LATTICE CONSENSUS: A PARTIAL ORDER ON PHYLOGENETIC TREES THAT INDUCES AN ASSOCIATIVELY STABLE CONSENSUS METHOD

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Abstract. There is a long tradition of the axiomatic study of consensus methods in phylogenetics that satisfy certain desirable properties. One recently-introduced property is associative stability, which is desirable because it confers a computational advantage, in that the consensus method only needs to be computed “pairwise”. In this paper, we introduce a phylogenetic consensus method that satisfies this property, in addition to being “regular”. The method is based on the introduction of a partial order on the set of rooted phylogenetic trees, itself based on the notion of a hierarchy-preserving map between trees. This partial order may be of independent interest. We call the method “lattice consensus”, because it takes the unique maximal element in a lattice of trees defined by the partial order. Aside from being associatively stable, lattice consensus also satisfies the property of being Pareto on rooted triples, answering in the affirmative a question of Bryant et al (2017). We conclude the paper with an answer to another question of Bryant et al, showing that there is no regular extension stable consensus method for binary trees.

1. Introduction

Phylogenetic consensus methods are functions that take a profile of trees with the same set of leaves as an input, and return a single phylogenetic tree on the same leaves as an output. They are widely used because there are many contexts in which many trees are generated that relate the same set of leaves $X$. For instance, if $X$ is a set of species, one might have a set of gene trees obtained by looking at the relationships between copies of the same gene that appears in each element of $X$ [5]. Or one might have a large set of trees on the same set of leaves, obtained as the result of MCMC experiments [7]. Or indeed one might have several trees arising from different data sources, or using different tree construction methods [12]. In each of these cases, methods that provide a consensus tree that attempts to summarize the set of trees are an important tool.

The question of how best to construct a consensus tree has been studied for several decades, and indeed there are many candidate properties that one might want a consensus method to possess. An axiomatic approach to phylogenetic consensus, inspired by approaches such as that of Arrow to voting systems [2], has been actively pursued since the mid-1980s [8] (see [4] for a summary), and continues to yield new insights, methods, and properties [6, 9, 3].

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In this paper we follow \cite{3} in studying regular consensus methods, that satisfy three very generic desirable properties (see Definition 2.1), and two additional properties that are studied in that paper: extension stability and associative stability. Extension stability, first introduced in \cite{11}, is a “future-proofing” property that considers the effect of adding new leaves to the trees in the profile, and sadly is impossible to achieve in a consensus method in general \cite{3}. In this paper we will only touch on this property in order to complete the impossibility result for binary trees (Section 7), and instead focus on associative stability.

Instead of the effect of adding new leaves to the trees in the profile, associative stability considers the effect of adding new trees to the profile. The principle is that the addition of a new tree to the profile ought not have the effect of requiring a reappraisal of the consensus of the original profile (see Section 2 for a formal definition). The effect of associative stability is that if one adds a new tree to the profile, one can obtain the consensus of the enlarged profile just by finding the consensus of two trees: the original consensus tree and the additional tree. This would be a significant computational advantage.

Most known regular phylogenetic consensus methods are not associatively stable. For instance “majority rule” is not (this is the method that makes a consensus tree from the clusters that are in the majority of the trees in the profile), and neither is Adams consensus \cite{1, 3}. Strict consensus on the other hand, is associatively stable, and so is the method that returns the star tree for any profile unless all the trees are equal, in which case it returns that tree.

None of the methods found to be associatively stable in \cite{3} are “Pareto on rooted triples” (see Definition 2.2), and so it was asked whether it was perhaps not possible for a method to be both associatively stable and also Pareto on rooted triples.

The main result of the present paper is the construction of a new consensus method that is both associatively stable and Pareto on rooted triples, and that we call “lattice consensus”, for the following reason. The method is based on a partial order that we define on the set of rooted phylogenetic trees. Essentially, we say \( T_1 \leq T_2 \) if there is a “hierarchy-preserving map” from \( T_1 \) to \( T_2 \). We show that the set of trees less than all elements of a given profile forms a bounded lattice; the maximum element of this lattice is then the consensus tree for the profile.

The partial order on rooted phylogenetic trees defined in this paper appears to be new and may be of independent interest, and so may be the definition of the hierarchy-preserving map. This consensus method is the first known associatively stable method that is Pareto on rooted triples. It can be computed by a fairly straightforward algorithm which we will publish separately. Because of its computational practicality, we hope that it will be useful, and that it will stimulate further investigation.

The paper is structured as follows. In Section 2 we provide background terminology for consensus methods, and derive some new results on associative stability in general (including some observations on dictator trees). In Section 3 we introduce a class of “hierarchy-preserving maps” between trees, that preserve features of the hierarchies given by the trees, and in Section 4 we use the existence of such maps to define a partial order on the set of rooted phylogenetic trees. We restrict this partial order to the set of trees below a given profile of trees in Section 5, and show that
once restricted it forms a bounded lattice. The new consensus method is introduced in Section 6 and shown to be associatively stable and Pareto on rooted triples (Theorems 6.3 and 6.4). We end with a short section answering in the negative another question left open in [3], about the existence of regular extension stable consensus methods when the profile is restricted to binary trees (Section 7).

2. Associatively Stable Consensus Methods

Let $X$ be a finite set, and let $RP(X)$ denote the set of rooted phylogenetic trees on $X$. A profile of trees is an ordered tuple $(T_1, \ldots, T_k) \in RP(X)^k$. Note, we do not assume the trees are binary.

**Definition 2.1.** A consensus method is a function $\varphi$ that, for every finite set $X$ and $k \geq 1$, associates with each profile of $k$ trees from $RP(X)$ a corresponding tree in $RP(X)$. We call a consensus method regular [3] if it obeys the following three properties:

1. **Unanimity:** If the trees in $P$ are all the same tree $T$, then $\varphi(P) = T$.
2. **Anonymity:** Changing the order of the trees in $P$ does not change $P$.
3. **Neutrality:** Changing the labels on the leaves of the trees in $P$ simply relabels the leaves of $\varphi(P)$ in the same way.

Unanimity reflects the natural assertion that if all evidence points to a particular tree on $X$, then the data are best represented by that tree. It additionally rules out useless consensus methods, such as $\varphi(P) = T$ for all $P$, given some $T$. Anonymity means that the order of trees in the profile does not affect the consensus method, preventing additional unrealistic consensus methods such as returning the first tree in the profile every time. Neutrality reflects the condition that the labels should not affect the outcome - if you swap cat and dog in every tree in the profile, the only outcome should be that cat and dog are swapped in the consensus tree.

Some further desirable properties for consensus methods are associative stability and Pareto on rooted triples.

**Definition 2.2.** Let $\varphi$ be a consensus method. If, for any profile $P$, each tree $T \in P$ displaying $ab|c$ implies that $\varphi(P)$ displays $ab|c$, then $\varphi$ is termed Pareto on rooted triples.

The property of being Pareto on rooted triples reflects the intuition that if a particular rooted triple is represented in every tree in the profile, it is considered unlikely to happen by chance, so ought to be represented in the consensus tree.

**Definition 2.3.** Let $\varphi$ be a consensus method on $RP(X)$. If, for all $k \geq 1$, and every profile $P = (T_1, \ldots, T_k) \in RP(X)^k$, we have

$$\varphi(\varphi(P - T_k), T_k) = \varphi(P),$$

then $\varphi$ is referred to as associatively stable.

Associative stability is a computationally desirable property, as the consensus method may be computed by comparing each tree in a pairwise manner. It also means that if an additional tree is found after calculating the consensus tree, one
does not need to perform the whole operation again, and can just find the consensus of the old output tree and the newly-found tree.

We now consider associatively stable, regular consensus methods in general, deriving some results about dictator trees.

**Definition 2.4.** Let $\varphi$ be a consensus method on $RP(X)$ and $T$ some tree on $X$. If $T \in P$ implies $\varphi(P) = T$ for any profile $P$, $T$ is referred to as a dictator tree on $RP(X)$.

Clearly, if a dictator tree exists it is unique, as if $T_1$ and $T_2$ are both dictator trees then take $P = (T_1, T_2)$ and $\varphi(P) = T_1$ as $T_1$ is a dictator, and $\varphi(P) = T_2$ as $T_2$ is a dictator, whence $T_1 = T_2$. Recall the following theorem.

