BOUNDEDNESS OF THE BERGMAN PROJECTION ON $L^p$ SPACES WITH EXPONENTIAL WEIGHTS

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Abstract. Let $v(r) = \exp \left( -\frac{\alpha}{1-r^2} \right)$ with $\alpha > 0$, and let $D$ be the unit disc in the complex plane. Denote by $A^p_v$ the subspace of analytic functions of $L^p(D, v)$ and let $P_v$ be the orthogonal projection from $L^2(D, v)$ onto $A^2_v$. In 2004, Dostanic revealed the intriguing fact that $P_v$ is bounded from $L^p(D, v)$ to $A^p_v$ only for $p = 2$, and he posed the related problem of identifying the duals of $A^p_v$ for $p \geq 1$, $p \neq 2$. In this paper we propose a solution to this problem by proving that that $P_v$ is bounded from $L^p(D, v^{p/2})$ to $A^{p/2}_v$ whenever $1 \leq p < \infty$, and, consequently, the dual of $A^{p/2}_v$ for $p \geq 1$ can be identified with $A^{q/2}_v$, where $1/p + 1/q = 1$. In addition, we also address a similar question on some classes of weighted Fock spaces.

1. Introduction

Let $D$ be the unit disc in the complex plane, $dm(z) = \frac{dx \, dy}{\pi}$ be the normalized area measure on $D$, and denote by $H(D)$ the space of all analytic functions in $D$. A function $\omega : D \to (0, \infty)$, integrable over $D$, is called a weight function or simply a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in D$.

For $0 < p < \infty$, the weighted Bergman space $A^p_w(D)$ is the space of functions $f \in H(D)$ such that

$$\|f\|^p_{A^p_w} = \int_D |f(z)|^p w(z) \, dm(z) < \infty.$$ 

We shall write $L^\infty_w$ for the weighted growth space of measurable functions $f$ such that

$$\|f\|_{L^\infty_w} = \sup_{z \in D} |f(z)| w(z) < \infty.$$ 

We denote $A^\infty_w = H(D) \cap L^\infty_w$. For $w(r) = (\alpha + 1)(1 - r^2)^\alpha$, $\alpha > -1$, we obtain the standard Bergman spaces $A^p_\alpha$ [7, 18].

Whenever $w$ is a continuous weight, the $\|\cdot\|_{A^p_w}$ convergence implies uniform convergence on compact subsets of $D$, which in particular gives that $A^p_w$ is a

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closed subspace of \( L^p(\mathbb{D}, w) \). In particular, the point evaluations \( L_z \) (at the point \( z \in \mathbb{D} \)) are bounded linear functionals on \( A^2_w \). Therefore, there are reproducing kernels \( K_z \in A^2_w \) with \( \| L_z \| = \| K_z \|_{A^2_w} \) such that

\[
L_z f = f(z) = \langle f, K_z \rangle = \int_{\mathbb{D}} f(\zeta) \overline{K_z(\zeta)} w(\zeta) \, dm(\zeta), \quad f \in A^2_w.
\]

Since \( A^2_w \) is a closed subspace of the Hilbert space \( L^2_w \), we may consider the orthogonal projection \( P_w : L^2(\mathbb{D}, w) \to A^2_w \) that is usually called the Bergman projection. It is precisely the integral operator

\[
P_w(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{K_z(\zeta)} w(\zeta) \, dm(\zeta), \quad f \in L^2(\mathbb{D}, w).
\]

The boundedness of projections on \( L^p \)-spaces is an intriguing topic which has attracted a lot attention in recent years [4, 5, 7, 17, 18]. For the class of standard weights, the Bergman projection

\[
P_{\alpha}(f)(z) = (\alpha + 1) \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^\alpha}{(1 - \overline{z}\zeta)^{2+\alpha}} \, dm(\zeta),
\]

is bounded from \( L^p(\mathbb{D}, (1 - |z|^2)^\alpha) \) onto \( A^p_w \) if and only if \( 1 < p < \infty \) [7, 18]. This result is the key for the description of dual spaces \( (A^p_w)^* \approx A^q_w \), \( p > 1 \) \( \frac{1}{p} + \frac{1}{q} = 1 \), and can be used for obtaining Littlewood-Paley formulas in this context [18, Chapter 4]. Further results along this line were obtained whenever \( \omega \) is a non (necessarily) radial weight such that \( \frac{\omega(z)}{(1 - |z|^2)^\eta} \) belongs to some class of Bekollé weights \( B_p(\eta), \ p > 1, \eta > -1 \) [1, 2]. However, Dostanic [4, 5] revealed a behaviour of the Bergman projection in the case of Bergman spaces with the exponential type weights \( w(r) = (1 - r^2)^A \exp \left( \frac{-B}{(1 - r^2)^\eta} \right) \), \( A \in \mathbb{R}, B, \alpha > 0 \), that is in stark contrast with the situation encountered on standard Bergman spaces. More precisely, boundedness only occurs for \( p = 2 \). This phenomenon has implications for duality issues, as pointed out in [4], where the description of the dual of \( A^p_w \) for \( p \neq 2 \), with \( w \) as above, is stated as an open problem. Dostanic’s result [4] was recently generalized in [17] for a larger class of weights. This behaviour is reminiscent of that of the Bergman projection on Fock spaces [16, 19]. Other similarities between Fock spaces and Bergman spaces with rapidly decreasing weights were pointed out in [12, 3]. Inspired by the Fock space setting, we propose a positive solution for the boundedness of the Bergman projection for the canonical example of a rapidly decreasing weight \( w(z) = e^{-\frac{|z|^2}{1-|z|^2}}, \alpha > 0 \). Our main result is the following:

**Theorem 1.** Let \( v(r) = \exp \left( -\frac{\alpha r^2}{1-r^2} \right), \alpha > 0, \text{ and } 1 \leq p < \infty. \) Then, the Bergman projection

\[
P_v(f)(z) = \int_{\mathbb{D}} f(\zeta) K(z, \zeta) v(\zeta) \, dm(\zeta)
\]

is bounded from \( L^p(\mathbb{D}, v^{p/2}) \) to \( A^p_{v^{\alpha/2}} \). Moreover, \( P_v : L^\infty_{v^{1/2}} \to A^\infty_{v^{1/2}} \) is bounded.
As a consequence of Theorem 1 we identify the dual of \( A^p_{q/2} \) for \( p \geq 1 \) as \( A^q_{p/2} \), where \( 1/p + 1/q = 1 \). The approach to prove our main result relies on accurate estimates for

\[
||K_r||_{H^1} = M_1(r, K) = \int_0^{2\pi} |K(re^{it})| dt,
\]

the integral means of the reproducing kernel of \( A^2_w \), on circles of radius \( r < 1 \) centered at the origin, (see Proposition 5 below). These are obtained using two key tools; the sharp asymptotic estimates obtained in [9] for the moments of our weight in terms of the Legendre-Fenchel transform, and an upper estimate of \( M_1(r, K) \) by the \( l^1 \)-norm of the \( H^1 \)-norms of the Hadamard product of \( K_r \) with certain smooth polynomials.

Concerning the unboundedness of the Bergman projection, we highlight that the decay of the weight plays a role in this problem. In fact, whenever the weight is smooth and decreases rapidly enough, it enters into the framework of the result in [17, Theorem 1.2], so the Bergman projection is bounded from \( L^p(\mathbb{D}, w) \) to \( A^p_w \) only for \( p = 2 \).

