We study one of the key tools in data approximation and optimization: low-discrepancy colorings. Formally, given a finite set system \((X, S)\), the discrepancy of a two-coloring \(\chi : X \rightarrow \{−1, 1\}\) is defined as \(\max_{S \in S} |\chi(S)|\), where \(\chi(S) = \sum_{x \in S} \chi(x)\).

We propose a randomized algorithm which, for any \(d > 0\) and \((X, S)\) with dual shatter function \(\pi^∗(k) = O(k^d)\), returns a coloring with expected discrepancy \(O\left(\sqrt{|X|^{1−1/d} \log |S|}\right)\) (this bound is tight) in time \(\tilde{O}\left(|S| \cdot |X|^{1/d} + |X|^{2+1/d}\right)\), improving upon the previous-best time of \(O(|S| \cdot |X|^3)\) by at least a factor of \(|X|^{2−1/d}\) when \(|S| \geq |X|\). This setup includes many geometric classes, families of bounded dual VC-dimension, and others. As an immediate consequence, we obtain an improved algorithm to construct \(\varepsilon\)-approximations of sub-quadratic size.

Our method uses primal-dual reweighing with an improved analysis of randomly updated weights and exploits the structural properties of the set system via matchings with low crossing number—a fundamental structure in computational geometry. In particular, we get the same \(|X|^{2−1/d}\) factor speed-up on the construction time of matchings with crossing number \(O\left(|X|^{1−1/d}\right)\), which is the first improvement since the 1980s.

The proposed algorithms are very simple, which makes it possible, for the first time, to compute colorings with near-optimal discrepancies and near-optimal sized approximations for abstract and geometric set systems in dimensions higher than 2.

**Keywords:** discrepancy, approximations, low-crossing matchings, VC-dimension, MWU
1 Introduction and Main Result

A set system is a pair \((X, S)\), where \(X\) is a set and \(S\) is a collection of subsets of \(X\). The elements of \(S\) are called ranges. We consider finite set systems, where both \(|X|\) and \(|S|\) are finite, and use the notation \(n = |X|\) and \(m = |S|\) throughout this paper. We study the discrepancy problem, which asks for a two-coloring \(\chi : X \to \{-1, 1\}\) that minimizes the discrepancy

\[
\text{disc}_S (\chi) = \max_{S \in S} \left| \sum_{x \in S} \chi (x) \right|
\]

Starting from 1950s, the study of low-discrepancy colorings has been an active area of research with applications in various branches of mathematics and computer science. As it is often termed, the ‘discrepancy method’ inspired many approximation algorithms for discrete optimisation problems. For instance, it is an important tool in rounding fractional solutions of a linear system of equations to integral ones [Lovász et al., 1986, Rothvoss, 2012, Bansal and Nagarajan, 2017, Rothvoss, 2013] and in recent combinatorial algorithms proposed for problems such as bin-packing [Rothvoss, 2013, Eisenbrand et al., 2013, Hoberg and Rothvoss, 2017], or scheduling problems [Bansal et al., 2014, Berndt et al., 2022]. In data approximation, a coloring with discrepancy \(o(\sqrt{n})\) can be used to construct \(o(1/\varepsilon^2)\)-sized \(\varepsilon\)-approximations (or \(\varepsilon\)-samples), outperforming the \(\Theta(1/\varepsilon^2)\) guarantee of a single random sample [Matoušek et al., 1993]. Discrepancy is also closely connected to the sample complexity of learning. For instance, in the paper of Bartlett et al. [2002], discrepancy of a random balanced coloring is used to construct penalized empirical risk minimization algorithms, leading to improved statistical guarantees. Furthermore, the study of Rademacher complexity can be seen as a study of discrepancy of a random coloring. In more recent works, the discrepancy method had a key role in core-set constructions for kernel density estimation [Phillips, 2013, Phillips and Tai, 2020, Tai, 2020] and quantizing neural networks [Lybrand and Saab, 2021]. For further details and other examples of applications, we refer the interested reader to dedicated books on discrepancy [Chazelle, 2000, Chen et al., 2014, Matoušek, 1999].
**Previous algorithms**

Spencer [1985] showed that for any set system \((X, S)\) there exists a two-coloring of \(X\) with discrepancy \(O\left(\sqrt{n \ln(m/n)}\right)\), which is tight for \(m = O(n)\). However, his original proof only demonstrated the existence of such a coloring, without any efficient algorithm to construct it. Finding a polynomial-time algorithm to construct colorings with optimal discrepancy had remained a major open problem for 25 years, until a breakthrough result of Bansal [2010], who gave a randomized polynomial-time algorithm with near-optimal discrepancy guarantees. Since then, several researchers have proposed new polynomial-time algorithms with optimal discrepancy guarantees [Harvey et al., 2014, Lovett and Meka, 2015, Levy et al., 2017, Bansal et al., 2018]. At the core of these methods is a random walk algorithm which starts with the uniformly 0 coloring, and at each step, updates the color of an element by adding a small increment to its coloring. If a variable reaches ‘−1’ or ‘1’, its value is fixed. The increment is determined by either solving an appropriate SDP [Bansal, 2010, Lovett and Meka, 2015, Harvey et al., 2014], or Gram-Schmidt orthogonalization [Bansal et al., 2018], or by a deterministic algorithm using the multiplicative weights update (MWU) method [Levy et al., 2017]. The next table contains a summary:

| Set system | Discrepancy | Time | Method | Citation |
|------------|-------------|------|--------|---------|
| arbitrary  | \(O\left(\sqrt{n \ln \left(\frac{m}{n}\right)}\right)\) | \(O(n^3 + m^3)\), \(O(n^4m)\), \(O(n^3 + mn^2 + 2^n)\) | random walk, random walk via MWU, Gram-Schmidt walk | Lovett and Meka [2015], Levy et al. [2017], Bansal et al. [2018] |
| \(\pi_S(k) = O(k^d)\) | \(O(\sqrt{n^{1-d}/d})\) | \(O(n^3 + m^3)\) | partial coloring | Matoušek [1995], Lovett and Meka [2015] |
| \(\pi_S^*(k) = O(k^d)\) | \(O(\sqrt{n^{1-d}/d} \ln m)\) | \(O(mn^d)\) | MWU | Matoušek et al. [1993] |
| \(\pi_S^*(k) = O(k^d)\) | \(O(\sqrt{n^{1-d}/d} \ln m)\) | \(O(mn^d + n^{1+2+1/d})\) | Sampling + Primal-Dual + MWU | This Paper |
| \(\pi_S^*(k) = O(k^d)\) | \(O(\sqrt{n^{1-d}/d} \ln m)\) | \(O(mn^d + n^{1+2+1/d})\) | Sampling + Pruning + Primal-Dual + MWU | This Paper |

Despite heavy interest for the past decades, still no efficient implementations with these guarantees exist. Indeed, that remains one of the open questions; see here.

In this work, we consider set systems with polynomially bounded dual shatter function.

**Definition 1** (Dual-shatter function). Let \((X, S)\) be a set system. For any \(R \subseteq S\), we say that the elements \(x, y \in X\) are equivalent with respect to \(R\) if \(x\) belongs to the same sets of \(R\) as \(y\). Then \(\pi_S^*(k)\), where \(\pi_S^*\) is called the dual-shatter function of \(S\), is defined to be the maximum number of equivalence classes on \(X\) defined by any \(k\)-element subfamily \(R \subseteq S\).

The class of set systems with polynomially bounded \(\pi_S^*(k)\) contains several fundamental cases:

- set systems with dual VC-dimension at most \(d\) (it implies \(\pi_S^*(k) \leq \left(\frac{ek}{d}\right)^d\) [Sauer, 1972, Shelah, 1972]);
- geometric set systems induced by (unions or intersections of) half-spaces, balls, etc;
- geometric set systems where \(X\) is a set of \(n\) points in \(\mathbb{R}^d\) and each range in \(S\) can be obtained as an intersection of \(X\) with a semialgebraic set of bounded complexity;
- set systems \((X, S)\) with the property that the common intersection of any \(d\) ranges from \(S\) has size at most \(c\), for given constants \(c\) and \(d\) [Matoušek, 1997].
We now present our five main algorithms. A highlight of our algorithms, besides near-quadratic improvement over previous-best running times, is that they avoid any input-specific tools, such as spatial partitioning. Thus we get improved practical constructions for many fundamental geometric set systems, narrowing the gap between theory and practice.

1. **Discrepancy.** Our main result on low-discrepancy colorings is the following.

**Main Theorem.** Let \((X, S)\) be a finite set system, \(n = |X|, m = |S|\), and \(c, d\) be constants such that \(\pi^*_S(k) \leq c \cdot k^d\). Then there is a randomized algorithm that constructs a coloring \(\chi\) of \(X\) with expected discrepancy at most

\[
3 \sqrt[4]{\frac{9c^{1/d}}{2} \cdot n^{1-1/d} \ln m + 19 \ln^2 m \ln n}
\]

with at most

\[
\frac{34n^{2+1/d} \ln n}{e^{1/d}} + \frac{25mn^{1/d} \ln(mn) \cdot \log n}{e^{1/d}}
\]

expected calls to the membership Oracle of \((X, S)\).

Our algorithm is very simple and does not use any advanced subroutines or data structures:

**Algorithm 1:** \textsc{LowDiscColor-DualShatter}((\(X, S\), \(d\))

\[
\begin{align*}
\text{while } & |X| \geq 4 \text{ do} \\
& n \leftarrow |X|, E \leftarrow \binom{X}{2} \quad // E \text{ is the set of all edges on } X \\
& \text{set the weight of each element in } E, S \text{ to 1} \\
& \text{for } i = 1, \ldots, n/4 \text{ do} \\
& \quad \text{sample } e_i = \{x_i, y_i\} \text{ from } E \text{ and } S_i \text{ from } S \text{ (according to their weights)} \\
& \quad \text{set } \chi(x_i) = \pm 1 \text{ with prob. } 1/2; \text{ set } \chi(y_i) = -\chi(x_i) \\
& \quad \text{set } X \leftarrow X \setminus \{x_i, y_i\}, \text{ and the weight of } e_i \text{ and its adjacent edges to zero} \\
& \quad E_i \subseteq E : \text{ uniform sample of size } \tilde{O}(\frac{|E|}{n^{1-1/d}}) \\
& \quad \text{halve weight of each } e \in E_i \text{ satisfying } |e \cap S_i| = 1 \\
& \quad S_i \subseteq S : \text{ uniform sample of size } \tilde{O}(\frac{|S|}{n^{1-1/d}}) \\
& \quad \text{double weight of each } S \in S_i \text{ satisfying } |S \cap e_i| = 1 \\
& \text{return } \chi \text{ (color the remaining at most 4 elements of } X \text{ arbitrarily)}
\end{align*}
\]

Importantly, the improved running time and the simplicity of this new method make it possible to perform an empirical study of low-discrepancy colorings of abstract and high-dimensional geometric set systems. As an illustration, the figures below show the average discrepancies in set systems induced by half-spaces in dimensions 2, 3, and 4, observed over 10 repetitions of our method, compared with a purely random coloring (the shaded areas denote \(\pm 1\) standard deviation from the mean).
2. Matchings with Low Crossing Number. The key property which guarantees that the output of \textsc{LowDiscColor-DualShatter} has low discrepancy is that any range in $S$ crosses\(^1\) at most $O\left(\frac{n^{1-1/d}}{d}\right)$ of the selected edges $\{\{x_i, y_i\}\}_{i=1}^{n/2}$.

In general, given a set system $(X, S)$ and a perfect matching\(^2\) $M$ of $X$, we define the crossing number of $M$ with respect to $S$ as the maximum number of edges of $M$ crossed by a single range $S \in S$. The study of perfect matchings (along with spanning paths and spanning trees) with low crossing number was originally introduced for geometric range searching [Welzl, 1988, Chazelle and Welzl, 1989]. Since then, they have found applications in various fields, for instance, discrepancy theory [Matoušek et al., 1993], learning theory [Alon et al., 2016], or algorithmic graph theory [Ducoffe et al., 2020].

The core of \textsc{LowDiscColor-DualShatter} can be generalized to construct low-crossing matchings in set systems satisfying the following assumption:

\begin{tcolorbox}
\textbf{Assumption (MainAssumption($a, b, \gamma$)).} $(X, S)$ is a finite set system with $m \geq n$, $m \geq 34$, such that any $Y \subseteq X$ has a perfect matching with crossing number at most $a|Y|^\gamma + b$ with respect to $S$.

It is known that if $\pi_*^X(k) \leq ck^d$, then $(X, S)$ satisfies the MainAssumption with parameters $a = \frac{(2\varepsilon)^{1/d}}{2\ln(2(1-1/d))}$, $b = \frac{\ln m}{\ln 2}$, and $\gamma = 1 - 1/d$. [Matoušek, 1999]
\end{tcolorbox}

The main technical ingredient of this work—of independent interest and improving the previous best construction time of $O(mn^3)$ known for several decades—is the following.

\textbf{Theorem 2.} Let $(X, S)$ be a set system satisfying MainAssumption($a, b, \gamma$). Then there is a randomized algorithm that returns a perfect matching of $X$ with expected crossing number at most

$$\frac{3a}{\gamma} n^\gamma + \frac{3b \log n}{2} + 18 \ln (mn) \log n$$

with at most

$$\min\left\{\frac{24n^{3-\gamma} \ln n}{a} + \frac{18mn^{1-\gamma} \ln mn}{a}, \min \left\{\frac{2}{1-\gamma}, \log n\right\}, \frac{n^3}{7} + \frac{mn}{2}\right\}.$$

expected calls to the membership Oracle of $(X, S)$.

Figure 1 illustrates the matchings constructed by our algorithm and random sampling on different input point arrangements and range types. It is clear that our method explicitly takes into account the information about ranges, which leads to different outcomes for set systems induced by half-spaces and balls. On the other hand, random sampling fails to preserve the intrinsic structure of the point set. We find it surprising that our method, that is based only on (non-uniform) sampling, gives a matching that adapts so well to each specific instance.

3. Approximations. An immediate consequence of our algorithms is an efficient construction algorithm of $\varepsilon$-approximations of sub-quadratic size. Given a finite set system $(X, S)$ and a parameter $\varepsilon \in (0, 1)$, a set $A \subseteq X$ is an $\varepsilon$-approximation of $(X, S)$ if the following holds for all $S \in S$:

$$\left|\frac{S}{|X|} - \frac{|A \cap S|}{|A|}\right| \leq \varepsilon.$$

Furthermore, let $\varepsilon(A, X, S)$ denote the smallest $\varepsilon$ for which $A$ is an $\varepsilon$-approximation of $(X, S)$:

$$\varepsilon(A, X, S) = \max_{S \in S} \left|\frac{|S|}{|X|} - \frac{|A \cap S|}{|A|}\right|.$$

\(^1\)We say that $S \in S$ crosses an edge $\{x, y\}$ if $|S \cap \{x, y\}| = 1$.

\(^2\)Partition of $X$ into $n/2$ disjoint pairs (edges).
Our method, Ranges: half-planes
Our method, Ranges: disks
Random Matching

Figure 1: Matchings of 5000 points. **Left column:** our method with half-plane ranges. **Middle column:** our method with disk ranges. **Right column:** random sampling. We emphasize that each of these figures contain exactly 2500 edges, which correspond to the matching of 5000 points.

The iterative application of Main Theorem implies the following on $\varepsilon$-approximations.

**Corollary 3.** Let $\varepsilon \in (0, 1)$, $(X, S)$ be a set system and $c, d$ be constants such that $\pi_S^*(k) \leq ck^d$. Then there is a randomized algorithm which returns a set $A \subset X$ of size

$$O \left( \max \left\{ \left( \frac{e^{1/d} \ln m}{\varepsilon^2} \right)^{\frac{d}{d+1}}, \frac{\ln n \ln m}{\varepsilon} \right\} \right)$$

with $E[\varepsilon(A, X, S)] \leq \varepsilon$, and with an expected

$$O \left( mn^{1/d} \ln (mn) \log^2 n + n^{2+1/d} \ln n \right)$$
calls to the membership Oracle of $(X, S)$.

