A LINEAR IMPLICIT FINITE DIFFERENCE DISCRETIZATION
OF THE SCHröDINGER-HIROTA EQUATION

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ABSTRACT. A linear implicit finite difference method is proposed for the approximation of the
solution to a periodic, initial value problem for a Schrödinger-Hirota equation. Optimal, second
order convergence in the discrete $H^1$–norm is proved, assuming that $\tau$, $h$ and $\tau^4/h^4$ are sufficiently
small, where $\tau$ is the time-step and $h$ is the space mesh-size. The efficiency of the proposed
method is verified by results from numerical experiments.

1. INTRODUCTION

1.1. Formulation of the problem. For $T > 0$ and $L > 0$, we consider the following periodic initial
value problem: find $\phi = \phi(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{C}$ which is $L$–periodic on $\mathbb{R}$ and such that:

\begin{align}
\phi_t &= i \rho \phi_{xx} - \sigma \phi_{xxx} - 3 \alpha |\phi|^2 \phi_x + i \delta |\phi|^2 \phi + f \quad \text{on } (0, T) \times \mathbb{R}, \tag{1.1} \\
\phi(0, x) &= \phi_0(x) \quad \forall x \in \mathbb{R}, \tag{1.2}
\end{align}

where: $\rho$, $\sigma$, $\alpha$ and $\delta$ are real constants, $\phi_0 = \phi_0(x) : \mathbb{R} \rightarrow \mathbb{C}$ is an $L$–periodic function and $f = f(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{C}$ is a function which is $L$–periodic on $\mathbb{R}$.

The nonlinear partial differential equation (1.1) is known as: ‘the cubic nonlinear Schrödinger
equation’ (cNLS) [3] when $\alpha = \sigma = 0$, ‘the Hirota equation’ (H) [10] when
\begin{equation}
\rho \alpha = \sigma \delta \tag{1.3}
\end{equation}

and ‘the complex modified Korteweg-de Vries equation’ (cmKdV) [12] when $\rho = \delta = 0$. Since the
(cmKdV) equation is a special case of the (H) equation, we adopt, for equation (1.1), the name
Schrödinger-Hirota (SH) equation (cf. [2]). The (SH) equation is widely used in the description
of the propagation of optical solitons in a dispersive optical fiber (see, e.g., [9], [11]), and in the
modeling of the motion of vortex filaments (see, e.g., [7], [11]).

For existence and uniqueness results in the homogeneous case we refer the reader to [11]. There, it
is shown that: i) if $\phi_0 \in H^4_{\text{per}}(0, L)$ for any integer $\ell \geq 2$, then there exists $T > 0$ such that the problem
has a unique solution $\phi \in C([0, T], H^2_{\text{per}}(0, L))$ with $\phi_t \in C([0, T], H^{\ell-3}_{\text{per}}(0, L))$ (see Theorem 2.1
in [11]), and ii) if $\phi_0 \in H^2_{\text{per}}(0, L)$, $\rho \neq 0$ and (1.3) is satisfied, then $\phi \in C([0, +\infty), H^2_{\text{per}}(0, L)) \cap
C([0, +\infty), H^3_{\text{per}}(0, L))$ (see Theorem 2.5 in [11]), i.e., there is no finite time blow up for the solution
and its first space derivative. The latter results, are based on the following conservation properties:

\begin{equation}
\int_0^L |\phi(t, x)|^2 \, dx = \int_0^L |\phi_0(x)|^2 \, dx \quad \forall t \in [0, T], \tag{1.4}
\end{equation}

and, when (1.3) holds,

\begin{equation}
\rho \int_0^L |\phi_x(t, x)|^2 \, dx - \frac{4}{3} \int_0^L |\phi(t, x)|^4 \, dx = \rho \int_0^L |\phi_0'(x)|^2 \, dx - \frac{4}{3} \int_0^L |\phi_0(x)|^4 \, dx \quad \forall t \in [0, T]. \tag{1.5}
\end{equation}

However, it is easily seen that there exist unique, smooth, special solutions to the homogeneous
problem (1.1)–(1.2) for any choice of the parameters $\rho$, $\alpha$, $\delta$ and $\sigma$ (see Section 5.2).

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1
In the paper at hand, we focus on the numerical approximation of the solution to the problem (1.1)-(1.2). In particular, we propose a new linear implicit finite difference method, the convergence of which is ensured by providing an optimal, second order, error estimate. For the needs of the convergence analysis, we will assume that the problem above admits a unique solution that is sufficiently smooth.

1.2. The Finite Difference Method (FDM). Let $\mathbb{N}$ be the set of all positive integers, $x_a, x_b \in \mathbb{R}$ with $x_b - x_a = L$, and $N, J \in \mathbb{N}$. Then, we define a uniform partition of the time interval $[0,T]$ with time-step $\tau := \frac{T}{N}$ and nodes $t_n := n \tau$ for $n = 0, \ldots, N$, and a uniform partition of $\mathbb{R}$ with mesh-width $h := \frac{L}{J}$ and nodes $x_j := x_a + jh$ for $j \in \mathbb{Z}$. Also, we introduce the discrete space:

$$\mathcal{X}_h := \{ (\psi_j)_{j \in \mathbb{Z}} \in \mathbb{C}^\infty : \psi_j = \psi_{j+1} \quad \forall \ j \in \mathbb{Z} \},$$

a discrete product operator $\cdot \otimes \cdot : \mathcal{X}_h \times \mathcal{X}_h \rightarrow \mathcal{X}_h$ by

$$(v \otimes w)_j = v_j w_j \quad \forall \ j \in \mathbb{Z}, \quad \forall \ v, w \in \mathcal{X}_h,$$

a discrete Laplacian operator $\Delta_h : \mathcal{X}_h \rightarrow \mathcal{X}_h$ by

$$\Delta_h v_j := \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} \quad \forall \ j \in \mathbb{Z}, \quad \forall \ v \in \mathcal{X}_h,$$

a discrete space derivative operator $\partial_h : \mathcal{X}_h \rightarrow \mathcal{X}_h$ by

$$\partial_h v_j := \frac{v_j - v_{j-1}}{h} \quad \forall \ j \in \mathbb{Z}, \quad \forall \ v \in \mathcal{X}_h$$

and a discrete space average operator $A_h : \mathcal{X}_h \rightarrow \mathcal{X}_h$ by

$$A_h v_j := \frac{1}{2} (v_{j+1} + v_{j-1}) \quad \forall \ j \in \mathbb{Z}, \quad \forall \ v \in \mathcal{X}_h.$$

In addition, we define the space $C_{\text{per}} := \{ v \in C(\mathbb{R}; \mathbb{C}) : v(x + L) = v(x) \quad \forall \ x \in \mathbb{R} \}$ and the operator $\Lambda_h : C_{\text{per}} \rightarrow \mathcal{X}_h$ by $(\Lambda_h v)_j := v(x_j)$ for $j \in \mathbb{Z}$ and $v \in C_{\text{per}}$, and we set $\phi^n := \Lambda_h(\phi(t_n, \cdot))$ for $n = 0, \ldots, N$. For $\ell \in \mathbb{N}$ and for any function $g : C^\ell \rightarrow C$ and any $w = (w^1, \ldots, w^j) \in (\mathcal{X}_h)^j$, we define $g(w) \in \mathcal{X}_h$ by $(g(w))_j := g(w^1, \ldots, w^j)$ for $j \in \mathbb{Z}$.

For $n = 0, \ldots, N$, the proposed linear implicit finite difference method constructs, recursively, an approximation $\Phi^n \in \mathcal{X}_h$ of $\phi^n$ following the steps below:

**Step 1:** Set

$$\Phi^0 := \phi^0.$$

**Step 2:** Find $\Phi^1 \in \mathcal{X}_h$ such that

$$\frac{\Phi^1 - \Phi^0}{\tau} = i \rho \Delta_h \left( \frac{\Phi^1 + \Phi^0}{2} \right) - \frac{\Delta_h}{2} \left( \frac{\Phi^1 + \Phi^0}{2} \right)$$

$$- 3 \alpha A_h \left[ \left[ \Phi^0 \right]^2 \otimes \partial_h \left( \frac{\Phi^1 + \Phi^0}{2} \right) \right] + i \delta \left[ \left[ \Phi^0 \right]^2 \otimes \left( \frac{\Phi^1 + \Phi^0}{2} \right) \right] + F^\frac{1}{2},$$

where $F^\frac{1}{2} \in \mathcal{X}_h$ with $(F^\frac{1}{2})_j := f(t_j, x_j)$ for $j \in \mathbb{Z}$.

