Energy condition and cosmic censorship conjecture in the perfect fluid collapse

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Abstract
The hypothesis of cosmic censorship plays a crucial role in classical general relativity, as an aspect of the strong version it expresses that naked singularities would never occur from the black hole singularity. In this paper, we will present how energy conditions prohibit forming the local naked singularity in the spherical perfect fluid collapse, and thus we get one step closer to the strong cosmic censorship conjecture proof. We also show that this result can be extended to the cosmological constant background.

Keywords Black hole · Cosmic censorship conjecture · Perfect fluid · Energy condition

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1 Introduction
Hawking and Penrose have shown spacetime singularities are unavoidable when the trapped region form in general relativity and suitable energy conditions are satisfied
The key issue is whether these singularities are covered by the event horizon as in the Schwarzschild case or not, that is whether they are black holes or naked singularities. Although Penrosés cosmic censorship conjecture (CCC) [2] states that physically reasonable initial data cannot end in naked singularities visible to distant observers, various counterexamples have been proposed. The issue is whether these are astrophysically plausible counterexamples to the CCC. These counterexamples usually challenged the strong cosmic censorship conjecture that, asserts that, generically, timelike singularities never occur, so that even an observer who falls into a black hole will never “see” the singularity. In counterpoint, when singularities (or locally naked singularity) of gravitational collapse are contained in black holes is known as the weak cosmic censorship conjecture [3].

Indeed CCC counterexamples have been proposed in the case of pressure-free spherically symmetric (LTB) collapse, which have attracted major interest [4–10] (see [11, 12] for review). It seems that naked singularity could occur in cases where there is no pressure, and although pressure will occur and could alter the situation in more realistic cases, these examples provide useful toy models as a basis for further investigation. Most of the attempts are focused on finding timelike singularities from which the null geodesics emerge (at least locally) [13, 14] though there were some attempts to show the importance of energy conditions for the validity of the (weak) cosmic censorship hypothesis is pointed out [15–17]. In this letter, we focus on the trapping horizon region after singularity formation which the apparent horizon is its boundary. We will present how energy conditions prohibit forming the naked singularity.

Our approach to show the CCC in the perfect fluid collapse is based on the fact that the apparent horizon which is coincident with the singularity at the center cannot be future timelike. We assume that the weak energy condition, which states that the energy density is not negative, is satisfied and will see how this is playing an important role in this context.

This paper is organized as follows: In Sect. 2 we introduce the spherical perfect fluid collapse model and investigate the naked singularity formation in both shell-crossing and shell-focusing singularity. In Sect. 3 we study the cosmic censorship conjecture in presence of the cosmological constant. Section 4 is devoted to studying a numerical example to verify the results. Concluding remarks follow in Sect. 5. In the following, we use units in which $\text{G} = \text{c} = 1$ and our reasoning is based on the radial null geodesics.

### 2 Perfect fluid spherical collapse

A collapsing perfect fluid within a compact spherically symmetric spacetime region can be described [22] by the following metric in the comoving coordinates $(t, r, \theta, \varphi)$:

$$
\begin{align*}
\,\mathrm{d}s^2 &= -e^{2\nu(t,r)}\,\mathrm{d}t^2 + e^{2\psi(t,r)}\,\mathrm{d}r^2 + R(t,r)^2\,\mathrm{d}\Omega^2, \\
u(t,r) &= e^{-\nu(t,r)}\delta^\mu_0.
\end{align*}
$$

Assume the energy momentum tensor has the perfect fluid form

$$
\begin{align*}
\dot{u}^\mu &= e^{-\nu(t,r)}\delta^\mu_0.
\end{align*}
$$
\[ T^i_t = -\rho(t, r), \quad T^r_r = T^{\theta}_\theta = T^\phi_\phi = p(t, r). \]  

