THE A-MÖBIUS FUNCTION OF A FINITE GROUP

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Abstract. The Möbius function of the subgroup lattice of a finite group $G$ has
been introduced by Hall and applied to investigate several different questions.
We propose the following generalization. Let $A$ be a subgroup of the automorphism
group $\text{Aut}(G)$ of a finite group $G$ and denote by $C_A(G)$ the set of
$A$-conjugacy classes of subgroups of $G$. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A \}$
be the element of $C_A(G)$ containing $H$. We may define an ordering in $C_A(G)$
in the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$. We consider
the Möbius function $\mu_A$ of the corresponding poset and analyse its properties
and possible applications.

1. Introduction

The Möbius function of a finite partially ordered set (poset) $P$ is the map
$\mu_P : P \times P \to \mathbb{Z}$ satisfying $\mu_P(x, y) = 0$ unless $x \leq y$, in which case it is
defined inductively by the equations $\mu_P(x, x) = 1$ and $\sum_{x \leq z \leq y} \mu_P(x, z) = 0$ for
$x < y$.

In a celebrated paper [7], P. Hall used for the first time the Möbius function
$\mu$ of the subgroup lattice of a finite group $G$ to investigate some properties of
$G$, in particular to compute the number of generating $t$-tuples of $G$. A detailed
investigation of the properties of the function $\mu$ associated to a finite group $G$ is
given by T. Hawkes, I. M. Isaacs and M. Özaydın in [8]. In that paper, the authors
also consider the Möbius function $\lambda$ of the poset of conjugacy classes of subgroups
of $G$, where $[H] \leq [K]$ if $H \leq K^g$ for some $g \in G$ (see [8, Section 7]). In particular,
they propose the interesting and intriguing question of comparing the values of $\mu$
and $\lambda$.

In this paper we aim to generalize the definitions and main properties of the func-
tions $\mu$ and $\lambda$ to a more general contest. Let $G$ and $A$ be a finite group and a sub-
group of the automorphism group $\text{Aut}(G)$ of $G$, respectively. Denote by $C_A(G)$ the
set of $A$-conjugacy classes of subgroups of $G$. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A \}$
be the element of $C_A(G)$ containing $H$. We may define an ordering in $C_A(G)$ in
the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$; we consider
the Möbius function $\mu_A$ of the corresponding poset. We will write $\mu_A(H, K)$ in place of
$\mu_A([H]_A, [K]_A)$. When $A = \text{Inn}(G)$, we write $\mathcal{C}(G)$ and $[H]$, in place of $C_{\text{Inn}(G)}(G)$
and $[H]_{\text{Inn}(G)}$. When $A = 1$, $\mu_A = \mu$ is the Möbius function in the subgroup
lattice of $G$, introduced by P. Hall. In the case when $A = \text{Inn}(G)$ is the group of the inner
automorphism, then $\mu_{\text{Inn}(G)} = \lambda$. Note that for any subgroup $A$ of $\text{Aut}(G)$, we get
$[G]_A = \{ G \}$.

In Section 2 we prove some general properties of $\mu_A$. In particular we prove the
following result:
Proposition 1. Let $G$ be a finite solvable group. If $G' \leq K \leq G$ and $A$ is the subgroup of $\text{Inn}(G)$ obtained by considering the conjugation with the elements of $K$, then $\lambda(H, G) = \mu_A(H, G)$ for any $H \leq G$.

To illustrate the meaning of the previous proposition, consider the following example. Let $G = A_4$ be the alternating group of degree 4 and $A$ the subgroup of $\text{Inn}(G)$ induced by conjugation with the elements of $G' \cong C_2 \times C_2$. The posets $\mathcal{C}(G)$ and $\mathcal{C}_A(G)$ are different. For example there are three subgroups of $G$ of order 2, which are conjugated in $G$, but not $A$-conjugated. However $\lambda(H, G) = \mu_A(H, G)$ for any $H \leq G$.

In Section 3 we generalize some result given by Hall in [7], about the cardinality $\phi(G, t)$ of the set $\Phi(G, t)$ of $t$-tuples $(g_1, \ldots, g_t)$ of group elements $g_i$ such that $G = \langle g_1, \ldots, g_t \rangle$. As observed by P. Hall, using the Möbius inversion formula, it can be proved that

$$\phi(G, t) = \sum_{H \leq G} \mu(H, G)|H|^t.$$  

We generalize this formula, showing that $\phi(G, t)$ can be computed with a formula involving $\mu_A$ for any possible choice of $A$.

Theorem 2. For any finite group $G$ and any subgroup $A$ of $\text{Aut}(G)$,

$$\phi(G, t) = \sum_{[H, A] \in \mathcal{C}_A(G)} \mu_A(H, G)|\cup_{a \in A} (H^a)|^t.$$  

If $G$ is not cyclic, then $\phi(G, 1) = 0$, so we obtain the following equality, involving the values of $\mu_A$.

Corollary 3. If $G$ is not cyclic, then

$$0 = \sum_{[H, A] \in \mathcal{C}_A(G)} \mu_A(H, G)|\cup_{a \in A} H^a|.$$  

Further generalizations are given in Section 4 where we consider the function $\phi^*(G, t)$, which is an analogue of $\phi(G, t)$: actually, $\phi^*(G, t)$ denotes the cardinality of the set of of $t$-tuples $(H_1, \ldots, H_t)$ of subgroups of $G$ such that $G = \langle H_1, \ldots, H_t \rangle$. As a corollary of our formula for computing $\phi^*(G, t)$, we obtain we following unexpected result.