**Proposition 2.** Assume that \( w(r) = e^{-2\phi(r)} \) is a radial weight such that \( \phi : [0,1) \rightarrow \mathbb{R}^+ \) is a \( C^\infty \)-function, \( \phi' \) is positive on \( [0,1) \), \( \lim_{r \to 1^-} \phi(r) = \lim_{r \to 1^-} \phi'(r) = +\infty \) and

\[
\lim_{r \to 1^-} \frac{\phi^{(n)}(r)}{(\phi'(r))^n} = 0, \quad \text{for any } n \in \mathbb{N} \setminus \{1\}.
\]

Then, the Bergman projection is bounded from \( L^p(\mathbb{D}, w) \) to \( L^p(\mathbb{D}, w) \) only for \( p = 2 \).

In particular, any weight in [12, section 7] satisfies the hypotheses of Proposition 2, as well as triple exponential weights of the form \( \omega(z) = \exp(-e^{-|z|}) \).

We also investigate some analogous problems in the setting of Fock spaces. Given \( \phi : \mathbb{C} \rightarrow \mathbb{R}^+ \) a (nonharmonic) subharmonic function, we consider the weighted Fock spaces,

\[
\mathcal{F}_p^\phi = \left\{ f \in H(\mathbb{C}) : ||f||_{\mathcal{F}_p^\phi}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} dm(z) < \infty \right\}, \quad 0 < p < \infty,
\]

and

\[
\mathcal{F}_\infty^\phi = H(\mathbb{C}) \cap L_\infty^\phi,
\]

where \( H(\mathbb{C}) \) is the space of entire functions and \( L_\infty^\phi \) is the weighted growth space of measurable functions \( f \) such that

\[
||f||_{L_\infty^\phi} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\phi(z)} < \infty.
\]

We shall say that \( \phi \) is radial if \( \phi(z) = \phi(|z|) \) for all \( z \in \mathbb{C} \).

First, we discuss some positive results. The estimates of the Bergman kernel proved in [15] for a large class of Fock spaces induced by radial weights \( e^{-2\phi} \),
respectively those obtained in [11] for Fock spaces $F^\phi_2$ with nonradial weights, where $\Delta \phi$ is a doubling measure, imply that the Bergman projection

\[
P_\phi(f)(a) = \int_C f(z) \overline{K_a(z)} e^{-2\phi(z)} \, dm(z), \quad a \in \mathbb{C},
\]

is bounded from $L^p(C, e^{-p\phi})$ to $F^\phi_p$ for $1 \leq p < \infty$, and from $L^\infty_\phi$ to $F^\phi_\infty$. From these results we deduce that the dual of $F^\phi_p$ can be identified with $F^\phi_q$, where $1/p + 1/q = 1$.

Finally, regarding the unboundedness of the Bergman projection on “weighted $L^p$ spaces on $\mathbb{C}$”, we note that the analogue of Proposition 2 works.

**Proposition 3.** Assume that $\phi : [0, \infty) \to \mathbb{R}^+$ is a $C^\infty$-function, $\phi'$ is positive on $[0, +\infty)$ and $\lim_{r \to \infty} \phi(r) = \lim_{r \to \infty} \phi'(r) = +\infty$. If

\[
\lim_{r \to \infty} \frac{\phi^{(n)}(r)}{(\phi'(r))^n} = 0, \quad \text{for any } n \in \mathbb{N} \setminus \{1\},
\]

then, Bergman projection is bounded from $L^p(C, e^{-2\phi})$ to $L^p(C, e^{-2\phi})$ only for $p = 2$.

Throughout the paper, the letter $C$ will denote an absolute constant whose value may change at different occurrences. We also use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leq Cb$, and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

## 2. Weighted Bergman spaces

We begin with a general result that may be known to specialists, but we include a proof for the sake of completeness.

**Lemma 4.** Assume that $w = e^{-2\phi}$ is a weight such that its associated kernel function $K(z, \zeta)$ satisfies

\[
M = \sup_{z \in \mathbb{D}} \int_\mathbb{D} |K(z, \zeta)| e^{-\phi(\zeta) - \phi(z)} \, dm(\zeta) < \infty.
\]

Then

\[
P_w : L^p(\mathbb{D}, w^{p/2}) \to A^p_{w^{p/2}}
\]

is bounded for $1 \leq p < \infty$. Moreover, Moreover,

\[
P_w : L^\infty_{w^{1/2}} \to A^\infty_{w^{1/2}}
\]

is a bounded operator.
Proof. For simplicity, we write $P$ instead of $P_w$ throughout the proof. Using (1.1) and Hölder’s inequality we deduce

$$\left| Pf(z)e^{-\phi(z)} \right| \leq \int_{\mathbb{D}} |f(\zeta)e^{-\phi(\zeta)}||K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta)$$

$$\leq \left( \int_{\mathbb{D}} |f(\zeta)e^{-\phi(\zeta)}|^p |K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta) \right)^{\frac{1}{p}} \left( \int_{\mathbb{D}} |K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta) \right)^{\frac{1}{p'}}$$

$$\leq M^{\frac{1}{p'}} \left( \int_{\mathbb{D}} |f(\zeta)e^{-\phi(\zeta)}|^p |K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta) \right)^{\frac{1}{p'}}.$$

So, by the previous inequality, Fubini’s theorem and (2.1), we obtain

$$\int_{\mathbb{D}} \left| Pf(z)e^{-\phi(z)} \right|^p \, dm(z)$$

$$\leq M^{\frac{p}{p'}} \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f(\zeta)e^{-\phi(\zeta)}|^p |K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta) \right) \, dm(z)$$

$$= M^{\frac{p}{p'}} \int_{\mathbb{D}} |f(\zeta)e^{-\phi(\zeta)}|^p \left( \int_{\mathbb{D}} |K(z, \zeta)| e^{-\phi(\zeta)-\phi(z)} \, dm(\zeta) \right) \, dm(\zeta)$$

$$\leq M_{L^p(\mathbb{D}, w^{p/2})}^p.$$ 

The inequality

$$||Pf||_{A^\infty_{\frac{p}{2}}} \leq M||f||_{L^\infty_{\frac{p}{2}}},$$

is immediate, while a simple application of Fubini’s theorem yields

$$||Pf||_{A^1_{\frac{p}{2}}} \leq M||f||_{L^1\left(\mathbb{D}, w^{\frac{p}{2}}\right)}.$$ 

This finishes the proof. 

Bearing in mind Lemma 4, Theorem 1 will been proved once we get the following inequality

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |K(z, \zeta)| \exp \left( -\frac{\alpha}{2(1-|z|)} - \frac{\alpha}{2(1-|\zeta|)} \right) \, dm(\zeta) < \infty,$$

where $K$ is the reproducing kernel of the weighted Bergman space $A^2 \left( \exp \left( -\frac{\alpha}{(1-|\zeta|)} \right) \right)$. 

