Energy convexity estimates for non-degenerate ground states of nonlinear 1D Schrödinger systems

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Abstract

We study the spectral structure of the complex linearized operator for a class of nonlinear Schrödinger systems, obtaining as byproduct some interesting properties of non-degenerate ground state of the associated elliptic system, such as being isolated and orbitally stable.

1 Introduction and main results

In the last few years, the interest in the study of Schrödinger systems has considerably increased, in particular, for the following class of two weakly coupled nonlinear Schrödinger equations

\[
\begin{cases}
    i\partial_t \phi_1 + \frac{1}{2} \partial_{xx} \phi_1 + (|\phi_1|^{2p} + \beta |\phi_2|^{p-1} |\phi_1|^{p-1}) \phi_1 = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+; \\
    i\partial_t \phi_2 + \frac{1}{2} \partial_{xx} \phi_2 + (|\phi_2|^{2p} + \beta |\phi_1|^{p-1} |\phi_2|^{p-1}) \phi_2 = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+; \\
    \phi_1(0,x) = \phi_0^1(x), \quad \phi_2(0,x) = \phi_0^2(x) & \text{in } \mathbb{R},
\end{cases}
\]

(1.1)

where \( \Phi = (\phi_1, \phi_2) \) and \( \phi_i : [0,\infty) \times \mathbb{R} \to \mathbb{C}, \phi_i^0 : \mathbb{R} \to \mathbb{C} \), \( 0 < p < 2 \). Usually the coupling constant \( \beta > 0 \) models the birefringence effects inside a given anisotropic material (see e.g. [13], [14]). A soliton or standing wave solution is a solution of the form \( \Phi(x,t) = (u_1(x)e^{it}, u_2(x)e^{it}) \) where \( U(x) = (u_1(x), u_2(x)) \) solves the elliptic system

\[
\begin{cases}
    -\frac{1}{2} \partial_{xx} r_1 + r_1 = r_1^{2p+1} + \beta r_1^p r_2^{p+1} & \text{in } \mathbb{R}, \\
    -\frac{1}{2} \partial_{xx} r_2 + r_2 = r_2^{2p+1} + \beta r_2^p r_1^{p+1} & \text{in } \mathbb{R}.
\end{cases}
\]

(1.2)

Among all the solutions of (1.2) there are the ground states, namely least energy solutions. It is known (see e.g. [11], [17]) that for \( p \geq 1 \) there exists a ground state \( R = (r_1, r_2) \in C^2(\mathbb{R}) \cap W^{2,s}(\mathbb{R}) \) for any positive \( s \); Moreover, \( R \) has nonnegative components \( r_i \) which are even, decreasing on \( \mathbb{R}^+ \) and exponentially...
decaying. In [12] it is shown that $R$ can be characterized as a solutions of the following minimization problem

\begin{equation}
\mathcal{E}(R) = \inf_{M} \mathcal{E}(V) \quad \text{where} \quad M := \{ V \in H^1(\mathbb{R}) \times H^1(\mathbb{R}), \|V\|_2 = \|R\|_2 \},
\end{equation}

and

\begin{equation}
\mathcal{E}(V) = \mathcal{E}(v_1, v_2) = \frac{1}{2} \|\partial_x V\|_2^2 - \frac{1}{p+1} \int \left( |v_1|^{2p+2} + |v_2|^{2p+2} + 2\beta |v_1 v_2|^{p+1} \right),
\end{equation}

when the exponent $p$ satisfies

\begin{equation}
1 \leq p < 2.
\end{equation}

The interest in finding ground states is also motivated by their properties with respect of the analysis of the dynamical system (1.1), such as stability properties. For the single Schrödinger equation many notions of stability have been introduced and proved, among all, we recall [5] and [19, 20]; in the former it is proved that the ground state, which is unique, of the equation

\begin{equation}
-\frac{1}{2} \partial_{xx} z + z = z^{2p+1} \quad \text{in} \; \mathbb{R},
\end{equation}

is orbitally stable, that is, roughly speaking, if $\phi^0$ is a function close to $z$ with respect to the $H^1$ norm then the solution of the Cauchy problem

\begin{equation}
\begin{cases}
\iota \phi + \frac{1}{2} \partial_{xx} \phi + |\phi|^{2p} \phi = 0 & \text{in} \; \mathbb{R} \times \mathbb{R}^+, \\
\phi(0, x) = \phi^0(x) & \text{in} \; \mathbb{R},
\end{cases}
\end{equation}

where $\phi : [0, \infty) \times \mathbb{R} \to \mathbb{C}, \phi^0 : \mathbb{R} \to \mathbb{C}$ and $1 \leq p < 2$, remains close to $z$ up to phase rotations and translations. In [19, 20] the study becomes deeper assuming that $z$ is non-degenerate, that is the linearized operator for (1.6) has a 1-dimensional kernel which is spanned by $\partial_x z$. More precisely, it is proved that for every $\phi \in H^1(\mathbb{R})$ such that $\|\phi\|_{L^2} = \|z\|_{L^2}$, the following inequality holds

\begin{equation}
\mathcal{E}(\phi) - \mathcal{E}(z) \geq C \inf_{\phi \in C[0, 2\pi]} \inf_{\theta \in \mathbb{R}} \|\phi - e^{\beta z (-x_0)}\|_{H^1}^2,
\end{equation}

for some positive constant $C$, provided that the energy $\mathcal{E}(\phi)$ is sufficiently close to $\mathcal{E}(z)$. Here, $\mathcal{E}$ is the energy defined in (1.4) once we consider $V = (z, 0)$. Inequality (1.8) allows to provide not only the same orbital stability result proved in [5], but it also permits to derive explicit differential equation to which the phase and position adjustment have to obey for the ground state to be linearly stable. Moreover, (1.8) tells us that the energy functional can be seen as a Lyapunov functional, as it measures the deviation of the solution of (1.1) from the ground state orbit.

The main goal of this paper is to extend inequality (1.8) to the more general framework of 1D vector Schrödinger problems. In order to do this we are lead to consider non-degenerate ground state for system (1.2). This notion is introduced in the following definition.

**Definition 1.1.** We will say that a ground state solution $R = (r_1, r_2)$ of system (1.2) is non-degenerate if the set of solutions of the linearized system

\begin{equation}
\begin{cases}
-\frac{1}{2} \partial_{xx} \phi + \phi = \left( (2p + 1)r_1^{2p} + \beta p r_1^{p-1} r_2^{p+1} \right) \phi + \beta (p + 1) r_1^{p} r_2^{p} \psi & \text{in} \; \mathbb{R}, \\
-\frac{1}{2} \partial_{xx} \psi + \psi = \left( (2p + 1)r_2^{2p} + \beta p r_1^{p+1} r_2^{p-1} \right) \psi + \beta (p + 1) r_1^{p} r_2^{p} \phi & \text{in} \; \mathbb{R},
\end{cases}
\end{equation}

is an 1-dimensional vector space and any solution $(\phi, \psi)$ of (1.9) is given by $\theta \partial_x R$, for some $\theta \in \mathbb{R}$. 

The main result of the paper is stated in the following

**Theorem 1.2.** Let \( R \) be non-degenerate and assume (1.5). Then, for every \( \Phi \in H^1 \times H^1 \) with

\[
\|\Phi\|_{L^2 \times L^2} = \|R\|_{L^2 \times L^2},
\]

the following inequality holds

\[
E(\Phi) - E(R) \geq \inf_{x \in \mathbb{R}} \inf_{\delta, \Phi} \|\Phi - (e^{i\delta_1} r_1(\cdot - x), e^{i\delta_2} r_2(\cdot - x))\|_{H^1 \times H^1}^2
\]

\[
+ o\left( \inf_{x \in \mathbb{R}} \inf_{\delta, \Phi} \|\Phi - (e^{i\delta_1} r_1(\cdot - x), e^{i\delta_2} r_2(\cdot - x))\|_{H^1 \times H^1}^2 \right)
\]

where \( o(x) \) satisfies \( o(x)/x \to 0 \) as \( x \to 0 \).