Proposition 4. Let $\sigma(X)$ denote the number of subgroups of a finite group $X$. For any finite group $G$, the following equality holds:

$$1 = \sum_{H \leq G} \mu(H, G)\sigma(H).$$  

Finally, in Section 5 we consider one question originated from a result given by Hawkes, Isaacs and Özyaydin in [8]: they proved that the equality

$$\mu(1, G) = |G'| \lambda(1, G)$$  

holds for any finite solvable group $G$; later Pahlings [10] generalized the result proving that

$$\mu(H, G) = |N_{G'}(H) : G' \cap H| \cdot \lambda(H, G)$$
 holds for any \( H \leq G \) whenever \( G \) is finite and solvable. Following [5], we say that \( G \) satisfies the \((\mu, \lambda)\)-property if [2] holds for any \( H \leq G \). Several classes of non-solvable groups satisfy the \((\mu, \lambda)\)-property, for example all the minimal non-solvable groups (see [5]). However it is known that the \((\mu, \lambda)\)-property does not hold for every finite group. For instance, it does not hold for the following finite almost simple groups: \( A_9, S_9, A_{10}, S_{10}, A_{11}, S_{11}, A_{12}, S_{12}, A_{13}, S_{13}, J_2, PSU(3,3), PSU(5,2), M_{12}, M_{23}, M_{24}, PSL(3,11), HS, Aut(HS), He, Aut(He), McL, PSL(5,2), G_2(4), Co_3, PSU^-\(8,2\), PSU^+(8,2). It is somehow intriguing to notice that although the \((\mu, \lambda)\)-property fails for the sporadic groups \( M_{12}, J_2, McL \), it holds for their automorphism groups.

We prove the following generalization of Pahlings’s result.

**Theorem 5.** Let \( N \) be a solvable normal subgroup of a finite group \( G \). If \( G/N \) satisfies the \((\mu, \lambda)\)-property, then \( G \) also satisfies the \((\mu, \lambda)\)-property.

An almost immediate consequence of the previous theorem is the following.

**Corollary 6.** \( PSU(3,3) \) is the smallest group which does not satisfy the \((\mu, \lambda)\) property.

In the last part of Section 5 we use Theorem 2 to deduce some consequences of the \((\mu, \lambda)\)-property. In particular we prove the following theorem.

**Theorem 7.** Suppose that a finite group \( G \) satisfies the \((\mu, \lambda)\)-property. Then, for every positive integer \( t \), the following equality is satisfied:

\[
\sum_{[H] \in C(G)} \lambda(H,G) \left( \frac{|H^{t-1}|G|G'H|}{|G'N_G(H)|} - |\cup_{a \in A(H')} (H'^t)| \right) = 0.
\]

Some open questions are proposed along the paper.

2. **Applying some general properties of the Möbius function**

Given a poset \( P \), a closure on \( P \) is a function \( : P \to P \) satisfying the following three conditions:

a) if \( x, y \in P \) with \( x \leq y \), then \( \bar{x} \leq \bar{y} \);

b) \( \bar{\bar{x}} = \bar{x} \) for all \( x \in P \).

If \( \bar{\cdot} \) is a closure map on \( P \), then \( \bar{P} = \{ x \in P | \bar{x} = x \} \) is a poset with order induced by the order on \( P \). We have:

**Theorem 8** (The closure theorem of Crapo [3]). Let \( P \) be a finite poset and let \( : P \to P \) be a closure map. Fix \( x, y \in P \) such that \( y \in \bar{P} \). Then

\[
\mu_P(x, z) = \begin{cases} 
\mu_P(x, y) & \text{if } x = \bar{x} \\
0 & \text{otherwise}.
\end{cases}
\]

In [7], P. Hall proved that if \( H < G \), then \( \mu(H, G) \neq 0 \) only if \( H \) is an intersection of maximal subgroups of \( G \). Using the previous theorem, the following more general statement can be obtained.

**Proposition 9.** If \( H < G \) and \( \mu_A(H, G) \neq 0 \), then \( H \) can be obtained as intersection of maximal subgroups of \( G \).
Proof. Let $H$ be a proper subgroup of $G$ and let $\mathcal{P}$ be the intersection of the maximal subgroups of $G$ containing $H$. The map $[H]_A \mapsto [\mathcal{P}]_A$ is a well defined closure map on $\mathcal{C}_A(G)$, so the conclusion follows immediately from Theorem \ref{thm:main}. \hfill \Box

An element $a$ of a poset $\mathcal{P}$ is called conjunctive if the pair $\{a, x\}$ has a least upper bound, written $a \lor x$, for each $x \in \mathcal{P}$.

Lemma 10. \cite{[G]} Lemma 2.7] Let $\mathcal{P}$ be a poset with a least element $0$, and let $a > 0$ be a conjunctive element of $\mathcal{P}$. Then, for each $b > a$, we have

$$\sum_{a \lor x = b} \mu(x) = 0.$$ 

From the above Lemma \ref{lem:conjunctive} it follows easily the following Lemma \ref{lem:conjunctive2}, which, together with Lemma \ref{lem:conjunctive3} and Lemma \ref{lem:conjunctive4} allow us to prove Proposition \ref{prop:main}.

