REFINEMENTS OF LOWER BOUNDS FOR POLYGAMMA FUNCTIONS

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Abstract. In the paper, some lower bounds for polygamma functions are refined. Moreover, several open problems are posed.

1. Introduction and main results

It is well-known that the classical Euler gamma function

\[ \Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt \]

for \( x > 0 \), the psi function \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \), and the polygamma functions \( \psi^{(i)}(x) \) for \( i \in \mathbb{N} \) are a series of important special functions and have many extensive applications in many branches such as statistics, probability, number theory, theory of 0-1 matrices, graph theory, combinatorics, physics, engineering, and other mathematical sciences.

We begin by summarizing several results which motivated this paper.

In [6, Corollary 2], the inequality

\[ \psi'(x) e^{\psi(x)} < 1 \]

for \( x > 0 \) was deduced. We observe that the inequality [2] can be rearranged as

\[ \psi'(x) < e^{-\psi(x)}. \]

In [4, Lemma 1.2], the inequality [2] was recovered.

In [1, Theorem 4.8], by the aid of the inequality

\[ [\psi'(x)]^2 + \psi''(x) > 0 \]

for \( x > 0 \) or, say,

\[ \sqrt{\frac{|\psi''(x)|}{(2-1)!}} < \psi'(x), \]

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the inequality (2) was generalized to

\[(n-1)! \exp(-n\psi(x+1)) < |\psi^{(n)}(x)| < (n-1)! \exp(-n\psi(x))\]

for \(x > 0\) and \(n \in \mathbb{N}\), which can be restated as

\[e^{-\psi(x+1)} < \sqrt[2]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(x)}\]

for \(x > 0\) and \(n \in \mathbb{N}\).

In [3, Theorem 2.1], the left-hand-side inequality in (6) was refined to

\[(n-1)! \exp\left(-n\psi\left(x + \frac{1}{2}\right)\right) < |\psi^{(n)}(x)| < (n-1)! \exp(-n\psi(x)),\]

which can be reformulated as

\[e^{-\psi(x+1/2)} < \sqrt[2]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(x)}\]

for \(x > 0\) and \(n \in \mathbb{N}\).

Furthermore, the function \(\psi^{(n)}(x)\) was alternatively bounded in [3, Theorem 2.2] by

\[(n-1)! \left[ \frac{\psi^{(k)}(x + 1/2)}{(-1)^{k-1}(k-1)!} \right]^{n/k} < |\psi^{(n)}(x)| < (n-1)! \left[ \frac{\psi^{(k)}(x)}{(-1)^{k-1}(k-1)!} \right]^{n/k},\]

which can be rewritten as

\[k \sqrt[2]{\frac{|\psi^{(k)}(x + 1/2)|}{(k-1)!}} < \sqrt[2]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < k \sqrt[2]{\frac{|\psi^{(k)}(x)|}{(k-1)!}}\]

for \(x > 0\) and \(1 \leq k \leq n-1\).

For more information on this topic, please refer to the expository and survey article [15] and plenty of closely related references therein.

The main aim of this paper is to further refine the left-hand-side inequalities in (8) and (10) or their variants (9) and (11).

Our main results are stated in the following theorem.

**Theorem 1.** For \(n = 1, 2\), the inequality

\[\sqrt[2]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} > e^{-\psi(1/\ln(1+1/x))}\]

holds on \((0, \infty)\). For \(n \in \mathbb{N}\) and \(1 \leq k \leq n-1\), the inequality

\[\sqrt[2]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} > k \sqrt[2]{\frac{|\psi^{(k)}(1/\ln(1+1/x))|}{(k-1)!}}\]

is valid on \((0, \infty)\).
2. Lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([7, 10, 12, 21]). For $k \in \mathbb{N}$, the inequalities

\begin{align*}
\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \\
and\\
\frac{(k-1)!}{x^k} + \frac{k!}{2x^k+1} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^k+1}
\end{align*}

are valid on $(0, \infty)$.

Lemma 2. The inequality

\begin{equation}
\psi'(t) < e^{1/t} - 1
\end{equation}

holds on $(0, \infty)$.