**Theorem 2.5** ([3], Lemma 5). If $\varphi$ is an associatively stable, regular consensus method, then $\varphi(P)$ depends only on the set of trees present in $P$ and not on their frequency.

**Theorem 2.6.** Let $\varphi$ be a regular consensus method on $RP(X)$ with a dictator tree $T_D$ on $RP(X)$. Then $T_D$ is the star tree.

**Proof.** Seeking a contradiction, suppose $T_D$ is not the star tree. Then there exists an order 2 permutation $\sigma$ such that $T_\sigma \neq T_D$ — if you take some proper cluster $C$ of $T_D$, let $a \in C$ and $b \notin C$ and the permutation $(a \ b)$ fits the requirements. Now, let $P = (T_D, T_\sigma)$. As $\varphi$ is neutral, $\varphi(P) = \varphi(P) = T_D$. But $P_\sigma = (T_D, T_D)$, so $\varphi(P_\sigma) = T_D$ as $T_D$ is a dictator. But this is a contradiction, as we stated $T_D \neq T_D$. It follows that $T_D$ must be the star tree. □

**Theorem 2.7.** Let $\varphi$ be a regular, associatively stable consensus method on $RP(X)$. Then $\varphi$ must have a dictator tree, specifically the star tree.

**Proof.** Let $P$ be an ordered tuple whose entries are every tree in $RP(X)$, and suppose $\varphi(P) = T_D$. Then we claim $T_D$ is a dictator tree. In fact, for any tree $T'$:

\[
\varphi(T_D, T') = \varphi(P, T') = \varphi(P) = T_D.
\]

By anonymity and associativity, $T_D$ is a dictator tree, and by Theorem 2.6 it must be the star tree. □

### 3. Hierarchy-preserving maps

In this section we introduce hierarchy-preserving maps on the set of trees $RP(X)$. These are used to define a partial order on $RP(X)$ in Section 4.

Recall the following standard definitions in phylogenetics (see for example [10]).

**Definition 3.1.** A hierarchy $H$ on a set $X$ is a collection of subsets of $X$ with the following properties:

1. $H$ contains both $X$ and all singleton sets $\{x\}$ for $x \in X$.
2. Let $H_1, H_2$ be a pair of subsets of $H$. Then $H_1 \cap H_2 = \emptyset$, $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. 

Definition 3.2. Let $T \in RP(X)$ be a tree and $v$ be a vertex of $T$. Then a cluster of $T$ associated with $v$ is the subset of $X$ consisting of the descendants of $v$ in $T$. The clusters other than $X$ and the singletons are referred to as the proper clusters.

A collection of subsets of $X$ is a hierarchy if and only if it is the set of clusters of some rooted phylogenetic tree $T$ taken over all vertices of $T$ (see [10] for instance). For this reason we refer to the set of clusters of $T$ as the hierarchy of $T$, denoted $H(T)$.

Definition 3.3. Let $T_1, T_2 \in RP(X)$ with hierarchies $H(T_1)$ and $H(T_2)$. Then $\delta : H(T_1) \to H(T_2)$ is a hierarchy-preserving map if $\delta$ is the identity on singletons and $X$, and the following properties hold for all $A, B \in H(T_1)$:

1. Semi-Injective: $\delta(A) = \delta(B)$ implies either $A = B$ or $A \cap B = \emptyset$,
2. Enveloping: $A \subseteq \delta(A)$, and
3. Subset-Preserving: $A \subset B$ implies $\delta(A) \subset \delta(B)$.

Example 3.4. Let $T_1, T_2 \in RP(X)$ where $X = \{a, b, c, d, e, f\}$ as depicted in Figure 1. The hierarchies consist of the proper clusters $\{ab, cd, abcd\}$ and $\{abcd, abcde\}$ respectively. Then there exists a hierarchy-preserving map $\varphi$ from $H(T_1)$ to $H(T_2)$ that is the identity on singletons and $X$, maps $ab$ and $cd$ to $abcd$ and maps $abcd$ to $abcde$. One can easily confirm the necessary properties hold.

![Figure 1](image.png)

Figure 1. A pair of trees $T_1$ and $T_2$ with a hierarchy-preserving map from $H(T_1)$ to $H(T_2)$ that maps $ab$ and $cd$ to $abcd$, and maps $abcd$ to $abcde$.

4. A PARTIAL ORDER ON THE SET OF TREES

In this section we use the hierarchy-preserving maps just introduced, to define a partial order $\leq_{HP}$ on $RP(X)$. We then explore a lattice structure on the set of trees less than a profile, using this order.

We say $T \leq_{HP} T'$ if there is a hierarchy-preserving map from $H(T)$ to $H(T')$.

Theorem 4.1. The set $RP(X)$ forms a poset under the relation $\leq_{HP}$.

Proof. The observation that the identity map from the hierarchy of a tree to itself is a hierarchy-preserving map gives reflexivity, and the transitivity of hierarchy-preserving maps is also easy to check. It remains to show antisymmetry.

Suppose $T_1 \leq_{HP} T_2$ and $T_2 \leq_{HP} T_1$. Then there exist hierarchy-preserving maps $\varphi_1 : H(T_1) \to H(T_2)$ and $\varphi_2 : H(T_2) \to H(T_1)$.
Take $C_1 \in H(T_1)$, and suppose $\varphi_1(C_1) = D_1$ and $\varphi_2(D_1) = C_2$ so that $C_1 \subseteq D_1 \subseteq C_2$. Then we can consider the sequences $C_1 \subseteq C_2 \subseteq \ldots$ and $D_1 \subseteq D_2 \subseteq \ldots$, where $\varphi_1(C_i) = D_i$ and $\varphi_2(D_i) = C_{i+1}$. As $X$ is finite, eventually there is some $i$ for which $\varphi_1(C_i) = D_i = \varphi_1(C_{i+1})$, or $\varphi_2(D_i) = C_{i+1} = \varphi_2(D_{i+1})$. But since $C_i \subseteq C_{i+1}$ and $D_i \subseteq D_{i+1}$, by semi-injectivity it follows that $C_{i+1} = C_i$ or $D_{i+1} = D_i$.

Without loss of generality, suppose $C_i = C_{i+1}$, which implies that $D_i = \varphi(C_i) = \varphi(C_{i+1}) = D_{i+1}$. Then, as both $\varphi_1$ and $\varphi_2$ are enveloping, $D_i \subseteq C_{i+1} = C_i$ and $C_i \subseteq D_i$, so $D_i = C_i$.

We now consider $D_{i-1}$, which is mapped to $C_i$. As $D_{i-1}$ and $D_i$ both map to $C_i$, the semi-injectivity of $\varphi_2$ implies $D_i = D_{i-1}$, because their intersection is non-empty (indeed $D_{i-1} \subseteq D_i$). Continuing in this manner, we see that all of the $C_i$’s are equal and all of the $D_i$’s are equal, so $C_i \subseteq D_1$ and $D_i \subseteq C_i$, and hence $C_i = D_i$. It follows that $H(T_1) = H(T_2)$ and thus $T_1 = T_2$, giving antisymmetry, and completing the proof. \hfill \qed

For several results in the remainder of this section, and the next, we will show that a tree $T$ with certain properties is not maximal under $\leq_{HP}$. In general, this will be done by constructing another tree $T'$ and showing that there exists a hierarchy-preserving map from $T$ into $T'$. The tree we construct will be a “binding” of $T$.

**Definition 4.2.** Let $T \in RP(X)$, and let $A_1, \ldots, A_m \in H(T)$ (with $m \geq 2$) be maximal subclusters of a cluster $C \in H(T)$ such that $A = \cup_{i=1}^m A_i \neq C$. Take $H(T)$, delete all $A_i$ for which $|A_i| > 1$ from $H(T)$, and add $A$, forming a new set of clusters, $$\mathcal{H} := (H(T) \setminus \{A_i \mid |A_i| > 1\}) \cup \{A\}.$$ Then $\mathcal{H}$ is a hierarchy (see Lemma 4.3), and the corresponding tree is termed a binding of $T$ at $A$, and denoted $T_A^C$.

