Dual Shapiro-Virasoro amplitudes in the QCD dipole picture

R. Peschanski

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Service de Physique Théorique
CEA-Saclay
F-91191 Gif-sur-Yvette
FRANCE

Abstract

Using the QCD dipole picture of BFKL dynamics and the conformal invariance properties of the BFKL kernel in transverse coordinate space, we show that the $1 \rightarrow p$ dipole densities can be expressed in terms of dual Shapiro-Virasoro amplitudes $B_{2p+2}$ and their generalization including non-zero conformal spins. We discuss the possibility of an effective closed string theory of interacting QCD dipoles.
1 BFKL equation and QCD dipole picture

The QCD “hard Pomeron” is understood as the solution of perturbative QCD expansion at high energy ($W$) after resumming the leading $(\alpha \ln W^2)^n$ terms. It is known to obey the BFKL equation. It has recently attracted a lot of interest in relation with the experimental results obtained at HERA for deep-inelastic scattering reactions at very low value of $x_{Bj} \approx Q^2/W^2$, where $Q^2$ is the virtuality of the photon probe $\gamma^*$ and $W$ is, in this case, the c.o.m energy of the $\gamma^*$-proton system. Interestingly enough, the proton structure functions increase with $W$ at fixed $Q^\ast$ in a way qualitatively compatible with the prediction of the BFKL equation. However, the phenomenological discussion is still under way, since scattering of a “hard” probe on a proton is not a fully perturbative QCD process and moreover, alternative explanations based on renormalization group evolution equations do exist. On the other hand, the phenomenological success of parametrizations based on the BFKL evolution in the framework of the QCD dipole model is quite encouraging for a further study of its properties.

Beyond these phenomenological motivations, there exist interesting related theoretical problems which we want to address in the present paper. In its 2-dimensional version, the BFKL equation expresses the leading-order resummation result for the elastic (off-mass-shell) gluon-gluon scattering amplitude in the $2-d$ transverse plane $f(k_0k_1; k'_0k'_1|Y)$, which depends on the energy $W = e^{Y/2}$ and on the 2-momenta of the incoming and outgoing gluons with $q = k'_0 - k_0 = k'_1 - k_1$ is the transferred 2-momentum. $Y$ is the total rapidity space available for the gluon-gluon reaction. Alternatively, one introduces the coordinate variables via 2-dimensional Fourier transforms and the gluon Green function $f(\rho_0\rho_1; \rho'_0\rho'_1|Y)$ where 2-momentum conservation in momentum space leads to global translational invariance in coordinate space. The amplitude $f(\rho_0\rho_1; \rho'_0\rho'_1|Y)$ is solution of the BFKL equation expressed in the 2-dimensional transverse coordinate space and explicit solutions can be obtained using conformal invariance properties of the BFKL kernel. We will heavily use these symmetry properties in the sequel.

In our paper, we will address the problem of finding the amplitudes $f^{(p)}$ solution of processes involving $2p + 2$ external gluon legs where $\rho_0\rho_1$, $\rho_{a_0}\rho_{a_1}$, $\ldots$, $\rho_{p_0}\rho_{p_1}$ are their arbitrary coordinates in the plane transverse to the initial gluon-gluon direction. Note that $f^{(1)} \equiv f(\rho_0\rho_1; \rho'_0\rho'_1|Y)$ is the origi-
nal BFKL amplitude. Recently, it has been shown that $f^{(1)}$ is equal, up to kinematical factors, to the number density $n_1$ of dipoles existing in the wave-function of an initial quark-antiquark pair (onium) after an evolution “time” $Y$ (such can be interpreted $Y$ in the BFKL equation written as a diffusion process). In the QCD-dipole picture, gluons are equivalent to a $q\bar{q}$ pairs (in the $N_c \to \infty$ limit) which recombine into a collection of independent and colorless dipoles. The elastic amplitude is thus obtained from the elementary dipole-dipole amplitude weighted by the dipole number densities of each initial state obtained after evolution time $Y$. Using conformal invariance properties of the BFKL kernel, it turns out that:

$$n_1^{n,\nu} (\rho_0 \rho_1; \rho'_0 \rho'_1) = \frac{2}{\pi^4 |\rho'_0 / \rho_1|^{n/2}} \int d\omega \frac{\nabla \psi (1)}{\omega - \omega (n, \nu)} f^{n,\nu} (\rho_0 \rho_1; \rho'_0 \rho'_1)$$

(1)

where $n_1^{n,\nu}$ (resp. $f^{n,\nu}$) are the components of the dipole density (resp. the gluon Green’s function) expanded upon the conformally invariant basis, namely:

$$n_1 (\rho_0 \rho_1; \rho'_0 \rho'_1 | Y) = \int d\omega \ e^{\omega Y} \sum_{n \in \mathbb{Z}} \int d\nu \ n_1^{n,\nu} (\rho_0 \rho_1; \rho'_0 \rho'_1) \frac{n_1^{n,\nu} (\rho_0 \rho_1; \rho'_0 \rho'_1)}{\omega - \omega (n, \nu)}$$

(2)

and

$$\omega (n, \nu) = \frac{2\alpha N_c}{\pi} \Re \left\{\psi (1) - \frac{1}{2} \psi (1/2 (1 + n) + i\nu)\right\},$$

