Three types of polynomials related to $q$–oscillator algebra

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Abstract

This work addresses a full characterization of three new $q$–polynomials derived from the $q$–oscillator algebra. Related matrix elements and generating functions are deduced. Further, a connection between Hahn factorial and $q$–Gaussian polynomials is established.

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1 Introduction

The $q$–deformed Lie algebras whose applications in the quantum field theory and quantum groups [1] possess an important and useful representation theory in connection to that of their classical limit algebra. The $q$–deformed harmonic oscillator algebra introduced by Arik and Coon [2], and Biedenharn [3] plays a similar role as the usual boson oscillator in nonrelativistic quantum mechanics. This is why various $q$–deformed boson oscillator commutation relations attracted more attention during the last few years [4]–[8]. Furthermore, quantum groups and their representations are closely related to the so-called $q$–calculus: $q$–numbers, $q$–factorials, $q$–differentiation, basic hypergeometric functions, special functions and $q$–orthogonal polynomials. The connection between special functions and group representations was first discovered by Cartan [9] in 1929. Vilenkin made a systematic account of the theory of classical special functions [10], while Koekoek R. et al gave a scheme on the hypergeometric orthogonal polynomials and their $q$–analogues [11]. In this scheme, all polynomials are characterized by a set of properties: (i) they are solutions of difference equations, (ii) they are generated by a

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recursion relation, (iii) they are orthogonal with respect to weight function and (iv) they obey the Rodrigues-type formulas.

Other polynomial families which do not obey the above characteristic properties, do not belong to the Askey $q$–scheme. In this work, we deal with the study of some properties of three types of polynomials: $q$–Gaussian, $q$–factorial and Hahn factorial polynomials.

The paper is organized as follows. In Section 2 we define the $q$–Gaussian polynomials. Matrix elements of the deformed exponential functions are computed and the generating function of the $q$–Gaussian polynomials is deduced. The inversion formula is derived. Section 3 is devoted to the $q$–factorial polynomials. Matrix elements are computed and used to deduce some associated properties. In Section 4 we recall some results on the Hahn calculus, define the Hahn factorial polynomials and compute the matrix elements of the new deformed exponential function $E_{q,\omega}(x)$. A connection between $q$–Gaussian and Hahn factorial polynomials is established. The related inversion formula and generating function are deduced in the Section 5. We end by some concluding remarks in Section 6.

2 $q$–Gaussian polynomials

**Definition 2.1** The $q$–Gaussian polynomials are defined as follows

$$
\phi_n(x) := \prod_{k=0}^{n-1} (x - q^k) = (x - 1)_q^n
$$

$$
= \sum_{k=0}^{n} \binom{n}{k}_q q^{k} (-1)^k x^{n-k}, \quad n \geq 1
$$

(1)

with $\phi_0(x) := 1$ and the $q$–binomials coefficients are given by

$$
\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k} \quad \text{for} \quad 0 \leq k \leq n,
$$

(2)

and zero otherwise,

$$
[n]_q := \frac{1 - q^n}{1 - q}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad (z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k), \quad (z; q)_0 := 1.
$$

(3)

**Proposition 2.1** The $q$–Gaussian polynomials obey the following recursion relation

$$
x \phi_n(x) = \phi_{n+1}(x) + q^n \phi_n(x), \quad \phi_0(x) := 1.
$$

(4)

**Proof.** Multiplying

$$
\binom{n+1}{k}_q = \binom{n}{k}_q + q^{n+1-k} \binom{n}{k-1}_q,
$$

(5)
by \( q^{(k)}(-1)^k x^{n+1-k} \) and summing over \( k = 0, 1, \cdots, n + 1 \), we get

\[
\phi_{n+1}(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} q^{(k)}(-1)^k x^{n+1-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} q^{(k)}(-1)^k x^{n+1-k} + q^{n+1} \sum_{k=0}^{n+1} \binom{n}{k-1} q^{(k)}(-1)^k x^{n+1-k} q^{-k}
\]

\[
= x \sum_{k=0}^{n} \binom{n}{k} q^{(k)}(-1)^k x^{n-k} + q^{n+1} \sum_{k=0}^{n} \binom{n}{k} q^{(k+1)}(-1)^{k+1} x^{n-k} q^{-k-1}
\]

\[
= x \phi_n(x) - q^n \sum_{k=0}^{n} \binom{n}{k} q^{(k)}(-1)^k x^{n-k}
\]

\[
= x \phi_n(x) - q^n \phi_n(x)
\]

which achieves the proof. \( \square \)

**Definition 2.2** Let \( a \) and \( a^\dagger \) be the operators defined as follows:

\[
a = \frac{1 - q^x \partial x}{x(1 - q)}, \quad a^\dagger = (x - 1)q^{-x \partial x}
\]

where \( \partial x := \frac{d}{dx} \) is the ordinary derivative and \( q^{\pm x \partial x} f(x) := f(q^{\pm 1} x) \).

