Liquidity in Competitive Dealer Markets∗

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July 22, 2018

Abstract

We study a continuous-time version of the intermediation model of Grossman and Miller [18]. To wit, we solve for the competitive equilibrium prices at which liquidity takers’ demands are absorbed by dealers with quadratic inventory costs, who can in turn gradually transfer these positions to an end-user market. This endogenously leads to a model with transient price impact. Smooth, diffusive, and discrete trades all incur finite but nontrivial liquidity costs, and can arise naturally from the liquidity takers’ optimization.

Mathematics Subject Classification: (2010) 91B26, 91B24, 91B51.

JEL Classification: C68, D43, G12.

Keywords: dealer market, dynamic equilibrium, endogenous liquidity.

1 Introduction

A basic paradigm of classical financial theory is that markets are “perfectly liquid”, in that arbitrary amounts can be traded immediately at the quoted market price. Yet, in reality liquidity is limited and generated by the interplay of strategic liquidity providers and consumers. Indeed, as succinctly summarized by [6] in the context of foreign exchange markets:

“Dealers in over-the-counter financial markets provide liquidity to customers on a principal basis and manage the risk position that arises out of this activity in one of two ways. They may internalize a customers trade by warehousing the risk in anticipation of future offsetting flow, or they can externalize the trade by hedging it out in the open market. […] The notion that dealers are either perfect internalizers or perfect externalizers is of course too constraining and in practice they will use to varying extent a mix of both to manage their risk.”

∗We thank Jan Kallsen for fruitful discussions.
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arXiv:1807.08278v1 [q-fin.TR]  22 Jul 2018
In the present study, we develop a dynamic equilibrium model for this kind of liquidity formation by extending the classical intermediation model of Grossman and Miller [18] to continuous time. To wit, we consider representative dealers, who intermediate between the demands of their clients and a group of end-users. The clients trade to track an exogenous, stochastically evolving target position as in [24, 31, 12] and are therefore willing to pay a premium for the immediacy the dealers provide. The end-users have no intrinsic trading needs, but are willing to trade the asset under consideration at its exogenous “fundamental value”.

With access to a perfectly liquid end-user market, competitive dealers would provide immediacy at the fundamental price. Nontrivial liquidity costs arise in the natural situation when the client order flow can only be passed on to the end-users gradually. We model this in a tractable manner by a quadratic cost levied on the dealers’ transfer rate. This can be interpreted either as a reduced form search cost, or as linear temporary price impact in the end-user market as in the optimal execution literature [1].

When limited liquidity in the end-user market makes it impossible to offload accumulated positions immediately, the dealers’ holding costs are reflected in the equilibrium prices at which they absorb their clients’ demands. In order to permit the analysis of this complex interaction, we assume that i) dealers act as price takers, ii) their holding costs are quadratic in the size of the positions held, and iii) the fundamental price has martingale dynamics. As in Garleanu, Pedersen, and Poteshman [14], the first assumption means that the equilibrium price is “competitive”, which is reasonable if the representative dealers correspond to a large number of small liquidity providers, whose individual actions cannot affect the overall market equilibrium. Quadratic holding costs are also used in [30, 26, 28], for example, because this “inventory aversion” is considerably more tractable than risk aversion, yet still penalizes the accumulation of large and thereby risky positions. The martingale assumption on the fundamental price, also made in [18, 15], ensures that the dealers focus on inventory management rather than speculative investment in the end-user market.

In this setting, our first main result identifies the unique equilibrium price at which the dealers absorb a given client demand:

$$\text{Equilibrium Price} = \text{Fundamental Price} + \text{Holding Cost} \times \text{Expected Future Inventory.}$$

This adjustment is consistent with the small-risk aversion limit obtained for the model of [14] in [22, 21]. However, whereas the price impact is permanent in these models without end-users, it becomes transient in our model. The reason is that the client order flow can be gradually passed on to the end-user market here, so that the expected future demand in the formula of [21] is replaced by the optimally controlled inventory above [3].

By appealing to results on quadratic tracking problems [3], the dynamics of the optimally-controlled inventory of the dealers can be computed explicitly in terms of the past, present, and (suitably discounted expectations of) the future client order flow. The comparative statics of these

1. Unlike the “market makers” in [19, 23] and many more recent studies, these liquidity providers are not obliged to absorb any order flow, but trade at their discretion. Investment banks providing liquidity in foreign exchange markets are a typical example, compare [6].

2. Evidently, a very important but challenging direction for further research is to study extensions to a game-theoretic setting, where dealers dynamically adjust their price quotes to account for their inventories while competing with each other for the clients’ order flow. For liquidation problems, a first step in this direction is undertaken in [7], where dealers strategically quote bid-ask spreads for randomly arriving clients with exogenous demand curves, and a strategic client who needs to liquidate a large position.

3. Put differently, the dealers in [14, 21, 22] correspond to “pure internalisers” in the terminology of [6], whereas our dealers employ both internalisation and externalisation in an optimal manner.
explicit but rather intricate dynamics can be better understood for highly-liquid end-user markets. In this regime, the dealers’ inventories can be kept small, so that the equilibrium price in the client market closely tracks the asset’s fundamental value. The clients’ liquidity costs compared to the corresponding frictionless wealth processes in turn admit simple, intuitive expressions that depend on the fluctuations of the client demand.

If the latter is smooth, trading through the dealer market is approximately equivalent to trading directly in the end-user market at twice the cost. This higher friction reflects the premium that is necessary to entice the dealers to provide the necessary liquidity. Since smooth client flow can be hedged rather efficiently by trading in the end-user market, the dealers’ holding costs do not appear in the leading-order term in this regime. This changes for diffusive client demands. Such more irregular order flow is much more difficult to pass on to the illiquid end-user market. Accordingly, the corresponding liquidity costs asymptotically scale with the square root of the trading cost in the end-user market, multiplied by the square root of the dealers’ holding cost. The order flow at hand enters through its quadratic variation, similarly as in the reduced form model of [11]. Here, however, the trading costs of such “rough” strategies cannot be avoided as in [11, 2] by approximating it with smooth strategies. More generally, the liquidity costs implied by our model are continuous in the client demand. As a consequence, strategies of different form are priced consistently and incur finite but nontrivial liquidity costs.

The results described so far pertain to the case of a fixed given client demand as in [14]. However, our model is tractable enough to also endogenize this order flow. To this end, we assume that the (representative) clients also act as price takers and trade to track an exogenous target position. For constant trading targets as in [12], this is similar to the optimal liquidation problems studied in [5, 12, 20] and many more recent papers. Diffusive trading targets correspond to the “high-frequency trading needs” considered in [24, 31].

Starting from an execution price corresponding to some fixed demand process, the clients’ optimization problem can readily be solved in closed form. In equilibrium, the optimal client demand in turn has to coincide with the one used to generate the execution price. This leads to an integral equation describing the clients’ optimal order flow. This integral equation can be reformulated as a linear forward-back stochastic differential equation. We show that it admits a unique explicit solution that can be described in closed form in a number of examples.

Indeed, for constant trading targets, the clients’ optimal trading strategies are of a similar form as in optimal liquidation models with transient price impact [20]: isolated bulk trades combined with otherwise smooth order flow. In contrast, diffusive trading targets as in [24, 31] lead to optimal client demands with nontrivial quadratic variation. Therefore, our model consistently combines the qualitative properties of standard models for optimal liquidation, while nevertheless allowing for rapidly fluctuating inventories in line with the empirical evidence documented by [9].

In both of these examples, we can compare the clients’ liquidity costs when trading through the dealers to the hypothetical costs they would incur if they could trade directly in the end-user market. At the leading-order for high liquidity in the end user market, the respective liquidity costs only differ by a universal function of the ratio of clients’ and dealers inventory costs. To wit, if the dealers’ inventory costs are sufficiently low, then the additional risk-sharing capacity they provide is more than worth the premium charged to clients in equilibrium. In contrast, dealers with small risk-bearing capacities require large premia, so that clients would be better off by implementing their trading strategies directly in the end-user market.

In the baseline version of our model, we consider price-taking clients, who do not internalize the impact of their order flow on the execution price. While this is reasonable for “representative” clients aggregating the demand of many small liquidity takers, this assumption is questionable for “large”
clients that have to liquidate a substantial asset position, for example. In this case, it is natural to study a variant of the above setup where clients internalize the dependence of the equilibrium price on their demand. This amounts to a more involved optimization problem that, remarkably, turns out to produce an equilibrium of exactly the same form. The only difference is that the clients’ inventory costs are halved in the corresponding formulas. This means that when internalizing their price impact, clients weigh their inventory concerns less heavily and accordingly trade more slowly. By correctly internalizing their price impact, large clients improve their performance compared to the aggregate of small clients, and our model allows to quantify the magnitude of this “price of anarchy” \[\text{[8]}\].

\textbf{Notation} Throughout, we fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\) satisfying the usual conditions. We say a predictable process \(X = (X_t)_{t \in [0,T]}\) belongs to \(\mathcal{L}^2\) if \(E[\int_0^T X_t^2 dt] < \infty\) and to \(\mathcal{S}^2\) if \(E[\sup_{t \in [0,T]} X_t^2] < \infty\). The set of square-integrable martingales is denoted by \(\mathcal{M}^2\), and we write \(\mathcal{H}^2\) for the semimartingales whose local martingale part belongs to \(\mathcal{M}^2\) and whose finite-variation part has square-integrable total variation.\[\text{[4]}\]

\section{The Dealers’ Problem}

\subsection{A Dynamic Model for Intermediation}

We consider dealers who intermediate between the demand of their clients for some financial asset, and end-users with whom they can trade the asset at its “fundamental value”. In this section, we suppose the clients’ demand is an exogenously given process \(K = (K_t)_{t \in [0,T]} \in \mathcal{S}^2\); it will be endogenized in Section \([3]\) below.

Clients trade with dealers at a competitive price \(S^K\) to be determined in equilibrium by matching the dealers’ supply \(K\) to the clients’ demand \(K\). The dealers also have access to a group of end-users, who are willing to trade the risky asset at its exogenous fundamental value \(V = (V_t)_{t \in [0,T]} \in \mathcal{M}^2\).\[\text{[6]}\]

However, the end-user market is not perfectly liquid, so that the dealers can transfer the assets only gradually to the end-users. We model this in a tractable manner by a quadratic cost levied on the dealers’ layoff rate. This can be interpreted as a reduced-form search cost. Alternatively, one can view this as a result of linear price impact in the end-user market as in the optimal execution literature \([1]\). To wit, if dealers trade \(dU_t\) units of the risky asset with the end-users over the (infinitesimal) time interval \([t, t + dt]\), then the trade is executed at the unit price \(V_t + \frac{\lambda}{2} dU_t\). Compared to the frictionless execution price, the dealers therefore incur costs \(\frac{\lambda}{2} u_t^2 dt\) quadratic in the trading rate \(u_t = dU_t/dt\). Accordingly, the dealers’ P&L from trading with the end-users is \(\int_0^T U_t dV_t - \frac{\lambda}{2} \int_0^T u_t^2 dt\) for a given trading rate \(u = (u_t)_{t \in [0,T]}\) and the corresponding positions \(U_t = \int_0^t u_s ds\).