(3)

is the value of the BFKL kernel in the (diagonal) conformal basis. The corresponding eigenvectors are explicitly known to be:

$$E^{n,\nu} (\rho_{0\gamma}; \rho_{1\gamma}) = (-1)^n \left(\frac{\rho_{0\gamma} \rho_{1\gamma}}{\rho_{01}}\right)^{\Delta} \left(\frac{\rho_{0\gamma} \rho_{1\gamma}}{\rho_{01}}\right)^{\tilde{\Delta}},$$

(4)

with $\rho_{ij} = \rho_i - \rho_j$ (resp. $\bar{\rho}_{ij} = \bar{\rho}_i - \bar{\rho}_j$) are the holomorphic (resp. antiholomorphic) components in the 2-d transverse plane considered as $\mathbb{C}$ and $\Delta = n/2 + 1/2 - i\nu$, $\tilde{\Delta} = -n/2 + 1/2 - i\nu$ ($n \in \mathbb{Z}$, $\nu \in \mathbb{R}$), are the quantum numbers defining the appropriate unitary representations of the conformal group $SL(2, \mathbb{C})$. Indeed the BFKL solution (i.e., also, the QCD dipole
solution is given by

\[ f^{n,\nu}(\rho_0, \rho_1; \rho_0', \rho_1') = \int_{\mathbb{R}^2} d^2 \rho \overline{E}^{n,\nu}(\rho_{\gamma}, \rho_{1\gamma}) E^{n,\nu}(\rho_0, \rho_1) \],

(6)

which can be explicitly calculated in terms of hypergeometric functions.

In order to generalize these investigations to an arbitrary number of gluons, we shall use the QCD dipole formalism allowing to express the probability of finding \( p \) dipoles in an initial one, i.e. the \( p \)-uple dipole density after an evolution ”time” \( Y \), \( n_p(\rho_0, \rho_1; \rho_0, \rho_1, \rho_0, \rho_1, \ldots, \rho_0, \rho_1 | Y) \). \( n_p \) is the solution of an integral equation which has been proposed in the paper of Ref.\[11\], and approximate solutions have been worked out and applied to problems like the double and triple-QCD Pomeron coupling\[11\], the QCD dipole production\[10\], hard diffraction\[12\] and, more generally, to the unitarization problem\[11\]. In particular, Monte-Carlo simulations of the unitarization series based on a numerical resolution of the \( n_p \) integral equations have been performed\[13\]. However a general expression for the solution of these equations and a physical interpretation of its properties are yet lacking. It is the purpose of our work to provide such a solution, which is intimately related, as we shall see, to dual string amplitudes emerging from the QCD dipole picture.

Our main result is the explicit expression of the \( p \)-uple dipole density distributions in the transverse coordinate plane as dual Virasoro-Shapiro amplitudes\[14, 15\] (for conformal spins all equal to 0). These are the dominant contributions at high \( Y \). We also give the expressions for arbitrary conformal spins (i.e. for all the conformal components) in terms of a well-defined generalization of Shapiro-Virasoro amplitudes.

The paper is organized as follows: In section 2, we derive the QCD dipole equation for \( n_2 \) (for zero conformal spins). The solution is found in a compact form in terms of integrals over explicit conformal eigenvectors. In section 3 we reformulate the obtained 3-dipole vertex in terms of the Koba-Nielsen projective-invariant parametrization of the Shapiro-Virasoro amplitude \( B_6 \). In the following section 4 we show that the solution can be iterated and

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1 There is a controversy\[8\] about the equivalence between the BFKL and the QCD dipole approach concerning a discrepancy in the separate evaluation of real and virtual corrections. In the processes we consider, both virtual and real contributions are included in the BKFL kernel and the conformal symmetry of the QCD dipole model calculations preserves this compensation in the calculations\[8\].
obtain the triple dipole density distribution in terms of $B_8$. This iteration
gives the p-uple dipole distribution in terms of $B_{2p+2}$ integrands. In section
5 we discuss the extension for arbitrary conformal spin and the possibility
of finding a target-space realization of the underlying closed string picture
emerging from the QCD dipole interactions.

2 Solution of the dipole equation for $n_2$

The integral equation satisfied by the dipole pair density $n_2$, see Fig.1, is
written as follows:

$$n_2 (\rho_0 \rho_1; \rho_a \rho_{a_1}; \rho_b \rho_{b_1} | Y) = \frac{\alpha N_c}{\pi} \int_{\mathbb{R}^2} \left| \frac{\rho_{01}}{\rho_{12} \rho_{02}} \right|^2 d^2 \rho_2 \int_0^Y dy \times$$

$$e^{-\frac{2 \alpha N_c}{\epsilon} (Y-y) \ln \frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}}} \times n_1 (\rho_0 \rho_2; \rho_a \rho_{a_1} | y) \times n_1 (\rho_1 \rho_2; \rho_b \rho_{b_1} | y) +$$

$$+ \frac{\alpha N_c}{\pi^2} \int \left| \frac{\rho_{01}}{\rho_{12} \rho_{02}} \right|^2 d^2 \rho_2 \int_0^Y dy \epsilon^{-\frac{2 \alpha N_c}{\epsilon} (Y-y) \ln \frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}}} \times$$

$$\times \frac{1}{2} \left\{ n_2 (\rho_0 \rho_2; \rho_a \rho_{a_1}; \rho_b \rho_{b_1} | y) + (\rho_0 \leftrightarrow \rho_1) \right\}, \quad (7)$$

where the integration domain $|\frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}}| > 1$ avoids the singularity at the
initial dipole end-points $\rho_0, \rho_1$. The physical meaning of this equation is trans-
parent: The probability of finding a pair of dipoles at given transverse coor-
dinates at time $Y$ is made of two terms; The last term in (7) is the survival
probability after $Y$ of two dipoles, whereas the former corresponds to the
probability of creating two new dipoles ($\rho_0 \rho_2$) and ($\rho_1 \rho_2$) which then are sur-
viving till $Y$, each with probability $n_1$. In both cases, the survival probability
is given by the $\epsilon$-dependent Sudakov-like form factor $e^{-\frac{2 \alpha N_c}{\epsilon} (Y-y) \ln \frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}}}$.

