On the relation between states and maps in infinite dimensions

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Abstract

Relations between states and maps, which are known for quantum systems in finite-dimensional Hilbert spaces, are formulated rigorously in geometrical terms with no use of coordinate (matrix) interpretation. In a tensor product realization they are represented simply by a permutation of factors. This leads to natural generalizations for infinite-dimensional Hilbert spaces and a simple proof of a

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generalized Choi Theorem. The natural framework is based on spaces of Hilbert-Schmidt operators $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$ and the corresponding tensor products $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ of Hilbert spaces. It is proved that the corresponding isomorphisms cannot be naturally extended to compact (or bounded) operators, nor reduced to the trace-class operators. On the other hand, it is proven that there is a natural continuous map $\mathcal{C} : \mathcal{L}_1(\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)) \rightarrow \mathcal{L}_\infty(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{H}_1))$ from trace-class operators on $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$ (with the nuclear norm) into compact operators mapping the space of all bounded operators on $\mathcal{H}_2$ into trace class operators on $\mathcal{H}_1$ (with the operator-norm). Also in the infinite-dimensional context, the Schmidt measure of entanglement and multipartite generalizations of state-maps relations are considered in the paper.

1 Introduction

Entanglement is one of the most counterintuitive features of quantum mechanics. Already in the early years of the birth of quantum theory, Erwin Schrödinger realized that this aspect is a consequence of the mathematical structure of the theory \cite{1}. It is a characteristic property of quantum mechanics, not present in other physical theories described by linear equations like, for instance, classical electrodynamics. Entanglement was considered with embarrassment in connection with non locality, as pointed out by the Einstein-Podolsky-Rosen gedanken experiment \cite{2}. Its role was clarified with the discovery of Bell’s inequalities \cite{3, 4}. It was shown that these inequalities can be violated in quantum mechanics but have to be satisfied by all local realistic theories. The violation of the inequalities demonstrates the presence of entanglement. In the near past it was realized that entanglement could be a great resource for quantum information theory. The promising applications of this peculiar quantum property has induced intensive experimental efforts to build entangled quantum states and major theoretical efforts to understand the mathematical structure of entanglement \cite{5}. To put the present paper into perspective, let us briefly consider how the problem arises.

In the Dirac-Schrödinger picture of quantum mechanics, one identifies the carrier space of quantum evolution with a complex separable Hilbert space $\mathcal{H}$. The probabilistic interpretation of quantum mechanics requires that states be identified with rays, points of the complex projective space of $\mathcal{H}$. By using the Hermitian inner product one defines an action of the unitary group with an associated momentum map \cite{6, 7, 8}. This map relates rays with rank-one projectors, i.e. operators, elements of the dual vector space of the Lie algebra of the unitary group. Thanks to this immersion, it becomes possible to consider convex combinations of elements in the image of this map and therefore to construct density states, also called density operators. In this way observables and states are
represented both by means of operators, even though with qualifying different properties to take into account their corresponding physical interpretations. A similar situation results in the $C^*$-algebraic approach to quantum mechanics, originated by Heisenberg and developed by Segal and Haag \[9, 10\]. Here one considers states as nonnegative normalized linear functionals on the space of observables, real elements of the $C^*$-algebra and associates with them density states by means of Gleason's theorem \[11\]. In either approach states are identified with appropriate operators.

Composite systems are mathematically formed as tensor products of the Hilbert spaces associated with the system we are composing, called subsystems. Similarly within the $C^*$-algebra approach, the consideration of states as maps has boosted a search for various procedures to characterize separability and entanglement of states by exploiting as much as possible what is available for the classification of maps \[12, 13, 14\]. The difficulties in a straightforward application of known classification procedures rely on the fact that the very definition of states as convex combination of rank-one projectors provides them with a positivity property which is not preserved under tensorial products, in general the product of positive maps does not result into a positive one. While the existing literature is concerned with the relation between maps and states restricted to finite dimensional Hilbert spaces or $C^*$-algebras, the aim of this paper is to present a careful analysis of these various relations between states and maps for composite quantum systems in the more realistic situation of infinite dimensions.

Relations between states and maps are well known for systems in finite-dimensional Hilbert spaces. In the second section of the paper we reformulate them without invoking any particular matrix realization of the states. This allows us to generalize in the following sections the known results to infinite-dimensional Hilbert spaces and Hilbert-Schmidt operators acting as maps between them. As a result we can describe in the infinite-dimensional setting connections between positivity and complete positivity of maps and separability properties of the corresponding states on the composite spaces proven by Jamiołkowski and Choi for the finite-dimensional case. We discuss briefly generalization to multipartite systems and show that the infinite-dimensional Jamiołkowski isomorphism can be neither sensibly extended to the larger class of bounded operators nor reduced to a smaller set of the trace-class operators.

2 The Jamiołkowski isomorphism

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. In a finite dimensional case, $\dim \mathcal{H}_{1,2} \leq \infty$, the Jamiołkowski isomorphism \[15, 16\] is a mapping:

$$\mathcal{J} : \mathcal{L}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{H}_1)) \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)), \tag{1}$$
where by \( \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \) we denote the space of all complex linear maps from \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \), with an abbreviation \( \mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H}) \) for the space of all linear endomorphisms of \( \mathcal{H} \).

We prefer to define objects in a basis-independent way, so we refer to speak about \( \mathcal{H} \) and \( \mathcal{L}(\mathcal{H}) \) instead of \( \mathbb{C}^n \) and complex matrices. Therefore we start with the following observations. First, there is a natural anti-isomorphism between \( \mathcal{H} \) and its dual complex vector space, \( \kappa_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^* \), induced by the scalar product on \( \mathcal{H} \), which in the Dirac notation reads

\[
\mathcal{H} \ni x : \equiv \langle x \rangle \mapsto x : = \langle x \rangle \in \mathcal{H}^*.
\]

The dual space \( \mathcal{H}^* \) is canonically a Hilbert space with the Hermitian product \( \langle x^1, x^2 \rangle_{\mathcal{H}^*} = \langle x^2, x^1 \rangle_{\mathcal{H}} \), where the latter is the scalar product on \( \mathcal{H} \), which shows that \( \kappa \) is an anti-unitary. In the following we usually skip the subscripts specifying the Hermitian products in various spaces, if this does not lead to a confusion.