**Example 4.3.** Let $X = \{a, b, c, d, e, f, g, h\}$ and let $T \in RP(X)$ correspond to the hierarchy with proper clusters $ab, abc, de, abdefg$. Let $A = abde$, $B = abdef$ and $C = abcede fg$. Then the binding of $T$ at $A$, denoted $T_A^C$, is the tree on $X$ corresponding to the hierarchy with proper clusters $ab, abde, abde f g$. The binding of $T$ at $B$, denoted $T_B^C$, is the tree on $X$ corresponding to the hierarchy with proper clusters $ab, abdef, abcede fg$; specifically, note that we do not delete $f$ as it is a singleton and the result would no longer be a hierarchy. These three trees can be seen in Figure 2.

**Lemma 4.4.** Let $T \in RP(X)$, and suppose $A_1, \ldots, A_m$ are maximal subclusters of some cluster $C \in H(T)$, where $C \neq A = \cup A_j$. Then the binding of $T$ at $A$ is a hierarchy. Moreover, if $T_A^C$ is the corresponding tree, then $T \leq_{HP} T_A^C$, with the inequality strict if $m > 1$.

**Proof.** In a minor abuse of notation, let $H(T_A^C)$ be the set of clusters corresponding to the binding of $T$ at $A$. To confirm that $H(T_A^C)$ is a hierarchy, it suffices to check that any $M \in H(T_A^C)$ for which $M \cap A \neq \emptyset$ is either contained in or contains $M \cap A$.

If $M \cap A$ is non-empty, then $M \cap A_j$ is non-empty for some $j$. Hence in $H(T)$, as $A_j$ is maximal in $C$ it follows that $M$ either contains $C$ (and so contains $A$), or is a subset of $A_j$ (and thus is contained in $A$). Thus $H(T_A^C)$ is a hierarchy.
Figure 2. Two potential bindings of the tree $T$, as described in Example 4.3, with $A = abcd$, $B = abcdef$, and $C = abcdefg$.

The map from $H(T)$ to $H(T_C^A)$ that is the identity on all clusters except for $A_1, \ldots, A_m$, which are all mapped to $A$, is clearly hierarchy-preserving. □

Lemma 4.5. Let $T, T' \in RP(X)$ with a hierarchy-preserving map $\delta : T \to T'$. Suppose $A$ and $B$ are maximal subclusters of some third cluster $C$ in $H(T)$, where $C \neq A \cup B$ and $\delta(A) \subseteq \delta(B)$. Then the binding $T^C_{A \cup B} \leq_{HP} T'$.

Proof. By Lemma 4.4, we know that the described set of clusters $H(T^C_{A \cup B})$ is a hierarchy. We need to show that there exists a hierarchy-preserving map $\delta' : H(T^C_{A \cup B}) \to H(T')$. Noting that all clusters in $H(T^C_{A \cup B})$ other than $A \cup B$ are also clusters in $H(T)$, for any cluster $M \in H(T^C_{A \cup B})$, define

$$\delta'(M) = \begin{cases} \delta(M), & \text{if } M \neq A \cup B \\ \delta(B), & \text{if } M = A \cup B. \end{cases}$$

We first show that $\delta'$ is semi-injective. Suppose $Y$ and $Z$ are two clusters in $H(T^C_{A \cup B})$ satisfying $\delta'(Y) = \delta'(Z)$. We need to show that either $Y = Z$ or $Y \cap Z = \emptyset$.

If neither $Y$ nor $Z$ are equal to $A \cup B$, then the definition of $\delta'$ implies $\delta(Y) = \delta(Z)$. The semi-injectivity of $\delta$ then implies that $Y = Z$ or $Y \cap Z = \emptyset$, as required.

Suppose on the other hand, without loss of generality, that $Z = A \cup B$, with $\delta'(Y) = \delta'(A \cup B)$. Then by definition of $\delta'$, we have $\delta'(Y) = \delta(Y)$ and $\delta'(Z) = \delta(B)$, and so $\delta(Y) = \delta(B)$. Therefore, since $\delta$ is a hierarchy-preserving map, we have either $Y = B$ or $Y \cap B = \emptyset$. However, $Y$ is a cluster in $H(T^C_{A \cup B})$, while $B$ is not, forcing us to conclude that $Y \cap B = \emptyset$.

This rules out $Y = Z$, since $Z = A \cup B$, and so we aim to show that $Y \cap (A \cup B) = \emptyset$. If this were not the case, then we must have $Y \cap A \neq \emptyset$, since $Y \cap B = \emptyset$. If
If  $A \subseteq B$, then $A$ is a maximal cluster in $C$, and therefore $B \subseteq Y$, which contradicts $Y \cap B = \emptyset$. But if $Y \subseteq A$, then $\delta(Y) \subseteq \delta(A)$ by subset-preservation, and this implies $\delta(B) \subseteq \delta(A)$, contradicting the assumption in the lemma statement. Therefore, we have $Y \cap A = \emptyset$, which completes the proof of semi-injectivity.

To verify that $\delta'$ is enveloping, note that it suffices to check that $A \cup B \subseteq \delta'(A \cup B)$, since all other clusters have this property due to $\delta$ being enveloping. We have $B \subseteq \delta(B)$ and $A \subseteq \delta(A)$ thanks again to $\delta$ being enveloping, and $\delta(A) \subseteq \delta(B)$ by the assumption in the lemma statement. Therefore $A$ and $B$ are both contained in $\delta(B)$, and so $A \cup B \subseteq \delta(B) = \delta'(A \cup B)$, as required.

Finally, we check subset preservation. For  $Y$ and $Z$ clusters in $H(T_{A \cup B}^C)$, we need to check $Y \subseteq Z$ implies $\delta'(Y) \subseteq \delta'(Z)$. If neither are equal to $A \cup B$, then this follows immediately from the definition of $\delta'$ and the properties of $\delta$. It remains to check the two cases: (i) $Y = A \cup B \subseteq Z$, and (ii) $Y \subseteq A \cup B = Z$.

In the first case, $A \cup B \subseteq Z$ implies $C \subseteq Z$, because $A$ and $B$ are maximal subclusters of $C$. Then $\delta'(A \cup B) = \delta(B)$ by definition of $\delta'$, and $\delta(B) \subseteq \delta(C) \subseteq \delta(Z)$ because $\delta$ is subset-preserving and $B, C, Z$ are all clusters in $H(T)$. Finally noting that $\delta'(Z) = \delta(Z)$ completes this case.

In the second case, $Y \subseteq A \cup B$ implies $Y \subseteq A$ or $Y \subseteq B$ because $Y, A, B$ are all part of a single hierarchy, $H(T)$. Assuming without loss of generality that $Y \subseteq A$, we have $\delta'(Y) = \delta(Y) \subseteq \delta(A)$ by subset-preservation of $\delta$, $\delta(A) \subseteq \delta(B)$ by assumption of the lemma, and $\delta(B) = \delta'(A \cup B)$ by definition of $\delta'$. Therefore $\delta'(Y) \subseteq \delta'(A \cup B) = \delta'(Z)$, as required. \hfill \Box

**Lemma 4.6.** Let $T, T' \in \text{RP}(X)$ with a hierarchy-preserving map $\delta : T \to T'$. Suppose $A_1, \ldots, A_n$ are maximal subclusters of some cluster $C$ in $H(T)$, where $C \neq \bigcup A_j$. Suppose further that $\delta(A_j) \cap \delta(A_k) = \emptyset$ for each pair $j, k$ and that there is some cluster $D \subseteq \delta(C)$ in $H(T')$ that contains $\bigcup \delta(A_j)$. Then the binding $T_{A}^C \leq_{HP} T'$, with $\leq_{HP}$ strict if $m > 1$.

**Proof.** By Lemma 4.4 we know that the described set of clusters $H(T_{A \cup B}^C)$ is a hierarchy. We need to show that there exists a hierarchy-preserving map $\delta' : H(T_{A \cup B}^C) \to H(T')$. Noting that all clusters in $H(T_{A \cup B}^C)$ other than $A \cup B$ are also clusters in $H(T)$, for any cluster $M \in H(T_{A \cup B}^C)$, define

$$\delta'(M) = \begin{cases} 
\delta'(M), & \text{if } M \neq A \cup B \\
D, & \text{if } M = \bigcup A_j.
\end{cases}$$

This map is immediately enveloping, so we just need to check semi-injectivity and subset-preservation, but these follow from the same logic as Lemma 4.4. \hfill \Box

Note that in the $m = 1$ case we do not change any of the clusters of $H(T)$, but are changing the image of $A_1$ to a larger cluster.