In fact, it is possible to reexpress this equation in a way which is
explicitly independent of $\epsilon$ when $\epsilon \to 0$. Multiplying both terms of Eqn.(7)
by $e^{-\frac{2 \alpha N_c}{\epsilon} \ln \frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}}} \left| \frac{\epsilon \rho_{01}}{\epsilon \rho_{12} \rho_{02}} \right|^Y$ and differentiating with respect to $Y$, one gets:

$$\frac{\partial n_2}{\partial Y} (\rho_0 \rho_1; \rho_a \rho_{a_1}; \rho_b \rho_{b_1} | Y) =$$

$$= \frac{\alpha N_c}{\pi} \int_{\mathbb{R}^2} \left| \frac{\rho_{01}}{\rho_{12} \rho_{02}} \right|^2 d^2 \rho_2 \ n_1 (\rho_0 \rho_2; \rho_a \rho_{a_1} | Y) \times n_1 (\rho_1 \rho_2; \rho_b \rho_{b_1} | Y) +$$
\[ \frac{\alpha N_c}{2\pi^2} \int_{\mathbb{R}^2} \{Lip\} d^2\rho_2 \left\{ n_2 (\rho_0 \rho_2; \rho_{a0} \rho_{a1}, \rho_{b0} \rho_{b1} | Y) + (\rho_0 \leftrightarrow \rho_1) \right\}, \tag{8} \]

where \( \{Lip\} \) denotes a parametrization\[11\] of the BFKL kernel in coordinate space, namely

\[ \{Lip\} = \left| \frac{\rho_{01}}{\rho_{12} \rho_{02}} \right|^2 - \frac{1}{2\pi} \ln \left| \frac{\rho_{01}}{\epsilon} \right| \left\{ \delta^{(2)} (\rho_{12}) + \delta^{(2)} (\rho_{02}) \right\}, \tag{9} \]

with the property\[4, 11\]

\[ \int_{\mathbb{R}^2} \{Lip\} d^2\rho_2 E_{n,\nu} (\rho_{01}, \rho_{21}) = \omega(n, \nu) E_{n,\nu} (\rho_{01}, \rho_{11}). \]

In equation (8), the \( \epsilon \to 0 \) limit can be taken without harm, thanks to the non-singular behaviour of the kernel \( \{Lip\} \) after regularization. The kernel \( \{Lip\} \) is conformally invariant\[5\] and its eigenvalues and eigenfunctions are given by equations (3) and (4), respectively.

In order to solve Eqn.(8), we shall expand the function \( n_2 \) on the basis (4) by writing:

\[ n_2 (\rho_0 \rho_1; \rho_{a0} \rho_{a1}, \rho_{b0} \rho_{b1} | Y) = \int_{\mathbb{R}} d\omega e^{i\omega Y} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} d\nu \int_{\mathbb{R}^2} d^2\rho_\gamma \times \]

\[ \times E_{n,\nu} (\rho_{0\gamma}, \rho_{1\gamma}) \bar{E}_{n,\nu} (\rho_{a0} \rho_{a1}, \rho_{b0} \rho_{b1} | \omega), \tag{10} \]

where \( (\omega, Y) \) are conjugate variables and \( \rho_\gamma \) is an auxiliary transverse coordinate variable which labels the set of eigenfunctions \( \bar{E}_{n,\nu} \). The solution of Eqn.(8) is obtained using the known\[3\] orthogonality relations of the \( E_{n,\nu} \) eigenfunctions

\[ \int \frac{d^2\rho_0 d^2\rho_1}{|\rho_{01}|^4} E_{n,\nu} (\rho_{0\gamma}, \rho_{1\gamma}) \bar{E}_{n',\nu'} (\rho_{0\gamma'}, \rho_{1\gamma'}) = \]

\[ = a_{n,\nu} \delta_{n,\nu} \delta_{\nu-\nu'} \delta^{(2)} (\rho_{\gamma\gamma'}) + (-1)^n b_{n,\nu} |\rho_{\gamma\gamma'}| \left( \frac{\rho_{\gamma\gamma'}}{\bar{\rho}_{\gamma\gamma'}} \right)^n \delta_{n,-n} \delta (\nu + \nu'), \tag{11} \]