**Step 3:** For $n = 1, \ldots, N - 1$, find $\Phi^{n+1} \in \mathcal{X}_h$ such that

$$\frac{\Phi^{n+1} - \Phi^n}{\tau} = i \rho \Delta_h \left( \frac{\Phi^{n+1} + \Phi^n}{2} \right) - \frac{\Delta_h}{2} \left( \frac{\Phi^{n+1} + \Phi^n}{2} \right)$$

$$- 3 \alpha A_h \left[ \left[ \Phi^n \right]^2 \otimes \partial_h \left( \frac{\Phi^{n+1} + \Phi^n}{2} \right) \right] + i \delta \left[ \left[ \Phi^n \right]^2 \otimes \left( \frac{\Phi^{n+1} + \Phi^n}{2} \right) \right] + F^n,$$

where $F^n \in \mathcal{X}_h$ with $(F^n)_j := f(t_n, x_j)$ for $j \in \mathbb{Z}$.

Thus, at every time step, the computation of the finite difference approximations above, requires the solution of a linear system of algebraic equations the matrix of which is cyclic pentadiagonal.
1.3. Overview and references. The finite difference method formulated and computationally tested in \cite{6} stands out among the known linear implicit methods for the discretization of the (cNLS) equation, since it satisfies a discrete analogue of \cite{14} and \cite{15}. Both discrete conservation laws ensure that the finite difference approximations are uniformly bounded in the discrete $L^\infty$-norm, which leads to an optimal order error estimate (see, e.g., \cite{8}, \cite{14}).

The (FDM) we propose, for the numerical treatment of the solution to the (SH) equation, is an extension of the method proposed in \cite{8}. However, we are not able to show that the (FDM) approximations are uniformly bounded in the discrete $W^{1,\infty}$-norm, which is necessary in handling the nonlinearities of the (SH) equation. Therefore, the only choice left is to work with an auxiliary modified scheme.

Using an idea from \cite{14}, first we define an operational mollifier depending on a positive parameter $\lambda$ and the discrete $W^{1,\infty}$-norm (see Section 3.1), and then we formulate a Modified Finite Difference Scheme (MFDS) by mollifying properly the nonlinear terms of the (FDM) (see Section 3.2). Assuming that $\tau$ is small enough and $\lambda$ large enough, for the non-computable (MFDS) approximations first we show that are well-defined (see Proposition 3.1) and then we establish an optimal, second order error estimate in a discrete $H^1$-norm (see Theorem 3.2), which, after applying a discrete Sobolev inequality, yields a convergence result in the discrete $\delta$-(FDM) approximations. Finally, we expose results from numerical experiments in Section 5.

We close this section by giving a brief overview of the paper. In Section 2 we introduce notation and we prove a series of auxiliary results that we will often use later in the analysis of the (MFDS) and the (FDM) approximations. Section 3 is dedicated to the construction and the analysis of the MFDS equation. Therefore, the only choice left is to work with an auxiliary modified scheme.

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\( v \in \mathcal{X}_h \), and a discrete \( W^{m,\infty} \)-norm \( \| \cdot \|_{m,\infty,h} \), by \( \|v\|_{m,\infty,h} := \max \{ |v|_{\infty,h}, \max_{1 \leq \ell \leq m} |v|_{\ell,\infty,h} \} \) for \( v \in \mathcal{X}_h \). For given norm \( \nu \) on \( \mathcal{X}_h \), \( v \in \mathcal{X}_h \) and \( \varepsilon > 0 \), we define the closed ball \( \mathcal{B}_\varepsilon(v; \nu) := \{ w \in \mathcal{X}_h : \nu(v-w) \leq \varepsilon \} \).

Below, we provide a series of auxiliary results that we will often use in the rest of the paper.

**Lemma 2.1.** For all \( v, z \in \mathcal{X}_h \), we have
\[
\begin{align*}
(2.1) \quad & \sigma^+ z = z, \\
(2.2) \quad & \partial_h \Delta_h v = \Delta_h \partial_h v, \\
(2.3) \quad & \partial_h A_h v = A_h \partial_h v, \\
(2.4) \quad & \partial_h (v \otimes z) = \partial_h v \otimes \sigma^+ z + \sigma^+ v \otimes \partial_h z.
\end{align*}
\]

*Proof.* The verification of the formulas above is straightforward. \( \square \)

**Lemma 2.2.** For all \( v, z \in \mathcal{X}_h \) it holds that
\[
\begin{align*}
(2.5) \quad & (\sigma^+ v, z)_{0,h} = (v, \sigma^- z)_{0,h}, \\
(2.6) \quad & (\sigma^+ v, z)_{0,h} = (v, \sigma^+ z)_{0,h}, \\
(2.7) \quad & (\partial_h v, z)_{0,h} = -(v, \partial_h z)_{0,h}, \\
(2.8) \quad & (\Delta_h v, z)_{0,h} = -(\partial_h v, \partial_h z)_{0,h} = (v, \Delta_h z)_{0,h}, \\
(2.9) \quad & (\Delta_h v, v)_{0,h} = -|\delta_h v|^2_{0,h}, \\
(2.10) \quad & \mathcal{R}(\partial_h \Delta_h v, v)_{0,h} = 0.
\end{align*}
\]

*Proof.* Let \( v, z \in \mathcal{X}_h \). First, we establish (2.5) proceeding as follows:
\[
(\sigma^+ v, z)_{0,h} = h \sum_{j=1}^j v_{j+1} \hat{s}_j = h \sum_{j=2}^{j+1} v_{j+1} \hat{s}_j = h \sum_{j=1}^j v_{j+1} \hat{s}_j = (v, \sigma^- z)_{0,h}.
\]

Then, we apply (2.5) to obtain
\[
(\sigma^- v, z)_{0,h} = (z, \sigma^- v)_{0,h} = (\sigma^+ z, v)_{0,h} = (v, \sigma^+ z)_{0,h}.
\]

To obtain (2.6), we combine (2.5) and (2.6) as follows:
\[
(\partial_h v, z)_{0,h} = \frac{1}{2h} (\sigma^+ v - \sigma^- v, z)_{0,h} = \frac{1}{2h} (v, \sigma^- z - \sigma^+ z)_{0,h} = -(v, \partial_h z)_{0,h}.
\]

Also, using (2.6), we have
\[
(\Delta_h v, z)_{0,h} = \frac{1}{m^2} (\sigma^+ v - 2v + \sigma^- v, z)_{0,h}
\]
\[
= \frac{1}{h} \left[ (\partial_h v, z)_{0,h} - (\sigma^- \partial_h v, z)_{0,h} \right]
\]
\[
= \frac{1}{h} (\partial_h v, z - \sigma^- z)_{0,h}
\]
\[
= - (\partial_h v, \partial_h z)_{0,h}
\]

which yields
\[
(2.12) \quad (v, \Delta_h z) = (\Delta_h z, v)_{0,h} = -(\partial_h z, \partial_h v)_{0,h} = -(\partial_h v, \partial_h z)_{0,h}.
\]

Thus, (2.8) is a simple consequence of (2.11) and (2.12). Relation (2.9) follows easily from (2.8) setting \( z = v \). Finally, we use (2.2), (2.8) and (2.7), to have
\[
(\partial_h \Delta_h v, v)_{0,h} = (\Delta_h \partial_h v, v)_{0,h} = (\partial_h v, \Delta_h v)_{0,h} = -(v, \partial_h \Delta_h v)_{0,h} = -(\partial_h \Delta_h v, v)_{0,h},
\]

which, obviously, yields (2.10). \( \square \)

**Lemma 2.3.** For all \( v, z \in \mathcal{X}_h \) it holds that
\[
\begin{align*}
(2.13) \quad & |\partial_h (|z|^2)|_{\infty,h} \leq 2 |z|_{\infty,h} |z|_{1,\infty,h}, \\
(2.14) \quad & \|z^2 - |v|^2\|_{0,h} \leq (|z|_{\infty,h} + |v|_{\infty,h}) \|z - v\|_{0,h}.
\end{align*}
\]
Proof. Let \( v, z \in \mathcal{X}_h \). Observing that

\[
|\partial_h (|z|^2)|_{\infty, h} = \max_{1 \leq j \leq J} \left| \text{Re} \left[ \partial_h z_j \left( z_{j+1} + z_{j-1} \right) \right] \right|
\]

and

\[
\left\| |z|^2 - |v|^2 \right\|_{0, h} = \left[ h \sum_{j=1}^{J} \left| \text{Re} \left[ (z_j - v_j) \left( z_j + v_j \right) \right] \right|^2 \right]^{1/2}
\]

(2.13) and (2.14) easily follow.

