EXISTENCE RESULTS AND BLOW-UP CRITERION OF COMPRESSIBLE RADIATION HYDRODYNAMIC EQUATIONS

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Abstract. In this paper, we consider the 3D compressible radiation hydrodynamic (RHD) equations with thermal conductivity in a bounded domain. The existence of unique local strong solutions is firstly established when the initial data are arbitrarily large and satisfy some initial layer compatibility condition. The initial mass density needs not be bounded away from zero and may vanish in some open set. Moreover, we show that if the initial vacuum domain is not so irregular, then the compatibility condition is necessary and sufficient to guarantee the existence of the unique strong solution. Finally, we show a Beal-Kato-Majda type blow-up criterion in terms of $(\nabla I, \rho, \theta)$.

1. Introduction

The RHD system appears in various astrophysical contexts [20] and high-temperature plasma physics [19]. Suppose the matter is in local thermodynamical equilibrium, the coupled system of Navier-Stokes-Boltzmann equations with heat conduction for the mass density $\rho(t,x)$, the fluid velocity $u(t,x) = (u^1, u^2, u^3)$, the specific internal energy $e(t,x)$, and the specific radiation intensity $I(v, \Omega, t, x)$ in a domain $\mathcal{V} \subset \mathbb{R}^3$ reads as [26]:

$$
\begin{aligned}
\frac{1}{c^2} I_t + \Omega \cdot \nabla I &= A_r, \\
\rho_t + \text{div}(\rho u) &= 0, \\
\left(\rho u + \frac{1}{c^2} F_r\right)_t + \text{div}(\rho u \otimes u + P_r) + \nabla P_m &= \text{div} T, \\
\left(\rho E_m + E_r\right)_t + \text{div}((\rho E_m + P_m)u + F_r) &= \text{div}(uT) + \kappa \Delta \theta.
\end{aligned}
$$

(1.1)

In this system, $t \geq 0$ is the time; $x \in \mathcal{V}$ is the spatial coordinate; $v \in \mathbb{R}^+$ is the frequency of photon; $\Omega \in S^2$ is the travel direction of photon, here $S^2$ is the unit sphere in $\mathbb{R}^3$; $E_m = \frac{1}{2} u^2 + e$ is the specific total material energy; $P_m$ is the material pressure satisfying the following equations of state

$$
P_m = R \rho \theta = (\gamma - 1) \rho e, \quad e = c_v \theta,
$$

(1.2)
depend on the mass density $\rho$ and the temperature $T$ contained in $d\Omega$, and travelling a distance $d\sigma$ from $\Omega$ to $\Omega'$ contained in $d\Omega$ and $d\Omega'$, and also proved the regular solutions with vacuum has been solved by many papers, and we refer the readers to [6][7].

For the existence of solutions with arbitrary data in 3D space, the main breakthrough is due to Lions [23], where he established the global existence of weak solutions for $\mathbb{R}^3$, periodic domains or bounded domains with homogenous Dirichlet boundary conditions provided $\gamma > 9/5$. The restriction on $\gamma$ is improved to $\gamma > 3/2$ by Feireisl [11]. Recently, Huang-Li-Xin obtained the global well-posedness of classical solutions with large oscillations and vacuum to Cauchy problem [16] for isentropic flow with small energy.

In general, studying the radiation hydrodynamic equations is challenging because of its complexity and mathematical difficulty. For Euler-Boltzmann equations, recently, Jiang-Zhong [19] obtained the local existence of $C^1$ solutions for the Cauchy problem away from vacuum. Jiang-Wang [18] showed that some $C^1$ solutions will blow up in finite time, regardless of the size of the initial disturbance. Li-Zhu [21] established the local existence of Makino-Ukai-Kawashima type’s regular solution (see [24]) with vacuum, and also proved that the regular solutions with compact density will blow up in finite time.

For Navier-Stokes-Boltzmann equations, Ducomet-Nečasová [2][10] studied the global weak solutions to the Navier-Stokes-Boltzmann equations and its large time behavior in
1-D space. Li-Zhu [22] considered the formation of singularities to classical solutions with compact density. Some special phenomenon has been observed, for example, it is known in contrast with the second law of thermodynamics, the associated entropy equation may contain a negative production term for RHD system, which has already been observed in Buet-Després [5]. Moreover, from Ducomet-Feireisl-Nečasová [8], in which they obtained the existence of global weak solution for some RHD model, we know that the velocity field \( u \) may develop uncontrolled time oscillations on vacuum zones.

However, in this paper, due to the radiation transfer equation (1.1), system (1.5)-(1.6), system (1.1) can be written as

\[
\begin{cases}
\frac{1}{c} I_t + \Omega \cdot \nabla I = A_r, \\
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P_m + Lu = \frac{1}{c} \int_0^\infty \int_{S^2} A_r \Omega d\Omega dv, \\
(\rho \theta)_t + \text{div}(\rho \theta u) + \frac{1}{c_v}(P_m \text{div} u - \kappa \Delta \theta) = \frac{1}{c_v}(Q(u) + N_r),
\end{cases}
\]

where \( N_r, Lu \) and \( Q(u) \) are defined by

\[
\begin{align*}
N_r &= \int_0^\infty \int_{S^2} \left( 1 - \frac{u \cdot \Omega}{c} \right) A_r d\Omega dv, \\
Lu &= -\mu \Delta u - (\lambda + \mu) \nabla \text{div} u, \quad Q(u) = \frac{\mu}{2} |\nabla u + (\nabla u)^\top|^2 + \lambda |\text{div} u|^2.
\end{align*}
\]

Now we consider the initial-boundary value problem for system (1.7) with initial data

\[
I|_{t=0} = I_0(v, \Omega, x), \quad (\rho, u, \theta)|_{t=0} = (\rho_0(x), u_0(x), \theta_0(x)), \quad (v, \Omega, x) \in \mathbb{R}^+ \times S^2 \times \mathbb{V},
\]

and one of the following two types of boundary conditions:

1. Transparency, Dirichlet and Neumann boundary conditions for \((I, u, \theta)\): \( \mathbb{V} \subset \mathbb{R}^3 \) is a bounded smooth domain and

\[
I|_{\partial \mathbb{V}} = 0, \quad \Omega \cdot n \leq 0; \quad u|_{\partial \mathbb{V}} = 0; \quad \nabla \theta \cdot n|_{\partial \mathbb{V}} = 0,
\]

where \( n = (n_1, n_2, n_3) \) is the unit outward normal to \( \partial \mathbb{V} \).

2. Transparency, Navier-slip and Neumann boundary conditions for \((I, u, \theta)\): \( \mathbb{V} \subset \mathbb{R}^3 \) is bounded, simply connected, smooth domain, and

\[
I|_{\partial \mathbb{V}} = 0, \quad \Omega \cdot n \leq 0; \quad (u \cdot n, (\nabla \times u) \cdot n)|_{\partial \mathbb{V}} = (0, 0); \quad \nabla \theta \cdot n|_{\partial \mathbb{V}} = 0.
\]

The first condition in (1.10) is the physical non-penetration boundary condition for radiation flow. While the first part \( u \cdot n = 0 \) of second one in (1.10) is the non-penetration boundary condition for the fluid, and \( (\nabla \times u) \cdot n = 0 \) is also known in the following form

\[
(D(u) \cdot n)_\tau = k \tau u,\]

where \( k \tau \) is the corresponding principal curvature of \( \mathbb{V} \). Then the second one in (1.10) implies the tangential component of \( D(u) \cdot n \) vanishes on flat portions of the boundary \( \partial \mathbb{V} \).

The purpose of our paper is to provide a local theory of strong solutions (see Definition 2.1) with vacuum state to IBVP (1.7)-(1.8) under (1.9) or (1.10) in the framework of Sobolev space. Via the arguments used in [7], we introduce a similar initial layer
compatibility condition which will be used to compensate the lower bound of \( \rho_0 \) when vacuum appears. First we give the existence of the unique local strong solution via establishing a priori estimate without dependence on the lower bound of \( \rho_1 \), which can be obtained via Minkowski’s inequality, the regularity estimate arguments introduced in [4] for incompressible Euler equations with Navier-slip boundary condition, the Poincaré type inequality on \( \| \theta_t \|_{L^6} \) introduced in [23], and the approximation process from non-vacuum to vacuum used in [7]. Moreover, we prove that if the initial vacuum is not so irregular, then the compatibility condition of the initial data is necessary and sufficient to guarantee the existence of the unique local strong solution. Finally, we give some blow-up criterion for the strong solution that we obtained via a subtle analysis on the viscosity coefficients \((\lambda, \mu)\) (see Lemma 6.2), which is in terms of \((\nabla I, \rho, \theta)\).

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

\[
D^{k,r} = \{ f \in L^1_{\text{loc}}(\Omega) : \| f \|_{D^{k,r}} = \| \nabla^k f \|_{L^r} < +\infty \}, \quad D^k = D^{k,2},
\]

\[
\| f \|_{W^{m,r}(\Omega)} = \| f \|_{W^{m,r}(\Omega)}, \quad \| f \|_s = \| f \|_{H^s(\Omega)}, \quad \| f \|_p = \| f \|_{L^p(\Omega)},
\]

\[
\| f \|_{D^{k,r}(\Omega)} = \| f \|_{D^{k,r}(\Omega)}, \quad \| f \|_{D^k} = \| f \|_{D^k(\Omega)}, \quad \| (f, g) \|_X = \| f \|_X + \| g \|_X.
\]

A detailed study of homogeneous Sobolev space may be found in [13].

Next we make some assumptions for the physical coefficients \( \sigma_a \) and \( \sigma_s \). First, let

\[
\sigma_s = \sigma_s(v' \to v, \Omega', \rho, \theta), \quad \sigma_s' = \sigma_s'(v \to v', \Omega', \rho, \theta), \quad \rho = \sigma_s(\rho),
\]

where the functions \( \sigma_s \geq 0 \) and \( \sigma_s' \geq 0 \) are \( C^1 \) for \((v', v, \Omega', \Omega, \rho, \theta)\), and satisfy

\[
\begin{align*}
\alpha_1 &= \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left[ \frac{1}{v'} \left( \nabla^2 + |\nabla_{t,x} \sigma_s|^2 \right) d\Omega' dv \right]^{\lambda_1} \right) d\Omega dv \leq C, \\
\alpha_2 &= \int_0^\infty \int_{S^2} \left( \int_0^\infty \int_{S^2} \left( \nabla_{t,x} \sigma_s' \right) d\Omega' dv \right)^{\lambda_2} d\Omega dv \leq C, \\
\alpha_3 &= \int_0^\infty \int_{S^2} \left( \nabla_{t,x} \sigma_s' \right) d\Omega' dv \leq C,
\end{align*}
\]

\[
S(v, t, x)|_{\partial \Omega} = \sigma_s(v' \to v, \Omega', \Omega, t, x)|_{\partial \Omega} = 0, \quad \text{when} \quad n \cdot \Omega < 0
\]

for \((t, x) \in [0, T] \times \Omega\), where \( \lambda_1 = 1 \) or \( \frac{1}{2} \), \( \lambda_2 = 1 \) or \( 2 \), and hereafter we denote by \( C \) a generic positive constant depending only on \( \mu, \lambda, \gamma, R, c_v, \kappa, q \) \((3 < q \leq 6 \text{ is a constant})\), \( T, \alpha_1, \alpha_2, \alpha_3, \Omega, \) and the norms of \( S \).

Second, let

\[
\sigma_a = \sigma(v, t, x, \rho, \theta) \rho = \sigma(\rho),
\]

then for \((\rho^i(t, x), \theta^i(t, x)) \quad (i = 1, 2)\) satisfying

\[
\begin{align*}
\| \rho^i(t, x) \|_{C([0, T]; W^{1,q}(\Omega, \rho))} + \| \rho^i_t(t, x) \|_{C([0, T]; L^q(\Omega, \rho))} &= \Lambda_1 < +\infty, \\
\| \theta^i(t, x) \|_{C([0, T]; H^2(\Omega, \rho))} + \| \theta^i_t(t, x) \|_{L^2([0, T]; D^1(\Omega, \rho))} &= \Lambda_2 < +\infty,
\end{align*}
\]
where $\Lambda_1$ and $\Lambda_2$ are both positive constants, we assume that
\[
\begin{align*}
&\|\sigma^i\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2, L^\infty(V))} \leq M(|\rho^i|_\infty + |\theta^i|_\infty), \\
&\|\sigma^i\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2, D^1,q(V))} \leq M(|\rho^i|_\infty + |\theta^i|_\infty)(|\rho|_{D^1,q} + |\theta|_{D^1,q} + 1), \\
&\|\sigma^i\|_{L^2(\mathbb{R}^+ \times S^2, L^2(V))} \leq M(|\rho^i|_\infty + |\theta^i|_\infty)(|\rho_t|_2 + |\theta_t|_2 + 1), \\
&|\sigma(v, \Omega, t, x, \rho^i, \theta^i) - \sigma(v, \Omega, t, x, \rho^2, \theta^2)| \leq \sigma(v, \Omega, t, x)|\theta^1 - \theta^2|, \\
&|\sigma(v, \Omega, t, x, \rho_1, \theta^1) - \sigma(v, \Omega, t, x, \rho_2, \theta^i)| \leq \sigma(v, \Omega, t, x)|\rho^1 - \rho^2|, \\
&\|\overline{\sigma}(v, \Omega, t, x)|\|_{L^\infty \cap L^2(\mathbb{R}^+ \times S^2, L^\infty(V))} \leq M(\Lambda_1 + \Lambda_2)
\end{align*}
\]
for $t \in [0, T]$ and $r \in [2, q]$, where $M = M(\cdot)$ denotes a strictly increasing continuous function from $[0, \infty)$ to $[1, \infty)$, and $\sigma(v, \Omega, t, x, \rho^i, \theta^i) \in C([0, T]; L^2(\mathbb{R}^+ \times S^2; L^\infty(V)))$.

**Remark 1.1.** Assumptions (1.11)-(1.12) are similar to those of [19-21] for the existence theory to Euler-Boltzmann equations. The evaluation of these physical coefficients is a difficult problem of quantum mechanics and their general form is not known. An general expression of $\sigma_a$ and $\sigma_s$ used for describing Compton Scattering process can be given as
\[
\sigma_a(v, t, x, \rho, \theta) = D_1\rho\theta^{-\frac{1}{2}}\exp\left(-\frac{D_2}{\theta^2}(v - v_0)^2\right), \quad \sigma_s = \overline{\sigma}_s(v \rightarrow v', \Omega \cdot \Omega', t, x)\rho,
\]
where $v_0$ is the fixed frequency, $D_i (i = 1, 2)$ are positive constants (see [26]).

The rest of this paper is organized as follows. We first show our main results in Section 2. In Section 3, we give some important lemmas which will be used frequently in our proofs. In Section 4, we prove the existence of the unique local strong solution under the boundary conditions (1.10) via establishing a priori estimate which is independent of the lower bound of $\rho_0$. In Section 5, we give the proof for the necessity and sufficiency of the initial layer compatibility condition. Finally in Section 6, we give the proof for the blow-up criterion that we claimed in Section 2.2.

### 2. Main results

#### 2.1. Existence results for strong solutions with vacuum.

First, we will give the definition of strong solutions to IBVP (1.7)-(1.8) with (1.9) or (1.10).

**Definition 2.1** (Strong solutions with vacuum to RHD). Functions $(I, \rho, u, \theta)$ is called a strong solution on $\mathbb{R}^+ \times S^2 \times [0, T] \times \mathbb{V}$ to IBVP (1.7)-(1.8) with (1.9) or (1.10), if

1. $(I, \rho, u, \theta)$ satisfies the system (1.7) a.e. in $\mathbb{R}^+ \times S^2 \times (0, T) \times \mathbb{V}$;
2. $(I, \rho, u, \theta)$ belongs to the following class with some regularity:
   \[
   \Phi = \{(I, \rho, u, \theta) | 0 \leq I \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; W^{1,q})), \\
   l_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; L^q)), \quad 0 \leq \rho \in C([0, T]; W^{1,q}), \\
   \rho_t \in C([0, T]; L^q), \quad (\theta, u) \in C([0, T]; H^2) \cap L^2([0, T]; D^2,q), \\
   (\theta_t, u_t) \in L^2([0, T]; H^1), \quad (\sqrt{\rho} \theta, \sqrt{\rho} u_t) \in L^\infty([0, T]; L^2)\};
   \]
3. $(I, \rho, u, \theta)$ satisfies the corresponding initial conditions a.e. in $\mathbb{R}^+ \times S^2 \times \{t = 0\} \times \mathbb{V}$, and also satisfies the corresponding boundary conditions in the sense of traces.
As has been observed in Navier-Stokes equations, the lack of a positive lower bound of \( \rho_0 \) should be compensated with some initial layer compatibility conditions on the initial data. Similarly, via denoting \( P_m^0 = R\rho_0\theta_0 \), and

\[
A_v^0 = S(v, \Omega, 0, x) - \sigma_a(v, \Omega, 0, x, \rho_0, \theta_0)I_0 + \int_0^\infty \int_{S^2} \left( \frac{\nabla}{\nabla'} \sigma_a(\rho_0)I_0' - \sigma_a'(\rho_0)I_0 \right) d\Omega' dv'.
\]

we have the following existence theorem:

**Theorem 2.1.** Let assumptions \([1.1]-[1.12]\) hold, and

\[
\|S(v, t, x)\|_{L^2(\mathbb{R}^+; C^1([0, \infty); W^{1,q}(\mathbb{V}) \cap C^1([0, \infty); L^1(\mathbb{R}^+; L^2(\mathbb{V}))) < +\infty.
\]

1. Assume the initial data \((I_0, \rho_0, u_0, \theta_0)\) satisfies the regularity

\[
0 \leq I_0(v, \Omega, x) \in L^2(\mathbb{R}^+ \times S^2; W^{1,q}), \quad 0 \leq \rho_0 \in W^{1,q}, \quad u_0 \in H^1 \cap H^2, \quad \theta_0 \in H^2,
\]

and the initial layer compatibility conditions

\[
Lu_0 + \nabla P_m^0 + \frac{1}{\epsilon} \int_0^\infty \int_{S^2} A_v^0 \frac{\Omega_d\Omega}{\partial} dv = \frac{1}{\epsilon} \int 1 - \frac{u_0 - \Omega}{c} A_v^0 \frac{\Omega_d\Omega}{\partial} dv = \frac{1}{\epsilon} g_1(\nabla \theta_0 + Q(u_0)) - \frac{1}{\epsilon} (k\nabla \theta_0 + Q(u_0)) - \frac{1}{\epsilon} (k\nabla \theta_0 + Q(u_0)) \]

for some \((g_1, g_2) \in L^2\). Then there exists a time \(T_*\) and a unique strong solution \((I, \rho, u, \theta)\) on \(\mathbb{R}^+ \times S^2 \times [0, T_*] \times \mathbb{V}\) to IBVP \([1.7]-[1.8]\) with \([1.7]\).

2. Assume the initial data \((I_0, \rho_0, u_0, \theta_0)\) satisfies the regularity

\[
0 \leq I_0(v, \Omega, x) \in L^2(\mathbb{R}^+ \times S^2; W^{1,q}), \quad 0 \leq \rho_0 \in W^{1,q}, \quad u_0 \in H^2, \quad \theta_0 \in H^2
\]

and \([2.3]\). Then there exists a small time \(T_*\) and a unique strong solution \((I, \rho, u, \theta)\) on \(\mathbb{R}^+ \times S^2 \times [0, T_*] \times \mathbb{V}\) to IBVP \([1.7]-[1.8]\) with \([1.7]\).

In particular, we have the following condition:

**Theorem 2.2 (Necessity and sufficiency of the compatibility condition).** Let conditions supposed in Theorem \([2.1]\) hold. We assume that the initial vacuum only appears in the far field, or \(V\) has zero 3-D Lebesgue measure, or the elliptic system

\[
\begin{aligned}
-\mu \Delta \phi - (\lambda + \mu) \nabla \text{div} \phi &= 0 \quad \text{in} \ V, \\
-\kappa \Delta h - Q(\phi) &= 0 \quad \text{in} \ V
\end{aligned}
\]

has only zero solution in \(D^1_0(V) \cap D^2(V)\). Then there exists a unique (local) strong solution \((I, \rho, u, \theta)\) with the regularity shown in Definition \([2.1]\) such that

\[
\|I(t) - I_0\|_{W^{1,q}(V)} \to 0, \quad \text{as} \ t \to 0, \quad \forall (v, \Omega) \in \mathbb{R}^+ \times S^2,
\]

\[
\|\rho(t) - \rho_0\|_{W^{1,q}(V)} + \|u(t) - u_0, \theta(t) - \theta_0\|_{L^2(V)} \to 0, \quad \text{as} \ t \to 0,
\]

if and only if initial data \((I_0, \rho_0, u_0, \theta_0)\) satisfies the compatibility condition \([2.3]\).
Remark 2.1. Because \((I, \rho, u, \theta)\) only satisfies (1.7)-(1.8) in distribution sense, we know
\[ I(v, \Omega, 0, x) = I_0, \quad \rho(0, x) = \rho_0, \quad pu(0, x) = \rho_0 u_0, \quad \rho \theta(0, x) = \rho_0 \theta_0. \]
In the vacuum domain \(V\), the relations \(u(t = 0, x) = u_0\) and \(\theta(t = 0, x) = \theta_0\) maybe not hold. The conclusions obtained in Theorem 2.2 tell us that if the vacuum domain \(V\) has a sufficient simple geometry, for instance, the Lipschitz continuous domain, we have \(u(t = 0, x) = u_0\) and \(\theta(t = 0, x) = \theta_0\) a.e. in \(V\).

2.2. Beal-Kato-Majda Blow-up criterion.

Next we naturally consider that the local strong solutions to IBVP (1.7)-(1.8) with (1.9) or (1.10) may cease to exist globally (see [22]), or what is the key point to make sure that the solution obtained in Theorem 2.1 could become a global one?

The similar question has been studied for the incompressible Euler equation by Beale-Kato-Majda (BKM) in their pioneering work [3], which showed that the \(L^\infty\)-bound of vorticity \(\nabla \times u\) must blow up. Later, Ponce [25] rephrased the BKM-criterion in terms of the deformation tensor \(D(u)\). The same result as [25] has been proved by Huang-Li-Xin [17] for the 3D compressible isentropic Navier-Stokes equations, which can be shown: if \(0 < T < +\infty\) is the maximum time for the strong solutions, then
\[
\limsup_{T \to T} \int_0^T |D(u)|_{L^\infty(\Omega)} dt = \infty. \tag{2.7}
\]
Later on, under the physical assumption (1.4) and \(\lambda < 7\mu\), Sun-Wang-Zhang [28] improved this result based on some inequalities in BMO space such that
\[
\limsup_{T \to T} |\rho|_{L^\infty([0, T]; L^\infty(\Omega))} = \infty.
\]

Our main result in the following theorem shows that the \(L^\infty\) norms of \((\rho, \theta)\) and \(L^2\) norm of \(\nabla I\) control the possible blow-up (see [21]) for strong solutions, which means that if a solution of the compressible RHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of \(\nabla I\), \(\rho\) or \(\theta\) as the critical time approaches. We first assume that
\[
\begin{aligned}
\sigma_a & = \sigma(v, \Omega, \theta) \rho = \sigma \rho, \quad \sigma_\theta = \frac{d\sigma}{d\theta},
\|
\| (\sigma, \sigma_\theta)(v, \Omega, \theta) \|_{L^2 \cap L^\infty(\mathbb{R}^+ \times S^2; L^\infty([0, T] \times V))} & \leq M(\|\theta\|_\infty),
\end{aligned}
\tag{2.8}
\]
which is similar to (1.13), and obviously satisfies the assumption (1.12).