As interesting consequences, we will obtain the property of being isolated, and of being orbitally stable for a non-degenerate ground state. In [12] it has been recently proved that the set of ground states of (1.2) enjoys the orbital stability property. To this respect, we have to recall that up to now it is not yet been proved a uniqueness result for ground state solutions of the system (1.2). Therefore, a solution of (1.1) which starts near a ground state \( R \), may leave the orbit around \( R \) and approach the orbit generated by another ground state. But, this is not the case, once we know that the ground states are isolated. This property is easily obtained as a consequence of Theorem 1.2 as stated in the following corollary.

**Corollary 1.3.** Let \( R \) be non-degenerate and assume (1.5). Then \( R \) is isolated, that is, if there exists a ground state of (1.2) \( S \) satisfying \( \|R - S\|_{H^1} < \delta \) for a \( \delta > 0 \) sufficiently small, then \( S = R \) up to a translation and a phase change.

Then, we can also prove the following

**Corollary 1.4.** Let \( R \) be non-degenerate and assume (1.5). Then \( R \) is orbitally stable.

We recall that a ground state \( R = (r_1, r_2) \) is said to be orbitally stable if for any given \( \epsilon > 0 \), there exist \( \delta(\epsilon) > 0 \) such that

\[
\sup_{t \in [0, \infty)} \inf_{x \in \mathbb{R}} \|\Psi(t, \cdot) - (e^{i\delta_1} r_1(\cdot - x), e^{i\delta_2} r_2(\cdot - x))\|_{H^1 \times H^1} < \epsilon,
\]

provided that

\[
\inf_{x \in \mathbb{R}} \|\Psi^0 - (e^{i\delta_1} r_1(\cdot - x), e^{i\delta_2} r_2(\cdot - x))\|_{H^1 \times H^1} < \delta,
\]

where \( \Psi \) is the solution of (1.1) with initial datum \( \Psi^0 \).

Theorem 1.2 plays a very important role also in the study of the so-called **soliton dynamics** for Schrödinger. More precisely, when one considers (1.1) when the Plank’s constant \( \hbar \) explicitly appears in the equations, and studies the evolution, in the semi-classical limit \( (\hbar \to 0) \), of the solution of (1.1) starting from a \( \hbar \)-scaling of a soliton, once the action of external forces appears. We refer the reader to [3, 9, 10] for the scalar case and to [15] for systems, where the authors have recently showed, in semi-classical regime, how the soliton dynamics can be derived from Theorem 1.2.

Finally, we have to point out that some of our results can be proved in general dimension \( n \geq 1 \) as well, with minor changes. Unfortunately, this is not the case for our main Theorem, since, in order to work
on the linearized equation, and to perform Taylor expansion on the energy functional $\mathcal{E}$, we need enough regularity on the nonlinear term and this forces us to restrict the range of $p$ because of the presence of the coupling term. Of course, it is a really interesting open problem, to prove the assertion of Theorem 1.2 for any $n \geq 1$ and any $0 < p < 2/n$.

In Section 2, we will study some delicate spectral properties of the linearized system introduced in Definition 1.1. The proofs of Theorem 1.2 and of Corollaries 1.3 and 1.4 will be carried out in Section 3. Finally, in Section 4, we shall prove that there exists a non-degenerate ground state for system (1.2).

# 2 Spectral analysis of the linearized operators

In this section we will prove some important properties concerning the linearized Schrödinger system associated with (1.1).

We will make use of the functional spaces $L^2 = L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C})$ and $H^1 = H^1(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C})$. We recall that the inner product between $u, v \in \mathbb{C}$ is given by $u \cdot v = \Re(u \bar{v}) = 1/2(u \bar{v} + v \bar{u})$. It is known (see [4, 18]) that (1.1) is well locally posed in time, for any $p$, in the space $H^1$ endowed with the norm $||\Phi||_{H^1}^2 = ||\partial_t \Phi||_2^2 + ||\Phi||_2^2$ for every $\Phi = (\phi_1, \phi_2) \in H^1$. Moreover we set the $L^q$ norm as $||\Phi||_q^q = ||\phi_1||_q^q + ||\phi_2||_q^q$ for any $q \in [1, \infty)$, we denote by $(U, V)$ the inner scalar product in $L^2$ and by $(U, V)_{H^1}$ the inner scalar product in $H^1$. In [7] it is proved that, for $p$ satisfying $0 < p < 2$ the solution of the Cauchy problem (1.1) exists globally in time and the mass of a solution and its total energy are preserved in time, that is having defined the total energy of system (1.1) as

\begin{equation}
E(\Phi(t)) = \frac{1}{2} ||\partial_t \Phi(t)||_2^2 - \int F(\Phi(t))
\end{equation}

where

\begin{equation}
F(U) = F(u_1, u_2) = \frac{1}{p+1}(|u_1|^{2p+2} + |u_2|^{2p+2} + 2\beta|u_1 u_2|^{p+1}),
\end{equation}

the following conservation laws hold (see [7]):

\begin{equation}
||\phi_1||_2^2 = ||\phi_1^0||_2^2, \quad ||\phi_2||_2^2 = ||\phi_2^0||_2^2, \quad E(\Phi(t)) = E(0) = \frac{1}{2} ||\partial_t \Phi^0||_2^2 - \int F(\Phi^0).
\end{equation}

Setting $\phi_i = r_i + \varepsilon w_i, i = 1, 2$, the linearized Schrödinger system at $r_i$ in $w_i$ is given by

\begin{equation}
\begin{cases}
i\partial_t w_1 + \frac{1}{2}\partial_{xx} w_1 - w_1 + G_1(w_1, w_2) = 0 \quad \text{in} \ \mathbb{R}, \\
i\partial_t w_2 + \frac{1}{2}\partial_{xx} w_2 - w_2 + G_2(w_1, w_2) = 0 \quad \text{in} \ \mathbb{R},
\end{cases}
\end{equation}

where we have set

\begin{align*}
G_1(w_1, w_2) &= \left[r_1^{2p} + \beta r_1^{p-1} r_2^{p+1}\right] w_1 + \left[2p r_1^{2p} + \beta(p-1) r_1^{p-1} r_2^{p+1}\right] \Re(w_1) + \beta(p+1) r_1^{p} r_2^{p} \Re(w_2), \\
G_2(w_1, w_2) &= \left[r_2^{2p} + \beta r_1^{p-1} r_2^{p-1}\right] w_2 + \left[2p r_2^{2p} + \beta(p-1) r_1^{p-1} r_2^{p+1}\right] \Re(w_2) + \beta(p+1) r_1^{p} r_2^{p} \Re(w_1).
\end{align*}

System (2.4) can be written down as $\partial_t W = LW$, for $L : L^2 \times L^2 \to L^2 \times L^2$ defined by

\begin{equation}
L = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \quad W \in \mathbb{C}^2, \ W = (w_1, w_2)
\end{equation}
and where the operators $L_-, L_+ : L^2(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$ acting respectively on the real and imaginary parts of $w_j$ are the following

\begin{equation}
(2.5) \quad L_+ = \begin{pmatrix} L_+^{11} & L_+^{12} \\ L_+^{21} & L_+^{22} \end{pmatrix} \quad L_- = \begin{pmatrix} L_-^{11} & 0 \\ 0 & L_-^{22} \end{pmatrix}
\end{equation}

where $L_{+,-}^{ij} : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R})$ are defined by

\begin{align*}
L_+^{11} &= \frac{1}{2} \partial_{xx} + 1 - H^{11}(R) \quad L_+^{12} = L_+^{21} = -H^{12}(R) \\
L_+^{22} &= \frac{1}{2} \partial_{xx} + 1 - H^{22}(R) \\
L_-^{11} &= \frac{1}{2} \partial_{xx} + 1 - \left[ r_1^{2p} + \beta r_1^{p-1} r_1^{p+1} \right] \quad L_-^{22} = \frac{-1}{2} \partial_{xx} + 1 - \left[ r_2^{2p} + \beta r_1^{p} r_2^{p-1} \right]
\end{align*}

and the Hessian matrix $H_L(U) = (H^{ij}) : (\mathbb{R}^+)^2 \to M_{2 \times 2}(\mathbb{R})$ is given by

\begin{align*}
H^{11} &= (2p + 1) u_1^{2p} + p \beta u_1^{p-1} u_1^{p+1} \quad H^{12} = H^{21} = (p + 1) \beta u_1^p u_1^{p+1} \\
H^{22} &= (2p + 1) u_2^{2p} + p \beta u_2^{p-1} u_1^{p+1}.
\end{align*}