**Remark.** Previous-best algorithms for constructing $o(1/\varepsilon^2)$-sized $\varepsilon$-approximations of set systems with polynomially bounded dual shatter functions were based on the low-discrepancy coloring approach of Matoušek et al. [1993], with time complexity $O(mn^3)$.

For set systems where uniform sampling yields small-sized $\varepsilon$-approximations, for instance set systems with bounded VC-dimension, the guarantees can be improved.
**Corollary 4.** Let $\varepsilon \in (0, 1)$, $(X, S)$ be a set system with VC-dimension $d_{VC} \geq 2$ and $c, d$ be constants such that $\pi^*_k(k) \leq ck^d$. Then there is a randomized algorithm that returns a set $A \subset X$ of size at most $O\left(\max \left\{ \left( \frac{c^{1/d_{VC}} \ln \left( \frac{d_{VC}}{\varepsilon^2} \right)}{\varepsilon^2} \right)^{\frac{d_{VC}}{2}}, \frac{d_{VC} \ln^{3/2} \left( \frac{d_{VC}}{\varepsilon^2} \right)}{\varepsilon} \right\} \right)$ with expected approximation guarantee $E[\varepsilon(A_j, X, S)] \leq \varepsilon$, and with an expected $O\left( d_{VC} \left( \frac{d_{VC}}{\varepsilon^2} \right)^{d_{VC}+1/d} \ln^3 \left( \frac{d_{VC}}{\varepsilon^2} \right) \right)$ calls to the membership Oracle of $(X, S)$.

The figures above present a visual comparison of the approximations created by our method (Top row) and random sampling (Bottom row). Both methods are applied to the same set system on 10,000 points with ranges induced by disks.

### 4. AN EVEN FASTER ALGORITHM FOR DISCREPANCY.

We also propose and analyse an accelerated version of our method, where instead of maintaining the weights on all the $O(n^2)$ edges induced by $X$, we use further random sampling. The following theorem describes the trade-off between the expected running time and discrepancy guarantee as function of a pre-sampling parameter $\alpha$.

**Theorem 5.** Let $(X, S)$ be a set system and $d$ be a constant such that $\pi^*_k(k) = O \left( k^d \right)$. For any $0 < \alpha \leq 1$, there is a randomized algorithm which constructs a coloring $\chi$ of $X$ with expected discrepancy

$$O \left( \sqrt{n^{1-\alpha/d} \ln m + \ln^2 m \log n} \right),$$

with at most

$$O \left( n^{1+\alpha(1+1/d)} \ln^2 n + mn^{\alpha/d} \ln(mn) \cdot \log n \right)$$

expected calls to the membership Oracle of $(X, S)$.

The randomized algorithm that achieves the guarantees of Theorem 5 is presented in Section 3.5 (**LowDiscColorPresampled**). It is essentially the algorithm **LowDiscColorDualShatter** run on an initial random sample of edges with a small modification: to incorporate the pre-sampling step in the analysis, we need to recurse slightly more often (after $n/16$ steps instead of $n/4$). The proof of Theorem 5 relies on the following theorem on matchings in random edge-sets, which might be of independent interest.
Theorem 6. Let \((X,S)\) be a set system with dual shatter function \(\pi_\alpha^*(k) = O(k^d)\), \(\alpha \in (0,1]\), and \(\delta \in (0,1)\). Let \(E\) be a uniform sample from \(\binom{X}{2}\), where each edge is picked i.i.d. with probability
\[
p = \min \left\{ \frac{2\ln n}{n^{1-a}} + \frac{4\ln(2/\delta)}{n^{2-a}}, \ 1 \right\}.
\]
Then with probability at least \(1 - \delta\), \(E\) contains a matching of size \(n/4\) with crossing number
\[
O\left(n^{1-a/d} + \ln |S|\right).
\]
Moreover, for any \(d \geq 2\), and \(n_0 \in \mathbb{N}\) there is a set system \((X,S)\) with \(|X| = n \geq n_0\) and dual shatter function \(\pi_\alpha^*(k) = O(k^d)\) such that for any \(\alpha \in (0,1]\) and \(p(n) = o\left(n^{\alpha-1}\right)\) if \(E\) is a random edge-set obtained by selecting each edge in \(\binom{X}{2}\) i.i.d. with probability \(p(n)\), then with probability at least \(1/2\), every matching in \(E\) of size \(n/4\) has crossing number \(\omega\left(n^{1-a/d}\right)\) with respect to \(S\).

5. Geometric systems. Set systems of bounded dual shatter function and bounded dual VC dimension arise naturally in many geometric scenarios. Previous works on the above three problems—discrepancy, matchings approximations—heavily relied on spatial partitioning techniques, which essentially blocked any further progress and limited their practical applicability for the past decades. We refer the reader to Section 2.3 for a detailed explanation. The precise guarantees for several geometric set systems and their proofs are presented in Section 4.

2 Previous Results

2.1 Discrepancy

A first bound on the combinatorial discrepancy of \((X,S)\) follows immediately from Chernoff’s bound, which implies that a random two-coloring \(\chi\) of \(X\) satisfies \(\text{disc}_S(\chi) = O\left(\sqrt{n \ln m}\right)\) with probability at least \(1/2\). This also gives a randomized algorithm to obtain such a coloring, and it is possible to derandomize the method yielding a deterministic algorithm with running time \(O\left(n m\right)\) [Chazelle, 2000].

Spencer [1985] showed that for any set system \((X,S)\), there exists a coloring of \(X\) with discrepancy \(O\left(\sqrt{n \ln(m/n)}\right)\), which is tight and improves the general bound for \(m = O(n)\). A series of algorithms for its construction started with the work of Bansal [2010], who gave the first polynomial-time randomized algorithm (using SDP rounding) to compute a coloring with discrepancy \(O\left(\sqrt{n \ln(m/n)}\right)\), which matches the bound of Spencer for \(m = O(n)\). Later Lovett and Meka [2015] gave a combinatorial randomized algorithm for constructing colorings with discrepancy \(O\left(\sqrt{n \ln(m/n)}\right)\) and improved the expected running time to \(O\left(n^3 + m^3\right)\); see also Rothvoss [2017] for a different proof. The algorithm of Bansal was de-randomized by Bansal and Spencer [2013] (but still used a non-constructive method to prove the feasibility of an underlying SDP), and later, Levy et al. [2017] used the multiplicative weights update technique to give a deterministic \(O\left(n^4 m\right)\)-time algorithm to compute a two-coloring with discrepancy \(O\left(\sqrt{n \ln(m/n)}\right)\) for an arbitrary set system. See also Bansal et al. [2018] for a random-walk algorithm for Banaszczyk’s discrepancy bound, with running time \(O\left(n^{3.3728-} + nm^{2.3728-}\right)\) (the exponent depends on the running time for matrix multiplication).

For general set systems, one cannot hope to have polynomial-time algorithms with better guarantees: it was shown by Charikar et al. [2011] that there exists a set system with \(m = O(n)\) for which it is NP-hard to decide whether discrepancy zero or \(\Omega(\sqrt{n})\).
Lastly, we mention that there is an active line of research considering sparse set systems\(^3\) [Beck and Fiala, 1981, Levy et al., 2017, Bansal et al., 2019] and the (stochastic) online setting [Spencer, 1977, Bárány, 1979, Swanepoel, 2000, Bansal and Spencer, 2020, Bansal et al., 2020, Alweiss et al., 2021, Bansal et al., 2021, Gupta et al., 2022].

Improved discrepancy bounds can be obtained if the set system satisfies additional constraints. In particular, we have the following result for set systems with polynomially bounded shatter function:

**Theorem 7** ([Alon et al. [1999], Matoušek et al. [1993], Matoušek [1997]]). Let \((X, S)\) be a finite set system and \(d\) be a constant such that \(\pi_S^*(k) = O\left(k^d\right)\). Then there exists a polynomial-time algorithm to compute a two-coloring of \(X\) with discrepancy \(O\left(\sqrt{n^{1-1/d}\ln m}\right)\). Furthermore, for any \(d\), there exists a set system with dual shatter function \(\pi_S^*(k) = O\left(k^d\right)\) such that any two-coloring of \(X\) has discrepancy \(\Omega\left(\sqrt{n^{1-1/d}\ln n}\right)\).

If \(d\) is considered as a constant, the upper and lower bounds of Theorem 7 match: if the dual-shatter function of \((X, S)\) is \(\pi_S^*(k) = O\left(k^d\right)\), then \(VC\)–\(\dim(X, S)\) \(\leq 2^d\) and thus by the Sauer-Shelah lemma, \(m = |S| \leq \left(\frac{en}{2^d}\right)^{2^d}\); see [Matoušek, 1999, Chapter 5] for further details.

**Algorithms.** The classical proof of the upper-bound in Theorem 7 uses the multiplicative weights update (MWU) technique\(^4\) as follows. The algorithm maintains a weight function \(\pi\) on \(S\), with initial weights set to 1. For any pair \(\{x, y\} \in X\), let \(\Delta S(x, y)\) denote the set of those sets \(S \in S\) which satisfy \(|S \cap \{x, y\}| = 1\) and let \(\tilde{\pi}(x, y) = \sum_{S \in \Delta_S(x, y)} \pi(S)\). The algorithm colors two elements of \(X\) at a time (for simplicity, we assume that \(X\) is even) and proceeds as follows for \(i = 1, \ldots, \frac{n}{2}\)

1. Find a pair \(\{x_i, y_i\} \in X\) that minimizes \(\tilde{\pi}(x, y)\).
2. Set \(\chi(x_i) = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}\), \(\chi(y_i) = -\chi(x_i)\), remove \(x_i, y_i\) from \(X\).
3. Update \(\pi\) by doubling the weight of each set in \(\Delta S(x_i, y_i)\).

The reweighing scheme ensures the key property that for each \(S \in S\),

\[
|\{i : S \in \Delta S(x_i, y_i)\}| = O\left(n^{1-1/d}\right).
\]

This implies, using Chernoff’s bound and the union bound, that the discrepancy of the resulting coloring is \(O\left(\sqrt{n^{1-1/d}\ln m}\right)\) with probability at least \(1/2\) Matoušek et al. [1993]. The algorithmic bottleneck is finding the pair \(\{x_i, y_i\}\) that minimizes \(\tilde{\pi}\) at each iteration. Using the incidence matrix for \(S\), this can be done in \(O\left(n^2 m\right)\) steps, and thus the algorithm has overall running time \(O\left(n^3 m\right)\).

### 2.2 Matchings with low crossing number.

Given a set \(X\), a **matching** in \(X\) is a set of disjoint edges (pairs) from \(X\). A (perfect) **matching of \(X\)** is a matching of size \([n/2]\) plus a loop (an edge \(\{x, x\}\)) if \(n\) is odd. The size of a matching is the number of its edges. We say that a range \(S \in S\) **crosses** a pair \(\{x, y\}\) if and only if

---

\(^3\)Where every element is contained in at most \(t\) ranges for some constant \(t\).

\(^4\)For an excellent survey on the MWU technique see Arora et al. [2012].
constructs a spanning tree with expected crossing number at most $\frac{\kappa}{\gamma}$ for some $\gamma \in [1/\log n, 1]$, then a spanning tree of crossing number $O(\kappa/\gamma)$ can be found by solving an LP on $\binom{n}{2}$ variables and $m + n$ constraints. There also exists an algorithm using a general framework of rounding fractional solutions of minimax integer programs with matroid constraints. This method gives a randomized algorithm that constructs a spanning tree with expected crossing number at most $\kappa + O(\sqrt{\kappa \log m})$ in time $\tilde{O}(mn^4 + n^8)$ [Chekuri et al., 2009].

### 2.3 Geometric set systems.

Now we turn to the case where $X$ is a set of $n$ points in $\mathbb{R}^d$ and $S$ consists of subsets of $X$ that are induced by geometric objects. In this setting, improved bounds are made possible using spatial partitioning. The current-best algorithms for geometric set systems induced by half-spaces recursively construct simplicial partitions, stored in a hierarchical structure called the partition tree, which then at its base level gives a matching with low crossing number. This approach is used in the breakthrough result of Chan [2012] who gave an $O(n \log n)$ time algorithm to build partition trees with respect to half-spaces in $\mathbb{R}^d$, which then implies the same for computing matchings with crossing number $O(n^{1-1/d})$.

For set systems where $X$ is a set of $n$ points in $\mathbb{R}^d$ and $S$ consists of subsets of $X$ that are induced by certain geometric objects, improved bounds are made possible using spatial partitioning. For instance, if $S$ consists of the subsets of $X$ that are induced by half-spaces, one can apply the algorithm of Chan [2012] to construct a perfect matching with crossing number $O(n^{1-1/d})$ in time $O(n \ln n)$, which then implies the same running-time for computing coloring with discrepancy $O\left(\sqrt{n^{1-1/d} \ln m}\right)$. While the use of spatial partitioning gives $o(mn^3)$ running times, progress remains blocked in several ways:

a) Spatial partitioning only exist in certain geometric settings; it is not possible when dealing with abstract set systems such as those arising in learning theory or graph theory. Indeed, as shown by Alon et al. [1987], they do not always exist in settings satisfying the requirements of Theorem 7 (e.g., the projective plane).

b) Optimal bounds for constructing simplicial partitions are only known for the case of half-spaces; this is one of the main problems left open by Chan [2012]. Despite a series of research for semi-algebraic set systems (using linearization, cuttings, and more recently, polynomial partitioning [Agarwal et al., 2013]), the bounds are still sub-optimal for polynomials of degree larger than 2, with exponential dependence on the dimension.
c) There are large constants in the asymptotic notation depending on the dimension $d$ both in the running time as well as the crossing number bounds, due to the use of cuttings (see Ezra et al. [2020]).

d) Practical implementation of spatial partitioning in $\mathbb{R}^d$, $d > 2$, remains an open problem in geometric computing, even for half-spaces. In particular, for $d > 2$, we know of no implementations for low-crossing matchings; nor for constructing $o\left(\frac{d^2}{\varepsilon^2}\right)$-sized $\varepsilon$-approximations even for half-spaces in $\mathbb{R}^3$.

**Remark.** In $\mathbb{R}^2$, an algorithm to create optimal cuttings (the main tool in the construction of spatial partitioning for half-space ranges) was implemented Har-Peled [2000] and was recently used for computing near-optimal $\varepsilon$-approximations with respect to half-spaces in $\mathbb{R}^2$ [Matheny and Phillips, 2018].

Thus, one of our main objectives was to find an efficient algorithm that does not use spatial partitioning; continuing the recent theme of such algorithms proposed for $\varepsilon$-nets and $\varepsilon$-approximations [Varadarajan, 2010, Chan et al., 2012, Mustafa et al., 2018, Mustafa, 2019].

## 3 Proofs

### 3.1 Outline and ideas

Our methods rest on the following three key ideas:

1. We replace the bottleneck algorithmic step of finding a minimum weight pair (with respect to $\tilde{\pi}$) in the multiplicative weights update technique (see Section 2.1) by simply sampling a pair according to a carefully maintained distribution. In particular, we maintain weights not only on the ranges in $S$, but also on $\left(\frac{X}{2}\right)$. At each iteration we sample a range $S$ and an edge (pair) $e$ according to the current weights. Then we color the endpoints of $e$ and update the weights by doubling the weight of each range that crosses $e$ and halving the weight of each edge that is crossed by $S$. In comparison to previous MWU-based solutions for constructing low-discrepancy colorings [Levy et al., 2017], our method is much simpler and faster.

2. The idea of maintaining ‘primal-dual’ weights has been used earlier to approximately solve matrix games [Grigoriadis and Khachiyan, 1995] and in geometric optimization [Agarwal and Pan, 2014]. In our case, the process is more elaborate as we are constructing a proper coloring at the same time as reweighing. Therefore, at the end of each iteration, as we color the endpoints $e$, we are forced to set the weights of $e$ and all edges adjacent to $e$ to 0. This breaks down the reweighing scheme, as the removal of the edges amplifies the error introduced in later iterations and thus our maintained weights degrade over time. However, we prove that restarting the algorithm by ‘resetting’ all the weights a logarithmic number of times suffices to ensure the required low discrepancy.

3. Finally, updating the weights of all edges and sets crossing the randomly picked set and edge would be too expensive. Instead, we show that updating the weights of a uniform sample of edges and ranges at each iteration is sufficient for our purposes. The key observation here is that the standard multiplicative weights proof has an additive smaller-order term; we take advantage of this gap to improve the running time at the cost of amplifying this term, just enough so that it is still within a constant factor of the optimal solution.

