Using weak measurements to extract the \( Z_2 \) index of a topological insulator

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Recently there has been an interest in applying the concept of weak values and weak measurements to condensed matter systems. Here a weak measurement protocol is proposed for obtaining the \( Z_2 \) index of a topological insulator. The setup consists of a topological insulator with a hole pierced by a time dependent Aharonov-Bohm flux. A certain weak value \( (A) \) associated with the time-integrated magnetization in the hole has a universal response to a small ambient magnetic field \( (B) \), namely \( BA = 2\hbar \). This result is unaffected by disorder, interactions, and, to a large extent, the speed of the flux threading. It hinges mainly on preventing the flux from leaking outside the hole, as well as being able to detect magnetization at a resolution of a few spins. A similar result may be obtained using only charge measurements, in a setup consisting of double quantum dot weakly coupled to an LC circuit. The universality of these results suggests that they may be used as a test for the use of weak values in condensed matter physics.

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Topological insulators (TIs) have attracted much attention in recent years due to their novel bulk and surface properties. In their bulk, these materials are insulating and certain twists in the bulk’s band structure are characterized by topological indices. This implies, via a bulk-edge correspondence, that the surfaces of these materials are metallic, and have a strong coupling between momentum and spin. Such profound spin dependent effects, that do not require external magnetic fields, are potentially useful in the field of spintronics.

The topological character of a material may correspond to a quantized bulk response function, e.g., the quantized Hall conductance in the integer quantum Hall effect (QHE). This allows for the direct detection of the topological index of the material. However TIs, which in two dimensions (2D) can be pictured as two stacked QHE layers with opposite magnetic fields, are not known to bear any quantized bulk response function or observable. As a result, experimental identification of a TI material is a more subtle task that relies on indirect evidence. For example analysis for surface ARPES spectrum and measurement of edge conductance.

Recently there has been an interest in applying a different measuring scheme, based on the idea of weak values, to condensed matter systems. This measurement scheme allows one to measure off-diagonal matrix elements of operators directly, and hence extract more information than is available from a standard measurement. Also, under certain circumstances, weak values can be used to amplify a weak signal. A typical setup is a double quantum dot on which one applies various perturbation to induce either a Zener transition or Rabi oscillations between two charge states of the device. The signal of a charge detector which is weakly coupled to the device, can then achieve values which exceed the classically allowed ones, provided that one post-selects only the measurements in which an unlikely outcome occurred. From an entirely different direction, a certain weak value (called the “strange correlator”) has been used to identify power law correlations in symmetry protected topological phases. However, measuring this weak value is unfeasible, as it would require waiting for an extremely unlikely event in which a quantum fluctuation in the topological insulator makes it appear as the ground state of a trivial insulator. In contrast, below we propose a more physical weak measurement which can be used to identify a TI.

In a geometry with closed boundary conditions, TIs host Zener transitions driven by Aharonov-Bohm (AB) fluxes. These transitions occur between the ground state and a magnetic excitation which resides on the boundary. Considering for instance a Corbino-disk geometry, the threading of a single AB flux quantum through the hole in the disk results in a single level crossing, which occurs exactly when half the flux is threaded. The final state after the threading is orthogonal to the ground state of a trivial insulator. In contrast, below we consider a detector which is weakly coupled to the boundary magnetization and measure its signal during a threading of the AB flux. At the end of the threading, a regular (strong) magnetization measurement is performed to determine the final state. The weak detector signal is then conditionally averaged on having a non-magnetized final state. The result \( (A) \), also known as a weak value (WV), shows a quantized response to a small ambient magnetic field \( (B_s) \) which corresponds...
to the \(Z_2\) invariant directly (see Eq. \(10\)). Similarly quantized results are obtained for a double quantum dot coupled an LC circuit. Thus besides providing an alternative route to measure the topology of a band insulator, the quantized nature of these WVs suggest that they can be used to test the concept of weak measurements in condensed matter systems.

Let us begin with some background on the aforementioned Zener transition. For many purposes, a TI can be thought of as a double layer system wherein one layer consists of only spin-up \((s = 1)\) electrons and is in an integer quantum Hall effect with a Hall conductance \((\sigma_{xy})\) of \(e^2/\hbar\) and the other layer consists of only spin-down \((s = -1)\) electrons and is in an integer quantum Hall effect with an opposite Hall conductance \([2]\). We focus on one layer in a Corbino-disk geometry, initially with no interactions and no disorder, such that the edge conserves the momentum parallel to it \((k_i)\). One then finds a branch of chiral modes confined to the boundary \((E_{k_i}(n))\). The sign of the slope of these chiral modes \(\text{sign}(\partial_{k_x} E_{k_i}(n))\), is determined by the sign of \(\sigma_{xy}\), or equivalently in our setup, by \(s\). For a finite boundary of length \(L\), the allowed momenta along this chiral branch are quantized to \(k_{||}(n) = \frac{2\pi n}{L} + \frac{s}{2}\) with \(n\) being an integer and \(\phi\) is the AB flux through the Corbino disk (see Fig. 1b, red branch, and imagine that \(\Delta_h\) consists of only spin-up \((\text{a double layer system wherein one layer})\). Below we will always consider the limit \(\Delta_h^\text{−1} \ll t_f/\hbar \ll \Delta_h^\text{−1}\) where \(\Delta\) in the bulk gap of the TI.

