A Sound and Complete Hoare Logic for Dynamically-typed, Object-Oriented Programs – Extended Version –

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Abstract. A simple dynamically-typed, (purely) object-oriented language is defined. A structural operational semantics as well as a Hoare-style program logic for reasoning about programs in the language in multiple notions of correctness are given. The Hoare logic is proved to be both sound and (relative) complete and is – to the best of our knowledge – the first such logic presented for a dynamically-typed language.

1 Introduction & Related Work

While dynamic typing itself was introduced with the advent of LISP decades ago and more and more dynamically-typed programs are written as languages like JavaScript, Ruby and Python are gaining popularity, to the current day, no sound and complete program logic has been published for any such language.

In an attempt to bridge this Gap between static- and dynamically-typed languages, we focus our inquiry on completeness (for closed programs) and on studying the proof-theoretic implications of dynamic typing. This differentiates our work from other axiomatic semantics published mainly for JavaScript [15,9] as their focus lies more on soundness and direct applicability to real-world programming languages.

Hence, to avoid getting tangled in the details of any real-world programming language, we introduce a small dynamically-typed object-oriented (OO) language called dyn1.

Additionally, in previous work [8] the authors developed a technique for reducing the effort of verifying a dynamically-typed program to the level of verifying an equivalent statically-typed one. This technique, however, assumed the existence of a sound and complete program logic for the dynamically-typed language. The current work hence substantiates this assumption.

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1 One may ask whether it is at all possible to obtain a sound and relatively complete Hoare logic for dyn in light of Clarke’s incompleteness result [6]. However, Clarke’s argument is not applicable to dyn for various reasons elaborated in appendix C.
Besides presenting the Hoare logic, there are further technical contributions:

1) Tagged Hoare Logic, a novel notation for Hoare triples making the notion of correctness explicit and thereby allowing the (previously separated) Hoare logics for partial correctness, strong (= failsafe) partial correctness, typesafe partial correctness and total correctness to be merged into a single proof system and to concisely express the rules of this system.

2) A novel technique to specify loop variants circumventing a common incompleteness issue in Hoare logics for total correctness (see proof of Theorem 5).

As detailed in Section 7, we consider our results as a stepping stone towards similar proof systems for real-world languages.

Our paper is organized as follows. In Section 2, we introduce the language \texttt{dyn}. In Section 3, its operational semantics is defined. In Section 4, its axiomatic semantics (Hoare logic) is introduced. In Section 5, we briefly touch upon soundness of this Hoare logic, and in Section 6, we prove its (relative) completeness for closed programs.

\textbf{Notation:} $\mathbb{N}_n \equiv \{n, ..., m\}$, $\mathbb{N}_m \equiv \mathbb{N}_0^m$, $S_1 S_2$ denotes concatenation of the sequences $S_1$ and $S_2$, $\{S\}$ is the set of all elements of the sequence $S$.

2 Dynamically-Typed Programs

We will study a language called \texttt{dyn}, whose syntax is depicted in Figure 1. Like its popular real-world siblings JavaScript, Ruby and Python, \texttt{dyn} is a dynamically-typed purely OO-programming language. However, to focus our inquiry on dynamic typing, we chose not to model other features commonly found in these languages like method update, closures or \texttt{eval}().

As customary in such languages, \texttt{dyn} desugars operations to method calls. Consequently, the only built-in operation in \texttt{dyn} is object equality. Everything else is defined in \texttt{dyn} itself. However, a syntactic distinction between built-in operations and method calls is necessary for the convenient distinction between (side-effect-free) expressions and (side-effecting) statements. In order to make \texttt{dyn} programs resemble their real-world counterparts, we had to allow method calls as well as assignments in expressions. For example, $a := b := 5$ is a valid \texttt{dyn} expression with the side-effect of assigning 5 to both $a$ and $b$.

Since types in \texttt{dyn} are a property of values rather than variables, there is no need to declare the latter. Following its real-world counterparts, both local- and instance variables in \texttt{dyn} are created upon their first assignment. Accessing a variable that has not been assigned before results in a (runtime) type error.

Other reasons for type errors are non-boolean conditions in conditionals or while-loops and method call receivers whose class does not support a method matching name and arity of the call (MethodNotFound).

3 Operational Semantics

In Figure 2, we define an operational semantics of \texttt{dyn} in the style of Hennessy and Plotkin [11,14]. It is based on a set \textit{Conf of configurations}, which are pairs
$C = \langle s, \sigma \rangle$ consisting of a statement $s$ of dyn and a state $\sigma$, assigning values to variables. By syntax-directed rules, the operational semantics defines which transitions $\langle s, \sigma \rangle \rightarrow \langle s', \sigma' \rangle$ are possible between configurations.

As dyn is a purely OO-language, the value domain is the set $\mathbb{O}$ of objects, including the special objects null (the usual OO-null value) and $\bot$ (marking non-existing variables). The definition of states and state updates is standard and therefore omitted (see e.g. [2]).

For a given program, we denote the set of all variables as $\mathcal{V} = \mathcal{V}_L \sqcup \mathcal{V}_I \sqcup \mathcal{V}_S$ where $\mathcal{V}_L$ is the set of local variables, $\mathcal{V}_I$ the set of instance variables and $\mathcal{V}_S = \{\text{self, } r\}$ the set of special variables. self is special because it cannot be assigned to in programs and $r$ will be explained below. We also use the set of all classes $\mathcal{C}$ with each class $C \in \mathcal{C}$ having a set of methods $\mathcal{M}_C$ and $\mathcal{M} = \bigcup_{C \in \mathcal{C}} \mathcal{M}_C$.

Usually, in a structural operational semantics, expressions are assumed to be side-effect-free and the effect of assignments can hence be expressed as an axiom $\langle v := e, \sigma \rangle \rightarrow \langle \emptyset, \sigma[v := \sigma(e)] \rangle$. In dyn, however, expressions are side-effecting. We hence need to evaluate the assignment $v := e$ in two steps: first evaluating the expression $e$ and then assigning its resulting value to the variable $v$. Furthermore, we need an interface between these two steps: A way by which the assignment can determine the result of the previously evaluated expression $e$. For this purpose, we introduce a special variable $r$ of type $\mathbb{O}$ as well as the convention that every expression or statement will store its result in $r$. Note that this construction works only due to dynamic typing: In a statically-typed programming language, expressions would evaluate to values of different types which could not well be assigned to a single variable. The choice of object as the unifying supertype of all values is common in pure OO-languages: When everything is an object, clearly every expression will evaluate to one. Furthermore, as $r$ is the only statement that does not change anything (not even $r$), we define the empty program as $r$, stipulate $(r; s) \equiv (s; r) \equiv s$ for all statements $s$ and call the configurations $\langle r, \sigma \rangle$ for some state $\sigma$ final.

For dyn, we use class-based OO and model object creation as activation\(^2\). We introduce a “representative” object $\theta_C$ for each class $C$ as well as a special instance variable $@c$ not allowed to occur in programs for maintaining both the instance-class relation and the activation state of each object.

We call an object $o$ with $o.@c = \text{null inactive}$, meaning it is “not yet created”. Initially, all objects (except null and the representatives $\theta_C$ for each class $C$) are inactive. We suppose an infinite enumeration of objects $o_1, o_2, ...$ containing every object (both active and inactive) exactly once and introduce a function $\gamma : \Sigma \mapsto \mathbb{O}$ mapping every state $\sigma \in \Sigma$ to the object $o_k$ with the least index $k$ that is inactive in $\sigma$.

Upon its creation, an object $o$ is assigned a class $C$ and is henceforth regarded an instance of $C$. Technically, this is achieved by resetting the value of $o.@c$ to $\theta_C$ (see the rule for object creation). We use $\text{init}_C$ to denote the initial (internal)

\(^2\) Assuming an infinite sequence of already existing, but deactivated objects, object creation boils down to picking the next one and marking it as “activated”.
state of an object of class $C$: $\text{init}_C.\varnothing c = \theta_C$ and $\text{init}_C.\varnothing v = \emptyset$ for all $\varnothing v \in \mathcal{V}_I \setminus \{\varnothing c\}$.

We can then formally define the predicate $\text{bool}(o)$ and $\text{bool}(o, b)$ used in Figure 2 to check for boolean values as

$$\text{bool}(o) \equiv o.\varnothing c = \theta_{\text{bool}} \text{ for all } o \in \mathbf{O} \text{ and }$$

$$\text{bool}(o, b) \equiv \text{bool}(o) \land b \leftrightarrow o.\varnothing \text{ref} \neq \text{null}^3 \text{ for all } o \in \mathbf{O}, b \in \mathbb{B}.$$

Note how the rule for assignment uses the two-step idea to handle side-effecting expressions. The rules for conditionals and while loops also use it to evaluate the condition first and then branch on its result. Since no type system guarantees this result to be boolean, further distinguished behaviors for failures and type errors are necessary. The same holds for receivers of method calls.

Additionally, the rules for method call (or better: begin local-blocks) and object creation instantiate all local- and instance variables to $\emptyset$, which marks them as “not yet created” and causes typeerror in the rule of variable access.

Note also the handling of special variables in method calls: on entry, self is set to the receiver of the method call while on exit $\mathbf{r}$ intentionally remains unmodified to pass the return value back to the caller.

Constructors are normal methods conventionally named init that are called on newly created instances directly after they were created. The instance creation (activation) itself is called new$_C$. Note that new$_C(...) \,$ returns the constructor’s return value which is not necessarily the newly created instance. Also note that calling new$_{C'}(...) \,$ for a class $C'$ that does not have a method init results in a typeerror.

