Ascertaining the uncertainty relations via quantum correlations

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Abstract
We propose a new scheme to express the uncertainty principle in the form of inequality of the bipartite correlation functions for a given multipartite state, which provides an experimentally feasible and model-independent way to verify various uncertainty and measurement disturbance relations. By virtue of this scheme, the implementation of experimental measurement on the measurement disturbance relation to a variety of physical systems becomes practical. The inequality in turn, also imposes a constraint on the strength of correlation, i.e. it determines the maximum value of the correlation function for two-body system and a monogamy relation of the bipartite correlation functions for multipartite system.

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The uncertainty principle lies at the heart of quantum mechanics and is one of the most fundamental features that distinguish it from the classical mechanics. The original form, \( p_1 q_1 \sim h \), stems from a heuristic discussion of Heisenberg on Compton scattering [1] where \( p_1, q_1 \) are the determinable precisions of position and momentum, \( h \) is the Planck constant. A generalization to arbitrary pairs of observables is \( \Delta A \Delta B \geq |[A, B]|/2 \), where the standard deviation is \( \Delta X = (\langle X^2 \rangle - \langle X \rangle^2)^{1/2} \), \( X = A \) or \( B \), \( \langle \cdots \rangle \) stands for expectation value, and the commutator is defined as \([A, B] = AB - BA\). This is the so-called Heisenberg–Robertson uncertainty relation [2]. A stronger version is the Robertson–Schrödinger uncertainty relation [3] which takes the form of \((\Delta A)^2 (\Delta B)^2 \geq \left( \left( \langle A, B \rangle \right)/2 - \langle A \rangle \langle B \rangle \right)^2 + |<[A, B]>|^2/4 \) where the anticommutator is defined as \([A, B] = AB + BA\).

Note that in the form involving standard deviations, the uncertainty relation represents the property of the ensemble of arbitrary quantum state in Hilbert space and is not concerned with...
the specific measurements. Thus such an uncertainty relation is not related to the precision of measurement on one observable and the disturbance to its conjugate.

If we assume $\epsilon(A)$ to be the precision of the measurement on $A$ and $\eta(B)$ to be the disturbance of the same measurement on $B$, the Heisenberg-type relation with regard to measurement and disturbance would read

$$\epsilon(A) \eta(B) \geq \frac{1}{2} |\langle [A, B] \rangle|.$$  \hfill (1)

Recently, Ozawa found that this form of measurement disturbance relation (MDR) (1) is not a universal one, and a new MDR was proposed [4], which was thought to be generally valid, i.e.

$$\epsilon(A) \eta(B) + \epsilon(A) \Delta B + \Delta A \eta(B) \geq \frac{1}{2} |\langle [A, B] \rangle|.$$  \hfill (2)

Equation (2) is of fundamental importance, i.e. it leads to a totally different accuracy limit $\epsilon(A)$ for non-disturbing measurements ($\eta(B) = 0$) compared to the Heisenberg-type MDR. In quantum information science, the uncertainty principle in general is also crucial to the security of certain protocols in quantum cryptography [5], and additionally, it plays an important role in the quantum metrology [6].

Despite the importance of the uncertainty principle, only the uncertainty relation in the form of standard deviations has been well verified in various situations, e.g., see [7] and the references therein. Experiments concerning both Heisenberg-type and Ozawa’s MDRs have just been performed with neutrons [8] and photons [9]. For neutrons in a given polarization state, the error and disturbance can be statistically determined based on a method proposed by Ozawa [10]. In the photon experiment, the weak measurement model introduced in [11] was employed for the measurement. Large samples of data are necessary due to the sensitivity to the measurement strength of a weak measurement process, which is used for gathering information of the system prior to the actual measurement [12]. The results of [8] and [9] exhibit the validation of Ozawa’s MDR but rather the Heisenberg-type. Since the uncertainty principle limits our ultimate ability to reduce noise when gaining information from the state of a physical system, its experimental verification in various systems and different measurement interactions is still an important subject.

Here in this work, we present a general scheme from which both the uncertainty relation and MDR turn to the forms involving only bipartite correlation functions. In this formalism, whilst the uncertainty relation becomes an inequality imposed on the correlation functions of bipartite states, the different forms of MDR transform into strong constraints on the shareability (monogamy) of the bipartite correlations in multipartite state. This directly relates the key element of quantum information, i.e., the nonlocal correlation, to the fundamental principle of quantum mechanics, i.e., uncertainty principle, in a quantitative way. And most importantly, it enables us to test the MDR in a variety of physical systems.