The Einstein equations for this energy momentum tensor give,

\[
\rho(t, r) = \frac{2M'(t, r)}{R^2(t, r)R'(t, r)}, \\
p(t, r) = -\frac{2M(t, r)}{R^2(t, r)\dot{R}(t, r)}, \\
v'(t, r) = -\frac{p'(t, r)}{\rho(t, r) + p(t, r)},
\]

where \( \dot{\cdot} \) and \( \cdot' \) mean the partial derivative relative to \( t \) and \( r \) respectively. Four velocity of this comoving fluid is \( u^\mu = (e^{-\nu}, 0, 0, 0) \) and after contraction this vector with the energy momentum tensor, \( \nabla_\nu T^{\mu\nu} = 0 \), we get the energy-momentum conservation in the form

\[
\frac{d\rho}{d\tau} + \theta(\rho + p) = 0,
\]

where \( \theta = \nabla_\mu u^\mu \) and \( \frac{d}{d\tau} = e^{-\nu} \frac{d}{dt} \). As a result, one gets

\[
e^{-\nu} \frac{d\rho}{dt} + \left( \dot{\nu} + \dot{\psi} + 2\frac{\dot{R}}{R} \right)(\rho + p) = 0 \\
-2\dot{R}' + R' \frac{\dot{G}}{G} + \dot{R} \frac{H'}{H} = 0,
\]

where

\[
G = e^{-2\psi}(R')^2, \quad H = e^{-2\nu}(\dot{R})^2,
\]

and \( M(t, r) \) quantity is defined as

\[
G(t, r) - H(t, r) = 1 - \frac{2M(t, r)}{R(t, r)}.
\]

It is obvious from the equation (7) that \( G, H > 0 \) are non-negative quantities. It is known that the apparent horizon (that is the boundary of the trapped region) is located at \( R = 2M \) [11, 12, 23, 24]. The \( M \) quantity is the Misner-Sharp Energy

\[
E_{MS} = M(R) = \frac{1}{2} \int_0^R \rho R^2 dR,
\]
or

\[ E_{MS} = \frac{1}{8\pi} \int_0^r \rho \sqrt{1 + \left( \frac{dR}{d\tau} \right)^2 - \frac{2M}{R}} d^3V, \]  

(10)

where \( d^3V = 4\pi e^\psi R' dr \), and \( \frac{d}{d\tau} = e^{-\nu} \frac{d}{dt} \). The last form of the function \( M \) indicates that when considered as energy, it includes contribution terms the kinetic energy and the gravitational potential energy.

Equation (9) shows that non-singular density gives \( M(R \to 0) = 0 \). This equation can be held just before the singularity formation (onset of singularity formation) when \( \rho/\rho_0 \gg 1 \) where \( \rho_0 \) can be the average density of collapsing object.

2.1 Shell-crossing singularities

During the gravitational collapse of a cloud of matter, a singularity will form when physical quantities like the energy density \( \rho \) and the Ricci scalar \( R \) will diverge. There can be two kinds of singularities in this metric: a big bang (big crunch) singularity (shell-focusing singularity) at \( R(t, r) = 0 \), and a shell-crossing one at \( R'(t, r) = 0 \). Shell-crossing singularities are gravitationally weak which there are proposals for extending the spacetime through such singularities and show a breakdown of the coordinate system being used, rather than a genuine physical singularity. These singularities are generally not considered as being serious counterexamples to the cosmic censorship conjecture [18–20]. Indeed it has been suggested one need not consider the shell-crossing singularity because even if they occur in the dust case, they are likely to not occur in the fluid case because of pressure effects. To show that the shell-crossing singularity cannot be naked, suppose a shell-crossing singularity first forms at \( R_0(t_{sc}) \neq 0 \). To examine whether a point is in the trapped surface or not, first, we must calculate the ratio \( \frac{R}{2M} \) to see that it is greater than one or not. If \( \frac{R}{2M} < 1 \) this point will be trapped and no null or timelike geodesic can escape from it to infinity.