Apart from the transfers to the end-user market, the dealers also choose their supply \((K_t)_{t \in [0,T]}\) to clients. That is, they are not contractually obliged to absorb all client orders like traditional “market makers” studied in \([19]\), for example. Instead, as in \([13]\), the dealers choose how much liquidity they want to provide at each time, as is typical for modern foreign exchange markets, for example.

\[\text{[4]}\]For an Itô process with dynamics \(dX_t = \mu_t dt + \sigma_t dW_t\), this holds if \(E \left[ (\int_0^T |\mu_t| dt)^2 + \int_0^T \sigma_t^2 dt \right] < \infty\).

\[\text{[5]}\]As in \([19]\ [15]\), we assume that the fundamental value has martingale dynamics, so that the dealers only engage in intermediation rather than speculative investment.
If the trades between dealers and clients are settled at a competitive price $S = (S_t)_{t \in [0,T]} \in \mathcal{H}^2$, then supplying $K_t$ shares of the risky asset at time $t$ gives the dealers a P&L of $\int_0^T -K_t dS_t$.\footnote{This means that as in [14], the dealers act as price takers. This is natural if they are viewed as a continuum of infinitesimally small agents. Then, the individual actions of each of them will not alter the aggregate supply. Thus, offering more favorable prices to clients would only reduce individual profits. Conversely, no clients would be willing to trade at less favorable than market prices.}

All the interactions between dealers, clients, and end-users are illustrated in Figure 1. Now, we combine the dealers’ P&L from trading with the end-users (whose first component $\int_0^T U_t dV_t$ has expectation zero because $V \in \mathcal{M}^2$ and $U \in \mathcal{S}^2$) with the P&L from the client transactions, add a running cost $\gamma_d/2$ for the dealers’ squared inventories, and value the terminal position at its fundamental value. Together, this leads to the following linear-quadratic goal functional for the dealers:

$$J_d(u, K; S) = \mathbb{E} \left[ \int_0^T (\gamma_c \zeta_t - K_t) dS_t - (V_T - S_T)K_T \right] - \mathbb{E} \left[ \int_0^T \left( \frac{\lambda}{2} u_t^2 + \frac{\gamma_d}{2} (U_t - K_t)^2 \right) dt \right]. \tag{2.1}$$

Here, we focus on admissible controls $(u, K) \in \mathcal{L}^2 \times \mathcal{S}^2$; for these, we have $U = \int_0^T u_s ds \in \mathcal{S}^2$ so that the expectation in (2.1) is well-defined for any price process $S \in \mathcal{H}^2$ in the client market.

Remark 2.1. (i) For constant asset volatilities, the goal functional (2.1) can be interpreted as a mean-variance tradeoff over wealth changes as in [20, 25, 15]. In general, the reduced-form holding cost in (2.1) is more tractable than models with risk aversion. Similar criteria are used in [30, 28, 27], for example.

(ii) We assume for simplicity that the risky asset is valued at its fundamental value at the terminal time $T$. This could be replaced by a quadratic liquidation cost or avoided altogether by postponing the planning horizon to infinity under suitable transversality conditions.
2.2 Optimal Trading in the End-User Market

For any given supply $K \in \mathcal{S}^2$ to the client market, the dealers' optimal transfer rate $u$ to the end-user market only affects the second expectation in (2.1). It thus amounts to the solution of a tracking problem:

$$E \left[ \int_0^T \left( \frac{\lambda}{2} u_t^2 + \frac{\gamma_d}{2} (U_t - K_t)^2 \right) dt \right] \rightarrow \min_{u \in \mathcal{L}^2}$$  \hfill (2.2)

Due to its quadratic structure, this problem has the following explicit solution [3]:

**Lemma 2.2.** Given $K \in \mathcal{S}^2$, the dealers' transfer rate $u^K_t \in \mathcal{L}^2$ to the end-user market that minimizes (2.2) solves the (random) linear ODE

$$u^K_t := \frac{d}{dt} U^K_t = \overline{K}_t - F^\Delta(t) U^K_t, \quad 0 \leq t \leq T, \quad \text{with} \quad U^K_0 = 0.$$  \hfill (2.3)

Here,

$$F^\Delta(t) = \sqrt{\Delta} \tanh(\sqrt{\Delta}(T-t)), \quad 0 \leq t \leq T, \quad \text{for} \quad \Delta = \frac{\gamma_d}{\lambda},$$

and $\overline{K}$ is a weighted average of the expected future supply:

$$\overline{K}_t = E_t \left[ \int_t^T D^\Delta(t,s) K_s ds \right] \quad \text{with} \quad D^\Delta(t,s) = \frac{\Delta \cosh(\sqrt{\Delta}(T-s))}{\cosh(\sqrt{\Delta}(T-t))}.$$  \hfill (2.4)

In explicit form, the dealers' optimal position in the end-user market is

$$U^K_t = \int_0^t e^{-\int_s^t F^\Delta(\tau) d\tau} \overline{K}_s ds = \frac{1}{\Delta} \int_0^t D^\Delta(s,t) \overline{K}_s ds, \quad 0 \leq t \leq T,$$  \hfill (2.5)

and the optimal transfer rate balances the expected future net position, in that

$$\lambda u^K_t = -\gamma_d E \left[ \int_t^T (U^K_s - K_s) ds \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$  \hfill (2.6)

**Proof.** The optimal $u^K$ is computed in [3, Theorem 3.1] and presented here with slightly different notation. Identity (2.6) is the first-order condition for the optimality of $u^K$, cf. [3, Lemma 5.2]. □

**Remark 2.3.** As observed in [3, (2.3)] determines the dealers’ optimal transfer rate $u^K_t$ at time $0 \leq t \leq T$ by how far their current cumulative transfers $U^K_t$ deviate from a weighted average of expected future demand:

$$\hat{K}_t := \frac{\overline{K}_t}{F^\Delta(t)} = E_t \left[ \int_t^T \frac{\sqrt{\Delta} \cosh(\sqrt{\Delta}(T-s))}{\sinh(\sqrt{\Delta}(T-t))} K_s ds \right].$$

The function $F^\Delta(t)$ describes the urgency with which the dealers seek to reduce their displacement $U^K_t - \hat{K}_t$ from the target $\hat{K}$. In particular, if $K$ is constant then $\hat{K} \equiv K$ and the dealers’ inventory $U^K_t - K$ decreases at the exponential rate $F^\Delta(t)$.

In view of Lemma 2.2, the dealers’ goal functional (2.1) can be rewritten as a functional only depending on their supply choice $K \in \mathcal{S}^2$:

$$J_d(K; S) = E \left[ \int_0^T \left( (-K_t) dS_t - \frac{\lambda}{2} (u^K_t)^2 dt - \frac{\gamma_d}{2} (U^K_t - K_t)^2 dt \right) - (V_T - S_T) K_T \right]$$  \hfill (2.7)

with $u^K$ and $U^K$ as in (2.3).
2.3 Equilibrium Price in the Client Market

Our goal now is to determine the competitive equilibrium price at which the dealers’ optimal supply $K$ matches a given demand $\mathcal{K}$ in the client market:

**Definition 2.4.** For a given client demand $\mathcal{K} \in \mathcal{S}^2$, $S \in \mathcal{H}^2$ is called equilibrium price, if the dealers’ optimal supply matches the clients’ demand:

$$\mathcal{K} \in \arg\max_{K \in \mathcal{S}^2} J_d(K; S).$$

Our first main result shows that a unique equilibrium price exists. It adjusts the fundamental value of the risky asset by the dealers’ expected future inventories, weighted by their holding costs:

**Theorem 2.5.** For a given client demand $\mathcal{K} \in \mathcal{S}^2$, the unique equilibrium price is

$$S^K_t = V_t - \gamma_d E_t \left[ \int_t^T (U^K_s - \bar{K}_s) \, ds \right], \quad 0 \leq t \leq T.$$  \hfill (2.8)

Here, $U^K$ is given by (2.5) for $K = \mathcal{K}$.

**Proof.** Observe that, for any $S \in \mathcal{H}^2$, the functional $(u, K) \mapsto J_d(u, K; S)$ of (2.1) is concave on $\mathcal{S}^2 \times \mathcal{S}^2$. Hence, the functional $K \mapsto J_d(K; S) = \sup_{u \in \mathcal{L}^2} J_d(u, K; S)$ is concave on $\mathcal{S}^2$. As it is quadratic in $K$, this goal functional is furthermore Gâteaux differentiable in any direction $L \in \mathcal{S}^2$.

For $S$ to be an equilibrium price, it is thus necessary and sufficient that at $K = \mathcal{K}$ all directional derivatives of $K \mapsto J_d(K; S)$ vanish:

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (J_d(\mathcal{K} + \varepsilon L; S) - J_d(\mathcal{K}; S))$$

$$= - E \left[ \int_0^T \left( \lambda u^K_t + \gamma_d \int_t^T (U^K_s - \bar{K}_s) \, ds \right) u^L_t \, dt \right]$$

$$+ E \left[ \int_0^T (-L_t \, dS_t + \gamma_d (U^K_t - \bar{K}_t) L_t \, dt) - (V_T - S_T) L_T \right].$$

Now, note that the first expectation on the right-hand side vanishes by (2.6). Next, specialize to simple processes $L = \ell 1_{[t, T]}$, where $\ell$ is an $\mathcal{F}_T$-measurable, square-integrable random variable, to rewrite the above condition as

$$0 = E \left[ \ell \left( S_t + \gamma_d \int_t^T (\bar{U}^{\bar{K}}_s - \bar{K}_s) \, ds - V_T \right) \right].$$

As $\ell \in L^2(\Omega, \mathcal{F}_t, P)$ is arbitrary and the fundamental value $V \in \mathcal{H}^2$ is a martingale, it follows from definition of the conditional expectation and the preceding relation that the unique equilibrium price $S$ is

$$S_t = V_t - \gamma_d E_t \left[ \int_t^T (U^{\bar{K}}_s - \bar{K}_s) \, ds \right],$$

exactly as asserted in (2.8) since $K = \bar{K}$. \hfill $\square$

\footnotetext{An analogous result obtains in the small risk-aversion limit for the model of [14] studied by [21] – for small risk aversion, their exponential utility is equivalent to the mean-inventory goal functional we consider here. The advantage of the present approach is that the much more tractable linear-quadratic setting readily allows us to incorporate the dealers’ hedging trades in the end-user market and, in addition, to endogenize the clients’ demand; see Section 3.}
In view of the first-order condition for the dealers’ optimal trading rate in the end-user market (2.6), the equilibrium price from Theorem 2.5 admits the following concise representation:

\[ S_t^K = V_t + \lambda u_t^K. \]  

(2.9)

This means that the adjustment of the equilibrium price compared to the fundamental value is the product of the trading cost in the end-user market, and the dealers’ current trading rate in this market. To wit, suppose the dealers are selling to the end-users \((u^K_t > 0)\) because they have already accumulated a positive net position. Then they will only be willing to purchase further risky assets from clients at a premium, but will conversely be willing to sell at a discount – exactly as mandated by (2.9).

Despite this appealingly simple interpretation, the dependence of the equilibrium price (2.8) on the model parameters is generally rather involved. Indeed, it typically depends on the past, present, and future client demands, cf. Lemma 2.2. In particular, it exhibits resilience in that it reverts towards the fundamental price \(V\) when \(K\) remains constant; cf. Remark 2.3.

2.4 Large-Liquidity Asymptotics

In order to better understand how equilibrium prices depend on the clients’ demand, we now discuss the properties of the equilibrium price from Theorem 2.5 in the case of highly liquid end-user markets \((\lambda \approx 0)\). For better readability, the proofs are delegated to Appendix A.

