which is the union of two Morse–Bott Lefschetz thimbles (a Morse–Bott matching cycle) fibering over that path. Let $W_i$ denote the restriction of $W$ to $S_i$. For any $c \in \gamma_i$, the fibers $W_i^{-1}(c)$ are orbits of the group action and moreover this action is free unless $c \in \{-1, 0, 1\}$. Over these endpoints, the action on $S_i$ has $S^1$ stabilizer. We have that $S_1 \cap S_2 = W_1^{-1}(0) = W_2^{-1}(0) \cong S^1$. Taking a Weinstein tubular neighborhood of the configuration $S_1 \cup S_1, S_2 \hookrightarrow X$ gives the desired embedding of $W_1 \hookrightarrow X$. □

REMARK 6.31. It is not difficult to see that the inclusion $W_1 \hookrightarrow M \setminus D$ is a homotopy equivalence.\footnote{Evans–Smith–Wemyss in their forthcoming work shows more generally that there is a unique normal identification for which $H_1(K, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. They denote by $W_n(\kappa_1, \kappa_2)$, the resulting plumbing.}

\[ \text{In fact, Evans–Smith–Wemyss show that } W_1 \text{ is deformation equivalent to } M \setminus D \text{ as Stein manifolds, but we do not need this stronger statement.} \]
It is worth mentioning at this stage that $D$ is a del Pezzo surface of degree 6 (this will account for the use of characteristic 3 coefficients in our arguments). To check this, note that by the adjunction formula, we have that $\mathcal{O}(D)_{|D} = K_D^{-1}$ and so to calculate the degree we need to calculate $D_N \cdot D_N$ where $D_N$ is a smooth divisor representing $\mathcal{O}(D)_{|D}$. Let $f : M \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ be the embedding of $M$ into $\mathbb{P}^2 \times \mathbb{P}^2$. Set $H_1 := H \times [\mathbb{P}^2] \in H_6(\mathbb{P}^2 \times \mathbb{P}^2)$ and $H_2 := [\mathbb{P}^2] \times H \in H_6(\mathbb{P}^2 \times \mathbb{P}^2)$ ($H$ denotes the hyperplane class in $H_2(\mathbb{P}^2)$) so that $[M] = H_1 + H_2 \in H_6(\mathbb{P}^2 \times \mathbb{P}^2)$. We have that $f_*(|D|) = f_*(H_1 + H_2)_{|M} = (H_1 + H_2)^2 \in H_4(\mathbb{P}^2 \times \mathbb{P}^2)$ and so

$$D_N \cdot D_N = (H_1 + H_2)^2 \cdot (H_1 + H_2)^2 = 6.$$  

(6.49)

Because $W_1$ embeds in $M \setminus D$, we can use a variant of the argument of Theorem 6.2 (compare also with [75, Conjecture 18.6]) to prove:

**Proposition 6.32.** $W_1$ admits a dilation over any field $k$ of characteristic 3.

**Proof.** By Viterbo functoriality (see (2.33)), it suffices to prove this for $M \setminus D$. We explain how to modify Theorem 6.2 and use the notation from that Theorem and its proof. We have that $m = 2$ and we let $\alpha_0 = 1 \in H^0(M)$. Choose $D_N$ as before and we set $L := Bl_{D_N} D \subset S_D$ to be the real-oriented blow-up along $D_N$ whose boundary is $\partial L := \pi^{-1}(D_N) \subset S_D$. Then $\mathcal{O}(D)$ restricted to $D_N$ has degree 6 by equation (6.49). Choose a point $p$ on $D_N$ and let $K' := Bl_p(D_N)$ be the real-oriented blow-up of $D_N$ at $p$, which is again a submanifold inside of $SO(p)$, the circle bundle associated to $\mathcal{O}(p)$. Because $(S_D)_{|D_N}$ has degree 6, there is a natural 6:1 fiberwise covering $w : SO(p) \rightarrow (S_D)_{|D_N}$ over $D_N$:

$$\xymatrix{ SO(p) \ar[r]^w & (S_D)_{|D_N} \ar[d]_{\pi_D} \\
& D_N \ar[u]_{\pi_f} }$$

We let $K$ be the pushforward of $K'$ to $S_D$ along this map.