We clearly have \( \kappa_{\mathcal{H}^*} \circ \kappa_{\mathcal{H}} = id_{\mathcal{H}} \) up to an obvious identification \( (\mathcal{H}^*)^* = \mathcal{H} \). Moreover, the anti-isomorphism \( \kappa_{\mathcal{H}} \) induces an anti-isomorphism \( \mathcal{L}(\mathcal{H}) \ni A \mapsto \bar{A} \in \mathcal{L}(\mathcal{H}^*) \) of the corresponding spaces of complex linear operators, where \( \bar{A} = (A^\dagger)^* \). Here, clearly, the adjoint operator \( A^\dagger \in \mathcal{L}(\mathcal{H}) \) is defined by \( \langle A^\dagger x, y \rangle_{\mathcal{H}_1} = \langle x, Ay \rangle_{\mathcal{H}_1} \) and \( A^* \in \mathcal{L}(\mathcal{H}^*) \) is the dual map. By definition, \( \kappa_{\mathcal{H}} \) intertwines \( A \) with \( \bar{A} \), i.e. \( \bar{A}(x) = \kappa_{\mathcal{H}}(A) \). The notation is consistent, because \( \bar{A} = \kappa_{\mathcal{L}(\mathcal{H})}(A) \) (up to an obvious identification \( \mathcal{L}(\mathcal{H})^* \simeq \mathcal{L}(\mathcal{H}^*) \)) for the Hermitian product \( \langle A, B \rangle = \text{Tr}(A^\dagger \circ B) \) on \( \mathcal{L}(\mathcal{H}) \).

The point here is that \( A \mapsto \bar{A} \) respects the composition, \( \bar{A} \circ \bar{B} = \bar{A} \circ B \), while \( (AB)^* = B^* A^* \). This means that, restricting ourselves to the groups of invertible complex operators on the Hilbert spaces, we have a canonical group isomorphism \( \text{GL}(\mathcal{H}) \ni A \mapsto \bar{A} \in \text{GL}(\mathcal{H}^*) \), while \( \text{GL}(\mathcal{H}) \ni A \mapsto A^* \in \text{GL}(\mathcal{H}^*) \) is an anti-isomorphism. This group isomorphism restricts to an isomorphism of the unitary groups \( \text{U}(\mathcal{H}) \ni A \mapsto \bar{A} \in \text{U}(\mathcal{H}^*) \), as in this case

\[
\langle \bar{A}(x), \bar{A}(y) \rangle_{\mathcal{H}^*} = \langle Ay, Ax \rangle_{\mathcal{H}} = \langle y, x \rangle_{\mathcal{H}} = \langle \bar{x}, \bar{y} \rangle_{\mathcal{H}^*}.
\]

Note that in the physics literature one usually identifies \( \mathcal{H} \) with \( \mathcal{H}^* \) by fixing an orthonormal basis \( (e_i) \) in \( \mathcal{H} \) and putting

\[
\sum_i c_i|e_i\rangle \simeq \sum_i c_i\langle e_i|.
\]
It is a true isomorphism which, however, depends on the choice of the basis, and not the canonical anti-isomorphism we speak about.

We will use the following canonical identification of Hilbert spaces,

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) = \mathcal{H}_1 \otimes \mathcal{H}_2^*.$$  \hfill (4)

Under this identification \((x \otimes y) (y') = \langle y, y' \rangle x\) for \(x \in \mathcal{H}_1\) and \(y, y' \in \mathcal{H}_2\). Moreover, the Hilbert-Schmidt scalar product \(\langle A, B \rangle = \text{Tr} (A^\dagger \circ B)\) on \(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)\) coincides with the standard scalar product \(\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle_{\mathcal{H}_1} \langle y_2, y_1 \rangle_{\mathcal{H}_2}\) on \(\mathcal{H}_1 \otimes \mathcal{H}_2^*\). Here, the adjoint operator

$$A^\dagger \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{H}_2 \otimes \mathcal{H}_1^*$$  \hfill (5)

is defined in an obvious way by \(\langle A^\dagger x, y \rangle_{\mathcal{H}_2} = \langle x, Ay \rangle_{\mathcal{H}_1}\) (or, in the tensor product realization, \((x \otimes y)^\dagger = y \otimes x\)). Indeed, for \(\{f_\alpha\}\) being an arbitrary orthonormal basis in \(\mathcal{H}\),

$$\text{Tr}((x_1 \otimes y_1)^\dagger \circ (x_2 \otimes y_2)) = \text{Tr}((x_1, x_2) y_1 \otimes y_2) = \sum_\alpha \langle f_\alpha, (y_1 \otimes y_2) f_\alpha \rangle = \langle x_1, x_2 \rangle \sum_\alpha \langle f_\alpha, y_1 \rangle \langle y_2, f_\alpha \rangle = \langle x_1, x_2 \rangle \langle y_2, y_1 \rangle.$$

We have a canonical \(GL(\mathcal{H}_1) \times GL(\mathcal{H}_2)\)-action on \(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)\):

$$GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \ni (A, B, T) \mapsto A \circ T \circ B^\dagger \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$$  \hfill (6)

which in the tensor product realization takes the form

$$GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times \mathcal{H}_1 \otimes \mathcal{H}_2^* \ni (A, B, x \otimes y) \mapsto A(x) \otimes B(y) \in \mathcal{H}_1 \otimes \mathcal{H}_2^*.$$  \hfill (7)

This action can be reduced to an \(U(\mathcal{H}_1) \times U(\mathcal{H}_2)\)-action which is unitary, as in this case

$$\langle Ax \otimes B y, A x' \otimes B y' \rangle = \langle Ax, Ax' \rangle \overline{\langle B y, B y' \rangle} = \langle x, x' \rangle \langle y', y \rangle = \langle x \otimes y, x' \otimes y' \rangle.$$

If \(\mathcal{H}_1 = \mathcal{H}_2\), then one can reduce the above action to a diagonal action of \(GL(\mathcal{H})\) (or \(U(\mathcal{H})\): \((A, T) \mapsto A \circ T \circ A^\dagger\).