Our first result shows that the equilibrium price approaches the fundamental value as the end-user market becomes more and more liquid for \(\lambda \to 0\):

**Proposition 2.6.** For any demand \(K \in S^2\), the equilibrium price \(S^K\) from Theorem 2.5 converges to the fundamental price \(V\) in the particularly strong sense that

\[
\sup_{-1 \leq H \leq 1 \text{ predictable}} E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t H_s d(V_s - S^K_s) \right|^2 \right] \to 0, \quad \text{as } \lambda \to 0.
\]

In particular \(S^K\) converges to \(V\) in the Emery topology as \(\lambda \to 0\) and the corresponding wealth processes generated by the clients’ demand satisfy

\[
\int_0^T K_t dS^K_t = \int_0^T K_t dV_t + o(1), \quad \text{in } L^1 \text{ as } \lambda \to 0.
\]

Proposition 2.6 asserts that, as the end-user market becomes more and more liquid, the dealer price approaches the fundamental value of the risky asset. Given additional structure of the client demand, we can also identify the leading-order correction term for the client’s wealth process, i.e., the liquidity costs implied by the dealers’ nontrivial but finite risk-bearing capacity. The form of this leading-order correction term depends on the variability of the clients’ demand. To illustrate this, we discuss the two examples that appear most frequently in applications: smooth demands \(K_t = \int_0^t \mu_s^K ds\) and diffusive demands with Itô dynamics \(K_t = \int_0^t \mu_s^K ds + \int_0^t \sigma_s^K dW_s\).

\[8\] For simplicity, we focus here on the case of a fixed client demand that does not vary with the liquidity parameter \(\lambda\). The more involved case where the client optimizes given the equilibrium pricing rule is subsequently treated in Section 3.
Smooth Demands  We first discuss demands that accumulate at a finite, absolutely continuous rate. In this case, the dealers could hedge their exposure perfectly by passing on their positions immediately to the end-users subject to the quadratic cost \(\lambda/2\) imposed on the corresponding trading rate. Therefore, the dealers could break even using this strategy for an execution price equal to the fundamental value plus the trading cost in the end-user market. However, due to the quadratic nature of the trading cost, they could achieve strictly positive profits in this case by absorbing only a fraction of the clients’ demands, so that the market would not clear at this price. Accordingly, in equilibrium, the dealers need to be paid an additional premium.

The subsequent result identifies the leading-order term of this liquidity cost as twice the trading cost in the end-user market. This term is independent of the dealers’ inventory costs, since smooth client demands can be hedged very efficiently by trading in the end-user market.

Lemma 2.7. Suppose that \(K_t = \int_0^t \mu^K_s ds\) for a continuous process \(\mu^K \in S^2\). Then, the liquidity costs generated by \(K\) are
\[
\int_0^T K_t dV_t - \int_0^T K_t dS^K_t = \lambda \int_0^T (\mu^K_t)^2 dt + o(\lambda), \quad \text{in } L^1 \text{ as } \lambda \to 0. \quad (2.10)
\]

Diffusive demands  Next, we turn to trading strategies with nontrivial Brownian fluctuations. These could not be implemented directly in the end-user market, but can be traded at a finite cost through the dealers. Since the dealers can hedge these more irregular order flows less efficiently than the smooth flows considered above, the corresponding trading costs are of a higher asymptotic order, namely \(O(\sqrt{\lambda})\) instead of \(O(\lambda)\) as \(\lambda \to 0\). Moreover, the dealers’ inventory cost now becomes visible in the leading-order term. The asymptotically crucial feature of client demand turns out to be its quadratic variation, which is a sufficient statistic for highly liquid end-user markets:

Lemma 2.8. Suppose that the underlying filtration is generated by a Brownian motion \(W\) and assume that the client demand has Itô dynamics,
\[
K_t = \int_0^t \mu^K_s ds + \int_0^t \sigma^K_s dW_s, \quad t \in [0, T].
\]
Here, \(|K|^2, |\mu^K|^2, |\sigma^K|^2 \in H^2\) and these processes are Malliavin differentiable in the sense of [27, p. 27]): \(\mu^K, \sigma^K \in D^{1/2}\), with continuous Malliavin derivatives \(s \mapsto (D_t(K_s), D_t(\mu^K_s), D_t(\sigma^K_s))\), \(0 \leq t \leq s \leq T\). Finally, suppose that \(\sup_{0 \leq t \leq T} E[\sup_{t \leq s \leq T} \{(D_t(K_s))^2 + (D_t(\mu^K_s))^2\}] < \infty\). Then, the liquidity costs generated by the demand \(K\) are
\[
\int_0^T K_t dV_t - \int_0^T K_t dS^K_t = \sqrt{\lambda} \int_0^T (\sigma^K_t)^2 dt + o(\sqrt{\lambda}), \quad \text{in } L^1 \text{ as } \lambda \to 0.
\]

Remark 2.9. The regularity conditions of Proposition 2.8 are satisfied, in particular, if the demand \(K\) is the solution of a scalar stochastic differential equation whose drift and diffusion coefficients are twice continuously differentiable with bounded derivatives of orders 0, 1, 2. In this case, the required bounds for the Malliavin derivatives follow from [27, Theorem 2.2.1].

3 The Clients’ Problem

So far, the client demand was assumed to be given exogenously as in [18, 14, 17, 22]. We now consider how to endogenize this demand for clients that behave optimally. To this end, we first
specify the clients’ optimization problem for a given price process \((S^K_t)_{t \in [0,T]}\) of the form \((2.8)\) for some demand \(\tilde{K}\). The clients then compute their optimal demand \(\hat{K}\). Finally, the competitive equilibrium price is determined by the consistency condition \(\hat{K} = \tilde{K}\), which ensures that the equilibrium pricing rule indeed corresponds to the clients’ aggregate demand. As for the dealers in Section 2, this notion of competitive equilibrium assumes that the “clients” aggregate a continuum of infinitesimal agents. Each of these then indeed acts as a price taker, since individual decisions are negligible for the dealers’ pricing rule \((2.8)\), which only reflects the aggregate demand.

To carry out this program, we first formulate the clients’ optimization problem. As in [24, 31, 12], we assume that they track an exogenous trading target \(\zeta = (\zeta_t)_{t \in [0,T]} \in \mathcal{S}^2\). For a fixed execution price \((S^K_t)_{t \in [0,T]}\), holding \(K_t\) shares of the risky asset at time \(t\) gives the clients the standard overall P&L \(\int_0^T K_t dS^K_t\). The corresponding local mean-variance goal functional for a running displacement cost \(\gamma_c > 0\) in turn reads as follows:

\[
J_c(K; S^K) = E \left[ \int_0^T \left( K_t dS^K_t - \frac{\gamma_c}{2} (K_t - \zeta_t)^2 dt \right) \right] = E \left[ \int_0^T \left( K_t \gamma_d(U^K_t - \hat{K}_t) - \frac{\gamma_c}{2} (K_t - \zeta_t)^2 \right) dt \right].
\]  

(3.1)

Pointwise maximization of the integrand directly gives

\[
\hat{K}_t = \zeta_t + \frac{\gamma_d}{\gamma_c} (U^K_t - \hat{K}_t).
\]  

(3.2)

This pins down the clients’ optimal demand \(\hat{K} \in \mathcal{S}^2\), taking the price \(S^K\) as given. The consistency condition \(\hat{K} = \hat{K}\) in turn requires that

\[
\hat{K}_t = \frac{\gamma_d}{\gamma_d + \gamma_c} U^K_t + \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t, \quad 0 \leq t \leq T.
\]  

(3.3)

In view of the representation of the optimal transfers to the end-user market from Lemma 2.2

\[
U^K_t = \frac{1}{\Delta} \int_0^t D^\Delta(s, t) \tilde{K}_s ds \quad \text{with} \quad \tilde{K}_t = E_t \left[ \int_t^T D^\Delta(t, v) \tilde{K}_v dv \right], \quad 0 \leq t \leq T,
\]  

(3.4)

we observe that \((3.3)\) is an integral equation for \(\hat{K}\). It can be reformulated as a linear FBSDE, and in turn solved explicitly:

**Proposition 3.1.** The clients’ optimal demand for \((3.1)\) is the unique solution \(\hat{K} \in \mathcal{S}^2\) to \((3.3) - (3.4)\), given by

\[
\hat{K}_t = \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t + \tilde{U}^{\frac{\gamma_d}{\gamma_d + \gamma_c}}_t, \quad 0 \leq t \leq T.
\]  

(3.5)

Here, \(\tilde{U}^{\frac{\gamma_d}{\gamma_d + \gamma_c}}\) is the minimizer of \((2.2)\) for \(K = \frac{\gamma_d}{\gamma_d + \gamma_c} \zeta\) and for \(\bar{\gamma} = (\gamma_d^{-1} + \gamma_c^{-1})^{-1}\) instead of \(\gamma_d:\)

\[
\tilde{U}^{\frac{\gamma_d}{\gamma_d + \gamma_c}}_t = \frac{1}{\Delta} \int_0^t D^\Delta(s, t) \frac{\gamma_d}{\gamma_d + \gamma_c} \zeta_s ds,
\]  

\[\footnote{A diffusive trading target corresponds to the “high-frequency trading needs” studied in [24]. A constant target \(\zeta \equiv \text{const.}\) resembles the optimal execution problems studied by [5, 1, 29] and many more recent studies. \(\zeta\) For the second step, the drift rate of the execution price \((2.8)\) is inserted. The expectation of the corresponding stochastic integral vanishes because the martingale part of the execution price is square integrable and the clients’ demand belongs to \(\mathcal{S}^2\).} \]
where \( \bar{\Delta} = \bar{\gamma}/\lambda \) and

\[
\zeta_t = E \left[ \int_t^T \bar{D}(t,s) \zeta_s \, ds \mid F_t \right] \quad \text{with} \quad \bar{D}(t,s) = \bar{\Delta} \text{cosh} \left( \sqrt{\bar{\Delta}} (T-s) \right) / \text{cosh} \left( \sqrt{\bar{\Delta}} (T-t) \right), \quad 0 \leq t \leq T.
\]

Hence, the clients’ share \( \gamma_c/(\gamma_d + \gamma_c) \) of the total holding costs determines the fraction of the exogenous risk \( \zeta \) that they directly pass on to the dealers, taking advantage of the immediacy the dealers provide. The clients transfer the remaining fraction \( \gamma \) exogenous risk \( \bar{\rho} \). This resembles how the dealers in turn pass on their exposure to the end-users, except that, trading through their dealers, the clients can draw on both their own as well as their dealers’ inventory capacities so that the relevant holding costs parameter for them is \( 1/\gamma = 1/\gamma_d + 1/\gamma_c \) rather than \( 1/\gamma_d \).

**Proof of Proposition 3.1.** (i) For uniqueness in the integral equation (3.3), observe that the difference \( \delta \in \mathcal{S} \) of any two solutions solves \( \delta = \bar{\rho}_d U^\delta \). Recalling the definition of \( U^\delta \) from (2.5), we thus can estimate

\[
E \left[ \sup_{0 \leq t \leq T} |\delta_t| \right] \leq E \left[ \sup_{0 \leq t \leq T} \left\{ \frac{\gamma_d}{\gamma_d + \gamma_c} \int_0^t D^\Delta(s,t) \int_s^T D^\Delta(v,s)E \{ \delta_v \} \, dv \, ds \right\} \right]
\]

\[
\leq E \left[ \sup_{0 \leq t \leq T} |\delta_t| \right] \frac{\gamma_d}{\gamma_d + \gamma_c} \sup_{0 \leq t \leq T} \left\{ \int_0^t D^\Delta(s,t) \int_s^T D^\Delta(v,s) \, dv \, ds \right\}
\]

\[
= E \left[ \sup_{0 \leq t \leq T} |\delta_t| \right] \frac{\gamma_d}{\gamma_d + \gamma_c} \sup_{0 \leq t \leq T} \left\{ 1 - \frac{\text{cosh} \left( \sqrt{\Delta} (T-t) \right)}{\text{cosh} \left( \sqrt{\Delta} T \right)} \right\}
\]

\[
\leq E \left[ \sup_{0 \leq t \leq T} |\delta_t| \right] \frac{\gamma_d}{\gamma_d + \gamma_c}.
\]

Since \( \frac{\gamma_d}{\gamma_d + \gamma_c} < 1 \), this implies \( E \left[ \sup_{0 \leq t \leq T} |\delta_t| \right] = 0 \), establishing uniqueness.