To test the validity of the various MDRs, one has to measure the physical observable quantities for which the different MDRs exhibit distinct responses. Here we present our method of constructing such quantities for qubit systems. Although the generalization to arbitrary systems is not trivial, the various MDRs have already shown the essential differences in two-dimensional Hilbert spaces within our scheme. The qubit systems include spin 1/2 particle, polarizations of photons, two level atoms, etc. For the sake of convenience we take the measurable observables to be the spin components. A measurement of spin along arbitrary vector $\vec{a}$ in three-dimensional Euclidean space can be represented by the following operator

$$A = \vec{\sigma} \cdot \vec{a} = |\vec{a}| \vec{\sigma} \cdot \vec{n}_a.$$  \hfill (3)

Here $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices, $\vec{n}_a = \vec{a}/|\vec{a}|$, and a general commutative relation holds for such operators

$$[A, B] = 2iC.$$  \hfill (4)
Figure 1. Illustration of the detection of measurement precision and disturbance. P, D, M stand for the function of projection, disturbance, and measuring. A meter system $|\psi_s\rangle$ interacts with the signal state $|\psi_{12}^m\rangle$, which is prepared by projecting a bipartite entangled state $|\psi_{12}^{en}\rangle$ at P. The measurement result can be obtained from M, and the measurement disturbance on signal $|\psi_{12}^d\rangle$ will be detected at D.

where $B = \vec{a} \cdot \vec{b}, C = \vec{a} \cdot \vec{c}, \vec{c} = \vec{a} \times \vec{b}$. Let $|n_p^\pm\rangle$ be the two eigenvectors of operator $P = \vec{a} \cdot \vec{n}_p$ with eigenvalues $\pm 1$, the following complete relations hold

$$|n_p^+\rangle|n_p^-\rangle + |n_p^-\rangle|n_p^+\rangle = 1, \quad |n_p^+\rangle|n_p^-\rangle - |n_p^-\rangle|n_p^+\rangle = \vec{a} \cdot \vec{n}_p = P.$$

Here $\vec{n}_p$ is a unit vector, and $|n_p^\pm\rangle|n_p^\pm\rangle = P^\pm$ are the projection operators. Using the Schmidt decomposition, any bipartite pure state is unitarily equivalent to the state [13]:

$$|\psi_{12}\rangle = \alpha|+\rangle|+\rangle + \beta|--\rangle|--\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$, and $\alpha \geq 0, \beta \geq 0$. The correlation function between two operators $A$ and $B$ for an arbitrary quantum state $|\psi\rangle$ is defined as $E(A, B) = \langle \psi | A_1 \otimes B_2 | \psi \rangle$. Here the subscripts of $A, B$ stand for the corresponding partite upon which they are acting.

For the Robertson–Schrödinger uncertainty relation we have the following theorem:

**Theorem 1.** The Robertson–Schrödinger uncertainty relation implies the following inequality on the correlation functions of an arbitrary bipartite quantum state

$$|E(A_1, P_2)|B - E(B_1, P_2)|A|^2 + |E(C_1, P_2)|^2 \leq S^2,$$

where $X_i = \vec{a}_i \cdot \vec{X}, X = A, B, \text{ or } C, \vec{c} = \vec{a} \times \vec{b}, P_i = \vec{a}_i \cdot \vec{n}_p, \vec{n}_p$ is unit vector, $i = 1, 2$ denotes the corresponding partite, $S$ is the parallelogram area formed by $\vec{a}$ and $\vec{b}$. The maximal attainable value of the bipartite correlation function is $E(A_1, A_2) = |\vec{a}|^2$, which is the area of a square with length $|\vec{a}|$. A proof of this theorem is given in appendix A.

As for the MDR, it is a subtle problem in quantum theory. In order to detect the influence (disturbance) on quantity $B$ introduced in measuring $A$, one needs to measure $B$ before and after the measurement on $A$. If the initial state is not $B$’s eigenstate, the acquisition of information on $B$ prior to the measurement $A$ will inevitably change the initial state and makes the subsequent measurement process irrelevant to the initial state. To illustrate this, a simple measurement scheme is presented in figure 1 where the measurement is performed via the interaction of the signal system $|\psi_{12}^+\rangle$ with a meter system $|\psi_s\rangle$ [11].

The Ozawa’s precision and disturbance quantities in equation (2) are defined as [4]

$$\epsilon(A)^2 = \langle (U_{13}^\dagger (I_1 \otimes M_3)U_{13} - A_1 \otimes I_3)^2 \rangle,$$  

(6)
Here the expectation values in equations (6), (7) are evaluated with the same compound state $|ψ_1⟩|ψ_3⟩$, where $|ψ_1⟩$ can be arbitrary, i.e., $|ψ_1⟩^+; |ψ_2⟩$ is the quantum state of the measurement apparatus; $U_{13}$ is a unitary measurement interaction. If the measurement process is carried out via spin dependent interaction with a qubit state (partite 3) and we regard the measurement read out of the spin of partite 3 to be the measurement result of the signal state $|ψ_1⟩$, we can have $M_3 → A_3$. It is obvious that in determining $η(B)$ (equation (7)), we have to measure $B_1$ before and after the measurement interaction $U_{13}$.

