ON ATTAINABILITY OF MOSER-TRUDINGER INEQUALITY WITH LOGARITHMIC WEIGHS IN HIGHER DIMENSIONS

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ABSTRACT. Moser-Trudinger inequality was generalised by Calanchi-Ruf to the following version: If $\beta \in [0,1)$ and $w_0(x) = |\log |x||^{\beta(n-1)}$ or $|\log e|^{\beta(n-1)}$ then
$$\sup_{f \in W^{1,n}_{0,rad}(w_0,B)} \int_B \exp\left(\alpha |u|^{n/(n-\beta)} \right) dx < \infty$$
if and only if $\alpha \leq \alpha_\beta = n \left[ \omega_{n-1}^\frac{1}{n-\beta} (1-\beta) \right]^{\frac{1}{n-\beta}}$ where $\omega_{n-1}$ denotes the surface measure of the unit sphere in $\mathbb{R}^n$. The primary goal of this work is to address the issue of existence of extremal function for the above inequality. A non-existence (of extremal function) type result is also discussed, for the usual Moser-Trudinger functional.

1. Introduction. Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^n$. Let $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_p := \left( \int_\Omega |\nabla u|^p \right)^\frac{1}{p}$. The $L^p$ norm of any function $u$ will be denoted by $|u|_p := \left( \int_\Omega |u|^p \right)^\frac{1}{p}$. For $p < n$ it is well known, from from Sobolev embedding theorem that the space $W_0^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$, if $1 \leq q \leq p^* (= np/(n-p))$. The value of $p^*$ is called the critical exponent for this embedding. In fact, the above embedding is compact whenever $q < p^*$. The Moser-Trudinger inequality concerns about the borderline case, that is when $p = n$. In this case
$$W_0^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$$
for all $p \in [1,\infty)$. For $p = \infty$ the above embedding does not hold. To observe this fact, take $n = 2$ and the function $f(x) = \log (1 - \log |x|)$ in unit ball $B$ centered at origin, then it is easy to check that $f \in W_0^{1,2}(B)$, but clearly $f$ is not in $L^\infty(B)$. Hence, one may look for maximal growth function $g$ such that
$$\int_\Omega g(u) dx < \infty, \; \forall u \in W_0^{1,n}(\Omega).$$

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Throughout this article $e^t$ or $\exp(t)$ will denote the exponential function. The following result is known as Moser-Trudinger inequality [24], [31]:

**Theorem 1.1.** Let $|\Omega| < \infty$, then

$$\sup_{u \in W^{1,n}_0(\Omega), ||u||_{\alpha} \leq 1} \int_{\Omega} \exp \left( \alpha |u|^{\frac{n}{n-1}} \right) dx < \infty,$$

iff $\alpha \leq \alpha_n := n \omega_{n-1}^{\frac{1}{n-1}}$ (1)

where $w_{n-1}$ is the $(n-1)$ dimensional surface measure of the unit sphere and $|\Omega|$ denotes the $n$ dimensional Lebesgue measure of the set $\Omega$.

Regarding the issues of existence of extremal functions for the above inequality, Carleson-Chang [6] showed that there exist at least one radial function for which the above supremum in (1) is attained, when the domain $\Omega$ is a ball. In [30], Struwe proved the case when the domain $\Omega$ is close to a ball. Flucher, in [11], provided a positive answer for the case of any general bounded domain $\Omega$ in 2 dimension. The higher dimension case for general bounded domains were done by Lin in [15]. To the best of our knowledge there exist no work that deals with the case when the domain is not bounded, but $|\Omega| < \infty$ (though we think that Flucher’s method goes through). Moser-Trudinger type inequality had been an interesting topic of research for several authors. For Moser-Trudinger inequality with singular weight we refer to [7, 8, 14, 19, 27]. For the study of this inequality in entire space and on manifolds we refer to [17, 23, 29, 33, 35]. Blow up analysis for Moser-Trudinger functional is dealt in [19, 25, 26, 36]. We list a few other related works in this direction [1, 2, 3, 10, 12, 13, 18, 16, 20, 22, 30, 32, 34] and the references there in for the available literatures in this direction.

Through out this article $B$ will denote the unit ball centered at origin. Consider the functions

$$w_0 = |\log |x||^{\beta(n-1)} \quad \text{or} \quad w_1 = \left( \log \frac{e}{|x|} \right)^{\beta(n-1)}$$
on $B$.

Let $W_0^{1,n}(w_i, B)$ denotes the usual weighted Sobolev space defined as the completion of $C^\infty_c(B)$ (the space of smooth functions with compact support) functions with respect to the norm

$$||u||_{w_i} := \left( \int_B |\nabla u|^n w_i dx \right)^\frac{1}{n}, \quad \text{for} \ i = 0, 1.$$

Note that $w_0$ and $w_1$ depends on the parameter $\beta$ as well, but to keep the notation simple we do not mention it in the definition. The subspace of radial functions in $W_0^{1,n}(w_i, B)$ is denoted by $W_0^{1,n,rad}(w_i, B)$ for $i = 0, 1$. The following theorem by Calanchi-Ruf in [4] generalizes the Moser-Trudinger inequality on balls. The work of this paper is based on this generalization.

**Theorem 1.2.** [Calanchi-Ruf] Let $\beta \in [0, 1)$ then for $i = 0, 1$

$$\sup_{||u||_{w_i} \leq 1, u \in W_0^{1,n,rad}(w_i, B)} \int_B \exp \left( \alpha |u|^{\frac{n}{(n-1)(1-\beta)}} \right) dx < \infty,$$

iff $\alpha \leq \alpha_\beta = n \left[ \omega_{n-1}^{\frac{1}{n-1}} (1 - \beta) \right]^{\frac{1}{1-\beta}}$ (2)

The two dimensional version of the above theorem was first established in [5]. For $n = 2$ and $w_0 = |\log |x||^{\beta(n-1)}$, the issue of existence of extremal function is
addressed in [28]. Unfortunately, there is a mistake in [28]. The main theorem is true for $\beta$ in some small positive neighborhood of 0, after a slight modification. The required modification will be clear from the proof of our main theorem. When $w_0 = \left(\log \frac{e}{|x|}\right)^{\beta(n-1)}$, Calanchi-Ruf (in [4]) have also obtained the optimal Moser-Trudinger type inequality (of type double exponential growth) when $\beta = 1$. We will discuss some issues related to this embedding in Section 4 [see, Theorem 4.3]. In this work we are mainly concerned with the existence of extremal function for the inequality in (2) for the critical case i.e. $\alpha = \alpha_\beta$. In the sub critical case ($\alpha < \alpha_\beta$) the issue of existence of an extremal function is not very difficult, as one can use Vitali’s convergence theorem to pass through the limit. We will provide a proof of this fact in the next section.