4 Axiomatic Semantics

4.1 Tagged Hoare Logic

The original paper of Hoare [12] considers partial correctness. Other “notions of correctness” like strong partial correctness and total correctness were added later as separate proof systems. While termination as a liveness property might justify this special handling, there seem to be little reason to grant this special place also to properties like failsafely and typesafety. They do, however, affect the proof rules (mostly by adding additional preconditions) and hence triggered the creation of new proof systems for new “notions of correctness”. Additionally, the term “total correctness” was interpreted as “the absence of any kind of fault” and hence strongly depends on what other faults the authors are considering. Furthermore, in this abundance of available proof systems, tool designers are forced to choose which one to implement, depriving their users of the choice which properties they actually want to verify. From a tool-design perspective, it would be much better to make all properties part of the specification, have a single proof system dealing with them and allowing the users to choose which

3 Other methods to distinguish the values true and false are conceivable.
Syntax of dyn:

Fig. 1. Syntax of dyn

1. \( \langle \text{null}, \sigma \rangle \rightarrow \langle r, \sigma[r := \text{null}] \rangle \)
2. \( \langle i, \sigma \rangle \rightarrow \langle r, \sigma[r := \sigma(v)] \rangle \) where \( v \in \mathbb{U} \) and \( \sigma(v) \neq \Box \)
3. \( \langle v, \sigma \rangle \rightarrow \langle r, \text{typeerror} \rangle \) where \( v \in \mathbb{U} \) and \( \sigma(v) \equiv \Box \)
4. \( \langle e, \sigma \rangle \rightarrow \langle r, \tau \rangle \) where \( v \in \mathbb{U} \)
5. \( \langle s_1; s, \sigma \rangle \rightarrow \langle s_2; s, \tau \rangle \)
6. \( \langle e, \sigma \rangle \rightarrow \langle r, \tau \rangle \) \( \text{bool}(\tau(r), \text{true}) \)
7. \( \langle e, \sigma \rangle \rightarrow \langle r, \tau \rangle \) \( \text{null} \)
8. \( \langle e, \sigma \rangle \rightarrow \langle r, \text{typeerror} \rangle \)
9. \( \langle \text{while } e \text{ do } s \text{ done}, \sigma \rangle \rightarrow \langle e \rangle \) \( \langle \text{if then } s_1 \text{ else } s_2 \text{ end}, \sigma \rangle \rightarrow \langle s_1, \tau \rangle \)
10. \( \langle \text{begin local } \bar{u} := \bar{v}, \text{end}, \sigma \rangle \rightarrow \langle u \rangle \) \( \langle s, \sigma \rangle \) where \( \bar{u} = \mathbb{U} \setminus \{ \bar{v} \} \) and \( \bar{v} \) is a fitting sequence of \( \Box \) values.
11. \( \langle e_1, \sigma_i \rangle \rightarrow \langle r, \sigma_{i+1} \rangle \) for all \( i \in \mathbb{N} \)
12. \( \langle e_0.\text{meth}(e_1, \ldots, e_n), \sigma_0 \rangle \rightarrow \langle \text{begin local } \bar{u} := \sigma_1(r), \ldots, \sigma_{n+1}(r); s \text{ end}, \sigma_{n+1} \rangle \)
13. \( \langle e_0, \sigma_0 \rangle \rightarrow \langle r, \sigma_1 \rangle \) \( \sigma_1(r) = \text{null} \)
14. \( \langle \text{new } C(e_1, \ldots, e_n), \sigma \rangle \rightarrow \langle \text{new } C.\text{init}(e_1, \ldots, e_n), \sigma \rangle \)
15. \( \langle \text{new } C, \sigma \rangle \rightarrow \langle r, \sigma[o := \text{init}(C)[r := o]] \rangle \) where \( o = \gamma(\sigma) \)
16. \( \exists h : \mathbb{B} \bullet b \leftrightarrow \sigma_1(r) = \sigma_2(r) \land s_r \equiv \text{true} \lor b \lor s_r \equiv \text{false} \land \neg b \)
17. \( \langle e, \sigma \rangle \rightarrow \langle r, \sigma_1 \rangle \) \( \exists h : \mathbb{B} \bullet b \leftrightarrow \sigma_1(r, \theta(c)) = \theta_C, s_r \equiv \text{true} \lor b \lor s_r \equiv \text{false} \land \neg b \)

Fig. 2. dyn’s structural operational semantics.
guarantees to derive for which part of the program. We hence propose the formalism of *tagged Hoare logic*, a uniform framework for all these properties featuring a single proof system to treat them.

A (big step) program semantics maps programs and initial states to sets of final states. Traditionally, each notion of correctness needs its own program semantics as they differ in what characteristics of a computation they guarantee. We define the (infinite) set of (finite or infinite) computations as 
\[
\text{Comp} = \text{Conf}^* \cup \text{Conf}^\omega
\]
and those of a program \(s\) starting in an initial state \(\sigma\) as 
\[
\text{Comp}(s, \sigma) = \{C_0, C_1, ... \mid C_0 = (s, \sigma) \land \forall i \cdot C_i \rightarrow C_{i+1}\} \subset \text{Comp}.
\]

We use the symbol \(\rho\) to denote elements of \(\text{Comp}\) and define the following tags along with their respective error states:
\[
\text{Tags} = \{\text{terminates, typesafe, failsafe}\}, \Sigma_\bot = \{\bot, \text{typeerror, fail}\},
\]
\(\triangleright: \text{Tags} \mapsto \Sigma_\bot\), \(\Sigma^+ = \Sigma \uplus \Sigma_\bot\)
\(\triangleright (\text{terminates}) = \bot, \triangleright (\text{typesafe}) = \text{typeerror}, \triangleright (\text{failsafe}) = \text{fail}\)

Their behaviour may be defined as a selector for their respective characteristic:
\[
S: \text{Tags} \cup \{\emptyset\} \mapsto \text{Comp} \mapsto \mathcal{P}(\Sigma^+)
\]
\[
S_\emptyset(\rho) = \begin{cases}
\{\tau\} & \text{if } \rho = C_0, ..., C_n \land C_n = (r, \tau) \land \tau \in \Sigma \\
\{\} & \text{otherwise}
\end{cases}
\]
\[
S_{\text{terminates}}(\rho) = \begin{cases}
\{\bot\} & \text{if } \rho \text{ is infinite} \\
\{\} & \text{otherwise}
\end{cases}
\]
\[
S_{\text{tag}}(\rho) = \begin{cases}
\{\triangleright (\text{tag})\} & \text{if } \rho = C_0, ..., C_n \land C_n = (r, \triangleright (\text{tag})) \\
\{\} & \text{otherwise}
\end{cases}
\]
for all other tags. Finally, we are able to define tagged program semantics
\[
\mathcal{M}: \mathcal{P}(\text{Tags}) \mapsto \text{Prog} \mapsto \Sigma \mapsto \mathcal{P}(\Sigma^+)
\]
allowing arbitrary combinations of correctness notions. Let \(\text{tags} \subseteq \text{Tags}\), then
\[
\mathcal{M}_{\text{tags}}\|s\|(\sigma) = \bigcup\{S_{\text{tag}}(\rho) \mid \rho \in \text{Comp}(s, \sigma), \text{tag} \in \text{tags} \uplus \{\emptyset\}\}
\]
which is certainly the most central ingredient of a Tagged Hoare Logic. However, we first need to extend the semantics of our assertions to also include tags
\[
\llbracket p \rrbracket_{\text{tags}} = \{\sigma \mid \sigma \in \Sigma \land \sigma, \text{tags} \models p\}
\]
before we can properly define the meaning of our Tagged Triples:

**Definition 1 (Tagged Hoare Triples).**
\[
\models \{p\}s\{\text{tags} \land q\} \iff \mathcal{M}_{\text{tags}}\|s\|(\llbracket p \rrbracket_{\text{tags}}) \subseteq \llbracket q \rrbracket_{\text{tags}}
\]
where \(\models\) denotes semantic truth of Tagged Hoare Triples.
4.2 Assertion Language

Before going into details of the program logic, we introduce the assertion language \textbf{AL}. Its syntax is depicted in Figure 3. Essentially, it is predicate logic with quantification over finite sequences of typed elements – weak second order logic. We extend the logic with constants \(c\) and operations \(op(\overrightarrow{l})\) corresponding to \texttt{dyn}'s syntactic sugar for boolean values, natural numbers, strings and lists (which includes the usual arithmetic operations on both booleans and natural numbers). Also, \(c\) contains constants \(\theta_C\) denoting the representative objects of all classes \(C \in \mathcal{C}\). Note that our assertion language is statically typed, as usual.

Its type system however is simplistic: basic types \(T = \{N, O, B, S\}\) form a flat lattice with \(\top\) and \(\bot\) and a type constructor \(\tau^*\) for finite sequences of elements of type \(\tau\).

Assertions contain typed logical expressions (\(l\)). Such expressions consist of accesses to logical variables (of some type \(t \in T\)), local program variables (of type \(O\)) including the self-reference \(self\), instance variables (\(l.\_@x\) where both \(l\) and the result are of type \(O\)), typed constants and typed operations. Note that contrary to programming expressions, logical expressions are able to access instance variables of objects other than self.

Assertions are then constructed from equations between logical expressions of identical type, boolean connectives and quantification over finite sequences. Following [5], undefined operations like dereferencing a \texttt{null} value or accessing a sequence with an index that is out of bounds (\(l[n]\) with \(n \geq |l|\)) yield a \texttt{null} value and equality is non-strict with respect to such values (\(null = null\) is \texttt{true}) in order to keep assertions two-valued. Also, for logical expressions \(l \in LExp\), we extend the state-access to \(\sigma(l)\) in the canonical way.

To link programming language-objects with assertion-values, we define

\begin{align*}
N(o) &\equiv o \neq \texttt{null} \land o.\_@c = \theta_{\text{num}} \\
N(o, n) &\equiv N(o) \land n = 0 \rightarrow o.\_@pred = \texttt{null} \land n > 0 \rightarrow N(o.\_@pred, n - 1) \\
B(o) &\equiv o \neq \texttt{null} \land o.\_@c = \theta_{\text{bool}} \\
B(o, b) &\equiv B(o) \land b \leftrightarrow o.\_@to\_ref \neq \texttt{null}
\end{align*}

To see that mapping predicates are necessary for completeness, consider the intermediate assertion \(p\) in the following program

\begin{verbatim}
P \equiv if b then x := 5 else x := \texttt{true} end\{p\};

if x is\_a\_bool then if x then x := 10 end else x := x * 2 end
\end{verbatim}

Since \textbf{AL} is statically typed, we must also give a type to the program variable \(x\). Now, giving it the type \(N\) would allow us to express \(x = 5\), but not \(x = \texttt{true}\) while giving it the type \(B\) raises the converse problem. However, using mapping predicates, it is possible to accurately describe the set of intermediate states as \(N(x, 5) \lor B(x, \texttt{true})\). From this observation it is not hard to see that \(\{\texttt{true}\} P\{x = 10\} \lor \{\texttt{true}\} P\{N(x, 10)\}\) is not derivable without mapping predicates. We use

\begin{itemize}
\item \texttt{null}\footnote{The predicate \(N(o, n)\) is recursive. However, the technique used for proving the case for primitive recursion in Lemma 5 allows expressing it in \textbf{AL}.}
\item \(\_@pred\) and \(\_@to\_ref\) are instance variables of the classes num and bool respectively.
\end{itemize}
the notation \( \sigma, \text{tags} \models \top \) to denote the fact that the assertion \( \top \) is true in the state \( \sigma \) under the tags \( \text{tags} \). The definition of \( \models \) is standard except for the case \( \sigma, \text{tags} \models \text{tag} \) iff \( \text{tag} \in \text{tags} \).

