INTERSECTION PAIRINGS ON SINGULAR MODULI SPACES OF BUNDLES OVER A RIEMANN SURFACE AND THEIR PARTIAL DESINGULARISATIONS

L.C. JEFFREY, Y.-H. KIEM, F.C. KIRWAN, J. WOOLF

Abstract. This paper studies intersection theory on the compactified moduli space \( \mathcal{M}(n, d) \) of holomorphic bundles of rank \( n \) and degree \( d \) over a fixed compact Riemann surface \( \Sigma \) of genus \( g \geq 2 \) where \( n \) and \( d \) may have common factors. Because of the presence of singularities we work with the intersection cohomology groups \( IH^*(\mathcal{M}(n, d)) \) defined by Goresky and MacPherson and the ordinary cohomology groups of a certain partial resolution of singularities \( \tilde{\mathcal{M}}(n, d) \) of \( \mathcal{M}(n, d) \). Based on our earlier work \[25\], we give a precise formula for the intersection cohomology pairings and provide a method to calculate pairings on \( \tilde{\mathcal{M}}(n, d) \).

The case when \( n = 2 \) is discussed in detail. Finally Witten’s integral is considered for this singular case.

1. Introduction

This paper studies intersection theory on the compactified moduli spaces \( \mathcal{M}(n, d) \) and \( \mathcal{M}_\Lambda(n, d) \) of holomorphic bundles of rank \( n \) and degree \( d \) over a fixed compact Riemann surface \( \Sigma \) of genus \( g \geq 2 \) (and with fixed determinant line bundle \( \Lambda \) in the case of \( \mathcal{M}_\Lambda(n, d) \)), and their partial desingularisations \( \tilde{\mathcal{M}}(n, d) \) and \( \tilde{\mathcal{M}}_\Lambda(n, d) \) in the sense of \[37\] \[40\]. Here \( n \) and \( d \) may have common factors so that \( \mathcal{M}(n, d) \) and \( \mathcal{M}_\Lambda(n, d) \) may be singular; when \( n \) and \( d \) are coprime then \( \tilde{\mathcal{M}}(n, d) = \mathcal{M}(n, d) \) and \( \tilde{\mathcal{M}}_\Lambda(n, d) = \mathcal{M}_\Lambda(n, d) \) and the results of this paper have already been obtained in \[29\]. Indeed, intersection theory on these moduli spaces when \( n \) and \( d \) are coprime has been studied intensively for several decades \[3\] \[5\] \[6\] \[7\] \[13\] \[21\] \[22\] \[29\] \[31\] \[33\] \[34\] \[57\] \[54\] \[61\] \[64\] \[69\]; more recently work has also been done on the singular moduli spaces, in particular \( \mathcal{M}(2, d) \) when \( d \) is even \[32\] \[33\] \[34\]. Our aim here is to extend the results of \[29\] on intersection pairings in the cohomology of \( \mathcal{M}_\Lambda(n, d) \) to the case when \( n \) and \( d \) are not coprime, by using the methods of \[25\]. Because of the presence of singularities we work with the intersection cohomology groups \( IH^*(\mathcal{M}(n, d)) \) and \( IH^*(\mathcal{M}_\Lambda(n, d)) \) defined by Goresky and MacPherson \[16\] \[17\] and the ordinary cohomology groups of the partial resolutions of singularities \( \tilde{\mathcal{M}}(n, d) \) and \( \tilde{\mathcal{M}}_\Lambda(n, d) \) of \( \mathcal{M}(n, d) \) and \( \mathcal{M}_\Lambda(n, d) \) \[37\] \[40\]. The formulas we obtain for the intersection pairings in \( IH^*(\mathcal{M}(n, d)) \) and \( IH^*(\mathcal{M}_\Lambda(n, d)) \) (see Theorem \[35\]) follow easily from the results of \[29\] \[32\] \[34\] and are essentially the same as in the coprime case studied in \[29\]. As was shown in \[32\] these pairings can be regarded as providing some of the intersection pairings in \( H^*(\mathcal{M}(n, d)) \) and \( H^*(\mathcal{M}_\Lambda(n, d)) \). The complete picture

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1The moduli spaces \( \mathcal{M}(n, d) \) and \( \mathcal{M}_\Lambda(n, d) \) are singular if and only if \( n \) and \( d \) have a common factor, except in one special case: when \( g = 2 \) then \( \mathcal{M}(2, d) \) and \( \mathcal{M}_\Lambda(2, d) \) are nonsingular when \( d \) is even as well as when \( d \) is odd.
of the pairings in $H^*(\tilde{M}(n,d))$ and $H^*(\tilde{M}_\Lambda(n,d))$ is much more complicated to describe; we give a method for calculating these (see Theorem 35) and carry out the case when $n=2$ in detail (see Theorem 42).

The original motivation for both [29] and this article (as well as much other work) was [64], where Witten studied the moduli spaces $\mathcal{M}(n,d)$ as symplectic reductions of infinite dimensional affine spaces by infinite dimensional groups following [3]. He found formulas for intersection pairings on these moduli spaces for coprime $n$ and $d$ from asymptotic expansions of certain infinite dimensional integrals as a parameter $\epsilon$ tends to 0. He did this by showing that each integral is a sum of terms tending to 0 exponentially fast with $\epsilon$, together with a polynomial in $\epsilon$ whose coefficients are intersection pairings on the moduli space. Witten’s formulas were later proved using finite-dimensional methods in [29] for $\mathcal{M}(n,d)$ and $\mathcal{M}_\Lambda(n,d)$ with coprime $n$ and $d$, and for moduli spaces of principal bundles for more general compact groups in [48]. Witten also gave formulas for the asymptotic expansions of his integrals in the case of bundles of rank two and even degree, and noted that powers of $\epsilon^{1/2}$ appeared. In fact (cf. [25, 56, 65, 4]) when $n$ and $d$ are not coprime Witten’s integrals are always sums of polynomials in $\epsilon^{1/2}$ rather than $\epsilon$, together with terms tending to 0 exponentially fast with $\epsilon$. Some of the coefficients of integral powers of $\epsilon$ in these expressions can be interpreted as intersection pairings; however it is not clear whether there is a geometrical interpretation of the coefficients of the half-integral powers of $\epsilon$.

The layout of this paper is as follows. In §2 we recall the results we shall need from [25] on intersection pairings on geometric invariant theoretic quotients and symplectic reductions. §3 recalls facts about moduli spaces of vector bundles over curves and their partial desingularisations. In §4 we review the finite-dimensional methods used in [29] to rederive Witten’s formulas in the case when $n$ and $d$ are coprime, and in §5 we combine these methods with the results of [25] to calculate the pairings in $IH^*(\mathcal{M}(n,d))$ in Theorem 35. The pairings in the intersection homology $IH^*(\mathcal{M}_\Lambda(n,d))$ of the moduli space of bundles with fixed determinant can be obtained from these calculations. Indeed, the latter part of the paper deals only with $\mathcal{M}(n,d)$ and $\tilde{M}(n,d)$ since computations for $\mathcal{M}_\Lambda(n,d)$ and $\tilde{M}_\Lambda(n,d)$ can be easily derived from these, see Remark 32. §6 begins the much more laborious task of extending these calculations to cover all the pairings in $H^*(\tilde{M}(n,d))$, by expressing such pairings as sums of formulas like those already seen together with certain ‘wall-crossing terms’ given by integrals over symplectic quotients of projective bundles (Theorem 35). §7 completes the calculation of pairings on $\tilde{M}(n,d)$ by explaining how to use the techniques of [25] to compute the wall-crossing terms inductively via integration over the fibres; unfortunately this becomes very cumbersome in practice for large $n$. In §8 we consider the case when $n=2$ in detail; here it is not hard to give explicit formulas for the pairings in $H^*(\tilde{M}(2,d))$ as well as in $IH^*(\mathcal{M}(2,d))$ (see Theorem 42). Finally in §9 we look at Witten’s integrals.

2. Pairings on singular quotients

Let $M//G$ be the quotient in the sense of Mumford’s geometric invariant theory [51] of a nonsingular connected complex projective variety $M$ by a linear action of a connected complex reductive group $G$. In [25] we gave formulas, under certain conditions on the group action, for the pairings of intersection cohomology classes of complementary degrees in the intersection cohomology $IH^*(M//G)$ of $M//G$. (Intersection cohomology is defined with respect to the middle perversity throughout this paper, and all cohomology and homology
groups have complex coefficients). We also gave formulas for intersection pairings on resolutions $\tilde{M} // G$ (or more precisely partial resolutions, since orbifold singularities are allowed) of the quotients $M // G$.

Recall that if every semistable point of $M$ is stable then $0$ is a regular value of the moment map and the stabiliser in the maximal compact subgroup $K$ of $G$ of every point of $\mu^{-1}(0)$ is finite. This implies that the cohomology $H^*(M // G)$ of the quotient $M // G$ is naturally isomorphic to the equivariant cohomology $H^*_K(\mu^{-1}(0)) \cong H^*_K(M^{ss})$ (recall that we are working with cohomology with complex coefficients). The restriction map $H^*_K(M) \to H^*_K(M^{ss})$ is surjective (36 5.4), and so the composition of the restriction map $H^*_K(M) \to H^*_K(M^{ss})$ and the isomorphism $H^*_K(M^{ss}) \to H^*(M // G)$ gives us a natural surjective ring homomorphism

$$\kappa_M : H^*_K(M) \to H^*(M // G).$$

Since $H^*(M // G)$ satisfies Poincaré duality, the kernel of this surjection is then determined by the formula obtained in [26] (see (2.3) below) for pairings of cohomology classes of complementary dimensions in $M // G$ in terms of equivariant cohomology classes in $M$ which represent them.

If there are semistable points of $M$ which are not stable (we assume only that there do exist some stable points, or equivalently that there exist some points in $\mu^{-1}(0)$ where the derivative of $\mu$ is surjective) then there is no longer a natural surjection from $H^*_K(M)$ to $H^*(M // G)$, and $M // G$ is in general singular so its cohomology $H^*(M // G)$ may not satisfy Poincaré duality. However its intersection cohomology groups satisfy Poincaré duality, and there is a surjection from $H^*_K(M)$ to the intersection cohomology $IH^*(M // G)$, which we will call $\kappa_M$ since it coincides with (2.1) when semistability is the same as stability. This surjection $\kappa_M : H^*_K(M) \to IH^*(M // G)$ arises as follows.

We can construct a canonical partial resolution of singularities $\tilde{M} // G$ of the quotient $M // G$ (see [37]), by blowing $M$ up along a sequence of nonsingular $G$-invariant subvarieties, all contained in the complement $M - M^s$ of the set $M^s$ of stable points of $M$. This eventually gives us a nonsingular projective variety $\tilde{M}$ with a linear $G$-action, lifting the action on $M$, for which every semistable point of $\tilde{M}$ is stable. The quotient $\tilde{M} // G$ has only orbifold singularities, and the blowdown map $\pi_G : \tilde{M} // G \to M // G$

which is an isomorphism over the dense open subset $M^s // G$ of $M // G$. The intersection cohomology $IH^*(M // G)$ of $M // G$ is a direct summand of the cohomology of this partial resolution of singularities $\tilde{M} // G$, and the composition

$$\kappa_M : H^*_K(M) \to H^*_K(\tilde{M}) \to H^*(\tilde{M} // G) \to IH^*(M // G)$$

is surjective ([39 67]). In fact this surjection is the composition of the restriction map from $H^*_K(M)$ to $H^*_K(M^{ss})$ and a surjection $\kappa^ss_M : H^*_K(M^{ss}) \to IH^*(M // G)$.

The work of the second author [32] allows us to understand pairings in $IH^*(M // G)$ of intersection cohomology classes on the singular quotient $M // G$ in terms of this surjection $\kappa^ss_M : H^*_K(M^{ss}) \to IH^*(M // G)$. It is shown in [32] that if the action of $G$ on $M$ is weakly balanced (a condition satisfied in the case of moduli spaces of bundles over Riemann surfaces: see [32]), then there is a naturally defined subset $V_M$ of $H^*_K(M^{ss})$ such that $\kappa^ss_M : H^*_K(M^{ss}) \to IH^*(M // G)$ restricts to an isomorphism

$$\kappa^ss_M : V_M \to IH^*(M // G).$$
It is also shown in [32] that the intersection pairing of two elements \( \kappa_M(\alpha) \) and \( \kappa_M(\beta) \) of complementary degrees in \( IH^*(M//G) \) is equal to the evaluation of the image in \( H^*(M//G) \) of the product \( \alpha \beta \in H^*_K(M^{ss}) \) on the fundamental class \([\tilde{M}//G]\), provided that \( \alpha \) and \( \beta \) lie in \( V_M \). This means that we can study intersection pairings in \( IH^*(M//G) \) via intersection pairings in the ordinary cohomology of the quotient \( \tilde{M}//G \) for which semistability is the same as stability.

The residue formula of [26] is a formula, in the case when semistability equals stability, for pairings of cohomology classes \( \kappa_M(\alpha) \) and \( \kappa_M(\beta) \) of complementary dimensions in \( M//G \) in terms of equivariant cohomology classes \( \alpha \) and \( \beta \) in \( M \) which represent them. Let \( T \) be a maximal torus in \( K \); its complexification \( T_c \) is then a maximal complex torus of \( G \). Let \( \Gamma \) be the set of roots of \( K \) regarded as elements of the dual \( t^* \) of the Lie algebra \( t \) of \( T \), and let \( \Gamma_+ \) and \( \Gamma_- \) be the subsets of \( \Gamma \) consisting of the positive and negative roots of \( K \). Let \( \omega = \omega + \mu \) be the standard\(^3\) extension of the symplectic form \( \omega \) to an equivariantly closed differential form on \( M \). We shall assume for simplicity throughout that the stabiliser in \( K \) of a generic point of \( \mu^{-1}(0) \) is trivial. Then if \( F \) is the set of components of the fixed point set \( M^T \) of \( T \) on \( M \), the residue formula is

\[
(2.4) \quad \kappa_M(\alpha)\kappa_M(\beta)[M//G] = \frac{(-1)^{s+n_+}}{|W| \mathrm{vol}(T)} \mathrm{res}(D(X)^2 \sum_{F \in \mathcal{F}} \int_{F} i_F^*(\alpha \beta e^{\omega})(X) e_F(X)[dX]),
\]

where \( \mathrm{vol}(T) \) and \( [dX] \) are the volume of \( T \) and the measure on its Lie algebra \( t \) induced by the restriction to \( t \) of the fixed inner product on \( k \), while \( W \) is the Weyl group of \( K \), the polynomial function \( D(X) = \prod_{\gamma \in \Gamma_+} \gamma(X) \) of \( X \in t \) is the product of the positive roots\(^4\) of \( K \) and \( n_+ = (s-l)/2 \) is the number of those positive roots; \( s \) is the dimension of \( K \) and \( l \) is the dimension of \( T \). Also if \( F \in \mathcal{F} \) is a component of the fixed point set \( M^T \) then \( i_F : F \to M \) is the inclusion and \( e_F \) is the equivariant Euler class of the normal bundle to \( F \) in \( M \). The multivariable residue map \( \mathrm{res} \) which appears in the formula is a linear map, but in order to apply it to the individual terms in the formula some choices have to be made which do not affect the residue of the whole sum. Once the choices have been made, many of the terms in the sum have residue zero and the formula can be rewritten as a sum over a subset \( \mathcal{F}_+ \) of \( \mathcal{F} \). When \( T \) has dimension one (which is the only case we shall need explicitly in this paper, we shall be using an inductive argument modelled on that of \([20]\) we can take \( \mathcal{F}_+ \) to consist of those \( F \in \mathcal{F} \) on which the constant value taken by the \( T \)-moment map \( \mu_T : \tilde{M} \to t^* \cong \mathbb{R} \) is positive, and then res applied to those terms in the sum labelled by \( F \in \mathcal{F}_+ \) is the usual one-variable residue \( \mathrm{res}_{X=0} \) at \( 0 \). Indeed for \( K = U(1) \) we have

\[
(2.5) \quad \kappa_M(\alpha)\kappa_M(\beta)[M//G] = \mathrm{res}_{X=0} \left( \sum_{F \in \mathcal{F}_+} \int_{F} i_F^*\alpha(X)\beta(X) e_F(X) \right)
\]

\(^2\)See Theorem 3.1 of [20] for a corrected version.

\(^3\)Here we follow the conventions of [29] [25] which differ slightly from those used in [26] and by Witten in [32]; in particular we have no factors of \( i \).

\(^4\)In this paper, as in [29], we adopt the convention that weights \( \beta \in t^* \) send the integer lattice \( \Lambda^T = \ker(\exp : t \to T) \) to \( \mathbb{Z} \) rather than to \( 2\pi \mathbb{Z} \), and that the roots of \( K \) are the nonzero weights of its complexified adjoint action. This is one reason why the constant in the residue formula above differs from that of [20] Theorem 8.1.

\(^5\)Notice that \((-1)^{n_+}(D(X))^2 = \prod_{\gamma \in \Gamma} \gamma(X) \).
(see [26, 31, 68]) where \( \text{res}_{X=0} \) denotes the coefficient of \( 1/X \) when \( X \in \mathbb{R} \) has been identified with \( 2\pi iX \in k \).

Thus, in principle, when \( M \) has semistable points which are not stable we can apply the residue formula [2.4] above to \( \tilde{M} \) to obtain pairings on the partial desingularisation \( \tilde{M}/G \) and in the intersection cohomology \( IH^*(M//G) \) of the singular quotient \( M//G \). When the action of \( G \) on \( M \) is weakly balanced (which will be the case for the actions to be considered in this paper), so that \( \kappa_{ss}^M : H^*_K(M//G) \to IH^*(M//G) \) restricts to an isomorphism \( \kappa_{ss}^M : V_M \to IH^*(M//G) \) as at \((2.3)\), and if \( \alpha|_{M//G} \) and \( \beta|_{M//G} \) lie in \( V_M \) and \( \kappa_M(\alpha) \) and \( \kappa_M(\beta) \) have complementary degrees, then their intersection pairing in \( IH^*(M//G) \) is given by

\[
\langle \kappa_M(\alpha), \kappa_M(\beta) \rangle_{IH^*(M//G)} = \frac{(-1)^{n+1}}{|W|} \kappa_M^{(\delta)}(\alpha \beta D^2)[\mu_T^{-1}(\delta)/T]
\]

and thus by

\[
\langle \kappa_M(\alpha), \kappa_M(\beta) \rangle_{IH^*(M//G)} = \frac{(-1)^{s+n+1}}{|W| \text{vol}(T)} \text{res}(D(X)^2 \sum_{F \in \mathcal{F}} \int_F \frac{i^*_F(\alpha \beta e^{\omega-\delta})(X)}{e_F(X)} [dX]),
\]

for any sufficiently small \( \delta \in \mathfrak{t}^* \) which is a regular value of the moment map \( \mu_T \) (see [20] Theorem 18). Here \( \kappa_M^{(\delta)} \) is defined as at \((2.1)\) but with \( K \) replaced by \( T \) and the moment map \( \mu \) replaced by \( \mu_T - \delta \). Moreover in fact \((2.6)\) is valid even when \( M \) is not compact and has singularities away from \( \mu^{-1}(0) \), provided that \( \mu^{-1}(0) \) is compact and \( \tilde{M} \) is smooth near \( \mu^{-1}(0) \).

Thus the intersection pairings in \( IH^*(M//G) \) are given by a very simple modification of the residue formula [2.4]. However when we try to apply \((2.4)\) to \( \tilde{M} \) to obtain pairings on the partial desingularisation \( \tilde{M}/G \) then complications arise. The main difficulty is that, although the construction of \( \tilde{M}^{ss} \) and of \( \tilde{M}/G \) from the linear \( G \)-action on \( M \) is canonical and explicit, the construction of \( \tilde{M} \) is not. In [37] the set \( M^{ss} \) of semistable points of \( M \) is blown up along a sequence of nonsingular \( G \)-invariant closed subvarieties \( V \), and after each blow-up any points which are not semistable are thrown out, so that eventually we arrive at \( \tilde{M}^{ss} \) and thus obtain \( \tilde{M}/G = \tilde{M}^{ss}/G \). If necessary \( \tilde{M} \) itself could be constructed by resolving the singularities of the closures \( \tilde{V} \) of these subvarieties \( V \) and blowing up along their proper transforms, but in practice this is not usually simple. Unfortunately the residue formula of [26] involves the set of components of the fixed point set of the action of the maximal torus \( T \) of \( K \), so applying it directly to \( \tilde{M} \) would be very complicated, and knowledge of the set of semistable points \( \tilde{M}^{ss} \) alone would not suffice. However there is an alternative way to calculate the pairings which only requires information about \( \tilde{M}^{ss} \). This makes use of the method of reduction to the maximal torus \([18, 15, 16]\).

When \( M^{ss} = M^s \) one can reduce to the maximal torus as follows. Let \( \mu_T : M \to \mathfrak{t}^* \) be the \( T \)-moment map given by composing \( \mu : M \to k^* \) with the natural map \( k^* \to \mathfrak{t}^* \). As Guillemin and Kalkman observe in [18], it follows immediately from the residue formula [2.4] that if \( 0 \) is a regular value\(^6\) of \( \mu_T \) then we have a surjection \( \kappa_M^T : H^*_T(M) \to H^*(M/T_c) = H^*(\mu_T^{-1}(0)/T) \)

\(^6\)We are assuming that 0 is a regular value of \( \mu \), or equivalently that \( K \) acts with finite stabilisers on \( \mu^{-1}(0) \).
defined as at (2.11), and if $\alpha, \beta \in H^*_K(M)$ then

\[(2.8) \quad \kappa_M(\alpha\beta)[M//G] = \frac{1}{|W|}\kappa^T_M(\alpha\beta \prod_{\gamma \in \Gamma} \gamma)[\mu_T^{-1}(0)/T].\]

This formula requires some interpretation since $\alpha, \beta \in H^*_K(M)$ and $\prod_{\gamma \in \Gamma} \gamma \in H^*_T = H^*(BT)$, which we think of as the equivariant cohomology of a point. We regard $\alpha$ and $\beta$ as elements of $H^*_T(M)$ via the natural identification of $H^*_K(M)$ with the Weyl invariant part $(H^*_T(M))^W$ of $H^*_T(M)$ and $\gamma$ as an element of $H^*_T(M)$ via the natural inclusion of the equivariant cohomology of a point in $H^*_T(M)$. In fact Martin [45, 46] gives a direct proof of (2.8) without appealing to the residue formula. His proof shows also that, provided $\mu^{-1}(0)/T$ is oriented appropriately,

\[(2.9) \quad \kappa_M(\alpha\beta)[M//G] = \frac{1}{|W|}\kappa^T_M(\alpha\beta \prod_{\gamma \in \Gamma_+} \gamma)[\mu^{-1}(0)/T]\]

where the product is now over only the positive roots of $K$, and his argument shows in addition that it is possible to represent the cohomology classes $\kappa_M(\prod_{\gamma \in \Gamma_+} \gamma)$ and $\kappa_M(\prod_{\gamma \in \Gamma_-} \gamma)$ (of course only differ by a sign $(-1)^{n^+}$) by closed differential forms on $\mu_T^{-1}(0)/T$ with support in an arbitrarily small neighbourhood of $\mu^{-1}(0)/T$. Thus there is in fact no need to assume in (2.8) and (2.9) that $0$ is a regular value of $\mu_T$; it is enough to have $0$ a regular value of $\mu$, and $M$ itself may have singularities away from $\mu^{-1}(0)$. This is important in (2.10) when these ideas are applied to the moduli spaces $\mathcal{M}(n,d)$ when $n$ and $d$ are coprime, and it will be similarly important in this paper.