We start by proving the main technical ingredient of this work: Theorem 2.

### 3.2 Proof of Theorem 2

The algorithm that achieves the guarantees of Theorem 2 is presented in Algorithm 2.
Algorithm 2: BuildMatching((X, S), a, b, γ)

\[ M \leftarrow \emptyset \]
while \(|X| \geq 4\) do
\[ E \leftarrow \binom{X}{2} \]
\[ M \leftarrow M \cup \text{PartialMatching}((X, S), E, a, b, γ, |X|/4) \] // see Algorithm 3
\[ X \leftarrow X \setminus \text{endpoints}(M) \] // remove the elements covered by \( M \)
match the remaining elements of \( X \) randomly and add the edges to \( M \)
\[ \text{// if } |X| \text{ is odd, we allow one edge to be a loop} \]
return \( M \)

Algorithm 3: PartialMatching((X, S), E, a, b, γ, t)
\[ \omega_1(e) \leftarrow 1, \quad \pi_1(S) \leftarrow 1 \quad \forall e \in E, S \in S \]
\[ p \leftarrow \min \left\{ \frac{48\ln(|E| \cdot t)}{a|X|^\gamma \cdot b} + 1 \right\} \]
\[ q \leftarrow \min \left\{ \frac{72\ln(|S| \cdot t)}{a|X|^\gamma \cdot b} + 1 \right\} \]
for \( i = 1, \ldots, t \) do
\[ \omega_i(E) \leftarrow \sum_{e \in E} \omega_i(e) \]
\[ \pi_i(S) \leftarrow \sum_{S \in S} \pi_i(S) \]
choose \( e_i \sim \omega_i \) // \( \Pr[e_i = e] = \frac{\omega_i(e)}{\omega_i(E)} \quad \forall e \in E \)
choose \( S_i \sim \pi_i \) // \( \Pr[S_i = S] = \frac{\pi_i(S)}{\pi_i(S)} \quad \forall S \in S \)
\[ E_i \leftarrow \text{sample from } E \text{ with probability } p \] // \( \Pr[e \in E_i] = p \quad \forall e \in E \)
\[ S_i \leftarrow \text{sample from } S \text{ with probability } q \] // \( \Pr[S \in S_i] = q \quad \forall S \in S \)
\[ I(e,S) = 1 \text{ if } e \text{ crosses } S, \quad I(e,S) = 0 \text{ otherwise} \]
for \( e \in E_i \) do
\[ \omega_{i+1}(e) \leftarrow \omega_i(e) \left( 1 - \frac{1}{2} I(e, S_i) \right) \] // halve weight if \( S_i \) crosses \( e \)
for \( S \in S_i \) do
\[ \pi_{i+1}(S) \leftarrow \pi_i(S) \left( 1 + I(e_i, S) \right) \] // double weight if \( S \) crosses \( e_i \)
set the weight in \( \omega_{i+1} \) of \( e_i \) and of each edge adjacent to \( e_i \) to zero
return \( \{e_1, \ldots, e_t\} \)

The main result of this section is the following theorem:

**Theorem 8.** Let \((X, S)\) be a set system, which satisfies \textbf{MainAssumption}(a, b, γ) and let \( E \) denote the set of all pairs (edges) from \( X \). Then \textbf{PartialMatching} \((X, S), E, a, b, γ, n/4\) returns a matching of size\(^5\) \( n/4 \) with expected crossing number at most
\[ a \left( \frac{n}{2} \right)^\gamma + b + \max \left\{ a \left( \frac{n}{2} \right)^\gamma + \frac{b}{2}, \frac{18\ln(n/4)}{2} \right\}, \]
with an expected number of Oracle calls at most
\[ \min \left\{ \frac{6}{a} \left( n^{3-\gamma} \ln \frac{n^3}{4} + 3mn^{1-\gamma} \ln \frac{mn}{4} \right), \frac{n^3 + 2mn}{8} \right\}. \]

Before we present its proof, we first show how Theorem 8 implies Theorem 2. The algorithm \textbf{BuildMatching} makes \( \log n \) calls to \textbf{PartialMatching} with exponentially decreasing input

\(^5\)The size of a matching is the number of its edges.
sizes. In particular, the overall expected number of membership Oracle calls of $\text{BuildMatching}$ can be bounded as

$$\sum_{i=0}^{\log n} \min \left\{ 6 \left( \frac{n}{2^i} \right)^{3-\gamma} \ln \frac{n^3}{2^{3i+2}} + 3m \left( \frac{n}{2^i} \right)^{1-\gamma} \ln \frac{mn}{2^{i+2}} \right\}, \left( \frac{n}{2^i} \right)^{3} + \frac{mn}{2^{i+1}} \right\}$$

$$\leq \min \left\{ \sum_{i=0}^{\log n} \frac{6}{a} \left( \frac{n^3}{2^i} \right)^{\gamma} \ln \frac{n^3}{2^{3i+2}} + 3m \left( \frac{n}{2^i} \right)^{1-\gamma} \ln \frac{mn}{2^{i+2}} \right\}, \sum_{i=0}^{\log n} \frac{n^3}{2^i} + \frac{mn}{2^{i+1}} \right\}$$

As for the crossing number, Theorem 8 implies that $\text{BuildMatching}((X, S), a, b, \gamma)$ returns a matching with expected crossing number at most

$$\sum_{i=1}^{\log n} \left[ \frac{3a}{2} \left( \frac{n}{2^i} \right)^{\gamma} + \frac{3b}{2} + 18 \ln \frac{mn}{2^{i+1}} \right] < \frac{3a n^{\gamma}}{2} + \frac{3b}{2} + 18 \ln (mn) \log n$$

$$< \frac{3a n^{\gamma}}{\gamma} + \left( \frac{3b}{2} + 18 \ln (mn) \right) \log n.$$ 

Hence, we have shown that Theorem 2 is a consequence of Theorem 8. \qed

**Proof of Theorem 8.**

For an edge $e$ and a set $S$, we define

$$I(e, S) = \begin{cases} 
1 & \text{if } S \text{ crosses } e, \\
0 & \text{otherwise.}
\end{cases}$$

We will deduce Theorem 8 from the next lemma, which is proved later in this section.

**Lemma 9.** Let $t \in [1, |X|/4]$ be an integer and let $\{e_1, \ldots, e_t\}$, $\{S_1, \ldots, S_t\}$, $p$, and $q$ as in Algorithm 3. Furthermore let $E_t \subseteq E$ denote the set of edges that have non-zero weight when $\text{PartialMatching}((X, S), E, a, b, \gamma, t)$ terminates. Then

$$\mathbb{E} \left[ \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \right] \leq \frac{1}{2} \mathbb{E} \left[ \min_{e \in E_t} \sum_{i=1}^{t} I(e, S_i) \right] + \frac{128}{13} + \frac{8}{15} \ln(|E|t) \frac{3p}{3} + \frac{16}{4} \ln(|S|t) \frac{3q}{3}.$$ 

Setting $t = n/4$, the left-hand side of Equation (2) is precisely the expected crossing number of the edges $\{e_1, \ldots, e_{n/4}\}$ returned by $\text{PartialMatching}((X, S), E, a, b, \gamma, n/4)$. To bound the expectation in the right-hand side of Equation (2), we use the following lemma.

**Lemma 10.** Let $(Y, \mathcal{R})$ be a set system, $w : \mathcal{R} \rightarrow \mathbb{R}_{\geq 0}$, and $\kappa$ be such that $Y$ has a perfect matching with crossing number at most $\kappa$ with respect to $\mathcal{R}$. Then there is an edge $\{x, y\}$ in $\binom{Y}{2}$ such that

$$\sum_{R \text{ crosses } \{x, y\}} w(R) \leq \frac{2w(\mathcal{R}) \cdot \kappa}{|Y|}.$$ 

**Proof.** Let $M$ be a perfect matching of $Y$ such that any set of $\mathcal{R}$ crosses at most $\kappa$ edges of $M$. Then if we consider the weighted sum there are at most $w(\mathcal{R}) \cdot \kappa$ crossings between the edges of $M$ and sets in $\mathcal{R}$ counted with weights. By the pigeonhole principle, there is an edge in $M$ that is crossed by sets of total weight at most

$$\frac{w(\mathcal{R}) \cdot \kappa}{|M|} = \frac{w(\mathcal{R}) \cdot \kappa}{|Y|/2} = \frac{2w(\mathcal{R}) \kappa}{|Y|}$$

sets of $\mathcal{R}$. \qed
Let $\tilde{X}_{n/4} \subset X$ denote the set of points that are not covered by the edges $\{e_1, \ldots, e_{n/4}\}$. Note that $|\tilde{X}_{n/4}| = n/2$ and that since BuildMatching calls PartialMatching with $E = (\tilde{X}_{n/4})$, we have $E_{n/4} = (\tilde{X}_{n/4})$. Moreover, since $(X, S)$ satisfies MainAssumption$(a, b, \gamma)$, there exists a perfect matching of $\tilde{X}_{n/4}$ with crossing number at most $a|\tilde{X}_{n/4}|^\gamma + b$. Applying Lemma 10 to $Y = \tilde{X}_{n/4}$ and $R = \{S_1, \ldots, S_{n/4}\}$ with weights $w(S_i) = 1$, we get that there is an edge $e \in \tilde{E}_{n/4}$ that satisfies

$$\sum_{i=1}^{n/4} I(e, S_i) \leq 2 \cdot \frac{a}{4} \frac{|\tilde{X}_{n/4}|^\gamma + b}{|X_{n/4}|} = 2 \cdot \frac{n}{4} \frac{a(n/2)^\gamma + b}{n/2} = a \left( \frac{n}{2} \right)^\gamma + b. \quad (3)$$

Since Equation (3) holds for any choice of $\{S_1, \ldots, S_{n/4}\}$ and $\tilde{X}_{n/4}$, we can conclude that

$$\mathbb{E} \left[ \min_{e \in \tilde{E}_{n/4}} \sum_{i=1}^{n/4} I(e, S_i) \right] \leq a \left( \frac{n}{2} \right)^\gamma + b. \quad (4)$$

Now Equations (2) and (4) imply that the expected crossing number of the edges returned by PartialMatching$(X, S, E, a, b, \gamma, n/4)$ can be bounded as

$$\mathbb{E} \left[ \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \right] \leq a \left( \frac{n}{2} \right)^\gamma + b + \frac{128}{39} \frac{a \ln(\frac{|E|}{n})}{\ln(\frac{|S|}{n})} + \frac{16 + \frac{4}{m^2} \ln(\frac{|S|}{n})}{3} + \frac{1}{3} \min_{\gamma} \left\{ \frac{48}{a n^{\gamma+b}} \cdot \ln(\frac{|E|}{n}), 1 \right\}$$

$$\leq a \left( \frac{n}{2} \right)^\gamma + b + \frac{128 + 104}{39} \max \left\{ \frac{a n^{\gamma+b}}{48}, \frac{\ln(\frac{|S|}{n})}{\ln(\frac{|S|}{n})} \right\} + \frac{16 + \frac{4}{m^2} \ln(\frac{|S|}{n})}{3} + \frac{3}{3} \min_{\gamma} \left\{ \frac{48}{a n^{\gamma+b}} \cdot \ln(\frac{|E|}{n}), 1 \right\}$$

$$= a \left( \frac{n}{2} \right)^\gamma + b + \frac{128 + 104}{26} \max \left\{ \frac{a n^{\gamma+b}}{48}, \frac{\ln(\frac{|S|}{n})}{\frac{48}{\ln(\frac{|S|}{n})}} \right\} + \frac{16 + \frac{4}{m^2} \ln(\frac{|S|}{n})}{3} + \frac{3}{3} \min_{\gamma} \left\{ \frac{48}{a n^{\gamma+b}} \cdot \ln(\frac{|S|}{n}) \right\}$$

Finally, we bound the number of membership Oracle calls. At each iteration $i = 1, \ldots, n/4$, we update the weights of at most $\frac{n}{2} p + m q$ elements in expectation, each requiring one call to the membership Oracle. Thus in expectation, the total number of membership Oracle calls is at most

$$\frac{n}{4} \left( \frac{n}{2} \min \left\{ \frac{48 \ln \frac{n^3}{a n^{\gamma+b}}}{48}, 1 \right\} + m \min \left\{ \frac{72 \ln \frac{m n}{48}}{a n^{\gamma+b}}, 1 \right\} \right)$$

$$\leq \min \left\{ \frac{6}{a} \left( n^{3-\gamma} \ln \frac{n^3}{4} + 3 m n^{1-\gamma} \ln \frac{m n}{4} \right), \frac{n^3 + 2 mn}{8} \right\}.$$
Lemma 11.

$$E \left[ \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \right] \leq \frac{4}{3 \ln 2} \sum_{i=1}^{t} E \left[ \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \right] + \frac{(16 + \frac{4}{m^2}) \ln(\|S\|t)}{3q}$$

Proof. Let $\pi_{t+1}(S)$ denote the total weight of the sets of $S$ in $\pi_{t+1}$. We bound $\pi_{t+1}(S)$ in two different ways. On the one hand, $\pi_{t+1}(S)$ is clearly lower-bounded by the weight of the set of maximum weight in $\pi_{t+1}$. Recall that the weight of a set $S$ is doubled in iteration $i$ if and only if $S \in \mathcal{S}_i$ and $S$ crosses $e_i$, therefore

$$\pi_{t+1}(S) \geq \max_{S \in \mathcal{S}} \pi_{t+1}(S) = 2^{\max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) 1\{e_i \in S_i\}},$$

where $1_A$ denotes the indicator random variable of the event $A$. On the other hand, we can express $\pi_{t+1}(S)$ using the update rule of the algorithm

$$\pi_{t+1}(S) = \sum_{S \in \mathcal{S}} \pi_{t+1}(S) = \sum_{S \in \mathcal{S}} \pi_t(S) (1 + 1(e_t, S) \cdot 1\{S \in \mathcal{S}_t\})$$

$$= \sum_{S \in \mathcal{S}} \pi_t(S) + \sum_{S \in \mathcal{S}} \pi_t(S) I(e_t, S) \cdot 1\{S \in \mathcal{S}_t\}$$

$$= \pi_t(S) + \pi_t(S) \sum_{S \in \mathcal{S}} \pi_t(S) I(e_t, S) \cdot 1\{S \in \mathcal{S}_t\}$$

$$= \pi_t(S) \left( 1 + \sum_{S \in \mathcal{S}} \frac{\pi_t(S)}{\pi_t(S)} I(e_t, S) \cdot 1\{S \in \mathcal{S}_t\} \right).$$

Unfolding this recursion and using the fact that $1 + a \leq \exp(a)$, we get

$$\pi_{t+1}(S) = \pi_1(S) \prod_{i=1}^{t} \left( 1 + \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \cdot 1\{S \in \mathcal{S}_i\} \right)$$

$$\leq |S| \cdot \exp \left( \sum_{i=1}^{t} \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \cdot 1\{S \in \mathcal{S}_i\} \right).$$

Putting together the obtained upper and lower bounds on $\pi_{t+1}(S)$, we get

$$2^{\max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) 1\{e_i \in S_i\}} \leq |S| \cdot \exp \left( \sum_{i=1}^{t} \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \cdot 1\{S \in \mathcal{S}_i\} \right).$$

Taking the logarithm of each side yields

$$\ln(2) \cdot \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \cdot 1\{e_i \in S_i\} \leq \sum_{i=1}^{t} \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \cdot 1\{S \in \mathcal{S}_i\} + \ln|S|.$$  \hspace{1cm} (5)

If $q = 1$, then $1\{e \in S_i\} = 1$ for all $i$ and $S \in \mathcal{S}$, thus taking total expectation we conclude

$$E \left[ \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \right] \leq \frac{1}{\ln 2} \sum_{i=1}^{t} E \left[ \sum_{S \in \mathcal{S}} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \right] + \frac{\ln|S|}{\ln 2}.$$

Assume that $q < 1$. Since $\max f(x) - \max g(x) \leq \max(f(x) - g(x))$, Equation (5) implies

$$\ln(2) \cdot \frac{3}{4} \cdot \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \cdot q \leq \ln(2) \cdot \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \cdot \left( \frac{3q}{4} - 1\{S \in \mathcal{S}_i\} \right)$$
Therefore, Lemma 12 with Claim 1.

