SUBELLIPTIC BIHARMONIC MAPS

Sorin Dragomir and Stefano Montaldo

ABSTRACT. We study subelliptic biharmonic maps i.e. smooth maps \( \phi : M \rightarrow N \) from a compact strictly pseudoconvex CR manifold \( M \) into a Riemannian manifold \( N \) which are critical points of the energy functional \( E_{2,\theta}(\phi) = \frac{1}{2} \int_M \|\tau_\theta(\phi)\|^2 \theta \wedge (d\theta)^n \). We show that \( \phi : M \rightarrow N \) is a subelliptic biharmonic map if and only if its vertical lift \( \phi \circ \pi : C(M) \rightarrow N \) to the (total space of the) canonical circle bundle \( S^1 \rightarrow C(M) \xrightarrow{\pi} M \) is a biharmonic map with respect to the Fefferman metric \( F_\theta \) on \( C(M) \).

1. Introduction

Biharmonic maps were introduced by J. Eells & L. Lemaire, [18], as critical points \( \phi \in C^\infty(M, N) \) of the bienergy functional

\[
E_2(\phi) = \frac{1}{2} \int_M \|\tau(\phi)\|^2 \, dv_g
\]

where \( M \) and \( N \) are Riemannian manifolds and \( \tau(\phi) \) is the tension field of \( \phi : M \rightarrow N \). Biharmonic maps were further investigated by G. Jiang, [22], who derived the first and second variation formulas for \( E_2(\phi) \) and gave several applications to the geometry of the second fundamental form of a submanifold in a Riemannian manifold. In the last decade a large amount of work has been devoted to biharmonic maps with particular attention to the constructions and classifications of proper biharmonic maps and proper biharmonic submanifolds, see e.g. [2]-[3], [6]-[5], [13]-[14], [21], [25]-[27].

A program aiming to extending results on nonlinear elliptic systems of variational origin to the hypoelliptic case was started by J. Jost & C-J. Xu, [23]. Given a Hörmander system of vector fields \( \{X_1, \cdots, X_p\} \) on an open set \( U \subset \mathbb{R}^m \) and a map \( \phi \in C^\infty(U, N) \) the function

\[
e(\phi) = \frac{1}{2} \sum_{a=1}^p X_a(\phi^j) X_a(\phi^k)(h_{jk} \circ \phi)
\]
SUBELLIPTIC BIHARMONIC MAPS

is globally defined and generalizes the ordinary energy density of \( \phi \). Here \( h_{jk} \) is the Riemannian metric on \( N \) in a local coordinate system \((V, y^j)\) and \( \phi^j = y^j \circ \phi \). A subelliptic harmonic map is a critical point \( \phi \in C^\infty(U, N) \) of the functional (cf. [23])

\[
E_X(\phi) = \int_\Omega e(\phi) \, dx
\]

where \( \Omega \subset \mathbb{R}^m \) is a bounded domain such that \( \overline{\Omega} \subset U \). The Euler-Lagrange equations of the variational principle \( \delta E_X(\phi) = 0 \) are

\[
-H\phi^i + \left( \Gamma^i_{jk} \circ \phi \right) \sum_{a=1}^p X_a(\phi^j)X_a(\phi^k) = 0
\]

where \( H \equiv \sum_{a=1}^{2n} X_a \ast X_a \) is the Hörmander operator and \( \Gamma^i_{jk} \) are the Christoffel symbols of \( h_{jk} \). Subelliptic harmonic maps were recognized (cf. E. Barletta & S. Dragomir & H. Urakawa, [8]) as the local manifestation of pseudoharmonic maps i.e. critical points \( \phi \in C^\infty(M, N) \) of the functional

\[
E_{1,b}(\phi) = \frac{1}{2} \int_M \text{trace}_{G_\theta}(\Pi_H \phi^* h) \, \theta \wedge (d\theta)^n.
\]

Here \( M \) is a compact strictly pseudoconvex CR manifold, of CR dimension \( n \), and \( \theta \) is a contact form on \( M \) such that the Levi form \( G_\theta \) is positive definite. Also \( h \) is the Riemannian metric on \( N \) and \( \Pi_H \phi^* h \) is the restriction of the bilinear form \( \phi^* h \) to the Levi, or maximally complex, distribution \( H(M) \). The Euler-Lagrange equations of \( \delta E_{1,b}(\phi) = 0 \) may be written as \( \tau_b(\phi) = 0 \) where the field \( \tau_b(\phi) \in C^\infty(\phi^{-1}T(N)) \) is locally given by

\[
\tau_b(\phi)^i = \Delta_b \phi^i + \sum_{a=1}^{2n} X_a(\phi^j)X_a(\phi^k) \left( \Gamma^i_{jk} \circ \phi \right).
\]

Here \( \{X_a : 1 \leq a \leq 2n\} \) is a local \( G_\theta \)-orthonormal frame of \( H(M) \). Also \( \Delta_b \) is the sublaplacian i.e. the second order differential operator given by

\[
\Delta_b u = \text{div} \left( \nabla^H u \right), \quad u \in C^\infty(M).
\]

The divergence operator is meant with respect to the volume form \( \theta \wedge (d\theta)^n \) while the horizontal gradient is given by \( \nabla^H u = \Pi_H \nabla u \) and \( g_\theta(\nabla u, X) = X(u) \) for any \( X \in \mathfrak{X}(M) \). CR manifolds occur mainly as boundaries \( M = \partial \Omega \) of domains \( \Omega \subset \mathbb{C}^{n+1} \) and boundary values of Bergman-harmonic maps \( \Phi : \Omega \rightarrow N \) may be shown (cf. [16]) to be pseudoharmonic provided \( \Phi \) has vanishing normal derivatives (thus motivating our use of the index \( b \) for ”boundary” analogs to geometric objects such as the tension field \( \tau(\Phi) \), the second fundamental form
The similar boundary behavior of biharmonic maps from a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$ endowed with the Bergman metric is unknown.

The approaches in [23] and [8] overlap partially, as follows. For any local $G_{\theta}$-orthonormal frame \( \{X_a : 1 \leq a \leq 2n\} \) in $H(M)$ defined on a local coordinate neighborhood $\varphi : U \to \mathbb{R}^{2n+1}$ the push-forward \( \{(d\varphi)X_a : 1 \leq a \leq 2n\} \) is a Hörmander system on $\varphi(U)$ and $\Delta_{\theta} = -H$. However in [23] and in general in the theory of Hörmander vector fields the Euclidean dimension $m$ is arbitrary (as opposed to $m = 2n + 1$ in the CR case), the vector fields forming the given system are allowed to be linearly dependent (at particular points), and the formal adjoints $X^*_a$ are meant with respect to the Euclidean metric on $\Omega$ (rather than the Webster metric - a non flat Riemannian metric springing from the given CR structure in the presence of a contact form). It is therefore a natural problem, within J.Jost & C.-J. Xu’s program, to study critical points $\phi \in C^\infty(M, N)$ of the functional

\[ E_{2,\theta}(\phi) = \frac{1}{2} \int_M \|\tau_{\theta}(\phi)\|^2 \theta \wedge (d\theta)^n. \]

These are referred to as subelliptic biharmonic maps. We give a geometric interpretation of subelliptic biharmonic maps in terms of ordinary biharmonic maps from a Lorentzian manifold (the total space of the canonical circle bundle $S^1 \to C(M) \to M$ endowed with the Fefferman metric $F_{\theta}$, [17]). The paper is organized as follows. In §2 we consider ordinary biharmonic maps from a Fefferman space-time. The rough Laplacian (a degenerate elliptic operator appearing in the principal part of the subelliptic biharmonic map system) is discussed in §3. The first variation formula for the functional $E_{2,\theta} : C^\infty(M, N) \to \mathbb{R}$ is derived in §4 while the main result (cf. Theorem 1 below) is proved in §5. A few open problems are outlined in §6.

2. Biharmonic maps from Fefferman space-times

Let $(M, T_{1,0}(M))$ be a compact orientable CR manifold, of CR dimension $n$, where $T_{1,0}(M)$ is its CR structure. Let us assume that $M$ is strictly pseudoconvex and $\theta$ a contact form on $M$ such that the Levi form $G_{\theta}(X,Y) = (d\theta)(X,JY)$, $X,Y \in H(M)$, is positive definite. Here $H(M) = \text{Re} \{T_{1,0}(M) \oplus T_{0,1}(M)\}$ is the Levi distribution of $M$ and $J : H(M) \to H(M)$, $J(Z + \overline{Z}) = i(Z - \overline{Z})$, $Z \in T_{1,0}(M)$, its complex structure ($i = \sqrt{-1}$). For all needed notions of CR and pseudohermitian geometry we rely on [17]. Let $T$ be the Reeb vector of $(M, \theta)$ i.e. the nowhere zero globally defined tangent vector field transverse to $H(M)$ determined by $\theta(T) = 1$ and $T \mid d\theta = 0$. Let $\nabla$ be
the Tanaka-Webster connection of \((M, \theta)\) (cf. Theorem 3.1 in \[17\], p. 25, for the axiomatic description of \(\nabla\)).

