PLASMON RESONANCES OF NANORODS IN TRANSVERSE ELECTROMAGNETIC SCATTERING

Youjun Deng¹, Hongyu Liu² and Guang-Hui Zheng³

Abstract. Plasmon resonance is the resonant oscillation of conduction electrons at the interface between negative and positive permittivity material stimulated by incident light, which forms the fundamental basis of many cutting-edge industrial applications. We are concerned with the quantitative theoretical understanding of this peculiar resonance phenomenon. It is known that the occurrence of plasmon resonance as well as its quantitative behaviours critically depend on the geometry of the material structure, the corresponding material parameters and the operating wave frequency, which are delicately coupled together. In this paper, we study the plasmon resonance for a 2D nanorod structure, which presents an anisotropic geometry and arises in the transverse electromagnetic scattering. We present delicate spectral and asymptotic analysis to establish the accurate resonant conditions as well as sharply characterize the quantitative behaviours of the resonant field.

Keywords: Plasmon resonance, electromagnetism, nanorod, Neumann-Poincaré operator, spectral analysis, asymptotic analysis,

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1. INTRODUCTION

1.1. Mathematical setup and summary of major findings. Initially focusing on the mathematics, but not the physics, we present the mathematical setup as well as summarize the major findings of our study.

Consider the following Helmholtz system in $\mathbb{R}^2$:

$$
\begin{aligned}
\nabla \cdot \left( \frac{1}{\varepsilon(x)} \nabla u(x) \right) + \omega^2 \mu(x) u(x) &= 0, & x &\in \mathbb{R}^2, \\
u(x) &= u^i(x) + u^s(x), & x &\in \mathbb{R}^2, \\
\lim_{|x| \to \infty} |x|^{1/2} \left( \frac{x}{|x|} \cdot \nabla u^s(x) - i\omega u^s(x) \right) &= 0,
\end{aligned}
$$

where $i := \sqrt{-1}, \omega \in \mathbb{R}_+$ signifies the temporal frequency and the last limit holds uniformly in $\hat{x} := x/|x| \in S^1, x = (x_1, x_2) \in \mathbb{R}^2$. The PDE system (1.1) describes the transverse electromagnetic scattering [22]. Here, $\varepsilon$ and $\mu$ are respectively the (relative) electric permittivity and magnetic permeability, which characterize the medium configuration of the space. By a standard normalization, throughout the rest of the paper, we assume that $\mu \equiv 1$ and moreover $(\varepsilon(x) - 1)$ is compactly supported which shall be fixed shortly. In (1.1), $u^i$ signifies an incident wave field satisfying $\Delta u^i + \omega^2 u^i = 0$ in $\mathbb{R}^2$ and, $u^s$ and $u$ are respectively referred to as the scattered and total wave fields.

Suppose $\varepsilon(x) = (\varepsilon_c - 1)\chi(D_\delta) + 1$ with $\varepsilon_c \in \mathbb{C}$, where $\chi(D_\delta)$ is the indicator/characteristic function for $D_\delta$. $\varepsilon_c$ is the material parameter of $D_\delta$, which is a key and subtle ingredient in our resonance study and shall be delicately determined in what follows. We consider

¹School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, China. Email: youjundeng@csu.edu.cn; dengyijun001@163.com
²Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong SAR, China. Email: hongyu.liuip@gmail.com; hongyliu@cityu.edu.hk
³School of Mathematics, Hunan Provincial Key Laboratory of Intelligent Information Processing and Applied Mathematics, Hunan University, Changsha, Hunan, China. Email: zhenggh2012@hnu.edu.cn
that $D_δ$ is a nanorod shape. To define such a shape, let $Γ_0$ be a straight line with the parametrization $Γ_0(t)$, $t ∈ (t_0, t_1)$. Here we set $t_0 = −L/2$ and $t_1 = L/2$, where $L$ is a positive constant. Define $Γ_0(t_0) := P$, $Γ_0(t_1) = Q$, and let $n(t)$ be the normal direction of $Γ_0(t)$, respectively. The nanorod $D_δ$ is defined by $D_δ = \overline{D_δ^0} ∪ D_δ^f ∪ \overline{D_δ^t}$, where $D_δ^f$ is defined by

$$D_δ^f := \{ x(t) | x(t) = Γ_0(t) ± δn(t), \; t ∈ (t_0, t_1) \}.$$  \hspace{1cm} (1.2)

The two caps $D_δ^0$ and $D_δ^t$ are two half disks with radius $δ$ and centering at $P$ and $Q$, respectively. Let $S_δ^0$ and $S_δ^t$ be the surface of $D_δ^0$ and $D_δ^t$, respectively. It can be verified that $D_δ$ is of class $C^{1,α}$ for a certain $α ∈ ℝ_+$, which depends on $δ$. In what follows, we define $S_δ^0 := ∂D_δ^0 = ∂(D_δ^0 ∪ D_δ^f)$, and $S_δ^t := ∂D_δ^t$. As we are considering the straight nanorod, $S_δ^t = Γ_{1,δ} ∪ Γ_{2,δ}$, where $Γ_{1,δ}$, $j = 1, 2$ are defined by

$$Γ_{1,δ} = \{ x : x = Γ_0 − δn \}, \; Γ_{2,δ} = \{ x : x = Γ_0 + δn \}.$$  \hspace{1cm} (1.3)

In particular, when $δ = 1$, we also set $∂D := ∂D_1$, $S^l := S^l (l = a, b, c, f)$, $Γ_j := Γ_{j,1}$ ($j = 1, 2$) for simplicity. Moreover, we shall always use $z_x$ and $z_y$ to signify the projections of $x ∈ S_δ^0$ and $y ∈ S_δ^t$ on $Γ_0$, respectively.

We shall consider our study in the quasi-static regime, which signifies that size of $D_δ$ is much smaller than the operating wavelength $2π/ω$. Noting that in our study, we require that $L ≈ 1$ and $δ ≪ 1$, and hence in general we shall always assume that $ω ≪ 1$. Nevertheless, at this point, we would like to emphasize that the two asymptotic parameters $ω$ and $δ$ are also delicately related, which shall be observed later.

With the above preparation, the first main result of our study is the following asymptotic representation of the scattered wave field $u^s$ from a nanorod material structure.

**Theorem 1.1.** Let $u$ be the solution to (1.1), where $D_δ$ is the nanorod described above. Let the incident wave $u^i$ be the plane wave, i.e. $u^i = e^{iωd · x}$, $d = (d_1, d_2) ∈ S^1$. Then for $x ∈ ℝ^2 \setminus D_δ$, it holds that

$$u^s(x) = ωd \frac{i}{2π} d_2 \int_{-L/2}^{L/2} \frac{x_2}{(x_1 − y_1)^2 + x_2^2}(λ(ε_ε)I + A_δ)^{-1}[1](y_1)dy_1 \hspace{1cm} (1.4)$$

\hspace{1cm} + ωd \frac{i}{2π}(λ(ε_ε) − 1) d_1 \ln \frac{(x_1 + L/2)^2 + x_2^2}{(x_1 − L/2)^2 + x_2^2} + ω · o(δ) + O(ω^2 ln ω),

where

$$λ(ε_ε) = \frac{1}{2} \cdot \frac{1 + ε_ε}{1 − ε_ε},$$

and the operator $A_δ$ shall be defined in (3.22).

**Remark 1.1.** The formula (1.4) presents a neat and concise representation of the wave field from the scattering of a nanorod material structure, which is of significant practical interest for its own sake; see e.g. [17] for the related discussion in a different physical context. It is interesting to see from (1.4) that, if the incident wave is propagating parallel to the nanorod, i.e., $d_2 = 0$, then the formula yields:

$$u^s(x) ∼ ωd \frac{i}{2π}(λ(ε_ε) − 1)^{-1} d_1 \ln \frac{(x_1 + L/2)^2 + x_2^2}{(x_1 − L/2)^2 + x_2^2}.$$  \hspace{1cm} (1.5)

From (1.5), it is readily observed that the scattered field becomes much stronger if one approaches the two ends of the nanorod $P_0$ and $Q_0$.

Theorem 1.1 paves the way for our study of the plasmon resonance associated with the nanorod material structure $(ε_ε, D_δ)$. Before that, we first present the notion of plasmon resonance mathematically.
Definition 1.1. Consider the Helmholtz system (1.1) associated with the nanorod $(D_\delta, \varepsilon_\rho)$. Then plasmon resonance occurs if the following condition is fulfilled:

$$\|\nabla u^s\|_{L^2(\mathbb{R}^3, |D\delta|)} \gg 1.$$  \hspace{1cm} (1.6)

We mention that there are different definitions of plasmon resonance but with essentially similar formulation, see e.g. [5, 14]. The energy blowup is a hallmark feature of the plasmon resonance which implies that the resonant field exhibits highly oscillatory patterns. In the occurrence of plasmon resonance, the highly oscillating behaviour of the resonant field happens near the boundary of the nanostructure, and hence plasmon resonance is also referred to as the Surface Localized Resonance (SLR) or Surface Plasmon Resonance (SPR).

Remark 1.2. Two remarks are in order. First, in Theorem 1.2, we mention that both $\mathbb{R}^\varepsilon$ and $u^s$ should be properly given. Indeed, they are delicately and subtly connected to the spectral properties of certain integral operators. The technical details will be given in Theorem 4.1 in what follows. Second, the condition $\omega^2 \ln |\rho|^{-1} \leq c_1$, where $\rho := \Im \left( \frac{1}{\varepsilon_\rho} \right) < 0$, for a sufficiently small $c_1$, and moreover $\Re \varepsilon_\rho \leq 0$ is properly given. If the incident wave $u^i$ is well chosen and the parameter $\rho$ fulfils that $|\rho|^{-1} \omega \delta \to \infty$ (as $\omega \to 0$, $\delta \to 0$, and $|\rho| \to 0$), then it holds

$$\|\nabla u^s\|_{L^2(\mathbb{R}^3, |D\delta|)} \to \infty.$$  \hspace{1cm} (1.8)

1.2. Background and discussion. Plasmon resonance is the resonant oscillation of conduction electrons at the interface between negative and positive permittivity material stimulated by incident light, which forms the fundamental basis of many cutting-edge applications. The plasmonic technology is revolutionizing many industrial applications including enhancing the brightness of light, confining strong electromagnetic fields, medical therapy, invisibility cloaking and biomedical imaging; see e.g. [2, 4, 7–9, 11, 20–24, 28, 32–34] and the references cited therein.

Metallic nanostructures are often used to construct various plasmonic devices. The metallic nanoparticle exhibits nanoscale uniformly in all dimensions, a.k.a. isotropic geometry, which is the simplest nanostructure. Recently, there are extensive and intensive studies on mathematically characterizing the plasmon resonances associated with nanoparticles;
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see [2–6,8,9,11,16,21–24,28,32,34] and the references cited therein. The metallic nanorod is another important nanostructure, which has a long aspect ratio and possesses different size scales in different dimensions, a.k.a. anisotropic geometry. Metallic nanorods have been widely used in real applications including semiconductor materials, microelectromechanical systems, food packaging, catalysis, energy storage, biomedicine and cloaking [12,18,19,25,31]. In particular, some nanorods (such as $\text{CeO}_2$) display higher catalytic activity compared to the nanoparticles, which could potentially increase their usage [25]. However, to our best knowledge, there is little mathematical study in the literature on theoretically characterizing the plasmon resonances associated with nanorods. In [27], the authors study the plasmon resonance associated with a certain axisymmetric slender body in the quasi-static regime via the matched asymptotics method. In [15], the authors investigate the plasmon resonance of curved nanorod for 3D Helmholtz system, which describes the acoustic scattering.

In this article, we focus on the mathematical analysis on the plasmon resonances associated with a 2D straight nanorod, which arises in the transverse electromagnetic scattering. The technical novelty of our study can be summarized as follows. Compared to the 3D study in [15], we give more delicate asymptotic and spectral analysis of the layer potential operators, especially the single layer potential operator and Neumann-Poincaré operator. In fact, by looking at the aspect ratios, the nanorod in 2D present more severe geometric singularities and challenges than that in 3D. This can be partly evidenced by the severe logarithmic singularity of the fundamental solution of the 2D Helmholtz equation, compared to the weaker singularity of the 3D fundamental solution. Nevertheless, through delicate and subtle asymptotic and spectral analysis, we manage to derive an accurate asymptotic formula of the scattering field, from which we can further establish the sharp conditions that ensure the occurrence of the plasmon resonance. In [15], the 3D nanorod can be curved, whereas in the current study, we mainly consider the straight nanorod. This enables to derive much more accurate understandings of the plasmon resonance as well as its quantitative relationships to the metamaterial parameters, the wave frequency and the nanorod geometry.

The rest of the paper is organized as follows. In Section 2, we present the layer potential theory and several technical asymptotic expansions. The quantitative analysis of the scattered field is derived in Section 3. Finally, we establish the mathematical analysis of the plasmon resonance for the nanorod in Section 4.

2. Auxiliary results

In this section, we establish several technical auxiliary results for our subsequent use.

2.1. Layer potential operators. Our analysis heavily relies on the layer potential technique. We briefly present some preliminary knowledge on the layer potential theory and refer to [2,4–6,13,16] and the references cited therein for more related results.

