Recent progress in string theory has led to a reformulation of quantum-group polynomial invariants for knots and links into new polynomial invariants whose coefficients can be understood in topological terms. We describe in detail how to construct the new polynomials and we conjecture their general structure. This leads to new conjectures on the algebraic structure of the quantum-group polynomial invariants. We also describe the geometrical meaning of the coefficients in terms of the enumerative geometry of Riemann surfaces with boundaries in a certain Calabi-Yau threefold.

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1. Introduction

During the last two decades the theory of knot and link invariants has experienced important progress. In the eighties a series of new polynomial invariants were discovered leading to a unified picture provided by quantum-group polynomial invariants, and by vacuum expectation values of operators in Chern-Simons gauge theory. The progress in the nineties was characterized by the discovery of Vassiliev invariants or invariants of finite type. It was realized soon that these types of invariants were related. It is now known that the coefficients of the power series expansion of the quantum-group invariants and the coefficients of the perturbative series expansion of the vacuum expectation values of operators in Chern-Simons gauge theory are Vassiliev invariants. These connections inspired important developments in the theory of finite-type invariants.

Besides the great progress achieved during the last two decades, there are still many unanswered questions in the theory of knots and links. One of these questions is about the topological meaning of the integer coefficients of the quantum-group polynomial invariants (see, for example, [13]). The discovery of the relation between these polynomial invariants and Vassiliev invariants did not provide important progress in this direction. The main goal of this paper is to point out that the situation has changed dramatically in the last two years. A new point of view to study knot and link invariants is now available. In this new approach the integer coefficients of a reformulation of the quantum-group polynomial invariants carry topological content.

At the heart of this development is the idea that quantum gauge theories may have a string theory description. In the case of Chern-Simons theory, a first step in this direction was taken by Witten in 1992 [18]. The final picture emerged in 1998, when R. Gopakumar and C. Vafa [21] found a description of Chern-Simons theory in terms of a closed, topological string theory. In 1999 H. Ooguri and C. Vafa [22] showed how to describe Wilson loops of knots in Chern-Simons theory by introducing an open-string sector in the topological string theory of [21]. They also showed that the string description of Chern-Simons Wilson loops involved a reformulation of quantum-group invariants in terms of new integer invariants. The connection between the quantum-group invariants and the string description

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1 For reviews on these developments see [13] and [14].
2 This idea has been also explored in [19] and [20].
was spelled out in detail in [23]. The integer invariants of [22] were further refined in [24], where the string description was also extended to the case of links.

In this paper we will present a summary of these developments and their mathematical implications for the theory of quantum-group invariants of knots and links. The new point of view described in this paper first reformulates the quantum-group polynomial invariants in terms of new polynomials, and then assigns topological content to their coefficients. The relation establishes a connection between quantum-group invariants and the geometry of the moduli spaces of Riemann surfaces with holes holomorphically embedded in a specific Calabi-Yau manifold. The embedding has Lagrangian boundary conditions, and the Lagrangian submanifold which specifies them is conjectured to be determined by the link [22]. This connection has to be considered at the level of conjecture. The integer invariants are difficult to compute from the topological side and up to date the conjecture has been fully tested only for the unknot [25] [26]. Nevertheless, the conjecture also predicts a particular structure for the reformulated polynomial invariants which has been verified in many cases [22] [23] [27] [24].

The integer coefficients of the reformulated polynomial invariant of a given link can be also interpreted in terms of a generalization of Gromov-Witten invariants which involve Riemann surfaces with boundaries. These integer invariants turn out to be a resummation of generalized Gromov-Witten invariants much in the same spirit as the Gopakumar-Vafa invariants are for the ordinary ones [28] [29].

The paper is organized as follows. In section 2 we review the polynomial invariants for knots and links from a quantum group perspective and we make a slight refinement for the case of links to make them more suitable for our purposes. In section 3 we reformulate these polynomials in terms of new ones and we conjecture their general structure. We also present some simple consequences of the conjecture for the structure of the HOMFLY polynomial of links. In section 4 we describe the topological content of the integer coefficients of the new polynomials. Finally, in section 5 we state our conclusions and comment on future developments.

2. Quantum-group polynomial invariants of knots and links

The quantum-group invariants that we will be considering are multicolored generalizations of the HOMFLY polynomial (with an important subtlety in the case of links), and their definition is as follows.
Let $\mathcal{L}$ be a link of $L$ components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$. The linking number of the components $\mathcal{K}_\alpha$, $\mathcal{K}_\beta$ will be denoted by $\text{lk}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$. We will represent the link $\mathcal{L}$ by the closure of a braid. The braid will have $N$ strands, and $N_\alpha$ will denote the number of strands of the component $\mathcal{K}_\alpha$. We will also assume that the strands are ordered in such a way that the first $N_1$ strands correspond to the component $\mathcal{K}_1$, and so on.

We associate to each component of the link an irreducible representation $R_\alpha$ of the quantized universal enveloping algebras $U_q(\text{sl}(N, \mathbb{C}))$. These representations are labeled by highest weights $\Lambda_\alpha$, and as usual (see for example [30]) we will associate to them a Young diagram with $\ell_\alpha$ boxes. The corresponding module will be denoted by $V_{\Lambda_\alpha}$. Therefore, to the $j$-th strand in the braid we will associate an irreducible module $V_j$. If the $j$-th strand belongs to the component $\mathcal{K}_\alpha$, then $V_j = V_{\Lambda_\alpha}$. The total module associated to the braid is then $V = \otimes_{j=1}^{N} V_j$, which is also given by:

$$V = \otimes_{\alpha=1}^{L} V_{\Lambda_\alpha}^{N_\alpha}. \quad (2.1)$$

It is well-known that each solution to the Yang-Baxter equation provides a representation of the braid group of $N$ strands $\mathcal{B}_N$. The solution that we will use is the universal $R$-matrix of $U_q(\text{sl}(N, \mathbb{C}))$ [31]:

$$R = q^{\frac{1}{2}} \sum_{i,j} C_{ij}^{-1} H_i \otimes H_j \prod_{\beta} \exp_q[(1-q^{-1})X^+_{\beta} \otimes X^-_{\beta}]. \quad (2.2)$$

In this equation, $C_{ij}$ is the Cartan matrix of $SU(N)$, and the $q$-exponential has the form,

$$\exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \frac{x^k}{[k]_q!}, \quad (2.3)$$

where the $q$-numbers are defined as:

$$[n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (2.4)$$

In the expression (2.2) for $R$, the product over $\beta$ denotes a product over positive roots of $\text{sl}(N, \mathbb{C})$, and the $X^\pm_{\beta}$ are certain elements in $U_q(\text{sl}(N, \mathbb{C}))$ (see [31], chapter 8 for details). Notice that $R$ acts in a natural way on the tensor product of two $U_q(\text{sl}(N, \mathbb{C}))$-modules. Finally, we denote $\tilde{R} = P_{12}R$, where $P_{12}$ is the exchange operator between the two factors of the tensor product.
The representation of $B_N$ on $V$ is defined as follows. If $\sigma_{i}^{\pm 1}$ is an elementary braid, then

$$\pi(\sigma_{i}^{\pm 1}) = I_{V_i} \otimes \cdots \otimes \check{R}^{\pm 1} \otimes \cdots \otimes I_{V_N},$$

(2.5)

where $\check{R}^{\pm 1}$ acts on $V_i \otimes V_{i+1}$. Therefore, every braid word $\xi$ gives an operator on the total module $V$ (2.1) that we will denote by $\pi(\xi)$. Since $R$ is a solution of the quantum Yang-Baxter equation, the above representation of the braid group is well-defined, i.e., it respects the relations between the generators [5].