with \( b_{n,\nu} \) given in ref.(5) and

\[ a_{n,\nu} \equiv \frac{|b_{n,\nu}|^2}{2\pi^2} = \frac{\pi^4/2}{\nu^2 + n^2/4}. \tag{12} \]
Inserting the decomposition (10) into Eqn.(3), we integrate both sides of the equation by
\[
\int dY \ e^{-\omega Y} \frac{d^2 \rho_0 d^2 \rho_1}{|\rho_0|^{4}} \ E^{n,\nu}(\rho_{0\gamma}, \rho_{1\gamma})
\]
(13)
\[
\] one finds using relations (11)
\[
\begin{align*}
n^2_{n,\nu} \left( \rho; \rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1} \right) &= \frac{1}{2a_{n,\nu} \left( \omega(n, \nu) \right)} \times \\
& \times \frac{\alpha N_c}{\pi^2} \int \frac{d^2 \rho_0 d^2 \rho_1 d^2 \rho_2}{|\rho_0 |^{2} |\rho_2 |^{2}} \ E^{n,\nu}(\rho_{0\gamma}, \rho_{1\gamma}) \int dY \ e^{-\omega Y} \\
& \times n_1 \left( \rho_{0\gamma}, \rho_{a0}, \rho_{a1} \right) Y \right) \ n_1 \left( \rho_{0\gamma}, \rho_{b0}, \rho_{b1} \right) Y \right),
\end{align*}
\]
(14)
\[
\] where the normalisation factor takes into account the overcompleteness of the eigenbasis, since \( E^{-n,\nu} \) and \( E^{n,\nu} \) correspond to the same component \( n^2_{n,\nu} \).

Introducing finally the components \( n^2_{n,\nu} \), see (2), for the single density distributions with their corresponding eigenvalues \( \omega(n, \nu) \), \( \omega(n, \nu) \) and eigenvectors \( E_{n,a,\nu}^{n,a,\nu}(\rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1}) \), one writes
\[
\begin{align*}
n^2_{n,\nu} \left( \rho; \rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1} \right) &= \frac{1}{2a(n, \nu) \left( \omega(n, \nu) \right)} \times \\
& \times \sum_{n,\nu} \int \frac{d\nu_a d\nu_b}{a(n, \nu_a) a(n, \nu_b)} \frac{1}{\omega(n, \nu_a) + \omega(n, \nu_b) - \omega(n, \nu)} \\
& \times \int d^2 \rho_0 d^2 \rho_1 d^2 \rho_2 \ E_{n,a,\nu}^{n,a,\nu}(\rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1}) \ E_{n,b,\nu}^{n,b,\nu}(\rho_{b0}, \rho_{b1}) \\
& \times \int d^2 \rho_0 d^2 \rho_1 d^2 \rho_2 \ E^{n,\nu}(\rho_{0\gamma}, \rho_{1\gamma}) \ E_{n,a,\nu}^{n,a,\nu}(\rho_{0a}, \rho_{2a}) E_{n,b,\nu}^{n,b,\nu}(\rho_{1b}, \rho_{2b}),
\end{align*}
\]
(15)
\[
\] where \( \rho_{a} = \rho_{a0} - \rho_{a1} \), \( \rho_{b} = \rho_{b0} - \rho_{b1} \), and \( \rho_{a}, \rho_{b} \) are auxiliary coordinates which play for the dipoles \( (\rho_{0}\rho_{2}) \) and \( (\rho_{1}\rho_{2}) \) the same rôle played by \( \rho_{\gamma} \) for the dipole \( (\rho_{0}\rho_{1}) \), see Fig.1. The expression (15) gives the formal solution of \( n_2 \) in terms of the eigenfunctions of the BFKL kernel which are given in (4).

Let us comment one after the other the 3 different building blocks appearing in (15)
i) The summation $\sum_{n_a, n_b} \int d\nu_a d\nu_b$ is related to the dipoles $(\rho_{a\alpha} \rho_{a\beta})$, $(\rho_{b\alpha} \rho_{b\beta})$ and will be used for inserting $n_2$ in cross-section calculations. It will also restrict the final integration over the quantum numbers $(n, \nu)$ through the poles appearing at $\omega = \omega(n, \nu) = \omega(n_a, \nu_a) + \omega(n_b, \nu_b)$ in the denominators of (15).

ii) The factors $\bar{E}_{n_a, \nu_a} \rho_a \alpha \rho_{a1} \beta$ are the only ones depending on the coordinates $(\rho_{a\alpha} \rho_{a\beta})$ and $(\rho_{b\alpha} \rho_{b\beta})$. Interestingly enough, the subsequent interaction terms involving these dipoles, for instance when computing multi-Pomeron interactions in the QCD dipole picture[13], will be greatly simplified by using factorization and the orthogonality relations (11). These features are already at the root of the derivation[6] of the equivalence (see formula (1)) between the elastic BFKL and QCD-dipole amplitudes.

iii) The last factor in the solution (15), is the triple-dipole vertex $V_{\alpha\beta\gamma}$ in the QCD dipole picture. It depends on the conformal quantum numbers and auxiliary coordinates of the 3 dipoles which are coupled together in $n_2$. It is the purpose of the next sections to give an interpretation of these vertices in terms of dual amplitudes.

3 The triple-dipole vertex as a dual Shapiro-Virasoro amplitude

Let us first consider the vertex $V$ for conformal spins $n_a = n_b = n = 0$. This vertex is physically relevant since it corresponds to the dominant contribution at high $Y$. In fact, the method used for its evaluation will be valid for any conformal spin. Inserting the definitions (4) in the expression of $V_{\alpha\beta\gamma}$ obtained from the solution (15), one gets:

$$V_{\alpha\beta\gamma} = \int_{C^3} \frac{d^2 \rho_0 d^2 \rho_1 d^2 \rho_2}{|\rho_0 \rho_1 \rho_2|^2} \left| \begin{array}{c}
\rho_0 \rho_1 \\
\rho_1 \rho_2 \\
\rho_2 \rho_0 
\end{array} \right| \left[ \begin{array}{c}
-2i\nu-1 \\
-2i\nu-1 \\
-2i\nu-1 
\end{array} \right].$$