\[ Asrt \ni a ::= l = l \mid [l] \in \{ CL \} \mid \lnot a \mid a \land a \mid \exists v : t^* \bullet a \mid \top \quad t \in T, \text{tag} \in \text{Tags} \]

\[ LExp \ni \nu ::= v \mid u \mid \nu \cdot x \mid \text{null} \mid \otimes \mid \text{self} \mid \text{if} \ l \ \text{then} \ \nu \ \text{else} \ \nu \ \text{end} \mid c \mid \text{op}(\overrightarrow{\tau}) \]

\( \{+, -, *, \text{div}, \text{mod}, <, =, \&\&, |, [\cdot, \cdot] \} \subseteq \text{op} \) (brackets are used for disambiguation)

\( CL ::= \emptyset \mid CL \]

\( C L ::= C \mid C, C L \]

\( C \in \mathcal{C} \)

with the usual abbreviations: \( a_1 \lor a_2 \equiv \neg (\neg a_1 \land \neg a_2) \), \( a_1 \rightarrow a_2 \equiv \neg a_1 \lor a_2 \), \( a_1 \leftrightarrow a_2 \equiv a_1 \rightarrow a_2 \land a_2 \rightarrow a_1 \), \( \forall v : t^* \bullet a \equiv \neg \exists v : t^* \bullet \lnot a \), \( \text{true} \equiv (\text{null} = \text{null}) \), \( \text{false} \equiv \neg \text{true} \), \( Qv : t \bullet a \equiv Qv : t^* \bullet |v| = 1 \land a[v[0]/v] \) for \( Q \in \{\forall, \exists\} \), \( l \equiv \text{true} \) if \( l \) is of type \( \mathbb{B} \).

**Fig. 3.** Syntax of the assertion language AL

### 4.3 (Tagged) Hoare Logic for Dynamically Typed Programs

Our exposition of the proof rules of \( \mathcal{H} \) will use three substitutions on assertions. Proper definitions for all three can be found in Appendix B.

The special variable \( r \) may appear in both pre- and postconditions. In preconditions it references some initial value, in postconditions the return value of the last executed expression. Note that it is important that \( r \) can appear in preconditions. Otherwise the weakest precondition \( WP(r, r = \text{null}) \) would not be expressible which would induce incompleteness.

For a \texttt{dyn} statement \( s \) let \( \text{var}(s) \) (change\( (s) \)) denote the set of variables accessed in \( s \) (appearing on the left of an assignment in \( s \)). For an assertion \( p \) let \( \text{free}(p) \) denote the set of free variables of \( p \) and \( p[v := t] \) the result of substituting \( t \) for \( v \) in \( p \).

**AXIOM: VAR**

\[ \{ p[r := v] \} v \{ p \} \quad \{ \text{typesafe} \rightarrow v \neq \otimes \} v \{ \text{tags} \} \]

Note: includes the case of \( v \equiv \text{self} \).

**AXIOM: IVAR**

\[ \{ p[r := \text{self} \cdot @v] \} @v \{ p \} \quad \{ \text{typesafe} \rightarrow \text{self} \cdot @v \neq \otimes \} @v \{ \text{tags} \} \]

**RULE: ASGN** (both normal and instance variables)

\[ p[e] \{ \text{tags} \land q[v := r] \} \]

where \( v \in \mathcal{V} \)

\[ p[v := e] \{ \text{tags} \land q \} \quad \{ p[r := \text{null}] \} \text{null} \{ \text{tags} \land p \} \]

**RULE: SEQ**
RULE: COND

\[
\begin{align*}
\{p\} s_1 \{\text{tags} \land r \}\quad \{\text{r} \} s_2 \{\text{tags} \land q\}
\end{align*}
\]

\[
\{p\} s_1; s_2 \{\text{tags} \land q\}
\]

RULE: LOOP

\[
\begin{align*}
\{p\} e \{\text{tags} \land r \land \text{failsafe} \rightarrow r \neq \text{null} \land \text{typesafe} \rightarrow r \neq \text{null} \rightarrow \mathbb{B}(r)\}
\end{align*}
\]

\[
\{r \land \mathbb{B}(r, \text{true})\} s_1 \{\text{tags} \land q\}
\]

\[
\{r \land \mathbb{B}(r, \text{false})\} s_2 \{\text{tags} \land q\}
\]

\[
\{p\} \text{ if } e \text{ then } s_1 \text{ else } s_2 \text{ end } \{\text{tags} \land q\}
\]

RULE: CONS

\[
p \rightarrow p_1, \{p_1\} s\{\text{tags} \land q_1\}, q_1 \rightarrow q, \text{ tags} \supset \text{ tags}
\]

\[
\{p\} s\{\text{tags} \land q\}
\]

RULE: BLCK

AXIOM: PASGN

\[
\begin{align*}
\{p\} \overrightarrow{u} \overrightarrow{u} := \overrightarrow{t} p; s\{\text{tags} \land q\}
\end{align*}
\]

\[
\{p[\overrightarrow{u} := \overrightarrow{t}]\} \overrightarrow{u} := \overrightarrow{t} \{\text{tags} \land p\}
\]

\[
\{p\} \text{ begin local } \overrightarrow{u} := \overrightarrow{t}; s \text{ end } \{\text{tags} \land q\}
\]

where \(\overrightarrow{u}\) \(\subseteq\) \(\mathcal{U}_L\) and \(\{\overrightarrow{t}\}\) \(\subseteq\) \(\mathcal{U}_L\cup\{\text{null}\}\), \(\overrightarrow{u}\) \(=\) \(\mathcal{U}_L\setminus(\{\overrightarrow{u}\}\cup\mathcal{U}_S)\) and \(\overrightarrow{t}\) is a fitting sequence of \(\overrightarrow{u}\) constants.

RULE: METH

\[
\{p_1\} e_i\{\text{tags} \land p_{i+1}[v_i := r]\} \text{ for } i \in \mathbb{N}_n
\]

\[
\{p_{n+1}\} v_0.\text{m}(v_1, \ldots, v_n)\{\text{tags} \land q\}
\]

\[
\{p_0\} e_0.\text{m}(e_1, \ldots, e_n)\{\text{tags} \land q\}
\]

where the \(v_i\) are fresh local variables that do not occur in any \(e_j\) for all \(i, j \in \mathbb{N}_n\).

RULE: REC

\[
A \vdash \{p\} s\{\text{tags} \land q\},
\]

\[
\begin{align*}
A' \vdash \{p_i \land r_i(z)\} \text{ begin local self, } \overrightarrow{u}_i^2 := i, \overrightarrow{v}_i^2; s_i \text{ end } \{\text{tags} \land q_i\}, i \in \mathbb{N}_n
\end{align*}
\]

\[
p_i \rightarrow \text{ (failsafe) } \rightarrow l_i \neq \text{ null } \land \text{ typesafe } \rightarrow l_i \neq \text{ null } \rightarrow l_i.\theta(C_i), i \in \mathbb{N}_n
\]

\[
\{p\} s\{\text{tags} \land q\}
\]

where \text{method } \text{mu}(\overrightarrow{u}_i)\{s_i\} \in \mathcal{M}_C, A = \{\{p_i\} \l_i.\text{m}_i(\overrightarrow{v}_i)\} \{\text{tags} \land q_i\} \mid i \in \mathbb{N}_n\}, A' = \{\{p_i \land \text{terminates} \rightarrow \forall \overrightarrow{z}' : N \land r_i(z') \rightarrow z' < z\} l_i.\text{m}_i(\overrightarrow{v}_i)\} \{\text{tags} \land q_i\} \mid i \in \mathbb{N}_n\}, \overrightarrow{v}_i\) is a logical variable of type \(N\) that does not occur in \(p_i, q_i\) and \(s_i\) for \(i \in \mathbb{N}_n\) and is treated
in the proofs as a constant, \( r_i(z) \) for \( i \in \mathbb{N}_0^1 \) are predicates with \( z \) among their free variables such that \( \forall \sigma \cdot \sigma \models p_i \rightarrow \exists z' : \mathbb{N} \cdot r_i(z') \) for all \( i \in \mathbb{N}_0^1 \) and \( r_i(z') \) denotes the result of substituting \( z' \) for \( z \) in \( r_i(z) \).

**AXIOM: EQUAL**

\[
\{ \text{true} \} u_0 \equiv u_1 \{ \text{tags} \wedge \mathbb{B}(r, u_0 = u_1) \}
\]

**AXIOM: IS A**

\[
\{ true \} u_0 \text{ is } A \{ true \} u_0
\]

**RULE: CNST**

\[
\{ p \}\text{new}_C.\text{init}(\overline{\varepsilon})\{ \text{tags} \wedge q \}
\]

**RULE: NEW**

\[
\{ p[r := \text{new}_C]\}\text{new}_C\{ \text{tags} \wedge p \}
\]

### 4.4 Auxiliary Rules

**RULE: DISJ**

\[
\frac{\{ p \} s\{ \text{tags} \wedge q \} \quad \{ r \} s\{ \text{tags} \wedge q \}}{\{ p \lor r \} s\{ \text{tags} \wedge q \}}
\]

**RULE: CONJ**

\[
\frac{\{ p_1 \} s\{ \text{tags} \wedge q_1 \} \quad \{ p_2 \} s\{ \text{tags}' \wedge q_2 \}}{\{ p_1 \land p_2 \} s\{ \text{tags} \wedge \text{tags}' \wedge q_1 \wedge q_2 \}}
\]

**RULE: \( \exists \)-INT**

\[
\frac{\{ p \} s\{ \text{tags} \wedge q \}}{\{ \exists x.p \} s\{ \text{tags} \wedge q \}}
\]

where \( x \not\in \text{var}(M) \cup \text{var}(s) \cup \text{free}(q) \).

**RULE: INV**

\[
\frac{\{ r \} s\{ \text{tags} \wedge q \}}{\{ p \land r \} s\{ \text{tags} \wedge p \wedge q \}}
\]

where \( \text{free}(p) \cap (\text{change}(M) \cup \text{change}(s)) = \emptyset \) and \( p \) does not contain quantification over objects.