The canonical isomorphism \(\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_2^* \otimes \mathcal{H}_1\) gives rise to an identification

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \simeq \mathcal{L}(\mathcal{H}_1^*, \mathcal{H}_2^*) \simeq \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$  \hfill (8)

Moreover,

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)^* \simeq \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2).$$  \hfill (9)
with the obvious pairing
\[ \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \times \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \ni (A, B) \mapsto \text{Tr}(A \circ B) \in \mathbb{C}. \] (10)

In particular,
\[ \mathcal{L}(\mathcal{H})^* \simeq \mathcal{L}(\mathcal{H}) \simeq \mathcal{L}(\mathcal{H}^*). \] (11)

Note that we have further natural identifications
\[ \mathcal{L}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{H}_1)) = \mathcal{H}_1 \otimes \mathcal{H}_1^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \]
and
\[ \mathcal{L}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)) = \mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2^*)^* = \mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_1^*. \]

**Definition 1.** The Jamiołkowski isomorphism \([\mathcal{J}],\) up to above natural identifications, is defined as a natural transposition in the tensor products
\[ \mathcal{J} : \mathcal{H}_1 \otimes \mathcal{H}_1^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_1^* \] (12)
consisting of interchanging of the second and fourth factors, i.e.
\[ \mathcal{J} : x_1 \otimes \overline{x}_2 \otimes y_1 \otimes \overline{y}_2 \mapsto x_1 \otimes \overline{y}_2 \otimes y_1 \otimes \overline{x}_2. \] (13)

The twisted Jamiołkowski isomorphism
\[ \widetilde{\mathcal{J}} : \mathcal{L}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{H}_1)) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \] (14)
comes in a similar way from the permutation
\[ \widetilde{\mathcal{J}} : \mathcal{H}_1 \otimes \mathcal{H}_1^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_1^* \] (15)
\[ x_1 \otimes \overline{x}_2 \otimes y_1 \otimes \overline{y}_2 \mapsto x_1 \otimes y_1 \otimes \overline{y}_2 \otimes \overline{x}_2. \] (16)

As Jamiołkowski isomorphisms are simply permutations in the tensor product, they are automatically unitary. Moreover, it is completely clear that Jamiołkowski isomorphisms intertwine the canonical actions of the group \(GL(\mathcal{H}_1) \times GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times GL(\mathcal{H}_2)\) on the tensor product \(\mathcal{H}_1 \otimes \mathcal{H}_1^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^*:\)
\[ (A, A', B, B', x_1 \otimes \overline{x}_2 \otimes y_1 \otimes \overline{y}_2) \mapsto Ax_1 \otimes \overline{A}x_2 \otimes By_1 \otimes \overline{B}y_2 \] (17)
and its corresponding permutations, so the following is immediate.

**Theorem 1.** The Jamiołkowski isomorphisms are unitary and intertwine the canonical \(GL(\mathcal{H}_1) \times GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times GL(\mathcal{H}_2)\)-actions.
3 Infinite dimensions

The above definitions can be extended to infinite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows. In this case we define the Hilbert-Schmidt tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the closure of the algebraic tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$ with respect to the scalar product which on simple tensors reads $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle_{\mathcal{H}_1} \langle y, y' \rangle_{\mathcal{H}_2}$. In this way, elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ represent Hilbert-Schmidt operators from $\mathcal{H}_2$ into $\mathcal{H}_1$ and can be viewed as infinite combinations $A = \sum_{i,\alpha} a_{i\alpha} (e_i \otimes f_{\alpha})$, where $(e_i)$ and $(f_{\alpha})$ are orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\|A\|^2 = \sum_{i,\alpha} |a_{i\alpha}|^2 < \infty$ is the squared Hilbert-Schmidt norm of $A$ (it does not depend on the choice of bases). It is well-known \cite{17} that every compact operator $A: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ admits the so called Schmidt decomposition, $A = \sum_j \lambda_j (a_j, \cdot) b_j$, with $(a_j)$ and $(b_j)$ being (not necessarily complete) orthonormal sets, and $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. The Hilbert-Schmidt norm can be equivalently defined as $\|A\|^2 = \sum_j |\lambda_j|^2$. In fact, the coefficients $\lambda_j$ can be chosen positive. In the following we will denote $\mathcal{H}_1 \otimes \mathcal{H}_2$ simply as $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since the Hermitian conjugation is also a transposition of the tensor product, $A^\dagger$ is Hilbert-Schmidt if $A$ is.

Now, the (Hilbert-Schmidt) tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2^\ast$ represents the space $L_2(\mathcal{H}_2, \mathcal{H}_1)$ of the Hilbert-Schmidt operators, i.e. the Hilbert space of those complex linear maps $A$ from $\mathcal{H}_2$ to $\mathcal{H}_1$ such that $\sum_i \langle Af_{\alpha}, Af_{\alpha} \rangle_{\mathcal{H}_1} < \infty$, for some (thus all) orthonormal basis $\{f_{\alpha}\}$ in $\mathcal{H}_2$, and with the Hermitian form

$$\langle A, B \rangle = \text{Tr}(A^\dagger B) = \sum_i \langle Af_{\alpha}, Bf_{\alpha} \rangle_{\mathcal{H}_1}.$$ 

Note that the trace is well-defined, since any composition of Hilbert-Schmidt operators is known to be a trace-class operator \cite{17}.

We will abbreviate $L_2(\mathcal{H}, \mathcal{H})$ to $L_2(\mathcal{H})$ for an arbitrary Hilbert space $\mathcal{H}$. The symbol $\mathcal{L}$ is now reserved for all bounded complex linear maps, so that $L(\mathcal{H}_2, \mathcal{H}_1)$ is the space of all bounded operators from $\mathcal{H}_2$ to $\mathcal{H}_1$ with the operator-norm topology. The latter makes sense also in the Banach category.

In complete analogy with the finite-dimensional case we have also natural canonical identifications

$$L_2(L_2(\mathcal{H}_2), L_2(\mathcal{H}_1)) = (\mathcal{H}_2 \otimes \mathcal{H}_1^\ast) \otimes (\mathcal{H}_2 \otimes \mathcal{H}_2^\ast)^\ast = \mathcal{H}_2 \otimes \mathcal{H}_1^\ast \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^\ast$$

and

$$L_2(L_2(\mathcal{H}_2, \mathcal{H}_1)) = (\mathcal{H}_2 \otimes \mathcal{H}_1^*) \otimes (\mathcal{H}_1 \otimes \mathcal{H}_2^*)^* = \mathcal{H}_2 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_1^*.$$
Thus, the Jamiołkowski isomorphism, defined on the level of tensor products by the same transposition (13), is now interpreted as
\[ J : \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)) \to \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{H}_1), \]
and the twisted Jamiołkowski isomorphism, as
\[ \tilde{J} : \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)) \to \mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2). \]
Both isomorphisms are clearly unitary. Moreover, since the Hilbert-Schmidt operators form an operator ideal, the Hilbert-Schmidt tensor products are invariant with respect to the canonical \( GL(\mathcal{H}_1) \times GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times GL(\mathcal{H}_2) \)-actions and the Jamiołkowski isomorphisms intertwines these actions.

Note also that the original definitions are obviously equivalent to the following properties of the Jamiołkowski isomorphisms.