Our observation is that $V_{\alpha\beta\gamma}$ can be expressed as follows:

$$V_{\alpha\beta\gamma} = \nu_{\alpha\beta\gamma} \nu_b \ B_6 (\nu, \nu_a, \nu_b),$$

where $B_6$ is a Shapiro Virasoro[14] amplitude with 6 external legs and $\nu_{\alpha\beta\gamma}$ is a known conformally-invariant tensor of the coordinates, i.e. it is completely fixed[16] by the symmetry. In order to derive the expression (17), let
us recall the Koba-Nielsen formulation\cite{17} of $B_6$, namely.

$$B_6 = \int \frac{d^2 \rho_0 d^2 \rho_1 d^2 \rho_2}{|\rho_{\alpha \beta} \rho_{\beta \gamma} \rho_{\gamma \alpha}|^{-2}} \sum_{i<j} |\rho_{ij}|^{-2p_{ij}}, \quad (18)$$

where the integration measure is the conformally-invariant one and the powers $p_{ij}$ correspond to scalar products of external momenta on the target space in the closed string realizations of the Shapiro-Virasoro amplitudes. Note that the string tension can be chosen to be equal to 1 by scale invariance of the solution. In our case, the powers $p_{ij}$ are not specified in terms of external momenta but stringent constraints are required on $p_{ij}$ in order to satisfy the requirements of duality and conformal symmetry\cite{15} namely:

$$\forall i: \quad p_{ii} = -2 \sum_{j=1}^{6} p_{ij} = 0; \quad (19)$$

It is tedious but straightforward to verify that the constraints (19) are verified for the set of $p_{ij}$ corresponding to formula (16) given in Table I. From that, one easily obtains formula (17) with

$$\nu_{\alpha \beta \gamma}^{\nu, \nu_a, \nu_b} = |\rho_{\alpha \beta}|^{2i(\nu - \nu_a - \nu_b) - 1} |\rho_{\beta \gamma}|^{2i(\nu_a - \nu_b - \nu) - 1} |\rho_{\gamma \alpha}|^{2i(\nu_b - \nu_a - \nu) - 1}. \quad (20)$$

Formula (20) does not come as a surprise, since it is the universal form\cite{16} of a 3-point correlation function for field theories obeying global conformal invariance. Indeed it corresponds to a correlation function

$$\langle O^{\Delta}(\rho_\gamma) O^{\Delta_a}(\rho_\alpha) O^{\Delta_b}(\rho_\beta) \rangle$$

where $O^{\Delta}$, $O^{\Delta_a}$, $O^{\Delta_b}$ are so-called “quasi-primary” fields\cite{16} of conformal dimensions $\frac{i}{2} - i\nu_a \frac{i}{2} - i\nu_b \frac{i}{2} - i\nu_b$, respectively. It is interesting to note that such fields already appear in the field-theoretical interpretation of the BFKL equation and its relation to conformal invariance properties\cite{3}, where correlation functions

$$\langle \varphi(\rho_a) \varphi(\rho_b) O^{\Delta}(\rho_\gamma) \rangle \propto E^{n,\nu}(\rho_{a\gamma}, \rho_{b\gamma}) \quad (21)$$

are introduced, with $\varphi$ being a scalar (i.e. of conformal dimension 0) field representing external (reggeized) gluons\cite{4}. In this sense our result (17) can

\footnote{Reggeized gluons are known to have no spin degree of freedom in the high-energy limit of QCD.}
be considered as a QCD dipole model realization of the triple hard-Pomeron vertex. It is a stimulating (but non-trivial) challenge to confront the solution we obtain (in particular Eqns.(15-20)) with the known results\cite{18} on the \((2 \rightarrow 4)\) gluon vertex in the BFKL approach. This would allow to go deeper in the equivalence of BFKL to the QCD dipole approach$^3$.

The result obtained in Eqn.(20) for $\nu_{\alpha\beta\gamma}^{\mu_\alpha\nu_\beta}$ is a consequence of the global conformal invariance of the BFKL equation. Indeed, the $B_6$ function is conformally invariant by construction (provided the constraints (19) are satisfied) and the overall vertex $V_{\alpha\beta\gamma}$ of formula (17) should reflect the general structure$^{16}$

$$\langle \mathcal{O}^{\Delta}(\rho_\gamma)\mathcal{O}^{\Delta a}(\rho_\alpha)\mathcal{O}^{\Delta b}(\rho_\beta) \rangle \equiv \frac{C(\Delta, \Delta_a, \Delta_b)}{|\rho_{\alpha\beta}|^{2(\Delta_a+\Delta_b-\Delta)}|\rho_{\beta\gamma}|^{2(\Delta_b+\Delta-\Delta_a)}|\rho_{\gamma\alpha}|^{2(\Delta_a+\Delta-\Delta_b)}} ;$$

where $C(\Delta, \Delta_a, \Delta_b)$ has to be coordinate-independent while its dependence on the conformal dimensions is not specified by the global $SL(2;\mathbb{C})$ symmetry. This result could be expected for any conformally invariant kernel. By contrast, the specific BFKL kernel $\{L_{ip}\}$ of formula (9) allows the determination of $C(\Delta, \Delta_a, \Delta_b)$ and its relation to the Shapiro-Virasoro amplitude $B_6$.