Lemma 2.4. Let \( \mathcal{A}_h \) be the space average operator defined by (1.7). Then, for \( w, z \in \mathcal{X}_h \), it holds that

\[
\mathcal{A}_h \mathcal{C}_w = (w, \mathcal{A}_h z)_{0, h}
\]

(2.15)

\[
|\sigma^+ z|_{0, h} = |z|_{0, h},
\]

(2.16)

\[
|\sigma^- w|_{0, h} = |w|_{0, h},
\]

(2.17)

\[
|\mathcal{A}_h z|_{0, h} \leq |z|_{0, h},
\]

(2.18)

\[
\text{Re} \left( \mathcal{A}_h (|z|^2 \otimes \partial_h w), w \right)_{0, h} = -\frac{1}{2} \left( \partial_h (|z|^2), |w|^2 \right)_{0, h}.
\]

Proof. Let \( w, z \in \mathcal{X}_h \). Using (2.15) and (2.16), we obtain (2.15) proceeding as follows

\[
(\mathcal{A}_h w, z)_{0, h} = \frac{1}{2} (\sigma^+ w + \sigma^- w, z)_{0, h} = \frac{1}{2} (w, \sigma^- z + \sigma^+ z)_{0, h} = (w, \mathcal{A}_h z)_{0, h}.
\]

Combining (2.15) and (2.16), we have

\[
(\sigma^+ z, \sigma^+ z)_{0, h} = (z, \sigma^- \sigma^+ z)_{0, h} = (z, z)_{0, h},
\]

which, obviously yields (2.16). To arrive at (2.17), we use (2.16) and (2.1) as follows

\[
|\sigma^- w|_{0, h} = |\sigma^+ \sigma^- w|_{0, h} = |w|_{0, h}.
\]

Also, we apply (2.16) and (2.17) to get

\[
|\mathcal{A}_h z|_{0, h} = \frac{1}{2} |\sigma^+ z + \sigma^- z|_{0, h}
\]

\[
\leq \frac{1}{2} (|\sigma^+ z|_{0, h} + |\sigma^- z|_{0, h})
\]

\[
\leq |z|_{0, h},
\]

which is (2.18). Finally, we use (2.15) and (2.7) to obtain

\[
\text{Re} \left( \mathcal{A}_h (|z|^2 \otimes \partial_h w), w \right)_{0, h} = \text{Re} \left( |z|^2 \otimes \partial_h w, \mathcal{A}_h w \right)_{0, h}
\]

\[
= \frac{1}{2} \sum_{j=1}^{J} |z|^2 \frac{|w_{j+1}^2 - |w_{j-1}^2|}{2h}
\]

\[
= \frac{1}{2} \left( |z|^2, \partial_h (|w|^2) \right)_{0, h}
\]

\[
= -\frac{1}{2} \left( \partial_h (|z|^2), |w|^2 \right)_{0, h}
\]

which establishes (2.19).

Lemma 2.5. Let \( J \geq 3 \). Then, we have that

\[
|\psi|_{\infty, h} \leq \frac{\sqrt{3 \times 2J - 2}}{\sqrt{L}} |\psi|_{1, h} \quad \forall \psi \in \mathcal{X}_h.
\]

Proof. Let \( \psi \in \mathcal{X}_h \). It is easily seen that there exists \( m \in \{ J + 1, \ldots, 2J \} \) such that \( |\psi_m| = |\psi|_{\infty, h} \). Then, we consider the following cases:

CASE 1: \( m \) is odd, i.e. there exists \( \tilde{m} \in \mathbb{N} \) such that \( m = 2 \tilde{m} + 1 \).
Let \( A := \{ \ell \in \mathbb{N} : \ell \leq J, \ \ell \equiv 1 \mod 2 \} \). Then, for \( \kappa \in A \) there exists \( \rho(\kappa) \in \mathbb{N} \) such that \( \kappa = 2\rho(\kappa) - 1 \). Thus, we have

\[
|\psi_m| = \left| \psi_\kappa + 2h \sum_{\ell=\rho(\kappa)}^{\bar{m}} \partial_h \psi_{2\ell} \right|
\leq |\psi_\kappa| + 2h \sum_{\ell=2\rho(\kappa)}^{\bar{m}} |\partial_h \psi_\ell| \quad \forall \ \kappa \in A,
\]

which yields

\[
\text{card}(A) \ |\psi|_{\infty,h} \leq \sum_{\kappa \in A} |\psi_\kappa| + 2h \sum_{\kappa \in A} \left( \sum_{\ell=\kappa+1}^{m-1} |\partial_h \psi_\ell| \right)
\leq \sum_{\kappa \in A} |\psi_\kappa| + 2h \text{card}(A) \sum_{\ell=2}^{m-1} |\partial_h \psi_\ell|.
\]

Observing that

\[
\text{card}(A) = \begin{cases} \frac{J}{2} & \text{if } J \text{ is even} \\ \frac{J-1}{2} & \text{if } J \text{ is odd} \end{cases},
\]

and using (2.21) along with the Cauchy-Schwarz inequality, we obtain

\[
|\psi|_{\infty,h} \leq \frac{\bar{m}}{2} \sum_{\ell=A} |\psi_\ell| + 2h \sum_{\ell=1}^{J} |\partial_h \psi_\ell|
\leq \frac{\bar{m}}{2} \sum_{\ell=1}^{J} |\psi_\ell| + 4h \sum_{\ell=1}^{J} |\partial_h \psi_\ell|
\leq \frac{\bar{m}}{2} \|\psi\|_{0,h} + 4\sqrt{L} |\psi|_{1,h}
\leq 2 \sqrt{L+\frac{1}{2}} \|\psi\|_{1,h}.
\]

**CASE 2:** \( m \) is even, i.e. there exists \( \bar{m} \in \mathbb{N} \) such that \( m = 2\bar{m} \).

Let \( B := \{ \ell \in \mathbb{N} : \ell \leq J, \ \ell \equiv 0 \mod 2 \} \). Then, for \( \kappa \in B \) there exists \( \rho(\kappa) \in \mathbb{N} \) such that \( \kappa = 2\rho(\kappa) \). First we observe that

\[
|\psi_m| = \left| \psi_\kappa + 2h \sum_{\ell=\rho(\kappa)+1}^{\bar{m}-1} \partial_h \psi_{2\ell+1} \right|
\leq |\psi_\kappa| + 2h \sum_{\ell=2\rho(\kappa)+1}^{\bar{m}-1} |\partial_h \psi_\ell| \quad \forall \ \kappa \in B.
\]

Then, we sum over \( \kappa \in B \) to get

\[
\text{card}(B) \ |\psi|_{\infty,h} \leq \sum_{\kappa \in B} |\psi_\kappa| + 2h \sum_{\kappa \in B} \left( \sum_{\ell=\kappa+1}^{\bar{m}-1} |\partial_h \psi_\ell| \right)
\leq \sum_{\kappa \in B} |\psi_\kappa| + 2h \text{card}(B) \sum_{\ell=3}^{m-1} |\partial_h \psi_\ell|.
\]

Observing that

\[
\text{card}(B) = \begin{cases} \frac{J}{2} & \text{if } J \text{ is even} \\ \frac{J-1}{2} & \text{if } J \text{ is odd} \end{cases},
\]
Lemma 2.6. The following discrete inverse inequality holds

\[ |\psi|_{\infty, h} \leq \frac{2}{\sqrt{\lambda}} \sum_{\ell=1}^{2} |\psi_{\ell}| + 2h \sum_{\ell=1}^{2} |\partial_{h} \psi_{\ell}| \]

(2.24)

\[ \leq \frac{2}{\sqrt{\lambda}} |\psi|_{0, h} + 4h \sum_{\ell=1}^{2} |\partial_{h} \psi_{\ell}| \]

\[ \leq \frac{4}{\sqrt{\lambda}} |\psi|_{0, h} + 4 \sqrt{L} |\psi|_{1, h} \]

\[ \leq \frac{\sqrt{9+16L^{2}}}{\sqrt{\lambda}} |\psi|_{1, h}. \]

The desired inequality (2.24) is a simple consequence of (2.22) and (2.23). \qed

Lemma 2.6. The following discrete inverse inequality holds

(2.25) \[ |\psi|_{\infty, h} \leq h^{\frac{1}{2}} |\psi|_{0, h} \quad \forall \psi \in \mathcal{X}_{h}. \]

Proof. Let \( \psi \in \mathcal{X}_{h} \) and \( m \in \{1, \ldots, J\} \) such that \( |\psi|_{\infty, h} = |\psi_{m}| \). Then, we have

\[ |\psi|_{0, h}^{2} = h \sum_{\ell=1}^{N} |\psi_{\ell}|^{2} \geq h |\psi_{m}|^{2}, \]

which easily yields (2.25). \qed

3. A Modified Finite Difference Scheme

We will carry out the convergence analysis of the proposed (FDM) by investigating the convergence of a properly defined Modified Finite Difference Scheme (MFDS) that derives noncomputable finite difference approximations of the exact solution \( \phi \) to the problem (1.1)–(1.2) (cf. [13]). In particular, we will construct the (MFDS) using an operational mollification of the nonlinear terms in (FDM), which is based on a given real parameter \( \lambda > 0 \) and the norm \( \parallel \cdot \parallel_{1, \infty, h} \) on \( \mathcal{X}_{h} \). The goal of this construction is to provide the (MFDS) with the following key property: ‘when the (MFDS) approximations have \( \parallel \cdot \parallel_{1, \infty, h} \)-distance from the exact solution to the problem lower than \( \lambda \), then they are also (FDM) approximations’.