We will study $L_+$ on $\mathcal{V}$, namely the closed subspace of $\mathbb{H}^1$ defined as

\begin{equation}
(2.6) \quad \mathcal{V} = \left\{ U \in \mathbb{H}^1 : (U, R) = 0 \right\}.
\end{equation}

The first important property of $L_+$ on $\mathcal{V}$ is proved in the following proposition.

**Proposition 2.1.** Assume (1.5) and that $R$ a ground state of (1.2). Then $\inf_{\mathcal{V}} (L_+(U), U) = 0$.

**Proof.** First notice that $U_+ = (r_1', r_2')$ belongs to $\mathcal{V}$ and $U_+$ satisfies $(L_+(U_+), U_+) = 0$, showing that the infimum is less or equal than zero. On the other hand, since $R$ solves problem (1.3), of course $R$ is also a minimum point of $I = \mathcal{E}(\Phi) + \|\Phi\|^2_2$ on $M$. Consequently, for any smooth curve $\psi : [-1, 1] \to M$ such that $\psi(0) = R$, it follows

\[
\frac{d^2 I(\psi(s))}{ds^2} \Big|_{s=0} \geq 0.
\]

Therefore, taking into account that $I'(R) = 0$, we get

\[
0 \leq \langle I''(\psi(s)) \psi'(s), \psi'(s) \rangle \Big|_{s=0} + \langle I'(\psi(s)), \psi''(s) \rangle \Big|_{s=0} = \langle I''(R) \psi'(0), \psi'(0) \rangle + \langle I'(R), \psi''(0) \rangle = \langle I''(R) \psi'(0), \psi'(0) \rangle.
\]

Now, taking into account that the map $s \mapsto \|\psi(s)\|^2_2$ is constant, it readily follows that $\psi'(0)$ belongs to $\mathcal{V}$, which yields the assertion by the arbitrariness of $\psi$.

The above result is the first step to show that $L_+$ is coercive once we restrict it on a closed subspace of $\mathcal{V}$, as shown in the following proposition.

**Proposition 2.2.** Assume (1.5) and that $R$ is a ground state of (1.2) satisfying Definition 1.1. Then

\begin{equation}
(2.7) \quad \inf_{U \in \mathcal{V}_0} \frac{(L_+(U), U)}{\|U\|^2_2} > 0, \quad \mathcal{V}_0 = \left\{ U \in \mathbb{H}^1 : (U, R) = (U, H_R(\partial_x R)) = 0 \right\}.
\end{equation}
Proof. Denoting with $\alpha$ the infimum

$$\alpha = \inf_{\|V\|_{L^2(\Omega)} = 1, V \in V_0} (L_+(V), V),$$

first notice that Proposition 2.1 implies that $\alpha$ is nonnegative, so that we only have to show that $\alpha$ is not zero. Let us argue by contradiction and suppose that $\alpha = 0$. Taken $U_n$ a minimizing sequence, from the regularity properties of $R$ it follows that $U_n$ is bounded in $H^1$. These gives us a function $U \in H^1$, such that $U_n \rightharpoonup U$ weakly (up to a subsequence) in $H^1$, implying that $U \in \mathcal{V}_0$. From Proposition 2.1 and (2.7), we get

$$0 \leq (L_+(U), U) \leq \liminf_{n \to \infty} \left\{ \|U_n\|^2_{H^1} - (U_n, H_F(R)U_n) \right\} = \lim_{n \to \infty} (L_+(U_n), U_n) = 0.$$

So that $U$ solves $(L_+(U), U) = 0$ and $(L_+(U_n), U_n) \to (L_+(U), U)$. Moreover,

$$\|U\|^2_{H^1} \leq \liminf_{n \to \infty} \|U_n\|^2_{H^1} \leq \limsup_{n \to \infty} \|U_n\|^2_{H^1} = \lim_{n \to \infty} \left\{ (L_+(U_n), U_n) + (U_n, H_F(R)U_n) \right\} = (L_+(U), U) + (U, H_F(R)U) = \|U\|^2_{H^1},$$

from which $U_n \to U$ strongly in $H^1$, so that $\|U\|_{H^1} = 1$ and $U$ solves the constrained minimization problem (2.7). When we derive the functional $(L_+(V), V)/\|V\|^2_{L^2}$ and use that $(L_+(U), U) = 0$ we obtain that there exists Lagrange multipliers $\mu, \gamma \in \mathbb{R}$ such that

$$(L_+(U), V) = \mu (R, V) + (\gamma \cdot H_F(R)\partial_{\bar{r}} R, V), \quad \text{for every } V \in H^1.\tag{2.8}$$

Choosing as test function $V = \partial_{\bar{r}} R$ and taking into consideration that $(R, \partial_{\bar{r}} R) = 0$, gives

$$0 = (L_+(U), \partial_{\bar{r}} R) = (\gamma \cdot H_F(R)\partial_{\bar{r}} R, \partial_{\bar{r}} R) = \gamma (H_F(R)\partial_{\bar{r}} R, \partial_{\bar{r}} R),$$

where we have taken into account that $L_+$ is a self-adjoint operator and $\partial_{\bar{r}} R = (\partial_{x_1} R_1, \partial_{x_2} R_2)$ is a solution of $L_+ V = 0$. Since $R$ has even components the summands on the right hand side are nonzero, so that $\gamma = 0$. As a consequence, $U$ solves $L_+ U = \mu R$. Moreover, we consider the vector $x \cdot \partial_{\bar{r}} R$, whose components are $x \cdot \partial_{x} R = (x \partial_{x_1} R_1, x \partial_{x_2} R_2)$ and we compute $L_+(x \cdot \partial_{\bar{r}} R)$. After some simple calculations, one reaches

$$L_+(x \cdot \partial_{\bar{r}} R) = (-\partial_{x_1} R_1, -\partial_{x_2} R_2) \quad \text{and} \quad L_+(R/p) = -2(r_1^{2p+1} + \beta r_2^{p+1} r_1^p r_2^{2p+1} + \beta r_1^{p+1} r_2^p).$$

Then, in turn, we get $L_+(R/p + x \cdot \partial_{\bar{r}} R) = -2R$, and by linearity

$$L_+(-\mu/2(R/p + x \cdot \partial_{\bar{r}} R)) = \mu R.$$

Then, Definition 1.1 (nondegeneracy) immediately yields

$$(2.9) \quad U = -\mu/2(R/p + x \cdot \partial_{\bar{r}} R) + \theta \cdot \partial_{\bar{r}} R$$

for some constant $\theta \in \mathbb{R}$. Now we have to show that $\theta = 0$, by using the available constraints. By applying to equation (2.9) the self-adjoint operator $H_F = H_F(R)$, we get

$$H_F U = -\frac{\mu}{2p} H_F R - \frac{\mu}{2} H_F x \cdot \partial_{\bar{r}} R + H_F \theta \cdot \partial_{\bar{r}} R.$$

