Analysis of mass difference of the \( \pi \) and \( \rho \) with Bethe-Salpeter equation

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Abstract

In this article, we take into account the one-pion exchange force besides the one-gluon exchange force to study the mass difference of the \( \pi \) and \( \rho \) mesons with the Bethe-Salpeter equation. After projecting the Bethe-Salpeter equation into an simple form, we can see explicitly that the bound energy \(|E_\pi| \gg |E_\rho|\).

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1 Introduction

The constituent quark models have given many successful descriptions of the hadron spectroscopy, the simple constituent quark mass plus hyperfine spin-spin interaction model works well for the ground state mesons \[1\],

\[
M_m = M_1 + M_2 + C \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{M_1 M_2},
\]

(1)

where the coefficient \( C \) can be fitted phenomenologically. The masses of the ground state pseudoscalar and vector mesons are \( M_\pi = 140 \) MeV, \( M_K = 494 \) MeV, \( M_\eta = 548 \) MeV, \( M_{\eta'} = 958 \) MeV, \( M_\rho = 775 \) MeV, \( M_\omega = 783 \) MeV, \( M_{K^*} = 892 \) MeV, and \( M_\phi = 1019 \) MeV \[2\]. The mass difference between the \( \pi \) and \( \rho \) mesons are huge, we have to resort to large hyperfine spin-spin interactions for explanation. The fine spin-orbit interactions and the hyperfine spin-spin interactions are usually studied in the relativized quark model based on the one-gluon exchange plus linear confinement potential motivated by QCD \[3\], for more literatures, one can consult the comprehensive review Ref.\[4\]. One may wander why the contributions from the hyperfine interactions in Eq.\,(1) are so large, and how to understand them in the quantum field theory.

At the energy scale \( \mu = 4\pi f_\pi \approx 1 \) GeV, the approximate chiral \( SU_L(3) \times SU_R(3) \) symmetry is spontaneously broken to the \( SU_V(3) \) symmetry by the small current quark masses \( m_u, m_d \) and \( m_s \), and there appear eight Nambu-Goldstone bosons (in the following we will neglect the word Nambu for simplicity). The masses of the Goldstone bosons are related with the current quark masses through the Gell-Mann-Oakes-Renner relation \[5\]. The quark fields \( q(x) \) are usually decomposed as

\[
q(x) = \exp \left[-i\gamma_5\xi^a(x)\lambda^a\right] \tilde{q}(x),
\]

(2)

where the \( \tilde{q}(x) \) and \( \xi^a(x) \) denote the constituent quark fields (or Goldstone free fields) and the octet Goldstone boson fields, respectively \[6\], there exist interactions among the quarks and the Goldstone bosons. For example, the spectra of the nucleons, \( \Delta \) resonances and

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the strange hyperons are well described by the constituent quark model with the harmonic
confinement potential plus the one-Goldstone-boson exchanges induced potential $[7]$. In
the energy region between the confinement and the spontaneous chiral symmetry breaking,
the elementary degrees of freedom are quarks, gluons and Goldstone bosons $[8]$.

On the other hand, the mass breaking effects in the chiral doublets are very large $[2, 9]$,
for example,

$$
\begin{align*}
\pi(0^-, 140 \text{ MeV}) & \leftrightarrow f_0(600)(0^+, 400 - 1200 \text{ MeV}), \\
\rho(1^-, 775 \text{ MeV}) & \leftrightarrow a_1(1260)(1^+, 1230 \text{ MeV}), \\
\rho(1^+, 938 \text{ MeV}) & \leftrightarrow N(1535)(1^-, 1525 - 1545 \text{ MeV}),
\end{align*}
$$

(3)

which requires that the chiral symmetry should be badly broken, here we use $\leftrightarrow$ to denote
the chiral rotations. The chiral massive quark dresses itself with the gluon cloud and
quark-antiquark pairs, and acquires a dynamically generated large mass. We usually carry
out the re-summation of the loops nonperturbatively with the Dyson-Schwinger equation,
and obtain the Euclidean constituent quark masses by the definition $p^2 = M^2(p^2)$, which
are compatible with the values used in the constituent quark models $[10, 11]$. The light
pseudoscalar mesons play a double role, as both Goldstone bosons and $q\bar{q}$ bound states.