We propose to monitor the system evolution by coupling a weak detector to the magnetization at the inner edge. The detector is modeled as a Harmonic oscillator with a low frequency, \(\omega\), whose momentum, \(P\), is weakly coupled to the system, namely

\[
H_{\text{detector}} = \frac{M\omega^2 X^2}{2} + \frac{P^2}{2M} + \lambda PM_i. \quad (2)
\]

In the setup suggested in Fig. 1b, \(P\) would correspond to the vertical position operator of the cantilever. Later we comment on how to choose \(\lambda\) and calibrate our measurement of \(X\).

The measurement protocol begins with both system and detector in their respective ground states, i.e. the initial state is \(|i\rangle = |i_S\rangle|i_D\rangle\). Pictorially, the system’s ground state \(|i_S\rangle\) corresponds to Fig. 1b. Next \(\phi(t)\) is scanned, at constant rate, from \(0\) to \(\phi_0\). The resulting state of the system and detector, at time \(t_f\), can be expressed as

\[
|f\rangle = \mathcal{T} e^{i\int_{t_0}^{t_f} \frac{\phi(t)}{\hbar} H} |i\rangle, \quad (3)
\]

where \(\mathcal{T}\) denotes time ordering.

A weak measurement means one uses first order perturbation in the system-detector coupling, and so the final state can be expressed as

\[
|f\rangle = \left( U_{t_f}^{\dagger} + i\lambda \int_0^{t_f} \frac{dt}{\hbar} U_{t_f}^{\dagger} M_i PU_0^\dagger \right) |i\rangle, \quad (4)
\]

\[
U_{t_0}^{\dagger} = \mathcal{T} e^{i\int_{t_0}^{t_1} \frac{\phi(t)}{\hbar} H_{TI}[\phi(t)]},
\]

where, for simplicity, we have assumed the time scales of the detector to be much longer than \(t_f\), allowing us to ignore the free evolution of the detector. The final state

\[
H = H_{TI}[\phi(t)] + H_{\text{pert}} + H_{\text{detector}}. \quad (1)
\]

We have in mind a 2D TI in a Corbino disk geometry (see Fig. 1b). Its Hamiltonian \(H_{TI}[\phi(t)]\) depends on the AB-flux, \(\phi(t) = \phi_0 t_f/\hbar\), that is generated by a solenoid situated within the hole (see Fig. 1b). As is, each threading of a \(\phi_0\), would induces the aforementioned magnetic excitations on both the inner and outer edges. Applying a small magnetic field \(B_i\) at the inner edge introduces the perturbation, \(H_{\text{pert}} = B_i M_i\); where \(M_i\) is the total magnetization on the edge in the \(i\)-direction. Generically, this would cause a small gap, \(\Delta_h\), between the counter-propagating spin edges (see Fig. 1b). Below we will always consider the limit \(\Delta_h^\text{−1} \ll t_f/\hbar \ll \Delta_h^\text{−1}\) where \(\Delta\) in the bulk gap of the TI.
Notably in the last equality we exploited the fact that exponentiation \(e^{\lambda P}\) can be readily evaluated on the product basis consisting of detector position basis \([|x\rangle]\) and many-body edge excitation spectrum \([|m\rangle]\),

\[
\langle x| \langle m| f \rangle = \langle m| U_{0}^{f^T} |i_S\rangle \langle x| e^{\lambda h^{-1} A_m P} |i_D\rangle + O(\lambda^2),
\]

where

\[
A_m = i \frac{\langle m| \int dt U^{f^T}_{t} \bar{M}_{t} U_{0}^{f^T} |i_S\rangle}{\langle m| U_{0}^{f^T} |i_S\rangle} = h\theta_B \log \left[ \langle m| U_{0}^{f^T} |i_S\rangle \right].
\]

Notably in the last equality we exploited the fact that \(B_i\) couples to the same operator as \(P\) does. The \(A_m\)'s are known as weak-values (WVs), and for the re-exponentiation \(e^{\lambda h^{-1} A_m P} = 1 + \lambda h^{-1} A_m P\) we have assumed that higher-order WVs are negligible \([8, 10, 26]\).