Setting \( \pi_t(S) \) as the martingale for each \( S \in \mathcal{S} \), we will need the following Azuma-type inequality for martingales.

**Lemma 12** ([Koufogiannakis and Young, 2014, Lemma 10]). Let \( X = \sum_{i=1}^{T} x_i \) and \( Y = \sum_{i=1}^{T} y_i \) be sums of non-negative random variables, where \( T \) is a random stopping time with finite expectation, and, for all \( i \), \( |x_i - y_i| < 1 \) and

\[
\mathbb{E} \left[ x_i - y_i \bigg| \sum_{s<i} x_s, \sum_{s<i} y_s \right] \leq 0.
\]

Let \( \varepsilon \in [0, 1] \) and \( A \in \mathbb{R} \), then

\[
\mathbb{P} \left[ (1 - \varepsilon)X \geq Y + A \right] \leq \exp(-\varepsilon A).
\]

**Claim 1.**

\[
\mathbb{P} \left[ \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \cdot \left( \frac{3q}{4} - 1_{\{S \in \mathcal{S}_i\}} \right) \geq 3 \ln|S| \right] \leq \frac{1}{t}.
\]

**Proof.** For each \( i \in [1, t] \) and \( S \in \mathcal{S} \), consider the random variables \( x_i(S) = I(e_i, S) \cdot q \) and \( y_i(S) = I(e_i, S) \cdot 1_{\{S \in \mathcal{S}_i\}} \), which are measurable with respect to \( e_i \) and \( \mathcal{S}_i \). For any \( i \) and \( S \in \mathcal{S} \), we have \( |x_i(S) - y_i(S)| \leq 1 \). Since \( \mathcal{S}_i \) is independent of \( e_i \), \( \sum_{k<i} x_k(S) \), and \( \sum_{k<i} y_k(S) \), we have

\[
\mathbb{E} \left[ x_i(S) - y_i(S) \bigg| \sum_{k<i} x_k(S), \sum_{k<i} y_k(S) \right] = 0
\]
as \( \mathbb{E} \left[ q - 1_{\{S \in \mathcal{S}_i\}} \right] = 0 \) for all \( i \in [1, t] \) and \( S \in \mathcal{S} \).

Therefore, Lemma 12 with \( \varepsilon = 1/4 \), combined with the union bound implies for any \( A \in \mathbb{R}, \)

\[
\mathbb{P} \left( \max_{S \in \mathcal{S}} \sum_{i=1}^{t} I(e_i, S) \cdot \left( \frac{3q}{4} - 1_{\{S \in \mathcal{S}_i\}} \right) \geq A \right) \leq |\mathcal{S}| \exp\left(-\frac{A}{4}\right).
\]

Setting \( A = 4 \ln(|\mathcal{S}|) \), we conclude the proof of Claim 1. \( \square \)
Applying Claim 1 and using that $\sum_{i=1}^{t} I(e_i, S) \cdot \left(\frac{3q}{4} - 1_{\{S \in S_i\}}\right) \leq t$ always holds, we get

$$E\left[\max_{S \in S} \sum_{i=1}^{t} I(e_i, S) \cdot \left(\frac{3q}{4} - 1_{\{S \in S_i\}}\right)\right] \leq 4 \ln(|S|) + t \cdot \frac{1}{t} \leq 4 \ln(|S|) + 1. \quad (8)$$

Hence Equations (6), (7), and (8) imply

$$\frac{3 \ln 2}{4} - q \cdot E\left[\max_{S \in S} \sum_{i=1}^{t} I(e_i, S)\right] \leq \frac{4}{3} \ln 2 \cdot \sum_{i=1}^{t} E\left[\sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S)\right] + \ln(2) \cdot (4 \ln(|S|) + 1) + \ln |S|.$$

Dividing both sides by $q \frac{3 \ln 2}{4}$, we obtain

$$E\left[\max_{S \in S} \sum_{i=1}^{t} I(e_i, S)\right] \leq \frac{4}{3} \ln 2 \cdot \sum_{i=1}^{t} E\left[\sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S)\right] + \frac{16 \ln(|S|) + 4 + \frac{4 \ln|S|}{\ln 2}}{3q} \cdot \ln(|S|).$$

This concludes the proof of Lemma 11. \qed

The next lemma is proven by applying analogous arguments for the total weight of edges in $\omega_{t+1}$ with a small adjustment as in each iteration we set some edge weights to zero. Recall that $E_t$ denotes the set of edges that have non-zero when PartialMatching($(X, S), E, a, b, \gamma, t$) terminates, in other words, $E_t$ is the set of edges that have non-zero weight in $\omega_{t+1}$.

**Lemma 13.**

$$\sum_{i=1}^{t} \sum_{e \in E} E\left[\frac{\omega_i(e)}{\omega_i(E)} I(e, S_i)\right] < \frac{3 \ln 2}{8} \cdot \sum_{e \in E} E\left[\min_{S \in S_i} \sum_{i=1}^{t} I(e, S_i)\right] + \frac{(32 \ln 2 + 2) \ln(|E|)}{p}.$$

**Proof.** Let $\omega_{t+1}(E)$ denote the total weight of edges in $\omega_{t+1}$. Again, we lower-bound $\omega_{t+1}(E)$ by the largest edge-weight in $\omega_{t+1}$, which is now attained at some edge of $E_t$.

$$\omega_{t+1}(E) \geq \max_{e \in E} \omega_{t+1}(e) = \max_{e \in E_t} \omega_{t+1}(e) = \left(\frac{1}{2}\right) \min_{e \in E_t} \sum_{i=1}^{t} I(1_{e}, S_i) \cdot 1_{\{e \in E_i\}}.$$

The upper bound is obtained by using the algorithm’s weight update rule. Since $e_t$ has positive weight in $\omega_t$, but its weight in $\omega_{t+1}$ is set to 0, we have a strict inequality

$$\omega_{t+1}(E) = \sum_{e \in E} \omega_{t+1}(e) < \sum_{e \in E} \omega_{t}(e) \left(1 - \frac{1}{2} I(e, S_t) \cdot 1_{\{e \in E_t\}}\right)$$

$$= \sum_{e \in E} \omega_{t}(e) - \frac{1}{2} \sum_{e \in E} \omega_{t}(e) I(e, S_t) \cdot 1_{\{e \in E_t\}}$$

$$= \omega_{t}(E) \left(1 - \frac{1}{2} \sum_{e \in E} \omega_{t}(e) I(e, S_t) \cdot 1_{\{e \in E_t\}}\right).$$

Unfolding this recursion and using the fact that $1 + a \leq \exp(a)$, we get

$$\omega_{t+1}(E) < |E| \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^{t} \sum_{e \in E} \omega_{t}(e) I(e, S_t) \cdot 1_{\{e \in E_t\}}\right).$$
Combining the obtained upper and the lower bounds on $\omega_{t+1}(E)$ and taking the logarithm of each side, we get

$$
\ln \left( \frac{1}{2} \right) \cdot \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \mathbb{1}_{\{e \in E_i\}} < -\frac{1}{2} \sum_{i=1}^t \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \cdot \mathbb{1}_{\{e \in E_i\}} + \ln |E|,
$$

which is equivalent to

$$
\sum_{i=1}^t \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \cdot \mathbb{1}_{\{e \in E_i\}} < 2 \ln(2) \cdot \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \mathbb{1}_{\{e \in E_i\}} + 2 \ln |E|. \quad (9)
$$

If $p = 1$, then $\mathbb{1}_{\{e \in E_i\}} = 1$ for all $i$ and $e \in E$, thus taking total expectation we conclude

$$
\sum_{i=1}^t \mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \right] < 2 \ln(2) \cdot \mathbb{E} \left[ \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \right] + 2 \ln |E|.
$$

Assume that $p < 1$. Since $\min f(x) - \min g(x) \leq \max(f(x) - g(x))$, Equation (9) implies

$$
\sum_{i=1}^t \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \cdot \mathbb{1}_{\{e \in E_i\}} < 2 \ln(2) \cdot \max_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \left( \mathbb{1}_{\{e \in E_i\}} - \frac{3p}{16} \right)
$$

$$
+ 2 \ln(2) \cdot \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \frac{3p}{16} + 2 \ln |E|.
$$

Taking total expectation of each side, and using that for each fixed $i$, the random variables $\{\omega_i, S_i\}$ and $E_i$ are independent, we get

$$
p \cdot \sum_{i=1}^t \mathbb{E} \left[ \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \right] < 2 \ln(2) \cdot \mathbb{E} \left[ \max_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \left( \mathbb{1}_{\{e \in E_i\}} - \frac{3p}{16} \right) \right]
$$

$$
+ 2 \ln(2) \cdot \mathbb{E} \left[ \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \frac{3p}{16} \right] + 2 \ln |E|. \quad (10)
$$

We need the following claim whose proof uses Lemma 12 and is similar to Claim 1.

**Claim 2.**

$$
\mathbb{P} \left[ \max_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \left( \mathbb{1}_{\{e \in E_i\}} - \frac{3p}{16} \right) \geq \frac{16}{13} \ln(|E|) \right] \leq \frac{1}{t}.
$$

This, together with the fact that $\sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \left( \mathbb{1}_{\{e \in E_i\}} - \frac{3p}{16} \right) \leq t$ always holds imply

$$
\mathbb{E} \left[ \max_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \cdot \left( \mathbb{1}_{\{e \in E_i\}} - \frac{3p}{16} \right) \right] \leq \frac{16}{13} \ln(|E|) + t \cdot \frac{1}{t} \leq \frac{16}{13} \ln(|E|) + 1.
$$

Hence Equation (10) yields

$$
\sum_{i=1}^t \mathbb{E} \left[ \frac{\omega_i(e)}{\omega_i(E)} \mathbb{I}(e, S_i) \right] < \frac{6 \ln 2}{16} \cdot \mathbb{E} \left[ \min_{e \in E_t} \sum_{i=1}^t \mathbb{I}(e, S_i) \right] + 2 \ln(2) \cdot \left( \frac{16}{13} \ln(|E|) + 1 \right) + 2 \ln |E|.
$$
Proof. Let \( \omega, \sigma \) be in \( [1, t] \). We will prove a more general statement of Main Theorem using Main Assumption \((a, b, \gamma)\):

\[
\sum_{i=1}^{t} \sum_{e \in E} \mathbb{E} \left[ \frac{\omega_i(e)}{\omega_i(E)} I(e, S_i) \right] < \frac{3 \ln 2}{8} \cdot \mathbb{E} \left[ \min_{e \in E} \sum_{i=1}^{t} I(e, S_i) \right] + \frac{32 \ln 2}{3q} + \frac{2}{3} \mathbb{E} \left[ \frac{\ln(|S|)}{3q} \right].
\]

We need one more lemma to tie the previous two together.

Lemma 14. For any \( i \in [1, t] \), we have

\[
\mathbb{E} \left[ \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \right] = \mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e_i, S) \right].
\]

Proof. Let \( F_i = \sigma(e_1, e_i, S_1, \ldots, S_i, E_1, \ldots, E_i, S_1, \ldots, S_i) \). We have

\[
\mathbb{E} \left[ \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \right] = \mathbb{E} \left[ \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e, S) \bigg| F_i \right] \quad \text{and}
\]

\[
\mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e_i, S) \right] = \mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e, S) \bigg| F_i \right].
\]

Observe that \( \omega_i \) and \( \pi_i \) are measurable with respect to \( F_i \), thus

\[
\mathbb{E} \left[ \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \bigg| F_i \right] = \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} \left( \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e, S) \right)
\]

\[
= \sum_{e \in E} \sum_{S \in S} \frac{\omega_i(e)}{\omega_i(E)} \frac{\pi_i(S)}{\pi_i(S)} I(e, S)
\]

\[
= \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} \left( \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e, S) \right) = \mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e, S_i) \bigg| F_i \right].
\]

Finally, we combine Lemmas 11, 13, and 14 in the following way

\[
\mathbb{E} \left[ \max_{S \in S} \sum_{i=1}^{t} I(e_i, S) \right] \leq \frac{4 \ln 2}{3} \cdot \sum_{i=1}^{t} \mathbb{E} \left[ \sum_{S \in S} \frac{\pi_i(S)}{\pi_i(S)} I(e_i, S) \right] + \frac{(16 + \frac{4}{m^2}) \ln(|S|)}{3q}
\]

\[
= \frac{4 \ln 2}{3} \cdot \sum_{i=1}^{t} \mathbb{E} \left[ \sum_{e \in E} \frac{\omega_i(e)}{\omega_i(E)} I(e, S_i) \right] + \frac{(16 + \frac{4}{m^2}) \ln(|S|)}{3q}
\]

\[
< \frac{1}{2} \cdot \mathbb{E} \left[ \min_{e \in E} \sum_{i=1}^{t} I(e, S_i) \right] + \frac{(128 + \frac{8}{m^2}) \ln(|E|)}{3p} + \frac{(16 + \frac{4}{m^2}) \ln(|S|)}{3q}
\]

This concludes the proof of the Lemma 9 and thus completes the proof of Theorem 2.

3.3 Proof of Main Theorem

We will prove a more general statement of Main Theorem using \textsc{Main Assumption}(a, b, \gamma):
Theorem 15. Let \((X, S)\) be a set system that satisfies \textsc{MainAssumption}(a, b, \gamma). The algorithm \textsc{LowDiscColor}((X, S), a, b, \gamma) constructs a coloring \(\chi\) of \(X\) of with expected discrepancy at most
\[
3 \sqrt{\frac{a n^\gamma \ln m}{\gamma} + \frac{b \ln m \log n}{2} + 12 \ln^2 m \log n},
\]
with an expected number of Oracle calls at most
\[
\min \left\{ \frac{24 n^{3 - \gamma} \ln n}{a} + \frac{18 mn^{1 - \gamma} \ln mn}{a}, \frac{\min \left\{ \frac{2}{1 - \gamma}, \log n \right\}}{\frac{n^3}{7}} + \frac{mn}{2} \right\}.
\]
The algorithm \textsc{LowDiscColor} is presented in Algorithm 4. It is easy to check that Main Theorem follows immediately from Theorem 15 by substituting \(a = \frac{(2c)^{1/d}}{2 \ln (2(1 - 1/d))}, b = \frac{\ln m}{2},\) and \(\gamma = 1 - 1/d\).

Algorithm 4: \textsc{LowDiscColor}((X, S), a, b, \gamma)

\[
n \leftarrow |X|
\]
\[
\{e_1, \ldots, e_{\lceil n/2 \rceil}\} \leftarrow \text{BuildMatching}((X, S), a, b, \gamma) \quad \text{// see Algorithm 2}
\]
for \(i = 1, \ldots, \lfloor n/2 \rfloor\) do
\[
\{x_i, y_i\} \leftarrow \text{endpoints}(e_i)
\]
\[
\chi(x_i) = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}
\]
\[
\chi(y_i) = -\chi(x_i) \quad \text{// we skip this step if } y_i = x_i
\]
return \(\chi\)

We will prove Theorem 15 using Theorem 2 and the following lemma.

Lemma 16. Let \((X, S)\) be a set system, \(n = |X|, m = |S| \geq 34,\) and let \(M\) be a perfect matching of \(X\) with crossing number \(\kappa\) with respect to \(S\) and for each edge \(\{x, y\} \in M\) define
\[
\chi_M(x) = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}
\]
and \(\chi_M(y) = -\chi_M(x).\) Then the expected discrepancy of \(\chi_M\) is at most \(\sqrt{3\kappa \ln m}\).

Remark. A ‘high probability version’ of Lemma 16 is well-known [Matoušek et al., 1993, Lemma 2.5] and implies the above bound through basic probabilistic calculations, see Appendix for the precise proof.