In order to define an invariant of links, we need an enhancement of the $R$-matrix [3]. We take $\mu = q^{\rho^*}$,

$$\mu = q^{\rho^*},$$

(2.6)

where $\rho^*$ is the element in the Cartan subalgebra $h \subset U_q(h)$ which corresponds to the Weyl vector (i.e., the sum of fundamental weights) under the natural isomorphism $h \simeq h^*$ induced by the Killing form.

The quantum-group invariant that we will consider is defined as follows:

$$W(R_1, \ldots, R_L)(\mathcal{L}) = q^{d(\mathcal{L})} \text{Tr}_V (\mu^{\otimes N} \pi(\xi));$$

(2.7)

where $d(\mathcal{L})$ is given by:

$$d(\mathcal{L}) = -\frac{1}{2} \sum_{\alpha=1}^{L} w(K_\alpha)(\Lambda_\alpha, \Lambda_\alpha + 2\rho) + \frac{1}{N} \sum_{\alpha < \beta} \text{lk}(K_\alpha, K_\beta)\ell_\alpha \ell_\beta.$$  

(2.8)

In this expression $w(K_\alpha)$ is the writhe of the $\alpha$-th component of the link $\mathcal{L}$, and $\ell_\alpha$ and $\ell_\beta$ are the number of boxes of the Young diagrams associated to the irreducible representations $\Lambda_\alpha$ and $\Lambda_\beta$. The first term in (2.8) guarantees, by the usual arguments [5], that (2.7) is an ambient isotopy invariant of the link. The second term in (2.8) cancels overall powers of $q^{1/N}$ that appear after taking the trace in (2.7). The resulting quantum-group invariant is in general a rational function of $q^{\pm 1/2}$ and $\lambda^{\pm 1/2}$, where

$$\lambda = q^N.$$  

(2.9)

\footnote{Often, following standard usage, we will refer to this invariant (as well as to the reformulated one below) as polynomial invariant though, strictly speaking, it is in general a rational function.}
Remarks:

- If $L$ is the trivial link of $L$ components, with attached representations $R_\alpha$, $\alpha = 1, \cdots, L$, then the quantum-group invariant is,

$$W_{(R_1, \cdots, R_L)}(\mathcal{L}) = \prod_{\alpha=1}^{L} \dim_q(R_\alpha), \quad (2.10)$$

where $\dim_q(R_\alpha)$ is the quantum dimension of $R_\alpha$.

- When all the components of the link are in the fundamental representation, i.e., $R_\alpha = \Box$, the above quantum-group invariant is related to the HOMFLY polynomial of the link, $P_\mathcal{L}(q, \lambda)$, in the following way:

$$W_{(\Box, \cdots, \Box)}(\mathcal{L}) = \lambda^{\text{lk}(\mathcal{L})} \left( \frac{\lambda^2 - \lambda^{-2}}{q^{1/2} - q^{-1/2}} \right) P_\mathcal{L}(q, \lambda), \quad (2.11)$$

where

$$\text{lk}(\mathcal{L}) = \sum_{\alpha<\beta} \text{lk}(K_\alpha, K_\beta) \quad (2.12)$$

is the total linking number of $\mathcal{L}$. From this relation we observe that the quantum-group invariant defined in (2.7) for a link with the fundamental representation attached to all its components is not the unnormalized HOMFLY polynomial, but differs from it in an overall factor $\lambda^{\text{lk}(\mathcal{L})}$. Usually, when all the components of a link are in the same representation $\Lambda$, its quantum-group invariants are defined as in (2.7), with the only difference that the overall power of $q$ is not $d(\mathcal{L})$ but,

$$-\frac{1}{2}(\Lambda, \Lambda + 2\rho)w(\mathcal{L}), \quad (2.13)$$

where $w(\mathcal{L})$ the total writhe of the link,

$$w(\mathcal{L}) = \sum_{\alpha=1}^{L} w(K_\alpha) + 2\text{lk}(\mathcal{L}). \quad (2.14)$$

The difference between (2.13) and (2.8) when $\Lambda = \Box$ gives precisely the extra factor $\lambda^{\text{lk}(\mathcal{L})}$, which will be crucial for our considerations.

- In (2.7), we have assumed that none of the representations $R_1, \cdots, R_L$ is the trivial one. It will be useful to extend the definition to include the trivial representation as follows: let $\{\alpha_1, \cdots, \alpha_s\}$ be a subset of $\{1, \cdots, L\}$, with $s > 0$. The complementary set will be
denoted by \{\alpha_{s+1}, \ldots, \alpha_L\}. The sublink \(L_{\alpha_1, \ldots, \alpha_s}\) of \(L\) is obtained by considering only the components \(K_{\alpha_i}\) of \(L\), with \(i = 1, \ldots, s\), and “deleting” the rest of the components. If we take \(R_{\alpha_{s+1}} = \cdots = R_{\alpha_L} = \cdot\) to be the trivial representation, we define:

\[
W_{(R_1, \ldots, R_L)}(L) = W_{(R_{\alpha_1}, \ldots, R_{\alpha_s})}(L_{\alpha_1, \ldots, \alpha_s}).
\]

(2.15)

Examples:

• For the trefoil knot 3_1, the quantum-group invariants for the lowest representations are [23]:

\[
W_{\Box} = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}(-2\lambda^\frac{1}{2} + 3\lambda^\frac{3}{2} - \lambda^5) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(-\lambda^\frac{1}{2} + \lambda^3),
\]

\[
W_{\Diamond} = \frac{(\lambda - 1)(\lambda q - 1)}{\lambda(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(1 + q)}
\times \left((\lambda q^{-1})^2(1 - \lambda q^2 + q^3 + \lambda q^3 + q^4 - \lambda q^5 + \lambda^2 q^5 + q^6 - \lambda q^6)\right),
\]

(2.16)

\[
W_{\Box \Box} = \frac{(\lambda - 1)(\lambda - q)}{\lambda(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(1 + q)}
\times \left((\lambda q^{-2})^2(1 - \lambda - \lambda q + \lambda^2 q + q^2 + q^3 - \lambda q^3 - \lambda q^4 + q^6)\right).
\]

• For the Hopf link, one finds:

\[
W_{(\Box, \Box)} = \left(\frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right)^2 - \lambda^{-1}(\lambda - 1).
\]

(2.17)

3. Reformulated polynomial invariants and a conjecture on their structure

In this section we will introduce a generating functional for the quantum-group polynomial invariants which will lead to their reformulation in terms of new polynomials. In addition, we will state a conjecture on their structure.

