ON GENERALIZATIONS OF BAER'S THEOREMS ABOUT THE HYPERCENTER OF A FINITE GROUP

V. I. Murashka

{mvimath@yandex.net}

Francisk Skorina Gomel State University, Gomel

Abstract. We investigate the intersection of normalizers and \( F \)-subnormalizers of different types of systems of subgroups (\( F \)-maximal, Sylow, cyclic primary). We described all formations \( F = \prod_{i \in I} F_{\pi_i} \) for which the intersection of normalizers of all \( F_{\pi_i} \)-maximal subgroups of \( G \) is the \( F \)-hypercenter of \( G \) for every group \( G \). Also we described all formations \( F \) for which the intersection of \( F \)-subnormalizers of all Sylow (cyclic primary) subgroups of \( G \) is the \( F \)-hypercenter of \( G \) for every group \( G \).

Keywords: saturated formation, hereditary formation, \( F \)-hypercenter, \( F \)-subnormalizer, intersection of subgroups.

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1 Introduction

All considered groups are finite. In [1] R. Baer showed that from one hand the hypercenter \( Z_{\infty}(G) \) of a group \( G \) coincides with the intersection of all maximal nilpotent subgroups of \( G \) and from another hand \( Z_{\infty}(G) \) coincides with the intersection of normalizers of all Sylow subgroups of \( G \).

The concept of hypercenter was extended on classes of groups (see [2, p. 127–128] or [3, p. 6–8]). Let \( \mathcal{X} \) be a class of groups. A chief factor \( H/K \) of a group \( G \) is called \( \mathcal{X} \)-central if \( (H/K) \leq G/C_G(H/K) \in \mathcal{X} \). A normal subgroup \( N \) of \( G \) is said to be \( \mathcal{X} \)-hypercentral in \( G \) if \( N = 1 \) or \( N \neq 1 \) and every chief factor of \( G \) below \( N \) is \( \mathcal{X} \)-central. The \( \mathcal{X} \)-hypercenter \( Z_{\mathcal{X}}(G) \) is the product of all normal \( \mathcal{X} \)-hypercentral subgroups of \( G \). So if \( \mathcal{X} = \mathcal{N} \) is the class of all nilpotent groups then \( Z_{\infty}(G) = Z_{\mathcal{N}}(G) \) for every group \( G \).

In [4] A. V. Sidorov showed that for a soluble group \( G \) the intersection of all maximal subgroups of nilpotent length at most \( r \) is \( Z_{\mathcal{N}_r}(G) \). Beidleman and Heineken [5] studied the properties of the intersection \( \text{Int}_{\mathcal{F}}(G) \) of \( \mathcal{F} \)-maximal subgroups of a group \( G \) in case when \( G \) is soluble and \( \mathcal{F} \) is a hereditary saturated formation.

Let \( F \) be the canonical local definition of a local formation \( \mathcal{F} \). Then \( \mathcal{F} \) is said to satisfy the boundary condition [6] if \( \mathcal{F} \) contains every group \( G \) whose all maximal subgroups belong to \( F(p) \) for some prime \( p \).

A. N. Skiba [6] showed that the equality \( \text{Int}_{\mathcal{F}}(G) = Z_{\mathcal{F}}(G) \) holds for every group \( G \) if and only if a hereditary saturated formations \( \mathcal{F} \) satisfies the boundary condition. This and further results was included in the first chapter of [3].

The intersection of normalizers of different systems of subgroups is the main theme of many papers. In [7] Baer considered the intersection of normalizers of all subgroups of a group. Wielandt [8] studied the intersection of normalizers of all subnormal subgroups of a group. Li and Shen [9] considered the intersection of normalizers of all derived subgroups of all subgroups of a group.

Let \( \sigma = \{ \pi_i | i \in I \} \) be a partition of \( \mathbb{P} \) into disjoint subsets, \( \mathcal{X}_i \) be a class of groups such that \( \pi(\mathcal{X}_i) = \pi_i \). Then \( \times_{i \in I} \mathcal{X}_{\pi_i} = (G = \times_{i \in I} O_{\pi_i}(G)|O_{\pi_i}(G) \in \mathcal{X}_i) \). Recall that \( \mathcal{G}_\pi \) is the class of all \( \pi \)-groups. Hence \( \mathfrak{H} = \times_{p \in \mathbb{P}} \mathcal{G}_p \).
In [10] author showed that if $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_\pi$, then for any group $G$ the intersection of all normalizers of all $\pi_i$-maximal subgroups of $G$ for all $i \in I$ coincides with the $\mathfrak{F}$-hypercenter. So the general problem is

**Problem A.** Let $\Sigma$ be a subgroup functor. What can be said about the intersection of normalizers of subgroups from $\Sigma(G)$?

Recall [11, p. 206] that a subgroup functor is a function $\tau$ which assigns to each group $G$ a possibly empty set $\tau(G)$ of subgroups of $G$ satisfying $f(\tau(G)) = \tau(f(G))$ for any isomorphism $f : G \to G^*$.

**Definition 1.** Let $X$ be a class of groups and $G$ be a group. Then $\text{NI}_X(G)$ is the intersection of all normalizers of $\mathfrak{X}$-maximal subgroups of $G$.

The following proposition shows that if $\mathfrak{F}$ is a hereditary saturated formation and $\pi(\mathfrak{F}) = \mathbb{P}$ then the equality $\text{NI}_\mathfrak{F}(G) = \text{Int}_\mathfrak{F}(G)$ holds for every group $G$.

**Proposition 1.** Let $\mathfrak{F}$ be a hereditary saturated formation and $\pi = \pi(\mathfrak{F})$. Then $\text{O}^\pi(\text{NI}_\mathfrak{F}(G)) = \text{Int}_\mathfrak{F}(G)$ for every group $G$.