Along its boundary, $K$ is a 6:1 covering onto the fibers of $(S_D)_{|D_N}$ over $p$ and hence $K$ represents a class in $H_2(S_D, k) \cong H^3(S_D, k)$. Moreover, the pushforward $j_{D,N}(D_N) = 3(A_1 + A_2) \in H_3(M)$, where $A_1$ and $A_2$ are the line classes coming from the two $\mathbb{P}^2$ factors. This implies, by a variant of Lemma 6.4, that equation (6.3) holds as well with $k$ coefficients. Thus we can choose some bounding cochain $Z_b$ to define a symplectic cohomology class $\text{PSS}^4_{\text{log}}(Kt, Z_b)$. Finally, we note that $GW_M(\text{PD}(D), \alpha_1, pt) = 2$ (see, for example, [45, § 4] for a convenient reference) and so following Lemma 6.3, we have

$$\Delta(\text{PSS}^4_{\text{log}}(Kt, Z_b)) = -\text{PSS}^4_{\varphi_8}(GW_{\varphi_4}(L)) = -2.$$  

□

**Remark 6.33.** We note that although our typical ground ring convention in this paper is $k = \mathbb{Z}$, $\mathbb{C}$ or $\mathbb{Q}$, the above argument works with $k$ a field of characteristic 3. To explain both the reason for our usual convention and why the above argument encounters no related issues; see Remark 4.9.

**Lemma 6.34.** Let $i : Q \hookrightarrow W_1$ be an exact Lagrangian embedding of a closed orientable manifold. Then $Q$ admits a prime decomposition with prime summands $S^1 \times S^2$ and spherical spaces forms $Q$, such that $|\pi_1(Q)|$ is not divisible by 3.

**Proof.** This follows from Viterbo functoriality by combining Proposition 6.32 together with the claim (2) of Lemma 5.17 (with $p = 3$). □
Remark 6.35. In particular, none of the summands \(Q_i\) can be a Poincaré homology sphere \(S^3/\pi\) with \(|\pi| = 120\). This is sufficient to rule out the case of the trefoil from Proposition 6.28 because in the case of a trefoil the Polterovich surgery of \(\tilde{S_1}\) and \(\tilde{S_2}\) would be a Poincaré homology sphere. Note that Proposition 6.29 is slightly stronger as it concerns a priori arbitrary Lagrangian spheres \(\tilde{S_1}\) and \(\tilde{S_2}\) meeting cleanly in a circle.

We record one final preparatory lemma before proceeding with the proof of Proposition 6.29.

Lemma 6.36. The only spherical space forms \(Q\) such that \(|\pi_1(Q)|\) is not divisible by 3 and \(|H_1(Q)| \leq 2\) are \(S^3\) and \(\mathbb{R}P^3\).

Proof. This statement is obvious if one restricts to lens spaces. The general case follows from the classification of spherical space forms; see [83, Theorem 2.2] for a table listing the fundamental groups of all spherical space forms which are not lens spaces. In particular, inspecting this classification shows that all of the space forms (other than lens spaces) such that \(|\pi_1(Q)|\) is not divisible by 3 are 'prism manifolds.' There are two types of prism manifolds. The first type have fundamental group

\[ Q_{4n} := \langle x, y | x^n = y^2, yxy = y \rangle \] (6.50)

for some \(n \geq 1\). The abelianization (= \(H_1(Q)\)) of \(Q_{4n}\) has order 4 (see the beginning of [83, §3]). In the second case, the fundamental group is

\[ B_{2k(2n+1)} := \langle x, y | x^{2k} = y^{2n+1} = 1, yxy^{-1} = y^{-1} \rangle \] (6.51)

for some \(k \geq 2, n \geq 0\). As discussed in [83, §5.2], the abelianization of \(B_{2k(2n+1)}\) has order \(2^k > 2\).