**RULE: SUBST**

\[
\frac{\{ p \} s\{ \text{tags} \wedge q \}}{\{ p[\overline{\varepsilon} := \overline{\varepsilon}] \} s\{ \text{tags} \wedge q[\overline{\varepsilon} := \overline{\varepsilon}] \}}
\]

where \( \text{var}(\overline{\varepsilon}) \cap (\text{var}(M) \cup \text{var}(s)) = \text{var}(\overline{\varepsilon}) \cap (\text{change}(M) \cup \text{change}(s)) = \emptyset \).

The fact that dyn-expressions have side effects is mirrored in several rules: Like their corresponding rules in the operational semantics, the usual axiom for assignment is turned into a rule and the COND and LOOP rules both evaluate the condition before branching on its result in an intermediate state.

The rules PASGN, BLCK, METH and REC are needed to handle method calls. After handling side effecting expressions in arguments beforehand (METH) and ensuring that methods are only called on receivers supporting them (last
premise of REC), method calls are assumed to satisfy the same properties as
a block executing the body of the called method in an environment with local
variables suitably initialized by parallel assignment (BLCK,PASGN).

The rules CNST and NEW handle object creation using the respective sub-
stitution defined in appendix B.

The LOOP and REC rules feature a novel form of loop variants / recursion
bound. The basic idea is to use a predicate \( r(z) \) instead of the usual integer
expression \( t \) in order to allow quantification within loop variants / recursion
bounds. While this was primarily introduced to circumvent a common incom-
pleteness issue in Hoare logics for total correctness (see proof of Theorem 5
for details), note that it also allows using mapping predicates directly in loop
variants / recursion bounds, i.e. proving

\[
\{N(i)\} \text{while } i > 0 \text{ do } i := i - 1 \text{ done} \{\text{terminates}\}
\]

with \( r(z) \equiv N(i,z) \).

5 Soundness

Soundness follows from a standard inductive argument. We will only present the
case for the LOOP rule as the idea of using a predicate \( r \) as a loop variant for
total correctness is novel.

**Induction Hypothesis:** \( \vdash \{p\} s\{\text{tags} \land q\} \rightarrow \models \{p\} s\{\text{tags} \land q\} \) for all assertions
\( p \) and \( q \) and all dyn statements \( s \).

**Induction Step:**

**Partial Correctness:** Given \( \vdash \{p\} e\{\text{tags} \land p\}' \) and \( \vdash \{p' \land \mathbb{B}(r, true)\} s\{\text{tags} \land p\}' \), by the induction hypothesis \( \models \{p\} e\{\text{tags} \land p\}' \) and \( \models \{p' \land \mathbb{B}(r, true)\} s\{\text{tags} \land p\}' \) follow.

Hence, when executing the program **while e do s done** in a state \( \sigma \models p \), the operational semantics will first apply rule 9 yielding the configuration
\( \langle \text{if } e \text{ then } s; \text{while } e \text{ do } s \text{ done } \text{else null end } \rangle, \sigma \), then apply whatever rules
necessary to evaluate \( \langle e, \sigma \rangle \) to a final configuration \( \langle r, \tau \rangle \). From \( \models \{p\} e\{p'\} \)
we can deduce \( \tau \models p' \). Furthermore, the operational semantics uses rules 6-8 to
branch on the value of \( \tau(r) \). Now, for the case of partial correctness, we are only
interested in normal program termination, the cases yielding **fail** or **typeerror**
will be handled below. Hence there are really only two cases to consider:

1) \( \tau \models \mathbb{B}(r, true) \): In this case, rule 6a) is the only one applicable and \( \langle s, \tau \rangle \)
will be evaluated next. From \( \{p' \land \mathbb{B}(r, true)\} s\{p\} \) we can deduce that the
resulting state \( \sigma' \) will again satisfy \( p \). We are hence again in a configuration
\( \langle \text{while } e \text{ do } s \text{ done } \rangle, \sigma' \) with \( \sigma' \models p \). When regarding \( \sigma' \) as equivalent to \( \sigma \), then
this configuration is equivalent to the one before applying rule 9. Now this loop
in the (abstract) transition system raises the possibility of divergence. However,
for partial correctness we may disregard this possibility, as we only provide guar-
antees for finite computations. The case of divergence will be discussed below.
2) \( r \models B(r, false) \): In this case, rule 6b) is the only one applicable and \((null, r)\) is the only statement left to evaluate. Applying rule 1 leaves us in a final configuration \( \langle r, \tau' \rangle \) with \( \tau' \models \bar{p}/[r/b] \land B(b, false) \land r = null \). As this is the only way for our program to terminate normally, \( \models \{p\} \) while \( e \) do \( s \) done \{p'[r/b] \land B(b, false) \land r = null \} holds. □

Termination: For partial correctness, the premise \( \{p' \land B(r, true)\}; e\{p'\} \) can be derived from the two other premises by an application of the SEQ rule. It hence does not strengthen the premises in any way. However, for total correctness, it requires an additional predicate \( r(z) \) with \( z \) among its free variables, such that \( \forall \sigma \bullet \sigma \models p' \rightarrow \exists z': N \bullet r(z') \) and \( \{p' \land B(r, true) \land r(z)\}; e\{p' \land \forall z': N \bullet r(z') \rightarrow z' < z\} \) hold. \( r(z) \) may be understood as mapping states to sets of natural number values for \( z \). The first requirement thus ensures that the “mapping” \( r(z) \) is (conditionally) total on all states in \( \llbracket p' \rrbracket \), while the second requires the loop body \( s \) together with the condition \( e \) to decrease its supremum. Since the state \( \tau \) reached after evaluating \( e \) the first time satisfied \( p' \), by the conditional totality of \( r(z) \) we deduce that there must be an “initial” non-empty set of natural numbers \( \mathbb{Z} \) such that for all \( z_i \in \mathbb{Z} \), \( \tau \models r(z_i) \) holds. Let \( z_{\text{max}} \) be the supremum of \( \mathbb{Z} \). Then, since \( z_{\text{max}} \) is a natural number and since the supremum of \( \mathbb{Z} \) is required to strictly decrease on each loop iteration, there must be a finite number of iterations after which \( z_{\text{max}} = 0 \). Since there is no natural number smaller than zero, there is no way by which the second requirement for \( r(z) \) can be satisfied on the next iteration. Consequently, the loop has to terminate after finitely many iterations.

Failsafety: A failure might occur either in evaluating \( e \) or \( s \) or by rule 7 when \( e \) evaluates to \( null \). Requiring \( e \) and \( s \) to both be failsafe as well as \( \{p\} e\{r \neq null\} \) hence covers all these cases. □

Typesafety: Same argument as for failsafety applies here, only with the requirement \( r \neq null \rightarrow B(r) \) instead of \( r \neq null \). Note that the case for failure is intentionally left open as typesafe partial correctness only needs to guarantee the absence of type errors and too strong a premise would lead to incompleteness. □

6 Completeness

In this Section, we will prove the axiomatic semantics of \texttt{dyn} (relative) complete [7] with respect to its operational semantics following the seminal completeness proof of Cook and Gorelick [7,10] as well as its extension to OO-programs due to de Boer and Pierik [5]. That is, given a closed program \( \pi \) with a finite set of class definitions, we prove that \( \models \{p\} \pi \{q\} \) implies \( \models_{H,T} \{p\} \pi \{q\} \) assuming a complete proof system \( \mathcal{F} \) for the assertion language \texttt{AL}.

Traditionally, completeness proofs are structured into 3 steps. First, the assertion language is shown to be expressive, then the system is proven complete for all statements of the programming language and finally, it is shown to be complete for recursive methods using the concept of most general correctness formulas. Since both the first and the last step rely on techniques for “freezing”
program states and for evaluating assertions on such frozen states, we follow [5] in prepending a step for developing adequate freezing techniques for dyn.

Completeness proofs for Hoare Logics have been extended and refined for several decades now. Unfortunately, due to space restrictions we will not be able to give a proper account to the numerous ideas and intriguing details in the works of our predecessors, but must assume a certain familiarity with such proofs on the side of the reader. For the same reason, we will not be able to present the proof as a whole, but will concentrate on those parts we had to adapt.

6.1 Freezing the Initial State

As noticed by Gorelick [10], achieving completeness requires that the assertion language is able to capture every aspect of a program state in logical variables, in order to “freeze” this information during program execution and allow the postcondition to compare the initial- to the final state. Pierik and de Boer [5] pointed out that in OO-contexts this additionally requires freezing the internal states of all objects existing in the state, necessitating a more sophisticated freezing-strategy.

While their approach stores objects and the values of their instance variables class-wise, which is difficult in a dynamically-typed language like dyn, the basic idea is fortunately still applicable. We use a logical variable obj of type $O^*$ to store a (finite) sequence of all existing objects:

$$all(obj) \equiv \forall o : O \bullet \exists i : N \bullet i < |obj| \land obj[i] = o$$

Since obj establishes a bijection from natural numbers to objects, its allows encoding states as sequences of natural numbers. For convenience, we introduce a polymorphic$^6$ pos function satisfying $\forall \tau : T \bullet \forall e : \tau, s : \tau^* \bullet s[pos(s, e)] = e$.

We introduce an enumeration $ivar : \mathcal{B}^\tau_1$ of all instance variables and define the following predicate for freezing states:

$$code(\pi, obj, \varsigma) \equiv |\varsigma| = |ivar| + 1 \land |\varsigma[0]| = |\pi| \land obj[0] = \Xi \land
\forall i : N \bullet i < |\pi| \rightarrow \varsigma[0][i] = pos(obj, x_i) \land
\forall i, j : N \bullet (i < |ivar| \land j < |obj|) \rightarrow ivar[i] = @v \land obj[j] = o \rightarrow
\varsigma[i + 1][j] = pos(obj, o, @v)$$

where $\pi = x_1, ..., x_n$ is a sequence of local variables. The predicate $code(\pi, obj, \varsigma)$ uses the sequence obj to capture the state of all local variables in $\pi$ as well as all objects in obj in the frozen state $\varsigma$ of type $(\mathbb{N}^*)^\tau$. Note that $\varsigma$ can capture the internal states of all existing objects without referencing any of them.

Also note that this is indeed satisfiable for all states as $\Xi \in O$ and $\Xi \in obj$. Furthermore, we say that $\varsigma$ encodes $\sigma$ and write

$$\sigma \sim \varsigma \text{ if } \sigma = \exists obj : O^* \bullet all(obj) \land code(\pi, obj, \varsigma)$$

with $\{\pi\} = \mathcal{M}_L \cup \mathcal{M}_S$.