The operators \( a \) and \( a^\dagger \) act on the \( q \)–Gaussian polynomials \( \{ \phi_n(x) \} \) as follows:

**Proposition 2.2**

\[
a \phi_n(x) = [n]_q \phi_{n-1}(x), \quad a^\dagger \phi_n(x) = q^{-n} \phi_{n+1}(x).
\]

\[
\phi_n(x) = q^{n(n-1)/2} (a^\dagger)^n \cdot 1.
\]

Besides,

\[
a a^\dagger \phi_n(x) = q^{-n} [n + 1]_q \phi_n(x), \quad a^\dagger a \phi_n(x) = q^{-n+1} [n]_q \phi_n(x)
\]

and

\[
[a, a^\dagger] \phi_n(x) = q^{-n} \phi_n(x), \quad [a, a^\dagger] q \phi_n(x) = \phi_n(x)
\]

where \( [A, B] := AB - BA, \quad [A, B]_q := AB - q^{-1} BA \).

**Proof.** See appendix A.

Therefore, the set of polynomials \( \{ \phi_n(x) | n = 0, 1, \cdots \} \) provides a basis for a realization of the \( q \)–deformed harmonic oscillator algebra given by

\[
a a^\dagger - a^\dagger a = q^{-N}, \quad aa^\dagger - q^{-1} a^\dagger a = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger
\]

where the operator \( N \) is such that

\[
N \phi_n(x) := n \phi_n(x).
\]
From the Definition 2.2, we deduce the following differential equation

\[
\left( (x - 1)q^{-x^\partial x} D_x^q - [n]_{q^{-1}} \right) \phi_n(x) = 0
\]

(14)

for the \(q\)-Gaussian polynomials where

\[
D_x^q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.
\]

(15)

In order to derive their generating function, let us start defining the following \((q, \mu)\)-exponential function \([12, 13]\)

\[
E_{\mu}^q(x) = \sum_{n=0}^{+\infty} \frac{\mu^n q^n}{[n]_q} x^n, \quad \mu \geq 0, \quad 0 < q < 1.
\]

(16)

In the limit \(q \to 1\), \(E_{\mu}^q(x)\) tends to the ordinary exponential, i.e., \(\lim_{q \to 1} E_{\mu}^q(x) = e^x\), and for some specific values of \(\mu\), it corresponds to the standard \(q\)-exponentials, i.e. for \(\mu = 0\) and \(\mu = 1/2\) one has \([11]\)

\[
E_{0}^q(x) = e^q(x) = \sum_{n=0}^{+\infty} \frac{q^n}{[n]_q} x^n, \quad E_{\frac{1}{2}}^q(x) = e^q(x) = \sum_{n=0}^{+\infty} \frac{q^n}{[n]_q} x^n = (q^{1/2}x; q)_\infty
\]

(17)

and

\[
E_{0}^q(x) E_{\frac{1}{2}}^q(-q^{-1/2}x) = 1.
\]

(19)

Besides, let us introduce the following operator \([12]\)

\[
\mathcal{L}_{\mu, \nu}(\alpha, \beta) = E_{\mu}^q(\alpha a^\dagger) E_{\nu}^q(\beta a).
\]

(20)

In the limit case when \(q \to 1\), it goes into the Lie group element \(\exp(\alpha a^\dagger) \exp(\beta a)\). Then, the matrix elements, in the representation space spanned by the \(q\)-Gaussian polynomials \(\phi_n(x)\), are defined by

\[
\mathcal{L}_{\mu, \nu}(\alpha, \beta) \phi_n(x) = \sum_{r=0}^{+\infty} \mathcal{L}_{\mu, \nu}^{(\alpha, \beta)}(\alpha, \beta) \phi_r(x)
\]

(21)

where \(\mathcal{L}_{\mu, \nu}^{(\alpha, \beta)}(\alpha, \beta)\) is explicitly given in this work by

\[
\mathcal{L}_{n, r}^{(\mu, \nu)}(\alpha, \beta) = \beta^{n-r} q^{\nu(n-r)^2} \left[ \begin{array}{c} n \\ r \end{array} \right]_q U_{\mu, \nu}^{(\alpha, \beta)}(\alpha \beta (q - 1) q^{1+2\nu(n-r)}; q^{1+n-r}|q)
\]

(22)

if \(r \leq n\), and

\[
\mathcal{L}_{n, r}^{(\mu, \nu)}(\alpha, \beta) = \alpha^{-n} \frac{\mu^{(r-n)^2} + (n-r)(n+r-1)}{[r-n]_q} U_{n, \nu}^{(\alpha, \beta)}(\alpha \beta (q - 1) q^{1+2\mu(r-n)}; q^{1+r-n}|q)
\]

(23)
if \( n \leq r \).

The underlying polynomials \( U_n^{(\mu,\nu)}(x; q^{1+\theta}|q) \) are defined by the expression

\[
U_n^{(\mu,\nu)}(x; q^{1+\theta}|q) := \sum_{k=0}^{n} \frac{q^{k^2(\mu+\nu)}(q^{-n}; q)_{k}}{(q^{1+\theta}, q; q)_k} x^k
\]  

(24)

generating the standard \( q \)-polynomials for particular values of \( \mu \) and \( \nu \). Indeed,

1. For \( \mu = 0 = \nu \),

\[
U_n^{(0,0)}(x; q^{1+\theta}|q) = 2\phi_1 \left( \frac{q^{-n}, 0}{q^{1+\theta}} \mid q; x \right) = p_n \left( xq^{-1}; q^\theta, 0|q \right),
\]

(25)

where \( p_n(x; \gamma, \sigma|q) \) is the little \( q \)-Jacobi polynomials [11].