Consider a sphere with radius \( R_0 + \epsilon \) which the \( \epsilon \) can has any small positive value before the shell-crossing singularity form \( t < t_{sc} \). The ratio of \( \frac{R}{2M} \) for this point gets

\[ \frac{R}{2M} = \frac{R}{\int_0^{R=R_0+\epsilon} \rho R'^2 dR'}. \]

(11)

Here, we wrote the Misner–Sharp mass in the \((t, R)\) coordinate. One must show that \( \int_0^{R=R_0+\epsilon} \rho R'^2 dR' > R \). If the left-hand side tends to infinity the proof is clear, but if the left-hand side has a finite value, since the \( \epsilon \) is a small value, one can do the Taylor expansion for the left-hand side term as:

\[ \int_0^{R=R_0} \rho R'^2 dR' + \rho(R_0)R_0^2 \epsilon + ... \]  

(12)
Since these two terms have positive values, we have \( \int_{0}^{R=R_0} \rho R^2 dR' + \rho(R_0) R_0^2 \epsilon + \cdots > \rho(R_0) R_0^2 \epsilon \). We know that the \( \epsilon \) have a small positive value, but \( \rho(R_0) \) tend to infinity at the shell crossing singularity then \( \rho(R_0) R_0^2 \epsilon > R \). Thus

\[
\int_{0}^{R=R_0} \rho R^2 dR' + \rho(R_0) R_0^2 \epsilon + \cdots > \rho(R_0) R_0^2 \epsilon > R, \tag{13}
\]

which give the (11) equation.

As a result, the trapped surface will form just before the shell-crossing singularity formation and outside it. Consequently, no naked singularity forms from these shell-crossing singularities.

### 2.2 Shell-focusing singularity

Note that in contrast to the dust collapse which always ends in a singularity at the center [21], the perfect fluid collapse can be stopped by pressure at the center and the collapse ends to a compact star, not a black hole [26]. Note that for *future directed radial null geodesic (fdnul)* we have

\[
\left. \frac{dt}{dr} \right|_{fdnul} = \frac{e^\psi}{e^\nu} > 0. \tag{14}
\]

On the apparent horizon where the expansion of the outgoing null geodesics are zero, equation (8) gives \( G = H \rightarrow R(t, r) = 2M(t, r) [11, 12, 24] \). This surface is the 3-dimension trapping horizon hypersurface at 4-dimension spacetime. In the \( (t, r) \) plane the apparent horizon is a curved line. Taking the derivative along this curved line we get \( \dot{R} dt + R' dr = 2M dt + 2M' dr \). Thus, the apparent horizon slope in the \( (t, r) \) plane is

\[
\left. \frac{dt}{dr} \right|_{AH} = \frac{R' − 2M'}{2M − R}, \tag{15}
\]

The tangent vector on the apparent horizon can be presented as

\[
k^\mu = c \left( 1, \frac{dr}{dt}, 0, 0 \right), \tag{16}
\]

where \( c \) is a constant. To see whether this tangent vector is timelike or spacelike, we have to calculate:

\[
k_\mu k^\mu = c^2 \left( -e^{2\nu} + e^{2\psi} \left( \frac{dr}{dt} \right)^2 \right). \tag{17}
\]
Using the equation (15), we have

\[ k_\mu k^\mu = c^2 e^{2\nu} \left( -1 + \frac{e^{2\psi}}{e^{2\nu}} \left( \frac{2\dot{M} - \dot{R}}{R' - 2M'} \right)^2 \right). \]  

(18)

Since on the apparent horizon, we have \( G = H \rightarrow \frac{e^{2\psi}}{e^{2\nu}} = \frac{R^2}{R^2} \), and applying the Einstein equation (3) we get \( 2\frac{M'}{R} = \rho \), \( -2\frac{M}{R} = p \), \( > 0 \) in the case of that \( p, \rho > 0 \). Using these equations, the above equations will be simplified as

\[ k_\mu k^\mu = c^2 e^{2\nu} \left( \frac{(1 + \rho R^2)^2}{(1 - \rho R^2)^2} - 1 \right), \]  