Theorem 2.3. Let (2.2)-(2.3) and (2.8) hold, \((\mu, \lambda)\) satisfy
\[
\mu > 0, \quad 3\lambda + 2\mu \geq 0, \quad \lambda < 3\mu. \tag{2.9}
\]
If \((I, \rho, u, \theta)\) is a strong solution obtained in Theorem 2.1 to IBVP (1.7)-(1.8) with boundary condition (1.9), and \(0 < T < \infty\) is the maximal time of its existence, then we have
\[
\limsup_{T \to T} \left( |\nabla I|_{L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T]; L^2(\Omega)))} + |(\rho, \theta)|_{L^\infty([0, T] \times V)} \right) = \infty. \tag{2.10}
\]

Remark 2.2. All results shown in Theorems 2.1,2.3 can be obtained for \(V = \mathbb{R}^3\) via some slight modifications. (2.10) also holds for isentropic flow in terms of \(\nabla I\) and \(\rho\).
Remark 2.3. We introduce the main ideas of our proof for Theorems 2.1-2.3, some of which are inspired by the arguments used in [1,7,13,17,28,33].

I) In Theorem 2.1, in order to get a a priori estimate which is independent of the lower bound of \( \rho_0 \) under boundary condition (1.10), some new arguments have been introduced compared with [7]. First, due to \( I_t = 0 \) on \( \partial V \) when \( n \cdot \Omega \leq 0 \), we observe that \( I_t = 0 \) and \( \nabla I = (\nabla I \cdot n)n \), on \( \partial V \) when \( n \cdot \Omega \leq 0 \), which will be used to deal with boundary terms for the estimate on \( \nabla I \). Second, in order to deal with terms \( (|\theta_t|^6,|\theta|^6) \) under (1.10), a Poincaré type inequality is introduced as

\[
|\theta_t|^6 \leq C(\sqrt{|\rho_0|} |\theta_t|^2 + (1 + |\rho|^2) |\nabla \theta|^2).
\]

Finally, the Minkowski’s inequality (see Section 4.1) and some regularity theory (see Section 3) introduced in [4] for incompressible Euler equations has been applied to the energy estimate for the velocity \( u \) of the fluid under the Navier-slip boundary condition.

II) In Theorem 2.3, in order to get a restriction of \( \mu \) and \( \lambda \) as better as possible, the crucial ingredient to relax the additional restrictions to \( \lambda < 3\mu \) has been observed that

\[
|\nabla u|^2 = |u|^2 |\nabla \left( \frac{u}{|u|} \right)|^2 + |\nabla |u||^2.
\]

for \( |u| > 0 \), and thus we deduce that

\[
\int_{|\nabla |u| > 0} |u|^{r-2} |\nabla u|^2 dx \geq (1 + \phi(\epsilon_0, \epsilon_1, r)) \int_{|\nabla |u| > 0} |u|^{r-2} |\nabla |u||^2 dx,
\]

if the following assumption holds:

\[
\int_{|\nabla |u| > 0} |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2 dx \geq \phi(\epsilon_0, \epsilon_1, r) \int_{|\nabla |u| > 0} |u|^{r-2} |\nabla |u||^2 dx
\]

for some positive function \( \phi(\epsilon_0, \epsilon_1, r) \) near \( r = 4 \). The details can be seen in Lemma 6.2. III) As it was pointed out in [33], we need to deal with the essential difficulties caused by the nonlinear term \( Q(u) \) in (1.14). However, a important fact has been observed that

\[
(P_m)_t = (\rho E_m)_t - \left( \frac{1}{2} \rho |u|^2 \right)_t,
\]

and integration by parts such that

\[
- \int_V (P_m)_t G dx = - \int_V (\rho E_m)_t G dx + \ldots = - \int_V T : \nabla G dx \leq C |\nabla |u||^2 |\nabla G||^2 + \ldots,
\]

where \( G = (2\mu + \lambda) \text{div} u - P_m \) is the effective viscous flux, which plays an important role in our proof for the blow-up criterion.

3. Preliminary

In this section, we give some important Lemmas which will be used frequently in our proof. The first one comes from Gagliardo-Nirenberg inequality and Poincaré inequality:
Lemma 3.1. For \( q \in (3, 6] \), there exist constants \( C > 0 \) (depend on \( q \)) and \( C_1 > 0 \) (depending on \( \mathbb{V} \)) such that for
\[
f \in D_0^1(\mathbb{V}), \quad g \in D_0^1 \cap D^2(\mathbb{V}), \quad h \in W^{1, q}(\mathbb{V})
\]
and \( w \in H^1(\mathbb{V}) \) with \( w \cdot n_{|\partial \mathbb{V}} = 0 \), we have
\[
|f|_6 \leq C|f|_{D_0^1}, \quad |g|_\infty \leq C|g|_{D_0^1 \cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{1, q}}, \quad \|w\|_1 \leq C_1|\nabla w|_2.
\tag{3.1}
\]

Then we will introduce some Poincaré type inequality (see Chapter 8 in [23]):

Lemma 3.2. There exists a constant \( C \) depending only on \( \mathbb{V} \) and \( |\rho|_r \) (\( r \geq 1 \)) (\( \rho \geq 0 \) is a real function satisfying \( |\rho|_1 > 0 \)), such that for every \( F \geq 0 \) satisfying
\[
\rho F \in L^1(\mathbb{V}), \quad \sqrt{\rho} F \in L^2(\mathbb{V}), \quad \nabla F \in L^2(\mathbb{V}),
\]
we have
\[
|F|_6 \leq C\left(|\sqrt{\rho} F|_2 + (1 + |\rho|_2)|\nabla F|_2\right).
\]

Proof. We first denote that
\[
\mathcal{F} = \frac{1}{|\mathbb{V}|} \int_{\mathbb{V}} F(y) dy,
\]
then via the classical Poincaré inequality, we quickly deduce that
\[
\mathcal{F} \int_{\mathbb{V}} \rho dx = \int_{\mathbb{V}} \rho(F - \mathcal{F}) dx + \int_{\mathbb{V}} \rho F dx
\leq C \left(|\rho F|_1 + |\rho|_2|\nabla F|_2\right) \leq C \left(|\rho|_1^\frac{1}{2} \sqrt{\rho F}_2 + |\rho|_2|\nabla F|_2\right),
\]
which implies that
\[
\mathcal{F} \leq C\left(|\sqrt{\rho} F|_2 + |\rho|_2|\nabla F|_2\right). \tag{3.2}
\]

Second, we consider that
\[
\|F\|_1 = |\nabla F|_2 + |F|_2 \leq |\nabla F|_2 + |F - \mathcal{F}|_2 + \mathcal{F} |\mathbb{V}|^\frac{1}{2}
\leq C\left(|\sqrt{\rho} F|_2 + (1 + |\rho|_2)|\nabla F|_2\right), \tag{3.3}
\]
then via (3.2)-(3.3) and the classical Sobolev imbedding theorem, we easily obtain:
\[
|F|_6 \leq C\|F\|_1 \leq C\left(|\sqrt{\rho} F|_2 + (1 + |\rho|_2)|\nabla F|_2\right).
\]

Next we consider the following homogenous Dirichlet boundary value problem for the Lamé operator \( L \): let \( U = (U^1, U^2, U^3), \quad F = (F^1, F^2, F^3) \) and
\[
Lu = -\mu \Delta U - (\mu + \lambda) \nabla \text{div} U = F \text{ in } \Omega, \quad U|_{\partial \Omega} = 0. \tag{3.4}
\]
If \( F \in W^{-1,2}(\mathbb{V}) \), then there exists a unique weak solution \( U \in H^1_0(\mathbb{V}) \). We begin with recalling various estimates for this system in \( L^1(\mathbb{V}) \) spaces, which can be seen in [2].

Lemma 3.3. Let (3.1) hold and \( \mathbb{V} \) be a bounded, smooth domain and \( l \in (1, +\infty) \). There exists a constant \( C \) depending only on \( \lambda, \mu, l \) and \( \mathbb{V} \) such that:

(1) If \( F \in L^1(\mathbb{V}) \), then we have
\[
\|U\|_{L^{2, l}} \leq C|F|_l; \tag{3.5}
\]
Lemma 3.7. If \( F \in W^{-1,1}(\mathcal{V}) \) (i.e., \( F = \text{div} \ f \) with \( f = (f_{ij})_{3 \times 3}, f_{ij} \in L^1(\mathcal{V}) \)), then we have
\[
\|U\|_{W^{1,1}} \leq C|f|_l.
\] (3.6)

Moreover, if \( \Delta U = F \) with \( \nabla U(t,x) \cdot n|_{\partial \mathcal{V}} = 0 \), for weak solution \( U \in H^1 \), we also know:
\[
\|U\|_{W^{2,1}} \leq C|F|_l, \quad \text{for } F \in L^1(\mathcal{V}).
\] (3.7)

In the next lemma, we give the following Beale-Kato-Majda type inequality, which was proved in [15]. And will be used later to estimate \( |\nabla u|_\infty \) and \( |\nabla p|_q \).

Lemma 3.4. [15] Let \( \mathcal{V} \) be a bounded and smooth domain and \( \nabla u \in L^2 \cap D^{1,q}(\mathcal{V}) \) with \( q \in (3, \infty) \). There exists a constant \( C \) depending on \( q \) such that
\[
|\nabla u|_{L^\infty(\mathcal{V})} \leq C(|\text{div} u|_\infty + |\nabla \times u|_\infty) \ln(1 + |\nabla^2 u|_q) + C|\nabla u|_2 + C.
\] (3.8)

The next two lemmas will be used to deal with the Navier-slip boundary conditions.

Lemma 3.5. [4] Let \( \mathcal{V} \) be a bounded, smooth domain and \( u \in H^s \) be a vector valued function satisfying \( u \cdot n|_{\partial \mathcal{V}} = 0 \), where \( n \) is the unit outer normal of \( \partial \mathcal{V} \), then
\[
\|u\|_s \leq C(|\text{div} u|_{s-1} + |\text{rot} u|_{s-1} + \|u\|_{s-1}),
\] (3.9)
for \( s \geq 1 \) and the constant \( C \) only depends on \( s \) and \( \mathcal{V} \).

Lemma 3.6. [32] Let \( \mathcal{V} \) be a bounded, smooth domain and \( u \in D^1 \) be a vector valued function satisfying
\[
u \cdot n|_{\partial \mathcal{V}} = 0 \quad \text{or} \quad u \times n|_{\partial \mathcal{V}} = 0,
\]
where \( n \) is the unit outer normal of \( \partial \mathcal{V} \), then for any \( l \in (1, +\infty) \), there exists a constant \( C \) only depends on \( l \) and \( \mathcal{V} \):
\[
|\nabla u|_l \leq C(|\text{div} u|_l + |\nabla \times u|_l).
\] (3.10)

Finally we show \( W^{2,p} \)-estimate for Lamé operator under Navier-slip boundary condition.

Lemma 3.7. [14] For any simply connected, bounded and smooth domain \( \mathcal{V} \subset \mathbb{R}^3 \), \( 1 < p < +\infty \), and if \( f \in L^p(\mathcal{V}; \mathbb{R}^3) \). If \( u \in H^2(\mathcal{V}; \mathbb{R}^3) \) is a weak solution of
\[
Lu = f \quad \text{in } \mathcal{V}, \quad (u \cdot n, (\nabla \times u) \cdot n)|_{\partial \mathcal{V}} = (0,0).
\] (3.11)
Then \( u \in W^{2,p}(\mathcal{V}) \), and there exists \( C > 0 \) depending on \( p, \mathcal{V} \) and \( L \) such that
\[
|u|_{W^{2,p}} \leq C(|f|_p + |\nabla u|_2).
\] (3.12)

4. Well-posedness of strong solutions

In this section, we always assume that
\[
\|S(v,t,x)\|_{L^2(\mathbb{R}^+;C^1([0,T];W^{1,q}))} \cap C^1([0,T];L^1(\mathbb{R}^+;L^2)) < +\infty.
\]
In order to prove Theorem 2.1, it is sufficient to prove the existence of the IBVP (1.7)-(1.8) with (1.10). Then next we need to consider the following linearized problem:

\[
\begin{aligned}
\rho_t + \text{div}(\rho w) &= 0, \\
\frac{1}{c_t} I_t + \Omega \cdot \nabla I &= \overline{A}_r, \\
(\rho \theta)_t + \text{div}(\rho \theta w) + \frac{1}{c_v} (P_m \text{div} w - k \Delta \theta) &= \frac{1}{c_v} (Q(w) + \overline{N}_r), \\
(\rho u)_t + \text{div}(\rho w \otimes u) + \nabla P_m + Lu &= -\frac{1}{c} \int_0^\infty \int_{S^2} \overline{C}_r \Omega d\Omega dv,
\end{aligned}
\]

(4.1)

where the terms \( \overline{A}_r, \overline{B}_r, \overline{C}_r \) and \( \overline{N}_r \) are given by

\[
\begin{aligned}
\overline{A}_r &= S - \sigma_a(\rho, \phi) I + \int_0^\infty \int_{S^2} \left( \frac{v}{w} \sigma_s(\rho) \psi - \sigma'_s(\rho) I \right) d\Omega dv', \\
\overline{B}_r &= S - \sigma_a(\rho, \phi) I + \int_0^\infty \int_{S^2} \left( \frac{v}{w} \sigma_s(\rho) I' - \sigma'_s(\rho) I \right) d\Omega dv', \\
\overline{C}_r &= S - \sigma_a(\rho, \theta) I + \int_0^\infty \int_{S^2} \left( \frac{v}{w} \sigma_s(\rho) I' - \sigma'_s(\rho) I \right) d\Omega dv', \\
\overline{N}_r &= \int_0^\infty \int_{S^2} \left( 1 - \frac{w \cdot \Omega}{c} \right) \overline{B}_r d\Omega dv, \quad \sigma_a(\rho, \phi) = \sigma_a(v, \Omega, t, x, \rho, \phi).
\end{aligned}
\]

And \( w(t, x) \in \mathbb{R}^3 \) is a known vector, \((\phi(t, x), \psi(\nu', \Omega', t, x))\) are known functions. Assume

\[
(w, \phi, \psi)|_{t=0} = (u_0, \theta_0, I_0), \quad (w, \phi, \psi) \in C([0, T]; H^2) \cap L^2([0, T]; D^2), \quad (w_t, \phi_t) \in L^2([0, T]; H^1),
\]

(4.2)

\[
0 \leq \psi \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; W^{1, q})), \quad \psi_t \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; L^q)),
\]

and \( \forall \subset \mathbb{R}^3 \) is bounded, simply connected, smooth domain with (1.10) and

\[
\psi|_{\partial \forall} = 0, \quad \Omega \cdot n \leq 0; \quad (w \cdot n, (\nabla \times w) \cdot n)|_{\partial \forall} = (0, 0); \quad \nabla \phi \cdot n|_{\partial \forall} = 0.
\]

(4.3)

### 4.1. A priori estimate to the linearized problem away from vacuum.

We immediately have the global existence of a unique strong solution \((I, \rho, u, \theta)\) to (4.1)-(4.3) by the standard methods at least for the case that \( \rho_0 \) is away from vacuum.

**Lemma 4.1.** Assume in addition to (4.2) that \( \rho_0 \geq \delta > 0 \) for some constant \( \delta > 0 \) and (2.3)-(2.4). Then there exists a unique strong solution \((I, \rho, u, \theta)\) to IBVP (4.1)-(4.3) such that \((I, \rho, u, \theta)\) belongs to class \( \Phi \) (see Definition 2.1) in \( \mathbb{R}^+ \times S^2 \times [0, T] \times \forall \) and

\[
I \in C([0, T]; L^2(\mathbb{R}^+ \times S^2; L^q(\forall))), \quad \rho \geq \delta
\]

for some constant \( \delta > 0 \).

**Proof.** Firstly, the existence and regularity of a unique solution \( \rho \) to (4.1) can be obtained essentially according to Lemma 6 in [7]. And \( \rho \) can be written as

\[
\rho(t, x) = \rho_0(U(0, 0, x)) \exp \left( -\int_0^t \text{div}w(s, U(s, 0, x))ds \right),
\]

(4.4)
where $U \in C([0,T] \times [0,T] \times \mathbb{V})$ is the solution to the initial value problem

$$\begin{cases}
\frac{d}{dt} U(t;0,x) = w(t,U(t;0,x)), & 0 \leq t \leq T, \\
U(0;0,x) = x, & x \in \mathbb{V},
\end{cases} \tag{4.5}$$

so we can get the lower bound of $\rho$.

Secondly, \((4.1)_2\) can be written into

$$\frac{1}{c} I_t + \Omega \cdot \nabla I + \left( \sigma_a + \int_0^\infty \int_{S^2} \sigma_s'(\rho) d\Omega' dv' \right) I = F(v, \Omega, t, x), \tag{4.6}$$

where

$$F = S + \int_0^\infty \int_{S^2} v \sigma_s(\rho) \psi d\Omega' dv' \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; W^{1,q})), \tag{4.7}$$

then we easily get the existence and regularity of a unique solution $I$ to \((4.6)\) that

$I \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; W^{1,q})), \quad I_t \in L^2(\mathbb{R}^+ \times S^2; C([0,T]; L^q))$,

according to the classical imbedding theory for Sobolev spaces, we easily have

$I \in C([0,T]; L^2(\mathbb{R}^+ \times S^2; L^q(\mathbb{V}))).$

And the radiation transfer equation \((4.1)_2\) can be written as

$$\frac{1}{c} I_t + c \Omega \cdot \nabla I + \mathbb{H} I = F, \quad \mathbb{H} = \sigma_a + \int_0^\infty \int_{S^2} \sigma_s'(\rho) d\Omega' dv'. \tag{4.8}$$

We denote by $x(t; x_0)$ the photon path starting from $x_0$ when $t = \tau$, i.e.,

$$\frac{d}{dt} x(t; \tau, x_0) = c \Omega, \quad x(\tau; \tau, x_0) = x_0.$$

Along the photon path, we obtain

$$I(v, \Omega, t, x(t; \tau, x_0)) = \exp \left( \int_{\tau}^t -c \mathbb{H}(v, \Omega, s, x(s; \tau, x_0), \rho, \theta) ds \right) \left( I(v, \Omega, \tau, x_0) \right. \left. + \int_{\tau}^t F(v, \Omega, s, x(s; \tau, x_0), \rho, \theta) \exp \left( \int_{\tau}^s c \mathbb{H}(v, \Omega, l, x(l; \tau, x_0), \rho, \theta) dl \right) ds \right) \geq 0, \tag{4.9}$$

where $x_0 = x - c \Omega(t - \tau)$. Then we obtain that $I$ is nonnegative.

Finally, \((4.1)_3\) and \((4.1)_4\) can be written into

$$\theta_t + w \nabla \theta + \frac{R}{c_v} \theta \text{div} w - \frac{\kappa}{c_v} \rho^{-1} \Delta \theta = \frac{1}{c_v} \rho^{-1} (Q(w) + \bar{N}_z),$$

$$u_t + w \cdot \nabla u + \rho^{-1} Lu = -\rho^{-1} \nabla P_m - \frac{1}{c_v} \rho^{-1} \int_0^\infty \int_{S^2} \mathcal{C}_r \Omega d\Omega dv, \tag{4.10}$$

the existence and regularity of solutions $\theta$ and $u$ to the corresponding linear parabolic problems can be obtained by classical methods such as in [6].

We first fix a positive constant $c_0$ that

$$1 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; W^{1,q})} + \|\rho_0\|_{W^{1,q}} + \|((\theta_0, u_0))\|_2$$

$$+ \|g_1\|_2 + \|S(v, t, x)\|_{L^2(\mathbb{R}^+ \times C^1([0,T]; W^{1,q}) \cap C^1([0,T]; L^1(\mathbb{R}^+; L^2)))} \leq c_0,$$
and a positive constant $c_1$ that
\[
\|\psi\|_{L^2([0,T];W^{1,q})} + \|\psi_t\|_{L^2([0,T];L^q)} + \sup_{0 \leq t \leq T^*} ((\phi, w)(t))_2 + \int_0^{T^*} \left( |(\phi, w)|_{L^2}^2 + |(\phi_t, w_t)|_{L^2}^2 \right) dt \leq c_1
\]
for some time $T^* \in (0, T)$. Moreover, we assume $1 \leq c_0 \leq c_1$. Then we will get a priori estimate for the solution $(I, \rho, u, \theta)$ obtained in Lemma [4.1], which is independent of $\delta$.

**Lemma 4.2.** (Estimates for mass density $\rho$)
\[
\|\rho(t)\|_{W^{1,q}} + |\rho_t(t)|_{q} \leq Cc_1^2
\]
for $0 \leq t \leq T_1 = \min(T^*, (1 + c_1)^{-1})$.

**Proof.** From standard energy estimate shown in [7] and (1.10), we have
\[
\|\rho(t)\|_{W^{1,q}} \leq \|\rho_0\|_{W^{1,q}} \exp \left( C \int_0^t \|\nabla w(s)\|_{W^{1,q}} ds \right),
\]
where we have used the fact that $w \cdot n|_{\partial V} = 0$. Therefore, observing that
\[
\int_0^t \|\nabla w(s)\|_{W^{1,q}} ds \leq t^{\frac{1}{4}} \left( \int_0^t \|\nabla w(s)\|_{W^{1,q}}^2 ds \right)^{\frac{1}{2}} \leq C(c_1 t + (c_1 t)^{\frac{1}{2}}).
\]
Then desired estimate for $\rho$ is available. The estimate for $\rho_t$ follows from $\rho_t = -\text{div}(\rho w)$. \qed

**Lemma 4.3.** (Estimates for specific radiation intensity $I$)
\[
\|I\|_{L^2([0,T];W^{1,q})} + \|I_t\|_{L^2([0,T];L^q)} \leq M(c_1)c_1^4
\]
for $T_2 = \min(T^*, (M(c_1)c_1^4)^{-1})$.

**Proof.** Firstly, multiplying (4.2) by $q|I|^{q-2}I$ and integrating over $V$, via (1.10) and Hölder’s inequality, we have
\[
\frac{d}{dt}|I_q| + \int_{\partial V \cap \{n \geq 0\}} |I|^q n \cdot \Omega d\nu \leq C|S_q| + C|\rho|_{\infty} \int_0^\infty \int_{S^2} \frac{v}{|v^2|} |\psi_q| \sigma_s d\Omega' dv', \quad (4.11)
\]
where we used the fact $\sigma_n \geq 0$ and $\sigma'_s \geq 0$.