The exchanges of the one-gluon and one-Goldstone-boson between the two constituent
quarks result in the hyperfine interactions $H_C$ and $H_F$, respectively $[7, 12]$,

\begin{align*}
H_C & \sim \frac{1}{M_i M_j} \vec{\lambda}_i^C \cdot \vec{\lambda}_j^C \vec{\sigma}_i \cdot \vec{\sigma}_j, \\
H_F & \sim \frac{1}{M_i M_j} \vec{\lambda}_i^F \cdot \vec{\lambda}_j^F \vec{\sigma}_i \cdot \vec{\sigma}_j, 
\end{align*}

(4)

and they both contribute to the spin-spin interactions. In this article, we take into account
the contributions from the one-Goldstone-boson exchange force besides the one-gluon ex-
change force, study the $\pi$ and $\rho$ mass difference with the Bethe-Salpeter equation, and try
to understand the difference in the quantum field theory.

The Bethe-Salpeter equation is a conventional approach in dealing with the two-body
relativistic bound state problems $[13]$, and has given many successful descriptions of the
hadron properties in a Poincare covariant way $[11, 14]$.

The article is arranged as follows: we solve the Bethe-Salpeter equation for the $u\bar{d}$
bound states in Sec.2; in Sec.3, we present the numerical results; and Sec.4 is reserved for
our conclusions.
2 Bethe-Salpeter equation

We write down the ladder Bethe-Salpeter equation for the $u\bar{d}$ bound states in the Euclidean spacetime:

$$ S^{-1}_u(q + \xi_u P) \chi(q, P) S^{-1}_d(q - \xi_d P) = -\int \frac{d^4k}{(2\pi)^4} \left[ \frac{\lambda^\alpha}{2} \gamma_\mu \chi(k, P) \frac{\lambda^\beta}{2} g_\alpha^2(q - k) \right. \\
+ i\gamma_5 \chi(k, P) i\gamma_5 \frac{g_\pi^2(q - k)}{(q - k)^2 + m_\pi^2} ] \right. \\
\left. + \frac{\xi_{u/d} \cdot \gamma \cdot P}{M_u + M_d}, \right. \tag{5}$$

The $P_\mu$ is the four-momentum of the center of mass of the $u\bar{d}$ bound states, the $q_\mu$ is the relative four-momentum between the $u$ and $\bar{d}$ quarks, the $\chi(q, P)$ is the Bethe-Salpeter amplitude of the $u\bar{d}$ bound states, and the $g_\pi(q - k)$ and $g_\rho(q - k)$ are the energy dependent $\pi$-quark and gluon-quark coupling constants, respectively. In this article, we take the $g_\rho^2(k)$ as a modified Gaussian distribution, $g_\rho^2(k) = A \left( \frac{k^2}{M^2} \right)^2 \exp \left( -\frac{k^2}{M^2} \right)$, where the strength $A$ and the distribution width $M^2$ are free parameters. The ultraviolet behavior of the modified Gaussian distribution warrants that the integral in the Bethe-Salpeter equation is convergent.

The Euclidean Bethe-Salpeter amplitudes of the $u\bar{d}$ bound states can be decomposed as

$$ \chi^\pi(q, P) = \gamma_5 \{ F_\pi(q, P) + i P F_\pi^T(q, P) + i q \cdot \vec{P} F_\pi^0(q, P) + \frac{1}{2} \{ P, q \} F_\pi^2(q, P) \}, \tag{6} \right. \\
\chi^\rho(q, P) = \{ P \cdot \vec{P} F_\rho(q, P) + i q \cdot \vec{P} F_\rho^T(q, P) - i q \vec{P} F_\rho^0(q, P) + \frac{1}{2} \{ P, q \} F_\rho^2(q, P) \}$$

due to Lorentz covariance \[14,15\], where the $\gamma_5$ is the polarization vector of the $\rho$ meson, the $F_\pi(q, P)$, $F_\pi^T(q, P)$, $F_\rho(q, P)$ and $F_\rho^T(q, P)$ are the components of the Bethe-Salpeter amplitudes, which can be expanded in terms of Tchebychev polynomials $T_n^\frac{1}{2}(\cos \theta)$ \[16\], where $\theta$ is the included angle between $q_\mu$ and $P_\mu$. Numerical calculations indicate that taking only the terms $T_n^\frac{1}{2}(\cos \theta) = 1$ can give satisfactory results \[17\]. If we take into account the small terms with $n \geq 1$, the predictions may be improved mildly. In the following, we use the amplitudes $F_{\pi/\rho}(q^2, P^2)$ and $F^{\pi/\rho}_n(q^2, P^2)$ to denote the $n = 0$ terms of the Bethe-Salpeter amplitudes $F_{\pi/\rho}(q, P)$ and $F^{\pi/\rho}_n(q, P)$, respectively. Then the Bethe-Salpeter equations can be projected into four and eight coupled integral equations for the $\pi$ and $\rho$ mesons respectively, and it is very difficult to solve them numerically. Furthermore, we cannot obtain physical insight from those involved integral equations.