As $U \in \mathcal{V}_0$, it results $(H_F U, \partial_{\bar{r}} R) = (U, H_F \partial_{\bar{r}} R) = 0$. Furthermore, since $R$ is a radial solution of (1.2), we also have that $(H_F R, \partial_{\bar{r}} R) = (H_F x \cdot \partial_{\bar{r}} R, \partial_{\bar{r}} R) = 0$. On the other hand

$$(H_F \theta \cdot \partial_{\bar{r}} R, \partial_{\bar{r}} R) = \theta (H_F \partial_{\bar{r}} R, \partial_{\bar{r}} R) = c \theta$$

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with \( c \neq 0 \), so it has to be \( \theta = 0 \). Then \((2.9)\) reduces to

\[
U = -\frac{\mu}{2p}R - \frac{\mu}{2} x \cdot \partial_x R.
\]

Computing the \( L^2 \)-scalar product with \( R \) and keeping in mind that \( U \in \mathcal{V}_0 \) yields

\[
0 = (U, R) = -\frac{\mu}{2} \left[ \frac{1}{p} ||R||_2^2 + (x \cdot \partial_x R, R) \right].
\]

As far as concern the last term in the previous relation, we integrate by parts and obtain

\[
(x \cdot \partial_x R, R) = -\frac{1}{2} ||R||_2^2.
\]

The last two equations and \((1.5)\) give the desired contradiction. \( \square \)

**Remark 2.3.** The argument in the proof of the previous Proposition shows that there exists a positive constant \( \alpha_0 \) such that

\[
(L_+ V, V) \geq \alpha_0 ||V||^2_{\mathcal{H}^1}, \quad \text{for all } V \in \mathcal{V}_0.
\]

Moreover, if we consider \( |||U||| = \sqrt{(L_+ U, U)} \) for every \( U \in \mathcal{V}_0 \), we obtain that \( ||| \cdot ||| \) satisfies all the required properties of a norm, by \((2.10)\) and by the self-adjointness property of \( L_+ \). In addition, every Cauchy sequence \( \{U_n\} \) with respect to \( ||| \cdot ||| \) has a strong limit \( U \) belonging \( L^2 \); moreover \( U \) satisfies all the orthogonality relations required in \( \mathcal{V}_0 \). Besides, computing \((L_+(U_n - U_m), U_n - U_m)\) gives that also \( \{\partial_x U_n\} \) is a Cauchy sequence in \( L^2 \) then \( U \) is necessarily the strong limit of \( \{U_n\} \) in \( \mathcal{H}^1 \). Finally, \( |||U_n - U||| \to 0 \) by the definition of \( L_+ \). As a consequence, \( \mathcal{V}_0 \) is a Banach space with respect to this norm, and we get the equivalence with the standard \( \mathcal{H}^1 \) norm, namely there exists \( \alpha > 0 \) such that

\[
(L_+ V, V) \geq \alpha ||V||^2_{\mathcal{H}^1}, \quad \text{for all } V \in \mathcal{V}_0.
\]

Before stating our next result let us prove the following lemma.

**Lemma 2.4.** Let us take \( \Phi \in L^2 \) such that \( |||U||| = ||R||_2 \) and consider the difference \( W = \Phi - R \). Denoting with \( U \) and \( V \) the real and imaginary part of \( W \), it results

\[
(R, U) = -\frac{1}{2} \left[ ||U||_2^2 + ||V||_2^2 \right] = -\frac{1}{2} ||W||_2^2
\]

**Proof.** The above identity immediately follows by imposing \( ||R + W||_2^2 = ||R||_2^2 \) and by recalling that \( R \) is a real function. \( \square \)

**Proposition 2.5.** Assume \((1.5)\) and that \( R \) satisfies Definition 1.1. Moreover, let us take \( W = U + iV \) satisfying \((2.11)\) with \( U \) verifying

\[
(U, H_F(R)\partial_x R) = 0.
\]

Then, there exists positive constants \( D, D_i \) such that

\[
(L_+ U, U) \geq D||U||^2_{\mathcal{H}^1} - D_1||W||_2^4 - D_2||W||_2^2||\partial_x W||_2^2
\]
We also obtain from (2.11) that

\[ (L_+ U, U) = (R, U) (L_+ U, R) = -\frac{1}{2} \|W\|^2 (L_+ U, R) \geq -d_2 \|W\|^2 \|\partial_x W\|_2. \]

As far as concern the last term in (2.14), it results

\[ (L_+ U, U) = (R, U) (L_+ U, R) = \frac{1}{4} \|W\|^2 (L_+ R, R) \geq -d_3 \|W\|^4. \]

This last equation, joint with (2.16) and (2.17) yields the conclusion. \(\blacksquare\)

Proposition 2.6. It results \(\inf_{V \neq 0, (V, r)_{H^1} = 0} \frac{(L_-(V), V)}{\|V\|_2^2} > 0.\)

Proof. Let us first prove that \(L_-\) is a positive operator. Denoting with \(\sigma_d(L_-)\) the discrete spectrum of the operator \(L_-\) it results

\[ \sigma_d(L_-) = \sigma_d(L^{11}_-) \cup \sigma_d(L^{22}_-). \]

Indeed, if \(\lambda \in \sigma_d(L^{11}_-)\) we get that \(L^{11}_-(u) = \lambda u\), then \(\lambda \in \sigma_d(L_-)\) with eigenfunction \(U = (u, 0)\), analogous argument holds for \(\lambda \in \sigma_d(L^{22}_-)\), proving that \(\sigma_d(L^{11}_-) \cup \sigma_d(L^{22}_-) \subseteq \sigma_d(L_-)\). On the other hand, if \(\lambda \in \sigma_d(L_-)\) there exists \(U = (u_1, u_2) \neq (0, 0)\) such that

\[ L^{11}_- u_1 = \lambda u_1, \quad L^{22}_- u_2 = \lambda u_2 \]

so that, if \(u_1 \neq 0\), \(\lambda \in \sigma_d(L^{11}_-)\), otherwise \(u_2 \neq 0\) and \(\lambda \in \sigma_d(L^{22}_-)\), showing (2.18). Moreover, since \(L_- R = 0\), with \(R = (r_1, r_2) \neq (0, 0), r_i \geq 0\), we get that \(\lambda = 0\) is the first eigenvalue of \(L^{11}_-\) and \(L^{22}_-\) when both \(r_1, r_2 \neq 0\). Besides, if for example \(r_1 \equiv 0, \lambda = 0\) is the first eigenvalue of \(L^{22}_-\), while \(L^{11}_- = -\partial_{xx} + 1\) and its discrete spectrum is empty (see e.g. Chapter 3 in [2]), yielding that \(\lambda = 0\) is the first eigenvalue of \(L_-\). Then \((L_-(V), V) \geq 0\) for every function \(V \in H^1\), proving that \(L_-\) is a positive operator. Arguing now as in the proof of Proposition 2.2, and considering the (nonnegative) infimum

\[ \alpha = \inf_{\|V\|_{L^2} = 1, (V, r)_{H^1} = 0} (L_-(V), V), \]
assuming by contradiction that $\alpha = 0$, we find that there exists a nonzero minimizer $U$ (satisfying the constraints) for the problem such that

\[(L_- U, U) = 0\]