Our procedure to settle the measurement problem under Ozawa’s definitions goes as follows. Suppose we want to measure the MDR with respect to any given pair of spin components of $A_1 = ⃗σ_1 · ⃗a$ and $B_1 = ⃗σ_1 · ⃗b$ for arbitrary state $|ψ_1⟩$. This state can be prepared via the following entangled state

$$|ψ_{12}^{(m)}⟩ = \frac{1}{\sqrt{2}}(|+⟩_c|−⟩_c + (−1)^m|−⟩_c|+⟩_c).$$

Here, $m \in \{0, 1\}$; $⃗c = ⃗a × ⃗b$ and $|±⟩_c$ are the spin eigenstates along $⃗c$ ($(±)$ stands for the eigenstates along $z$ if not specified). Without loss of generality, we can set the $⃗a-⃗b$ plane as $x-z$ plane then $⃗c$ is along the $y$ axis

$$|ψ_{12}^{(1)}⟩ = \frac{1}{\sqrt{2}}(|+⟩ − |−⟩),$$

$$|ψ_{12}^{(0)}⟩ = \frac{1}{\sqrt{2}}(|+⟩ + |−⟩).$$

$|ψ_{12}^{(m)}⟩$ have the following property

$$V_i \otimes V_2^{-1}|ψ_{12}^{(m)}⟩ = (−1)^m|ψ_{12}^{(m)}⟩, \ m \in \{0, 1\},$$

where $V_i = ⃗σ_i · ⃗v$ is an operator acting on the $i$th partite and $⃗v$ is a unit vector in the $⃗a-⃗b$ (i.e., $x-z$) plane. With the definition of projection operators in equation (5), an arbitrary quantum state ($|ψ_1⟩$) of partite 1 can be obtained via a projective measurement $P$ on partite 2 (see figure 1)

$$|ψ_1^{+}⟩ = \frac{2|ψ_1^{+}⟩|ψ_{12}^{(m)}⟩}{|⟨ψ_1^{+} |ψ_{12}^{(m)}⟩|}.$$

Here in the present situation $|⟨ψ_1^{+} |ψ_{12}^{(m)}⟩| = 1/\sqrt{2}$ and the arbitrariness of $|ψ_1^{+}⟩$ is guaranteed by the arbitrariness of $⃗n_p$.

The measurement precision of quantity $A$ for quantum state $|ψ_1^{+}⟩$ and the corresponding disturbance on another quantity $B$ now can be written as

$$ε^±(A)^2 = ⟨ψ_1|(|ψ_1^{+}⟩|U_{13}^† (I_1 \otimes I_3) U_{13} − A_1 \otimes I_3^2|ψ_1^{+})⟩|ψ_3⟩,$$

$$η^±(B)^2 = ⟨ψ_1|(|ψ_1^{+}⟩|U_{13}^† (B_1 \otimes I_3) U_{13} − B_1 \otimes I_3^2|ψ_1^{+})⟩|ψ_3⟩.$$  

With these definitions, we can derive the following relation (see the appendix B)

$$|⃗a|^2 + |⃗b|^2 − (−1)^m[E(A_2, A_3) + E(B_1, B_2)] = \frac{1}{4}[ε^+(A)^2 + η^+(B)^2 + ε^−(A)^2 + η^−(B)^2].$$

(15)

where the correlation function $E(X_i, X_j) = ⟨ψ_{132}|X_i \otimes X_j|ψ_{132}⟩$, $X = A$ or $B$, $|ψ_{132}⟩ \equiv U_{13}|ψ_{12}^{(m)}⟩|ψ_3⟩$, $i, j \in \{1, 2, 3\}$, the subscripts of operators stand for the corresponding partite upon which they are acting. The precision and disturbance of the measurements are now directly
related to the bipartite correlation functions of a tripartite state. Equation (15) is universally valid regardless of the measurement interaction $U_{13}$ which brings about the tripartite state.

For the arbitrary given state $|\psi_1^+\rangle$, the Heisenberg-type and Ozawa’s MDRs read

$$\epsilon^\pm(A)\eta^\pm(B) \geq \frac{1}{2}|\langle [A, B]|\psi_1^+ \rangle|,$$  \hspace{1cm} (16)

$$\epsilon^\pm(A)\eta^\pm(B) + \epsilon^\pm(A)\Delta^\pm(B) + \eta^\pm(B)\Delta^\pm(A) \geq \frac{1}{2}|\langle [A, B]|\psi_1^+ \rangle|,$$  \hspace{1cm} (17)

An intuitive view of the above equations tells that the allowed regions for $\epsilon$ and $\eta$ lie above the hyperbolic curves of $\epsilon^\pm(A)$ and $\eta^\pm(B)$ in the quadrant I. The constraints equations (16), (17) are then transferred to the bipartite correlation functions via equation (15). Thus we have the following theorem