If $M^{ss} \neq M^s$ then we can apply (2.8) to the blow-up $\tilde{M}$ of $M$ to get

\[(2.10) \quad \kappa_M(\alpha\beta)[\tilde{M}/T_c] = \frac{(-1)^{n^+}}{|W|}\kappa^T_M(\alpha\beta D^2)[\tilde{M}/T_c].\]

It is shown in [25] §8 that it is possible to choose a value $\xi \in \mathfrak{t}^*$ which is regular for both $\mu_T$ and $\tilde{\mu}_T$, such that the difference between $\kappa^T_M(\alpha\beta D^2)[\tilde{M}/T_c]$ and the evaluation on the fundamental class of $\tilde{\mu}_T^{-1}(\xi)/T = M//\xi T_c$ of the cohomology class induced by $\alpha\beta D^2 \in H^*_T(M)$ can be calculated in terms of data determined purely by the construction of $M^{ss}$ from $M^{ss}$, which is canonical and explicit, instead of the construction of $\tilde{M}$ from $M$, and moreover this evaluation on $[\tilde{M}/\xi T_c]$ equals the evaluation on the fundamental class of $\mu_T^{-1}(\xi)/T = M//\xi T_c$ of the cohomology class induced by $\alpha\beta D^2$, which can be calculated by using the residue formula (2.4) applied to the action of $T$ on $M$ with the moment map $\mu_T - \xi$. Combining all these calculations enables us to calculate pairings in the cohomology of the partial desingularisation $\tilde{M}/T$ of $M//G$.

Once we have reduced to calculating pairings on $[\tilde{M}/\xi T_c]$, an alternative strategy to the use of the residue formula (2.4) is to follow the approach taken by Guillemin and Kalkman in [18] and Martin in [45, 46], which was applied to the moduli spaces $\mathcal{M}(n,d)$ when $n$ and $d$ are coprime in (2.10). This is to consider the change in

\[\kappa^\xi_M(\alpha\beta)[\mu_T^{-1}(\xi)/T],\]

for fixed $\alpha, \beta \in H^*_T(M)$, as $\zeta$ varies through the regular values of $\mu_T$. This is sufficient, if $M$ is a compact symplectic manifold, because the image $\mu_T(M)$ is bounded, so if $\zeta$ is far enough
from 0 then $\mu_T^{-1}(\zeta)/T$ is empty and thus $\kappa_M^{(\zeta)}(\alpha\beta)[\mu_T^{-1}(\zeta)/T] = 0$. Now the image $\mu_T(M)$ is a convex polytope $[2, 19]$; it is the convex hull in $t^*$ of the set

$$\{\mu_T(F) : F \in \mathcal{F}\}$$

of the images $\mu_T(F)$ (each a single point of $t^*$) of the connected components $F$ of the fixed point set $M^T$. This convex polytope is divided by codimension-one walls into subpolytopes, themselves convex hulls of subsets of $\{\mu_T(F) : F \in \mathcal{F}\}$, whose interiors consist entirely of regular values of $\mu_T$. When $\zeta$ varies in the interior of one of these subpolytopes there is no change in $\kappa_M^{(\zeta)}(\alpha\beta)[\mu_T^{-1}(\zeta)/T]$, so it suffices to understand what happens as $\zeta$ crosses a codimension-one wall.

Any such wall is the image $\mu_T(M_1)$ of a connected component $M_1$ of the fixed point set of a circle subgroup $T_1$ of $T$. The quotient group $T/T_1$ acts on $M_1$, which is a symplectic submanifold of $M$, and the restriction of the moment map $\mu_T$ to $M_1$ has an orthogonal decomposition

$$\mu_T|_{M_1} = \mu_{T/T_1} \oplus \mu_{T_1}$$

where $\mu_{T/T_1} : M_1 \to (t/t_1)^*$ is a moment map for the action of $T/T_1$ on $M_1$ and $\mu_{T_1} : M_1 \to t_1^*$ is constant (because $T_1$ acts trivially on $M_1$). If $\zeta_1$ is a regular value of $\mu_{T/T_1}$ then there is a symplectic quotient

$$\mu_{T_{T_1}}^{-1}(\zeta_1)/(T/T_1),$$

and it is shown in $[18]$ that the change in $\kappa_M^{(\zeta)}(\alpha\beta)[\mu_T^{-1}(\zeta)/T]$ as $\zeta$ crosses the wall $\mu_T(M_1)$ is

$$(\text{res}_{M_1}(\alpha\beta))_{\zeta_1}[\mu_{T/T_1}^{-1}(\zeta_1)/(T/T_1)]$$

for a suitable residue operation

$$\text{res}_{M_1} : H_T^*(M) \to H_{T/T_1}^{*-d_1}(M_1)$$

where $d_1 = \text{codim}M_1 - 2$. When $T$ is itself a circle, this residue operation is given by restricting to $M_1$, dividing by the equivariant Euler class of the normal bundle to $M_1$ in $M$, and taking the ordinary residue $\text{res}_{X=0}$ at 0 on $\mathbb{C}$. This gives an inductive method for calculating the change in $\kappa_M^{(\zeta)}(\alpha\beta)[\mu_T^{-1}(\zeta)/T]$ as the wall is crossed, in terms of data on $M$ localised near $M^T$; it is essentially equivalent to the residue formula $[2, 4]$ when $\dim(T) = 1$, but differs from it for groups of higher rank.

**Remark 1.** The advantage of this method over the residue formula $[2, 4]$ for our purposes is that it can be applied in situations when $M$ is not compact, and indeed when the fixed point set of the action of $T$ on $M$ has infinitely many components so that the residue formula cannot be applied directly. This method was used in $[29]$ with $M$ as the extended moduli space of $[23]$ to obtain formulas for the pairings on $\mathcal{M}(n, d)$ when $n$ and $d$ are coprime (see §4 below), and exactly the same arguments will provide us with formulas for pairings on $M//\xi T_c$ when $\xi$ is a regular value of $\mu$ sufficiently close to 0.

**Remark 2.** When it is unlikely to cause confusion we will simplify the notation a little and write $\kappa$ instead of $\kappa_M$, $\kappa^{ss}_M$ etc.
3. Moduli spaces of bundles and their partial desingularisations

Recall that a holomorphic vector bundle $E$ of rank $n$ and degree $d$ over the compact Riemann surface $\Sigma$ of genus $g \geq 2$ is called semistable (respectively stable) if every proper subbundle $E'$ of $E$ satisfies $\deg(E')/\text{rank}(E') \leq d/n$ (respectively $\deg(E')/\text{rank}(E') < d/n$). There is a moduli space $\mathcal{M}^s(n, d)$ of isomorphism classes of stable bundles of rank $n$ and degree $d$ over $\Sigma$, which is a nonsingular quasi-projective variety with a natural compactification $\mathcal{M}(n, d)$ whose points are represented by semistable bundles of rank $n$ and degree $d$ over $\Sigma$. The compactified moduli space $\mathcal{M}(n, d)$ is a projective variety which is in general singular, although if $d$ and $n$ have no common factors then $\mathcal{M}(n, d)$ coincides with $\mathcal{M}^s(n, d)$ and is a nonsingular projective variety.

The spaces $\mathcal{M}(n, d)$ can be represented in several different ways as quotients in the sense of Mumford’s geometric invariant theory [51] or as quotients in an analogous sense for infinite dimensional group actions, leading to constructions of partial desingularisations $\tilde{\mathcal{M}}(n, d)$ of $\mathcal{M}(n, d)$ [40], which are projective varieties with only orbifold singularities. In this section we shall follow the infinite-dimensional point of view taken by Atiyah and Bott in [3].

Let $E$ be a fixed $C^\infty$ complex hermitian vector bundle of rank $n$ and degree $d$ over $\Sigma$. Let $\mathcal{C}$ be the space of all holomorphic structures on $E$, let $\mathcal{C}^s$ (respectively $\mathcal{C}^{ss}$) be the open subset of $\mathcal{C}$ consisting of all stable (respectively semistable) holomorphic structures on $E$, let $\mathcal{G}$ denote the gauge group of $E$ (the group of all $C^\infty$ unitary automorphisms of $E$) and let $\mathcal{G}_c$ denote its complexification which is the group of all $C^\infty$ complex automorphisms of $E$. When it is necessary for clarification we shall write $\mathcal{C}(n, d)$ and $\mathcal{G}(n, d)$ instead of $\mathcal{C}$ and $\mathcal{G}$. The moduli space $\mathcal{M}^s(n, d)$ can be identified naturally with $\mathcal{C}^s/\mathcal{G}_c$ and $\mathcal{M}(n, d)$ can be identified naturally with the quotient of $\mathcal{C}^{ss}$ by the equivalence relation for which semistable structures are equivalent if and only if the closures of their $\mathcal{G}_c$-orbits meet in $\mathcal{C}^{ss}$. Thus we can think of $\mathcal{M}(n, d)$ as a quotient $\mathcal{C}/\mathcal{G}_c$ in a sense analogous to geometric invariant theoretic quotients, and so construct a partial desingularisation $\tilde{\mathcal{M}}(n, d)$ of $\mathcal{M}(n, d)$ [40]. In fact in [40] $\tilde{\mathcal{M}}(n, d)$ is not constructed using the representation of $\mathcal{M}(n, d)$ as the geometric invariant theoretic quotient of $\mathcal{C}$ by $\mathcal{G}_c$, although it is noted at [40], p.246 that this representation of $\mathcal{M}(n, d)$ would lead to the same partial desingularisation. Instead in [40] $\mathcal{M}(n, d)$ is represented as a geometric invariant theoretic quotient of a finite-dimensional nonsingular quasi-projective variety $R(n, d)$ by a linear action of $SL(p; \mathbb{C})$ where $p = d + n(1 - g)$ with $d \gg 0$.

Alternatively $\mathcal{M}(n, d)$ can be thought of as a symplectic quotient of $\mathcal{C}$ by the gauge group $\mathcal{G}$, where the rôle of the normsquare of the moment map is played by the Yang-Mills functional.

The construction of $\tilde{\mathcal{M}}(n, d)$ involves a set $\mathcal{R}$ of representatives $R$ of the conjugacy classes of reductive subgroups of $\mathcal{G}_c$ which occur as the connected components of stabilisers in $\mathcal{G}_c$ of semistable points of $\mathcal{C}$, together with their fixed point sets $Z_R$ in $\mathcal{C}$. Equivalently we look for automorphism groups of semistable bundles over $\Sigma$. Such conjugacy classes correspond to unordered sequences $(m_1, n_1), ..., (m_q, n_q)$ of pairs of positive integers such that $m_1n_1 + ... + m_qn_q = n$ and $n$ divides $nd$ for each $i$ (cf. [40], pp. 248-9)). An element $R$ of the corresponding conjugacy class is given by

\begin{equation}
R = GL(m_1; \mathbb{C}) \times \ldots \times GL(m_q; \mathbb{C}).
\end{equation}

In the notation of [37] and [40] we construct $\tilde{\mathcal{C}}^{ss}$ from $\mathcal{C}^{ss}$ by blowing up along the subvarieties $\mathcal{G}_cZ_R^{ss}$ (or rather their proper transforms), in decreasing order of dim $R$, and removing the
points which are not semistable at each stage, where $Z_R^{ss}$ is the set of semistable holomorphic structures fixed by $R$. Equivalently we construct $\mathcal{M}(n, d)$ from $\mathcal{M}(n, d)$ by blowing up along the images $Z_R//N^R$ in $\mathcal{M}(n, d)$ of the subvarieties $\mathcal{G}_c Z_R^{ss}$, where $N = N^R$ is the normaliser of $R$ in $\mathcal{G}_c$. A holomorphic structure fixed by $R$ is semistable (equivalently belongs to $Z_R^{ss}$) if and only if it is semistable for the induced action of $N/R$ on $Z_R$; we denote by $Z_R^{ss}$ the open subset of $Z_R^{ss}$ consisting of those holomorphic structures fixed by $R$ which are stable for the induced action of $N/R$ on $Z_R$. When $R$ is as at (3.1) above then $\mathcal{G}_c Z_R^{ss}$ consists of all those holomorphic structures $E$ with
\begin{equation}
E \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)
\end{equation}
with $D_1, \ldots, D_q$ all semistable and $D_i$ of rank $n_i$ and degree $d_i = n_i d/n$, while $\mathcal{G}_c Z_R^{ss}$ consists of all those holomorphic structures $E$ as above where $D_1, \ldots, D_q$ are all stable and not isomorphic to one another. Moreover the normaliser $N$ of $R$ in $\mathcal{G}_c$ has connected component
\begin{equation}
N_0 \cong \prod_{1 \leq i \leq q} (GL(m_i; \mathbb{C}) \times \mathcal{G}_c(n_i, d_i))/\mathbb{C}^*
\end{equation}
where $\mathbb{C}^*$ is the diagonal central one-parameter subgroup of $GL(m_i; \mathbb{C}) \times \mathcal{G}_c(n_i, d_i)$. The group $\pi_0(N) = N/N_0$ is the product
\begin{equation}
\pi_0(N) = \prod_{j \geq 0, k \geq 0} \text{Sym}(\# \{ i : m_i = j \text{ and } n_i = k \})
\end{equation}
where $\text{Sym}(b)$ denotes the symmetric group of permutations of a set with $b$ elements. Furthermore the semistable holomorphic structures which become unstable after the blow-up corresponding to the conjugacy class of $R$ are those with a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$ such that $E$ is not isomorphic to $\bigoplus_{1 \leq k \leq s} E_k/E_{k-1}$ but
\begin{equation}
\bigoplus_{1 \leq k \leq s} E_k/E_{k-1} \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)
\end{equation}
where $D_1, \ldots, D_q$ are all stable and not isomorphic to one another, and $D_i$ has rank $n_i$ and degree $d_i$ [10, p. 248].

In [43] the action of $R$ on the normal $\mathcal{N}_R$ to $\mathcal{G}_c Z_R^{ss}$ at a point represented by a holomorphic structure $E$ of the form (3.2) and the induced action on $\mathfrak{p}(\mathcal{N}_R)$ are studied. If a $C^\infty$ isomorphism of our fixed $C^\infty$ bundle $\mathcal{E}$ with $(\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)$ is chosen, then we can identify $\mathcal{C}$ with the infinite-dimensional vector space
\[\Omega^{0,1}(\text{End} ((\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_q} \otimes D_q)))\]
and the normal to the $\mathcal{G}_c$-orbit at $E$ is given by $H^1(\Sigma, \text{End}E)$, where $\text{End}E$ is the bundle of holomorphic endomorphisms of $E$ [3, §7]. The normal to $\mathcal{G}_c Z_R^{ss}$ can then be identified with
\[H^1(\Sigma, \text{End}^0 \mathcal{E}) \cong \bigoplus_{i_1, i_2 = 1}^q \mathbb{C}^{m_{i_1} m_{i_2} - \delta_{i_1}^{i_2}} \otimes H^1(\Sigma, D_{i_1}^{*} \otimes D_{i_2})\]
where $\delta_{i_1}^{i_2}$ denotes the Kronecker delta and $\text{End}^0 \mathcal{E}$ is the quotient of the bundle $\text{End}E$ of holomorphic endomorphisms of $E$ by the subbundle $\text{End}_0 E$ consisting of those endomorphisms which preserve the decomposition (3.2). The action of $R = \prod_{i=1}^q GL(m_i; \mathbb{C})$ on this is given by the natural action on $\mathbb{C}^{m_{i_1} m_{i_2} - \delta_{i_1}^{i_2}}$ identified with the set of $m_{i_1} \times m_{i_2}$ matrices if
Let $\Delta_1$ and directed graph $G$ semistable set $P_i$. Moreover the conditions on the function $S$ its indexing can be recovered from the coefficients of $\beta$ of set of the weights of the action of $R$ on $\oplus_{i=1}^q \mathbb{C}^{m_i}$.

The linear action of $R$ on the normal $\mathcal{N}_R$ induces a stratification of $\mathbb{P}(\mathcal{N}_R)$ with the semistable set $\mathbb{P}(\mathcal{N}_R)^{ss}$ as its open stratum [30]. An element $\beta$ of the indexing set $\mathcal{B}_R$ of this stratification is represented by the closest point to 0 of the convex hull of some nonempty set of the weights of the action of $R$ on $\mathcal{N}_R$, and two such closest points can be taken to represent the same element of $\mathcal{B}_R$ if and only if they lie in the same $\text{Ad}(N)$-orbit, where $N$ is the normaliser of $R$ (see [36] or [42]). This indexing set $\mathcal{B}_R$ is described more explicitly in [43] as follows. Let us take our maximal compact torus $T_R$ in $R$ to be the product of the standard maximal tori of the unitary groups $U(m_1), \ldots, U(m_q)$ consisting of the diagonal matrices, and let $t_R$ be its Lie algebra. Let

$$M = m_1 + \ldots + m_q$$

and let $e_1, \ldots, e_M$ be the weights of the standard representation of $T_R$ on $\mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_q}$. We use the usual invariant inner product on the Lie algebra $\mathfrak{u}(m_i)$ of $U(m_i)$ for $1 \leq i \leq q$ given by $\langle A, B \rangle = -\text{tr}AB^t$ and multiply by a positive scalar factor $\rho_i$ (to be chosen later) to induce an inner product on the Lie algebra of $T_R$ such that $e_1, \ldots, e_M$ are mutually orthogonal and $|e_j|^2 = \rho_i$ if $m_1 + \ldots + m_{i-1} < j \leq m_1 + \ldots + m_i$. Note that in [43] $\rho_i$ is chosen to be $(n_i + d_i(1-g))^{-1}$, but this does not affect the proof of the following result which is [43] Proposition 5.1.

**Proposition 3.** Let $\beta$ be any nonzero element of the Lie algebra $t_R$ of the maximal compact torus $T_R$ of $R$. Then $\beta$ represents an element of $\mathcal{B}_R \setminus \{0\}$ if and only if there is a partition

$$\{ \Delta_{h,m} : (h, m) \in J \}$$

of $\{1, \ldots, M\}$, indexed by a rectangle $J$ in $\mathbb{Z} \times \mathbb{Z}$, with the following properties. If

$$r_{h,m} = \sum_{j \in \Delta_{h,m}} |e_j|^2$$

and

$$\epsilon(h) = \left( \sum_m m r_{h,m} \right) \left( \sum_m r_{h,m} \right)^{-1},$$

then $-1/2 \leq \epsilon(h) < 1/2$ and $\epsilon(1) > \epsilon(2) > \ldots$, and

$$\frac{\beta}{\|\beta\|^2} = \sum_{(h, m) \in J} \sum_{j \in \Delta_{h,m}} \frac{(\epsilon(h) - m) e_j}{\|e_j\|^2}.$$

Moreover the conditions on the function $\epsilon$ ensure that the partition $\{ \Delta_{h,m} : (h, m) \in J \}$ and its indexing can be recovered from the coefficients of $\beta$ with respect to the basis

$$e_1/\|e_1\|^2, \ldots, e_M/\|e_M\|^2$$

of $t_R$.

The proof of this proposition involves studying the convex hull of

$$\{ e_i - e_j : (i, j) \in S \}$$

for a nonempty subset $S$ of \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq i, j \leq M \}. From S we can construct a directed graph $G(S)$ with vertices 1, ..., $M$ and directed edges from $i$ to $j$ whenever $(i, j) \in S$. Let $\Delta_1, \ldots, \Delta_s$ be the connected components of this graph. Then $\{ e_i - e_j : (i, j) \in S \}$ is
the disjoint union of its subsets \( \{e_i - e_j : (i, j) \in S \text{ and } i, j \in \Delta_h\} \) for \( 1 \leq h \leq t \), and \( \{e_i - e_j : (i, j) \in S \text{ and } i, j \in \Delta_h\} \) is contained in the vector subspace of \( \mathfrak{t}_R \) spanned by the basis vectors \( \{e_k : k \in \Delta_h\} \). Since these subspaces are mutually orthogonal for \( 1 \leq h \leq s \), the closest point to 0 in the convex hull of \( \{e_i - e_j : (i, j) \in S\} \) is

\[
\beta = \left( \sum_{h=1}^{s} \frac{1}{\|\beta_h\|^2} \right)^{-1} \sum_{h=1}^{s} \beta_h
\]

where \( \beta_h \) is the closest point to 0 of the convex hull of \( \{e_i - e_j : (i, j) \in S \text{ and } i, j \in \Delta_h\} \) for \( 1 \leq h \leq s \). It is shown in the proof of [43] Proposition 5.1 that we can express each \( \Delta_h \) as a disjoint union

\[
\Delta_h = \bigcup_m \Delta_{h,m}
\]

such that

\[
\frac{\beta_h}{\|\beta_h\|^2} = \sum_m \sum_{j \in \Delta_{h,m}} (e(h) - m) e_j / |e_j|^2
\]

where \( e(h) \) has the required properties, and from this the result follows.

Recall from Remark 1 that in order to compute intersection pairings on \( \tilde{M}(n, d) \) we will consider the change in pairings as walls are crossed between convex subpolytopes whose interiors consist of regular values of a torus moment map. To deal with the blow-up corresponding to the subgroup

\[
\mathcal{R} = \text{GL}(m_1; \mathbb{C}) \times \cdots \times \text{GL}(m_q; \mathbb{C})
\]

as above, we will fix a ray

\[
\mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M)
\]

in \( \mathfrak{t}_R \) and consider the wall crossings needed to approach 0 along this ray. Here the walls are convex hulls of subsets of the set of weights \( \{e_i - e_j : 1 \leq i, j \leq M\} \) for the action of \( \mathcal{R} \) on \( \mathcal{N}_R \), and hence all lie within the codimension 1 subspace in \( \mathfrak{t}_R \) given by \( \{\sum_i \lambda_i e_i : \sum_i \lambda_i = 0\} \).

Any wall which needs to be crossed lies in a hyperplane in this subspace obtained by intersecting the subspace with one of the affine hyperplanes \( \beta + \beta^\perp \) determined by some \( \beta \) representing an element of \( \mathcal{B}_R \setminus \{0\} \). Such a \( \beta \) corresponds to a partition \( \{\Delta_{h,m} : (h, m) \in J\} \) satisfying the conditions in Proposition 3 or equivalently is the closest point to zero in the convex hull of a nonempty subset \( \{e_i - e_j : (i, j) \in S\} \) of weights of the \( \mathcal{R} \) action on \( \mathcal{N}_R \). The subset \( S \) determines a directed graph \( G(S) \) as above.

**Lemma 4.** The directed graph \( G(S) \) corresponding to a non-zero \( \beta \) contains no directed loops.

**Proof:** Suppose we have a directed loop in \( G(S) \), that is a sequence of edges corresponding to weights \( e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \ldots, e_{i_r} - e_{i_1} \). Then \( \frac{1}{r} ((e_{i_1} - e_{i_2}) + \ldots + (e_{i_r} - e_{i_1})) = 0 \) lies in the convex hull of the weights. This contradicts the assumption that \( \beta \) is the closest point to the origin of this convex hull. \( \square \)

**Lemma 5.** If the ray \( \mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M) \) meets the wall determined by \( \beta \) then \( M \) is the only vertex of the directed graph \( G(S) \) with no outgoing edges.
Proof: Since the ray (3.6) meets the wall determined by $\beta$ we have
\[
\lambda (e_1 + e_2 + \cdots + e_{M-1} - (M - 1)e_M) = \sum_{(i,j) \in S} \lambda_{ij} (e_i - e_j)
\]
for some $\lambda > 0$ and $\lambda_{ij} \geq 0$ with $\sum_{(i,j) \in S} \lambda_{ij} = 1$. Equating coefficients of $e_i$ for $i \leq M - 1$ gives
\[
\lambda = \sum_{j: (i,j) \in S} \lambda_{ij} - \sum_{j: (j,i) \in S} \lambda_{ji}
\]
and hence
\[
\sum_{j: (i,j) \in S} \lambda_{ij} = \lambda + \sum_{j: (j,i) \in S} \lambda_{ji} > 0
\]
so there is some $j$ for which $(i, j) \in S$ and hence $i$ has an outgoing edge.

Suppose $M$ has an outgoing edge. Then all the vertices would have outgoing edges and $G(S)$ would contain a directed loop, contradicting Lemma 4.

Lemma 6. If the ray $\mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M - 1)e_M)$ meets the wall determined by $\beta$ then the directed graph $G(S)$ is connected.

Proof: If $G(S)$ is not connected there is a component not containing $M$. Every vertex in this component has an outgoing edge by Lemma 5 and so the component contains a directed loop contradicting Lemma 4.

Remark 7. Since $G(S)$ is connected the index $h$ for the partition $\{\Delta_{h,m} : (h, m) \in J\}$ can be omitted. We will also relabel the partition $\{\Delta_m\}$ by adding a constant to $m$ so that it is indexed by $m \in \{1, \ldots, t\}$; the only difference this makes is that we can no longer assume that $\epsilon$ lies in $[-1/2, 1/2]$. We also know from Lemma 5 that $M$ is the only 'top' element of the graph. It follows from the definition of the $\Delta_{h,m}$ in the proof of [43] Proposition 5.1 that $M$ is then the only element in $\Delta_t$.