3.1. Generating functional for quantum-group invariants

To state the conjecture about the structure of the quantum-group polynomial invariants of links, it is useful to package these invariants into a generating functional. The quantum-group invariants are labeled by representations of \(SU(N)\), which can be also regarded as representations of the permutation group. It is useful to introduce a related set
of quantum-group invariants which are labeled by conjugacy classes of the permutation group. We will specify these conjugacy classes by vectors $\vec{k} = (k_1, k_2, \cdots)$ with,

$$\ell = \sum_j j k_j, \quad |\vec{k}| = \sum_j k_j,$$  \hspace{1cm} (3.1)

and $\ell$ is then the order of the permutation group. The conjugacy class associated to a vector $\vec{k}$ will be denoted by $C(\vec{k})$, and it has $k_j$ cycles of length $j$. The number of elements in the class, denoted by $|C(\vec{k})|$, is given by the formula,

$$|C(\vec{k})| = \frac{\ell!}{\prod k_j! \prod j^{k_j}}.$$  \hspace{1cm} (3.2)

Given a link $L$ with $L$ components, and given $L$ vectors $\vec{k}^{(1)}, \cdots, \vec{k}^{(L)}$, we define the following linear combination of quantum-group invariants:

$$W_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})} = \sum_{\{R_{\alpha}\}} \prod_{\alpha=1}^L \chi_{R_{\alpha}}(C(\vec{k}^{(\alpha)})) W_{(R_1, \cdots, R_L)},$$  \hspace{1cm} (3.3)

where $\chi_{R_{\alpha}}$ are characters of the symmetric group $S_{\ell_{\alpha}}$, and $\ell_{\alpha} = \sum_j j k^{(\alpha)}_j$ equals the number of boxes in the Young tableau of $R_{\alpha}$. If $\vec{k}^{(\alpha)}$ is the zero vector for some $\alpha$, then a $R_{\alpha}$ will be the trivial representation.

We now introduce generic $SU(N)$ elements, $V_{\alpha}, \alpha = 1, \cdots, L$, one for each component of the link, and denote:

$$\Upsilon_{\vec{k}}(V_{\alpha}) = \prod_{\alpha=1}^L (\text{Tr } V^{j}_{\alpha})^{k_j}, \quad \alpha = 1, \cdots, L.$$  \hspace{1cm} (3.4)

If $x^{(\alpha)}_j, j = 1, \cdots, N$ are the eigenvalues of $V_{\alpha}$, then (3.4) are symmetric polynomials in these eigenvalues (the Newton polynomials) and they provide a basis for $Q[x^{(\alpha)}_1, \cdots, x^{(\alpha)}_N]^{S_N}$, the ring of symmetric functions in $x^{(\alpha)}_1, \cdots, x^{(\alpha)}_N$ with rational coefficients. It is convenient in fact to consider $V_{\alpha} \in SU(\infty)$, for all $\alpha = 1, \cdots, L$, and correspondingly to consider the ring of symmetric functions (in fact power series) in infinitely many variables, $x^{(\alpha)}_1, x^{(\alpha)}_2, \cdots$, which will be denoted by $\Lambda^{(\alpha)}$. Armed with this machinery we make the following definition.

**Definition:** Let $L$ be a link with $L$ components. The *generating functional of quantum-group polynomial invariants* is formally defined as follows:

$$Z(V_1, \cdots, V_L) = 1 + \sum_{\{\vec{k}^{(\alpha)}\}} W_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})} \prod_{\alpha=1}^L \frac{|C(\vec{k}^{(\alpha)})|}{\ell_{\alpha}!} \Upsilon_{\vec{k}^{(\alpha)}}(V_{\alpha}).$$  \hspace{1cm} (3.5)
where the sum is over all vectors \( \vec{k}^{(\alpha)} \) such that \( \sum_{\alpha=1}^{L} |\vec{k}^{(\alpha)}| > 0 \). Notice that, if \( \vec{k}^{(\alpha)} \) is the zero vector for some \( \alpha \), the invariant defined by (3.3) will involve quantum group invariants with trivial representations. Therefore, the generating functional (3.5) of the quantum group invariants of a link also includes all the quantum group invariants of all its sublinks. Also notice that \( Z(V_1, \cdots, V_L) \) can be regarded as as element in the ring \( \Lambda^L(q^{\pm 1/2}, \lambda^{\pm 1/2}) \) of rational functions in \( q^{\pm 1/2}, \lambda^{\pm 1/2} \) with coefficients in \( \Lambda^L = \bigotimes_{\alpha=1}^{L} \Lambda^{(\alpha)} \).

We will also use the logarithm of this generating functional (also known as free energy), which can be written as:

\[
F(V_1, \cdots, V_L) = \log Z(V_1, \cdots, V_L) = \sum_{\{\vec{k}\}} W^{(c)}_{\vec{k}(1), \cdots, \vec{k}(L)} \prod_{\alpha=1}^{L} \frac{|C(\vec{k}^{(\alpha)})|}{\ell^{\alpha}} Tr_{\vec{k}^{(\alpha)}}(V_{\alpha}),
\]

and defines the “connected” invariants \( W^{(c)}_{\vec{k}(1), \cdots, \vec{k}(L)} \).

**Examples:** In the case of a knot, one has for the simplest cases,

\[
W^{(c)}_{(1,0,\cdots)} = W_{(1,0,\cdots)} = W_{\square},
\]

\[
W^{(c)}_{(2,0,\cdots)} = W_{(2,0,\cdots)} - W_{(1,0,\cdots)}^2 = W_{\sqcap \sqcup} + W_{\sqcap} - W_{\sqcup},
\]

while for a 2-component link \( L \) with components \( K_1 \) and \( K_2 \),

\[
W^{(c)}_{(1,0,\cdots),(1,0,\cdots)}(L) = W_{(1,0,\cdots),(1,0,\cdots)}(L) - W_{(1,0,\cdots)}(K_1)W_{(1,0,\cdots)}(K_2)
\]

\[
= W_{\square \square}(L) - W_{\square}(K_1)W_{\square}(K_2).
\]

**3.2. Reformulated polynomial invariants**

Our next goal is to reformulate the quantum-group polynomial invariants. Let us begin introducing rational functions \( f_{(R_1, \cdots, R_L)}(q, \lambda) \), labeled by representations of \( SU(N) \), as follows:

\[
F(V_1, \cdots, V_L) = \sum_{d=1}^{\infty} \sum_{\{R_{\alpha}\}} \frac{1}{d} f_{(R_1, \cdots, R_L)}(q^d, \lambda^d) \prod_{\alpha=1}^{L} \text{Tr}_{R_{\alpha}} V_{\alpha}^d.
\]

Using Frobenius formula and (3.6), the relation (3.9) is equivalent to the following equation:

\[
W^{(c)}_{\vec{k}(1), \cdots, \vec{k}(L)} = \sum_{d \mid \vec{k}} d \sum_{\alpha \mid \vec{k}^{(\alpha)}} \frac{1}{d^{\alpha}} \prod_{\alpha=1}^{L} \chi_{R_{\alpha}}(C(\vec{k}^{(\alpha)})_1/d) f_{(R_1, \cdots, R_L)}(q^d, \lambda^d).
\]

In this equation, the vector \( \vec{k}_{1/d} \) is defined as follows. Fix a vector \( \vec{k} \), and consider all the positive integers \( d \) that satisfy the following condition: \( d \mid j \) for every \( j \) with \( k_j \neq 0 \). When
this happens, we will say that “$d$ divides $\vec{k}$”, and we will denote this as $d|\vec{k}$. We can then define the vector $\vec{k}_{1/d}$ whose components are:

$$(\vec{k}_{1/d})_i = k_{di}. \quad (3.11)$$

The vectors which satisfy the above condition and are “divisible by $d$” have the structure $(0, \ldots, 0, k_{d}, 0, \ldots, 0, k_{2d}, 0, \ldots)$, and the vector $\vec{k}_{1/d}$ is then given by $(k_{d}, k_{2d}, \ldots)$. In (3.10), the integer $d$ has to divide all the vectors $\vec{k}(\alpha), \alpha = 1, \cdots, L$.