For each $\beta \in [0, 1)$, $i = 0, 1$, let $J_\beta : W^{1,n}_{0,\text{rad}}(w_i, B) \to \mathbb{R}$ defined by

$$J_\beta(u) := \frac{1}{|B|} \int_B \exp(\alpha|u|^\eta) \, dx,$$

where $
 = \frac{n}{n-1}(1-\beta)$, denotes logarithmic Moser-Trudinger functional. The following two theorems are our main results when the weight functions are $w_0$ and $w_1$ respectively.

**Theorem 1.3.** [Main Result] Let $n \geq 2$ and $w_0 = |\log |x||^{\beta(n-1)}$. Then there exist $\beta^* = \beta^*(n) \in (0, 1)$ and $u_\beta \in W^{1,n}_{0,\text{rad}}(w_0, B)$, $\|u_\beta\|_{w_0} \leq 1$ such that

$$J_\beta(u_\beta) = \sup_{\|u\|_{w_0} \leq 1, \beta \in [0, \beta^*)} J_\beta(u), \ \forall \beta \in [0, \beta^*).$$

**Theorem 1.4.** Let $w_1 = \left(\log \frac{e}{|x|}\right)^{\beta(n-1)}$ and $n \geq 2$, then there exist $\beta^* = \beta^*(n) \in (0, 1)$ and $u_\beta \in W^{1,n}_{0,\text{rad}}(w_1, B)$, $\|u_\beta\|_{w_1} \leq 1$ such that

$$J_\beta(u_\beta) = \sup_{\|u\|_{w_1} \leq 1, \beta \in [0, \beta^*)} J_\beta(u), \ \forall \beta \in [0, \beta^*).$$

Let $i = 0, 1$. The main difficulty in proving the existence of extremal function lies in the fact that the functional $J_\beta$ is not continuous with respect to the weak convergence of the space $W^{1,n}_{0,\text{rad}}(w_i, B)$. One can construct (taking powers of Moser sequence) sequence of functions $v_k \in W^{1,n}_{0,\text{rad}}(w_i, B)$, with the property that $v_k \rightharpoonup 0$ in $W^{1,n}_{0,\text{rad}}(w_i, B)$, but $J_\beta(v_k) \to J_\beta(0)$. We refer to Section 2 for the proof of the last statement when $i = 0$. The same fact when $i = 1$ is done in [4].

A sequence of functions $u_k \in W^{1,n}_{0,\text{rad}}(w_i, B)$ is said to concentrate at $x = 0$, denoted by $|\nabla u_k|^{\eta} |w_i |dx \rightharpoonup \delta_0$, if $\|u_k\|_{w_i} \leq 1$ and for any given $1 > \delta > 0$, $\int_{B_\delta, B_{\frac{3}{4}\delta}} |\nabla u_k|^{\eta} w_i \to 0$, where $B_\delta$ denotes the ball of radius $\delta$ at origin.

Concentrating sequences are the only problem in passing through the limit of the maximizing sequences or in other words if one can show that the maximizing sequence does not concentrate at $x = 0$, then one can pass through the limit in $J_\beta$. This follows from P. L. Lions concentration compactness lemma (Lemma 2.1). The method of the proof of our main result (Theorem 1.3) follows similar idea as it is done in [6] and [28]. In Lemma 2.1, we show that a maximizing sequence can loose compactness only if it concentrates at the point $x = 0$. Then in Lemma 3.3 the concentration level $J^\beta_{\beta,w_0}(0)$ (see (12) for the formal definition) is explicitly calculated. It is shown that for all $\beta \in [0, 1)$

$$(1) \quad J^\beta_{\beta,w_0}(0) \leq 1 + \exp \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right).$$
Note that the right hand side of the above inequality is independent of $\beta$. Then the idea is to provide an explicit function $v_\beta \in W^{1,n}_{0,rad}(w_0, B)$, $\|v_\beta\|_{w_0} \leq 1$ such that
\begin{equation}
J_\beta(v_\beta) > 1 + \exp \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right),
\end{equation}
which shows that any maximizing sequence can never concentrate.

As an immediate application of Theorem 1.3, one obtains the existence of radial solution of the following nonlinear elliptic Dirichlet (mean field type) problem, for $\alpha < \alpha_\beta$ if $\beta > 0$ close to zero.

The paper is arranged in the following way: In the next section we develop some basic tools that are required for the proof of our main result (Theorem 1.3). In section 3, we present the proof of Theorem 1.3. In section 4, we present the proof of Theorem 1.4 and concentration compactness principal related to limiting case $\beta = 1$. In section 5, we provide a non-existence type result concerning the existence of extremal function for Moser-Trudinger inequality for a class of simply connected domain.