3.1. Constructing an Operational Mollification. For \( \lambda > 0 \), let \( \xi(\lambda; \cdot) : \mathbb{R} \to [0, 1] \) be a continuous function defined by

\[ \xi(\lambda; x) := \begin{cases} 
1, & \text{if } x \leq \lambda, \\
\frac{x}{\lambda}, & \text{if } x \in (\lambda, 2\lambda], \\
0, & \text{if } x > 2\lambda,
\end{cases} \]

(3.1)

Then, for \( \lambda > 0 \) and \( t \in [0, T] \), we construct an operational mollifier \( m(\lambda, t; \cdot) : \mathcal{X}_{h} \to \mathcal{X}_{h} \) by

\[ m(\lambda, t; w) := w \xi(\lambda; \|w - \Lambda_{h}(\phi(t, \cdot))\|_{1, \infty, h}) \]

(3.2)

\[ + \Lambda_{h}(\phi(t, \cdot)) \left[ 1 - \xi(\lambda; \|w - \Lambda_{h}(\phi(t, \cdot))\|_{1, \infty, h}) \right] \quad \forall w \in \mathcal{X}_{h}, \]

where \( \phi \) is the solution to the problem (1.1).

In the lemmas below, we establish some usefull properties of the map \( m(\lambda, t; \cdot) \).

Lemma 3.1. It holds that

(3.3) \[ m(\lambda, t; v) = v \quad \forall v \in B_{c}(\Lambda_{h}(\phi(t, \cdot)), \lambda; \| \cdot \|_{1, \infty, h}), \quad \forall \lambda > 0, \quad \forall t \in [0, T], \]

and

(3.4) \[ \|m(\lambda, t; w)\|_{1, \infty, h} < 3\lambda \quad \forall w \in \mathcal{X}_{h}, \quad \forall \lambda \geq \lambda_{*}, \quad \forall t \in [0, T], \]

where \( \lambda_{*} := \max_{0 \leq \ell \leq 1} \left( \max_{[0,1] \times [0,T]} \left| \frac{\partial^{\ell} \phi}{\partial t^{\ell}} \right| \right). \]
Proof. Let \( t \in [0, T] \), \( \lambda > 0 \) and \( v \in \mathcal{B}_c \left( \Lambda_h \left( \phi(t, \cdot) \right), \lambda; \| \cdot \|_{1, \infty, h} \right) \). Then, it holds that
\[
\| v - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \leq \lambda
\]
which, along with (3.1), yields
\[
(3.5) \quad \xi \left( \lambda; \| v - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \right) = 1.
\]
The equality (3.3) follows easily combining (3.2) and (3.5).

Now, let us assume that \( \lambda \geq \lambda^* \). Then, we use (3.2) to get
\[
\| m(\lambda, t; w) \|_{1, \infty, h} \leq \| w \|_{1, \infty, h} \quad \xi \left( \lambda; \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \right) + \| \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \quad [1 - \xi \left( \lambda; \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \right)].
\]

If \( \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \geq 2\lambda \), then (3.1) and (3.6) yield
\[
(3.7) \quad \| m(\lambda, t; w) \|_{1, \infty, h} \leq \lambda^* \quad \leq \lambda.
\]

If \( \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} < 2\lambda \), then
\[
\| m(\lambda, t; w) \|_{1, \infty, h} \leq \max \{ \| w \|_{1, \infty, h}, \| \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \}
\]
\[
\leq \max \{ \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} + \| \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h}, \| \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \}
\]
\[
< 2\lambda + \| \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h}
\]
\[
< 2\lambda + \lambda^*
\]
\[
< 3\lambda.
\]

Thus, (3.4) follows easily from (3.7) and (3.8). \( \square \)

Lemma 3.2. Let \( \nu \) be a seminorm on \( \mathcal{X}_h \). Then, it holds that
\[
(3.9) \quad \nu \left( m(\lambda, t; w) - \Lambda_h \left( \phi(t, \cdot) \right) \right) \leq \nu \left( w - \Lambda_h \left( \phi(t, \cdot) \right) \right) \quad \forall w \in \mathcal{X}_h, \quad \forall \lambda > 0, \quad \forall t \in [0, T].
\]

Proof. Let \( t \in [0, T] \), \( \lambda > 0 \) and \( w \in \mathcal{X}_h \). Then, (3.2) and (3.1) yield
\[
\nu \left( m(\lambda, t; w) - \Lambda_h \left( \phi(t, \cdot) \right) \right) = \nu \left( w - \Lambda_h \left( \phi(t, \cdot) \right) \right) \quad \xi \left( \lambda; \| w - \Lambda_h \left( \phi(t, \cdot) \right) \|_{1, \infty, h} \right)
\]
\[
\leq \nu \left( w - \Lambda_h \left( \phi(t, \cdot) \right) \right).
\]

\( \square \)

3.2. The Modified Finite Difference Scheme (MFDS). Here, we introduce a modified finite difference scheme which, for \( \lambda > 0 \), derives non-computable approximations \( (S^n(\lambda))_{n=0}^N \subset \mathcal{X}_h \) of the solution \( \phi \) to (1.1), (1.2), following the steps below:

**Step A**: First, set
\[
(3.10) \quad S^0(\lambda) := \phi^0.
\]

**Step B**: Find \( S^1(\lambda) \in \mathcal{X}_h \) such that
\[
\frac{S^1(\lambda)-S^0(\lambda)}{\tau} = i \rho \Delta h \left( \frac{S^1(\lambda)+S^0(\lambda)}{2} \right) - \sigma \partial_h \Delta h \left( \frac{S^1(\lambda)+S^0(\lambda)}{2} \right)
\]
\[
+ i \delta \left[ \left| m(\lambda, t_0; S^0(\lambda)) \right|^2 \otimes \left( \frac{S^1(\lambda)+S^0(\lambda)}{2} \right) \right]
\]
\[
- 3 \alpha A_h \left[ \left| m(\lambda, t_0; S^0(\lambda)) \right|^2 \otimes \partial_h \left( \frac{S^1(\lambda)+S^0(\lambda)}{2} \right) \right] + F^\tau.
\]

**Step C**: For \( n = 1, \ldots, N-1 \), find \( S^{n+1}(\lambda) \in \mathcal{X}_h \) such that
\[
\frac{S^{n+1}(\lambda)-S^n(\lambda)}{2\tau} = i \rho \Delta h \left( \frac{S^{n+1}(\lambda)+S^n(\lambda)}{2} \right) - \sigma \partial_h \Delta h \left( \frac{S^{n+1}(\lambda)+S^n(\lambda)}{2} \right)
\]
\[
+ i \delta \left[ \left| m(\lambda, t_n; S^n(\lambda)) \right|^2 \otimes \left( \frac{S^{n+1}(\lambda)+S^n(\lambda)}{2} \right) \right]
\]
\[
- 3 \alpha A_h \left[ \left| m(\lambda, t_n; S^n(\lambda)) \right|^2 \otimes \partial_h \left( \frac{S^{n+1}(\lambda)+S^n(\lambda)}{2} \right) \right] + F^n.
\]
3.3. Existence and uniqueness of the (MFDS) approximations. Below we show that, if \( \tau \) is small enough, then the (MFDS) approximations are well-defined.

**Proposition 3.1.** Let \( \lambda_\ast := \max_{0 \leq t \leq 1} \left( \max_{0 \leq \teta \leq \tau} |\partial_t^\teta \phi| \right) \) and \( \lambda \geq \lambda_\ast \). Then, there exists a constant \( C_1 > 0 \), depending only on \( \alpha \), such that: if \( \tau C_1 \lambda^2 < 1 \), then modified finite difference approximations (3.10) - (3.12) are well-defined.

**Proof.** Let \( t \in [0, T] \), \( \zeta > 0 \) and \( \chi \in \mathcal{X}_h \). Then, we define linear operators \( Q(t, \zeta, \chi) : \mathcal{X}_h \to \mathcal{X}_h \) and \( T(t, \zeta, \chi) : \mathcal{X}_h \to \mathcal{X}_h \) by

\[
T(t, \zeta, \chi)v := 2v + Q(t, \zeta, \chi)v \quad \forall v \in \mathcal{X}_h,
\]

where

\[
Q(t, \zeta, \chi)v := -i \rho \zeta \Delta_h v + \sigma \tau \zeta \partial_h \Delta_h v - i \delta \tau \zeta \left[ |m(\lambda, t; \chi)|^2 \otimes v \right] + 3 \alpha \tau \zeta A_h \left[ |m(\lambda, t; \chi)|^2 \otimes \partial_h v \right] \quad \forall v \in \mathcal{X}_h.
\]

Using (2.9), (2.10), (2.19), the Cauchy-Schwarz inequality, (2.13) and (3.4), we have

\[
(3.16)
\]

Let \( \tau \in (0, h) \), such that:

\( \forall v \in \{0\} \) and \( T(t, \zeta, \chi) \) is invertible, since \( \mathcal{X}_h \) has finite dimension.