It is easy to show that the rational functions $f_{(R_1, \cdots, R_L)}$ are uniquely determined in terms of the connected invariants $W^{(c)}_{(\vec{k}(1), \cdots, \vec{k}(L))}$ by equation (3.10). Moreover, one can invert this equation and write an explicit formula for $f_{(R_1, \cdots, R_L)}$ in terms of the quantum group invariants $\mathbf{4}$. Following [35], if $F \in \Lambda^L(q^{\pm 1/2}, \lambda^{\pm 1/2})$, we can define an operation $\psi_d$ on this ring by

$$\psi_d \circ F(q, \lambda; \Upsilon_{\vec{k}(\alpha)}(V_\alpha)) = F(q^d, \lambda^d; \Upsilon_{\vec{k}(\alpha)}(V_\alpha^d)). \quad (3.12)$$

We can also define the so-called plethystic exponential [33],

$$\text{Exp}(F) = \exp \left( \sum_{d=1}^{\infty} \frac{\psi_d}{d} \circ F \right). \quad (3.13)$$

This exponential has an inverse given by [33]:

$$\text{Log}(F) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log(\psi_d \circ F), \quad (3.14)$$

where $\mu(d)$ is the Moebius function. Recall that this function is zero if $d$ is not square-free, and it is $(-1)^s$ otherwise, where $s$ is the number of primes in the decomposition of $d$.

Using these results, we can write an explicit formula for the $f_{(R_1, \cdots, R_L)}(q, \lambda)$. From (3.12) and (3.13) it follows that

$$Z(V_1, \cdots, V_L) = \text{Exp} \left( \sum_{\{R_\alpha\}} f_{(R_1, \cdots, R_L)}(q, \lambda) \prod_{\alpha=1}^{L} \text{Tr}_{R_\alpha} V_\alpha \right). \quad (3.15)$$

\footnote{We are grateful to Ezra Getzler for explaining to us how to find this explicit expression. A similar inversion formula has been obtained for the Gopakumar-Vafa invariants in [34].}
Using now the inverse of the plethystic exponential and the Frobenius formula, we finally obtain:

\[
\begin{align*}
  f_{(R_1, \ldots, R_L)}(q, \lambda) &= \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{dm} \sum_{\{k(\alpha j), R_{\alpha j}\}} \prod_{\alpha=1}^{L} \chi_{R_{\alpha}} \left( C \left( \sum_{j=1}^{m} k(\alpha j)_d \right) \right) \\
  &\quad \times \prod_{j=1}^{m} \frac{|C(k(\alpha j))|}{\ell_{\alpha j}!} \chi_{R_{\alpha}} \left( C(k(\alpha j)) \right) W_{(R_{1j}, \ldots, R_{Lj})}(q^d, \lambda^d). 
\end{align*}
\]  

(3.16)

The second sum runs over all vectors \( \vec{k}(\alpha j) \), with \( \alpha = 1, \ldots, L \) and \( j = 1, \ldots, m \), such that \( \sum_{\alpha=1}^{L} |k(\alpha j)| > 0 \) for any \( j \), and over representations \( R_{\alpha j} \). In (3.16), the vector \( \vec{k}_d \) is defined as follows: \( \vec{k}_d = (k_1, k_2, \ldots) \), where \( k_1 \) is in the \( d \)-th entry, \( k_2 \) in the \( 2d \)-th entry, and so on. Notice that \( (\vec{k}_d)_{1/d} = \vec{k} \), so this is the inverse operation to (3.11). Using (3.16), it is easy to show that the \( f_{(R_1, \ldots, R_L)} \) are equal to the quantum-group invariants \( W_{(R_1, \ldots, R_L)} \), plus some extra terms that involve the \( W_{(R_1, \ldots, R_L)} \) with a lower total number of boxes \( \sum_{\alpha} \ell'_{\alpha} \). These functions \( f_{(R_1, \ldots, R_L)} \) are indeed the reformulated polynomial invariants.

**Definition:** The reformulated quantum-group polynomial invariants are the functions \( f_{(R_1, \ldots, R_L)} \) entering (3.9). They can be expressed in terms of quantum group invariants through (3.16).

**Examples:** In the case of knots, one has, for representations of up to three boxes:

\[
\begin{align*}
  f_{\square}(q, \lambda) &= W_{\square}(q, \lambda), \\
  f_{\square}(q, \lambda) &= W_{\square}(q, \lambda) - \frac{1}{2} \left( W_{\square}(q, \lambda)^2 + W_{\square}(q^2, \lambda^2) \right), \\
  f_{\blacktriangle}(q, \lambda) &= W_{\blacktriangle}(q, \lambda) - \frac{1}{2} \left( W_{\square}(q, \lambda)^2 - W_{\square}(q^2, \lambda^2) \right), \\
  f_{\blacktriangle}(q, \lambda) &= W_{\blacktriangle}(q, \lambda) - \frac{1}{2} \left( W_{\square}(q, \lambda)^2 - W_{\square}(q^2, \lambda^2) \right), \\
  f_{\blacktriangle}(q, \lambda) &= W_{\blacktriangle}(q, \lambda) - W_{\square}(q, \lambda) W_{\square}(q, \lambda) + \frac{1}{3} \left( W_{\square}(q, \lambda)^3 - W_{\square}(q^3, \lambda^3) \right), \\
  f_{\blacktriangle}(q, \lambda) &= W_{\blacktriangle}(q, \lambda) - W_{\square}(q, \lambda) (W_{\square}(q, \lambda) + W_{\square}(q, \lambda)) + \frac{2}{3} W_{\square}(q, \lambda)^3 + \frac{1}{3} W_{\square}(q^3, \lambda^3), \\
  f_{\blacktriangle}(q, \lambda) &= W_{\blacktriangle}(q, \lambda) - W_{\square}(q, \lambda) W_{\square}(q, \lambda) + \frac{1}{3} \left( W_{\square}(q, \lambda)^3 - W_{\square}(q^3, \lambda^3) \right). 
\end{align*}
\]  

(3.18)
On the other hand, for a link $\mathcal{L}$ with two components, $K_1$ and $K_2$, one finds:

\[
\begin{align*}
 f(\bullet,\bullet)(\mathcal{L}) &= W(\bullet,\bullet)(\mathcal{L}) - W(\bullet)(K_1)W(\bullet)(K_2), \\
 f(\square,\bullet)(\mathcal{L}) &= W(\square,\bullet)(\mathcal{L}) - W(\square,\bullet)(\mathcal{L})W(\bullet)(K_1) - W(\square)(K_1)W(\bullet)(K_2) + W(\bullet)(K_1)^2W(\bullet)(K_2), \\
 f(\blacklozenge,\bullet)(\mathcal{L}) &= W(\blacklozenge,\bullet)(\mathcal{L}) - W(\blacklozenge,\bullet)(\mathcal{L})W(\bullet)(K_1) - W(\blacklozenge)(K_1)W(\bullet)(K_2) + W(\bullet)(K_1)^2W(\bullet)(K_2). 
\end{align*}
\]

(3.19)

**Remark:** The functions $f_{(R_1,\ldots,R_L)}$ were introduced by Ooguri and Vafa [22] through the relation (3.9). A recursive procedure to obtain these functions in terms of quantum-group polynomial invariants was spelled out in detail in [23] [24].