The following theorem generalizes two above mentioned Baer’s theorems about the hypercenter:

**Theorem A.** Let $\sigma = \{\pi_i | i \in I\}$ be a partition of $\mathbb{P}$ into disjoint subsets and $\mathfrak{F}_i$ be a hereditary saturated formation such that $\pi(\mathfrak{F}_i) = \pi_i$ and $\mathfrak{F} = \times_{i \in I} \mathfrak{F}_i$. The following statements are equivalent:

1. $\mathfrak{F}_i$ satisfies the boundary condition in the universe of all $\pi_i$-groups for all $i \in I$.
2. For every group $G$ holds $\bigcap_{i \in I} \text{NI}_{\mathfrak{F}_i}(G) = Z_{\mathfrak{F}}(G)$.

**Corollary A.1** [11]. The hypercenter of a group $G$ is the intersection of all normalizers of all Sylow subgroups of $G$.

**Corollary A.2** [10]. Let $\sigma = \{\pi_i | i \in I\}$ be a partition of $\mathbb{P}$ into disjoint subsets, $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_\pi_i$ and $G$ be a group. Then the intersection of all normalizers of all $\pi_i$-maximal subgroups of $G$ for all $i \in I$ is the $\mathfrak{F}$-hypercenter of $G$.

From proposition 1 and theorem A when $|I| = 1$ it follows that our theorem A extends theorem A from [9]:

**Corollary A.3.** Let $\mathfrak{F}$ be a hereditary saturated formation and $\pi(\mathfrak{F}) = \mathbb{P}$. The equality $\text{NI}_\mathfrak{F}(G) = Z_{\mathfrak{F}}(G) = \text{Int}_\mathfrak{F}(G)$ holds for every group $G$ if and only if $\mathfrak{F}$ satisfies the boundary condition.

**Corollary A.4** [11]. The hypercenter of a group $G$ is the intersection of all maximal nilpotent subgroups of $G$.

Let $\mathfrak{X}$ be a class of groups. Recall that a subgroup $H$ of a group $G$ is called $\mathfrak{F}$-subnormal if either $H = G$ or there is a maximal chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{X}$ for all $i = 1, \ldots, n$.

Let $\mathfrak{X}$ be a class of groups. A $\mathfrak{X}$-subnormalizer [12, p. 380] of a subgroup $H$ of a group $G$ is a subgroup $T$ of $G$ such that $H$ is $\mathfrak{X}$-subnormal in $T$ and if $H$ is $\mathfrak{X}$-subnormal in $M$ and $T \leq M$ then $T = M$. It is clear that a $\mathfrak{X}$-subnormalizer always exists but may be not unique.

**Problem B.** Let $\Sigma(G)$ be a subgroup functor and $\mathfrak{F}$ be a formation. What can be said about the intersection $\text{SI}_{\mathfrak{F}}(G)$ of $\mathfrak{F}$-subnormalizers of subgroups from $\Sigma(G)$?

If $\Sigma(G)$ is the set of all maximal subgroups of $G$ then this intersection coincides with $\Delta_{\mathfrak{F}}(G)$ where $\Delta_{\mathfrak{F}}(G)$ is the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of $G$. According to [13] p. 96 if $\mathfrak{F}$ is a hereditary saturated formation then $\Delta_{\mathfrak{F}}(G)/\Phi(G) = Z_{\mathfrak{F}}(G)/\Phi(G)$.

**Proposition 2.** Let $\mathfrak{F}$ be a hereditary formation and $\Sigma$ be a subgroup functor. Then $\text{SI}_{\mathfrak{F}}(G)$ is the product of normal subgroups $N$ of a group $G$ such that $H$ is $\mathfrak{F}$-subnormal in $HN$ for every $H \in \Sigma(G)$.

A.F. Vasil’ev and T.I. Vasil’eva [14] studied a class of groups $\mathfrak{u}\mathfrak{F}$ whose all Sylow subgroups are $\mathfrak{F}$-subnormal for a given hereditary saturated formation $\mathfrak{F}$. Let us note that in this case
$Z_{v\mathfrak{F}}(G)$ lies in the intersection of all $\mathfrak{F}$-subnormalizers of all Sylow subgroups of a group $G$. Author [15] studied a class of groups $v\mathfrak{F}$ whose all cyclic primary subgroups are $\mathfrak{F}$-subnormal for a given hereditary saturated formation $\mathfrak{F}$. Again $Z_{v\mathfrak{F}}(G)$ lies in the intersection of all $\mathfrak{F}$-subnormalizers of all cyclic primary subgroup of a group $G$.

In this paper we count the unit group as cyclic primary subgroup and also as Sylow subgroup.

**Theorem B.** Let $\mathfrak{F}$ be a hereditary saturated formation. The following statements are equivalent:

1. There exists a partition $\sigma = \{\pi_i|i \in I\}$ of $\pi(\mathfrak{F})$ into disjoint subsets such that $\mathfrak{F} = \times_{i \in I} \mathfrak{S}_{\pi_i}$.

2. The intersection of all $\mathfrak{F}$-subnormalizers of all cyclic primary subgroups of $G$ is the $\mathfrak{F}$-hypercenter of $G$ for every group $G$.

3. The intersection of all $\mathfrak{F}$-subnormalizers of all Sylow subgroups of $G$ is the $\mathfrak{F}$-hypercenter of $G$ for every group $G$.

Note that in the universe of all soluble groups the concepts of a subnormal subgroup and a $\mathfrak{N}$-subnormal subgroup coincides. It is well known that if a Sylow subgroup $P$ of $G$ is subnormal in $G$ then it is normal in $G$. Hence a $\mathfrak{N}$-subnormalizer of a Sylow subgroup $P$ of a soluble group $G$ is just the normalizer of $P$ in $G$. So theorem B can be viewed as the generalization of R. Baer’s theorem about the intersection of normalizers of Sylow subgroups.