2. For \( \mu = 0, \nu = 1/2 \) or vice-versa,

\[
U_n^{(0,1/2)}(x; q^{1+\theta}|q) = 1\phi_1 \left( \frac{q^{-n}}{q^{1+\theta}} \mid q; -xq^{1/2} \right) = \frac{(q; q)_n}{(q^{1+\theta}, q)_n} L_n^{(\theta)}(-xq^{-n-\theta-1/2}|q)
\]

(26)

where \( L_n^{(\gamma)}(x|q) \) are the \( q \)-Laguerre polynomials [11].

3. For \( \mu = 1/2 = \nu \),

\[
U_n^{(1/2,1/2)}(x; q^{1+\theta}|q) = 1\phi_2 \left( \frac{q^{-n}}{q^{1+\theta}}, 0 \mid q; qx \right).
\]

(27)

Using the relations (8), (9) and the operator (20), we derive the generating function of the \( q \)-Gaussian polynomials.

**Theorem 2.1** The generating function of the \( q \)-Gaussian polynomials satisfies the relation

\[
\frac{(t(1-q); q)_{\infty}}{(tx(1-q); q)_{\infty}} = \sum_{n=0}^{\infty} \frac{\phi_n(x)}{[n]_q!} t^n.
\]

(28)

**Proof.** See Appendix B.

Note that the generating function (28) can be used to find different forms of formulas appearing in this work. From (11), we deduce the inversion formula for the \( q \)-Gaussian polynomials

\[
x^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \phi_k(x)
\]

(29)

**Corollary 2.1**

\[
2\phi_0 \left( q^{-n}, x^{-1} \mid q; xq^n \right) = x^n
\]

(30)

and

\[
\sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q \frac{(-1)^j}{q^{j}} 2\phi_0 \left( q^{-n+j}, 0 \mid q; xq^{n-j} \right) = x^n.
\]

(31)
From the Definition 2.1, the $q-$Gaussian polynomials can be also determined under the form
\[ \phi_n(x) = E_q^{(1/2)} \left( -q^{-1/2} D_x^q \right) x^n. \] (32)

Let $|\psi\rangle := \sum_{n=0}^{\infty} c_n(x)\phi_n(x)$ be the eigenvector of the position operator $X := a^\dagger + a$ with the the eigenvalue $x$, i.e.,
\[ X|\psi\rangle = x|\psi\rangle. \] (33)

Since (33) is satisfied, equating coefficients of the polynomials $\phi_n(x)$ in both the sides of (33), we obtain the following three-term recursion relation for the coefficients $c_n(x)$:
\[ xc_n(x) = [n + 1]_q c_{n+1}(x) + q^{1-n} c_{n-1}(x), \quad n \geq 1 \] (34)
with $c_0(x) := 1$. Since $[n]_q \neq q^{1-n}$, the position operator is not symmetric. Immediately, one can see that
\[ c_{2n}(0) = (-1)^n \frac{q^{n(1-n)}}{[2n]_q !}, \quad c_{2n+1}(0) = 0. \] (35)

As a matter of illustration, we compute the first five coefficients as follows:
\[ c_1(x) = x \] (36)
\[ c_2(x) = \frac{1}{[2]_q} (x^2 - 1) \] (37)
\[ c_3(x) = \frac{1}{[3]_q} \left( x^3 - x (1 + q^{-1}[2]_q) \right) \] (38)
\[ c_4(x) = \frac{1}{[4]_q} \left( x^4 - x^2 (1 + q^{-1}[2]_q + q^{-2}[3]_q) + q^{-2}[3]_q \right) \] (39)
\[ c_5(x) = \frac{1}{[5]_q} \left( x^5 - x^3 (1 + q^{-1}[2]_q + q^{-2}[3]_q + q^{-3}[4]_q) + x (q^{-2}[3]_q + q^{-3}[4]_q + q^{-4}[2]_q [4]_q) \right). \] (40)

3 $q-$Factorial polynomials

**Definition 3.1** The $q-$factorial polynomials are defined as follows:
\[ \hat{\phi}_n(x) = \frac{\Gamma_q(x+1)}{\Gamma_q(x+1-n)} = \prod_{k=0}^{n-1} [x-k]_q, \quad n \geq 1 \] (41)
with $\hat{\phi}_0(x) := 1$.

The $q-$factorial polynomials behave as ordinary monomials under the action of the operators
\[ \hat{a}^\dagger = [x]_q e^{-\partial_x}, \quad \hat{a} = q^{-x-1} (e^{\partial_x} - 1) \] (42)
where $e^{\pm \partial_x} f(x) := f(x \pm 1)$. Indeed, $a$ and $a^\dagger$ are called step operators when they appear in the $q-$deformed quantum theory and they satisfy the following relation.
Proposition 3.1

\[ \hat{a}^\dagger \hat{\phi}_n(x) = \hat{\phi}_{n+1}(x), \quad \hat{a} \hat{\phi}_n(x) = q^{-n}[n] \hat{\phi}_{n-1}(x). \] (43)