2. Some preliminary results. In this section we will only consider the function $w_0$.

2.1. Subcritical case. First, let us discuss the question of the existence of extremal function in the subcritical case, that is when $\alpha < \alpha_\beta$. Let $v_k$ denotes a maximizing sequence, that is
\begin{equation}
\int_B \exp (\alpha|v_k|^{\gamma_n, \beta}) \, dx \to \sup_{\|u\|_{w_0} \leq 1, u \in W^{1,n}_{0,rad}(w_0, B)} \int_B \exp (\alpha|u|^{\gamma_n, \beta}) \, dx.
\end{equation}
Since $v_k$ lies in the unit ball of $W^{1,n}_{0,rad}(w_0, B)$, we have weak and pointwise convergence (up to a subsequence) of $v_k$ to some function $v$. Using weak lower semi continuity of the norm one can easily obtain that $v$ also lies in the unit ball of $W^{1,n}_{0,rad}(w_0, B)$. Let $E$ be a measurable subset in $B$. Fix $\epsilon > 0$ small, then for all $k$,
\begin{equation}
\int_E \exp (\alpha|v_k|^{\gamma_n, \beta}) \, dx \leq \left(\int_E \exp (\alpha|v_k|^{\gamma_n, \beta}) \, dx\right)^{\frac{\alpha}{\alpha_\beta}} |E|^{1 - \frac{\alpha}{\alpha_\beta}} \leq C|E|^{1 - \frac{\alpha}{\alpha_\beta}} \leq \epsilon
\end{equation}
whenever $|E| \leq \left(\frac{\epsilon}{C}\right)^{\frac{\alpha}{\alpha_\beta}} := \delta$. Now passing through the limit follows from the Vitali’s convergence theorem. The above argument works if $w_0 = |\log |x||^{\beta(n-1)}$ is replaced by $\left(\log \frac{|x|}{|B|}\right)^{\beta(n-1)}$.

2.2. Concentration-compactness lemma. From now, we will be in the critical case $\alpha = \alpha_\beta$ and we will work with $w_0 = (|\log |x||)^{\beta(n-1)}$ till the end of section 3. At first, we will deduce an equivalent formulation of the problem with which we will work in this paper. For $u \in W^{1,n}_{0,rad}(w_0, B)$ first change the variable as
\begin{equation}
|x| = \exp \left(\frac{-t}{n}\right) \quad \text{and set } \psi(t) = \alpha\frac{1}{\alpha_\beta} u(x),
\end{equation}
Then the functional changes as
\[ I_\beta(\psi) := \int_0^\infty \exp \left( |\psi(t)|^{\gamma \beta} - t \right) dt = \frac{1}{|B|} \int_B \exp (\alpha \beta |u|^{\gamma \beta}) \, dx = J_\beta(u) \quad (4) \]
and the weighted gradient norm changes as
\[ \Gamma(\psi) := \int_0^\infty \frac{|\psi(t)|^{n \beta (n-1)}}{(1-\beta)^{(n-1)}} \, dt = \int_B |\nabla u|^n \log |x|^\beta (n-1) \, dx. \]
For \( \delta \in (0, 1] \), define
\[ \tilde{\Lambda}_\delta := \{ \phi \in C^1[0, \infty) \mid \phi(0) = 0, \ \Gamma(\phi) \leq \delta \}. \]
Since (after using an approximation argument)
\[ \sup_{\|u\|_{W_0^{1,n}} \leq 1, u \in W_0^{1,n}(w_0, B)} \int_B \exp (\alpha \beta |u|^{\gamma \beta}) \, dx = \sup_{\|u\|_{W_0^{1,n}} \leq 1, u \in W_0^{1,n}(w_0, B), \text{smooth}} \int_B \exp (\alpha \beta |u|^{\gamma \beta}) \, dx, \]
the problem reduces in finding \( \psi_0 \in \tilde{\Lambda}_1 \) such that
\[ M_\beta := I_\beta(\psi_0) = \sup_\psi I_\beta(\psi). \quad (5) \]
As mentioned in the introduction, we give the construction of the sequence \( \psi_k \in W_0^{1,n}(w_0, B) \) such that \( \psi_k \to 0 \) in \( W_0^{1,n}(w_0, B) \) but \( J_\beta(\psi_k) \to J_\beta(0) \). Define
\[ \psi_k(t) = \begin{cases} t^{1-\beta} k^{\frac{\beta-1}{\gamma \beta}} & \text{in } 0 \leq t \leq k, \\ k^{\gamma \beta} & \text{in } k \leq t \leq \infty. \end{cases} \]
Finally, define \( \psi_k \) via the change of variable \( \psi_k(t) = \alpha \beta^{-\frac{1}{\gamma \beta}} \psi_k(x) \) and \( |x| = \exp \left( -\frac{1}{n} \right) \).
It is clear that \( \psi_k \to 0 \) in \( W_0^{1,n}(w_0, B) \). Then in view of (4), it is enough to show that \( \liminf_{k \to \infty} I_\beta(\psi_k) > 1 = J_\beta(0) \). Then
\[ I_\beta(\psi_k) = \int_0^k \exp \left( \frac{t^{\frac{n}{k^{1/\gamma \beta}}} - t}{k^{1/\gamma \beta}} \right) + \int_k^\infty \exp (k - t) \, dt = k \int_0^1 \exp (k(t^{\frac{n}{k^{1/\gamma \beta}}} - t)) \, dt + 1. \]
Note that the function \( \exp (t^{\frac{n}{k^{1/\gamma \beta}}} - t) \) is monotonically decreasing in the neighbourhood of 0. Therefore one has
\[ k \int_0^1 \exp (k(t^{\frac{n}{k^{1/\gamma \beta}}} - t)) \, dt > k \exp \left( k \left( \frac{1}{k^{1/\gamma \beta}} - \frac{1}{k} \right) \right) \frac{1}{k} = \exp \left( \frac{1}{k^{1/\gamma \beta}} \right) \exp (-1) \to \exp (-1). \]
This implies that \( \liminf_{k \to \infty} I_\beta(\psi_k) \geq 1 + \exp (-1) \).
Let \( \tilde{g}_m \) be a maximizing sequence, that is \( J_\beta(\tilde{g}_m) \to M_\beta \). Since
\[ \int_B |\nabla \tilde{g}_m|^n \log |x|^\beta (n-1) \, dx \leq 1, \]
one can find a subsequence (which we again denote by \( \tilde{g}_m \)) and a function \( \tilde{g}_0 \in W_0^{1,n}(w_0, B) \)
\[ \tilde{g}_m \to \tilde{g}_0 \text{ in } W_0^{1,n}(w_0, B), \tilde{g}_m \to \tilde{g}_0 \text{ pointwise.} \quad (6) \]
For any \( v \in C^1[0, \infty) \) and \( t \geq A \geq 0 \), we get after using Hölder’s inequality and fundamental theorem of Calculus, that

\[
|v(t) - v(A)| \leq \left( \int_A^t \left| \frac{v'(s)}{s^\beta(n-1)} \right|^{1 \over 1 - \beta} \frac{1}{s^\beta(n-1)} ds \right)^{1 \over \beta} \left( t^{1-\beta} - A^{1-\beta} \right)^{n-1 \over n}. \tag{7}
\]

We will recall this inequality several times. The next lemma is equivalent to concentration-compactness alternative, for Moser-Trudinger case, by P. L. Lions in [16]. As a consequence of the next lemma it would be enough to prove that the sequence \( g_m \) does not concentrates at 0, in order to pass to the limit in the functional.