Lemma 11. Let $N$ be an $A$-invariant normal subgroup of $G$ and $H \leq G$. If $H < HN < G$, then

$$\mu_A(H, G) = -\sum_{|Y|_A \in \mathcal{S}_A(H, N)} \mu_A(H, Y),$$

with $\mathcal{S}_A(H, N) = \{|Y|_A \in \mathcal{C}_A(G) \mid [H]_A \leq |Y|_A < |G|_A \text{ and } YN = G\}$.

Proof. Let $\mathcal{P}$ be the interval $\{[K]_A \in \mathcal{C}_G(A) \mid [H]_A \leq [K]_A \leq [G]_A\}$. Notice that $[HN]_A$ is a conjunctive element of $\mathcal{P}$. Indeed $[HN]_A \lor [K]_A = [KN]_A$ for every $[K]_A \in \mathcal{P}$. So the conclusion follows immediately from Lemma \ref{lem:conjunctive} \hfill \Box

Lemma 12. Let $K$ and $A$ be a subgroup of $G$ and the subgroup of $\text{Inn}(G)$ induced by the conjugation with the elements of $K$, respectively. Assume that $N$ is an abelian minimal normal subgroup of $G$ contained in $K$ and $H < HN \leq G$. Then

$$\mu_A(H, G) = -\mu_A(HN, G)\gamma_A(N, H),$$

where $\gamma_A(N, H)$ is the number of $A$-conjugacy classes of complements of $N$ in $G$ containing $H$.

Proof. If $HN = G$, then $H$ is a maximal subgroup of $G$, hence $\mu_A(H, G) = -1$, while $\mu_A(HN, G) = \mu_A(G, G) = 1$ and $\gamma_A(N, H) = 1$, so the statement is true. So we may assume $HN < G$ and apply Lemma \ref{lem:conjunctive}. Suppose $|Y|_A \in \mathcal{S}_A(H, N)$. Let

$\mathcal{C} = \{J \leq G \mid H \leq J \leq Y\}$, \hspace{1cm} $\mathcal{D} = \{L \leq G \mid HN \leq L\}$

$\mathcal{C}_A = \{|J|_A \in \mathcal{C}_A(G) \mid [H]_A \leq |J|_A \leq |Y|_A\}$, \hspace{1cm} $\mathcal{D}_A = \{|L|_A \in \mathcal{C}_A(G) \mid [HN]_A \leq |L|_A\}$.

The map $\eta : \mathcal{C} \to \mathcal{D}$ sending $J$ to $JN$ is an order preserving bijection. Clearly, if $J_2 = J_1^x$ for some $x \in K$, then $\gamma(J_2) = N J_2 = N J_1^x = (N J_1)^x = (\gamma(J_1))^x$. Conversely assume $\gamma(J_2) = (\gamma(J_1))^x$ with $x \in K$. Since $YN = G$, $x = y n$ with $n \in N$ and $y \in Y \cap K$. Thus $J_2 N = (J_1 N)^y = (J_1 N)^y$ and consequently $J_2 = J_2 N \cap Y = (J_1 N)^y \cap Y = (J_1 N \cap Y)^y = J_1^y$. It follows that $\eta$ induces an order preserving bijection from $\mathcal{C}_A$ to $\mathcal{D}_A$, but then $\mu_A(H, Y) = \mu_A(HN, YN) = \mu_A(HN, G)$. \hfill \Box

The statement of the previous lemma leads to the following open question.

Question 13. Let $G$ be a finite group, $A \leq \text{Aut}(G)$ and $N$ an $A$-invariant normal subgroup of $G$. Does $\mu_A(HN, G)$ divide $\mu_A(H, G)$ for every $H \leq G$?

The following lemma is straightforward.
**Lemma 14.** Let $A$ be a subgroup of $\text{Aut}(G)$ and $N$ an $A$-invariant normal subgroup of $G$. Every $a \in A$ induces an automorphism $\overline{a}$ of $G/N$. Let $\overline{A} = \{ \overline{a} | a \in A \}$. Then, for any $H \leq G$, $\mu_A(HN,G) = \mu_{\overline{A}}(HN/N,G/N)$.

**Proof of Proposition 7**. We work by induction on $|G| \cdot |G : H|$. The statement is true if $G$ is abelian. Assume that $G'$ contains a minimal normal subgroup, say $N$, of $G$. If $N \leq H$, then, by Lemma 14

$$\lambda(H,G) = \lambda(H/N,G/N) = \mu_{\overline{A}}(H/N,G/N) = \mu_A(H,G).$$

So we may assume $N \not\leq H$. If $H$ is not an intersection of maximal subgroups of $G$, then $\lambda(H,G) = \mu_A(H,G) = 0$. Suppose $H = M_1 \cap \ldots \cap M_i$ where $M_1, \ldots, M_i$ are maximal subgroups of $G$. In particular $N$ is not contained in $M_i$ for some $i$, so $M_i$ is a complement of $N$ in $G$ containing $H$ and $N \cap H = 1$. By Lemma 12 we have

$$\lambda(H,G) = -\lambda(HN,G)\gamma(N,H), \quad \mu_A(H,G) = -\mu_A(HN,G)\gamma_A(N,H),$$

where $\gamma(N,H)$ is the number of conjugacy classes of complements of $N$ in $G$ containing $H$ and $\gamma_A(N,H)$ is the number of $A$-conjugacy classes of these complements. Suppose that $K_1, K_2$ are two conjugated complements if $N$ in $G$ containing $H$. Then $K_2 = K_1^n$ for some $n \in N_N(H)$. Since $N \leq G' \leq K$, it follows $\gamma(N,H) = \gamma_A(N,H)$. Moreover, by induction, $\lambda(HN,G) = \mu_A(HN,G)$, hence we conclude $\lambda(H,G) = \mu_A(H,G)$. \hfill \Box

### 3. Generalizing a formula of Philip Hall

We begin with introducing the functions $\Psi_A(H,t)$ and $\psi_A(H,t)$, analogue of $\Phi(H,t)$ and $\phi(H,t)$ in the general case of any possible subgroup $A$ of $\text{Aut}(G)$.