Proof of Proposition 6.29. Form a Polterovich surgery \(\tilde{Q}\) of \(\tilde{S_1}\) and \(\tilde{S_2}\), which as we have seen will be an exact, orientable Lagrangian. If both knots are non-trivial, then \(Q\) contains an incompressible torus and is irreducible by [37, Proposition 2.3]. Lemma 6.34 shows that the only possible irreducible Lagrangians in \(W_1\) are spherical space forms. As none of the spherical space forms contain an incompressible torus, it follows that at least one of the knots is the unknot. Without a loss of generality we can assume this is \(\kappa_1\), and the surgery is then a Dehn surgery on \(\kappa_2\).

It is known that a manifold \(Q\) which arises from surgery on a knot has a prime decomposition with at most three summands. If there are three summands, then two of the summands are lens spaces and the third a homology sphere [41, Corollary 5.3]. Thus, because none of the summands arising in Lemma 6.34 are homology spheres (see Remark 6.35), there are at most two summands. Note as well that \(H_1(Q) \cong \mathbb{Z}/n\mathbb{Z}\) where \(n\) is the surgery slope, which means that \(Q\) cannot be one of the manifolds in Lemma 6.34 with an \(S^1 \times S^2\) summand and a non-trivial spherical space form. It follows that \(Q\) is either

1. a spherical space form with \(|\pi_1(Q)|\) not divisible by 3;
2. a connected sum of two of these types of spherical manifolds;
3. \(Q = S^1 \times S^2\).

We will discuss each of these cases individually, beginning with

Case (1): Throughout the rest of the proof we let \(k\) be an algebraically closed field of characteristic 3. Suppose that \(Q\) is a spherical space form with \(|\pi_1(Q)|\) not divisible by 3 (note that this implies that \(Q\) is a \(k\)-homology sphere) and fix a generator for \(H_1(Q, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}\). We

\[ \text{For instance because an incompressible torus gives rise to an injective map } \pi_1(T^2) \to \pi_1(Q). \]
may then equip $Q$ with $n$ rank one $k$-local systems $\psi_m$ corresponding to $\rho^m, m \in \{0, \ldots, n - 1\}$ where $\rho$ is a primitive $n$th root of unity.

For $m \neq 0$, the cohomology groups $H^i(Q, \psi_m) = 0$. This is obvious for $i = 0$ and follows for $i = 3$ using Poincaré duality in the form $H^i(Q, \psi_m) \cong H^{3-i}(Q, \psi_m^*)$. For $i = 1, 2$, we can again reduce the claim to $i = 1$ using duality. To check this case, recall that cohomology with coefficients in this local coefficient system is by definition the (hyper-)ext spectral sequence

$$\text{Ext}^p_{\mathbb{Z}[\pi_1(Q)]}(H_q(\hat{Q}), k) \Rightarrow H^{p+q}(Q, \psi_m)$$

which vanishes when $p = 0, q = 1$ because $\hat{Q}$ is a universal cover and vanishes when $p = 1, q = 0$ because the group cohomology $H^1(\pi_1, \psi_m) = 0$ (again we use that 3 does not divide $|\pi_1|$).

In the Fukaya category of $W_1$, we have that

$$\text{Hom}((Q, \psi_i), (Q, \psi_j)) \cong H^*(Q, \psi_{j-i}) = 0 \text{ if } i \neq j.$$

Thus, the objects $\{(Q, \psi_i)\}$ are pairwise disjoineable in the sense of [76]; hence their homology classes must span a subspace of rank at least $n/2$ by [76, Theorem 1.7], which applies because $Q$ is a $k$-homology sphere. However, these objects all represent the same homology class and hence $n \leq 2$; so by Lemma 6.36 we have that $Q = S^3$ or $\mathbb{R}P^3$. It follows from the main result of [50] that in either case $\kappa_2$ must be the unknot.