**Remark.** Formations $\mathfrak{F} = \times_{i \in I} \mathfrak{S}_{\pi_i}$ are lattice formations, i.e. formations were $\mathfrak{F}$-subnormal subgroups form a sublattice of the subgroup’s lattice of every group (for example see chapter 6.3 of [11]). Also properties of the $\mathfrak{F}$-hypercenter and the $\mathfrak{F}$-residual for such formations was studied by author in [10]. A. N. Skiba extends the theory of nilpotent groups on such classes (for example see [16]).

2 Preliminaries

We use standard notation and terminology that if necessary can be found in [12]. Recall some of them that are important in this paper. By $\mathbb{P}$ is denoted the set of all primes; $\pi(G)$ is the set of all prime divisors of the order of $G$; $\pi(\mathfrak{F}) = \bigcup_{G \in \mathfrak{F}} \pi(G)$; a group $G$ is called $\pi$-group if $\pi(G) \subseteq \pi$; $Z_p$ is the cyclic group of order $p$; $O_\pi(G)$ is the greatest normal $\pi$-subgroup $G$; $O^\pi(G)$ is the smallest subgroup of $G$ such that $\pi(G/O^\pi(G)) \subseteq \pi$; $G^\pi$ is the derived subgroup of $G$; $G^\mathfrak{F}$ is the $\mathfrak{F}$-residual for a formation $\mathfrak{F}$; $O_{p',p}^{G}(G)$ is the $p$-nilpotent radical of $G$ for $p \in \mathbb{P}$, it also can be defined by $O_{p',p}^{G}(G) = O_{p}(G)/O_{p'}(G)$; $\Phi(G)$ is the Frattini subgroup of a group $G$; $G = N \rtimes M$ is the semidirect product of groups $M$ and $N$ ($N \triangleleft G$ and $N \cap M = 1$); $\mathfrak{S}_\pi$ is the class of (soluble, nilpotent) $\pi$-groups, where $\pi \subseteq \mathbb{P}$.

A class of groups $\mathfrak{X}$ is called hereditary if from $G \in \mathfrak{X}$ and $H \leq G$ it follows that $H \in \mathfrak{X}$. A class of groups $\mathfrak{X}$ is called saturated if from $G/\Phi(G) \in \mathfrak{X}$ it follows that $G \in \mathfrak{X}$.

By well known Gashutz-Lubeseder-Shmid Theorem saturated formations are exactly local formations, i.e. formations $\mathfrak{F} = LF(f)$ defined by a formation function $f$: $LF(f) = \{G \in \mathfrak{S} | H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K) \text{ then } G/C_G(H/K) \in f(p)\}$. Among all possible local definitions of a local formation $\mathfrak{F}$ there is exactly one, denoted by $F$, such that $F$ is integrated $(F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P})$ and full $(\mathfrak{N}_p, F(p) = F(p)$ for all $p \in \mathbb{P})$. $F$ is called the canonical local definition of $\mathfrak{F}$.

Let $\mathfrak{F}$ be a local formation, $F$ be its canonical local definition and $G$ be a group. Then a chief factor $H/K$ of a group $G$ is $\mathfrak{F}$-central if and only if $G/C_G(H/K) \in F(p)$ for all $p \in \pi(H/K)$ (see [3, p. 6]).
Let $\mathfrak{F} = LF(F)$ be a hereditary local formation, $F$ be its canonical local definition and $\pi = \pi(\mathfrak{F})$. Then $\mathfrak{F}$ is said to satisfy the boundary condition in the universe of all $\pi$-groups if $\mathfrak{F}$ contains every $\pi$-group whose all maximal subgroups belong to $F(p)$ for some prime $p$.

The following lemma can be found in [13] p. 239). For reader’s convenience, we give a direct proof.

**Lemma 2.1.** Let $\mathfrak{X}$ be a saturated homomorph and $N$ be a normal subgroup of a group $G$. Then for every $\mathfrak{X}$-subgroup $H/N$ of $G/N$ there exists a $\mathfrak{X}$-subgroup $M$ of $G$ such that $MN/N = H/N$.

**Proof.** Let $H/N$ be a $\mathfrak{X}$-subgroup of $G/N$. Let us show that there exists a $\mathfrak{X}$-subgroup $K$ of $G$ such that $KN/N = H/N$. Let $M$ be a minimal subgroup of $H$ such that $MN = H$ (i.e.

if $M_1 < M$ then $M_1N < H$). Assume that there is a maximal subgroup $M_1$ of $M$ such that $M_1(M \cap N) = M$. Then $M_1N = H$, a contradiction. Hence $M \cap N \leq \Phi(M)$. Since $\mathfrak{X}$ is saturated and $H/N = MN/N \simeq M/M \cap N \in \mathfrak{X}$, we see that $M \in \mathfrak{X}$. It means that there is a $\mathfrak{X}$-subgroup $M$ of $G$ such that $H/N = MN/N$. □

**Lemma 2.2.** Let $\mathfrak{F}$ be a hereditary saturated formation, $N$ be a normal subgroup of a group $G$, $H$ be a subgroup of $G$ then

1. $\mathfrak{N}_\mathfrak{F}(G)/N \leq \mathfrak{N}_\mathfrak{F}(G/N)$.
2. $\mathfrak{N}_\mathfrak{F}(G) \cap H \leq \mathfrak{N}_\mathfrak{F}(H)$.
3. Let $N \leq \text{Int}_\mathfrak{F}(G)$ then $N \leq \mathfrak{N}_\mathfrak{F}(G)$ and $\mathfrak{N}_\mathfrak{F}(G)/N = \mathfrak{N}_\mathfrak{F}(G/N)$.