(ii) Let us next argue that (3.3) is equivalent to a linear FBSDE. We start by deriving the FBSDE. For the forward component, let

\[
k_t := \bar{K}_t - \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t, \quad 0 \leq t < T, \tag{3.6}
\]

and observe that (3.3) yields \( k_0 = 0 \) and

\[
dk_t = \frac{\gamma_d}{\gamma_d + \gamma_c} dU_t \bar{K}_t = \frac{\gamma_d}{\gamma_d + \gamma_c} \left( \bar{K}_t - F^\Delta(t) U_t \bar{K}_t \right) dt = \left( \frac{\gamma_d}{\gamma_d + \gamma_c} \bar{K}_t - F^\Delta(t) k_t \right) dt.
\]

Here, we first used (2.3) and then (3.3) together with (3.6).

The backward component emerges from the observation that for any \( \bar{K} \in \mathcal{S}^2 \) the corresponding \( \bar{K} \) of (2.4) is the unique solution \( \kappa \) in \( \mathcal{S} \) of the linear BSDE

\[
d\kappa_t = \left( F^\Delta(t) \kappa_t - \Delta \bar{K}_t \right) dt + dM_t^\kappa, \quad 0 \leq t \leq T, \quad \kappa_T = 0,
\]

where \( M^\kappa \in \mathcal{M}^2 \) is a suitable square-integrable martingale uniquely determined by the BSDE.
In summary, the integral equation (3.3) therefore leads to the following linear FBSDE for the processes \( k = \hat{K} - \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta \) and \( \kappa = \tilde{K} \):

\[
k_0 = 0, \quad dk_t = \left( \frac{\gamma_d}{\gamma_d + \gamma_c} \kappa_t - F^\Delta(t)k_t \right) dt, \quad dk_t = \left( F^\Delta(t)\kappa_t - \Delta \left( k_t + \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t \right) \right) dt + dM^*_t, \quad \kappa_T = 0. \tag{3.7}
\]

Backtracking these steps, we also see that a solution \((k, \kappa, M^\kappa)\) to this BSDE yields via (3.6) a solution \( \hat{K} \) of (3.3).

(iii) To verify that (3.5) indeed is the solution of our integral equation (3.3), observe that the corresponding

\[
k_t := \tilde{K}_t - \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t = \frac{\gamma_d}{\gamma_d + \gamma_c} \int_0^t D\tilde{\Delta}(s, t)\tilde{\zeta}_s ds, \quad 0 \leq t \leq T,
\]

satisfies \( k_0 = 0 \) and

\[
\frac{dk_t}{dt} = \frac{\gamma_d}{\gamma_d + \gamma_c} \tilde{\zeta}_t - F^\tilde{\Delta}(t)k_t.
\]

With

\[
\tilde{\kappa}_t := \tilde{\zeta}_t + \frac{\gamma_d + \gamma_c}{\gamma_d} (F^\Delta(t) - F^\tilde{\Delta}(t))k_t, \quad 0 \leq t \leq T,
\]

this can be written as

\[
\frac{d\tilde{\kappa}_t}{dt} = \frac{\gamma_d}{\gamma_d + \gamma_c} \tilde{\kappa}_t - F^\Delta(t)k_t. \tag{3.9}
\]

By comparison with (3.7), it thus remains to show that the above \( \tilde{\kappa} \) is actually the solution \( \kappa \) of the backward equation (3.8). Clearly, \( \tilde{\kappa}_T = 0 \) since, by definition, \( \tilde{\zeta}_T = 0 \) and \( F^\tilde{\Delta}(T) = F^\Delta(T) = 0 \). So we only need to determine the \( dt \)-component of \( \tilde{\kappa} \)'s dynamics. Observe that

\[
d\tilde{\kappa}_t = (F^\Delta \tilde{\zeta} - \tilde{\Delta} \zeta)dt + dM^\tilde{\kappa}, \quad d(F^\Delta - F^\tilde{\Delta}) = \{(F^\Delta)^2 - (F^\tilde{\Delta})^2 - (\Delta - \tilde{\Delta})\}dt
\]

for some martingale \( M^\tilde{\kappa} \in \mathcal{M}^2 \). Also using (3.9), we can in turn readily compute the \( dt \)-component of \( d\tilde{\kappa}_t \) as

\[
F^\Delta \tilde{\zeta} - \tilde{\Delta} \zeta + \frac{\gamma_d + \gamma_c}{\gamma_d} \left( (F^\Delta)^2 - (F^\tilde{\Delta})^2 - (\Delta - \tilde{\Delta}) \right) k + \frac{\gamma_d + \gamma_c}{\gamma_d} (F^\Delta - F^\tilde{\Delta}) \left( \frac{\gamma_d}{\gamma_d + \gamma_c} \kappa - F^\Delta \kappa \right)
\]

\[
= F^\Delta \tilde{\zeta} - \tilde{\Delta} \zeta + F^\Delta \tilde{\kappa} - F^\tilde{\Delta} \tilde{\kappa} + \frac{\gamma_d + \gamma_c}{\gamma_d} \left( (F^\Delta)^2 - (F^\tilde{\Delta})^2 - (\Delta - \tilde{\Delta}) - (F^\Delta - F^\tilde{\Delta})F^\Delta \right) k
\]

\[
= F^\Delta \tilde{\kappa} - \tilde{\Delta} \zeta + F^\Delta \left( \tilde{\zeta} + \frac{\gamma_d + \gamma_c}{\gamma_d} (F^\Delta - F^\tilde{\Delta})k - \tilde{\kappa} \right) - \frac{\gamma_d + \gamma_c}{\gamma_d} (\Delta - \tilde{\Delta})k
\]

\[
= F^\Delta \tilde{\kappa} - \Delta \left( k + \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta \right).
\]

where the last identity holds by definition of \( \tilde{\kappa} \) and \( \tilde{\Delta} = \frac{\gamma_c}{\gamma_d + \gamma_c} \Delta \). This completes the proof. \( \square \)
To illustrate the implications of this result, we now consider two examples: optimal execution as in [5,1,29] and diffusive trading targets as in [24,31,12]. In both cases we will compare the clients’ situation when trading through the dealers with the one when they directly trade with end-users seeking to maximize

\[ J_c(u) = E \left[ \int_0^T U_s dV_s - \frac{\lambda}{2} \int_0^T u_s^2 ds - \frac{\gamma_c}{2} \int_0^T (U_s - \zeta_s)^2 ds \right] \]

\[ = -E \left[ \frac{\lambda}{2} \int_0^T u_s^2 ds + \frac{\gamma_c}{2} \int_0^T (U_s - \zeta_s)^2 ds \right] \rightarrow \max_{u \in L^2, U = \int_0^T \cdot} \]

Example 3.2 (Optimal Liquidation). We first consider the simplest example where the clients’ target is a constant position, \( \zeta_t \equiv \zeta \in \mathbb{R}, t \in [0,T] \). In this case, an elementary integration shows that \( \zeta_t = F(\Delta(t)) \zeta \). Thus, (3.5) and another elementary integration imply that the liquidating clients’ optimal demand is

\[ \hat{K}_t = \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta + \frac{\gamma_d}{\gamma_d + \gamma_c} \zeta \left( 1 - \frac{\cosh(\sqrt{\Delta(T-t)})}{\cosh(\sqrt{\Delta T})} \right). \quad \text{(3.10)} \]

This means that the clients use a bulk trade at time \( t = 0 \) to sell a fraction of their trading target equal to their share of the total holding costs. Subsequently, they continue selling at an absolutely continuous rate.

Let us now compare this optimal execution path to the one that would obtain if the clients could trade directly in the end-user market:

\[ \hat{K}_{\text{direct}}(t) = \left( 1 - \frac{\cosh(\sqrt{\Delta_{\text{direct}}(T-t)})}{\cosh(\sqrt{\Delta_{\text{direct}} T})} \right) \zeta, \quad \Delta_{\text{direct}} = \frac{\gamma_c}{\lambda}. \]

In this case, the optimal execution path is evidently smooth, since bulk trades have an infinite cost in the end-user market. Accordingly, optimal liquidation is initially faster in the dealer market (that can absorb an initial bulk trade), but depending on the model parameters, the total number of shares sold directly can (but does not have to) eventually surpass the number of shares sold through the dealers. This is illustrated in Figure 2 for \( \gamma_d = \gamma_c = 1, \lambda = 1, \zeta = -1 \) and \( T = 1 \) (left panel) and \( T = 5 \) (right panel).

This raises the question whether it is more efficient for the clients to trade through the dealers or directly in the end-user market. To compare the clients’ performance in the two different markets, we compute the optimal value of their goal functional in each case. When trading directly in the end-user market, it follows from [3, Theorem 3.1] that

\[ \max J_{\text{direct}} = -\frac{\sqrt{\gamma_c} \lambda}{2} \tanh \left( \frac{\sqrt{\gamma_c T}}{\lambda} \right) \zeta^2 = -\frac{\sqrt{\gamma_c}}{2} \sqrt{\lambda} \zeta^2 + o(\sqrt{\lambda}), \quad \text{as } \lambda \to 0. \]

When trading through the intermediating dealers, the first-order condition (3.2) and (3.10) show

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11 Complete liquidation as in [3,1,29] could be promoted using a quadratic liquidation penalty as in [10,4] or enforced by a hard terminal constraint as in [3]. To ease notation, we do not pursue this here.

12 See [10,3] or Equation (2.5) in Lemma 2.2 with \( K_t = \zeta, \gamma_d = \gamma_c, \) and \( S_t = V_t \) so that \( \tilde{K}_t = F(t) \zeta \).
that the clients optimal performance in (3.1) is

\[ J_c(\hat{K}) = \frac{\gamma_c}{2} \int_0^T (\hat{K}_t^2 - \zeta^2) dt \]

\[ = \gamma_c \zeta^2 \quad \int_0^T \left( \frac{T}{2} \frac{\cosh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}T)} - \frac{2\gamma_d}{\gamma_d + \gamma_c} \frac{\cosh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}T)} \right) dt \]

\[ = \gamma_c \zeta^2 \quad \left( \frac{T}{2} \frac{\tanh(\sqrt{\Delta}T)}{2\cosh(\sqrt{\Delta}T)} - \frac{2\gamma_d}{\gamma_d + \gamma_c} \frac{\tanh(\sqrt{\Delta}T)}{\sqrt{\Delta}} \right) \]

\[ = -\frac{\sqrt{\gamma_d^2 c(3\gamma_d + 4\gamma_c)}}{4(\gamma_d + \gamma_c)^{3/2}} \lambda \zeta^2 + o(\lambda). \]

We see that for a highly liquid end-user market ($\lambda \approx 0$) the liquidity costs are of the same asymptotic order $O(\sqrt{\lambda})$ in both markets. Indeed, the leading-order optimal performances in both markets are the same up to replacing $\sqrt{\gamma_c}$ (for direct trading in the highly liquid end-user market) with

\[ \frac{\sqrt{\gamma_d^2 c(3\gamma_d + 4\gamma_c)}}{4(\gamma_d + \gamma_c)^{3/2}} \lambda \zeta^2 + o(\lambda). \]

(3.11)

when trading through the dealers. In the large-liquidity limit, the relative performance in both markets therefore only depends on the ratio of the clients' and dealers' inventory costs. A short calculation shows that the dealer market becomes strictly more attractive if $\gamma_c/\gamma_d$ is larger.