Subpolytopes of the wall determined by $\beta$ are the intersections of convex hulls of subsets of $\{e_i - e_j : (i, j) \in S\}$ with exactly $M - 1$ elements which are linearly independent. These correspond to subgraphs of $G(S)$ with precisely $M - 1$ edges. If the ray (3.6) meets the subpolytope it must correspond to a connected subgraph with $M$ as the only vertex with no outgoing edge (by the same arguments as for Lemmas 4 and 5). Thus these subgraphs are trees; they are the minimal connected subgraphs of $G(S)$ with the same vertices as $G(S)$ and having $M$ as the only vertex with no outgoing edges.

Lemma 8. If the connected graph $G(S)$ has the property that its only vertex with no outgoing edges is $M$, then $G(S)$ has a minimal connected subgraph with the same property.

Proof: If $1 \leq i \leq M - 1$ then we can pick some $j(i) \in \{1, \ldots, M\}$ such that $(i, j(i)) \in S$. Since $G(S)$ contains no directed loops we can check that $\{(i, j(i)) : 1 \leq i \leq M - 1\}$ determines a minimal connected subgraph with the required property.

Lemma 9. Suppose that $G(S)$ has the property that its only vertex with no outgoing edges is $M$, and let $G_0 = G(S_0)$ be a minimal connected subgraph with the same property. Then the ray $\mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M - 1)e_M)$ meets the interior of the convex hull of $\{e_i - e_j : (i, j) \in S_0\}$.
Proof: Without loss of generality we can assume that the vertices $i$ such that $(i, M) \in S_0$ are precisely $M - 1, M - 2, \ldots, M - k$. If we remove the vertex $M$ and the edges joining $M - 1, M - 2, \ldots, M - k$ to $M$ from the graph $G(S_0)$, then the resulting graph has $k$ connected components $G_1 = G(S_1), \ldots, G_k = G(S_k)$, say, where for $1 \leq j_0 \leq k$ the vertex $M - j_0$ is the only vertex in $G_{j_0}$ with no outgoing edges. Let $M_{j_0}$ be the number of vertices in the connected component $G_{j_0}$ and let the other vertices of $G_{j_0}$ apart from $M - j_0$ be $v_1^{j_0}, \ldots, v_{M_{j_0}}^{j_0}$. Then $G_{j_0}$ has $M_{j_0} - 1$ edges and is a minimal connected graph on its vertices with the property that $M - j_0$ is the only vertex with no outgoing edges. Hence by induction on $M$ we can assume that there exist $\lambda^{(j_0)} > 0$ and $\lambda^{(j_0)}_{ij} > 0$ for $(i, j) \in S_{j_0}$ with

$$\sum_{(i, j) \in S_{j_0}} \lambda^{(j_0)}_{ij} = 1$$

and

$$\lambda^{(j_0)} \left( e_{i_1}^{j_0} + e_{i_2}^{j_0} + \cdots + e_{i_{M_{j_0}}^{j_0}}^{j_0} - (M_{j_0} - 1)e_{M - j_0} \right) = \sum_{(i, j) \in S_{j_0}} \lambda^{(j_0)} (e_i - e_j).$$

Dividing by $\lambda^{(j_0)}$ (which is strictly positive) gives

$$e_{i_1}^{j_0} + e_{i_2}^{j_0} + \cdots + e_{i_{M_{j_0}}^{j_0}}^{j_0} - (M_{j_0} - 1)e_{M - j_0} = \sum_{(i, j) \in S_{j_0}} \lambda^{(j_0)} (e_i - e_j)$$

and summing over $j_0 = 1, \ldots, k$ gives

$$e_1 + e_2 + \cdots + e_{M - k - 1} - (M_k - 1)e_{M_k} - \cdots - (M_1 - 1)e_{M_1} - (M - 1)e_{M - 1} = \sum_{j_0 = 1}^k \sum_{(i, j) \in S_{j_0}} \lambda^{(j_0)} (e_i - e_j).$$

Adding $M_k(e_{M - k} - e_M) + \cdots + M_1(e_{M - 1} - e_M)$ to each side and using the equality $M_1 + \cdots + M_k = M - 1$ gives

$$e_1 + e_2 + \cdots + e_{M - 1} - (M - 1)e_M = \sum_{j_0 = 1}^k \sum_{(i, j) \in S_{j_0}} \lambda^{(j_0)} (e_i - e_j) + \sum_{j_0 = 1}^k M_{j_0}(e_{M - j_0} - e_M).$$

Since $S_{j_0} \subseteq S_0$ and $(M - j_0, M) \in S_0$ for $1 \leq j_0 \leq k$, and in addition $\lambda^{(j_0)}_{ij}/\lambda^{(j_0)} > 0$ and $M_{j_0} > 0$ for $1 \leq j_0 \leq k$ and $(i, j) \in S_{j_0}$, we can rewrite this as

$$e_1 + e_2 + \cdots + e_{M - 1} - (M - 1)e_M = \sum_{(i, j) \in S_0} \lambda'_{ij} (e_i - e_j)$$

where $\lambda'_{ij} \geq 0$ for all $(i, j) \in S_0$. Indeed since

$$S_0 = \{ (M - j_0, M) : 1 \leq j_0 \leq k \} \cup \bigcup_{j_0 = 1}^k S_{j_0}$$

we have $\lambda'_{ij} > 0$ for all $(i, j) \in S_0$. Finally dividing each side by

$$\sum_{(i, j) \in S_0} \lambda'_{ij}$$
\[ \lambda(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M) = \sum_{(i,j) \in S_0} \lambda_{ij}(e_i - e_j) \]

where \( \lambda > 0 \) and \( \lambda_{ij} > 0 \) for all \((i, j) \in S_0\) and

\[ \sum_{(i,j) \in S_0} \lambda_{ij} = 1 \]

as required.

Combining Lemmas 5, 6, 8 and 9 gives us

**Proposition 10.** The ray \( \mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M) \) meets the wall determined by \( \beta \) if and only if \( M \) is the only vertex of \( G(S) \) with no outgoing edges.

**Remark 11.** We now know which minimal connected subgraphs of \( G(S) \) determine subpolytopes of this wall through which the ray \( \mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M) \) passes. They are the minimal connected subgraphs with the property that \( M \) is the only vertex with no outgoing edges. The case \( M = 4 \) is illustrated in Figure 1.

**Figure 1.** The case \( M = 4 \). The 12 weights \( \{e_i - e_j : 1 \leq i, j \leq 4, i \neq j\} \) are shown as dots forming the midpoints of a hexahedron. A wall, corresponding to the subset \( S = \{(1, 4), (2, 4), (3, 1), (3, 2)\} \), which meets the ray \( \mathbb{R}_+(e_1 + e_2 + e_3 - 3e_4) \) is shown shaded. The connected graph \( G(S) \), in which 4 is the only vertex with no outgoing edges, is shown on the right. The subpolytopes of the wall which meet the ray are shown below, and on their right are the corresponding minimal connected subgraphs of \( G(S) \), again in which 4 is the only vertex with no outgoing edges.

The procedure described in §8 of [25] for calculating intersection pairings works most efficiently if the wall crossing takes place at a point \( \beta^* \) of the affine hyperplane \( \beta + \beta^\perp \) in the
Lie algebra of the maximal torus of $R$ which satisfies $\text{Stab}\beta \subseteq \text{Stab}\beta^*$ where $\text{Stab}\beta$ denotes the stabiliser of $\beta$ under the adjoint action of $G_c$ (cf. [25] Remark 27). Recall that

$$\frac{\beta}{\|\beta\|^2} = \sum_{m=1}^{t} \sum_{j \in \Delta_m} (\epsilon - m) \frac{e_j}{\|e_j\|^2}$$

where $\epsilon$ is a constant. Recall also that we chose an invariant inner product on the Lie algebra of $R$ such that $e_1, \ldots, e_M$ are mutually orthogonal and $\|e_j\|^2 = \rho_i$ if $m_1 + \ldots + m_{i-1} < j \leq m_1 + \ldots + m_i$ where $\rho_i$ can be any strictly positive scalar for $1 \leq i \leq q$. Thus for generic choices of $\rho_1, \ldots, \rho_q$ we will have the required condition

$$(3.7) \quad \text{Stab}\beta \subseteq \text{Stab}\beta^*$$

provided that $\beta^*$ is of the form

$$(3.8) \quad \beta^* = \sum_{m=1}^{t} \sum_{j \in \Delta_m} f(m) \frac{e_j}{\|e_j\|^2}$$

for some function $f(m)$ of $m \in \{1, \ldots, t\}$. But by Proposition 10 the condition for the ray $\mathbb{R}_+(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M)$ to meet the wall determined by $\beta$ is that $M$ should be the only vertex of $G(S)$ with no outgoing edges, and hence the only element of $\Delta_t$ (see Remark 7). If this condition is satisfied, then the ray meets the wall in a point $\beta^*$ of the form (3.8) above, since

$$\lambda(e_1 + e_2 + \cdots + e_{M-1} - (M-1)e_M) = \sum_{m=1}^{t} \sum_{j \in \Delta_m} f(m) \frac{e_j}{\|e_j\|^2}$$

where $f(m) = -\lambda(M-1)\|e_M\|^2$ if $m = t$ and $f(m) = \lambda\|e_m\|^2$ if $1 \leq m < t$.

**Remark 12.** With such generic choices of $\rho_1, \ldots, \rho_q$ we get $R \cap \text{Stab}\beta = \prod GL(m^k_i)$ where $m^k_i = |\Delta^k_i|$ and

$$\Delta^k_i = \Delta_k \cap \{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i\}$$

cf. Definition 5.5 in [43]. We will make use of this in §6.

### 4. Intersection theory on nonsingular moduli spaces of bundles

In this section we review the results of [29] (see also [30]). Throughout we will use a fixed invariant inner product on the Lie algebra $k$ of a compact Lie group $K$ to identify $k$ with $k^*$.

#### 4.1. Generators of the cohomology ring

A set of generators for the cohomology of $H^*(\mathcal{M}(n, d))$ of the moduli space $\mathcal{M}(n, d)$ of stable holomorphic vector bundles of coprime rank $n$ and degree $d$ on a compact Riemann surface $\Sigma$ of genus $g \geq 2$ is given in [3] by Atiyah and Bott. It may be described as follows. There is a universal rank $n$ vector bundle

$$U \to \Sigma \times \mathcal{M}(n, d)$$

which is unique up to tensor product with the pullback of any holomorphic line bundle on $\mathcal{M}(n, d)$; for definiteness Atiyah and Bott impose an extra normalizing condition which determines the universal bundle up to isomorphism, but this is not crucial to their argument

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7In this paper, all cohomology groups are assumed to be with complex coefficients.
(see [3], p582). Then by [3] Proposition 2.20 the following elements of $H^*(\mathcal{M}(n,d))$ for $2 \leq r \leq n$ make up a set of generators:

\begin{equation}
(4.1) \quad f_r = c_r(\mathbb{U})/\alpha_j, \quad 2 \leq r \leq n
\end{equation}

\begin{equation}
(4.2) \quad b_r^j = c_r(\mathbb{U})/\alpha_j, \quad 1 \leq r \leq n
\end{equation}

\begin{equation}
(4.3) \quad a_r = c_r(\mathbb{U})/1, \quad 2 \leq r \leq n
\end{equation}

Here, $[\Sigma] \in H_2(\Sigma)$ and $\alpha_j \in H_1(\Sigma)$ ($j = 1, \ldots, 2g$) form standard bases of $H_2(\Sigma, \mathbb{Z})$ and $H_1(\Sigma, \mathbb{Z})$, and / represents the slant product $H^N(\Sigma \times \mathcal{M}(n,d)) \otimes H_j(\Sigma) \to H^{N-j}(\mathcal{M}(n,d))$.

**Remark 13.** These generators lift to equivariant cohomology: there is a $\mathcal{G}(n,d)$-equivariant universal bundle over $\Sigma \times \mathcal{C}(n,d)$ and the slant product of its equivariant Chern classes with $1 \in H_0(\Sigma)$, $[\Sigma] \in H_2(\Sigma)$ and the $\alpha_j \in H_1(\Sigma)$ give classes which (abusing notation) we denote respectively $a_1, \ldots, a_r, f_2, \ldots, f_r$ and $b_r^j$ for $1 < r \leq n$ and $1 \leq j \leq 2g$. These classes generate $H^*_\mathcal{G}(n,d)\mathcal{C}(n,d))$.

The quotient $\mathcal{G}(n,d) \to \overline{\mathcal{G}}(n,d)$ by the central subgroup $S^1$ induces an inclusion

$$H^*_\mathcal{G}(n,d)\mathcal{C}(n,d)) \hookrightarrow H^*_\overline{\mathcal{G}}(n,d)\mathcal{C}(n,d))$$

such that identifying $H^*(BS^1)$ with the polynomial subalgebra generated by $a_1$ gives an isomorphism

$$H^*_\mathcal{G}(n,d)\mathcal{C}(n,d)) \cong H^*_\overline{\mathcal{G}}(n,d)\mathcal{C}(n,d)) \otimes H^*(BS^1).$$

Hence $a_2, \ldots, a_r, f_2, \ldots, f_r$ and $b_r^j$ for $1 < r \leq n$ and $1 \leq j \leq 2g$ determine generators of

$$H^*_\mathcal{G}(n,d)\mathcal{C}(n,d))$$

When $n$ and $d$ are coprime the latter is isomorphic to $H^*(\mathcal{M}(n,d))$ and these equivariant classes correspond to those defined in (4.1), (4.2) and (4.3). When $n$ and $d$ are not coprime we think of $a_2, \ldots, a_r, f_2, \ldots, f_r$ and $b_r^j$ for $1 < r \leq n$ and $1 \leq j \leq 2g$ as equivariant cohomology classes.

If we replace $\mathcal{M}(n,d)$ by the moduli space $\mathcal{M}_\Lambda(n,d)$ of stable holomorphic vector bundles of coprime rank $n$ and degree $d$ and fixed determinant line bundle, the set of generators is given by (4.1) and (4.3), while (4.2) is replaced by

\begin{equation}
(4.4) \quad b_r^j = c_r(\mathbb{U})/\alpha_j, \quad 2 \leq r \leq n.
\end{equation}

The paper [29] treats intersection numbers in the cohomology of $\mathcal{M}_\Lambda(n,d)$ rather than that of $\mathcal{M}(n,d)$, but the two sets of pairings are very closely related since

$$H^*(\mathcal{M}(n,d)) \cong H^*(\mathcal{M}_\Lambda(n,d)) \otimes H^*(\text{Jac})$$

where $\text{Jac} \cong U(1)^{2g}$ is the Jacobian (see Remark 32 below).
4.2. **Extended moduli space.** Let $K = SU(n)$. We recall the construction of the extended moduli space from [23]. This is a finite-dimensional (but non-compact) Hamiltonian $K$-space $M_K^{\text{ext}}$ with moment map $\mu$ such that the symplectic quotient $\mu^{-1}(0)/K$ is homeomorphic to the moduli space $\mathcal{M}_\Lambda(n, d)$.

First consider the space $K^{2g}$. We define the map $\Phi : K^{2g} \to K$ by

$$\Phi(h_1, \ldots, h_{2g}) = \prod_{j=1}^{g} h_{2j-1} h_{2j}^{-1} h_{2j-1}^{-1}$$

and $\omega \in \Omega^2(K^{2g})^K$ is the 2-form defined in [23] §8 (8.28)-(8.30) (cf. [1], §9.1), Theorem 9.1). The space $K^{2g}$ is a quasi-Hamiltonian $K$-space in the terminology of [1]. We recall the definition:

**Definition 14.** ([1]) A manifold $M$ with a $K$-action and a 2-form $\omega$ and a map $\Phi : M \to K$ is a quasi-Hamiltonian $K$-space if it satisfies the following three axioms.

1. The differential of $\omega$ is given by:

$$d\omega = -\Phi^* \chi_K$$

where $\chi_K = (\theta_K, [\theta_K, \theta_K])$ is the differential form on $K$ which represents the generator of $H^3(K, \mathbb{Z})$, in terms of the left invariant Maurer-Cartan form $\theta_K \in \Omega^1(K) \otimes k$.

2. The map $\Phi$ satisfies

$$\iota(\nu_\xi)\omega = \frac{1}{2} \Phi^*(\theta_K + \bar{\theta}_K, \xi)$$

where $\theta_K$ is the left invariant Maurer-Cartan form and $\bar{\theta}_K$ is the corresponding right invariant Maurer-Cartan form. Here, for $\xi \in k$ we denote by $\nu_\xi$ the vector field on $M$ arising from the action of $K$.

3. At each $x \in M$, the kernel of $\omega_x$ is given by

$$\ker \omega_x = \{ \nu_\xi(x) \mid \xi \in \ker(Ad_{\Phi(x)} + 1) \}$$

We can construct a corresponding Hamiltonian $K$-space $M_K^{\text{ext}}$ as follows. We choose an element $c \in Z(K)$, by setting

$$c = e^{2\pi i d/n} I$$

where $I$ is the identity matrix. Let

$$M_K^{\text{ext}} = \{(m, \Lambda) \in K^{2g} \times k : \Phi(m) = c \exp(\Lambda)\}$$

and let $\mu : M_K^{\text{ext}} \to k$ be defined by

$$\mu(m, \Lambda) = -\Lambda.$$

Then we get the following commutative diagram.

$$\begin{array}{ccc}
M_K^{\text{ext}} & \xrightarrow{\mu} & k \\
\pi_1 \downarrow & & \downarrow c \exp \\
K^{2g} & \xrightarrow{\Phi} & K
\end{array}$$

(4.5)
Since $d \exp^* \chi_K = 0$ (where $\chi_K \in \Omega^2(K)$ represents the generator of $H^3(K, \mathbb{Z})$; see Definition 14 above) we can find some $\sigma \in \Omega^2(k)$ such that $d\sigma = \exp^* \chi_K$. We see that $d(\pi^* \omega - \mu^* \sigma) = 0$. The space $M_K^\text{ext}$ defined by (4.5) becomes a Hamiltonian $K$-space with moment map $\mu$ and the invariant 2-form

$$\tilde{\omega} = \pi^* \omega - \mu^* \sigma \in \Omega^2(M_K^\text{ext})^K.$$

The space $M_K^\text{ext}$ is the extended moduli space defined in [23].

There are classes $\tilde{a}_r \in H^2_K(M_K^\text{ext})$ (for $r = 2, \ldots, n$) which pass to $a_r \in H^{2r} (\mathcal{M}_\Lambda(n,d))$ under the composition of the restriction map $H^*_K(M_K^\text{ext}) \to H^*_K(K^{-1}(0))$ and the isomorphism $H^*_K(K^{-1}(0)) \cong H^*(\mathcal{M}_\Lambda(n,d))$ (this composition is sometimes referred to as the Kirwan map). Likewise (for $r = 2, \ldots, n$ and $j = 1, \ldots, 2g$) there are classes $\tilde{b}_r^j \in H^{2r-2}_K(M_K^\text{ext})$ and $\tilde{f}_r \in H^{2r-2}_K(M_K^\text{ext})$ which pass to $b_r^j$ and $f_r$ under the Kirwan map (see [29]). The classes $\tilde{a}_r$ and $\tilde{b}_r^j$ are invariant under translation in $k$.

**Remark 15.** We can modify the infinite dimensional description used in §3 so it applies to the space $\mathcal{M}_\Lambda(n,d)$. It is also possible to treat the space $\mathcal{M}(n,d)$ using the finite dimensional methods of the present section, with $U(n)$ replacing $SU(n)$. The extended moduli space $M_K^\text{ext}$ may be constructed as in [23] by a partial reduction of the infinite-dimensional space $\mathcal{C}$ by the based gauge group. Hence, for our purposes working with the infinite dimensional description using the gauge group and the space of all complex structures is equivalent to working with the finite dimensional description via the extended moduli space for both $\mathcal{M}(n,d)$ and $\mathcal{M}_\Lambda(n,d)$.

### 4.3. Equivariant Poincaré Dual.

Since we know that $K^{2g} \times k$ is always smooth, we will work with integration over $K^{2g} \times k$, instead of working with integration over its subset $M_K^\text{ext}$.

We work with the Cartan model of equivariant cohomology, for which if the space $Y$ is equipped with an action of $K$

$$H^*_K(Y) = H^*(\Omega^*_K(Y), d_K)$$

Here,

$$\Omega^*_K(Y) = \left(\Omega^*(Y) \otimes S(k^*)\right)^K$$

and the equivariant differential $d_K$ is given by

$$d_K(\alpha)(\xi) = d(\alpha(\xi)) - \nu_\xi \alpha$$

for $\xi \in k$ and $\alpha \in \Omega^*_K(Y)$ where $\nu_\xi$ is the vector field on $Y$ generated by $\xi$.

**Lemma 16.** ([29] Corollary 5.6) Let $T$ be the maximal torus of $K = SU(n)$ acting on $K$ by conjugation. If $c \in T$ then we can find a $T$-equivariantly closed differential form $\hat{\alpha} \in \Omega^*_T(K)$ on $K$ with support arbitrarily close to $c$ such that

$$\int_K \eta \hat{\alpha} = \eta|_c \in H^*_T$$

for all $T$-equivariantly closed differential forms $\eta \in \Omega^*_T(K)$.

**Proposition 17.** ([29], Proposition 5.7) Let $P : K^{2g} \times k \to K$ be defined by

$$P : (m, \Lambda) \mapsto \Phi(m) \exp(-\Lambda)$$

$$\left(\sum_{r=1}^{2g} e^a_r \sum_{j=1}^{2g} \sum_{i=1}^{2g} e^{b_r^j} + \sum_{r=1}^{2g} e^{f_r}\right).$$
so that \( M^\text{ext}_K = P^{-1}(c) \). Let \( \alpha = P^*\hat{\alpha} \). Then
\[
\int_{K^{2g} \times_k} \eta_\alpha = \int_{M^\text{ext}_K} \eta
\]
for all \( T \)-equivariantly closed differential forms \( \eta \in \Omega^*_T(K^{2g} \times k) \).

From now on we shall use \( \mu \) to denote the map
\[
\mu : K^{2g} \times k \to k, \quad \mu : (m, \Lambda) \mapsto -\Lambda
\]
as well as its restriction to \( M^\text{ext}_K \). Then \( M^\text{ext}_K = P^{-1}(c) \subset K^{2g} \times k \) and \( M^\text{ext}_K / K = (P^{-1}(c) \cap \mu^{-1}(0))/K \).

Let \( V \) be a small neighbourhood of \( c \) in \( K \). In fact, if \( V' \) is any neighbourhood of \( c \) in \( K \) containing the closure of \( V \) then
\[
\int_{P^{-1}(V')} \eta_\alpha = \int_{M^\text{ext}_K} \eta \in H^*_T
\]

**Lemma 18.** If 0 is a regular value of \( \mu \), then \( (P^{-1}(V) \cap \mu^{-1}(0))/T \) is an orbifold.

**Proof:** Our first observation is that near \( \Phi^{-1}(c) \), the manifold \( K^{2g} \) is endowed with a symplectic structure, as there is a \( K \)-invariant neighbourhood \( V \subset K \) containing \( c \) such that the restriction of the closed 2-form \( \tilde{\omega} \) to \( (\Phi \circ \pi_1)^{-1}(V) \) is nondegenerate (where \( \tilde{\omega} \) was defined at (4.9)). This is true for the following reason. Consider the diagram (4.3). The space \( K^{2g} \) is a smooth manifold, so the space \( M^\text{ext}_K \) is smooth whenever \( d(\exp) \) is surjective (a condition satisfied on a neighbourhood \( V \subset K \) of \( c \)). The two-form \( \tilde{\omega} \) defined by (4.6) is closed on \( M^\text{ext}_K \). The map \( \mu \) satisfies the moment map condition \( d\nu_\xi = \tilde{\omega}(\nu_\xi, \cdot) \) on \( M^\text{ext}_K \), where (for \( \xi \in k \)) we denote by \( \nu_\xi \) the vector field on \( K^{2g} \) arising from the action of \( K \) (see [1] and [23]). Furthermore the 2-form \( \tilde{\omega} \) descends under symplectic reduction from \( (\Phi \circ \pi_1)^{-1}(V) \) to the standard symplectic form on \( M(n, d) \). It follows that \( \tilde{\omega} \) is nondegenerate on \( (\Phi \circ \pi_1)^{-1}(V) \), which is an open neighbourhood of \( \mu^{-1}(0) \) in \( M^\text{ext}_K \).