### 3.3. A conjecture on the structure of the reformulated polynomial invariants

In this subsection we present a conjecture on the algebraic structure of $f_{(R_1,\ldots,R_L)}$, which in turn implies a structure result for the $W_{(R_1,\ldots,R_L)}$. To state it we need to introduce the Clebsch-Gordan coefficients of the symmetric group. These are given by,

\[
C_{R'R''R} = \sum_{k} \frac{|C(k)|}{\ell!} \chi_{R}(C(k))\chi_{R'}(C(k))\chi_{R''}(C(k)).
\]

(3.20)

Finally, we also need to introduce the monomial $S_R(q)$, defined as follows. If $R$ is a hook representation, i.e., a representation whose Young tableau is of the form,

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
\end{array}
\]

(3.21)

with $\ell$ boxes in total and $\ell - d$ boxes in the first row, then

\[
S_R(q) = (-1)^d q^{-\ell^2/2 + d},
\]

(3.22)

and $S_R(q) = 0$ otherwise. Now we are ready to formulate the conjecture.

**Conjecture:** Given a link $\mathcal{L}$, the reformulated quantum-group polynomial invariants, $f_{(R_1,\ldots,R_L)}(q,\lambda)$, have the following structure:

\[
\begin{align*}
f_{(R_1,\ldots,R_L)}(q,\lambda) &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{L-2} \sum_{g \geq 0} \sum_{Q} \sum_{\{R'_\alpha, R''_\alpha\}} \left( \prod_{\alpha=1}^{L} C_{R'_\alpha, R''_\alpha} S_{R''_\alpha}(q) \right) N_{(R'_1,\ldots,R'_L),g,Q}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g} \chi_{Q},
\end{align*}
\]

(3.23)
where \( N_{(R_1,\ldots,R_L),g,Q} \) are integer numbers\(^5\), and \( Q \) are either all integers or all semi-integers.

**Remark:** the conjecture (3.23) was proposed in [24], and refines and generalizes a previous conjecture by Ooguri and Vafa. It can be regarded as a definition of the integer invariants \( N_{(R_1,\ldots,R_L),g,Q} \). The fact that they can be extracted from the \( W_{(R_1,\ldots,R_L)}(q,\lambda) \) in the way described is far from obvious. For example, according to the conjecture, \( f_{(R_1,\ldots,R_L)} \) must be a polynomial in \( q^{\pm \frac{1}{2}}, \lambda^{\pm \frac{1}{2}} \) with integer coefficients, times \((q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{L-2}\).

**Examples:**

- For a knot \( K \) with HOMFLY polynomial \( P_K \), one has,

\[
P_K(q,\lambda) = \sum_{g,j} a_{g,j} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g} \lambda^j,
\]

and the integer invariants are:

\[
N_{\square,g,j+1/2} = a_{g,j+1} - a_{g,j}.
\]

- For the trefoil knot, after using the known quantum-group invariants, one finds [23]:

\[
f_{\bigcircle}(q,\lambda) = \frac{q^{-\frac{1}{2}}\lambda(\lambda - 1)^2 (1 + q^2) (q + \lambda^2 q - \lambda (1 + q^2))}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},
\]

\[
f_{\bigtriangleleft}(q,\lambda) = -\frac{1}{q^3} f_{\bigcircle}(q,\lambda).
\]

The corresponding integer invariants are listed in the following tables:

**Table 1:** The integers \( N_{\square,g,Q} \) for the trefoil knot.

| \( g \) | \( Q = 1 \) | 2 | 3 | 4 | 5 |
|-------|-------|---|---|---|---|
| 0     | -2    | 8 | -12| 8 | -2|
| 1     | -1    | 6 | -10| 6 | -1|
| 2     | 0     | 1 | -2 | 1 | 0 |

**Table 2:** The integers \( N_{\bigtriangleleft,g,Q} \) for the trefoil knot.

| \( g \) | \( Q = 1 \) | 2 | 3 | 4 | 5 |
|-------|-------|---|---|---|---|
| 0     | -4    | 16| -24| 16| -4|
| 1     | -4    | 20| -32| 20| -4|
| 2     | -1    | 8 | -14| 8 | -1|
| 3     | 0     | 1 | -2 | 1 | 0 |

\(^5\) These integers differ in a sign from the integers introduced in [24]. More precisely, the \( N_{(R_1,\ldots,R_L),g,Q} \) are \((-1)^{L-2}\) times the integers denoted by \( \hat{N}_{(R_1,\ldots,R_L),g,Q} \) in [24].
3.4. Some consequences for the algebraic structure of the HOMFLY polynomial of links

In this subsection, we will explore some of the consequences of (3.23) for the algebraic structure of the HOMFLY polynomial of links. To do this, it is convenient to introduce some notation. Let us consider a link $L$ of $L$ components $K_\alpha$, $\alpha = 1, \cdots, L$. For simplicity we will denote

$$W((1,0,\cdots),(1,0,\cdots)) = W(L).$$

(3.27)

To write the “connected” invariant defined by (3.6), we have to take into account the invariants of all the sublinks of $L$. In section 2, given a subset $\{\alpha_1, \cdots, \alpha_s\} \subset \{1, \cdots, L\}$, we defined $L_{\alpha_1,\cdots,\alpha_s}$ as the sublink of $s$ components which is obtained from the link $L$ by keeping the components $K_{\alpha_i}$, $i = 1, \cdots, s$, and by “deleting” the remaining $L-s$ components. One can easily see from the definition of the “connected” invariants that

$$W(c)(L) = W(L) - \sum_{\alpha_1,\cdots,\alpha_{L-1}} W(L_{\alpha_1,\cdots,\alpha_{L-1}}) W(L_{\alpha_L}) + \cdots.$$  

(3.28)

For example, for a link of two components $K_1$, and $K_2$ one simply has:

$$W(c)(L) = W(L) - W(K_1)W(K_2).$$  

(3.29)

Using (3.10), we see that the conjecture (3.23) states that,

$$W(c)(L) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{L-2} \sum_{Q, g \geq 0} N_{\{1,\cdots,\}} g, Q \lambda Q (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g}.$$  

(3.30)

To analyze (3.30), we will first consider the simple case of a link of two components. Using (3.29) and (2.11), we find that the HOMFLY polynomial of the link has the following structure:

$$P_L(q, \lambda) = \sum_{g \geq 0} p_{2g-1}^{L}(\lambda)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g-1},$$  

(3.31)

i.e. the lowest power of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ is $-1$, and the powers are congruent to $-1$ mod 2. Moreover, if we denote the HOMFLY polynomial of the component knots by,

$$P_{K_\alpha}(\lambda, q) = \sum_{g \geq 0} p_{2g}^{K_\alpha}(\lambda)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g},$$  

(3.32)
for $\alpha = 1, 2$, we find,

$$p^L_{-1}(\lambda) = \lambda^{-lk(L)}(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})p_0^K_1(\lambda)p_0^K_2(\lambda).$$

(3.33)

The last equation comes from the requirement that there are no powers of $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-2}$ in $W^{(c)}(L)$. The results $(3.31)$ and $(3.33)$ capture completely the algebraic structure of the HOMFLY polynomial of a two-component link, and reproduce the results of Lickorish and Millett [36].