Taking into account that the constraints $(U, r_i)_H = 0$ can be written in the $L^2$ form

\[(q_-^{11}(R)R_1, U) = 0, \quad (q_-^{22}(R)R_2, U) = 0,\]

where we have set

\[q_-^{11}(R) = r_1^{2p} + \beta r_1^{p-1} r_2^{p+1}, \quad q_-^{22}(R) = r_2^{2p} + \beta r_1^{p+1} r_2^{p-1}, \quad R_1 = (r_1, 0), \quad R_2 = (0, r_2).\]

we have three lagrange parameters $\lambda, \gamma_1, \gamma_2 \in \mathbb{R}$ such that

\[(L_- U, V) = \lambda(U, V) + \gamma_1(q_-^{11}(R)R_1, V) + \gamma_2(q_-^{22}(R)R_2, V)\]

for all $V \in \mathbb{H}^1$. Hence, by choosing $V = U$ and taking into account (2.19) and that $U$ satisfies the constraints (2.20), we immediately get $\lambda = 0$. Choosing now $V = R_1$ and $V = R_2$ and taking into account $L_-$ is self-adjoint and that $L_- R_i = 0$ we obtain $\gamma_1 = \gamma_2 = 0$. Therefore, we conclude that

\[L_- U = 0,\]

namely $L_-^1 u_1 = 0$ and $L_-^2 u_2 = 0$ where we set $U = (u_1, u_2)$. In turn, $u_i$ is a first eigenfunction of $L_-^i$, which yields $u_i \in \text{span}(r_i)$ since the first eigenvalue is simple (see e.g. Theorem 3.4 in [2]). This is of course a contradiction with (2.20). Hence $\alpha > 0$ and the proof is complete. 

\[\square\]

**Remark 2.7.** Arguing as in Remark 2.3, it is possible to find a positive constant $\alpha > 0$ such that

\[(L_- V, V) \geq \alpha ||V||_{\mathbb{H}^1}^2, \quad \text{for all } V \in \mathbb{H}^1 \text{ with } (v_i, r_i)_H = 0, i = 1, 2.\]

### 3 Proofs of the main results

In order to prove Theorem 1.2, the following characterization will be crucial.

**Proposition 3.1.** Let us consider $y_0 \in \mathbb{R}$ and $\Gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ be such that

\[
\min_{\phi \in \mathbb{R}^2} \|((\phi_1(\cdot + x_0)e^{iy_1}, \phi_2(\cdot + x_0)e^{iy_2}) - R\|^2_{\mathbb{H}^1} = \|((\phi_1(\cdot + y_0), \phi_2(\cdot + y_0)e^{iy_2}) - R\|^2_{\mathbb{H}^1},
\]

Then, writing

\[\phi_1(\cdot + y_0), \phi_2(\cdot + y_0)e^{iy_2}) = R + W,\]

where $W = U + iV$, the following orthogonality condition are satisfied

\[(U, H_F(R)\partial_x R) = 0, \quad (v_1, r_1)_H = (v_2, r_2)_H = 0.\]
Proof. Let us introduce the functions \( P, Q : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
P(x_0, \Theta) = P(x_0, \theta_1, \theta_2) = ||(\phi_1(\cdot + x_0)e^{i\theta_1}, \phi_2(\cdot + x_0)e^{i\theta_2}) - R||_2^2
\]
\[
Q(x_0, \Theta) = Q(x_0, \theta_1, \theta_2) = ||\partial_x \phi_1(\cdot + x_0)e^{i\theta_1}, \partial_x \phi_2(\cdot + x_0)e^{i\theta_2} - \partial_x R||_2^2.
\]
Writing down the partial derivatives of \( P \) and \( Q \) and integrating by parts, give us
\[
\partial_{x_0} P(x_0, \Theta) = 2 \sum_{j=1}^2 \int \left( \phi_j e^{i\theta_j} - r_j \right) e^{-i\theta_j} \partial_{x_0} \phi_j + \left( \phi_j e^{i\theta_j} - r_j \right) e^{i\theta_j} \partial_{x_0} \phi_j
\]
\[
= -2 \sum_{j=1}^2 \int r_j \Re \left( e^{i\theta_j} \partial_{x_0} \phi_j \right);
\]
\[
\partial_{x_0} Q(x_0, \Theta) = 2 \sum_{j=1}^2 \int \partial_x \left( \phi_j e^{i\theta_j} - r_j \right) \partial_x \partial_{x_0} \phi_j e^{-i\theta_j} + \partial_x \left( \phi_j e^{i\theta_j} - r_j \right) \partial_x \partial_{x_0} \phi_j e^{i\theta_j}
\]
\[
= -2 \sum_{j=1}^2 \int \partial_x r_j \Re \left( \partial_x \partial_{x_0} \phi_j e^{i\theta_j} \right);
\]
\[
\frac{\partial P}{\partial \theta_j}(x_0, \Theta) = \int \left[ -\left( \phi_j e^{i\theta_j} - r_j \right) e^{-i\theta_j} \phi_j + \left( \phi_j e^{i\theta_j} - r_j \right) e^{i\theta_j} \phi_j \right]
\]
\[
= 2 \int r_j \Im \left( e^{i\theta_j} \phi_j \right);
\]
\[
\frac{\partial Q}{\partial \theta_j}(x_0, \Theta) = \int \left[ -\partial_x \left( \phi_j e^{i\theta_j} - r_j \right) \partial_x \phi_j e^{-i\theta_j} + \partial_x \left( \phi_j e^{i\theta_j} - r_j \right) \partial_x \phi_j e^{i\theta_j} \right]
\]
\[
= 2 \int \partial_x r_j \Im \left( \partial_x \phi_j e^{i\theta_j} \right).
\]
If \( x_0 = y_0 \) and \( \Gamma = (\gamma_1, \gamma_2) \) realize the minimum in (3.1), the following equations are satisfied
\[
\frac{\partial(P + Q)}{\partial x_0}(x_0, \Theta) = -2 \sum_{j=1}^2 \int \left[ r_j(x) \Re \left( e^{i\gamma_j} \frac{\partial \phi_j}{\partial x_0}(x - y_0) \right) + \partial_x r_j(x) \Re \left( e^{i\gamma_j} \partial_x \frac{\partial \phi_j}{\partial x_0}(x - y_0) \right) \right] = 0
\]
\[
\frac{\partial(P + Q)}{\partial \theta_j}(x_0, \Theta) = 2 \int \left[ r_j(x) \Im \left( e^{i\gamma_j} \phi_j(x - y_0) \right) + \partial_x r_j(x) \Im \left( e^{i\gamma_j} \partial_x \phi_j(x - y_0) \right) \right] = 0.
\]
Denoting with \( U \) and \( V \) the real and imaginary (respectively) part of \( W = \Phi(x - y_0)e^{i\Gamma} - R(x) \) and taking into account that \( R \) is real and does not depend on \( x_0 \), it follows
\[
\frac{\partial(P + Q)}{\partial x_0}(x_0, \Theta) = \sum_{j=1}^2 \int \left[ r_j \frac{\partial u_j}{\partial x_0} + \partial_x r_j \frac{\partial u_j}{\partial x_0} \right] = -\sum_{j=1}^2 \int \left[ u_j \frac{\partial r_j}{\partial x_0} + \partial_x u_j \frac{\partial r_j}{\partial x_0} \right] = 0
\]
\[
\frac{\partial(P + Q)}{\partial \theta_j}(x_0, \Theta) = \int \left[ r_j \nu_j + \partial_x r_j \partial_x \nu_j \right] = 0, \quad j = 1, 2.
\]
The second line of the above equations can be read as the orthogonality conditions on \( V \) in (3.2). As far as regards \( U \), we only have to notice that \( \partial_x R \) satisfies the linearized system of (1.2) so that all the conditions in (3.2) are proved.