Case (2): Essentially the same argument works for a connected sum of two of these types of manifolds $Q = Q_1 \# Q_2$. Namely, let $\psi_m$ denote a rank one $k$ representation of $\pi_1(Q_1)$ considered in Case (1). Note that $\pi_1(Q) \cong \pi_1(Q_1) \ast \pi_1(Q_2)$ and extend $\psi_m$ to a representation of $\pi_1(Q)$ by letting $\pi_1(Q_2)$ act trivially. We denote the resulting representation of $\pi_1(Q)$ by $\hat{\psi}_m$. As in Case (1), the key vanishing of the groups $H^i(Q, \hat{\psi}_m)$ can be reduced to the vanishing of the group cohomology $H^1(\pi_1, \hat{\psi}_m)$ using Poincaré duality and the hyper-ext spectral sequence. We have that $K(\pi_1(Q))$ is homotopy equivalent to the wedge sum:

$$K(\pi_1(Q), 1) \cong K(\pi_1(Q_1), 1) \vee K(\pi_1(Q_2), 1).$$

By the Mayer–Vietoris sequence

$$H^1(\pi_1(Q, \hat{\psi}_m) \cong H^1(\pi_1(Q_1), \psi_m) \oplus H^1(\pi_1(Q_2), k) = 0.$$  

(The point is that the restriction map $H^0(\pi_1(Q_2), k) \to H^0(\psi, k)$ is an isomorphism because we have taken the representation to be trivial on the $\pi_1(Q_2)$ factor.) The same argument using [76, Theorem 1.7] shows that $Q_1$ is $\mathbb{R}P^3$ (or $S^3$ but then $Q$ is irreducible). By symmetry, we also deduce that $Q_2$ is $\mathbb{R}P^3$. However, then $H_1(Q)$ is not cyclic which is a contradiction.

Case (3): Finally, if $Q = S^1 \times S^2$, then again using the result of [50], $\kappa_2$ must be the unknot (and the surgery slope is zero). \hfill \Box

Remark 6.37. The idea of studying which diffeomorphism types of Lagrangians may arise as the surgery $Q$ of $S_1$ and $S_2$ is also borrowed from forthcoming work of Evans–Smith–Wemyss, who use a detailed classification of objects in the Fukaya category of $W_1$ (together with similar results from 3-manifold theory to those used above) to prove Proposition 6.28.

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1Here, as in Theorem 6.20, we use the fact that our dilation arises from a Hamiltonian which wraps around the divisor once; hence our manifold has ‘property (H)’ in the terminology of [76, Definition 2.11].
Appendix A. Examples of Lagrangian connected sums

In this Appendix, we give some specific examples of exact Lagrangian connected sums in affine conic bundles. Let \( X^o := (\mathbb{C}^*)^2 \) and \( Z^o \) be the zero locus of \( f(\bar{x}) = x_1 + x_2 + \frac{1}{x_1 x_2} - c \) for \( c > 3 \) (recall (6.30)). In this case, we can take our good compactification (recall Definition 6.12) so that \( \hat{\cal M} \) is a toric surface and \( \cal D \) is its toric boundary. We will need to choose a specific Kähler form on the conic bundle. To do this, note that \( \bar{\cal M} \) is a toric surface and \( \bar{\cal D} \) is its toric boundary. We will need to choose a specific Kähler form on the conic bundle. To do this, note that \( \hat{\cal M} \) is a toric surface and \( \cal D \) is its toric boundary. We will need to choose a specific Kähler form on the conic bundle. To do this, note that \( \cal M \) is a hypersurface in the \( \mathbb{C}P^1 \) bundle, \( \mathbb{P}(\mathcal{O}(Z) \oplus \mathcal{O}) \) over \( M := \cal M \times \mathbb{C}P^1 \) (in an abuse of notation we let \( \mathcal{O}(Z) \) be the line bundle associated to the hypersurface \( Z \times \mathbb{C}P^1 \subset \cal M \times \mathbb{C}P^1 \)). Let \( \bar{\omega} \) be a toric Kähler form on \( \cal M \) and lift this to a Kähler form on \( \hat{\cal M} \) by taking product with the standard symplectic structure on \( \mathbb{C}P^1 \). Then a choice of Hermitian metric on the line bundle \( \mathcal{O}(Z) \) induces a Kähler form on \( \mathbb{P}(\mathcal{O}(Z) \oplus \mathcal{O}) \) so that the \( \mathbb{C}P^1 \) fibers have area \( \epsilon_Z \). We can then restrict this form to the hypersurface \( \cal M \) to obtain a Kähler form which we denote by \( \hat{\omega} \). Away from \( \epsilon \) we have that

\[
\hat{\omega} = \pi^* \omega_M + i \epsilon_Z \partial \bar{\partial} \psi
\]  

(A.1)

for some potential \( \psi \).