We can generalize the above results for links with an arbitrary number of components $L$. By induction on the number of components, and using $(3.28)$ and $(3.30)$, it is easy to prove that the HOMFLY polynomial of the link has the following structure:

$$P_L(q, \lambda) = \sum_{g \geq 0} p_{2g+1-L}(\lambda)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g+1-L},$$

(3.34)

i.e. the lowest power of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ is $1 - L$. This has been proved in [36]. Due to the relation $(2.11)$, it is convenient to introduce the following polynomials in $\lambda$:

$$\tilde{p}^L_{\alpha_1, \cdots, \alpha_s}(\lambda) = \lambda^{lk(L, \alpha_1, \cdots, \alpha_s)}p^L_{\alpha_1, \cdots, \alpha_s}(\lambda).$$

(3.35)

Finally, we will write,

$$W^{(c)}(L) = \left(\frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right)\sum_{g \geq 0} \tilde{p}^{(c),L}_{2g+1-L}(\lambda)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g+1-L}.$$ 

(3.36)

The conjecture $(3.30)$ then states that,

$$\tilde{p}^{(c),L}_{1-L}(\lambda) = \tilde{p}^{(c),L}_{3-L}(\lambda) = \cdots = \tilde{p}^{(c),L}_{L-3}(\lambda) = 0.$$ 

(3.37)

This implies, in particular, that the polynomials $p^L_k(\lambda)$ of the HOMFLY polynomial of a link, for $k = 1 - L, 3 - L, \cdots, L - 3$, are completely determined by the HOMFLY polynomial of its sublinks. As a first consequence of $(3.37)$, it is easy to show the following proposition.

**Proposition** (Lickorish and Millett [36]). The polynomial in $\lambda$, $p_{1-L}(\lambda)$, in the HOMFLY polynomial of a link $(3.34)$ is given by

$$p^L_{1-L}(\lambda) = \lambda^{-lk(L)}(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^{L-1} \prod_{\alpha=1}^{L} p_0^K_{\alpha}(\lambda).$$

(3.38)
This result is a consequence of $\tilde{p}^{(c)}_{1-L}(\lambda) = 0$, and it is easily proven by induction on the number of components of the link: since $\tilde{p}^{(c)}_{1-L}(\lambda) = 0$, one can extract the coefficient of the lowest power of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ in $W(L)$ from the terms in the expansion of $W^{(c)}(L)$ that only involve products of invariants of sublinks. One sees immediately that the relevant part of these invariants is again the coefficient of the lowest power of $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. But because of the induction hypothesis, these in turn can be evaluated by factorization into their knots. This means that the coefficient $\tilde{p}^{E}_{1-L}$ can be evaluated from $\prod_{\alpha=1}^{L} W(K_{\alpha})$, and this proves (3.38).

Notice that (3.38) is just the simplest consequence of (3.37), which gives much more relations. For example, for links with $L = 3$, the equality $\tilde{p}^{(c)}_{0-L}(\lambda) = 0$ implies that

$$\tilde{p}^{E}_{0}(\lambda) = (\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})(p_{0}^{K_{1}}(\lambda) p_{1}^{L=23}(\lambda) + \text{perms})$$
$$- 2(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^{2}(p_{2}^{K_{1}}(\lambda) p_{0}^{K_{2}}(\lambda) p_{0}^{K_{3}}(\lambda) + \text{perms}).$$

(3.39)

For links with more components, one obtains more complicated equations which can be summarized as in (3.37), providing a new set of results on the algebraic structure of the HOMFLY polynomial of links.

4. Topological content of the new integer invariants

In the previous sections, we have presented the conjecture on the structure of the reformulated quantum-group invariants of knots and links, and we have introduced a new set of integer invariants. In this section, we will describe the topological content of the latter. The starting point is the connection between quantum-group polynomial invariants and Chern-Simons gauge theory. Quantum-group invariants can be expressed as vacuum expectation values of Wilson loops. As in any gauge theory these vacuum expectation values admit a large-$N$ expansion [37], which in turn can be interpreted as a string theory expansion [38]. The string theory description of Wilson loops in Chern-Simons gauge theory is given in terms of a topological open string theory, of the kind that in the closed case leads to Gromov-Witten invariants. The first step to provide a geometrical meaning to the new integer invariants is to express the reformulated quantum-group polynomial invariants in terms of Gromov-Witten invariants generalized to the open string case.

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6 The large-$N$ expansion of Wilson loops in Chern-Simons gauge theory was studied from a field theory point of view in [38].
4.1. 1/N expansion and Gromov-Witten invariants for open strings

The geometric interpretation first appears in the context of the so-called 1/N expansion of the invariants. The “connected” invariants that we introduced in $(3.6)$ are rational functions of $q^{±1/2}$ and $λ^{±1/2}$. If we put $q = e^{ix}$ but keep $λ$ fixed, and we formally expand in $x$, we find a series with the structure:

$$\left( \prod_{α=1}^{L} \frac{|C(\vec{k}(α))|}{ℓ_α!} \right) W^{(c)}_{\vec{k}(1), \ldots, \vec{k}(L)} = i \sum_{α=1}^{L} |\vec{k}(α)| \sum_{g=0}^{∞} x^{2g-2} \sum_{α=1}^{L} |\vec{k}(α)| F_{g,(\vec{k}(1), \ldots, \vec{k}(L))}(λ).$$ (4.1)

Notice that the generating functional $(3.6)$ can be written in terms of the functions $F_{g,(\vec{k}(1), \ldots, \vec{k}(L))}(λ)$ as,

$$F(V_1, \ldots, V_L) = \sum_{\{\vec{k}(α)\}} i \sum_{α=1}^{L} |\vec{k}(α)| \sum_{g=0}^{∞} x^{2g-2} \sum_{α=1}^{L} |\vec{k}(α)| F_{g,(\vec{k}(1), \ldots, \vec{k}(L))}(λ) \prod_{α=1}^{L} \Upsilon_{\vec{k}(α)}(V_α).$$ (4.2)

**Remarks:**

- Since we are not expanding $λ = e^{iNx}$, we are keeping the variable $t = iNx$ fixed, and therefore we can equivalently write the above series as a “1/N expansion” by putting $x = −it/N$. The parameter $t$ is also called the ‘t’ Hooft parameter.
- The structure of the above expansion can be proved in the context of Chern-Simons theory by using standard 1/N analysis. We are not aware of a proof relying on the quantum-group definition of the invariants.

