QCD tests in $\tau$ decays with optimized perturbation expansion

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The next-to-next-to-leading order perturbative QCD corrections to $R_\tau$ and the higher moments of the invariant mass distribution in the hadronic $\tau$ decays are considered. The renormalization scheme dependence of these corrections is discussed. The optimized predictions are obtained, using the principle of minimal sensitivity as a guide to select the preferred renormalization scheme. A simplified fit is performed, using $R_\tau$ and $R_{12}^\tau$, to see how the use of the optimized expansion may affect the determination of the $\alpha_s$ and the dimension six condensates from the experimental data.

Recently there has been considerable interest in the QCD predictions for the total hadronic width of the $\tau$ and the higher order moments of the invariant mass distribution in the hadronic $\tau$ decays. These quantities have been used to obtain tight experimental constraints on $\alpha_s$ and the condensates. In spite of the high precision which may be obtained in fits to the experimental data it is important to investigate in detail how the results may be affected by the renormalization scheme (RS) dependence of the perturbative QCD predictions. Usually the $\overline{MS}$ scheme is used to evaluate the perturbative QCD corrections. However, there is a two-parameter freedom in the choice of the RS in the next-to-next-to-leading order (NNLO), and there is no a priori theoretical or phenomenological reason why the $\overline{MS}$ scheme should be preferred. The difference in predictions obtained in various schemes is formally of higher order in the coupling, but numerically it may be significant, particularly at the energy scale of $m_\tau = 1.777$ GeV.

The QCD prediction for the $R_\tau$ ratio

$$R_\tau = \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau e^- \overline{\nu_e})},$$

has the form

$$R_\tau = 3S_{CKM}S_{EW} (1 + \delta_{pt}^{\text{tot}} + \delta_{m}^{\text{tot}} + \delta_{SVZ}^{\text{tot}}),$$

where $S_{CKM} = (|V_{ud}|^2 + |V_{us}|^2) \approx 1$. The factor $S_{EW} = 1.0194$ represents the corrections from electroweak interactions. The $\delta_{pt}^{\text{tot}}$ contribution denotes the purely perturbative QCD correction, evaluated for three massless quarks. The $\delta_{m}^{\text{tot}}$ contribution denotes the correction from quark masses ($\delta_{m}^{\text{tot}} \approx 0.009$). In the case of the $R_{kl}^{\tau}$ moments of the invariant mass distribution $d\Gamma_{ud}/ds$ of the Cabibbo allowed decays, which are defined by the relation

$$R_{kl}^{\tau} = \frac{1}{\Gamma_\tau} \int_0^{m_{\tau}^2} ds \left(1 - \frac{s}{m_\tau^2}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{d\Gamma_{ud}}{ds},$$

where $\Gamma_\tau$ is the electronic width of $\tau$, the QCD prediction has the form:

$$R_{kl}^{\tau} = 3 |V_{ud}|^2 S_{EW} R_0 (1 + \delta_{kl}^{pt} + \delta_{SVZ}^{kl}),$$

where $R_0$ in $R_{kl}^{\tau}$ denotes the parton model prediction. The $\delta_m^{kl}$ contribution is negligible in the case of $R_{kl}^{\tau}$.

The $\delta_{SVZ}^{kl}$ contributions in $R_{kl}^{\tau}$ and $R_{kl}^{\tau}$ denote nonperturbative QCD corrections calculated using the SVZ approach

$$\delta_{SVZ} = \sum_{D=4,6...} c_D \frac{O_D}{m_{\tau}^D},$$

The parameters $O_D$ in Eq. (5) denote vacuum expectation values of the gauge invariant operators of dimension $D$. The $c_D$ coefficients are in principle power series in the strong coupling constant, which depend on the considered quantity.

The contribution from the $D = 4$ term in the SVZ expansion for $R_{kl}^{\tau}$ may be reliably expected.
to be small, since $O_4$ is well constrained by the sum rules phenomenology, and the relevant coefficient function starts at $O(\alpha_s^2)$. However, the $D = 6$ contributions to $R_{\tau}$ is not suppressed, and there is little information on the value of $O_6$. It was therefore proposed to treat $O_D$ as free parameters, which are to be extracted together with $\alpha_s$ from a fit to the experimental data for $R_{\tau}$ and the higher moments of the invariant mass distribution.

The analysis reported in involved the $R_{\tau}^{kl}$ moments with $(k,l)$ equal to $(1,0)$, $(1,1)$, $(1,2)$ and $(1,3)$. However, if we are interested primarily in the possible effect of RS dependence, we may simplify the discussion by considering only the $R_{\tau}^{kl}$ moment, for which — similarly as for the $R_{\tau}$ — the $D = 4$ contribution is suppressed and there is significant contribution from the $D = 6$ term. Retaining in the SVZ expansion only the $D = 6$ term, which appears to be a dominant source of the uncertainty in the nonperturbative sector, we obtain a simplest set of the QCD predictions which allows for a self-consistent extraction of $\alpha_s$ and $O_6$ from tau decays.