Let $G_k$ be the outgoing fundamental solution to the PDE operator $\Delta + k^2$ in $\mathbb{R}^2$, which is given by:

$$G_k(x) = -\frac{i}{4}H_0^{(1)}(k|\mathbf{x}|),$$

where $H_0^{(1)}(k|\mathbf{x}|)$ is the Hankel function of first kind of order zero. For any bounded Lipschitz domain $B \subset \mathbb{R}^2$, we denote by $S^k_B : H^{-1/2}(\partial B) \to H^1(\mathbb{R}^d \setminus \partial B)$ the single layer potential operator given by

$$S^k_B[\phi](\mathbf{x}) := \int_{\partial B} G_k(\mathbf{x} - \mathbf{y})\phi(\mathbf{y}) \, d\sigma(\mathbf{y}),$$

2.2
and \((K^k_B)^*: H^{-1/2}(\partial B) \to H^{-1/2}(\partial B)\) the Neumann-Poincaré operator
\[
(K^k_B)^*[\varphi](x) := \text{p.v.} \int_{\partial B} \frac{\partial G_k(x - y)}{\partial \nu(x)} \varphi(y) \, d\sigma(y),
\]
where p.v. stands for the Cauchy principle value. In (2.3) and also in what follows, unless otherwise specified, \(\nu\) signifies the exterior unit normal vector to the boundary of the concerned domain. It is known that the single layer potential operator \(S^k_B\) is continuous across \(\partial B\) and satisfies the following trace formula
\[
\frac{\partial}{\partial \nu} S^k_B[\varphi] \bigg|_\pm = (\pm i + (K^k_B)^*)[\varphi] \quad \text{on} \quad \partial B,
\]
where \(\frac{\partial}{\partial \nu}\) stands for the normal derivative and the subscripts \(\pm\) indicate the limits from outside and inside of a given inclusion \(B\), respectively. In the following, if \(k = 0\), we formally set \(G_0\) introduced in (2.1) to be \(G_0\), which has the form
\[
G_0(x) := \frac{1}{2\pi} \ln |x|,
\]
and the other integral operators introduced above can also be formally defined when \(k = 0\).

In what follows, for the sake of simplicity, we also denote by \(K_B\) and \(K_B^*\) be the Neumann-Poincaré operators \(K^0_B\) and \((K^0_B)^*\), respectively.

### 2.2. Asymptotic expansions of layer potentials

In the part, we shall first present some asymptotic expansions of layer potential operators with respect to the size parameter \(\delta \ll 1\). Note that the single layer potential operator \(S^0_B: H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B)\) is not invertible in \(\mathbb{R}^2\). We introduce a substitute of \(S^0_B\) as follows
\[
\tilde{S}^0_B[\psi] = \begin{cases} S^0_B[\varphi], & \text{if } \langle \psi, \chi(\partial B) \rangle_{-\frac{1}{2}, \frac{1}{2}} = 0, \\ \chi(\partial B), & \text{if } \psi = \varphi_0, \end{cases}
\]
where \(\langle \cdot, \cdot \rangle_{-\frac{1}{2}, \frac{1}{2}}\) is the duality pairing between \(H^{\frac{1}{2}}(\partial B)\) and \(H^{-\frac{1}{2}}(\partial B)\). \(\varphi_0\) is the unique eigenfunction of static Neumann-Poincaré operator \((K^0_B)^*\) associated with eigenvalue \(\frac{1}{2}\) such that
\[
\langle \varphi_0, \chi(\partial B) \rangle_{-\frac{1}{2}, \frac{1}{2}} = 1.
\]
It can be found that \(\tilde{S}^0_B: H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B)\) is invertible. Thanks to the Calderón identity:
\[
K^0_B \tilde{S}^0_B = \tilde{S}^0_B (K^0_B)^*,
\]
from the invertibility and positivity of \(-\tilde{S}^0_B\), we can define the inner product
\[
\langle u, v \rangle_{H^*(\partial B)} = -\langle u, \tilde{S}^0_B[v] \rangle_{-\frac{1}{2}, \frac{1}{2}}.
\]

Then, we have the following results

**Lemma 2.1.** Let \(B\) be of class \(C^{1,\alpha}\). Then

1. \((K^0_B)^*\) is a compact self-adjoint operator in the Hilbert space \(H^*(\partial B)\) equipped with the inner product (2.7), which is equivalent to the original one;
2. Let \(\lambda_j, \varphi_j\), \(j = 0, 1, 2, \cdots\), be the eigenvalue and eigenfunction pair of \((K^0_B)^*\), here \(\lambda_0 = \frac{1}{2}\). Then, \(\lambda_j \in (-\frac{1}{2}, \frac{1}{2})\), and \(\lambda_j \to 0\) as \(j \to \infty\);
3. \(H^*(\partial B) = \mathcal{H}^*_0(\partial B) \oplus \{c\varphi_0\}, \quad c \in \mathbb{C}\), where \(\mathcal{H}^*_0(\partial B) = \{\varphi \in H^*(\partial B) : \int_{\partial B} \varphi d\sigma = 0\} \).
(4) For any \( \psi \in H^{-\frac{1}{2}}(\partial B) \), it hold:
\[
(K_B^0)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j a_j^{-1} \langle \psi, \varphi_j \rangle_{H^*(\partial B)} \varphi_j, \tag{2.8}
\]
where
\[
a_j = \langle \varphi_j, \varphi_j \rangle_{H^*(\partial B)}.
\]

**Remark 2.1.** Note that \((\delta B)^{-1}[\chi(\partial B)] = \varphi_0\), and \(-\frac{1}{2}I + (K_B^0)^* = (-\frac{1}{2}I + (K_B^0)^*) \mathcal{P}_{\mathcal{H}_0^*(\partial B)}\), where \(\mathcal{P}_{\mathcal{H}_0^*(\partial B)}\) is the orthogonal projection onto \(\mathcal{H}_0^*(\partial B)\). Furthermore, it follows that
\[
\left(-\frac{1}{2}I + (K_B^0)^*) \right) (\delta B)^{-1}[\chi(\partial B)] = 0. \tag{2.9}
\]

We also introduce the function space \(\mathcal{H}(\partial B)\) which is the space \(H^\frac{1}{2}(\partial B)\) equipped with the following inner product
\[
(u, v)_{\mathcal{H}(\partial B)} = -((\delta B)^{-1}[u], v)_{-\frac{1}{2}}. \tag{2.10}
\]
It can be directly verified that \(\tilde{\delta B}\) is an isometry between \(\mathcal{H}^*(\partial B)\) and \(\mathcal{H}(\partial B)\), and \(\mathcal{H}^*(\partial B)\) is the dual space of \(\mathcal{H}(\partial B)\). From now on, we use \((\cdot, \cdot)\) as the standard inner product in \(\mathbb{R}^2\). The inner product (2.7) and the corresponding norm on \(\partial D_\delta\) are denoted by \((\cdot, \cdot)\) and \(\| \cdot \|\) in short, respectively. \(A \lesssim B\) means \(A \leq CB\) for some generic positive constant \(C\). \(A \approx B\) means that \(A \lesssim B\) and \(B \lesssim A\).

In what follows, we always suppose that \(\delta \ll 1\) and each eigenvalue of the operator Neumann-Poincaré operator \(\mathcal{K}_{D_\delta}\) is simple. We shall first present some asymptotic expansions of the Neumann-Poincaré operator with respect to \(\delta\). Recalling that \(\partial D_\delta = S^o_\delta \cup S^b_\delta\), we decompose the Neumann-Poincaré operator into several parts accordingly. To that end, we introduce the following boundary integral operator:
\[
\mathcal{K}_{S,S'}[\varphi](x) := \chi(S') \frac{1}{2\pi} \int_S \frac{(x - y, \nu_x)}{|x - y|^2} \varphi(y) d\sigma(y), \quad \text{for } S \cap S' = \emptyset. \tag{2.11}
\]

It is obvious that \(\mathcal{K}_{S,S'}\) is a bounded operator from \(L^2(S)\) to \(L^2(S')\). For the subsequent use, we also introduce the following regions:
\[
\iota_\delta(P) := \{ x; \ |P - x| \in \mathcal{O}(\delta), \ x \in S^o_\delta \}, \tag{2.12}
\]
\[
\iota_\delta(Q) := \{ x; \ |Q - x| \in \mathcal{O}(\delta), \ x \in S^b_\delta \}. \tag{2.13}
\]
Define \(\tilde{\varphi}(\tilde{x}) := \varphi(x)\), where \(x \in S^o_\delta, S^b_\delta\) and \(\tilde{x} \in S^o, S^b\).

The asymptotic expansion of the Neumann-Poincaré operator is given in [17].

**Lemma 2.2.** The Neumann-Poincaré operator \(\mathcal{K}_{D_\delta}^*\) admits the following asymptotic expansion:
\[
\mathcal{K}_{D_\delta}^*[\varphi](x) = \mathcal{K}_0[\varphi](x) + \delta \mathcal{K}_1[\varphi](x) + \mathcal{O}(\delta^2), \tag{2.14}
\]
where \(\mathcal{K}_0\) is defined by
\[
\mathcal{K}_0[\varphi](x) = \chi(S^o_\delta) \left( \mathcal{K}_{S^o_\delta, S^b_\delta}[\varphi](x) + \frac{1}{4\pi} \int_{S^o_\delta} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) \right) + \chi(S^b_\delta) \left( \mathcal{K}_{S^o_\delta, S^b_\delta}[\varphi](x) + \frac{1}{4\pi} \int_{S^b_\delta} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) \right)
+ A_{\mathcal{F}_2, \mathcal{F}_1}[\varphi] + A_{\mathcal{F}_1, \mathcal{F}_2}[\varphi] + \chi(\iota_\delta(P))\mathcal{K}_{S^o_\delta, S^b_\delta}[\varphi](x) + \chi(\iota_\delta(Q))\mathcal{K}_{S^o_\delta, S^b_\delta}[\varphi](x), \tag{2.15}
\]
and

\[ K_1[\varphi] = \chi(S_0^b) \frac{(x - P, \nu_x)}{2\pi |x - P|} \int_{S_0^b} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + \chi(S_0^b) \frac{(x - Q, \nu_x)}{2\pi |x - Q|} \int_{S_0^b} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) \]

\[ + \chi(S_0^\delta \setminus \iota_\delta(P)) \left( \frac{\delta}{|x - P|^2} \int_{S_0^\delta} (1 - \langle \tilde{y} - P, \nu_x \rangle) \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + o\left( \frac{\delta}{|x - P|^2} \right) \right) \]

\[ + \chi(S_0^\delta \setminus \iota_\delta(Q)) \left( \frac{\delta}{|x - Q|^2} \int_{S_0^\delta} (1 - \langle \tilde{y} - Q, \nu_x \rangle) \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + o\left( \frac{\delta}{|x - P|^2} \right) \right). \]

(2.16)

Here, the operators \( A_{\Gamma_1, \delta, \Gamma_2, \delta} \) and \( A_{\Gamma_2, \delta, \Gamma_1, \delta} \) are defined by

\[ A_{\Gamma_1, \delta, \Gamma_2, \delta}[\varphi](x) = \frac{1}{\pi} \chi(\Gamma_{1, \delta}) \int_{\Gamma_{1, \delta}} \frac{\delta}{|x - y|^2} \varphi(y) d\sigma(y), \]

\[ A_{\Gamma_2, \delta, \Gamma_1, \delta}[\varphi](x) = \frac{1}{\pi} \chi(\Gamma_{1, \delta}) \int_{\Gamma_{1, \delta}} \frac{\delta}{|x - y|^2} \varphi(y) d\sigma(y). \]

(2.17)

Remark 2.2. We mention that the operator \( K_1 \) appeared in (2.14) still depends on \( \delta \). In fact, if \( x \in \Gamma_{1, \delta} \setminus (\iota_\delta(P) \cup \iota_\delta(Q)) \), \( j = 1, 2 \). Then the operator \( \delta K_1 \) admits the following asymptotic form:

\[ \delta K_1 = \chi(\iota_{\delta^*}(P) \cup \iota_{\delta^*}(Q))O(\delta^{2(1-\epsilon)}) + O(\delta^2), \quad 0 < \epsilon < 1. \]

(2.18)

Next, we consider the asymptotic expansions for single layer potential operator \( S_{D, \delta}^0 \) with respect to \( \delta \). To begin with, we first define the following subregions

\[ \iota_{1, \delta^*}(\bar{x}) := \{\bar{y} | |z_{\bar{x}} - z_{\bar{y}}| < \delta^*, \bar{y} \in \partial D\}, \]

\[ \iota_{1, \delta^*}(\bar{x}) := \{\bar{y} | |z_{\bar{x}} - z_{\bar{y}}| < \delta^*, \bar{y} \in \Gamma_j\}, \quad j = 1, 2, \]

\[ \iota_{1, \delta^*}(P) := \{\bar{y} | |P - z_{\bar{y}}| < \delta^*, \bar{y} \in \partial D\}, \]

\[ \iota_{1, \delta^*}(Q) := \{\bar{y} | |Q - z_{\bar{y}}| < \delta^*, \bar{y} \in \partial D\}, \]

\[ \iota_{2, \delta^*}(\bar{x}) := \{\bar{y} | |z_{\bar{x}} - z_{\bar{y}}| < \delta^*, \bar{y} \in \Gamma_j\}, \quad j = 1, 2, \]

\[ \iota_{2, \delta^*}(P) := \{\bar{y} | |P - z_{\bar{y}}| < \delta^*, \bar{y} \in \Gamma_j\}, \quad j = 1, 2, \]

\[ \iota_{2, \delta^*}(Q) := \{\bar{y} | |Q - z_{\bar{y}}| < \delta^*, \bar{y} \in \Gamma_j\}, \quad j = 1, 2, \]

and

\[ \iota_{\delta^*}(x) := \{y | |x - y| < \delta^*, y \in \partial D\}, \]

\[ \iota_{\delta^*}(x) := \{y | |x - y| < \delta^*, y \in \Gamma_j\}, \quad j = 1, 2, \]

\[ \iota_{\delta^*}(P) := \{y | |P - y| < \delta^*, y \in \partial D\}, \]

\[ \iota_{\delta^*}(Q) := \{y | |Q - y| < \delta^*, y \in \partial D\}, \]

\[ \iota_{\delta^*}(P) := \{y | |P - y| < \delta^*, y \in \Gamma_j\}, \quad j = 1, 2, \]

\[ \iota_{\delta^*}(Q) := \{y | |Q - y| < \delta^*, y \in \Gamma_j\}, \quad j = 1, 2, \]