We know that \( c \) is a regular value for \( P : K^{2g} \times k \to K \), and therefore we can choose the neighbourhood \( V \) so that all points of \( V \) are also regular values of \( P \) (by standard properties of the rank of a differentiable map). Because \( \Phi^{-1}(V) \) is symplectic with moment map \( \mu \) related to \( \Phi \) as in diagram (4.5), the action of \( K \) has finite stabilizers at all points of \( \Phi^{-1}(V) \). This implies that \( T \) also acts with finite stabilizers at all points of \( \Phi^{-1}(V) \). Hence \( P^{-1}(V) \cap \mu^{-1}(0)/T \) is an orbifold.

**Remark 19.** In fact in the case when \( K = SU(n) \) and \( c = e^{2\pi i d/n}I \) generates \( Z(K) \), we may choose \( V \) small enough to guarantee that \( T/Z(G) \) acts freely on \( P^{-1}(V) \cap \mu^{-1}(0) \) (since the action of \( Z(G) \) is trivial), so the quotient \( P^{-1}(V) \cap \mu^{-1}(0)/T \) is a smooth manifold (see [29], Lemma 5.10).

We extend the definition of the composition
\[
\kappa : H_T^*(P^{-1}(c)) \to H_T^*(P^{-1}(c) \cap \mu^{-1}(0)) \cong H^*(P^{-1}(c) \cap \mu^{-1}(0)/T)
\]
to
\[
\kappa : H_T^*(P^{-1}(V)) \to H_T^*(P^{-1}(V) \cap \mu^{-1}(0)) \cong H^*(P^{-1}(V) \cap \mu^{-1}(0)/T)
\]
By Proposition 17 if \( \eta \in H^*_T(K^{2g} \times k) \) we have
\[
\int_{P^{-1}(c) \cap \mu^{-1}(0)/T} \kappa(\eta) = \int_{P^{-1}(V) \cap \mu^{-1}(0)/T} \kappa(\eta \alpha)
\]
where the class \( \alpha \) is the equivariant Poincaré dual of \( P^{-1}(c) \) in \( P^{-1}(V) \). The quantity \( \int_{P^{-1}(V) \cap \mu^{-1}(0)/T} \kappa(\eta \alpha) \) is given by a formula involving iterated residues, as we will see below.

4.4. Periodicity. We define a one dimensional torus \( \hat{T}_1 \cong S^1 \) in \( K \) generated by the element \( \hat{e}_1 = (1, -1, 0, \ldots, 0) \) in the Lie algebra of the standard maximal torus \( T \). Then \( T_1 \) is identified with \( S^1 \) via
\[
e^{2\pi i \theta} \in S^1 \mapsto (\exp \theta \hat{e}_1) \in \hat{T}_1.
\]
The one dimensional Lie algebra \( \hat{t}_1 \) is spanned by \( \hat{e}_1 \). Its orthocomplement in \( t \) is denoted by \( t_{n-1} \). Define \( T_{n-1} \) to be the torus given by \( \exp(t_{n-1}) \). Explicitly this is
\[
T_{n-1} = \{(t_1, t_1, t_3, \ldots, t_{n-1}, t_n) \in U(1)^n : (t_1)^2(n_{j=3} t_j) = 1\}.
\]
Then \( T_{n-1} \) is isomorphic to the maximal torus of \( SU(n-1) \) (i.e. \( T_{n-1} \cong (S^1)^{n-2} \)). We have
\[
t_{n-1} = \{(X_1, \ldots, X_n) \in \mathbb{R}^n : X_1 = X_2, \sum_{j=1}^n X_j = 0\}.
\]
The torus \( T_{n-1} \) has the same Lie algebra as the torus \( T/\hat{T}_1 \).

Lemma 20. (29, Lemma 6.1) Let \( W \) be the Weyl group of \( K = SU(n) \) so that the order of \( W \) is \( |W| = n! \) and let \( c = \text{diag}(e^{2\pi i d/n}, \ldots, e^{2\pi i d/n}) \) where \( d \) is coprime to \( n \). If \( V \) is a sufficiently small neighbourhood of \( c \) in \( K \) that the quotient \( T/Z_n \) of \( T \) by the centre \( Z_n \) of \( K = SU(n) \) acts freely on \( P^{-1}(V) \cap \mu^{-1}(0) \), then for any \( \eta \in H^*_K(M^{ext}_K) \) we have
\[
\int_{M_{\Lambda(n,d)}} \kappa(\eta e^{\omega}) = \frac{1}{|W|} \int_{N(c)} \kappa(D\eta e^{\omega}) = \frac{1}{|W|} \int_{N(V)} \kappa(D\eta e^{\omega})
\]
where
\[
N(c) = (M^{ext}_K \cap \mu^{-1}(0))/T
\]
for \( \mu : K^{2g} \times k \to k \) given by minus the projection onto \( k \) and
\[
N(V) = (P^{-1}(V) \cap \mu^{-1}(0))/T.
\]
Also \( \alpha \) is a \( T \)-equivariantly closed form on \( K^{2g} \times k \) representing the \( T \)-equivariant Poincaré dual to \( M^{ext}_K \), which is chosen as in Proposition 17 so that the support of \( \alpha \) is contained in \( P^{-1}(V) \) and has compact intersection with \( \mu^{-1}(0) \).

Proposition 21. (29, Proposition 6.3; 15) For any symplectic manifold \( M \) equipped with a Hamiltonian action of \( T = T_n \) such that \( T_{n-1} \) acts locally freely on \( \mu^{1}_{T_{n-1}}(0) \), the symplectic quotient \( \mu^{1}_{T_{n-1}}(0)/T_n \) may be identified with the symplectic quotient of \( \mu^{1}_{T_{n-1}}(0)/T_{n-1} \) by the induced Hamiltonian action of \( \hat{T}_1 \). Moreover if in addition \( T_n \) acts locally freely on \( \mu^{1}_{T_n}(0) \) then the ring homomorphism \( \kappa : H^*_T(M) \to H^*(\mu^{1}_{T_n}(0)/T_n) \) which is the composition of restriction with the natural isomorphism \( \kappa : H^*_T(\mu^{1}_{T_n}(0)) \cong H^*(\mu^{1}_{T_n}(0)/T_n) \) factors as
\[
\kappa = \hat{\kappa}_1 \circ \kappa_{n-1}
\]
where
\[ \kappa_{n-1} : H^*_T(M) \to H^*_{T_n}(\mu^{-1}_{T_n}(0)) \cong H^*_{T_1 \times T_{n-1}}(\mu^{-1}_{T_n}(0)) \cong H^*_{\hat{T}_1}(\mu^{-1}_{T_n}(0)/T_{n-1}) \]
and
\[ \hat{\kappa}_1 : H^*_{\hat{T}_1}(\mu^{-1}_{T_n}(0)/T_{n-1}) \to H^*(\mu^{-1}_{T_n}(0)/T_{n-1}) \cong H^*(\mu^{-1}_{T_n}(0)/T) \]
are the corresponding compositions of restriction maps with similar isomorphisms.

**Remark 22.** Let \( T = T_n \) and \( T_{n-1} \) be as in Proposition 21. Note that it is also true that (4.13)
\[ \kappa = \hat{\kappa}_{n-1} \circ \kappa_1 \]
where
\[ \kappa : H^*_T(M) \to H^*_\hat{T}(\mu^{-1}_{\hat{T}}(0)/\hat{T}_1) \]
and
\[ \hat{\kappa}_{n-1} : H^*_\hat{T}(\mu^{-1}_{\hat{T}}(0)/\hat{T}_1) \to H^*(\mu^{-1}_{\hat{T}}(0)/T) \]
are defined in a similar way to \( \hat{\kappa}_1 \) and \( \kappa_{n-1} \).

**Proposition 23. (Dependence of symplectic quotients on parameters) [Guillemin-Kalkman [18], S. Martin [45]]** Let \( M \) be a Hamiltonian \( T \)-space (where \( T = U(1) \)). If \( n_0 \) is the order of the stabilizer in \( T \) of a generic point of \( M \) then
\[ \int_{\mu^{-1}(\xi_1)/T} (\eta e^{\omega})_{\xi_1} - \int_{\mu^{-1}(\xi_0)/T} (\eta e^{\omega})_{\xi_0} = n_0 \sum_{E \in \mathcal{E}, \xi_0 < \mu_T(E) < \xi_1} \text{res}_{X=0} e^{\mu_T(E)X} \int_E \eta(X) e^{\omega} \]
where \( X \in \mathbb{C} \) has been identified with \( 2\pi i X \in t \otimes \mathbb{C} \) and \( \xi_0 < \xi_1 \) are two regular values of the moment map. Here \( \mathcal{E} \) is the set of components of the fixed point set of \( \hat{T}_1 \) on \( M \).

Using Remark 22 the following lemma is proved exactly as Lemma 6.7 of 29.

**Lemma 24.** Suppose that 0 is a regular value of \( \mu_T \) and that \( \eta \) is an equivariant cohomology class on \( M^\text{ext}_K \), which is a polynomial in the classes \( \hat{\alpha}_r \) and \( \hat{\beta}_r \). Suppose also that 0 is a regular value of \( \mu_{T/\hat{T}_1} : E \to t/\hat{T}_1 \) for all components \( E \) of the fixed point set of \( \hat{T}_1 \). If \( V \) is a sufficiently small \( T \)-invariant neighbourhood of \( c \) in \( K \) so that \( P^{-1}(V) \cap \mu^{-1}(\hat{\xi}_1)/T_{n-1} \) is an orbifold and we define \( N(V) = P^{-1}(V) \cap \mu^{-1}(0)/T \), then
\[ \int_{N(V)} \kappa(\eta e^{\omega} e^{-Y_1} \alpha) = \int_{P^{-1}(V) \cap \mu^{-1}(\hat{\xi}_1)/T_n} \kappa(\eta e^{\omega} \alpha) \]
\[ = \int_{N(V)} \kappa(\eta e^{\omega} \alpha) - n_0 \sum_{E \in \mathcal{E}, ||\hat{\xi}_1|| < ||\hat{\xi}_1, \mu(T) < 0} \int_{E/\hat{T}_{n-1}} \kappa_{T/\hat{T}_1} \text{res}_{Y_1=0} \eta e^{\omega} \alpha \]
where \( \mathcal{E} \) is the set of components of the fixed point set of the action of \( \hat{T}_1 \) on \( K^{2g} \times t \), and \( e_E \) is the \( \hat{T}_1 \)-equivariant Euler class of the normal to \( E \) in \( K^{2g} \), while \( n_0 \) is the order of the subgroup of \( \hat{T}_1/(\hat{T}_1 \cap T_{n-1}) \) that acts trivially on \( K^{2g} \times t \). Here \( Y_1 \) is a complex variable defined by \( < \hat{\xi}_1, X > = Y_1 \), where \( X \in t \otimes \mathbb{C} \), and \( \alpha \) is the \( T \)-equivariantly closed differential form on \( K^{2g} \times t \) given by Proposition 17, which represents the equivariant Poincaré dual of \( M^\text{ext}_K \), chosen so that the support of \( \alpha \) is contained in \( P^{-1}(V) \).
Remark 25. The proof of this lemma uses the fact that the restriction of \( P : K^{2g} \times k \to K \) to \( \mu^{-1}(t) = K^{2g} \times t \) is invariant under the translation \( s_{\Lambda_0} : K^{2g} \times k \to K^{2g} \times k \) defined by \( s_{\lambda_0} : (m, \Lambda) \mapsto (m, \Lambda + \lambda_0) \) for \( \lambda_0 \in \Lambda^I = \ker(\exp) \) in \( t \), and so is the polynomial \( \eta \) in the classes \( \bar{a}_r \) and \( \bar{b}_r^j \) (see [23]).

Remark 26. If \( \mu_T : M \to t^* \) is a moment map for the action of a torus \( T \) on a symplectic manifold \( M \), then we can add any constant \( \xi \) in \( t^* \) to \( \mu_T \) and get another moment map. Then by applying Proposition 21 to \( \mu_T - \xi \) we can generalise the proposition to apply to \( \mu_{T_n-1}(\xi_{n-1}) \) and \( \mu_{T_1}(\xi_1) \) where \( \xi_{n-1} \) and \( \xi_1 \) are the projections of \( \xi \) into \( t_{n-1} \) and \( t_1 \). A similar generalisation is available for Lemma 24.

From Lemma 24 we get
\[
\int_{N(V)} \kappa(\eta e^\omega \alpha) - \int_{N(V)} \kappa(\eta e^\omega e^{Y_1} \alpha) = \int_{N(V)} \kappa(\eta e^\omega (1 - e^{Y_1}) \alpha)
\]
(4.14)
\[
= n_0 \sum_{E \in E : -||\tilde{\eta}|^2 < \langle \tilde{\eta}, \mu(E) \rangle < 0} \int_{E/T_{n-1}} \kappa_{n-1} \operatorname{res}_{Y_1=0} \frac{\eta e^\omega \alpha}{e_E}
\]
Therefore
\[
\int_{N(V)} \kappa(\eta e^\omega \alpha) = n_0 \sum_{E \in E : -||\tilde{\eta}|^2 < \langle \tilde{\eta}, \mu(E) \rangle < 0} \int_{E/T_{n-1}} \kappa_{n-1} \operatorname{res}_{Y_1=0} \frac{(\eta e^\omega \alpha)}{e_E (1 - e^{Y_1})}
\]

4.5. Formulas for intersection pairings. From [30] we have

Lemma 27. Let \((M, \omega, \mu)\) be a quasi-Hamiltonian \( K \)-space and \( \tilde{T}_1 \cong S^1 \) be a circle subgroup of \( K \). If \( H \) is the fixed point set of the adjoint action of \( \tilde{T}_1 \) on \( K \), then \( H \) is a Lie subgroup of \( K \) and the fixed point set \( M_{\tilde{T}_1} \) is a quasi-Hamiltonian \( H \)-space.

Remark 28. In our particular case we take \( M = K^{2g} \), and \( M_{\tilde{T}_1} \) then has the form \( H^{2g} \) where \( T \subset H \) and \( \tilde{T}_1 \subset Z(H) \). Thus \( T_{n-1} = T/\tilde{T}_1 \) is a group of rank \( n-2 \) with a quasi-Hamiltonian action on \( H^{2g} \). This enables us to perform an inductive argument.

The main result of [29] is the following.

Theorem 29. ([29], Theorem 8.1) Let \( c = \operatorname{diag}(e^{2\pi i d/n}, \ldots, e^{2\pi i d/n}) \) where \( d \in \{1, \ldots, n-1\} \)
is coprime to \( n \), and suppose that \( \eta \in H^*_K(\mu_{K_0}^*) \) is a polynomial \( Q(\tilde{a}_2, \ldots, \tilde{a}_n, \tilde{b}_2^1, \ldots, \tilde{b}_n^{2g}) \) in the equivariant cohomology classes \( \tilde{a}_r \) and \( \tilde{b}_r^j \) for \( 2 \leq r \leq n \) and \( 1 \leq j \leq 2g \) introduced in \( \S 4.2 \). Then the pairing
\[
Q(a_2, \ldots, a_n, b_2^1, \ldots, b_n^{2g}) \exp(f_2)[\mathcal{M}_{\Lambda}(n, d)]
\]
is given by
\[
\int_{\mathcal{M}_{\Lambda}(n, d)} \kappa(e^\omega) = (-1)^{n_+} (g-1)! \operatorname{res}_{Y_1=0} \ldots \operatorname{res}_{Y_{n-1}=0} \left( \sum_{w \in W_{n-1}} \frac{1}{D^{2g-2}} \prod_{1 \leq j \leq n-1} (\exp(Y_j^w) - 1) \right),
\]
where \( n_+ = \frac{1}{2} n(n-1) \) is the number of positive roots of \( K = SU(n) \) and \( X \in t \) has coordinates \( Y_1 = X_1 - \tilde{X}_2, \ldots, Y_{n-1} = X_{n-1} - X_n \) defined by the simple roots, while \( W_{n-1} \cong S_{n-1} \) is...
Let Proposition 30. The proof reduces to proving the following Proposition 30, which is in fact more general because we are no longer assuming that \( c = \text{diag}(e^{2\pi i d/n}, \ldots, e^{2\pi i d/n}) \) (and therefore this proposition provides formulas for pairings in moduli spaces of parabolic bundles \([10]\)).

Theorem 29 can be proved inductively using Lemma 20 and (4.15), together with Lemma 27. The proof reduces to proving the following Proposition 30, which is in fact more general because we are no longer assuming that \( c = \text{diag}(e^{2\pi i d/n}, \ldots, e^{2\pi i d/n}) \) (and therefore this proposition provides formulas for pairings in moduli spaces of parabolic bundles \([10]\)).

**Proposition 30.** Let \( c = \text{diag}(c_1, \ldots, c_n) \in T \) be such that the product of no proper subset of \( c_1, \ldots, c_n \) is 1. If \( \eta(X) \) is a polynomial in the \( \tilde{a}_r \) and \( \tilde{b}_r \), then

\[
\int_{N(c)} \kappa(D_n \eta e^{\omega}) = (-1)^{n+(g-1)} \text{res}_{Y_1=0} \cdots \text{res}_{Y_{n-1}=0} \left( \frac{\sum_{w \in W_{n-1}} e([\gamma_w]) X}{D^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(Y_j) - 1)} \right),
\]

where \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n) \in \mathbf{t}_n \) satisfies \( e^{2\pi i \tilde{c}} = c \) and belongs to the fundamental domain defined by the simple roots for the translation action on \( \mathbf{t}_n \) of the integer lattice \( \Lambda^I \) and

\[
N(c) = (\mu^{-1}(0) \cap M_K) / T.
\]

The other notation is as in Theorem 29.

### 4.6. Extension to general pairings.

So far we have considered pairings of powers of the classes \( a_r, b'_r \) and the Kähler class \( f_2 \). We now explain the general case.

We define \( q \in S(k^*)^K \) to be an invariant polynomial, which is given in terms of the elementary symmetric polynomials \( \tau_j \) by

\[(4.16) \quad q(X) = \tau_2(X) + \sum_{r=3}^n \delta_r \tau_r(X).\]

The associated element \( \tilde{f}_q \) of \( H^*_K(M_K^{\text{ext}}) \) is defined by

\[(4.17) \quad \tilde{f}_q = f_2 + \sum_{r=3}^n \delta_r \tilde{f}_r.\]

Here, the \( \delta_r \) are formal nilpotent parameters: we expand \( \exp \tilde{f}_q \) as a formal power series in the \( \delta_r \).

**Theorem 31.** (29, Theorem 9.11(a)) For \( q \) and \( \tilde{f}_q \) defined as above and

\[ \eta = Q(\tilde{a}_2, \ldots, \tilde{a}_n, \tilde{b}_2, \ldots, \tilde{b}_n) \]

as in Theorem 29, the pairing

\[ Q(a_2, \ldots, a_n, b'_2, \ldots, b'_n) \exp(f_2 + \sum_{r=3}^n \delta_r f_r) [\mathcal{M}_\Lambda(n, d)] \]
is given by
\[
\frac{1}{n!} \int_{N(c)} \kappa(e^{\tilde{f}_{(a)i}} \mathcal{D}\eta) = \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \prod_{Y_1=0}^{n-1} \prod_{Y_{n-1}=0}^{\infty} \int_{T^{2g} \times \{-\lfloor \omega c \rfloor\}} \frac{e^{\tilde{f}_{(a)}(X) \eta}(X)}{D(X)^{2g-2} \prod_{j=1}^{n-1} \exp(-B(-X)_j - 1)},
\]
when \( c = \text{diag} \left( e^{2\pi i d_1/n}, \ldots, e^{2\pi i d/n} \right) \) and
\[d \in \{1, \ldots, n-1\}\]
is coprime to \( n \). In addition (4.18) is true more generally for any \( c = \text{diag}(c_1, \ldots, c_n) \in T \) such that the product of no proper subset of \( c_1, \ldots, c_n \) is 1. Here \( B(X)_j = -(dq)_X(\hat{e}_j) \); we have used the fixed invariant inner product on \( k \) to identify \( dq_X : t \to \mathbb{R} \) with an element of \( t \) and thus define the map \( B : t \to t \). The other notation \([\gamma] \) is as in Theorem 29 and Proposition 30.

**Remark 32.** There are exact sequences
\[
1 \to SU(n) \to U(n) \xrightarrow{\text{det}} U(1) \to 1
\]
and thus a finite covering map
\[
\mathcal{M}_\Lambda(n, d) \times \text{Jac} \to \mathcal{M}(n, d)
\]
with fiber \( \mathbb{Z}_{n}^{2g} \) which induces an isomorphism
\[
H^*(\mathcal{M}(n, d)) \cong H^*(\mathcal{M}_\Lambda(n, d)) \otimes H^*(\text{Jac})
\]
(see 3). As a result the cohomology of the space \( \mathcal{M}(n, d) \) is related to that of \( \mathcal{M}_\Lambda(n, d) \) by introducing the additional generators \( b_j^1 \in H^1(\mathcal{M}(n, d)) \) (corresponding to the generators of the cohomology of the Jacobian) where \( j = 1, \ldots, 2g \). Intersection pairings on \( \mathcal{M}(n, d) \) are related to the corresponding pairings on \( \mathcal{M}_\Lambda(n, d) \) by a factor \( n^{2g} \) corresponding to the order of the fiber in (4.21).

Similarly we have a covering map
\[
\mu_{SU(n)}^{-1}(\tilde{c})/T_{SU(n)} \times \text{Jac} \to \mu_{U(n)}^{-1}(\tilde{c})/T_{U(n)}
\]
with fiber \( \mathbb{Z}^{2g}_n \).

The results of the remainder of this paper could be phrased equally well in terms of the moduli spaces \( \mathcal{M}(n, d) \) or the moduli spaces \( \mathcal{M}_\Lambda(n, d) \) of holomorphic vector bundles with fixed determinant, though some care is needed when comparing the partial desingularisations \( \tilde{\mathcal{M}}(n, d) \) and \( \tilde{\mathcal{M}}_\Lambda(n, d) \): the covering map
\[
\tilde{\mathcal{M}}_\Lambda(n, d) \times \text{Jac} \to \tilde{\mathcal{M}}(n, d)
\]
does not induce an isomorphism from \( H^*(\tilde{\mathcal{M}}(n, d)) \) to \( H^*(\tilde{\mathcal{M}}_\Lambda(n, d)) \otimes H^*(\text{Jac}) \).

For simplicity, from now on we have chosen to restrict our treatment to the moduli spaces \( \mathcal{M}(n, d) \).
5. Pairings in intersection cohomology

In this section, we show that the pairings in the intersection cohomology $IH^*(\mathcal{M}(n,d))$ are given by essentially the same formulas as in the nonsingular case reviewed in §4, but with a small shift.