We are now ready to complete the proof of the main result, Theorem 1.2.
Proof of Theorem 1.2 concluded. Let us consider \( \Phi \in \mathcal{H}^1 \) with \( \|\Phi\|_2 = \|R\|_2 \) and \( W(x) = \Phi(x - y_0)e^{iT - R(x)} \), where \( y_0 \in \mathbb{R} \) and \( \Gamma \in \mathbb{R}^2 \) satisfy the minimality conditions (3.1). We want to control the \( \mathcal{H}^1 \) norm of \( W \) in terms of the difference \( I(\Phi) - I(R) \), being \( I \) the action functional associated to the system and defined as

\[
I(\Phi) = E(\Phi) + \|\Phi\|_2^2.
\]

To this aim, we first compute the difference \( I(\Phi) - I(R) \) and we use scale invariance, obtaining \( I(\Phi) - I(R) = I(R + W) - I(R) \). Then, recalling that \( \langle I'(R), W \rangle = 0 \), Taylor expansion gives

\[
I(\Phi) - I(R) = I(R + W) - I(R) = \langle I'(R), W \rangle + \langle I''(R + \vartheta W)W, W \rangle
\]

\[
= \langle I''(R)W, W \rangle + \langle I''(R + \vartheta W)W, W \rangle - \langle I''(R)W, W \rangle.
\]

In order to evaluate the difference on the right hand side we will use the \( C^2 \) regularity of \( I \), at this point it is crucial (1.5). For simplicity, let us consider separately the nonlinear terms in \( I \). The term \( G : \mathcal{H}^1 \to \mathbb{R} \) defined by

\[
G(U) = G(u_1, u_2) = \|u_1\|_{2p+2}^2 + \|u_2\|_{2p+2}^2,
\]

is of class \( C^3 \), as \( p \geq 1 \), so that

\[
\langle G''(R + \vartheta W)W, W \rangle - \langle G''(R)W, W \rangle \geq -c_1\|W\|_{\mathcal{H}^1}^3.
\]

As far as concern the coupling term \( \Upsilon : \mathcal{H}^1 \to \mathbb{R} \) defined by \( \Upsilon(U) = \Upsilon(u_1, u_2) = \|u_1u_2\|_{p+1}^2 \), it results

\[
\langle \Upsilon''(U)W, W \rangle = (p^2 - 1) \int |u_1|^{p-3}|u_2|^{p-3}\left[|u_2|^4\Re^2(u_1)|w_1|^2 + |u_1|^4\Re^2(u_2)|w_2|^2\right]
\]

\[
+ (p + 1) \int |u_1|^{p-1}|u_2|^{p-1}\left[|u_2|^2|w_1|^2 + |u_1|^2|w_2|^2\right]
\]

\[
+ 2(p + 1)^2 \int |u_1|^{p-1}|u_2|^{p-1}\Re(u_1)\Re(u_2)\Re(w_1\overline{w_2}).
\]

When we write the difference \( \langle \Upsilon''(R)W, W \rangle - \langle \Upsilon''(R + \vartheta W)W, W \rangle \) we use that \( R \) is a real function and we control the first two terms with the real parts by the modulus; finally we use the inequality

\[
||r_j + \vartheta w_j||^{p-1} - |r_j|^{p-1}| \leq C|w_j|^{p-1},
\]

to get

\[
\langle \Upsilon''(R)W, W \rangle - \langle \Upsilon''(R + \vartheta W)W, W \rangle \geq -c_1\|W\|_{\mathcal{H}^1}^{2+\mu},
\]

for some \( \mu > 0 \).

This inequality joint with (3.3) implies that

\[
\langle I''(R + \vartheta W)W, W \rangle - \langle I''(R)W, W \rangle \geq -C\|W\|_{\mathcal{H}^1}^{2+\mu}.
\]

Therefore,

\[
I(\Phi) - I(R) \geq \langle I''(R)W, W \rangle - C\|W\|_{\mathcal{H}^1}^{2+\mu} = \langle L_V, V \rangle + \langle L_U, U \rangle - C\|W\|_{\mathcal{H}^1}^{2+\mu}.
\]

Taking into account the orthogonality conditions of Proposition 3.1, the assertion now follows from Proposition 2.5 and Remark 2.7. 

\[\blacksquare\]
Proof of Corollary 1.3 Let $\delta$ be a positive number to be chosen later. Moreover, let $R = (r_1, r_2) \in \mathbb{H}^1$ and $S = (s_1, s_1) \in \mathbb{H}^1$ be two given non-degenerate ground state solutions to system (1.2) such that
\[
\|R - S\|_{\mathbb{H}^1} < \delta.
\]
Then, taking into account the variational characterization (1.3) for ground states, we learn that
\[
\mathcal{E}(R) = \mathcal{E}(S), \quad \|R\|_{L^2} = \|S\|_{L^2}.
\]
Notice also that
\[
\inf_{\theta \in \mathbb{R}} \|R - (e^{\theta_1} r_1(-x_0), e^{\theta_2} s_2(-x_0))\|_{\mathbb{H}^1}^2 \leq \|R - S\|_{\mathbb{H}^1}^2 < \delta.
\]
Therefore, by applying Theorem 1.2, if $\delta > 0$ is chosen sufficiently small, we get
\[
\inf_{\theta \in \mathbb{R}} \|R - (e^{\theta_1} r_1(-x_0), e^{\theta_2} s_2(-x_0))\|_{\mathbb{H}^1}^2 \leq 0.
\]
In turn we conclude that $R = S$, up to a suitable translation and phase change.

Proof of Corollary 1.4 Let $T > 0$ and let us fix $\epsilon > 0$ sufficiently small. Consider the solution $\Psi$ of system (1.1) with initial datum $\Psi^0$. By the conservation laws, we have
\[
\|\Psi(t)\|_{L^2} = \|\Psi^0\|_{L^2}, \quad \mathcal{E}(\Psi(t)) = \mathcal{E}(\Psi^0), \quad \text{for all } t \in [0, \infty).
\]
By the continuity of the energy $\mathcal{E}$, there exists $\delta = \delta(\epsilon) > 0$ such that
\[
\mathcal{E}(\Psi(t)) - \mathcal{E}(R) = \mathcal{E}(\Psi^0) - \mathcal{E}(R) < \epsilon, \quad \text{for all } t \in [0, \infty),
\]
provided that
\[
\inf_{\theta \in \mathbb{R}} \|\Psi^0(\cdot) - (e^{\theta_1} r_1(-x), e^{\theta_2} r_2(-x))\|_{\mathbb{H}^1}^2 < \delta.
\]
Then, if we define for any $t > 0$ the positive number
\[
\Gamma_{\Psi(t)} = \inf_{\theta \in \mathbb{R}} \|\Psi(t) - (e^{\theta_1} r_1(-x), e^{\theta_2} r_2(-x))\|_{\mathbb{H}^1}^2,
\]
we learn from Theorem 1.2 that there exist two positive constants $\mathcal{A}$ and $C$ such that
\[
\Gamma_{\Psi(t)} \leq C(\mathcal{E}(\Psi(t)) - \mathcal{E}(R)),
\]
provided that $\Gamma_{\Psi(t)} < \mathcal{A}$. Let us define the value
\[
T_0 := \sup \{t \in [0, T] : \Gamma_{\Psi(s)} < \mathcal{A} \text{ for all } s \in [0, t]\}.
\]
Of course, it holds $T \geq T_0 > 0$ by means of (3.6) (up to reducing the size of $\delta$, if necessary) and the continuity of $\Psi(t)$. Hence, we deduce that
\[
\sup_{t \in [0, T_0]} \inf_{\theta \in \mathbb{R}} \|\Psi(t, \cdot) - (e^{\theta_1} r_1(-x), e^{\theta_2} r_2(-x))\|_{\mathbb{H}^1}^2 \leq C(\mathcal{E}(\Psi(t)) - \mathcal{E}(R)) = C(\mathcal{E}(\Psi^0) - \mathcal{E}(R)) < C\epsilon.
\]
On the other hand, it is readily seen that, from this inequality, one obtains $T_0 = T$. In fact, assume by contradiction that $T_0 < T$. Then, since by (3.8)
\[
\Gamma_{\Psi(T_0)} = \inf_{t \in \mathbb{R}} \|\psi(t, \cdot) - (a^2 \varphi_1(-x), e^{i\varphi_2(-x)})\|_{L^1}^2 < C\varepsilon,
\]
equality holds true by continuity for any $t \in [T_0, T_0 + \rho)$, for some small $\rho > 0$, which is a contradiction by the definition of $T_0$. Hence $T_0 = T$ and, for any $T > 0$, from (3.8) we get
\[
\sup_{t \in [0, T]} \inf_{\theta \in \mathbb{R}} \|\psi(t, \cdot) - (a^2 \varphi_1(-x), e^{i\varphi_2(-x)})\|_{L^1}^2 < C\varepsilon,
\]
which is the desired property on $[0, T]$. By the arbitrariness of $T$ the assertion follows.