**Proposition 1.** The Jamiołkowski unitary isomorphisms (18) and (19) can be uniquely characterized, respectively, by the identities
\[ \langle x \otimes y, J(\Phi)(x' \otimes y') \rangle = \langle x \otimes x', \Phi(y \otimes y') \rangle, \]
\[ \langle x \otimes y, \tilde{J}(\Phi)(x' \otimes y') \rangle = \langle x \otimes x', \Phi(y' \otimes y) \rangle, \]
which must be fulfilled for all \( x, x' \in \mathcal{H}_1 \), \( y, y' \in \mathcal{H}_2 \). Equivalent formulations of the above identities are, respectively,
\[ \langle y \otimes \Phi \otimes x' \otimes y', J(\Phi) \rangle = \langle x' \otimes \Phi \otimes y \otimes y', \Phi \rangle, \]
\[ \langle y \otimes \overline{x} \otimes x' \otimes y', \tilde{J}(\Phi) \rangle = \langle x' \otimes \overline{x} \otimes y' \otimes y, \Phi \rangle. \]

**Example 1.** For \( \rho \in \mathcal{L}_2(\mathcal{H}_2) \), orthonormal bases \((x_l) \) and \((y_\alpha) \) in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), and for
\[ A, B \in \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1) = \mathcal{H}_1 \otimes \mathcal{H}_2^*:\]
\[ A = \sum_{l, \alpha} A^{l \alpha} \cdot x_l \otimes \overline{y}_\alpha, \quad B = \sum_{l, \alpha} B^{l \alpha} \cdot x_l \otimes \overline{y}_\alpha, \]
let us consider the map
\[ M^B_A \in \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)), \quad M^B_A : \rho \mapsto A\rho B^\dagger, \]
\[ M^B_A : y_\alpha \otimes y_\beta \mapsto \sum_{l,m,\alpha,\beta} A^{l\alpha} B^{m\beta} \cdot x_l \otimes x_m. \]  

(26)

Hence, the Jamiołkowski isomorphism applied to \( M^B_A \) is a map \( \mathcal{J}(M^B_A) \in \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)) \) represented by

\[
\mathcal{J}(M^B_A) = \mathcal{J} \left( \sum_{l,m,\alpha,\beta} A^{l\alpha} B^{m\beta} \cdot x_l \otimes y_\alpha \otimes y_\beta \otimes x_m \right)
\]

(27)

\[
= \sum_{l,m,\alpha,\beta} (A^{l\alpha} \cdot x_l \otimes y_\alpha) \otimes (B^{m\beta} \cdot x_m \otimes y_\beta) = A \otimes B,
\]

(28)

i.e. \( \mathcal{J}(M^B_A) \) is just the one-dimensional operator \( |A\rangle\langle B| \). In particular, if \( A = B \), the operator \( M^A_A = K_A \) is just the standard Kraus map \( K_A(\rho) = A\rho A^\dagger \) and its Jamiołkowski image \( \mathcal{J}(K_A) \) is the Hermitian "projection" \( p_A = |A\rangle\langle A| \) on the vector \( A \in \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1) \). This is a true projection if the length of \( A \) is 1. The fact that we deal with a unitary isomorphism implies easily that there is an orthonormal basis in \( \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)) \) consisting of operators of the form \( M^A_{A_j} \) for a basis \((A_j)\) in \( \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1) \).

Since \( \mathcal{J} \) is unitary, the Hilbert-Schmidt norm of \( K_A \) equals the Hilbert-Schmidt norm of this projection, i.e. \( ||A||^2_2 \). Recall that the space of Hilbert-Schmidt operators is an operator ideal in the space of all bounded operators as \( ||A \circ \rho||_2 \leq ||A||_\infty ||\rho||_2 \), where \( ||\cdot||_\infty \) is the operator-norm. Since \( ||K_A(\rho)||_2 \leq ||A||^2_\infty ||\rho||_2 \), the operator-norm \( \|K_A\|_\infty \) of \( K_A \) is not bigger than the square of the operator-norm of \( A \), i.e. \( \|K_A\|_\infty \leq \|A\|^2_\infty \). But the operator-norm of the projection \( A \otimes A \) is still \( ||A||^2_2 \). Since we can easily find a sequence \((A_n)\) with \( ||A_n||_2 = 1 \) such that \( ||A_n||_\infty \to 0 \), this shows that the Jamiołkowski isomorphism is not continuous in the operator-norm topology. In other words, \( \mathcal{J} \) does not admit a natural extension to a map

\[ \mathcal{J} : \mathcal{L}_\infty(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)) \to \mathcal{L}_\infty(\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)), \]

where \( \mathcal{L}_\infty \) denotes the space of compact operators – the operator-norm closure of the space of all Hilbert-Schmidt operators. One can think that the above suggests that the inverse Jamiołkowski isomorphism \( \mathcal{J}^{-1} \) is continuous in the norm topology, as

\[ ||J^{-1}(p_A)||_\infty = ||K_A||_\infty \leq ||p_A||_2. \]
But this is also not true, since \( \left\| \sum_{i=1}^{n} p_{A_i} \right\|_{\infty} \) is 1 for \( A_i = e \otimes f_i, \|e\| = 1 \), and the operator-norm of the corresponding Kraus map

\[
L_2(\mathcal{H}_2) \ni \rho \mapsto \sum_{i=1}^{n} A_i \rho A_i^\dagger \in L_2(\mathcal{H}_1)
\]

is at least \( \sqrt{n} \). Indeed, the projection \( P_n \) on the subspace spanned by \( f_1, \ldots, f_n \) has the Hilbert-Schmidt norm \( \sqrt{n} \), while its image

\[
\sum_{i=1}^{n} A_i P A_i^\dagger = \sum_{i=1}^{n} (f_i, P f_i) e \otimes \overline{e}
\]

has the Hilbert-Schmidt norm \( n \).

Any composition of Hilbert-Schmidt operators is well known to be a trace-class operator, called also nuclear operator (see e.g. [18, Chapter VII]). The space of nuclear operators \( T : \mathcal{H}_2 \to \mathcal{H}_1 \) consists of operators admitting a decomposition into one-dimensional operators: \( Tx = \sum_{i} (a_i, x) b_i \) (in \( L_2(\mathcal{H}_2, \mathcal{H}_1) = \mathcal{H}_1 \otimes \mathcal{H}_2^* \) they are represented as tensors that can be written in the form \( \sum_i b_i \otimes \overline{a_i} \) with \( \sum_i \|a_i\| \|b_i\| < \infty \). They can be equivalently described as these operators for which \( T^*T \) is Hilbert-Schmidt on \( \mathcal{H}_2 \). The nuclear norm can be defined as \( \|T\|_1 = \sum_a \|T f_a\| = \text{Tr} \left( \sqrt{T T^*} \right) \), as \( \|T\|_1 = \sum_i \lambda_i \) for any Schmidt decomposition \( T = \sum_i \lambda_i (a_i, \cdot) b_i \) with \( (a_i) \) and \( (b_i) \) being (not necessarily complete) orthonormal sets, or as the infimum of \( \sum_i \|a_i\| \|b_i\| \) for all possible realizations. The latter has the advantage that it applies also in the Banach space context.