Let \(S^1 \to C(M) \xrightarrow{\pi} M\) be the canonical circle bundle over \(M\) and \(F_\theta\) the Fefferman metric on \(C(M)\) (cf. J.M. Lee, \[24\], or Definition 2.15 in \[17\], p. 128) i.e. the Lorentzian metric on \(C(M)\) given by (cf. (2.30) in \[17\], p. 127)

\[
\begin{align*}
F_\theta &= \pi^* \tilde{G}_\theta + 2 (\pi^* \theta) \odot \sigma, \\
\sigma &= \frac{1}{n+2} \left\{ d\gamma + \pi^* \left( i \omega^\alpha - \frac{i}{2} g^{\beta\bar{\beta}} dg_{\alpha\beta} - \frac{\rho}{4(n+1)} \theta \right) \right\}.
\end{align*}
\]

As to the notation in (1)-(2), given a local frame \(\{T_\alpha : 1 \leq \alpha \leq n\}\) of \(T_{1,0}(M)\) we set \(g_{\alpha\beta} = G_\theta(T_\alpha, T_\beta)\) (with \(T_\alpha = \overline{T}_\alpha\)). Also \(\omega_\beta^\alpha\) are the connection 1-forms of \(\nabla\) with respect to \(\{T_\alpha : 1 \leq \alpha \leq n\}\) i.e. \(\nabla T_\beta = \omega_\beta^\alpha \otimes T_\alpha\). Moreover \(\rho = g^{\alpha\beta} R_{\alpha\beta}\) is the pseudohermitian scalar curvature of \((M, \theta)\) (cf. e.g. \[17\], p. 50). The \((0,2)\)-tensor field \(\tilde{G}_\theta\) in (1) is obtained by extending the Levi form \(G_\theta\) to a degenerate form defined on the whole of \(T(M)\). By definition \(\tilde{G}_\theta = G_\theta\) on \(H(M) \otimes H(M)\) and \(\tilde{G}_\theta(X, T) = 0\) for any \(X \in T(M)\). By a result of C.R. Graham, \[20\], the (globally defined) 1-form \(\sigma\) is a connection form in the canonical circle bundle. Let \(X^\dagger \in \mathfrak{X}(C(M))\) be the horizontal lift of \(X \in \mathfrak{X}(M)\) with respect to \(\sigma\). If \(S \in \mathfrak{X}(C(M))\) is the tangent to the \(S^1\) action then \(T^\dagger - S\) is a nowhere vanishing globally defined timelike vector field hence \((C(M), F_\theta)\) is a time-oriented Lorentzian manifold, referred to as the Fefferman space-time. \(F_\theta\) was discovered by C. Fefferman, \[19\], in connection with the study of the boundary behavior of the Bergman kernel of a bounded domain \(\Omega \subset \mathbb{C}^n\) (as a Lorentzian metric on \(C(\partial \Omega) \approx \partial \Omega \times S^1\)). An array of popular nonlinear problems arise from \(F_\theta\) e.g. the CR Yamabe problem (cf. \[17\], p. 159-160) is the projection via \(\pi : C(M) \to M\) of the ordinary Yamabe problem for \(F_\theta\). While the principal part of the Yamabe equation on \(C(M)\) is the wave operator \(\Box\) (hence the Yamabe equation on \(C(M)\) is not elliptic) the principal part of the projected equation is the sublaplacian \(\Delta_b\) (hence subelliptic theory applies, cf. \[17\], p. 176-210). Also pseudoharmonic maps from strictly pseudoconvex CR manifolds may be characterized as base maps of \(S^1\)-invariant harmonic maps from \((C(M), F_\theta)\), thus suggesting that Theorem \[1\] below should hold.

Several basic facts in harmonic map theory are known to extend in a straightforward manner from the Riemannian to the semi-Riemannian setting (cf. \[3\], p. 427-452) e.g. a smooth map of semi-Riemannian manifolds has a well defined tension tensor field. In particular the
applications we seek for are to maps from Lorentzian to Riemannian manifolds. A $C^\infty$ map $\Phi : C(M) \to N$ into a real $\nu$-dimensional Riemannian manifold $(N,h)$ is biharmonic if $\Phi$ is a critical point of the functional

\begin{equation}
E_2(\Phi) = \frac{1}{2} \int_{C(M)} \|\tau(\Phi)\|^2 \, d\text{vol}(F_\theta).
\end{equation}

The tension field $\tau(\Phi)$ is the $C^\infty$ cross-section in the pullback bundle $\Phi^{-1}TN \to C(M)$ locally given by

$$
\tau(\Phi) = \left(\Box \Phi^i + (\Gamma^i_{jk} \circ \Phi) \frac{\partial \Phi^j}{\partial u^p} \frac{\partial \Phi^k}{\partial u^q} F^{pq}\right) X_i^\Phi
$$

where $\Box$ is the Laplace-Beltrami operator of $F_\theta$ (the wave operator as $F_\theta$ is Lorentzian) and $\Phi^i = y^i \circ \Phi$. Also if $(U, x^A)$ is a local coordinate system on $M$ such that $\Phi^{-1}(V) \subset \pi^{-1}(U)$ and $\gamma : \pi^{-1}(U) \to \mathbb{R}$ is a local fibre coordinate on $C(M)$ then $(\pi^{-1}(U), \ u^A = x^A \circ \pi, \ u^{2n+2} = \gamma)$ are the naturally induced local coordinates on $C(M)$ and $[F^{pq}] = [F_{pq}]^{-1}$ while $F_{pq} = F_\theta(\partial_p, \partial_q)$. Here $\partial_p$ is short for $\partial/\partial u^p$. Finally $X_i^\Phi$ is the natural lift of $\partial_i = \partial/\partial y^i$ i.e. the local smooth section in $\Phi^{-1}TN \to C(M)$ given by $X_i^\Phi(z) = (\partial_i)_{\Phi(z)}$ for any $z \in \Phi^{-1}(V)$.

The bundle metric $h^\Phi$ appearing in (3) is naturally induced by $h$ in $\Phi^{-1}TN \to C(M)$ so that $h^\Phi(X_i^\Phi, X_j^\Phi) = h_{ij} \circ \Phi$. Our main result is

**Theorem 1.** Let $M$ be a compact strictly pseudoconvex CR manifold, of CR dimension $n$, and $\theta$ a contact form on $M$ with $G_\theta$ positive definite. Let $\Phi : C(M) \to N$ be a smooth $S^1$-invariant map and $\phi : M \to N$ the corresponding base map. Then $E_2(\Phi) = 2\pi E_{2,b}(\phi)$.

Consequently if $\Phi$ is biharmonic then $\phi$ is a critical point of $E_2$. The Euler-Lagrange equations of the variational principle $\delta E_{2,b}(\phi) = 0$ are

\begin{equation}
BH_b(\phi) \equiv \Delta_b^\phi \tau_b(\phi) + \text{trace}_{G_\theta} \left\{ \Pi_H \hat{R}^h(\tau_b(\phi), \phi^*) \phi^* \cdot \right\} = 0
\end{equation}

where $\Delta_b^\phi$ is the rough sublaplacian and $R^h$ the curvature tensor field of $N$. Consequently the vertical lift to $C(M)$ of any $C^\infty$ solution $f$ to (4) is a biharmonic map (with respect to the Fefferman metric $F_\theta$).

Here $\tau_b(\phi) = \tau(\phi; \theta, \nabla^h)$ is the subelliptic tension field of $\phi : M \to N$ i.e. the $C^\infty$ section in $\phi^{-1}TN \to M$ locally given by

$$
\tau_b(\phi) = \left\{ \Delta_b \phi^j + 2g^{\alpha\beta} (\Gamma^i_{jk} \circ \phi) T_\alpha(\phi^j) T_{\beta}(\phi^k) \right\} X_i^\phi
$$

where $\Delta_b$ is the sublaplacian of $(M, \theta)$ (cf. the Introduction or Definition 2.1 in [17], p. 134), $\phi^i = y^i \circ \phi$ and $X_i^\phi(x) = (\partial_i)_{\phi(x)}$ for any $x \in \phi^{-1}(V)$ (the natural lift of $\partial/\partial y^i$ as a section in $\phi^{-1}TN \to M$).
Note that \( X_i^\phi = X_i^\psi \circ \pi \). The rough sublaplacian is the second order differential operator locally given by
\[
\Delta_b^\phi V = \sum_{a=1}^{2n} \{(\phi^{-1} \nabla^h)_{X_a}(\phi^{-1} \nabla^h)_{X_a} V - (\phi^{-1} \nabla^h)_{X_a X_a} V\}
\]
for any \( V \in C^\infty(\phi^{-1} T N) \), where \( \{X_a : 1 \leq a \leq 2n\} \) is a local orthonormal (i.e. \( G_\theta(X_a, X_b) = \delta_{ab} \)) frame of \( H(M) \). Moreover \( \phi^{-1} \nabla^h \) is the connection in \( \phi^{-1} T N \to M \) defined as the pullback of the Levi-Civita connection \( \nabla^h \) of \( (N, h) \) by \( \phi \) i.e.
\[
(\phi^{-1} \nabla^h)_{\partial/\partial x^i} X_k^\phi = \frac{\partial \phi^i}{\partial x^A} (\Gamma^i_{jk} \circ \phi) X_k^\phi.
\]
Given a bilinear form \( B \) on \( T(M) \) we denote by \( \Pi_H B \) the restriction of \( B \) to \( H(M) \otimes H(M) \). We shall need the following

**Theorem 2.** For any smooth map \( \phi : M \to N \) the rough sublaplacian \( \Delta_b^\phi \) is a formally self adjoint (with respect to the \( L^2 \) inner product \( \langle V, W \rangle = \int_M h^\phi(V, W) \theta \wedge (d\theta)^n \) \( , V, W \in C^\infty(\phi^{-1} T N) \) ) second order differential operator locally expressed as
\[
(5) \quad \Delta_b^\phi V = \left\{ \Delta_b V^i + 2 \sum_{a=1}^{2n} X_a(\phi^i) (\Gamma^i_{jk} \circ \phi) X_a(V^k) + 
\right.
\]
\[
+ \left[ (\Gamma^i_{jk} \circ \phi) \Delta_b \phi^j + \sum_{a=1}^{2n} X_a(\phi^i) X_a(f^\ell) \left( \frac{\partial \Gamma^i_{jk}}{\partial y^\ell} + \Gamma^j_{k\ell} \Gamma^i_{js} \right) \circ \phi \right] V^k \right\} X_i^\phi
\]
where \( V = V^i X_i^\phi \), for any local orthonormal frame \( \{X_a : 1 \leq a \leq 2n\} \) of \( H(M) \) defined on the open set \( U \subset M \) and any local coordinate system \( (V, y^i) \) on \( N \) such that \( \phi^{-1}(V) \subset U \). Let \( D^* \) be the formal adjoint of \( D = (\phi^{-1} \nabla^h)^H \) i.e. \( (D^* \phi, V) = (\phi, DV) \) for any \( \phi \in C^\infty(H(M)^* \otimes \phi^{-1} T N) \) and any \( V \in C^\infty_0(\phi^{-1} T N) \). Then
\[
(6) \quad \Delta_b^\phi = - D^* D.
\]
In particular \( (\Delta_b^\phi V, V) \leq 0 \).