For any $H \in \mathcal{C}_A(G)$ and any positive integer $t$, let

1. $\Omega_A(H,t) = \bigcup_{a \in A} (H^a)^t$;
2. $\omega_A(H,t) = |\Omega_A(H,t)|$;
3. $\Psi_A(H,t) = \{ (g_1, \ldots, g_i) \in G^t : (g_1, \ldots, g_i) = H^a \text{ for some } a \in A \}$;
4. $\psi_A(H,t) = |\Psi_A(H,t)|$.

If $(x_1, \ldots, x_t) \in \Omega_A(H,t)$, then $(x_1, \ldots, x_t) \leq H^a$ for some $a \in A$, hence $(x_1, \ldots, x_t) = K$ for some $K \leq G$ with $|K|_A \leq |H|_A$. Thus

$$\sum_{|K|_{\leq A} |H|} \psi_A(K,t) = \omega_A(H,t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{|H| \in \mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t) = \psi_A(G,t).$$

On the other hand $\psi_A(G,t) = \phi(G,t)$ so we have proved the following formula.

**Theorem 15.** For any finite group $G$ and any subgroup $A$ of $\text{Aut}(G)$,

$$\phi(G,t) = \sum_{|H| \in \mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t).$$

Notice that if $A = 1$, then $\omega_A(H,t) = |H|^t$, so that the result by Hall given in (1.1) is a particular case of the previous theorem.
Corollary 16. If $G$ is not cyclic, then

$$0 = \phi(G, 1) = \sum_{[H] \in C_A(G)} \mu_A(H, G) \omega_A(H, 1).$$

Taking $A = \text{Inn}(G)$, we deduce in particular that if $G$ is not cyclic, then

$$\sum_{H \in C(H)} \lambda(H, G) \omega_{\text{Inn}(G)}(H, 1) = \sum_{H \in C(H)} \lambda(H, G) | \cup_g H^g | = 0.$$  

For example, if $G = S_4$, then the values of $\lambda(H, G)$ and $| \cup_g H^g |$ are as in the following table and $24 - 12 - 16 - 15 + 4 + 9 + 7 - 1 = 0$.

| $H$         | $\lambda(H, G)$ | $| \cup_g H^g |$ |
|------------|-----------------|----------------|
| $S_4$      | 1               | 24             |
| $A_4$      | -1              | 12             |
| $D_4$      | -1              | 16             |
| $S_3$      | -1              | 15             |
| $K$        | 1               | 4              |
| $\langle (1, 2, 3, 4) \rangle$ | 0               | 10             |
| $\langle (1, 2, 3) \rangle$ | 1               | 9              |
| $\langle (1, 2) \rangle$ | 1               | 7              |
| $\langle (1, 2)(3, 4) \rangle$ | 0               | 4              |
| 1          | -1              | 1              |

If $G = A_5$, then the values of $\lambda(H, G)$, $\omega_{\text{Inn}(G)}(H, 1) = | \cup_g H^g |$, $\omega_{\text{Inn}(G)}(H, 2) = | \cup_g (H^g)^2 |$ (taking only the subgroups $H$ with $\lambda(H, G) \neq 0$) are as in the following table and $60 - 36 - 36 - 40 + 21 + 32 - 1 = 0$.

| $H$         | $\lambda(H, G)$ | $| \cup_g H^g |$ | $| \cup_g (H^g)^2 |$ |
|------------|-----------------|----------------|----------------|
| $A_5$      | 1               | 60             | 3600           |
| $A_4$      | -1              | 36             | 636            |
| $S_3$      | -1              | 36             | 306            |
| $D_5$      | -1              | 40             | 550            |
| $\langle (1, 2, 3) \rangle$ | 1               | 21             | 81             |
| $\langle (1, 2)(3, 4) \rangle$ | 2               | 16             | 46             |
| 1          | -1              | 1              | 1              |

Moreover

$$3600 - 636 - 306 - 550 + 81 + 2 \cdot 46 - 1 = 2280 = \frac{19}{30} \cdot 3600 = \phi(A_5, 2).$$

If $G = D_p = \langle a, b \mid a^p = 1, b^2 = 1, ab = a^{-1} \rangle$ and $p$ is an odd prime, then the behaviour of the subgroups in $C(G)$ is described by the following table.

| $H$         | $\lambda(H, G)$ | $| \cup_g H^g |$ |
|------------|-----------------|----------------|
| $D_p$      | 1               | $2p$           |
| $\langle a \rangle$ | -1              | $p$           |
| $\langle b \rangle$ | -1              | $p + 1$       |
| 1          | -1              | 1              |
Another interesting example is given by considering $G = C_p^n$ and $A = \text{Aut}(G)$. Let $H \cong C_p^{n-1}$ be a maximal subgroup of $G$. Then, for $K \leq G$, $\mu_A(K, G) \neq 0$ if and only if either $[K]_A = [G]_A$ or $[H]_A = [H]_A$. Clearly $\cup_{\omega \in \text{Aut}(G)}H^\omega = G$ so $\mu_A(G, G)\omega_A(G, 1) - \mu_A(H, G)\omega_A(H, 1) = |G| - |G| = 0$. More in general $\Omega_A(H, t)$ is the set of $t$-tuples $(x_1, \ldots, x_t)$ such that $(x_1, \ldots, x_t) \in K^t$ for some maximal subgroup $K$ of $G$, so $\mu_A(G, G)\omega_A(G, t) - \mu_A(H, G)\omega_A(H, t) = |G|^t - \omega_A(H, t)$ is the number of generating $t$-tuples of $G$.