**Proof.** (1) If $K/N$ is a $\mathfrak{F}$-maximal subgroup of $G/N$ then by lemma 2.1 there exists a $\mathfrak{F}$-maximal subgroup $Q$ of $G$ such that $QN/N = K/N$. If $x \in N_G(Q)$ then $xN \in N_{G/N}(QN/N) = N_{G/N}(K/N)$. Thus $\mathfrak{N}_\mathfrak{F}(G)/N \leq \mathfrak{N}_\mathfrak{F}(G/N)$.

(2) If $M$ is a $\mathfrak{F}$-maximal subgroup of $H$ then there exists a $\mathfrak{F}$-maximal subgroup $Q$ of $G$ such that $Q \cap H = M$. So if $x \in \mathfrak{N}_\mathfrak{F}(G) \cap H$ then $Q^x \cap H^x = Q \cap H = M$. Hence $x \in \mathfrak{N}_\mathfrak{F}(H)$. Thus $\mathfrak{N}_\mathfrak{F}(G) \cap H \leq \mathfrak{N}_\mathfrak{F}(H)$.

(3) Let $N \leq \text{Int}_\mathfrak{F}(G)$. It is clear that $N \leq \mathfrak{N}_\mathfrak{F}(G)$. Note that $M$ is a $\mathfrak{F}$-maximal subgroup of $G$ if and only if $M/N$ is a $\mathfrak{F}$-maximal subgroup of $G/N$. Now $N_G(M)/N = N_{G/N}(M/N)$. Thus $\mathfrak{N}_\mathfrak{F}(G)/N = \mathfrak{N}_\mathfrak{F}(G/N)$. □

Let $\mathfrak{F}$ be a saturated formation. Then in every group exists a $\mathfrak{F}$-projector [12, p. 292]. Recall that a $\mathfrak{F}$-projector of a group $G$ is a $\mathfrak{F}$-maximal subgroup $H$ of $G$ such that $HN/N$ is a $\mathfrak{F}$-maximal subgroup of $G/N$ for every normal subgroup $N$ of $G$.

Recall that a group $G$ is called semisimple if $G$ is the direct product of simple groups. A chief factor of a group is the example of a semisimple group.

**Lemma 2.3.** Let $\mathfrak{F}$ be a hereditarily saturated formation and a group $G = HK$ be a product of normal $\mathfrak{F}$-subgroups. If $K$ is semisimple then $G \in \mathfrak{F}$.

**Proof.** Assume the contrary. Let a group $G$ be a counterexample of a minimal order. Then $G = HK$ is a product of normal $\mathfrak{F}$-subgroups $H$ and $K$ where $K$ is semisimple. Let $N$ be a normal subgroup of $G$. Then $G/N = (HN/N)(KN/N)$ where $HN/N$ and $KN/N$ are normal $\mathfrak{F}$-subgroups of $G/N$ and $KN/N$ is semisimple. So $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, we see that $\Phi(G) = 1$ and $G$ has an unique minimal normal subgroup that equals $K$. Now $K \leq H$. So $G = H \in \mathfrak{F}$, the contradiction. □

The following lemma is well known.

**Lemma 2.4.** Let $\mathfrak{F}$ be a hereditarily saturated formation and $H$ be a $\mathfrak{F}$-subgroup of a group $G$. Then $Z_{\mathfrak{F}}(G)H \in \mathfrak{F}$.

Recall that if $\mathfrak{F}$ is a hereditary formation then a subgroup $H$ of a group $G$ is called $\mathfrak{F}$-subnormal if either $H = G$ or there is a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, \ldots, n$. We will need the following facts about $\mathfrak{F}$-subnormal subgroups.

**Lemma 2.5** [11, p. 236]. Let $\mathfrak{F}$ be a hereditary formation, $N$ be a normal subgroup of a group $G$ and $H$, $K$ be subgroups of $G$. Then:
(1) If \( H \) is \( \mathfrak{F} \)-subnormal in \( G \) then \( HN/N \) is \( \mathfrak{F} \)-subnormal in \( G/N \).
(2) If \( H/N \) is \( \mathfrak{F} \)-subnormal in \( G/N \) then \( H \) is \( \mathfrak{F} \)-subnormal in \( G \).
(3) If \( H \) is \( \mathfrak{F} \)-subnormal in \( K \) and \( K \) is \( \mathfrak{F} \)-subnormal in \( G \) then \( H \) is \( \mathfrak{F} \)-subnormal in \( G \).

**Lemma 2.6** [11, p. 239]. Let \( \mathfrak{F} \) be a saturated formation and a group \( G = HF^*(G) \) where \( H \) is a \( \mathfrak{F} \)-subnormal \( \mathfrak{F} \)-subgroup of \( G \). Then \( G \in \mathfrak{F} \).

**Lemma 2.7** [12, p. 390]. Let \( \mathfrak{F} \) be a hereditary saturated formation then \([G^\mathfrak{F}, Z_{\mathfrak{F}}(G)] = 1 \) for any group \( G \).

3 Proves of the main results

3.1 Proof of proposition 1

Let \( \mathfrak{F} \) be a hereditary saturated formation and \( G \) be a group. According to lemma 2.2 all \( \mathfrak{F} \)-maximal subgroups of \( NI_{\mathfrak{F}}(G) \) are normal in \( NI_{\mathfrak{F}}(G) \). Among this \( \mathfrak{F} \)-maximal subgroups there is a \( \mathfrak{F} \)-projector \( H \). Now \( NI_{\mathfrak{F}}(G)/H \) does not contain any \( \mathfrak{F} \)-subgroup. Hence \( NI_{\mathfrak{F}}(G)/H \in \pi(\mathfrak{F})' \).