**Example 2.** For an orthonormal base \( (x_i) \) in \( \mathcal{H}_1 \), a vector \( y \in \mathcal{H}_2 \) of length 1, and for a sequence of complex numbers \( a = (a_i) \in l^2 \), the operator \( T : \mathcal{H}_2 \to \mathcal{H}_1 \),

\[
T = \left\langle \sum_i a_i \cdot x_i \otimes \overline{x_i} \right\rangle y \otimes \overline{y} = \sum_i a_i \cdot x_i \otimes \overline{x_i} \otimes y \otimes \overline{y},
\]

is nuclear with the nuclear norm \( \|T\|_1 = \|a\|_2 = \sqrt{\sum_i |a_i|^2} \). Its Jamiołkowski image is Hermitian

\[
J(T) = \sum_i a_i |x_i \otimes \overline{y}\rangle \langle x_i \otimes \overline{y}| = \sum_i a_i \cdot x_i \otimes \overline{y} \otimes y \otimes \overline{x_i}
\]

with eigenvalues \( (a_i) \), so \( \|J(T)\|_1 = \|a\|_1 = \sum_i |a_i| \). Since there are sequences \( a \in l^2 \) with infinite \( l^1 \)-norm, the Jamiołkowski isomorphism does not map nuclear operators into nuclear ones. A similar fact can be proved for the inverse Jamiołkowski isomorphism.
Let us summarize the conclusions of the above examples in the following proposition.

**Proposition 2.** The Jamiołkowski isomorphism [18] and its inverse cannot be extended to all compact operators nor restricted to nuclear (trace-class) operators.

Despite of the above negative result, it is obvious that the map $M_B^A : \rho \mapsto A\rho B^\dagger$, associated with $A, B \in \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$, can be viewed as a map $M_B^A : \mathcal{L}(\mathcal{H}_2) \rightarrow \mathcal{L}_1(\mathcal{H}_1)$.

**Theorem 2.** There is a unique continuous map

$$C : \mathcal{L}_1(\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)) \rightarrow \mathcal{L}_\infty(\mathcal{L}(\mathcal{H}_2), \mathcal{L}_1(\mathcal{H}_1))$$

(29)

such that $C(A \otimes B)(\rho) = A\rho B^\dagger$ for all $A, B \in \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$.

**Proof.** Let us start with computing the operator-norm of $M_B^A$. First of all, for an orthonormal basis $(e_j)$ in $\mathcal{H}_1$, we have

$$\rho B^\dagger x = \sum_j \langle e_j, \rho B^\dagger x \rangle e_j,$$

so

$$A\rho B^\dagger x = \sum_j \langle B\rho^\dagger e_j, x \rangle A e_j.$$

Hence,

$$\|A\rho B^\dagger\|_1 \leq \sum_j \|B\rho^\dagger e_j\| \cdot \|A e_j\| \leq \left( \sum_j \|B\rho^\dagger e_j\|^2 \right)^{\frac{1}{2}} \left( \sum_j \|A e_j\|^2 \right)^{\frac{1}{2}} = \|B\rho^\dagger\|_2 \|A\|_2.$$

But, as easily seen, $\|B\rho^\dagger\|_2 \leq \|B\|_2 \|\rho\|_\infty$, so $\|A\rho B^\dagger\|_1 \leq \|A\|_2 \|B\|_2 \|\rho\|_\infty$ and $\|M_B^A\|_\infty \leq \|A\|_2 \|B\|_2 = \|A \otimes B\|_1$. Moreover, if $(A_j)$ is an orthonormal basis in $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$, then

$$\| \sum_{j,k} \lambda_j \lambda_k^* \| \leq \sum_{j,k} |\lambda_j||\lambda_k^*| \|M_{A_k}^A\|_\infty \leq \sum_{j,k} |\lambda_j|^2 = \| \sum_{j,k} \lambda_j^* A_k \otimes A_j \|_1$$

which shows that $C$ is bounded (continuous) with the operator-norm $\leq 1$. One can easily see that this norm is actually 1. Let us see that the operators $M_B^A$ are compact. Indeed,

$$M_B^A = \sum_{l,m,\alpha,\beta} A^{\alpha \beta} \overline{B}^{m\beta} \cdot x_l \otimes \overline{x}_m \otimes y_\beta \otimes \overline{y}_\alpha$$

$$= \sum_{l+\alpha \leq N} \sum_{m,\beta} A^{\alpha \beta} \overline{B}^{m\beta} \cdot x_l \otimes \overline{x}_m \otimes y_\beta \otimes \overline{y}_\alpha + \sum_{l+\alpha > N} \sum_{m,\beta} A^{\alpha \beta} \overline{B}^{m\beta} \cdot x_l \otimes \overline{x}_m \otimes y_\beta \otimes \overline{y}_\alpha.$$

But the operator

$$\sum_{l+\alpha \leq N} \sum_{m,\beta} A^{\alpha \beta} \overline{B}^{m\beta} \cdot x_l \otimes \overline{x}_m \otimes y_\beta \otimes \overline{y}_\alpha$$
is finite-dimensional and
\[ R_N = \sum_{l+\alpha>N} \sum_{m,\beta} A^{l\alpha} B^{m\beta} \cdot x_l \otimes \overline{x_m} \otimes y_\beta \otimes \overline{y_\alpha} \]
has the operator-norm
\[ \| R_N \|_\infty \leq \left( \sum_{l+\alpha>N} |A^{l\alpha}|^2 \right)^{1/2} \| B \|_2 \]
which is arbitrary small for large \( N \), so \( M_B^A \) is an operator-norm limit of finite-dimensional operators. Further,
\[ \sum_{j,k} \lambda^j_k M^j_k A^j_k \]
is clearly compact for \( A^j_k, B^j_\beta \) of length 1 if \( \sum_{j,k} |\lambda^j_k| < \infty \).

The operator \( C \) we will call the Choi map.