**Theorem 2.** For $A = \vec{\sigma} \cdot \vec{a}$, $B = \vec{\sigma} \cdot \vec{b}$, a tripartite state can be obtained by interacting one partite of $|\psi_1^{(m)}\rangle$ with a third partite 3. The Heisenberg-type and Ozawa’s MDRs imply the following different relations on the resulting tripartite state

$$E(A_2, A_3) + E(B_1, B_2) \leq |\vec{a}|^2 + |\vec{b}|^2 - \kappa_{h, o} |\vec{n}_p \cdot (\vec{a} \times \vec{b})|.$$  \hspace{1cm} (18)

Here $E(X_i, X_j)$ are the bipartite correlation functions of the tripartite state, $\kappa_h = 1$ and $\kappa_o = (\sqrt{2} - 1)^2$ for Heisenberg-type and Ozawa’s MDR respectively, $\vec{n}_p$ is an arbitrary unit vector.

The experiments to test the validity of the MDRs become straightforward due to theorem 2. Here we present an example of the measurement model of qubit system with the measurement interaction $U_{13}$ being the CNOT gate [11] within our method. Suppose we want to measure the precision of $Z = \sigma_z$ and the disturbance on $X = \sigma_x$ for an arbitrary qubit state $|\psi_1\rangle$. Following theorem 2, on choosing $|\psi_{12}^{(1)}\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$, the measurement interaction CNOT gate between one partite of $|\psi_{12}^{(1)}\rangle$ and the meter system $|\psi_3\rangle = \cos \theta_3|+\rangle + \sin \theta_3|--\rangle$ will lead to the following tripartite state

$$|\psi_{123}\rangle = \frac{1}{\sqrt{2}}(|++\rangle(\cos \theta_3|+\rangle + \sin \theta_3|--\rangle) + |--\rangle(\cos \theta_3|--\rangle + \sin \theta_3|+\rangle)).$$  \hspace{1cm} (19)

According to theorem 2, the Heisenberg-type and Ozawa’s MDRs impose the following constraints on the bipartite correlation functions of $|\psi_{123}\rangle$

Heisenberg-type MDR: $E(Z_2, Z_3) + E(X_1, X_2) \leq 2 - |\cos \theta_p|,$ \hspace{1cm} (20)

Ozawa’s MDR : $E(Z_2, Z_3) + E(X_1, X_2) \leq 2 - (\sqrt{2} - 1)^2 |\cos \theta_p|$, \hspace{1cm} (21)

for arbitrary $\theta_p$, the angle between $\vec{n}_p$ and $\vec{c}$. The tightest bound happens when $\theta_p = 0$. Thus a measurement of bipartite correlation function of $E(Z_2, Z_3)$, $E(X_1, X_2)$ in the tripartite state would be capable of verifying the Heisenberg-type and Ozawa’s MDR (see figure 2). That is, the Heisenberg-type MDR will be validated provided that the experimental result agrees with the solid line of $E(Z_2, Z_3) + E(X_1, X_2)$ in figure 2.

From the above example, the procedure of our scheme can be summarized as: (1) prepare a bipartite entangled state, (2) interact one partite of the entangled state with a third partite, and (3) measure the bipartite correlation functions of the resulting tripartite state. The generation of the bipartite entangled state has already been realized in various systems, e.g. photons [14, 15], atoms [16, 17], and high energy particles [18, 19]. The further interaction of one
To the sum of two particular CHSH type correlations [23], measuring the precision of Bell correlations [20–22] in the tripartite entangled state. According to theorem 2, a more important physical consequence of theorem 2 is that it reveals a monogamy relation MDRs. Hence, our scheme could be applied to a large number of systems in the verification of the disturbance, which may not be easy to quantify for some types of measurement interactions. Correlation functions of the obtained tripartite state rather than the measurement precision and collisions, or via optical cavities, etc. More importantly, we need only to measure the bipartite part of the entangled state with a third partite can also be arbitrary, i.e., elastic or inelastic collisions, or via optical cavities, etc. Adding equations (18) and (22), taking equations (23), (24), we have

\[
|E(B_2, B_3) + E(A_1, A_2)| \leq |\vec{a}|^2 + |\vec{b}|^2 - \kappa_{h,o} \vec{n}_p \cdot (\vec{a} \times \vec{b}),
\]

Introducing two new vectors \(\vec{a}' = \frac{1}{2}(\vec{a} + \vec{b})\), \(\vec{b}' = \frac{1}{2}(\vec{b} - \vec{a})\), we can similarly define \(A' = \vec{\sigma} \cdot \vec{a}'\), \(B' = \vec{\sigma} \cdot \vec{b}'\). Following the definition of correlation function in equation (15), we can get

\[
E(A_1, A_j) = E(A_1, A'_j) - E(A_1, B'_j),
\]

\[
E(B_1, B_j) = E(B_1, A'_j) + E(B_1, B'_j).
\]