Secondly, we consider $|\nabla I_q|$. Differentiating (4.1) $\beta$-times ($|\beta| = 1$) with respect to $x$, multiplying the resulting equation by $q|D^\beta|^{q-2}I|D^\beta I$ and integrating over $V$, we have
\[
\frac{d}{dt}|D^\beta I_q| + \int_{\partial V} |D^\beta I|^q n \cdot \Omega d\nu \leq C \left( |D^\beta S_q| + |\sigma_a|_{D^\beta q} |I|_{\infty} \right) + C|\nabla \rho|_{q} |I|_{\infty} \int_0^\infty \int_{S^2} \sigma_s d\Omega' dv' + C|\rho|_{\infty} |I_q| \int_0^\infty \int_{S^2} |D^\beta \sigma_s| d\Omega' dv' + C \int_0^\infty \int_{S^2} \frac{v}{|v|} |\psi_q| |\rho|_{\infty} |D^\beta \sigma_s| d\Omega' dv' + C \int_0^\infty \int_{S^2} \frac{v}{|v|} \sigma_s \left( |\psi|_{D^\beta q} |\rho|_{\infty} + \|\psi\|_{W^{1,q}} |\nabla \rho|_{q} \right) d\Omega' dv', \quad (4.12)
\]
where we also used $\sigma_n \geq 0$ and $\sigma'_s \geq 0$.\]
In (4.12) we need to consider the boundary term $I_{\partial V} = \int_{\partial V \cap \{n, \Omega \leq 0\}} |\nabla I|^2 n \cdot \Omega d\nu$. Due to $I = 0$ on $\partial V$ when $n \cdot \Omega \leq 0$, we obtain that

$$I_t = I = 0, \quad \nabla I = (\nabla I \cdot n)n \text{ on } \partial V, \text{ when } n \cdot \Omega \leq 0,$$

which, together with the assumption (1.11) and (4.12), implies that

$$\Omega \cdot \nabla I = S + \int_0^\infty \int_{S^2} \frac{v}{v'} |\sigma|_q d\Omega' dv' = 0, \text{ on } \partial V \text{ when } n \cdot \Omega \leq 0. \quad (4.14)$$

Then, via (4.13), $I_{\partial V}$ can be written as

$$I_{\partial V} = \int_{\partial V \cap \{n, \Omega \leq 0\}} |\nabla I|^2 n \cdot \Omega d\nu = \int_{\partial V \cap \{n, \Omega \leq 0\}} |\nabla I|^{q-2} |\nabla I| |\nabla I \cdot n| d\nu = 0. \quad (4.15)$$

Next, we consider the terms on the right-hand side of (4.12). From Sobolev’s imbedding theorem, we know $|I|_\infty \leq C|I|_{W^{1,q}}$. So via (4.12) and assumption (1.12), we have,

$$\frac{d}{dt} |D^\beta I|^2 \leq C(c_1^2 \alpha_1 + |\sigma|_{D^{1,q}} + \alpha_3 |\nabla \rho|_q) |I|^2_{W^{1,q}} + C|D^\beta S\|q^2$$

$$+ Cc_1^4 \int_0^\infty \int_{S^2} \frac{|v|}{v'} |D^\beta \sigma|_q^2 d\Omega' dv' \cdot \int_0^\infty \int_{S^2} |\psi|_q^2 d\Omega' dv'$$

$$+ Cc_1^4 \int_0^\infty \int_{S^2} \frac{|v|}{v'} \sigma_d \Omega' dv' \cdot \int_0^\infty \int_{S^2} |\psi|_{D^{1,q}}^2 d\Omega' dv'$$

$$+ Cc_1^4 \int_0^\infty \int_{S^2} \frac{|v|}{v'} \sigma_s \Omega' dv' \cdot \int_0^\infty \int_{S^2} |\psi|_{W^{1,q}}^2 d\Omega' dv'$$

$$\leq M(c_1) c_1^2 |I|_{W^{1,q}}^2 + C|D^\beta S\|q^2 + Cc_1^4 \int_0^\infty \int_{S^2} \frac{|v|}{v'} |\sigma_s|_q^2 d\Omega' dv', \quad (4.16)$$

where we have used the fact that

$$|\sigma|_{D^{1,q}} \leq (|\rho|_\infty |\nabla \sigma|_q + |\sigma|_\infty |\nabla \rho|_q) \leq M(c_1) c_1^2.$$

Then combining (4.11) and (4.16), we have

$$\frac{d}{dt} \|I(v, \Omega, t, \cdot)\|_{W^{1,q}}^2 \leq M(c_1) c_1^2 \|I\|_{W^{1,q}}^2 + C\|S\|_{W^{1,q}}^2 + Cc_1^6 \int_0^\infty \int_{S^2} \frac{|v|}{v'} |\sigma_s|_q^2 d\Omega' dv'. \quad (4.17)$$

From Gronwall’s inequality, we have

$$\|I(v, \Omega, t, x)\|_{C([0, T_2]; W^{1,q})} \leq \exp(M(c_1) c_1^2 T_2) \left( \|I_0\|_{W^{1,q}}^2 + \int_0^{T_2} \|S\|_{W^{1,q}}^2 ds + c_1^6 T_2 \int_0^\infty \int_{S^2} \frac{|v|}{v'} |\sigma_s|_q^2 d\Omega' dv' \right). \quad (4.18)$$

Then integrating above inequality in $\mathbb{R}^+ \times S^2$ with respect to $(v, \Omega)$, via (1.12), we have

$$\|I\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_2]; W^{1,q}))} \leq Cc_1^4, \quad \text{for } T_2 = \min(T^*, (M(c_1) c_1^2)^{-1}).$$

Finally, due to $I_t = -c\Omega \cdot \nabla I + c\Delta I$, then, the desired conclusions for $I_t$ are obvious. \square

Next we give the estimate for $\theta$, for simplicity, we denote

$$\int_0^\infty \int_{S^2} \int_{S^2} f(v, \Omega, v', \Omega', t, x) d\Omega' dv' d\Omega dv = \int f d\mathbb{H}.$$
Lemma 4.4.

\[ \|\theta(t)\|_2^2 + |\sqrt{\rho} \theta_t(t)|_2^2 + \int_0^t \left( |\theta(s)|_2^2 + |\theta_t(s)|_2^2 \right) ds \leq M(c_1 c_1^{20}) \]

for \( 0 \leq t \leq T_3 = \min(T^*, \frac{1}{2}(M(c_1 c_1^{40})^{-1}) \).

Proof. First differentiating (4.13) with respect to \( t \), we have

\[
\rho \theta_{tt} - \frac{\kappa}{c_v} \triangle \theta_t = -\rho_t \theta_t - (\rho w \cdot \nabla \theta)_t - \frac{1}{c_v} \left( (P_m \text{div} w)_t - Q(w)_t - (\nabla r)_t \right). \tag{4.19}
\]

Multiplying (4.19) by \( \theta_t \) and integrating over \( \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\theta_t|^2 \, dx + \int_\Omega |\nabla \theta_t|^2 \, dx 
\leq C \int_\Omega \left( |\rho_0 w \cdot \nabla \theta_t| + |\rho \theta_t| + |\rho \text{div} w \theta| \right) \, dx + \int_0^\infty \int_{S^2} \int_\Omega \left( \frac{1}{c_v} \left( 1 - \frac{w \cdot \Omega}{c} \right) (B_r)_t \theta_t \, dx \right) \, dt 
\leq \sum_{i=1}^6 I_i + E_I,
\]

where the radiation source term:

\[
E_I = \frac{1}{c_v} \int_0^\infty \int_{S^2} \int_\Omega \left( \left( 1 - \frac{w \cdot \Omega}{c} \right) (B_r)_t \right) \, dx \, dt = \sum_{j=1}^8 J_j.
\]

Second, we estimate \( \sum_{i=1}^6 I_i \). According to Lemma 3.1, Lemmas 4.2-4.3, Gagliardo-Nirenberg inequality and Young’s inequality, we have

\[
I_1 \leq C |\rho_t|_3 w_\infty \| \nabla \theta_t \|_2 \| \theta_t \|_6 \leq C c_1^6 \| \nabla \theta_t \|_2^2 + \frac{1}{20 c_1^4} (|\sqrt{\rho} \theta_t|_2^2 + c_1^4 \| \nabla \theta_t \|_2^2),
\]

\[
I_2 + I_5 \leq C |\rho \frac{1}{2} w_\infty \| \nabla \theta_t \|_2 \| \sqrt{\rho} \theta_t \|_6 \leq C c_1^6 \| \nabla \theta_t \|_2^2 + \frac{1}{20 c_1^4} (|\sqrt{\rho} \theta_t|_2^2 + c_1^4 \| \nabla \theta_t \|_2^2) + |\nabla w_t|_2^2,
\]

\[
I_3 \leq C |\rho \frac{1}{2} w_\infty \| \nabla \theta_t \|_2 \| \sqrt{\rho} \theta_t \|_6 \leq C c_1^6 \| \nabla \theta_t \|_2^2 + \frac{1}{20 c_1^4} (|\sqrt{\rho} \theta_t|_2^2 + c_1^4 \| \nabla \theta_t \|_2^2),
\]

\[
I_4 \leq C |(P_m)_t|_2 \| \nabla w_t \|_2 \| \theta_t \|_6 \leq C c_1^4 \| \nabla \theta_t \|_2^2 + \frac{1}{20 c_1^4} (|\sqrt{\rho} \theta_t|_2^2 + c_1^4 \| \nabla \theta_t \|_2^2) + C c_1^6 \| \nabla \theta_t \|_2^2,
\]

\[
I_6 \leq C |\theta_t|_6 \| \nabla w_t \|_2 \| \nabla w_t \|_3 \leq \frac{1}{20 c_1^4} (|\sqrt{\rho} \theta_t|_2^2 + c_1^4 \| \nabla \theta_t \|_2^2) + C c_1^2 \| \nabla w_t \|_2^2,
\]

where we have used the Poincaré type inequality (see Chapter 8 in [23]):

\[
|\theta_t|_6 \leq C (|\sqrt{\rho} \theta_t|_2 + (1 + |\rho|_2)|\nabla \theta_t|_2) \leq C (|\sqrt{\rho} \theta_t|_2 + c_1^2 \| \nabla \theta_t \|_2). \tag{4.21}
\]
Next considering $E_l$, from H"{o}lder’s inequality, (4.12) and Young’s inequality, we have

$$J_1 = \frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(1 - \frac{w \cdot \Omega}{c}\right) S_1 \theta t d\Omega dv$$

$$\leq C(1 + |w|_3) |\theta t|_6 \int_0^\infty \int_{S^2} |S_1|_2 d\Omega dv \leq \frac{1}{20c_1} (|\sqrt{\rho \theta t}|_2^3 + c_4 |\nabla \theta t|_2^3) + Cc_4^1,$$

$$J_2 = -\frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(1 - \frac{w \cdot \Omega}{c}\right) (\sigma_{a I})_t \theta t d\Omega dv$$

$$\leq C(1 + |w|_3) \int_0^\infty \int_{S^2} \left(\frac{1}{2} |\sigma_{\infty} \sqrt{\rho \theta t}|_2 |\sigma t|_2 |I|_3 + |\sigma|_\infty |I|_3 |\rho t|_6 |\theta t|_6\right) d\Omega dv$$

$$\leq C(1 + |w|_3) |\rho|_\infty |\sqrt{\rho \theta t}|_2 \int_0^\infty \int_{S^2} |\sigma|_\infty |I|_2 d\Omega$$

$$\leq \frac{1}{20c_1^2} (|\sqrt{\rho \theta t}|_2^3 + c_4 |\nabla \theta t|_2^3) + M(c_1)c_1^{12} (|\sqrt{\rho \theta t}|_2 + 1) + \epsilon \int_0^\infty \int_{S^2} |\sigma t|_2^3 d\Omega dv$$

$$\leq \frac{1}{20} |\theta t|_{2D_1}^2 + M(c_1)c_1^{12} |\sqrt{\rho \theta t}|_2^3 + \epsilon \int_0^\infty \int_{S^2} |\sigma t|_2^3 d\Omega dv + M(c_1)c_1^{12},$$

where we used $(\sigma_{a I})_t = \rho \sigma_t + \rho t \sigma$ and $\epsilon$ is a sufficiently small constant. And similarly,

$$J_3 = \frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(1 - \frac{w \cdot \Omega}{c}\right) \left(\sigma_{a I}\right)_t \theta t d\Omega d\nu$$

$$\leq C(1 + |w|_\infty) \int_0^\infty \int_{S^2} \left(\frac{1}{2} |\sigma_{\infty} \sqrt{\rho \theta t}|_2 |\sigma t|_2 |I|_3 + |\rho|_\infty |\sqrt{\rho \theta t}|_2 \sigma_t |I|_2\right) d\Omega d\nu$$

$$\leq C\alpha_1^2 \int_0^\infty \int_{S^2} |I|_2^3 d\Omega dv' + C\alpha_1^2 c_1^4 \int_0^\infty \int_{S^2} |I t|_2^3 d\Omega dv' + C|\sqrt{\rho \theta t}|_2^3$$

$$+ \frac{1}{20c_1^2} (|\sqrt{\rho \theta t}|_2^3 + c_4 |\nabla \theta t|_2^3) \leq \frac{1}{20} |\sqrt{\rho \theta t}|_2^3 + C|\sqrt{\rho \theta t}|_2^3 + M(c_1)c_1^{14},$$

$$J_4 = \frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(1 - \frac{w \cdot \Omega}{c}\right) \left(\sigma_{a I}\right)_t \theta t d\Omega d\nu$$

$$\leq C(1 + |w|_\infty) \int_0^\infty \int_{S^2} \left(\frac{1}{2} |\sigma_{\infty} \sqrt{\rho \theta t}|_2 |\sigma t|_2 |I|_3 + |\rho|_\infty |\sqrt{\rho \theta t}|_2 \sigma_t |I|_2\right) d\Omega d\nu$$

$$\leq C\alpha_2 c_1^0 \int_0^\infty \int_{S^2} |I|_2^2 d\Omega dv + C\alpha_2 c_1^4 \int_0^\infty \int_{S^2} |I t|_2^2 d\Omega dv + C|\sqrt{\rho \theta t}|_2^3$$

$$+ \frac{1}{20c_1^2} (c_1^4 |\nabla \theta t|_2 |\sqrt{\rho \theta t}|_2) \leq \frac{1}{20} |\theta t|_{2D_1}^2 + C|\sqrt{\rho \theta t}|_2^3 + M(c_1)c_1^{14},$$

$$J_5 = \frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(-\frac{w t}{c}\right) S \theta t d\Omega dv$$

$$\leq C|w t|_\infty |\theta t|_6 \int_0^\infty \int_{S^2} |S|_2^3 d\Omega dv \leq \frac{1}{20c_1^2} (c_1^4 |\theta t|_{2D_1}^3 + |\sqrt{\rho \theta t}|_2^3) + Cc_0^3 |w t|_{2D_1}^2,$$

$$J_6 = \frac{1}{c_w} \int_0^\infty \int_{S^2} \int_{\nu} \left(-\frac{w t}{c}\right) \left(\sigma_{a t}\right)_t d\Omega d\nu \leq C|w t|_\infty |\rho|_\infty \int_0^\infty \int_{S^2} |I|_2^2 d\Omega dv \leq C|\sqrt{\rho \theta t}|_2^3 + M(c_1)c_1^{10} |w t|_{2D_1}^2.$$
\[ J_7 = \frac{1}{c_v} \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \frac{w_t \cdot \Omega}{c} \sigma_a I_{\theta t} \, dx \, d\Omega \, dv \]

\[ \leq C |w_t|_\infty |\rho|^{\frac{2}{3}}  \sqrt{\rho} |\theta_t|_2 \int_0^\infty \int_{S^2} |\sigma|_\infty \|I\|_1 \, d\Omega \, dv \]

\[ \leq C |\sqrt{\rho} \theta_t|_2^2 + C |w_t|_{D_1}^2 |\rho|_\infty \int_0^\infty \int_{S^2} |\sigma|_\infty^2 \, d\Omega \, dv \int_0^\infty \int_{S^2} \|I\|_1^2 \, d\Omega \, dv \]

\[ \leq C |\sqrt{\rho} \theta_t|_2^2 + M(c_1) c_1^1 |w_t|_{D_1}^2, \quad (4.24) \]

\[ J_8 = \frac{1}{c_v} \int_1^\infty \int_{\mathcal{V}} - \frac{v \cdot w_t \cdot \Omega}{c} \sigma_s I_{\theta t} \, dx \, d\Omega \leq C |w_t|_\infty |\rho|^{\frac{2}{3}}  \sqrt{\rho} |\theta_t|_2 \int_1^\infty \frac{v}{v'} |\theta_s| |I'\|_1 \, d\Omega \]

\[ \leq \lambda \theta_1^2 |w_t|_{D_1}^2 |\rho|_\infty \int_0^\infty \int_{S^2} |\theta'|^2_1 \, d\Omega \, dv' + C |\theta|_2^2 \]

\[ \leq C |\sqrt{\rho} \theta_t|_2^2 + M(c_1) c_1^1 |w_t|_{D_1}^2, \]

Then combining the above estimates for \( I_i \) and \( J_j \), from (4.20) we quickly have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathcal{V}} \rho|\theta_t|^2 \, dx + \int_{\mathcal{V}} \left| \nabla \theta_t \right|^2 \, dx \]

\[ \leq M(c_1) c_1^2 (1 + |\nabla \theta|^2_2) |\sqrt{\rho} \theta_t|^2 + \epsilon \int_0^\infty \int_{S^2} |\sigma|_\infty^2 \, d\Omega \, dv + M(c_1) c_1^3 (1 + c_1^4 |w_t|_{D_1}^2). \quad (4.25) \]

Notice that, via the assumption (1.12), we have

\[ \epsilon \int_0^\infty \int_{S^2} |\sigma|_\infty^2 \, d\Omega \, dv \leq \epsilon M(c_1) (c_1^4 + |\theta_0|^2) \leq \epsilon M(c_1) (c_1^4 + |\sqrt{\rho} \theta_t|^2 + c_1^4 |\nabla \theta_t|_2), \]

then integrating (4.25) over \((\tau, t)\) with \( \tau \in (0, t) \), letting \( \epsilon \) be sufficiently small, we have

\[ |\sqrt{\rho} \theta_t(t)|_2^2 + \int_\tau^t |\theta_t|^2_1 \, ds \]

\[ \leq |\sqrt{\rho} \theta_t(\tau)|_2^2 + M(c_1) c_1^2 \int_\tau^t (1 + |\nabla \theta|^2_2) |\sqrt{\rho} \theta_t|^2 \, ds + M(c_1) c_1^4 (1 + t). \quad (4.26) \]

From (4.13), we have

\[ |\sqrt{\rho} \theta_t|^2_2 \leq |\rho|_\infty \left| \nabla w_0 \right|^2 \left| \nabla \theta_0 \right|^2 + \int_{\mathcal{V}} |\Phi|^2 / \rho \, dx, \quad (4.27) \]

where

\[ \Phi = \kappa \Delta \theta + Q(w) + \int_0^\infty \int_{S^2} \left( 1 - \frac{w \cdot \Omega}{c} \right) \mathcal{B}_x \, d\Omega \, dv. \]

Via the assumption (1.12), Lemma 4.1 the regularity of \( S(v, t, x) \) and Minkowski’s inequality, we easily have

\[ \lim_{t \to 0} \int_{\mathcal{V}} \left( \frac{|\Phi(t)|^2}{\rho} - \frac{|\Phi(0)|^2}{\rho_0} \right) \, dx \]

\[ \leq \lim_{t \to 0} \left( \frac{1}{\delta} \int_{\mathcal{V}} |\Phi(t) - \Phi(0)|^2 \, dx + \frac{1}{\delta} |\rho(t) - \rho_0| \int_{\mathcal{V}} |\Phi(0)|^2 \, dx \right) = 0. \]

According to the compatibility condition (2.3) and equation (4.13), we have

\[ \lim_{t \to 0} \left| \sqrt{\rho} \theta_t(\tau) \right|^2_2 \leq |\rho_0|_\infty \left| \nabla w_0 \right|^2 \left| \nabla \theta_0 \right|^2_2 + |g_1|^2 \leq C c_1^6. \quad (4.28) \]
Therefore, letting $\tau \to 0$ in (4.26), we have
\[
|\sqrt{\rho \theta_t}(t)|^2 + \int_0^t |\theta_t|^2 \, ds \leq M(c_1)c_1^{12} \int_0^t (1 + |\nabla \theta|^2)|\sqrt{\rho \theta_t}|^2 \, ds + M(c_1)c_1^{14}(1 + t). \tag{4.29}
\]

On the other hand, due to (4.21), we have
\[
|\theta(t)|^2 + |\nabla \theta(t)|^2 \\
\leq |\theta_0|^2 + C \int_0^t (|\theta|_2|\theta_t|_2 + |\nabla \theta_t|_2) \, ds \\
\leq |\theta_0|^2 + \int_0^t (C(|\theta(t)|^2 + |\nabla \theta|^2 + |\sqrt{\rho \theta_t}(t)|^2) + \frac{1}{20}|\nabla \theta|^2) \, ds. \tag{4.30}
\]

Combining (4.29) and (4.30), we have
\[
|\theta(t)|^2 + |\sqrt{\rho \theta_t}(t)|^2 + \int_0^t |\theta_t|^2 \, ds \\
\leq M(c_1)c_1^{12} \int_0^t (1 + |\nabla \theta|^2)(1 + |\theta|^2 + |\sqrt{\rho \theta_t}|^2) \, ds + M(c_1)c_1^{14}
\]
for $0 \leq t \leq T_2$. Then, let $L(t) = 1 + |\theta(t)|^2 + |\sqrt{\rho \theta_t}|^2$, via solving
\[
L(t) \leq M(c_1)c_1^{14} + M(c_1)^{12} \int_0^t L^3(s) \, ds,
\]
we conclude that
\[
M(c_1)c_1^{14} + M(c_1)c_1^{12} \int_0^t L^3(s) \, ds \leq M(c_1)c_1^{14}(1 - M(c_1)c_1^{10}t)^{-\frac{1}{2}}.
\]

So when $0 \leq t < T_3 = \frac{1}{2}(M(c_1)c_1^{10})^{-1}$,
\[
|\theta(t)|^2 + |\sqrt{\rho \theta_t}(t)|^2 + \int_0^t |\theta_t|^2 \, ds \leq M(c_1)c_1^{14}.
\]

The further estimates can be obtained by Lemma 3.3. From
\[
-\frac{\kappa}{c_v} \Delta \theta = -\rho \theta_t - \rho w \cdot \nabla \theta + \frac{1}{c_v} (\rho \theta - \nabla \cdot \rho w + Q(w) + \int_0^\infty \int_{S^2} (1 - \frac{w \cdot \Omega}{c}) |B_r| d\Omega dv), \tag{4.32}
\]
via Minkowski’s inequality, we have
\[
|\theta|_{D^2} \leq C (|\rho \theta_t|_2 + |\rho w \cdot \nabla \theta|_2 + |\rho \theta \nabla w|_2 + |Q(w)|_2) \\
+ (1 + |w|_\infty) \int_0^\infty \int_{S^2} |\bar{B}_r|_2 d\Omega dv \leq M(c_1)c_1^{10} \tag{4.33}
\]
for $0 \leq t \leq T_3$. Similarly, we also have
\[
\int_0^t |\theta|_{D^2,q}^2 \, ds \leq C \int_0^t \left( |\rho \theta_t|_q^2 + |\rho w \cdot \nabla \theta|_q^2 + |\rho \theta \nabla w|_q^2 + |Q(\nabla w)|_q^2 \right) \, ds \\
+ \int_0^t (1 + |w|_\infty)^2 \left( \int_0^\infty \int_{S^2} |\bar{B}_r|_q d\Omega dv \right)^2 \, ds \leq M(c_1)c_1^{16} \tag{4.34}
\]
for \(0 \leq t \leq T_3\). According to \(P_m = R \rho \theta\), for \(0 \leq t \leq T_3\), we easily obtain that
\[
|\nabla P_m|_2 \leq M(c_1) c_1^{12}, \quad |\nabla P_m|_q \leq C c_1^{12}, \quad |(P_m)_t|_2 \leq C c_1^{12}.
\] (4.35)

Next we give the estimate for the velocity \(u\).