(2.19)

(2.20)

where \( \gamma \in (0, 1) \) is a parameter which can be selected arbitrarily.
Meanwhile, we introduce the following boundary integral operators:

\[
S_{\Lambda_1, \Lambda_2}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \ln |z_{\tilde{x}} - z_{\tilde{y}}| \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
S_{\Lambda_1(0)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \ln |\delta(\tilde{x} - \tilde{y}) + (1 - \delta)(z_{\tilde{x}} - z_{\tilde{y}})| \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
S_{\Lambda_1, \Lambda_2}[\tilde{\varphi}](\tilde{x}) := \delta \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \ln |\delta(\tilde{x} - \tilde{y}) + (1 - \delta)(z_{\tilde{x}} - z_{\tilde{y}})| \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
\Pi_{\Lambda_1}^{(0)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
S_{\Lambda_1, \Lambda_2}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \frac{(z_{\tilde{x}} - z_{\tilde{y}}, \tilde{x} - \tilde{z})}{|z_{\tilde{x}} - z_{\tilde{y}}|^2} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

and

\[
S_{\Lambda_1, \Lambda_2}^{(1)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \ln |\tilde{x} - \tilde{y}| \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
S_{\Lambda_1, \Lambda_2}^{(0,0)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \frac{1 + (\tilde{x} - z_{\tilde{y}}, \tilde{y} - \tilde{y})}{|\tilde{x} - z_{\tilde{y}}|^2} \frac{1}{|\tilde{x} - \tilde{y}|^4} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

\[
S_{\Lambda_1, \Lambda_2}^{(2)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \frac{\tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y}{|\tilde{x} - z_{\tilde{y}}|^2},
\]

\[
S_{\Lambda_1, \Lambda_2}^{(2,0)}[\tilde{\varphi}](\tilde{x}) := \frac{1}{2\pi} \chi(\Lambda_2) \int_{\partial \Omega_1} \frac{(z_{\tilde{x}} - z_{\tilde{y}}, \tilde{x} - \tilde{z})}{|z_{\tilde{x}} - z_{\tilde{y}}|^2} \tilde{\varphi}(\tilde{y}) d\tilde{\sigma}_y,
\]

where \( \Lambda_j \subset \partial D \) (\( j = 1, 2 \)) are open surfaces.

**Lemma 2.3.** Let \( \varphi \in H^s(\partial D_\delta) \) and \( \tilde{\varphi}(\tilde{x}) = \varphi(x) \) for \( x \in \partial D_\delta \) and \( \tilde{x} \in \partial D \). Then the following asymptotic results hold:

\[
S_{D_0}[\varphi](x) = S_{D,0}[\tilde{\varphi}](\tilde{x}) + (\delta \ln \delta) S_{D,\ln}[\tilde{\varphi}](\tilde{x}) + \delta S_{D,1}[\tilde{\varphi}](\tilde{x}) + \delta^2 S_{D,2}[\tilde{\varphi}](\tilde{x}) + O(\delta^3),
\]

where

\[
S_{D,0}[\tilde{\varphi}](\tilde{x}) := \left( S_{S_{\iota_1, \iota_7}(P), S^n} + S_{S_{\iota_1, \iota_7}(Q), S^n} + S_{S_{\iota_1, \iota_7}(\tilde{x}), \Gamma_1} + S_{S_{S_{\iota_1, \iota_7}(\tilde{x})}, \Gamma_2} + S_{S_{\iota_1, \iota_7}(\tilde{x}), S^n} + S_{S_{\iota_1, \iota_7}(\tilde{x}), \Gamma_1} + S_{S_{S_{\iota_1, \iota_7}(\tilde{x})}, \Gamma_2} + S_{S^n, S_{\iota_1, \iota_7}(P)} + S_{S^n, S_{\iota_1, \iota_7}(Q)} \right) [\tilde{\varphi}](\tilde{x}),
\]

\[
S_{D,\ln}[\tilde{\varphi}](\tilde{x}) := \left( \Pi_{S^n, S^n}^{(0)} + \Pi_{S^n, S^n}^{(0)} \right) [\tilde{\varphi}](\tilde{x}),
\]

\[
S_{D,1}[\tilde{\varphi}](\tilde{x}) := \left( S_{S^n, S_{\iota_1, \iota_7}(P)} + S_{S^n, S_{\iota_1, \iota_7}(Q)} + S_{S^n, S^n} + S_{S^n, S_{\iota_1, \iota_7}(P)} + S_{S^n, S_{\iota_1, \iota_7}(Q)} \right) [\tilde{\varphi}](\tilde{x}),
\]

\[
S_{D,2}[\tilde{\varphi}](\tilde{x}) := \left( S_{S^{(2,0)}, S_{\iota_1, \iota_7}(P)} + S_{S^{(2,0)}, S_{\iota_1, \iota_7}(Q)} + S_{S^{(2,1)}, S_{\iota_1, \iota_7}(\tilde{x}), \Gamma_1} + S_{S^{(2,1)}, S_{\iota_1, \iota_7}(\tilde{x}), \Gamma_2} + S_{S^n, S_{\iota_1, \iota_7}(P)} + S_{S^n, S_{\iota_1, \iota_7}(Q)} \right) [\tilde{\varphi}](\tilde{x}).
\]
Proof. Owing to the definition formula of single layer potential operator (2.2) and the Taylor’s expansion of \(\ln(1 + x)\), we have that

**Case 1:** For \(y \in \Gamma_{1,\delta}\). Since \(\partial D_\delta = S_0^a \cup \Gamma_{1,\delta} \cup \Gamma_{2,\delta} \cup S_0^b\), it implies

\[
\frac{1}{2\pi} \chi(S_0^a) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(P)} \ln \|x - y\| \varphi(y) d\sigma_y
= \frac{1}{4\pi} \chi(S_0^a) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(P)} \ln \left|P - z_\gamma\right|^2 \left(1 + \frac{2\delta(P - z_\gamma, \bar{x} - P) + 2\delta^2(1 + (\bar{x} - P, z_\gamma - \bar{y}))}{|P - z_\gamma|^2}\right) \tilde{\varphi}(\bar{y}) d\bar{y}
\]

\[
= \frac{1}{2\pi} \chi(S_0^a) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(P)} \ln \|P - z_\gamma\| \tilde{\varphi}(\bar{y}) d\bar{y}
+ \frac{1}{4\pi} \chi(S_0^a) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(P)} \ln \left(1 + \frac{2\delta(P - z_\gamma, \bar{x} - P) + 2\delta^2(1 + (\bar{x} - P, z_\gamma - \bar{y}))}{|P - z_\gamma|^2}\right) \tilde{\varphi}(\bar{y}) d\bar{y}
\]

\[
= \left(S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0} + \delta S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0} + \delta^2 S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^3).
\]

Similarly,

\[
\frac{1}{2\pi} \chi(S_0^b) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(P)} \ln \|x - y\| \varphi(y) d\sigma_y
= \left(S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0} + \delta S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0} + \delta^2 S_{\Gamma_{1,\delta} \setminus \partial \gamma(P),S_0}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^3).
\]

Clearly,

\[
\frac{1}{2\pi} \chi(\Gamma_{1,\delta}) \int_{\Gamma_{1,\delta}} \ln \|x - y\| \varphi(y) d\sigma_y = \frac{1}{2\pi} \chi(\Gamma_1) \int_{\Gamma_1} \ln \|z - z_\gamma\| \tilde{\varphi}(\bar{y}) d\bar{y}.
\]

Furthermore,

\[
\frac{1}{2\pi} \chi(\Gamma_{2,\delta}) \int_{\Gamma_{2,\delta} \setminus \partial \gamma(x)} \ln \|x - y\| \varphi(y) d\sigma_y
= \frac{1}{4\pi} \chi(\Gamma_2) \int_{\Gamma_{1,\delta} \setminus \partial \gamma(\bar{x})} \ln \left|z - z_\gamma\right|^2 \left(1 + \frac{4\delta^2}{|z - z_\gamma|^2}\right) \tilde{\varphi}(\bar{y}) d\bar{y},
\]

\[
= \left(S_{\Gamma_{1,\delta} \setminus \partial \gamma(\bar{x}),\Gamma_2} + \delta^2 S_{\Gamma_{1,\delta} \setminus \partial \gamma(\bar{x}),\Gamma_2}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^4).
\]

**Case 2:** For \(y \in \Gamma_{2,\delta}\). Noticing the symmetry of geometry of \(\partial D\), we only need to exchange \(\Gamma_{2,\delta}\) with \(\Gamma_{1,\delta}\) in Case 1 and get the corresponding expansion formula. That is,

\[
\frac{1}{2\pi} \chi(S_0^a) \int_{\Gamma_{2,\delta} \setminus \partial \gamma(P)} \ln \|x - y\| \varphi(y) d\sigma_y
= \left(S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0} + \delta S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0} + \delta^2 S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^3),
\]

\[
\frac{1}{2\pi} \chi(S_0^b) \int_{\Gamma_{2,\delta} \setminus \partial \gamma(P)} \ln \|x - y\| \varphi(y) d\sigma_y
= \left(S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0} + \delta S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0} + \delta^2 S_{\Gamma_{2,\delta} \setminus \partial \gamma(P),S_0}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^3),
\]

\[
\frac{1}{2\pi} \chi(\Gamma_{2,\delta}) \int_{\Gamma_{2,\delta}} \ln \|x - y\| \varphi(y) d\sigma_y = \frac{1}{2\pi} \chi(\Gamma_2) \int_{\Gamma_2} \ln \|z - z_\gamma\| \tilde{\varphi}(\bar{y}) d\bar{y},
\]

\[
= \left(S_{\Gamma_{2,\delta} \setminus \partial \gamma(\bar{x}),\Gamma_2} + \delta^2 S_{\Gamma_{2,\delta} \setminus \partial \gamma(\bar{x}),\Gamma_2}\right) [\varphi](\bar{x}) + \mathcal{O}(\delta^4).
\]
and
\[
\frac{1}{2\pi} \chi(\Gamma_{1,\delta}) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( S_{\Gamma_2,\Gamma_1}^{(1)} + \delta^2 S_{\Gamma_2,\Gamma_1}^{(2,1)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3) \]

**Case 3:** For \( y \in S^3_\delta \).

\[
\frac{1}{2\pi} \chi(\Gamma_{1,\delta} \setminus \iota_{\delta^\gamma}(P)) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \frac{1}{4\pi} \chi(\Gamma_1 \setminus \iota_{\delta^\gamma}(P)) \int_{S^3} \ln \left( |P - z_x|^2 \left( 1 + \frac{2\delta(P - z_x, y - P) + 2\delta^2(1 + (y - P, z_x - \tilde{x}))}{|P - z_x|^2} \right) \right) \varphi(y) d\sigma_y = \frac{1}{2\pi} \chi(\Gamma_1 \setminus \iota_{\delta^\gamma}(P)) \delta \int_{S^3} \ln \left( 1 + \frac{2\delta(P - z_x, y - P) + 2\delta^2(1 + (y - P, z_x - \tilde{x}))}{|P - z_x|^2} \right) \varphi(y) d\sigma_y
\]

By exchanging \( \Gamma_{2,\delta} \) with \( \Gamma_{1,\delta} \), we have

\[
\frac{1}{2\pi} \chi(\Gamma_{2,\delta} \setminus \iota_{\delta^\gamma}(P)) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta S_{\Gamma_2,\Gamma_1} + \delta^2 S_{\Gamma_2,\Gamma_1}^{(2,2)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3).
\]

Moreover, we also obtain

\[
\frac{1}{2\pi} \chi(S^3_\delta) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta S_{S^3,\Gamma_2} + \delta^2 S_{S^3,\Gamma_2}^{(2,3)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3),
\]

and

\[
\frac{1}{2\pi} \chi(S^3_\delta) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta \ln \delta \Pi_{S^3,\Gamma_2}^{(0)} + \delta S_{S^3,\Gamma_2} \right) [\tilde{\varphi}](\tilde{x}) + \varphi(\tilde{x}).
\]