Adding equations (18) and (22), and taking equations (23), (24), we have

\[
|E(A_2, A'_3) - E(A_2, B'_3) + E(B_2, A'_3) + E(B_2, B'_3)| + E(A_1, A'_j) - E(A_1, B'_j) + E(B_1, A'_j) + E(B_1, B'_j) \leq 2K_{H,O}.
\]

where \(K_{H,O} = |\vec{a}|^2 + |\vec{b}|^2 - \kappa_{h,o} \vec{n}_p \cdot (\vec{a} \times \vec{b})\). When \(|\vec{a}| = |\vec{b}| = 1\), \(\vec{a} \perp \vec{b}\), equation (25) leads to the sum of two particular CHSH type correlations [23]

\[
|B^{(23)}_{\text{CHSH}} + B^{(12)}_{\text{CHSH}}| \leq 2\sqrt{2}K_{H,O}.
\]

Here \(B^{(ij)}_{\text{CHSH}} = E(A_i, A'_j) - E(A_i, B'_j) + E(B_i, A'_j) + E(B_i, B'_j)\). The tightest bound also happens when \(\theta_p = 0\), which leads to the following

Heisenberg-type MDR: \(|B^{(23)}_{\text{CHSH}} + B^{(12)}_{\text{CHSH}}| \leq 2\sqrt{2}.
\]
Ozawa’s MDR: $|P_{\text{CHSH}}^{23} + B_{\text{CHSH}}^{12}| \leq 2\sqrt{2}(2\sqrt{2} - 1)$.

The above monogamy relations on quantum nonlocality are direct results of the MDRs according to our theorem. Note, there are also discussions in the literature on Bell correlations based on the entropic measures of uncertainty relation [24, 25].

It should be noted that the definitions of measurement precision and disturbance in equations (6), (7) by Ozawa involve comparison of the same physical observable before and after the measurement, thus are based on practical physical motivations. However, the exact definitions that capture the full physical contents of the measurement error and disturbance are still under study [26–28]. Nevertheless, Ozawa’s definitions and the resulted MDRs may be regarded as one of the best attempts to capture the quantitative descriptions of the measurement and its back action in quantum mechanics. The method we presented provides a powerful tool to study the physical consequences of the MDRs that is meaningful in judging their usefulness. For example, our method transforms the MDRs into inequalities of correlation functions of tripartite entangled state. In this way the importance of the MDRs manifests in their connections with the quantum entanglement, which is a key physical resource in quantum information science and has a close relation with quantum metrology [6]. Meanwhile, in principle the idea of our scheme may also be applied to other definitions of error and disturbance. This would enable the use of this method to examine the meaningfulness of the variant definitions.

In conclusion, we proposed in this work a general scheme to express the uncertainty principle in terms of bipartite correlation functions, by which the essential differences between the MDRs are characterized by the inequalities constraining the correlation functions of multipartite state. This not only builds a bridge between the MDRs and the quantum entanglement but also provides a way to study the direct physical consequences of such fundamental relations. The resulted inequalities reveal that both the strength and the shareability (monogamy) of the quantum correlation are determined by the uncertainty principle. Further studies on the uncertainty relation and MDRs with, e.g., atoms, ions, or even high energy particles, become possible due to our scheme. The connections between MDRs and entanglement revealed in our scheme may also shed new light on the studies of the relations between the MDRs and the quantum cryptography, quantum metrology, etc.

Note: after the completion of this manuscript, there has been some progress in the study of MDRs, i.e., [29, 30], etc. Our method may apply in such cases as well and these MDRs would also give distinct constraints on quantum correlations [31].

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Appendix A. Proof of theorem 1

Proof of the equation of theorem 1:

$$|E(A_1, P_2)\vec{b} - E(B_1, P_2)\vec{a}|^2 + |E(C_1, P_2)|^2 \leq S^2.$$ 

Proof. Following the definition of the standard deviation, the Robertson–Schrödinger uncertainty relation takes the following form

$$((\langle A^2 \rangle - \langle A \rangle^2)(\langle B^2 \rangle - \langle B \rangle^2) \geq \left(\frac{1}{2}(\langle AB + BA \rangle - \langle A \rangle \langle B \rangle)^2 + \frac{1}{4}|[A, B]|^2\right). \quad (A.1)$$
With the definition of operators as in equation (3) and the basic commutator equations (4), (A.1) can be written as
\[ |\bar{a}|^2|\bar{b}|^2 - (A)^2|\bar{b}|^2 - (B)^2|\bar{a}|^2 \geq (\bar{a} \cdot \bar{b})^2 - 2(\bar{a} \cdot \bar{b})(A)(B) + (C)^2. \]

After rearranging the terms, we have
\[ |\langle A\rangle \bar{b} - \langle B\rangle \bar{a}|^2 + (C)^2 \leq |\bar{a}|^2|\bar{b}|^2 - (\bar{a} \cdot \bar{b})^2 = S^2. \]