The proof of Theorem 2 will be given in §3. If \( N = \mathbb{R}^n \) then (by (4)–(5)) the subelliptic biharmonic map equations become \( L \phi^i = 0 \) where \( L \equiv \Delta_b \circ \Delta_b \) (the bi-sublaplacian) is a fourth order hypoelliptic operator. The analysis of the scalar case \( N = \mathbb{R} \) (maximum principles, existence of Green functions for \( L \) , Harnack inequalities for positive solutions to \( Lu = 0 \), etc.) is however open. The calculation of the Green function for \( \Delta^2 \equiv \Delta \circ \Delta \) (where \( \Delta \) is the ordinary Laplacian on \( \mathbb{R}^n \)) is due to T. Boggio, [10]. The existence of the Green function for \( \Delta_b \) follows from
work by J.M. Bony, [12], while estimates (on the Green function and its derivatives) were got by A. Sánchez-Calle, [29], yet the problem of adapting their techniques to the bi-sublaplacian is unsolved.

3. The rough sublaplacian

The second order differential operator $\Delta^\phi_h$ is similar to the rough Laplacian on vector fields due to G. Wiegmink, [30], and C.M. Wood, [31]. Let $\phi : M \to N$ be a smooth map and $\phi^{-1}TN \to M$ the pullback bundle. Let $h^\phi$ (respectively $\phi^{-1}\nabla^h$) be the pullback of the Riemannian metric $h$ (respectively of the Levi-Civita connection $\nabla^h$) by $\phi$. Then $h^\phi$ is parallel with respect to $\phi^{-1}\nabla^h$. We shall establish

**Lemma 1.** For any $V, W \in C^\infty(\phi^{-1}TN)$ there is a smooth tangent vector field $X_\phi$ on $M$ such that

$$
\Delta^\phi_h(V, W) = \Delta^h [h^\phi(V, W)] + h^\phi(V, \Delta^\phi_h W) - 2\text{div}(X_\phi)
$$

where the divergence is taken with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$ i.e. $\mathcal{L}_{X_\phi} \Psi = \text{div}(X_\phi)\Psi$.

**Proof.** Let $\{X_a : 1 \leq a \leq 2n\}$ be a local orthonormal frame of $H(M)$. As $(\phi^{-1}\nabla^h)h^\phi = 0$

$$
h^\phi((\phi^{-1}\nabla^h)X_a(\phi^{-1}\nabla^h)X_a V, W) = 0
$$

$$
= X_a (h^\phi((\phi^{-1}\nabla^h)X_a V, W)) - h^\phi((\phi^{-1}\nabla^h)X_a V, (\phi^{-1}\nabla^h)X_a W) = 0
$$

$$
= X_a^2 (h^\phi(V, W) - 2X_a (h^\phi(V, (\phi^{-1}\nabla^h)X_a W)) + h^\phi(V, (\phi^{-1}\nabla^h)^2X_a W)
$$

$$
= (\nabla_{X_a}X_a)(h^\phi(V, W)) - h^\phi(V, (\phi^{-1}\nabla^h)\nabla_{X_a}X_a W).
$$

Let us recall that $\Delta_h u = \text{div} (\nabla^H u), u \in C^2(M)$, where $\nabla^H u \in C^\infty(H(M))$ (the horizontal gradient of $u$) is given by $\nabla^H u = \Pi_H \nabla u$ and $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. Also $\Pi_H : T(M) \to H(M)$ is the natural projection associated to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ while $g_\theta$ is the Webster metric of $(M, \theta)$ (cf. Definition 1.10 in [17], p. 9). A large amount of the existing subelliptic theory is built on the Heisenberg group (cf. [11], p. 155) i.e. on the noncommutative Lie group $\mathbb{H}_n \equiv \mathbb{C}^n \times \mathbb{R}$ with the multiplication law

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\text{Im}(z \cdot \overline{w}))
$$

for any $(z, t), (w, s) \in \mathbb{H}_n$. Let $Z_\alpha$ be the Lewy operators i.e.

$$
Z_\alpha \equiv \frac{\partial}{\partial z^\alpha} + i\overline{z^\alpha} \frac{\partial}{\partial t}, \quad 1 \leq \alpha \leq n.
$$
Then $T_{1,0}(\mathbb{H}_n) = \sum_{a=1}^{n} \mathbb{C}Z_\alpha$ is a CR structure on $\mathbb{H}_n$, of CR dimension $n$, making $\mathbb{H}_n$ into a CR manifold (cf. e.g. [17]). The Levi distribution $H(\mathbb{H}_n)$ is spanned by the left invariant vector fields $X_\alpha \in \mathfrak{X}(\mathbb{H}_n)$ given by

$$X_\alpha \equiv \frac{\partial}{\partial x^\alpha} + 2y^\alpha \frac{\partial}{\partial t}, \quad X_{n+\alpha} \equiv \frac{\partial}{\partial y^\alpha} - 2x^\alpha \frac{\partial}{\partial t},$$

and the horizontal gradient is familiar (cf. [11], p. 68) in subelliptic theory as the horizontal $H$-gradient i.e. $\nabla^H u = \sum_{a=1}^{2n} X_a(u)X_a$. Note also that for the Heisenberg group $H = -\sum_{a=1}^{2n} X_a^2$ (Hörmander's sum of squares of vector fields) as $X^*_a = -X_a$.

Let $\nabla$ be the Tanaka-Webster connection of $(M,\theta)$. As $\nabla \Psi = 0$ the divergence of a vector field may also be computed as the trace of its covariant derivative with respect to $\nabla$. Hence $\Delta_b u$ may be locally written as

$$\Delta_b u = \sum_{a=1}^{2n} \{ X_a^2 u - (\nabla X_a u) \}.$$

Consequently

$$h^\phi \left( \Delta_b^\phi V, W \right) = \Delta_b \left[ h^\phi (V, W) \right] + h^\phi \left( V, \Delta_b^\phi W \right) + 2 \sum_a \{ h^\phi (V, (\phi^{-1}\nabla^h)_{X_a} W) - X_a(h^\phi (V, (\phi^{-1}\nabla^h)_{X_a} W)) \}.$$

Let $X_\phi \in H(M)$ be the vector field determined by

$$G_\theta(X_\phi, Y) = h^\phi (V, (\phi^{-1}\nabla^h)_Y W)$$

for any $Y \in H(M)$. Then (by $\nabla g_\theta = 0$)

$$\sum_a X_a(h^\phi (V, (\phi^{-1}\nabla^h)_{X_a} W)) = \sum_a X_a(G_\theta(X_\phi, X_a)) =$$

$$= \sum_a \{ g_\theta(\nabla X_a X_f, X_a) + g_\theta(X_f, \nabla X_a X_a) \} =$$

$$= \text{div}(X_\phi) + \sum_a h^\phi (V, (\phi^{-1}\nabla^h)_{X_a} X_a W).$$

Indeed $g_\theta(\nabla_T X_\phi, T) = 0$ (as $H(M)$ is parallel with respect to $\nabla$). Together with (8) this leads to (7). Lemma 1 is proved.

Let us assume that either $M$ is compact or at least one of the sections $V, W$ has compact support. At this point we may integrate (7) over $M$ and use Green's lemma to show that $(\Delta_b^\phi V, W) = (V, \Delta_b^\phi W)$. Moreover, if $V = V^i X^\phi_i$ is a $C^\infty$ section in $\phi^{-1}TN \rightarrow M$ then

$$\left( \phi^{-1}\nabla^h \right)_X V = \{ X(V^i) + X(\phi^i)V^k (\Gamma^i_{jk} \circ \phi) \} X^\phi_i$$
for any $X \in \mathcal{X}(M)$. The proof of [5] follows from
\[
(\phi^{-1}\nabla^h)^2 V = \left\{ X^2(V^i) + 2X(\phi^j) \left( \Gamma^i_{jk} \circ f \right) X(V^k) + \right.
\left. + \left[ X^2(\phi^j) \left( \Gamma^i_{jk} \circ \phi \right) + X(\phi^j)X(f^i) \left( \frac{\partial \Gamma^j_{ik}}{\partial y^l} + \Gamma^j_{kl}\Gamma^i_{jm} \right) \circ \phi \right] \left. \right\} \mathbf{X}_X. \]