**Case 4:** For \( y \in S^3_\delta \). Similar to Case 3, it follows that

\[
\frac{1}{2\pi} \chi(\Gamma_{1,\delta} \setminus \iota_{\delta^\gamma}(Q)) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta S_{S^3,\Gamma_1} + \delta^2 S_{S^3,\Gamma_1}^{(2,2)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3),
\]

\[
\frac{1}{2\pi} \chi(\Gamma_{2,\delta} \setminus \iota_{\delta^\gamma}(Q)) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta S_{S^3,\Gamma_2} + \delta^2 S_{S^3,\Gamma_2}^{(2,2)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3),
\]

\[
\frac{1}{2\pi} \chi(S^3_\delta) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta S_{S^3,\Gamma_1} + \delta^2 S_{S^3,\Gamma_1}^{(2,3)} \right) [\tilde{\varphi}](\tilde{x}) + O(\delta^3),
\]

and

\[
\frac{1}{2\pi} \chi(S^3_\delta) \int_{S^3} |x - y| \varphi(y) d\sigma_y = \left( \delta \ln \delta \Pi_{S^3,\Gamma_1}^{(0)} + \delta S_{S^3,\Gamma_1} \right) [\tilde{\varphi}](\tilde{x}).
\]
Next, we deal with the following terms:

\[
\left\| S^{(0,1)}_{\psi^{(1)}(x), S^0_\delta} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} = \sup_{\varphi \neq 0} \frac{\int_{S^0_\delta} \ln |x - y| |\varphi(y)dy|_{H^\frac{1}{2}(S^0_\delta)}}{\int S^0_\delta |\varphi|^2_{H^\frac{1}{2}(S^0_\delta)}} \geq \frac{1}{2\pi} \int S^0_\delta \ln |x - y| |d\sigma_y| \left( x \right) d\sigma_x
\]

Then, by using the polar coordinate transformation and mean-value theorem, it deduces

\[
\frac{1}{2\pi} \int S^0_\delta \ln |x - y| |d\sigma_y| \left( x \right) d\sigma_x
\]

\[
= \frac{1}{2\pi} \sqrt{\frac{1}{\delta^\gamma + \pi \delta}} \int S^0_\delta \int_{S^0_\delta} \ln \left( \frac{L}{2} + \frac{\delta \cos \theta - y_1}{2} \right)^2 + \delta^2 (1 + \sin \theta)^2 |d\sigma_y| \left( x \right) d\sigma_x
\]

Thus, by using the integral formula of \( \int \ln(a^2 + x^2)dx \), we obtain

\[
\left\| S^{(0,1)}_{\psi^{(1)}(x), S^0_\delta} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} \geq O \left( \delta^{\frac{1}{2}} \ln \delta \right).
\]

Similarly, we can estimate \( \left\| S^{(0,1)}_{\psi^{(2)}(x), S^0_\delta} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} \), \( \left\| S^{(0,1)}_{\psi^{(1)}(x), S^0_\delta} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} \), and \( \left\| S^{(0,1)}_{\psi^{(2)}(x), S^0_\delta} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} \) is also no less than \( O \left( \delta^{\frac{1}{2}} \ln \delta \right) \).

Moreover,

\[
\left\| S^{(0,2)}_{S^0_\delta, \psi^{(1)}(x)} \right\|_{\mathcal{L}(H^{-\frac{1}{2}}(S^0_\delta), H^\frac{1}{2}(S^0_\delta))} = \sup_{\varphi \neq 0} \frac{\int_{S^0_\delta} |x - y| |\varphi(y)dy|_{H^\frac{1}{2}(S^0_\delta)}}{\int S^0_\delta |\varphi|^2_{H^\frac{1}{2}(S^0_\delta)}} \geq \frac{1}{2\pi} \sqrt{\frac{1}{\delta^\gamma + \pi \delta}} \int S^0_\delta \ln |x - y| |d\sigma_y| \left( x \right) d\sigma_x
\]

\[
\geq O \left( \delta^{\frac{1}{2}} \ln \delta \right).
\]
Likewise, the terms: 
\[ \left\| S^{(0,2)} \right\| \mathcal{L}\left( H^{-\frac{1}{2}}(S^b_\gamma, H^\frac{1}{2}(\partial\Omega^{(2)})) \right) \]
and
\[ \left\| S^{(0,2)} \right\| \mathcal{L}\left( H^{-\frac{1}{2}}(S^b_\gamma, H^\frac{1}{2}(\Omega^{(2)})) \right) \]
can be also estimated and no less than \( O\left( \delta^{7 - \frac{1}{2}} \ln \delta \right) \).

Finally,
\[ \left\| S^{(1,1)} \left( \partial\Omega^{(1)}(x), \Gamma_{1,2} \right) \right\| \mathcal{L}\left( H^{-\frac{1}{2}}(S^b_\gamma, H^\frac{1}{2}(\Omega^{(1)})) \right) \]
\[ = \sup_{\varphi \neq 0} \frac{\int_{\Omega^{(1)}}(\ln|x - y|\varphi(y))d\sigma(y)}{\| \varphi \|_{H^{-\frac{1}{2}}(S^b_\gamma, \Omega^{(1)})}} \]
\[ \geq \frac{1}{2\pi} \left\| \int_{\Omega^{(1)}}(\ln|x - y|)d\sigma(y) \right\|_{L^2(\Omega^{(1)})} \]
\[ \geq O\left( \delta^{\frac{7}{2}} \ln \delta \right) . \]

By using the same method, we can also find that
\[ \left\| S^{(0,1)} \left( \partial\Omega^{(1)}(x), \Gamma_{1,2} \right) \right\| \geq O\left( \delta^{\frac{7}{2}} \ln \delta \right) . \]

Hence, (2.23) holds. \qed
where
\[
S_{D,1,0}[\tilde{\varphi}](\tilde{x}) = S_{S^t,D,0}[\tilde{\varphi}](\tilde{x}),
\]
\[
S_{D,1,1}[\tilde{\varphi}](\tilde{x}) = \left(S_{S^t,D,S^u,0} + S_{S^t,D,S^v,0} + S_{S^t,S^v,1}\right)[\tilde{\varphi}](\tilde{x}),
\]
\[
K_{D,1,0}[\tilde{\varphi}](\tilde{x}) = K_{S^t,S^v,0}[\tilde{\varphi}](\tilde{x}),
\]
\[
K_{D,1,1}[\tilde{\varphi}](\tilde{x}) = \left(K_{S^t,S^v,0} + K_{S^s,S^u,0} + K_{S^t,S^v,1} + S_{S^v,S^t,1}\right)[\tilde{\varphi}](\tilde{x}).
\]

**Proof.**

(i) From the definition of $S_{D,1,1}$, we have

**Case 1:** For $y \in \Gamma_{1,\delta}$, it follows that
\[
|x - y|^2 = (P - z_y + \delta(\tilde{x} - P) + \delta(z_y - \tilde{y}), P - z_y + \delta(\tilde{x} - P) + \delta(z_y - \tilde{y}))
\]
\[
= |P - z_y|^2 + 2\delta(P - z_y, \tilde{x} - P) + 2\delta^2(1 + (\tilde{x} - P, z_y - \tilde{y})).
\]

Since $\partial D_\delta = S^a_\delta \cup \Gamma_{1,\delta} \cup \Gamma_{2,\delta} \cup S^b_\delta$, it implies
\[
- \frac{1}{8\pi} \chi(S^a_\delta) \int_{\Gamma_{1,\delta}} |x - y|^2 \phi(y) d\sigma(y)
\]
\[
= - \frac{1}{8\pi} \chi(S^a_\delta) \int_{\Gamma_1} |P - z_y|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) - \frac{1}{4\pi} \delta(S^a_\delta) \int_{\Gamma_1} (P - z_y, \tilde{x} - P) \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + O(\delta^2).
\]

Similarly,
\[
- \frac{1}{8\pi} \chi(S^b_\delta) \int_{\Gamma_{1,\delta}} |x - y|^2 \phi(y) d\sigma(y)
\]
\[
= - \frac{1}{8\pi} \chi(S^b_\delta) \int_{\Gamma_1} |Q - z_y|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) - \frac{1}{4\pi} \delta(S^b_\delta) \int_{\Gamma_1} (Q - z_y, \tilde{x} - Q) \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + O(\delta^2).
\]

Clearly,
\[
- \frac{1}{8\pi} \chi(\Gamma_{1,\delta}) \int_{\Gamma_{1,\delta}} |x - y|^2 \phi(y) d\sigma(y) = - \frac{1}{8\pi} \chi(\Gamma_1) \int_{\Gamma_1} |z_\bar{x} - z_\bar{y}|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}).
\]

Furthermore,
\[
- \frac{1}{8\pi} \chi(\Gamma_{2,\delta}) \int_{\Gamma_{2,\delta} \setminus \partial \gamma(x)} |x - y|^2 \phi(y) d\sigma(y) = - \frac{1}{8\pi} \chi(\Gamma_2) \int_{\Gamma_1} |z_\bar{x} - z_\bar{y}|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + O(\delta^2).
\]

**Case 2:** For $y \in \Gamma_{2,\delta}$. Notice the symmetry of geometry of $\partial D_\delta$, we only need to exchange $\Gamma_{2,\delta}$ with $\Gamma_{1,\delta}$ in Case 1 and get the corresponding expand formula. Namely,
\[
- \frac{1}{8\pi} \chi(S^a_\delta) \int_{\Gamma_{2,\delta}} |x - y|^2 \phi(y) d\sigma(y)
\]
\[
= - \frac{1}{8\pi} \chi(S^a_\delta) \int_{\Gamma_2} |P - z_y|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) - \frac{1}{4\pi} \delta(S^a_\delta) \int_{\Gamma_2} (P - z_y, \tilde{x} - P) \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + O(\delta^2),
\]
\[
- \frac{1}{8\pi} \chi(S^b_\delta) \int_{\Gamma_{2,\delta}} |x - y|^2 \phi(y) d\sigma(y)
\]
\[
= - \frac{1}{8\pi} \chi(S^b_\delta) \int_{\Gamma_2} |Q - z_y|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) - \frac{1}{4\pi} \delta(S^b_\delta) \int_{\Gamma_2} (Q - z_y, \tilde{x} - Q) \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}) + O(\delta^2),
\]
\[
- \frac{1}{8\pi} \chi(\Gamma_{2,\delta}) \int_{\Gamma_{2,\delta}} |x - y|^2 \phi(y) d\sigma(y) = - \frac{1}{8\pi} \chi(\Gamma_2) \int_{\Gamma_2} |z_\bar{x} - z_\bar{y}|^2 \tilde{\phi}(\tilde{y}) d\tilde{\sigma}(\tilde{y}),
\]
where
\[ S \]

Clearly,\n\[ S \]

Similarly,\n\[ S \]

Further,\n\[ S \]

Case 3: For \( y \in S^B_\delta \).
\[ -\frac{1}{8\pi} \chi(\Gamma_{1,\delta}) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = -\frac{1}{8\pi} \delta \chi(\Gamma_1) \int_{S^B_\delta} |P - z \varphi(y) d\sigma(y) + O(\delta^2). \]

Clearly,\n\[ -\frac{1}{8\pi} \chi(S^B_\delta) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = -\frac{1}{8\pi} \delta \chi(S^B) \int_{S^B_\delta} |P - Q| \varphi(y) d\sigma(y) + O(\delta^2). \]

Furthermore,\n\[ -\frac{1}{8\pi} \chi(S^B_\delta) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = O(\delta^3). \]

Case 4: For \( y \in S^B_\delta \). Owing to the symmetry of geometry of \( \partial D_\delta \), and then exchange \( S^B_\delta \) with \( S^B_\delta \) in Case 3, we get
\[ -\frac{1}{8\pi} \chi(\Gamma_{1,\delta}) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = -\frac{1}{8\pi} \delta \chi(\Gamma_1) \int_{S^B_\delta} |P - z \varphi(y) d\sigma(y) + O(\delta^2), \]
\[ -\frac{1}{8\pi} \chi(\Gamma_{2,\delta}) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = -\frac{1}{8\pi} \delta \chi(\Gamma_2) \int_{S^B_\delta} |P - z \varphi(y) d\sigma(y) + O(\delta^2), \]
\[ -\frac{1}{8\pi} \chi(S^B_\delta) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = -\frac{1}{8\pi} \delta \chi(S^B) \int_{S^B_\delta} |P - Q| \varphi(y) d\sigma(y) + O(\delta^2), \]
and
\[ -\frac{1}{8\pi} \chi(S^B_\delta) \int_{S^B_\delta} |x - y|^2 \varphi(y) d\sigma(y) = O(\delta^3). \]

Hence, by combining above four case, (2.26) is proved.