$^6$ We use the polymorphic version for the sake of readability although the type system of AL does not allow polymorphism. However, polymorphic functions can be emulated using one version for each element type.
Lemma 1 (Left-Totality of $\sim$). $\forall \sigma : \Sigma \cdot \exists \varsigma : (\mathbb{N}^*)^* \cdot \sigma \sim \varsigma$.

Finally, we are ready to define a predicate transformer $\Theta$ (called the “freezing function” in [5]). However, while in their work, $\Theta$ also bounds all quantification and replaces instance variable dereferencing by lookups in sequences, we additionally translate all object expressions into expressions of type $\mathbb{N}$ to allow simulating computations directly on the frozen states.

We hence have the following main cases for our predicate transformer $\Theta^\pi_{obj}(\varsigma)$:

- $(l \cdot \forall v) \Theta^\pi_{obj}(\varsigma) \equiv \varsigma[\text{pos}(\text{ivar}, \forall v) + 1][l \Theta^\pi_{obj}(\varsigma)]$
- $u \Theta^\pi_{obj}(\varsigma) \equiv \varsigma[0][\text{pos}(x, u)]$ where $u$ is a logical variable of type $\mathbb{O}$ and $u'$ is a fresh logical variable of type $\mathbb{N}$
- $(l_1 = l_2) \Theta^\pi_{obj}(\varsigma) \equiv l_1 \Theta^\pi_{obj}(\varsigma) = l_2 \Theta^\pi_{obj}(\varsigma)$ where $l_1$ and $l_2$ are of type $\mathbb{O}$.
- $(\exists o : \mathbb{O} \cdot p) \Theta^\pi_{obj}(\varsigma) \equiv (\exists o' : \mathbb{N} \cdot 0 \leq o' < |\text{obj}| \rightarrow p \Theta^\pi_{obj}(\varsigma))$

$\Theta^\pi_{obj}(\varsigma)$ transforms any assertion $p$ in such a way that it operates on the frozen state $\varsigma$ instead of the real program variables. Like the $\Theta$ in [5], our $\Theta^\pi_{obj}(\varsigma)$ hence satisfies the following property

**Theorem 1 (Invariance).** $\vdash \{p \Theta^\pi_{obj}(\varsigma)\} s \{p \Theta^\pi_{obj}(\varsigma)\}$ for all statements $s$ and assertions $p$ as long as $\pi$ contains all program variables used and $\text{obj}$ contains all objects accessed in $p$.

It can hence replace $\Theta$ in the remaining argument. Note that $p \Theta^\pi_{obj}(\varsigma)$ is a property of $\varsigma$ as its truth value is independent of any particular state. We hence write $\models p \Theta^\pi_{obj}(\varsigma)$ if its truth value is true. Also observe

**Lemma 2 (Freezing).** $\sigma \models q$ iff $\sigma \sim \varsigma$ and $\varsigma \models q \Theta^\pi_{obj}(\varsigma)$

**Proof.** By induction over the structure of $q$.

### 6.2 Expressivity

Cook [7] first discussed the importance of an expressive assertion language for the completeness of a Hoare logic. In essence, the assertion language must be able to express the strongest postcondition $SP(s, p)$ for all statements $s$ and preconditions $p$.

In the last Section, we already established that it is possible to capture all information about a state in a structure consisting of finite sequences of natural numbers. Using Gödelization, one can take this a step further and encode these sequences themselves as a single natural number. Then, we consider a predicate $\text{comp}_s$ of type $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ simulating $\text{dyn}$ computations on such frozen states and note that, since such computations are by definition computable, it can be defined as a $\mu$-recursive function.

By Theorem 6, it is hence expressible in our assertion language and we can use it within our assertions without any loss of generality. For convenience, we
will omit the Gödelization step and instead use a version of \( \text{comp}_s \) operating on frozen states as defined above. To formalize the idea that \( \text{comp}_s \) simulates \( \text{dyn} \) computations on frozen states, we stipulate

**Lemma 3.** \( \text{comp}_s = \{ (\varsigma, \varsigma') \mid \forall \sigma, \sigma' \cdot (\sigma \sim \varsigma \land \sigma' \sim \varsigma') \rightarrow \sigma' \in \mathcal{M}[s](\sigma) \} \)

Using \( \text{comp}_s \) we can show the following:

**Theorem 2 (Definability of Weakest Preconditions).** For all postconditions \( q \) and statements \( s \), the precondition

\[
p \equiv \forall \varsigma, \varsigma' \cdot (\text{all}(\text{obj}) \land \text{code}(\pi, \text{obj}, \varsigma') \land \text{comp}_s(\varsigma', \varsigma)) \rightarrow q \Theta_{\text{obj}}(\varsigma)
\]

satisfies \( \llbracket p \rrbracket = \{ \sigma \mid \mathcal{M}[s](\sigma) \subseteq \llbracket q \rrbracket \} \).

The proof can be found in appendix A. Since definability of weakest preconditions is equivalent to the definability of strongest postconditions [13], we have

**Theorem 3 (Expressiveness).** The assertion language \( \mathcal{A}L \) is expressive with respect to its standard interpretation and the programming language \( \text{dyn} \).

### 6.3 Completeness for Statements

As usual [7, 10], the core of our completeness proof consists of an induction over the structure of a statement \( s \). Since several of our rules deviate from theirs, we need to exchange these cases in argument. We will concentrate on the most interesting cases.

**Induction Basis:**

\( s = \text{null} \): Assume \( \models \{ p \}\text{null}\{ q \} \). Then, by the operational semantics, \( p \rightarrow q[r := \text{null}] \) must also be true. It is hence derivable in \( T \) and the desired result follows from the CONST axiom followed by applying the rule of consequence (CONS). **Typesafety:** The CONST axiom always derives typesafety. Should \text{typesafe} not be required, it can be omitted in the rule of consequence. The same holds for \text{failsafe} and \text{terminates}.

\( s = u \): Assume \( \models \{ p \}\text{u}\{ q \} \). Then, by the operational semantics, \( p \rightarrow q[r := u] \) must also be true. It is hence derivable in \( T \) and the desired result follows from the VAR axiom followed by applying the rule of consequence (CONS). **Typesafety:** Assume \( \models \{ p \}\text{u}\{ \text{typesafe} \land q \} \). Then \( \{ p \}\text{u}\{ q \} \) and \( \{ p \}\text{u}\{ \text{typesafe} \} \) must also be true. The former can thus be derived using above argumentation and the latter implies \( p \rightarrow u \neq \Xi \), which is hence derivable in \( T \) and the axiom VAR-TAG followed by applying the rule of consequence (CONS) derives \( \{ p \}\text{u}\{ \text{typesafe} \} \). Now the rule of conjunction (CONJ) followed by the rule of consequence (CONS) derives the desired result. **failsafe** and **terminates** can be derived using the axiom VAR-TAG without any preconditions.

\( s = @v \): Just like the case for \( u \), applying IVAR instead of VAR and IVAR-TAG instead of VAR-TAG.
Induction Hypothesis: \( \vdash \{ p \} s \{ q \} \rightarrow^*_{\mathcal{H}, T} \{ p \} s \{ q \} \) for all assertions \( p, q \) and all statements \( s \) of a program \( \pi \) containing no recursive method calls.

Induction Step:

- \( p := e \): Assume \( \vdash \{ p \} p := e \{ \text{tags} \land q \} \). Then according to the operational semantics, \( \vdash \{ p \} e \{ \text{tags} \land q[p := r] \} \) must also be. By the induction hypothesis, it is hence derivable. An application of the rule \text{ASGN} derives the desired result.

- \( s \equiv s_1; s_2 \): Assume \( \vdash \{ p \} s_1; s_2 \{ \text{tags} \land q \} \). Then by the expressibility of the assertion language, there is an intermediate assertion \( r \) such that \( \vdash \{ p \} s_1 \{ \text{tags} \land r \} \) and \( \{ r \} s_2 \{ \text{tags} \land q \} \) are also true. Hence by the induction hypothesis both are derivable and an application of the rule \text{SEQ} derives the desired result.

- \( s \equiv \text{if } e \text{ then } s_1 \text{ else } s_2 \text{ end} \): Assume \( \vdash \{ p \} e \{ \text{tags} \land q \} \). Then by the expressibility of the assertion language and the operational semantics, there is an intermediate assertion \( r \) such that \( \vdash \{ p \} e \{ \text{tags} \land r \} \), \( \{ r \land \mathbb{B}(r, \text{true}) \} s_1 \{ \text{tags} \land q \} \) and \( \{ r \land \mathbb{B}(r, \text{false}) \} s_2 \{ \text{tags} \land q \} \) are also true and hence derivable by the induction hypothesis. Now an application of the rule \text{COND} derives the desired result. Failsafety: Since above argumentation already ensured that \( e, s_1 \) and \( s_2 \) are all failsafe, the only additional requirement is \( \vdash \{ p \} e \{ r \neq \text{null} \} \). However, since the case \( r = \text{null} \) leads to failure in the operational semantics, this must hold for any execution of \( s \) in order to be failsafe and hence must be derivable by the induction hypothesis.

Typesafety: The same argumentation as for failsafety applies here, only the additional requirement is \( \vdash \{ p \} e \{ r \neq \text{null} \} \). Note that the case of \( r = \text{null} \) can be deliberately allowed, since it leads to a failure in the operational semantics and thus does not affect typesafety.

- \( s \equiv \text{while } e \text{ do } s_1 \text{ done} \): Assume \( \vdash \{ p \} e \{ \text{tags} \land q \} \) is true. Then, by the standard argument for while loops due to Cook [7] (and explained particularly well by Apt [1]), the expressiveness of the assertion language and the operational semantics, there are two assertions \( i \) and \( i' \) such that \( p \rightarrow i, \{ i \} e \{ \text{tags} \land i' \} \), \( \{ i' \land \mathbb{B}(r, \text{true}) \} s_1 \{ \text{tags} \land i \} \) and \( i'[b/r] \land \mathbb{B}(b, \text{false}) \land r = \text{null} \rightarrow q \) are true and hence derivable by the induction hypothesis and the completeness of \( T \). While \( i \) is the loop invariant of \( s, i' \) is an intermediate state necessary because in \text{dyn}, \( e \) could have side-effects. Now, an application of the \text{LOOP} rule followed by the rule of consequence derives the desired result.