The perturbative QCD corrections $\delta^\text{pt}$ and $\delta^\text{kl}$ may be expressed as a contour integral in the complex energy plane, with the so called Adler function under the integral. (Actually $\delta^\text{tot} = \delta^\text{pt}$. We have:

$$\delta^\text{kl} = \frac{i}{\pi} \int_{C} \frac{d\sigma}{\sigma} f^{kl}(\frac{\sigma}{m_{\tau}^2})d\sigma_{D,V}(-\sigma),$$

(6)

where $C$ is a contour running clockwise from $\sigma = m_{\tau}^2 + i\epsilon$ to $\sigma = m_{\tau}^2 - i\epsilon$ away from the region of small $|\sigma|$. In the actual calculation we assume that $C$ is a circle $|\sigma| = m_{\tau}^2$. The Adler function is defined by the relation:

$$(-12\pi^2)\frac{d}{d\sigma}\Pi^{V(1)}_{V}(\sigma) = 3S_{C,KM}[1 + \delta_{D,V}(-\sigma)],$$

(7)

where $\Pi^{V(1)}_{V}$ denotes the transverse part of the vector current correlator. The function $f^{12}(\sigma/m_{\tau}^2)$ has the form:

$$f^{00}(x) = \frac{1}{2} - x + x^3 - \frac{1}{2} x^4.$$  

(9)

The function $f^{00}(x/m_{\tau}^2)$ has the form:

$$f^{00}(x) = 1 - x + x^3 - \frac{1}{2} x^4.$$  

(9)

The NNLO renormalization group improved perturbative expansion for $\delta_{D,V}$ may be written in the form:

$$\delta_{D,V}^{(2)}(-\sigma) = a(-\sigma)[1 + r_1 a(-\sigma) + r_2 a^2(-\sigma)],$$

where $a = \alpha_s/\pi = g^2/(4\pi^2)$ denotes the running coupling constant that satisfies the NNLO renormalization group equation:

$$\sigma \frac{da}{d\sigma} = -b \sigma^2 (1 + c_1 a + c_2 a^2),$$

(11)

In the $\overline{MS}$ scheme we have $r_1^{\overline{MS}} = 1.63982$ and $r_2^{\overline{MS}} = 6.37101$. The renormalization group coefficients for $n_f = 3$ are $b = 4.5$, $c_1 = 16/9$ and $c_2^{\overline{MS}} = 3863/864 \approx 4.471$.

If the Adler function is expanded in terms of $a(m_{\tau}^2)$, then the $\sigma$ dependence appears through the powers of $\ln(-\sigma/m_{\tau}^2)$. The contour integration is then straightforward and the conventional NNLO expansion of $\delta^\text{pt}$ in terms of $a(m_{\tau}^2)$ is easily obtained. However, it was observed in that one may also keep under the integral the renormalization group improved expression for the Adler function. In this case the contour integral has to be evaluated numerically. This results in the essential improvement of the conventional expansion, corresponding to the all-order resummation of some of the corrections arising from analytic continuation from spacelike to timelike momenta.

The QCD predictions calculated in the next-to-next-to-leading order (NNLO) approximation with massless quarks depend on two RS parameters, which in principle may be arbitrary. The two-parameter freedom in NNLO arises because in each order of perturbation expansion we are free to choose independently the finite parts of the coupling constant renormalization constant. Different choices of the finite parts of the renormalization constants result in different definitions of the coupling constant, which are related by finite renormalization. (The formulas describing how the redefinition of the coupling affects the
coefficients \( r_i \) and \( c_2 \) are collected for example in [13]. Also the dimensional QCD parameter \( \Lambda \) depends on the choice of the RS. In the NNLO there exists however a RS invariant combination of the expansion coefficients [15,16]:
\[
\rho_2 = c_2 + r_2 - c_1 r_1 - r_1^2. \tag{12}
\]

For the Adler function we have \( \rho_2 = 5.23783 \).

The change in the expansion coefficients and the change in the coupling constant compensate each other, but of course in the finite order of perturbation expansion such compensation may only be approximate, which results in the numerical RS dependence of the perturbative predictions. There has been intensive discussion on the prescriptions for making an optimal choice of the RS [17]. One of the most attractive propositions is the choice based on the so called principle of minimal sensitivity (PMS) [13], which singles out the scheme parameters for which the finite order prediction is least sensitive to the change of RS, similarly to what we expect from the actual physical quantity. (It should be emphasized that in our case the algebraic PMS optimization equations [15] do not apply and a nontrivial numerical analysis of the perturbative prediction is required.)