Let $D = (\phi^{-1}\nabla^h)^{H}$ i.e. $DV \in C^\infty(H(M)^{*} \otimes \phi^{-1}TN)$ is the restriction of $(\phi^{-1}\nabla^h)V$ to $H(M)$. An $L^2$ inner product on $C^\infty(H(M)^{*} \otimes \phi^{-1}TN)$ is given by
\[
(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \Psi, \quad \langle \varphi, \psi \rangle|_U = \sum_{a=1}^{2n} h^\phi(\varphi X_a, \psi X_a),
\]
for any $\varphi, \psi \in C^\infty(H(M)^{*} \otimes \phi^{-1}TN)$ and any local orthonormal frame $\{X_a : 1 \leq a \leq 2n\}$ of $H(M)$ on $U \subseteq M$. Then for any $V \in C^\infty_0(\phi^{-1}TN)$
\[
(D^* \varphi, V) = \int \sum_{a} h^\phi(\varphi X_a, (\phi^{-1}\nabla^h)_{X_a} V) \Psi = \int \sum_{a} \{ X_a(h^\phi(\varphi X_a, V)) - h^\phi((\phi^{-1}\nabla^h)_{X_a} \varphi X_a, V) \} \Psi.
\]

Let $X_{\varphi, V} \in H(M)$ be determined by $G_\theta(X_{\varphi, V}, Y) = h^\phi(\varphi Y, V)$ for any $Y \in H(M)$. Then
\[
\sum_{a} X_a(h^\phi(\varphi X_a, V)) = \sum_{a} X_a(g_\theta(X_{\varphi, V}, X_a)) = \sum_{a} \{ g_\theta(\nabla_{X_a} X_{\varphi, V}, X_a) + g_\theta(X_{\varphi, V}, \nabla_{X_a} X_a) \} = \text{div}(X_{\varphi, V}) + \sum_{a} h^\phi(\varphi \nabla_{X_a} X_a, V).
\]

We may conclude that
\[
D^* \varphi = -\sum_{a=1}^{2n} \{ (\phi^{-1}\nabla^h)_{X_a} \varphi X_a - \varphi \nabla_{X_a} X_a \}
\]
on $U$ and then $D^* DV = -\Delta_b^\phi V$ for any $V \in C^\infty(\phi^{-1}TN)$.

**Proposition 1.** The symbol of the rough sublaplacian is
\[
\sigma_2\left(\Delta_b^\phi \right) \omega = [\omega(T_x)^2 - \|\omega\|^2] \nu
\]
for any $\omega \in T^*_x(M) \setminus \{0\}$, $v \in (\phi^{-1}TN)_x$ and $x \in M$. Therefore $\Delta_b^\phi$ is a degenerate elliptic operator and its ellipticity degenerates precisely in the cotangent directions spanned by $\theta$. 

Proof. Let $T'(M) = T^*(M) \setminus \{0\}$ and let $\Pi : T'(M) \to M$ be the projection. If $E \to M$ and $F \to M$ are vector bundles we set

$$\text{Smbl}_k(E, F) = \{ \sigma \in \text{Hom}(\Pi^*E, \Pi^*F) : \sigma_{\rho \omega} = \rho \sigma_{\omega}, \; \rho > 0 \}$$

(with $k \in \mathbb{Z}$). Let $\sigma_k(L) \in \text{Smbl}_k(E, F)$ be the symbol of the $k$-th order differential operator $L \in \text{Diff}_k(E, F)$. We wish to compute $\sigma_2(\Delta^b \phi) \in \text{Smbl}_2(\phi^{-1}TN, \phi^{-1}TN)$. To this end let $\omega \in T'(M)$ such that $\Pi(\omega) = x$ and let $f \in C^\infty(M)$ such that $(df)_x = \omega$. Also let $v \in (\phi^{-1}TN)_x$ and $V \in C^\infty(\phi^{-1}TN)$ such that $V_x = v$. Then

$$\sigma_2(\Delta^b \phi) = -\frac{1}{2} \Delta^b \phi [(f - f(x))^2V] (x).$$

Then (10) follows from the identities

$$\Delta_b(u^2) = 2u \Delta_b u + 2\|\nabla^H u\|^2,$$

$$\Delta^b \phi(gV) = g \Delta^b \phi V + (\Delta_b g)V + 2(\phi^{-1}\nabla^h)_{\nabla^H g}V,$$

where $g = u^2$ and $u = f - f(x)$. The norm in (10) is $\|\omega\| = g_{\theta,x}(\omega, \omega)$. Hence for each $v \in \ker \sigma_2(\Delta^b \phi) \omega$ either $v = 0$ or $\omega = \lambda \theta_x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Q.e.d.

4. THE FIRST VARIATION FORMULA

Let $\tilde{M} = M \times (-\epsilon, \epsilon)$, $\epsilon > 0$, and let $F : \tilde{M} \to N$ be a smooth 1-parameter variation of $\phi$ by smooth maps i.e. $\phi_0 = \phi$ where $\phi_t = F \circ \alpha_t$ and $\alpha_t : M \to \tilde{M}$ is the injection $\alpha_t(x) = (x, t)$ for any $x \in M$. Let $V \in C^\infty(\phi^{-1}TN)$ be the corresponding infinitesimal variation

$$V_x = (d(x, 0)F) \frac{\partial}{\partial t}_{(x, 0)}, \; x \in M.$$ 

Let us consider the second fundamental form of $\phi$ (cf. R. Petit, [28])

$$\beta_b(\phi)(X, Y) = (f^{-1}\nabla^h)_X \phi_* Y - \phi_* \nabla_X Y, \; X, Y \in \mathfrak{X}(M).$$

Here $\phi_* X$ denotes the cross-section in $\phi^{-1}TN \to M$ given by

$$(\phi_* X)(x) = (d_x \phi)X_x, \; x \in M,$$

for each $X \in \mathfrak{X}(M)$. Then (cf. e.g. [8])

$$\tau_b(\phi) = \text{trace}_{G_a} \Pi_H \beta_b(\phi) = \sum_{a=1}^{2n} \beta_b(\phi)(X_a, X_a)$$

on $U \subset M$. We shall establish the following
Theorem 3. Let \( M \) be a compact strictly pseudoconvex CR manifold and \( \phi : M \to N \) a smooth map into a Riemannian manifold. Then for any smooth 1-parameter variation \( F : \bar{M} \to N \) of \( \phi \)

\[
\frac{d}{dt} \{ E_{2,b}(\phi_t) \}_{t=0} = \int_M \hat{h}^b (V, BH_b(\phi)) \Psi 
\]

where \( \hat{h}^b \equiv \phi^{-1} R^h \) is given by \( \hat{R}^h(u,v)w \) for any \( u,v,w \in C^\infty(\phi^{-1}TN) \) and any \( x \in N \).

If \( M \) is not compact one may, as usual, integrate over an arbitrary relatively compact domain \( \Omega \subset M \) and consider only smooth 1-parameter variations supported in \( \Omega \).

Proof of Theorem 3. Given \( X \in \mathfrak{x}(M) \) we define \( \tilde{X} \in \mathfrak{x}(\bar{M}) \) by setting

\[
\tilde{X}(x,t) = (d_x \alpha_t) X_x, \quad (x,t) \in \bar{M}.
\]

Then

\[
((\phi_t)_* X)_x = (d_x \phi_t) X_x = (d_{(x,t)} F)(d_x \alpha_t) X_x = (d_{(x,t)} F) \tilde{X}_{(x,t)}
\]

hence

\[
(\phi_t)_* X = (F_* \tilde{X}) \circ \alpha_t, \quad |t| < \epsilon,
\]

with the obvious meaning of \( F_* \tilde{X} \) as a section in \( F^{-1}TN \to \bar{M} \). Similarly the identities

\[
X_i^{\phi_t} = X_i^F \circ \alpha_t, \quad 1 \leq i \leq \nu,
\]

relate the local frames \( \{ X_i^{\phi_t} : 1 \leq i \leq \nu \} \) and \( \{ X_i^F : 1 \leq i \leq \nu \} \) in the pullback bundles \( \phi_t^{-1}TN \to M \) and \( F^{-1}TN \to \bar{M} \) respectively. Let \( \{ X_a : 1 \leq a \leq 2n \} \) be a local orthonormal frame of \( H(M) \). We wish to compute

\[
\tau_b(\phi_t) = \sum_{a=1}^{2n} \{ (\phi_t^{-1} \nabla^h)_{X_a}(\phi_t)_* X_a - (\phi_t)_* \nabla_{X_a} X_a \}
\]

for any \( |t| < \epsilon \). Note that

\[
(\phi^{-1} \nabla^h)_{X} \phi_* Y = \{ X(Y \phi^i) + X(\phi^j) Y(\phi^k) (\Gamma^l_{jk} \circ \phi) \} X_i^{\phi}
\]

where \( \phi^i = y^i \circ \phi \). As \( \phi_t^i = F_t^i \circ \alpha_t \) it follows that

\[
X(\phi_t^i) = \tilde{X}(F_t^i) \circ \alpha_t, \quad X(Y \phi_t^i) = [\tilde{X}(\tilde{Y} F_t^i)] \circ \alpha_t,
\]

for any \( X,Y \in \mathfrak{x}(M) \). Therefore (by (14)–(15))

\[
(\phi_t^{-1} \nabla^h)_{X} \phi_* Y = \left[ (F^{-1} \nabla^h)_{X} F_* \tilde{Y} \right] \circ \alpha_t
\]
where $F^{-1} \nabla^h$ is the connection in $F^{-1}TN \to \tilde{M}$ induced by $\nabla^h$. Then (by (12) and (16))

$$\tau_\beta(\phi_t) = \tau_\beta(F) \circ \alpha_t, \quad |t| < \epsilon,$$

where $\tau_\beta(F) \in C^\infty(F^{-1}(V), F^{-1}TN)$ is defined by

$$\tau_\beta(F) = \sum_{a=1}^{2n} \left\{ (F^{-1} \nabla^h)_{\tilde{X}_a} F_* \tilde{X}_a - F_* \nabla_{\tilde{X}_a} \tilde{X}_a \right\}.$$ 