**Lemma 2.1.** [Concentration-Compactness alternative] For any sequence \( v_m, \tilde{v} \in W_{0, rad}(w_0, B) \) such that \( v_m \to \tilde{v} \) in \( L^1(\mathbb{R}^n, B) \), then for a subsequence, either (1) \( J_\beta(v_m) \to J_\beta(\tilde{v}) \), or (2) \( v_m \) concentrates at \( x = 0 \).

**Proof.** Let us assume that (2) does not hold, then it is enough to show that (1) holds in order to prove the lemma. There exist \( A > 0 \) and \( \delta \in (0, 1) \), as \( m \to \infty \), it implies that

\[
\int_{B \setminus B_{\delta}} |\nabla v_m|^n |\log |x||^{\beta(n-1)} dx = \int_0^A \frac{|v_m|^n t^\beta(n-1)}{(1-\beta)(n-1)} dt \geq \delta, \quad \text{for all } m \geq m_0
\]

where

\[
\alpha_{\beta, n} \, v_m(x) = v_m(t), \quad \text{and } |x| = \exp \left( -\frac{t}{n} \right).
\]

Using (7),

\[
|v_m(t) - v_m(A)| \leq (1 - \delta)^{1 \over n} \left( t^{1-\beta} - A^{1-\beta} \right)^{n-1 \over n} \leq (1 - \delta)^{1 \over n} t^{(1-\beta)(n-1) \over n}. \tag{8}
\]

Now using the inequality \( |v_m(A)| \leq A \left( \frac{1-\beta}{n} \right) \) for all \( m \), we have for \( t \geq N \), (for sufficiently large \( N \))

\[
|v_m(t)|^{1 \over (1-\beta)(n-1)} \leq \left\{ A \left( \frac{1-\beta}{n} \right) + (1-\delta)^{1 \over n} t^{(1-\beta)(n-1) \over n} \right\}^{1 \over (1-\beta)(n-1)} \\
\leq A + \left( 1 - \delta \right)^{1 \over 2} \left( t^{1 \over (1-\beta)(n-1)} \right).
\]

In the last step we have used the following inequality: If \( \mu > \gamma > 0 \), \( p > 1 \) then for sufficiently large \( y \in \mathbb{R} \), one has \( (1+\gamma y)^p \leq 1 + \mu y^p \). Split \( I_\beta(v_m) = I_1(v_m) + I_2(v_m) \) where

\[
I_1(v_m) := \int_0^N \exp \left( |v_m(t)|^{n \over (1-\beta)(n-1)} - t \right) dt.
\]

Using the bound \( |v_m(t)| \leq t^{(1-\delta)(n-1)} \) for all \( m \) and dominated convergence theorem, one obtains \( I_1(v_m) \to I_1(v) \) (Since we know that \( v_m \) converges pointwise to \( \tilde{v} \), this implies that \( v_m \) also converges point wise to \( v \)). From (8), we have

\[
I_2(v_m) := \int_N^\infty \exp \left( |v_m(t)|^{n \over (1-\beta)(n-1)} - t \right) dt \leq \exp (A) \int_N^\infty \exp \left( \left[ 1 - \delta \right]^{\frac{1}{(1-\beta)(n-1)}} - t \right) dt, \tag{9}
\]

which can be made less than any arbitrary positive number \( \epsilon \), after choosing \( N \) large enough. Therefore \( I_\beta(v_m) \to I_\beta(v) \) that is \( J_\beta(v_m) \to J_\beta(\tilde{v}) \).
3. Proof of the main result (Theorem 1.3). The following lemma is proved in [6, Lemma 2]. Here we will use it without giving the proof. Let for $\delta, a > 0$,

$$\Lambda^a_\delta := \left\{ \phi \in C^1[0, \infty) \mid \phi(0) = 0, \int_0^\infty |\phi'|^n dt \leq \delta \right\}.$$ 

Lemma 3.1. [Carleson-Chang] For each $a > 0$ and $\phi \in \Lambda^a_\delta$, we have

$$\int_a^\infty \exp \left( \phi \frac{n}{n-1} - t \right) dt \leq \frac{\exp \left( \phi(a) \frac{n}{n-1} - a \right)}{1 - \delta \frac{n}{n-1}} \exp \left( C_n \frac{n-1}{n} \beta_n \right) \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right),$$

and where $\beta_n = \delta(1 - \delta \frac{n}{n-1})^{-n+1}$ and $c = \frac{n}{n-1} \phi \frac{n}{n-1} (a)$. The inequality above tends asymptotically to an equality if $c^n \beta_n \to \infty$, $a \to \infty$ and $\delta \to 0$.

Let $\tilde{f}_m \in W^{1,n}_{0,rad}(w_0, B)$ such that $\|f_m||_n \leq 1$, $\|\nabla \tilde{f}_m\|_{0,w_0} \to \delta_0$ and define $f_m$, from $\tilde{f}_m$, using the same transformation introduced in (3). Then since $\tilde{f}_m$ concentrates, this implies that $\tilde{f}_m \to 0$ in $W^{1,n}_{0,rad}(w_0, B)$ and also pointwise.