(ii) By using the formula of \( (x - y, \nu_x) \) in \( K_{D,1} \) and the same method in (i), (2.27) can be proved. \( \square \)

Finally, we give the asymptotic expansions of layer potential operators with respect to the frequency. Suppose \( k \ll 1 \), one can first find out that [5]
\[ H_0^{(1)}(k|x|) = c_k + i \frac{2}{\pi} \ln |x| - \frac{i}{2\pi} (k^2 \ln k)|x|^2 - \frac{1}{2\pi} k^2 |x|^2 (\ln |x| + \tau_k) + O(k^4 \ln k), \] (2.28)
where \( c_k := 1 + i \pi \left( \ln \frac{k}{2} + \gamma \right) \), \( \tau_k := 1 + \ln 2 + i \pi - \gamma \), and \( \gamma = 0.5772 \ldots \) is the Euler-Mascheroni constant. Based on (2.28), one can then easily find the asymptotic expansions for single layer potential operator \( S^k_B \) and Neumann-Poincaré operator \( (K^k_B)^* \) for \( k \ll 1 \) as follows:
\[ S^k_B[\varphi] = S^k_B[\varphi] - \frac{1}{4} c_k (1, \varphi) \partial B + (k^2 \ln k) S^k_B[\varphi] + k^2 S^k_B[\varphi] + k^4 \ln k O(||\varphi||), \] (2.29)
\[ (K^k_B)^*[\varphi] = (K^0_B)^*[\varphi] + (k^2 \ln k) K^k_B[\varphi] + k^2 O(||\varphi||), \quad \varphi \in H^* (\partial B), \] (2.30)
where \( S^k_B, S^k_B \) and \( K^k_B \) are defined as (2.24).
3. Quantitative Analysis of the Scattered Field

In this section, we focus on the quantitative analysis of the scattered field \( u^s \). Define \( k_c := \omega \sqrt{\varepsilon_c} \). With the help of layer potential techniques, one has the following integral representation for the solution to (1.1):

\[
  u(x) = \begin{cases} 
    u'(x) + S_{D_0}^k[\psi](x), & x \in \mathbb{R}^2 \setminus \overline{D_0}, \\ 
    S_{D_0}^k[\phi](x), & x \in D_0, 
  \end{cases}
\]  

(3.1)

where by using the jump formula (2.4), \((\phi, \psi) \in H^{-\frac{1}{2}}(\partial D_0) \times H^{-\frac{1}{2}}(\partial D_0)\) satisfy the following integral system:

\[
\begin{aligned}
& \left\{ \begin{array}{ll}
S_{D_0}^k[\phi] - S_{D_0}^k[\psi] = u' & \text{on } \partial D_0, \\
\frac{1}{\varepsilon_c} \left( -\frac{1}{2} \mathcal{I} + (K_{D_0}^k)^* \right)[\phi] - \left( \frac{1}{2} \mathcal{I} + (K_{D_0}^k)^* \right)[\psi] = \frac{\partial \phi}{\partial n} & \text{on } \partial D_0.
\end{array} \right.
\end{aligned}
\]

(3.2)

Lemma 3.1. For \( k \ll 1 \), the operator \( S_{D_0}^k : H^s(\partial D_0) \to H(\partial D_0) \) is invertible. Furthermore, it holds

\[
\left( S_{D_0}^k \right)^{-1} = \mathcal{L}_{D_0} + \mathcal{U}_k - k^2 \ln k \mathcal{L}_{D_0} \mathcal{S}_{D_0,1} \mathcal{L}_{D_0} - k^2 (\mathcal{L}_{D_0} \mathcal{S}_{D_0,2} \mathcal{L}_{D_0} + k (\mathcal{U}_k \mathcal{S}_{D_0,1} \mathcal{L}_{D_0} + \mathcal{L}_{D_0} \mathcal{S}_{D_0,1} \mathcal{U}_k)) + \mathcal{O}(k^2 \ln^{-1} k),
\]

(3.3)

where \( \mathcal{L}_{D_0} = \mathcal{P}_{H^s(\partial D_0)}(S_{D_0}^k)^{-1} \) and \( \mathcal{U}_k = (\mathcal{S}_{D_0}^k)^{-1}[\psi, \varphi_0] \). In particular, \( \mathcal{U}_k = \mathcal{O}((\ln k)^{-1}) \).

Proof. We set \( \tilde{S}_{D_0}^k[\psi] = S_{D_0}^k[\psi] - \frac{1}{4} c_k(1, \psi)_{L^2(\partial D_0)}, \forall \psi \in H^s(\partial D_0) \). Firstly, we prove that, for \( k \) small enough, \( \tilde{S}_{D_0}^k : H^s(\partial D_0) \to H(\partial D_0) \) is invertible. In fact, for any \( \psi \in H^s(\partial D_0) \)

\[
(\tilde{S}_{D_0}^k)^{-1} = (S_{D_0}^k)^{-1} \mathcal{P}_{H^s(\partial D_0)}[\psi] + (\psi, \varphi_0) \varphi_0
\]

and it deduces

\[
\tilde{S}_{D_0}^k[\psi] = S_{D_0}^k[\psi] + (\psi, \varphi_0) (S_{D_0}^k[\varphi_0] - \chi(\partial D_0))
\]

\[
- \frac{1}{4} c_k \int_{\partial D_0} \left( \mathcal{P}_{H^s(\partial D_0)}[\psi] + (\psi, \varphi_0) \varphi_0 \right) d\sigma(y)
\]

\[
= \tilde{S}_{D_0}^k[\psi] + \Upsilon_k[\psi],
\]

(3.4)

where

\[
\Upsilon_k[\psi] = (\psi, \varphi_0) \left( S_{D_0}^k[\varphi_0] - \chi(\partial D_0) \frac{1}{4} c_k a_{0,\delta} \right).
\]

Owing to the invertibility of \( \tilde{S}_{D_0}^k \), we see \( \tilde{S}_{D_0}^k(\tilde{S}_{D_0}^k)^{-1} = \mathcal{I} + \Upsilon_k(\tilde{S}_{D_0}^k)^{-1} \). From the compactness of \( \Upsilon_k \) and the Fredholm alternative theorem, we only need to prove the injectivity of \( \mathcal{I} + \Upsilon_k(\tilde{S}_{D_0}^k)^{-1} \).

In fact, if \( \psi \in H^s(\partial D_0) \) satisfies \( (\mathcal{I} + \Upsilon_k(\tilde{S}_{D_0}^k)^{-1})[\psi] = 0 \). By the definition of \( \tilde{S}_{D_0}^k \) and \( \Upsilon_k \), if \( (\tilde{S}_{D_0}^k)^{-1}[\psi] \in H^s(\partial D_0) \), it implies \( (\mathcal{I} + \Upsilon_k(\tilde{S}_{D_0}^k)^{-1})[\psi] = 0 \) and then \( \psi = 0 \). If
\[(S^0_{D_d})^{-1}[v] \in \{\mu \varphi_0, \mu \in \mathbb{C}\},\] we find
\[(I + \Upsilon_k(S^0_{D_d})^{-1})[v] = v + \mu \left(S^0_{D_d}[\varphi_0] - \chi(\partial D_\delta) + \frac{i}{4} c_k a_{0,\delta}\right) = \mu \left(S^0_{D_d}[\varphi_0] + \frac{i}{4} c_k a_{0,\delta}\right),\]
since we can always find a small enough \(k\) such that \(S^0_{D_d}[\varphi_0] \neq -\frac{i}{4} c_k a_{0,\delta}\), it follows that \(\mu = 0\) and then \(u = 0\).

Since, for \(k\) small enough, \(\hat{S}^k_{D_d} - S^k_{D_d}\) is a compact operator and \(\hat{S}^k_{D_d}\) is invertible. Furthermore, it is easy to prove that \(S^k_{D_d}\) is injective for \(k\) small enough. In fact, we consider \(\psi \in H^{-\frac{1}{2}}(\partial D_\delta)\) such that \(S^k_{D_d}[\psi] = 0\). Since \(u = S^k_{D_d}[\psi]\) satisfies Helmholtz equation \(\Delta u + k^2 u = 0\) in \(D_\delta\) and \(\mathbb{R}^2 \setminus \overline{D_\delta}\). Therefore, if \(k\) is sufficiently small such that \(k^2\) is neither an eigenvalue of \(-\Delta\) in \(D_\delta\) with the Dirichlet boundary condition on \(\partial D_\delta\) nor in \(\mathbb{R}^2 \setminus \overline{D_\delta}\) with the Dirichlet boundary condition on \(\partial D_\delta\) and the Sommerfeld radiation condition. It follows that \(u = 0\) and thus, \(\psi = \frac{\partial u}{\partial n} \bigg|_+ - \frac{\partial u}{\partial n} \bigg|_- = 0\), as desired. By using the Fredholm alternative theorem, we see that, as \(k\) is small enough, \(S^k_{D_d}\) is invertible.

Next, we verify (3.3). Obviously, (2.29) can be written as
\[S^k_{D_d} = \hat{S}^k_{D_d} + G_k,\]
where \(G_k = (k^2 \ln k) S_{D_d,1} + k^2 S_{D_d,2} + \mathcal{O}(k^4 \ln k)\). Since \(S^k_{D_d}\) is invertible, it follows that
\[(S^k_{D_d})^{-1} = (I + (S^k_{D_d})^{-1}G_k)(S^k_{D_d})^{-1} - \mathcal{O}(k^4 \ln^2 k).\]
Noting that \(\|(S^k_{D_d})^{-1}\|_{\mathcal{L}(\mathcal{H}(\partial D_\delta), \mathcal{H}^*(\partial D_\delta))}\) is bounded for every \(k\). Thereby, when \(k\) is small enough, we obtain
\[\begin{align*}
(S^k_{D_d})^{-1} &= (S^k_{D_d})^{-1} - (S^k_{D_d})^{-1}G_k(S^k_{D_d})^{-1} + \mathcal{O}(k^4 \ln^2 k). \quad (3.5)
\end{align*}\]
Moreover, set \(\Lambda_k = (S^k_{D_d})^{-1}(S^k_{D_d} + \Upsilon_k)\), then
\[\Lambda_k = (S^k_{D_d})^{-1}(S^k_{D_d} + \Upsilon_k) = (S^k_{D_d})^{-1}(S^k_{D_d} + \varphi_0(S^0_{D_d}[\varphi_0] - \chi(\partial D_\delta) + \frac{i}{4} c_k a_{0,\delta}))\]
\[= I + \langle \cdot, \varphi_0 \rangle(S^0_{D_d}[\varphi_0] - \chi(\partial D_\delta) + \frac{i}{4} c_k a_{0,\delta})\varphi_0.\]
And then
\[\begin{align*}
(\Lambda_k)^{-1} &= I - \langle \cdot, \varphi_0 \rangle(\frac{S^0_{D_d}[\varphi_0] - \chi(\partial D_\delta) + \frac{i}{4} c_k a_{0,\delta}}{S^0_{D_d}[\varphi_0] + \frac{i}{4} c_k a_{0,\delta}})\varphi_0.
\end{align*}\]
Therefore, we see that
\[\left(S^k_{D_d}\right)^{-1} = (\Lambda_k)^{-1}(S^0_{D_d})^{-1} = (S^0_{D_d})^{-1} - ((S^0_{D_d})^{-1}[\cdot], \varphi_0) \varphi_0 + \frac{((S^0_{D_d})^{-1}[\cdot], \varphi_0) \varphi_0 + \frac{((S^0_{D_d})^{-1}[\cdot], \varphi_0)}{S^0_{D_d}[\varphi_0] + \frac{i}{4} c_k a_{0,\delta}}\varphi_0. \quad (3.6)\]
then, substituting (3.6) into (3.5), it deduces (3.3). Furthermore, from the definition of \(\epsilon_k\), \(\mathcal{U}_k = \mathcal{O}(\ln k)^{-1}\) holds obviously.

\[\square\]

Remark 3.1. It is easy to find the operator \((S^0_{D_d})^{-1}\) is uniformly bounded with respect to \(\delta\). In fact, from the definition of norm \(\mathcal{H}(\partial D_\delta)\) and \(\mathcal{H}^*(\partial D_\delta)\), we have
\[\|S^0_{D_d}[\psi]\|^2 = -\left(S^0_{D_d}^{-1}[\cdot, \psi]\right)_{\mathcal{H}(\partial D_\delta)} = \|\psi\|^2_{\mathcal{H}(\partial D_\delta)}.\]
Thus, we write \((S^0_{D_d})^{-1}\) as \((S^0_{D})^{-1}\) occasionally in what follows.
Thanks to the invertibility of $S_{D,k}^\delta$, together with the first equation in (3.2), one can directly obtain that
\[ \varphi = \left(S_{D,k}^\delta\right)^{-1}(S_{D,k}^\delta[\varphi] + u^\delta). \] (3.7)

Then, from the second equation in (3.2), we have that
\[ A_{D,k}(\omega)[\varphi] = f, \] where
\[ A_{D,k}(\omega) = \left(\frac{1}{2}I + (K_{D,k})^*\right) + \frac{1}{\varepsilon_c} \left(\frac{1}{2}I - (K_{D,k}^0)^*\right) (S_{D,k}^\delta)^{-1}S_{D,k}^\delta, \] (3.9)
\[ f = -\frac{\partial u^\delta}{\partial \nu} - \frac{1}{\varepsilon_c} \left(\frac{1}{2}I - (K_{D,k}^0)^*\right) (S_{D,k}^\delta)^{-1}[u^\delta]. \] (3.10)

Clearly,
\[ A_{D,k}(0) = A_{D,k,0} = \left(\frac{1}{2}I + (K_{D,k}^0)^*\right) + \frac{1}{\varepsilon_c} \left(\frac{1}{2}I - (K_{D,k}^0)^*\right) = \frac{1}{2} \left(1 + \frac{1}{\varepsilon_c}\right) I + \left(1 - \frac{1}{\varepsilon_c}\right) (K_{D,k}^0)^*. \] (3.11)

Denote by \( \{\lambda_{j,\delta}, \varphi_{j,\delta}\}, \ j = 0, 1, 2, \ldots, \) the eigenvalue and eigenfunction pair of \( (K_{D,k}^0)^* \), owing to (2.8) and (3.11), we have
\[ A_{D,k,0}[\varphi] = \sum_{j=0}^{\infty} \tau_{j,\delta} \langle \varphi, \varphi_{j,\delta} \rangle \varphi_{j,\delta}. \] (3.12)

where
\[ \tau_{j,\delta} = \frac{1}{2} \left(1 + \frac{1}{\varepsilon_c}\right) + \left(1 - \frac{1}{\varepsilon_c}\right) \lambda_{j,\delta}. \] (3.13)

First, we give the asymptotic expansion formula of \( A_{D,k}(\omega) \) as \( \omega \to 0 \) and \( \delta \to 0 \).