## 4 Existence of a non-degenerate ground state

In the following section we will show that there exists a non-degenerate ground state $Z$. More precisely, let us consider $Z$ be the unique positive radial least energy solution of (1.6) and let $a$ be given by
\[(4.1)\quad a = (1 + \beta)^{-1/2}.
\]
We will prove the following result.

**Theorem 4.1.** Let $a$ be given in (4.1), then the vector $Z = a(z, z)$ is a non-degenerate ground state of system (1.2) for every $p > 0$, $\beta > 1$ and $p \neq \beta$.

**Remark 4.2.** In [11] it is proved that for $\beta \leq 1$ every ground state of (1.2) necessarily has one trivial component, that is the reason of the assumption $\beta > 1$. Moreover, it can been easily seen that for $p = \beta$ the ground state $Z$ is a degenerate solution that is why we assume $p \neq \beta$.

This result will be a consequence of the two following results.

**Theorem 4.3.** Let $a$ be given in (4.1), then the vector $Z = a(z, z)$ is a ground state of system (1.2) for every $p > 0$, $\beta > 1$.

**Theorem 4.4.** Let $a$ be given in (4.1), then the vector $Z = a(z, z)$ is a non-degenerate ground state of system (1.2) for every $p > 0$, $\beta > 1$ and $p \neq \beta$.

**Remark 4.5.** In [7] it is studied the global existence for the Cauchy problem (1.1) and it is proved that the solution exists for any time if $p < 2/n$, while it can blow up if $p \geq 2/n$. In the critical case $p = 2/n$ it is given a bound on the $L^2$-norm of the initial data which guarantees the global existence of the solution (see Theorem 2). Since Theorem 4.3 shows that the test functions used in [7] to estimate the blow-up threshold belong to the set of ground state solutions, as a by product, we obtain that the bound given in [7] is the exact threshold value.

**Remark 4.6.** The above results have been proved for $p = 1$, respectively, in [17] and [6] in any dimension. Actually, the same arguments work for any $p > 0$. In the following we include the details for completeness. Let us notice that the same proof of Theorem 4.3 holds in dimension greater than one; in addition, the arguments used in [6] hold for $p \in (0, 2/n)$ for every $n \geq 1$. Thus, the vector $Z$ is a non-denerate ground state solution of (1.2) in any dimension $n \geq 1$, our conjecture is that it is the only one if $\beta > 1$. Here our interest, is restricted to the one dimension setting so that we will see the proof of Theorem 4.1 in this case.
4.1 Proof of Theorem 4.3

First, we recall this simple facts.

**Proposition 4.7.** Let us set

\[ S_1 = \inf_{H^1(\mathbb{R}), 0} \frac{\|u\|_{H^1}^2}{\|u\|_{2p+2}^2}, \quad T_1 = \inf_{N_1} \left\{ \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{2p+2} \|u\|_{2p+2}^{2p+2} \right\}, \]

where

\[ N_1 = \{ u \in H^1(\mathbb{R}) : \|u\|_{H^1}^2 = \|u\|_{2p+2}^{2p+2} \}. \]

Then, the following equality holds

\[ T_1 = \frac{1}{2} \frac{p}{p+1} (S_1)^{(p+1)/p}. \]

**Proof.** As \( z \) solves the minimization problems that defines \( S_1 \) and \( T_1 \), using (1.6) we get

\[ S_1 = \frac{\|z\|_{H^1}^2}{\|z\|_{2p+2}^2} = \frac{\|z\|_{H^1}^2}{\|z\|_{2(p+1)}^2} = \|z\|_{H^1}^{2p/(p+1)} = \|z\|_{2p+2}^{2p}, \]

namely

\[ (4.2) \quad \|z\|_{H^1}^2 = S_1^{(p+1)/p} \quad \text{and} \quad \|z\|_{2p+2} = S_1^{1/2p}. \]

Using these equalities in the definition of \( T_1 \) permits to conclude the proof. \( \blacksquare \)

Define now the sets

\[ N_0 = \{ U \in H^1 : U \neq (0, 0), \|U\|_{H^1}^2 = \|U\|_{2p+2}^{2p+2} + 2\beta \|u_1u_2\|_{p+1}^{p+1} \}, \]

\[ N = \{ U \in H^1 : u_i \neq 0, \|u_i\|_{H^1}^2 = \|u_i\|_{2p+2}^{2p+2} + \beta \|u_1u_2\|_{p+1}^{p+1}, \quad i = 1, 2 \}. \]

Moreover, if \( H^1_r \) is the set of radial function of \( H^1 \), we introduce the numbers

\[ (4.3) \quad A_0 = \inf_{U \in N_0} I(U), \quad A = \inf_{U \in N} I(U), \quad A_r = \inf_{U \in N \cap H^1_r} I(U), \]

where

\[ I(U) = \frac{1}{2} \|U\|_{H^1}^2 - \frac{1}{2p+2} \|U\|_{2p+2}^{2p+2} - \frac{1}{p+1} \beta \|u_1u_2\|_{p+1}^{p+1}. \]

Let \( a \) be a positive number. Writing down the equations that define \( N \) and recalling that \( z \) satisfies (1.6) it is easy to see that \( a(z, z) \in N \) if \( a \) satisfies (4.1).

Concerning the infimum problems \( A_0, A, A_r \), in [17] the following result is proved for \( p = 1 \); actually the same proof holds for any \( p \) satisfying (1.5), we include some details.

**Proposition 4.8.** Let \( a \) satisfies (4.1). Then the following inequalities hold

\[ (4.4) \quad 0 < A_0 \leq A \leq A_r \leq \frac{P}{p+1} a^2 S_1^{(p+1)/p}, \]

where the values \( A_0 \) and \( A_r \) are defined in (4.3).
Proof. First note that, taken any \( U = (u_1, u_2) \in \mathcal{N}_0 \), the value \( I(U) \) is equal to

\[
I(U) = \frac{1}{2} \left( \frac{p}{p+1} \right) \| U \|_{2p+2}^2 + 2\beta \| u_1 u_2 \|_{p+1}^2 = \frac{1}{2} \left( \frac{p}{p+1} \right) \| U \|_{H^1}^2.
\]

Moreover, since \( a(z, z) \in \mathcal{N} \) and has radial components, recalling (4.2) we get

\[
A_r \leq I(a z, a z) = \frac{1}{2} \left( \frac{p}{p+1} \right) \| (a z, a z) \|_{H^1}^2 = \left( \frac{p}{p+1} \right) a^2 S_1^{(p+1)/p},
\]

which is the last inequality on the right-hand side in (4.4). It just remains to show that \( A_0 > 0 \). To this aim, take \( U \in \mathcal{N}_0 \) and observe that Hölder and Sobolev inequalities imply that there exist positive constants \( C_0, C_1 \) such that

\[
\| U \|_{H^1}^2 = \| U \|_{2p+2}^2 + 2\beta \| u_1 u_2 \|_{p+1}^2 \leq C_0 \| U \|_{2p+2}^2 \leq C_1 \| U \|_{H^1}^2.
\]

so that the norm \( \| U \|_{H^1} \) remains uniformly away from zero. Hence, recalling formula (4.5), we conclude the proof.

We are now ready to complete the proof of Theorem 4.3.

Proof of Theorem 4.3 concluded. We will obtain Theorem 4.3 by showing that the infimum \( A \) equals \( A_r \) and it is achieved at the couple \((a(z), z)\), which is thus a ground state solution of (1.2).