The PMS optimization has to some extent a heuristic character and therefore it is important to investigate the stability of the predictions also with respect to non-infinitesimal changes of the scheme parameters. This may be done by calculating the variation of the predictions over a set of \textit{a priori} acceptable schemes. A condition for selecting a class of acceptable schemes in NNLO has been proposed in [13]:
\[
|c_2| + |r_2| + c_1 |r_1| + r_1^2 \leq l |\rho_2|. \tag{13}
\]

This condition is based on the observation, that the schemes with unnaturally large expansion coefficients would give rise to extensive cancellations in the expression for the RS invariant \( \rho_2 \). The constant \( l \) controls the degree of cancellation in \( \rho_2 \) that we want to allow. In particular, for the conventional QCD expansion the PMS scheme lies approximately at the boundary of the region corresponding to \( l = 2 \) [13].

A detailed discussion of the RS dependence of \( \delta_{pt}^{12} \) has been presented in [19] and would not be repeated here. An important conclusion from [19] is that the contour integral resummation of higher order analytic continuation corrections is very important for ensuring the RS stability of the predictions. (This has been also discussed in [12].)

The instability of the conventional expansion for \( \delta_{pt}^{12} \) has been discussed in [13,14,12]. Below we concentrate on the RS dependence of \( \delta_{pt}^{12} \).

![Figure 1. The contour plot of \( \delta_{pt}^{12} \) as a function of the scheme parameters \( r_1 \) and \( c_2 \), for \( \Lambda^{(3)\text{MS}} = 325 \text{ MeV} \). For technical reasons we use \( c_2 - c_1 r_1 \) on the vertical axis instead of \( c_2 \). The boundary of the region of scheme parameters satisfying the Eq. 13 is also indicated.](image-url)
sentially non-polynomial character, the PMS parameters are well approximated by $r_1 = 0$ and $c_2 = 1.5 \rho_2$. We shall choose these values as our optimized parameters. (The exact PMS parameters have some dependence on the value of $\Lambda_{\overline{MS}}$. Also, for very large values of $\Lambda_{\overline{MS}}$ the pattern of RS dependence is more complicated than this shown in figure 1. However, for all values of $\Lambda_{\overline{MS}}$ the RS dependence in the vicinity of $r_1 = 0$ and $c_2 = 1.5 \rho_2$ is very small.) The set of scheme parameters, that involve the same — or smaller — degree of cancellation in $\rho_2$ than our preferred parameters, satisfies the condition (13) with $l = 2$. The boundary of this set is indicated on the figure 1. By calculating the variation of the predictions over this set of scheme parameters we may estimate in a quantitative way the sensitivity of the NNLO predictions to the change of RS. Thus obtained estimate may then be compared with a similar estimate for other quantities, for which the NNLO predictions are known. It should be noted that although the $\overline{MS}$ parameters lie outside of the “allowed” region shown in the figure 1, the numerical value of the prediction in the $\overline{MS}$ scheme is close to the lowest value attained in the “allowed” region.

It is interesting to note, that the RS dependence pattern of $\delta_{12}^{pt}$ is quite similar to the pattern found in [19] for $\delta_{tot}^{pt}$, despite the fact that the conventional expansions for these quantities appear to be quite different. Indeed, in the conventional expansion for $\delta_{tot}^{pt}$ we have $r_1^{\overline{MS}} = 5.2023$ and $r_2^{\overline{MS}} = 26.3659$, which gives $\rho_2 = -5.4757$, while for $\delta_{pt}^{12}$ we have $r_1^{\overline{MS}} = 3.5795$ and $r_2^{\overline{MS}} = 4.3441$, which gives $\rho_2 = -10.3614$. This seems to suggest that the improved predictions obtained with contour integral expressions are more natural, reflecting the common origin of the two corrections in a better way.

In figure 2 we show the RS dependence of the next-to-leading order (NLO) prediction for $\delta_{pt}^{12}$. By numerical optimization we find that in NLO $r_1^{\overline{MS}} \approx -0.64$. (Again, this depends to some extent on $\Lambda_{\overline{MS}}$, but this dependence has negligible effect on the numerical value of the prediction.)

In figure 3 we show the NNLO PMS predictions for $\delta_{pt}^{12}$ as a function of $m_\tau/\Lambda_{\overline{MS}}$ (3), together with the minimal and maximal values obtained by varying the scheme parameters within the region determined by the condition (13) with $l = 2$. We see that the NNLO predictions for $\delta_{pt}^{12}$, obtained by numerically evaluating the contour integral expression (6), are free from potentially dangerous RS instabilities, even for large values of $\Lambda_{\overline{MS}}$. This situation is similar to that encountered for $\delta_{tot}^{pt}$. For comparison we also show the PMS predictions obtained in the next-to-leading order (NLO). We see that RS dependence of the NNLO expression within the region defined by the condition (13) is smaller than the difference between NNLO and NLO PMS predictions. In a separate figure we show the NNLO and NLO predictions in the $\overline{MS}$ scheme (figure 4).