It is known that

$$K(z, \zeta) = \sum_{n=0}^{\infty} \frac{(z\zeta)^n}{v_{2n+1}},$$

where, for any positive $\lambda$, $v_\lambda = \int_0^1 r^\lambda \exp \left( -\frac{\alpha}{1-r} \right) \, dr$. Our approach begins with writing the integral in (2.2) in polar coordinates. So the next result will be a key ingredient in this procedure. Its proof will be presented in Section 2.1.
Proposition 5. Let \( v(r) = \exp\left(-\frac{\alpha}{1-r}\right), \alpha > 0 \), and let \( K(z) = \sum_{n=0}^{\infty} \frac{z^n}{v^{2n+1}} \). Then, there is a positive constant \( C \) such that

\[
M_1(r, K) \approx \exp\left(\alpha\sqrt{1-r}\right)(1-r)^{3/2}, \quad r \to 1^-.
\]

where

\[
M_1(r, K) = \int_{0}^{2\pi} |K(re^{it})| dt, \quad 0 < r < 1.
\]

Proof of Theorem 1. By Proposition 5,

\[
\int_{D} |K(z, \zeta)| \exp\left(-\frac{\alpha}{2(1-|z|)} - \frac{\alpha}{2(1-|\zeta|)}\right) dm(\zeta)
\]

\[
\times \int_{0}^{1} \exp\left(\left(\frac{\alpha}{1-\sqrt{s}|z|} - \frac{\alpha}{2(1-|z|)} - \frac{\alpha}{2(1-s)}\right) ds
\]

Put \( r = |z| \) and denote the integrand in the last relation above by \( f_r(s) \). Now, a calculation shows that

\[
\left(\frac{1}{1-\sqrt{sr}}\right) - \frac{1}{2(1-r)} - \frac{1}{2(1-s)} = \frac{1}{2} \left(\sqrt{s} - \sqrt{r}\right) \left(\frac{\sqrt{r}}{1-\sqrt{sr}} - \frac{\sqrt{s}}{1-s}\right).
\]

In particular, the above expression is negative. We assume without loss of generality \( r > \frac{1}{2} \). We provide an upper bound for the last integral in (2.3) by means of a sum of integrals on four subintervals of \([0, 1] \). Let us first notice that

\[
\int_{0}^{1/2} f_r(s) ds \leq \frac{1}{2(1 - \frac{1}{\sqrt{2}})^{3/2}}.
\]

and that

\[
\int_{r-(1-r)^{3/2}}^{r+(1-r)^{3/2}} f_r(s) ds \leq \frac{1}{(1-\sqrt{r})^{3/2}} \int_{r-(1-r)^{3/2}}^{r+(1-r)^{3/2}} ds \leq C.
\]

It remains to deal with the integrals \( \int_{r+(1-r)^{3/2}}^{1} f_r(s) ds \) and \( \int_{1/2}^{1-(1-r)^{3/2}} f_r(s) ds \). If \( N = N(r) \) is the largest positive integer such that

\[
2^{2N} < \frac{1}{1-r},
\]

then

\[
\int_{r+(1-r)^{3/2}}^{1} f_r(s) ds = \sum_{k=0}^{N} \int_{r+2^k(1-r)^{3/2}}^{r+2^{k+1}(1-r)^{3/2}} f_r(s) ds.
\]

For \( r + 2^k(1-r)^{3/2} \leq s \leq \min(r + 2^{k+1}(1-r)^{3/2}, 1) \) we have

\[
\frac{\sqrt{s} - \sqrt{r}}{1-\sqrt{sr}} = \frac{1 + \sqrt{sr}}{\sqrt{s} + \sqrt{r}} \frac{s-r}{1-r} \geq C \frac{s-r}{1-r} \geq C 2^k (1-r)^{1/2}.
\]
and, by the mean value theorem, there exists \( r < x < s \) such that

\[
(2.8) \quad \frac{\sqrt{s}}{1-s} - \frac{\sqrt{r}}{1-r} = \frac{1 + x}{2\sqrt{x(1-x)^2}}(s-r) \geq C \frac{2^k(1-r)^{3/2}}{(1-r)^2} = C' 2^k(1-r)^{-1/2}.
\]

Using (2.7) and (2.8) in (2.6) we obtain

\[
\int_{r+(1-r)^{3/2}}^{1} f_r(s) \, ds \leq \sum_{k=0}^{N} \int_{r+2^k(1-r)^{3/2}}^{\min(r+2^k+1(1-r)^{3/2},1)} e^{-C4^k} \frac{1}{(1-\sqrt{sr})^{3/2}} \, ds
\]

\[
\leq \sum_{k=0}^{N} \int_{r+2^k(1-r)^{3/2}}^{r+2^{k+1}(1-r)^{3/2}} e^{-C4^k} \frac{1}{(1-r)^{3/2}} \, ds
\]

\[
\leq \sum_{k=0}^{\infty} 2^k e^{-C4^k} < \infty.
\]

Finally, if \( N \) is the biggest positive integer such that

\[
2^N < \frac{r - 1/2}{(1-r)^{3/2}}.
\]

we have

\[
(2.10) \quad \int_{1/2}^{r-(1-r)^{3/2}} f_r(s) \, ds \leq \sum_{k=0}^{N} \int_{r-2^k(1-r)^{3/2}}^{r-2^{k+1}(1-r)^{3/2}} f_r(s) \, ds.
\]

Consider now \( s \in [r - 2^{k+1}(1-r)^{3/2}, r - 2^k(1-r)^{3/2}] \). Then

\[
(2.11) \quad \frac{\sqrt{r} - \sqrt{s}}{1-\sqrt{sr}} \geq C \frac{r-s}{1-sr} \geq C \frac{2^k(1-r)^{3/2}}{1-r(r-2^{k+1}(1-r)^{3/2})} = C \frac{2^k(1-r)^{3/2}}{(1-r)(1+r+2^{k+1}(1-r)^{1/2})} \geq C \frac{2^k(1-r)^{1/2}}{2+2^{k+1}(1-r)^{1/2}} \geq C \frac{2^k(1-r)^{1/2}}{1+2^k(1-r)^{1/2}}.
\]

Moreover, we use the fact that the function \( x \mapsto \frac{\sqrt{x}}{1-x} \) is increasing for \( x > 0 \) to deduce

\[
(2.12) \quad \frac{\sqrt{r}}{1-r} - \frac{\sqrt{s}}{1-s} \geq \frac{\sqrt{r} - \sqrt{s}}{1-\sqrt{sr}} \geq \frac{\sqrt{r - 2^k(1-r)^{3/2}}}{1-r + 2^k(1-r)^{3/2}} = \frac{\sqrt{r(1+2^k(1-r)^{1/2})} - \sqrt{r - 2^k(1-r)^{3/2}}}{(1-r)(1+2^k(1-r)^{1/2})} \geq \frac{2^{k-1}(1-r)^{-1/2}}{1+2^k(1-r)^{1/2}}.
\]
since
\[
\sqrt{r}(1 + 2^k (1 - r)^{1/2}) - \sqrt{r - 2^k (1 - r)^{3/2}} = \frac{(1 - r)^{1/2} [r 2^{k+1} + r 2^{2k} (1 - r)^{1/2} + 2^k (1 - r)]}{\sqrt{r}(1 + 2^k (1 - r)^{1/2}) + \sqrt{r - 2^k (1 - r)^{3/2}}} \geq \frac{(1 - r)^{1/2} 2^{k+1} + r 2^{2k} (1 - r)^{1/2}}{2 + 2^k (1 - r)^{1/2}} \geq 2^{k-1} (1 - r)^{1/2}.
\]