We will make use of the notion of a “maximal vertical subhierarchy”, as defined below.
**Definition 4.7.** Let $T \in RP(X)$. If $C_1, \ldots, C_k$ are clusters in $H(T)$ with the property that $C_1 \subset \cdots \subset C_k$ and there are no other clusters $D$ for which $C_i \subset D \subset C_{i+1}$, then we say $\{C_1, \ldots, C_k\}$ is a maximal vertical subhierarchy of $H(T)$.

We finish this section with a result describing the maximal elements under this partial order (noting that the minimal element is the star tree).

**Proposition 4.8.** The set of maximal elements of $(RP(X), \leq_{HP})$ is precisely the set of binary trees.

**Proof.** First, if a tree is non-binary, then its hierarchy has a cluster with at least three maximal subclusters. Therefore, by Lemma 4.3 we can bind two of them to create a tree that is strictly greater in the partial order. So non-binary trees are not maximal.

Second, if two trees $T$ and $T'$ are binary and there is a hierarchy-preserving map between them, they must be equal, as follows.

Let $\varphi : H(T) \to H(T')$ be a hierarchy-preserving map. Observe that $\varphi$ maps $X$ to $X$, and let $Y$ be a cluster of $T$ such that every element of the maximal vertical subhierarchy of $Y$ is fixed under $\varphi$. As $T$ and $T'$ are binary, $Y$ has two maximal subclusters in each of $H(T)$ and $H(T')$. Let $C_1$ and $C_2$ be the maximal clusters of $Y$ in $H(T)$, and $D_1$ and $D_2$ be the maximal clusters of $Y$ in $H(T')$. As $\varphi$ is subset-preserving, $C_1$ and $C_2$ must each be mapped to some subcluster of $D_1$ and $D_2$. As $\varphi$ is enveloping, this implies that each of $C_1$ and $C_2$ are subsets of $D_1$ or $D_2$. Additionally, $C_1 \cup C_2 = Y$, which is only possible if $C_1 = D_1$ and $C_2 = D_2$ or $C_1 = D_2$ and $C_2 = D_1$. It follows that $\varphi$ is the identity on all elements of $H(T)$, so $T = T'$.

**5. The partial order on trees below a given profile**

In this section we restrict the partial order defined in Section 4 to the set of trees below every element of a profile of trees $P$. While the poset $(RP(X), \leq_{HP})$ is not a bounded lattice (there is not a unique least upper bound for every pair of trees), it turns out that this restricted poset is, and this is crucial for allowing us to define a consensus method in Section 6.

If $P = (T_1, \ldots, T_k)$ is a profile of trees, then the set of trees $T$ for which there exists a hierarchy-preserving map $\delta_i : T \to T_i$ for each $i$ is denoted by $HP(P)$. In other words,

$$HP(P) := \{ T \in RP(X) \mid T \leq_{HP} T_i {\text{ for all }} T_i \in P \}.$$ 

In particular, observe that if $T_1$ is the star tree $S$ or $T_1 = T_2$ then a hierarchy-preserving map from $T_1$ to $T_2$ will always exist, namely the identity map on clusters in $T_1$. Therefore $HP(P)$ is always non-empty, as it will certainly contain $S$. We further note that if $P$ consists of the single tree $T$, then $HP(P)$ can immediately be seen to be a bounded lattice, with least element $S$ and greatest element $T$, as every element of $HP(P)$ has a hierarchy-preserving map into $T$ by definition. It follows that if $P = (T_1, \ldots, T_k)$, then $HP(P)$ forms the poset obtained by taking the intersection of the bounded lattices corresponding to each tree in $P$. Later we will show that this intersection is also a bounded lattice (Corollary 5.7).
Definition 5.1. Let $P = (T_1, \ldots, T_k) \in RP(X)^k$ and let $T \in HP(P)$ with a set of hierarchy-preserving maps $\delta_i : T \to T_i$. If a cluster $C \in H(T)$ satisfies $C \not= \bigcap_{i=1,\ldots,k} \delta_i(C)$, then $C$ is termed an unfulfilled cluster. A cluster that is not unfulfilled is called fulfilled.

Note that for any tree in $HP(P)$ there may be many alternative sets of hierarchy-preserving maps into the trees in $P$, and we say a cluster is unfulfilled if there is at least one of these choices satisfying the relevant property.

The term unfulfilled is intentionally evocative of the fact that we are potentially ‘wasting’ elements in the cluster. If there is some element in every image of $\delta_i(C)$, then it seems intuitively plausible that we could add it to $C$. This is in fact the case, as we will soon see (Theorem 5.3). Note also that $X$ and the singletons are always fulfilled clusters.

Example 5.2. Let $X = \{a, b, c, d, e, f\}$ and let $T \in RP(X)$ be a tree on $X$ with proper cluster $ab$. Let $P = (T_1, T_2)$ where $T_1$ and $T_2$ have the proper clusters $abcd$ and $abce$ respectively. Then there exist hierarchy-preserving maps $\delta_1 : T \to T_1$ and $\delta_2 : T \to T_2$, where $\delta_1(ab) = abcd$ and $\delta_2(ab) = abce$. Then $ab$ is an unfulfilled cluster of $T$ as $ab \subset \varphi_1(ab) \cap \varphi_2(ab) = abc$. In this case we can easily see that $T^+$, the tree with proper cluster $abc$, has the property that $T \leq_{HP} T^+$, as $T$ was ‘wasting’ the $c$.

We now link the notions of unfulfilled clusters (Definition 5.1) and maximality of trees under $\leq_{HP}$.

Theorem 5.3. Let $P = (T_1, \ldots, T_k) \in RP(X)^k$, and let $T \in HP(P)$. If there is an unfulfilled cluster of $T$, then $T$ is not maximal in $(HP(P), \leq_{HP})$.

Proof. Suppose that $C_1$ is an unfulfilled cluster of $T$ that is maximal with respect to being unfulfilled. That is, all clusters except $C_1$ in the maximal vertical subhierarchy $C_1, C_2, \ldots, C_n, X$ are fulfilled. Let $\{\delta_i : T \to T_i \mid i = 1, \ldots, k\}$ be a set of hierarchy-preserving maps such that $C_1 \not= \bigcap_{i=1,\ldots,k} \delta_i(C_1)$ (such a set exists because $C_1$ is unfulfilled). Note that $C_1$ is contained in the intersection of all its images $\delta_i(C_1)$, that is,

$$C_1 \subseteq C = \bigcap_{i=1,\ldots,k} \delta_i(C_1),$$

because $C_1 \subseteq \delta_i(C_1)$ for all $i \in \{1, \ldots, k\}$.

We now consider two cases. Either: (1) there is some cluster $M \in H(T)$ such that $M \cap C \neq \emptyset$, but $M$ is not a subset of $C_1$ or a superset of $C_2$; or (2) the only clusters that intersect with $C$ are either contained in $C_1$ or contain $C_2$.

For Case (1), suppose that there is some cluster $M \in H(T)$ such that $M \cap C \neq \emptyset$, but $M$ is not a subset of $C_1$ or a superset of $C_2$. Observe that since $C_2$ is fulfilled, $C_2 = \bigcap \delta_i(C_2)$, and therefore $C \subset C_2$, as $\bigcap \delta_i(C_2)$ always contains $C = \bigcap \delta_i(C_1)$. Hence $M$ and $C_2$ intersect non-trivially, because $M$ and $C$ intersect non-trivially. It follows that $M \subset C_2$, because they are in the same hierarchy $H(T)$, and $M$ is not a superset of $C_2$. 
Without loss of generality, suppose $M$ is a maximal subset of $C_2$ in $H(T)$, and let $\mathcal{D} = C_1 \cup M$. By Lemma 4.3 we know that the binding $T_{D_2}^{C_2}$ is a hierarchy and that there exists a hierarchy-preserving map from $H(T)$ to $H(T_{D_2}^{C_2})$, so that $T \leq_{HP} T_{D_2}^{C_2}$.

To finish the proof, it suffices to check that $T_{D_2}^{C_2} \in HP(P)$, which requires showing that there exists a hierarchy-preserving map $H(T_{D_2}^{C_2}) \rightarrow H(T_i)$ for each $i$.