4 \((1 \rightarrow p)\) dipole vertex

Let us generalize the vertex calculation to the \((1 \rightarrow p)\)-dipole amplitude by considering first the example of $n^{n,\nu}_{3}(\rho_0 \rho_1; \rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1}, \rho_{c0}, \rho_{c1} | \omega)$, see Fig.2. In much the same way as for $n^{n,\nu}_{2}$, one writes:

$$n^{n,\nu}_{3}(\rho_\gamma; \rho_{a0}, \rho_{a1}, \rho_{b0}, \rho_{b1}, \rho_{c0}, \rho_{c1} | \omega) = \frac{1}{2a(n, \nu) (\omega - \omega(n, \nu))} \frac{1}{|\rho_{a0} \rho_{b0} \rho_{c1}|} \times$$

$$\sum_{n_a, n_b, n_c} \int \frac{d\nu_a d\nu_b d\nu_c}{a(n_a, \nu_a) a(n_b, \nu_b) a(n_c, \nu_c)} \frac{1}{\omega(n_a, \nu_a) + \omega(n_b, \nu_b) + \omega(n_c, \nu_c) - \omega}$$

$$\int d^2\rho_a d^2\rho_b d^2\rho_c E^{n_a,\nu_a}(\rho_{a0\alpha}, \rho_{a1\alpha}) E^{n_b,\nu_b}(\rho_{b0\beta}, \rho_{b1\beta}) E^{n_c,\nu_c}(\rho_{c0\delta}, \rho_{c1\delta}) \times \hspace{1cm}$$

$^3$See footnote 1.
\[
\int \frac{d^2\rho_0}{\rho_{01} \rho_{12} \rho_{23} \rho_{30}} \quad E_{n,\nu}(\rho_{01})E_{n_a,\nu_a}(\rho_{12})E_{n_b,\nu_b}(\rho_{23})E_{n_c,\nu_c}(\rho_{30}), \tag{23}
\]

where we have used the fact that the probability of finding three dipoles at \(Y\) can be expressed by two equivalent iterations of the BFKL kernel, namely:

\[
\left| \frac{\rho_{01}}{\rho_{03}} \frac{\rho_{13}}{\rho_{32}} \frac{\rho_{02}}{\rho_{12}} \frac{\rho_{23}}{\rho_{30}} \right|^2 \equiv \left| \frac{\rho_{01}}{\rho_{12}} \frac{\rho_{02}}{\rho_{13}} \frac{\rho_{03}}{\rho_{30}} \right|^2. \tag{24}
\]

Hence, the intermediate step (the \(\rho_0\rho_2\) segment or, equivalently, the \(\rho_1\rho_3\) segment in the case of Fig.2) is not relevant in the final integration kernel.

From the symmetry of (24), it is easy to realize that this iteration procedure will give a symmetric kernel for any number \(p\) of produced dipoles.

The integral over the four eigenfunctions appearing in the last term of expression (23) gives the \(1 \rightarrow 3\) dipole vertex and can be cast in the following form (for zero conformal weights):

\[
V_{(\alpha_i)} = \prod_{i<j}^4 \left| p_{\alpha_i,\alpha_j} \right|^2 \left| \sum_{i,j} (\Delta) - \Delta_i - \Delta_j \right| \times C_8 \left( \rho_{\alpha_\beta,\gamma,\delta}, \rho_{\alpha_\gamma,\beta,\delta}, \rho_{\alpha_\delta,\beta,\gamma} \right), \tag{25}
\]

where the first factor is determined\[16, 19\] by conformal invariance constraints on 4-point correlation functions for quasi-primary fields of conformal dimension \(\Delta_i\). It is a matter of tedious but straightforward verification that the vertex term in equation (23) can be cast into the generic form (25) with:

\[
\Delta = \frac{1}{2} - i\nu, \quad \Delta_a = \frac{1}{2} - i\nu_a, \quad \Delta_b = \frac{1}{2} - i\nu_b, \quad \Delta_c = \frac{1}{2} - i\nu_c. \tag{26}
\]

Indeed the functional dependence (25) is obtained from the constrained relations \(p_{ii} = -2\), \(\sum_{j=1}^8 p_{ij} = 0\), see (19). The corresponding values of \(p_{ij}\) are displayed in table II.

As well-known\[16\] global conformal invariance of 4-point correlation functions does not determine the residual function \(C_8\), as an arbitrary function of the \(SL(2,\mathbb{C})\)-invariant harmonic ratios of the 8 coordinates involved in the problem, see Fig.2. However the explicit integrand (24) induced by the BFKL kernels leads to a well-defined connection with the (Koba-Nielsen form of) the Shapiro-Virasoro \(B_8\) function. Indeed, one may write:

\[
C_8 = \int \prod_{i=1}^8 d\rho_i \left[ \frac{d\rho_\alpha d\rho_\beta d\rho_\gamma d\rho_\delta}{|\rho_{\alpha_\beta,\gamma,\delta}|^2} \right]^{-1} \prod_{i<j}^8 \rho_{ij}^{-2p_{ij}}, \tag{26}
\]

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where the factor $[...]^{-1}$ conventionally means that 4 integration variables out of 8 have to be omitted. Formula (26) can be identified by comparison with the Koba-Nielsen formulation of $B_8$, namely:

$$B_8 = \int \prod_{i=1}^{8} d\rho_i \left[ \frac{d\rho_{\alpha}d\rho_{\beta}d\rho_{\delta}}{|\rho_{\alpha\beta}\rho_{\beta\delta}\rho_{\delta\alpha}|^2} \right]^{-1} \prod_{i<j} \rho_{ij}^{-2p_{ij}} \equiv \int d\rho_\delta \left| \frac{\rho_{\beta\delta}}{\rho_{\gamma\delta}\rho_{\delta\alpha}} \right|^2 C_8, \quad (27)$$

where the three coordinates in the $SL(2, \mathbb{C})$ volume factor have been chosen in order to match with the variables of $C_8$. It is clear from formulae (26,27), that $C_8$ is the $SL(2, \mathbb{C})$-invariant integrand of the Shapiro–Vivasoro amplitude $B_8$.