Set \( C_1 := 27 |\alpha| \) and require \( \tau C_1 \lambda^2 < 1 \). Then, according to the discussion above, the element \( S^1(\lambda) = T^{-1}(t_0, 1, \phi^0) \psi^1 \) is the solution to (3.11), with \( \psi^1 := 2 \phi^0 + 2 \tau F^\frac{1}{2} - Q(t_0, 1, \phi^0) \phi^0 \). Let \( \kappa \in \{1, \ldots, N-1\} \), \( S^1(\lambda) \) be the solution to the linear system (3.11) and \( (S^m(\lambda))_{n=2} \) be the solutions to the linear system (3.12) for \( n = 1, \ldots, \kappa - 1 \), respectively. Then, the element \( S^{\kappa+1}(\lambda) = T^{-1}(t_\kappa, 2, S^\kappa(\lambda)) \psi^\kappa \) is a solution of (3.12) for \( \kappa = n \), with

\[
psi^\kappa := 2 S^{\kappa-1}(\lambda) + 4 \tau F^\kappa - Q(t_\kappa, 2, S^\kappa(\lambda)) S^{\kappa-1}(\lambda).
\]

3.4. Consistency of (MFDS) and (FDM) approximations. Let \( (\eta^n)_{n=0}^{n=N-1} \subset \mathcal{X}_h \) be defined by

\[
(3.14)
\]

and

\[
(3.15)
\]

for \( n = 1, \ldots, N - 1 \). Then, assuming enough space and time regularity for the solution \( \phi \) and using the Taylor formula, we conclude that there exists positive real constants \( C_1 \) and \( C_2 \), which are independent of \( \tau \) and \( h \), such that:

\[
\|\eta^0\|_{1, \infty, h} \leq C_1 (h^2 + \tau),
\]

\[
\max_{1 \leq n \leq N-1} \|\eta^n\|_{1, \infty, h} \leq C_2 (h^2 + \tau^2).
\]

Since the property (3.3) yields that \( |\phi^n|^2 = |m(\lambda, t_n; \phi^n)|^2 \) for \( n = 0, \ldots, N \), the consistency result described above for the (FDM) approximations is, also, a consistency result for the (MFDS) approximations introduced in Section 3.2.
3.5. Convergence of the (MFDS) approximations. In the theorem below, we show convergence of the (MFDS) approximations in the discrete $H^1$–norm.

**Theorem 3.2.** Let $\mu_* := \max_{0 \leq t \leq 2} \left( \max_{[0,1]} | \partial_t \phi | \right)$, $\lambda_c := \mu_* + 1$ and $C_1$ be the constant specified in Proposition 3.1. Also, we assume that $\tau C_1 \lambda_c^2 < 1$ and denote by $(Z^m)_{n=0}^\infty$ the modified finite difference approximations defined by (3.10)–(3.12) for $\lambda = \lambda_c$, i.e., $Z^m = S^t(\lambda_c)$ for $t = 0, \ldots, N$. Then, there exist positive constants $C_2$ and $C_3 \geq C_1$, independent of $\tau$ and $h$, such that: if $\tau C_3 \lambda_c^2 < 1$, then

\[
\max_{0 \leq m \leq N} \| \phi^m - Z^m \|_{1,h} \leq C_2 (\tau^2 + h^2).
\]

**Proof.** To simplify the notation, we set $E^m := \phi^m - Z^m$, $D^m := \partial_t E^m$ and $m_m := m(\lambda_c, t_m; Z^m)$ for $m = 0, \ldots, N$. Also, we set $t(0) := 1$, $t(n) = n + 1$ for $n = 1, \ldots, N - 1$, $r(0) = 0$, $r(n) = n - 1$ for $n = 1, \ldots, N - 1$. We note that, since $\lambda_* < \lambda_c$ and $\tau C_1 \lambda_c^2 < 1$, Proposition 3.1 yields the existence and uniqueness of $(Z^m)_{n=1}^\infty$. In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\tau$ and $h$, and may changes value from one line to the other.

**Step 1.** We subtract (3.11) from (3.14) and (3.12) from (3.15), to obtain the following error equations:

\[
\begin{align*}
E^1 - E^0 & = i \rho \Delta_h \left( \frac{E^1 + E^0}{2} \right) - \sigma \partial_h \Delta_h \left( \frac{E^1 + E^0}{2} \right) + \sum_{k=1}^{5} A_{k,0}, \\
\frac{E^{n+1} - E^{n-1}}{2\tau} & = i \rho \Delta_h \left( \frac{E^{n+1} + E^{n-1}}{2} \right) - \sigma \partial_h \Delta_h \left( \frac{E^{n+1} + E^{n-1}}{2} \right) + \sum_{k=1}^{5} A_{k,n}, \quad n = 1, \ldots, N - 1,
\end{align*}
\]

where

\[
\begin{align*}
A_{1,n} & := - \frac{3 \alpha}{2} A_h \left( | \phi^n |^2 - | m_n |^2 \right) \otimes \partial_h \left( \phi^n + \phi^r(n) \right), \\
A_{2,n} & := - \frac{3 \alpha}{2} A_h \left( | m_n |^2 \otimes \partial_h (E^{\ell(n)} + E^{r(n)}) \right), \\
A_{3,n} & := \frac{i}{\rho} \left[ | \phi^n |^2 - | m_n |^2 \right) \otimes \left( \phi^n + \phi^r(n) \right), \\
A_{4,n} & := \frac{i}{2} \left[ | m_n |^2 \otimes (E^{\ell(n)} + E^{r(n)}) \right], \\
A_{5,n} & := \eta^n.
\end{align*}
\]

**Step 2.** We take the inner product $(\cdot, \cdot)_{0,h}$ of (3.19) with $(E^1 + E^0)$ and of (3.20) with $(E^{n+1} + E^{n-1})$. Then, we keep the real part of the obtained relation and use (2.15) and (2.11), to have

\[
\begin{align*}
\| E^1 \|_{0,h}^2 - \| E^0 \|_{0,h}^2 & = \tau \sum_{k=1}^{5} a_{k,0}, \\
\| E^{n+1} \|_{0,h}^2 - \| E^{n-1} \|_{0,h}^2 & = 2 \tau \sum_{k=1}^{5} a_{k,n}, \quad n = 1, \ldots, N - 1,
\end{align*}
\]

where

\[
a_{k,n} := \text{Re} \left( A_{k,n} \right) = \left( E^{\ell(n)} + E^{r(n)} \right)_{0,h}.
\]

Let $n \in \{0, \ldots, N - 1\}$. First, we observe that

\[
\begin{align*}
a_{4,n} & = 0, \\
a_{5,n} & \leq | \eta^n |_{0,h} \left[ \left( E^{\ell(n)} \right)_{0,h} + \left( E^{r(n)} \right)_{0,h} \right].
\end{align*}
\]

Next, we apply the Cauchy-Schwarz inequality, (2.18) and (2.11) to get

\[
\begin{align*}
a_{1,n} + a_{3,n} & \leq \left( \frac{3 \alpha}{2} + \frac{\mu}{2} \right) \mu_* \left( \| \phi^n \|_{0,h}^2 - | m_n |^2 \right) \left( \| E^{\ell(n)} \|_{0,h} + \| E^{r(n)} \|_{0,h} \right) \\
& \leq C \mu_* \left( | \phi^n |_{0,h} + | m_n |_{0,h} \right) \left( \| \phi^n \|_{0,h} - | m_n |_{0,h} \right) \left( \| E^{\ell(n)} \|_{0,h} + \| E^{r(n)} \|_{0,h} \right),
\end{align*}
\]
which, along with (3.24) and (3.29) (with $\nu(\cdot) = \| \cdot \|_{0,h}$), yields

$$a_{1,n} + a_{3,n} \leq C \mu_* (\mu_* + 3 \lambda_c) \| E^n \|_{0,h} \| E^{r(n)} \|_{0,h}$$

$$\leq C \lambda_c^2 \| E^n \|_{0,h} \left[ \| E^{r(n)} \|_{0,h} + \| E^{r(n)} \|_{0,h} \right].$$

(3.25)

Also, we use (2.19), (2.18) and (3.21) to obtain

$$a_{2,n} \leq \frac{\| \nu \|_{0,h}}{4} \left( \partial_h (|m_n|^2), |E^{r(n)} + E^{r(n)}|^2 \right)_{0,h}$$

$$\leq C \| \partial_h (|m_n|^2) \|_{0,h} \| E^{r(n)} + E^{r(n)} \|_{0,h}^2$$

$$\leq C \| m_n \|_{1,\infty,h}^2 \left[ \| E^{r(n)} \|_{0,h} + \| E^{r(n)} \|_{0,h} \right]^2$$

(3.26)