The geometric picture for the reformulated quantum-group invariants is based on the proposals made in [21][22]. Before stating it we need to introduce some machinery.

It was conjectured in [22] that to every link $L$ in $S^3$ one can associate a Lagrangian submanifold $C_L$ in the non-compact Calabi-Yau $X$,

$$\mathcal{O}(-1) ⊕ \mathcal{O}(-1) → P^1,$$ (4.3)

also called the resolved conifold. The assignment implies that $b_1(C_L) = L$, the number of components of $L$.

The quantities $F_{g,(\vec{k}(1), \ldots, \vec{k}(L))}(λ)$ are then interpreted in terms of an appropriate generalization of the Gromov-Witten invariants for Riemann surfaces with boundaries. Let $γ_α, α = 1, \ldots, L$, be one-cycles representing a basis for $H_1(C_L, \mathbb{Z})$, and let $Q ∈ H_2(X, C_L, \mathbb{Z})$ be
a relative two-homology class (i.e., a two-cycle of $X$ that ends on $C_L$). Then, one considers the holomorphic maps $f : \Sigma_{g,h} \to X$ from a Riemann surface $\Sigma_{g,h}$ of genus $g$ and with $h = \sum_{\alpha=1}^{L} |\vec{k}(\alpha)|$ holes, which satisfy the following conditions: first, $f_*[\Sigma_{g,h}] = Q$; second, $k_j^{(\alpha)}$ of the $h$ (oriented) boundaries of $\Sigma_{g,h}$ map to the cycle $\gamma_\alpha$ with winding number $j$, i.e., $f_*[C] = j[\gamma_\alpha]$ for $k_j^{(\alpha)}$ oriented circles $C$ of the boundary. The “number” of such maps will be denoted by $N_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}^Q$. These numbers are the open-string analog of Gromov-Witten invariants, and a precise definition would involve the construction of a compact moduli space for the maps $f$. The invariant $N_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}^Q$ would be then given by the degree of the virtual fundamental class of the moduli space, as in Gromov-Witten theory (see [23][26] for more details).

We can now describe the geometrical content of the coefficients in the $1/N$ expansion. Given a link $\mathcal{L}$, the functions $F_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}(\lambda)$ appearing in the expansion (4.1) are expressed in terms of the Gromov-Witten invariants for open strings in the following way:

$$F_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}(\lambda) = \sum_Q N_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}^Q \frac{\int_Q \omega}{Q!} (\lambda^d)^Q,$$  

where $\omega$ is the Kähler class of the Calabi-Yau manifold $X$ and $\lambda = e^t$, with

$$t = \int_{\mathbb{P}^1} \omega. \quad (4.5)$$

For any $Q$, one can always write $\int_Q \omega = Qt$, where $Q$ is in general a half-integer number, and therefore $F_{g,(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}(\lambda)$ is a polynomial in $\lambda^\pm \frac{1}{2}$ with rational coefficients.

**Example:** In the case of the trivial knot or unknot, it is easy to compute the functions $F_{g,\vec{k}}(\lambda)$ from the quantum-group invariants. These functions are nonvanishing only for vectors of the form $\vec{k} = (0,\ldots,0,1,0,\ldots,0)$ with the nonzero entry in the $d$-th position, and:

$$F_{g,(0,\ldots,0,1,0,\ldots,0)}(\lambda) = \frac{(1-2^{1-2g}|B_{2g}|)(2^{2g-2}(\lambda^d - \lambda^{-d})). \quad (4.6)$$

According to [22], the Lagrangian submanifold $\mathcal{C}_K$ associated to the unknot is a sphere bundle over the equator of $\mathbb{P}^1$, and the non-trivial one-cycle of $\mathcal{C}_K$ is precisely this equator. The relative homology $H_2(X,\mathcal{C}_K,\mathbb{Z})$ has two primitive generators $[N]$, $[S]$, corresponding to the northern and southern hemisphere of $\mathbb{P}^1$. The terms with $\lambda^\pm \frac{d}{2}$ correspond to holomorphic maps satisfying $f_*[\Sigma_{g,1}] = d[N]$ or $f_*[\Sigma_{g,1}] = d[S]$, respectively. The expression (4.6) was first obtained in [22] by using the above conjectured relation with Chern-Simons theory, and it has been computed directly in Gromov-Witten theory in [23][26].
Remark: As we said above, the correspondence that associates a Lagrangian submanifold \( C_L \) to a \( L \) is conjectural, and so far there is no well-defined procedure to construct \( C_L \) once \( L \) is given. In [22], Ooguri and Vafa showed that given a link \( L \) in \( S^3 \), one can canonically associate to it a Lagrangian submanifold \( \hat{C}_L \) in \( T^* S^3 \). \( C_L \) should be obtained from \( \hat{C}_L \) after a “conifold transition” from the \( T^* S^3 \) geometry (the deformed conifold) to the resolved conifold. There is however a proposal for \( C_L \) in [24] for a class of torus knots and links.

4.2. The integer invariants

To describe the geometrical content of the new integer invariants introduced in (3.23), \( N(R_1, \ldots, R_L, g, Q) \), we have to resum the Gromov-Witten invariants for open strings. It was shown in [22][24] that the generating functional \( F(V_1, \ldots, V_L) \) can be written as:

\[
F(V_1, \ldots, V_L) = \sum_{g, Q} \sum_{d > 0} \sum_{\{R_\alpha, R'_\alpha, R''_\alpha\}} N(R_1, \ldots, R_L, g, Q, \frac{1}{d} (2i \sin(dx/2)))^{2g + L - 2} \times \left( \prod_{\alpha=1}^L C_{R_\alpha R'_\alpha R''_\alpha} S_{R''_\alpha} (e^{dix}) \text{Tr}_{R_\alpha} V_d^a \right) \lambda^{dQ}.
\]

(4.7)

The conjecture (3.23) follows from this equation. From (4.7) one can also extract the expression of the \( F_{g, (\vec{k}(1), \ldots, \vec{k}(L))}(\lambda) \) in terms of integer invariants \( N(R_1, \ldots, R_L, g, Q) \), by simply combining (3.10), (3.23), and (4.1).