Proof. For the sake of convenience, denote $e^{1/t} - \psi'(t)$ by $h(x)$. It is clear that

\begin{equation}
\lim_{t \to \infty} h(t) = 1.
\end{equation}

A direct calculation reveals that

\begin{align*}
h(t+1) - h(t) &= e^{1/(t+1)} - e^{1/t} + \psi'(t) - \psi'(t+1) \\
&= e^{1/(t+1)} - e^{1/t} + \frac{1}{t^2}
\end{align*}

and

\begin{align*}
e^{1/t} - e^{1/(t+1)} &= \int_0^1 \frac{1}{(t+u)^2} e^{1/(t+u)} \, du \\
&> \int_0^1 \frac{1}{(t+u)^2} \left[ 1 + \frac{1}{t+u} + \frac{1}{2(t+u)^2} \right] \, du \\
&> \frac{6t(t+1)^3 + 1}{6t^2(t+1)^3} \\
&> \frac{1}{t^2}
\end{align*}

for $t \in (0, \infty)$. Hence, by the limit (17) and mathematical induction, we have

\begin{equation}
h(t) > h(t+1) > h(t+2) > \cdots > h(t+k) > \lim_{k \to \infty} h(t+k) = 1,
\end{equation}

which is equivalent to the inequality (16). $\Box$

3. Proof of Theorem 1

Now we turn our attention to proving Theorem 1. Letting $\frac{1}{\ln(1+1/x)} = t$ in (12) and rearranging yield

\begin{equation}
e^{n\psi(t)} \left| \psi^{(n)} \left( \frac{1}{e^{1/t} - 1} \right) \right| > (n-1)!
\end{equation}

for $t \in (0, \infty)$. 
Utilizing the left-hand-side inequalities in (11) and (13) gives
\[ e^{n\psi(t)} \left[ n \left( 1 - \frac{1}{e^{1/t} - 1} \right) \right] > e^{n(\ln t - 1/t)} \left[ (n - 1)! \left( e^{1/t} - 1 \right)^n + \frac{n!}{2} \left( e^{1/t} - 1 \right)^{n+1} \right] \]
\[ = (n - 1)! \left( e^{1/t} - 1 \right)^n \left[ \frac{n}{2} (e^{1/t} - 1) + 1 \right] \]
\[ = (n - 1)! \left( e^{u - 1} \right)^n \left[ \frac{n}{2} (e^{u - 1} - 1) + 1 \right], \]
where \( u = \frac{1}{t} > 0 \). So, in order to prove (19), it is sufficient to show
\[ \frac{(e^u - 1)^n}{u^n e^{nu}} \left[ \frac{n}{2} (e^u - 1) + 1 \right] \geq 1, \quad u > 0, \]
that is,
\[ (e^u - 1)^n [n(e^u - 1) + 2] \geq 2u^n e^{nu}, \quad u > 0. \]
Let
\[ f_n(u) = (e^u - 1)^n [n(e^u - 1) + 2] - 2u^n e^{nu} \]
on \((0, \infty)\). A straightforward differentiation gives
\[ f_1'(u) = 2e^u (e^u - 1 - u) > 0, \]
\[ f_2'(u) = 2e^u \left[ 3e^{2u} - 2e^u (u^2 + u + 2) + 1 \right], \]
\[ \left[ \frac{f_2'(u)}{2e^u} \right]' = 2e^u (3e^u - u^2 - 3u - 3) > 0. \]
Hence, the derivative \( f_2'(u) \) is also positive on \((0, \infty)\). Since \( f_n(0) = 0 \) and the functions \( f_1(u) \) and \( f_2(u) \) are strictly increasing on \((0, \infty)\), it is readily obtained that the functions \( f_1(u) \) and \( f_2(u) \) are strictly positive on \((0, \infty)\). This shows that inequalities (20) and (21) are valid on \((0, \infty)\) for \( n = 1, 2 \). As a result, the inequality (12) is valid on \((0, \infty)\) for \( n = 1, 2 \).