By simple iteration in the number of produced dipoles, it is not difficult to find the generalization of formulae (23)-(27) to the $(1 \rightarrow p)$ dipole distribution. The key observation to determine the powers $p_{ij}$ appearing in (26) – which are known combinations of conformal dimensions $\Delta_i$ obeying the constraints (19) – is to notice that the coordinates in transverse plane are nearest neighbours along the polygonal perimeter $\rho_0, \rho_1, ..., \rho_p$, (see Fig.2 for the case $p = 3$). The auxiliary points $\rho_{\alpha_i}$ are also connected via nearest neighbours to the previously mentioned perimeter, except for their conformal-dimension factors $p_{\alpha_i\alpha_j}$ which are fixed by the constraints. Indeed, we note that the 4 legs connected to a given coordinate $\rho_i$ in the polygonal perimeter are such that the corresponding sum $\sum p_{ij}$ is always zero. The powers attached to the auxiliary points are fully determined by $SL(2, \mathbb{C})$ invariance (for the set of constraints (19)).

5 A closed string theory for the QCD dipole picture?

The Shapiro-Virasoro amplitudes which are obtained for $(1 \rightarrow p)$ dipole distributions lead naturally to the question of a closed-string interpretation of the high-energy limit of perturbative QCD. Indeed, these amplitudes appear in the context of a closed string moving in a Minkowskian $(1, d-1)$ target space[15]. More generally, such amplitudes appear as a consequence of vertex operator constructions in conformal field theories[19] and are related to the existence of an (anomalous) infinite-dimensional algebra associated with
local conformal invariance, namely the Virasoro algebra. Moreover in the case of a critical target-space dimension $d_c$, the Fock Space on which the quantum string theory is realized is spanned by positive normed states (no ghosts) with full reparametrization invariance. This connection has both a practical and conceptual interest for QCD calculations. First, the many and much explored mathematical properties of dual amplitudes may lead to a simplification of QCD dipole computations for given processes, e.g. “hard” diffraction, multi-Pomeron contributions, etc. Second, there is a possibility of building an effective theory of QCD in the high-energy limit, which could be based on a string theory (instead of a field theory). This would allow the computation of string loop contributions and thus induce an effective theory of interacting QCD Pomerons.

However, the variables which appear as conformal exponents $p_{ij}$ of the integrands are not directly expressed as scalar products of momenta in a Minkowskian $(1,d-1)$ target space. They are complex numbers, see, e.g. (26) obeying constraints which are not directly expressed as on-mass shell and momentum conservation constraints as for the closed string\cite{[15]}. Even if such a target-space interpretation is possible, an analytic continuation in the imaginary direction (implied by the quantum numbers of the conformal eigenvectors (4)) is to be performed. It is thus useful to pass in review the properties of Shapiro-Virasoro amplitudes in this context and to see which are those to be completed for a full closed string theory to be valid.

i) Duality

A first consequence of the solutions (17), (25), and their iterations, is that duality properties exist in the dipole formulation of QCD vertices. Indeed, by construction, Shapiro-Virasoro amplitudes are meromorphic but forbid the existence of multiple pole singularities coming from dual channels. For instance, in Fig.2, the $(\rho_0, \rho_2)$ and $(\rho_1, \rho_3)$ channels cannot both together bring singularities to the amplitude. As usual in dual theories, there exists intricate relations between different ways of describing the amplitudes depending on the series of pole contributions which are chosen for their expansion. An interesting example of such a duality property has been provided by the equivalence of the “t-channel” BFKL elastic 4-gluon amplitude with the “s-channel” QCD dipole description of the same amplitude\cite{[8]}. Further application of this fruitful concept are expected from our results.

ii) Non-zero conformal spins

As a practical consequence of our identification of the multiple-dipole
vertices with integrands of standard Shapiro-Virasoro amplitudes in the case of zero conformal spins, one may use some tools which are developed in the string theoretical formalism to generalize our investigations to the case of general (integer or half-integer) conformal spins as follows; One can consider in general amplitudes of the form:

\[
B_N = \int \prod_{i=1}^{N} d\rho_i \left[ \frac{d\rho_\alpha d\rho_\beta d\rho_\delta}{|\rho_\alpha \beta \rho_\delta \rho_\delta\alpha|^2} \right]^{-1} \prod_{i<j}^{N} \rho_{ij}^{-p_{ij} + \frac{n_{ij}}{2}} \rho_{ij}^{-p_{ij} + \frac{\tilde{n}_{ij}}{2}},
\]  

(28)

where \(n_{ij}, \tilde{n}_{ij}\) are integers\(^4\). Interestingly enough, in the framework of string theory, this corresponds to consider external excited states of the bosonic string\(^20\). Moreover, the same techniques allow to connect closed string to open string tree amplitudes which may allow to extend to the multiple-vertex calculations the conformal-block structure initially identified in the BFKL 4-point amplitudes\(^3\).