**Lemma 3.2.** The operator \( A_{D,k}(\omega) : \mathcal{H}^*(\partial D_k) \to \mathcal{H}^*(\partial D_k) \) has the following expansion formula:
\[ A_{D,k}(\omega) = A_{D,k,0} + \omega^2 \ln \omega A_{D,k,1} + O(\omega^2). \] (3.14)

Furthermore,
\[ A_{D,k}(\omega) = A_{D,0} + \omega^2 \ln \omega A_{D,1} + o(\omega^2 \ln \omega) + o(1)\delta, \] (3.15)
where
\[ A_{D,k,0} = \frac{1}{2} \left(1 + \frac{1}{\varepsilon_c}\right) I + \left(1 - \frac{1}{\varepsilon_c}\right) (K_{D,k}^0)^*; \] (3.16)
\[ A_{D,0} = \frac{1}{2} \left(1 + \frac{1}{\varepsilon_c}\right) I + \left(1 - \frac{1}{\varepsilon_c}\right) K_0; \] (3.17)
\[ A_{D,k,1} = \frac{1}{\varepsilon_c} \left(\frac{1}{2}I - (K_{D,k}^0)^*\right) (S_{D,k}^0)^{-1}S_{D,k,1} \left(I - \varepsilon_c P_{H_0^0(\partial D_k)}\right) \] \[ + K_{D,k,1}(I - \varepsilon_c P_{H_0^0(\partial D_k)}), \] (3.18)
\[ A_{D,1} = \frac{1}{\varepsilon_c} \left(\frac{1}{2}I - K_0\right) (S_{D}^0)^{-1}S_{D,1,0} \left(I - \varepsilon_c P_{H_0^0(\partial D)}\right) \] \[ + K_{D,1,0}(I - \varepsilon_c P_{H_0^0(\partial D)}), \] (3.19)
and \( o(1)\delta \) denotes an infinitesimal with respect to \( \delta \).
Proof. From (2.29), (3.3) and (3.4), we have that
\[ S_{D_δ}^k = \tilde{S}_{D_δ}^k + \Upsilon_k + (\omega^2 \ln \omega)S_{D_δ,1} + \mathcal{O}(\omega^2), \]
\[ (S_{D_δ}^k)^{-1} = \mathcal{L}_{D_δ} + U_{k_δ} - (\omega^2 \ln \omega)\varepsilon_c\mathcal{L}_{D_δ}S_{D_δ,1}\mathcal{L}_{D_δ} + \mathcal{O}(\omega^2). \]
Noting the definition of \( \Upsilon_k \) in (3.4), we see \( \mathcal{L}_{D_δ} \Upsilon_k = 0 \), and then
\[ (S_{D_δ}^k)^{-1}S_{D_δ}^k = \mathcal{P}_h(\partial D_δ) + U_{k_δ}\tilde{S}_{D_δ}^0 + U_{k_δ}\Upsilon_k \]
\[ + (\omega^2 \ln \omega)\mathcal{L}_{D_δ}S_{D_δ,1}\left( \mathcal{I} - \varepsilon_c\mathcal{P}_h(\partial D_δ) \right) + \mathcal{O}(\omega^2). \]
By using (2.9), it yields \((-\frac{1}{2}\mathcal{I} + (K_{D_δ})^*)U_{k_δ} = 0\). Since \(-\frac{1}{2}\mathcal{I} + (K_{D_δ})^* = (-\frac{1}{2}\mathcal{I} + (K_{D_δ}^0)^*) + \mathcal{O}(k^2)\), it deduces (3.14). Furthermore, from Lemma 2.2 and Lemma 2.4, we substitute the corresponding expansion back into (3.14) and obtain (3.15).

Second, we have the following asymptotic expansion of \( f \) given by (3.10) with respect to \( \omega \) and \( \delta \).

Lemma 3.3. Let \( f \) be defined by (3.10), we have
\[ f = -i\omega(1 - \varepsilon_c^{-1})d \cdot \nu(x) + \mathcal{O}(\omega^2 \ln \omega). \]  
(3.20)

Proof. From (3.10), for \( x \in \partial D_δ \), and using the expansion \( e^{i\omega d \cdot x} = 1 + i\omega d \cdot x + \mathcal{O}(\omega^2) \), we have
\[ f(x) = -i\omega d \cdot \nu(x) + \mathcal{O}(\omega^2) \]
\[ - \varepsilon_c^{-1}\left( \left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) - \omega^2 \varepsilon_c \ln(\omega \sqrt{\varepsilon_c})K_{D_δ,1} + \mathcal{O}(\omega^2 \varepsilon_c) \right) \mathcal{L}_{D_δ} + U_{k_δ} \]
\[ - (\omega^2 \ln \omega)\varepsilon_c\mathcal{L}_{D_δ}S_{D_δ,1}\mathcal{L}_{D_δ} + \mathcal{O}(\omega^2)[1 + i\omega d \cdot x + \mathcal{O}(\omega^2)] \]
\[ = -i\omega d \cdot \nu(x) - \varepsilon_c^{-1}\left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) \mathcal{L}_{D_δ}[1] + U_{k_δ}[1] \]
\[ + \mathcal{L}_{D_δ}[i\omega d \cdot x] + U_{k_δ}[i\omega d \cdot x] + \mathcal{O}(\omega^2 \ln \omega). \]
Since \( \left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) \left( \tilde{S}_{D_δ}^0 \right)^{-1}[1] = 0 \) and \( \left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) U_{k_δ} = 0 \), by applying Lemma 2.2, we get that
\[ f = -i\omega \left( d \cdot \nu(x) + \varepsilon_c^{-1} \left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) \left( \tilde{S}_{D_δ}^0 \right)^{-1}[d \cdot x] \right) + \mathcal{O}(\omega^2 \ln \omega). \]
Note that
\[ \left( \frac{1}{2}\mathcal{I} - (K_{D_δ}^0)^* \right) \left( \tilde{S}_{D_δ}^0 \right)^{-1}[d \cdot x] = -d \cdot \nu, \]
we obtain the result (3.20).

From the operator equation (3.8), using Lemma 3.2 and Lemma 3.3, there holds
\[ (\lambda(\varepsilon_c)\mathcal{I} - K_0 - \delta K_1 + \mathcal{O}(\delta^2)) |\psi\rangle = \omega f_1 + \mathcal{O}(\omega^2 \ln \omega), \]
(3.21)
where \( f_1 = id \cdot \nu \). Before proceeding, we need the spectral result of \( K_0 \). Define the operator \( A_δ \) by
\[ A_δ|\psi\rangle(x_1) := \frac{1}{\pi} \int_{-L/2}^{L/2} \frac{\delta}{(x_1 - y_1)^2 + 4\delta^2} \psi(y_1)dy_1, \quad \psi \in L^2(-L/2, L/2). \]
(3.22)
We shall see that the operator \( A_δ \) is strongly related to the operator \( K_0 \). The following result on the spectral of \( A_δ \) is given in [17]:
Lemma 3.4 (Lemma 3.2 in [17]). Suppose $A_{\delta}$ is defined in (3.22), then it holds that
\begin{equation}
A_{\delta}[y_{n}^{\ast}](x_{1}) = \frac{1}{2}x_{1}^{n} + o(1), \quad x \in \Gamma_{j} \setminus (\iota_{\delta}(P) \cup \iota_{\delta}(Q)), \quad n \geq 0.
\end{equation}

Remark 3.2. By following the proof in [17], together with asymptotic analysis, one can get more accurate result than (3.23), which is
\begin{equation}
A_{\delta}[y_{n}^{\ast}](x_{1}) = \frac{1}{2}x_{1}^{n} + \chi(\iota_{\delta}(P) \cup \iota_{\delta}(Q))\mathcal{O}(\delta^{1-\epsilon}) + \mathcal{O}(\delta), \quad 0 < \epsilon < 1,
\end{equation}
\[x \in \Gamma_{j} \setminus (\iota_{\delta}(P) \cup \iota_{\delta}(Q)), \quad n \geq 0.
\]

With the above results on hand, we can show the following asymptotic result for the density $\psi$.

Lemma 3.5. Suppose $\psi$ is the solution of (3.8), then one has
\begin{equation}
\psi(x) = \begin{cases}
(-1)^{j}\omega_{d_{2}}(\lambda(\varepsilon_{c})I + A_{\delta})^{-1}[1] + \omega\chi(\iota_{\delta}(P) \cup \iota_{\delta}(Q))\mathcal{O}(\delta^{2(1-\epsilon)}) & \text{if } x \in \Gamma_{j} \setminus (\iota_{\delta}(P) \cup \iota_{\delta}(Q)), \\
\omega_{i}(\lambda(\varepsilon_{c})I - K_{1}^{\ast})^{-1}[d \cdot \nu(x)] + \omega \cdot o(1)_{\delta} + \mathcal{O}(\omega^{2} \ln \omega), & \text{if } x \in S_{\delta}^{\ast} \cup \iota_{\delta}(P), \\
\omega_{i}(\lambda(\varepsilon_{c})I - K_{2}^{\ast})^{-1}[d \cdot \nu(x)] + \omega \cdot o(1)_{\delta} + \mathcal{O}(\omega^{2} \ln \omega), & \text{if } x \in S_{\delta}^{\ast} \cup \iota_{\delta}(Q),
\end{cases}
\end{equation}

where the operators $K_{1}^{\ast}$ and $K_{2}^{\ast}$ are defined by
\begin{equation}
K_{1}^{\ast}[\varphi_{1}](x) := \int_{S_{\delta}(\iota_{\delta}(P))} \frac{(x - y, \nu_{x})}{|x - y|^{2}} \varphi_{1}(y)d\sigma(y),
\end{equation}
\begin{equation}
K_{2}^{\ast}[\varphi_{2}](x) := \int_{S_{\delta}(\iota_{\delta}(Q))} \frac{(x - y, \nu_{x})}{|x - y|^{2}} \varphi_{2}(y)d\sigma(y),
\end{equation}
respectively.

Proof. From (3.21), by using (2.15), (2.14) and (2.18), one can show
\begin{equation}
\lambda(\varepsilon_{c})\psi(x_{1}, -\delta) - \frac{1}{\pi} \int_{L/2} L/2 \frac{\delta}{(x_{1} - y_{1})^{2} + 4\delta^{2}} \psi(y_{1}, -\delta)dy_{1} + \mathcal{O}(\omega^{2} \ln \omega), \quad 0 < \epsilon < 1, \quad x \in \Gamma_{1} \setminus (\iota_{\delta}(P) \cup \iota_{\delta}(Q)),
\end{equation}

and
\begin{equation}
\lambda(\varepsilon_{c})\psi(x_{1}, \delta) - \frac{1}{\pi} \int_{-L/2} L/2 \frac{\delta}{(x_{1} - y_{1})^{2} + 4\delta^{2}} \psi(y_{1}, \delta)dy_{1} + \mathcal{O}(\omega^{2} \ln \omega), \quad 0 < \epsilon < 1, \quad x \in \Gamma_{2} \setminus (\iota_{\delta}(P) \cup \iota_{\delta}(Q)),
\end{equation}

One can thus decompose the density $\psi$ into
\begin{equation}
\psi(x_{1}, -1)^{j} \delta = \omega \left(\psi^{(0)}(x_{1}, (-1)^{j} \delta)ight.
\end{equation}
\begin{equation}
+ \delta^{2(1-\epsilon)} \chi(\iota_{\delta}(P) \cup \iota_{\delta}(Q))\psi^{(1)}(x_{1}, (-1)^{j} \delta) + \mathcal{O}(\delta^{2}) \right) + \mathcal{O}(\omega^{2} \ln \omega),
\end{equation}
for $|x_{1}| \leq L/2 - \mathcal{O}(\delta)$. Furthermore, from (3.27) and (3.28), one has
\begin{equation}
\psi^{(0)}(x_{1}, -\delta) = -\psi^{(0)}(x_{1}, \delta), \quad |x_{1}| \leq L/2 - \mathcal{O}(\delta).
\end{equation}
Thus one can derive that
\[
\psi(x_1, -\delta) = \omega i d_2 (\lambda(\varepsilon_c) I + A_\delta)^{-1} [1] + \omega \chi(\iota_\delta (P) \cup \iota_\delta (Q)) O(\delta^{2(1-\epsilon)}) \\
+ O(\omega \delta^2) + O(\omega^2 \ln \omega), \quad \text{for } |x_1| \leq L/2 - \mathcal{O}(\delta).
\] (3.31)

Similarly, we obtain
\[
\psi(x_1, \delta) = - \omega i d_2 (\lambda(\varepsilon_c) I + A_\delta)^{-1} [1] + \omega \chi(\iota_\delta (P) \cup \iota_\delta (Q)) O(\delta^{2(1-\epsilon)}) \\
+ O(\omega \delta^2) + O(\omega^2 \ln \omega), \quad \text{for } |x_1| \leq L/2 - \mathcal{O}(\delta).
\] (3.32)

Thus one obtains the first equation in (3.25). For \( x \in S^a \cup \iota_\delta (P) \), by making use of (3.21), (3.23), together with the first equation in (3.25) one has
\[
(\lambda(\varepsilon_c) I - K^*_1)[\varphi](x) = \omega d \cdot \nu(x) + \omega \cdot o(1)_\delta + O(\omega^2 \ln \omega), \quad x \in S^a_\delta \cup \iota_\delta (P).
\]

In a similar manner, one can show that
\[
(\lambda(\varepsilon_c) I - K^*_2)[\varphi](x) = \omega d \cdot \nu(x) + \omega \cdot o(1)_\delta + O(\omega^2 \ln \omega), \quad x \in S^b_\delta \cup \iota_\delta (Q).
\]

and so the last two equations in (3.25) follows. \(\square\)

We are now in the position of proving the first main result Theorem 1.1.