Another generalization of (1.1), essentially due to Gaschütz, has been described by Brown in [7, Section 2.2]. Let $N$ be a normal subgroup of $G$ and suppose that $G/N$ admits $t$ generators for some integer $t$. Let $y = (y_1, \ldots, y_t)$ be a generating $t$-tuple of $G/N$ and denote by $P(G, N, t)$ the probability that a random lift of $y$ to a $t$-tuple of $G$ generates $G$. Then $P(G, N, t) = \phi(G, N, t)|N|^t$, where $\phi(G, N, t)$ is the number of generating $t$-tuples of $G$ lying over $y$. Using again the Möbius inversion formula it can be proved:

$$\phi(G, N, t) = \sum_{H \leq G, HN = G} \mu(H, G)|H \cap N|^t.$$ 

This formula can be generalized in our contest in the following way:

**Theorem 17.** Let $N$ be an $A$-invariant normal subgroup of $G$ and fix $g_1, \ldots, g_t \in G$ with the property that $G = \langle g_1, \ldots, g_t \rangle N$. Define

- $\Omega_A(H, N, t) = \{ (n_1, \ldots, n_t) \mid \langle g_1n_1, \ldots, g_tn_t \rangle \leq H^a \text{ for some } a \in A \}$;
- $\omega_A(H, N) = |\Omega_A(H, N, t)|$

and let $\mathcal{C}_A(G, N) = \{ [H]_A \in \mathcal{C}_A(G) \mid HN = G \}$. Then

$$\phi(G, N, t) = \sum_{[H]_A \in \mathcal{C}_A(G, N)} \mu_A(H, G)\omega_A(H, N, t).$$

**Proof.** Fix $g_1, \ldots, g_t \in G$ with the property that $G = \langle g_1, \ldots, g_t \rangle N$. Then $\phi(G, N, t)$ is the cardinality of the set

$$\Phi(G, N, g_1, \ldots, g_t) = \{ (n_1, \ldots, n_t) \in N^t \mid \langle g_1n_1, \ldots, g_n n_t \rangle = G \}.$$

Set:

$$\Psi_A(H, N, g_1, \ldots, g_t) = \{ (n_1, \ldots, n_t) \in N^t \mid \langle g_1n_1, \ldots, g_n n_t \rangle = H^a \text{ for some } a \in A \};$$

$$\psi_A(H, N, t) = |\Psi_A(H, N, g_1, \ldots, g_t)|.$$

Notice that $\omega_A(H, N, t) \neq 0$ if and only if $[H]_A \in \mathcal{C}_A(G, N)$. If $(n_1, \ldots, n_t) \in \Omega_A(H, N, t)$, then $\langle g_1n_1, \ldots, g_n n_t \rangle \leq H^a$ for some $a \in A$, and $\langle g_1n_1, \ldots, g_n n_t \rangle = K$ for some $K \leq G$ with $[K]_A \leq [H]_A$. Thus

$$\sum_{[K]_A \leq [H]_A} \psi_A(K, N, t) = \omega_A(H, N, t)$$

and therefore, by the Möbius inversion formula

$$\sum_{[H] \in \mathcal{C}_A(G, N)} \mu_A(H, G)\omega_A(H, N, t) = \psi_A(G, N, t) = \phi(G, N, t) \quad \square$$
4. Another application of Möbius inversion formula

Denote by $\Phi^*(G, t)$ the set of $t$-tuples $(H_1, \ldots, H_t)$ of subgroups of $G$ such that $G = \langle H_1, \ldots, H_t \rangle$ and by $\phi^*(G, t)$ the cardinality of this set. For any $H \in C_A(G)$ and any positive integer $t$, let

1. $\Sigma_A(H, t) = \{(H_1, \ldots, H_t) \mid \langle H_1, \ldots, H_t \rangle \leq H^a \text{ for some } a \in A\}$;
2. $\sigma_A(H, t) = |\Sigma_A(H, t)|$;
3. $\Gamma_A(H, t) = \{(H_1, \ldots, H_t) \mid \langle H_1, \ldots, H_t \rangle = H^a \text{ for some } a \in A\}$;
4. $\gamma_A(H, t) = |\Gamma_A(H, t)|$.