The trace-class operators appear in Quantum Mechanics as quantum states. The convex set \( \mathcal{D}(\mathcal{H}) \) of quantum states consists of trace-class non-negative Hermitian operators with trace 1. It follows from the spectral theorem that each quantum state \( \rho \) can be written in a form \( \rho = \sum_i \lambda_i \rho_i \), where \( \{\rho_i\} \) is a sequence of one-dimensional orthogonal projectors, with \( \langle \rho_i, \rho_j \rangle = 0 \) for \( i \neq j \), and \( \lambda_i \geq 0 \), \( \sum_i \lambda_i = 1 \). In other words, \( \mathcal{D}(\mathcal{H}) \) is the smallest convex set in \( \mathcal{L}(\mathcal{H}) \) closed in the nuclear topology which contains all pure states — one-dimensional orthogonal projectors. The Choi map associates with quantum states \( \sum_j \lambda_j |A_j\rangle\langle A_j| \), with \( \| A_j \|_2 = 1 \), on \( \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1) \) a Kraus maps from \( \mathcal{L}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{H}_1)) \) with the operator sum representation \( \rho \mapsto \sum_j \lambda_j A_j \rho A_j^\dagger \).

If \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), a quantum state \( \rho \in \mathcal{D}(\mathcal{H}) \) we call separable if one can find a decomposition \( \rho = \sum_i \lambda_i \rho_i \) as above but with \( \rho_i \) being simple tensors, \( \rho_i = \rho_i' \otimes \rho_i'' \), where \( \rho_i' \) and \( \rho_i'' \) are one-dimensional orthogonal projectors in, respectively, \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). Composite quantum states we call entangled if they are not separable. Replacing \( \mathcal{H}_2 \) with \( \mathcal{H}_2^* \) we can speak, in an obvious sense, about separable and entangled quantum states on \( \mathcal{L}_2(\mathcal{H}_2^*, \mathcal{H}_1) \).

4 Basic features of \( \mathcal{J} \)

To proceed we shall need some further observations. First, let us see that maps from a linear subspace \( V \) of \( \mathcal{L}(\mathcal{H}_2) \), closed with respect to the Hermitian conjugation, into \( \mathcal{L}(\mathcal{H}_1) \), which preserve hermiticity, commute with the operation of taking the adjoint. Indeed, assume that \( \Phi : \mathcal{L}(\mathcal{H}_2) \supset V \rightarrow \mathcal{L}(\mathcal{H}_1) \) is a linear map that maps Hermitian operators into Hermitian ones, and define, for an arbitrary \( A \in V \),
\[ \Psi(A) = \Phi(A)^\dagger - \Phi(A^\dagger) \].
Clearly, $\Psi$ is additive, $\Psi(A + B) = \Psi(A) + \Psi(B)$, and antilinear, $\Psi(\alpha A) = \overline{\alpha} \Psi(A)$. Now,

$$\Psi(A + A^\dagger) = \Phi(A + A^\dagger)^\dagger - \Phi((A + A^\dagger)^\dagger) = \Phi(A + A^\dagger) - \Phi(A + A^\dagger) = 0,$$

where we use the hermiticity of $A + A^\dagger$ and the assumed property that $\Phi$ maps Hermitian operators into Hermitian ones. Since $A$ was arbitrary, we can take $iA$ instead of $A$, hence

$$0 = \Psi(iA + (iA)^\dagger) = \Psi(iA - iA^\dagger) = -i\Psi(A - A^\dagger),$$

which, upon additivity of $\Psi$ and together with (30) gives $\Psi(A) = 0$. Thus we get the following proposition.

**Proposition 3.** If $V \subset \mathcal{L}(\mathcal{H}_2)$ is a linear subspace, closed with respect to Hermitian conjugation, and $\Phi : V \to \mathcal{L}(\mathcal{H}_1)$ is a linear map that maps Hermitian operators into Hermitian ones, then $\Phi$ commutes with Hermitian conjugation, $\Phi(A^\dagger) = \Phi(A)^\dagger$.

**Theorem 3.** A Hilbert-Schmidt operator $\Phi : \mathcal{L}_2(\mathcal{H}_2) \to \mathcal{L}_2(\mathcal{H}_1)$ preserves hermiticity if and only if $\mathcal{J}(\Phi)$ is Hermitian.

**Proof.** According to the above proposition, preserving hermiticity means commuting with the Hermitian conjugation. Since, fixing orthonormal bases $(x_j)$ and $(y_a)$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $(y_a \otimes \overline{y}_b)^\dagger = \overline{y}_b \otimes y_a$, etc., $\Phi = \sum_{i,j,a,b} \lambda_{ijab} x_i \otimes \overline{x}_j \otimes y_a \otimes \overline{y}_b$ commutes with the Hermitian conjugation if and only if $\lambda_{ijab} = \overline{\lambda_{jiba}}$. On the other hand, $\mathcal{J}(\Phi)$ is Hermitian if and only if

$$\mathcal{J}(\Phi) = \sum_{i,j,a,b} \lambda_{ijab} \cdot x_i \otimes \overline{y}_b \otimes y_a \otimes \overline{x}_j = \left( \sum_{i,j,a,b} \lambda_{ijab} \cdot x_i \otimes \overline{y}_b \otimes y_a \otimes \overline{x}_j \right)^\dagger,$$

i.e., as above, if and only if $\lambda_{ijab} = \overline{\lambda_{jiba}}$.

We say that $\Phi$ as above preserves positivity (this property is usually called also positivity that might be confused with positivity of a Hermitian operator), if it maps non-negatively defined Hermitian operators on $\mathcal{H}_2$ (we will call them simply positive) into positive ones on $\mathcal{H}_1$. Using (20) we can prove now the following.

**Theorem 4.** A Hilbert-Schmidt operator $\Phi : \mathcal{L}_2(\mathcal{H}_2) \to \mathcal{L}_2(\mathcal{H}_1)$ preserves positivity if and only if $\mathcal{J}(\Phi)$ is a Hermitian operator on $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$ which is non-negatively defined on separable states, i.e. $\text{Tr}(\mathcal{J}(\Phi) \rho) \geq 0$ for separable states $\rho$ on $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$.

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Proof.- We have to prove that $\Phi$ preserves the positivity if and only if, for arbitrary $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$,

$$\langle x \otimes y, J(\Phi) (x \otimes y) \rangle \geq 0. \quad (31)$$

Indeed, assume that (31) holds. Then from (20)

$$\langle x \otimes x, \Phi (y \otimes y) \rangle \geq 0. \quad (32)$$

Hence, for each projection $x \otimes x$ its Hilbert-Schmidt scalar product with $\Phi$ evaluated on arbitrary $y \otimes y$ is positive, so $\Phi(y \otimes y)$ is positive for all $y$ and then positivity of $\Phi(A)$ for arbitrary positive-definite $A$ follows from the spectral decomposition of $A$.