**Lemma 4.5.**
\[
\|u(t)\|_2^2 + |\sqrt{\rho}u_t(t)|_2^2 + \int_0^t \left( |u(s)|_{D^{2,q}}^2 + |u_t(s)|_{D^{2,q}}^2 \right) ds \leq M(c_1) c_1^{32}, \quad \text{for} \ 0 \leq t \leq T_3.
\]

**Proof.** Differentiating (1.14) with respect to \(t\), we have
\[
\rho u_{tt} + Lu_t = -\rho_t u_t - (\rho w \cdot \nabla) u_t - (\nabla P_m)_t - \frac{1}{c} \int_{\Omega} (\overline{C}_r) t \Omega d\Omega dv,
\] (4.36)

multiplying (4.36) by \(u_t\) and integrating over \(\mathbb{V}\), via \(\triangle u = \nabla \text{div} u - \nabla \times \text{curl} u\) and the boundary condition (1.10) we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{V}} |\rho w|_2^2 dx + \int_{\mathbb{V}} ((\mu + \lambda)|\text{div} u_t|^2 + \mu(\nabla \times u_t)^2) dx
\]
\[
\leq C \int_{\mathbb{V}} \left( |\rho w \cdot \nabla u \cdot u_t| + |\rho w_t \cdot \nabla u \cdot u_t| + |\rho w \cdot \nabla u_t \cdot u_t| + |(P_m)_t \text{div} u_t| \right) dx - \frac{1}{c} \int_{\mathbb{V}} \left( \int_0^\infty \int_{\Omega} (\overline{C}_r) t u_t \cdot \Omega d\Omega dv \right) dx \equiv \sum_{i=7}^{10} I_i + E_{II},
\] (4.37)

where the radiation source term:
\[
E_{II} = -\frac{1}{c} \int_{\mathbb{V}} \left( \int_0^\infty \int_{\Omega} (\overline{C}_r) t u_t \cdot \Omega d\Omega dv \right) dx
\]
\[
= -\frac{1}{c} \int_0^\infty \int_{\mathbb{V}} \int_{\Omega} u_t \cdot \Omega (S_t - (\sigma_a I)_t) + \int_0^\infty \int_{S^2} \frac{\nu}{\nu'} (\sigma_a I')_{t} d\Omega dv' \right) dx d\Omega dv
\]
\[
+ \frac{1}{c} \int_{\mathbb{V}} \int_{\Omega} \Omega (\sigma_a I)_t d\Omega d\Omega = \sum_{j=9}^{12} J_j.
\]

Due to Lemma 3.6 we have \(|\nabla u_t|_2 \leq C(|\text{div} u_t|_2 + |\nabla \times u_t|_2)\), then
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{V}} |\rho u_t|_2^2 dx + C \int_{\mathbb{V}} |\nabla u_t|_2^2 dx \leq \sum_{i=7}^{10} I_i + E_{II}.
\] (4.38)

First, we estimate \(\sum_{i=7}^{10} I_i\). According to Lemma 3.1 Lemmas 4.2.4.4 Gagliardo-Nirenberg inequality and Young’s inequality, we have
\[
I_7 \leq C |\rho_t|_3 |w|_\infty |\nabla u_t|_2 |u_t|_6 \leq C c_1^{6} |\nabla u_t|_2^2 + \epsilon |\nabla u_t|_2^2,
\]
\[
I_8 \leq C |\rho|_\infty |w_t|_6 |\nabla u_t|_2 \sqrt{\rho u_t} \leq C c_1^{2} |\nabla u_t|_2^2 + C |\nabla u_t|_2^2 \sqrt{\rho u_t}^2,
\]
\[
I_9 \leq C |\rho|_\infty |w|_\infty |\nabla u_t|_2 \sqrt{\rho u_t} \leq C c_1^{2} \sqrt{\rho u_t}^2 + \epsilon |\nabla u_t|_2^2,
\]
\[
I_{10} \leq C |(\rho u_t)|_2 \sqrt{\rho u_t} \leq M(c_1) c_1^{24} + \epsilon |\nabla u_t|_2^2.
\]
Next we need to consider the radiation term $E_{II}$,

$$J_9 = -\frac{1}{c} \int_0^\infty \int_{S^2} I_t u_t \cdot \Omega \, dx \, d\Omega$$

$$\leq C|u_t|^2 \int_0^\infty \int_{S^2} |I_t|^2 d\Omega dv \leq \epsilon|\nabla u_t|^2 + Cc_0^2,$$

$$J_{10} = \frac{1}{c} \int_0^\infty \int_{S^2} (\sigma_a)_{l} u_t \cdot \Omega \, dx \, d\Omega$$

$$\leq C|u_t|^6 \int_0^\infty \int_{S^2} |(\sigma_a)_{l}|^2 I_t^3 d\Omega dv + C|\rho|^\frac{1}{2} \sqrt{\rho} u_t |^2 \int_0^\infty \int_{S^2} |\sigma_a| |I_t|^2 d\Omega dv$$

$$\leq \epsilon|\nabla u_t|^2 + C|\sqrt{\rho} u_t|^2 + M(c_1)c_1^{12} \int_0^\infty \int_{S^2} |\sigma_a|^2 d\Omega dv + M(c_1)c_1^{12},$$

where we used the fact $(\sigma_a)_l = \rho \sigma_t + \rho_t \sigma$. And similarly

$$J_{11} = -\frac{1}{c} \int_0^\infty \int_{S^2} v(\sigma_s I)_l u_t \cdot \Omega \, dx \, d\Omega$$

$$\leq C|u_t|^6 \int_0^\infty \int_{S^2} |I'_l|^2 d\Omega dv + C|\rho|^\frac{1}{2} \sqrt{\rho} u_t |^2 \int_0^\infty \int_{S^2} |\sigma_s| |I_t|^2 d\Omega dv$$

$$\leq C\alpha_1^2 c_1^4 \int_0^\infty \int_{S^2} |I'_l|^2 d\Omega dv + C\alpha_2^2 c_1^4 \int_0^\infty \int_{S^2} |I_t|^2 d\Omega dv + \epsilon|u_t|^2_{D}\ + C|\sqrt{\rho} u_t|^2$$

$$\leq \epsilon|u_t|^2_{D} + C|\sqrt{\rho} u_t|^2 + M(c_1)c_1^{12},$$

$$J_{12} = \frac{1}{c} \int_0^\infty \int_{S^2} (\sigma_s I)_l u_t \cdot \Omega \, dx \, d\Omega$$

$$\leq C|u_t|^6 \int_0^\infty \int_{S^2} |I'_l|^2 d\Omega dv + C|\rho|^\frac{1}{2} \sqrt{\rho} u_t |^2 \int_0^\infty \int_{S^2} |\sigma_s| |I_t|^2 d\Omega dv$$

$$\leq C\alpha_1^2 c_1^4 \int_0^\infty \int_{S^2} |I'_l|^2 d\Omega dv + C\alpha_2^2 c_1^4 \int_0^\infty \int_{S^2} |I_t|^2 d\Omega dv + \epsilon|u_t|^2_{D}\ + C|\sqrt{\rho} u_t|^2$$

$$\leq \epsilon|u_t|^2_{D} + C|\sqrt{\rho} u_t|^2 + M(c_1)c_1^{12}.$$
Then we have to estimate \( \| \nabla u \|_1 \), due to Lemma 3.3, we have
\[
\| \nabla u \|_1 \leq C \left( |\rho u_t|_2 + |\rho w \cdot \nabla u|_2 + |\nabla P_m|_2 + \int_0^\infty \int_{S^2} |C_r|_2 d\Omega dv \right) 
\]
\[\leq C(c_1|\sqrt{\rho} u_t|_2 + c_1^2|\nabla u|_2 + M(c_1)c_1^{12}). \tag{4.40}
\]
Therefore, from (4.39)-(4.40), we have
\[
|\nabla u(t)|^2 + |\sqrt{\rho} u_t(t)|_2^2 + \mu \int_0^t |u_t|_{D_2^1}^2 ds 
\]
\[\leq Cc_1^{12} \int_0^t |\nabla u|^2 ds + Cc_1^8 \int_0^t (1 + |\nabla u|^2)|\sqrt{\rho} u_t|_2 ds + M(c_1)c_1^{24}(t + 1)
\]
for \( 0 \leq t \leq T_3 \). From Gronwall’s inequality and (4.40), we have
\[
\| \nabla u(t) \|_1^2 + |\sqrt{\rho} u_t(t)|_2^2 + \mu \int_0^t |u_t|_{D_2^1}^2 ds \leq M(c_1)c_1^{30}
\]
for \( 0 \leq t \leq T_3 \). Then similarly, we have
\[
\int_0^t |u|^2_{D_2,q} ds \leq \int_0^t \left( |\rho u_t + \rho w \cdot \nabla u + \nabla P_m|^2 + \left( \int_0^\infty \int_{S^2} |C_r|_q d\Omega dv \right)^2 \right) ds \leq M(c_1)c_1^{32}.
\]

4.2. The unique solvability of the linearized problem with vacuum.
Based on the results of Lemmas 4.2-4.5, we conclude that
\[
\| I \|_{L^2(\mathbb{R}^+ \times S^2;C([0,T_*];W^{1,q}))} + \| I_t \|_{L^2(\mathbb{R}^+ \times S^2;C([0,T_*];L^q))} \leq M(c_1)c_1^4,
\]
\[
\sup_{0 \leq t \leq T_*} \left( |\rho|_{W^{1,q}} + |p_t(t)|_q + \|(u(t), \theta(t))\|_2 \right) \leq M(c_1)c_1^{16},
\]
\[
\| (\sqrt{\rho} u_t(t), \sqrt{\rho} \theta_t(t))\|_2 + \int_0^T \left( \|(u, \theta)\|_{D_2,q}^2 + \|(u_t, \theta_t)\|_1^2 \right) ds \leq M(c_1)c_1^{32},
\]
where \( T_* = \min\{T^*, 1/2(M(c_1)c_1^{40})^{-1} \} \). Now we give the key lemma to prove our main result.

**Lemma 4.6.** Let (4.2) and (2.3)-(2.4) hold. Assume further, there exists positive constants \( c_0, c_1 \) and \( T_* \) chosen as before such that
\[
2 + \| I_0 \|_{L^2(\mathbb{R}^+ \times S^2;W^{1,q})} + \| \rho_0 \|_{W^{1,q}} + \| \theta_0, u_0 \|_2 + \|(g_1, g_2)\|_2 \\
+ \| S(v, t, x) \|_{L^2(\mathbb{R}^+ ; C^1([0,T_*];W^{1,q}))} \leq c_0,
\]
and
\[
\| \psi \|_{L^2(\mathbb{R}^+ \times S^2;C([0,T_*];W^{1,q}))} + \| \psi_t \|_{L^2(\mathbb{R}^+ \times S^2;C([0,T_*];L^q))} \\
+ \|(\phi, w)\|_2 + \int_0^{T_*} \left( \|(\phi, w)\|_{D_2,q}^2 + \|(\phi_t, w_t)\|_1^2 \right) ds \leq c_1,
\]
then \( \exists \) unique strong solution \((I, \rho, u, \theta)\) to (4.1)-(4.3) such that \((I, \rho, u, \theta)\) belongs to class \( \Phi \) in \( \mathbb{R}^+ \times S^2 \times [0, T_*] \times \nabla \). Moreover, \((I, \rho, u, \theta)\) satisfies the local estimate (4.43).
Proof. Step 1: Existence of strong solution. We define \( \rho_0 = \rho_0 + \delta \) for each \( \delta \in (0, 1) \). Then from the compatibility conditions \([2,3]\), we have

\[
Lu_0 + R\nabla (\rho_0^\delta \theta_0) + \frac{1}{c} \int_0^\infty \int_{S^2} A^0_{\rho,\delta} \Omega d\Omega dv = (\rho_0^\delta)^{\frac{1}{2}} g_1^\delta,
\]

\[
- \frac{1}{c_v} (\kappa \Delta \theta_0 + Q(u_0)) - \int_0^\infty \int_{S^2} \frac{1}{c_v} \left(1 - \frac{u_0 \cdot \Omega}{c}\right) A^0_{\rho,\delta} d\Omega dv = (\rho_0^\delta)^{\frac{1}{2}} g_2^\delta,
\]

where

\[
g_1^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_2 + R\delta \frac{\nabla \theta_0}{(\rho_0^\delta)^{\frac{1}{2}}} - \frac{1}{c} \int_0^\infty \int_{S^2} \frac{(A^0_{\rho,\delta} - A^0_{\rho,\delta^\delta})}{(\rho_0^\delta)^{\frac{1}{2}}} \Omega d\Omega dv,
\]

\[
g_2^\delta = \left(\frac{\rho_0}{\rho_0^\delta}\right)^{\frac{1}{2}} g_1 + \int_0^\infty \int_{S^2} \frac{1}{c_v} \left(1 - \frac{u_0 \cdot \Omega}{c}\right) \frac{(A^0_{\rho,\delta} - A^0_{\rho,\delta^\delta})}{(\rho_0^\delta)^{\frac{1}{2}}} d\Omega dv.
\]

Then according to the assumption \([1,12]\), for all \( \delta > 0 \) small enough,

\[
1 + \|\rho_0^\delta\|_{W^{1,q}} + \|\theta_0, u_0\|_2 + \|(g_1^\delta, g_2^\delta)\|_2 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; W^{1,q})}
\]

\[
+ \|S(v, t, x)\|_{L^2(\mathbb{R}^+; C^1([0,T]\); W^{1,q}) \cap C^1([0,T]; L^1(\mathbb{R}^+; L^2))} \leq c_0.
\]

Therefore, corresponding to the initial data \((I_0, \rho_0^\delta, \theta_0, u_0)\), there exists a unique strong solution \((I^\delta, \rho^\delta, u^\delta, \theta^\delta)\) satisfying \([4,43]\). Then there exists a subsequence of solutions \((I^\delta, \rho^\delta, u^\delta, \theta^\delta)\) converges to a limit \((I, \rho, u, \theta)\) in weak or weak* sense. Due to the compact property in \([29]\), there exists a subsequence of solutions \((I^\delta, \rho^\delta, u^\delta, \theta^\delta)\) satisfying:

\[
I^\delta \to I \text{ weakly in } L^2(\mathbb{R}^+ \times S^2 \times [0, T_s] \times \mathcal{V}),
\]

\[
\rho^\delta \to \rho \text{ in } C([0, T_s]; L^2(K)), (u^\delta, \theta^\delta) \to (u, \theta) \text{ in } C([0, T_s]; H^1(K)),
\]

where \(K\) is any compact subset of \(\mathcal{V}\). Combining the lower semi-continuity of norms and \([4,44]\), we know that \((I, \rho, u, \theta)\) also satisfies the local estimates \([4,43]\). For arbitrary \(\varphi \in C_c^\infty(\mathbb{R}^+ \times S^2 \times [0, T_s] \times \mathcal{V})\), from \([4,43]\), \([4,44]\) and \([1,12]\), we easily have

\[
\int_0^{T_s} \int_1^2 \left((\sigma_a(\rho^\delta, \phi)I^\delta - \sigma_a(\rho, \phi)I) + \frac{u}{v} \left(\sigma_a(\rho^\delta)I^\delta - \sigma_a(\rho)I\right)\right) \varphi dt d\Omega \to 0, \text{ as } \delta \to 0,
\]

where \(\sigma = \sigma_a + \int_0^\infty \int_{S^2} \sigma_a d\Omega dv\), so it is easy to show that \((I, \rho, u, \theta)\) is a weak solution in distribution sense and satisfies

\[
I \in L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T_s]; W^{1,q})), I_t \in L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T_s]; L^q)),
\]

\[
\rho \in L^\infty([0, T_s]; W^{1,q}), \quad \rho_t \in L^2([0, T_s]; L^\infty([0, T_s]; L^q)),
\]

\[
(\theta, u) \in L^\infty([0, T_s]; H^2) \cap L^2([0, T_s]; D^2,q),
\]

\[
(\theta_t, u_t) \in L^2([0, T_s]; H^1), \quad (\sqrt{\rho} \theta_t, \sqrt{\rho} u_t) \in L^\infty([0, T_s]; L^2).
\]

Step 2: Uniqueness can be obtained by the same method used in Lemma \([4,41]\).

Step 3: The time-continuity. The continuity of \(\rho\) can be obtained in Lemma 6 of \([7]\). For \(I\), due to \([4,45]\), for \(\forall (v, \Omega) \in R^+ \times S^2\), we have

\[
I(v, \Omega, \cdot, \cdot) \in C([0, T_s]; L^q) \cap C([0, T_s]; W^{1,q} - \text{weak}).
\]
Then taking a small time \(0 < \varepsilon\leq 1\), we have

\[
\limsup_{t \to 0} \|I(v, \Omega, \cdot, \cdot)\|_{W^{1,q}}^2 \leq \|I_0\|_{W^{1,q}}^2,
\]

which implies that \(I(v, \Omega, t, x)\) is right-continuous at \(t = 0\) (see [91]), we easily get the conclusion that we need. Similarly, from (4.45), we have

\[
(u, \theta) \in C([0, T_*]; H^1) \cap C([0, T_*]; D^2 - \text{weak}).
\]

From equations (4.1) and Lemmas 4.2-4.5, we know that \((\rho t, \rho u_t) \in L^2([0, T_*]; L^2)\), and \((\rho t, \rho u_t) \in L^2([0, T_*]; H^{-1})\), via Aubin-Lions lemma, we have \((\rho t, \rho u_t) \in C([0, T_*]; L^2)\). Due to (4.32) and

\[
Lu = -\rho u_t - \rho(w \cdot \nabla)u - \nabla P_m = -\frac{1}{c} \int_0^\infty \int_{S^2} \Omega d\Omega dv,
\]

and the elliptic regularity estimate in Lemma [3.3] we have \((u, \theta) \in C([0, T_*]; D^2)\). \(\square\)

### 4.3. Proof of Theorem 2.1

Our proof is based on the classical iteration scheme and the existence results for the linearized problem in Section 4.2. Let us denote as in Section 4.2 that

\[
2 + \|I_0\|_{L^2(\mathbb{R}^+ \times S^2; W^{1,q})} + \|\rho_0\|_{W^{1,q}} + \| (\theta_0, u_0)\|_2 + \| (g_1, g_2)\|_2
\]

\[
+ \| S(v, t, x)\|_{L^2(\mathbb{R}^+ \times C([0,T]; W^{1,q}))} \leq c_0.
\]

Next, let \(u^0 \in C([0, T]; H^2) \cap L^2([0, T]; D^2 q), \quad \theta^0 \in C([0, T]; H^2) \cap L^2([0, T]; D^2 q)\) and \(I^0 \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; W^{1,q}))\) be the solutions to the following linear problems

\[
u_t^0 - \Delta u^0 = 0; \quad u^0(0) = u_0 \text{ in } \mathbb{V}; \quad (u \cdot n, (\nabla \times u) \cdot n)|_{\partial \Omega} = (0, 0),
\]

\[
\theta_t^0 - \Delta \theta^0 = 0; \quad \theta^0(0) = \theta_0 \text{ in } \mathbb{V}; \quad \nabla \theta \cdot n|_{\partial \Omega} = 0.
\]

\[
I^0_t + c\Omega \cdot \nabla I^0 = 0; \quad I^0(0) = I_0 \text{ in } \mathbb{R}^+ \times S^2 \times \mathbb{V}; \quad I^0|_{\partial \Omega} = 0, \quad \Omega \cdot n \leq 0.
\]

Then taking a small time \(T_1 \in (0, T)\), we have

\[
\|I^0\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_1]; W^{1,q}))} + \|I^0\|_{L^2(\mathbb{R}^+ \times S^2; C([0, T_1]; L^q))}
\]

\[
+ \sup_{0 \leq t \leq T_1} \| (\theta^0, u^0)\|_2 + \int_0^{T_1} \left( \| (\theta^0, u^0)\|_{L^2}^2 + \| (\theta^0, u^0)\|_1^2 \right) dt \leq c_1.
\]

**Proof.** Step 1. Existence. Let \((w, \phi, \psi) = (u^0, \theta^0, I^0)\), we can get \((I^1, \rho^1, u^1, \theta^1)\) as a strong solution to (4.1)-(4.3). Then we construct approximate solutions \((I^{k+1}, \rho^{k+1}, u^{k+1}, \theta^{k+1})\) inductively as follows: assuming \((I^k, u^k, \theta^k)\) was defined for \(k \geq 1\), let \((I^{k+1}, \rho^{k+1}, u^{k+1}, \theta^{k+1})\)
be the solution to (4.1)-(4.3) with \((\psi, w, \phi)\) replaced by \((I^k, u^k, \theta^k)\) as following:

\[
\begin{aligned}
\rho_t^{k+1} + \text{div}(\rho^{k+1}u^k) &= 0, \\
\frac{1}{c} I_t^{k+1} + \Omega \cdot \nabla I^{k+1} &= \overline{A}_r^k, \\
(\rho^{k+1} I^{k+1})_t + \text{div}(\rho^{k+1} \theta^{k+1} u^k) + \frac{1}{c_v} (P_m^{k+1} \text{div}u^k - k \Delta \theta^{k+1}) &= \frac{1}{c_v} (Q(\nabla u^k) + \overline{N}_r^k), \\
(\rho^{k+1} u^{k+1})_t + \text{div}(\rho^{k+1} u^k \otimes u^{k+1}) + \nabla P_m^{k+1} + Lu^{k+1} &= -\frac{1}{c} \int_0^\infty \int_{S^2} \overline{C}_r^k \Omega d\Omega dv,
\end{aligned}
\]

(4.48)

where

\[
\begin{aligned}
\overline{A}_r^k &= S - \sigma_a^{k+1,k} I^{k+1} + \int_0^\infty \int_{S^2} \left( \frac{v}{u} \sigma_s^{k+1} I^k - (\sigma_s')^{k+1} I^{k+1} \right) d\Omega' dv', \\
\overline{B}_r^k &= S - \sigma_a^{k+1,k} I^{k+1} + \int_0^\infty \int_{S^2} \left( \frac{v}{u} \sigma_s^{k+1} I^{k+1} - (\sigma_s')^{k+1} I^{k+1} \right) d\Omega' dv', \\
\overline{C}_r^k &= S - \sigma_a^{k+1,k} I^{k+1} + \int_0^\infty \int_{S^2} \left( \frac{v}{u} \sigma_s^{k+1} I^{k+1} - (\sigma_s')^{k+1} I^{k+1} \right) d\Omega' dv', \\
P_m^{k+1} &= R \rho^{k+1} \theta^{k+1}, \quad N_r^k = \int_0^\infty \int_{S^2} \left( 1 - \frac{u_k}{c} \right) \overline{B}_r^k d\Omega dv,
\end{aligned}
\]