**Theorem 18.**

$$\phi^*(G, t) = \sum_{[H] \in C_A(G)} \mu_A(H, G) \sigma_A(H, t).$$

**Proof.** If $(H_1, \ldots, H_t) \in \Sigma_A(H, t)$, then $\langle H_1, \ldots, H_t \rangle = K$ for some $K \leq G$ with $[K]_A \leq [H]_A$. Thus

$$\sum_{[K] \leq [H]} \gamma_A(K, t) = \sigma_A(H, t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{[H] \in C_A(G)} \mu_A(H, G) \sigma_A(H, t) = \gamma_A(G, t) = \phi^*(G, t). \quad \Box$$

In the particular case when $A = 1$, $\sigma_A(H, t) = \sigma(H)^t$, denoting with $\sigma(H)$ the number of subgroups of $H$. So we obtain the following corollary:

**Corollary 19.**

$$\phi^*(G, t) = \sum_{H \leq G} \mu(H, G) \sigma(H)^t.$$  

Clearly $\Sigma^*(G, t) = \{G\}$, so $\phi^*(G, 1) = 1$ and therefore it follows:

**Corollary 20.**

$$1 = \sum_{H \in H_A} \mu_A(H, G) \sigma_A(H, 1).$$

In particular:

**Corollary 21.**

$$1 = \sum_{H \leq G} \mu(H, G) \sigma(H).$$

For example, if $G = A_5$ then the subgroups of $G$ with $\mu(H, G) \neq 0$ are listed in the following table (where $\kappa(H, G)$ denote the numbers of conjugate of $H$ in $G$).

| $H$         | $\mu(H, G)$ | $\kappa(H, G)$ | $\sigma(H)$ |
|-------------|-------------|----------------|-------------|
| $A_5$       | 1           | 1              | 59          |
| $A_4$       | -1          | 5              | 10          |
| $S_5$       | -1          | 10             | 6           |
| $D_5$       | -1          | 6              | 8           |
| $\langle(1, 2, 3)\rangle$ | 2          | 10             | 2           |
| $\langle(1, 2)(3, 4)\rangle$ | 4          | 15             | 2           |
| 1           | -60         | 1              | 1           |
According with Corollary [21] 1 = 59 – 5 · 10 – 10 · 6 – 6 · 8 + 2 · 10 · 2 + 4 · 15 · 2 – 60.

For a finite group $G$, denote by $P(G, t)$ and $P^*(G, t)$ the probability of generating $G$ with, respectively, $t$ elements or $t$ subgroups. It can be easily seen that $P(G, t) = P(G/\text{Frat}(G), t)$, but in general $P^*(G, t) \neq P^*(G/\text{Frat}(G), t)$. For example, if $G \cong C_{p^a}$, then $G$ and $H \cong C_{p^{a-1}}$ are the unique subgroups of $G$ with non trivial Möbius number and therefore

$$P(G, t) = \frac{|G|^t - |H|^t}{|G|^t} = 1 - \frac{1}{p^t},$$

$$P^*(G, t) = \frac{\sigma(G)^t - \sigma(H)^t}{\sigma(G)^t} = 1 - \frac{a^t}{(a + 1)^t}.$$

So $P(G, t)$ is independent on $a$, while $P^*(G, t)$ tends to 0 when $a$ tends to infinity.

5. The $(\mu, \lambda)$-property

Proof of Theorem [3]. Working by induction on the order of $G$, it suffices to prove the statement in the particular case when $N$ is an abelian minimal normal subgroup of $G$. Let $H$ be a subgroup of $G$. If $N \leq H$, then

$$\mu(H, G) = \mu(H/N, G/N) = \lambda(H/N, G/N)|N_{G', N/N}(H/N) : H/N \cap G'N/N|$$

$$\lambda(H, G)|N_{G', N}(H) : H \cap G'| = \lambda(H, G)|N_{N_{G'}(H)} : N(H \cap G')|$$

$$= \lambda(H, G)|N_{G'}(H) : H \cap G'| = \lambda(H, G)|N_{G'}(H) : H \cap G'|$$

$$\lambda(H, G)|N_{G'}(H) : H \cap G'|.$$

So we may assume $N \not\subseteq H$. If $H$ is not an intersection of maximal subgroups of $G$, then $\mu(G, H) = \lambda(G, H) = 0$. So we may assume $H = M_1 \cap \cdots \cap M_i$, where $M_1, \ldots, M_i$ are maximal subgroups of $G$. Since $N$ is not contained in $H$, then $N$ is not contained in $M_i$ for some $i$, but then $M_i$ is a complement of $N$ in $G$ containing $H$ and $N \cap H = 1$. If $g \in N_G(HN)$, then $g = x\delta$ with $x \in M_i$ and $\delta \in N$. In particular $H^\delta \leq HN \cap M_i = H(N \cap M_i) = H$, so $N_{G'}(HN) = N_{G'}(H)N$. By Lemma [12] we have

$$\mu(H, G) = \mu(H/N, G/N) \lambda(H, G),$$

$$\lambda(H, G) = \lambda(H/N, G/N)\delta = |N_{G', N}(HN) : HN \cap G'N|^{\kappa/\delta} = |N_{N_{G'}(H)} : HN \cap G'|^{\kappa/\delta}$$

where $k$ is the number of complements of $N$ in $G$ containing $H$ and $\delta$ is the number of conjugacy classes of these complements. First assume that $N \leq Z(G)$. Then $\kappa = \delta$, $G' = M_i \leq M_i$, $N \cap G' = 1$ and

$$\mu(H, G) = |N_{N_{G'}(H)} : HN \cap G'|^{\kappa/\delta} = |N_{N_{G'}(H)} : HN \cap G'|^{\kappa/\delta}$$

$$= |N_{G'}(H) : H \cap G'|.$$

Finally assume $N \not\subseteq Z(G)$. Then $N \leq G'$, $\kappa/\delta = |N_N(H)|$ and

$$\mu(H, G) = |N_{N_{G'}(H)} : HN \cap G'|^{\kappa/\delta} = |N_{N_{G'}(H)} : HN \cap G'|^{\kappa/\delta}$$