Proof of Theorem 15

Let \(M\) be the matching returned by \textsc{BuildMatching}((X, S), a, b, \gamma) during the run of \textsc{LowDiscColor}((X, S), a, b, \gamma) and let \(\kappa(M)\) denote its crossing number with respect to \(S\). By Theorem 2,
\[
\mathbb{E} \left[ \kappa(M) \right] \leq \frac{3a}{\gamma} n^\gamma + \frac{3b \log n}{2} + 18 \ln (mn) \log n
\]
Using Lemma 16, taking total expectation over the matchings returned by \textsc{BuildMatching}, and applying Jensen’s inequality, we get
\[
\mathbb{E} \left[ \text{disc}_S(\chi_M) \right] \leq \mathbb{E} \left[ \sqrt{3\kappa(M) \ln m} \right] \leq \sqrt{3 \mathbb{E} \left[ \kappa(M) \right] \ln m}.
\]
Therefore, the expected discrepancy of the coloring returned by \texttt{LOWDISCColor}((X,S),a,b,\gamma) is at most 
\[
\sqrt{3 \left( \frac{3a}{\gamma} n^{\gamma} + \frac{3b \log n}{2} + 18 \ln (nm) \log n \right) \ln m}.
\]
Each call of the membership Oracle is performed during the call of \texttt{BUILDMatching}, thus the bound on the expected number of membership Oracle calls follows immediately from Theorem 2. This concludes the proof of Theorem 15 and thus of Main Theorem.

\[\square\]

### 3.4 Proof of Corollaries 3 and 4

**Proof of Corollary 3**

The problems of low-discrepancy colorings and \(\varepsilon\)-approximations are naturally connected: finding a set of \(|X|/2\) elements with low approximation error is essentially equivalent to finding a low-discrepancy coloring of \((X,S)\):

**Lemma 17** ([Matoušek et al., 1993, Lemma 2.1]). Let \((X,S)\) be a set system with \(|X| = n\), \(X \in S\) and let \(\chi\) be a coloring with discrepancy \(\text{disc}_S(\chi) = \Delta\) and let \(A \subseteq X\) be a set of \([n/2]\) elements from the larger color class of \(\chi\). Then \(A\) is a \((2\Delta/n)\)-approximation of \((X,S)\).

One can obtain lower order approximations by iteratively halving the point-set along a low-discrepancy colorings. The final approximation error can be bound using Lemma 17 and the following basic fact.

**Fact 18.** If \(A_1\) is an \(\varepsilon_1\)-approximation of \((X,S)\) and \(A_2\) is an \(\varepsilon_2\)-approximation of \((A_1,S|_{A_1})\), then \(A_2\) is an \((\varepsilon_1 + \varepsilon_2)\)-approximation of \((X,S)\).

These ideas yield the following corollary of Theorem 15, which immediately implies Corollary 3 by substituting \(a = \frac{(2\varepsilon)^1/d}{2 \ln 2(1-1/d)}\), \(b = \frac{\ln m}{\ln 2}\), and \(\gamma = 1 - 1/d\).

**Corollary 19.** Let \((X,S)\) be a set system that satisfies \texttt{MainAssumption}(a,b,\gamma) and let \(\varepsilon \in (0,1)\). Then \texttt{APPROXIMATE}((X,S),a,b,\gamma,\varepsilon) returns a set \(A \subseteq X\) of size at most

\[
2 \max \left\{ \left(30 \sqrt{a \ln m \gamma} \frac{1}{\varepsilon} \right)^{2\gamma}, \left(12 \sqrt{\frac{b}{2} + 12 \ln m} \ln m \log n \right)^{2\gamma} \right\} + 1,
\]

with expected approximation guarantee \(\mathbb{E}[\varepsilon(A,X,S)] \leq \varepsilon\), and with an expected

\[
\min \left\{ \frac{8\varepsilon^{2-\gamma} \ln n}{a} + \frac{18mn^{1-\gamma} \ln (mn)}{a} \min \left\{ \frac{4}{(1-\gamma)^2}, \log^2 n \right\}, \frac{n^3}{49} + \frac{mn}{2} \right\}
\]

calls to the membership Oracle of \((X,S)\).

The precise analysis of the ‘halving process’ (used in the algorithm \texttt{APPROXIMATE} and in the deduction of Corollary 19 from Theorem 15) is well-known, therefore we only present it in the Appendix (Section 5.2).

**Proof of Corollary 4**

For set systems with bounded VC-dimension, one can obtain small-sized \(\varepsilon\)-approximations via uniform sampling:
Algorithm 5: \textsc{Approximate}\((X, \mathcal{S}), a, b, \gamma, \varepsilon\)

\[ A_0 \leftarrow X \]

\[ j = \left\lfloor \log |X| + \min \left\{ \frac{2}{2-\gamma} \log \frac{\varepsilon \sqrt{T}}{30 \sqrt{a \ln(|S|)}}, \log \frac{\varepsilon}{12 \sqrt{\left( \frac{2}{2+12 \ln(|S|)} \right) \ln|S| \ln|X|}} \right\} \right\rfloor \]

for \(i = 1, \ldots, j\) do

\[ \chi \leftarrow \text{LowDiscColor}\((A_{i-1}, \mathcal{S}_{|A_{i-1}|}), a, b, \gamma\) \]

\[ A_i \leftarrow \chi^{-1}(1) \]

return \(A_j\)

Theorem 20 ([Vershynin, 2018, Theorem 8.3.23]). There is a universal constant \(C_{\text{apx}}\) such that for any constant \(d_{\text{VC}}\) and any set system \((X, \mathcal{S})\) with VC-dimension at most \(d_{\text{VC}}\), a uniform random sample \(A\) of \(X\) satisfies

\[ \mathbb{E}[\varepsilon(A, X, \mathcal{S})] \leq \sqrt{\frac{C_{\text{apx}} \cdot d_{\text{VC}}}{|A|}}. \]

Let \(A_0\) be a uniform random sample of \(\frac{4C_{\text{apx}} d_{\text{VC}}}{\varepsilon^2}\) elements from \(X\). By Corollary 19, the algorithm \textsc{Approximate}\((A_0, \mathcal{S}_{|A_0|}), a, b, \gamma, \varepsilon/2\) returns a set \(A_1\) with \(\mathbb{E}[\varepsilon(A_1, A_0, \mathcal{S}_{A_0})] \leq \varepsilon/2\). By Theorem 20, \(A_0\) satisfies \(\mathbb{E}[\varepsilon(A_0, X, \mathcal{S})] \leq \varepsilon/2\), and thus, by Fact 18,

\[ \mathbb{E}[\varepsilon(A_1, X, \mathcal{S})] \leq \mathbb{E}[\varepsilon(A_1, A_0, \mathcal{S}_{A_0})] + \mathbb{E}[\varepsilon(A_0, X, \mathcal{S})] \leq \varepsilon. \]

The resulting guarantees are summarized in the next corollary.

Corollary 21. Let \((X, \mathcal{S})\) be a set system that satisfies \textsc{MainAssumption}\((a, b, \gamma)\), let \(d_{\text{VC}}\) denote the VC-dimension of \((X, \mathcal{S})\), and let \(\varepsilon \in (0, 1)\). Let \(A_0\) be a uniform random sample of \(\frac{C_{\text{apx}} d_{\text{VC}}}{(\varepsilon/2)^2}\) elements from \(X\). Then \textsc{Approximate}\((A_0, \mathcal{S}_{|A_0|}), a, b, \gamma, \varepsilon/2\) returns a set \(A \subset X\) of size at most

\[ 2 \max \left\{ \left( \frac{30 \sqrt{a \ln|\mathcal{S}_{|A_0|}}}{\gamma} \right)^{2/\gamma}, \frac{24 \sqrt{\left( \frac{2}{2 + 12 \ln|\mathcal{S}_{|A_0|}|} \right) \ln|\mathcal{S}_{|A_0|}| \log|A_0|}}{\varepsilon} \right\} + 1, \]

with expected approximation guarantee \(\mathbb{E}[\varepsilon(A, X, \mathcal{S})] \leq \varepsilon\), and with an expected \(\min \left\{ \frac{8|A_0|^{3-\gamma} \ln|A_0|}{a} + \frac{18|S|_{|A_0|} |A_0|^{1-\gamma} \ln(|S|_{|A_0|} ||A_0||)}{\ln|\mathcal{S}_{|A_0|}| |A_0|} \right\} \),

\[ \min \left\{ \frac{|A_0|^3}{49} + \frac{|S|_{|A_0|} ||A_0||}{2} \right\} \]

calls to the membership Oracle of \((X, \mathcal{S})\).

We can deduce Corollary 4 from Corollary 21 by using that \(|\mathcal{S}_{|A_0|}| = O(|A_0|^{d_{\text{VC}}})\) by the Sauer-Shelah lemma [Sauer, 1972, Shelah, 1972] and substituting \(a = \frac{(2\gamma)^{1/d}}{2 \ln 2 (1-1/d)}, b = \frac{\ln m}{\ln 2}, \) and \(\gamma = 1 - 1/d\).
3.5 Proof of Theorem 5

We will deduce Theorem 6 implies Theorem 5.

Proof of Theorem 5

The randomized algorithm that achieves the guarantees of Theorem 5 is presented in Algorithm 6.

Algorithm 6: LOWDiscColorPresampled((X, S), d, α)

\[
n \leftarrow |X|
\{e_1, \ldots, e_{[n/2]}\} \leftarrow \text{MATCHINGPresampled}((X, S), d, \alpha)
\]

for \( i = 1, \ldots, [n/2] \) do

\[
\{x_i, y_i\} \leftarrow \text{endpoints}(e_i)
\]

\[
\chi(x_i) = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 
\end{cases}
\]

\[
\chi(y_i) = -\chi(x_i)
\]

// we skip this step if \( y_i = x_i \)

return \( \chi \)

Algorithm 7: MATCHINGPresampled((X, S), d, α)

\[
M \leftarrow \emptyset
\]

while \( |X| > 16 \) do

\[
n \leftarrow |X|
\]

\[
E \leftarrow \text{sample of } O(n^{1+\alpha} \ln n) \text{ edges from } \binom{X}{2}
\]

\[
\{e_1, \ldots, e_{[n/16]}\} \leftarrow \text{PARTIALMatching}((X, S), E, (2c)^{1/d}, \ln m, 1 - \alpha/d, [n/16])
\]

\[
M \leftarrow M \cup \{e_1, \ldots, e_{[n/16]}\}
\]

\[
X \leftarrow X \setminus \text{endpoints}(M)
\]

match the remaining elements of \( X \) randomly and add the edges to \( M \)

return \( M \)

Recall the following lemma from Section 3.3:

Lemma 16. Let \((X, S)\) be a set system, \( n = |X|, m = |S| \geq 34 \), and let \( M \) be a perfect matching of \( X \) with crossing number \( \kappa \) with respect to \( S \) and for each edge \( \{x, y\} \in M \) define

\[
\chi_M(x) = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 
\end{cases}
\]

and \( \chi_M(y) = -\chi_M(x) \). Then the expected discrepancy of \( \chi_M \) is at most \( \sqrt{3\kappa \ln m} \).

By Lemma 16, it is sufficient to show that the algorithm MATCHINGPresampled constructs a matching with expected crossing number \( O\left(n^{1-\alpha/d} + \ln |S| \log n\right) \). To prove this, recall the following statement on PARTIALMATCHING.

Lemma 9. Let \( t \in [1, |X|/4] \) be an integer and let \( \{e_1, \ldots, e_t\}, \{S_1, \ldots, S_{t}\}, p, \) and \( q \) as in Algorithm 3. Furthermore let \( E_t \subseteq E \) denote the set of edges that have non-zero weight when PARTIALMATCHING\((X, S), E, a, b, \gamma, t\) terminates. Then

\[
E\left[ \max_{S \in S} \sum_{i=1}^{t} I(e, S_i) \right] \leq \frac{1}{2} E \left[ \min_{e \in E_t} \sum_{i=1}^{t} I(e, S_i) \right] + \frac{\frac{128}{13} + \frac{8}{\ln 2}}{3p} \ln(|E| t) + \frac{(16 + \frac{4}{\ln 2}) \ln(|S| t)}{3q}.
\]  

(2)
Substituting \((a, b, \gamma, t) = \left((2c)^{1/d}, \ln m, 1 - \alpha/d, [n/16]\right)\) and the proper values for \(p\) and \(q\), we get the following bound on the expected crossing number of \(\{e_1, \ldots, e_{[n/16]}\}\):

\[
E \left[ \max_{S \in \mathcal{S}} \frac{[n/16]}{n} \sum_{i=1}^{[n/16]} I(e_i, S) \right] \leq \frac{1}{2} E \left[ \min_{e \in \mathcal{E}_{[n/16]}} \frac{[n/16]}{n} \sum_{i=1}^{[n/16]} I(e, S_i) \right] + O \left( n^{1-\alpha/d} \right) .
\] (11)

It remains to bound the expectation on the right-hand side of Equation (11). By Theorem 6, with probability at least \(1 - \frac{1}{n}\), the initial sample \(E\) contains a matching \(M_0\) of size \([n/4]\) with crossing number

\[
c_0 \cdot \left( n^{1-\alpha/d} + \ln |S| \right)
\]

for some fixed constant \(c_0\). Assume that it happens, then clearly \(M_0 \cap \tilde{E}_{[n/16]}\) also has crossing number at most \(c_0 \cdot (n^{1-\alpha/d} + \ln |S|)\) with respect to \(S\). Moreover, since we only zeroed the weights of edges adjacent to \(2 \cdot [n/16]\) distinct vertices of \(X\), there are at least \([n/8 - 2]\) edges of \(M_0\) with positive weight when \textsc{PartialMatching} terminates. That is, \([M_0 \cap \tilde{E}_{[n/16]}] \geq [n/8 - 2]\) and \([n/8 - 2] > 0\) since \(n > 16\) at each call of \textsc{PartialMatching}. By the pigeonhole principle, there is an edge in \(M_0 \cap \tilde{E}_{[n/16]}\) which is crossed by at most

\[
\frac{c_0 \cdot (n^{1-\alpha/d} + \ln |S|) \cdot [n/16]}{[n/8 - 2]} = O \left( n^{1-\alpha/d} + \ln |S| \right)
\]

sets from \(S_1, \ldots, S_{[n/16]}\). Therefore, we have

\[
E \left[ \max_{S \in \mathcal{S}} \frac{[n/16]}{n} \sum_{i=1}^{[n/16]} I(e_i, S) \right] \leq \frac{1}{2} E \left[ \min_{e \in \mathcal{E}_{[n/16]}} \frac{[n/16]}{n} \sum_{i=1}^{[n/16]} I(e, S_i) \right] + O \left( n^{1-\alpha/d} \right) = O \left( n^{1-\alpha/d} + \ln |S| \right),
\]

where the last bound holds with with probability at least \(1 - \frac{1}{n}\). Since the crossing number of any matching of \(X\) is \(O(n)\), the expected crossing number of the matching returned by the subroutine \textsc{PartialMatching}((\(X, S\), \(E\), \((2c)^{1/d}, \ln m, 1 - \alpha/d, [n/16]\)) is \(O \left( n^{1-\alpha/d} + \ln |S| \right)\). The algorithm \textsc{MatchingPresampled} makes \(\log n\) calls to \textsc{PartialMatching} with exponentially decreasing input sizes. It can easily be deduced (with calculations analogous to the ones in Section 3.2) that the expected crossing number of the matching returned by \textsc{MatchingPresampled} is \(O \left( n^{1-\alpha/d} + \ln |S| \log n \right)\). Hence we have shown that Theorem 5 is a consequence of Theorem 6. \(\square\)

**Proof of Theorem 6**

Our starting point is the following algorithm which is a variant of the classical MWU method [Welzl, 1988, Chazelle and Welzl, 1989, Welzl, 1992].