On the other hand, if $\Phi$ preserves positivity, then evaluated on a positive operator $y \otimes y$ it gives a positive operator for which the Hilbert-Schmidt scalar product with an arbitrary projection $x \otimes x$ is non-negative, hence (32) and a fortiori (31) hold. □

A natural question now is: what Hilbert-Schmidt operators $\Phi : L_2(\mathcal{H}_2) \to L_2(\mathcal{H}_1)$ correspond, via the Jamiołkowski isomorphism, to Hermitian operators which are positive on the whole $L_2(\mathcal{H}_2, \mathcal{H}_1)$.

Definition 2. A Hilbert-Schmidt operator $\Phi : L_2(\mathcal{H}_2) \to L_2(\mathcal{H}_1)$ we call completely positive, if $J(\Phi)$ is Hermitian positive on $L_2(\mathcal{H}_2, \mathcal{H}_1)$.

We will show now that the above natural definition is equivalent to the standard concepts of complete positivity. Note however that we cannot consider tensor products with the identity on an infinite-dimensional Hilbert space, as the latter is not a Hilbert-Schmidt operator. Therefore, for an auxiliary Hilbert space $\mathcal{H}$ with an orthonormal basis $(u_i)$, consider the Hilbert-Schmidt operator $K_A$ on $L_2(\mathcal{H})$ associated with a diagonal Hilbert-Schmidt matrix $A = \sum_j \lambda_j \cdot u_j \otimes \overline{u}_j$, $\sum_j |\lambda_j|^2 < \infty$. In other words,

$$K_A = \sum_{i,j} \lambda_i \overline{\lambda}_j u_i \otimes \overline{u}_j \otimes u_j \otimes \overline{u}_i. \quad (33)$$

We know that $J(K_A)$ is Hermitian positive,

$$J(K_A) = A \otimes \overline{A} = \sum_{i,j} \lambda_i \overline{\lambda}_j u_i \otimes \overline{u}_i \otimes u_j \otimes \overline{u}_j. \quad (34)$$

For $\Phi \in L_2(L_2(\mathcal{H}_2), L_2(\mathcal{H}_1))$, we can consider $\Phi \otimes K_A \in L_2(L_2(\mathcal{H}_2'), L_2(\mathcal{H}_1'))$ with $\mathcal{H}_i' = \mathcal{H}_i \otimes \mathcal{H}$, $i = 1, 2$, with its Jamiołkowski image $J(\Phi \otimes K_A) \in L_2(L_2(\mathcal{H}_2', \mathcal{H}_1'))$. 

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Take now arbitrarily chosen \( x_1, \ldots, x_m \in \mathcal{H}_1 \), \( y_1, \ldots, y_m \in \mathcal{H}_2 \). We have
\[
\left\langle \left( \sum_{k=1}^{m} x_k \otimes u_k \right) \otimes \left( \sum_{k=1}^{m} x_k \otimes u_k \right), (\Phi \otimes K_A) \left( \sum_{k=1}^{m} y_k \otimes u_k \right) \otimes \left( \sum_{k=1}^{m} y_k \otimes u_k \right) \right\rangle
= \left\langle \sum_{k} x_k \otimes \lambda_k y_k, J(\Phi) \left( \sum_{p} x_p \otimes \lambda_p y_p \right) \right\rangle.
\] (35)

Indeed, for
\[
X = \sum_{k=1}^{m} x_k \otimes u_k \in \mathcal{H}_1 \otimes \mathcal{H}, \quad Y = \sum_{k=1}^{m} y_k \otimes u_k \in \mathcal{H}_2 \otimes \mathcal{H},
\]
one can write
\[
\left\langle X \otimes X, (\Phi \otimes \Lambda) (Y \otimes Y) \right\rangle = \left\langle X \otimes Y, J(\Phi) (X \otimes Y) \right\rangle
= \sum_{k,l,p,q} \left\langle x_k \otimes \lambda_l y_l, J(\Phi) x_p \otimes \lambda_q y_q \right\rangle \left\langle u_k \otimes \lambda_l u_l, J(\Lambda) (u_p \otimes \lambda_q u_q) \right\rangle
= \sum_{k,l,p,q} \left\langle x_k \otimes \lambda_l y_l, J(\Phi) x_p \otimes \lambda_q y_q \right\rangle \sum_{i,j} \lambda_i \lambda_j \delta_{p,i} \delta_{q,j} \delta_{k,i} \delta_{l,j}
= \sum_{k,p} \lambda_k \lambda_p \left\langle x_k \otimes \lambda_p y_p, J(\Phi) x_p \otimes \lambda_p y_p \right\rangle
= \left\langle \sum_{k} x_k \otimes \lambda_k y_k, J(\Phi) \left( \sum_{p} x_p \otimes \lambda_p y_p \right) \right\rangle.
\]

Note that any vector in \( \mathcal{H}_1 \otimes \mathcal{H}_2^* \), thus any map in \( L_2(\mathcal{H}_2, \mathcal{H}_1) \), can be approximated by vectors of the form \( Z = \sum_{p} x_p \otimes \lambda_p y_p \). Similarly, vectors from \( \mathcal{H}_1 \otimes \mathcal{H} \) and \( \mathcal{H}_2 \otimes \mathcal{H} \) can be approximated by vectors of the form \( X \) and \( Y \) as above. If \( \mathcal{H}_i \), \( i = 1, 2 \), are finite-dimensional, then we can actually get all these vectors taking the number of \( u_i \) not exceeding the maximum of these dimensions. Since, according to the formula (35), \( \left\langle X \otimes x, (\Phi \otimes \Lambda) (Y \otimes y) \right\rangle \geq 0 \) if and only if \( \left\langle Z, J(\Phi) (Z) \right\rangle \geq 0 \), we can derive the following characterization of complete positivity that can be viewed as an infinite-dimensional version of Choi Theorem [19], cf. also [20].
Theorem 5. Let $\Phi \in \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1))$. The following are equivalent:

(a) $\mathcal{J}(\Phi)$ is Hermitian positive;

(b) For any finite-dimensional Hilbert space $\mathcal{H}$ the operator $\Phi \otimes I \in \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2 \otimes \mathcal{H}), \mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}))$ preserves positivity;

(c) For an infinite-dimensional Hilbert space $\mathcal{H}$ and for a Hermitian positive $A \in \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1)$ with trivial kernel, the operator $\Phi \otimes K_A \in \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2 \otimes \mathcal{H}), \mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}))$ preserves positivity.

If $\mathcal{H}_i$, $i = 1, 2$, are finite-dimensional, then the dimensions of above auxiliary Hilbert spaces $\mathcal{H}$ can be restricted to the maximum of the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Of course, all the above has the corresponding counterpart for the other Jamiołkowski isomorphism $\tilde{\mathcal{J}}$. This version fits sometimes better to the language of bi-partite systems.