This means that the dealer market is preferable to direct trading if the dealers’ risk-bearing capacity is sufficiently large compared to the clients’.

The universal cutoff point is $\gamma_c/\gamma_d \approx 2.44$.

**Example 3.3 (Diffusive Trading Targets).** To explore the other end of the spectrum of potential target strategies, suppose as in [24] [31] [12] that the (aggregate) clients have a “high-frequency trading need” modeled by a target position $\zeta$ following Brownian motion with volatility $\sigma_\zeta$. For such a martingale $\zeta$, we also have $\tilde{\zeta}_t = F^\Delta(t)\zeta_t$. Together with $\zeta_0 = 0$ it in turn follows from Formula (3.9) in the proof of Proposition 3.1 that the clients’ optimal position has the following dynamics:

\[ d\hat{K}_t = F^\Delta(t)(\zeta_t - \hat{K}_t) dt + \frac{\gamma_c}{\gamma_d + \gamma_c} d\zeta_t. \]

As a consequence, the deviation $\zeta_t - \hat{K}_t$ from the target position has Ornstein-Uhlenbeck-type dynamics,

\[ d(\zeta_t - \hat{K}_t) = -F^\Delta(t)(\zeta_t - \hat{K}_t) dt + \frac{\gamma_d}{\gamma_d + \gamma_c} d\zeta_t, \quad \zeta_0 - \hat{K}_0 = 0. \]
Figure 3: Optimal tracking strategies for a diffusive target (blue), when trading through dealers (orange, left panel) or directly with end-users (orange, right panel).

The displacement from the target position therefore mean reverts around zero. The corresponding mean reversion speed is given by the function

\[ F(\Delta_t) = \sqrt{\Delta_t \tanh(\sqrt{\Delta_t}(T - t))}, \]

which interpolates between stopping trading at the terminal time and the long-run relative trading speed \( \sqrt{\Delta} = \sqrt{\gamma_c/\lambda} \). The latter also arises as the leading order term in the small-cost expansion of \( F(\Delta_t) \) (that is, as \( \lambda \approx 0 \)).

Let us now again compare the equilibrium with intermediaries to a model where the clients trade directly in the end-user market as in [3]. Then, their optimal trading rate is smooth:

\[ d\hat{K}_{\text{direct}}^t = F_{\Delta_{\text{direct}}}(t)(\zeta_t - \hat{K}_{\text{direct}}^t)dt. \]

However, even though the optimal position is now smooth, the corresponding displacement from the target position again has Ornstein-Uhlenbeck-type dynamics:

\[ d(\zeta_t - \hat{K}_{\text{direct}}^t) = -F_{\Delta_{\text{direct}}}(t)(\zeta_t - \hat{K}_{\text{direct}}^t)dt + d\zeta_t, \]

\[ \zeta_0 - \hat{K}_{\text{direct}}^0 = 0. \]

The relative trading speed now is

\[ F_{\Delta_{\text{direct}}}(t) = \sqrt{\Delta_{\text{direct}}(T - t)} \]

which approximately equals \( \sqrt{\gamma_c/\lambda} \) for a long time horizon \( T \) or small trading costs \( \lambda \). Compared to trading through the dealers, we see that this trading speed is larger, but also that the Brownian shocks in the Ornstein-Uhlenbeck dynamics are larger because they cannot be hedged by trading in an absolutely continuous manner. This is illustrated in Figure 3.

When trading directly in the end-user market, [3, Theorem 3.1] as well as an elementary integration and Taylor expansion show that the clients’ optimal performance is

\[ \max J_{\text{direct}}^t = -\gamma_c \sigma_\zeta^2 \log(\cosh(\sqrt{\Delta_{\text{direct}}}T)) \]

\[ \Delta_{\text{direct}} \]

\[ = -\frac{\gamma_c}{2} \sqrt{\lambda} \sigma_\zeta^2 T + o(\sqrt{\lambda}) \quad \text{as} \quad \lambda \to 0. \]

When trading through the intermediating dealers, the first-order condition (3.2) and integration of the hyperbolic functions in (3.5) shows that the clients’ optimal performance in (3.1) is

\[ J_c(\hat{K}) = \frac{\gamma_c}{2} E \left[ \int_0^T (\hat{K}_t^2 - \zeta_t^2) dt \right] \]

\[ = -\frac{\sigma_\zeta^2}{\Delta} \frac{\gamma_d \gamma_c}{\gamma_d + \gamma_c} \left( \log \left( \cosh(\sqrt{\Delta}T) \right) - \frac{1}{4} \frac{\gamma_d}{\gamma_d + \gamma_c} \sqrt{\Delta T \tanh(\sqrt{\Delta}T)} \right) \]

\[ = -\frac{\sqrt{\gamma_d \gamma_c}(3 \gamma_d + 4 \gamma_c)}{4(\gamma_d + \gamma_c)^{3/2}} \sqrt{\lambda} \sigma_\zeta^2 T + o(\sqrt{\lambda}) \quad \text{as} \quad \lambda \to 0. \]

\[ ^{14} \text{Note that this term is always negative since the function} \ x \mapsto \log(\cosh(x)) - \frac{1}{4} x \tanh(x) \text{is positive.} \]
Whence, exactly as in Example 3.2, the leading-order optimal performances in both markets are the same up to replacing $\sqrt{\gamma_c}$ (for direct trading in the highly liquid end-user market) with the term from (3.11). Thus, the dealer market is again preferable to direct trading if the dealers’ risk-bearing capacity is sufficiently large compared to the clients. For highly-liquid end-user markets, the corresponding cutoff also remains invariant.

4 Large Clients

In Section 3 we have considered a continuum of infinitesimal clients, each acting as a price taker. We now turn to a single large client, who internalizes the price impact of his trades. The analogue of the goal functional (3.1) in turn is

$$\tilde{J}_c(K) = E \left[ \int_0^T \left( K_t dS_t^K - \frac{\gamma_c}{2} (K_t - \zeta_t)^2 dt \right) \right] = E \left[ \int_0^T \left( K_t \gamma_d(U^K_t - K_t) - \frac{\gamma_c}{2} (K_t - \zeta_t)^2 \right) dt \right].$$

Since the large trader takes into account how his demand $K$ affects the execution price $S^K$ from Theorem 2.5, the new criterion $\tilde{J}_c$ can no longer be optimized by pointwise maximization as was still possible for $J_c$ of (3.1). The new target functional is, however, still concave:

**Lemma 4.1.** The bilinear form

$$B : S^2 \times S^2 \to \mathbb{R}$$

$$(\mathcal{L}, K) \mapsto -E \left[ \int_0^T \mathcal{L}_t dS_t^K \right]$$

is symmetric and positive definite. In particular, the induced quadratic form $K \mapsto B(K, K)$ is convex and $\tilde{J}_c$ is concave.

**Remark 4.2.** Note that $B(K, K) \geq 0$ describes the dealers’ expected profits and, accordingly, $-B(K, K) \leq 0$ the large client’s expected costs. The equilibrium price impact functional $K \mapsto S^K$ therefore does not allow for price manipulation strategies as discussed for reduced form price impact models in [16].

**Proof.** Observe that

$$B(\mathcal{L}, K) = -E \left[ \int_0^T \mathcal{L}_t dS_t^K \right] = E \left[ \int_0^T \mathcal{L}_t \gamma_d(K_t - U^K_t) dt \right].$$

Whence, for symmetry it suffices to verify

$$E \left[ \int_0^T K_t U^K_t dt \right] = E \left[ \int_0^T \mathcal{L}_t U^K_t dt \right], \text{ for all } K, \mathcal{L} \in \mathcal{F}^2.$$

By (2.5) and (2.4), we have

$$E \left[ \int_0^T K_t U^K_t dt \right] = \frac{1}{\Delta} E \left[ \int_0^T \int_0^T \int_0^T 1_{\{s \leq r\}} D^K(s, t) D^K(s, r) K_t E_s [\mathcal{L}_r] dr ds dt \right]. \quad (4.1)$$

Now notice that the tower property of the conditional expectation implies that

$$E \left[ K_t E_s [\mathcal{L}_r] \right] = E \left[ \mathcal{L}_r E_s[K_t] \right], \text{ on } \{0 \leq s \leq t \leq T\} \cap \{0 \leq s \leq r \leq T\}.$$
Together with Fubini’s theorem, it follows that (4.1) can indeed be rewritten as
\[
E \left[ \int_0^T K_t U_t^L dt \right] = \frac{1}{\Delta} E \left[ \int_0^T \int_0^T \int_0^T 1_{\{s \leq t\}} 1_{\{s \leq r\}} D^\Delta(s, t) D^\Delta(s, r) \mathcal{L}_r E_s[K_t] dt ds dr \right] \\
= \frac{1}{\Delta} E \left[ \int_0^T \left( \mathcal{L}_r \int_0^r \left( D^\Delta(s, r) E_s \left[ \int_s^T D^\Delta(s, t) K_t dt \right] \right) ds \right) dr \right] \\
= E \left[ \int_0^T \mathcal{L}_r U_r^K dr \right].
\]

Here, we have again used (2.4) and (2.5) for the least equality.

As for the positive definiteness of \( B \), recall that at equilibrium prices \( S^K \) the dealers’ optimal supply is \( \tilde{K} = K \). So, in particular,
\[
0 = J_d(0; S^K) \leq J_d(K; S^K) = B(K, K) - \frac{1}{2} E \left[ \int_0^T \lambda(u_t^K)^2 + \gamma_d (U_t^K - K_t)^2 dt \right]
\]
where we used (2.7) and (2.8). Since the latter expectation is nonnegative, this proves \( B(K, K) \geq 0 \) as claimed.

Due to the concavity of \( \tilde{J}_c \), we can optimize it the same way as for the dealers’ problem in the proof of Theorem 2.5. To wit, the necessary and sufficient first-order condition for optimality is that its Gâteaux derivative vanishes at the optimum \( \tilde{K} \) in any direction \( \mathcal{L} \in \mathcal{S}^2 \):
\[
0 = \frac{d}{de} \tilde{J}_c(\tilde{K} + e \mathcal{L}) \bigg|_{e=0} = \frac{d}{de} \left( -B(\tilde{K} + e \mathcal{L}, \tilde{K} + e \mathcal{L}) - \frac{\gamma_c}{2} E \left[ \int_0^T (\tilde{K}_t + e \mathcal{L}_t - \zeta_t)^2 dt \right] \right) \bigg|_{e=0}
\]
\[
= -2B(\mathcal{L}, \tilde{K}) - \gamma_c E \left[ \int_0^T (\tilde{K}_t - \zeta_t) \mathcal{L}_t dt \right],
\]
where we used the symmetry of \( B \). In view of \(-B(\mathcal{L}, K) = E \left[ \int_0^T \mathcal{L}_t \gamma_d (U_t^K - K_t) dt \right]\), this is tantamount to
\[
0 = E \left[ \int_0^T \mathcal{L}_t \left( 2\gamma_d U_t^K + 2\gamma_d U_t^K + 2\gamma_c \tilde{K}_t + 2\gamma_c \zeta_t \right) dt \right].
\]

As this condition has to hold for any perturbation \( \mathcal{L} \in \mathcal{S}^2 \), it is equivalent to
\[
\tilde{K}_t = \frac{\gamma_d}{\gamma_d + \gamma_c/2} U_t^K + \frac{\gamma_c/2}{\gamma_d + \gamma_c/2} \zeta_t, \quad 0 \leq t \leq T.
\]
Remarkably, this is almost exactly the same integral equation as its counterpart (3.3) for a continuum of small clients – the only difference is that the large client’s holding costs are halved here. In particular, existence, uniqueness, and explicit formulas for the large client’s optimal trading strategy follow from Proposition 3.1. The structure of the equilibrium therefore remains unchanged. The only quantitative difference is that, since large clients internalize their price impact, they weigh transaction costs more heavily compared to displacement from their trading targets, and therefore trade more slowly.