**Proof.** By acting \( \hat{a}^\dagger \) on the \( q \)-factorial polynomials \( \hat{\phi}_n(x) \), we have

\[ \hat{a}^\dagger \hat{\phi}_n(x) = [x]_q \hat{\phi}_n(x - 1) \]
\[ = [x]_q \prod_{k=0}^{n-1} [x - (1 + k)]_q \]
\[ = [x]_q [x - 1]_q \cdots [x - n]_q \]
\[ = \hat{\phi}_{n+1}(x). \] (44)

In the same way, by acting \( \hat{a} \) on the \( q \)-factorial polynomials \( \hat{\phi}_n(x) \), one gets

\[ \hat{a} \hat{\phi}_n(x) = q^{-x-1}(\hat{\phi}_n(x + 1) - \hat{\phi}_n(x)) \]
\[ = q^{-x-1} \left( \prod_{k=0}^{n-1} [x + 1 - k]_q - \prod_{k=0}^{n-1} [x - k]_q \right) \]
\[ = q^{-x-1} \left( [x + 1]_q - [x - n + 1]_q \right) \prod_{k=0}^{n-2} [x - k]_q \]
\[ = q^{-n}[n] \hat{\phi}_{n-1}(x) \] (45)

which achieves the proof. \( \square \)

Since (43) holds, we have

\[ \hat{a} \hat{a}^\dagger \hat{\phi}_n(x) = q^{-n-1}[n + 1] \hat{\phi}_n(x), \quad \hat{a}^\dagger \hat{a} \hat{\phi}_n(x) = q^{-n}[n] \hat{\phi}_n(x) \] (46)

and

\[ [\hat{a}, \hat{a}^\dagger] \hat{\phi}_n(x) = q^{-n-1} \hat{\phi}_n(x), \quad [\hat{a}, \hat{a}^\dagger] q \hat{\phi}_n(x) = q^{-1} \hat{\phi}_n(x). \] (47)

Therefore, the set of polynomials \( \{ \hat{\phi}_n(x) | n = 0, 1, \cdots \} \) provides a basis for a realization of the \( q \)-deformed harmonic oscillator algebra given by

\[ \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = q^{-N-1}, \quad \hat{a} \hat{a}^\dagger - q^{-1} \hat{a}^\dagger \hat{a} = q^{-1}, \quad [N, \hat{a}] = -\hat{a}, \quad [N, \hat{a}^\dagger] = \hat{a}^\dagger. \] (48)

Following the previous development for the \( q \)-Gaussian polynomials, the introduction of the operator

\[ \mathcal{L}^{(\mu, \nu)}(\alpha, \beta) = E_q^{(\mu)}(\alpha \hat{a}^\dagger) E_q^{(\nu)}(\beta \hat{a}), \] (49)

leads to the matrix elements, in the representation space spanned by the \( q \)-factorial polynomials \( \hat{\phi}_n(x) \) defined by

\[ \mathcal{L}^{(\mu, \nu)}(\alpha, \beta) \hat{\phi}_n(x) = \sum_{r=0}^{+\infty} \mathcal{L}^{(\mu, \nu)}_{n,r}(\alpha, \beta) \hat{\phi}_r(x) \] (50)
where $\hat{L}_{n,r}^{(\mu,\nu)}(\alpha, \beta)$ are here explicitly given by
\[
\hat{L}_{n,r}^{(\mu,\nu)}(\alpha, \beta) = \beta^{r-n} q^{(r-n)^2 + \frac{r-n(q-n+1)}{2}} \left[ \frac{n}{r} \right] U_n^{(\nu)} (\alpha; q, q^{2(r-n)}; q^{1+n-r} | q) \tag{51}
\]
if $r \leq n$, and
\[
\hat{L}_{n,r}^{(\mu,\nu)}(\alpha, \beta) = \frac{\alpha^{r-n} q^{(r-n)^2}}{[r-n]_q !} U_n^{(\alpha,\nu)} (\alpha; q, q^{2(r-n)}; q^{1+r-n} | q) \tag{52}
\]
if $n \leq r$.

Therefore, the following statement holds.

**Theorem 3.1** The generating function for the $q$–factorial polynomials is defined by
\[
2\phi_0 \left( q^{-x}, 0 \mid q; tq \right) = \sum_{n=0}^{\infty} \frac{\hat{\phi}_n(x)}{[n]_q !} t^n. \tag{53}
\]
or
\[
1\phi_0 \left( q^{-x} \mid q; tq \right) = \sum_{n=0}^{\infty} \frac{q^{(2)} \hat{\phi}_n(x)}{[n]_q !} t^n. \tag{54}
\]

**Proof.** From the definition of the matrix elements (49), we have
\[
\hat{L}^{(\mu,0)}(\alpha, 0).1 = E_{q}^{(\mu)} (\alpha ; q) \tag{55}
\]
\[
\phi_0 \left( q^{-x}, 0 \mid q; tq \right) = \sum_{n=0}^{\infty} \frac{\hat{\phi}_n(x)}{[n]_q !} t^n.
\]

- If $\mu = 0$, we arrive at the generating function of the $q$–factorial polynomials
\[
E_q^{(0)} (\alpha ; q) \tag{56}
\]