Multiplying both sides of the Bethe-Salpeter equations of the $\pi$ and $\rho$ mesons by $\gamma_5 \{ q, P \}$ \[18\], and $\ell \{ q, P \} + [q, P] \ell$, $q \cdot \epsilon q \cdot P$ $P$ respectively, completing the trace in the
Dirac spinor space, carrying out the integrals for the included angle $\theta$, and neglecting the small components $F_3^\pi$, $F_3^\rho$, $F_5^\rho$ and $F_6^\rho$, we can obtain the following three relations,

$$
F_\pi(q^2, P^2) - (M_u + M_d) F_1^\pi(q^2, P^2) = 0, \\
F_\rho(q^2, P^2) + (M_u + M_d) F_1^\rho(q^2, P^2) = 0, \\
2F_\rho(q^2, P^2) - (M_u + M_d) F_4^\rho(q^2, P^2) = 0. \\
$$

(7)

The Bethe-Salpeter amplitudes can be approximated as

$$
\chi^\pi(q, P) = \gamma_5 \left(1 + \frac{iP}{M_u + M_d}\right) F_\pi(q^2, P^2), \\
\chi^\rho(q, P) = \left\{ \eta \left( i - \frac{P}{M_u + M_d} \right) + \frac{2q \cdot \epsilon}{M_u + M_d} \right\} F_\rho(q^2, P^2),
$$

(8)

if we also neglect the small components $F_2^\pi$, $F_2^\rho$ and $F_7^\rho$. Then the involved Bethe-Salpeter equations can be projected into the following simple form,

$$
\begin{align*}
\left\{ q^2 + M_u M_d \left[ 1 + \frac{P^2}{(M_u + M_d)^2} \right] \right\} F_\pi(q^2, P^2) &= \int \frac{d^4k}{(2\pi)^4} F_\pi(k^2, P^2) \\
&= \left\{ \frac{16 g_\pi^2(q - k)}{3 (q - k)^2} + \frac{g_\pi^2(q - k)}{(q - k)^2 + m_\pi^2} \right\}, \\
\left\{ q^2 + M_u M_d \left[ 1 + \frac{P^2}{(M_u + M_d)^2} \right] \right\} F_\rho(q^2, P^2) &= \int \frac{d^4k}{(2\pi)^4} F_\rho(k^2, P^2) \\
&= \left\{ \frac{8 g_\rho^2(q - k)}{3 (q - k)^2} - \frac{g_\rho^2(q - k)}{(q - k)^2 + m_\rho^2} \right\}.
\end{align*}
$$

(9)

If we take $q^2 = 0$ and $g_\pi^2 = 0$, and assume that there exists a physical solution, then

$$
M_u M_d \left[ 1 - \frac{M_{\pi/\rho}^2}{(M_u + M_d)^2} \right] F_{\pi/\rho}(0, -M_{\pi/\rho}^2) = \int \frac{d^4k}{(2\pi)^4} F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2) G_{\pi/\rho}(0 - k),
$$

(10)

where the $G_{\pi/\rho}(k)$ denotes the interacting kernels. In numerical calculations, we observe that the Bethe-Salpeter amplitude $F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2)$ has the same sign in the region $k^2 \geq 0$,

$$
1 - \frac{M_{\pi/\rho}^2}{(M_u + M_d)^2} = \int \frac{d^4k}{(2\pi)^4} \frac{F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2)}{M_u M_d F_{\pi/\rho}(0, -M_{\pi/\rho}^2)} G_{\pi/\rho}(0 - k) > 0,
$$

(11)

and obtain an simple relation (or constraint),

$$
M_{\pi/\rho}^2 < (M_u + M_d)^2,
$$

(12)

which survives for $q^2 > 0$ (although the relation is not explicit for $q^2 > 0$), i.e. the bound energy $E_{\pi/\rho}$ originates from the interacting kernel $G_{\pi/\rho}(k)$ and should be negative, $E_{\pi/\rho} = M_{\pi/\rho} - M_u - M_d < 0$. The numerical calculations indicate that above arguments survive in the case $g_\pi^2 \neq 0$. 