$$= \frac{|N_{N_{G'}(H)}|}{|N_N(H)|} \frac{|N_N(H)|}{|N_{N_{G'}(H)}|} = |N_{G'}(H) : H \cap G'|. \quad \Box$$
Proof of Corollary 6. Suppose that $G$ has minimal order with respect to the property that $G$ does not satisfy the $(\mu, \lambda)$ property. By the previous proposition, $G$ contains no abelian minimal normal subgroup and therefore $\text{soc}(G) = S_1 \times \cdots \times S_t$ is a direct product of nonabelian finite simple groups. If $|G| \leq |PSU(3, 3)| = 6048$, then either $t = 1$ or $G = \text{soc}(G) = A_5 \times A_5$. So it suffices to check that $A_5 \times A_5$ and any almost simple group of order at most 6048 satisfies the $(\mu, \lambda)$ property. Since, for every $H \leq G$, $\lambda(H, G)$ and $\mu(H, G)$ can be computed from the table of marks of $G$ (see [10, Proposition 1]), this task can be easily completed using the library of table of marks available in GAP [4]. □

We may use Theorem 15 to deduce some consequences of the $(\mu, \lambda)$-property.

**Theorem 22.** Suppose that a finite group $G$ satisfies the $(\mu, \lambda)$-property. Then

\[(5.1) \quad \sum_{H \in \mathcal{C}(G)} \lambda(H, G) \left( \frac{|H|^{-1}|G||G' H|}{|G' \mathcal{N}_G(H)|} - \omega(H, t) \right) = 0.\]

**Proof.** By Theorem 15

\[\sum_{H \in \mathcal{C}(G)} \lambda(H, G) \omega(H, t) = \phi(G, t) = \sum_{H \leq G} \mu(H, G)|H|^t\]

\[= \sum_{H \in \mathcal{C}(G)} \mu(H, G)|H : \mathcal{N}_G(H)||H|^t\]

\[= \sum_{H \in \mathcal{C}(G)} \lambda(H, G)|\mathcal{N}_{G'}(H) : G' \cap H||G : \mathcal{N}_G(H)||H|^t\]

\[= \sum_{H \in \mathcal{C}(G)} \lambda(H, G) \frac{|H|^t|G||\mathcal{N}_{G'}(H)|}{|G' \cap H||\mathcal{N}_G(H)|} = \sum_{H \in \mathcal{C}(G)} \lambda(H, G) \frac{|H|^{-1}|G||G' H|}{|G' \mathcal{N}_G(H)|}. \square\]

A natural question is whether (5.1) is also a sufficient condition for the $(\mu, \lambda)$-property. For any $H \leq G$, set $\mu^*(H, G) = |\mathcal{N}_{G'}(H) : G' \cap H| \lambda(H, G)$. The validity of (5.1) is equivalent to

\[\sum_{H \in \mathcal{C}(G)} \lambda(H, G) \omega(H, t) - \sum_{H \in \mathcal{C}(G)} \mu^*(H, G)|H|^t|G : \mathcal{N}_G(H)| = 0.\]

In any case we must have

\[\sum_{H \in \mathcal{C}(G)} \lambda(H, G) \omega(H, t) - \sum_{H \in \mathcal{C}(G)} \mu(H, G)|H|^t|G : \mathcal{N}_G(H)| = 0.\]

So (5.1) is equivalent to

\[\sum_{H \in \mathcal{C}(G)} \frac{(\mu(H, G) - \mu^*(H, G))|H|^t}{|\mathcal{N}_G(H)|} = 0.\]

Let $T = \{ [H] \in \mathcal{C}(G) \mid \mu(H, G) \neq \mu^*(H, G) \}$. Then (5.1) is true if and only if

\[(5.2) \quad \sum_{[H] \in T} \frac{(\mu(H, G) - \mu^*(H, G))|H|^t}{|\mathcal{N}_G(H)|} = 0.\]
For example, if \( G = PSU(3,3) \), then \( \mathcal{T} \) consists of four conjugacy classes of subgroups and the corresponding values are given by the following table:

| \( \mu(H,G) \) | \( \mu^*(H,G) \) | \( |H| \) | \( |N_G(H)| \) |
|----------------|----------------|-------|----------|
| -48            | 0              | 2     | 96       |
| 3              | 0              | 6     | 18       |
| 0              | -4             | 8     | 32       |
| 1              | 2              | 24    | 24       |

In this case (5.2) is equivalent to

\[
2^{t-1} - 6^{t-1} - 8^{t-1} + 24^{t-1} = 0
\]

which is true only if \( t = 1 \).

For any positive integer \( n \) let

\[
\tau(n) = \sum_{H \in \mathcal{T}, |H|=n} \frac{\mu(H,g) - \mu^*(H,G)}{|N_G(H)|}.
\]

**Proposition 23.** A finite group \( G \) satisfies (5.1) for any positive integer \( t \) if and only if \( \tau(n) = 0 \) for any \( n \in \mathbb{N} \).

**Question 24.** Does \( \tau(n) = 0 \) for any \( n \in \mathbb{N} \) implies \( \mu^*(H,G) = \mu(H,G) \) for any \( H \leq G \)?