In [24] a geometric interpretation of these integer invariants was given in terms of the Calabi-Yau geometry described in the previous subsection. Let \( R_1, \ldots, R_L \) be representations of \( SU(N) \), where \( R_\alpha \) has \( \ell_\alpha \) boxes, and let \( C_L \) be the Lagrangian submanifold associated to the link \( L \) with \( L \) components. Let us denote by \( M_{g, \ell, Q} \) the moduli space of Riemann surfaces of genus \( g \) and \( \ell \) holes embedded in the resolved conifold, where \( \ell = \sum_{\alpha=1}^L \ell_\alpha \). The embedding is such that \( \ell_\alpha \) holes end on the the non-trivial cycles \( \gamma_\alpha \), for \( \alpha = 1, \ldots, L \), and the relative class \( H_2(X, C_L) \) is labeled by \( Q \) in the way explained after (4.5). The group,

\[
\prod_{\alpha=1}^L S_{\ell_\alpha},
\]

(4.8)

acts naturally on the Riemann surfaces by exchanging the \( \ell_\alpha \) holes that end on \( \gamma_\alpha \). The action of (4.8) lifts to the moduli space \( M_{g, \ell, Q} \) and therefore to the cohomology group \( H^*(M_{g, \ell, Q}) \). We can then project this cohomology group into the subspace which is invariant under the symmetry associated to the Young tableaux of \( R_1, \ldots, R_L \). This projection is made through the operator,

\[
S_{R_1, \ldots, R_L} = \otimes_{\alpha=1}^L S_{R_\alpha},
\]

(4.9)
where the $S_{R_α}$ are the usual Schur functors (see for example [30]). According to [24], the integer invariants $N_{(R_1,\ldots,R_L),g,Q}$ in [3.23] have to be interpreted as Euler characteristics of the projected cohomologies,

$$N_{(R_1,\ldots,R_L),g,Q} = \chi(S_{R_1,\ldots,R_L}(H^*(M_{g,\ell,Q}))). \quad (4.10)$$

This gives the geometrical content of the integer coefficients that appear in the reformulated polynomial invariants.

Remarks:
• The equation (4.7) encodes the multicovering and bubbling phenomena for the Gromov-Witten invariants of open strings. A comparison with the closed string case is very illuminating. If $X$ is a Calabi-Yau manifold, the Gromov-Witten invariants $N^g_\beta$ associated to genus $g$ curves in the two-homology class $\beta$ can be organized in the Gromov-Witten potential,

$$F(x,t) = \sum_{g\geq 0} \sum_{\beta \in H_2(X)} N^g_\beta x^{2g-2} e^{-t\beta}. \quad (4.11)$$

This potential can be rewritten in terms of Gopakumar-Vafa invariants $n^g_\beta$ [28] as follows:

$$F(x,t) = \sum_{g,\beta} \sum_{d>0} n^g_\beta d \left(2\sin(dx/2)\right)^{2g-2} e^{-td\beta}. \quad (4.12)$$

The Gopakumar-Vafa invariants can be computed in terms of Euler characteristics of moduli spaces of holomorphically embedded surfaces in the Calabi-Yau manifold [28] [29]. The integer invariants $N_{(R_1,\ldots,R_L),g,Q}$ can then be regarded as the open string version of the Gopakumar-Vafa invariants, and the relation (4.7) encodes all the multicovering and bubbling phenomena associated to the Gromov-Witten invariants for open strings, as (4.12) does in the context of closed strings. It is worth mentioning that both (4.7) and (4.12) are based on an analysis in terms of D-branes, and the Jacobian of the Riemann surface plays a crucial role.

• The characterization of the integer invariants given in (4.10) should be studied more carefully. It was obtained in [24] with the simplifying assumption that there is no degeneration of the Riemann surface along the moduli. Although this assumption gives the right

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7 The above formula should hold only up to an overall sign. This is related to the analytic continuation that one has to perform in order to compare Chern-Simons invariants to enumerative invariants [22] [24] [24].
structure of $F(V_1, \cdots, V_L)$, the definition of the integer invariants should be analyzed in more detail along the lines of [29].

- As a final remark, it is interesting to observe how the structure theorem (4.7) encodes the structure of the $1/N$ expansion (4.1). If we define

$$f_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}(q, \lambda) = \sum_{\{R_\alpha\}} \prod_{\alpha=1}^{L} \chi_{R_\alpha}(C(\vec{k}^{(\alpha)})) f_{(R_1, \cdots, R_L)}(q, \lambda)$$

(4.13)

and

$$n_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}, g, Q = \sum_{\{R_\alpha\}} \prod_{\alpha=1}^{L} \chi_{R_\alpha}(C(\vec{k}^{(\alpha)})) N_{(R_1, \cdots, R_L), g, Q},$$

(4.14)

one can show [24] that (3.23) implies,

$$f_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}(q, \lambda) = \left( \prod_j \frac{1}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2} \right)^{2g} \sum Q \sum_{g \geq 0} n_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}, g, Q (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g} \lambda^Q.$$ 

(4.15)

On the other hand, the “connected” invariants defined in (3.6) are related to the functions defined in (4.13) as follows:

$$W_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}^{(c)}(q, \lambda) = \sum d \sum_{d| \vec{k}^{(\alpha)}} f_{(\vec{k}^{(1)}/d, \cdots, \vec{k}^{(L)}/d)}(q^d, \lambda^d).$$

(4.16)

Using (4.15) and (4.16), it is easy to check that $W_{(\vec{k}^{(1)}, \cdots, \vec{k}^{(L)})}^{(c)}(q, \lambda)$ has a $1/N$ expansion with the structure (4.1). From this point of view, the expression in terms of the integer invariants $N_{(R_1, \cdots, R_L), g, Q}$ provides a resummation of the $1/N$ expansion.

5. Conclusions and open problems

In this paper we have described a new set of polynomial invariants for knots and links which are closely related to the familiar quantum-group polynomial invariants. We have also stated a conjecture on their general algebraic structure, and described the topological content of their coefficients: the integer invariants appearing in the new polynomials are interpreted as a resummation of the Gromov-Witten invariants, and are identified in terms of topological properties of the moduli space of Riemann surfaces with holes embedded in a particular way, fixed by the knot or link under consideration, into the Calabi-Yau manifold (4.3) or resolved conifold.
Up to now, the interpretation of the new invariants in terms of enumerative geometry has been fully tested only for the unknot \([25][26]\). The conjecture (3.23) on the structure of the reformulated polynomial invariants, however, has been shown to be satisfied in a variety of cases, and since this structure result is a consequence of the geometric formulation, this test can be regarded as a further support for this formulation. Unfortunately, not very much is known about the properties of quantum-group polynomial invariants for higher dimensional representations (at least for \(SU(N)\) with \(N\) generic) and no test has been carried out beyond representations whose Young tableau possesses four boxes. For lower representations the conjecture (3.23) has been tested for a series of knots and links \([23][24][27]\).

Further studies should be done on the topological side to compute the integer invariants (4.10). Recent work \([25][26]\) has presented a firm path towards the computation of these quantities. A good starting point could be the consideration of torus knots, a case for which a proposal for the corresponding Lagrangian submanifold is already available \([24]\).

Another important issue is the search for more structure. Quantum-group invariants satisfy skein relations which must have some implications on the reformulated polynomial invariants. The properties behind these relations have a different nature than the ones contained in the conjecture (3.23). It would be very important to work out the conditions that this additional structure imposes on the new integer invariants. In turn, one should answer also the question about its significance taking into account their topological origin.

The work summarized in this paper should be extended to take into consideration quantized universal enveloping algebras different than \(U_q(sl(N, \mathbb{C}))\). The extension of \([21]\) to other gauge groups has been already done in \([39]\), and it involves non-orientable Riemann surfaces in a crucial way.

Finally, one should rephrase many of the unanswered questions in the theory of knots and links in terms of these new integer invariants. The approach certainly opens a new perspective to face these problems. However, much work is first needed to study these invariants from their topological origin, obtaining some familiarity with their properties and developing approaches towards their computation.
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