Letting \( \frac{1}{\ln(1+1/x)} = t \) in (13) leads to
\[ (22) \quad \sqrt[n]{\frac{\psi^{(n)}(1/(e^{1/t} - 1))}{(n - 1)!}} > \sqrt[k]{\frac{\psi^{(k)}(t)}{(k - 1)!}} \]
for \( t > 0 \), where \( n \in \mathbb{N} \) and \( 1 \leq k \leq n - 1 \). In [3, Lemma 1.2], the inequality
\[ (23) \quad (-1)^n \psi^{(n+1)}(x) < \frac{n}{\sqrt[n]{(n - 1)!}} \left[ (-1)^{n-1} \psi^{(n)}(x) \right]^{1+1/n} \]
for \( x > 0 \) and \( n \in \mathbb{N} \) was turned out, which can be restated more significantly as
\[ (24) \quad \sqrt[n+1]{\frac{\psi^{(n+1)}(x)}{n!}} < \sqrt[n]{\frac{\psi^{(n)}(x)}{(n - 1)!}}, \]
an equivalence of the right-hand-side inequalities in (10) and (11). Therefore, it is sufficient to show
\[ (25) \quad \lim_{n \to \infty} \sqrt[n]{\frac{\psi^{(n)}(1/(e^{1/t} - 1))}{(n - 1)!}} \geq \psi'(t) \]
for \( t > 0 \).
Making use of the double inequality (15), it is easy to acquire that
\[
(e^{1/t} - 1)^k \sqrt{\frac{k}{2} (e^{1/t} - 1)} + 1 < \sqrt[k]{\frac{\psi(k)(1/(e^{1/t} - 1))}{(k - 1)!}} < (e^{1/t} - 1)^k \sqrt{k(e^{1/t} - 1)} + 1
\]
for \( t \in (0, \infty) \) and \( k \in \mathbb{N} \). Hence,
\[
\lim_{k \to \infty} \sqrt[k]{\frac{\psi(k)(1/(e^{1/t} - 1))}{(k - 1)!}} = e^{1/t} - 1.
\]
By virtue of inequality (16), inequality (25) follows, so inequality (13) is proved.

4. Remarks

In this section, we would like to supply several remarks on Theorem 1.

Remark 1. Since
\[
x < \frac{1}{\ln(1 + 1/x)} < x + \frac{1}{2},
\]
the psi function \( \psi(x) \) is strictly increasing, and \( |\psi^{(n)}(x)| \) for \( n \in \mathbb{N} \) are strictly decreasing on \((0, \infty)\). Then the left-hand-side inequalities in (8) and (9) for \( n = 1, 2 \) and the left-hand-side inequalities in (10) and (11) are respectively refined, say nothing of the left-hand-side inequality in (6) for \( n = 1, 2 \).

Remark 2. The inequality (12) would be invalid if \( n \) were big enough. In other words, the inequality (12) is not valid for all \( n \in \mathbb{N} \). Otherwise, the inequality
\[
\lim_{n \to \infty} \sqrt[n]{\frac{\psi^{(n)}(x)}{(n - 1)!}} = \frac{1}{x} \geq e^{-\psi(1/\ln(1+1/x))}
\]
would be valid on \((0, \infty)\). However, the reversed inequality of (28) holds on \((0, \infty)\).
(Why? See Remark 4 below.) This situation motivates us to pose an open problem: What is the largest positive integer \( n \) such that inequality (12) holds on \((0, \infty)\)?

Remark 3. Rewriting (12) and (12) for \( n = 1 \) leads to
\[
e^{-\psi(L(x,x+1))} < \psi'(x) < e^{-\psi(L(x,x))}
\]
for \( x > 0 \), where
\[
L(a,b) = \begin{cases} 
\frac{b - a}{\ln b - \ln a}, & a \neq b, \\
\frac{1}{a}, & a = b
\end{cases}
\]
stands for the logarithmic mean for positive numbers \( a \) and \( b \). Since the logarithmic mean \( L(a,b) \) is strictly increasing with respect to both \( a > 0 \) and \( b > 0 \) and the psi function \( \psi(x) \) is also strictly increasing on \((0, \infty)\), inequalities (6), (8), (9), (12), and (29) stimulate us to naturally ask the following question: What are the best scalars \( p(n) \geq 0 \) and \( q(n) > 0 \) such that the inequality
\[
e^{-\psi(L(x,x+q(n)))} < \sqrt[n]{\frac{\psi^{(n)}(x)}{(n - 1)!}} < e^{-\psi(L(x,x+p(n)))}
\]
is valid on \((0, \infty)\)?
Similarly, inequalities (10), (11), and (13) motivate us to pose the following open problem: What are the best constants $p(n, k) \geq 0$ and $0 < q(n, k) \leq 1$ such that the inequality

$$
\sqrt[k]{\frac{\psi^{(k)}(L(x, x + q(n, k)))}{(k-1)!}} < \sqrt[n]{\frac{\psi^{(n)}(x)}{(n-1)!}} < \sqrt[k]{\frac{\psi^{(k)}(L(x, x + p(n, k)))}{(k-1)!}}
$$

holds on $(0, \infty)$ for $1 \leq k \leq n - 1$.

**Remark 4.** Letting $\frac{1}{\ln(1+1/x)} = t$ in the reversed version of the inequality (28) and taking the logarithm yield

$$
\psi(t) + \ln(e^{1/t} - 1) < 0
$$
on $(0, \infty)$, an inequality established in [3, Theorem 2.8] and [4, Theorem 2]. The increasing monotonicity of the function in the left-hand-side of the inequality (33) was presented in [2, 11, 17] respectively. The strict concavity and some other generalizations of the function in the inequality (33) was discussed in [11] recently.