By using the identity
\[
\prod_{k=0}^{n-1} [x - k] = (-1)^n q^{n^{2}(x-\frac{1}{2})} \frac{(q^{-x}; q)_n}{(1-q)^n} \tag{57}
\]
we get
\[
\sum_{n=0}^{\infty} \frac{\hat{\phi}_n(x)}{[n]_q !} \alpha^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\frac{1}{2}} (q^{-x}; q)_n}{(q; q)_n} (\alpha q^{x})^n = 2\phi_0 \left( q^{-x}, 0 \mid q; \alpha q^{x} \right). \tag{58}
\]

- If $\mu = 1/2$, (55) takes the form
\[
\hat{L}^{(\mu,0)}(\alpha, 0).1 = \sum_{n=0}^{\infty} \frac{(q^{-x}; q)_n}{(q; q)_n} (-\alpha q^{x+1/2})^n
\]
\[
= 1\phi_0 \left( q^{-x} \mid q; -\alpha q^{x+1/2} \right). \tag{59}
\]

The rest is achieved by setting $t = \alpha q^{1/2}$ on the right hand-side of (59). □
Hahn calculus: Hahn factorial polynomials

Quantum difference operators are receiving an increasing interest in applied mathematics and theoretical physics because of their numerous applications [14]-[18]. Further, the quantum calculus generates the ordinary derivative by a difference operator, which allows us to treat sets of non-differentiable functions. Since Jackson [19] introduced the first expression of difference operator, called the Jackson derivative, several expressions of the difference operator appeared. Among them, the most famous one is Hahn’s difference operator [20], which has two deformation parameters \( \omega \) and \( q \). Hahn’s operator reduces to Jackson’s \( q \)-derivative when the parameter \( \omega \) goes to 0.

4.1 Hahn’s calculus

Definition 4.1 The Hahn’s derivative is defined as follows

\[
D_{q, \omega} f(x) = \begin{cases} 
\frac{f(qx + \omega) - f(x)}{(q-1)x + \omega} & x \neq \omega_0 \\
 f'(x) & x = \omega_0,
\end{cases}
\]  

(60)

where \( \omega_0 = \frac{\omega}{1-q} \).

Theorem 4.1 The Hahn’s derivative satisfies the following deformed Leibniz rule:

\[
D_{q, \omega}(f(x)g(x)) = (D_{q, \omega} f(x))g(x) + f(qx + \omega)D_{q, \omega} g(x)
\]  

(61)

\[
D_{q, \omega} \left( \frac{f(x)}{g(x)} \right) = \frac{(D_{q, \omega} f(x))g(x) - f(x)D_{q, \omega} g(x)}{g(x)g(qx + \omega)}.
\]  

(62)

The proof is straightforward. See Appendix C.

The Hahn integral is introduced as the inverse operation of the Hahn derivative, i.e.,

Theorem 4.2 The Hahn integral is defined by

\[
\int_{x_0}^{x} f(x')d_{q, \omega}x' = ((1-q)x - \omega) \sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q).
\]  

(63)

Proof. If \( D_{q, \omega} F(x) = f(x) \), we have

\[
F(x) - F(qx + \omega) = ((1-q)x - \omega)f(x)
\]  

(64)

\[
F(qx + \omega) - F(q^2x + [2]_q \omega) = ((1-q)qx - q\omega)f(qx + \omega)
\]  

(65)

\[ \vdots \]

\[
F(q^n x + \omega) - F(q^{n+1}x + [n+1]_q \omega) = ((1-q)q^n x - q^n \omega)f(q^n x + [n]_q \omega)
\]  

(66)

Summing (64) to (66), we arrive at

\[
F(x) - F(q^{n+1}x + [n+1]_q \omega) = ((1-q)x - \omega) \sum_{k=0}^{n} q^k f(q^k x + [k]_q \omega).
\]  

(67)
When \( n \to \infty \), the latter expression takes the form

\[
F(x) - F(\omega_0) = \int_{\omega_0}^x f(x')d_{q,\omega}x' = ((1-q)x - \omega) \sum_{k=0}^{\infty} q^k f(q^k x + [k]_q \omega)
\]  

(68)

what achieves the proof. □

4.2 Hahn factorial polynomials

**Definition 4.2** The Hahn factorial polynomials are defined by

\[
\hat{\phi}_n(x) = \prod_{k=0}^{n-1} (x - [k]_q \omega), \quad n \geq 1
\]  

(69)

with \( \hat{\phi}_0(x) := 1 \), satisfying the following recursion relation

\[
x \hat{\phi}_n(x) = \hat{\phi}_{n+1}(x) + \omega [n]_q \hat{\phi}_n(x), \quad n \geq 1, \quad \hat{\phi}_0(x) := 1.
\]  

(70)

**Theorem 4.3** Let \( \hat{a} \) and \( \hat{a}^\dagger \) be the operators defined as follows:

\[
\hat{a} = D_{q,\omega}, \quad \hat{a}^\dagger = xq^{-x\partial x - \omega \partial \omega}e^{-\omega \partial x}
\]  

(71)

where \( e^{\pm \omega \partial x}f(x) := f(x \pm \omega) \). Then,

\[
\hat{a} \hat{\phi}_n(x) = [n]_q \hat{\phi}_{n-1}(x), \quad \hat{a}^\dagger \hat{\phi}_n(x) = q^{-n} \hat{\phi}_{n+1}(x).
\]  

(72)

and

\[
\hat{\phi}_n(x) = q^{n(n-1)/2} (\hat{a}^\dagger)^n. \quad 1.
\]  

