A localization theorem for equivariant connective K theory

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Abstract

We identify Greenlees’ $C_n$-equivariant connective K theory spectrum $kU_{C_n}$ as an $RO(C_n)$-graded localization of the actual connective cover of $KU_{C_n}$.

1 Introduction

For a cyclic group $C_n$, there are two important $C_n$-equivariant forms of connective K theory. One is the connective cover $ku_{C_n}$ of Atiyah’s periodic K theory $KU_{C_n}$. As its name suggests, $ku_{C_n}$ is obtained from $KU_{C_n}$ by killing all homotopy Mackey functors of degree $< 0$, so that the coefficients of $ku_{C_n}$ are

$$\pi_k(ku_{C_n}) = \begin{cases} RU & k \geq 0 \text{ is even} \\ 0 & \text{else}. \end{cases}$$

While the definition of this spectrum is quite natural, it fails to satisfy some useful properties. For example, $ku_{C_n}$ does not have Thom isomorphisms for all equivariant complex vector bundles, and $ku_{C_n}$ does not satisfy the “completion theorem”, in that the comparison map

$$ku_{C_n}^* \rightarrow ku^{-*}(BC_n)$$

is not completion at the augmentation ideal

$$J = \ker \left( ku_{C_n}^* \xrightarrow{\text{res}_{C_n}^C} ku^* \right).$$

For these reasons, $ku_{C_n}$ has been viewed as a pathological $C_n$-equivariant form of connective K theory.

In his pioneering work ([2], [3], [4], [5]), Greenlees’ constructs a different $C_n$-equivariant form $kU_{C_n}$ of connective K theory via the homotopy pullback square

$$\begin{array}{ccc}
kU_{C_n} & \rightarrow & KU_{C_n} \\
\downarrow & & \downarrow \\
F(EC_{n+}, ku) & \rightarrow & F(EC_{n+}, KU).
\end{array}$$
This $C_n$-spectrum is better behaved than $ku_{C_n}$, in that

- $kU_{C_n}$ is complex oriented, so it has Thom isomorphisms for all $C_n$-equivariant complex vector bundles,
- $kU_{C_n}^C \to ku^*(BC_n)$ is completion at the augmentation ideal $J \subset kU_{C_n}^C$, and
- $kU_{C_n}^C$ classifies multiplicative $C_n$-equivariant formal group law, in the sense of [1].

The purpose of this note is to identify $kU_{C_n}$ as a localization of $ku_{C_n}$. Recall that $kU_{C_n}$ is a complex oriented $C_n$-spectrum, meaning that $kU_{C_n}$ has Thom isomorphisms for all $C_n$-equivariant complex vector bundles. In particular, if we consider an irreducible complex $C_n$-representation $\gamma$ as a vector bundle over a point, we obtain a Thom class $u_\gamma \in kU_{C_n}^2 - \gamma$. The class $u_\gamma$ is a unit in $kU_{C_n}^*$, which means that multiplication by $u_\gamma$ determines an equivalence of $C_n$-spectra

$$kU_{C_n} \xrightarrow{\approx} \Sigma^{-2} kU_{C_n}.$$  

It turns out that for any such $\gamma$, the class $u_\gamma$ lifts to $ku_{C_n}^C$, but is not a unit in $ku_{C_n}^C$. Our main result is that, by inverting the class $u_\gamma \in ku_{C_n}^C$ associated to a generator $\alpha$ of the character group

$$\widehat{C}_n \cong \langle \alpha \mid \alpha^n - 1 \rangle,$$

one recovers Greenlees’ spectrum $kU_{C_n}$.

**Theorem 1.1** If $C_n$ is a cyclic group, then the comparison map

$$ku_{C_n} \to kU_{C_n}$$

induces an equivalence of $C_n$-spectra

$$ku_{C_n} [u_{\alpha}^{-1}] \approx kU_{C_n}.$$

## 2 Notation

We work in the $C_n$-equivariant setting for a fixed cyclic group $C_n$ of order $n \geq 0$. If $V$ is an orthogonal or unitary $C_n$-representation, we write $S(V)$ for the unit sphere in $V$, and $S^V$ for the one point compactification of $V$. If $E_{C_n}$ is a $C_n$ spectrum, we write

$$\Sigma^V E_{C_n} = E_{C_n} \wedge S^V$$

for the $V$th suspension of $E_{C_n}$. The homotopy groups $\pi^H_*(E_{C_n})$ (as $H$ varies over subgroups of $C_n$) assemble to form a $C_n$ Mackey functor, which we denote $\underline{\pi}_*$. We say $E_{C_n}$ is *connective* if $\underline{\pi}_k(E_{C_n}) = 0$ for $k < 0$. More generally, given an integer $m$, we say $E_{C_n}$ is $m$-connective if $\underline{\pi}_k(E_{C_n}) = 0$ for $k < m$. Every $C_n$ spectrum $E_{C_n}$ has a connective cover $\tau_{\geq 0} E_{C_n} \to E_{C_n}$, which is a connective spectrum such that $\tau_{\geq 0}(E_{C_n}) \to \tau_{\geq 0}(E_{C_n})$ is an isomorphism for $k \geq 0$. 

2
3 Proof of the theorem

Recall that $kU_C^n$ is defined by the homotopy pullback square

$$
\begin{array}{ccc}
  kU_C^n & \longrightarrow & KU_C^n \\
  \downarrow & & \downarrow \\
  F(EC_{n+}, ku) & \longrightarrow & F(EC_{n+}, KU).
\end{array}
$$

The canonical map $ku_C^n \rightarrow KU_C^n$ and the homotopy completion map $ku_C^n \rightarrow F(EC_{n+}, ku_C^n) \simeq F(EC_{n+}, ku)$ determine a comparison map $ku_C^n \rightarrow kU_C^n$.