The first part of Theorem 6 is implied by the following two properties of \textsc{RelaxedMWU}:

1. For any halting time \(T = t\), the edges returned by \textsc{RelaxedMWU} have crossing number \(O \left( t^{1-\alpha/d} \right)\);
2. If \(E \subseteq \binom{X}{2}\) is an i.i.d. sample where each edge is picked with probability

\[
p = \min \left\{ \frac{2 \ln n}{n^{1-\alpha}} + \frac{4 \ln(2/\delta)}{n^2}, 1 \right\},
\]

then \(T \geq n/4\) with probability at least \(1 - \delta\). In other words, \textsc{RelaxedMWU}((\(X, S\), \(\alpha\), \(E\)) returns at least \(n/4\) edges with probability at least \(1 - \delta\).
Algorithm 8: RELAXEDMWU((X, S), α, E)

\[\begin{align*}
\omega_1(S) &\leftarrow 1 \text{ for all } S \in S \\
X_1 &\leftarrow X \\
\text{for } i = 1, \ldots, n/2 \text{ do} \\
&\quad \mathcal{E}_i \leftarrow \text{ the } |X_i|^{2-\alpha} \text{ lightest edges in } \left(\frac{X_i}{2}\right) \text{ w.r.t. } \omega_i \\
&\quad \text{if } E \cap \mathcal{E}_i = \emptyset \text{ then} \\
&\quad \quad \text{set } T = i - 1 \text{ and return } \{e_1, \ldots, e_{i-1}\} \\
&\quad \text{else} \\
&\quad \quad \text{Pick an edge } e_i \text{ from } E \cap \mathcal{E}_i \text{ uniformly at random} \\
&\quad \quad \text{Define } \omega_{i+1} \text{ from } \omega_i \text{ by doubling the weights of each set crossing } e_i \\
&\quad \quad X_{i+1} \leftarrow X_i \setminus \text{endpoints}(e_i) \\
&\quad \text{set } T = n/2 \text{ and return } \{e_1, \ldots, e_{n/2}\}
\end{align*}\]

1. Bounding the crossing number of the output. Assume that \(\tau : \mathbb{N} \times \mathbb{R} \to \mathbb{R}\) is a function such that at iteration \(i\), RELAXEDMWU picks an edge which is crossed by ranges of total weight at most \(\tau(|X_i|, \omega_i(S))\) or it terminates. Then at each iteration, the total weight of ranges in \(S\) changes as

\[
\omega_{i+1}(S) \leq \omega_i(S) + \tau(|X_i|, \omega_i(S)) = \omega_i(S) \left( 1 + \frac{\tau(|X_i|, \omega_i(S))}{\omega_i(S)} \right) \\
\leq \omega_i(S) \prod_{j=1}^{i} \left( 1 + \frac{\tau(|X_j|, \omega_j(S))}{\omega_j(S)} \right) = |S|^i \prod_{j=1}^{i} \left( 1 + \frac{\tau(|X_j|, \omega_j(S))}{\omega_j(S)} \right)
\]

Let \(t \in [1, n/2]\) be a stopping time and let \(\kappa_t\) denote the maximum number of edges in \(\{e_1, \ldots, e_t\}\) that are crossed by any set in \(S\), then by the update rule,

\[
\omega_{t+1}(S) \geq \max_{S \in \mathcal{S}} \omega_{t+1}(S) = 2^{\kappa_t}.
\]

We get that

\[
2^{\kappa_t} \leq \omega_{t+1}(S) \leq |S|^t \prod_{j=1}^{t} \left( 1 + \frac{\tau(|X_j|, \omega_j(S))}{\omega_j(S)} \right) \leq |S|^t \exp \left( \sum_{j=1}^{t} \frac{\tau(|X_j|, \omega_j(S))}{\omega_j(S)} \right)
\]

which implies

\[
\kappa_t \leq \frac{1}{\ln 2} \left( \ln |S| + \sum_{j=1}^{t} \frac{\tau(|X_j|, \omega_j(S))}{\omega_j(S)} \right).
\]

We use the following lemma to bound the function \(\tau(\cdot, \cdot)\) for set systems with polynomially bounded dual shatter function.

**Lemma 22.** Let \((X, S)\) be a set system with dual shatter function \(\pi^*(k) \leq c_1 \cdot k^d\). Then for any \(Y \subset X\), \(w : S \to \mathbb{N}\), and parameter \(|Y| \leq \ell \leq \binom{|Y|}{2}\) there are at least \(\ell\) distinct edges in \(\binom{Y}{2}\) such that any of these edges are crossed by sets of total weight at most \(\tau_\ell(|Y|, w(S)) = (10c_1)^{1/d} \cdot \frac{w(S)}{|Y|^{2/d}}\).

**Proof.** Let \((S_w, R_Y)\) denote the set system where \(S_w\) contains \(w(S)\) copies of each \(S \in S\), \(R_Y = \{R_y : y \in Y\}\), and \(R_y = \{S \in S_w : y \in S\}\). Note that \(|S_w| = w(S)\) and the shatter function of \((S_w, R_X)\) is the dual shatter function of \((Y, S)\). Recall the following lemma of Haussler [1995].
The Packing Lemma. Let $(X, S)$ be a set system with shatter function $\pi_S(k) \leq c_1 \cdot k^d$ and $1 < \delta < |X|$ be a parameter. Furthermore, let $\mathcal{P} \subset S$ be a $\delta$-separated set, that is, $|S_1 \Delta S_2| \geq \delta$ for any $S_1, S_2 \in \mathcal{P}$. Then

$$|\mathcal{P}| \leq 2c_1 \left(\frac{|X|}{\delta}\right)^d.$$ 

For the choice of

$$\delta_\ell = \left(10c_1 \cdot \frac{w(S)\delta_\ell}{|Y|^2}\right)^{1/d},$$

the Packing Lemma implies that any $\delta_\ell$-separated subset of ranges in $\mathcal{R}_Y$ has cardinality at most

$$C_\ell = 2c_1 \left(\frac{w(S)}{\delta_\ell}\right)^d = \frac{|Y|^2}{5\ell},$$

Observe that for any pair $x, y \in Y$, the set $R_x \Delta R_y$ contains precisely the sets in $S_w$ that cross the edge $xy$. Consider the graph $G_Y$ on $Y$ defined by the edges that are crossed by at least $\delta_\ell$ sets in $S_w$. The Packing Lemma implies that $G_Y$ does not contain a clique on $C_\ell + 1$ vertices. Thus by the classical theorem of extremal graph theory Turán [1941], the number of pairs that are not edges in $G_Y$ is at least

$$C_\ell \left(\frac{|Y|/C_\ell}{2}\right) \geq C_\ell \cdot \frac{(|Y|/C_\ell - 1) (|Y|/C_\ell - 2) - 3|Y|}{2} = \frac{|Y|^2}{2C_\ell} - \frac{3|Y|}{2} = \frac{5\ell}{2} - \frac{3|Y|}{2} = \ell,$$

where we used that $|Y| \leq \ell$. That is, there are at least $\ell$ edges which cross ranges of total weight at most $\delta_\ell$. This concludes the proof of Lemma 22.

At iteration $i$, we have $|X_i| = n - 2i + 2$ and we pick one of the $|E_i| = |X_i|^{2-\alpha}$ lightest edges of $\binom{X_i}{2}$. By Lemma 22, each edge in $E_i$ crosses ranges of total weight at most

$$(10c_1)^{1/d} \cdot \frac{w_i(S) \cdot |E_i|^{1/d}}{|X_i|^{2/d}} = \frac{(10c_1)^{1/d}w_i(S)}{|X_i|^{\alpha/d}} = \frac{(10c_1)^{1/d}w_i(S)}{(n-2i+2)^{\alpha/d}},$$

which bounds $\tau(|X_i|, \omega_i(S))$. Thus Equation (12) implies that for any stopping time $t \in [1, n/2]$, the matching $\{e_1, \ldots, e_t\}$ returned by RELAXEDMWU has crossing number at most

$$\frac{\ln |S|}{\ln 2} + \frac{(10c_1)^{1/d}}{\ln 2} \sum_{j=1}^{t} \frac{1}{(n-2j+2)^{\alpha/d}} \leq \frac{\ln |S|}{\ln 2} + \frac{(10c_1)^{1/d}}{\ln 2} \cdot \frac{t^{1-\alpha/d}}{1-\alpha/d}.$$ (13)

2. Halting time on a random input. Now we show that if $E \subseteq \binom{X}{2}$ is a random edge-set, where each edge is picked i.i.d with probability

$$p = \min\left\{\frac{2 \ln n}{n^{1-\alpha}} + \frac{4 \ln (2/\delta)}{n^{2-\alpha}}, 1\right\}$$

then with probability at least $1 - \delta$, the algorithm RELAXEDMWU($\mathcal{E}, \alpha, E$) satisfies

$$\mathbb{P}[T \leq n/4] \leq \delta.$$ 

If $p = 1$, then the statement is trivially true, therefore we assume that $p < 1$. We will bound the probabilities $\mathbb{P}[T = i]$ for each $i = 1, \ldots, n/4$. Since $E$ is an i.i.d. uniform random sample of $\binom{X}{2}$,

$$\mathbb{P}[T = 1] = \mathbb{P}[E \cap \mathcal{E}_1 = \emptyset] = (1 - p)^{|E_1|} = (1 - p)^{n^{2-\alpha}}.$$ 

Observe that in iteration $i \geq 2$ of the algorithm RELAXEDMWU, the edge-set $\mathcal{E}_i$ depends on the previously picked edges. To signify this, for any set of edges $e^1, \ldots, e^{i-1}$, we denote the set of
\((n - 2i + 2)^{2-\alpha}\) shortest edges of \(X \setminus \text{endpoints}(e^1, \ldots, e^{i-1})\) as \(\mathcal{E}_i(e^1, \ldots, e^{i-1})\). We say that a vector of edges \((e^1, \ldots, e^i)\) is feasible if \(e^1 \in \mathcal{E}_1, e^2 \in \mathcal{E}_2(e^1), \ldots, e^i \in \mathcal{E}_i(e^1, \ldots, e^{i-1})\). For brevity, we write \(e^i = (e^1, \ldots, e^i)\) for a vector of edges with the agreement \(e^0 = \emptyset\) and \(e_i = (e_1, \ldots, e_i)\) for the vector of random variables from \textsc{RelaxedMWU}. Observe that

\[
P[T = i + 1] = \sum_{e^i \text{ feasible}} P[E \cap \mathcal{E}_i \neq \emptyset, E \cap \mathcal{E}_{i+1} = \emptyset] = \sum_{e^i \text{ feasible}} P[E \cap \mathcal{E}_{i+1}(e^i) = \emptyset, E \cap \mathcal{E}_j(e^{j-1}) \neq \emptyset \quad \forall j \in [1, i] \mid e_i = e^i] \cdot P[e_i = e^i]
\]

\[(14)\]

Note that \(\mathcal{E}_{i+1}(e^1, \ldots, e^i)\) is a fixed, non-random set. Using Bayes' theorem we can express the conditional probabilities in the right hand side of Equation (14) as

\[
P[E \cap \mathcal{E}_{i+1}(e^i) = \emptyset \mid e_i = e^i] = \frac{P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] \cdot P[E \cap \mathcal{E}_{i+1}(e^i) = \emptyset]}{P[e_i = e^i]}.
\]

Substituting this back to Equation (14), we get

\[
P[T = i + 1] = \sum_{e^i \text{ feasible}} P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] \cdot P[E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] = \sum_{e^i \text{ feasible}} P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] \cdot (1 - p)^{|\mathcal{E}_{i+1}(e^i)|} = \sum_{e^i \text{ feasible}} P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] \cdot (1 - p)(n - 2i)^{2-\alpha}.
\]

as we have \(|\mathcal{E}_{i+1}(e^i)| = (n - 2i)^{2-\alpha}\) for any \(e^i = (e^1, \ldots, e^i)\). We proceed by bounding the probability \(P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset]\). Observe that

\[
P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] = P[e_i = e^i \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset, e_{i-1} = e^{i-1}] \cdot P[e_{i-1} = e^{i-1} \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset]
\]

\[= \prod_{j=2}^i P[e_j = e^j \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset, e_{j-1} = e^{j-1}] \cdot P[e_1 = e^1 \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset]
\]

Recall that \(e_1\) was picked uniformly at random from \(\mathcal{E}_1 \cap E\), where \(\mathcal{E}_1\) is a fixed set such that \(e^1 \in \mathcal{E}_1\) for any feasible \(e^1\), and that \(E\) is a uniform random sample. This implies the following

\[
P[e_1 = e^1 \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset] = \sum_{S' \subseteq \mathcal{E}_1} P[e_1 = e^1 \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset, E \cap \mathcal{E}_1 = S'] \cdot P[E \cap \mathcal{E}_1 = S' \mid E \cap \mathcal{E}_{i+1}(e^i) = \emptyset]
\]

\[= \sum_{S' \subseteq \mathcal{E}_1 : |E \setminus \mathcal{E}_{i+1}(e^i)| = |S'|} \frac{1}{|S'|} \cdot p^{|S'|} \cdot (1 - p)^{|E \setminus \mathcal{E}_{i+1}(e^i)| - |S'|}
\]

\[= \sum_{\ell=1}^{(|E_1 \setminus \mathcal{E}_{i+1}(e^i))} \left(\frac{|E_1 \setminus \mathcal{E}_{i+1}(e^i)|}{\ell} - 1\right) \cdot \frac{1}{\ell} \cdot p^\ell \cdot (1 - p)^{|E_1 \setminus \mathcal{E}_{i+1}(e^i)| - \ell}
\]

\[= \sum_{\ell=1}^{(|E_1 \setminus \mathcal{E}_{i+1}(e^i))} \frac{1}{|E_1 \setminus \mathcal{E}_{i+1}(e^i)|} \left(\frac{|E_1 \setminus \mathcal{E}_{i+1}(e^i)|}{\ell} - 1\right) \cdot p^\ell \cdot (1 - p)^{|E_1 \setminus \mathcal{E}_{i+1}(e^i)| - \ell}
\]

\[= \frac{1}{|E_1 \setminus \mathcal{E}_{i+1}(e^i)|} (p + (1 - p)\varepsilon_{E_1 \setminus \mathcal{E}_{i+1}(e^i)}) - (1 - p)\varepsilon_{E_1 \setminus \mathcal{E}_{i+1}(e^i)}
\]
Putting everything together and using the notation

The last step is to bound the probabilities \( \mathbb{P} \left[ e_j = e^j \mid E \cap \mathcal{E}_{i+1}(e^j) = \emptyset, \ e_{j-1} = e^{j-1} \right] \) for \( j \geq 2 \). Note that, given the realization \( e_{j-1} = e^{j-1} \), the set \( \mathcal{E}_j(e^{j-1}) \) is not random, and thus we have a similar relation as before

\[
\begin{align*}
\mathbb{P} \left[ e_j = e^j \mid E \cap \mathcal{E}_{i+1}(e^j) = \emptyset, \ e_{j-1} = e^{j-1} \right] &= \mathbb{P} \left[ e_j = e^j \mid E \cap \mathcal{E}_{i+1}(e^j) = \emptyset \right] \\
&= \sum_{e^j \in S' \subseteq \mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)} \frac{1}{|S'|} \cdot p_j |S'| \cdot (1 - p) |\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| - |S'| \\
&= \sum_{\ell=1}^{i+1} \left( |\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| - 1 \right) \cdot \frac{1}{\ell} \cdot p^\ell \cdot (1 - p) |\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| - \ell \\
&= \frac{1}{|\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)|} \left( 1 - (1 - p) |\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| \right).
\end{align*}
\]

Recall that \( |\mathcal{E}_i(e^{i-1})| = (n - 2i + 2)^{2-\alpha} \) for all \( i \in [1, T] \). Thus for each \( 1 \leq j \leq i \) we have

\[(n - 2(j - 1))^{2-\alpha} - (n - 2i)^{2-\alpha} \leq |\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| \leq (n - 2(j - 1))^{2-\alpha} - (n - 2i)^{2-\alpha}\]

and so the probability \( \mathbb{P} \left[ e_j = e^j \mid E \cap \mathcal{E}_{i+1}(e^j) = \emptyset, \ e_{j-1} = e^{j-1} \right] \) is maximized if

\[|\mathcal{E}_j(e^{j-1}) \setminus \mathcal{E}_{i+1}(e^j)| = (n - 2(j - 1))^{2-\alpha} - (n - 2i)^{2-\alpha}\]

Putting everything together and using the notation \( k_i = |\mathcal{E}_i(e^{i-1})| = (n - 2(i - 1))^{2-\alpha} \), we get

\[
\begin{align*}
\mathbb{P} \left[ T = i + 1 \right] &\leq (1 - p)^{k_{i+1}} \cdot \sum_{e^j \text{ feasible}} \prod_{j=1}^{i} \frac{1}{k_j - k_{i+1}} \left( 1 - (1 - p)^{k_j - k_{i+1}} \right) \\
&= (1 - p)^{k_{i+1}} \cdot k_1 \cdot k_2 \cdots k_i \cdot \prod_{j=1}^{i} \frac{1}{k_j - k_{i+1}} \left( 1 - (1 - p)^{k_j - k_{i+1}} \right) \\
&= (1 - p)^{k_{i+1}} \cdot \prod_{j=1}^{i} \frac{k_j}{k_j - k_{i+1}} \left( 1 - (1 - p)^{k_j - k_{i+1}} \right)
\end{align*}
\]