The right-hand side of the inequality is just the determinant of Gram matrix of the vector \(\bar{a}, \bar{b}\), which is the square of the area of parallelogram formed by \(\bar{a}, \bar{b}\). The expectation value is evaluated for a certain quantum state which can be prepared by projecting one partite of the bipartite entangled state onto a specific quantum state. For example, for the entangled state \(|\psi_{12}\rangle = a|+1\rangle|+2\rangle + b|-1\rangle|-2\rangle\), by projecting the partite 2 onto a specific state \(|n^p_2\rangle = \cos \frac{\theta}{2}|+\rangle + e^{\psi} \sin \frac{\theta}{2}|-\rangle\) (Eigenstate of \(\sigma_2 \cdot n^p\), where \(n^p = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\)), we can get arbitrary quantum state \(|\psi^+_1\rangle\)
\[ |\psi^+_1\rangle = \frac{1}{\sqrt{2}}(n^+_2|\psi_{12}\rangle) = \frac{1}{\sqrt{2}}(n^+_2|\psi_{12}\rangle) \left( e^{\cos \frac{\theta}{2}} + e^{-\psi} \sin \frac{\theta}{2} \right). \quad (A.2) \]

Similar expression holds for \(|\psi^+_1\rangle\) when projecting with \(|n^p_2\rangle\). The uncertainty relation holds for arbitrary state, so for \(|\psi^+_n\rangle\)
\[ |\langle A\rangle \bar{b} - \langle B\rangle \bar{a}|^2 + (C)^2 \leq S^2 \Rightarrow |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 + (\psi^+_n|C_1|\psi^+_n)^2 \leq S^2. \quad (A.3) \]

Here the subscript 1 standards for partite 1. Multiplying \(\sqrt{2}n^+_2|\psi_{12}\rangle\) to equation (A.3) with the corresponding superscript \(\pm\) and adding the two inequalities we have
\[ |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 + |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 + |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \]
\[ + 2|n^+_2|\psi_{12}\rangle|^2|\psi^+_1\rangle|A_1|\psi^+_n\rangle|B_1|\psi^+_n\rangle|\bar{a}|^2 \]
\[ + 2|n^+_2|\psi_{12}\rangle|^2|\psi^+_1\rangle|B_1|\psi^+_n\rangle|\bar{a}|^2 \leq S^2. \quad (A.4) \]

With Cauchy’s inequality \(\sum_i p_i \sum_j p_j a_i^2 \geq (\sum_i p_i a_i)^2\), equation (A.2), and the following relation
\[ |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 + |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \]
\[ + 2|\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \geq |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \]
\[ = |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \geq |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \]
\[ \geq |\langle A\rangle |\psi^+_n\rangle| \bar{b} - \langle B\rangle |\psi^+_n\rangle| \bar{a}|^2 \]
\[ = |\langle A\rangle |\psi^+_n\rangle| |P_2| |\psi_{12}\rangle| = |E(A_1, P_2)|, \quad (A.5) \]
we can get
\[ |E(A_1, P_2)\bar{b} - E(B_1, P_2)\bar{a}|^2 + |E(C_1, P_2)|^2 \leq S^2. \quad (A.6) \]

\[ \square \]

**Appendix B. Proof of equation (15)**

Proof of equation (15):
\[ |\bar{a}|^2 + |\bar{b}|^2 - (-1)^m[E(A_2, A_3) + E(B_1, B_2)] = \frac{1}{4}[\epsilon^+(A)^2 + \eta^+(B)^2 + \epsilon^-(A)^2 + \eta^-(B)^2]. \]
Proof. For the particular state $|\psi_{12}^{\pm}\rangle$, taking the definitions of equation (12), the measurement precisions turn to

$$\frac{1}{2}(n_p^2|\psi_{12}^{(m)}\rangle)^2 e^\pm(A)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|P_2^k U_{13}(I_1 \otimes I_2 \otimes A_3) U_{13} - A_1 \otimes I_2 \otimes I_3|2P_2^k |\psi_{12}^{(m)}\rangle |\psi_3\rangle.$$  

The corresponding disturbances are

$$\frac{1}{2}(n_p^2|\psi_{12}^{(m)}\rangle)^2 \eta^\pm(B)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|P_2^k U_{13}(B_1 \otimes I_2 \otimes I_3) U_{13} - B_1 \otimes I_2 \otimes I_3|2P_2^k |\psi_{12}^{(m)}\rangle |\psi_3\rangle.$$  