It is clear that \( \text{Int}_{\mathfrak{F}}(G) \leq O^\pi(NI_{\mathfrak{F}}(G)) \in \mathfrak{F} \). Let us show by induction the the equality \( \text{Int}_{\mathfrak{F}}(G) = O^\pi(NI_{\mathfrak{F}}(G)) \) holds. It is clear that it holds for the unit group. Assume that we prove our statement for groups whose order is less then the order of a group \( G \). Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \leq O^\pi(NI_{\mathfrak{F}}(G)) \) and \( M \) be a \( \mathfrak{F} \)-maximal subgroup of \( G \). So \( M < MN \) and \( N \) is a normal semisimple \( \mathfrak{F} \)-subgroup of \( MN \). By lemma 2.3 \( MN \in \mathfrak{F} \) and hence \( MN = M \) for all \( \mathfrak{F} \)-maximal subgroups \( M \) of \( G \). Hence \( N \leq \text{Int}_{\mathfrak{F}}(G) \).

By induction \( \text{Int}_{\mathfrak{F}}(G/N) = O^\pi(NI_{\mathfrak{F}}(G/N)) \). According to [15] \( \text{Int}_{\mathfrak{F}}(G)/N = \text{Int}_{\mathfrak{F}}(G/N) \). By (3) of lemma 2.2 \( NI_{\mathfrak{F}}(G)/N = NI_{\mathfrak{F}}(G/N) \). From \( \pi(N) \subseteq \pi(\mathfrak{F}) \) it follows that \( O^\pi(NI_{\mathfrak{F}}(G)/N) = O^\pi(NI_{\mathfrak{F}}(G)/N) \). Now \( \text{Int}_{\mathfrak{F}}(G)/N = \text{Int}_{\mathfrak{F}}(G/N) = O^\pi(NI_{\mathfrak{F}}(G/N)) = O^\pi(NI_{\mathfrak{F}}(G)/N) \).

Thus \( \text{Int}_{\mathfrak{F}}(G) = O^\pi(NI_{\mathfrak{F}}(G)) \). □

3.2 Proof of theorem A

The following result directly follows from the proof of the main result of [6].

Let \( \mathfrak{F} \) be a hereditary saturated formation and \( \pi = \pi(\mathfrak{F}) \). Then for every \( \pi \)-group \( G \) the intersection of all \( \mathfrak{F} \)-maximal subgroups of \( G \) is the \( \mathfrak{F} \)-hypercenter of \( G \) if and only if \( \mathfrak{F} \) satisfies the boundary condition in the universe of all \( \pi \)-groups.

According to [11, p. 96] \( \mathfrak{F} \) is a hereditary saturated formation. So \( \mathfrak{F} \) is a local formation. Let \( F \) be the canonical local definition of \( \mathfrak{F} \).

(1) \( \Rightarrow \) (2) Assume that \( \mathfrak{F}_i \) satisfies the boundary condition in the universe of all \( \pi_i \)-groups for all \( i \in I \). Let us show that \( \bigcap_{i \in I} NI_{\mathfrak{F}_i}(G) = Z_{\mathfrak{F}_i}(G) \) holds for every group \( G \).

Let \( G \) be a group and \( D = \bigcap_{i \in I} NI_{\mathfrak{F}_i}(G) \). By proposition 1 \( NI_{\mathfrak{F}_i}(G) \) has the Hall \( \pi_i \)-subgroup that belongs to \( \mathfrak{F}_i \), and is normal in \( G \). Hence \( D \) has the normal Hall \( \pi_i \)-subgroup that belongs to \( \mathfrak{F}_i \) for every \( i \in I \). Thus \( D \in \mathfrak{F} \).

Let \( H/K \) be a chief factor of \( G \) below \( D \). Then \( \pi(H/K) \subseteq \pi_n \) for some \( n \in I \).

(a) \( O^\pi_n(G/K) \leq C_G(H/K) \).

By (1) of lemma 2.2 \( H/K \) normalizes all \( \mathfrak{F}_i \)-maximal subgroups of \( G/K \). Hence \( H/K \) normalizes all \( \mathfrak{F}_i \)-projectors of \( G/K \) for all \( i \in I \setminus \{n\} \). Let \( F/K \) be a \( \mathfrak{F}_n \)-projector of \( G/K \) for some \( i \in I \setminus \{n\} \). From \( \pi_n \cap \pi_i = \emptyset \) it follows that \( F/K \cap H/K = K/K \). Let \( hK \in H/K \) and \( fK \in F/K \). Then from one hand \( [fK, hK] = (fK)^{-1}(fK)(hK) \) and from another hand \( [fK, hK] = (hK)^{-1}(fK)(hK) \) in \( H/K \). So \( [fK, hK] = 1 \). Hence \( [H/K, F/K] = 1 \). Thus \( H/K \) centralizes all \( \mathfrak{F}_i \)-projectors of \( G/K \) for all \( i \in I \setminus \{n\} \). Since \( \mathfrak{F}_i \) is a hereditary saturated
formation for all \(i \in I\), we see that \(G/C_G(H/K)\) does not contain any \(\pi_i\)-subgroups for all \(i \in I \setminus \{n\}\). Thus \(O^{\pi_n}(G/K) \leq C_G(H/K)\).

(b) \(H/K \leq Z_{\tilde{\mathfrak{S}}_n}(R/K)\) for every \(\mathfrak{S}_{\pi_n}\)-maximal subgroup \(R/K\) of \(G/K\).