The price impact is correctly internalized in the large client’s optimization problem \( \tilde{J}_c \) unlike for its counterpart \( J_c \) for the price-taking clients. Accordingly, the optimal performance of the large client is always at least as good as for the aggregation of small (decentralized) clients:
\[
\tilde{J}_c(\tilde{K}) \geq \tilde{J}_c(\tilde{K}) = J_c(\tilde{K}).
\]
Figure 4: Ratio of optimal performances for large client and aggregation of small clients.

The magnitude of this “price of anarchy” can be analyzed by computing the large client’s optimal performance exactly as for the small clients in Examples 3.2 and 3.3. For a highly liquid end-user market ($\lambda \approx 0$), the respective performances merely differ by a universal factor, which is a function of the ratio $\gamma_c/\gamma_d$ of the dealers’ and clients’ holding costs only:

\[ \frac{4 + 2\gamma_c/\gamma_d}{2(2 + \gamma_c/\gamma_d)^{3/2}} / \frac{3 + 4\gamma_c/\gamma_d}{4(1 + \gamma_c/\gamma_d)^{3/2}}. \]

The ratio $\rho$ of these terms is plotted against $\gamma_c/\gamma_d$ in Figure 4. Since the clients’ performance is negative, values smaller than one correspond to the better performance of the large client. Maybe surprisingly, the maximal asymptotic performance gain by correctly internalizing the price impact is only about 7%, independent of the other model parameters.

A Proofs for Section 2.4

Proof of Proposition 2.6. Write

\[ V_t - S_t^K = M_t - A_t \quad \text{with} \quad A_t := \gamma_d \int_0^t (U^K_s - K_s) ds, \quad M_t := E_t[A_T], \quad 0 \leq t \leq T. \]

Then, for any predictable $H$ with values in $[-1, 1]$, Doob’s maximal inequality, the Itô isometry, and Jensen’s inequality show that

\[
E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t H_s d(V_s - S^K_s) \right|^2 \right] \leq 2 \left( E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t H_s dM_s \right|^2 \right] + E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t H_s dA_s \right|^2 \right] \right) \\
\leq 2 \left( 4E \left( \int_0^T H_t^2 d(M)_t \right) + E \left[ \left( \int_0^T |H| \gamma_d |U^K_t - K_t| dt \right)^2 \right] \right) \\
\leq 2 \left( 4E[M_T^2] + T\gamma_d^2 E \left[ \int_0^T (U^K_t - K_t)^2 dt \right] \right) \\
\leq 10T\gamma_d^2 E \left[ \int_0^T (U^K_t - K_t)^2 dt \right]. \tag{A.1}
\]

Since $\{U = \int_0^t u_s ds : u \in \mathcal{L}^2\}$ is dense in $\mathcal{L}^2$, there exists a sequence $u^n \in \mathcal{L}^2$ such that

\[ E \left[ \int_0^T \left( \int_0^t u^n_s ds - K_t \right)^2 dt \right] \leq \frac{1}{n}, \quad n = 1, 2, \ldots \]
Due to the minimality condition \(2.2\), we have
\[
E \left[ \int_0^T \frac{\gamma d}{2} (U_t^K - K_t)^2 dt \right] \leq E \left[ \int_0^T \left( \frac{\lambda}{2} (u_t^n)^2 + \frac{\gamma d}{2} \left( \int_0^t u_s^n ds - K_t \right)^2 \right) dt \right]
\]
\[
\leq \frac{\lambda}{2} E \left[ \int_0^T (u_t^n)^2 dt \right] + \frac{\gamma d}{2n}
\]
Thus, \(E[\int_0^T (U_t^K - K_t)^2 dt] \to 0\) as \(\lambda \to 0\). Together with \(A.1\), this establishes the first convergence asserted in Proposition \(2.6\).

To also show convergence in \(L^1\), we apply the inequalities of Burkholder-Davis-Gundy and Hölder to obtain
\[
E \left[ \int_0^T K_t (dS_t^K - dV_t) \right] \leq E \left[ \sup_{0 \leq s \leq T} \left| \int_0^s K_s dM_s \right| \right] + E \left[ \sup_{0 \leq s \leq T} \left| \int_0^s K_s dA_s \right| \right]
\]
\[
\leq CE \left[ \left( \int_0^T K_s^2 d\langle M \rangle_s \right)^{1/2} \right] + E \left[ \int_0^T |K_s| |U_s^K - K_s| ds \right]
\]
\[
\leq E \left[ \sup_{s \in [0,T]} |K_s|^2 \right] \left( C \langle M \rangle_{1T}^{1/2} + \int_0^T |U_s^K - K_s|^2 ds \right)
\]
\[
\leq E \left[ \sup_{s \in [0,T]} |K_s|^2 \right] \left[ 2(C^2T + T) \int_0^T |U_s^K - K_s|^2 ds \right]^{1/2},
\]
where \(C > 0\) is the constant of the Burkholder-Davis-Gundy inequality. \(L^1\) convergence now follows, since we have already verified above that the last term converges to 0 as \(\lambda \to 0\).

**Proof of Lemma \(2.7\)** By \(2.8\), \(2.9\), and integration by parts (using \(K_0 = u_T^K = 0\)), we have
\[
\int_0^T K_t dV_t - \int_0^T K_t dS_t^K = \lambda \int_0^T K_t du_t^K = \lambda \int_0^T \mu_t^K u_t^K dt.
\]
(A.2)

To establish \(2.10\), it therefore suffices to show that
\[
\int_0^T |\mu_t^K - u_t^K| dt = o(1) \quad \text{in } L^1 \text{ as } \lambda \to 0.
\]
(A.3)

To this end, first notice that \(2.3\), \(2.5\), and integration by parts give
\[
\mu_t^K - u_t^K = \mu_t^K + F(\Delta(t)) U_t^K - K_t
\]
\[
= \mu_t^K - F(\Delta(t))(K_t - U_t^K) + F(\Delta(t))K_t - \frac{\Delta}{\cosh(\sqrt{\Delta}(T-t))} E_t \left[ \int_t^T \cosh(\sqrt{\Delta}(T-s))K_s ds \right]
\]
\[
= \mu_t^K - F(\Delta(t))(K_t - U_t^K) - \frac{\sqrt{\Delta}}{\cosh(\sqrt{\Delta}(T-t))} E_t \left[ \int_t^T \sinh(\sqrt{\Delta}(T-s))\mu_s^K ds \right].
\]
(A.4)

Now, note that
\[
\frac{\sqrt{\Delta} \int_t^T \sinh(\sqrt{\Delta}(T-s)) ds}{\cosh(\sqrt{\Delta}(T-t))} = 1 - \frac{1}{\cosh(\sqrt{\Delta}(T-t))}.
\]
Together with (A.4), it follows that $\mathcal{K} - U^K$ satisfies the linear ODE
\[
\frac{d(K_t - U^K_t)}{dt} = \mu^K_t - u^K_t = -F^\Delta(t)(K_t - U^K_t) + w^\Delta(t) + \frac{\mu^K_t}{\cosh(\sqrt{\Delta}(T-t))}, \quad (A.5)
\]
where
\[
w^\Delta(t) = E_t \left[ \int_t^T \frac{\sqrt{\Delta} \sinh(\sqrt{\Delta}(T-s))}{\cosh(\sqrt{\Delta}(T-t))} (\mu^K_s - \mu^K_t) ds \right].
\]

Since $K_0 = U^K_0 = 0$ and by definition of the function $F^\Delta$ in Lemma 2.2, the explicit solution of (A.5) is
\[
K_t - U^K_t = \int_0^t e^{\int_s^t F^\Delta(u) du} \left( u^K_s + \frac{\mu^K_s}{\cosh(\sqrt{\Delta}(T-s))} \right) ds,
\]

Together with (A.5), it follows that
\[
\int_0^T |\mu^K_t - u^K_t| dt \leq \sup_{u \in [0,T]} |w^\Delta_u| \int_0^T \left( 1 + \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} ds \right) dt + \sup_{u \in [0,T]} |\mu^K_u| \int_0^T \left( 1 + \sqrt{\Delta} \int_0^t e^{-\sqrt{\Delta}(t-s)} ds \right) dt + \sup_{u \in [0,T]} |w^\Delta_u| \int_0^T 2e^{-\sqrt{\Delta}t} dt + \int_0^T \sqrt{\Delta} \int_0^T \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh^2(\sqrt{\Delta}(T-s))} dt ds
\]
\[
\leq 2T \sup_{u \in [0,T]} |w^\Delta_u| + \frac{3}{\sqrt{\Delta}} \sup_{u \in [0,T]} |\mu^K_u|.
\quad (A.6)
\]

Write
\[
\omega(\delta) = \sup_{t,s \in [0,T],|t-s| \leq \delta} \left| \mu^K(s) - \mu^K(t) \right|
\]
for the modulus of continuity of $\mu^K$. Since $t \mapsto \mu^K_t$ is continuous on the compact set $[0,T]$,
\[
\omega(\delta) \to 0, \quad \text{a.s. as } \delta \to 0.
\quad (A.7)
\]

The definitions of $\omega^\Delta$ and $\omega$ and a change of variables yield the following estimate:
\[
0 \leq |w^\Delta(t)| \leq \frac{1}{2} E_t \left[ \int_t^T \sqrt{\Delta} e^{-\sqrt{\Delta}(s-t)} \omega(s-t) ds \right] \leq \frac{1}{2} E_t \left[ \int_0^{\infty} e^{-u \omega \left( \frac{u}{\sqrt{\Delta}} \right) du} \right] := M^\Delta_t. \quad (A.8)
\]
Here, $(M^\Delta_t)_{t \in [0,T]}$ is a martingale for each $\Delta > 0$ since $|\omega(\delta)| \leq 2 \sup_{s \in [0,T]} |\mu^K_s|$ is integrable by assumption. Also note that by definition of the modulus of continuity $\omega$, the mapping $\Delta \mapsto M^\Delta_t$ is decreasing for each $t$. Define
\[
M^* := \lim_{\Delta \to \infty} \sup_{t \in [0,T]} M^\Delta_t \geq 0.
\]

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Fix $\epsilon > 0$. Then, by the monotonicity in $\Delta$, we have

$$P[M^* \geq \epsilon] \leq \lim_{\Delta \to \infty} P \left[ \sup_{t \in [0,T]} M^\Delta_t \geq \epsilon \right] \leq \lim_{\Delta \to \infty} \frac{E[M^\Delta_T]}{\epsilon} = 0.$$ 

Here, the last equality is a consequence of (A.7), another application of the monotone convergence theorem and the integrability of right-hand side in (A.8). As a result,

$$0 \leq \lim_{\Delta \to \infty} \sup_{t \in [0,T]} |w^\Delta(t)| \leq M^* = 0 \text{ a.s.}$$

In view of (A.6), it follows that the asserted convergence (A.3) holds in the almost-sure sense.