Observe that $\delta_i(M)$ and $\delta_i(C_1)$ have non-empty intersection, as $M$ intersects with $\delta_i(C_1)$ and $\delta_i$ is enveloping. As $\delta_i(M)$ and $\delta_i(C_1)$ are in the same hierarchy, therefore either $\delta_i(M) \supset \delta_i(C_1)$ or $\delta_i(M) \subseteq \delta_i(C_1)$. Define the map $\delta'_i : H(T_{D_2}^{C_2}) \rightarrow H(T_i)$, by letting

$$\delta'_i(A) = \begin{cases} 
\delta_i(A), & \text{if } A \neq \mathcal{D}, \\
\delta_i(M), & \text{if } A = \mathcal{D} \text{ and } \delta_i(M) \supset \delta_i(C_1), \text{ or} \\
\delta_i(C_1), & \text{if } A = \mathcal{D} \text{ and } \delta_i(M) \subseteq \delta_i(C_1).
\end{cases}$$

Then each $\delta'_i$ is hierarchy-preserving by Lemma 4.5. It follows that $T_{D_2}^{C_2}$ is in $HP(P)$ and thus $T$ is not maximal in $HP(P)$.

Now suppose for Case (2) that the only clusters that intersect with $\mathcal{C}$ are either contained in $C_1$ or contain $C_2$. Then the tree $T^+$ corresponding to the hierarchy obtained by taking $H(T)$, and replacing $C_1$ with $\mathcal{C}$ has the property that $T \leq_{HP} T^+$, which can be seen by considering the hierarchy-preserving map from $T$ to $T^+$ that is the identity on all elements except for $C_1$, which is mapped to $\mathcal{C}$. This is immediately semi-injective and enveloping, and as $C$ and $C_1$ both have the same containment relations with all other elements of $H(T)$, it is subset-preserving as well.

Now we show that $T^+$ is in $HP(P)$ by considering the map $\delta'_i : H(T^+) \rightarrow H(T_i)$ defined by

$$\delta'_i(A) = \begin{cases} 
\delta_i(A), & \text{if } A \neq \mathcal{C}, \text{ and} \\
\delta_i(C_1), & \text{if } A = \mathcal{C}.
\end{cases}$$

By assumption, no cluster of $H(T)$ that can be mapped to $\delta_i(C_1)$ intersects with $\mathcal{C}$, so we get semi-injectivity immediately. Secondly, $\mathcal{C}$ is the intersection of all $\delta_i(C_1)$ and thus a subset of $\delta_i(C_1)$, so certainly $\delta'_i$ is enveloping. Finally, as $\mathcal{C}$ and $C_1$ both have the same containment relations with all other elements of $H(T)$, it follows that $\delta'_i$ is subset-preserving. Therefore $\delta'_i$ is a hierarchy-preserving map, so $T^+$ is in $HP(P)$ and it follows that $T$ is not maximal in $HP(P)$ under $\leq_{HP}$.

\begin{theorem}
Let $P = (T_1, \ldots, T_k) \in RP(X)^k$. Then $(HP(P), \leq_{HP})$ contains a unique maximal tree. We denote this tree $T_P$.
\end{theorem}

\textbf{Proof.} First note that $HP(P)$ is not empty, because at the least it contains the star tree.

Suppose there are two maximal trees $T, T' \in (HP(P), \leq_{HP})$, and for each, we choose sets of hierarchy-preserving maps

$$\varphi_i : H(T) \rightarrow H(T_i) \quad \delta_i : H(T') \rightarrow H(T_i).$$

There is a cluster $C$ with a maximal vertical subhierarchy $C = C_1 \subseteq C_2 \subseteq \cdots \subseteq X$ common to both trees, such that the maximal subclusters of $C$ in $T$ and $T'$ are
different (noting that we could have $C = X$). Call these sets of maximal subclusters of $C$, $G$ and $H$ respectively.

We can assume that $\varphi_i(C_j) = \delta_i(C_j)$ for each $i$ and each cluster $C_j$, because if there were some $C_j$ with $\varphi_i(C_j) \neq \delta_i(C_j)$, we can replace $\varphi_i(C_j)$ with $\tilde{\varphi}_i(C_j) := \delta_i(C_j)$ while maintaining the hierarchy-preserving nature of $\varphi_i$, due to repeated application of the single cluster case of Lemma 4.6. In other words, it is possible to choose sets of hierarchy-preserving maps into $P$ for $T$ and $T'$ that have this property.

Because $G$ and $H$ are distinct sets, there is a cluster $G \in G$ that intersects non-trivially with more than one element of $H$. Let $\{H_j \mid G \cap H_j \neq \emptyset\}$ be this set of clusters in $H$, and note that

$$G \subseteq \bigcup_j H_j.$$  

Claim: There is a choice of $\{\varphi_i\}$ for which we have $\bigcup_j H_j \subseteq \varphi_i(G)$ for each $i$.

Proof of Claim: For each $j$ and any choice of $\{\varphi_i\}$, we know that $\varphi_i(G) \cap \delta_i(H_j) \neq \emptyset$ because $G \cap H_j \neq \emptyset$ and $\varphi_i$ and $\delta_i$ are enveloping. Therefore, for each $j$, either $\varphi_i(G) \subseteq \delta_i(H_j)$ or vice versa, because they are both in the same hierarchy $H(T_i)$.

If $\delta_i(H_j) \subseteq \varphi_i(G)$ for every $j$, then we have $\bigcup_j \delta_i(H_j) \subseteq \varphi_i(G)$, and so $\bigcup_j H_j \subseteq \varphi_i(G)$ because $\delta_i$ is enveloping, which proves the Claim. So we need to show that there is a set $\{\varphi_i\}$ for which $\delta_i(H_j) \subseteq \varphi_i(G)$ for every $j$.

So suppose on the other hand that there is a $j$ for which $\varphi_i(G) \cap \delta_i(H_j)$. If any other $H_k$ satisfies $\varphi_i(G) \cap \delta_i(H_k)$, then $\delta_i(H_k)$ and $\delta_i(H_j)$ are nested, since they are elements of the same hierarchy and have non-trivial intersection. It follows that those $\delta_i(H_j)$ that contain $\varphi_i(G)$ are totally ordered by inclusion. Since all other $\delta_i(H_j)$ are subsets of $\varphi_i(G)$, we have that the set $\{\delta_i(H_j)\}$ has a unique maximal element, say $\delta_i(H_m)$, ordered by set containment. Changing the choice of $\varphi_i$ to $\tilde{\varphi}_i(G) = \delta_i(H_m)$ retains the property of hierarchy-preserving, since $\tilde{\delta}_i(H_m) \subseteq \delta_i(C)$, and the other maximal clusters of $C$ are disjoint from $G$ (so they may also map to $\delta_i(H_m)$ without violating the definition of hierarchy-preserving). This gives a choice of $\{\varphi_i\}$ for which $\delta_i(H_j) \subseteq \varphi_i(G)$ for every $j$, as required.

This completes the proof of the Claim.

Using this choice of $\{\varphi_i\}$, we now have $\bigcup_j H_j \subseteq \varphi_i(G)$ for each $i$, and therefore

$$\bigcup_j H_j \subseteq \bigcap \varphi_i(G),$$

which implies $\bigcup_j H_j \subseteq G$, since $G$ is fulfilled (it must be fulfilled since $T$ is maximal, by Theorem 5.3). It then follows from Equation (1) that

$$G = \bigcup_j H_j.$$
Consider the binding of $T'$ using $G = \bigcup H_j$, that is, $(T')^G_C$. This binding satisfies
$T' \leq_{HP} (T')^G_C$, by Lemma 4.4. Define a set of maps $\delta'_i : H((T')^G_C) \rightarrow H(T_i)$ by

\[ M \mapsto \begin{cases} 
\delta_i(M) & \text{if } M \neq G \\
\varphi_i(M) & \text{if } M = G.
\end{cases} \]

This is hierarchy-preserving, by Lemma 4.6, contradicting the maximality of $T'$, and completing the proof of the Theorem. \qed

**Example 5.5.** Let $P = (T_1, T_2, T_3)$ be the profile of trees depicted in Figure 3, with non-trivial clusters $\{abef\}$, $\{ab, cd, abcd\}$ and $\{abcd, abcde\}$ respectively. We shall derive $T_P$ for this profile. Consider a tree $T'$ such that $H(T')$ has at least one proper cluster, and a hierarchy-preserving map into the hierarchies corresponding to all three trees in $P$ (if no such tree exists then $T_P$ will be the star tree). In order to have a hierarchy-preserving map $\varphi_1$ that maps into $H(T_1)$, as $\varphi_1$ is enveloping it follows that $H(T')$ can only have proper clusters that are non-singleton subsets of $abef$. But we can immediately see that the only non-singleton subcluster of $abef$ that can be mapped into clusters in $H(T_2)$ and $H(T_3)$ is $ab$. It follows that $T' = T_{ab}$, the tree with proper cluster $ab$. As the only trees in $HP(P)$ are the star tree $S$ and the non-star tree $T_{ab}$ and there certainly exists a hierarchy-preserving map from $H(S)$ into $H(T_{ab})$, it follows that $T_P = T_{ab}$ (shown in Figure 3).