Combining relations (2.10)-(2.12) we obtain

\[
\int_{1/2}^{r-(1-r)^{3/2}} f_r(s) \, ds 
\leq \sum_{k=0}^{N} \exp \left( -\frac{C}{(1 + 2^k (1 - r)^{1/2})^2} \right) \int_{r - 2^k (1 - r)^{3/2}}^{1 - 2^k (1 - r)^{3/2}} \frac{1}{(1 - sr)^{3/2}} \, ds 
\leq \sum_{k=0}^{N} \exp \left( -\frac{C}{(1 + 2^k (1 - r)^{1/2})^2} \right) \frac{2^{k+3/2}}{(1 + 2^k (1 - r)^{1/2})^{3/2}} 
\leq 2^{5/2} \int_{1/2}^{(1-r)^{3/2}} e^{-\frac{C x^2}{(1 + (1-r)^{1/2}x)^2}} \frac{1}{(1 + (1-r)^{1/2}x)^{3/2}} \, dx,
\]

where the last step follows in view of the inequalities

\[
\frac{2^k}{(1 + 2^k (1 - r)^{1/2})^{3/2}} \leq 2 \int_{2^{k-1}}^{2^k} \frac{dx}{(1 + (1-r)^{1/2}x)^{3/2}}
\]

and

\[
\frac{2^{2k}}{(1 + 2^k (1 - r)^{1/2})^2} \geq \frac{2^{2k}}{4(1 + 2^{k-1} (1 - r)^{1/2})^2} \geq \frac{x^2}{4(1 + (1-r)^{1/2}x)^2}, \quad 2^{k-1} \leq x \leq 2^k.
\]

Now

\[
\int_{1/2}^{(1-r)^{3/2}} \exp \left( -\frac{C x^2}{(1 + (1-r)^{1/2}x)^2} \right) \frac{dx}{(1 + (1-r)^{1/2}x)^{3/2}} 
\leq \int_{1/2}^{(1-r)^{3/2}} e^{-\frac{C x^2}{(1 + (1-r)^{1/2}x)^2}} \, dx \leq \int_{1/2}^{\infty} e^{-\frac{C x^2}{(1 + (1-r)^{1/2}x)^2}} \, dx < \infty.
\]

On the other hand, performing the substitution \( t = (1 - r)^{1/2} x \) we obtain

\[
\int_{(1-r)^{3/2}}^{(1-r)^{1/2}} \exp \left( -\frac{C x^2}{(1 + (1-r)^{1/2}x)^2} \right) \frac{dx}{(1 + (1-r)^{1/2}x)^{3/2}} 
= \int_{1}^{1/(1-r)} \exp \left( -\frac{C t^2}{(1 - r)(1 + t)^2} \right) \frac{dt}{\sqrt{1 - r(1+t)^{3/2}}} 
\leq (1 - r)^{-1/2} e^{-\frac{C}{4(1-r)}} \int_{1}^{1/(1-r)} \frac{dt}{(1 + t)^{3/2}} \to 0,
\]

as \( r \to 1^- \).
as $r \to 1$. Taking into account relations (2.14)-(2.15) we now return to (2.13) to deduce

\[(2.15) \sup_{1/2<r<1} \int_{1/2}^{r-(1-r)^{3/2}} f_r(s) \, ds < \infty,\]

and, with this, the proof is complete.\[\square\]

**Remark 6.** It follows from the proof of Theorem 1 that the sublinear operator

$$\tilde{P}_v(f)(z) = \int_{C} |f(\zeta)| \left| K(z, \zeta) \right| v(z) \, dm(\zeta)$$

is bounded from $L^p(D, v^{p/2})$ to $A^p v^{p/2}$ for $p \geq 1$, respectively from $L^\infty v^{1/2}$ to $A^\infty v^{1/2}$.

### 2.1. Integral means of the reproducing kernel.

We need some preliminary results.

**Lemma 7.** For each $\alpha, \lambda \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$, let us consider

$$v_{\lambda, \log^n} = \int_{0}^{1} r^\lambda \left( \log \frac{1}{r} \right)^n \exp \left( -\frac{\alpha}{1-r} \right) \, dr.$$ 

Then,

$$v_{\lambda, \log^n} \asymp \lambda^{-\frac{2n+3}{4}} \exp \left( -2\sqrt{\alpha\lambda} \right), \quad \lambda \to \infty,$$

where the constants involved may depend on $\alpha$ and $n$.

**Proof.** Let us observe that

$$\exp \left( -\frac{\alpha}{1-r} \right) \leq \exp \left( -\frac{\alpha}{\log \frac{1}{r}} \right) \leq \exp \left( \alpha - \frac{\alpha}{1-r} \right), \quad 0 < r < 1,$$

so it is enough to estimate

$$\tilde{w}_{\lambda, \log^n} = \int_{0}^{1} r^\lambda \left( \log \frac{1}{r} \right)^n \exp \left( -\frac{\alpha}{\log \frac{1}{r}} \right) \, dr.$$ 

A change of variables $\log \frac{1}{r} = t$,

$$\tilde{w}_\lambda = \int_{0}^{\infty} e^{-\frac{\alpha}{t} - (\lambda+1)t + n \log t} \, dt.$$ 

Then, take $v(t) = \frac{\alpha}{t} - n \log t$ and consider its Legendre-Fenchel transform

\[(2.17) \quad L(x) = \inf_{0 < t < \infty} [v(t) + xt].\]

A simple calculation shows that $L(x) = \sqrt{n^2 + 4\alpha x} + n \log \frac{2x}{\sqrt{n^2 + 4\alpha x} + n}$, so

$$e^{-L(x)} \asymp x^{-\frac{\alpha}{4}} \exp \left( -2\sqrt{\alpha x} \right), \quad x \to \infty.$$ 

Moreover,

$$-L''(x) \asymp x^{-\frac{\alpha}{4}}, \quad x \to \infty.$$
So by [9, Theorem 1] we obtain the estimate for $v_{\lambda, \log n}$. This finishes the proof.

Now let us consider two sequence of polynomials with useful properties with respect to smooth partial sums.

Firstly, given a $C^\infty$-function $\Phi : \mathbb{R} \to \mathbb{C}$ with compact support $\text{supp}(\Phi)$, we set

$$A_{\Phi, m} = \max_{s \in \mathbb{R}} |\Phi(s)| + \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)|,$$

and we consider the polynomials

$$W_n^\Phi(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \Phi\left(\frac{k}{n}\right) e^{ik\theta}, \quad n \in \mathbb{N}.$$ 

For any function $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(D)$, we write

$$(W_n^\Phi \ast f)(z) = \sum_{k=0}^{\infty} \Phi\left(\frac{k}{n}\right) a_k z^k$$

for the Hadamard product of $W_n^\Phi$ and $f$. With this notation we can state the next result, which follows from the considerations in [13, p.111-113]. We denote by $H^p$ the classical Hardy spaces on $\mathbb{D}$ (see [6]).