Letting $\nu^m := \| E^m \|_{0,h} + \| E^{m-1} \|_{0,h}$ for $m = 1, \ldots, N$, observing that $E^0 = 0$ and combining (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.16) and (3.17), we conclude that there exist positive constants $C_{E,1}$ and $C_{E,2}$ such that

$$\nu^1 \leq C_{E,1} \lambda_c^2 \tau (\tau^2 + \tau h^2),$$

(3.27)

$$\nu^{m+1} \leq \nu^m + C_{E,2} \lambda_c^2 \tau \left( (\nu^{m+1} + \nu^m) + 2 C_2 \tau (\tau^2 + \tau^2) \right), \quad m = 1, \ldots, N - 1.$$

(3.28)

**Step 3.** Apply the operator $\partial_h$ on (3.19) and (3.20), and then use (2.2), (2.3) and (2.4) to get

$$\frac{\partial^{1+D^0}}{2\tau} = i \rho \Delta_h \left( \frac{\partial^{1+D^0}}{2} \right) - \sigma \partial_h \Delta_h \left( \frac{\partial^{1+D^0}}{2} \right) + \sum_{\kappa=1}^9 B_{\kappa,0},$$

(3.29)

$$\frac{\partial^{n+1+D^{n-1}}}{2\tau} = i \rho \Delta_h \left( \frac{\partial^{n+1+D^{n-1}}}{2} \right) - \sigma \partial_h \Delta_h \left( \frac{\partial^{n+1+D^{n-1}}}{2} \right) + \sum_{\kappa=1}^9 B_{\kappa,n}, \quad n = 1, \ldots, N - 1,$$

(3.30)

where

$$B_{1,n} := - \frac{3\alpha}{2} \partial_h \left[ \sigma^- \left( |\phi^0|^2 - |m_n|^2 \right) \otimes \partial_h \left( \phi^{r(n)} + \phi^{r(n)} \right) \right],$$

$$B_{2,n} := - \frac{3\alpha}{2} \partial_h \left[ \partial_h \left( |\phi^0|^2 - |m_n|^2 \right) \otimes \sigma^+ \left( \phi^{r(n)} + \phi^{r(n)} \right) \right],$$

$$B_{3,n} := - \frac{3\alpha}{2} \partial_h \left[ \sigma^- (|m_n|^2) \otimes \partial_h \left( D^{r(n)} + D^{r(n)} \right) \right],$$

$$B_{4,n} := - \frac{3\alpha}{2} \partial_h \left[ \partial_h (|m_n|^2) \otimes \sigma^+ \left( D^{r(n)} + D^{r(n)} \right) \right],$$

$$B_{5,n} := \frac{i}{2} \left[ \sigma^- (|\phi^0|^2 - |m_n|^2) \otimes \partial_h \left( \phi^{r(n)} + \phi^{r(n)} \right) \right],$$

$$B_{6,n} := \frac{i}{2} \left[ \partial_h (|\phi^0|^2 - |m_n|^2) \otimes \sigma^+ \left( \phi^{r(n)} + \phi^{r(n)} \right) \right],$$

$$B_{7,n} := \frac{i}{2} \left[ \sigma^- (|m_n|^2) \otimes \left( D^{r(n)} + D^{r(n)} \right) \right],$$

$$B_{8,n} := \frac{i}{2} \left[ \partial_h (|m_n|^2) \otimes \sigma^+ \left( E^{r(n)} + E^{r(n)} \right) \right],$$

$$B_{9,n} := \partial_h \eta^n.$$

**Step 4.** First, we take the inner product $(\cdot, \cdot)_{0,h}$ of (3.29) with $(D^1 + D^0)$ and of (3.30) with $(D^{n+1} + D^{n-1})$. Then, we take real parts and use the properties (2.9) and (2.10), to get

$$\| D^1 \|_{0,h}^2 - \| D^0 \|_{0,h}^2 = \tau \sum_{\kappa=1}^9 b_{\kappa,0},$$

(3.31)

$$\| D^{n+1} \|_{0,h}^2 - \| D^{n-1} \|_{0,h}^2 = 2\tau \sum_{\kappa=1}^9 b_{\kappa,n}, \quad n = 1, \ldots, N - 1,$$

(3.32)

where

$$b_{\kappa,n} := \text{Re} \left( B_{\kappa,n}, D^{r(n)} + D^{r(n)} \right)_{0,h},$$

11
Let $n \in \{0, \ldots, N - 1\}$. It is obvious that

\begin{equation}
(3.33) \quad b_{7,n} = 0,
\end{equation}

\begin{equation}
(3.34) \quad b_{3,n} \leq |\eta^n|_{1,h} \left[ \|D^{(f^n)}\|_{0,h} + \|D^{(r^n)}\|_{0,h} \right].
\end{equation}

Now, using the Cauchy-Schwarz inequality, (2.18), (2.13), (2.16), (3.4), (2.17), (2.14) and (3.9) (with $\nu = \| \cdot \|_{0,h}$), we have

\begin{equation}
(3.35) \quad b_{4,n} + b_{8,n} \leq \frac{|\alpha|}{2} \left| \partial_h (|m_n|^2) \right|_{|\infty,h} \|\phi^+(E^{(f^n)} + E^{(r^n)})\|_{0,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
+ \frac{3|\alpha|}{2} \left| \partial_h (|m_n|^2) \right|_{|\infty,h} \|\phi^+(D^{(f^n)} + D^{(r^n)})\|_{0,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
\leq C \|m_n^2\|_{|1,\infty,h} \|E^{(f^n)} + E^{(r^n)}\|_{0,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
+ C \|m_n^2\|_{|1,\infty,h} \|D^{(f^n)} + D^{(r^n)}\|^2_{0,h}
\end{equation}

\begin{equation}
\leq C \lambda_c^2 \left[ \|E^{(f^n)} + E^{(r^n)}\|_{0,h} + \|D^{(f^n)} + D^{(r^n)}\|_{0,h} \right] \left[ \|D^{(f^n)}\|_{0,h} + \|D^{(r^n)}\|_{0,h} \right]
\end{equation}

and

\begin{equation}
(3.36) \quad b_{1,n} + b_{5,n} \leq \frac{3|\alpha|}{2} \left| \partial_h (\phi^+(\phi^n) + \phi^+(r^n)) \right|_{|\infty,h} \|\phi^-(|\phi^n|^2 - |m_n|^2)\|_{0,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
+ \frac{|\alpha|}{2} \left| \partial_h (\phi^+(\phi^n) + \phi^+(r^n)) \right|_{|\infty,h} \|\phi^-(|\phi^n|^2 - |m_n|^2)\|_{0,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
\leq C \mu \|\phi^n\|^2_{|1,\infty,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h}
\end{equation}

\begin{equation}
\leq C \lambda_c \left[ \|\phi^n\|_{|1,\infty,h} \|D^{(f^n)} + D^{(r^n)}\|_{0,h} \right]
\end{equation}

and

\begin{equation}
\left\| \partial_h (|\phi^n|^2 - |m_n|^2) \right\|_{0,h} \leq \left\| \partial_h (\phi^n - m_n) \right\|_{0,h} \leq \left\| \partial_h (\phi^n - m_n) \right\|_{0,h} + \left\| \sigma^-(\phi^n - m_n) \right\|_{0,h}
\end{equation}

\begin{equation}
\leq \left( |\phi^n|_{|1,\infty,h} + |m_n|_{|1,\infty,h} \right) \left[ \left\| \partial_h (\phi^n - m_n) \right\|_{0,h} + \left\| \sigma^-(\phi^n - m_n) \right\|_{0,h} \right]
\end{equation}

\begin{equation}
\leq C (\mu + 3 \lambda_c) \left[ |\phi^n - m_n|_{|1,\infty,h} + \left\| \phi^n - m_n \right\|_{0,h} \right]
\end{equation}

\begin{equation}
\leq C \lambda_c \left( \|\phi^n\|_{|1,\infty,h} + \|E^n\|_{0,h} \right)
\end{equation}

\begin{equation}
\leq C \lambda_c \left( \|D^n\|_{0,h} + \|E^n\|_{0,h} \right),
\end{equation}

which, finally, yield

\begin{equation}
(3.37) \quad b_{2,n} + b_{6,n} \leq C \lambda_c^2 \left( \|D^n\|_{0,h} + \|E^n\|_{0,h} \right) \left[ \|D^{(f^n)}\|_{0,h} + \|D^{(r^n)}\|_{0,h} \right].
\end{equation}