Let $h_F$ be the metric induced by $h$ in $F^{-1}TN \to \tilde{M}$ so that

$$h^F(X_i^{\phi_t}, X_j^{\phi_t}) = h^F(X_i^F, X_j^F) \circ \alpha_t, \quad 1 \leq i, j \leq \nu.$$ 

Consequently

$$\|\tau_\beta(\phi_t)\|^2 = \sum_{a=1}^{2n} h^F \left( (F^{-1} \nabla^h)_{\tilde{X}_a} F_* \tilde{X}_a - F_* \nabla_{\tilde{X}_a} \tilde{X}_a, \tau_\beta(F) \right) \circ \alpha_t$$

so that

$$\frac{d}{dt} \{ E_{2,\beta}(\phi_t) \}_{t=0} = \frac{1}{2} \int_M \frac{\partial}{\partial t} \{ h^F(\tau(F), \tau(F)) \}_{(x,0)} \Psi(x) =$$

$$= \int_M \sum_{a=1}^{2n} h^F \left( (F^{-1} \nabla^h)_{\partial/\partial t} \left[ (F^{-1} \nabla^h)_{\tilde{X}_a} F_* \tilde{X}_a - F_* \nabla_{\tilde{X}_a} \tilde{X}_a, \tau_\beta(F) \right] \right)_{(x,0)} \Psi(x).$$

Let $R^{F^{-1} \nabla^h}$ be the curvature tensor field of $F^{-1} \nabla^h$. As

$$\left[ \tilde{X}_a, \frac{\partial}{\partial t} \right] = 0, \quad 1 \leq a \leq 2n,$$

it follows that

$$(F^{-1} \nabla^h)_{\partial/\partial t} (F^{-1} \nabla^h)_{\tilde{X}_a} F_* \tilde{X}_a =$$

$$= (F^{-1} \nabla^h)_{\tilde{X}_a} (F^{-1} \nabla^h)_{\partial/\partial t} F_* \tilde{X}_a - R^{F^{-1} \nabla^h}(\tilde{X}_a, \frac{\partial}{\partial t}) F_* \tilde{X}_a.$$

On the other hand

$$(F^{-1} \nabla^h)_{\partial/\partial t} F_* \tilde{X} - (F^{-1} \nabla^h)_{\tilde{X}} F_* \frac{\partial}{\partial t} F_* \tilde{X} =$$

$$= \left\{ \frac{\partial}{\partial t} (\tilde{X} F^j) + \tilde{X} (F^j) \frac{\partial F^k}{\partial t} (\Gamma^i_{jk} \circ F) \right\} X_i^F -$$

$$- \{ \tilde{X} \left( \frac{\partial F^i}{\partial t} \right) + \frac{\partial F^j}{\partial t} \tilde{X} (F^k) (\Gamma^i_{jk} \circ F) \} X_i^F =$$

$$= \left[ \frac{\partial}{\partial t}, \tilde{X} \right] (F^i) X_i^F = F_* \left[ \frac{\partial}{\partial t}, \tilde{X} \right] = 0.$$
so that
\begin{equation}
(F^{-1}\nabla^h)_{\partial/\partial t} F_* \tilde{X} = (F^{-1}\nabla^h)\tilde{X} F_* \frac{\partial}{\partial t}.
\end{equation}
By (18)-(19)
\begin{equation}
\frac{d}{dt} \left\{ E_{2b}(\phi_t) \right\}_{t=0} = 2n \sum_{a=1}^n \int_M h^F \left( (F^{-1}\nabla^h)\tilde{X}_a (F^{-1}\nabla^h)\tilde{X}_a F_* \frac{\partial}{\partial t} - (F^{-1}\nabla^h)\nabla_{\tilde{X}_a \tilde{X}_a} F_* \frac{\partial}{\partial t} - R^F (F^{-1}\nabla^h) \tilde{X}_a, \frac{\partial}{\partial t} \right) F_* \tilde{X}_a, \tau_b(F) \right\}_{(x,0)} \Psi(x).
\end{equation}
We compute separately the integrand in the right hand side of (20) as follows. First (as \((F^{-1}\nabla^h) h^F = 0)\n\begin{align*}
h^F \left( (F^{-1}\nabla^h)\tilde{X}_a (F^{-1}\nabla^h)\tilde{X}_a F_* \frac{\partial}{\partial t}, \tau_b(F) \right) &= \\
&= \tilde{X}_a \left( h^F \left( (F^{-1}\nabla^h)\tilde{X}_a F_* \frac{\partial}{\partial t}, \tau_b(F) \right) \right) - \\
&- h^F \left( (F^{-1}\nabla^h)\tilde{X}_a F_* \frac{\partial}{\partial t}, (F^{-1}\nabla^h)\tilde{X}_a \tau_b(F) \right) = \\
&= \tilde{X}_a \tilde{X}_a \left( h^F \left( F_* \frac{\partial}{\partial t}, \tau_b(F) \right) \right) - \\
&- 2\tilde{X}_a \left( h^F \left( F_* \frac{\partial}{\partial t}, (F^{-1}\nabla^h)\tilde{X}_a \tau_b(F) \right) \right) + \\
&+ h^F \left( F_* \frac{\partial}{\partial t}, (F^{-1}\nabla^h)\tilde{X}_a (F^{-1}\nabla^h)\tilde{X}_a \tau_b(F) \right).
\end{align*}
Moreover, as \(\tilde{X}(\varphi) \circ \alpha_t = X(\varphi \circ \alpha_t)\) for any \(X \in \mathcal{X}(M)\) and any \(\varphi \in C^\infty(\tilde{M})\)
\begin{equation}
\left[\tilde{X}_a \tilde{X}_a \left( h^F \left( F_* \frac{\partial}{\partial t}, \tau(F) \right) \right) \right] \circ \alpha_t = \\
= X_a X_a \left( h^F \left( F_* \frac{\partial}{\partial t}, \tau_b(F) \right) \circ \alpha_t \right).
\end{equation}
Another tautology we shall need is
\[ V = \left( F_* \frac{\partial}{\partial t} \right) \circ \alpha_0. \]
The following calculation
\[ [(F^{-1}\nabla^h)\tilde{X} \tau_b(F)]_{(x,t)} = \]
\begin{align*}
&= \{ \tilde{X}(\tau_b(F)^i) + \tilde{X}(F^j) \tau_b(F)^k (\Gamma^i_{jk} \circ F) \}_{(x,t)} X^F_i (x, t) = \\
&\quad \{ \tilde{X}(\tau_b(F)^i) + \tilde{X}(F^j) \tau_b(F)^k (\Gamma^i_{jk} \circ F) \}_{(x,t)} X^F_i (x, t) =
\end{align*}
\[ \{ X(\tau_b(\phi_t)^i) + X(\phi_t^j)\tau_a(\phi_t)^k (\Gamma^l_{jk} \circ \phi_t) \}_x X_t^{\phi_t}(x) \]

shows that

\[(22) \quad [(F^{-1} \nabla^h) \tilde{X}_a \tau_b(F)] \circ \alpha_t = (\phi_t^{-1} \nabla^h) X_a \tau_b(\phi_t).\]

Let \( X_t \in H(M) \) be the horizontal tangent vector field determined by

\[ G_\theta(X_t, Y) = h^F \left( F_\ast \frac{\partial}{\partial t}, (F^{-1} \nabla^h) \tilde{X}_a \tau_b(F) \right) \circ \alpha_t \]

for any \( Y \in H(M) \). Then (by \( \nabla g_\theta = 0 \))

\[ \tilde{X}_a \left( h^F \left( F_\ast \frac{\partial}{\partial t}, (F^{-1} \nabla^h) \tilde{X}_a \tau_b(F) \right) \right) \circ \alpha_t = X_a(G_\theta(X_t, X_a)) =
\]

\[ = g_\theta(\nabla_{X_a} X_t, X_a) + g_\theta(X_t \nabla_{X_a} X_a) \]

that is

\[(23) \quad \tilde{X}_a \left( h^F \left( F_\ast \frac{\partial}{\partial t}, (F^{-1} \nabla^h) \tilde{X}_a \tau_b(F) \right) \right) \circ \alpha_t = g_\theta(\nabla_{X_a} X_t, X_a) +
\]

\[ + h^F \left( F_\ast \frac{\partial}{\partial t}, (F^{-1} \nabla^h) \tilde{X}_a \tau_b(F) \right) \circ \alpha_t. \]

A calculation similar to that leading to (22) furnishes

\[(24) \quad [(F^{-1} \nabla^h) \tilde{X}_a (F^{-1} \nabla^h) \tilde{Y} \tau_b(F)] \circ \alpha_t = (\phi_t^{-1} \nabla^h) X (\phi^{-1} \nabla^h) Y \tau_b(\phi_t).\]

Summing up the information in (21) and (23)-(24)

\[(25) \quad \sum_{a=1}^{2n} h^F \left( (F^{-1} \nabla^h) \tilde{X}_a (F^{-1} \nabla^h) \tilde{X}_a F_\ast \frac{\partial}{\partial t}, \tau_b(F) \right) \circ \alpha_0 =
\]

\[ = h^\phi \left( V, \Delta^\phi \tau_b(\phi) \right) - 2\text{div}(X_0) +
\]

\[ + \sum_{a} \left\{ X_a X_a \left( h^\phi(V, \tau_b(\phi)) \right) - h^\phi \left( V, (\phi^{-1} \nabla^h) \nabla_{X_a} X_a \tau_b(\phi) \right) \right\}. \]