Using the complete relation of projection operators, the summation of the precision and disturbance for $|\psi_1^+\rangle$ and $|\psi_1^-\rangle$ gives

$$|\alpha_m|^2 e^+(A)^2 + |\beta_m|^2 e^-(A)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (I_1 \otimes I_2 \otimes A_3) U_{13} - A_1 \otimes I_2 \otimes I_3]^2 |\psi_{12}^{(m)}\rangle |\psi_3\rangle, \quad (B.1)$$

$$|\alpha_m|^2 \eta^+(B)^2 + |\beta_m|^2 \eta^-(B)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (B_1 \otimes I_2 \otimes I_3) U_{13} - B_1 \otimes I_2 \otimes I_3]^2 |\psi_{12}^{(m)}\rangle |\psi_3\rangle. \quad (B.2)$$

where $\alpha_m \equiv 2(n_p^2 |\psi_{12}^{(m)}\rangle$, $\beta_m \equiv 2(n_p^2 |\psi_{12}^{(m)}\rangle$ and $|\alpha_m|^2 + |\beta_m|^2 = 1$. Due to the properties of equation (11), we have

$$|\alpha_m|^2 e^+(A)^2 + |\beta_m|^2 e^-(A)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (I_1 \otimes I_2 \otimes A_3) U_{13} - (-1)^m I_1 \otimes A_2 \otimes I_3]^2 |\psi_{12}^{(m)}\rangle |\psi_3\rangle, \quad (B.3)$$

$$|\alpha_m|^2 \eta^+(B)^2 + |\beta_m|^2 \eta^-(B)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (B_1 \otimes I_2 \otimes I_3) U_{13} - (-1)^m I_1 \otimes B_2 \otimes I_3]^2 |\psi_{12}^{(m)}\rangle |\psi_3\rangle. \quad (B.4)$$

The measurement interaction involves only particles of 1, 3, thus it commutates with operators acting on part 2, so we have

$$|\alpha_m|^2 e^+(A)^2 + |\beta_m|^2 e^-(A)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (I_1 \otimes I_2 \otimes A_3) - (-1)^m I_1 \otimes A_2 \otimes I_3]^2 U_{13} |\psi_{12}^{(m)}\rangle |\psi_3\rangle, \quad (B.5)$$

$$|\alpha_m|^2 \eta^+(B)^2 + |\beta_m|^2 \eta^-(B)^2 = \langle\psi_3|\langle\psi_{12}^{(m)}|[U_{13}^\dagger (B_1 \otimes I_2 \otimes I_3) - (-1)^m I_1 \otimes B_2 \otimes I_3]^2 U_{13} |\psi_{12}^{(m)}\rangle |\psi_3\rangle. \quad (B.6)$$

Define $|\psi_{123}\rangle = U_{13}|\psi_{12}^{(m)}\rangle |\psi_3\rangle$, equations (B.5), (B.6) turn to

$$|\alpha_m|^2 e^+(A)^2 + |\beta_m|^2 e^-(A)^2 = \langle\psi_{123}|(A_3 - (-1)^m A_2)^2 |\psi_{123}\rangle, \quad (B.7)$$

$$|\alpha_m|^2 \eta^+(B)^2 + |\beta_m|^2 \eta^-(B)^2 = \langle\psi_{123}|(B_3 - (-1)^m B_2)^2 |\psi_{123}\rangle. \quad (B.8)$$

From the definition of operators $A = \vec{a} \cdot \vec{a}$, $B = \vec{a} \cdot \vec{b}$, and the wave function $|\psi_{12}^{(m)}\rangle$ we have chosen (this gives $|\alpha_m|^2 = |\beta_m|^2 = 1/2$), the above equations reduce to

$$\frac{1}{2}[e^+(A)^2 + e^-(A)^2] = 2|\vec{a}|^2 - (-1)^m 2E(A_2, A_3), \quad (B.9)$$

$$\frac{1}{2}[\eta^+(B)^2 + \eta^-(B)^2] = 2|\vec{b}|^2 - (-1)^m 2E(B_1, B_2). \quad (B.10)$$

This gives the relation equation (15). □
Appendix C. Proof of theorem 2

Proof. Here we present the proof for $m = 0$, the case of $m = 1$ can be derived similarly. For the Heisenberg-type MDR, taking $[A, B] = 2iC$ we have

$$
\epsilon^+ (A) \eta^+ (B) \geqslant |\langle \psi^+_1 | C | \psi^+_1 \rangle|, \quad \epsilon^- (A) \eta^- (B) \geqslant |\langle \psi^-_1 | C | \psi^-_1 \rangle|.
$$

These hyperbolic form constraints on $\epsilon (A)$ and $\eta (B)$ with given asymptotes are totally characterized by the distances from the vertices to the origin of the coordinates. That is, the essence of the above inequalities is characterized by

$$
\epsilon^+ (A)^2 + \eta^+ (B)^2 \geqslant 2|\langle \psi^+_1 | C | \psi^+_1 \rangle|, \quad \epsilon^- (A)^2 + \eta^- (B)^2 \geqslant 2|\langle \psi^-_1 | C | \psi^-_1 \rangle|.
$$