![Figure 3](image.png)

**Figure 3.** A profile of trees $P = (T_1, T_2, T_3)$, together with the unique maximal element $T_P$ of $HP(P)$, $T_{ab}$, the tree with proper cluster $ab$.

**Theorem 5.6.** For $P \in RP(X)^k$, $(HP(P), \leq_{HP})$ forms a bounded lattice in $RP(X)$.

**Proof.** We need to show that each pair of elements in $(HP(P), \leq_{HP})$ has a unique least upper bound and unique greatest lower bound in $HP(P)$.

Let $T_1$ and $T_2$ be two elements of $HP(P)$ (note that they may not necessarily be elements of $P$). A greatest lower bound of $T_1$ and $T_2$ is a tree that has a hierarchy-preserving map into both $T_1$ and $T_2$, and that is maximal under this condition. We therefore consider $HP(T_1, T_2)$, which has a unique maximal element by Theorem 5.4 and is certainly an element of $HP(P)$. It follows that $T_1$ and $T_2$ have a unique greatest lower bound in $HP(P)$. 

\[ \text{□} \]
A least upper bound of $T_1$ and $T_2$ is a tree $T_U \in HP(P)$ for which $T_1, T_2$ both have a hierarchy-preserving map into $T_U$, and that is minimal under this condition. We note that an upper bound for $T_1$ and $T_2$ must exist, because every element of $HP(P)$ has a hierarchy-preserving map into $T_P$ at the very least, and therefore there must be minimal such upper bounds.

Suppose there were two such least upper bounds for $T_1$ and $T_2$, namely $T_U$ and $T_U'$. We then consider $HP(T_U, T_U')$. Certainly $T_1$ and $T_2$ are contained in $HP(T_U, T_U')$, and as there is not a hierarchy-preserving map from $T_U$ to $T_U'$ or vice versa (by their minimality), the maximal element $T_P$ of $HP(T_U, T_U')$ is not equal to either of them. However, by Theorem 5.6, $T_1$ and $T_2$ both have a hierarchy-preserving map into $T_P$, and $T_P$ has a hierarchy-preserving map into both $T_U$ and $T_U'$. This is a contradiction, as we assumed $T_U$ and $T_U'$ were both least upper bounds for $T_1$ and $T_2$. It follows that there is a unique least upper bound for $T_1$ and $T_2$, and therefore that $HP(P)$ is a lattice.

Finally, note that $HP(P)$ is bounded, since it has a greatest element (Theorem 5.4), and a least element (the star tree).

This implies an interesting property of the intersections of the lattices $HP(P_1)$ and $HP(P_2)$, for $P_1$ and $P_2$ two profiles of trees in $RP(X)$.

**Corollary 5.7.** Let $P_1$ and $P_2$ be two profiles from $RP(X)^{k_1}$ and $RP(X)^{k_2}$ respectively. Then the intersection of the two bounded lattices $(HP(P_1), \leq_{HP})$ and $(HP(P_2), \leq_{HP})$ is itself a bounded lattice under the same partial order.

**Proof.** Let $P^+ = (P_1, P_2)$. Then $HP(P^+)$ consists of the trees in $RP(X)$ that have a hierarchy-preserving map into all elements of $P_1$ and all elements of $P_2$, that is, it is the intersection of the bounded lattices $(HP(P_1), \leq_{HP})$ and $(HP(P_2), \leq_{HP})$. By Theorem 5.6 $(HP(P^+), \leq_{HP})$ is itself a bounded lattice.

This is a surprising property; it is certainly not true in general that the intersection of two bounded lattices is a bounded lattice (see Example 5.8).

**Example 5.8.** The intersection of bounded lattices of sets under subset containment is not necessarily a bounded lattice. For instance, the bounded lattice of subsets of $Y = \{w, x, y, z\}$, ordered by containment, has bounded sublattices $L_1$, the subsets of $\{w, x, y\}$ ordered by inclusion, and $L_2$, the subsets of $Y$ containing an even number of elements, ordered by inclusion. Then $L_1 \cap L_2$ is the set $\{\emptyset, \{w, x\}, \{w, y\}, \{x, y\}\}$ ordered by inclusion, which has three maximal elements and is therefore not a bounded lattice. Hence the intersection of two bounded lattices is not necessarily a bounded lattice.

6. **AN ASSOCIATIVELY STABLE CONSENSUS METHOD THAT IS PARETO ON ROOTED TRIPLES**

The previous Section has shown that the presence of a hierarchy-preserving map between two trees defines a partial order on the set of trees, and that the set of trees less than all the elements of a given profile $P$ has a unique maximal element $T_P$. We now use this to define a consensus method and derive some of its properties.
Definition 6.1 (Lattice consensus). Let $\psi : \bigcup_{k \geq 1} RP(X)^k \to RP(X)$ be given by $\psi(P) = T_P$, the unique maximal element of $HP(P)$. We call this lattice consensus.

We will now show that lattice consensus is regular (Definition 2.1), Pareto on rooted triples (Definition 2.2) and associatively stable (Definition 2.3).

Theorem 6.2. Lattice consensus is regular.

Proof. Certainly $\psi$ obeys unanimity, as if all trees in $P$ are $T$, then $T \in HP(P)$, as we can take $\delta_i$ to be the identity map on clusters in $T$. Then, by definition, every tree in $HP(P)$ has a hierarchy-preserving map into $T$, so $T_P = T$.

Further, $\psi$ obeys anonymity, as changing the order of trees in $P$ does not change the existence of a hierarchy-preserving map into them.

Finally, observe that if $T^\alpha$ is the tree $T$ with some permutation $\alpha$ rearranging the leaves, and $T$ has a hierarchy-preserving map into some tree $T_i$, then $T^\alpha$ has a hierarchy-preserving map into $T_i^\alpha$, whence $\psi$ obeys neutrality. □

Theorem 6.3. Lattice consensus is Pareto on rooted triples.

Proof. Let $P = (T_1, \ldots, T_k)$ be a profile of trees in $RP(X)^k$, and let $\psi(P) = T_P$.

Suppose $ab|c$ is a triple in all trees of $P$. That is, there exists a cluster in the hierarchy of each tree in the profile that contains $a$ and $b$ but not $c$. Each hierarchy $H(T_i)$ must have a maximum such cluster $C_i$ since by the definition of hierarchies, for any pair of clusters containing both $a$ and $b$, one must be a subset of the other.

Let $D$ be a minimal cluster of $H(T_P)$ that contains $a$, $b$ and $c$. We now consider how $a$, $b$ and $c$ are partitioned in the maximal clusters of $D$; the possibilities being that all are in the same cluster, two are in the same cluster and one a different cluster, or all three are in different clusters. We will show that the only possibility is that, specifically, $a$ and $b$ share a cluster while $c$ is in a second cluster. First note that $a$, $b$, $c$ cannot be in the same subcluster of $D$ as this contradicts the minimality of $D$.

Suppose that $a$ and $b$ are not in the same maximal subcluster of $D$, so that $A$ and $B$ are maximal clusters of $D$ with $a \in A$ and $b \in B$. Seeking a contradiction, suppose $c \in A$. Let $\delta_i$ be a hierarchy-preserving map from $H(T_P)$ to $H(T_i)$. Then each $\delta_i(A)$ must also contain $a$ and $c$, because $\varphi_i$ is enveloping, and therefore each $C_i \subset \delta_i(A)$ (because they are both clusters in the same hierarchy $H(T_i)$ with non-empty intersection, but $\delta_i(A)$ contains $c$ while $C_i$ does not). But this argument implies that $b \in \delta_i(A)$ for all $i$, and thus that $A$ is an unfulfilled cluster. This contradicts the maximality of $T_P$, by Theorem 5.3. It follows that $A$ cannot contain $c$, and by symmetry, that $B$ cannot contain $c$ either.