Combining (2.19), (2.13) and (3.4), we have

\begin{equation}
(3.38) \quad b_{3,n} = \frac{-3|\alpha|}{2} \text{Re} \left( A_h \left[ (\phi^-(m_n)^2) \otimes \partial_h (D^{(f^n)} + D^{(r^n)}) \right] \right)_{0,h}
\end{equation}

\begin{equation}
\leq \frac{3|\alpha|}{2} \left\| \partial_h (\phi^-(m_n)^2) \right\|_{0,h} \left\| D^{(f^n)} + D^{(r^n)} \right\|_{0,h}^2
\end{equation}

\begin{equation}
\leq C \left\| \partial_h (\phi^-(m_n)^2) \right\|_{|\infty,h} \left\| D^{(f^n)} + D^{(r^n)} \right\|_{0,h}^2
\end{equation}

\begin{equation}
\leq C \left\| D^{(f^n)} + D^{(r^n)} \right\|_{0,h}^2
\end{equation}

\begin{equation}
\leq C \lambda_c^2 \left[ \|D^{(f^n)} + D^{(r^n)}\|_{0,h} \right] \left[ \|D^{(f^n)}\|_{0,h} + \|D^{(r^n)}\|_{0,h} \right].
\end{equation}
Letting $\nu_0^m := |D^m|_{0,h} + |D^{m-1}|_{0,h}$ for $m = 1, \ldots, N$, observing that $E^0 = D^0 = 0$, and using (3.31) and (3.32) along with (3.33), (3.34), (3.35), (3.36), (3.37), (3.38), (3.39), (3.40), (3.41), (3.42), and (3.43), we conclude that there exist positive constants $C_{0,1}$ and $C_{0,2}$ such that

\begin{align}
\nu_0^1 &\leq C_{0,1} \lambda^2 \tau \left( \nu_0^1 + \nu_0^1 \right) + C_1 (\tau^2 + \tau h^2), \\
\nu_0^{m+1} &\leq \nu_0^m + C_{0,2} \lambda^2 \tau \left( \nu_0^{m+1} + \nu_0^m + \nu_0^m + \nu_0^{m+1} \right) + 2 C_2 \tau (\tau^2 + h^2), \quad m = 1, \ldots, N - 1.
\end{align}

Step 5. Let $C_\ast = \max\{C_1, C_{e,1}, C_{e,2}, C_{0,1}, C_{0,2}\}$, and $\nu_0^m := \nu_0^m + \nu_0^m$ for $m = 1, \ldots, N$. Assuming that $4 \tau \lambda^2 C_\ast < 1$, and using the inequalities (3.37), (3.38), (3.41), and (3.42), we conclude that

\begin{align}
\nu_0^1 &\leq 4 C_1 (\tau^2 + \tau^2), \\
\nu_0^{m+1} &\leq 1 + 2 C_2 \lambda^2 \tau \nu_0^m + 8 C_2 \tau (\tau^2 + h^2), \quad m = 1, \ldots, N - 1.
\end{align}

The estimate (3.18) follows easily by employing a standard discrete Gronwall argument based on (3.41) and (3.42).

4. Convergence of the (FDM) Approximations

Using the convergence result of Theorem 3.2, we are able to find a mild mesh condition, which when satisfied ensures that the $\| \cdot \|_{1,\infty,h}$-distance of the (MFDS) approximations from the exact solution to the continuous problem is bounded by $\lambda$, for a given value of $\lambda$.

**Proposition 4.1.** Let $\mu_\ast := \max_{0\leq l \leq 2} \left( \max_{0 \leq j \leq \lfloor l / 2 \rfloor} |\partial_x^j \phi| \right)$, $\lambda_\ast := \mu_\ast + 1$, $C_1$ be the constant specified in Proposition 3.1 and $C_2, C_3$ be the constants specified in Proposition 3.2, where $C_3 \geq C_2$. Also, we assume that $\tau C_3 \lambda^2 < 1$ and let $(Z^\ell)_{\ell=0}^N$ be the modified finite difference approximations defined by (3.10)–(3.12) for $\lambda = \lambda_\ast$, i.e., $Z^\ell = S^\ell(\lambda_\ast)$ for $\ell = 0, \ldots, N$. If

\begin{equation}
C_2 (\tau^2 h^{-2} + h^2) \leq \lambda_\ast,
\end{equation}

then, it holds that

\begin{equation}
\max_{0 \leq m \leq N} \| Z^m - \phi^m \|_{1,\infty,h} \leq \lambda_\ast
\end{equation}

and

\begin{equation}
m(\lambda_\ast, t_m ; Z^m) = Z^m, \quad m = 0, \ldots, N.
\end{equation}

**Proof.** The convergence estimate (3.18), the inverse inequality (2.18) and (4.1) yield that

\begin{align*}
\| Z^m - \phi^m \|_{1,\infty,h} &\leq \max \{ \| Z^m - \phi^m \|_{\infty,h}, \partial_h(\| Z^m - \phi^m \|_{\infty,h}) \} \\
&\leq h^{-2} \max \{ \| Z^m - \phi^m \|_{0,h}, \| \partial_h(\| Z^m - \phi^m \|_{0,h}) \|_{0,h} \} \\
&\leq h^{-2} \| Z^m - \phi^m \|_{1,h} \leq C_2 (\tau h^{-2} + h^2) \leq \lambda_\ast,
\end{align*}

which establish (4.2). Finally, (4.3) follows as a simple consequence of (4.2) and of definitions (3.1) and (3.2).

Now, we are ready to show that the (FDM) approximations are well-defined and have second order convergence with respect to $\| \cdot \|_{\infty,h}$-norm.

**Theorem 4.2.** Let $\mu_\ast := \max_{0 \leq l \leq 2} \left( \max_{0 \leq j \leq \lfloor l / 2 \rfloor} |\partial_x^j \phi| \right)$, $\lambda_\ast := \mu_\ast + 1$, $C_1$ be the constant specified in Proposition 3.1 and $C_2, C_3$ be the constants specified in Proposition 3.2, where $C_3 \geq C_2$. Also, we assume that $\tau C_3 \lambda^2 < 1$ and that (4.1) holds. Then, the finite difference method (1.8)–(1.10) is well-defined and

\begin{equation}
\max_{0 \leq m \leq N} \| \phi^m - \Phi^m \|_{1,h} \leq C_2 (\tau^2 + h^2).
\end{equation}
Proof. Since we have \( \lambda_* \leq \mu_* < \lambda_c \) and \( \tau C_1 \lambda_*^2 \leq \tau C_3 \lambda_c^2 < 1 \), Proposition 3.1 yields that the (MFDS) approximations (3.10) + (3.12) are well-defined when \( \lambda = \lambda_c \). Then, we simplify the notation by setting \( Z^\ell := S^\ell (\lambda_c) \) for \( \ell = 0, \ldots, N \). Also, by assuming that \( \tau C_3 \lambda_c^2 < 1 \) and (4.11) hold, Proposition 3.1 gives that (4.1) and (4.2) are satisfied.

Combining (1.3) with (3.11) and (3.12), we conclude that the (MFDS) approximations \((Z^\ell)_{\ell=0}^N\) are also (FDM) approximations, i.e., (1.8), (1.9) and (1.10) hold, after setting \( \Phi^\ell = Z^\ell \) for \( \ell = 0, \ldots, N \).

Let \((\Psi^m)_{m=0}^N\) be an outcome of the finite difference method (1.8)-(1.10), and set \( e^m := Z^m - \Psi^m \) for \( m = 0, \ldots, N \). We will show, by induction, that \( e^m = 0 \) for \( m = 0, \ldots, N \). Since the initial value is common, we have \( e^0 = 0 \). Thus, using (1.9), we obtain

\[
2 e^1 = i \rho \tau \Delta_h e^1 - \sigma \tau \partial_h \Delta_h e^1 - 3 \alpha \tau A_h \left( |\phi^0|^2 \otimes \partial_h e^1 \right) + i \delta \tau \left( |\phi^0|^2 \otimes e^1 \right).
\]

First, take the \((\cdot, \cdot)_{0,h}\)-inner product of (4.5) with \( e^1 \), and then, keep real parts and use (2.4), (2.10), (2.19) and (2.13), to obtain

\[
2 \|e^1\|_{0,h}^2 = \frac{3 \alpha}{\tau} \left( \partial_h (|\phi^0|^2), |e^1|^2 \right)_{0,h} 
\leq 3 |\alpha| \tau |\phi^0|_{1,\infty,h} |e^1|_{0,h}^2 
\leq 3 |\alpha| \tau \|\phi^0\|_{1,\infty,h} \|e^1\|_{0,h}^2 
\leq 3 |\alpha| \lambda_c^2 \tau \|e^1\|_{0,h}^2,
\]

which yields

\[
\|e^1\|_{0,h}^2 \left( 2 - 3 |\alpha| \lambda_c^2 \tau \right) \leq 0.
\]

Recalling from the proof of Proposition 3.1 that \( C_1 = 27 |\alpha| \), we have

\[
3 |\alpha| \lambda_c^2 \tau \leq \tau C_1 \lambda_c^2 \leq \tau C_3 \lambda_c^2 < 1.
\]

Thus, from (4.6), follows that \( e^1 = 0 \). Let \( \kappa \in \{1, \ldots, N - 1\} \) and that \( e^\ell = 0 \) for \( \ell = 0, \ldots, \kappa \). Then, using (1.10), we have

\[
e^{\kappa + 1} = i \rho \tau \Delta_h e^{\kappa + 1} - \sigma \tau \partial_h \Delta_h e^{\kappa + 1} - 3 \alpha \tau A_h \left( |Z^{\kappa}|^2 \otimes \partial_h e^{\kappa + 1} \right) + i \delta \tau \left( |Z^{\kappa}|^2 \otimes e^{\kappa + 1} \right).
\]