(73)

**Proof.**

\[
\hat{a} \hat{\phi}_n(x) = D_{q,\omega} \hat{\phi}_n(x) = \frac{\hat{\phi}_n(qx + \omega) - \hat{\phi}_n(x)}{(q-1)x + \omega}
\]

\[
= \frac{1}{(q-1)x + \omega} \left( \prod_{k=0}^{n-1} (xq + \omega - [k]_q \omega) - \prod_{k=0}^{n-1} (x - [k]_q \omega) \right)
\]

\[
= \frac{1}{(q-1)x + \omega} \left( (qx + \omega)q^n \prod_{k=1}^{n-1} (x - [k-1]_q \omega) - \prod_{k=0}^{n-1} (x - [k]_q \omega) \right)
\]

\[
= \frac{1}{(q-1)x + \omega} \left( (q-1)[n]_q x + \omega [n]_q \prod_{k=0}^{n-2} (x - [k]_q \omega) \right)
\]

\[
= [n]_q \hat{\phi}_{n-1}(x).
\]  

(74)

In the same way, we have

\[
\hat{a}^\dagger \hat{\phi}_n(x) = xq^{-x\partial x - \omega \partial \omega}e^{-\omega \partial x} \hat{\phi}_n(x)
\]

\[
= x \hat{\phi}_n(q^{-1}(x - \omega))
\]
\[
\begin{align*}
= xq^{-n} \prod_{k=0}^{n-1} (x - \omega - q[k]\omega) \\
= xq^{-n} \prod_{k=0}^{n-1} (x - [k+1]\omega) \\
= q^{-n}\phi_{n+1}(x)
\end{align*}
\] (75)

which achieves the proof. □

Since (72) holds, we get
\[\dot{a}\dot{a}^\dagger\phi_n(x) = q^{-n}[n+1]q\dot{\phi}_n(x), \quad \dot{a}^\dagger\dot{a}\phi_n(x) = q^{-n+1}[n]\dot{\phi}_n(x)\] (76)

and
\[\{\dot{a}, \dot{a}^\dagger\}\phi_n(x) = q^{-n}\dot{\phi}_n(x), \quad \{\dot{a}, \dot{a}^\dagger\}q\dot{\phi}_n(x) = \dot{\phi}_n(x)\] (77)

Therefore, the set of polynomials \{\phi_n(x)|n = 0, 1, \cdots\} provides a basis for a realization of the \(q\)-deformed harmonic oscillator algebra given by
\[\dot{a}\dot{a}^\dagger - \dot{a}^\dagger\dot{a} = q^{-N}, \quad \dot{a}\dot{a}^\dagger - q^{-1}\dot{a}^\dagger\dot{a} = 1, \quad [N, \dot{a}] = -\dot{a}, \quad [N, \dot{a}^\dagger] = \dot{a}^\dagger.\] (78)

**Definition 4.3** The Hahn exponential function \(e_{q,\omega}(x)\) is defined as
\[D_{q,\omega}e_{q,\omega}(x) = e_{q,\omega}(x).\] (79)

Thus, we have the following.

**Theorem 4.4**
\[e_{q,\omega}(x) = \frac{e_{q,\omega}(\omega_0)}{\prod_{k=0}^{\infty}(1 + q^k((q-1)x + \omega))}.\] (80)

**Proof.** Since (79) is satisfied, we have
\[\frac{e_{q,\omega}(qx + \omega) - e_{q,\omega}(x)}{(q-1)x + \omega} = e_{q,\omega}(x)\] (81)

equivalent to
\[e_{q,\omega}(x) = \frac{e_{q,\omega}(qx + \omega)}{1 + (q-1)x + \omega} = \frac{e_{q,\omega}(q^2x + (1 + q)\omega)}{(1 + (q-1)x + \omega)(1 + q(q-1)x + q\omega)}\]
\[= \frac{e_{q,\omega}(q^3x + (1 + q + q^2 + q^{s-1})\omega)}{(1 + (q-1)x + \omega)(1 + q(q-1)x + q\omega)(1 + q^2((q-1)x + \omega))}\]
\[= \frac{e_{q,\omega}(q^s x + [s]_{q}\omega)}{\prod_{k=0}^{s-1}(1 + q^k((q-1)x + \omega))},\] (82)
When \( s \to \infty \), \(|q| < 1\), the latter expression takes the form

\[
e_{q,\omega}(x) = \frac{e_{q,\omega}(\omega_0)}{\Pi_{k=0}^{\infty}(1 + q^k((q - 1)x + \omega))}
\]

which achieves the proof. \( \square \)

In the limit when \( \omega_0 \to 0 \), we recover the well known exponential function \( E_q^{(0)}(x) \).

The definition of the matrix elements for the Hahn factorial polynomials requires the construction of the \((q,\omega,\mu)\)-exponential function in the form:

\[
E_{q,\omega}^{(\mu)}(x) = \sum_{n=0}^{+\infty} \frac{q^{n^2}}{(q; q)_n} (1 - q)x - \omega)^n, \quad \mu \geq 0, \quad 0 < q < 1,
\]

(84)

giving, for \( \omega = 0 \), the \((q,\mu)\)-exponential function \( E_q^{(\mu)}((1 - q)x) \) investigated in [12, 13]. For \( \mu = 0 \),

\[
e_{q,\omega}(x) = e_{q,\omega}(\omega_0) E_{q,\omega}^{(0)}(x),
\]

(85)

while in the limit \( q \to 1, \omega = 0 \), \( E_{q,\omega}^{(\mu)}(x) \) tends to the ordinary exponential: \( \lim_{q \to 1} E_{q,0}^{(\mu)}(x) = e^x \).