For any \( i \geq 1 \), we conclude the following bound on the probability of \( T \leq i + 1 \)

\[
\mathbb{P} \left[ T \leq i + 1 \right] = \sum_{j=1}^{i+1} \mathbb{P} \left[ T = j \right] \leq (1 - p)^{k_i} + \sum_{i=1}^{i+1} (1 - p)^{k_{i+1}} \cdot \prod_{j=1}^{\ell} \frac{k_j}{k_j - k_{\ell+1}} \left( 1 - (1 - p)^{k_j - k_{\ell+1}} \right)
\]

Using the bounds \( k_1 \geq k_2 \geq \cdots \geq k_{n/4} \geq k_{n/4} \) and \( k_j - k_{j+1} \geq 1 \) for all \( 1 \leq j \leq i \), we can bound the probability of \( T \leq n/4 \) as

\[
\mathbb{P} \left[ T \leq n/4 \right] \leq (1 - p)^{k_{n/4}} + (1 - p)^{k_{n/4}} \cdot \sum_{\ell=1}^{n/4} \prod_{j=1}^{\ell} k_j \cdot p \leq (1 - p)^{k_{n/4}} \cdot (1 - p)^{k_{n/4}} \cdot (k_1 \cdot p)^\ell
\]

\[
= (1 - p)^{k_{n/4}} \cdot \frac{1 - (pk_1)^{n/4 + 1}}{1 - pk_1} \leq 2(1 - p)^{k_{n/4}} \cdot (pk_1)^{n/4} \leq 2 \exp(-pk_{n/4}) \cdot k_1^{n/4}
\]

Substituting \( k_1 = n^{2-\alpha} \), \( k_{n/4} \geq (n/2)^{2-\alpha} \geq n^{2-\alpha}/4 \) and \( p = \frac{2 \ln n}{n^{2-\alpha} + \frac{4 \ln(2/\delta)}{n^{2-\alpha}}} \), we conclude

\[
\mathbb{P} \left[ T \leq n/4 \right] \leq 2 \exp \left( -\frac{n \ln n}{2} - \ln \frac{2}{\delta} \right) \cdot \left( n^{2-\alpha}/4 \right)^{n/4} = 2 \cdot n^{n/2-\alpha n/4} \cdot \frac{\delta}{n^{n/2}} \leq \delta
\]

Therefore, with probability at least \( 1 - \delta \), \textsc{RelaxedMWU} returns a matching of size \( n/4 \). This, together with Equation (13) implies the first part (upper bound) of Theorem 6.
Lower bound construction. The example is a geometric set system induced by half-spaces on a subset of the integer grid, more precisely, let $X$ be the set of $n = d \cdot \left\lceil \frac{n^{1/d}}{16p(n)} \right\rceil^d$ points defined as $\times_{i=1}^d \left[ 1, \left\lceil \frac{n^{1/d}}{16p(n)} \right\rceil \right] \subset \mathbb{Z}^d$ and let $\mathcal{S}$ consist of the $d \cdot \left\lceil \frac{n^{1/d}}{16p(n)} \right\rceil$ subsets of $X$ induced by half-spaces of the form

$$\{ x_i \leq j + 1/2 \mid i = 1, \ldots, d, j = 1, \ldots, \left\lceil \frac{n^{1/d}}{16p(n)} \right\rceil \}.$$ 

Observe that for any edge $\{x, y\} \in \left( \frac{X}{2} \right)$, the number of ranges in $\mathcal{S}$ that crosses $\{x, y\}$ is precisely the $\ell_1$-distance of $x$ and $y$, which is defined as

$$\ell_1(x, y) = \sum_{i=1}^d |x_i - y_i|.$$ 

Using this observation, it is easy to see that for any fixed $k$, the number of edges crossed by at most $k$ sets from $\mathcal{S}$ is at most $nk^d$. We refer to these edges as $k$-good and denote their set with $\mathcal{G}_k$. Let $p(n) = o(n^{a-1})$ be a function and define $k_p(n) = \left( \frac{1}{16p(n)} \right)^{1/d}$. The expected number of $k_p(n)$-good edges in $E$ is

$$\mathbb{E} \left[ |E \cap \mathcal{G}_{k_p(n)}| \right] \leq n \cdot (k_p(n))^d \cdot p(n) = \frac{n}{16}.$$ 

Thus, by Markov’s inequality, we have $|E \cap \mathcal{G}_{k_p(n)}| \leq \frac{n}{8}$ with probability at least $1/2$. Assume that $|E \cap \mathcal{G}_{k_p(n)}| \leq \frac{n}{8}$ holds and let $M \subset E$ be any subset of size $n/4$. Then $M$ contains at least $n/8$ edges which are not $k_p(n)$-good. Therefore, the number of crossings between the edges of $M$ and the sets of $\mathcal{S}$ is at least

$$\frac{n}{8} \cdot \left( \frac{1}{16p(n)} \right)^{1/d}.$$ 

Recall that $|\mathcal{S}| = d \cdot \left\lceil \frac{n^{1/d}}{16p(n)} \right\rceil \leq dn^{1/d}$ and so by the pigeonhole principle, we get that there is a range in $\mathcal{S}$ that crosses at least

$$\frac{n}{8} \cdot \left( \frac{1}{16p(n)} \right)^{1/d} \geq \frac{n}{8} \cdot \frac{1}{dn^{1/d}} \cdot \left( \frac{1}{p(n)} \right)^{1/d} = \omega \left( n^{1-\alpha/d} \right)$$

edges of $M$, which concludes the proof of Theorem 6 and thus completes the proof of Theorem 5. 

\[ \square \]

4 Geometric Set Systems

In this section, we apply our algorithms for set systems induced by geometric objects. We will show that Theorem 15 implies improved constructions of low-discrepancy colorings in several geometric set systems, see Table 1. Formally, given a set $X$ of $n$ points and a collection $\mathcal{C}$ of geometric objects in $\mathbb{R}^d$, we say that a set $Y \subset X$ is induced by $\mathcal{C}$ if $Y = X \cap C$ for some $C \in \mathcal{C}$. We say that a set system $(X, \mathcal{S})$ is induced by $\mathcal{C}$ if each range in $\mathcal{S}$ is induced by $\mathcal{C}$.

4.1 Semialgebraic set systems.

Let $\Gamma_{d,\Delta,s}$ denote the collection of semialgebraic sets in $\mathbb{R}^d$ that can be defined as the solution set of a Boolean combination of at most $s$ polynomial inequalities of degree at most $\Delta$. First, we give a bound on the VC-dimension and dual shatter function of set systems induced by $\Gamma_{d,\Delta,s}$.
Table 1: Summary of guarantees for geometric set systems. We use the notation $H_d$ for half-spaces in $\mathbb{R}^d$, $B_d$ for balls in $\mathbb{R}^d$, and $\Gamma_{d,\Delta,s}$ for semialgebraic sets in $\mathbb{R}^d$ described by at most $s$ equations of degree at most $\Delta$.

**Lemma 23.** Let $X$ be a set of points in $\mathbb{R}^d$ and $(X, S)$ be a set system induced by $\Gamma_{d,\Delta,s}$. Then $\text{VC-dim}(X, S) \leq 2s \log(\epsilon s) (\Delta + d)$ and the dual shatter function of $(X, S)$ can be upper-bounded as $\pi^*_d(k) \leq (4\epsilon \Delta s)^d \cdot k^d$.

**Proof.** The bound on the VC-dimension can be deduced from Propositions 10.3.2 and Proposition 10.3.3 in Matoušek [2013]. To bound $\pi^*_d(k)$, let $R \subseteq \Gamma_{d,\Delta,s}$ be a set of $k$ ranges with defining polynomials $\mathcal{P} = \{p_{ij} : 1 \leq i \leq k, 1 \leq j \leq s\}$, where each $p_{ij}$ is a $d$-variate polynomial of degree at most $\Delta$. Observe that if $\text{sign}[p(x)] = \text{sign}[p(y)]$ for all $p \in \mathcal{P}$, then $x, y$ are equivalent with respect to $R$. Therefore, $\pi^*_{\Gamma_{d,\Delta,s}}(k)$ can be upper-bounded by the number of different sign patterns in $\{-1, 1\}^k$ induced by $k s$ $d$-variate polynomials of degree at most $\Delta$. This quantity is bounded by $(4\epsilon \Delta s)^d \cdot k^d$, see [Warren, 1968, Theorem 3].

By Lemma 23, we get that set systems induced by $\Gamma_{d,\Delta,s}$ satisfy **Main Assumption** $(a, b, \gamma)$ with parameters $a = \frac{4\epsilon \Delta s}{\ln 2(1-1/d)}$, $b = \frac{\ln |S|}{\ln 2}$, and $\gamma = 1 - 1/d$. Furthermore, any set system induced by $\Gamma_{d,\Delta,s}$ has a membership Oracle of time complexity $O\left(s(\frac{\Delta + d}{d})\right)$. Thus, we can apply Theorems 15, 2, and Corollary 19 and obtain the following.

**Corollary 24.** Let $X$ be a set of $n$ points in $\mathbb{R}^d$ and $(X, S)$ be a set system with $m$ ranges induced by $\Gamma_{d,\Delta,s}$. Then

i) **LowDiscColor** $((X, S), \frac{4\epsilon \Delta s}{\ln 2(1-1/d)}, \frac{\ln m}{\ln 2}, 1 - \frac{1}{d})$ constructs a coloring $\chi$ of $X$ of with expected discrepancy at most

$$3\sqrt{\frac{4\epsilon \Delta s \ln m}{\ln 2(1-1/d)^2} \cdot n^{1-1/d} + 19 \ln^2 m \ln n}$$

in expected time $O\left(s(\frac{\Delta + d}{d}) (mn^{1/d} \ln(mn) \ln n + n^{2+1/d} \ln n)\right)$.

ii) **BuildMatching** $((X, S), \frac{4\epsilon \Delta s}{\ln 2(1-1/d)}, \frac{\ln m}{\ln 2}, 1 - \frac{1}{d})$ returns a perfect matching of $X$ with expected crossing number at most

$$\frac{12\epsilon s \Delta}{(1-1/d)^2 \ln 2} \cdot n^{1-1/d} + O\left(\ln m \ln n\right)$$

in expected time $O\left(s(\frac{\Delta + d}{d}) (mn^{1/d} \ln(mn) \ln n + n^{2+1/d} \ln n)\right)$.

iii) if $\epsilon \in (0, 1)$, $d_{\text{VC}} := \text{VC-dim}(X, S)$, and $A_0$ is a uniform random sample of $X$ of size $\frac{4\epsilon \Delta s \cdot d_{\text{VC}}}{\ln 2(1-1/d)}$, then **Approximate** $((A_0, S|_{A_0}), \frac{4\epsilon \Delta s}{\ln 2(1-1/d)}, \frac{\ln |S|_{A_0}}{\ln 2}, 1 - \frac{1}{d}, \epsilon/2)$ returns a set
A ⊂ X of size

\[ O \left( \max \left\{ \left( \frac{\Delta s \cdot \text{VC}}{\varepsilon^2} \ln \frac{1}{\varepsilon}, \frac{\text{VC}}{\varepsilon} \ln^{3/2} \left( \frac{\text{VC}}{\varepsilon} \right) \right) \right\} \right) \]

with expected approximation guarantee satisfying \( \mathbb{E}[\varepsilon(A, X, S)] \leq \varepsilon \), and in expected time

\[ O \left( n + s \left( \frac{\Delta + d}{d} \right)^2 \ln \frac{\text{VC}}{\varepsilon} + \left( \frac{\text{VC}}{\varepsilon} \right)^{d+1} \ln \left( \frac{\text{VC}}{\varepsilon} \right)^{d+1} \ln^2 \frac{\text{VC}}{\varepsilon} \right) \).

**Remark.** The previous best algorithm for constructing matchings with low crossing numbers with respect to \( \Gamma_{d, \Delta, s} \) relies on the polynomial partitioning technique [Agarwal et al., 2013]. It computes a perfect matching of \( A \) in Corollary 24 for \( \alpha \). There exists a set \( H \) in Lemma 25 such that if a matching has low crossing number with respect to this family (of half-spaces and balls respectively): (of half-spaces and balls) such that if a matching has low crossing number with respect to this family, then it is guaranteed to have low crossing number with respect to any member of the family (of half-spaces and balls respectively):

**Lemma 25** (Test-set lemma [Matoušek, 1992]). Let \( X \) be a set of \( n \) points in \( \mathbb{R}^d \) and \( t \) be a parameter. There exists a set \( \mathcal{T}(t) \) of at most \( (d+1)^t \) hyperplanes such that if a perfect matching of \( X \) has crossing number \( \kappa \) with respect to \( \mathcal{T}(t) \), then its crossing number with respect to any half-space in \( \mathbb{R}^d \) is at most \( (d+1)\kappa + \frac{6d^2n}{t} \).

We will use Lemma 25 as black-box to obtain a test-set lemma for balls. It is well known that there are mappings \( \alpha : \mathbb{R}^d \to \mathbb{R}^{d+1} \) and \( \beta : B_d \to \mathcal{H}_{d+1} \) such that for any \( p \in \mathbb{R}^d \) and \( B \in B_d \), we have \( p \in B \) if and only if \( \alpha(p) \in \beta(B) \), see e.g. [Matoušek, 2013, Chap. 10]. This mapping and Lemma 25 applied in \( \mathbb{R}^{d+1} \) with \( t = n^{1/d} \) give the following lemma.

**Lemma 26.** Let \( X \) be a set of \( n \) points in \( \mathbb{R}^d \). There exists a set \( \mathcal{Q} \) of at most \( (d+2)^n n^{1+1/d} \) balls such that if a perfect matching of \( X \) has crossing number \( \kappa \) with respect to \( \mathcal{Q} \), then its crossing number with respect to any ball in \( \mathbb{R}^d \) is at most \( (d+2)\kappa + 6(d+1)^2 n^{1-1/d} \).

In contrast to previous setups (where we required to have a finite set of \( m \) ranges as an input), test-sets allow us to efficiently construct matchings with low crossing number with respect to any half-space or ball in \( \mathbb{R}^d \). For half-spaces, we have a membership Oracle of time complexity \( O(d) \), thus Corollary 24 and Lemma 25 imply the following.

**Corollary 27.** Let \( X \) be a set of \( n \) points in \( \mathbb{R}^d \) and \( \mathcal{T} = \mathcal{T}(n^{1/d}) \) be the set of half-spaces provided by Lemma 25. Then \( \text{BuildMatching}((X, \mathcal{T}), \frac{4e}{(1-1/d)\ln 2}, \frac{\ln((d+1)n)}{\ln 2}, 1 - \frac{1}{d}) \) returns a perfect matching of \( X \) with expected crossing number at most

\[ \left( 6d^2 + \frac{12e(d+1)}{(1-1/d)^2\ln 2} \right) n^{1-1/d} + O(\ln(dn) \ln n) \]

with respect to half-spaces in \( \mathbb{R}^d \), in expected time \( O(dn^{2+1/d} \ln n) \).

Similarly, we can apply Corollary 24 and Lemma 26 to set systems induced by balls. Note that in case of balls, the Oracle complexity can be improved to \( O(d) \) from the \( O(d^2) \) bound used in Corollary 24 for \( \Gamma_{d,2,1} \).
Corollary 28. Let \( X \) be a set of \( n \) points in \( \mathbb{R}^d \) and let \( \mathcal{Q} \) be the set of balls provided by Lemma 26. Then \( \text{BuildMatching}((X, \mathcal{Q}), \frac{8e}{1-1/d} \ln 2, \frac{\ln((d+1)n^{1/d})}{\ln 2}, 1 - \frac{1}{d}) \) returns a matching \( \{e_1, \ldots, e_{n/2}\} \) with expected crossing number at most

\[
\left(6(d+1)^2 + \frac{24e(d+2)}{(1-1/d)^2 \ln 2}\right) n^{1-1/d} + O(\ln(dn) \ln n)
\]

with respect to balls in \( \mathbb{R}^d \), in expected time \( O\left(dn^{2+1/d} \ln n\right) \).