**Theorem A.** Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a $C^\infty$-function with compact support $\text{supp}(\Phi)$. The following assertions hold:

(i) There exists a constant $C > 0$ such that

$$\left| W_n^\Phi(e^{i\theta}) \right| \leq C \min \left\{ n \max_{s \in \mathbb{R}} |\Phi(s)|, n^{1-m} |\theta|^{-m} \max_{s \in \mathbb{R}} |\Phi^{(m)}(s)| \right\},$$

for all $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $0 < |\theta| < \pi$.

(ii) If $0 < p \leq 1$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C(p) > 0$ such that

$$\left( \sup_{n} \left| (W_n^\Phi \ast f)(e^{i\theta}) \right| \right)^p \leq C A_{\Phi, m} M(|f|^p)(e^{i\theta})$$

for all $f \in H^p$. Here $M$ denotes the Hardy-Littlewood maximal-operator

$$M(|f|)(e^{i\theta}) = \sup_{0 < h < \pi} \frac{1}{2h} \int_{\theta-h}^{\theta+h} |f(e^{it})| dt.$$ 

(iii) Then, for each $p \in (0, \infty)$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that

$$\|W_n^\Phi \ast f\|_{H^p} \leq C A_{\Phi, m} \|f\|_{H^p}$$

for all $f \in H^p$ and $n \in \mathbb{N}$.

Secondly, let us consider a useful particular case of the previous construction. We define the sequence of polynomials $\{V_n\}_{n=0}^{\infty}$ [8, Section 2] (see also [14, p. 9]) as follows. Let $\Psi$ be a $C^\infty$-function on $\mathbb{R}$ such that

(1) $\Psi(t) = 1$ for $t \leq 1$, 

(2) $\Psi(t) = 0$ for $t > 1$.
(2) $\Psi(t) = 0$ for $t \geq 2$,
(3) $\Psi$ is decreasing and positive on the interval $(1, 2)$.

Set $\psi(t) = \Psi\left(\frac{t}{2}\right) - \Psi(t)$. Let $V_0(z) = 1 + z$ and for $n \geq 1$, let

$$V_n(z) = W_{2^n-1}^{\psi}(z) = \sum_{k=0}^{\infty} \psi\left(\frac{k}{2^n-1}\right) z^k = \sum_{k=2^n-1}^{2^n-1} \psi\left(\frac{k}{2^n-1}\right) z^k.$$  

(2.18)

These polynomials have the following properties with regard to smooth partial sums \cite{8} (see also \cite{13, p.111-113}):

$$f(z) = \sum_{n=0}^{\infty} (V_n * f)(z), \quad \text{for any } f \in H(\mathbb{D}),$$  

(2.19)

$$||V_n * f||_{H^p} \leq C||f||_{H^p}, \quad \text{for any } f \in H^p \text{ and } 0 < p < \infty,$$

$$||V_n||_{H^p} \propto 2^{n(1-1/p)}, \quad \text{for any } 0 < p < \infty.$$  

With these preparations we are ready to prove the estimate for the integral means of the kernel.

**Proof of Proposition 5.**

First, we shall prove the upper estimate

$$M_1(r, K) \lesssim \frac{\exp\left(\frac{r}{1-\rho}\right)}{1 - \rho}, \quad \rho = \sqrt{r}.$$  

(2.20)

**Step 1.** It follows from (2.19) that

$$M_1(r, K) = ||f||_{H^1} \leq \sum_{n=0}^{\infty} ||V_n * f||_{H^1},$$  

(2.21)

where $f(z) = \sum_{n=0}^{\infty} \frac{r^n}{v_{2n+1}} z^n$. Next each $n = 3, 4, \ldots$, let us consider

$$F_n(x) = \frac{r^x}{v_{2n+1}}, \quad x \in [2^{n-1}, 2^n].$$  

Applying Lemma 7 for $n = 0, 1, 2$ we have that

$$\int_0^{1} s^{2x+1} \log \left(1 + \frac{1}{s}\right) v(s) ds \lesssim v_{2x+1}, \quad \text{and} \quad \int_0^{1} s^{2x+1} \left(\log \left(1 + \frac{1}{s}\right)\right)^2 v(s) ds \lesssim x^{-1}, \quad x \to \infty$$  

so there is a positive constant $C$ such that

$$A_{F_n,2} = \max_{x \in [2^{n-1}, 2^n]} |F_n(x)| + \max_{x \in [2^{n-1}, 2^n]} |F_n''(x)|$$

$$\leq C \max_{x \in [2^{n-1}, 2^n]} F_n(x) = CM_n, \quad n \in \mathbb{N},$$  

where $M_n = \max_{x \in [2^{n-1}, 2^n]} F_n(x)$.

For each $n = 3, 4, \ldots$, choose $\Phi_n$ a $C^\infty$-function with compact support contained in $[2^{n-2}, 2^{n+2}]$ such that $\Phi_n(x) = F_n(x) = \frac{r^x}{v_{2x+1}}$ if $x \in [2^{n-1}, 2^n]$ and

$$A_{\Phi_n,2} = \max_{x \in \mathbb{R}} |\Phi_n(x)| + \max_{x \in \mathbb{R}} |\Phi_n''(x)| \leq CM_n, \quad n = 3, 4, \ldots$$  

(2.22)
Since
\[ W_1^Φ (e^{iθ}) = \sum_{k \in Z} Φ_n (k) e^{ikθ} = \sum_{k \in \mathbb{Z} \cap [2n-2, 2n+2]} Φ_n (k) e^{ikθ} \]
and by (2.18) \( \text{supp} \hat{V}_n \subset [2^{n-1}, 2^{n+1} - 1] \subseteq [2^{n-2}, 2^{n+2}] \), then
\[
(V_n * f) (z) = \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi \left( \frac{k}{2^{n-1}} \right) \frac{r_k}{v_{2k+1}} z^k
\]
\[
= \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi \left( \frac{k}{2^{n-1}} \right) Φ_n (k) z^k
\]
\[
= \left( W_1^Φ * V_n \right) (z)
\]
This together with Theorem A, (2.22) and (2.19), implies that
\[
\| V_n * f \|_{H^1} = \| W_1^Φ * V_n \|_{H^1}
\]
\[
\leq C A_{Φ_n, 2} \| V_n \|_{H^1}
\]
\[
\leq C M_n \| V_n \|_{H^1} \leq C M_n \quad n = 3, 4, \ldots
\]
So, bearing in mind (2.21)
\[
(2.23) \quad M_1 (r, K) \leq C + \sum_{n=3}^{∞} M_n.
\]

**Step 2.** Recall from Lemma 7 that
\[
(2.24) \quad \frac{r^x}{v_{2x+1}} = \frac{ρ^{2x}}{v_{2x+1}} \asymp \rho^{2x} (2x)^\frac{x}{4} e^{2\sqrt{2a}x} = e^{h(2x)},
\]
where \( h(x) = \frac{3}{4} \log(x) + 2\sqrt{ax} + x \log ρ \), \( x \in (0, ∞) \). A calculation shows that
\( h \) increases in \((0, x_ρ)\) and decreases in \((x_ρ, ∞)\), where \( x_ρ = \left( \frac{\sqrt{α+3 \log \frac{1}{ρ} + \sqrt{α}}}{2 \log \frac{1}{ρ}} \right)^2 \).
Moreover,
\[
(2.25) \quad \sup_{x \in (0, ∞)} e^{h(x)} = e^{h(x_ρ)} \asymp \left( \frac{1}{\log \frac{1}{ρ}} \right)^{3/2} \exp \left( \frac{α}{\log \frac{1}{ρ}} \right) \asymp \left( \frac{1}{1 - ρ} \right)^{3/2} \exp \left( \frac{α}{1 - ρ} \right).
\]