Lemma 3.2. Let $f_m$ be as above. Then one of the following always holds. (1) There exists a sequence of points $\{a_m\}_m \in [1, \infty)$, such that

$$|f_m(a_m)|^{\gamma n - \beta} - a_m = -2 \log(a_m).$$

(2) If such $a_m$ does not exist, then

$$\limsup_{m \to \infty} \int_0^\infty \exp \left( |f_m|^{\gamma n - \beta} - t \right) dt = 1.$$

Further more, whenever the first case holds, we can choose $a_m$ to be the first point in $[1, \infty)$ satisfying (11) and such point always satisfy the property that $a_m \to \infty$.

Proof. Since $|f_m(t)| \leq \left( \frac{1 - \delta \frac{n}{n-1}}{a} \right)^{\left( \frac{n}{n-1} - 1 \right)}$, this implies that $|f_m(t)|^{\left( \frac{n}{n-1} - 1 \right)} - t \leq 0$ if $t \in [0, 1)$, while $-2 \log(t) > 0$ if $t \in [0, 1)$. Therefore $|f_m(t)|^{\left( \frac{n}{n-1} - 1 \right)} - t < -2 \log(t)$ which implies non existence of such $a_m$ satisfying (11) on the interval $[0, 1)$.

Now let us assume (1) does not hold, that is the non existence of such $a_m$’s in the interval $[1, \infty)$. This implies that $|f_m(t)|^{\left( \frac{n}{n-1} - 1 \right)} - t < -2 \log(t)$ on $[1, \infty)$. Or in other words we have

$$\exp \left( |f_m(t)|^{\left( \frac{n}{n-1} - 1 \right)} - t \right) \leq t^{-2}, \text{ if } t \in [1, \infty).$$

One can use dominated convergence theorem, with the dominating function

$$g(t) = \begin{cases} 1 & \text{ in } (0, 1), \\ \frac{1}{t^2} & \text{ on } [1, \infty), \end{cases}$$

to show that $I_{\beta}(f_m) \to 1$.

Let (1) holds, then we can choose the first $a_m \geq 1$ satisfying (11). We have to show that $a_m \to \infty$. Given $K$ arbitrary large number. It is sufficient to show that there exist $m_0 = m_0(K)$ such that $a_m \geq K$, for all $m \geq m_0$. First choose $\eta$ small, such that

$$\eta t < t - 2 \log(t), \text{ for all } t \in [0, K).$$
Now using the fact that the sequence $f_m$ concentrates, we get for $t \in [0, K)$ and $\forall \ m \geq m_0$,

$$|f_m(t)|^{\frac{1}{1+n \beta}} \leq \left( \int_0^K \left| f_m \right|^n t^{\beta(n-1)(1-\beta)} \frac{1}{n-1} dt \right)^{\frac{1}{1+n \beta}} t < \eta t \leq t - 2 \log(t).$$

This says that $\alpha_m > K$ for all $m \geq m_0$.

Let us first define the concentration level at 0,

$$J_{\beta}^{\delta, u_0}(0) := \sup_{\{v_m\}_m \in W^{1, n}_{\text{rad}}(w_i, B)} \left\{ \limsup_{m \to \infty} J_{\beta}(v_m) \mid |\nabla v_m|^n u_i \to \delta_0 \right\}, \text{ for } i = 0, 1. \quad (12)$$

**Theorem 3.3.** [Estimate for Concentration level] For $\beta \in [0, 1)$ it implies that

$$J_{\beta, u_0}(0) \leq 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right). \quad (13)$$

**Proof.** First note that it is sufficient to consider concentrating sequences $\tilde{f}_m$ such that the first alternative is satisfied in Lemma 3.2, because in the second case the required inequality (13) is already satisfied.

In the first step we will show that

$$\lim_{m \to \infty} \int_0^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt = 1,$$

where $f_m$ and $a_m$ are as in the previous lemma. Using $|\nabla \tilde{f}_m|^n w_0 \to 0$ and (7) we notice that $f_m \to 0$ uniformly on compact subsets of $\mathbb{R}^+$. Therefore for each $A, \epsilon > 0$, we have $|f_m(t)|^{\gamma_n, \beta} \leq \epsilon$ for all $t \leq A$ and sufficiently large $m$. Using the property of $a_m$, that is for all $t \leq a_m$ one has $|f_m(t)|^{\gamma_n, \beta} \leq t - 2 \log(t)$, we get

$$\int_0^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt = \int_0^{A} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt + \int_{A}^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt \leq \exp (\epsilon) \int_0^{A} \exp (-t) dt + \int_{A}^{a_m} \exp (-2 \log(t)) dt = \exp (\epsilon) (1 - \exp (-A)) + \left( \frac{1}{A} - \frac{1}{a_m} \right).$$

Therefore

$$\limsup_{m \to \infty} \int_0^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt \leq \exp (\epsilon) (1 - \exp (-A)) + \frac{1}{A}.$$ 

Now as $\epsilon \to 0$ and $A \to \infty$ we have

$$\limsup_{m \to \infty} \int_0^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt \leq 1.$$

For the other way round

$$\int_0^{a_m} \exp (|f_m(t)|^{\gamma_n, \beta} - t) dt \geq \int_0^{a_m} \exp (-t) dt = 1 - \exp (-a_m) \to 1.$$
In the second step we claim that
\[
\lim_{m \to \infty} \int_{a_m}^{\infty} \exp (|f_m(t)|^{\gamma_{n, \beta}} - t) \, dt \leq \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right).
\]

Set \( \delta_m = \int_{a_m}^{\infty} \frac{|f_m|^n t^{\beta(n-1)}}{(1 - \beta)^{n-1}} \, dt \). Then using (7) with \( A = 0 \) and \( t = a_m \), one obtains
\[
\delta_m := 1 - \int_{0}^{a_m} \frac{|f_m|^n t^{\beta(n-1)}}{(1 - \beta)^{n-1}} \, dt \leq 1 - \left( \frac{|f_m(a_m)|^{\gamma_{n, \beta}}}{a_m^{\beta(n-1)}} \right)^{(1-\beta)(n-1)} \int_{a_m}^{\infty} \exp \left( - \frac{|g_m|^n}{1 - \beta} \right) \, dt.
\]