**Proof of Theorem 1.1.** For the sake of simplicity we use the notation \( \lambda \) and omit its dependence. By using (2.28), (3.1) and Taylor’s expansion along with \( \Gamma_0 \), we have that
\[
u(x) = u^*(x) + \int_{S^a_\delta \setminus (\iota_\delta (P) \cup S^a_\delta (Q))} G_0(x - z_y) \psi(y) d\sigma(y) \\
+ \delta \int_{S^a_\delta \setminus (\iota_\delta (P) \cup S^a_\delta (Q))} \nabla_y G_0(x - z_y) \cdot \nu(y) \psi(y) d\sigma(y) \\
+ \int_{S^b_\delta \cup S^a_\delta (P)} G_0(x - z_y) \psi(y) d\sigma(y) + \int_{S^b_\delta \cup S^a_\delta (Q)} G_0(x - z_y) \psi(y) d\sigma(y) + O(\delta^2 \omega^2 \ln \omega).
\] (3.33)

Furthermore, by substituting the asymptotic expansion of \( \psi \) in (3.25) into above equation, and using the fact that (see Lemma 3.3 in [17])
\[
\int_{S^b_\delta \cup S^a_\delta (P)} \psi(y) d\sigma(y) = - 2i \omega \delta \left( \lambda(\varepsilon_c) - \frac{1}{2} \right)^{-1} d_1 + \mathcal{O}(\delta), \\
\int_{S^b_\delta \cup S^a_\delta (Q)} \psi(y) d\sigma(y) = 2i \omega \delta \left( \lambda(\varepsilon_c) - \frac{1}{2} \right)^{-1} d_1 + \mathcal{O}(\delta),
\]

one can derive that
\[
in_{S^b_\delta \cup S^a_\delta (P)} G_0(x - z_y) \psi(y) d\sigma(y) = \int_{S^b_\delta \cup S^a_\delta (P)} (G_0(x - P) + \mathcal{O}(\delta)) \psi(y) d\sigma(y) \\
= - \omega \delta \frac{i}{\pi} \left( \lambda(\varepsilon_c) - \frac{1}{2} \right)^{-1} d_1 \ln |x - P| + \mathcal{O}(\delta)
\] (3.34)
\[
= - \omega \delta \frac{i}{2\pi} \left( \lambda(\varepsilon_c) - \frac{1}{2} \right)^{-1} d_1 \ln((x_1 - L/2)^2 + x_2^2) + \mathcal{O}(\delta),
\]

and
\[
\int_{S^b_\delta \cup S^a_\delta (Q)} G_0(x - z_y) \psi(y) d\sigma(y) \\
= \omega \delta \frac{i}{2\pi} \left( \lambda(\varepsilon_c) - \frac{1}{2} \right)^{-1} d_1 \ln((x_1 + L/2)^2 + x_2^2) + \mathcal{O}(\delta),
\] (3.35)
Finally, by using (3.25), there holds
\[
\int_{S_{\delta}^{\delta}(P) \setminus S_{\delta}(Q)} G_0(x - z_y)\psi(y) \, d\sigma(y) = \omega \delta,
\] (3.36)
and
\[
\int_{S_{\delta}^{\delta}(P) \setminus S_{\delta}(Q)} \nabla_y G_0(x - z_y) \cdot \nu_y \psi(y) \, d\sigma(y) = \frac{\omega}{2\pi} \int_{-L/2}^{L/2} \frac{x_2}{(x_1 - y_1)^2 + x_2^2} (\lambda(\varepsilon_c)I + A_\delta)^{-1}[1](y_1) \, dy_1 + \omega \delta, \] (3.37)

By combing the above results one can finally obtain (1.4).

\[\square\]

4. RESONANCE ANALYSIS OF THE NANOROD

In this section, according to definition 1.1, we proceed to analyze the plasmon resonance of the scattering system (1.1). It is worth emphasizing that we do not analyze the plasmon resonance by utilizing the expansion formula of the scattering field (1.4) directly. We just use some estimation of scattering field and the relevant operators. First, we derive the gradient estimate of the scattering field \(u^s\) outside the nanorod \(D_\delta\).

**Lemma 4.1.** Let \(\psi = \psi_c + a_{0,\delta}(\psi, \varphi_\delta, \varphi_\delta, \psi \in \mathcal{H}_0^2(\partial D). Then for the scattering solution of (1.1), we have the following estimate
\[
\left| \| \nabla u^s \|^2_{L^2(\mathbb{R}^2 \setminus D_\delta)} - \| \psi \|^2 \right| \lesssim \omega^2 |\ln \omega|^{2} a_{0,\delta}^{-1} |\langle \psi, \varphi_\delta \rangle|^2.
\] (4.1)

**Proof.** Suppose \(B_D\) is a sufficiently large disk such that \(D_\delta \subset B_D\), and \(S_{D_\delta}^{k^m}[\psi] = S_{D_\delta}^{k^m}[\psi] + \frac{1}{4} c_{k^m}(1, \psi \partial D_\delta). By utilizing the divergence theorem in \(B_D \setminus D_\delta\), the jump relation (2.4) and the Sommerfeld radiation condition, we have that
\[
\int_{B_D \setminus D_\delta} \| \nabla u^s \|^2 \, dx = k^m \int_{B_D \setminus D_\delta} |u^s|^2 \, dx - \int_{\partial D_\delta} u^s \frac{\partial u^s}{\partial \nu} \, d\sigma + \int_{B_D} u^s \frac{\partial u^s}{\partial \nu} \, d\sigma
\]
\[
= k^m \int_{B_D \setminus D_\delta} \| S_{D_\delta}^{k^m}[\psi] \|^2 \, dx - \int_{\partial D_\delta} S_{D_\delta}^{k^m}[\psi] \left( \frac{1}{2} I + (K_{D_\delta}^{k^m})^* \right) [\psi] \, d\sigma
\]
\[
+ \int_{\partial B_D} S_{D_\delta}^{k^m}[\psi] \cdot i k^m S_{D_\delta}^{k^m}[\psi] + \mathcal{O}(R^{-\frac{3}{2}}) \, d\sigma.
\]

Notice (2.29) and (2.30), it implies
\[
\left| \int_{\partial D_\delta} S_{D_\delta}^{k^m}[\psi] \left( \frac{1}{2} I + (K_{D_\delta}^{k^m})^* \right) [\psi] \, d\sigma \right| \leq \int_{\partial D_\delta} S_{D_\delta}^{k^m}[\psi] \left( \frac{1}{2} I + (K_{D_\delta}^{k^m})^* \right) [\psi] \, d\sigma + |E|, \] (4.2)

where
\[
E = \int_{\partial D_\delta} S_{D_\delta}^{k^m}[\psi] \omega^2 \ln \omega K_{D_\delta,1}[\psi] + \omega^2 \mathcal{O}(\|\psi\|) \, d\sigma
\]
(4.3)
\[
+ \int_{\partial D_\delta} (\omega^2 \ln \omega S_{D_\delta,1}[\psi] + \omega^2 S_{D_\delta,2}[\psi] + \omega^4 \ln \omega \mathcal{O}(\|\psi\|)) \left( \frac{1}{2} I + (K_{D_\delta}^{k^m})^* \right) [\psi] \, d\sigma.
\]
Since \(\omega\) is small enough, it is easy to see that \(|E| \lesssim |\omega \ln \omega|^2 \|\psi\|^2\).
Next, we estimate \( \int_{\partial D_\delta} S_{D_\delta}^0[\psi] \left( \frac{1}{2} I + (K_{D_\delta})^* \right) [\psi] d\sigma \). Since \( (K_{D_\delta})^*[\varphi_{0,\delta}] = \frac{1}{2} \varphi_{0,\delta} \) and \( S_{D_\delta}^0[\varphi_{0,\delta}] = 0 \), for \( \psi = \psi_c + a_{0,\delta}^{-1}(\psi, \varphi_{0,\delta}) \varphi_{0,\delta} \), it follows that

\[
\int_{\partial D_\delta} S_{D_\delta}^0[\psi] \left( \frac{1}{2} I + (K_{D_\delta})^* \right) [\psi] d\sigma \\
= \int_{\partial D_\delta} S_{D_\delta}^0[\psi_c] \left( a_{0,\delta}^{-1}(\psi, \varphi_{0,\delta}) \varphi_{0,\delta} + \frac{1}{2} I + (K_{D_\delta})^* \right) [\psi_c] d\sigma \\
= \int_{\partial D_\delta} \left[ \sum_{j=1}^{\infty} a_{j,\delta}^{-1}(\psi_c, \varphi_{j,\delta}) \varphi_{j,\delta} \right] \left( a_{0,\delta}^{-1}(\psi, \varphi_{0,\delta}) \varphi_{0,\delta} + \frac{1}{2} I + (K_{D_\delta})^* \right) \left( \sum_{l=1}^{\infty} a_{l,\delta}^{-1}(\psi_c, \varphi_{l,\delta}) \varphi_{l,\delta} \right) d\sigma \\
= \sum_{j=1}^{\infty} \frac{1}{2} a_{0,\delta}^{-1}(\varphi_{0,\delta}) a_{j,\delta}^{-1}(\psi_c, \varphi_{j,\delta}) \int_{\partial D_\delta} S_{D_\delta}^0[\varphi_{j,\delta} ] \varphi_{0,\delta} d\sigma \\
+ \sum_{j=1}^{\infty} \left( \frac{1}{2} + \lambda_{j,\delta} \right) a_{l,\delta}^{-1}(\psi_c, \varphi_{l,\delta}) a_{j,\delta}^{-1}(\psi_c, \varphi_{j,\delta}) \int_{\partial D_\delta} S_{D_\delta}^0[\varphi_{l,\delta} ] \varphi_{l,\delta} d\sigma.
\]

Noting that

\[
\int_{\partial D_\delta} S_{D_\delta}^0[\varphi_{l,\delta} ] \varphi_{l,\delta} d\sigma = -\langle \varphi_{l,\delta}, \varphi_{j,\delta} \rangle = \begin{cases} 1, & l = j, \\ 0, & l \neq j, \end{cases}
\]

it deduces

\[
\int_{\partial D_\delta} S_{D_\delta}^0[\psi] \left( \frac{1}{2} I + (K_{D_\delta})^* \right) [\psi] d\sigma = -\sum_{j=1}^{\infty} \left( \frac{1}{2} + \lambda_{j,\delta} \right) a_{j,\delta}^{-1} |\langle \psi_c, \varphi_{j,\delta} \rangle|^2.
\]

Since \( \lambda_{j,\delta} \in (-\frac{1}{2}, \frac{1}{2}) \) \( j \geq 1 \), we can get

\[
\left| \int_{\partial D_\delta} S_{D_\delta}^0[\psi] \left( \frac{1}{2} I + (K_{D_\delta})^* \right) [\psi] d\sigma \right| \approx \| \psi_c \|^2.
\]

Furthermore, by Cauchy’s inequality, it finds

\[
\left| \int_{\partial D_\delta} u^* \cdot k_m u^+ \mathcal{O}(R^{-\frac{3}{2}}) d\sigma \right| \lesssim \omega \| u^+ \|^2_{L^2(\partial B_R)} + \int_{\partial B_R} |u^* \cdot \mathcal{O}(R^{-\frac{3}{2}})| d\sigma \\
\lesssim \omega \| \psi \|^2 + \omega \mathcal{O}(R^{-1}) \cdot \| \psi \|.
\]

By combing (4.4) and (4.5), for \( \omega \ll 1 \), we have

\[
\| \nabla u^+ \|^2_{L^2(\partial B_R)} \lesssim \| \psi_c \|^2 + |\omega \ln \omega| \| a_{0,\delta}^{-1} |\langle \psi, \varphi_{0,\delta} \rangle|^2 + \omega \mathcal{O}(R^{-1}) \cdot \| \psi \|.
\]

Similarly, by (4.4) and (4.5), we also deduce the inverse inequality as

\[
\| \nabla u^+ \|^2_{L^2(\partial B_R)} \gtrsim \| \psi_c \|^2 - |\omega \ln \omega| \| a_{0,\delta}^{-1} |\langle \psi, \varphi_{0,\delta} \rangle|^2 - \omega \mathcal{O}(R^{-1}) \cdot \| \psi \|.
\]

Thus, by letting \( R \to \infty \), we see that the estimate (4.1) holds.

Before proceeding with the gradient analysis of the scattering field \( u^+ \) outside the nanorod \( D_\delta \), we consider the parameter choice of the permittivity with an imaginary part. In fact, in real applications, nano-metal materials always contain losses, which are reflected in the
Lemma 4.2. Let $\psi_0$ be given by (4.7) and has the decomposition $\psi_0 = \psi_{0,c} + c\varphi_0$, ($\psi_{0,c} \in H_0^1(\partial D_0)$, $c$ is a constant). Then, for sufficiently small $\delta$, $|\rho|$, and $\epsilon \in (0, 1)$, it holds that

(1) $|A^{-1}_{D_0,0}||L(H^r(\partial D_0), H^s(\partial D_0))| \lesssim (|\rho| \cdot O(\delta^{-1})^{-1})$. 