(4.49)

\[
\begin{aligned}
\sigma_a^{k+1} &= \sigma(v, \Omega, t, x, \rho^{k+1}, \theta^k) \rho^{k+1} = \sigma^{k+1,k} \rho^{k+1}, \quad \sigma_s^{k+1} = \overline{S} \rho^{k+1}, \\
\sigma_a^{k+1} &= \sigma(v, \Omega, t, x, \rho^{k+1}, \theta^{k+1}) \rho^{k+1} = \sigma^{k+1} \rho^{k+1}, \quad \sigma_s^{k+1} = \overline{S} \rho^{k+1},
\end{aligned}
\]

where \((I^{k+1}, \rho^{k+1}, u^{k+1}, \theta^{k+1})|_{t=0} = (I_0, \rho_0, u_0, \theta_0)\). Via Section 4.2, we know \((I^k, \rho^k, u^k, \theta^k)\) also satisfy the estimate (4.48). Next, we show that \((I^k, \rho^k, u^k, \theta^k)\) converges to a limit in a strong sense which will be used to prove the existence of the strong solution, let

\[
I^{k+1} = I^{k+1} - I^k, \quad \rho^{k+1} = \rho^{k+1} - \rho^k, \quad u^{k+1} = u^{k+1} - u^k, \quad \theta^{k+1} = \theta^{k+1} - \theta^k.
\]

Then we have

\[
\begin{aligned}
\frac{1}{c} \rho_t^{k+1} + \text{div}(\rho^{k+1} u^k) + \text{div}(\rho^{k+1} u^k) &= 0, \\
\frac{1}{c} I_t^{k+1} + \Omega \cdot \nabla I^{k+1} &= \overline{A}_r^k, \\
(\rho^{k+1} I^{k+1})_t + \text{div}(\rho^{k+1} \theta^{k+1} u^k) + \frac{1}{c_v} (P_m^{k+1} \text{div}u^k - k \Delta \theta^{k+1}) &= \frac{1}{c_v} (Q(\nabla u^k) + \overline{N}_r^k), \\
(\rho^{k+1} u^{k+1})_t + \text{div}(\rho^{k+1} u^k \otimes u^{k+1}) + \nabla P_m^{k+1} + Lu^{k+1} &= -\frac{1}{c} \int_0^\infty \int_{S^2} \overline{C}_r^k \Omega d\Omega dv,
\end{aligned}
\]

(4.49)
where $L_1$ and $L_2$ are given via

$$L_1 = \int_0^\infty \int_{S^2} \left( 1 - \frac{u_k^L \cdot \Omega}{c} \right) \left( -\sigma_a^{k+1,k} T^{k+1} - I^k (\sigma_a^{k+1,k} - \sigma_a^{k,k-1}) \right) d\Omega dv$$

$$+ \int_0^\infty \int_{S^2} \left( 1 - \frac{u_k^L \cdot \Omega}{c} \right) \left( I^k (\sigma_s^{k+1} - \sigma_s^k) + \sigma_s^{k+1} T^{k+1} \right) d\Omega dv$$

$$- \int_0^\infty \int_{S^2} \left( 1 - \frac{u_k^L \cdot \Omega}{c} \right) \left( I^k ((\sigma_s')^{k+1} - (\sigma_s')^k) + (\sigma_s')^{k+1} T^{k+1} \right) d\Omega dv$$

$$- \int_0^\infty \int_{S^2} \left( 1 - \frac{u_k^L \cdot \Omega}{c} \right) \left( S - \sigma_a^{k,k-1} I^k + \int_0^\infty \int_{S^2} \left( \frac{v}{\nu} \sigma_s^{k+1} I^k - (\sigma_s')^k I^k \right) d\Omega dv' \right) d\Omega dv,$$

$$L_2 = \int_0^\infty \int_{S^2} \left( -\sigma_a^{k+1} T^{k+1} - I^k (\sigma_a^{k+1} - \sigma_a^k) \right) d\Omega dv$$

$$+ \int_0^\infty \int_{S^2} \left( \sigma_s^{k+1} T^{k+1} + I^k (\sigma_s^{k+1} - \sigma_s^k) \right) d\Omega dv - \int_1^T \left( I^k ((\sigma_s')^{k+1} - (\sigma_s')^k) + (\sigma_s')^{k+1} T^{k+1} \right) d\Omega dv.$$
Then considering $\theta$. Multiplying (4.12), by $\overline{\theta}^{k+1}$ and integrating over $\mathcal{V}$, we have
\[
\frac{1}{2} \frac{d}{dt} |\sqrt{\rho^{k+1}}\overline{\theta}^{k+1}|_2^2 + \frac{\kappa}{c_v} |\nabla \overline{\theta}^{k+1}|_2^2 = \int_{\mathcal{V}} \left( \frac{1}{c_v} (Q(\nabla u^k) - Q(\nabla u^{k-1}))\overline{\theta}^{k+1} - \overline{\rho}^{k+1} \theta^{k+1} \overline{\theta}^{k+1} - \overline{\rho}^{k+1} \theta^{k+1} \cdot \nabla \theta^{k+1} \right) dx
\]
\[
+ \int_{\mathcal{V}} \left( - \frac{1}{c_v} R_{\rho^{k+1}} \theta^{k+1} \div u^{k-1} \theta^{k+1} - \frac{1}{c_v} R_{\theta^{k+1}} \theta^{k+1} \rho^{k+1} \theta^{k+1} \div u^k - \rho^{k+1} \overline{\rho}^{k+1} \cdot \nabla \theta^{k+1} \right) dx
\]
\[
+ \int_{\mathcal{V}} \left( - \frac{1}{c_v} R_{\rho^{k+1}} \theta^{k+1} \div u^{k-1} \overline{\theta}^{k+1} + L_1 \overline{\theta}^{k+1} \right) dx =: \sum_{i=1}^{26} I_i.
\]
According to Gagliardo-Nirenberg inequality and Minkowski’s inequality we have
\[
I_{11} \leq C |\nabla u^{k+1}|_6 (|\nabla u^k|_3 + |\nabla u^{k-1}|_3), I_{12} \leq C |\overline{\rho}^{k+1}|_2 |\overline{\theta}^{k+1}|_6, I_{13} + I_{14} \leq C |\overline{\rho}^{k+1}|_2 |\overline{\theta}^{k+1}|_6 \|\nabla \theta^{k+1}\|_1 \|\nabla u^{k-1}\|_1, I_{15} \leq C |\overline{\rho}^{k+1}|_2 |\rho^{k+1}|_6 \|\nabla \theta^{k+1}\|_1, I_{16} + I_{17} \leq C |\overline{\rho}^{k+1}|_2 |\rho^{k+1}|_6 \|\nabla \theta^{k+1}\|_1,
\]
and from the assumption (4.12) and the definition of $L_1$,
\[
I_{18} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \int_{S^2} \left( 1 - \frac{u^k \cdot \Omega}{c} \right) \overline{\theta}^{k+1} \left( - \sigma_a^{k+1} + \sigma_s^{k+1} \right) d\Omega dv dx
\]
\[
\leq C \left( 1 + \|\nabla u^k\|_1 \right) \|\sqrt{\rho^{k+1}} \overline{\theta}^{k+1}\|_6 \|\rho^{k+1}\|_6 \|\theta^{k+1}\|_6 \|\nabla \theta^{k+1}\|_6 \|\nabla u^{k-1}\|_6 \|\div u^k\|_6 \|\nabla \theta^{k+1}\|_6 \|\nabla \theta^{k+1}\|_6,
\]
\[
I_{19} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \int_{S^2} \left( 1 - \frac{u^k \cdot \Omega}{c} \right) \overline{\theta}^{k+1} \left( - I^k (\sigma_a^{k+1} - \sigma_s^{k+1}) \right) d\Omega dv dx
\]
\[
\leq C \left( 1 + \|\nabla u^k\|_1 \right) \|\sqrt{\rho^{k+1}} \overline{\theta}^{k+1}\|_6 \|\rho^{k+1}\|_6 \|\theta^{k+1}\|_6 \|\nabla \theta^{k+1}\|_6 \|\nabla \theta^{k+1}\|_6 \|\nabla u^{k-1}\|_6 \|\div u^k\|_6 \|\nabla \theta^{k+1}\|_6 \|\nabla \theta^{k+1}\|_6,
\]
\[
I_{20} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \int_{S^2} \left( 1 - \frac{u^k \cdot \Omega}{c} \right) \overline{\theta}^{k+1} I^k (\sigma_s^{k+1} - \sigma_s^{k+1}) d\Omega dv dx
\]
\[
\leq C \alpha_1 \left( 1 + \|\nabla u^k\|_1 \right) \|\overline{\theta}^{k+1}\|_6 \|\rho^{k+1}\|_6 \|I^k\|_L^2(\mathbb{R}^+ \times S^2; H^1),
\]
\[
I_{21} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \int_{S^2} \left( 1 - \frac{u^k \cdot \Omega}{c} \right) \overline{\sigma_s^{k+1}} \overline{T}^{k+1} d\Omega dv dx
\]
\[
\leq C \alpha_1 \left( 1 + \|\nabla u^k\|_1 \right) \|\overline{\theta}^{k+1}\|_6 \|\rho^{k+1}\|_6 \|I^k\|_L^2(\mathbb{R}^+ \times S^2; L^2),
\]
\[
I_{22} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \left( 1 - \frac{u^k \cdot \Omega}{c} \right) I^k (\sigma_s^{k+1} - \sigma_s^{k+1}) d\Omega dv dx
\]
\[
\leq C \alpha_2 \left( 1 + \|\nabla u^k\|_1 \right) \|\overline{\theta}^{k+1}\|_6 \|\rho^{k+1}\|_6 \|I^k\|_L^2(\mathbb{R}^+ \times S^2; H^1),
\]
\[
I_{23} = \frac{1}{c_v} \int_{\mathcal{V}} \int_0^\infty \int_{S^2} \left( - \frac{\rho^k \cdot \Omega}{c} \right) S \overline{\theta}^{k+1} d\Omega dv dx \leq C \left|\overline{\theta}^{k+1}\|_6 \|\nabla u^k\|_2 \|S\|_{L^1(\mathbb{R}^+; \mathcal{F}(\mathcal{V}))}^2,
\]
where in $I_{19}$ we have used the fact \[4.52\], and similarly we have

\[
I_{24} = \frac{1}{c_v} \int_{t_0}^t \int_{\mathbb{V}} \int_{S^2} \frac{\theta^{k+1}}{\theta} \left( -\frac{\theta^k \cdot \Omega}{c} \right) \left( -\sigma_{a}^{k,k-1} I^k \right) d\Omega dv dx
\]

\[\leq C \left| \theta^{k+1} \right|_6 \left| \sqrt{\rho^k \cdot v} \right|_2 \left| \theta^k \right|_{1/2} \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)} \left| \sigma^{k+1,k} \right|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)},\]

\[
I_{25} = \frac{1}{c_v} \int_{t_0}^t \int_{\mathbb{V}} \int_{S^2} \frac{\theta^{k+1}}{\theta} \left( -\frac{\theta^k \cdot \Omega}{c} \right) \left( \int_{S^2} \int_{S^2} v \sigma_{a}^{k,k} I^k d\Omega' dv' \right) d\Omega dv dx
\]

\[\leq C \alpha_1 \left| \rho^k \right|_\infty \left| \theta^{k+1} \right|_6 \left| \sqrt{\rho^k \cdot v} \right|_2 \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)},\]

\[
I_{26} = \frac{1}{c_v} \int_{t_0}^t \int_{\mathbb{V}} \int_{S^2} \left( -\frac{\theta^k \cdot \Omega}{c} \right) \theta^{k+1} \left( \int_{S^2} \left( \sigma_{a}^{j,k} \right) I^k d\Omega' dv' \right) d\Omega dv dx
\]

\[\leq C \alpha_2 \left| \rho^k \right|_\infty \left| \theta^{k+1} \right|_6 \left| \sqrt{\rho^k \cdot v} \right|_2 \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)}.\]

Then combining the estimates for $I_i$ ($i = 11, \ldots, 26$), Young’s inequality and Lemma 3.2, we have

\[
\begin{align*}
\frac{d}{dt} \left| \sqrt{\rho^k + \theta^{k+1}} \theta \right|_2^2 + |\nabla \theta^{k+1}|_2^2 &\leq E_1^k (t) \left| \sqrt{\rho^k + \theta^{k+1}} \theta \right|_2^2 + E_2^k (t) \left| \sqrt{\rho^k + \theta^{k+1}} \theta \right|_2^2 \\
&+ E_3^k (t) \left( T^{k+2} \right)^2_{L^2(\mathbb{R}^+ \times S^2; L^2(\mathbb{V}))} + C \left( |\nabla u^{k+1}|_2^2 + |\sqrt{\rho^k + \theta^{k+1}} \theta |_2^2 \right) + \eta \left( |\nabla \theta^{k+1}|_2^2 + |\sqrt{\rho^k + \theta^{k+1}} \theta |_2^2 \right),
\end{align*}
\]

(4.53)

where

\[
E_1^k (t) = C + C \left| \rho^{k+1} \right|_1 \left( |\nabla \theta^{k+1}|_2^2 + |\nabla u^k|_1^2 \right)
\]

\[+ 1/\eta \left( 1 + \left| \nabla u^k \right|_2^2 \right) \left| \rho^{k+1} \right|_1 \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; W^{1,q})} \left| \nabla \theta^{k+1} \right|_2^2 \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)},\]

\[
E_2^k (t) = C \left( |\nabla \theta^{k+1}|_2^2 + |\nabla u^k|_2^2 + \alpha_1 + \alpha_2 \right) \left( 1 + \left| \nabla u^k \right|_2^2 \right) \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)} \left| \sigma^{k+1,k} \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)}
\]

\[+ C \left( 1 + \left| \nabla u^k \right|_2^2 \right) \left| I^k \right|_{L^2(\mathbb{R}^+ \times S^2; H^1)} \left| \sigma^{k+1,k} \right|_{L^2(\mathbb{R}^+ \times S^2; L^\infty)} + C \left| \theta^k \right|_3^2;\]

\[
E_3^k (t) = C \left( 1 + \left| \nabla u^k \right|_2^2 \right) \left| \rho^{k+1} \right|_1 \left( |\sigma^{k+1,k} \theta^k|_2^2 + \alpha_1^2 + \alpha_2 \right),
\]

and we have $\int_0^t E_1^k (s) \leq \hat{C} + \hat{C} t$ and $\int_0^t (E_2^k (s) + E_3^k (s)) ds \leq \hat{C} + \hat{C} t$, for $t \in [0, T_1]$. Finally, multiplying (4.49) by $\bar{u}^{k+1}$ and integrating over $\mathbb{V}$, we have

\[
\frac{1}{2} \frac{d}{dt} \left| \sqrt{\rho^k + \theta^{k+1}} \theta \right|_2^2 + \mu \left| \nabla \theta \right|_2^2 + (\lambda + \mu) \left| \div \bar{u}^k \right|_2^2
\]

\[
= \int_{\mathbb{V}} \left( -\bar{u}^k \cdot \nabla \bar{u}^k \right) \cdot \bar{u}^{k+1} - \bar{u}^k \cdot \left( \rho^k \cdot \nabla u^k \right) \cdot \bar{u}^{k+1} - \rho^k \cdot \left( \bar{u}^k \cdot \nabla u^k \right) \cdot \bar{u}^{k+1}
\]

\[- R \nabla \left( \rho^k \cdot \theta^{k+1} \right) \cdot \bar{u}^{k+1} - R \nabla \left( \rho^k \cdot \theta^k \right) \cdot \bar{u}^{k+1} + L_2 \bar{u}^{k+1} \right) dx \equiv \sum_{i=27}^{37} I_i.
\]

According to the Gagliardo-Nirenberg inequality, Minkowski’s inequality and Hőlder’s inequality, we easily have

\[
I_{27} \leq C \left| \bar{u}^{k+1} \right|_2 \left| \theta^k \right|_3 \left| \bar{u}^{k+1} \right|_6, \quad I_{28} \leq C \left| \bar{u}^{k+1} \right|_2 \left| \nabla \bar{u}^{k+1} \right|_2 \left| \nabla u^k \right|_1 \left| \nabla u^{k-1} \right|_1,
\]

\[
I_{29} \leq C \left| \rho^k \right|_\infty \left| \sqrt{\rho^k + \theta^{k+1}} \theta \right|_2 \left| \nabla u^k \right|_1 \left| \nabla \bar{u}^k \right|_2,
\]
\[
I_{30} \leq C |\rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2}, \quad I_{31} \leq C |\bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2} |\sigma |_{\infty}, \\
I_{32} = - \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \Omega \cdot \bar{\nu}^{k+1} \left( - \sigma^{k+1} \bar{T}^{k+1} \right) d\Omega d\nu dx \\
\leq C |\rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2}^{2} |\sigma^{k+1} |_{L^2(\mathbb{R}^+ \times S^2; L^2(\nu))} \| \sigma^{k+1} |_{L^2(\mathbb{R}^+ \times S^2; L^\infty)}^2, \\
I_{33} = - \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \Omega \cdot \bar{\nu}^{k+1} \left( - I^{k} \left( \sigma^{k+1} - \sigma^{k} \right) \right) d\Omega d\nu dx \\
\leq C |\rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; L^2(\nu))} \| \bar{T}^{k+1} |_{L^2(\mathbb{R}^+ \times S^2; L^\infty)}^2 \\
+ C |\rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; W^{1,q}(\nu))} \| \bar{T}^{k+1} |_{L^2(\mathbb{R}^+ \times S^2; L^\infty)}^2 \\
+ C |\rho^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; H^1(\nu))} \| \sigma^{k} |_{L^2(\mathbb{R}^+ \times S^2; L^\infty)}^2,
\]

where in \(I_{33}\) we have used the fact
\[
\sigma^{k+1} - \sigma^{k} = \rho^{k+1} \sigma^{k+1} - \rho^{k} \sigma^{k} = \rho^{k+1} \left( \left| \sigma^{k+1} - \sigma^{k+1} \right| + \left| \sigma^{k+1} - \sigma^{k} \right| \right) + \bar{\nu}^{k+1} \sigma^{k+1},
\]
and similarly, we have
\[
I_{34} = - \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \frac{v \cdot \Omega \cdot \bar{\nu}^{k+1} \sigma^{k+1} \bar{T}^{k+1} d\Omega d\nu dx \\
\leq C \alpha_1 |\rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; L^2(\nu))}, \\
I_{35} = - \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \frac{v \cdot \Omega \cdot \bar{\nu}^{k+1} I^{k} \left( \sigma^{k+1} - \sigma^{k} \right) d\Omega d\nu dx \\
\leq C \alpha_1 |\nabla \bar{\nu}^{k+1} |_{2} |\rho^{k+1} \frac{1}{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; H^1(\nu))}, \\
I_{36} = \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \bar{\nu}^{k+1} I^{k} \left( \left| \sigma^{k+1} - \sigma^{k} \right| \right) d\Omega d\nu dx \\
\leq C \alpha_2 \left| \nabla \bar{\nu}^{k+1} \right|_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; H^1)}, \\
I_{37} = \frac{1}{c} \int_{0}^{\infty} \int_{S^2} \left( \sigma^{k+1} \right)^{k+1} \bar{T}^{k+1} d\Omega d\nu dx \\
\leq C \alpha_2 \left| \nabla \bar{\nu}^{k+1} \right|_{2} |\nabla \bar{\nu}^{k+1} |_{2} |I^{k} |_{L^2(\mathbb{R}^+ \times S^2; L^2)}.
\]

Then via Lemma 3.6 and the above estimates for \(I_i \ (i = 27, \ldots, 37)\), we have
\[
\frac{d}{dt} \left| \rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} + |\nabla \bar{\nu}^{k+1} |_{2} \leq F_{\eta}(t) \left| \rho^{k+1} \frac{1}{2} \rho \bar{\nu}^{k+1} |_{2} + F_{\eta}(t) \right| \bar{\nu}^{k+1} |_{2} \\
+ F_{3}(t) \left| \nabla \bar{\nu}^{k+1} |_{2} \bar{T}^{k+1} |_{L^2(\mathbb{R}^+ \times S^2; L^2)} + F_{4}(t) \bar{\nu}^{k+1} \left| \nabla \bar{\nu}^{k+1} |_{2} \right|^{2} + \eta |\nabla \bar{\nu}^{k+1} |_{2} \right|,
\]
and also \( \int_0^t \left( F_0^k(s) + F_2^k(s) + F_3^k(s) + F_4^k(s) \right) ds \leq \hat{C} + \hat{C}_\eta t \), for \( t \in [0, T_1] \).

Next, we denote that
\[
\Lambda^{k+1}(T_1, \epsilon) = \sup_{0 \leq t \leq T_1} \left\| \tilde{T}^{k+1}(t) \right\|_{L^2(\mathbb{R}^+ \times S^2)}^2 + \sup_{0 \leq t \leq T_1} \left| \rho^{k+1}(t) \right|_2^2 + \epsilon \sup_{0 \leq t \leq T_1} \left| \sqrt{\rho^{k+1}} \tilde{\theta}^{k+1}(t) \right|_2^2 + \eta \sup_{0 \leq t \leq T_1} \left| \sqrt{\rho^{k+1}} \tilde{\nu}^{k+1}(t) \right|_2^2,
\]
then from (4.50)-(4.51), we have
\[
\Lambda^{k+1}(T_1, \epsilon) + \int_0^{T_1} \left( \epsilon \left| \nabla \tilde{\theta}^{k+1} \right|_2^2 + \left| \nabla \tilde{\nu}^{k+1} \right|_2^2 \right) dt \leq \int_0^{T_1} G_\epsilon,\eta \Lambda^{k+1}(t, \epsilon) dt + \int_0^{T_1} \hat{C} \left( (\eta + \epsilon) |\nabla \tilde{\theta}|_2^2 + (\epsilon + \eta) |\nabla \tilde{\nu}|_2^2 \right) dt + \eta \int_0^{T_1} \sup_{0 \leq t \leq T_1} \left| \sqrt{\rho^{k+1}} \tilde{\nu}^{k+1}(t) \right|_2^2 + \eta \int_0^{T_1} \sup_{0 \leq t \leq T_1} \left| \tilde{T}^{k}(t) \right|_{L^2(\mathbb{R}^+ \times S^2)}^2 \right) \exp \left( (1 + \epsilon)(\hat{C} + \hat{C}_\eta t) \right).
\]

Due to \( 0 < T_1 \leq 1 \), firstly, we can choose \( 0 < \epsilon = \epsilon_0 < 1 \) small enough such that
\[
(1 + \hat{C})\epsilon_0 \exp \left( (1 + \epsilon_0)\hat{C} \right) \leq \frac{1}{8},
\]
secondly, we can choose \( \eta = \eta_0 \) small enough such that
\[
(\hat{C} + 1)(\epsilon_0 \eta_0 + \eta_0) \exp \left( (1 + \epsilon_0)\hat{C} \right) \leq \frac{1}{8},
\]
then finally, we can choose \( T_1 = T_2 \) small enough such that
\[
\exp \left( (1 + \epsilon_0)\hat{C}_\eta T_2 \right) \leq 2.
\]

So, when \( \Lambda^{k+1} = \Lambda^{k+1}(T_1, \epsilon_0, \eta_0) \), we have
\[
\sum_{k=1}^{\infty} \left( \Lambda^{k+1} + \int_0^{T_2} \left( \left| \nabla \tilde{T}^{k+1} \right|_2^2 + \left| \nabla \tilde{\nu}^{k+1} \right|_2^2 \right) dt \right) \leq \hat{C} < +\infty.
\]

So we know that the full consequence \( \left( T^k, \rho^k, u^k, \theta^k \right) \) converges to a limit \( (I, \rho, u, \theta) \) in the following strong sense:
\[
I^k \rightarrow I \text{ in } L^\infty([0, T_2]; L^2(\mathbb{R}^+ \times S^2; L^2(\mathbb{V}))), \quad \rho^k \rightarrow \rho \text{ in } L^\infty([0, T_2]; L^2(\mathbb{V})), \quad (\theta^k, u^k) \rightarrow (\theta, u) \text{ in } L^2([0, T_2]; D^1(\mathbb{V})).
\]
Due to the local uniform estimate (4.43) and the strong convergence in (4.55), we obtain

\[
\text{ess sup}_{t \in [0, T_\ast]} (\|\rho(t)\|_{W^{1,q}} + |\rho_\ast(t)|_q) + \text{ess sup}_{t \in [0, T_\ast]} \|\theta(t), u(t)\|_2
\]

\[
+ \|I\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T_\ast]; W^{1,q}))} + \|I_t\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T_\ast]; L^q))}
\]

\[
+ \int_0^{T_\ast} (\|\theta(t), u(t)\|_2^2) ds + \text{ess sup}_{t \in [0, T_\ast]} \|\sqrt{\rho\theta}(t), \sqrt{\rho u}(t)\|_2 \leq C.
\]

Then it is easy to see that \((I, \rho, u, \theta)\) is a weak solution in distribution sense. So we have given the existence of the strong solution.