The only possibilities left are that $a$, $b$ are in one maximal subcluster of $D$ and $c$ another, or that $a$, $b$, $c$ are each in different maximal subclusters. We will show that the latter is impossible.

Suppose that $a$, $b$, $c$ are in distinct maximal subclusters of $D$, with $a \in A$ and $b \in B$. As $H(T_i)$ is a hierarchy, observe that either $\delta_i(A)$ and $\delta_i(B)$ are disjoint and both are contained in $C_i$, or one of $\delta_i(A)$ and $\delta_i(B)$ is contained in the other.
In either case, though, we claim that $T_P \leq_{HP} T^D_{P,A}$, where $T^D_{P,A}$ is a binding of $T_P$ at $A = A \cup B$. As neither $A$ nor $B$ contain $c$, it follows that $A \subseteq D$, so by Lemma $4.3$ we know that $H(T^D_{P,A})$ is a hierarchy and that there exists a hierarchy-preserving map from $H(T_P)$ to $H(T^D_{P,A})$. Noting that $T_P \neq T^D_{P,A}$, we have $T_P <_{HP} T^D_{P,A}$ as required.

Finally we claim that $T^D_{P,A} \in HP(P)$, which will contradict the maximality of $T_P$ in $HP(P)$.

For each $i = 1, \ldots, k$, define $\delta'_i : H(T^D_{P,A}) \to H(T_i)$ as follows:

$$\delta'_i(M) = \begin{cases} 
\delta_i(M), & \text{if } M \neq A, \\
\delta_i(A), & \text{if } M = A \text{ and } \delta_i(A) \subseteq \delta_i(B), \\
\delta_i(B), & \text{if } M = A \text{ and } \delta_i(B) \subset \delta_i(A) \\
C_i, & \text{if } M = A, \delta_i(A) \cap \delta_i(B) = \emptyset.
\end{cases}$$

By Lemma $4.5$ and Lemma $4.6$ $\delta'_i$ is hierarchy-preserving and so $T^D_{P,A} \in HP(P)$, which implies that $T_P$ is not maximal in $P$, a contradiction. It follows that the maximal clusters of $D$ in $T_P$ have $a, b$ in one cluster and $c$ in another, so $T_P$ displays $ab|c$. Thus $\psi$ is Pareto on rooted triples, as required.

**Theorem 6.4.** Lattice consensus is associatively stable.

**Proof.** Let $P = (T_1, \ldots, T_{k-1})$ be a profile of trees and $T_k$ a tree, all from $RP(X)$. Suppose $\psi(P) = T_P$. That is, $T_P$ is the maximal tree for which there exists a hierarchy-preserving map $\delta_i$ from the hierarchy $H(T_P)$ into all $H(T_i)$ for $i < k$. Now, suppose $\psi(T_P, T_k) = T$, and $\psi(T_1, \ldots, T_k) = T'$. To prove the theorem it suffices to show that $T = T'$.

Because $\psi(T_P, T_k) = T$, there are hierarchy-preserving maps $\gamma_1 : H(T) \to H(T_P)$ and $\gamma_2 : H(T) \to H(T_k)$. Thus there exists a hierarchy preserving map $\gamma_1 \circ \delta_i$ from $H(T)$ to all $H(T_i)$ for $i < k$, which together with the existence of $\gamma_2$ implies $T \in HP(T_1, \ldots, T_k)$.

Now, observe that as $H(T')$ has a hierarchy-preserving map into each of $H(T_i)$, for $1 \leq i \leq k$, $T'$ is in $HP(P)$, and so $H(T')$ has a hierarchy-preserving map into $H(T_P)$ (since it certainly maps into $T_1, \ldots, T_{k-1}$, and $T_P$ is maximal in $HP(P)$). As there are hierarchy-preserving maps from $H(T)$ into both $H(T_P)$ and $H(T_k)$, it follows that $T' \in HP(T_P, T_k)$.

As $T \in HP(T_1, \ldots, T_k)$, there exists a hierarchy-preserving map from $H(T)$ into $H(T')$ as $T'$ is maximal in $HP(T_1, \ldots, T_k)$, and similarly as $T' \in HP(T_P, T_k)$, there exists a hierarchy-preserving map from $H(T')$ into $H(T)$. Therefore by Lemma $4.1$, $T = T'$, and $\psi$ is associatively stable as required.

**Remark 6.5.** As lattice consensus is a regular, associatively stable consensus method, it must have a dictator tree, by Theorem $2.7$. By Theorem $2.6$ the dictator tree must be the star tree.

We will end this section by noting some additional properties of lattice consensus. The first is an immediate consequence of the fact that the maximal trees under the order $\leq_{HP}$ are precisely the binary trees.
Proposition 6.6. Let $P$ be a profile of trees on $RP(X)$. Then $\psi(P)$ is a binary tree $T$ if and only if all trees in $P$ are $T$.

Proof. Proposition 6.8 shows that binary trees are maximal under $\leq_{HP}$, so if $T$ is binary there are no trees $T' \neq T$ for which $T \leq_{HP} T'$. But a consensus tree $T = \psi(P)$ has the property that $T \leq_{HP} T'$ for all $T' \in P$, which forces $T' = T$ for all $T'$ if $T$ is binary.

Proposition 6.7. Suppose $P \in RP(X)^k$ so that each tree in $P$ has up to $m$ proper maximal clusters, where $|X| \geq m^k + 1$. Then $\psi(P)$ is not the star tree.

Proof. Let $P_1, \ldots, P_k$ be $k$ partitions of $X$ into up to $m$ parts, so that each partition is of form $P_i = \{C_{i1}, C_{i2}, \ldots, C_{in}\}$, where $n \leq m$. Observe that every element must be contained in exactly one of $C_{i1}$, one $C_{i2}$, one $C_{i3}$, and so on, and therefore it is in exactly one intersection of one part from each $P_i$, $\bigcap_{i=1}^k C_{ij}$, where $j = 1, \ldots, m$. As there are up to $m^k$ possible intersections, by the pigeonhole principle if $|X| \geq m^k + 1$ there is some intersection that contains at least 2 elements.

It follows that if $P = (T_1, \ldots, T_k)$ is a profile of trees fulfilling the assumptions of the proposition, and we consider each intersection consisting of a maximal proper cluster from each tree then there is some selection that contains 2 elements, say $a$ and $b$. Therefore the hierarchy consisting of singletons, $X$ and $ab$ has a hierarchy-preserving map into all trees in the profile. It follows that $\psi(P)$ is not the star tree.

Remark 6.8. As lattice consensus is associatively stable and regular, by Theorem 2.5 it is a different consensus method to majority rule, majority-$(+)$, frequency-difference, loose, Adams and Aho consensus methods. As it is Pareto on rooted triples, it differs from the strict consensus method.

7. Extension stability on binary trees

We end the paper by addressing another open question from [3] about extension stability on binary trees. The notion of extension stability was first introduced in [11], where it was described as a “weak independence” condition.

Definition 7.1. Suppose that $T_1$ and $T_2$ are two trees so that all clusters of $T_1$ are clusters of $T_2$. Then we say that $T_2$ refines $T_1$, denoted as $T_1 \preceq T_2$. A consensus method $\varphi$ is called extension stable if, for every profile $P$ on any $X$ and any $Y \subseteq X$,

$$\varphi(P|_Y) \preceq \varphi(P)|_Y.$$  

In [3] it was shown that in general there are no regular extension stable consensus methods, but it was left open as to whether a method might exist that worked when
restricted to binary trees. In the section we will show that the answer to the question for binary trees is also negative. We begin with a lemma.

**Lemma 7.2.** Let $P = (T_1, T_2)$ be the profile shown in Figure 4 and let $\varphi$ be an extension stable, regular consensus method. Then $\varphi(P)$ must contain the cluster $abc$.