In the last inequality we have used the property of the points \( a_m \). Define the function \( g_m(t) = |f_m(t)|^{\frac{n}{n-1}} \), then it is easy to note that
\[
\int_{a_m}^{\infty} \exp (|f_m(t)|^{\gamma_{n, \beta}} - t) \, dt = \int_{a_m}^{\infty} \exp \left( g_m(t) \frac{n}{n-1} - t \right) \, dt.
\]

Using \( |f_m(t)|^{\gamma_{n, \beta}} \leq t \) one has
\[
\int_{a_m}^{\infty} |f_m'|^n = \frac{1}{(1 - \beta)^n} \int_{a_m}^{\infty} \frac{n}{n-1} |f_m'|^n \leq \frac{1}{(1 - \beta)^n} \int_{a_m}^{\infty} t^{n-1, \beta} |f_m(t)|^n \leq \frac{\delta_m}{1 - \beta} = \delta_m^* \to 0.
\]

Now applying Lemma 3.1 with \( \delta = \delta_m^* \) and \( a = a_m \), we get
\[
\int_{a_m}^{\infty} \exp (|f_m(t)|^{\gamma_{n, \beta}} - t) \, dt \leq \frac{\exp \left( g_m(a_m) \frac{n}{n-1} - a_m \right)}{1 - |\delta_m^*|^{\frac{n}{n-1}}} \exp \left( \frac{c^n}{n} \left( \frac{n-1}{n} \right)^{n-1} \beta n + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right),
\]

and where \( \beta_n = \delta_m^*(1 - |\delta_m^*|^{\frac{1}{n-1}})^{-n+1} \) and \( c = \frac{n}{n-1} g_m^*(a_m) \). Therefore it is left to show that
\[
\limsup_{m \to \infty} K_m := \limsup_{m \to \infty} \left\{ g_m(a_m) \frac{n}{n-1} - a_m + \frac{g_m(a_m) \frac{n}{n-1} \delta_m^*}{(n-1)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})} \right\} \leq 0.
\]

\[
K_m = -2 \log a_m + \frac{(a_m - 2 \log a_m) \delta_m}{(n-1)(1 - \beta)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})} = -2 \log a_m + \frac{a_m \delta_m}{(n-1)(1 - \beta)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})} - \frac{2 \log a_m \delta_m}{(n-1)(1 - \beta)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})}.
\]

\[
= \left\{ -2 \log a_m + \frac{a_m \delta_m}{(n-1)(1 - \beta)} \right\} + \frac{a_m \delta_m (1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1}}{(n-1)(1 - \beta)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})} - \frac{2 \log a_m \delta_m}{(n-1)(1 - \beta)((1 - |\delta_m^*|^{\frac{1}{n-1}})^{n-1})} := I_1^m + I_2^m - I_3^m.
\]
We will make use of Maclaurin series expansion for this purpose.
\[ \delta_m := 1 - \left( 1 - \frac{2 \log a_m}{a_m} \right)^{(1 - \beta)(n - 1)} \]
\[ = (1 - \beta)(n - 1) \frac{2 \log a_m}{a_m} + C \left( \frac{2 \log a_m}{a_m} \right)^2 + o \left( \left( \frac{2 \log a_m}{a_m} \right)^2 \right), \tag{19} \]
for some constant \( C = C(\beta, n) > 0 \). Using the above expansion formula, we estimate
\[ I_1^m = 4C \frac{(\log a_m)^2}{a_m} + a_m o \left( \left( \frac{\log a_m}{a_m} \right)^2 \right) \to 0 \text{ as } m \to \infty. \]
Also
\[ |I_3^m| \leq C_1 \frac{(\log a_m)^2}{a_m} + C_2 \frac{(\log a_m)^3}{a_m^2} + (\log a_m) o \left( \left( \frac{\log a_m}{a_m} \right)^2 \right) \to 0 \text{ as } m \to \infty. \]
For estimating the term \( I_2^n \), we first use binomial expansion of the \( \left( 1 - |\delta_m|^\nu \right)^{(n - 1)} \) to obtain
\[ |I_2^m| \leq C(\log a_m) \frac{\pi^\nu}{a_m^\nu} \to 0 \text{ as } m \to \infty. \]
This finishes the proof of the theorem. \( \square \)

Proof of Theorem 1.3. If possible let \( J_\beta(\tilde{g}_m) \) does not converges to \( J_\beta(\tilde{g}_0) \), where \( \tilde{g}_m, \tilde{g}_0 \) is as in (6). Then from previous lemma’s we know that
\[ M_\beta = \lim_{m \to \infty} J_\beta(\tilde{g}_m) \leq 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right). \]
If we show that, there exist some \( \phi \in \tilde{\Lambda}_1 \) such that
\[ I_\beta(\phi) > 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right), \]
then clearly \( M_\beta > 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right) \) and this would be a contradiction. Consider the function \( f_n \) defined as
\[
 f_n(t) = \begin{cases} 
 \frac{(n-1)}{n} (n-1)^{-1} t & \text{in } 0 \leq t \leq n, \\
 (n-1)^{\frac{n-1}{n}} t - 1 & \text{on } n \leq t \leq N_n, \\
 (N_n - 1)^{\frac{n-1}{n}} & \text{on } t \geq N_n,
\end{cases}
\]
where \( N_n = (n-1) \exp \left( (\frac{n}{n-1})^{n - \frac{n}{n-1}} \right) + 1 \). Set \( \phi^{\lambda, \beta} = (\lambda f_n)^{1-\beta} \) for \( \lambda \in (0,1) \). It has been verified in [6] that \( f_n^\infty |f_n|^n \leq 1 \) and
\[
 \int_0^\infty \exp \left( f_n(t)^{\frac{n}{n-1}} - t \right) dt = 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right) + \zeta^*(n) > 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1} \right). \tag{20}
\]
Hence we can choose \( \beta \) as follows: \( \beta = \beta_*(n) \) such that \( I_1(\beta_*) + I_2(\beta_*) < 1 \). This finishes the proof of the theorem.
4. Proof of Theorem 1.4. Let \( w_1 = \left( \log \frac{r}{|x|} \right)^{\beta(n-1)} \). Under the same change of variable as introduced in (3), the functional is not affected and stays the same as in (4), that is
\[
I_\beta(\psi) := \int_0^\infty \exp\left( (\psi(t))^{\gamma_{n,\beta}} - t \right) dt = \frac{1}{|B|} \int_B \exp\left( (\alpha_\beta |u|^{\gamma_{n,\beta}} \right) dx = J_\beta(u). \tag{22}
\]
The new weighted norm becomes
\[
\Gamma_\beta(\psi) := \int_0^\infty |\psi'|^n (t + n)^{\beta(n-1)} \frac{1}{(1 - \beta)(n-1)} dt = \int_B |\nabla u|^n \left( \log \frac{e}{|x|} \right)^{\beta(n-1)} dx.
\]
For any \( v \in C^1[0,\infty) \) and \( t \geq A \geq 0 \), we get after using Hölder’s inequality and fundamental theorem of Calculus, that
\[
|v(t) - v(A)| \leq \left( \int_A^t |v'|^n (s + n)^{\beta(n-1)} \frac{1}{(1 - \beta)(n-1)} ds \right)^\frac{1}{n} \left( (t + n)^{1-\beta} - (A + n)^{1-\beta} \right)^{n-1}. \tag{23}
\]
Using this inequality and following exactly in similar way the steps done in Lemma 2.1, one has the following lemma:

**Lemma 4.1.** [Concentration-Compactness alternative] For any sequence \( \tilde{v}_m, \tilde{v} \in W^{1,n}_{0,rad}(w_1, B) \) such that \( \tilde{v}_m \rightharpoonup \tilde{v} \) in \( W^{1,n}_{0,rad}(w_1, B) \), then for a subsequence, either (1) \( J_\beta(\tilde{v}_m) \to J_\beta(\tilde{v}) \), or (2) \( \tilde{v}_m \) concentrates at \( x = 0 \).

**Lemma 4.2.** [Concentration level] For each \( \beta \in [0,1) \),
\[
J_\beta(w_1(0)) \leq 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right).
\]
Proof. Using the embedding \( W^{1,n}_{0,rad}(w_1, B) \hookrightarrow W^{1,n}_{0,rad}(w_0, B) \) (note that one can take 1 as the constant while writing down the norm inequality for the above embedding) and from Theorem 3.3 the lemma follows. \( \square \)

Now we present the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Take \( \phi_n^{\lambda,\beta} \) as in the proof of Theorem 1.3. Then it is possible to choose \( \lambda = \lambda^* \) (independent of \( \beta \)) close to 1 such that
\[
I_\beta(\phi_n^{\lambda^*,\beta}) > 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right).
\]
This is because we already know from [6] that
\[
I_0(\phi_n^{1,0}) > 1 + \exp \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \right).
\]
It is easy to check that \( \Gamma_0(\phi_n^{\lambda^*,0}) \leq |\lambda^*|^n < 1 \) after using \( \int_0^\infty |f_n'|^n \leq 1 \). Therefore it is possible to find \( \lambda^* \) close to 0, such that \( \Gamma_\beta(\phi_n^{\lambda^*,\beta}) \leq 1 \) whenever \( \beta \in (0, \beta^*) \).

This proves that there exist a function in unit ball of \( W^{1,n}_{0,rad}(w_1, B) \) whose functional value is strictly larger than the concentration level. This implies that a maximizing sequence can never concentrate. Therefore from the previous lemma the theorem follows.

**Concentration Comapctness alternative for the limiting case \( \beta = 1 \).** For the weight function \( w_1 = \left( \log \frac{r}{|x|} \right)^{\beta(n-1)} \) the following embedding is obtained for
the critical case $\beta = 1$. The growth function is of double exponential type, which is natural to expect.

**Theorem 4.3.** [Calanchi-Ruf] Let $w_1 = \left(\log \frac{e}{|x|}\right)^{(n-1)}$, then

$$
\sup_{\|u\|_{w_1} \leq 1, \text{rad} J_B} \int_B \exp \left( a \exp \left( \frac{\alpha}{\alpha-1} \frac{u}{u^{\alpha-1}} \right) \right) < \infty \iff a \leq n. \quad (24)
$$

The existence of extremal function for the sub critical case ($a < n$) follows from Vitali’s convergence theorem. The following concentration compactness alternative is in the same spirit as originally due to P. L. Lions [16], and also like Lemma 2.1 and Lemma 4.1 of this paper.

**Lemma 4.4.** Let $\beta = 1$ and $f_m \rightharpoonup f$ in $W^{1,n}_{0,\text{rad}}(w_1, B)$, then either

1. $\int_B \exp \left( n \exp \left( \frac{\gamma}{\gamma-1} |f_m|^\frac{n}{\gamma-1} \right) \right) dx \rightarrow \int_B \exp \left( n \exp \left( \frac{\gamma}{\gamma-1} |f|^\frac{n}{\gamma-1} \right) \right) dx$

or

2. $|\nabla f_m|^n \rightharpoonup \delta_0$.

**Proof.** First introduce the following change of variable (as done in [5]): $g_m(t) = \frac{1}{\omega_{n-1}} f_m(x)$, $|x| = \exp(-t)$. Then it is easy to see that

$$
\int_B |\nabla f_m|^n \log \frac{e}{|x|} \beta(n-1) dx = \int_0^\infty |g_m|^n (1 + t)^{n-1} dt,
$$

and

$$
\int_B \exp \left( a \exp \left( \frac{\gamma}{\gamma-1} |f_m|^\frac{n}{\gamma-1} \right) \right) dx = \frac{\omega_{n-1}}{n} \int_0^\infty \exp \left( n \exp \left( \frac{\gamma}{\gamma-1} g_m^{\frac{n}{\gamma-1}} \right) - nt \right) dt.
$$

Like in Lemma 2.1, let us assume that (2) does not hold true, then it is enough to show that (1) holds in order to prove the lemma. There exist a $\delta > 0$, $A > 0$ such that as $m \rightarrow \infty$, it implies that

$$
\int_0^A |g_m|^n (1 + t)^{(n-1)} dt \geq \delta, \quad \text{for all } m \geq m_0.
$$