Step 2. The uniqueness. Let \((I_1, \rho_1, u_1, \theta_1)\) and \((I_2, \rho_2, u_2, \theta_2)\) be two strong solutions to IBVP (1.7)-(1.8) with (1.10) satisfying the regularity (4.56). We denote that

\[
\mathcal{T} = I_1 - I_2, \quad \mathcal{P} = \rho_1 - \rho_2, \quad \mathcal{U} = u_1 - u_2, \quad \mathcal{Q} = \theta_1 - \theta_2.
\]

Via the same method to the derivations of (4.50)-(4.54), let

\[
\Lambda(t) = \|\mathcal{T}\|_{L^2(\mathbb{R}^+ \times S^2; L^2(V))} + \|\mathcal{P}\|_2^2 + \|\sqrt{\mathcal{P}} \mathcal{Q}\|_2^2 + \|\sqrt{\mathcal{P}} \mathcal{Q}\|_2^2,
\]

we similarly have

\[
\frac{d}{dt} \Lambda(t) + |\nabla \mathcal{P}|_2^2 + |\nabla \mathcal{Q}|_2^2 \leq H(t) \Lambda(t),
\]

where \(\int_0^t H(s) ds \leq \tilde{C}\), for \(t \in [0, T_\ast]\). Then from the Gronwall’s inequality and \(u \cdot n_{|\partial V} = 0\), we conclude that

\[
\mathcal{T} = \mathcal{P} = \mathcal{U} = \mathcal{Q} = 0,
\]

then the uniqueness is obtained.

Step 3. Time-continuity can be obtained by the same method used in Lemma 4.1. \(\square\)

5. necessity and sufficiency of the compatibility condition

In this section, we prove Theorem 2.2. Due to the strong solution \((I, \rho, u, \theta)\) only satisfies our problem in distribution sense, then we only have

\[
I(\nu, \Omega, t = 0, x) = I_0, \quad \rho(t = 0, x) = \rho_0, \quad x \in V,
\]

\[
\rho u(t = 0, x) = \rho_0 u_0, \quad \rho \theta(t = 0, x) = \rho_0 \theta_0, \quad x \in V.
\]

So, the key point of the proof is to make sure that the relations \(u(t = 0, x) = u_0\) and \(\theta(t = 0, x) = \theta_0\) hold in the vacuum domain.

Proof. Step 1. Necessity. Let \((I, \rho, u, \theta)\) be a strong solution to (1.7)-(1.8) with (1.9) or (1.10) and the regularity shown in Definition 2.1. Then due to (1.7), we have

\[
Lu(t) + \nabla P_m(t) + \frac{1}{c} \int_0^\infty \int_{S^2} A_r(t) \Omega d\Omega dv = \sqrt{\rho(t)} G_1(t),
\]

\[
- \frac{1}{c_v} (\kappa \Box \theta + Q(u)) - \int_0^\infty \int_{S^2} \frac{1}{c_v} \left(1 - \frac{u \cdot \Omega}{c}\right) A_r d\Omega dv = \sqrt{\rho(t)} G_2(t),
\]

for \(0 \leq t \leq T_\ast\), where

\[
G_1(t) = \sqrt{\rho}(-u_\| - u \cdot \nabla u), \quad G_2(t) = \sqrt{\rho}(-\theta_\| - u \cdot \nabla \theta - R \theta \text{div} u).
\]

Since

\[
(\sqrt{\rho} u_t, \sqrt{\rho} \theta_t, \sqrt{\rho} u \cdot \nabla u, \sqrt{\rho} u \cdot \nabla \theta, \sqrt{\rho} \theta \text{div} u) \in L^\infty([0, T]; L^2),
\]
we have \((G^1, G^2) \in L^\infty([0, T]; L^2)\). So there exists a sequence \(\{t_k\} \ (t_k \to 0)\) such that
\[
(G^1(t_k), G^2(t_k)) \to (f, g) \quad \text{in} \quad L^2 \quad \text{for some} \ (f, g) \in L^2.
\]
So, let \(t = t_k \to 0\) in (5.1), we obtain
\[
Lu(0) + \nabla P_m(\rho(0)) + \frac{1}{c} \int_0^\infty \int_{S^2} A_r(0) \Omega d\Omega dv = \sqrt{\rho(0)} f,
\]
\[
- \frac{1}{c_v} (\kappa \triangle \theta(0) + Q(u(0))) - \int_0^\infty \int_{S^2} \frac{1}{c_v} (1 - \frac{u(0) \cdot \Omega}{c}) A_r(0) \Omega d\Omega dv = \sqrt{\rho(0)} g.
\]

Combining with the strong convergence (2.6) and (5.2), we know that the necessity of the compatibility condition is obtained. Moreover, from the construction of our strong solutions in Section 4, we easily deduce that \(f = g_1\) and \(g = g_2\).

Step 2. To prove the sufficiency. Let \((I_0, \rho_0, u_0, \theta_0)\) be the initial data satisfying (2.2)-(2.3). Then there exists a unique solution \((I, \rho, u, \theta)\) to (1.7)-(1.8) with (1.9) or (1.10):
\[
I(v, \Omega, t, x) \in L^2(\mathbb{R}^+ \times S^2; C([0, T]; W^{1,q}(V))),
\]
\[
\rho(t, x) \in C([0, T]; W^{1,q}), \quad (\theta, u)(t, x) \in C([0, T]; H^2(V)).
\]

Then we only need to make sure that
\[
I(v, \Omega, 0, x) = I_0, \quad \rho(0, x) = \rho_0, \quad u(0, x) = u_0(x), \quad \theta(0, x) = \theta_0, \quad x \in V.
\]

From the weak formulation of the strong solution, we easily have
\[
I(v, \Omega, 0, x) = I_0, \quad \rho(0, x) = \rho_0, \quad (\rho(0, x) u(0, x) = \rho_0 u_0, \quad \rho(0, x) \theta(0, x) = \rho_0 \theta_0, \quad x \in V.
\]

It remains to prove that \(u(0, x) = u_0(x)\) and \(\theta(0, x) = \theta_0(x)\) when \(x \in V\). Let \(\overline{u}_0 = u_0 - u(0, x)\) and \(\overline{\theta}_0 = \theta_0 - \theta(0, x)\). According to the proof of the necessity, we know that \((I(v, \Omega, 0, x), \rho(0, x), u(0, x), \theta(0, x))\) also satisfies the relation (2.3) for \((g_1, g_2) \in L^2\). Then we quickly know that \((\overline{u}_0, \overline{\theta}_0) \in D^1(V) \cap D^2(V)\) is the unique solution of the elliptic problem (2.3) in \(V\) and thus \(\overline{u}_0 = 0\) and \(\overline{\theta}_0 = 0\) in \(V\), which implies that \(u(0, x) = u_0(x), \quad \theta(0, x) = \theta_0(x), \quad x \in V\). \(\square\)

6. BEAL-KATO-MAJDA BLOW-UP CRITERION.

Now we prove (2.10). Let \((I, \rho, u, \theta)\) be the strong solution obtained in Theorem 2.1 to (1.7)-(1.8) with (1.9) in \(\mathbb{R}^+ \times S^2 \times [0, T] \times V\). We assume the opposite of (2.10) holds, i.e.,
\[
\lim_{T \to T^*} \sup \left( \|\nabla I\|_{L^2(\mathbb{R}^+ \times S^2; L^\infty([0, T]; L^2(V)))} + |\rho|_{L^\infty([0, T] \times V)} + |\theta|_{L^\infty([0, T] \times V)} \right) = C_0 < \infty. \quad (6.1)
\]

Before our proof, we denote
\[
\text{div}(f \otimes u) = \sum_{j=1}^3 \partial_j (f u^j), \quad A \otimes f = (a_{ij} f^k)_{3 \times 3 \times 3}, \quad \text{div}(A \otimes f) = \sum_{k=1}^3 \partial_k (a_{ij} f^k). \quad (6.2)
\]
6.1. The lower order estimate for \( |u|_{L^\infty([0,T];D^1(V))} \).

We first have the \( L^\infty \) bound of \( I \) and some classical energy estimates.

**Lemma 6.1.**

\[
\int_0^\infty \int_{S^2} |A_r(t)|_{L^\infty(V)} d\Omega dv + |\sqrt{\rho}u, \theta(t)|_2 + ||(\nabla u, \nabla \theta)||_{L^2([0,T] \times V)} \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).

**Proof.** Firstly, let \( 2 \leq \rho \), multiplying \((6.7)\) by \( r |I|^{r-2} I \) and integrating over \( V \), we have

\[
\frac{d}{dt} |I|^r + \int_{\partial V \{n \geq 0\}} |I|^r \cdot \Omega ds \leq C |S|^r + C |\rho|^\infty \int_0^\infty \int_{S^2} \frac{v}{\sqrt{v}} |I|^r \sigma_s d\Omega dv' . \tag{6.3}
\]

According to the assumption \((1.12)\) and \((6.3)\), we deduce

\[
\frac{d}{dt} |I|^r \leq C \left( |I|^r + |S|^r + |I|^2_{L^2(\mathbb{R}^+ \times S^2)} \right) \int_0^\infty \int_{S^2} \frac{v}{\sqrt{v}} |I|^2 \sigma_s d\Omega dv' . \tag{6.4}
\]

From Gronwall’s inequality, we have

\[
\|I(v, \Omega, t, x)\|^2_{C([0,T]; L^r)} \leq \exp(CT) \left( |I|_{1}^{2} + \int_0^T |S|^{2} ds + T \int_0^\infty \int_{S^2} \frac{|v|}{\sqrt{v}} |I|^{2} \sigma_s^{2} d\Omega dv' \right) . \tag{6.5}
\]

Integrating above inequality in \( \mathbb{R}^+ \times S^2 \) with respect to \((v, \Omega)\), we have

\[
\|I\|^2_{L^2(\mathbb{R}^+ \times S^2, C([0,T]; L^r))} \leq C, \quad 0 \leq t < T,
\]

where \( C \) is independent of \( r \). Letting \( r \to \infty \), we obtain the bound of \( \|I\|^2_{L^2(\mathbb{R}^+ \times S^2, C([0,T]; L^\infty))} \).

Moreover, the estimate for \( I_t \) follows quickly from \( I_t = -c\Omega \cdot \nabla I + cA_r \).

Secondly, multiplying \((6.7)\) by \( u \), \((6.8)\) by \( \theta \) respectively and integrating the resulting equations over \( V \) by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_V \rho |u|^2 dx + \int_V (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) dx
\]

\[
= \int_V P_m u v dx - \frac{1}{c} \int_V \int_0^\infty \int_{S^2} A_r \Omega \cdot ud\Omega dx \leq \frac{\mu}{2} |\nabla u|^2 + C,
\]

\[
\frac{1}{2} \frac{d}{dt} \int_V \rho |\theta|^2 dx + \int_V \kappa |\nabla \theta|^2 dx
\]

\[
\leq \int_V C (\rho \theta^2 |\text{div} u| + |\nabla u|^2 |\theta| + N_r \theta) dx \leq C |\nabla u|^2 + C,
\]

which, via Gronwall’s inequality, immediately implies the desired conclusions. \( \Box \)

Now we improve the energy estimate obtained in Lemma \(6.1\).

**Lemma 6.2.** If \( \lambda < 3\mu \), it holds that

\[
\int_V \rho |u(t)|^4 dx + \int_0^T \int_V |u|^2 |\nabla u|^2 dx dt \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).
**Proof.** For any \( \lambda \) satisfying that \( \lambda < 3\mu \), there must exist a sufficiently small constant \( \alpha_\lambda > 0 \) such that:

\[
\lambda < (3 - \alpha_\lambda)\mu < 3\mu.
\]  

(6.8)

So we only need to show that (6.2) holds under the assumption (6.8).

Firstly, multiplying (1.7) by \( r|u|^{-2}u \) \( (r \geq 3) \) and integrating the resulting equation over \( V \) by parts, then we have

\[
\frac{d}{dt} \int_V \rho|u|^r dx + \int_V H_r dx
\]

\[
= -r(r-2)(\mu + \lambda) \int_V \text{div} |u|^{r-3}u \cdot \nabla |u| dx
\]

\[
+ \int_V rP_m \text{div} (|u|^{r-2}u) dx - \frac{1}{c} \int_0^\infty \int_{S^2} r|u|^{r-2} A_r u \cdot \Omega d\Omega d\nu dx,
\]

where

\[
H_r = r|u|^{-2}(\mu|\nabla u|^2 + (\mu + \lambda)|\text{div} u|^2 + \mu(r - 2)|\nabla |u||^2).
\]

For any given \( \varepsilon_1 \in (0, 1) \), we define a nonnegative function which will be determined in Step 2 as follows

\[
\phi(\varepsilon_0, \varepsilon_1, r) = \begin{cases} 
\frac{\mu_1(r-1)}{3(-\frac{\mu(\lambda+\varepsilon_0)}{4} - \frac{\mu(\lambda+\mu)}{4})}, & \text{if } \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{\mu(\lambda+\mu)}{3} - \lambda > 0, \\
0, & \text{otherwise}.
\end{cases}
\]

**Step 1:** We assume that

\[
\int_{V \cap |u| > 0} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 dx > \phi(\varepsilon_0, \varepsilon_1, r) \int_{V \cap |u| > 0} |u|^{r-2} |\nabla |u||^2 dx.
\]  

(6.10)

A direct calculation gives for \( |u| > 0 \):

\[
|\nabla |u|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2,
\]

(6.11)

which plays a important role in the proof. By (6.9) and the Cauchy’s inequality, we have

\[
\frac{d}{dt} \int_V \rho|u|^r dx + \int_{V \cap |u| > 0} H_r dx
\]

\[
= -r(r-2)(\mu + \lambda) \int_{V \cap |u| > 0} \text{div} |u|^{r-3}u \cdot \nabla |u| dx
\]

\[
+ \int_V rP_m \text{div} (|u|^{r-2}u) dx - \frac{1}{c} \int_0^\infty \int_{S^2} r|u|^{r-2} A_r u \cdot \Omega d\Omega d\nu dx
\]  

(6.12)

\[
\leq r(\mu + \lambda) \int_{V \cap |u| > 0} |u|^{r-2} |\text{div} u|^2 dx + \frac{r(r-2)^2(\mu + \lambda)}{4} \int_{V \cap |u| > 0} |u|^{-2} |\nabla |u||^2 dx
\]

\[
+ \int_V rP_m \text{div} (|u|^{r-2}u) dx - \frac{1}{c} \int_0^\infty \int_{S^2} r|u|^{r-2} A_r u \cdot \Omega d\Omega d\nu dx.
\]
Via Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we have

\[ M_1 = r \int_V P_m |u|^{r-2} |\nabla u| dx \leq C \left( \int_V |u|^{r-2} |\nabla u|^2 dx \right)^\frac{1}{2} \left( \int_V |u|^{r-2} P_m dx \right)^\frac{1}{2} \]

\[ \leq C \left( \int_V |u|^{r-2} |\nabla u|^2 dx \right)^\frac{1}{2} \left( \int_V \left| \frac{u}{|u|} \right|^{\frac{6}{2}} P_m \frac{1}{|u|^{r-2}} \right)^\frac{1}{2} \]

\[ \leq \frac{1}{4} \mu r \epsilon_0 \int_V |u|^{r-2} |\nabla u|^2 dx + C(\mu, r, \epsilon_0), \tag{6.13} \]

and similarly, for \( A_r \), we have

\[ M_2 = -\frac{1}{c} \int_V \int_0^\infty \int_{S^2} r |u|^{r-2} A_r u \cdot \Omega d\Omega dv dx \]

\[ \leq C \left| |u|^{\frac{r-2}{2}} \right| \int_0^\infty \int_{S^2} |A_r| \frac{1}{|r|} d\Omega dv \leq \frac{1}{4} \mu r \epsilon_0 \int_V |u|^{r-2} |\nabla u|^2 dx + C(\mu, r, \epsilon_0), \tag{6.14} \]

where \( \epsilon_0 \in (0, \frac{1}{r}) \) is independent of \( r \). Then combining (6.10)-(6.14), we quickly have

\[ \frac{d}{dr} \int_V \int_{\mathbb{R}^3} \rho |u|^r dx + rf(\epsilon_0, \epsilon_1, \epsilon_2, r) \int_{\mathbb{R}^{3-n}} |u|^{r-2} |\nabla u|^2 dx \]

\[ + \int_{\mathbb{R}^{3-n}} \mu r \epsilon_2 |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2 dx \leq C(\mu, r, \epsilon_0), \tag{6.15} \]

where

\[ f(\epsilon_0, \epsilon_1, \epsilon_2, r) = \mu(1 - \epsilon_0)(1 - \epsilon_2)\phi(\epsilon_0, \epsilon_1, r) + \mu(\gamma - 1 - \epsilon_0) - \frac{(r - 2)^2(\mu + \lambda)}{4}. \tag{6.16} \]

**Subcase 1:** If \( 4 \notin \left\{ \frac{r^2(\mu + \lambda)}{4(r - 1)} - \frac{\mu(4 - \epsilon_0)}{3} - \lambda > 0 \right\} \), i.e., \( \lambda + \epsilon_0 \mu > 0 \), it is easy to get

\[ [4, +\infty) \in \left\{ r \left| \frac{r^2(\mu + \lambda)}{4(r - 1)} - \frac{\mu(4 - \epsilon_0)}{3} - \lambda > 0 \right\} \].

Therefore, we have

\[ \phi(\epsilon_0, \epsilon_1, r) = \frac{\mu \epsilon_1 (r - 1)}{3 \left( - \frac{\mu(4 - \epsilon_0)}{3} - \lambda + \frac{r^2(\lambda + \mu)}{4(r - 1)} \right)} \]. \tag{6.17}

for any \( r \in [4, \infty) \). Substituting (6.17) into (6.16), for \( r \in [4, \infty) \), we have

\[ f(\epsilon_0, \epsilon_1, \epsilon_2, r) = \mu^2 \epsilon_1 (1 - \epsilon_0)(1 - \epsilon_2)(r - 1) \left( - \frac{\mu(4 - \epsilon_0)}{3} - \lambda + \frac{r^2(\lambda + \mu)}{4(r - 1)} \right) + \mu(r - 1 - \epsilon_0) - \frac{(r - 2)^2(\mu + \lambda)}{4}. \tag{6.18} \]

For \( (\epsilon_0, \epsilon_1, \epsilon_2, r) = (0, 1, 0, 4) \), we have

\[ f(0, 1, 0, 4) = \frac{3\mu^2}{\lambda + \epsilon_0 \mu} + 2\mu - \lambda - \epsilon_0 \mu = -C_1(\lambda - a_1 \mu)(\lambda - a_2 \mu), \tag{6.19} \]

where, according to \( \lambda + \epsilon_0 \mu > 0 \), we have \( C_1 = \frac{1}{\lambda + \epsilon_0 \mu} > 0 \) and

\[ a_1(\epsilon_0) = 3 - \epsilon_0, \quad a_2(\epsilon_0) = -1 - \epsilon_0. \]

Then if we want to make sure that \( f(0, 1, 0, 4) > 0 \), we need to assume that

\[ -\epsilon_0 \mu < \lambda < (3 - \epsilon_0) \mu. \]
Due to the continuity of \( f(\epsilon_0, \epsilon_1, \epsilon_2, r) \), we can choose \( \epsilon_0 \) small enough in \((0, \frac{1}{4})\) such that \( \epsilon_0 = \alpha \lambda \) and:

\[
\lambda < (3 - \alpha \lambda) \mu < 3\mu,
\]

and \((\epsilon_1, \epsilon_2) \in (0, 1) \times (0, 1)\) such that \( f(\epsilon_0, \epsilon_1, \epsilon_2, 4) > 0\), then we have

\[
\frac{d}{dt} \int_{V} \rho |u|^4 dx + 4f(\epsilon_0, \epsilon_1, \epsilon_2, 4) \int_{V \cap |u|>0} |u|^2 |\nabla u|^2 dx + 4\mu \epsilon_2 \int_{V \cap |u|>0} |u|^4 |\nabla \left( \frac{u}{|u|} \right)|^2 dx \leq C
\]

(6.20)

under the assumption that \(-\epsilon_0 \mu < \lambda < (3 - \alpha \lambda) \mu\).