Since the characteristic scale of \(P\) is \(\sqrt{hM\omega}\), the condition for a weak measurement is

\[
\lambda A_m \ll \frac{1}{\sqrt{hM\omega}}.
\]

Following the weak measurement protocol \([8]\) one now applies a second strong measurement that determines the final state of the system \(|m_f\rangle\) (postselection) in order to measure a specific WV. In our case, we consider a postselection on the ground state of the system at time \(t_f\) with \(\phi = \phi_0\), i.e., \(|m_f\rangle = |i_S\rangle \equiv G_{\phi_0} |i_S\rangle\), where \(G_{\phi_0}\) is a gauge transformation which inserts a flux quantum through the disk \((G_{\phi_0} = e^{i\theta}, \text{where } \theta \text{ is the angle along the disk})\). The weak measurement outcomes are collected conditional on the postselection outcome, i.e., if the system is not found to be in its ground state, the experiment outcome is ignored. We denote \(A_{m_f}\) simply as \(A\) from now on.

Repeating the procedure, we obtain the average of detector position, \(X\), conditioned on the postselection from Eq. (5) with \(m = i_S\),

\[
\bar{X}_{i_S} = \frac{\langle i_S | U_{0}^{f^T} | i_S \rangle^2 \langle i_D | e^{-\lambda h^{-1} A_P} X e^{\lambda h^{-1} A_P} | i_D \rangle}{\langle i_S | U_{0}^{f^T} | i_S \rangle^2}
\]

assuming for the moment that \(A\) is purely imaginary, the average value obtained would be \(\bar{X} = \lambda A\).

Let us turn to evaluate \(A_{m_f}\). Following Eq. (6), this simply amounts to evaluating \(\langle i_S | U_{0}^{f^T} | i_S \rangle\). Taking the simplest level crossing model and assuming \(\Delta_h t_f / \hbar \ll 1\) one may use the well known result \([27]\)

\[
\langle i_S | U_{0}^{f^T} | i_S \rangle^2 = c_0 B_i^2 + O(B_i^4),
\]

where \(c_0\) is some non-universal constant which depends, in particular, on the rate of flux threading, and the direction of the magnetic field \((\hat{i})\). Taking a more realistic descriptions of this transition, allowing for example several nearby energy levels, would not alter this result \([18, 19]\). Comparatively, for a trivial band insulator, with no edge modes, the only energy scale is \(\Delta\) with respect to which the experiment is adiabatic and consequently, the above probability changes to \(1 - O(B_i^2)\).

Plugging these expressions into Eq. (6), the non-universal contributions decouple and one obtains

\[
A = \frac{2\hbar \nu_2}{B_i} + O(1)
\]

where \(\nu_2 = 1(0)\) for a topological insulator (band insulators). Roughly speaking, this follows from viewing the effective magnetic perturbation as \((B_i + \lambda P)\). Postselecting for an avoided Zener transition, means that as \(B_i\) decreases a strong fluctuation in \(\lambda P\) is required to assist an avoided transition.
Notably however, $A$ turned out real. Consequently, one extra procedure is needed to witness its effect on the detector position. Examining Eq. (5), one finds that
\[
e^{\lambda\hbar^{-1}AP}|\hat{D}\rangle = \left(\frac{\hbar}{2\pi M\omega}\right)^{1/2} \int dk e^{-\frac{k^2}{2\pi \nu} + \lambda A}\langle k |\]
(11)
\[
= \left(\frac{\hbar}{2\pi M\omega}\right)^{1/2} \int dk e^{-\frac{\nu}{2\pi \omega} [k - \hbar^{-1} A M^2]^2} + o(\lambda^2) |k|.
\]
Up to corrections in $\lambda^2$, which we consistently neglect, the weak value simply shifts the momentum ($P$) by $\omega M\hbar^{-1} A$. Waiting for the resulting coherent state to evolve for a quarter period ($\pi/2$) and then measuring $X$, yields
\[
B_i \frac{\bar{X}_{\pi/2}}{\lambda} = 2\hbar \nu_2.
\]
Equation (7) ensures that $\bar{X}_{\pi/2}$ is much smaller than the standard deviation of $X$. Notably all the quantities on the l.h.s. are system independent quantities associated with the detector and external perturbation, while the r.h.s is quantized in units of $\hbar$. This is the key result of this work.

A few comments are in order regarding the observability of the above result. First, obtaining a divergent weak value does not imply that the detector signal actually diverges, as that would mean that the measurement ceases to be weak (see Eq. (7)). Instead as $A$ diverges, $\lambda$ must be reduced and the detector read-out must be recalibrate using an independent classical source of magnetization. Alternatively stated, we treat $\bar{X}_{\pi/2}$ as the calibrated detector read-out and only in terms of this value would the signal appear divergent.

Second, the quantization depends on an accurate detection of the final state. Final state detection errors would cut-off the divergent nature of the weak value [11]. Since we require single spin levels of detection, such errors are unavoidable with current technology although the field is progressing rapidly [24].