**Step 3.** Choose \( n_0 \in \mathbb{N} \) such that \( 2^{n_0} \leq x_ρ < 2^{n_0+1} \) and split the sum in (2.23) as follows
\[
(2.26) \quad \sum_{n=3}^{∞} M_n = \sum_{n=3}^{n_0-3} M_n + \sum_{n=n_0-2}^{n_0+1} M_n + \sum_{n=n_0+2}^{∞} M_n
\]
Bearing in mind (2.24) and the monotonicity of \( h \), we deduce for \( 3 \leq n \leq n_0 - 3 \)
\[
(2.27) \quad M_n = \max_{x \in [2^{n-1}, 2^{n+1}]} \frac{ρ^{2x}}{v_{2x+1}} \asymp \max_{x \in [2^{n-1}, 2^{n+1}]} e^{h(2x)} = e^{h(2^{n+2})}.
\]
It follows from (2.24) and (2.25) that
\begin{equation}
\sum_{n=n_0-2}^{n_0+1} M_n \lesssim \left( \frac{1}{1-\rho} \right)^{3/2} \exp \left( \frac{\alpha}{1-\rho} \right).
\end{equation}

Using again (2.24) and the monotonicity of \( h \), whenever \( n \geq n_0 + 2 \)
\begin{equation}
M_n = \max_{x \in [2^{n-1},2^n+1]} v_{2x+1} \times \max_{x \in [2^{n-1},2^n+1]} \exp h(2x) = \exp h(2^n),
\end{equation}
which together with (2.26), (2.27) and (2.28) gives that
\begin{equation}
\sum_{n=3}^{\infty} M_n \lesssim \sum_{n=4}^{n_0-2} \exp h(2^{n+1}) + \sum_{n=n_0+2}^{\infty} \exp h(2^n) + \left( \frac{1}{1-\rho} \right)^{3/2} \exp \left( \frac{\alpha}{1-\rho} \right).
\end{equation}

Next, if \( n \leq n_0 - 2 \),
\begin{equation}
\exp h(2^{n+1}) \leq \frac{1}{2^{n+1}} \sum_{k=2^n}^{2^{n+1}-1} \exp h(2k) \leq 4 \sum_{k=2^n}^{2^{n+1}-1} \frac{\exp h(2k)}{k+1}
\end{equation}
and for \( n \geq n_0 + 2 \)
\begin{equation}
\exp h(2^n) \leq \frac{1}{2^{n-2}} \sum_{k=2^{n-2}}^{2^{n-1}-1} \exp h(2k) \leq 4 \sum_{k=2^{n-2}}^{2^{n-1}-1} \frac{\exp h(2k)}{k+1}.
\end{equation}

Joining the above two inequalities with (2.29) and (2.23), we get
\begin{equation}
M_1(r,K) \lesssim \left( \frac{1}{1-\rho} \right)^{3/2} \exp \left( \frac{\alpha}{1-\rho} \right) + \sum_{k=1}^{\infty} \frac{\exp h(2k)}{k+1}
\times \left( \frac{1}{1-\rho} \right)^{3/2} \exp \left( \frac{\alpha}{1-\rho} \right) + \sum_{k=1}^{\infty} (2k)^{-1/4} \exp \left( 2\sqrt{2} \alpha k \right) \rho^{2k}
\leq \left( \frac{1}{1-\rho} \right)^{3/2} \exp \left( \frac{\alpha}{1-\rho} \right) + \sum_{n=2}^{\infty} n^{-1/4} \exp \left( 2\sqrt{\alpha n} \right) \rho^n.
\end{equation}

Next, let us consider \( N(x) = 2\sqrt{\alpha x} - \frac{1}{6} \log(x) \). A new calculation shows that its inverse Legendre-Fenchel transform (see [9]) is
\( u(t) = \frac{2\alpha + t + 2\sqrt{\alpha(\alpha - t)}}{4t} - \frac{1}{2} \log \left( \frac{\sqrt{\alpha + \sqrt{\alpha - t}}}{2t} \right), \)
so it follows that
\( e^{u(t)} \asymp \sqrt{t} e^{\frac{\alpha}{t}}, \quad t \to 0^+, \)
and
\( u''(t) \asymp \frac{1}{t^3}, \quad t \to 0^+. \)

So, by [9, Corollary 1]
\[ \sum_{n=2}^{\infty} n^{-\frac{1}{4}} \exp \left( 2\sqrt{\alpha n} \right) \rho^n \asymp \frac{\exp \left( \frac{\alpha}{1-\rho} \right)}{1-\rho}, \]
which together with (2.30) finishes the proof of the upper estimate for $M_1(r, K)$.

On the other hand, choose $N_0 \in \mathbb{N}$ such that $N_0 \leq \frac{2N_0}{2} < N_0 + 1$. Then, by Lemma 7
\[
\frac{r_0^N}{v_2N_0+1} \propto \rho^{2N_0} (2N_0)^{\frac{3}{2}} e^{2\sqrt{2\alpha N_0}} = e^{h(2N_0)} \propto e^{h(x_{\rho})},
\]
which together with the well known estimate on coefficients of $H^1$-functions [6, Theorem 6.4], implies that
\[
M_1(r, K) \geq \frac{r_0^N}{v_2N_0+1} \propto \left(\frac{1}{1-\rho}\right)^{3/2} \exp \left(\frac{\alpha}{1-\rho}\right).
\]
This concludes the proof. \(\square\)

2.2. Dual spaces. As an immediate consequence of Theorem 1 we have

**Corollary 8.** Assume $v(r) = \exp \left(-\frac{\alpha}{1-r}\right)$, $\alpha > 0$. For $p > 1$, the dual of $A^p_{v^p/2}$ is $A^q_{v^q/2}$, under the pairing
\[
\langle f, g \rangle = \int_{\mathbb{D}} f \bar{g} vdm,
\]
where $1/p + 1/q = 1$. Moreover, the dual of $A^1_{v^1/2}$ is $A^\infty_{v^1/2}$ under the pairing (2.31).

**Proof.** The fact that any function in $A^q_{v^q/2}$ induces a bounded linear functional on $A^p_{v^p/2}$ with the pairing (2.31) follows by a simple application of Hölder’s inequality in the case $p > 1$, while the case $p = 1$ is obvious.