5 Schmidt rank and Schmidt measure

We know already that any element $v \in \mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition $\varphi = \sum_j \lambda_j \cdot a_j \otimes b_j$ with $(a_j)$ and $(b_j)$ being (not necessarily complete) orthonormal sets, and $\lambda_j$ being positive. The Hilbert-Schmidt norm can be equivalently defined as $\|A\|_2^2 = \sum_j |\lambda_j|^2$. The number of summands in this decomposition (which can be infinite if both Hilbert spaces are infinite-dimensional) we call the Schmidt rank $S(\varphi)$ of $\varphi$. Directly by definition, a pure state $p_\varphi = |\varphi\rangle\langle\varphi|$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable if and only if the Schmidt rank of $\varphi$ is 1.

Proposition 4. The Schmidt rank of $\varphi$ is $m = 1, 2, \ldots, \infty$ if and only if $p_\varphi = \tilde{\mathcal{J}}(\Phi)$ for an operator $\Phi : \mathcal{L}_2(\mathcal{H}_2) \to \mathcal{L}_2(\mathcal{H}_1)$ of rank $m^2$.

Proof.- Since $\Phi = \sum_{j,k} \lambda_j \lambda_k \cdot a_j \otimes \overline{a_k} \otimes b_j \otimes \overline{b_k}$ and $\lambda_j \lambda_k > 0$ for $j, k = 1, \ldots, m$, the image of $\Phi$ is spanned by $a_j \otimes \overline{a_k}$, $j, k = 1, \ldots, m$, thus its rank is $m^2$.

This suggests the following extension of the concept of Schmidt rank.

Definition 3. The Schmidt rank of $\rho \in \mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is the operator-rank of $\tilde{\mathcal{J}}^{-1}(\rho)$.

In these terms we can state the following corollary, where we admit infinite-dimensional Hilbert spaces.
Corollary 1. A pure state $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable if and only if the Schmidt rank of $\rho$ is 1.

This easy characterization of separable pure states has been used by Terhal and Horodecki [21] to develop the concept of Schmidt number of an arbitrary density matrix $\rho$ (quantum state in finite dimensions). This number characterizes the minimum Schmidt rank of the pure states that are needed to construct such density matrix. The Schmidt number is non-increasing under local operations and classical communications, i.e. it provides a legitimate entanglement measure. We can construct an entanglement measure – Schmidt measure $\mu_S$ – which is additionally convex using the convex roof construction (see e.g. [22]). This construction, proposed as a general tool for entanglement measures (see e.g. [23, 6, 7]), can be repeated in infinite dimensions as

$$\mu_S(\rho) = \inf \left\{ \sum_j \lambda_j S(\varphi_j) \right\}, \tag{36}$$

where the infimum is taken over all possible realizations of $\rho$ as infinite-convex combinations $\rho = \sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j|$ with $0 \leq \lambda_j \leq 1$, $\sum_j \lambda_j = 1$ and $\varphi_j \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Every quantum state admits such a realization and a reasoning analogous to the one in [6] shows that $\mu_S$ is infinite-convex, non-negative and vanishes exactly on separable states.

6 Multipartite generalizations

The diagram of the Jamiołkowski isomorphisms

\[
\begin{array}{c}
\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1^* \\
\downarrow \mathcal{J} \\
\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_1^* \\
\downarrow \mathcal{J} \\
\mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1), \mathcal{L}_2(\mathcal{H}_2)) \\
\downarrow \mathcal{J} \\
\mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2)
\end{array}
\] \tag{37}

interpreted also as

\[
\begin{array}{c}
\mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1), \mathcal{L}_2(\mathcal{H}_2)) \\
\downarrow \mathcal{J} \\
\mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2)
\end{array}
\] \tag{38}
can be easily generalized to multipartite cases, where we replace $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Of course, the number of possible permutations grows quickly with $n$. Part of them can be obtained by a trivial induction. For instance, we can replace $\mathcal{H}_2$ with $\mathcal{H}_2 \otimes \mathcal{H}_3$ (or $\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_3)$) in (38), but we can also get

$$\mathcal{L}_2(\mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2), \mathcal{H}_3), \mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_3)) \longleftrightarrow \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_3), \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_3))$$

or

$$\mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_3 \otimes \mathcal{H}_4)) \longleftrightarrow \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_1 \otimes \mathcal{H}_3), \mathcal{L}_2(\mathcal{H}_2 \otimes \mathcal{H}_4))$$

etc. We will not study here these isomorphisms in details, as the choice of a particular one depends on our interests in possible applications.

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References

[1] E. Schrödinger, Naturwissenschaften 23, 807 (1935).

[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).

[3] J. Bell, Physics 1, 195 (1964).

[4] J. Bell, Rev. Mod. Phys. 38, 447 (1966).

[5] D. Bouwmeester, A. Ekert, and A. Zeilinger, The Physics of Quantum Information (Springer, Berlin, 2000).

[6] J. Grabowski, M. Kuś, and G. Marmo, J. Phys. A: Math. Gen. 38, 10217 (2005).
[7] J. Grabowski, M. Kuś, and G. Marmo, Open Sys. Information Dyn. 13, 343 (2006).
[8] J. F. Carinena, J. Clemente-Gallardo, and G. Marmo, arXiv:math-ph/0701053.
[9] I. E. Segal, Ann. Math. 48, 930 (1947).
[10] R. Haag, Local Quantum Physics (Springer, Berlin, 1992).
[11] A. Gleason, J. Math. Mech. 6, 885 (1957).
[12] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. 121, 920 (1961).
[13] K. Życzkowski and I. Bengtsson, Open Sys. Information Dynamics 11, 3 (2004).
[14] M. Asorey, A. Kossakowski, G. Marmo, and G. Sudarshan, Open Sys. Information Dyn. 12, 319 (2005).
[15] A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972).
[16] A. Jamiołkowski, Rep. Math. Phys. 5, 415 (1975).
[17] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis (Academic Press, New York, 1972).
[18] K. Maurin, Methods of Hilbert spaces, Mathematical Monographs, vol. 45, second revised ed. (PWN–Polish Scientific Publishers, Warsaw, 1972).
[19] M. D. Choi, Linear Alg. Appl. 10, 285 (1975).
[20] D. Salgado, J. L. Sánchez-Gómez, and M. Ferrero, Open Sys. Information Dyn. 12, 55 (2005).
[21] B. M. Terhal and P. Horodecki, Phys. Rev. A 61, 040301 (2000).
[22] J. Eisert and H. J. Briegel, Phys. Rev. A 64, 022306 (2001).
[23] A. Uhlmann, Phys. Rev. A 62, 032307 (2000).