Using Holder’s inequality one can easily obtains, for $t > A$,

$$
|g_m(t)| \leq |g_m(A)| + (1 - \delta)^{\frac{n}{\gamma}} |\log(1 + t) - \log(1 + A)|^{\frac{n}{\gamma-1}}.
$$

If $\mu > \gamma > 0$, $p > 1$ then for sufficiently large $y \in \mathbb{R}$, one has $(1 + \gamma y)^p \leq 1 + \mu y^p$. Using this we obtain

$$
|g_m(t)|^{\frac{n}{\gamma-1}} \leq C(\delta, A, n) + (1 - \frac{\delta}{2})^{\frac{n}{\gamma-1}} \log(1 + t).
$$

Therefore, $\exp(|g_m|^{\frac{n}{\gamma-1}} - t) \leq C(\delta, A, n) \gamma t - t$. Here we have used the following inequality: Let $\alpha \in (0, 1)$ then for each $\gamma > 0$ there exist large $t > t(\gamma)$, such that $(1 + t)^{\alpha} \leq \gamma t$. Therefore

$$
\exp \left( n \exp \left( |g_m|^{\frac{n}{\gamma-1}} - t \right) \right) \leq \exp \left( n(C\gamma - 1)t \right).
$$

First choose $\gamma$ small such that $C\gamma < 1$ and then choose $N$ large enough such that

$$
\int_N^\infty \exp \left( n \exp \left( |g_m|^{\frac{n}{\gamma-1}} - t \right) \right) dt < \epsilon.
$$

On the interval 0 to $N$ one can easily pass through the limit using dominated convergence theorem. □
The existence of extremal function for this embedding still remains open.

5. A non existence result. In two dimension Moser-Trudinger inequality is generalized by Mancini-Sandeep [22] to the following form:

**Theorem 5.1.** [Mancini-Sandeep] Let $\Omega \subset \mathbb{R}^2$ be a simply connected, then

\[ \sup_{u \in W^{1,2}_0(\Omega), ||u||_{L^2(\Omega)} \leq 1} \int_{\Omega} \left( \exp\left(4\pi u^2\right) - 1 \right) dx < \infty \]  

(25)

if and only if $\Omega$ satisfies finite ball condition (see below for the definition).

The question of existence of extremal function is largely open to the best of our knowledge for the above result. The only paper that deals with such issue is by Mancini-Battaglia [21]. They proved that when $\Omega = (\mathbb{R}, \mathbb{R}) \times (-1,1)$ then there exist an extremal function for the above inequality. For bounded domains concentrating sequence is the only problem in passing through the limit, but in this case there is an extra problem of vanishing sequence. They could overcome this difficulty by using symmetrization argument, after exploiting the special symmetries of the domain. We give a class of domain, more precisely domains of class $T$ for which there does not exist extremal function for (25).

**Definition.** (Finite Ball Condition) We say $\Omega$ satisfies finite ball condition if it cannot contain ball of arbitrary large radius.

**Definition.** (Domain of Class $T$) We say $\Omega$ is of class $T$ if there exist a translation map $T$ such that $T(\Omega)$ is strictly contained in $\Omega$.

An example of such domain(satisfies both finite ball condition and of class $T$) would be $\Omega = (0, \infty) \times (-1,1)$. If one drops the simply connected assumption then another example of a domain, satisfying both the above properties would be the following: Consider the first quadrant of $\mathbb{R}^2$, then remove a circle of radius $\frac{1}{1+i}$ with center at $(i,j)$, for all $i, j \in \mathbb{Z}^+$. $\mathbb{Z}^+$ denotes the set of positive integers.

**Theorem 5.2.** Let $\Omega$ be a simply connected domain, of class $T$ and satisfying finite ball condition. Then the following supremum is never achieved:

\[ \sup_{u \in W^{1,2}_0(\Omega), ||u||_{L^2(\Omega)} \leq 1} \int_{\Omega} \left( \exp\left(4\pi u^2\right) - 1 \right) dx < \infty. \]  

(26)

**Proof.** Let us argue by contradiction. If possible let us assume that there exist a function $u_0 \in W^{1,2}_0(\Omega)$ with $\int_{\Omega} |\nabla u_0|^2 \leq 1$ such that

\[ \sup_{u \in W^{1,2}_0(\Omega), ||u||_{L^2(\Omega)} \leq 1} \int_{\Omega} \left( \exp\left(4\pi u^2\right) - 1 \right) dx = \int_{\Omega} \left( \exp\left(4\pi u_0^2\right) - 1 \right) dx. \]

Replacing $u_0$ by $|u_0|$ we can always assume that $u_0 \geq 0$ in $\Omega$. Since $\Omega$ is of class $T$, there exist a translation map $T$ such that $T(\Omega)$ is strictly contained in $\Omega$. Define the function $v_0 \in W^{1,2}_0(\Omega)$ in the following way:

\[ v_0 = \begin{cases} T(u_0) & \text{in } T(\Omega), \\ 0 & \text{on } \Omega \setminus T(\Omega). \end{cases} \]

$v_0$ being a translate of $u_0$ and extension by zero, it is easy to verify that

\[ \int_{\Omega} |\nabla v_0|^2 = \int_{\Omega} |\nabla u_0|^2 \leq 1, \]
and
\[ \int_{\Omega} \left( \exp \left( 4\pi u_0^2 \right) - 1 \right) \, dx = \int_{\Omega} \left( \exp \left( 4\pi v_0^2 \right) - 1 \right) \, dx. \]
That is, \( v_0 \) is also an extremiser. Then \( v_0 \) satisfies the following Euler-Lagrange equation:
\[
\begin{aligned}
-\Delta v_0 &= \lambda v_0 \left( \exp \left( 4\pi v_0^2 \right) - 1 \right) \quad &\text{in } \Omega, \\
v_0 &= 0 \quad &\text{on } \partial \Omega, \\
v_0 &\geq 0,
\end{aligned}
\]
where \( \lambda \) is a real constant. Then by the application of maximum principle gives that \( v_0 > 0 \) in \( \Omega \) which contradicts that \( v_0 = 0 \) on \( \Omega \setminus T(\Omega) \) (by construction). This finishes the proof of the theorem.

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