(19)

and

\[ \frac{dt}{dr} \big|_{AH} = \frac{dt}{dr} \big|_{f_{\text{null}}} \frac{(1 - \rho R^2)}{(1 + \rho R^2)}. \]  

(20)

This equation can be written as

\[ \frac{dt}{dr} \big|_{AH} = \frac{dt}{dr} \big|_{f_{\text{null}}} \frac{(1/R^2 - \rho)}{(1/R^2 - \rho) + (\rho + p)}. \]  

(21)

If the weak energy conditions, \( \rho \geq 0 \) and \( \rho + p > 0 \), are held in the model, we have two cases: the first case is that \( 0 < \rho < 1/R^2 \); In this case the slope of the apparent horizon in negative \( \frac{dt}{dr} \big|_{AH} < 0 \). This case happens when we have a high dense density on the apparent horizon which matters fall into it (see [25]). Since in the spherical symmetric case the apparent horizon and singularity joined in the center of the black hole, then the singularity in the center of the black hole can not be naked (any emitting ray from the center of singularity falls into the trapped region see Fig. 1). A similar case happens for the Vaidya black holes when radiating fluid falls into it [28].

The second case is that \( \rho < 1/R^2 \). From the equation (19) we get \( k_\mu k^\mu > 0 \), so the tangent vector of the apparent horizon is spacelike. Thus this condition does not allow the apparent horizon to be future-timelike. In the case that \( \rho = 1/R^2 \) and \( (\rho + p) > 0 \) the equation (19) leads \( k_\mu k^\mu > 0 \), hence the tangent vector of the apparent horizon is spacelike. From the equation (20) the apparent horizon in this case at \((t, r)\) plane is a horizontal spacelike line. There is also an interesting case that the fluid equation of state is like cosmological constant or vacuum energy, \( p = -\rho \), where the weak energy condition is violated. In this case the apparent horizon tangent vector is null surface, and locally naked singularity can happen in this case[27].

This argument does not alter by changing the coordinate because the causal nature of spacetime does not change by using another coordinate.
3 Collapse in presence of the cosmological constant

To have a more relativistic model, consider a black hole which is formed in dark energy background with the cosmological constant. If we solve the Einstein equation with the spherical symmetric metric and the perfect fluid energy momentum tensor, we get

\[ \rho(t, r) = \frac{2M'(t, r)}{R^2(t, r)R'(t, r)} - \Lambda, \]
\[ p(t, r) = -\frac{2M(t, r)}{R^2(t, r)\dot{R}(t, r)} + \Lambda, \]  \hspace{1cm} (22)

where \( M(t, r) \) quantity is defined in the following equation

\[ G(t, r) - H(t, r) = 1 - \frac{2M(t, r)}{R(t, r)} + \frac{\Lambda R(t, r)^2}{3}. \]  \hspace{1cm} (23)

There is a view that considers the cosmological constant as a matter with the positive density \( \rho = \frac{\Lambda}{3} \) and the negative pressure \( -\frac{\Lambda}{3} \). Thus we can attribute the total density \( \rho_t = \rho + \Lambda \) and total pressure \( p_t = p - \Lambda \) for the collapsing matter. The \( M(R) \) quantity is the Misner–Sharp Energy

\[ E_{MS} = \frac{1}{2} \int_0^R \rho R^2 dR, \]  \hspace{1cm} (24)

or

\[ E_{MS} = \frac{1}{8\pi} \int_0^r \rho \sqrt{1 + \left( \frac{dR}{d\tau} \right)^2 - \frac{2M}{R} + \frac{\Lambda R^2}{3}} d^3V. \]  \hspace{1cm} (25)