(2) If $\frac{\delta \rho}{2\gamma - 1} \neq \lambda_j, (\forall j \geq 0)$, for any fixed $\delta$, then $|A^{-1}_{D_0,0}||L(H^r(\partial D_0), H^s(\partial D_0))| \lesssim C$ for some positive constant $C$. 

(3) If $\lambda_j, \delta = \frac{\delta \rho}{2\gamma - 1} - \rho \frac{1}{\gamma - 1}$, for some $j_* \geq 1$, then $\|\psi_{0,c}\| \gtrsim |\rho|^{-1} a_{j_*,\delta}^{-1} \|f, \varphi_{j_*,\delta}\|$. 

Proof. (1) For $j \neq 0$, since $|\tau_{j,\delta}^{-1}| \lesssim \frac{1}{|\rho| \cdot O(\delta^{-1})^{-1}} \sum_{j=0}^{\infty} a_{j_*,\delta}^{-1} \|f, \varphi_{j_*,\delta}\|^2 \lesssim (|\rho| \cdot O(\delta^{-1})^{-1})^{-2} \|f\|^2$. 

Hence, $|A^{-1}_{D_0,0}||L(H^r(\partial D_0), H^s(\partial D_0))| \lesssim (|\rho| \cdot O(\delta^{-1})^{-1})^{-1}$. 

(2) If $\frac{\delta \rho}{2\gamma - 1} \neq \lambda_j, \delta$, then, for $j \geq 0$, we have $\left|\frac{\delta \rho}{2\gamma - 1} - \lambda_j, \delta\right| \geq c_0$, where $c_0$ is a positive constant. Therefore, $|\tau_{j,\delta}^{-1}| \lesssim 1$ and $\|\psi_{0,c}\|^2 \lesssim \|f\|^2$, i.e., $|A^{-1}_{D_0,0}||L(H^r(\partial D_0), H^s(\partial D_0))| \lesssim C$. 

(3) When $\lambda_j, \delta = \frac{\delta \rho}{2\gamma - 1} - \rho \frac{1}{\gamma - 1}$, for some $j_* \geq 1$, by (4.6), one has $\tau_{j,\delta} = \rho (1 - O(\delta^{-1}) \cdot i)$, it then follows that 

$\|\psi_{0,c}\| \gtrsim a_{j_*,\delta}^{-1} \|\psi_{0,c}, \varphi_{j_*,\delta}\| \gtrsim \frac{a_{j_*,\delta}^{-\frac{1}{2}} \|f, \varphi_{j_*,\delta}\|}{|\rho(1 - O(\delta^{-1}) \cdot i)|} \gtrsim |\rho|^{-1} a_{j_*,\delta}^{-\frac{1}{2}} \|f, \varphi_{j_*,\delta}\|$, 

which completes the proof. 

With Lemma 4.2, we can establish the following key result for estimating the gradient of the scattering field $u^*$, which provide resonant and non-resonant conditions for the scattering system (1.1) associated with the nanorod $D_\delta$ according to criterion (1.6) in Definition 1.1.

Theorem 4.1. Let $u^*$ be the scattering solution of (1.1). Assume that 

$\omega^2 \ln \omega (1 + O((\ln \omega)^{-1}))|\rho|^{-1} \leq c_1$, 

for a sufficiently small $c_1$, then we have that

(1) If there holds $\frac{\delta \rho}{2\gamma - 1} \neq \lambda_j, \delta$ for any $j \geq 0$ and $\delta$, then there exists a constant $C$ independent of $\delta$ such that 

$\|\nabla u^*\|_{L^2(\mathbb{R}^3 \setminus \overline{D_\delta})} \leq C$. 

imaginary part of a complex electric permittivity $\varepsilon_c$. As shall be shown, like the frequency $\omega$ and the size $\delta$, the lossy parameter also plays a key role in the plasmon resonance of the scattering field of the nanorod. Let $\theta = 4 \left(\frac{1}{\varepsilon_c}\right)$, $\rho = 3 \left(\frac{1}{\varepsilon_c}\right) < 0$. Then $\tau_{j,\delta}$ given by (3.13) can be written as

$\tau_{j,\delta} = \frac{1}{\rho} \left(\frac{\delta \rho}{2\gamma - 1} - \lambda_j, \delta\right) + \rho \left(\frac{1}{\gamma - 1} - \lambda_j, \delta\right)i$. 

(4.6) 

Notice that $\lambda_j, \delta = 1/2 + O(\delta^{-1})$, $(0 < \epsilon < 1)$, then $\tau_{j,\delta} = 1 + (1 - \theta) \cdot O(\delta^{-1}) - O(\delta^{-1}) \cdot \rho \pi$. By considering the principal equation $A_{D_0,0}|\psi_0\rangle = f$, where $A_{D_0,0}$ is defined by (3.11) and $\psi_0, f \in H^s(\partial D_0)$. Then, applying the eigenfunction expansion, it follows that 

$\psi_0 = A^{-1}_{D_0,0}[f] = \sum_{j=0}^{\infty} a_{j,\delta}^{-1} (f, \varphi_{j,\delta}) \tau_{j,\delta}$. 

(4.7)
(2) Suppose $d_2 = 0$. If there holds $\lambda_{j*,\delta} = \frac{\delta + 1}{2} - \rho \frac{1}{\sigma - 1}$ and
\[ d_1 \langle (1, \tilde{\varphi}_{j*}) \rangle_{SI} \neq 0 \tag{4.12} \]
for some $j_* \geq 1$, it holds
\[ \| \nabla u^* \|_{L^2(\mathbb{R}^2 \setminus \overline{D}_0)} \gtrsim \| \rho \|^{-1} \omega \delta + o(\omega \delta) + O(\omega^2 \ln \omega). \tag{4.13} \]
Furthermore, assuming that $|\rho|^{-1} \omega \delta \to \infty$ (as $\omega \to 0$, $\delta \to 0$, and $|\rho| \to 0$), then it holds
\[ \| \nabla u^* \|_{L^2(\mathbb{R}^2 \setminus \overline{D}_0)} \to \infty. \tag{4.14} \]
Here,
\[ \langle (1, \tilde{\varphi}_{j*}) \rangle_{SI} := \int_{-L/2}^{L/2} \ln \left( \frac{(x_1 + L/2)^2 + \delta^2 \varphi_{j*}(x_1)dx_1}{(x_1 - L/2)^2 + \delta^2 \varphi_{j*}(x_1)} \right), \]
and
\[ \tilde{\varphi}_{j*} (x_1) = \frac{\varphi_{j*} (x_1, 1) + \varphi_{j*} (x_1, -1)}{2}. \]

Proof. (1) From (3.14), it finds
\[ A_{D_0} (\omega) = A_{D_0,0} \left( I + A_{D_0,1}^{-1} (\omega^2 \ln \omega A_{D_0,1} + O(\omega^2)) \right), \tag{4.15} \]
and then
\[ \psi = \left( I + A_{D_0,0}^{-1} (\omega^2 \ln \omega A_{D_0,1} + O(\omega^2)) \right)^{-1} A_{D_0,0}[f]. \tag{4.16} \]
By Lemma 4.2 (1), one deduce
\[ \| A_{D_0,0}^{-1} \omega^2 \ln \omega (A_{D_0,1} + O((\ln \omega)^{-1})) \|_{L(H^s(\partial D_0), H^s(\partial D_0))} \lesssim (|\rho| \cdot O(\delta^{-1} - \epsilon))^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})). \tag{4.17} \]
Thereby, it holds
\[ \| \psi - \psi_0 \| = \left\| \left( I + A_{D_0,0}^{-1} (\omega^2 \ln \omega (A_1 + O((\ln \omega)^{-1})) \right)^{-1} A_{D_0,0}[f] - A_{D_0,0}^{-1}[f] \right\| \]
\[ \lesssim (|\rho| \cdot O(\delta^{-1} - \epsilon))^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \| A_{D_0,0}^{-1}[f] \| \]
\[ = (|\rho| \cdot O(\delta^{-1} - \epsilon))^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \| \psi_0 \|. \tag{4.18} \]
If $\frac{\delta + 1}{2} \neq \lambda_{j,\delta}$, by Lemma 4.2 (2), we get
\[ \| \psi \| \lesssim (1 + (|\rho| \cdot O(\delta^{-1} - \epsilon))^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \| \psi_0 \| \lesssim (1 + c_1) \| f \|. \]
Then, from Lemma 4.1, it yields
\[ \| \nabla u^* \|_{L^2(\mathbb{R}^2 \setminus \overline{D}_0)} \lesssim \| \psi \|^2 \lesssim C. \]

(2) If $\lambda_{j,\delta} = \frac{\delta + 1}{2} - \rho \frac{1}{\sigma - 1}$, $(j_* \geq 1)$, then, by using Lemma 4.2 (3), we have that
\[ \| \psi_{0,c} \| \gtrsim |\rho|^{-1} \left( \frac{1}{\sigma - 1} \right) \| \langle f, \varphi_{j,\delta} \rangle \|. \tag{4.19} \]
Moreover, following a similar estimate as in (4.18) and use the results in Lemma 4.2 (2) and (3), we see
\[ |\rho|^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \| \psi_0 \| \gtrsim \| \psi - \psi_0 \| \gtrsim \| \psi_c - \psi_{0,c} \| \gtrsim \| \psi_{0,c} \| - \| \psi_c \|. \tag{4.20} \]
Combining now (4.19) and (4.20), and notice that $|\rho|^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \leq c_1$ for a sufficiently small $c_1$, we have
\[ \| \psi_c \| \gtrsim \| \psi_{0,c} \| - |\rho|^{-1} \omega^2 \ln \omega (1 + O((\ln \omega)^{-1})) \| \psi_0 \| \gtrsim |\rho|^{-1} \left( \frac{1}{\sigma - 1} \right) \| \langle f, \varphi_{j,\delta} \rangle \|. \tag{4.21} \]
Thus, we obtain from Lemma 4.1 that
\[
\|\nabla u^s\|^2_{L^2(\mathbb{R}^3)} \geq \|\psi_c\|^2 - \omega^2 \ln \omega a_{\delta,\theta}^{-1} |\langle \psi, \varphi_{0,\delta} \rangle|^2
\]
\[
\geq |\rho|^{-2} a_{\delta,\theta}^{-1} |\langle f, \varphi_{j,\delta} \rangle|^2 - \omega^2 \ln \omega a_{\delta,\theta}^{-1} |\langle \psi, \varphi_{0,\delta} \rangle|^2.
\]
Furthermore, note that \( d_2 = 0 \), by (3.20), and applying Lemma 2.3, it follows that
\[
\langle f, \varphi_{j,\delta} \rangle = \int_{\partial D} (-i\omega(1 - \varepsilon_c^{-1})d \cdot \nu(x) + O(\omega^2 \ln \omega)) (-S_{D,0}[\tilde{\varphi}_{j,\delta}](\tilde{x}) + O(\delta \ln \delta)) d\sigma(\tilde{x})
\]
\[
= i\omega(1 - \varepsilon_c^{-1}) \int_{(S^\delta \cup d(P)) \cup (S^\delta \cup d(Q))} d \cdot \nu(x) S_{D,0}[\tilde{\varphi}_{j,\delta}](\tilde{x}) d\sigma(\tilde{x})
\]
\[
+ i\omega(1 - \varepsilon_c^{-1}) \int_{S^\delta \setminus (d(P) \cup d(Q))} d \cdot \nu(x) S_{D,0}[\tilde{\varphi}_{j,\delta}](\tilde{x}) d\sigma(\tilde{x}) + o(\omega \delta) + O(\omega^2 \ln \omega)
\]
\[
= \frac{i}{2\pi} \omega(1 - \varepsilon_c^{-1}) d_1 \int_{-L/2}^{L/2} \frac{(x_1 + L/2)^2 + \delta^2}{(x_1 - L/2)^2 + \delta^2} \tilde{\varphi}_{j,\delta}(x) dx_1 + o(\omega \delta) + O(\omega^2 \ln \omega),
\]
and
\[
a_{\delta,\theta} = - \int_{S^\delta} \tilde{\varphi}_{j,\delta}(\tilde{x}) S_{D,0}[\tilde{\varphi}_{j,\delta}](\tilde{x}) d\sigma(\tilde{x}) + O(\delta \ln \delta).
\]

The proof is complete. \( \square \)

Remark 4.1. Note that case (2) in Theorem 4.1 is the resonance condition, i.e. \( \lambda_{j,\delta} = \frac{1}{2} \frac{1}{\varepsilon_\delta - \sigma} - \frac{1}{\varepsilon_\delta - 1} \). If the lossy parameter of the nanorod \( 3(\varepsilon_c) \to 0 \), and when \( \lambda_{j,\delta} \in (-1/2, 1/2) \), it implies \( \Re(\varepsilon_c) < 0 \), which is the negative permittivity materials (see [29, 30]). Furthermore, if \( 3(\varepsilon_c) \to 0 \), and \( \delta \to 0 \), the resonance condition is consistent with \( \lambda(\varepsilon_c) - \frac{1}{\varepsilon_\delta} = 0 \), which appeared in (1.4). In particular, it further deduces that \( \Re(\varepsilon_c) \to 0 \) and \( \lambda_{j,\delta} \to 1/2 \), which is called the epsilon-near-zero materials (ENZM). Recently, the ENZM have drawn much attention for their intriguing electromagnetic properties (cf. [1, 10]).

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