Conversely, assume that $L$ is a bounded linear functional on $A^p_{v^p/2}$. By the Hahn-Banach theorem we can extend $L$ to a functional $\tilde{L}$ on $L^p(\mathbb{D}, v^h/2)$ with $\|L\| = \|\tilde{L}\|$. Hence there exists a function $h \in L^q(v^h/2)$ for $p > 1$, respectively $h \in L^\infty_{v^1/2}$ for $p = 1$, such that
\[
L f = \int_{\mathbb{D}} f \bar{h} vdm, \quad f \in A^p_{v^p/2},
\]
and $\|\tilde{L}\| = \|h\|_{L^q(v^h/2)}$. Then, in view of Remark 6, we can use Fubini’s theorem to deduce
\[
L f = \int_{\mathbb{D}} f \bar{h} vdm = \int_{\mathbb{D}} P_v f \bar{h} vdm
= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} f(\zeta) K(z, \zeta) v(\zeta) dm(\zeta) \right) \bar{h}(z) v(z) dm(z)
= \int_{\mathbb{D}} f(\zeta) P_v h(\zeta) v(\zeta) dm(\zeta).
\]
By Theorem 1 we obtain $P_v h \in A^q_{v^q/2}$ and
\[
\|P_v h\|_{A^q_{v^q/2}} \lesssim \|h\|_{L^q(v^h/2)} = \|L\|, \quad \text{if } p > 1.
\]
Moreover, for \( p = 1 \) we have \( P_v h \in A_{1/2}^{\infty} \) and
\[
\| P_v h \|_{A_{1/2}^{\infty}} \lesssim \| h \|_{L_{1/2}^{\infty}} = \| L \|
\]
and the proof is complete. \( \square \)

2.3. \( L^p \)-unboundedness. This section is mainly devoted to prove Proposition 2. For this aim, we shall use [17, Theorem 1.2].

**Theorem B.** If \( w \) is a radial weight on \( \mathbb{D} \) which satisfies

(i) \( w \) is \( C^\infty([0,1)) \),

(ii) \( \lim_{r \to 1^-} w^{(n)}(r) = 0 \) for any \( n \in \mathbb{N} \),

(iii) for any \( n \in \mathbb{N} \) there exists \( a_n \in (0,1) \) such that \( (-1)^n w^{(n)} \) is non-negative on the interval \( (a_n, 1) \).

Then, if \( 0 < p < \infty \), the Bergman projection is bounded from \( L^p(\mathbb{D}, w) \) to \( L^p(\mathbb{D}, w) \) only for \( p = 2 \).

**Definition 9.** Given \( \phi \in C^\infty([0,1)) \), we shall say that \( Q_n(\phi) \) is a product of level \( n \) (for \( \phi \)) if \( Q_n(\phi) = \Pi_{j=1}^n (\phi(j))^{m(j)} \) where \( m(j) \in \mathbb{N} \cup \{0\} \) and \( \Sigma_{j=1}^n jm(j) = n \).

We shall also use the next lemma which proof will be omitted.

**Lemma 10.** Assume that \( \phi \in C^\infty([0,1)) \). Then

(1) If \( Q_n(\phi) = \Pi_{j=1}^n (\phi(j))^{m(j)} \) is a product of level \( n \), then \( \phi' Q_n(\phi) = (\phi')^{m(1)+1} \Pi_{j=2}^n (\phi(j))^{m(j)} \) is a product of level \( (n+1) \).

(2) The derivative of a product of level \( n \) is a finite linear combination of products of level \( (n+1) \).

**Proof of Proposition 2.**

First we observe that
\[
(2.32) \quad \lim_{r \to 1^-} (\phi'(r))^n e^{-2\phi(r)} = 0, \quad \text{for any } n \in \mathbb{N}.
\]

This holds if and only if
\[
\lim_{r \to 1^-} 2\phi(r) - n \log \phi'(r) = +\infty.
\]

So it suffices to prove
\[
\lim_{r \to 1^-} \frac{\phi(r)}{\log \phi'(r)} = +\infty,
\]
but this follows from L’Hospital’s rule and (1.2) for \( n = 2 \).

Next, we are going to see that \( w = e^{-2\phi} \) satisfies the hypotheses of Theorem B. Since \( \phi \in C^\infty([0,1)) \), Theorem B (i) is clear. Observe that \( w' = -2\phi'e^{-2\phi} \) and \( w'' = [4(\phi')^2 - 2\phi''] e^{-2\phi} \). Indeed reasoning by induction, it follows from Lemma 10 that
\[
(2.33) \quad w^{(n)} = P_n(\phi)e^{-2\phi}, \quad \text{where } P_n(\phi) = (-1)^n 2^n (\phi')^n + R_n(\phi),
\]
where the harmless term \( R_n(\phi) \) is a finite linear combination of products of level \( n \) such that the exponent of \( \phi' \) is at most \((n - 1)\). So, by (1.2) and (2.33)

\[
(2.34) \quad \lim_{r \to 1^-} P_n(\phi)(r) (\phi'(r))^{-n} = (-1)^n 2^n,
\]

which together (2.32) gives

\[
\lim_{r \to 1^-} w^{(n)}(r) = \lim_{r \to 1^-} \frac{P_n(\phi)(r)}{(\phi'(r))^{-n}} (\phi'(r))^{-n} e^{-2\phi(r)} = 0,
\]

that is, Theorem B (ii) holds.

On the other hand, by (2.33),

\[
(-1)^n w^{(n)} = \left[ 2^n (\phi')^{-n} + (-1)^n R_n(\phi) \right] e^{-2\phi} = \left[ 2^n + \frac{(-1)^n R_n(\phi)}{(\phi')^{-n}} \right] (\phi')^{-n} e^{-2\phi}.
\]

Since \( \phi' \) is positive on \([0,1)\) and \( \lim_{r \to 1^-} \frac{(-1)^n R_n(\phi)}{(\phi'(r))^{-n}} = 0 \), we deduce that there is \( a_n(\phi) \in (0,1) \) such that \((-1)^n w^{(n)}(r) > 0\) on the interval \((a_n,1)\). This fact together with Theorem B finishes the proof. \( \square \)

3. Weighted Fock spaces

We begin by introducing a couple of definitions.

**Definition 11.** We shall say that \( \phi : \mathbb{C} \to \mathbb{R}^+ \) belongs to the class \( \mathcal{D} \) if it is a subharmonic function (not necessarily radial) having the property that \( \mu = \Delta \phi \) is a doubling measure.

**Definition 12.** We shall say that \( \phi : \mathbb{C} \to \mathbb{R}^+ \) belongs to the class \( \mathcal{S} \) if it is radial and the function \( \Psi(x) = 2\phi(\sqrt{x}) \) satisfies that

\[
\Psi'(x) > 0, \quad \Psi''(x) \geq 0, \quad \Psi'''(x) \geq 0
\]

and there exists a real number \( \eta < \frac{1}{2} \) such that

\[
2\Psi''(x) + x\Psi'''(x) = O\left( \left(\frac{\Psi'(x) + x\Psi''(x)}{\sqrt{x}}\right)^{1+\eta} \right), \quad x \to \infty.
\]

It is worth to notice that \( \mathcal{D} \cap \mathcal{S} \) contains functions, as shown by the examples \( \phi(z) = |z|^m \) with \( 2 < m < 4 \). However, roughly speaking, if \( \phi \) is radial, smooth enough an increases very rapidly, then \( \phi \in \mathcal{S} \setminus \mathcal{D} \).

**Theorem 13.** Assume that \( 1 \leq p < \infty \) and \( \phi : \mathbb{C} \to \mathbb{R}^+ \) belongs to \( \mathcal{D} \cup \mathcal{S} \). Then, the Bergman projection

\[
P_\phi(f)(z) = \int_\mathbb{C} f(\zeta) K(z, \zeta) e^{-2\phi(\zeta)} dm(\zeta)
\]

is bounded from \( L^p(\mathbb{C}, e^{-p\phi}) \) to \( F^\phi_p \). Moreover, \( P_\phi : L^\infty \to F^\phi_\infty \) is bounded.
Proof. We shall split the proof in two cases.