For \( \omega = 0 = \mu \) and \( \omega = 0, \mu = 1/2 \) one has [11]

\[
E_{q,0}^{(0)}(x/(1 - q)) = e_q(x) = \sum_{n=0}^{+\infty} \frac{1}{(q; q)_n} x^n = \frac{1}{(x; q)_{\infty}}
\]

(86)

\[
E_{q,0}^{(1/2)}(x/(1 - q)) = E_q(q^{-1/2})x = \sum_{n=0}^{+\infty} \frac{q^n(n-1/2)}{(q; q)_n} x^n = (-q^{-1/2}; q)_{\infty}
\]

(87)

satisfying

\[
E_{q,0}^{(0)}(x) E_{q,0}^{(1/2)}(-q^{1/2}x) = 1.
\]

(88)

Introduce the previous operator

\[
\tilde{L}^{(\mu,\nu)}(\alpha, \beta) = E_{q,\omega(\alpha \hat{a}^\dagger)}^{(\mu)}(\alpha \hat{a}^\dagger) E_{q,\omega(\beta \hat{a})}^{(\nu)}(\beta \hat{a})
\]

(89)

going into the Lie group element \( \exp(\alpha \hat{a}^\dagger) \exp(\beta \hat{a}) \) in the limit \( \omega = 0, q \to 1 \). Their matrix elements, in the representation space spanned by the Hahn factorial polynomials \( \hat{\phi}_n(x) \), are defined by

\[
\tilde{L}^{(\mu,\nu)}(\alpha, \beta) \hat{\phi}_n(x) = \sum_{r=0}^{+\infty} \tilde{L}^{(\mu,\nu)}_{n,r}(\alpha, \beta) \hat{\phi}_r(x)
\]

(90)

where \( \tilde{L}^{(\mu,\nu)}_{n,r}(\alpha, \beta) \) is explicitly given by

\[
\tilde{L}^{(\mu,\nu)}_{n,r}(\alpha, \beta) = (\beta(1 - \omega_0))^{n-r} q^{\nu(n-r)} q^{-\nu(n-r)} \left[ \begin{array}{c} n \\ r \end{array} \right] q^{\nu} q^{1+2\nu(n-r)} q^{1+n-r} |q|
\]

(91)
if $r \leq n$, and
\[
\hat{L}_{n,r}^{(\mu,\nu)}(\alpha, \beta) = \frac{\left(\alpha(1 - \omega_0)\right)^{r-n}q^{\mu(r-n)^2+(n-r)(\alpha+r-1)}}{[r-n]_q!} \times U_n^{(\nu,\mu)}(\alpha \beta - 1)(1 + \omega_0)^2q^{1+2\mu(r-n)}; q^{1+r-n}|q)
\] (92)

if $n \leq r$.

In the limit when $\omega_0$ goes to 0, the matrix elements (91) and (92) are reduced to (22) and (23), respectively.

5 Connection between the Hahn factorial and the $q$-Gaussian polynomials

In this section, we establish a connection between the Hahn factorial and the $q$-Gaussian polynomials. The inversion formula and generating function related to the Hahn factorial polynomials are given.

Let us start with an alternative definition of the Hahn factorial polynomials (69) as follows:

**Definition 5.1**
\[
\hat{\phi}_n(x) := \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} q^{k}\omega_0^k(x-\omega_0)^{n-k}, \quad n \geq 1
\] (93)

with $\hat{\phi}_0(x) := 1$ and $\omega_0 = \omega/(1 - q)$, satisfying the recursion relation
\[
(x - \omega_0)\hat{\phi}_n(x) = \hat{\phi}_{n+1}(x) - \omega_0 q^n \hat{\phi}_n(x).
\] (94)

**Proposition 5.1** The connection formula between the Hahn factorial polynomials $\hat{\phi}_n(x)$ (93) and the $q$-Gaussian polynomials $\phi_n(x)$ is given by
\[
\hat{\phi}_n(x) = (-1)^n \omega_0^n \phi_n(1 - x/\omega_0).
\] (95)

**Proof.** Immediately, one can see that
\[
x - \omega[k]_q = -\omega_0 \left(1 - \frac{x}{\omega_0} - q^k\right).
\] (96)

Then, the relation (93) takes the form
\[
\hat{\phi}_n(x) = \prod_{k=0}^{n-1} -\omega_0 \left(1 - \frac{x}{\omega_0} - q^k\right)
= (-1)^n \omega_0^n \phi_n(1 - x/\omega_0).
\] (97)

The rest of the proof is achieved by combining (11) and (95). \hfill \Box

Since (29) is valid, the inversion formula for the Hahn factorial polynomials is
\[
(x - \omega_0)^n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} (-1)^{n-k} \omega_0^{n-k} \hat{\phi}_k(x).
\] (98)
Theorem 5.1 The generating function for the Hahn factorial polynomials is defined by
\[
\frac{(-t\omega; q)_\infty}{(-t((q - 1)x + \omega); q)_\infty} = \sum_{n=0}^\infty \frac{\phi_n(x)}{[n]_q!}t^n.
\] (99)