\(\text{iii) Extended conformal symmetry and Virasoro algebra}\)

In the seminal paper of Ref.\(^5\), it has been noticed that it was not straightforward to extend the (already beautiful) global conformal symmetry \(SL(2, \mathbb{C})\) to the infinitely dimensional conformal group in 2 dimensions. In other words, only the 6 generators \(\mathbb{L}_{-1}, \mathbb{L}_1, \mathbb{L}_0\) (3 holomorphic and 3 non-holomorphic) of the Virasoro algebra were expected to generate the symmetry algebra of the BFKL kernel. The results we obtain indicate that the algebra can probably be extended to the infinite series of locally conformal generators, i.e. the whole Virasoro algebra, at least in the QCD dipole representation\(^5\), as usual, the symmetry is expected to be anomalous due to the possibility of a central charge\(^15\) (conformal anomaly) at the quantum level of consistency. This issue will depend on the interpretation of a suitable \(p\)–independent target-space representation of the exponents \(p_{ij}\) (see, for \(p = 2, 3, \) Table I and II). For instance, If an embedding \(p_{ij} \to p_i \cdot p_j\) in a Minkowskian \((1, d-1)\) space is allowed, this will determine the central charge to be related to \(d\) and the critical dimension to be \(d_c = 26\), by compensation of the ghost contribution due to reparametrization symmetry\(^15\). This interesting issue deserves certainly more study.

\(^4\)The possibility of extending the calculation of Ref.\(^{20}\) to amplitudes with half-integer conformal-spin has been noticed and indicated to me by H. Navelet, I thank him for this information prior to publication.

\(^5\)see footnote 1.
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Figure 1.

3 - dipole vertex configuration
Figure 2.
4 - dipole vertex configuration
|   | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\rho_\alpha$ | $\rho_\beta$ | $\rho_\gamma$ |
|---|---------|---------|---------|-------------|-------------|-------------|
| $\rho_0$ | -4      | $1-2i\nu$ | $1-2i\nu_a$ | $1+2i\nu_a$ | 0           | $1+2i\nu$  |
| $\rho_1$ | $1-2i\nu$ | -4      | $1-2i\nu_b$ | 0           | $1+2i\nu_b$ | $1+2i\nu$  |
| $\rho_2$ | $1-2i\nu_a$ | $1-2i\nu_b$ | -4      | $1+2i\nu_a$ | $1+2i\nu_b$ | 0           |
| $\rho_\alpha$ | $1+2i\nu_a$ | 0      | $1+2i\nu_a$ | -4          | $1+2i(\nu - \nu_a - \nu_b)$ | $1+2i(\nu_b - \nu_a - \nu)$ |
| $\rho_\beta$ | 0 | $1+2i\nu_b$ | $1+2i\nu_b$ | $1+2i(\nu_b - \nu_a - \nu)$ | -4          | $1+2i(\nu_a - \nu_b - \nu)$ |
| $\rho_\gamma$ | $1+2i\nu$ | $1+2i\nu$ | 0      | $1+2i(\nu_b - \nu_a - \nu)$ | $1+2i(\nu_a - \nu_b - \nu)$ | -4          |

Symmetrix Matrix of $B_6$ powers\{2p_{ij}\}
|     | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\rho_3$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| $\rho_0$ | -4 | $2\Delta + 2$ | 0 | $2\Delta_c + 2$ | 0 | 0 | $-2\Delta_c$ | $-2\Delta$ |
| $\rho_1$ | $2\Delta + 2$ | -4 | $2\Delta_a + 2$ | 0 | $-2\Delta_a$ | 0 | 0 | $-2\Delta$ |
| $\rho_2$ | 0 | $2\Delta_a + 2$ | -4 | $2\Delta_b + 2$ | $-2\Delta_a$ | $-2\Delta_b$ | 0 | 0 |
| $\rho_3$ | $2\Delta_c + 2$ | 0 | $2\Delta_b + 2$ | -4 | 0 | $-2\Delta_b$ | $-2\Delta_c$ | 0 |
| $\rho_\alpha$ | 0 | $-2\Delta_a$ | $-2\Delta_a$ | 0 | -4 | $2/3 \{ 2\Delta_a + 2\Delta_b \}$ | $2/3 \{ 2\Delta_a + 2\Delta_c \}$ | $2/3 \{ 2\Delta_a + 2\Delta \}$ |
| $\rho_\beta$ | 0 | 0 | $-2\Delta_b$ | $-2\Delta_b$ | $2/3 \{ 2\Delta_a + 2\Delta_b \}$ | -4 | $2/3 \{ 2\Delta_b + 2\Delta_c \}$ | $2/3 \{ 2\Delta_b + 2\Delta \}$ |
| $\rho_\gamma$ | $-2\Delta_c$ | 0 | 0 | $-2\Delta_c$ | $2/3 \{ 2\Delta_a + 2\Delta_c \}$ | $2/3 \{ 2\Delta_b + 2\Delta_c \}$ | -4 | $2/3 \{ 2\Delta_c \}$ |
| $\rho_\delta$ | $-2\Delta$ | $-2\Delta$ | 0 | 0 | $2/3 \{ 2\Delta_a + 2\Delta \}$ | $2/3 \{ 2\Delta_b + 2\Delta \}$ | $2/3 \{ 2\Delta_c \}$ | -4 |

Symmetrix Matrix of B₈ powers \(\{2p_{ij}\}\)