Termination: Assuming \( \vdash \{ p \} e \{ \text{terminates} \land q \} \), then there is a \( \mu \)-recursive function \( v(z) \) simulating the execution of \( s \) using \text{comp}, and determining the least number of iterations it takes to reach a state \( \tau \) from the current state such that \( e \) evaluates to false in \( \tau \). Note that by Theorem 6 \( v(z) \) can expressed in \text{AL}(). Also, by our assumption that the loop \( s \) terminates, the function \( v(z) \) is well-defined on all states in \( p' \) and thus \( r(z) \equiv \text{all(obj)} \land \text{code}(\tau, \text{obj}, z) \land z = v(z) \) is a canonical loop variant satisfying \( \forall \sigma; \sigma = p' \rightarrow \exists z': \mathbb{N} \cdot r(z') \). Since \( v(z) \) determines the number of iterations until reaching a target state, executing \( s; e \) clearly decreases it and thus \( \vdash \{ p' \land \mathbb{B}(r, \text{true}) \land r(z) \} s; e \{ p' \land \forall z': \mathbb{N} \cdot r(z') \rightarrow z' < z \} \) holds. By
the induction hypothesis, it is thus derivable. An application of the LOOP rule derives the desired result.

Falsafety & Typesafety: the exact same argument as for conditionals applies here as well.

6.4 Completeness for Recursive Methods

The methodology for proving a Hoare logic complete for recursive procedures by using most general correctness formulas is due to Gorelick [10]. It was extended to OO-programs by De Boer and Pierik [5].

A curious implication of dynamic dispatch under dynamic typing is that the lack of type information prohibits pinpointing the exact target of a method call.

For instance, the weakest precondition of the call x.size() with respect to the postcondition N(r, 5) must include all possibilities like the case of the variable x referring to a string of length 5 as well as x referring to a list of size 5. In general, the weakest precondition of a method call l.m(v1, ..., vn) is the disjunction of all weakest preconditions derivable as described in the proof of Theorem 4 from the most general correctness formulas of all methods C.m of arity n of all classes C ∈ C, each conjoined with the corresponding type assumption [l] ∈ {C}. Note that this methodology introduces an implicit closed world assumption as it fails when using a method with a different set of classes. However, we regard this problem as one of modularity rather than completeness and thus out of scope.

As our tagged Hoare logic incorporates different notions of correctness, we generalize Gorelick’s idea to a set of most general correctness formulas. The most general correctness formulas of all methods C.m of arity

\[ MGF(s) = \{WP(s, init)\}s\{init\} \cup \{WP_{tag}(s, true)\}s\{tag\} | tag \in Tags \]

with init \( \equiv all(obj) \land code(x, obj, \ldots) \). The reason for this is obvious: From MGF(s), we can deduce \( WP_{tag}(s, q)\)s\{tags \and q\} with \( tags \subseteq Tags \) using the conjunction rule. The converse is not in all cases possible.

The results from Section 6.3 imply that above set can be derived for any dyn statement s given that they are true. Should, e.g., s raise a type error on all inputs then \( WP_{typesafe}(s, true) \equiv false \) and \{false\}s\{typesafe\} is derivable.

Theorem 4 (MGFs). \( \vdash \{p\}s\{tags \and q\} \rightarrow MGF(s) \vdash H, T \{p\}s\{tags \and q\} \)

Proof. Assume \( \models \{p\}s\{tags \and q\} \). Then \( \{p\}s\{q\} \) and \( \{p\}s\{tag\} \) for all \( tag \in tags \) are also all true.

1) \( \vdash \{p\}s\{q\} \): For technical convenience only we assume that p and q do not contain free occurrences of the logical variables used to freeze states. If they do, these need to be renamed using the substitution rule. By Theorem 1 we have \( \{q\Theta_{obj}\}s\{q\Theta_{obj}\} \). An application of the conjunction rule yields

\[ \{q\Theta_{obj} \land WP(s, init)\}s\{q\Theta_{obj} \land init\} \]

Next, we have to prove \( p \rightarrow q\Theta_{obj} \land WP(s, init) \). Assume \( \sigma \models p \). Then by \( \models \{p\}s\{q\} \), for all \( \sigma' \in M[s]|(\sigma) \), we have \( \sigma' \models q \). By Lemma 2, we have \( \sigma' \models q\Theta_{obj} \land init \). Now, by Theorem 1, we have \( \vdash \{q\Theta_{obj}\}s\{q\Theta_{obj}\} \), and by soundness of our proof system \( \models \{q\Theta_{obj}\}s\{q\Theta_{obj}\} \). Hence, \( \sigma \models q\Theta_{obj} \) and by
the definition of $WP$, $\sigma \models WP(s, init)$. Therefore, $p \rightarrow q \theta_\text{obj}^\sigma \land WP(s, init)$ holds and since $q \theta_\text{obj}^\sigma \land \text{init} \rightarrow q$ follows directly from Lemma 2, an application of the rule of consequence derives $\{p\}s\{q\}$.

2) $\vdash \{p\}s\{\text{tag}\}$: if true, then $p \rightarrow WP_{\text{tag}}(s, true)$ must also be and is hence derivable by the completeness of $T$. Since $\{WP_{\text{tag}}(s, true)\}s\{\text{tag}\} \in MGF(s)$, an application of the consequence rule derives the desired result.

3) $\vdash \{p\}s\{\text{tags} \land q\}$: One application of the conjunction rule per tag in $\text{tags}$ completes the proof. □

Finally, since our recursion rule is identical to the one devised by Gorelick [10] for this purpose, we are now able to apply the same inductive argument used by Gorelick for proving our Hoare logic complete for recursive methods.

**Lemma 4.** Let $M_i \equiv l_i.m_i(\overrightarrow{v_i})$ denote the $i$th (possibly recursive) method call occurring in a closed program $\pi$ and let $A = \bigcup_{n=1}^{\infty} MGF(M_i)$ be the set of most general correctness formulas about these method calls then for all statements $s$ of $\pi$ and all assertions $p$ and $q$: $\models \{p\}s\{q\} \rightarrow A \vdash_{\mathcal{H}, \mathcal{T}} \{p\}s\{q\}$

**Proof.** By induction over the structure of $s$. Most cases are as in the proof for the non-recursive case. Most interesting is the new case for method calls: $s \equiv l_i.m_i(\overrightarrow{v_i})$: Assuming $\models \{p\}s\{q\}$ and $s$ is the $i$th method call $M_i$ in our program, then $MGF(s) \subseteq A$ and hence $A \vdash \{p\}s\{q\}$ by Theorem 4. As Gorelick [10] pointed out, this also holds for recursive method calls.

**Theorem 5 (Completeness for Recursive Methods).**

$\models \{p\}s\{\text{tags} \land q\} \rightarrow A \vdash_{\mathcal{H}, \mathcal{T}} \{p\}s\{\text{tags} \land q\}$

for any statement $s$ of a closed program $\pi$ containing possibly recursive method calls and all assertions $p$ and $q$.

**Proof.** Expressiveness of AL guarantees the expressibility of $WP_{\text{tags}}(s, q)$ for any statement $s$ and postcondition $q$. Hence by setting $q \equiv \text{init}$ and $s \equiv M_i$ for any $i \in \mathbb{N}_n^1$ we can see that the set $A$ of most general correctness formulas of all method calls is expressible in our logic. Now, since by definition of $WP_{\text{tags}}$, these formulas are true, we have by Lemma 4

$A \vdash_{\mathcal{H}, \mathcal{T}} \{p_i\}_{\text{begin local self, } \overrightarrow{u_i} := l_i, \overrightarrow{v_i}; s_i \text{ end}\{q_i\}}$ as well as

$A \vdash_{\mathcal{H}, \mathcal{T}} \{p_{\text{tag}, i}\}_{\text{begin local self, } \overrightarrow{u_i} := l_i, \overrightarrow{v_i}; s_i \text{ end}\{\text{tag}\}}$ for all $\text{tag} \in \mathcal{T}_{\text{tags}}$

with $p_i \equiv WP(M_i, \text{init})$, $q_i \equiv \text{init}$, $p_{\text{tag}, i} \equiv WP_{\text{tag}}(M_i, \text{true})$ and $s_i$ denoting the method body of the method called in $M_i$ for all $i \in \mathbb{N}_n^1$. Note that above statements establish the assumptions in the set $A$ and together allow deriving the assumptions for the REC rule of the form

$A \vdash_{\mathcal{H}, \mathcal{T}} \{p_i \land r_i(z)\}_{\text{begin local self, } \overrightarrow{u_i} := l_i, \overrightarrow{v_i}; s_i \text{ end}\{\text{tags}, \land q_i\}}$

for all $i \in \mathbb{N}_n^1$. As for the case not concerned with termination, we can simply set $r_i(z) \equiv z = z$. Furthermore, assuming $\models \{p\}s\{q\}$, by Lemma 4 we have

$A \vdash \{p\}s\{\text{tags} \land q\}$

Now these are just the premises of the REC rule. Note that in the case not concerned with termination, the set of assumptions $A$ is derivable from $A'$ by applying the consequence rule to each element. Hence, an application of the REC
Termination: for proving termination of \textbf{dyn} programs, the rules LOOP and REC must be altered to support so-called loop-variants or recursion bounds. Usually, these take the form of an integer expression \textit{t} whose value a) must be $> 0$ whenever the loop / recursive method is entered (thus forcing termination when reaching zero) and b) must decrease on every iteration / recursive call. Note that this methodology syntactically restricts the loop variant / recursion bound to be an integer expression of the assertion language. Now, as observed by Apt, De Boer and Olderog in [2], this method introduces incompleteness in the case of total correctness, since it assumes the integer expressions of the assertion language to be able to express any necessary loop-variant / recursion bound. However, while-loops and recursive methods allow \textbf{dyn}-programs to calculate any \(\mu\)-recursive function and hence obviously also to bound the number of loop iterations by any \(\mu\)-recursive function, while the set of integer operations available in the assertion language might be quite limited (e.g. in our case lacking exponentiation). We circumvent this problem by introducing a new form of loop-variants and recursion bounds, which allow the use of quantifiers. The old form used a logical variable \textit{z} of type \texttt{N} to store the value of \textit{t} before a loop iteration \((t = z \text{ in the precondition})\) and compare it to the new value in the postcondition \((t < z)\). Our new form uses a predicate \(r(z)\) with \textit{z} among its free variables instead of \(t = z\) and the logical expression \(\forall z': \texttt{N} \bullet r(z') \rightarrow z' < z\) where \(r(z')\) denotes the result of substituting \(z'\) for \textit{z} in \(r\) instead of \(t < z\). Firstly, observe that this is a conservative extension as one may set \(r \equiv t = z\) for some integer expression \(t\). Secondly, note that by Lemma 5, \(r\) may compute any \(\mu\)-recursive function and is thus contrary to integer expressions able to express any function computable by \textbf{dyn}-programs including exponentiation.