For any \( H \leq G \), consider

\[
\alpha(H,t) = \frac{|H|^{t-1}|G'||G'H|}{|G'N_G(H)|}, \quad \beta(H,t) = \alpha(H,t) - \omega(H,t).
\]

Let \( \mathcal{C}^*(G) = \{ [H] \in \mathcal{C}(H) \mid |H| < |G| \} \) and \( \lambda(H,G) \neq 0 \}. If \( G \) satisfies the \((\lambda,\mu)\)-property, then for any \( t \in \mathbb{N} \), the vector

\[
\beta_t(G) = (\beta(H,t))_{[H] \in \mathcal{C}^*(G)}
\]

is an integer solution of the linear equation

\[
(5.3)
\]

\[
\sum_{[H] \in \mathcal{C}^*(G)} \lambda(H,G)x_H = 0.
\]

One could investigate about the dimension of the vector space generate by the vectors \( \beta_t(G) \), \( t \in \mathbb{N} \). For example, if \( G = A_5 \), then we may order the elements of \( \mathcal{C}^*(G) \) so that \( H_1 = A_4 \), \( H_2 = S_3 \), \( H_3 = D_5 \), \( H_4 = \langle (1,2,3) \rangle \), \( H_5 = \langle (1,2)(3,4) \rangle \), \( H_6 = 1 \). Then (5.3) can be written in the form

\[
\sum_{[H] \in \mathcal{C}^*(G)} \lambda(H,G)x_H = -x_{H_1} - x_{H_2} - x_{H_3} + x_{H_4} + 2x_{H_5} - x_{H_6}
\]

and

\[
\beta_1(G) = (24,24,20,39,44,59),
\]

\[
\beta_2(G) = (84,54,50,99,74,59),
\]

\[
\beta_3(G) = (264,114,110,279,134,59),
\]

\[
\beta_4(G) = (804,234,230,819,254,59),
\]

\[
\beta_5(G) = (2424,474,470,2439,494,59),
\]

\[
\beta_6(G) = (7284,954,950,7299,974,59).
\]
The first three vectors $\beta_1(G), \beta_2(G), \beta_3(G)$ are linearly independent, while $\beta_4(G), \beta_5(G)$ and $\beta_6(G)$ can be obtained as linear combinations of $\beta_1(G), \beta_2(G), \beta_3(G)$.

The situation is completely different when $G = S_3$. We may order the elements of $C^*(G)$ so that $H_1 = \langle (1, 2, 3) \rangle$, $H_2 = \langle (1, 2) \rangle$, $H_3 = 1$. The equation (5.3) has in this case the form $x_{H_1} + x_{H_2} - x_{H_3} = 0$ and $\beta_1(G) = (0, 2, 2)$ independently on the choice of $t$.

Some properties of the vectors $\beta_t(G)$ are described in the following propositions.

**Proposition 25.** If $H \in C^*(G)$, then $\beta(H, t) \geq 0$ with equality if and only if $G' \leq H$. In particular $\beta_1(G)$ is a non-negative vector and $\beta_1(G) = 0$ if and only if $G$ is nilpotent.

**Proof.** Notice that $\omega(H, t) \leq |G : N_G(H)|(|H|^t - 1) + 1$. So

$$\beta(H, t) \geq \frac{|H|^{t-1}|G| |G'| H}{|G'| N_G(H)} - |G : N_G(H)|(|H|^t - 1) - 1$$

$$= |H|^t |G : N_G(H)| \frac{|G' \cap N_G(H)|}{|G' \cap H|} - |G : N_G(H)|(|H|^t - 1) - 1 \geq 0$$

with equality if and only if $H \geq G'$.

**Proposition 26.** The vector $\beta_t(G)$ is independent on the choice of $t$ if and only if $G$ is a nilpotent group or a primitive Frobenius group, with cyclic Frobenius complement.

**Proof.** By the previous proposition, if $G$ is nilpotent then $\beta_t(G)$ is the zero vector for any $t \in \mathbb{N}$, so we may assume that $G$ is not nilpotent. Assume that $\beta_t(G)$ is independent on the choice of $t$. Let $H$ be a maximal non-normal subgroup of $G$. Then $\alpha(H, t) = |H|^t \cdot u$ with $u = |G : H|$. Let $H_1, \ldots, H_u$ be the conjugates of $H$ in $G$. For any $J \subseteq \{1, \ldots, u\}$, let $\alpha_J = |\cap_{j \in J} H_j|$. Then

$$\beta(H, t) = \sum_{J \neq \{1, \ldots, u\}} (-1)^{|J|+1} |\alpha_J|^t.$$

We must have $\alpha_J = 1$ for every choice of $J$, otherwise $\lim_{t \to \infty} \beta(H, t) = \infty$. Hence $H$ is a Frobenius complement and, since $H$ is a maximal subgroup, the Frobenius kernel $V$ is an irreducible $H$-module. Since $\beta(V, t) = |V|^t(|H'|^t - 1)$ does not depend on $t$, $H$ must be abelian, and consequently cyclic. So if $\beta_t(G)$ is independent of the choice of $t$, then $G$ is a primitive Frobenius group with a cyclic Frobenius complement. Conversely assume $G = V \rtimes H$, where $H$ is cyclic and $V$ and irreducible $H$-module. If $X \in C^*(G)$, then $\lambda(X, G) \neq 0$, so $X$ is an intersection of maximal subgroups of $G$ and therefore either $V = G' \leq X$, or $X$ is conjugate to a subgroup of $H$. In the first case $\beta(H, t) = 0$. Assume $X = K^v$ for some $K \leq H$ and $v \in V$. Then $\beta(H, t) = |K|^t |V| - \omega(K, t) = |K|^t |V| - (|V|(|K|^t - 1) + 1) = |V| - 1$. □

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