In [32] it is shown that the weakly balanced condition ([32] §7 or [25] §5) is satisfied for the geometric invariant theoretic construction of $\mathcal{M}(n,d)$. This means that the subspace $V(n,d) = V^\text{M}_{\text{ext}}(\mu^{-1}(0)) \cong H^*_{\mathcal{G}(n,d)}(C(n, d)^{ss})$ defined in [32] is isomorphic to $IH^*(\mathcal{M}(n,d))$ where $K = SU(n)$ and $\overline{\mathcal{G}}(n,d) = \mathcal{G}(n,d)/U(1)$. To define $V(n,d)$ we consider as in §3 a set of representatives

\begin{equation}
R = \prod_{i=1}^{q} GL(m_i; \mathbb{C}) \quad \text{where} \quad \sum_{i=1}^{q} m_i n_i = n \quad \text{and} \quad n_i d = d_i n
\end{equation}

of the conjugacy classes of reductive subgroups of $\mathcal{G}_c(n,d)$ which occur as stabilizer groups of semistable bundles. As in §3 we have

$Z_R^{ss} \cong \prod_{i=1}^{q} C(n_i, d_i)^{ss}$

and if $N_0^R = N_0$ is the connected component of the normaliser $N^R = N$ of $R$ in $\mathcal{G}_c(n,d)$ then

$N_0^R = \prod_{i=1}^{q} (GL(m_i; \mathbb{C}) \times \mathcal{G}_c(n_i, d_i))/\mathbb{C}^*$,

$N_0^R/R = \prod_{i=1}^{q} \mathcal{G}_c(n_i, d_i)/\mathbb{C}^* = \prod_{i=1}^{q} \overline{\mathcal{G}}(n_i, d_i),$

$\pi_0 N_R^R = \prod_{j \geq 0, k \geq 0} \text{Sym}(\# \{i : m_i = j, n_i = k\})$

and hence

$H^*_{N_0^R/R}(Z_R^{ss}) \cong \bigotimes_{i=1}^{q} H^*_{\overline{\mathcal{G}}(n_i, d_i)}(C(n_i, d_i)^{ss}).$

The obvious maps

$\overline{\mathcal{G}}_c \times_{N_R} Z_R^{ss} \rightarrow \overline{\mathcal{G}}_c Z_R^{ss} \hookrightarrow C(n,d)^{ss}$

give rise to a map

$H^*_{\overline{\mathcal{G}}(n,d)}(C(n,d)^{ss}) \rightarrow H^*_{N_R^R}(Z_R^{ss}) \cong [H^*_{N_0^R/R}(Z_R^{ss}) \otimes H^*_{\pi_0^R N^R} \hookrightarrow \bigotimes_{i=1}^{q} H^*_{\overline{\mathcal{G}}(n_i, d_i)}(C(n_i, d_i)^{ss})] \otimes H^*_{R}$

where $H^*_{R} = H^*(BR)$ denotes the $R$-equivariant cohomology of a point. The subspace $V(n,d)$ is defined to be the intersection of the kernels of the compositions

$H^*_{\overline{\mathcal{G}}(n,d)}(C(n,d)^{ss}) \rightarrow \bigotimes_{i=1}^{q} H^*_{\overline{\mathcal{G}}(n_i, d_i)}(C(n_i, d_i)^{ss}) \otimes H^*_{R} \rightarrow \bigotimes_{i=1}^{q} H^*_{\overline{\mathcal{G}}(n_i, d_i)}(C(n_i, d_i)^{ss}) \otimes H^*_{R}^{-n_R}$

for all $R$ where $H^*_{R}^{-n_R} = \bigoplus_{j \geq n_R} H^j_{R}$ and $n_R$ is given by

$n_R = \dim_{\mathbb{C}} H^1(\Sigma, \text{End}^\vee_{\mathbb{C}} E) - \dim_{\mathbb{C}} R$
which is easily computable using Riemann–Roch. If we denote the equivariant universal bundle over \( C(n, d)^{ss} \times \Sigma \) by \( \mathbb{U}(n, d) \) then the restriction of the generators of the equivariant cohomology rings can also be easily computed from the equation of Chern classes

\[
c(\mathbb{U}(n, d))|_{Z^*_R} = \prod_{i=1}^q c(\mathbb{C}^{m_i} \otimes \mathbb{U}(n_i, d_i))
\]

(see §7 for the computation in the rank 2 case). Therefore, given an equivariant cohomology class in \( H^*_K(\mu^{-1}(0)) \cong H^*_U(n, d)(C(n, d)^{ss}) \), it is straightforward to check if it belongs to \( V(n, d) \) or not. In particular, when the rank \( n \) is 2, we can write down \( V(2, d) \) explicitly as worked out in [33] (Theorem 5.3) using the results of [34].

Let \( \alpha, \beta \) be two classes in \( H^*_K(M^\text{ext}_{U(n)}) \) with \( \deg \alpha + \deg \beta = \dim \mathcal{M}(n, d) \) such that their restrictions to \( \mu^{-1}(0) \) lie in \( V(n, d) \cong IH^*(\mathcal{M}(n, d)) \). Then the restriction of their cup product \( \alpha \beta \) to \( \mu^{-1}(0) \) also lies in \( V(n, d) \) by [25] Theorem 14 or [32] Theorem 5.3. By the de Rham model for intersection cohomology, any top degree class in \( IH^*(\mathcal{M}(n, d)) \) is represented by a differential form \( \eta \), \textit{compactly supported} on the smooth part \( \mathcal{M}(n, d)^s \) of \( \mathcal{M}(n, d) \); the pairing \( \langle \kappa(\alpha), \kappa(\beta) \rangle \) in \( IH^*(\mathcal{M}(n, d)) \) of the classes \( \kappa(\alpha) \) and \( \kappa(\beta) \) represented by \( \alpha \) and \( \beta \) is the integral of any such differential form representing \( \alpha \beta \). By pulling back \( \eta \) to \( \mu^{-1}(0)^s \), \( \alpha \beta \) is represented by an equivariant differential form compactly supported on \( \mu^{-1}(0)^s \) where \( \mu^{-1}(0)^s \) denotes the smooth part in \( \mu^{-1}(0) \) so that \( \mu^{-1}(0)^s/K = \mathcal{M}(n, d)^s \).

By Martin’s argument (see (2.9) above) using the fibration

\[
\begin{array}{ccc}
K/T & \longrightarrow & \mu^{-1}(0)^s/T \\
\pi \downarrow & & \downarrow \\
\mu^{-1}(0)^s/K & = & \mathcal{M}(n, d)^s
\end{array}
\]

the pairing \( \langle \kappa(\alpha), \kappa(\beta) \rangle \) is given by

\[
\int_{\mathcal{M}(n, d)} \kappa(\alpha) \wedge \kappa(\beta) = \frac{1}{n!} \int_{\mu^{-1}(0)/T} \kappa(\alpha \beta \mathcal{D}).
\]

Let \( \varepsilon \in \mathfrak{t}^* \) be a regular value sufficiently close to 0. Then there is a surjective map

\[
\mu^{-1}(\varepsilon)/T \to \mu^{-1}(0)/T
\]

induced by the gradient flow of minus the normsquare \(-|\mu|^2\) of the moment map, which is a diffeomorphism over the smooth part \( \mu^{-1}(0)^s/T \).\(^8\) Hence we have

\[
\frac{1}{n!} \int_{\mu^{-1}(0)/T} \kappa(\alpha \beta \mathcal{D}) = \frac{1}{n!} \int_{\mu^{-1}(\varepsilon)/T} \kappa(\varepsilon)(\alpha \beta \mathcal{D}).
\]

Therefore we deduce that the pairing \( \langle \kappa(\alpha), \kappa(\beta) \rangle \) in \( IH^*(\mathcal{M}(n, d)) \) is given by

\[
\langle \kappa(\alpha), \kappa(\beta) \rangle = \frac{1}{n!} \int_{\mu^{-1}(\varepsilon)/T} \kappa(\varepsilon)(\alpha \beta \mathcal{D}) \tag{5.2}
\]

for any \( \varepsilon \in \mathfrak{t}^* \) sufficiently close to 0.

\(^8\)One way to see the diffeomorphism is to view the \( T \)-quotients as the moduli spaces of parabolic bundles (see [23]). When the underlying vector bundle of a parabolic bundle is stable and the parabolic weight \( \varepsilon \) is sufficiently small, the stability of the parabolic bundle does not change as we move \( \varepsilon \) around 0. This gives us the diffeomorphism.
To compute the right hand side of (5.2) by using the formulas described in §4 from [29] §8 and §9 (where $M_{SU(n)}^{\text{ext}}$ is used instead of $M_{U(n)}^{\text{ext}}$, i.e. the determinant of semistable bundles is fixed there), we consider the fibration

$$
\mu^{-1}_A(\varepsilon)/T \longrightarrow \mu^{-1}(\varepsilon)/T
$$

where $\mu_A : M_{SU(n)}^{\text{ext}} \to k^*$ is the moment map for $M_{SU(n)}^{\text{ext}}$. Integrating over the fiber first, we get

$$
\frac{1}{n!} \int_{\mu^{-1}(\varepsilon)/T} \kappa^{(\varepsilon)}(\alpha \beta D) = \frac{1}{n!} \int_{(S^1)^{2g}} \int_{\mu^{-1}(\varepsilon)/T} \kappa^{(\varepsilon)}(\alpha \beta D).
$$

The inner integral is given by Theorem 31 and so we have proved the following.

**Theorem 33.** Let $\alpha, \beta$ be two classes in $H_{\mathbb{Z}}^*(\mathcal{C}(n, d))$ with deg $\alpha + \deg \beta = \dim_k \mathcal{M}(n, d)$ such that their restrictions to $\mathcal{C}(n, d)^{ss}$ lie in $\mathcal{V}(n, d) \cong IH^*(\mathcal{M}(n, d))$. For $k = (k_2, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n$, let $f^k = \prod_{r=2}^n f^{r, r}$, let $k! = \prod_{r=2}^n k_r!$ and let $\delta^k = \prod_{r=3}^n \delta^{r, r}$. Write

$$
\alpha \beta = \sum_k Q_k(a_2, \ldots, a_n, b_1^1, \ldots, b_n^{2g}) \frac{r^k}{k!}
$$

where $Q_k$ is a polynomial of the Atiyah–Bott classes $a_r, b^j_r \in H^*_{\mathbb{Z}}(\mathcal{C}(n, d))$ defined in Remark 13. Here, and later, for simplicity of notation we think of this as representing the class

$$
\sum_k Q_k(\tilde{a}_2, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n) \frac{\tilde{r}^k}{k!}
$$

in the equivariant cohomology $H^*_{\mathbb{Z}}(\mathcal{M}^{\text{ext}})$ of the extended moduli space. Then, using Theorem 37 the pairing $\langle \kappa(\alpha), \kappa(\beta) \rangle$ of $\kappa(\alpha)$ and $\kappa(\beta)$ in $IH^*(\mathcal{M}(n, d))$ is given by

$$
\frac{(-1)^{n+g-1}}{n!} \sum_k \text{Coeff}_{\delta^k} \left( \sum_{w \in W_{n-1}} \text{res}_{Y_1=0} \ldots \text{res}_{Y_{n-1}=0} \frac{\int_{(S^1)^{2g}} \int_{T^{2g} \times \{[w^\varepsilon]\}} (e^{\tilde{f}_2(X)}Q_k(X))}{D(X)^{2g-2} \prod_{j=1}^{n-1} \exp(-B(X)j - 1)} \right)
$$

for any $\tilde{c}$ sufficiently close to $\tilde{c}_0$ where $\tilde{c}_0$ is the unique element of $t_n$ which satisfies $e^{2\pi i \tilde{c}_0} = \text{diag}(e^{2\pi i \tilde{c}_0/n}, \ldots, e^{2\pi i \tilde{c}_0/n})$ and belongs to the fundamental domain defined by the simple roots for the translation action on $t_n$. Here

$$
\tilde{f}_2 = \hat{f}_2 + \sum_{r=3}^n \delta^r \hat{f}_r
$$

as at (4.17) where the $\delta_r$ are formal nilpotent parameters, and we expand $\exp \hat{f}_2$ as a formal power series in the $\delta_r$; the coefficient of $\delta^k$ is denoted by $\text{Coeff}_{\delta^k}$ and $B(X)j = -(d|q)_X(\tilde{c}_j)$ as in Theorem 37.

In particular, if $\alpha \beta$ is a polynomial $Q$ in the classes $a_2, \ldots, a_n, b_1^1, \ldots, b_n^{2g}$ then by Proposition 30 we have

$$
\langle \kappa(\alpha), \kappa(\beta) \rangle = \frac{(-1)^{n+g-1}}{n!} \text{res}_{Y_1=0} \ldots \text{res}_{Y_{n-1}=0} \left( \sum_{w \in W_{n-1}} e^{\langle [w^\varepsilon]\rangle X} \frac{\int_{(S^1)^{2g}} \int_{T^{2g}} Qe^\varepsilon}{D^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(Y_j) - 1)} \right)
$$
as in the coprime case, while if $\alpha \beta$ is the product of $(f_2)^k$ with a polynomial $Q$ in the classes $a_2, \ldots, a_n, b_1, \ldots, b_{2g}^n$ then
\[
\langle \kappa(\alpha), \kappa(\beta) \rangle = \frac{(-1)^{n+g-1}k!}{n!} \sum_{y_1=0}^{\text{res} Y_1=0} \cdots \sum_{y_{n-1}=0}^{\text{res} Y_{n-1}=0} \left( \frac{\sum_{w \in W_{n-1}} c((w,\xi),X) \int_{(S^1)^{2g}} \int_{(T^2)^g} \sum_{y_{n-1}=0}^{\text{res} Y_{n-1}=0} \frac{\exp(Y_j) - 1}{\prod_{1 \leq j \leq n-1} \exp(B(-X)j - 1)} \right).
\]

6. Pairings on the partial desingularisation $\tilde{\mathcal{M}}(n,d)$

In this section we will study pairings on the partial desingularisation $\tilde{\mathcal{M}}(n,d)$ using the method described in §8 of $\cite{25}$. In the notation of §2 above, $\cite{25}$ §8 provides a method for calculating pairings $\kappa_{\tilde{\mathcal{M}}}(\alpha \beta)[\tilde{\mathcal{M}}/G]$ in the cohomology $H^*(\tilde{\mathcal{M}}/G)$ of classes $\kappa_{\tilde{\mathcal{M}}}(\alpha)$ and $\kappa_{\tilde{\mathcal{M}}}(\beta)$ in the image of the composition
\[
H^*_K(M) \to H^*_K(\tilde{\mathcal{M}}) \to H^*(\tilde{\mathcal{M}}/G)
\]
of the pullback from $M$ to $\tilde{\mathcal{M}}$ and the map $\kappa_{\tilde{\mathcal{M}}}$. The first observation is that
\[
(6.1) \quad \kappa_{\tilde{\mathcal{M}}}(\alpha \beta)[\tilde{\mathcal{M}}//G] = \frac{1}{|W|} \kappa^T_{\tilde{\mathcal{M}}}(\alpha \beta D)[\mu^{-1}(0)/T] = \frac{(-1)^{n+g-1}}{|W|} \kappa^T_{\tilde{\mathcal{M}}}(\alpha \beta D^2)[\tilde{\mathcal{M}}/T_c]
\]
by (2.8) and (2.9), and the next is that if $\xi$ is any regular value of the $T$-moment map $\mu_T$ for $M$ and we choose the symplectic structure appropriately (as a sufficiently small perturbation of the pullback of the symplectic structure on $M$; cf. $\cite{25}$ §4) then
\[
(6.2) \quad \kappa^T_{\tilde{\mathcal{M}},\xi}(\alpha \beta D)[\mu^{-1}(\xi)/T] = \kappa^T_{\tilde{\mathcal{M}},\xi}(\alpha \beta D)[\mu^{-1}(\xi)/T],
\]
where the latter expression can in our case (for $M = M_{\text{ext}}^{\text{ext}}(n)$) be calculated as in §5.

**Remark 34.** Indeed the proof of Theorem $\cite{33}$ tells us that if $\alpha$ and $\beta$ are two classes in $H^*_y(\mathcal{C}(n,d))$ with $\deg \alpha + \deg \beta = \dim_{\mathbb{R}} \mathcal{M}(n,d)$ and
\[
\alpha \beta = \sum_k Q_k(a_2, \ldots, a_n, b_1, \ldots, b_{2g}^n) \frac{j^k}{k!}
\]
for polynomials $Q_k$ where the sum runs over $k = (k_2, \ldots, k_n) \in \mathbb{Z}^{n-1}_{\geq 0}$, then the pairing $\kappa^T_{\tilde{\mathcal{M}},\xi}(\alpha \beta D)[\mu^{-1}(\xi)/T]$ is given by
\[
(6.3) \quad \frac{(-1)^{n+g-1}}{n!} \sum_k \text{Coeff}_{j^k} \left( \sum_{w \in W_{n-1}} \prod_{y_{n-1}=0}^{\text{res} Y_{n-1}=0} \frac{\int_{(S^1)^{2g}} \int_{(T^2)^g} \sum_{y_{n-1}=0}^{\text{res} Y_{n-1}=0} \frac{\exp(-B(-X)j - 1)}{\prod_{1 \leq j \leq n-1} \exp(B(-X)j - 1)} \right)
\]
where $\tilde{c} = \tilde{c}_0 + \xi$ and $\tilde{c}_0$ is the unique element of $\mathfrak{t}_n$ which satisfies $e^{2\pi i \tilde{c}_0} = \text{diag}(e^{2\pi i / n}, \ldots, e^{2\pi i / n})$ and belongs to the fundamental domain defined by the simple roots for the translation action on $\mathfrak{t}_n$ of the integer lattice $\Lambda^T$. Here $j^k = \prod_{r=2}^n j^k_r$, $k! = \prod_{r=2}^n k_r!$ and $\delta^k = \prod_{r=3}^n \delta_r^k$ where the $\delta_r$ are formal nilpotent parameters, and
\[
\tilde{f}(q) = \tilde{f}_2 + \sum_{r=3}^n \delta_r \tilde{f}_r
\]
as at (1.14). Finally $B(X)j = -(dq)_X(\tilde{e}_j)$ as in Theorem $\cite{31}$. 

This means that to calculate the pairing of the cohomology classes induced by $\alpha$ and $\beta$ in $H^*(\mathcal{M}(n, d))$ it suffices to calculate the difference between

\begin{equation}
\kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\mu^{-1}(\xi)/T]
\end{equation}

and

\begin{equation}
\kappa^T_{\mathcal{M}}(\alpha\beta D)[\mu^{-1}(0)/T]
\end{equation}

when $M = M^{\text{ext}}_{\psi(\xi)}$.

Since the partial desingularisation process takes place in stages, it is easiest to consider a single stage of the construction, when a blow-up along the proper transform $\hat{Z}_R//N$ of $Z_R//N$ with

$$R = GL(m_1; \mathbb{C}) \times \ldots \times GL(m_q; \mathbb{C})$$

as at \ref{3.1} results in $\hat{M} // G$ which in our case can be written as $\hat{\mathcal{M}}(n, d) = \hat{\mathcal{C}}//\mathcal{G}_c$. (Note that in \cite{25} the superscript $^*$ in $\hat{Z}_R//N$ is omitted since, for simplicity, it is assumed there that the blow up along $\hat{Z}_R//N$ is the first in the partial desingularisation process and hence $\hat{Z}_R//N = Z_R//N$.) Let $\mu$ and $\hat{\mu}_T$ be the moment maps for the actions of $K$ and $T$ on $\hat{M}$; then as at \ref{6.2} when $\xi$ is a regular value of $\mu_T$ we have

\begin{equation}
\kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\hat{\mu}^{-1}(\xi)/T] = \kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\mu^{-1}(\xi)/T]
\end{equation}

provided that we have made a sufficiently small perturbation of the pullback to $\hat{M}$ of the symplectic form on $M$ (where ‘sufficiently small’ depends on $\xi$). We choose $\xi \in \mathfrak{t}^*$ to lie in a connected component $\Delta_{(i)}$ of the set of regular values of $\mu_T$ for which $0 \in \Delta_{(i)}$, and choose $\hat{\xi} \in \mathfrak{t}^*$ to lie in the intersection of $\Delta_{(i)}$ and a connected component of the set of regular values of $\hat{\mu}_T$ which contains $0$ in its closure. We cannot necessarily choose $\hat{\xi} = \xi$ because the choices of symplectic structure on $\hat{M}$ and the moment maps $\mu$ and $\hat{\mu}_T$ depend on $\xi$, but it is enough to calculate the difference

\begin{equation}
\kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\hat{\mu}^{-1}(\xi)/T] - \kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\mu^{-1}(\xi)/T],
\end{equation}

since repeating this for each stage and using \ref{6.2} and \ref{6.6} will give us the difference between

$\kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\mu^{-1}(\xi)/T]$ and $\kappa^T_{\mathcal{M}, \xi}(\alpha\beta D)[\hat{\mu}^{-1}(\xi)/T]$ for any $\xi$ in a connected component of the set of regular values of $\mu_T$ which contains $0$ in its closure. Since $0$ is itself a regular value of $\hat{\mu}_T$, we can choose $\xi$ to be $0$ and thus calculate the pairing \ref{6.1}.

Recall that the image $\hat{\mu}_T(\hat{M})$ of the moment map $\hat{\mu}_T$ is a convex polytope which is divided by walls of codimension one into subpolytopes whose interiors consist of regular values of $\hat{\mu}_T$. The pairings we wish to calculate are unchanged as $\xi$ varies within a connected component of the set of regular values of $\hat{\mu}_T$, so it is enough to be able to calculate the change as $\xi$ crosses a wall of codimension one. Any such wall is of the form

$$\hat{\mu}_T(\hat{M}_1)$$

where $\hat{M}_1$ is a connected component of the fixed point set in $\hat{M}$ of a circle subgroup $T_1$ of $T$, and it is shown in \cite{25} Lemma 23 that in order to calculate the difference \ref{6.7} the only wall crossing terms we need to consider correspond to components $\hat{M}_1$ of fixed point sets of circle subgroups $T_1$ satisfying

$$\emptyset \neq \pi(\hat{M}_1) \cap M^{ss} \subset G\hat{Z}_R^{ss}$$
so that $\hat{M}_1$ is contained in the exceptional divisor of $\pi : \hat{M} \to M$. Moreover if $N^T_0$ is the identity component of the normaliser of $T_1$ then the subset
\[ \{ g \in G : T_1 \subseteq gRg^{-1} \} \]
of $G$ is the disjoint union
\[ \bigsqcup_{1 \leq i \leq m} N^T_0 g_i N^R \]
of finitely many double cosets for the left $N^T_0$ action and the right $N^R$ action, and the $T_1$-fixed point set in $GZ^ss_R$ is the disjoint union
\[ \bigsqcup_{1 \leq i \leq m} N^T_0 \hat{Z}^ss_{R_i} \]
where $R_i = g_i R g_i^{-1} \supset T_1$. Let the $T_1$-eigenbundles of the restriction to $N^T_0 \hat{Z}^ss_{R_i}$ of the normal bundle to $G\hat{Z}^ss_R$ be denoted by
\[ W_{i,j} \to N^T_0 \hat{Z}^ss_{R_i} \]
for $1 \leq j \leq i$. If $\hat{M}_1$ is a component of a fixed point set of a circle subgroup $T_1$ satisfying $\emptyset \neq \pi (M_1) \cap M^ss \subset GZ^ss_R$ then
\[ \hat{M}_1 \cap \pi^{-1}(M^ss) = \mathbb{P}W_{i,j} \]
for some $i, j$, and it is shown in [25] that the corresponding wall crossing term is
\[ \int_{\mathbb{P}W_{i,j} // \xi_1 (T/T_1)c} e^{\sum_{X_1=0}^{\alpha \beta D^2}} \left( \frac{\beta X}{\alpha \beta} \right) \]
where $e_{\mathbb{P}W_{i,j}}$ is the $T$-equivariant Euler class of the normal bundle to $\mathbb{P}W_{i,j}$, and the wall is crossed at $\xi_1 + \xi_2$ where $\xi_2$ is the constant $T_1$-component of $\mu$ on $\hat{M}_1$ and $\xi_1$ is orthogonal to the Lie algebra of $T_1$.

From (6.1), (6.2) and (6.3), we deduce the following.

**Theorem 35.** Let $\alpha, \beta$ be two classes in $H^2_{g(n,d)} (\mathcal{C}(n,d))$ with $\deg \alpha + \deg \beta = \dim_{\mathbb{R}} \mathcal{M}(n,d)$ and
\[ \alpha \beta = \sum_k Q_k(a_1, \ldots, a_n, b_1, \ldots, b_{2g}) \frac{k^k}{k!} \]
where $Q_k$ is a polynomial. Then using the notation of Remark 34, the intersection pairing on the partial desingularisation $\mathcal{M}(n,d)$ is given by
\[ \int_{(S^1)^{2g} \times [-[w]]} \left( \frac{e^{\sum_{X_1=0}^{\alpha \beta D^2}} Q_k(X)}{D(X)^{2g-2} \prod_{j=1}^{n-1} (\exp - B(-X)_{j-1})} \right) \]
\[ -(-1)^{n(n-1)/2} \sum_{R \in \mathcal{R}} \sum_{T_1} \sum_{i,j} e^{\sum_{X_1=0}^{\alpha \beta D^2}} \left( \frac{\alpha \beta X}{\alpha \beta} \right) \int_{\mathbb{P}W_{i,j} // \xi_1 (T/T_1)c} \left( \frac{\beta X}{\alpha \beta} \right) \]
\[ \sum_{w \in W_{n-1}} \text{res}_{Y_1=0} \cdots \text{res}_{Y_{n-1}=0} \int_{(S^1)^{2g} \times [-[w]]} \left( \frac{e^{\sum_{X_1=0}^{\alpha \beta D^2}} Q_k(X)}{D(X)^{2g-2} \prod_{j=1}^{n-1} (\exp - B(-X)_{j-1})} \right) \]
where the last sum runs over all reductive groups \( R \) coming from partitions of \( n \) (see (3.1)), over all circle subgroups \( T_1 \) of \( T \) that appear as the stabiliser of a point in the exceptional divisor of the blowup of \( G \hat{Z}_{R_i}^{ss} \) and over all \( i, j \) such that \( \hat{\mu}_T(\mathbb{P}W_{i,j}) \) is a wall between \( 0 \) and \( \hat{\xi} \).