Following [7, § 3.2], we will modify this Kähler form to a Kähler form \( \omega \) so that \( \omega \) agrees with \( \pi^* \omega_M \) outside of (the preimage of) a tubular neighborhood \( U \) of \( Z \times 0 \subset \cal M \times \mathbb{C}P^1 \) of \( \cal M \). Namely, we choose a cutoff function \( \chi : \cal M \to [0, 1] \) which is supported in \( U \) and which is \( S^1 \) invariant with respect to the rotation action on \( \mathbb{C}P^1 \). We require that \( \chi = 1 \) in a smaller open set about \( Z \times 0 \). We will also assume that the support of \( \chi \) is disjoint from the (preimage of the) exact Lagrangian torus \( \hat{\cal L}_0 \subset (\mathbb{C}^*)^2 \) given by

\[
|x_1| = |x_2| = 1.
\]  

(A.2)

On the complement of \( \epsilon \), we set

\[
\hat{\omega} = \pi^* \omega_M + i \epsilon_Z \partial \bar{\partial} (\chi \psi).
\]  

(A.3)

This extends to a form on all of \( \hat{\cal M} \) which again Kähler (after possibly shrinking \( \epsilon_Z \)).

Let \( \hat{\cal L}_1 \) denote be the ‘positive real Lefschetz thimble’, that is the portion of \( (\mathbb{R}_{\geq 0})^2 \) which fibers over the interval \([3, \epsilon]\),

\[
\hat{\cal L}_1 := \{(\mathbb{R}_{\geq 0})^2, f(\bar{x}) \in [3, \epsilon]\}.
\]  

(A.4)

\( \hat{\cal L}_0 \) and \( \hat{\cal L}_1 \) intersect at exactly one point \( p \) where \( x_1 = x_2 = 1 \) and we can form surgery at this intersection point to obtain an exact Lagrangian \( \Sigma_{1,1} \subset (\mathbb{C}^*)^2 \) with boundary on the hypersurface \( Z^o \). We let \( \hat{\cal L}_0, \hat{\cal L}_1, Q \) be the subspaces obtained using the constructions (6.37) from \( \hat{\cal L}_0, \hat{\cal L}_1, \Sigma_{1,1} \).

**Lemma A.1.** For a suitable choice of primitive \( \theta \), \( \hat{\cal L}_0, \hat{\cal L}_1, Q \) are all exact Lagrangian submanifolds of \( \cal X \).

**Proof.** It is obvious that \( \hat{\cal L}_0 \) is an embedded submanifold and the fact that \( \hat{\cal L}_1 \) and \( Q \) are embedded submanifolds is a local calculation near the boundary of \( \hat{\cal L}_1 \). Because \( \chi \) is supported away from \( \hat{\cal L}_0 \), it is also obvious that it is a Lagrangian submanifold (diffeomorphic to \( T^3 \)). It is also not difficult to make the Lagrangian \( \hat{\cal L}_0 \) exact by choosing a primitive which agrees with a suitable product primitive on \( \mathbb{C}^* \times (\mathbb{C}^*)^2 \). Let \( \cal X \setminus E \parallel S^1 \) denote the symplectic reduction of \( \cal X \setminus E \) at moment level set \( \mu = \epsilon_Z \). \( \cal X \setminus E \parallel S^1 \) is holomorphically identified with \( (\mathbb{C}^*)^2 \setminus Z^o \) by the projection map, and is equipped with a Kähler form \( \omega_{red} \) on \( (\mathbb{C}^*)^2 \setminus Z^o \).