**Subcase 2:** If \( 4 \notin \{ r \frac{r^2(\mu + \lambda)}{4(r-1)} - \frac{\mu(4-\epsilon_0)}{3} - \lambda > 0 \} \), i.e., \( \lambda < -\epsilon_0 \mu \). In this case, for \( r = 4 \), it is easy to get

\[
4 \left[ \frac{\mu(1 - \epsilon_0)(1 - \epsilon_2) \phi(\epsilon_0, \epsilon_1, r) + \mu(r - 1 - \epsilon_0) - \frac{(r - 2)^2(\mu + \lambda)}{4} \right] > 4 \left( \frac{7\mu}{4} - \lambda \right) > 4 \left( \frac{7\mu}{4} + \epsilon_0 \mu \right) > 7\mu,
\]

which, together with (6.15) - (6.16), implies that

\[
\frac{d}{dt} \int_{V} \rho |u|^4 dx + 4f(\epsilon_0, \epsilon_1, \epsilon_2, 4) \int_{V \cap |u|>0} |u|^2 |\nabla u|^2 dx + 4\mu \epsilon_2 \int_{V \cap |u|>0} |u|^4 |\nabla \left( \frac{u}{|u|} \right)|^2 dx \leq C.
\]

(6.22)

**Step 2:** We assume that

\[
\int_{V \cap |u|>0} |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2 dx \leq \phi(\epsilon_0, \epsilon_1, r) \int_{V \cap |u|>0} |u|^{r-2} |\nabla u|^2 dx.
\]

(6.23)

A direct calculation gives for \( |u| > 0\),

\[
\text{div} u = |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}.
\]

(6.24)

Then combining (6.12) - (6.14) and (6.24), we quickly have

\[
\frac{d}{dt} \int_{V} \rho |u|^r dx + \int_{V \cap |u|>0} \mu r(1 - \epsilon_0) \left( |u|^{r-2} |\nabla u|^2 + |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2 \right) dx + \int_{V \cap |u|>0} r(\lambda + \mu) |u|^{r-2} |\text{div} u|^2 dx
\]

\[
+ \int_{V \cap |u|>0} \mu r(r - 2)|u|^{r-2} |\nabla u|^2 dx = - r(r - 2)(\mu + \lambda) \int_{V \cap |u|>0} \left( |u|^{r-2} u \cdot \nabla u \text{div} \left( \frac{u}{|u|} \right) + |u|^{r-4} u \cdot \nabla |u|^2 \right) dx,
\]

which implies that

\[
\frac{d}{dt} \int_{V} \rho |u|^r dx + \int_{V \cap |u|>0} r|x|^{r-4} G^* dx \leq C,
\]

(6.26)
where \( G^* \) is given as:
\[
G^* = \mu(1 + \epsilon_0)|u|^2\nabla |u|^2 + \mu(1 - \epsilon_0)|u|^4|\nabla \left( \frac{u}{|u|} \right)|^2 \\
+ (r - 2)(\mu + \lambda)|u|^2u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (\mu + \lambda)|u|^2|\text{div} u|^2 \\
+ \mu(r - 2)|u|^2\nabla |u|^2 + (r - 2)(\mu + \lambda)|u \cdot \nabla |u|^2. \tag{6.27}
\]

Now we consider how to make sure that \( G^* \geq 0 \).
\[
G^* = \mu(1 - \epsilon_0)|u|^2\nabla |u|^2 + \mu(1 - \epsilon_0)|u|^4|\nabla \left( \frac{u}{|u|} \right)|^2 \\
+ (r - 2)(\mu + \lambda)|u|^2u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + \mu(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 \\
+ (r - 1)(\mu + \lambda)|u \cdot \nabla |u|^2 \\
+ r(\mu + \lambda)|u|^2u \cdot \nabla |u| \text{div} \left( \frac{u}{|u|} \right) + (\mu + \lambda)|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 \\
= \mu(1 - \epsilon_0)|u|^4|\nabla \left( \frac{u}{|u|} \right)|^2 + \mu(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 \\
+ (r - 1)(\mu + \lambda)\left( u \cdot \nabla |u| + \frac{r}{2(r - 1)}|u|^2\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 \right) \\
+ (\mu + \lambda)|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 \\
\geq \mu(1 - \epsilon_0)|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 + (\frac{4 - \epsilon_0}{3} \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)})|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2, \tag{6.28}
\]

which, combining with the fact \(|\text{div} \left( \frac{u}{|u|} \right) |^2 \leq 3|\nabla \left( \frac{u}{|u|} \right)|^2\), implies that
\[
G^* \geq \frac{\mu(1 - \epsilon_0)}{3}|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 + \mu(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 \\
+ \left( \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right)|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 \geq \mu(r - 1 - \epsilon_0)|u|^2|\nabla |u|^2 + \left( \frac{4 - \epsilon_0}{3} \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right)|u|^4\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2. \tag{6.29}
\]

Thus
\[
\int_{\mathbb{V} | |u| > 0} r|u|^{r+4}G^* \, dx \\
\geq r\left( \frac{4 - \epsilon_0}{3} \mu + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) \int_{\mathbb{V} | |u| > 0} |u|^r\left( \text{div} \left( \frac{u}{|u|} \right) \right)^2 \, dx \\
+ \mu(r - 1 - \epsilon_0) \int_{\mathbb{V} | |u| > 0} |u|^{r-2}|\nabla |u|^2 \, dx. \tag{6.30}
\]
are the material derivative of $f$
where $\phi(\epsilon_0, \epsilon_1, r) \int_{|u| > 0} |u|^{r-2} |\nabla u|^2 \, dx$
$+ \mu r (r - 1 - \epsilon_0) \int_{|u| > 0} |u|^{r-2} |\nabla u|^2 \, dx \geq g(\epsilon_0, \epsilon_1, r) \int_{|u| > 0} |u|^{r-2} |\nabla u|^2 \, dx,$
where
\begin{align*}
g(\epsilon_0, \epsilon_1, r) &= \left[ 3r \left( \frac{4 - \epsilon_0}{3} + \lambda - \frac{r^2(\mu + \lambda)}{4(r - 1)} \right) \phi(\epsilon_0, \epsilon_1, r) + \mu r (r - 1 - \epsilon_0) \right]. 
\end{align*}
(6.31)

Then combining (6.11), (6.23), (6.26) and (6.30)-(6.31), for $r = 4$, we quickly have
\begin{align*}
\int_{\Omega} \rho |u(t)|^4 \, dx + \int_{0}^{T} \int_{\Omega} |u|^2 |\nabla u|^2 \, dt \leq C, \quad 0 \leq t < T. 
\end{align*}
(6.32)

So combining (6.20)-(6.22) and (6.32) for Step:1 and Step:2, we conclude that if $\lambda < (3 - \alpha \lambda) \mu$, there exists some constant $C > 0$ such that
\begin{align*}
\int_{\Omega} \rho |u(t)|^4 \, dx + \int_{0}^{T} \int_{\Omega} |u|^2 |\nabla u|^2 \, dt \leq C, \quad 0 \leq t < T.
\end{align*}
(6.33)

The next lemma will give a key estimate on $\nabla u$.

**Lemma 6.3.**
\begin{align*}
|\nabla u(t)|^2 + \int_{0}^{T} |\sqrt{\rho u}|^2 \, dt \leq C, \quad 0 \leq t < T,
\end{align*}
where $C$ only depends on $C_0$ and $T$ (any $T \in (0, T]$).

**Proof.** Via the momentum equations (1.7), we have
\begin{align*}
\Delta G &= \text{div} \left( \rho \frac{d}{dt} + \frac{1}{c} \int_{s}^{t} A_v \Omega \, dv \right), \quad \mu \Delta \omega = \nabla \times \left( \rho \frac{d}{dt} + \frac{1}{c} \int_{s}^{t} A_v \Omega \, dv \right),
\end{align*}
where
\begin{align*}
\dot{f} &= f_t + u \cdot \nabla f = f_t + \text{div}(fu) - f \text{div}u, \quad G = (2\mu + \lambda) \text{div}u - P_m, \quad \text{and} \quad \omega = \nabla \times u,
\end{align*}
are the material derivative of $f$, the effective viscous flux, and the vorticity, respectively. It follows from Lemmas 3.3 and 6.1 that
\begin{align*}
|\nabla G|_2 + |\nabla \omega|_2 &\leq C(|\rho u_t|_2 + |\rho u \cdot \nabla u|_2 + 1) \leq C(|\sqrt{\rho u_t}|_2 + |\sqrt{\rho u}| |\nabla u|_2 + 1).
\end{align*}
(6.34)

Multiplying (1.7) by $u_t$ and integrating over $\Omega$ gives
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div}u)^2) \, dx + \int_{\Omega} \rho |u_t|^2 \, dx
= \int_{\Omega} \left( P_m \text{div}u_t - \rho u \cdot \nabla u \cdot u_t - \frac{1}{c} \int_{s}^{t} A_v \Omega \cdot u_t \, dv \right) \, dx = A + B + C.
\end{align*}
(6.35)
For the first term on the right-hand side of (6.35), one has
\[ A = \int_V P_m \text{div} u_t \, dx = \frac{d}{dt} \int_V P_m \text{div} u \, dx - \int_V (P_m)_t \text{div} u \, dx \]
\[ = \frac{d}{dt} \int_V P_m \text{div} u \, dx - \frac{1}{2\mu + \lambda} \int_V (P_m)_t G \, dx - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_V P_m^2 \, dx \]
\[ = A_1 + A_2 + A_3. \quad (6.36) \]

We first consider the second term on the right-hand side of (6.36) that
\[ A_2 = -\frac{1}{2\mu + \lambda} \int_V (P_m)_t G \, dx = -\frac{\gamma - 1}{2\mu + \lambda} \int_V \left( (\rho E_m)_t - \frac{P_m}{\gamma - 1} u \cdot \nabla G \right) \, dx \]
\[ = A_{21} + A_{22}, \]
\[ A_{21} = -\frac{\gamma - 1}{2\mu + \lambda} \int_V \left( \frac{1}{2} \rho |u|^2 u \cdot \nabla G + P_m u \cdot \nabla G + \frac{P_m}{\gamma - 1} u \cdot \nabla G \right) \, dx \]
\[ - \frac{\gamma - 1}{2\mu + \lambda} \int_V (N_r G - \kappa \nabla \theta \cdot \nabla G) \, dx + \frac{\gamma - 1}{2\mu + \lambda} \int_V (u \nabla \theta) \cdot \nabla G \, dx \]
\[ \leq -\frac{\gamma - 1}{2\mu + \lambda} \int_V \frac{1}{2} \rho |u|^2 u \cdot \nabla G \, dx + C |G|_2 |N_r|_2 \]
\[ + C |\nabla G|_2 (|u P_m|_2 + ||u|| \nabla |u|_2 + |\nabla \theta|_2). \quad (6.37) \]

For the second term on the right-hand side of (6.35), Cauchy’s inequality yields
\[ \int_V \rho u \cdot \nabla u \cdot u_t \, dx \leq \frac{1}{6} |\sqrt{\rho} u_t|_2^2 + \int_V \rho |u \cdot \nabla u|^2 \, dx. \quad (6.38) \]

Then according to (6.34), we obtain that
\[ |\sqrt{\rho} |u||\nabla u|_2 \leq C |\rho^\frac{1}{4} u|_4 |\nabla u|_4 \leq C (|G|_4 + |\omega|_4 + 1) \]
\[ \leq C (|G|_2^\frac{3}{4} |\nabla G|_2 \frac{3}{4} + |\omega|_2^\frac{1}{2} |\omega|_2^\frac{3}{2} + 1) \]
\[ \leq \epsilon (|\nabla G|_2 + |\nabla \omega|_2) + C(\epsilon)(|G|_2 + |\omega|_2) + C \]
\[ \leq C \epsilon (|\sqrt{\rho} u_t|_2 + |\sqrt{\rho} |u||\nabla u|_2 + C(\epsilon)(|\nabla u|_2 + 1), \quad (6.39) \]

which immediately means that
\[ |\sqrt{\rho} u \cdot \nabla u|_2 \leq C \epsilon |\sqrt{\rho} u_t|_2 + C(\epsilon)(|\nabla u|_2 + 1). \quad (6.40) \]

Then substituting (6.40) into (6.34), we have
\[ |\nabla G|_2 + |\nabla \omega|_2 \leq C (|\sqrt{\rho} u_t|_2 + |\nabla u|_2 + 1), \quad (6.41) \]

which, together with (6.37), we have
\[ A_{21} \leq -\frac{1}{4\mu + 2\lambda} \int_V \rho |u|^2 u \cdot \nabla G \, dx + C (|\nabla u|_2^2 + |\nabla \theta|_2^2 + ||u|| \nabla |u|_2^2 + 1) + \epsilon |\sqrt{\rho} u_t|_2^2. \quad (6.42) \]
where we used the fact that $|N_r|_2 \leq C(1 + |\nabla u|_2)$ via Lemma \[6.1\]. Next we consider $A_{22}$,

$$A_{22} = \frac{\gamma - 1}{2\mu + \lambda} \int_V \frac{1}{\rho} \nabla u \cdot G dx + \frac{\gamma - 1}{2\mu + \lambda} \int_V \rho u \cdot u G dx$$

$$\leq -\frac{\gamma - 1}{2\mu + \lambda} \int_V \frac{1}{\rho} \text{div}(\rho u) u G dx + \epsilon |\sqrt{\rho} u|_2^2 + C(\epsilon) \int_V |\rho u|^2 |G| dx$$

$$\leq \frac{\gamma - 1}{2\mu + \lambda} \int_V p u \cdot \nabla u u G dx + \frac{\gamma - 1}{2\mu + \lambda} \int_V \frac{1}{\rho} |u|^2 u \cdot \nabla G dx$$

$$+ \epsilon |\sqrt{\rho} u|_2^2 + C \int_V |\rho u|^2 |\nabla u| dx + C \leq |\sqrt{\rho} u|_2 + C |\nabla u|_2^2 + C |\nabla u|_2^2 + C,$$  \hspace{1cm} (6.43)

which, together with (6.40), we have

$$A_{22} \leq \epsilon |\sqrt{\rho} u|_2 + \frac{\gamma - 1}{2\mu + \lambda} \int_V \frac{1}{\rho} |u|^2 u \cdot \nabla G dx + C |\nabla u|_2^2 + C |\nabla u|_2^2 + C.$$  \hspace{1cm} (6.44)

Then combining (6.42) and (6.44), we deduce that

$$A_2 \leq \epsilon |\sqrt{\rho} u|_2 + C |\nabla u|_2^2 + C |\nabla \theta|_2^2 + C |\nabla u|_2^2 + C.$$  \hspace{1cm} (6.45)

Next we consider the term $B$ and $C$. Via (6.38) and (6.40), we have

$$B = \int_V -\rho u \cdot \nabla u d x \leq C |\sqrt{\rho} u|_2 |\nabla u|_2 + \epsilon |\sqrt{\rho} u|_2^2 \leq \epsilon |\sqrt{\rho} u|_2^2 + C |\nabla u|_2^2 + C,$$

$$C = -\frac{1}{c} \int_V \int_0^\infty \int_{S^2} A_r \Omega \cdot u d \Omega dv dx$$

$$\leq -\frac{1}{c} \frac{d}{dt} \int_V \int_0^\infty \int_{S^2} S \Omega \cdot u d \Omega dv dx + \frac{1}{c} \int_V \int_0^\infty \int_{S^2} S \Omega \cdot u d \Omega dv dx$$

$$\leq \frac{1}{c} \int_V \int_0^\infty \int_{S^2} S \Omega \cdot u d \Omega dv dx + C |\sqrt{\rho} u|_2^2 + C |\nabla u|_2^2 + C.$$ \hspace{1cm} (6.46)

Then from (6.35)-(6.36) and (6.45)-(6.46), letting $\epsilon$ sufficiently small, we have

$$\frac{1}{2} \int_V \int_0^\infty \int_{S^2} (\mu |\nabla u|^2 + (\lambda + \mu) (\text{div} u)^2) dx + \int_V \rho |u|^2 dx$$

$$\leq C (|\nabla u|_2^2 + |\nabla \theta|_2^2) + C |\nabla u|_2^2 + C.$$ \hspace{1cm} (6.47)

From Gronwall’s inequality and Lemma \[6.2\], we obtain the desired conclusions.

\[\square\]

6.2. The lower order estimate for $|\theta|_{L^\infty([0, T]; D^1_V)}$.

Lemma 6.4 (Lower order estimate of the temperature $\theta$).

$$|\theta(t)|_{D^1} + \int_0^T (|\theta|_2^2 + |\dot{u}|_2^2 + |\sqrt{\rho} \theta|_2^2 + |\text{div} u|_\infty) dt \leq C, \quad 0 \leq t \leq T,$$

$$|u(t)|_\infty + |\nabla u(t)|_6 + |(G, \omega)(t)|_{D^1} + |(\sqrt{\rho \dot{u}}, \sqrt{\rho u})|_2 \leq C, \quad 0 \leq t \leq T,$$
where $C$ only depends on $C_0$ and $T$ (any $T \in (0, T_1)$).

**Proof.** Step 1. Applying $\dot{u}[\partial/\partial t + \text{div}(\mathbf{u})]$ to (1.7) and integrating by parts give

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\dot{u}|^2 \, dx = -\int_\Omega \dot{u} \cdot (\nabla (P_m) + \text{div}(\nabla P_m \otimes \mathbf{u})) \, dx$$

$$- \int_\Omega \dot{u} \cdot (\Delta u_t + \text{div}(\Delta u \otimes u)) \, dx + (\lambda + \mu) \int_\Omega \dot{u} \cdot (\nabla \text{div} u_t + \text{div}(\nabla \text{div} u \otimes u)) \, dx$$

(6.48)

$$- \frac{1}{c} \int_\Omega \int_0^t \int_{S^2} \dot{u} \cdot \left((A_r)_{ij} \Omega + \text{div}(A_r \otimes u)\right) \, d\Omega \, dv \, dx \equiv \sum_{i=3}^6 M_i.$$  

According to the continuity equation (1.7)2, Lemmas 6.1-6.3, Hölder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

$$M_3 = \int_\Omega (\text{Rdiv} \dot{u} (\rho \theta) - P_m (\nabla u)^T : \nabla \dot{u} - R \rho \theta u \cdot \nabla \text{div} \dot{u}) \, dx$$

$$= \int_\Omega \left(\frac{1}{2} |\nabla \dot{u}|^2_{L^2} + |\nabla u|_{L^2} \right) \, dx$$

$$\leq C \rho_{\infty} |\nabla \theta|_{L^2} |\nabla \dot{u}|_{L^2} + C |\nabla |_{L^2} |\nabla \dot{u}|_{L^2} \leq \frac{\mu}{20} |\nabla \dot{u}|^2_{L^2} + C |\nabla \dot{u}|^2_{L^2},$$

(6.49)

$$M_4 = -\int_\Omega \mu \left(\partial_i \dot{u} \partial_{ij} \partial_{ij}^2 + \Delta u - \nabla \text{div} \dot{u}^i \right) \, dx$$

$$= -\int_\Omega \mu \left(\|\nabla \dot{u}\|^2 - \partial_i \dot{u} \partial_{ij} \partial_{ij} - \partial_i \dot{u} \partial_{ij} \partial_{ij} - \partial_i \dot{u} \partial_{ij} \partial_{ij} - \partial_i \dot{u} \partial_{ij} \partial_{ij} \right) \, dx$$

$$\leq -\frac{\mu}{2} |\nabla \dot{u}|^2_{L^2} + C |\nabla \dot{u}|^2_{L^2},$$

and similarly, we have

$$M_5 = (\lambda + \mu) \int_\Omega \left(\dot{u} \cdot (\nabla \text{div} u_t + \text{div}(\nabla \text{div} u \otimes u)) \right) \, dx \leq -\frac{\mu + \lambda}{2} |\nabla \dot{u}|^2_{L^2} + C |\nabla \dot{u}|^2_{L^2}.$$  

(6.50)

Next we consider the radiation term $M_6$:

$$M_6 = -\frac{1}{c} \int_\Omega \int_0^t \int_{S^2} \dot{u} \cdot \left((A_r)_{ij} \Omega + \text{div}(A_r \otimes u)\right) \, d\Omega \, dv \, dx \equiv \sum_{i=1}^5 -\frac{1}{c} M_{6i}.$$  

(6.51)

Then via (2.8) and $\sigma_a = \sigma(v, \Omega, \theta) \rho$, we have

$$M_{61} = \int_0^t \int_{S^2} \int_\Omega \dot{u} \cdot \Omega S_i \, dx \, d\Omega \, dv \leq |\dot{u}|_2 \int_{S^2} |S_i|_{L^2} \, d\Omega \, dv \leq \frac{\mu}{20} |\nabla \dot{u}|^2_{L^2} + C,$$

$$M_{62} = \int_0^t \int_{S^2} \int_\Omega \dot{u} \cdot \Omega (\sigma_\theta \dot{\theta} I - \sigma \text{div}(\rho \dot{u}) I + \sigma \rho I) \, dx \, d\Omega \, dv$$

$$\leq C \left( |\sqrt{\rho} \dot{u}|^2_{L^2} + |\nabla \dot{u}|^2_{L^2} \right) + \frac{\mu}{20} |\nabla \dot{u}|^2_{L^2} + C |\nabla \dot{u}|^2_{L^2} + C,$$  

(6.52)
where we used the fact \((\sigma_a)_1 = \sigma_\theta \rho + \sigma \rho_t\) and
\[
C_r = \int_0^\infty \int_{S^2} |\sigma |_{\infty} (|I|_\infty + |I_t|_2 + |\nabla I|_2) + |\nabla \sigma|_{\infty} |I|_6 + |\sigma|_{\infty} |I_t|_2) d\Omega \leq C,
\]

And similarly, we have
\[
M_{64} = -\frac{1}{c} \int_0^\infty \int_{S^2} \int_V \left( \int_0^\infty \int_{S^2} \dot{u} \cdot \Omega (\sigma' \rho I_t + (\sigma')_t \rho I + \sigma \rho_t I) d\Omega d\Omega \right) dx dv
\leq C E_r (|\rho|_{\infty} |\sqrt{\rho \dot{u}}|_2 + |\sqrt{\rho \dot{u}}|_2 |\nabla \dot{u}|_2) \leq C |\sqrt{\rho \dot{u}}|_2^2 + \frac{\mu}{20} |\nabla \dot{u}|_2^2 + C,
\]

\[
M_{65} \leq C |\nabla \dot{u}|_2 |u|_6 \int_0^\infty \int_{S^2} |\sigma' \rho I_t + (\sigma')_t \rho I + \sigma \rho_t I| d\Omega \leq C |\nabla u|_2^2 + \frac{\mu}{20} |\nabla \dot{u}|_2^2,
\]

where
\[
E_r = \int_V (|\sigma|_{\infty} (|I|_\infty + |I_t|_2 + |\nabla I|_2) + |\nabla \sigma|_{\infty} |I|_6 + |\sigma|_{\infty} |I_t|_2) d\Pi \leq C.
\]

Together with \((6.43)-(6.54)\), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_V \rho\dot{u}^2 dx + |\dot{u}|_{D1}^2 \leq C |\nabla u|_4^4 + C |\sqrt{\rho \dot{u}}|_2^2 + C |\sqrt{\rho \theta}|_2^2 + C |\nabla \theta|^2 + C. \tag{6.55}
\]

Step 2. Multiplying \((1.7)\) by \(\dot{\theta}\) and integrating over \(V\), we have
\[
\frac{\kappa}{2c_v} \frac{d}{dt} \int_V \nabla \theta^2 dx + \int_V \rho \dot{\theta}^2 dx = \frac{1}{c_v} \int_V \left( - P \nabla \nabla u \dot{\theta} + Q(u) \theta_t + Q(u) \nabla \theta + \kappa \Delta \theta \nabla u + N_v \dot{\theta} \right) dx = 11 \sum_{i=1}^{11} M_i. \tag{6.56}
\]

Then from Cauchy inequality and Young’s inequality, we have
\[
M_7 \leq C |\rho|_{\infty} |\sqrt{\rho \theta}|_2 |\nabla \dot{u}|_2 \leq \frac{1}{20} |\sqrt{\rho \theta}|_2^2 + C,
\]
\[
M_8 = \frac{1}{c_v} \frac{d}{dt} \int_V Q(u) \theta dx - \frac{4\mu}{c_v} \int_V D(u) : D(u_t) \theta dx - \frac{2\lambda}{c_v} \int_V \text{div} \text{div} u \theta dx
\leq \frac{1}{c_v} \frac{d}{dt} \int_V (\frac{1}{2} |\nabla \theta|^2 + C |\nabla \theta|_2^2 + \frac{4\mu}{c_v} \int_V \left( D(u) : (\nabla \theta + (\nabla \theta) \nabla \theta) \right) \theta dx
\leq 11 \sum_{i=1}^{11} M_i. \tag{6.57}
\]
Next, we consider the radiation terms.