The computation of the intersection pairing on the partial desingularisation \( \hat{M}(n, d) \) is completed by computing the last terms (the wall crossing terms) in the above theorem. In the subsequent section, we will see how we can calculate the wall crossing terms.

### 7. Wall crossing terms

The purpose of this section is to compute the wall crossing term

\[
\int_{\mathbb{P}W_{i,j} // / T / T_1} \kappa_{\mathbb{P}W_{i,j}, \xi_1}^{T/T_1} \left( \text{res}_{X_1=0} \frac{\alpha \beta D^2}{e_{\mathbb{P}W_{i,j}}} \right)
\]

which appears in Theorem 35 and hence to complete our calculation of the intersection pairing on the partial desingularisation \( \hat{M}(n, d) \).

**Remark 36.** It is noted in [25] Remarks 26 and 27 that this wall crossing term is an integral over a quotient of the form \( \hat{\mu}^{-1}(\xi_1 + \xi_2) \cap \hat{M}_1 \cap T \) where \( \xi_1 \) and \( \xi_2 \) can be taken to be arbitrarily close to 0. Since \( \mathbb{P}W_{i,j} \) is a projective bundle over \( N_0^{T_1} \hat{Z}_{R_i}^{ss} \), it is useful to use the method of reduction to a maximal torus (cf. (2.8)) to relate integrals over quotients of \( \mathbb{P}W_{i,j} \) by \( T \) to integrals over quotients of \( \mathbb{P}W_{i,j} \) by \( N_0^{T_1} \), provided that \( \xi_1 \) is centralised by \( N_0^{T_1} \), so that the quotient

\[
\mathbb{P}W_{i,j} // / T_1 \cap (N_0^{T_1} \cap K)
\]

is well defined. Luckily in our situation we can use (3.7) to allow us to assume that \( \xi_1 \) is centralised by \( N_0^{T_1} \), and this simplifies the calculations described in [25] §8 considerably. The wall crossing term (3.9) becomes

\[
(-1)^{n_{+}^{N_0^{T_1}}} \int_{\mathbb{P}W_{i,j} // / T_1} \kappa_{\mathbb{P}W_{i,j}, \xi_1}^{T/T_1} \left( \text{res}_{X_1=0} \frac{\alpha \beta D^2}{e_{\mathbb{P}W_{i,j}}} \right)
\]

where \( n_{+}^{N_0^{T_1}} \) is the number of positive roots of \( N_0^{T_1} \), while \( D_{N_0^{T_1}} \) is the product of the positive roots of \( N_0^{T_1} \) and \( W_{N_0^{T_1}} \) is the Weyl group of \( N_0^{T_1} \). Moreover by [25] (8.13) and Lemma 31 there is a fibration

\[
\mathbb{P}W_{i,j} // / T_1 \cap N_0^{T_1} \cap N_{R_i} \rightarrow \hat{Z}_{R_i}^{ss} // / N_{R_i} \rightarrow \tilde{Z}_{R_i}^{ss} \cap N_{R_i}
\]

with fiber \( \mathbb{P}(W_{i,j})_x // / \text{Stab}(x) \cap N_{R_i} \cap N_{R_i} \). Thus we can calculate (3.11) by integrating over the fibers of \( \Psi \).

Recall from (5.3) that \( W_{i,j} \rightarrow N_0^{T_1} \hat{Z}_{R_i}^{ss} \) is a \( T_1 \)-eigenbundle of the restriction to \( N_0^{T_1} \hat{Z}_{R_i}^{ss} \) of the normal bundle to \( G \hat{Z}_{R_i}^{ss} \), and \( R_i = g_i R g_i^{-1} \supset T_1 \). Let us drop the indices \( i \) and \( j \) and consider one wall crossing term (3.11) for

\[
R_i = R \cong GL(m_1; \mathbb{C}) \times \ldots \times GL(m_q; \mathbb{C})
\]

We have

\[
\text{Stab}(x) = R \text{ for every } x \in \hat{Z}_{R_i}^{ss}
\]
and by (3.3) and (3.4) the normaliser $N = N^R$ of $R$ in $G_c$ has identity component

$$ N_0^R \cong \prod_{1 \leq i \leq q} (GL(m_i; \mathbb{C}) \times G_c(n_i, d_i))/\mathbb{C}^* $$

and

$$ \pi_0(N^R) = \prod_{j \geq 0, k \geq 0} \text{Sym} (\# \{i : m_i = j \text{ and } n_i = k\}). $$

Moreover

$$(7.4) \quad \hat{Z}_R/\!/N^R \cong \left[\prod_{i=1}^q \tilde{M}(n_i, d_i)\right]/\pi_0(N^R)$$

where $\left[\prod_{i=1}^q \tilde{M}(n_i, d_i)\right]$ is the ‘blow up along diagonals’ of $\prod_{i=1}^q \tilde{M}(n_i, d_i)$, defined as follows (cf. [38] Definition 3.9 and Lemma 3.11, but noting that the definition given in [38] of the ‘blow up along diagonals’ is incorrect).

**Definition 37.** Let $Y_1, \ldots, Y_s$ be quasi-projective varieties and let $\Pi = \{I_1, \ldots, I_k\}$ be a partition of $\{1, \ldots, s\}$ with associated equivalence relation $\sim$ such that $Y_i = Y_j$ if and only if $i \sim j$. Given any partition $\Pi' = \{I'_1, \ldots, I'_{s'}\}$ of $\{1, \ldots, s\}$ which refines $\Pi$, there is a nonsingular closed subvariety $A_{\Pi'}$ of the product $\prod_{i=1}^s Y_i$ defined by

$$ A_{\Pi'} = \{(y_1, \ldots, y_s) \in \prod_{i=1}^s Y_i : y_i = y_j \text{ if } i \sim' j\} $$

where $\sim'$ is the equivalence relation on $\{1, \ldots, s\}$ induced by the partition $\Pi'$. Let

$$ \left[\prod_{i=1}^s Y_i\right] $$

denote the result of blowing up $\prod_{i=1}^s Y_i$ along the proper transforms of the subvarieties $A_{\Pi'}$ (where $\Pi'$ runs over all refinements of $\Pi$) in increasing order of dimension. We call $[\prod_{i=1}^s Y_i]$ the product of $Y_1, \ldots, Y_s$ blown up along all diagonals.

We saw in §3 that the wall $\mu_T(\hat{M}_1)$ lies in an affine hyperplane $\beta + \beta^\perp$ in $t_R$ where $\beta$ generates $T_1$ and is determined by a partition

$$ \{\Delta_{h,m} : (h, m) \in J\} $$

of $\{1, \ldots, M\}$, a nonempty subset $S$ of $\{(i,j) : 1 \leq i, j \leq M\}$, and a directed graph $G(S)$ with vertices $1, \ldots, M$ and directed edges from $i$ to $j$ whenever $(i,j) \in S$, as in Proposition 3. Recall from Remark 4 that the graph $G(S)$ is connected and so we can omit the index $h$ and take $J$ to be of the form

$$ J = \{1, \ldots, t\}. $$

Our aim is to calculate the wall crossing term $\langle 7.1 \rangle$ by integrating over the fibers

$$ P(W)//\text{Stab}(x) \cap N_0^{T_1} \cap N^R = P(W)//N_0^{T_1} \cap N^R $$

of the fibration

$$ \Psi : P(W)//\xi_1(N_0^{T_1}/(T_1)_c) \cong P(W)//\tilde{Z}_R/\!/N_0^{T_1} \cap N^R $$
defined at (7.2) (see Remark 36 and (7.3)), where $W = W_{i,j}$ and $N_{0}^{T_{1}}$ is the identity component of the normaliser $N^{T_{1}}$ in $G_{c}$ of the one-parameter subgroup $T_{1}$ generated by $\beta$, so that

\begin{equation}
N_{0}^{T_{1}} = N^{T_{1}} = \text{Stab}\beta.
\end{equation}

From [25] Lemma 29 and Corollary 1 and from (7.4) above we know that the connected component $R(N_{0}^{T_{1}} \cap N^{R})_{0}$ has finite index in $N^{R}$ and that there are isomorphisms

\[
\hat{Z}_{R}/(N_{0}^{T_{1}} \cap N^{R})_{0} \cong \hat{Z}_{R}/N_{0}^{R} \cong \prod_{i=1}^{q} \tilde{M}(n_{i}, d_{i})
\]

and

\[
\hat{Z}_{R}/N_{0}^{T_{1}} \cap N^{R} \cong \prod_{i=1}^{q} \tilde{M}(n_{i}, d_{i}) \big/ \pi_{0}(N_{0}^{T_{1}} \cap N^{R})
\]

where

\[
\pi_{0}(N_{0}^{T_{1}} \cap N^{R}) = \prod_{j_{1} \geq 0, j_{2} \geq 0, j_{3} \geq 0} \text{Sym}(\#\{(i, k) : m_{i} = j_{1} \text{ and } m_{i}^{k} = j_{2} \text{ and } n_{i} = j_{3}\})
\]

(cf. (3.5)). Thus once we have reduced to integrals over $\hat{Z}_{R}/N_{0}^{T_{1}} \cap N^{R}$ by integrating over the fibers of $\Psi$ we can complete the calculation by using induction on $n$ to compute integrals over the product $\prod_{i=1}^{q} M(n_{i}, d_{i})$ and its blowup along diagonals $[\prod_{i=1}^{q} \tilde{M}(n_{i}, d_{i})]$.

We can integrate (7.4) over the fibers $\mathbb{P}(W)_{x}/N_{0}^{T_{1}} \cap N^{R}$ of $\Psi$ by using the formula (2.4) for pairings on the symplectic quotient of a projective space. In this calculation the terms $\alpha$, $\beta$, $D$ and $D_{N_{0}^{T_{1}}}$ all restrict to equivariant classes on the projective space $\mathbb{P}(W)_{x}$ which are pulled back from the equivariant cohomology of a point and are easy to calculate. The remaining term to consider is the equivariant Euler class $e_{\mathbb{P}W}$ of the normal bundle to $\mathbb{P}W$.

Let $E \cong (\mathbb{C}^{m_{1}} \otimes D_{1}) \oplus \cdots \oplus (\mathbb{C}^{m_{q}} \otimes D_{q})$ represent an element of $G_{c}Z_{R}^{ss}$ as at (3.2) above, with $D_{1}, \ldots, D_{q}$ all stable and not isomorphic to one another, and $D_{i}$ of rank $n_{i}$ and degree $d_{i}$. Recall from §3 that $C$ is an infinite dimensional affine space, and if we fix a $C^{\infty}$ identification of the fixed $C^{\infty}$ hermitian bundle $E$ with $\bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}$ then we can identify $C$ with the infinite dimensional vector space

$$
\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}))
$$

in such a way that the zero element of $\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}))$ corresponds to the given holomorphic structure on $E = \bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}$. With respect to this identification, the action of $R = \prod_{i=1}^{q} GL(m_{i}; \mathbb{C})$ on $C$ is the action induced by the obvious action of $R$ on $\bigoplus_{i=1}^{q} \mathbb{C}^{m_{i}} \otimes D_{i}$. The tangent space to the $G_{c}$-orbit through this holomorphic structure is the image of the differential

$$
\Omega^{0}(\text{End} E) \rightarrow \Omega^{0,1}(\text{End} E),
$$

and the normal $N_{R}$ to $G_{c}Z_{R}^{ss}$ at $E$ is naturally isomorphic to the cokernel of the restriction of this differential to

$$
\Omega^{0}(\text{End} E) \rightarrow \Omega^{0,1}(\text{End} E),
$$

where
which is
\[ H^1(\Sigma, \text{End}'_e E) \cong \bigoplus_{i_1, i_2=1}^q \mathbb{C}^{m_{i_1} m_{i_2} - \delta_{i_1}^2} \otimes H^1(\Sigma, D_{i_1}^r \otimes D_{i_2}). \]

We have \( H^0(\Sigma, \text{End}'_e E) = 0 \), so the normal bundle to \( G_c Z^s_R \) in \( C \) is
\[ -\pi_! \left( \bigoplus_{i_1, i_2=1}^q \mathbb{C}^{m_{i_1} m_{i_2} - \delta_{i_1}^2} \otimes \mathbb{U}_{i_1} \otimes \mathbb{U}_{i_2} \right) \]
where \( \mathbb{U}_{i_1} \) and \( \mathbb{U}_{i_2} \) are the appropriate universal bundles on \( \mathcal{C}(n_{i_1}, d_{i_1}) \times \Sigma \) and \( \mathcal{C}(n_{i_2}, d_{i_2}) \times \Sigma \) pulled back to \( \mathcal{C}(n_{i_1}, d_{i_1}) \times \mathcal{C}(n_{i_2}, d_{i_2}) \times \Sigma \), and \( \pi \) denotes the projection from \( \mathcal{C}(n_{i_1}, d_{i_1}) \times \mathcal{C}(n_{i_2}, d_{i_2}) \times \Sigma \) to \( \mathcal{C}(n_{i_1}, d_{i_1}) \times \mathcal{C}(n_{i_2}, d_{i_2}) \).

We need to extend this description to a description of the normal bundle \( N_R \) to the proper transform \( G_c Z^ss_R \) of \( G_c Z^ss_R \). For simplicity we consider the case when \( Z^ss_R \) is obtained from \( Z^ss_R \) via a single blow up along its intersection with \( G_c Z^ss_R \) for some \( R' \) containing \( R \); the general case can then be obtained inductively (cf. [37] §8). From [37] Corollary 8.11 we know that \( G_c Z^ss_R \cap Z^ss_R \) is the disjoint union
\[ G_c Z^ss_R \cap Z^ss_R = \bigsqcup_{r=1}^{m'} \mathcal{M}_0 R' g_r^* Z^ss_R \]
where the subset \( \{ g \in G_c : R \subseteq g R' g^{-1} \} \) of \( G_c \) is the disjoint union
\[ \{ g \in G_c : R \subseteq g R' g^{-1} \} = \bigsqcup_{r=1}^{m'} \mathcal{M}_0 R' g_r N R' \]
of finitely many double \((\mathcal{M}_0 R, N R')\) cosets\(^9\). Moreover if \( E \) represents an element of one of the components \( \mathcal{M}_0 R g_r Z^ss_R \) of \( G_c Z^ss_R \cap Z^ss_R \) where \( R \subseteq g R' g^{-1} \) then we have
\[ E = (\mathbb{C}^{m'_1} \otimes D'_1) \oplus \cdots \oplus (\mathbb{C}^{m'_Q} \otimes D'_Q) \]
with \( D'_1, \ldots, D'_Q \) all stable and not isomorphic to each other and \( D'_j \) of rank \( n'_j \) and degree \( d'_j \) where
\[ R' \cong \prod_{j=1}^Q GL(m'_j; \mathbb{C}). \]

But also since \( E \in Z^ss_R \) we have a decomposition
\[ E \cong (\mathbb{C}^{m_1} \otimes D_1) \oplus \cdots \oplus (\mathbb{C}^{m_Q} \otimes D_Q) \]
where \( D_i \) is semistable of rank \( n_i \) and degree \( d_i \). As the decomposition \((7.7)\) of \( E \) is canonical, for \( 1 \leq i \leq Q \) we must have
\[ D_i \cong (\mathbb{C}^{M_{i,1}} \otimes D'_1) \oplus \cdots \oplus (\mathbb{C}^{M_{i,Q}} \otimes D'_Q) \]
for some \( M_{ij} \geq 0 \) satisfying
\[ m'_j = \sum_{i=1}^q m_i M_{ij} \]
\(^9\)Strictly speaking here we should apply [37] Corollary 8.11 to a finite dimensional description of \( \mathcal{M}(n, d) \) as a quotient (cf. §3 above and [37]).
for \(1 \leq j \leq Q\), and we can assume that \(R \cong \prod_{i=1}^{q} GL(m_i; \mathbb{C})\) is embedded in \(gR'g^{-1} \cong \prod_{j=1}^{Q} GL(m'_j; \mathbb{C})\) via decompositions

\[
C_{m'_j} = \bigoplus_{i=1}^{q} C_{m_i} \otimes C_{M_{ij}}
\]

coming from (7.8). Note that the normal to \(N\) in \(Z\) is then

\[
-\pi! \left( \bigoplus_{i=1}^{q} \bigoplus_{j_1,j_2=1}^{Q} C_{m_{i,j_1}m_{i,j_2}} \otimes (U'_{j_1})^* \otimes U'_{j_2} \right)
\]

where \(U'_j\) denotes the pullback of the appropriate universal bundle on \(C(n'_j, d'_j) \times \Sigma\).

**Lemma 38.** There are short exact sequences of sheaves over \(\hat{Z}_R\) as follows, where \(N_{A|B}\) denotes the normal bundle to \(A\) in \(B\) when \(A\) is a smooth submanifold of a manifold \(B\):

\[
i \quad 0 \to N_{\hat{Z}_R|\hat{c}_s}\to p^*(N_{Z_R|c_s}) \to p^*t_1 \left( \frac{TC}{Tg_cZ_R^s + T\hat{Z}_R^s} \right) \to 0
\]

where \(p: \hat{Z}_R \to Z_R\) is the blow down map and \(\iota: Z_R \cap G_cZ_R \to Z_R^s\) is the inclusion, and

\[
t_1 \left( \frac{TC}{Tg_cZ_R^s + T\hat{Z}_R^s} \right)
\]

is shorthand for the extension by zero over \(Z_R^s\) of the vector bundle

\[
\frac{TC}{Tg_cZ_R^s|Z_R \cap G_cZ_R^s + T\hat{Z}_R^s|Z_R \cap G_cZ_R^s}
\]

which fits into an exact sequence

\[
0 \to T(N^R_0g'_jZ_R^s) \to T\hat{Z}_R^s|N^R_0g'_jZ_R^s \to \hat{N}_R' \to \frac{TC}{Tg_cZ_R^s|Z_R \cap G_cZ_R^s + T\hat{Z}_R^s|Z_R \cap G_cZ_R^s} \to 0
\]

on each component \(N^R_0g'_jZ_R^s\) of \(Z_R^s \cap G_cZ_R^s\);

\[
(ii) \quad 0 \to \text{Lie}(G_c)/\text{Lie}(N^R) \to N_{\hat{Z}_R|\hat{c}_s} \to N_{G_c\hat{Z}_R|\hat{c}_s} = N_R|\hat{Z}_R^s \to 0
\]

where \(\text{Lie}(G_c)/\text{Lie}(N^R)\) is shorthand for the trivial vector bundle on \(\hat{Z}_R^s\) whose fiber is

\[
\text{Lie}(G_c)/\text{Lie}(N^R) \cong \Omega^0(\text{End}'_{\hat{c}}E).
\]

**Proof:** (i) follows directly from [12] Lemma 15.4 (i) and (iv), while (ii) is an immediate consequence of the fact that

\[
(7.10) \quad G_c\hat{Z}_R^s \cong G_c \times_{N^R} \hat{Z}_R^s
\]

by [37] Corollary 5.6. \(\square\)

**Corollary 39.** The equivariant Chern polynomial of the normal bundle \(N_R\) to \(G_c\hat{Z}_R^s\) in \(\hat{c}_s\) is given by

\[
c(N_R)(t) = p^* \left( c(-\pi! \left( \bigoplus_{i=1}^{q} \bigoplus_{j_1,j_2=1}^{Q} C_{m_{i1}m_{i2}}c^{m_{i1}m_{i2}} - \delta_{i1}^2 \otimes U'_{j_1} \otimes U'_{j_2} \right)(t)) \right)
\]

\[
c(t_1 \left( \frac{TC}{Tg_cZ_R^s + T\hat{Z}_R^s} \right))(t)
\]
with
\[
c(t_1 \left( \frac{TC}{T_G Z_R^{ss} + T Z_R^{ss}} \right))(t) = \prod_{r=1}^{m'} \left( c \left( t_1 \left( -\pi_1 \left( \bigoplus_{j_1, j_2 = 1}^{Q} C^{m'_1, m'_2} \otimes (U'_1)^* \otimes U'_2 \right) \right) \right)(t) \right)
\]
in the notation of Lemma 38 and the preceding paragraph.

**Proof:** Since \( H^*_G(\mathcal{G}_c \hat{Z}_R^{ss}) \cong H^*_N(\hat{Z}_R^{ss}) \) by (7.10), it suffices to consider the restriction of \( \mathcal{N}_R \) to \( \hat{Z}_R^{ss} \). When \( E = (C^{m_1} \otimes D_1) + \cdots + (C^{m_q} \otimes D_q) \) represents an element of \( Z_R^{ss} \) as above the normal to \( Z_R^{ss} \) in \( C^{ss} \) at \( E \) is naturally isomorphic to \( \Omega^{0,1}(\text{End}_E) \). The result now follows from (7.9), Lemma 38 and the exact sequence
\[
0 \rightarrow H^0(\Sigma, \text{End}_E) \rightarrow \Omega^0(\text{End}_E) \rightarrow \Omega^{0,1}(\text{End}_E) \rightarrow H^1(\Sigma, \text{End}_E) \rightarrow 0.
\]

The one-parameter subgroup \( T_1 \) of \( R \) generated by \( \beta \) acts diagonally on \( C^{m_1} \oplus \cdots \oplus C^{m_q} \) with weights \( \beta \cdot e_j \) for \( j \in \{1, \ldots, M\} \) where \( M = m_1 + \cdots + m_q \), and so it acts on
\[
\Omega^{0,1}(\text{End}(\bigoplus_{i=1}^{q} C^{m_i} \otimes D_i)) = \bigoplus_{i_1, i_2 = 1}^{q} \Omega^0(\bigotimes_{i=1}^{m_i} \otimes (C^{m_{i_2}})^* \otimes D_{i_1} \otimes D_{i_2}^*)
\]
with weights \( \beta \cdot (e_i - e_j) \) for \( i, j \in \{1, \ldots, M\} \). By Proposition 3 we have a partition
\[
\{\Delta_m : m \in J = \{1, \ldots, t\}\}
\]
of \( \{1, \ldots, M\} \) such that
\[
\frac{\beta}{\|\beta\|^2} = \sum_{m=1}^{t} \sum_{j \in \Delta_m} (e - m) \frac{e_j}{\|e_j\|^2}
\]
and \( \beta \cdot (e_i - e_j) = \|\beta\|^2 \) if and only if \( i \in \Delta_k \) and \( j \in \Delta_{k+1} \) for some \( k \in \{1, \ldots, t-1\} \) by Lemma 5.3. Recall that if \( 1 \leq i \leq q \) then \( e_{m_1+\ldots+m_{i-1}+1}, \ldots, e_{m_1+\ldots+m_i} \) are the weights of the standard representation on \( C^{m_i} \) of the component \( GL(m_i; \mathbb{C}) \) of \( R = \prod_{i=1}^{q} GL(m_i; \mathbb{C}) \). If \( 1 \leq i \leq q \) and \( k \in J = \{1, \ldots, t\} \), then let
\[
\Delta_i = \Delta_k \cap \{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i\}
\]
and let \( m_i^k \) denote the size of \( \Delta_i \), so that \( \sum_{k=1}^{t} m_i^k = m_i \). If we make the induced identifications
\[
C^M = \bigoplus_{i=1}^{q} C^{m_i} = \bigoplus_{i=1}^{q} \bigoplus_{k=1}^{t} C^{m_i^k}
\]
then the \( T_1 \)-eigenbundle \( \mathcal{W} = W_{i,j} \rightarrow N_{0}^{T_1} \hat{Z}_R^{ss} \) of the restriction to \( N_{0}^{T_1} \hat{Z}_R^{ss} \) of the normal bundle to \( \mathcal{G}_c \hat{Z}_R^{ss} \) which corresponds to the affine hyperplane \( \beta + \beta^\perp \) is given by the image of
\[
-\pi_1 \left( \bigoplus_{i_1, i_2}^{q} C^{m_i^k} \otimes (C^{m_{i_2}^{k+1}})^* \otimes U_{i_1} \otimes U_{i_2}^* \right)
\]
and has equivariant Chern polynomial equal to the pullback of
\[
c(\mathcal{W})(t) = p^* \left( \frac{c \left( -\pi_1 \left( \bigoplus_{i_1, i_2}^{Q} C^{m_i^k} \otimes (U_{i_1} \otimes U_{i_2}^*) \right) \right)(t)}{\prod_{r=1}^{m'} c \left( t_1 \left( -\pi_1 \left( \bigoplus_{j_1, j_2}^{Q} C^{m_i^k} \otimes (U_{j_1} \otimes (U_{j_2}^*)) \right) \right) \right)(t)} \right)
\]
where \( m_{\delta}^k = \sum_{i=1}^{q} m_i^k M_{i,j} \).