Using (25) and

\[ h^F \left( (F^{-1} \nabla^h) \nabla_{X_a} X_a F_\ast \frac{\partial}{\partial t}, \tau_b(F) \right) =
\]

\[ = \nabla_{X_a} X_a \left( h^F \left( F_\ast \frac{\partial}{\partial t}, \tau_b(F) \right) \right) - h^F \left( F_\ast \frac{\partial}{\partial t}, (F^{-1} \nabla^h) \tilde{X}_a \tau_b(F) \right) \]

we may conclude that

\[(26) \quad \sum_{a=1}^{2n} h^F \left( (F^{-1} \nabla^h) \tilde{X}_a (F^{-1} \nabla^h) \tilde{X}_a F_\ast \frac{\partial}{\partial t}, \tau_b(F) \right) \circ \alpha_0 =
\]
we obtain

\[ = h^\phi \left( V, \Delta_b \tau_b(\phi) \right) - 2 \text{div}(X_0) + \Delta_b \left[ h^\phi(V, \tau_b(\phi)) \right]. \]

Given local coordinates \((U, \tilde{x}^A)\) on \(M\) we set \(x^A = \tilde{x}^A \circ p\) where \(p : \tilde{M} \to M\) is the natural projection. To compute the curvature term in the right hand side of (20) we conduct

\[
\left( R^{F^{-1}\nabla_h}(\tilde{X}, \frac{\partial}{\partial t})F_i \tilde{Y} \right)_{(x,0)} =
\]

\[
= X^A(x) Y(F^i)_{(x,0)} \left( R^{F^{-1}\nabla_h} \left( \frac{\partial}{\partial x^A}, \frac{\partial}{\partial t} \right) X^F_i \right)_{(x,0)} =
\]

\[
= X^A(x) Y(\phi^i)_{x} \{ (F^{-1}\nabla_h)_{\partial/\partial x^A}(F^{-1}\nabla_h)_{\partial/\partial t} X^F_i -
\]

\[
- (F^{-1}\nabla_h)_{\partial/\partial t}(F^{-1}\nabla_h)_{\partial/\partial x^A} X^F_i \} \}_{(x,0)} =
\]

\[
= X^A(x) Y(\phi^i)_{x} \left\{ \left( \frac{\partial^2 F^k}{\partial x^A \partial t} (\Gamma^i_{kj} \circ F) + \frac{\partial F^k}{\partial t} \left( \frac{\partial \Gamma^i_{kj}}{\partial y^l} \circ F \right) \right) X^F_i +
\]

\[
+ \frac{\partial F^k}{\partial t} \left( \frac{\partial \Gamma^i_{kj}}{\partial x^A} \circ F \right) \frac{\partial F^\ell}{\partial x^A} (\Gamma^m_{\ell i} \circ F) X^F_m -
\]

\[
- \left( \frac{\partial^2 F^k}{\partial t \partial x^A} (\Gamma^i_{kj} \circ F) + \frac{\partial F^k}{\partial x^A} \left( \frac{\partial \Gamma^i_{kj}}{\partial y^l} \circ F \right) \frac{\partial F^\ell}{\partial t} \right) X^F_i -
\]

\[
- \frac{\partial F^k}{\partial x^A} (\Gamma^i_{kj} \circ F) \frac{\partial F^\ell}{\partial t} (\Gamma^m_{\ell i} \circ F) X^F_m \right\} \}_{(x,0)} =
\]

\[
= X^A(x) Y(\phi^i)_{x} \left( \frac{\partial F^k}{\partial x^A}(x,0) \frac{\partial F^\ell}{\partial x^A}(x,0) \times
\]

\[
\times \left( \frac{\partial \Gamma^m_{k_j}}{\partial y^l} - \frac{\partial \Gamma^m_{l_j}}{\partial y^k} + \Gamma^i_{kj} \Gamma^m_{\ell i} - \Gamma^i_{\ell j} \Gamma^m_{k_i} \right) \phi(x) \right) X^F_m(x) =
\]

\[
= X^A(x) Y(\phi^i)_{x} (R^h)^m_{l_j}(\phi(x)) \frac{\partial F^k}{\partial t}(x,0) \frac{\partial F^\ell}{\partial x^A}(x,0) X^F_m(x)
\]

where \(X = X^A \partial/\partial \tilde{x}^A\) on \(U\). Noticing that

\[ X(\phi^i)_{x} = X^A(x) \frac{\partial F^i}{\partial x^A}(x,0), \quad x \in U, \]

we obtain

\[
(27) \quad \left( R^{F^{-1}\nabla_h}(\tilde{X}, \frac{\partial}{\partial t})F_i \tilde{Y} \right)_{(x,0)} =
\]
\[ X(\phi^j)_x Y(\phi^j)_x \frac{\partial F}{\partial t}(x, 0) (R^h)^m_{ikj}(\phi(x))X^\phi_m(x). \]

Finally
\[ V_x = \frac{\partial F}{\partial t}(x, 0) X^\phi_i(x), \quad X \in U, \]
yields
\[ (28) \left( R^{F^{-1}V^h}(\tilde{X} - \frac{\partial}{\partial t})F, Y \right)_{(x, 0)} = R^h_{\phi(x)}((\phi_* X)_x, V_x)(\phi_* Y)_x \]
for any \( x \in M \). Next (by (28) and the symmetries of the Riemann-Christoffel 4-tensor of \( (N, h) \))
\[ h^F \left( R^{F^{-1}V^h}(\tilde{X} - \frac{\partial}{\partial t})F, \tau_b(F) \right)_{(x, 0)} = \]
\[ = h_{\phi(x)}(R^h_{\phi(x)}((\phi_* X)_x, V_x)(\phi_* X)_x, \tau_b(\phi)_x) = \]
\[ = -h_{\phi(x)}(R^h_{\phi(x)}(\tau_b(\phi)_x, (\phi_* X)_x)((\phi_* X)_x, V_x). \]
Together with (26) and Green’s lemma this leads to the first variation formula (11) for \( E_{2,b} \) in Theorem 3.

The following concept is central to this paper. A smooth map \( \phi : M \to N \) is said to be a subelliptic biharmonic map if \( \phi \) is a critical point of the functional \( E_{2,b} : C^\infty(M, N) \to \mathbb{R} \). By Theorem 3 a smooth map \( \phi : M \to N \) is subelliptic biharmonic if and only if \( \phi \) is a solution to (4). A pseudoharmonic map is trivially subelliptic biharmonic.

5. Subelliptic biharmonic maps and Fefferman’s metric

We start by proving the identity \( \mathbb{E}_2(\phi \circ \pi) = 2\pi E_{2,b}(\phi) \) in Theorem 1. Let us set \( \partial_A = \lambda^A_B T_B \) with \( \lambda^B_A \in C^\infty(U) \). As to the range of indices, we adopt the convention
\[ A, B, C, \cdots \in \{0, 1, \cdots, n, \bar{1}, \cdots, \bar{n}\} \]
with \( T_0 = T \). Then (by (11))
\[ F_{AB} = \tilde{G}_{\phi}(\partial A, \partial B) + \theta(\partial A)\sigma(\partial B) + \theta(\partial B)\sigma(\partial A) = \]
\[ = g_{\alpha\beta}(\lambda_A^\alpha \lambda_B^\beta + \lambda_B^\alpha \lambda_A^\beta) + \lambda_A^0 \sigma_B + \lambda_B^0 \sigma_A \]
where \( \sigma_A = \sigma(\partial A) \). A calculation based on (2) shows that
\[ \sigma_A = \frac{1}{n+2} \left\{ i\lambda_A^\alpha (1_B^\alpha - \frac{1}{2} g^{\alpha\beta} T_B(g_{\alpha\beta})) - \frac{\rho}{4(n+1)} \lambda_A^0 \right\} \circ \pi \]
where $\Gamma^\beta_{BA}$ are given by $\nabla_B T_\alpha = \Gamma^\beta_{BA} T_\beta$. Moreover (by (11))

$$F_{A,2n+2} = 2[(\pi^* \theta) \otimes \sigma](\partial A, \partial/\partial \gamma) = \frac{1}{n+2} \lambda^0_A,$$

$$F_{2n+2,2n+2} = 0.$$ 

Next, using $F^{ab} F_{bc} = \delta^a_c$ (with $a, b, c, \ldots \in \{1, \ldots, 2n+2\}$) we find

$$(29) \begin{cases} F^{AB} F_{BC} + \frac{\lambda^0_C}{n+2} F^{A,2n+2} = \delta^A_C, \\ F^{AB} \lambda^0_B = 0, \\ F^{2n+2,B} F_{BC} + \frac{\lambda^0_C}{n+2} F^{2n+2,2n+2} = 0, \\ F^{2n+2,B} \lambda^0_B = n + 2. \end{cases}$$

Taking into account that $\partial \Phi^j/\partial \gamma = 0$ and $F^{AB} \lambda^0_B = 0$ we have

$$(30) F^{pq}(\Gamma^i_{jk} \circ \Phi) \frac{\partial \Phi^j}{\partial u^A} \frac{\partial \Phi^k}{\partial u^B} =

= F^{AB}(\Gamma^i_{jk} \circ \Phi) \left\{ \lambda^\alpha_A \lambda^\beta_B T_\alpha(\phi^j) T_\beta(\phi^k) + \lambda^\alpha_A \lambda^\beta_B T_\alpha(\phi^j) T^-_\beta(\phi^k) + \lambda^\alpha_A \lambda^\beta_B T^-_\alpha(\phi^j) T_\beta(\phi^k) + \lambda^\alpha_A \lambda^\beta_B T^-_\alpha(\phi^j) T^-_\beta(\phi^k) \right\} \circ \pi.$$