As discussed in [24], the cosmological constant contributes to the Misner-Sharp Energy, but we can discriminate the matter-energy in a comoving sphere with radius \( r \) and time \( t \) with \( M(t, r) \). The apparent horizon location will be changed by cosmological constant as \( R = 2M - \frac{\Lambda R^3}{3} \). This equation shows that the positive cosmological constant will reduce the horizon radius for a given matter mass, \( M \). Nevertheless, using the present cosmological constant value for the astrophysical black hole in this equation leads to negligible correction for the astrophysical apparent horizon radius. In the presence of the cosmological constant, the apparent horizon slope in the \((t, r)\) plane is

\[ \frac{dt}{dr} |_{AH} = \frac{R' - 2M' + \Lambda R^2 R'}{2M - R - \Lambda R^2 R}, \]  \hspace{1cm} (26)

To see whether this tangent vector is timelike or spacelike in presence of cosmological constant, we have to calculate the magnitude of the apparent horizon tangent vector:
\[ k_\mu k^\mu = c^2 \left( -e^{2\nu} + e^{2\psi} \left( \frac{dt}{dr} \right)^2 \right). \quad (27) \]

Now, using the equation (26)

\[ k_\mu k^\mu = c^2 e^{2\nu} \left( -1 + e^{2\psi} \left( \frac{2\dot{M} - \dot{R} - \Lambda R^2 \dot{R}}{R' - 2M' + \Lambda R^2 R'} \right)^2 \right). \quad (28) \]

On the apparent horizon, we have \( G = H \rightarrow e^{2\psi} = \frac{R'^2}{R^2} \), and using the Einstein equation (22) we get \( 2\dot{M}/R = \rho R^2 + \Lambda R^2 \) and \( -2\dot{M}/R = \rho R^2 - \Lambda R^2 > 0 \) (we can consider a total pressure, \( \rho_t = p - \Lambda \), for the fluid which is sum of the fluid pressure and cosmological constant negative pressure). Applying these equations to the apparent horizon tangent vector

\[ k_\mu k^\mu = c^2 e^{2\nu} \left( \frac{(1 + \rho_t R^2)^2}{(1 - \rho_t R^2)^2} - 1 \right), \quad (29) \]

and

\[ \left. \frac{dt}{dr} \right|_{AH} = \left. \frac{dt}{dr} \right|_{\text{radial}} \frac{(1 - \rho_t R^2)}{(1 + \rho_t R^2)}. \quad (30) \]

Similar to the previous section result if the weak energy conditions, \( \rho_t \geq 0 \) and \( \rho_t + \rho_t > 0 \), are held in this model, we have two cases: the first case is that \( \rho_t R^2 > 1 \); in this case the slope of the apparent horizon in negative \( \left. \frac{dt}{dr} \right|_{AH} < 0 \).

The second case is that \( \rho_t R^2 < 1 \). From the equation (29) we get \( k_\mu k^\mu > 0 \), thus this condition does not allow the apparent horizon to become future timelike. With the current small value for the cosmological constant and high-pressure value around the black hole, we can be sure that the total pressure is positive around all astrophysical black holes.

As shown in Fig. 1, only the point which has a chance to be a naked singularity is the center of the black hole. Since the apparent horizon cannot be future timelike, then no light can escape from this point at the singularity, and therefore no naked singularity happens in the perfect fluid collapse which has the central singularity \( R = 0 \). To have a timelike apparent horizon in the center of the black hole we need to have \( k_\mu k^\mu > 0 \) with \( \left. \frac{dt}{dr} \right|_{AH} > 0 \) which is obviously contradiction with the weak energy condition in equation(29). The timelike apparent horizon anywhere except the center of the black hole cannot help to have a naked singularity in perfect fluid spherical collapse as shown in Fig. 1.
4 A numerical study

In this part, we consider the Eardely and Smarr model [4] which is claimed that contains the naked singularity. They consider a flat LTB model whose mass function is in the form $M(r) = r^3$ and the singularity time for each layer is $t_0(r) = \zeta r^p$. In Fig. 2, the density profile of collapse at the onset of singularity formation for the parameters, $p = 2$ and $\zeta = 10$ is depicted. In Fig. 3, a typical null geodesics after $t > 0$ (before singularity formation) near the singularity are shown. These three geodesics cross the apparent horizon and end to the singularity. This diagram verifies some points:

- The apparent horizon is spacelike.
- Singularity is spacelike.
- This behavior will be held even in the case that the null geodesic is close to the crossing point i.e $R = M = 0$.
- Another reason for holding the CCC is that we see that a typical null geodesic near the singularity point will fall into the trapped region (and then into a singularity). Since null geodesic congruences do not cross each other then all null geodesics originated from the center and after this null geodesics will also fall into the trapped region and then it holds CCC.

These confirm that no naked singularity occurs in this model.

To see analytically how the singularity is spacelike in this model let us calculate the tangent vector on the singularity as

$$S^\mu = l \left( 1, \frac{dr}{dt}, 0, 0 \right), \quad (31)$$

where $l$ is a constant. To see whether this tangent vector is timelike or spacelike, we have to calculate:

$$S_\mu S^\mu = l^2 \left( -e^{2\nu} + e^{2\psi} \left( \frac{dr}{dt} \right)^2 \right). \quad (32)$$
Fig. 2 Density profile of collapse at onset of singularity formation

![Density profile](image)

Fig. 3 Null geodesic behaviour near singularity formation

![Null geodesic](image)

Note that we have $e^{2\nu} = 1$, $e^{2\psi} = \frac{R'^2}{1 + f(r)}$ in dust case. The singularity two-surface in located at $R(t, r) = 0$. Hence we have $\left. \frac{dt}{dr} \right|_{\text{sin}} = \frac{R'}{R}$. Using the Einstein equation for the dust case [11, 12] where we have $-\dot{R} = \sqrt{\frac{2M}{R}} + f(r)$ where $f(r) < 1$ in the collapsing region, we get

$$S^\mu_\mu = l^2 e^{2\psi} \left( \frac{2M}{R} - \frac{1}{1 + f(r)} \right).$$  \hspace{1cm} (33)

This equation shows that in the trapped region where $R < 2M$ we have $S^\mu_\mu > 0$. This equation indicates that the singularity is spacelike and verifies the above numerical result.
5 Conclusion

The main issue of the present investigation was to examine whether the end state of the collapse is a naked singularity or a black hole in the perfect fluid collapse. Our results show that having the weak energy condition causes the slope of the apparent horizon can not be future timelike. As a result, no local naked singularity can occur in the spherical perfect fluid collapse. Since the fluid pressure can be negative in other energy conditions, only the weak energy condition can prevent the apparent horizon to be future timelike. This feature will not change if the black holes are located in the cosmological constant background. This result will be held by dust (LTB model) collapse which is a special case of the perfect fluid. As an illustrative example, we took a known model in dust collapse and numerically depict its horizon and singularity curve, and it was shown how a typical null geodesic from the central singularity is trapped. The above result provides an intriguing perspective on black hole physics which tells that not only do energy conditions play an important role in forming the singularity but also cover it by forming the apparent horizon. All these results lead to the point that the strong cosmic censorship conjecture holds in the perfect fluid spherical collapse. Related to this research, several directions for future research exist; It is known that formal developments in trapped surface research suggest that a rephrasing in terms of trapping horizons is potentially more fruitful. Especially, some non-symmetric foliations of spherically symmetric spacetimes can lead to rather “unexpected” apparent horizons [29]. The question is whether it is possible to choose a time slice (appropriate Cauchy surface) that comes arbitrarily close to the singularity, yet for which no trapped surfaces exist in its past like Schwarzschild spacetime. Showing that there exist no future-directed null geodesics whatsoever (e.g., including those with non-zero angular momentum or non-radial null geodesic) which get to the singularity lying outside this apparent horizon affirm the CCC in the general case.

Data availability This manuscript has no associated data or the data will not be deposited, and this research is a theoretical analysis.

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