Remark. The previous-best algorithm to construct matchings with crossing number \( O(n^{1-1/d}) \) with respect to \( \mathcal{B}_d \) had time complexity \( \tilde{O}(mn^3) \) [Matoušek et al. 1993], which combined with Lemma 26 yields an \( \tilde{O}(n^{1+1/d}) \) time algorithm. Alternatively, one can obtain a matching with sub-optimal crossing number \( O\left(n^{1-1/(d+1)}\right) \) by lifting \( X \) into \( \mathbb{R}^{d+1} \), where the image of each range in \( \mathcal{B}_d \) can be represented by a range in \( \mathcal{H}_{d+1} \) and applying the algorithm of Chan [2012] with time complexity \( \tilde{O}(n) \).

We can show that test-sets can also be used as an input the algorithms \text{LowDiscColor} and \text{Approximate} using Lemma 16:

Lemma 16. Let \((X, \mathcal{S})\) be a set system, \( n = |X|, m = |\mathcal{S}| \geq 34 \), and let \( M \) be a perfect matching of \( X \) with crossing number \( \kappa \) with respect to \( \mathcal{S} \) and for each edge \( \{x, y\} \in M \) define

\[
\chi_M(x) = \begin{cases} 1 & \text{with probability } 1/2 \\
-1 & \text{with probability } 1/2
\end{cases}
\]

and \( \chi_M(y) = -\chi_M(x) \). Then the expected discrepancy of \( \chi_M \) is at most \( \sqrt{3\kappa \ln m} \).

Notice that the algorithm \text{LowDiscColor} creates a coloring from the output of \text{BuildMatching} precisely as it is defined in Lemma 16. Therefore, a matching \( M \) returned by \text{BuildMatching} on a test-set (with low expected crossing number with respect to \( \mathcal{H}_d \)) can be used to construct a coloring \( \chi_M \) with low expected discrepancy with respect to \( \mathcal{H}_d \). Similarly, \( \chi_M \) can be used to construct a small-sized \( \varepsilon \)-approximation of \((X, \mathcal{H}_d)\). These observations lead to the last two corollaries of this section.

Corollary 29. Let \( X \) be a set of \( n \) points in \( \mathbb{R}^d \) and \( T = T(n^{1/d}) \) be the set of \((d+1)n\) half-spaces provided by Lemma 25. Then

i) \text{LowDiscColor}((X, T), \frac{4e}{\ln(2(1-1/d))}, \frac{\ln((d+1)n)}{\ln 2}, 1 - \frac{1}{d}) \) constructs a coloring \( \chi \) of \( X \) of with expected discrepancy at most

\[
3\sqrt{\left(6d^2 + \frac{12e(d+1)}{(1-1/d)^2 \ln 2}\right) n^{1-1/d} \ln m + O(\ln(dn) \ln n \ln m)}
\]

with respect to half-spaces in \( \mathbb{R}^d \), in expected time \( O\left(dn^{2+1/d} \ln n\right) \).

ii) if \( \varepsilon \in (0,1) \) and \( A_0 \) is a uniform random sample of \( X \) of size \( \frac{4C_{\text{opt}}(d+1)}{\varepsilon^2} \), then \text{Approximate}((A_0, T|_{A_0}), \frac{4e}{\ln(2(1-1/d))}, \frac{\ln(|T|_{A_0})}{\ln 2}, 1 - \frac{1}{d}, \varepsilon) \) returns a set \( A \subset X \) of size

\[
O\left(\max\left\{ \left(\frac{d}{\varepsilon^2} \ln \frac{1}{\varepsilon}\right)^{d+1}, \frac{d}{\varepsilon} \ln^{3/2} \left(\frac{d}{\varepsilon}\right)\right\}\right)
\]

with expected approximation guarantee satisfying \( \mathbb{E}[\varepsilon(A, X, \mathcal{H}_d)] \leq \varepsilon \), and in expected time \( O\left(n + d \left(\frac{d}{\varepsilon^2}\right)^{2+1/d} \ln \frac{d}{\varepsilon}\right) \).

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Corollary 30. Let $X$ be a set of $n$ points in $\mathbb{R}^d$ and let $Q$ be the set of balls provided by Lemma 26. Then

i) \textbf{LowDiscColor}\left((X, Q), \frac{8e}{\ln 2(1-1/d)}, \frac{\ln((d+2)n^{1+1/d})}{\ln 2}, 1 - \frac{1}{d}\right) constructs a coloring $\chi$ of $X$ of with expected discrepancy at most

$$3\sqrt{\left(6(d+1)^2 + \frac{2\epsilon(d+2)}{(1-1/d)^2}\ln 2\right) n^{1-1/d} \ln m + O\left(\ln (dn) \ln n \ln \ln m\right)}$$

with respect to balls in $\mathbb{R}^d$, in expected time $O\left( dn^{2+1/d} \ln n \right)$.

ii) if $\epsilon \in (0, 1)$, $A_0$ is a uniform random sample of $X$ of size $\frac{4\text{Cap}(d+2)}{\epsilon^2}$, then the algorithm \textbf{Approximate}\left((A_0, Q|_{A_0}), \frac{8e}{\ln 2(1-1/d)}, \frac{\ln |Q|_{A_0}}{\ln 2}, 1 - \frac{1}{d}, \epsilon\right) returns a set $A \subset X$ of size at most

$$O\left(\max\left\{\frac{d}{\epsilon^2} \ln \frac{1}{\epsilon}, \frac{d}{\epsilon} \ln 3/2 \left(\frac{d}{\epsilon}\right)\right\}\right)$$

with expected approximation guarantee satisfying $\mathbb{E}[\epsilon(A, X, B_d)] \leq \epsilon$, and in expected time $O\left( n + d \left(\frac{d}{\epsilon^2}\right)^{2+1/d} \ln \frac{d}{\epsilon}\right)$.

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5 Appendix

5.1 Proof of Lemma 16

Let $S \in \mathcal{S}$ be a fixed range. We express the sum $\chi_M(S)$ of colors over elements of $S$ as

$$\chi_M(S) = \sum_{\{x,y\} \in M \cap \{x,y\} \in S} (\chi_M(x) + \chi_M(y)), \quad \chi_M(x) = \sum_{x \in \text{cr}(S,M)} \chi_M(x),$$

where $\text{cr}(S,M) = \{x \in S : \{x,y\} \in M, y \not\in S\}$. Since $\text{cr}(S,M) \leq \kappa$ for any $S \in \mathcal{S}$, $\text{disc}(S,\chi_M)$ is a sum of at most $\kappa$ independent random variables. We use the following concentration bound from Alon and Spencer [2016]

**Claim 3** (Theorem A.1.1 from Alon and Spencer [2016]). Let $X_1, \ldots, X_k$ be independent $\{-1,1\}$-valued random variables with $P[X_i = -1] = P[X_i = 1] = 1/2$. Then for any $\alpha \geq 0$

$$P\left[ \left| \sum_{i=1}^k X_i \right| > \alpha \right] \leq 2e^{-\alpha^2/2k}.$$ 

Applying Claim 3, we get that for any fixed $S \in \mathcal{S}$ and $\alpha > 0$,

$$P\left[ |\chi_M(S)| > \alpha \right] \leq 2e^{-\alpha^2/2\kappa}.$$ 

By the union bound, we get

$$P\left[ \text{disc}_S(\chi_M) > \alpha \right] = P\left[ \max_{S \in \mathcal{S}} |\chi_M(S)| > \alpha \right] \leq m \cdot 2e^{-\alpha^2/2\kappa}.$$ 

Finally, we bound the expected discrepancy by applying Fubini’s theorem

$$E\left[ \text{disc}_S(\chi_M) \right] \overset{\text{def}}{=} \int_0^\infty P\left[ \text{disc}_S(\chi_M) > \alpha \right] d\alpha \leq \int_0^\infty \min \left\{ 2m \cdot e^{-\alpha^2/2\kappa}, 1 \right\} d\alpha$$

$$= \int_0^{\sqrt{2\kappa \ln(2m)}} 1 d\alpha + \int_{\sqrt{2\kappa \ln(2m)}}^\infty 2m \cdot e^{-\alpha^2/2\kappa} d\alpha$$

$$= \sqrt{2\kappa \ln(2m)} + 2m \sqrt{2\kappa} \int_{\sqrt{\ln(2m)}}^\infty e^{-t^2} dt$$

$$= \sqrt{2\kappa \ln(2m)} + 2m \sqrt{2\kappa} \int_{\sqrt{\ln(2m)}}^\infty \frac{t}{2} e^{-t^2} dt$$

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Thus for

Recall that if constructs a sequence of sets \( \mathcal{S} \) iteratively. In particular, it sets \( \mathcal{A}_0 = \mathcal{X} \) and for \( i = 1, \ldots, j \), \( \mathcal{A}_i \subseteq \mathcal{A}_{i-1} \) is defined as \( \chi_{\mathcal{A}_i}^{-1}(1) \), where \( \chi_i : \mathcal{A}_i \to \{-1, 1\} \) is the coloring provided by \( \text{LowDiscColor}(\mathcal{A}_i, \mathcal{S}|\mathcal{A}_i), a, b, \gamma) \). Note that \( |\mathcal{A}_{i+1}| = \lceil |\mathcal{A}_i|/2 \rceil = \lceil n/2^{i+1} \rceil \). We bound the approximation guarantee using the following lemma.

**Lemma 17** ([Matoušek et al., 1993, Lemma 2.1]). Let \( (\mathcal{X}, \mathcal{S}) \) be a set system with \( |\mathcal{X}| = n \), \( \mathcal{X} \in \mathcal{S} \) and let \( \chi \) be a coloring with discrepancy \( \text{disc}(\chi) = \Delta \) and let \( \mathcal{A} \subseteq \mathcal{X} \) be a set of \( \lceil n/2 \rceil \) elements from the larger color class of \( \chi \). Then \( \mathcal{A} \) is a \( (2\Delta/n) \)-approximation of \( (\mathcal{X}, \mathcal{S}) \).

By Theorem 15,

\[
E \left[ \text{disc}_{\mathcal{S}|\mathcal{A}_i}(\chi_i) \right] \leq 3 \sqrt{\frac{a}{\gamma} |\mathcal{A}_i| \ln |\mathcal{S}|_{\mathcal{A}_i} | + \left( \frac{b}{2} + 12 \ln |\mathcal{S}|_{\mathcal{A}_i} \right) \log |\mathcal{A}_i| \cdot \ln |\mathcal{S}|_{\mathcal{A}_i}.
\]

Thus for \( i = 0, \ldots, j - 1 \), by Lemma 17,

\[
E \left[ \varepsilon(\mathcal{A}_{i+1}, \mathcal{A}_i, \mathcal{S}|\mathcal{A}_i) \right] \leq \frac{6}{n^{2\gamma}} \sqrt{\frac{a}{\gamma} \cdot \lceil n/2 \rceil \gamma \ln m + \left( \frac{b}{2} + 12 \ln m \right) \log \lceil n/2 \rceil \ln m. \quad (16)
\]

Recall that if \( \mathcal{A}_1 \) is an \( \varepsilon_1 \)-approximation of \((\mathcal{X}, \mathcal{S})\) and \( \mathcal{A}_2 \) is an \( \varepsilon_2 \)-approximation of \((\mathcal{A}_1, \mathcal{S}|\mathcal{A}_1)\), then \( \mathcal{A}_2 \) is an \((\varepsilon_1 + \varepsilon_2)\)-approximation of \((\mathcal{X}, \mathcal{S})\). Therefore,

\[
\varepsilon(\mathcal{A}_1, \mathcal{X}, \mathcal{S}) \leq \varepsilon(\mathcal{A}_2, \mathcal{A}_1, \mathcal{S}|\mathcal{A}_1) + \varepsilon(\mathcal{A}_1, \mathcal{A}_2, \mathcal{S}|\mathcal{A}_2) + \cdots + \varepsilon(\mathcal{A}_2, \mathcal{A}_1, \mathcal{S}|\mathcal{A}_1) + \varepsilon(\mathcal{A}_1, \mathcal{X}, \mathcal{S}),
\]

which by linearity of expectation and Equation (16) yield

\[
E \left[ \varepsilon(\mathcal{A}_j, \mathcal{X}, \mathcal{S}) \right] \leq \sum_{i=0}^{j-1} \frac{6}{n^{2\gamma}} \sqrt{\frac{a}{\gamma} \cdot \lceil n/2 \rceil \gamma \ln m + \left( \frac{b}{2} + 12 \ln m \right) \log \lceil n/2 \rceil \ln m
\]

\[
\leq \frac{15}{n^{1-\gamma/2}} \sqrt{\frac{a}{\gamma} \cdot \left( \frac{2^j}{n} \right)^{1-\gamma/2} + 6 \frac{b \log n \ln m}{2} + 12 \log n \ln^2 m}
\]

\[
\leq \frac{15}{n^{1-\gamma/2}} \sqrt{\frac{a}{\gamma} \cdot \left( \frac{2^j}{n} \right)^{1-\gamma/2} + 6 \frac{b \log n \ln m}{2} + 12 \log n \ln^2 m}.
\]

Substituting

\[
j = \left\lfloor \log n + \min \left\{ \frac{2 - \gamma}{2} \log \frac{\varepsilon \sqrt{\gamma}}{30 a \ln m}, \log \frac{\varepsilon}{12 \sqrt{\left( \frac{b}{2} + 12 \ln m \right) \ln m \log n}} \right\} \right\rfloor,
\]

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we get that $\mathbb{E}[\epsilon(A_j, X, S)] \leq \epsilon$ and

$$|A_j| = \frac{n}{2^j} \leq 2 \max\left\{ \left( \frac{30\sqrt{a \ln m}}{\epsilon \sqrt{\gamma}} \right)^{\frac{2}{\gamma}}, \frac{12\sqrt{(\frac{b}{2} + 12 \ln m) \ln(m) \log n}}{\epsilon} \right\}.$$  

By Theorem 15, constructing a the coloring $\chi_i$ requires at most

$$\min \left\{ \frac{24|A_i|^{3-\gamma} \ln |A_i|}{a} + \frac{18m|A_i|^{1-\gamma} \ln (m|A_i|)}{a} \min \left\{ \frac{2}{1 - \gamma}, \log |A_i| \right\}, \frac{1}{7}|A_i|^3 + \frac{m|A_i|}{2} \right\}$$

calls to the membership Oracle, in expectation. Since $|A_i| = \left\lceil \frac{n}{2^j} \right\rceil$, the expected number of membership Oracle calls that $\text{APPROXIMATE}((X, S), a, b, \gamma, j)$ performs is at most

$$\sum_{i=0}^{j} \min \left\{ \frac{24 \left( \frac{n}{2^i} \right)^{3-\gamma} \ln \frac{n}{2^i}}{a} + \frac{18m \left( \frac{n}{2^i} \right)^{1-\gamma} \ln \frac{mn}{2^i}}{a} \min \left\{ \frac{2}{1 - \gamma}, \log \frac{n}{2^i} \right\}, \frac{1}{7} \left( \frac{n}{2^i} \right)^3 + \frac{mn}{2^{i+1}} \right\} \leq \min \left\{ \sum_{i=0}^{j} \left( \frac{24 \left( \frac{n}{2^i} \right)^{3-\gamma} \ln \frac{n}{2^i}}{a} + \frac{18m \left( \frac{n}{2^i} \right)^{1-\gamma} \ln \frac{mn}{2^i}}{a} \min \left\{ \frac{2}{1 - \gamma}, \log \frac{n}{2^i} \right\} \right), \sum_{i=0}^{j} \left( \frac{n^3}{7 \cdot 2^{3i}} + \frac{mn}{2^{i+1}} \right) \right\} \leq \min \left\{ \frac{32n^{3-\gamma} \ln n}{a} + \frac{18mn^{1-\gamma} \ln(mn)}{a} \left( \min \left\{ \frac{2}{1 - \gamma}, \log n \right\} \right)^2, \frac{8n^3}{49} + mn \right\}.$$  

This concludes the proof of Corollary 19. \(\square\)