In order to see how the optimization of the scheme parameters affects the fits to the experimental data, we first test the accuracy of the approximation in which one only retains the $O_4$, $O_6$ and $O_8$ contributions in the SVZ expansion. To this end we make a fit of $\alpha_s$ and $Q_0$ in the $\overline{MS}$ scheme and compare the results with the fit performed by ALEPH [3], in which the $O_4$, $O_6$ and $O_8$ contributions have been taken into account in the (1,0), (1,1), (1,2) and (1,3) moments. To make
the fits we use the following expressions:

\[ R_\tau = 3 \times 1.0194 \left( 0.991 + \delta_{\text{tot}} - 3.75 \, O_6 \right), \]  

(14)

and

\[ D_{12}^{\tau} = R_{12}^{\tau} / R_{00} = \frac{13 \left( 1 + \delta_{12}^{\tau} + 20.16 \, O_6 \right)}{210 \left( 1 + \delta_{\text{tot}} - 3.75 \, O_6 \right)}, \]  

(15)

If we take, following ALEPH [3], \( R_\tau = 3.645 \pm 0.024 \) and \( D_{12}^{\tau} = 0.0570 \pm 0.0013 \), we obtain from the fit in the \( \overline{\text{MS}} \) scheme \( \alpha_s(M_\tau^2) = 0.1209 \pm 0.0013 \) and \( O_6 = -0.0010 \pm 0.0012 \). This appears to be remarkably close to the values 0.121 and -0.0016 obtained in the full fit by ALEPH. This gives us confidence that the “\( O_6 \) approximation” captures the essential features of QCD corrections in \( \tau \) decays.

Let us now study how the RS dependence may affect the fit to the experimental data. Let us take \( R_\tau = 3.635 \pm 0.016 \), which is a weighted average of three possible determinations [20], involving \( B_e = 0.1783 \pm 0.0008 \), \( B_\mu = 0.1735 \pm 0.0010 \) and \( \tau_\tau = (291.0 \pm 1.5) \times 10^{-15} \text{sec.} \) (These are the so-called “our fit” values. Using a set of “our average” values given in [20] we would obtain \( R_\tau = 3.643 \).) Let us also take [4] \( D_{12}^{\tau} = 0.0559 \pm 0.0007 \), which is the most precise published value up to date. Using the NNLO PMS expression we obtain then \( \alpha_s(M_\tau^2) = 0.1188 \pm 0.0008 \) \( (\alpha_s(m_\tau^2) = 0.330 \pm 0.008) \) and \( O_6 = -0.0021 \pm 0.0006 \). Performing the same fit, but using now the NNLO \( \overline{\text{MS}} \) expression, we obtain \( \alpha_s(M_\tau^2) = 0.1198 \) and the same value for \( O_6 \) as in NNLO. Taking the difference of the NNLO and

\[ \delta_{12}^{\tau} \] as a function of \( m_\tau / \Lambda_{\overline{\text{MS}}}^{(3)} \), obtained in NNLO (upper solid curve) and NLO (lower solid curve). The dashed lines indicate variation of the predictions when the scheme parameters are changed within the region satisfying the Eq. 13.
NLO PMS fits is perhaps the best way of estimating the accuracy of the perturbative contribution in this problem, since we found the QCD corrections to be quite stable with respect to change of the scheme. We see that thus obtained uncertainty of the fitted value of $\alpha_s$ is of the order 0.0022, i.e. it is quite large, compared for example to the experimental uncertainty. (This estimate of uncertainty involves only the perturbative uncertainty — to have estimate of theoretical uncertainty for the total QCD prediction one should also consider the accuracy of the SVZ expansion itself.) It should be noted that the NNLO-NLO difference is strongly RS dependent so it is essential to optimize the choice of the scheme before comparing the predictions in successive orders.

Concluding, the renormalization scheme dependence of the perturbative QCD corrections to $R_\tau$ and to the $R_{12}^{12}$ moment of the invariant mass distribution in hadronic tau decays has been studied in detail. The optimized predictions have been obtained using the principle of minimal sensitivity as a guide to select the preferred renormalization scheme. The stability of the predictions obtained via the contour integral expression (6) has been verified, using a specific condition to eliminate the schemes that have unnaturally large expansion coefficients. However, the difference between predictions in the conventionally used $\overline{\text{MS}}$ scheme and the optimized predictions obtained in the PMS scheme was found to be phenomenologically significant. Also, the difference between the NNLO and NLO predictions in the PMS scheme was found to be significant, indicating perhaps that the uncertainty in the perturbative expression is larger than previously expected.

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