**Proof.** This proof closely follows the approach of that of [3, Lemma 4]. Suppose, seeking a contradiction, that $\varphi(P)$ did not contain the cluster $abc$. Then, as both trees in $P$ display the triple $ab | d$, by extension stability and unanimity, $\varphi(P)$ must contain $ab | d$. As $\varphi(P)$ does not contain the cluster $abc$, it must be $T_A, T_B$ or $T_C$ in Figure 4.

Now, let $P'$ be the profile $(T'_1, T'_2)$ obtained by mapping $(a, b, c, d)$ to $(c, b, a, e)$. By neutrality, if $\varphi(P) = T_A, T_B$ or $T_C$, then $\varphi(P')$ must be $T'_A, T'_B$ or $T'_C$ respectively. In all three cases, the triple $abc$ is resolved differently, so $\varphi(P)$ and $\varphi(P')$ are incompatible.

Now, let $P^+ = (T^+_1, T^+_2)$ from Figure 4. One can see that $P = P^+|_{\{a,b,c,d\}}$, and if we reverse the order of trees in $P'$, then $P' = P^+|_{\{a,b,c,e\}}$, so by extension stability and anonymity $\varphi(P^+)$ must resolve both of them, which is impossible as they resolve $abc$ incompatibly.

It follows that $\varphi(P)$ must contain the cluster $abc$. □

**Figure 4.** The three profiles of trees used in Lemma 7.2: $P = (T_1, T_2), P' = (T'_1, T'_2)$, and $P^+ = (T^+_1, T^+_2)$.

We can now prove the main result of this section.
**Theorem 7.3.** There are no extension stable, regular, consensus methods on pairs of binary trees.

**Proof.** Let \( \varphi \) be an extension stable, regular consensus method, let \( P \) be a profile consisting of \( T_1 \) and \( T_2 \) from Figure 5 and let \( T = \varphi(P) \). Observe that \( T_1 \) and \( T_2 \) both display the triple \( bd|e \), and so by extension stability \( T \) must contain \( bd|e \).

![Figure 5. Counterexample.](image)

We first consider \( P|_{\{a,b,e,f\}} \). Observe that this profile is identical to the profile in Lemma 7.2 under the map from \( (a,b,e,f) \) to \( (a,b,c,d) \). It follows, by extension stability, that \( \varphi(P)|_{\{a,b,e,f\}} \) displays the cluster \( abe \), and in particular that \( T \) displays the triple \( bd|e \). As we have noted, \( T \) also contains \( bd|e \), so \( T \) must therefore display \( de|f \) as well.

Now, consider \( P|_{\{c,d,e,f\}} \). Observe that under the map from \( (c,d,e,f) \) to \( (a,b,d,c) \), \( P|_{\{c,d,e,f\}} \) becomes \( (T_2, T_1) \) from Lemma 7.2. By anonymity and extension stability, it follows that \( \varphi(P)|_{\{c,d,e,f\}} \) displays the cluster \( cdf \), and in particular that \( T \) displays the triple \( df|e \).

However, we have already shown that \( T \) must display \( de|f \), so this is a contradiction. Therefore the consensus method \( \varphi \) cannot exist. \( \Box \)

8. Discussion

Consensus methods in phylogenetics have been widely used, and will continue to be used as long as multiple methods are used to obtain estimates of the relationships among taxa — a practice only likely to continue as technology and supplies of data improve. However, consensus methods that satisfy specific desirable properties have proved to be difficult, and in some cases impossible, to obtain.

The regular method that we introduce in this paper, *lattice consensus*, satisfies some properties that may prove to be useful in practice: associative stability; and being Pareto on rooted triples. Being Pareto on rooted triples is a basic property that is desirable because it represents the preservation of agreed relationships: if every tree in the profile agrees on a relationship on a subset of the leaves, then the consensus tree ought to respect that agreement. Associative stability is a “stability” property, that asks that the consensus on a profile of trees need not be recomputed when an additional tree is added to the profile. Effectively, it means that a consensus tree for a profile can be computed pairwise — two trees at a time — which should provide a computational advantage.
Lattice consensus is the first regular consensus method known to be both associatively stable and Pareto on rooted triples. As such, there are opportunities to investigate it and related properties further. For instance one might ask whether this method is the unique consensus method with these properties? Or if not, whether it satisfies some optimality criterion (is it maximal, or minimal, in some sense?).

One might also ask how robust lattice consensus is to certain errors in phylogenetic inference. For instance, if an error causes some leaf in a tree in the profile to be relocated, how much is the consensus affected? In some methods, such as strict consensus, this can easily result in the star tree. But being Pareto on rooted triples guarantees that triples that are agreed on across the profile are nevertheless preserved when an unrelated leaf is moved.

Lattice consensus has a tendency to push uncertainty into the past, towards the root. This can be best seen by considering the results on binding from Section 4, which show that a binding of a tree is greater than the tree in the partial order: binding pushes branchings further towards the leaves. If one thinks of the entire poset \((RP(X), \leq_{HP})\), with least element the star tree and maximal elements the binary trees, binding pushes trees “up” towards the binary trees. Lattice consensus then takes a set of trees and identifies the closest tree below the set — thus pushing the uncertainty back towards the root. As an example, consider \(T_1\) and \(T_2\) in Figure 6. The second tree is a binding of the first, and so the consensus tree is \(\psi(T_1, T_2) = T_1\), where all of the uncertainty is concentrated at the root.

There are of course some potential issues with lattice consensus, some of which arise as a direct result of favourable properties. For instance, it is not affected by the number of copies of a tree in the profile, as a result of the associative stability property. This may not be desirable in circumstances where having 100 or more copies of a given tree \(T\) is an indication that the consensus tree should look closer to \(T\) than otherwise, but there are circumstances where this may not be relevant, such as producing species trees from gene trees. Additionally, even one occurrence of the star tree in a profile is enough to turn the consensus tree into a star tree (as the star tree is a dictator tree), but this is a direct result of regularity and associative stability as well.

In developing lattice consensus, we rely on a partial order on the set \(RP(X)\) of rooted phylogenetic trees on \(X\), and the notion of a hierarchy-preserving map. This partial order also appears to be new, and so it would be interesting to explore...
whether it has any other useful applications in phylogenetics, or indeed whether it can be extended to phylogenetic networks.

Finally we might also ask whether lattice consensus itself can be generalised to a consensus method for rooted phylogenetic networks, or alternatively, whether it could be generalised to an associatively stable supertree method (where the trees in the profile may not have the same leaf set).

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References

1. Edward N Adams, N-trees as nestings: complexity, similarity, and consensus, Journal of Classification 3 (1986), no. 2, 299–317.
2. Kenneth J Arrow, A difficulty in the concept of social welfare, Journal of Political Economy 58 (1950), no. 4, 328–346.
3. David Bryant, Andrew Francis, and Mike Steel, Can we “future-proof” consensus trees?, Systematic Biology 66 (2017), no. 4, 611–619.
4. William HE Day and FR McMorris, Axiomatic consensus theory in group choice and biometrics, vol. 29, SIAM, 2003.
5. JH Degnan, M DeGiorgio, D Bryant, and NA Rosenberg, Properties of consensus methods for inferring species trees from gene trees, Systematic Biology 58 (2009), no. 1, 35–54.
6. Jianrong Dong, David Fernández-Baca, Fred R McMorris, and Robert C Powers, An axiomatic study of majority-rule (+) and associated consensus functions on hierarchies, Discrete Applied Mathematics 159 (2011), no. 17, 2038–2044.
7. Mark T. Holder, Jeet Sukumaran, and Paul O. Lewis, A justification for reporting the majority-rule consensus tree in Bayesian phylogenetics, Systematic Biology 57 (2008), 814–821.
8. FR McMorris, Axioms for consensus functions on undirected phylogenetic trees, Mathematical Biosciences 74 (1985), no. 1, 17–21.
9. FR McMorris and Robert C Powers, Some axiomatic limitations for consensus and supertree functions on hierarchies, Journal of Theoretical Biology 404 (2016), 342–347.
10. Mike Steel, Phylogeny: Discrete and random processes in evolution, Society for Industrial and Applied Mathematics, 2016.
11. Mike Steel and Joel Velasco, Axiomatic opportunities and obstacles for inferring a species tree from gene trees., Systematic Biology 63 (2014), no. 5, 772–778.
12. David L Swofford. When are phylogeny estimates from molecular and morphological data, ch. 14, p. 295, Oxford University Press, 1991.

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