Proof. From the definition of the matrix elements (90) and the relation (73), we have
\[
\hat{L}^{(\mu, 0)}(\alpha, 0) \cdot 1 = E^{(\mu)}_{q, \alpha \omega \hat{a}^\dagger} (\alpha \hat{a}^\dagger) \cdot 1 = \sum_{n=0}^\infty \frac{q^{\mu n^2 - \frac{n}{2}} \phi_n(x)}{(q; q)_n}(\alpha(1 - q - \omega))^n.
\] (100)

If \( \mu = 1/2 \), we arrive at the generating function of the Hahn factorial polynomials
\[
E^{(1/2)}_{q, \alpha \omega \hat{a}^\dagger} (\alpha \hat{a}^\dagger) \cdot 1 = \sum_{n=0}^\infty \frac{\phi_n(x)}{(q; q)_n}(\alpha(1 - q - \omega)q^{1/2})^n
\]
\[
= \sum_{n=0}^\infty \frac{\phi_n(1 - x/\omega_0)}{[n]_q!}(\alpha \omega_0(\omega_0 - 1)q^{1/2})^n.
\] (101)

The proof is achieved by using (28) and setting \( t = \alpha(1 - \omega_0)q^{1/2} \) on the right-hand side of (101). \( \square \)

6 Concluding remarks

In this work, we have studied three types of \( q \)-polynomials related to the \( q \)-oscillator algebra. Matrix elements of each family of polynomials are computed and the associated generating functions are deduced. Finally, a connection between Hahn factorial and the \( q \)-Gaussian polynomials is established.

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Appendix A

From the definition, we have
\[
a\phi_n(x) = \frac{1 - q^x \partial_x}{x(1 - q)} \prod_{k=0}^{n-1}(x - q^k)
\]
\[
= \frac{1}{x(1 - q)} \left( \prod_{k=0}^{n-1}(x - q^k) - \prod_{k=0}^{n-1}(qx - q^k) \right)
\]
\[
\begin{aligned}
&= \frac{1}{x(1-q)} \left( \prod_{k=0}^{n-1} (x - q^k) - q^n \prod_{k=0}^{n-1} (x - q^{k-1}) \right) \\
&= \frac{1}{x(1-q)} \left( (x - q^{n-1}) - q^n (x - q^{-1}) \right) \prod_{k=0}^{n-2} (x - q^k) \\
&= \frac{1-q^n}{1-q} \phi_{n-1}(x). \\
\end{aligned}
\]

(102)

Similarly,
\[
\begin{aligned}
a^\dagger \phi_n(x) &= (x-1)q^{-x} \prod_{k=0}^{n-1} (x - q^k) \\
&= (x-1) \prod_{k=0}^{n-1} (q^{-1}x - q^k) \\
&= q^{-n}(x-1) \prod_{k=1}^{n} (x - q^k) \\
&= q^{-n} \prod_{k=0}^{n-1} (x - q^k) \\
&= q^{-n} \phi_{n+1}(x)
\end{aligned}
\]

(103)

which achieves the proof. □

**Appendix B**

From the definition of the matrix elements (20), we have
\[
\mathcal{L}^{(\mu,0)}(\alpha,0).1 = E_q^{(\mu)}(\alpha a^\dagger).1 = \sum_{n=0}^{\infty} \frac{q^{\mu n^2 - \binom{n}{2}}}{[n]_q!} \alpha^n \phi_n(x)
\]

(104)

If \( \mu = 1/2 \), we arrive at the generating function of the \( q \)-Gaussian polynomials
\[
E_q^{(1/2)}(\alpha a^\dagger).1 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[n-k]_q [k]_q} (q^{1/2} \alpha)^n x^{n-k}
\]

(105)

By introducing the new summation index \( m = n-k \) on the right-hand side of (105), one obtains
\[
\mathcal{L}^{(1/2,0)}(\alpha,0).1 = c_q(q^{1/2} \alpha x(1-q)) E_q(-q^{1/2} \alpha (1-q)).
\]

(106)

The proof is achieved by taking \( t = \alpha q^{1/2} \) on the right-hand side of (106). □

**Appendix C**

**Proof.** Using the definition,
\[
D_{q,\omega}(f(x)g(x)) = \frac{(f(qx + \omega)g(qx + \omega) - (f(x)g(x))}{(q-1)x + \omega}
\]
In the same way,\[ D_{q,\omega} \left( \frac{f(x)}{g(x)} \right) = \frac{f(qx+\omega) - f(x)}{g(qx+\omega) - g(x)} \frac{g(qx+\omega)}{(q-1)x + \omega} \]
\[ = \frac{g(x)}{(q-1)x + \omega} \frac{(g(qx+\omega) - f(x)}{f(qx+\omega) - f(x)} - \frac{f(x)}{(q-1)x + \omega} \frac{g(qx+\omega) - g(x)}{g(qx+\omega)} \]
\[ = \frac{D_{q,\omega}f(x)}{g(x)g(qx+\omega)} \]
which achieves the proof. □

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