We need the following

**Lemma 2.** The (reciprocal) Fefferman metric is related to the (reciprocal) Levi form by

$$(31) F^{AB} \lambda^\alpha_A \lambda^\beta_B = g^{\alpha\beta},$$

$$(32) F^{AB} \lambda^\alpha_A \lambda^\beta_B = 0.$$

**Proof.** The identities (29) may be written

$$F^{AB} g_{\alpha\overline{\beta}}(\lambda^\alpha_A \lambda^\overline{\beta}_C + \lambda^\alpha_C \lambda^\overline{\beta}_B) + F^{AB} \lambda^0_B \sigma_B + \frac{1}{n+2} F^{A,2n+2} \lambda^0_C = \delta^A_C,$$

$$F^{AB} \lambda^0_B = 0,$$

$$F^{2n+2,B} g_{\alpha\overline{\beta}}(\lambda^\alpha_B \lambda^\overline{\beta}_C + \lambda^\alpha_C \lambda^\overline{\beta}_B) + (n+2) \sigma_C +$$

$$+ F^{2n+2,B} \lambda^0_C \sigma_B + \frac{1}{n+2} F^{2n+2,2n+2} \lambda^0_C = 0,$$

$$F^{2n+2,B} \lambda^0_B = n + 2.$$ 

If $\mu := \lambda^{-1}$ then (by the first of the previous four identities)

$$\mu^A_D = \left( \frac{1}{n+2} F^{A,2n+2} + F^{AB} \sigma_B \right) \delta^0_D + F^{AB} g_{\alpha\overline{\beta}}(\lambda^\alpha_B \delta^\overline{\beta}_D + \lambda^\overline{\beta}_B \delta^\alpha_D)$$

...
yielding

\[
\begin{aligned}
\mu_0^A &= \frac{1}{n+2} F^{A,2n+2} + F^{AB} \sigma_B \\
\mu_0^A &= F^{AB} g^{\alpha\beta} \lambda_B^\alpha \\
\mu_0^A &= F^{AB} g^{\alpha\beta} \lambda_B^\alpha.
\end{aligned}
\]  

(33)

The second and third of the identities (33) lead to (31) and (32), respectively. Lemma 2 is proved.

By Lemma 2 we may write (30) as

\[
\begin{aligned}
F_{pq} (\Gamma^i_{jk} \circ \Phi) \frac{\partial \Phi^j}{\partial u^A} \frac{\partial \Phi^k}{\partial u^B} = 2 \left\{ (\Gamma^i_{jk} \circ \phi) g^{\alpha\beta} T^j_\alpha T^k_\beta (\phi^i) \right\} \circ \pi.
\end{aligned}
\]  

(34)

By a result of J.M. Lee (cf. [24], or Proposition 2.8 in [17], p. 140)

\[
\Box(u \circ \pi) = (\Delta_b u) \circ \pi, \quad u \in C^2(M).
\]

Hence (by (34))

\[
\tau(\Phi) = \tau_b(\phi) \circ \pi.
\]

(35)

On the other hand if \( \pi^{-1}(U) \approx U \times S^1 \) is a local trivialization chart of the canonical circle bundle and \( u \in C^\infty(M) \) is a function supported in \( U \) then

\[
\int_{C(M)} u \circ \pi \ d\text{vol}(F_\theta) = 2\pi \int_M u \ \Psi
\]

(by integration along the fibres of \( S^1 \to C(M) \to M \), cf. e.g. (2.49) in [17], p. 141). Hence (by a partition of unity argument)

\[
\int_{C(M)} \|\tau(\Phi)\|^2 d\text{vol}(F_\theta) = 2\pi \int_M \|\tau_b(\phi)\|^2 \Psi.
\]

(36)

To prove the next statement in Theorem 1 let \( \{\phi_t\}_{|t|<\epsilon} \) be a smooth 1-parameter variation of \( \phi \) \( (\phi_0 = \phi) \) so that \( \Phi_t = \phi_t \circ \pi \) is a 1-parameter variation of \( \Phi \). Therefore (by (36)) if \( \Phi \) is biharmonic then \( \phi \) is a critical point of \( E_{2,b} \).

The converse doesn’t follow from (36) but rather from the first variation formula for \( E_{2,b} \). Indeed let \( \phi : M \to N \) be a smooth solution to (4). A slight modification of Jiang Guoying’s arguments (cf. [22]) leads to the following

**Lemma 3.** A smooth map \( \Phi : C(M) \to N \) is biharmonic if and only if \( \Phi \) is a solution to

\[
BH(\Phi) \equiv \Box^\Phi \tau(\Phi) + \text{trace}_F_h \left\{ (\Phi^{-1} R^h) (\tau(\Phi), \Phi_* \cdot) \Phi_* \cdot \right\} = 0
\]

(37)
where

\begin{equation}
(38) \quad \square^\phi \Upsilon = \sum_{\ell=1}^{2n+2} \epsilon_p \left\{ (\Phi^{-1} \nabla^h)_{X_p} - (\Phi^{-1} \nabla^h)_{\nabla C(M) X_p} \right\} \Upsilon
\end{equation}

is the rough Laplacian on \( C(M) \). Here \( \Upsilon \in C^\infty(\Phi^{-1}TN) \) and \( \{ X_p : 1 \leq p \leq 2n+2 \} \) is a local \( F_\theta \)-orthonormal (i.e. \( F_\theta(X_p, X_q) = \epsilon_p \delta_{pq} \) with \( \epsilon_1 = \cdots = \epsilon_{2n+1} = -\epsilon_{2n+2} = 1 \)) frame in \( T(C(M)) \). Also \( \nabla C(M) \) is the Levi-Civita connection of \( (C(M), F_\theta) \).

Let \( \{ X_a : 1 \leq a \leq 2n \} \) be a local \( G_\theta \)-orthonormal frame in \( H(M) \), defined on the open set \( U \subseteq M \). Then \( \{ X_a^\uparrow, T^\uparrow : 1 \leq a \leq 2n \} \) is a local \( F_\theta \)-orthonormal frame of \( T(C(M)) \). Here for any \( X \in \mathfrak{X}(M) \) we denote by \( X^\uparrow \in \mathfrak{X}(C(M)) \) the horizontal lift of \( X \) with respect to the connection 1-form \( \sigma \) on the canonical circle bundle (thought of as a principal \( S^1 \)-bundle over \( M \)). We recall that \( X^\uparrow \in \text{Ker}(\sigma)_z \) and \( (dz^a)X^\uparrow_z = X_{\pi(z)} \) for any \( z \in C(M) \). Also

\[ S = \frac{n+2}{2} \frac{\partial}{\partial \gamma}. \]

Let \( V \in C^\infty(\phi^{-1}TN) \) and \( \Upsilon = V \circ \pi \). Let \( \Phi = \phi \circ \pi \). The identities

\[ (\Phi^{-1} \nabla^h)_{\partial/\partial \gamma} X^\Phi_k = 0, \]

\[ (\Phi^{-1} \nabla^h)_{\partial/\partial a} X^\Phi_k = \left[ (\phi^{-1} \nabla^h)_{\partial/\partial x^k} X^\Phi_k \right] \circ \pi, \]

imply

\begin{equation}
(39) \quad (\Phi^{-1} \nabla^h)_{X^\uparrow} \Upsilon = \left[ (\phi^{-1} \nabla^h)_X V \right] \circ \pi,
\end{equation}

\begin{equation}
(40) \quad (\Phi^{-1} \nabla^h)_{X^\uparrow} (\Phi^{-1} \nabla^h)_{Y^\uparrow} \Upsilon = \left[ (\phi^{-1} \nabla^h)_X (\phi^{-1} \nabla^h)_Y V \right] \circ \pi,
\end{equation}

for any \( X, Y \in T(M) \). At this point we need

**Lemma 4.** For any \( X, Y \in H(M) \)

\[ \nabla C(M) X^\uparrow = (\nabla_X Y)^\uparrow - (d\theta)(X,Y)T^\uparrow - [A(X,Y) + (d\sigma)(X^\uparrow, Y^\uparrow)]S, \]

\[ \nabla C(M) T^\uparrow = (\tau X + q X)^\uparrow, \]

\[ \nabla C(M) Y^\uparrow = (\nabla_T X + q X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)S, \]

\[ \nabla C(M) S = \nabla C(M) X^\uparrow = (JX)^\uparrow, \]

\[ \nabla C(M) T^\uparrow = Q^\uparrow, \quad \nabla C(M) S = 0, \]

\[ \nabla C(M) T^\uparrow = \nabla C(M) S = 0, \]