\[
\leq \frac{1}{c_v} \frac{d}{dt} \int_{\mathcal{V}} Q(u) \theta dx + C |\nabla \hat{u}|_2 + C |\nabla u|_3^3 + \frac{2\mu}{c_v} \int_{\mathcal{V}} |D(u)|^2 \text{div} u \theta dx \\
+ \frac{2\mu}{c_v} \int_{\mathcal{V}} |D(u)|^2 u \cdot \nabla \theta dx + \lambda \int_{\mathcal{V}} (|\text{div} u|^3 \theta + |\text{div} u|^2 u \cdot \nabla \theta) dx
\]

(6.58)

\[
\leq \frac{1}{c_v} \frac{d}{dt} \int_{\mathcal{V}} Q(u) \theta dx + C |\nabla \hat{u}|_2 + C |\nabla u|_3^3 + C \int |\nabla u|^2 |u||\theta| dx.
\]

From Lemmas 3.3 and 6.1, we quickly have

\[
|\theta|_{D^2} \leq C (|\rho \hat{\theta}|_2 + |\rho \text{div} u|_2 + |\nabla u|_4^2 + 1) \leq C |\sqrt{\rho} \hat{\theta}|_2 + C |\nabla u|_4^2 + C.
\]

(6.59)

Then via Hölder’s inequality, Cauchy inequality and Gagliardo-Nirenberg inequality,

\[
\int_{\mathcal{V}} |\nabla u|^2 |u||\theta| dx \leq C |\nabla u|_4^2 |u|_{\mathcal{V}}^3 |\nabla \theta|_2^\frac{1}{2} |\nabla \theta|_6^\frac{1}{2} \leq \frac{1}{20} |\sqrt{\rho} \hat{\theta}|_2 + C |\nabla u|_4^2 + C |\nabla \theta|_2^2 + C,
\]

which, together with (6.57), implies that

\[
M_9 \leq \frac{1}{c_v} \frac{d}{dt} \int_{\mathcal{V}} Q(u) \theta dx + C |\nabla u|_4^4 + C |\nabla \hat{u}|_2 + C |\nabla \theta|_2^2 \leq \frac{1}{20} |\sqrt{\rho} \hat{\theta}|_2^2 + C.
\]

(6.61)

And similarly, we have

\[
M_9 \leq C \int_{\mathcal{V}} |\nabla u|^2 |u||\theta| dx \leq \frac{1}{20} |\sqrt{\rho} \hat{\theta}|_2^2 + C |\nabla u|_4^4 + C |\nabla \theta|_2^2 + C,
\]

(6.62)

Next, we consider the radiation terms

\[
M_{11} = \frac{1}{c_v} \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \hat{\theta} \left( 1 - \frac{u \cdot \Omega}{c} \right) (S + \sigma_u I + \int_0^\infty \int_{S^2} \left( \frac{u \cdot \hat{\sigma}_u I + \sigma_u I} {c} \right) \text{d} \Omega \cdot \text{d} \Omega') \text{d} x \text{d} \Omega \text{d} v
\]

\[
\leq \frac{1}{c_v} \frac{d}{dt} \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \left( 1 - \frac{u \cdot \Omega}{c} \right) S \theta \text{d} x \text{d} \Omega \text{d} v + \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \left( \frac{\hat{u} - u \cdot \nabla u}{c} \right) \cdot \frac{\Omega}{c} \theta \text{d} x \text{d} \Omega \text{d} v
\]

\[
+ C |\rho \hat{\theta}|_2 |I|_{L^2(\mathbb{R}^+ \times S^2; L^2(\mathcal{V}))} \left( \alpha_1 + \alpha_2 + ||\sigma||_{L^2(\mathbb{R}^+ \times S^2; L^2(\mathcal{V}))} \right)
\]

\[
\leq \frac{1}{c_v} \frac{d}{dt} \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \left( 1 - \frac{u \cdot \Omega}{c} \right) S \theta \text{d} x \text{d} \Omega \text{d} v + C |\sqrt{\rho} \hat{\theta}|_2
\]

\[
+ C(|\hat{u}|_6|\theta|_3 + |u \cdot \nabla u|_2|\theta|_\infty) ||S||_{L^1(\mathbb{R}^+ \times S^2; L^2(\mathcal{V}))}
\]

\[
+ C(1 + ||u||_1)(|\theta|_\infty ||S||_{L^1(\mathbb{R}^+ \times S^2; L^2(\mathcal{V}))} + |u \cdot \nabla \theta|_3 ||S||_{L^1(\mathbb{R}^+ \times S^2; L^2(\mathcal{V}))})
\]

\[
\leq \frac{1}{c_v} \frac{d}{dt} \int_0^\infty \int_{S^2} \int_{\mathcal{V}} \left( 1 - \frac{u \cdot \Omega}{c} \right) S \theta \text{d} x \text{d} \Omega \text{d} v
\]

\[
+ C |\nabla \hat{u}|_2 + C |\sqrt{\rho} \hat{\theta}|_2 + C |\nabla \theta|_2^2 + C |u \cdot \nabla u|_2 + C.
\]

(6.63)
Then combining (6.57)–(6.63), we have
\[ \frac{1}{c_v} \frac{d}{dt} \int_V \left( \frac{\kappa}{2} |\nabla \theta|^2 - Q(u) \theta + \int_0^\infty \int_{S^2} \left( 1 - \frac{u \cdot \Omega}{c} \right) S \theta d\Omega dv \right) dx + \int_V \rho |\dot{\theta}|^2 dx \]
\[ \leq C(|\nabla u|_4 + |\nabla u|^3_2 + |\nabla u|^3_1 + |\nabla u|^3_1 + 1). \]
(6.64)

Now multiplying (6.64) by 2C, and adding the resulting inequality into (6.55), we have
\[ \frac{d}{dt} \int_V \left( \frac{\kappa}{2c_v} |\nabla \theta|^2 - \frac{1}{c_v} Q(u) \theta + \frac{1}{c_v} \int_0^\infty \int_{S^2} \left( 1 - \frac{u \cdot \Omega}{c} \right) S \theta d\Omega dv + \frac{1}{2} \rho |\dot{\theta}|^2 \right) dx \]
\[ + \int_V \rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2 \right) dx \leq C(|\nabla u|_4 + |\nabla \theta|^3_2 + |\nabla u|^3_3 + |u \cdot \nabla u|^3_2 + 1). \]
(6.65)

Due to
\[ |\nabla u|^4_4 + |\nabla u|^3_3 \leq C(|G|^4_4 + |\omega|^4_4 + |G|^3_3 + |\omega|^3_3 + 1) \leq C(|\omega|^3_2 |\nabla \omega|^3_2 + |G|^3_2 |\nabla G|^3_2 + |\omega|^3_2 |\nabla \omega|^3_2 + |G|^3_2 |\nabla G|^3_2 + 1), \]
(6.66)

(6.34), (6.41), (6.65), we quickly have
\[ |\nabla \theta|^2_2 + |\sqrt{\rho} \dot{u}|^2_2 + \int_0^t (|\sqrt{\rho} \dot{\theta}|^2_2 + |\nabla \dot{u}|^2_2) ds \leq C \int_0^t |\sqrt{\rho} \dot{u}|^2_2 ds + C. \]
(6.67)

According to (6.40), we have
\[ |\sqrt{\rho} \dot{u}|_2 \leq C(|\sqrt{\rho} \dot{u}|_2 + |\sqrt{\rho} \cdot \nabla u|) \leq C(|\sqrt{\rho} \dot{u}|_2 + |\nabla u|_2 + 1) + \epsilon |\sqrt{\rho} \dot{u}|_2. \]
(6.68)

Then substituting (6.68) into (6.65), via Gronwall’s inequality, we obtain
\[ |\dot{\theta}(t)|^2_{D^1} + |\sqrt{\rho} \dot{u}(t)|^2_2 + \int_0^T (|\dot{u}|^2_{D^1} + |\sqrt{\rho} \dot{\theta}|^2_2) dt \leq C, \quad 0 \leq t \leq T, \]
which, together with Sobolev imbedding theorem, (6.41), (6.59), (6.68) and
\[
\begin{aligned}
|\text{div} u|_\infty & \leq C(||G||_\infty + 1) \leq C(||G||_{W^{1,6}} + 1) \leq C(|\nabla u|_6 + |\nabla \dot{u}|_2 + 1), \\
|\omega|_\infty & \leq C(||\omega||_{W^{1,6}} + 1) \leq C(|\nabla u|_6 + |\nabla \dot{u}|_2 + 1), \\
|\nabla u|_6 & \leq C(|\text{div} u|_6 + |\omega|_6) \leq C(||G||_1 + ||\omega||_1 + 1),
\end{aligned}
\]
implies that, for $0 \leq t \leq T$,
\[ |\sqrt{\rho} u(t)|_2 + |u(t)|_\infty + |\nabla u(t)|_6 + |(G, \omega)(t)|_{D^1_2} + \int_0^T (|\dot{\theta}|^2_{D^2} + |\text{div} u|_\infty + |\omega|_\infty) dt \leq C. \]

\[ \square \]

6.3. The higher order estimate for $|\nu(t, \theta)|_{L^\infty([0, T]; D^2(\nu))}$.

Lemma 6.5.
\[ |\dot{\theta}(t)|_{D^2} + |\sqrt{\rho} \dot{u}(t)|_2 + \int_0^T |\dot{\theta}|^2_{D^1_2} dt \leq C, \quad 0 \leq t < T, \]
where $C$ only depends on $C_0$ and $T$ (any $T \in (0, T)$).
Proof. Differentiating (6.14) with respect to $t$, multiplying by $\theta_t$ and integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\theta_t|^2 \, dx + \frac{\kappa}{c_v} \int_{\Omega} |\nabla \theta_t|^2 \, dx
\]
\[
= \int_{\Omega} \left( - \rho (\frac{\theta_t}{2} + u \cdot \nabla \theta + \rho \nabla \phi) - \rho (u_t \cdot \nabla \theta + u \cdot \nabla \theta_t + R\nabla \phi) \theta_t 
\right.
\]
\[
- \frac{1}{c_v} P_m \nabla \phi \theta_t \theta + \int_{\Omega} \theta_t \Theta(u) \theta_t \, dx + \int_{\Omega} \int_{\Omega} \frac{1}{c_v} \left( 1 - \frac{u \cdot \nabla \theta}{c} \right) (B_r) \theta_t \, dx \, dv
\]
\[
+ \int_{\Omega} \int_{\Omega} \int_{\Omega} \frac{1}{c_v} \left( - \frac{u \cdot \nabla \theta}{c} \right) \Omega_r \theta_t \, dx \, dv \equiv: \sum_{j=1}^{15} M_i + E^* ,
\]
where
\[
E^* = \frac{1}{c_v} \int_{\Omega} \int_{\Omega} \int_{\Omega} \left( \left( 1 - \frac{u \cdot \nabla \theta}{c} \right) (B_r) \theta_t + \left( - \frac{u \cdot \nabla \theta}{c} \right) \Omega_r \theta_t \right) \, dx \, dv = \sum_{j=1}^{8} G_j .
\]
For $M_{12}$, via (6.34), (6.41) and Lemma 6.4, we have
\[
M_{12} = \int_{\Omega} \div (\rho u) \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \rho \nabla \phi \right) \theta_t \, dx
\]
\[
= - \int_{\Omega} \rho u \cdot \nabla \theta_t \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \rho \nabla \phi \right) \, dx - \int_{\Omega} \rho u \cdot \nabla \theta_t \theta_t \, dx
\]
\[
- \int_{\Omega} \rho u \cdot (\nabla u \cdot \nabla \theta + u \cdot \nabla \nabla \phi) \theta_t \, dx - R \int_{\Omega} \rho u \cdot (\nabla \phi \nabla u + \nabla \phi \nabla \phi) \theta_t \, dx
\]
\[
\leq \frac{\kappa}{20c_v} |\theta_t|^2_{D_1} + C|\rho u \theta_t|^2_{D_2} + C|\rho u \nabla \theta|^2_{D_2} + C|\rho \nabla \phi \nabla u|^2_{D_2} + C|\rho \theta_t|^2_{D_2} \quad (6.70)
\]
\[
+ C||\nabla u|||\nabla \phi||^2_{D_2} + C|\rho u \cdot \nabla \theta|^2_{D_2} - \frac{R}{2\mu + \lambda} \int_{\Omega} \rho u \cdot (\nabla \nabla G + \nabla (\nabla \phi \theta)) \theta_t \, dx
\]
\[
\leq \frac{\kappa}{10c_v} |\theta_t|^2_{D_1} + \frac{R^2}{2\mu + \lambda} \int_{\Omega} \frac{1}{2} \rho^2 \theta^2 (\div u \theta_t + u \cdot \nabla \theta_t) \, dx + C(1 + |\sqrt{\theta_t}|^2_{D_2})
\]
\[
+ C(|\nabla^2 \theta|^2_{D_2} + |\nabla G|^2_{D_2}) \leq C + \frac{\kappa}{10c_v} |\theta_t|^2_{D_1} + C|\sqrt{\theta_t}|^2_{D_2} + C|\nabla \theta|^2_{D_2} .
\]
For $M_{13}$ and $M_{14}$, we have
\[
M_{13} = \int_{\Omega} - \rho (u_t \cdot \nabla \theta + u \cdot \nabla \theta_t + R \nabla \phi \theta_t) \, dx \]
\[
\leq \int_{\Omega} \rho (u_t \cdot \nabla \theta + (u \cdot \nabla) u \cdot \nabla \theta) \, dx + \frac{\kappa}{20c_v} |\theta_t|^2_{D_1} + (|\div u|_{\infty} + 1)|\sqrt{\theta_t}|^2_{D_2}
\]
\[
\leq C(|\nabla u|_{\infty} + |\nabla u|_{\infty})|\nabla \theta|_{D_3} + \frac{\kappa}{20c_v} |\theta_t|^2_{D_1} + (|\div u|_{\infty} + 1)|\sqrt{\theta_t}|^2_{D_2}
\]
\[
\leq \frac{\kappa}{10c_v} |\theta_t|^2_{D_1} + C(|\div u|_{\infty} + |\nabla u|_{\infty})|\nabla \theta|_{D_3} + C|\theta_t|^2_{D_2} + C , \quad (6.71)
\]
\[
M_{14} = \int_{\Omega} - \frac{R}{c_v} \rho \theta t \, dx + \int_{\Omega} \frac{R}{c_v} \rho \theta t \, dx + \int_{\Omega} \frac{R}{c_v} \rho \theta_t \cdot \nabla u \, dx
\]
\[
\leq C|\rho \theta |^2_{D_2} + C|\nabla u|_{\infty}^2 + C \int_{\Omega} \rho |\nabla u|^2 |\theta_t| \, dx + \int_{\Omega} \frac{R}{c_v} \rho \theta_t u \cdot \nabla \phi \, dx .
\]
\[
\begin{align*}
\leq & C |\sqrt{\rho_D}|^2 + C |\text{div}u|^2 + C |\nabla u|^4 + \frac{R}{(2\mu + \lambda)c_v} \int_V \rho \theta_t u \cdot \nabla G \, dx \\
& + \frac{R}{(2\mu + \lambda)c_v} \int_V \rho^2 \theta_t u \cdot \nabla \theta dx + \frac{R}{(2\mu + \lambda)c_v} \int_V \rho^2 \theta_t u \cdot \nabla \rho dx \\
\leq & C + C |\sqrt{\rho_D}|^2 + C |\text{div}u|^2 + \frac{R}{(2\mu + \lambda)c_v} \int_V \rho^2 \theta_t u \cdot \nabla \rho dx.
\end{align*}
\]
\begin{equation}
G_4 = \frac{1}{c_v} \int_1^\infty \int_\mathcal{V} - \left(1 - \frac{u \cdot \Omega}{c}\right) (\bar{\sigma}_s' \rho I_t + (\bar{\sigma}_s')_t \rho I + \bar{\sigma}_s' \rho I) \theta_t \, dx \, d\Omega \leq J_t \left(1 + |u|_\infty^2 \right) \left(1 + |\sqrt{\rho} \theta_t|_2^2 + |\nabla \theta_t|_2^2 \right), \tag{6.78}
\end{equation}

where

\begin{equation}
G_r = \int_1^\infty \int_\mathcal{V} \left(\bar{\sigma}_s' \rho I_t^2 + |I_t^2|_2 + |\nabla I_t^2|_2 + |I_t^2|_\infty \right) + |I_t^2|_2 ((\bar{\sigma}_s)_{t \infty} + |\nabla \bar{\sigma}_s|_2) \, d\Omega, \tag{6.79}
\end{equation}

Finally, we consider the terms \(G_5\) to \(G_8\):

\begin{align}
G_5 & = \frac{1}{c_v} \int_0^\infty \int_\mathcal{V} \int_\mathcal{V} - \frac{u_t \cdot \Omega}{c} S \sigma_s I \theta_t \, dx \, d\Omega \, dv \\
& \leq C |u_t|_6 |\theta_t|_6 \int_0^\infty \int_\mathcal{V} |S|_2 I d\Omega \, dv \leq \frac{\kappa}{20 c_v} (|\theta_t|_{D^1}^2 + |\sqrt{\rho} \theta_t|_2^2) + C |u_t|_{D^1}^2, \\
G_6 & = \frac{1}{c_v} \int_0^\infty \int_\mathcal{V} \int_\mathcal{V} \frac{u \cdot \Omega}{c} S \sigma_s I \theta_t \, dx \, d\Omega \, dv \\
& \leq C |u_t|_6 |\rho|_{\mathcal{V}}^2 \sqrt{\rho} \theta_t|_2 \int_0^\infty \int_\mathcal{V} |\sigma_{t \infty}| |I|_3 \, d\Omega \, dv \leq C |\sqrt{\rho} \theta_t|_2^2 + C |u_t|_{D^1}^2, \\
G_7 & = \frac{1}{c_v} \int_\mathcal{V} \int_\mathcal{V} \frac{v \cdot u_t \cdot \Omega}{c} \sigma_s I \theta_t \, dx \, d\Omega \\
& \leq C |u_t|_6 |\rho|_{\mathcal{V}}^3 \sqrt{\rho} \theta_t|_2 \int_1^\infty \frac{v \cdot \sigma_s}{\sigma_s} |I|_3 d\Omega \leq C |\sqrt{\rho} \theta_t|_2^2 + C |u_t|_{D^1}^2, \\
G_8 & = \frac{1}{c_v} \int_\mathcal{V} \int_\mathcal{V} \frac{u \cdot \Omega}{c} \sigma_s I \theta_t \, dx \, d\Omega \\
& \leq C |u_t|_6 |\rho|_{\mathcal{V}}^3 \sqrt{\rho} \theta_t|_2 \int_1^\infty \frac{\sigma_s}{\sigma_s'} |I|_3 d\Omega \leq C |\sqrt{\rho} \theta_t|_2^2 + C |u_t|_{D^1}^2. \tag{6.80}
\end{align}

Then combining (6.69) to (6.80), via Gronwall's inequality, we quickly have

\begin{equation}
|\theta(t)|_{D^2}^2 + |\sqrt{\rho} \theta(t)|_2 + \int_0^T |\theta_t|_{D^1}^2 \, dt \leq C, \quad 0 \leq t < T. \tag{6.81}
\end{equation}

Finally, we give the upper bounds of \(|\rho, I|_{D^{1,q}}\), \(|(u, \theta)|_{D^{2,q}}\), \(|u_t|_{D^1}^2\) and so on.

**Lemma 6.6.**

\begin{equation}
|(\rho, I)(t)|_{D^{1,q}} + |(\rho_t, I_t)(t)|_q + |u(t)|_{D^2} + \int_0^T (|u_t|_{D^1}^2 + |(u, \theta)|_{D^{2,q}}^2) \, dt \leq C, \quad 0 \leq t < T, \quad \text{where } C \text{ only depends on } C_0 \text{ and } T \text{ (any } T \in (0, T]).
\end{equation}

**Proof.** In the following estimates we will use

\begin{align}
|\nabla^2 u|_q & \leq C(|\nabla \rho|_q + |\nabla \bar{u}|_2 + 1), \quad |\nabla^2 u|_2 \leq C(|\nabla \rho|_2 + 1), \\
|\nabla u|_\infty & \leq C(|\text{div} u|_\infty + |\omega|_\infty) \ln(e + |\nabla^2 u|_q) + C |\nabla u|_2 + C, \\
& \leq C(|\text{div} u|_\infty + |\omega|_\infty) \ln(e + |\nabla \rho|_q + \ln(e + |\nabla \bar{u}|_2 + 1), \tag{6.82}
\end{align}

\begin{align}
|\theta|_{D^2} & \leq C |\nabla \theta_t|_2 + C |u|_{D^2} + C |\nabla u|_\infty |\nabla u|_q + C,
\end{align}
where we have used the equations (1.7) and Lemmas 5.3 and 6.1-6.5.

Firstly, applying \( \nabla \) to (1.7), multiplying the resulting equations by \( q|\nabla \rho|^q - 2 \nabla \rho \), and integrating over \( V \), via (6.82) we immediately obtain

\[
\frac{d}{dt} |\nabla \rho|_q \leq C|\nabla u|_\infty |\nabla \rho|_q + C |\nabla^2 u|_q + C|\nabla \dot{u}|_2 + C. 
\]  

(6.83)

Via (6.82), (6.83) and notations:

\[
f = e + |\nabla \rho|_q, \quad g = 1 + (|\text{div} u|_\infty + |\omega|_\infty) \ln(e + |\nabla \dot{u}|_2),
\]

we quickly have

\[
f_t \leq Cg f + Cf \ln f + Cg,
\]

which, together with Lemma 6.4 and Gronwall’s inequality, implies that

\[
\ln{f(t)} \leq C, \quad 0 \leq t < T.
\]

Then we obtained the desired estimate for \( |\nabla \rho|_q \). And the upper bound of \( |\nabla I|_q \) can be deduced easily via the similar argument used in Lemma 1.3. The estimates for \( \rho_t \) and \( I_t \) can be obtained via equations (1.7). Finally, via (6.82), we only need to check that

\[
|\nabla u_t|_2 \leq |\nabla \dot{u}|_2 + |\nabla(u \cdot \nabla u)|_2 \leq C(|\nabla \dot{u}|_2 + 1),
\]

which, combining with Lemma 6.4, implies the desired conclusions. \( \square \)

And this will be enough to extend the strong solution \((I, \rho, u, \theta)\) beyond \( t \geq T \).

In truth, via the estimates obtained in Lemmas 6.1-6.6, we quickly know that the functions \((I, \rho, u, \theta)|_{t=T} = \lim_{t \to T} (I, \rho, u, \theta)\) satisfy the conditions imposed on the initial data (2.2)-(2.3). Therefore, we can take \((I, \rho, u, \theta)|_{t=T}\) as the initial data and apply Theorem 2.1 to extend the local solution beyond \( t \geq T \). This contradicts the assumption on \( T \).

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