Case 1. Assume that $\phi \in D$. We shall follow ideas and the notation from [11]. Adapting the approach used in the proof of Lemma 4 to the complex plane setting, we see that the result will follow once we can show that there is a positive constant $M$ such that

$$\sup_{z \in \mathbb{C}} \int_{\mathbb{C}} |K(z, \zeta)| e^{-\phi(\zeta) - \phi(z)} \, dm(\zeta) \leq M. \tag{3.1}$$

In order to prove (3.1) we need to introduce a technical tool. We denote by $\rho(z)$ the positive radius for which we have $\mu(D(z, \rho(z))) = 1$, $z \in \mathbb{C}$. The function $\rho^{-2}$ can be regarded as a regularized version of $\Delta \phi$ (see [10]). For $z \in \mathbb{C}$, we introduce the notation $\tilde{D}_z = \{ \zeta \in \mathbb{C} : |z - \zeta| < \rho(z) \}$. Then we have

$$\int_{\mathbb{C}} |K(z, \zeta)| e^{-\phi(\zeta) - \phi(z)} \, dm(\zeta) = I + II. \tag{3.2}$$

Estimating $II$ is the easy part. Combining [11, Proposition 2.11] with [10, Corollary 3] we deduce

$$II \lesssim \int_{D(z, \rho(z)) \cup \tilde{D}_z} \frac{1}{\rho(z) \rho(\zeta)} \, dm(\zeta) < C < \infty. \tag{3.3}$$

On the other hand, using [11, Theorem 1.1], [10, Lemmas 2 and 4], together with [11, Lemma 2.7], we get

$$I \lesssim \int_{|z - \zeta| \geq \max\{\rho(z), \rho(\zeta)\}} \frac{dm(\zeta)}{\rho(z) \rho(\zeta) \exp(d^e_\phi(z, \zeta))}, \quad \text{for some } \varepsilon > 0$$

$$= \int_{|z - \zeta| \geq \max\{\rho(z), \rho(\zeta)\}} \frac{\rho(\zeta) \, dm(\zeta)}{\rho(z) \rho^2(\zeta) \exp(d^e_\phi(z, \zeta))}$$

$$\lesssim \int_{|z - \zeta| \geq \max\{\rho(z), \rho(\zeta)\}} \frac{d^e_\phi(z, \zeta) \, dm(\zeta)}{\rho^2(\zeta) \exp(d^e_\phi(z, \zeta))}, \quad \text{for some } \delta \in (0, 1)$$

$$\lesssim \int_{|z - \zeta| \geq \max\{\rho(z), \rho(\zeta)\}} \frac{\rho^2(\zeta) \, dm(\zeta)}{\rho^2(\zeta) \exp(d^e_\phi(z, \zeta))}, \quad \text{for some } \varepsilon' < \varepsilon$$

$$\lesssim \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\exp(d^e_\phi(z, \zeta))} \leq C < \infty,$$

which together (3.2) and (3.3) gives (3.1).
Case 2. Assume that \( \phi \in \mathcal{S} \). We shall follow ideas and the notation from [15]. Again, it is enough to prove (3.1) that is equivalent to

\[
(3.4) \quad \sup_{z \in \mathbb{C}} \int_{\mathbb{C}} |K(z, \zeta)|e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta) \leq C_2 < \infty.
\]

where \( C_2 \) is an absolute constant.

By [15, (5.1)],

\[
(3.5) \quad \int_{\mathbb{C}} \rho(z, \zeta)|K(z, \zeta)|e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta) \leq C_1 < \infty.
\]

where \( C_1 \) is an absolute constant and \( \rho \) is defined by [15, (1.2)].

So, by (3.5), for any \( \delta > 0 \)

\[
(3.6) \quad \int_{\{\rho(z, \zeta) \geq \delta\}} |K(z, \zeta)|e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta) \leq \frac{C_1}{\delta}.
\]

On the other hand, by [15, Lemma 7.2], \( \{\zeta : \rho(z, \zeta) < \delta\} \subset \{\zeta : |z - \zeta| \leq \delta M(\Delta(\Psi(|z|^2)))^{-1/2}\} \) if \( \delta \) is small enough. So, by [15, Lemma 3.1, Lemma 3.2]

\[
\int_{\{\rho(z, \zeta) < \delta\}} |K(z, \zeta)|e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta)
\]

\[
\leq \int_{\{\zeta : |z - \zeta| \leq \delta M(\Delta(\Psi(|z|^2)))^{-1/2}\}} |K(z, \zeta)|e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta)
\]

\[
\leq \int_{\{\zeta : |z - \zeta| \leq \delta M(\Delta(\Psi(|z|^2)))^{-1/2}\}} (K(z, z))^{1/2} (K(\zeta, \zeta))^{1/2} e^{-1/2(\Psi(|z|^2)+\Psi(|\zeta|^2))} dA(\zeta)
\]

\[
\leq \int_{\{\zeta : |z - \zeta| \leq \delta M(\Delta(\Psi(|z|^2)))^{-1/2}\}} (\Delta(\Psi(|z|^2)))^{1/2} (\Delta(\Psi(|\zeta|^2)))^{1/2} dA(\zeta)
\]

\[
\leq C,
\]

which together (3.6) gives (3.4). This finishes the proof.

Bearing in mind the above result, the dual spaces of weighted Fock spaces \( F_p^\phi \) can be described.

**Corollary 14.** Asume that \( \phi : \mathbb{C} \rightarrow \mathbb{R}^+ \) belongs to \( \mathcal{D} \cup \mathcal{S} \). For \( p > 1 \), the dual of \( F_p^\phi \) is \( F_q^\phi \), under the pairing

\[
(3.7) \quad \langle f, g \rangle = \int_{\mathbb{C}} f \overline{g} e^{-2\phi} dm,
\]

where \( 1/p + 1/q = 1 \). Moreover, the dual of \( F_1^\phi \) is \( F_\infty^\phi \) under the pairing (3.7).

The proof is analogous to that of Corollary 8, it will be omitted.

Finally, we observe that the analogue of Zeytuncu's unboundedness result for the Bergman projection [17, Theorem 1.2] remains true in the complex plane. More precisely, we have
Theorem 15. If $w$ is a radial weight on $\mathbb{C}$ which satisfies

(i) $w \in C^\infty([0, \infty))$,

(ii) $\lim_{r \to \infty} w^{(n)}(r) = 0$ for any $n \in \mathbb{N}$,

(iii) for any $n \in \mathbb{N}$ there exists $a_n \in (0, \infty)$ such that $(-1)^n w^{(n)}$ is non-negative on the interval $(a_n, \infty)$.

Then, for $0 < p < \infty$, the Bergman projection is bounded from $L^p(\mathbb{C}, w)$ to $L^p(\mathbb{C}, w)$ only for $p = 2$.

A proof can be obtained by mimicking that of Theorem [17, Theorem 1.2], where $\int_0^1 r^{2x+1}w(r)dr$ must be replaced by $\int_0^\infty r^{2x+1}w(r)dr$, so it will be omitted.

With this result in our hands, Proposition 3 can be proved analogously to Proposition 2.

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