Since $u_\alpha \in ku_C^n$ maps to a unit in $kU_C^n$, this induces a map $f$ as shown below.

$$
\begin{array}{ccc}
  ku_C^n & \longrightarrow & kU_C^n \\
  \downarrow & & \downarrow \\
  ku_C^n [u_\alpha^{-1}] & \longrightarrow & kU_C^n.
\end{array}
$$

Our goal is to prove that $f$ is an equivalence. Recall that the localization $ku_C^n[u_\alpha^{-1}]$ is defined to be the homotopy colimit of the sequence

$$
ku_C^n \xrightarrow{u_\alpha} \Sigma^{\alpha-2} ku_C^n \xrightarrow{u_\alpha} \Sigma^{2(\alpha-2)} ku_C^n \xrightarrow{u_\alpha} \cdots.
$$

Since $ku_C^n$ is connective, so is $\Sigma^m ku_C^n$ for any $m \geq 0$. This implies that $\Sigma^{m(\alpha-2)} ku_C^n$ is $(-2m)$-connective, and so the composite

$$
\Sigma^{m(\alpha-2)} ku_C^n \rightarrow ku_C^n[u_\alpha^{-1}] \rightarrow kU_C^n
$$

factors through the $(-2m)$-connective cover of $kU_C^n$ as shown below.

$$
\begin{array}{ccc}
  \Sigma^{m(\alpha-2)} ku_C^n & \longrightarrow & ku_C^n [u_\alpha^{-1}] \\
  \downarrow f_m & & \downarrow f \\
  \tau_{\geq -2m} kU_C^n & \longrightarrow & kU_C^n.
\end{array}
$$

Altogether, we have a diagram

$$
\begin{array}{ccc}
  ku_C^n & \xrightarrow{u_\alpha} & \Sigma^{\alpha-2} ku_C^n & \xrightarrow{u_\alpha} & \Sigma^{2(\alpha-2)} ku_C^n & \xrightarrow{u_\alpha} & \cdots \\
  \downarrow f_\alpha & & \downarrow f_1 & & \downarrow f_2 & & \cdots \\
  \tau_{\geq 0} kU_C^n & \xrightarrow{\tau_{\geq 0}} & \tau_{\geq -2} kU_C^n & \xrightarrow{\tau_{\geq -2}} & \tau_{\geq -4} kU_C^n & \xrightarrow{\cdots},
\end{array}
$$

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which, upon taking homotopy colimits, yields the map \( f \). For this reason, if we want to show that \( f \) is an equivalence, it suffices to prove that each \( f_m \) is an equivalence. We do so by induction on \( m \), with the base case \( m = 0 \) holding by the defining property of the connective cover.

Supposing we have proved that \( f_0, \ldots, f_m \) are equivalences, we will prove that \( f_{m+1} \) is an equivalence. In order to do so, we smash the map

\[
\Sigma^{m+1} ku_{C_n} \to \Sigma^{m+1} kU_{C_n}
\]

with the cofiber sequence

\[
S(\alpha)_+ \to S^0 \to S^\alpha.
\]

Our first claim is that

\[
\Sigma^{m+1} ku_{C_n} \wedge S(\alpha)_+ \to \Sigma^{m+1} kU_{C_n} \wedge S(\alpha)_+
\]

is an equivalence. We verify this by observing that \( \Sigma^{m+1} ku_{C_n} \to \Sigma^{m+1} kU_{C_n} \) is an equivalence of non-equivariant spectra, hence it is equivalence after smashing with the free \( C_n \) orbit \((C_n)_+\). The existence of the cofiber sequence

\[
(C_n)_+ \to (C_n)_+ \to S(\alpha)_+
\]

proves that \( \Sigma^{m+1} ku_{C_n} \to \Sigma^{m+1} kU_{C_n} \) is also an equivalence after smashing with \( S(\alpha)_+ \), as claimed.

Using our inductive hypothesis and the 5 lemma, we deduce that the homotopy Mackey functor

\[
\pi_*(\Sigma^{m+1}\alpha ku_{C_n} \wedge S^\alpha) = \pi_*(\Sigma^{(m+1)\alpha} ku_{C_n})
\]

is 0 in negative degrees, and

\[
\pi_*(\Sigma^{(m+1)\alpha} ku_{C_n}) \to \pi_*(\Sigma^{(m+1)\alpha} kU_{C_n})
\]

is an isomorphism in non-negative degrees. In other words,

\[
\Sigma^{(m+1)\alpha} ku_{C_n} \to \Sigma^{(m+1)\alpha} kU_{C_n}
\]

is a connective cover, hence so is the composite

\[
\Sigma^{(m+1)\alpha} ku_{C_n} \xrightarrow{\Sigma^{(m+1)\alpha} ku_{C_n}} \Sigma^{(m+1)\alpha} kU_{C_n} \xrightarrow{u_{(m+1)\alpha}} \Sigma^{2(m+1)\alpha} kU_{C_n}.
\]

This implies that the adjoint map

\[
\Sigma^{(m+1)(\alpha-2)} ku_{C_n} \to kU_{C_n}
\]

is a \((-2(m+1))\)-connective cover, and so

\[
\Sigma^{(m+1)(\alpha-2)} ku_{C_n} \xrightarrow{f_{m+1}} \tau_{\geq -2(m+1)} kU_{C_n}
\]

is an equivalence. This completes our inductive step, and we conclude that

\[
ku_{C_n}[u_{-1}^{-1}] \xrightarrow{f} kU_{C_n}
\]

is an equivalence.
References

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[5] S. Schwede, *Global homotopy theory*, New Mathematical Monographs **34** (2018), Cambridge University Press, Cambridge.