To show convergence in $L^1$, it therefore suffices to establish uniform integrability of (A.2).

By (A.6) and (A.8),

$$\int_0^T |(\mu^K_t)^2 - u^K_t \mu^K_t| dt \leq \sup_{s \in [0,T]} |\mu^K_s| \int_0^T |\mu^K_t - u^K_t| dt \leq T \sup_{s \in [0,T]} |w^\Delta_s|^2 + (T + \frac{3}{\sqrt{\Delta}}) \sup_{s \in [0,T]} |\mu^K_s|^2 \leq T \sup_{s \in [0,T]} |M^\Delta_s|^2 + (T + \frac{3}{\sqrt{\Delta}}) \sup_{s \in [0,T]} |\mu^K_s|^2.$$

Observe that the right-hand side is decreasing in $\Delta$, and integrable for, e.g., $\Delta = 1$ by Doob’s maximal inequality:

$$E \left[ T \sup_{t \in [0,T]} |M^1_t|^2 + (T + 3) \sup_{t \in [0,T]} |\mu^K_t|^2 \right] \leq E \left[ 2T|M^1_T|^2 + (T + 3) \sup_{t \in [0,T]} |\mu^K_t|^2 \right] < \infty.$$ 

Therefore, the family, $\{ \int_0^T |(\mu^K_t)^2 - u_t \mu^K_t| ds, \Delta \geq 1 \}$ is uniformly integrable. Since

$$\int_0^T |(\mu^K_t)^2 - u_t \mu^K_t| ds \leq \sup_{s \in [0,T]} |\mu^K_s| \int_0^T |\mu^K_s - u_s| ds,$$

this implies that the almost sure convergence we have established for (A.3) also holds in $L^1$. \qed

**Proof of Lemma 2.8.** In view of (2.8) (2.9), as well as integration by parts (using that $K_0 = u^K_T = 0$), the liquidity costs of the clients can be written as

$$\int_0^T K_t dV_t - \int_0^T K_t dS^K_t = -\lambda \int_0^T K_t du^K_t + \lambda \int_0^T u^K_t (\mu^K_t dt + \sigma^K_t dW_t). \tag{A.9}$$

By (2.3), we have $u^K_t = K_t - F^\Delta(t)U^K_t$, which implies that the covariation of $u^K$ is the same as the one of $K$. Note that by definition (cf. (2.4)),

$$\cosh(\sqrt{\Delta}(T - t))K_t - \Delta \int_0^t \cosh(\sqrt{\Delta}(T - s))K_s ds = \Delta E_t \left[ \int_0^T \cosh(\sqrt{\Delta}(T - s))K_s ds \right] \tag{A.10}$$

is a square-integrable martingale. By the martingale representation theorem, it therefore can be written as a stochastic integral with respect to the Brownian motion generating the underlying filtration. The integrand in this representation can be computed using the Clark-Ocone formula. Indeed, setting

$$\Phi = \Delta \int_0^T \cosh(\sqrt{\Delta}(T - s))K_s ds$$

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and using the Malliavin differentiability of $\Phi$ that we prove below, the Clark-Ocone formula \[27\] Proposition 1.3.14] yields

$$\Phi = E[\Phi] + \int_0^T E_t[D_t \Phi] dW_t.$$  

By inserting this into (A.10) and integrating by parts, we in turn obtain

$$\langle u^K, K \rangle_T = \int_0^T \frac{E_t[D_t \Phi]}{\cosh(\sqrt{\Delta}(T - t))} \sigma_t^K dt.$$  \hspace{1cm} (A.11)

We now show that we indeed have $\Phi \in \mathbb{D}^{1,2}$, so that the Clark-Ocone formula can be applied. Given our assumption on the square-integrability of the supremum of its Malliavin derivative, $K \in L^{1,2,f}$, cf. [27, p. 45]. Thus, by [27, p. 45], $\Phi$ is Malliavin differentiable and it follows from the product rule that

$$D_t \Phi = \int_t^T \cosh(\sqrt{\Delta}(T - s)) D_s K_s ds.$$  \hspace{1cm} (A.12)

We now expand $D_t \Phi$ and $E_t[D_t \Phi]$ for $\lambda \to 0$ or, equivalently, $\Delta \to \infty$. First note that by (A.12) and the definition of the hyperbolic cosine,

$$\frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} = \sqrt{\Delta} \int_t^T \frac{e^{\sqrt{\Delta}(t-s)} + e^{-\sqrt{\Delta}(2T-(s+t))}}{1 + e^{-2\sqrt{\Delta}(T-t)}} D_s K_s ds.$$  \hspace{1cm} (A.13)

By continuity of $s \mapsto D_t(K_s)$ on $[t, T]$, some elementary integrations show that the above expression converges to $D_t K_t$ as $\Delta \to \infty$. In view of [27, Proposition 1.3.8], we have $D_t(\int_0^t \sigma_s^K dW_s) = \sigma_t^K$. Moreover, $D_t(\int_0^t \mu_s^K ds) = 0$, so that

$$\frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} \to \sigma_t^K, \quad P\text{-a.s. as } \Delta \to \infty.$$  

Next, observe that for every $t \in [0, T]$, it follows from (A.13) that

$$\sup_{\Delta > 1} \left| \frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} \right| \leq \sup_{s \in [t, T]} |D_t K_s| \sup_{\Delta > 1} \left\{ \sqrt{\Delta} \int_t^T e^{\sqrt{\Delta}(t-s)} ds \right\} \leq 2 \sup_{t \leq \Delta \leq T} |D_t K_s|. \hspace{1cm} (A.14)$$

Since the right-hand side is integrable by assumption, the dominated convergence theorem in turn shows

$$E_t \left[ \frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} \right] \to \sigma_t^K, \quad dP \times dt\text{-a.s. as } \Delta \to \infty.$$  

We now show that this expansion of $D_t \Phi$ is inherited by its conditional expectation and in turn the covariation \(A.11\). To this end, we first use \(A.14\) and Young’s inequality to obtain that

$$\sup_{\Delta > 1} |\sigma_t^K| \left| E_t \left[ \frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} \right] \right| \leq \frac{1}{3} \sup_{t \in [0, T]} |\sigma_t^K|^3 + \frac{2}{3} \sup_{\Delta > 1} E_t \left[ \frac{\sqrt{\Delta}^{-1} D_t \Phi}{\cosh(\sqrt{\Delta}(T - t))} \right]^{3/2}$$

$$\leq \frac{1}{3} \sup_{t \in [0, T]} |\sigma_t^K|^3 + \frac{\sqrt{32}}{3} E_t \left[ \sup_{t \leq s \leq T} |D_t K_s|^{3/2} \right].$$

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Jensen’s inequality and the integrability assumption for the supremum of the Malliavin derivative of $K$ yield

$$E \left[ \int_0^T E_t \left[ \sup_{t \leq s \leq T} |D_t K_s|^{3/2} \right]^{4/3} dt \right] \leq E \left[ \int_0^T \sup_{t \leq s \leq T} |D_t K_s|^2 dt \right] < \infty.$$ 

Moreover, $(\sup_{t \in [0,T]} |\sigma_t^K|^3)^{4/3} = \sup_{t \in [0,T]} |\sigma_t^K|^4$ is also integrable by assumption. Together, these two estimates show that

$$E \left[ \int_0^T \left( \sup_{t \leq s \leq T} \left| \sigma_t^K \right| \right) \left( \frac{\sqrt{\Delta^{-1}}} {\cosh(\sqrt{\Delta}(T-t))} \right)^{4/3} dt \right] < \infty. \tag{A.15}$$

Since the term inside this expectation is finite, the dominated convergence theorem implies that, as $\lambda \to 0$ and in turn $\Delta \to \infty$,

$$\sqrt{\Delta^{-1}} \langle u^K, K \rangle_T = \int_0^T \frac{\sqrt{\Delta^{-1}} E_t[D_t \Phi]} {\cosh(\sqrt{\Delta}(T-t))} \sigma_t^K dt \to \int_0^T (\sigma_t^K)^2 dt, \quad P\text{-a.s.}$$

Finally, (A.15) also shows that the $4/3$-th moment of $\int_0^T \frac{\sqrt{\Delta^{-1}} E_t[D_t \Phi]} {\cosh(\sqrt{\Delta}(T-t))} \sigma_t^K dt$ is bounded, uniformly for all $\Delta > 1$. Therefore, this family indexed by $\Delta > 1$ is uniformly integrable and the almost sure convergence for $\Delta \to \infty$ also holds in $L^1$.

To complete the proof, we now show that the other terms in (A.9) do not contribute at the leading order $O(\sqrt{\lambda})$, that is,

$$\lambda \int_0^T u_t^K (\mu_t^K dt + \sigma_t^K dW_t) = o(\sqrt{\lambda}), \quad \text{in } L^1 \text{ as } \lambda \to 0.$$ 

Since $\Delta = \gamma d/\lambda$, this is implied by a bound for $\Delta^{-1/2} u^K$. To this end, observe that the inequalities of Jensen and Burkholder-Davis-Gundy show

$$E \left[ (K_t - K_s)^4 \right] \leq C \left( E \left[ \left( \int_s^t \mu^K_r dr \right)^4 \right] + E \left[ \left( \int_s^t \sigma^K_r dW_r \right)^4 \right] \right) \leq C' E \left[ \sup_t |\mu_t^K|^4 + |\sigma_t^K|^4 \right] (t-s)^2,$$

for some universal constants $C, C' > 0$. For $\alpha < \frac{1}{4}$, write $R_{\alpha}$ for the modulus of $\alpha$-Hölder continuity of $K$. This quantity is well defined and satisfies $E[R_{\alpha}^4] < \infty$ by [13, Theorem 3.1]. Define

$$M^\alpha := \sup_{s \in [0,T]} E_s[R_{\alpha}] \left( 1 + \int_0^\infty 2e^{-u}|u|^\alpha ds \right) + \sup_{s \in [0,T]} |K_s| < \infty.$$ 

\textsuperscript{14}This theorem requires an additional assumption on the iterated integral of the process $K$. However, a careful inspection of the proof reveals that this extra assumption is only needed to establish additional path regularity of the iterated integral and not for the path regularity of the process $K$ itself.
Then, we can estimate
\[
\left| \Delta^{-1/2} \mathcal{K}_t + \mathcal{K}_t \right| \leq E_t \left[ R_\alpha \int_t^T \Delta^{-1/2} D^\Delta(t, s)|t - s|^\alpha ds \right] + \left| 1 - \int_t^T \Delta^{-1/2} D^\Delta(t, s) ds \right| |\mathcal{K}_t|
\]
\[
\leq E_t[R_\alpha] \Delta^{(1-\alpha)/2} \int_t^T \frac{2e^{\sqrt{\Delta}(T-s)}}{e^{\sqrt{\Delta}(T-t)}} \Delta(t-s) \alpha ds + \left| 1 - \tanh \left( \sqrt{\Delta}(T-t) \right) \right| |\mathcal{K}_t|
\]
\[
\leq E_t[R_\alpha] \Delta^{-\alpha/2} \int_0^\infty 2e^{-u} |u|^\alpha du + 2e^{-2\sqrt{\Delta}(T-t)} |\mathcal{K}_t|
\]
\[
\leq \left( \Delta^{-\alpha/2} + 2e^{-2\sqrt{\Delta}(T-t)} \right) M^\alpha := C_{t, T, \alpha, \lambda}.
\]