**Remark 40.** Recall that the final ingredient of the wall crossing term (7.1) which is needed is the equivariant Euler class \( e_{\mathbb{P}W} \) of the normal bundle in \( \hat{G}^{ss} \) to the projectivisation \( \mathbb{P}W \) of \( W \). This can of course be obtained from the equivariant Chern polynomial which we are now in a position to calculate as follows.

The projective bundle \( \mathbb{P}W \to N_0^{T_1} \hat{Z}_R^{ss} \) is a subbundle of the restriction to \( N_0^{T_1} \hat{Z}_R^{ss} \) of the exceptional divisor \( \mathbb{P}N_R \) for the blow-up along \( G_c \hat{Z}_R^{ss} \). The normal to \( \mathbb{P}N_R \) is \( \mathcal{O}(-1) \), and by \([12]\) Lemma 15.4(ii) there is an exact sequence

\[
0 \to \mathcal{O} \to p^* N_R \otimes \mathcal{O}(1) \to T_{\mathbb{P}N_R} \to p^* T_{G_c \hat{Z}_R^{ss}} \to 0
\]

where \( T_X \) denotes the tangent bundle to \( X \) and \( p : \mathbb{P}N_R \to G_c \hat{Z}_R^{ss} \) is the natural projection. Similarly we have

\[
0 \to \mathcal{O} \to p^* \mathcal{W} \otimes \mathcal{O}(1) \to T_{\mathbb{P}W} \to (p|_{\mathbb{P}W})^* T_{N_0^{T_1} \hat{Z}_R^{ss}} \to 0
\]

so the equivariant Chern polynomial of the normal to \( \mathbb{P}W \) in \( \mathbb{P}N_R \) is

\[
(7.13) \quad c(p^*(N_0^{T_1} \hat{Z}_R^{ss}|G_c \hat{Z}_R^{ss}))(t)c(p^*(N_R/\mathcal{W}) \otimes \mathcal{O}(1))(t),
\]

where \( N_0^{T_1} \hat{Z}_R^{ss}|G_c \hat{Z}_R^{ss} \) is the normal bundle to \( N_0^{T_1} \hat{Z}_R^{ss} \) in \( G_c \hat{Z}_R^{ss} \). We have

\[
G_c \hat{Z}_R^{ss} \cong G_c \times_{N_R} \hat{Z}_R^{ss}
\]

and

\[
N_0^{T_1} \hat{Z}_R^{ss} \cong N_0^{T_1} \times_{N_0^{T_1} \cap N_R} \hat{Z}_R^{ss},
\]

so the normal to \( N_0^{T_1} \hat{Z}_R^{ss} \) in \( G_c \hat{Z}_R^{ss} \) is isomorphic to \( G_c/N_0^{T_1} N_R \), and therefore its equivariant Chern roots are the weights of the natural action on \( G_c/N_0^{T_1} N_R \).

Let \( \lambda_1, \ldots, \lambda_l \) be the equivariant Chern roots of \( N_R/\mathcal{W} \), so that its equivariant Chern polynomial is given by

\[
c(N_R/\mathcal{W})(t) = \prod_{j=1}^{l} (1 + \lambda_j t).
\]

Then the equivariant Chern polynomial of \( p^*(N_R/\mathcal{W}) \otimes \mathcal{O}(1) \) is

\[
(7.14) \quad \prod_{j=1}^{l} (1 + \lambda_j t + \zeta t) = (1 + \zeta t)^l c(N_R/\mathcal{W}) \left( \frac{t}{1 + \zeta t} \right)
\]

where \( \zeta \) is the standard generator of the cohomology of the projective bundle over the cohomology of the base. But by Corollary \([39]\) we have

\[
c(N_R)(t) = p^* \left( \frac{c(-\pi_1 \left( \bigoplus_{i_1,i_2=1}^{q} \mathbb{C}^{m_{i_1} m_{i_2} - \delta_{i_1}^{i_2}} \otimes U_{i_1}^* \otimes U_{i_2} \right))(t)}{c(\iota_1 \left( \frac{TC}{T G_c \hat{Z}_R^{ss} + T Z_R^{ss}} \right))(t)} \right)
\]

with

\[
c(\iota_1 \left( \frac{TC}{T G_c \hat{Z}_R^{ss} + T Z_R^{ss}} \right))(t) =
\]
\[ \prod_{r=1}^{m'} \left( \frac{c(t_1 \left( -\pi_1 \left( \bigoplus_{j_1,j_2=1}^{Q} \mathbb{C}^{m'_{j_1} m'_{j_2}} \otimes (U'_{j_1})^* \otimes U'_{j_2} \right) \right) \right)}{c(t_1 \left( -\pi_1 \left( \bigoplus_{i=1}^{Q} \bigoplus_{j_1,j_2=1}^{M_{i,j_1} M_{i,j_2}} \otimes (U'_{j_1})^* \otimes U'_{j_2} \right) \right) \right)}(t) \right) . \]

Moreover by (7.12) we have

\[ c(W)(t) = p^* \left( \frac{c(\pi V_0(t)c(t_1(-\pi)V_0')(t))}{c(W)(t)} \right) \]

so

\[ c(N_R/W)(t) = \frac{c(N_R)(t)}{c(W)(t)} = \frac{p^*\left( c(-\pi V_0)(t)c(t_1(-\pi)V_0')(t) \right)}{p^*\left( c(-\pi V_1)(t)c(t_1(-\pi)V_1')(t) \right)} \]

where

\[ V_0 = \bigoplus_{i_1,i_2=1}^{i_2} \mathbb{C}^{m_{i_1} m_{i_2}} \otimes U_{i_1}^* \otimes U_{i_2} \]

and

\[ V_1 = \bigoplus_{i_1,i_2=1}^{i_2} \mathbb{C}^{m_{i_1} m_{i_2}} \otimes U_{i_1} \otimes U_{i_2}^* \]

while

\[ V'_0 = \bigoplus_{r=1}^{m'} \left( \bigoplus_{i=1}^{Q} \bigoplus_{j_1,j_2=1}^{M_{i,j_1} M_{i,j_2}} \mathbb{C}^{m_{j_1} m_{j_2}} \otimes (U'_{j_1})^* \otimes U'_{j_2} \right) \bigoplus_{j_1,j_2=1}^{Q} t-1 \bigoplus_{k=1}^{m'_{j_1} m'_{j_2}} \mathbb{C}^{m_{j_1} m_{j_2}} \otimes U'_{j_1} \otimes (U'_{j_2})^* \]

and

\[ V'_1 = \bigoplus_{r=1}^{m'} \bigoplus_{j_1,j_2=1}^{Q} \mathbb{C}^{m'_{j_1} m'_{j_2}} \otimes (U'_{j_1})^* \otimes U'_{j_2} . \]

By the Grothendieck–Riemann–Roch theorem ([12], Theorem 15.2) we have

\[ \text{ch}(f_0 \alpha) \text{td}(T_Y) = f_*(\text{ch}(\alpha) \text{td}(T_X)) \]

when \( f : X \rightarrow Y \) is proper. When \( f = \pi \) is a fibration with fiber \( \Sigma \) this gives us

\[ \text{ch}(\pi_1 \alpha) = \pi_* \left( \text{ch}(\alpha) \text{td}(\Sigma) \right) = \pi_* \left( \text{ch}(\alpha)(1 - (g - 1)\omega) \right) \]

where \( \omega \) is the standard generator of \( H^2(\Sigma) \). When \( f = \iota : Z^{ss}_{R} \cap G_c Z^{ss}_{R} \rightarrow Z^{ss}_{R} \) is a closed embedding we get

\[ \text{ch}(\iota_1 \alpha) = \iota_* \left( \text{ch}(\alpha) \left( \text{td} \left( N_{Z_R^{ss} \cap G_c Z_R^{ss}} | Z^{ss}_{R} \right) \right)^{-1} \right) \]

(see [12] §15.2; in particular the formula immediately before Corollary 15.2.1). Then we use (7.6) and (7.9) to compute the Chern polynomials.

By putting all this together we can (at least in principle) calculate the equivariant Chern polynomial of the normal bundle to \( \mathbb{P}W \), and hence calculate the equivariant Euler class \( e_{\mathbb{P}W} \) (cf. similar calculations in [10], in particular [10] Proposition 9.2).
We now have all the ingredients needed to calculate the wall crossing terms (7.1) by integrating over the fibers $\mathbb{P}(\mathcal{W}_{i,j}/\xi_1(N_0^{T_1}/(T_1)_c) \cong \mathbb{P}\mathcal{W}_{i,j} |_{\tilde{Z}_{R_i}} // N_0^{T_1} \cap N_{R_i} \rightarrow \tilde{Z}_{R_i} // N_0^{T_1} \cap N_{R_i}$ defined at (7.2). The results can be expressed (as in [25]) in terms of integrals over projective subbundles of $\mathbb{P}(\mathcal{W}_{i,j})$ which can be calculated with the following standard lemma.

**Lemma 41.** Let $E$ be a rank $r$ complex vector bundle over a manifold $M$ and let $\eta \in H^*(\mathbb{P}E) \cong H^*(M)[y]/(p(y))$ where $p(y) = y^r + c_1(E)y^{r-1} + \cdots + c_r(E)$. Then

$$\int_{\mathbb{P}E} \eta = \text{res}_{y=0} \int_M \frac{\eta}{p(y)}.$$

This lemma reduces the wall crossing terms (7.1) to integrals over spaces of the form $\tilde{Z}_{R_i} // N_{R_i} \cong \left[ \prod_{i=1}^q \tilde{\mathcal{M}}(n_i, d_i) \right] / \pi_0(N_{R_i})$ (see (7.4)) which can be calculated using induction on $n$.

In the next section we will carry out the details of all these calculations in the case when $n = 2$.

### 8. The Case when the Rank $n$ is Two

In this section we explicitly compute intersection numbers in the partial desingularization $\tilde{\mathcal{M}}(2, 0)$. Let $M_{\text{ext}} = M_{U(2)}^{\text{ext}}$ be the extended moduli space with the group $U(2)$ on which $K = SU(2)$ acts by conjugation, i.e. $M_{\text{ext}}$ is the fiber product

$$M_{\text{ext}} \xrightarrow{\mu} \mathfrak{su}(2)$$

$$\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \text{exp}$$

$$U(2)^{2g} \xrightarrow{\Phi} SU(2)$$

Let $T$ denote the maximal torus of $K = SU(2)$ consisting of diagonal matrices. Fix a sufficiently small positive number $\varepsilon$. From (2.10) we have

$$\kappa_{\tilde{\mathcal{M}}_{\text{ext}}} (\eta e^{\omega}) \left[ \tilde{\mathcal{M}}(2, 0) \right] = \frac{1}{2} \left( -\int_{\tilde{\mu}_T^{-1}(0)/T} \kappa_{\tilde{\mathcal{M}}_{\text{ext}}} (\eta e^{\omega} D^2) + \int_{\tilde{\mu}_T^{-1}(\varepsilon)/T} \kappa_{\tilde{\mathcal{M}}_{\text{ext}}} (\eta e^{\omega} D^2) \right)$$

where $\mu_T$ and $\tilde{\mu}_T$ are moment maps for the $T$-action on $M_{\text{ext}}$ and $\tilde{\mathcal{M}}_{\text{ext}}$ respectively. Notice that the second term in (8.2) may be computed using periodicity as in [29], as explained in §4 of the present paper.

To compute the first term, we need to examine the walls (images under $\tilde{\mu}_T$ of components of the fixed point set of $T$) crossed in passing from 0 to $\varepsilon$ in $t^* = \mathbb{R}$. The components of the fixed point set that are relevant to us are those that meet the exceptional divisors in $\tilde{\mathcal{M}}_{\text{ext}}$ (see [25], Lemma 23).
8.1. **Wall crossing term from the first blow-up.** To form $\tilde{M}^{\text{ext}}$ from $M^{\text{ext}}$, we first blow up along the set $\Delta$ of the points with stabilizer $K = SU(2)$ which is $(S^1)^{2g}$ where $S^1$ represents the center of $U(2)$. With the natural identification of $(S^1)^{2g}$ with the Jacobian $\text{Jac}$ of $\Sigma$, the normal bundle of the $K$-fixed point locus $\Delta$ is

$$R^1\pi_* \text{End}(\mathcal{L} \oplus \mathcal{L})_0 \cong R^1\pi_* \mathcal{O} \otimes \mathfrak{sl}(2)$$

where $\mathcal{L} \to \Sigma \times \text{Jac}$ is the Poincaré bundle and $\pi : \Sigma \times \text{Jac} \to \text{Jac}$ is the projection. Hence, the exceptional divisor of the first blow-up in the partial desingularisation process is $\mathbb{P}(R^1\pi_* \mathcal{O} \otimes \mathfrak{sl}(2))$. The $T$-fixed point component with positive moment map value in the exceptional divisor is thus the projectivization

$$\mathbb{P} \left[ R^1\pi_* \mathcal{O} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] \cong \mathbb{P}T_{\text{Jac}} \cong \text{Jac} \times \mathbb{P}^{g-1}$$

of the tangent bundle of the Jacobian which is trivial. The normal bundle to this fixed point component is thus

$$\mathcal{O}_{\mathbb{P}^{g-1}}(-1) \oplus \left[ R^1\pi_* \mathcal{O} \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \otimes \mathcal{O}_{\mathbb{P}^{g-1}}(1) \right] \oplus \left[ R^1\pi_* \mathcal{O} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathcal{O}_{\mathbb{P}^{g-1}}(1) \right].$$

The first summand is normal to the exceptional divisor and the last two are normal in the exceptional divisor. The torus $T$ acts on the three summands with weights $2, -2, -4$ respectively. As $R^1\pi_* \mathcal{O} \cong T_{\text{Jac}}$ is trivial, the equivariant Euler class of the normal bundle is

$$(-y + 2Y)(y - 2Y)^g(y - 4Y)^g$$

where $y = c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(1))$ is the generator of $H^*(\mathbb{P}^{g-1})$ and $Y$ is the generator of $H^*_T(pt)$. Hence the wall crossing term from the first blow-up in the partial desingularization process is

$$-\frac{1}{2} \int_{\mathbb{P}^{g-1} \times \text{Jac}} \res_{Y=0} \eta e^\omega|_\Delta \mathcal{D}^2 \frac{\eta e^\omega}{(-y + 2Y)(y - 2Y)^g(y - 4Y)^g}$$

(8.3)

$$= \frac{1}{2} \int_{\Delta} \int_{\mathbb{P}^{g-1}} \res_{Y=0} \eta e^\omega|_\Delta \frac{\eta e^\omega}{(-2)^{g+1}(-4)^g Y^{2g+1}} \frac{1}{2Y} \left(1 - \frac{y}{2Y}\right)^{-g-1} \left(1 - \frac{y}{4Y}\right)^{-g}$$

$$= -\frac{1}{2^{2g}} \int_{\Delta} \res_{Y=0} \res_{\Delta} \eta e^\omega \frac{\eta e^\omega}{y^{2g}} \left(1 - \frac{y}{2Y}\right)^{-g} \left(1 - \frac{y}{4Y}\right)^{-g}.$$
The last expression in (8.3) is nonzero only when \( \eta e^\omega |_\Delta \in H^*_T(\Delta) = H^*_T(Jac) \) is a constant multiple of the product of the fundamental class \( \frac{g}{g!} \) of \( \Delta \) and \( Y^{3g-3} \), in which case the wall crossing term is computed as follows:

\[
-\frac{1}{2^{2g}} \int_{\Delta} \text{res}_{y=0} \text{res}_{y'}=0 \frac{g^2 Y^{3g-3}}{y^g Y^{2g-1}} (1 - \frac{y}{2Y})^{-g-1}(1 - \frac{y}{4Y})^{-g} \]

(8.4)

\[
= -\frac{1}{2^{2g}} \text{Coeff}_{y^g - 1} Y^{-g+1} (1 - \frac{y}{2Y})^{-g-1}(1 - \frac{y}{4Y})^{-g} \\
= -\frac{1}{2^{2g}} \text{Coeff}_{t^g - 1} (1 - \frac{t}{2})^{-g-1}(1 - \frac{t}{4})^{-g}.
\]

8.2. Wall crossing term from the second blow-up. Let \( \Gamma \) denote the set of points in \( M^{\text{ext}} \) fixed by the action of \( T \) and \( \tilde{\Gamma} \) be the proper transform after the first blow-up. To get the partial desingularization we blow up along \( \tilde{\Gamma} \) for \( G = K^C \).

Note that \( \Gamma = (S^1 \times S^1)^{2g} \) where \( (S^1 \times S^1) \subset U(2) \) is the set of diagonal matrices in \( U(2) \). \( \Gamma \) can be naturally identified with the product \( Jac \times Jac \) of the Jacobian of \( \Sigma \) and \( \tilde{\Gamma} \) is the blow-up of \( Jac \times Jac \) along the diagonal \( \Delta \cong Jac \). The normal bundle \( \mathcal{N} \) to \( \tilde{\Gamma} \) splits as the direct sum \( W_+ \oplus W_- \) on which \( T \) acts with weights 2 and -2 respectively. Hence the wall to be crossed is exactly \( \mathbb{P}W_+ \).

Let us compute the equivariant Euler class of the normal bundle of the wall \( \mathbb{P}W_+ \). Let \( \mathcal{L}_1 \) (resp. \( \mathcal{L}_2 \)) be the pull-back of the Poincaré bundle \( \mathcal{L} \rightarrow \Sigma \times Jac \) via \( \pi_{12} \) (resp. \( \pi_{13} \)) where \( \pi_{12} : \Sigma \times Jac \times Jac \rightarrow \Sigma \times Jac \) (resp. \( \pi_{23} : \Sigma \times Jac \times Jac \rightarrow \Sigma \times Jac \)) is the projection onto the first and second (resp. third) components. Let \( \pi_{23} : \Sigma \times Jac \times Jac \rightarrow Jac \times Jac \) and \( \pi : \Sigma \times Jac \rightarrow Jac \) denote the obvious projections.

As observed in §6, the Chern class of the normal bundle \( \mathcal{N} \) of \( \tilde{\Gamma} \) restricted to \( \tilde{\Gamma} \) is

\[
c(\mathcal{N}|_{\tilde{\Gamma}}) = \frac{c(- (\pi_{23})(\mathcal{L}_1^\vee \otimes \mathcal{L}_2 \oplus \mathcal{L}_2^\vee \otimes \mathcal{L}_1)))}{c(\iota_!(R^1 \pi_{\ast}(\mathcal{O})) \oplus R^1 \pi_{\ast}(\mathcal{O})))}
\]

where \( \iota : \Delta \hookrightarrow \Gamma \) is the diagonal embedding. Similarly, we have

\[
c(W_-|_{\tilde{\Gamma}}) = \frac{c(- (\pi_{23})(\mathcal{L}_1^\vee \otimes \mathcal{L}_2)))}{c(\iota_!(R^1 \pi_{\ast}(\mathcal{O})))}
\]

\[
c(W_+|_{\tilde{\Gamma}}) = \frac{c(- (\pi_{23})(\mathcal{L}_2^\vee \otimes \mathcal{L}_1)))}{c(\iota_!(R^1 \pi_{\ast}(\mathcal{O})))}.
\]

After normalization, the ordinary first Chern classes of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) can be written as

\[
\sum_{i=1}^{2g} d_i^1 \otimes \sigma_i, \quad \sum_{i=1}^{2g} d_i^2 \otimes \sigma_i
\]

where \( \sigma_i \) is a symplectic basis of \( H^1(\Sigma) \) as before. By Grothendieck–Riemann–Roch, the Chern characters are

\[
ch(-(\pi_{23})(\mathcal{L}_1^\vee \otimes \mathcal{L}_2)) = ch(-(\pi_{23})(\mathcal{L}_2^\vee \otimes \mathcal{L}_1)) = (g - 1) + \hat{\gamma}
\]
where
\[ \hat{\gamma} = \sum_{i=1}^{g} d_{i}^{1}d_{i}^{+} + d_{2}^{1}d_{2}^{+} + d_{1}^{g}d_{1}^{+} + d_{2}^{g}d_{1}^{+}. \]

From a well-known combinatorial argument relating the Chern characters with the Chern classes, we have
\[ c(-\pi_{23})(\mathcal{L}_{1}^{\vee} \otimes \mathcal{L}_{2}) = c(-\pi_{23})(\mathcal{L}_{2}^{\vee} \otimes \mathcal{L}_{1}) = \exp(\hat{\gamma}) \]

Since \( R^{1}\pi_{*}\mathcal{O} \cong \mathcal{O}^{3g} \), we have
\[ c(n_{i}(R^{1}\pi_{*}(\mathcal{O}))) = c(n_{i}(\mathcal{O}))^{g} = (1 + h)^{-g} \]

by [12] Example 15.3.5 where \( h = c_{1}(\mathcal{O}_{F}(-E)) \) for the exceptional divisor \( E \) of the blow-up \( \hat{\Gamma} \rightarrow \Gamma \). Therefore we have
\[ (8.5) \quad c(W_{\pm}|_{\hat{\Gamma}}) = (1 + h)^{g} \exp(\hat{\gamma}). \]

It is an illuminating exercise to check that the right hand side indeed lies in \( H^{\leq 2g-2}(\hat{\Gamma}) \). Let \( y = c_{1}(\mathcal{O}_{p_{W}}(1)) \). Since the normal bundle to \( p_{W} \) in \( \mathbb{P}_{N} \) is the pull-back of \( W \) with \( \mathcal{O}_{p_{W}}(1) \) and \( T \) acts with weights \(-4\), the equivariant Euler class of the normal bundle to \( p_{W} \) in \( \mathbb{P}_{N} \) is
\[ (y - 4Y)^{g-1} \left( 1 + \frac{h}{y - 4Y} \right)^{g} \exp \left( \frac{\hat{\gamma}}{y - 4Y} \right). \]

The normal bundle of \( \mathbb{P}_{N} \) restricted to \( p_{W} \) is \( \mathcal{O}_{p_{W}}(-1) \) on which \( T \) acts with weight \( 2 \) so that the equivariant Euler class is \((y - 2Y)^{g-1}\). Therefore the equivariant Euler class of the normal bundle to \( p_{W} \) in \( \hat{M}^{\text{ext}} \) is
\[ (8.6) \quad e_{p_{W}} = (y - 2Y)(y - 4Y)^{g-1} \left( 1 + \frac{h}{y - 4Y} \right)^{g} \exp \left( \frac{\hat{\gamma}}{y - 4Y} \right). \]

As observed in the previous subsection, we can easily compute the restriction to \( G\hat{\Gamma} \) of \( \eta e_{\mathcal{O}} \) and express as a linear combination of classes of the form \( \xi Y^{n} \) for some nonnegative integer \( n \) and \( \xi \in H^{*}(\Gamma) \subset H^{*}(\hat{\Gamma}) \). Since \( \dim_{\mathbb{R}} \mathcal{M}(2,0) = 8g - 6 \), the intersection pairing is nonzero only for classes of degree \( 8g - 6 \). So it suffices to consider the case when \( \xi \in H^{8g-6-2n}(\Gamma) \).