Let \(R/K\) be a \(\mathfrak{S}_{\pi_n}\)-maximal subgroup of \(G/K\). Then \(\pi_n((R/K)(H/K)) \leq \pi_n\). Hence \((R/K)(H/K) = R/K\). So \(H/K \leq R/K\). By (2) of lemma 2.2 \(H/K\) normalizes all \(\mathfrak{S}_{\pi_n}\)-maximal subgroups of \(R/K\). Note that \(H/K\) is semisimple \(\mathfrak{S}_{\pi_n}\)-subgroup. So \((H/K)(F/K) \in \mathfrak{S}_{\pi_n}\) for every \(\mathfrak{S}_{\pi_n}\)-maximal subgroup \(F/K\) of \(R/K\) by lemma 2.3. Hence \((H/K)(F/K) = F/K\) for every \(\mathfrak{S}_{\pi_n}\)-maximal subgroup \(F/K\) of \(R/K\). Thus \(H/K \leq \text{Int}_{\mathfrak{S}_n}(R/K)\). Since \(\mathfrak{S}_n\) satisfies the boundary condition in the universe of all \(\pi_n\)-groups, \(H/K \leq Z_{\mathfrak{S}_n}(R/K)\).

(c) Let \(R/K\) be a \(\mathfrak{S}_{\pi_n}\)-maximal subgroup of \(G/K\) such that \((R/K)O^{\pi_n}(G/K) = G/K\). Then \(H/K\) is a chief factor of \(R/K\).

Assume that \(N/K\) is a minimal normal subgroup of \(R/K\) such that \(K/K \neq N/K < H/K\). From \(O^{\pi_n}(G/K) \leq C_G(H/K)\) it follows that \(O^{\pi_n}(G/K) \leq C_G(N/K)\). From \((R/K)O^{\pi_n}(G/K) = G/K\) it follows that \(N/K\) is normal in \(G/K\). Hence \(H/K\) is not a chief factor of \(G\), a contradiction.

(d) \((R/K)^{F(p)} \leq C_G(H/K)\) for all \(p \in \pi(H/K)\).

From \(H/K \leq Z_{\mathfrak{S}_n}(R/K)\) and \(\mathfrak{S}_n \leq \mathfrak{S}\) it follows that a chief factor \(H/K\) of \(R/K\) lies in \(Z_{\mathfrak{S}}(R/K)\). Now \((R/K)/C_{R/K}(H/K) \in F(p)\) for all \(p \in \pi(H/K)\). Thus \((R/K)^{F(p)} \leq C_G(H/K)\) for all \(p \in \pi(H/K)\).

(e) \(H/K\) is a \(\mathfrak{S}\)-central chief factor of \(G\).

From \(O^{\pi_n}(G/K) \leq C_G(H/K)\), \((R/K)^{F(p)} \leq C_G(H/K)\) for all \(p \in \pi(H/K)\) and \((R/K)O^{\pi_n}(G/K) = G/K\) it follows that \(G/C_G(H/K) \in F(p)\) for all \(p \in \pi(H/K)\). Thus \(H/K\) is a \(\mathfrak{S}\)-central chief factor of \(G\).

(f) \(D \leq Z_{\mathfrak{S}}(G)\).

We showed that every chief factor of \(G\) below \(D\) is \(\mathfrak{S}\)-central. Hence \(D \leq Z_{\mathfrak{S}}(G)\).

(g) \(D \geq Z_{\mathfrak{S}}(G)\) and hence \(D = Z_{\mathfrak{S}}(G)\).

Let \(H\) be a \(\mathfrak{S}_n\)-maximal subgroup of \(G\) for some \(i \in I\). Then \(HZ_{\mathfrak{S}}(G) \in \mathfrak{S}\) by lemma 2.4. Since \(H\) is a \(\mathfrak{S}_n\)-maximal subgroup of \(G\), \(H\) is a \(\mathfrak{S}_n\)-maximal subgroup of \(HZ_{\mathfrak{S}}(G)\). So \(H < HZ_{\mathfrak{S}}(G)\). Hence \(D \geq Z_{\mathfrak{S}}(G)\). Thus \(D = Z_{\mathfrak{S}}(G)\).

(2) \(\Rightarrow\) (1) Suppose now that \(\bigcap_{i \in I} N_{\mathfrak{S}_n}(G) = Z_{\mathfrak{S}}(G)\) holds for every group \(G\). Let us show that \(\mathfrak{S}_n\) satisfies the boundary condition in the universe of all \(\pi_i\)-groups for all \(i \in I\).

Assume the contrary. Then some \(\mathfrak{S}_n\) does not satisfy the boundary condition in the universe of all \(\pi_i\)-groups. So there is \(\pi_i\)-group \(G\) such that \(\text{Int}_{\mathfrak{S}_n}(G) \neq Z_{\mathfrak{S}_n}(G)\). Note that \(\text{Int}_{\mathfrak{S}_n}(G) = N_{\mathfrak{S}_n}(G)\) by proposition 1. Since \(G\) is a \(\pi_i\)-group, \(N_{\mathfrak{S}_n}(G) = \bigcap_{i \in I} N_{\mathfrak{S}_n}(G)\). From \(\mathfrak{S}_{\pi_i} \cap \mathfrak{S} = \mathfrak{S}_n\) it follows that \(Z_{\mathfrak{S}_n}(G) = Z_{\mathfrak{S}}(G)\).

Hence \(\bigcap_{i \in I} N_{\mathfrak{S}_n}(G) = N_{\mathfrak{S}_n}(G) = \text{Int}_{\mathfrak{S}_n}(G) \neq Z_{\mathfrak{S}_n}(G) = Z_{\mathfrak{S}}(G)\), the contradiction.

3.3 Proof of proposition 2

Let \(N\) be a normal subgroup of a group \(G\) such that \(H\) is \(\mathfrak{S}\)-subnormal in \(HN\) for every \(H \in \Sigma(G)\). Let \(S\) be a \(\mathfrak{S}\)-subnormalizer in \(G\) of \(H \in \Sigma(G)\). Then \(HN/N\) is \(\mathfrak{S}\)-subnormal in \(SN/N\) by (1) of lemma 2.5. So \(HN\) is \(\mathfrak{S}\)-subnormal in \(SN\) by (2) of lemma 2.5. Hence \(H\) is \(\mathfrak{S}\)-subnormal in \(SN\) by (3) of lemma 2.5. Thus \(SN = N\). It means that \(N \leq \text{SL}_n(G)\). So every normal subgroup of \(G\) that \(\mathfrak{S}\)-subnormalize all subgroups from \(\Sigma(G)\) lies in \(\text{SL}_n(G)\).