First, let \((U_m) = (u_m, 1, u_m, 2) \subset \mathcal{N}\) be a minimizing sequence for \( A \), namely \( I(U_m) = A + o(1) \) as \( m \to \infty \). Let us set \( y_{m,i} = \| u_{m,i} \|_{2p+2}^2 \) for any \( m \in \mathbb{N} \) and \( i = 1, 2 \). Hence, by the definition of \( S_1 \) and Hölder inequality, it follows that, for all \( m \in \mathbb{N} \),

\[
S_1 y_{m,1} \leq \| u_{m,1} \|_{H^1}^2 = \| u_{m,1} \|_{2p+2}^2 + \beta \| u_{m,1} u_{m,2} \|_{p+1}^2 \leq \gamma_{y_{m,1}} + \beta \gamma_{y_{m,2}}.
\]

for all \( m \in \mathbb{N} \). Of course, for all \( m \in \mathbb{N} \), the analogous inequality holds

\[
S_1 y_{m,2} \leq \| u_{m,2} \|_{H^1}^2 = \| u_{m,2} \|_{2p+2}^2 + \beta \| u_{m,1} u_{m,2} \|_{p+1}^2 \leq \gamma_{y_{m,2}} + \beta \gamma_{y_{m,1}}.
\]

Furthermore, taking into account formula (4.5), by addition of the first inequalities in (4.7) and (4.8) one obtains

\[
S_1 (y_{m,1} + y_{m,2}) \leq 2 \frac{p+1}{p} I(U_n) = \frac{p+1}{p} A + o(1), \quad \text{as } m \to \infty.
\]

By combining this inequality with Proposition 4.8 gives

\[
S_1 (y_{m,1} + y_{m,2}) \leq 2a^2 S_1^{(p+1)/p} + o(1), \quad \text{as } m \to \infty.
\]

Hence, defining \( z_{m,i} = y_{m,i}/S_1^{1/p} \), we derive \( z_{m,1} + z_{m,2} \leq 2a^2 + o(1) \), as \( m \) tends to infinity. Also, by dividing (4.7) by \( S_1 y_{m,1} \) and (4.8) by \( S_1 y_{m,2} \) and using \( S_1 = S_1^{(p-1)/2} S_1^{(p+1)/2p} \) we obtain that, as \( m \to \infty \), \((z_{m,1}, z_{m,2})\) satisfies the following system of inequalities

\[
\begin{align*}
z_{m,1} + z_{m,2} & \leq 2a^2 + o(1), \\
z_{m,1}^p + \beta z_{m,1}^{(p-1)/2} z_{m,2}^{(p+1)/2} & \geq 1, \\
z_{m,2}^p + \beta z_{m,1}^{(p+1)/2} z_{m,2}^{(p-1)/2} & \geq 1.
\end{align*}
\]
Taking into account (4.1) we are lead to the study of the associated algebraic system of inequalities

\[
\begin{align*}
    &x + y \leq 2a^2, \\
    &x^p + \beta x^{(p-1)/2} y^{(p+1)/2} \geq (1 + \beta) a^{2p}, \\
    &y^p + \beta y^{(p+1)/2} x^{(p-1)/2} \geq (1 + \beta) a^{2p},
\end{align*}
\]

(4.10)

for which we refer to Figure 1.

Then, for \( \beta > 1 \) and any \( i = 1, 2 \), the sequence \((z_{m,i})\) remains bounded away from zero and it has to be \( z_{m,1} \to a^2 \) and \( z_{m,2} \to a^2 \) as \( m \to \infty \), so that looking at the first (in)equality of (4.10) with \( x = y \) (by figure 1) yields \( x = y = a^2 \), so that \( y_{m,1} \to a^2 S_1^{1/p} \), and \( y_{m,2} \to a^2 S_1^{1/p} \), as \( m \) diverges. Whence, passing to the limit in formula (4.9), in light of Proposition 4.8 we obtain

\[
2S_1^{(p+1)/p} a^2 \leq \frac{2p + 1}{p} A \leq 2a^2 S_1^{(p+1)/p}
\]

so that, (4.6), gives

\[
A \leq A_r \leq I(az, az) \leq \left( \frac{p}{p + 1} \right) a^2 (S_1^{(p+1)/p}) = A,
\]

which gives \( A = A_r = I(az, az) \), concluding the proof.

\[ \square \]

### 4.2 Proof of Theorem 4.4

According to Section 4.1, let us consider \( Z = a(\xi, \eta) \) the particular ground state solution of (1.2), with \( a \) given in (4.1); we will now show the non-degeneracy property of \( Z \). First, notice that the linearized system (1.9) can be obtained using the operator \( L_+ \) acting on \( Z \), and by the explicit expression of \( Z \) we get

\[
L_+ = \begin{pmatrix}
    \frac{1}{2} \partial_{xx} + 1 & 0 \\
    0 & \frac{1}{2} \partial_{xx} + 1
\end{pmatrix} - \begin{pmatrix}
    \frac{p(2 + \beta) + 1}{1 + \beta} S_{2p} & \frac{\beta(p + 1)}{1 + \beta} S_{2p} \\
    \frac{\beta(p + 1)}{1 + \beta} S_{2p} & \frac{p(2 + \beta) + 1}{1 + \beta} S_{2p}
\end{pmatrix}.
\]

In accordance with Section 2, we denote with \( H_F(Z) \) the second matrix on the right hand side. The quadratic form related to \( H_F(Z) \) can be diagonalized by an orthonormal change of coordinates, introducing

\[
w_1 = \frac{\sqrt{2}}{2}(\phi_1 + \phi_2), \quad w_2 = \frac{\sqrt{2}}{2}(\phi_1 - \phi_2).
\]

Since we have

\[
\text{Tr}(H_F(Z)) = 2 \frac{(2 + \beta)p + 1}{1 + \beta} = (2p + 1) + \frac{2p + 1 - \beta}{1 + \beta}, \quad \text{Det}(H_F(Z)) = \frac{(2p + 1)(2p + 1 - \beta)}{1 + \beta},
\]

it follows that its eigenvalues are

\[
\lambda_1 = 2p + 1, \quad \lambda_2 = \frac{2p + 1 - \beta}{1 + \beta} \in (-1, 2p + 1)
\]

(4.12)
so the linear elliptic system $L_x \Phi = 0$ decouples and reduces to

\[
\begin{aligned}
&-\frac{1}{2} \partial_{xx} w_1 + w_1 = (2p + 1)z^{2p}(x)w_1, \quad \text{in } \mathbb{R} \\
&-\frac{1}{2} \partial_{xx} w_2 + w_2 = \frac{2p + 1 - \beta}{1 + \beta}z^{2p}(x)w_2, \quad \text{in } \mathbb{R}.
\end{aligned}
\]

(4.13)

Taking into account that the weight $z$ is exponentially decaying, the spectrum of the linear self-adjoint operator $-\frac{1}{2} \partial_{xx} + \text{Id} - \mu z^{2p}$ is discrete. Furthermore, from [19, (a) and (b) of Proposition 2.8] with proofs for $n = 1$ in [19, Appendix A], we learn that the eigenvalues of

\[
-\frac{1}{2} \partial_{xx} w - \mu z^{2p}(x)w = 0 \quad \text{in } \mathbb{R},
\]

(4.14)

are given by $\mu_1 = 1$, $\mu_2 = 2p + 1$, $\mu_3 > 2p + 1$, and, denoting by $V_{\mu_i}$ the eigenspace corresponding to the eigenvalue $\mu_i$, we have $V_{\mu_1} = \text{span}[z]$, $V_{\mu_2} = \text{span}[\partial_x z]$. Therefore, from the first equation of (4.13) we deduce $w_1 \in \text{span}[\partial_x z]$. From (4.12) we also deduce, from the second equation of (4.13), that $w_2 = 0$. In turn, by the orthonormal change of coordinates (4.11) we obtain $\phi_1 = \phi_2 = c\partial_x z$, for some coefficient $c \in \mathbb{R}$. Whence $\text{Ker}(L_x) = \langle \partial_x Z \beta \rangle$, which concludes the proof.

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