Together with the formulas for \(u^\mathcal{K}, U^\mathcal{K}\) and the definition of the function \(F^\Delta\) from Lemma 2.2, this estimate yields
\[
\left| \Delta^{-1/2} u^\mathcal{K}_t \right| = -\Delta^{-1/2} F^\Delta(t) U^\mathcal{K}_t + \Delta^{-1/2} \mathcal{K}_t
\]
\[
= \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} \Delta^{-1/2} \mathcal{K}_s ds + \Delta^{-1/2} \mathcal{K}_t
\]\n\[
\leq \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} \mathcal{K}_s ds - \mathcal{K}_t + C_{t, T, \alpha, \lambda} + \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} C_{s, T, \alpha, \lambda} ds
\]\n\[
\leq \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} \left( R_\alpha |t - s|^\alpha + C_{s, T, \alpha, \lambda} \right) ds
\]\n\[
+ C_{t, T, \alpha, \lambda} + \left| 1 - \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} ds \right| |\mathcal{K}_t|
\]\n\[
\leq C_{t, T, \alpha, \lambda} + \Delta^{-\alpha/2} R_\alpha \int_0^\infty e^{-u} |u|^\alpha du + \sqrt{\Delta} \int_0^t e^{-\sqrt{\Delta}(t-s)} C_{s, T, \alpha, \lambda} ds
\]\n\[
+ \left| 1 - \sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} ds \right| |\mathcal{K}_t|.
\] (A.16)

Recall the addition formula \(\arctan(x) - \arctan(y) = \arctan \left( \frac{x-y}{1+xy} \right)\) for \(x, y \geq 0\) and observe that \(|\arctan(x)| \leq |x|\). As a consequence:
\[
\sqrt{\Delta} \int_0^t \frac{\sinh(\sqrt{\Delta}(T-t))}{\cosh(\sqrt{\Delta}(T-s))} ds = \sinh(\sqrt{\Delta}(T-t)) \left( \arctan \left( \sinh(\sqrt{\Delta}T) \right) - \arctan \left( \sinh(\sqrt{\Delta}(T-t)) \right) \right)
\]
\[
= \sinh(\sqrt{\Delta}(T-t)) \arctan \left( \frac{\sinh(\sqrt{\Delta}T) - \sinh(\sqrt{\Delta}(T-t))}{1 + \sinh(\sqrt{\Delta}T) \sinh(\sqrt{\Delta}(T-t))} \right)
\]
\[
\leq \sinh(\sqrt{\Delta}(T-t)) \arctan \left( \frac{\sinh(\sqrt{\Delta}T)}{1 + \sinh(\sqrt{\Delta}T) \sinh(\sqrt{\Delta}(T-t))} \right)
\]
\[
\leq \frac{\sinh(\sqrt{\Delta}(T-t)) \sinh(\sqrt{\Delta}T)}{1 + \sinh(\sqrt{\Delta}T) \sinh(\sqrt{\Delta}(T-t))} \leq 1,
\]
as well as

\[
\sqrt{\lambda} \int_0^t \frac{\sinh(\sqrt{\lambda}(T-t))}{\cosh(\sqrt{\lambda}(T-t))} ds = \sqrt{\lambda} \int_0^t \frac{e^{-\sqrt{\lambda}(t-s)} - e^{-\sqrt{\lambda}(2T-t-s)}}{1 + e^{-2\sqrt{\lambda}(T-s)}} ds \\
\geq \sqrt{\lambda} \int_0^t \left( e^{-\sqrt{\lambda}(t-s)} - e^{-\sqrt{\lambda}(2T-s-t)} \right) \left( 1 - e^{-2\sqrt{\lambda}(T-s)} \right) ds \\
\geq \sqrt{\lambda} \int_0^t e^{-\sqrt{\lambda}(t-s)} - e^{-\sqrt{\lambda}(2T-2t+(t-s))} - e^{-\sqrt{\lambda}(2T-2t+3(t-s))} ds \\
\geq 1 - e^{-\sqrt{\lambda}t} - 2e^{-2\sqrt{\lambda}(T-t)}.
\]

In view of these two estimates, (A.16) yields

\[
\left| \Delta^{-1/2} u_t^K \right| \leq C_{t,T,\alpha,\lambda} + \Delta^{-\alpha/2} R_\alpha \int_0^\infty e^{-u} |u|^\alpha du + \sqrt{\lambda} \int_0^t e^{-\sqrt{\lambda}(t-s)} C_{s,T,\alpha,\lambda} ds \\
+ \left( e^{-\sqrt{\lambda}t} + 2e^{-2\sqrt{\lambda}(T-t)} \right) \sup_{s \in [0,T]} |K_s| \\
\leq 4 \left( e^{-\sqrt{\lambda}t} + \Delta^{-\alpha/2} + e^{-2\sqrt{\lambda}(T-t)} \right) M^\alpha. \tag{A.17}
\]

(Here, the last inequality follows from the definition of $C_{t,T,\alpha,\lambda}$.) In particular, there exists a constant $C_T > 0$ only depending on $T$ such that

\[
\left| \int_0^T \Delta^{-1/2} u_t^K \mu_t^K dt \right| \leq C_T M^\alpha \sup_{t \in [0,T]} |\mu_t^K| (\Delta^{-\alpha/2} + \Delta^{-1/2}) \\
\leq \frac{C_T}{2} \left( |M^\alpha|^2 + \sup_{t \in [0,T]} |\mu_t^K|^2 \right) (\Delta^{-\alpha/2} + \Delta^{-1/2}) \to 0,
\]

as $\lambda \to 0$ and in turn $\Delta \to \infty$. By the dominated convergence theorem, this pointwise convergence also holds in $L^1$, since the upper bound in this estimate is integrable under our assumptions. This shows that the Lebesgue integral in (A.9) is indeed of order $o(\sqrt{\lambda})$ as claimed.

The argument for the stochastic integral in (A.9) is similar. By the Burkholder-Davis-Gundy inequality, choosing $C_T > 0$ larger if necessary, we obtain

\[
E \left[ \left| \int_0^T \Delta^{-1/2} u_t^K \sigma_t^K dW_t \right| \right] \leq C_T E \left[ \left( \int_0^T |\Delta^{-1/2} u_t^K \sigma_t^K|^2 dt \right)^{1/2} \right] \\
\leq 4C_T E \left[ |M^\alpha|^2 \sup_{t \in [0,T]} |\sigma_t^K|^2 \left( \int_0^T e^{-\sqrt{\lambda}t} + \Delta^{-\alpha/2} + e^{-2\sqrt{\lambda}(T-t)} dt \right)^{1/2} \right] \\
\leq 2C_T (\Delta^{-\alpha/2} + \Delta^{-1/2})^{1/2} E \left[ |M^\alpha|^2 + \sup_{t \in [0,T]} |\sigma_t^K|^2 \right] \to 0,
\]

as $\lambda \to 0$ and in turn $\Delta \to \infty$. Here, we have used (A.17) for the second inequality. Therefore, the stochastic integral in (A.9) also is of order $o(\sqrt{\lambda})$ in $L^1$, as $\lambda \to 0$ and the proof is complete. \hfill \Box

**References**

[1] R. F. Almgren and N. Chriss. Optimal execution of portfolio transactions. *J. Risk*, 3:5–40, 2001.
[2] P. Bank and D. Baum. Hedging and portfolio optimization in financial markets with a large trader. *Math. Finance*, 14(1):1–18, 2004.
[3] P. Bank, M. Soner, and M. Voß. Hedging with temporary price impact. *Math. Fin. Econ.*, 11(2):215–239, 2017.
[4] P. Bank and M. Voß. Linear quadratic stochastic control problems with singular stochastic terminal constraint. *SIAM J. Control Optim.*, 56(2):672–699, 2018.
[5] D. Bertsimas and A. Lo. Optimal control of execution costs. *J. Fin. Markets*, 1(1):1–50, 1998.
[6] M. Butz and R. Oomen. Internalisation by electronic FX spot dealers. To appear in *Quant. Finance*, 2017.
[7] A. Capponi, A. J. Menkveld, and H. Zhang. The term structure of liquidity: a liquidation game approach. Preprint.
[8] P. Cardaliaguet and C. Rainer. On the (in) efficiency of MFG equilibria. Preprint, 2018.
[9] R. Carmona and K. Webster. The self-financing condition in high-frequency markets. Preprint, 2013.
[10] ´A. Cartea and S. Jaimungal. A closed-form execution strategy to target volume weighted average price. *SIAM J. Fin. Math.*, 7(1):760–785, 2016.
[11] U. Cetin, R. A. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance Stoch.*, 8(3):311–341, 2004.
[12] J.-H. Choi, K. Larsen, and D. Seppi. Smart TWAP trading in continuous-time equilibria. Preprint, 2018.
[13] P. K. Friz and M. Hairer. *A course on rough paths*. Springer, Berlin, 2014.
[14] N. Garleanu, L. Pedersen, and A. Poteshman. Demand-based option pricing. *Rev. Fin. Stud.*, 22(10):4259–4299, 2009.
[15] N. Gärleanu and L. H. Pedersen. Dynamic portfolio choice with frictions. *J. Econ. Theory.*, 165:487–516, 2016.
[16] J. Gatheral, A. Schied, and A. Slynko. Transient linear price impact and Fredholm integral equations. *Math. Finance*, 22(3):445–474, 2012.
[17] D. German. Pricing in an equilibrium based model for a large investor. *Math. Fin. Econ.*, 4(4):287–297, 2011.
[18] S. J. Grossman and M. H. Miller. Liquidity and market structure. *J. Finance*, 43(3):617–633, 1988.
[19] T. Ho and H. R. Stoll. Optimal dealer pricing under transactions and return uncertainty. *J. Fin. Econ.*, 9(1):47–73, 1981.
[20] J. Kallsen. Derivative pricing based on local utility maximization. *Finance Stoch.*, 6(1):115–140, 2002.
[21] D. Kramkov and S. Pulido. Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model. *SIAM J. Fin. Math.*, 7(1):567–587, 2016.
[22] D. Kramkov and S. Pulido. A system of quadratic BSDEs arising in a price impact model. *Ann. Appl. Probab.*, 26(2):794–817, 2016.
[23] A. S. Kyle. Continuous auctions and insider trading. *Econometrica*, 53(6):1315–1335, 1985.
[24] A. W. Lo, H. Mamaysky, and J. Wang. Optimal execution with nonlinear impact functions and trading-enhanced risk. *J. Pol. Econ.*, 112(5):1054–1090, 2004.
[25] R. Martin and T. Schöneborn. Mean reversion pays, but costs. *RISK*, February:96–101, 2011.
[26] J. Muhle-Karbe and K. Webster. Information and inventories in high-frequency trading. *Market Microstructure Liq.*, 3(02):1750010, 2017.
[27] D. Nualart. *The Malliavin calculus and related topics*. Springer, Berlin, 2006.
[28] M. Nutz and J. A. Scheinkman. Shorting in speculative markets. Preprint, 2017.
[29] A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. *J. Fin. Markets*, 16(1):1–32, 2013.
[30] I. Rosu. Fast and slow informed trading. Preprint, 2016.
[31] Y. Sannikov and A. Skrzypacz. Dynamic trading: price inertia and front-running. Preprint, 2016.