4
From Eq.(9), we can see explicitly that the one-gluon exchange force in the π channel is more attractive than that in the ρ channel due to the factors $\frac{16}{3}$ and $\frac{8}{3}$; furthermore, the one-pion exchange force is attractive in the π channel and repulsive in the ρ channel, the bound energy $|E_\pi| \gg |E_\rho|$ can be accounted for naturally.

We can introduce a parameter $\lambda(P^2)$ and solve above equations as an eigenvalue problem, the masses of the π and ρ mesons can be determined by the condition $\lambda(P^2 = -M^{2}_{\pi/\rho}) = 1$,

$$
\left\{ q^2 + M_u M_d \left[ 1 + \frac{P^2}{(M_u + M_d)^2} \right] \right\} F_\pi(q^2, P^2) = \lambda(P^2) \int \frac{d^4k}{(2\pi)^4} F_\pi(k^2, P^2) = \left\{ \begin{array}{l} 16 \frac{g^2_\pi(q - k)}{3 (q - k)^2} + \frac{g^2_\pi(q - k)}{(q - k)^2 + m^2_\pi} \end{array} \right.,
$$

$$
\left\{ q^2 + M_u M_d \left[ 1 + \frac{P^2}{(M_u + M_d)^2} \right] \right\} F_\rho(q^2, P^2) = \lambda(P^2) \int \frac{d^4k}{(2\pi)^4} F_\rho(k^2, P^2) = \left\{ \begin{array}{l} 8 \frac{g^2_\rho(q - k)}{3 (q - k)^2} - \frac{g^2_\rho(q - k)}{(q - k)^2 + m^2_\rho} \end{array} \right.,
$$

(13)

3 Numerical results

The constituent quark masses of the $u$ and $d$ quarks are taken as $M_u = M_d = 400$ MeV. The strength parameter $A$ and the distribution width $\Lambda$ are free parameters, we take the values $\Lambda = 200$ MeV and $A = 146(105)$ for the one-gluon (one-pion) exchange. Other values of the $\Lambda$ and $A$ also work, we choose the present parameters for illustration.

We solve the Bethe-Salpeter equations as an eigenvalue problem numerically by direct iterations, and observe the convergent behaviors are very good. The numerical results for the Bethe-Salpeter amplitudes are shown in Fig.1. From the figure, we can see that the Bethe-Salpeter amplitudes center around zero momentum and extend to the energy scale about $q = 0.4$ GeV and 0.7 GeV for the ρ and π mesons respectively, the stronger interactions in the π channel result in more stable bound state than that in the ρ channel, as the bound energies are $E_\pi = -660$ MeV and $E_\rho = -25$ MeV, respectively. In numerical calculations, we observe that the one-pion exchange force plays an important role and should be taken into account.

With the following simple replacements in Eq.(13),

$$
M_{\bar{g}} \rightarrow M_\bar{s} = 536 \text{ MeV},
$$

$$
g^2_\pi(k^2) \rightarrow g^2_\pi(k^2) = A \left( \frac{k^2}{\Lambda^2} \right)^2 \exp \left( -\frac{k^2}{\Lambda^2} \right),
$$

$$
M_\pi \rightarrow M_K = 494 \text{ MeV},
$$

(14)

where the $\tilde{\Lambda} = \Lambda \left( \frac{M_K}{M_\pi + M_\rho} \right)^2$ denotes the $SU_1(3)$ breaking effect for the coupling constant, we can obtain the corresponding solutions for the pseudoscalar meson $K$ and vector meson $K^*$ with the eigenvalues $\lambda(P^2 = -M^2_K = -(494 \text{ MeV})^2) = 1$ and $\lambda(P^2 = -(494 \text{ MeV})^2) = 1$. 

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−M_{K^*}^2 = −(892 \text{ MeV})^2 = 1, \text{ respectively. The bound energies are } E_K = −442 \text{ MeV and } E_{K^*} = −44 \text{ MeV, respectively, which indicate that there exists a more stable bound state in the pseudoscalar channel than that in the vector channel.}

4 Conclusion

In this article, we take into account the one-pion exchange force besides the one-gluon exchange force to study the mass difference of the π and ρ mesons with the Bethe-Salpeter equation. After simplifying the involved Bethe-Salpeter equations, we observe that the one-gluon exchange force in the π channel is more attractive than that in the ρ channel, while the one-pion exchange force is attractive in the π channel and repulsive in the ρ channel, the bound energy |E_π| ≫ |E_ρ| can be accounted for naturally.

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