**Remark 5.** The case $n = 2$ and $k = 1$ in (13) is

$$
\psi''(x) + \left[\psi'(1/\ln(1+1/x))\right]^2 < 0
$$
on $(0, \infty)$. This refines the inequality

$$
\psi''(x) + \left[\psi\left(x + \frac{1}{2}\right)\right]^2 < 0
$$
on $(0, \infty)$, the special case $n = 2$ and $k = 1$ of the inequality (10). The inequality (35) was also refined and generalized in [19] in another direction.

The inequality (34), a special case with $n = 1$ of the inequality (23), has been generalized to the complete monotonicity and many other cases. For more information, please refer to [8, 13, 18, 19] and closely related references therein.

**Remark 6.** The generalized logarithmic mean $L(p; a, b)$ of order $p \in \mathbb{R}$ for positive numbers $a$ and $b$ with $a \neq b$ is defined in [5, p. 385] by

$$
L(p; a, b) = \begin{cases} 
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p}, & p \neq -1, 0; \\
\frac{b-a}{\ln b - \ln a}, & p = -1; \\
\frac{1}{e} \left( \frac{b}{a} \right)^{1/(b-a)}, & p = 0.
\end{cases}
$$

It is known from [22, 23] that $L(p; a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$. See also [9, 16] and closely related references therein. Furthermore, we can pose the following more general open problem: What are the best scalars $\lambda(n)$, $\mu(n)$, $p(n)$ and $q(n)$ such that the inequality

$$
e^{-\psi(L(\lambda(n);x,x+q(n))))} < \sqrt[n]{\frac{\psi^{(n)}(x)}{(n-1)!}} < e^{-\psi(L(\mu(n);x,x+p(n))))}
$$
is valid on $(0, \infty)$? What are the best constants $\lambda(n,k), \mu(n,k), p(n,k)$ and $q(n,k)$ such that the inequality

\[
\sqrt[k]{\frac{\psi(k)(L(\lambda(n,k);x,x+q(n,k)))}{(k-1)!}} < \sqrt[n]{\frac{\psi(n)(x)}{(n-1)!}} < \sqrt[k]{\frac{\psi(k)(L(\mu(n,k);x,x+p(n,k)))}{(k-1)!}}
\]

(38)

holds on $(0, \infty)$ for $1 \leq k \leq n-1$?

Remark 7. Finally, an alternative proof of the inequality (12) for $n = 1$ is provided as follows. Letting $1 \ln(1+x)/x = t$ in (12) results in

\[
\sqrt[n]{\frac{\psi(n)(1/(e^{1/t} - 1))}{(n-1)!}} > e^{-\psi(t)}
\]

(39)

for $t > 0$ and $n \in \mathbb{N}$. By the inequality

\[
1 + \frac{\alpha x}{1 + (1-\alpha)x} \leq (1+x)^\alpha \leq 1 + \alpha x
\]

(40)

for $x > -1$ and $0 \leq \alpha \leq 1$ (see [14] p. 128 and [24] p. 533) we have

\[
\sqrt[k]{\frac{k}{2}(e^{1/t} - 1) + 1} \geq 1 + \frac{e^{1/t} - 1}{2 + (k-1)(e^{1/t} - 1)}, \quad t > 0.
\]

(41)

Combining this with the left-hand-side inequality in (26) reveals that it suffices to show

\[
1 + \frac{e^{1/t} - 1}{2 + (k-1)(e^{1/t} - 1)} > \frac{e^{-\psi(t)}}{e^{1/t} - 1}, \quad t > 0,
\]

(42)

that is,

\[
k < \frac{1}{e^{-\psi(t)}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} + 1, \quad t > 0.
\]

(43)

By the left-hand-side inequality in (14), it follows that

\[
\frac{1}{e^{-\psi(t)}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} > \frac{1}{e^{-\ln(t-1)/t}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} = \frac{1}{e^{1/t}/t(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} = \frac{e^{2u} - 2e^u - 1}{(e^u - 1)(ue^u - e^u + 1)} > 0
\]

and

\[
\lim_{u \to \infty} \frac{e^{2u} - 2e^u - 1}{(e^u - 1)(ue^u - e^u + 1)} = 0,
\]

where $u = \frac{1}{t}$. Hence, we obtain that $k \leq 1$. The inequality (12) for $n = 1$ is proved.

Remark 8. This is a revised version of the preprint [20].
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