where \( q : H(M) \rightarrow H(M) \) is given by \( G_\theta(qX,Y) = (d\sigma)(X^\uparrow, Y^\uparrow) \) and \( Q \in H(M) \) is given by \( G_\theta(Q, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow) \). Also \( \tau \) is the pseudohermitian torsion of \( \nabla \) and \( A(X,Y) = G_\theta(\tau X,Y) \).
Cf. Lemma 2 in E. Baletta et al., [9], p. 083504-26. As a consequence of Lemma 4
\[ \nabla^{C(M)}_{X^\uparrow} X^\uparrow = (\nabla_X X)^\uparrow - A(X, X) S, \]
\[ \nabla^{C(M)}_{T^\uparrow \pm S}(T^\uparrow \pm S) = Q^\uparrow, \]
hence (by (39))
\[ \sum_a (\Phi^{-1} \nabla^h)_{X_a} \mathcal{W} = \]
\[ = \sum_a \{(\Phi^{-1} \nabla^h)(\nabla_{X_a} X_a)^\uparrow - A(X_a, X_a)(\Phi^{-1} \nabla^h)_S \mathcal{W} = \]
\[ = \sum_a [(\phi^{-1} \nabla^h)\nabla_{X_a} X_a V] \circ \pi \]
as \text{trace}_{G_\phi} A = \text{trace}(\tau) = 0 \text{ (cf. e.g. (1.59) in [17], p. 37). Together with (40) and}
\[ (\Phi^{-1} \nabla^h)_S \mathcal{W} = 0 \]
this allows one to conduct the following calculation
\[ \square \Phi \mathcal{W} = \sum_{a=1}^{2n} \{(\Phi^{-1} \nabla^h)^2_{X_a} - (\Phi^{-1} \nabla^h)_{X_a} \mathcal{W} + \]
\[ + \{(\Phi^{-1} \nabla^h)^2_{T^\uparrow + S} - (\Phi^{-1} \nabla^h)_{T^\uparrow + S} \mathcal{W} - \]
\[ - \{(\Phi^{-1} \nabla^h)^2_{T^\uparrow - S} - (\Phi^{-1} \nabla^h)_{T^\uparrow - S} \mathcal{W} = \]
\[ = \sum_a \{(\phi^{-1} \nabla^h)^2_{X_a} V - (\phi^{-1} \nabla^h)\nabla_{X_a} X_a V \mid \circ \pi \]
that is
\[ (41) \]
\[ \square \Phi \mathcal{W} = (\Delta_\phi^G V) \circ \pi. \]
It remains that we compute the curvature term in (37). As \( \Phi_\ast X^\uparrow = (\phi_\ast X) \circ \pi \) for any \( X, Y \in \mathcal{X}(M) \)
\[ (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast X^\uparrow) \Phi_\ast Y^\uparrow = [(\phi^{-1} R^h)(\tau_\phi, \phi_\ast X) \phi_\ast Y] \circ \pi \]
hence
\[ \text{trace}_{F_\phi} \{(\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast \cdot) \Phi_\ast \cdot \} = \]
\[ = \sum_a (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast X_a^\uparrow) \Phi_\ast X_a^\uparrow + \]
\[ + (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast(T^\uparrow + S)) \Phi_\ast(T^\uparrow + S) - \]
\[ - (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast(T^\uparrow - S)) \Phi_\ast(T^\uparrow - S) = \]
\[ = \text{trace}_{G_\phi} \pi_H \{(\phi^{-1} R^h)(\tau_\phi, \phi_\ast \cdot) \phi_\ast \cdot \} + \]
\[ + 2 (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast T^\uparrow) \Phi_\ast S + (\Phi^{-1} R^h)(\tau(\Phi), \Phi_\ast S) \Phi_\ast T^\uparrow \} \]
and $\Phi_* S = 0$ so that
\begin{equation}
\text{trace}_{F_{\theta}} \{(\Phi^{-1} R^h) (\tau(\Phi), \Phi_* \cdot) \Phi_* \cdot \} = \\
= \text{trace}_{G_{\theta}} \pi_H \{(\phi^{-1} R^h) (\tau_b(\phi), \phi_* \cdot) \phi_* \cdot \}.
\end{equation}
Finally (by (41)-(42)) $\phi \circ \pi$ is biharmonic. Theorem 1 is proved.

6. Conclusions and open problems

We introduced the new concept of a subelliptic biharmonic map as a $C^\infty$ solution to the system (4). This is a quasilinear system of variational origin whose principal part is the bi-sublaplacian $\Delta^2_b$. $\Delta^2_b$ is a fourth order hypoelliptic operator, though not elliptic, so that our work is part of the program outlined in [23]. Higher order degenerate elliptic equations, say of the form $\Delta^k_b u = 0$, were not studied in the present day PDEs literature (cf. e.g. [1] for the elliptic case). Our main result is a geometric interpretation of subelliptic biharmonic maps within Lorentzian geometry i.e. each $C^\infty$ solution to (4) may be characterized as the base map corresponding to a $S^1$-invariant biharmonic map from the total space $C(M)$ of the canonical circle bundle endowed with the Fefferman metric. Although, as shown in §4, it makes sense to look for weak solutions to (4) our methods in this paper are purely geometric and a study of local properties of weak subelliptic biharmonic maps appears nowhere in the mathematical literature. Neither may the partial regularity theory be naively reduced to that of $S^1$-invariant biharmonic maps, as $C(M)$ is Lorentzian so that no natural distance function on $C(M)$ is available a priori. If $\Omega \subset M$ is a bounded domain the functionals $E_{2,b}(\phi)$ and $\int_\Omega |\Delta_b \phi|^2 \Psi$ (as introduced by S-Y.A. Chang & L. Wang & P.C. Yang, [15], in the elliptic case for maps $\phi : \Omega \to S^\nu$) have not been compared so far (and of course the corresponding regularity for $\delta \int_\Omega |\Delta_b \phi|^2 \Psi = 0$ is unknown).

References

[1] N. Aronszajn & T.M. Creese & L.J. Lipkin, Polyharmonic functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1983.
[2] P. Baird & D. Kamissoko, On constructing biharmonic maps and metrics, Ann. Global Anal. Geom. 23 (2003), 65–75.
[3] P. Baird, A. Fardoun, and S. Ouakkas, Conformal and semi-conformal biharmonic maps, Ann. Global Anal. Geom. 34 (2008), 403–414.
[4] P. Baird & J.C. Wood, Harmonic morphisms between Riemannian manifolds, London Mathematical Society Monographs, New Series, Vol. 29, Clarendon Press, Oxford, 2003.
[5] A. Balmuş & S. Montaldo & C. Oniciuc, Biharmonic hypersurfaces in 4-dimensional space forms, Math. Nachr. 283 (2010), 1696–1705.
[6] A. Balmuş & S. Montaldo & C. Oniciuc, Classification results for biharmonic submanifolds in spheres, Israel J. Math. 168 (2008), 201–220.
[7] E. Barletta & S. Dragomir, Sublaplacians on CR manifolds, Bull. Math. Soc. Sci. Math. Roumanie, Tome 52 (100), 2009, 3-32.
[8] E. Barletta & S. Dragomir & H. Urakawa, Pseudoharmonic maps from a nondegenerate CR manifold into a Riemannian manifold, Indiana University Math. J., (2) 50 (2001), 719-746.
[9] E. Barletta & S. Dragomir & H. Urakawa, Yang-Mills fields on CR manifolds, Journal of Mathematical Physics, (8) 47 (2006), 1-41.
[10] T. Boggio, Sulle funzioni di Green d’ordine m, Rend. Circ. Mat. Palermo, 20 (1905), 97-135.
[11] A. Bonfiglioli & E. Lanconelli & F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians, Springer Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg, 2007.
[12] J.M. Bony, Principe du maximum, inégalité de Harnak et unicité du problème de Cauchy pour les opérateurs elliptiques dégénéré, Ann. Inst. Fourier, Grenoble, (1) 19 (1969), 277-304.
[13] R. Caddeo & S. Montaldo & C. Oniciuc, Biharmonic submanifolds of $S^3$, Internat. J. Math., 12 (2001), 867-876.
[14] R. Caddeo & S. Montaldo & C. Oniciuc, Biharmonic submanifolds in spheres, Israel J. Math., 130 (2002), 109-123.
[15] S-Y.A. Chang & L. Wang & P.C. Yang, A regularity theory of biharmonic maps, Communications on Pure and Applied Mathematics, LI (1999), 0001-0025.
[16] S. Dragomir & Y. Kamishima, Pseudoharmonic maps and vector fields on CR manifolds, J. Math. Soc. Japan, (1) 62 (2010), 269-303.
[17] S. Dragomir & G. Tomassini, Differential geometry and analysis on CR manifolds, Progress in Mathematics, Vol. 246, Ed. by H. Bass & J. Oesterlé & A. Weinstein, Birkhäuser, Boston-Basel-Berlin, 2006.
[18] J. Eells and L. Lemaire, Selected topics in harmonic maps, CBMS, Vol. 50, Amer. Math. Soc, 1983.
[19] C. Fefferman, Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, Ann. of Math., (2) 103 (1976), 395-416, 104 (1976), 393-394.
[20] C.R. Graham, On Sparling’s characterization of Fefferman metrics, American J. Math., 109 (1987), 853-874.
[21] T. Ichiyama & J-I. Inoguchi & H. Urakawa, Biharmonic maps and bi-Yang-Mills fields, Note Mat. 28 (2009), 233–275.
[22] G. Jiang, 2-Harmonic maps and their first and second variational formulas, Chinese Ann. Math., Ser. A, 7 (1986), 389-402, in Chinese. English translation and notes by H. Urakawa in Note di Matematica, 28 (2009), 209-232, suppl. n. 1, Proceedings of the meeting Recent Advances in Differential Geometry, June 13-16, 2007, Lecce, Italy.
[23] J. Jost & C-J. Xu, Subelliptic harmonic maps, Trans. of A.M.S., (11) 350 (1998), 4633-4649.
[24] J.M. Lee, The Fefferman metric and pseudohermitian invariants, Trans. A.M.S., (1) 296 (1986), 411-429.
[25] Y.-L. Ou. *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math. 248 (2010), 217–232.
[26] Y.-L. Ou, *On conformal biharmonic immersions*, Ann. Global Analysis and Geometry, 36 (2009), 133–142.
[27] Y.-L. Ou & L. Tang, *The generalized Chen’s conjecture on biharmonic submanifolds is false*, arXiv:1006.1838v1.
[28] R. Petit, *Harmonic maps and strictly pseudoconvex CR manifolds*, Communications in Analysis and Geometry, (3) 10 (2002), 575-610.
[29] A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Invent. Math., 78 (1984), 143-160.
[30] G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann., 303 (1995), 325-344.
[31] C.M. Wood, *On the energy of a unit vector field*, Geometriae Dedicata, 64 (1997), 319-330.