Taking the \((\cdot, \cdot)_{0,h}\)-inner product of (4.7) by \( e^{\kappa + 1} \) and then keeping real parts and using (2.4), (2.10), (2.19) and (2.13), we get

\[
\|e^{\kappa + 1}\|_{0,h}^2 = \frac{3 \alpha}{\tau} \left( \partial_h (|Z^{\kappa}|^2), |e^{\kappa + 1}|^2 \right)_{0,h} 
\leq 3 |\alpha| \tau \|Z^{\kappa}\|_{1,\infty,h}^2 \|e^{\kappa + 1}\|_{0,h}^2 
\leq 3 |\alpha| \left( \|Z^{\kappa} - \phi^{\kappa}\|_{1,\infty,h} + \|\phi^{\kappa}\|_{1,\infty,h} \right)^2 \|e^{\kappa + 1}\|_{0,h}^2 
\leq 12 |\alpha| \lambda_c^2 \|e^{\kappa + 1}\|_{0,h}^2,
\]

which yields

\[
\|e^{\kappa + 1}\|_{0,h}^2 \left( 1 - 12 |\alpha| \lambda_c^2 \tau \right) \leq 0.
\]

Observing that

\[
12 |\alpha| \lambda_c^2 \tau \leq \tau C_1 \lambda_c^2 \leq \tau C_3 \lambda_c^2 < 1,
\]

(4.8) yields that \( e^{\kappa + 1} = 0 \), which closes the induction argument.

Under our assumptions on \( \tau \) and \( h \), we have shown that the (FDM) approximations are well-defined and, in particular, that \( \Phi^m = Z^m \) for \( m = 0, \ldots, N \). Thus, the error estimate (4.4) follows from the error bound (5.18) for the (MFDS) approximations. \( \square \)
5. Numerical Results

We implemented the (FDM) in a FORTRAN 90 program named FD. The program FD uses double precision complex arithmetic and solves, at every time-step, the resulting cyclic penta-diagonal linear system of algebraic equations using a direct method based on the well-known Gauss elimination for banded matrices. For graph drawing, we used the gnuplot command-line program [13].

When the exact solution is known, we measured the approximation error in the space-time discrete maximum norm:

$$E_{\infty}(N, J) := \max_{0 \leq n \leq N} \max_{1 \leq j \leq J} |\Phi_n^j - \phi^j|. $$

Also, letting $N$ be proportional to $J$ (i.e. $N = qJ$ for a given $q \in \mathbb{Q}$), we computed the experimental order of convergence for successive values values $J_1$ and $J_2$ of $J$, using the formula

$$\log \left( \frac{E_{\infty}(qJ_1, J_1)}{E_{\infty}(qJ_2, J_2)} \right) / \log(J_2/J_1).$$

5.1. Example 1. We consider the problem (1.1)-(1.2) with: $L = 1$, $(\rho, \alpha, \sigma, \delta) = (2, \frac{1}{2}, 4, \frac{1}{4})$ or $(\rho, \alpha, \sigma, \delta) = (0, \frac{1}{2}, 4, 0)$, and load $f$ such that the function $\phi(t, x) = e^{i(t+2\pi x)}$ to be its exact solution. In the performed numerical experiments we have set $T = 5$, $[x_a, x_b] = [0, 1]$, $(N, J) = (5\nu, \nu)$ for $\nu = 20, 40, 80, 160, 320, 640$, and computed the approximation error $E_{\infty}(N, J)$. The results shown on Table 1 confirm that the experimental order of convergence with respect to $\frac{1}{\nu}$ is equal to 2, which is in agreement with Theorem 4.2.

| $\nu$  | $E_{\infty}(5\nu, \nu)$ | Rate |
|--------|------------------------|------|
| 20     | 6.54162(-2)            | —    |
| 40     | 1.61352(-2)            | 2.019|
| 80     | 4.01170(-3)            | 2.008|
| 160    | 9.91777(-4)            | 2.016|
| 320    | 2.39596(-4)            | 2.049|
| 640    | 5.63974(-5)            | 2.087|

Table 1. Discrete maximum norm convergence for Example 1.

5.2. Example 2. For $\kappa \in \mathbb{R}$, it is easily seen that the function $\phi : [0, T] \times \mathbb{R} \to \mathbb{C}$ given by

$$\phi(t, x) = e^{i(\kappa x + \omega t)} \quad \forall \ (t, x) \in [0, T] \times \mathbb{R}$$

with

$$\omega = \kappa^2 (\sigma \kappa - \rho) + \delta - 3\alpha \kappa,$$

solves the homogeneous (SH) equation (1.1) and is $L$-periodic on $\mathbb{R}$ when

$$\frac{\pi \kappa}{2} \in \mathbb{Z}.$$

According to [1], the smooth function (5.1) under (5.2) and (5.3) is the unique $L$-periodic solution to the problem (1.1)-(1.2) with initial condition $\phi_0(x) = e^{i\kappa x}$. Observe that the solution does not requires the condition $\rho \alpha = \sigma \delta$ to be valid.

We performed numerical experiments choosing $L = 1, \kappa = 2\pi, T = 1, [x_a, x_b] = [0, 1], (\rho, \alpha, \sigma, \delta) = (0, \frac{1}{2}, \frac{1}{2}, 0)$ or $(\rho, \alpha, \sigma, \delta) = (0, \frac{1}{2}, \frac{1}{2}, 1)$, $(N, J) = (5\nu, \nu)$ for $\nu = 80, 160, 320, 640, 1280$, and computing the approximation error $E_{\infty}(N, J)$. The results exposé on Table 2 show that the experimental order of convergence with respect to $\frac{1}{\nu}$ is equal to 2, which is in agreement with the convergence result of Theorem 4.2. Also, we observe that the numerical method is efficient when $\rho \alpha \neq \sigma \delta$, which is due to the existence of a unique smooth solution.
5.3. **Bright Soliton solutions.** A Bright Soliton (BS) solution to the (SH) equation (1.1) is a function \( \phi, : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{C} \) of the form

\[
\phi_\ast(t, x) = A \operatorname{sech}(B(x - v t)) e^{i(\kappa x + \omega t)} \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R},
\]

where \( A \neq 0, B \neq 0, \kappa \) and \( \omega \) are real constants (see, e.g. [1], [2]). It is easily seen that a (BS) solution exists, iff,

\[
\rho \alpha = \sigma \delta, \quad v = \sigma (B^2 - 3 \kappa^2) + 2 \rho \kappa, \quad \omega = \rho (B^2 - \kappa^2) + \kappa \sigma (\kappa^2 - 3B^2), \quad |B| = |A| \left( \frac{\rho}{\sigma} \right)^{\frac{1}{2}},
\]

where \( A, \kappa \) are parameters. Observing that

\[
|\phi_\ast(t, x)| \leq 2|A| e^{-|B| |x-\nu t|} \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R},
\]

and taking into account the properties of the hyperbolic function \( \operatorname{sech} \), we conclude that, for \( t \in [0, +\infty) \), the modulus of a (BS) solution tends monotonically to 0 when \( x \rightarrow \pm \infty \). Thus \( \phi_\ast \) has no periodic structure and it cannot be an \( L \)-periodic solution to the problem (1.1) - (1.2). However, using our finite difference method, we are able to simulate a (BS) solution on a set \( [0, T] \times [x_a, x_b] \) in which it has almost compact support with respect to the space variable.

We consider problem (1.1) - (1.2) with: \( T = 5, L = 40, [x_a, x_b] = [-20, 20], (\rho, \alpha, \sigma, \delta) = (\frac{1}{2}, 2, 1, 1), \) and initial condition \( \phi_0(x) = \phi_\ast(0, x) \), where the bright soliton parameters are given by \( A = \frac{1}{2} \) and \( \kappa = \frac{1}{2} \). In the numerical experiment, we have computed the approximation error \( E_\infty(N, J) \) with \( (N, J) = (2\nu, \nu) \) for \( \nu = 200, 400, 800, 1600, 3200 \). The results presented on Table 3 confirm, again an experimental order of convergence equal to 2. Finally, Figure 1 shows a good final time graph agreement of the (BS) solution along with its finite difference approximation obtained for \( (N, J) = (400, 300) \).

| \( \nu \) | \( E_\infty(2\nu, \nu) \) | Rate |
|---|---|---|
| 200 | 2.75915(-2) | — |
| 400 | 7.14955(-3) | 1.948 |
| 800 | 1.80531(-3) | 1.985 |
| 1600 | 4.53750(-4) | 1.992 |
| 3200 | 1.15731(-4) | 1.971 |

Table 3. Errors and discrete maximum norm convergence rates to a (BS) solution.

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