As \( \hat{\cal L}_1 \) is fixed by an anti-holomorphic involution, it defines a Lagrangian with respect to \( \omega_{red} \). It is then an elementary fact in the theory of symplectic reduction that \( L_1 \) is a Lagrangian

\footnote{The Kähler form does not extend smoothly to all of \( (\mathbb{C}^*)^2 \); see [7, § 4.1]. However, this is not relevant for us.}
in the total space $X$ (which is automatically exact because it is diffeomorphic to $S^3$). The same argument applies to $Q$ because it agrees with $L_1$ when $\chi \neq 0$. Finally, the fact that $Q$ is exact as well for this choice of primitive follows from the fact that we have an isomorphism $H_1(L_0 \setminus p) \cong H_1(Q)$ and in particular we can choose representatives for the generators of $H_1(Q)$ which lie far away from the surgery locus (which is a local construction). Exactness of $Q$ therefore follows from exactness of $L_0$. \qed

We can construct examples with $n = k + 1$ by passing to $k$-fold covers of this conic bundle.

**Remark A.2.** Tropical geometry provides a general method for constructing Lagrangian surfaces in $(\mathbb{C}^*)^2$ (see, for example, [53]). It should be possible to ‘suspend’ some of these tropical Lagrangians to three-dimensional Lagrangians in conic bundles, potentially giving rise to a wide class of examples.

**Appendix B. More on $\Psi$-equivariant structures**

In general, unlike Hamiltonian Floer cohomology, Lagrangian Floer cohomology is not commutative. However, classes in the image of the maps (6.33) are central elements of Lagrangian Floer cohomology. To be precise, for any pair of exact Lagrangians $Q_0, Q_1$ (with brane structure) there is a canonically defined homotopy ([78, equation (2.12)]):

$$\phi_1 : CF^*(X \subset M; H_m^{2\lambda}) \otimes CF^*(Q_0, Q_1) \rightarrow CF^*(Q_0, Q_1)[-1]$$

such that for any $b \in CF^*(X \subset M; H_m^{2\lambda})$ and $a \in CF^*(Q_0, Q_1)$,

$$\mu_2(\phi_{Q_1}(b), a) - (-1)^{|a||b|}\mu_2(a, \phi_{Q_0}(b)) = \mu_1(\phi^1(b, a)) + \phi^1(db, a) + (-1)^{|b|}\phi^1(b, \mu_1(a)).$$

Given $\Psi$-equivariant structures on $Q_0, Q_1$ (recall Definition 6.17), we can, following [78, equation (4.16)], use this chain homotopy to define a chain map:

$$\Phi_{Q_0, Q_1} : CF^*(Q_0, Q_1) \rightarrow CF^*(Q_0, Q_1)$$

$$\Phi_{Q_0, Q_1}(a) = \phi_1(\Psi, a) - \mu_2(c_{Q_1}, a) + \mu_2(a, c_{Q_0}).$$

If $Q_0 = Q_1 = Q$, it is possible to calculate the endomorphism induced by (B.2) on cohomology (which we will in a slight abuse of notation also denote by $\Phi_{Q, Q}$) in certain degrees. Namely, by [76, Corollary 3.6], we have that $\Phi_{Q, Q}$ acts by 0 on $HF^0(Q, Q)$. More interesting is the action on $HF^*(Q, Q)$, where we have by [78, equation (4.16)] (see also discussion following [75, Lemma 18.1]) that for any class $a \in HF^*(Q, Q)$

$$\Phi_{Q, Q}(a) = \mu_2(\Phi^0(\Delta \Psi), a).$$

(B.3)

Note that when $\hat{Q}$ is an equivariant sphere, these two calculations determine $\Phi_{Q, Q}$ completely. In particular, by (B.3), we have that if $\hat{Q}$ is a sphere

$$\hat{Q} \cdot \hat{Q} = (-1)^n \phi^0_Q(\Delta \Psi) \in k,$$

(B.4)

where $\hat{Q} \cdot \hat{Q}$ is defined in (6.35).

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