Third, an error in the quantization of the pole's residue would be induced by tilting the direction of perturbing field ($B_i$) with respect to that of the magnetization being measured ($M_i$). Less restrictively, all is required is that the TRS breaking perturbation couples to same TRS breaking operator which is being weakly measured. This operator can be any TRS breaking operator and, in particular, may vary in space. To achieve such coupling, one may use an invasive magnetometer, such as a magnetized AFM tip, to both generate the perturbation and measure it. Lastly, to preserve TRS, it is important that the flux threading is done symmetrically around $\phi(t_0) = \phi_0/2$ meaning that $\phi(t_0 + t) = -\phi(t_0 - t)$.

Our main result, being the quantized residue of a particular WV, relied mainly on having an un-avoided Zener transition and detector-perturbation alignment. Consequently, it can be generalized to different setups. For instance, one may consider a gate-voltage driven Zener transition between two charge states of a weakly coupled double quantum dot [11]. These two states being one electron on the left dot and no electrons on the right dot, and vice versa. Obtaining residue quantization in this setup requires a detector which couples to an operator ($T$) that transfers charge between the two dots. Formally this requires $H_{\text{detector}}$ with $M_i$ replaced by $T$. Furthermore, one should control the average value of the detector’s momentum ($\langle P \rangle = \lambda^{-1} B_i$) as this effectively generates the analogous term to $H_{\text{pert}}$. Once this is achieved, all previous arguments follow. Potentially, such a detector could be realized using an LC circuit whose charging ($P$ in our notations) controls the opening of a quantum point contact (QPC) between the two dots, or similarly a charged cantilever coupled to the QPC. Postselection must again be done for the unlikely outcome, being that the electron hopped between the dots. The advantage here is that postselection requires a detection of a single charge rather than a single spin.

To conclude, a protocol for measuring a weak value associated with the $Z_2$ index [11, 2] of a TI was proposed. This weak value is formally infinite for a TI with no TRS breaking and diverges as the inverse of the strength of TRS breaking perturbation. In contrast, for a band insulator it is typically zero and has no such divergent behavior. If more stringent limitations are imposed, the residue of this divergence becomes quantized in units of $2\hbar$. The two main experimental difficulties involved appear to be making a precision measurement of a few spins, and threading a flux through a hole in a TI such that it does not appreciably leak to the TI edges. Alternatively, similar physics can be observed in a double quantum dot setup with a charged AFM tip or an LC circuit which measure and control the coupling between the dots. Such quantized residues may thus provide a test for the application of weak values in condensed matter physics.

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[3] R. Ilan, F. de Juan, and J. E. Moore, ArXiv e-prints (2014), 1410.5823.
[4] Z. Wu, F. M. Peeters, and K. Chang, Applied Physics Letters 98, 162101 (2011), URL http://scitation.aip.org/content/aip/journal/apl/98/16/10.1063/1.3581887?
[5] C. Ojeda-Aristizábal, M. S. Fuhrer, N. P. Butch, J. Paglione, and I. Appelbaum, Applied Physics Letters 101, 023102 (2012), URL http://scitation.aip.org/
In the simplistic model described below, this magnetization has a magnitude of two spins and points in the $\hat{z}$ direction. Generically it may point in any direction, and should typically be of the order of, but smaller than, two spins.

The rate at which flux quanta are threaded must be smaller than the bulk gap over $\hbar$. From Ref. (20) one obtains a threshold of roughly 10 Terahertz.

[20] B. I. Halperin, Phys. Rev. B 25, 2185 (1982), URL http://link.aps.org/doi/10.1103/PhysRevB.25.2185

[21] Y. Martin and H. K. Wickramasinghe, Applied Physics Letters 50, 1455 (1987), URL http://scitation.aip.org/content/aip/journal/apl/50/20/10.1063/1.97800

[22] O. Zger and D. Rugar, Applied Physics Letters 63, 2496 (1993), URL http://scitation.aip.org/content/aip/journal/apl/63/18/10.1063/1.110460

[23] M. S. Grinolds, S. Hong, P. Maletinsky, L. Luan, M. D. Lukin, R. L. Walsworth, and A. Yacoby, Nat Phys 9, 215 (2013), URL http://dx.doi.org/10.1038/nphys2543

[24] J. R. Kirtley, M. B. Ketchen, K. G. Stawiasz, J. Z. Sun, W. J. Gallagher, S. H. Blanton, and S. J. Wind, Applied Physics Letters 66, 1138 (1995), URL http://scitation.aip.org/content/aip/journal/apl/66/9/10.1063/1.113838

[25] J. Dressel, D. Rugar, Applied Physics Letters 63, 2496 (1993), URL http://scitation.aip.org/content/aip/journal/apl/63/18/10.1063/1.110460