From the other hand \(H\text{SL}_n(G)\) belongs to every \(\mathfrak{S}\)-subnormalizer of \(H\) in \(G\) for every \(H \in \Sigma(G)\). Hence \(H\) is \(\mathfrak{S}\)-subnormal in \(H\text{SL}_n(G)\) for every \(H \in \Sigma(G)\).
3.4 Proof of theorem B

(1) \Rightarrow (2) Assume that there exists a partition \( \sigma = \{ \pi_i | i \in I \} \) of \( \pi(\mathfrak{H}) \) into disjoint subsets such that \( \mathfrak{H} = \times_{i \in I} \mathfrak{H}_{\pi_i} \). Let us show that the intersection of all \( \mathfrak{H} \)-subnormalizers of all cyclic primary subgroups of a group \( G \) is the \( \mathfrak{H} \)-hypercenter of \( G \) for every group \( G \).

Note that \( \mathfrak{H} \) is local formation with the canonical local definition \( F \) where \( F(p) = \mathfrak{S}_{\pi_p} \) for \( p \in \pi_i \) for all \( i \in I \).

Let \( D \) be the intersection of all \( \mathfrak{H} \)-subnormalizers of all cyclic primary subgroups of a group \( G \) and \( H/K \) be a chief factor of \( G \) below \( D \).

(a) \( H/K \) lies in the intersection of all \( \mathfrak{H} \)-subnormalizers of all cyclic primary subgroups of a group \( G/K \).

Let \( C/K \) be a cyclic primary subgroup of \( G/K \). According to lemma 2.1 we may assume that \( C \) is a cyclic primary subgroup of \( G \). Now \( C \) is \( \mathfrak{H} \)-subnormal in \( HC/K \) by proposition 2. So \( C/K \) is \( \mathfrak{H} \)-subnormal in \( HC/K \) by (1) of lemma 2.5. Hence \( H/K \) lies in the intersection of \( \mathfrak{H} \)-subnormalizers of all cyclic primary subgroups of \( G/K \).

(b) \( H/K \in \mathfrak{H} \).

Now \( K/K \) is a \( \mathfrak{H} \)-subnormal \( \mathfrak{H} \)-subgroup of a quasinilpotent group \( H/K \). By lemma 2.6 \( H/K \in \mathfrak{H} \). Hence \( \pi(H/K) \subseteq \pi_n \) for some \( n \in I \).

(c) \( C/K \leq C_G(H/K) \) for every cyclic primary \( \pi(\mathfrak{H})' \)-subgroup of \( G/K \).

Let \( C/K \) be a cyclic primary \( \pi(\mathfrak{H})' \)-subgroup of \( G/K \). Since \( C/K \) is a \( \mathfrak{H} \)-subnormal \( \pi(\mathfrak{H})' \)-subgroup of \( HC/K \), \( (C/K)^{\mathfrak{H}} = (C/K) \) is subnormal in \( HC/K \) by (1) of lemma 6.1. Hence \( H/K \in \mathfrak{H} \) by lemma 2.6. So \( C/K \leq C_G(H/K) \).

(d) \( C/K \leq C_G(H/K) \) for every cyclic primary \( \pi(\mathfrak{H}) \cap (\pi_p') \)-subgroup of \( G/K \).

Let \( C/K \) be a cyclic primary \( \pi(\mathfrak{H}) \cap (\pi_p') \)-subgroup of \( G/K \). Since \( C/K \) is \( \mathfrak{H} \)-subnormal in \( HC/K \), \( HC/K \in \mathfrak{H} \) by lemma 2.6. So \( C/K \leq C_G(H/K) \).

(e) \( H/K \) is a \( \mathfrak{H} \)-central chief factor of \( G \) and \( D \leq Z_{\mathfrak{H}}(G) \).

From (c) and (d) it follows that \( O^{\pi_n}(G) \leq C_G(H/K) \). Hence \( G/C_G(H/K) \in \mathfrak{S}_{\pi_n} = F(p) \) for all \( p \in \pi(H/K) \). So \( H/K \) is a \( \mathfrak{H} \)-central chief factor of \( G \). It means that \( D \leq Z_{\mathfrak{H}}(G) \).

(f) \( Z_{\mathfrak{H}}(G) \leq D \) and hence \( D = Z_{\mathfrak{H}}(G) \).

Let \( \mathcal{C} \) be a cyclic \( p \)-subgroup of a group \( G \). If \( p \in \pi(\mathfrak{H}) \) then \( CZ_{\mathfrak{H}}(G) \in \mathfrak{H} \) by lemma 2.4. Hence \( C \) is \( \mathfrak{H} \)-subnormal in \( CZ_{\mathfrak{H}}(G) \). If \( p \notin \pi(\mathfrak{H}) \) then \( C \leq G^\delta \). By lemma 2.7 \( C \leq C_G(Z_{\mathfrak{H}}(G)) \). Hence \( (CZ_{\mathfrak{H}}(G))^\delta = C \). So \( C \) is \( \mathfrak{H} \)-subnormal in \( CZ_{\mathfrak{H}}(G) \). Hence \( Z_{\mathfrak{H}}(G) \leq D \). Thus \( D = Z_{\mathfrak{H}}(G) \).

(2) \Rightarrow (3) Let \( P \) be a Sylow \( p \)-subgroup of \( G \). If \( p \in \pi(\mathfrak{H}) \) then \( P \in \mathfrak{H} \) and hence \( PZ_{\mathfrak{H}}(G) \in \mathfrak{H} \) by lemma 2.4. So \( P \) is \( \mathfrak{H} \)-subnormal in \( PZ_{\mathfrak{H}}(G) \).