7 The Translational Approach

The translational approach was introduced by Apt, De Boer and Olderog in [3] to facilitate the availability of sound and complete axiomatic proof systems for different programming languages. The basic idea is to transfer soundness and completeness results for their proof systems from language to language by means of a (more intuitive and usually much simpler) semantics-preserving translation between the programming languages. The program logic presented in this work handles the fundamental issues of dynamically typed languages and hence opens the gate for using the translational approach to prove program logics for such programming languages sound and complete in future. In the following, we will list some ideas on how more advanced dynamic features found in real-world dynamically-typed languages like JavaScript, Ruby and Python might be translated to \textbf{dyn}.

7.1 Method Update

Languages combining class-based object orientation with dynamic typing (like Ruby and Python) often support a feature we call “method update” allowing
programs to override methods at runtime - most often it is not even required for the arity of the new method to match the old version.

**Translation:** First, for each method \( C.m \) in the original program having multiple versions, the corresponding \texttt{dyn} program must have a global state \( g_{C.m} \) for storing the information which version is the current one. Since \texttt{dyn} program usually do not have any global state, we accomplish this by introducing a class \textit{Global} encapsulating this state information and passing a reference \( g \) to its only instance into each and every method in the program. Second, let there be versions \( C.m_1, ..., C.m_k \) of a method \( C.m \) with arities \( n_1, ..., n_k \) within the original program, then the function \( a_{C.m}(n) = \{ C.m_i \mid n_i = n \} \) groups all versions having the same arity. For each arity \( n \), such that \( |a_{C.m}(n)| > 0 \), the corresponding \texttt{dyn} program contains a method \( C.m_n \) with arity \( n \) whose body is structured as follows

\[
\begin{align*}
\text{if } g.\text{version}_{C.m}() &= 1 \text{ then} \\
\text{# body of } C.m_1 \text{ here} \\
\text{else} \\
\text{if } g.\text{version}_{C.m}() &= 2 \text{ then} \\
\text{# body of } C.m_2 \text{ here} \\
\text{else} \\
\text{...} \\
\text{else} \\
\text{typeerror} \\
\text{end} \\
\text{end} \\
\text{end}
\end{align*}
\]

where \( a_{C.m}(n) = \{ C.m_1, C.m_2, ..., C.m_l \} \).

Then, we only have to treat the updating of a method \( C.m \) as setting its global state to a new value \( v \) by \( g.\text{set\_version}_{C.m}(v) \).

Furthermore, when translating every method call to a method \( C.m \) of arity \( n \) in the original program as a call to \( C.m_n \) in the corresponding \texttt{dyn} program, the result should be behaviourally equivalent.

### 7.2 Closures

Another feature of functional languages that is often found in dynamically typed languages are closures (E.g. JavaScript functions or ruby blocks). Closures are characterized by two properties: Firstly, they allow passing around (a reference to) code as data and secondly, they capture the values of all free variables within their body upon creation.

**Translation:** In \texttt{dyn}, we can emulate both properties by introducing a new class \( C_c \) for each closure \( c \). This class defines a method \texttt{do}() of the same arity as the closure and whose method body is just \( c \)'s body with all free variables replaced by corresponding instance variables as well as a constructor \texttt{init} taking all variables as arguments that occur free in \( c \)'s body and storing their value.
in corresponding instance variables. Now, the closure definition \( c = \lambda p_1, \ldots, p_n.s \) can be replaced by
\[
c = \text{new } C_c(v_1, \ldots, v_k)
\]
where \( v_1, \ldots, v_k \) are all variables occurring free in \( s \) and each call \( c(a_1, \ldots, a_n) \) can be replaced by the method call
\[
c.do(a_1, \ldots, a_n)
\]
The resulting program should be behaviourally equivalent. Note that a finite program can only contain a finite number of closures and this replacement hence will only introduce a finite number of additional classes \( C_c \).

### 7.3 Multiple Return Values

In Ruby, methods are allowed to have multiple return values. Under the hood, this is realized by converting the return values into a list and assigning the list elements to their respective variables.

**Translation:** Since \( \text{dyn} \) also supports heterogeneous lists, one can create a similar mechanism by translating
\[
\text{return } e_1, \ldots, e_n \quad \text{into} \quad \text{return } [e_1, \ldots, e_n]
\]
and
\[
v_1, \ldots, v_n = C.m() \quad \text{into} \quad l = C.m(); v_1 = l[0]; \ldots; v_n = l[n-1]
\]
where \( l \) is a fresh local variable.

### 8 Conclusions & Outlook

We presented a sound and (relative) complete Hoare logic for \( \text{dyn} \). Open are the issues of modularity (applicability to open programs) and allowing tags carrying additional information (to incorporate extensions like De Boer’s footprints [4]).

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### References

1. Apt, K.R.: Ten Years of Hoare’s Logic: A Survey – Part I. ACM Trans. Program. Lang. Syst. 3(4), 431–483 (1981), [link](#)
2. Apt, K.R., de Boer, F.S., Olderog, E.R.: Verification of Sequential and Concurrent Programs, 3rd Edition. Texts in Computer Science, Springer-Verlag (2009), 502 pp
3. Apt, K.R., de Boer, F.S., Olderog, E.R., de Gouw, S.: Verification of Object-Oriented Programs: A Transformational Approach. J. Comp. Sys. Sci. 78(3), 823–852 (2012), [link](#)
A Omitted Proofs

Notation: We sometimes use $p^v_t$ to denote the result of substituting $t$ for $v$ in $p$.

Proof of Theorem 2:

Proof. We have to prove the equality $LHS = RHS$ where $LHS \equiv \llbracket p \rrbracket$ and $RHS \equiv \{\sigma \mid M[s](\sigma) \subseteq \llbracket q \rrbracket\}$. We will first prove the direction $LHS \subseteq RHS$ and then turn to the converse question.

1) $LHS \subseteq RHS$: Assuming a state $\sigma \in LHS$, then by left-totality of $\sim$, there is a $\zeta'$ such that $\sigma \sim \zeta'$. Furthermore, from $\sigma \models \forall \varsigma \bullet \text{comp}_s(\zeta', \varsigma) \rightarrow q\Theta^{\text{obj}}_s(\varsigma)$ and Lemma 3 we can deduce that every state $\sigma' \in M[s](\sigma)$ has a $\zeta$ satisfying $\sigma' \sim \zeta$ as well as $\text{comp}_s(\zeta', \zeta)$. Since all premises of the implication on the left-hand side are satisfied, $\models q\Theta^{\text{obj}}_s(\varsigma)$ must hold as well. Note that the latter two are properties of $\varsigma$ and $\zeta'$ rather than any particular state. Using Lemma 2 we can then deduce $\sigma' \models q$ and since our only assumption about $\sigma'$ is that it is a post-state of $\sigma$, it follows that $M[s](\sigma) \subseteq \llbracket q \rrbracket$ and hence that $\sigma \in RHS$. 

2) **RHS \subseteq LHS:** Assume \( \sigma \in RHS \). \( \sigma \) is hence an initial state and all its post-states \( \sigma' \in M[s](\sigma) \) satisfy the assertion \( q \). Then, by left-totality of \( \sim \), there is a frozen state \( \varsigma' \) such that \( \sigma \sim \varsigma' \) and for every post-state \( \sigma' \) there is a frozen state \( \varsigma \) such that \( \sigma' \sim \varsigma \). Now, by Lemma 3, every pair of (frozen) pre- and post-state \( (\varsigma', \varsigma) \in \text{comp}_\sigma \). Also, since the post-states \( \sigma' \) satisfy \( q \) and \( \sigma' \sim \varsigma \), by Lemma 2 we know that \( \models q_{\sigma'}(\varsigma) \). Therefore, the entire assertion \( p \) is true in \( \sigma \) and hence \( \sigma \in LHS \).

**Lemma 5.** For every \( \mu \)-recursive \( k \)-ary function \( f \), there exists a formula \( p \) in \( AL \) with free variables \( r, x_1, \ldots, x_k \), such that

\[
f(a_1, \ldots, a_k) = z \iff p_{r, x_1, \ldots, x_k}^z, a_1, \ldots, a_k
\]

Proof. By induction over the structure of \( \mu \)-recursive functions.

- If \( f \) is a constant function \( f(x_1, \ldots, x_k) = n \), then the formula \( p \equiv r = n \) satisfies the Lemma.
- If \( f \) is the successor function \( f(x_1) = x_1 + 1 \), then the formula \( p \equiv r = x_1 + 1 \) satisfies the Lemma.
- If \( f \) is the projection \( f(x_1, \ldots, x_n) = x_i \), then the formula \( p \equiv r = x_i \) satisfies the Lemma.
- If \( f \) is a composition of a \( k \)-ary function \( h \) and \( k \)-ary functions \( g_1, \ldots, g_k \), then by the induction hypothesis, there are formulas \( p_h, p_{g_1}, \ldots, p_{g_k} \) corresponding to the functions \( h, g_1, \ldots, g_k \) as described in the Lemma. Then, \( p \equiv \exists v_1, \ldots, v_k : \mathbb{N} \bullet p_h^{x_1 \ldots x_k} \wedge p_{g_1}^{v_1} \wedge \ldots \wedge p_{g_k}^{v_k} \) satisfies the Lemma.
- If \( f \) is a primitive recursion \( \rho(g, h) \) with a \( k \)-ary function \( g \) and a \( k + 2 \)-ary function \( h \), then according to the induction hypothesis, there are formulas \( p_g \) and \( p_h \) corresponding to the functions \( g \) and \( h \) as described in the Lemma. Now, \( p \equiv \exists s : (\mathbb{N}^k)^* \bullet |s| = x_1 \wedge p_g^{r, x_1 \ldots x_k} \wedge p_h^{s[x+1], s|x, s|, x_1, \ldots, x_k} \) satisfies the Lemma.
- If \( f \) is a minimization \( \mu f \) of a \( k \)-ary function \( f \), then according to the induction hypothesis, there is a formula \( p_f \) corresponding to \( f \) as described in the Lemma. Now, \( p \equiv \exists v : \mathbb{N} \bullet p_f^{r, x_1 \ldots x_k} \wedge \forall v' : \mathbb{N} \bullet \exists v_r : \mathbb{N} \bullet v' < v \rightarrow p_f^{r, x_1 \ldots x_k} \wedge v_r > 0 \) satisfies the Lemma.