The summation over the above two equations gives

$$
\epsilon^+ (A)^2 + \eta^+ (B)^2 + \epsilon^- (A)^2 + \eta^- (B)^2 \geqslant 2(|\langle \psi^+_1 | C | \psi^+_1 \rangle|^2 + |\langle \psi^-_1 | C | \psi^-_1 \rangle|^2)
$$

(C.1)

The left-hand side of the above inequality can be represented as correlation functions via equation (15). The right-hand sides of the inequality can be written as

$$
(|\langle \psi^+_1 | C | \psi^+_1 \rangle|^2 + |\langle \psi^-_1 | C | \psi^-_1 \rangle|^2) = 2\left[|\langle \psi^{(0)}_1 | C_1 \otimes P_2^+ | \psi^{(0)}_1 \rangle|^2 + |\langle \psi^{(0)}_1 | C_1 \otimes P_2^- | \psi^{(0)}_1 \rangle|^2\right]
$$

$$
\geqslant 2\left[|\langle \psi^{(0)}_1 | C_1 \otimes P_2^+ | \psi^{(0)}_1 \rangle|^2 - |\langle \psi^{(0)}_1 | C_1 \otimes P_2^- | \psi^{(0)}_1 \rangle|^2\right]
$$

$$
= 2|\langle \psi^{(0)}_1 | C_1 \otimes P_2 | \psi^{(0)}_1 \rangle|^2 = 2E_{12}(C_1, P_2),
$$

(C.2)

where we have used equation (12) and $P_{12}^\pm = |\eta_{12}^\pm|^2 |\eta_{12}^\pm|$. It is clear that the essence of the Heisenberg-type MDR, combining equations (15) and (16), is characterized by the following inequalities

$$
E(A_2, A_3) + E(B_1, B_2) + |E_{12}(C_1, P_2)| \leqslant |\tilde{a}|^2 + |\tilde{b}|^2.
$$

(C.3)

Here the bipartite correlation function $E_{12}$ is written with subscript explicitly. Equation (C.3) must be satisfied for any given $P_2$

$$
E(A_2, A_3) + E(B_1, B_2) \leqslant |\tilde{a}|^2 + |\tilde{b}|^2 - |\tilde{n}_p \cdot \tilde{c}|.
$$

(C.4)

This is just the Heisenberg upper bound for the correlations and its lower limit is 0 for $m = 0$.

From the Ozawa’s MDR, we have

$$
\epsilon^+ (A) \eta^+ (B) + \epsilon^- (A) \Delta^-(B) + \eta^- (B) \Delta^+(A) \geqslant |\langle \psi^+_1 | C | \psi^+_1 \rangle|^2
$$

$$
\Rightarrow [\epsilon^+ (A) + \Delta^+(B)] |\eta^+ (B) + \Delta^- (A)| \geqslant |\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2 + \Delta^- (A) \Delta^+(B),
$$

where $\Delta^+(A), B$ are the standard deviations evaluated with $|\psi^+_1 \rangle$. We see that the Ozawa’s MDR is just a displaced hyperbolic curve compared to the Heisenberg-type MDR. The characterization distance of its vertices to the origin can be formulated as

$$
\epsilon^+ (A)^2 + \eta^+ (B)^2 \geqslant \sqrt{f(\Delta^+(A), \Delta^-(B), |\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2)},
$$

(C.5)

where $f$ is a function of $\Delta(A), \Delta(B)$. In order to make this inequality universally valid the left-hand side has to be greater than or equal to the maximum value of the right-hand side. Function $f$ gets the maximum value of $(2-\sqrt{2})^2 |\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2$ at $\Delta^+(A)^2 = \Delta^-(B)^2 = |\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2$. As in the case of Heisenberg-type MDR, we will get

$$
(|\tilde{a}|^2 + |\tilde{b}|^2) - [E(A_2, A_1) + E(B_1, B_2)] = \frac{1}{2} (\epsilon^+ (A)^2 + \eta^+ (B)^2 + \epsilon^- (A)^2 + \eta^- (B)^2)
$$

$$
\geqslant \frac{1}{2} (\sqrt{2} - 1)^2 (|\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2 - |\langle \psi^+_1 | C_1 \otimes P_2^- | \psi^-_1 \rangle|^2)
$$

$$
\geqslant (\sqrt{2} - 1)^2 |E_{12}(C_1, P_2)|.
$$

(C.6)

Thus the essence of the Ozawa’s MDR is characterized by the following inequalities

$$
E(A_2, A_3) + E(B_1, B_2) \leqslant |\tilde{a}|^2 + |\tilde{b}|^2 - (\sqrt{2} - 1)^2 |\tilde{n}_p \cdot \tilde{c}|.
$$

(C.7)
It should be noted here that the above constraint on correlations has no lower limit because the MDRs (both Heisenberg-type and Ozawa’s) do not specify the upper limits. In the qubit systems, the upper bound for the measurement precision and disturbance of the observables may be obtained from the finite spectrums of the observable operators.

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