If \( p \notin \pi(\mathfrak{H}) \) then \( P \leq G^\delta \). By lemma 2.7 \([G^\delta, Z_{\mathfrak{H}}(G)] = 1 \). So \( PZ_{\mathfrak{H}}(G) = P \times Z_{\mathfrak{H}}(G) \). Hence \( P \) is \( \mathfrak{H} \)-subnormal in \( PZ_{\mathfrak{H}}(G) \).

Thus \( Z_{\mathfrak{H}}(G) \) lies in the intersection \( D \) of all \( \mathfrak{H} \)-subnormalizers of all Sylow subgroups of \( G \). Since the unit subgroup is \( \mathfrak{H} \)-subnormal in \( D \), we see that \( \pi(D) \subseteq \pi(\mathfrak{H}) \).

Now let \( \mathcal{C} \) be a cyclic primary \( p \)-subgroup of \( G \). Then there is a Sylow \( p \)-subgroup \( P \) of \( G \) such that \( C \leq P \). If \( p \in \pi(\mathfrak{H}) \) then \( C \) is \( \mathfrak{H} \)-subnormal in \( P \) and \( P \) is \( \mathfrak{H} \)-subnormal in \( PD \). Hence \( C \) is \( \mathfrak{H} \)-subnormal in \( PD \) and also in \( CD \) by lemma 2.5.

If \( p \notin \pi(\mathfrak{H}) \) then \( C \) is subnormal in \( P \) and \( P \) is normal in \( PD \). Hence \( C \) is subnormal in \( PD \) and also in \( CD \). So \( C \) is the normal Sylow subgroup of \( CD \). By our assumption the unit group is a Sylow subgroup. Hence \( 1 \) is \( \mathfrak{H} \)-subnormal in \( D \). Now \( C/C \) is \( \mathfrak{H} \)-subnormal in \( CD/C \). Hence \( C \) is \( \mathfrak{H} \)-subnormal in \( CD \).

Thus \( D \) lies in the intersection of all \( \mathfrak{H} \)-subnormalizers of all cyclic primary subgroups of \( G \). Hence \( Z_{\mathfrak{H}}(G) \leq D \leq Z_{\mathfrak{H}}(G) \). Thus \( D = Z_{\mathfrak{H}}(G) \).

Consider the following statement:
\((4)\) $\mathfrak{F}$ has the canonical local definition $F$ such that for every prime $p$, $F(p)$ contains every group $G$ whose all Sylow subgroups belong to $F(p)$.

\((3)\Rightarrow(4)\) Let the intersection of all $\mathfrak{F}$-subnormalizers of all Sylow subgroups of $G$ be the $\mathfrak{F}$-hypercenter of $G$ for every group $G$. Assume that there exist a prime $p$ and groups $G$ such that $G \not\in F(p)$ but for every Sylow subgroup $P$ of $G$, $P \in F(p)$. Let us chose the minimal order group $G$ from such groups.

It is clear that $O_p(G) = 1$ and $G$ has an unique minimal normal subgroup. Then by lemma 2.6 from [8] there exists a faithful irreducible $\mathfrak{F}$-module $N$ over the field $F_p$. Let $H$ be the semidirect product of $N$ and $G$. Note that $NP \in \mathfrak{F}$ for every Sylow subgroup $P$ of $H$. Hence $N$ lies in the intersection of all $\mathfrak{F}$-subnormalizers of Sylow subgroups of $H$ by proposition 2. But $H/C_H(N) \not\in F(p)$. So $N \not\subseteq Z_\mathfrak{F}(H)$, the contradiction.

\((4)\Rightarrow(1)\). Assume that $Z_q \in F(p)$ for primes $p \neq q$. Suppose that $F(q) \cap \mathfrak{N}_p \neq \mathfrak{N}_p$. Let $P$ be the minimal order $p$-group from $\mathfrak{N}_p \setminus (F(q) \cap \mathfrak{N}_p)$. Then $P$ has an unique minimal normal subgroup and $P \in F(p)$. There exists a faithful irreducible $P$-module $Q$ over the field $F_q$. Note that $Q \in F(p)$. Hence the semidirect product $G = Q \rtimes P \in F(p) \subseteq \mathfrak{F}$. Now $G/O_{q,p}(G) = G/Q \simeq P \in F(q)$, a contradiction.

So from $Z_q \in F(p)$ it follows that $F(q) \cap \mathfrak{N}_p = \mathfrak{N}_p$ and hence $F(p) \cap \mathfrak{N}_q = \mathfrak{N}_q$. So $\mathfrak{N}_{r(F(p))} \subseteq F(p)$. Let a group $G$ be a $s$-critical for $F(p)$. Since $F(p)$ is hereditary, we see that $G$ is $r$-group for some prime $r$. Now $r \not\in \pi(F(p))$. Hence $G \simeq Z_r$. It means that $F(p) = \mathfrak{G}_{\pi(F(p))}$ for all $p \in \pi(\mathfrak{F})$.

Assume now that for three different primes $p$, $q$, and $r$ we have that $\{p, q\} \subseteq \pi(F(r))$. Let us show that $q \in \pi(F(p))$. By theorem 10.3B [12] there exists a faithful irreducible $Z_q$-module $P$ over the field $F_p$. Let $G$ be the semidirect product $P$ and $Z_q$. Then $T \in F(r) \subseteq \mathfrak{F}$. Thus $G/O_{p,q}(G) = G/P \simeq Z_q \in F(p)$.

It means that there exists a partition $\sigma = \{\pi_i|i \in I\}$ of $\pi(\mathfrak{F})$ into disjoint subsets such that $F(p) = \mathfrak{G}_{\pi_i}$ for all $p \in \pi_i$ and for all $i \in I$. Now $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$.

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