**Theorem 6.** Every \( \mu \)-recursive function is expressible in \( AL \).

Proof. Direct consequence of Lemma 5.

**B Substitutions**

Analogous to the state update operation \( [u := e] \), the program logic uses 3 different kinds of substitutions on assertions.
### B.1 Substitutions

1. **Substitution of local variables** \( p[x := e] \)

   The substitution for local variables (or multiple local variables in parallel) is straightforward.

   It is defined by induction on the structure of \( p \):

   - \( y[x := e] \equiv \begin{cases} e & \text{if } x = y \\ y & \text{otherwise} \end{cases} \) (includes \( y \equiv \text{self} \))
   - \( v[x := e] \equiv v \)
   - \( n, \text{true}, \text{false} \) (constants - unaffected)
   - \( l.@v[x := e] \equiv l[x := e].@v \)
   - if \( l \) then \( l_1 \) else \( l_2 \) end \( x := e \) \( \equiv \) if \( l[x := e] \) then \( l_1[x := e] \) else \( l_2[x := e] \) end
   - \( l_1 = l_2 \equiv l_1[x := e] = l_2[x := e] \)
   - \( l_1 < l_2 \equiv l_1[x := e] < l_2[x := e] \)
   - \( [l] \in \{C_1, ..., C_n\} \equiv [l[x := e]] \in \{C_1, ..., C_n\} \)
   - \( l_1 \land l_2 \equiv l_1[x := e] \land l_2[x := e] \)
   - \( l_1 \lor l_2 \equiv l_1[x := e] \lor l_2[x := e] \)
   - \((\neg l_1)[x := e] \equiv \neg l_1[x := e] \)

   - \((\exists y : T.l_1)[x := e] \equiv \begin{cases} \exists y : T.l_1 & \text{if } x = y \\ \exists y : T.l_1[x := e] & \text{otherwise} \end{cases} \)

   - \((\forall y : T.l_1)[x := e] \equiv \begin{cases} \forall y : T.l_1 & \text{if } x = y \\ \forall y : T.l_1[x := e] & \text{otherwise} \end{cases} \)

2. **Substitution of instance variables** \( p[l.@v := e] \)

   The substitution for instance variables needs to take aliasing into account.

   For this, it is handy to have conditionals in the assertion language.

   It is defined by induction on the structure of \( p \):

   - \( l[l.@v := e] \equiv l \) for \( l \equiv x, v, \text{self}, n, \text{true}, \text{false} \) (constants - unaffected)
   - \( l'.@v'[l.@v := e] \equiv \begin{cases} \text{if } l' = l & \text{then } e \text{ else } l'.@v' \text{ end} & \text{if } @v = @v' \\ l'.@v' & \text{otherwise} \end{cases} \)
   - if \( l' \) then \( l_1 \) else \( l_2 \) end \( l.@v := e \) \( \equiv \) if \( l'[l.@v := e] \) then \( l_1[l.@v := e] \) else \( l_2[l.@v := e] \) end
   - \( l_1 = l_2 \equiv l_1[l.@v := e] = l_2[l.@v := e] \)
   - \( l_1 < l_2 \equiv l_1[l.@v := e] < l_2[l.@v := e] \)
   - \( [l'] \in \{C_1, ..., C_n\} \equiv [l'[l.@v := e]] \in \{C_1, ..., C_n\} \)
   - \( l_1 \land l_2 \equiv l_1[l.@v := e] \land l_2[l.@v := e] \)
   - \( l_1 \lor l_2 \equiv l_1[l.@v := e] \lor l_2[l.@v := e] \)
   - \((\neg l_1)[l.@v := e] \equiv \neg l_1[l.@v := e] \)
   - \((\exists y : T.l_1)[l.@v := e] \equiv \exists y : T.l_1[l.@v := e] \)
   - \((\forall y : T.l_1)[l.@v := e] \equiv \forall y : T.l_1[l.@v := e] \)
Lemma 6 (Substitution of instance variables). For all logical expressions $s$ and $t$, all assertions $p$, all instance variables $@u$ and all proper states $\sigma$

$$\sigma(s[@u := t]) = \sigma(@u := t)(s)$$

$$\sigma \models p[@u := t] \iff \sigma(@u := \sigma(t)) \models p.$$  \hspace{1cm} (1)  \hspace{1cm} (2)

Proof. By induction on the structure of $s$ and $p$. $\square$

2. Substitution for object creation $p[x := \text{new}_C]$

The substitution for object creation calculates the weakest precondition of an object creation statement. For a slightly simpler case without classes, [2, page 221] defines a substitution $[x := \text{new}].$ This substitution, however, is only applicable to so-called “pure” assertions. Fortunately, except for conditionals, our logical expressions satisfy all requirements and [2, page 223] gives a Lemma that allows eliminating conditionals like ours by substituting them for logically equivalent expressions. We can thus use the substitution and only need to modify it slightly to reflect the addition of classes.

The substitution is then defined by induction on the structure of $p$:

- $l[x := \text{new}_C] = l$ for $l \equiv \text{self, null, v, x, n, true, false}$
- $l.[v][x := \text{new}_C] \equiv \begin{cases} \text{init}_C.@v \text{ if } l \equiv x \\ l.@v \text{ otherwise} \end{cases}$
- $(\{x\} \in \{C_1, \ldots, C_n\})[x := \text{new}_C] \equiv C \in \{C_1, \ldots, C_n\}$
- $(l_1 = l_2)[x := \text{new}_C] \equiv l_1[x := \text{new}_C] = l_2[x := \text{new}_C]$ if $l_1 \neq x$ and $l_1 \neq \text{if...end}$ and $l_2 \neq x$ and $l_2 \neq \text{if...end}$
- $(x = l_2)[x := \text{new}_C] \equiv \text{false}$ if $l_2 \neq x$ and $l_2 \neq \text{if...end}$ (also the symmetric case)
- $(x = x)[x := \text{new}_C] \equiv \text{true}$
- $(\text{if } l_0 \text{ then } l_1 \text{ else } l_2 \text{ end } = l')[x := \text{new}_C] \equiv \text{if } l_0[x := \text{new}_C] \text{ then } (l_1 = l')[x := \text{new}_C] \text{ else } l_2[x := \text{new}_C] \text{ end}$
- $(\text{if } l' \text{ then } l_1 \text{ else } l_2 \text{ fi})[x := \text{new}_C] \equiv \text{if } l'[x := \text{new}_C] \text{ then } l_1[x := \text{new}_C] \text{ else } l_2[x := \text{new}_C] \text{ fi}$

Note: conditionals can always be moved outwards to be the outmost operation in an assertion.

- $l_1 < l_2 \equiv l_1[x := \text{new}_C] < l_2[x := \text{new}_C]$
- $(l_1 \land l_2)[x := \text{new}_C] \equiv l_1[x := \text{new}_C] \land l_2[x := \text{new}_C]$
- $(l_1 \lor l_2)[x := \text{new}_C] \equiv l_1[x := \text{new}_C] \lor l_2[x := \text{new}_C]$
- $(\neg l_1)[x := \text{new}_C] \equiv \neg l_1[x := \text{new}_C]$
- $(\exists y : T.l_1)[x := \text{new}_C] \equiv \exists y : T.l_1[x := \text{new}_C] \lor l_1[x/y] : x := \text{new}_C$
- $(\forall y : T.l_1)[x := \text{new}_C] \equiv \forall y : T.l_1[x := \text{new}_C] \land l_1[x/y] : x := \text{new}_C$

C Clarke’s Incompleteness Result & Turing Completeness

Clarke’s Incompleteness Result [6] demonstrates that there are programming languages for which no sound and complete Hoare Logic can exist. Since no sound and complete Hoare Logic was proposed for a dynamically-typed programming language before, it is interesting to study whether this is at all possible.

However, the argument of Clarke is not applicable to $\text{dyn}$ for three reasons:
1. **dyn** does not satisfy the assumption that the expressions used in the programming language are a subset of those used in the assertion language. This is only the case for statically-typed languages.
2. **dyn** does not fulfill the language requirements Clarke bases his argument on. In particular, it features neither global variables nor internal procedures nor does it allow passing procedure names as parameters.
3. Indeed, **dyn** ceases to be Turing complete under a finite interpretation\(^7\) which will be explained in the following section.

### C.1 Turing completeness

**dyn** is of course Turing complete. Writing a **dyn** program simulating a Turing machine is a straightforward exercise. However, it is not that easy to see that this expressiveness stems only from the fact that **dyn** programs are allowed to create an unbounded number of objects. In particular, while the stack depth in **dyn** is also unbounded, it is only possible to access a finite number of variables at the top of the stack (the local variables of the current method) without pop'ing (exiting the current method) which only yields the expressive power of push-down automata rather than that of queue- or Turing-machines. To see that this is the case, consider the following construction:

1. bounding the number of objects on the Heap to some limit \(k \in \mathbb{N}\) can be achieved by introducing a global counter and letting object creation fail once the limit is reached.
2. It is straightforward to rewrite **dyn**’s operational semantics (given in Section 3) in such a way that it uses a stack to handle method calls instead of `begin-local-blocks`.
3. Now the states can be separated into Stack and Heap. By identifying states with the same Heap, the labelled transition system defined by the operational semantics becomes a directed graph. Note that there are only finitely many possible Heaps containing \(\leq k\) objects. This implies that diverging programs must have state cycles. Annotating the edges not only by computation steps, but also by stack frames being pushed and popped on method calls / returns is possible since we only have a finite number of objects and hence also a finite number of possible stack frames to be pushed. This way, we can for every **dyn** program obtain a push-down automaton that accepts all finite computations of the original program with an empty stack.
4. Since emptyness is decidable for push-down automata, we hence have a method to check whether or not a given **dyn** program has a finite computation. If is does not, it will surely not halt. Thus the halting problem for **dyn** with bounded Heap is decidable and **dyn** hence ceases to be Turing-complete when bounding the Heap.

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7 Clarke’s programming language uses Integer variables. In this case “finite interpretation” means restricting the program variables to the finite subset \(\{0, 1\}\) of their original domain. Since in **dyn** variables are of object type, interpreting them finitely means bounding the number of objects on the Heap to a finite number.