The wall crossing term from the second blow-up is then
\[ (8.7) \quad \int_{p_{W}} \text{res}_{y = 0} \frac{\xi Y^{n}(4Y)^{2}}{\eta_{W}^{\vee}} = \int_{p_{W}} \text{res}_{y = 0} \frac{-(-y + 2Y)(y - 4Y)^{g-1}(1 + \frac{h}{y - 4Y})^{g} \exp \left( \frac{\hat{\gamma}}{y - 4Y} \right)}{2(-4)^{-1}} \]

where \( A_{r,s,l} \) are defined by the power series expansion in \( t \)
\[ \left( 1 + 2t \right) \sum_{k=0}^{g-1} (1 + t)^{g-1-k} \sum_{r+s=k} \frac{g!}{r!s!(g-s)!} z^{r}x^{s} \right)^{-1} = \sum_{r,s,l} A_{r,s,l} \cdot z^{r}x^{s} t^{l}. \]

So it suffices to compute
\[ \int_{p_{W}} \xi \hat{\gamma}^{r} h^{s} y^{l} \]
for $r + s + l = n - g + 1$. By the residue formula applied to $W_+|_{\hat{\Gamma}}$, whose Chern class is $(1 + h)^g \exp(\hat{\gamma})$, with respect to the action of $U(1)$ by usual complex multiplication on fibers, we have

$$
\int_{pW_+|_{\hat{\Gamma}}} \xi^r \gamma^s \eta^l = \int_{\hat{\Gamma}} \text{res}_{Y=0} \frac{\xi^r \gamma^s \eta^l}{y^{g-1} (1 + \frac{x}{y})^g \exp(z)}
$$

(8.8)

$$
= \int_{\hat{\Gamma}} \text{res}_{Y=0} \frac{\xi^r \gamma^s \eta^l}{y^{g-1}} \sum_{a,b \geq 0} B_{a,b} (\frac{\gamma}{y})^a (\frac{h}{y})^b
$$

$$
= \sum_{a,b} \int_{\hat{\Gamma}} \text{res}_{Y=0} B_{a,b} y^{a+b-1} \exp(\hat{\gamma}) \int_{\hat{\Gamma}} \xi^r \gamma^s \eta^l h^{s+b}
$$

where $B_{a,b}$ are defined by the power series expansion

$$
\frac{1}{(1 + x)^g \exp(z)} = \sum B_{a,b} x^a y^b.
$$

Since $\hat{\gamma}, \xi \in H^*(\Gamma)$ and $-h$ is the Poincaré dual of the exceptional divisor $E$ in $\hat{\Gamma}$, it is easy to see that the integral

$$
\int_{\hat{\Gamma}} \xi^r \gamma^s \eta^l h^{s+b}
$$

is nonzero only if $s + b = 0$ or $s + b = g$.

First suppose $b = -s$ so that $a = n - 2g + 3 - r$ since $r + s + l = n - g + 1$ and $a + b = l - g + 2$. Then we have

$$
\int_{\hat{\Gamma}} \xi^r \gamma^s \eta^l h^{s+b} = \int_{\hat{\Gamma}} \xi^r \gamma^{n-2g+3}.
$$

Since $\xi \in H^{8g-6-2n}(\Gamma)$, $\xi \gamma^{n-2g+3}$ is a constant multiple of $\prod_{j=1}^g d_1^i d_2^i + g = \prod_{j=1}^g d_1^i d_2^i + g$ in which case we can complete the computation from

$$
\int_{\hat{\Gamma}} \prod_{j=1}^g d_1^i d_2^i + g = 1
$$

Next suppose $b = g - s$ so that $a = n - 3g + 3 - r$. In this case,

$$
\int_{\hat{\Gamma}} \xi^r \gamma^{s+b} = \int_{\hat{\Gamma}} \xi^r \gamma^{n-3g+3} h^g = - \int_{E} \xi (4\gamma)^{n-3g+3} h^{g-1} = - \int_{\Delta} \xi (4\gamma)^{n-3g+3}
$$

where $E$ is the exceptional divisor in $\hat{\Gamma}$ which is a projective bundle over $\Delta$ and $\gamma \in H^2(\Delta)$ is the class introduced in the previous subsection.

Since $\xi \in H^{8g-6-2n}(\Gamma)$, the integral

$$
\int_{\Delta} \xi (4\gamma)^{n-3g+3} = 4^{n-3g+3} \int_{\Delta} \xi^r \gamma^{n-3g+3}
$$

is nonzero only when $\xi|_{\Delta}$ is a constant multiple of $\gamma^{4g-3-n}$. Hence the computation is now complete from

$$
\int_{\Delta} \gamma^g = g!
$$

To be quite explicit, let us consider the classes $a_2^n f_2^n$ with $2m + n = 4g - 3$. Recall that $a_2|_{\Delta} = Y^2$ and $f_2|_{\Delta} = -2\gamma$. Similarly, from

$$
c_2(U)|_{\Gamma} = c_2(L_1 \oplus L_2) = c_1(L_1) c_1(L_2) = Y^2 \otimes 1 + \sum_{i=1}^{2g} Y(d_1^i + d_2^i) \otimes \sigma_i - \sum_{j=1}^{g} (d_1^i d_2^i + d_2^i d_1^i) \otimes \rho
$$
we see that \( a_2 | \Gamma = Y^2 \), \( f_2 | \Gamma = -\gamma_{12} \) where \( \gamma_{12} = \sum_{j=1}^g d_1^j d_2^{j+g} + d_2^j d_1^{j+g} \). Therefore
\[
a_2^m f_2^n | \Delta = Y^{2m}(-2\gamma)^n, \\
a_2^m f_2^n | \Gamma = Y^{2m}(-\gamma_{12})^n.
\]
Notice that \( \gamma_{12} | \Delta = 2\gamma \). Let \( \gamma_1 = \sum_{j=1}^g d_1^j d_2^{j+g} \) and \( \gamma_2 = \sum_{j=1}^g d_2^j d_1^{j+g} \) so that \( \hat{\gamma} = \gamma_1 + \gamma_2 + \gamma_{12} \).

The wall crossing term from the first blow-up is zero if \( g \) is even so that \( 3g - 3 \) is odd or if \( n \neq g \). If \( g \) is odd and \( n = g \),
\[
a_2^n f_2^n | \Delta = Y^{3g-3}(-2\gamma)^g = (-2)^g g! \frac{\gamma^g}{g!} Y^{3g-3} = -2^g g! \frac{\gamma^g}{g!} Y^{3g-3}
\]
and thus from (8.3) the wall crossing term is
\[
\frac{g!}{2^{2g}} \text{Coeff}_{i g - 1}(1 - t)^{g-1}(1 - \frac{t}{4})^{-g}.
\]

From (8.7) and (8.8), the wall crossing term from the second blow-up is
\[
- \frac{1}{2} \sum_{r+s+l=2m-g+1} A_{r,s,l} \sum_{a+b=l-g+2} B_{a,b} \int_{\Gamma} (-\gamma_{12})^n \gamma^r + a h^{r+b}
\]
\[
= - \frac{1}{2^{2m+1}} \sum_{r+s+l=2m-g+1} A_{r,s,l} \sum_{a+b=l-g+2} (B_{2m-2g+3-r-s} \int_{\Gamma} (-\gamma_{12})^n \gamma^{2m-2g+3} + B_{2m-3g+3-r,g-s} \int_{\Delta} (-2\gamma)^n (4\gamma)^{2m-3g+3}).
\]
The last integral is just
\[
\int_{\Delta} (-2\gamma)^n (4\gamma)^{2m-3g+3} = (-1)^n 2^{2m-2g+3} \int_{\Delta} \gamma^g = (-1)^n 2^{2m-2g+3} g!
\]
Also, by combinatorial computation, we have
\[
\int_{\Gamma} (-\gamma_{12})^n \gamma^{2m-2g+3} = (-1)^n \int_{\Gamma} \gamma^{n} \gamma_{12} + \gamma_1 + \gamma_2)^{2g-n}
\]
\[
= (-1)^n \sum_{k=0}^{\(\frac{2g-n}{2}\)} \frac{(-1)^k (2g-2k)! (2g-n)! g!}{(2g-2k-n)! k! (g-k)!}.
\]
So the computation is complete and we have proved the following.

**Theorem 42.** For a pair \((m, n)\) of nonnegative integers satisfying \(2m + n = 4g - 3\),
\[
\kappa(a_2^m f_2^n)[\widehat{\mathcal{M}}(2, 0)] = (-1)^{g-1-m-n} \frac{1}{2^{2m-g+1}} \text{Res}_{Y=0} Y^{2g-2-2m}(eY - 1)
\]
\[
+ \left( \frac{g!}{2^{2g}} \text{Coeff}_{i g - 1}(1 - \frac{t}{2})^{-g-1}(1 - \frac{t}{4})^{-g} \right) \delta_{g,n} - \frac{1}{2^{4m+1}} \sum_{r+s+l=2m-g+1} A_{r,s,l} \sum_{a+b=l-g+2}
\]
\[
\left[ (-1)^n \sum_{k=0}^{\(\frac{2g-n}{2}\)} \frac{(-1)^k (2g-2k)! (2g-n)! g!}{(2g-2k-n)! k! (g-k)!} B_{2m-2g+3-r-s} + (-1)^n 2^{2m-2g+3} g! B_{2m-3g+3-r,g-s} \right]
\]
where \( \delta_{g,n} \) is Kronecker’s delta and the constants \( A_{r,s,l} \) and \( B_{a,b} \) are defined as after (8.7) and (8.8) respectively.
In the case when 0 is a regular value of the moment map \( \mu \) Witten [64] relates the intersection pairings of two classes \( \kappa_M(\alpha), \kappa_M(\beta) \) of complementary degrees in \( H^*(M/\!/G) \) coming from \( \alpha, \beta \in H^*_K(M) \) to the asymptotic behaviour of the integral \( T^*(\alpha\beta e^{i\omega}) \) given by

\[
T^*(\alpha\beta e^{i\omega}) = \frac{1}{(2\pi)^s \text{vol}(K)} \int_{X \in \mathfrak{k}} [dX] e^{-\epsilon <X,X>/2} \int_M \kappa(\alpha)\beta(X) e^{i\omega} e^{i\mu(X)}
\]

(9.1)

\[
= \frac{1}{(2\pi)^{t|W|} \text{vol}(T)} \int_{X \in \mathfrak{t}} [dX] e^{-\epsilon <X,X>/2} \int_M \alpha(X)\beta(X) D^2(X) e^{i\omega} e^{i\mu(X)},
\]

(9.2)

where as before \( \omega = \omega + \mu \) and the notations \( s \) and \( l \) are as after (2.4). \S 2. He expresses the integral as a sum of contributions, one of which is localized near \( \mu^{-1}(0) \) and reduces to the intersection pairing required, while the rest tends to 0 exponentially fast as \( \epsilon \) tends to 0. These results were described in [25], \S 9 and extended there to the case when 0 is not a regular value of \( \mu \).

**Remark 43.** Note that in the conventions of \S 4.3 \( \bar{\omega} = \omega + \mu \) satisfies \( d_K \bar{\omega} = 0 \). On the other hand, Witten uses the convention that

\[
d_K = d - i\nu, e
\]

for \( \xi \in \mathfrak{k} \) so that in his convention \( \omega + i\mu \) is equivariantly closed. For this reason we have chosen to retain \( e^{i\omega} \) in our integral (9.1), whereas Witten writes the integral (9.1) without the factor \( i \) multiplying \( \omega \).

Even when 0 is not a regular value of \( \mu \), Witten’s integral \( T^*(\alpha\beta e^{i\omega}) \) decomposes into the sum of a term \( T^*_0(\alpha\beta e^{i\omega}) \) determined by the action of \( K \) on an arbitrarily small neighbourhood of \( \mu^{-1}(0) \), and other terms which tend to zero exponentially fast as \( \epsilon \to 0 \). Moreover there is a residue formula for \( T^*_0(\alpha\beta e^{i\omega}) \) which is a sum over components of the fixed point set of \( T \) on \( M \) and reduces to the residue formula (2.4) when 0 is a regular value of \( \mu \) (see [25] \S 9). When 0 is not a regular value of \( \mu \) then this residue formula is related to, but not quite the same as, the formulas for intersection pairings given in previous sections; it is not in general a polynomial in \( \epsilon \) but instead it is a polynomial in \( \sqrt{\epsilon} \) (as was proved by Paradan in [56] Cor.5.2).

**Remark 44.** Let \( \Theta \) denote the equivariant class given by the invariant polynomial function \( \Theta(X) = \langle X, X \rangle \) on \( \mathfrak{t} \). It is shown in [25] \S 9 that if a component \( F \in \mathcal{F} \) of the fixed point set \( MT \) is such that \( \mu_T(F) \) does not lie on a wall through 0 (or a wall such that the affine hyperplane spanned by the wall passes through 0), then the contribution of \( F \) to the residue formula for \( T^*_0(\alpha\beta e^{i\omega}) \) is the same as the contribution of \( F \) to the pairing

\[
\kappa_{M,\xi}^T(\alpha\beta D e^{i\omega - \epsilon\Theta/2})[\mu^{-1}(\xi)/T]
\]

for \( \xi \) sufficiently close to 0, which was calculated in Remark 34 it is zero when \( F \notin \mathcal{F}_+ \) and

\[
\frac{(-1)^{s+n}}{|W| \text{vol}(T)} \text{res}(D(X)^2) \int_F i_F^* (\alpha\beta e^{i(\bar{\omega} - \delta)}) (X) e^{-\epsilon (X, X)/2} \frac{e_F(X)}{e_F(X)} [dX])
\]

for any sufficiently small \( \delta \in \mathfrak{t}^* \) which is a regular value of the moment map \( \mu_T \) when \( F \in \mathcal{F}_+ \). When the degrees of \( \alpha \) and \( \beta \) sum to the real dimension of \( \mathcal{M}(n, d) \) and \( \alpha \) and \( \beta \) both restrict to elements of the subspace \( V(n, d) \) of \( H^*_g(n, d) \) which is isomorphic
to $IH^*(\mathcal{M}(n,d))$ (see §5 above) then this contribution is also the same as the contribution of $F$ to the residue formula for the pairing $\langle \kappa(\alpha), \kappa(\beta) \rangle$ in the intersection cohomology of $\mathcal{M}(n,d)$. If however $\mu_T(F)$ does lie on a wall through $0$ then the contribution of $F$ to the residue formula for $T_0^*(\alpha\beta e^{i\omega})$ involves Gaussian integrals over cones $C$ in $t^*$ of the form (cf. \[25\], (9.5))

\begin{equation}
(9.4) \quad \frac{i^j(2\pi)^{-1/2}}{|W| \text{vol}(T)} e^{\gamma/2} \int_{y \in C} d[y] \mathcal{D}(y) e^{-<y,g>/2\pi} \text{res} \left( \mathcal{D}(X) \int_F i^*_F(\alpha(X)\beta(X)e^{i\omega}) e^{i<\mu(F)-y,X>} \right)
\end{equation}

which reduces to the expression above when $C$ is the whole of $t^*$.

We now study $T^*(\alpha\beta e^{i\omega})$ when $M = M^\text{ext}_K$, for simplicity we will only consider the case when $n = 2$ and $d = 0$. As before $T$ denotes the maximal torus $U(1)$ of $SU(2)$, and we identify the Lie algebra of $U(1)$ with $\mathbb{R}$ by equating $X \in \mathbb{R}$ with $iX\gamma$ where $\gamma = (1,-1)$ is a chosen root of $SU(2)$. Note that $<\gamma, \gamma> = 2$ in the Euclidean inner product.

$M^\text{ext}_K$ is of course not compact and there are infinitely many components $F$ of the fixed point set of the action of $T$ on $M^\text{ext}_K$, which are indexed by the value of the moment map (or equivalently by the integers), each diffeomorphic to $T^{2g}$. But by Theorem 34 and Remark 36 of \[25\] (cf. Remark 14 above), the difference between

\begin{equation}
T^*(\alpha\beta e^{i\omega})
\end{equation}

and the pairing $\kappa^T_{M,F}(\alpha\beta \mathcal{D} e^{i\omega-e\Theta/2})[\mu^{-1}(\xi)/T]$ for any $\xi$ sufficiently close to 0 (which can be calculated as at Remark 34) is a sum of contributions corresponding to those components $F$ of the fixed point set of $T$ for which $\mu_T(F)$ lies on a wall through 0 in $t^* \cong \mathbb{R}$, i.e. for which $\mu_T(F) = 0$. In fact there is only one such component $F_0 = T^{2g} \times \{0\}$ in the extended moduli space $M^\text{ext}_K$ (see §4.2). The Euler class $e_{F_0}(X)$ is given by $(2X)^{2g}$ as it is the product of $2g$ copies of the root $\gamma(X) = 2X$, and we have $\mathcal{D}^2(X) = (2X)^2$ since $\mathcal{D}(X) = \gamma(X)$. Hence we can conclude that

\begin{equation}
(9.5) \quad T^*(\alpha\beta e^{i\omega}) = E_0 + E_1 + \frac{(-1)^{g-1}}{2} \text{res}_{X=0} \left( \sum_{j \geq 0} \frac{(-\epsilon X^2)^j}{j!} \int_{T^{2g}} \alpha \beta e^{i\omega} \right)
\end{equation}

where $E_0$ is the contribution of $F_0$ to (9.4), while $E_1$ is a sum of terms vanishing exponentially in $\epsilon$ as $\epsilon \to 0$ (see \[26\] Theorem 4.1 for a more precise definition of “vanishing exponentially”) and the last term is given by Remark 34.

For simplicity we take $\alpha = \beta = 1$ from now on. Then the contribution $E_0$ to the integral $T^*(e^{i\omega})$ from the component $F_0$ is given by

\begin{equation}
(9.6) \quad E_0 = C_0 \int_{X \in \mathbb{R} - i\delta} \frac{e^{-\epsilon X^2}}{X^{2g-2}} dX
\end{equation}

where

\begin{equation}
C_0 = \frac{1}{4\pi} \frac{\int_{F_0} e^{i\omega}}{2^{2g-2}}
\end{equation}

and $\int_{F_0} e^{i\omega} = (2i)^g$ (see \[29\], Lemma 10.10). We have treated overall normalization constants using the conventions of Corollary 8.2 of \[26\], with the correction made in Footnote 9 of \[29\]. Here $\delta > 0$ is a small real parameter which ensures convergence of the integral (9.6) despite the pole at $X = 0$. 


Remark 45. One readily sees (by changing variables from $X$ to $Y = \epsilon X^2$) that the integral in (9.6) is homogeneous in $\epsilon$. In fact

$$\int_{X \in \mathbb{R} - i\delta} \frac{e^{-\epsilon X^2}}{X^{2g-2}} dX = \frac{\epsilon^{g-3/2}}{2} \int_{\epsilon Y^2 \in \mathbb{R} - i\delta} e^{-Y^{1/2} i/\epsilon} Y^{1/2 - 9} dY.$$  

In the situation studied in [64], Witten found that the integral analogous to $I^\epsilon(e^{i\omega})$ (which Witten denotes by $Z(\epsilon)$) is computed when $n = 2$ and $d = 0$ as

$$Z(\epsilon) = \frac{1}{(2\pi)^{g-1}} \sum_{n=1}^{\infty} \exp(-\pi^2 n^2) \frac{1}{n^{2g-2}}$$  

(see [64], (4.43)). It follows (using the Poisson summation formula as in (4.53) of [64]) that

$$\left( \frac{\partial}{\partial \epsilon} \right)^{g-1} Z(\epsilon) = C_1 \epsilon^{-1/2} + E'_1$$  

where $C_1 = \frac{(-1)^{g-1}}{2^{g-1} \sqrt{\pi}}$ and $E'_1$ vanishes exponentially as $\epsilon \to 0$. Thus we have (integrating with respect to $\epsilon$) that (see [64], (4.54))

$$Z(\epsilon) = C_1 C_2 \epsilon^{g-3/2} + E_1 + E_2$$  

where

$$C_2 = 2 \cdot (2/3) \cdot (2/5) \ldots (2/(2g - 3))$$  

and $E_1$ vanishes exponentially as $\epsilon \to 0$ while $E_2$ is a polynomial in $\epsilon$ of degree $g - 2$.

We can now compare our expression for $I^\epsilon(e^{i\omega})$ to Witten’s $Z(\epsilon)$. It is easy to see that if $j \geq g - 1$ then

$$\operatorname{res}_{X = 0} \left( \frac{(-X^2)^j}{j!(2X)^{2g-2}(e^{2X} - 1)} \right) = 0.$$  

Hence it follows from (9.6) and (9.7) that

$$\left( \frac{\partial}{\partial \epsilon} \right)^{g-1} I^\epsilon(e^{i\omega}) = (-1)^{g-1} C_0 \int_{X \in \mathbb{R} - i\delta} \frac{dX \epsilon e^{-\epsilon X^2}}{2g \sqrt{\pi}} = (-1)^{g-1} C_0 \sqrt{\pi} \epsilon^{-1/2} + E'_1$$  

where $E'_1$ vanishes exponentially as $\epsilon \to 0$ (cf. [26] Theorem 4.1). From this and Remark 44 we obtain

$$I^\epsilon(e^{i\omega}) = \frac{(-1)^{g-1} i^g}{2g \sqrt{\pi}} C_2 \epsilon^{g-3/2} + E_1 + \sum_{j=0}^{g-2} \operatorname{res}_{X = 0} \left( \frac{(-\epsilon X^2)^j}{j!(2X)^{2g-2}(e^{2X} - 1)} \right)$$  

where $E'_1$ is a sum of terms vanishing exponentially in $1/\epsilon$ as $\epsilon \to 0$ and $C_2$ was introduced in (9.10) above.

Up to multiplication by the constant $i^g$, the coefficient of $\epsilon^{g-3/2}$ in (9.12) agrees with the coefficient of $\epsilon^{g-3/2}$ in Witten’s expression (9.9) for $Z(\epsilon)$. The factor $i^g$ is accounted for because Witten computes

$$Z(\epsilon) = \frac{1}{(2\pi)^{s}} \operatorname{vol}(K) \int_{X \in \mathbb{R}^k} e^{-\epsilon X^2} e^{\omega + \mu X}$$  

which can be transformed into our $I^\epsilon$ by the substitution $\omega \mapsto i\omega$. 

We can also investigate the relation between the polynomial part of $Z(\epsilon)$ and our expression

\[(9.13) \quad \sum_{j=0}^{g-2} \text{res}_{X=0} \left( \frac{(-\epsilon X^2)^j}{j!(2X)^{2g-2}(e^{2X} - 1)} \right)\]

for the polynomial part of $I^i(e^{i\omega})$. The formula (8.7) for $Z(\epsilon)$ may be expanded to find the coefficient of $\epsilon^k$ for $0 \leq k \leq g - 2$ (cf. (4.49) of [64]) as

\[(9.14) \quad \left( \frac{1}{2\pi^2} \right)^{g-1} \sum_{n=1}^{\infty} \frac{(-\pi^2)^k}{n^{2g-2-2k}}\]

which equals

\[(9.15) \quad \left( \frac{1}{2\pi^2} \right)^{g-1} (-\pi^2)^k \zeta(2g - 2 - 2k)\]

where $\zeta$ denotes the Riemann zeta function.

By an elementary contour integral argument (cf. [28] Lemma 5.12) we see that

**Lemma 46.** For any positive integer $m$, we have

\[\zeta(2m) = \sum_{n>0} \frac{1}{n^{2m}} = \pi i \text{res}_{Y=0} \left( \frac{1}{2^{2m}(e^{2\pi i Y} - 1)} \right)\]

\[= (-1)^m (\pi)^{2m} \text{res}_{X=0} \left( \frac{1}{X^{2m}(e^{2X} - 1)} \right).\]

Using Lemma 46 to express the residue in (9.13) as a multiple of a zeta function, together with the fact from [29], Lemma 10.10 that $\int_{T^{2g}} e^{i\omega} = (2i)^g$, we see that (9.13) agrees with (9.15).

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Lisa C. Jeffrey, Department of Mathematics, University of Toronto, Toronto ON, M5S 2E4, Canada
E-mail address: jeffrey@math.toronto.edu

Young-Hoon Kiem, Department of Mathematics, Seoul National University, Seoul 151-747, Korea
E-mail address: kiem@math.snu.ac.kr

Frances C. Kirwan, Mathematical Institute, Oxford University OX1 3LB, UK
E-mail address: frances.kirwan@balliol.oxford.ac.uk

Jonathan M. Woolf, Department of Mathematical Sciences, Liverpool, L69 7ZL, UK
E-mail address: jonathan.woolf@liverpool.ac.uk